Repeated differentiation suppresses superoscillations

M V Berry

H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, United Kingdom
E-mail: asymptotico@bristol.ac.uk

Received 25 August 2021, revised 21 October 2020
Accepted for publication 23 October 2020
Published 4 November 2020

Abstract
Two mathematical phenomena with applications in physics are: superoscillations, in which band-limited functions oscillate more rapidly than their fastest Fourier component; and the transformation of almost any smooth function into a monochromatic oscillation under repeated differentiation. These are opposite phenomena, and one mutates into the other, i.e. superoscillations are destroyed, as the number of derivatives increases. This behaviour is explained, and illustrated with an example.

Keywords: Fourier, monochromaticity, high derivatives, asymptotics

1. Introduction
Two counterintuitive properties of functions $f(x)$ are in a sense opposites. The first is superoscillation: if $f(x)$ is band-limited, it can nevertheless vary arbitrarily faster than its highest Fourier component [1, 2]. Many applications are being studied, including optics, quantum weak measurement and signal processing [3]. The second property is that for a wide class of functions $f(x)$, sufficiently high derivatives $f^{(p)}(x)$ ($p \gg 1$) tend, up to scaling and phase shift, to $\exp(ix)$, or $\cos x$ for real functions: $\exp(ix)$ is the universal attractor of the derivative operator [4–6]. Applications include high-order corrections to geometric phases [7], behaviour of high cumulants of counting statistics [8, 9], and properties of Riemann’s zeta function [6].

The pure monochromatic oscillation $\exp(ix)$ that emerges during repeated differentiation of $f(x)$ contrasts with superoscillation, which can be interpreted as the ultimate polychromaticity: scales of oscillation not only inside but far outside the Fourier content of $f(x)$. The implication is that repeated differentiation of a superoscillatory $f(x)$ will suppress the superoscillations,
leading to functions with modest oscillations that are no faster than the highest Fourier component. In section 2, the mechanism of this suppression will be described, and illustrated with an example. The concluding section 3 includes a brief discussion of how the suppression appears in phase-space representations of the derivatives $f^{(p)}(x)$.

2. Suppression theory

It is convenient to consider band-limited functions that are periodic in $x$ and can therefore be represented as a finite Fourier series:

$$f(x) = \sum_{n=N_-}^{N_+} f_n \exp(inx). \quad (2.1)$$

To save writing, and with no loss of essential generality, we consider $f_n$ real, and assume $N_+ > |N_-|$. The $x$ period is $2\pi$ and the highest Fourier component is $n = N_+$ (we consider later the case where $N_+ = -N_- = N$). For non-periodic functions, the theory to follow is similar, with incommensurate frequencies replacing $n$ in the sum, or an integral replacing the sum.

A convenient measure of the local oscillations is the phase gradient of $f(x)$. This has several interpretations [10]: local wavenumber, local expectation value of momentum, weak value of momentum with position post-selected, and the mean momentum of the Wigner or Husimi function at position $x$. Convenient forms are

$$k(x) = \partial_x \arg f(x) = \text{Im} \frac{\partial_x f(x)}{f(x)} = \text{Re} \left[ \frac{\sum_{n=N_-}^{N_+} n f_n \exp(inx)}{\sum_{n=N_-}^{N_+} f_n \exp(inx)} \right]. \quad (2.2)$$

The function is superoscillatory at $x$ if $k(x) > N_+$ or $k(x) < N_-$. This property is associated with the zeros of $f(x)$, and can be interpreted as a feature of almost-perfect destructive interference between the Fourier components $n$. (Alternative measures of superoscillations are possible, for example the number of superoscillations, or their strength. These are implicit in the behaviour of $k(x)$, as will be described later for the explicit example (2.8)).

The $p$th derivative is

$$f^{(p)}(x) = \partial_x^p f(x) = \sum_{n=N_-}^{N_+} (in)^p f_n \exp(inx). \quad (2.3)$$

Thus $f(x) = f^{(0)}(x)$. For these band-limited functions, the high derivatives provide a very simple illustration of the more general universal monochromatisation phenomenon:

$$f^{(p)}(x) \rightarrow (iN_+)^p f_{N_+} \exp(iN_+x). \quad (2.4)$$

The local wavenumber of the $p$th derivative is

$$k_p(x) = \partial_x \arg f^{(p)}(x) = \text{Re} \left[ \frac{\sum_{n=-N_+}^{N_+} n^{p+1} f_n \exp(inx)}{\sum_{n=-N_+}^{N_+} n^p f_n \exp(inx)} \right] \rightarrow N_+. \quad (2.5)$$

This shows that even if $f(x)$ is superoscillatory its high derivatives are not: their local oscillations are consistent with the Fourier content of $f(x)$. 

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A possible exception to (2.5) is when $N_+ = -N_- = N$. Then the highest and lowest Fourier components both contribute to the high derivatives, and

$$f^{(p)}(x) \to (iN)^p (f_N + (-1)^p f_{-N} \exp(-iNx)).$$

(2.6)

For $p \gg 1$, the local wavenumber is

$$k_p(x) \to \frac{f_N^2 - f_{-N}^2}{f_N^2 + f_{-N}^2 \pm 2f_N f_{-N} \cos(2Nx)} \left(\begin{array}{c} +p \text{ even} \\ -p \text{ odd} \end{array}\right).$$

(2.7)

In this case, $k_p(x)$ varies between $N(f_N \mp f_{-N})/(f_N \pm f_{-N})$; one of these extremes always exceeds $N$, so the high derivatives exhibit the modest superoscillations of functions with two Fourier components [10]. Even in this case, the magnitudes $|f_N|$ and $|f_{-N}|$ are often very different, and the difference increases with $N$; then one of the two oscillatory exponentials dominates for all $p$, and again the superoscillations are suppressed: $|k_p(x)| \to N$ as $p \to \infty$. This happens in the example to be considered in the paragraphs that follow. But an important further exception is the case of real functions; their high derivatives behave like $\cos(Nx)$, and superoscillations sometimes require slightly different treatment, as explained elsewhere [11].

We illustrate the suppression of superoscillations with what has become the canonical function [2], written in the $2\pi$ periodic form (2.1) with $N_+ = -N_- = N$:

$$f(x) = \left(\cos\left(\frac{x}{2}\right) + ia \sin\left(\frac{x}{2}\right)\right)^{2N}, \quad (a > 1)$$

(2.8)

For small $|x|$, $f(x) = \exp(iNax + O(x^3))$,

(2.9)

indicating that $f(x)$ is superoscillatory at the origin, with the local oscillations faster by a factor $a$ than the band-limit $N$. The function is illustrated in figure 1(a), using the familiar device of plotting the real part, to show the oscillations, using its logarithm, i.e. $\log[\Re f^{(p)}(x)]$, because the function is very small near $x = 0$ relative to its maximum at $x = \pi$; $|f(0)/f(\pi)| = a^{-2N})$. The suppression of the fast oscillations near $x = 0$ as $p$ increases is clear from figures 1(a), (c), (e) and (g).

For the undifferentiated function, the local wavenumber $k(x)$ is, from (2.5),

$$k(x) = \frac{Na}{\cos^2(2x) + a^2 \sin^2(2x)}.$$

(2.10)

As $x$ varies from 0 to $\pm \pi$, $k(x)$ decreases from its superoscillatory value $aN$ to the value $Na$ within the Fourier spectrum. Expansion to order $x^2$ shows that the size of the region of superoscillations is $O(1/\sqrt{N})$. Therefore $a$ governs the strength of the superoscillations, and $N$ governs their number, which is $O(\sqrt{N})$. Figure 1(b) illustrates this behaviour. As $p$ increases, the superoscillations get weaker, as illustrated in figures 1(b),(d),(f) and (h).

In the expansion (2.1), the coefficients, and their behaviour at the band-limits of the spectrum, are

$$f_n = \frac{(2N)!}{4^n (N+n)! (N-n)!} \frac{(a+1)^{N+n}}{(a-1)^n},$$

(2.11)

$$f_{\pm N} = \frac{(a \pm 1)^{2N}}{4^n}.$$
Figure 1. Suppression of superoscillations (monochromatisation) for the $p$th derivative of the function (2.8), for $a = 4$, $N = 3$, and (a, b) $p = 0$, (c, d) $p = 2$, (e, f) $p = 4$, (g, h) $p = 8$. The left panels (a, c, e, g) illustrate the function, and the right panels (b, d, f, h) illustrate the local wavenumber, with the horizontal lines indicating the band limit $N$. In (g) and (h) the dashed curves show the large $p$ approximations (2.6) and (2.7). Note the very different values of the ordinates in each panel as $p$ increases.

The alternating signs $(-1)^{N+n}$ represent the almost-destructive interference near $x = 0$ responsible for the superoscillations. The limiting coefficients $n = \pm N$ shows that it is the component $n = +N$, rather than $n = -N$, that dominates the high derivatives for $N \gg 1$, and then the approximation (2.7) takes the simple form

$$k_p(x) \xrightarrow{p \gg 1, N \gg 1} \frac{N}{1 \pm 2 \left(\frac{a}{p+1}\right)^2 N \cos(2Nx)} \left(\begin{array}{c} N \gg 1, \frac{a}{p+1}N \approx \frac{a}{p+1}N \approx 1, +p \text{ even} \end{array}\right).$$

(2.12)
Figure 2. Suppression of superoscillations with increasing differentiation \( p \) for the function (2.8) with \( a = 4 \) and (a) \( N = 2 \), (b) \( N = 3 \), (c) \( N = 4 \). The horizontal lines indicate the band limits \( N \), about which the asymptotic value of \( k_p(0) \) alternates as \( p \) increases, according to the + and − signs in (2.12), with the alternations weaker for larger \( N \).

with the local wavenumber oscillating weakly about the large \( p \) limit \( k_p = N \).

A quantitative measure of the suppression of the superoscillations as \( f(x) \) is repeatedly differentiated is the decrease of the maximum wavenumber \( k_p(0) \) as \( p \) increases. Expansion in powers of \( N \) shows that the initial decrease is linear in \( p \), with a coefficient independent of \( N \):}

\[
k_p(0) = aN - \frac{a^2 - 1}{2a}p + \frac{a^2 - 1}{4a^3}Np(p - 1) + O\left(\frac{1}{N^2}\right).
\]  

(2.13)

This behaviour is illustrated in figure 2.

How many derivatives guarantee that superoscillations will be suppressed? From the general theory, this requires \( p > p^* \) such that the contribution \( N^pf_{N+} \) from the band limit dominates all other terms in the Fourier series for \( f^{(p)}(x) \). It often happens that the highest Fourier coefficient \( f_{N+} \) is not the largest, and this must be taken into account when estimating \( p^* \). As an illustration, for the canonical superoscillatory function (2.8) the distribution of the \( |f_N| \) in (2.11) is closely approximated by a Gaussian centred on \( n^* = N/a \) [12]: well within the Fourier spectrum. Using Stirling’s formula, the maximum coefficient is

\[
|f_{n^*}| \approx \frac{a^{2N+1}}{\sqrt{\pi N (a^2 - 1)}},
\]  

(2.14)

As \( N \) increases, this rapidly dominates the band-limiting coefficient \( f_N \) in (2.11). For example,

\[
\begin{align*}
  a = 4, N = 3 : & \quad |f_{-N}| = 11.4, \quad |f_n| = 1318.4, \quad |f_N| = 244.1, \\
  a = 4, N = 4 : & \quad |f_{-N}| = 25.6, \quad |f_n| = 18457.0, \quad |f_N| = 15259, \\
  a = 4, N = 10 : & \quad |f_{-N}| = 3325.7, \quad |f_n| = 1.924 \times 10^{11}, \quad |f_N| = 9.095 \times 10^7.
\end{align*}
\]  

(2.15)
An estimate of $p^*$ is

$$N^p f_N = n^p p^* f_p^* \Rightarrow p^* \approx \frac{2N \log \left( \frac{2n}{n+1} \right)}{\log a}.$$  

(2.16)

So, more derivatives are required for increasing range $\sqrt{N}$ of the superoscillations, and fewer as their strength $a$ increases.

3. Concluding remarks

The foregoing analysis has shown how under repeated differentiation superoscillatory functions transform into their opposite—functions that are monochromatic, oscillating with the frequency of the band limit. This behaviour applies not just to band-limited functions such as the canonical (2.8), but more widely. Two examples are: backflow in waves [13, 14], where all Fourier components (momenta) are positive, corresponding to $N_- = 0$ and $N_+ > 0$ in (2.1), yet the local wavenumber (momentum) $k(x)$ can be negative; and relativistic waves, where the group velocities in plane-wave solutions of the Klein–Gordon or Dirac equation superpositions are all subluminal [15], yet there can be regions of spacetime where the local group velocity is superluminal. Repeated differentiation will destroy such behaviour.

Although modest superoscillations are surprisingly common, for example in waves [16, 17], strong superoscillations such as those considered here are exotic, and delicate because they are always exponentially weak in comparison to regions where functions are not superoscillatory. Therefore they are vulnerable to more than repeated differentiation. For example, noise also suppresses superoscillations, as explained elsewhere [18, 19].

Phase space representations of functions, such as Wigner [20–25] and Husimi [26, 27], can also display superoscillations [28], so such representations should also encode their suppression under repeated differentiation. A brief discussion will suffice. In the Wigner function $W(x, k)$ representing $f(x)$, the local wavenumber $k(x)$ is the mean value of $k$, with weight $W(x, p)$, for fixed $x$. In this formalism, superoscillations are not immediately discernible, because $W(x, k)$ inherits the band-limitedness (in $k$) of $f(x)$; the average, when this is superoscillatory, can lie outside the spectrum because $W(x, k)$ can be negative and so is not a phase-space probability density.

More interesting is the Husimi function [26, 27]. This is the square of the Gauss-windowed Fourier transform of $f(x)$. For the periodic functions (2.1) considered here, it is convenient to define the periodised window

$$\exp \left( -\frac{(x - y)^2}{4L^2} \right) \bigg|_{\text{per}} = \sum_{n=-\infty}^{\infty} \exp \left( -\frac{(x - y - 2\pi n)^2}{4L^2} \right),$$

(3.1)

and hence the Husimi function representing $f^{(p)}(x)$:

$$H_p(x, k) = \frac{1}{4\pi L^2} \left| \int_{-\pi}^{\pi} dy \ f^{(p)}(y) \exp(-iky) \left[ \exp \left( -\frac{(x - y)^2}{4L^2} \right) \right]_{\text{per}} \right|^2$$

(3.2)

$$= \left| \sum_{n=-N}^{N} f_n n^p \exp \left( \text{i}x - L^2(n-k)^2 \right) \right|^2.$$

(Alternatively interpreted, $H_p$ is the Gauss-smoothed version of the Wigner function representing $f^{(p)}(x)$, with $x$ width $L$ and $k$ width $1/L$, or as the square of the overlap of $f^{(p)}$ with a coherent...
state with the same $x$ and $k$ widths). The Husimi function is never negative, and the windowing destroys the band-limited property of $f^{(p)}(x)$, so in $H_p(x,k)$ momenta $k$ appear that are outside the band limits of $f(x)$, including those representing superoscillations. The $k$ mean of $H_p(x,k)$ is simply the local wavenumber $k_p(x)$ weighted by $|f_p(x)|^2$ and $L$-smoothed in $x$ [28]:

$$
k_{H,p}(x) = \frac{\sum_{k=-\infty}^{\infty} k H_p(x,k)}{\sum_{k=-\infty}^{\infty} H_p(x,k)} = \frac{\int_{-\pi}^{\pi} dy \, k_p(y) |f^{(p)}(y)|^2 \left[ \exp \left( -\frac{(y-x)^2}{2L^2} \right) \right]_{\text{per}}}{\int_{-\pi}^{\pi} dy \, |f^{(p)}(y)|^2 \left[ \exp \left( -\frac{(y-x)^2}{2L^2} \right) \right]_{\text{per}}}.
$$

(3.3)

As $L \to 0$, $k_{H,p}(x) \to k_p(x)$, revealing the increasingly $p$-suppressed superoscillations. As $L \to \infty$, $k_{H,p}(x)$ tends to the spectral mean, i.e. the average of $n$ with density $|n^{(p)} f_n|^2$, which lies within the spectrum and so is never superoscillatory.

Superoscillation and its suppression can be discerned in 3D or contour plots of the Husimi function (not shown here), but such pictures are not very informative because the large values of $k$ contributing to the mean in the $x$ regions where superoscillation occurs correspond to extremely small values of $H_p(x,k)$, and are easily masked by the much larger values for surrounding values of $x$, even in logarithmic plots.

Acknowledgments

I thank David Farmer for sending me reference [6], thereby stimulating the work reported here. My research is supported by an Emeritus Fellowship from the Leverhulme Trust.

ORCID iDs

M V Berry https://orcid.org/0000-0001-7921-2468

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