Explicit coverings of families of elliptic surfaces by squares of curves

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Abstract
We show that, for each \( n > 0 \), there is a family of elliptic surfaces which are covered by the square of a curve of genus \( 2n + 1 \), and whose Hodge structures have an action by \( \mathbb{Q}(\sqrt{-n}) \). By considering the case \( n = 3 \), we show that one particular family of K3 surfaces are covered by the squares of curves of genus 7. Using this, we construct a correspondence between the square of a curve of genus 7 and a general K3 surface in \( \mathbb{P}^4 \) with 15 ordinary double points up to a map of finite degree of K3 surfaces. This gives an explicit proof of the Kuga–Satake–Deligne correspondence for these K3 surfaces and any K3 surfaces related to them by maps of finite degree, and further, a proof of the Hodge conjecture for the squares of these surfaces. We conclude that the motives of these surfaces are Kimura-finite. Our analysis gives a birational equivalence between a moduli space of curves with additional data and the moduli space of these K3 surfaces with a specific elliptic fibration.

Keywords K3 surfaces · Motives · Kuga–Satake · Hodge conjecture · Kimura-finite · Pure motives · Coverings by curves

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1 Statement of results

The work of Kuga and Satake [25] and Deligne [9] demonstrates that, if one assumes the Hodge conjecture, every K3 surface over \( \mathbb{C} \) is of abelian Hodge type, i.e., there exists an algebraic correspondence between any K3 surface and an associated abelian variety. This would imply that the motive of any such K3 surface is abelian, and hence that the variety, by an appropriate Torelli-type argument, is completely determined by linear data associated to it.

More specifically (see, e.g., [40, section 10]), given a polarized Hodge structure \( V \) of weight 2 with \( \dim V(2,0) = 1 \), there exists an abelian variety \( A \), the Kuga–Satake variety of \( V \), such that \( V \) is a sub-Hodge structure of \( H^2(A \times A, \mathbb{Q}) \). When there is another variety \( X \) with \( V \hookrightarrow H^2(X, \mathbb{Q}) \), the Hodge conjecture on \( A^2 \times X \) predicts the existence of an algebraic cycle \( Z \subset A^2 \times X \), the Kuga–Satake–Deligne correspondence, which realizes the morphism of Hodge structures

\[
H^2(A^2, \mathbb{Q}) \to V \to H^2(X, \mathbb{Q}).
\]

This morphism is particularly nice when \( X \) is a K3 surface, or more generally when \( \dim H^2(2,0)(X, \mathbb{Q}) = 1 \). In this case, \( V \) can be identified with the orthogonal complement of the Néron-Severi group of \( X \), i.e.

\[
H^2(X, \mathbb{Q}) = V \oplus NS(X) \mathbb{Q}
\]

and \( Z \) induces an isomorphism of \( V \subset H^2(A^2, \mathbb{Q}) \) with \( V \subset H^2(X, \mathbb{Q}) \).

In [36], Paranjape gives a method for computing an explicit cover, by the squares of curves of genus 5, of K3 surfaces \( X \) of Picard rank 16 with an action of \( \mathbb{Q}(\sqrt{-1}) \) on their Hodge lattices, and shows that this product computes the abelian variety predicted by Kuga and Satake. In turn this not only gives a constructive proof of the Kuga–Satake–Deligne correspondence in this special case, as one can take \( Z \) to be the square of the aforementioned curve of genus 5, but also a proof of the Hodge conjecture for \( X \times X \) [38]. Note that to construct the curve and this cover of his given K3, Paranjape is first forced to construct intermediate curves and the surfaces their squares cover, so the construction is more delicate than it might at first appear. The question of which other families of K3 surfaces his method can be generalized to is left open.

In [17], Garbagnati and Sarti characterize Picard lattices of K3 surfaces in \( \mathbb{P}^n \) with 15 nodes. The simplest examples of such surfaces, namely, those K3s given by a double cover of the plane branched along six lines, were already known classically and satisfy the condition studied by Paranjape above. The family of K3 surfaces in \( \mathbb{P}^4 \) with 15 ordinary double points seems to have first appeared in their paper [17].

In this paper we show that a variant of Paranjape’s method does generalize: indeed, for each \( n \), we show (Theorem 3.56) that there are families of elliptic surfaces which are covered by the square of a curve of genus \( 2n + 1 \). The properties of these surfaces are listed in Definition 2.24. In particular, applying the work of Garbagnati and Sarti, we show the following results:

1. up to maps of K3 surfaces of finite degree, there is an explicitly computable correspondence between the square of a curve of genus 7 and a K3 surface \( Y \) in \( \mathbb{P}^4 \) with 15 ordinary double points (Propositions 6.2, 6.3);
2. the construction of the curve is unique in the sense of Theorem 1.10 below;
3. the construction does in fact produce the family of abelian varieties predicted by Kuga and Satake and constructively compute the Kuga–Satake–Deligne correspondence (Propositions 4.3, 6.3); and, moreover,
(4) the construction gives a proof of the Hodge conjecture for $Y \times Y$ and the square of any K3 surface $X$ for which there is a sequence of K3 surfaces $X = X_0, X_1, \ldots, X_n = Y$ such that for all $1 \leq i \leq n$ there is either a rational map of finite degree from $X_{i-1}$ to $X_i$ or from $X_1$ to $X_{i-1}$ (Theorems 4.8, 6.4 Corollary 6.15).

It would be interesting to use the methods of Schlickewei [38] to prove the Hodge conjecture for $A^2 \times Y'$ as well, where $Y'$ is any K3 surface related to $Y$ by maps of finite degree.

Interestingly, our variant of Paranjape’s construction only yields an explicit covering of K3 surfaces by squares of curves in the cases $n = 2, 3, 4$, and 6. Numerical calculations prove that the case $n = 5$ does not produce a cover of K3 surfaces (Remark 3.44), and we suspect that the cases $n = 2, 3, 4$, and 6 are the only $n$ for which the elliptic surfaces so covered admit maps to K3 surfaces, and hence for which we can constructively prove the Kuga–Satake–Deligne correspondence. Indeed, in the cases $n = 2, 4$ we recover special cases of Paranjape’s construction (see Remarks 3.42, 3.43), and we expect (Remark 3.44) that the case $n = 6$ recovers a special case of our construction (Theorem 1.6 below).

We have focused on the special case of K3 surfaces in part because they are well known surfaces. We are cognizant of the fact that there may be other surfaces that have interesting geometric or arithmetic properties that come up in (generalizations of) our construction. In particular, it would be interesting to see if the elliptic surfaces we construct for arbitrary geometric or arithmetic properties that come up in (generalizations of) our construction. In particular, it would be interesting to see if the elliptic surfaces we construct for arbitrary values of $n$ encode any useful arithmetic data.

We now state our main results with some preliminary definitions. Throughout the paper we work over a field $k$ of characteristic not 2 or 3.

**Definition 1.1** [37, Sections 6.4, 8.6, 11.1] Let $Y$ be a K3 surface and suppose that $Y$ admits an elliptic fibration $\pi : Y \to \mathbb{P}^1$ with section $\sigma_\pi$. Let $f_\pi$ be a fibre of $\pi$. The **hyperbolic plane** $\mathbb{H}$ is the lattice generated by two vectors $x, y$ such that $(x, x) = (y, y) = 0$ and $(x, y) = 1$. The **frame** $W(\pi)$ is the orthogonal complement of the sublattice of $\text{Pic } Y$ generated by the classes of $\sigma_\pi$ and $f_\pi$. The **trivial lattice** $T(\pi)$ is the sublattice of $\text{Pic } Y$ generated by the classes of $\sigma_\pi$ and all components of fibres. The **essential lattice** $L(\pi)$ is the orthogonal complement of $T(\pi)$ in $\text{Pic } Y$.

**Remark 1.2** The isomorphism classes of the lattices introduced in Definition 1.1 are independent of the choice of $\sigma_\pi$. Note also that $\text{Pic } Y = \langle W(\pi), [\sigma_\pi], [f_\pi] \rangle$, since the lattice generated by $[\sigma_\pi], [f_\pi]$ is isomorphic to $\mathbb{H}$ and is therefore unimodular.

**Definition 1.3** Let $L_1$ be the Picard lattice of a K3 surface with an elliptic fibration with one $\tilde{D}_4$ and nine $\tilde{A}_1$ fibres and Mordell–Weil group $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$, such that there is a generator of the Mordell–Weil group modulo torsion that passes through the zero component of the $\tilde{D}_4$ fibre and the nonzero component of every $\tilde{A}_1$ fibre. Let $\Lambda_1$ be the frame of $L_1$.

**Definition 1.4** Let $X, X'$ be K3 surfaces. We say that $X$ and $X'$ are **isogenous** if there is a finite sequence $X = X_0, X_1, \ldots, X_n = X'$ of K3 surfaces such that, for all $0 \leq i < n$, there exists a rational map of finite degree from $X_i$ to $X_{i+1}$ or from $X_{i+1} \to X_i$. For a prime $p$ we say that $X$ and $X'$ are $p$-isogenous if there exist elliptic fibrations $\pi : X \to \mathbb{P}^1$ and $\pi' : X \to \mathbb{P}^1$ such that the generic fibres are $p$-isogenous elliptic curves; this implies that there exist maps of degree $p$ from $X$ to $X'$ and from $X'$ to $X$. In the case $p = 2$, the existence of a rational 2-torsion section on $X$ gives an involution $\iota : X \to X$ by which the quotient is birationally equivalent to $X'$; this is sometimes known as a **van Geemen–Sarti involution**.

**Remark 1.5** Sometimes two K3 surfaces $X, X'$ over $\mathbb{C}$ are called isogenous under the strictly weaker condition that there is an isomorphism of integral or rational Hodge structures $\phi :$
By results of Mukai (see for example the introduction of [3]) it is known that the class of any Hodge isometry (under this weaker definition) is algebraic. This in particular implies the Hodge conjecture for self-products of K3 surfaces over \( \mathbb{C} \) with complex multiplication (and elliptic surfaces over \( \mathbb{C} \) with complex multiplication more generally by the work of Nikulin). However, the results described in this paper are stronger and more general. In particular, they do not depend on the surface in question being defined over the complex numbers, are based on explicit constructions, and have applications (such as to motive-finiteness) not deducible from the above results. Thus in this paper we will work only with isogenies in the strong sense, even though some of our results extend to this more general context.

**Theorem 1.6** A general K3 surface whose Picard lattice has a primitive sublattice isometric to \( L_1 \) is covered by the square of a curve of genus 7.

We will see in Sect. 6 that this implies the following:

**Theorem 1.7** A general K3 surface in \( \mathbb{P}^4 \) with 15 ordinary double points is isogenous to a K3 surface \( K \) which realizes the Kuga–Satake–Deligne correspondence between \( K \) and the square of a curve of genus 7 constructed in Theorem 1.6.

**Theorem 1.6** can also be expressed in terms of a map of moduli spaces as follows:

**Definition 1.8** For a curve \( C \) of genus 1, let \( T_2(C) \) be the group of translations of \( C \) of order 1 or 2. Let \( \mathcal{M}_C \) be the moduli space that parametrizes curves \( C_1 \) of genus 1 with an unordered set of 4 distinct points \( \{p_1, \ldots, p_4\} \), together with \( O \in C_1/T_2(C_1) \) such that \( 4O \sim p_1 + p_2 + p_3 + p_4 \) and an irreducible unramified cover of \( C_1 \) of degree 3. Let \( \mathcal{M}_{K,1} \) be the moduli space of marked \( L_1 \)-polarized K3 surfaces as in [19, Theorem 9], together with a choice of elliptic fibration with one reducible fibre of type \( \tilde{D}_4 \) and 9 of type \( \tilde{A}_1 \) and Mordell–Weil group \( \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 \), such that a generator of infinite order has intersection 1 with the zero section and 0 with the sections of order 2. The choice of the elliptic fibration is equivalent to the choice of an embedding of \( \mathbb{H} \) into \( \text{Pic} \ L \) such that the orthogonal complement is isometric to \( \Lambda_1 \).

**Remark 1.9** Although \( C_1 \) and \( C_1/T_2(C_1) \) are isomorphic varieties, it is necessary to distinguish them here, because replacing \( O \) by a point \( t_2(O) \) for \( t_2 \) a nonidentity element of \( T_2(C_1) \) would ultimately lead to the same K3 surface.

**Theorem 1.10** Given a general point \( P \) of \( \mathcal{M}_C \), there is an explicit construction of a point \( f(P) \) of \( \mathcal{M}_{K,1} \), such that the data parametrizing \( f(P) \) can be built from the data parametrizing \( P \), and vice versa. In particular, these constructions give a modular birational equivalence \( f: \mathcal{M}_C \to \mathcal{M}_{K,1} \).

Theorems 1.6 and 1.10 follow directly from Remark 3.4 and Theorem 3.54, while Theorem 1.7 follows from Propositions 4.3 and 6.3.

Our work shows that generic members of our families of elliptic surfaces are covered by the square of a curve. It follows that they, and generic members of any family of K3 surfaces that are related (via isogenies and correspondences) to these families of elliptic surfaces, also have Kimura-finite (and hence Schur-finite) motive. (See [10, 21] or [30] for a definition of Kimura finiteness. See for example [27] for a discussion of motives known to be Kimura-finite.) Since the Kimura-finiteness of a motive is invariant under maps of K3 surfaces of
finite degree, as in the proof of [27, Theorem 3.1], this implies Kimura-finiteness and Schur-finiteness for a large class of K3 surfaces. In this paper, we will refer to this property as motive-finiteness.

The property of motive-finiteness has received some attention in previous work, but for K3 surfaces it is not well understood. Certainly every K3 surface isogenous to an algebraic Kummer surface is motive-finite, but such Kummer surfaces all have Picard number at least 17. The work of Paranjape [36] and Laterveer [27] give some examples with Picard number 16. In addition, Garbagnati and Penegini [16] studied the families of K3 surfaces that are of the form \((C_1 \times C_2)/G\), where \(C_1, C_2\) are curves and \(G\) acts diagonally. We remark that our construction is different from theirs in that our K3 surfaces are of the form \(((C_1 \times C_2)/G_1)/G_2\), where \((C_1 \times C_2)/G_1\) is not a K3 surface.

We thank the referee for showing us that the surface \(W\) constructed in this paper is a “surface isogenous to a product of mixed type”, as studied in [5, 13, 14]. These surfaces are minimal resolutions of a quotient of a surface of the form \(C \times C\) by a subgroup of the wreath product of Aut \(C\) by \(S_2\), where \(C\) is a curve. In particular, the description of the singularities of \(W\) in Prop. 3.16 follows from these works.

One new feature of our work is that we show that a K3 surface in our family arises from an essentially unique curve. This means that the fields of moduli of the curve and the K3 surface are the same, a fact that is useful in arithmetical applications. We expect that it would be possible to prove an analogous result for the construction of [36].

The plan for the paper is as follows. In Sect. 2 we define the various moduli spaces of curves and elliptic surfaces, including K3 surfaces, that we will use in the paper, and indicate some birational equivalences between them. In Sect. 3 we give an alternative, and explicit, construction of the birational equivalence for \(n = 3\); namely, starting with a rational curve together with a cover of degree 3 and a point of ramification of the cover, we construct a curve of genus 7 whose square covers a K3 surface. In Sect. 4, we reinterpret the construction of Sect. 3 in terms of Hodge theory (Proposition 4.5), and in particular show that in characteristic zero we have constructed the Kuga–Satake variety for our K3 surfaces (Proposition 4.3). In Sect. 5 we explore the limits of our construction from Sect. 3. In particular, we show two results by explicit calculation. First, not every K3 of rank 16 whose Picard lattice has determinant \(-12t^2\) is coverable by the square of a curve using our construction (Example 5.7). Second, our construction, though it induces a covering of some Picard rank 17 and 18 non-Kummer K3 surfaces not previously known to be covered by squares of curves, does not determine coverings for all Picard rank 17 and 18 K3s not isomorphic to Kummers. Finally, in Sect. 6, we discuss Picard lattices of various families of K3 surfaces, and in particular relate our construction from Sect. 3 to K3 surfaces of degree 6 with 15 singularities of type \(A_1\) (Propositions 6.2, 6.3, Theorem 6.4).

2 The construction in terms of moduli spaces

In this section, we introduce some moduli spaces of curves and of elliptic surfaces, including K3 surfaces, with additional structure and indicate some birational equivalences among them. Our main goal is to pave the way for the construction in Sect. 3, in which we will show how to use a small amount of starting data on a rational curve, namely a cover of degree 3 and a point of ramification of the cover, to construct a curve of genus 7 whose square covers a K3 surface. This will give an alternative birational equivalence between two of our moduli spaces, and will show that the K3 surfaces in question have Kimura-finite motive and that the
Kuga–Satake–Deligne correspondence in this case is realized by a correspondence between the surface and the square of an abelian variety.

\section{2.1 Moduli spaces of elliptic K3 surfaces}

We begin by defining the moduli spaces of K3 surfaces that are of interest.

\textbf{Definition 2.1} Let $\mathcal{M}_{K,1}$ be the moduli space of K3 surfaces together with an elliptic fibration $\phi_1$ with a reducible fibre of type $\tilde{D}_4$ and nine of type $\tilde{A}_1$, full level-2 structure, and a section of infinite order that passes through the zero component of the $\tilde{D}_4$ fibre and the nonzero components of all $\tilde{A}_1$ fibres while meeting the 0 section once. Let $\mathcal{M}_{K,1}'$ be the 6-to-1 cover of $\mathcal{M}_{K,1}$ that parametrizes the same data but with an additional choice of labelling of the sections of order 2.

\textbf{Remark 2.2} When we discuss K3 surfaces in terms of their Picard lattices, as mostly in this section and in Sect. 6, we use the ADE notation for reducible fibres, since the Picard lattice does not contain the more refined information present in the Kodaira classification. On the other hand, in studying a K3 surface as a variety, as we will in Sect. 3, it is better to keep track of this information. We recall the correspondence between the two notations, as in [31, Table I.4.1]:

| ADE type | Kodaira symbol |
|----------|----------------|
| $\tilde{D}_4$ | $I_0, I_1, I_2$ |
| $\tilde{A}_1$ | $I_3, I_4$ |
| $\tilde{A}_2$ | $I_5, I_6$ |
| $\tilde{A}_n$ ($n > 2$) | $I_{n+1}$ |
| $\tilde{E}_6$ | $I_{n-4}$ |
| $\tilde{E}_7$ | $I_7$ |
| $\tilde{E}_8$ | $I_8$ |

\textbf{Remark 2.3} Now let us perform the standard calculation to determine the intersection of the torsion sections with the reducible fibres and the reducible fibres on the quotients of the surface by the 2-torsion translation automorphisms. The $\tilde{D}_4$ fibre goes to a $\tilde{D}_4$ fibre, and the $\tilde{A}_1$ fibres go to $\tilde{A}_3$ fibres if the section passes through the zero component and singular irreducible fibres otherwise: see [11, Table 1].

As in [37, Lemma 7.3] each 2-torsion section passes through a different nonzero component of the $\tilde{D}_4$ fibre. The quotient is a K3 surface, so its Euler characteristic is 24. It follows that each 2-torsion section passes through the zero component of exactly three $\tilde{A}_1$ fibres. (Alternatively, this follows from [8, Lemma 1.15]: the height of a torsion section must be 0, and from [8, Table 1.14] it is equal to $4 - 1 - n_1/2$, where $n_1$ is the number of $\tilde{A}_1$ fibres such that the section passes through the nonzero component.) By considering the group structure on such a fibre, we see that two of the 2-torsion sections pass through the nonzero component and one through the zero component. Hence we have partitioned the $\tilde{A}_1$ fibres into three sets of size 3. For $\mathcal{M}_{K,1}'$ this becomes an ordered partition.

We now define another moduli space $\mathcal{M}_{K,2}'$ of elliptic surfaces and show that it is birationally equivalent to $\mathcal{M}_{K,1}'$. In terms of lattices, we may describe the situation as follows.
The frames $\Lambda_1$, $\Lambda_2$ of the two types of fibration are in the same genus. In addition, the images of the automorphism groups of $\Lambda_1$, $\Lambda_2$ in the automorphism group of the discriminant group are conjugate, so the two types of fibration are determined by each other; cf. [15, Theorem 2.8]. For our purposes, we need to make the equivalence explicit; it is not enough to know that the fibrations of the two types are in canonical bijection.

**Remark 2.4** The statement that the genera of $\Lambda_1$, $\Lambda_2$ are the same follows by exhibiting an isomorphism of the discriminant groups together with their quadratic forms, or more simply by enumerating the genus of $\Lambda_1$ and noticing that it contains a lattice isometric to $\Lambda_2$. This statement also follows from Proposition 2.7. However, there are many possible frames of elliptic fibrations on K3 surfaces with Picard lattice $L_1$; if we had not done the lattice-theoretic calculation first, we would have found the statement of Proposition 2.7 quite unmotivated.

**Definition 2.5** Let $\mathcal{M}_{K,2}$ be the moduli space of K3 surfaces with a specified elliptic fibration with three $D_4$ fibres and one $\tilde{A}_2$ fibre (and generically trivial Mordell–Weil group). Let $\mathcal{M}'_{K,2}$ be the cover of $\mathcal{M}_{K,2}$ that also keeps track of a labelling of the $D_4$ fibres. We will denote a point of $\mathcal{M}'_{K,2}$ by $(S, \phi_2, F_1, F_2, F_3)$ where $\phi_2$ is the fibration and the $F_i$ are the $\tilde{D}_4$ fibres in order.

**Remark 2.6** The dimensions of $\mathcal{M}_{K,1}$ and $\mathcal{M}_{K,2}$ are 4, because they parametrize K3 surfaces of Picard number 16 together with a finite amount of additional data. The same holds for the $\mathcal{M}'_{K,i}$, because these are finite covers of the $\mathcal{M}_{K,i}$.

**Proposition 2.7** There is an $S_3$-equivariant birational equivalence between $\mathcal{M}'_{K,1}$ and $\mathcal{M}'_{K,2}$.

**Proof** First let $(S, \phi_1, T_1, T_2, T_3, G)$ be the data of a point of $\mathcal{M}'_{K,1}$, where the $T_i$ are the labelled torsion sections and $G$ is the generator of infinite order. Let $D_i$ be the set of curves on $S$ consisting of $T_i$, the zero components of $\tilde{A}_1$ fibres of $\phi_1$ that $T_i$ meets, and the component of the $\tilde{D}_4$ that $T_i$ meets. Since the $\tilde{D}_4$ and $\tilde{A}_1$ are distinct fibres of $\phi_1$, the curves in them are disjoint, and so there is a $\tilde{D}_4$ fibre supported on $D_i$ whose nonreduced component is $T_i$. We now find an $\tilde{A}_2$ fibre. Indeed, consider the zero component of the $\tilde{D}_4$ fibre of $\phi_1$ together with the curves $G$, $-G$. From our description of the generator in Proposition 2.15, which pulls back to $S$, we see that $G$ and its inverse meet the zero section $0_{\phi_1}$ once each and pass through the nonzero components of all $\tilde{A}_1$ fibres and the zero component of the $\tilde{D}_4$ fibre. Let us denote the height pairing on sections by $(x, y)$ and the intersection pairing on the surface by $x \cdot y$. Applying the formulas of [8], in particular Lemma 1.18 and Table 1.19, we find that $(G, G) = -(0_{\phi_1} - G)^2 - \sum D(G) = 4 + 2 - 9(1/2) = 3/2$, where the $D(G)$ are the local correction terms of [8], so that $(G, -G) = -3/2$. Then we have $(G, -G) = -(0_{\phi_1} - G) - (0_{\phi_1} - (-G)) - \sum G \cdot D(-G)$ and so $-3/2 = 2+1+1-G \cdot -G/2$ and $G \cdot -G = 1$. It is easily checked that the Picard classes of the three $\tilde{D}_4$ fibres and the $\tilde{A}_2$ fibre just mentioned are equal, so there is a genus-1 fibration with these reducible fibres. These fibrations meet the central component of the $\tilde{D}_4$ fibre of $\phi_1$ in a single point, so it is an elliptic fibration. This gives a map $\mathcal{M}'_{K,1} \rightarrow \mathcal{M}'_{K,2}$.

Conversely, suppose given $(S, \phi_2, F_1, F_2, F_3)$, a point of $\mathcal{M}'_{K,2}$. To find its image in $\mathcal{M}'_{K,1}$, first consider the zero section and the zero components of the reducible fibres of $\phi_2$; these constitute a $\tilde{D}_4$ configuration $D$. This gives an elliptic fibration $\phi_1$, since there are sections such as the nonzero components of the $\tilde{A}_2$ fibre. For a general point of $\mathcal{M}'_{K,2}$, the surface $S$ has Picard lattice isometric to $\mathbb{H} + D_4^3 + A_2$, and one computes that the orthogonal
complement of the lattice spanned by $D$ and a section has root sublattice $D_4 + A_1^g$ and that this root lattice is embedded in its saturation with quotient $(\mathbb{Z}/2\mathbb{Z})^2$.

The reduced nonzero components of the $D_4$ fibres of $\phi_2$ are disjoint from $D$, so they are vertical for $\phi_1$. Since any two of them are disjoint and no reducible fibre of $\phi_1$ other than the $D_4$ contains two disjoint curves, they must be components of the nine different $A_1$ fibres of $\phi_1$. Further, the nonreduced components of the $D_4$ fibres of $\phi_2$ meet $D$ once, so they are sections, and it is easily computed that the difference of any two is of order 2. The labelling of these is given by the labelling of the $D_4$ fibres of $\phi_1$. Finally, the section of infinite order and its inverse are given by the two nonzero components of the $A_2$ fibre. This is not really a choice, because there is an automorphism of $S$ that takes $G$ to $-G$ and preserves $\phi_1$ and the $T_i$. Thus we have constructed a map $\mathcal{M}'_{K,1} \rightarrow \mathcal{M}'_{K,2}$, and it is clear that this is a birational inverse to the map $\mathcal{M}'_{K,2} \rightarrow \mathcal{M}'_{K,1}$ constructed above. \hfill \Box

We illustrate the construction in figures. In Fig. 1, we have drawn curves in an elliptic fibration $\phi_1 : S \rightarrow \mathbb{P}^1$. The 0-section $0_{\phi_1}$ is the black horizontal curve, and we have also indicated the 0-section of the fibration $\phi_2 : S \rightarrow \mathbb{P}^1$ as $0_{\phi_2}$. We have indicated the three $D_4$ fibres of $\phi_2$ with their central components $T_1, T_2, T_3$ in green, blue and red respectively. The $A_2$ fibre has been drawn in orange. In addition, the nonzero components of the $9$ $A_1$ fibres have been drawn in black.

In Fig. 2, we have drawn curves in an elliptic fibration $\phi_2 : S \rightarrow \mathbb{P}^1$. We have indicated the three $D_4$ fibres $F_1, F_2, F_3$ with their central components $T_1, T_2, T_3$ in green, blue and red respectively. The $A_2$ fibre has been drawn in orange. The 0-section, $0_{\phi_2}$, is the black horizontal curve. Note that the sets of curves of the same colour in each figure are equal; this reflects the bijection between the elliptic fibrations of each type on $S$ illustrated in the two figures.

**Proposition 2.8** Let $\phi_1$ be the fibration associated to a point of $\mathcal{M}'_{K,1}$, let $0_{\phi_1}$ be its zero section, and let $\phi_2$ be the fibration constructed in the proof of Proposition 2.7. Then the intersection of $0_{\phi_1}$ with a fibre of $\phi_2$ is 3, and $\phi_2$ restricted to $0_{\phi_1}$ is ramified at the intersection of $0_{\phi_1}$ with the $A_2$ fibre of $\phi_2$.

**Proof** The first statement is a routine calculation in light of the description of the reducible fibres of $\phi_2$. As for the second, two of the components of the $A_2$ fibre are a generating section of $\phi_1$ and its inverse. These meet the zero section at the same point: on the fibre of $\phi_1$ where the generating section meets the zero section, translation by the generating section acts as the identity, and applying its inverse to the zero section gives the intersection with the negative of the generating section. \hfill \Box

### 2.2 Moduli spaces of covers of curves of genus 0

We now shift our attention to some moduli spaces of covers of curves of genus 0 that will turn out to be connected to $\mathcal{M}'_{K,1}$. We use the standard notation $\overline{\mathcal{M}}_{g,n}$ [23] for the usual compactification of the moduli space of stable curves of genus $g$ with $n$ marked points. We now introduce a subvariety of $\overline{\mathcal{M}}_{0,10}$ whose quotients by finite groups we will relate to $\mathcal{M}'_{K,1}$ and $\mathcal{M}'_{K,1}$, as well as a moduli space of covers of curves of genus 0 that will be connected to $\mathcal{M}'_{K,1}$.

**Convention 2.9** We view $\overline{\mathcal{M}}_{0,4}$ not as an abstract $\mathbb{P}^1$ but rather as a $\mathbb{P}^1$ with three marked points $b_1, b_2, b_3$, namely the boundary divisors; as a result $\text{Aut} \overline{\mathcal{M}}_{0,4}$ is trivial. Thus when we
Fig. 1 Elliptic fibration in $\mathcal{M}'_{K,1}$

Fig. 2 Elliptic fibration in $\mathcal{M}'_{K,2}$
speak of a moduli space of covers of \( \overline{M}_{0,4} \), we do not take the quotient by automorphisms of the target, but only of the source. In addition, there is a natural isomorphism from \( \overline{M}_{0,4} \) to the modular curve \( X(2) \) parametrizing elliptic curves with a basis for the 2-torsion given on \( M_{0,4} \) by taking a set of 4 points to the double cover of \( \mathbb{P}^1 \) branched there, with the first marked point going to the origin and the second and third to the basis. In turn, there is a 6-to-1 map from \( X(2) \) to the \( j \)-line \( \overline{M}_{1,1} \) given by forgetting the basis. It is a Galois cover of curves with Galois group \( S_3 \). The support of the fibre of the composed map \( \overline{M}_{0,4} \to \overline{M}_{1,1} \) above \( \infty \) in the \( j \)-line is the set of boundary points of \( \overline{M}_{0,4} \); in particular these are points of ramification of the cover.

**Definition 2.10** Let \( U \subset \overline{M}_{0,10} \) be the subvariety of \( \overline{M}_{0,10} \) parametrizing stable curves \( C_0 \) of genus 0 with marked points \( p_1 \) and \( q_{ij} \) for \( 1 \leq i, j \leq 3 \) such that there is a 3-to-1 cover \( \phi : C_0 \to \overline{M}_{0,4} \) for which the \( q_{ij} \) constitute the fibre above the boundary points \( b_i \), while \((p_1, q, q)\) is a scheme-theoretic fibre for some \( q \in C_0 \). Note that \( S_3 \wr S_3 \) acts on \( U \) by permuting the \( q_{ij} \). Let \( V = U/S_3 \wr S_3 \) and let \( V' = U/S_3^3 \).

Fix a positive integer \( d \). Let \( \mathcal{M}_d \) be the moduli space parametrizing \( d \)-to-1 covers \( \phi : C_0 \to \overline{M}_{0,4} \) together with a point \( p \in \overline{M}_{0,4} \).

**Remark 2.11** The dimension of \( U \) is 4, because a point of \( U \) is specified by a 3-to-1 cover \( C_0 \to \overline{M}_{0,4} \), which is determined by a choice of two sections of \( O_{C_0}(3) \) up to scaling and the action of \( PGL_2 = \text{Aut} C_0 \), together with a finite amount of additional data. The same follows for \( V, V' \). In general we have \( \dim \mathcal{M}_d = 2d - 1 \).

First we will describe a relation between points of \( \overline{M}_{0,4} \) and certain rational elliptic surfaces. By taking covers, we will relate this to \( \mathcal{M}_d \). We start by recalling the standard description of \( \overline{M}_{0,5} \) [6, Exercise 1.3.10, Lemma 2.1].

**Definition 2.12** We identify \( \overline{M}_{0,5} \) with \( \mathbb{P}^2 \) blown up in four general points \( p_0, \ldots, p_3 \). To do so, let \( E_i \) be the exceptional divisor above \( p_i \) and \( E_{ij} \) the strict transform of the line joining \( p_i, p_j \). Then the \( E_i, E_{ij} \) are the ten \( -1 \)-curves on \( \overline{M}_{0,5} \). Let \( \pi_{5,4} \) be the projection away from \( p_0 \); it gives a map to \( \mathbb{P}^1 \) whose general fibre is a smooth rational curve and that has three singular reducible fibres \( E_i \cup E_0 \) with \( 1 \leq i \leq 3 \). In other words, the map \( \pi_{5,4} \) makes \( \overline{M}_{0,5} \) into a conic bundle over \( \overline{M}_{0,4} \). There are four sections \( E_0, E_{12}, E_{13}, E_{23} \), and this family is isomorphic to the universal curve over \( \overline{M}_{0,4} \).

**Construction 2.13** Fix \( p \in \overline{M}_{0,4} \), and let \( F = \pi_{5,4}^{-1}(p) \) be the corresponding fibre. Consider the double cover \( \mathcal{E}_p \) of \( \overline{M}_{0,5} \) branched along \( F \) and the four sections. (It is easy to see that the sum of the classes of these curves is divisible by 2 in \( \text{Pic}(\overline{M}_{0,5}) \); since \( \text{Pic}(\overline{M}_{0,5}) \) is torsion-free, the double cover is unique up to isomorphism and quadratic twist. The properties that we will discuss do not depend on the choice of twist.) This is an elliptic surface, because the general fibre is a double cover of \( \mathbb{P}^1 \) branched at 4 points. The surface \( \mathcal{E} \) has fibres of type \( \tilde{A}_1 \) above the three reducible fibres of \( \overline{M}_{0,5} \to \overline{M}_{0,4} \) and a \( \tilde{D}_4 \) above \( F \). Also the four sections of \( \overline{M}_{0,5} \to \overline{M}_{0,4} \) pull back to sections; taking \( E_0 \) as the zero section, the other three are the 2-torsion sections.

**Proposition 2.14** The association \( p \to \mathcal{E}_p \) gives a birational equivalence between \( \overline{M}_{0,4} \) and the moduli space of rational surfaces with an elliptic fibration with a \( \tilde{D}_4 \) and three labelled \( \tilde{A}_1 \) fibres.

**Proof** First note that there is an obvious one-sided inverse to the map just constructed, namely the map that takes a fibration to the point \( p \) such that \( p, b_1, b_2, b_3 \) are the locations of the

\[ \square \text{ Springer} \]
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\[ \tilde{D}_4 \text{ and } \tilde{A}_1 \text{ fibres. Given such an elliptic surface } \mathcal{E}, \text{ we obtain a map } \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,4} \text{ from the family of curves } \mathcal{E}/\pm 1 \to \mathbb{P}^1, \text{ where } \mathbb{P}^1 \text{ is the base of the fibration. This map determines the } j\text{-invariant of the fibration. It is clear that the inverse images of the } b_i \text{ have degree } 1, \text{ so this map is an isomorphism and is therefore uniquely determined (recall Convention 2.9 on automorphisms of } \overline{\mathcal{M}}_{0,4}). \text{ Thus all such surfaces are twists of a fixed one; in particular, given one such surface with a } \tilde{D}_4 \text{ fibre above } p_0, \text{ we obtain all of them by twisting by a function with divisor } (p_0) - (p). \text{ Clearly these are parametrized by points of } \overline{\mathcal{M}}_{0,4}. \]

**Proposition 2.15** The Mordell–Weil group of \( \mathcal{E}_p \) is isomorphic to \( \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 \), and the components of the pullback of the conic through \( p_0, \ldots, p_3 \) tangent to \( F \) at \( p_0 \) are a generator modulo torsion and its inverse. These curves pass through the zero component of the \( \tilde{D}_4 \) fibre and the nonzero components of the \( \tilde{A}_1 \).

**Proof** Since \( \mathcal{E}_p \) is a rational elliptic surface, its Picard lattice has rank 10 and discriminant \(-1\). It follows by the Shioda-Tate formula that the rank of the Mordell–Weil group is 1. We know that the torsion subgroup contains \((\mathbb{Z}/2\mathbb{Z})^2\); by [7, Proposition 2.1], it is no larger. Thus the canonical height of a generator is \( 4^2/(4 \cdot 2^3) = 1/2 \). Let \( C \) be the conic in the statement: then the intersection of \( C \) with the ramification locus of \( \mathcal{E}_p \to \overline{\mathcal{M}}_{0,5} \) is twice a divisor, so \( C \) pulls back to the union of two curves. The sum of these meets a fibre twice, so each one must be a section. Taking the curve above \( E_0 \) as the origin, we see that the two sections intersect every good fibre in a point and its negative; it follows that they are inverses of each other. It is clear that these sections meet the zero component of the \( \tilde{D}_4 \) (coming from the intersection of \( F \) with the exceptional divisor above \( p_0 \)) and the nonzero components of the \( \tilde{A}_1 \) (the \( E_{ij} \) with \( 1 \leq i < j \leq 3 \)). In addition, they meet the zero section once. It now follows easily from the results of [8] that they have height 1/2. \( \square \)

Given a point of \( \mathcal{M}_d \), we thus obtain an elliptic surface \( \mathcal{E}_p \times \overline{\mathcal{M}}_{0,4} \cdot C_0 \), which generically has \( 3d \) fibres of type \( \tilde{A}_1 \) and \( d \) of type \( \tilde{D}_4 \). In particular, for \( d = 2 \) these are special double covers of \( \mathbb{P}^2 \) branched along six lines, which are K3 surfaces. The elliptic fibrations on a double cover of \( \mathbb{P}^2 \) ramified along six lines in general position are classified in [22, Theorem 1.1, Corollary 1.3] (we thank the referee for calling this to our attention).

### 2.3 Relating the two types of moduli space

We now indicate how the two types of moduli spaces of elliptic surfaces are related.

**Definition 2.16** Let \( \mathcal{M}'_3 \subset \mathcal{M}_3 \) be the subvariety of pairs \((\phi, p)\) where \( \phi : C_0 \to \overline{\mathcal{M}} \) is a cover of degree 3 with nonreduced fibre above \( p \). Let \( \mathcal{N}'_4 \subset \mathcal{M}_4 \) be the subvariety of pairs \((\phi, p)\) where \( p \) lies under a ramification point of type \((2, 2)\) or \((4)\).

**Remark 2.17** The definition of \( \mathcal{N}'_4 \) ensures that the map is of degree 4 in that case. Later, in Definition 3.1, we will define \( \mathcal{M}'_n \) in general; our definition will coincide with \( \mathcal{M}'_3 \) for \( n = 3 \) but not with \( \mathcal{N}'_4 \) for \( n = 4 \).

**Proposition 2.18** The elliptic surface corresponding to a point of \( \mathcal{M}'_3 \) or \( \mathcal{N}'_4 \) is a K3 surface.

**Proof** The ramification means that one or two pairs of \( \tilde{D}_4 \) fibres coalesce and therefore become a smooth fibre, so we have one \( \tilde{D}_4 \) and nine \( \tilde{A}_1 \) fibres, resp. 12 \( \tilde{A}_1 \) fibres, in the two cases. The result follows from [31, Lemma III.4.6 (a)]. \( \square \)
Remark 2.19 Our primary concern in this paper is with the case \( n = 3 \). Nevertheless we mention that in the case \( n = 4 \) we may obtain Mordell–Weil rank 2, and thus Picard number 16, by requiring an additional ramification point of type \((2, 2)\). It turns out that these surfaces admit 4-isogenies to double covers of \( \mathbb{P}^2 \) branched along six lines which are the composition of two quotients by van Geemen–Sarti involutions: see Proposition 3.55. When \( n > 4 \) we cannot obtain a K3 surface from this construction, but we do obtain some interesting elliptic surfaces. These will be described in Definition 2.24 and a basic property given in Proposition 2.26. Unfortunately they do not appear to admit correspondences to K3 surfaces.

Note in particular that we have a partition of the \( \tilde{A}_1 \) fibres into three sets of 3 according to the fibres of \( \mathcal{E}_p \) above which they lie (cf. Remark 2.3). Thus we have constructed a map \( \kappa : \mathcal{M}'_3 \to \mathcal{M}'_{K,1} \) (Definition 2.1):

Definition 2.20 Let \( \kappa \) be the map \( \mathcal{M}'_3 \to \mathcal{M}'_{K,1} \) that takes a pair \((\phi, p)\) to \( \mathcal{E}_p \times \overline{\mathcal{M}}_{0,4} C_0 \), where the induced fibration and section of infinite order are those pulled back from \( \mathcal{E}_p \to \overline{\mathcal{M}}_{0,4} \) and the order on 2-torsion sections is that of \( \mathcal{E}_p \).

In addition, there is an obvious map \( \nu : \mathcal{M}'_{K,1} \to \mathcal{V}' \) given by mapping to the locations of the singular fibres:

Definition 2.21 Suppose we are given the data of a point of \( \mathcal{M}'_{K,1} \). We obtain a point of \( \mathcal{V}' \) as \((p, \{q_{ij}\})\), where \( p \) lies under the \( \tilde{D}_4 \) fibre and the \( q_{ij} \) lie under the \( \tilde{A}_1 \) fibres, in such a way that the torsion section \( T_i \) passes through the zero components of the fibres above the \( q_{ij} \).

It is not difficult to see that \( \mathcal{M}'_3 \) and \( \mathcal{V}' \) are birationally equivalent. Indeed, define \( \rho : \mathcal{M}'_3 \to \mathcal{V}' \) to send a point \((\phi, p)\) to the third point above \( p \) and the fibres above \( b_1, b_2, b_3 \); this is a birational equivalence because a map \( \mathbb{P}^1 \to \mathbb{P}^1 \) is determined up to scaling on the target by its fibres at 0, \( \infty \), while the scaling is fixed by the fibre at 1. Also, it is clear from the construction that \( \nu \circ \kappa = \rho \).

Proposition 2.22 The map \( \nu \) is a birational equivalence. Hence \( \kappa \) is also.

Proof Since the dimensions are equal, it suffices to show that the degree of \( \nu \) is 1. This will be done by an argument much like that of Proposition 2.14. Indeed, given a K3 surface parametrized by a point of \( \overline{\mathcal{M}}_{K,1} \), we obtain a family of 4-pointed stable curves of genus 0 by taking the quotient by the negation. The map to \( \overline{\mathcal{M}}_{0,4} \) is of degree 3, because there are 3 of each type of reducible fibre, and such a map is determined by the fibres at 0, 1, \( \infty \). In addition, the location of the single \( \tilde{D}_4 \) fibre is determined once we choose the image of a ramification point in \( \overline{\mathcal{M}}_{0,4} \). Hence the \( j \)-invariant of the fibration and the location of fibres with starred Kodaira type are determined by the image in \( \mathcal{V} \); but this is enough to recover the fibration.

The introduction of \( \mathcal{M}'_3 \) allows us to give a further relation between \( \mathcal{M}_{K,1} \) and \( \mathcal{M}_{K,2} \).

Proposition 2.23 The degree-3 map from \( 0_{\phi_1} \) to the base of \( \phi_2 \) coincides with the map \( \phi \) in the corresponding point of \( \mathcal{M}'_3 \).

Proof For \( i = 1, 2, 3 \), we consider the intersection \( T_i \cap 0_{\phi_i} \). By definition of \( \phi_i \), each of these is the fibre of the map \( C_0 \to \overline{\mathcal{M}}_{0,4} \) above a boundary point. On the other hand, since the \( T_i \)
are fibres of \( \phi_2 \), the map \( \phi_2|_{\phi_1} \) is constant on each of the \( T_i \cap 0_{\phi_1} \), and in fact these are fibres because their degree is equal to that of the map.

We have seen that the two maps from \( \mathbb{P}^4 \) to a curve of genus 0 have three fibres in common. Therefore they are equal. (We did not identify the base of \( \phi_2 \) with \( \overline{\mathcal{M}}_{0,4} \), but it would be natural to do so by choosing the points under the \( \tilde{D}_4 \) fibres to be the boundary points.) \( \square \)

### 2.4 Some elliptic surfaces of Kodaira dimension 1

In most cases this construction does not give K3 surfaces, but with larger \( n \) and more general ramification type we still obtain specific families of elliptic surfaces. These satisfy conditions given in Definition 2.24 below. To do so, we fix \( n > 2 \) and let \( R \) be a subset of the ramification data for a map of degree \( n \) of rational curves. In other words, \( R \) is a sequence \( (r_i)^k_{i=1} \) of partitions of \( n = \sum_{i=1}^{m_1} a_{ij} \) such that \( \sum_i \sum_j a_{ij} - 1 \leq 2n - 2 \), corresponding to maps of rational curves with fibres \( F_1, \ldots, F_k \) such that the multiplicities of the points in \( F_i \) are given by \( (r_i) \). Let \( t \) be the sum of the numbers of parts of the \( r_i \), and for \( 1 \leq i \leq k \) let \( c_i = \sum_{j=1}^{m_1} a_{ij} - 1 \) be the number of conditions imposed by the ramification data at \( F_i \). We assume that \( 2n - 2 - \sum_{i=1}^{n} c_i \geq 0 \).

**Definition 2.24** Given an elliptic curve \( E/k \) with labelled 2-torsion points \( t_1, t_2, t_3 \), let \( \lambda(E) \in k \) be the unique element such that there is an isomorphism \( E \to E_\lambda \) taking \( t_1, t_2, t_3 \) to \((0, 0), (1, 0), (\lambda, 0)\) respectively, where \( E_\lambda \) is the elliptic curve \( y^2 = x(x - 1)(x - \lambda) \). Given a family of elliptic curves over a smooth curve \( C \) with labelled 2-torsion sections, we will refer to the map \( C \to \overline{\mathcal{M}}_{0,4} \) taking a fibre \( F \) to \( \lambda(F) \) as the \( \lambda \)-invariant map. A priori it is only defined over an open subset of \( C \), but since \( C \) is a smooth curve it extends to all of \( C \).

Let \( \mathcal{M}_{n,R} \) be the moduli space of pairs consisting of a map of degree \( n \) from \( C_0 \) to \( \overline{\mathcal{M}}_{0,4} \) and \( k \) points \( p_1, \ldots, p_k \in \overline{\mathcal{M}}_{0,4} \) where the ramification above \( p_i \) is as specified by the partition \( r_i \). Let \( V'_{n,R} \) be the quotient of the subset \( U \subset \overline{\mathcal{M}}_{0,t+3n}/S \times S^3_n \) that parametrizes collections of \( t + 3n \) points \( p_{11}, \ldots, p_{1m_1}, \ldots, p_{km_k}, q_{11}, \ldots, q_{1n}, q_{21}, \ldots, q_{2n}, q_{31}, \ldots, q_{3n} \) on a rational curve \( C_0 \) and maps \( \phi \) of degree \( n \) from \( C_0 \) to \( \overline{\mathcal{M}}_{0,4} \) and points \( p_1, \ldots, p_k \in \overline{\mathcal{M}}_{0,4} \) for which \( \phi^{-1} \) of the divisor \( (p_i) \) is \( \sum_{j=1}^{m_1} a_{ij}(p_{ij}) \), and such that the \( \{q_{ij} : 1 \leq j \leq n\} \) are the fibres of \( \phi \) above the boundary points of \( \overline{\mathcal{M}}_{0,4} \). Here \( S \) is the subgroup of \( \prod_{i=1}^{k} S_{m_i} \) that preserves the partitions, while \( S^3 \) acts on the second subscripts of the \( q_{ij} \).

Let \( \mathcal{M}_{E,n,R} \) be the coarse moduli space of elliptic surfaces \( S \to C_0 \) with full level-2 structure and a labelling \((t_1, t_2, t_3)\) of the sections of order 2, together with a collection of \( t + 3n \) points up to permutation as above, such that:

- the pair \((C_0, p_{11}, \ldots, p_{km_k}, q_{11}, \ldots, q_{3n})\) corresponds to a point of \( U \) as in the last paragraph;
- the surface has \( \tilde{D}_4 \) fibres above the points \( p_{1j} \) for the \( j \) such that \( r_{1j} \) is odd and has \( \tilde{A}_1 \) fibres above the \( q_{ij} \);
- for all \( i \), the \( \lambda \)-invariant of the fibre above \( p_{ij} \) does not depend on \( j \);
- for \( 1 \leq i \leq 3 \), the torsion section \( t_i \) passes through the zero component of the fibres above the \( q_{ij} \) and the nonzero component of the other \( \tilde{A}_1 \) fibres (in other words, the \( \lambda \)-invariant map takes the \( q_{ij} \) to 0, 1, \( \infty \) for \( i = 1, 2, 3 \) respectively).

**Remark 2.25** Let \( m \) be the number of odd parts of the partition \( r_1 \). Then the general point of \( \mathcal{M}_{E,n,R} \) describes an elliptic surface of Euler characteristic \( 6(m + n) \) and hence \( h^{2,0} = (m + n - 2)/2 \) by \([31, (III.4.2), (III.4.3)]\). So, as in Remark 2.19, we do not obtain K3 surfaces for \( n > 4 \).
Proposition 2.26 The moduli spaces $M'_{n,R}$, $M'_{E,n,R}$, $V'_{n,R}$ are birationally equivalent.

Proof First we consider the map $\mu_1 : V'_{n,R} \to M'_{n,R}$ taking the pair consisting of the given map $\phi$ and the collection of points to the same map $\phi$ and the points $\phi(p_{ij})$ for $1 \leq i \leq n$. This is a birational equivalence. The sets $\{q_{ij} : 1 \leq j \leq n\}$ are determined by $\phi$ and the set $\{p_{ij} : 1 \leq j \leq m_k\}$ modulo the subgroup of $S_m$ preserving the partition uniquely determines, and is uniquely determined by its image $\phi(p_1)$.

We now define a map $\mu_2 : V'_{n,R} \to M'_{E,n,R}$. Given a point $\alpha = (\phi, \{p_{ij}\}, \{q_{ij}\}) \in V'_{n,R}$, we construct a point $\mu_2(\alpha)$ whose underlying surface (unique up to twist) has $\lambda$-invariant $\phi$. Identify the 2-torsion sections by means of the ordering of the $\{q_{ij}\}$ and choose the quadratic twist such that starred fibres occur only at the $p_{ij}$ where $r_{ij}$ is odd (this is unique up to an element of $k$; it exists because the number of such points is congruent mod 2 to the contribution to the characteristic from fibres with multiplicative reduction divided by 6). It is clear that this surface has the desired properties, and that the forgetful map taking a point of $M'_{E,n,R}$ to $(\lambda, \{p_{ij}\}, \{q_{ij}\})$ is a birational equivalence. $\square$

Remark 2.27 We will show in Theorem 3.56 that, for certain special choices of $n$ and $R$, the surfaces parametrized by $M'_{E,n,R}$ are motive-finite. We do not know whether this is to be expected in other cases. The case $n = 4$, $R = ((2, 2))$, corresponding to a 5-dimensional family of K3 surfaces of generic Picard rank 15, is perhaps the most interesting.

Remark 2.28 In contrast to the case of K3 surfaces, we see no reason why the elliptic surfaces parametrized by the $M'_{E,n,R}$ should be determined by their Picard groups. Consider, for example, the case $n = 5$, $R = ((2, 1, 1, 1))$. Then $M'_{E,n,R}$ parametrizes covers of degree 5 of a curve with a point of ramification, so it is of dimension 8. On the other hand, the points of $M'_{E,n,R}$ correspond to elliptic surfaces with 15 fibres of type $A_1$ and 3 of type $D_4$, leading to a generic Picard number of 29. In the 38-dimensional moduli space of elliptic surfaces over $\mathbb{P}^1$ with $h^{2,0} = 3$, surfaces with these reducible fibres should be a family of dimension 11. We see no way to construct 3 independent sections on such an elliptic surface and specialization does not often produce elliptic curves over $\mathbb{Q}$ of rank 3 or greater, so we believe that the points of $M'_{E,n,R}$ do not exhaust the locus of elliptic surfaces of a given generic Picard group.

3 The construction in terms of curves

In this section, we will give an entirely different construction of a map $M'_3 \to M'_{K,1}$, based on the ideas of Paranjape [36, Section 3]. We begin by using a point $p_0 \in M'_3$ to construct an elliptic curve together with some auxiliary data similar but not identical to that used in [36, Section 3]. We will then use this to construct another curve $C_3$ such that $S_{p_0}$ is a quotient $C_3 \times C_3$, where $S_{p_0}$ is the K3 surface associated to the point of $M'_{K,1}$ which is the image of $p_0$ by the construction of the last section.

Although the construction of a K3 surface quotient of the square of a curve is special to $n = 2, 3$, most of the geometry can be studied without this assumption. Accordingly we will work with general $n$ for as long as possible and specialize to $n = 3$ only at the end. With $n = 2$ our construction is less interesting from this point of view, because the K3 surfaces that we obtain are isogenous (Definition 1.4) to Kummer surfaces. Hence motive-finiteness and the strong form of the Kuga–Satake construction are already known for them. In the case $n = 4$ our construction also gives a correspondence to a certain moduli space of K3 surfaces. We do not obtain results on the Kuga–Satake conjecture or motive-finiteness directly, because we do not realize these surfaces as quotients. However, we will show in Proposition 3.55 that
these surfaces are isogenous to K3 surfaces of the type introduced at the beginning of [36, Section 1] and shown to be quotients of the square of a curve at the end of [36, Section 3].

In this section, we are concerned with moduli spaces and so for ease of exposition we will assume that the ground field \( k \) is algebraically closed when constructing any map of moduli spaces. The reason for this is that in order to find the image of a point corresponding to a K3 surface defined over \( k \), we may need to make a choice of one of a set of data, where the set is defined over \( k \) but the individual elements are not. It will turn out that the image represents a point in the moduli space defined over \( k \), so that the map of moduli spaces takes \( k \)-points to \( k \)-points for all \( k \) and is therefore defined over the prime subfield of \( k \). Further, we will see that our map is a birational equivalence over an algebraically closed field; this property descends to \( k \).

3.1 Moduli spaces related to curves of genus 0 and 1

We start by defining a moduli space \( \mathcal{M}_n' \) that generalizes the construction of \( \mathcal{M}_3' \) in Definition 2.16. Our definition will capture the ramification properties of the quotient of a cyclic \( n \)-isogeny of elliptic curves by \( \pm 1 \). That is to say, a cyclic \( n \)-isogeny \( E_n \rightarrow E \) of elliptic curves descends to a map of quotients \( E_n/\pm 1 \rightarrow E/\pm 1 \), which gives a point of \( \mathcal{M}_n' \).

**Definition 3.1** Fix an integer \( n > 2 \). Let \( \mathcal{M}_n' \) be the moduli space parametrizing degree-\( n \) covers \( \phi_n : C_0 \rightarrow \overline{\mathcal{M}}_{0,4} \) such that there are 4 points where the ramification is of type \((2, \ldots, 2, 1)\) if \( n \) is odd, or \((2, \ldots, 2)\) if \( n \) is even, together with a point \( q_0 \in C_0 \) in a fibre with \( \left\lfloor \frac{n+3}{2} \right\rfloor \) distinct points.

For \( n = 2 \) we make a slightly different definition. Namely, we fix a degree-2 cover \( C_0 \rightarrow \overline{\mathcal{M}}_{0,4} \), where \( C_0 \) is a rational curve, and two additional points of \( \overline{\mathcal{M}}_{0,4} \), and single out a point of \( C_0 \) above one of these.

**Remark 3.2** Note that in every case we have \( \dim \mathcal{M}_n' = 4 \): the dimension of the space of degree-\( n \) covers is \( 2n - 2 \), while the ramification imposes \( 4(n - 3)/2 \) conditions for odd \( n \) or \( 2(n - 2)/2 + 2(n - 4)/2 \) for even \( n > 2 \), and the choice of \( q_0 \) is from a finite set. For \( n = 2 \) this is clear. We also point out that the genus of \( C_0 \) is 0 for \( n > 2 \), not only for \( n = 2 \): this follows from Riemann-Hurwitz.

**Definition 3.3** Let \( E \) be the double cover of \( \overline{\mathcal{M}}_{0,4} \) branched at the ramification points of \( \phi_n \) (if \( n = 2 \), the ramification points and the two additional points) and define \( \overline{E}_n \) to be the normalization of \( E \times \overline{\mathcal{M}}_{0,4} C_0 \). Choose the origin \( O \) on \( E \) to be the point of \( E \) lying above \( \phi_n(q_0) \in \overline{\mathcal{M}}_{0,4} \) (if \( n = 2 \), the point that was specified in defining the cover).

**Remark 3.4** It is clear that \( \mathcal{M}_n' \) is birational to the moduli space whose points correspond to a cyclic \( n \)-isogeny \( E_n \rightarrow E \), an unordered set of 4 points of \( E \), and a double cover branched at those 4 points. (Recall that the double cover is not uniquely determined by its branch locus; rather, the double covers with a given branch locus constitute a torsor for \( \text{Pic}[2](E) \). Concretely, if one double cover corresponds to the extension of function fields \( K(E)(\sqrt{f})/K(E) \), then the others are given by the extensions \( K(E)(\sqrt{fg}) \), where \( g \) belongs to the group of functions with divisor divisible by 2 modulo squares.) In particular, taking \( n = 3 \), we have shown that \( \mathcal{M}_3' \) is birationally equivalent to \( \mathcal{M}_C \) (Definition 1.8).

Our first task is to define an auxiliary double cover \( D_3 \rightarrow E \) branched at 4 points and hence of genus 3. Let \( \pm B_i \) be the points of \( E \) lying above the boundary points \( b_i \in \overline{\mathcal{M}}_{0,4} \).
Proposition 3.5  Up to translation and negation, there is a unique set \( \{p_0, \ldots, p_3\} \) of 4 points on \( E \) such that \( \{p_i - p_j : 0 \leq i \neq j \leq 3\} = \{ \pm B_i \pm B_j : 1 \leq i < j \leq 3\} \), counted with multiplicity.

Proof  For existence, let \( p_0 = O \) and \( p_i = B_1 + B_2 + B_3 - B_i \) for \( 1 \leq i \leq 3 \). For uniqueness, we may again let \( p_0 = O \), since translations are permitted. We then have \( p_i = \pm B_{j_1} \pm B_{j_2} \) for suitable choices; it is not possible to have a plus and a minus sign in each point, so, negating and reordering if necessary, we take \( p_1 = B_2 + B_3 \). Then, since \( p_2, p_1 - p_2 \in \{ \pm B_i \pm B_j\} \), we must have \( p_2 \in \{ B_1 + B_2, -B_1 + B_2, B_1 + B_3, -B_1 + B_3\} \); by symmetry we take \( p_2 \in \{ B_1 + B_3, -B_1 + B_3\} \) and changing the sign gives the same four points. \( \square \)

Definition 3.6  Let \( f \) be a function on \( E \) with divisor \( (p_0) + (p_1) + (p_2) + (p_3) - 2(O) - 2(B_1 + B_2 + B_3) \), and let \( D \) be the smooth curve with function field \( k(E)(\sqrt{f}) \) (where \( k \) is the ground field and \( k(E) \) is the function field of \( E \) as usual). Let the \( P_i \) be the unique points of \( D \) lying above the \( p_i \).

Remark 3.7  Note that the choice of \( O \) determines the double cover of \( E \) ramified above \( p_1, \ldots, p_4 \). Indeed, replacing \( O \) by \( O + T \), where \( T \) is a 2-torsion point, would add \( 2(O + T) - 2(O) \) to the divisor of \( f \). This is a principal divisor and twice a divisor but not twice a principal divisor, so it gives a different double cover.

Since we cannot distinguish \( B_i \) from \( -B_i \), we must explain why our construction does not depend on the choice. Indeed, let us replace \( B_1 \) by \( -B_1 \), so that the points become \( p'_0 = O, p'_1 = B_2 + B_3, p'_2 = -B_1 + B_3, p'_3 = -B_1 + B_2 \) and the divisor of the function is \( (p'_0) + (p'_1) + (p'_2) + (p'_3) - 2(O) - 2(-B_1 + B_2 + B_3) \). Translating by \( -B_2 - B_3 \) and changing the sign, we restore the previous points and obtain a function \( f' \) with divisor \( (p_0) + (p_1) + (p_2) + (p_3) - 2(B_1) - 2(B_2 + B_3) \), which is equal to the previous \( f \) up to a square. Thus \( D \) is unaltered.

For \( D \) to be unique up to isomorphism we must assume that \( k \) has no nontrivial quadratic extensions. But even without this assumption the point of \( \mathcal{M}_3 \) corresponding to \( D \) is well-defined.

3.2 A curve of genus \( 2n + 1 \) whose square covers an elliptic surface

We now define the curve whose square will cover an elliptic surface over the elliptic curve \( E_n \). We will then show that this surface has a quotient which is an elliptic surface \( W \) over \( \mathbb{P}^1 \); in the case \( n = 3 \), this will be a K3 surface. Following this, we compare our construction to Paranjape’s.

Definition 3.8  Let \( C_n = D_3 \times_E E_n \). This is a curve of genus \( 2n + 1 \) with an action of \( \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). Let \( \iota \) be the involution of \( D_3 \) induced by the double cover, let \( \beta \) be its lift to \( C_n \) (the quotient being \( E_n \)), and let \( \gamma \) be a generator of the group of automorphisms of \( C_n \) over \( D_3 \). Let \( G_n = \langle \gamma, \beta \rangle \), so that \( G_n \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).

We summarize our definitions in Fig. 3.

Remark 3.9  Paranjape fixes \( p_0, \ldots, p_3 \in E \) (in his notation, \( p_1, \ldots, p_4 \)) and considers a double cover of \( E_n \) (in fact he only works with \( n = 2 \)) branched along the pullback of \( \sum_{i=1}^4 (p_i) \) from \( E \) to \( E_n \). Thus in his construction the Galois group is \( \mathbb{Z}/2n\mathbb{Z} \) rather than...
$\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The Prym variety of the cover $C_n \to E$ is an abelian variety of dimension $2n$ with an automorphism of order $2n$, and we may view an appropriate component of its Hodge structure as a module of rank 2 over $\mathbb{Z}[\zeta_{2n}]$. In our construction the relevant module associated to a summand of the Hodge structure of the Prym variety of $C_n \to E$ is over $\mathbb{Z}[\zeta_n]$ instead, although this is a distinction without a difference for odd $n$.

**Definition 3.10** Let $\psi : D_3 \to E$ be the natural quotient map $C_n/\langle \gamma \rangle \to C_n/G_n$, as in Fig. 3.

**Lemma 3.11** Suppose that $D_3$ is not hyperelliptic, and consider it with its canonical embedding in $\mathbb{P}^2$. Let $P_i$ be the point above $p_i$ in $D_3$, and let $T_i,1, T_i,2$ be the residual intersection of the tangent line to $D_3$ at $P_i$. Then $\iota(T_i,1) = T_i,2$ and $p_i + \psi(T_i,1) \sim 2O$ on $E$. In other words, if $O$ is chosen as the origin of $E$, then $\psi(T_i,1) = -p_i$.

**Proof** Since $P_i$ is $\iota$-invariant, the same is true of the tangent line and of its intersection with $D_3$, so the first claim follows. For the second one, note that the divisor $2(p_i) + (-p_i) + 2O$ is principal on $E$. Let $S_i,1, S_i,2$ be the points above $-p_i$ in $D_3$; then $2(p_i) + (S_i,1) + (S_i,2) - 2(O_1) - 2(O_2)$ is the pullback of this principal divisor, so it is principal on $D_3$. In addition, $(p_0) + (p_1) + (p_2) + (p_3) - 2(O_1) - 2(O_2)$ is principal, since it is the divisor on $D_3$ of the element $\psi$ as defined in Definition 3.6.

By the Riemann-Hurwitz formula, the divisor $(p_0) + (p_1) + (p_2) + (p_3)$ is in the canonical class; the same follows for the linearly equivalent divisor $2(p_i) + (S_i,1) + (S_i,2)$. But $2(p_i) + (T_i,1) + (T_i,2)$ is also in the canonical class, being a hyperplane section of a canonically embedded curve, and $D_3$ is not hyperelliptic, so these divisors must be equal. \(\square\)

No power of $\gamma$ not equal to the identity has fixed points. The quotient of $C_n$ by $\beta$ has genus 1, so $\beta$ has $4n$ fixed points. To determine the number of fixed points of $\beta \gamma^j$, consider the group generated by $\beta, \gamma^j$: the quotient of genus 1 and $\beta$ has $4n$ fixed points. So by Riemann-Hurwitz no other nonidentity element of $G_n$ has any fixed points.

We now introduce the surface $C_n \times C_n$. It admits an action of the group $G_n \ltimes S_2$ in which $G_n$ acts on each copy of $C_n$ and $S_2$ interchanges the factors.

**Definition 3.12** Following [36, Section 3], let $G$ be the subgroup of $G_n \ltimes S_2$ generated by $(\gamma^{-1}, \gamma), (\beta, \beta)$, and the nonidentity element of $S_2$. Denote the generators by $g$, $b$, $\sigma$ respectively. Let $W$ be the surface $C_n \times C_n/G$, and let $\omega$ be the map $C_n \times C_n \to W$. Let $W$ be the minimal desingularization of $W$ (we will describe the singularities of $W$ in Proposition 3.13). For a surface of the form $C \times C$, where $C$ is a curve, we use $\sigma$ for the involution $(x, y) \to (y, x)$. We record some subgroups of $G_n \ltimes S_2$ whose quotients define surfaces of interest. In particular, as in Fig. 4 we define $V := C_n \times C_n/(\langle \gamma, (\beta, \beta) \rangle)$ and $S := C_n \times C_n/(\langle (\gamma, \gamma^{-1}), (\beta, 1), \sigma \rangle)$.
Proposition 3.13 The surface \( W \) is an elliptic surface over \( E_n \) with full level-2 structure. It has \( 6n \) ordinary double points; resolving these creates \( 6n \) fibres of type \( \tilde{A}_1 \), and these are the only singular fibres of \( \tilde{W} \).

Proof We first claim that

\[
W = V \times_{\text{Sym}^2 E} S = V \times_{\text{Sym}^2 E} \text{Sym}^2 E \times E \cong V \times E E_n.
\]

The first equality follows from the fact that \( W, V, \text{Sym}^2 E, S \) are all quotients of \( C_n \times C_n \) for which the corresponding diagram of groups is a pushout. (This is illustrated in Fig. 4.)

The second results from the fact that \( E_n \to E \) are unramified, so that the fibres of \( W \to E_n \) are fibres \( V \to E \). The third is immediate.

Since \( \pi : V \to \text{Sym}^2 E \) has degree 2 and \( \text{Sym}^2 E \to E \) is a ruled surface, we see that the generic fibre of \( V \to E \) is a double cover of a smooth rational curve, so we need to compute the ramification locus of \( \pi : V \to \text{Sym}^2 E \). Because \( V = C_n \times C_n/\langle (\gamma, 1), (\beta, \beta), \sigma \rangle \) and \( \text{Sym}^2 E = C_n \times C_n/\langle (\gamma, 1), (\beta, 1), \sigma \rangle \), the map \( \pi : V \to \text{Sym}^2 E \) is given by the quotient by \( (\beta, 1) \) and so \( \pi \) is branched along the curves \( \{ p_i, E \} \subset \text{Sym}^2 E \) for \( i = 0, \ldots, 3 \). The curves \( \{ p_i, E \} \) are sections of the addition map \( \text{Sym}^2 E \to E \). Since the general fibre of \( V \to E \) is a double cover of the fibre of \( \text{Sym}^2 E \to E \) branched at 4 points, and the fibre of \( \text{Sym}^2 E \to E \) is a rational curve, we see that the general fibre of \( V \to E \) is an elliptic curve. Furthermore, we see that \( \{ p_i, E \} \cap \{ p_j, E \} = \{ p_i, p_j \} \), so that the singular fibres of \( \pi : V \to E \) are above the points \( p_i + p_j \in E \) where the sections meet. These give 6 nodes in \( V \). Resolving these nodes give 6 fibres of type \( \tilde{A}_1 \) in the minimal resolution \( \tilde{V} \) of \( V \).
Since $W = V \times_E E_n$, we obtain its singular points and the singular fibres of $\tilde{W} \to E_n$ by pulling back those of $V, \tilde{V}$, obtaining $6n$ of each. In addition, the differences of the given sections of $V \to E$ are of order 2, so the same is true of their pullback to sections of $\tilde{W} \to E_n$.

Corollary 3.14 The topological Euler characteristic of $W$ is $6n$, while that of $\tilde{W}$ is $12n$.

Proof The Euler characteristic of an elliptic surface is the sum of those of the singular fibres. On $W$ these are nodal rational curves, so each has Euler characteristic 1, and there are $6n$ of them. On $\tilde{W}$, each one is an $I_2$ fibre, whose Euler characteristic is 2 (alternatively, we obtain $\tilde{W}$ from $W$ by blowing up the $6n$ singular points, replacing $6n$ points by $6n$ smooth rational curves).

We now study some curves on the surfaces $D_3 \times D_3$ in order to construct sections of the elliptic surface structure of $\tilde{W}$. This will be essential to match our construction in this section with that of Sect. 2.

Definition 3.15 Recall that the $p_{i1}$ are the points of $E$ at which the map $D_3 \to E$ is ramified. Let the $P_i$ be their inverse images in $D_3$ and the $(Q_{i,r})_{r=0}^{d-1}$ their inverse images in $C_n$, chosen such that $\gamma(Q_{i,r}) = Q_{i,r+1 \mod n}$.

Let $P \in D_3$. Define $V_P, H_P$ to be the curves on $D_3 \times D_3$ obtained by pulling back $P$ through the first and second projections respectively, and let $V_{NS}, H_{NS}$ be their divisor classes up to algebraic equivalence (which do not depend on $P$). Let $V_i = V_{P_i}$ and $H_i = H_{P_i}$ for $0 \leq i \leq 3$. In addition, let $\Delta, \Gamma_i$ be the classes of the divisors of the diagonal and the graph of $r$ respectively. For $i \in \{0, 1, 2, 3\}$, pull back $V_P$ to a curve on $C_n \times C_n$ and let $S_i$ be its image under $\omega$ (the curve on $C_n \times C_n$ has $n$ components but they all have the same image in $W$).

We now use the definition of $W$ as a quotient to identify its singularities. The following proposition can be derived from previous work on surfaces isogenous to a product of mixed type, as in [5, 13, 14], but we include a proof for completeness.

Proposition 3.16 The singularities of $W$ are the images of the points $(Q_{i,r}, Q_{j,s}) \in C_n \times C_n$ for $i \neq j$ and $r, s \in \{0, \ldots, n-1\}$.

Proof A singularity of $W$ can only occur at the image of a point of $C_n \times C_n$ with nontrivial $G$-stabilizer. Further, if $P$ is a point such that the fixed locus of its $G$-stabilizer is a divisor that is smooth at $P$, then the image of $P$ is smooth.

In particular, the fixed locus of an element of $G$ of the form $g^k\sigma$ is the graph of $\gamma^k$ or $\gamma^k\beta$ on $C_n \times C_n$, which is a smooth curve, so singularities are only possible at points of intersection of two of these, which are fixed points of some $g^k$ or $g^k\beta$. For $k \neq 0$ these have no fixed points, so it suffices to consider the $16n^2$ fixed points of $\beta$, which are exactly the pairs $(Q_{i,r}, Q_{i',r'})$. This map acts on the tangent space of such a point by $-1$. If $i \neq i'$ it is the full stabilizer, so the quotient is étale-locally isomorphic to $\mathbb{A}^2/\pm 1$ and we obtain an $A_1$ singularity.

Since there are $12n^2$ pairs $(Q_{i,r}, Q_{i',r'})$, each with stabilizer of order 2, they fall into orbits of size $2n$ and we obtain $6n$ singular points on the quotient. We showed in Proposition 3.13 that there are no more.

Remark 3.17 Suppose that $P_i + P_j \neq P_k + P_l$ on $E$ for all permutations $(i, j, k, l)$ of $0, 1, 2, 3$. Then the $6n$ singular points of $W$ all lie on distinct fibres of the map $W \to E_n$. 

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and conversely. Note also that any two of the \( S_i \) intersect in \( n \) points; since there are four \( S_i \), this gives \( n(\binom{4}{2}) = 6n \) points of intersection, which are precisely the singularities of \( W \). Each of the three equalities of the form \( P_i + P_j = P_k + P_\ell \), if it holds, causes two \( \tilde{A}_1 \) fibres to merge to form a \( \tilde{A}_3 \) fibre.

The \( H_i, V_i \) will give us the identity and 2-torsion sections on \( W \) from another point of view. In [36, Section 3] that is sufficient, because the K3 surfaces considered there have Picard number 16 while the elliptic fibration constructed has six \( \tilde{A}_1 \) fibres and two \( \tilde{D}_4 \), so the Mordell–Weil group of the fibration is torsion. In our situation with \( n = 3 \), however, we will find that \( \widetilde{W} \) has a quotient K3 surface that comes with an elliptic fibration with nine \( \tilde{A}_1 \) fibres and one \( \tilde{D}_4 \) fibre, and so we need to construct a section of infinite order.

**Proposition 3.18** For each \( i \) with \( 0 \leq i \leq 3 \), the linear system \( |K_{D_3 \times D_3} - V_i - H_i - \Delta| \) has projective dimension 0. In other words, there is a unique effective divisor in the canonical class of \( D_3 \times D_3 \) whose support includes \( V_i, H_i \), and the diagonal.

**Proof** The diagonal is defined in \( D_3 \times D_3 \subseteq \mathbb{P}^2 \times \mathbb{P}^2 \) by \( x_0y_1 - x_1y_0 = x_0y_2 - x_2y_0 = x_1y_2 - x_2y_1 = 0 \). In this representation, the canonical class of \( D_3 \times D_3 \) is \( \mathcal{O}(1,1) \). Choose coordinates on \( \mathbb{P}^2 \) so that the first coordinate of every \( P_i \) is 0. Let \( P_i = (0 : a : b) \), and let \( r, s, t \) be such that

\[
(r(x_0y_1 - x_1y_0) + s(x_0y_2 - x_2y_0) + t(x_1y_2 - x_2y_1))
\]

vanishes on \( V_i \) and \( H_i \). In particular, setting \( y_0 = 0, y_1 = a, y_2 = b \) we find that \((ra + sb)x_0 + t(bx_1 - ax_2) = 0\) for all points \((x_0 : x_1 : x_2) \in D_3\). This means that all of the coefficients of the \( x_i \) in this linear form must be 0: in particular, \( t = 0 \), while \( ra = -sb \). Up to scaling the only possibility is that \( r = b, s = -a, t = 0 \), so \( E_i \) can only be the divisor cut out by \( b(x_0y_1 - x_1y_0) - a(x_0y_2 - x_2y_0) \).

**Remark 3.19** The same statement is true with the diagonal replaced by the graph, \( \Gamma_i \), of the involution \( \iota \), up to the changes of sign between the equations defining the diagonal and those defining the graph.

**Definition 3.20** For \( 0 \leq i \leq 3 \), let \( U_i \) be the corresponding divisor from Proposition 3.18. Let the \( R_i \) be the residual component of the \( U_i \). Let \( G_i \subset W \) be the images of the pullbacks of the \( R_i \) to \( C_n \times C_n \) under \( \omega \). Similarly, let the \( U'_i \) be the effective divisors, the \( R'_i \) the residuals, and the \( G'_i \) the curves on \( W \) that are obtained by the analogous construction with the diagonal replaced by \( \Gamma_i \). Let \( H_{NS}, V_{NS}, D_{NS} \) be the classes of \( D_3 \times p_0, p_0 \times D_3 \), and the diagonal in the Néron-Severi group of \( D_3 \times D_3 \) (sometimes \( D \) will also denote the diagonal itself).

In order to obtain further information about the elliptic surface structure on \( W \), we make a closer study of \( D_3 \times D_3 \) and the \( R_i \). First we observe that \( R_i \) was obtained from the canonical by removing one divisor of each of the classes \( H_{NS}, V_{NS}, D_{NS} \). Since \( K_{D_3 \times D_3} = 4H_{NS} + 4V_{NS} \), we see that \( R_i \sim 3H_{NS} + 3V_{NS} - D_{NS} \). By the adjunction formula we have \( D_{NS}^2 = -4 \). It is now easily checked that \( R_i \cdot D_{NS} = 10 \), while \( R_i \cdot V_{NS} = R_i \cdot H_{NS} = R_i \cdot \Gamma_i = 2 \), where \( \Gamma_i \) is the graph of the involution. Likewise \( R'_i \sim 3H_{NS} + 3V_{NS} - \Gamma_i \), so that \( R_i \cdot R'_i = 9 + 9 - 6 - 6 + 4 = 10 \).

**Proposition 3.21** The point \((P_j, P_k) \in D_3 \times D_3 \) belongs to \( R_i \) if and only if: (1) \( \#(i, j, k) = 3 \), or (2) \( i = j = k \) and \( P_i \) is a flex of \( D_3 \). The same holds with \( R'_i \) in place of \( R_i \).
Proof. We give the proof for $R_i$, that for $R_i'$ being identical. We use the formula for the section of $\mathcal{O}(1, 1)$ defining $U_i$ from Proposition 3.18. Clearly all points of the form $(P_j, P_k)$ belong to $U_i$. It is routine to prove "only if" in case (1): if $j \neq k$ then $(P_j, P_k) \notin \Delta$, while $i \neq j$ implies that $(P_j, P_k) \notin V_i$ and $i \neq k$ tells us that $(P_j, P_k) \notin H_i$. Thus if all three conditions hold, then $(P_j, P_k) \in R_i$.

Conversely, let us start with a point $(P_j, P_k)$, where $k \neq i$ (and either $j = i$ or $j = k$). Now, $U_i$ and $V_k$ have no components in common and their intersection number is 4. Since $U_i \cap V_k$ contains the 4 points $(P_j, P_k)$ for $1 \leq j \leq 4$, the local intersection multiplicity at each of them is 1. Thus $U_i$ must be smooth at each of these points. In particular, it is not possible for more than one component of $U_i$ to pass through $(P_j, P_k)$. But $H_i$ does, so $R_i$ cannot. Similarly for $(P_j, P_i)$ where $j \neq i$.

Finally, we consider the point $(P_i, P_j)$. We have $U_i \cdot \Gamma_i = 8$. Three of the points of intersection are the $(P_j, P_j)$ for $i \neq j$. Each of $H_i, V_i, D$ passes through $(P_i, P_j)$. The two remaining points of intersection are the $(t_m, (t_m))$, where the $t_m$ for $m = 1, 2$ are the residual intersection points of the tangent line to $C$ at $P_i$ with $C$ (this is easily checked: letting $P_i = (0 : a : b)$ as before, the tangent line to $C$ at $P_i$ is defined by $bx_1 - ax_2 = 0$). So $U_i \cap \Delta$ consists of these two points, which are equal to $P_i$ if and only if $P_i$ is a flex. (If $P_i$ is a flex it must be a hyperflex, in view of Lemma 3.11.)

\[ \text{Corollary 3.22} \quad U_i \cap V_j = \{(P_j, P_k), (P_j, P_\ell)\}, \text{ where } \{i, j, k, \ell\} = \{0, 1, 2, 3\}. \text{ Similarly for } U_i \cap H_j. \]

Let $\tilde{S}_i, \tilde{G}_j, \tilde{G}'_k$ be the strict transforms of $S_i, G_j, G'_k$ on $\tilde{W}$. In fact we have already met the $\tilde{S}_i$. 

\[ \text{Proposition 3.23} \quad \text{The } \tilde{S}_i \text{ coincide with the pullback of the ramification locus of the map } V \to \text{Sym}^2(E) \text{ (Proposition 3.13) to } \tilde{W}. \text{ In particular, they are sections and the difference of any two of them is } 2\text{-torsion.} \]

\[ \text{Proof} \quad \text{Indeed, both } \tilde{S}_i \text{ and the component above } \{p_i, E\} \text{ map to } \{p_i, E\} \subset \text{Sym}^2(E) \text{ and are irreducible. The result follows.} \]

\[ \text{Proposition 3.24} \quad \text{If } i \neq j \text{ then } \tilde{S}_i \cap \tilde{G}_j = \tilde{S}_i \cap \tilde{G}'_j = \emptyset. \]

\[ \text{Remark 3.25} \quad \text{Here and in the following proofs the arguments required for the } \tilde{G}_j \text{ are identical to those for the } \tilde{G}_j, \text{ so we will not state them separately.} \]

\[ \text{Proof} \quad \text{The intersection } U_i \cap V_j \text{ is transverse, so the same is true after pulling back by the étale map } C_n \times C_n \to D_3 \times D_3. \text{ The intersection of the inverse images consists of the points of the form } (Q_{k,r}, Q_{\ell,s}), \text{ where } \{i, j, k, \ell\} = \{1, 2, 3, 4\} \text{ and } r, s \in \{0, \ldots, n - 1\}. \text{ So if we blow up the preimages of the } 2n^2 \text{ singular points of } W \text{ on } C_n \times C_n, \text{ the strict transforms of the inverse images do not meet. The quotient of this blowup by } G \text{ is exactly } \tilde{W}. \text{ By push-pull and the fact that the inverse image of } V_j \text{ consists of a complete } G\text{-orbit, it follows that the images of these inverse images on } \tilde{W} \text{ do not meet either.} \]

\[ \text{Proposition 3.26} \quad \text{We have } \tilde{S}_i \cdot \tilde{G}_i = \tilde{S}_i \cdot \tilde{G}'_i = n. \text{ In fact, the image of } \tilde{S}_i \cap \tilde{G}_i \text{ is the inverse image of } O \in E \text{ under the unramified cover } E_n \to E; \text{ in particular it is independent of } i. \]

\[ \text{Proof} \quad \text{We have already seen that } R_i \cdot V_i = 2, \text{ so when we pull back to } C_n \times C_n \text{ the intersection number is } 2n^2 \text{ (this being an étale map of degree } n^2). \text{ The intersection locus is preserved} \]
by the action of $b$ and $g$ on $C_n \times C_n$, while it is disjoint from its image under $\sigma$. The only fixed points of a nonidentity element of the group generated by $b$, $g$ are the $Q_i,j$, which are generically not in the intersection, as we have seen. Thus this group acts on the set of $2n^2$ points with orbits of size $2n$, and so there are $n$ orbits. It also follows that the intersection points are smooth, so that the number of points of intersection of $\tilde{S}_i$ and $\tilde{G}_i$ on $\tilde{W}$ is the same as that of $S_i$ and $G_i$ on $W$.

To obtain the more precise information in the second statement, we appeal to Lemma 3.11. The intersection $R_i \cdot H_i$ is linearly equivalent to $O_{H_i}(1) - (H_i + V_i + D) \cdot H_i$. Here the first is the canonical, while the second consists of two copies of the point $(P_i, P_i) \in H_i$. Since it is an effective divisor of degree 2, and plane quartics are canonically embedded, it can only be the residual intersection of the tangent line, so it consists of the two points $(P_i, P_i)$ for $j = 1, 2$, where the $P_{i,j}$ are the two points of $D_3$ with image $-p_i$ on $E$. When we pull back to $C_n$, therefore, the first coordinate is an inverse image of $P_i$ and the second is an inverse image of $P_{i,j}$, so their images on $E_n$ are inverse images there of $\pm p_i$. The sums of these are the points of $E_n$ in the kernel of the isogeny $E_n \to E$.

**Proposition 3.27** We have $\tilde{G}_i \cdot \tilde{G}'_i = n$, while $\tilde{G}_i \cdot \tilde{G}'_j = 2n$ for $i \neq j$.

**Proof** First, for $\tilde{G}_i \cdot \tilde{G}'_i$, we computed just before Proposition 3.21 that $R_i \cdot R'_i = 10$, and in that proposition that 6 of the special points $(P_j, P_k)$ lie on the intersection. Thus we have $10n^2$ points of intersection when we pull back to $C_n \times C_n$, of which $6n^2$ map to singular points of $W$. When we blow up to pass to $\tilde{W}$, these intersections are pulled apart and do not contribute to $\tilde{G}_i \cdot \tilde{G}'_i$, leaving $4n^2$ points of intersection on $C_n \times C_n$. The degree-4$n$ map to $W$ combines these into $4n^2/4n = n$ points, since as in the argument for Proposition 3.26 just above they are not fixed by any element of $G$ other than the identity.

Likewise, for $\tilde{G}_i \cdot \tilde{G}'_j$, we again have $R_i \cdot R'_j = 10$, but this time only 2 of the special points are on the intersection. Thus we are left with $10n^2 - 2n^2$ points of intersection on $C_n \times C_n$ that give rise to intersection points on $\tilde{W}$, and they are combined in sets of 4$n$, so the intersection number is 2$n$.

**Proposition 3.28** The $S_i$, the $G_i$, and the $G'_i$ are sections of the map $W \to E_n$.

**Proof** We already explained this for the $S_i$ in Proposition 3.23, so we consider the $G_i$. Let $F$ be a fibre of the map $D_3 \times D_3 \to E$, i.e., a curve whose points are the $(x, y)$ such that the images of $x, y$ on $E$ have a given sum. Since the map $D_3 \to E$ has degree 2, we have $F \cdot H_{NS} = F \cdot D_{NS} = 2$, while $F \cdot \Delta = F \cdot \Gamma_i = 8$ (here using the fact that doubling has degree 4 on $E$ and that $i$ preserves the map to $E$). Thus, if $NS(D_3 \times D_3)$ is generated by $H_{NS}, V_{NS}, \Delta, \Gamma_i$, we easily find that $F$ has class $4(H_{NS} + V_{NS}) - \Delta - \Gamma_i$, and this follows in general by writing down the equation defining $F \cup \Delta \cup \Gamma_i$. We now compute that $F \cdot R_i = 4$. Thus the pullbacks to $C_n \times C_n$ meet in $4n^2$ points, and, since the pullback of $F$ consists of complete $G$-orbits, their images on $W$ meet in $4n^2/\#G = n$ points. That is to say, the curve $G_i$ meets a fibre of the map $W \to E$ in $n$ points. Such a fibre consists of $n$ fibres of the map $W \to E_n$, and the intersection with each of them must be the same since the fibres are algebraically equivalent: it is therefore 1.

Henceforth we assume the condition of Remark 3.17, which holds on an open subset of the moduli space.

**Remark 3.29** The sections $\tilde{G}'_i$ are the negatives of the sections $\tilde{G}_i$ if one of the $\tilde{S}_j$ is taken as origin. We will prove this in Proposition 3.45.
Proposition 3.30  The curves $\tilde{G}_i, \tilde{S}_i$ pass through different components of every one of the $A_i$ fibres, and likewise $\tilde{G}_j, \tilde{S}_j$. Under the assumption of Remark 3.17, for $i \neq j$, there are $2n$ fibres of type $A_i$ for which $\tilde{S}_i$ and $\tilde{S}_j$ meet the same component. On these fibres the other two $\tilde{S}$ pass through the other component.

Proof  For the first statement, it suffices to use the fact (Proposition 3.21) that $(P_j, P_k) \in R_i$ if and only if $i \neq \{j, k\}$, which implies that each singular point lies on exactly one of $\tilde{G}_i, \tilde{S}_i$ for each $i$.

For the second statement, note that $\tilde{S}_i, \tilde{S}_j$ can be on the same component in two ways: either they both pass through the singular point or neither does. As in Remark 3.17, there are $n$ of each type, and if $\tilde{S}_i, \tilde{S}_j$ meet in a point then that point is not on $\tilde{S}_k, \tilde{S}_\ell$ (where $\{i, j, k, \ell\} = \{0, 1, 2, 3\}$).

Remark 3.31  The map induced by the canonical divisor class on $\tilde{W}$ is precisely the fibration $\phi : \tilde{W} \to E_n$ for an appropriate embedding of $E_n$ into $\mathbb{P}^{n-1}$. Indeed, let $F$ be a fibre. Then the adjunction formula shows that $(K_{\tilde{W}} + F) \cdot F = K_F$, and both $K_F$ and $F \cdot F$ are 0 as divisor classes on $F$. On the other hand, the fundamental line bundle $\mathcal{L}$ [31, Definition II.4.1] has degree $n$, since $\tilde{W}$ has Euler characteristic $12n$. The canonical bundle of $\tilde{W}$ is $\phi^*(K_{E_n} \otimes \mathcal{L}) = \phi^*(\mathcal{L})$ [31, III.1.1], and so its intersection with a section of $\phi$ (such as one of the $\tilde{G}_i$ or $\tilde{S}_i$ defined earlier) is $n$. It therefore embeds the section as an elliptic normal curve in $\mathbb{P}^{n-1}$ isomorphic to $E_n$, and, since the fibres of $\phi$ are contracted to points, the image of $W$ is the same curve.

3.3 A K3 quotient of $W$

We have constructed a singular surface $W$ whose minimal desingularization $\tilde{W}$ is an elliptic surface over $E_n$. We would like to construct an elliptic K3 surface that is a quotient of $\tilde{W}$. In order to do this, we need to construct a group of automorphisms of $\tilde{W}$ that includes an element that acts as negation on the base $E_n$, since a K3 surface does not have a nonconstant map to an elliptic curve. We will do this by relating $W$ to the symmetric square of $E$, which is a $\mathbb{P}^1$-bundle over $E$, and to its pullback to $E_n$.

Definition 3.32  Let $\tilde{\Sym}^2(E)$ be the blowup of $\Sym^2(E)$ at the points $(p_i, p_j)$ for $i \neq j$. Let the $\Sigma_i$ be the sections $x \to \{p_i, x\}$ of $\Sym^2(E) \to E$ and the $\tilde{\Sigma}_i$ their proper transforms on $\tilde{\Sym}^2(E)$ (which are still sections of $\Sym^2(E) \to E$). Let $E_{ij} \subset \Sym^2(E)$ be the exceptional divisor above the point of intersection of $\tilde{\Sigma}_i \cap \tilde{\Sigma}_j$ and let $F_{ij}$ be the other component of the fibre there.

We will construct a birational automorphism on the double cover $\tilde{V}$ (Definition 3.12, Proposition 3.13) of $\tilde{\Sym}^2(E)$ and pull it back to $W$. The idea is that $\tilde{\Sym}^2(E)$ is itself a double cover of $\overline{\Sym}^2_{0,5}$, and so $V$ covers $\overline{\Sym}^2_{0,5}$ with degree 4. The construction is closely related to that of Construction 2.13. We will show that the Galois group of the cover is the Klein four-group, so that there is an additional involution that acts nontrivially on the base and by which the quotient is not $\Sym^2(E)$.

Lemma 3.33  The self-intersection of $\Sigma_i$ on $\Sym^2(E)$ is the divisor class $(p_i, p_i)$ on $\Sigma_i$. On $\tilde{\Sym}^2(E)$, the self-intersection of $\tilde{\Sigma}_i$ is $(p_i, p_i) - \sum_{j=0, i \neq j}^3 (p_i, p_j)'$, where $(p_i, p_j)'$ is the point of the exceptional divisor above $(p_i, p_j) \in \Sym^2(E)$ that lies on $\tilde{\Sigma}_i$.  

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Proof On $\text{Sym}^2(E)$ one checks easily that the self-intersection number is 1, so it suffices to identify the point. Let $p' \neq p_i \in E$ correspond to the section $x \to \{p', x\}$. Clearly these two sections intersect at the point $\{p_i, p'\}$, so the same remains true in the limit $p' \to p_i$.

We consider the map $\hat{\text{Sym}}^2(E) \to \text{Sym}^2(E)$ and apply the push-pull formula, finding that $\hat{\Sigma}_i \cdot (\hat{\Sigma}_i + \sum_j E_{ij}) = \Sigma_i^2$. But $\hat{\Sigma}_i \cdot E_{ij} = \{p_i, p_j\}'$. The result follows. □

Lemma 3.34 Let $q$ be a point of $E$ such that $2q = p_0 + p_1 + p_2 + p_3$ in the group law of $E$. Let $F_q$ be the fibre of $\text{Sym}^2(E) \to E$ lying above $q$. Then the divisor

$$D_g = \sum_{i=0}^3 \hat{\Sigma}_i - 2(\hat{\Sigma}_0 + \hat{\Sigma}_1 + E_{01} + F_{23} - F_q)$$

on $\hat{\text{Sym}}^2(E)$ is principal.

Proof The Picard group of $\hat{\text{Sym}}^2(E)$ is an extension of the Néron-Severi group by $\text{Pic}^0$. The Néron-Severi group is generated by the classes of a section, a fibre, and the exceptional curves. On the other hand, $\text{Pic}^0$ is a 1-dimensional abelian variety and is easily seen to be isomorphic to $E$ by the map $\text{Sym}^2(E) \to \text{Sym}^2(E) \to E$.

The $\hat{\Sigma}_i$ have self-intersection $-2$ and $\hat{\Sigma}_i$ meets $E_{jk}$ in degree 1 if $i \in \{j, k\}$ or 0 otherwise. Also $(E_{ij}, F_{k_l}) = 1$ for $(i, j) = (k, \ell)$, else 0. Since the $E_{ij}$ are $-1$-curves, it is now clear that $(D_g, E_{ij}) = 0$ for all $i, j$; that $D_g$ has degree 0 on fibres is also obvious.

In addition, we easily compute that $(D_g, \hat{\Sigma}_0) = 0$, but we need to check that the intersection is the principal divisor class on $\Sigma_0$. The self-intersection of $\Sigma_0$ on $\text{Sym}^2(E)$ is the divisor $\mathcal{O}(P)$, where $P$ is the point $\{p_0, p_0\}$ on $\Sigma_0$. When we blow up the points $\{p_0, p_i\}$ for $1 \leq i \leq 3$, the self-intersection becomes $\mathcal{O}(p_0 - p_1 - p_2 - p_3)$. On the other hand, the other $\hat{\Sigma}_i$ are disjoint from $\hat{\Sigma}_0$, and $E_{01}, F_{23}$ intersect it in $\{p_0, p_1\}$ and $\{p_0, p_2 + p_3 - p_0\}$. Finally, $F_q, F_0$ intersect in $\{p_0, q - p_0\}, \{p_0, -p_0\}$. Thus the intersection of $D_g$ with $\Sigma_0$ is given by the divisor $-(p_0) + (p_1) + (p_2) + (p_3) - 2(p_1) - 2(p_2 + p_3 - p_0) + (2q - p_0)$. This is a divisor of degree 0 and the points add to 0, so it is a principal divisor. □

We would like to say that $V$ is the double cover of $\hat{\text{Sym}}^2(E)$ ramified along the $\hat{\Sigma}_i$ obtained by adjoining a square root of the function with divisor $D_g$. However, this is only well-defined when the point $q$ is specified. In fact this is the same choice that we made in Definition 3.6 where we defined a double cover of $E$ branched at $p_0, \ldots, p_3$: choosing such a double cover is equivalent to defining a function with divisor $(p_0) + (p_1) + (p_2) + (p_3) - 2D$, where $D \in \text{Pic}^2(E)$. This is equivalent to choosing the origin (as we did there) or $q$ (as we do here).

We now observe that the $\hat{\Sigma}_i$ are the pullbacks of the special sections of $\hat{M}_{0,5} \to \hat{M}_{0,4}$, while the map $E \to \hat{M}_{0,4}$ is induced by the family maps both $p_i + p_j$ and $p_k + p_{\ell}$ to the boundary point $\delta_{ij} = \delta_{k\ell}$. Accordingly it is the quotient by the negation map $p \to \sum_{i=0}^3 p_i - p$. The fibres fixed as sets by this negation map are those at the four points $(\sum p_i)/2$. Comparing the canonical divisors, we see that all four of these are ramified in the cover.

Proposition 3.35 The extension of function fields corresponding to the cover $V \longrightarrow \hat{M}_{0,5}$ is Galois and its Galois group is the Klein four-group.

Proof The $\hat{\Sigma}_i$ are pullbacks of the special sections $\Sigma_i : \hat{M}_{0,4} \to \hat{M}_{0,5}$, while $E_{01} + F_{23}$ is the pullback of $E_{01} \subset \hat{M}_{0,5}$; the discussion just above shows that $F_q$ is the pullback of a single fibre above the image $q'$ of $q$ in $\hat{M}_{0,4}$. Accordingly the divisor $D_g$ above is the
pullback of the divisor $\sum_{i=0}^{3} E_i - 2(E_0 + E_1 + E_{01}) + F_{q'}$ on $\overline{M}_{0,5}$ (recall the notation introduced in Definition 2.12), which has intersection 0 with all $E_{ij}$ and $E_1$ and is therefore principal. Let $g \in k(\overline{M}_{0,5})$ be a function with this divisor (in terms of the interpretation of $\mathcal{M}_{0,5}$ as a blowup of $\mathbb{P}^2$, the fibre $F_{q'}$ is defined by a quadratic form $Q$, while $E_{01}$ is defined by a linear form $L; the pullback of $Q/L^2$ to $\mathcal{M}_{0,5}$ is the desired function). Then the cover $\tilde{V} \to \Sigma^2(E)$ is given by adjoining a square root of $g$. Let the $F_{p_i}$ be the fibres above the $p_i$, let $D$ be a divisor which is the sum of two arbitrary fibres, and let $f$ be a function with divisor $\sum_{i=0}^{3} F_{p_i} - 2D$. Then we have

$$k(V) = k(\Sigma^2(E))(\sqrt{g}) = k(\overline{M}_{0,5})(\sqrt{\mathcal{F}}, \sqrt{\mathcal{G}}).$$

Since $f, g \in k(\overline{M}_{0,5})$, and clearly their ratio is not a square, the result follows. □

Let $v_V$ be the negation automorphism of the elliptic surface $\tilde{V}$ (the quotient of this is $\Sigma^2(E)$, whose function field is $k(\overline{M}_{0,5})(\sqrt{\mathcal{F}})$, so at the level of fields it is given by $\sqrt{\mathcal{F}} \to \sqrt{\mathcal{F}}$, $\sqrt{\mathcal{G}} \to -\sqrt{\mathcal{G}}$. Let $\lambda_V$ be the involution of $\tilde{V}$ induced by the automorphism $\sqrt{\mathcal{F}} \to -\sqrt{\mathcal{F}}$, $\sqrt{\mathcal{G}} \to \sqrt{\mathcal{G}}$.

**Remark 3.36** Since the divisor of $g$ has odd multiplicity along $F_{q'}$, the fibre above $q'$ is unramified in the quotient map $\tilde{V} \to V/\lambda_V$, while the other three fibres above points ($\sum p_i)/2$ are ramified in that extension. The opposite is true for $\lambda_V v_V$.

We are now ready to consider $\tilde{W}$. This involves changing the base from $\overline{M}_{0,5}$ to $\mathcal{C}_0 \times \overline{M}_{0,4}$, the covers are $\Sigma^2(E) \times E$ and $\tilde{W} = \tilde{V} \times E$ (cf. Fig. 4). The choice of $\lambda_V$ determines an origin on $E$, which will be denoted $O_\lambda$.

**Definition 3.37** We define an automorphism $\lambda = \lambda_{\tilde{W},Q}$ of $\tilde{W}$ as the map corresponding to the maps $\tilde{W} \to \tilde{V} : \lambda_V \circ \pi_1$ and $\tilde{W} \to E_n : (-1)_{E_n} \circ \pi_2$ by the universal property of a product, where $(-1)_{E_n}$ is the negation on $E_n$ with origin $Q$. (Note that $\lambda$ induces the negation on $E$, so we must compose with $(-1)_{E_n}$ to obtain maps that make the diagram commute.)

We restate this, together with information on the fixed fibres, as a theorem.

**Theorem 3.38** For all lifts $Q$ of $O_\lambda$ to $E_n$, the negation map of $E_n$ with origin at $Q$ lifts to an automorphism $\lambda_Q$ of $\tilde{W}$ that preserves the elliptic fibration and the sections $\tilde{S}_i$. The map $\lambda$ acts as $-1$ on the fibres at the 2-torsion points of $E_n$ relative to the origin $Q$ that are not in the kernel of the isogeny $E_n \to E$ and as $+1$ on those that are.

**Proof** The automorphism $\lambda_Q$ is that of Definition 3.37. The automorphism of $E_n$ induced by $\lambda_W$ lifts the negation on $E$ induced by $\lambda$. The only fibres of $\tilde{W} \to E_n$ that are fixed as sets by $\lambda$ are those above the 2-torsion of $E_n$ relative to the chosen origin, and these are fixed pointwise or not according as the fibres over the images of these points in $E$ are fixed pointwise or not (clear from the description of the map as a fibre product). □

**Remark 3.39** In particular, the automorphism of $E_n$ given by translation by the difference of two points mapped to the same point by the isogeny to $E$ also lifts to an automorphism of $\tilde{W}$.

Let $v = v_W$ be the negation automorphism of $\tilde{W}$ with respect to one of the $\tilde{S}_i$ as zero section (in light of Proposition 3.23 they all give the same negation automorphism). Let $\alpha_P : \phi^{-1}(P) \to \phi^{-1}(-P)$ be an isomorphism that takes the intersection of the zero section...
with $\phi^{-1}(P)$ to its intersection with $\phi^{-1}(-P)$. Since $\lambda$ preserves the zero section, it must identify $\phi^{-1}(P)$ with $\phi^{-1}(P)$ either by $\alpha_P$ or by $-\alpha_P$. In either case it commutes with $v$. Because $\lambda$ and $v$ commute on an open subset of $\tilde{W}$, they commute on all of $\tilde{W}$ and generate a Klein four-group.

**Proposition 3.40** Both $\tilde{W}/(\lambda\nu)$ and $\tilde{W}/(\lambda)$ have $3n$ fibres of type $I_2$. If $n$ is even then both have two fibres of type $I_0^*$, and if $n$ is odd then $\tilde{W}/(\lambda)$ has one and $\tilde{W}/(\lambda\nu)$ has three. There are no other singular fibres.

**Proof** Recall that $\tilde{W}$ has $6n$ fibres of type $I_2$, of which $n$ contain the strict transform of a point of $\Sigma_i \cap \Sigma_j$ for each subset $\{i, j\} \subset \{0, 1, 2, 3\}$ of order 2. Both $\lambda$ and $\lambda\nu$ identify these fibres for a given $\{i, j\}$ with those for the complement, so the quotient has $3n$ such fibres. These are the only singular fibres of $\tilde{W}$ and the image of a smooth fibre cannot be singular in the quotient unless it is fixed setwise and the involution acts as negation, in which case we obtain an $I_0^*$ fibre. (The quotient of an elliptic curve by negation is a rational curve, which will be double in the quotient fibration, and we need to blow up the four fixed points.)

As in Theorem 3.38, then, the number of points of order 2 of $E_n$ that are (respectively, are not) in the kernel of the map $E_n \to E$ is the number of $I_0^*$ fibres of $\tilde{W}/(\lambda)$ (resp. $\tilde{W}/(\lambda\nu)$). Clearly this is as claimed. \qed

**Corollary 3.41** The desingularizations of $\tilde{W}/(\lambda\nu)$ and $\tilde{W}/(\lambda)$ have Euler characteristics $12\left[\frac{n-1}{2}\right], 12\left[\frac{n+1}{2}\right]$ respectively, and they have $h^{2,0} = \left[\frac{n-1}{2}\right], \left[\frac{n+1}{2}\right]$. In particular $\tilde{W}/(\lambda)$ is a K3 surface for $n = 2, 3$.

**Proof** For the first statement, we recall that the Euler characteristic of an elliptic surface is the sum of those of the singular fibres and use the fact that $I_0^*$ and $I_2$ fibres have Euler characteristic 6, 2 respectively. Alternatively, we could compute this by determining the Euler characteristics of $\tilde{W}$ and of the fixed loci of $\lambda, \lambda\nu$. The second follows from the first by applying [31, (III.4.2)]. \qed

**Remark 3.42** In the case $n = 2$, our construction gives two K3 surfaces. In addition, we have two abelian surfaces, namely the Prym varieties of the double covers $C_2 \to D_3$ and $D_3 \to E$. It is natural to expect a relation between these, and indeed we verified in some examples that the K3 surfaces are isogenous in the sense of Definition 1.4 to the Prym varieties of these two covers. In any case, they come with elliptic fibrations that have two $\tilde{D}_4$ and six $\tilde{A}_1$ fibres, and so the results of [36] already show that they are covered by the square of a curve of genus 5 and construct an explicit correspondence between them and the square of the Kuga–Satake varieties.

The $\tilde{A}_1$ fibres are above the same points of $\mathbb{P}^1$. If we twist to construct a surface $E_2$ with $\tilde{D}_4$ fibres above the four points where one of the surfaces $\tilde{W}/(\lambda), \tilde{W}/(\lambda\nu)$ has such a fibre, we obtain an elliptic surface with $h^{2,0} = 2$ and Kodaira dimension 1. It turns out that the double cover $E_2 \to E$ in this case induces an involution $\mu$ of $W_2$ that acts nontrivially on the base. Letting $\nu$ be the negation map for the fibration, we find that $W_2/(\mu), W_2/(\mu\nu)$ are both K3 surfaces that come with an elliptic fibration with (generically) three fibres of type $I_0^*$ and three of type $I_2$, full level-2 structure, and Picard rank 17. The quotient of such a fibration by a 2-torsion translation has three $I_0^*$ fibres, an $I_4$, and a 2-torsion point. Under the genericity assumption that the rank is 17, this determines the Picard lattice completely. Since the same reducible fibres and torsion subgroup arise for fibration type 4 in the list in [24], the quotients are the Kummer surfaces of principally polarized abelian surfaces. These surfaces are not quotients of $\tilde{W}$ and there appears to be no obvious connection between the
corresponding abelian surfaces and the curves of our construction; nevertheless, we do obtain a correspondence on a subvariety of the square of the moduli space of curves of genus 2.

**Remark 3.43** In the case \( n = 4 \), we obtain, as quotients of \( \tilde{W} \), two surfaces with \( h^{2,0} = 2 \) and Kodaira dimension 1. These admit involutions by which the quotients are isogenous to Kummer surfaces as above. If we twist to remove the \( \tilde{D}_4 \) fibres, we obtain a K3 surface, with an elliptic fibration with 12 fibres of type \( I_2 \) and generic Mordell–Weil rank 1; compare Remarks 2.19 and 2.27. Such a K3 surface has Picard number 15, and we would not expect it to arise from a construction such as this one. In Sect. 2 we constructed a special family of surfaces of this type with generic Mordell–Weil rank 2; however, we will show in Proposition 3.55 that these are isogenous to double covers of \( \mathbb{P}^2 \) branched along six lines, so the fact that they are quotients of the square of a curve already follows from the main result of [36].

**Remark 3.44** In [36], in which \( n = 2 \), the choice of \( \lambda \) is less relevant because \( n\lambda = 2 \) for both possibilities and Paranjape is not concerned with the degree of the map of moduli spaces. Thus it is not necessary for him to distinguish between \( \lambda \) and \( \lambda \nu \), as it is for us.

For \( n = 5 \), we obtain an elliptic surface over \( \mathbb{P}^1 \) with 15 fibres of type \( I_2 \) and one of type \( I_0^* \). We proved by checking a single example that there is no automorphism of the base that permutes the \( I_2 \) fibres, so there is no automorphism of the surface that could have a K3 surface as its quotient (recall that the elliptic fibration is the map associated to the canonical divisor, so it is preserved by all automorphisms of the surface). Furthermore, we have found by numerical calculation for small primes that the characteristic polynomial of Frobenius acting on the transcendental lattice, whose dimension is 12, is often absolutely irreducible. That is to say, it remains irreducible when any power of the variable is substituted for the variable; it follows that the Galois representation on the transcendental lattice over any finite field of the appropriate characteristic is irreducible and hence that there can be no nontrivial correspondence to a K3 surface.

In the case \( n = 6 \), we obtain a curve \( C_n \) of genus 13 which is an unramified cover of degree 6 of \( D_3 \). The intermediate covers \( C_{3,2}, C_{3,3} \) have genus 5, 7, so the quotient \( \text{Jac}(C_n)/(\text{Jac}(C_{3,2}) + \text{Jac}(C_{3,3})) \) is of dimension \( 13 - 5 - 7 + 3 = 4 \) and we expect to obtain a family of K3 surfaces of Picard number 16. In terms of our construction here, we obtain surfaces with \( h^{2,0} = 3 \). These surfaces should map to those given by the \( n = 2, n = 3 \) constructions, which suggests that there should be a third quotient which has \( h^{2,0} = 1 \) and is therefore a K3 surface. We expect that it will be a surface already covered by our main result, Theorem 1.6, since the Hodge structure will admit an action of an order of \( \mathbb{Q}(\sqrt{-3}) \) rather than of some other quadratic ring (cf. Sect. 4).

We will apply the following alternative characterization of \( \lambda \) in the case \( n = 3 \). However, the proof is valid for all \( n > 2 \).

**Proposition 3.45** For \( n > 2 \), the \( \tilde{G}_i \) and \( \tilde{G}'_i \) (Definition 3.20) are preserved by \( \lambda \), whereas \( \lambda \nu \tilde{G}_i = \tilde{G}'_i \) and vice versa.

**Proof** First we show that \( \lambda \nu \tilde{G}_i \neq \tilde{G}'_i \): this is the only part of the proof that requires \( n > 2 \). If they were equal, then \( \tilde{G}_i \) would have to meet the fibres fixed as sets but not pointwise by \( \lambda \nu \) in a point on one of the \( \tilde{S}_i \), these being the only points on such fibres fixed by \( \lambda \nu \). Since \( \tilde{G}_i \) and \( \tilde{S}_j \) are disjoint for \( i \neq j \), and \( \tilde{G}_i \cdot \tilde{S}_i = n \), this would only be possible if all \( n \) points of intersection were on these fibres. This cannot happen, because the fibres differ by 2-torsion whereas the images of the points of intersection in \( E_n \) differ by \( n \)-torsion.

Next we show that \( \nu \tilde{G}_i = \tilde{G}'_i \) and \( \nu \tilde{G}'_i = \tilde{G}_i \). Note that \( \nu \) lifts to the automorphism of \( C_n \times C_n \) that acts as the identity on one copy of \( C_n \) and \( \beta \) on the other. This in turn descends
to the automorphism of $D_3 \times D_3$ that acts as the identity on one copy and $\iota$ on the other. This automorphism exchanges $R_i$ and $R'_i$ and the claim follows.

Finally we prove that the set $\{\tilde{G}_i, \tilde{G}'_i\}$ is preserved by $\lambda$ and $\lambda_v$. Let $L_i = \lambda(\tilde{G}_i)$ and consider the intersection matrix $M_n$ of the divisors $\tilde{S}_i, \tilde{G}_i, \tilde{G}'_i, L_i, F, A_1, \ldots, A_{6n}$, as summarized in Table 1, where $F$ is the class of a fibre and the $A_j$ are the nonzero components of the $\tilde{A}_1$ fibres of the fibration. We know all of the intersections among these divisors, except for $\tilde{G}_i \cdot L_i$ and $\tilde{G}'_i \cdot L_i$, as follows:

- We saw in Proposition 3.26 that $\tilde{S}_i \cdot \tilde{G}_i = \tilde{S}_i \cdot \tilde{G}'_i = n$. Since $\lambda(S_i) = S_i$, it follows that $\tilde{S}_i \cdot L_i = n$ as well.
- In Proposition 3.27 we found that $\tilde{G}_i \cdot \tilde{G}'_i = n$.
- In Remark 3.31 we showed that $K_{\tilde{W}} \sim nF$. By the adjunction formula, we conclude that the self-intersection of every section is $-n$ (recall that the base is an elliptic curve, so the canonical divisor of a section is $0$). Likewise the self-intersection of a fibral rational curve is $-2$.
- As usual we have $F^2 = F \cdot A_i = A_i \cdot A_j = 0$ for $i \neq j$.
- By definition $\tilde{S}_i \cdot A_j = 0$ for all $j$. On the other hand $\tilde{G}_i \cdot A_j = \tilde{G}'_i \cdot A_j = 1$ for all $j$ by Proposition 3.30.

Let $\tilde{G}_i \cdot L_i = a$, $\tilde{G}'_i \cdot L_i = b$. The divisor $\tilde{S}_i + \tilde{G}_i + \tilde{G}'_i$ has self-intersection $3n$, so by the Hodge index theorem $M_n$ cannot have negative determinant. We now show that $\det M_n = -n \cdot 2^{6n} \cdot (a + b)^2$. Add $1/2$ the sum of the last $6n$ rows to rows $2, 3, 4$, which does not change the determinant but makes the matrix block triangular. The bottom right block is $-2I_{6n}$ of determinant $2^{6n}$, and the top left block is a $5 \times 5$ matrix whose determinant is checked to be $-n(a + b)^2$. Since the determinant must be nonnegative, the conclusion is that the determinant is $0$ and so $a \leq 0$ or $b \leq 0$.

It is not possible for $\tilde{G}_i \cdot L_i$ to be $0$: these are irreducible curves with nonempty intersection (at the points of $\tilde{G}_i \cap \tilde{S}_i$), so that would require $\tilde{G}_i = L_i$ and $\tilde{G}'_i = 0$. The second of these is false. Therefore either $a < 0$ and $L_i = \tilde{G}_i$, or $b < 0$ and $L_i = \tilde{G}'_i$. We ruled out the second choice above, so the first must hold: that is, $\lambda(\tilde{G}_i) = \tilde{G}_i$.

**Definition 3.46** Let $K = \tilde{W}/(\lambda)$. Let $\pi_K : K \rightarrow \mathbb{P}^1$ be the induced elliptic fibration on $K$ arising from the fibration $\tilde{W} \rightarrow E_n$. Let $T_i, J_i, J'_i$ be the images of the curves $\tilde{S}_i, \tilde{G}_i, \tilde{G}'_i$, the strict transforms of the $S_i, G_i, G'_i$ on $\tilde{W}$, on $K$ (recall that the $S_i, G_i, G'_i$ were defined in Definitions 3.15 and 3.20).
By construction, relative to the origin $O$ on $E$ we have $p_i + p_j = -(p_k + p_l)$ when \( \{i, j, k, l\} = \{1, 2, 3, 4\} \), so for all lifts $q_i, q_j, q_k$ of $p_i, p_j, p_k$ to $E_n$ there is a lift $q_l$ of $p_l$ with $q_i + q_l = -(q_k + q_j)$. As these are the points lying under the singular fibres of $\tilde{W} \to E_n$, these fibres are identified in pairs by the quotient map $\tilde{W} \to \tilde{W}/(\lambda)$. In addition, the quotient map takes every fibre that is negated to a rational curve along which there are four singularities: we thus introduce fibres of type $I^*_0$ there. So we have proved:

**Theorem 3.47** The elliptic fibration $\pi_K$ has $3n$ reducible fibres of type $I_2$. It also has one of type $I^*_0$ when $n$ is odd and two when $n$ is even.

**Proposition 3.48** The $J_i, J'_i$ are sections of the induced fibration $K \to \mathbb{P}^1$.

**Proof** More generally, we consider the following situation. Let $X_1 \to C_1$ be an elliptic surface, and let $\alpha$ be an involution of $X_1$ preserving the set of fibres and acting nontrivially on $C_1$: let $X_2 = X_1/\alpha$, so that $X_2$ has an induced elliptic fibration to $C_1/\alpha$, and let $\pi : X_1 \to X_2$ be the quotient map. Let $F_1, F_2$ be the classes of a fibre on $X_1, X_2$ respectively. Then $\pi_* (F_1) = F_2$ and $\pi^* (F_2) = 2F_1$ up to algebraic equivalence. Thus if $S_1$ is a section on $X_1$, we have $2 = (S_1, \pi^* F_2) = (\pi_* S_1, F_2)$. In other words, $S_1$ descends to a section of $X_2 \to C_1/\alpha$ if and only if $\alpha(S_1) = S_1$ (the condition that $\alpha$ act nontrivially on $C_1$ prevents a section from being ramified in the quotient).

The claim now follows from Proposition 3.45.

We now choose $T_i$ as the zero section of $\pi_K$. Let $z_i, a_i$ be the curves in the $I_2$ fibres of $K$ through which $T_i$ does (respectively does not) pass.

**Lemma 3.49** (1) For $j = 2, 3, 4$, the curves $T_j$ pass through $2n$ of the $a_i$ and $n$ of the $z_i$.

(2) The curves $J_i$ pass through exactly those of the $z_i, a_i$ that are disjoint from $T_j$.

**Proof** This follows from Proposition 3.30. The $6n$ singular fibres are identified by $\lambda$ in pairs in such a way that a fibre containing the exceptional divisor above a point in $S_i \cap S_j$ goes to one coming from an intersection point of $\tilde{S}_k \cap \tilde{S}_\ell$ (where again $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$). Thus $T_i$ passes through the same component as $T_i$ for exactly $2n/2 = n$ of the singular fibres. For the second statement, it suffices to recall from Corollary 3.22 that $E_i$ contains the points $(p_j, p_k)$ of $D_3 \times D_3$ when $\#\{i, j, k\} = 3$, since that implies that $G_j$ passes through the points of $S_j \cap S_k$ and hence that $J_i$ meets the exceptional divisors above these.

**Remark 3.50** The symmetry of the situation with respect to translation by the 2-torsion divisors $T_j - T_i$ implies that any two $T_i$ and any two $J_i$ pass through the same component of exactly $n$ of the $\tilde{A}_i$ fibres, while $T_i$ and $J_j$ pass through the same component of $2n$ if $i \neq j$.

(Again this follows from Proposition 3.30 as well.)

**Proposition 3.51** We have $J_i \cdot T_j = 0$ for $i \neq j$ and $J_i \cdot T_i = \lfloor \frac{n-1}{2} \rfloor$.

**Proof** For $i \neq j$ the curves $\tilde{S}_j$ and $\tilde{G}_i$ are disjoint, while $\lambda$ fixes $S_j$ as a set: the proves the first equality. For the second, we would like to use the push-pull formula for the map $q_\lambda : \tilde{W} \to K$. However, this requires that we blow up the isolated fixed points of $\lambda$ on $\tilde{W}$: let the blowup be $\hat{W}$. These include the points on the fixed fibres above $O$ at which $S_1$ meets $G_i$ (cf. Proposition 3.26). For $n$ odd there is one such fibre and for $n$ even there are 2, so the intersection on $\hat{W}$ is 1 or 2 less than on $\tilde{W}$ and is therefore $n-1$ or $n-2$ (Corollary 3.26).

Now applying the push-pull formula and observing that $S_j$ maps to $T_j$ with degree 2, we see that $2T_j \cdot J_i = \tilde{S}_j \cdot \tilde{G}_i$, and the result follows.

\( \square \)
Theorem 3.52 The sections $J_i$ are of infinite order in the Mordell–Weil group of $\pi_K$, and further we have $[J_i] - [J_j] = [T_i] - [T_j]$. 

Proof In view of the second statement, it suffices to prove the first for $i = 0$. If it were not true, then the Picard class of $J_i$ would be in the subspace of Pic $K \otimes \mathbb{Q}$ spanned by curves in reducible fibres and the $T_j$. We will obtain a contradiction by solving for $[J_i]$ under the assumption that it is in this subspace of Pic $K \otimes \mathbb{Q}$. Note that a section $S$ of $\pi_K$ satisfies $S(K_K + S) = -2$; since $K_K$ is $\lfloor \frac{n+3}{2} \rfloor$ times the fibre class, this means that $S^2 = \frac{3}{2} - \lfloor \frac{n}{2} \rfloor$. 

We first consider the case of $n$ odd. Choose a basis consisting of the fibre class $F$, the 0 section $S_0$, the nonzero components $d_1, \ldots, d_4$ of the $I_0^n$ fibre, and the nonzero components $a_1, \ldots, a_{3n}$ of the $3n$ fibres of type $I_2$. The class of $J_i$ has intersection 1 with all of these basis vectors except for the $d_i$, with which its intersection is 0. One easily computes that if $[J_i]$ is in the given subspace its class is $C = \frac{n+3}{2} F + S_0 - \sum_{i=1}^{3n} a_i/2$ and that $C^2 = -\frac{5}{2} - n \notin \mathbb{Z}$, a contradiction.

Now let $n$ be even. In this case there is no $I_0^n$ fibre and our basis has $3n + 2$ elements, each of which intersects the putative class $[J_i]$ once. Again one sees that $[J_i] = nF + S_0 - \sum_{i=1}^{3n} a_i/2$. One of the torsion sections has the class $(n - 1)F + S_0 - \sum_{i=1}^{2n} a_i/2$; the intersection of these is $1 - \frac{n}{2}$, which is negative for $n \geq 4$, a contradiction. One also checks that $[J_i]^2 = 2 - n$. This contradicts our calculation of the self-intersection of a section for $n \neq 4$ and in particular for the last remaining possibility $n = 2$.

To show that $[J_i] - [J_j] = [T_i] - [T_j]$ in the Mordell–Weil group of the fibration, we return to the definitions. Recall that $J_i$ was obtained by pulling back a divisor $R_i$ from $D_2 \times D_3$ to $C_n \times C_n$ and mapping down to $W$ and then $K$. Now, on $D_2 \times D_3$ we have the linear equivalence $[R_i] + [H_j] + [V_i] \sim [J_i] + [H_j] + [V_j]$, since both of these are the class $O(1, 1) - [D]$, where $O(1)$ refers to the embedding of $D_3$ as a plane quartic (cf. Proposition 3.18) and $D$ is the diagonal on $D_2 \times D_3$. We pull this back to $C_n \times C_n$ and then map down to $K$. The inverse image of $R_i$ on $C_n \times C_n$ maps with degree $2n$ to $G_i \subset W$, while those of $H_i$ and $V_i$ map with degree $n$ to $S_i \subset W$. Thus, by pushing forward to $W$ and back to $\tilde{W}$, we obtain the relation $2n([J_i] + [T_i]) = 2n([J_j] + [T_j])$ up to a linear combination of the exceptional divisors of $\tilde{W}$ above the singular points of $W$. These divisors are vertical for the fibration, so this relation holds in the Mordell–Weil group of $\tilde{W}$ (as elliptic surface over $E$) up to torsion.

So $[J_i] + [T_i] = [J_j] + [T_j]$ up to torsion, and the two sides agree in the component groups of all $A_1$ fibres. Thus, in the Mordell–Weil group, the class of $[J_i] - [J_j] + ([T_i] - [T_j])$ is a torsion section $T$ passing through the zero component of every reducible fibre. The height of a torsion section is 0, but if we apply [8, Lemma 1.18] to compute the height pairing of such a section with itself, there are no correction terms and we obtain $0 = -(T_0 - T)^2 = -T_0^2 - T^2 + 2T_0 \cdot T$. For $T_0 \neq T$ we have $T_0 \cdot T = 0$; but this is not possible since all sections have the same self-intersection and this cannot be 0 for any fibration that is not a product. □

We now specialize to $n = 3$ so as to study the situation of a K3 surface with Picard lattice $L_1$. In this case we have a complete description of the Mordell–Weil group, at least for a generic choice of initial data.

Theorem 3.53 Suppose that the Picard number of $K$ is 16. Then the Mordell–Weil group of $K$ is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ and any one class $J_i - T_j$ generates the group modulo torsion.

Proof The given description of the singular fibres together with the assumption on the Picard number shows (by means of the Shioda-Tate formula) that the Mordell–Weil rank is 1. Further, the discriminant of the lattice spanned by the vertical curves and the given sections is 192,
so if its index in the full Picard group is \( d \) then \( 192/d^2 \in \mathbb{Z} \), so that \( d \) is a power of 2. Thus the only question is whether there is a nonempty subset of \( \{ T_1, T_2, J_0 \} \) whose sum can be divided by 2. To see that this is not the case, note simply that the component groups of the singular fibres are all of exponent 2, so that no section can be divided by 2 if it passes through a nonzero component on any singular fibre. But, as already noted, the \( T_j \) pass through six of the \( b_i \) for \( i \neq j \) and \( J_0 \) through all nine; it follows that \( T_1 + T_2 \) also passes through six, while \( J_0 + T_1 + T_2 \) and the \( J_0 + T_j \) pass through three.

\[ \square \]

### 3.4 Proof of the main theorem and two variants

At this point we have constructed a new map \( \kappa_{\text{mod}} : \mathcal{M}'_3 \to \mathcal{M}'_{K,1} \) (recall that these are respectively the moduli space of degree-3 covers \( C_0 \to \mathcal{M}_{0,4} \) with a choice of ramification point and of K3 surfaces with a suitable elliptic fibration and a labelling of the 2-torsion sections; we take the torsion sections in the order \( T_1, T_2, T_3 \)). To reiterate, starting from a point of \( \mathcal{M}'_3 \) we successively construct \( E, E_3, D_3, C_3, W, \tilde{W}, K \), of which the last is a K3 surface with the desired additional data (the generator \( J_0 \) matches the section of infinite order denoted by \( G \) in Fig. 1). As pointed out at the beginning of this section, the choice of \( D_3 \) is not determined by the initial data and the field of moduli of \( D_3 \) may not be the same as that of \( E \) and the four points on it, but since \( K \) is determined up to isomorphism by the initial data its field of moduli is the same.

We are now in a position to prove the main theorem (Theorem 1.6) of the paper. In view of Remark 3.4, this is equivalent to proving Theorem 1.10.

**Theorem 3.54** The map \( \kappa_{\text{mod}} \) coincides with the map \( \kappa \) defined in Definition 2.20.

**Proof** We showed in Proposition 2.22 that a fibration of this type is determined by the locations of its singular fibres, so it suffices to show that these are the same for the two constructions. Recall that the definition of \( \kappa \) started from a triple cover \( \phi : C_0 \to \mathcal{M}_{0,4} \) and associated to it an elliptic surface with a \( D_4 \) fibre at the third point in a chosen ramified fibre and with \( \tilde{A}_1 \) fibres above the boundary points of \( \mathcal{M}_{0,4} \). To see that this construction does the same, we start by locating the \( D_4 \) fibre. It is above the chosen origin \( q'' \) of \( E_3 \) above \( q' \), which in turn lies above the point of \( \mathcal{M}_{0,4} \) under the chosen ramified fibre. In the associated map \( E_3/\pm \to E/\pm \), the fibre above the image of the origin of \( E \) consists of \( q'', q'' + T, q'' - T \), where \( T \) is a point of order 3. Thus the map is ramified there and the origin of \( E_3 \) is the third point of the fibre, which is \( q'' \).

The \( \tilde{A}_1 \) fibres of \( V \) lie above the points \( p_i + p_j \) of \( E \), as we saw in Proposition 3.13. Recall that \( p_0 = O \) and \( p_i = B_j + B_k \) where \( \{ i, j, k \} = \{ 1, 2, 3 \} \). So the \( \tilde{A}_1 \) fibres lie above the points \( B_1 + B_2 + B_3 \pm B_i \) in \( E \), and therefore above their inverse images in \( E_3 \). These are paired by the negation map to which \( \lambda \) descends on \( E_3 \), which must therefore fix a point above \( B_1 + B_2 + B_3 \). Composing with the translation by \( -(B_1 + B_2 + B_3) \) on \( E \), we find that the \( A_1 \) fibres are above the points \( \pm B_i \) and that the \( D_4 \) fibre is above \( O \). Then the \( D_4 \) fibre in this structure is at the origin of \( E_3 \), which maps to the unramified point of \( E_3/\pm \) in the fibre above the image of \( q' \) in \( E/\pm \). So the locations of the singular fibres match in the two constructions.

\[ \square \]

Now we briefly discuss the case \( n = 4 \) to indicate why it does not receive more detailed attention in this paper. As in the case \( n = 3 \), we express a 4-parameter family of K3 surfaces as quotients of squares of curves, but this essentially follows from the results of [36] and does not require our construction.
Proposition 3.55 The K3 surfaces of Picard number 16 obtained from a point of $\mathcal{M}_4'$ admit maps of degree 4 to and from double covers of $\mathbb{P}^2$ branched along six lines.

Proof Generically, the Picard lattice is generated by the following curves: the components of the 12 fibres of type $\tilde{A}_1$; the zero section and the 2-torsion sections; two disjoint sections $C_1, C_2$ whose classes together with the torsion generate a subgroup of the Mordell–Weil group of index 2, each passing through the nonzero components of all reducible fibres and meeting the zero section twice; and a section whose class in the Mordell–Weil group is $(C_1 + C_2)/2$. (The sections $C_1, C_2$ are obtained by pulling back conics as in Proposition 2.15, like the section $G$ in the case $n = 3$. We know that the section of class $C_1 + C_2$ is divisible by 2: some linear combination of those already mentioned must be, since otherwise the discriminant group would need 8 generators, which is not possible for the Picard lattice of a K3 surface of rank 16. On the other hand, no other class in the Mordell–Weil group mod 2 can be divisible by 2, because this is the only one other than 0 that passes through the zero components of all the reducible fibres.)

By enumerating the genus of the Picard lattice, we find that this surface also admits a fibration with eight fibres of type $\tilde{A}_1$ and two of type $\tilde{A}_3$ with full level-2 structure, such that there is a 2-torsion section passing through the identity component of both $A_3$ fibres and no $\tilde{A}_1$ fibre. The quotient by translation by this section is therefore a fibration with two $\tilde{A}_7$ fibres and a 2-torsion section disjoint from the identity component on both reducible fibres. By [39] this is the full torsion subgroup, so we have determined the lattice.

In turn, this quotient surface admits a fibration with reducible fibres of type $\tilde{A}_3, \tilde{A}_3, \tilde{D}_8$ and a 2-torsion section. Since this section has height 0 (or by direct computation) it must pass through the far component of each of the reducible fibres, which means that the quotient of the corresponding isogeny has a $\tilde{D}_6$ fibre and eight of type $\tilde{A}_1$ (the other six being the images of singular irreducible fibres). Again by [39] the quotient must have full level-2 structure and no further torsion; this determines the lattice, and it is in the same genus as the frame of the standard fibration on a double cover of $\mathbb{P}^2$ branched along six lines. (The fibration we have shown to exist is of class 2.7 in [22, Table 2].)

Thus we have shown the existence of a pair of 2-isogenies—more precisely, quotients by van Geemen–Sarti involutions—whose composition is a map from a surface obtained by our construction to a suitable double cover of $\mathbb{P}^2$. Considering the dual isogenies, we also obtain a map in the other direction.

To close this section, we remark that a general statement can be made using Proposition 2.26. In terms of Definition 2.24, we have shown that the elliptic surfaces parametrized by $\mathcal{M}_{\varepsilon, 3, ((2,1),(2,1))}$ are motive-finite. It is not necessary to introduce $\mathcal{M}_{\varepsilon, n, R}$ to discuss the case $n = 3$, because the ramification data associated to the cover $E_3/\pm \to E/\pm$ are generic. However, this is necessary for larger $n$; applying our construction in general, we conclude:

Theorem 3.56 Let $R$ be the ramification data consisting of two copies of $(2, ..., 2)$ and two of $(2, ..., 2, 1, 1)$ for $n$ even, or four copies of $(2, ..., 2, 1)$ for $n$ odd. Then the elliptic surfaces parametrized by $\mathcal{M}_{\varepsilon, n, R}$ are covered by the square of a curve, and in particular have Kimura-finite motive.

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4 Hodge theory

Following the discussion in [36, Sections 1–2], we reinterpret the construction of Sect. 3 in terms of Hodge theory. In the case $n = 3$ that we are considering, the construction can be summarized as follows:

1. The cover $C_3 \rightarrow D_3$ has a generalized Prym variety $P_4$ which is an abelian variety of dimension $7 - 3 = 4$.
2. The automorphism of order 3 on $C_3$ acts on $H^1(P_4, \mathbb{Q})$ to make it a 4-dimensional vector space over $\mathbb{Q}(\sqrt{-3})$.
3. The symplectic form on $H^1$ gives an hermitian form with respect to the $\mathbb{Q}(\sqrt{-3})$-structure whose invariants can be computed.
4. The decomposition of the $\mathbb{Q}(\sqrt{-3})$-vector space $H^1(P_4, \mathbb{Q}(\sqrt{-3}))$ into components of dimensions 2, 2 gives a decomposition of $\Lambda^2(H^1(P_4, \mathbb{Q}))$ into components of dimensions 1, 4, 1, which we interpret as the transcendental lattice of a K3 surface.

Remark 4.1 The only cases in which we obtain a generalized Prym variety of dimension 4, either for Paranjape’s original construction or in our variant, are those with $n = 2, 3, 4, 6$. For $n = 4$ our construction produces a curve of genus 9 which is an unramified cover of degree 4 of a curve of genus 3; the intermediate cover of degree 2 has genus 5. Again, the moduli space is a finite cover of that of the base curve of genus 1 with 4 points, so it has dimension 4.

In the case $n = 6$, we obtain a curve $C_6$ of genus 13 which is an unramified cover of degree 6 of a curve of genus 3; the intermediate covers have genus 5, 7. Thus $\text{Jac}(C_6)$ has abelian variety factors of dimension 3, 5 – 3, 7 – 3, and the quotient by the sum of these has dimension 4. These considerations are related to those of Remark 3.44.

Definition 4.2 As in Definition 4.4, the hyperbolic lattice $\mathbb{H}$ is the lattice of rank 2 with a basis $x, y$ such that $x^2 = y^2 = 0, xy = 1$. Given a lattice $L$, let $L(n)$ be the lattice of the same rank whose Gram matrix is $n$ times that of $L$.

In this section, we will prove:

Proposition 4.3 The abelian variety $A$ associated to a K3 surface $K$ with transcendental lattice isometric to $\Lambda_3 = \langle -2 \rangle + \langle -2n \rangle + \mathbb{H} + \mathbb{H}$ by the Kuga–Satake–Deligne correspondence is a power of the Prym variety of the double cover $C_3 \rightarrow D_3$ up to isogeny. Further, there is a correspondence between the Shimura varieties corresponding to $SO((-2) + (-6) + \mathbb{H} + \mathbb{H})$ and the unitary group of $\mathbb{H} + \mathbb{H}$ for $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$ that parametrize the appropriate Hodge structures.

The discussion in [36, Section 2] up to the end of the proof of [36, Lemma 2] applies to our situation almost word for word, given the differences we previously pointed out in Remark 3.9. Since our automorphism is of order 3 rather than 4, the action on the spaces $V$ and $W$ is by $\zeta_3, \zeta_3^2$ rather than by $\pm i$. In addition, the discriminant of the Hermitian structure $H$ in our case is a power of 3, but again this is enough to ensure that it is a norm from $\mathbb{Q}(\sqrt{-3})$ to $\mathbb{Q}$. The result of Landherr [26] used in [36, Section 2] does not depend on the particular quadratic extension that arises.

One might speculate that for other quadratic fields $\mathbb{Q}(\sqrt{-n})$ there would be an analogous construction with the automorphism replaced by a correspondence, perhaps based on Paranjape’s idea or our variant. One would expect that the discriminant of the Hermitian structure...
would be a power of \( n \) and therefore a norm in the extension \( \mathbb{Q}(\sqrt{-n})/\mathbb{Q} \) and that the argument would still apply. (See [28, 29] for interesting discussions of the Hodge structure for the associated abelian Kuga–Satake variety and for computations similar to the ones below.)

In any case, even in the absence of a geometric construction beyond the simplest cases \( n = 1, 3 \), let \( F \) be a \( \mathbb{Q}(\sqrt{-n}) \)-vector space of dimension 4 equipped with a basis \( e_1, f_1, e_2, f_2 \) and an hermitian form with \( H(e_1, f_1) = H(f_1, e_1) = 1 \) and all other pairings equal to 0. We carry through this calculation in detail in order to obtain a precise statement at the end. We consider \( \Lambda^2(F) \) with the basis

\[
a_1 = e_1 \wedge f_1, \quad a_2 = e_1 \wedge e_2, \quad a_3 = e_1 \wedge f_2, \\
b_1 = e_2 \wedge f_2, \quad b_2 = f_2 \wedge f_1, \quad b_3 = f_1 \wedge e_2
\]

(chosen such that \( a_i \wedge b_i \) is independent of \( i \)). We define \( H \) on \( \Lambda^2 F \) by defining \( H(p \wedge q, r \wedge s) = H(p, s)H(q, r) - H(p, r)H(q, s) \), which is linear in \( p, q \) and antilinear in \( r, s \). It also satisfies \( H(p \wedge q, r \wedge s) = -H(q \wedge p, r \wedge s) = -H(p \wedge q, s \wedge r) \); thus it gives a well-defined form on \( \Lambda^2 F \), which is easily seen to be Hermitian. For example,

\[
H(a_1, a_1) = H(e_1 \wedge f_1, e_1 \wedge f_1) \\
= H(e_1, f_1)H(f_1, e_1) - H(e_1, e_1)H(f_1, f_1) \\
= 1 \cdot 1 - 0 \cdot 0 = 1.
\]

Similar calculations show that \( H(b_1, b_1) = H(a_2, b_2) = -H(a_3, b_3) = 1 \), while all other products are 0. Likewise, we may extend \( H \) to a symmetric form on the 1-dimensional space \( \Lambda^4 F = \langle g \rangle \), where \( g = a_i \wedge b_i \). Doing so, we find that \( H(g, g) = H(a_1, b_1)H(b_1, a_1) - H(a_1, a_1)H(b_1, b_1) = 1 \).

Still following Paranjape, we now define a \( \mathbb{Q}(\sqrt{-n}) \)-antilinear automorphism \( t \) of \( \Lambda^2 F \) by the condition that \( H(u_1, u_2)g = -u_1 \wedge tu_2 \) for all \( u_1, u_2 \in \Lambda^2 F \). For example, the only nonzero evaluation of \( H \) involving \( a_1 \) is \( H(a_1, a_1) = 1 \), while the only nonzero wedge product is \( a_1 \wedge b_1 = g \). Thus we must have \( t(a_1) = -b_1 \), and similarly \( t(b_1) = -a_1 \), \( t(a_2) = -a_2 \), and \( t(b_2) = -b_2 \) while \( t(a_3) = a_3 \) and \( t(b_3) = b_3 \). The invariant subspace for this involution is generated by

\[
a_1 - b_1, \quad (a_1 + b_1)\sqrt{-n}, \quad a_2\sqrt{-n}, \quad b_2\sqrt{-n}, \quad a_3, \quad b_3.
\]

By an easy calculation, the matrix \( M_H \) of \( H \) on this basis is

\[
M_H = \begin{pmatrix}
-2 & 0 & 0 & 0 & 0 \\
0 & -2n & 0 & 0 & 0 \\
0 & 0 & -2n & 0 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
\]

**Definition 4.4** Let \( M_n \) be the lattice \( \langle -2 \rangle + \langle -2n \rangle + \mathbb{H} + \mathbb{H} \).

Note that, for all \( k \neq 0 \), the quadratic space \( \mathbb{H} \otimes \mathbb{Q} \) is isomorphic to the quadratic space over \( \mathbb{Q} \) with Gram matrix \( \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \), by multiplying one of the generators by \( k \). So, by a suitable change of basis, the matrix \( M_H \) can be rewritten to coincide with the Gram matrix of the standard basis of the lattice \( M_n \).
In this section, we have shown the following statement.

**Proposition 4.5** Over $\mathbb{Q}$, the quadratic form $H$ is isomorphic to the quadratic form associated to $M_n$.

Combining Proposition 4.5 with the results of the previous section, we obtain a proof of Proposition 4.3.

The following result, due to Lombardo, gives additional information:

**Proposition 4.6** ([28, Corollary 6.3 and Theorem 6.4]) Notation as in Proposition 4.3.

1. There is an isogeny $A \sim P_4^4$;
2. $\mathbb{Q}(\sqrt{-3}) \subseteq \text{End}_\mathbb{Q}(P_4)$;
3. $P_4$ admits a polarization $H$ such that $(P_4, H, \mathbb{Q}(\sqrt{-3}))$ is a polarized abelian variety of Weil type with $\text{disc}(P_4, H, \mathbb{Q}(\sqrt{-3})) = 1$.

**Corollary 4.7** There is an inclusion of Hodge structures $T(K) \hookrightarrow H^2(P_4^4 \times P_4^4, \mathbb{Q})$, where $K$ is as in Proposition 4.3.

This corollary follows directly from Proposition 4.3.

**Theorem 4.8** The Hodge conjecture is true for $K \times K$.

**Proof** The proof is the same as for [38, Theorem 2], mutatis mutandis as in Proposition 4.6 and Corollary 4.7 above.

**Remark 4.9** To close this section, we remark that a K3 surface $S$ with Picard lattice isometric to $L_1$ is isogenous to a K3 surface $S'$ with transcendental lattice $\mathbb{Q}$-isometric to $\langle -2 \rangle + \langle -6 \rangle + \mathbb{H} + \mathbb{H}$. The signature is correct: the Hodge index theorem shows that the signature of the transcendental lattice of a K3 surface of Picard rank $\rho$ is $(2, 20 - \rho)$. Indeed, we will show in Corollary 6.10 that $S$ is 2-isogenous to a K3 surface whose Picard lattice is $\mathbb{Q}$-isometric to $\mathbb{H} + D_8 + A_5 + A_1$, and hence to $\mathbb{H} + E_8 + A_5 + A_1$. We embed this lattice primitively into $\mathbb{H}^3 + E_8^2$ by matching copies of $\mathbb{H}$ and $E_8$ and embedding $A_5 + A_1$ into $E_8$ in one of the obvious ways; it is routine to calculate the Gram matrix of the orthogonal complement of $A_5 + A_1$ in $E_8$ and show that it is equivalent to the diagonal matrix with diagonal $(-2, -6)$.

## 5 Scope of the construction

We have just shown that $U(\mathbb{H} + \mathbb{H}, \mathbb{Q}(\sqrt{-n}))$ is a double cover of $SO(\langle -2 \rangle + \langle -2n \rangle + \mathbb{H} + \mathbb{H})$; in the case $n = 1$ this was already remarked by Paranjape. More generally, let $P$ be a Hodge structure of type $(4, 4)$ with an action of an order $\mathcal{O} = \mathcal{O}_K$ in an imaginary quadratic field. We may then view $P$ as an Hermitian module $P'$ of rank 4, which is determined by its signature (necessarily $(2, 2)$) and its discriminant up to norms from $K$.

If the discriminant is a square, then as above we have the exceptional isogeny $SU(P', \mathcal{O}/\mathbb{Z})/\pm 1 \to SO(M_n)$, where $M_n \cong \langle -2 \rangle + \langle -2n \rangle + \mathbb{H} + \mathbb{H}$. If it is not, then as in [18, 2.13] we do not have an isogeny defined over $\mathbb{Q}$ to the special orthogonal group of any lattice; the analogue of Paranjape’s map still has all eigenvalues equal up to sign and square equal to the discriminant, but that now means that the eigenspaces are not defined over $\mathbb{Q}$. 
Suppose that there is a curve $C$ for which the Hodge structure on $H^1(C)$ admits $P$ as a quotient. We might then expect that $C^2$ admits a correspondence to a K3 surface whose transcendental lattice is isometric to $L$ after tensoring with $\mathbb{Q}$. Thus, in order to determine the limitations of this construction, we would like to know which lattices of rank 6 arise from the exceptional automorphism in this way.

Our starting point is the following observation. Let $S$ be a K3 surface. Define the projective variety $Q(S) = Q(T(S)) \subset \mathbb{P}^n(\mathbb{Q})$, where $n = 21 - \rho$, by the vanishing of the discriminant quadratic form. The relation of the transcendental lattices of isogenous K3 surfaces is well understood. Before giving the result, we introduce an important lattice.

**Definition 5.1** Let $V = \mathbb{H}^3 + E_8^2$, where $E_8$ is the exceptional root lattice with negative sign. Sometimes $V$ is known as the *K3 lattice*.

**Remark 5.2** It is well known that $H^2$ of a K3 surface, with the usual intersection form on cohomology given by cup product, is isometric to $V$: see, for example, [32, Section 1].

**Theorem 5.3** [1, Section 1.8] Let $\phi : S \rightarrow S'$ be a rational map of degree $n$ of K3 surfaces. Then $T(S')$ is isometric to a sublattice of $T(S)(n)$ for which the quotient has exponent dividing $n$.

**Corollary 5.4** Let $S, S'$ be isogenous K3 surfaces (as always, in the sense of Definition 1.4). Then $Q_S \cong Q_{S'}$ as varieties over $\mathbb{Q}$.

**Proof** This follows immediately from Theorem 5.3 by induction on $k$, because rescaling does not affect the quadric at all, while passing to a sublattice amounts to a linear change of coordinates.

We now note:

**Proposition 5.5** Let $L$ be the family of K3 surfaces with a given marked Picard lattice $L$ and transcendental lattice $L^\perp$, and let $S \in L$. Then $Q(S)$ is isomorphic to a linear section of $Q(L^\perp)$.

**Proof** Indeed, the hypothesis implies that $L$ is embedded in $\text{Pic} S$, whence $T(S)$ is primitively embedded in $L^\perp$. Thus $Q(S)$ is obtained from $Q(L^\perp)$ by restricting to the linear subspace on which elements of $T(S)^\perp$ vanish.

**Corollary 5.6** The same conclusion holds if $S$ is isogenous to a member of $L$.

**Proof** This follows immediately from the theorem and proposition just above.

### 5.1 Examples and counterexamples

In this section, we give some examples to indicate the limits of our construction. First we describe some families of K3 surfaces of Picard number 16 for which the discriminant of the Picard lattice is $-12$ times a square but to which our construction does not apply. Following that, we show that no finite union of families of K3 surfaces of Picard number 16 includes all K3 surfaces of Picard number 17 or 18 in its closure. It follows that no finite collection of constructions for K3 surfaces of rank 16 suffices to prove that every K3 surface of Picard number 17 or 18 is covered by the square of a curve.
Example 5.7 We consider the example \( n = 3 \), the primary concern of this paper. One readily computes that the Hasse–Minkowski invariant \([4, Chapter 4.1]\) of \( M_3 \) is 1 at all finite primes. Multiplying the form by a prime \( p \equiv 0, 1 \mod 3 \) does not change the invariants, while multiplying by \( p \equiv 2 \mod 3 \) changes the invariants at 3 and \( p \).

Let \( L' \) be a lattice of discriminant 3 times a square, signature \((2, 4)\), and nontrivial Hasse–Minkowski invariant at a prime \( p \equiv 1 \mod 3 \). It follows from Corollary 5.4 that a K3 surface with transcendental lattice isometric to \( L' \) is not isogenous to one with transcendental lattice isometric to \( M_3 \). In particular, such a K3 surface cannot be shown by our construction to be covered by a product of curves.

For example, we may take \( L' \) to have an orthogonal basis with elements that square to \(-1, -1, -2, -6, 7, 7\), so that the Hasse–Minkowski invariant is nontrivial at 2 and 7. Proving that a K3 surface with transcendental lattice commensurable with \( L' \) is motive-finite would seem to require a significant new idea.

We now study K3 surfaces of Picard number 17 and 18. The following result is due to Nikulin:

Theorem 5.8 \([34, Theorem 3]\) There is a lattice \( K \) of rank 16 and discriminant 64 such that a K3 surface \( S \) is a Kummer surface if and only if there is a primitive embedding of \( K \) into Pic \( S \).

Nikulin describes \( K \) as follows \([34, Definition 1]\). Let \( G = (\mathbb{Z}/2\mathbb{Z})^4 \), and start with the lattice \( A_G^4 \) (that is, the lattice generated by 16 vectors indexed by \( G \) of norm \(-2\) of which any two distinct ones are orthogonal). Adjoin the vectors of the form \( \sum_{i \in C} v_i / 2 \), where \( C \) is a coset of a subgroup of \( G \) of index 2.

It is easy to see that \( K^\perp \) in \( V \) is isometric to \( \mathbb{H}(2)^3 \). It follows that \( Q(K) \) is \( \mathbb{Q} \)-isomorphic to the quadric in \( \mathbb{P}^5 \) defined by \( x_0 x_1 + x_2 x_3 + x_4 x_5 = 0 \), since \( \mathbb{H}(2) \cong \mathbb{H} \), as pointed out after Definition 4.4.

Corollary 5.9 Let \( S \) be a K3 surface of Picard number 17 (respectively 18). If there are no rational lines (resp. rational points) on \( Q(S) \), then \( S \) is not isogenous to a Kummer surface.

Proof Both statements follow immediately from the existence of rational planes on \( Q(K) \), in light of Proposition 5.5. \( \square \)

Corollary 5.10 Let \( S \) be a K3 surface of Picard number 17 (respectively 18), and let \( p \) be a prime congruent to 1 mod 4. If there are no \( \mathbb{Q}_p \)-rational lines (resp. \( \mathbb{Q}_p \)-rational points) on \( Q(S) \), then \( S \) is not isogenous to a double cover of \( \mathbb{P}^2 \) branched along six lines.

Proof Paranjape shows \([36, Lemma 1]\) that the generic transcendental lattice of a double cover of \( \mathbb{P}^2 \) branched along six lines is isometric to \( \langle -2 \rangle + \langle -2 \rangle + \mathbb{H} + \mathbb{H} \), which is isometric over \( \mathbb{Q}(i) \) and hence over \( \mathbb{Q}_p \) to \( \mathbb{H}(2) + \mathbb{H} + \mathbb{H} \). In turn this is isometric to \( \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H} \), as noted after Definition 4.4. \( \square \)

We recall that every K3 surface of Picard number 19 or 20 is isogenous to a Kummer surface \([32, Corollary 6.4 (i)]\). For completeness, we include an example to illustrate the well-known fact that this is not true in Picard number 18.

Example 5.11 We describe an example of a family of K3 surfaces of Picard number 18 that are not isogenous to Kummer surfaces. Let \( T \) be the lattice with Gram matrix

\[
\begin{pmatrix}
-2 & -1 & 0 & -1 \\
-1 & 2 & 1 & -1 \\
0 & 1 & -2 & 1 \\
-1 & -1 & 1 & 2
\end{pmatrix}
\]
Thus a K3 surface with transcendental lattice Gram in Magma and then change basis by a suitable combination of signs and a permutation. This lattice has norm form \( \mathbb{Z} \) (this was a random choice of two orthogonal vectors of norm 6). By direct calculation this lattice has norm form \( \mathbb{Z} \)-equivalent to \(-2x^2 - 2y^2 + 6z^2 + 6w^2 = 0\). Adjoining \((1/2, 1/2, 1/2, 1/2)\) to this lattice produces a lattice isometric to \(T\) (to check this, use LLL-Gram in Magma and then change basis by a suitable combination of signs and a permutation). Thus a K3 surface with transcendental lattice \(T'\) can be written as a double cover of \(\mathbb{P}^2\) branched on six very general lines after tensoring with \(\mathbb{Q}\) [36, Lemma 1]. Inside this lattice, consider the orthogonal complement \(T' \cap \mathbb{P}^2\) of the subspace spanned by the vectors
\[(0, 0, 1, -2, -1, 1), (0, 0, 1, -1, 1, -2)\]
(this was a random choice of two orthogonal vectors of norm 6). By direct calculation this lattice has norm form \(\mathbb{Z}\)-equivalent to \(-2x^2 - 2y^2 + 6z^2 + 6w^2 = 0\). Adjoining \((1/2, 1/2, 1/2, 1/2)\) to this lattice produces a lattice isometric to \(T\) (to check this, use LLL-Gram in Magma and then change basis by a suitable combination of signs and a permutation).

It is not difficult to construct such K3 surfaces. To do so, we observe that \(T \perp\) (relative to a primitive embedding \(T \hookrightarrow V\)) is isometric to \(E_6 + E_6 + D_4\). For any \(c \neq 0\), the elliptic surface \(y^2 = x^3 + sx + (s + cs^2)\) has an \(IV^*\) fibre at \(s = \infty\) and a type \(II\) fibre at \(s = 0\), so a quadratic twist by \(s(s - d)\), where \(d \neq 0\), is a surface of the desired form, and this gives 2 moduli of such surfaces as expected.

**Example 5.12** In [2, Section 9.4.4], Boxer, Calegari, Gee and Pilloni present a construction of a certain family of K3 surfaces due to Nori. Let \(L_0, \ldots, L_4\) be five lines in \(\mathbb{P}^2\) and let \(C\) be the conic through the five points \(L_i \cap L_i+1\) (indices read mod 5). Let \(L\) be a sixth line tangent to \(C\), and let \(S\) be the double cover of \(\mathbb{P}^2\) branched along the \(L_i\) and \(L\). They show that these K3 surfaces are not isogenous to Kummer surfaces for very general choices of \(L_i, L\).

We verify this claim in a different way from that of [2]. First we note that the Picard lattice is generated by the classes of the hyperplane, the nodes, the lines, and the components of the pullback of \(C\), and that it is of rank 17. We compute that the discriminant is 96 and that the discriminant group is the same as for \(\mathbb{H} + E_8 + A_2 + A_1^5\), so that the lattices are in fact isometric by [32, Theorem 2.8]. To describe \((\text{Pic} S)\perp\), we embed this into \(V\). We embed \(\mathbb{H} + E_8\) into one copy of \(\mathbb{H} + E_8\) in \(V\). Then \(A_2 + A_1^5\) can be embedded primitives into \(E_8\) by embedding the Dynkin diagram: for example, we may take the three roots corresponding to the neighbours of the vertex of index 3 and the two that are not adjacent to any of these. The complement is then \(A_1 + A_2(2)\). Finally, the other \(A_1\) components are embedded in \(\mathbb{H}\), each with complement \(\langle 2 \rangle\). So the transcendental lattice is isometric to \(A_1 + A_2(2) + (2) + (2)\). The Fano variety of lines on the corresponding quadric in its Plücker embedding is a smooth variety of dimension 3 and degree 8 and has no points over \(\mathbb{Q}\). So by Corollary 5.9, \(S\) is not isogenous to a Kummer surface.

On the other hand, as pointed out in [2], the transcendental lattice can be rationally embedded in \(\mathbb{H} + \langle -6 \rangle + \langle -2 \rangle + \langle -2 \rangle\) and hence in \(\mathbb{H} + \mathbb{H} + \langle -6 \rangle + \langle -2 \rangle\). Thus we have constructed the fake abelian surface associated to the generalized Prym variety with endomorphisms by an order in the division algebra \(D = (-1, 3)\mathbb{Q}\) whose existence was suggested in [2, 9.4.4].
More generally, the same ideas can be used to prove that no finite collection of types of K3 surfaces corresponding to a rank-16 lattice suffices to describe all K3 surfaces of rank 17 or 18.

**Theorem 5.13** Let $L = \{L_1, \ldots, L_n\}$ be a finite set of lattices of rank 16 and let $r \in \{17, 18\}$. Then there is a K3 surface $S_r$ of Néron-Severi rank $r$ such that no element of $L$ can be rationally embedded into $\text{Pic} S_r$.

Note that the theorem does not require the $L_i$ to have signature $(1, 15)$, as they would for the Picard lattice of a projective surface; the signature $(0, 16)$ is also permitted.

**Proof** We may assume that the $L_i$ can be primitively embedded in $V$. Let the $Q_i$ be the quadric hypersurfaces in $\mathbb{P}^5(\mathbb{Q})$ defined by the norm forms of the orthogonal complements of the $L_i$ in $V$. Since the $Q_i$ are defined by indefinite quadratic forms, they have real points; also, smooth quadrics of dimension $\geq 4$ over local fields always have rational points, so the $Q_i$ are everywhere locally solvable and hence have rational points.

Recall that the two families of $k$-planes on a smooth quadric of dimension $2k$ over a field $F$ of characteristic not equal to 2 are defined over $\mathbb{Q}((\sqrt{-1})^k D)$, where $D$ is the determinant of the symmetric matrix associated to the quadric. Let the fields of definition of the families of 2-planes on the $Q_i$ be the $\mathbb{Q}((\sqrt{D_i}))$. Since the $Q_i$ have rational points, they have planes defined over the $\mathbb{Q}((\sqrt{D_i}))$. In particular, every hyperplane section over this field has rational lines and every $\mathbb{P}^3$-section has rational points. Let $p$ be an odd prime that splits in every $\mathbb{Q}((\sqrt{D_i}))$, let $N_p$ be a lattice of signature $(2, 2)$ with norm form $x^2 - ny^2 + pz^2 - npw^2$, where $n$ is a quadratic nonresidue mod $p$, and let $T_p$ be an even sublattice of $N_p$ of full rank. By [32, Theorem 2.8], we can embed $T_p$ primitively into $V$. However, the quadric associated to $T_p$ has no $\mathbb{Q}_p$-points and therefore none over any of the $\mathbb{Q}((\sqrt{D_i}))$, because the completion of any of these at a prime over $p$ is isomorphic to $\mathbb{Q}_p$. Thus a K3 surface of Picard number 18 with transcendental lattice $T_p$ is not isogenous to any surface into whose Picard lattice any of the $L_i$ embeds.

Similarly, if we take $T_p$ to be an even lattice of rank 5 admitting a rational embedding of $N_p$, then the corresponding quadric has no $\mathbb{Q}_p$-rational lines, because it has a hyperplane section with no $\mathbb{Q}_p$-points, and hence it is not a hyperplane section of any of the $Q_i$. □

**Corollary 5.14** With notation as in the theorem, no element of $L$ can be embedded into $\text{Pic} S'$ for any K3 surface $S'$ isogenous to $S_r$.

**Proof** This follows by combining Theorem 5.13 with Corollary 5.6. □

**Theorem 5.15** There exist K3 surfaces of Picard number 17 that are covered by the square of a curve but that are not isogenous to a Kummer surface or a double cover of $\mathbb{P}^2$ branched along 6 lines.

**Proof** Fix a prime $p \equiv 17 \mod 24$, and let $L_p$ be the lattice with Gram matrix $\langle -2 \rangle + \langle -6 \rangle + \mathbb{H} + \langle 4p \rangle$. Again, this is the transcendental lattice of a K3 surface, which we will call $S_p$. One easily checks that the local invariant at $p$ of the associated quadratic form is $-1$; this means that the quadric defined by this form has no lines over $\mathbb{Q}_p$. It follows from Proposition 5.5 and Corollary 5.9 that $S_p$ is not isogenous to a Kummer surface, and from Proposition 5.5 and Corollary 5.10 that $S_p$ is not isogenous to a double cover branched along 6 lines.

On the other hand, since $\mathbb{H}$ has a primitive vector of norm $4p$, the lattice $L_p$ can be embedded primitively in $\langle -2 \rangle + \langle -6 \rangle + \mathbb{H} + \mathbb{H}$, and thus $S_p$ belongs to the family of K3
surfaces whose general member we have shown to be covered by the square of a curve (Theorem 1.6). Since our map of moduli spaces is only a birational equivalence and not an isomorphism we cannot conclude that the $S_p$ are covered in this way. However, a birational equivalence cannot fail to be defined on infinitely many divisors, so the general $S_p$ can be covered for all but finitely many $p$.

Example 5.16 In order to apply this argument to show that a K3 surface of rank 18 is isogenous neither to a Kummer surface nor to a double cover branched along 6 lines, we need a transcendental lattice whose norm form is not solvable at a prime congruent to 1 mod 8. The smallest examples appear to have determinant $2^2 \cdot 17^2$; one possible Gram matrix is

$$\begin{pmatrix} 6 & 5 & 3 & -3 \\ 5 & 6 & -2 & 4 \\ 3 & -2 & -6 & -2 \\ -3 & 4 & -2 & 6 \end{pmatrix}.$$ 

Note that a K3 surface with this transcendental lattice would be expected from our results to be covered by the square of a curve of genus 7. We do not know this for certain because we have only constructed a birational equivalence of moduli spaces, not an isomorphism. (The method used to prove the last theorem does not apply here, because an infinite set of codimension-2 loci could all be contained in a single divisor.) Indeed, this transcendental lattice tensored with $\mathbb{Q}$ is isometric to $(-2) + (-6) + (17) + (51)$, as one sees by computing Hasse–Minkowski invariants. This embeds into $((-2) + (-6) + \mathbb{H} + \mathbb{H}) \otimes \mathbb{Q}$ by a map that takes the first two basis vectors to the first two basis vectors and the last two to vectors in the two copies of $\mathbb{H}$ of the appropriate norm.

6 K3 surfaces

We now study the family of K3 surfaces with Picard lattice isometric to $L_1$ (Definition 1.3) in order to relate them to other interesting families.

Definition 6.1 Let $K$ be a K3 surface with Picard lattice isometric to $L_1$.

Our work in previous sections provides a proof of the following proposition:

Proposition 6.2 There is an explicitly computable algebraic correspondence between $C_3^2$ and $K$, where $C_3$ depends on $K$ and is as defined in Sect. 3. This correspondence induces a morphism of Hodge structures $H^2(C_3, \mathbb{Z}) \to H^2(K, \mathbb{Z})$ when the base field of $K$ has characteristic zero.

In this section, among other things, we prove the following statements:

Proposition 6.3 $K$ is isogenous to a K3 surface $K_{gs}$ in $\mathbb{P}^4$ with 15 ordinary double points.

Theorem 6.4 The Hodge conjecture is true for fourfolds $K_{gs} \times K_{gs}$.

The proofs appear at the end of the paper. The notation $K_{gs}$ is in honour of Garbagnati and Sarti, who first discussed such surfaces in [17].

Combining Proposition 6.2 and Proposition 6.3 gives an explicitly computable correspondence between $C_3^2$ and $K_{gs}$.
Remark 6.5 Laterveer proved [27, Theorem 3.1] that K3 surfaces of degree 8 with a faithful symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$ are motive-finite, by showing that the quotients are double covers of $\mathbb{P}^2$ branched along six lines and invoking the results of [36]. He suggested [27, Remark 3.3] that it would be interesting to prove an analogous result for other families of K3 surfaces with a faithful symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$. In light of the result of Garbagnati and Sarti [17, Theorem 8.6 (2)], there is such a family of K3 surfaces that admits a map to the $K_g$, and Laterveer’s arguments now show that these are also motive-finite.

We believe, as suggested by Laterveer, that this should be possible for all K3 surfaces with an action of $(\mathbb{Z}/2\mathbb{Z})^4$, and in fact we have a construction for one more family. We may return to this in future work.

Remark 6.6 Recall that in Definition 1.3 we defined $L_1$ as a certain lattice of rank 16 and signature $(1, 15)$ containing a sublattice $L_0$ isometric to $\mathbb{H} + D_4 + A_1^5$ with quotient $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$. We suppose given a fibration as in Definition 2.1 (cf. Fig. 1: Theorem 3.47), with nine singular fibres of type $I_2$, one of type $I_3^*$, and Mordell–Weil group $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$. Recall the notation: the section $0_{\phi_1}$ has been chosen as the origin, the $T_i$ are the torsion sections, and $G$ is a generator of the Mordell–Weil group modulo torsion that passes through the nonzero component of all the $A_1$ fibres. Let the $a_i$, $b_i$ be respectively the zero and nonzero components of the $I_2$ fibres, and the $d_i$ for $0 \leq i \leq 3$ the reduced components of the $I_2^*$, where $d_0$ is the zero component. By [8, Lemma 1.18], the canonical height of the section $G$ is 3/2. Note that the intersection of $G$ with any of the 2-torsion sections is 0. This can be checked from [8, Lemma 1.18] or directly by writing down the classes of the 2-torsion sections in $L_0 \otimes \mathbb{Q}$.

Proposition 6.7 On the surface $K$ there is an elliptic fibration $\phi_3$ with one singular fibre of type $I_2^*$, one of type $I_3$ and six of type $I_2$. Its Mordell–Weil group is $\mathbb{Z}/2\mathbb{Z}$, and the nonzero section passes through the zero component of the $I_3$ the nonzero component of every $I_2$, and the reduced component of the $I_2^*$ that is at distance 2 from the zero component. There is one $I_1$ fibre.

Proof We consider the sections $G, -G$ together with one of the $b$, say $b_j$. We saw in Proposition 2.7 that $G \cdot -G = 1$. So any two of $G$, $-G$, $b_j$ have intersection 1 and they constitute an $I_1$ fibre $F$, and we obtain a genus-1 fibration. (This is not at all the same as constructing a fibration with $G \cup -G \cup 0_{\phi_1}$ as a fibre, as we did in Definition 2.5 and Fig. 2.) The $T_j$ are disjoint from $\pm G$, as is seen by calculating the height pairing. So if $j$ is one of the two indices such that $T_j$ meets the chosen $b_i$, then $T_j$ is a section of $\phi_3$. Computing the trivial lattice of the fibration 1.1, we find the given singular fibre types and torsion. (Each of the $I_2$ fibres contains one of the zero components of an $I_2$ fibre of the fibration $\phi_1$ that meets $T_2$ or $T_3$, shown in blue and red in Fig. 1, while the components of the $I_2^*$ include the zero components of the two remaining $I_2$ fibres and $T_1$.) The rank of the Mordell–Weil group is 0, by the Shioda-Tate formula [37, Corollary 6.13]. As a consistency check we note that the discriminant of the lattice $\mathbb{H} + D_6 + A_2 + A_1^5$ is $-4 \cdot 3 \cdot 2^6$, and dividing this by 4 for the 2-torsion point gives $-192$, which we know to be the discriminant of $L_1$. There cannot be a fibre of type $IV$ because there is 2-torsion (recall that we are assuming that the characteristic of the ground field is not 2), so the fibre whose $ADE$ type is $I_2$ must be an $I_3$. The reducible fibres contribute 23 to the Euler characteristic. This implies the statement about singular irreducible fibres. □

The elliptic fibration $\phi_3$ described in Proposition 6.7 is depicted in Fig. 5 for the choice of $b_j = b_1$. The colours red, blue, green and orange match those in Figs. 1 and 2. The horizontal red and blue curves are the sections $T_2$, $T_3$. The other curves in the figure are the components...
of the singular fibres. The blue curve labelled $O_{I_2^*}$ marks the identity component of the $I_2^*$ singular fibre when we choose $T_2$ to be the zero section of the fibration.

**Remark 6.8** Using Nishiyama’s method [35], or by enumerating the lattices in the genus, one finds that there are 25 possible frames for elliptic fibrations on $K$ up to isomorphism. When the trivial lattice corresponding to a frame is not saturated, there is an isogeny to another K3 surface $K'$, and we have also shown that $K'$ is covered by the square of a curve. This occurs for 8 of the 25 frames. This can be continued indefinitely, finding elliptic fibrations with torsion on the quotient surfaces and passing to the quotient. For the state of the art in determining the elliptic fibrations on a K3 surface we refer the reader to [15].

It turns out that the discriminant of the Picard lattice of the quotient of $K$ by the 2-torsion translation is much smaller than that for $K$, which makes the quotient easier to work with in some ways. In particular, it has only a few types of genus-1 fibration, as we will see in Remark 6.14.

**Definition 6.9** Let $L$ be the quotient of $K$ by the 2-torsion translation for the fibration of Proposition 6.7 and let $L_1'$ be its Picard lattice.

**Corollary 6.10** The fibration on $L$ induced by that of Proposition 6.7 has reducible fibres of types $I_4^*$, $I_6$, $I_2$, and Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$. In particular $\text{disc } L_1' = -12$.

**Proof** The quotient of the given $I_2^*$ is of type $I_4^*$, since the 2-torsion section passes through the reduced component that meets the same nonreduced component as the zero section. The $I_3$ fibre produces an $I_6$ on the quotient, and the Shioda-Tate formula again shows that there must be one more reducible fibre, which can only arise from an $I_1$ fibre of $K$ (computing the Euler characteristic shows that there is exactly one such fibre).

Since $K$ has a 2-torsion point, so does the 2-isogenous surface $L$. On the other hand, it is easily checked that $\mathbb{H} + D_8 + A_5 + A_1$ is not a sublattice of index 4 in any even lattice. Thus $\text{disc Pic } L = -4 \cdot 6 \cdot 2/2^2 = -12$.

**Proposition 6.11** The surface $L$ admits a fibration in curves of genus 1 with no section and reducible fibres of type $II^*$, $IV^*$.
Proof We define an $II^*$ fibre whose support consists of the zero section and all curves in the $I_4^*$ fibre except for the reduced curve adjacent to the zero component, and an $IV^*$ fibre whose support consists of the 2-torsion section and the nonzero components of the $I_2$ and $I_6$ fibres. It is readily checked that these sets of curves have the correct topology to support the given fibres and that these fibres are linearly equivalent. There cannot be a section, for if there were the discriminant of the Picard lattice would be $-3$ rather than $-12$. \(\square\)

Lemma 6.12 The property of having finite motive is invariant under maps of finite degree of K3 surfaces.

Proof Let $S \to T$ be such a map. If $S$ has finite motive, then $T$ is covered by the same power of a curve as $S$. The other direction is proved in the course of proving [27, Theorem 3.1]. \(\square\)

Corollary 6.13 A general K3 surface with Picard lattice isometric to $L_1'$ or $\mathbb{H} + E_8 + E_6$ is covered by curves, and hence has finite-dimensional motive.

Proof We have seen that a general K3 surface with Picard lattice $L_1'$ is covered by one with Picard lattice $L_1$ and hence by curves. In turn, there is a primitive class of self-intersection 0 in $L_1$ with even intersection with all of $L_1$, and this class can be chosen to be that of a curve of genus 1 by a standard result [20, Corollary 8.2.9]. The Picard lattice of the Jacobian of the associated fibration is then an even lattice containing $L_1'$ with index 2. Using the results of Nikulin it is easy to check that this lattice is isometric to $\mathbb{H} + E_8 + E_6$.

Clearly the map from the moduli space of K3 surfaces with marked Picard lattice $L_1$ to that of surfaces with lattice $\mathbb{H} + E_8 + E_6$ has finite fibres and is therefore surjective. We have shown in Sect. 2 that a general K3 surface with Picard lattice $L_1$ has finite-dimensional motive, so the result now follows from Lemma 6.12. \(\square\)

Remark 6.14 The surfaces with Picard lattice isometric to $\mathbb{H} + E_8 + E_6$ are very convenient for Nishiyama’s method for enumerating the types of elliptic fibration on a K3 surface [35]. There is an embedding of $E_8 + E_6$ into $E_8^3$ with complement $E_8 + A_2$. The only Niemeier lattices into which $E_8 + A_2$ can be embedded are those with an $E_8$ component, and one readily checks that the embedding of $A_2$ into the complement is unique up to isometry in both cases. Thus there are only two types of elliptic fibration on such a surface; their Mordell–Weil ranks are 0 and 1. It is noteworthy that neither type admits nontrivial torsion, and so there is no obvious way to map such a surface to any K3 surface not isomorphic to it; we do not know whether such a map exists.

The surface $L$ admits elliptic fibrations with 8 different frames, in addition to 2 types of genus-1 fibration without a section. The only type with nontrivial torsion is the one arising from the description of $L$ as a quotient of $K$, so there are no obvious maps from $L$ to other K3 surfaces that do not factor through a surface with Picard lattice isometric to $\mathbb{H} + E_8 + E_6$ or that of $K$.

Corollary 6.15 The Hodge conjecture is true for $Y \times Y$, where $Y$ is a general K3 surface with Picard lattice isometric to $L_1'$ or $\mathbb{H} + E_8 + E_6$.

Proof This is an immediate consequence of the proof of Corollary 6.13 and Theorem 4.8. \(\square\)

These observations allow us to relate our construction to certain families of K3 surfaces with 15 nodes studied in [17]. In particular, in the notation of [17, Theorem 8.3], we consider the case $d = 3$. First we recall the definition of the lattice $M(Z/2Z)^4$, which is of fundamental importance in [17].
**Definition 6.16** Let $A = (\mathbb{Z}/2\mathbb{Z})^4$, let $N$ be the set of nonzero elements of $A$, and let $M_0$ be the lattice $A_1^{15}$ with basis $b_i$ indexed by $N$. For every subset $C \subset N$ of order 8 which is the complement of a subgroup, we adjoin the vector $\sum_{i \in N} b_i$ to $M_0$. The result of this is the lattice $M((\mathbb{Z}/2\mathbb{Z})^4)$ of discriminant $-128$.

**Definition 6.17** We now consider the lattice $(6) + M((\mathbb{Z}/2\mathbb{Z})^4)$, whose first generator will be denoted $h$. It can be enlarged by adjoining a vector $(h, v)/2$, where $v = \sum_{i \in G \setminus \{0\}}$ for $G$ a subgroup of order 4, or by adjoining $(h, w)/2$, where $w = \sum_{i \in N} b_i$. Let us denote these two lattices by $N_1, N_2$ (for definiteness, and following [17], we use the subgroup of elements with first two components 0 to define $N_1$).

**Remark 6.18** The lattices $N_1, N_2$ are the two lattices on the list of Garbagnati and Sarti [17, Theorem 8.3] of possible Picard lattices of K3 surfaces with 15 nodes for the case $d = 3$. Note that these lattices are not isometric to $L_1$, although they have the same rank and discriminant.

We will prove the following theorem.

**Theorem 6.19** Let $S$ be a K3 surface with Picard lattice isometric to $N_1$ or $N_2$. Then $S$ admits a finite map to a K3 surface with Picard lattice $\mathbb{H} + E_8 + E_6$.

Before proving Theorem 6.19, we first note a simple corollary.

**Corollary 6.20** Let $S$ be a general K3 surface of degree 6 with 15 singularities of type $A_1$. Then $S$ is motive-finite.

**Proof of corollary** First suppose that $S$ has Picard rank 16. Then by [17, Theorem 8.3], its Picard lattice is isometric to $N_1$ or $N_2$. Combining this with Theorem 6.19 and Lemma 6.12, as in the proof of Corollary 6.13, yields the desired result.

In the general case, we consider a 1-parameter family of K3 surfaces with 15 singularities of type $A_1$ and Picard rank 16 whose limit is $S$. We invert the map of moduli spaces of Theorem 1.10: this is well-defined elsewhere on the curve, so it has a limit at $S$, which is generally smooth. Lifting arbitrarily to the cover of $\mathcal{M}_C$ that additionally parametrizes a square root of the line bundle $O(p_1 + p_2 + p_3 + p_4)$ allows us to construct a curve of arithmetic genus 7 (the stable limit of smooth curves in the family) whose square covers $S$; again, it is generically smooth. □

**Remark 6.21** We expect that these surfaces should be covered by the square of the same curves that cover those with Picard lattice $L_1$, and that this can be proved by finding chains of maps of finite degree leading from surfaces with Picard lattice $L_1$ to surfaces with Picard lattice $N_1$ and $N_2$.

In order to prove Theorem 6.19, we will identify some rational curves on $N_1, N_2$ and use them to construct genus 1 fibrations without a section, whose Jacobians can thus be used to move toward surfaces with discriminant $-3$.

**Definition 6.22** For $S = N_1$ or $N_2$, let $H$ be a vector on the boundary of the ample cone of $S$ that is in the orbit of $\pm h$ under the reflection group (as in, for example, [20, Corollary 8.2.9]). Further, let the $C_i$ be the images of the $b_i$ under the reflection.

The map to projective space given by the linear system $|H|$ exhibits $S$ either as a complete intersection of hypersurfaces of degree 2, 3 in $\mathbb{P}^4$ or as a double cover of a cubic scroll. In either case, every Picard class has intersection a multiple of 3 with the hyperplane class, and the curves of class $C_i$ are nodes.
Definition 6.23 Let $S_1$ be a K3 surface with Picard lattice $N_1$.

Proposition 6.24 There are at least 15 smooth rational curves on the minimal desingularization of $S_1$ whose intersection with the strict transform of $H$ is 3.

Proof For every subgroup $G \subset A$ of order 8, consider the Picard class $C_G = (H - \sum_{i \in G \setminus \{0\}} C_i)/2 \in N_1$. It has self-intersection $-2$ and positive intersection with $H$, so it is effective. Repeatedly subtracting the classes of the nodes with negative intersection from $C_G$ until there are none left, we obtain classes $C_G'$ of self-intersection $-2$ and intersection 3 with $H$. These must represent irreducible curves: by Riemann–Roch they are effective, and if they were reducible one of the components would have degree 0 and negative intersection with $C_G'$, which is not possible since the only curves of degree 0 are the 15 nodes. One checks that the $C_G'$ are all distinct. $\square$

Proposition 6.25 Fix a subgroup $G \subset A$ of order 4 and let $F = H - \sum_{i \in G \setminus \{0\}} C_i$. Then $F$ is not divisible by 2, but $F$ has even intersection with every curve on $S_1$.

Proof This is a simple calculation. $\square$

Proposition 6.26 Let $S$ be a K3 surface and $v \in \text{Pic} \ S$ a vector not divisible by any integer $n > 1$ such that $(v, v) = 0$ and $d | (v, w)$ for all $w \in \text{Pic} \ S$. Then there is a map from $S$ to a K3 surface $S'$ with Picard lattice $\text{Pic} \ S'[v/d]$.

Proof By [20, Corollary 8.2.9], there is a sequence of reflections whose product $\rho$ is such that $\pm \rho(v)$ is nef. It thus suffices to assume that $v$ is nef. In this case it is the class of a fibre of a genus 1 fibration with minimal multisection degree $d$, and the map from this fibration to its Jacobian [20, Section 11.4] given on smooth fibres by $P \mapsto dP - F \cap M_d$, is the desired map, where $F$ is the fibre containing $P$ and $M_d$ is the multisection. (The description of the Picard lattice of the target follows from the discussion after [20, Corollary 11.4.7].) $\square$

Remark 6.27 This proposition cannot be used to prove the existence of a map between K3 surfaces whose Picard lattices are not isometric after tensoring with $\mathbb{Q}$. In particular it cannot be used to construct a map from a surface with Picard lattice $L_1$ to one with Picard lattice $\mathbb{H} + E_8 + E_6$ directly: the Hasse–Minkowski invariants are different. Indeed, for every map between such K3 surfaces of degree $k$, we must have $v_2(k) + v_3(k)$ odd, where $v_p$ denotes the $p$-adic valuation. This follows from much the same reasoning as is used to prove [1, Theorem 1.1]. On the other hand, the map described above, taking $P$ to $dP - F \cap M_d$, has degree $d^2$, and of course $v_2(k^2) + v_3(k^2)$ is even.

Corollary 6.28 A K3 surface with Picard lattice $N_1$ admits a finite map to a surface with Picard lattice $\mathbb{H} + E_8 + E_6$.

Proof We take $G_1, G_2$ to be subgroups whose pairwise intersections with each other and with the subgroup $G$ used to define $N_1$ have order 2, but such that $G \cap G_1 \cap G_2 = \{0\}$. Again one defines $F_i = H - \sum_{i \in G_i} C_i$ and verifies that $(F_i, F_i) = 0$ and $F_1, F_2 = 4$, so that we obtain a map from the given surface to one with Picard lattice $N_1[F_1/2, F_2/2]$. It is a straightforward matter to find a vector in this lattice of norm 0, not divisible by 2, and having even intersection with all vectors: for example, letting $G_1$ and $G_2$ be generated by $C_{0001}, C_{0100}$ and $C_{0010}, C_{0100}$, we may take $2H - 2C_{0111} - 2C_{1011} - \sum_{i,j=0}^1 C_{11ij}$. The rest of the proof is the same as for Corollary 6.20. $\square$

It turns out that $N_2$ is more manageable than $N_1$.
Corollary 6.29 A K3 surface with Picard lattice $N_2$ admits a finite map to a surface with Picard lattice $\mathbb{H} + E_8 + E_6$.

Proof Let $G_1, G_2, G_3$ be subgroups of $A$ of order 4 with pairwise intersection of order 2 whose union generates $A$ (for example, let $a_i$ be the generators of $A$ and take $G_i = \langle a_i, a_4 \rangle$). Let $F_1, F_2, F_3$ be the corresponding vectors $F$ as in Proposition 6.25: we then have $(F_i, F_j) = 4$ for $i \neq j$, so that $F_2$ is still even in $N_1[F_1/2]$ and $F_3$ in $N_1[F_1/2, F_2/2]$. Thus we may apply Proposition 6.26 successively to the $F_i$, obtaining lattices of discriminant $-48, -12, -3$. To see that the lattice of discriminant $-3$ is isometric to $\mathbb{H} + E_8 + E_6$, it suffices to compute the discriminant group and apply a result of Nikulin [32, Corollary 2.10 (ii)]. Alternatively this can be checked directly by embedding $\mathbb{H}$ into $N_2[F_1/2, F_2/2, F_3/2]$ and checking that the orthogonal complement is one of the two lattices in the same genus as $E_8 + E_6$. 

More generally, we can use Proposition 6.26 to clarify the relation among the different families of K3 surfaces with 15 singularities considered by Garbagnati and Sarti. Recall that in [17, Theorem 8.3] they give a complete description of all rank-16 lattices that can be the Picard lattice of such a surface. In particular, these lattices contain the lattice $(2d) + M_{(\mathbb{Z}/2\mathbb{Z})^4}$ with index 2; there are 2 possibilities if $d \equiv 3$ mod 4 and 1 otherwise. If $d/d' \notin (\mathbb{Q}^*)^2$, then there can be no relation between the surfaces corresponding to $d$ and $d'$. This can be seen from Theorem 5.3: since the rank of the Picard group is even, we have disc $T(S)(n)/$ disc $T(S) \in (\mathbb{Q}^*)^2$, and the same conclusion follows for disc $T(S')/\text{disc} T(S)$. On the other hand, we show:

Proposition 6.30 Let $d, d'$ be positive integers with $d/d' \in (\mathbb{Q}^*)^2$ and let $L_d, L_{d'}$ be lattices from the list in [17, Theorem 8.3] of possible Picard lattices of K3 surfaces with 15 singularities of type $A_1$. Let $S_d$ be a K3 surface with Picard lattice $L_d$. Then there is a correspondence between $S_d$ and a surface $S_{d'}$ with Picard lattice $L_{d'}$: more precisely, there is a sequence of finite maps $(\pi_i)_{i=0}^{n}$ of K3 surfaces such that the domain of $\pi_0$ is $S_d$, the codomain of $\pi_n$ is $S_{d'}$, and for all $i$ with $0 \leq i < n$ either the domains or the codomains of $\pi_i, \pi_{i+1}$ coincide.

Proof It suffices to prove this under the assumption that $d/d'$ is the square of a prime $p$, since the existence of a correspondence as in the statement of the proposition is an equivalence relation. We begin with a bit of notation. In the lattice $(2d) + M_{(\mathbb{Z}/2\mathbb{Z})^4}$ that is contained in $L_d$ with index 2, let $\ell_d$ be the first generator, of norm $2d$ and orthogonal to $M_{(\mathbb{Z}/2\mathbb{Z})^4}$; recall that we designate the generators of $A_1^{15}$ and their images in $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ by $b_i$.

We first dispose of the easy case of odd $p$. We may assume, if $d \equiv 3$ mod 4, that we are in the same case (iii) or (iv) for both $d$ and $d'$: otherwise, we reduce separately to $d = d'$ squarefree and relate both lattices to $L_{4d}$ as shown below. In this case $L_{d'} = L_d[\ell_d/p]$. Since $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ contains 4 pairwise orthogonal elements of norm 2, it suffices to write $d'/2$ as a sum $\sum_{i=1}^{4} a_i^2$ of 4 squares and to note that $\ell_d - p \sum_{i=1}^{4} a_i b_i$ is a vector of norm 0 to which Proposition 6.26 applies. Thus $L_{d'}[\ell_d - p \sum_{i=1}^{4} a_i b_i]/p = L_d[\ell_d/p]$ is the Picard lattice of a K3 surface that admits a finite map from the surface with Picard lattice $L_d$. The same argument works for $p = 2$ if $4|d'$.

Now we consider the cases with $p = 2$, and in particular the cases (i), (ii), (iii) of [17, Theorem 8.3] (we will treat (iv) separately and (v) was taken care of just above). In each of these cases it turns out that $L_d[\ell_d/2]$ is isometric to $L_{d'}$. This is most easily seen in terms of the descriptions of the discriminant forms at the beginning of the proof of [17, Theorem 8.3]. Indeed, the vector $\ell_d$ corresponds to the generator $1/2d$ of $q_2 + q_2(1/2) + (1/2d)$, so the discriminant form of $L_d[\ell_d/2]$ is $q_2 + q_2 + (1/2) + (4/2d)$, the same as that of $L_{d'}$. 

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By [32, Corollary 2.10 (ii)], the Picard lattice is uniquely determined by its signature and discriminant form in this situation, so it must be that \( L_d[\ell_d / 2] \cong L_d / 2 \).

In the case (iv), no overlattice of \( L_d \) is isometric to \( L_d' \). Nevertheless, we may proceed as follows. First we divide \( \ell_d \) by 2 as before to obtain the lattice \( L_{iii,d'} \) of case (iii); as before, this comes from a map of K3 surfaces.

To complete the proof, we show that there is a common overlattice \( M_{d'} \), containing both of the two lattices \( L_{iii,d'} \), \( L_{d'} \) of discriminant \(-64d\) with index 2, that can be reached by dividing vectors of norm 0 by 2. In both cases we will obtain it by dividing a vector congruent to \( x = \sum_{i \in (\mathbb{Z}/2\mathbb{Z})^4 \setminus H} b_i \) by 2, where \( H \) is the subgroup of order 4 used to define \( L_{iii,d'} \). This vector has norm \(-24\) and so the vector \( 2\ell_{d'} - b_i - x \) has norm \( 8(d' - 5) \). Since \( d' - 5 \equiv 2 \mod 4 \), it is a sum of three squares, and so \(-8(d' - 5)\) is the norm of some integral combination of the \( 2b_i \) (whose norms are \(-8\)) for \( h \in H \setminus \{0\} \), except in the case \( d' = 3 \) which can be treated directly as in Corollary 6.29 above or by replacing \( 2\ell_{d'} \) by \( 6\ell_{d'} \).

Thus, the K3 surfaces with Picard lattice \( L_{d'} \) are in correspondence with those of Picard lattice \( M_{d'} \), and then with those of Picard lattice \( L_{iii,d'} \) and \( L_d \) as claimed.

The proof of this proposition provides a proof of Proposition 6.3. We close with the proof of Theorem 6.4.

**Proof** By Proposition 6.30, we have a correspondence between \( K \) and any surface \( S' \) with Picard lattice \( L_{d'} \) with \( \frac{2}{\pi} \in \mathbb{Q}^{\times^2} \). This gives an isomorphism of Hodge structures \( T(K) \cong T(S') \). Thus Theorem 6.4 follows from [28, Theorem 2] in the same way as in the proof of Theorem 4.8.

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**References**

1. Boissière, S., Sarti, A., Veniani, D.C.: On prime degree isogenies between K3 surfaces. Rend. Circolo Mat. Palermo Ser. 2 66(1), 3–18 (2017)
2. Boxer, G., Calegari, F., Gee, T., Pilloni, V.: Abelian surfaces over totally real fields are potentially modular. arXiv:1812.09269
3. Buskin, N.: Every rational Hodge isometry between two K3 surfaces is algebraic. J. Reine Angew. Math. 755, 127–150 (2019). https://doi.org/10.1515/crelle-2017-0027
4. Cassels, J.W.S.: Rational Quadratic Forms. Academic Press, London (1968)
5. Catanese, F.: Fibred surfaces, varieties isogenous to a product and related moduli spaces. Am. J. Math. 122, 1–44 (2000)
6. Cavalieri, R.: Moduli spaces of pointed rational curves, notes of lectures given at the Fields Institute. https://www.math.colostate.edu/~renzo/teaching/Moduli16/Fields.pdf
7. Cox, D.: Mordell–Weil groups of elliptic curves over \( \mathbb{C}(t) \) with \( pg = 0 \) or 1. Duke Math. J. 49(3), 677–689 (1982)
8. Cox, D., Zucker, S.: Intersection numbers of sections of elliptic surfaces. Inv. Math. 53, 1–44 (1979)
9. Deligne, P.: La Conjecture de Weil pour les surfaces K3. Inventiones Math. 15, 206–226 (1972)
10. Deligne, P.: Catégories tensorielles. Mosc. Math. J. 2(2), 227–248 (2002)
11. Dokchitser, T., Dokchitser, V.: Local invariants of isogenous elliptic curves. Trans. Am. Math. Soc. 367(6), 4339–4358 (2015)
12. Eklund, D.: Curves on Heisenberg invariant quartics in projective 3-space. Eur. J. Math. 4(3), 931–995 (2018)
13. Frapporti, D.: Mixed quasi-étale surfaces, new surfaces of general type with $p_g = 0$ and their fundamental group. Collect. Math. 64(3), 293–311 (2013)
14. Frapporti, D., Pignatelli, R.: Mixed quasi-étale quotients with arbitrary singularities. Glasg. Math. J. 57(1), 143–165 (2015)
15. Festi, D., Veniani, D.C.: Counting elliptic fibrations on K3 surfaces. arXiv:2102.09411
16. Garbagnati, A., Penegini, M.: K3 surfaces with a non-symplectic automorphism and product-quotient surfaces. Rev. Mat. Iberoam. 31, 1277–1310 (2015). (vol. 4)
17. Garbagnati, A., Sarti, A.: Kummer surfaces and K3 surfaces with $(\mathbb{Z}/2\mathbb{Z})^4$ symplectic action. Rocky Mt. J. Math 46(4), 1141–1205 (2016)
18. Garrett, P.: Sporadic isogenies to orthogonal groups. Unpublished notes. http://www.math.umn.edu/~garrett/m/v/sporadic_isogenies.pdf
19. Harder, A., Thompson, A.: The geometry and moduli of K3 surfaces. In: Laza, R., Schütt, M., Yui, N. (eds.) Calabi-Yau varieties: arithmetic, geometry, and physics, Fields Institute Monographs, vol. 34. Springer, Berlin (2015)
20. Huybrechts, D.: Lectures on K3 Surfaces. Cambridge Studies in Advanced Mathematics, vol. 158. Cambridge University Press, Cambridge (2016)
21. Kimura, S.I.: Chow motives can be finite-dimensional, in some sense. Math. Annal. 331(1), 173–201 (2005)
22. Kloosterman, R.: Classification of all Jacobian elliptic fibrations on certain K3 surfaces. J. Math. Soc. Jpn. 58(3), 665–680 (2006)
23. Knudsen, F.: The projectivity of the moduli space of stable curves, II: The stacks. Math. Scand. 52, 161–199 (1983)
24. Kumar, A.: Elliptic fibrations on a generic Jacobian Kummer surface. J. Algebr. Geom. 23(4), 599–667 (2014)
25. Kuga, M., Satake, I.: Abelian Varieties attached to polarized K3-surfaces. Math. Annal. 169, 239–242 (1967)
26. Landherr, W.: Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper. Abh. Math. Sem. Hamburg Univ. 11, 245–248 (1936). https://www.maths.ed.ac.uk/~v1ranick/papers/landherr.pdf
27. Laterveer, R.: A family of K3 surfaces having finite-dimensional motive. Arch. Math. 106, 515–524 (2016)
28. Lombardo, G.: Abelian varieties of Weil type and Kuga-Satake varieties. Tohoku Math. J. Sec. Ser. 53(3), 453–466 (2001)
29. Lombardo, G., Peters, C., Schütt, M.: Abelian fourfold of Weil type and certain K3 double planes. Rend. Semin. Mat. Univ. Politec. Torino 71(3–4), 339–383 (2013)
30. Mazza, C.: Schur functors and motives. arXiv:1010.3932
31. Miranda, R.: The basic theory of elliptic surfaces. Dissertation for the degree of Dottorato di Ricerca in Matematica, Pisa (1989). https://www.math.colostate.edu/~miranda/BTES-Miranda.pdf
32. Morrison, D.R.: On K3 surfaces with large Picard number. Inv. Math. 75, 105–121 (1984)
33. Mukai, S.: On the moduli space of bundles on K3 surfaces, I. In: Vector Bundles on Algebraic Varieties. Tata Institute of Fundamental Research. Oxford University Press, Oxford (1987)
34. Nikulin, V.: On Kummer surfaces. Izv. Akad. Nauk SSSR Ser. Mat. 39(2), 278–293 (1975). (Translation: Math. USSR Izv. 9(2), 1975)
35. Nishiyama, K.: The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups. Jpn. J. Math. (N.S.) 22, 293–347 (1996)
36. Paranjape, K.: Abelian varieties associated to certain K3 surfaces. Comput. Math. 68, 11–22 (1988)
37. Schütt, M., Shioda, T.: Elliptic surfaces. arXiv:0907.0298
38. Schlickewei, U.: The Hodge conjecture for self-products of certain K3 surfaces. J. Algebra 324(3), 507–529 (2010)
39. Shimada, I.: Table of all ADE-types of singular fibers of elliptic K3 surfaces and the torsion parts of their Mordell-Weil groups. http://www.math.sci.hiroshima-u.ac.jp/~shimada/preprints/EllipticK3/Table.pdf
40. van Geemen, B.: Kuga–Satake varieties and the Hodge conjecture. In: The Arithmetic and Geometry of Algebraic Cycles, pp. 51–82. Springer, Dordrecht (2000)