Does quantum theory apply to observers? A resurgence of interest in the long-standing Wigner’s friend paradox has shed new light on this fundamental question. Brukner introduced a scenario with two separated but entangled “friends”. Here, building on that work, we rigorously prove that if quantum evolution is controllable on the scale of an observer, then one of the following three assumptions must be false: “freedom of choice”, “locality”, or “observer-independent facts” (i.e. that every observed event exists absolutely, not relatively). We show that although the violation of Bell-type inequalities in such scenarios is not in general sufficient to demonstrate the contradiction between those assumptions, new inequalities can be derived, in a theory-independent manner, which are violated by quantum correlations. We demonstrate this in a proof-of-principle experiment where a photon’s path is deemed an observer. We discuss how this new theorem places strictly stronger constraints on quantum reality than Bell’s theorem.

I. INTRODUCTION

Wigner’s friend is a thought experiment that illustrates what is perhaps the thorniest foundational problem in quantum theory: the measurement problem. In a nutshell, the problem is how to reconcile the two rules for state evolution found in every textbook on quantum mechanics: the (unitary, deterministic) evolution of isolated systems, and the (non-unitary, probabilistic) state update after a measurement (the “collapse” of the wave function). In the thought experiment, we consider an observer who performs a measurement on a quantum system inside an isolated laboratory. In accordance with the state-update rule, the friend assigns the eigenstate corresponding to their observed outcome to the system. Meanwhile, Wigner, who is outside the laboratory, takes the perspective of a superobserver, describing the isolated laboratory and all of its contents as a unitarily evolving quantum state, in accordance with the first rule. This process, however, leads to a quantum state that does not ascribe a well-defined value to the outcome of the friend’s observation, in apparent contradiction with the friend’s perspective.

Recently, there has been a surge of renewed interest in the Wigner’s friend problem. In particular, Brukner introduced an extended Wigner’s friend scenario (EWFS) as follows. There are two spatially separated laboratories, each containing a friend, and each accompanied by a superobserver. The two superobservers perform various measurements on their associated laboratories and obtain binary outcomes. The two laboratories are prepared in a macroscopic entangled state (involving the friends), and all correlations between the results of the superobservers arise from this entanglement.

In the context of this EWFS, Brukner considered three assumptions, namely: freedom of choice, locality (in the sense of “parameter independence”) and observer-independent facts (OIF). The last of these means that every observed event exists in an absolute sense, not just relative to the observer who observes it. For convenience, we will call the conjunction of these three metaphysical assumptions Local Friendliness. Brukner then presented an argument purporting to prove a no-go theorem that we reformulate as follows.

Theorem 1 (No Local Friendliness). If a superobserver can perform arbitrary quantum operations on an observer and its environment, then no physical theory can satisfy Local Friendliness.

Brukner’s argument proceeded by claiming that Local Friendliness (LF) leads to a deterministic local hidden variable (LHV) model, and thereby to Bell inequalities. Assuming that quantum mechanics is universally valid and that each superobserver can perform suitable operations on the associated friend, the correlations of the superobservers’ results can violate those inequalities. According to Brukner’s argument, this would contradict the LF assumptions. A recent experiment, using a setup where the role of each friend is played by a single photon, demonstrated the violation of a Bell inequality in the scenario considered by Brukner.

However, as discussed by Healey, Brukner’s argument relied on an additional claim, as mentioned above, namely, that OIF in the Wigner’s friend scenario entails the following (which Brukner calls a postulate): that there is a matter of fact about the results of all measurements, even ones Wigner chose not to perform. Assuming
this postulate is equivalent to assuming that all possible measurement outcomes are predetermined by hidden variables—a very strong assumption. The claim that this postulate follows from the OIF assumption is not justified in Brukner’s paper, which places the no-go theorem put forward by him into jeopardy.

Here, we rigorously derive the implications of the LF assumptions in a theory-independent manner—that is, without relying on any assumptions specific to quantum mechanics—and prove Theorem 1. We do this without using Brukner’s postulate, and indeed show that OIF, properly formulated, does not imply this postulate. In particular, Local Friendliness does not, in an EWFS, imply hidden variables determining values for all measurement results. Therefore, the violation of a Bell inequality by correlations between these results is not always sufficient to demonstrate the failure of LF.

For the specific EWFS Brukner considered—involved two binary-outcome measurement choices per superobserver—the two sets of correlations (LF and LHV) are equivalent, but they are not equivalent in general. We derive the set of inequalities for LF for a different EWFS, with three binary-outcome measurement choices per superobserver, and show that it is possible for quantum correlations to violate a Bell inequality while satisfying all of the LF inequalities. This demonstrates that, in a general EWFS, the set of LHV correlations is a strict subset of the set of LF correlations, as illustrated in Fig. 1. We also prove that new LF inequalities we derive can nevertheless be violated by quantum correlations. We demonstrate these facts in an experimental simulation where the friends are represented by photon paths.

We now proceed to explain the background theory in more detail, before presenting our results in Sec. II and discussing them in Sec. III.

A. Wigner’s friend thought experiment

In the Wigner’s friend thought experiment we consider an observer, whom we call the friend, who performs a measurement on a quantum system S. The friend is assumed to be inside a laboratory that can be coherently controlled by a second experimenter, Wigner, who is capable of performing arbitrary quantum operations on the friend’s laboratory and all of its contents. Although this may be possible, in principle, according to quantum mechanics, it would of course be a truly Herculean task if the friend were a macroscopic observer like a human, as we have chosen for our illustrations and discussions below. For this reason Wigner is often called a superobserver. However, there is good reason to think that quantum mechanics would allow control of the type required if the friend (plus the laboratory) were an artificial intelligence algorithm in a simulated environment running in a large quantum computer. It is not out of the question that an open-minded Wigner could genuinely consider such a system to be a friend, and therefore the Wigner’s friend paradox is not necessarily doomed to forever remain a thought experiment.

Wigner associates a Hilbert space $\mathcal{H}_S$ with the system S, and a Hilbert space $\mathcal{H}_F$ with all the contents of the lab excluding S; that is, $F$ represents not just the friend and her measurement apparatus, but also—for the case that she is a human rather than a simulation—all the air molecules inside the lab, etc. Wigner initially assigns a product quantum state $|\psi_F\rangle_S \otimes |\psi_F\rangle$ in $\mathcal{H}_F \otimes \mathcal{H}_S$ to the system composed of F and S. For this example, let us suppose that the system is a spin-1/2 particle, and the friend measures the operator corresponding to spin projection along the $z$ direction, with eigenstates $|\uparrow\rangle_S$ and $|\downarrow\rangle_S$.

From Wigner’s perspective, the friend’s measurement in the $z$ basis is described as a unitary evolution $U_Z$ acting on $\mathcal{H}_F \otimes \mathcal{H}_S$ that correlates the friend (and the display on her measurement apparatus, etc.) to system S in the appropriate way. That is, if the initial state of S is $|\uparrow\rangle_S$, the final state of the joint system is $U_Z(|\phi_0\rangle_F \otimes |\uparrow\rangle_S) = |\text{up}\rangle_F \otimes |\uparrow\rangle_S$, and likewise $U_Z(|\phi_0\rangle_F \otimes |\downarrow\rangle_S) = |\text{down}\rangle_F \otimes |\downarrow\rangle_S$. In these cases, the state $|\phi_0\rangle_F$ will change to one of two values according to the result of the friend’s experiment: the states $|\text{up}\rangle_F$ and $|\text{down}\rangle_F$ will describe the system F after the states $|\uparrow\rangle_S$ and $|\downarrow\rangle_S$ are detected, respectively.

The interesting case is when, instead, S is prepared
in a superposition state, for example $\frac{1}{\sqrt{2}}(|\uparrow\rangle_S + |\downarrow\rangle_S)$. Then standard textbook quantum mechanics predicts that the friend will observe one or another outcome with equal probability, and the state of the system after measurement (and therefore that of the friend) will be one or another of the corresponding states above. On the other hand, due to the linearity of the unitary map, from Wigner’s perspective the final joint state will be $|\Phi^+\rangle_{FS} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_F |\uparrow\rangle_S + |\downarrow\rangle_F |\downarrow\rangle_S)$. This entangled state does not assign well-defined values to the states of system $S$ or $F$ separately, and therefore seems to be in direct contradiction with standard textbook quantum mechanics. This contradiction is called the measurement problem.

Indeed, if Wigner had the control over $F$ that quantum mechanics in principle allows, then he could perform a measurement of the POVM $\{ |\Phi^+\rangle_{FS}, I_{SF} - |\Phi^+\rangle_{FS} \}$. If quantum mechanics does apply at the macroscopic scale, as we are assuming, then Wigner would always get the outcome corresponding to state $|\Phi^+\rangle_{FS}$, confirming the state assignment after the unitary interaction. Had the state of $FS$ before this measurement been an equal mixture of the post-measurement states $|\uparrow\rangle_F \otimes |\uparrow\rangle_S$ and $|\downarrow\rangle_F \otimes |\downarrow\rangle_S$, Wigner would have obtained, with equal probability, either of the above outcomes.

The contradiction arises from the assumptions that (i) quantum theory is universal and can be applied at any scale, even to a macroscopic observer, and that (ii) there is an objective collapse after a measurement [8]. Thus no contradiction arises if quantum mechanics does not describe objects as large as the friend, or if the collapse of system $S$ is not an objective physical process affecting the wavefunction described by Wigner.

The latter case poses new questions, however. If wavefunction collapse is not objective, is there nevertheless an objective fact corresponding to the friend’s observed outcome? Brukner proposed a no-go theorem [2] that aims at demonstrating a contradiction between the (metaphysical) assumptions of observer-independent facts, locality and freedom of choice, and the (empirical) hypothesis that quantum mechanics is valid, and in principle allows coherent operations (such as the above measurements by Wigner) to be implemented, on the scale of a friend $F$.

It is important to emphasize that, unlike the formulation of the measurement problem sketched above, and unlike another recent no-go theorem by Frauchiger and Renner [3], this no-go theorem is like Bell’s theorem in that it is based on inequalities that are derived in a theory-independent way. The three metaphysical assumptions above are used to derive constraints on black-box (device-independent [3]) observations in the extended Wigner’s friend scenario (EWFS) mentioned earlier. The constraints do not depend on any assumption about quantum mechanics, or on the description of measurements within the theory. The hypothesis relating to the universality of quantum theory is used only to show that quantum mechanics predicts that the derived inequalities can be violated. One could instead take the ability to violate the inequalities to be an empirically determinable fact that may or may not be true. However, to make the description of the bipartite scenario easier to follow, we will use a (universally valid) quantum mechanical formalism below, and later abstract from it when deriving the Local Friendliness constraints.

**B. The extended Wigner’s friend scenario**

Following Brukner’s reasoning, we consider a bipartite version of the Wigner’s friend experiment: we now have two superobservers, Alice and Bob, and their respective friends, Charlie and Debbie (Fig. 2). Charlie and Debbie are space-like separated and in possession of systems $S_A$ and $S_B$ respectively, with associated Hilbert spaces $\mathcal{H}_{S_A}$ and $\mathcal{H}_{S_B}$, and initially prepared in a (possibly entangled) state $\rho_{S_A, S_B}$. For simplicity, we will once again suppose these systems are spin-1/2 particles. They perform a measurement of the $z$-spin of their particles, and we label their outcomes $c$ and $d$, respectively (Fig. 2). We denote everything in Charlie’s lab except $S_A$ as system $F_A$, with Hilbert space $\mathcal{H}_{F_A}$, and $F_B, \mathcal{H}_{F_B}$ for Debbie’s lab. According to Alice and Bob, Charlie’s and Debbie’s measurements have obtained, with equal probability, either of the above outcomes.

The contradiction arises from the assumptions that (i) quantum mechanics is valid, and in principle allows coherent operations (such as the above measurements by Wigner) to be implemented, on the scale of a friend $F$. Wick’s theorem predicts that the derived inequalities can
measurements are described by unitary evolutions $U_{ZA}$ and $U_{ZB}$ acting on $\mathcal{H}_{FA} \otimes \mathcal{H}_{SA}$ and $\mathcal{H}_{FA} \otimes \mathcal{H}_{SB}$, respectively.

In each iteration of the experiment, Alice and Bob randomly and independently choose one out of $N \geq 2$ measurements to be performed. The settings are respectively labelled $x \in \{1,...,N\}$ and $y \in \{1,...,N\}$, with corresponding outcomes $a$ and $b$ (we do not assume anything about the number of possible outcomes at this stage). If $x = 1$, Alice simply opens Charlie’s laboratory and directly asks him for his outcome $c$, then assigns her own outcome as $a = c$. Within quantum mechanics, Alice’s measurement for $x = 1$ could be described by the POVM \{\langle c |_{FA} \otimes I_{SA} \rangle \}, where $|c\rangle_{FA}$ represents the state of Charlie and his lab after seeing outcome $c$ and $I_{SA}$ is the identity operator on $\mathcal{H}_{SA}$. For $x \in \{2,...,N\}$, Alice will perform a measurement on $\mathcal{H}_{FA} \otimes \mathcal{H}_{SA}$. Bob and Debbie operate in a similar fashion.

From this experiment we can reproduce the empirical probabilities $\psi(ab|xy)$, that can be obtained using only the information available at the end of the experiment, namely, the values for $a$, $b$, $x$, and $y$. Unless $x = 1$, all records for the value of $c$ are erased when Alice performs her measurement, so in general that information can not be accessed at the end of the experiment, and likewise with the value of $d$.

With the above bipartite scenario in mind, we consider the following assumptions, previously described in [2]. Here we give the exact wording as in Brukner’s Theorem 1; in the following section we give more precise formulations which we believe reflect Brukner’s intended meanings. Observer-independent facts: One can jointly assign truth values to the propositions about observed outcomes (“facts”) of different observers. Locality: The choice of measurement settings has no influence on the outcomes of distant measurements. Freedom of choice: The choice of measurement settings is statistically independent from the rest of the experiment.

We call the conjunction of the three assumptions above “Local Friendliness”. Brukner’s Theorem 1 states that Local Friendliness contradicts the empirical hypothesis of universal validity of quantum theory: Quantum predictions hold at any scale, even if the measured system contains objects as large as an “observer” (including her laboratory, memory etc.).

Again we have reproduced Brukner’s words here. As explained earlier, what is actually required is not just that quantum theory is universally valid at the scale of an observer, but that the (quantum) physical laws of our universe allow, in principle, a physical observer to implement coherent quantum operations on that scale. More specifically, the hypothesis is that an LF inequality can be violated in an extended Wigner’s friend scenario.

II. RESULTS

A. Formalization of the Local Friendliness assumptions

Within a bipartite Wigner’s experiment, what constraints do the LF assumptions imply for the probabilities $\psi(ab|xy)$ observed by Alice and Bob for outcomes $a$ and $b$, given settings $x$ and $y$? To determine this rigorously we need to formalize Brukner’s three assumptions.

1. Observer-independent facts

In [2], Brukner considered the following question: “Is there a theoretical framework, potentially going beyond quantum theory, in which one can account for observer-independent facts, ones that hence can be called ‘facts of the world’? In such a framework, one could assign jointly truth values to both the observational statement $A_1$: ‘The pointer of Wigner’s friend’s apparatus points to result ↑’ and $A_2$: ‘The pointer of Wigner’s apparatus points to result $\Phi^+$.’ He then claimed that to capture the assumption of observer-dependent facts “we require an assignment of truth values to statements $A_1$ and $A_2$ independently of which measurement Wigner performs”.

As pointed out by Healey in [6], this claim by Brukner is not justified by any reasonable interpretation of OIF. The claim amounts to saying that OIF requires that there exists a non-contextual hidden-variable model [9], and this is what allowed Brukner to immediately derive the Bell-type inequality in [2].

Here we do not assume that all statements about results have truth values independently of which measurement ‘Wigner’ (whom we call Alice) performs. Instead, we note (contrary to Brukner’s claim above) that the assumption of observer-independent facts only entails assigning truth values to propositions about observed outcomes. In particular, Alice’s measurement outcome $A_2$ (which in our notation corresponds to the value of $a$ when she performs measurement $x = 2$) has a value only when she performs that measurement. This is in keeping with Peres’ dictum “unperformed experiments have no results” [10]; observer-independent facts is the assumption that performed experiments have observer-independent (i.e. absolute) results.

Assumption 1 (Observer-Independent Facts (OIF)). An observed event is a real single event, and not ‘relative’ to anything or anyone.

In [11], this assumption was called “macroreality”, in the context of the derivation of Bell inequalities. In an EWFS, the assumption of observer-independent facts implies that in each run of the experiment—i.e. given that Alice has performed measurement $x$ and Bob has performed measurement $y$ on a given pair of systems—there exists a well-defined value for all observed outcomes $a$, $b$, $...$
c and d. Formally, this implies that there exists a theoretical joint probability distribution \( P(\text{abcd}|\text{xy}) \) from which the empirical probability \( \varphi(ab|xy) \) can be obtained while also ensuring that the observed outcomes for \( x, y = 1 \) are consistent between the superobservers and the friends.

- **Observer-independent facts (in the EWFS):**
  \[ \exists P(\text{abcd}|\text{xy}) \text{ s.t.} \]
  
  i) \( \varphi(ab|xy) = \sum_{c,d} P(\text{abcd}|\text{xy}) \forall a, b, x, y, \)
  
  ii) \( P(a|cd, x = 1, y) = \delta_{a,c} \forall a, c, d, y, \)
  
  iii) \( P(b|cd, x = 1, y) = \delta_{b,d} \forall b, c, d, x. \)

2. **Freedom of choice**

The Freedom of Choice assumption here is intended to play the same role as it does in derivations of Bell inequalities. It is the assumption that the experimental settings can be chosen freely, that is, uncorrelated with any relevant variables prior to that choice. For added clarity, here we formulate it, following [11] (where it was called “No Superdeterminism”), as

**Assumption 2 (Freedom of Choice (FC)).** Any set of events on a space-like hypersurface is uncorrelated with any set of freely chosen actions subsequent to that space-like hypersurface.

In the EWFS, this formally implies that \( c \) and \( d \) are independent of the choices \( x \) and \( y \):

- **Freedom of choice (in the EWFS and under assumption 1):**
  \( P(\text{cd}|\text{xy}) = P(\text{cd}) \forall c, d, x, y. \)

3. **Locality**

Finally, the assumption of locality prohibits the influence of a local setting (such as \( x \)) on a distant outcome (such as \( b \)). Brukner does not formally define this, but he clearly intends the assumption that Bell in 1964 [12], and many others subsequently, also called locality [11], and which Shimony called “parameter independence” [7]. That is, in the formalization of [11], the assumption that

**Assumption 3 (Locality (L)).** The probability of an observable event \( e \) is unchanged by conditioning on a space-like-separated free choice \( z \), even if it is already conditioned on other events not in the future light-cone of \( z \).

In the EWFS, this formally implies:

- **Locality (in the EWFS and under assumption 1):**
  \( P(a|cdxy) = P(a|cdx) \forall a, c, d, x, y, \)
  \( P(b|cdxy) = P(b|cdy) \forall b, c, d, x, y. \)

Note that \( c, d \) play the formal role, within the definitions of locality and freedom of choice, of the hidden variables \( \lambda \) in the usual derivation of Bell inequalities. However, we emphasize once more that those correspond to observed events, and note that we make no assumption about hidden variables predetermining all measurement outcomes.

We call the set of correlations \( \varphi(ab|xy) \) that satisfy Assumptions 1–3 the Local Friendliness correlations.

B. **Properties of LF correlations**

In this section, we summarize our main results on the properties of LF correlations, with the detailed derivations provided in the Supplementary Materials. In the Supplementary Materials, section S.2, we derive the general form of \( \varphi(ab|xy) \) for an LF model, from which it is shown that an LHV correlation for a bipartite Wigner’s friend scenario will also be an LF correlation. The opposite is not necessarily true, however. In fact, in Sec. II C, we will show an example of LF correlations that are not LHV correlations. Thus, LHV correlations always form a subset of LF correlations, and sometimes a strict subset.

Next, we tackle the problem of characterizing the set of LF correlations. In the Supplementary Materials, section S.2 we show that in a general scenario with \( N \) measurement settings per party and \( O \) outcomes for each measurement, the set of LF correlations is the convex hull of a finite number of extreme points. Therefore, it is a polytope.

For the scenario of two measurement settings \( (N = 2) \), we can recover an LHV model for any value of \( O \) (see the Supplementary Materials, section S.2). By contrast, for the scenario of three measurement settings and binary measurement outcomes \( (N = 3, O = 2) \), the LF polytope is considerably more difficult to construct than the corresponding LHV polytope. We derive in the Supplementary Materials, section S.2 that the set of LF correlations is a polytope with 932 facets. The facets can be grouped into 9 inequivalent classes, each represented by a different inequality (provided in the Supplementary Materials, section S.2). These classes can be further grouped into categories, according to the measurement settings involved, and whether the facets are Bell facets [13]. In Table I we list the categories of LF facets, ignoring all positivity facets, i.e., the constraints that probabilities cannot be negative.

We call inequalities that are not facets of the LHV polytope for this scenario “Genuine LF” inequalities. An example is “Genuine LF”:

\[-\langle A_1 \rangle - \langle A_2 \rangle - \langle B_1 \rangle - \langle B_2 \rangle - \langle A_1 B_1 \rangle - 2\langle A_1 B_2 \rangle - 2\langle A_2 B_1 \rangle + 2\langle A_2 B_2 \rangle \leq 0. \]

Here and throughout the remainder of the article, we
use $A_x \in \{+1, -1\}$ as the random variable for Alice’s outcome $a$ when she chooses setting $x$, and similarly $B_y$. That is, the expectation values are calculated from the empirical probabilities $\rho(ab|xy)$.

The Bell $I_{3322}$ inequalities are a class of facet-defining Bell inequalities for the case of three binary-outcome measurement settings per party [14], for example “$I_{3322}$”:

$$
\begin{align*}
-\langle A_1 \rangle + \langle A_2 \rangle + \langle B_1 \rangle - \langle B_2 \rangle \\
+\langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle - \langle A_2 B_3 \rangle - \langle A_2 B_1 \rangle \\
+\langle A_2 B_2 \rangle - \langle A_2 B_3 \rangle - \langle A_3 B_1 \rangle - \langle A_3 B_2 \rangle - 4 \leq 0.
\end{align*}
$$

Brukner’s inequality is the Clauser-Horne-Shimony-Holt (CHSH)-type inequality [15] that appeared in [2]. “Brukner”:

$$
\begin{align*}
\langle A_1 B_1 \rangle - \langle A_1 B_3 \rangle - \langle A_2 B_1 \rangle - \langle A_2 B_3 \rangle - 2 \leq 0.
\end{align*}
$$

Interestingly, the inequality we call “Semi-Brukner” has a simpler experimental realization than Brukner’s inequality, as it only requires one of the parties to measure a friend (setting 1), yet its violation also demonstrates the failure of Local Friendliness. “Semi-Brukner”:

$$
\begin{align*}
-\langle A_1 B_2 \rangle + \langle A_1 B_3 \rangle - \langle A_2 B_2 \rangle - \langle A_3 B_3 \rangle - 2 \leq 0.
\end{align*}
$$

To facilitate a comparison of LF models with LHV models, we also list the additional category of “Bell non-LF” in Table I. This is the category of conventional Bell facets that are not facets of LF. An example is “Bell non-LF”:

$$
\begin{align*}
\langle A_2 B_2 \rangle - \langle A_2 B_3 \rangle - \langle A_3 B_2 \rangle - \langle A_3 B_3 \rangle - 2 \leq 0.
\end{align*}
$$

### C. Quantum violations

We now search for quantum violations of the LF inequalities. Following the description in Sec. [13] Charlie and Debbie initially share an entangled state $\rho_{S_A S_B}$ of two qubits. Charlie’s measurement of $S_A$ in the basis $\{|-1\rangle_{S_A}, |+1\rangle_{S_A}\}$ is described by a unitary $U_{Z_A}$ acting on $\mathcal{H}_{FA} \otimes \mathcal{H}_{SA}$. Alice’s $x = 1$ measurement (corresponding to opening the box and asking Charlie what he saw) can be described by a POVM $\{|c\rangle\langle c|_{FA} \otimes I_{S_A}\}_c$, where $|c\rangle_{FA} (c \in \{-1,+1\})$ represents the state of Charlie after seeing outcome $c$ and $I_{S_A}$ is the identity operator on $\mathcal{H}_{SA}$. The theorem makes no assumption about the form of the measurements that Alice performs for $x \in \{2,3\}$, but in our experimental realization, we consider the class of measurements that reverse the evolution $U_{Z_A}$ that tangled $FA$ with $S_A$ (Fig. 3B), followed by a measurement on $S_A$ alone (Fig. 3C). This can be described by a POVM with elements $U_{Z_A} (I_{FA} \otimes E_{SA}^{b|x} U_{Z_A}^{-1})$, where $I_{FA}$ is the identity on $\mathcal{H}_{FA}$ and $E_{SA}^{b|x}$ is the positive operator associated with outcome $a$ for measurement $x$ that Alice performs directly on $S_A$.

Bob’s POVM elements $E_{SA}^{b|x}$ are defined analogously. Thus, the maximum violation of the inequalities can be sought simply in measurements acting on the Hilbert spaces $\mathcal{H}_{SA}$ and $\mathcal{H}_{SB}$; since Charlie and Debbie start in a known product state, there is no advantage in considering arbitrary measurements on $\mathcal{H}_{FA} \otimes \mathcal{H}_{SA}$ and $\mathcal{H}_{FB} \otimes \mathcal{H}_{SB}$.

To demonstrate that the set of LF correlations is strictly larger than the LHV correlations, we are interested in finding a state and measurement choices such that a violation of a “Bell non-LF” inequality is exhibited while having no violation in any of the LF inequalities. For reasons of experimental convenience, we consider two-qubit photon polarization states of the form

$$
\rho_\mu = \mu |\Phi^-\rangle\langle \Phi^-| + \frac{1-\mu}{2} (|HV\rangle\langle HV| + |HV\rangle\langle VV|),
$$

where $|\Phi^-\rangle = (|HV\rangle - |VV\rangle)/\sqrt{2}$, $0 \leq \mu \leq 1$, and $H$ and $V$ denote horizontal and vertical polarizations, respectively. We restrict ourselves to projective measurements confined to the $XY$ plane of the Bloch sphere (with states $|H\rangle$ and $|V\rangle$ on the $z$-axis). In particular, Alice’s measurement results are represented by operators of the form $A_x = 2\Pi_{x=1}^{z} - |H\rangle\langle H| - |V\rangle\langle V|$, with $\Pi_{x=1}^{z} = |\phi_x\rangle\langle \phi_x|$ being the projector onto the state

$$
|\phi_x\rangle = \frac{1}{\sqrt{2}} (|H\rangle + e^{i\phi_x}|V\rangle).
$$

Bob’s corresponding operators are chosen to be $B_y = 2\Pi_{y=1}^{b} - |H\rangle\langle H| - |V\rangle\langle V|$, with $\Pi_{y=1}^{b} = |\beta_y\rangle\langle \beta_y|$ being the projector onto

$$
|\beta_y\rangle = \frac{1}{\sqrt{2}} (|H\rangle + e^{i\beta - \phi_y}|V\rangle).
$$

### Table I. Categorization of LF and Bell inequalities for the scenario of three binary-outcome measurement settings per party.

| Label | Measurement Settings | LF inequality? | Bell facet? |
|-------|----------------------|----------------|------------|
| Brukner | $(1,i,1)$ | Yes | Yes |
| Semi-Brukner | $(1,i,23)$ | Yes | Yes |
| Bell non-LF | $(23,23)$ | No | Yes |
| $I_{3322}$ | $(123,123)$ | Yes | Yes |
| Genuine LF | $(123,123)$ | Yes | No |
The bipartite Wigner’s friend experiment. (A) When $x = 1$, Alice opens Charlie’s laboratory and asks him his outcome. (B) Alternatively, for $x = 2, 3$, she may restore the laboratory to a previous state. (C) She then proceeds to ignore Charlie, and performs a measurement directly on the particle.

We performed a numerical search where, for each value of the tetrad $(\phi_1, \phi_2, \phi_3, \beta)$, and for each category in Table I, we find the smallest value of $\mu$ for which one of the inequalities in that category is violated. We then picked a tetrad (see the caption of Fig. 4) that makes the gap between these values of $\mu$ conveniently large. For the inequality in each category that is violated first, we display the values of the left-hand side as a function of $\mu$ in Fig. 4.

We study the EWFS with three measurement settings $(N = 3)$ in an experiment where the systems distributed between the two laboratories are polarization-encoded photons, the friends are photon paths within the setup, and the measurements by the superobservers are photon-detection measurements. Since the qubit comprised of the two photon paths that represents each of our friends would not typically be considered a macroscopic, sentient observer as originally envisioned by Wigner, our experiment is best described as a microscopic proof-of-principle version of the EWFS. The experiment lets us demonstrate the key properties of LF inequalities and its results generalize provided that quantum evolution is in principle controllable on the scale of an observer. A fully rigorous demonstration that the LF assumptions are untenable would require, in addition to a plausibly sentient friend, closing separation, efficiency, and freedom-of-choice loopholes, similarly to the case of Bell inequality violations [16–18].

Our experimental setup, which comprises a photon source and a measurement section, is illustrated in Fig. 5. The photon source, shown in the left half of Fig. 5, is designed to generate the quantum state $\rho_{\mu}$ of Eq. (6) with a tunable $\mu$ parameter. Details about this spontaneous parametric down-conversion source are provided in Materials and Methods.

The measurement section of the experimental setup, shown in the right half of Fig. 5, consists of two copies of an apparatus, one belonging to Alice and Charlie, and the other to Bob and Debbie. The measurement section serves two purposes. The first is to perform quantum state tomography in order to characterize the generated quantum state. To allow tomography, the motorized mir-
FIG. 5. **Experimental setup.** The source is depicted on the left-hand side, and the measurement section on the right-hand side. The desired quantum state is generated via type-I spontaneous parametric down-conversion using two orthogonally oriented BiBO crystals. The crystals are pumped with a mixture of a diagonally polarized state, coming from the short arm of the interferometer, and a decohered state, coming from the long arm. In the measurement section, tomography can be performed when the motorized mirrors are removed such that the photon pairs pass through the beam displacer (BD) interferometers. The tomography stages also transform into the projective measurement stages of Alice and Bob when the quarter-wave plates (QWP) are removed. Charlie and Debbie’s projective measurements correspond to the beam paths within the interferometers, so that Alice and Bob can ask their respective friends for their measurement outcomes by inserting the motorized mirrors. Abbreviations used: Non-polarizing beam splitter (NPBS), potassium titanyl phosphate (KTP), half-wave plate (HWP), avalanche photodiode (APD), polarization control (PC), polarizing beam splitter (PBS).

The second purpose of the measurement section is to perform the projective measurements of the observers. The friend’s projective polarization measurement corresponds to the photon path after the QWP, HWP, and BD1. Hence, when measurement setting 1 is chosen, the motorized mirror is inserted to reveal the photon path within the interferometer, which corresponds to Alice asking Charlie his measurement outcome (or Bob asking Debbie on the other side). When measurement settings 2 or 3 are chosen, the polarization is measured after the interferometer, with the motorized mirror and the QWP after BD2 removed. This corresponds to Alice (Bob) implementing one of her (his) other two measurements, depending on which one of two settings of the last HWP is used. Single photons are detected with avalanche photodiodes (APDs) and coincidences are recorded with counting modules.

To obtain the expectation values required for the inequalities being tested at each $\mu$ value, we performed the nine sets of measurements that arise from combining the three independent measurement settings on Alice’s and Bob’s sides.

The experimental results are shown in Fig. 4. The $\mu$ values cover the full range of interest, from none of the inequalities being violated (at low $\mu$), to the violation of all inequalities (at high $\mu$). The experimental data demonstrates the sequential violations of the Bell non-LF, Semi-Brukner, and Genuine LF inequalities. The data points corresponding to $\mu = 0.80$ and $\mu = 0.81$ are of particular significance, as they demonstrate that it is possible to violate Bell inequalities without violating any LF inequalities. (We can be confident of this because we verified that the remaining 928 LF inequalities are also not violated.) This means that the correlations allowed by an LHV model are a subset of the correlations allowed by LF assumptions. The case of $\mu = 0.87$ is the first of the plotted data sets where a contradiction with the LF
assumptions occurs, through the first violation of an inequality associated with LF. Finally, the two highest $\mu$ values verify that the genuine LF inequality can also be violated. All the experimental data points, except for the case of $\mu = 0.81$, are at least two standard deviations from away 0, thus attesting the violation of non-violation of the inequalities with statistical significance. This covers all the different regions we show in terms of (non-)violation of different inequalities, since the data set at $\mu = 0.81$ belongs to the same region as $\mu = 0.80$. Along with the experimental data, the results predicted for the design measurement directions and input states of Eq. (6) are shown by the solid lines. We note, however, that the inequalities are device-independent, therefore our conclusions are independent of which states and measurement directions were actually employed in the experiment.

### III. DISCUSSION

In this work, we have provided the first rigorous proof of the theorem stated in [2]. That is, we have proven that the joint assumption of observer-independent facts, locality and freedom of choice (which we call Local Friendliness) is incompatible with the empirical predictions of quantum mechanics if one observer (a ‘superobserver’) can manipulate the quantum state ascribed to another observer (a ‘friend’).

To establish this, we considered Brukner’s extended Wigner’s friend scenario [2], with two superobservers and two friends. We derived the correlations compatible with the Local Friendliness assumptions in a theory-independent manner. We showed that an LF model does not entail the same correlations as an LHV model. In particular, we proved that all LHV correlations are LF correlations, but not vice versa for the case of three measurement choices for each superobserver. We have further established that quantum correlations can violate the new LF inequalities we discovered for this case, giving a novel proof of the LF no-go theorem.

It is interesting to compare the assumptions that go into the LF no-go theorem with those for Bell’s theorem. First, we note that the assumption of observer-independent facts is implicit in the derivation of Bell inequalities (for a derivation in which it is explicitly included see [11], where it is called “macroreality”). If, as is common, we also formulate Bell’s theorem using the other two assumptions of LF, namely freedom of choice and locality (or “parameter independence”), then an additional assumption is also required. The minimal extra assumption required is “outcome independence” [7], which in the bipartite scenario is the requirement that $P(a|x y \lambda) = P(a|x \lambda), P(b|a x y \lambda) = P(b|x y \lambda)$ for all $a, b, x, y, \lambda$ (c.f. the definition of locality in Sec. II.A). The above makes it clear that the LF assumption is strictly weaker than the set of assumptions for Bell inequalities. Thus, the conclusions we could derive from an empirical violation of the LF inequalities are strictly stronger.

In particular, one popular way to accommodate the violation of Bell inequalities is to reject outcome independence (which is violated by operational quantum theory [11]) while maintaining locality and freedom of choice. The present theorem shows that this strategy does not extend to Brukner’s scenario. If the LF inequalities were violated empirically, then in order to maintain locality and freedom of choice, one would have to reject observer-independent facts.

It is important to keep in mind that it is much harder to satisfy the conditions for an experimental violation of the LF inequalities than of Bell inequalities. A fully convincing demonstration would require a strong justification for the attribution of a “fact” to the friend’s measurement. This, of course, depends on what counts as an “observer”. Since conducting this kind of experiment with human beings is physically impractical, what do we learn from experiments with simpler “friends”?

Wigner’s own conclusion from his thought experiment was that the collapse of the wave function should happen at least before it reaches the level of an “observer”. The concept of an “observer”, however, is a fuzzy one. In objective collapse theories [19], a necessary condition for a physical system to count as an “observer” is that it causes objective collapse of the quantum state after measurement. In that case, the LF inequalities would not be violated with actual observers. Clearly, our experiment (and that of [4]) did not probe collapse theories. Therefore, an open possibility is that the Local Friendliness assumptions are valid, but that nature forever forbids the observation of violation of LF inequalities with sentient friends, whether because of objective collapse or some other limitation on coherent quantum control.

A challenge to the above resolution of the LF no-go theorem could come from experiments involving AI agents in a quantum computer. If universal quantum computation and strong AI are both physically possible, it should be possible to realize quantum coherent simulations of an observer and its (virtual) environment, and realize an extended Wigner’s friend experiment. While there could be ethical implications for the realization of this experiment with agents that we would regard as sentient, this prospect is physically much more plausible than that of realizing the experiment with human beings. Towards that goal, experiments can test agents of increasing complexity; an experimental violation of LF inequalities with a given class of physical systems as “friends” implies that either the LF assumption is false, or that class of friends is not an “observer”. At the same time, more prosaic loopholes in our experiment could be overcome, by implementing shot-by-shot randomized measurement settings, space-like separation between the parties’ events, and high detection efficiencies.

Among interpretations of quantum mechanics that allow, in principle, the violation of LF inequalities, Theorem 1 can be accommodated in different ways. In-
interpretations that reject observer-independent facts include Quantum Bayesianism \cite{20}, the relational interpretation \cite{21}, and the Many Worlds interpretation \cite{22}. Bohmian mechanics \cite{23, 24} violates locality but not the other assumptions. Some authors advocate giving up freedom of choice (either due to retrocausality \cite{25}, superdeterminism \cite{26}, or other mechanisms); however, as yet, no such interpretation has been proposed that reproduces all the predictions of quantum mechanics.

For the quantum state and the set of measurements that we have studied, the violation of Genuine LF inequalities occurs after some of the other LF inequality categories, i.e. at a higher $\mu$ value. It remains an open question whether any of the Genuine LF inequalities can be violated by quantum states while none of the other LF facets are violated.

Finally, we note that after one of us presented the results of this article in a recent conference, it was brought to our attention that the Local Friendliness polytopes have been independently studied in \cite{27} under the name of “partially deterministic polytopes”, from an information-theoretic motivation: they are connected to the problem of device-independent randomness certification (see, e.g., \cite{28, 30} and references therein) in the presence of non-signalling adversaries.

IV. MATERIALS AND METHODS

A. Spontaneous parametric down-conversion source

The source is made up of an imbalanced pump-beam interferometer (one arm of the interferometer is longer than the other) and two orthogonally oriented (sandwiched) bismuth triborate (BiBO) crystals \cite{31}, which are pumped by a 404 nm continuous wave laser diode to produce spontaneous parametric down-conversion. The relative pump power in the interferometer arms determines the $\mu$ parameter of the state and is controlled by the half-wave plate (HWP) after the laser. When all the pump power is in the short arm, the first term of the quantum state, the singlet state, is generated (after a local polarization rotation in the fiber). Conversely, when all the pump power is in the long arm, only the second term, a mixed state, is generated. The beams in both arms are recombined in the non-polarizing beam splitter to pump the sandwiched crystal, generating the desired quantum state.

The polarization in the short arm is rotated to diagonal by a HWP and an additional birefringent element is used to pre-compensate the temporal walk-off in the down-conversion. The polarization in the long arm is also rotated to diagonal by a HWP and a birefringent crystal decoheres the horizontal and vertical polarization components completely, which is necessary to generate the mixed part of the state.

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S.1. LHV CORRELATIONS AS A SUBSET OF LF CORRELATIONS

Recall that a set of correlations has a LHV model if and only if there exists a probability distribution $P(\lambda)$ over a set of variables $\lambda \in \Lambda$ such that

$$\varphi(ab|xy) = \sum_{\lambda \in \Lambda} P(a|x\lambda)P(b|y\lambda)P(\lambda),$$  \hspace{1cm} (S.1)

for all values of the variables $a, b, x, y$. We now derive the general form for an LF model. From observer-independent facts and freedom of choice, we have that:

$$\varphi(ab|xy) = \sum_{c,d} P(ab|cxy) = \sum_{c,d} P(ab|cdxy)P(cd).$$  \hspace{1cm} (S.2)

From locality, we can decompose the first term on the right-hand side in two ways:

$$P(ab|cdxy) = P(a|bcdxy)P(b|cdy)$$
$$= P(a|cdxy)P(b|acy).$$  \hspace{1cm} (S.3)

or

$$P(ab|cdxy) = P(ab|cdxy)P(b|acdx)$$
$$= P(a|cdx)P(b|acdx).$$  \hspace{1cm} (S.4)

Note, however, that we cannot further reduce these expressions with locality alone—reinforcing the fact that locality is a weaker assumption than local causality (which leads to a LHV model). However, by construction, when $x = 1$ we have $a = c$, and when $y = 1$, $b = d$. Then, if $x = 1$, $P(a|bcx) = \delta_{a,c}$, and if $y = 1$, $P(b|acdx) = \delta_{b,d}$. When taking this, along with Eqs. (S.3) and (S.4), into account, we obtain from Eq. (S.2)

$$\varphi(ab|xy) = \begin{cases} 
\sum_{c,d} \delta_{a,c} P(b|cdy)P(cd) & \text{if } x = 1 \\
\sum_{c,d} \delta_{b,d} P(a|cdx)P(cd) & \text{if } y = 1 \\
\sum_{c,d} P_{NS}(ab|cdxy)P(cd) & \text{if } x \neq 1, y \neq 1
\end{cases},$$  \hspace{1cm} (S.5)

where $P_{NS}(ab|cdxy)$ denotes some joint probability distribution that satisfies the condition of locality. For any fixed values of $c$ and $d$, it is easy to see that the set of $P_{NS}(ab|cdxy)$ is simply the no-signalling polytope with one less measurement setting for both Alice and Bob [32] (thus the NS subscript). In general, because of the additional structure given by the first two lines of Eq. (S.5), the set of LF correlations only forms a subset of the no-signalling polytope.

To see that LHV correlations are also LF correlations, we first recall from [13] that correlations of the form of Eq. (S.1) can always be decomposed in terms of the extreme points of the set of such correlations. To this end, it is expedient to write the hidden variable as $\lambda = (\lambda^A_1, \lambda^B_1, \lambda^A_2, \ldots \lambda^B_N)$, with $\lambda^A_1$ and $\lambda^B_y$ parameterizing all possible local deterministic strategies, i.e.,

$$P(a|x\lambda) = \delta_{a,\lambda^A_1}, \quad P(b|y\lambda) = \delta_{b,\lambda^B_y}.$$  \hspace{1cm} (S.6)

We may now rewrite Eq. (S.1) as:

$$\varphi(ab|xy) = \delta_{a,\lambda^A_1} \delta_{b,\lambda^B_y} P(\lambda).$$  \hspace{1cm} (S.7)
This is now readily cast in the form of Eq. (S.5) if we set \(\lambda_1^A = c\) and \(\lambda_1^B = d\). For example, if \(x = 1\), we get

\[
\psi(ab|x = 1, y) = \sum_{c,d,\lambda_2^A,\ldots,\lambda_N^A} \delta_{a,c} \delta_{b,\lambda_2^y} P(cd\lambda_2^A \ldots \lambda_N^B),
\]

\[
= \sum_{c,d,\lambda_2^A} \delta_{a,c} \delta_{b,\lambda_2^y} P(cd\lambda_2^B),
\]

\[
= \sum_{c,d} \delta_{a,c} \left[ \sum_{\lambda_2^y} \delta_{b,\lambda_2^y} P(\lambda_2^B|cd) \right] P(cd),
\]

\[
= \sum_{c,d} \delta_{a,c} P(b|cdy) P(cd),
\]

which is clearly of the form given in the first line of Eq. (S.5). The proof for the \(y = 1\) case is completely analogous.

Similarly, for the case where \(x \neq 1\), \(y \neq 1\), we can again make use of \(\lambda_1^A = c\), \(\lambda_1^B = d\) and Eq. (S.7) to arrive at:

\[
\psi(ab|xy) = \sum_{c,d,\lambda_2^A,\ldots,\lambda_N^B} \delta_{a,\lambda_2^A} \delta_{b,\lambda_2^y} P(cd\lambda_2^A \cdots \lambda_N^B),
\]

\[
= \sum_{c,d,\lambda_2^A,\lambda_2^B} \delta_{a,\lambda_2^A} \delta_{b,\lambda_2^B} P(cd\lambda_2^A \lambda_2^B),
\]

\[
= \sum_{c,d} \left[ \sum_{\lambda_2^A,\lambda_2^B} \delta_{a,\lambda_2^A} \delta_{b,\lambda_2^B} P(\lambda_2^A \lambda_2^B|cd) \right] P(cd),
\]

\[
= \sum_{c,d} P(ab|cdxy) P(cd).
\]

From the second last line of Eq. (S.9) and the fact that \(a\) (or \(b\)) is entirely decided by \(\lambda_2^A\) (or \(\lambda_2^B\)), we see that \(P(ab|cdxy)\) in the last expression satisfies the condition of locality (i.e., \(\sum_a P(ab|cdxy)\) does not depend on \(y\) while \(\sum_b P(ab|cdxy)\) does not depend on \(x\)). Thus, starting from LHV correlations for \(x \neq 1\), \(y \neq 1\), we recover the last line of Eq. (S.5).

Hence any correlation that satisfies Eq. (S.1) will also satisfy Eq. (S.5). Yet, the opposite is not necessarily true. Therefore, LHV correlations are a subset of LF correlations.

### S.2. Characterization of LF Correlations

Consider first a general scenario with \(N\) measurement settings per party, with \(O\) outcomes each. Note that we can always rewrite Eq. (S.5) in the form

\[
\psi(ab|xy) = \begin{cases} 
\sum_{\lambda} \delta_{a,c(\lambda)} \psi_{\text{ext}}^{(j(\lambda))}(b|y) P(\lambda) & \text{if } x = 1 \\
\sum_{\lambda} \delta_{b,d(\lambda)} \psi_{\text{ext}}^{(j(\lambda))}(a|x) P(\lambda) & \text{if } y = 1 \\
\sum_{\lambda} \psi_{\text{ext}}(ab|xy) P(\lambda) & \text{otherwise,}
\end{cases}
\]

where \(\lambda\) is a variable that determines the values of \(c(\lambda), d(\lambda)\), and that of a variable \(j(\lambda)\) that labels the (finitely many) extreme points of the no-signalling polytope with \(N - 1\) inputs and \(O\) outputs per party, and \(\psi_{\text{ext}}^{(j)}(ab|xy) = \sum_b \psi_{\text{ext}}^{(j)}(a,b|xy)\) and \(\psi_{\text{ext}}^{(j)}(b|y) = \sum_a \psi_{\text{ext}}^{(j)}(a,b|xy)\) are the marginal distributions of these extremal boxes.

It is easy to see from the above that this set of correlations is convex. That is, for any two points \(\psi_1(ab|xy)\) and \(\psi_2(ab|xy)\), both satisfying the LF conditions, any convex combination \(\psi'(ab|xy) = \alpha \psi_1(ab|xy) + (1 - \alpha) \psi_2(ab|xy)\), with \(0 < \alpha < 1\), also satisfies those conditions. The set of LF correlations is therefore a polytope.

For the two-measurement-setting case (\(N = 2\)), the \(\psi_{\text{ext}}^{(j)}(a,b|xy)\) now refer only to the case \(x = y = 2\), and the extreme points are now simply deterministic functions for \(a, b\). Thus, we recover an LHV model for any value of \(O\), in agreement with Brukner’s result for \(N = O = 2\).

Next, we consider the LF polytope for the \(N = 3, O = 2\) scenario. Without loss of generality, we label the outcomes as \(a, b \in \{+1, -1\}\). From Eq. (S.10), the set of LF correlations \(\tilde{\psi} = \{\psi(a, b|xy)\}_{a,b=\pm1,x,y=1,2,3}\) is the convex hull of
the extreme points \(\{\varphi^\lambda(a,b|x,y)\}_\lambda\) defined by

\[
P^{(\lambda)}(a, b|x, y) = \begin{cases} 
  \delta_{a,c(\lambda)}\delta_{b,d(\lambda)} : x = y = 1 \\
  \delta_{a,c(\lambda)}\varphi^{(j(\lambda))}(b|y) : x = 1, y \neq 1 \\
  \varphi^{(j(\lambda))}(a|x)\delta_{b,d(\lambda)} : x \neq 1, y = 1 \\
  \varphi^{(j(\lambda))}(a, b|x, y) : x \neq 1, y \neq 1
\end{cases}
\] (S.11)

Since there are four combinations of \((c, d)\) corresponding to \(2^2\) local deterministic strategies for the first inputs, and 24 extreme points for the aforementioned no-signalling polytope \([32]\), we thus end up with 96 points in this set.

By writing the components of these points in a text file and feeding the latter into the freely available software PANDA—which allows one to transform between the two representations of a polytope using the parallel adjacency decomposition algorithm \([33]\)—we obtain the complete set of 932 LF facets for this scenario. Many of these inequalities can be transformed from one to another under a relabeling of parties (Alice ↔ Bob), inputs \((x = 2 \leftrightarrow x = 3\) and/or \(y = 2 \leftrightarrow y = 3\), and/or outputs \((a = +1 \leftrightarrow a = -1\) and/or \(b = +1 \leftrightarrow b = -1\)). With the exception of the settings for \(x = 1\) and \(y = 1\), the rest of these labelings are arbitrary. Taking advantage of this arbitrariness, we may group the obtained facets into the following 9 inequivalent classes (written in terms of correlators, where \(A_i\) is a random variable representing measurement \(x = i\) and taking values \(-1, +1\), and similarly for \(B_j\)):

1. Genuine LF facet 1 (appearing 256 times among the 932 facets):

\[
-\langle A_1 \rangle - \langle A_2 \rangle - \langle B_1 \rangle - \langle B_2 \rangle
-\langle A_1 B_3 \rangle - 2\langle A_1 B_2 \rangle - 2\langle A_2 B_1 \rangle + 2\langle A_2 B_2 \rangle
-\langle A_2 B_3 \rangle - \langle A_3 B_2 \rangle - \langle A_3 B_3 \rangle \leq 6
\] (S.12a)

2. Genuine LF facet 2 (appearing 256 times):

\[
-\langle A_1 \rangle - \langle A_2 \rangle - \langle A_3 \rangle - \langle B_1 \rangle
-\langle A_1 B_1 \rangle - \langle A_2 B_1 \rangle - \langle A_3 B_1 \rangle - 2\langle A_1 B_2 \rangle
+\langle A_2 B_2 \rangle + \langle A_3 B_2 \rangle - \langle A_2 B_3 \rangle + \langle A_3 B_3 \rangle \leq 5
\] (S.12b)

3. Bell \(I_{3322}\) \([14]\) with marginals over input 1 and 2 (appearing 256 times):

\[
-\langle A_1 \rangle + \langle A_2 \rangle + \langle B_1 \rangle - \langle B_2 \rangle
+\langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle - \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle
+\langle A_2 B_3 \rangle - \langle A_3 B_2 \rangle - \langle A_3 B_3 \rangle \leq 4
\] (S.12c)

4. Bell \(I_{3322}\) with marginals over input 2 and 3 (appearing 64 times):

\[
-\langle A_2 \rangle - \langle A_3 \rangle - \langle B_2 \rangle - \langle B_3 \rangle
-\langle A_1 B_2 \rangle + \langle A_1 B_3 \rangle - \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle
-\langle A_2 B_3 \rangle + \langle A_3 B_1 \rangle - \langle A_3 B_2 \rangle - \langle A_3 B_3 \rangle \leq 4
\] (S.12d)

5. “Brukner inequality”: Bell-CHSH for input 1 and 2 (appearing 32 times):

\[
\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \leq 2
\] (S.12e)

6. “Semi-Brukner” inequality: Bell-CHSH for input 2, 3 of Alice, and input 1, 2 of Bob (appearing 32 times):

\[
\langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle + \langle A_3 B_1 \rangle - \langle A_3 B_2 \rangle \leq 2
\] (S.12f)

7. Positivity for input 1 of Alice and input 1 of Bob (appearing 4 times):

\[
1 + \langle A_1 \rangle + \langle B_1 \rangle + \langle A_1 B_1 \rangle \geq 0
\] (S.12g)

8. Positivity for input 1 of Alice and input 2 of Bob (appearing 16 times):

\[
1 + \langle A_1 \rangle + \langle B_2 \rangle + \langle A_1 B_2 \rangle \geq 0
\] (S.12h)
9. Positivity for input 2 of Alice and input 2 of Bob (appearing 16 times):

\[ 1 + \langle A_2 \rangle + \langle B_2 \rangle + \langle A_2 B_2 \rangle \geq 0 \]  

(S.12i)

Note that some Bell facets for this scenario are not facets of LF and thus do not appear in the list above, e.g., the Bell-CHSH inequalities that do not include any input 1 for either party:

\[ \langle A_2 B_2 \rangle + \langle A_2 B_3 \rangle + \langle A_3 B_2 \rangle - \langle A_3 B_3 \rangle \leq 2 \]  

(S.13)

### S.3. MAXIMAL QUANTUM VIOLATIONS OF THE GENUINE LF INEQUALITIES

By implementing a see-saw type algorithm (see, e.g., [34–36] and references therein), one finds that the Genuine LF inequality 1 (S.12a), with an LF upper bound of 6, can be violated by quantum correlations up to 7.345 using a partially entangled two-qubit state (with Schmidt coefficients 0.776 and 0.631) and rank-1 projective measurements. Moreover, it can be verified by solving a converging hierarchy [37–39] of semidefinite programs that this quantum violation is (within a numerical precision of $10^{-7}$) the maximum allowed in quantum theory. In terms of noise robustness, this quantum strategy can tolerate up to 18.3% of white noise before it stops beating the LF bound.

For Genuine LF inequality 2 (S.12b) (with an LF upper bound of 5), the best quantum violation that we have found is 5.880, which apparently can only be achieved using a partially entangled two-qutrit state (with Schmidt coefficients 0.645, 0.570, and 0.509) and a combination of rank-2 and rank-1 projectors in the optimal measurements. As with the case of Genuine LF inequality 1, this quantum violation is provably optimal (within a numerical precision of $10^{-7}$) using the solution obtained from solving some semidefinite programs. The white-noise tolerance of this inequality is somewhat worse than the other Genuine LF inequality, giving approximately 18.0%.

### S.4. FURTHER INFORMATION ABOUT FIGURE 1

Here, we provide further details on the 2-dimensional slice of the space of correlations presented in Fig. 1. Any such 2-dimensional slice is spanned by three affinely-independent correlations in this space (see, e.g., [40]). In our case, the chosen slice is spanned by the uniform (white-noise) distribution \( \bar{\varphi}_0 \)

\[ \varphi_0(ab|xy) = \frac{1}{4}, \quad \forall a, b, x, y, \]  

(S.14)

an extreme point of the LF polytope:

\[
\varphi_{LF}^{Ext}(ab|xy) = \delta_{xy,1}\delta_{a,-1}\delta_{b,1} \\
+ \frac{1}{2} \left[ \delta_{x,1}\delta_{a,-1}(1 - \delta_{y,1}) + \delta_{y,1}\delta_{b,1}(1 - \delta_{x,1}) \right] \\
+ \frac{1}{4} \left[ 1 + (-1)^{xy-x-y}ab \right] (1 - \delta_{x,1})(1 - \delta_{y,1}),
\]

(S.15)

and a symmetrical quantum correlation, written in the Collins-Gisin form (see, e.g., Eq. (9) of [14]):

\[
\varphi_Q^{Max} : \begin{bmatrix}
0.554 & 0.197 & 0.040 & 0.537 \\
0.040 & 0.021 & 0.311 & 0.150 \\
0.057 & 0.150 & 0.040 & 0.109
\end{bmatrix},
\]

(S.16)

i.e., the \( i \)-th row of the left-most column represent Alice’s marginal probability \( \varphi_Q^{Max}(+1|x = i - 1) \), the \( j \)-column of the top row represent Bob’s marginal probability \( \varphi_Q^{Max}(+1|y = j - 1) \), while the remaining entries at the \( i \)-th and \( j \)-th column represent the joint probability \( \varphi_Q^{Max}(+1,+1|x = i - 1, y = j - 1) \). The quantum correlation \( \varphi_Q^{Max} \) is the one that maximally violates Genuine LF inequality 1, giving a value of 7.345, as explained in section S.3.

In our plot, we have chosen the left-hand side of Eq. (S.12a) to label our horizontal axis and the negative of the left-hand side of Eq. (S.12b) to label our vertical axis. Different choices would lead to affine transformations of the plot. Also shown in the figure are a dashed vertical line and a dashed horizontal line intersecting at \( \bar{\varphi}_{LF}^{Ext} \). These dashed lines mark a projection of the boundary of the LF polytope—as given by inequality (S.12a) and a relabeling of inequality (S.12b) to give a lower bound of -2 as allowed by LF correlations—on the plane that we have chosen. Note also that the set of LHV correlations (coloured green in the figure) could also touch this boundary of -2, but this does not take place on the 2-dimensional plane that we have chosen.