#P-hardness proofs of matrix immanants evaluated on restricted matrices

István Miklós\textsuperscript{1,2} and Cordian Riener\textsuperscript{3}

\textsuperscript{1}Rényi institute, 1053 Budapest, Reáltanoda u. 13-15, Hungary
\textsuperscript{2}SZTAKI, 1111 Budapest, Lágymányosi u. 11, Hungary
\textsuperscript{3}Dept. of Mathematics and Statistics, UiT The Arctic University of Norway, Tromsø, Norway

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Abstract

#P-hardness of computing matrix immanants are proved for each member of a broad class of shapes and restricted sets of matrices. We prove #P-hardness of computing $\lambda$-immanants of 0-1 matrices when $\lambda$ has a large domino-tilable part and satisfying some technical conditions. We also give hardness proofs of some $\lambda$-immanants of weighted adjacency matrices of planarly drawable directed graphs, such that the shape $\lambda = (1 + \lambda_d)$ has size $n$ such that $|\lambda_d| = n^\varepsilon$ for some $0 < \varepsilon < \frac{1}{2}$, and for some $w$, the shape $\lambda_d/(w)$ is tilable with $1 \times 2$ dominos.

1 Introduction

A matrix functional assigns a scalar to a given $n \times n$ matrix $M := \{m_{i,j}\}$ with coefficients from a commutative ring with characteristic 0. Two well known examples of such functions are the determinant

$$\text{det}(M) := \sum_{\rho \in S_n} \text{sign}(\rho) \sum_{i=1}^{n} m_{i,\rho(i)},$$

and the permanent given by

$$\text{per}(M) := \sum_{\rho \in S_n} \sum_{i=1}^{n} m_{i,\rho(i)},$$

where $S_n$ is the permutation group of size $n$. Surprisingly, although these two matrix functionals have the same number of terms and are defined in a very similar manner they behave very differently and, on particular, have strikingly different computational complexity: While the determinant of an $n \times n$ matrix can be computed in $O(2^{2.373})$ time \cite{21, 23}, computing the permanent is #P-hard \cite{31}. Well-before the seminal paper by Valiant, mathematicians investigated the difference between computing the determinant and the permanent. For example, in 1913 Pólya \cite{27} asked for which matrices it is impossible to convert
the permanent into the determinant by just multiplying the entries of the matrix in any uniform way, and it was shown by Szegő [29] that this is generally impossible for matrices of size greater than 2 × 2.

Valiant conjectured that computing the permanent needs super-polynomial number of arithmetic operations over any field of characteristic other than 2. Bürgisser proved if Valiant’s conjecture fails over any field of positive characteristic or it fails over a field of characteristic zero and the Generalized Riemann Hypothesis holds, then $\text{NP} \subset \text{P/poly}$ [3].

Both, in the definition of the determinant and as well as in the definition of the permanent, the symmetric group plays an important role. Both the terms involved as well as the coefficients are defined via this group. Building on these similarities, Littlewood and Richardson defined matrix immanants as generalizations of the determinant and permanent [20] using notions from representation theory. The sign of a permutation as well as the function which assigns the constant 1 to every permutation are irreducible characters of the group $S_n$. Therefore, a natural generalization is to consider similar matrix functionals for any irreducible character of $S_n$. In this way one arrives at a family of functionals called matrix immanants. The determinant and the permanent, respectively, can be viewed as the matrix immanants with the sign and the trivial (constant 1) characters, respectively.

Before we are discussing the main results of this paper below, we give a short overview of irreducible characters of the symmetric group and the formal definition of immanants in the following. For sake of completeness, we also give the definition of permutations and their cycle structure.

### 1.1 Irreducible $S_n$ representations, characters, and immanants

**Definition 1.** A permutation of length $n$ is a bijective mapping from the set $[n]$ to itself. A convenient way to write permutations is with the cycle notation: Any permutation can be uniquely decomposed into products of disjoint cycles, where a cycle of length $k$ is denoted by $(c_1, c_2, \ldots, c_k)$ indicates that the permutation cyclically sends $c_i$ to $c_{i+1}$, for $1 \leq i \leq k-1$ and $c_k$ to $c_1$. The cycle structure of a permutation is the statistics of its cycle lengths in its cycle representation.

The $n!$ permutations on the set $[n]$ together with composition as the group operation form the symmetric group $S_n$.

Next, we collect notions from representation theory (readers unfamiliar with the notations might consult [16, 11] for more details)

**Definition 2.** Let $G$ be a (finite) group. Then a (complex) representation of $G$ is a (group) homomorphism

$$\varphi : G \rightarrow \text{GL}(V),$$

for some (complex) vector space $V$, i.e., an embedding of $G$ in the group of invertible linear transformations on $V$. Given a representation $\varphi$, one defines the associated character to be the map

$$\chi_\varphi : G \rightarrow \mathbb{C} \text{ mapping } g \in G \text{ to the trace } \text{Tr}(\varphi(g)) \text{ of the associated linear transformation.}$$

A representation is irreducible if $V$ does not contain a nontrivial $G$-invariant subspace and the characters
given by an irreducible representations are called irreducible characters.

Note that for matrices \( A, B \) we have \( \text{Tr}(AB) = \text{Tr}(BA) \), and therefore, \( \text{Tr}(A) = \text{Tr}(B^{-1}BA) = \text{Tr}(BAB^{-1}) \). That is, the trace is independent of a chosen basis. Furthermore,

\[
\text{Tr}(\varphi(g)) = \text{Tr}(\varphi(h)\varphi(g)\varphi(h^{-1})) = \text{Tr}(\varphi(hgh^{-1}))
\]

for any \( h, g \in G \). That is, characters not depending on the choice of basis and are constant on conjugacy classes of \( G \), i.e., they are class functions on \( G \). Moreover, the irreducible characters form a basis for the space of all class functions. In particular, the number of distinct irreducible characters equals the number of conjugacy classes.

**Example 1.1.** The function which assigns 1 to any element in a group \( G \) is the trivial character of \( G \). The function sign which maps any element \( \rho \in S_n \) to \((-1)^{e(\rho)}\), where \( e(\rho) \) is the number of even-length cycles in \( \rho \), is called the sign-character.

With these notations matrix immanants are a natural generalization of both the determinant and the permanent by using the irreducible characters of \( S_n \).

**Definition 3.** Let \( \varphi \) be an irreducible representation of the symmetric group \( S_n \), and let \( A = \{a_{i,j}\} \) be an \( n \times n \) matrix over \( \mathbb{C} \). Then the \( \varphi \)-immanant of \( A \) is defined as

\[
\text{Imm}_\varphi(A) := \sum_{\rho \in S_n} \chi_\varphi(\rho) \prod_{i=1}^{n} a_{i,\rho(i)}.
\]

It is easy to see that two permutations are in the same the conjugacy classes of \( S_n \) if they have the same cycle structure. Since characters are constant on conjugacy classes and since we mainly are interested in values of characters, we will slightly abuse the notation and use \( \rho \) both for the permutation and also for the cycle structure of \( \rho \). From the view point of combinatorics, every cycle structure can be identified with a partition of \( n \). Further, the number of conjugacy classes of \( S_n \) is therefore \( p(n) \), the number of partitions of the natural number \( n \). Since the number of irreducible representations equals the number of the conjugacy classes, \( S_n \) has \( p(n) \) irreducible representations. To describe partitions in a convenient way we will use the following definitions which we recall for the convenience of the reader.

**Definition 4.** A weakly decreasing sequence of integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \) is a partition of \( n \) if \( \sum_{i=1}^{h} \lambda_i = n \). The height of the partition is \( h \) and it is denoted by \( h(\lambda) \). The size of \( \lambda \) is \( n \), and it is denoted by \( |\lambda| \). Finally, we will denote \( n - h \) by \( b(\lambda) \).

For short, we will write \( c^k \) for a run of \( k \) number of \( c \)'s in a partition. For example, a partition \( (2,2,1,\ldots,1) \) of size \( n \) can be written as \((2^2,1^{n-4})\). For a series of sequences (with even possibly different lengths), \( \lambda \) and \( \mu \), we denote by \( \lambda + \mu \) the sequence that contains \( \lambda_i + \mu_i \) in each position where both \( \lambda_i \) and \( \mu_i \) exists and \( \lambda_i \) or \( \mu_i \) where only one of the numbers exists. We will use the addition operation in extending a partition. That is, for partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \) we define \((w, \lambda + \mu)\) as \((w, \lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)\).

**Definition 5.** A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \) can be represented by a Young diagram, which is a finite
collection of boxes, arranged into left-justified rows, such that for each $i$, row $i$ contains $\lambda_i$ boxes. A Young diagram is also called shape.

If $\lambda$ is a shape, then its conjugate shape $\lambda^*$ is obtained by swapping the rows and columns.

**Example 1.2.** Let $n = 4$ and $\lambda = (2, 1, 1)$. Then a Young diagram of shape $\lambda$ and its conjugate are given by

```
+---+---+---+
|   |   |   |
+---+---+---+
|   |   |   |
+---+---+---+
|   |   |   |
```

and

```
+---+---+---+
|   |   |   |
+---+---+---+
|   |   |   |
+---+---+---+
|   |   |   |
```

The conjugate is therefore $\lambda^* = (3, 1)$.

We will also be needing the notion of a skew-shape associated to a pair of partitions.

**Definition 6.** If $\lambda$ and $\mu$ are two partitions, such that $h(\lambda) \geq h(\mu)$, and furthermore, for each $i$ smaller or equal the height of $\mu$, it holds that $\mu_i \leq \lambda_i$, then we can define the skew-shape $\lambda/\mu$, which is the set-theoretic difference of the Young diagrams of $\lambda$ and $\mu$. The height of a skew-shape $\lambda/\mu$ is the height of $\lambda$. A skew-shape is connected if there is a rook path between any pair of boxes, that is, from any box any another one is reachable by horizontal and vertical steps.

**Example 1.3.** Consider the partitions $\lambda = (4, 3, 1)$ and $\mu = (1, 1)$. Then the skew shape $\lambda/\mu$ is

```
+---+---+---+
|   |   |   |
+---+---+---+
|   |   |   |
+---+---+---+
|   |   |   |
```

Note that this skew-shape is not rook-path connected, that is, it is not a connected skew-shape by definition.

Following the work of Frobenius[12] the irreducible characters of the symmetric group can be completely understood using these Young diagrams and thus one has a precise way to identify these characters with the partitions, respectively with the Young diagrams. For example, the trivial character of $S_n$ is identified with the partition $(n)$ or equivalently with the Young diagram consisting of one long row and the sign character with the partition $(1^n)$, or equivalent the Young diagram consisting of one long column. Therefore, we can call the immanant constructed with the irreducible character corresponding to the partition $\lambda$ the $\lambda$-immanant. In this way, the determinant is the $(1^n)$-immanant, and the permanent is the $(n)$-immanant.

Finally a classical result from combinatorics uses Young diagrams to define a partial ordering on the partitions.

**Definition 7.** Two partition $\lambda$ and $\mu$ of $n$ are comparable, and $\lambda \preceq \mu$ if the Young diagram of $\mu$ can be obtained by replacing some of the boxes of the Young diagram of $\lambda$ such that each box is moved only to right and up.

In this partial ordering the two partitions related to the determinant and permanent are extreme. Indeed, it is easy to see that $(n)$ is the minimal element in the poset and $(1^n)$ is the maximal one. From this
point of view, the determinant and the permanent are as far from each other as possible: the determinant corresponds to partitioning $n$ into $n$ parts, while the permanent corresponds to partitioning $n$ into one part. It is therefore natural to ask how the computational complexity of the immanants changes from the partition $(1^n)$ to the partition $(n)$.

Since we consider some restricted matrices defined as adjacency matrices of graphs, it is important to define certain graphs and their adjacency matrices.

**Definition 8.** A bipartite graph $G = (U, V, E)$ is a simple graph in which each vertex incident to 1-1 vertices in both vertex classes $U$ and $V$. The adjacency matrix $A$ of a bipartite graph $G = (U, V, E)$ is a $|U| \times |V|$ matrix, in which $a_{i,j} = 1$ if $(u_i, v_j) \in E$ and otherwise 0. If the bipartite graph $G = (U, V, E)$ is an edge-weighted graph by a weight function $w : E \to \mathbb{C}$, then in the weighted adjacency matrix of $G$, it holds that $a_{i,j} = w(e)$ for each edge $e = (u_i, v_j)$. A bipartite graph is planar if it has a drawing in the Euclidian plane without crossing edges.

A directed graph $\vec{G} = (V, E)$ is a graph in which each vertex has a direction, that is, each edge has a head and a tail. The edge whose tail is $v_1$ and head is $v_2$ is distinguished from the edge whose tail is $v_2$ and head is $v_1$. We also allow loops in a directed graph, that is a vertex going from some $v \in V$ to itself. The adjacency matrix of $A$ of a directed graph $\vec{G} = (V, E)$ is a $|V| \times |V|$ matrix in which $a_{i,j} = 1$ if there is an edge in $\vec{G}$ going from $v_i$ to $v_j$. A directed graph is planar if it has a drawing in the Euclidian plane without crossing edges.

There is a bijection between bipartite graphs $G = (U, V, E)$ with $|U| = |V| = n$ and directed graphs $\vec{G} = (V, E)$ on $|V| = n$ vertices. The bijection is that $G = (U, V, E)$ and $\vec{G} = (V, E)$ are images of each other if they have the same adjacency matrix. Note that this bijection does not keep the planarity property. Indeed, the complete bipartite graph $K_{3,3}$ is not planar while the complete directed graph $\vec{G}_{3}$ has a planar drawing, although both graphs have the same adjacency matrix, the $3 \times 3$ all-1 matrix.

### 1.2 Main results

We are now in the position to state the main results of this article which naturally arise from a sequence of important contributions by various authors in the last 40 years. In a fundamental work Hartmann [14] showed in the 1980s that the immanants corresponding to shapes from which the partition $(1^n)$ can be obtained by moving only constant many boxes can be computed in polynomial time. Furthermore, he showed that those immanants corresponding to a shape which arises from the partition $(n)$ by moving only constant many boxes are #P-hard to compute [14]. Barvinok [1] and Bürgisser [2] improved the polynomial running time of Hartmann’s algorithm for immanants close to the determinant, and Bürgisser in the same paper also proved #P-hardness for immanant families of irreducible characters corresponding to hook and rectangle partitions. Brylinski and Brylinski [4] proved VNP-completeness of computing immanants related to partitions of $n$ in which any two consecutive number differs by $\Omega(n^\alpha)$ for some $\alpha > 0$. Mertens and Moore [24] proved that computing all immanants related to partitions containing only 2’s and 1’s is #P-hard. The #P-hardness for immanants related to partitions of $n$ containing only $n - n^\alpha$ numbers, each of them upper bounded by a constant was proved by de Rugy-Altherre [8].
From these previous results, it was natural to conjecture that computing any immanant which is \( n^\varepsilon \) far from the determinant is \#P-hard. Our initial aim of this paper was to prove this conjecture for a large fraction of such immanants with \( 0 < \varepsilon \). However, during the writing of this manuscript, Radu Curticapean [5] provided a complete dichotomy result in a preprint uploaded to ArXiv. In that paper he is able to show the \#P-hardness of computing an immanant for any shape that is \( \Omega(n^\varepsilon) \) far from the determinant (the shape of the partition can be obtained from the shape of \( (n) \) by moving \( \Omega(n^\varepsilon) \) boxes, for a \( \varepsilon > 0 \)).

After the results of Curticapean, the next natural question to ask if computing the immanants remains hard for restricted matrices. Curticapean addressed these questions at the end of his final paper [6] citing an earlier arXiv version of our manuscript. Indeed, we do know that some of the hard immanants are easy to compute for some restricted matrices. The most curious example is the following. If a matrix \( A \) is the (possibly weighted) adjacency matrix of a planar bipartite graph, then the permanent of \( A \) can be computed in polynomial time. On the other hand, computing the permanent of a general 0−1 matrix (that is, the unweighted adjacency matrix of a general bipartite graph with equal size of vertex classes) is already \#P-complete [31]. Observe that based on the partial ordering of shapes, the permanent is the furthermost immanant from the determinant, and computationally expected to be the hardest. Still, it is easy to compute for a large class of matrices. Another example is the \( \mu \)-immanant of adjacency matrices of directed, bipartite graphs for the staircase shape \( \mu = (k, k−1, k−2, \ldots, 1) \). Computing this immanant is trivial since \( \chi_\mu(\rho) \) is vanishing for all permutations \( \rho \) that contains an even cycle [5]. Observe that a bipartite graph can contain only even cycles. The matrix class – shape class pairs for which computing the immanant is “accidentally” easy might have a rich structure.

In this paper, we infer the computational complexity of computing the immanants of restricted matrices. In the first part of the paper, we restrict the values of the matrix to 0s and 1s. We further restrict the matrices to be adjacency matrices of directed graphs that contains only even cycles. Curticapean proved \#P-hardness of computing the \( \lambda \)-immanant of 0-1-matrices only for such \( \lambda \) shapes in which the number of black and white boxes in a checkerboard coloring differ by \( \Omega(n^\varepsilon) \) for some \( \varepsilon > 0 \). Indeed, he showed that for any shape \( \lambda \) for which \( b(\lambda) = \Omega(n^\varepsilon) \), \( \varepsilon > 0 \), either \( \Omega(n^\varepsilon) \) 1×2 dominos can be peeled out or after the peeling process, the remaining staircase shape \( \mu = (k, k−1, k−2, \ldots, 1) \) has \( \Omega(n^\varepsilon) \) size (or both cases holds). Then he proved \#P-hardness for both cases, using different graph gadgets. The graph gadget proving the \#P-hardness for the large remaining staircase case is unweighted, that is, the proof holds if the computation problem is restricted to 0-1 matrices. When there is a large domino-tilable part of the shape, Curticapean’s proof needs a gadget containing a −1 sign. That −1 sign seems to be essential in the proof as it provides a cancellation of terms, and it is unclear how that −1 could be omitted from his construction (personal communication with Radu Curticapean). Observe that the difference of the black and white boxes in a shape \( \mu \) is \( \Omega(k) = \Omega(\sqrt{|\mu|}) \). On the other hand, observe that any shape that has a 1×2 domino tiling has the same number of black and white boxes. We prove \#P-hardness for a large class of such domino-tilable shapes. The computation remains hard if it is restricted to adjacency matrices of directed graphs in which any cycle has an even length. We would like to recall that computing the staircase immanant for such matrices is trivial. More precisely, in section 3 we prove the following theorem:
Theorem 1.4. Let $\lambda$ be a partition of an even $n$ in form $(w, 1 + \lambda_d)$ such that $|\lambda_d|$ is $n^\varepsilon$ for some $0 < \varepsilon$, the shape of $\lambda_d$ has a $1 \times 2$ domino tiling and $(3w + 3h(\lambda_d) + 4)|\lambda_d| \leq n$. Then it is $\#P$-hard to compute the $\lambda$-immanant of a 0-1 matrix which is an adjacency matrix of a bipartite directed graph. The theorem also holds for the conjugate partition $\lambda^\ast$.

The proof given in section 3 is based on reducing the counting of perfect matchings in 3-regular graphs to computing the indicated immanants. The graph gadget designed for the proof is different from the graph gadgets used by Curticapean.

Planar bipartite graphs take a special place on the scene of computational complexity of immanants. It turns out that computing the permanent of the adjacency matrices of planar bipartite graphs needs only polynomial time. Therefore, it would be interesting to ask what other immanants of the adjacency matrices of planar bipartite graphs are easy to compute. However, there is no natural bijection between the vertices of the two vertex classes of a bipartite graph. The consequence is that the adjacency matrix of a bipartite graph is not well defined: changing the order of the vertices in one of the vertex classes causes permuting the rows of the adjacency matrix. While permuting the rows of a matrix does not change the permanent, it might change the immanant and even it might change the computational complexity of computing these immanants. To see this, consider the 1-regular bipartite graphs on $n + n$ vertices. Depending how the vertices in one of the class is ordered, its adjacency matrix $A$ might be a permutation matrix for an arbitrary permutation $\rho \in S_n$. Then $\text{Imm}_\lambda(A) = \chi_\lambda(\rho)$. If $\rho$ is the identity, then $\chi_\lambda(\rho)$ can be computed using the hook-length formula (see [10, 28]). On the other hand, even deciding for a general $\lambda$ and $\rho$ if $\chi_\lambda(\rho)$ is zero or not is already PP-hard [15].

The rows of the adjacency matrices of directed graphs cannot be permuted without changing the corresponding directed graph. Therefore it is more natural to ask the computational complexity of immanants of directed graphs with given properties. It is also a natural restriction to consider planar directed graphs. In the second part of the paper, we prove that for a large class of shapes, computing the immanant of the weighted adjacency matrices of planar directed graphs is $\#P$-complete. More specifically, in section 4 we prove the following theorem:

Theorem 1.5. Let $\lambda = (1 + \lambda_d)$ be a partition of $n$ such that $|\lambda_d| = n^\varepsilon$ for some $0 < \varepsilon < \frac{1}{2}$, and for some $w$, the shape $\lambda_d/(w)$ is tilable with $1 \times 2$ dominos. Then it is $\#P$-hard to compute the $\lambda$-immanant of adjacency matrices of edge-weighted, planar, directed graphs. The problem remains $\#P$-hard if the weights are small, non-negative integers given unary.

The proof is based on reducing the counting the (not necessarily perfect) matchings in planar graphs to computing the indicated immanants. In this proof, we combine the graph gadget presented in section 3 with a graph gadget appearing in Curticapean’s paper [5].

2 Irreducible characters and the Murnaghan-Nakayama rule

Below we give the necessary definitions and theorems to compute irreducible characters and also state properties of them we used in the $\#P$-hardness proofs. To be able to compute irreducible characters corresponding to a given shape, we have to define the notion of border-strip tableaux.
Definition 9. Let $\lambda$ and $\rho$ be two partitions of $n$. The border-strip tableau of shape $\lambda$ and type $\rho$ is a Young diagram of shape $\lambda$ filled in with positive integers satisfying the following rules:

1. The integers are weakly increasing in each row and in each column.
2. Each integer $i$ is presented $\rho_i$ number of times.
3. For each $i$, the set of squares filled with number $i$ form a border strip, that is, a connected skew-shape with no $2 \times 2$ squares.

The height of the border strip tableau is defined as the sum of the heights of the border strips minus the number of border strips.

Example 2.1. Consider the two partitions $\lambda = (5, 2, 1)$ and $\mu = (3, 3, 1, 1)$. Then there are the following six border strip tableaux:

\[
\begin{array}{cccc}
1 & 1 & 1 & 3 & 4 \\
2 & 2 & & & \\
2 & & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & 2 & 2 & 2 \\
1 & & 3 & & \\
4 & & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & 2 & 2 & 2 \\
1 & & 4 & & \\
3 & & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 2 & 2 & 4 \\
1 & 3 & & & \\
1 & & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 2 & 2 & 3 \\
1 & 4 & & & \\
1 & & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 2 & 3 & 4 \\
1 & 2 & & & \\
1 & & & & \\
\end{array}
\]

and the corresponding heights are 1, 1, 2, 2, 3.

As stated earlier, both the different irreducible characters of the symmetric group, as well as the conjugacy classes are naturally parametrized by partitions of $n$. The following so-called Murnaghan-Nakayama rule, which involves the skew-shapes defined above, allows to compute the value of an irreducible character corresponding to a partition $\lambda$ on the conjugacy class parametrized by a partition $\rho$ in a purely combinatorial way.

Theorem 2.2 (The non-recursive version of the Murnaghan-Nakayama rule). Let $\lambda$ and $\rho$ be two partitions of $n$. Then the irreducible character of the symmetric group $S_n$ corresponding to shape $\lambda$ evaluated on a permutation with cycle structure $\rho$ can be obtained as

\[
\chi_\lambda(\rho) = \sum_{T \in \text{BST}(\lambda, \rho)} (-1)^{\text{ht}(T)}
\]  

where $\text{BST}(\lambda, \rho)$ is the set of border-strip tableaux of shape $\lambda$ and type $\rho$, and $\text{ht}(T)$ is the height of the border-strip tableau $T$.

The Murnaghan-Nakayama rule was first proved by Littlewood and Richardson [20] based on the Frobenious formula [12], then also by Murnaghan [25] and Nakayama [26].

We can say even more on the irreducible characters of $S_n$. For this, we need the definition of $k$-spectrum
functions.

**Definition 10.** The $k$-spectrum of a permutation $\rho$ is a $k$-dimensional vector $(x_1, x_2, \ldots, x_k)$ such that for each $i$, $x_i$ is the number of cycles of length $i$ in $\rho$. The size of the $k$-spectrum is

$$\sum_{i=1}^{k} i x_i.$$ 

A function $f$ mapping from $\mathbb{C}^k$ to $\mathbb{C}$ is a $k$-spectrum function if $f$ is a polynomial of $k$ variables and for each monomial $\prod_{i=1}^{k} x_i^{\beta_i}$ in it, it holds that

$$\sum_{i=1}^{k} i \beta_i \leq k.$$ 

Although the following theorem (possibly in slightly different form) is known (for example, Hartmann already used it in his paper on complexity of immanants [14]), we give here a short proof.

**Theorem 2.3.** Let $\lambda$ be a partition of $n$ and let $k = n - h(\lambda)$. Then the irreducible character for $\lambda$ can be given in form

$$\chi_\lambda(\rho) = \text{sign}(\rho) \cdot f(c_1(\rho), c_2(\rho), \ldots, c_k(\rho))$$

where $f$ is a $k$-spectrum function and $(c_1(\rho), c_2(\rho), \ldots, c_k(\rho))$ is the $k$-spectrum of $\rho$. Furthermore, if $\lambda$ and $\lambda'$ are two shapes that differ only in the first column, and $\rho$ and $\rho'$ have the same $k$-spectrum, then

$$\text{sign}(\rho) \chi_\lambda(\rho) = \text{sign}(\rho') \chi_{\lambda'}(\rho').$$

**Proof.** Garsia and Golupil [13] proved that the irreducible character of a shape $(n - |\mu|, \mu)$ is a $|\mu|$-spectrum function on variables $(c_1(\rho), c_2(\rho), \ldots, c_{|\mu|}(\rho))$, and that it does not depend on $w = n - |\lambda|$. The statement of the theorem follows from their observation and from the fact that for any shape $\lambda$ and its conjugate $\lambda^*$ it holds that

$$\chi_{\lambda}(\rho) = \text{sign}(\rho) \chi_{\lambda^*}(\rho)$$

for any permutation $\rho$ [16, Equation 2.1.8].

Theorem 2.3 is in particular useful to evaluate the characters of certain partitions easier.

**Example 2.4.** Consider the character $\chi_{(2^{2n})}$ and evaluate it at the conjugacy class corresponding to $(2^{2n})$, which yields $\binom{2n}{n}$. Doing the necessary computations merely by using the non-recursive Murnaghan-Nakayama rule turns out to be fairly involved. However, it is easy to compute once we take into consideration that

$$\text{sign}(2^{2n}) \chi_{(2^{2n})}(2^{2n}) = \text{sign}(2n + 1, 2^{2n}) \chi_{(2^{2n}, 1^{2n+1})}(2n + 1, 2^{2n}).$$

Indeed, it is significantly easier to compute $\chi_{(2^{2n}, 1^{2n+1})}(2n + 1, 2^{2n})$. Any border-strip tableau of shape $(2^{2n}, 1^{2n+1})$ and type $(2n + 1, 2^{2n})$ must have $n$ border-strips of size 2 in its second column. Indeed, if the border strip of size $2n + 1$ would take 1 square from the second column, then the remaining skew-
shape would not be connected. Further, the two parts of the remaining skew shape both would contain odd number of squares making it impossible to tile with border-strips of size 2. Therefore, in each border strip tableaux, the border strip of size $2n + 1$ is placed in top of the first column, and the remaining skew-shape is not connected. Both of its parts are a $2n \times 1$ shape. The numbers of these border strips that go to the second column have to be selected freely from the set $\{1, 2, \ldots, 2n\}$. There is a unique way to arrange them into the second column: they must be put into order. Thus there are indeed $\binom{2n}{n}$ possible border-strip tableaux, and each of them has the same height.

Surprisingly, it turns out that the parity of the height is a useful invariant for some sets of border-strip tableaux as we show in the following theorem. In fact, this theorem is the key for the hardness result.

**Theorem 2.5.** Let $\lambda = (w, 1 + \lambda_d)$ be a partition where 1 is the all-1 vector. Furthermore, assume that $\lambda_d$ can be tiled with $1 \times 2$ dominos. Let $\rho$ be a permutation of size $|\lambda|$ such that it has exactly $|\lambda_d|/2$ cycles of length 2, it does not have any fixpoint and all other cycles has a length at least $w + h(\lambda)$. Then $\chi_{\lambda}(\rho)$ is not vanishing.

**Proof.** The only way to tile $\lambda$ with appropriate border-strips is that the border-strip for the largest cycle in $\rho$ must be in a hook in the top left corner of the shape, then all other long cycles must be in the first column, and the border-strips for the cycles of length 2 must tile the shape $\lambda_d$. By the conditions, at least one such a tiling exists. There might be multiple tilings thus border-strip tableaux. The sum in the equation 1 could be 0 if different border-strip tableaux could have different parities of their height, however, the parity of the height is invariant. Indeed, the parity depends on only the number of horizontal $1 \times 2$ dominos tiling $\lambda_d$. However, any such domino tiles exactly one square in every second column in $\lambda_d$ while each vertical dominos tile even number of squares in every second row. Thus, the parity of the number of horizontal dominos is the parity of the sum of the squares in every second row in $\lambda_d$. \qed

An important other consequence of the Murnaghan-Nakayama rule is that it allows to combinatorially describe sufficient conditions to decide when a given character has to vanish on certain permutations.

**Example 2.6.** Consider the partition $\lambda = (2^2, 1^n - 4)$. Then, in fact, $\chi_{\lambda}(\rho) = 0$ for any permutation in which each cycle has length at least 3. Indeed, any border-strip tableaux should be filled in with $\rho_1 > 2$ many 1s. Then in the second column of the shape, 1 or 2 squares remain empty while the first two squares in the first column are filled with 1s. Therefore, a border strip of size 1 or 2 should be filled with a number, but there is no such short cycle in $\rho$. That is, there is no border-strip tableau of shape $\lambda$ and type $\rho$, and thus, the sum in equation 1 is empty.

We generalize this observation and put into the theorem that some of the characters are vanishing for the shape $(w, 1 + \lambda_d)$.

**Theorem 2.7.** Let $\lambda = (w, 1 + \lambda_d)$ be a partition, and let $\rho$ be a permutation such that the size of its $|\lambda_d| + w - 1$-spectrum is less than $|\lambda_d|$. Then

$$\chi_{\lambda}(\rho) = 0.$$
Proof. Without loss of generality, we may assume that the longest cycle of $\rho$ is larger than $w + h(\lambda_d)$, based on Theorem 2.3. Then the border-strip containing 1s can cover at most $w - 1$ of the squares not in the first column. The remaining at least $|\lambda_d|$ squares should be covered with border-strips each of them covering at most $|\lambda_d| + w - 1$ squares. However, there are too few such border-strips. That is, there is no border-strip tableau of shape $(w, 1 + \lambda_d)$ and type $\rho$, therefore, the sum in equation 1 is empty and thus 0.

3 The #P-hardness result on computing immanants of 0-1 matrices

In this section we are proving the #P-hardness result stated in Theorem 1.4. Our proof proceeds by reducing the computation of the number of perfect matchings of 3-regular bipartite graphs to computing the 3-immanant of adjacency matrices of directed graphs. #P-completeness of counting perfect matchings of 3-regular bipartite graphs was proved by Dagum and Luby [7]. In order to arrive at the reduction we use the following construction.

Construction 1. Take a 3-regular bipartite graph $H = (V, E)$ with $|\lambda_d|/2$ many vertices in both vertex classes. Replace each vertex $v \in V$ with the gadget shown in Figure 1. There are distinguished vertices $v_c, v_1^+, v_1^-, v_2^+, v_2^-, v_3^+, v_3^-$. For each $i$, there are edges from $(v_i^-, v_i), (v_i, v_i^+), (v_i^-, v_c), (v_c, v_i^+)$. For each $i = 1, 2$, there is a path of length $p = w + h(\lambda_d) - 1$ or $p = w + h(\lambda_d)$, choosing the odd $p$, from $v_i^+$ to $v_i^-$, and there is also a path of length $p$ from $v_3^+$ to $v_1^-$. For each edge $(v, w) \in E$, if $w$ is the $i$th neighbor of $v$ and $v$ is the $j$th neighbor of $w$, then there are edges $(v_i, w_j)$ and $(w_j, v_i)$. So far there are at most $(3p + 4)|\lambda_d| = (3w + 3h(\lambda_d) + 4)|\lambda_d|$ vertices. If it is less than $n$, then in exactly one of the gadgets, in exactly one of the paths of length $p$, the path is elongated to get $n$ vertices. Since $\lambda_d$ is domino tilable, $|\lambda_d|$ is even, so $n - (3w + 3h(\lambda_d) + 4)|\lambda_d|$ is even. Therefore, the length of the elongated path remains odd as an odd path is elongated to get even number of vertices.

There are no more vertices or edges. In this way, one arrives at a directed graph $\tilde{G}$. In this graph, any cycle has even length. Indeed, it is easy to see that any cycle inside the gadget has even length. Since $H$ was a bipartite graph, any cycle in $\tilde{G}$ has also even length.

The following properties of the directed graph $\tilde{G}$ will be important.

Lemma 3.1. The directed graph $\tilde{G}$ resulting from Construction 1 has the following properties.

1. Any cycle of length 2 in any cycle cover of $\tilde{G}$ is on vertices $(v_i, v_j)$.
2. Any cycle longer than 2 in any cycle cover of $\tilde{G}$ has length at least $p + 1 \geq w + h(\lambda_d)$.
3. There is a 1-to-1 correspondence between perfect matchings of $G$ and cycle covers of $\tilde{G}$ with $|V|/2$ cycles of length 2.
4. Furthermore, there is no cycle cover of $\tilde{G}$ that contains more than $|V|/2$ cycles of length 2.

Proof. 1. Clearly, there are no other cycles of length 2 in $\tilde{G}$.
2. To see this, notice that a cycle which is not a 2-cycle must contain a \( v_i \). Thus it has a length at least \( p + 1 \).

3. Let \( M \subseteq E(H) \) be a perfect matching in \( H \). We claim that there is exactly one cycle cover in \( \vec{G} \) that for each \((v, w) \in M\) contains the cycle of length 2 between vertices \( v_i \) and \( w_j \). Indeed, if vertex \( v_i \) is covered by a cycle of length 2, then the remaining vertices of the gadget replacing vertex \( v \in H \) can be covered uniquely by a cycle. Furthermore, there is no cycle cover in which \( v_i \) is covered by a cycle of length 2, and any other cycle in the cycle cover does not contain all the remaining vertices of the gadget replacing vertex \( v \in H \). What also follows is that there is no cycle cover of \( \vec{G} \) that contains two cycles of length 2 covering vertex \( v_i \) and \( v_j \) in the same gadget replacing vertex \( v \in H \). Therefore, if a cycle cover of \( \vec{G} \) contains exactly \( |V|/2 \) cycles, then it corresponds to a perfect matching in \( H \).

4. From the fact, that there is no cycle cover of \( \vec{G} \) that contains two cycles of length 2 covering \( v_i \) an \( v_j \) it immediately follows that the maximum number of cycles of length 2 is \( |V|/2 \) in the cycle covers of \( \vec{G} \).

With the properties of cycle covers of \( \vec{G} \) established above we can show the \#P-hardness.

Proof of Theorem 1.4. Let \( A \) be the adjacency matrix of the graph \( \vec{G} \) resulting from Construction 1 from a 3-regular bipartite graph \( H \), which has \( |\lambda_d|/2 \) many vertices in both vertex classes. Consider the corresponding immanant

\[
Imm_\lambda(A) = \sum_{\rho \in S_n} \chi_\lambda(\rho) \prod_{i=1}^{n} a_{i,\rho(i)}.
\]

First, we observe which of the summands are actually vanishing: If \( \rho \) is not a cycle cover in \( \vec{G} \), then \( \prod_{i=1}^{n} a_{i,\rho(i)} = 0 \). Furthermore, if \( \rho \) is a cycle cover, but does not contain \( |\lambda_d|/2 \) 2-cycles, then it contains
less than \(|V|/2\) cycles of length 2, and thus \(\text{Imm}_\lambda(\rho) = 0\), according to Theorem 2.7. Indeed, observe that any cycles \(\rho\) which are not 2 cycles has size \(p + 1\) and \(p + 1 \geq w + |\lambda_d|\).

Finally, if \(\rho\) is a cycle cover in \(\vec{G}\) and contains \(|\lambda_d|/2\) 2-cycles, then by Lemma 3.1 (3) it corresponds to a perfect matching in \(H\), therefore, we get that

\[
\text{Imm}_\lambda(A) = \chi_\lambda(n - |\lambda_d|, 2^{|\lambda_d|/2}) \cdot \text{PM}(H),
\]

where \(\text{PM}(H)\) is the number of perfect matchings in \(H\). Since \(\chi_\lambda(n - |\lambda_d|, 2^{|\lambda_d|/2})\) is not vanishing, according to Theorem 2.5 we obtain

\[
\text{PM}(H) = \frac{\text{Imm}_\lambda(A)}{\chi_\lambda(n - |\lambda_d|, 2^{|\lambda_d|/2})}.
\]

Note that by construction the size of \(A\) is polynomial in the number of vertices of \(H\). Exhibit a directed graph \(\vec{G}_1\) consists of a directed cycle of length \(n - |\lambda_d|\) and \(2^{\lambda_d/2}\) disjoint cycles of length 2. Then it holds for its adjacency matrix \(A_1\) that \(\text{Imm}_\lambda(A_1) = \chi_\lambda(n - |\lambda_d|, 2^{|\lambda_d|/2})\). That is,

\[
\text{PM}(H) = \frac{\text{Imm}_\lambda(A)}{\text{Imm}_\lambda(A_1)}.
\]

From the orthonormality relation of the irreducible characters, any immanant of a 0–1 matrix is at most \(n!\) (see, for example, [28], Theorem 1.9.3). So it follows that the logarithm of any immanant of a 0–1 matrix is upper bounded by a polynomial of the size of the matrix. Therefore, \(\text{Imm}_\lambda(A)\) and \(\text{Imm}_\lambda(A_1)\) contains at most polynomial number of digits. Therefore, the fraction of them can be also computed in polynomial time. We can conclude that any algorithm computing \(\text{Imm}_\lambda(A)\) and \(\text{Imm}_\lambda(A_1)\) in polynomial time is applicable to compute \(\text{PM}(H)\) in polynomial time.

Finally, in order to show that the theorem also holds for \(\lambda^*\), observe that the immanant is non-vanishing only on one particular cycle structure. \(\square\)

There is an upper limit how large \(\lambda_d\) might be. Clearly, if \(w = O(|\lambda_d|)\), then \(\varepsilon\) might be any number smaller than \(\frac{1}{2}\). If \(w = O(\sqrt{|\lambda_d|})\) and \(\lambda_d\) has a square shape, then \(\varepsilon\) might be any number smaller than \(\frac{2}{3}\).

4 \#P-hardness result on computing immanants of adjacency matrices of edge-weighted, planar, directed graphs

Finally, we prove the hardness result stated in Theorem 1.5 by reducing the number of (not necessarily perfect) matchings of planar graphs to computing the \(\lambda\)-immanant of the adjacency matrix of edge-weighted, planar, directed graphs. Computing the number of matchings in planar graphs is \#P-complete [17, 18].

**Construction 2.** Let \(H = (V,E)\) be a planar graph on \(\frac{n^2}{2}\) vertices. We construct a corresponding directed, planar graph \(\vec{G}\) in the following way. Replace each edge \((u,v) \in E\) with the gadget represented
Figure 2: The match gadget replacing each edge in a planar graph. See the Construction 2 for details. In Figure 2, we call this gadget the match gadget. The indicated edge has weight $-1$, all other edges have weight 1. For each vertex $v \in H$, add a gadget represented in Figure 3. The indicated edge has a weight $x$, all other edges have weight 1. The paths between $u_1^+$ to $u_2^-$ as well as from $u_2^+$ to $u_1^-$, from $w_1^+$ to $w_2^-$ and from $w_2^+$ to $w_1^-$ are all have length $p$ such that $(4p + 7)|V| + 2|V| + 2|E| = n$, that is

$$p = \frac{n - 2|E| - 9|V|}{4|V|}.$$ 

Note that in any planar graph, $|E| \leq 3|V| - 6$, therefore

$$p \geq \frac{n^{1-\varepsilon}}{2} - \frac{15}{4} - \frac{12}{n^\varepsilon}.$$ 

We call this gadget the vertex covering gadget.

Finally, we add $2|V|$ additional vertices to $\vec{G}$ and create $|V|$ isolated 2-cycles on them. There are no more vertices and edges in $\vec{G}$.

Note that the gadget in Figure 2 is identical with the gadget in [5]. Here we state properties of this match gadget that are not hard to verify. Detailed proofs of these statements can be found in the cited manuscript.

**Proposition 4.1.** The match gadget shown in Figure 2 has the following properties.

1. If both $u$ and $v$ are in a cycle of a cycle cover not involving edges in the match gadget between $u$ and $v$, then the remaining two vertices can be covered in exactly one way, with weight $-1$.

2. If exactly one of the vertices $u$ and $v$ are in a cycle of a cycle cover not involving edges in the match gadget between $u$ and $v$, then the remaining three vertices in this gadget can be covered in two different ways, and the sum of the weights of these cycle covers cancel each other.

3. If both $u$ and $v$ are in one or two cycles of a cycle cover involving only edges in the match gadget between $u$ and $v$, then there are four such ways, two of them contain two 2-cycles, two of them contain one 4-cycle. In all cases, the weight of the cycle(s) is 1.

4. If both $u$ and $v$ are in one cycle of a cycle cover containing both edges in the match gadget between $u$ and $v$ and edges outside this gadget, then such cycles can be paired such that their weights cancel.
each other and the pair of cycle covers have exactly the same cycle length structures. What follows is that in any immanant evaluated on the adjacency matrix of $\vec{G}$, the cycle covers with any cycle that contains both edges from a match gadget between two vertices and edges outside of that gadget cancel each other.

We now examine the connection of matchings in the original graph $H$ to cycle covers in the resulting graph $\vec{G}$:

Consider a vertex $v \in H$ and its corresponding vertex $v \in \vec{G}$. If $v$ is covered with a cycle containing edges from the vertex covering gadget in Figure 3, then there is one way for it, furthermore, the remaining vertices in the vertex covering gadget can be covered in exactly one way: with a cycle containing vertices $w_c, w_2^+, w_1^-$ and a cycle containing vertices $u_c, u_2^+, u_1^-$. The product of the weights of these cycles is $x$.

If $v$ is covered with a cycle not containing edges from its vertex covering gadget, then the remaining vertices in the vertex covering gadget of $v$ can be covered in exactly one way, with three cycles: the first is the cycle between vertices $u_2$ and $w_1$, the second is the cycle that contains the vertices $w_c, w_1^+, w_1^-$ and the third is the cycle that contains the vertices $u_c, u_2^+, u_2^-$. 

Based on these observations we have the following.

**Proposition 4.2.** For the pair of graphs $H$ and $\vec{G}$ in Construction 2 there is a one-to-many correspondence between the matchings of $H$ and the cycle covers of $\vec{G}$. Moreover, to a matching in $H$ containing $k$ edges, there are $4^k$ corresponding cycle covers in $\vec{G}$. Each of these has a weight $x^{|V| - 2k(-1)^{|E| - k}}$, and for each $m = 0, 1, \ldots, k$ exactly $2^k \binom{k}{m}$ of them have the following cycle structure:

1. $2(k - m) + |E| - k + 2k + |V|$ cycles of length 2: two cycles in each of the $k - m$ match gadgets corresponding to edges in the matching of $H$, one cycle in each of the $|E| - k$ match gadgets corresponding to edges not in the match of $H$, one cycle in each of the $2k$ vertex cover gadgets corresponding to vertices incident to edges in the matching of $H$ and the additional $|V|$ 2-cycles,
2. $m$ cycles of length 4 (one cycle in each of the $m$ match gadgets corresponding to edges in the matching of $H$),

3. $4k$ cycles of length $2p+2$ (two cycles in each of the $2k$ vertex cover gadgets corresponding to vertices incident to edges in the matching of $H$),

4. $|V| - 2k$ cycles of length $2p+5$ (one cycle in each of the $|V| - 2k$ vertex cover gadgets corresponding to vertices not incident to edges in the matching of $H$),

5. and $2|V| - 4k$ cycles of length $p+1$ (two cycles in each of the $|V| - 2k$ vertex cover gadgets corresponding to vertices not incident to edges in the matching of $H$).

As a cross check, the number of vertices in such a cycle cover is

$$4m + 2(2(k-m) + |E| - k + 2k + |V|) + (2p+5)(|V| - 2k) + (p+1)(2|V| - 4k) + (2p+2)4k = (4p+7)|V| + 2|V| + 2|E|,$$

which is indeed the number of vertices in $\mathcal{G}$. Note that by construction one of the weights in the constructed directed graph $\mathcal{G}$ is $x$ and thus, we can view its adjacency matrix as a linear matrix polynomial in $x$. More concretely:

**Proposition 4.3.** Let $H = (V,E)$ be a planar graph on $\frac{n}{2}$ many vertices, let $A(\mathcal{G})$ denote the adjacency matrix of the directed graph $\mathcal{G}$ obtained from $H$ by Construction 2. Then, the immanent $\text{Imm}_\lambda(A(\mathcal{G}))$ is a polynomial in $x$ of degree $\lfloor|V|/2\rfloor$. More concretely, we have

$$\text{Imm}_\lambda(A(\mathcal{G})) = \sum_{k=0}^{\lfloor|V|/2\rfloor} M(H,k)x^{|V|-2k}(-1)^{|E|-k}2^k \sum_{m=0}^{k} \binom{k}{m} \chi_\lambda(\rho(|E|,|V|,k,m)),$$

where $M(H,k)$ denotes the number of $k$-matchings in $H$ and $\rho(|E|,|V|,k,m)$ is $(2p+5)^{|V|-2k}, (2p+2)^4k, (p+1)^2|V|-4k, 4^m, 2^{2(k-m)+|E|-k+2k+|V|})$.

With these preparations we are able to give the proof of the #P hardness for edge-weighted, planar, directed graphs.

**Proof of Theorem 1.5.** First we give the proof for edge-weighted graphs with possible $-1$ weights then we show how to eliminate the $-1$ weights.

We will use the gadget described in Construction 2 to show that a polynomial time algorithm to compute the immanent would also yield a polynomial time algorithm to calculate for all $k$ the number of $k$-matchings in a planar graph $H$. To see this, we observe first that according to Proposition 4.3 the immanent of the adjacency matrix of $A(\mathcal{G})$ is a polynomial in $x$ of degree $\lfloor|V|/2\rfloor$. In order to obtain the corresponding coefficients we can use interpolation: Assigning $\lfloor|V|/2\rfloor + 1$ different values to $x$ and computing the immanent of the resulting matrix we have access to all of the coefficients. Therefore,
if $Imm_{\lambda}(A(\bar{G}))$ is computable in a polynomial time we can also compute all of these coefficients in polynomial time.

In order to use Equation (2) to obtain the $k$-matchings in $H$ we have to make sure that

$$
\sum_{m=0}^{k} \binom{k}{m} \chi_{\lambda}(\rho(|E|, |V|, k, m) \tag{3}
$$

is not vanishing and can be computed in polynomial time. Indeed, we know from Theorem 2.3 that $\chi_{\lambda}$ is a $b(\lambda)$-spectrum function. Since $b(\lambda) < p + 1$ we have that

$$
\chi_{\lambda}(\rho(|E|, |V|, k, m)) = (-1)^{|V|} \chi_{\lambda'}(4^m, 2^{2(k-m)+|E|-k+2k+|V|})
$$

where $\lambda'$ obtained from $\lambda$ by deleting $((2p + 5)(|V| - 2k) + 4k(2p + 2) + (p + 1)(2|V| - 4k)$ squares from its first row. For such characters Curticapean proved that

$$
\sum_{k=0}^{m} \binom{k}{m} \chi_{\lambda'}(4^m, 2^{2(k-m)+|E|-k+2k+|V|})
$$

is not vanishing and can be computed in polynomial time [5], thus the value in equation (3) is not vanishing and also computable in polynomial time. Therefore, for each $k$, $M(H, k)$ could be computable in polynomial time if $Imm_{\lambda}(A(\bar{G}))$ could be computed in polynomial time. Indeed,

$$
M(H, k) = \frac{\alpha_{|V|-2k}}{(-1)^{|E|-k} 2^{k} \sum_{m=0}^{k} \binom{k}{m} \chi_{\lambda}(\rho(|E|, |V|, k, m))}
$$

where $\alpha_{|V|-2k}$ is the coefficient of $x^{2k}$ in $poly(x)$ as $Imm_{\lambda}(A(\bar{G}))$.

The $-1$ weights can be eliminated if for some appropriate sets of prime numbers $p = p_1, p_2, \ldots, p_s$, the $-1$ weight is replaced with $p - 1$, and then each $M(H, k)$ is computed modulo $p$. Using the Chinese Reminder Theorem, $M(H, k)$ can be computed modulo $\prod_{i=1}^{s} p_i$. Since $M(H, k)$ is definitely smaller than $2^{|E|} \leq 2^{\frac{3|V|}{2}}$, it provides an exact computation if $\prod_{i=1}^{s} p_i > 2^{\frac{3|V|}{2}}$. The appropriate prime numbers are those that are odd and does not divide any of the coefficients

$$
\sum_{m=0}^{k} \binom{k}{m} \chi_{\lambda}(\rho(|E|, |V|, k, m)).
$$

It is definitely possible to select such prime numbers from the first prime numbers whose product is larger than

$$
2 \times 2^{\frac{3|V|}{2}} \prod_{k=0}^{\frac{|V|}{2}} \left( \sum_{m=0}^{k} \binom{k}{m} \chi_{\lambda}(\rho(|E|, |V|, k, m)) \right).
$$

A very crude approximation

$$
\sum_{m=0}^{k} \binom{k}{m} \chi_{\lambda}(\rho(|E|, |V|, k, m)) \leq 2^k n^s \leq 2^k n \sqrt{s}
$$
suffices. To see this inequality, observe that in any border-strip tableaux, the \( \lambda_d \) part must be filled in with numbers varying between 1 and \( n \) and \( \binom{k}{m} \leq 2^k \). Then we are looking for a \( t \) such that the product of the prime numbers at most \( t \) be at least

\[
2 \times 2^{\lceil|V|/2\rceil} \prod_{k=0}^{|V|/2} 2^k n^{v/\sqrt{n}} \leq 2^{3n^r + n^r(n^r - 1)} n^{v/\sqrt{n}} \leq 2^{3/n + n + \log_2(n)n} \leq 2^{n^2}.
\]

We know that the product of the prime numbers smaller than \( t \) is at least \( 2^{t/2} \) [9]. Then \( t = 2n^2 \) suffices. It is possible to find all prime numbers below \( 2n^2 \) in polynomial time, also it is possible in polynomial time to select those that do not divide \( \sum_{m=0}^{k} \binom{k}{m} \chi_{\lambda}(p(|E|, |V|, k, m)) \) for any \( k \), compute the corresponding immanants and solve exactly \( M(H, k) \) for each \( k \) using the Chinese Reminder Theorem.

We would like to mention that extending Theorem 1.5 to 0-1 matrices has potential technical difficulties. A possible idea is to use Valiant’s approach to replace a directed edge \( e = (u, v) \) in a graph \( G = (V, E) \) with small natural number weight \( x \) by a gadget of size \( O(x) \) without edge weights [31]. In the so-obtained graph \( G' \), there is exactly one cycle cover covering all of the vertices except \( u \) and \( v \) (corresponding to cycle covers in \( G \) in which \( u \) and \( v \) are covered by edges from \( E \setminus \{e\} \), and there are exactly \( x \) ways to cover all vertices in the gadget including \( u \) and \( v \). Furthermore, there is no cycle cover in the gadget that covers only one of \( u \) and \( v \). However, the emerging cycle covers contain many cycles of length 2 and 1. Therefore, for each prime, \( p \), the immanants must be evaluated on permutations with cycle structures depending on \( p \). It is far not obvious that appropriate prime numbers \( p \) can be selected such that now the almost changing and not fixed

\[
\sum_{m=0}^{k} \binom{k}{m} \chi_{\lambda(p)}(p(p, k, m))
\]

is not divisible by \( p \) for any \( k \). Note that Valiant’s gadget changes the number of vertices, thus the corresponding shape depends on \( p \), and so the cycle structure of the relevant permutations on which the immanant must be evaluated.

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