Abstract. This paper derives and analyzes new diffusion synthetic acceleration (DSA) preconditioners for the $S_N$ transport equation when discretized with a high-order (HO) discontinuous Galerkin (DG) discretization. By expanding the $S_N$ transport operator in the mean free path $\varepsilon$ and employing a rigorous singular matrix perturbation analysis, we derive a DSA matrix that reduces to the symmetric interior penalty (SIP) DG discretization of the continuum diffusion equation when the mesh is first-order and the total opacity is constant. We prove that preconditioning the HO DG $S_N$ transport equation with the SIP DSA matrix results in an $O(\varepsilon)$ perturbation of the identity, and fixed-point iteration therefore converges rapidly for optically thick problems. However, the SIP DSA matrix is conditioned like $O(\varepsilon^{-1})$, making it difficult to invert for small $\varepsilon$. We further derive a new two-part, additive DSA preconditioner based on a continuous Galerkin discretization of diffusion-reaction, which has a condition number independent of $\varepsilon$, and prove that this DSA variant has the same theoretical efficiency as the SIP DSA preconditioner in the optically thick limit. The analysis is extended to the case of HO (curved) meshes, where so-called mesh cycles can result from elements both being upwind of each other (for a given discrete photon direction). In particular, we prove that performing two additional transport sweeps, with fixed scalar flux, in between DSA steps yields the same theoretical convergence. This paper focuses on DSA-type algorithms.

Some of the earliest work on accelerating transport equations with a diffusion-based preconditioner can be found in, for example, [10, 16, 17]. It was shown in [7, 21] that diffusion-based acceleration for source iteration is effective for fine spatial meshes ($\varepsilon \geq h_x \sigma_t$) but its performance can degrade for coarse meshes (that is, $\varepsilon < h_x \sigma_t$). Further seminal work in [3] contained a derivation and theory for a diffusion-equation accelerator whose discretization is consistent with the $S_N$ transport diamond-difference scheme (a finite-volume type scheme for transport), and which yields fast acceleration independent of the spatial mesh size.

1. Introduction. The $S_N$ transport equation forms a key component in modeling the interaction of radiation and a background medium, and its accurate solution is critical in the simulation of astrophysics and Interal Confinement Fusion. In this paper, we derive and analyze diffusion-based preconditioners for a high-order (HO) discontinuous Galerkin (DG) discretization of the monoenergetic $S_N$ transport equations in the challenging (but typical) case of scattering-dominated regimes. One motivation of this research is in the context of HO arbitrary Lagrangian-Eulerian (ALE) hydrodynamics on HO (curved) meshes [4], where standard diffusion-based preconditioners are inadequate.

The standard approach for solving the $S_N$ transport equations involves a fixed-point iteration, referred to as source iteration in the transport literature. It is well known that source iteration can converge arbitrarily slowly in the optically thick limit of large scattering and small absorption. To quantify this, it is useful to introduce the diffusion scaling. In particular, letting $\varepsilon$ denote the characteristic mean free path of the photons, $h_x$ the characteristic mesh spacing, and $\sigma_t$ the total opacity, the optically thick limit corresponds to $\varepsilon \ll h_x \sigma_t$. In this case, the matrix corresponding to source iteration has a condition number that scales like $(h_x \sigma_t / \varepsilon)^2$ and, therefore, will converge very slowly without specialized preconditioners. Such preconditioners typically involve a two-level acceleration scheme and fall within two broad classes: (i) using a diffusion equation to solve for a corrected scalar flux, referred to as diffusion synthetic acceleration (DSA), and (ii) solving the $S_N$ transport equations with a reduced number of angular quadrature points, referred to as transport synthetic acceleration (TSA). This paper focuses on DSA-type algorithms.
This work has been significantly refined and expanded to other spatial discretizations [1, 2, 12, 13, 14, 24, 26]. In the context of HO DG discretizations, the authors in [23] develop a modified symmetric interior penalty (MIP) DSA scheme for HO DG (on first-order meshes) and numerically demonstrate that source iteration converges rapidly with the MIP DSA preconditioner. An excellent discussion of the historical development of DSA can be found in [15].

Here we present a rigorous analysis of DSA in the context of HO DG, on potentially HO (curved) meshes. In general, the discrete source-iteration propagation operator has singular modes with singular values on the order of $O(\varepsilon^2)$, where $\varepsilon$ is the characteristic mean free path. These modes are referred to as the near nullspace of the source iteration matrix, and the corresponding error modes are extremely slow to converge when $\varepsilon \ll 1$. By directly expanding the discrete DG source-iteration operator in $\varepsilon$, we derive a DSA preconditioner that exactly represents the problematic $O(\varepsilon^2)$ error modes that are slow to decay. The proofs make it clear that the constants in the error bounds are small in the optically thick regime $\varepsilon \ll h_x \sigma_t$ (but break down for $\varepsilon \geq h_x \sigma_t$). For first order meshes and constant opacities, we also show that the DSA matrix exactly corresponds to the symmetric interior penalty (SIP) DG discretization of the diffusion equation. In Theorem 3, we prove that the corresponding DSA-preconditioned $S_N$ transport equations is an $O(\varepsilon)$ perturbation of the identity, and the resulting fixed-point iteration therefore converges rapidly for small enough mean free path. In the optically thick limit of small $\varepsilon \ll h_x \sigma_t$ (and assuming constant total opacity and a first order mesh), this diffusion discretization is identical to the MIP DSA preconditioner that is numerically analyzed in [23], and Theorem 3 provides a rigorous justification for its efficacy (see Section 3.2 for a further discussion of the MIP preconditioner and a heuristic argument for how the modified penalty coefficient in [23] stabilizes the SIP DSA preconditioner in the optically thin regime).

The resulting SIP DSA matrix is in the form of a singular matrix perturbation: the dominant term (the “penalty term”) is of order $1/\varepsilon$ relative to the other terms, and has a nullspace consisting of continuous functions with zero boundary values. This term acts as a large penalization and constrains the solution to be continuous in the limit of $\varepsilon \to 0$. This term also leads to the SIP DSA matrix having a condition number that scales like $O(1/\varepsilon)$. In addition, DG discretizations of diffusion such as the SIP DSA matrix can be difficult to precondition effectively (although see [19, 25] for several effective approaches to preconditioning these systems). Therefore, to get a better conditioned system, we derive an alternative two-part additive DSA preconditioner, where a single DSA step entails inverting four symmetric positive definite matrices whose condition numbers are independent of the mean free path $\varepsilon$. We prove (see Theorem 4) that the resulting preconditioned $S_N$ transport equation fixed-point iteration is also an $O(\varepsilon)$ perturbation of the identity and therefore has the same theoretical efficiency as the SIP DSA preconditioner in the optically thick limit. We note that the leading order term in this two-part additive DSA preconditioner corresponds to the Continuous Galerkin (CG) discretization of the diffusion equation obtained in [8].

We finally modify the analysis to account for HO curved meshes. In this case, neighboring mesh elements can both be upwind of each other, leading to so-called mesh cycles. With mesh cycles, the discrete streaming plus collision operator that is inverted in source iteration is no longer block lower triangular in any element ordering, and so it cannot be easily inverted through a forward solve. To get around this, it is common practice to find an element ordering that makes the upper triangular part of this matrix as small as possible, and lag this part in a block Guass-Seidel-type fixed-point iteration. We prove in Theorem 5 that performing two additional transport sweeps on the $S_N$ transport equations, with a fixed scalar flux, yields a preconditioner that has the same asymptotic efficiency as that obtained on cycle-free meshes.

The paper proceeds as follows. Section 2 introduces the DG discretization of the $S_N$ transport equations, as well as the standard fixed-point iteration to solve the discrete $S_N$ system, known as “source iteration.” This section also motivates the need for DSA preconditioning in the optically thick limit of small mean free path, which is formalized in Proposition 2. The primary theoretical contributions are formally stated in Section 3.1, with the proofs provided in Section 4. Section 3.2 relates our analysis to both the modified interior penalty (MIP) preconditioner [23] and the consistent DSA preconditioner [26]. In Section 5, the efficacy of the SIP DSA preconditioner is demonstrated for HO DG discretizations on highly curved 2D and 3D meshes generated by [4] (a HO ALE hydrodynamics code). In particular, it is shown that rapid fixed-point iteration convergence is obtained for small enough mean free path, as long as two additional $S_N$ transport sweeps are performed in between DSA steps (otherwise the iteration diverges); for large enough values of the mean free path, divergence of the preconditioned fixed-point iteration is observed which is also consistent with [23] and the need to modify the SIP DSA preconditioner outside of optically thick regimes.
We also numerically demonstrate that, by discarding one term in the SIP DSA bilinearform one obtains rapid fixed-point iteration convergence for all tested values of the mean free path. As discussed in Section 3.2.2, the resulting nonsymmetric interior penalty DSA matrix is equivalent to the consistent diffusion discretization [26] for linear DG discretizations and first-order meshes. Brief conclusions are given in Section 6.

2. High-Order (HO) Discontinuous Galerkin (DG) discretization of $S_N$ transport and the need for preconditioning in scattering dominated regimes.

2.1. DG discretization. Consider the mono-energetic, steady-state, discrete-ordinates linear Boltzmann equation, given by

\begin{equation}
\Omega_d \cdot \nabla_x \psi_d (x) + \frac{\sigma_t}{\varepsilon} \psi_d (x) = \frac{1}{4\pi} \left( \frac{\sigma_t}{\varepsilon} - \varepsilon \sigma_a (x) \right) \sum_{d'=1}^{N_{\Omega}} w_{d'} \psi_{d'} (x) + \varepsilon q_d (x), \quad x \in \mathcal{D}
\end{equation}

\[ \psi_d (x) = q_{d, \text{inc}} (x), \quad x \in \partial \mathcal{D} \quad \text{and} \quad n (x) \cdot \Omega_d < 0. \]

Here, the total opacity, $\varepsilon \sigma_t^{-1} (x)$, and the absorption opacity, $\varepsilon \sigma_a (x)$, are scaled according to the diffusion limit, where $\varepsilon$ is a non-dimensional parameter that goes to zero in the optically thick limit. The quadrature angle vectors $\Omega_d \in S^2$ and weights $w_d > 0$ are constructed to have desirable symmetry properties and integrate spherical harmonics up to a given degree that depends on the number of angles, $N_\Omega$. A few useful identities involving $w_d$ and $\Omega_d$ are

\[ \sum_d w_d = 4\pi, \quad \sum_d w_d \Omega_d \Omega_d^T = \frac{4\pi}{3} I, \]

\[ \sum_d w_d \Omega_d = 0, \quad \sum_d w_d \Omega_d |\Omega_d \cdot n| = 0. \]

The previous identities follow from the fact that the quadrature weights $w_d$ and directions $\Omega_d$ exactly integrate spherical harmonics $Y_l^m (\Omega)$ of at least degree $l = 2$, and have the symmetry property that each direction $\Omega_d$ and weight $w_d$ has a corresponding reflected direction $\Omega_{d'} = -\Omega_d$ with identical weight $w_{d'} = w_d$ for some $1 \leq d' \leq N_\Omega$.

We consider a discontinuous Galerkin (DG) discretization of the $S_N$ transport equation. To do so, we set some notation. First, consider a decomposition of the domain $\mathcal{D}$ in to a set of elements $\{ \kappa_e \}, e = 1 \ldots N_\kappa$, and let $\mathcal{F}$ denote the set of interior and boundary finite element faces $\Gamma \in \mathcal{F}$. The finite element space $\mathcal{U}$ corresponds to the collection of piecewise polynomial functions of fixed degree $r$ on each finite element $\kappa$, $\mathcal{P}_r (\kappa)$,

\[ \mathcal{U} = \{ u \in L^2 (\mathcal{D}) : u |_\kappa \in \mathcal{P}_r (\kappa) \}. \]

For an interior mesh face $\Gamma \in \mathcal{F}$ shared by two neighboring elements $\kappa_e$ and $\kappa_{e'}$, we let $n$ denote the normal vector that points from $\kappa_e$ to $\kappa_{e'}$. Given this (fixed but arbitrary) choice for the sign of the normal vector $n$ on each element face, the jump $\lVert u \rVert$ and average $\{ u \}$ for a function $u \in \mathcal{U}$ are defined by

\[ \lVert u \rVert = \begin{cases} u_e - u_{e'}, & \text{if } \Gamma \text{ is an interior face shared by elements } \kappa_e \text{ and } \kappa_{e'}, \\ u_e, & \text{if } \Gamma \text{ is a boundary face of element } \kappa_e, \end{cases} \]

and

\[ \{ u \} = \begin{cases} (u_e + u_{e'}) / 2, & \text{if } \Gamma \text{ is an interior face shared by elements } \kappa_e \text{ and } \kappa_{e'}, \\ u_e, & \text{if } \Gamma \text{ is a boundary face of element } \kappa_e. \end{cases} \]

Although the definitions of the jump $\lVert u \rVert$ and average $\{ u \}$ depend on arbitrarily choosing a sign for the normal vector $n$, it turns out that the bilinear forms below are invariant with respect to this choice.

Following the standard DG discretization procedure and using upwinding to define the numerical flux, (1) can be discretized as

\begin{equation}
\Omega_d \cdot G \psi^{(d)} + F^{(d)} \psi^{(d)} + \frac{1}{\varepsilon} M_t \psi^{(d)} - \frac{1}{4\pi} \left( \frac{1}{\varepsilon} M_l - \varepsilon M_a \right) \varphi = \frac{1}{4\pi} \left( q^{(d)}_{\text{inc}} + \varepsilon q^{(d)} \right).
\end{equation}
Here the vector $\varphi$ of coefficients for the scalar flux $\varphi$ is given by

$$\varphi = \sum_d w_d \psi^{(d)},$$  

the vectors $q^{(d)}_{\text{inc}}$ and $q^{(d)}$ on the right hand side of (2) correspond to the linear forms

$$\left[ q^{(d)}_{\text{inc}} \right]_m = \sum_{\Gamma \in F} \int_\Gamma \Omega_d \cdot n_m q^{d,\text{inc}} \, dS - \frac{1}{2} \sum_{\Gamma \in F} \int_\Gamma |\Omega_d \cdot n| v^{(d)} \, dS,$$

$$\left[ q^{(d)} \right]_m = \sum_{\kappa} \int_{\kappa} v_m q^{d} \, dx,$$

where $\{v_m\}^N_1$ is the finite element basis of $\mathcal{U}$, and $N$ is the total number of degrees of freedom in $\mathcal{U}$. We will also denote by $u$ and $v$ the vectors of coefficients corresponding to some discrete functions $u$ and $v$ in the finite element space $\mathcal{U}$. The matrices $\Omega_d \cdot G$, $F^{(d)}$, $M_i$, and $M_a$ in equation (2) correspond, respectively, to the bilinear forms,

$$v^T(\Omega_d \cdot G)u = \sum_{\kappa} \int_{\kappa} (\Omega_d \cdot \nabla_x u) \, v \, dx,$$

$$v^TF^{(d)}u = -\sum_{\Gamma \in F} \int_\Gamma \Omega_d \cdot n \{u\} \{v\} \, dS + \frac{1}{2} \sum_{\Gamma \in F} \int_\Gamma |\Omega_d \cdot n| \{u\} \{v\} \, dS,$$

$$v^TM_iu = \sum_{\kappa} \int_{\kappa} \sigma_{iuv} \, dx,$$

$$v^TM_au = \sum_{\kappa} \int_{\kappa} \sigma_{auv} \, dx.$$

Note that in our convention bold symbols indicate vectors and capital (from the Latin alphabet) symbols indicate matrices. In addition, the notation $G$ is shorthand for a vector with three matrix components, $G = (G_1, G_2, G_3)$, so that

$$\Omega_d \cdot G = \sum_{j=1}^3 (\Omega_d)_j G_j.$$

To reformulate equation (2), define the column vector $\psi = (\psi^{(1)}; \ldots; \psi^{(N_{\Omega})})$ and projection

$$(P_0\psi)_d = \frac{1}{4\pi} \sum_{d'} w_{dd'} \psi_{d'} = \frac{1}{4\pi} \varphi, \quad d = 1, \ldots, N_{\Omega}.$$

$P_0$ is a weighted average over direction $d$ that projects the average on to all vector blocks. In the matrix sense, $P_0$ is an $NN_{\Omega} \times NN_{\Omega}$ operator, where each block row takes the form $\frac{1}{4\pi} [w_0 I_N, w_1 I_N, \ldots, w_{N_{\Omega}} I_N]$. $P_0$ being a projection relies on the fact that $\sum_d w_d = 4\pi$. Defining

$$W = \text{diag} [w_0 I_N, w_1 I_N, \ldots, w_{N_{\Omega}} I_N], \quad \langle x, y \rangle_W = \langle W x, y \rangle,$$

$P_0$ is an orthogonal projection in the $W$-inner product. Letting $Q_0 := I - P_0$ denote the orthogonal complement to $P_0$, recall that for any vector $\psi$, $\|\psi\|_W = \|P_0\psi\|_W + \|Q_0\psi\|_W$. Now, rewrite (2) as

$$\left[ I + \varepsilon M_t^{-1} (\Omega_d \cdot G + F^{(d)}) \right] \psi^{(d)} = \frac{1}{4\pi} \left( I - \varepsilon^2 M_t^{-1} M_a \right) \varphi = \frac{1}{4\pi} \varepsilon M_t^{-1} \left( q^{(d)}_{\text{inc}} + \varepsilon q^{(d)} \right).$$

In matrix form, over all angles, the first term in (12) operating on $\psi^{(d)}$ is block diagonal in $d$, with each block corresponding to a fixed direction $\Omega_d$, and the second term a global angular coupling through projection $P_0$. A standard technique in transport is to invert the first, block-diagonal term. This approach corresponds to solving the linear transport equation independently, for all directions $d$, and is known as a transport
sweep. Define $T_ε$ as the block-diagonal operator over direction $d$, multiplied by $P_0$, when a transport sweep is applied:

$$T_ε = \text{diag}_d \left[ (I + εM_t^{-1} \left( Ω_d \cdot G + F^{(d)} \right))^{-1} (I - ε^2 M_t^{-1} M_a) \right] P_0. $$

Then, equation (2) can be re-written as the preconditioned linear system

$$ (I - T_ε)\bar{q}^{(d)} = \bar{q}^{(d)}, $$

for

$$ \bar{q}^{(d)} = \left( I + εM_t^{-1} \left( Ω_d \cdot G + F^{(d)} \right) \right)^{-1} \frac{1}{4π} εM_t^{-1} \left( q^{(d)}_{\text{inc}} + εq^{(d)} \right). $$

Multiplying equation (13) by the quadrature weight, $w_d$, and summing over direction index, $d$, yields a linear system for the scalar flux,

$$ (I - S_ε) \varphi = q, $$

where

$$ S_ε = \sum_d w_d \left( I + εM_t^{-1} \left( Ω_d \cdot G + F^{(d)} \right) \right)^{-1} \frac{1}{4π} (I - ε^2 M_t^{-1} M_a), $$

$$ q = \sum_d w_d \left( I + εM_t^{-1} \left( Ω_d \cdot G + F^{(d)} \right) \right)^{-1} εM_t^{-1} \left( q^{(d)}_{\text{inc}} + \frac{1}{4π} q^{(d)} \right). $$

We note that, in applying the operator $T_ε$, we need to invert $\left[ I + εM_t^{-1} \left( Ω_d \cdot G + F^{(d)} \right) \right]$. As it turns out, this is not always computationally tractable, particularly in the case of HO curved meshes. Theorem 5 and Section 4.3 analyze a more general case where this term is not inverted exactly.

**Remark 1.** Our analysis of equation (13) is valid under the assumption that

$$ ε\| M_t^{-1} \left( Ω_d \cdot G + F^{(d)} \right) \| < 1. $$

In particular, choosing numbers $h_x$, $\tilde{σ}_t$, and $\tilde{σ}_a$ for which

$$ h_x \left\| Ω_d \cdot G + F^{(d)} \right\| \leq 1, \quad \tilde{σ}_t \| M_t^{-1} \| \leq 1, \quad \tilde{σ}_a \| M_a \| \leq 1 $$

the error bounds in Theorems 3-5 below are small as long as

$$ η = \min \left\{ ε/h_x \tilde{σ}_t, ε/\sqrt{\tilde{σ}_a/\tilde{σ}_t} \right\} \ll 1. $$

The regime $η \ll 1$ corresponds to the standard optically thick limit.

### 2.2. Useful identities.

Here, two identities are presented that will be used regularly in further derivations. First, define the matrix $F^{(d)}$ corresponding to the bilinear form

$$ v^T \tilde{F}^{(d)} u = \sum_{Γ \in F} \int_{Γ} Ω_d \cdot \n \{ u \} \{ v \} dS + \sum_{Γ \in F} \int_{Γ} \frac{1}{2} \| Ω_d \cdot n \{ u \} \{ v \} dS. $$

Applying integration by parts to the term

$$ v^T Ω_d \cdot G u = \sum_k \int_{K} (Ω_d \cdot \nabla x u) v d\mathbf{x} $$

in $v^T (Ω_d \cdot G + F^{(d)}) u$ yields the identity

$$ Ω_d \cdot G + F^{(d)} = -Ω_d \cdot G^T + \tilde{F}^{(d)}. $$
Second, a key property follows immediately from equations (7) and (17). Let $P$ denote a projection on to the space of continuous functions with zero boundary values. Then, $P\psi$ corresponds to a continuous function with zero boundary value and, therefore, $\|\psi\| = 0$ on each interior mesh face $\Gamma$ and $\psi = 0$ on each boundary face. From expression (7), we see that
\begin{equation}
(P\psi)^T \hat{F}(d) u = \psi^T P^T \hat{F}(d) u = 0,
\end{equation}
for any $u$ and $\psi$. Since $u$ and $\psi$ are arbitrary, $P^T \hat{F}(d) = 0$. A similar identity follows from expression (17), yielding the two identities
\begin{equation}
P(d) P = 0, \quad P^T \hat{F}(d) = 0.
\end{equation}

2.3. Need for preconditioning in the optically thick limit. To motivate DSA and further analysis in this paper, we state the following Proposition which shows that preconditioning the linear system in (13) is important in the optically thick limit of small $\varepsilon$. The proof of Proposition 2 is given in the Appendix.

**Proposition 2.** Assume that the matrix $I - T_\varepsilon$ in the linear system (13) is invertible. Then the condition number of the matrix $I - T_\varepsilon$ from equation (13) satisfies
\begin{equation}
\text{cond}(I - T_\varepsilon) = \|I - T_\varepsilon\|_W \|(I - T_\varepsilon)^{-1}\|_W \geq O(\varepsilon^{-2}),
\end{equation}
where the norm $\| \cdot \|_W$ is defined by equation (11). In addition, suppose that $E_\varepsilon$ inverts $P_0 (I - T_\varepsilon) P_0$ on the range of $P_0$ to within $O(\varepsilon)$,
\begin{equation}
E_\varepsilon P_0 (I - T_\varepsilon) P_0 = P_0 + O(\varepsilon).
\end{equation}
Then the preconditioned matrix $((I - P_0) + E_\varepsilon P_0) (I - T_\varepsilon)$ is an $O(\varepsilon)$ perturbation of the identity,
\begin{equation}
((I - P_0) + E_\varepsilon P_0) (I - T_\varepsilon) = I + O(\varepsilon).
\end{equation}

The relationship
\begin{equation}
P_0 (I - T_\varepsilon) P_0 \psi(d) = (I - S_\varepsilon) P_0 \psi(d), \quad d = 1, \ldots, N_\Omega,
\end{equation}
connects the Theorems in Section 3.1 with Proposition 2.

3. DSA preconditioners for HO DG discretizations on curved meshes.

3.1. Overview of the DSA preconditioners and statement of the theorems. This section presents the main theoretical contributions of this paper, the proofs of which are contained in the following subsections.

First we present results on a symmetric interior penalty (SIP) DSA preconditioner. To do so, define the SIP DSA matrix,
\begin{equation}
D_\varepsilon = \frac{1}{\varepsilon} F_0 + D_0,
\end{equation}
where
\begin{equation}
D_0 = \frac{1}{3} G^T \cdot M_t^{-1} G - \hat{F}_1 \cdot M_t^{-1} G + G^T \cdot M_t^{-1} F_1 + M_a,
\end{equation}
and
\begin{equation}
F_0 = \frac{1}{4\pi} \sum_d w_d F(d), \quad F_1 = \frac{1}{4\pi} \sum_d w_d \Omega_d F(d), \quad \hat{F}_1 = \frac{1}{4\pi} \sum_d w_d \Omega_d \hat{F}(d).
\end{equation}
In the previous equations, $F_1$ and $\hat{F}_1$ correspond to vectors of matrices; for example, in three spatial dimensions
\begin{equation}
(F_1)_j = \frac{1}{4\pi} \sum_d w_d (\Omega_d)_j F(d), \quad j = 1, 2, 3.
\end{equation}
Assuming that the mesh is first order and that the opacities, $\sigma_t$ and $\sigma_a$, are constants, it turns out (see Section 4.2) that $D_\varepsilon$ corresponds to the bilinear form,

$$
\mathbf{v}^T D_\varepsilon \mathbf{u} = B_{\text{SIP}} (\cdot, \cdot),
$$

where

$$
B_{\text{SIP}} (u, v) := \frac{1}{\varepsilon} \sum_{\Gamma \in \mathcal{F}_{\Gamma}} \int_{\Gamma} \alpha (\|v\|) \|v\| dS + \sum_{\kappa \in \mathcal{E}} \int_{\kappa} \frac{1}{3\sigma_t} \nabla_x u \cdot \nabla_x v d\mathbf{x} + \sum_{\kappa \in \mathcal{E}} \int_{\kappa} \sigma_a uv d\mathbf{x} - \sum_{\Gamma \in \mathcal{F}_{\Gamma}} \int_{\Gamma} \|v\| \left\{ \mathbf{n} \cdot \frac{1}{3\sigma_t} \nabla_x u \right\} dS,
$$

Here, the function $\alpha (\cdot)$ in the first integral is defined as

$$
\alpha (\mathbf{x}) = \frac{1}{4\pi} \sum_d w_d \left| \Omega_d \cdot \mathbf{n} (\mathbf{x}) \right|, \quad \mathbf{x} \in \Gamma \in \mathcal{F},
$$

and converges to $1/4$ in the limit of a large number of angles, $\Omega_d$. The bilinear form in (27) corresponds to a variant of the symmetric interior penalty discretization of the reaction-diffusion operator,

$$
\nabla_x \cdot \left( \frac{1}{3\sigma_t} \nabla_x \right) - \sigma_a.
$$

Theorem 3 shows that preconditioning the fixed-point iteration based on $(I - S_\varepsilon)$ (14) with the DSA matrix $D_\varepsilon$ results in fast convergence in the optically thick limit.

**Theorem 3 (SIP DSA preconditioner).** Assume that the function $\alpha (\cdot)$ defined in equation (28) is uniformly bounded away from zero on each interior and boundary mesh faces. Then

$$
(\varepsilon^2 D_\varepsilon)^{-1} M_t (I - S_\varepsilon) = I + \mathcal{O} (\varepsilon).
$$

Theorem 3 states that the preconditioned iteration matrix looks like the identity plus an $\mathcal{O} (\varepsilon)$ perturbation. For small $\varepsilon$, this ensures a well-conditioned iteration matrix and fast convergence. Under the assumptions of Theorem 3, it follows from the identity (see Section 4.2)

$$
\mathbf{v}^T F_0 \mathbf{u} = \sum_{\Gamma \in \mathcal{F}_{\Gamma}} \int_{\Gamma} \alpha (\|u\|) \|v\| dS
$$

that $F_0$ has a nullspace consisting of continuous functions with zero boundary values. For example, if $\mathbf{u}$ is in the nullspace of $F_0$, then

$$
\mathbf{u}^T F_0 \mathbf{u} = \frac{1}{\varepsilon} \sum_{\Gamma \in \mathcal{F}_{\Gamma}} \int_{\Gamma} \alpha (\|u\|)^2 dS = 0,
$$

and so the jump $\|u\|$ must vanish on each interior mesh face and $u$ must vanish on each boundary face. It follows that the condition number of $D_\varepsilon$ scales like $\mathcal{O} (\varepsilon^{-1})$, and a good preconditioner is required to efficiently invert the interior penalty DSA matrix. Unfortunately, HO DG discretizations can prove difficult for fast linear solvers and preconditioners, such as algebraic multigrid (AMG), even when considering elliptic problems $[6, 18, 20, 22]$. This difficulty is compounded on highly unstructured grids, which are some of the motivating problems here.

Fortunately, the proof of Theorem 3 also yields a better-conditioned DSA preconditioner for optically thick problems. In fact, let $P$ denote a projection of functions in the DG space onto the continuous functions, and $\mathbf{Q} = I - P$ be its complement. Then Theorem 4 develops a two-part, additive DSA matrix; a single DSA step involves three applications of $P (P^T D_\varepsilon P)^{-1} P^T$ (that is, solving a continuous Galerkin diffusion discretization), and one application of $\mathbf{Q} (\mathbf{Q}^T F_0 \mathbf{Q})^{-1} \mathbf{Q}^T$ (solving in the complement). In the optically thick limit, this DSA matrix is proven to have the same theoretical iteration efficiency as the symmetric interior penalty DSA matrix discussed in Theorem 3, and its application requires inverting matrices with condition number independent of $\varepsilon$. 
THEOREM 4. Let $P$ denote a projection on to the subspace of $U$ containing continuous polynomials with zero boundary values, and let $Q = I - P$. Define the operators
\[
E_P = P \left( P^T D_0 P \right)^{-1} P^T, \quad E_Q = Q \left( Q^T F_0 Q \right)^{-1} Q^T,
\]
and
\[
E_\varepsilon = \frac{1}{\varepsilon} E_P + (I - E_P D_0) E_Q (I - D_0 E_P),
\]
with $D_0$ as in (24). Then
\[
\frac{1}{\varepsilon} E_\varepsilon M_\varepsilon (I - S_\varepsilon) = I + O(\varepsilon).
\]

As in Theorem 3, Theorem 4 proves that the preconditioned operator is an $O(\varepsilon)$ perturbation of the identity and is thus well-conditioned for small $\varepsilon$, and the corresponding fixed-point iteration will converge rapidly. Note that, using equation (27) for the bilinear form corresponding to $P \left( P^T D P \right)^{-1} P^T$, it is straightforward to see that the matrix $P \left( P^T D_0 P \right)^{-1} P^T$ corresponds to solving a continuous Galerkin discretization of the diffusion equation (3.1) (for constant opacities $\sigma_0$ and $\sigma_t$).

The final result of this paper regards applying DSA to HO (curved) meshes. In particular, consider the general linear system in (12), expressed as a single operator on $\psi$:
\[
\left([I + \varepsilon H] - (I - \varepsilon^2 M_t^{-1} M_a) P_0\right) \psi^{(d)} = q^{(d)}.
\]
Often it is possible to order the mesh elements so that $H = \text{diag}_d \left[ M_t^{-1} \left( \Omega_d \cdot \mathbf{G} + F^{(d)} \right) \right]$ is block lower triangular, with blocks corresponding to mesh elements. In such cases, $I + \varepsilon H$ can be inverted directly to give the equivalent (but better conditioned) system
\[
(I - T_\varepsilon) \psi^{(d)} = (I + \varepsilon H)^{-1} q^{(d)},
\]
where
\[
T_\varepsilon = (I + \varepsilon H)^{-1} (I - \varepsilon^2 M_t^{-1} M_a) P_0.
\]
However, for HO meshes, it is typically the case that $H$ is no longer block lower triangular, and cannot be easily inverted through a forward solve. Recently, a graph-based algorithm was developed to replace the inversion with a Gauss-Seidel type iteration in a pseudo-optimal ordering when mesh cycles are present [9]. To consider this more general case, suppose that we choose a mesh element ordering that leads to a decomposition,
\[
H = H_\leq + H_>,
\]
where
\[
H_\leq = \text{diag}_d \left[ M_t^{-1} \left( \Omega_d \cdot \mathbf{G} + F_\leq^{(d)} \right) \right], \quad H_\geq = \text{diag}_d \left[ M_t^{-1} F_\geq^{(d)} \right].
\]
Here, we invert $H_\leq$ exactly and move $H_\geq$ to the right-hand side. For example, $H_\leq$ corresponds to the lower-triangular part of the matrix ordering in [9], which is inverted in an ordered Gauss-Seidel iteration.

The following theorem shows that three transport sweeps with lagging—that is, three applications of $(I + \varepsilon H_\leq)^{-1}$—yields an efficient preconditioner using the DSA matrix from Theorem 3 or 4.

THEOREM 5. Let $I - T_\varepsilon$ be the preconditioned linear system in (30) that corresponds to applying $(I + \varepsilon H)^{-1}$ as a preconditioner. Define $I - \widetilde{T}_\varepsilon$ as the preconditioned linear system associated with applying three iterations of $(I + \varepsilon H_\leq)^{-1}$ to (29), while keeping the term $(I - \varepsilon^2 M_t^{-1} M_a) P_0 \psi$ fixed. Then
\[
\widetilde{T}_\varepsilon = T_\varepsilon + O(\varepsilon^3),
\]
and, letting $E_\varepsilon$ correspond to the DSA preconditioner in Theorem 3 or 4,
\[
E_\varepsilon P_0 (I - T_\varepsilon) P_0 = E_\varepsilon P_0 \left( I - \widetilde{T}_\varepsilon \right) P_0 + O(\varepsilon).
\]
Remark 6. Note that moving the term \((I - \varepsilon^2 M_r^{-1} M_a) P_\theta \psi\) in the linear system to the right-hand side and fixing it—that is, not updating \((I - \varepsilon^2 M_r^{-1} M_a) P_\theta \psi\) based on an updated \(\psi\)—is not typical in a fixed-point iterative method. One can also work out the error-propagation matrix for multiple iterations that include updating this term each iteration. For this variant, the asymptotics in \(\varepsilon\) do not clearly indicate a well-conditioned system for \(\varepsilon \ll 0\), as obtained in Theorem 5. However, numerically, updating \((I - \varepsilon^2 M_r^{-1} M_a) P_\theta \psi\) each iteration proves to be more robust for larger \(\varepsilon\), which is discussed in Section 5.

Proofs of Theorems 3-5 are given in Section 4.

3.2. Connection to previous work.

3.2.1. The modified interior penalty DSA preconditioner. We first connect our derivation and analysis of the SIP DSA preconditioner to the MIP DSA preconditioner in [23], and then relate the SIP DSA preconditioner to the consistent DSA preconditioner derived in [26] for linear DG discretizations.

In [23] the authors numerically demonstrate that using the modified interior penalty (MIP) DSA matrix yields uniformly good convergence in both optically thick and thin regimes. The corresponding bilinear form is similar to equation (27), but the penalty coefficient \(\kappa\) in the penalty term,

\[
\sum_{x \in F} \int_{\Gamma} \kappa \|u\| \|v\| \, dS,
\]

is modified outside of the optically thick limit. In particular, letting \(h_x\) denote the characteristic mesh spacing, the MIP penalty coefficient \(\kappa\) in [23] scales like max \((1/(4\varepsilon), C_p/(\sigma_t h_x))\), where \(C_p\) is a constant that depends on the finite element local polynomial order. Notice that, when \(\varepsilon \lesssim \sigma_t h_x\), the MIP penalty coefficient reduces to \(1/(4\varepsilon) \approx \alpha/\varepsilon\) (this inequality becomes an equality in the limit of an infinite number of quadrature angles), which is identical to the SIP DSA penalty coefficient in equation (27). Therefore, Theorem 3 justifies the numerically observed behavior in [23] when \(\varepsilon \lesssim \sigma_t h_x\).

When \(\varepsilon \gtrsim \sigma_t h_x\), the analysis in Theorem 3 breaks down. Nevertheless, at this point the mesh spacing \(h_x\) is small enough to numerically resolve the continuum transport equation (1). It is then expected that the analysis of DSA acceleration for the continuum \(S_N\) transport equation using the continuum diffusion equation can describe this situation (for example, see [15]). In particular, as long as the discrete diffusion equation remains a valid discretization of the continuum diffusion equation when \(\varepsilon \gtrsim \sigma_t h_x\), we expect rapid acceleration for both optically thick and thin regimes. However, it is well-known that the penalty parameter must be at least as large as \(O\left(\frac{\varepsilon}{\sigma_t h_x}\right)\) in order for the SIP discretization to remain a stable discretization of the continuum diffusion equation (for example, see [5]). This motivates choosing \(\kappa\) to scale like max \(\{1/(4\varepsilon), C_p/(\sigma_t h_x)\}\) to ensure that the MIP DSA matrix both approximates the near-nullspace in the optically thick (ill-conditioned) limit \(\varepsilon \lesssim \sigma_t h_x\), and also remains a good approximation to the continuum diffusion equation as \(\varepsilon \gtrsim \sigma_t h_x\) and \(h_x\) begins to resolve the mean free path.

3.2.2. The nonsymmetric interior penalty DSA preconditioner. In Section 5, the SIP DSA preconditioner is shown to be robust for \(\varepsilon \ll 1\), but does not converge for moderate \(\varepsilon\) (relative to the characteristic mesh spacing). Consider the nonsymmetric interior penalty (IP) version of the DSA matrix

\[
\frac{1}{\varepsilon} F_0 + \frac{1}{3} G^T M_r^{-1} G - \hat{F}_1 \cdot M_r^{-1} G + M_a,
\]

where we have neglected the term \(G^T \cdot M_r^{-1} F_1\) from the symmetric interior penalty DSA preconditioner defining \(D_s\) (see (24)). Dropping this term results in a nonsymmetric interior penalty (IP) discretization of the diffusion equation when the opacities are constant, and we observe empirically that uniformly good convergence is obtained for all tested values of \(\varepsilon\) using this DSA matrix (31). In fact, for linear DG discretizations and straight-edged meshes, the nonsymmetric interior penalty DSA matrix (31) reduces to the Warsa-Wareing-Morel consistent diffusion discretization [26].

Also, a straightforward (but tedious) calculation shows that one can obtain the SIP DSA preconditioner by taking the first two (discrete) angular moments of the discrete equation (2), and employing the following discrete version of Fick’s law

\[
\psi^{(d)} = \frac{1}{4\pi} \varphi - \frac{\varepsilon}{4\pi} M_r^{-1} \left(\Omega_d \cdot G + F^{(d)}\right) \varphi + O\left(\varepsilon^2\right).
\]
Equation (32) results from equation (2),

\[
\psi^{(d)} = \left( I + \varepsilon M_t^{-1} \left( \Omega_d \cdot G + F^{(d)} \right) \right)^{-1} \left( \frac{1}{4\pi} \varphi + \frac{1}{4\pi} q_{inc}^{(d)} \right)
\]

\[
= \frac{1}{4\pi} \varphi - \varepsilon M_t^{-1} \left( \Omega_d \cdot G + F^{(d)} \right) \frac{1}{4\pi} \varphi + \varepsilon \frac{1}{4\pi} q_{inc}^{(d)} + O \left( \varepsilon^2 \right),
\]

where the constant vector \( \varepsilon (4\pi)^{-1} q_{inc}^{(d)} \) is neglected for simplicity since it only contributes to the right-hand side. Similarly, by instead employing the following modified version of Fick’s law in the discrete moment equations,

\[
\psi^{(d)} \approx \frac{1}{4\pi} \varphi - \varepsilon M_t^{-1} (\Omega_d \cdot J),
\]

an analogous calculation shows that \( J = -G \varphi \), and leads to the nonsymmetric interior penalty DSA matrix (31). In particular, the modified version of Fick’s law (33) results from neglecting the term \( \varepsilon (4\pi)^{-1} M_t^{-1} F^{(d)} \varphi \) in equation (32).

4. Proofs of main results.

4.1. Proofs of the theorems. We first establish the following lemma.

**Lemma 7.** Consider the matrix

\[
\hat{D} = F_0 + \varepsilon D,
\]

where \( F_0 \) is a symmetric, singular matrix. Define \( P \) as a projection on to the nullspace of \( F_0 \), let \( Q = I - P \) denote its complement, and define

\[
E_P = P \left( P^T D P \right)^{-1} P^T, \quad E_Q = Q \left( Q^T F_0 Q \right)^{-1} Q^T.
\]

Then,

\[
\hat{D}^{-1} = \frac{1}{\varepsilon} E_P + (I - E_P D) E_Q (I - D E_P) + \varepsilon (I - E_P D) R_\varepsilon (I - D E_P),
\]

where

\[
R_\varepsilon = \varepsilon \left( I + \varepsilon E_Q (D - D E_P) D \right)^{-1} E_Q (I - D E_P) D E_Q.
\]

In addition, suppose that \( D = D_0 + D_1 \), where \( P^T D_1 = D_1 P = 0 \). Then,

\[
(F_0 + \varepsilon D)^{-1} = (F_0 + \varepsilon D_0)^{-1} + O (\varepsilon).
\]

**Proof.** Consider the equation \( \hat{D} \mathbf{x} = \mathbf{y} \), and let \( P \) be a projection onto the nullspace of \( F_0 \), and \( Q = I - P \) its complement. Similar to the proof of Proposition 2, \( \hat{D} \mathbf{x} = \mathbf{y} \) can be expanded based on \( P \) and \( Q \) as a \( 2 \times 2 \) system. First, note that \( \hat{D} \mathbf{x} = \mathbf{y} \) can be written as

\[
\hat{D} \begin{pmatrix} P \\ Q \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix} \mathbf{y}.
\]

Now, we can multiply on the left by the full column-rank operator \( \begin{pmatrix} P^T \\ Q^T \end{pmatrix} \) to yield

\[
\begin{pmatrix} P^T \\ Q^T \end{pmatrix} \hat{D} \begin{pmatrix} P \\ Q \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} P^T \\ Q^T \end{pmatrix} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} \mathbf{y},
\]

\[
\begin{pmatrix} P^T \hat{D} P \\ Q^T \hat{D} P \end{pmatrix} \begin{pmatrix} \mathbf{x}_P \\ \mathbf{Q} \mathbf{x} \end{pmatrix} = \begin{pmatrix} P^T \mathbf{y} \\ Q^T \mathbf{y} \end{pmatrix}.
\]

Denote \( \mathbf{x}_P = P \mathbf{x} \) and \( \mathbf{x}_Q := Q \mathbf{x} \). Using the equations \( P^T F_0 = F_0 P = 0 \), we can rewrite the linear system as

\[
\begin{pmatrix} \varepsilon P^T D P & \varepsilon P^T D Q \\ \varepsilon Q^T D P & Q^T \hat{D} Q \end{pmatrix} \begin{pmatrix} \mathbf{x}_P \\ \mathbf{x}_Q \end{pmatrix} = \begin{pmatrix} \varepsilon P^T \mathbf{y} \\ Q^T \mathbf{y} \end{pmatrix}.
\]
Then,
\[ x_P = \frac{1}{\varepsilon} (P^T DP)^{-1} P^T y - (P^T DP)^{-1} (P^T DQ) x_Q \]
\[ = \frac{1}{\varepsilon} E_P y - E_P DQ x_Q, \]
where the second equality follows from noting that \( x_P = Px \) and multiplying both sides by \( P \). Equation (36) also yields
\[ Q^T \hat{D} Q x_Q + \varepsilon Q^T D P x_P = Q^T y, \]
\[ (Q^T \hat{D} Q - \varepsilon (Q^T D P) (E_P D Q)) x_Q = Q^T y - (Q^T D E_P) y, \]
\[ (Q^T F_0 Q + \varepsilon Q^T (D - D E_P D) Q) x_Q = Q^T y - (Q^T D E_P) y. \]

Now, since the matrix \( Q^T F_0 Q \) above is invertible on the range of \( Q^T \), we can apply \((Q^T F_0 Q)^{-1} Q^T \) to both sides to get
\[ (I + \varepsilon Q (Q^T F_0 Q)^{-1} Q^T (D - D E_P D) Q) x_Q = (Q^T F_0 Q)^{-1} Q^T (I - D E_P) y. \]

Substituting \( E_Q = Q (Q^T F_0 Q)^{-1} Q^T \) in the left-hand side and applying the matrix identity \((I + A)^{-1} = I - (I + A)^{-1} A\) to \((I + \varepsilon E_Q (D - D E_P D) Q)^{-1}\) yields
\[ (I + \varepsilon E_Q (D - D E_P D) Q)^{-1} (Q^T F_0 Q)^{-1} Q^T = (Q^T F_0 Q)^{-1} Q^T - \hat{R}_\varepsilon, \]
where
\[ \hat{R}_\varepsilon = \varepsilon \left( I + \varepsilon E_Q (D - D E_P D) Q \right)^{-1} E_Q (D - D E_P D) Q \left( (Q^T F_0 Q)^{-1} Q^T \right) \]
\[ = \varepsilon \left( I + \varepsilon E_Q (D - D E_P D) Q \right)^{-1} E_Q (I - D E_P) D E_Q. \]

Therefore, using equation (38) in equation (37), and noting that \( Q x_Q = x_Q \),
\[ x_Q = \frac{Q^T F_0 Q}{Q^T I - D E_P} \]
\[ y - \hat{R}_\varepsilon (I - D E_P) y, \]
\[ = E_Q (I - D E_P) y + \hat{R}_\varepsilon (I - D E_P) y. \]

Solving for \( \hat{D}^{-1} y \) yields the final result
\[ \hat{D}^{-1} y = x_P + x_Q \]
\[ = \frac{1}{\varepsilon} E_P y + (I - E_P D Q) x_Q \]
\[ = \frac{1}{\varepsilon} E_P y + (I - E_P D Q) E_Q (I - D E_P) y + (I - E_P D) \hat{R}_\varepsilon (I - D E_P). \]

Equation (35) follows by noting that
\[ (I - E_P D Q) E_Q (I - D E_P) y = (I - E_P D_0 Q) E_Q (I - D_0 E_P) y, \]
and that terms involving \( D_1 \) only come up in the \( O(\varepsilon) \) remainder term, \( \hat{R}_\varepsilon \).

We can now use Lemma 7 to prove Theorems 3 and 4.
Proof (Proof of Theorem 3). Recall the definition from Lemma 7, \( H^{(d)} = M_t^{-1} (\Omega_d \cdot G + F^{(d)}) \), and the identities \( \sum_d w_d = 4\pi \), \( \sum_d w_d \Omega_d = 0 \) and \( F_0 = \frac{1}{4\pi} \sum_d w_d F^{(d)} \). Using the identity for \( (I + \varepsilon H^{(d)})^{-1} \) in (35), \( I - S_\varepsilon \) can be expanded as

\[
I - S_\varepsilon = I - \frac{1}{4\pi} \sum_d w_d \left( I + \varepsilon H^{(d)} \right)^{-1} \left( I - \varepsilon^2 M_t^{-1} M_a \right)
\]

(39)

where \( \tilde{D}_\varepsilon \) corresponds to the latter term in (39) and is given by

\[
\tilde{D}_\varepsilon = -\frac{1}{4\pi} M_t \sum_d w_d \left( M_t^{-1} (\Omega_d \cdot G + F^{(d)}) \right)^2 - M_t^{-1} M_a
\]

(40)

Recall the identity from (18), \( \Omega_d \cdot G + F^{(d)} = -\Omega_d \cdot G^T + \tilde{F}^{(d)} \), the definitions of \( F_1 = \frac{1}{4\pi} \sum_d w_d \Omega_d F^{(d)} \) and \( \tilde{F}_1 = \frac{1}{4\pi} \sum_d w_d \Omega_d \tilde{F}^{(d)} \) from equation (25), and also that because \( \Omega_d \) is a scalar vector, it commutes in a certain sense; for example,

\[
\Omega_d \cdot G^T M_t^{-1} F^{(d)} = (\Omega_d, \Omega_d, \Omega_d) \cdot (G_1^T, G_2^T, G_3^T) M_t^{-1} F^{(d)}
\]

\[
= \left[ (G_1^T, G_2^T, G_3^T) M_t^{-1} F^{(d)} \right] \cdot (\Omega_d, \Omega_d, \Omega_d)
\]

\[
= \Omega_d \cdot G^T M_t^{-1} \cdot F^{(d)} \Omega_d.
\]

Also recall the outer product summation \( \sum_d w_d \Omega_d \Omega_d^T = \frac{4\pi}{3} I \). Expanding the quadratic term in \( \tilde{D}_\varepsilon \) and plugging in these identities yields

\[
\tilde{D}_\varepsilon = -\frac{1}{4\pi} \sum_d w_d \left( \Omega_d \cdot G^T M_t^{-1} \Omega_d \cdot G + \tilde{F}^{(d)} M_t^{-1} \Omega_d \cdot G -
\right.

\[
\left. \Omega_d \cdot G^T M_t^{-1} F^{(d)} + F^{(d)} M_t^{-1} F^{(d)} - M_a \right)
\]

\[
= \frac{1}{3} \Omega_d \cdot G^T M_t^{-1} G - \tilde{F}_1 \cdot M_t^{-1} G + G^T M_t^{-1} F_1 + M_a -
\]

\[
\frac{1}{4\pi} \sum_d w_d \tilde{F}^{(d)} M_t^{-1} F^{(d)}.
\]

Decompose \( \tilde{D}_\varepsilon = D_0 + D_1 \), where

\[
D_0 = \left( \frac{1}{3} \Omega_d \cdot G^T M_t^{-1} G - \tilde{F}_1 \cdot M_t^{-1} G + G^T M_t^{-1} F_1 + M_a \right),
\]

(41)

\[
D_1 = -\sum_d w_d \tilde{F}^{(d)} M_t^{-1} F^{(d)}.
\]

Then,

\[
I - S_\varepsilon = \varepsilon M_t^{-1} (F_0 + \varepsilon (D_0 + D_1)) + O(\varepsilon^3).
\]

(42)

In the right-hand side of equation (42), the lower-order terms in \( \varepsilon \) exactly take the form of the operator in Lemma 7, where \( PD_1 = D_1 P = 0 \). To that end, from equation (35) in Lemma 7,

\[
(F_0 + \varepsilon (D_0 + D_1))^{-1} = \frac{1}{\varepsilon} \left( \frac{1}{\varepsilon} F_0 + D_0 \right)^{-1} + O(\varepsilon).
\]

(43)
Defining $D_\varepsilon = \frac{1}{\varepsilon} F_0 + D_0$, observe from equations (42) and (43) that

$$
(\varepsilon^2 D_\varepsilon)^{-1} M_t (I - S_\varepsilon) = \frac{1}{\varepsilon} \left( \frac{1}{\varepsilon} F_0 + D_0 \right)^{-1} (F_0 + \varepsilon (D_0 + D_1)) + O(\varepsilon)
$$

$$
= I + O(\varepsilon),
$$

which completes the proof.

\textbf{Proof (Proof of Theorem 4).} The proof of Theorem 4 follows naturally from that of Theorem 3 and Lemma 7. From Lemma 7,

$$
(F_0 + \varepsilon D_0)^{-1} = \frac{1}{\varepsilon} E_P + (I - E_P D_0) E_Q (I - D_0 E_P),
$$

where $E_P = P (P^T D_0 P)^{-1} P^T$ and $E_Q = Q (Q^T F_0 Q)^{-1} Q^T$. Defining $E_\varepsilon = (F_0 + \varepsilon D_0)^{-1}$ and appealing to (42) and (43), observe that

$$
\frac{1}{\varepsilon} E_\varepsilon M_t (I - S_\varepsilon) = \frac{1}{\varepsilon} E_\varepsilon M_t (I - S_\varepsilon) = (F_0 + \varepsilon D_0)^{-1} [(F_0 + \varepsilon (D_0 + D_1)) + O(\varepsilon^2)]
$$

$$
= (F_0 + \varepsilon D_0)^{-1} (F_0 + \varepsilon (D_0 + D_1)) + O(\varepsilon)
$$

$$
= I + O(\varepsilon).
$$

\hfill \Box

\section{4.2. Bilinear form for DG near-nullspace} This section proves the identity (26) relating the DSA matrix (23) to the symmetric interior penalty bilinear form (27). In matrix form, (23) corresponds to the bilinear form

$$
\mathbf{v}^T \left( \frac{1}{\varepsilon} F_0 + \frac{1}{3} G^T M_t^{-1} G - \tilde{\mathbf{F}}_1 \cdot M_t^{-1} \mathbf{G} + \mathbf{G}^T \cdot M_t^{-1} \mathbf{F}_1 + M_a \right) \mathbf{u}.
$$

Several of the relations are straightforward. The term $\mathbf{v}^T M_a \mathbf{u} = \sum_{\kappa \in \mathcal{E}} \int_{\kappa} \sigma_a w_d \text{d}x$ follows immediately from (9). Recalling the definitions of $\alpha(x)$ (28) and $\mathbf{v}^T F^{(d)} \mathbf{u}$ (6), along with the identity $\sum_d w_d \Omega_d = 0$,

$$
\mathbf{v}^T \frac{1}{\varepsilon} F_0 \mathbf{u} = \frac{1}{\varepsilon} \sum_d w_d \mathbf{v}^T F^{(d)} \mathbf{u} = \sum_{\Gamma \in \mathcal{F}} \int_{\Gamma} \alpha \| \mathbf{u} \| \| \mathbf{v} \| \text{d}S.
$$

The remaining terms are slightly more technical, and Section 4.2.1 proves that

$$
\mathbf{v}^T (G^T \cdot M_t^{-1} \mathbf{F}_1) \mathbf{u} = - \sum_{\Gamma \in \mathcal{F}} \int_{\Gamma} \frac{1}{3 \sigma_t} \left\{ \mathbf{n} \cdot \nabla_x \mathbf{v} \right\} \| \mathbf{u} \| \text{d}S,
$$

$$
\mathbf{v}^T \left( \tilde{\mathbf{F}}_1 \cdot M_t^{-1} \mathbf{G} \right) \mathbf{u} = \sum_{\Gamma \in \mathcal{F}} \int_{\Gamma} \frac{1}{3 \sigma_t} \left\{ \mathbf{n} \cdot \nabla_x \mathbf{u} \right\} \| \mathbf{v} \| \text{d}S.
$$

Together the above results combine to yield the identity in (26).

\subsection{4.2.1. Face matrix terms in bilinear form} This section starts with a lemma expressing the action of $M_t^{-1} \mathbf{G}$ in the context of bilinear forms.

\textbf{Lemma 8.} In the case of straight-edged meshes and constant opacities, $1/\sigma_t$, for each mesh element, $\kappa_e$, $(M_t^{-1} \mathbf{G}) \mathbf{u}$ is related to $\frac{1}{\sigma_t} \nabla_x \mathbf{u}$ via

$$
(M_t^{-1})_{\kappa_e, \kappa'_e} \mathbf{G}_{\kappa_e, \kappa'_e} [\mathbf{u}]_{\kappa'_e} = \frac{1}{\sigma_t} \left[ \left( \nabla_x \mathbf{u} \right)_{\kappa'_e, \kappa_e} \right]_{\kappa_e}.
$$

Moreover, for any bilinear form $\mathcal{B}(\mathbf{u}, \mathbf{v})$ with associated matrix $\mathcal{B}$, $\mathcal{B}(\mathbf{u}, \mathbf{v}) = \mathbf{v}^T \mathbf{B} \mathbf{u}$, it holds that

$$
\mathcal{B} \left( \frac{1}{\sigma_t} \partial_{\kappa_e} \mathbf{u}, \mathbf{v} \right) = \mathbf{v}^T \left( \mathcal{B} \left( M_t^{-1} \mathcal{G}^{(j)} \right) \right) \mathbf{u}.
$$
Proof. Without loss of generality, expand \( u(x) \) in a piecewise polynomial basis consisting of interpolating polynomials \( \{u_{e,m}(x)\}_{e,m} \),

\[
u(x) = \sum_m u(x_{e,m}) u_{e,m}(x), \quad x \in \kappa_e.
\]

Since the mesh transformation from the reference element \( \hat{k} \) to the physical element \( \kappa_e \) is linear, \( \partial_{x_j} u_e(x) \) is a polynomial of degree less than or equal to the degree of \( u_e(x) \), and so

\[
\partial_{x_j} u_e(x) = \sum_n u_e(x_{e,n}) \partial_{x_j} u_{e,n}(x) = \sum_n (\partial_{x_j} u_e) (x_{e,n}) u_{e,n}(x).
\]

Therefore,

\[
\int_{\kappa_e} (\partial_{x_j} u_e) u_{e,m} dx = \sum_n \partial_{x_j} u_e(x_{e,n}) \int_{\kappa_e} u_{e,n} u_{e,m} dx = \sum_n u_e(x_{e,n}) \int_{\kappa_e} (\partial_{x_j} u_{e,n}) u_{e,m} dx.
\]

Recalling that \( G^{(j)}_e = \left[ \int_{\kappa_e} u_{e,m} (\partial_{x_j} u_{e,n}) dx \right]_{mn} \) and \( M_{t,e} = \sigma_t \left[ \int_{\kappa_e} u_{e,n} u_{e,m} dx \right]_{mn} \), we can write the above identity as

\[
\sigma_t^{-1}(M_{t,e}) \partial_{x_j} \varphi_e(x_m) = G^{(j)}_e \varphi_e(x_m).
\]

Applying \( (M_{t,e}^{-1})_{e,e} \) to both sides above yields equation (47).

Now consider the bilinear form \( B(\partial_{x_j} u, v) \) and suppose that the matrix \( B \) is such that

\[
v^T B u = B(u, v),
\]

for any \( u(\cdot) \) and \( v(\cdot) \) in the DG space. Then we use the following: if \( \sigma_t \) is constant, then the bilinear form \( B(u, \partial_{x_j} v) \) corresponds to the matrix \( B(M_{t,e}^{-1}G^{(j)}) \). Indeed, letting \( B_{e',e} \) denote the submatrix of \( B \) corresponding to elements \( \kappa_{e'} \) and \( \kappa_e \),

\[
B \left( \frac{1}{\sigma_t} \partial_{x_j} u_e, v \right) = \sum_{e,e',m,n} [v_{e'}(x_{e',m})]_{e'} [B_{e',e} (M_{t,e}^{-1} G^{(j)})]_{e,e'} \frac{1}{\sigma_t} [\partial_{x_j} u_e](x_{e,n})_n
\]

\[
= \sum_{e,e'} v^T_{e'} B_{e',e} \left( (M_{t,e}^{-1} G^{(j)}) \right) u_e
\]

\[
= \sum_{e,e'} v^T_{e'} \left( B_{e',e} (M_{t,e}^{-1} G^{(j)}) \right) u_e
\]

\[
= B \left( (M_{t,e}^{-1} G^{(j)}) \right) u.
\]

Using equations (25) and (17), and the identities \( \sum_d w_d \Omega_d \Omega_d^T = \frac{4\pi}{d} I \) and \( \sum_d w_d \Omega_d |\Omega_d \cdot n| = 0 \),

\[
v^T (F)_{1,j} u = -\sum_{\Gamma \in \mathcal{F}} \int_{\Gamma} \left( \frac{1}{4\pi} \sum_d w_d (\Omega_d)_{j} \Omega_d^T \right) n \{u\} \{v\} dS +
\]

\[
\frac{1}{2} \sum_{\Gamma \in \mathcal{F}} \int_{\Gamma} \left( \frac{1}{4\pi} \sum_d w_d (\Omega_d)_{j} |\Omega_d \cdot n| \right) \|u\| \|v\| dS
\]

\[
= \frac{1}{3} \sum_{\Gamma \in \mathcal{F}} \int_{\Gamma} n_j \{u\} \{v\} dS,
\]

where \( n_j \) denotes the \( j \)th component of the normal vector \( n \). Similarly,

\[
v^T (\tilde{F})_{1,j} u = \sum_{\Gamma \in \mathcal{F}} \int_{\Gamma} \left( \frac{1}{4\pi} \sum_d w_d (\Omega_d)_{j} \Omega_d^T \right) n \{u\} \{v\} dS +
\]

\[
\frac{1}{2} \sum_{\Gamma \in \mathcal{F}} \int_{\Gamma} \left( \frac{1}{4\pi} \sum_d w_d (\Omega_d)_{j} |\Omega_d \cdot n| \right) \|u\| \|v\| dS
\]

\[
= \frac{1}{3} \sum_{\Gamma \in \mathcal{F}} \int_{\Gamma} n_j \{u\} \{v\} dS.
\]
Applying Lemma 8 yields equations (45) and (46).

4.3. Fixed-point iteration on HO meshes. Theorem 5 follows from the following Lemma.

Lemma 9. Consider a linear system of the form in (29), with condensed notation

\[(I + \varepsilon H - B)\psi^{(d)} = q^{(d)}.\]

Denote \(\mathcal{H} := I + \varepsilon H - B\), and consider a matrix splitting \(H = H_\leq + H_\succ\). Define \(\hat{M}^{-1} = (I + \varepsilon H)^{-1}\) as the preconditioner associated with inverting \(I + \varepsilon H\). Now, fix \(B\psi^{(d)}\) and move it to the right-hand side, for the modified linear system

\[(I + \varepsilon H)\psi^{(d)} = q^{(d)} + B\psi_0^{(d)}.\]

Define \(\hat{M}_k^{-1}\) as the preconditioner associated with performing \(k\) fixed-point iterations on (51), with approximate inverse \(\hat{M}_1^{-1} = (I + \varepsilon H_\leq)^{-1}\). Then, applying \(\hat{M}^{-1}\) and \(\hat{M}_k^{-1}\) as preconditioners for (50) is related via

\[\hat{M}_k^{-1}\mathcal{H} = \left(I - (-\varepsilon(I + \varepsilon H_\leq)^{-1}H_\succ)^k\right)\hat{M}^{-1}\mathcal{H}.\]

Proof (Proof of Theorem 5). Consider a problem of the form

\[(I + \varepsilon H_\leq + \varepsilon H_\succ - B)\psi^{(d)} = q^{(d)},\]

where \(\mathcal{H} := (I + \varepsilon H_\leq + \varepsilon H_\succ - B)\). Note the following identities, which will be used regularly:

\[
\begin{align*}
(I + \varepsilon H_\leq + \varepsilon H_\succ - B)^{-1} &= [I - (I + \varepsilon H_\leq + \varepsilon H_\succ)^{-1}B]^{-1}(I + \varepsilon H_\leq + \varepsilon H_\succ)^{-1}, \\
(I + \varepsilon H_\leq + \varepsilon H_\succ - B)^{-1} &= [I - (I + \varepsilon H_\leq)^{-1}(-\varepsilon H_\succ + B)]^{-1}(I + \varepsilon H_\leq)^{-1}, \\
(I + \varepsilon H_\leq + \varepsilon H_\succ)^{-1} &= (I + \varepsilon H_\leq)^{-1}(I + \varepsilon H_\succ)^{-1}. 
\end{align*}
\]

First, consider a single fixed-point iteration, where we invert \(I + \varepsilon H_\leq + \varepsilon H_\succ\). Define \(\hat{M}^{-1} = (I + \varepsilon H_\leq + \varepsilon H_\succ)^{-1}\). Then, the preconditioned linear system is given by \(\hat{M}^{-1}(\mathcal{H}\psi^{(d)} - q^{(d)}) = 0\), where

\[\hat{M}^{-1}\mathcal{H} = (I + \varepsilon H_\leq + \varepsilon H_\succ)^{-1}(I + \varepsilon H_\leq + \varepsilon H_\succ - B) = I - (I + \varepsilon H_\leq + \varepsilon H_\succ)^{-1}B.\]

Now suppose we only invert \(I + \varepsilon H_\leq\), that is, our preconditioner is given by \(\hat{M}_1^{-1} = (I + \varepsilon H_\leq)^{-1}\). This arises, for example, in the case of cycles in the mesh, where we can only directly invert the block lower triangular part. In the interest of asymptotics, additionally consider moving \(B\psi\) to the right-hand side and applying multiple iterations of \(\hat{M}_1^{-1}\) to the modified linear system, \(\hat{M}_1^{-1}\mathcal{H}\psi^{(d)} = q^{(d)} + B\psi_0^{(d)},\) given by

\[(I + \varepsilon H_\leq + \varepsilon H_\succ)\psi^{(d)} = q^{(d)} + B\psi_0^{(d)},\]

where \(\psi_0^{(d)}\) is fixed for all iterations. In a fixed-point sense, this is equivalent to

\[\psi_{k+1}^{(d)} = \psi_k^{(d)} + (I + \varepsilon H_\leq)^{-1}(q^{(d)} + B\psi_0^{(d)} - \varepsilon(\mathcal{H}_\leq + \mathcal{H}_\succ)\psi_k^{(d)}),\]

with error propagation given by

\[I - \hat{M}_1^{-1}\mathcal{H} = I - (I + \varepsilon H_\leq)^{-1}\mathcal{H} = -\varepsilon(I + \varepsilon H_\leq)^{-1}H_\succ.\]

Then, we are interested in the preconditioner \(\hat{M}_k\) that results from taking powers of \(I - \hat{M}_k^{-1}\mathcal{H} = (I - \hat{M}_1^{-1}\mathcal{H})^k\). Solving for \(\hat{M}_k^{-1}\),

\[\hat{M}_k^{-1}\mathcal{H} = I - (-\varepsilon(I + \varepsilon H_\leq)^{-1}H_\succ)^k,\]
\[
\hat{M}_k^{-1} = \hat{R}^{-1} - (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^k \hat{R}^{-1} \\
= (I - (-\varepsilon(I + \varepsilon H_<)^{-1} H_>))^{-1}(I - (-\varepsilon(I + \varepsilon H_<)^{-1} H_> + B))^{-1}(I + \varepsilon H_<)^{-1} \\
= \sum_{\ell=0}^{\infty} (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^\ell - \sum_{\ell=0}^{\infty} (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^\ell(I + \varepsilon H_<)^{-1} \\
= \sum_{\ell=0}^{k-1} (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^\ell(I + \varepsilon H_<)^{-1}.
\]

Now, suppose we apply \(\hat{M}_k^{-1}\) as a preconditioner for the original linear system, \(\mathcal{H}(\psi^{(d)}) = q^{(d)}\), and consider the difference between \(\hat{M}_k^{-1}\) and \(\hat{M}^{-1}\):

\[
\hat{M}_k^{-1} - \hat{M}^{-1} = \sum_{\ell=0}^{k-1} (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^\ell(I + \varepsilon H_<)^{-1} - (I + \varepsilon H_< + \varepsilon H_>)^{-1} \\
= \sum_{\ell=0}^{k-1} (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^\ell - \sum_{\ell=k}^{\infty} (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^\ell(I + \varepsilon H_<)^{-1} \\
= - \sum_{\ell=k}^{\infty} (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^\ell(I + \varepsilon H_<)^{-1} \\
= - (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^k \sum_{\ell=0}^{\infty} (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^\ell(I + \varepsilon H_<)^{-1} \\
= - (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^k \hat{M}^{-1}.
\]

Then,

\[
\hat{M}_k^{-1}\mathcal{H} = \left[I - (-\varepsilon(I + \varepsilon H_<)^{-1} H_>)^k\right]\hat{M}^{-1}\mathcal{H}.
\]

5. Numerical experiments of DSA preconditioning on a HO Lagrangian mesh. This section uses DSA to solve the discrete transport equations (2) on a HO hydrodynamics mesh generated from a purely Lagrangian simulation of the “triple point” problem [11], which is displayed in Figure 1. The mesh is 3rd-order mesh, that is, cubic polynomials are used to map the reference element to physical elements, and our DG discretization uses 3rd-order local basis functions.

![Fig. 1. A “triple point” 3rd-order Lagrangian mesh.](image)

For this problem, we use constant opacities

\[
\sigma_t(x_1, x_2) = \frac{1}{\varepsilon}, \quad \sigma_a = \varepsilon,
\]
and a smooth (but arbitrary) source term

\[ q(x_1, x_2) = \varepsilon \cos^2(2x_1 + x_2), \]

where \( \varepsilon \) is the characteristic mean free path. In our numerical experiments, we vary \( \varepsilon \) from relatively optically thin regimes \( \varepsilon = .75 \), to increasingly optically thick regimes \( \varepsilon = 10^{-j} \), for \( j = 1, 2, 3, 4 \). Last, constant inflow boundary conditions are applied,

\[ \psi_m(x) = 1 \quad \text{when} \quad \Omega_m \cdot n(x) < 0 \quad \text{and} \quad x \in \partial \mathcal{D}. \]

Figure 2 shows the iteration error estimate \( \|\psi_{j+1} - \psi_{j+1}\|_\infty \) as a function of iteration index \( j \), with and without DSA preconditioning. Recall, due to cycles in the mesh, the transport equation for a fixed angle cannot be easily inverted, so we invert the block lower triangular part of the matrix, and refer to this as a “transport sweep.” When DSA preconditioning is included, we consider using a single transport sweep with lagging between DSA steps, as well as using two “inner sweeps,” where the scalar flux is not updated, followed by one normal sweep with lagging between DSA steps (see Theorem 5). Finally, we also consider performing, between every DSA step, three transport sweeps, where the scalar flux is updated after each sweep.
To ensure a fair comparison, the iteration index \( j \) in all five cases displayed in Figure 2 accounts for the same number of transport sweeps; however, because of this, each case has a different interpretation. In the “no DSA” case, the iteration index \( j \) in Figure 2 corresponds to three applications of the fixed-point iteration without any DSA (i.e., (sweep, update flux))\(^3\): for example, \( j = 10 \) corresponds to 30 fixed-point iterations. In the “IP DSA, no inners” case, the iteration index \( j \) represents three applications of a transport sweep and nonsymmetric interior penalty (IP) DSA step (i.e., “(sweep, update flux, IP DSA)\(^3\)”). In the “IP DSA, 2 inners” case, the iteration index \( j \) corresponds to three applications of a transport sweep followed by a scalar flux update and a single IP DSA step (that is, “(sweep)\(^3\), update flux, IP DSA\(”\)). In the “IP DSA, 3 sweeps” case, the iteration index \( j \) represents three applications of both a transport sweep and scalar flux update, followed by a nonsymmetric interior penalty (IP) DSA step (i.e., “(sweep, update flux)\(^3\), IP DSA\(”\)). Finally, the “SIP DSA, 2 inners” case is the same as the “IP DSA, 2 inners” case, but with the symmetric interior penalty DSA matrix used instead. Because the sweep is typically computationally much more expensive than the DSA step, each iteration index in Figure 2 approximately represents the same computational work for each case. In particular, for small \( \epsilon \), Figure 2 provides numerical confirmation of the asymptotic result Theorem 5. Using three sweeps before each IP DSA step leads to a \( 4 \times \) speedup for \( \epsilon = 10^{-4} \) when using the nonsymmetric IP DSA matrix (at a slightly lesser cost as well, due to two less diffusion solves), although the cost increases in the optically thin regime relative to using no additional transport sweeps. In addition, although we didn’t show this in Figure 2, the SIP DSA variant actually diverges for \( \epsilon = 10^{-3} \) and \( \epsilon = 10^{-4} \) when the inner iterations are not performed.

Table 1 displays the \( L^\infty \) residuals of the final iterates,
\[
(52) \quad \max_d \left\| \left( \Omega_d \cdot G + F^{(d)} + \frac{1}{\epsilon} M_d \right) \psi^{(d)} - \frac{1}{4\pi} \left( \frac{1}{\epsilon} M_t - \epsilon M_a \right) \varphi - \frac{1}{4\pi} \left( q^{inc}_d + \epsilon q^{(d)} \right) \right\|_\infty,
\]
as well as the iterations counts. Together, Figure 2 and Table 1 confirm that DSA preconditioning on the HO mesh is effective across a wide range of characteristic mean free paths. Interestingly, although using three transport sweeps between DSA steps is more effective for small \( \epsilon \), for larger values of \( \epsilon \) it is best to apply a DSA step after each sweep.

Note that as \( \epsilon \) gets smaller than \( 10^{-4} \), the DSA preconditioner begins to degrade in effectiveness and ultimately leads to a divergent fixed-point iteration. This degradation in efficiency is likely due to the fact that the condition number of the system (13) scales like \( \mathcal{O}(\epsilon^{-2}) \), and for smaller values of \( \epsilon \) the delicate cancellations in the derivation of the DSA preconditioner can no longer be adequately captured in floating point arithmetic.

6. Conclusions. This paper derives a discrete analysis of DSA applied to high-order DG discretizations of the \( S_N \) transport equations. The basis for DSA is taking a simple fixed-point “source iteration,” which is slow to converge, and recognizing that the slowly decaying error modes can be represented by a certain diffusion operator. DSA then preconditions source iteration with an appropriate diffusion solve, as a correction for these slowly decaying error modes. When the mean free path of particles is very small, \( \epsilon \ll 1 \), conditioning of source iteration is \( \mathcal{O}(1/\epsilon^2) \), and DSA is critical for convergence.

Here, we derive a discrete representation of the slowly decaying error modes for small \( \epsilon \). This leads to the development of a DSA preconditioner that resembles a symmetric interior penalty DG discretization of diffusion-reaction, where the resulting (preconditioned) fixed-point iteration is conditioned like \( 1 + \mathcal{O}(\epsilon) \) (Theorem 3). However, applying this preconditioner requires inverting a DG matrix that is ill-conditioned, \( \kappa \sim \mathcal{O}(1/\epsilon) \), and, furthermore, elliptic DG discretizations are often difficult for fast preconditioners such
as multigrid. This motivates further analysis, where a two-part additive DSA preconditioner is developed based on solving a continuous Galerkin (CG) discretization of diffusion-reaction, in addition to a second term that involves two CG solves, and one solve of a mass-matrix-like term. These solves are now all conditioned independent of $\varepsilon$ and more amenable to fast solvers such as multigrid. Furthermore, the preconditioner leads to a larger fixed-point iteration that is well conditioned, $\kappa \sim 1 + \mathcal{O}(\varepsilon)$, and will converge rapidly for small $\varepsilon$ (Theorem 4).

Finally, there is larger interest in discretizing HO DG on HO (curved) meshes. Source iteration relies on the discretization of advection being block triangular in some ordering and, therefore, easily invertible. However, HO meshes lead to cycles in the mesh, and the resulting discretization of advection in the transport equations is no longer block triangular. When cycles are present, a method to approximate the inversion of advection in source iteration through a pseudo-optimal Gauss-Seidel has been developed in [9]. Theorem 5 extends the handling of cycles to cases where DSA is necessary, proving that cycles can be accounted for by performing an additional two source iterations for each larger DSA iteration. Numerical experiments confirm these results on test problems with highly distorted meshes, demonstrating a speedup of nearly $4\times$ for $\varepsilon = 10^{-4}$.

7. Appendix. To start, we introduce the following technical Lemma regarding the linear system $(I - T_\varepsilon)\psi = \tilde{q}$ (13). This then leads to Proposition 2, which proves that the conditioning of $(I - T_\varepsilon)$ is $\mathcal{O}(\varepsilon^{-2})$, making effective preconditioning critical for small $\varepsilon$.

**Lemma 10.** Define

$$H^{(d)} = M_t^{-1} \left( \Omega_d \cdot G + F^{(d)} \right),$$

$$c_0 = \max_d \left\{ \max \|H^{(d)}\|, \|M_t^{-1} M_a\| \right\},$$

and assume that $\varepsilon\|H^{(d)}\| < 1$. Then, the operator in (13) satisfies

$$[(I - T_\varepsilon)\psi]_d = \left( \psi_d - \frac{1}{4\pi} \varphi \right) + \varepsilon \frac{1}{4\pi} H^{(d)} \varphi - \frac{1}{4\pi} \varepsilon^2 \left( H^{(d)} \right)^2 - M_t^{-1} M_a \varphi + (R_\varepsilon)_d,$$

where the norm of the remainder $[R_\varepsilon]_d$ is bounded by

$$\|R_\varepsilon\| \leq \varepsilon^3 \left( \frac{c_0^3}{1 - \varepsilon c_0} (1 + \varepsilon^2 c_0) + (c_0^2 + \varepsilon c_0^3) \right) \|\varphi\|.$$

**Proof.** Note the matrix identity,

$$\left( I + \varepsilon H^{(d)} \right)^{-1} = I - \varepsilon H^{(d)} + \varepsilon^2 \left( H^{(d)} \right)^2 - \varepsilon^3 \left( H^{(d)} \right)^3 \left( I + \varepsilon H^{(d)} \right)^{-1}.$$

Plugging into the definition of $T_\varepsilon$ and expanding yields

$$[(I - T_\varepsilon)\psi]_d = \left[ I - \frac{1}{4\pi} \left( I + \varepsilon H^{(d)} \right)^{-1} \left( I - \varepsilon^2 M_t^{-1} M_a \right) P_0 \right] \psi$$

$$= \psi - \frac{1}{4\pi} \left[ I - \varepsilon H^{(d)} + \varepsilon^2 \left( H^{(d)} \right)^2 - M_t^{-1} M_a \right] - \varepsilon^3 \left( H^{(d)} \right)^3 \left( I + \varepsilon H^{(d)} \right)^{-1} - H^{(d)} M_t^{-1} M_a$$

$$= \varepsilon^4 \left( H^{(d)} \right)^2 M_t^{-1} M_a + \varepsilon^5 \left( H^{(d)} \right)^3 \left( I + \varepsilon H^{(d)} \right)^{-1} M_t^{-1} M_a \varphi.$$

Equation (54) consists of terms up to $\mathcal{O}(\varepsilon^2)$. Collecting higher-order terms yields the remainder term, $R_\varepsilon$, given by

$$R_\varepsilon = \frac{1}{4\pi} \varepsilon^3 \left( H^{(d)} \right)^3 \left( I + \varepsilon H^{(d)} \right)^{-1} \left( I - \varepsilon^2 M_t^{-1} M_a \right) \varphi +$$
\[ \varepsilon^3 H^{(d)} \left( I - \varepsilon H^{(d)} \right) M_{i-1}^{-1} M_{i\varphi} \cdot. \]

The bound on \( \|R_{\varepsilon}\| \) follows from the identity \( \left\| (I + \varepsilon H^{(d)})^{-1} \right\| \leq \frac{1}{1 - \|H^{(d)}\|} \).

Before stating Proposition 2, we set up preliminary notation. First, the linear system (13) can be written in the form

\[ I - T_{\varepsilon} = I - H_{\varepsilon} P_0, \]

where \( T_{\varepsilon} = H_{\varepsilon} P_0 \) and \( H_{\varepsilon} \) is defined via

\[ (H_{\varepsilon} \psi)_d = \left( I + \varepsilon M_{i-1}^{-1} \left( \Omega_d \cdot G + F^{(d)} \right) \right)^{-1} \frac{1}{4\pi} \left( I - \varepsilon^2 M_{i-1}^{-1} M_{i\varphi} \right) \psi_d, \]

for \( d = 1, \ldots, N_{\Omega} \).

Notice that

\[ (P_0 (I - T_{\varepsilon}) P_0 \psi)_d = \psi - P_0 H_{\varepsilon} \psi = (I - S_{\varepsilon}) P_0 \psi, \]

where \( S_{\varepsilon} \) is defined in equation (15). Also, from Lemma 10,

\[ I - T_{\varepsilon} = Q_0 + \varepsilon H_0 P_0 + \varepsilon^2 H_1 P_0 + O (\varepsilon^3), \]

where \( H_i \) denotes block-diagonal in \( d \) matrices, for \( i = 1, \ldots, N_{\Omega} \) as in (54); in particular,

\[ (H_0)_{d,d} = \frac{1}{4\pi} H^{(d)} = \frac{1}{4\pi} M_{i-1}^{-1} \left( \Omega_d \cdot G + F^{(d)} \right), \quad (H_1)_{d,d} = \frac{1}{4\pi} \left( \left( H^{(d)} \right)^2 - M_{i-1}^{-1} M_{i\varphi} \right). \]

Finally, define \( F_0 = \frac{1}{4\pi} \sum_d w_d F^{(d)} \). Then using \( \sum_d w_d \Omega_d = 0 \), it follows that

\[ (P_0 H_0 P_0 \psi)_d = M_{i-1} F_0 \varphi, \quad d = 1, \ldots, N_{\Omega}. \]

We now prove Proposition 2.

**Proof.** First, choose some unit norm vector \( \psi \) for which \( Q_0 \psi = \psi \). Then, using equation (56),

\[ \|I - T_{\varepsilon}\|_W \geq \|(I - T_{\varepsilon}) Q_0 \psi\|_W = \|Q_0 \psi\|_W = 1. \]

For the inverse, \( (I - T_{\varepsilon}) x = y \) can be decomposed based on \( P_0 \) and \( Q_0 \) via \( (I - T_{\varepsilon})(P_0 x + Q_0 x) = (P_0 y + Q_0 y) \).

Multiplying on the left by the full-column-rank operator \((P_0; Q_0)\) yields the equivalent linear system

\[ \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} (I - T_{\varepsilon}) \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} \begin{pmatrix} P_0 x \\ Q_0 x \end{pmatrix} = \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} (P_0 y + Q_0 y). \]

Denote \( x_P = P_0 x \) and \( x_Q = Q_0 x \), and likewise for \( y \). Then (60) yields a \( 2 \times 2 \) set of equations, which, noting the expansion from Lemma 10 and (58) and the orthogonality of \( P_0 \) and \( Q_0 \), reduces to

\[ \begin{pmatrix} P_0 (I - T_{\varepsilon}) P_0 \\ Q_0 (I - T_{\varepsilon}) P_0 \end{pmatrix} \begin{pmatrix} x_P \\ x_Q \end{pmatrix} = \begin{pmatrix} y_P \\ y_Q \end{pmatrix}. \]

Here, \( x_P \) is fully determined by inverting \( P_0 (I - T_{\varepsilon}) P_0 \) on the range of \( P_0 \). This is equivalent to inverting \( I - S_{\varepsilon} \) (57), which is assumed to be full rank. Now, choose some vector \( \hat{x} \) for which \( P_0 \hat{x} = \hat{x} \), where each direction block \( \hat{x}_d \) corresponds to a continuous function. From (19), we have that \( F_0 P_0 \hat{x} = 0 \) and from (59) \( P_0 H^{(d)} P_0 \hat{x} = 0 \). From equation (58), this yields

\[ \begin{pmatrix} P_0 (I - T_{\varepsilon}) P_0 \\ Q_0 (I - T_{\varepsilon}) P_0 \end{pmatrix} = O (\varepsilon^2) P_0 \hat{y}. \]

Recall by orthogonality, \( \|y\|_W = \|y_P\|_W + \|y_Q\|_W \). Now define a vector \( \tilde{y} \) such that \( \tilde{y}_P = P_0 \tilde{y} \) from (62) and \( \tilde{y}_Q = 0 \), and let \( \tilde{x} = (I - T_{\varepsilon})^{-1} \tilde{y} \). Then, in the notation of (61),

\[ \|(I - T_{\varepsilon})^{-1}\|_W = \sup_{\|y\|_W = 1} \|(I - T_{\varepsilon})^{-1} y\|_W = \sup_{\|y\|_W = 1} \|x_P\|_W + \|x_Q\|_W \]
As well as helping us understand many of the nuances of the nonsymmetric interior penalty method explored in this paper and his consistent P1 diffusion discretization, transport sweeps when there are mesh cycles.

To prove equation (21), note that

\[
\tilde{A}_\varepsilon = (E_\varepsilon P_0 + Q_0)(I - T_\varepsilon)
\]

\[
= E_\varepsilon P_0 (I - T_\varepsilon) P_0 + Q_0 (I - T_\varepsilon)
\]

\[
= P_0 + \mathcal{O}(\varepsilon) + Q_0 - Q_0 T_\varepsilon
\]

\[
= I + \mathcal{O}(\varepsilon) - Q_0 T_\varepsilon
\]

\[
= I + \mathcal{O}(\varepsilon).
\]

In the second equality, we used assumption (20) and the identity \(P_0 (I - T_\varepsilon) = P_0 (I - T_\varepsilon) P_0\). In the last equality, we used that \(Q_0 T_\varepsilon = \mathcal{O}(\varepsilon)\), which follows from equation (58).

Remark 11. Letting \(h_x\) denote the characteristic mesh spacing, the assumption \(\varepsilon \left\| H^{(d)} \right\| < 1\) in Lemma 10 holds if \(\varepsilon \lesssim \sigma_1 h_x\), which corresponds to the optically thick limit.

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