Complexity of the conditional colorability of graphs

Xueliang Li, Xiangmei Yao and Wenli Zhou
Center for Combinatorics and LPMC-TJKLC, Nankai University
Tianjin 300071, P.R. China. Email: lxl@nankai.edu.cn

Abstract

For an integer \( r > 0 \), a conditional \((k, r)\)-coloring of a graph \( G \) is a proper \( k \)-coloring of the vertices of \( G \) such that every vertex \( v \) of degree \( d(v) \) in \( G \) is adjacent to vertices with at least \( \min\{r, d(v)\} \) different colors. The smallest integer \( k \) for which a graph \( G \) has a conditional \((k, r)\)-coloring is called the \( r \)th order conditional chromatic number, denoted by \( \chi_r(G) \). It is easy to see that the conditional coloring is a generalization of the traditional vertex coloring for which \( r = 1 \). In this paper, we consider the complexity of the conditional colorings of graphs. The main result is that the conditional \((3, 2)\)-colorability is \( NP \)-complete for triangle-free graphs with maximum degree at most 3, which is different from the old result that the traditional 3-colorability is polynomial solvable for graphs with maximum degree at most 3. This also implies that it is \( NP \)-complete to determine if a graph of maximum degree 3 is \((3, 2)\)- or \((4, 2)\)-colorable. Also we have proved that some old complexity results for traditional colorings still hold for the conditional colorings.

Keywords. vertex coloring, conditional coloring, (conditional) chromatic number, \( NP \)-complete

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1 Introduction

We follow the terminology and notations of [1] and, without loss of generality, consider simple connected graphs only. \( \delta(G) \) and \( \Delta(G) \) denote the minimum degree and maximum degree of a graph \( G \), respectively. For a vertex \( v \in V(G) \), the neighborhood of \( v \) in \( G \) is \( N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\} \), and the degree of \( v \) is \( d(v) = |N_G(v)| \). Vertices in \( N_G(v) \) are called neighbors of \( v \). \( P_n \) denotes the path of \( n \) vertices. An edge \( e \) is said to be subdivided when it is deleted and replaced by

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a path of length two connecting its ends, the internal vertex of this path is a new vertex.

For an integer \( k > 0 \). A proper \( k \)-coloring of a graph \( G \) is a surjective map \( c : V(G) \rightarrow \{1, 2, \ldots, k\} \) such that if \( u, v \) are adjacent vertices in \( G \), then \( c(u) \neq c(v) \). The smallest \( k \) such that \( G \) has a proper \( k \)-coloring is the chromatic number of \( G \), denoted by \( \chi(G) \).

In the following we will consider a generalization of the traditional coloring. For integers \( k > 0 \) and \( r > 0 \), a proper \((k, r)\)-coloring of a graph \( G \) is a surjective map \( c : V(G) \rightarrow \{1, 2, \ldots, k\} \) such that both of the following two conditions hold:

(C1) if \( u, v \in V(G) \) are adjacent vertices in \( G \), then \( c(u) \neq c(v) \); and

(C2) for any \( v \in V(G) \), \( |c(N_G(v))| \geq \min\{d(v), r\} \), where and in what follows, \( c(S) = \{c(u) | u \in S \text{ for a set } S \subseteq V(G)\} \).

For a given integer \( r > 0 \), the smallest integer \( k > 0 \) such that \( G \) has a proper \((k, r)\)-coloring is the \( (r \text{th order}) \) conditional chromatic number of \( G \), denoted by \( \chi_r(G) \).

By the definition of \( \chi_r(G) \), it follows immediately that \( \chi(G) = \chi_1(G) \), and so \( \chi_r(G) \) is a generalization of the traditional graph coloring. The conditional chromatic number has very different behavior from the traditional chromatic number. For example, when \( r = 2 \), from [7] we know that for many graphs \( G \), \( \chi_2(G - v) > \chi_2(G) \) for at least one vertex \( v \) of \( G \), and there are graphs \( G \) for which \( \chi_r(G) - \chi(G) \) may be very large.

From [6] we know that if \( \Delta(G) \leq 2 \), for any \( r \) we can easily have an algorithm of polynomial time to give the graph \( G \) a \((k, r)\)-coloring. In [8], Lai, Montgomery and Poon got an upper bound of \( \chi_2(G) \) that if \( \Delta(G) \geq 3 \), then \( \chi_2(G) \leq \Delta(G) + 1 \). The proof is very long compared with the proof of a similar result for the traditional coloring. In [6] and [7], Lai, Lin, Montgomery, Shui and Fang got many new and interesting results on the conditional coloring. In the present paper, we are going to investigate the complexity of deciding if a graph is \((k, r)\)-colorable. We first give a simple proof that for any \( k \geq 3 \) and \( r \geq 2 \) it is \( NP \)-complete to check if a graph is \((k, r)\)-colorable. Then we give the main theorem in the paper that the conditional \((3, 2)\)-colorability is \( NP \)-complete for triangle-free graphs with maximum degree at most 3, which is different from the old result that the traditional 3-colorability is polynomial solvable for graphs with maximum degree at most 3. At last we show that the \((3, 2)\)-colorability is also \( NP \)-complete for some special classes of graphs, planar graphs, hamiltonian graphs and so on.

2 The complexity of the conditional colorings

In this section, we shall analyze the complexity of the \((k, r)\)-colorability of graphs. We refer to [3] for terminology, notations and basic results on complexity not given
If a connected graph $G$ has only one vertex, then $\chi_r(G) = 1$; if a connected graph $G$ has only two vertices, then $\chi_r(G) = 2$. For the other connected graphs $G$, we have $\chi_r(G) \geq 3$ for $r \geq 2$. But the following theorem show that for any $2 \leq r < k$ the $(k, r)$-Col is NP-complete.

The $(k, r)$-colorable problem, denoted by $(k, r)$-Col, is defined as follows:

**Input:** A graph $G = (V, E)$ and two integers $k > r \geq 2$.

**Question:** Can one assigns each vertex a color, so that only $k$ colors are used and the two conditions C1 and C2 are satisfied? i.e., Is $\chi_r(G) \leq k$?

**Theorem 2.1** For every fixed $(k, r)$, $2 \leq r < k$, $(k, r)$-Col is NP-complete.

**Proof.** First, it is easy to see that the problem $(k, r)$-Col is in NP.

Second, it is known that the traditional $k$-colorable problem is NP-complete. So, to show the NP-completeness, it is sufficient to reduce the traditional $k$-colorable problem to the $(k, r)$-Col. We want to relate any instance $G$ of the $k$-colorable problem to a graph $G'$, such that $G$ is $k$-colorable if and only if $G'$ is $(k, r)$-colorable.

For each vertex $v$ in $V(G)$, we add a new complete graph $K_r$ and add new edges such that $v$ and $K_r$ form a complete graph of order $r + 1$. The resultant graph is denoted by $G'$. So, $G'$ has $(r + 1)|V(G)|$ vertices, and every vertex in $G'$ is contained in a $K_{r+1}$. It is easy to see that $G$ is $k$-colorable if and only if $G'$ is $(k, r)$-colorable.

It is known that in traditional colorings, all graphs with maximum degree 3 are 3-colorable except for $K_4$ (by Brook’s theorem). So the 3-colorable problem is trivial for this class of graphs. But the next theorem tells us that the problem $(3, 2)$-Col remains NP-complete for triangle-free graphs with maximum degree 3.

**Theorem 2.2** The problem $(3, 2)$-Col remains NP-complete for triangle-free graphs with maximum degree at most 3.

**Proof.** First, the problem $(3, 2)$-Col for triangle-free graphs with maximum degree at most 3 is obviously in NP.

We want to modify the method given in [4] to prove the NP-completeness. We reduce the 3-SAT problem to the problem $(3, 2)$-Col for graphs with maximum degree at most 3. We want to relate any instance $I$ of the 3-SAT problem to a graph $G$ with $\Delta(G) \leq 3$, such that $I$ is satisfiable if and only if $G$ is $(3, 2)$-colorable. Let the set of literals, of the input $I$ to the 3-SAT problem, be $\{x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ and the clauses be $C_1, C_2, \ldots, C_m$.

The graph $G(V, E)$ with $\Delta(G) \leq 3$ is defined as follows:
First, for each clause \(C_i\) (1 \(\leq\) \(i\) \(\leq\) \(m\)) we construct the first kind of building-block \(H_i\) (see Figure 1). The second graph in Figure 1 is the shorthand notation of \(G\).

Now, it is easy to check that the graph constructed in Figure 1 has the following two properties:

1. If we use colors \(\{0, 1, 2\}\), and \(u_1, u_2, u_3\) are all colored by 0, then in every \((3, 2)-\)coloring, \(v\) is forced to be colored by 0;

2. If the three vertices \(u_1, u_2, u_3\) are colored only by 0 or 1 (the other vertices in \(G\) can also be colored by 0, 1 or 2), and not all the three vertices are colored by 0, then there is a \((3, 2)-\)coloring such that \(v\) can be colored by 1.

From the above properties we know that if \(v\) is not colored by 0, then one of \(u_1, u_2, u_3\) must be colored by 1, which means that if 1 represents ‘true’, 0 represents ‘false’, and the clause \(C = u_1 \lor u_2 \lor u_3\) must be satisfied.

Second, we construct two paths \(P_{6n-1} (P_{6n-1} = a_1a_2\cdots a_{6n-1})\) and \(P_{3m-2} (P_{3m-2} = b_1b_2\cdots b_{3m-2})\).

Third, for each pair \(x_i\) and \(\bar{x}_i\) (1 \(\leq\) \(i\) \(\leq\) \(n\)) we construct the second kind of building-block \(B_i\) (it is represented by a rectangle in Figure 2). The second kind of building-block \(B_i\) (1 \(\leq\) \(i\) \(\leq\) \(n\)) is constructed as follows:

1. Let \(t_i\) be the number of clauses which contain \(x_i\), and let \(\bar{t}_i\) be the number of clauses which contain \(\bar{x}_i\);

2. We construct a path \(P_{x_i}\) of \(3t_i - 2\) vertices corresponding to the vertex \(x_i\), and construct another path \(P_{\bar{x}_i}\) of \(3\bar{t}_i + 2\) vertices corresponding to the vertex \(\bar{x}_i\);

3. Let \(x_{ij}\) be the \((3j - 2)\)th (1 \(\leq\) \(j\) \(\leq\) \(t_i\)) vertex in the path \(P_{x_i}\), and let \(\bar{x}_{ij}\) be the \((3j - 2)\)th (1 \(\leq\) \(j\) \(\leq\) \(\bar{t}_i + 1\)) vertex in the path \(P_{\bar{x}_i}\);

4. Join \(x_{ij}\) with an edge to \(\bar{x}_{ij}\).
Finally, join $x_i$ with an edge to $a_{6i-5}$ and join $\bar{x}_i$ with an edge to $a_{6i-2}$ ($1 \leq i \leq n$). And each $v_i$ is joined with an edge to $b_{3i-2}$ ($1 \leq i \leq m$). The vertex $a_1$ is joined with an edge to the vertex $b_1$. Each $x_{ij}$ ($1 \leq j \leq t_i$) or $\bar{x}_{ij}$ ($2 \leq j \leq t_i + 1$) joins with an edge to some $H_l$ ($1 \leq l \leq m$) which represents the clause that contains the $j$th $x_i$ or the $(j-1)$th $\bar{x}_i$. The final resultant graph is shown in Figure 2. It is easy to see that the maximum degree of the graph is at most 3 and the graph is triangle-free.

![Figure 2: The entire construction](image)

The graphs $B_i$ ($1 \leq i \leq n$) in the final resultant graph (Figure 2) have the following properties:

1. If $G$ is $(3, 2)$-colorable, the vertices $x_{ij}$ ($1 \leq j \leq t_i$) must be colored by the same color, and the vertices $\bar{x}_{ij}$ ($1 \leq j \leq t_i + 1$) must also be colored by the same color. But

2. The color of the vertices $x_{ij}$ ($1 \leq j \leq t_i$) must be colored by different color from the color of the vertices $\bar{x}_{ij}$ ($1 \leq j \leq \bar{t}_i + 1$).

The two paths $P_{6n-1}$ and $P_{3m-2}$ in the final resultant graph have the following properties:

1. If $G$ is $(3, 2)$-colorable, the vertices $a_{3i-2}$ ($1 \leq j \leq 2n$) must be colored the same color, and the vertices $b_{3i-2}$ ($1 \leq i \leq m$) must also be colored by the same color. But

2. The color of the vertices $a_{3i-2}$ ($1 \leq i \leq 2n$) must be colored by different color from the color of the vertices $b_{3i-2}$ ($1 \leq i \leq m$).
Now if $I$ is satisfiable, we give a proper $(3,2)$-coloring of the graph in Figure 2 as follows: Let $c(x_{ij}) = 1$ (1 $\leq j \leq t_i$) if $x_i$ is true, and of course let $c(x_{ij}) = 0$ (1 $\leq j \leq t_i$); let $c(x_{ij}) = 0$ (1 $\leq j \leq t_i$) if $x_i$ is false, and of course let $c(x_{ij}) = 1$ (1 $\leq j \leq t_i + 1$). The rest vertices in $B_i$ can be colored easily to satisfy the conditions C1 and C2. By the properties given in the first step, we know that $v_i$ (i = 1, ..., m) can be colored by 1 since each clause is satisfied. Let $c(a_i) = 2$ (i = 1, 4, 7, ..., 6n-2), $c(a_i) = 1$ (i = 2, 5, 8, ..., 6n-1), $c(a_i) = 0$ (i = 3, 6, 9, ..., 6n-3); let $c(b_i) = 0$ (i = 1, 4, 7, ..., 3m-2), $c(b_i) = 1$ (i = 2, 5, 8, ..., 3m-3), $c(b_i) = 2$ (i = 3, 6, 9, ..., 3m-4). Then it is easy to check that $c$ is a proper $(3,2)$-coloring.

Conversely, if $G$ is $(3,2)$-colorable, there is a proper $(3,2)$-coloring $c$. Without loss of generality, suppose $c(a_1) = 2$ and $c(b_1) = 0$. Then, from the properties of the two paths we give above, all the vertices $a_{3i-2}$ (1 $\leq i \leq 2n$) are colored by 2, while all the vertices $b_{3i-2}$ (1 $\leq i \leq m$) are colored by 0. By the properties of the graph $B_i$ described above, $c(x_{ij})$ (1 $\leq j \leq t_i$) and $c(x_{i})$ (1 $\leq j \leq t_i + 1$) are colored by 1 or 0, and $c(v_i)$ cannot be colored by 0. Then, let $x_i$ be true if $c(x_{ij}) = 1$, and let $x_i$ be false if $c(x_{ij}) = 0$. Since $c(v_i)$ cannot be colored by 0, each $C_i$ is satisfiable, and thus $I$ is satisfiable. \hfill \blacksquare

From [6] we know that if $\Delta(G) = 1$ or 2, $\chi_2(G)$ can be determined in polynomial time. From [8] we know that if $\Delta(G) = 3$, then $\chi_2(G) = 3$ or 4. So by Theorem 2.2 we can get the following result.

**Corollary 2.3** When $\Delta(G) = 3$, it is NP-complete to determine whether $\chi_2(G) = 3$ or $\chi_2(G) = 4$.

Next we will consider the other special classes of graphs, hamiltonian graphs, planar graphs, claw-free graphs.

**Theorem 2.4** The problem $(3,2)$-Col is NP-complete when restricted to hamiltonian graphs with $\Delta(G) \leq 6$.

**Proof.** A known result is that to determine whether a hamiltonian graph with maximum degree at most 4 is 3-colorable is NP-complete. Now, given a hamiltonian graph $G$ with $V(G) = \{v_1, v_2, ..., v_n\}$ and, without loss of generality, $v_1v_2...v_nv_1$ is a hamiltonian cycle of $G$. We construct a new hamiltonian graph $G'$ as follows: For each $v_i$ we add two new vertices $x_{i1}$ and $x_{i2}$ and three new edges $v_ix_{i1}$, $x_{i1}x_{i2}$ and $x_{i2}v_i$ (a triangle). Then, add new edges $x_{12}x_{21}$, $x_{32}x_{41}$, $x_{52}x_{61}$, ..., $x_{(n-1)2}x_{n1}$ for $n$ even; add new edges $x_{12}x_{21}$, $x_{32}x_{41}$, $x_{52}x_{61}$, ..., $x_{(n-2)2}x_{(n-1)1}$ and add a new vertex $u$ and three edges $x_{11}u$, $x_{12}u$ and $x_{(n-1)2}u$ for $n$ odd.

It is easy to see that $G'$ is also a hamiltonian graph with $\Delta(G) \leq 6$. First, if $G'$ is $(3,2)$-colorable, then restrict a proper $(3,2)$-coloring to the vertices $v_1, v_2, ..., v_n$, it is a proper coloring for $G$. Second, if $G$ is 3-colorable, then the vertices $v_1, v_2, ..., v_n$ in $G'$ are colored by the same color as they are colored in $G$, and since there are 3 colors, the rest vertices of $G'$ can be colored properly, and so $G'$ is 3-colorable.
Since every vertex in $G'$ is contained in a triangle, $G'$ is 3-colorable means $G'$ is $(3, 2)$-colorable. Then $G$ is 3-colorable if and only if $G'$ is $(3, 2)$-colorable. Then we get the result.

Now we consider planar graphs. From [2] we know that the 3-colorable problem for planar hamiltonian graphs is $NP$-complete. Then we have the following theorem.

**Theorem 2.5** The problem $(3, 2)$-Col is $NP$-complete for planar hamiltonian graphs.

**Proof.** Given a planar hamiltonian graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and, without loss of generality, $v_1v_2 \ldots v_nv_1$ is a hamiltonian cycle $C_n$ of $G$. For any edge $v_iv_{i+1}$ of $C_n$, we do the local transformation to get a new graph $G'$ as follows: For each edge $v_iv_{i+1}$ in $C_n$, we add 4 new vertices $x_{i1}, x_{i2}, y_{(i+1)1}, y_{(i+1)2}$ and 7 new edges $x_{i1}v_i, x_{i2}v_i, x_{i1}x_{i2}, y_{(i+1)1}v_{i+1}, y_{(i+1)2}v_{i+1}, y_{(i+1)1}y_{(i+1)2}, x_{i2}y_{(i+1)1}$, two triangles with an edge connecting them. Then $G'$ has $5|V(G)|$ vertices, and every vertex is in a triangle, and moreover, each “two triangles with an edge joining them” can be drawn in the local space of $v_iv_{i+1}$ without crossing the boundary of any face, so that the new graph $G'$ remains planar. It is easy to see that is is also hamiltonian. By the same reason as in Theorem 2.4, it is easy to see that $G$ is 3-colorable if and only if $G'$ is $(3, 2)$-colorable. The proof is complete.

For claw-free graph $G$, we can similarly add vertices and edges to make every vertex $v \in V(G)$ in a triangle, and show that $(3, 2)$-Col is $NP$-complete for claw-free graphs.

To conclude the paper, we point out that there are polynomial algorithms to solve the $k, r$-Col problem for some special classes of graphs. From the proof of [8], one can design a polynomial algorithm to color the graph $G$ by $\Delta(G) + 1$ colors when $\Delta(G) \geq 3$. For some classes of perfect graphs, such as triangulated graphs and comparability graphs, there are polynomial algorithms to color the graph $G$ by $\chi(G)$ colors for traditional coloring in [5]. For these kinds of graphs, one can also design polynomial algorithms to get the conditional coloring number and the way to color these graphs, with a little change of the original algorithms in [5]. The details are omitted.

**References**

[1] J.A. Bondy and U.S.R Murty, Graph Theory with Applications, North-Holland, Elsevier, 1981.

[2] P. Bonsma, The complexity of the Matching-Cut problem for planar graphs and other graph classes, Lecture Notes in Computer Science, 2880(2003), 93–105.

[3] M.R. Garey and D.S. Johnson, Computers and Intractability, A Guide to the Theory of $NP$-Completeness, Freeman, 1979.

[4] M.R. Garey, D.S. Johnson and L.J. Stockmeyer, Some simplified $NP$-complete graph problems, Theor. Comput. Sci. 1(1976), 237–267.
[5] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Academic Press, 1980.

[6] H.J. Lai, J. Lin, B. Montgomery, T. Shui and S. Fan, Conditional colorings of graphs, Discrete Math. 306(2006), 1997–2004.

[7] H.J. Lai and B. Montgomery, Dynamic Coloring of Graphs, available in: http://jacobi.math.wvu.edu/~hjlai/Pdf/Dynamic-Gen.pdf.

[8] H.J. Lai, B. Montgomery and H. Poon, Upper bounds of dynamic chromatic number, Ars Combin. 68(2003), 193–201.