NOETHER-LEFSCHETZ THEORY WITH BASE LOCUS

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Abstract. Let $Z$ be a closed subscheme of a smooth complex projective variety $Y \subseteq \mathbb{P}^N$, with $\dim Y = 2r + 1 \geq 3$. We describe the intermediate Néron-Severi group (i.e. the image of the cycle map $A_r(X) \to H_{2r}(X; \mathbb{Z})$) of a general smooth hypersurface $X \subset Y$ of sufficiently large degree containing $Z$.

Keywords: Noether-Lefschetz Theory, Néron-Severi group, Borel-Moore Homology, Monodromy representation, Isolated singularities, Blowing-up.

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1. Introduction

The classical Noether-Lefschetz Theorem implies that the Néron-Severi group of a (very) general space surface $X \subset \mathbb{P}^3 = \mathbb{P}^3(\mathbb{C})$, with degree $\geq 4$, is generated by the hyperplane class. The proof rests on two main ingredients: a monodromy argument, showing that any class in the Néron-Severi group of $X$ can be lifted to $H^2(\mathbb{P}^3; \mathbb{Q})$ as a rational class; Lefschetz Hyperplane Theorem, saying that the map $H^2(\mathbb{P}^3; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ is injective with torsion-free cokernel.

What can be said, in general, about the $i$-th Néron-Severi group $\text{NS}_i(X; \mathbb{Z})$ (i.e. the image of the cycle map $A_i(X) \to H_{2i}(X; \mathbb{Z}) \cong H^{2\dim X - 2i}(X; \mathbb{Z})$, \cite{11}, §19.1), for a general hypersurface $X$ of a smooth projective variety $Y$? As far as we know the more general result in this direction is due to Moishezon (\cite{23}, Theorem 5.4, pag. 245), who provided a general monodromy-type argument concerning the rational Néron-Severi groups $\text{NS}_i(X; \mathbb{Q})$ ($:= \text{NS}_i(X; \mathbb{Z} \otimes \mathbb{Q})$), from which one deduces that the natural map $\text{NS}_{i+1}(Y; \mathbb{Q}) \to \text{NS}_i(X; \mathbb{Q})$ is surjective, as soon as $h^{m,0}(X) > h^{m,0}(Y)$ ($m + 1 := \dim Y$). Combining with Lefschetz Hyperplane Theorem, saying that the map $H^j(Y; \mathbb{Z}) \to H^j(X; \mathbb{Z})$ is an isomorphism when $j < m$ and injective with torsion-free cokernel when $j = m$, the best one can obtain in general is:

$$\text{NS}_i(X; \mathbb{Z}) \subseteq H^{2m-2i}(Y; \mathbb{Z}), \quad i \geq \frac{m}{2}. \quad (1)$$

So for instance one deduces:

$$\text{NS}_{i+1}(Y; \mathbb{Z}) \cong \text{NS}_i(X; \mathbb{Z}), \quad i \geq \frac{m}{2} \quad (2)$$

when the cohomology of $Y$ is algebraic. Let us say that the unique hard case to prove is the intermediate one: $m = 2i$.

By contrast, very few can be said in general about the Néron-Severi groups in dimensions lower than the intermediate one. For instance, it is still unknown whether the degree of a curve on a general threefold in $\mathbb{P}^4$ is a multiple of the degree.
of the threefold ([14], [22]). So there is no hope to have a general result in small dimension.

The main purpose of our paper is to extend [11] and [2] above to the general hypersurface containing a given base locus (compare with [20], [3], [7]). Let \( Y \subseteq \mathbb{P}^N \) be a smooth complex projective variety of dimension \( m + 1 = 2r + 1 \geq 3 \), \( Z \) be a closed subscheme of \( Y \), and \( \delta \) be a positive integer such that \( \mathcal{I}_{Z,Y}(\delta) \) is generated by global sections. Assume that for \( d \gg 0 \) the general divisor \( X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))| \) is smooth. This implies that \( \dim Z \leq r \) and that, for any \( d \geq \delta \), there exists a smooth hypersurface of degree \( d \) containing \( Z \) [24]. Our main result concerns the intermediate Néron-Severi group \( NS_r(X; \mathbb{Z}) \), the higher cases being rather trivial. It says, roughly speaking, that [11] and [2] above are corrected by a group which is freely generated by the components of the base locus:

**Theorem 1.1.** With notations as above, let \( X \in |H^0(Y, \mathcal{O}_Y(d))| \) be a general divisor containing \( Z \), with \( d \geq \delta + 1 \). Assume that the vanishing cohomology of \( X \) is not of pure Hodge type \((r, r)\). Denote by \( H^m(X; \mathbb{Z}) \) the subgroup of \( H^m(X; \mathbb{Z}) \) generated by the irreducible components \( Z_1, \ldots, Z_\rho \) of \( Z \) of dimension \( r \). Then we have:

(a) \( H^m(X; \mathbb{Z}) \) is a free subgroup of \( H^m(X; \mathbb{Z}) \) of rank \( \rho \);

(b) \( NS_r(X; \mathbb{Z}) = [NS_r(X; \mathbb{Z}) \cap H^m(Y; \mathbb{Z})] \oplus H^m(X; \mathbb{Z}) \);

(c) \( NS_r(X; \mathbb{Q}) = NS_{r+1}(Y; \mathbb{Q}) \oplus H^m(X; \mathbb{Q}) \).

In the case \( Z = \emptyset \), i.e. when \( X \) is simply a general hypersurface section of \( Y \), this result easily follows combining the quoted paper [23] with Lefschetz Hyperplane Theorem. It seems unknown whether the inclusion \( NS_{r+1}(Y; \mathbb{Z}) \subseteq NS_r(X; \mathbb{Z}) \cap H^m(Y; \mathbb{Z}) \) is an equality, but there is some evidence supporting this ([27], Remark 1, p. 490).

The line of proof of our Theorem is the following. Fix smooth divisors \( G \in |H^0(Y, \mathcal{O}_Y(\delta))| \) and \( X \in |H^0(Y, \mathcal{O}_Y(d))| \) containing \( Z \), and put \( W := G \cap X \) (by [19], p. 133, Proposition 4.2.6. and proof) one knows that \( W \) has only isolated singularities. By an inductive method (Theorem 3.3), in part already appearing in [6] and [7], one reduces the proof to identify the subgroup \( I_W(\mathbb{Z}) \subseteq H^m(X; \mathbb{Z}) \) of the invariant cocycles with respect to the monodromy representation on \( H^m(X_i; \mathbb{Z}) \) for the family of smooth divisors \( X_i \) in \( |H^0(Y, \mathcal{O}_Y(d))| \) containing \( W \). In the case of rational coefficients we already know that \( I_W(\mathbb{Q}) = H^m(Y; \mathbb{Q}) + H^m(X_i; \mathbb{Q}) \) [6], where \( H^m(X_i; \mathbb{Q})W \) denotes the image of the push-forward map \( H_m(W; \mathbb{Q}) \to H_m(X_i; \mathbb{Q}) \cong H^m(X_i; \mathbb{Q}) \). However, unlike the case in which \( Y \) is a complete intersection ([7], Theorem 2.3), in our general setting classical Lefschetz Theory is not enough to deduce that \( H^m(X_i; \mathbb{Z}) / [H^m(Y; \mathbb{Z}) + H^m(X_i; \mathbb{Z})W] \) is torsion free, hence that \( I_W(\mathbb{Z}) = H^m(Y; \mathbb{Z}) + H^m(X_i; \mathbb{Z})W \). We are able to overcome this difficulty combining a more refined Lefschetz Theory (see [13], [14], and Theorem 2.1, Lemma 2.9 and Lemma 3.2 below), with a topological description of the blowing-up \( P := Bl_W(Y) \) of \( Y \) along \( W \). This description relies on a sort of a decomposition theorem for the integral homology of \( P \) (Corollary 2.6) for which, even if many
similar results already appear in the literature ([11], [28], [17], [4]), we did not succeed in finding an appropriate reference.

2. The group of invariants $I_W(Z)$.

Let $Y \subseteq \mathbb{P}^N$ be a smooth complex projective variety in $\mathbb{P}^N$, of odd dimension $m+1 = 2r+1 \geq 3$. Fix integers $1 \leq k < d$, and smooth divisors $G \in |H^0(Y, \mathcal{O}_Y(k))|$ and $X \in |H^0(Y, \mathcal{O}_Y(d))|$. Put

$$W := G \cap X.$$  

By ([9], p. 133, Proposition 4.2.6. and proof) one knows that $W$ has only isolated singularities. Let $I_W(Z) \subseteq H^m(X; \mathbb{Z})$ be the group of the invariant cocycles with respect to the monodromy representation on $H^m(X; \mathbb{Z})$ for the family of smooth divisors $X_t$ in $|H^0(Y, \mathcal{O}_Y(d))|$ containing $W$. The aim of this section is to identify this group $I_W(Z)$. In fact we are going to prove that, at least under suitable assumptions (unnecessary in the case of rational coefficients), one has

$$I_W(Z) = H^m(Y; \mathbb{Z}) + H^m(X_t; \mathbb{Z})_W$$  

(see Proposition 2.8 below), where we denote by $H^m(X_t; \mathbb{Z})_W$ the image of the push-forward map $H_m(W; \mathbb{Z}) \to H_m(X_t; \mathbb{Z})$ composed with Poincaré duality $H_m(X_t; \mathbb{Z}) \cong H^m(X_t; \mathbb{Z})$, and we see $H^m(Y; \mathbb{Z})$ contained in $H^m(X_t; \mathbb{Z})$ via pull-back thanks to Lefschetz Hyperplane Theorem (one may give similar definition with $\mathbb{Q}$ instead of $\mathbb{Z}$). Equality (3) relies on the study of the rational map $Y \dashrightarrow \mathbb{P} := \mathbb{P}(H^0(Y, I_{W,Y}(d))^*)$ defined by the linear system $|H^0(Y, I_{W,Y}(d))|$. For a geometric description of this map, we refer to [4], p. 755, and [6], p. 525. Here we simply recall the main properties. Next we will turn to the topology of the blowing-up $P := Bl_W(Y)$ of $Y$ along $W$.

A geometric description of the rational map $Y \dashrightarrow \mathbb{P}$.

(a) Let $P$ be the blowing-up of $Y$ along $W$. For the strict transforms $\tilde{G}$ and $\tilde{X}_t$ of $G$ and $X_t$ in $P$, one has $\tilde{G} \cong G$, and $\tilde{X}_t \cong X_t$ when $G$ is not contained in $X_t$. By [11], 4.4, the rational map $Y \dashrightarrow \mathbb{P} := \mathbb{P}(H^0(Y, I_{W,Y}(d))^*)$ defines a morphism $P \to \mathbb{P}$. Denote by $Q$ the image of this morphism, i.e.:

$$Q := \mathbb{P}(P \to \mathbb{P})$$  

(compare with [10], [14]).

(b) Set $E := \mathbb{P}(\mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d))$. The surjections $\mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d) \to \mathcal{O}_Y(d)$ and $\mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d) \to \mathcal{O}_Y(k)$ give rise to divisors $\Theta \cong Y \subseteq E$ and $\Gamma \cong Y \subseteq E$, with $\Theta \cap \Gamma = \emptyset$. The line bundle $\mathcal{O}_E(\Theta)$ is base point free and the corresponding morphism $E \to \mathbb{P}(H^0(E, \mathcal{O}_E(\Theta))^*) \cong \mathbb{P}$ sends $E$ to the cone $CY$ over $Y$ embedded via $|H^0(Y, \mathcal{O}_Y(d-k))|$. This map contracts $\Gamma$ to the vertex $v_\infty$ of the cone, and $\Theta$ to a general hyperplane section of $CY$. There is a natural closed immersion $P \subset E$,
and the trace of $|\Theta|$ on $P$, giving the linear series spanned by the strict transforms $\tilde{X}_t$, induces the map $P \to Q$. Hence we have a natural commutative diagram:

$$
P \hookrightarrow E \\
\downarrow \searrow \searrow \\
Y \dashrightarrow Q \hookrightarrow CY \subseteq \mathbb{P}.
$$

(c) Moreover one has: $\Gamma \cap P = \tilde{G}$; the map $P \to Q$ contracts $\tilde{G}$ to $v_\infty \in Q$: $P \setminus \tilde{G} \cong Q \setminus \{v_\infty\}$; the hyperplane sections $Q_t$ of $Q$, not containing the vertex, are isomorphic, via $P \to Q$, to the corresponding divisors $X_t \in |H^0(Y, \mathcal{I}_{w_t}(d))|$; the monodromy representation on $H^m(X_t; \mathbb{Z})$ for the family of smooth divisors $X_t$ in $|H^0(Y, \mathcal{O}_Y(d))|$ containing $W_t$ identifies with the monodromy representation on $H^m(Q_t; \mathbb{Z})$ for the family of smooth hyperplane sections $Q_t$ of $Q$; so as $W$, also $P$ and $Q$ have only isolated singularities.

The following Theorem [24.1] applies to $Q$ (with $Q = R$, $m = n$, and $I_{W}(\mathbb{Z}) = I$). Recall that the inclusion $X_t \cong Q_t \subset Q$ induces a Gysin map $H_{m+2}(Q; \mathbb{Z}) \to H_{m}(X_t; \mathbb{Z})$ (see [1], or [11], p. 382, Example 19.2.1).

**Theorem 2.1.** Let $R \subseteq \mathbb{P}^n$ be an irreducible, reduced, non-degenerate projective variety of dimension $n + 1 \geq 2$, with isolated singularities, and let $R_t$ be a general hyperplane section of $R$. Denote by $i^*_t : H_{k+2}(R_t; \mathbb{Z}) \to H^{2n-k}(R_t; \mathbb{Z})$ the map obtained composing the Gysin map $H_{k+2}(R; \mathbb{Z}) \to H_k(R_t; \mathbb{Z})$ with Poincaré duality $H_k(R_t; \mathbb{Z}) \cong H^{2n-k}(R_t; \mathbb{Z})$. Then the following properties hold true.

(a) For any integer $n < k \leq 2n$ the map $i^*_t$ is an isomorphism, the map $i^*_t$ is injective with torsion-free cokernel, and $H_{n+2}(R_t; \mathbb{Z}) \cong I$ via $i^*_n$, where $I \subseteq H^n(R_t; \mathbb{Z})$ denotes the invariant subgroup given by the monodromy action on the cohomology of $R_t$.

(b) For any even integer $n < k = 2i \leq 2n$ the map $i^*_k \otimes_{\mathbb{Z}} \mathbb{Q}$ induces an isomorphism $NS_{i+1}(R; \mathbb{Q}) \cong NS_i(R_t; \mathbb{Q})$.

(c) If $k = 2i = n$ and the orthogonal complement $V$ of $I \otimes_{\mathbb{Z}} \mathbb{Q}$ in $H^n(R_t; \mathbb{Q})$ is not of pure Hodge type $(n/2, n/2)$, then $NS_i(R_t; \mathbb{Z}) \cong I$, and the map $i^*_k \otimes_{\mathbb{Z}} \mathbb{Q}$ induces an isomorphism $NS_{i+1}(R; \mathbb{Q}) \cong NS_i(R_t; \mathbb{Q})$.

**Proof.** (a) From Borel-Moore homology exact sequence: $0 = H^{BM}_{k+2}(\text{Sing}(R); \mathbb{Z}) \to H^{BM}_{k+2}(R; \mathbb{Z}) \to H^{BM}_{k+1}(\text{Sing}(R); \mathbb{Z}) \to H^{BM}_{k}(\text{Sing}(R); \mathbb{Z}) = 0$ ([11], p. 371, and [12], p. 219, Lemma 3) we see that $H_{k+2}(R; \mathbb{Z}) \cong H^{BM}_{k+2}(R; \mathbb{Z}) \cong H^{BM}_{k+1}(\text{Sing}(R); \mathbb{Z})$, and that in the projective case Borel-Moore and singular homology agree ([12], p. 217). On the other hand by ([12], p. 217, (26)) we have $H^{BM}_{k+2}(R; \mathbb{Z}) \cong H^{2n-k}(R; \mathbb{Z})$, and so $H_{k+2}(R; \mathbb{Z}) \cong H^{2n-k}(R; \mathbb{Z})$. Therefore $i^*_k$ identifies with the pull-back $H^{2n-k}(R; \mathbb{Z}) \to H^{2n-k}(R_t; \mathbb{Z})$. Now, by the Lefschetz Theorem with Singularities (see [13], p. 199 or also [10], p. 552) we know that the pair $(R; \text{Sing}(R), R_t)$ is $n$-connected ([20], p. 373). From the relative Hurewicz Isomorphism Theorem and the Universal Coefficient Theorem ([20], p. 397 and p. 243) it follows that $H^{2n-k}(R; \text{Sing}(R), R_t; \mathbb{Z}) = 0$.


for $2n - k \leq n$, and that $H^{n+1}(R; \text{Sing}(R), R_i; \mathbb{Z})$ is torsion-free. This implies that $H_{k+2}(R; \mathbb{Z}) \cong H^{2n-k}(R; \text{Sing}(R); \mathbb{Z}) \to H^{2n-k}(R_i; \mathbb{Z})$ is an isomorphism for $2n - k < n$, and injective with torsion-free cokernel when $k = n$.

It remains to prove that $H_{n+2}(R; \mathbb{Z}) = 1$. Since $i_n^*$ is injective with torsion free cokernel, it will suffice to prove that the space $I \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq H^n(R_i; \mathbb{Q})$ of invariants with rational coefficients is equal to the image of the injective map $i_n^* \otimes_{\mathbb{Z}} \mathbb{Q} : H_{n+2}(R; \mathbb{Q}) \to H^n(R_i; \mathbb{Q})$. This is a consequence of Deligne Invariant Subspace Theorem (25. p. 165) in view of the following reasoning.

Let $\tilde{R} \to R$ be a desingularization of $R$, and let $L$ be a general pencil of hyperplane sections of $R$. Denote by $\tilde{R}_L$ the blowing-up of $\tilde{R}$ along the base locus $B_L$ of $L$, and consider the induced maps $\tilde{R}_L \to \tilde{R} \to R$. By Decomposition Theorem (8, Proposition 5.4.4 p. 157, and Corollary 5.4.11 p. 161) we know that $H^{n+2}(R; \mathbb{Q})$ is naturally embedded in $H^{n+2}(\tilde{R}; \mathbb{Q})$ via pull-back. Therefore the push-forward $H_{n+2}(\tilde{R}; \mathbb{Q}) \to H_{n+2}(R; \mathbb{Q})$ is surjective. We deduce that the image of $H_{n+2}(R; \mathbb{Q})$ in $H^n(R_i; \mathbb{Q})$ is equal to the image of $H_{n+2}(\tilde{R}; \mathbb{Q})$ via Gysin map composed with Poincaré duality $H_{n+2}(\tilde{R}; \mathbb{Q}) \to H_n(R_i; \mathbb{Q}) \cong H^n(R_i; \mathbb{Q})$. On the other hand by the decomposition $H_{n+2}(\tilde{R}_L; \mathbb{Q}) = H_{n+2}(\tilde{R}; \mathbb{Q}) \oplus H_n(B_L; \mathbb{Q})$ (25, p. 170, Théorème 7.31) we see that the image of $H_{n+2}(\tilde{R}_L; \mathbb{Q})$ in $H^n(R_i; \mathbb{Q})$ is equal to the image of $H_{n+2}(\tilde{R}; \mathbb{Q})$ plus the image of the push-forward $H_n(B_L; \mathbb{Q}) \to H_n(R_i; \mathbb{Q}) \cong H^n(R_i; \mathbb{Q})$. But this last image is contained in the image of $H_{n+2}(R; \mathbb{Q})$: in fact by Poincaré duality and Lefschetz Hyperplane Theorem we have $H_n(B_L; \mathbb{Q}) \cong H^{n-2}(B_L; \mathbb{Q}) \cong H^{n-2}(R_i; \mathbb{Q}) \cong H_{n+2}(R_i; \mathbb{Q})$, so $H_n(B_L; \mathbb{Q})$ arrives in $H^n(R_i; \mathbb{Q})$ passing through the push-forward $H_{n+2}(R_i; \mathbb{Q}) \to H_{n+2}(R; \mathbb{Q})$. It follows that the image of $H_{n+2}(R; \mathbb{Q})$ in $H^n(R_i; \mathbb{Q})$ is equal to the image of $H_{n+2}(\tilde{R}_L; \mathbb{Q})$ in $H^n(R_i; \mathbb{Q})$. Taking into account the Poincaré duality $H_{n+2}(\tilde{R}_L; \mathbb{Q}) \cong H^n(R_i; \mathbb{Q})$, this image is the invariant space $I \otimes_{\mathbb{Z}} \mathbb{Q}$ by the quoted Deligne Theorem.

(b) In view of previous property, we only have to prove that the map $NS_{i+1}(R; \mathbb{Q}) \to NS_i(R_i; \mathbb{Q})$ is surjective for any $n < k = 2i \leq 2n$. We argue by induction on $n$ and by decreasing induction on $k$, the cases $n = 1$ and $k = 2n$ being trivial. Hence assume $n > 1$ and $n < k = 2i < 2n$. As before, let $L := \{R_i\}_{i \in \mathbb{Z}}$ be a general pencil of hyperplane sections of $R$, and denote by $R_L$ the blowing-up of $R$ at the base locus $B_L$. By previous property (a) we know that all cycles in $H_{2i}(R_i; \mathbb{Z})$ are invariant. Based on this, the following argument proves that $NS_{i+1}(R_L; \mathbb{Q})$ maps onto $NS_i(R_i; \mathbb{Q})$ (compare with 23, p. 242, Lemma 2 and proof).

In fact, fix any algebraic class $\xi \in NS_i(R_i; \mathbb{Q})$, which we may assume represented by some projective algebraic subvariety $S_1 \subseteq R_i$ of dimension $i$, and consider the Hilbert scheme $S$, with reduced structure, parametrizing pairs $(S, R')$, with $R'$ a hyperplane section of $R$, and $S \subseteq R'$ a projective subvariety of dimension $i$. Let $C \subseteq S$ be an irreducible projective curve passing through the point $(S_1, R_i)$. Since $R_i$ is Noether-Lefschetz general, we may assume $C$ dominating $L$ and such that $t$ is a regular value of the natural branched covering map $\pi : C \to L$. The fibres of $\pi$ sweep out a projective subvariety $T \subseteq R_L$ of dimension $i + 1$, whose intersection with $R_i$ is the union of all the subvarieties $S_h$, $h = 1, \ldots, p$, corresponding to the fibre of $\pi$ over the point $t \in L$ ($p = \text{degree of } \pi$). Since the monodromy of $\pi$ is
transitive, by (a) we deduce that all the \( S_k \) are homologous in \( R_t \), and therefore 
\( \xi \) comes from \( \frac{1}{2} \cdot T \) through the natural map \( NS_{i+1}(R_L; \mathbb{Q}) \to NS_i(R_t; \mathbb{Q}) \). This proves that \( NS_{i+1}(R_L; \mathbb{Q}) \) maps onto \( NS_i(R_t; \mathbb{Q}) \).

On the other hand \( NS_{i+1}(R; \mathbb{Q}) \oplus NS_{i+1}(B_L \times \mathbb{P}^1; \mathbb{Q}) \) maps onto \( NS_{i+1}(R_t; \mathbb{Q}) \) (\([11]\), Proposition 6.7, (e), p. 115). Hence \( NS_{i+1}(R; \mathbb{Q}) \oplus NS_i(B_L; \mathbb{Q}) \) maps onto \( NS_i(R_t; \mathbb{Q}) \). Now by induction \( NS_{i+1}(R_t; \mathbb{Q}) \) maps onto \( NS_i(B_L; \mathbb{Q}) \), and \( NS_{i+2}(R; \mathbb{Q}) \) maps onto \( NS_{i+1}(R_t; \mathbb{Q}) \). This means that the cycles of \( NS_i(B_L; \mathbb{Q}) \) arrive in \( NS_i(R_t; \mathbb{Q}) \) as cycles coming from \( NS_{i+2}(R; \mathbb{Q}) \), hence as cycles coming from \( NS_{i+1}(R; \mathbb{Q}) \).

\( \square \)

\( \text{(c)} \) Now assume \( k = 2i = n \). Since \( NS_{n/2}(R_t; \mathbb{Q}) \) is globally invariant (\([19]\), p. 207, Theorem 13.18 and proof), \( V \) is not of pure Hodge type \((n/2, n/2)\), and \( V \) is irreducible (\([8]\), Theorem 3.1), by a standard argument (compare e.g. with \([7]\), proof of Theorem 1.1) it follows that \( NS_{n/2}(R_t; \mathbb{Q}) \subseteq I \otimes_{\mathbb{Q}} \mathbb{Q} \). Then previous argument we used in proving (b) works well again to prove that \( NS_{n/2+1}(R; \mathbb{Q}) \cong NS_{n/2}(R_t; \mathbb{Q}) \) (in this case \( NS_{n/2+1}(R_t; \mathbb{Q}) \) maps onto \( NS_{n/2}(B_L; \mathbb{Q}) \), and \( NS_{n/2+2}(R; \mathbb{Q}) \) maps onto \( NS_{n/2+1}(R_t; \mathbb{Q}) \)) by (b). Finally we notice that the inclusion \( NS_{n/2}(R_t; \mathbb{Q}) \subseteq I \otimes_{\mathbb{Q}} \mathbb{Q} \) implies that \( NS_{n/2}(R_t; \mathbb{Z}) \subseteq I \) for \( H^0(R_t; \mathbb{Z})/I \) is torsion free by (a).

\textbf{Remark 2.2.} We will not need this fact but a similar argument as before shows that all cycles in \( H_k(R_t; \mathbb{Z}) \) are invariant also for \( 0 \leq k < n \), and if \( 0 \leq k = 2i \leq n \) is even and \( h^{n,0}(R_t) > h^{n,0}(\tilde{R}) \), then \( i_k \otimes_{\mathbb{Z}} \mathbb{Q} \) induces a surjection of \( NS_{i+1}(R; \mathbb{Q}) \) onto \( NS_i(R_t; \mathbb{Q}) \) (compare with \([23]\), p. 245, Theorem 5.4). Indeed for \( k < n \) \( H_k(B_L; \mathbb{Z}) \) maps onto \( H_k(R_t; \mathbb{Z}) \) by Lefschetz Hyperplane Theorem. Moreover, since \( h^{n,0}(R_t) > h^{n,0}(\tilde{R}) \) then \( h^{n-1,0}(B_L) > h^{n-1,0}(R_t) \), and therefore one may use induction as in (b).

\[ A \text{ topological description of the blowing-up } P = Bl_W(Y). \]

We are going to prove there is a natural isomorphism

\[ H_k(P; \mathbb{Z}) \cong H_{k-2}(W; \mathbb{Z}) \oplus H_k(Y; \mathbb{Z}) \]

for any \( k \) (see Corollary \([2,8]\) below), from which we deduce that \( H^n(Y; \mathbb{Z}) + H^n(X_t; \mathbb{Z})w \) is equal to the image of the map \( H_{m+2}(P; \mathbb{Z}) \to H_n(X_t; \mathbb{Z}) \) obtained composing the Gysin map \( H_{m+2}(P; \mathbb{Z}) \to H_m(X_t; \mathbb{Z}) \), induced by the natural inclusion \( X_t \subset P \), with Poincaré duality (Corollary \([2,7]\)). This is an intermediate step to prove \([3]\). Recall that \( P \) can have isolated singularities, so, to prove \((4)\), we cannot apply (\([28]\), p. 170, Théorème 7.31). For similar results compare also with (\([11]\), Proposition 6.7, (e), p. 115), (\([17]\), Proposition 4.5, p. 43), and (\([8]\), Corollary 3, p. 371). Next, comparing \( P \) with \( Q \), and using Theorem \([2,1]\) we will prove \((4)\) (see Proposition \([2,3]\) below).

In order to prove \((4)\), we need some preliminaries.
Notations 2.3. (i) Consider the natural commutative diagram:

\[ \begin{array}{ccc}
\widetilde{W} & \xrightarrow{i} & P \\
\downarrow g & & \downarrow f \\
W & \xrightarrow{j} & Y
\end{array} \]

where \( \widetilde{W} = P(\mathcal{O}_W(-k) \oplus \mathcal{O}_W(-d)) \) denotes the exceptional divisor on \( P \). Put \( U = P \setminus \widetilde{W} \cong Y \setminus W \), and consider, for any integer \( k \), the following natural commutative diagram in Borel-Moore Homology Theory ([12], p. 219, Exercise 5):

\[
\begin{array}{cccc}
H^{BM}_{k+1}(U; \mathbb{Z}) & \xrightarrow{\partial} & H_k(\widetilde{W}; \mathbb{Z}) & \cong H_k(P; \mathbb{Z}) & \rightarrow & H^{BM}_k(U; \mathbb{Z}) \\
\downarrow g & & \downarrow f & & \downarrow \circ \ \\
H^{BM}_{k+1}(U; \mathbb{Z}) & \xrightarrow{\partial} & H_k(W; \mathbb{Z}) & \xrightarrow{i} & H_k(Y; \mathbb{Z}) & \rightarrow & H^{BM}_k(U; \mathbb{Z})
\end{array}
\]  

(5)

(as before, in the projective case, we identify Borel-Moore and singular homology groups).

(ii) Besides the push-forward maps \( f_* \) and \( g_* \) we may consider the Gysin maps \( f^* : H_k(Y; \mathbb{Z}) \rightarrow H_k(P; \mathbb{Z}) \) and \( g^* : H_{k-2}(W; \mathbb{Z}) \rightarrow H_k(\widetilde{W}; \mathbb{Z}) \) (see [1], or [11], p. 382, Example 19.2.1), and the map \( g^* : H_k(\widetilde{W}; \mathbb{Z}) \rightarrow H_k(\tilde{W}; \mathbb{Z}) \) defined composing the Gysin map \( g^* : H_k(W; \mathbb{Z}) \rightarrow H_{k+2}(\tilde{W}; \mathbb{Z}) \) with the cap-product \( \smile [P]_{\tilde{W}} = H_{k+2}(\tilde{W}; \mathbb{Z}) \rightarrow H_k(\tilde{W}; \mathbb{Z}) \). Here \( \lfloor P \rfloor_{\tilde{W}} \) denotes the pull-back in \( H^2(\tilde{W}; \mathbb{Z}) \) of the orientation class \( [P] \in H^2(E, E \setminus P; \mathbb{Z}) \cong H_{2m+2}(P; \mathbb{Z}) \) defined by \( P \subset E \) ([11], p. 371 and p. 378).

(iii) More generally, consider a line bundle \( \mathcal{O}_E(D) \), and denote by \( [D] \in H^2(E; \mathbb{Z}) \) its cohomology class. We may define a map \( g^*_D : H_k(W; \mathbb{Z}) \rightarrow H_k(\tilde{W}; \mathbb{Z}) \) in a similar way as \( g^*_g \), using \( [D]_{\tilde{W}} \) instead of \( [P]_{\tilde{W}} \). Observe that \( g^*_D = g^*_{[D]} \) because \( \mathcal{O}_E(P) = \mathcal{O}_E(\Theta + k\Lambda) \) (\( \Lambda := \) pull-back of the hyperplane section of \( Y \subseteq \mathbb{P}^N \) through \( E \rightarrow Y \)).

(iv) We denote by \( \gamma : H_k(P; \mathbb{Z}) \rightarrow H_{k-2}(W; \mathbb{Z}) \) the map obtained composing the Gysin map \( H_k(P; \mathbb{Z}) \rightarrow H_{k-2}(W; \mathbb{Z}) \) with push-forward \( H_{k-2}(\tilde{W}; \mathbb{Z}) \rightarrow H_{k-2}(W; \mathbb{Z}) \), and by \( \mu_k : H_k(G; \mathbb{Z}) \rightarrow H_{k-2}(W; \mathbb{Z}) \) and \( \nu_k : H_k(G; \mathbb{Z}) \rightarrow H_k(Y; \mathbb{Z}) \) the Gysin map and the push-forward.

Lemma 2.4. The following properties hold true.

(a) \( j_* \circ g^* = f^* \circ i_* \), and \( f_* \circ f^* = \text{id}_{H_k(Y; \mathbb{Z})} \);

(b) for any \( D = \pm \Theta + l\Lambda, l \in \mathbb{Z} \), one has \( g_* \circ g^*_{[D]} = \pm \text{id}_{H_k(W; \mathbb{Z})} \); in particular \( g_* \circ g^* = \text{id}_{H_k(W; \mathbb{Z})} \), and \( g^*_{[D]} \) and \( g^* \) are injective;

(c) \( \exists (g^* : H_{k-2}(W; \mathbb{Z}) \rightarrow H_k(\tilde{W}; \mathbb{Z})) = \ker(g_* : H_k(\tilde{W}; \mathbb{Z}) \rightarrow H_k(W; \mathbb{Z})) \);

(d) \( \gamma \circ j_* \circ g^* = -\text{id}_{H_{k-2}(W; \mathbb{Z})} \); in particular \( j_* \circ g^* \) embeds \( H_{k-2}(W; \mathbb{Z}) \) into \( H_k(P; \mathbb{Z}) \);
(c) the diagram obtained from (6) replacing $g_*$ and $f_*$ with $g^!$ and $f^*$ is commutative:

$$
\begin{align*}
H_{k+1}^B(U; \mathbb{Z}) & \xrightarrow{g} H_k(W; \mathbb{Z}) & \xrightarrow{j} H_k(P; \mathbb{Z}) & \rightarrow H_k^B(U; \mathbb{Z}) \\
\| & \| & \| & \|
H_{k+1}^B(U; \mathbb{Z}) & \xrightarrow{g^!} H_k(W; \mathbb{Z}) & \xrightarrow{j^*} H_k(Y; \mathbb{Z}) & \rightarrow H_k^B(U; \mathbb{Z}).
\end{align*}
$$

Proof. (a) and (b) By functoriality, one may construct $f^*$ in a similar way as $g^!$, i.e. composing the Gysin map $H_k(Y; \mathbb{Z}) \rightarrow H_{k+2}(E; \mathbb{Z})$ with the cap-product $\smile [P] : H_{k+2}(E; \mathbb{Z}) \rightarrow H_k(P; \mathbb{Z})$ (p. 256). Therefore the equality $j_* \circ g^! = f^* \circ i_*$ follows from the commutativity of the two little squares in the following diagram:

$$
\begin{align*}
H_k(W; \mathbb{Z}) & \xrightarrow{j} H_k(P; \mathbb{Z}) \\
\sim \smile [P] & \rightarrow H_k(Y; \mathbb{Z}) \\
H_{k+2}(W; \mathbb{Z}) & \rightarrow H_{k+2}(E; \mathbb{Z}) \\
\uparrow & \uparrow \\
H_k(W; \mathbb{Z}) & \xrightarrow{i_*} H_k(Y; \mathbb{Z})
\end{align*}
$$

(the lower vertical maps are the Gysin maps, and $H_{k+2}(W; \mathbb{Z}) \rightarrow H_{k+2}(E; \mathbb{Z})$ is the push-forward).

As for the map $f_* \circ f^*$, first observe that it is equal to the composition of the map $H_k(Y; \mathbb{Z}) \rightarrow H_{k+2}(E; \mathbb{Z})$ with $\smile [P] : H_{k+2}(E; \mathbb{Z}) \rightarrow H_k(E; \mathbb{Z})$ and push-forward $H_k(E; \mathbb{Z}) \rightarrow H_k(Y; \mathbb{Z})$. Now pick any $y \in H_k(Y; \mathbb{Z})$ and denote by $\widetilde{y}$ its image in $H_{k+2}(E; \mathbb{Z})$. Since capping $\widetilde{y}$ with $[\Lambda]$ and pushing-forward it gives $0 \in H_k(Y; \mathbb{Z})$, then previous composition is the same as composing $H_k(Y; \mathbb{Z}) \rightarrow H_{k+2}(E; \mathbb{Z})$ with $\smile [\Theta] : H_{k+2}(E; \mathbb{Z}) \rightarrow H_k(E; \mathbb{Z})$ and push-forward $H_k(E; \mathbb{Z}) \rightarrow H_k(Y; \mathbb{Z})$ (recall that $\mathcal{O}_E(P) = \mathcal{O}_E(\Theta + k\Lambda)$). And this map is equal to $\text{id}_{H_k(Y; \mathbb{Z})}$ because $\Theta \equiv Y$ is a section of the projective bundle $E \rightarrow Y$.

For the same reason one has $g_* \circ g^![D] = g_* \circ g^!_{\pm\Theta} = \pm \text{id}_{H_k(W; \mathbb{Z})}$.

(c) Fix any $D = \pm \Theta + l\Lambda$. Since $[D]$ restricts to $\pm 1 \in H^2(g^{-1}(w); \mathbb{Z})$ for any $w \in W$, then it determines a cohomology extension of the fiber-bundle $g : \widetilde{W} \rightarrow W$ (p. 256). This in turn induces, by the Leray-Hirsh Theorem (p. 258), a decomposition

$$
H_k(\widetilde{W}; \mathbb{Z}) \cong H_k(W; \mathbb{Z}) \oplus H_{k-2}(W; \mathbb{Z})
$$

given by the isomorphism $\tilde{w} \in H_k(\widetilde{W}; \mathbb{Z}) \rightarrow (g_*(\tilde{w}), g_*(\tilde{w} \smile [D]_{\widetilde{W}})) \in H_k(W; \mathbb{Z}) \oplus H_{k-2}(W; \mathbb{Z})$. Using (b) one sees that the inverse map $H_k(W; \mathbb{Z}) \oplus H_{k-2}(W; \mathbb{Z}) \rightarrow H_k(\widetilde{W}; \mathbb{Z})$ is the sum of $g^![D] : H_k(W; \mathbb{Z}) \rightarrow H_k(\widetilde{W}; \mathbb{Z})$ with $g^* : H_{k-2}(W; \mathbb{Z}) \rightarrow H_k(\widetilde{W}; \mathbb{Z})$. It follows that $3\gamma g^* = \ker g_*$.

(d) Since $\mathcal{O}_P(\widetilde{W}) \otimes \mathcal{O}_{\widetilde{W}} = \mathcal{O}_E(-\Theta + d\Lambda) \otimes \mathcal{O}_W$ then composing $j_* \circ g^* : H_{k-2}(W; \mathbb{Z}) \rightarrow H_k(P; \mathbb{Z})$ with Gysin map $H_k(P; \mathbb{Z}) \rightarrow H_{k-2}(\widetilde{W}; \mathbb{Z})$ we get the map $g^![D]$ induced by $D = -\Theta + d\Lambda$. Therefore $\gamma j_* \circ g^* = g_* \circ g^![D] = -\text{id}_{H_k-2(W; \mathbb{Z})}$ by (b).

(e) The central square diagram in (11) commutes by (a). Moreover the right square commutes by functoriality of the Gysin maps. So we only have to prove the left square diagram commutes. To this purpose, denote by $h_* : H_{k+1}^B(U; \mathbb{Z}) \rightarrow$
$H^{BM}_{k+1}(U; \mathbb{Z})$ the push-forward isomorphism on the left of the diagram (5), so that we have $\partial \circ h_* = g_* \circ \partial$. We have to prove that $\partial = g^! \circ \partial \circ h_*$. Pick any $u \in H^{BM}_{k+1}(U; \mathbb{Z})$ and let $\tilde{w} := \partial(u) - (g^! \circ \partial \circ h_*)(u) \in H_k(\widetilde{W}; \mathbb{Z})$. By (b) and the commutativity of (5) we have $g_*(\tilde{w}) = g_* (\partial(u)) = g_* (\partial(h_*(u))) = \partial(h_*(u)) = 0$. Therefore by (c) we deduce that $\tilde{w} = g^*(w)$ for some $w \in H_{k-2}(W; \mathbb{Z})$. On the other hand by (a) and (5) we see that $j_* (g^*(w)) = j_* (\tilde{w}) = j_* (\partial(u)) = j_* (g^1 \circ \partial \circ h_*)(u) = -(f^* \circ i_* \circ \partial \circ h_*)(u) = 0$. This implies $\tilde{w} = 0$ because by (d) we know that $j_* \circ g^*$ is injective.

Proposition 2.5. The sequence

$$0 \rightarrow H_k(W; \mathbb{Z}) \xrightarrow{\alpha} H_k(W; \mathbb{Z}) \oplus H_k(Y; \mathbb{Z}) \xrightarrow{\beta} H_k(P; \mathbb{Z}) \rightarrow 0$$

with $\alpha(w) = (g^1(w), -i_* (w))$ and $\beta(\tilde{w}, y) = j_* (\tilde{w}) + f^*(y)$, is exact.

Proof. The map $\alpha$ is injective because $g^1$ is by Lemma 2.4 (b). To prove that $\beta$ is surjective, fix $p \in H_k(P; \mathbb{Z})$. From the commutativity of (5) and (9) (Lemma 2.4 (e)), it follows that the image of $(f^* \circ f_*)(p) - p$ in $H^{BM}_k(U; \mathbb{Z})$ is 0. Hence there exists $\tilde{w} \in H_k(W; \mathbb{Z})$ such that $j_* (\tilde{w}) = (f^* \circ f_*)(p) - p$, i.e. $p = (f^* \circ f_*)(p) - j_* (\tilde{w}) = \beta(-\tilde{w}, f_*(p))$. This proves that $\beta$ is surjective. Finally observe that $\beta \circ \alpha = 0$ by the commutativity of (5), and that $\ker(\beta) \subseteq \Im(\alpha)$ by diagram chasing in (9).

Corollary 2.6. The map $(w, y) \in H_{k-2}(W; \mathbb{Z}) \oplus H_k(Y; \mathbb{Z}) \rightarrow j_* (g^*(w)) + f^*(y) \in H_k(P; \mathbb{Z})$ is an isomorphism, whose inverse map is given by $p \in H_k(P; \mathbb{Z}) \rightarrow (\gamma(p), f_*(p)) \in H_{k-2}(W; \mathbb{Z}) \oplus H_k(Y; \mathbb{Z})$. In particular, via this isomorphism, the push-forward $H_k(G; \mathbb{Z}) \rightarrow H_k(P; \mathbb{Z})$ identifies with the map $x \in H_k(G; \mathbb{Z}) \rightarrow (\mu_k(x), \nu_k(x)) \in H_{k-2}(W; \mathbb{Z}) \oplus H_k(Y; \mathbb{Z})$.

Proof. By Proposition 2.5 we know that for any $p \in H_k(P; \mathbb{Z})$ there are $\tilde{w} \in H_k(W; \mathbb{Z})$ and $y \in H_k(Y; \mathbb{Z})$ such that $p = j_* (\tilde{w}) + f^*(y)$. By (9) we may write $\tilde{w} = g^1(w_1) + g^2(w_2)$ for suitable $w_1 \in H_k(W; \mathbb{Z})$ and $w_2 \in H_{k-2}(W; \mathbb{Z})$. Therefore $p = j_* (g^1(w_1)) + j_* (g^2(w_2)) + f^*(y)$, and by Lemma 2.4 (a), we get $p = j_* (g^1(w_1)) + f^*(y + i_* w_1)$. This proves that the given map is onto. Moreover if $j_* (g^1(w)) + f^*(y) = 0$ then by Proposition 2.5 we may write $(g^1(w), y) = (g^1(u), -i_* u)$ for some $u \in H_k(W; \mathbb{Z})$. Again by (9) we deduce $g^1(w) = g^1(u) = 0$, and so $u = 0$ by the injectivity of $g^1$ (Lemma 2.4 (b)). This proves that the given map is injective. As for the description of its inverse, it follows from Lemma 2.4 (a) and (d).

Corollary 2.7. Let $X_t \in |H^0(Y, \mathcal{I}_W(Y))|$ be a smooth hypersurface containing $W$. Let $H_{m+2}(P; \mathbb{Z}) \rightarrow H^m(X_t; \mathbb{Z})$ be the map obtained composing the Gysin map $H_{m+2}(P; \mathbb{Z}) \rightarrow H_m(X_t; \mathbb{Z})$ (induced by the natural inclusion $X_t \hookrightarrow P$) with Poincaré duality. Then the image of this map is equal to $H^m(Y; \mathbb{Z}) + H^m(X_t; \mathbb{Z})_W$.

Now consider the following commutative diagrams:

$$\begin{align*}
H_{m+2}(P; \mathbb{Z}) & \quad \downarrow \\
H_{m+2}(Q; \mathbb{Z}) & \quad \rightarrow \quad H^m(X_t; \mathbb{Q})
\end{align*}$$
\[ H_k(G; \mathbb{Z}) \to H_k(P; \mathbb{Z}) \to H_k(P, G; \mathbb{Z}) \to H_{k-1}(G; \mathbb{Z}) \]

(9)

In diagram (8) the vertical map is the push-forward corresponding to the natural projection \( P \to Q \), and the other maps are obtained composing the Gysin maps \( H_{m+2}(P; \mathbb{Z}) \to H_m(X_1; \mathbb{Z}) \) and \( H_{m+2}(Q; \mathbb{Z}) \to H_m(X_1; \mathbb{Z}) \) with Poincaré duality \( H_m(X_1; \mathbb{Z}) \cong H^m(X_1; \mathbb{Z}) \). The rows appearing in diagram (9) are the exact sequences of the pairs \( (P, G) \) and \( (Q, \{ v_\infty \}) \), and the vertical maps denote pushforward (compare with [18], p. 23).

Combining Theorem 2.6 (a), and Corollary 2.7 with (9), we see that (3) holds if and only if \( H_{m+2}(P; \mathbb{Z}) \) maps onto \( H_{m+2}(Q; \mathbb{Z}) \). On the other hand, by diagram (9) with \( k = m + 2 \), we deduce that \( H_{m+2}(P; \mathbb{Z}) \to H_{m+2}(Q; \mathbb{Z}) \) is surjective if and only if the push-forward \( H_{m+1}(G; \mathbb{Z}) \to H_{m+1}(P; \mathbb{Z}) \) is injective, and by Corollary 2.6 this is equivalent to say that \( \ker \mu_{m+1} \cap \ker \nu_{m+1} = 0 \). We notice that, in the case of rational coefficients, Hard Lefschetz Theorem implies that both maps \( \mu_{m+1} \otimes \mathbb{Q} \) and \( \nu_{m+1} \otimes \mathbb{Q} \) are injective, i.e. \( \ker \mu_{m+1} \) and \( \ker \nu_{m+1} \) are finite torsion groups.

Summing up we have the following:

**Proposition 2.8.** \( I_W(Z) = H^m(Y; \mathbb{Z}) + H^m(X_1; \mathbb{Z})_W \) if and only if \( \ker \mu_{m+1} \cap \ker \nu_{m+1} = 0 \). In particular \( I_W(Q) = H^m(Y; \mathbb{Q}) + H^m(X_1; \mathbb{Q})_W \).

Unfortunately we are not able to prove that \( \ker \mu_{m+1} \cap \ker \nu_{m+1} = 0 \) in this generality. This has prevented us from extending ([17, Theorem 1.2]) in the case \( Y \) is not necessarily a complete intersection. In general it may happen \( \ker \nu_{m+1} \neq 0 \), so we expect that \( \ker \mu_{m+1} = 0 \) (which holds true when \( W \) is smooth). Since \( \ker \nu_{m+1} \) is a torsion group, it would be sufficient to prove that \( \mu_{m+1} \) simply injects the torsion. We will overcome this difficulty later on, in the proof of Theorem 3.1 assuming \( X_1 \) varies in the linear system \( |H^0(Y, I_Z, Y \cup d)| \) (see Lemma 3.2 below).

We conclude this section identifying the kernel of the push-forward \( H_m(W; \mathbb{Z}) \to H_m(X_1; \mathbb{Z}) \), and the intersection \( H^m(Y; \mathbb{Z}) \cap H_m(X_1; \mathbb{Z})_W \) in \( H^m(X_1; \mathbb{Z}) \). We need again some preliminaries, the first one is the following:

**Lemma 2.9.** The Gysin map \( \mu_{m+2} : H_{m+2}(G; \mathbb{Z}) \to H_m(W; \mathbb{Z}) \) is injective.

**Proof.** Consider the following natural commutative diagram

\[
\begin{array}{ccc}
H^m(W; \mathbb{Z}) & \to & H^m(W'; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_m(W; \mathbb{Z}) & \to & H_{m-2}(W'; \mathbb{Z}) \cong H_{m-2}(W'; \mathbb{Z})
\end{array}
\]

where the vertical map \( H^m(W; \mathbb{Z}) \to H_m(W; \mathbb{Z}) \) is the duality morphism ([24], p. 150), \( W' \) is the general hyperplane section of \( W \) (\( W' \) is smooth), \( H^m(W; \mathbb{Z}) \to H^m(W'; \mathbb{Z}) \) is the pull-back, and \( H_m(W; \mathbb{Z}) \to H_{m-2}(W'; \mathbb{Z}) \) is the Gysin map \( H_m(W; \mathbb{Z}) \to H_{m-2}(W'; \mathbb{Z}) \) composed with with Poincaré duality \( H_{m-2}(W'; \mathbb{Z}) \cong
Now consider the following commutative diagram:

\[
\begin{array}{ccc}
H_{m+2}(G; \mathbb{Z}) & \overset{\nu_{m+2}}{\rightarrow} & H_{m+2}(Y; \mathbb{Z}) \\
\mu_{m+2} & & \downarrow \\
H_m(W; \mathbb{Z}) & \rightarrow & H_m(X_i; \mathbb{Z}) \cong H^m(X_i; \mathbb{Z}),
\end{array}
\]

where the bottom map denotes push-forward, and the right vertical map pull-back identified with Gysin map via Poincaré duality. By Lemma 2.9 we know that \(\mu_{m+2}\) is injective. Hence we obtain a natural inclusion

\[
(10) \quad \ker \nu_{m+2} \hookrightarrow \ker (H_m(W; \mathbb{Z}) \rightarrow H_m(X_i; \mathbb{Z})).
\]

Observe that, by Hard Lefschetz Theorem, \(\ker \nu_{m+2}\) is a torsion group.

Next consider the map \(H^{m-2}(Y; \mathbb{Z}) \rightarrow H^m(X_i; \mathbb{Z})\) given composing:

\[
H^{m-2}(Y; \mathbb{Z}) \cong H_{m+4}(Y; \mathbb{Z}) \overset{-[W]}{\rightarrow} H_{m+2}(Y; \mathbb{Z}) \cong H^m(Y; \mathbb{Z}) \subseteq H^m(X_i; \mathbb{Z}).
\]

We may obtain this map also composing

\[
H^{m-2}(Y; \mathbb{Z}) \cong H_{m+4}(Y; \mathbb{Z}) \overset{-[W]}{\rightarrow} H_m(W; \mathbb{Z}) \rightarrow H_m(X; \mathbb{Z}) \cong H^m(X_i; \mathbb{Z}).
\]

Hence \(\exists (H^{m-2}(Y; \mathbb{Z}) \rightarrow H^m(X_i; \mathbb{Z}))\) is a subgroup of \(H^m(Y; \mathbb{Z}) \cap H_m(X; \mathbb{Z})_W\).

On the other hand, by Lefschetz Hyperplane Theorem we have \(H^{m-2}(G; \mathbb{Z}) \cong H^{m-2}(Y; \mathbb{Z})\). Therefore, via Poincaré duality, there is a natural isomorphism \(\exists \nu_{m+2} \cong \exists (H^{m-2}(Y; \mathbb{Z}) \rightarrow H^m(X_i; \mathbb{Z}))\). Summing up we get another natural inclusion

\[
(11) \quad \exists \nu_{m+2} \hookrightarrow H^m(Y; \mathbb{Z}) \cap H_m(X; \mathbb{Z})_W.
\]

Observe that, tensoring with \(\mathbb{Q}\), the map \(H^{m-2}(Y; \mathbb{Q}) \rightarrow H^m(X_i; \mathbb{Q})\) becomes injective by Hard Lefschetz Theorem. So we have \(H^{m-2}(Y; \mathbb{Q}) \cong \exists (\nu_{m+2} \otimes \mathbb{Q})\) and therefore

\[
(12) \quad H^{m-2}(Y; \mathbb{Q}) \hookrightarrow H^m(Y; \mathbb{Q}) \cap H_m(X; \mathbb{Q})_W.
\]

Actually all previous inclusions \([10], [11]\) and \([12]\) are equalities. This is the content of the following Proposition 2.10.

**Proposition 2.10.** Let \(\nu_{m+2} : H_{m+2}(G; \mathbb{Z}) \rightarrow H_{m+2}(Y; \mathbb{Z})\) be the push-forward. Then there are canonical isomorphisms \(\ker \nu_{m+2} \cong \ker (H_m(W; \mathbb{Z}) \rightarrow H_m(X_i; \mathbb{Z}))\) and \(\exists \nu_{m+2} \cong H^m(Y; \mathbb{Z}) \cap H_m(X; \mathbb{Z})_W\). In particular the push-forward \(H_m(W; \mathbb{Q}) \rightarrow H_m(X_i; \mathbb{Q})\) is injective (so \(H_m(W; \mathbb{Q}) \cong H^m(X; \mathbb{Q})_W\), and \(H^{m-2}(Y; \mathbb{Q}) \cong H^m(Y; \mathbb{Q}) \cap H_m(X; \mathbb{Q})_W\).
Proof. By Theorem 2.1 and (8) we see that the maps $H_{m+2}(P; \mathbb{Z}) \to H^m(X_i; \mathbb{Z})$ and $H_{m+2}(P; \mathbb{Z}) \to H_{m+2}(Q; \mathbb{Z})$ have the same kernel. Therefore, by (11), Corollary 2.6 and Lemma 2.9 we deduce a natural isomorphism

$$H^m(Y; \mathbb{Z}) + H_m(X; \mathbb{Z})_W \cong [H^m(Y; \mathbb{Z}) \oplus H_m(W; \mathbb{Z})] / H_{m+2}(G; \mathbb{Z})$$

where the inclusion $H_{m+2}(G; \mathbb{Z}) \subseteq H^m(Y; \mathbb{Z}) \oplus H_m(W; \mathbb{Z})$ is defined via the map $x \to (\nu'_m(x), -\mu_m(x))$ (here $\nu'_m$ denotes the map $\nu_m$ composed with Poincaré duality $H_{m+2}(Y; \mathbb{Z}) \cong H^m(Y; \mathbb{Z})$). So the kernel of the push-forward $H_m(W; \mathbb{Z}) \to H_m(X_i; \mathbb{Z})$ identifies with the kernel of the map

$$w \in H_m(W; \mathbb{Z}) \to [(0, w)] \in [H^m(Y; \mathbb{Z}) \oplus H_m(W; \mathbb{Z})] / H_{m+2}(G; \mathbb{Z}).$$

Now if $[(0, w)] = [(0, 0)]$ then there is $x \in H_{m+2}(G; \mathbb{Z})$ such that $\nu'_m(x) = 0$ and $w = -\mu_m(x)$. This means that the inclusion (11) is also surjective.

As for the intersection, first observe that the inclusion (11) identifies with the image in $H^m(Y; \mathbb{Z}) \cap H_m(X; \mathbb{Z})_W$ of the following map:

$$(13) \quad x \in H_{m+2}(G; \mathbb{Z}) \to [(\nu'_m(x), 0)] \in H^m(Y; \mathbb{Z}) \cap H_m(X; \mathbb{Z})_W.$$  

If $[(y, 0)] \in H^m(Y; \mathbb{Z}) \cap H_m(X; \mathbb{Z})_W$ then for some $w \in H_m(X; \mathbb{Z})_W$ and $x \in H_{m+2}(G; \mathbb{Z})$ we have $(y, 0) = (0, w) + (\nu'_m(x), -\mu_m(x))$, hence $y = \nu'_m(x)$. This proves that the map (13) is surjective, so also (11) is. \qed

3. The proof of Theorem 1.1

Notations 3.1. (i) Let $Y \subseteq \mathbb{P}^N$ be a smooth complex projective variety of dimension $m + 1 = 2r + 1 \geq 3$, $Z$ be a closed subscheme of $Y$, and $\delta$ be a positive integer such that $\mathcal{I}_Z(Y)$ is generated by global sections. Assume that for any $k \geq \delta$ the general divisor in $|H^0(Y, \mathcal{I}_Z(Y))|$ is smooth. This is equivalent to say that $Z$ verifies condition (0.1) in [24]. In particular $2 \dim Z \leq m$.

(ii) Let $X, G_1, \ldots, G_r$ be general divisors with $X \in |H^0(Y, \mathcal{I}_Z(Y))|$ and $G_i \in |H^0(Y, \mathcal{I}_Z(Y))|$ for $1 \leq i \leq r$. Assume that $d > k_i \geq \delta$ for any $1 \leq i \leq r$. By (24, 1.2, Theorem) we know that $X, G_1, \ldots, G_r$ is a regular sequence, verifying the following conditions for any $2 \leq l \leq r$:

$$\dim \text{Sing}(G_1 \cap \cdots \cap G_l) \leq l - 2 \quad \text{and} \quad \dim \text{Sing}(X \cap G_1 \cap \cdots \cap G_l) \leq l - 1.$$ 

Put $\Delta := X \cap G_1 \cap \cdots \cap G_r$. Hence $\Delta$ is a complete intersection of dimension $r$ containing $Z$. Denote by $C_1, \ldots, C_\omega$ the irreducible components of $\Delta$. Observe that also $\Delta$ verifies condition (0.1) in [24]. Put $W := X \cap G_1$.

(iii) Let $I_Z(\mathbb{Z})$ be the subgroup of the invariant cocycles in $H^m(X_i; \mathbb{Z})$ with respect to the monodromy representation on $H^m(X_i; \mathbb{Z})$ for the family of smooth divisors $X_i \in |H^0(Y, \mathcal{O}_Y(d))|$ containing $Z$. Denote by $H^m(X_i; \mathbb{Z})_W$ the image of $H_m(Z; \mathbb{Z})$ in $H^m(X_i; \mathbb{Z})$ via the natural map $H_m(Z; \mathbb{Z}) \to H_m(X_i; \mathbb{Z}) \cong H^m(X_i; \mathbb{Z})$. One may give a similar definition for $I_\Delta(Z), H^m(X_i; \mathbb{Z})_\Delta, I_W(\mathbb{Z})$, and $H^m(X_i; \mathbb{Z})_W$, and also with $Q$ instead of $Z$. Observe that since $Z \subseteq \Delta \subseteq W$ then the monodromy
group of the family of smooth divisors in $|H^0(Y, O_Y(d))|$ containing $W$ is a subgroup of the monodromy group of the family of smooth divisors in $|H^0(Y, O_Y(d))|$ containing $\Delta$, which in turn is a subgroup of the monodromy group of the family of smooth divisors in $|H^0(Y, O_Y(d))|$ containing $Z$. Therefore we have

$$I_Z(Z) \subseteq I_\Delta(Z) \subseteq I_W(Z).$$

Denote by $V_\Delta := I_\Delta(Q) \perp$ the orthogonal complement of $I_\Delta(Q)$ in $H^m(X; Q)$.

(iv) For any $1 \leq l \leq r$ fix general divisor $H_l \in |H^0(Y, O_Y(\mu_l))|$, with $0 < \mu_1 < \cdots < \mu_{r-1}$, and for any $0 \leq l \leq r - 1$ define $(Y_l, Z_l, X_l, W_l, \Delta_l)$ as follows. For $l = 0$ put $(Y_0, Z_0, X_0, W_0, \Delta_0) := (Y, Z, X, W, \Delta)$. For $1 \leq l \leq r - 1$ put $Y_l := G_1 \cap \cdots \cap G_l \cap H_1 \cap \cdots \cap H_l$, $Z_l := Z \cap H_1 \cap \cdots \cap H_l$, $X_l := X \cap Y_l$, $W_l := X \cap Y_l \cap G_{l+1}$, and $\Delta_l := \Delta \cap Y_l$. Notice that dim $Y_{r-1} = 3$ and that $\Delta_{r-1} = W_{r-1}$.

**Lemma 3.2.** Let $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ be a general divisor containing $Z$. Then the Gysin map $\mu_{m+1} : H_{m+1}(G_1; Z) \to H_{m-1}(W; Z)$ is injective, and therefore one has $I_W(Z) = H^m(Y; Z) + H^m(X; \mathcal{I}_{Z,Y}; W)$.

**Proof.** Since $|H^0(Y, \mathcal{I}_{Z,Y}(d))|$ is very ample on $G_1 \setminus Z$, then by Lefschetz Theorem with Singularities ([13], p. 199) we know that the pull-back $H^m-1(G_1; Z) \to H^m-1(W; Z)$ is injective for a general $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ (recall that $W = G_1 \cap X$). Moreover by Bertini Theorem we have $\text{Sing}(W) \subseteq Z$, and therefore $W \setminus Z \subseteq W \setminus \text{Sing}(W)$. So we may consider the following natural commutative diagram

$$
\begin{array}{ccc}
H^m-1(G_1; Z) & \to & H^m-1(W \setminus \text{Sing}(W); Z) \\
\downarrow & & \downarrow \\
H^m-1(G_1 \setminus Z; Z) & \leftrightarrow & H^m-1(W \setminus Z; Z)
\end{array}
$$

where all maps are pull-back. Since also $H^m-1(G_1; Z) \to H^m-1(G_1 \setminus Z; Z)$ is injective ([2], Theorem 5.4.12, p. 162), we deduce the injectivity of $H^m-1(G_1; Z) \to H^m-1(W \setminus \text{Sing}(W); Z)$. On the other hand, from Borel-Moore homology exact sequence $0 = H^m_{BM}(\text{Sing}(W); Z) \to H_{m-1}(W; Z) \to H^m_{BM}(W \setminus \text{Sing}(W); Z)$ we see that $H_{m-1}(W; Z) \subseteq H^m_{BM}(W \setminus \text{Sing}(W); Z) \cong H^m-1(W \setminus \text{Sing}(W); Z)$ ([12], p. 217, (26)). So the injective map $H_{m+1}(G_1; Z) \cong H^m-1(G_1; Z) \to H^m-1(W \setminus \text{Sing}(W); Z)$ factors through $\mu_{m+1}$:

$$
H_{m-1}(W; Z) \xrightarrow{\mu_{m+1}} H_{m+1}(G_1; Z) \xrightarrow{\mu_{m+1}^{-1}} H^m-1(W \setminus \text{Sing}(W); Z).
$$

Last claim now follows by Proposition 2.8. \qed

**Theorem 3.3.** Let $X \in |H^0(Y, O_Y(d))|$ be a smooth divisor containing $\Delta$. Then we have:

(a) $H^m(X; Z)_{\Delta}$ is freely generated by $C_1, \ldots, C_\omega$;

(b) $I_{\Delta}(Z) = H^m(Y; Z) + H^m(X; Z)_{\Delta} = H^m(Y; Z) + H^m(X; W)$, and the monodromy representation on $V_{\Delta}$ for the family of smooth divisors in $|H^0(Y, O_Y(d))|$ containing $\Delta$ is irreducible;
(c) \( \dim [H^m(Y; \mathbb{Q}) \cap H^m(X; \mathbb{Q})] = 1 \).

Proof. (a) Since \( H_m(\Delta; \mathbb{Z}) \) is freely generated by \( C_1, \ldots, C_\omega \) then to prove (a) is equivalent to prove that the push-forward \( H_m(\Delta; \mathbb{Q}) \to H_m(X; \mathbb{Q}) \) is injective. When \( r = 1 \) this follows by Proposition \ref{prop:injection} because in this case \( W = \Delta \). Now argue by induction on \( r \geq 2 \). Since \( \Delta_1 = \Delta \cap H_1 \), then \( \Delta \) and \( \Delta_1 \) have the same number of components. Therefore the Gysin map \( H_m(\Delta; \mathbb{Q}) \to H_{m-2}(\Delta_1; \mathbb{Q}) \) is bijective, and its composition \( \varphi \) with the push-forward \( H_{m-2}(\Delta_1; \mathbb{Q}) \to H_{m-2}(X_1; \mathbb{Q}) \) is injective by induction. On the other hand \( \varphi \) is nothing but the composition of the push-forward \( H_m(\Delta; \mathbb{Q}) \to H_m(W; \mathbb{Q}) \) with the Gysin map \( H_m(W; \mathbb{Q}) \to H_{m-2}(X_1; \mathbb{Q}) \) (observe that \( X_1 = W \cap H_1 \)). Hence the map \( H_m(\Delta; \mathbb{Q}) \to H_m(W; \mathbb{Q}) \) is injective, and so is the map \( H_m(\Delta; \mathbb{Q}) \to H_m(X; \mathbb{Q}) \) by Proposition \ref{prop:injection} again.

(b) Since \( I_W(Z) \supseteq I_\Delta(Z) \) and \( I_\Delta(Z) \supseteq H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_\Delta \), by Lemma \ref{lem:generation} it suffices to prove that \( H^m(X; \mathbb{Z})_W \subseteq H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_\Delta \), and that \( V_\Delta \) is irreducible. So it is enough to show that for any \( 0 \leq l \leq r - 1 \) one has

\[
H^m(X; \mathbb{Z})_W \subseteq H^m(Y; \mathbb{Z}) + H^m(X; \mathbb{Z})_\Delta,
\]

\((m_l := m - 2l)\), and that the monodromy representation on \( V_{\Delta_l} \) for the family of smooth divisors \( X_l \in |H^0(Y_l, O_{Y_l}(d))| \) containing \( \Delta_l \) is irreducible. To this purpose we argue by decreasing induction on \( l \). When \( l = r - 1 \) we have \( \Delta_{r-1} = W_{r-1} \). In this case \((\ref{eq:generation})\) is obvious, and the irreducibility of \( V_{\Delta_{r-1}} \) follows from \cite[Theorem 1.1]{[6]} because, with the same notation as in \cite[Theorem 1.1]{[6]}, one has \( I_{W_{r-1}}(Q)^{-1} = H^{m_{r-1}}(X_{r-1}; \mathbb{Z})_{\Delta_{r-1}} \) (compare with \cite[p. 526, from line 27 to 38]{[6]}).

Now assume \( 0 \leq l < r - 1 \). By induction and Lemma \ref{lem:generation} (with \( Y_{l+1} \cap G_{l+2} \) instead of \( G_1 \)), we have

\[
I_{W_{l+1}}(Z) = I_{\Delta_{l+1}}(Z) = H^{m_{l+1}}(Y_{l+1}; \mathbb{Z}) + H^{m_{l+1}}(X_{l+1}; \mathbb{Z})_{\Delta_{l+1}}.
\]

Since \( X_{l+1} = W_l \cap H_{l+1} \), then the inclusion map \( i_{X_{l+1}}: X_{l+1} \to W_l \) defines a Gysin map \( i_{X_{l+1}}^*: H_m(W_l; \mathbb{Z}) \to H_{m_{l+1}}(X_{l+1}; \mathbb{Z}) \). Moreover since \( \Delta_l \subseteq W_l \), \( \Delta_{l+1} = \Delta_l \cap H_{l+1} \) (hence the Gysin map \( H_m(\Delta_l; \mathbb{Z}) \to H_{m_{l+1}}(\Delta_{l+1}; \mathbb{Z}) \) is bijective because both groups are freely generated by the irreducible components), and by Lefschetz Hyperplane Theorem we have \( H^{m_{l+1}}(Y_l; \mathbb{Z}) \cong H^{m_{l+1}}(Y_{l+1}; \mathbb{Z}) \), then

\[
(\text{PD means “Poincaré duality”} \quad H^{m_{l+1}}(X_{l+1}; \mathbb{Z}) \cong H^{m_{l+1}}(X_{l+1}; \mathbb{Z})) \). By \cite[Lemma 2.3]{[6]} \( \exists (\text{PD} \circ i_{X_{l+1}}^*) \supseteq H^{m_{l+1}}(X_{l+1}; \mathbb{Z}) \Delta_{l+1} \)

\((\text{PD means “Poincaré duality”} \quad H^{m_{l+1}}(X_{l+1}; \mathbb{Z}) \cong H^{m_{l+1}}(X_{l+1}; \mathbb{Z})) \). By \cite[Lemma 2.3]{[6]} \( \exists (\text{PD} \circ i_{X_{l+1}}^*) \supseteq H^{m_{l+1}}(X_{l+1}; \mathbb{Z}) \Delta_{l+1} \)

In fact, by Theorem \ref{thm:injection} (a), we know that \( H^{m_{l+1}}(X_{l+1}; \mathbb{Z})/I_{W_{l+1}}(Z) \) is torsion-free. Therefore if \((\ref{eq:generation})\) would be strict, by \((\ref{eq:generation})\) one would have \( \left( \exists (\text{PD} \circ i_{X_{l+1}}^*) \supseteq H^{m_{l+1}}(X_{l+1}; \mathbb{Z}) \Delta_{l+1} \right) \cap V_{\Delta_{l+1}} \neq \{0\} \), and since \( V_{\Delta_{l+1}} \) is irreducible, it would follow that \( H^{m_{l+1}}(X_{l+1}; \mathbb{Z}) = \)
3.2 and Theorem 3.3, we get
\[
\exists (PD \circ i_{X_{l+1}}^*) \otimes \mathbb{Q}. \quad \text{This is impossible because, for } 0 \ll \mu_1 \ll \cdots \ll \mu_{l+1}, \text{ } h_{m_{l+1}}(X_{l+1}; \mathbb{Q}) \text{ is arbitrarily large with respect to } h_{m_l}(W_l; \mathbb{Q}). \text{ Again by Theorem 2.11 (a), with } R := W_l, \text{ we know that } i_{X_{l+1}}^* \text{ is injective, hence } H_{m_l}(W_l; \mathbb{Q}) \cong \exists (PD \circ i_{X_{l+1}}^*). \text{ So from (15), for the map } H_{m_l}(W_l; \mathbb{Q}) \to H_{m_l}(X_l; \mathbb{Q}) \text{ sends } H_{m_{l+1}}(X_{l+1}; \mathbb{Q})_{\Delta_{l+1}} \text{ in } H_{m_l}(X_l; \mathbb{Q})_{\Delta_l} \text{ and } H_{m_{l+1}}(Y_{l+1}; \mathbb{Q}) \text{ in } H_{m_l}(Y_l; \mathbb{Q}). \text{ Finally note that by Proposition 2.8 and (15) it follows } I_{\Delta_l}(\mathbb{Q}) = I_{W_l}(\mathbb{Q}). \text{ So the irreducibility of } V_{\Delta_l} (= I_{W_l}(\mathbb{Q})^{'}) \text{ follows again from (15), Theorem 1.1).}
\]
\[c) \text{ Again we argue by decreasing induction on } l. \text{ When } l = r - 1 \text{ previous equality follows by Proposition 2.10 because in this case } \dim Y_{r-1} = 3 \text{ and } \Delta_{r-1} = W_{r-1}. \text{ Now assume } 0 \leq l < r - 1. \text{ By (a) the Gysin map } H_{m_l}(\Delta_l; \mathbb{Q}) \to H_{m_{l+1}}(\Delta_{l+1}; \mathbb{Q}) \text{ induces an isomorphism } \varphi : H_{m_l}(\Delta_l; \mathbb{Q}) \to H_{m_{l+1}}(\Delta_{l+1}; \mathbb{Q}). \text{ Suppose that } \xi \in H_{m_l}(Y_l; \mathbb{Q}) \cap H_{m_l}(X_l; \mathbb{Q})_{\Delta_l}. \text{ Then } \xi \in H_{m_l}(Y_l; \mathbb{Q}) \cap H_{m_l}(X_l; \mathbb{Q})_{\Delta_l}. \text{ By Proposition 2.10 } \xi \text{ comes from some } \eta \in H_{m_{l+1}}(Y_{l+1}; \mathbb{Q}) \cong H_{m_{l+1}}(Y_{l+1}; \mathbb{Q})_l. \text{ Therefore pulling-back } \eta \text{ in } H_{m_{l+1}}(X_{l+1}; \mathbb{Q}) \text{ we get } \varphi(\xi). \text{ This proves that } \varphi \text{ sends isomorphically } H_{m_l}(Y_l; \mathbb{Q}) \cap H_{m_l}(X_l; \mathbb{Q})_{\Delta_l} \text{ onto } H_{m_{l+1}}(Y_{l+1}; \mathbb{Q}) \cap H_{m_{l+1}}(X_{l+1}; \mathbb{Q})_{\Delta_{l+1}}. \quad \square
\]

Remark 3.4. From (a), (b) (with } \mathbb{Q} \text{ instead of } \mathbb{Z} \text{) of previous Theorem 3.3 and Proposition 2.10, we deduce that } \dim [H^m(Y; \mathbb{Q}) \cap H^m(X; \mathbb{Q})_\Delta] = \omega - h^m(W_l; \mathbb{Q}) + h^{m-2}(Y_l; \mathbb{Q}) = 1.

We are in position to prove our main result.

Proof of Theorem 3.3. (a) Let } G_1, \ldots, G_r \text{ be general divisors in } |H^0(Y, \mathcal{O}_Y(\delta))| \text{ containing } Z, \text{ and put } \Delta := X \cap G_1 \cap \cdots \cap G_r. \text{ Hence } \Delta \text{ is a complete intersection of dimension } r \text{ containing } Z, \text{ and then (a) follows by (a) of previous Theorem 3.3.}

(b) By (21), 1.2, Theorem) we know that } \Delta \setminus Z \text{ is smooth and connected. Observe that } \Delta \neq Z_1 \cup \cdots \cup Z_r, \text{ otherwise } Z = \Delta \text{ and this is in contrast with the assumption that } I_{Z,Y}(\delta) \text{ is generated by global sections. Therefore, apart from } Z_1, \ldots, Z_r, \text{ } \Delta \text{ has a unique residual irreducible component } C, \text{ and we have:}
\[
\Delta = Z_1 + \cdots + Z_r + C = \delta^r H^r_X \in H^m(X; \mathbb{Q}).
\]
\[\text{By (19), and (a) and (c) of Theorem 3.3, we deduce } \dim [H^m(Y; \mathbb{Q}) \cap H^m(X; \mathbb{Q})_Z] = 0. \text{ Hence, taking into account that } H^m(X; \mathbb{Q})_Z \subseteq NS_r(X; \mathbb{Q}), \text{ in order to prove (b) it suffices to prove that}
\]
\[NS_r(X; \mathbb{Q}) \subseteq H^m(Y; \mathbb{Q}) + H^m(X; \mathbb{Q})_Z.
\]
\[\text{To this aim, first notice that again by (19) we have } H^m(Y; \mathbb{Q}) + H^m(X; \mathbb{Q})_\Delta = H^m(Y; \mathbb{Q}) + H^m(X; \mathbb{Q})_Z, \text{ which is contained in } I_{Z}(\mathbb{Q}). \text{ Therefore, by (11), Lemma 3.2 and Theorem 3.3 we get}
\]
\[I_{Z}(\mathbb{Q}) = I_{\Delta}(\mathbb{Q}) = I_{W}(\mathbb{Q}) = H^m(Y; \mathbb{Q}) + H^m(X; \mathbb{Q})_Z
\]
\[(W := X \cap G_1). \text{ Hence } I_{Z}(\mathbb{Q})^{'}, (= V_{\Delta}) \text{ is irreducible, and the vanishing cohomology } H^m(Y; \mathbb{Q})^{'}, \text{ of } X \text{ is contained in } I_{Z}(\mathbb{Q})^{'}, + H^{m/2, m/2}(X; \mathbb{Q}). \text{ So, as } H^m(Y; \mathbb{Q})^{'},,
We deduce:

$$\alpha$$

The map $R$ applied to

To this purpose first observe that by Proposition 2.10 we have:

where all the maps are Gysin’s. It induces the following commutative diagram:

and so

Now consider the following natural commutative diagram:

(23)

NS

is also the blowing-up of NS

(21)

m

(\[15\], p. 163). So we may assume

rem 2.1 we know that

This is certainly true when dim $Y = 3$ in view of Lefschetz Theorem on (1, 1)-Classes (\[15\], p. 163). So we may assume $m \geq 4$ and argue by induction on dim $Y = m + 1$. First note that

(22)

(observe that NS

(21)

m

(\[15\], p. 115). Since

NS

\[2\] it follows that in order to prove (21) (i.e. (c)) it suffices to prove that

(23)

NS

To this purpose first observe that by Proposition2.10 we have:

$$\alpha$$

and so

Now consider the following natural commutative diagram:

$$\begin{array}{ccc}
H_{m+2}(G_1; \mathbb{Q}) & \to & H_m(W; \mathbb{Q}) \\
\downarrow & & \downarrow \\
H_m(Y_1; \mathbb{Q}) & \to & H_{m-2}(X_1; \mathbb{Q}),
\end{array}$$

where all the maps are Gysin’s. It induces the following commutative diagram:

$$\begin{array}{ccc}
NS_{r+1}(G_1; \mathbb{Q}) & \overset{\alpha}{\to} & NS_r(W; \mathbb{Q}) \cap H_{m-2}(Y; \mathbb{Q}) \\
\downarrow & & \downarrow \\
NS_r(Y_1; \mathbb{Q}) & \overset{\beta}{\to} & NS_{r-1}(X_1; \mathbb{Q}) \cap H_{m-2}(Y_1; \mathbb{Q}).
\end{array}$$

The map $\alpha_1$ is an isomorphism by Theorem 2.1. The map $\beta_1$ is an isomorphism by induction (compare with (21)). The map $\beta$ is an isomorphism by Theorem 2.1 applied to $R := W$ which only has isolated singularities. It follows that:

$$NS_r(W; \mathbb{Q}) \cap H_{m-2}(W; \mathbb{Q}) = NS_{r+1}(G_1; \mathbb{Q}).$$

We deduce:

$$NS_{r+1}(G_1; \mathbb{Q}) = NS_r(W; \mathbb{Q}) \cap H_{m-2}(W; \mathbb{Q})$$

$$= NS_r(W; \mathbb{Q}) \cap H_{m-2}(Y; \mathbb{Q}) \supseteq NS_r(W; \mathbb{Q}) \cap H_{m-2}(X_1; \mathbb{Q}) NS_{r+1}(Y; \mathbb{Q}).$$
But we also have
\[ \text{NS}_r(W; Q) \cap H_\infty(X; \mathbb{Q}) \cap \text{NS}_{r+1}(Y; \mathbb{Q}) \supseteq \text{NS}_{r+1}(G_1; \mathbb{Q}) \]
because the cycles of \( \text{NS}_{r+1}(G_1; \mathbb{Q}) \) pass through \( \text{NS}_{r+1}(Y; \mathbb{Q}) \) via push-forward \( \text{NS}_{r+1}(G_1; \mathbb{Q}) \subseteq H_{m+2}(G_1; \mathbb{Q}) \to H_{m+2}(Y; \mathbb{Q}) \). This proves (23), hence (c). □

**Corollary 3.5.** With the same assumptions as in Theorem 1.1, suppose also that \( \dim Y = 3 \). Then we have:
\[ \text{Pic}(X) = \text{Pic}(Y) \oplus H^2(X; \mathbb{Z}) \]

*Proof.* If \( \dim Y = 3 \) then \( m = 2, r = 1 \), and by Lefschetz Theorem on (1, 1)-Classes we have \( \text{NS}_2(Y; \mathbb{Z}) \subseteq \text{NS}_1(X; \mathbb{Z}) \cap H^2(Y; \mathbb{Z}) \subseteq H^{1,1}(Y; \mathbb{Z}) = \text{NS}_2(Y; \mathbb{Z}) \). So we get \( \text{NS}_1(X; \mathbb{Z}) = \text{NS}_2(Y; \mathbb{Z}) \oplus H^2(X; \mathbb{Z}) \). And this identification lifts to Picard groups in view of Lefschetz Hyperplane Theorem and the exponential sequence (compare with the proof of Corollary 2.3.4, p. 51, in [2]). □

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