Instantons and odd Khovanov homology

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ABSTRACT

We construct a spectral sequence from the reduced odd Khovanov homology of a link converging to the framed instanton homology of the double cover branched over the link, with orientation reversed. Framed instanton homology counts certain instantons on the cylinder of a 3-manifold connect-summed with a 3-torus. En route, we provide a new proof of Floer’s surgery exact triangle for instanton homology using metric stretching maps, and generalize the exact triangle to a link surgeries spectral sequence. Finally, we relate framed instanton homology to Floer’s instanton homology for admissible bundles.

1. Introduction

Given a closed, connected, oriented 3-manifold $Y$, we study the framed instanton homology $I^\#(Y)$, an absolutely $\mathbb{Z}/4$-graded abelian group that is an invariant of the oriented homeomorphism type of $Y$. This group is obtained by counting, in a suitable sense, $SO(3)$-instantons on a bundle over $\mathbb{R} \times (Y \# T^3)$ which is non-trivial when restricted to the 3-tori. These groups are a special case of Floer’s instanton homology for admissible bundles from [14, §1] and are considered by Kronheimer and Mrowka in [22, §4.3; 23, §4.1]. The terminology framed is from [23].

Given an oriented link $L$ in $S^3$, we relate the framed instanton homology of $\Sigma(L)$, the double cover of $S^3$ branched over $L$, to the reduced odd Khovanov homology of $L$. This latter invariant, written as $\text{Kh'}(L)$, was defined by Ozsváth, Rasmussen and Szabó in [33]. It is an abelian group bigraded by a quantum grading, $q$, and a homological grading, $t$. It is a variant of Khovanov homology, defined originally by Khovanov in [19].

**Theorem 1.1.** Given an oriented link $L$ in $S^3$, there is a spectral sequence whose second page is $\text{Kh'}(L)$ that converges to $I^\#(\Sigma(L))$. Each page of the spectral sequence comes equipped with a $\mathbb{Z}/4$-grading, which on $\text{Kh'}(L)$ is given by

$$\frac{1}{2}q - t + \frac{1}{2}(\sigma + \nu) \mod 4,$$

where $\sigma$ and $\nu$ are the signature and nullity of $L$, respectively, and the induced $\mathbb{Z}/4$-grading on $I^\#(\Sigma(L))$ is the usual one.

Our convention is that the signature of the right-handed trefoil is $+2$. The theorem immediately implies the four rank inequalities

$$\text{rk}_{\mathbb{Z}} \text{Kh'}(L)_j \geq \text{rk}_{\mathbb{Z}} I^\#(\Sigma(L))_j,$$

where $j \in \mathbb{Z}/4$ and the gradings are as in the theorem. Non-split alternating links, and more generally quasi-alternating links as introduced in [34], have the property that $\text{Kh'}(L)$ is supported in the even gradings of (1.1) and is a free abelian group of rank $\det(L)$; see [29, Theorem 1; 33, §5] and the remarks in [28, §9.3]. In these cases, the spectral sequence collapses.

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COROLLARY 1.2. If \( L \) is a quasi-alternating link, then \( I^\#(\Sigma(L)) \) is free abelian of rank \( \det(L) \) and is supported in even gradings. The rank in grading \( j \in \{0, 2\} \subset \mathbb{Z}/4 \) is given by

\[
\frac{1}{2} [\det(L) + (-1)^{j/2} 2^{|\#L| - 1}],
\]

where \( \#L \) is the number of components of \( L \).

If rational coefficients are assumed, of the 250 prime knots that have at most 10 crossings, only 7 of them have potentially non-trivial differentials after the \( E^2 \)-page of Theorem 1.1. This follows from the computations of odd Khovanov homology in [33].

To put Theorem 1.1 into context, we relate framed instanton homology to previously studied instanton invariants. To start, framed instanton homology is a special case of Floer’s instanton homology for admissible bundles. An \( SO(3) \)-bundle \( \mathbb{Y} \) over a connected 3-manifold \( Y \) is admissible if either \( Y \) is a homology 3-sphere, in which case \( \mathbb{Y} \) is trivial, or there is an oriented surface \( \Sigma \subset Y \) with \( \mathbb{Y}|_{\Sigma} \) non-trivial. This latter condition guarantees that \( \mathbb{Y} \) does not support any reducible flat connections.

A geometric representative for \( \mathbb{Y} \) is an unoriented, closed 1-manifold \( \omega \subset Y \) with \([\omega] \in H_1(Y; \mathbb{F}_2)\) Poincaré dual to \( w_2(\mathbb{Y}) \). The non-trivial admissibility condition for \( \mathbb{Y} \) amounts to the existence of an oriented surface \( \Sigma \subset Y \) that intersects \( \omega \) in an odd number of points, or, equivalently, to the conditions that \([\omega]\) is non-zero and lifts to a non-torsion class in \( H_1(Y; \mathbb{Z}) \).

For admissible \( \mathbb{Y} \), Floer defined in [13, 14] a relatively \( \mathbb{Z}/8 \)-graded abelian group \( I(\mathbb{Y}) \). When \( Y \) is a homology 3-sphere and \( \mathbb{Y} \) is trivial, we write \( I(Y) \) for this group, and it comes equipped with an absolute \( \mathbb{Z}/8 \)-grading. The isomorphism class of \( I(\mathbb{Y}) \) depends only on the oriented homeomorphism type of \( Y \) and \( w_2(\mathbb{Y}) \).

Now let \( \mathbb{Y} \) be any \( SO(3) \)-bundle over a closed, connected, oriented 3-manifold \( Y \) geometrically represented by \( \lambda \). Making some inessential choices, we can construct a bundle \( \mathbb{Y}^\# \) over \( Y \# T^3 \) by gluing together \( \mathbb{Y} \) and a non-trivial bundle over \( T^3 \). The bundle \( \mathbb{Y}^\# \) is always admissible, and the group \( I(\mathbb{Y}^\#) \) is always 4-periodic. The framed instanton homology twisted by \( \lambda \), written as \( I^\#(Y; \lambda) \), is relatively \( \mathbb{Z}/4 \)-graded and isomorphic to four consecutive gradings of \( I(\mathbb{Y}) \).

When \( \lambda \) is mod 2 null-homologous, we recover the framed instanton homology \( I^\#(Y) \).

When \( Y \) is a homology 3-sphere, we relate \( I^\#(Y) \) to Floer’s \( \mathbb{Z}/8 \)-graded \( I(Y) \). It is convenient to employ Frøyshov’s reduced groups \( \tilde{I}(Y) \) from [15], which are obtained from \( I(Y) \) by considering interactions with the trivial connection. They come equipped with an absolute \( \mathbb{Z}/8 \)-grading and a degree 4 endomorphism \( u \). Frøyshov’s Theorem 10 says \((u^2 - 64)^n = 0\) for some \( n > 0 \), when the coefficient ring used contains an inverse for 2.

If \( \mathbb{Y} \) is non-trivial and admissible, then the situation is simpler, as there is no trivial connection to worry about. In this case, \( u \) is a degree 4 endomorphism defined on \( I(\mathbb{Y}) \), and Frøyshov’s [15, Theorem 9] says \((u^2 - 64)^n = 0\) for some \( n > 0 \). The proof of the following is essentially an application of Fukaya’s connected sum theorem of [17].

THEOREM 1.3. Let \( F \) be a field with \( \text{char}(F) \neq 2 \), and suppose that all homology groups are taken with \( F \)-coefficients, unless indicated otherwise. If \( H_1(Y; \mathbb{Z}) = 0 \), then

\[
I^\#(Y) \simeq \ker(u^2 - 64) \otimes H_*(S^3) \oplus H_*(\text{pt.})
\]

as \( \mathbb{Z}/4 \)-graded \( F \)-modules, where \( u^2 - 64 \) is acting on \( \bigoplus_{j=0}^3 \tilde{I}(Y)_j \). If \( \mathbb{Y} \) is non-trivial and admissible with geometric representative \( \lambda \), then

\[
I^\#(Y; \lambda) \simeq \ker(u^2 - 64) \otimes H_*(S^3)
\]

as relatively \( \mathbb{Z}/4 \)-graded \( F \)-modules, where \( u^2 - 64 \) is acting on four consecutive gradings of the relatively \( \mathbb{Z}/4 \)-graded \( F \)-module \( I(\mathbb{Y}) \).
When $L$ is the $(3, 5)$ torus knot, the double cover of $S^3$ branched over $L$ is the Poincaré homology sphere $\Sigma(2, 3, 5)$. By the results in [15], Frøyshov’s reduced group for $\Sigma(2, 3, 5)$ is trivial. Theorem 1.3 implies that $I^#(\Sigma(2, 3, 5))$ has rank 1, supported in grading 0. This provides an example where the spectral sequence of Theorem 1.1 does not collapse, as the reduced odd Khovanov homology of $L$ has rank 3, as computed in [33].

As another application of Theorem 1.3, a simple inductive argument involving the exact triangle, which we present in §10, yields the following.

**Corollary 1.4.** For any $Y$ and $\lambda$, we have $\chi(I^#(Y; \lambda)) = |H_1(Y; \mathbb{Z})|$.  

For a set $S$, the notation $|S|$ is to be interpreted as the cardinality of $S$ if it is finite, and 0 otherwise.

Theorem 1.3 suggests that knowledge of the smallest positive integer $n$ such that $(u^2 - 64)^n = 0$, is useful in understanding the relationships between the various instanton groups. It is known (cf. [15, §6]) that if $Y$ is non-trivial admissible and restricts non-trivially to a surface of genus at most 2, then one can take $n = 1$. For the following, let $h : \Theta^3_\mathbb{Z} \to \mathbb{Z}$ be Frøyshov’s homomorphism from [15], where $\Theta^3_\mathbb{Z}$ is the integral homology cobordism group.

**Corollary 1.5.** Let $Y$ be the result of $\pm 1$-surgery on a knot $K \subset S^3$ with genus at most 2. Let $F$ be a field with char($F$) $\neq 2$. Then, with all homology taken with $F$-coefficients, we have an isomorphism

$$I^#(Y) \cong H_*^{pt.}(X) \oplus H_*(S^3) \otimes \bigoplus_{j=0}^{3} \hat{I}(Y)_j$$

as $\mathbb{Z}/4$-graded $F$-modules. In particular, if in addition $h(Y) = 0$, then the groups $\hat{I}(Y)_j$ on the right can be replaced by $I(Y)_j$.

We apply this result using computations from the literature. Frøyshov proves in [15] that, for the family of manifolds $\Sigma(2, 3, 6k + 1)$ with $k$ a positive integer, we have $h = 0$. Furthermore, $\Sigma(2, 3, 6k + 1)$ can be realized as 1-surgery on the twist knot $(2k + 2)_1$ with $k$ full twists, which is a knot of genus 1. Using the computation of $I(\Sigma(2, 3, 6k + 1))$ from [11], we obtain the following corollary.

**Corollary 1.6.** With coefficients in a field $F$ with char($F$) $\neq 2$, we have isomorphisms

$$I^#(\Sigma(2, 3, 6k + 1)) \cong F_0^{[k/2] + 1} \oplus F_1^{[k/2]} \oplus F_2^{[k/2]} \oplus F_3^{[k/2]}$$

where $k$ is a positive integer and the subscripts indicate the gradings.

We also consider the manifolds $\Sigma(2, 3, 6k - 1)$ with $k$ a positive integer, which are obtained from $-1$-surgery on twist knots with $2k - 1$ half-twists. In this case, $h(\Sigma(2, 3, 6k - 1)) = 1$, and we can again use the computations of [11] to obtain the following corollary.

**Corollary 1.7.** With coefficients in a field $F$ with char($F$) $\neq 2$, we have isomorphisms

$$I^#(\Sigma(2, 3, 6k - 1)) \cong F_0^{[k/2]} \oplus F_1^{[k/2] - 1} \oplus F_2^{[k/2]} \oplus F_3^{[k/2]}$$

where $k$ is a positive integer and the subscripts indicate the gradings.

As $\Sigma(2, 3, 6k \pm 1)$ is the branched double cover over the $(3, 6k \pm 1)$ torus knot, Corollaries 1.6 and 1.7 provide more examples in which the spectral sequence of Theorem 1.1 does not collapse. In [4], Bloom considers a spectral sequence in monopole Floer homology with $\mathbb{F}_2$-coefficients.
analogous to that of Theorem 1.1. For the \((3,6k \pm 1)\) torus knot, he conjectures that the spectral sequence collapses at the fourth page, and guesses the higher differentials. We note that if his speculation were to hold in our setting (also with \(\mathbb{F}_2\)-coefficients), then we would recover, using Bloom’s computations and our grading \((1.1)\), the formulae of Corollaries 1.6 and 1.7, but with \(F = \mathbb{F}_2\).

1.1. Background and motivation

In the setting of Heegaard Floer homology, Ozsváth and Szabó in \([34]\) constructed a spectral sequence with \(\mathbb{F}_2\)-coefficients from the reduced Khovanov homology of a link \(L\) to the Heegaard Floer hat homology of \(\Sigma(L)\), with orientation reversed. This was the first instance of a structural relation between a Floer homology and a combinatorial link homology. For this result, we write

\[ \overline{Kh}(L; \mathbb{F}_2) \leadsto \widehat{HF}(\Sigma(L); \mathbb{F}_2). \]

The notation \(A \leadsto B\) is an abbreviation for the existence of a spectral sequence with some starting page \(A\), converging to \(B\). An essential ingredient for their construction was a link surgeries spectral sequence. See \(\S 2\) for the instanton analog. Subsequently, Bloom \([4]\) constructed a link surgeries spectral sequence in the setting of monopole Floer homology, and from this obtained a spectral sequence

\[ \overline{Kh}(L; \mathbb{F}_2) \leadsto \widehat{HM}(\Sigma(L); \mathbb{F}_2). \]

It has since been shown, after much work, that the monopole Floer group \(\widehat{HM}(Y)\) is isomorphic to \(\widehat{HF}(Y)\); cf. \([6, 25, 38]\).

Ozsváth and Szabó speculated in \([34]\) that their spectral sequence, if lifted to \(\mathbb{Z}\)-coefficients, would not have reduced Khovanov homology as the \(E^2\)-page, but would have some other link homology with altered signs in the differentials. With this in mind, Ozsváth, Rasmussen and Szabó \([33]\) defined an abelian group \(\text{Kh}'(L)\) called the \textit{odd Khovanov homology of} \(L\). With \(\mathbb{F}_2\)-coefficients, it coincides with Khovanov homology; but they are very different with \(\mathbb{Z}\)-coefficients.

Odd Khovanov homology is bigraded by a homological grading, called \(t\), and a quantum grading, called \(q\). There is a splitting

\[ \text{Kh}'(L)_{t,q} \simeq \overline{\text{Kh}'(L)}_{t,q-1} \oplus \overline{\text{Kh}'(L)}_{t,q+1}, \]

where \(\overline{\text{Kh}'(L)}\) is called the \textit{reduced} odd Khovanov homology. The bigraded group \(\overline{\text{Kh}'(L)}_{t,q}\) categorifies the Jones polynomial \(J_L\) in the sense that

\[ J_L(x) = \sum_{t,q} (-1)^t \text{rk}_\mathbb{Z}(\overline{\text{Kh}'(L)}_{t,q}) x^q. \]

Here, \(J_{\text{unknot}}(x) = 1\). It was conjectured in \([33]\) that there is a spectral sequence

\[ \overline{\text{Kh}'(L)} \leadsto \widehat{HF}(\Sigma(L)) \]

with \(\mathbb{Z}\)-coefficients. Our Theorem 1.1 provides such a spectral sequence with instanton homology in place of Heegaard Floer homology.

The correct version of instanton homology for the task was suggested in the work of Kronheimer and Mrowka. The framed instanton homology \(I^\#(Y)\) is isomorphic to the sutured instanton group \(\text{SHI}(M, \gamma)\) introduced in \([21]\), where \(M\) is the complement of an open 3-ball in \(Y\) and \(\gamma\) is a suture on the 2-sphere boundary. Restating a conjecture of Kronheimer and Mrowka, transferred from the sutured setting (cf. \([21, \text{Conjecture 7.24}]\)), it is expected that

\[ \widehat{HF}(Y; \mathbb{C}) \simeq I^\#(Y; \mathbb{C}) \simeq \widetilde{HM}(Y; \mathbb{C}). \]
Theorem 1.1 was inspired by this conjecture, and offers some validation for it, as does Corollary 1.2. Note that Corollary 1.4 confirms that the Euler characteristics of the three Floer groups under consideration agree.

Recently, Kronheimer and Mrowka [22] introduced singular instanton homology groups. We only mention the groups $I^\#(Y,L)$, where $L$ is a link in $Y$, and from which $I^\#(Y)$ is obtained by taking $L$ to be empty. The construction of this group involves counting instantons on $R \times Y$ singular along $R \times L$. Writing $I^\#(L) = I^\#(S^3, L)$, Kronheimer and Mrowka produced a spectral sequence

$$\text{Kh}(L) \Rightarrow I^\#(L),$$

where the left-hand side is unreduced Khovanov homology. Their spectral sequence respects $\mathbb{Z}/4$-gradings. Using related singular instanton groups, they proved in [22] that Khovanov homology detects the unknot. This paper adapts many details from the aforementioned paper.

1.2. Outline and acknowledgments

Many of the proofs in this paper are partially adapted from the papers so far mentioned, especially [22], and are credited throughout. For example, the proof of the exact triangle in §5 is inspired mostly by [22], although the analysis of instanton counts is somewhat different.

To work out the signs in the differential of the spectral sequence in Theorem 1.1, we introduce an algebro-topological way of composing homology orientations in §8.2. The composition rule is associative and has distinguished units. Indeed, homology orientations are part of the morphism data in an appropriate category on which the framed instanton homology is a functor.

The proof of Theorem 1.3 follows from versions of Fukaya’s connected sum theorem of [17] where non-trivial admissible bundles are involved, which to the author’s knowledge have not appeared in the literature. These versions are simpler than Fukaya’s original theorem, as there are fewer trivial connections with which to deal. We outline Donaldson’s proof of Fukaya’s theorem from [7, §7.4], more or less verbatim (except for notation), and show how the versions of interest to us follow, making the necessary modifications.

The structure of this paper is as follows. In §2, we outline the proof of Theorem 1.1, introducing the surgery exact triangle and the link surgeries spectral sequence. In §3, we construct the bundles that are relevant to the surgery exact triangle. In §4, we review the construction of instanton homology for admissible bundles. In §5, we prove Floer’s exact triangle, and in §6, we prove the link surgeries spectral sequence. In §7, we define framed instanton homology and discuss its basic properties. In §8, we define reduced odd Khovanov homology and complete the proof of Theorem 1.1 and Corollary 1.2. In §9, we prove Theorem 1.3 and Corollaries 1.5–1.7. In §10, we prove Corollary 1.4.

2. The surgery story

In this section, we outline the proof of Theorem 1.1. To begin, we recall the instanton surgery exact sequence, or exact triangle, introduced by Floer [14]. Let $K$ be a framed knot in $Y$. That is, $K$ has a preferred meridian and longitude. Let $\omega$ be a geometric representative for $Y$ disjoint from $K$. Denote by $Y_0$ and $Y_1$ the results of performing 0- and 1-surgery on $K$, respectively. We can view $\omega$ inside $Y_0$ and $Y_1$ by keeping it away from the surgery neighborhood. Let $\omega_0 = \omega \cup \mu \subset Y_0$ where $\mu$ is a core for the induced framed knot in $Y_0$, and let $\omega_1 = \omega \subset Y_1$.

Finally, for $i = 0, 1$ choose a bundle $\mathcal{Y}_i$ over $Y_i$ geometrically represented by $\omega_i$. If the ordered triple of bundles $\mathcal{Y}, Y_0, Y_1$ can be geometrically represented in this way, we say they form a surgery triad.
**Figure 1** (colour online). *Local surgery diagrams. The slanted line in each case is the knot $K$. Each row represents a possible construction for a surgery triad.*

**Theorem 2.1** (Floer). There is an exact sequence

$$\cdots I(Y) \rightarrow I(Y_0) \rightarrow I(Y_1) \rightarrow I(Y) \rightarrow \cdots$$

provided all three bundles are admissible and form a surgery triad.

The loop $\mu$ in $Y_0$, pushed out of the surgery solid torus, becomes a small meridional loop around the surgered neighborhood of $K$ in $Y_0$. This is depicted in the top row of Figure 1 in a local surgery diagram for $Y_0$. One can view $Y_1$ (respectively, $Y$) as obtained from $0$-surgery on the induced framed knot in $Y_0$ (respectively, $Y_1$); see §3. Thus we obtain two more local surgery diagram depictions of where $\mu$ may be placed, listed in the bottom two rows of Figure 1. See also §3.7.

Floer’s exact triangle was studied by Braam and Donaldson in [5], where a detailed proof following Floer’s ideas can be found. In this paper, we provide an alternative proof. The proof relies on an algebraic lemma which was first used by Ozsváth and Szabó [34]. The lemma requires the input of maps between the three relevant chain complexes satisfying certain properties. The maps we choose count instantons on families of metrics that are parameterized by convex polytopes. This approach was used by Kronheimer, Mrowka, Ozsváth and Szabó to prove a surgery exact sequence in the monopole case [24]. Our proof is largely an adaptation of Kronheimer and Mrowka’s proof in [22] of an analogous exact triangle in singular instanton knot homology.

This method of proof leads to a generalization of Floer’s theorem to a so-called link surgeries spectral sequence, as was first done by Ozsváth and Szabó in Heegaard Floer homology [34]. Let $L$ be a framed link in $Y$ with components $L_1, \ldots, L_m$. For each $v \in \{0, 1, \infty\}^m$, let $Y'_v$ be the result of $v_i$ surgery on $L_i$ for $1 \leq i \leq m$. Briefly, we say $Y'_v$ is the result of $v$-surgery on $L$. Choose a geometric representative $\omega$ for $Y$ disjoint from $L$. Let $\omega_v \subset Y'_v$ be $\omega$ together with a core for the knot in $Y'_v$ induced by $L_i$ for each $i$ with $v_i = 0$. Let $Y_v$ be bundles over $Y_v$ geometrically represented by the $\omega_v$. If the bundles $Y_v$ can be geometrically represented according to these rules, then we say that they form a *surgery cube*. 

| $Y$ | $Y_0$ | $Y_1$ |
|-----|-------|-------|
| $\infty$ | $\mu$ | $0$ |
| | | $1$ |
| $\infty$ | | $\mu$ |
| | | $1$ |
| $\mu$ | $\infty$ | $0$ |
| | | $1$ |
Theorem 2.2. Suppose that the bundles $Y_v$ for $v \in \{0, 1, \infty\}^m$ are admissible and that they form a surgery cube. Then there is a spectral sequence

$$\bigoplus_{v \in \{0, 1\}^m} I(Y_v) \leadsto I(Y).$$

That is, the left-hand side is the $E_1$-page and the sequence converges to the right-hand side.

A more detailed statement is provided in Theorem 6.1. An analogous result in monopole Floer homology was proved by Bloom [4] with $\mathbb{F}_2$-coefficients, and in singular instanton knot homology by Kronheimer and Mrowka [22].

From this, we obtain a surgery spectral sequence for the groups $I^#(Y)$, which generally must involve the twisted groups $I^#(Y; \lambda)$. The group $I^#(Y; \lambda)$ is four consecutive gradings of $I(Y^#)$, where $Y^#$ is a bundle over $Y#T^3$ geometrically represented by an $S^1$-factor of $T^3$ together with $\lambda \subset Y$. In this setting, the surgeries on the link $L$ are performed away from the 3-tori, and every bundle is automatically admissible. To minimize the number of non-trivial bundles in the mix, we refer the reader to the bottom row of Figure 1. Using this, we can ensure that the bundles for $v$-surgeries with $v \in \{0, 1\}^m$ are geometrically represented only by the $S^1$-factor of $T^3$. The trade-off is that the geometric representative for the bundle over $Y$ is the $S^1$-factor together with the link $L$. We obtain the following theorem.

Theorem 2.3. Let $L$ be a framed $m$-component link in $Y$. There is a spectral sequence

$$\bigoplus_{v \in \{0, 1\}^m} I^#(Y_v) \leadsto I^#(Y; L).$$

That is, the left-hand side is the $E_1$-page and the sequence converges to the right-hand side.

A more detailed statement is given in Theorem 7.2.

Now we introduce branched double covers, following Ozsváth and Szabó [34]. Let $L$ be a link in $S^3$ and $\Sigma(L)$ be the double cover of $S^3$ branched over $L$. Let $D$ be a planar diagram for $L$ with $m$ crossings. For each $v \in \{0, 1\}^m$, there is a resolution diagram $D_v$ which is a disjoint union of circles, obtained by performing 0- and 1-resolutions according to Figure 2. Each branched cover $\Sigma(D_v)$ is diffeomorphic to $#^kS^1 \times S^2$ where $D_v$ has $k + 1$ circles. Further, there is a link $L' \subset \Sigma(L)$ and a framing on $L'$ such that $\Sigma(D_v)$ is the result of $v$-surgery on $L'$. If we draw a small arc between each crossing in $D$, then the preimages in the branched cover $\Sigma(L)$ are loops, and the link $L'$ is the union of these preimages.

With this setup, from Theorem 2.3 we have a spectral sequence

$$\bigoplus_{v \in \{0, 1\}^m} I^#(\Sigma(D_v)) \leadsto I^#(\Sigma(L); L').$$

(2.1)

We claim that $[L'] \in H_1(\Sigma(L); \mathbb{F}_2)$ is zero, so that the target of this spectral sequence is in fact $I^#(\Sigma(L))$. The diagram $D$ divides the plane into regions. To show $[L'] = 0$, it suffices to color the regions black and white in a way such that each crossing touches exactly one black region.

Figure 2. From the diagram $D$ to a resolution diagram $D_v$. 
See Figure 3. For then the black regions can be lifted to a surface in $\Sigma(L)$ whose boundary is $L'$, implying $[L']=0$.

To color the regions, we follow an argument communicated to the author by Jianfeng Lin. We proceed as if performing the algorithm to construct a Seifert surface, as in [35, §5.4]. First, we orient $L$. Then we resolve each crossing as in Figure 3. We assign to each circle $z$ in the resolved diagram two signs, $a_z$ and $b_z$. The first sign $a_z$ is $+1$ if $z$ is oriented counter-clockwise in the plane, and $-1$ otherwise. The second sign $b_z$ is given by $(-1)^N$ where $N$ is the number of circles that surround $z$. Now color, with black, the regions that are directly interior to each circle $z$ with $a_z b_z = +1$. Transferring the coloring back to the unresolved diagram, each crossing touches exactly one such region.

This reduces the proof of Theorem 1.1 to identifying the $E^1$-page of (2.1) and then understanding the gradings. We can compute the groups $I^\#(\Sigma(D_v))$ because each $\Sigma(D_v)$ is of the form $\#^kS^1 \times S^2$, and we can compute the $E^1$-differential because the cobordism maps involved are topologically simple. This is carried out in §8.3, where we identify the $E^1$-page as the chain complex used to compute $\overline{Kh}'(L)$ from the diagram $D$. We then check in §8.4 that the relevant gradings are preserved, completing the proof of Theorem 1.1.

### 3. Bundles in the exact triangle

In this section, we introduce the manifolds and bundles that feature in the proof of Theorem 2.1.

We take a systematic approach to the bundles $Y_i$ that appear in Floer’s exact triangle by extending Dehn surgery to $\text{SO}(3)$-bundles. This viewpoint was Floer’s [14], and is expanded upon in [5]. The construction of surgery cobordism bundles $X_{ij}$ in §3.3 is straightforward in this setting. These bundles induce the maps in the exact triangle. We then introduce some hypersurfaces in $X_{ij}$ that yield useful metric families; these were used in [4, 22, 24]. In §3.7, we relate our new setup to that of the statement of Theorem 2.1 in §2.

In this section, we write $A \cup_f B$ for the space obtained from the disjoint union of $A$ and $B$, with points identified using the map $f$. Our convention is that the gluing map $f$ is always from a subset of $B$ to a subset of $A$. We freely use isomorphisms of the form $A \cup_f B \simeq A \cup_f g C$, where $g$ is an isomorphism from a subset of $C$ to a subset of $B$. All constructions that are not smooth have a canonical smoothing, as mentioned in [18, Remark 1.3.3]. All (principal) $\text{SO}(3)$-bundles have right actions. Thus our bundle gluing maps, in order to be equivariant, always involve left multiplication on trivialized fibers.

#### 3.1. Dehn surgery with bundles

Let $\mathcal{Y}$ be an $\text{SO}(3)$-bundle over a closed, oriented 3-manifold $Y$. Let $K : S^1 \times D^2 \to Y$ be an embedding. We refer to $K$ as a framed knot in $Y$. We consider equivariant embeddings $\mathbb{K} : S^1 \times D^2 \times \text{SO}(3) \to \mathcal{Y}$ that lie above $K$, that is, $\mathbb{K}/\text{SO}(3) = K$. We refer to $\mathbb{K}$ as a framed knot in $\mathcal{Y}$. The space of bundle automorphisms of $S^1 \times D^2 \times \text{SO}(3)$ fixing the base space has
two connected components. An automorphism \( \tau \) not isotopic to the identity is

\[
\tau(w, z, a) = (w, z, c(w)a),
\]

where \((w, z) \in S^1 \times D^2, a \in SO(3)\) and \(c\) is a standard inclusion \(S^1 \to SO(3)\) of a maximal torus. In particular, \(c\) is a homomorphism and generates \(\pi_1(SO(3)) \cong \mathbb{Z}/2\). If \(\mathbb{K}\) is one embedding, then another embedding lying above \(K\) is given by \(\mathbb{K}\tau\).

We generalize Dehn surgery to surgery on the framed knots \(\mathbb{K}\). For \(\Omega = (A, b) \in SL(2, \mathbb{Z}) \ltimes (\mathbb{Z}/2)^2\), we define an automorphism \(\psi_\Omega\) of \(S^1 \times \partial D^2 \times SO(3)\) by

\[
\psi_\Omega(w, z, a) = (w^A_{11}z^{A_{12}}, w^A_{21}z^{A_{22}}, c(w)^b, c(z)^b, a).
\]

Let \(\mathbb{K}'\) be the interior of the image of \(\mathbb{K}\). The result of \(\Omega\)-surgery on \(K\) is then defined to be the identification space

\[
\mathbb{Y}_\Omega(\mathbb{K}) = (\mathbb{Y} \setminus \mathbb{K}') \cup_{\mathbb{K}' \psi_\Omega} (S^1 \times \partial D^2 \times SO(3)).
\]

There is an induced framed knot \(\Omega(\mathbb{K})\) in \(\mathbb{Y}_\Omega(\mathbb{K})\) given by the inclusion of \(S^1 \times \partial D^2 \times SO(3)\) into the above expression for \(\mathbb{Y}_\Omega(\mathbb{K})\). The product of elements in \(G = SL(2, \mathbb{Z}) \ltimes (\mathbb{Z}/2)^2\) is given by \((A', b')(A, b) = (A'A, b'A + b)\). The assignment \(\Omega \mapsto \psi_\Omega\) induces an isomorphism from \(G\) to the group of isotopy classes of orientation-preserving equivariant automorphisms of \(S^1 \times \partial D^2 \times SO(3)\). We have an associativity rule

\[
\mathbb{Y}_{\Omega'\Omega}(\mathbb{K}) \cong (\mathbb{Y}_{\Omega'}(\mathbb{K}))_{\Omega}(\mathbb{K}').
\]

The space \(\mathbb{Y}_\Omega(\mathbb{K})\) is naturally a bundle over \(Y_{p/q}(K)\), the result of \(p/q\) Dehn surgery on the framed knot \(K\) in \(Y\), where \(\Omega = (A, b)\), \(p = A_{22}\), \(q = A_{12}\) and of course \(K = \mathbb{K}/SO(3)\). Note that the automorphism \(\tau\) above restricts to \(\psi_\Theta\) where \(\Theta = (1_{2 \times 2}, (1, 0)) \in G\). We have the transformation rule \(\mathbb{Y}_{\Omega}(\mathbb{K}\tau) \cong \mathbb{Y}_{\Theta\Omega}(\mathbb{K})\).

3.2. The surgery bundle \(\mathbb{Y}_i\)

There is a particular choice of surgery parameter \(\Omega\) that Floer used in the setting of his exact triangle:

\[
\Lambda = \left( \begin{array}{cc} \begin{array}{cc} -1 & 1 \\ -1 & 0 \end{array} \end{array} \right), (1, 0) \right).
\]

To understand this, write \(\Lambda = \Psi\Lambda'\), where

\[
\Psi = (1_{2 \times 2}, (0, 1)), \quad \Lambda' = \left( \begin{array}{cc} \begin{array}{cc} -1 & 1 \\ -1 & 0 \end{array} \end{array} \right), (0, 0) \right).
\]

First, \(\Psi\) twists the trivialization around \(\partial D^2\). Then, \(\Lambda'\) performs 0-surgery on \(K\), leaving bundles alone. Note that \(\Lambda^3 = 1\). With \(\mathbb{Y}\) and \(\mathbb{K}\) fixed, we define for \(i \in \mathbb{Z}\) the surgery bundles \(\mathbb{Y}_i = \mathbb{Y}_{\Lambda^i+1}(\mathbb{K})\), the surgery base manifolds \(Y_i = \mathbb{Y}_i/SO(3)\) and the induced embeddings \(\mathbb{K}_i = \Lambda_{i+1}(\mathbb{K})\). The index offset is here so that \(Y_0\) and \(Y_1\) are simply 0- and 1-surgery on \(K \subset Y\), respectively. Because \(\Lambda^3 = 1\), there are isomorphisms \(\mathbb{Y}_i \cong \mathbb{Y}_{i+3}\).

3.3. The surgery cobordism \(X_{ij}\)

Our goal is to construct cobordism bundles \(X_{ij} : Y_i \to Y_j\) for \(i < j\). Each \(X_{ij}\) will be an \(SO(3)\)-bundle over a standard surgery cobordism \(X_{ij} : Y_i \to Y_j\). We first construct \(X_{ij}\) when \(j - i = 1\).
and use these as building blocks for the general construction. Write $\mathbb{H} = D^2 \times D^2 \times SO(3)$. We view $\mathbb{H}$ as a 2-handle thickened by $SO(3)$. Also write
\[
\partial \mathbb{H} = \mathbb{H}_1 \cup \mathbb{H}_2, \\
\mathbb{H}_1 = \partial D^2 \times D^2 \times SO(3), \\
\mathbb{H}_2 = D^2 \times \partial D^2 \times SO(3).
\]

Viewing $\mathbb{K}_0$ as a map $\mathbb{H}_1 \to \{1\} \times \mathbb{Y}_0$, we define $X_{01}$ by setting
\[
X_{01} = ([0, 1] \times \mathbb{Y}_0) \cup_{\mathbb{K}_0} \mathbb{H}.
\]
The definition of $X_{ij}$ for general $j - i = 1$ is similar. We want to define $X_{02}$ as $X_{01} \cup_1 X_{12}$. To make sense of this expression, we give an explicit identification of $\partial X_{01} \setminus \mathbb{Y}_0$ with $\mathbb{Y}_1$. Let the interior of the image of $\mathbb{K}_0$ in $\mathbb{Y}_0$ be denoted as $\mathbb{K}_0'$. Note that
\[
\partial \mathbb{H}_1 = \mathbb{H}_1 \cap \mathbb{H}_2 = \partial \mathbb{H}_2
\]
is a trivial bundle over a 2-torus. Now we write
\[
\partial X_{01} \setminus \mathbb{Y}_0 = (\mathbb{Y}_0 \setminus \mathbb{K}_0') \cup \mathbb{H}_0|_{\mathbb{H}_1 \cap \mathbb{H}_2} \cup \mathbb{H}_2.
\]
Let $\psi : \mathbb{H}_1 \to \mathbb{H}_2$ be an isomorphism. Then
\[
\partial X_{01} \setminus \mathbb{Y}_0 \simeq (\mathbb{Y}_0 \setminus \mathbb{K}_0') \cup \mathbb{H}_0|_{\mathbb{H}_1 \cap \mathbb{H}_2} \cup \mathbb{H}_2 = (\mathbb{Y}_0)|_{\mathbb{H}_1 \cap \mathbb{H}_2}(\mathbb{K}_0).
\]
To identify this bundle with $\mathbb{Y}_1 = (\mathbb{Y}_0)|_{\mathbb{H}_1 \cap \mathbb{H}_2}(\mathbb{K}_0)$, we need $\psi$ such that $\psi|_{\mathbb{H}_1 \cap \mathbb{H}_2} = \psi_\Lambda$; we choose
\[
\psi : \mathbb{H}_1 \to \mathbb{H}_2, \quad \psi(w, z, a) := (wz, w, c(w)a).
\]
Making this choice, we have identified $X_{01} \setminus \mathbb{Y}_0$ with $\mathbb{Y}_1$. Finally, to construct $X_{ij}$ for $j - i > 1$, we inductively define $X_{ij} = X_{i,j-1} \cup_{Y_{j-1}} X_{j-1,j}$, where the gluing is done according to the same identification process.

3.4. The bundle $S_i$

We construct a subset $S_1 \subset X_{02}$ which is a bundle over a 3-sphere $S_1 \subset X_{02}$. One gets $S_i$ inside $X_{i-1,i+1}$ for each $i$ in a similar fashion. Write
\[
X_{02} = ([0, 1] \times \mathbb{Y}_0 \cup_{\mathbb{K}_0} \mathbb{H}) \cup_1 ([0, 1] \times \mathbb{Y}_1 \cup_{\mathbb{K}_1} \mathbb{H}) \simeq ([0, 1] \times \mathbb{Y}_0) \cup_{\mathbb{K}_0} \mathbb{H} \cup_{\psi} \mathbb{H}
\]
with notation as in the construction of $X_{01}$. Introduce the subset
\[
\mathbb{H}(r, s) = D^2(r) \times D^2(s) \times SO(3) \subset \mathbb{H},
\]
where $D^2(r)$ is the disk of radius $r$, $0 < r \leq 1$, and consider the following restriction bundles of $X_{02}$:
\[
U = \mathbb{H}(1/2, 1) \cup \mathbb{H}(1, 1/2) \subset \mathbb{H} \cup_{\psi} \mathbb{H}, \quad S_1 = \partial U.
\]
It is well known that the base space $U$ of $U$ is diffeomorphic to $-\mathbb{CP}^2$ minus an embedded 4-ball; cf. [18, Example 4.2.4]. It follows that $S_1$ is a trivial bundle over a 3-sphere $S_1$. We see that we can decompose $X_{02}$ along $S_1$ into a connected sum of $-\mathbb{CP}^2$ with a manifold whose boundary is $Y_1 \cup \overline{Y}_0$. The intersection $S_1 \cap Y_1$ is a 2-torus. This decomposition is depicted in Figure 4.

We claim that $U$ is a non-trivial bundle. We check that the restriction of $U$ to an essential sphere is non-trivial. Define $D_1 = D^2 \times \{0\} \times SO(3)$ and $D_2 = \{0\} \times D^2 \times SO(3)$ as subsets of $\mathbb{H}$. Consider
\[
D_2 \cup_{\psi \partial D_2} D_1 \subset U.
\]
The two hypersurfaces $Y_1$ and $S_1$ in the interior of $X_{02}$. The 3-sphere $S_1$ separates off a copy of $-\mathbb{C}P^2$ minus a 4-ball.

The intersections of the five hypersurfaces in the interior of $X_{03}$. The $S^1 \times S^2$ hypersurface $T$ divides $X_{03}$ into two pieces, $V$ and $E$. This picture first appeared in [24].

This is isomorphic to $D^2 \times \text{SO}(3) \cup_f D^2 \times \text{SO}(3)$ where $f$ is the automorphism of $\partial D^2 \times \text{SO}(3)$ given by $f(z, a) = (\tau, c(z)a)$. This is a non-trivial bundle over a 2-sphere.

3.5. The bundle $T$

We construct a subset $T \subset X_{03}$ which is a trivial bundle over $T \subset X_{03}$ where $T$ is diffeomorphic to $S^1 \times S^2$. By iterating (3.3) and stretching the ends, we write $X_{03}$ as

$$X_{03} \simeq ([0, 1] \times Y_0) \cup_{\psi} \mathbb{H} \cup_{\psi} \mathbb{H} \cup_{\psi} \mathbb{H} \cup_{\psi} ([0, 1] \times Y_0).$$

Identifying $X_{03}$ with the expression on the right, we define the restriction bundles

$$E = \mathbb{H} \cup_{\psi} \mathbb{H} \cup_{\psi} \mathbb{H}, \quad T = \partial E,$$

and their respective base spaces $E$ and $T$. We have an isomorphism

$$f : T = \mathbb{H}_1 \cup_{\psi} \mathbb{H}_2 \rightarrow S^1 \times S^2 \times \text{SO}(3), \quad (3.4)$$

where viewing $S^2 \subset S^1 \times S^2$ as $\mathbb{C} \cup \infty$, we set

$$f|_{\mathbb{H}_1} = \text{id}, \quad f|_{\mathbb{H}_2}(z, w, a) = (\overline{w}, w/\tau, c(w)a).$$

The triviality of the bundle $T$ is also seen from the observation that it is the restriction of a bundle on a space in which $T$ is contractible. We note that we could have also trivialized $T$ by using a similar isomorphism in which $f|_{\mathbb{H}_2} = \text{id}$. These two isomorphisms determine trivializations that differ, in the terminology of §4, by a non-even gauge transformation.

We remark that the intersections $T \cap Y_1$ and $T \cap Y_2$ are 2-tori. We illustrate the arrangement of intersections in Figure 5. We note that $T$ may be described as the boundary of a regular neighborhood of the union of the two essential spheres inside the copies of $-\mathbb{C}P^2$ divided off by $S_1$ and $S_2$. The hypersurface $T$ separates $X_{03}$ into two 4-manifolds, $E$ and $V$, where $E$ is
diffeomorphic to $-\mathbb{CP}^2$ minus a neighborhood of an unknotted circle, and $V$ is diffeomorphic to $[0,1] \times Y_0$ minus a neighborhood of $\{1/2\} \times K$ (Figure 5).

3.6. An involution of $E$

We construct an involution $\sigma : E \to E$. We write $E = \mathbb{H}^{i-1} \cup \psi \mathbb{H}^0 \cup \psi \mathbb{H}^{i+1}$, where the superscripts have been added to distinguish the copies of $\mathbb{H}$. We write $[w,z,a,i] \in E$ for the point represented by $(w,z,a) \in \mathbb{H}^i$. We define our involution by

$$\sigma[w,z,a,i] = [z,w,c(w)d(z)i, -i].$$

Here we have extended $c^2 : S^1 \to \text{SO}(3)$ to a map $c^2 : D^2 \to \text{SO}(3)$ such that $c^2(w) = (c^2(w))^{-1}$. Note that $\sigma$ interchanges the outer copies of $\mathbb{H}$ and fixes the middle copy of $\mathbb{H}$. It is straightforward that $\sigma$ is well defined: writing $\sigma$ as three maps $\sigma_i : \mathbb{H}^{i+1} \to \mathbb{H}^{-i}$, one uses the relations

$$\sigma_0^2 = \text{id}, \quad \sigma_{\pm 1} = \psi_{\pm 1} \sigma_0 \psi_{\pm 1}, \quad \psi^3 = \text{id},$$

whenever these compositions are defined. The involution $\sigma$ is a bundle automorphism that restricts to an orientation-preserving diffeomorphism of $E$. It fixes $T$ and swaps $S_1$ with $S_2$. The involution $\sigma$ is depicted in Figure 6.

Let us look at how the involution affects $T$. Recall the isomorphism (3.4). We have

$$f \sigma f^{-1}(w,z,a) = (w,w/z,c(w)d(z)a),$$

(3.5)

where $w \in S^1$, $z \in \mathbb{C} \cup \infty$, $a \in \text{SO}(3)$ and $d : S^2 \to \text{SO}(3)$ is the double of $c^2 : D^2 \to \text{SO}(3)$. It is easily seen that $f \sigma f^{-1}$ is isotopic to a composition, $\theta \circ v$, where

$$\theta(w,z,a) = (w,w/z,a), \quad v(w,z,a) = (w,z,c(w)a).$$

The map $\theta$ is a diffeomorphism of $S^1 \times S^2$ that, with respect to our trivialization, is extended in a trivial way to the overlying bundle. In the terminology of §4, $v$ is a non-even gauge transformation of the trivial bundle over $S^1 \times S^2$. The involution $\sigma$ will be useful in the proof of the exact triangle.

3.7. Geometric representatives

Let $\omega$ be an embedded loop in $Y$. Extend this to an embedding $\mathbb{K}_{\omega} : S^1 \times D^2 \times \text{SO}(3) \to Y \times \text{SO}(3)$. Let $\Psi = (1_{2 \times 2}, (0,1))$ as in (3.2). Then the result of $\Psi$-surgery on $\mathbb{K}_{\omega}$ as a framed knot in $Y \times \text{SO}(3)$ is a bundle geometrically represented by $\omega$. More generally, $\omega$ can be a collection of embedded loops, and $\Psi$-surgery for each component gives a bundle geometrically represented by $\omega$.

This relates our current framework to the statement of Theorem 2.1 in §2. Let $\omega$ be a closed, unoriented 1-manifold in $Y$, and $K$ be a framed knot in $Y$ disjoint from $\omega$. We set

$$Y = (Y \times \text{SO}(3))_{\psi} (\mathbb{K}_{\omega}),$$

(3.6)

where it is understood that if $\omega$ has multiple components, we do $\Psi$-surgery for each component. This description of $Y$ gives a preferred trivialization away from a neighborhood of $\omega$. We let $K$
be the $SO(3)$-thickening of $K$ using these preferred data, precomposed with $\tau$. That is,

$$K = (K \times \text{id}_{SO(3)})\tau.$$  

Recall that $\tau$ restricts to $\psi_{\Theta}$ where $\Theta = (1_{2\times 2}, (1, 0))$, and that $Y_0$ is defined as $Y_\Lambda(K)$. Using $\Theta\Lambda = \Lambda'\Psi$ with notation as in (3.2), we have

$$Y_0 \simeq Y_{\Lambda'\Psi}(K \times \text{id}_{SO(3)}).$$

Because $\Lambda'$ is 0-surgery without bundle-twisting, we see that $Y_0$ is of the form (3.6), where $Y$ is replaced by $Y_0$ and $\omega$ is replaced by $\omega \cup K_0$, where $K_0$ is the induced knot in $Y_0$. Thus $Y_0$ is geometrically represented by $\omega \cup K_0$. Pushing $K_0$ away from the surgered neighborhood makes it a small meridional loop $\mu$ as in Figure 1, by the nature of 0-surgery.

We may deduce that $Y_1$ is geometrically represented by $\omega \subset Y_1$ by either of two ways. First, we may interpret $Y_1$ as 0-surgery on the induced knot $K_0 \subset Y_0$ and iterate the rule already established, forgetting about bundles altogether. Alternatively, we can repeat the above argument for $\Lambda^2$ in place of $\Lambda$. The difference in this case is that $\Theta\Lambda^2 = (\Lambda')^2$. This is 1-surgery on $K$ without bundle-twisting.

## 4. Instanton homology

In this section, we review the relevant aspects of instanton homology for admissible bundles. Our main technical references are [7, 22]. Other useful references include [5, 13–15, 36]. In §4.3, we define the index $\mu$ that encodes the expected dimension of instanton moduli spaces. Subsection 4.4 is an adaptation of some results from [22, §3.9] regarding maps obtained from families of metrics on cobordisms. In §4.5, we discuss how to use the index to put constraints on the existence of instantons, and in §4.6 we discuss the $\mathbb{Z}/2$-grading on $I(Y)$.

### 4.1. Instanton groups

Let $Y$ be an $SO(3)$-bundle over a closed, connected, oriented Riemannian 3-manifold $Y$. The group $I(Y)$ is heuristically a Morse homology group computed using a suitably perturbed Chern–Simons functional $cs_\pi : \mathcal{C}(Y) \to \mathbb{R}$ modulo a group of gauge transformations:

$$cs_\pi(a) = -\frac{1}{8\pi^2} \int_{[0,1] \times Y} \text{tr}(F_A^2) + f_\pi(a).$$

Here $A$ is a connection on $[0,1] \times Y$ which restricts to a base connection $a_0$ on $\{0\} \times Y$ and the connection $a$ on $\{1\} \times Y$, and $f_\pi$ is a small perturbation; see [22, §3.4]. We have written $\mathcal{C}(Y)$ for the space of smooth connections on $Y$, an affine space modeled on $\Omega^1(\mathfrak{g}_{ad})$, where $\mathfrak{g}_{ad} = Y \times_{\text{ad}} \mathfrak{so}(3)$ is the adjoint bundle of $Y$.

Let $X$ be an $SO(3)$-bundle over an $n$-dimensional manifold $X$. In our constructions, we do not use the full automorphism group $G(X)$ of $X$, but rather, following the terminology of [15], we use the subgroup $G_{ev} = G_{ev}(X)$ of even gauge transformations. Elements of $G_{ev}$ are called determinant-1 gauge transformations in [22] and restricted gauge transformations in [5]. Viewing gauge transformations as sections of the bundle $X \times_{\text{Ad}} SO(3)$, the even transformations are the ones that lift to sections of $X \times_{\text{Ad}} SU(2)$. There is an exact sequence

$$1 \longrightarrow G_{ev}(X) \longrightarrow G(X) \xrightarrow{\eta} H^1(X; F_2) \longrightarrow 1,$$

where $\eta$ measures the obstruction to deforming a gauge transformation over the 1-skeleton of $X$. For a connection $A$ on $X$, we write $h^0(A)$ for the dimension of its $G_{ev}$-stabilizer. The possible values of $h^0(A)$ are 0, 1, 3. For us, the only stabilizers that will appear will be 1, $S^1$, $SO(3)$. We call $A$ irreducible if $h^0(A) = 0$, and reducible otherwise.
We will write \(a, b, c, \ldots\) for typical connections on bundles over 3-manifolds and \(a, b, c, \ldots\) for their respective \(\mathcal{G}_{ev}\)-classes. A typical connection on a bundle over a 4-manifold \(X\) is written as \(A\), and simply \([A]\) for its \(\mathcal{G}_{ev}\)-class.

Let \(\mathcal{B}(Y)\) denote the quotient \(\mathcal{C}(Y)/\mathcal{G}_{ev}\). The functional \(\text{cs}_\pi\) induces a map \(\text{cs}_\pi': \mathcal{B}(Y) \to \mathbb{R}/\mathbb{Z}\). The set of critical points of \(\text{cs}_\pi'\) is denoted by \(\mathcal{C}\) or \(\mathcal{C}(Y)\); when the perturbation \(\pi\) is zero, this is the set of flat connection classes on \(Y\). We write \(h^1(a)\) for the dimension of the Zariski tangent space of \(a\) in \(\mathcal{C}\). Following [7], when \(h^0(a) = h^2(a) = 0\), the connection \(a\) is called acyclic. Let \(\mathcal{C}_{\text{irr}}\) denote the subset of irreducibles in \(\mathcal{C}\). When \(Y\) is admissible and a suitable perturbation is chosen, \(\mathcal{C}_{\text{irr}}\) is a finite set of acyclic classes, and it is fact in all of \(\mathcal{C}\) or is missing only the trivial class, according to whether \(b_1(Y) \neq 0\) or not. Assume that such a perturbation is chosen.

Fix a base connection \(a_0\) on \(Y\). We define the chain group

\[
C(Y) = \bigoplus_{\Lambda(a) \in \mathcal{C}_{\text{irr}}} \mathbb{Z}\Lambda(a),
\]

where \(\Lambda(a)\) is the 2-element set of orientations of the real line \(\det(D_A)\), where \(A\) is a connection on \(\mathbb{R} \times Y\) with \(A|_{\mathbb{R} \times \{t\}} \equiv a_0\) for \(t \to 0\) and in the class \(a\) for \(t \to \infty\), and \(D_A\) is the Fredholm operator \(-d_A \pm d_A^1\) defined on suitable Sobolev spaces in \(\S 4.3\); see also [22, \S 3.6]. Here \(\mathbb{Z}\Lambda(a)\) means the infinite cyclic group with generators the elements of \(\Lambda(a)\). We often think of \(C(Y)\) as generated by \(\mathcal{C}_{\text{irr}}\); when doing this it is understood that we have chosen distinguished elements from each set \(\Lambda(a)\).

A connection \(A\) on an \(SO(3)\)-bundle over a Riemannian 4-manifold is an instanton or is anti-self-dual if its curvature \(F_A\) satisfies

\[
* F_A = -F_A,
\]

where \(*\) is the Hodge star. The energy of a connection \(A\) is given by \(\|F_A\|_{L^2}^2 = -\int \text{tr}(F_A \wedge * F_A)\). Instantons on \(X = \mathbb{R} \times Y\) may be interpreted as gradient flow lines for the Chern–Simons functional. In actuality, we consider a perturbed instanton equation involving \(\pi\), and call the solutions instantons as well. Given acyclic \(a, b \in \mathcal{C}(Y)\), we let \(M(a, b)\) be the space of \(\mathcal{G}_{ev}\)-classes of finite-energy instantons on \(X\) asymptotic at \(-\infty\) to \(a\) and at \(+\infty\) to \(b\).

When the perturbation is zero, elements \([A] \in M(a, b)\) are distinguished by the property

\[
(1/8\pi^2)\|F_A\|_{L^2}^2 = \text{cs}(b) - \text{cs}(a).
\]

For a small, generic perturbation \((a, b)\) is a smooth manifold, and we write

\[
\mu(a, b) = \dim M(a, b).
\]

Passing to \(\mathcal{G}_{ev}\)-classes, the number \(\mu(a, b)\) is well-defined modulo 8, and equips \(C(Y)\) with a relative \(\mathbb{Z}/8\)-grading given by \(\text{gr}(a) - \text{gr}(b) \equiv \mu(a, b)\). The space \(M(a, b)\) has an \(\mathbb{R}\)-action by translation along the \(\mathbb{R}\)-factor of \(\mathbb{R} \times Y\), and we write

\[
\hat{M}(a, b) = M(a, b)/\mathbb{R}.
\]

The data of \(a, b\) and the lift of \(\mu(a, b) \in \mathbb{Z}/8\) to \(d \in \mathbb{Z}\) are sufficient to describe \(M(a, b)\); viewing \([A] \in M(a, b)\) as a path in \(\mathcal{B}(Y)\), the index \(d\) faithfully records the homotopy class of \([A]\) relative to the endpoints \(a, b\). That said, if \(d = \mu(a, b)\), then we also write \(M(a, b)_d\) for the space \(M(a, b)\), and similarly \(\hat{M}(a, b)_d\) for \(\hat{M}(a, b)\). Thus \(M(a, b)_d\) is a \(d\)-dimensional component of instanton classes whose limits are in the classes \(a\) and \(b\).

Suppose \(a, b \in \mathcal{C}_{\text{irr}}\) with \(\mu(a, b) \equiv 1\). With suitable perturbation, \(\hat{M}(a, b)_0\) is a finite set, and, as explained in [22, \S 3.6], each of its elements determines an isomorphism \(\Lambda(a) \to \Lambda(b)\). Denoting the induced isomorphism \(\mathbb{Z}\Lambda(a) \to \mathbb{Z}\Lambda(b)\) corresponding to \([A] \in \hat{M}(a, b)_0\) by the symbol \(\epsilon[A]\), the differential \(\partial\) for \(C(Y)\) is defined in pieces by

\[
\partial|_{\mathbb{Z}\Lambda(a) \to \mathbb{Z}\Lambda(b)} = \sum_{[A] \in \hat{M}(a, b)_0} \epsilon[A].
\]
If we choose an element from each $\Lambda(a)$, then we may view $\partial$ as a map on $C^{\text{irr}}$ and write $(\partial a, b) = \#M(a, b)_0$, where $\#$ indicates a signed count. The differential lowers the relative $\mathbb{Z}/8$-grading by 1. The identity $\partial^2 = 0$ is obtained by interpreting the boundary of a one-dimensional moduli space $M(a, b)_1$ as a disjoint union of broken trajectories $M(a, c)_0 \times M(c, b)_0$. The relatively $\mathbb{Z}/8$-graded abelian group $I(Y)$ is defined to be $H_*(C(Y), \partial)$.

In defining the complex $C(Y)$, we have chosen a Riemannian 3-manifold $Y$, an admissible $SO(3)$-bundle $Y$ over $Y$, a perturbation $\pi$ and a base connection $a_0$ on $Y$. When working with the chain group, we always assume that such data are chosen. The isomorphism class of the relatively $\mathbb{Z}/8$-graded group $I(Y)$ depends only on the oriented homeomorphism type of $Y$ and $w_2(Y)$.

4.2. Maps from cobordisms

Let $X : Y_1 \to Y_2$ be a cobordism from $Y_1$ to $Y_2$. That is, $X$ is a compact, connected, oriented 4-manifold with an orientation-preserving diffeomorphism $\partial X \simeq Y_2 \sqcup Y_1$. As before, each $Y_i$ is connected. Assume that $X$ is equipped with a metric that is product-like near its boundary. Suppose further that $X$ is an $SO(3)$-bundle over $X$ with $X|_{Y_i} = Y_i$ where each $Y_i$ is admissible. We abbreviate this setup as $X : Y_1 \to Y_2$. To obtain a chain map

$$m(X) : C(Y_1) \to C(Y_2),$$

first form the bundle $(\mathbb{R}_{\leq 0} \times Y_1) \cup X \cup (\mathbb{R}_{\geq 0} \times Y_2)$ over the Riemannian 4-manifold obtained from $X$ by attaching cylindrical ends to the boundary. We define $M(a, X, b)$ to be the space of $\mathcal{G}_\nu$-classes of finite-energy instantons on this bundle. With suitable perturbations chosen, $M(a, X, b)$ is a smooth manifold, and we write $\mu(a, X, b) = \dim M(a, X, b)$. As before, $\mu(a, X, b)$ is well-defined modulo 8, and we write $M(a, X, b)_d = M(a, X, b)$ for $d = \mu(a, X, b)$. Now suppose $a \in C^{\text{irr}}(Y_1)$ and $b \in C^{\text{irr}}(Y_2)$ with $\mu(a, X, b) = 0$. With suitable perturbations, $M(a, X, b)_0$ is a finite set of points. In defining $C(Y_i)$, basepoint connections $a_{i,0}$ are chosen. Let $A$ be a connection on $X$ (with cylindrical ends attached) with limits at the ends equivalent to the $a_{i,0}$. An orientation of the line $\det(D_A)$ will be called an $I$-orientation of $X$, following [22, Definition 3.9]. With an $I$-orientation of $X$, an element $[A] \in M(a, X, b)_0$ determines an isomorphism $\epsilon[A] : \Lambda(a) \to \Lambda(b)$, and $m(X)$ is defined in pieces by

$$m(X)|_{\Lambda(a) \to \Lambda(b)} = \sum_{[A] \in M(a, X, b)_0} \epsilon[A].$$

In shorthand, $\langle m(X)a, b \rangle = \#M(a, X, b)_0$. When $\mu(a, X, b) \neq 0$, this part of the differential is zero. Different choices of $I$-orientations only affect the overall sign of the map $m(X)$. The notation we use for composing bundle cobordisms is given by

$$X_2 \circ X_1 = X_1 \cup_{Y_2} X_2 : Y_1 \to Y_3,$$

where $X_i : Y_i \to Y_{i+1}$ for $i = 1, 2$. We write $I(X_1)_i : I(Y_1) \to I(Y_2)$ for the map on homology induced by $m(X_1)$. Having assumed that $Y_i$ is connected for $i = 1, 2$, we have the composition law

$$I(X_2 \circ X_1) = I(X_2) \circ I(X_1).$$

There is a well-defined notion of composing $I$-orientations using (4.2), and this is needed to make sense of this expression. For a general discussion of the composition law involving disconnected 3-manifolds, see [22, §5.2]. We mention that the composition law follows from the homotopy formula (4.5) below, using a one-dimensional family of metrics that stretches along $Y_2$. 


4.3. Index formulas

The numbers $\mu(a, b)$ and $\mu(a, X, b)$ above are more properly described as the indices of certain Fredholm operators. Let $X : Y_1 \to Y_2$ as above. The $Y_i$ are not assumed to be admissible. Let $a$ and $b$ be connections on $Y_1$ and $Y_2$, respectively. Attach cylindrical ends to $X$ as above and call the result $X$ as well. Choose a connection $A$ on $X$ with $A|_{Y_1 \times \{t\}}$ equal to $a$ for $t \leq 0$ and $A|_{Y_2 \times \{t\}}$ equal to $b$ for $t \geq 0$, and consider the operator

$$D_A = -d_A^* \otimes d_A^+ : L^p_{s, \phi}(A^1 \otimes X_{ad}) \to L^p_{s-1, \phi}((A^0 \oplus A^+) \otimes X_{ad}),$$

where $L^p_{s, \phi} = \phi L^p_s$ are Sobolev spaces weighted by the real function $\phi$, equal to $e^{-\epsilon t}$ for some sufficiently small $\epsilon > 0$ on the ends $\mathbb{R}_{<0} \times Y_1$ and $\mathbb{R}_{>0} \times Y_2$, and equal to 1 otherwise. This operator arises from linearizing the instanton equation and using a Coulomb gauge condition. If $X' : Y_2 \to Y_3$ and $A'$ is a connection on $X'$ with limit $b$ over $Y_2$, then there is a natural isomorphism

$$\det(D_A) \otimes \det(D_{A'}) \simeq \det(D_{A \cup A'})$$

and the index relation $\text{ind}(D_A) + \text{ind}(D_{A'}) = \text{ind}(D_{A \cup A'})$ holds; see, for example, [7, Proposition 5.11]. In the definition of $C(Y)$ in §4.4, we take $X = [0, 1] \times Y$ to define $D_A$.

Note that the two ends $Y_1$ and $Y_2$ of the cobordism $X$ have opposite Sobolev weights in the description of $D_A$. If we instead view $X : \emptyset \to Y_2 \sqcup Y_1$, then the construction yields a different operator $D'_A$. That is, $D'_A$ differs from $D_A$ by using the weight function $\phi'$ in place of $\phi$, where $\phi'$ is obtained by altering $\phi$ over $\mathbb{R}_{<0} \times Y_1$ from $e^{-\epsilon t}$ to $e^{+\epsilon t}$. We have the relation

$$\text{ind}(D'_A) - \text{ind}(D_A) = h^0(a) + h^1(a);$$

cf. [7, Proposition 3.10]. When there is one cylindrical end, the number $\text{ind}(D'_A)$ is the same as $\text{ind}^-(A)$ in the notation of [5] and $\text{ind}^+(A)$ in the notation of [7].

The index $\text{ind}(D'_A)$ is the expected dimension of the moduli space $M(a, X, b)^{\text{irr}}$ of irreducible instanton classes. It is this number which we refer to in computations, so we define

$$\mu(a, X, b) = \mu(A) = \text{ind}(D'_A),$$

and this agrees with our earlier usage of $\mu(a, X, b)$. Note that the order of the symbols $a, X, b$ does not matter, and is only suggestive of the situation in mind. If $X_1$ and $X_2$ are bundles over cobordisms and are composable, then we have the gluing formula

$$\mu(a, X_2 \circ X_1, c) = \mu(a, X_1, b) + \mu(b, X_2, c) + h^0(b) + h^1(b).$$

(4.3)

If $X$ is over a closed 4-manifold $X$; then we also have

$$\mu(X) = -2p_1(X) - 3(1 - b_1(X) + b_+(X)).$$

(4.4)

Here $b_+(X)$ is the dimension of a maximal positive-definite subspace for the intersection form on $H_2(X; \mathbb{R})$. The term $1 - b_1(X) + b_+(X)$ may also be written as $(\chi(X) + \sigma(X))/2$, where $\chi$ is the Euler characteristic and $\sigma$ the signature.

4.4. Maps from families of metrics on cobordisms

This section extracts formulæ due to Kronheimer and Mrowka from [22, §3.9]. We first consider families of metrics in a general context. Let $X$ be any smooth manifold and $S$ be a hypersurface in the interior of $X$. We assume that $S$ has a neighborhood $N \subset X$ diffeomorphic to $(-1, 1) \times S$. A metric on $X$ cut along $S$ is a Riemannian metric $g$ on $X \setminus S$ that on the neighborhood $N$ is of the form

$$dr^2 + r^2 + g_0.$$
ends that along the cut hypersurface $S$. We may regard a Riemannian manifold with a cut metric as one with two opposing cylindrical ends that along the cut hypersurface $S$ meet only at infinity.

Given a collection of hypersurfaces $\mathcal{H} = \{S_i\}$ in the interior of $X$ with similar neighborhoods, we construct a set of metrics $G = G(\mathcal{H})$ on $X$ that are cut along various subsets of $\mathcal{H}$. The construction is intuitively simple: stretch an initial metric in all possible ways along each hypersurface.

First, suppose that $\mathcal{H}$ has no intersecting hypersurfaces. We will parameterize the family $G$ by $[0,1]^d$ where $d = |\mathcal{H}|$. Let $b_t$ be a family of positive smooth functions on $[-1,1]$ parameterized smoothly by $t \in [0,1)$ such that $b_t(r)$ approaches $1/r^2$ as $t$ goes to 1. For some fixed $\varepsilon$, $0 < \varepsilon < 1$, we require that $b_t(r) = 1$ for $|r| > \varepsilon$. We also require $b_t \neq b_s$ when $t \neq s$. We choose the initial metric $G(0)$ on $X$ so that it is of the form $dr^2 + g_i$ in the neighborhood of $S_i \subset X$ diffeomorphic to $(-1,1) \times S_i$. Here $S_i \in \mathcal{H}$ and $g_i$ is a metric on $S_i$. For $t \in [0,1]^d$, we define $G(t)$ on $X$ by changing $G(0)$ in the neighborhood of $S_i$ to $b_i(r)dr^2 + g_i$.

Now consider an arbitrary set of hypersurfaces $\mathcal{H}$. Let $\mathcal{H}_0$ be a subset of $\mathcal{H}$ with no intersecting hypersurfaces. We have constructed a family $G(\mathcal{H}_0)$ for each such $\mathcal{H}_0$. We glue the hypercubes $[0,1]^{d_0}$ where $d_0 = |\mathcal{H}_0|$ together to form a space in the obvious way: when two points correspond to the same metric, identify them. This defines the family $G(\mathcal{H})$.

Now suppose that $\mathcal{X} : Y_1 \to Y_2$ as in §4.2. Let $G = G(\mathcal{H})$ be a family of metrics on $X$ constructed as above. We extend $G$ to a family of metrics on $X$ with cylindrical ends attached, product-like on the ends, which we also call $G$. Let $M_G(a,\mathcal{X},b)$ be the moduli space of pairs $([A],g)$ where $g \in G$ and $A$ is a finite-energy instanton with respect to $g$. The meaning of this is straightforward if $g$ is an uncut, smooth metric. An instanton with a metric cut along $S \subset X$ is an instanton on the complement of $S$, with its limits on the two cylindrical ends $[0,\infty) \times S$ agreeing. More details can be found in [22, §3.9].

Let $G = G(\mathcal{H})$ be a family of metrics on $X$ as constructed above. In the cases in which we are interested, $G$ will have the structure of a convex polytope. The metrics parameterized by a face of $G$ consist of cut metrics, cut along a hypersurface in $\mathcal{H}$. The expected dimension of $M_G(a,\mathcal{X},b)$ is $\mu(a,\mathcal{X},b) + \dim G$. A map

$$m_G(\mathcal{X}) : C(Y_1) \to C(Y_2)$$

is defined just as for cobordisms. To fix the sign of $m_G(\mathcal{X})$, in addition to an $I$-orientation of $\mathcal{X}$, we must orient the metric family $G$. The following three formulae are due to Kronheimer and Mrowka, [22, §3.9], and arise from understanding the compactification and gluing of certain moduli spaces. First,

$$(-1)^{\dim G} m_G(\mathcal{X}) \partial - \partial m_G(\mathcal{X}) = m_{\partial G}(\mathcal{X}). \tag{4.5}$$

In writing this, we have inherited the orientation conventions of [22], with the exception that the quotients $\tilde{M}(a,b)$ are oriented oppositely, changing the signs of the maps $\partial$. For the polytopes $G$ that we will consider, $\partial G$ decomposes into a union of faces $G(S)$, one for each hypersurface $S \in \mathcal{H}$. In this case,

$$m_{\partial G}(\mathcal{X}) = \sum_{S \in \mathcal{H}} m_{G(S)}(\mathcal{X}). \tag{4.6}$$

Finally, suppose that $\mathcal{X}$ is the composite of two bundle cobordisms: $\mathcal{X} = \mathcal{X}_2 \circ \mathcal{X}_1$. Also suppose that $G = G_1 \times G_2$ where $G_1$ is a family of metrics that only varies on $X_1$ and $G_2$ on $X_2$, and all metrics are cut along $X_1 \cap X_2$. Then

$$m_G(\mathcal{X}) = (-1)^{\dim G_1 \dim G_2} m_{G_2}(\mathcal{X}_2) m_{G_1}(\mathcal{X}_1), \tag{4.7}$$

where we interpret $G_1$ as a family of metrics on $X_1$ and $G_2$ as a family on $X_2$. Here the metric families are oriented, and $G = G_1 \times G_2$ is an orientation-preserving identification.
4.5. **Index bounds**

The following discussion is based on [5, § 3.4; 7, § 4], with the material of [22, § 3.9] in mind. So far we have only mentioned moduli spaces for which the limiting connections are acyclic. This guarantees, in particular, that all instantons are irreducible.

For simplicity, suppose that $X$ has one cylindrical end. We consider moduli spaces $M(X, a)$ where $a$ is any almost flat connection (that is, an element of $\mathcal{C}$), where the finite-energy instantons exponentially approach $a$ over the cylindrical end. Then, with suitable perturbation, the subset of irreducibles $M(X, a)^{\text{irr}}$ is a smooth manifold of dimension $\mu(X, a)$. In this case, the existence of $[A] \in M(X, a)^{\text{irr}}$ implies $\mu(X, a) = \mu(A) \geq 0$. On the other hand, if all the instantons are reducible with common isotropy group $\Gamma$, the space $M(X, a)$ has dimension $\mu(X, a) + \dim \Gamma$. Recall $h^0(A) = \dim \Gamma$. In this case, after perturbation, the existence of an instanton $[A]$ in the moduli space implies the bound

$$\mu(A) + h^0(A) \geq 0. \quad (4.8)$$

More generally, suppose $([A], g) \in M_G(X, a)$ for a family of metrics $G$. Then we obtain

$$\mu(A) + h^0(A) + \dim G \geq 0. \quad (4.9)$$

We also consider the case in which some of the limiting connections are allowed to vary. Suppose that $[0, \infty) \times Y$ is the cylindrical end of $X$, and consider a smooth manifold $\mathfrak{F} \subset \mathcal{C}(Y)$ of critical points to which the Chern–Simons functional is non-degenerate transverse. We consider $M(X, \mathfrak{F})$, the instanton classes that exponentially approach the set $\mathfrak{F}$. The irreducibles within typically form a smooth manifold whose components have dimensions mod 8 congruent to $\mu(X, a) + \dim \mathfrak{F}$, where $a \in \mathfrak{F}$. We write $M(X, \mathfrak{F})^{\text{irr}}$ for the $d$-dimensional component.

We can introduce metrics into all of these situations. The most general situation we consider is the following. Suppose that $\mathfrak{F}$ is as above, and consider the moduli space $M_G(X, \mathfrak{F})$. If $([A], g)$ is a member, then in the generic case we obtain a bound

$$\mu(A) + h^0(A) + \dim G + \dim \mathfrak{F} \geq 0. \quad (4.10)$$

We write $M_G(X, \mathfrak{F})^d$ for the $d$-dimensional moduli space of instantons $([A], g)$ with $d$ equal to the left-hand side of (4.10) and where $\circ = \text{irr}$, red, flat describes the respective stabilizer-types $h^0(A) = 0, 1, 3$. One can drop the assumption that $\mathfrak{F}$ is smooth and obtain moduli spaces that are stratified according to the structure of $\mathfrak{F}$. Such spaces have been studied in [30, 37].

4.6. **Gradings**

In addition to the relative $\mathbb{Z}/8$-grading on $I(\mathbb{Y})$, we can define an absolute $\mathbb{Z}/2$-grading following [7, § 5.6; 15, § 2.1]. It is more generally defined on the critical sets $\mathcal{C}$. If $a \in \mathcal{C}$, then its grading is given by

$$\text{gr}(a) = b_1(E) + b_+(E) + \mu(E, a) \mod 2,$$

where $E : \emptyset \to Y$ is an SO(3)-bundle over a connected 4-manifold $E$ with $\partial E = Y$ that restricts to $\mathbb{Y}$ over $Y$. The differential of $C(\mathbb{Y})$ shifts this grading by 1. A map $m(X) : C(\mathbb{Y}_1) \to C(\mathbb{Y}_2)$ shifts the grading by the parity of

$$\text{deg}(X) = -\frac{3}{2}(\chi(X) + \sigma(X)) + \frac{1}{2}(b_1(Y_2) - b_1(Y_1)). \quad (4.11)$$

cf. [22, § 4.5]. More generally, a map $m_G(X)$ shifts the grading by $\text{deg}(X) + \dim G$. As an example, suppose that $T^3$ is the bundle over $T^3$ with $w_2(T^3)$ Poincaré dual to an $S^1$-factor. Then $I(T^3)$ is two copies of $\mathbb{Z}$ supported in the even grading. Note that the trivial connection $\theta$ on $S^3$ has $\text{gr}(\theta) \equiv 1$. We note that $I(\mathbb{Y})_i$ is the same as the cohomology group $I(\mathbb{Y})_{b_1(Y)+i+1}$, where $\mathbb{Y}$ means the orientation of the base space $Y$ is reversed. For our conventions regarding the absolute $\mathbb{Z}/8$-grading in the case where $Y$ is a homology 3-sphere, see § 9.
5. Proving the exact triangle

In this section, we prove Theorem 2.1, Floer’s exact triangle. We use an algebraic lemma first used in [34] by Oszváth and Szabó to prove an exact sequence in Heegaard Floer homology. The use of metric stretching maps in this context was applied in [24] by Kronheimer, Mrowka and the previous two authors to prove an exact triangle in monopole Floer homology. Bloom [4] also treats the monopole case. Our proof is largely an adaptation of Kronheimer and Mrowka’s proof [22] in the singular instanton knot homology setting. In particular, while § 5.2 is essentially part of Floer’s original proof (see [5, § 4]) the contents of § 5.3, notably the idea for Lemma 5.4 and its proof, are based on ideas from [22, § 7.1].

5.1. The triangle detection lemma

The following statement is adapted from [22, § 7.1] and first appeared in [34].

**Lemma 5.1.** Let \((C_i, \partial_i)\) be a sequence of complexes, \(i \in \mathbb{Z}\). Suppose that there are chain maps \(f_i : C_i \rightarrow C_{i+1}\) and maps \(h_i : C_i \rightarrow C_{i+2}\) satisfying

\[
f_{i+1}f_i + \partial_{i+2}h_i + h_i \partial_i = 0.
\]

Suppose further that each sum

\[
f_{i+2}h_i + h_{i+1}f_i
\]

induces an isomorphism \(H(C_i) \rightarrow H(C_{i+3})\). Then

\[
\cdots \rightarrow H(C_i) \xrightarrow{H(f_i)} H(C_{i+1}) \xrightarrow{H(f_{i+1})} H(C_{i+2}) \rightarrow \cdots
\]

is an exact sequence. Furthermore, the anti-chain map \(f_i \oplus h_i : C_i \rightarrow \text{Cone}(f_{i+1})\) is a quasi-isomorphism for each \(i \in \mathbb{Z}\).

To apply this lemma, we use the notation of § 3, so that we have a 3-periodic sequence of surgery bundles \(\mathbb{Y}_i, i \in \mathbb{Z}\), and surgery cobordism bundles \(\mathbb{X}_{ij} : \mathbb{Y}_i \rightarrow \mathbb{Y}_j\) whenever \(j > i\). We let \((C_i, \partial_i)\) be the instanton chain complex \(\mathbb{C}(\mathbb{Y}_i)\) with its differential. We take \(f_i\) to be \(m(\mathbb{X}_{i,i+1}) : \mathbb{C}(\mathbb{Y}_i) \rightarrow \mathbb{C}(\mathbb{Y}_{i+1})\). The map \(h_i\) is defined in § 5.2, and in § 5.3 we define a chain homotopy \(k_i\) from \(f_{i+2}h_i + h_{i+1}f_i\) to an intermediate map, and then show that this intermediate map is chain homotopic to the identity map of \(C_i\) up to sign. All maps are of the form \(m_G(\mathbb{X})\).

5.2. The \(h_i\) maps

We define \(h_0 : C_0 \rightarrow C_2\) in this section. Recall from § 3.4 that we can write

\[
X_{02} = W \cup_{S_1} U,
\]

where \(U\) is diffeomorphic to \(-\mathbb{CP}^2\) minus a 4-ball, and \(W\) has boundary \(Y_2 \sqcup \overline{\mathbb{Y}}_0 \sqcup S_1\). The map \(h_0\) is taken to be \(m_G(\mathbb{X}_{02})\) where \(G\) is a family of metrics on \(X_{02}\) induced by the set of two intersecting hypersurfaces \(\mathcal{H} = \{S_1, Y_1\}\). Thus \(G\) is parameterized by an interval, with endpoint metrics \(G(S_1)\) and \(G(Y_1)\), cut along \(S_1\) and \(Y_1\), respectively, as depicted in Figure 7. Equations (4.5) and (4.6) yield

\[
-h_0 \partial_0 - \partial_2 h_0 = m_{G(S_1)}(\mathbb{X}_{02}) + m_{G(Y_1)}(\mathbb{X}_{02}).
\]

By equation (4.7), we also have \(m_{G(Y_1)}(\mathbb{X}_{02}) = m(\mathbb{X}_{12})m(\mathbb{X}_{01}) = f_1 f_0\). It remains to show that \(m_{G(S_1)}(\mathbb{X}_{02}) = 0\).

Let \(a\) and \(b\) be given with \(\mu(a, \mathbb{X}_{02}, b) = 0\). To show that \(m_{G(S_1)}(\mathbb{X}_{02}) = 0\), it suffices to show that \(M_{G(S_1)}(a, \mathbb{X}_{02}, b)\) is empty for any such \(a, b\). We prove this by contradiction. Suppose \([A] \in M_{G(S_1)}(a, \mathbb{X}_{02}, b)\). Write \(U\) and \(W\) for the restriction of \(\mathbb{X}_{02}\) to \(U\) and \(W\), respectively.
Because $G(S_1)$ is cut along $S_1$, $[A]$ is a pair $[A_W], [A_U]$ in $M(a, W, b, c) \times M(c, U)$ for some flat connection $c$ on $S_1$. We arrange that the perturbation data near $S_1$ is $0$. The gluing formula (4.3) says
\[ \mu(A) = \mu(A_W) + \mu(A_U) + h^0(c) + h^1(c). \]

The flat connection $c$ is on a 3-sphere, so $h^1(c) = 0$ and $h^0(c) = 3$. Since $a$ and $b$ are irreducible, so is $A_W$. It follows that $\mu(A_W) \geq 0$; see inequality (4.9). The connection $A_U$ may be reducible to $S^1$, but no further, because $U$ is non-trivial, so $h^0(A_U) \leq 1$. It follows from (4.8) that $\mu(A_W) \geq -1$, implying $\mu(A) = \mu(a, X_{02}, b) \geq 2$, which is a contradiction.

5.3. The $k_i$ maps

We define $k_0 : C_0 \to C_0$ in this section. Recall from §3.5 that we have five hypersurfaces $Y_1, Y_2, S_1, S_2, T$ in $X_{03}$ that intersect one another as in Figure 5. We define $k_0$ to be $m_G(X_{03})$ where $G$ is the family of metrics on $X_{03}$ induced by the set of hypersurfaces $\mathcal{H} = \{Y_1, Y_2, S_1, S_2, T\}$. The family $G$ is parameterized by a pentagon and has faces $G(Y_1), G(Y_2), G(S_1), G(S_2), G(T)$, each of which is an interval of metrics broken along the indicated hypersurface. See Figure 8. Equations (4.5) and (4.6) yield
\[ k_0 \partial_0 - \partial_0 k_0 = \sum_{S \in \mathcal{H}} m_G(S)(X_{03}) \]

and the argument from §5.2 shows that $m_G(S_1)(X_{03}) = m_G(S_2)(X_{03}) = 0$. We also have $m_G(Y_1)(X_{03}) = h_1 f_0$ and $m_G(Y_2)(X_{03}) = f_2 h_0$ by (4.7). Thus
\[ k_0 \partial_0 - \partial_0 k_0 = m_G(T)(X_{03}) + f_2 h_0 + h_1 f_0, \]
or in other words, $k_0$ is a chain homotopy from $m_G(T)(X_{03})$ to $f_2 h_0 + h_1 f_0$. The proof is thus complete if we establish the following lemma.

**Lemma 5.2.** The map $m_G(T)(X_{03})$ is chain homotopic to $\pm id : C_0 \to C_0$.

The remainder of this section goes toward proving this lemma. From §3.5, we know the hypersurface $T$ induces a decomposition
\[ X_{03} = V \cup_T E, \]

where $E$ is diffeomorphic to $-\mathbb{CP}^2$ minus a regular neighborhood of an unknotted $S^1$. Let $V, E$ be the restrictions of $X_{03}$ to $V, E$, respectively. The restriction of $G(T)$ to $V$ is a single metric. On the other hand, the restriction of $G(T)$ to $E$ is an interval of metrics, and we denote this
family by $G_T$, see Figure 9. We arrange that the perturbations used near $T$ are zero, so that the relevant limiting connections are flat.

The map $m_{G(T)}(\mathcal{X}_{03})$ is defined by counting isolated points $[A] \in M_{G(T)}(a, \mathcal{X}_{03}, b)_0$. That is,

$$\langle m_{G(T)}(\mathcal{X}_{03})a, b \rangle = \#M_{G_T}(a, \mathcal{X}_{03}, b)_0,$$

where $\#$ means a signed count determined by orienting moduli spaces. Note that $\mu(A) = -1$ since $\dim G(T) = 1$. Let $a$ and $b$ be the limiting connections of $A$ on $\mathcal{Y}_0$ and $\mathcal{Y}_3$, respectively, so $[a] = a$ and $[b] = b$. Each such $A$ can be written as a pair

$$A_V, (A_E, g),$$

where $A_V$ is an instanton on $V$ with limit $a$ over $\mathcal{Y}_0$, $b$ over $\mathcal{Y}_3$ and some flat limit $c$ over $T$; and $A_E$ is a $g$-instanton on $E$ where $g \in G_T$, and $A_E$ has the same flat limit $c$ over $T$.

First, let us understand $\mathcal{T} = \mathcal{C}(T)$, the space of $\mathcal{G}_{ev}$-classes of flat connections on $T$. Recall that $T$ is a trivial $SO(3)$-bundle over an $S^1 \times S^2$. Choose a spin structure for $T$, that is, a lift to an $SU(2)$-bundle. Lifting connections sets up a bijection between flat $SO(3)$-connections modulo $G_{ev}$ on $T$ with flat $SU(2)$-connections modulo $SU(2)$ gauge transformations. It is well known that this latter set is in correspondence with $\text{Hom}(\pi_1(T), SU(2))$ modulo conjugation, which is essentially the set of conjugacy classes of $SU(2)$. The space of conjugacy classes of $SU(2)$ is $[-1, 1]$, given by the trace map divided by 2.

The isomorphism $\mathcal{T} \simeq [-1, 1]$ depends on the spin structure of $T$ chosen. There are two such choices, and they are related by any non-even gauge transformation of $T$; using such a transformation the isomorphisms $\mathcal{T} \simeq [-1, 1]$ are related by reflecting $[-1, 1]$ about 0. The choice of isomorphism can also be determined by choosing a trivial holonomy flat connection on $T$; this choice corresponds to $1 \in [-1, 1]$. We record the following.
Lemma 5.3. A choice of spin structure for $\mathcal{T}$ determines an isomorphism $\mathcal{T} \simeq [-1, 1]$. The action on $\mathcal{T}$ by $\mathcal{G}/\mathcal{G}_{ev} \simeq \mathbb{Z}/2$ under this isomorphism is reflection about 0.

We can now understand the structure of the relevant moduli space following basic index computations. Write $\mathcal{T}^0$ for the interior of $\mathcal{T}$, and $\mathcal{G}_T^0$ for the interior of $\mathcal{G}_T$.

Lemma 5.4. The moduli space $M(a, X_{03}, b)_0$ can be identified with the fiber product

$$M(a, V, b, \mathcal{T}^0)_0 \times_{\mathcal{T}^0} M_{G_T^0}(\mathcal{T}^0, \mathcal{E})_1^{\text{red}}$$

after a suitable perturbation.

The moduli space on the right is the space of pairs $([A_E], g)$ where $g \in G_T^0$ and $A_E$ is a $g$-instanton on $\mathcal{E}$ (exponentially decaying over the ends), such that the flat limit class of $A_E$ over $T$ lies in the interior of $\mathcal{T}$: $h^0(A_E) = 1$, that is, $A_E$ has gauge-stabilizer $S^1$; and $\mu(A_E) = 1 - h^0(A_E) - \dim G_T^0 - \dim \mathcal{T}^0 = -2$. In other words, the lemma says that, in the pair $(5.1)$ representing $[A] \in M(a, X_{03}, b)_0$, we have the constraints

$$\mathcal{C} = [c] \in \mathcal{T}^0, \quad g \in G_T^0, \quad \mu(A_V) = -1, \quad \mu(A_E) = -2.$$  

(5.2)

The fiber product is taken with respect to limit maps $\lambda : M \to \mathcal{T}^0$ that send an instanton class to its flat limit class over $\mathcal{T}$, where $M$ is one of the two moduli spaces appearing in the lemma. This fiber product description is an application of the Morse–Bott gluing theory as discussed in [7, § 4.5.2; 30, 31, 37]. Our situation, that of instantons broken along $S^1 \times S^2$ with flat limits in $\mathcal{T} \simeq [-1, 1]$, is similar to that of Fintushel and Stern’s in [12], where results of Mrowka’s thesis [31] are used, and we will refer the reader to these sources for more details. We mention that, for the above fiber product, it is important that the stabilizers of $c$ and $A_E$, each a circle, can be identified. In general, one must record a gluing parameter in $\Gamma_c/\Gamma_{AV} \times \Gamma_{AE}$ where $\Gamma_A$ is the stabilizer of $A$. For instance, if both $[A_V]$ and $[A_E]$ were irreducible, then there would be more than one choice of such a parameter. We proceed to prove that the constraints (5.2) characterize the possible gluing data.

Proof of Lemma 5.4. We first show $\mathcal{C} \in \mathcal{T}^0$. For convenience, we set

$$h(c) = (h^0(c) + h^1(c))/2.$$ 

We note that $h(c) = 1$ or $3$, depending on whether $\mathcal{C}$ is in the interior or boundary of $\mathcal{T}$, respectively; cf. [12, § 3]. By assumption, $\mu(A) = -1$, so (4.3) yields

$$-1 = \mu(A) = \mu(A_V) + \mu(A_E) + 2h(c).$$

Let $A_{S^1 \times D^3}$ be a connection on the trivial bundle over $S^1 \times D^3$ with one cylindrical end attached. We identify the bundle over cross sections of the end with $\mathcal{T}$, with the base having the opposite orientation of $T$. Suppose that $A_{S^1 \times D^3}$ has flat limit $c$. We glue $A_{S^1 \times D^3}$ to $A_E$ to obtain a connection $A_{-\mathbb{C}P^2}$ on a non-trivial bundle $\mathcal{E}'$ over $-\mathbb{C}P^2$. The isomorphism class of $\mathcal{E}'$ depends on $c$, but we know $p_1(\mathcal{E}') = 4k - 1$ for some $k \in \mathbb{Z}$; cf. [8, § 4.1.4]. We have

$$\mu(A_E) + \mu(A_{S^1 \times D^3}) + 2h(c) = \mu(A_{-\mathbb{C}P^2}).$$

We compute $\mu(A_{S^1 \times D^3})$. Two copies of $S^1 \times D^3 \times SO(3)$, each with a cylindrical end, glue, overlapping the ends, to give $S^1 \times S^3 \times SO(3)$. Index additivity yields

$$2\mu(A_{S^1 \times D^3}) + 2h(c) = \mu(S^1 \times S^3 \times SO(3)).$$

On the other hand, (4.4) says the right-hand side is

$$-3(1 - b_1 + b_2^+)(S^1 \times S^3) = 0.$$
Thus $\mu(A_{S^1 \times D^3}) = -h(c)$. This can also be deduced from the Atiyah–Patodi–Singer index theorem; cf. [1, Theorem 3.10]. From (4.4), we obtain $\mu(A_{\mathbb{CP}^2}) = -8k - 1$, and then

$$\mu(A_V) = 8k - h(c), \quad \mu(A_E) = -8k - 1 - h(c).$$

Suppose for a contradiction that $c$ is on the boundary of $\mathcal{T}$, so that $h(c) = 3$. Since $A_V$ is irreducible and the boundary of $\mathcal{T}$ has dimension 0, we have

$$8k - 3 = \mu(A_V) \geq 0$$

in the generic case, so $k > 0$. Since $E'$ is non-trivial, $h^0(A_E) \in \{0, 1\}$. Using (4.9), we find

$$-8k - 4 = \mu(A_E) \geq -\dim G_T - \dim \partial \mathcal{T} - h^0(A_E) \geq -2.$$

Then $k < 0$, which is a contradiction. Thus $h(c) = 1$ and $c \in \mathcal{T}^0$. It follows that $\mu(A_V) = 8k - 1$ and $\mu(A_E) = -8k - 2$. Applying (4.9) in this case,

$$\mu(A_E) \geq -\dim \mathcal{T}^0 - \dim G_T - h^0(A_E) \geq -3,$$

so $k \leq 0$. Similarly, $\mu(A_V) \geq -\dim \mathcal{T}^0 = -1$ gives $k \geq 0$. Thus $k = 0$, yielding $\mu(A_V) = -1$ and $\mu(A_E) = -2$, as claimed.

Next, we rule out the possibility that $h^0(A_E) = 0$, that is, that $[A] \in M_G(T)(a, \mathbb{Z}_{03}, b)_0$ can be written as a gluing of $[A_V]$ and $([A_E], g)$ where $A_E$ is irreducible, that is,

$$([A_E], g) \in M_{G_T}(\mathcal{T}^0, E)_0.$$ 

Note that if there were such a gluing, then we would have to keep track of a gluing parameter, as mentioned earlier. However, this moduli space of irreducibles and $M(a, \mathbb{Z}, \mathcal{T}^0)_0$ are both finite sets after perturbation, by standard compactness results; cf. [12, §5]. Further, the intersection of their flat limits in $\mathcal{T}^0$ can be made transverse, in which case they have empty intersection. Thus, after a suitable perturbation, $h^0(A_E) = 1$.

Finally, we show $g \in G^0_T$. Suppose for a contradiction that $g \in \partial G_T$. Then $g$ is one of two metrics on $E$, $G_T(S_1)$ or $G_T(S_2)$, cut along $S_1$ or $S_2$, respectively. See Figure 9. Suppose $g = G_T(S_1)$; the other case is similar. Write

$$E = X \cup_{S_1} U,$$

where $U \simeq -\mathbb{CP}^2 \setminus \text{int}(D^4)$ and $X \simeq D^2 \times S^3 \setminus \text{int}(D^4)$. Note that the restriction of $E$ over $X$ is trivial, while the restriction over $U$, as in §5.2, is non-trivial; write $A_X$ and $A_U$ for the restriction of $A_E$ over these respective bundles. They have a common flat limit $d$ on $S_1$. In particular, $h^0(d) = 3$ and $h^1(d) = 0$. The connection $A_X$ has the limit $c$ over $\mathcal{T}$ from before.

We compute $\mu(A_X)$ and $\mu(A_U)$. There is only one instanton class on $X$: the trivial class; cf. [7, §7.4.1]. Thus $A_X$ is trivial, so $h(c) = 3$. Let $A_{S^1 \times D^3}$ be a connection on the trivial bundle over $S^1 \times D^3$ with one cylindrical end attached whose flat limit is $c$. Then $A_X$ and $A_{S^1 \times D^3}$ glue, overlapping ends, to give a connection $A_{D^4}$ over $D^4$ with one cylindrical end attached. Then (4.3) and (4.4) yield

$$\mu(A_X) + \mu(A_{S^1 \times D^3}) + 2h(c) = \mu(A_{D^4}) = -3.$$

From above, $\mu(A_{S^1 \times D^3}) = -h(c) = -3$. Thus $\mu(A_X) = -6$. With $\mu(A_U) = 8k - 1$ for some $k \in \mathbb{Z}$, we apply (4.3) once more to get

$$\mu(A_E) = \mu(A_X) + \mu(A_U) + 2h(d) = 8k - 4.$$

It follows that $\mu(A_E) \neq -2$, which is a contradiction. 

\begin{lemma}
The projection $M_{G_T^0}(\mathcal{T}^0, E)^{sim}_1 \rightarrow G^0_T$ is a smooth homeomorphism.
\end{lemma}
Proof. The moduli space here is topologized as a subset of $\mathcal{B} \times G^0_T$, so the projection map is a continuous, open map. It is also smooth, in the transverse case, by general theory. It suffices to show bijectivity. The argument is a standard account of counting reducible instantons.

Let $([A_E], g)$ be such that $\mu(A_E) = -2, h^0(A_E) = 1$ and $g \in G^0_T$. Because $H^1(E; \mathbb{R}) = 0$, $E$ admits no non-trivial real line bundles. Thus $h^0(A_E) = 1$ implies that $A_E$ is compatible with a splitting $\mathbb{L} \oplus \mathbb{R}$ of the associated vector bundle of $E$, where $\mathbb{L}$ is a complex line bundle and $\mathbb{R}$ is a trivial real line bundle. Gluing $A_E$ to a connection $A_{S^1 \times D^3}$ on a trivial bundle over $S^1 \times D^3$ with one cylindrical end attached, gives an instanton $A_{-\mathbb{CP}^2}$ on a bundle $\mathbb{R}^\ell \oplus \mathbb{L}'$ over $-\mathbb{CP}^2$ where $\mathbb{R}^\ell$ and $\mathbb{L}'$ are extensions of $\mathbb{R}$ and $\mathbb{L}$. The gluing formula says

$$\mu(A_E) + \mu(A_{S^1 \times D^3}) + h^0(c) + h^1(c) = \mu(A_{-\mathbb{CP}^2}) = -2p_1(\mathbb{R}^\ell \oplus \mathbb{L}') - 3.$$  \hspace{1cm} (5.3)

Using that $p_1(\mathbb{R}^\ell \oplus \mathbb{L}') = c_1(\mathbb{L})^2$, we have $\mu(A_E) = -2c_1(\mathbb{L})^2 - 4$. Since $\mu(A_E) = -2$, we conclude that $c_1(\mathbb{L})^2 = -1$. Let $P(E)$ denote the image of the map $H^2(E, \partial E; \mathbb{Z}) \to H^2(\mathbb{E}; \mathbb{Z})$. Note that inclusion $E \to -\mathbb{CP}^2$ induces an isomorphism of intersection forms from $H^2(-\mathbb{CP}^2; \mathbb{Z})$ to $P(E)$, both negative definite of rank 1, under which $c_1(\mathbb{L}')$ is sent to $c_1(\mathbb{L})$. It follows that $c_1(\mathbb{L})$ is a generator of $H^2(\mathbb{E}; \mathbb{Z})$.

There are thus two choices of $\mathbb{L}$ corresponding to the choices of generator for $H^2(\mathbb{E}; \mathbb{Z})$. To get one from the other, take the conjugate $\mathbb{L}^*$. The choice we make does not matter in the end, as we can relate the two by an even gauge transformation, by combining the conjugation map $\mathbb{L} \to \mathbb{L}^*$ with the involution of $\mathbb{R}$ that reflects each fiber. Note that $\mathcal{B} = \mathcal{B}_e$ for $E$.

We are left with the problem of finding $g$-instantons on $\mathbb{L}$. According to [1, Proposition 4.9], the space of $L^2$-harmonic 2-forms on $E$ is isomorphic to the image of $H^2(E, \partial E; \mathbb{R}) \to H^2(\mathbb{E}; \mathbb{R})$, and under this isomorphism a harmonic form $x$ corresponds to its de Rham class $[x]$. In our case, this map is an isomorphism $\mathbb{R} \to \mathbb{R}$. Further, any such harmonic $x$ satisfies $\ast x = -x$, as follows: $\ast x$ is $L^2$-harmonic, so $\ast x = cx$ for some $c \in \mathbb{R}$; then $\ast^2 = 1$, $\int x \wedge x < 0$, and $0 < |x|^2_{L^2} = \int x \wedge x = c \int x \wedge x$ imply that $c = -1$. Conversely, a closed $L^2$-2-form $x$ satisfying $\ast x = -x$ is easily seen to be $L^2$-harmonic.

The arguments from [8, §2.2.1] easily adapt here, since $H^1(E; \mathbb{R}) = 0$, to show that, given a closed $L^2$-2-form $x$ on $E$, there is a connection $A$ on $\mathbb{L}$ with curvature $ix$ which is unique up to gauge equivalence. In this way, the unique $L^2$-harmonic 2-form representing $-2\pi c_1(\mathbb{L})$ specifies a unique $g$-instanton class on $\mathbb{L}$. \hfill $\square$

**Lemma 5.6.** The moduli space $M_{\partial^G_T}(\partial \Xi, \mathbb{E})^{\text{red}}$ consists of two points, and is the natural boundary of the open interval $M_{\mathcal{B}^G_T}(\Xi, \mathbb{E})^{\text{red}}$.

**Proof.** The previous lemma tells us that the ends of the latter moduli space are essentially the ends of $G_T$. There are two endpoint metrics of $G_T$, labeled $G_T(S_1)$ and $G_T(S_2)$, each broken along the indicated 3-sphere. Any instanton $A$ on $E$ compatible with $G_T(S_1)$ is a gluing of the trivial instanton on the trivial bundle over $X \simeq D^2 \times S^1 \setminus \text{int}(D^4)$ with two cylindrical ends attached and an instanton $A_U$ on $U \simeq -\mathbb{CP}^2 \setminus \text{int}(D^4)$ with one cylindrical end attached.

By the removable singularities theorem of Uhlenbeck (cf. [8, Theorem 4.4.12]), the instanton $A_U$ uniquely extends to an instanton $A$ on a bundle $\mathbb{W}$ over $-\mathbb{CP}^2$. If $A$ is to be a limit of elements in $M_E$, then $p_1(\mathbb{W}) = -1$. There is only one such instanton class on $\mathbb{W}$; cf. [22, §2.7]. Thus $[A]$ is uniquely determined. Similarly, there is one instanton class to add for $G_T(S_2)$. That $A$ is trivial over $X$ implies that the flat limits over $\mathcal{T}$ of these two instanton classes lie in $\partial \Xi$. \hfill $\square$

Note that the map in Lemma 5.5 extends to a homeomorphism of closed intervals. We write $M_{G_T}(\Xi, \mathbb{E})^{\text{red}}$ for the completed closed interval moduli space. We call a map between closed intervals **proper** if it sends boundary to boundary. A proper map between oriented, closed
intervals has a well-defined degree, which is 0 or \(\pm 1\). Indeed, one can define the degree by looking at the induced map \(S^1 \to S^1\) obtained by identifying boundary points.

**Lemma 5.7.** The map \(\lambda : M_{G_T}(\Sigma, \Sigma_1^\text{red}) \to \Sigma\) defined by sending an instanton class to its flat limit class over \(T\) has degree \(\pm 1\).

**Proof.** We use the involution \(\sigma : E \to E\) of §3.6. Write \(M\) for the moduli space in the lemma. We see that \(\sigma\) induces an action on \(M\), and because \(\sigma(T) = T\), an action on \(\Sigma\). We can arrange the family of metrics \(G_T\) so that \(\sigma\) restricts to an isometry of the base space and reflects \(G_T\), in turn swapping the endpoints of the interval \(M\). If we establish that \(\sigma\) also swaps the endpoints of the interval \(\Sigma\), we are done, because the limit map \(\lambda\) respects the action of \(\sigma\). From §3.6, we know that with respect to a fixed trivialization \(\Sigma \cong S^1 \times S^2 \times SO(3)\), \(\sigma\) is isotopic to a composition \(\theta \circ \nu\), where \(\theta\) is a diffeomorphism of \(S^1 \times S^2\) lifted in a trivial way to \(S^1 \times S^2 \times SO(3)\). The diffeomorphism under consideration acts trivially on \(\pi_1(T)\), and hence \(\theta\) acts trivially on \(\Sigma\). The map \(\nu\) is a non-even gauge transformation, and so, by Lemma 5.3, it reflects the interval \(\Sigma\). It follows that \(\sigma\) reflects \(\Sigma\).

**Proof of Lemma 5.2.** By our fiber product description of \(M_{G(T)}(a, \mathcal{X}_03, b)\), we can write

\[
\# M_{G_T}(a, \mathcal{X}_03, b)_0 = \pm \sum \varepsilon(x) \varepsilon(y),
\]

where the sum is over pairs

\[(x, y) \in M(a, \mathcal{V}, b, \mathcal{X}_0) \times M_{G_T}^\text{red}(\mathcal{X}_0, E)_0\]

having equal flat limit class \(\lambda(x) = \lambda(y) \in \mathcal{X}_0\). Each \(x\) and \(y\) has a sign, \(\varepsilon(x)\) and \(\varepsilon(y)\), respectively, prescribed by orienting moduli spaces. In the generic case, the sum of the \(\varepsilon\) for a fixed value \(\varepsilon(y)\) equals \(\pm \deg(\lambda) = \pm 1\). In this way, we obtain

\[
\# M_{G_T}(a, \mathcal{X}_03, b)_0 = \pm \# M(a, \mathcal{V}, b, \mathcal{X}_0)_0,
\]

where the sign does not depend on the pair \((a, b)\). Thinking of cobordisms as morphisms, we abbreviate \([0, 1] \times \mathcal{Y}_0\) to \(Y_0\). Write \(1_{Y_0} = \mathcal{V} \cup T \mathcal{W}\) where \(\mathcal{W}\) is a trivial bundle over \(W = S^1 \times D^3\). We choose the perturbation data for \(\mathcal{W}\) to be 0. Let \(Q\) be the family of metrics on \([0, 1] \times Y_0\) induced by \(\mathcal{H} = \{T\}\). The boundary of \(Q\) consists of an initial product metric on \([0, 1] \times Y_0\) and a metric \(Q(T)\) cut along \(T\). Thus (4.5) and (4.6) yield

\[-m_Q(1_{Y_0}) \partial_0 - \partial_0 m_Q(1_{Y_0}) = m_{Q(T)}(1_{Y_0}) + m(1_{Y_0}).\]

Of course, \(m(1_{Y_0})\) is the identity. It remains to show \(m_{Q(T)}(1_{Y_0}) = \pm m_{G(T)}(\mathcal{X}_03)\), or

\[
\# M_{Q(T)}(a, b)_0 = \pm \# M(a, \mathcal{V}, b, \mathcal{X}_0)_0,
\]

where again the sign does not depend on the pair \((a, b)\). In the spirit of our previous arguments, we establish this by arguing that \(M(a, \mathcal{V}, b, \mathcal{X}_0)_0\) can be written as a fiber product

\[
M(a, \mathcal{V}, b, \mathcal{X}_0)_0 \times \mathcal{X}_0 M(\mathcal{X}_0, \mathcal{W})_1^\text{flat}.
\]

Here \(M(\mathcal{X}_0, \mathcal{W})_1^\text{flat}\) is the one-dimensional family of flat connection classes on \(\mathcal{W}\) with arbitrary flat limit class in \(\mathcal{X}_0\). Indeed, any flat connection class on \(T\) uniquely extends to a flat connection class on \(\mathcal{W}\) over \(S^1 \times D^3\). We conclude that all instantons on \(\mathcal{W}\) are flat; cf. [7, §7.4]. In particular, the limit map \(\lambda : M(\mathcal{X}_0, \mathcal{W})_1^\text{flat} \to \mathcal{X}_0\) is a smooth homeomorphism. Now suppose
that \([A] \in M_Q(T)(a, b)_0\) restricts to a pair \([A_V], [A_W]\) of instantons on \(V\) and \(W\), respectively, with equal limit \(c\) over \(T\). Then
\[
0 = \mu(A) = \mu(A_V) + \mu(A_W) + 2h(c).
\]
We saw in Lemma 5.4 that \(\mu(A_W) = -h(c)\), so \(\mu(A_V) = -h(c)\). The space of \([A_V]\) with \(\mu(A_V) = -2\) is generically empty, so we conclude that \(\mu(A_V) = -1\). It follows that \(c \in \mathbb{T}^0\).

Because the stabilizer of each \(A_W\) is \(SU(2)\), the gluing parameter space is trivial, and our fiber product description is verified; cf. [12, §4]. Because the limit map \(\lambda : M(\mathbb{T}^0, W)^{\text{flat}} \to \mathbb{T}^0\) is a homeomorphism, our fiber product yields (5.4). This completes the proof of Lemma 5.2, and consequently the proof of Theorem 2.1.

6. A link surgeries spectral sequence

In this section, we prove Theorem 2.2. We follow \([4, 22]\). Kronheimer and Mrowka work over \(\mathbb{Z}\), taking care with signs, and we adapt many of the details from their setup. Bloom’s paper \([4]\) is especially descriptive of the combinatorics involved here, and provides many illustrations. As mentioned in §1, the idea for this spectral sequence originates from Ozsváth and Szabó’s paper [34].

6.1. The cobordisms and metric families

Let \(Y\) be an admissible bundle over \(Y\) and \(L \subset Y\) be a framed link with \(m\) components \(L_1, \ldots, L_m\). Suppose that we have admissible bundles \(Y_v\) for \(v \in \{-\infty, 0, 1\}^m\) that form a surgery cube as in §2. We confine the subscript \(\infty\) with \(-1\) and write \(Y_v\) for \(v \in \{-1, 0, 1\}^m\). Further, we write \(Y_v\) for \(v \in \mathbb{Z}^m\) by taking the modulo 3 reduction of \(v\). Define the norms
\[
|v|_1 = \sum_{i=1}^m |v_i|, \quad |v|_\infty = \max_{1 \leq i \leq m} \{|v_i|\}.
\]
We use the partial order on \(\mathbb{Z}^m\) that says \(v \leq w\) whenever \(v_i \leq w_i\) for \(i = 1, \ldots, m\).

Since the \(Y_v\) form a surgery cube, they can be generated by the data of \(Y\) and a framed link \(L = L_1 \cup \cdots \cup L_m\) in \(Y\) as in §3.1, where each \(L_i\) is an equivariant embedding of \(S^1 \times D^2 \times SO(3)\) into \(Y\). For \(v < w\), we have surgery bundle cobordisms \(X_{vw} : Y_v \to Y_w\) constructed by iterating the construction for \(X_{ij}\) from §3.3 for each \(L_i\). To give a definition, first set \(k = |w - v|_1\). We choose a maximal chain \(v = v(0) < v(1) < \cdots < v(k) = w\). Each \(X_{v(i)v(i+1)}\) may be viewed as a surgery bundle as defined in §3.3, and we may set
\[
X_{vw} = X_{v(k-1)v(k)} \circ \cdots \circ X_{v(0)v(1)}.
\]
The choice of maximal chain does not affect the isomorphism type of \(X_{vw}\). In fact, the identification of (3.3) lends a more invariant interpretation: we may view \(X_{vw}\) as \(Y_v \times [0, 1]\) with, for each \(i = 1, \ldots, m\), a copy of \(\mathbb{H} \cup \psi \cdots \cup \psi\) \(\mathbb{H}(w_i - v_i\) copies of \(\mathbb{H}\)) attached to \(Y_v \times \{1\}\) via the framed knot \(\Lambda^{v_i+1}(L_i)\). We have the isomorphism
\[
X_{vw} \simeq X_{uvw} \circ X_{vw}
\]
whenever \(v < u < w\). We write \(0\) for the element of \(\mathbb{Z}^m\) with all zeros, and similarly \(\mathbf{n}\) for the element with all elements equal to \(n \in \mathbb{Z}\). Note that \(X_{03}\) is not \(X_{00} = Y_0 \times [0, 1]\), but, for instance, \(X_{01} \simeq X_{34}\). The base space of \(X_{vw}\) is written as \(X_{vw}\). In the sequel, we will only consider \(X_{vw}\) with \(|w - v|_\infty \leq 3\).

As in the case when \(L\) had one component, we have distinguished hypersurfaces in the interior of \(X_{vw}\). Of course, the 3-manifolds \(Y_u \subset X_{vw}\) for \(v < u < w\) are the first examples.
Note that $Y_u$ and $Y_{u'}$ are disjoint if and only if $u < u'$ or $u' < u$. For each $i \in \{1, \ldots, m\}$ and $k$ with $v_i < k < w_i$ we have a 3-sphere $S_k^i$ in $X_{vw}$ which generalizes $S_1 \subset X_{02}$ from §3.4. The spheres $S_k^i$ and $S_{k'}^i$ intersect if and only if $i = j$ and $|k - j| \leq 1$, and $S_k^i$ intersects $Y_u$ if and only if $u_i = k$. For $v, w \in \mathbb{Z}^m$ with $v < w$ and $|w - v|_\infty \leq 2$, we define a set of hypersurfaces in $X_{vw}$:

$$H_{vw} = \{ Y_u : v < u < w \} \cup \{ S_k^i : 1 \leq i \leq m, \ v_i < k < w_i \}.$$ 

Note that the second set is empty if $|w - v|_\infty > 2$.

We obtain a family of metrics $G_{vw} = G(H_{vw})$ on $X_{vw}$ as constructed in §4.4. The space of metrics $G_{vw}$ is a convex polytope called a graph-associahedron, and

$$\dim G_{vw} = |w - v|_1 - 1,$$

as Bloom explains in [4, Theorem 5.3]. In fact, when $|w - v|_\infty < 2$, $G_{vw}$ is the permutahedron $P_N$, the convex polytope defined as the convex hull in $\mathbb{R}^N$ of all permutations of $(1, 2, \ldots, N) \in \mathbb{R}^N$ where $N = |w - v|_1$. For example, $P_3$ is a hexagon, and the polytope $P_4$ is shown (hollowed out) in Figure 10. Write $m_{vw} = m_{G_{vw}}(X_{vw})$ and $\partial_v$ for the differential of $G(Y_v)$. From the formulae in §4.4, we obtain

$$( -1)^{|w - v|_1 - 1} m_{vw} \partial_v - \partial_w m_{vw} = \sum_{v < u < w} m_{G(Y_u)}(X_{vw}) + \sum_{1 \leq i \leq m} m_{G(S_k^i)}(X_{vw}).$$

As in §5.2, each $m_{G(S_k^i)}(X_{vw}) = 0$. Also, the family $G(Y_v)$ can be identified with the product $G_{vu} \times G_{uw}$. Before we apply equation (4.7), we discuss the arrangement of signs.

It is possible to choose I-orientations $\mu_{uw}$ for $X_{vw}$ such that $\mu_{vw} = \mu_{uw} \circ \mu_{vu}$ whenever $v < u < w$, and we do so. For a proof, see [22, Lemma 6.1]. We can orient each $G_{vw}$ such that the identification of $G_{vu} \times G_{uw}$ with $G(Y_u) \subset \partial G_{vw}$ has orientation deficiency $(-1)^{\dim G_{vw}}$. That is, the product orientation for $G_{vu} \times G_{uw}$ using our chosen orientations differs from the boundary orientation as induced from $G_{uw}$ by the sign $(-1)^{\dim G_{uw}}$. This essentially follows from the discussion in [22] following Proposition 6.4. With this understood, equation (4.7) yields

$$m_{G(Y_u)}(X_{vw}) = ( -1)^{\dim G_{uw} + 1} \dim G_{vw} m_{uw} m_{vu}.$$ 

Writing $m_{vw} = \partial_v$, equation (6.1) becomes

$$\sum_{v \leq u \leq w} ( -1)^{|w - u|_1 (|u - v|_1 - 1)} m_{uw} m_{vu} = 0.$$ 

We remind the reader that this holds under the assumptions that $v < w$ and $|w - v|_\infty \leq 2$. The case $v = w$ also holds, encoding the relation $\partial_v^2 = 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure10.png}
\caption{(colour online). The permutahedron $P_4$.}
\end{figure}
6.2. Constructing the spectral sequence

We now construct the spectral sequence of Theorem 2.2. We define a chain complex \((C, \partial)\) with a filtration \(F^i C\). The filtration will induce the spectral sequence we desire. To begin, set

\[
C = \bigoplus_{v \in \{0,1\}^m} C(Y_v), \quad \partial = \sum_{v \leq w} \partial_{vw}, \quad \text{(6.2)}
\]

where \(\partial_{vw} = (-1)^{s(v,w)} m_{vw}\). The sign here is given by

\[
s(v, w) = (|w - v|^2 - |w - v|)/2 + |v|_1,
\]

as lifted from [22, Equation 38]. We compute the \(C(Y_v) \to C(Y_w)\) component of \(\partial^2\) to be

\[
(-1)^{s(v,w)+|w|} \sum_{v \leq u \leq w} (-1)^{|w-u|_1(|u-v|_1-1)} m_{uw}, m_{vu} = 0.
\]

We call \((C, \partial)\) the link surgeries complex associated to \((Y, L)\), with the understanding that the necessary auxiliary choices we have made have been fixed.

We define the filtration on \((C, \partial)\) by setting

\[
F^i C = \bigoplus_{|v| \geq i} C(Y_v) \subseteq C. \quad \text{(6.3)}
\]

Since \(\partial\) involves only terms with \(v \leq w\), it is immediate that \(\partial F^i C \subseteq F^i C\). This filtered complex induces a spectral sequence whose \(E^1\)-page and \(E^1\)-differential \(d^1\) are given by

\[
E^1 = \bigoplus_{v \in \{0,1\}^m} I(Y_v), \quad d^1 = \sum_{v \leq w} (-1)^{\delta(v,w)} m(X_{vw}),
\]

where \(\delta(v, w) \equiv \sum_{1 \leq i \leq j} v_i\), in which \(j\) is the unique index where \(v\) and \(w\) differ. This carries over from the discussion following [22, Corollary 6.9]. To prove Theorem 2.2, it remains to identify the \(E^\infty\)-page: we must show that the homology of \((C, \partial)\) is the instanton homology \(I(Y)\).

6.3. Convergence

Let \((C, \partial)\) be the link surgeries complex associated to \((Y, L)\). For \(i \in \mathbb{Z}\), define the chain complex \((C_i, \partial_i)\) to be the link surgeries complex associated to \((Y_{\Lambda^{i+1}}(L_1), L \setminus L_1)\). Recall that the notation \(Y_{\Lambda^{i+1}}(L_1)\) is from §3.2, and stands for \(\Lambda^{i+1}\)-surgery on \(L_1\) in \(Y\). We conflate \(\infty\) and \(-1\) in the following. Note that, for \(i = \infty, 0, 1\) and \(a, b \in \mathbb{Z}^{m-1}\), we have \((\partial_i)_{ab} = \partial_{uv}\) where \(v = (i, a)\) and \(w = (i, b)\). Thus we can work exclusively with the maps \(\partial_{vw}\) with \(v, w \in \mathbb{Z}^m\).

Consider the map \(f_0 : C_0 \to C_1\) given by

\[
f_0 = \sum_{v, w \in \{0,1\}^m \atop v_i = 0, w_i = 1} \partial_{vw}.
\]

It should be understood that \(\partial_{vw} = 0\) if \(v \not\leq w\). In words, \(f_0\) is the sum of the components in the differential \(\partial\) that correspond to surgery-cobordisms that include surgery on \(L_1\). This is an anti-chain map, and the larger complex \((C, \partial)\) is the cone-complex of \(f_0\). That is,

\[
C = C_0 \oplus C_1, \quad \partial = \begin{pmatrix} \partial_0 & 0 \\ f_0 & \partial_1 \end{pmatrix}.
\]

Define a map \(F : C_\infty \to C\) by

\[
F = \sum_{v_i = -1 \atop w_i \in \{0,1\}} \partial_{vw}.
\]
This is an anti-chain map: the relation $\mathbf{F} \partial_\infty + \partial \mathbf{F} = 0$ is an encoding of (6.1) via

$$\sum_{v_1 = u_1 = -1 \atop v_1, w_1 \in \{0,1\}} \partial_{uv_1} \partial_{wu} + \sum_{v_1 = -1 \atop u_1, w_1 \in \{0,1\}} \partial_{uw} \partial_{vu} = 0.$$ 

Equip $\mathbf{C}$ and $\mathbf{C}_\infty$ with filtrations as in (6.3) but using the sum $\sum_{i=2}^m v_i$ instead of $|v_1|$. Then $\mathbf{F}$ respects these filtrations, and on the $E^0_p$-components of the induced spectral sequences, the map induced by $\mathbf{F}$ takes the form

$$\mathbf{F}^0_p : \bigoplus_{v_1 = -1 \atop \sum_{i>2} v_i = p} \mathbf{C}(\mathbb{Y}_v) \longrightarrow \bigoplus_{v_1 \in \{0,1\} \atop \sum_{i>2} v_i = p} \mathbf{C}(\mathbb{Y}_v)$$

and for $v$ with $v_1 = -1$ is given by

$$\mathbf{F}^0_p|_{\mathbf{C}(\mathbb{Y}_v)} = \partial_{vv'} \oplus \partial_{vv''},$$

where $v'$, $v''$ have $v'_1 = 0$ and $v''_1 = 1$, and otherwise agree with $v$. But $\partial_{vv'}$ is the map $f_{-1}$ in §5.2 for the surgery triangle involving $\mathbb{Y}_v$ and $L_1$; and likewise $\partial_{vv''}$ is the map $h_{-1}$. It follows from Lemma 5.1 that $\mathbf{F}^0$ is a quasi-isomorphism, and hence so is $\mathbf{F}$. By removing each link component as we have just done for $L_1$, and composing the $m$ maps $\mathbf{F}$ associated to each removal, we get a quasi-isomorphism $\mathbf{Q}$ from $(\mathbf{C}(\mathbb{Y}), \partial)$ to $(\mathbf{C}, \partial)$, completing the proof of Theorem 2.2.

6.4. Gradings

We follow Bloom’s [4] treatment of gradings for the spectral sequence. We refer the reader to the mod 2 grading on the complex $\mathbf{C}(\mathbb{Y})$ defined in §4.6 as $\text{gr}[\mathbb{Y}]$. We define a grading $\text{gr}[\mathbf{C}]$ on the complex $\mathbf{C}$ in (6.2). For $x \in \mathbf{C}(\mathbb{Y}_v) \subset \mathbf{C}$ with homogeneous $\text{gr}[\mathbb{Y}_v]$ grading, we define

$$\text{gr}[\mathbf{C}](x) \equiv \text{gr}[\mathbb{Y}_v](x) + \deg(X_\infty) + |v_1| \mod 2.$$  

(6.4)

Recall that we conflate $\infty$ with $-1 \in \mathbb{Z}$. Let $\pi_w : \mathbf{C} \rightarrow \mathbf{C}(\mathbb{Y}_w)$ be the projection. Note that

$$\text{gr}[\mathbf{C}][\pi_w(\partial(x))] \equiv \text{gr}[\mathbb{Y}_w](m_{vw}(x)) + \deg(X_\infty) + |w_1| \mod 2.$$  

(6.5)

We have the additivity relation $\deg(X_\infty) \equiv \deg(X_\infty) + \deg(X_{vw})$, and also

$$\text{gr}[\mathbb{Y}_w](m_{vw}(x)) = \text{gr}[\mathbb{Y}_v](x) + \dim(G_{vw}) + \deg(X_{vw}).$$

Knowing $\dim(G_{vw}) = |w - v| - 1$ shows that the expressions (6.4) and (6.5) differ by 1 mod 2, and thus the differential $\partial$ alters $\text{gr}[\mathbf{C}]$ by 1.

The quasi-isomorphism $\mathbf{Q} : \mathbf{C}(\mathbb{Y}) \rightarrow \mathbf{C}$ is a composition of $m$ maps $\mathbf{F}$ as in the previous section. Thus it is a sum of maps of the form $m_G(X_\infty)$, where $v \in \{0,1\}^m$ and $G = G_1 \times \cdots \times G_m$. Here $G_i = G_i(v_i(v_{i+1}))$ only varies on $X_{v_i(v_{i+1})} \subset X_\infty$, and $\infty = -1 = v(1) < v(2) < \cdots < v(m + 1) = v$. Using $\dim(G_{vw}) = |w - v| - 1$ for $v < w$, we find $\dim(G) = |v_1|$. Since the $\text{gr}[\mathbb{Y}]$ to $\text{gr}[\mathbb{Y}_v]$ degree of $m_G(X_\infty)$ is $\dim(G) + \deg(X_\infty)$, it follows that $\mathbf{Q}$ preserves the $\mathbb{Z}/2$-gradings.

There is also a $\mathbb{Z}$-grading on $\mathbf{C}$ given by the vertex weight $|v|_1$ for a homogeneous element in $\mathbb{C}(\mathbb{Y}_v) \subset \mathbf{C}$, and by construction $\partial$ increases this by 1. We summarize a more detailed statement of Theorem 2.2; cf. [4, Theorem 1.1; 22, Corollaries 6.9, 6.10].

**Theorem 6.1.** Let $L$ be an oriented, framed link with $m$ components in $Y$. For each $v \in \{\infty, 0, 1\}^m$ denote by $Y_v$ the result of $v$-surgery on $L$ and let $\mathbb{Y}_v$ be an admissible bundle over $Y_v$ such that the total collection of $\mathbb{Y}_v$ forms a surgery cube. For $v < w$, there are surgery cobordism bundles $X_{vw}$ from $\mathbb{Y}_v$ to $\mathbb{Y}_w$ with $I$-orientations $\mu_{vw}$ satisfying $\mu_{uw} \circ \mu_{vu} = \mu_{vw}$.
whenever $v < u < w$, such that there is a spectral sequence $(E^r, d^r)$ with

$$E^1 = \bigoplus_{v \in \{0, 1\}^m} I(Y_v), \quad d^1 = \sum_{v < w} (-1)^{\delta(v,w)} I(X_{vw}),$$

where $\delta(v, w) = \sum_{1 \leq j \leq m} v_j$, in which $j$ is the unique index where $v$ and $w$ differ. The spectral sequence is graded by $\mathbb{Z}/2 \times \mathbb{Z}$, where $d^r$ has bi-degree $(1, r)$. The $\mathbb{Z}/2$-grading is given by (6.4) while the $\mathbb{Z}$-grading is by vertex weight. The spectral sequence converges by the $E^{m+1}$-page to $I(Y)$, and it induces the usual $\mathbb{Z}/2$-grading on $I(Y)$.

7. Framed instanton homology

In this section, we discuss the basic constructions and properties of the groups $I^\#(Y)$. These are a special case of the groups $I^\#(Y, K)$ introduced by Kronheimer and Mrowka in [22]. Here $Y$ is a 3-manifold and $K$ is a knot or link in $Y$, and we have $I^\#(Y) = I^\#(Y, \emptyset)$. The name framed instanton homology comes from [23]. The group $I^\#(Y)$ is isomorphic to the sutured instanton group $SHI(M, \gamma)$ from [21], where $M$ is the complement of an open 3-ball in $Y$ and $\gamma$ is a suture on the 2-sphere boundary.

7.1. Framed instanton groups

Let $Y$ be a connected, oriented, closed 3-manifold. Consider an SO(3)-bundle $\mathbb{Y}^\#$ over $Y \# T^3$ with $\mathbb{Y}^\#$ trivial over $Y$ and non-trivial over $T^3$. To make the construction of $\mathbb{Y}^\#$ from $Y$ more precise, we can once and for all pick a point $x \in T^3$, a bundle $T^3$ over $T^3$ geometrically represented by an $S^1$-factor, and an isomorphism $T^3_x \simeq SO(3)$. Then, up to inessential choices, $\mathbb{Y}^\#$ can be constructed from $Y$ and a basepoint $y \in Y$. Indeed, we can perform the connected sum $Y \# T^3$ between 3-balls around $y$ and $x$, and glue the bundles $Y \times SO(3)$ and $T^3$ by expanding the isomorphism $SO(3) \simeq T^3_x$ near $x$.

We describe a useful operation for cobordisms in this context. Let $X : Y_1 \to Y_2$ be a cobordism and let $\gamma : [0, 1] \to X$ be a properly embedded path with $\gamma(0)$ and $\gamma(1)$ being the chosen basepoints in $Y_1$ and $Y_2$, respectively. Given another such pair $X', \gamma'$ where $X' : Y'_1 \to Y'_2$, we form a cobordism

$$X \ltimes X' : Y_1 \# Y'_1 \to Y_2 \# Y'_2$$

as follows: let $\Gamma$ be a neighborhood of $\gamma$ diffeomorphic to $int(D^3) \times [0, 1]$, and write

$$\partial(X \setminus \Gamma) \setminus (Y_1 \cup Y_2 \setminus \partial \Gamma) = S^2 \times [0, 1];$$

do the same for $X'$, and identify the copies of $S^2 \times [0, 1]$ by an orientation-reversing homeomorphism. See Figure 11. We omit the paths from the notation $X \ltimes X'$ because, for all of our cobordisms, there will be a natural choice of path up isotopy relative to the boundaries. The operation $\ltimes$ extends to glue together cobordisms of bundles $X$ and $X'$ if a path of isomorphisms $X_\gamma(t) \simeq X'_\gamma(t)$ is chosen.

Let $g$ be a gauge transformation of $\mathbb{Y}^\#$ with $\eta(g) \in H^1(Y \# T^3; \mathbb{F}_2)$ Poincaré dual to a 2-torus $\Sigma \subset T^3$ over which $\mathbb{Y}^\#$ is non-trivial. Here $\eta : \mathcal{V}(X) \to H^1(X; \mathbb{F}_2)$ is from the exact sequence (4.1). Such a transformation may be constructed explicitly as in [9, Lemma A.2]. Define the framed gauge transformations $\mathcal{V}^\#$ to be the subgroup of $\mathcal{V}(\mathbb{Y}^\#)$ generated by $\mathcal{V}_c(\mathbb{Y}^\#)$ and $g$. We let $\mathcal{C}^\#$ denote the critical set of a perturbed Chern–Simons functional $c_{\mathcal{C}}$ on $\mathcal{C}/\mathcal{V}^\#$. Note that $\mathcal{C}^\#$ is obtained from $\mathcal{C}(\mathbb{Y}^\#)$ by modding out by the $\mathbb{Z}/2$-action of degree 4 induced by the gauge transformation $g$.

We define the chain complex $C^\#(Y)$ for $I^\#(Y)$ following ideas from [22, §4.4]. This definition transparently replaces the notion of an I-orientation with that of a homology orientation. Fix
once and for all a bundle $\mathbb{W} : S^3 \times SO(3) \to \mathbb{T}^3$ over $T^2 \times D^2 \setminus \text{int}(D^4) : S^3 \to T^3$ extending $T^3$. Fix a path $\gamma$ in $W$ beginning in $S^3$, and ending at $x \in T^3$, and a path of isomorphisms $\mathbb{W}_{\gamma(t)} \simeq SO(3)$, the isomorphisms at the ends being the natural choices. We define

$$C^\#(Y) = \bigoplus_{a \in \mathbb{C}^\#} \mathbb{Z}A^\#(a),$$

where $A^\#(a)$ is the 2-element set of orientations of the line $\det(D_A)$; here $A$ is a connection on $[0,1] \times Y \times \mathbb{W}$ (with cylindrical ends attached) where the limit of $A$ over the $\mathbb{R} \times Y$ cylindrical end is equivalent to the trivial connection, and the limit of $A$ over the $\mathbb{R} \times Y\mathbb{#}$ end is in the class $a$. The operator $D_A$ is as in §4.3.

The differential for $C^\#(Y)$ is straightforward to define, following the construction of the differential for $I(Y)$ in §4.1, which followed [22, §3.6]. Note that a base connection as in the definition for $C(Y)$ is no longer needed. In summary, given $Y$ with a basepoint, with suitable metric and perturbation, the complex $C^\#(Y)$ and hence the group $I^\#(Y)$ are determined. The isomorphism class of $I^\#(Y)$ depends only on $Y$.

7.2. Maps from cobordisms

We describe how a cobordism $X : Y_1 \to Y_2$ with a path $\gamma$ as above gives rise to a map $I^\#(X) : I^\#(Y_1) \to I^\#(Y_2)$. Again, we omit $\gamma$ from the notation because there will always be a natural choice for us. We always assume that $X$ and $Y_1, Y_2$ are connected. Take the path in $T^3 \times [0,1]$ given by $t \mapsto (x, t)$. Using this, we form a cobordism

$$X^\# = X \times (T^3 \times [0,1]) : Y_1 \#T^3 \to Y_2\#T^3.$$

Further, there is a natural choice for bundle $X^\#$ over $X^\#$ by performing the $\times$ operation between $X \times SO(3)$ and $T^3 \times [0,1]$ using the constant path of isomorphisms $SO(3) \simeq \mathbb{T}^3_x$. We enlarge the even gauge transformation group used for $X^\#$ to include gauge transformations whose restriction to each $T^3$ is of the form $g$ from §7.1. See [22, §5.1] for a general discussion. Then, in the usual way, we obtain a chain map $m^\#(X) : C^\#(Y_1) \to C^\#(Y_2)$ and an induced map $I^\#(X)$ on homology.

The data of an $I$-orientation may be replaced by an orientation of the line $\det(D_A)$ where $A$ is the trivial connection on $X \times SO(3)$. Following [23, §3.8], but using homology instead of cohomology, this amounts to an orientation of the vector space

$$\mathcal{L}(X) := H_1(Y_1; \mathbb{R}) \oplus H_1(X; \mathbb{R}) \oplus H_2^+(X; \mathbb{R}),$$

where $H_2^+(X; \mathbb{R})$ is a maximal positive-definite subspace for the intersection form on $H_2(X; \mathbb{R})$. A choice of such an orientation is called a homology orientation for the cobordism $X$, and is typically denoted by $\mu_X$. In summary, given $X : Y_1 \to Y_2$, a path $\gamma$ from the basepoint of $Y_1$ to the basepoint of $Y_2$, a suitable perturbation and metric, and a homology orientation

Figure 11 (colour online). A schematic depiction of the $\times$ operation. The thicker lines represent actual boundary components.
More generally, with the 4-ball filled in, so that it is a non-trivial bundle over $T$ because the characteristic classes of the bundles are uniformly controlled in this case. We give the case of the absolute mod 2 grading for $Z$.

We now define the absolute grading $I$ of two 4-manifolds with boundary, as used in [18]; one deletes a model half-4-ball along the boundaries of $W$ and $W'$ and glues them together with an orientation-reversing homeomorphism, so that $\partial(W\#W') = \partial W\#\partial W'$. We have

$$(X \otimes X') \circ (W\#W') \simeq (X \circ W)\#(X' \circ W')$$

where compositions involved are of course assumed to make sense, and the same relation holds with the compositions reversed. See Figure 12.

7.3. Grading

We now define the absolute $\mathbb{Z}/4$-grading on $I^*(Y)$. Let $\mathbb{W}'$ be a completion of $\mathbb{W}$ from §7.1 with the 4-ball filled in, so that it is a non-trivial bundle over $T^2 \times D^2$, and we may write $\mathbb{W}' : \emptyset \to \mathbb{D}$. Fix an integer $k$. For $a \in C^*(Y)$, we define

$$\text{gr}(a) := -\mu(E \sharp \mathbb{W}', a) - b_1(E) + b_+(E) - b_1(Y) + k \text{ mod } 4,$$

where $E : \emptyset \to Y$ is a 4-manifold with boundary $Y$ and $E = E \times SO(3)$. We choose $k$ such that $I^*(S^3)$ is supported in grading 0. The proof that this grading is well defined is the same as the case of the absolute mod 2 grading for $I(Y)$ as, for example, in [7]: we get $\mathbb{Z}/4$ instead of $\mathbb{Z}/2$ because the characteristic classes of the bundles are uniformly controlled in this case. We give the argument for completeness, and compute the degrees of cobordism maps. We have chosen our conventions so that the degree formula aligns with that of [22, Proposition 4.4].

**Proposition 7.1.** The assignment $a \mapsto \text{gr}(a)$ gives a well-defined $\mathbb{Z}/4$-grading on $C^*(Y)$ for which the differential has degree $-1$ and thus descends to a $\mathbb{Z}/4$-grading on $I^*(Y)$. Given a cobordism $X : Y_1 \to Y_2$ equipped with the data to form $X^\#$ as in §7.2, the degree of the induced map $I^*(X) : I^*(Y_1) \to I^*(Y_2)$ is given by the expression for $\text{deg}(X)$ in (4.11) taken modulo 4. More generally, if $\mathbb{Y}_i = Y_i \times SO(3)$ and $X : \mathbb{Y}_1 \to \mathbb{Y}_2$ is possibly non-trivial and comes equipped with the data to form $X^\#$, then the degree of the induced map $I^*(X) : I^*(Y_1) \to I^*(Y_2)$ is...
given by
\[ -\frac{3}{2}(\chi(X) + \sigma(X)) + \frac{1}{2}(b_1(Y_2) - b_1(Y_1)) + 2\mathcal{P}(X) \mod 4, \]  
where the invariant \( \mathcal{P}(X) \in \mathbb{Z}/2 \) is defined by
\[ \mathcal{P}(X) \equiv [S] \cdot [\bar{S}] \mod 2. \]

Here \( S \subset X \) is a surface in the interior of \( X \), \([S] \in H_2(X; \mathbb{F}_2)\) and the image of \([S]\) in \( H_2(X, \partial X; \mathbb{F}_2)\) is Poincaré dual to \( w_2(X) \).

\[ \text{Proof.} \quad \text{Let } E' : Y \to \emptyset \text{ and } E' = E' \times SO(3), \text{ and let } \mathbb{W}' \text{ be the reverse of } \mathbb{W}. \text{ In particular, we may write } \mathbb{W}' : T^3 \to \emptyset. \text{ Then, by (4.3), we have}
\]
\[ \mu(\mathbb{E} \natural \mathbb{W}, a) + \mu(\mathbb{E} \natural \mathbb{W}') = \mu((E' \circ E)\#(\mathbb{W}' \circ \mathbb{W}')). \]  
By (4.3), we may write the right-hand side as
\[ \mu(E' \circ E) + 3 + \mu(\mathbb{W}' \circ \mathbb{W}). \]
Note \( \mathbb{W}' \circ \mathbb{W} \) is a bundle over \( T^2 \times S^2 \), which necessarily has \( p_1 \) congruent to 0 mod 4. Also, \((1 - b_1 + b_+)(T^2 \times S^2) = 0\). By (4.4), we conclude that \( \mu(\mathbb{W}' \circ \mathbb{W}') \) is congruent to 0 mod 4. Noting that \( E' \circ E \) is a trivial bundle, (7.2) is mod 4 congruent to
\[ \mu(E' \circ E) + 3 = 3(b_1 - b_+)(E' \circ E), \]
which by a Mayer–Vietoris argument (see §8.2) is mod 4 congruent to
\[ -b_1(E) - b_1(E') + b_+ (E') + b_1 (Y). \]
It follows that the expression
\[ \text{gr}(a) - \mu(a, E' \natural \mathbb{W}') = b_1(E') - b_+ (E') - 2b_1(Y) + k \mod 4 \]
is independent of \( E \), and thus so is \( \text{gr}(a) \). In other words, \( \text{gr}(a) \) is a well-defined \( \mathbb{Z}/4 \)-grading on \( C^\#(Y) \). Suppose \( a, b \in C^\#(Y) \) with \( \mu(a, R \times \mathbb{Y}^\#, b) = 1 \). Then
\[ \mu(E \natural \mathbb{W}', a) + \mu(a, R \times \mathbb{Y}^\#, b) = \mu(E \natural \mathbb{W}', b) \]
yields \( \text{gr}(b) - \text{gr}(a) = -1 \). It follows that the differential lowers the grading by 1. Now we compute the degree of a map \( I^\#(X) \) induced by a cobordism \( X : Y_1 \to Y_2 \). Let \( X = X \times SO(3) \) and form \( V = X \times (T^3 \times [0, 1]) \). Let \( a \in C^\#(Y_1) \) and \( b \in C^\#(Y_2) \) with \( \mu(a, V, b) = 0 \). Let \( E : \emptyset \to Y_1 \) and \( E = E \times SO(3) \). Then (4.3) and \( \mu(a, V, b) = 0 \) yield \( \mu(V \circ (E \natural \mathbb{W}'), b) = \mu(E \natural \mathbb{W}', a) \). Thus \( \text{deg}(X) \equiv \text{gr}(b) - \text{gr}(a) \) is given by
\[ -b_1(X \circ E) + b_+ (X \circ E) - b_1(Y_2) + b_1(E) - b_+ (E) + b_1 (Y_1). \]
From the discussion in §8.2, \( -b_1(X \circ E) + b_+ (X \circ E) \) is equal to
\[ -b_1(E) - b_1(X) + b_+ (X) + b_1 (Y). \]
We obtain the simplified expression
\[ \text{deg}(X) \equiv -b_1(X) + b_+ (X) + 2b_1(Y_1) - b_1(Y_2) \mod 4. \]  
Using the assumption that \( X, Y_1 \) and \( Y_2 \) are connected and non-empty, we have \( \chi(X) = 1 - b_1(X) + b_2(X) - b_3(X) \). Poincaré–Lefschetz duality tells us \( b_3(X) = b_1(X, \partial X) \), and by the long exact sequence for the pair \( (X, \partial X) \) with real coefficients, we obtain
\[ d - b_2(X) + b_1(\partial X) - b_1(X) + b_1(X, \partial X) - b_0(\partial X) + b_0(X) = 0, \]
where $d$ is the dimension of the image of the map $H_2(X) \to H_2(X, \partial X)$. Note $b_0(\partial X) = 2$ and $b_0(X) = 1$. On the other hand, $d = b_+(X) + b_-(X)$ and $\sigma(X) = b_+(X) - b_-(X)$. We obtain
\[
\chi(X) = -2b_1(X) + b_1(Y_1) + b_1(Y_2) + d, \quad \sigma(X) = 2b_+(X) - d.
\]

Plugging these data into expression (4.11), rewritten here as
\[
-\frac{3}{2}(\chi(X) + \sigma(X)) + \frac{1}{2}(b_1(Y_2) - b_1(Y_1)),
\]
yields, modulo 4, the expression for $\deg(X)$ in (7.3). Now we approach the more general statement, supposing that $X : Y_1 \to Y_2$ is possibly non-trivial. We write
\[
\deg(X) = \deg(Y) + 2\mathcal{P}(X) \mod 4,
\]
where $\mathcal{P}(X)$ is to be determined. Let $E_1 : \emptyset \to Y_1$ and $E_2 = E_1 \times SO(3)$. Given $a \in C^\#(Y_1)$, choose $b \in C^\#(Y_2)$ such that $\mu(a, X^\#, b) \equiv 0$. Write $X_{tr} = X \times SO(3)$. Then
\[
\deg(X) - \deg(Y) = \mu(X \circ E_1 \circ W', b) - \mu(X_{tr} \circ E_1 \circ W', b).
\]

After closing up bundles using some $E_2 : Y_2 \to \emptyset$ with $E_2 = E_2 \times SO(3)$ and canceling out the contribution from the bundle over $T^2 \times S^2$ as above, this difference is seen from (4.4) to be
\[
-2p_1(E_2 \circ X \circ E_1) = \frac{1}{4\pi^2} \int_{E_2 \circ X \circ E_1} \text{tr}(F_A^2),
\]
where $A$ is any connection. We can choose $A$ to be trivial away from the interior of $X$, thus
\[
\mathcal{P}(X) = \frac{1}{8\pi^2} \int_X \text{tr}(F_A^2) \mod 2,
\]
where $A$ is any connection on $X$ that restricts to trivial connections on each $Y_i$. In other words, $\mathcal{P}(X) \equiv p_1(X') \mod 2$, where $X'$ is any trivial extension of $X$ over a closed 4-manifold. Thus
\[
\mathcal{P}(X) \equiv \tilde{w}_2(X)^2 \mod 2,
\]
where $\tilde{w}_2(X)$ is a lift of $w_2(X)$ to $H^2(X, \partial X; F_2)$. The result follows. \qed

### 7.4. Duality

The chain group $C^\#(Y)$ is the same as $C^\#(Y)$ but with the differential maps transposed. It follows that $I^\#(Y)$ and $I^\#(Y)$ are isomorphic over $\mathbb{Q}$. More precisely, given a homology orientation of $Y$, that is, an orientation of $H_1(Y; \mathbb{R})$, we get an isomorphism
\[
I^\#(Y; \mathbb{Q})_i \cong I^\#(Y; \mathbb{Q})_{b_i(Y) - i}.
\]

The homology orientation is required to identify the chain groups. The grading shift in (7.4) is explained as follows. Let $E_1 : \emptyset \to Y$ and $E_2 : Y \to \emptyset$, and $a \in C^\#(Y)$. Write $\overline{a}$ for the corresponding class in $C^\#(Y)$. From (4.3) and (4.4), we obtain
\[
\mu(E_1 \circ W', a) + \mu(\overline{a}, E_2 \circ W'') = 3(b_1(E) - b_+(E)),
\]
where $E_i = E_i \times SO(3)$, $E = E_2 \circ E_1$ and $W''$ is the reverse of $W'$. The bundle $W'' \circ W'$ over $T^2 \times S^2$ has been removed from the expression just as in §7.3. Using that $b_1(E) - b_+(E)$ is equal to
\[
b_1(E_1) + b_1(E_2) - b_+(E_1) - b_+(E_2) - b_1(Y)
\]
(see §8.2), we obtain $\text{gr}(a) + \text{gr}(\overline{a}) \equiv b_1(Y) + 2k$. We claim that $k$ is even. Let $a$ be the generator of $I^\#(S^3)$, represented by a flat connection on $T^3 \simeq S^3 \# T^3$. Recall that $k$ is chosen so that
$I^\#(S^3)$ is supported in grading 0, so we have $\text{gr}(a) = 0$ (also see §7.6). In the definition of $\text{gr}(a)$, choose $\mathbb{E} : \emptyset \to S^3$ to be a trivial bundle over a 4-ball. Then

$$0 \equiv \text{gr}(a) \equiv -\mu(\mathbb{W}', a) + k \text{ mod } 4.$$ 

Recall from the proof of Proposition 7.1 that $\mu(\mathbb{W}'' \circ \mathbb{W}'') \equiv 0 \text{ mod } 4$, where $\mathbb{W}'' : T^3 \to \emptyset$ is the reverse bundle-cobordism of $\mathbb{W}'$. By the index gluing formula (4.3), we then have $-\mu(\mathbb{W}', a) \equiv \mu(a, \mathbb{W}'') \text{ mod } 4$. Since $\mathbb{W}'$ is diffeomorphic to its orientation reversal, which is $\mathbb{W}''$, we also have $\mu(\mathbb{W}', a) \equiv \mu(a, \mathbb{W}'') \text{ mod } 4$, as follows from the Atiyah–Patodi–Singer index formula [1, Theorem 3.10]. Thus $k \equiv \mu(\mathbb{W}', a) \equiv 0 \text{ mod } 2$. It follows that

$$\text{gr}(a) + \text{gr}(\bar{a}) \equiv b_1(Y) \text{ mod } 4,$$

establishing the grading shift in (7.4).

7.5. Exact triangles

In this section, we state a few exact triangles for framed instanton homology. For these it is necessary to allow non-trivial bundles. In the above constructions, take $\mathcal{Y}^\#$ to be geometrically represented by $\lambda \cup \omega$ where $\lambda \subset Y$ and $\omega$ is an $S^1$-factor of $T^3$. We obtain a group $I^\#(Y; \lambda)$ that is now only relatively $\mathbb{Z}/4$-graded. It is isomorphic to four consecutive gradings of the relatively $\mathbb{Z}/8$-graded group $I(\mathcal{Y}^\#)$. The isomorphism class of $I^\#(Y; \lambda)$ depends only on the oriented homeomorphism type of $Y$ and the class $[\lambda] \in H_1(Y; \mathbb{F}_2)$.

Let $Y$ be a closed, oriented 3-manifold and $\lambda \subset Y$ be a closed, unoriented 1-manifold as above. Let $K$ be a framed knot in $Y$ disjoint from $\lambda$. Denote by $Y_i$ the result of $i$-surgery on $K$. Let $\mu$ be the core of the knot $K$ as viewed in $Y_0$. Then we have an exact triangle

$$\cdots I^\#(Y; \lambda) \longrightarrow I^\#(Y_0; \lambda \cup \mu) \longrightarrow I^\#(Y_1; \lambda) \longrightarrow I^\#(Y; \lambda) \cdots.$$ 

There are two other exact triangles corresponding to the two other rows in Figure 1. For example, if we view $\mu$ as the core of the knot inside $Y_i$ where $i = \infty$ or $i = 1$, the exact sequence has $\mu$ appearing in the twisting for the group of $Y_i$, and not the other two. Each of these is an application of Floer’s original exact triangle, Theorem 2.1, obtained by connected summing each 3-manifold with $T^3$ and performing the surgeries away from $T^3$, with the appropriate overlying bundles.

By changing the framing of $K$, we obtain variants of the above triangles that are computationally handy. Let $l$ and $m$ be the longitude and meridian of $K$, respectively. Suppose that the meridian is unchanged but the longitude is changed to $-pm + l$. Then we have

$$\cdots I^\#(Y; \lambda) \longrightarrow I^\#(Y_p; \lambda \cup \mu) \longrightarrow I^\#(Y_{p+1}; \lambda) \longrightarrow I^\#(Y; \lambda) \cdots,$$

where again the core $\mu$ can be arranged in two other ways. Alternatively, keep the longitude the same but change the meridian to $m - ql$. Then we have

$$\cdots I^\#(Y_0; \lambda) \longrightarrow I^\#(Y_{1/(q+1)}; \lambda \cup \mu) \longrightarrow I^\#(Y_{1/q}; \lambda) \longrightarrow I^\#(Y_0; \lambda) \cdots,$$

where the same freedom with the placement of $\mu$ is understood. For other variants, we refer the reader to [20, §42.1].

For an alternative perspective, one can begin with a 3-manifold $Z$ with torus boundary and consider the possible ordered triplets of Dehn fillings of $Z$ that are compatible with a surgery triangle description. This is the viewpoint taken in [20, §42.1; 34].

We mention that the mod 2 degrees of the cobordism maps in these exact triangles is the same as the monopole case, and is explained in [20, §42.3]. There are always non-trivial bundles amongst the three cobordism maps, even if the three framed groups are untwisted. For in this case, the composite of three consecutive cobordism bundles, call it $\mathcal{X}_{03}$ as in §3.3, has $\mathcal{P}(\mathcal{X}_{03}) \equiv 1 \text{ mod } 2$. This is because $\mathcal{X}_{03}$ is trivial away from a copy of $-\mathbb{CP}^2$ minus a thickened
over this area it restricts to a non-trivial bundle $E$ which is easily seen to have $\mathcal{P}(E) \equiv 1$. Then, by the additivity of $\mathcal{P}(\mathcal{X})$, at least one of $\mathcal{X}_{i,i+1}$ has $\mathcal{P}(\mathcal{X}_{i,i+1}) \equiv 1$. Note that, after computing $\deg(X_{03}) = 1$, we see $\deg(X_{03}) \equiv -1 \mod 4$.

7.6. Examples

In this section, we consider the framed instanton homology of $S^3$ and $S^1 \times S^2$. To compute $I^\#(S^3)$ it suffices to compute $I(T^3)$. This is well known and elementary; see [5]. Let $N$ be a regular neighborhood of the geometric representative for $T^3$. The flat connections modulo even gauge on $T^3$ are in correspondence with the set

$$\{ \rho \in \text{Hom}(\pi_1(T^3 \setminus N), \text{SU}(2)) | \rho(\nu) = -1 \} / \text{SU}(2),$$

where $\nu$ is a small meridian around $N$, and the SU(2)-action is by conjugation. A computation shows that this set consists of two elements; these two elements are non-degenerate and irreducible. The two classes as generators for $C(T^3)$ differ by degree 4. It follows that $C^\#(S^3)$ has one generator, and we obtain

$$I^\#(S^3) \simeq \mathbb{Z}_0,$$

where, as usual, the subscript indicates the grading. We usually assume that a distinguished generator for $I^\#(S^3)$ has been fixed.

Next, we compute $I^\#(S^1 \times S^2)$. For this, we adapt [22, Lemma 8.3]. By placing the twisting $\mu$ at an $S^3$, we have an exact sequence

$$\cdots I^\#(S^3) \xrightarrow{\alpha} I^\#(S^1 \times S^2) \xrightarrow{\beta} I^\#(S^3) \xrightarrow{\gamma} I^\#(S^3) \cdots$$

We apply the grading formula (4.11). The map $\alpha$ comes from the cobordism $D^2 \times S^2 \setminus \text{int}(D^4)$ from $S^3$ to $S^1 \times S^2$. The overlying bundle is necessarily trivial. We compute $\deg(\alpha) \equiv -1$. The map $\beta$ is the same cobordism, but reversed, and $\deg(\beta) \equiv -2$. From the previous section, we know that the sum of the degrees of the three maps is $-1 \mod 4$, so $\deg(\gamma) \equiv 2$. This can be computed directly by observing that $\gamma$ comes from the cobordism $-\mathbb{CP}^2$ minus two 4-balls, from $S^3$ to $S^3$, with a non-trivial bundle. Because $\gamma : \mathbb{Z}_0 \to \mathbb{Z}_0$ has degree 2, it must be 0. By exactness, we conclude

$$I^\#(S^1 \times S^2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_3,$$

where, as usual, the subscripts indicate gradings. As is evident by the above computation, a canonical generator in grading 3 for $I^\#(S^1 \times S^2)$ is given by $[D^2 \times S^2]^\#$. Recall from §7.2 that $[D^2 \times S^2]^\#$ is the notation for the relative invariant induced by the cobordism $D^2 \times S^2 : \emptyset \to S^1 \times S^2$. A canonical homology orientation is used here. The element $[S^1 \times D^3]^\#$ generates the summand in grading 2. This is seen by identifying $S^1 \times S^2$ with its orientation-opposite in a standard way, and viewing $[D^2 \times S^2]^\#$ as a map $I^\#(S^1 \times S^2) \to \mathbb{Z}$. For then we have

$$[D^2 \times S^2]^\# [S^1 \times D^3]^\# \equiv \pm 1,$$

since $S^1 \times D^3$ and $D^2 \times S^3$ glue along $S^1 \times S^2$ to give $S^4$. However, the element $[S^1 \times D^3]^\#$ is not canonically homology oriented; it requires an orientation of $H_1(S^1 \times S^2 ; \mathbb{R})$. Thus a generator for $\mathbb{Z}_2 \subset I^\#(S^1 \times S^2)$ is distinguished by orienting $H_1(S^1 \times S^2 ; \mathbb{R})$.

7.7. The Künneth formula

Let $Y$ and $Y'$ be closed, oriented and connected 3-manifolds. If either one of $I^\#(Y)$ or $I^\#(Y')$ is torsion-free, then there is a graded isomorphism

$$I^\#(Y \# Y') \simeq I^\#(Y) \otimes I^\#(Y').$$
This is a special case of [22, Corollary 5.9], and follows from Floer’s original excision theorem. Further, this isomorphism is natural for split cobordisms, in the following sense. Let \( X : Y_1 \to Y_2 \) and \( X' : Y'_1 \to Y'_2 \) be cobordisms with paths chosen so that the composite \( X \simeq X' \) is defined. Suppose that the above product isomorphism holds for \( Y_1 \# Y'_1 \) and \( Y_2 \# Y'_2 \); then we have a commutative diagram

\[
\begin{array}{c}
I^\#(Y_1 \# Y'_1) \xrightarrow{\simeq} I^\#(Y_1) \otimes I^\#(Y'_1) \\
\downarrow I^\#(X \simeq X') \quad \downarrow I^\#(X) \otimes I^\#(X') \\
I^\#(Y_2 \# Y'_2) \xrightarrow{\simeq} I^\#(Y_2) \otimes I^\#(Y'_2).
\end{array}
\]

We do not address the arrangement of homology orientations here, as we will not require it.

7.8. A connected sum of several copies of \( S^1 \times S^2 \)

Let \( Y \) be a 3-manifold with \( Y \simeq \#^k S^1 \times S^2 \). From the Künneth formula, it is clear that \( I^\#(Y) \simeq \bigotimes^k (\mathbb{Z}_2 \oplus \mathbb{Z}_3) \). The subscripts here indicate gradings. Let \( \mu_Y \) be an orientation of \( H_1(Y; \mathbb{R}) \). In this section, we construct an isomorphism

\[
\phi : \bigwedge^*(H_1(Y; \mathbb{Z})) \to I^\#(Y),
\]

which only depends on \( Y \) and \( \mu_Y \), not the decomposition \( Y \simeq \#^k S^1 \times S^2 \). The choice of \( \mu_Y \) only affects the overall sign of \( \phi \). The exterior power here, and for most of the paper, is over the ring \( \mathbb{Z} \).

Choose oriented, closed, embedded curves \( c_1, \ldots, c_k \) in \( Y \) such that there exists a diffeomorphism \( Y \simeq \#_{i=1}^k S^1 \times S^2 \) sending \( c_i \) to \( S^1 \times \text{pt} \) in the \( i \)-th copy of \( S^1 \times S^2 \). Given \( J = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\} \), we define a cobordism \( X_J : \emptyset \to Y \) by starting with \( Y \times [0,1] \) and attaching a 2-handle to each \( c_i \times \{0\} \) if \( i \in J \), and on top of this, attaching 3-handles and a 4-handle in a way such that \( \partial X_J = Y \times \{1\} \) and

\[
X_J \simeq X_1 \cup \cdots \cup X_k, \quad X_i = \begin{cases} S^1 \times D^3 & \text{if } i \notin J, \\ D^2 \times S^2 & \text{if } i \in J \end{cases}
\]

with \( \partial X_i \) the \( i \)-th copy of \( S^1 \times S^2 \) in the decomposition \( Y \simeq \#_{i=1}^k S^1 \times S^2 \). Let \( \{1, \ldots, k\} \setminus J = \{i_{l+1}, \ldots, i_k\} \) be such that \( \mu_Y = [c_{i_1} \wedge \cdots \wedge c_{i_l}] \). To homology orient \( X_J \), we orient \( \mathcal{L}(X_J) = H_1(X_J; \mathbb{R}) \) by \( [c_{i_{l+1}} \wedge \cdots \wedge c_{i_k}] \). Define \( \psi : \bigwedge^*(c_1, \ldots, c_k) \to I^\#(Y) \) by

\[
\psi(c_{i_1} \wedge \cdots \wedge c_{i_l}) = [X_J]^\#.
\]

This map is an isomorphism by the case \( k = 1 \) and the Künneth formula.

With the help of the orientation \( \mu_Y \) of \( H_1(Y; \mathbb{R}) \), we can define a bilinear form \( \langle \cdot, \cdot \rangle : I^\#(Y) \otimes I^\#(Y) \to \mathbb{Z} \); see also (7.4). The elements \([X_J]^\#\) as \( J \) runs over subsets of \( \{1, \ldots, k\} \) form a basis for \( I^\#(Y) \), so it suffices to define the form on these. Given \( J, K \subset \{1, \ldots, k\} \), let \( X_J \) and \( X_K \) be as above with homology orientations \( \mu_J \) and \( \mu_K \), respectively. Then we have elements \([X_J]^\#, [X_K]^\# \in I^\#(Y)\). Consider \( X_K : Y \to \emptyset \) and homology orient it by \( \mu_Y \wedge \mu_K \). This yields \([X_K]^\# : I^\#(Y) \to \mathbb{Z} \).

Then the bilinear form \( \langle \cdot, \cdot \rangle \) is given by

\[
\langle [X_K]^\#, [X_J]^\# \rangle = [X_K]^\# [X_J]^\# = [X_K \circ X_J]^\# = A_{JK} \in \mathbb{Z}.
\]

Now observe that

\[
X_K \circ X_J \simeq X_1 \# \cdots \# X_k, \quad X_i \simeq \begin{cases} S^1 \times S^3 & \text{if } i \notin J \cup K, \\ S^2 \times S^2 & \text{if } i \in J \cap K, \\ S^4 & \text{otherwise.}
\end{cases}
\]

(7.6)
Note $[S^2 \times S^2]^\# = 0$, because the degree of the cobordism $S^3 \to S^3$ given by $S^2 \times S^2$ minus two 4-balls is odd, and similarly for $[S^1 \times S^3]^\#$. Using the naturality with respect to split cobordisms of the K"unneth formula, we conclude that $A_{JK} \neq 0$ if and only if $J$ and $K$ are complementary, and in this case $A_{JK} = \pm 1$. This sign may be determined by using Definition 8.1, but we will not need it. It is clear that this bilinear form is non-degenerate. Note that $\langle \cdot, \cdot \rangle$ depends on $c_1, \ldots, c_k$ (which determine an identification of $Y$ with $Y'$).

We argue that $\psi$ is independent of the 2-handle framings chosen to construct the $X_J$. First construct cobordisms $X_J$ for each subset $J \subset \{1, \ldots, k\}$ as above. Choose some $J$, and construct a cobordism $X'_J$ by attaching the 2-handles using possibly different framings as was done for $X_J$, subject to the constraint that $X'_J$ is of the form (7.5). Then

$$X_K \circ X'_J \simeq X_1 \# \cdots \# X_k$$

just as in (7.6), except now if $i \in J \cap K$, then $X_i$ is a possibly non-trivial $S^2$-bundle over $S^2$, in which case $[X_i]^\# = 0$. We homology orient $X'_J$ in the same way as $X_J$. It is easily seen that $[X'_J]^\#$ has all the same values $A_{JK}$ as $[X_J]^\#$ under the bilinear pairing, and thus $[X'_J]^\# = [X_J]^\#$.

Now we see how $\psi$ changes when we change the loops $c_i$. Consider replacing the oriented loop $c_1$ by an oriented connected sum $c_1 \# c_2$. There are many ways of forming this connected sum. Let $X_{c_1 \# c_2}$ be the cobordism $\emptyset \to Y$ obtained by attaching to $Y \times [0,1]$ a 2-handle along $c_1 \# c_2 \times \{0\}$ and 3-handles and a 4-handle as above. Supposing $\mu_Y = [c_1 \wedge \cdots \wedge c_k]$, we homology orient $X_{c_1 \# c_2}$ by $[c_2 \wedge \cdots \wedge c_k]$, just as we homology orient $X_{\{1\}}$ and $X_{\{2\}}$. Then

$$[X_{c_1 \# c_2}]^\# = [X_{\{1\}}]^\# + [X_{\{2\}}]^\#.$$

Viewing $X_{c_1 \# c_2} : \emptyset \to Y$ and $X_J : Y \to \emptyset$, this follows from computing

$$X_J \circ X_{c_1 \# c_2} \simeq X_3 \# \cdots \# X_k,$$

where each $X_i \simeq S^4$ if $i \not\in J$ and $\deg(X_i)$ is odd otherwise, and then appealing to the non-degeneracy of our bilinear form. A similar argument shows $[X_{c_1 \# c_2 \cup J}]^\# = [X_{\{1\} \cup J}]^\# + [X_{\{2\} \cup J}]^\#$ where $J$ is any subset of $\{3, \ldots, k\}$.

As a consequence, $\psi$ induces a well-defined isomorphism

$$\phi : \bigwedge^\infty(H_1(Y;\mathbb{Z})) \to \mathcal{I}^\#(Y).$$

This is because any two sets of loops $c_1, \ldots, c_k$ in $Y$ as above (having the property that there exists a diffeomorphism $Y \simeq \#^k S^1 \times S^2$ sending each $c_i$ to a factor $S^1 \times pt$) are related by sequences of connected sums (and the reverse operation) as in the previous paragraph. Indeed, these are just handle slides, and a result of Laudenbach and Poéna [26], as cited in [18, Remark 4.4.1], says that any self-diffeomorphism of $\#^k S^1 \times S^2$ extends to a diffeomorphism of $\#^k S^1 \times D^3$, a bounding 1-handlebody, which can be written as a composite of 1-handle slides. In fact, this result also says that the way in which the 3-handles and 4-handle are attached to construct $X_J$ above is essentially unique.

In summary, $\phi$ is defined by choosing an orientation $\mu_Y$ of $H_1(Y;\mathbb{R})$, a diffeomorphism $Y \simeq \#^k S^1 \times S^2$, oriented loops $c_1, \ldots, c_k$ corresponding to the $S^1 \times pt$ factors and setting

$$\phi([c_{i_1}] \wedge \cdots \wedge [c_{i_l}]) = [X_J]^\#,$$

where the element $[X_J]^\#$ is defined as above. The content of the above discussion is that this map is well defined and is an isomorphism. We mention that for $x \in \bigwedge^i(H_1(Y;\mathbb{Z}))$ with $b_1(Y) = k$, the grading of $\phi(x)$ in $\mathcal{I}^\#(Y)$ is given by $2k + i$ mod 4.
7.9. A spectral sequence

The spectral sequence of Theorem 6.1 leads to one for the groups $I^\# (Y)$. The setup is as follows. Again we have an $m$-component framed link $L$ in $Y$. We view $L$ as a link in $Y \# T^3$, and we choose a family of bundles over the surgered manifolds $Y_v \# T^3$ which, for $v \in \{0, 1\}^m$, are of the form $(Y_v \times SO(3)) \# T^3$, at the expense of having possibly non-trivial bundles (so twisted framed groups) for the indices $v \in \{0, 1, \infty\}^m \setminus \{0, 1\}^m$. We are using the third row of Figure 1 to achieve this setup. This forces the bundle over $Y \# T^3$ to be geometrically represented by the link $L$ together with an $S^1$-factor of $T^3$. More general spectral sequences may be obtained by allowing twisting in the $E^1$-page.

Before stating the resulting theorem, we discuss how to lift the previous $\mathbb{Z}/2$-grading $gr[C]$ for the $E^1$-page of the link surgeries spectral sequence to a $\mathbb{Z}/4$-grading, in the special case where $[L] = 0 \in H_1(Y; F_2)$. Write $gr[Y]$ for the $\mathbb{Z}/4$-grading on $I^\# (Y)$ and $Y^\# = Y_v \# T^3$, where, for $v \in \{0, 1\}^m \cup \{\infty\}$, we have $Y_v = Y_e \times SO(3)$. Recall that we conflate $\infty$ and $-1$. Also write $X_{vw}^\# = X_{vw} \# (T^3 \times [0, 1])$ for the surgery cobordism bundles. For $v \in \{0, 1\}^m \cup \{\infty\}$, we may view each $C(Y^\#)$ as two copies of $C(Y_v)$, $\mathbb{Z}/4$-graded by $gr[Y_v]$. For $v \in \{0, 1\}^m$ and $x \in C(Y^\#) \subset C$ of homogeneous $gr[Y_v]$ grading, we define

$$gr[C](x) = gr[Y_v](x) - deg(X_{v\infty}) - |v|_1 \mod 4. \quad (7.7)$$

The verification that $\partial$ lowers this grading by 1, and that the quasi-isomorphism $Q : C(Y^\#) \rightarrow C$ preserves the relevant $\mathbb{Z}/4$-gradings, is the same as in §6.4.

**Theorem 7.2.** Let $L$ be an oriented framed link with $m$ components in $Y$ and, for each $v \in \{\infty, 0, 1\}^m$, denote by $Y_v$ the result of $v$-surgery on $L$ in $Y$. There are surgery cobordisms $X_{vw}$ for $v < w$ from $Y_v$ to $Y_w$ with homology orientations $\mu_{vw}$ satisfying $\mu_{uw} \circ \mu_{vw} = \mu_{vw}$ whenever $v < u < w$, and an appropriate bundle $X_{vw}$ over each $X_{vw}$, such that there is a spectral sequence $(E^r, d^r)$ with

$$E^1 = \bigoplus_{v \in \{0, 1\}^m} I^\# (Y_v), \quad d^1 = \sum_{v < w \mid w - v = 1} (-1)^{\delta(v, w)} I^\# (X_{vw}),$$

where $\delta(v, w)$ is as in Theorem 6.1. The spectral sequence is graded by $\mathbb{Z}/2 \times \mathbb{Z}$, where $d^r$ has bi-degree $(1, r)$, and it converges by the $E^m+1$-page to the possibly twisted group

$I^\# (Y; L)$.

The $\mathbb{Z}/2$-grading induced by the spectral sequence agrees with the $\mathbb{Z}/2$-grading of $I^\# (Y; L)$. If $[L] = 0 \in H_1(Y; F_2)$, then we can lift the $\mathbb{Z}/2$-grading of the $E^1$-page to a $\mathbb{Z}/4$-grading by (7.7), such that the induced $\mathbb{Z}/4$-grading agrees with the one on $I^\# (Y)$. The differential for the $\mathbb{Z}/4 \times \mathbb{Z}$-grading has bi-degree $(-1, r)$.

8. Branched double covers

In this section, we complete the proof of Theorem 1.1. First, we define reduced odd Khovanov homology in §8.1, following Bloom\'s description from [3]. We give an alternative description of the differential that will suit our goals. We then discuss how to compose homology orientations in §8.2. This is the framework we use to understand the signs in our spectral sequence. In §8.3, we identify the $E^1$-page of the spectral sequence (2.1) with the reduced odd Khovanov chain complex. Finally, in §8.4, we discuss the $\mathbb{Z}/4$-grading of the spectral sequence and the conclusion of Corollary 1.5.
8.1. Odd Khovanov homology

Let $L$ be an oriented link and $D$ be a planar diagram for $L$. Suppose that $D$ has $m$ crossings. We assume that each crossing has an arrow drawn over it, as in Figure 13. Then, for each $v \in \{0, 1\}^m$, we can define a resolution diagram $D_v$ according to the rules of Figure 13. Each $D_v$ is a disjoint union of planar-embedded unoriented circles together with a disjoint union of planar-embedded oriented arcs, each arc beginning and ending at a circle. Suppose that $D_v$ has $k+1$ circles. Then we have a rank $k$ abelian group $V_v$ defined by

$$V_v = \mathbb{Z}\{\text{arcs}\}/\ker(\mathbb{Z}\{\text{arcs}\} \rightarrow \mathbb{Z}\{\text{circles}\}),$$

where the map involved sends an arc to the circle at which it begins minus the circle at which it ends. A basis for $V_v$ is given by any $k$ arcs that touch all $k+1$ circles in $D_v$, or equivalently, the edges of any spanning tree of the graph whose vertices are the circles of $D_v$ and edges are the arcs. We define

$$C_v = \bigwedge^*(V_v), \quad C = \bigoplus_{v \in \{0, 1\}^m} C_v.$$

For each $v, w \in \{0, 1\}^m$ with $v < w$ and $|w - v|_1 = 1$, we introduce a map $\partial_{vw}' : C_v \rightarrow C_w$. There is a single arc $x_{vw}$ in each of $D_v$ and $D_w$ that changes from a 0-resolution position to a 1-resolution position. There are two cases to consider, corresponding to two circles merging or splitting:

$$\partial_{vw}'(x) := \begin{cases} x_{vw} \land x & \text{if } 0 = x_{vw} \in C_v \text{ (split)}, \\ x & \text{if } 0 \neq x_{vw} \in C_v \text{ (merge)}. \end{cases}$$

In these expressions, we use the symbol $x_{vw}$ to stand both for an arc and its equivalence class in $V_v$. We call the collection of $\partial_{vw}'$ the predifferential. The differential for $C$ is defined by

$$\partial = \sum \partial_{vw} = \sum_{v \in \{0, 1\}^m} \varepsilon_{vw} \partial_{vw}' ,$$

where each $\varepsilon_{vw}$ is $+1$ or $-1$, and the sums are over $v, w$ with $v < w$ and $|w - v|_1 = 1$. The signs $\varepsilon_{vw}$ are chosen to satisfy two conditions. The first condition is that $\partial^2 = 0$. The second condition is as follows. Let $v < t, u$ with $|t - v|_1 = |u - v|_1 = 1$ be three vertices where the arcs $x_{vu}$ and $x_{vt}$ are arranged in $D_v$ as in the left of Figure 14. Let $w$ be the vertex with $w > t, u$ and $|w - t|_1 = |w - u|_1 = 1$. Any four such vertices $v, u, t, w$ will be called a type X face. A type Y face is obtained by reversing one of either $x_{vu}$ or $x_{vt}$. The second condition is that for a type X face, the sign

$$\varepsilon_{vu} \varepsilon_{vt} \varepsilon_{lw} \varepsilon_{uw}$$

is always $+1$ or always $-1$; and the same product for a type Y face is also always $+1$ or always $-1$, and is minus the type X sign. We call the collection of $\varepsilon_{vw}$ a valid edge assignment if it satisfies these two conditions. The reduced odd Khovanov homology of $L$ is then defined to be $\text{Kh}^\prime(L) = H_*(C, \partial)$. The well-definedness and invariance is proved in [33].

Figure 13 (colour online). Resolution conventions for the arc-decorated diagrams in reduced odd Khovanov homology. There are two choices for the placement of an arc at a given crossing; in the left-most picture, the arc can be pointing up (as depicted) or down. In the latter case, the arcs in the resolution pictures are correspondingly reversed.
Suppose that we are given \(8.2\). Composing homology orientations

where \(x\) element of framed instanton homology are associativity and the existence of units. In other words, the two most important formal properties of a composition rule compatible with a construction applications we prefer to have a concrete, algebro-topological description of such a rule. Perhaps orientations originates from the determinants of the relevant Fredholm operators, in our assumption that the \(3\) whenever \(2\) exists a distinguished homology orientation \(2\) and these compositions make sense. We will first define we describe the rule we use to orient \(3\) is a homology orientation of \(1\) and \(3\) is a homology orientation of \(1\) for \(2\) and \(3\), and for \(2\;\times\;[0,1]\) there exists a distinguished homology orientation \(2\) such that \(2\) whenever \(2\) is a homology orientation and these compositions make sense. We will first define a composition rule in an algebro-topological fashion and then show it has these two properties. At the end of this section, we will describe how the rule we have defined can be described using Fredholm determinant line bundles, using the setup of Kronheimer and Mrowka [20, §20.2], ensuring that our rule is compatible with a construction of framed instanton homology. In this section, all homology groups are assumed to have real coefficients.

Typically, an orientation of \(2\;\times\;[0,1]\) there

At the end of this section, we will describe how the rule we have defined can be described using Fredholm determinant line bundles, using the setup of Kronheimer and Mrowka [20, §20.2], ensuring that our rule is compatible with a construction of framed instanton homology. In this section, all homology groups are assumed to have real coefficients.

We proceed to construct the composition rule. As in the previous sections, we maintain the assumption that the \(2\) and \(3\) are connected. For background on the following setup, see [2, §7; 10, Theorem 27.5]. Let \(1\) be the map in the Mayer–Vietoris sequence. Consider the following exact sequences:

\[
0 \longrightarrow \operatorname{im}(f_{12}) \longrightarrow H_1(X_1) \oplus H_1(X_2) \longrightarrow H_1(X_{12}) \longrightarrow 0, \quad (8.2)
\]

\[
0 \longrightarrow \ker(f_{12}) \longrightarrow H_1(Y_2) \longrightarrow \operatorname{im}(f_{12}) \longrightarrow 0, \quad (8.3)
\]

\[
0 \longrightarrow H_2^+(X_1) \oplus H_2^+(X_2) \longrightarrow H_2^+(X_{12}) \longrightarrow \ker(f_{12}) \longrightarrow 0. \quad (8.4)
\]
The first exact sequence is extracted from the Mayer–Vietoris sequence, and the second is naturally associated to the map $f_{12}$. Our convention is that $f_{12}(x) = (x, -x)$ on the chain level. For the third sequence, we choose the positive-definite subspace $H^+_2(X_{12})$ so that it contains the image of $H^+_2(X_1) \oplus H^+_2(X_2)$ under the map $H_2(X_1) \oplus H_2(X_2) \to H_2(X_{12})$. The map $H^+_2(X_{12}) \to \ker(f_{12})$ is a restriction of the Mayer–Vietoris boundary map $H_2(X_{12}) \to H_1(Y_1)$. There is a concrete interpretation of (8.4). Upon splitting the sequence, it says we can write
\[ H^+_2(X_{12}) = H^+_2(X_1) \oplus H^+_2(X_2) \oplus \ker(f_{12}). \]
To interpret the summand $\ker(f_{12})$, we write down a section $s$ for the map $H^+_2(X_{12}) \to \ker(f_{12})$. We define $s : \ker(f_{12}) \to H^+_2(X_{12})$ on a basis of 1-cycle classes $[\gamma]$ in $\ker(f_{12}) \subset H_1(Y_2)$ as follows. For each such 1-cycle $\gamma$ in $Y_2$, choose a 2-cycle $\Sigma$ in $Y_2$ such that $\#(\gamma \cap \Sigma) = 1$, and extend $\gamma$ to a 2-cycle $\Gamma$ in $X_{12}$. Then $s[\gamma] = [\Gamma] + [\Sigma]$. Choosing splittings of the above three exact sequences, summing, canceling a copy of $\ker(f_{12})$ on both sides, and then moving summands around yields an identification
\[ \mathcal{L}(X_{12}) \oplus \im(f_{12})^{\otimes 2} = \mathcal{L}(X_1) \oplus \mathcal{L}(X_2). \tag{8.5} \]
Thus we can orient $\mathcal{L}(X_{12})$ by using given orientations of $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$ and equipping the two copies of $\im(f_{12})$ with the same orientation. We will give an explicit rule for doing this, designed so as to be associative. We choose splittings of the above exact sequences, in their respective order:
\begin{align*}
F_{12} : \im(f_{12}) \oplus H_1(X_{12}) &\sim H_1(X_1) \oplus H_1(X_2), \tag{8.6} \\
G_{12} : \ker(f_{12}) \oplus \im(f_{12}) &\sim H_1(Y_2), \tag{8.7} \\
H_{12} : H^+_2(X_1) \oplus H^+_2(X_2) \oplus \ker(f_{12}) &\sim H^+_2(X_{12}). \tag{8.8}
\end{align*}
The space of such splittings is contractible, so these particular choices do not matter for the following definition.

**Definition 8.1.** For $i = 1, 2$ let $X_i : Y_i \to Y_{i+1}$ be two connected cobordisms between connected, non-empty 3-manifolds. Write $X_{12} = X_2 \circ X_1$. Suppose that $\mu_i$ is a homology orientation of $X_i$, that is, an orientation of $\mathcal{L}(X_i)$, for $i = 1, 2$. Write $\mu_i = \beta_i \land \alpha_i \land \gamma_i$ where $\alpha_i$ is an orientation for $H_1(Y_i)$, $\beta_i$ for $H_1(X_i)$ and $\gamma_i$ for $H^+_2(X_i)$. Choose any orientation $d_{12}$ of $\im(f_{12})$. Choose splittings of the exact sequences (8.2)–(8.4) written as in (8.6)–(8.8). Equip $H_1(X_{12})$ with an orientation $\beta_{12}$ given by the condition
\[ F_{12}(\delta_{12} \land \beta_{12}) = \beta_1 \land \beta_2. \]
Similarly, equip $\ker(f_{12})$ with an orientation $\zeta_{12}$ which satisfies
\[ G_{12}(\zeta_{12} \land \delta_{12}) = \alpha_2. \]
Then define the composition of $\mu_1$ with $\mu_2$, which is an orientation of $\mathcal{L}(X_{12})$, by
\[ \mu_2 \circ \mu_1 = (-1)^{\beta_{12}} \land \alpha_1 \land H_1(\gamma_1 \land \gamma_2 \land \zeta_{12}), \]
\[ s = \frac{1}{2}(d_{12}^2 - d_{12}) + b_1(X_1)b_1(Y_2) + b_1(X_1)b_2^+(X_2) + b_1(Y_2)b_2^+(X_2). \]
Here $d_{12} = \dim[\im(f_{12})]$.

**Proposition 8.2.** This composition rule for homology orientations is associative.

**Proof.** We first rephrase the problem in terms of linear algebra. For $i = 1, 2$, consider quadruples $\mathcal{A}_i = (A_i, B_i, C_i, \mu_i)$ where $A_i, B_i, C_i$ are vector spaces and $\mu_i$ is an orientation of $A_i \oplus B_i \oplus C_i$. In our application, we have $A_i = H_1(Y_i)$, $B_i = H_1(X_i)$ and $C_i = H^+_2(X_i)$. 


Given a linear map
\[ f_{12} : A_2 \rightarrow B_1 \oplus B_2, \]
we can compose \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) along \( f_{12} \) to form
\[ \mathcal{A}_2 \circ f_{12} \circ \mathcal{A}_1 = (A_1, \text{coker}(f_{12}), C_1 \oplus C_2 \oplus \ker(f_{12}), \mu_2 \circ \mu_1). \]

The orientation \( \mu_{12} = \mu_2 \circ \mu_1 \) is adapted from Definition 8.1 as follows. Write \( \mu_i = \beta_i \wedge \alpha_i \wedge \gamma_i \), where \( \alpha_i, \beta_i, \gamma_i \) are respective orientations of \( A_i, B_i, C_i \). Choose an orientation \( \delta_{12} \) of \( \text{im}(f_{12}) \). Choose isomorphisms
\[
F_{12} : \text{im}(f_{12}) \oplus \text{coker}(f_{12}) \xrightarrow{\sim} B_1 \oplus B_2, \\
G_{12} : \ker(f_{12}) \oplus \text{im}(f_{12}) \xrightarrow{\sim} A_2
\]
that are splittings of the naturally associated exact sequences. Orient \( \text{coker}(f_{12}) \) by \( \beta_{12} \) and \( \ker(f_{12}) \) by \( \zeta_{12} \) using the conditions
\[ F_{12}(\delta_{12} \wedge \beta_{12}) = \beta_1 \wedge \beta_2, \quad G_{12}(\zeta_{12} \wedge \delta_{12}) = \alpha_2. \]

Then the composition \( \mu_{12} \) is given by
\[
\mu_{12} = (-1)^{s_{12}} \beta_{12} \wedge \alpha_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \zeta_{12}, \\
s_{12} = b_1a_2 + b_1c_2 + a_2c_2 + (d_{12}^2 - d_{12})/2,
\]
where \( a_i = \dim A_i, \ b_i = \dim B_i, \ c_i = \dim C_i \) and \( d_{12} = \dim[\text{im}(f_{12})] \). Now suppose that we have a third quadruple \( \mathcal{A}_3 = (A_3, B_3, C_3, \mu_3) \) and a linear map \( f_{23} : A_3 \rightarrow B_2 \oplus B_3 \). Consider
\[ f = f_{12} + f_{23} : A_2 \oplus A_3 \rightarrow B_1 \oplus B_2 \oplus B_3. \]
The map \( f \) induces further maps
\[ f_{1,23} : A_2 \rightarrow B_1 \oplus \text{coker}(f_{23}), \quad f_{12,3} : A_3 \rightarrow \text{coker}(f_{12}) \oplus B_3. \]
We write \( F_{23}, G_{23} \) for the isomorphisms associated to \( f_{23} \) as in (8.9) and (8.10); \( F_{12,3}, G_{12,3} \) associated to \( f_{12,3} \) and \( F_{12,3}, G_{12,3} \) to \( f_{1,23} \). We have identifications
\[
k \ker(f_{12}) \oplus \text{ker}(f_{12,3}) = \ker(f) = \ker(f_{12}) \oplus \ker(f_{12,3}). \]

The cokernel identifications are natural. The kernel identifications depend on some choices. For instance, \( \ker(f_{12}) \oplus \ker(f_{12,3}) = \ker(f) \) is established as follows. Clearly, \( \ker(f_{12}) \subset \ker(f) \). Now suppose \( a \in \ker(f_{12,3}) \subset A_3 \). Then \( \pi_{12}(f(a)) \in \text{im}(f_{12}) \) where \( \pi_{12} \) projects onto \( B_1 \oplus B_2 \). Thus \( \pi_{12}(f(a)) = f_{12}(b) \) for some \( b \in A_2 \). Let \( \sigma_{12} : \text{im}(f_{12}) \rightarrow A_2 \) be such that \( f_{12}\sigma_{12} = \text{id}_{\text{im}(f_{12})} \). Then we may take \( b = \sigma_{12}(\pi_{12}(f(a))) \), and the assignment \( a \mapsto (-b, a) \) injects \( \ker(f_{12,3}) \) into \( \ker(f) \). In this way, we obtain a map from \( \ker(f_{12}) \oplus \ker(f_{12,3}) \) to \( \ker(f) \) which is easily seen to be an isomorphism. With these identifications, the associativity of our rule in Definition 8.1 is nearly equivalent to
\[ \mathcal{A}_3 \circ f_{12,3} (\mathcal{A}_2 \circ f_{12} \mathcal{A}_1) = (\mathcal{A}_3 \circ f_{23} \mathcal{A}_2) \circ f_{1,23} \mathcal{A}_1. \]
We have only left out the roles of the \( H_i \) maps; these are not essential and we remark on their absence at the end of the proof. We proceed to establish (8.13). Let us write out \( \mu_{12,3} = \mu_3 \circ \mu_{12} \), the orientation associated to the left-hand side of (8.13). Let \( \mu_3 = \beta_3 \wedge \alpha_3 \wedge \gamma_3 \) where \( \alpha_3, \beta_3, \gamma_3 \) are orientations of \( A_3, B_3, C_3 \), respectively. Let \( \delta_{12,3} \) orient \( \text{im}(f_{12,3}) \). Orient \( \text{coker}(f_{12,3}) \) by \( \beta_{12,3} \) and \( \ker(f_{12,3}) \) by \( \zeta_{12,3} \), where
\[ F_{12,3}(\delta_{12,3} \wedge \beta_{12,3}) = \beta_1 \wedge \beta_2, \quad G_{12,3}(\zeta_{12,3} \wedge \delta_{12,3}) = \alpha_3. \]
Then we use our composition rule to obtain
\[
\mu_{12,3} = (-1)^{s_{12,3}}\beta_{12,3} \land \alpha_1 \land (\gamma_1 \land \gamma_2 \land \zeta_{12}) \land \gamma_3 \land \zeta_{12,3},
\]
\[
s_{12,3} = s_{12} + b_{12}a_3 + b_{12}c_3 + a_3c_3 + (d_{12,3}^2 - d_{12,3})/2.
\]
Here \(d_{12,3} = \dim[\dim(f_{12,3})]\) and \(b_{12} = \dim[\coker(f_{12})]\), and so in particular
\[
b_{12} = b_1 + b_2 - d_{12}.
\]
Now we write out the orientation associated to the right-hand side of (8.13). We first write
\[
\mu_{23} = \mu_3 \circ \mu_2 = (-1)^{s_{23}}\beta_{23} \land \alpha_2 \land \gamma_2 \land \gamma_3 \land \zeta_{23},
\]
\[
s_{23} = b_2a_3 + b_2c_3 + a_3c_3 + (d_{23}^2 - d_{23})/2,
\]
where, given an orientation \(\delta_{23}\) of \(\text{im}(f_{23})\), we have imposed
\[
F_{23}(\delta_{23} \land \beta_{23}) = \beta_2 \land \beta_3, \quad G_{23}(\zeta_{23} \land \delta_{23}) = \alpha_3.
\]
Now we can also write
\[
\mu_{1,23} = (-1)^{s_{1,23}}\beta_{1,23} \land \alpha_1 \land \gamma_1 \land (\gamma_2 \land \gamma_3 \land \zeta_{23}) \land \zeta_{1,23},
\]
\[
s_{1,23} = s_{23} + b_1a_2 + b_1c_2 + a_2c_2 + (d_{1,23}^2 - d_{1,23})/2,
\]
where \(c_{23} = \dim[C_2 \oplus C_3 \oplus \ker(f_{23})]\), so that
\[
c_{23} = c_2 + c_3 + a_3 - d_{23},
\]
and, given an orientation \(\delta_{1,23}\) of \(\text{im}(f_{1,23})\), we have the conditions
\[
F_{1,23}(\delta_{1,23} \land \beta_{1,23}) = \beta_1 \land \beta_2, \quad G_{1,23}(\zeta_{1,23} \land \delta_{1,23}) = \alpha_2.
\]
We will now show that \(\mu_{12,3} = \mu_{1,23}\). We choose identifications
\[
\text{im}(f_{12}) \oplus \text{im}(f_{12,3}) = \text{im}(f) = \text{im}(f_{23}) \oplus \text{im}(f_{12,3}).
\]
These depend on \(F_{12}\) and \(F_{23}\). For instance, let \(\tau_{12} : \coker(f_{12}) \to B_1 \oplus B_2\) be the map extracted from \(F_{12}\) (and conversely it may define \(F_{12}\)). Then \(\text{im}(f_{12,3})\) maps into \(\text{im}(f)\) by \(a \mapsto (\tau_{12}(\pi(a)), \pi_3(a))\) where \(\pi\) projects onto \(\coker(f_{12})\) and \(\pi_3\) onto \(B_3\). Since \(\text{im}(f_{12})\) is naturally a subset of \(\text{im}(f)\), we then obtain a map from \(\text{im}(f_{12}) \oplus \text{im}(f_{12,3})\) into \(\text{im}(f)\) which yields the above identification. We can thus orient \(\text{im}(f)\) by \(\delta_{12} \land \delta_{12,3}\) or by \(\delta_{23} \land \delta_{1,23}\). It suffices to show
\[
\delta_{12} \land \delta_{12,3} \land \mu_{12,3} \land \delta_{12} \land \delta_{12,3} = \delta_{23} \land \delta_{1,23} \land \mu_{1,23} \land \delta_{23} \land \delta_{1,23}
\]
(8.14)
as orientations of \(\text{im}(f) \oplus V \oplus \text{im}(f)\), where \(V\) is the total space of either side of (8.13) for which \(\mu_{1,23}\) and \(\mu_{12,3}\) are orientations. We compute the left-hand side of (8.14):
\[
(-1)^{s_{12,3} + d_{12}d_{12,3}}\delta_{12} \land \delta_{12,3} \land \beta_{12,3} \land \alpha_1 \land \gamma_1 \land \gamma_2 \land \zeta_{12} \land \gamma_3 \land \zeta_{12,3} \land \delta_{12,3} \land \delta_{12}
\]
\[
= (-1)^{s_{12,3} + d_{12}d_{12,3} + d_{12}(a_3 + c_3) + a_2c_2}[c_{\text{im}(f_{12})} \oplus F_{12,3}^{-1}(F_{12}^{-1} \oplus \text{id}_{B_3})]\](\beta_1 \land \beta_2 \land \beta_3)
\]
\[
\land \alpha_1 \land \gamma_1 \land \gamma_2 \land \gamma_3 \land [G_{12}^{-1} \oplus G_{12,3}^{-1}](\alpha_2 \land \alpha_3).
\]
Now, choose splitting isomorphisms
\[
F : \text{im}(f) \oplus \coker(f) \simto B_1 \oplus B_2 \oplus B_3,
G : \ker(f) \oplus \text{im}(f) \simto A_2 \oplus A_3
\]
for the naturally associated short exact sequences. We claim we have
\[
[(\text{id}_{\text{im}(f_{12})} \oplus F_{12,3}^{-1})(F_{12}^{-1} \oplus \text{id}_{B_3})](\beta_1 \land \beta_2 \land \beta_3) = F^{-1}(\beta_1 \land \beta_2 \land \beta_3),
\]
(8.15)
\[
[G_{12}^{-1} \oplus G_{12,3}^{-1}](\alpha_2 \land \alpha_3) = G^{-1}(\alpha_2 \land \alpha_3).
\]
(8.16)
We consider (8.15). To abstract the underlying problem, consider a linear map \( \phi : V \to W \) and distinguished subspaces \( V' \subset V \) and \( W' \subset W \) such that \( \phi(V') \subset W' \). In other words, we have a relative linear map \( \phi : (V, V') \to (W, W') \). Choose an isomorphism
\[
\Phi : \text{im}(\phi) \oplus \text{coker}(\phi) \xrightarrow{\sim} W
\]
associated to the natural short exact sequence. Similarly, choose
\[
\Phi' : \text{im}(\phi') \oplus \text{coker}(\phi') \xrightarrow{\sim} W',
\]
where \( \phi' : V' \to W' \) is a restriction of \( \phi \). Also, with \( \phi'' : V/V' \to \text{coker}(\phi') \oplus W/W' \) choose
\[
\Phi'' : \text{im}(\phi'') \oplus \text{coker}(\phi'') \xrightarrow{\sim} \text{coker}(\phi') \oplus W/W'.
\]
We can identify \( \text{coker}(\phi'') = \text{coker}(\phi) \) and \( \text{im}(\phi) = \text{im}(\phi') \oplus \text{im}(\phi'') \) just as we have done in our setting above. We also choose an identification \( W/W' \oplus W' = W \). Then (8.15) is equivalent to
\[
\det[(\Phi^{-1}(\Phi' \oplus \text{id}_{W/W'})(\Phi'' \oplus \text{id}_{\text{im}(\phi')}))] > 0,
\]
by setting \( \phi = f \), \( V = A_2 \oplus A_3 \), \( V' = A_2 \), \( W = B_1 \oplus B_2 \oplus B_3 \) and \( W' = B_1 \oplus B_2 \). In fact, we can choose the data so that, under these identifications,
\[
\Phi = (\Phi' \oplus \text{id}_{W/W'})(\Phi'' \oplus \text{id}_{\text{im}(\phi')}).
\]
(8.17)

This can be seen as follows. We may equip \( V \) and \( W \) with inner products so that we may freely take complements. In the following, we use the notation \( V_1^\bot \subset V_2 \) to mean that the complement \( V_1^\bot \) (with \( V_2 \) possibly inside a larger space) was taken inside \( V_2 \). We may then identify \( \text{coker}(\phi) = \text{im}(\phi)^\bot \subset W \), \( \text{coker}(\phi') = \text{im}(\phi')^\bot \subset W' \) and \( \text{im}(\phi'') = \text{im}(\phi'')^\bot \subset \text{im}(\phi) \). We also identify \( W/W' \) with \( W/W' \). We use these identifications to define \( \Phi, \Phi', \Phi'' \) in the natural way. Then \( \Phi \) is just the identification \( \text{im}(\phi) \oplus \text{im}(\phi')^\bot = W \). On the other hand, we view \( \Phi'' \oplus \text{id}_{\text{im}(\phi')} \) as a map
\[
\text{im}(\phi) \oplus \text{im}(\phi')^\bot \longrightarrow \text{im}(\phi') \oplus \text{im}(\phi'')^\bot \oplus W/W',
\]
where \( \text{im}(\phi')^\bot \subset W/W' \). This last expression uses the identification \( \text{im}(\phi) = \text{im}(\phi') \oplus \text{im}(\phi'') \) where \( \text{im}(\phi')^\bot \subset \text{im}(\phi) \), followed by the identification \( \text{im}(\phi')^\bot \oplus \text{im}(\phi'')^\bot = \text{im}(\phi')^\bot \oplus W/W' \), where on the left \( \text{im}(\phi')^\bot \subset \text{im}(\phi) \) but on the right we have the larger complement \( \text{im}(\phi')^\bot \subset W/W' \). These are just two different decompositions of \( \text{im}(\phi')^\bot \subset W \). Then, \( \Phi'' \oplus \text{id}_{W/W'} \), viewed as a map
\[
\text{im}(\phi') \oplus \text{im}(\phi'')^\bot \oplus W/W' \longrightarrow W,
\]
where again \( \text{im}(\phi')^\bot \subset W/W' \) first uses the identification \( \text{im}(\phi') \oplus \text{im}(\phi'')^\bot = W/W' \), and then the identification \( W/W' \oplus W/W' = W \). From this perspective, from which everything happens inside \( W \) and uses its various orthogonal decompositions, (8.15) is clear, and thus (8.15) is established; (8.16) is similar. We return to establishing (8.14). We now know the left-hand side is
\[
(-1)^{s_{12,3}+d_{12}d_{12,3}+d_{12}(a_3+c_3)+a_2c_3} F^{-1}(\beta_1 \land \beta_2 \land \beta_3)
\land \alpha_1 \land \gamma_1 \land \gamma_2 \land \gamma_3 \land G^{-1}(\alpha_2 \land \alpha_3).
\]

We can also compute the right-hand side of (8.14):
\[
(-1)^{s_{12,3}+d_{12}d_{12,3}+d_{12}(a_3+c_3)+a_2c_3} F^{-1}(\beta_1 \land \beta_2 \land \beta_3)
\land \alpha_1 \land \gamma_1 \land \gamma_2 \land \gamma_3 \land G^{-1}(\alpha_2 \land \alpha_3),
\]
\[
= (-1)^{s_{12,3}+d_{12}d_{12,3}+d_{12}(a_3+c_3)+a_2c_3} F^{-1}(\beta_1 \land \beta_2 \land \beta_3)
\land \alpha_1 \land \gamma_1 \land \gamma_2 \land \gamma_3 \land G^{-1}(\alpha_2 \land \alpha_3).
\]

We have used the necessary analogs of (8.15) and (8.16). Thus (8.14) holds if the quantity
\[
s_{12,3} + d_{23}(d_{12,3} + b_1 + a_2) + a_2(a_3 + c_3) + s_{12,3} + d_{12}(d_{12,3} + a_3 + c_3)
\]
is even. Using \( d_{23} + d_{1,23} = d_{12} + d_{12,3} \), this is easily verified. This establishes (8.13). Finally, we remark on the absence of the \( H_{12} \) maps in our setup. In our application, we can choose the relevant maps \( H_{12} \) and \( H_{12,3} \) so that

\[
(H_{12,3})(H_{12} \oplus \text{id}_{H^+_2(X_3) \oplus \ker(f_{12,3})}) = H,
\]

where we are using some chosen map

\[
H : H^+_2(X_2) \oplus H^+_2(X_2) \oplus H^+_2(X_3) \oplus \ker(f) \xrightarrow{\sim} H^+_2(X_{123})
\]

associated to the natural short exact sequence, and the identifications (8.12). This is established just as was (8.15). The maps \( H_{23} \) and \( H_{1,23} \) can be chosen similarly, and this compatibility allows the above argument to carry through.

Now we define distinguished identity homology orientations. If \( X = Y \times [0, 1] \), then \( \mathcal{L}(X) = H_1(Y) \oplus H_1(X) \). Let \( \alpha \) be any orientation of \( H_1(Y) \), and choose an orientation \( \beta \) of \( H_1(X) \) such that \( \alpha = \beta \) under the natural identification of \( H_1(X) \) with \( H_1(Y) \). Then define

\[
\mu_Y^{\text{id}} := (-1)^{(1/2)(b_1(Y)^2 + b_1(Y))} \beta \wedge \alpha
\]

to be the distinguished identity homology orientation of \( Y \times [0, 1] \).

**Proposition 8.3.** Whenever \( \mu \) is a homology orientation of a cobordism \( X \) with incoming boundary \( Y \), we have \( \mu \circ \mu_Y^{\text{id}} = \mu \). Similarly, if \( X \) has outgoing boundary \( Y \), then \( \mu_Y^{\text{id}} \circ \mu = \mu \).

**Proof.** Suppose that \( X \) has incoming boundary \( Y \), that is, \( X : Y \rightarrow Y' \). We let \( X_1 = Y \times [0, 1] \) and \( X_2 = X \) and use the notation of Definition 8.1. We have \( \text{im}(f_{12}) = H_1(Y) \) and thus \( d_{12} = b_1(Y) \). We identify \( X_{12} \) with \( X_2 = X \). Choose the section of the exact sequence (8.2), which is a map \( H_1(X) \rightarrow H_1(Y \times [0, 1]) \oplus H_1(X) \), to be of the form \( y \mapsto (0, y) \). The induced isomorphism \( F_{12} : H_1(Y) \oplus H_1(X) \rightarrow H_1(Y \times [0, 1]) \oplus H_1(X) \) is of the form \( (x, y) \mapsto (x, y - \pi(x)) \) where \( \pi : H_1(Y) \rightarrow H_1(X) \) is induced by inclusion. Let \( \mu = \mu_2 = \beta_2 \wedge \alpha_2 \wedge \gamma_2 \) where \( \beta_2, \alpha_2, \gamma_2 \) are respective orientations of \( H_1(X), H_1(Y), H_2^+ \). Write \( \mu_1 = \mu_Y^{\text{id}} = (-1)^{(1/2)(b_1(Y)^2 + b_1(Y))} \beta_1 \wedge \alpha_1 \) as above, where \( \alpha_1 = \alpha \) and \( \beta_1 = \beta \). Choose \( \delta_{12} = \alpha_1 \). Then

\[
F_{12}^{-1}(\beta_1 \wedge \beta_2) = \delta_{12} \wedge \beta_{12},
\]

where \( \beta_{12} = \beta_2 \). We can choose \( \alpha_2 = \alpha_1 \) so that the condition \( \zeta_{12} \wedge \delta_{12} = \alpha_2 \) \((G_{12} \text{ implicit})\) forces \( \zeta_{12} \) to be the canonical +1 orientation of the 0-vector space. Similarly, \( \gamma_1 \) is taken to be +1, and the expression \( H_{12}(\gamma_1 \wedge \gamma_2 \wedge \zeta_{12}) \) may be regarded as equal to \( \gamma_2 \). The sign \( s \) in Definition 8.1 is equal to \( \frac{1}{2}(b_1(Y)^2 + b_1(Y)) \), and so cancels with the sign in \( \mu_Y^{\text{id}} \). Altogether, Definition 8.1 yields

\[
\mu \circ \mu_Y^{\text{id}} = \beta_2 \wedge \alpha_2 \wedge \gamma_2 = \mu.
\]

Next, suppose that \( X \) has outgoing boundary \( Y \), that is, \( X : Y' \rightarrow Y \). Now we write \( X = X_1 = X_{12} \) and \( Y \times [0, 1] = X_2 \) and, correspondingly, we swap the indices for the above orientations and write \( \mu = \mu_1 = \beta_1 \wedge \alpha_1 \wedge \gamma_1 \) and \( \mu_Y^{\text{id}} = (-1)^{(1/2)(b_1(Y)^2 + b_1(Y))} \beta_2 \wedge \alpha_2 = \mu_2 \). Choose the section of the exact sequence (8.2), which is a map \( H_1(X) \rightarrow H_1(X) \oplus H_1(Y \times [0, 1]) \), to be of the form \( y \mapsto (y, 0) \). Now the induced map \( F_{12} : H_1(Y) \oplus H_1(X) \rightarrow H_1(X) \oplus H_1(Y \times [0, 1]) \) is of the form \( (x, y) \mapsto (y + \pi(x), -x) \). Choose \( \delta_{12} = \alpha_2 = \beta_2 \) and so on, just as above. Then

\[
F_{12}^{-1}(\beta_1 \wedge \beta_2) = \delta_{12} \wedge \beta_{12},
\]

where \( \beta_{12} = (-1)^{b_1(Y)b_1(X)} \beta_1 = (-1)^{s} \beta_1 \). The exponent \( s \) in Definition 8.1 is given by

\[
\frac{1}{2}(b_1(Y)^2 - b_1(Y)) + b_1(Y)b_1(X) \mod 2.
\]
We see that \( s + t \equiv \frac{1}{2}(b_1(Y)^2 + b_1(Y)) \) mod 2. This cancels with the sign put in front of \( \mu_1^\text{id} \), and we obtain from Definition 8.1 the identity \( \mu_2^\text{id} \circ \mu = \mu \), just as before.

In the remainder of this section, we describe how our composition rule can be described in the setting of Fredholm determinant line bundles, as in [20, §20.2], the purpose of which is to show that our rule is compatible with a construction of instanton homology. As such, the following details are not needed to understand the rest of the paper.

In the Fredholm setting, a homology orientation of \( X \) is an orientation of \( \det(D) \), where \( D \) is the operator \( \overset{-}{d}^* \oplus d^+ \) acting on suitably weighted Sobolev spaces over \( X \) with cylindrical ends attached. Recall that

\[
\det(D) = \bigwedge^{\text{max}}(\ker(D)) \otimes \bigwedge^{\text{max}}(\coker(D)^*).
\]

The Sobolev weights are chosen such that we have natural identifications

\[
\ker(D) = H^1(X), \quad \coker(D) = H^1(Y) \oplus H^2_+(X),
\]

where \( Y \) is the incoming end of \( X \); cf. [7, Proposition 3.15]. Note that an orientation of a vector space induces, in a natural way, an orientation of its dual space. Since we are working with real coefficients, homology and cohomology groups are dual to one another, so an orientation of \( \det(D) \) is the same as an orientation of \( \mathcal{L}(X) \).

Let us now suppose that we are in the situation of Definition 8.1, so that \( \mu_i \) is an orientation of \( \mathcal{L}(X_i) \), or equivalently \( \det(D_i) \) for \( i = 1, 2 \). We again write \( \mu_i = \beta_i \land \alpha_i \land \gamma_i \) where now we view \( \beta_i \) as orienting \( \ker(D_i) \) and \( \alpha_i \land \gamma_i \) as orienting \( \coker(D_i) \) (or its dual). We will denote the composition of \( \mu_1 \) and \( \mu_2 \) as given in this setting by

\[
\mu_2^\text{id} \circ \mu_1
\]

to distinguish it from our previous rule. The composition \( \mu_2^\text{id} \circ \mu_1 \) goes in two steps. First, we use the \( \mu_i \) to orient \( \det(D_1 \oplus D_2) \), which is identified with

\[
\bigwedge^{\text{max}}(\ker(D_1) \oplus \ker(D_2)) \otimes \bigwedge^{\text{max}}(\coker(D_1) \oplus \coker(D_2))^*.
\]

We use the following general rule for doing this: if \( K_i \land C_i \) is an orientation for \( \det(D_i) \) where \( K_i \) orients \( \ker(D_i) \) and \( C_i \) orients \( \coker(D_i) \) (or its dual), then we orient \( \det(D_1 \oplus D_2) \) by

\[
(-1)^{\dim \ker(D_2) \cdot \text{index}(D_1)} (K_2 \land K_1) \land (C_1 \land C_2).
\]

This is a slight modification of the rule in [20, Lemma 20.2.1] but is easily seen to be associative; the difference between the two rules is the sign \((-1)^s\) where

\[
s = \dim \ker(D_1) \cdot \dim \ker(D_2) + \text{index}(D_2) \cdot \dim \ker(D_1).
\]

Applying this procedure to \( \mu_1 \) and \( \mu_2 \), we obtain the orientation

\[
\mu' := (-1)^{(a_2 + c_2)(a_1 + b_1 + c_1)} (\beta_2 \land \beta_1) \land (\alpha_1 \land \gamma_1 \land \alpha_2 \land \gamma_2)
\]

of \( \det(D_1 \oplus D_2) \), where \( a_i = \dim H^1(Y_i), b_i = \dim H^1(X_i) \) and \( c_i = \dim H^2_+(X_i) \).

The second step in describing the composition rule in this setting involves relating \( \det(D_1 \oplus D_2) \) to \( \det(D_{12}) \) by means of a (Fredholm) homotopy from the operator \( D_1 \oplus D_2 \) to \( D_{12} \), where \( D_{12} \) is the operator associated to \( X_{12} \). We will use the notation of [20, §20.2]. Let \( P_s \) for \( s \in [0, 1] \) be such a homotopy, so that \( P_0 = D_1 \oplus D_2 \) and \( P_1 = D_{12} \). To be precise, we should understand these two aforementioned operators as having the same domain and codomain; this may be achieved using the finite cylinder setup as in [20]. Denoting our codomain by \( B \), choose \( J \subset B \) so that \( P_s^{-1} \circ J \circ B = B \) for all \( s \). We have for each \( s \) an exact sequence

\[
0 \longrightarrow \ker(P_s) \overset{j}{\longrightarrow} P_s^{-1} J \overset{k}{\longrightarrow} J \overset{l}{\longrightarrow} \coker(P_s) \longrightarrow 0.
\]

(8.18)
We use the following general rule for orienting $\det(P_s)$ given an orientation $\mu''$ of the line $\bigwedge^{\text{max}} P_s^{-1} J \otimes \bigwedge^{\text{max}} J^*$ using the exact sequence (8.18): write

$$\mu'' = (K \wedge D) \wedge (k(D) \wedge C),$$

(8.19)

where $K$ is an orientation of $\text{im}(j)$, $D$ of $\text{im}(j)^\perp$ and $C$ of $k(\text{im}(j)^\perp)^\perp$; then orient $\det(P_s)$ by

$$(-1)^{\phi(d)j^{-1}}(K) \wedge l(C),$$

(8.20)

where $\phi(x):=(x^2-x)/2$ and $d:=\dim(\text{im}(j)^\perp)$. In our situation, we choose $J$ to be a complement of $\text{im}(P_0) = \text{im}(D_1 \oplus D_2)$, and we make the identification

$$J = H^1(Y_1) \oplus H^2_+(X_1) \oplus H^1(Y_2) \oplus H^2_+(X_2).$$

We choose the homotopy so that $P_s^{-1} J = \ker(P_0) = \ker(D_1 \oplus D_2)$ for all $s$, so that

$$P_s^{-1} J = H^1(X_1) \oplus H^1(X_2).$$

In particular, we have an identification of $\bigwedge^{\text{max}} P_1^{-1} J \otimes \bigwedge^{\text{max}} J^*$ with $\det(D_1 \oplus D_2)$, which is oriented by $\mu'$. Noting that the maps in (8.18) for $s=1$ come from the Mayer–Vietoris maps as in Definition 8.1, we can write $\mu''$ from $\mu'$ as in (8.19):

$$\mu'' = (-1)^t(\beta_{12} \wedge \delta_{12}) \wedge (\delta_{12} \wedge \gamma_{12}).$$

In this expression, and in all to follow, the maps $F_{12}, G_{12}$ and $H_{12}$ from Definition 8.1 as well as the maps in (8.18) will be implicitly understood, for example, $F_{12}(\delta_{12} \wedge \beta_{12})$ is the same as $\delta_{12} \wedge \beta_{12}$. The orientation $\beta_{12}$ plays the role of $K$ above, $\gamma_{12}$ that of $C$ and $\delta_{12}$ that of $D$. The sign $(-1)^t$ is given by

$$t = (a_2 + c_2)(a_1 + b_1 + c_1) + d_{12}(b_1 + b_2 + d_{12}) + b_1 b_2,$$

where $d_{12}$ is as in Definition 8.1. The first term in $t$ is from $\mu'$ and the rest are added to ensure that $\beta_{12}$ is defined by the condition $\delta_{12} \wedge \beta_{12} = \beta_1 \wedge \beta_2$, to match Definition 8.1. The orientation $\gamma_{12}$ is defined by the condition $\delta_{12} \wedge \gamma_{12} = \alpha_1 \wedge \gamma_1 \wedge \alpha_2 \wedge \gamma_2$. The general rule that takes $\mu''$ to (8.20), applied to our $\mu''$, tells us the final orientation of $\det(D_{12})$:

$$\mu_2 \sigma \mu_1 = (-1)^{\phi(d_{12})+t} \beta_{12} \wedge \gamma_{12}.$$

Now write $\alpha_2 = \zeta_{12} \wedge \delta_{12}$ as in Definition 8.1. We compute

$$\mu_2 \sigma \mu_1 = (-1)^t \beta_{12} \wedge \alpha_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \zeta_{12},$$

$$r = \phi(d_{12}) + t + d_{12}(a_2 + d_{12} + c_1 + a_1) + c_2(d_{12} + a_2).$$

The sign given by $r$ does not match the sign given by $s$ in Definition 8.1, and so this composition rule is not the same as the one previously defined. However, there is an automorphism $\mu \mapsto \overline{\mu}$ on the class of all homology orientations that intertwines the two rules. Given a homology orientation $\mu$ of a cobordism $X$, we set

$$\overline{\mu} = (-1)^{\phi(b_1(X)) + \phi(b_1(Y) + b_1^+(X))} \mu,$$

where $Y$ is the incoming end of $X$. Then we have

$$\overline{(\mu_1 \sigma \mu_2)} = \mu_1 \circ \mu_2.$$

The verification is a straightforward computation that we omit. It follows that the composition rule $\mu_1 \circ \mu_2$ of Definition 8.1 is compatible with a construction of Floer homology, and it is this rule that we will use in our computations below.
8.3. The $E^1$-page

In this section, we identify the $E^1$-page of (2.1) with the chain complex that computes reduced odd Khovanov homology. We fix as before a diagram $D$ for the $m$-component link $L$ with crossings decorated by arcs as in §8.1. We let $Y_v = \Sigma(D_v)$ for each $v \in \{0,1\}^m$ so that $Y_v$ is homeomorphic to $\#^k S^1 \times S^2$ when $D_v$ has $k + 1$ circles. The $E^1$-page and differential of the spectral sequence we are considering is given by

$$E^1 = \bigoplus_{v \in \{0,1\}^m} I^\#(Y_v), \quad d^1 = \sum (-1)^{\delta(v,w)} I^\#(X_{vw}),$$

where the sum runs over $v < w$ with $|w - v|_1 = 1$ and $v, w \in \{0,1\}^m$. In writing $d^1$, we have chosen homology orientations $\mu_{vw}$ of the $X_{vw}$ so that $\mu_{vw} \circ \mu_{vu} = \mu_{tv} \circ \mu_{vt}$ always holds. We are also using that the relevant bundles $\pi_{vw}$ are trivial. This is because each such bundle lies over a cobordism which is $D^2 \times S^2 \setminus \text{int}(D^4)$ running along a product cobordism; see (8.21); since we have arranged that the restriction of each such bundle over the boundary is trivial, for topological reasons the bundle must be trivial.

Let $C = \bigoplus C_v$ be the reduced odd Khovanov chain group for the diagram $D$ and $\partial' = \sum \partial'_{vw}$ be its predifferential. For each $v \in \{0,1\}^m$, we define an isomorphism

$$\Phi_v : C_v \longrightarrow I^\#(Y_v)$$

defined as a composition $\Phi_v = \phi_v \circ \rho_v$, where $\phi_v : \bigwedge^* (H_1(Y_v;\mathbb{Z})) \longrightarrow I^\#(Y_v)$ is from §7.8 and $\rho_v : C_v \rightarrow \bigwedge (H_1(Y_v;\mathbb{Z}))$ is defined by lifting arcs in $D_v$ to loops in $Y_v$, and is explained in the following paragraph. For the $\phi_v$ maps, we of course fix orientations $\mu_v$ for each $H_1(Y_v;\mathbb{R})$. We write $\Phi : C \rightarrow E^1$ for the sum of the $\Phi_v$ maps.

Recall $C_v = \bigwedge^* (V_v)$, and that $Y_v$ is branched over $D_v \subset S^3$. Let $S$ be the union of disks in the plane enclosed by the circles in $D_v$. They can be pushed out so that they are disjoint and form a Seifert surface for the union of circles. Let $N$ be a neighborhood of the circles, a union of solid tori. Then $Y_v$ can be written as a gluing

$$Y_v = Y_- \cup N \cup Y_+,$$

where $Y_\pm = S^3 \setminus (S \cup N)$. Distinguishing one of the copies of $S^3 \setminus (S \cup N)$, say $Y_+$, allows us to lift an arc $x$ in $D_v$ to an oriented loop $\tilde{x}$ in $Y_v$: the orientation is obtained by locally lifting the orientation of $x$ to the part of $\tilde{x}$ in $Y_+$. Then $x \mapsto [\tilde{x}]$ is an isomorphism from $V_v$ to $H_1(Y_v;\mathbb{Z})$, and $\rho_v$ is taken to be the extension of this map to exterior algebras. We can construct the $\rho_v$ in this way so that it is uniform among all $v$, in the sense that there are natural ways of identifying $Y_v$ with $Y_w$ away from surgery (or resolution) areas, and in these areas we can lift arcs the same way.

In summary, the map $\Phi_v$ is described as follows. Let $x = x_1 \wedge \cdots \wedge x_i$ be a wedge of arcs in $C_v$. Lift the arcs to embedded loops $\tilde{x}_j$ in the branched double cover $\tilde{Y}_v$ as above. Choose $x_{i+1}, \ldots, x_k$ and their lifts such that $\mu_v = [\tilde{x}_1 \wedge \cdots \wedge \tilde{x}_k]$. Attach 2-handles to $\tilde{x}_1, \ldots, \tilde{x}_i$ and 3-handles and a 4-handle as in §7.8 to obtain a cobordism $X : \emptyset \rightarrow Y_v$ homology oriented by $[\tilde{x}_{i+1} \wedge \cdots \wedge \tilde{x}_k]$. Then $\Phi_v(x) = [X]^\#$. The following completes the proof of Theorem 1.1 up to gradings, which are dealt with in the next section.

**Lemma 8.4.** For some valid edge assignment $\varepsilon_{vw}$, we have $\Phi^{-1} d^1 \Phi = \sum \varepsilon_{vw} \partial'_{vw}$

**Proof.** Let $v, w \in \{0,1\}^m$ with $v < w$ and $|w - v|_1 = 1$. There are two cases to consider, depending on whether $D_{vw}$ is a split or a merge diagram. We retain the convention from §8.2 that singular homology $H_*(X)$ is taken with real coefficients. For most of the proof, we conflate the symbols $x$ and $\tilde{x}$, where $x$ is an arc (usually viewed as a class in $V_v$) and $\tilde{x}$ is its lift to $Y_v$ (usually viewed as a class in $H_1(Y_v)$). That is, the maps $\rho_v$ from above are implicit. Suppose
first that we are in the split case. Let \( k = b_1(X_{vw}) \). Note that \( b_2^+(X_{vw}) = 0 \), and \( X_{vw} : Y_v \to Y_w \) is homeomorphic to

\[ (Y_v \times [0, 1]) \rtimes (D^2 \times S^2 \setminus \text{int}(D^4)). \]

We note that we may also view \( X_{vw} \) as the branched double cover of a pair of pants properly embedded in \( S^3 \times [0, 1] \). We have \( \mathcal{L}(X_{vw}) = H_1(Y_v) \oplus H_1(X_{vw}) \). We will follow the notation of Definition 8.1, setting \( X_1 = X \) and \( X_2 = X_{vw} \). Choose orientations \( \alpha_2 \) and \( \beta_2 \) of \( H_1(Y_v) \) and \( H_1(X_{vw}) \), respectively. We can identify \( H_1(Y_v) = H_1(X_{vw}) \) using the map induced by inclusion, and we choose to impose the condition \( \mu_2 = \beta_2 \). Define \( \epsilon_{vw}' = \pm 1 \) by

\[ \mu_{vw} = \epsilon_{vw}' \beta_2^+ \wedge \alpha_2. \]

Let \( x = x_1 \wedge \cdots \wedge x_k \in C_v \). Recall that \( \Phi_v(x) = [X]^{\#} \) where \( X \) is obtained by attaching 2-handles to \( x_1, \ldots, x_k \) along with some 3-handles and a 4-handle. Choose \( x_{i+1}, \ldots, x_k \) so that \( \mu_v = [x_1 \wedge \cdots \wedge x_k] \). Then \( \mathcal{L}(X) = H_1(X) \) is generated by \( x_{i+1}, \ldots, x_k \), and \( X \) is homology oriented by \( \beta_1 := [x_{i+1} \wedge \cdots \wedge x_k] \). We can identify \( \mathcal{L}(X_{vw} \circ X) = H_1(X_{vw} \circ X) \). Note that \( \text{im}(f_{12}) = H_1(Y_v) \), so \( d_{12} = k \). Choose the section in the exact sequence (8.2), which in this case is a map \( H_1(X_{vw} \circ X) \to H_1(X) \oplus H_1(X_{vw}) \), to be of the form \( y \mapsto (y, 0) \). The induced isomorphism \( F_{12} : H_1(Y_v) \oplus H_1(X_{vw} \circ X) \to H_1(X) \oplus H_1(X_{vw}) \), written as in (8.6), can be written as

\[
F_{12} : \mathbb{R}\{x_1, \ldots, x_k\} \oplus \mathbb{R}\{x_{i+1}, \ldots, x_k\} \rightarrow \mathbb{R}\{x_{i+1}, \ldots, x_k\} \oplus \mathbb{R}\{x_1, \ldots, x_k\},
\]

where \( \pi : H_1(Y_v) \to H_1(X) \) is a projection induced by inclusion. Writing \( \beta_2 = \delta_{12} \), we have

\[
F_{12}^{-1}\beta_1 \wedge \beta_2 = (-1)^k \beta_1 \wedge \beta_2 = \delta_{12} \wedge \delta_{12},
\]

where \( \beta_{12} = (-1)^{(k-i)k+k} \beta_1 = (-1)^k \beta_1 \). Using Definition 8.1, we obtain

\[
I^\#(X_{vw}) \Phi_v(x) = (-1)^{(k^2+k)/2} \epsilon_{vw}' [X_{vw} \circ X]^\#,
\]

where \( X_{vw} \circ X \) is homology oriented by \( \beta_1 \). The sign \((-1)^{(k^2+k)/2}\) is obtained by computing

\[
ki + ((k^2 - k)/2 + (k - i)k),
\]

where the term \( ki \) is from \( \delta_{12} \), and the expression inside the parentheses is from Definition 8.1. We mention that the condition \( G_{12}(\zeta_{12} \wedge \delta_{12}) = \alpha_2 \) holds by \( \alpha_2 = \delta_{12} = \beta_2 \) and setting \( \zeta_{12} \) to be the canonical +1 orientation of the 0 vector space. Note that \( [X_{vw} \circ X]^\# = \Phi_w(x_{vw} \wedge x) \) if and only if \( \mu_{vw} = [x_{vw} \wedge x_1 \wedge \cdots \wedge x_k] = x_{vw} \wedge \mu_v \); otherwise they differ in sign. We record a sign \( \epsilon_{vw}'' = \pm 1 \) measuring this possible discrepancy between \( \mu_v \) and \( \mu_w \):

\[
\mu_w = \epsilon_{vw}' \varepsilon_{vw} \wedge \mu_v.
\]

Recalling that \( d_{12}^{\#} = (-1)^{\delta(v, w)} I^\#(X_{vw}) \) and \( \partial''_{vw}(x) = x_{vw} \wedge x \), we conclude

\[
\Phi_w(\partial''_{vw}(x)) = \varepsilon_{vw} d_{12}^{\#}(\Phi_v(x)),
\]

where \( \varepsilon_{vw} = \pm 1 \) is given by

\[
\varepsilon_{vw} = (-1)^{(k^2+k)/2+\delta(v, w)} \epsilon_{vw}' \epsilon_{vw}''.
\]

Now suppose that we are in the merge case. Again, let \( k = b_1(X_{vw}) \). As before, \( b_2^+(X_{vw}) = 0 \) and the cobordism \( X_{vw} : Y_v \to Y_w \) is now homeomorphic to

\[ (Y_w \times [0, 1]) \rtimes (D^2 \times S^2 \setminus \text{int}(D^4)). \]
We identify $H_1(X_{vw}) = H_1(Y_v)$, and write $\mathcal{L}(X_{vw}) = H_1(Y_v) \oplus H_1(Y_w)$. Note the natural codimension 1 inclusion $H_1(Y_v) \subset H_1(Y_w)$. A complement for $H_1(Y_w)$ is generated by $x_{vw}$. Let $\alpha_2$ be an orientation for $H_1(Y_v)$. Define $\varepsilon'_{vw} = \pm 1$ by

$$\mu_{vw} = \varepsilon'_v \beta_2 \wedge \alpha_2, \quad \beta_2 = \alpha_2 \cup x_{vw}.$$ 

The condition $\beta_2 = \alpha_2 \cup x_{vw}$ is equivalently expressed (or is defined) by $\beta_2 \cup x_{vw} = \alpha_2$. Let $x = x_1 \wedge \cdots \wedge x_i \in \mathcal{L}(Y_v)$. If $x_{vw}$ is among $x_1, \ldots, x_i$ (or linearly dependent on them), the 4-manifold $X$ constructed by attaching 2-handles to $x_1, \ldots, x_i$ and some 3-handles and a 4-handle, once paired with $X_{vw}$ to form $X_{vw} \cup X$, contains a non-trivial $S^2$-bundle over $S^2$ as in §7.8, so $[X_{vw} \cup X]^\# = 0$. Choose $x_{i+1}, \ldots, x_{k+1}$ so that $\mu_v = [x_1 \wedge \cdots \wedge x_{k+1}]$; we may assume that $x_{vw} = x_{k+1}$. We may also set $\alpha_2 = \mu_v$, so that $\beta_2 = [x_1 \wedge \cdots \wedge x_k]$. Recall $\Phi_v(x) = [X]^\#$ where $X$ is homology oriented by $\beta_1 = [x_{i+1} \wedge \cdots \wedge x_{k+1}]$. There is a codimension 1 inclusion $H_1(X_{vw} \cup X) \subset H_1(X)$. The vector space $H_1(X_{vw} \cup X)$ is generated by $x_{i+1}, \ldots, x_k$ and a complement for $H_1(X_{vw} \cup X)$ in $H_1(X)$ is generated by $x_{vw} = x_{k+1}$. Choose the section in the exact sequence (8.2), which is a map $H_1(X_{vw} \cup X) \rightarrow H_1(X) \oplus H_1(X_{vw})$, to be of the form $y \mapsto (y, 0)$. As in the split case, $\text{im}(\delta_1) = H_1(Y_v)$. We obtain an isomorphism $F_{12} : H_1(Y_v) \oplus H_1(X_{vw}) \rightarrow H_1(X) \oplus H_1(X_{vw})$ that takes the form

$$F_{12} : \mathbb{R}\{x_1, \ldots, x_{k+1}\} \oplus \mathbb{R}\{x_{i+1}, \ldots, x_k\} \longrightarrow \mathbb{R}\{x_{i+1}, \ldots, x_{k+1}\} \oplus \mathbb{R}\{x_1, \ldots, x_k\},$$

where $\pi_1 : H_1(Y_v) \rightarrow H_1(X)$ and $\pi_2 : H_1(Y_v) \rightarrow H_1(X_{vw})$ are projections induced by inclusion maps. In particular, $\pi_1(x_p) = x_p$ if $p \geq i + 1$ and is otherwise 0, and $\pi_2(x_p) = x_p$ if $p \neq k + 1$ and $\pi_2(x_{k+1}) = 0$. Recalling that $\beta_2 = \alpha_2 \cup x_{vw}$ and choosing $\delta_{12} = \alpha_2$, we have

$$F_{12}^{-1}(\beta_1 \wedge \beta_2) = (\beta_1 \cup x_{vw}) \wedge \alpha_2 = \delta_{12} \wedge \beta_{12},$$

where $\beta_{12} = (-1)^{ki+i}(\beta_1 \cup x_{vw})$. Note $\beta_1 \cup x_{vw} = [x_{i+1} \wedge \cdots \wedge x_k]$. From Definition 8.1, we obtain

$$I^#(X_{vw})[X]^\# = (-1)^{(k^2-k)/2+1} \varepsilon'_{vw}[X_{vw} \cup X]^\#,$$

where $X_{vw} \cup X$ is homology oriented by $\beta_1 \cup x_{vw}$. We have computed the sign from

$$ki + i + ((k+1)^2 - (k+1))/2 + (k+1-i)(k+1),$$

where $ki + i$ is from $\beta_{12}$, and the expression inside the parentheses is from the sign in Definition 8.1. On the other hand, $[X_{vw} \cup X]^\# = \Phi_w(x)$ exactly when $\mu_w = [x_1 \wedge \cdots \wedge x_k] = \mu_v \cup x_{vw}$. Accounting for this, we define $\varepsilon''_{vw} = \pm 1$ by

$$\mu_w = \varepsilon''_{vw} \mu_v \cup x_{vw}.$$ 

Recalling that $\partial'_{vw}(x) = x$, we conclude

$$\Phi_w(\partial'_{vw}(x)) = \varepsilon_{vw} d^3 \Phi_w(\Phi_v(x)),$$

where $\varepsilon_{vw} = \pm 1$ is given by

$$\varepsilon_{vw} = (-1)^{(k^2-k)/2+1+\delta(v,w)} \varepsilon'_{vw} \varepsilon''_{vw}.$$ 

In summary, we have shown that

$$\Phi^{-1} d^3 \Phi = \sum_{v < w, \frac{|w-v|}{3} = 1} \varepsilon_{vw} \partial'_{vw},$$

where we have determined $\varepsilon_{vw}$ in the split and merge cases separately. It remains to show that $\varepsilon_{vw}$ is a valid edge assignment. The first condition, that the total differential squares to zero,
already falls out from the spectral sequence. We now show that the \( \varepsilon_{vw} \) satisfy the second condition, that is, if \( v, u, t, w \) form a type X face, then the product

\[
\varepsilon_{vu} \varepsilon_{vt} \varepsilon_{uw} \varepsilon_{tw}
\]

is always +1 or −1, independently of the particular face chosen; and if they form a type Y face, the same is true, and the sign is the opposite of the type X case. We fix such a type X face. Note

\[
\delta(v, u) + \delta(v, t) + \delta(u, w) + \delta(t, w) \equiv 0 \mod 2.
\]

Next we consider the \( \varepsilon''_{vw} \) terms. We compute

\[
x_{vu} \wedge \mu_v = \varepsilon''_{vu} \mu_u = \varepsilon''_{vu} \varepsilon_{uw} \mu_w \wedge x_{uw}.
\]

Since \( x_{vu} = -x_{uw} \) in \( D_u \), the above can be abbreviated to \( \mu_v = (-1)^{k+1} \varepsilon''_{vu} \varepsilon_{uw} \mu_w \). Similarly, we obtain \( \mu_v = (-1)^{k+1} \varepsilon''_{vt} \varepsilon_{uw} \mu_w \), implying \( \varepsilon''_{vu} \varepsilon_{vt} \varepsilon_{uw} \varepsilon_{tw} = 1 \). A similar argument in the type Y case yields \( \varepsilon''_{vu} \varepsilon_{vt} \varepsilon_{uw} \varepsilon_{tw} = 1 \) as well. To summarize, we may reconsider the problem with (8.22) replaced by the expression \( \varepsilon''_{vu} \varepsilon_{vt} \varepsilon_{uw} \varepsilon_{tw} \).

Note \( \mathcal{L}(X_{vu}) = H_1(Y_v) \oplus H_1(X_{vu}) \), and, as this is a split cobordism, we have a natural identification \( H_1(Y_v) = H_1(X_{vu}) \). Choose respective orientations \( \alpha_1 \) and \( \beta_1 \) of \( H_1(Y_v) \) and \( H_1(X_{vu}) \) that agree under this identification. Recall that \( \varepsilon_{vu} \) has been defined by

\[
\mu_v = \varepsilon_{vu} \beta_1 \wedge \alpha_1.
\]

On the other hand, \( \mathcal{L}(X_{uw}) = H_1(Y_u) \oplus H_1(X_{uw}) \), and, as this is a merge cobordism, there is a codimension 1 inclusion \( H_1(X_{uw}) \subset H_1(Y_u) \) with a complement generated by \( x_{uw} \). Let \( \alpha_2 \) be an orientation of \( H_1(Y_u) \) and set \( \beta_2 = \alpha_2 \wedge x_{uw} \). Then \( \varepsilon_{uw} \) has been defined by

\[
\mu_{uw} = \varepsilon_{uw} \beta_2 \wedge \alpha_2.
\]

In this situation, the map \( f_{12} \) of (8.2) has a one-dimensional kernel spanned by \( x_{uw} \). In this way, \( \text{im}(f_{12}) \) can be identified with \( H_1(Y_v) \) and \( H_1(X_{vu}) \). Let a section for the exact sequence (8.2), here a map \( H_1(X_{vw}) \to H_1(X_{vu}) \oplus H_1(X_{uw}) \), be given by \( y \mapsto (y, 0) \). The map \( F_{12} : \text{im}(f_{12}) \oplus H_1(X_{vu}) \to H_1(X_{vu}) \oplus H_1(X_{uw}) \) of (8.6) can be written as

\[
F_{12} : \mathbb{R}\{x_1, \ldots, x_k\} \oplus \mathbb{R}\{x_1, \ldots, x_k\} \longrightarrow \mathbb{R}\{x_1, \ldots, x_k\} \oplus \mathbb{R}\{x_1, \ldots, x_k\},
F_{12}(x_p, x_q) = (x_q + x_p, -x_p).
\]

Proceeding with the conditions of Definition 8.1, we find

\[
F_{12}^{-1}(\beta_1 \wedge \beta_2) = \delta_{12} \wedge \beta_{12},
\]

where \( \delta_{12} = \alpha_1 = \beta_2 \) and \( \beta_{12} = \beta_1 \). We can arrange that \( \beta_2 = \beta_1 \) under the appropriate identification. The condition \( G_{12}(\zeta_{12} \wedge \delta_{12}) = \alpha_2 \), having that \( \alpha_2 = \beta_2 \wedge x_{uw} \), yields \( \zeta_{12} = (-1)^k x_{uw} \). Using Definition 8.1, we obtain

\[
\mu_{uw} \circ \mu_{vu} = (-1)^{(k^2-k)/2+k(k+1)+k'} \varepsilon_{vu} \varepsilon_{tw} \beta_1 \wedge \alpha_1 \wedge H_{12}^u(x_{uw}),
\]

where we have used \( k = b_1(X_{vu}) = d_{12} \). The superscript \( u \) in \( H_{12}^u \) distinguishes this map from the map \( H_{12}^w \) which appears when \( u \) is replaced by \( t \). We obtain a similar equation for \( \mu_{tu} \circ \mu_{vt} \) with \( x_{uw} \) replaced by \( x_{tv} \) and \( \varepsilon_{vu} \varepsilon_{uw} \) replaced by \( \varepsilon_{vt} \varepsilon_{tw} \). Because our setup includes the compatibility condition \( \mu_{tw} \circ \mu_{vt} = \mu_{uw} \circ \mu_{vu} \), we conclude

\[
\varepsilon_{vu} \varepsilon_{vt} \varepsilon_{uw} \varepsilon_{tw} = H_{12}^u(x_{uw})/H_{12}^t(x_{uw}) =: \varphi.
\]

In summary, we see that \( \varepsilon \) is the sign determined by comparing the result of orienting \( H_2^u(X_{vu}) \) by \( x_{uw} \) versus the result by using \( x_{tw} \). Using the interpretation of the splitting
Figure 15 (colour online). Local pictures for four diagrams appearing in a type X face, starting at the diagram $D_v$ and ending at $D_w$. The circles in each diagram are colored so as to distinguish their roles in Figure 16.

Figure 16 (colour online). This is an illustration (missing a dimension) of $S^4$ minus two 4-balls, with a properly embedded surface $F$, a torus with two disks removed, with the local portions of the diagrams of the type X face from Figure 15 embedded; the circles of the diagrams lie on $F$, while only the endpoints of the arcs lie on $F$. The cobordism $X_{vw}$ is the double cover over $S^4$ minus two 4-balls branched over $F$. The disk $S_u'$ lifts to a 2-sphere $S_u \subset X_{vw}$ intersecting $\tilde{x}_{uw}$ (the lift of $x_{uw}$) in one point. The disk $T_u'$ lifts to a 2-sphere $T_u \subset X_{vw}$ intersecting $Y_u$ in $\tilde{x}_{uw}$.

map (8.4) from §8.2, we obtain the following interpretation of $\varepsilon$. Here is a suitable moment to reintroduce the distinction between each arc $x$ and its lift $\tilde{x}$. Choose an oriented surface $S_u \subset Y_u$ transverse to $\tilde{x}_{uw}$ with intersection product $[S_u] \cdot [\tilde{x}_{uw}] = 1$. Choose an oriented surface $T_u$ with $T_u \cap Y_u = \tilde{x}_{uw}$. Then

$$[S_u] + [T_u] = H_1^u(\tilde{x}_{uw}).$$

To illustrate this, we supply Figures 15 and 16, where we use that $X_{vw}$ is a double cover of $S^4$ minus two 4-balls branched over a properly embedded torus with two disks removed. Similarly, we can write $[S_t] + [T_t] = H_1^u(\tilde{x}_{tw})$. The sign $\varepsilon$ is then the intersection product of these classes:

$$\varepsilon = ([S_u] + [T_u]) \cdot ([S_t] + [T_t]).$$

In fact, $[T_t] = \varepsilon [S_u]$. From this, it is clear that $\varepsilon$ only depends on the topology of the type X configuration. A type Y face is obtained from a type X face by reversing the direction of either $\tilde{x}_{uw}$ or $\tilde{x}_{tw}$, and $\varepsilon$ correspondingly changes sign. □
8.4. Gradings

In this section, we prove that the spectral sequence preserves the relevant $\mathbb{Z}/4$-gradings, completing the proof of Theorem 1.1. We then deduce Corollary 1.2. As usual, let $k = \dim(V_v)$. For $x \in \bigwedge^i(V_v) \subset C$, the grading of $\Phi(x)$ in $E^1$ is given in (7.7) by
\[
g_r[E^1](\Phi(x)) = gr[Y_v](\Phi(x)) - \deg(X_{\infty}) + |v|_1 \mod 4. \tag{8.23}
\]

We know, by the remark at the end of §7.8, that $gr[Y_v](\Phi(x)) = 2k + i$. We have $\deg(X_{\infty}) = \deg(X_{\infty}) - \deg(X_{\infty})$, since $X_{\infty}$ is trivial. From (4.11), we compute
\[
\deg(X_{\infty}) = -\frac{3}{2}(2m + \sigma(X_{\infty})) + \frac{1}{2}(b_1(Y_1) - b_1(\Sigma(L)))
\]
using $\chi(X_{vw}) = |w - v|_1$, $\sigma(X_{\infty}) = 0$ and $b_1(Y_v) = k$. We also compute
\[
\deg(X_{\infty}) = -\frac{3}{2}(2m + \sigma(X_{\infty})) + \frac{1}{2}(b_1(Y_1) - b_1(\Sigma(L)))
\]
knowing $\Sigma(L) = \overline{Y}_\infty$. Recall from (7.1) that $\deg(X_{\infty}) \equiv \deg(X_{\infty}) + 2\mathcal{P}(X_{\infty})$.

**Lemma 8.5.** \(\mathcal{P}(X_{\infty}) \equiv \sigma(X_{\infty}) \mod 2.\)

Before proving this lemma, we make our conclusion. In [4], Bloom computes $\sigma(X_{0,\infty}) = \sigma - n_+$ and $b_1(\Sigma(L)) = \nu$, where $\sigma$ and $\nu$ are the signature and nullity of $L$, respectively, and $n_+$ is the number of $\pm$ crossings of the diagram $D$. Note that $X_{\infty}$ and $X_1 \subset X_1$ compose along $Y_1$ to give a cobordism which, away from a manifold of signature 0, has $m$ copies of $-\mathbb{C}P^2$ connected to it (cf. $E$ from §3.4). In addition, since $\sigma(X_{0}) = 0$, we have $\sigma(X_{\infty}) = \sigma(X_{0})$. Additivity of the signature again implies that $\sigma(X_{\infty}) = -m - \sigma + n_+$. Note $m = n_+ + n_-$. Altogether, (8.23) computes to
\[
i + 2n_- + \frac{3}{2}(n_+ + k) + \frac{1}{2}(|v|_1 + \nu + \sigma) \mod 4,
\]
which is congruent to (8.1). This completes the proof of Theorem 1.1.

**Proof of Lemma 8.5.** By additivity and the fact that $\mathcal{P}(X_{0}) \equiv \sigma(X_{0}) \equiv 0 \mod 2$, it suffices to show that $\mathcal{P}(X_{\infty}) \equiv \sigma(X_{\infty})$. Write $X = X_{\infty}$ and $X$ for its base space. We have
\[
X = ([0, 1] \times \mathbb{Y}) \cup_L \left( \bigcup_{i=1}^m \mathbb{H} \right),
\]
where $L = L_1 \cup \cdots \cup L_m$ is an $SO(3)$-thickening of $L = L_1 \cup \cdots \cup L_m$, and each $L_i : \mathbb{H} \to \mathbb{Y} \times \{1\}$ is as in §3.3. Here we are viewing
\[
\mathbb{Y} = (Y \times SO(3))\Psi(L)
\]
as a bundle over $Y = Y_{\infty} = \Sigma(L)$ built from the geometric representative $L$ as in §3.7. In §2, we saw that $L$ is the boundary of a surface $S \subset Y$, and so, in fact, $\mathbb{Y}$ is a trivial bundle. Note $X$ is reducible to an $S^1$-bundle by its very construction. Let $\mathcal{L}$ be the associated complex line bundle. The Poincaré dual of a preimage of $c_1(\mathcal{L}) \in H^2(X; \mathbb{Z})$ in $H^2(X, \partial X; \mathbb{Z})$ is represented by the closed surface $S' \subset \text{int}(X)$, which is the union of the cores of the 2-handles together with $S \subset Y \times \{1\}$. Indeed, it is straightforward to define a section of $\mathcal{L}$ with zero set $S'$. By the definition of $\mathcal{P}(X)$, it suffices to show that
\[
[S'] \cdot [S'] \equiv \sigma(X) \mod 2,
\]
where $[S'] \cdot [S']$ is the intersection product. To do this, we write down a relative Kirby diagram for $(X, Y)$. We start by writing a surgery diagram for $Y = \Sigma(L)$ using the chosen diagram $D$. For this, we follow Bloom [4]. First, choose $v \in \{0, 1\}^m$ for which the resolution $D_v$ has 1 circle. We can always choose $D$ so that there is such a resolution. Then, in $D_v$, having placed arcs where crossings once were, cut the lone circle at an isolated point $p$ and unravel it, with
To obtain a relative Kirby diagram for \((X, Y)\) where \(Y = \Sigma(L)\), we borrow some constructions from Bloom [4]. With a diagram of the figure eight knot in (i) as an example, we first choose a resolution that yields one connected circle as in (ii), drawing small arcs where crossings used to be. We then cut the connected circle at the dot, and straighten it out, as in (iii). Reflecting this picture across the line, we obtain a surgery diagram (iv) for \(Y\) by choosing a +1 framing for each circle corresponding to a 0-resolution, and a −1 framing for 1-resolution circles. Finally, the relative Kirby diagram (v) is obtained by placing a small meridional circle on each circle in (iv) framed by 0 or −1, depending on whether the circle corresponds to a 0- or 1-resolution, respectively.

the arcs attached, into a horizontal segment; then double it as in Figure 17(iv). Place a +1 framing on a circle in the resulting picture if that circle came from a 0-resolution, and a −1 framing otherwise. This gives a surgery diagram for \(Y = \Sigma(L)\).

To turn this into a relative Kirby diagram for \((X, Y)\), we simply add small meridians around each circle, framed with a 0 if the circle is +1 framed and a −1 if the circle is −1 framed. The intersection number \([S'] \cdot [S']\) is concentrated at the attaching locations of the 2-handles, represented by the meridional circles in the relative Kirby diagram. Thus there is a −1 contribution to \([S'] \cdot [S']\) from each 1-resolution in \(D_v\). We conclude \([S'] \cdot [S'] = -|v|\).

We now discuss Corollary 1.2. As alluded to in §1, [29, Theorem 1], [33, §5] and the remarks in [28, §9.3] tell us that, for a quasi-alternating link \(L\), the gradings \(q\) and \(t\) of \(\overline{K\ell}(L)\) satisfy \(q/2 - t - \sigma/2 = 0\). Let us write \(\delta = q/2 - t - \sigma/2\); then we may say that \(\overline{K\ell}(L)\) is supported in \(\delta\)-grading 0. Note that \(\nu = 0\) when \(L\) is quasi-alternating. Further, as is described in [29], the rank of \(\overline{K\ell}(L)_{q,t}\) is given by \(|a_q|\), where

\[J_L(x) = \sum a_q x^q\]

is the Jones polynomial, with conventions as given in [33]. The grading (1.1) of Theorem 1.1, which we shall call \(\delta^\#\), is given by \(\delta^\# = \delta + q + \sigma\) for quasi-alternating links. Note that \(\delta\) and \(\delta^\#\) agree modulo 2, implying that the spectral sequence collapses at the \(E^2\)-page. Write \(\overline{K\ell}(L)_j\) with \(j \in \{0, 2\} \subset \mathbb{Z}/4\mathbb{Z}\) for the \(\delta^\#\)-grading. Then

\[\text{rk}_2 \overline{K\ell}(L)_j = \sum |a_q|,\]
where the congruence is modulo 4. We remark that \( \sum |a_q| = \det(L) \), and the sign of \( a_q \) is \((-1)^{q/2+\sigma/2} \), cf. [4, § 9.1]. It follows that

\[
\text{rk}_Z \text{Khi}^7(L)_j = \frac{1}{2} \left[ \sum |a_q| + (-1)^{j/2} \sum a_q \right].
\]

Now we obtain the result of Corollary 1.2 using the fact that \( J_L(1) = \sum a_q = 2^{m-1} \), where \( m \) is the number of components of \( L \).

9. Connected sum formulas

A closed, connected, oriented 3-manifold \( Y \) is called an integral homology 3-sphere if \( H_1(Y; \mathbb{Z}) = 0 \), or equivalently, if \( Y \) has the same integral homology as the 3-sphere. In this section, we study \( I^#(Y) \) when \( Y \) is an integral homology 3-sphere. We relate the \( \mathbb{Z}/4 \)-graded group \( I^#(Y) \) to Floer’s original \( \mathbb{Z}/8 \)-graded instanton homology \( I(Y) \) using the trivial connection and \( u \)-maps studied by Donaldson [7] and Frøyshov [15].

9.1. Gradings

Let \( Y \) be an integral homology 3-sphere. Let \( \theta \) be the distinguished trivial connection on \( Y \times \text{SO}(3) \). We can use \( \theta \) to fix an absolute \( \mathbb{Z}/8 \)-grading on \( I(Y) \) as was done in Floer’s original construction [13]. For a connection \( a \) on \( Y \times \text{SO}(3) \), we set

\[
\text{gr}(a) = -3 - \mu(\theta, a)
\]

and on \( \mathcal{G} \)-classes \( a \) this descends to a function with \( \text{gr}(a) \in \mathbb{Z}/8 \). Note that \( \mathcal{G}_\text{ev} = \mathcal{G} \) in this setting. When \( a \) is irreducible, \( \text{gr}(a) = \mu(a, \theta) \). The trivial connection has \( \text{gr}(\theta) = 0 \). The differential shifts this grading by \(-1\) and the grading descends to the \( \mathbb{Z}/2 \)-grading defined in § 4.6. We write \( t \) for the \( \mathcal{G} \)-class of \( \theta \). Our \( I(Y)_i \) agrees with Donaldson’s \( HF(Y)_i \), in [7]. Note that \( I(Y)_i \) is the same as the cohomology group \( I(Y)^{5-i}_i \). In particular, by the universal coefficients theorem, the vector spaces \( I(Y)_i \otimes \mathbb{Q} \) and \( I(Y)_{5-i} \otimes \mathbb{Q} \) are isomorphic. Our \( I(Y)_i \) is the same as Frøyshov’s \( HF(Y)^{5-i} = HF(Y)_i \) from [15].

9.2. Other boundary maps

From here on, we fix a field \( F \) which has \( \text{char}(F) \neq 2 \) and take all homology with \( F \)-coefficients. With an integral homology 3-sphere \( Y \) fixed, we write \( C_i = C(Y)_i \) and \( I_i = I(Y)_i \). Following [7, 15], we have maps

\[
\delta : C_1 \longrightarrow F, \quad \delta' : F \longrightarrow C_4
\]

defined using the trivial connection. For \( a \in \mathcal{C}^{\text{tr}}(Y) \) with \( \text{gr}(a) \equiv 1 \), we define \( \delta(a) = \# \hat{M}(a, t)_0 \), and for \( b \) with \( \text{gr}(b) \equiv -4 \), we define \( \delta'(1, b) = \# \hat{M}(t, b)_0 \). More precisely, one writes \( F = F \Lambda(t) \) and \( e[A] : \Lambda(t) \rightarrow \Lambda(a) \), and \( \delta = \sum e[A] \) for each \( [A] \in \hat{M}(a, t)_0 \), and so on, as in § 4.1. We will often conflate \( \delta' \) with \( \delta'(1) \in C_4 \). These are chain maps, in the sense that \( \delta \theta = \theta \delta' = 0 \), and we write

\[
\delta : I_1 \longrightarrow F, \quad \delta' : F \longrightarrow I_4
\]

for the induced maps on homology.

We also have maps that record data from the three-dimensional moduli spaces \( \hat{M}(a, b)_3 \),

\[
v : C_i \longrightarrow C_{i+4}.
\]

Our \( v \) is \( 1/2 \) times the \( v \) of Frøyshov, and four times the \( U \) of Donaldson. That is, it is defined, roughly, by evaluating the four-dimensional class \( 2\mu(pt) \) over four-dimensional moduli spaces \( M(a, b)_4 \). We refer the reader to [7, § 7.3.1; 15, § 3.1] for precise definitions of \( v \). We have in
mind the following interpretation. First suppose that $\hat{M}(a,b)_3$ is connected. We obtain a map $h : \hat{M}(a,b)_3 \to SO(3)$ by evaluating the holonomy of a connection along the path from $(\infty, y)$ to $(\infty, y)$ on the cylinder $\mathbb{R} \times Y$. With some modifications (see [7, § 7.3.2]), $\langle v(a), b \rangle = \deg(h)$.

If $\hat{M}(a,b)_3$ has more than one component, then the evaluation is done on each component, and then added together.

The map $v$ is not quite a chain map. As explained in [7, § 7.3.3], when $\text{gr}(a) \equiv 1$ and $\text{gr}(b) \equiv -4$, there are ends of $\hat{M}(a,b)_4$ modeled on $SO(3)$, that is, cylinders $\mathbb{R} \times SO(3)$, one for each pair of instantons in $\hat{M}(a,t)_{0} \times \hat{M}(t,b)_{0}$. Each copy of $SO(3)$ records the choices for gluing parameters. The holonomy at a cross section is captured by the gluing parameter and has degree 1. Accounting for the other usual ends, modeled on $\mathbb{R} \times \hat{M}(a,c)_{3} \times \hat{M}(c,b)_{j}$, where $i = 0$ and $j = 3$, or vice versa, one is led to the relation

$$\partial v - v \partial + \delta \delta = 0;$$

see [7, Proposition 7.8; 15, Theorem 4]. Here $\delta = 0$ in gradings different from $1 \in \mathbb{Z}/8$. In particular, we obtain the maps

$$v : I_{i} \to I_{i+4}, \quad i \neq 0, 1 \mod 8,$$

$$v : I_{0} \to \text{coker}(\delta'), \quad v : \ker(\delta) \to I_{5}.$$

### 9.3. Reduced instanton groups

Frøyshov defined a $\mathbb{Z}/8$-graded group $\hat{I} = \hat{I}(Y)$ by cutting down $I(Y)$ using the maps introduced above. Precisely,

$$\hat{I}_{i} = I_{i}, \quad i \neq 0, 1, 4, 5 \mod 8,$$

$$\hat{I}_{0} = I_{0}/\left( \sum \text{im}(v^{2k+1}\delta') \right),$$

$$\hat{I}_{4} = I_{4}/\left( \sum \text{im}(v^{2k}\delta') \right),$$

$$\hat{I}_{1} = \bigcap \ker(\delta v^{2k}) \subset I_{1},$$

$$\hat{I}_{5} = \bigcap \ker(\delta v^{2k+1}) \subset I_{5}.$$

Using these groups, Frøyshov defined his $h$-invariant by

$$h(Y) = -\frac{1}{2}(\chi(I(Y)) - \chi(\hat{I}(Y))).$$

This has several nice properties, among them

$$h(\bar{Y}) = -h(Y), \quad h(Y \# Y') = h(Y) + h(Y').$$

It also descends to a homomorphism $h : \Theta^{3}_{H} \to \mathbb{Z}$, where $\Theta^{3}_{H}$ is the integral homology cobordism group. Frøyshov showed that both $I$ and $\hat{I}$ are 4-periodic (recall that we are working with $F$-coefficients). By the chain level relation (9.1), either $\delta$ or $\delta'$ is zero. It follows that, over $\mathbb{Q}$, we can go between $I$ and $\hat{I}$ using only $h$. For example, if $h(Y) = 0$, then $\hat{I} = I$, whereas if $h(Y) > 0$, then $\hat{I}_{i} = I_{i}$ for $i \neq 0, 4$ and $\text{rk}(\hat{I}_{i}) = \text{rk}(I_{i}) - h(Y)$ for $i = 0, 4$.

The maps $v$ above induce maps $\hat{v} : \hat{I}(Y)_{i} \to \hat{I}(Y)_{i+4}$ for each grading $i \in \mathbb{Z}/8$. As mentioned, this is half of Frøyshov’s $u$ mentioned in § 1:

$$\hat{v} = u/2.$$

We have chosen this normalization to avoid writing in certain factors of 2. Frøyshov showed that each $\hat{v}$ is an isomorphism, and that $\hat{v}^2 - 16$ is nilpotent, that is,

$$(\hat{v}^2 - 16)^n = 0.$$
for some \( n > 0 \). If \( Y \) is admissible and \( b_1(Y) > 0 \), then there is no trivial connection to work with, and the maps \( v : C_i \to C_{i+4} \) are indeed chain maps, inducing maps \( \hat{v} : I(Y)_i \to I(Y)_{i+4} \) for each grading \( i \) (here we arbitrarily fix an absolute grading). Again, each \( \hat{v} \) is half of Frøyshov’s \( u \), is an isomorphism, and \( \hat{v}^2 - 16 \) is nilpotent. The hat notation in this case is used only for uniformity.

9.4. Connected sums

In this section, we recall the connected sum theorem of Fukaya [17], reviewing the proof exposited by Donaldson in [7, §7.4]. This problem was also considered in [27]. In the following sections, we will adapt the proof to the settings of interest to us. Let \( Y_1 \) and \( Y_2 \) be integral homology 3-spheres. For \( i = 1, 2 \), write \( C(i) = C(Y_i) \) and \( \partial(i) \) for the corresponding differentials, and \( \delta(i), \delta'(i), v(i) \) for the relevant boundary maps. For a graded \( F \)-module \( A \) define the shifted module \( A[n] \) by \( A[n]_i = A_{i-n} \). We define a chain complex

\[
C = (C(1) \otimes C(2)) \oplus (C(1)[3] \otimes C(2)) \oplus (C(1) \otimes F) \oplus (F \otimes C(2))
\]

\[
\partial = \begin{pmatrix}
\delta_{(12)} & 0 & 0 & 0 \\
\partial_{(12)} & -1 \otimes \delta'_{(2)} & \delta'_{(1)} \otimes 1 & 0 \\
-1 \otimes \delta_{(2)} & \partial_{(1)} \otimes 1 & 0 & 0 \\
\delta_{(1)} \otimes 1 & 0 & 0 & \epsilon \otimes \partial_{(2)}
\end{pmatrix},
\]

where \( \delta_{(12)} = \partial_{(1)} \otimes 1 + \epsilon \otimes \partial_{(2)}, \quad \partial_{(12)} = \partial_{(1)} \otimes 1 + \partial_{(2)} \otimes 1 \) and \( \epsilon \) is equal, in grading \( k \), to \((-1)^k \) times the identity map on \( C(1) \).

**THEOREM 9.1 (Fukaya).** As \( \mathbb{Z}/8 \)-graded \( F \)-modules, \( I(Y_1 \# Y_2) \simeq H_*(C, \partial) \).

For example, let \( Y \) be the Poincaré homology 3-sphere \( \Sigma(2, 3, 5) \). The reader can verify that

\[
I(Y \# Y) \simeq F_1^2 \oplus F_5^2, \quad I(Y \# \overline{Y}) = 0
\]

using that \( C(Y) = F_1 \oplus F_5 \) and \( \delta, v \) are isomorphisms. Recall that subscripts indicate gradings. These examples appear in [17]. Note that, generally, the \( \delta, \delta', \epsilon \) maps for \( \overline{Y} \) are the duals of the maps \( \delta', \delta, \epsilon \) for \( Y \), respectively.

We now review the proof that appears in [7, §7.4]. We mention at the outset that to avoid certain factors of 2 that appear in the composition law (since we will glue along a disconnected 3-manifold), we enlarge the gauge transformation group when necessary, as in [22, §5.1]. Let \( C' = C(Y_1 \# Y_2) \) and \( \delta' \) be its differential. Let \( X : Y_1 \sqcup Y_2 \to Y_1 \# Y_2 \) be the cobordism which is \( ([0, 1] \times Y_1) \cup ([0, 1] \times Y_2) \), where the boundary sum is taken near 1, and let \( W : Y_1 \# Y_2 \to Y_1 \sqcup Y_2 \) be the corresponding cobordism when the boundary sum is taken near 0 (see Figure 18). We define chain maps

\[
m_X : C \to C', \quad m_W : C' \to C
\]

![Figure 18 (colour online). The cobordism \( W : Y_1 \# Y_2 \to Y_1 \sqcup Y_2 \) and its reverse, \( X \).](image_url)
as follows. The map \( m_X \) is given by four components:

\[
\begin{align*}
\nu_X & : C(1) \otimes C(2) \longrightarrow C', \\
i_X & : C(1)[3] \otimes C(2) \longrightarrow C', \\
\delta'_{X}(2) & : C(1) \otimes F \longrightarrow C', \\
\delta'_{X}(1) & : F \otimes C(2) \longrightarrow C'.
\end{align*}
\]

In the following, \( a \in \mathfrak{c}^{\text{irr}}(Y_1) \), \( b \in \mathfrak{c}^{\text{irr}}(Y_2) \) and \( c \in \mathfrak{c}^{\text{irr}}(Y_1 \# Y_2) \). The map \( i_X \) counts zero-dimensional moduli spaces \( M(a, b, X, c)_{0} \). The map \( \nu_X \) evaluates the holonomy of three-dimensional moduli spaces \( M(a, b, X, c)_{3} \) along a curve \( \gamma_X \) running from \( Y_1 \) to \( Y_2 \) on the incoming end of \( X \). The map \( \delta'_{X}(2) \) counts zero-dimensional moduli spaces \( M(a, t, X, c)_{0} \) where \( t \) is a trivial connection class on \( Y_2 \), and \( \delta'_{X}(1) \) is defined similarly, with \( t \) on \( Y_1 \). Now, \( m_X \) is a chain map because of the following relations. First,

\[
i_X \partial(12) = \partial' i_X
\]

is the usual relation for the map involving only irreducibles. Second,

\[
i_X \nu(12) + \nu_X \partial(12) + \delta'_{X}(1)(\delta(1) \otimes 1) - \delta'_{X}(2)(1 \otimes \delta(2)) = \partial' \nu_X
\]

(9.2) records how the holonomy interacts with the ends of a four-dimensional moduli space \( M(a, b, X, c)_{4} \). This is essentially [15, Theorem 6] (see Figure 19). Third, the relation

\[
i_X (\delta'_{X}(1) \otimes 1) + \delta'_{X}(1)(\epsilon \otimes \partial(2)) = \partial' \delta'_{X}(1)
\]

and its analog with indices swapped record the ends of a one-dimensional moduli space \( M(t, b, X, c)_{1} \), where \( t \) is the trivial connection class on \( Y_1 \). This is a variation of [15, Lemma 1].

The map \( m_W \) is defined similarly, this time with components

\[
\begin{align*}
i_W & : C' \longrightarrow C(1) \otimes C(2), \\
\nu_W & : C' \longrightarrow C(1)[3] \otimes C(2), \\
\delta_W(2) & : C' \longrightarrow C(1) \otimes F, \\
\delta_W(1) & : C' \longrightarrow F \otimes C(2).
\end{align*}
\]

\[\begin{align*}
i_X(\nu(1) \otimes 1) & \quad i_X(1 \otimes \nu(2)) & \quad \nu_X(\delta(1) \otimes 1) & \quad \nu_X(1 \otimes \partial(2))
\end{align*}\]

\[\begin{align*}
\partial' \nu_X & \quad \delta'_{X}(1)(\delta(1) \otimes 1) & \quad \delta'_{X}(2)(1 \otimes \delta(3))
\end{align*}\]

Figure 19 (colour online). A representation of the terms appearing in (9.2). The pieces represent counts of isolated instantons, unless there is a curve present in the interior, indicating a contribution from a \( \nu \)-map. All limiting connections are irreducible, except in the last two diagrams, where trivial limits \( \theta \) are present. The first two diagrams make up \( i_X \nu(12) \) and the second two make up \( \nu_X \partial(12) \).
Now, we argue that \( m_X \) and \( m_W \) are chain homotopy inverse to one another. First consider \( m_X m_W \). We have
\[
m_X m_W = v_X i_W + i_X m_W + \delta_X^{(2)} \delta_W^{(2)} + \delta_X^{(1)} \delta_W^{(1)}.
\]
We claim that \( m_X m_W \) is chain homotopic to the map \( m(Z, \gamma) : C_\gamma \to C'_\gamma \) obtained by evaluating \( 2\mu(\gamma) \) on the composite \( Z = X \circ W \). This is the same as the map defined by taking the degrees of modified holonomy maps \( M(a, Z, d)_3 \to SO(3) \) along \( \gamma \); see [8, §5.1.2]. The chain homotopy is obtained by stretching the middle copies of \( Y_1 \) and \( Y_2 \). The three-dimensional space \( M(a, Z, d)_3 \) where \( a, d \) are irreducible has four components after stretching:
\[
\begin{align*}
M(a, X, b, c)_0 \times M(b, c, W, d)_3, \\
M(a, X, b, c)_3 \times M(b, c, W, d)_0, \\
M(a, X, b, t)_0 \times SO(3) \times M(b, t, W, d)_0, \\
M(a, X, t, c)_0 \times SO(3) \times M(t, c, W, d)_0.
\end{align*}
\]
As in (9.1), in the last two cases the holonomy is captured by the gluing space \( SO(3) \). The four components correspond, in order, to the four components of \( m_X m_W \) above. In this way, the chain homotopy from \( m_X m_W \) to \( m(Z, \gamma) \) may be defined as a map using the one-dimensional metric family that simultaneously stretches along \( Y_1, Y_2 \).

The next step is to use a surgery property, interesting in its own right, due to Donaldson. We state it in a form convenient for our purposes. Let \( X \) be a \( SO(3) \)-bundle over a cobordism which restricts to admissible bundles over its boundary components. Let \( \gamma \) be a loop in the interior of the base of \( X \). Let \( X_\gamma \) be the bundle obtained by cutting out a neighborhood \( S^1 \times D^3 \times SO(3) \) lying over \( \gamma \) and gluing back in a copy of \( D^2 \times S^2 \times SO(3) \). Denote by \( m(X, \gamma) : C(\gamma_1) \to C(\gamma_2) \) the map obtained by evaluating \( \mu(\gamma) \) on three-dimensional moduli spaces \( M(a, X, b)_3 \).

**Theorem 9.2** (see [7, Theorem 7.16]). The map \( m(X, \gamma) \) is chain homotopic to \( m(X_\gamma) \).

In our situation, observe that the surgered manifold \( Z_\gamma \) is the product \([0, 1] \times (Y_1 \# Y_2)\). It follows that \( m_X m_W \) is chain homotopic to the identity.

Now consider \( m_W m_X \). This has sixteen components
\[
i_W v_X, \quad v_W i_X, \quad \delta_W^{(1)} \delta_X^{(1)}, \ldots.
\]
It is chain homotopic to a map \( f \) that counts similar data on the cobordism \( W \circ X \) with metric stretched very long along the internal connected sum portion between \([0, 1] \times Y_1 \) and \([0, 1] \times Y_2 \). The map \( f \) has components corresponding to the components of \( m_W m_X \), but most of them vanish. For instance, the seven components of \( f \) corresponding to
\[
i_W i_X, \quad \delta_W^{(i)} i_X, \quad i_W \delta_X^{(i)}, \quad \delta_W^{(i)} \delta_X^{(i)} \quad (i \neq j)
\]
all vanish by index arguments. Each counts instantons \( A \) with \( \mu(A) = 0 \) obtained by gluing an instanton \( A_1 \) on \( \mathbb{R} \times Y_1 \) to an instanton \( A_2 \) over \( \mathbb{R} \times Y_2 \) along a 3-sphere. For \( i = 1, 2 \) at least one of the limits on \( \mathbb{R} \times Y_i \) is irreducible. Thus both \( A_1, A_2 \) are irreducible. It follows from \( 0 = \mu(A) = \mu(A_1) + \mu(A_2) + 3 \) and \( \mu(A_i) \geq 0 \) that no such \( A \) exist. Similarly, the four components of \( f \) corresponding to
\[
\delta_W^{(i)} v_X, \quad v_W \delta_X^{(i)}
\]
are zero. These components require three-dimensional moduli spaces. However, with the neck stretched, the relevant three-dimensional moduli spaces are \( M(a, b)_0 \times SO(3) \times M(c, d)_0 \) where one of \( a, b, c, d \) is the trivial class \( t \) and \( a, b \) are connection classes on \( Y_1 \) and \( c, d \) on \( Y_2 \).
But $M(\alpha, t)|_0$ is empty for any irreducible $\alpha$. Next, the four components of $f$ corresponding to
\begin{align*}
i_W i_X, & \quad v_W i_X, \\ \delta_W \delta_X, & \quad \delta_W \delta_X(i)
\end{align*}
are identity maps. For instance, the first one uses three-dimensional spaces modeled on $M(\alpha, b)|_0 \times SO(3) \times M(c, d)|_0$ from gluing; the holonomy map $v_X$ captures the gluing parameter just as in (9.1), leaving us to count $M(\alpha, b)|_0 \times M(c, d)|_1$. Of course, $M(\alpha, b)|_0$ forces $\alpha = b$ and has one translation invariant irreducible flat connection. Finally, we are left with one component of $f$ corresponding to
\[v_W v_X,\]
which may be non-zero. However, we know that $f$ is the identity plus this off-diagonal term, and thus induces an isomorphism on homology. So $m_W m_X$ also induces an isomorphism on homology. Because $m_X m_W$ induces the identity on $I(Y_1 \# Y_2)$, so does $m_W m_X$. This completes the proof.

9.5. Connected sum with non-trivial bundles

In this section, we state two variants of the connected sum theorem, when one or both of $Y_1$ and $Y_2$ are replaced by a non-trivial admissible bundle. We then explain how the proof above adapts to these cases. These are simpler than the above, having fewer trivial connections to deal with.

We first consider the case where $\mathcal{Y}_1$ is trivial and $Y_1$ is an integral homology 3-sphere, but $\mathcal{Y}_2$ is non-trivial and admissible. Let $C^{(1)} = C(\mathcal{Y}_1)$ with maps $\partial^{(1)}, \delta, \delta', v^{(1)}$. Let $C^{(2)} = C(\mathcal{Y}_2)$ with maps $\partial^{(2)}, v^{(2)}$. Define
\[
C = (C^{(1)} \otimes C^{(2)}) \oplus (C^{(1)}[3] \otimes C^{(2)}) \oplus (F \otimes C^{(2)}),
\]
\[
\partial = \begin{pmatrix}
\delta^{(12)} & 0 & 0 \\
\partial^{(12)} & -\partial^{(12)} & \delta' \otimes 1 \\
\delta \otimes 1 & 0 & \epsilon \otimes \partial^{(2)}
\end{pmatrix}
\]
with notation as before.

**Theorem 9.3.** Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be admissible bundles, with $\mathcal{Y}_1$ trivial and $\mathcal{Y}_2$ non-trivial. As $\mathbb{Z}/8$-graded $F$-modules, $I(\mathcal{Y}_1 \# \mathcal{Y}_2) \simeq H_*(C, \partial)$.

As before, we let $C' = C(\mathcal{Y}_1 \# \mathcal{Y}_2)$ and let $\partial'$ be its differential. Let
\[
X : \mathcal{Y}_1 \cup \mathcal{Y}_2 \longrightarrow \mathcal{Y}_1 \# \mathcal{Y}_2
\]
be the cobordism bundle obtained from a boundary sum between $[0, 1] \times \mathcal{Y}_1$ and $[0, 1] \times \mathcal{Y}_2$ near 1, making some inessential choices in gluing the bundles. Let $\mathcal{W}$ be the cobordism in the reverse direction obtained from the boundary sum near 0. We define chain maps
\[
m_X : C \longrightarrow C', \quad m_W : C' \longrightarrow C.
\]
The map $m_X$ is given by three components:
\[
v_X : C^{(1)} \otimes C^{(2)} \longrightarrow C', \\
i_X : C^{(1)}[3] \otimes C^{(2)} \longrightarrow C', \\
\delta_X : F \otimes C^{(2)} \longrightarrow C'.
\]
The map $v_X$ counts instantons in zero-dimensional moduli spaces on $X$ with all limits irreducible; the map $i_X$ evaluates holonomy along a path $\gamma_X$ from $Y_1$ to $Y_2$ on three-dimensional moduli spaces with irreducible limits on $X$; the map $\delta_X$ counts zero-dimensional moduli spaces over $X$ where the limit over $Y_1$ is trivial. The map $m_X$ is a chain map because of the following. First, we
have the usual relation for the map involving only irreducibles, \( i_X \partial(12) = \partial' i_X \). Second,
\[
i_X v(12) + v_X \partial(12) + \delta_X^2 (\delta \otimes 1) = \partial' v_X.
\]
These relations are the same as before, except that all terms involving a trivial connection on \( Y_2 \) do not arise. In particular, all diagrams in Figure 19 are relevant except the last. Third, we have the relation
\[
i_X (\delta' \otimes 1) = \partial' \delta_X,
\]
which again is the same as before but with the term involving a trivial connection on \( Y_2 \) absent. The map \( m_W \) is defined similarly to \( m_W \), with the component involving the trivial connection on \( Y_2 \) thrown out.

We proceed as before. The first composite is \( m_X m_W = i_X v_W + v_X i_W \), and this is chain homotopic to \( m(Z, \gamma) \) where \( Z = X \circ W \) and \( \gamma = \gamma_W \cup \gamma_X \) by stretching along \( Y_1, Y_2 \). The surgery Theorem 9.2 applies, so \( m_W m_X \) is chain homotopic to \( m(Z, \gamma) \), which is the identity. The other composite \( m_W m_X \) now has only nine components. We stretch the neck as before, so that \( m_W m_X \) is chain homotopic to a map \( f \); the terms of \( f \) corresponding to the nine components all vanish except the diagonal ones, which are the identity, and possibly \( v_W v_X \). As before, \( m_W \) and \( m_X \) are chain homotopy inverses, and the proof follows through.

Next, we consider the case where both \( Y_1 \) and \( Y_2 \) are non-trivial. For \( i = 1, 2 \), let \( C(i) = C(Y_i) \) with maps \( \partial(i), v(i) \). Define
\[
C = (C(1) \otimes C(2)) \oplus (C(1)[3] \otimes C(2)),
\]
\[
\partial = \begin{pmatrix} \partial(12) & 0 \\ v(12) & -\partial(12) \end{pmatrix}
\]
with notation as before.

**Theorem 9.4.** Let \( Y_1 \) and \( Y_2 \) be non-trivial admissible bundles. As \( \mathbb{Z}/8 \)-graded \( F \)-modules, \( I(Y_1 \# Y_2) \simeq H_*(C, \partial) \).

This is the simplest case of all. Let \( C' = C(Y_1 \# Y_2) \) with differential \( \partial' \). Let
\[
X : Y_1 \cup Y_2 \to Y_1 \# Y_2
\]
be the cobordism bundle obtained from a boundary sum between \([0, 1] \times Y_1\) and \([0, 1] \times Y_2\) near 1, making some inessential gluing choices. Let \( W \) be the cobordism in the reverse direction obtained from the boundary sum near 0. As before, we can define chain maps \( m_X \) and \( m_W \). Here \( m_X \) is given by two components, \( v_X : C(1) \otimes C(2) \to C' \) and \( i_X : C(1)[3] \otimes C(2) \to C' \). As usual, \( i_X \) counts zero-dimensional moduli spaces on \( X \) with all limits irreducible, and \( v_X \) takes holonomy along a path \( \gamma_X \) from \( Y_1 \) to \( Y_2 \) on three-dimensional moduli spaces with irreducible limits on \( X \). The relations that make \( m_X \) a chain map are just \( i_X \partial(12) = \partial' i_X \) and \( i_X v(12) + v_X \partial(12) = \partial' v_X \), and follow from the previous cases, with the terms involving trivial connections thrown out. This latter relation is represented by Figure 19 with the last two diagrams omitted. The rest of the argument is the same as before.

### 9.6. Framed homology for integral homology 3-spheres

Now we apply the above results to compute \( I^*(Y) \) with \( F \)-coefficients for an integral homology 3-sphere \( Y \), proving Theorem 1.3. Recall that \( F \) is a field with \( \text{char}(F) \neq 2 \). Let \( T^3 \) be a non-trivial bundle over \( T^3 \) geometrically represented by an \( S^1 \)-factor of \( T^3 \). Let \( V = F_0 \oplus F_4 \) be the chain complex that computes \( I(T^3) \). Write \( \tau : V \to V \) for the \( \nu \)-map on \( T^3 \), with which our normalization may be written as the degree 4 involution that multiplies by 4. Write \( C = C(Y) \)
Theorem 9.3 tells us that

\[ H\ast(C, \partial) \simeq I(Y \# T^3) = I\#(Y)[4] \oplus I\#(Y), \]

where \( Y \) is the trivial bundle over \( Y \). Consider the filtration on \((C, \partial)\) given by

\[ 0 \subset C[3] \otimes V \subset (C[3] \otimes V) \oplus (F \otimes V) \subset C. \]

This induces a spectral sequence with \( E^2 \)-page

\[ (\ker(\delta) \otimes V) \oplus (\ker(\delta')/\im(\delta) \otimes V) \oplus (\coker(\delta')[3] \otimes V) \]

with the only non-zero component of the differential coming from

\[ \phi := \nu \otimes 1 + 1 \otimes \tau : \ker(\delta) \otimes V \longrightarrow \coker(\delta')[3] \otimes V. \]

We are writing all modules as \( \mathbb{Z}/8 \)-graded modules; for example, \( \ker(\delta)_i = I(Y)_i \) when \( i \neq 1 \), and similarly \( \coker(\delta')_i = I(Y)_i \) when \( i \neq 4 \). Also note that the component \( \ker(\delta')/\im(\delta) \) is supported in grading 0 and is either \( F \) or 0. Write

\[ \phi_i = (\nu \oplus \nu + \sigma)_i : \ker(\delta)_i \oplus \ker(\delta)_{i+4} \longrightarrow \coker(\delta')_{i+4} \oplus \coker(\delta')_i, \]

where \( \sigma(x, y) = (4y, 4x) \) is the degree 4 involution induced by \( \tau \). Then

\[ I\#(Y)_0 \simeq \ker(\phi_0) \oplus \coker(\phi_1) \oplus \ker(\delta')/\im(\delta), \]

\[ I\#(Y)_i \simeq \ker(\phi_i) \oplus \coker(\phi_{i+1}), \quad i = 1, 2, 3. \]

Recall that \( \hat{v} \) is a degree 4 automorphism of \( \hat{I}(Y) \). We claim that

\[ \ker(\phi_0) \simeq \ker(\hat{v}^2 - 16)_0 \oplus \im(\delta'), \]

\[ \ker(\phi_i) \simeq \ker(\hat{v}^2 - 16)_i, \quad i = 1, 2, 3. \]

To prove Theorem 1.3, it suffices to consider the case in which \( h(Y) \leq 0 \), so that \( \delta' = 0 \) and \( \hat{I}_i = I_i \) for \( i \neq 1, 5 \). For if \( h(Y) > 0 \), then the theorem applies for \( Y \), which has \( h(Y) = -h(Y) < 0 \), and the \( F[\nu] \)-module \( \hat{I} \) dualizes upon orientation reversal. Thus, for \( i = 0, 2, 3 \), we have

\[ \phi_i : I_i \oplus I_{i+4} \longrightarrow I_{i+4} \oplus I_i, \]

and each \( \hat{I}_i = I_i \) with \( v = \hat{v} \) an isomorphism. The isomorphisms \( \ker(\phi_i) \simeq \ker(\hat{v}^2 - 16)_i \) for \( i = 0, 2, 3 \) are given by \( (x, y) \mapsto x \) with inverse \( x \mapsto (x, -v^{-1}x/4) \). Next, consider

\[ \phi_1 : \ker(\delta)_1 \oplus I_5 \longrightarrow I_1 \oplus I_5. \]

We have an isomorphism \( \ker(\phi_1) \simeq \ker(v^2 - 16)_1 \subset \ker(\delta)_1 \) given by \( (x, y) \mapsto x \) with inverse \( x \mapsto (x, -v^{-1}x/4) \), using the isomorphism \( v : I_5 \rightarrow I_1 \). The natural inclusion \( \hat{I}_1 \rightarrow I_1 \) induces an injection \( \ker(\hat{v}^2 - 16)_1 \rightarrow \ker(v^2 - 16)_1 \). Note here that \( v \) is just the restriction of \( v \) to \( \hat{I} \). We show that this is surjective and hence an isomorphism. Given \( x \in I_1 \) with \( \delta x = 0 \) and \( v^2 x = 16x \), we must show \( x \in \ker(\delta v^{2k}) \) for all \( k > 0 \). But \( v^2 x = 16x \) implies \( \delta v^{2k} x = 4^k \delta x = 0 \). Having computed \( \ker(\phi_1) \), dimension counting then yields

\[ \coker(\phi_1) \simeq \ker(\hat{v}^2 - 16)_1 \oplus \im(\delta), \]

\[ \coker(\phi_1) \simeq \ker(\hat{v}^2 - 16)_i, \quad i = 0, 2, 3. \]
Using in our case that \( \dim(\text{im}(\delta)) + \dim(\ker(\delta')/\text{im}(\delta)) = 1 \), we deduce that

\[
I^\#(Y) \simeq \ker(\hat{\nu}^2 - 16) \otimes H_*(S^3) \oplus H_*(\text{pt.}),
\]

where it is understood that \( \hat{\nu}^2 - 16 \) is acting on \( \bigoplus_{i=0}^3 \hat{I}_i \). This proves the first part of Theorem 1.3.

### 9.7. Framed homology for non-trivial bundles

Let \( \mathcal{Y} \) be a non-trivial admissible bundle over \( Y \) geometrically represented by \( \lambda \subset Y \). We now write \( I^\#(Y; \lambda) \) in terms of \( I(\mathcal{Y}) \). Let \( V = F_0 \oplus F_1 \) and \( \tau : V \to V \) be as before. Write \( C = C(\mathcal{Y}) \) and \( \partial, v \) for its maps, and set

\[
C = (C \otimes V) \oplus (C[3] \otimes V),
\]

\[
\partial = \begin{pmatrix}
\partial \otimes 1 & 0 \\
v \otimes 1 & 1 \otimes \tau & -\partial \otimes 1
\end{pmatrix}.
\]

Theorem 9.4 tells us that

\[
H_*(C, \partial) \simeq I(\mathcal{Y} \# \mathbb{T}^3) = I^\#(Y; \lambda)[4] \oplus I^\#(Y; \lambda).
\]

This is a degeneration of the computation in §9.6. We want the kernel and cokernel of

\[
\hat{\nu} \otimes \hat{\nu} + \sigma : I[4] \oplus I \longrightarrow I \oplus I[4].
\]

The kernel is isomorphic to \( \ker(\hat{\nu}^2 - 16) \) by the assignment \((x, y) \mapsto x\), inverse to \( x \mapsto (x, -\hat{\nu}x/4) \). The cokernel is of course the same, and we obtain

\[
I^\#(Y; \lambda) \simeq \ker(\hat{\nu}^2 - 16) \otimes H_*(S^4)
\]  
(9.3)

as relatively \( \mathbb{Z}/4 \)-graded \( F \)-modules, where it is understood that \( \hat{\nu}^2 - 16 \) is acting on four consecutively graded summands of \( I(\mathcal{Y}) \). This proves the second part of Theorem 1.3.

### 9.8. Degenerations

In this section, we briefly consider a few cases in which the isomorphisms obtained degenerate into stricter relations between framed instanton homology and Floer’s instanton homology, and in particular, we prove Corollaries 1.5–1.7.

First, suppose that \( \mathcal{Y} \) is a non-trivial admissible bundle over \( Y \) geometrically represented by \( \lambda \). As mentioned in [15, §6], when there exists a surface \( \Sigma \subset Y \) of genus at most 2 with \( \mathcal{Y} \mid \Sigma \) non-trivial, then \( u^2 = 64 \) on \( I(\mathcal{Y}) \). In this case, (9.3) yields

\[
I^\#(Y; \lambda) \otimes H_*(S^4) \simeq I(\mathcal{Y}) \otimes H_*(S^3)
\]

as relatively \( \mathbb{Z}/4 \)-graded \( F \)-modules. The term \( H_*(S^4) \) appears because we take the full \( \mathbb{Z}/8 \)-graded group \( I(\mathcal{Y}) \) on the right, instead of four consecutive summands as before. Now suppose that \( K \) is a knot in \( S^3 \) of genus at most 2. Denote the result of \( r \)-surgery on \( K \) by \( Y_r \). For \( r = 1 \), the exact triangle, combined with passing to the reduced groups \( \hat{I} \), yields a map

\[
I(\mathcal{Y}_0) \longrightarrow \hat{I}(Y_1),
\]  
(9.4)

where \( \mathcal{Y}_0 \) is a non-trivial bundle over \( Y_0 \). This map is an surjection. This follows from Frøyshov’s observation after [15, Theorem 10] that, in this situation, when passing to the reduced groups \( \hat{I} \), the surgery triangle retains exactness at the homology 3-spheres (but not \( I(\mathcal{Y}_0) \)). If \( r = -1 \), then we similarly obtain a injection \( \hat{I}(Y_{-1}) \to I(\mathcal{Y}_0) \). In either case, we can form a surface \( \Sigma \) in \( Y_0 \) by capping off a Seifert surface for \( K \) of genus at most 2 by a meridional disk for the new framed knot in \( Y_0 \). The bundle \( \mathcal{Y}_0 \mid \Sigma \) is non-trivial, and so \( u^2 = 64 \) on \( I(\mathcal{Y}_0) \). Thus \( u^2 = 64 \) on \( \hat{I}(Y_{\pm 1}) \). With Theorem 1.3, this implies Corollary 1.5.
Now we consider the proofs of Corollaries 1.6 and 1.7. By the remarks in the introduction of [15], we have
\[ h(\Sigma(2, 3, 6k + 1)) = 0, \quad h(\Sigma(2, 3, 6k - 1)) > 0. \]
Fintushel and Stern [11] compute
\[ I(\Sigma(2, 3, 6k + 1)) = F_1^{[k/2]} \oplus F_3^{[k/2]} \oplus F_5^{[k/2]} \oplus F_7^{[k/2]}, \]
from which Corollary 1.6 follows, as \( \Sigma(2, 3, 6k + 1) \) is +1-surgery on a twist knot with \( k \) full twists, a knot of genus 1. On the other hand, \( \Sigma(2, 3, 6k - 1) \) is 1-surgery on a twist knot \( K \) with \( 2k - 1 \) half-twists. Since \( K \) is also genus 1, the inequality of [16, Corollary 1] yields \( h(\Sigma(2, 3, 6k - 1)) = 1 \). Combined with Fintushel and Stern’s computation from [11], we obtain
\[ \tilde{I}(\Sigma(2, 3, 6k - 1)) = F_1^{[k/2]-1} \oplus F_3^{[k/2]} \oplus F_5^{[k/2]-1} \oplus F_7^{[k/2]} \]
Now Corollary 1.7 follows from Corollary 1.5.

10. The Euler characteristic

In this section, we prove Corollary 1.4, which computes the Euler characteristic of \( I^\#(Y; \lambda) \) where \( Y \) is any closed, oriented 3-manifold and \( \lambda \) is any unoriented closed 1-manifold in \( Y \). The claim is that \( \chi(I^\#(Y; \lambda)) = |H_1(Y; \mathbb{Z})| \), where the expression on the right-hand side means the cardinality of \( H_1(Y; \mathbb{Z}) \) if it is finite, and is zero otherwise.

**Proof of Corollary 1.4.** We make the abbreviations
\[ i(Y; \lambda) = \chi(I^\#(Y; \lambda)), \quad |Y| = |H_1(Y; \mathbb{Z})|. \]
Note that § 7.7 implies the multiplicativity
\[ i(Y; \lambda)i(Y'; \lambda') = i(Y^\#; \lambda \cup \lambda'). \quad (10.1) \]
We also note that \( i(Y) = 1 \) when \( |Y| = 1 \) by Theorem 1.3.

Next, we claim that the result is true for rational homology 3-spheres \( Y \) that are obtained by integral surgery on an algebraically split link. That is, \( Y \) is the result of \((p_1, \ldots, p_k)\)-surgery on a framed link \( L = L_1 \cup \cdots \cup L_k \) in \( S^3 \) whose pairwise linking numbers vanish. Thus \( |Y| = |p_1 \cdots p_k| \). Assume that the result is true for \( |Y| < n \). Since the case \( |Y| = 1 \) has already been established, we may assume that \( Y \) is not an integral homology 3-sphere, and (by reordering) that \( |p_1| > 1 \). Let \( Z_p \) be \((p, p_2, \ldots, p_k)\)-surgery on \( L \). We have an exact sequence
\[ \cdots \rightarrow I^\#(Z_\infty; \lambda) \rightarrow I^\#(Z_{p_1-1}; \lambda \cup \mu) \rightarrow I^\#(Z_{p_1}; \lambda) \rightarrow I^\#(Z_{p_1}; \lambda) \rightarrow \cdots. \]
The degree of the first map is odd, while the other two are even; cf. [20, § 42.3]. Observing that \( Z_{p_1} = Y \), we obtain
\[ i(Y; \lambda) = i(Z_{p_1-1}; \lambda \cup \mu) + i(Z_\infty; \lambda). \]
By the induction hypothesis, the right-hand side is
\[ |(p_1 - 1)p_2 \cdots p_k| + |p_2 \cdots p_k| = n, \]
establishing the result for all rational homology 3-spheres which are obtained by integral surgeries on algebraically split links.

We now prove the result for all rational homology 3-spheres \( Y \). We use the fact that, for any such 3-manifold, there is a framed algebraically split link \( L \subset S^3 \) such that some integral surgery on \( L \) yields \( Z = Y^\#Y' \), where \( Y' \) is a connected sum of lens spaces of type \( L(p, 1) \); cf. [32, Corollary 2.5]. Since \( Y' \) is integral surgery on an algebraically split link, \( i(Y') = |Y'|. \)
Then (10.1) yields
\[ i(Y; \lambda) = i(Z; \lambda)/i(Y') = |Z|/|Y'| = |Y|, \]
establishing the result for all rational homology 3-spheres.

Finally, we consider the case in which \( b_1(Y) > 0 \). We can always find \( Z \) and a framed knot \( K \subset Z \) such that \( Y \) is 0-surgery on \( K \) and \( b_1(Z) + 1 = b_1(Y) \). We have an exact sequence
\[
\cdots \to I^\#(Y; \lambda) \to I^\#(Z_1; \lambda \cup \mu) \to I^\#(Z; \lambda) \to I^\#(Y; \lambda) \cdots,
\]
where \( Z_1 \) is the result of 1-surgery on \( K \). The degree of the first two maps are even, while the third is odd; again cf. [20, § 42.3]. Thus
\[ i(Y; \lambda) = i(Z_1; \lambda \cup \mu) - i(Z; \lambda). \]
The proof is again by induction. If \( b_1(Y) = 1 \), then the right-hand side is known, because \( Z_1 \) and \( Z \) are rational homology spheres; we have \( |Z_1| = |Z| \), so the right-hand side vanishes. Now suppose that the result has been proved for \( 0 < b_1 < n \). If \( b_1(Y) = n \), then both terms on the right-hand side vanish by the induction hypothesis, and the proof is complete.

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