Strong Topological Rigidity of Non-Compact Orientable Surfaces

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Abstract

We show that every orientable infinite-type surface is properly rigid as a consequence of a more general result. Namely, we prove that if a homotopy equivalence between any two non-compact orientable surfaces is a proper map, then it is properly homotopic to a homeomorphism, provided surfaces are neither the plane nor the punctured plane. Thus all non-compact orientable surfaces, except the plane and the punctured plane, are topologically rigid in a strong sense.

1 Introduction

All manifolds will be assumed to be second countable and Hausdorff. A surface is a two-dimensional manifold with an empty boundary. Throughout this note, all surfaces will be considered connected and orientable. We say a surface is of infinite-type if its fundamental group is not finitely generated; otherwise, we say it is of finite-type.

A fundamental question in topology is that if two closed $n$-manifolds are homotopy equivalent, are they homeomorphic? This has a positive answer in dimension two, as two closed surfaces having isomorphic fundamental groups are homeomorphic. But the same doesn’t happen in other dimensions; for example, there are homotopy equivalent lens spaces (a particular type of spherical 3-manifolds) that are not homeomorphic. A closed topological $n$-manifold $M$ is said to be topologically rigid if any homotopy equivalence $N \to M$ with a closed topological $n$-manifold $N$ as the source is homotopic to a homeomorphism. The Borel conjecture [30, Conjecture (A. Borel)] asserts that every closed aspherical (i.e., $\pi_k = 0$ if $k \neq 1$) manifold is topologically rigid. In dimension two, every closed surface is topologically rigid; it is known as the Dehn-Nielsen-Baer theorem [9, Appendix]. The Borel conjecture is known to be true in other dimensions under some additional hypotheses; for example, see [36, Theorem 6.1.] and [18, Theorem 0.1. i)] for dimension three, and for high dimensions, consider [15, Proof of Theorem 3.2].

Though non-compact manifolds are not rigid in the above sense, for example, in [25, Theorem 2], the author has constructed (generalizing a construction given by J. H. C. Whitehead) uncountably many contractible open subsets of $\mathbb{R}^3$ such that any two of them are not homeomorphic. Similarly, for non-compact surfaces, we have several examples. In the case of finite-type surfaces, for example, we may consider the once-punctured torus and thrice-punctured sphere, which are homotopy equivalent but non-homeomorphic, as any homomorphism preserves the cardinality of the puncture set as well as the genus. On the other hand, up to homotopy equivalence, there is precisely one infinite-type surface, but up to homeomorphism, there are $2^{\aleph_0}$ many infinite-type surfaces (see Proposition 3.1.11). This note considers only non-compact surfaces and discusses their topological rigidity in the proper category; here, proper category means the category of spaces with proper maps (recall that a map from a space $X$ to a space $Y$ is called a proper map if the inverse image of each compact subset of $Y$ is a compact subset of $X$). At first, we define the analogs of homotopy, homotopy equivalence, etc., in the proper category.
If a homotopy $H : X \times [0, 1] \to Y$ is a proper map, then we call $H$ a proper homotopy. Two proper maps from $X$ to $Y$ are said to be properly homotopic if there is a proper homotopy between them. We say that a proper map $f : X \to Y$ is a proper homotopy equivalence if there exists a proper map $g : Y \to X$ such that both $g \circ f$ and $f \circ g$ are properly homotopic to the identity maps (when such a $g$ exists, we say that $g$ is a proper homotopy inverse of $f$). Two spaces $X$ and $Y$ are said to have the same proper homotopy type if there is a proper homotopy equivalence between them. It is worth noting that homotopy through proper maps is a weaker notion than proper homotopy. For example, consider $H : \mathbb{C} \times [0, 1] \to \mathbb{C}$ given by $H(z, t) := tz^2 - z$. Being a polynomial, each $H(\cdot, t)$ is proper. But, $H$ itself is not proper as $H(n, \frac{1}{n}) = 0$ for all integers $n \geq 1$.

The analog of topological rigidity in the proper category is defined as follows: A non-compact topological manifold $M$ without boundary is said to be properly rigid if, whenever $N$ is another boundaryless topological manifold of the same dimension and $h : N \to M$ is a proper homotopy equivalence, then $h$ is properly homotopic to a homeomorphism. The analog of the Borel conjecture in the proper category, often called proper Borel conjecture [8, Conjecture 3.1.], asserts that every non-compact aspherical topological manifold without boundary is properly rigid.

It is known that non-compact finite-type surfaces are properly rigid. Further, using the algebraic tools of classification of non-compact surfaces [19, Theorem 4.1.], Goldman showed that two non-compact surfaces of the same proper homotopy type are homeomorphic; see [20, Corollary 11.1]. We show that infinite-type surfaces are also properly rigid. In fact, we show the rigidity of all non-compact surfaces, except for the plane and the punctured plane, under a weaker assumption, namely only assuming the existence of homotopy inverse, which a priori may or may not be proper. For brevity, define a weaker version of proper homotopy equivalence:

**Definition** A homotopy equivalence is said to be pseudo proper homotopy equivalence if it is proper.

Indeed, a proper homotopy equivalence is a pseudo proper homotopy equivalence, though not conversely: a pseudo proper homotopy equivalence has an “ordinary” homotopy inverse but may not have a proper homotopy inverse, for example, consider below given $\varphi$ and $\psi$. Our main theorem is the following:

**Theorem** Let $f : \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact surfaces. Then $\Sigma'$ is homeomorphic to $\Sigma$. If we further assume that $\Sigma$ is homeomorphic to neither the plane nor the punctured plane, then $f$ is a proper homotopy equivalence, and there exists a homeomorphism $g_{\text{homeo}} : \Sigma \to \Sigma'$ as a proper homotopy inverse of $f$.

The reason for the exclusion of the plane and the punctured plane from the hypothesis is almost immediate; for example, consider $\varphi : \mathbb{C} \ni z \mapsto z^2 \in \mathbb{C}$ and $\psi : S^1 \times \mathbb{R} \ni (z, x) \mapsto (z, |x|) \in S^1 \times \mathbb{R}$; each of these proper maps is a homotopy equivalence, but none of them is a proper homotopy equivalence as the degree of a proper homotopy equivalence is $\pm 1$ (see Section 2.6), though $\deg(\varphi) = \pm 2$ (as $\varphi$ is a two-fold branched covering) and $\deg(\psi) = 0$ (as $\psi$ is not surjective; see Lemma 3.6.4.1).

In general, additional assumptions must be imposed on a pseudo proper homotopy equivalence to become a proper homotopy equivalence. For example, using the binary symmetry of the Cantor tree $\mathcal{T}_{\text{Cantor}}$, we have a two-fold branched covering $f_{\text{Cantor}} : \mathcal{T}_{\text{Cantor}} \to \mathcal{T}_{\text{Cantor}}$, which is undoubtedly a pseudo proper homotopy equivalence (trees are contractible) but not a proper homotopy equivalence (the induced map on $\text{Ends}(\mathcal{T}_{\text{Cantor}})$ by $f_{\text{Cantor}}$ is non-injective; see part (1) and (3) of Proposition 2.3.1). Here is another example. Let $M$ be a connected, non-compact, contractible, boundaryless manifold of dimension $n \geq 2$; and let $f : M \to M$ be the composition of the two proper maps: a proper map $M \to [0, \infty)$ (using partition of unity) and a non-surjective proper map $[0, \infty) \to M$ corresponding to an end of $M$ (using compact exhaustion by connected codimension 0-submanifolds; see [21, Exercise 3.3.18]). Then $f$ is a
pseudo proper homotopy equivalence ($M$ is contractible) but not a proper homotopy equivalence (a proper homotopy equivalence is a surjective map as its degree is $\pm 1$; see Lemma 3.6.4.1).

Brown showed that a pseudo proper homotopy equivalence between two connected, finite-dimensional, locally finite simplicial complexes is a proper homotopy equivalence if and only if it induces a homeomorphism on the spaces of ends and isomorphisms on all proper homotopy groups; see [5, Whitehead theorem]. In [16, Corollary 4.10.], the authors have shown that if $f: M \to N$ is a pseudo proper homotopy equivalence between two simply-connected, non-compact, boundaryless $n$-dimensional smooth manifolds, where both $M$ and $N$ both are simply-connected at infinity, then $f$ is a proper homotopy equivalence if and only if $\deg(f) = \pm 1$. Another interesting statement in this context is that a proper map $f: X \to Y$ between two locally finite, infinite, connected, 1-dimensional CW-complexes is a proper homotopy equivalence if $\text{Ends}(f)$ is a homeomorphism and $f$ is an extension of a proper homotopy equivalence $X_g \to Y_g$ (where $X_g$ (resp. $Y_g$) denotes the smallest connected sub-complex of $X$ (resp. $Y$) that contains all immersed loops of $X$ (resp. $Y$)); see [1, Corollary 3.7.].

We conclude this section by citing a few more related results of two different flavors: when does a proper homotopy equivalence exist, and if it does exist, whether it determines the space up to homeomorphism. Similar to KerékJártó’s classification theorem (see Theorem 2.4.1), there exists a classification of graphs up to proper homotopy type: Two locally finite, infinite, connected, 1-dimensional CW-complexes $X$ and $Y$ have the same proper homotopy type if and only if $\text{rank}(\pi_1(X)) = \text{rank}(\pi_1(Y))$ and there exists a homeomorphism $\varphi: \text{Ends}(X) \to \text{Ends}(Y)$ with $\varphi(\text{Ends}(X_g)) = \text{Ends}(Y_g)$; see [3, Theorem 2.7.].

As stated earlier, any two non-compact surfaces of the same proper homotopy type are homeomorphic. Sometimes this happens also in other dimensions; for instance, a boundaryless topological manifold of dimension $n \geq 3$ with the same proper homotopy type of $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$; see [11, Theorem 1] for $n = 3$, [17, Corollary 1.2.] for $n = 4$, and [32, Corollary 1.4.] for $n \geq 5$. In contrast, there are exotic pairs, for example, two non-compact, connected, boundaryless manifolds $N$ and $M$ of the dimension $n \geq 5$ exist, where $N$ is smoothable, and $M$ is a nonuniform arithmetic manifold, such that $M$ and $N$ has the same proper homotopy type, but $M$ is not homeomorphic $N$; see [7, Theorem 2.6.] with [8, Pages 137 and 138].

### 1.1 Main results

The analog of [14, First proof of Theorem 8.9.] in the proper category is Theorem 2, which follows almost directly from our main result Theorem 1. Indeed, Theorem 1 is more general.

**Theorem 1** (Strong topological rigidity) Let $f: \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact surfaces. Suppose $\Sigma'$ is homeomorphic to neither $\mathbb{R}^2$ nor $S^1 \times \mathbb{R}$. Then $\Sigma'$ is homeomorphic to $\Sigma$, and $f$ is properly homotopic to a homeomorphism.

**Theorem 2** (Proper rigidity) If $f: \Sigma' \to \Sigma$ is a proper homotopy equivalence between two non-compact surfaces, then $\Sigma'$ is homeomorphic $\Sigma$ and $f$ is properly homotopic to a homeomorphism.

A theorem of Edmonds [10, Theorem 3.1.] says that any $\pi_1$-injective map of degree one between two closed surfaces is homotopic to a homeomorphism. The analog fact for non-compact surfaces is Theorem 3, which classifies all $\pi_1$-injective degree one maps between two non-compact surfaces and also follows almost directly from Theorem 1.

**Theorem 3** (Classification of $\pi_1$-injective degree one maps) Let $\Sigma, \Sigma'$ be any two non-compact oriented surfaces. Suppose there exists a $\pi_1$-injective proper map $f: \Sigma' \to \Sigma$ of degree $\pm 1$. Then $\Sigma$ is homeomorphic to $\Sigma'$, and $f$ is properly homotopic to a homeomorphism.

Proofs of Theorem 1, Theorem 2, and Theorem 3 can be found in Section 4.
1.2 Outline of the proof of Theorem 1

Let \( f : \Sigma' \to \Sigma \) be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. Suppose \( \Sigma \) is homeomorphic to neither \( \mathbb{R}^2 \) nor \( \mathbb{S}^1 \times \mathbb{R} \).

1.2.1 Decomposition and transversality

Let \( \mathcal{C} \) be a locally finite pairwise disjoint collection of smoothly embedded circles on \( \Sigma \) such that \( \mathcal{C} \) decomposes \( \Sigma \) into bordered sub-surfaces, and a complementary component of this decomposition is homeomorphic to either the one-holed torus or the pair of pants or the punctured disk (see Theorem 3.1.5). Properly homotope \( f \) to make it smooth as well as transverse to \( \mathcal{C} \). Thus, \( f^{-1}(\mathcal{C}) \) is either empty or a pairwise disjoint finite collection of smoothly embedded circles on \( \Sigma' \) for each component \( C \) of \( \mathcal{C} \). See Theorem 3.2.3.

1.2.2 Removing unnecessary circles

Now, following the three steps given below, we properly homotope \( f \) further so that for each component of \( \mathcal{C} \) of \( \mathcal{C} \), either \( f^{-1}(\mathcal{C}) \) is empty or \( f|_{f^{-1}(\mathcal{C})} \to \mathcal{C} \) is a homeomorphism.

1. (1) Notice that \( f^{-1}(\mathcal{C}) \) may have infinitely many disk-bounding components. But, in such a case, an arbitrarily large disk in \( \Sigma' \) bounded by a component of the locally finite collection \( f^{-1}(\mathcal{C}) \) is not possible as \( \Sigma' \neq \mathbb{R}^2 \) (see Lemma 3.3.1), i.e., there always exists an “outermost disk” bounded by some component of \( f^{-1}(\mathcal{C}) \). Now, properly homotope \( f \) to remove all disk bounding components of \( f^{-1}(\mathcal{C}) \) upon considering all these outermost disks simultaneously. See Theorem 3.3.5.

2. (2) Thereafter, using \( \pi_1 \)-bijectivity of \( f \), properly homotope \( f \) to map each (primitive) component of \( f^{-1}(\mathcal{C}) \) onto a component of \( \mathcal{C} \) homeomorphically. See Theorem 3.4.3.

3. (3) Since \( f \) has homotopy left inverse, any two components of \( f^{-1}(\mathcal{C}) \) co-bound an annulus in \( \Sigma' \) if and only if their \( f \)-images are the same, i.e., arbitrarily large annulus in \( \Sigma' \) co-bound by two components of \( f^{-1}(\mathcal{C}) \) is impossible. So, considering all these “outermost annuli” simultaneously, we complete the goal, as stated in the beginning. See Theorem 3.5.3.

1.2.3 Showing \( f \) is a degree \( \pm 1 \) map (see Theorem 3.6.3.1)

To rule out the possibility that \( f^{-1}(\mathcal{C}) \) is empty, where \( \mathcal{C} \) is a component of \( \mathcal{C} \), we prove \( \text{deg}(f) = \pm 1 \); this is because \( \text{deg}(f) \) remains the same after any proper homotopy of \( f \), and a map of non-zero degree is surjective; see Lemma 3.6.4.1 and Lemma 3.6.4.3. Our aim is to properly homotope \( f \) to obtain a closed disk \( D \subseteq \Sigma \) so that \( f|_{f^{-1}(D)} \to D \) becomes a homeomorphism, and thus we show \( \text{deg}(f) = \pm 1 \); see Theorem 2.6.1. The argument is based on finding a smoothly embedded finite-type bordered surface \( S \) in \( \Sigma \) such that for each component \( c \) of \( \partial S \); we have \( f^{-1}(c) \neq \varnothing \), even after any proper homotopy of \( f \). Depending on the nature of \( S \), we consider two cases.

1. (1) If \( \Sigma \) is either an infinite-type surface or any \( S_{g,0,p} \) with high complexity (i.e., \( g + p \geq 4 \) or \( p \geq 6 \)), then using \( \pi_1 \)-surjectivity of \( f \), we can choose \( S \) as a smoothly embedded pair of pants in \( \Sigma \) such that \( \Sigma \setminus S \) has at least two components and every component of \( \Sigma \setminus S \) has a non-abelian fundamental
group; see Lemma 3.6.1.2 and Lemma 3.6.1.4. Properly homotope \( f \) so that it becomes transverse to \( \partial S \). Then remove unnecessary components from the transversal pre-image \( f^{-1}(\partial S) \), i.e., after a proper homotopy, we may assume \( f \mid f^{-1}(c) \to c \) is a homeomorphism for each component \( c \) of \( \partial S \).

Now, since \( f \) is \( \pi_1 \)-injective, by the rigidity of pair of pants (see Theorem 3.6.1.9), after a proper homotopy, one can show that \( f \mid f^{-1}(S) \to S \) is a homeomorphism; see Lemma 3.6.1.10. Therefore, the required \( \mathcal{D} \) can be any disk in \( \text{int}(S) \).

(2) If \( \Sigma \) is a finite-type surface, then we choose a smoothly embedded punctured disk in \( \Sigma \) as \( S \) so that the puncture of \( S \) is an end \( e \in \text{Im}(\text{Ends}(f)) \subseteq \text{Ends}(\Sigma) \). By Theorem 3.6.2.1, it means every deleted neighborhood of \( e \) in \( \Sigma \) intersects \( \text{Im}(f) \), even after any proper homotopy of \( f \). Now, properly homotope \( f \) so that it becomes transverse to \( \partial S \). Then remove unnecessary components from the transversal pre-image \( f^{-1}(\partial S) \), i.e., after a proper homotopy, we may assume \( f \mid f^{-1}(\partial S) \to \partial S \) is a homeomorphism (as \( \Sigma \neq S^1 \times \mathbb{R} \), the fundamental group of \( \Sigma \setminus S \) is non-abelian; and so \( \pi_1 \)-surjectivity of \( f \) says \( f^{-1}(\partial S) \neq \emptyset \), even after any proper homotopy of \( f \)). Now, since \( f \) is \( \pi_1 \)-injective, by the proper rigidity of the punctured disk (see Theorem 3.6.2.4), after a proper homotopy, one can show that \( f \mid f^{-1}(S) \to S \) is a homeomorphism; see Lemma 3.6.2.3. Therefore, the required \( \mathcal{D} \) can be any disk in \( \text{int}(S) \).

### 1.2.4 Inverse decomposition

By the last three parts, after a proper homotopy, removing unnecessary components from the transversal pre-image \( f^{-1}(\mathcal{C}) \), we may assume that \( f \mid f^{-1}(\mathcal{C}) \to \mathcal{C} \) is a homeomorphism for each component \( \mathcal{C} \) of \( \mathcal{C} \). Thus, \( \mathcal{C} \) and \( f^{-1}(\mathcal{C}) \) decompose \( \Sigma \), \( \Sigma' \), respectively; and there is a shape-preserving bijective-correspondence between these two collections of complementary components (see Lemma 3.6.1.10 and Lemma 3.6.2.3). On each complementary component, apply either the rigidity of compact bordered surfaces (see Theorem 3.6.1.9) or the proper rigidity of the punctured disk (see Theorem 3.6.2.4). Thus, we have a collection of boundary-relative proper homotopies so that by pasting them, a proper homotopy from \( f \) to a homeomorphism \( \Sigma' \to \Sigma \) can be constructed; see the proof of Theorem 1 in Section 4.

### 2 Background

#### 2.1 Conventions

A **bordered surface** (resp. **surface**) is a connected, orientable two-dimensional manifold with a non-empty (resp. an empty) boundary. For integers \( g \geq 0 \), \( b \geq 0 \), \( p \geq 0 \), denote the 2-manifold of genus \( g \) with \( b \) boundary components by \( S_{g,b} \); and let \( S_{g,b,p} \) be the 2-manifold after removing \( p \) points from \( \text{int}(S_{g,b}) \). Note that for a manifold \( M \), we use \( \text{int}(M) \) to denote the interior of \( M \). Sometimes \( S_{0,0}, S_{0,2}, S_{0,3}, S_{1,2} \), and \( S_{0,1,1} \) will be called a disk, an annulus, a pair of pants, a two-holed torus, and a punctured disk, respectively.

We say a connected 2-manifold with or without boundary is of **infinite-type** if its fundamental group is not finitely generated; otherwise, we say it is of **finite-type**.

#### 2.2 Simple closed curves on two-manifolds

**Definition 2.2.1** Let \( S \) be a connected, orientable two-dimensional manifold with or without boundary. A **circle** (resp. **smoothly embedded circle**) on \( S \) is the image of an embedding (resp. a smooth embedding) of \( S^1 \) into \( S \). We say a circle \( \mathcal{C} \) on \( S \) is a **trivial circle** if there is an embedded disk \( \mathcal{D} \) in \( S \) such that \( \partial \mathcal{D} = \mathcal{C} \); and, we say a circle \( \mathcal{C} \) on \( S \) is a **primitive circle** if it is not a trivial circle.
The following theorem justifies naming a non-disk bounding circle as a primitive circle: a primitive circle represents a primitive element of the fundamental group. Recall that an element $g$ of a group $G$ is primitive if there does not exist any $h \in G$ so that $g = h^k$, where $|k| > 1$.

**Theorem 2.2.2** [12, Theorems 1.7. and 4.2.] Let $S$ be a connected, orientable two-dimensional manifold with or without boundary. Let $C$ be a primitive circle on $S$, and let $f : S^1 \hookrightarrow S$ be an embedding with $f(S^1) = C$. Then $[f] \in \pi_1(S)$ is a primitive element. In particular, $[f]$ is a non-trivial element of $\pi_1(S)$.

Recall that for a path-connected space $X$, there is a bijective correspondence between the set of all conjugacy classes of $\pi_1(X, \ast)$ and the set of all free homotopy classes of maps $S^1 \to X$. The next theorem says that two pairwise disjoint freely homotopic primitive circles on a two-manifold co-bound an annulus.

**Theorem 2.2.3** [12, Lemma 2.4.] Let $S$ be a connected, orientable two-dimensional manifold with or without boundary. Let $\ell_0, \ell_1 : S^1 \hookrightarrow S$ be two embeddings such that $\ell_0(S^1)$ is a smoothly embedded submanifold of $S$ and $\ell_0(S^1) \cap \ell_1(S^1) = \emptyset$. If $\ell_0$ and $\ell_1$ represent the same non-trivial conjugacy class in $\pi_1(S, \ast)$, then there is a embedding $\mathcal{L} : S^1 \times [0, 1] \hookrightarrow S$ so that $\mathcal{L}(-, 0) = \ell_0$ and $\mathcal{L}(-, 1) = \ell_1$.

### 2.3 Ends of spaces

Let $X$ be a connected, separable, locally compact, locally connected Hausdorff ANR (absolute neighborhood retract) space. For example, $X$ can be any connected topological manifold. We say $X$ admits an **efficient exhaustion by compacta** if there is a nested sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact, connected subsets of $X$ such that $\bigcup_i K_i = X$, $K_i \subseteq \text{int}(K_{i+1})$, $\cap_i (X \setminus K_i) = \emptyset$, and closure of each component of any $X \setminus K_i$ is non-compact. For the existence of efficient exhaustion of $X$ by compacta, see [21, Exercise 3.3.4].

Let $\text{Ends}(X)$ be the set of all sequences $(V_1, V_2, \ldots)$, where $V_i$ is a component of $X \setminus K_i$ and $V_1 \supseteq V_2 \supseteq \cdots$. Give $X^\uparrow := X \cup \text{Ends}(X)$ with the topology generated by the basis consisting of all open subsets of $X$, and all sets $V_i^\uparrow$, where

$$V_i^\uparrow := V_i \cup \{(V_i^1, V_2^2, \ldots) \in \text{Ends}(X) | V_i^j = V_i\}.$$ 

Then $X^\uparrow$ is separable, compact, and metrizable such that $X$ is an open dense subset of $X^\uparrow$; it is known as the **Freudenthal compactification** of $X$ (recall that we say a space $X_e$ is a **compactification** of $X$ if $X_e$ is compact Hausdorff space, and $X$ is a dense subset of $X_e$). The subspace $\text{Ends}(X)$ of $X^\uparrow$ is a totally-disconnected space; hence $\text{Ends}(X)$ is a closed subset of the Cantor set.

The Freudenthal compactification **dominates** any other compactification: If $\widetilde{X}$ is a compactification of $X$ such that $\widetilde{X} \setminus X$ is totally-disconnected, then there exists a map $f : X^\uparrow \to \widetilde{X}$ extending $\text{Id}_X$.

Also, the Freudenthal compactification is unique in the following sense: If $X^{\uparrow\uparrow}$ is a compactification of $X$ such that $X^{\uparrow\uparrow} \setminus X$ is totally-disconnected and $X^{\uparrow\uparrow}$ dominates any other compactification, then there exists a homeomorphism $X^{\uparrow\uparrow} \to X^\uparrow$ extending $\text{Id}_X$; see [20, Theorem 3.1]. Thus, the definition of $\text{Ends}(X)$ is independent of the choice of efficient exhaustion of $X$ by compacta.

Now, we consider a relationship between Ends and proper maps.

**Proposition 2.3.1** [21, Proposition 3.3.12] Let $X$ and $Y$ be two connected, separable, locally compact, locally connected Hausdorff ANRs. Then we have the following:

1. Every proper map $f : X \to Y$ induces a map $\text{Ends}(f) : \text{Ends}(X) \to \text{Ends}(Y)$ that can be used to extend $f : X \to Y$ to a map $f^{\uparrow\uparrow} : X^{\uparrow\uparrow} \to Y^{\uparrow\uparrow}$ between the Freudenthal compactifications.
(2) If two proper maps \( f_0, f_1 : X \to Y \) are properly homotopic, then \( \text{Ends}(f_0) = \text{Ends}(f_1) \).

(3) If \( f : X \to Y \) is a proper homotopy equivalence, then \( \text{Ends}(f) : \text{Ends}(X) \to \text{Ends}(Y) \) is a homeomorphism.

More about the ends of spaces and proper homotopy can be found in [27] and [23].

2.4 KerékJártó’s classification theorem and Ian Richard’s representation theorem

Let \( \Sigma \) be a non-compact surface with an efficient exhaustion \( \{ K_i \}_1^\infty \). Let \( e := (V_1, V_2, \ldots) \in \text{Ends}(\Sigma) \) be an end, where \( V_i \) is a component of \( X \setminus K_i \). We say \( e \) is a planar end if \( V_i \) is embeddable in \( \mathbb{R}^2 \) for some positive integer \( i \). An end is said to be non-planar if it is not planar. Denote the subspace of \( \text{Ends}(\Sigma) \) consisting of all planar (resp. non-planar) ends by \( \text{Ends}_{\text{pl}}(\Sigma) \) (resp. \( \text{Ends}_{\text{np}}(\Sigma) \)). Note that \( \text{Ends}_{\text{pl}}(\Sigma) \) is an open subset of \( \text{Ends}(\Sigma) \). Define the genus of \( \Sigma \) as \( g(\Sigma) := \sup g(S) \), where \( S \) is a compact bordered subsurface of \( \Sigma \). Therefore, the genus counts the number of handles of a surface, i.e., the number of embedded copies of \( S_{1,1} \) in a surface, which may be any non-negative integer or countably infinite.

**Theorem 2.4.1** (KerékJártó’s classification of non-compact surfaces [29, Theorem 1]) Let \( \Sigma \) and \( \Sigma' \) be non-compact surfaces of genus \( g, g' \), respectively. Then \( \Sigma \) is homeomorphic to \( \Sigma' \) if and only if \( g = g' \) and there is homeomorphism \( \varphi : \text{Ends}(\Sigma) \to \text{Ends}(\Sigma') \) with \( \varphi(\text{Ends}_{\text{pl}}(\Sigma)) = \text{Ends}_{\text{pl}}(\Sigma') \).

**Theorem 2.4.2** (Realization of ends and representation of a non-compact surface [29, Theorems 2, 3]) Let \( \mathcal{E}_{\text{np}} \subseteq \mathcal{E} \) be two closed totally-disconnected subsets of \( S^1 \), and let \( \mathcal{G} \) be an at most countable set with the following properties: \( \mathcal{E} \neq \emptyset \) and \( \mathcal{E}_{\text{np}} \neq \emptyset \) if and only if \( \mathcal{G} \) is infinite. Define \( \mathbb{D} := \{ z \in \mathbb{C} : 0 \leq |z| \leq 1 \} \).

Then there exists a pairwise disjoint collection \( \{ D_i : i \in \mathcal{G} \} \) of disks in \( \text{int}(\mathbb{D}) \) such that a point \( p \in \mathbb{D} \) is an element of \( \mathcal{E}_{\text{np}} \) if and only if every neighborhood of \( p \) in \( \mathbb{D} \) contains infinitely many elements of \( \{ D_i : i \in \mathcal{G} \} \). Moreover, \( S := (\mathbb{D} \setminus \mathcal{E'}) \setminus \cup_{i \in \mathcal{G}} \text{int}(D_i) \) is a non-compact bordered surface, and

\[
DS := \frac{(S \times 0) \cup (S \times 1)}{(p, 0) \sim (p, 1), p \in \partial S}
\]

is a \( |\mathcal{G}| \)-genus non-compact surface with \( \text{Ends}(DS) \cong \mathcal{E} \) and \( \text{Ends}_{\text{np}}(DS) \cong \mathcal{E}_{\text{np}} \).

Thus, given any non-compact surface \( \Sigma \), in this procedure, if we assume \( \mathcal{E}_{\text{np}} \subseteq \mathcal{E} \) is homeomorphic to the pair \( \text{Ends}_{\text{np}}(\Sigma) \subseteq \text{Ends}(\Sigma) \), and \( |\mathcal{G}| \) is equal to \( g(\Sigma) \); then \( DS \cong \Sigma \), by Theorem 2.4.1.

**Remark 2.4.3** The classification of non-compact bordered surfaces is also possible: When the boundary is compact, it follows from Theorem 2.4.1 together with [34, Proposition A.3.]. When each boundary component is compact, this follows from [4] (based on the classification of their interiors) or [34, Theorem A.7] (based on the classification of non-compact surfaces obtained from gluing a disk along each boundary component). For arbitrary boundary, see [6, Theorem 2.2].

2.5 Goldman’s inductive procedure of constructing all non-compact surfaces

A non-compact surface \( \Sigma_{\text{std}} \) is said to be in standard form if it is built up from four building blocks, \( S_{0,1}, S_{0,2}, S_{0,3}, \) and \( S_{1,2} \), in the following inductive manner: Start with \( S_{0,1} \). Suppose the \( i \)-th step of the induction has already been done. Let \( K_i \) be the compact bordered subsurface of \( \Sigma_{\text{std}} \) after the \( i \)-th step of induction. In particular, \( K_1 \cong S_{0,1} \). Now, to obtain \( K_{i+1} \) from \( K_i \), consider one of the last three building blocks, say \( S \) (i.e., \( S \) is homeomorphic to either \( S_{0,2}, S_{0,3}, \) or \( S_{1,2} \)); finally, suitably identify one boundary circle of \( S \) with a boundary circle of \( K_i \). See Figure 1.
Theorem 2.5.1  [19, Section 2.6.] and [35, Page 173] Let $\Sigma$ be a non-compact surface. Then $\Sigma$ is homeomorphic to a non-compact surface $\Sigma_{\text{std}}$ in standard form. Thus every non-compact surface is homeomorphic to a non-compact surface constructed using an inductive procedure as above, though two non-compact surfaces obtained from two different inductive procedures may be homeomorphic.

Fig. 1: Inductive construction of any non-compact surface and its spine uses four compact bordered surfaces: disk, annulus, pair of pants, and torus with two holes.

Theorem 2.5.2  [19, Section 2.6. and Section 7.3.] The graph in Figure 1 consisting of blue straight line segments and red circles is a deformation retract of the surface $\Sigma$. Thus, $\Sigma$ is homotopy equivalent to the wedge of at most countably many circles. In particular, $\pi_1(\Sigma)$ is free.

Remark 2.5.3  An alternative way of proving the last two sentences of Theorem 2.5.2 is given in [31, Lemma 3.2.2].

2.6 The degree of a proper map

We use singular cohomology with compact support to define the notion of the degree of a proper map. Recall that for a topological manifold $X$, the $r$-th singular cohomology with compact support $H^r_c(X, \partial X; \mathbb{Z})$ is equal to the direct limit $\lim_{\rightarrow} \mathcal{H}^r(X, \partial X \cup (X \setminus K); \mathbb{Z})$, where $K$ is a compact subset of $X$ and various maps to
define this direct system are inclusion induced maps. It is worth noting that when $X$ is compact topological manifold, $H^r_c(X, \partial X; \mathbb{Z}) = H^r(X, \partial X; \mathbb{Z})$ for all $r$.

Let $X$ and $Y$ be two topological manifolds. If $f : X \to Y$ is a proper map with $f(\partial X) \subseteq \partial Y$, then for each $r$, $f$ induces a map $H^r_c(f) : H^r_c(Y, \partial Y; \mathbb{Z}) \to H^r_c(X, \partial X; \mathbb{Z})$ so that $H^r_c(f)$ becomes a functor in the following sense: the induced map of the identity is the identity, and the induced map of a (well-defined) composition of two proper maps (each of which sends boundary into boundary) is the composition of their induced maps. Moreover, if $\mathcal{H} : X \times [0, 1] \to Y$ is a proper homotopy such that $\mathcal{H}(\partial X, t) \subseteq \partial Y$ for each $t \in [0, 1]$, then $H^r_c(\mathcal{H}(-, 0)) = H^r_c(\mathcal{H}(-, 1))$ for all $r$. For more details, see [33, Pages 320, 322, 323, 339, 341].

Let $M$ be a connected, orientable, topological $n$-manifold. Then $H^n_c(M, \partial M; \mathbb{Z})$ is an infinite cyclic group; see [33, Page 342]. If we choose an orientation of $M$ (i.e., $M$ is oriented), then there exists a unique element $[M] \in H^n_c(M, \partial M; \mathbb{Z})$ such that the following hold: (1) $[M]$ generates $H^n_c(M, \partial M; \mathbb{Z})$, and (2) for each $x \in M \setminus \partial M$, the unique generator of $H^n(M, M \setminus x; \mathbb{Z})$, which comes from the chosen orientation of $M$, is sent to $[M]$ by the inclusion-induced isomorphism $H^n(M, M \setminus x; \mathbb{Z}) \to H^n_c(M, \partial M; \mathbb{Z})$; see [13, Proof of Lemma 2.1]. Thus, if $f : M \to N$ is a proper map between two connected, oriented, topological $n$-manifolds with $f(\partial M) \subseteq \partial N$, then the (compactly supported cohomological) degree of $f$ is the unique integer $\text{deg}(f)$ defined as follows: $H^n_c(f)([M]) = \text{deg}(f) \cdot [M]$.

By the previous two paragraphs, we have the following: (i) When manifolds are compact, the notion of compactly supported cohomological degree agrees with the notion of the usual degree defined by singular cohomology. (ii) The degree is proper homotopy invariant: If $f, g : M \to N$ are proper maps between two connected, oriented, topological $n$-manifolds with $f(\partial M) \cup g(\partial M) \subseteq \partial N$ such that there is a proper homotopy $\mathcal{H} : M \times [0, 1] \to N$ with $\mathcal{H}(\partial M \times [0, 1]) \subseteq \partial N$, then $\text{deg}(f) = \text{deg}(g)$. (iii) The degree is multiplicative: The degree of the (well-defined) composition of two proper maps (each of which sends boundary into boundary) is the product of their degrees.

Therefore, the degree of a proper homotopy equivalence between two oriented, connected, boundaryless $n$-manifolds is $\pm 1$ due to (ii) and (iii) above. We use the following well-known characterizations of a map of degree $\pm 1$. In the below two theorems, $D$ is a disk in a smooth $n$-manifold $X$ means $D$ is the image of $\{z \in \mathbb{R}^n : |z| \leq 1\}$ under a smooth embedding $\{z \in \mathbb{R}^n : |z| \leq 2\} \to X$.

**Theorem 2.6.1** [13, Lemma 2.1b.] Let $f : M \to N$ be a proper map between two connected, oriented, smooth manifolds of the same dimension such that $f^{-1}(\partial N) = \partial M$. Suppose for a disk $D$ in $\text{int}(N), f^{-1}(D)$ is a disk in $\text{int}(M)$ such that $f$ maps $f^{-1}(D)$ homeomorphically onto $D$. Then $\text{deg}(f) = +1$ or $-1$ according as $f|f^{-1}(D) \to D$ is orientation-preserving or orientation-reversing.

The following theorem is due to Hopf, which says that for a degree one map, we can achieve such a disk with nice properties, as mentioned in Theorem 2.6.1 above, after a proper homotopy.

**Theorem 2.6.2** [13, Theorems 3.1 and 4.1] Let $f : M \to N$ be a proper map between two connected, oriented, smooth manifolds of the same dimension such that $f^{-1}(\partial N) \subseteq \partial M$. Suppose $\text{deg}(f) = \pm 1$. Then there is a proper map $g : M \to N$ with $g(\partial M) \subseteq \partial N$ and a homotopy $\mathcal{H} : M \times [0, 1] \to N$ with the following properties:

- There exists a compact subset $K \subseteq \text{int}(M)$ such that $\mathcal{H}(x, t) = f(x)$ for all $(x, t) \in (M \setminus K) \times [0, 1]$. In particular, $\mathcal{H}$ is a proper homotopy and $\mathcal{H}(\partial M, t) \subseteq \partial N$ for all $t \in [0, 1]$.
- There exists a disk $D \subseteq \text{int}(N)$ such that $g^{-1}(D)$ is a disk in $\text{int}(M)$ and $g|g^{-1}(D) \to D$ is a homeomorphism.
The theorem below is due to Olum, which roughly says that when there is a degree one map, the domain is more massive than the co-domain.

**Theorem 2.6.3** [13, Corollary 3.4] Let \( f : M \to N \) be a proper map between two connected, oriented, topological manifolds of the same dimension such that \( f(\partial M) \subseteq \partial N \). If \( \deg(f) = \pm 1 \), then \( \pi_1(f) : \pi_1(M) \to \pi_1(N) \) is surjective.

### 3 Ingredients For Proving Theorem 1

#### 3.1 Decomposition of a non-compact surface into pair of pants and punctured disks

Every compact surface of genus \( g \geq 2 \) is the union (with pairwise disjoint interiors) of \( 2g - 2 \) many copies of the pair of pants. But the same thing doesn’t happen for non-compact surfaces; for example, the thrice punctured sphere is not a union (with pairwise disjoint interiors) of copies of the pair of pants; we need copies of the punctured disk. The main aim of this section is to prove that every non-compact surface, except the plane and the once punctured torus, decomposes into copies of the pair of pants and copies of the punctured disk when we cut it along a collection of circles, where each circle of this collection has an open neighborhood that does not intersect with any other circles of this collection.

First, we define a few terminologies.

**Definition 3.1.1** Let \( X \) be a space, and let \( \{X_\alpha : \alpha \in \mathcal{I}\} \) be a collection of subsets of \( X \). We say \( \{X_\alpha : \alpha \in \mathcal{I}\} \) is a *locally finite collection* and write \( X_\alpha \to \infty \) if, for each compact subset \( K \) of \( X \), \( X_\alpha \cap K = \emptyset \) for all but finitely many \( \alpha \in \mathcal{I} \).

**Definition 3.1.2** Let \( \mathcal{A} \) be a pairwise disjoint collection of smoothly embedded circles on a surface \( \Sigma \). We say \( \mathcal{A} \) is a *locally finite curve system* (in short, LFCS) on \( \Sigma \) if \( \mathcal{A} \) is a locally finite collection.

**Remark 3.1.3** Let \( \mathcal{A} \) be an LFCS on a surface \( \Sigma \). Notice that \( \cup \mathcal{A} \) (i.e., the union of all elements of \( \mathcal{A} \)) is a closed subset of \( \Sigma \) as well as a smoothly embedded submanifold of \( \Sigma \) so that the set of all components of \( \cup \mathcal{A} \) is \( \mathcal{A} \). But to avoid too many notations, whenever needed, we will think of \( \mathcal{A} \) and \( \cup \mathcal{A} \) as the same without any harm.

**Definition 3.1.4** Let \( \mathcal{A} \) be an LFCS on a surface \( \Sigma \). Suppose there exists an at most countable collection \( \{\Sigma_n\} \) of bordered sub-surfaces of \( \Sigma \) such that the following hold: (1) each \( \Sigma_n \) is a closed subset of \( \Sigma \); (2) \( \text{int}(\Sigma_n) \cap \text{int}(\Sigma_m) = \emptyset \) if \( n \neq m \); (3) \( \bigcup_n \Sigma_n = \Sigma \); and (4) \( \bigcup_n \partial \Sigma_n = \cup \mathcal{A} \). In this case, we say \( \mathcal{A} \) *decomposes \( \Sigma \) into bordered sub-surfaces*, where complementary components are \( \{\Sigma_n\} \). Also, we call each component of \( \mathcal{A} \) a decomposition circle.

The following theorem asserts that any non-compact surface other than the plane has a decomposition, where each complementary part is either a pair of pants, a one-holed torus, or a punctured disk. This way of decomposition of the co-domain of a pseudo proper homotopy equivalence will be used in all cases.

**Theorem 3.1.5** Let \( \Sigma \) be a non-compact surface not homeomorphic to \( \mathbb{R}^2 \). Then there is an LFCS \( \mathcal{C} \) on \( \Sigma \) such that \( \mathcal{C} \) decomposes \( \Sigma \) into bordered sub-surfaces, and a complementary component of this decomposition is homeomorphic to either \( S_{1,1} \) (used at most once), \( S_{0,3} \), or \( S_{0,1,1} \).
**Proof.** Enough to find a collection \( \{ \Sigma_n \} \) of bordered sub-surfaces of \( \Sigma \) with four properties, as mentioned in Definition 3.1.4, so that each \( \Sigma_n \) is homeomorphic to either \( S_{0,3} \), \( S_{1,1} \), or \( S_{0,1,1} \). For that, consider an inductive construction of \( \Sigma \); see Theorem 2.5.1. Now, a finite sequence of annuli, when added to the compact bordered surface used just before it, can be ignored. Thus, we may assume \( S_{0,3} \) or \( S_{1,2} \) is used after \( S_{0,1} \) without loss of generality because of \( \Sigma \not\cong \mathbb{R}^2 \), and hence pushing \( S_{0,1} \) into \( \text{int}(S_{0,3}) \) or \( \text{int}(S_{1,2}) \), we end up with \( S_{0,2} \) (which can be ignored) or \( S_{1,1} \). Now, the proof will be completed by observing the following: \( S_{1,2} \) can be decomposed into two copies of \( S_{0,3} \), and \( S_{0,1,1} \) is the union (with pairwise disjoint interiors) of countably many copies of \( S_{0,2} \).

**Remark 3.1.6** A statement closely related to Theorem 3.1.5 is in [2, Theorem 1.1.], which says that “every surface except for the sphere, the plane, and the torus is the union (with pairwise disjoint interiors) of copies of the pair of pants and copies of the punctured disk”. But due to the part (4) of Definition 3.1.4, if we want that any complementary component is homeomorphic to only either \( S_{0,3} \) or \( S_{0,1,1} \), then \( \Sigma \not\cong S_{1,0,1} \) also needs to consider; see Figure 2 and Theorem 3.1.7 below.

**Theorem 3.1.7** Let \( \Sigma \) be a non-compact surface that is not homeomorphic to either \( \mathbb{R}^2 \) or \( S_{1,0,1} \). Then there is an LFCS \( \mathcal{C}' \) on \( \Sigma \) such that \( \mathcal{C}' \) decomposes \( \Sigma \) into bordered sub-surfaces, and a complementary component of this decomposition is homeomorphic to either \( S_{0,3} \) or \( S_{0,1,1} \).

![Fig. 2: On the top: Decomposition of Loch Ness Monster into countably infinitely many copies of the pair of pants. At the bottom: Decomposition of \( S_{3,0,1} \) into five copies of pair of pants and a copy of the punctured disk.](image)

**Proof.** Enough to find a collection \( \{ \Sigma_n \} \) of bordered sub-surfaces of \( \Sigma \) with four properties, as mentioned in Definition 3.1.4, so that each \( \Sigma_n \) is homeomorphic to either \( S_{0,3} \) or \( S_{0,1,1} \). For that, consider an inductive construction of \( \Sigma \); see Theorem 2.5.1. We will divide the whole proof into two cases, depending on whether \( \Sigma \) has at least two ends.

At first, suppose the number of ends of \( \Sigma \) is at least two. Now, the definition of the space of ends tells us that we need to use at least one pair of pants in the inductive construction of \( \Sigma \). By Lemma 3.1.8, we may assume that in this inductive construction, a pair of pants is used just after the disk. Now, an argument similar to before (see the proof of Theorem 3.1.5) concludes this case.
Next, consider the case when the number of ends of $\Sigma$ is precisely one. That is, $\Sigma$ can be either Loch Ness Monster (the infinite genus surface with one end) or $S_{g,0,1}$ with $g \geq 2$. Loch Ness Monster decomposes into countably infinitely many copies of the pair of pants, and $S_{g,0,1}$ with $g \geq 2$ decomposes into $2g - 1$ many copies of the pair of pants and a copy of the punctured disk. See Figure 2.

To prove Theorem 3.1.7, we used Lemma 3.1.8 below, which says that in an inductive construction of a non-compact surface, interchanging the positions of the compact bordered surfaces used in the first few inductive steps doesn’t change the homeomorphism type, and its proof is based on the observation that the portions outside compact subsets determine the space of ends.

**Lemma 3.1.8** Let $\Sigma$ be a non-compact surface with some inductive construction $\mathcal{I}$. Denote the compact bordered subsurface of $\Sigma$ after the $i$-th step of $\mathcal{I}$ by $K_i$. Suppose $\{B_1, \ldots, B_n :$ each $B_\ell$ is homeomorphic to either $S_{0,2}$, $S_{0,3}$, or $S_{1,2}\}$ is a finite collection of compact bordered surfaces such that $B_\ell$ is used to construct $K_{i_\ell+1}$ from $K_{i_\ell}$ for each $\ell = 1, \ldots, n$. Then there exists a non-compact surface $\Sigma'$ with an inductive construction $\mathcal{I}'$ such that $\Sigma' \approx \Sigma$ and $B_\ell$ is used to construct $K'_{i_\ell+1}$ from $K'_{i_\ell}$ for each $\ell = 1, \ldots, n$; where $K'_{i_\ell}$ denotes the compact bordered subsurface of $\Sigma$ after the $i$-th step of $\mathcal{I}'$.

![Figure 3](image-url)

**Proof.** Let $n_0$ be a positive integer such that $K_{n_0}$ contains each $B_\ell$. Define $S := \Sigma \setminus \text{int}(K_{n_0})$. Thus $S$ is a bordered subsurface of $\Sigma$ with $\partial S = \partial K_{n_0}$. Now, consider all copies of different building blocks used up to the $n_0$-th step of $\mathcal{I}$, and inside $K_{n_0}$ interchange them so that $B_1, \ldots, B_n$ comes just after the initial disk $K_1$ one by one following the increasing order of their indices. Denote the resultant of this interchange process by $K'_{n_0}$. So $K_{n_0} \approx K'_{n_0}$ as $g(K_{n_0}) = g(K'_{n_0})$ and $\partial K_{n_0} \approx \partial K'_{n_0}$. Define a non-compact surface $\Sigma'$ as $\Sigma' := K'_{n_0} \cup \partial K'_{n_0} \approx \partial S$. Therefore, $\Sigma \setminus K_{n_0} = \text{int}(S) = \Sigma' \setminus K'_{n_0}$ (notice that we are thinking $S$ as a subset of $\Sigma'$ using the obvious embedding $S \hookrightarrow \Sigma'$).

Choose an inductive construction $\mathcal{I}'_{\leq n_0}$ of $K'_{n_0}$ such that $i$-th element of the ordered sequence $K_1, B_1, \ldots, B_\ell$ is used in the $i$-th step of $\mathcal{I}'_{\leq n_0}$. Also, $\mathcal{I}$ gives a truncated inductive construction $\mathcal{I}|S$ on $S$ starting from...
the \((n_0 + 1)\)-th step. Now, \(I'_{\leq n_0}\) followed by \(I|S\) together gives an inductive construction \(I'\) of \(\Sigma'\). Roughly it means \(I'\) is the same as the inductive construction of \(\Sigma\), except for the first few steps. Denote the compact bordered subsurface of \(\Sigma'\) after the \(i\)-th step of \(I'\) by \(K'_i\). To complete the proof, we show \(\Sigma' \cong \Sigma\) using Theorem 2.4.1.

Consider the efficient exhaustion \(\{K_i\}\) (resp. \(\{K'_i\}\)) of \(\Sigma\) (resp. \(\Sigma'\)) by compacta to define \(\text{Ends}(\Sigma)\) (resp. \(\text{Ends}(\Sigma')\)). Recall that the space of ends remains the same up to homeomorphism even if we choose a different efficient exhaustion by compacta; see Section 2.3. By \(\Sigma \setminus K_{n_0} = \text{int}(S) = \Sigma' \setminus K'_{n_0}\), for every sequence \((V_1, V_2, \ldots) \in \text{Ends}(\Sigma)\), there exists a unique sequence \((V'_1, V'_2, \ldots) \in \text{Ends}(\Sigma')\) such that \(V_i = V'_i\) for all integer \(i \geq n_0\), and conversely. Thus, there exists a homeomorphism \(\varphi: \text{Ends}(\Sigma) \to \text{Ends}(\Sigma')\) with \(\varphi(\text{Ends}_{np}(\Sigma)) = \text{Ends}_{np}(\Sigma')\). Also, \(\Sigma \setminus K_{n_0} = \text{int}(S) = \Sigma' \setminus K'_{n_0}\) and \(K_{n_0} \cong K'_{n_0}\) together imply \(g(\Sigma) = g(\Sigma')\). Therefore, \(\Sigma' \cong \Sigma\) by Theorem 2.4.1.

The spine construction of Goldman’s inductive procedure shows that every non-compact surface \(\Sigma\) (\(\Sigma\) may be of infinite-type) is the interior of a bordered surface: consider the graph \(G\) consisting of blue straight line segments and red circles, as given in Figure 1. Any thickening [19, Definition 7.2.] of \(G\) in \(\Sigma\) is the interior of a bordered subsurface \(S\) of \(\Sigma\). Now, [19, Corollary 7.2. and section 7.3.] says that \(\text{int}(S) \cong \Sigma\). When \(\Sigma\) is of finite-type, we prove the same thing differently in the following theorem.

**Theorem 3.1.9** A non-compact finite-type surface is the interior of a compact bordered surface. In particular, if a non-compact surface has infinite cyclic (resp. trivial) fundamental group, then it is homeomorphic to \(S^1 \times \mathbb{R}\) (resp. \(\mathbb{R}^2\)).

**Proof.** Let \(\Sigma\) be a finite-type non-compact surface. Consider an inductive construction of \(\Sigma\); see Theorem 2.5.1. Since \(\pi_1(\Sigma)\) is finitely generated, Theorem 2.5.2 says that \(\Sigma\) is homotopy equivalent to \(\bigvee_{2r+s} S^1\), where in this inductive construction, \(r \in \mathbb{N}\) is the total number of copies of \(S_{1,2}\), and \(s \in \mathbb{N}\) is the total number of copies of \(S_{0,3}\); see Figure 1. Thus there is an integer \(n\) such that \(\Sigma \setminus K_n\) (where \(K_n\) is the compact bordered subsurface of \(\Sigma\) after \(n\)-th inductive step) is a finite collection of punctured disks. Now, \(g(\Sigma) = g(\text{int}(K_n))\). Also, each end of \(\Sigma\) (resp. \(\text{int}(K_n)\)) is planar, and the total number of ends of \(\Sigma\) (resp. \(\text{int}(K_n)\)) is the same as the number of components of \(\partial K_n\). By Theorem 2.4.1, \(\Sigma \cong \text{int}(K_n)\).

If \(\Sigma\) is a non-compact surface with the infinite-cyclic fundamental group, then any inductive construction of \(\Sigma\) contains no copy of \(S_{1,2}\) but precisely one copy of \(S_{0,3}\), i.e., \(\Sigma \cong S^1 \times \mathbb{R}\). Similarly, if \(\Sigma\) is a non-compact surface with the trivial fundamental group, then any inductive construction of \(\Sigma\) has no copy of \(S_{0,3}\) as well as no copy of \(S_{1,2}\), i.e., \(\Sigma \cong \mathbb{R}^2\).

The proposition below follows directly from Goldman’s inductive construction, so we quote it without proof. It says that an infinite-type surface has a finite genus only if it has infinitely many ends. On the other hand, Theorem 2.4.2 guarantees the existence of an infinite-type surface of the infinite genus with infinitely many ends.

**Proposition 3.1.10** A non-compact surface is of a finite genus if and only if the total number of copies of \(S_{1,2}\) used in any inductive construction of \(\Sigma\) is finite. Thus, if an infinite-type surface has a finite genus, then it must have infinitely many ends.

This section’s final fact (as promised in the introduction) says that the fundamental group alone can’t detect the homeomorphism type of an infinite-type surface.

**Proposition 3.1.11** Up to homotopy equivalence, there is exactly one infinite-type surface, but up to homeomorphism, there are \(2^{\aleph_0}\) many infinite-type surfaces.
for the first step, in each step of Goldman’s inductive procedure, we use either deg(∅, even if these proper homotopies start with a surjective proper map. A remedy for this is: Assume f is surjective; see Lemma 3.6.4.1 and Lemma 3.6.4.3. If f is a proper homotopy equivalence, then we don’t know (at least till this stage) whether f has a proper homotopy inverse or not (though it has a homotopy inverse). Later in Section 3.6, using π₁-bijictivity, we will show that most pseudo proper homotopy equivalence is a map of degree ±1.

The following theorem says that the transversal pre-image of an LFCS under a proper map is an LFCS.

**Theorem 3.2.3** Let f: Σ' → Σ be a smooth proper map between two non-compact surfaces, and let ℳ be an LFCS on Σ such that f −1 ℳ. Then for each component C of ℳ, either f −1(C) is empty or a pairwise disjoint finite collection of smoothly embedded circles on Σ'. Therefore, f −1(ℳ) is an LFCS on Σ'.

**Proof.** By the definition of transversality, f −1 ℳ implies f −1 C for each component C of ℳ. Thus f −1(C) is either empty or is a compact (since f is proper) one-dimensional boundaryless smoothly embedded submanifold of Σ'. Now, by classification of closed one-dimensional manifolds, we complete the first part.
Next, if possible, let \( K' \) be a compact subset of \( \Sigma' \) such that \( K' \) intersects infinitely many components of \( f^{-1}(\mathcal{A}) \). By the first part, it means the compact set \( f(K') \) intersects infinitely many components of \( \mathcal{A} \), which contradicts the fact that \( \mathcal{A} \) is a locally finite collection.

\[\square\]

### 3.3 Disk removal

Previously, as observed, after a proper homotopy, the number of components in the collection of transversal pre-images of all decomposition circles can be infinite, and many components (possibly infinitely many) of this collection, maybe trivial circles. Here, at first, our goal is to group all these trivial circles in terms of the size of the disk bounded by a trivial circle and then remove all groups of trivial circles simultaneously by a proper homotopy.

At first, our intended grouping requires a technical lemma, which asserts that on a non-simply connected surface, an LFCS consisting of concentric trivial circles doesn’t exist. Roughly it means, on a non-simply connected surface, arbitrarily large disks bounded by components of an LFCS don’t exist.

**Lemma 3.3.1** Let \( \Sigma \) be a surface, and let \( \mathcal{A} := \{ C_i : i \in \mathbb{N} \} \) be an LFCS on \( \Sigma \) such that for each \( i \) the circle \( C_i \) bounds a disk \( D_i \subset \Sigma \) with \( C_i \subset \text{int}(D_{i+1}) \). Then \( \Sigma \) is homeomorphic to \( \mathbb{R}^2 \).

**Proof.** At first, notice that \( \Sigma \) must be non-compact as \( \mathcal{A} \) is a locally finite, pairwise disjoint, infinite collection of circles. Using inductive construction (see **Theorem 2.5.1**), we have a sequence \( \{ S_j : j \in \mathbb{N} \} \) of compact bordered sub-surfaces of \( \Sigma \) such that \( \bigcup_j S_j = \Sigma \) and for each \( j \in \mathbb{N} \), \( S_j \subset \text{int}(S_{j+1}) \). Consider any \( p \in \Sigma \). So, there exists \( j_0 \in \mathbb{N} \) such that \( p \in S_{j_0} \) and \( S_{j_0} \cap (\bigcup_i C_i) \neq \emptyset \). Since \( \mathcal{A} \) is a locally finite collection, only finitely many components of \( \mathcal{A} \) intersect the compact set \( S_{j_0} \). Let \( C_{i_1}, ..., C_{i_t} \) be the only components of \( \mathcal{A} \) intersecting \( S_{j_0} \), where \( i_1 < \cdots < i_t \). Pick an integer \( i_0 > i_t \). Then \( C_{i_0} \cap S_{j_0} = \emptyset \). Now, since \( C_{i_0} \subset \text{int}(D_{j_0}) \), \( S_{j_0} \) is connected, and \( \Sigma \) is locally Euclidean, we can say that \( S_{j_0} \subseteq \text{int}(D_{j_0}) \). Thus, every point \( x \in \Sigma \) has an open neighborhood \( U_i \) in \( \Sigma \) such that \( U_i \subseteq D_i \) for some \( i \in \mathbb{N} \). Therefore, every loop on \( \Sigma \) is contained in a disk \( D_i \) for some large \( i \in \mathbb{N} \), i.e., \( \Sigma \) is simply-connected. By **Theorem 3.1.9**, \( \Sigma \cong \mathbb{R}^2 \).

The following lemma is the primary tool for showing that a homotopy is proper. It tells how a proper map can be properly homotoped so that it changes on infinitely many pairwise disjoint compact sets.

**Lemma 3.3.2** Let \( f : \Sigma' \to \Sigma \) be a proper map between two non-compact surfaces, and let \( \{ \Sigma'_n : n \in \mathbb{N} \} \) be a pairwise disjoint collection of compact bordered sub-surfaces of \( \Sigma' \). For each \( n \in \mathbb{N} \), suppose \( H_n : \Sigma'_n \times [0, 1] \to \Sigma \) is a homotopy relative to \( \partial \Sigma'_n \) such that \( H_n(-,0) = f|\Sigma'_n \) and \( \text{im}(H_n) \to \infty \). Then \( \mathcal{H} : \Sigma' \times [0, 1] \to \Sigma \) defined by

\[
\mathcal{H}(p, t) := \begin{cases} 
H_n(p, t) & \text{if } p \in \Sigma'_n \text{ and } t \in [0, 1], \\
 f(p) & \text{if } p \in \Sigma' \setminus (\bigcup_{n \in \mathbb{N}} \Sigma'_n) \text{ and } t \in [0, 1]
\end{cases}
\]

is a proper map.

**Proof.** Let \( \mathcal{K} \) be a compact subset of \( \Sigma \). By continuity of \( \mathcal{H} \), \( \mathcal{H}^{-1}(\mathcal{K}) \) is closed in \( \Sigma' \). Since \( \text{im}(H_n) \to \infty \), there exists \( n_0 \in \mathbb{N} \) such that \( \text{im}(H_n) \cap \mathcal{K} = \emptyset \) for all integers \( \ell \geq n_0 + 1 \). Now, \( f^{-1}(\mathcal{K}) \) is compact as \( f \) is proper. Also, the domain of each \( H_n \) is compact. Hence, the closed subset \( \mathcal{H}^{-1}(\mathcal{K}) \) of \( \Sigma' \) is contained in the compact set \( f^{-1}(\mathcal{K}) \cup \bigcup_{n=1}^{n_0} H_n^{-1}(\mathcal{K}) \). So, we are done. \[\square\]
To remove trivial components from the transversal pre-image of an LFCS with infinitely many components, we need to impose some conditions on this LFCS. One such preferred LFCS is given in Theorem 3.1.5. But for future use, not only this type of LFCS, we require other kinds of LFCS on the co-domain. So, here is the list of different preferred LFCS.

**Definition 3.3.3** Let \( \Sigma \) be a non-compact surface such that \( \Sigma \not\cong \mathbb{R}^2 \). Suppose, \( \mathcal{A} \) is a given LFCS on \( \Sigma \). We say \( \mathcal{A} \) is a preferred LFCS on \( \Sigma \) if either of the following happens: (i) \( \mathcal{A} \) is a finite collection of primitive circles on \( \Sigma \); (ii) \( \mathcal{A} \) decomposes \( \Sigma \) into bordered sub-surfaces, and a complementary component of this decomposition is homeomorphic to either \( S_{1,1}, S_{0,3}, S_{0,2}, \) or \( S_{0,1,1} \).

**Remark 3.3.4** The only use of case (i) of Definition 3.3.3 is in Section 3.6, where we consider the process of removing unnecessary circles from the transversal pre-image of the boundary of an essential pair of pants or an essential punctured disk. It is worth noting that by a finite LFCS, one can’t decompose an infinite-type surface into finite-type bordered surfaces.

In the theorem below, we construct a proper homotopy, which removes all trivial components keeping a neighborhood of each primitive component stationary from the transversal pre-image of a preferred LFCS. Recall that a homotopy \( H: X \times [0, 1] \to Y \) is said to be stationary on a subset \( A \) of \( X \) if \( H(a, t) = H(a, 0) \) for all \( (a, t) \in A \times [0, 1] \).

**Theorem 3.3.5** Let \( f: \Sigma' \to \Sigma \) be a smooth proper map between two non-compact surfaces, where \( \Sigma' \not\cong \mathbb{R}^2 \not\cong \Sigma \); and let \( \mathcal{A} \) be a preferred LFCS on \( \Sigma \) such that \( f \not\cong \mathcal{A} \). Then we can properly homotope \( f \) to remove all trivial components of the LFCS \( f^{-1}(\mathcal{A}) \) such that each primitive component of \( f^{-1}(\mathcal{A}) \) has an open neighborhood on which this proper homotopy is stationary.

**Proof.** Since \( \Sigma' \not\cong \mathbb{R}^2 \) and \( f^{-1}(\mathcal{A}) \) is an LFCS (see Theorem 3.2.3), by Lemma 3.3.1, there don’t exist infinitely many components \( C'_1, C'_2, \ldots \) of \( f^{-1}(\mathcal{A}) \) bounding the disks \( D'_1, D'_2, \ldots \), respectively such that \( C'_n \) is contained in the interior of \( D'_{n+1} \) for each \( n \). Thus, if \( f^{-1}(\mathcal{A}) \) has a trivial component, we can introduce the notion of an outermost disk bounded by a component of \( f^{-1}(\mathcal{A}) \) in the following way: A disk \( D' \subset \Sigma' \) bounded by a component of \( f^{-1}(\mathcal{A}) \) is called an outermost disk; if given another disk \( D'' \subset \Sigma \) bounded by a component of \( f^{-1}(\mathcal{A}) \), then either \( D'' \subset D' \) or \( D' \cap D'' = \emptyset \).

Let \( \{D'_n\} \) be the pairwise disjoint collection (which may be an infinite collection) of all outermost disks. Assume \( C_n \) represents that component of \( \mathcal{A} \) for which \( f(\partial D'_n) \subseteq C_n \). Note \( C_n \) may equal to \( C_m \) even if \( m \neq n \).

Now, for each integer \( n \), we will construct a compact bordered subsurface \( Z_n \) with \( f(D'_n) \subseteq Z_n \) such that \( Z_n \to \infty \). Roughly, \( Z_n \) will be obtained from taking the union of all those complementary components of \( \Sigma \) (if a punctured disk appears, truncate it), which are hit by \( f(D'_n) \).

Fix an integer \( n \). Let \( \chi'_{n,1}, \ldots, \chi'_{n,k_n} \) be the all connected components of \( D'_n \backslash f^{-1}(\mathcal{A}) \). By continuity of \( f \), for each \( \chi'_{n,\ell} \), there exists a complementary component \( Y_{n,\ell} \) of \( \Sigma \) decomposed by \( \mathcal{A} \) such that \( f(\chi'_{n,\ell}) \subseteq Y_{n,\ell} \) and \( \partial \chi'_{n,\ell} \subseteq f^{-1}(\partial Y_{n,\ell}) \). See Figure 4. For each \( \ell \), define a compact bordered subsurface \( Z_{n,\ell} \) of \( \Sigma \) as follows: If \( Y_{n,\ell} \) is homeomorphic to either \( S_{1,1}, S_{0,3}, S_{0,2} \); define \( Z_{n,\ell} := Y_{n,\ell} \). On the other hand, if \( Y_{n,\ell} \) is homeomorphic to \( S_{0,1,1} \), then let \( Z_{n,\ell} \) be an annulus in \( Y_{n,\ell} \) such that \( \partial Z_{n,\ell} \cap \partial Y_{n,\ell} = \partial Y_{n,\ell} \) and \( f(\chi'_{n,\ell}) \subseteq Z_{n,\ell} \). Finally, define \( Z_n := Z_{n,1} \cup \cdots \cup Z_{n,k_n} \).

Now, we show \( Z_n \to \infty \). So, consider a compact subset \( K \) of \( \Sigma \). Let \( S_1, \ldots, S_m \) be a collection of complementary components of \( \Sigma \) decomposed by \( \mathcal{A} \) such that \( K \subseteq \text{int} \left( \bigcup_{i=1}^{m} S_i \right) \). Define \( S := \bigcup_{\ell=1}^{m} S_\ell \). Thus, for an integer \( n \), \( f(D'_n) \cap S \neq \emptyset \) if and only if \( D'_n \) contains at least one component of \( \bigcup_{\ell=1}^{m} f^{-1}(\partial S_\ell) \). This is due to the construction of each \( Z_n \); see Figure 4. For each component \( C \) of \( \mathcal{A} \), Theorem 3.2.3 tells
that $f^{-1}(C)$ has only finitely many components. So $D_n'$ doesn’t contain any component of $\bigcup_{t=1}^n f^{-1}(\partial S_t)$ for all sufficiently large $n$, i.e., $f(D_n') \cap S = \emptyset$ for all sufficiently large $n$. Since $K \subseteq \text{int}(S)$ and each $Z_n$ is obtained from taking the union of all those complementary components of $\Sigma$ (if a punctured disk appears, truncate it), which are hit by $f(D_n')$, we can say that $Z_n \cap K = \emptyset$ for all sufficiently large $n$. Therefore, $Z_n \to \infty$ as $K$ is an arbitrary compact subset of $\Sigma$.

Fig. 4: Each component of $D_n' \setminus f^{-1}(\mathcal{A})$ maps into a component of $\Sigma \setminus \mathcal{A}$. This fact, together with Theorem 5.2.5, provides $\Sigma_n$. A black circle denotes a component of either $\mathcal{A}$ or a component of $f^{-1}(\mathcal{A})$.

For each $n$, adding a small external collar to one of the boundary components of $Z_n$ (if needed), we can construct a compact bordered surface $\Sigma_n$ with $C_n \subseteq \text{int}(\Sigma_n)$, $f(D_n') \subseteq \Sigma_n$ such that $\{\Sigma_n\}$ is a locally finite collection, i.e., $\Sigma_n \to \infty$. See Figure 4.

For each $n$, write $C_n' := \partial D_n'$. Thus $f(C_n') \subseteq C_n$. Since $C_n \subseteq \text{int}(\Sigma_n)$, using Theorem 5.2.1, choose a one-sided tubular neighborhood $C_n \times [0, \varepsilon_n)$ of $C_n$ in $\Sigma$ with $C_n \times 0 \equiv C_n'$ such that $f \upharpoonright (C_n \times t_n)$ for each $t_n \in [0, \varepsilon_n]$ and $C_n \times [0, \varepsilon_n] \subseteq \Sigma_n$. Without loss of generality, we may further assume that $f(x') \in C_n \times [0, \varepsilon_n]$ for each $x' \in \Sigma' \setminus D_n'$ sufficiently near to $C_n'$. Next, since $f^{-1}(\mathcal{A})$ is an LFCS, for each $n$, Theorem 5.2.3 gives a one-sided compact tubular neighborhood $U_n'$ of $C_n'$ such that the following hold: $U_n' \cap D_n' = C_n' = U_n' \cap f^{-1}(\mathcal{A})$, $f(U_n') \subseteq C_n \times [0, \varepsilon_n]$ for each $n$; and $U_n' \cap U_m' = \emptyset$ for $m \neq n$. Finally, Theorem 5.2.5 gives $\delta_n \subseteq (0, \varepsilon_n)$ and a component $C_{\delta_n,n}$ of $f^{-1}(C_{\delta_n,n})$ such that $C_{\delta_n,n}$ bounds a disk $D_{\delta_n,n}$ in $\Sigma'$ with $(U_n' \cup D_n') \supseteq D_{\delta_n,n} \supset \text{int}(D_{\delta_n,n}) \supset D_n'$ (equivalently, $U_n'$ contains the annulus co-bounded by $C_{\delta_n,n}$ and $C_n'$) and $f(D_{\delta_n,n} \setminus \text{int}(D_{\delta_n,n})) \subseteq C_n \times [0, \varepsilon_n]$. Thus, $D_{\delta_n,n} \cap f^{-1}(\mathcal{A}) = D_n' \cap f^{-1}(\mathcal{A})$, $f(D_{\delta_n,n}) \subseteq \Sigma_n$ for each $n$; and $D_{\delta_n,n} \cap D_{\delta_m,m} = \emptyset$ when $m \neq n$.

Since $C_{\delta_n,n}$ co-bounds an annulus with the primitive circle $C_n$, the inclusion $C_{\delta_n,n} \hookrightarrow \Sigma_n$ is $\pi_1$-injective (see
Theorem 2.2.2). Also, \( \Sigma_{\delta} \) is homotopy equivalent to \( \vee_{\text{finite}} S^1 \), which implies that the universal cover of \( \Sigma_{\delta} \) is contractible, and thus \( \pi_2(\Sigma_{\delta}) = 0 \). Therefore, exactness of

\[
\cdots \rightarrow \pi_2(\Sigma_{\delta}) \rightarrow \pi_2(\Sigma_{\delta}, C_{\delta, n}) \rightarrow \pi_1(C_{\delta, n}) \rightarrow \pi_1(\Sigma_{\delta}) \rightarrow \cdots
\]
gives \( \pi_2(\Sigma_{\delta}, C_{\delta, n}) = 0 \), i.e., we have a homotopy \( H_{\delta}: \Sigma_{\delta, n} \times [0,1] \rightarrow \Sigma_{\delta} \) relative to \( C_{\delta, n} \) from \( f([\Sigma_{\delta, n}, C_{\delta, n}]) \rightarrow (\Sigma_{\delta}, C_{\delta, n}) \) to a map \( \Sigma_{\delta, n} \rightarrow C_{\delta, n} \) for each \( n \); see [22, Lemma 4.6.]. Now, to conclude, apply Lemma 3.3.2 on \( \{H_{\delta}\} \).

Remark 3.4.2 In Theorem 3.3.5, the number of components of \( \mathcal{A} \) can be infinite; thus, the number of trivial components of \( f^{-1}(\mathcal{A}) \) can be infinite. That’s why we have removed all trivial components of \( f^{-1}(\mathcal{A}) \) by a single proper homotopy upon considering all outermost disks simultaneously. This process is in contrast to the finite-type surface theory, where the number of decomposition circles is finite, and therefore all trivial circles in the collection of transversal pre-images of all decomposition circles can be removed one by one, considering the notion of an innermost disk.

3.4 Homotope a degree-one map between circles to a homeomorphism

Previously, we have removed all trivial components keeping a neighborhood of each primitive component stationary from the transversal pre-image \( f^{-1}(\mathcal{A}) \) of a preferred LFCS \( \mathcal{A} \). In this section, we properly homotope our pseudo proper homotopy equivalence \( f: \Sigma' \rightarrow \Sigma \) to send each component \( C' \) of \( f^{-1}(\mathcal{A}) \) homeomorphically onto a component \( C \) of \( \mathcal{A} \) so that the restriction of \( f \) to a small one-sided tubular neighborhood \( C' \times [1,2] \) of \( C' \) (on either side of \( C' \)) can be described by the following homeomorphism:

\[
C' \times [1,2] \ni (z,t) \mapsto (f(z),t) \in C \times [1,2].
\]

First, we fix a few notations. Define \( \partial_{\varepsilon} := S^1 \times \varepsilon \) for \( \varepsilon \in \mathbb{R} \) and \( I := [0,1] \). Let \( p: S^1 \times \mathbb{R} \rightarrow S^1 \) be the projection. The following lemma roughly says that a self-map of \( S^1 \times [0,2] \) can be homotoped rel. \( S^1 \times 0 \) to map \( S^1 \times [1,2] \) into itself by the product \( \theta \times \text{Id}_{[1,2]} \), where \( \theta \) is a self-map of \( S^1 \).

Lemma 3.4.1 Let \( \Phi \) be a self-map of \( A := S^1 \times [0,2] \) such that \( \Phi^{-1}(\partial_{b}) = \partial_{b} \) for each \( b \in \{0,2\} \). If we are given a map \( \varphi_2: \partial_2 \rightarrow \partial_2 \) and a homotopy \( h_{(2)}: \partial_2 \times I \rightarrow \partial_2 \) from \( \Phi \mid \partial_2 \rightarrow \partial_2 \) to \( \varphi_2 \), then \( \Phi \) can be homotoped relative to \( \partial_0 \) to map \( S^1 \times [0,1] \) into \( S^1 \times [0,1] \) and to satisfy \( \Phi(-,r) = (p \circ \varphi_2(-,2), r) \) for each \( r \in [1,2] \).

Remark 3.4.2 In Lemma 3.4.1, up to homotopy, \( \varphi_2 \) is either a constant map or a covering map.

Proof. Homotope \( \Phi \) relative to \( \partial_0 \cup \partial_2 \) so that \( \Phi(S^1 \times [0,1]) \subseteq S^1 \times [0,1] \) and \( \Phi(z,r) = (p \circ \Phi(z,2), r) \) for all \( (z,r) \in S^1 \times [1,2] \). For each \( r \in [1,2] \), \( h_{(2)}(r) \) provides a homotopy \( h_{(r)}: \partial_r \times I \rightarrow \partial_r \). Let \( H: (\partial_0 \cup \partial_1) \times I \rightarrow \partial_0 \cup \partial_1 \) be the homotopy defined as follows: \( H|\partial_1 \times I = h_{(1)} \) and \( H(-,0)\partial_0 = \Phi|\partial_0 \) for any \( t \in [0,1] \). Homotopy extension theorem gives a homotopy \( \tilde{H}: S^1 \times [0,1] \times I \rightarrow S^1 \times [0,1] \) such that \( \tilde{H}|(\partial_0 \cup \partial_1) \times I = H \). Finally, paste \( \tilde{H} \) with the collection \( h_{(r)} \), \( 1 \leq r \leq 2 \).

The following theorem is the simple modification (in the proper category) of the analog theorem for closed surfaces.

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Theorem 3.4.3 Let $f: \Sigma' \to \Sigma$ be a smooth pseudo proper homotopy equivalence between two non-compact surfaces, where $\Sigma' \not\cong \mathbb{R}^2 \not\cong \Sigma$; and let $\mathcal{A}$ be a preferred LFCS on $\Sigma$ such that $f \not\sim \mathcal{A}$. Then $f$ can be properly homotopied to remove all trivial components of the $f^{-1}(\mathcal{A})$ as well as to map each primitive component of $f^{-1}(\mathcal{A})$ homeomorphically onto a component of $\mathcal{A}$. Moreover, after this proper homotopy, near each component of $f^{-1}(\mathcal{A})$, the map $f$ can be described as follows:

Let $C_p'$ (resp. $C$) be a component of $f^{-1}(\mathcal{A})$ (resp. $\mathcal{A}$) such that $f|C_p' \to C$ is a homeomorphism. Then $C_p'$ (resp. $C$) has two one-sided tubular neighborhoods $M'$ and $N'$ (resp. $M$ and $N$) with some specific identifications $(M', C_p') \cong (C_p' \times [1, 2], C_p' \times 2) \cong (N', C_p')$ (resp. $(M, C) \cong (C \times [1, 2], C \times 2) \cong (N, C)$) such that the following hold:

- $M' \cup N'$ is a (two-sided) tubular neighborhood of $C_p'$;
- $f|M' \to M$ and $f|N' \to N$ are homeomorphisms described by $C_p' \times [1, 2] \ni (z, t) \mapsto (f(z), t) \in C \times [1, 2]$.

Remark 3.4.4 In Theorem 3.4.3, though $M' \cup N'$ is a (two-sided) tubular neighborhood of $C_p'$, both $M$ and $N$ may lie on the same side of $C$, i.e., $M \cup N$ may not be a two-sided tubular neighborhood of $C$.

Proof. Let $\{C_{p_n}'\}$ be the collection of all primitive components of $f^{-1}(\mathcal{A})$. Assume $C_n$ represents the component of $\mathcal{A}$ for which $f(C_{p_n}') \subseteq C_n$. Note $C_n$ may equal to $C_m$ even if $m \neq n$.

Claim 3.4.4.1 There are one-sided compact tubular neighborhoods $U_n', V_n'(\subseteq \Sigma')$ of $C_{p_n}'$, and there are one-sided compact tubular neighborhoods $U_n, V_n(\subseteq \Sigma)$ of $C_n$ such that after defining $T_n' := U_n' \cup V_n$, the following hold:

1. $\tilde{\mathcal{A}} := \mathcal{A} \cup \{(\partial U_n \cup \partial V_n) \setminus C_n\}$ is an LFCS and $f \not\sim \tilde{\mathcal{A}}$;
2. $\partial U_n' \setminus C_{p_n}'$ (resp. $\partial V_n' \setminus C_{p_n}'$) is the only component of $f^{-1}(\partial U_n \setminus C_n) \cap U_n'$ (resp. $f^{-1}(\partial V_n \setminus C_n) \cap V_n'$) that co-bounds an annulus with $C_{p_n}'$ (see Figure 5);
3. each point of $\text{int}(U_n)$ (resp. $\text{int}(V_n)$) that is sufficiently near to $C_{p_n}'$ is mapped into $\text{int}(U_n)$ (resp. $\text{int}(V_n)$);
4. $T_n'$ is a two-sided tubular neighborhood of $C_{p_n}'$ with $f^{-1}(\mathcal{A}) \cap T_n' = C_{p_n}'$; and
5. $T_n' \cap T_m' = \emptyset$ if $m \neq n$, and $(U_n \cup V_n) \to \infty$.

Proof of Claim 3.4.4.1. For any positive integer $n_0$, Theorem 3.2.3 says that the set $\{m \in \mathbb{N} : C_m = C_{n_0}\}$ is finite. Also, $\mathcal{A}$ is locally finite. Thus $\{C_n : n \in \mathbb{N}\}$ is locally finite. So, for each $n$, there exists a two-sided tubular neighborhood $C_n \times [-\epsilon_n, \epsilon_n]$ of $C_n$ with $C_n \times 0 \equiv C_n$ such that $\{C_n \times [-\epsilon_n, \epsilon_n] : n \in \mathbb{N}\}$ is a locally finite collection. Further, for each $n \in \mathbb{N}$, we may assume that $f \not\sim (C_n \times t_n)$ whenever $t_n \in [-\epsilon_n, \epsilon_n]$ by Theorem 5.2.1.

Now, since $f^{-1}(\mathcal{A})$ is a locally finite collection, for each $n$, there are one-sided compact tubular neighborhoods $U_n', V_n' \subseteq C_{p_n}'$ in $\Sigma'$ such that after defining $T_n' := U_n' \cup V_n'$, the following hold: $T_n'$ is a two-sided tubular neighborhood of $C_{p_n}'$, $f^{-1}(\mathcal{A}) \cap T_n' = C_{p_n}'$, and $T_m' \cap T_m' = \emptyset$ if $m \neq n$. Moreover, using Theorem 5.2.3, $f(T_n') \subseteq C_n \times [-\epsilon_n, \epsilon_n]$ can also be assumed for each $n$.

Next, by Theorem 5.2.5, we may further assume $\partial U_n' \setminus C_{p_n}'$ (resp. $\partial V_n' \setminus C_{p_n}'$) is a component of $f^{-1}(C_n \times x_n)$ (resp. $f^{-1}(C_n \times y_n)$) for some $x_n, y_n \in (-\epsilon_n, 0) \cup (0, \epsilon_n)$ such that after defining $U_n$ (resp. $V_n$) as the annulus in $C_n \times [-\epsilon_n, \epsilon_n]$ co-bounded by $C_n \times 0$ and $C_n \times x_n$ (resp. $C_n \times y_n$), both (2) and (3) of Claim 3.4.4.1 do hold. Finally, $C_n \times [-\epsilon_n, \epsilon_n] \to \infty$ implies $(U_n \cup V_n) \to \infty$. 

\[ \square \]
Using Theorem 3.3.5, keeping stationary a neighborhood of each primitive component of \( f^{-1}(\mathcal{A}) \), we can properly homotope \( f \) to remove all trivial components from \( f^{-1}(\mathcal{A}) \). So, after this proper homotopy, (2) and (3) of Claim 3.4.4.1 imply that \( f(\mathcal{U}_n') \subseteq \mathcal{U}_n \), \( f^{-1}(\partial \mathcal{U}_n) \cap \mathcal{U}_n' = \partial \mathcal{U}_n' \) and \( f(\mathcal{V}_n') \subseteq \mathcal{V}_n \), \( f^{-1}(\partial \mathcal{V}_n) \cap \mathcal{V}_n' = \partial \mathcal{V}_n' \). Notice the abuse of notation, the initial and final maps of this proper homotopy both are denoted by \( f \).

Next, being an isomorphism, \( \pi_1(f) = \pi_1(f^{-1}(\mathcal{A})) \) preserves primitiveness, i.e., \( h_n(-, 1) = H(-, 1)C_{\mathcal{P}_n} \to C_n \) must be a homeomorphism. Thus, \( \mathcal{H} \) is our ultimate required homotopy.

Finally, we need to describe \( f \) near each component of \( f^{-1}(\mathcal{A}) \) after the proper homotopy \( \mathcal{H} \). Abusing notation, the final map of \( \mathcal{H} \) will be denoted by \( f \). Since Lemma 3.4.1 is being used, we have \( \mathcal{M}_n' \subseteq \mathcal{U}_n' \) and \( \mathcal{M}_n \subseteq \mathcal{U}_n \) with the identifications \( (\mathcal{M}_n', C_{\mathcal{P}_n}') \cong (\mathcal{C}_{\mathcal{P}_n} \times [1, 2], C_{\mathcal{P}_n} \times 2) \), \( (\mathcal{M}_n, C_n) \cong (\mathcal{C}_n \times [1, 2], C_n \times 2) \) such that after the proper homotopy \( \mathcal{H} : \Sigma' \times [0, 1] \to \Sigma \), the map \( f \) sends \( C_{\mathcal{P}_n} \times r \) onto \( C_n \times r \) using the homeomorphism \( f|C_{\mathcal{P}_n}' \to C_n \) for all \( r \in [1, 2] \). See Figure 5. Similar reasoning for \( f|\mathcal{V}_n' \to \mathcal{V}_n \).

The following proposition, which we don’t need to use anywhere, tells what happens if we drop the phrase “homotopy equivalence” in the statement of Theorem 3.4.3. Its proof is almost the same.

**Proposition 3.4.5** Let \( f : \Sigma' \to \Sigma \) be a smooth proper map between two non-compact surfaces, where \( \Sigma' \not\cong \mathbb{R}^2 \not\cong \Sigma \); and let \( \mathcal{A} \) be a preferred LFCS on \( \Sigma \) such that \( f \sim \mathcal{A} \). Then \( f \) can be properly homotoped to remove all trivial components of the \( f^{-1}(\mathcal{A}) \) as well as to map each primitive component of \( f^{-1}(\mathcal{A}) \) into a component of \( \mathcal{A} \) so that for any component \( \mathcal{C} \) of \( \mathcal{A} \) and any primitive component \( C_p' \) of \( f^{-1}(\mathcal{C}) \), after this proper homotopy, \( f|C_p' \to C \) is either a constant map or a covering map.
3.5 Annulus removal

In the previous two sections, after removing all trivial components from the transversal pre-image of a decomposition circle, the remaining primitive circles have been mapped homeomorphically to that decomposition circle. This section aims to remove all these primitive circles except one from the inverse image of each decomposition circle using the following three steps: annulus bounding, then annulus compression, and finally, annulus pushing.

At first, annulus bounding: Consider the collection of inverse images of all decomposition circles. The following lemma says that any two circles in this collection co-bound an annulus if and only if their images are the same. In other words, in the domain, by pasting all small annuli, we get the outermost annulus corresponding to a decomposition circle.

**Lemma 3.5.1** Let \( f: \Sigma' \to \Sigma \) be a homotopy equivalence between two non-compact surfaces, and let \( \mathcal{A}, \mathcal{A}' \) be two LFCS on \( \Sigma', \Sigma \), respectively, such that \( f \) maps each component of \( \mathcal{A}' \) homeomorphically onto a component of \( \mathcal{A} \). Suppose each component of \( \mathcal{A} \) is primitive, and any two distinct components of \( \mathcal{A} \) don’t co-bound an annulus in \( \Sigma \). Let \( C'_0, C'_1 \) be two distinct components of \( \mathcal{A}' \). Then \( C'_0, C'_1 \) co-bound an annulus in \( \Sigma' \) if and only if \( f(C'_0) = f(C'_1) \).

**Proof.** To prove only if part, let \( \Phi: S^1 \times [0, 1] \to \Sigma' \) be an embedding such that \( \Phi(S^1, k) = C'_k \) for \( k = 0, 1 \). Note that \( f \) maps each component of \( \mathcal{A}' \) homeomorphically onto a component of \( \mathcal{A} \), and each component of \( \mathcal{A} \) is a primitive circle on \( \Sigma \). Thus, the embeddings \( f\Phi(-, 0), f\Phi(-, 1): S^1 \to \Sigma \) are freely homotopic; and hence \( f\Phi(-, 0), f\Phi(-, 1): S^1 \to \Sigma \) represent the same non-trivial conjugacy class in \( \pi_1(\Sigma, *) \). Since any two distinct components of \( \mathcal{A} \) don’t co-bound an annulus in \( \Sigma \), by Theorem 2.2.3, \( f(C'_0) = f(C'_1) \).

To prove if part, let \( g: \Sigma \to \Sigma' \) be a homotopy inverse of \( f \), and let \( C \) be the component of \( \mathcal{A} \) defined by \( C := f(C'_0) = f(C'_1) \). Now, \( f|C_0' \to f(C'_1) \) is a homeomorphism for \( k = 0, 1 \). Thus, for a homeomorphism \( j: S^1 \cong C \), there are homeomorphisms \( \ell_0: S^1 \cong C'_0 \) and \( \ell_1: S^1 \cong C'_1 \) such that \( f\ell_0 = j = f\ell_1 \). Since \( \ell_0 \simeq gf\ell_0 = gj = gf\ell_1 \simeq \ell_1 \), applying Theorem 2.2.3 to \( \ell_0, \ell_1 \), we are done. \( \square \)

The following theorem, which will be used to compress each annulus bounded by two primitive circles of the domain, roughly says that most homotopies of a circle embedded in a surface are trivial.

**Theorem 3.5.2** [31, Lemma 4.9.15.] Let \( S \) be a compact bordered surface other than the disk, and let \( \Phi \) be a map from \( A := S^1 \times [0, 1] \) to \( S \) such that \( \Phi(\text{int}(A)) \subseteq \text{int}(S) \) and there is a boundary component \( C \) of \( S \) for which \( \Phi(-, 0), \Phi(-, 1): S^1 \cong C \) are the same homeomorphisms. Then \( \Phi \) can be homotoped relative to \( \partial A \) to map \( A \) onto \( C \).

The following theorem considers the last two steps - annulus compressing and annulus pushing. At first, by a proper homotopy, each outermost annulus will be mapped onto its decomposition circle; after that, by another proper homotopy, each outermost annulus will be pushed into a one-sided tubular neighborhood of one of its boundary components.

**Theorem 3.5.3** Let \( f: \Sigma' \to \Sigma \) be a smooth pseudo proper homotopy equivalence between two non-compact surfaces, where \( \Sigma' \neq \mathbb{R}^2 \neq \Sigma \); and let \( \mathcal{A} \) be a preferred LFCS on \( \Sigma \) such that \( f \not\simeq \mathcal{A} \). Suppose any two distinct components of \( \mathcal{A} \) don’t co-bound an annulus in \( \Sigma \). In that case, \( f \) can be properly homotoped to a proper map \( g \) such that for each component \( C \) of \( \mathcal{A} \), either \( g^{-1}(C) \) is empty or \( g^{-1}(C) \) is a component of \( f^{-1}(\mathcal{A}) \) that is mapped homeomorphically onto \( C \) by \( g \).
Proof. Using Theorem 3.4.3, we may assume each component of \( f^{-1}(\mathcal{A}) \) is primitive and also mapped homeomorphically onto a component of \( \mathcal{A} \). So for simplicity, we may drop the subscript \( p \) to indicate a primitive component of \( f^{-1}(\mathcal{A}) \). Let \( \{C_n\} \) be the pairwise disjoint collection of all those components of \( \mathcal{A} \) so that for each \( n, f^{-1}(C_n) \) has more than one component. By Lemma 3.5.1, for each \( n \), an annulus \( A'_n \) (say the \( n \)-th outermost annulus) exists with the following properties: (i) \( \partial A'_n \subseteq f^{-1}(C_n) \), (ii) \( A'_n \) is not contained in the interior of an annulus bounded by any two components of \( f^{-1}(\mathcal{A}) \). Thus \( A'_n \cap f^{-1}(\mathcal{A}) = f^{-1}(C_n) \) and \( A'_n \cap A'_m = \emptyset \) for \( m \neq n \). Now, using Theorem 2.2.3, find a parametrization \( \tau_n : \mathbb{S}^1 \times \{0, \ldots, k_n\} \to A'_n \) for some integer \( k_n \geq 1 \) so that \( \tau_n(\mathbb{S}^1 \times \{0, \ldots, k_n\}) = f^{-1}(C_n) \) and \( f\tau_n(-, \ell) : \mathbb{S}^1 \to C_n \) represents the same homeomorphism of \( C_n \) for each \( \ell = 0, \ldots, k_n \).

Claim 3.5.3.1 The proper map \( f : \Sigma' \to \Sigma \) can be properly homotoped relative to \( \Sigma' \setminus \bigcup_n \text{int}(A'_n) \) so that \( f(A'_n) = C_n \) for each \( n \).

![Diagram](image)

Fig. 6: Illustration of parts (1) and (3) of the definition of \( \Sigma_n \) given in the proof of Claim 3.5.3.1. Only black circles denote a component of either \( \mathcal{A} \) or a component of \( f^{-1}(\mathcal{A}) \).

Proof of Claim 3.5.3.1. For each integer \( n \), we will construct a compact bordered sub-surface \( \Sigma_n \) of \( \Sigma \) with \( f(A'_n) \subseteq \text{int}(\Sigma_n) \) such that \( \Sigma_n \to \infty \). Roughly, \( \Sigma_n \) will be obtained from taking the union of all those complementary components of \( \Sigma \) (if a punctured disk appears, truncate it), which are hit by \( f(A'_n) \).

Using continuity of \( f|\Sigma' \setminus f^{-1}(\mathcal{A}) : \Sigma \setminus \mathcal{A} \to \Sigma \setminus \mathcal{A} \), we can say that \( f(A'_n) \subseteq X_n \cup Y_n \), where \( X_n \) and \( Y_n \) are complementary components of \( \Sigma \) decomposed by \( \mathcal{A} \) such that \( C_n \subseteq \partial X_n \cap \partial Y_n \).

1. We define \( \Sigma_n \) as \( \Sigma_n := X_n \cup Y_n \) if either of the following happens: (i) \( X_n \cong S_{0,3} \cong Y_n \); or (ii) \( X_n \cong S_{1,1} \) and \( Y_n \cong S_{0,3} \); or (iii) \( Y_n \cong S_{1,1} \) and \( X_n \cong S_{0,3} \). See Figure 6.

2. If \( X_n \cong S_{0,1,1} \cong Y_n \) (in this case, \( \Sigma \) is homeomorphic to the punctured plane), then using compactness of \( f(A'_n) \), let \( \Sigma_n \) be an annulus in \( X_n \cup Y_n \) so that \( f(A'_n) \subseteq \text{int}(\Sigma_n) \).
(3) If \( \mathcal{X}_n \cong S_{0,1,1} \), and \( \mathcal{Y}_n \) is homeomorphic to either \( S_{0,3} \) or \( S_{1,1} \), then using compactness of \( f(\mathcal{A}'_n) \), find an annulus \( \mathcal{A}_n \) in \( \mathcal{X}_n \) so that \( f(\mathcal{A}'_n) \subseteq \text{int}(\mathcal{A}_n \cup \mathcal{Y}_n) \). Define \( \Sigma_n := \mathcal{A}_n \cup \mathcal{Y}_n \). See Figure 6.

(4) If \( \mathcal{Y}_n \cong S_{0,1,1} \), and \( \mathcal{X}_n \) is homeomorphic to either \( S_{0,3} \) or \( S_{1,1} \), define \( \Sigma_n \) similarly, as given in (3).

Thus, \( f(\mathcal{A}'_n) \subseteq \text{int}(\Sigma_n) \) for each \( n \). Now, we show \( \Sigma_n \to \infty \). So, consider a compact subset \( \mathcal{K} \) of \( \Sigma \). Let \( S_1, \ldots, S_m \) be a collection of complementary components of \( \Sigma \) decomposed by \( \mathcal{A} \) such that \( \mathcal{K} \subseteq \text{int}( \bigcup_{\ell=1}^m S_\ell ) \). Define \( S := \bigcup_{\ell=1}^m S_\ell \). Notice that for an integer \( n, f(\mathcal{A}'_n) \cap S \neq \emptyset \) if and only if \( C_n \) is a component of \( \bigcup_{\ell=1}^m \partial S_\ell \). This is due to the construction of each \( \Sigma_n \); see Figure 6. Since \( C_n \to \infty \) and \( \bigcup_{\ell=1}^m \partial S_\ell \) is compact, we can say that \( f(\mathcal{A}'_n) \cap S = \emptyset \) for all sufficiently large \( n \). Now, \( \mathcal{K} \subseteq \text{int}(S) \) and each \( \Sigma_n \) is obtained from taking the union of all those complementary components of \( \Sigma \) (if a punctured disk appears, truncate it), which are hit by \( f(\mathcal{A}'_n) \). Thus, \( \Sigma_n \cap \mathcal{K} = \emptyset \) for all sufficiently large \( n \). Therefore, \( \Sigma_n \to \infty \), as \( \mathcal{K} \) is an arbitrary compact subset of \( \Sigma \).

Next, for each \( \ell \in \{1, \ldots, k \} \), applying Theorem 3.5.2 to each \( f \tau_n | S^1 \times [\ell - 1, \ell] \to Z_n \), where \( Z_n \) can be either \( \Sigma_n \cap \mathcal{X}_n \) or \( \Sigma_n \cap \mathcal{Y}_n \), we have a homotopy \( H_n : A'_n \times [0, 1] \to \Sigma_n \) relative to \( \partial A'_n \) such that \( H_n(-, 0) = f| A'_n \) and \( H_n(\cdot, 1) = C_n \). Finally, apply Lemma 3.3.2 on \( \{H_n\} \) to complete proof of the Claim 3.5.3.1.

![Figure 7](image)

Fig. 7: Description of \( f| A'_c \to M_n \) (resp. \( H_n(-, 1) : A'_c \to M_n \)) using Theorem 3.4.3, Claim 3.5.3.1 (resp. Lemma 3.5.4). Only black circles denote a component of either \( \mathcal{A} \) or a component of \( f^{-1}(\mathcal{A}) \).

Now, consider Figure 7, where \( M'_{n,\alpha}, M_n \) are provided by Theorem 3.4.3 such that after defining \( A'_c \) as \( A'_n \cup M'_{n,\alpha} \), we can think

\[
(\mathcal{A}'_c, M'_{n,\alpha}, A'_n) \cong (S^1 \times [1, 3], S^1 \times [1, 2], S^1 \times [2, 3]) \quad \text{and} \quad (M_n, C_n) \cong (S^1 \times [1, 2], S^1 \times 2)
\]

resulting in the following description of \( f \): If \( \theta : S^1 \to S^1 \) describes the homeomorphism \( f| C'_{n,\alpha} \to C_n \) under the above identification, then \( f(z, t) = (\theta(z), t) \) for \( z \in S^1 \times [1, 2] \) and \( f(z, t) \in S^1 \times 2 \) for \( (z, t) \in S^1 \times [2, 3] \).

Consider Claim 3.5.3.1 to see why \( f(S^1 \times [2, 3]) = S^1 \times 2 \).

Now, use Lemma 3.5.4 to construct a homotopy \( H_n : A'_c \times [0, 1] \to M_n \) relative to \( \partial A'_c \) from \( f| A'_c \to M_n \) to the map \( H_n(-, 1) \) so that \( (H_n(-, 1))^{-1}(C_n) = C'_{n,\beta} \) and \( H_n(-, 1)|C'_{n,\beta} \to C_n \) is a homeomorphism.

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Notice that we are using the setup of proof of Theorem 3.4.3. By (4) and (5) of Claim 3.4.4.1 given in the proof of Theorem 3.4.3 show that $A'_m \cap f^{-1}(\mathcal{A}) = f^{-1}(C_n)$, $A'_m \cap A'_m = \emptyset$ if $m \neq n$, and $\mathcal{M}_n \to \infty$. Now, consider Lemma 3.3.9 with $\{H_n\}$ to obtain the desired homotopy.

Now, we prove the annulus-pushing lemma used in the proof of the previous theorem.

**Lemma 3.5.4** Any map $\varphi: S^1 \times [1, 3] \to S^1 \times [1, 2]$ which sends $S^1 \times r$ into $S^1 \times r$ for $1 \leq r \leq 2$ and sends $S^1 \times 2$ into $S^1 \times 2$ for $2 \leq r \leq 3$; can be homotoped relative to $S^1 \times \{1, 3\}$ to satisfy $\varphi^{-1}(S^1 \times 2) = S^1 \times 3$.

**Proof.** Let $\varphi_1: S^1 \times [1, 3] \to S^1$ and $\varphi_2: S^1 \times [1, 3] \to [1, 2]$ be the components of $\varphi$. Consider a homeomorphism $\ell: [1, 3] \to [1, 2]$ with $\ell(1) = 1$, $\ell(3) = 2$. Now, $H: S^1 \times [1, 3] \times [0, 1] \to S^1 \times [1, 2]$ defined by

$$H((z, s), t) := (\varphi_1(z, s), (1-t)\varphi_2(z, s) + t\ell(s))$$

for $(z, s) \in S^1 \times [1, 3]$ and $t \in [0, 1]$ is our required homotopy.

**Remark 3.5.5** In Theorem 3.5.3, the number of components of $\mathcal{A}$ can be infinite; thus, the number of outermost annuli (one outermost annulus for each component of $\mathcal{A}$, if any) can be infinite. That’s why we have removed all outermost annuli simultaneously by a single proper homotopy, not one by one. Also, to prove the topological rigidity of closed surfaces, one may ignore the annulus removal process considering induction on the genus; see [10, Theorem 3.1.] or [31, Theorems 4.6.2 and 4.6.3]. But, since the genus of a non-compact surface can be infinite, we can’t ignore the annulus removal process here.

### 3.6 Is pseudo proper homotopy equivalence a map of degree $\pm 1$?

Let $f: \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact oriented surfaces, where surfaces are homeomorphic to neither the plane nor the punctured plane. Our aim in this section is to properly homotope $f$ to obtain a closed disk $D \subseteq \Sigma$ so that $f|f^{-1}(D) \to D$ becomes a homeomorphism, and thus we show $\deg(f) = \pm 1$; see Theorem 2.6.1. Having got this and then using Lemma 3.6.4.1, it can be said that $f$ is surjective, which further implies that after a proper homotopy for removing unnecessary components from the transversal pre-image of a decomposition circle $C$, a single circle will still be left that can be mapped onto $C$ homeomorphically; see Theorem 3.6.4.4.

The argument for finding such a disk $D$ is based on finding a finite-type bordered surface $S$ in $\Sigma$ such that for each component $C$ of $\partial S$, we have $f^{-1}(C) \neq \emptyset$, even after any proper homotopy of $f$. Once we get $S$, after a proper homotopy, we may assume that $f|f^{-1}(\partial S) \to \partial S$ is a homeomorphism; see Theorem 3.5.3. Now, since $f$ is $\pi_1$-injective, by the topological rigidity of pair of pants together with the proper rigidity of the punctured disk, after a proper homotopy, one can show that $f|f^{-1}(S) \to S$ is a homeomorphism. Therefore, the required $D$ can be any disk in $\text{int}(S)$.

Now, to find such an $S$, we consider two cases: If $\Sigma$ is either an infinite-type surface or any $S_{g,0,p}$ with high complexity (to us, high complexity always means $g + p \geq 4$ or $p \geq 6$), then using $\pi_1$-surjectivity of $f$, we can choose $S$ as a pair of pants in $\Sigma$ so that $\Sigma \setminus S$ has at least two components and every component of $\Sigma \setminus S$ has a non-abelian fundamental group. On the other hand, if $\Sigma$ is a finite-type surface, then we choose a punctured disk in $\Sigma$ as $S$ so that the puncture of $S$ is an end $e \in \text{im}(\text{Ends}(f)) \subseteq \text{Ends}(\Sigma)$.

We can recall our earlier two examples to show that the plane and the punctured plane are the only surfaces for which our theory fails, i.e., consider the pseudo proper homotopy equivalences $\varphi: C \ni z \mapsto z^2 \in C$
and $\psi: S^1 \times \mathbb{R} \ni (z, x) \mapsto (z, |x|) \in S^1 \times \mathbb{R}$. The local-homeomorphism $\varphi$ is a map of $\deg = \pm 2$ by [13, Lemma 2.1b.] (note that for any local-homeomorphism $p: X \to Y$ between two manifolds, an orientation of $Y$ can be pulled back to give an orientation on $X$ so that $p$ becomes an orientation-preserving map). On the other hand, $\deg(\psi) = 0$ as $\psi$ is not surjective; see Lemma 3.6.4.1.

### 3.6.1 Essential pair of pants and the degree of a pseudo proper homotopy equivalence

**Definition 3.6.1.1** A smoothly embedded pair of pants $P$ in a surface $\Sigma$ is said to be an *essential pair of pants* of $\Sigma$ if $\Sigma \setminus P$ has at least two components and every component of $\Sigma \setminus P$ has a non-abelian fundamental group.

Finding an essential pair of pants in a non-compact surface will be divided into two cases: when the genus is at least two and when the space of ends has at least six elements.

**Definition 3.6.1** Let $P$ be a smoothly embedded copy of the pair of pants in a two-holed torus $S$ (i.e., $S$ is a copy of $S_{1,2}$). We say $P$ is obtained from decomposing $S$ into two copies of the pair of pants if there exists another smoothly embedded copy $\hat{P}$ of the pair of pants in $S$ such that $P \cup \hat{P} = S$ and $P \cap \hat{P} = \partial P \cap \partial \hat{P}$ is the union of two smoothly embedded disjoint circles in the interior of $S$ (i.e., $\partial P$ shares exactly two of its components with $\partial \hat{P}$).

The following lemma says that every non-compact surface with a genus of at least two has an essential pair of pants with some additional properties.

**Lemma 3.6.1.2** Let $\Sigma$ be a non-compact surface of the genus of at least two. Then $\Sigma$ has an essential pair of pants $P$ with the following additional properties: (1) $\Sigma$ contains a smoothly embedded copy $S$ of $S_{1,2}$ such that $\Sigma \setminus S$ has precisely two components and each component of $\Sigma \setminus S$ has a non-abelian fundamental group, (2) $P$ is a smoothly embedded copy of the pair of pants in $S$ obtained by decomposing $S$ into two copies of the pair of pants.

![Fig. 8](image.png)

**Fig. 8:** Finding an essential pair of pants $P$ in each of $S_{2,0,1}$ and Loch Ness Monster by decomposing a two-holed torus into two copies of the pair of pants.

**Proof.** Consider an inductive construction of $\Sigma$; see Theorem 2.5.1. Since $g(\Sigma) \geq 2$, at least two smoothly embedded copies of $S_{1,2}$ are used in this inductive construction. By Lemma 3.1.8, without loss of generality, we may assume that two smoothly embedded copies of $S_{1,2}$ are used successively just after the initial disk; see Figure 8. Among these two copies of $S_{1,2}$, breaking the last one (i.e., that copy of $S_{1,2}$ which we just used to construct $K_3$ from $K_2$) into two copies of the pair of pants, as illustrated in Figure 8, we get the required essential pair of pants.

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Lemma 3.6.1.3. Let \( f : \Sigma' \to \Sigma \) be a \( \pi_1 \)-surjective map between two non-compact surfaces, where \( \Sigma \) has the genus of at least two. Consider an essential pair of pants \( P \) in \( \Sigma \) given by Lemma 3.6.1.2. Then \( f^{-1}(\text{int } P) \neq \emptyset \) and \( f^{-1}(c) \neq \emptyset \) for each component \( c \) of \( \partial P \).

**Proof.** Let \( S \) be a smoothly embedded copy of \( S_{1,2} \) in \( \Sigma \) such that \( P \) is obtained from decomposing \( S \) into two copies of the pair of pants. If possible, let \( f^{-1}(\text{int } P) \neq \emptyset \). By continuity of \( f \), the image of \( f \) is contained in precisely one of the two components of \( \Sigma \setminus \text{int}(P) \). But each component of \( \Sigma \setminus \text{int}(P) \) has a non-abelian fundamental group, i.e., \( \pi_1(f) : \pi_1(\Sigma') \to \pi_1(\Sigma) \) is not surjective, a contradiction. Therefore, \( f^{-1}(\text{int } P) \) must be non-empty.

To prove the second part, let \( c_1, c_2, \) and \( c_3 \) denote all three components of \( P \) such that both \( \Sigma \setminus c_1 \) and \( \Sigma \setminus (c_2 \cup c_2) \) are disconnected, but neither \( \Sigma \setminus c_2 \) nor \( \Sigma \setminus c_3 \) is disconnected. In Figure 8, \( c_2 \) and \( c_3 \) are blue circles, whereas the color of the third component \( c_1 \) is black. Notice that we have a smoothly embedded primitive circle \( C \subseteq \text{int}(S) \) (in Figure 8, each red circle denotes \( C \)) so that for each \( k = 2, 3 \), \( c_k \cap C \) is a single point, where \( c_k \) intersects \( C \) transversally. Therefore, for each \( k = 2, 3 \), using the bigon criterion [14, Proposition 1.7], any loop belonging to class \( [C] \in \pi_1(\Sigma) \) must intersect \( c_k \). That is, if any of \( f^{-1}(c_2) \) or \( f^{-1}(c_3) \) were empty, then \( [C] \) would not belong to the image of \( \pi_1(f) : \pi_1(\Sigma') \to \pi_1(\Sigma) \). But \( f \) is \( \pi_1 \)-surjective. Thus \( f^{-1}(c_2) \neq \emptyset \neq f^{-1}(c_3) \). On the other hand, \( \Sigma \setminus c_1 \) has precisely two components, and each component of \( \Sigma \setminus c_1 \) has a non-abelian fundamental group, i.e., by continuity and \( \pi_1 \)-surjectivity of \( f \), we can say that \( f^{-1}(c_1) \neq \emptyset \).

Now, we consider the second case of finding an essential pair of pants in a non-compact surface, namely when the space of ends has at least six elements.

Lemma 3.6.1.4. Let \( \Sigma \) be a non-compact surface with at least six ends. Then \( \Sigma \) has an essential pair of pants \( P \) such that \( \Sigma \setminus P \) has precisely three components and each component of \( \Sigma \setminus P \) has a non-abelian fundamental group.

**Proof.** Consider an inductive construction of \( \Sigma \); see Theorem 2.5.1. Since \( |\text{Ends}(\Sigma)| \geq 6 \), at least five smoothly embedded copies of \( S_{0,3} \) are used in this inductive construction. By Lemma 3.1.8, without loss of generality, we may assume that five smoothly embedded copies of \( S_{0,3} \) are used successively just after the initial disk. Let \( P \) be the copy that shares all three boundary components with three other copies of this sequence of five copies of \( S_{0,3} \); see Figure 9. Thus, \( \Sigma \setminus P \) has precisely three components, and each component of \( \Sigma \setminus P \) has a non-abelian fundamental group.

In Figure 9, inductive constructions (up to a sufficient number of steps) of two surfaces have been given: the surface at the top contains a copy of \( \{ z \in \mathbb{C} : |z| \geq 1, z \notin \mathbb{N} \times 0 \} \), and the bottom is the Cantor tree surface (i.e., the planar surface whose space of ends is homeomorphic to the Cantor set). In each surface, an essential pair of pants \( P \) is contained in the shaded compact bordered subsurface.

We can prove the following Lemma by a similar argument given the proof of Lemma 3.6.1.3.

Lemma 3.6.1.5. Let \( f : \Sigma' \to \Sigma \) be a \( \pi_1 \)-surjective proper map between two non-compact surfaces, where \( \Sigma \) has at least six ends. Consider an essential pair of pants \( P \) in \( \Sigma \) given by Lemma 3.6.1.4. Then \( f^{-1}(\text{int } P) \neq \emptyset \) and \( f^{-1}(c) \neq \emptyset \) for each component \( c \) of \( \partial P \).

The following theorem completes the whole process of finding an essential pair of pants, which will be used to find the degree of a pseudo-proper homotopy equivalence.
Theorem 3.6.1.6 Let $f: \Sigma' \to \Sigma$ be a $\pi_1$-surjective proper map between two non-compact surfaces. Suppose $\Sigma$ is either an infinite-type surface or a finite-type surface $S_{g,0,p}$ with high complexity (i.e., $g+p \geq 4$ or $p \geq 6$). Then $\Sigma$ contains an essential pair of pants $P$ such that $f^{-1}(\text{int } P) \neq \emptyset$ and $f^{-1}(c) \neq \emptyset$ for each component $c$ of $\partial P$.

Proof. If an infinite-type surface has a finite genus, then it must have infinitely many ends; see Proposition 3.1.10. Thus using Lemma 3.6.1.2, Lemma 3.6.1.3, Lemma 3.6.1.4, and Lemma 3.6.1.5, the proof is complete in all cases, except when $\Sigma$ is homeomorphic to either $S_{1,0,3}$ or $S_{1,0,4}$ or $S_{1,0,5}$. We consider the case when $\Sigma \cong S_{1,0,3}$; the other cases are similar.

Define an inductive construction of $S_{1,0,3}$ in the following way: Start with a copy of $S_{0,1}$, then consecutively add two copies of $S_{0,3}$, and after that, a copy of $S_{1,2}$; and finally, add three sequences of annuli to obtain three planar ends; see Figure 1. Therefore, in this inductive construction, $K_4$ is obtained from $K_3$, adding a copy $S$ of $S_{1,2}$. Let $P$ be a smoothly embedded copy of the pair of pants in $S$ such that $P$ is obtained from decomposing $S$ into two copies of the pair of pants and $P \cap K_3 \neq \emptyset$. Now, an argument similar to that given in Lemma 3.6.1.3 completes the proof.

At this stage, we need a couple of lemmas. The first one, Lemma 3.6.1.7, is well-known; its proof has been given for reader convenience.
Lemma 3.6.1.7 Let $\Sigma$ be a surface, and let $S$ be a smoothly embedded bordered sub-surface of $\Sigma$. Then the inclusion induced map $\pi_1(S) \to \pi_1(\Sigma)$ is injective if either of the following satisfies:

1. $\partial S$ is a separating primitive circle on $\Sigma$ and $S$ is one of the two sides of $\partial S$ in $\Sigma$;
2. $S$ is compact and each component of $\partial S$ is a primitive circle on $\Sigma$.

Proof of part (1) of Lemma 3.6.1.7. Since $\pi_1(\Sigma) \cong \pi_1(S) \ast_{\pi_1(\partial S)} \pi_1(\Sigma \setminus \text{int } S)$ (by Seifert-van Kampen theorem) and the inclusions $\partial S \hookrightarrow S$, $\Sigma \setminus \text{int}(S)$ are $\pi_1$-injective, we are done.

Proof of part (2) of Lemma 3.6.1.7. It is enough to construct a sequence $\Sigma = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n = S$ of sub-surfaces of $\Sigma$, where $n$ is the number of components of $\partial S$, such that for each $k = 1, \ldots, n$, the following hold:

1. $S_k$ is a bordered sub-surface of $S_{k-1}$ and the inclusion map $S_k \hookrightarrow S_{k-1}$ is $\pi_1$-injective;
2. $\partial S_k \setminus \partial S_{k-1}$ is either a component of $\partial S_k$ or union of two components of $\partial S_k$. In either case, $\partial S_k \setminus \partial S_{k-1}$ shares only one component with $\partial S$.

We construct this sequence inductively as follows: To construct $S_k$ from $S_{k-1}$, pick a component $c$ of $\partial S \setminus \partial S_{k-1}$. If $c$ separates $S_{k-1}$, define $S_k$ as that side of $c$ in $S_{k-1}$, which contains $S$; then consider an argument similar to the proof of part (1) of Lemma 3.6.1.7. Now, if $c$ doesn’t separate $S_{k-1}$, pick a smoothly embedded annulus $A \subset \text{int}(S_{k-1})$ such that $A \cap S = c$. Define $S_k := S_{k-1} \setminus \text{int}(A)$. Now, $S_{k-1}$ is obtained from $S_k$ identifying $c$ with $\partial A \setminus c$ by an orientation-reversing diffeomorphism $\varphi : c \to \partial A \setminus c$. By HNN-Seifert-van Kampen theorem, $\pi_1(S_{k-1}) \cong \pi_1(S_k) \ast_{\pi_1(\partial c)} \pi_1(c)$, where the map $\pi_1(S_k) \to \pi_1(S_{k-1})$ (which is inclusion induced) is injective due to Britton’s lemma. This completes the proof.

The following lemma roughly says that the degree of a map between two compact bordered surfaces can be determined from the degree of its restriction on the boundaries.

Lemma 3.6.1.8 Let $\varphi : S_{g_1,b_1} \to S_{g_2,b_2}$ be a map between two compact bordered surfaces. If $\varphi|\partial S_{g_1,b_1} \hookrightarrow \partial S_{g_2,b_2}$ is an embedding, then $\varphi(\partial S_{g_1,b_1}) = \partial S_{g_2,b_2}$ and $\deg(\varphi) = \pm 1$.

Proof. Notice that $\varphi$ maps each component of $\partial S_{g_1,b_1}$ homeomorphically onto a component of $\partial S_{g_2,b_2}$, and any two distinct components of $\partial S_{g_1,b_1}$ have distinct $\varphi$-images. Now, naturality of homology long exact sequences of $(S_{g_1,b_1}, \partial S_{g_1,b_1})$ and $(S_{g_2,b_2}, \partial S_{g_2,b_2})$ give following commutative diagram:

$$
\begin{array}{ccc}
H_2(S_{g_1,b_1}, \partial S_{g_1,b_1}) & \cong & \mathbb{Z} \\
\times \deg(\varphi) & \downarrow & \oplus \mathbb{Z} \cong H_1(\partial S_{g_1,b_1}) \\
H_2(S_{g_2,b_2}, \partial S_{g_2,b_2}) & \cong & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \oplus & \mathbb{Z} \cong H_1(\partial S_{g_2,b_2})
\end{array}
$$

The horizontal maps are the connecting homomorphisms for homology long exact sequences, and for their description, see [22, Exercise 31 of Section 3.3]. Now, commutativity of this diagram gives $b_2 = b_1$ (the integer $\deg(\varphi)$ can’t be simultaneously 0 as well as $\pm 1$), and thus $\deg(\varphi) = \pm 1$. ☐

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The proof of \textbf{Theorem 3.6.1.9} below can be found in [31, Theorem 4.6.2.]. It also follows from the much more general result, [10, Theorem 3.1.]. Since compact bordered surfaces are aspherical, an application of the Whitehead theorem says that the assumption \(\varphi: S' \to S\) is a homotopy equivalence” in \textbf{Theorem 3.6.1.9} is equivalent to the assumption \(\pi_1(\varphi)\) is an isomorphism.

\textbf{Theorem 3.6.1.9} (Rigidity of compact bordered surfaces) Let \(\varphi: S' \to S\) be a homotopy equivalence between two compact bordered surfaces such that \(\varphi^{-1}(\partial S) = \partial S'\). If \(\varphi|\partial S' \to \partial S\) is a homeomorphism, then \(\varphi\) is homotopic to a homeomorphism relative to \(\partial S'\).

The following lemma gives some sufficient conditions so that the pre-image of a compact bordered subsurface under a proper map becomes a compact bordered subsurface of the same homeomorphism type. Its usage is two-fold: firstly, in \textbf{Theorem 3.6.1.11}, to find the degree of a pseudo proper homotopy equivalence; and secondly, in the proof of \textbf{Theorem 1}.

\textbf{Lemma 3.6.1.10} Let \(f: \Sigma' \to \Sigma\) be a \(\pi_1\)-injective proper map between two non-compact oriented surfaces, and let \(S\) be a smoothly embedded compact bordered subsurface of \(\Sigma\) with \(f^{-1}(\text{int}\ S) \neq \emptyset\). Suppose \(f^{-1}(\partial S)\) is a pairwise disjoint collection of smoothly embedded primitive circles on \(\Sigma'\) such that \(f\) sends \(f^{-1}(\partial S)\) homeomorphically onto \(\partial S'\). Then \(f^{-1}(S)\) is a copy of \(S\) in \(\Sigma'\) with \(\partial f^{-1}(S) = f^{-1}(\partial S)\), and \(\deg(f) = \pm 1\).

\textit{Proof.} Since \(f^{-1}(\text{int}\ S) \neq \emptyset\) and \(f\) is proper, the continuity of \(f|\Sigma' \setminus f^{-1}(\partial S) \to \Sigma \setminus \partial S\) tells that \(\Sigma' \setminus f^{-1}(\partial S)\) is disconnected. Let \(S' \subseteq \Sigma'\) be a bordered sub-surface obtained as a complementary component of the decomposition of \(\Sigma'\) by \(f^{-1}(\partial S)\) such that \(f(S') \subseteq S\). That is, \(S'\) is the closure of one of the components of \(\Sigma' \setminus f^{-1}(\partial S)\) and \(S'\) is contained in the compact set \(f^{-1}(\Sigma)\). So, \(S'\) is a compact bordered subsurface of \(\Sigma'\), and each component of \(\partial S'\) is a component of \(f^{-1}(\partial S)\). In the following few lines, we will show that each component of \(f^{-1}(\partial S)\) is also a component of \(\partial S'\). Anyway, since \(f|f^{-1}(\partial S) \to \partial S\) is a homeomorphism, we can say that \(f|\partial S' \to \partial S\) is an embedding. Now, by \textbf{Lemma 3.6.1.8}, \(\partial S' = f^{-1}(\partial S)\) and \(\deg(f|S' \to S) = \pm 1\). Next, by \textbf{Theorem 2.6.3}, \(f|S' \to S\) is \(\pi_1\)-surjective. Since the inclusion \(S' \subseteq \Sigma'\) and \(f\) are \(\pi_1\)-injective, \(f|S' \to S\) is also so (see part (2) of \textbf{Lemma 3.6.1.7}). Thus, \(f|S' \to S\) is \(\pi_1\)-bijective, and so \textbf{Theorem 3.6.1.9} tells that \(S' \cong S\). Finally, if \(S''\) is another bordered sub-surface obtained as a complementary component of decomposition of \(\Sigma'\) by \(f^{-1}(\partial S)\) with \(f(S'') \subseteq S\), then similarly, \(S'' \cong S\). Since \(f|f^{-1}(\partial S) \to \partial S\) is a homeomorphism and \(\Sigma'\) is connected, \(S'' = S'\) (otherwise, \(\Sigma'\) would be the compact surface \(S' \cup_{\partial S'' = \partial S'} S''\)). Therefore, \(f^{-1}(S) = S' \cong S\), and thus the proof of the first part is completed.

Now, we will prove that \(\deg(f) = \pm 1\). Since \(\deg(f)\) remains invariant after any proper homotopy of \(f\), we can properly homotope \(f\) as we want. So, apply \textbf{Theorem 3.6.1.9} to \(f|S' \to S\). Thus, \(f: \Sigma' \to \Sigma\) can be properly homotoped relative to \(\Sigma' \setminus \text{int}(S')\) to map \(S' = f^{-1}(S)\) is homeomorphically onto \(S\). Now, by \textbf{Theorem 2.6.1}, \(\deg(f) = \pm 1\).

We are now ready to prove that a pseudo proper homotopy equivalence is a map of degree \(\pm 1\) if the co-domain contains an essential pair of pants, as said before.

\textbf{Theorem 3.6.1.11} Let \(f: \Sigma' \to \Sigma\) be a pseudo proper homotopy equivalence between two non-compact oriented surfaces, where \(\Sigma\) is either an infinite-type surface or a finite-type non-compact surface \(S_{g,0,p}\) with high complexity (to us, high complexity means \(g + p \geq 4\) or \(p \geq 6\)). Then \(\deg(f) = \pm 1\).

\textit{Proof.} Since \(\deg(f)\) remains invariant after any proper homotopy of \(f\), we can properly homotope \(f\) as we want. Now, \textbf{Theorem 3.6.1.6} gives an essential pair of pants \(P\) in \(\Sigma\) such that \(f^{-1}(\text{int}(P)) \neq \emptyset\) and \(f^{-1}(c) \neq \emptyset\) for each component \(c\) of \(\partial P\), even after any proper homotopy of \(f\). Using \textbf{Theorem 3.2.1}
and then Theorem 3.5.3, after a proper homotopy, we may assume that \( f^{-1}(\text{int} P) \neq \emptyset \) and \( f^{-1}(\partial P) \) is a pairwise-disjoint collection of three smoothly embedded circles on \( \Sigma \) such that \( f|f^{-1}(\partial P) \rightarrow \partial P \) is a homeomorphism.

Now, if possible, let \( c' \) be a component of \( f^{-1}(\partial P) \) such that there is an embedding \( i': \mathbb{D}^2 \hookrightarrow \Sigma' \) with \( c' = i'(S^1) \). Then the embedding \( f \circ i'|S^1 \hookrightarrow \Sigma \) is null-homotopic and \( e := f \circ i'(S^1) \) is a component of \( \partial P \). But \( P \) is an essential pair of pants in \( \Sigma \) implies each component of \( \partial P \) is a primitive circle on \( \Sigma \). Now, Theorem 2.2.2 tells us we have reached a contradiction. Hence, each component of \( f^{-1}(\partial P) \) is a primitive circle on \( \Sigma' \). Finally, applying Lemma 3.6.1.10, we complete the proof.

\[ \square \]

### 3.6.2 An essential punctured disk of a proper map and the degree of a pseudo proper homotopy equivalence

We first build up notations for Section 3.6.2. Let \( \Sigma \) be a non-compact surface. Since the \( \text{Ends}(\Sigma) \) is independent of the choice of efficient exhaustion of \( \Sigma \) by compacta, we will use Goldman’s inductive construction to define \( \text{Ends}(\Sigma) \); see Section 2.3. So, consider an inductive construction of \( \Sigma \). For each \( i \geq 1 \), define \( K_i \) to be the compact bordered subsurface of \( \Sigma \) after the \( i \)-th step of the induction. Then \( \{K_i\}_{i=1}^{\infty} \) is an efficient exhaustion of \( \Sigma \) by compacta. Also, notice that \( \text{int}(K_1) \subseteq \text{int}(K_2) \subseteq \cdots \) is an increasing sequence of open subsets of \( \Sigma \) such that \( \bigcup_{i=1}^{\infty} \text{int}(K_i) = \Sigma \); and thus every compact subset of \( \Sigma \) is contained in some \( \text{int}(K_i) \).

Suppose \( \Sigma' \) is another non-compact surface and \( f: \Sigma' \rightarrow \Sigma \) is a proper map. Let \( (V_1, V_2, \ldots) \) be an end of \( \Sigma \), i.e., \( V_i \) is a component of \( \Sigma \setminus K_i \) and \( V_1 \supseteq V_2 \supseteq \cdots \). With this setup, we are now ready to state a lemma that is more or less related to Proposition 2.3.1.

**Theorem 3.6.2.1** Assume that \( f^{-1}(V_i) \neq \emptyset \) for each \( i \geq 1 \). Then for every proper map \( g: \Sigma' \rightarrow \Sigma \), which is properly homotopic to \( f \), we have \( g^{-1}(V_i) \neq \emptyset \) for each \( i \geq 1 \).

**Proof.** Let \( g: \Sigma' \rightarrow \Sigma \) be a proper map, and let \( \mathcal{H}: \Sigma' \times [0, 1] \rightarrow \Sigma \) be a proper homotopy from \( f \) to \( g \). Notice that \( V_i \rightarrow \infty \): If \( \mathcal{X} \) is a compact subset of \( \Sigma \), then \( \mathcal{X} \subset \text{int}(K_{i_0}) \) for some positive integer \( i_0 \), i.e., \( \mathcal{X} \cap V_i = \emptyset \) for all \( i \geq i_0 \). Therefore, \( f^{-1}(V_i) \rightarrow \infty \): If \( \mathcal{X}' \) is a compact subset of \( \Sigma' \), then \( f(\mathcal{X}') \) is compact, so \( f(\mathcal{X}') \cap V_i = \emptyset \) for all but finitely many \( i \), i.e., \( \mathcal{X}' \cap f^{-1}(V_i) = \emptyset \) for all but finitely many \( i \).

Let \( n \) be any positive integer. Consider the compact subset \( p(\mathcal{H}^{-1}(K_n)) \) of \( \Sigma' \), where \( p: \Sigma' \times [0, 1] \rightarrow \Sigma' \) is the projection. Since \( f^{-1}(V_i) \rightarrow \infty \), we have an integer \( i_n > n \) such that \( f^{-1}(V_{i_n}) \subseteq \Sigma \setminus p(\mathcal{H}^{-1}(K_n)) \).

Now, consider any \( x_{i_n} \in f^{-1}(V_{i_n}) \). Then \( \mathcal{H}(x_{i_n} \times [0, 1]) \subseteq \Sigma \setminus K_n \), i.e., the connected set \( \mathcal{H}(x_{i_n} \times [0, 1]) \) is contained in one of the components of \( \Sigma \setminus K_n \). But \( \mathcal{H}(x_{i_n}, 0) = f(x_{i_n}) \in V_{i_n} \subseteq V_n \), i.e., \( \mathcal{H}(x_{i_n} \times [0, 1]) \subseteq V_n \).

In particular, this means \( g(x_{i_n}) = \mathcal{H}(x_{i_n}, 1) \in V_n \). Since \( n \) is an arbitrary positive integer, we are done. \( \square \)

**Definition 3.6.2.2** Let \( e = (V_1, V_2, \ldots) \) be an end of \( \Sigma \) such that for some non-negative integer \( i_e \), \( V_i \cong S_{0, 1, 1} \) for all \( i \geq i_e \) (i.e., \( e \) is an isolated planar end of \( \Sigma \)). If \( f^{-1}(V_i) \neq \emptyset \) for all \( i \geq 1 \), then for each integer \( i \geq i_e \), we say \( V_i \) is an essential punctured disk of \( f \).

Theorem 3.6.2.1 says that the notion of an essential punctured disk is invariant under the proper homotopy. In Theorem 3.6.2.6, we show that, after a proper homotopy, the pre-image of the boundary of an essential punctured disk under a pseudo proper homotopy equivalence bounds a planar end of the domain. But before moving into its proof, we need to prove the following lemma, which gives some sufficient conditions so that the pre-image of a punctured disk in the co-domain under a proper map becomes a punctured disk in the domain.
Lemma 3.6.2.3 Let \( f : \Sigma' \to \Sigma \) be a \( \pi_1 \)-injective proper map between two non-compact oriented surfaces, and let \( C \) be a smoothly embedded separating circle on \( \Sigma \) such that one of the two sides of \( C \) in \( \Sigma \) is a punctured disk \( D_* \). Also, let \( \Sigma' \) is homeomorphic to neither \( S^1 \times \mathbb{R} \) nor \( \mathbb{R}^2 \). If \( f^{-1}(C) \) is a smoothly embedded primitive circle on \( \Sigma' \) so that \( f|f^{-1}(C) \to C \) is a homeomorphism and \( f^{-1}(\text{int } D_*) \neq \emptyset \), then \( f^{-1}(D_*) \) is a copy the punctured disk in \( \Sigma' \) with \( \partial f^{-1}(D_*) = f^{-1}(C) \) and \( \deg(f) = \pm 1 \).

Proof. Notice that \( \Sigma' \not\cong \mathbb{R}^2, S^1 \times \mathbb{R} \), i.e., \( \pi_1(\Sigma') \) is non-abelian by Theorem 3.1.9. Since \( f^{-1}(\text{int } D_*) \neq \emptyset \) and \( \pi_1(f)(\pi_1(\Sigma')) \) is non-abelian, by continuity of \( f|\Sigma' \setminus f^{-1}(C) \to \Sigma \setminus C \), we can say that \( \Sigma' \setminus f^{-1}(C) \) is disconnected. Let \( S' \) be a side of \( f^{-1}(C) \) in \( \Sigma' \) for which \( f(S') \subseteq D_* \). Since \( f \) is \( \pi_1 \)-injective, by part (1) of Lemma 3.6.1.7, \( f|S' \to D_* \) is also so. Thus, \( \pi_1(S') \) is a subgroup of \( \mathbb{Z} \). Now, \( \text{int}(S') \) is homotopy equivalent to \( S' \) and bounded by the primitive circle \( f^{-1}(C) \) on \( \Sigma' \); so, using Theorem 3.1.9, \( S' \cong S_{0,1,1} \). Next, if \( S'' \) is another side of \( f^{-1}(C) \) in \( \Sigma' \) for which \( f(S'') \subseteq D_* \), then similarly, \( S'' \cong S_{0,1,1} \). Since \( f|f^{-1}(C) \to C \) is homeomorphism and \( \Sigma' \) is connected, \( S'' = S' \); otherwise, \( \Sigma' \) would be \( S' \cup C S'' \cong S^1 \times \mathbb{R} \). Therefore, \( f^{-1}(D_*) = S' \cong D_* \), and thus the proof of the first part is completed.

Now, we will prove that \( \deg(f) = \pm 1 \). Since \( \deg(f) \) remains invariant after any proper homotopy of \( f \), we can properly homotope \( f \) as we want. So, apply Theorem 3.6.2.4 to \( f|S' \to D_* \). Thus, \( f : \Sigma' \to \Sigma \) can be properly homotoped relative to \( \Sigma' \setminus \text{int}(S') \) to map \( S' = f^{-1}(D_*) \) is homeomorphically onto \( D_* \). Now, by Theorem 2.6.1, \( \deg(f) = \pm 1 \).

Now, we prove a well-known theorem used in the previous lemma.

Theorem 3.6.2.4 (Proper rigidity of the punctured disk) Let \( D_* \) be a punctured disk, and let \( \varphi : D_* \to D_* \) be a proper map such that \( \varphi^{-1}(\partial D_*) = \partial D_* \) and \( \varphi|\partial D_* \to \partial D_* \) is a homeomorphism. Then \( \varphi \) is properly homotopic to a homeomorphism \( D_* \to D_* \) relative to the boundary \( \partial D_* \).

Proof. Without loss of generality, we may assume \( D_* = \{ z \in \mathbb{C} : 0 < |z| \leq 1 \} \). Define \( \mathcal{H} : D_* \times [0, 1] \to D_* \) by

\[
\mathcal{H}(z, t) := \begin{cases} (1 - t) \cdot \varphi \left( \frac{z}{1-t} \right) & \text{if } 0 < |z| \leq 1 - t, \\ |z| \cdot \varphi \left( \frac{z}{1-t} \right) & \text{if } 1 - t < |z| \leq 1. \end{cases}
\]

Notice that \( \varphi \cong \mathcal{H}(-, 1) \) relative to \( \partial D_* \), and \( \mathcal{H}(-, 1) : D_* \to D_* \) is a homeomorphism.

Now, we prove \( \mathcal{H} \) is a proper map. So let \( \{(z_n, t_n)\} \) is a sequence in \( D_* \times [0, 1] \) with \( z_n \to 0 \). We need to show that \( \mathcal{H}(z_n, t_n) \to 0 \). Define \( \mathcal{A} := \{ n \in \mathbb{N} : 1 - t_n < |z_n| \} \) and \( \mathcal{B} := \{ n \in \mathbb{N} : |z_n| \leq 1 - t_n \} \). Then \( \mathbb{N} = \mathcal{A} \cup \mathcal{B} \). Therefore, it is enough to show \( \{\mathcal{H}(z_n, t_n) : n \in \mathcal{A}\} \to 0 \) (resp. \( \{\mathcal{H}(z_n, t_n) : n \in \mathcal{B}\} \to 0 \)) whenever \( \mathcal{A} \) (resp. \( \mathcal{B} \)) is infinite.

If \( \mathcal{A} \) is infinite, then \( \{\mathcal{H}(z_n, t_n) : n \in \mathcal{A}\} \to 0 \), since \( |\mathcal{H}(z_n, t_n)| = |z_n| \cdot |\varphi\left(\frac{z_n}{1-t_n}\right)| \leq |z_n| \) for all \( n \in \mathcal{A} \).

Next, assume \( \mathcal{B} \) is infinite. We will prove that \( \{\mathcal{H}(z_n, t_n) : n \in \mathcal{B}\} \to 0 \). So consider any \( \varepsilon > 0 \).

We need to show \( |\mathcal{H}(z_n, t_n)| < \varepsilon \) for all but finitely many \( n \in \mathcal{B} \). Let \( \mathcal{B}_\varepsilon := \{ n \in \mathcal{B} : 1 - t_n \leq \varepsilon \} \).

Therefore \( |\mathcal{H}(z_n, t_n)| = (1 - t_n) \cdot |\varphi\left(\frac{z_n}{1-t_n}\right)| \leq (1 - t_n) < \varepsilon \) for all \( n \in \mathcal{B}_\varepsilon \). Also, if \( \mathcal{B} \setminus \mathcal{B}_\varepsilon \) is infinite, then \( \left\{ \frac{z_n}{1-t_n} : n \in \mathcal{B} \setminus \mathcal{B}_\varepsilon \right\} \to 0 \), which implies \( \left\{ \varphi\left(\frac{z_n}{1-t_n}\right) : n \in \mathcal{B} \setminus \mathcal{B}_\varepsilon \right\} \to 0 \) (as \( \varphi \) is proper), and thus \( |\mathcal{H}(z_n, t_n)| < \varepsilon \) for all but finitely many \( n \in \mathcal{B} \setminus \mathcal{B}_\varepsilon \). Now, the previous two lines together imply that \( |\mathcal{H}(z_n, t_n)| < \varepsilon \) for all but finitely many \( n \in \mathcal{B} \).

Example 3.6.2.5 Notice that Theorem 3.6.2.4 is obtained from a straightforward modification of the Alexander trick [14, Lemma 2.1].
**Theorem 3.6.2.6** Let \( f: \Sigma' \to \Sigma \) be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. Suppose \( \pi_1(\Sigma) \) is finitely-generated non-abelian group (equivalently \( \Sigma \cong S_{g,0,p} \) for some \((g,p) \neq (0,1),(0,2))\). Then \( \deg(f) = \pm 1 \).

**Proof.** Since \( \deg(f) \) remains invariant after any proper homotopy of \( f \), we can properly homotope \( f \) as we want. Now, \( \Sigma \) is a finite-type non-compact surface implies each end of \( \Sigma \) is an isolated planar end, i.e., for every \( e = (V_1, V_2, \ldots) \in \text{Ends}(\Sigma) \), we have an integer \( i_e \) such that \( V_i \) is homeomorphic to the punctured disk for each \( i \geq i_e \). Next, \( f \) is proper implies there exists \( \mathcal{E} = (\mathcal{W}_1, \mathcal{W}_2, \ldots) \in \text{Ends}(\Sigma) \) such that \( f^{-1}(\mathcal{W}_i) \neq \emptyset \) for each \( i \geq 1 \). Notice that \( \mathcal{W}_i \) is an essential punctured disk and \( C_{i_e} := \partial \mathcal{W}_i \) is a smoothly embedded circle separating \( \Sigma \). Also, \( C_{i_e} \) is a primitive circle on \( \Sigma' \) as \( C_{i_e} \) bound the punctured disk \( \mathcal{W}_i \) on \( \Sigma' \neq \mathbb{R}^2 \).

We aim to use Lemma 3.6.2.3, but some observations are needed before that. Let \( g: \Sigma' \to \Sigma \) be a proper map such that \( g \) is properly homotopic to \( f \) (note that \( f \) is properly homotopic to itself, i.e., \( g \) can be \( f \)). If possible, assume \( g^{-1}(C_{i_e}) = \emptyset \). Then continuity of \( g \) implies \( g(\Sigma') \) is contained in one of the two components of \( \Sigma \setminus C_{i_e} \). By Theorem 3.6.2.1, \( g(\Sigma') \) must be contained in \( \mathcal{W}_i \). But, then \( \pi_1(f) = \pi_1(g) \) is non-surjective as \( \pi_1(\Sigma \setminus \mathcal{W}_i) = \pi_1(\Sigma) \) is non-abelian. Therefore, \( g^{-1}(C_{i_e}) \neq \emptyset \). Also, by Theorem 3.6.2.1, \( g^{-1}(\mathcal{W}_i) \neq \emptyset \) for each \( i \geq 1 \), and thus \( g^{-1}(\mathcal{W}_i) \neq \emptyset \).

Now, we are ready to apply Lemma 3.6.2.3 after the observation given in the previous paragraph. At first, notice that \( \Sigma' \) is homeomorphic to neither the plane nor the punctured plane as \( \pi_1(\Sigma') = \pi_1(\Sigma) \) is non-abelian. After a proper homotopy of \( f \), we may assume that \( f \not\sim C_{i_e} \); see Theorem 3.2.1. By the previous paragraph, \( f^{-1}(C_{i_e}) \) is a pairwise disjoint non-empty collection of finitely many smoothly embedded circles on \( \Sigma' \). Now, by Theorem 3.5.3 and the previous paragraph, after a proper homotopy of \( f \), we may further assume that \( C_{i_e} := f^{-1}(C_{i_e}) \) is a (single) smoothly embedded circle on \( \Sigma' \) and \( f|C_{i_e} \to C_{i_e} \) is a homeomorphism. The previous paragraph also tells that after all these proper homotopies, \( f^{-1}(\mathcal{W}_i) \) remains non-empty.

We show that \( C_{i_e}' \) is a primitive circle on \( \Sigma' \). On the contrary, let there be an embedding \( i': \mathbb{D}^2 \hookrightarrow \Sigma' \) with \( C_{i_e}' = i'(S^1) \). Then the embedding \( f \circ i'|S^1 \hookrightarrow \Sigma \) is null-homotopic and \( C_{i_e} = f \circ i'(S^1) \). But \( C_{i_e} \) is a primitive circle on \( \Sigma \). Now, Theorem 2.2.2 tells us we have reached a contradiction. Finally, applying Lemma 3.6.2.3, we can say that \( \deg(f) = \pm 1 \).

\[ \square \]

### 3.6.3 Most pseudo proper homotopy equivalences between non-compact surfaces are maps of degree \( \pm 1 \)

**Theorem 3.6.3.1** Let \( f: \Sigma' \to \Sigma \) be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. If \( \Sigma \not\cong S^1 \times \mathbb{R}, \mathbb{R}^2 \) (equivalently \( \Sigma' \not\cong S^1 \times \mathbb{R}, \mathbb{R}^2 \)), then \( \deg(f) = \pm 1 \).

**Proof.** Combining Theorem 3.6.1.11 and Theorem 3.6.2.6, we complete the proof. \[ \square \]

The following proposition, which we don’t need to use anywhere, says that if either of the integers 1 and \( -1 \) appears as the degree of a pseudo proper homotopy equivalence between two non-compact oriented surfaces, then the other also appears.

**Proposition 3.6.3.2** Let \( f: \Sigma' \to \Sigma \) be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. Then there exists another pseudo proper homotopy equivalence \( \tilde{f}: \Sigma' \to \Sigma \) such that \( \deg(\tilde{f}) = -\deg(f) \).
Proof. Write $\Sigma$ as the double of a bordered surface $S$; see Theorem 2.4.2. Define a homeomorphism $\varphi: \Sigma \to \Sigma$ by sending $[p, t] \in \Sigma$ to $[p, 1 - t] \in \Sigma$ for all $(p, t) \in S \times \{0, 1\}$. Then $\varphi$ is an orientation-reversing homeomorphism. Therefore, the degree of $f := \varphi \circ f$ is $-\deg(f)$ as the degree is multiplicative; see Section 2.6.

3.6.4 An application of the non-vanishing degree of a pseudo proper homotopy equivalence

Consider a non-surjective map $\varphi: \mathcal{M} \to N$ between two closed, oriented, connected $n$-manifolds. Then for any $p \in N \setminus \text{im}(\varphi)$, the map $H^0(\varphi)$ factors through the inclusion-induced zero map $H^0(\mathcal{N}) \cong \mathbb{Z} \to 0 \cong H^0(N \setminus p)$ (recall that top integral singular cohomology of any connected, non-compact, boundaryless manifold is zero), i.e., $\deg(\varphi) = 0$. The lemma below generalizes this phenomenon in the proper category.

**Lemma 3.6.4.1** Let $\Phi: M \to N$ be a proper map between two connected, oriented, boundaryless, smooth $k$-dimensional manifolds. If $\deg(\Phi) \neq 0$, then $\Phi$ is surjective.

*Proof.* Being a proper map between two manifolds, $\Phi$ is a closed map; see [26]. Now, if possible, let $\Phi$ be non-surjective. Therefore, $N \setminus \Phi(M)$ is a non-empty open subset of $N$. Pick a point $y \in N \setminus \Phi(M)$. Since $N$ is locally Euclidean, there is a smoothly embedded closed ball $B \subset N$ such that $B \subset N \setminus \Phi(M)$. Notice that $N \setminus \text{int}(B)$ is a smoothly embedded co-dimension zero submanifold of $N$ with $\partial(N \setminus \text{int}(B)) = \partial B$. By Poincaré duality (see [22, Exercise 35 of Section 3.3]), $H^k_c(N \setminus \text{int}(B); \mathbb{Z}) \cong H_0(N \setminus \text{int}(B), \partial B; \mathbb{Z})$. Also, $H_0(N \setminus \text{int}(B), \partial B; \mathbb{Z}) = 0$ as $N$ is path-connected; see [22, Exercise 16.(a) of Section 2.1]. Now, $\Phi: M \to N$ can be thought as the composition $M \xrightarrow{\Phi^1} N \setminus \text{int}(B) \xrightarrow{i} N$, where $i$ is the inclusion map and $\Phi^1(m) := \Phi(m)$ for all $m \in M$. Certainly, $\Phi^1$ and $i$ are both proper maps. Therefore, $H^k_c(\Phi)$ is the composition

$$H^k_c(N; \mathbb{Z}) \xrightarrow{H^k_c(i)} H^k_c(N \setminus \text{int}(B); \mathbb{Z}) \xrightarrow{H^k_c(\Phi^1)} H^k_c(M; \mathbb{Z}),$$

i.e., $H^k_c(\Phi) = 0$, which contradicts $\deg(\Phi) \neq 0$. Thus, $\Phi$ must be a surjective map.

The above lemma, together with Theorem 3.6.3.1, gives the following corollary.

**Corollary 3.6.4.2** A pseudo proper homotopy equivalence between two non-compact surfaces is a surjective map, provided surfaces are homeomorphic to neither the plane nor the punctured plane.

The following lemma tells that one way to achieve the surjectivity throughout a proper homotopy is to assume that the initial map of this proper homotopy is a map of non-zero degree. Note that any proper map $f: X \to Y$ is properly homotopic to itself due to the proper homotopy $X \times [0, 1] \ni (x, t) \mapsto f(x) \in Y$.

**Lemma 3.6.4.3** Let $\Phi: M \to N$ be a proper map of non-zero degree between two connected, oriented, boundaryless, smooth $k$-dimensional manifolds, and let $\Psi: M \to N$ be a proper map such that $\Psi$ is properly homotopic to $\Phi$. Then $\Psi$ is a surjective map.

*Proof.* Since $\Psi$ is properly homotopic to $\Phi$, $\deg(\Psi) = \deg(\Phi) \neq 0$; see Section 2.6. Now, to conclude, consider Lemma 3.6.4.1.

Here is the main application of the non-vanishing degree of a pseudo proper homotopy equivalence.
Theorem 3.6.4.4 Let \( f : \Sigma' \to \Sigma \) be a smooth pseudo proper homotopy equivalence between two non-compact surfaces, where \( S^1 \times \mathbb{R} \not\cong \Sigma \not\cong \mathbb{R}^2 \); and let \( \mathcal{A} \) be a preferred LFCS on \( \Sigma \) such that \( f \not\cong \mathcal{A} \). Suppose any two distinct components of \( \mathcal{A} \) don’t co-bound an annulus in \( \Sigma \). In that case, \( f \) can be properly homotoped to a proper map \( g \) such that for each component \( C \) of \( \mathcal{A} \), \( g^{-1}(C) \) is a component of \( f^{-1}(\mathcal{A}) \) that is mapped homeomorphically onto \( C \) by \( g \).

**Proof.** Theorem 3.5.3 gives a proper map \( g : \Sigma' \to \Sigma \) such that the following hold: (1) \( g \) is properly homotopic to \( f \), and (2) for each component \( C \) of \( \mathcal{A} \), if \( g^{-1}(C) \not= \emptyset \), then \( g^{-1}(C) \) is a component of \( f^{-1}(\mathcal{A}) \) such that \( g|_{g^{-1}(C)} : C \to g^{-1}(C) \) is a homeomorphism. But \( \deg(f) = \pm 1 \), by Theorem 3.6.3.1. Thus, the map \( g \) is surjective since it is properly homotopic to the non-zero degree map \( f \); see Lemma 3.6.4.3. So, for each component \( C \) of \( \mathcal{A} \), \( g^{-1}(C) \) is a component of \( f^{-1}(\mathcal{A}) \) such that \( g|_{g^{-1}(C)} : C \to g^{-1}(C) \) is a homeomorphism.

Remark 3.6.4.5 For closed surfaces, the analog of Theorem 3.6.4.4 can be stated far before, exactly in the “annulus removal” section, as every homotopy equivalence between two closed manifolds has a homotopy inverse, hence is a map of degree \( \pm 1 \), and hence is surjective. But, before Section 3.6, we didn’t know the degree of a pseudo proper homotopy equivalence; even in this stage, we don’t know whether a pseudo proper homotopy equivalence has a proper homotopy inverse or not.

4 Finishing the proofs of Theorem 1, Theorem 2, and Theorem 3

**Proof of Theorem 1.** Consider an LFCS \( \mathcal{C} \) on \( \Sigma \) provided by Theorem 3.1.5. Using Theorem 3.2.1, assume \( f \) is smooth as well as \( f \not\cong \mathcal{C} \). Thus \( f^{-1}(\mathcal{C}) \) is a non-empty LFCS on \( \Sigma' \); see Corollary 3.6.4.2 and Theorem 3.2.3. By Theorem 3.6.4.4, \( f \) can be properly homotoped to a proper map \( g \) such that for each component \( C \) of \( \mathcal{C} \), \( g^{-1}(C) \) is a component of \( f^{-1}(\mathcal{C}) \) that is mapped homeomorphically onto \( C \) by \( g \). Thus, \( g^{-1}(\mathcal{C}) \) decomposes \( \Sigma' \) into bordered sub-surfaces and each component of \( \Sigma \setminus \mathcal{C} \) has non-empty pre-image; see Corollary 3.6.4.2. Let \( S \subset \Sigma \) be a bordered sub-surface obtained as a complementary component of the decomposition of \( \Sigma \) by \( \mathcal{C} \). Now, \( S \cong g^{-1}(S) \); by Lemma 3.6.1.10 (see its proof also) and Lemma 3.6.2.3. Since \( g \) sends \( \text{int}(g^{-1}(S)) \) onto \( \text{int}(S) \) and \( \partial g^{-1}(S) \) homeomorphically onto \( \partial S \), we can properly homotope \( g|_{g^{-1}(S)} : g^{-1}(S) \to S \) relative to \( \partial g^{-1}(S) \) to a homeomorphism \( g^{-1}(S) \to S \); see Theorem 3.6.1.9 and Theorem 3.6.2.4. Finally, vary \( S \) over different complementary components of \( \Sigma \) decomposed by \( \mathcal{C} \) to collect these boundary-relative proper homotopies and then paste them to get a proper homotopy from \( g \) to a homeomorphism \( \Sigma' \to \Sigma \). Since \( g \) is properly homotopic to \( f \), we are done.

The proof of Theorem 1 shows that we are using the non-zero degree assumption of the pseudo proper homotopy equivalence (which is gifted by Theorem 3.6.3.1) to ensure surjectivity after each proper homotopy. Thus, by a similar argument, we can prove the Theorem 4.1 below.

**Theorem 4.1** Let \( f : \Sigma' \to \Sigma \) be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. Suppose \( \Sigma \) is not homeomorphic to \( \mathbb{R}^2 \) and \( \deg(f) \neq 0 \). Then \( \Sigma' \) is homeomorphic to \( \Sigma \) and \( f \) is properly homotopic to a homeomorphism.

**Theorem 4.2** Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a proper map of degree \( \pm 1 \). Then \( f \) is properly homotopic to a homeomorphism \( \mathbb{R}^2 \to \mathbb{R}^2 \).

**Proof.** By Theorem 2.6.2, \( f \) can be properly homotoped to get smoothly embedded closed disks \( D, D' \subset \mathbb{R}^2 \) such that \( D' = f^{-1}(D) \) and \( f|D' \to D \) is a homeomorphism. Using the Jordan-Schönflies theorem, \( f|\mathbb{R}^2 \setminus D' \to \mathbb{R}^2 \setminus D \) resembles a map between two punctured disks, on which applying Theorem 3.6.2.4 we conclude.
where 1.1.2 terminology, X the proper transversality homotopy theorem [24, Theorem 6.36] says that for any smooth map
F: X → M and for any smoothly embedded boundaryless submanifold X of M, the smooth map F can be homotoped to another smooth map  ~F: X → M such that ~F ⊥ X. We modify these two theorems in the proper category. Our interest is in the properness of homotopies; the extra stuff not related to properness is in [24, Theorems 6.26, 6.36].

**Theorem 5.1.1** (Proper Whitney approximation theorem) Let f: N → M be a continuous proper map. Then f is properly homotopic to a smooth proper map.

**Theorem 5.1.2** (Proper transversality homotopy theorem) Let f: N → M be a smooth proper map, and let X be a smoothly embedded boundaryless submanifold of M. Then f is properly homotopic to a smooth proper map g: N → M which is transverse to X.

5 Appendix

5.1 Approximation and transversality in the proper category

Throughout Section 5.1, M, N will denote two smooth boundaryless manifolds, possibly non-compact. Let F: N → M be a smooth map, and let X be a smoothly embedded boundaryless submanifold of M. We say F is transverse to X, and write F ⊥ X, if for every p ∈ F−1(X), we have TF(p)X + DF(p)(T,N) = TF(p)M. If F is transverse to S, then F−1(X) is a smoothly embedded boundaryless submanifold of N such that dim(N) − dim(F−1(X)) = dim(M) − dim(X); see [24, Theorem 6.30.(a)].
We start by summarizing key facts in and around the tubular neighborhood theorem. Let $M \hookrightarrow \mathbb{R}^\ell$ be a smooth proper embedding; see [24, Theorems 6.15]. For each $x \in M$, define the normal space $N_xM$ to $M$ at $x$ as $N_xM := \{v \in \mathbb{R}^\ell : v \perp T_xM\}$. Then $N M := \{(x, v) \in \mathbb{R}^\ell \times \mathbb{R}^\ell : x \in M, v \perp T_xM\}$ is a smoothly embedded $\ell$-dimensional submanifold of $\mathbb{R}^\ell \times \mathbb{R}^\ell$ and $\pi : N M \ni (x, v) \mapsto x \in M$ is vector bundle of rank $\ell - \dim(M)$, called the normal bundle of $M$ in $\mathbb{R}^\ell$; see [24, Corollary 10.36].

Consider the smooth map $E : N M \ni (x, v) \mapsto x + v \in \mathbb{R}^\ell$. One can show that $dE_{(x, 0)}$ is bijective for each $x \in M$. Thus, for each $x \in M$, we have $\delta > 0$ such that $E$ maps diffeomorphically $V_\delta(x) := \{(x', v') \in N M : \|x - x'\| < \delta, \|v'\| < \delta\}$ onto an open neighborhood of $x$ in $\mathbb{R}^\ell$. Now, the map $\rho : M \to (0, 1]$ defined by

$$\rho(x) := \sup \left\{ \delta \leq 1 : E \text{ maps } V_\delta(x) \text{ diffeomorphically onto an open neighborhood of } x \text{ in } \mathbb{R}^\ell \right\}$$

is continuous. Further, $V_\delta := \{(x, v) \in N M : \|v\| < \frac{1}{2}\rho(x)\}$ is an open subset of $NM$ and $E$ maps diffeomorphically $V$ onto an open subset $U$ of $\mathbb{R}^\ell$ with $M \subseteq U$, i.e., $U$ is a tubular neighborhood of $M$ in $\mathbb{R}^\ell$; see [24, Theorem 6.24]. Note that the map $r : U \to M$ defined by $r := \pi \circ (E|V \to U)^{-1}$ is a retraction and submersion; see [24, Proposition 6.25]. Denote $\{y \in \mathbb{R}^\ell : \|y - x\| < \delta\}$ by $B_\delta(x)$. By an argument similar to showing the continuity of $\rho$, one can prove that $\delta : M \to (0, 1]$ defined by $\delta(x) := \sup \{\varepsilon \leq 1 : B_\varepsilon(x) \subseteq U\}$ for any $x \in M$, is also continuous.

With this setup, we are now ready to state a crucial lemma, which in particular says that if two points are at the most unit distance, then the distance between their images under the tubular neighborhood retraction can be at, most, 2.

**Lemma 5.1.3** Let $\varepsilon > 0$. If $y, y' \in U$ with $|y - y'| < \varepsilon$, then $|r(y) - r(y')| \leq \varepsilon + 1$.

**Proof.** Notice $|r(y) - r(y')| - |y - y'| \leq |y - r(y)| + |y' - r(y')| \leq \frac{1}{2} \rho \circ r(y) + \frac{1}{2} \rho \circ r(y')$ to conclude. $\square$

Consider another smooth proper embedding $N \hookrightarrow \mathbb{R}^k$ for the proof of the following three facts. The following lemma says that a homotopy lying in a $\lambda$-neighborhood (where $\lambda$ is a fixed positive number) of a proper map is a proper homotopy.

**Lemma 5.1.4** Let $h : N \to M$ be a continuous proper map, and let $\mathcal{H} : N \times [0, 1] \to M$ be a homotopy. If there exists a constant $\lambda$ so that $|\mathcal{H}(p, t) - h(p)| \leq \lambda$ for each $(p, t) \in N \times [0, 1]$, then $\mathcal{H}$ is proper.

**Proof.** Note that the embeddings $M \hookrightarrow \mathbb{R}^\ell$ and $N \hookrightarrow \mathbb{R}^k$ are closed maps as they are proper maps; see [26]. Consider the induced metric $d_M$ on $M$ inherited from $\mathbb{R}^\ell$, i.e., $d_M(m, m') = |m - m'|$ for all $m, m' \in M$. Also, we have the induced metric $d_N \times [0, 1]$ on $N \times [0, 1]$ inherited from $\mathbb{R}^k \times [0, 1]$, i.e., $d_N \times [0, 1](m, t), (n', t')) = |n - n'| + |t - t'|$ for all $(n, t), (n', t') \in N \times [0, 1]$. Thus, a subset of $N \times [0, 1]$ (respectively, $M$) is compact if and only if it is closed and bounded in $N \times [0, 1]$ (respectively, $M$).

Let $C$ be a compact subset of $M$. Continuity of $\mathcal{H}$ implies $\mathcal{H}^{-1}(C)$ is closed in $N \times [0, 1]$. Also, if there were an unbounded sequence $\{(n_j, t_j)\} \subseteq \mathcal{H}^{-1}(C)$, then $\{n_j\}$, and hence $\{h(n_j)\}$ would be unbounded (as $h$ is proper); and thus the unbounded set $\{h(n_j)\}$ would be inside the $\lambda$-neighborhood of the bounded set $C$, a contradiction. Therefore, $\mathcal{H}^{-1}(C)$ is closed and bounded in $N \times [0, 1]$; and hence $\mathcal{H}^{-1}(C)$ is compact. Since $C$ is an arbitrary compact subset of $M$, we are done. $\square$

Now we are ready to prove the analogs of the Whitney approximation theorem and transversality homotopy theorem in the proper category.
Proof of Theorem 5.1.1. Whitney Approximation theorem gives a smooth function \( \tilde{f} : N \to \mathbb{R}^\ell \) such that \( |\tilde{f}(y) - f(y)| < \delta(f(y)) \) for each \( y \in M \); see [24, Theorem 6.21]. Now, define \( \mathcal{H} : N \times [0, 1] \to M \) as \( \mathcal{H}(p, t) := r \left( (1 - t)f(p) + t\tilde{f}(p) \right) \) for all \( (p, t) \in N \times [0, 1] \). If \( (p, t) \in N \times [0, 1] \), then
\[
|\left( (1 - t)f(p) + t\tilde{f}(p) \right) - f(p)| \leq t \cdot |\tilde{f}(p) - f(p)| \leq 1.
\]
Therefore, \( |\mathcal{H}(p, t) - r \circ f(p)| = |\mathcal{H}(p, t) - f(p)| \leq 2 \) for all \( (p, t) \in N \times [0, 1] \) by Lemma 5.1.3. Now, Lemma 5.1.4 tells \( \mathcal{H} \) is proper. Therefore, \( \mathcal{H}(-, 1) = r \circ f \) is a smooth proper map that is properly homotopic to \( f \) (recall that \( r \) is a smooth retraction). So, we are done. \( \square \)

Proof of Theorem 5.1.2. Whitney Approximation theorem gives a smooth function \( e : N \to (0, \infty) \) with \( 0 < e < \delta \circ f \); see [24, Corollary 6.22]. Let \( B^\ell := \{ s \in \mathbb{R}^\ell : |s| < 1 \} \). Define \( F : N \times B^\ell \to M \) as \( F(p, s) := r(\tilde{f}(p) + e(p)s) \) for any \( (p, s) \in N \times B^\ell \). If \( p \in N \), the restriction of \( F \) to \( \{p\} \times B^\ell \) is the composition of the local diffeomorphism \( s \mapsto \tilde{f}(p) + e(p)s \) followed by the smooth submersion \( r \), so \( F \) is a smooth submersion and hence transverse to \( X \).

By parametric transversality theorem [24, Theorem 6.35], \( F(-, s_0) \) is transverse to \( X \) for some \( s_0 \in B^\ell \). Now, define \( \mathcal{H} : N \times [0, 1] \to M \) as \( \mathcal{H}(p, t) := r(\tilde{f}(p) + te(p)s_0) \) for all \( (p, t) \in N \times [0, 1] \). If \( (p, t) \in N \times [0, 1] \), then
\[
|\left( \tilde{f}(p) + te(p)s_0 \right) - f(p)| \leq te(p) \cdot |s_0| < \delta(f(p)) \leq 1.
\]
Therefore, \( |\mathcal{H}(p, t) - r \circ f(p)| = |\mathcal{H}(p, t) - f(p)| \leq 2 \) for all \( (p, t) \in N \times [0, 1] \) by Lemma 5.1.3. Now, Lemma 5.1.4 tells that \( \mathcal{H} \) is proper. Define \( g := \mathcal{H}(-, 1) \), i.e., \( g = r(\tilde{f}(\cdot) + e(\cdot)s_0) = F(-, s_0) \) is properly homotopic to \( f \) (recall that \( r \) is a smooth retraction) as well as transverse to \( X \). \( \square \)

5.2 Transversality of a proper map between two surfaces with respect to a circle

Here are a couple of notations that will be used throughout Section 5.2. Let \( f : \Sigma' \to \Sigma \) be a smooth proper map between two surfaces, and let \( C \) be a smoothly embedded circle on \( \Sigma \) such that \( f \) is transverse to \( C \). Also, let \( \varphi : C \times [-1, 1] \to \Sigma \) be a smooth embedding with \( \varphi(C, 0) = C \), i.e., \( \text{im}(\varphi) \) is a two-sided (trivial) tubular neighborhood of \( C \). We call each of \( \varphi(C \times [-1, 0]) \) and \( \varphi(C \times [0, 1]) \) a one-sided tubular neighborhood of \( C \) (in short, a side of \( C \)). By scaling, we may replace \([-1, 0]\) and \([0, 1]\) with other closed intervals.

The following theorem says that \( f \) is transverse to all circles, which are parallel to \( C \) and sufficiently near to \( C \).

**Theorem 5.2.1** There exists \( \varepsilon_0 \in (0, 1) \) such that \( f \) is transverse to \( C_\varepsilon := \varphi(C, \varepsilon) \) for each \( \varepsilon \in [-\varepsilon_0, \varepsilon_0] \). Thus for any \( \varepsilon \in [-\varepsilon_0, \varepsilon_0] \), \( f^{-1}(C_\varepsilon) \) is either empty or a pairwise disjoint collection of finitely many smoothly embedded circles on \( \Sigma' \).

At first, we need a lemma to prove Theorem 5.2.1.

**Lemma 5.2.2** Let \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) be a smooth map and \( x_n \to x \) in \( \mathbb{R}^2 \) with \( r_n := |g(x_n)| \to 1 \). Write \( S_r := \{ z \in \mathbb{R}^2 : |z| = r \} \) and assume \( \text{im}(dg_{x_n}) = T_{g(x_n)}(S_{r_n}) \) for all \( n \). If \( dg_x \neq 0 \), then \( \text{im}(dg_x) = T_{g(x)}(S_1) \).

**Proof.** The derivative map \( dg : \mathbb{R}^2 \to \text{L}(\mathbb{R}^2, \mathbb{R}^2) \) is continuous implies \( dg_{x_n} \to dg_x \), and this convergence can be thought as convergence of \( 2 \times 2 \)-matrices. In particular, if \( \mathbf{i}, \mathbf{j} \in \mathbb{R}^2 \) are two perpendicular unit vectors, then \( dg_{x_n}(\mathbf{i}) \to dg_x(\mathbf{i}) \) and \( dg_{x_n}(\mathbf{j}) \to dg_x(\mathbf{j}) \).
Recall that the tangent space at any point of a circle is the vector space of all points perpendicular to this point. So, \( \langle dg_{s_0}(x), g(x) \rangle = 0 = \langle dg_{s_1}(y), g(x) \rangle \) by hypothesis, and now \( \langle dg_{s_0}(x), g(x) \rangle = 0 = \langle dg_{s_1}(y), g(x) \rangle \) by the convergence of inner-product. Hence, \( \text{im}(dg_x) \subseteq T_{g(x)}(S_1) \). Since \( dg_x \neq 0 \) and \( \dim T_{g(x)}(S_1) = 1 \), we are done.

Proof of Theorem 5.2.1. Suppose not. So, a sequence \( \epsilon_n \to 0 \) and points \( x_n \in f^{-1}(C_{\epsilon_n}) \) exist such that \( \text{im}(df_{x_n}) + C_{\epsilon_n} \subseteq T_{f(x_n)} \Sigma \) for all \( n \). Hence, \( \text{im}(df_{x_n}) \subseteq T_{f(x_n)}C_{\epsilon_n} \) as \( T_{f(x_n)}C_{\epsilon_n} = N_{f(x_n)}C_{\epsilon_n} = T_{f(x_n)} \Sigma \) for all \( n \). Now, \( \{x_n\} \) is contained in the compact set \( f^{-1}(\text{im}(\varphi)) \) (recall that \( f \) is a proper map), i.e., passing to sub-sequence, if needed, assume \( x_n \to x \in f^{-1}(C) \).

The continuity of the derivative map says \( df_{x_n} \to df_x \). After discarding first few terms, we may assume \( df_{x_n} \neq 0 \) for all \( n \) (otherwise, we would have \( df_x = 0 \), i.e., \( T_{f(x)}C + \text{im}(df_x) = T_{f(x)}C \) wouldn’t be equal to \( T_{f(x)} \Sigma \), i.e., \( f \) wouldn’t be transverse to \( C \)). So, \( \text{im}(df_{x_n}) = T_{f(x_n)}(C_{\epsilon_n}) \) for all \( n \) (a non-zero vector subspace of a one-dimensional vector space is equal to the whole space).

Now, restricting \( f \) to a coordinate ball containing \( x \) and then post composing with \( \varphi^{-1} \), we can consider Lemma 5.2.2 above, which gives \( \text{im}(df_x) = T_{f(x)}(C) \), a contradiction to the assumption \( f \not\subseteq C \).

The previous theorem guarantees transversality near \( C \). In the rest part of Section 5.2, we aim to prove that every small one-sided tubular neighborhood of a component of \( f^{-1}(C) \) maps into a small one-sided tubular neighborhood of \( C \).

At first, we fix some notations. So, let \( C' \) be a component of \( f^{-1}(C) \). Also, consider an \( \epsilon_0 \in (0, 1) \) such that \( f \not\subseteq C \) for every \( \epsilon \in [-\epsilon_0, \epsilon_0] \); see Theorem 5.2.1.

Theorem 5.2.3 Let \( \epsilon \in (0, \epsilon_0) \), and let \( T' \) be a two-sided compact tubular neighborhood of \( C' \) in \( \Sigma'_1 \). Then there exist two one-sided compact tubular neighborhoods \( U'_l, U'_r \) of \( C' \) in \( \Sigma'_1 \) such that \( U'_l \cup U'_r \) is a two-sided tubular neighborhood of \( C' \) with \( U'_l \cup U'_r \subseteq T' \), and for each \( s \in \{l, r\} \) the following hold: (i) \( f^{-1}(C) \cap U'_s = C' \), (ii) either \( f(U'_{l}) \subseteq \varphi(C \times [0, \epsilon]) \) or \( f(U'_{r}) \subseteq \varphi(C \times [-\epsilon, 0]) \).

Proof. By Theorem 5.2.1, \( f^{-1}(C_{\epsilon}) \cup f^{-1}(C) \cup f^{-1}(C_{\epsilon}) \) is a pairwise disjoint collection of finitely many smoothly embedded circles on \( \Sigma'_1 \). Now, consider two one-sided compact tubular neighborhoods \( U'_l, U'_r \) of \( C' \) in \( \Sigma'_1 \) such that \( U_l \cup U_r \) is a two-sided tubular neighborhood of \( C' \) with \( U_l \cup U_r \subseteq T' \), and for each \( s \in \{l, r\} \) the following hold: (i) \( f^{-1}(C) \cap U'_s = C' \), (ii) \( U'_s \cap f^{-1}(C_{\epsilon}) = \emptyset = U'_l \cap f^{-1}(C_{\epsilon}). \)

Now, fix \( s \in \{l, r\} \). Since \( U'_s \setminus C' \) is connected and \( f \) is continuous, \( f(U'_s \setminus C') \) is contained in one of the components of \( \Sigma \setminus (C_{-\epsilon} \cup C \cup C_{\epsilon}) \). But \( f(C') \subseteq \emptyset \) implies either \( f(U'_l) \subseteq \varphi(C \times [0, \epsilon]) \) or \( f(U'_r) \subseteq \varphi(C \times [-\epsilon, 0]) \). So, we are done.

Remark 5.2.4 In Theorem 5.2.3, it is possible that \( f(U'_l \cup U'_r) \) is contained in either \( \varphi(C \times [0, \epsilon]) \) or \( \varphi(C \times [-\epsilon, 0]) \), i.e., \( f \) may map both sides of \( C' \) in one of the two sides of \( C \).

Consider the one-sided compact tubular neighborhoods \( U_l, U_r \) of \( C' \) in \( \Sigma' \) given by Theorem 5.2.3. Notice that for some \( s \in \{l, r\} \), it is possible that \( f((\partial U'_l \setminus C') \not\subseteq \varphi(C \times t) \) for any \( t \in [-\epsilon, \epsilon] \). A remedy for this is given in the following theorem.

Theorem 5.2.5 Let \( \epsilon \in (0, \epsilon_0) \), and let \( U' \) be a one-sided compact tubular neighborhood of \( C' \) such that \( f^{-1}(C) \cap U' = C' \), and \( f(U') \subseteq \varphi(C \times [0, \epsilon]) \). Then there is a \( \delta \in (0, \epsilon) \) and there is a component \( C'_\delta \) of \( f^{-1}(C_{\delta}) \) such that the following hold:

1. \( C'_\delta \) together with \( C' \) co-bounds an annulus \( A' \subseteq U' \) so that any other component of \( f^{-1}(C_{\delta}) \) in \( \text{int}(A') \), if any, bounds a disk inside \( A' \).

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(2) The map $f$ sends $A'$ into $\varphi(C \times [0, \varepsilon])$. Also, after removing the interiors of all disks bounded by components of $f^{-1}(C_\delta)$ from $A'$, we can send it to $\varphi(C \times [0, \delta])$ by $f$.

**Proof of part (1) of Theorem 5.2.5.** Choose a $\delta \in (0, \varepsilon)$ such that $\varphi(C \times [0, \delta]) \cap f((\partial U') \setminus C') = \emptyset$. Note that such a $\delta$ exists; otherwise, using the compactness of $(\partial U') \setminus C'$, we would have a sequence $\{x'_n\} \subseteq (\partial U') \setminus C'$ converging to some $x' \in (\partial U') \setminus C'$ such that $f(x'_n) \in \varphi(C \times [0, \delta/n])$, i.e., $f(x')$ would belong to $C$, a contradiction to the assumption $f^{-1}(C) \cap U' = C'$. Define an open set $W'$ as follows

$$W' := \text{int}(U') \cap f^{-1}(\varphi(C \times (0, \delta))).$$

Notice that no sequence in $W'$ converges to some point of $(\partial U') \setminus C'$. Otherwise, if we assume $W'_n \ni w'_n \longrightarrow x' \in (\partial U') \setminus C'$, then $\varphi(C \times (0, \delta)) \ni f(w'_n) \longrightarrow f(x')$. Since $\varphi(C \times [0, \delta])$ is a closed set containing the sequence $\{f(w'_n)\}$, we can say that $f(x') \in f((\partial U') \setminus C') \cap \varphi(C \times [0, \delta])$, which is impossible by our choice of $\delta$.

Therefore, $\overline{W'} \subseteq U'$ (as $U'$ is compact) but $(f((\partial U') \setminus C')) \cap \overline{W'} = \emptyset$. In particular, $\partial W' \subseteq U'$ but $(f((\partial U') \setminus C')) \cap \partial W' = \emptyset$.

**Claim 5.2.5.1** $\partial W' \subseteq C' \cup f^{-1}(C_\delta)$. Thus $\partial W'$ is contained in a finite union of pairwise disjoint circles.

**Proof of Claim 5.2.5.1.** Let $y \in \partial W'$ and consider a sequence $\{y'_n\} \subseteq \partial W'$ converging to $y'$. Then $\varphi(C \times (0, \delta)) \ni f(y'_n) \longrightarrow f(y') \in \varphi(C \times [0, \delta])$. If $f(y') \in \varphi(C \times (0, \delta)) = C \cup C_\delta$, then we are done since $f^{-1}(C) \cap U' = C'$. On the other hand, if $f(y') \in \varphi(C \times (0, \delta))$, then the definition of $W'$ and $W' \cap \partial W' = \emptyset$ (as $W'$ is open) together imply $y' \in U' \setminus \text{int}(U') = \partial U'$, i.e., $y' \in C'$ as $((\partial U') \setminus C') \cap \partial W' = \emptyset$. Since $y \in \partial W'$ is arbitrary, we are done. 

The definition of $W'$ tells that each point of $\text{int}(U')$ that is sufficiently near to $C'$ must belong to $W'$. Now, using Claim 5.2.5.1, we can say that there is at least one component of $f^{-1}(C_\delta)$, which co-bounds an annulus with $C'$ inside $U'$. Of all the $C'$-parallel components of $f^{-1}(C_\delta)$, we consider the closest to $C'$ as $C_\delta$. 

**Proof of part (2) of Theorem 5.2.5.** Certainly, $f(A') \subseteq f(U') \subseteq \varphi(C \times [0, \varepsilon])$. Now, the rest follows, once we observe that removing the interiors of all disks bounded by components of $f^{-1}(C_\delta)$ from $A'$, $A'$ remains connected, so by continuity of $f|_{\Sigma} \setminus f^{-1}(C \cup C_\delta) \rightarrow \Sigma \setminus (C \cup C_\delta)$, it maps into $\varphi(C \times (0, \delta))$. 

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