Ground state angular momenta of random nuclei

R. Bijker$^{1,2}$ and A. Frank$^{2,3}$

$^1$ Dipartimento di Fisica, Università degli Studi di Genova, Via Dodecaneso 33, I-16146 Genova, Italy *
$^2$ Instituto de Ciencias Nucleares, UNAM, AP 70-543, 04510 México, DF, México
$^3$ Centro de Ciencias Físicas, UNAM, AP 139-B, 62251 Cuernavaca, Morelos, México

Abstract

We investigate the phenomenon of emerging regular spectral features from random interactions. In particular, we address the dominance of $L = 0$ ground states in the context of the vibron model and the interacting boson model. A mean-field analysis links different regions of the parameter space with definite geometric shapes. The results that are, partly, obtained in closed analytic form, provide a clear and transparent interpretation of the distribution of ground state angular momenta as observed before in numerical studies.

1 Introduction

Recent shell model calculations of even-even nuclei in the $sd$ shell and the $pf$ shell with random interactions showed a remarkable statistical preference for ground states with $L = 0$, despite the random nature of the two-body matrix elements, both in sign and in relative magnitude [4]. A similar preponderance of $L = 0$ ground states was found in an analysis of the Interacting Boson Model (IBM) with random interactions [4]. In addition, in the IBM evidence was found for both vibrational and rotational band structures [2, 3]. These are surprising results in the sense that, according to the conventional ideas in the field, the occurrence of $L = 0$ ground states and the existence of vibrational and rotational bands are due to very specific forms of the interactions. The studies with random interactions show that the class of Hamiltonians that lead to these regular features

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is much larger than usually thought. These unexpected results have sparked a large number of investigations to explain and further explore the properties of random nuclei, both for the nuclear shell model [4, 5, 6, 7, 8, 9, 10, 11] and the IBM [2, 3, 12, 13, 14].

In this contribution, we investigate the origin of the dominance of \( L = 0 \) ground states that emerges from random interactions in two different models: the vibron model and the IBM. The vibron model is mathematically simpler than the IBM, but exhibits many of the same qualitative features. In this case, most of the results can be obtained in closed analytic form. For both models, we suggest to use a mean-field analysis to address the problem of the probability distribution of the ground state angular momenta [14].

2 The vibron model

The vibron model was introduced to describe the rotational and vibrational excitations of diatomic molecules [15]. Its building blocks are a dipole boson \( p^\dagger \) with \( L^P = 1^- \) and a scalar boson \( s^\dagger \) with \( L^P = 0^+ \). The total number of bosons \( N \) is conserved by the vibron Hamiltonian. We only consider one- and two-body interactions

\[
H = \frac{1}{N} \left[ H_1 + \frac{1}{N-1} H_2 \right],
\]

with

\[
H_1 = \epsilon_s s^\dagger \cdot \tilde{s} - \epsilon_p p^\dagger \cdot \tilde{p},
\]

\[
H_2 = u_0 \frac{1}{2} (s^\dagger \times s^\dagger)^{(0)} \cdot (\tilde{s} \times \tilde{s})^{(0)} + u_1 (s^\dagger \times p^\dagger)^{(1)} \cdot (\tilde{p} \times \tilde{s})^{(1)}
+ \sum_{\lambda=0,2} c^\lambda \frac{1}{2} (p^\dagger \times p^\dagger)^{(\lambda)} \cdot (\tilde{p} \times \tilde{p})^{(\lambda)}
+ v_0 \frac{1}{2\sqrt{2}} [(p^\dagger \times p^\dagger)^{(0)} \cdot (\tilde{s} \times \tilde{s})^{(0)} + H.c.] ,
\]

Here \( \tilde{s} = s \) and \( \tilde{p}_m = (-1)^{1-m} p_{-m} \). We have scaled \( H_1 \) by \( N \) and \( H_2 \) by \( N(N-1) \) in order to remove the \( N \) dependence of the matrix elements. The seven parameters of the Hamiltonian, \( \epsilon_s, \epsilon_p, u_0, u_1, c_0, c_2, v_0 \), altogether denoted by \( \vec{x} \), are taken as independent random numbers on a Gaussian distribution with zero mean. In this way, the interaction terms are arbitrary and equally likely to be attractive or repulsive.

The connection between the vibron Hamiltonian, potential energy surfaces, equilibrium configurations and geometric shapes can be investigated by means of mean-field methods [16]. For the vibron model, the coherent state can be written as a condensate of deformed bosons, which are superpositions of scalar and dipole bosons

\[
| N, \alpha \rangle = \frac{1}{\sqrt{N!}} \left( \cos \alpha s^\dagger + \sin \alpha p^\dagger_0 \right)^N | 0 \rangle ,
\]
with $0 \leq \alpha \leq \pi/2$. The potential energy surface is then given by the expectation value of the vibron Hamiltonian of Eqs. (1) and (2) in the coherent state

$$E(\alpha) = a_4 \sin^4 \alpha + a_2 \sin^2 \alpha + a_0 .$$

The coefficients $a_i$ are linear combinations of the parameters of the Hamiltonian

$$a_4 = \vec{r} \cdot \vec{x} = \frac{1}{2} u_0 + u_1 + \frac{1}{6} c_0 + \frac{1}{3} c_2 + \frac{1}{\sqrt{6}} v_0 ,$$
$$a_2 = \vec{s} \cdot \vec{x} = -\epsilon_s + \epsilon_p - u_0 - u_1 - \frac{1}{\sqrt{6}} v_0 ,$$
$$a_0 = \vec{r} \cdot \vec{r} = \epsilon_s + \frac{1}{2} u_0 .$$

For random interaction strengths, we expect the trial wave function of Eq. (3) and the energy surface of Eq. (4) to provide information on the distribution of shapes that the model can acquire. The value of $\alpha_0$ that characterizes the equilibrium configuration of the potential energy surface only depends on the coefficients $a_4$ and $a_2$. The $a_2 a_4$ plane can be divided into different areas according to the three possible equilibrium configurations: $S_1$ for the $s$-boson or spherical condensate ($\alpha_0 = 0$), $S_2$ for the deformed condensate ($0 < \alpha_0 < \pi/2$), and $S_3$ for the $p$-boson condensate ($\alpha_0 = \pi/2$). The distribution of shapes for an ensemble of Hamiltonians depends on the joint probability distribution of the coefficients $a_4$ and $a_2$ which is given by a bivariate normal distribution

$$P(a_4, a_2) = \frac{1}{2\pi \sqrt{|\det M|}} \exp \left[ -\frac{1}{2} \begin{pmatrix} a_4 & a_2 \end{pmatrix} M^{-1} \begin{pmatrix} a_4 \\ a_2 \end{pmatrix} \right] ,$$

with

$$M = \begin{pmatrix} \vec{r} \cdot \vec{r} & \vec{r} \cdot \vec{s} \\ \vec{r} \cdot \vec{s} & \vec{s} \cdot \vec{s} \end{pmatrix} = \frac{1}{18} \begin{pmatrix} 28 & -30 \\ -30 & 75 \end{pmatrix} .$$

The vectors $\vec{r}$ and $\vec{s}$ are defined in Eq. (5). The probability that the equilibrium shape of an ensemble of Hamiltonians is spherical can be obtained by integrating $P(a_4, a_2)$ over the appropriate range $S_1 (a_2 > 0$ and $a_4 > -a_2$)

$$P_1 = \frac{1}{4\pi} \left[ \pi + 2 \arctan \sqrt{\frac{27}{16}} \right] = 0.396 .$$

Similarly, the probability for the occurrence of a deformed shape is obtained by integrating $P(a_4, a_2)$ over $S_2 (-2a_4 < a_2 < 0)$

$$P_2 = \frac{1}{2\pi} \arctan \sqrt{\frac{64}{3}} = 0.216 .$$
Table 1: Percentages of ground states with $L = 0$, 1 and $N$, obtained in a mean-field analysis of the vibron model.

| Shape         | $L = 0$ | $L = 1$ | $L = N$ |
|---------------|---------|---------|---------|
| $\alpha_0 = 0$ | 39.6 %  | 0.0 %   | 0.0 %   |
| $0 < \alpha_0 < \pi/2$ | 21.6 %  | 13.3 %  | 8.3 %   |
| $\alpha_0 = \pi/2$ | 38.8 %  | 17.9 %  | 20.9 %  |
|                | 0.0 %   | 20.9 %  | 20.9 %  |
| $N = 2k$      |         |         |         |
| $N = 2k + 1$  |         |         |         |
| Total         | 100.0 % | 70.8 %  | 29.2 %  |
|               | 52.9 %  | 17.9 %  | 29.2 %  |

Finally, the probability for finding the third solution, a $p$-boson condensate, is

$$P_3 = 1 - P_1 - P_2 = 0.388.$$  \hspace{1cm} (10)

The rotational spectrum can be obtained from the angular momentum content of the condensate in combination with the Thouless-Valatin formula for the moment of inertia

$$E_{\text{rot}} = \frac{1}{2I}L(L+1).$$  \hspace{1cm} (11)

In Table 1 we show the probability distribution of the ground state angular momenta as obtained in the mean-field analysis. There is a statistical preference for $L = 0$ ground states. This is largely due to the occurrence of a spherical shape (whose angular momentum content is just $L = 0$) for almost 40% of the cases (see Eq. (8)). The deformed shape corresponds to a rotational band with $L = 0, 1, \ldots, N$, whose ground state has $L = 0$ for positive values of the moment of inertia $I > 0$, and $L = N$ for $I < 0$. The third solution, the $p$-boson condensate, has angular momenta $L = N, N-2, \ldots, 1$ or 0. For $I > 0$, the ground state has $L = 0$ or $L = 1$, depending whether the total number of vibrons $N$ is even or odd, whereas for $I < 0$ it has the maximum value $L = N$. The sum of the $L = 0$ and $L = 1$ percentages is constant.

Fig. 1 shows that the mean-field results for the percentages of $L = 0$ and $L = 1$ ground states are in excellent agreement with the exact ones. As is clear from the results
Figure 1: Percentages of ground states with $L = 0$ and $L = 1$ in the vibron model with random one- and two-body interactions calculated exactly for 100000 runs (solid lines) and in mean-field approximation (dashed lines).

Presented in Table [4], the fluctuations in the percentages of $L = 0$ and $L = 1$ ground states with $N$ are due to the contribution from the $p$-boson condensate solution.

3 The interacting boson model

The interacting boson model (IBM) describes collective excitations in nuclei in terms of a system of $N$ interacting bosons [18]. Its building blocks are a quadrupole boson $d^{\dagger}$ with $L^P = 2^+$ and a scalar boson $s^{\dagger}$ with $L^P = 0^+$. The total number of bosons $N$ is conserved by the IBM Hamiltonian. We consider the one- and two-body Hamiltonian of Eq. (1) with

$$
H_1 = \epsilon_s s^{\dagger} \cdot \bar{s} + \epsilon_d d^{\dagger} \cdot \bar{d},
$$

$$
H_2 = \frac{1}{2} u_0 (s^{\dagger} \times s^{(0)}) \cdot (\bar{s} \times \bar{s})^{(0)} + u_2 (s^{\dagger} \times d^{(2)}) \cdot (\bar{d} \times \bar{s})^{(2)}
$$

$$
+ \sum_{\lambda=0,2,4} \frac{1}{2} c_\lambda (d^{\dagger} \times d^{(\lambda)}) \cdot (\bar{d} \times \bar{d})^{(\lambda)}
$$

$$
+ \frac{1}{2\sqrt{2}} v_0 \left[ (d^{\dagger} \times d^{(0)}) \cdot (\bar{s} \times \bar{s})^{(0)} + H.c. \right]
$$
\[ + \frac{1}{2} v_2 \left[ (d^\dagger \times d^\dagger)^{(2)} \cdot (\tilde{d} \times \tilde{s})^{(2)} + H.c. \right] . \] (12)

The nine coefficients \( \epsilon_s, \epsilon_d, u_0, u_2, c_0, c_2, c_4, v_0, v_2 \), are chosen independently from a Gaussian distribution of random numbers with zero mean [2, 3]. Just as for the vibron model, the connection between the IBM, potential energy surfaces, equilibrium configurations and geometric shapes, can be studied with mean-field Hartree-Bose techniques by means of coherent states [17, 19]. The coherent state can be written as an axially symmetric condensate

\[ |N, \alpha\rangle = \frac{1}{\sqrt{N!}} \left( \cos \alpha s^\dagger + \sin \alpha d^\dagger \right)^N |0\rangle , \] (13)

with \(-\pi/2 < \alpha \leq \pi/2\). The angle \( \alpha \) is related to the deformation parameters in the intrinsic frame, \( \beta \) and \( \gamma \) [18, 19]. The potential energy surface is then given by the expectation value of the Hamiltonian in the coherent state

\[ E_N(\alpha) = a_4 \sin^4 \alpha + a_3 \sin^3 \alpha \cos \alpha + a_2 \sin^2 \alpha + a_0 . \] (14)

For the IBM, the structure of the energy surface is a bit more complicated than for the vibron model due to the presence of the \( a_3 \) term. This precludes an analytic treatment as presented for the vibron model, but qualitatively the results are very similar. In practice, for each Hamiltonian the minimum of the energy surface \( E(\alpha) \) is determined numerically. Again, the equilibrium configurations can be divided into three different classes: an \( s \)-boson or spherical condensate \( (\alpha_0 = 0) \), a deformed condensate with prolate \( (0 < \alpha_0 < \pi/2) \) or or oblate symmetry \( (-\pi/2 < \alpha_0 < 0) \), and a \( d \)-boson condensate \( (\alpha_0 = \pi/2) \). Each equilibrium configuration has its own characteristic angular momentum content. Even though we do not explicitly project the angular momentum states from the coherent state, the angular momentum of the ground state can, to a good approximation, be obtained from the rotational structure of the condensate in combination with the Thouless-Valatin formula for the corresponding moments of inertia [17]. The results are summarized in Table 2.

(i) The \( s \)-boson condensate corresponds to a spherical shape. Whenever such a condensate occurs (in 39.4 % of the cases), the ground state has \( L = 0 \).

(ii) The deformed condensate corresponds to an axially symmetric deformed rotor. The ordering of the rotational energy levels \( L = 0, 2, \ldots, 2N \) is determined by the sign of the moment of inertia

\[ E_{\text{rot}} = \frac{1}{2I_3} L(L + 1) . \] (15)

The deformed condensate occurs in 36.8 % of the cases. For \( I_3 > 0 \) the ground state has \( L = 0 \) (23.7 %), while for \( I_3 < 0 \) the ground state has the maximum value of the angular momentum \( L = 2N \) (13.1 %).
Table 2: Percentages of ground states with $L = 0$, 2 and $2N$, obtained in a mean-field analysis of the interacting boson model.

| Shape            | $L = 0$ | $L = 2$ | $L = 2N$ |
|------------------|---------|---------|----------|
| $\alpha_0 = 0$  | 39.4 %  | 0.0 %   | 0.0 %    |
| $0 < |\alpha_0| < \pi/2$ | 36.8 %  | 0.0 %   | 13.1 %   |
| $\alpha_0 = \pi/2$ | 23.8 %  | 0.0 %   | 10.3 %   |
|                  | 0.2 %   | 13.2 %  | 10.4 %   |
|                  | 4.4 %   | 9.0 %   | 10.5 %   |
|                  | 9.3 %   | 4.0 %   | 10.5 %   |
|                  | 13.2 %  | 10.4 %  | $N = 6k$ |
|                  | 10.4 %  | 10.4 %  | $N = 6k + 2, 6k + 4$ |
|                  | 10.5 %  | 10.5 %  | $N = 6k + 3$ |
| Total            | 100.0 % | 0.0 %   | 23.4 %   |
|                  | 63.3 %  | 13.2 %  | $N = 6k + 1, 6k + 5$ |
|                  | 67.5 %  | 9.0 %   | $N = 6k + 2, 6k + 4$ |
|                  | 72.4 %  | 4.0 %   | $N = 6k + 3$ |

(iii) The $d$-boson condensate corresponds to a quadrupole oscillator with $N$ quanta. Its rotational structure has a more complicated structure than the previous two cases. It is characterized by the labels $\tau$, $n_\Delta$ and $L$. The boson seniority $\tau$ is given by $\tau = 3n_\Delta + \lambda = N, N-2, \ldots, 1$ or 0 for $N$ odd or even, and the values of the angular momenta are $L = \lambda, \lambda+1, \ldots, 2\lambda-2, 2\lambda$ \[18\]. In this case, the rotational excitation energies depend on two moments of inertia

$$E_{\text{rot}} = \frac{1}{2I_5} \tau(\tau + 3) + \frac{1}{2I_3} L(L + 1). \quad (16)$$

The $d$-boson condensate occurs in 23.8 % of the cases. For $I_5 > 0$ the ground state has $\tau = 0$ for $N$ even or $\tau = 1$ for $N$ odd ($\sim 4 \%$), while for $I_5 < 0$ the ground state has the maximum value of the boson seniority $\tau = N$ ($\sim 19 \%$). For $\tau = 0$ and $\tau = 1$ there is a single angular momentum state with $L = 0$ and $L = 2$, respectively. For the $\tau = N$ multiplet, the angular momentum of the ground state depends on the sign of the moment of inertia $I_3$. For $I_3 > 0$ the ground state has $L = 0$ for $N = 3k$ or $L = 2$ for $N \neq 3k$ ($\sim 9 \%$), while for $I_3 < 0$ the ground state has the maximum value of the angular momentum $L = 2N$ ($\sim 10 \%$).
Figure 2: Percentages of ground states with $L = 0$ and $L = 2$ in the IBM with random one- and two-body interactions calculated exactly for 10000 runs (solid lines) and in mean-field approximation (dashed lines).

In Fig. 2 we show the percentages of ground states with $L = 0$ and $L = 2$ as a function of the total number of bosons $N$. A comparison of the results of the mean-field analysis (dashed lines) and the exact ones (solid lines) shows a good agreement. There is a dominance of ground states with $L = 0$. The oscillations of the $L = 0$ and $L = 2$ percentages with $N$ are due to the contribution of the $d$-boson condensate (see Table 2). For $N = 3k$ we see an enhancement for $L = 0$ and a corresponding decrease for $L = 2$. In the mean-field analysis, the sum of the two hardly depends on the number of bosons, in agreement with the exact results.

4 Summary and conclusions

In this contribution, we have studied the origin of the regular features that were obtained before in numerical studies of the IBM with random interactions. In particular, we addressed the dominance of $L = 0$ ground states in the context of the vibron model and the IBM. In a mean-field analysis it was found that different regions of the parameter space can be associated with definite geometric shapes: a spherical shape, a deformed shape and a condensate of dipole (quadrupole) bosons for the vibron model (IBM). For both models, we obtained a good description of the probability distribution of ground state
angular momenta.

The studies with random interactions indicate that there is a significantly larger class of Hamiltonians that leads to regular, ordered behavior at the low excitation energies than was commonly assumed. The fact that these properties are shared by different models, seems to exclude an explanation solely in terms of the angular momentum algebra, the connectivity of the model space, or the many-body dynamics of the model, as has been suggested before. The present analysis points, at least for systems of interacting bosons, to a more general phenomenon that does not depend so much on the details of the angular momentum coupling, but rather on the occurrence of definite, robust geometric phases such as spherical and deformed shapes. For the nuclear shell model the situation is less clear. Although a large number of investigations to explain and further explore the properties of random nuclei have shed light on various aspects of the original problem, i.e. the dominance of \( L = 0 \) ground states, in our opinion, no definite answer is yet available, and the full implications for nuclear structure physics are still to be clarified.

In conclusion, the results presented in this article for the vibron model and the IBM may help to understand the origin of ordered spectra arising from random interactions, as has been observed in numerical studies of nuclear structure models.

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