Multifractal Measures Characterized by the Iterative Map with Two Control Parameters

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Abstract

A one-dimensional iterative map with two control parameters, i.e. the Kim-Kong map, is proposed. Our purpose is to investigate the characteristic properties of this map, and to discuss numerically the multifractal behavior of the normalized first passage time. Especially, based on the Monte Carlo simulation, the normalized first passage time to arrive at the absorbing barrier after starting from an arbitrary site is mainly obtained in the presence of both absorption and reflection on a two-dimensional Sierpinski gasket. We also discuss the multifractal spectra of the normalized first passage time, and the numerical result of the Kim-Kong model presented will be compared with that of the Sinai and logistic models.

1 Introduction

Recently, increasing interest has been paid to the behavior of the disorder system in a variety of contexts in condensed matter physics. The stochastic process for the anomalous diffusion on fractal structure and Lévy flights has extensively been the subject of several literatures. It provides a dynamic framework for a precise connection between fractional diffusion equations and

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fractal walks,\textsuperscript{9,10} while the transport phenomena for the motion of a test particle have largely been extended to the reaction kinetics\textsuperscript{5,11} and the strange kinetics.\textsuperscript{12}

One of the well-known problems associated mainly with the theory of the random walk is the mean first passage time that defines the average time arriving at the absorbing barrier for the first time after a walker initially starts from an arbitrary lattice point. Particularly, this theoretical framework has extensively been applied in the Sinai model related intimately to the random barrier with the absorbing and reflecting barriers. In previous work, the Sinai model with asymmetric transition probabilities was studied for the mean and mean square displacements dependent anomalously on time.\textsuperscript{2−5} The transport process for the mean first passage time has been addressed by Noskowicz et al.\textsuperscript{13} who obtained the upper and lower bounds using the recursion - relation procedure.\textsuperscript{14−17} In a one - dimensional lattice with the periodic boundary condition, Kozak et al.\textsuperscript{18} have been treated with the first passage time, for the specific case of using the logistic map as a pseudorandom number generator.\textsuperscript{19} On the other hand, there have recently been a few heuristical and intuitive investigations exhibited for one - dimensional iterative maps. In this paper we will focus on the so - called Kim - Kong map which takes the form of a new map with two control parameters. Such a map is meant to be a well - defined dissipative map of the interval \((0, 1)\), and it can be applied to present the transition of a test particle from the chaotic sequences of our map. In particular, the Kim - Kong map proposed may be very useful to extensively research the transport of quantum excitations, directed polymer problems, and the electron localization in quantum mechanics.\textsuperscript{20−22} Interestingly, it may be expected that our iterative map will be relevant to a nonlinear map on simple models of spatial - temporal chaos in a nonconserving coupling map lattice.\textsuperscript{23−26}

Since the multifractals is directly related to rigorously analyzed statistical property of the normalized first passage time, the multifractal quantities have been intensively studied in connection with the random walk problem in a one -dimensional lattice with both absorbing and reflecting barriers.\textsuperscript{15,27} In general, multifractal investigations for chaotic disorder systems have essentially discussed the transport phenomena\textsuperscript{28,29} and the random fractal structure with the time - dependent random potential in a diffusive motion.\textsuperscript{30−33} Very recently Kim et al.\textsuperscript{34} argued on multifractal behavior of the normalized first passage time for the case where both absorbing and reflecting barriers exist in
a two-dimensional Sierpinski gasket. To our knowledge, it is of fundamental importance for the multifractals to be dealt with the transport process described by the transition probability such as the Kim-Kong map. In reality, the Sinai, logistic, and Kim-Kong models studied in this paper have not been fully explored up to now, as we will see.

In this study, we present the transport process of a given particle executed on random walk with symmetric and asymmetric transition probabilities on a two-dimensional Sierpinski gasket existing with both absorbing and reflecting barriers. Especially, the distribution of the normalized first passage time arriving at absorbing barrier is investigated primarily on the Kim-Kong model performing a random walk by the transition probability such as the Kim-Kong map. In Sec.2, we discuss, in great detail, the characteristic feature of the Kim-Kong map. The convenient formula of the normalized first passage time and the multifractals are also introduced. In Sec.3, we use the statistical value of the normalized first passage time to relate the generalized dimension and the spectra. The numerical results of the Monte Carlo simulation for the generalized dimension and the spectra are compared with those reported earlier. Finally, we end with some results and conclusions.

2 Kim-Kong map and the multifractal feature for the normalized first passage time

First of all, in this section we explain in detail the characteristic properties of the Kim-Kong map and the multifractal feature of the normalized first passage time. For our case the well-known quantity which defines the motion is the transition probability of a test particle performing the symmetric or asymmetric random walk. For simplicity we assume that transition probability takes the form of a Kim-Kong map. A new map that we consider is expressed in terms of

\[ x(n + 1) = \gamma \exp\left[-\beta (\log x(n))^2\right][1 - \exp\left[-\beta (\log x(n))^2\right]], \quad (1) \]

where \( \beta \) and \( \gamma \) stand for the control parameters. The transition probability can then be considered as chaotic orbits of the Kim-Kong map, i.e. a se-
quence of pseudorandom numbers. However, in next section we will use our map of Eq.(1) as the transition probability, when a particle performs a random walk on a two-dimensional Sierpinski gasket. Specifically Eq.(1) is a well-defined dissipative map with the interval $(0, 1)$, since it generates $x(n+1)$ from $x(n)$. As parameter $\beta$ varies slightly higher than $0.2$ at $\gamma = 3.78$, the maximum value of $x(n+1)$ proceeds from the left to the right side of $x(n)$, as noted in the results plotted in Fig.1. It is as well obtained that the maximum value of $x(n+1)$ exists near $x(n) = 0.5$ for $\beta = 1.5$ and $\gamma = 3.78$, although it is no longer exact. Hence we can describe higher chaotic sequences of iterated points for $\beta = 0.2$ and $\gamma = 3.78$ in Eq.(1), while $x(n+1)$ takes on smaller chaotic values for $\beta = 2.0$ and $\gamma = 3.78$, independent of the initial value of $x(n)$. As another characteristics, we now interpret the bifurcation structure, and discuss numerically the value of two convergence ratios in a one-dimensional Kim-Kong map. When the control parameter $\gamma$ is varied in the range $0 < \gamma < 4$, it leads to the onset of chaotic behavior, and the bifurcation diagram of the Kim-Kong map for a period-$2^n$ point can be shown in Fig. 2. It should be noted that the $\gamma_n$ and $\alpha_n$ sequences converge to the value $\delta = \lim_{n \to \infty} \delta_n = 4.66920$ and $\alpha = \lim_{n \to \infty} \alpha_n = 2.50290$, which are really the same values as the Feigenbaum constants.$^{35,36}$

Next we will assume that a test particle performs a random walk of nearest-neighboring transition on a one-dimensional lattice. Then, using the well-known results of the generating function technique, the mean first passage time $T$ satisfies the following equation:

$$T = \sum_{k=1}^{N-1} \frac{1}{p_k} + \sum_{k=0}^{N-2} \frac{1}{p_k} \sum_{i=k+1}^{N-1} \prod_{j=k+1}^{i} q_j p_j,$$

(2)

where it means that a given particle jumps to right site with $p_j$ or left site with $q_j$ after one step, since it starts at site $j$. The transition probabilities $p_j$ and $q_j$ lead us to have arbitrarily different values at the $j$-th site when a particle jumps one step from a given site to the nearest-neighbor site. The derivation of Eq.(2) can be found in details elsewhere.$^{14,15,19}$ There are $2^N$ realizations of the random walks possible for a chain of size $N$. We can explicitly calculate the mean first passage time of Eq.(2), after enumerating all the realizations. Now let us consider the normalized first passage time $T_n$ defined by

$$T_n = \frac{T_i - T_{\text{min}}}{T_{\text{max}} - T_{\text{min}}}$$

(3)
where $T_i$ is the first passage time required for the $i$-th particle to reach an absorbing barrier starting at an arbitrary site. $T_{max}$ and $T_{min}$ are respectively the maximum and minimum values for the first passage time arriving at the absorbing barrier. From Eq.(3) the normalized first passage time can be taken into account as the rescaled real values on $(0,1)$. However, we shall be interested in this nature employing multifractal formula.

In order to exhibit the multifractal feature for the normalized first passage time, we briefly repeat the definition of multifractals in the following. If we divide the normalized first passage time into a set of $\epsilon$ as $\epsilon \to 0$, then the generalized dimension in the multifractal structure$^{1,15}$ is represented as

$$D_q = \lim_{\epsilon \to 0} \frac{\ln \sum_i n_i p_i^q}{(q-1) \ln \epsilon}.$$  \hspace{1cm} (4)

Here $n_i$ is the number of configurations of the $i$-th particle, and $p_i$ is the probability for the number of configurations of the $i$-th particle arriving at the absorbing barrier. Owing to Eq.(4), the explicit relations between $D_q$ and $f_q$, and between $\alpha_q$ and $D_q$ are respectively given by the following Legendre transform:

$$f_q = q \frac{d}{dq} [(q-1)D_q] - (q-1)D_q$$ \hspace{1cm} (5)

and

$$\alpha_q = \frac{d}{dq} [(q-1)D_q].$$ \hspace{1cm} (6)

In our scheme, we will make use of Eqs.(4) – (6) to find out the multifractal behavior for the normalized first passage time in a two-dimensional Sierpinski gasket, and these mathematical techniques lead us to more general results.

### 3 Result and conclusions

For the investigations of the generalized dimension and the spectrum for the normalized first passage time, we take into account the two-dimensional random walk of a given particle, where the number of sites having a stage $n$ in the $d$-dimensional Sierpinski gasket$^{37}$ takes on only $N_n = (d+1)(1+(d+1)^n)/2$. 


Although the stage \( n \) can be extended to large number in our computer simulation studies, we conveniently restrict ourselves to \( n = 7 \) stages in a two-dimensional Sierpinski gasket. We will assume that an absorbing barrier is at an ended site on the right side, while the reflecting barriers are on all the boundary in Fig. 3. After a particle starts at the initial point \( i_0 = (11111233) \) at \( n = 7 \) stages on a two-dimensional Sierpinski gasket lattice, this absorbs at a site \((33333333)\), where the \( n = 0 \) stage is located at (1), (2), and (3). By using the transition probability of Eq. (1), a particle jumps to the right site with \( p_{1j} + \theta \) or the left site with \( q_{1j} - \theta \), and to the top site with \( p_{2j} - \theta \) or the bottom site with \( q_{2j} + \theta \) in four directions after a particle starts at a site \( j \) on a two-dimensional Sierpinski gasket. The disorder parameter \( \theta \) is the quantity for the biased value described by the direction of a given particle on fractal lattice model, and we have \( p_{1j} + q_{1j} + p_{2j} + q_{2j} = 1 \) by using the normalized condition on two-dimensional lattice model. For this case the random walk of a particle is symmetric for the disorder parameter \( \theta = 0 \), and will be a biased random walk that jumped to the direction of the absorbing barrier for \( 0 < \theta < \frac{1}{4} \).

For the sake of clarity we can obtain the multifractal behavior of the normalized first passage time from our simulation because of the apparent simplicity of our models. Our simulations are performed for \( 3 \times 10^6 \) particles and averaged over \( 10^4 \) configurations in the three models as follows. The first is the Sinai model that for Sinai disorder it has the symmetric transition probability with \( p_{1j} = q_{1j} = p_{2j} = q_{2j} = \frac{1}{4} \) and \( \theta = 0 \), and the two asymmetric probabilities with \( \theta = 0.05 \) and \( \theta = 0.1 \). The second is the logistic model for the transition probability represented in terms of the logistic function, and the numerical simulation of logistic model can be treated in a similar way to that of the Sinai model. In this paper the logistic map is given by \( x(n + 1) = R x(n) [1 - x(n)] \), where \( R = 3.9999 \), and the multifractal quantities of the normalized first passage time we found was also referred to Ref.[34], in detail. Lastly, we concentrate on the Kim-Kong model carrying out efficiently a random walk in the interesting case of a Kim-Kong map for \( \beta = 0.2 \) and \( \gamma = 3.78 \), where the numerical generated sequences of the Kim-Kong map appear to be wildly chaotic, as shown in Fig. 2. We find in the Kim-Kong model that the three asymmetric cutoff values are 0.026145, 0.147489, and 0.492946 with \( \theta = 0.05 \), while we obtain three asymmetric cutoff values of 0.042893, 0.147489, and 0.379262 with \( \theta = 0.1 \), respectively. Here
the asymmetric cutoff value is defined as the quantity used to independently determine the jumping direction of a random walker.

4From now on we estimate numerically the generalized dimension and the spectrum after finding the normalized first passage time from Eq.(3) via the Monte Carlo simulation. The results of these calculations are summarized in Table I. At present, of further interest is the fractal dimension among multifractal quantities. Here the fractal dimension $D_0$ (i.e., the maximum value of $D_q$ or $f_q$) and the scaling exponent $\alpha_{\pm\infty}$, $D_{\pm\infty}$ for $\epsilon = 10^{-2}$ are calculated in the Kim-Kong, logistic, and Sinai models, ultimately based on the theoretical expression Eqs.(4) – (6). The normalized first passage time in the Kim-Kong model is found to be infinite for $\theta = 0$, while the fractal dimension in the logistic model is expected to take a value near zero as $\theta \to 0$. It is also found that the fractal dimension anomalously changes as the disorder parameter $\theta$ increases for both the Kim-Kong and logistic models. Fig. 4 is the plot of the spectrum $f_q$ versus $\alpha_q$ for the Sinai model on a two-dimensional Sierpinski gasket.

In conclusion, we have discussed in this paper the characteristic properties of the Kim-Kong map with two control parameters. We also have investigated the multifractals from the distribution of the normalized first passage time arriving at an absorbing barrier for the first time. For completeness we have also included the results for three models, as summarized in Table I. In the future, the decay process can be presented analytically and numerically from the randomness of our map in a reaction-diffusion system with multi-species reactants. It is expected that the detail description of the multifractal behavior will be used to study the higher-dimensional extensions of the fractal lattice models, for the case of the transition probability as a chaotic orbits of the Kim-Kong map.

Acknowledgments

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Figure 1: A plot of \( x(n+1) \) verse \( x(n) \) graph in the Kim - Kong map. This map is given by
\[
x(n+1) = \gamma \exp[-\beta(\log x(n))^2][1 - \exp[-\beta(\log x(n))^2]],
\]
where \( \beta = 0.2, 0.6, 1.0, 1.5, \) and \( 2.0 \) for \( \gamma = 3.78 \) are respectively represented by the thick solid, thin solid, the dot, thin dashed, and thick dashed lines.
Figure 2: A bifurcation diagram of the Kim-Kong map where a control parameter $\gamma$ is varied in the range $0 < \gamma < 4$ at $\beta = 0.2$, and the bifurcation values $\gamma_1 = 1.673505$, $\gamma_2 = 2.502200$, and $\gamma_3 = 2.699329$. 
Figure 3: The trajectory for a particle performing a random walk in the logistic model with $\theta = 0$, where the logistic map is given by $x(n+1) = Rx(n)[1 - x(n)]$ with $R = 3.9999$. A particle is initially started from $A = (11111233)$, and absorbed finally at $B = (33333333)$ in a two-dimensional Sierpinski gasket at $n = 7$ stages.
Figure 4: Plots of $f_q$ and $\alpha_q$ for the disorder parameter $\theta = 0$(solid line), 0.05(dashed line), and 0.1(dot line) in the Sinai model on a two-dimensional Sierpinski gasket from Ref.[34].
Summary of values of the fractal dimension, the generalized dimension, and the spectrum in the Kim-Kong, logistic, and Sinai models on a two-dimensional Sierpinski gasket.

| Model       | Disorder Parameter | $D_0 = f_0$ | $D_{+\infty}$ | $D_{-\infty}$ | $\alpha_{+\infty}$ | $\alpha_{-\infty}$ |
|-------------|--------------------|-------------|---------------|---------------|---------------------|---------------------|
| Kim-Kong    | 0.10               | 0.79548     | 0.48462       | 1.99326       | 0.46998             | 1.99999             |
| Model       | 0.05               | 0.65897     | 0.29661       | 1.92748       | 0.19918             | 1.99999             |
| Logistic    | 0.10               | 0.57306     | 0.35095       | 1.96418       | 0.32672             | 1.99999             |
| Model       | 0.05               | 0.15015     | 6.00002       | 1.90476       | 0.00002             | 1.99999             |
| Sinai       | 0.10               | 0.94881     | 0.73094       | 1.92140       | 0.70446             | 1.99999             |
| Model       | 0.05               | 0.84424     | 0.76908       | 1.92328       | 0.73877             | 1.99999             |
|             | 0                  | 0.96213     | 0.60004       | 1.91935       | 0.57132             | 1.99999             |