THE GROMOV NORM OF THE PRODUCT OF TWO SURFACES

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Abstract. We make an estimation of the value of the Gromov norm of the Cartesian product of two surfaces. Our method uses a connection between these norms and the minimal size of triangulations of the products of two polygons. This allows us to prove that the Gromov norm of this product is between 32 and 52 when both factors have genus 2. The case of arbitrary genera is easy to deduce from this one.

1. Introduction

Gromov defined the simplicial volume (also now known as the Gromov norm) of a closed orientable manifold $M$ [Gromov 1982] as the infimum of the $l^1$-norms of all singular chains representing the top homology class of $M$. It is an invariant that quantifies the topological complexity of $M$. For example, if $\pi_1(M)$ is amenable then its Gromov norm, denoted $||M||$, vanishes. But if $M$ admits a metric of negative curvature, then $||M||$ is positive and finite. In fact, the Gromov-Thurston theorem [Gromov 1982, Thurston] states that if $M$ is a hyperbolic manifold then

\begin{equation}
||M|| = \text{volume}(M)/\text{volume}(s_n),
\end{equation}

where $s_n$ is an ideal simplex of maximum volume in $\mathbb{H}^n$ (and $\dim(M) = n$). This result was then used to give a topological proof of Mostow rigidity for hyperbolic manifolds.

More recently, while studying some data structure problems, Sleator, Tarjan, and Thurston [Sleator et al. 1988] made explicit computational connection of the Gromov norm to the size of minimal triangulations of polytopes and balls. They used this relation to compute the exact combinatorial diameter of the associahedron (or “Stasheff polytope”), one of the most important polytopes in combinatorics. In this paper we continue the inspection of this interrelation between topology and polyhedral combinatorics. We
investigate the Gromov norm of the product of two surfaces by relating it to triangulations of the product of two polygons.

More precisely, in Section 2 we define the polytopal Gromov norm of a convex polytope, and show that the Gromov norm of the product of two surfaces can be computed from the polytopal Gromov norm of the product $P(n, m)$ of an $n$-gon with an $m$-gon, for $n$ and $m$ asymptotically big. $P(n, m)$ is a four-dimensional polytope with $m+n$ facets: $m$ prisms over an $n$-gon and $n$ prisms over an $m$-gon. In Figure 1 we present the Schlegel diagram of $P(3, 4)$.

**Theorem 1.1.** Let $||P||$ denote the polytopal Gromov norm of a polytope $P$. Then, the Gromov norm of the product $\Sigma_g \times \Sigma_h$ of two surfaces of genera $g$ and $h$ equals

$$||\Sigma_g \times \Sigma_h|| = 16 \lim_{n,m \to \infty} \frac{||P(n,m)||}{nm} = 16 \inf_{n,m} \frac{||P(n,m)||}{nm}. \tag{2}$$

The case $g = h = 2$ of Theorem 1.1 is proved in Section 2. The general case follows from the following well-known lemma, since $\Sigma_g \times \Sigma_h$ is a $(g - 1)(h - 1)$-fold covering of $\Sigma_2 \times \Sigma_2$.

**Lemma 1.2.** If $f : M \to N$ is a degree $\deg(f)$ map between closed orientable manifolds $M$ and $N$ then

$$||M|| \geq \deg(f)||N|| \tag{3}$$

If $f$ is a covering map then the above inequality is an equality.

The polytopal Gromov norm of a polytope $P$ is, roughly speaking, the minimal cardinality of an affine triangulation of $P$ “with real coefficients” (see the precise definition in Section 2). By definition, it is at most equal to the minimum number of simplices needed to (affinely) triangulate $P$. For this reason, and for its intrinsic interest, we try in section 3 to compute the size of a minimal triangulation of $P(m, n)$. Our main results in this direction are:
Theorem 1.3. Let $T(m,n)$ denote the minimal number of simplices in a triangulation of $P(n,m)$. Then:

(4) $\forall m$ odd: $\left\lfloor \frac{9m-15}{2} \right\rfloor \leq ||P(3,m)|| \leq \left\lceil \frac{9m-15}{2} \right\rceil = T(3,m)$,

(5) $\forall m$ even: $\frac{15}{2}(m-2) \leq ||P(4,m)|| \leq T(4,m) \leq 8(m-2)$,

(6) $\forall m, n$ even: $T(m,n) \leq \frac{7}{2}mn - 6(m+n) + 8$.

In Section 3 some of these statements are more detailed and do not require any parity condition on $m$ and $n$. For example, for even $m$ we know that $9m/2 - 9 \leq ||P(m,3)|| \leq 9m/2 - 8 = T(3,m)$. Observe, however, that in order to apply Theorem 1.1 we only need to know $||P(m,n)||$ for a sequence of values of $(m,n)$ with both $m$ and $n$ going to infinity. In particular, the result in equation (6) already gives an upper bound for the Gromov norms we are interested in. In Section 4 we get a better upper bound by constructing a binary cover with asymptotically fewer simplices, in the case $m = n$. A general lower bound for $P(m,n)$ is computed in Section 5:

Theorem 1.4.

(7) $||P(m,m)|| \leq \frac{13}{4}m^2 - \frac{19}{2}m$, for even $m$,

(8) $||P(n,m)|| \geq 2mn - \frac{8}{3}(m+n)$.

Theorems 1.4 and 1.1 together give the following corollary, which is our main result:

Corollary 1.5. For every positive integers $g$ and $h$, the Gromov norm of the product of $\Sigma_g$ and $\Sigma_h$ is bounded by: $32 \leq \frac{||\Sigma_g \times \Sigma_h||}{(g-1)(h-1)} \leq 52$.

Let us compare this result with earlier bounds. It is well-known that the Gromov norm of a genus $g$ surface is zero if $g \leq 1$, and is equal to $4(g - 1)$ otherwise. Hoster and Kotschick [HOSTER AND KOTSCHEICK 2000] proved that whenever $M$ is a surface bundle over a surface $B$, with fiber $||F||$, then $||M|| \geq ||F||||B||$. This implies that $\frac{||\Sigma_g \times \Sigma_h||}{(g-1)(h-1)} \geq 16$. In contrast our new lower bound is 32. On the other end, from any triangulation of the surface $\Sigma_2$ with $T$ triangles, the product tiling of $\Sigma_2 \times \Sigma_2$ via the Cartesian product of two triangles can be subdivided into $6T^2$ triangles. This gives the easy (and known) result $\frac{||\Sigma_2 \times \Sigma_2||}{(g-1)(h-1)} \leq 96$, but now we have 52 as an upper bound.

2. Polytopal Gromov norm

We start by recalling the detailed definition of the Gromov norm of a closed orientable manifold $||M||$. Let $S(M)$ be the singular chain complex of $M$ (with real coefficients). For each nonnegative integer $k$, $S_k(M)$ is a
real vector space with basis consisting of all continuous maps $\sigma : \Delta^k \rightarrow M$ where $\Delta^k$ is a $k$-simplex. The norm of an element

\begin{equation}
(9) \quad c = \sum_{\sigma} r_\sigma, \quad \sigma \in S_k(M)
\end{equation}

is defined by

\begin{equation}
(10) \quad ||c|| := \sum_{\sigma} |r_\sigma|.
\end{equation}

If $\alpha$ is a homology class in $H_k(M)$, its simplicial norm is by definition the infimum of $||c||$ over all $k$–chains $c \in S_k(M)$ representing $\alpha$. The Gromov norm of $M$ is the simplicial norm of the fundamental class $[M] \in H_n(M)$.

In this section we relate the Gromov norm of $\Sigma_2 \times \Sigma_2$ with what we call the polytopal Gromov norm of $P(n, m)$:

**Definition 2.1.** Let $P$ be a polytope. For each $k \in \mathbb{N}$, let $S_k(P)$ be the $\mathbb{R}$-vector space with basis equal to the set of all affine maps $\sigma : \Delta^k \rightarrow P$. We call a chain in $S_k(P)$ an affine chain. We let $S(M)$ denote the resulting singular chain complex and

\begin{equation}
(11) \quad S(P, \partial P) = S(M)/S(\partial M)
\end{equation}

be the relative chain complex of $(P, \partial P)$. We denote the resulting homology by $H_*(P, \partial P)$. As before, we define a norm on the chains in $S(P)$ which induces a pseudonorm on the chains of $S(P, \partial P)$. In turn, this induces a pseudonorm on $H_*(P, \partial P)$. The polytopal Gromov norm (or polytopal simplicial volume) of $P$ is the pseudonorm of the fundamental class of $[P, \partial P] \in H_n(P, \partial P)$. We denote it by $||P||$.

**Remark 2.2.** If we do not require $\sigma$ to be affine, then we would be left with the usual definition of the relative Gromov norm. But the relative Gromov norm of $P$ (with respect to its boundary) is zero since it is homeomorphic to a ball, which admits self-maps of arbitrary degree. Instead of requiring that each map $\sigma$ is affine, we could require only that $\sigma$ takes every face of $\Delta^k$ into a single face of $P$. The resulting norm gives the same value since any such map can be “straightened” into an affine map that agrees on the vertices.

**Theorem 2.3.**

\begin{equation}
(12) \quad ||\Sigma_2 \times \Sigma_2|| = \lim_{n,m \to \infty} \frac{16||P(n, m)||}{(n-2)(m-2)}.
\end{equation}

We prove this in the next four lemmas.

**Lemma 2.4.**

\begin{equation}
(13) \quad \lim_{n,m \to \infty} \frac{||P(n, m)||}{(n-2)(m-2)} = \inf_{n,m} \frac{||P(n, m)||}{(n-2)(m-2)}.
\end{equation}
Proof. It suffices to show that for every fixed positive integers \( j \) and \( k \) and for sufficiently large \( m \) and \( n \),

\[
\frac{||P(j, k)||}{(j - 2)(k - 2)} \geq \frac{||P(n, m)||}{(n - 2)(m - 2)} + O((j + k)/mn).
\]

To start, suppose that \( j - 2 \) divides \( n - 2 \) and \( k - 2 \) divides \( m - 2 \). Divide the \( n \)-gon into \( \frac{n - 2}{j - 2} \) \( j \)-gons and the \( m \)-gon into \( \frac{m - 2}{k - 2} \) \( k \)-gons. Taking the product, we obtain a partition of \( P(n, m) \) into \( (n - 2)(m - 2)(j - 2)(k - 2) \) copies of \( P(j, k) \). From any chain \( c \) of \( S(P(j, k), \partial P(j, k)) \) representing the fundamental class we can construct a fundamental chain \( \tilde{c} \) on \( S(P(n, m), \partial P(n, m)) \) by combinatorially reflecting the chain \( c \) of any particular copy of \( P(j, k) \) to the adjacent ones. The new chain satisfies

\[
||\tilde{c}|| = ||c|| \frac{(n - 2)(m - 2)}{(j - 2)(k - 2)}.
\]

Equation (14) above follows in this case by choosing a sequence of chains \( c_i \) so that \( ||c_i|| \to ||P(j, k)|| \).

If \( n \) and \( m \) are large but \( j - 2 \) does not divide \( n - 2 \) and/or \( k - 2 \) does not divide \( m - 2 \), let \( n' \) and \( m' \) be the first integers after \( n \) and \( m \) and such that \( j - 2 \) divides \( n' - 2 \) and \( k - 2 \) divides \( m' - 2 \). Clearly

\[
n \leq n' < n + j - 2, \quad m \leq m' < m + k - 2.
\]

Since \( ||P(a, b)|| \) is an increasing function of \( a \) and \( b \) (because we can always collapse chains from bigger to smaller polytopes), we conclude that

\[
||P(n, m)|| \leq ||P(n', m')|| \leq \frac{(n + j - 4)(m + k - 4)}{(j - 2)(k - 2)} ||P(j, k)||,
\]

from which equation (14) follows. \( \square \)

In the next step we need the following standard results from hyperbolic geometry. See [Ratcliffe 1994].
Lemma 2.5. For every pair of positive integers \( n, j \) such that \((n-2)-n/j > 0\) there is a regular geodesic \( n \)-gon in the hyperbolic plane with interior angles equal to \( \pi/j \). The area of this \( n \)-gon is \( ((n-2)-n/j)\pi \). The group \( G_{n,j} \) generated by reflections in its sides is discrete and acts cocompactly on the hyperbolic plane. There is a torsion-free finite index subgroup \( G'_{n,j} < G_{n,j} \) consisting of orientation-preserving isometries.

Lemma 2.6. For any pair of integers \( n, m \geq 3 \),

\[
||\Sigma_2 \times \Sigma_2|| \leq \frac{16}{(n-2)(m-2)}||P(n,m)||. \tag{17}
\]

Proof. Let \( c \) be any chain in \( S(P(n,m), \partial P(n,m)) \) and suppose that \( c \) represents \([P(n,m), \partial P(n,m)]\). Choose \( j, k \) large enough so that

\[
(n-2) - n/j > 0 \quad \text{and} \quad (m-2) - m/k > 0. \tag{18}
\]

We represent \( P(n,m) \) as the polytope in \( \mathbb{H}^2 \times \mathbb{H}^2 \) formed from a regular \( n \)-gon with interior angles equal to \( \pi/j \times \pi/k \). Because the \( n \)-gon and the \( m \)-gon both tile \( \mathbb{H}^2 \) by reflection, \( P(n,m) \) tiles \( \mathbb{H}^2 \times \mathbb{H}^2 \) by reflections. Using these reflections, the chain \( c \) induces a chain \( \tilde{c} \) on \( \mathbb{H}^2 \times \mathbb{H}^2 \) that is invariant under \( G := G_{n,j} \times G_{m,k} \). But this group has a finite index subgroup \( G' := G'_{n,j} \times G'_{m,k} \) that is torsion-free with no orientation-reversing elements. Hence the chain \( \tilde{c} \) pushes forward to a chain \( c_M \) on the quotient manifold \( M \). Let \( c_M \) represent the fundamental class \([M]\). From this construction, we have

\[
||c_M|| = [G : G'][||c||]. \tag{20}
\]

It is easy to see that \( M \) is equal to the Cartesian product of two surfaces with both surfaces orientable and of genus at least two. Therefore, there is a covering map \( \pi : M \to \Sigma_2 \times \Sigma_2 \). Let \( c_\Sigma = \pi_*([c_M])/\text{deg}(\pi) \). It represents the fundamental class \([\Sigma_2 \times \Sigma_2]\). Its norm satisfies \( ||c_\Sigma|| = ||c_M||/\text{deg}(\pi) \). So,

\[
||c_\Sigma|| = ||c_M||/\text{deg}(\pi) \tag{21}
\]
\[
= ||c||[G : G']/\text{deg}(\pi) \tag{22}
\]
\[
= ||c||\text{vol}(\Sigma_2 \times \Sigma_2)/\text{vol}(P(n,m)) \tag{23}
\]
\[
= ||c||\frac{16\pi^2}{(n-2-n/j)(m-2-m/k)\pi^2}. \tag{24}
\]

The third equality above follows by observing that the volume of \( M \) is equal to \([G : G'] \text{vol}(P(n,m))\) since \( M \) is tiled by \([G : G']\) copies of \( P(n,m) \). Similarly, the volume of \( M \) is equal to \( \text{deg}(\pi) \) times the volume of \( \Sigma_2 \times \Sigma_2 \).
Letting \( j \) and \( k \) tend to infinity in the above and letting \( ||c|| \) tend towards \( ||P(n, m)|| \) finishes the lemma.

\[ \square \]

**Lemma 2.7.**

\[
||\Sigma_2 \times \Sigma_2|| \geq \inf_{n,m} \frac{16}{(n-2)(m-2)}||P(n, m)||.
\]

**Proof.** Let \( G \) be a discrete group of hyperbolic isometries such that \( \mathbb{H}^2/G \equiv \Sigma_2 \). Let \( c \) be a chain in \( S(\Sigma_2 \times \Sigma_2) \) representing the fundamental class. There is a finite chain \( c_0 \) of simplices in \( \mathbb{H}^2 \times \mathbb{H}^2 \) such that \( c = \pi_\ast(c_0) \) where \( \pi: \mathbb{H}^2 \times \mathbb{H}^2 \to \Sigma \times \Sigma \) is the universal covering map. It will be convenient to use Thurston’s smearing construction [Thurston]. Let

\[
c_0 = \sum_{i=1}^{k} r_i \sigma_i.
\]

For each \( i \), we let \( smear(\sigma_i) \) denote the real-valued measure supported on all orientation-preserving isometric translates of \( \sigma_i \) that is induced from Haar measure on \( Isom^+(\mathbb{H}^2 \times \mathbb{H}^2) \). We define \( smear(c_0) \) by

\[
smear(c_0) = \Sigma_{i=1}^{k} smear(\sigma_i).
\]

If \( G' \) is any discrete torsion-free cocompact group of isometries, \( smear(c_0) \) induces a measure \( smear_{G'}(c_0) \) on singular simplices of the quotient \( (\mathbb{H}^2 \times \mathbb{H}^2)/G' \) by setting \( smear_{G'}(c_0)(S) = smear(c_0)(\tilde{S}) \) where \( \tilde{S} \) is any Borel set of singular simplices in \( \mathbb{H}^2 \times \mathbb{H}^2 \) that projects down to \( S \) injectively. From the construction, the total mass of \( smear_{G'}(c_0) \) is given by

\[
(25) || smear_{G'}(c_0) || = ||c|| \frac{\text{vol}(\mathbb{H}^2 \times \mathbb{H}^2)/G'}{\text{vol}(\Sigma_2 \times \Sigma_2)}.
\]

The benefit of this formula is that we can pass from the original chain \( c \) defined on \( \Sigma_2 \times \Sigma_2 \) to a measure-chain on a more convenient manifold. To define that manifold, let \( d \) be the maximum diameter of the image of \( \sigma_i \) in \( \mathbb{H}^2 \times \mathbb{H}^2 \) \((i = 1, \ldots, k) \). For every \( h > 0 \), there is a regular \( 4h \)-gon \( F_h \) with all interior angles equal to \( 2\pi/4h \). We choose \( h \) large enough so that every pair of nonadjacent sides of \( F_h \) is at least a distance \( d \) apart. We let \( T \) be a tiling of the plane \( \mathbb{H}^2 \) with copies of \( F_h \).

We will need the concept of a **straight simplex**. We use the Lorenz model of the hyperbolic plane [Ratcliffe 1994]. The Lorenz inner product on \( \mathbb{R}^3 \) is defined by

\[
x \circ y = -x_1 y_1 + x_2 y_2 + x_3 y_3.
\]

\( \mathbb{H}^2 \) is identified with the set of vectors \( x \) satisfying \( x \circ x = -1 \) and \( x_1 > 0 \). We let \( |||x||| \) denote the absolute value of \( \sqrt{x \circ x} \). We say that a simplex \( \sigma: \Delta^4 \to \mathbb{H}^2 \) is straight if for every \( x = \Sigma_i x^i e_i \in \Delta^4 \)

\[
\sigma(x) = \Sigma_i x^i \sigma(e_i)/|||\Sigma_i x^i \sigma(e_i)|||.
\]

We say that a simplex \( \sigma: \Delta^4 \to \mathbb{H}^2 \times \mathbb{H}^2 \) is straight if composing \( \sigma \) with a projection to either of the \( \mathbb{H}^2 \) factors results in a straight simplex. Such
a simplex is uniquely determined by its vertices. Its image is equal to the convex hull of its vertices.

Our last operation is called “snapping”. If \( \sigma \) is a singular simplex in \( \mathbb{H}^2 \times \mathbb{H}^2 \), we let \( \text{snap}(\sigma) \) be the straight simplex satisfying the following. For all \( i = 0, \ldots, 4 \), \( \text{snap}(\sigma)(e_i) \) is the closest vertex of the tiling \( T \times T \) to \( \sigma(e_i) \) if there is only one closest vertex and \( \text{snap}(\sigma)(e_i) = \sigma(e_i) \) otherwise. The map \( \text{snap} \) on singular simplices induces a map \( \text{snap}^* \) on measures on the set of singular simplices. We will use \( \text{snap}^*(\text{smear}(c_0)) \) to construct a chain on \( (F \times F, \partial(F \times F)) \). But first, if \( \sigma \) has diameter at most \( d \) then we need to show that \( \text{snap}(\sigma) \) is contained in single tile of \( T \times T \).

To see this, consider the dual tiling \( T^* \) of \( T \). The vertices of \( T^* \) are the centers of the tiles of \( T \) and there is an edge between two dual vertices if and only if the corresponding tiles in \( T \) are adjacent. In our case, the tiles of \( T^* \) are copies of \( F \). Since every two nonadjacent sides of \( F \) have a distance at least \( d \) apart, the image of \( \pi_i \sigma \) does not overlap any nonadjacent edges of the dual tiling (where \( \pi_i \) is projection from \( \mathbb{H}^2 \times \mathbb{H}^2 \) onto the \( i \)-th \( \mathbb{H}^2 \) factor). Hence there is a vertex of the dual tiling contained in all the dual tiles that contain the image. Since vertices of the dual correspond to faces of the domain tiling \( T \) this implies that there is a single tile \( \tau \) of \( T \) such that: for every point \( x \) in the image of \( \pi_i \sigma \), the closest vertex \( v \) of \( T \) to \( x \) is contained in \( \tau \). By construction, this implies that the projection of \( \text{snap}(\sigma) \) to this \( \mathbb{H}^2 \)-factor is contained in \( \tau \). Since this is true for both \( \mathbb{H}^2 \) factors, \( \text{snap}(\sigma) \) is contained in a single tile of \( T \times T \).

Let \( c_F \) denote the restriction of \( \text{snap}^*(\text{smear}(c_0)) \) to the set of simplices that map into a chosen fixed tile of \( T \times T \). By construction, \( \text{snap}^*(\text{smear}(c_0)) \) is invariant under the symmetries of tiling \( T \times T \) so it is irrelevant which tile we use. \( c_F \) is supported on a finite set of simplices by construction, so we may identify it with a finite chain. It is a cycle representing \( (F \times F, \partial(F \times F)) \) because snapping and smearing commute with the boundary map. The norm of \( c_F \) is

\[
||c_F|| = ||\text{smear}_{G'}(c_0)|| = \frac{\text{vol}(F \times F)}{\text{vol}(\Sigma_2 \times \Sigma_2)} ||c|| = (h - 1)^2 ||c||,
\]

where \( G' \) is a group having \( F \times F \) as its fundamental domain. The first equation holds because the snapping operation preserves the total mass of a measure under a quotient by a group that stabilizes the tiling \( T \times T \). The second equality is equation 25. The third equality is from Lemma 2.5. Since \( F \times F \) is a realization of the polytope \( P(4h, 4h) \), \( ||P(4h, 4h)|| \leq ||c_F|| \). Equation 26 now implies (by taking \( ||c|| \) arbitrarily close to \( ||\Sigma_2 \times \Sigma_2|| \)) that

\[
||\Sigma_2 \times \Sigma_2|| \geq \frac{||P(4h, 4h)||}{(h - 1)^2}.
\]

By Lemma 2.4 as \( h \) tends to infinity, the right-hand side approaches
\[
\inf_{n,m} \frac{16 \| P(n, m) \|}{(n - 2)(m - 2)}.
\]

3. Small Triangulations of \( P(m, n) \)

Since the size of any triangulation of \( P(m, n) \) is an upper bound for its polytopal Gromov norm, in this section we give bounds, and in some cases exact values, for the size \( T(n, m) \) of a minimal triangulation of \( P(n, m) \). We consider this optimization over all possible coordinatizations of \( P(n, m) \). In this section we look at the combinatorics of the polytopes \( P(n, m) \) and their triangulations. We begin with a table of known sizes of minimal triangulations in specific instances.

|   | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|---|----|----|----|----|----|----|----|
| 3 | 6  | 10 | 15 | 19 | 24 | 28 | 33 |
| 4 | 10 | 16 | 26 | 32 | 42 | ≤48| ≤58 |
| 5 | 15 | 26 | 38 | ≤49| ≤61| ≤72|
| 6 | 19 | 32 | ≤49| ≤60| ≤77| ≤90|
| 7 | 24 | 42 | ≤61| ≤77|    |    |
| 8 | 28 | ≤48| ≤72| ≤90|    |    |
| 9 | 33 | ≤58|    |    |    |    |

Table 1. Minimal size triangulations for \( n \)-gons times \( m \)-gons.

For computing Table 1 we followed the approach of [De Loera et al. 1996], based on the solution of an integer programming problem. We think of the triangulations of a polytope as the vertices of the following high-dimensional polytope: Let \( A \) be a \( d \)-dimensional polytope with \( n \) vertices. Let \( N \) be the number of \( d \)-simplices in \( A \). We define \( P_A \) as the convex hull of the set of incidence vectors of all triangulations of \( A \). For a triangulation \( T \) the incidence vector \( v_T \) has coordinates \( (v_T)_\sigma = 1 \) if \( \sigma \in T \) and \( (v_T)_\sigma = 0 \) if \( \sigma \not\in T \). The polytope \( P_A \) is the universal polytope defined in general by Billera, Filliman and Sturmfels [Billera et al. 1990] although it appeared in the case of polygons in [Dantzig et al. 1985]. In [De Loera et al. 1996], it was shown that the vertices of \( P_A \) are precisely the integral points inside a polyhedron that has a simple description. The rational vertices of this polytope are in correspondence with the fractional face-to-face covers. The concrete integer programming problems were solved using C-plex Linear Solver\textsuperscript{TM}. The program to generate the linear constraints is a small C++ program written by De Loera and Peterson.

In the rest of the paper we will often use the following result, first proved (for triangulations) in [De Loera et al. 2001]. The same result (rounded up), and with almost the same proof, holds for odd \( m \) but we do not need it.
Theorem 3.1. Let \( m \geq 4 \) be an even number. The minimum triangulation of the prism over an \( m \)-gon, in any coordinatization, has size \( \frac{5}{2}(m-2) \). This number equals also the polytopal Gromov norm of the prism.

Proof. To see that \( \frac{5}{2}(m-2) \) is an upper bound for both numbers it suffices to describe a triangulation of that size. This goes as follows: first, chop alternate vertices of the \( m \)-prism, using \( m \) tetrahedra, to obtain an \((m/2)\)-antiprism. This has \( m \) triangular faces and two polygons of size \( m/2 \). Triangulate one of them arbitrarily, and triangulate the antiprism by coning from a vertex in the opposite face. One needs \( m - 3 + m/2 - 2 \) tetrahedra for this.

For the lower bound, we show that every affine chain in the fundamental class has norm at least \( \frac{5}{2}(m-2) \). Without loss of generality, we assume that the chain has all its vertices on vertices of the prism (see Remark 2.2). Also, since we are dealing with homology relative to the boundary, we assume that the chain has no tetrahedron contained in the boundary. In particular, each tetrahedron is of one of three types: a “bottom tetrahedron” with three vertices in the bottom \( m \)-gon and a vertex in the top, a “top tetrahedron” (the converse), or a “middle tetrahedron” with two vertices on each. It is obvious that we need at least \( m-2 \) bottom and \( m-2 \) top tetrahedra to cover the bottom and top \( m \)-gons. The result then follows if we prove that the number of middle tetrahedra is at least half that number. This holds because each middle tetrahedron has two “bottom triangles” (the ones with two vertices in the bottom) and the projections of these must also cover the bottom \( m \)-gon: any vertical (but otherwise generic) line must be covered by a sequence of tetrahedra in the chain that starts with a bottom tetrahedron and finishes with a top tetrahedron, which implies that in between necessarily some middle tetrahedra is used. \( \square \)

In the case of a triangle times an \( m \)-gon, the patterns shown in Table 1 suggested the following result, which is a rephrasing of equation (4) in Theorem 1.3.

Theorem 3.2. In any coordinatization, the minimum-size triangulations and the polytopal Gromov norm of \( P(3, m) \) satisfy:

1. If \( m \) is odd, \( \|P(3, m)\| = T(3, m) = 9m/2 - 15/2 \).
2. If \( m \) is even, \( T(3, m) = 9m/2 - 8 \) and \( \|P(3, m)\| \) lies between that number and \( 9m/2 - 9 \).

Proof. Let \( C_3 \) and \( C_m \) denote the triangle and \( m \)-gon of which \( P(3, m) \) is the product. Let \( A, B \) and \( C \) denote the three vertices of \( C_3 \).

We first prove the lower bound for the norm of an affine simplicial chain (and, hence, for the size of a triangulation). We assume without loss of generality that all the vertices in the chain are vertices of \( P(3, m) \) and that no 4-simplex is contained in the boundary of \( P(3, m) \). Then every maximal simplex in the chain falls into one of the following types:
(1) A “type A” simplex, with three vertices on $A \times C_m$ and one in each of $B \times C_m$ and $C \times C_m$. There are at least $m - 2$ of them (counted with their coefficients) in every affine simplicial chain. Similarly, there will be $m - 2$ simplices of types $B$ and $C$.

(2) A “type AB” simplex, with two vertices on $A \times C_n$, two on $B \times C_n$ and one on $C \times C_n$. Together with the type A and type C simplices, these must cover the prism $AB \times C_m$. Hence, as in the proof of Theorem 3.1 there are at least $\lceil (m - 2)/2 \rceil$ of them (there can certainly be more). Similarly, there are at least $\lceil (m - 2)/2 \rceil$ simplices of types $AC$ and $BC$.

Adding up these numbers gives $3m + 3\lceil m/2 \rceil - 9$, which coincides with the stated lower bound for $||P(3, m)||$ in both the odd and even cases. In the even case, however, no triangulation with exactly $9m/2 - 9$ simplices exists, hence increasing the lower bound by one. This is so because such a triangulation must triangulate each of the three facets $AB \times C_m$, $AC \times C_m$, and $BC \times C_m$ in its minimal way. But, by the analysis in [De Loera et al. 2001], every minimum-size triangulation of an $m$-prism with even $m$ must be obtained (as in the proof of Theorem 3.1) by first cutting alternate corners and then triangulating the remaining anti-prism. In particular, the three $m$-gons $A \times C_m$, $B \times C_m$ and $C \times C_m$ are triangulated by first cutting half the corners, and the corners cut should be opposite in the three $m$-gons, which is impossible. This proves $T(3, m) \leq 9m/2 - 8$ in this case.

The proof of the upper bound for $T(3, m)$ is via the explicit construction of a triangulation with the stated size. The triangulation is depicted for $P(3, m)$ in Figure 3 in a “Cayley Trick view”. The Cayley Trick is a simple but clever construction that, in our case, gives a natural bijection between the triangulations of $P(3, m)$ and the “mixed subdivisions” of the Minkowski sum of three equal copies of $C_m$ (see [Huber et al. 2000, Santos 2004] for details).

The triangulation displayed has the number of simplices of types “A”, “B”, “C”, “AB” and “AC” predicted in the above paragraphs, and only one more than predicted simplex of type “AC”. Exactly the same construction can be done for every even $m$, and produces $9m/2 - 8$ simplices. For odd $m$, we show on the right part of the figure how to obtain the minimal triangulation of $P(3, m)$ from that of $P(3, m + 1)$.

The above result suggests a simple and relatively efficient way of triangulating $C_n \times C_m$: triangulate $C_n$ into $n - 2$ triangles and triangulate each of the resulting $C_3 \times C_m$’s in the optimal way. It is easy to make the triangulations match in common boundaries: just label the vertices of $C_n$ with $A$, $B$ and $C$ in such a way that every triangle gets the three labels (as in Figure 4) and replicate the triangulation of Figure 3 so that the labels match. This procedure produces approximately $9mn/2$ maximal simplices. But this number can be decreased, as follows:
Figure 3. The minimal triangulation of $C_{12} \times C_3$ (left) and how to get the one of $C_{11} \times C_3$ from it.

Figure 4. Gluing triangulations in several copies of $C_3 \times C_m$.

**Theorem 3.3.** If $m$ and $n$ are both even, then $P(m, n)$ can be triangulated with $rac{7}{2}mn - 6(m + n) + 8$ simplices.

Observe that this coincides with the empirical values Table 1, except for (6,6), (6,8) and (8,6), where it is two units above the value in the table.

**Proof.** We start with the triangulation $K$ obtained by replicating $n-2$ times the triangulation of $P(3, m)$ with $\frac{9}{2} - 8$ simplices constructed in the previous theorem. It will be important later that the triangulation (and labeling) we choose for $C_n$ is exactly the one shown in Figure 4 with $n/2 - 2$ interior edges labeled $AB$, $n/2 - 1$ interior edges labeled $BC$, and no interior edge labeled $AC$.

The proof consists on repeatedly using the following trick: let us concentrate on two triangles of $C_n$ glued along a prism labeled, say, $AB$. We denote $C$ and $C'$ the vertices of $C_n$ opposite to the particular edge $AB$ we are considering (see again Figure 4).
Suppose there is a convex sub-polytope of the common $AB$-prism that is triangulated in $K$. Suppose also that all its simplices are joined to the same vertex $(C, i)$ in $C \times C_m$ (hence also to $(C', i)$ in $C' \times C_m$). Then, we have a bipyramid (a suspension) over $Q$, let us call it $SQ$, triangulated by first decomposing it into its two pyramids. One would expect that a more efficient way of triangulating $SQ$ is to join its axis to all the “equatorial” boundary simplices (that is to say, to all the triangles in $\partial Q$).

Things are actually a bit more complicated than suggested by the above sentence. In the sentence we are implicitly assuming that the segment joining the two apices of the pyramids intersects the interior of $Q$ (otherwise we do not have a geometric bipyramid). But it is not easy to guarantee that this is indeed the case and, moreover, it is not the most efficient way of doing things.

Indeed, if the axis intersects the interior of $Q$, then the number of simplices that we get when we retriangulate equals the number of triangles in $\partial Q$. But suppose, instead, that the axis intersects a boundary point $x$ of $Q$. Then, we can retriangulate by joining the axis to the triangles in facets of $\partial Q$ that do not contain $x$. One problem with this is then we have to take care that the retriangulation of $SQ$ matches the rest of the triangulation of $C_n \times C_m$ that we had. The way we guarantee this is as follows: $Q$ is going to contain the segment $[(A, i), (B, i)]$, and all the boundary faces of $Q$ containing $x$ are going to be boundary faces of $AB \times C_m$ as well. In the Cayley picture of Figure 3 this property corresponds to $Q$ containing a vertex of Minkowski sum and part of the two boundary segments incident to it. The consequence of this is that the segment $[(C, i), (C', i)]$ intersects $Q$ in a relative interior point of the edge $[(A, i), (B, i)]$. We will call that edge the distinguished edge in the following discussion.

Figures 5 and 6 show how we implement this idea in the $AB$ and $BC$ prisms, respectively. The shaded areas are the polygons $Q$ that we take in each of the prisms.

In the $AB$-prism, the region $Q$ we consider is the the antiprism obtained from $AB \times C_m$ by cutting alternate corners (this appears as a regular 12-gon in Figure 5) together with one corner tetrahedron of the prism (the small triangle in the top of the figure). With that small triangle included, $Q$ is triangulated into $3m/2 - 4$ simplices, so $SQ$ is triangulated into $3m - 8$ simplices. The boundary of $Q$ has $2m - 2$ triangles (two more than the antiprism would have), but two of them are incident to the distinguished edge. Hence, we can retriangulate $SQ$ into $2m - 2$ simplices, saving us $m - 4$ simplices in total. Since we have $n/2 - 2$ edges of type $AB$ we save $(n/2 - 2)(m - 4)$ simplices.

In each prism $BC \times C_n$, we take several different polytopes $Q$ to apply the trick, as shown in Figure 6. There are $m/2 - 2$ of a certain type and two of another. The type of each is very easy to deduce from the figure: the two special ones are triangular prisms (Cayley embedding of two equal triangles, one of type $B$ and one of type $C$) and the other ones are 3-dimensional.
total savings: $m - 4$

**Figure 5.** The region $Q$ in the facet $AB$.

total savings: $2(m/2) = m$

**Figure 6.** The regions $Q$ in the facet $BC$.

cubes with one vertex truncated (Cayley embedding of a quadrilateral and a triangle made with three of its four vertices. In the first type the original triangulation of $SQ$ has 6 simplices, and we substitute them by four simplices: the distinguished edge of the triangular prism is one connecting the two opposite triangles, so there are four triangles in facets not containing the
point \( x \). In the other type we originally have a triangulation into 8 simplices and substitute it by one with only 6: the distinguished edge is incident to two quadrilateral facets of \( Q \), and there remain another quadrilateral one (two triangles) plus four triangular ones. In total, we are decreasing the number of simplices by \( m \). Since we have \( n/2 - 1 \) prisms of this type, we save \( m(n/2 - 1) \) simplices in total.

Summing up, our final triangulation has

\[
(n - 2)(9m/2 - 8) - (n/2 - 2)(m - 4) - (n/2 - 1)m = 7nm/2 - 6n - 6m + 8
\]

simplices, as claimed.

There is still one more thing that needs to be said in order to justify correctness of the construction. In the triangulation of (almost all of) each copy of \( C_3 \times C_n \) we have done changes to some pyramids with base on the \( AB \) side and some on the \( BC \) side. Of course, for this to be possible we need these pyramids to be disjoint. That they indeed are disjoint is easy to check in Figures 5 and 6. It just amounts to observing that the shaded regions in the two pictures do not overlap.

To finish the proof of Theorem 1.3, only the equations for \( P(4, m) \) remain. The upper bound is just the substitution of \( n = 4 \) in equation (6). The idea for the lower bound is similar to the one in Theorem 3.1.

**Theorem 3.4.** The polytopal Gromov norm of \( P(4, m) \) is at least \( 3\lceil 5(m - 2)/2 \rceil \).

**Proof.** We regard \( P(4, m) \) as a prism over the prism over an \( m \)-gon. That is to say, we regard its vertices as lying in a “bottom prism” and a “top prism”. Let \( \alpha \) be an affine simplicial chain representing the top relative homology class. As usual, we assume that the vertices of \( \alpha \) are vertices of \( P(4, m) \) and that no four simplex in \( \alpha \) lies in the boundary. Then, the simplices in \( \alpha \) are of four types, depending on the number of vertices they have on the top prism: we call them “bottom”, “half-bottom”, “half-top” and “top” simplices. The bottom and top simplices need to cover the bottom and top prisms. By Theorem 3.1 there are at least \( \lceil 5(m - 2)/2 \rceil \) of each type in \( \alpha \). Also, by the same argument as in the proof of Theorem 3.1, the numbers of half-bottom and half-top simplices are each equal to at least that number, giving the total of \( 3\lceil 5(m - 2)/2 \rceil \) (rounded up to an even number). \( \square \)

4. A Binary Cover of \( P(m, m) \) with \( \frac{13m^2}{4} - \frac{19m}{2} \) Simplices

Clearly, the definition of \( \|P\| \) allows for much more freedom than using triangulations of \( P \), in order to get upper bounds. Here we use binary covers of \( P \) for this purpose.

Recall that a pseudo-manifold is a simplicial complex of pure dimension in which every codimension-one simplex lies in at most two full-dimensional ones. Its boundary consists of the codimension-one simplices that lie only in one full-dimensional simplex. A binary cover of an \( n \)-dimensional polytope
$P$ is a continuous map $f : K \to P$ from an oriented pseudo-manifold $K$ of dimension $n$ with the property that $f$ is linear on every simplex and it restricts to a degree 1 map from $\partial K$ to $\partial P$.

**Remark 4.1.** Every binary cover can be homotoped to one that sends vertices of $K$ to vertices of $P$. Just choose, for each vertex $v$ of $K$, a vertex of the minimal face of $P$ containing $f(v)$.

**Lemma 4.2.** If $f : K \to P$ is a binary cover of the polytope $P$, then $||P||$ is at most equal to the number of full-dimensional simplices in $K$.

**Proof.** Since $K$ is a simplicial complex, there is an obvious chain associated to it in which every top-dimensional simplex has weight 1 (the fact that $K$ is oriented is important here). We denote this chain by $K$. The induced chain $f_*(K)$ is an affine chain of the polytope $P$. Because every codimension-one simplex of $K$ lies in at most two full-dimensional ones, $f_*(K)$ is a cycle in $S(P, \partial P)$. Because $f$ restricted to the boundary has degree 1 it follows (via Mayer-Vietoris) that $f$ itself has degree 1, so $f_*(K)$ represents the fundamental class $[P, \partial P]$. Therefore, $||P||$ is at most equal to the number of simplices of $K$.  

In this section, we exhibit two binary covers of $P(m, m)$, for $m$ even. One has $\frac{13m^2}{4}$ and the other one slightly less. Instead of describing the pseudo-manifold $K$, we list the images of its simplices in $P(m, m)$. The pseudo-manifold structure will be discussed later. We label the vertices of $P(m, n)$ by $(i, j)$ for $i, j = 1, \ldots, m$, in the obvious way. Indices are regarded modulo $m$, and to list each simplex we give its vertices. The first list is:

1. For each of the $m^2/4$ values of $(i, j)$ with $i$ even and $j$ odd, the following six simplices, all of which contain the vertices $(i-1,j)$ and $(i+1,j)$:
   - (A) The corner simplex at $(i, j)$. A corner simplex consists of $(i, j)$ and its four neighbors $(i-1,j)$, $(i+1,j)$, $(i,j-1)$, $(i,j+1)$.
   - (B) The simplex $(i-1,j)$, $(i+1,j)$, $(i,j-1)$, $(i,j+1)$, $(i,i)$.
   - (C) The simplex $(i-1,j)$, $(i,j+1)$, $(i,j+1)$, $(i,j+1)$, $(i,i)$.
   - (D) The simplex $(i-1,j)$, $(i+1,j)$, $(i,j-1)$, $(j-1,j-1)$, $(i,i)$.
   - (E) The simplex $(i-1,j)$, $(i+1,j)$, $(j+1,j+1)$, $(j-1,j-1)$, $(i,i)$.
   - (F) The simplex $(i-1,j)$, $(i+1,j)$, $(j+1,j+1)$, $(j-1,j-1)$, $(j,j)$.
2. Symmetrically, for each of the $m^2/4$ values of $(j,i)$ with $j$ odd and $i$ even, the six simplices $(A')$, $(B')$, $(C')$, $(D')$, $(E')$ and $(F')$ obtained from the previous six by exchanging $i$ and $j$.
3. Finally, for each of the $m^2/4$ values of $(i,j)$ with $i$ and $j$ odd, the simplex $(i,j)$, $(i-1,i-1)$, $(i+1,i+1)$, $(j-1,j-1)$, $(j+1,j+1)$.

This gives 13 types of simplices, which we will refer to as $(A)$, $(B)$, $(C)$, $(D)$, $(E)$, $(F)$, $(A')$, $(B')$, $(C')$, $(D')$, $(E')$, $(F')$ and $(G)$. There are $m^2/4$
Figure 7. A small binary cover of $P(m,m)$. 
of each type. Figure 7 schematically shows the construction. It depicts
one simplex of each of the types (A) through (G), each drawn as a set of
two points in a 2-dimensional grid. The dashed diagonal line in six of the
simplices represents the set of vertices \((i, i)\) of \(P(m, m)\).

Observe that some of the simplices are degenerate (they are not full-
dimensional or they even have repeated vertices). This happens, for exam-
ple, for the simplices (B) through (F) if \(i - j = \pm 1\), or for any simplex \(G\) if
the factor polygons of \(P(m, m)\) are equal (the diagonal of the grid represents
then a 2-plane in \(\mathbb{R}^4\), and the “4-simplex” \(G\) has a 3-face lying in it).

To help check that this is indeed a binary cover, the incidences between
simplices are marked in the figure. More precisely, each simplex has five
“bonds” to either other simplices, or to the symbol \(\partial\), representing the
boundary of \(K\). The five vertices in each simplex are labeled 1 through
5 and the bond labeled \(i\) on one side represents the facet opposite to vertex
\(i\) on that simplex. A bond is drawn solid if it joins exactly the simplices
in the figure (or if the corresponding facet lies in the boundary of \(P(m, m)\))
and is drawn dashed if it is between the simplex in the picture and one not
in the picture, (but of the same type). It is left to the reader to check that,
with the glueings specified by the bonds in the picture, the list of simplices
is indeed an oriented pseudo-manifold with boundary.

Now we try to understand how the boundary of the pseudo-manifold
covers the boundary of \(P(m, m)\). The first check is that, indeed, all the facets
of simplices with bonds to the symbol \(\partial\) lie in the boundary of \(P(m, m)\).
Next, we concentrate on a facet of \(P(m, m)\), say the prism consisting of the
vertices \((i, \ast)\) and \((i + 1, \ast)\) for some (say, even) \(i\). Each simplex of type (A)
or (A') “centered” at a point \((i, j)\) or \((i + 1, j)\) contains a facet on our prism,
and it cuts a corner of it. After all of them are removed, what remains is an
\(m/2\)-antiprism consisting of the vertices \((i, j)\) for even \(j\) and \((i + 1, j)\) for odd
\(j\). The other simplices with facets in our prism are those of types (B'), (C)
and (F). This is so because (B), (C'), (D') and (F') only contain facets on
“vertical” prisms, and (D) contains a facet in every other horizontal prism,
but not the one we are considering. It can be easily checked that the facets
that (B'), (C) and (F) have in our antiprism produce the following degree
one cover of it: consider the cover of the \(m/2\)-gon \((i, \ast)\) (where “\(\ast\)” is meant
to be even) obtained by coning \((i, i)\) to the boundary. Then join this cover,
while the \(m\) triangular faces of the antiprism, to \((i + 1, i + 1)\).

With all this we conclude that:

**Theorem 4.3.** The above list of \(13m^2/4\) simplices forms a binary cover of
\(P(m, m)\).

Our next goal is to show that this binary cover, call it \(\alpha\), contains as a
proper subset an even smaller binary cover. This is obtained by deleting
from the initial binary cover all the simplices with repeated vertices (but
this condition is not enough to guarantee that they can be removed. The
reader should check that after the removal, and with some minor regluing, we still have an oriented pseudo-manifold):

1. The simplices of types (B) and (B') for which \( i - j = \pm 1 \) (2m of them).
2. The simplices of types (C) and (C') for which \( i - j = 1 \) (m of them).
3. The simplices of types (D) and (D') for which \( i - j = -1 \) (m of them).
4. The simplices of types (E) and (E') for which \( i - j = \pm 1 \) (2m of them).
5. The simplices of types (F) and (F') for which \( i - j = \pm 1 \) (2m of them).
6. The simplices of type (G) and (E') for which \( i - j = 0 \) or \( \pm 2 \) (3m/2 of them, if \( m \geq 6 \)).

This deletes 19m/2 simplices from the initial list. We leave it to the reader to check that indeed this is a binary cover.

**Corollary 4.4.** For every even \( m \geq 6 \), \( P(m, m) \) has a binary cover with \( 13m^2/4 - 19m/2 \) simplices.

**Remark 4.5.** Observe that this new binary cover still has some degenerate simplices, at least if we assume the two \( C_m \) factors in \( P(m, m) \) to be equal. For example, the \( m^2/4 - 3m/2 \) simplices of type G all have a 3-face contained in a 2-plane. Even though they do not cover any “space”, their removal would leave some interior tetrahedra unmatched. In other words, the \( 3m^2 - 8m \) simplices of types A through F' form a cover of \( P(m, m) \) without overlaps, but this cover is insufficient to make a statement about the Gromov norm because some faces are unmatched.

## 5. A Lower Bound for the Polytopal Gromov norm

In this section we prove a lower bound for the polytopal Gromov norm of \( P \) by counting (with weights) certain incidences in affine chains of \( S(P, \partial P) \).

Each affine 4-simplex \( \sigma \in P(m, n) \) has 20 triangle-tetrahedron incidences. We say that one of these incidences is a titap incidence if the tetrahedron is contained in a facet (prism) of \( P(m, n) \) and the triangle is interior to that facet. (“Titap” is short for “triangle interior to a prism”). We denote the number of titap incidences in \( \sigma \) as titap(\( \sigma \)). Similarly, for an affine chain \( c = \Sigma_i w_i \sigma_i \in S(P, \partial P) \) we define

\[
\text{titap}(c) = \Sigma_i |w_i| \text{titap}(\sigma_i).
\]

**Lemma 5.1.** For every affine chain \( c \in S(P, \partial P) \),

\[
\text{titap}(c) \geq 12mn - 16m - 16n.
\]
Proof. Clearly, the titap incidences in $c$ can be counted by adding the titap incidences in the restrictions of $c$ to the individual boundary prisms of $P(m,n)$ (because each titap incidence belongs to one and only one prism).

Let $c$ be an affine chain and let $c'$ be its restriction to a certain $m$-prism. As in the proof of Theorem 3.1, we classify the tetrahedra in $c'$ as “bottom”, “middle” and “top”, depending on their number of vertices in the bottom and top $m$-gons of the prism. We count titap incidences in the three groups of tetrahedra separately.

Each bottom tetrahedron $\tau$ has a unique triangle $\rho$ in the bottom $m$-gon of the prism. Clearly, if an edge of $\rho$ is interior to the $m$-gon, the corresponding triangle in $\tau$ is a titap incidence. Since the bottom triangles must cover the bottom $m$-gon, they produce at least $2(m-3)$ of these incidences (because a binary cover of the bottom $m$-gon has at least $m-3$ interior edges, each in at least two triangles). Similarly, there are at least $2(m-3)$ titap incidences in top tetrahedra.

Each middle tetrahedron has two bottom triangles (with 2 vertices in the bottom $m$-gon and one in the top $m$-gon) and two top triangles. Some of these triangles may be in vertical faces of the prism, but (as in the proof of Theorem 3.1) we at least know that the bottom triangles cover the $m$-gon, when projected to it, and the same for the top triangles. Hence, there are at least $m-2$ of each type that are not vertical. Hence, middle tetrahedra produce at least $2m-4$ titap incidences. Adding this up, we conclude that an $m$-prism contains at least $2(m-3) + 2(m-3) + 2m-4 = 6m-16$ titap incidences. Adding over the $n$ $m$-prisms plus $m$ $n$-prisms gives the statement. □

**Lemma 5.2.** Let $\sigma$ be an affine simplex in $P(m,n)$, not contained in the boundary. Then, $\text{titap}(\sigma) \leq 6$.

*Proof.* Let $k$ be the number of facets of $\sigma$ that lie in the boundary of $P(m,n)$. Since $\sigma$ is not contained in the boundary, the $k$ boundary tetrahedra in $\sigma$ lie each in a different facet of $P(m,n)$. In particular, the common triangle to two of them is not interior to a prism, and does not produce a titap incidence. Then, each of the $k$ tetrahedra produces at most $4-(k-1)=5-k$ titap incidences, because $k-1$ of its four triangles are used in adjacencies to other boundary tetrahedra. Hence, $\sigma$ has at most $k(5-k)$ titap incidences. The maximum of $k(k-1)$ is 6, achieved for $k=2$ or 3. □

**Corollary 5.3.** $T(m,n) \geq ||P(m,n)|| \geq 2mn - 8(m+n)/3$.

*Proof.* For every affine chain $c = \Sigma_i w_i \sigma_i \in S(P, \partial P)$,

$$12mn - 16m - 16n \leq \text{titap}(c) = \Sigma_i |w_i| \text{titap}(\sigma_i) \leq 6\Sigma_i |w_i| = 6||c||,$$

where the two inequalities come from the previous two lemmas. □

**Remark 5.4.** We do not believe our lower bound to be very close to the real value of $||P(m,n)||$ or $T(m,n)$, because it is based in a very specific
type of incidence. Our conjecture is that $||P(m, n)||$ is closer to the upper bound obtained in Section 2, perhaps in $3mn \pm O(m + n)$.

Observe also that our lower bound can be slightly improved if we restrict our attention to corner-cutting triangulations, that is to say, triangulations that first cut $mn/2$ vertices of $P(m, n)$ via corner 4-simplices, for $m$ and $n$ even. The $mn/2$ corner simplices produce only $2mn$ ttip incidences, and we need at least another $(10mn - 16n - 16m)/6$ simplices to produce the rest, giving a total of at least $13mn/6 - O(m + n)$ simplices.

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ERRATUM

Michelle Bucher-Karlsson has communicated to us that our proof of Lemma 2.7 is wrong. We say:

For every $h > 0$, there is a regular $4h$-gon $F_h$ with all interior angles equal to $2\pi/4h$. We choose $h$ large enough so that every pair of nonadjacent sides of $F_h$ is at least a distance $d$ apart.

In this sentence, $d$ could be arbitrarily large, since it was defined as the maximum diameter of an (a-priori, arbitrary) simplex in $\mathbb{H}^2 \times \mathbb{H}^2$. Our choice of $h$ is then in contradiction with the following statement, communicated to us by Bucher-Karlsson:

**Proposition.** For any $h > 0$, the distance between the midpoints of two adjacent edges in $F_h$ is smaller than $\text{arccosh}(3) \sim 1.763$. In particular, $F_h$ contains two non-adjacent edges at distance smaller than $2\text{arccosh}(3)$.

**Proof.** Given a hyperbolic geodesic triangle with angles $\alpha$, $\beta$, $\gamma$ and opposite sides of lengths $a$, $b$, $c$ respectively, the second cosine rule for hyperbolic triangles states that

$$\sin(\beta) \sin(\gamma) \cosh(a) = \cos(\alpha) + \cos(\beta) \cos(\gamma).$$

Consider the geodesic triangle with vertices the midpoints of two adjacent edges and the center of $F_h$. The angle at the center is equal to $2\pi/4h$ and, by a symmetry argument, the angles at the two other corners are both equal to $\pi/4$. Thus, by the second cosine rule the distance between the midpoints of two adjacent edges is equal to

$$\text{arccosh} \left(2 \cos \left(\frac{2\pi}{4h}\right) + 1\right),$$

and is hence bounded by $\text{arccosh}(3)$. \qed

Without Lemma 2.7, one direction of the equality in our Theorem 2.3, and in its generalization Theorem 1.1, is invalid. The correct statements must now be:

**Theorem 1.1.** Let $\|P\|$ denote the polytopal Gromov norm of a polytope $P$. Then, the Gromov norm of the product $\Sigma_g \times \Sigma_h$ of two surfaces of genera $g$ and $h$ equals

$$\frac{\|\Sigma_g \times \Sigma_h\|}{(g-1)(h-1)} \leq 16 \lim_{n,m \to \infty} \frac{\|P(n,m)\|}{nm} = 16 \inf_{n,m} \frac{\|P(n,m)\|}{nm}.$$  

**Theorem 2.3.**

$$\|\Sigma_2 \times \Sigma_2\| \leq \lim_{n,m \to \infty} \frac{16\|P(n,m)\|}{(n-2)(m-2)}.$$
The lower bound $32(g-1)(h-1) \leq \| \Sigma_g \times \Sigma_h \|$ that we gave in Corollary 1.5 is also invalid. In fact, Bucher-Karlsson has computed exactly the value of $\Sigma_g \times \Sigma_h$:

**Theorem.** (Bucher-Karlsson [BK1]) Let $M$ be a closed, oriented Riemannian manifold whose universal cover is isometric to $\mathbb{H}_2 \times \mathbb{H}_2$. Then

\[ \|M\| = 6\chi(M). \]

As an update, in a more recent paper [BK2] the same author has improved our lower bound for the polytopal Gromov norm of the product of two polygons. We proved $\|P(m,n)\| \geq 2mn - O(m + n)$ and she gets $\|P(m,n)\| \geq 3.125mn - 5(m + n) + 6$.

This confirms what we said in Remark 5.4: “We do not believe our lower bound to be very close to the real value of $\|P(m,n)\|$ or $T(m,n)$. Our conjecture is that $\|P(m,n)\|$ is closer to the upper bound$^1$ obtained in Section 4”. We were only wrong in our final guess “perhaps in $3mn \pm O(m + n)$”.

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$^1\|P(m,m)\| \leq 3.25m^2 - \Omega(m + n)$