Glueball masses and other physical properties of SU(N) gauge theories in D=3+1:
a review of lattice results for theorists

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Abstract

We summarise what lattice simulations have to say about the physical properties of continuum SU(N) gauge theories in 3+1 dimensions. The quantities covered are: the glueball mass spectrum, the confining string tension, the temperature at which the theory becomes deconfined, the topological susceptibility, the value of the scale $\Lambda_{\overline{MS}}$ that governs the rate at which the coupling runs and the $r_0$ parameter that characterises the static quark potential at intermediate distances.
1 Introduction

In recent years there has been a resurgence of interest in developing analytic techniques for calculating the physics of non-Abelian gauge theories. Examples include approaches using light cone quantisation [1] and the AdS calculations [2] based on the conjecture by Maldacena [3]. In practice all such calculations make approximations and so, in order to evaluate the accuracy of the approach being used, comparison with known results is very useful. In this context ‘known results’ means whatever has been learned from the computer simulation of these theories. The purpose of this review is to provide a readily accessible summary for such theorists of what has been learned from lattice calculations about the continuum properties of non-Abelian gauge theories in 3+1 dimensions.

The quantities I shall focus upon are the glueball mass spectrum, the confining string tension $\sigma$, the deconfining temperature $T_c$, the $\Lambda_{\overline{MS}}$ parameter that sets the scale in perturbative calculations, a parameter called $r_0$ that is a characteristic scale of the static quark potential at intermediate distances, and the topological susceptibility $\chi_t$.

There exists substantial information on these quantities for SU(2) and SU(3) gauge theories. For many of these quantities lattice calculations exist for a large enough range of lattice spacings that a controlled extrapolation to the continuum limit is possible. This continuum physics is summarised in Section 2. For some other quantities accurate calculations do exist but only for one or two values of the lattice spacing, $a$, and so a reliable continuum extrapolation is not possible. Where such calculations have been performed for values of $a$ that are very small, the lattice corrections to mass ratios should also be small. This applies in particular to some of the heavier states of the mass spectrum, and we list these in Section 3. The reader should be cautious in using the latter numbers: most of them ‘should’ be close to their corresponding continuum values but they might not be.

While SU(2) and SU(3) gauge theories are of obvious relevance, many analytic approaches find it simplest to calculate the $N \to \infty$ limit of SU($N$) gauge theories; and this limit is, in any case, of particular theoretical interest. So in the last section I will mention some very preliminary SU(4) calculations and discuss what all this tells us about the large-$N$ limit.

Although this review restricts itself to 3+1 dimensions, we note that from the point of view of most of these analytic approaches, non-Abelian gauge theories in 2+1 dimensions are equally interesting and challenging. Here there is a recent lattice calculation of the glueball mass spectrum and string tension, in units of the scale provided by the coupling $g^2$, which provides accurate extrapolations to SU($N = \infty$) [4]. There also exist calculations of $T_c$ for $D = 2 + 1$ SU(2), SU(3) and SU(4) gauge theories [5].

For the sake of clarity I have relegated to appropriate Appendices the details of what I have done, and what lattice calculations I have used, in order to obtain the final continuum numbers that I quote in Section 2. I have tried to perform the continuum extrapolations in a coherent way throughout and I apologise to those authors who may feel that I have not done their calculations full justice in the process. For example, calculations with improved actions on coarse lattices raise particular difficulties; and, whenever in doubt, I have chosen to err on the conservative side.
2 SU(2) and SU(3) in the continuum

The spectrum of SU(N) gauge theories is composed of states that are composed entirely of gluons and so it is usual to call the single particle states (including resonances) by the generic name ‘glueballs’. Their masses we will denote by $m_G$ or by $m_{J^PC}$ where $J^PC$ denotes the quantum numbers of the state. These theories appear to be confining at low temperature, with a linear potential at large distances between static sources in the fundamental representation. We label the corresponding string tension by $\sigma$. While this characterises the large-$r$ part of the potential, one can characterise its behaviour at intermediate distances by the distance $r_0$ at which the force, $F$, has a particular value. It has become customary to use the particular definition $r_0^2 F(r_0) = 1.65$ (which corresponds to a value that can be calculated with precision on the lattice and which can be estimated with some reliability from the observed spectrum of heavy quark systems). At sufficiently high temperatures the theory must become deconfined, and we label the temperature at which this happens by $T_c$. At short distances the theory becomes asymptotically free and the scale controlling the running of the coupling can be chosen to be $\Lambda_{\overline{MS}}$. SU(N) gauge fields in 3+1 dimensions possess a topological charge, $Q$, and a corresponding topological susceptibility $\chi_t = \langle Q^2 \rangle / V$ in a space-time volume $V$. There are other condensates, in particular the gluon condensate, but we choose to stop here.

In a lattice calculation the overall scale is set by the lattice spacing $a$ which is determined implicitly by the choice made for the inverse bare coupling $\beta = 2N/g^2(a)$. So, for example, a mass calculation will produce a number for the quantity $am_G(a)$, where the first $a$ tells us that the mass is in lattice units, and the other $a$ indicates that the mass will be afflicted by lattice discretisation errors. We can rid ourselves of the former by calculating the ratio of two such masses: $am_G(a)/am_G'(a) = m_G(a)/m_G'(a)$. For small $a$ one can expand the lattice action in powers of $a$ and hence show that the leading corrections are [6]:

$$m_G(a)/m_G'(a) = m_G(0)/m_G'(0) + c_1(a\mu)^2 + O(a^4)$$

where any convenient mass scale $\mu$, such as $m_G$ itself, may be used and where the coefficients $c_1 \ldots$ are power series in the (bare) coupling.

In the results I present in this paper I have usually extrapolated lattice masses to the continuum limit using the string tension

$$m_G(a)/\sqrt{\sigma(a)} = m_G(0)/\sqrt{\sigma(0)} + ca^2 \sigma + \ldots$$

since that quantity is known with some accuracy over a very wide range of lattice spacings. I could have used $r_0$ equally well and the reader can easily translate from one continuum ratio to another using the values I tabulate. ($r_0/a$ is more accurately calculable than $a\sqrt{\sigma}$ and so is likely to become the scale of choice in the future.)

I have attempted to include all the useful ‘modern’ lattice calculations of these physical quantities and have extrapolated them to the continuum limit using either a simple $O(a^2)$ correction, as in eqn(2), or using both $a^2$ and $a^4$ corrections. Details are in the Appendices.
Table 1: Glueball masses, in units of the string tension, in the continuum limit. Values in brackets have been obtained by extrapolating from only two lattice values and so should be treated with caution.

| state   | SU(2)   | SU(3)   |
|---------|---------|---------|
| 0^{++}  | 3.74±0.12 | 3.64±0.09 |
| 0^{+++} | 6.58±0.36 |         |
| 2^{++}  | 5.62±0.26 | 5.17±0.17 |
| 2^{+++} | 6.73±0.73 |         |
| 0^{--}  | 6.53±0.56 | 4.97±0.57 |
| 2^{--}  | 7.46±0.50 | 7.05±0.65 |
| 1^{++}  | 10.2±0.5  |         |
| 1^{+-}  |         | 6.32±0.31 |
| 1^{--}  | 1.78±0.24 | 1.34±0.18 |
| 1^{-+}  | 2.03±0.20 | 1.94±0.20 |
| 3^{++}  | 2.75±0.15 |         |
| 3^{+-}  | [3.03±0.31] |         |
| 3^{--}  | 2.91±0.47 |         |

Table 2: Glueball masses, in units of the lightest scalar glueball mass, in the continuum limit. Values in brackets have been obtained by extrapolating from only two lattice values and so should be treated with caution.

| state   | SU(2)   | SU(3)   |
|---------|---------|---------|
| 0^{+++} | 1.78±0.12 |         |
| 2^{++}  | 1.46±0.09 | 1.42±0.06 |
| 2^{+++} | 1.85±0.20 |         |
| 0^{--}  | 1.78±0.24 | 1.34±0.18 |
| 2^{--}  | 2.03±0.20 | 1.94±0.20 |
| 1^{++}  | 2.75±0.15 |         |
| 1^{+-}  |         | 1.73±0.09 |
| 1^{--}  | [3.03±0.31] |         |
| 3^{++}  | 2.46±0.23 |         |
| 3^{+-}  | [2.91±0.47] |         |
Table 3: The scale $r_0$, the deconfining temperature, $T_c$, the topological susceptibility, $\chi^{1/2}$, and the (zero-flavour) $\Lambda$ parameter in the $\overline{MS}$ scheme. All in the continuum and in units of the string tension.

|               | SU(2)           | SU(3)           |
|---------------|-----------------|-----------------|
| $r_0/\sqrt{\sigma}$ | $1.201\pm0.055$ | $1.195\pm0.010$ |
| $T_c/\sqrt{\sigma}$ | $0.694\pm0.018$ | $0.640\pm0.015$ |
| $\Lambda_{\overline{MS}}/\sqrt{\sigma}$ | $0.586\pm0.010$ | $0.454\pm0.012$ |

In Table 1 I list the continuum glueball masses (in units of the string tension) that I obtain, in this way, for both SU(2) and SU(3). Another way to present this ‘data’ is as a ratio of glueball masses. This may be convenient for people who don’t calculate the string tension. I present the masses in this way in Table 2. (For reasons explained in the Appendix these masses are not precisely what one would obtain by dividing the appropriate entries in Table 1.)

So much for what is known about the mass spectrum. In Table 3 I present the continuum values of the other promised quantities.

The error estimates in these Tables are supposed to be realistic. They include, albeit often indirectly, not only the statistical errors but also some of the important systematic errors. For example, in the mass calculations sufficiently large lattices are used for the finite-volume corrections to be very small. Using large lattices is slower and so leads to larger statistical errors than using small lattices. Similarly the process of extrapolating to the continuum limit with one or two correction terms magnifies the statistical errors on the component lattice calculations. In this sense it would be misleading to say that the errors in the Table are ‘only statistical’. The precautions taken during the calculation to control systematic errors typically enlarge the statistical errors so that one should think of the former as being included, albeit imperfectly, within the quoted errors. Perhaps the main caveat here concerns the identification of the value of the spin $J$. A given irreducible representation of the lattice rotational group will contain states with different values of $J$ in the continuum limit. One usually labels the lattice masses by the minimum such value of $J$. For calculations using operators smeared over physical length scales this can usually be justified for the lightest states with given quantum numbers (most clearly for $P = +$). Thus nearly all the spin assignments in the Tables should be correct. Nonetheless this still leaves open the possibility that, for example, what we have called the $0^{++ \ast}$ is in reality the $4^{++}$. In principle it is straightforward to deal with this ambiguity but, for now, the ambiguity remains. Apart from this caveat, it should be the case that the above numbers will withstand the test of time; at least within, say, 2 of the quoted standard deviations.
| state | SU(2) ; $a \simeq 0.03\, fm$ | SU(3) ; $a \simeq 0.05\, fm$ |
|-------|-------------------------------|-------------------------------|
| $0^{++}$ | $3.60 \pm 0.20$ | $3.52 \pm 0.12$ |
| $2^{++}$ | $5.55 \pm 0.27$ | $5.16 \pm 0.21$ |
| $1^{++}$ | $9.29 \pm 0.92$ | $9.0 \pm 0.7$ |
| $3^{++}$ | $9.75 \pm 0.85$ | $8.9 \pm 1.1$ |
| $0^{-+}$ | $7.04 \pm 0.57$ | $5.3 \pm 0.6$ |
| $2^{-+}$ | $7.70 \pm 0.50$ | $6.9 \pm 0.4$ |
| $1^{--}$ | $10.3 \pm 0.9$ | $9.5 \pm 1.7$ |
| $3^{--}$ | $7.7 \pm 1.0$ | |
| $2^{+-}$ | | $6.6 \pm 0.6$ |
| $1^{+-}$ | | $8.9 \pm 0.8$ |
| $2^{--}$ | | $9.9 \pm 1.1$ |

Table 4: Glueball masses in units of the string tension for fixed but small values of the lattice spacing, as described in the text. No SU(3) estimates are given for the $0^{-+}, 3^{-+}, 1^{--}, 3^{--}$ glueballs which appear to be too heavy to give a useful signal.

3 SU(2) and SU(3) at finite lattice spacing

The most comprehensive SU(3) lattice spectrum calculation is to be found in [7]. (This title will be usurped by [8] once the latter ceases to be preliminary.) The calculations were on a $32^3 64$ lattice at $\beta = 6.4$, which corresponds to a lattice spacing $a \simeq 0.1216/\sqrt{\sigma} \sim 0.05\, fm$. Similarly, in SU(2) there is a calculation [9] on a $48^3 56$ lattice at $\beta = 2.85$ which corresponds to a lattice spacing $a \simeq 0.0636/\sqrt{\sigma} \sim 0.03\, fm$. The values in physical units come from using $\sqrt{\sigma} \sim 450\, MeV$ [10], and are introduced solely to make the point that these are very small lattice spacings; they should not be taken more seriously than that.

We present the mass ratios from these papers in Table 4. Since the lattice spacings are very small it is reasonable to expect that the $O(a^2)$ lattice spacing corrections to the continuum values will also be small. However the reader should be aware that this represents an extra unquantified systematic error. It is clear that in those cases where the states also appear in Table 1, the reader should use the latter values.

The reason that our knowledge of the glueball mass spectrum is relatively incomplete is that since the late 1980’s there have been almost no dedicated glueball mass calculations; the few that have appeared have usually come as a side-product of other calculations. Fortunately this is now changing and we can hope to have accurate values for all the above states (and more) quite soon. For a fore-taste see [8], although these are still ‘preliminary’ and so have not been included in this review.
Table 5: Various continuum mass ratios after extrapolation to $N = \infty$ from $N = 2$ and $N = 3$; assuming the validity of eqn(3) all the way down to $N = 2$. Errors correspond to $\chi^2 = 1$.

| $SU(\infty)$ | $m_{0^{++}}/\sqrt{\sigma}$ | $m_{2^{++}}/\sqrt{\sigma}$ | $T_c/\sqrt{\sigma}$ | $\chi_t^+/\sqrt{\sigma}$ | $r_0\sqrt{\sigma}$ |
|---------------|-----------------------------|-----------------------------|----------------------|---------------------------|------------------|
|               | $3.56 \pm 0.18$             | $4.81 \pm 0.35$             | $0.597 \pm 0.030$    | $0.428 \pm 0.022$         | $1.190 \pm 0.043$ |

4. $SU(4)$ and large $N$

We see from Tables 1 - 4 that the physical properties of $SU(2)$ and $SU(3)$ gauge theories are very similar (apart from the absence of any $C = -$ states in the former case). It is tempting to see this as evidence that we are already close to the $N = \infty$ limit even with $N = 2, 3$. Since the leading corrections are expected to be $O(1/N^2)$ this suggestion is not absurd; and indeed it is exactly what is found to occur for $SU(N)$ theories in 2+1 dimensions [4]. If we assume that this is the case then we can extrapolate mass ratios to the $N = \infty$ limit using

$$\frac{m}{\sqrt{\sigma}}_N = \frac{m}{\sqrt{\sigma}}_\infty + \frac{c}{N^2} \quad (3)$$

and then we obtain the values displayed in Table 5.

Of course, since the fits have no degrees of freedom, we have no information on the reliability of using eqn(3) all the way down to $SU(2)$, and the numbers in Table 5 must remain speculative. Clearly we need calculations for at least one further value of $N$; in that case we could test the goodness of the fits based on eqn(3). Unfortunately, while some exploratory $SU(4)$ calculations do exist [11], as summarised in Table 6, they are not accurate enough to permit a useful continuum extrapolation. However all is not lost: the large-$N$ arguments apply not only to the continuum limit but also to the theory at finite values of the lattice spacing. This enables us to squeeze some interesting information from Table 6 as we shall now see.

The question we want to ask is whether there is a smooth large-$N$ limit and whether this is obtained by varying the coupling as $g^2 \propto 1/N$. This is what one expects from the usual analysis [12] of Feynman diagrams (and what one finds to occur in 2+1 dimensions [4]).

Since the coupling runs the latter part of the question needs some elaboration. The usual $\beta$-function will ensure that $g^2 \propto 1/N$ for large $N$ provided that $\Lambda_{\overline{MS}}$ becomes independent of $N$ when expressed in units of a physical length scale in the $SU(N)$ theory, such as $\sqrt{\sigma}$. Equivalently suppose we define the running coupling, $g_\lambda^2(l)$, on a length scale $l \equiv c_l \xi_\sigma$ where $\xi_\sigma \equiv 1/\sqrt{\sigma}$ and $c_l$ is small enough for the coupling to be small. Then what we are looking for
Table 6: SU(4) masses and string tensions calculated in [11], on 10^4, 12^4 and 16^4 lattices at β = 10.7, 10.9 and 11.1 respectively.

|                | β = 10.7          | β = 10.9          | β = 11.1          |
|----------------|-------------------|-------------------|-------------------|
| a√σ           | 0.296 ± 0.015     | 0.228 ± 0.007     | 0.197 ± 0.008     |
| am_{0++}      | 0.98 ± 0.17       | 0.75 ± 0.07       | 0.77 ± 0.06       |
| am_{0++]      | 1.54 ± 0.44       | 1.39 ± 0.16       | 1.03 ± 0.14       |
| am_{2++}      | 1.28 ± 0.27       | 1.21 ± 0.11       | 1.04 ± 0.12       |

is

$$g^2_N(c_l\xi) = \frac{N_0}{N} g^2_{N_0}(c_l\xi) + O\left(\frac{1}{N^2}\right)$$

for all N larger than some N_0. At the same time we want physical mass ratios, bar some obvious exceptions, to have finite non-zero limits as N → ∞.

We can test eqn(4) using the SU(4) string tensions in Table 6. We note that the bare coupling can be used to define a running coupling on the scale of the cut-off a: $g^2(a) = 2N/\beta$. However it is well-known that this is a ‘bad’ way to define a coupling and that one can do much better with a mean-field [13] or tadpole-improved [14] coupling defined by

$$g^2_I(a) = \frac{2N}{\beta \times \langle \text{Tr}U_p \rangle}$$

where $\langle \text{Tr}U_p \rangle$ is the average plaquette. We can extract three SU(4) couplings in this way corresponding to the three values of β in Table 6. The scales on which these couplings are defined are given by the corresponding values of $a\sqrt{\sigma}$. We now go to the SU(2) and SU(3) lattice string tension values tabulated in the Appendices and find (by interpolation) the mean-field improved lattice couplings corresponding to these three values of $a\sqrt{\sigma}$. We wish to test the hypothesis that $Ng^2$ is becoming independent of N at large N, so we list these couplings, multiplied by N, in Table 7. We observe that these values are indeed consistent with being independent of N. This provides some evidence that eqn(4) is indeed valid all the way down to SU(2). Presumably the $O(1/N)$ corrections would become visible in more accurate calculations.

Of course this analysis only makes sense if the theory has a smooth limit at large N. We can investigate this at each value of a separately. In Table 8 we plot the values of $m_{0++}/\sqrt{\sigma}$ for N = 2, 3, 4 at each value of $a\sqrt{\sigma}$. In Table 9 we do the same for $m_{2++}/\sqrt{\sigma}$. We observe that these ratios are indeed consistent with having a smooth, confining large-N limit although the errors are too large for this to be a taxing test.

We conclude that there is significant evidence that D=3+1 SU(N) gauge theories are ‘close’ to SU(∞) all the way down to SU(2). However it is also clear that very much better SU(4) calculations are needed (and indeed are in progress).


\[
\begin{array}{|c|c|c|c|}
\hline
N g_f^2(l) & c_l = a \sqrt{\sigma} & SU(2) & SU(3) & SU(4) \\
\hline
0.197 \pm 0.008 & 4.966 \pm 0.050 & 4.938 \pm 0.045 & 4.943 \\
0.228 \pm 0.007 & 5.116 \pm 0.036 & 5.097 \pm 0.036 & 5.149 \\
0.296 \pm 0.015 & 5.434 \pm 0.076 & 5.397 \pm 0.063 & 5.396 \\
\hline
\end{array}
\]

Table 7: Bare couplings multiplied by \( N \) on three different length scales \( l = c_l \xi_\sigma \), each scale being the same for the SU(2), SU(3) and SU(4) gauge theories when expressed in physical units (here the string tension).

\[
\begin{array}{|c|c|c|c|}
\hline
m_{0^{++}} / \sqrt{\sigma} & a \sqrt{\sigma} & SU(2) & SU(3) & SU(4) \\
\hline
0.197 \pm 0.008 & 3.67 \pm 0.10 & 3.33 \pm 0.05 & 3.91 \pm 0.33 \\
0.228 \pm 0.007 & 3.62 \pm 0.07 & 3.22 \pm 0.04 & 3.28 \pm 0.30 \\
0.296 \pm 0.015 & 3.49 \pm 0.03 & 2.94 \pm 0.08 & 3.31 \pm 0.57 \\
\hline
\end{array}
\]

Table 8: The scalar glueball as a function of \( N \), for lattice spacings that are independent of \( N \), in units of the string tension.

\[
\begin{array}{|c|c|c|c|}
\hline
m_{2^{++}} / \sqrt{\sigma} & a \sqrt{\sigma} & SU(2) & SU(3) & SU(4) \\
\hline
0.197 \pm 0.008 & 5.59 \pm 0.09 & 5.05 \pm 0.03 & 5.23 \pm 0.61 \\
0.228 \pm 0.007 & 5.58 \pm 0.03 & 5.02 \pm 0.04 & 5.31 \pm 0.48 \\
0.296 \pm 0.015 & 5.54 \pm 0.14 & 4.92 \pm 0.21 & 4.30 \pm 0.90 \\
\hline
\end{array}
\]

Table 9: The tensor glueball as a function of \( N \), for lattice spacings that are independent of \( N \), in units of the string tension.
5 Comments and Prospects

Although it might seem, from Table 1, that we already have a great deal of information about the mass spectra of SU(2) and SU(3) gauge theories, this appearance is deceptive. As soon as one tries to match these spectra with predictions of theoretical calculations (for example [1, 2]) or models (for example [13]), one soon realises that it is important to have predictions for several excitations of the same quantum numbers as well as for spins which are different in the continuum but which appear in the same representation of the lattice symmetry group. For example, \( J = 0 \) and \( J = 4 \). Thus one needs to develop techniques [10] that will resolve such different continuum spins, and one needs to be able to calculate accurately heavier excited states. The latter will clearly entail developing techniques for dealing with resonances and continuum states.

In order to take the continuum limit one needs a substantial range of lattice spacings, \( a \), for which accurate calculations are available; extending from relatively coarse \( a \) to as fine \( a \) as one can go. At the coarser end, the masses of heavier excited states are so large in lattice units, \( a m_G \), as to be virtually incalculable. Here the use of actions with different timelike and spacelike lattice spacings will be crucial: \( a_s \) can be large while \( a_t m_G \) remains calculably small [7, 8]. As we have seen in Appendix B, given the statistical accuracy of such calculations, it is important that the renormalisation of the asymmetry parameter be determined explicitly. Since part of the calculation must be performed for coarse \( a \), there is an argument for using improved actions which have smaller lattice discretisation errors. If one does so then one might be tempted to perform the whole calculation with coarse lattice spacings. However we have seen that even if such actions reduce the leading \( O(a^2) \) errors, it is dangerous to assume that these are so small as to be negligible compared to the sub-leading \( O(a^4) \) errors. That is to say, both terms need to be included in the continuum extrapolation. In that case it is useful to have calculations at smaller values of \( a \) to accurately fix the coefficients of such fits.

In addition to mass spectra, we have reviewed calculations of several other interesting physical quantities. Here too there are questions that need answering. Most pressing is a convincing demonstration that the effective string theory that governs the long-distance physics of the confining ‘flux tube’ is indeed in the universality class of the simple bosonic string (leading to the usual Lüscher correction in eqn(3) and the corresponding correction for the static potential). Whether this is in fact so depends on the effective degrees of freedom (central charge) of the effective string theory. There is numerical evidence for the usual assumption but it is not yet accurate enough to be really convincing. Also needed is a resolution of the \( \sim 4\% \) discrepancy between the two groups of calculations of the SU(3) deconfining temperature, as discussed in Appendix B.

A major gap at present is the lack of an accurate SU(4) calculation. While we have managed to squeeze a surprising amount out of the crude lattice calculations [11] that are currently available, a proper quantitative discussion of the large-\( N \) limit in 3+1 dimensions must await a much better set of calculations.

**Acknowledgements:** I have benefitted from discussions with many of the authors I quote. I am grateful to Rainer Sommer for providing me with the unpublished SU(2) \( r_0 \) values.
Appendix A : SU(2)

In this Appendix I describe in some detail how the SU(2) continuum numbers in the main body of the text were arrived at.

(a) the mass spectrum and string tension

The most extensive calculations are for the string tension and the lightest scalar and tensor glueballs. The values that I have used for my continuum extrapolations are listed in Table 10. There are also some calculations for other glueball quantum numbers and these are listed in Table 11. All these calculations use the standard Wilson plaquette action \[18\]. The sources for these calculations are as follows: \(\beta = 2.0\) and \(\beta = 2.1\) are from \[19\]; \(\beta = 2.2\) to \(\beta = 2.6\) are from \[20\]; except for the 32\(^4\) lattice at \(\beta = 2.5\) which comes from \[21\] as does the \(\beta = 2.7\) calculation; \(\beta = 2.85\) is from \[22\]. There are some other string tension calculations that I have not used. Notably \[23\] where one finds \(a\sqrt{\sigma} = 0.1871(10), 0.1871(32)\) on 16\(^4\), 32\(^4\) lattices at \(\beta = 2.5\); \(a\sqrt{\sigma} = 0.1207(3)\) on a 48\(^3\)64 lattice at \(\beta = 2.635\); and \(a\sqrt{\sigma} = 0.0911(4)\) on a 32\(^4\) lattice at \(\beta = 2.74\). (This last lattice is probably too small, in physical units, to be entirely trustworthy.) These values are consistent with the ones we list.

Some technical asides. The values listed are effective masses from \(t = a\) to \(t = 2a\) for \(\beta \leq 2.4\) for the string tension, for \(\beta \leq 2.5\) for the \(J = 0, 2\) glueballs, and for all \(\beta\) for the \(J = 1, 3\) glueballs. For other (higher) \(\beta\) I list effective masses from \(t = 2a\) to \(t = 3a\). When I need a single value of the string tension at \(\beta = 2.4\) or at \(\beta = 2.5\), I use \(a\sqrt{\sigma} = 0.2673(15)\) and \(a\sqrt{\sigma} = 0.186(3)\) respectively. The string tensions are all obtained from the masses of torelons. These are loops of fundamental flux closed through the periodic spatial boundary and hence neither contractible nor, in a confining phase, breakable. For a loop of length \(L\) in lattice units, one obtains \(\sigma\) from the mass \(m_l\) of the loop using

\[
am_l = a^2\sigma L - \frac{\pi}{3L}.
\]

The (leading) correction in eqn(6) is a universal string ‘Casimir energy’, and the coefficient used in eqn(6) assumes that the effective confining string belongs to the simplest bosonic string universality class \[24\]. There is some numerical evidence for this apparently reasonable assumption but it is not so precise as to be convincing. This correction typically contributes \(\sim 4\%\) to the values in Table 10.

To obtain the continuum limits listed in Table 11 I have taken the lattice values in the above Tables and used eqn(2). If the fit is bad then the lowest-\(\beta\) values are dropped till a good fit is obtained. I use conservative 25\% confidence level (CL) limits for the errors; less where the best fit has \(CL \leq 70\%\). In the case of the \(0^+\) and of the \(2^+\) one can fit all the way down to quite coarse lattice spacings (\(\beta = 2.0\) and \(\beta = 2.2\) respectively) and so there I also fit with an extra \(O(a^4)\) correction to check that the resulting continuum extrapolations are consistent with the values I quote. I follow a similar procedure to obtain the mass ratios in Table 2.

In addition to the above calculations, which use the Wilson plaquette action, there are a number of calculations using other actions. These provide a useful universality check and I list in Table 12 my extrapolations to the continuum limit of these lattice calculations. The first
SU(2)

| β  | lattice | $a\sqrt{\sigma}$ | $am_{0+}$ | $am_{2+}$ | $am_{0-}$ | $am_{2-}$ |
|----|---------|------------------|-----------|-----------|-----------|-----------|
| 2.0 | $6^312$ | 0.69(5)          | 1.76(16)  |           |           |           |
| 2.1 | $6^312$ | 0.608(16)        | 1.74(13)  |           |           |           |
| 2.2 | $8^310$ | 0.467(10)        | 1.44(8)   | 2.76(45)  |           |           |
| 2.3 | $10^4$  | 0.3687(22)       | 1.216(23) | 1.94(8)   |           |           |
| 2.4 | $14^4$  | 0.2686(21)       | 0.959(20) | 1.487(33) |           |           |
| 2.4 | $16^4$  | 0.2660(21)       | 0.936(23) | 1.520(30) |           |           |
| 2.5 | $16^4$  | 0.1834(26)       | 0.660(22) | 1.028(27) |           |           |
| 2.5 | $20^4$  | 0.1881(28)       | 0.723(23) | 1.051(26) | 1.241(63) | 1.34(13)  |
| 2.5 | $32^4$  | 0.187(12)        | 0.70(4)   | 0.99(4)   | 1.37(14)  | 1.42(11)  |
| 2.6 | $20^4$  | 0.1326(30)       | 0.51(3)   | 0.73(4)   |           |           |
| 2.7 | $32^4$  | 0.1005(20)       | 0.39(3)   | 0.55(3)   | 0.63(6)   | 0.72(4)   |
| 2.85| $48^356$| 0.0636(12)       | 0.229(12) | 0.353(16) | 0.430(35) | 0.49(3)   |

Table 10: Various SU(2) lattice calculations of the string tension and lightest $J = 0$ and $J = 2$ masses. Errors in brackets.

SU(2)

| β  | lattice | $am_{3+}$ | $am_{3-}$ | $am_{1+}$ | $am_{1-}$ |
|----|---------|-----------|-----------|-----------|-----------|
| 2.5 | $32^4$  | 1.88(43)  |           | 1.70(20)  |           |
| 2.7 | $32^4$  | 0.90(5)   | 1.32(11)  | 1.02(3)   | 1.11(4)   |
| 2.85| $48^356$| 0.588(36) | 0.71(5)   | 0.626(27) | 0.677(21) |

Table 11: Various SU(2) lattice calculations of the lightest $J = 1$ and $J = 3$ masses. Errors in brackets.

three calculations are from [25]. All three use an anisotropic lattice action with the ratio of timelike and spacelike lattice spacings chosen to be small: $\xi \equiv a_t/a_s \ll 1$. This is a trick [17] that enables one to work with coarse $a_s$ and yet have masses, $a_t m_G$, that are not so heavy as to be incalculable. The first two actions are variants of tadpole-improvement [14] while the third is an (anisotropic) Wilson action. The calculations take advantage of the improvement by being performed on quite coarse lattices. For this reason we have extrapolated these results to the continuum with a more complicated $c_1 a^2 \sigma + c_2 (a^2 \sigma)^2$ correction term. A potential problem with these anisotropic lattice calculations is that they have been performed with values of $\xi$ that vary strongly with $\beta$. This makes it tricky to motivate a simple uniform continuum extrapolation. The Manton [26] and Symanzik tree-level improved action [6] calculations are from [20]. These calculations suffer from having been performed at only two values of $\beta$, but these are large enough that a simple $O(a^2)$ correction should suffice. The errors on the extrapolations correspond to $\chi^2 = 1$ and are not really reliable. Despite these caveats, it is
clear that we have significant confirmation of universality from all these results.

\[
\text{SU}(2) - \text{improved actions} \\
| \text{action} | m_0^+ / \sqrt{\sigma} | m_2^+ / \sqrt{\sigma} | m_0^- / \sqrt{\sigma} | m_2^- / \sqrt{\sigma} |
|----------------|----------------|----------------|----------------|----------------|
| tadpole1(a_s \neq a_t) | 3.97 \pm 0.31 | 5.0 \pm 1.1 | | |
| tadpole2(a_s \neq a_t) | 4.4 \pm 0.5 | | | |
| Wilson(a_s \neq a_t) | 3.76 \pm 0.22 | 5.91 \pm 0.44 | | |
| Symanzik(a_s = a_t) | 3.93 \pm 0.21 | 5.67 \pm 0.31 | 6.77 \pm 0.53 | 6.78 \pm 0.80 |
| Manton(a_s = a_t) | 4.01 \pm 0.28 | 5.72 \pm 0.25 | 6.71 \pm 0.56 | 7.49 \pm 0.60 |

Table 12: SU(2) mass spectra, in the continuum limit, from calculations using a variety of 'improved' lattice actions (see text).

(b) the \( r_0 \) parameter

The SU(2) \( r_0 \) parameter has been calculated for a range of lattice spacings in [27]. In Table 13 I present the raw numbers [28] and the corresponding string tensions, which are taken from Table 10 either directly or by interpolation. One can extrapolate to the continuum limit using either an \( a^2\sigma \) or an \((a/r_0)^2\) correction. These give essentially identical best fits: for example

\[
\frac{r_0}{\sqrt{\sigma}} = 1.201(55) - 0.56(2.48)a^2\sigma. \tag{7}
\]

with 80% confidence level.

| \( \beta \) | \( r_0/a \) | \( a\sqrt{\sigma} \) |
|-------|-------|-------|
| 2.50  | 6.39 \pm 0.09 | 0.186 \pm 0.003 |
| 2.55  | 7.55 \pm 0.11 | 0.157 \pm 0.003 |
| 2.60  | 8.82 \pm 0.08 | 0.1326 \pm 0.0030 |
| 2.70  | 12.05 \pm 0.22 | 0.1005 \pm 0.0020 |
| 2.85  | 19.08 \pm 0.88 | 0.0636 \pm 0.0012 |

Table 13: Values of the \( r_0 \) parameter and string tension.

(c) the deconfining temperature \( T_c \)

A periodic Euclidean lattice of size \( N_s^3 N_r \) with \( N_s \gg N_r \) may be used to calculate the static properties of the gauge theory at a temperature \( T = 1/aN_r \). In Table 14 I list the critical coupling \( \beta_c \) corresponding to the SU(2) deconfining temperature for various \( N_r \) (see
and references therein) together with my interpolated values of the string tension. (These values have been extrapolated to $N_s = \infty$ using a finite-size scaling analysis.) Thus

$$T_c = \frac{1}{N_s a(\beta_c)}.$$  

(8)

and we can extrapolate to the continuum limit with

$$\frac{T_c}{\sqrt{\sigma}} = 0.694(18) - 0.28(33)(aT_c)^2$$  

(9)

or using an $a^2\sigma$ correction. The best fit has an excellent 90% confidence level.

| $N_s$ | $\beta_c$       | $a\sqrt{\sigma}$ |
|------|-----------------|------------------|
| 2    | 1.8800 ± 0.0030 | 0.4950 ± 0.0110  |
| 3    | 2.1768 ± 0.0030 | 0.3704 ± 0.0023  |
| 4    | 2.2986 ± 0.0006 | 0.2920 ± 0.0045  |
| 5    | 2.3726 ± 0.0045 | 0.2435 ± 0.0032  |
| 6    | 2.4265 ± 0.0030 | 0.1787 ± 0.0040  |
| 8    | 2.5115 ± 0.0040 | 0.0891 ± 0.0034  |
| 16   | 2.7395 ± 0.0100 |                  |

Table 14: Values of the deconfining critical coupling and string tension.

(d) the topological susceptibility $\chi_t$

If $Q$ is the topological charge of a gauge field then the susceptibility is defined as $\chi_t = \langle Q^2 \rangle / V$ where $V$ is the space-time volume. So on an $L^4$ lattice, $a^4 \chi_t = \langle Q^2 \rangle / L^4$. How to calculate the topological charge $Q$ of a lattice gauge field is a problem that is now well-understood \[10\]. Here we will summarise several calculations and use them to extract a continuum value for $\chi_t^{1/4} / \sqrt{\sigma}$. Our description of the methods used is inadequately brief and the reader needs to consult the literature for answers to a variety of natural questions.

The oldest reliable way to calculate the topological charge $Q$ of a lattice gauge field is to locally smoothen the gauge field, for example by minimising its action density, and then to calculate a lattice topological charge density $Q_L(x) \simeq a^4 Q(x)$ at every site, and finally to sum that up over all lattice sites. This approach goes by the name of ‘cooling’ \[30\] \[10\]. I will call the cooled total charge $Q_c$. The fields will contain some charges that are narrow and these will be particularly influenced by the presence of the lattice cut-off. So it is useful to define a charge $Q_{c'}$ from which such narrow charges have been excluded. The corresponding two susceptibilities, $\chi_c$ and $\chi_{c'}$, will differ at finite-$a$ but should converge in the continuum limit. (Note that, for the sake of clarity, I drop the subscript $t$ on $\chi$.) We shall use a cut-off
on the instanton size that corresponds to all charges with a peak height \(Q_L(x_{peak}) \geq 1/16\pi^2\) being excluded. The calculations I shall use are from \([31, 32, 19]\). In these calculations the number of cooling sweeps is typically 15 or 20. Since the topological charge varies negligibly with cooling in this range, we can safely average such slightly different calculations. A more recent ‘improved’ cooling method, whose charges we label by \(Q_{ic}\), is to be found in \([33]\). An approach using a renormalisation group inspired form of cooling has been developed in \([34]\) and the corresponding charges we label \(Q_{RG}\). (For the latter we shall use the values after ‘0 smears’.) There also exist methods that do not smoothen the gauge fields. In particular one can, instead, form smooth smeared operators and apply them directly to the original gauge fields, correcting explicitly for the additive and multiplicative renormalisations, as is done in \([35]\). We shall refer to this susceptibility as \(\chi_{sm}\). (We use the values after 2 ‘smears’.) Finally one can use a geometric form of the total topological charge \([32]\). To avoid problems with dislocations this is calculated on blocked lattice fields and averaged over all 16 blockings. This charge we call \(Q_G\). (As one takes \(a \to 0\) the number of blockings will, in principle, need to be increased.) All these methods will give different results at a fixed value of \(a\), because they deal differently with instantons whose sizes are near the cut-off. However they should converge to the same continuum value.

In Table 15 I list the 6 different susceptibilities. It is apparent that the variation between the different calculations decreases with increasing \(\beta\). In Table 16 I give the continuum extrapolation of \(\chi_i^{1/4}/\sqrt{\sigma}\) for each of these six calculations. (I have used an \(a^2\sigma\) lattice correction.) They are all pretty much consistent with each other and the final value I infer, and quote in Table 3, is an average value \(\chi_i^{1/4}/\sqrt{\sigma} = 0.486(10)\).

| \(\beta\) | \(a\chi_i^{1/4}\) | \(a\chi_{c}'^{1/4}\) | \(a\chi_G^{1/4}\) | \(a\chi_{ic}^{1/4}\) | \(a\chi_{RG}^{1/4}\) | \(a\chi_{sm}^{1/4}\) |
|---|---|---|---|---|---|---|
| 2.20 | 0.1481(53) | | | | | |
| 2.30 | 0.1420(21) | 0.1024(25) | 0.1211(30) | | | |
| 2.40 | 0.1163(13) | 0.0937(27) | 0.1026(25) | 0.1216(18) | 0.1299(21) | |
| 2.44 | | | | | | |
| 2.50 | 0.0844(11) | 0.0760(12) | 0.0800(18) | 0.0913(30) | 0.0957(21) | 0.1027(19) |
| 2.5115 | | | | | | |
| 2.57 | 0.0627(13) | 0.0585(14) | 0.0610(16) | 0.0605(46) | 0.0674(15) | |
| 2.60 | | | | | | |

Table 15: The topological susceptibility, using various methods as defined in the text.

**Appendix B : SU(3)**

In this Appendix I will describe the SU(3) lattice calculations which I use to obtain the continuum numbers in the main body of this paper.
\[
\begin{array}{cccccc}
\text{SU(2)} & & & & & \\
n = e & n = e' & n = G & n = Ic & n = RG & n = sm \\
0.486(12) & 0.468(15) & 0.480(18) & 0.501(45) & 0.528(21) & 0.480(23)
\end{array}
\]

Table 16: The continuum topological susceptibility, from the values in Table 15, in units of the string tension.

(a) the mass spectrum and the string tension

In Table 17 I present the lattice calculations of the string tension that I use. Where there is more than one reference listed, I have taken an average of the different values (checking, of course, that they are consistent with each other). For later use, values of the string tension have also been estimated by interpolation for a variety of intermediate \( \beta \) values at which there are no direct calculations.

\[
\begin{array}{cccc}
\text{SU(3)} & & & \\
\beta & a\sqrt{\sigma} & \text{sources} \\
5.50 & 0.5830 \pm 0.0130 & 36 \\
5.54 & 0.5727 \pm 0.0052 & 37 \\
5.60 & 0.5064 \pm 0.0028 & 37 \\
5.70 & 0.3879 \pm 0.0039 & 37 \\
5.80 & 0.3176 \pm 0.0016 & \text{int} \\
5.83 & 0.2991 \pm 0.0009 & \text{int} \\
5.85 & 0.2874 \pm 0.0007 & 37 \\
5.90 & 0.2613 \pm 0.0028 & 38 \\
5.93 & 0.2485 \pm 0.0012 & \text{int} \\
5.95 & 0.2391 \pm 0.0014 & \text{int} \\
6.00 & 0.2188 \pm 0.0008 & 37, 38, 39, 41, 44 \\
6.07 & 0.1964 \pm 0.0010 & \text{int} \\
6.10 & 0.1876 \pm 0.0012 & \text{int} \\
6.17 & 0.1684 \pm 0.0012 & \text{int} \\
6.20 & 0.1608 \pm 0.0010 & 38, 39, 40, 42, 44 \\
6.30 & 0.1398 \pm 0.0012 & \text{int} \\
6.40 & 0.1216 \pm 0.0011 & 4, 41, 44 \\
6.50 & 0.1068 \pm 0.0009 & 43 \\
6.57 & 0.0975 \pm 0.0012 & \text{int} \\
\end{array}
\]

Table 17: The string tension with sources (‘int’ means obtained by interpolation).

The calculations of the SU(3) mass spectrum fall into two groups which are sufficiently
different that they need separate treatment. The first and older category, involves calculations that use the standard plaquette action and are performed down to small values of the lattice spacing $a$. These allow accurate continuum extrapolations for the lightest glueballs. For the heavier glueballs they are much poorer. This is partly because many of the earlier calculations did not attempt to calculate them and, as a result, there is often too short a lever arm for a controlled continuum extrapolation. In addition heavy glueballs have larger values of $am_G$ and so the correlation function will often drop into the statistical noise before an effective mass plateau is really clear – except at the very smallest values of $a$, where the lattices have to be very large and hence computationally expensive.

The second category of calculations is recent and uses improved lattice actions so that useful calculations can be performed for coarse lattice spacings. To avoid large values of $am_G$ and the consequent danger of having no useful signal, they use an anisotropic action where the space and time lattice spacings differ and satisfy $a_t \ll a_s$. In that case one can have a small and easily measured value of $a_t m_G$, even when the glueball is heavy, while at the same time $a_s$ is large so that the spatial lattice can be small. I will now discuss these two kinds of calculations with $a_s = a_t$ and $a_s \neq a_t$, in turn.

• $a_s = a_t$ calculations

The lattice mass values that I use are listed in Tables 18, 19. This is an (almost) complete collection of ‘modern’ mass calculations where systematic errors should be negligible. The calculations extend to small values of $a$, as we can see by substituting $\sqrt{\sigma} \sim 0.5$ fm in Table 17. Thus one expects that over at least part of the range of $a$ in which calculations exist, the leading $O(a^2)$ corrections should dominate and we can perform extrapolations using

$$\frac{m_G(a)}{\sqrt{\sigma(a)}} = \frac{m_G(0)}{\sqrt{\sigma(0)}} + ca^2 \sigma. \quad (10)$$

We find that this functional form fits the 0$^{++}$ lattice values for $\beta \geq 5.7$ and the 2$^{++}$ lattice values for $\beta \geq 5.9$. For the 0$^{-+}$ and 2$^{-+}$ we include all the lattice values in the fits, but for the 1$^{-+}$ we find that we have to drop the $\beta = 5.9$ value. For the 0$^{+++}$ we have no degrees of freedom so we have no feedback about the validity of the fit in that case. The resulting continuum values are shown, a little later, in Table 22.

• $a_s \neq a_t$ calculations

This type of calculation is very recent and so the important questions have not all been resolved in the way that they have for the older and now well-established $a_s = a_t$ calculations. On the other hand these calculations certainly embody features that will be part of future mass calculations, so it is worth discussing them in some detail.

At the moment there is only one complete published glueball spectrum calculation of this kind [47]. To be precise, this paper contains two separate calculations, one with a tree-level anisotropy of $\xi = (a_s/a_t)_{\text{tree}} = 3$ and a second with $\xi = (a_s/a_t)_{\text{tree}} = 5$. An improved action is used which should reduce the $O(a^2_s)$ errors down to $O(a_s^2a^2_t)$. The $O(a^4_t)$ errors should be small because $a_t$ is (very) small, and so are ignored.

The lattice masses we use here are precisely the ones that are highlighted in the tables of [47] and there is no point reproducing all those readily accessible numbers here. However, to
Table 18: The lightest $0^{++}$ and $2^{++}$ glueball masses and the first scalar excitation, using lattice actions with $a_s = a_t$.

Table 19: The lightest $0^{-+}$, $2^{-+}$ and $1^{+-}$ glueball masses, using lattice actions with $a_s = a_t$.

give the reader a flavour of their results we provide in Table 20 the lightest $0^{++}$, $2^{++} (=E^{++})$ and $1^{+-}$ masses together with the scale $r_0$ in lattice units, all for the $\xi = 3$ calculation. Also shown is the quantity $r_0^2\sigma$ which, being a dimensionless ratio of physical quantities, is in a different category from the other quantities listed. Table 21 lists the same quantities for the $\xi = 5$ calculation. We note that by using $a_t \ll a_s$ it is possible to get accurate lattice mass estimates even for the heavier states, despite the fact that the spatial lattice spacing is large: $a_s \sim 0.2 - 0.4 fm$. If we use the listed values of $a_s/r_0$ and $r_0^2\sigma$ and compare to the values in Table 17 we see that the smallest $a_s$ in these calculations corresponds to $\beta \sim 5.60$ in the symmetric lattice calculations. These are indeed very coarse (spatial) lattices.

There are at least two questions that need answering if we are to be able to extract continuum numbers from these calculations:
1) what is $\xi_r \equiv a_s/a_t$?
2) what are the lattice corrections to the continuum limit of mass ratios?
SU(3) : $\xi = 3$

| $\beta$ | $a_t m_{0^{++}}$ | $a_t m_{2^{++}}$ | $a_t m_{1^{+-}}$ | $a_s / r_0$ | $r_0^2 \sigma$ |
|--------|-----------------|-----------------|-----------------|------------|--------------|
| 1.7    | 1.05(2)         | 1.65(5)         | 2.09(9)         | 0.861(2)   | 1.58(1)      |
| 1.9    | 0.87(1)         | 1.47(4)         | 1.80(6)         | 0.773(2)   | 1.52(2)      |
| 2.0    | 0.797(9)        | 1.40(2)         | 1.68(3)         | 0.7271(8)  | 1.462(7)     |
| 2.2    | 0.649(8)        | 1.19(2)         | 1.48(3)         | 0.6192(8)  | 1.362(8)     |
| 2.4    | 0.548(6)        | 0.995(9)        | 1.24(1)         | 0.505(1)   | 1.329(6)     |
| 2.6    | 0.464(7)        | 0.781(8)        | 0.97(1)         | 0.4021(9)  | 1.340(2)     |

Table 20: The lightest $0^{++}$, $2^{++}$ (= $E^{++}$) and $1^{+-}$ glueball masses, the parameter $r_0$ and its relation to the string tension, all for the $\xi = (a_t/a_s)_{tree} = 3$ calculation.

SU(3) : $\xi = 5$

| $\beta$ | $a_t m_{0^{++}}$ | $a_t m_{2^{++}}$ | $a_t m_{1^{+-}}$ | $a_s / r_0$ | $r_0^2 \sigma$ |
|--------|-----------------|-----------------|-----------------|------------|--------------|
| 1.7    | 0.578(5)        | 1.103(8)        | 1.19(1)         | 0.8169(9)  | 1.473(9)     |
| 1.9    | 0.475(4)        | 0.918(7)        | 1.053(8)        | 0.727(1)   | 1.45(1)      |
| 2.2    | 0.362(3)        | 0.686(4)        | 0.819(4)        | 0.5680(5)  | 1.356(4)     |
| 2.4    | 0.303(3)        | 0.542(2)        | 0.652(5)        | 0.459(1)   | 1.342(4)     |

Table 21: The lightest $0^{++}$, $2^{++}$ (= $T_2^{++}$) and $1^{+-}$ glueball masses, the parameter $r_0$ and its relation to the string tension, all for the $\xi = (a_t/a_s)_{tree} = 5$ calculation.

Let us start with the first question. The tree level value of $a_s/a_t = \xi$ will be renormalised,

$$\left( \frac{a_s}{a_t} \right) = \left( \frac{a_s}{a_t} \right)_{tree} \times (1 + O(g^2)).$$

(11)

The corrections here are potentially large and are not suppressed by powers of $a$, so they cannot be absorbed into the lattice corrections used in the continuum extrapolation of mass ratios. So, wherever $\xi_r \equiv a_s/a_t$ is important it should be calculated explicitly. This can be done straightforwardly (see Appendix D of [4] for example) if necessary. How necessary is it to do so? It is known [48, 25] that $\xi_r \simeq \xi$ for the type of improved actions used here. So one will have an idea of the spatial volume that is sufficient for control of finite volume effects, just from the tree-level estimate. Moreover one does not need to know $\xi_r$ for ratios of glueball masses, since the scale $a_t$ disappears in such ratios and the correction terms will be affected in a minor way. However there is at least one place where it is important and that is for calculations of how the potential (which is obtained in units of $a_t$) varies with distance (which is in units of $a_s$). So we need to know $\xi_r$ if we wish to express the glueball masses in units of $r_0$ or $\sqrt{\sigma}$. The quantity $r_0^2 \sigma$, on the other hand, is a dimensionless ratio of quantities derived from the potential and so the precise value of $\xi_r$ should be less important for it. Since $\xi_r$ is
not explicitly calculated in [47] this means that we have to be cautious about quantities that combine glueball masses with \( r_0 \) or \( \sigma \). We shall see below that such caution is indeed justified.

Now to the second question. The tadpole-improvement is supposed to ensure that the leading correction is \( O(\alpha_s a_s^2) \) rather than \( O(a_s^4) \). For calculations where \( a_s \) is large, as it is here, one might argue that the notionally sub-leading \( O(a_s^4) \) correction will be much more important than the notionally leading \( O(\alpha_s a_s^2) \) correction, so that the latter can be neglected and the continuum extrapolation can be performed using just the former. This has an obvious danger however: it will produce very small statistical errors at the price of potentially large systematic errors.

There is no à priori obvious answer to this second question so we turn to what the calculations themselves tell us. The most accurately calculated mass is the lightest \( 0^{++} \) and so we can ask what fit it prefers. We consider the dimensionless quantity \( r_0 m_{0^{++}} \) and try to perform a continuum extrapolation with corrections that are powers of \( a_s/r_0 \). For the \( \xi = 3 \) calculations we find that we can get a very good fit for \( \beta \geq 2.0 \),

\[
    r_0 m_{0^{++}} = 4.10(20) - (5.0 \pm 1.2) \left( \frac{a_s}{r_0} \right)^2 + (6.5 \pm 1.6) \left( \frac{a_s}{r_0} \right)^4, \tag{12}
\]

and poorer but still acceptable fits even if we include all the values in Table 20. For \( \xi = 5 \) we can fit the values in Table 21 equally well:

\[
    r_0 m_{0^{++}} = 3.83(15) - (3.45 \pm 0.75) \left( \frac{a_s}{r_0} \right)^2 + (4.50 \pm 0.80) \left( \frac{a_s}{r_0} \right)^4 \tag{13}
\]

In neither case are statistically acceptable fits with just a \( O(a_s^2) \) or just a \( O(a_s^4) \) correction possible, as should be apparent from eqns(12,13). Note also that the coefficient of the \( O(a_s^2) \) term is not much smaller than that of the \( O(a_s^4) \) term despite the fact that it presumably comes with an overall factor of \( \alpha_s \). This is presumably because, for these very coarse lattice spacings, \( \alpha_s \) is not small.

Although this already indicates that any continuum fit needs to incorporate both \( O(a_s^2) \) and \( O(a_s^4) \) correction terms, there is the possibility that the lightest \( 0^{++} \) glueball is special. For example, its mass vanishes at the nearby critical point in the fundamental-adjoint coupling plane. (Although it is not apparent why this should lead to a large \( O(a_s^2) \) correction of the kind that we find.) Let us then consider the continuum extrapolation of \( r_0 \sqrt{\sigma} \). For \( \xi = 3 \) we obtain a good fit, for \( \beta \geq 1.9 \), as follows:

\[
    r_0 \sqrt{\sigma} = 1.192(10) - (0.32 \pm 0.08) \left( \frac{a_s}{r_0} \right)^2 + (0.66 \pm 0.11) \left( \frac{a_s}{r_0} \right)^4 \tag{14}
\]

As we shall see below, this continuum limit is consistent with what one obtains from the standard \( a_s = a_t \) calculations. Once again fits with just a \( O(a_s^2) \) or a \( O(a_s^4) \) correction term are not possible.

We infer from the above analysis that continuum extrapolations from this region of lattice spacings should include both \( O(a_s^2) \) and \( O(a_s^4) \) corrections. We can see from eqns(12,14) that the coefficient of the \( a_s^2 \) term is somewhat smaller than that of the \( a_s^4 \) term, but that
this difference is not enough to make the $a_s^4$ correction the dominant one. Thus we shall extrapolate all mass ratios using two correction terms as in eqn(14).

In the above discussion we ignored the fact that we do not actually know the true value of $a_s/a_t$. For the $r_0\sqrt{\sigma}$ extrapolation that does not matter much, but for $r_0m$ it does. The fact that it does is apparent from the fits in eqns(12,13). If we take those values and translate to the string tension using eqn(14) we find

\[
\frac{m_{0^{++}}}{\sqrt{\sigma}} = \begin{cases} 
3.44(17) & \xi = 3 \\
3.21(13) & \xi = 5 
\end{cases}
\]

Comparing with the value of $3.64 \pm 0.09$ in Table 1, it is clear that there is a problem with the values in eqn(15). The discrepancies are only $\sim 5 - 10\%$ but this is large at the level of statistical accuracy being achieved in these calculations.

In conclusion, to minimise the systematic errors associated with the continuum extrapolation, we shall use two correction terms, as in eqn(14), in all our continuum extrapolations. To minimise the error associated with deviations of $\xi_r \equiv a_s/a_t$ from the tree-level value, $\xi$, we shall extrapolate to the continuum limit either ratios of glueball masses, $a_t m_G/a_t m_{0^{++}} = m_G/m_{0^{++}}$, or dimensionless ratios of quantities from the potential, such as $r_0\sqrt{\sigma}$. To obtain the continuum value of $m_G/\sqrt{\sigma}$ we multiply the extracted value of $m_G/m_{0^{++}}$ by the value of $m_{0^{++}}/\sqrt{\sigma}$ as obtained in the lattice calculations with $a_s = a_t$. If we do so we obtain the masses listed in Table 22. (The $a_s \neq a_t$ values are obtained by averaging the $\xi = 3,5$ results, and for the $2^{++}$ we use both lattice representations.) Averaging these masses leads to the final mass estimates in Table 1 and Table 4.

| SU(3) | $m_G/m_{0^{++}}$ | $m_G/\sqrt{\sigma}$ |
|-------|-----------------|-------------------|
| $G$   | $a_s \neq a_t$ | $a_s = a_t$      | $a_s \neq a_t$ | $a_s = a_t$ |
| 0$^{++}$ | 1.43(8) | 1.40(9) | 5.21(32) | 5.16(20) |
| 2$^{++}$ | 1.70(10) | 1.81(17) | 6.19(39) | 6.56(52) |
| 0$^{--}$ | 1.34(18) | 1.94(20) | 4.97(57) | 7.05(65) |
| 2$^{--}$ | 1.78(14) | 1.77(21) | 6.48(54) | 6.65(47) |
| 1$^{++}$ | 1.85(20) | 1.85(20) | 6.73(73) | 6.73(73) |

Table 22: Continuum mass ratios for the two different kinds of calculation. Brackets denote continuum extrapolations from only two values of $\beta$.

To gain the full benefit of this kind of calculation, one clearly needs to calculate $\xi_r \equiv a_s/a_t$ at every value of $\beta$. This will have its own statistical error and is likely to increase the final errors on the masses by a significant amount. In addition it would be useful to have calculations down to smaller values of the lattice spacing (perhaps down to the equivalent of
\( \beta = 5.9 \) or \( \beta = 6.0 \) in the isotropic action calculations) so as to achieve a much better control of the \( c_1 a^2 + c_2 a^4 \) extrapolation. A calculation possessing these features would produce a very accurate mass spectrum.

**(b) the \( r_0 \) parameter**

In addition to the calculations of \( r_0/a \) from [17] which were listed in Table 20 and Table 21 there have been two recent calculations using the Wilson plaquette action [37, 49] and also a scattering of older calculations, including [10, 41, 44, 50, 51]. These are listed in Table 23.

| \( \beta \) | ref[49] | ref[37] | ref[50, 41, 44, 51] |
|-----------|--------|--------|-------------------|
| 5.54      | 2.054(13) |        |                   |
| 5.60      | 2.344(8) |        |                   |
| 5.70      | 2.922(9) | 2.990(24) |                   |
| 5.80      | 3.673(5) | 4.103(12) |                   |
| 5.95      | 4.898(12) |        |                   |
| 6.00      | 5.369(9) | 5.35(3)  |                   |
| 6.07      | 6.033(17) |        |                   |
| 6.20      | 7.380(26) |        | 7.37(3)          |
| 6.40      | 9.74(5)  | 9.89(16) |                   |
| 6.40      |          | 9.75(17) |                   |
| 6.50      |          | 11.23(21)|                   |
| 6.57      | 12.38(7) |        |                   |

Table 23: Values of \( r_0/a \) from calculations with the Wilson plaquette action.

We now separately extrapolate to the continuum limit the lattice values in each of the three columns of Table 23, using an \( a^2/r_0^2 \) correction. (One gets almost identical fits using an \( a^2 \sigma \) correction.) The continuum limits are listed in Table 24. We also show the continuum limits for the anistropic calculations listed in Tables 20, 21. In this case we use a \( c_1 (a/r_0)^2 + c_2 (a/r_0)^4 \) correction since, as we remarked earlier, we cannot get statistically acceptable fits with just an \( a^2 \) or an \( a^4 \) term. The \( \xi = 5 \) calculation involves a fit with 3 parameters to 3 points and so we cannot tell how good a fit it is. I have therefore placed it in brackets. In the last column I show the range of \( \beta \) fitted in each case, as well as the confidence level of the best fit and, in brackets, the CL corresponding to the error.

We note that all the fits are very good apart from that using the calculations of [37]. One might imagine that this is because this calculation goes down to lower values of \( \beta \) where the \( O(a^2) \) correction becomes insufficient. However if we just fit the \( \beta \geq 5.70 \) part of the data we get a value of 1.179(13) with an even worse confidence level, \( \sim 20\% \). The fact that the fit is worse closer to the continuum limit is worrying and we therefore do not give this fit as much
weight as the three other fits. From these it seems that a final value of 1.195(10) is a safe one and so this is what we have quoted in Table 3.

\[
\begin{array}{|c|c|c|c|}
\hline
SU(3) & \lim_{a \to 0} r_0 \sqrt{\sigma} & \text{CL\%} & \text{range} \\
\hline
[49] & 1.197(11) & 70(20) & 5.70 \leq \beta \leq 6.57 \\
[31] & 1.175(7) & 40(10) & 5.54 \leq \beta \leq 6.00 \\
[40, 11, 14, 50, 51] & 1.204(26) & 95(25) & 6.0 \leq \beta \leq 6.5 \\
[17] \xi = 3 & 1.192(10) & 75(25) & 1.9 \leq \beta \leq 2.6 \\
[17] \xi = 5 & [1.177(13)] & -({\chi^2} = 1) & 1.9 \leq \beta \leq 2.4 \\
\hline
\end{array}
\]

Table 24: Continuum values of \(r_0 \sqrt{\sigma}\) from various lattice calculations.

(c) the deconfining temperature \(T_c\)

As described in Appendix A, a periodic Euclidean lattice of size \(N_s^3 N_r\) with \(N_s \gg N_r\) may be used to calculate the properties of the gauge theory at a temperature \(T = 1/aN_r\). Thus the deconfining temperature \(T_c\) may be obtained at fixed \(N_r\) by varying \(\beta\) and finding the value \(\beta_c(N_r)\) at which the system undergoes the deconfining phase transition. The corresponding temperature will be \(a(\beta_c)T_c = 1/N_r\) in lattice units. There now exist calculations of this quantity using a variety of actions.

The most extensive calculations are for the standard plaquette action and these are listed in Table 25. As well as the critical couplings, \(\beta_c\), I list my interpolated values of the string tension. The \(N_r = 4, 6\) calculations are from [52] while the \(N_r = 8, 12\) calculations are from [53]. The values I give are those obtained after an extrapolation to infinite volume using the finite-size scaling relation [52, 54]

\[
\beta_c(N_r, N_s) = \beta_c(N_r, \infty) - h \left( \frac{N_r}{N_s} \right)^3.
\]  

For \(N_r = 4, 6\) this relation has been fitted to a range of spatial volumes and the coefficient \(h\) explicitly determined [52, 54]. For \(N_r = 8, 12\) there is no such direct calculation but a value \(h = 0.06 \pm 0.04\) seems safe and this is what I have used in going from the values calculated in [53] for \(N_s = 32\), to the values listed in Table 25. (Aside: the values of the string tension used in [53] were, in retrospect, incorrect. This led to the continuum fit quoted there having an unacceptably low confidence level, and so it should be disregarded. This has also been noted in [53] and their alternative continuum extrapolation is consistent with ours.) I have extrapolated these lattice values to the continuum limit using a \((aT_c)^2\) correction. The result, with confidence level, is given in Table 26.

There are in addition four other recent calculations that use different improved actions [55, 56, 57, 58]. I have extrapolated these separately to the continuum limit and the results are presented in Table 26. I have used an \(O(a^2)\) correction in all cases. If I use instead an
Table 25: Values of the deconfining critical coupling, $\beta_c$, and the string tension.

| $N_\tau$ | $\beta_c(N_\tau, V = \infty)$ | $a\sqrt{\sigma}$ |
|----------|--------------------------------|-----------------|
| 4        | 5.6925(2)                     | 0.395(4)        |
| 6        | 5.8941(5)                     | 0.265(2)        |
| 8        | 6.0618(11)                    | 0.1975(18)      |
| 12       | 6.3364(25)                    | 0.1325(13)      |

Table 26: Continuum value of the deconfining temperature in units of the string tension, from the calculations as shown.

| refs | $\lim_{a \to 0} T_c/\sqrt{\sigma}$ | CL% | range  |
|------|-------------------------------------|-----|--------|
| [54] | 0.629(6)                            | 75(25) | $N_\tau = 4, 6, 8, 12$ |
| [55] | 0.634(8)                            | 55(15) | $N_\tau = 3, 4, 5, 6$ |
| [56] | 0.627(12)                           | $(\chi^2 = 1)$ | $N_\tau = 4, 6$ |
| [57] | 0.659(8)                            | 85(25) | $N_\tau = 2, 3, 4$ |
| [58] | 0.651(12)                           | 45(15) | $N_\tau = 3, 4, 6$ |

(d) the topological susceptibility $\chi_t$

As for SU(2) there are a number of quite different ways of calculating the topological charge, $Q$, and the resulting topological susceptibility, $\chi_t \equiv \langle Q^2 \rangle/V$. The older calculations [59] use cooling; after typically 20 cooling sweeps they calculate the lattice topological charge density, $Q_L(x)$, whose sum over $x$ provides an estimate for $Q$. Once again we can either take the full susceptibility, $\chi_c$, or we can try to remove the narrow instantons that might be lattice artifacts. As in the case of SU(2) we shall present results where all instantons with $Q_L(x_{peak}) \geq 1/16\pi^2$ have been removed; the corresponding susceptibility we call $\chi_{c'}$. (As for
SU(2) we suppress the subscript t on χ for clarity.) There is also a recent calculation that uses cooling [60]. As for SU(2) there are calculations that work directly with the Monte Carlo generated gauge fields, but use smeared operators and correct explicitly for the additive and multiplicative renormalisations [61]. We call the resulting susceptibility χ_{sm}. There are also calculations [62] that use a smoothing of the gauge fields that is motivated by renormalisation group considerations. This gives χ_{RG}. At β = 6.0 two of the above calculations provide values on two lattice sizes. The values of aχ^{1/4} corresponding to all these calculations are listed in Table 27.

| β   | aχ_{c}^{1/4} | aχ_{c'}^{1/4} | aχ_{s}^{1/4} | aχ_{sm} | aχ_{RG} |
|-----|--------------|---------------|--------------|---------|---------|
| 5.70| 0.1595(23)   | 0.1244(33)    |              |         |         |
| 5.80| 0.1347(25)   | 0.1147(28)    |              |         | 0.1179(28) |
| 5.85|              |               |              | 0.1019(21) |         |
| 5.90| 0.1135(17)   | 0.1059(16)    |              |         | 0.0937(12) |
| 6.00| 0.0915(21)   | 0.0880(22)    | 0.0977(34)   | 0.0901(19) | 0.0961(33) |
| 6.10| 0.0722(36)   | 0.0713(35)    | 0.0701(25)   | 0.0799(17) | 0.0816(20) |
| 6.40|              |               | 0.0537(48)   |         |         |

Table 27: The topological susceptibility, using various methods as defined in the text.

Extrapolating to the continuum limit using an O(a^2σ) correction I obtain the continuum values shown in Table 28. These are clearly consistent with each other. In addition, apart from the first two columns which come from the same configurations, these calculations are statistically independent (apart from the string tension). This provides the final continuum estimate in Table 3.

| n = c | n = c' | n = c | n = sm | n = RG |
|-------|--------|-------|--------|--------|
| 0.439(16) | 0.459(17) | 0.448(50) | 0.464(23) | 0.456(23) |

Table 28: The continuum topological susceptibility, extrapolating the values in Table 27, in units of the string tension.

(e) the Λ_{MS} scale

To calculate Λ_{MS} one needs an accurate calculation of the running coupling over a large range of length scales. Some of the first realistic attempts involved calculations of the static
potential, with the running coupling being obtained from the Coulomb interaction at short
distances. (See for example [33].) The most recent calculation uses a different technique that
allows a fine control of the various systematic errors [64] and this is the value we have quoted
in Table 3.

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