ALMOST-PERIODIC PERTURBATIONS OF NON-HYPERBOLIC EQUILIBRIUM POINTS VIA PÖSCHL-RÜSSMANN KAM METHOD

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Abstract. This paper focuses on almost-periodic time-dependent perturbations of a class of almost-periodically forced systems near non-hyperbolic equilibrium points in two cases: (a) elliptic case, (b) degenerate case (including completely degenerate). In elliptic case, it is shown that, under suitable hypothesis of analyticity, nonresonance and nondegeneracy with respect to perturbation parameter $\epsilon$, there exists a Cantor set $\mathcal{E} \subset (0, \epsilon_0)$ of positive Lebesgue measure with sufficiently small $\epsilon_0$ such that for each $\epsilon \in \mathcal{E}$ the system has an almost-periodic response solution. In degenerate case, we prove that, firstly, the almost-periodically perturbed degenerate system in one-dimensional case admits an almost-periodic response solution under nonzero average condition on perturbation and some weak non-resonant condition; Secondly, imposing further restriction on smallness of the perturbation besides nonzero average, we prove the almost-periodically forced degenerate system in $n$-dimensional case has an almost-periodic response solution under small perturbation without any non-resonant condition; Finally, almost-periodic response solution can still be obtained with weakened nonzero average condition by used Herman method but non-resonant condition should be strengthened. Some proofs of main results are based on a modified Pöschel-Rüssmann KAM method, our results show that Pöschel-Rüssmann KAM method can be applied to study the existence of almost-periodic solutions for almost-periodically forced non-conservative systems. Our results generalize the works in [14, 13, 23, 20] from quasi-periodic case to almost-periodic case and also give rise to the reducibility of almost-periodic perturbed linear differential systems.

1. Introduction. It is well known that the question of the existence of a.p. (almost periodic) solutions of ordinary differential equations is a hard problem. Much literature exists [6] on the a.p. solutions of small perturbed almost-periodically forced systems with hyperbolic linear part, but the real difficulty is non-hyperbolic case, especially the case where all eigenvalues of linear part have zero real part. In this case, there is no quality of exponential dichotomies and center manifold to take

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advantage of. However, many physical real word nonlinear models are related to
the above case. For example, the forced Duffing-van der Pol oscillator
\[ \ddot{x} + bx + h(t, x, \dot{x}) = \epsilon f(t, x, \dot{x}), \]
where \( b > 0 \) and \( h = O((x, \dot{x})^2)((x, \dot{x}) \to (0, 0)) \). One can check that all the
eigenvalues of \( \ddot{x} + bx = 0 \) are pure imaginary which is called elliptic case. By applying
KAM skill, in 1996 Jorba and Simó ([14]) considered non-degenerate system with a quasi-periodic perturbation
\begin{align*}
\omega \end{align*}
where the average of linearization of (2) along the unique quasi-periodic solution \( \tilde{x} \)
small \( \epsilon \)
\begin{align*}
\tau > d
\end{align*}
x
In this paper, we first consider \n-dimensional system
\begin{align*}
\dot{x} = Ax + h(t, x) + f(t, x, \epsilon), \quad x \in \mathbb{R}^n,
\end{align*}
where \( A \) is a nonsingular matrix, \( h \) and \( f \) are both quasi-periodic in \( t \) with the
same frequency \( \omega \in \mathbb{R}^d \), \( h = O(x^2)(x \to 0) \), and \( f \) is a small perturbation for
small \( \epsilon \), and assumed that \( A \) has \( n \) distinct eigenvalues \( \lambda_i \)'s and all eigenvalues
are pure imaginary. They proved that the existence of quasi-periodic solutions
highly depends on the relationship among the frequency \( \omega \), the tuple \( (\lambda_1, \ldots, \lambda_n) \) of
eigenvalues of \( A \) and the tuple \( (\lambda_1, \ldots, \lambda_n) \) of eigenvalues of the matrix
\( \tilde{A}(\epsilon) := A + [D_x f(\cdot, 0, \epsilon) + D_x h(\cdot, \dot{x}(\cdot, \epsilon), \epsilon)] \)
the average of linearization of (2) along the unique quasi-periodic solution \( \tilde{x}(t, \epsilon) \)
of the equation \( \dot{x} = Ax + f(\omega t, 0, \epsilon) \), where \( [F] \) denotes the average of \( F \) and
\( \tilde{h}(t, x, \epsilon) := h(t, x) + f(t, x, \epsilon) - f(t, 0, \epsilon) - D_x h(t, x, \epsilon) \).
Under the non-resonance conditions
\begin{align*}
|i, k, \omega| - \lambda_i |\geq \gamma|k|^{-\tau}, \quad |i, k, \omega| + \lambda_i - \lambda_j |\geq \gamma|k|^{-\tau}, \quad \forall \ k \in \mathbb{Z}^d \setminus \{0\}, \ 1 \leq i, j \leq n,
\end{align*}
where \( \tau > d + 1 \) is a certain real number, and the non-degeneracy conditions
\begin{align*}
\delta_1 |\epsilon_1 - \epsilon_2| < |\lambda_i(\epsilon_1) - \hat{\lambda}_i(\epsilon_2)| < \delta_2 |\epsilon_1 - \epsilon_2|,
\delta_1 |\epsilon_1 - \epsilon_2| < |\hat{\lambda}_i(\epsilon_1) - \lambda_i(\epsilon_1) - (\lambda_i(\epsilon_2) - \hat{\lambda}_j(\epsilon_2))| < \delta_2 |\epsilon_1 - \epsilon_2|,
\end{align*}
for \( |\epsilon_1| \) and \( |\epsilon_2| \) near 0, where \( \delta_2 > \delta_1 > 0 \) are constant, they proved that for
\( 0 < \epsilon_0 \ll 1 \) there exists a Cantor set \( E \subset (0, \epsilon_0) \) of almost full Lebesgue measure
such that for each \( \epsilon \in E \) the system (2) has a quasi-periodic solution with the same
frequency \( \omega \in \mathbb{R}^d \) as forcing, called response solution, near equilibrium point \( O \).
But as far as we knows, it seems very few results to study existence of response
solution of almost-periodically forced systems in elliptic case because it is not easily
to deal with 'small divisor' problem for infinite-dimensional frequencies. Thus, the
first goal of this paper is to consider the following question.

• Question 1: For almost-periodically forced system (2), can we get the existence
of a.p. response solutions with the same frequency \( \omega \) as forcing?

In this paper, we first consider \( n \)-dimensional system
\begin{align*}
\dot{x} = Ex + h(t, x) + f(t, x, \epsilon), \quad x \in \mathbb{R}^n,
\end{align*}
where \( E \) has \( n \) distinct pure imaginary eigenvalues \( \lambda_i \)'s and \( \lambda_i \neq 0 \) for all \( 1 \leq i \leq n \),
\( h = O(x^2)(x \to 0) \), and \( f \) is a small perturbation for small \( \epsilon \) and \( C^1 \)-smooth with
respect to \( \epsilon \). Here, \( f \) and \( h \) are both almost-periodic in \( t \) with same frequency
\( \omega \in \mathbb{R}^2 \) and admit the same spatial series expansion, the conception of which was
first presented by Pöschel in [15]. In section 3, by Pöschel-Rüssmann KAM method
we will prove that system (3) can be reduced to a suitable normal form with zero
as equilibrium point by an almost-periodic transformation. Hence, the almost-periodically
forced system (3) has an a.p. response solution near the equilibrium point.
Another direction of non-hyperbolic case is to study perturbations of degenerate equilibrium points. In degenerate case, this problem becomes more difficult to handle because the linear term cannot control the shift of equilibrium point very well, and so the standard KAM theory is not directly applicable to this perturbation problem. However, the degenerate systems are also relevant in view of applications to physical real world nonlinear models, such as system (1) with $b = 0$. In 1998, You ([27]) first considered the hyperbolic-type degenerate Hamiltonian systems

$$H = \langle \omega, y \rangle + \frac{1}{2} v^2 - u^2 + P(x, y, u, v),$$  \hspace{1cm} (4)

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. He proved there is an $\omega^*$ close to $\omega$ such that an $n$-dimensional invariant torus exits provided that the perturbation $P$ is analytic and small enough. In above work, the Hamiltonian system (4) is partially degenerate. Subsequently, Han, Li and Yi ([9]) investigated the degenerate Hamiltonian system

$$H = \langle \omega, y \rangle + \frac{1}{2} \langle z, M(\omega) z \rangle + \epsilon P(x, y, z, \omega),$$  \hspace{1cm} (5)

where $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2$. They proved the persistence of lower-dimensional tori in Hamiltonian systems of the form (5). In which, they imposed some restrictions on the perturbation $P$ but their result can be applied to the both hyperbolic and elliptic cases and is suitable for partially or completely degenerate case.

In recent years, many authors are devoted to investigate quasi-periodic perturbations of degenerate systems. In 2010 and 2013, Xu ([24, 25]) considered the perturbation of 2-dimensional nonlinear quasi-periodic system with partially degenerate equilibrium point, that is

$$\begin{align*}
\dot{x} &= y + h_1(x, y, t) + f_1(x, y, t), \\
\dot{y} &= x^{m+1} + h_2(x, y, t) + f_2(x, y, t),
\end{align*}$$  \hspace{1cm} (6)

for $m = 1$ or $m > 1$, respectively. It was proved that, by a nonlinear quasi-periodic transformation, system (6) can be reduced to a normal form with an equilibrium point at the origin under the Diophantine condition. Hence, a quasi-periodic response solution of the system (6) is obtained. But its method seems not valid to completely degenerate systems, i.e., systems with vanished linear part. The existence of quasi-periodic response solutions for quasi-periodically forced systems with completely degenerate equilibrium point, under small quasi-periodic perturbations, was also investigated recently. In 2016, Zhang, Jorba and Si ([28]) considered some specific completely degenerate differential system’s Poincaré map

$$\begin{align*}
\bar{x} &= x + y^m + f_1(\theta, x, y, \epsilon) + h_1(\theta, x, y, \epsilon), \\
\bar{y} &= y + x^n + f_2(\theta, x, y, \epsilon) + h_2(\theta, x, y, \epsilon),
\end{align*}$$

where $mn > 1$, $n \geq m$, $h_1$ and $h_2$ are higher order terms, $f_1$ and $f_2$ are lower order perturbations. It is shown that if $f = (f_1, f_2)^T$ satisfies some non-zero average hypotheses and the frequency vector $\omega$ is Diophantine, then the above map exists weakly hyperbolic invariant torus. In 2017 and 2018, Si and Si ([18, 19]) considered the perturbations of 4-dimensional quasi-periodically forcing systems with completely hyperbolic type degenerate equilibrium point and completely elliptic type degenerate equilibrium point respectively, in which we proved that these two
kinds of completely degenerate systems have a quasi-periodic response solution by means of the Pöschl-Rüssmann KAM method.

More recently, Si and Si ([20]) considered quasi-periodic perturbations of a general \( n \)-dimensional quasi-periodically forced systems with degenerate equilibrium point (including completely degenerate equilibrium point) like follows

\[
\dot{x} = \phi(x) + h(t, x) + f(t, x, \epsilon), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad (7)
\]

where \( \phi(x) = (a_1x_1^{l_1}, \ldots, a_{n-1}x_{n-1}^{l_{n-1}}, a_nx_n^{l_n})^T \) with \( 0 \neq a_i \in \mathbb{R}, \ l_i(i = 1, 2, \ldots, n) \) is positive integer, \( l_n \geq \bar{l} := \max\{l_1, \ldots, l_{n-1}\} \), \( h = (h_1, \ldots, h_n)^T \) with \( h = O(x^{l_n+1}) \) is a high order term and \( f = (f_1, \ldots, f_n)^T \) with \( f(\theta, x, 0) = 0 \) is a low order perturbation term. The above system is a special case of system (3) with \( E = D\phi(0) \).

Obviously, system (7) includes two kinds of degeneration, for example,

1. It is partially degenerate (i.e. \( E \neq 0 \) but \( \det E = 0 \)) if some \( l_i(i = 1, 2, \ldots, n-1) \) is one and \( l_n > \bar{l} \).
2. It is completely degenerate (i.e. \( E = 0 \)) if all of \( l_i(i = 1, 2, \ldots, n-1) \) are greater than one and \( l_n > \bar{l} \).

Moreover, system (7) has strong physical background which can describe oscillations of \( n \) weakly connected oscillators (Please see Remark 3.13 in the paper [20]). In [20], we proved that system (7) admits a quasi-periodic response solution if the frequency vector \( \omega \) satisfies Brjuno-Rüssmann’s non-resonant condition. And specially, system (7) in one-dimensional case admits a quasi-periodic response solution under some non-resonant condition weaker than Brjuno-Rüssmann’s and even without any non-resonant condition in some special perturbations. However, in almost-periodic case, the existence of a.p. response solutions of system (7) becomes more difficult because it is not only necessary to control the shift of equilibrium point with vanished linear part but also necessary to deal with the ‘small divisor’ produced by the integer combination of infinite many frequencies. Thus, the second goal of this paper is to consider the following question.

- **Question 2:** For almost-periodically forced degenerate system (7), can we get the existence of a.p. solutions with the same frequency \( \omega \) as forcing ?

In order to deal with the ‘small divisor’ in almost-periodic case, as shown in paper [15] the infinite-dimensional frequencies \( \omega \) should satisfy the following non-resonant condition

\[
|\langle k, \omega \rangle| \geq \frac{\gamma}{\Delta(|k|)\Delta(|k|)}, \quad 0 \neq k \in \mathbb{Z}^n, \quad (8)
\]

where the concrete definition of (8) can be seen later in next section. It can be seen that non-resonant condition (8) plays very important role in the process of KAM iteration. Thus, the third goal of this paper is to consider the following question.

- **Question 3:** Can almost-periodically forced degenerate system (7) admit a.p. solutions with the same frequency \( \omega \) as forcing under some non-resonant condition weaker than (8) or without any non-resonant condition ?

We give the answers of Questions 2 and 3 in section 4. We will apply the non-zero average hypothesis on perturbation to prove the almost-periodically forced degenerate system (7) in one-dimensional case admits an a.p. response solution under non-resonant condition weaker than (8). We also prove that the almost-periodically forced degenerate system (7) can admit an a.p. response solution without any non-resonant condition but smallness of perturbation must be restricted beside non-zero average condition. Finally, we prove that non-zero average hypothesis on perturbation can also be weakened under non-resonant condition (8).
In this paper, our main results are Theorem 3.1 and Theorems 4.1-4.3 below. Let us make some comments on the results:

1. A novelty of KAM skill in this paper is using Pöschel-Rüssmann KAM method in almost-periodic case. The Pöschel-Rüssmann KAM method is developed by Rüssmann [17] and Pöschel [16] who considered a class of dynamical systems of polynomial character and a constant vector field on n-torus, respectively. Compared with the traditional KAM iteration, the advantage of Pöschel-Rüssmann KAM method is that this KAM iteration containing an artificial parameter \( q, 0 < q < 1 \), makes the steps of KAM iteration infinitely small in the speed of the exponential function \( q^n e \) rather than super exponential function like classical KAM theory, which greatly simplifies the calculation process and as stated in [16], such result may be the shortest complete KAM proof for perturbations of integrable vector fields available.

Our results show that the Pöschel-Rüssmann KAM method can also be applied to study the perturbation problem of linear vector field. And even for the Pöschel-Rüssmann KAM method itself, we also have new improvements. It is worth to mention that we first use Pöschel-Rüssmann KAM method to almost-periodic case. The most important idea of Pöschel-Rüssmann KAM method in quasi-periodic case is sufficiently taking advantage of the polynomial structure in the truncation skill. Different from that, in almost-periodic case, we should find a new polynomial structure in the truncation (see (14) and (15) in the later). Furthermore, non-resonance condition in this paper is Brjuno-Rüssmann type. As far as we know, Xu and You had considered, in an old paper [22], almost periodic reducibility problem via the traditional KAM method and also used similar Brjuno-Rüssmann non-resonance conditions.

2. Our Theorem 3.1 and Theorems 4.1-4.3 are the direct generalization of works in [14, 23, 20] from quasi-periodic case to almost-periodic case. Actually, using the same method, we believe that the results in literatures [18, 19, 24, 25, 28] can also be generalized from quasi-periodic case to almost-periodic case. It is worth pointing out that the proof of Theorem 4.1 in this paper requires only that the frequency vector satisfies some weaker non-resonant conditions in which the approximate function does not necessary satisfies integral condition (see (11)). And the proof of Theorem 4.2 does not require any non-resonant condition on the frequency vector.

3. An important part of this paper is to consider the almost-periodic perturbations of degenerate systems. Since the systems we consider include completely degenerate case, new skill should adopted. A technical breakthrough to overcome completely degeneracy is that we make a translation transformation and suitable scalings for completely degenerate systems on the basis of equilibrium point of the averaged system to reduce the system into a normal form

\[
\dot{x} = (\eta^{l_1} A(\eta) + \eta^{l_2} Q(\theta, \eta)) x + \eta^{l_3} F(\theta, \eta) + G(\theta, x, \eta),
\]

where \( \eta \) is small perturbation parameter and \( G(\theta, x, \eta) = O(x^2)(x \to 0) \). Only when \( l_2 > l_1 \) and \( l_3 > 2l_1 \) hold, can the KAM method be valid. Generally speaking, it is hard to reach the order with \( l_2 > l_1 \) and \( l_3 > 2l_1 \) unless we impose more restrictions on the perturbation like non-zero average condition and smallness condition (see Theorems 4.1 and 4.2). Herman method is also used in this paper to weaken non-zero average condition (see Theorem 4.3). Herman method is a well-known KAM technique that introduces an artificial external parameter to make the unperturbed system highly non-degenerate.
4. In the last years, many authors have been devoted to studies of quasi-periodic bifurcation for conservative and dissipative dynamical systems and obtained many important results, see [1, 2, 3, 4, 5, 7, 8, 20, 21] and references therein. We think it is also important and interesting to study almost-periodic degenerate bifurcations for almost-periodically forced systems. We believe that the method used in the present paper can also be used to consider almost-periodic degenerate bifurcations for almost-periodically forced conservative and dissipative systems. This is one of the subjects of future work.

5. It worth to mention that almost-periodic solution in asymmetric oscillation and invariant curves of smooth quasi-periodic mappings have already been obtained in work [10, 11] by the authors P. Huang, X. Li and B. Liu.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions applied in the sequent. In Section 3, we give the existence of a.p. response solutions in elliptic case. In Section 4, we give the existence of a.p. response solutions in degenerate case.

2. Preliminaries. In this section, we first introduce some notations and definitions. The first thing is to define analytic functions on some infinite dimensional space

**Definition 2.1.** Let $X$ be a complex Banach space. A function $f : U \subseteq X \to C$, where $U$ is an open subset of $X$, is called analytic if $f$ is continuous on $U$, and $f|_{U \cap X_1}$ is analytic in the classical sense as a function of several complex variables for each finite dimensional subspace $X_1$ of $X$.

For infinite dimensional integer vector $k = (\cdots, k_\lambda, \cdots)_{\lambda \in \mathbb{Z}}$, we denote its support by

$$\text{supp} k = \{\lambda : k_\lambda \neq 0\}.$$

**Definition 2.2.** A function $f : \mathbb{R} \to \mathbb{R}$ is called almost periodic with the frequency $\omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}} \in \mathbb{R}^2$, if there exists a continuous function

$$F : \theta = (\cdots, \theta_\lambda, \cdots) \in \mathbb{R}^2/(2\pi \mathbb{Z})^2 \to \mathbb{R}$$

which admits a rapidly converging Fourier series expansion

$$F(\theta) = \sum_{A \in \mathcal{S}} F_A(\theta),$$

where

$$F_A(\theta) = \sum_{\text{supp} k \subseteq A} f_{A,k} e^{i(k,\theta)}$$

and $\mathcal{S}$ is a family of finite subsets $A$ of $\mathbb{Z}$ with $\mathbb{Z} \subseteq \bigcup_{A \in \mathcal{S}} A$, $(k, \theta) = \sum_{\lambda \in \mathbb{Z}} k_\lambda \theta_\lambda$, such that $f(t) = F(\omega t)$ for all $t \in \mathbb{R}$, where $F$ is $2\pi$-periodic in each variable. If $F(\theta)$ is analytic on $T^2_s := \{\theta = (\cdots, \theta_\lambda, \cdots) \in \mathbb{C}^2/(2\pi \mathbb{Z})^2, \sup_{\lambda \in \mathbb{Z}} |\Im \theta_\lambda| \leq s\}$ for some $s > 0$, then we call $f(t)$ is analytic on $T^2_s$.

Here $F(\theta)$ is the shell function of $f(t)$. We remark that this family $\mathcal{S}$ is not totally arbitrary. It is because for an almost-periodic function $f(t)$ which has the Fourier exponents $\{\Lambda_\lambda : \lambda \in \mathbb{Z}\}$, its basis is $\{\omega_\lambda : \lambda \in \mathbb{Z}\}$. For any $\lambda \in \mathbb{Z}$, $\Lambda_\lambda$ can be expressed into $\Lambda_\lambda = r_{\lambda_1} \omega_{\lambda_1} + \cdots + r_{\lambda_j(\lambda)} \omega_{\lambda_j(\lambda)}$, where $r_{\lambda_1}, \cdots, r_{\lambda_j(\lambda)}$ are rational numbers. Therefore, $\mathcal{S} = \{(\lambda_1, \cdots, \lambda_j(\lambda)) : \lambda \in \mathbb{Z}\}$. Rather, $\mathcal{S}$ has to be a spatial
structure on \( \mathbb{Z} \) characterized by the property that the union of any two sets in \( S \) is again in \( S \), if they intersect:

\[
A, B \in S, \ A \cap B \neq \emptyset \Rightarrow A \cup B \in S.
\]

Further, we define

\[
\mathbb{Z}_S^k = \{ k = (\cdots, k_x, \cdots) \in \mathbb{Z}^k : \text{supp} k \subset A, A \in S \}.
\]

Thus, \( f(t) \) can be represented as a Fourier series

\[
f(t) = \sum_{A \in S} \sum_{\text{supp} k \subseteq A} f_{A,k} e^{i(k,\omega)t} = \sum_{k \in \mathbb{Z}_S^k} f_{A,k} e^{i(k,\omega)t}.
\]

Define a nonnegative weight function

\[
[\cdot] : A \mapsto [A] = 1 + \sum_{i \in A} \log^q (1 + |i|),
\]

where \( q > 2 \) is a constant.

Let \( M \) be a compact set in \( \mathbb{C} \) without zero and \( B_r := \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z| \leq r \} \) be the \( n \)-dimensional sphere with radius \( r \) in \( \mathbb{C}^n \). Define

\[
\Delta_{s,r} = \mathbb{T}_{s}^n \times B_r,
\]

for each given \( r > 0 \) and \( s > 0 \). For an analytic function \( F(\theta, x, \xi) : \Delta_{s,r} \times M \to \mathbb{C} \), which admits the following spatial series expansion

\[
F(\theta, x, \xi) = \sum_{A \in S} F_A(\theta, x, \xi) = \sum_{A \in S} \sum_{\text{supp} k \subseteq A} f_{A,k}(x,\xi) e^{i(k,\theta)},
\]

we define the norm of \( F(\theta, x, \xi) \) by

\[
\| F \|_{s,r,m,M} = \sum_{A \in S} \sum_{\text{supp} k \subseteq A} |f_{A,k}(x,\xi)|_{r,M} e^{k,|A|m},
\]

where \( |f_{A,k}(x,\xi)|_{r,M} = \sup_{(x,\xi) \in B_r \times M} |f_{A,k}(x,\xi)| \). Thus, for all above bounded analytic functions, we can define a Banach space

\[
C_m^\omega(\Delta_{s,r} \times M, \mathbb{C}) = \{ F : \| F \|_{s,r,m,M} < \infty \}.
\]

Similarly, we can denote \( C_m^\omega(\Delta_{s,r} \times \mathbb{C}) \), \( C_m^\omega(\mathbb{T}_{s}^n \times M, \mathbb{C}) \) and \( C_m^\omega(\mathbb{T}_{s}^n, \mathbb{C}) \) the space of analytic function \( F : \Delta_{s,r} \to \mathbb{C} \), the space of analytic function \( F : \mathbb{T}_{s}^n \to \mathbb{C} \) and the space of analytic function \( F : \mathbb{T}_{s}^n \to \mathbb{C} \), respectively. Their norms are represented by \( \| \cdot \|_{s,r,m} \), \( \| \cdot \|_{s,m,M} \) and \( \| \cdot \|_{s,m} \) respectively. Let \( C_m^\omega(\Delta_{s,r} \times M, gl(n, \mathbb{C})) \) and \( C_m^\omega(\Delta_{s,r} \times M, \mathbb{C}^n) \) be the set of all analytic \( n \times n \) matrix functions and \( n \)-dimensional vector functions mapping from \( \Delta_{s,r} \times M \) to \( gl(n, \mathbb{C}) \) and \( \mathbb{C}^n \) respectively.

For each \( F = (F_1, \ldots, F_n) \in C_m^\omega(\Delta_{s,r} \times M, \mathbb{C}^n) \), we define its norm \( \| F \|_{s,r,m,M} = \sum_{1 \leq i \leq n} \| F_i \|_{s,r,m,M} \). For each \( F = (F_{i,j})_{1 \leq i,j \leq n} \in C_m^\omega(\Delta_{s,r} \times M, gl(n, \mathbb{C})) \), we define its norm \( \| F \|_{s,r,m,M} = \sup_{1 \leq i,j \leq n} \| F_{i,j} \|_{s,r,m,M} \). Similarly, we can consider \( C_m^\omega(\Delta_{s,r}, gl(n, \mathbb{C})) \), \( C_m^\omega(\Delta_{s,r}, \mathbb{C}^n) \), \( C_m^\omega(\mathbb{T}_{s}^n \times M, gl(n, \mathbb{C})) \), \( C_m^\omega(\mathbb{T}_{s}^n \times M, \mathbb{C}^n) \), \( C_m^\omega(\mathbb{T}_{s}^n, gl(n, \mathbb{C})) \) and \( C_m^\omega(\mathbb{T}_{s}^n, \mathbb{C}^n) \) in the same way.

A function \( f \in C_m^\omega(\Delta_{s,r} \times M, \mathbb{C}) \) is called a real analytic function if it gives real values to real arguments. Denote by \( C_m^\omega(\Delta_{s,r} \times M, \mathbb{R}) \) the set of all such real analytic functions in \( C_m^\omega(\Delta_{s,r} \times M, \mathbb{C}) \). Then \( C_m^\omega(\Delta_{s,r} \times M, \mathbb{R}) \) is a subspace of \( C_m^\omega(\Delta_{s,r} \times M, \mathbb{C}) \) under \( \| \cdot \|_{s,r,m,M} \). Similarly, we can consider \( C_m^\omega(\Delta_{s,r}, \mathbb{R}) \), \( C_m^\omega(\mathbb{T}_{s}^n, \mathbb{R}) \), \( C_m^\omega(\Delta_{s,r} \times M, gl(n, \mathbb{R})) \), \( C_m^\omega(\Delta_{s,r} \times M, \mathbb{R}^n) \), \( C_m^\omega(\Delta_{s,r}, gl(n, \mathbb{R})) \), \( C_m^\omega(\mathbb{T}_{s}^n \times M, gl(n, \mathbb{R})) \), \( C_m^\omega(\mathbb{T}_{s}^n \times M, \mathbb{R}^n) \), \( C_m^\omega(\Delta_{s,r}, \mathbb{R}^n) \) and \( C_m^\omega(\mathbb{T}_{s}^n, \mathbb{R}^n) \).
For $F(\theta) \in C^\omega_m(T^2, \mathbb{C})$, we denote the average of $F(\theta)$ by $[F(\theta)] = \sum_{A \in \mathcal{S}} [F_A(\theta)]$ where $[F_A(\theta)]$ represents the average of quasi-periodic function $F_A(\theta)$.

Next, we will define the strongly non-resonant condition on the frequency vector $\omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}}$. For $k \in \mathbb{Z}_S^2$, we define the weight of its support

$$[[k]] = \min_{\text{supp} k \subseteq A \in \mathcal{S}} [A].$$

Then the non-resonant condition reads

$$|(k, \omega)| \geq \frac{\gamma}{\Delta([|k|])\Delta(|k|)},$$

(10)

where $\gamma > 0$, $|k| = \sum \lambda |k_\lambda|$ and $\Delta$ is some fixed approximation function. A function $\Delta : [1, \infty) \to [1, \infty)$ is called an approximation function, if $\Delta$ is non-decreasing, $\Delta(1) = 1$ and

$$\int_1^\infty \frac{\ln \Delta(t)}{t^2} dt < \infty.$$  

(11)

In the following we will give a criterion for the existence of strongly nonresonant frequencies. It is based on the growth conditions on the distribution function

$$N_i(t) = \text{card}\{A \in \mathcal{S} : \text{card}(A) = i, |A| \leq t\}$$

for $i \geq 1$ and $t \geq 0$.

**Lemma 2.3.** There exist a constant $N_0$ and an approximation function $\Phi$ such that

$$N_i(t) = \begin{cases} 0, & t < t_i, \\ N_0\Phi(t), & t \geq t_i \end{cases}$$

with a sequence of real numbers $t_i$ satisfying

$$i \log^{q-1} i \leq t_i \sim i \log^q i$$

for $i$ large with some exponent $q - 1 > 1$. Here we say $a_i \sim b_i$ if there are two constant $c, C$ such that $ca_i \leq b_i \leq Ca_i$ and $c, C$ are independent of $i$.

The proof of this Lemma can be found in [15]. We omit it here.

Throughout this paper, we denote by $c, C$ the universal positive constants if we do not care their values, denote the absolute value (or norm of vector, or norm of matrix) by $|\cdot|$. In the sequel, we will denote the shell of an almost-periodic function $h(t)$ by $h(\omega t)$, for the sake of simplicity.

### 3. Elliptic case.

In this section, we will be devoted to the construction of a.p. response solutions of system (3) by adapting Pöschel-Rüssmann KAM method. To this end, we introduce the extended phase space $T^2 \times \mathbb{R}^n$, and so system (3) can be written as

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{x} = (E + Q(\theta, \epsilon))x + G(\theta, x, \epsilon) + F(\theta, \epsilon), \end{cases}$$

(12)

where $Q(\theta, \epsilon) = \frac{\partial f(\theta, 0, \epsilon)}{\partial x}$ and $F(\theta, \epsilon) = f(\theta, 0, \epsilon)$ and $G(\theta, x, \epsilon) = h(\theta, x) + f(\theta, x, \epsilon) - f(\theta, 0, \epsilon) - \frac{\partial f(\theta, 0, \epsilon)}{\partial x} x$, which are analytic on $\Delta_{s,r}$ for some positive numbers $s, r$ and admit spatial series expansions

$$G(\theta, x, \epsilon) = \sum_{A \in \mathcal{S}} G_A(\theta, x, \epsilon) = \sum_{A \in \mathcal{S}} \sum_{\text{supp} k \subseteq A} G_{A,k}(x, \epsilon)e^{i(k, \theta)},$$

$$F(\theta, \epsilon) = \sum_{A \in \mathcal{S}} F_A(\theta, \epsilon) = \sum_{A \in \mathcal{S}} \sum_{\text{supp} k \subseteq A} F_{A,k}(\epsilon)e^{i(k, \theta)},$$
\[ Q(\theta, \epsilon) = \sum_{A \in S} Q_A(\theta, \epsilon) = \sum_{A \in S} \sum_{\supp \subseteq A} Q_{A,k}(\epsilon)e^{i(k,\theta)}. \]

Since there exists an invertible matrix \( B \) such that \( B^{-1}EB \) is diagonal in complex field, we can assume that, without loss of generality, \( E = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \).

Moreover, we write \( \tilde{\lambda}(\epsilon), i = 1, \cdots, n, \) as the eigenvalues of

\[ E = \tilde{\lambda}(\epsilon) + D_\epsilon G(\cdot, x(\cdot, \epsilon), \epsilon), \]

where \( \tilde{x}(t, \epsilon) \) is the a.p.solution of the equation \( \dot{x} = E x + \tilde{F}(t, \epsilon) \), existence of \( \tilde{x}(t, \epsilon) \) can be obtained by solving the first homological equation in the later (see (21)), the definition of \( \tilde{F}(\cdot) \) can be seen in (14) and the definition of \( \tilde{P}(\cdot) \) can be seen in (13).

We have the following theorem.

**Theorem 3.1.** Consider analytic system (12) defined in \( \Delta_{s, r} \). Assume that the following hypotheses hold:

(i) (non-resonant conditions)

\[
|i(k, \omega) - \lambda_i| \geq \frac{\gamma}{\Delta([k])\Delta([k])}, \quad \forall \; k \in \mathbb{Z}^2 \setminus \{0\}, \; 1 \leq i \leq n,
\]

\[
|i(k, \omega) - \lambda_i + \lambda_j| \geq \frac{\gamma}{\Delta([k])\Delta([k])}, \quad \forall \; k \in \mathbb{Z}^2 \setminus \{0\}, \; 1 \leq i, j \leq n,
\]

(ii) (non-degeneracy conditions)

\[
\left| \frac{d\lambda_i(\epsilon)}{d\epsilon} \right| > \delta, \quad 1 \leq i \leq n,
\]

\[
\left| \frac{d(\lambda_i(\epsilon) - \lambda_j(\epsilon))}{d\epsilon} \right| > \delta, \quad 1 \leq i \neq j \leq n,
\]

where \( \delta > 0 \). Then, for sufficiently small constant \( \bar{\epsilon} > 0 \) there exist a Cantor set \( \mathcal{E} \subset (0, \bar{\epsilon}) \) of almost full Lebesgue measure and \( s_* < s, r_* < r \) such that for each \( \epsilon \in \mathcal{E} \) there exists a analytic transformation

\[ x = (I + U_*(\theta, \epsilon))z + V_*(\theta, \epsilon), \quad \theta = \theta, \]

where \( I \) is the identity matrix, \( U_* = O(\bar{\epsilon}) \in C^m_{m_*}(\mathbb{R}^2_{s_*}, gl(n, \mathbb{C})) \), \( V_* = O(\bar{\epsilon}) \in C^m_{m_*}(\mathbb{R}^2_{s_*}, \mathbb{C}^n) \) and \( C^1 \)-smooth with respect to \( \epsilon \), which transforms system (12) into

\[ \dot{z} = E_*z + h_*(\theta, z, \epsilon), \quad \dot{\theta} = \omega, \]

where \( E_* \) is a constant matrix, \( h_*(\theta, z, \epsilon) \in C^m_{m_*}(\Delta_{s, r_*}, \mathbb{C}^n) \) \( h_* = O(z^2)(z \to 0) \) is \( C^1 \)-smooth with respect to \( \epsilon \).

**Remark 1.** Linear non-autonomous differential systems

\[ \dot{x} = A(t)x, \; x \in \mathbb{R}^n \]

is called reducible, if there exists a non-singular change of variables \( x = \Phi(t)y \), for which \( \Phi(t) \) and \( \Phi(t)^{-1} \) are bounded, such that system is transformed to \( \dot{y} = By \), where \( B \) is a constant matrix. The reducibility of linear differential systems has been studied widely by many scholars. The earliest result in this field is the well-known Floquet theory ([6]), which states that any linear periodic system is always reducible. Reducibility for quasi-periodic differential systems can also be obtained not for all \( A(t) \) (see [12, 13]). In 1996, reducibility for almost-periodic differential
systems was obtained by Xu and You in [22]. Actually, if $G$ and $F$ in system (12) are both zero, then the system (12) can be written as follows,

$$
\begin{cases}
\dot{\theta} = \omega, \\
\dot{x} = (E + \epsilon Q(\theta, \epsilon))x.
\end{cases}
$$

Then, our Theorem 3.1 can reduce to Theorem A shown in the paper [22].

### 3.1. The proof of Theorem 3.1.

#### 3.1.1. Outline of our proof.

We will adapt the Pöschl-Rüssmann KAM method (see [16] and [17]), which was used in the perturbation problem of a constant vector field on $n$-torus, to prove Theorem 3.1. The result shows that the Pöschl-Rüssmann KAM method is also feasible for the perturbation problem of linear vector field with elliptic equilibrium point.

Before stating and proving the step lemma accurately, we first present a quick overview describing how Pöschl-Rüssmann KAM method works in the proof.

For any matrix function $B(\theta) = (b_{ij}(\theta))_{1 \leq i,j \leq n} \in C^\omega_m(T^n_\tau, gl(n, \mathbb{C}))$, we denote

$$
\overline{B}(\theta) := \text{diag}(b_{11}(\theta), \cdots, b_{nn}(\theta)), \quad |B(\theta)| = B(\theta) - \overline{B}(\theta).
$$

(13)

The most important idea of Pöschl-Rüssmann KAM method in quasi-periodic case is sufficiently taking advantage of the polynomial structure in the truncation skill. Different from that, in almost-periodic case, we should find a new polynomial structure in the truncation skill. Let $P(\theta) \in C^\omega_m(T^n_\tau, \mathbb{C})$ admits a spatial series expansion

$$
P(\theta) = \sum_{A \in S} P_A(\theta) = \sum_{A \in S} \sum_{\text{supp} \subseteq A} P_{A,k}e^{i(k, \theta)}.
$$

For $e^{-\tau_\sigma} = 1 - a$, we can truncate $P(\theta) = \tilde{P}(\theta) + \widehat{P}(\theta)$ with

$$
\tilde{P}(\theta) = \sum_{A \in S} \sum_{\text{supp} \subseteq A, \sigma[A] \leq \tau} (1 - (1 - a)e^{\sigma[A]}) (1 - (1 - a)e^{\sigma[k]}) P_{A,k}e^{i(k, \theta)}
$$

and

$$
\widehat{P}(\theta) = \sum_{A \in S} \sum_{\text{supp} \subseteq A, |A| > \tau} P_{A,k}e^{i(k, \theta)} + \sum_{A \in S} \sum_{\text{supp} \subseteq A, |A| \leq \tau} (1 - a)e^{\sigma[A]} P_{A,k}e^{i(k, \theta)}
\quad + \sum_{A \in S} \sum_{\text{supp} \subseteq A, \sigma[A] > \tau} (1 - (1 - a)e^{\sigma[A]}) P_{A,k}e^{i(k, \theta)}
\quad + \sum_{A \in S} \sum_{\text{supp} \subseteq A, |A| \leq \tau} (1 - (1 - a)e^{\sigma[A]}) (1 - a)e^{\sigma[k]} P_{A,k}e^{i(k, \theta)}.
$$

(14)

(15)

We have the following estimate

$$
||\tilde{P}||_{s-\sigma, m-\sigma} \leq \sum_{A \in S} \sum_{\text{supp} \subseteq A} |P_{A,k}e^{[k]e^{[A]}(m-\sigma)} + \sum_{\text{supp} \subseteq A} (1 - a)e^{\sigma[A]} |P_{A,k}e^{[k]e^{[A]}(m-\sigma)}
\quad + \sum_{\text{supp} \subseteq A} (1 - (1 - a)e^{\sigma[A]}) |P_{A,k}e^{[k]e^{[A]}m}
\quad + \sum_{\text{supp} \subseteq A} (1 - (1 - a)e^{\sigma[A]}) (1 - a)e^{\sigma[k]} |P_{A,k}e^{[k]e^{[A]}m}.
$$
The equations needs the non-resonant conditions

\[ \text{where } \theta, \lambda \text{ are the eigenvalues of the system } \]

\[ \lambda_i(i = 1, 2, \ldots, n) \text{ are the eigenvalues of } E + \sqrt{Q(\theta)} + \frac{dG}{d\theta}(\theta, V(\theta)) \text{ and } ] \]

\[ \text{in the next subsubsection, we will know above equations are solvable and } \]

\[ \text{we can perform the change of variables } z = V(\theta) + (I + U(\theta))z^+ \text{ to (18) to obtain } \]

\[ z^+ = (E_+ + Q_+(\theta))z^+ + F_+(\theta) + G_+(\theta, z^+) \]

\[ E_+ = E + \sqrt{Q(\theta)} + \frac{dG}{d\theta}(\theta, V(\theta)), \]

\[ Q_+(\theta) = (I + U(\theta))^{-1}\left\{ \frac{dG}{d\theta}(\theta, V(\theta)) + \sqrt{Q(\theta)} + \left( \frac{dG}{d\theta}(\theta, V(\theta)) \right) \right\} \]
In this way, after the proof of Theorem 4.3 can be arbitrary value in (0, Q), where we do not need the persistence of non-degeneracy conditions.

\[ F_+ (\theta) = (I + U(\theta))^{-1} \left( Q(\theta)V(\theta) + \tilde{F}(\theta) + G(\theta, V(\theta)) \right), \]

\[ G_+ (\theta, z^+) = (I + U(\theta))^{-1} \left( G(\theta, V(\theta) + (I + U(\theta))z^+) \right) - G(\theta, V(\theta)) - \frac{\partial G(\theta, V(\theta))}{\partial z}(I + U(\theta))z^+. \]

One can check that the term \( Q(\theta)V(\theta) + G(\theta, V(\theta)) \) in \( F_+ (\theta) \) is much smaller than \( F(\theta) \), whose order is \( O(\epsilon^2) \). And the term \( \tilde{F}(\theta) \) can also be smaller than \( F(\theta) \), whose order is \( O(2(1 - a)\epsilon) \) (see (16)). Finally, \( F_+ (\theta) = O((2(1 - a) + b)\epsilon) = O(q\epsilon) \) because \( \epsilon \ll b \). Here \( a, b, q \) are defined as in next subsection. One can also check \( \frac{\partial G(\theta, V(\theta))}{\partial z} \left( Q(\theta)V(\theta) + G(\theta, V(\theta)) \right) \) is much smaller than \( Q(\theta) \), whose order is \( O(\epsilon^2) \). And the term \( \tilde{Q}(\theta) \) can also be smaller than \( Q(\theta) \), whose order is \( O(2(1 - a)\epsilon) \) (see (16)). Finally, \( Q_+ (\theta) = O((2(1 - a) + b)\epsilon) = O(q^{1/2}\epsilon) \).

We can also check \( \frac{dQ_+ (\theta)}{d\epsilon} = O(q^{1/2}) \), \( \frac{dF_+ (\theta)}{d\epsilon} = O(q) \) and

\[
\left\| \frac{\partial^2 G_+ (\theta, z^+)}{\partial z^2} \right\|_{s-r-s,m-s} \leq \left\| (I + U(\theta))^{-1} \right\|_{s-r-s,m-s} \left\| \frac{\partial^2 G(\theta, V(\theta) + (I + U(\theta))z^+)}{\partial z^2} \right\|_{s-r-s,m-s} \times \left\| (I + U(\theta))^2 \right\|_{s-r-s,m-s}.
\]

\[
\leq \left( \frac{1}{1 - 4q^2} \right) \left\| \frac{\partial^2 G(\theta, V(\theta) + (I + U(\theta))z^+)}{\partial z^2} \right\|_{s-r-s,m-s}(1 + 4q^2)
\]

In this way, after \( l \) steps, system will look like

\[ \dot{z} = (E_1 + Q_1 (\theta))z + F_1 (\theta) + G_1 (\theta, z), \quad \dot{\theta} = \omega, \]

where \( Q_1 (\theta) = O(\epsilon^q), F_1 (\theta) = O(\epsilon^q), \quad \frac{Q_1 (\theta)}{d\epsilon} = O(\epsilon^q), \quad \frac{dF_1 (\theta)}{d\epsilon} = O(\epsilon^q) \) and \( G_1 (\theta, z) = O(z^2)(z \to 0) \). The scheme will be convergent to an equation like

\[ \dot{z} = E_\infty z + G_\infty (\theta, z), \quad \dot{\theta} = \omega. \]

where \( G_\infty (\theta, z) = O(z^2)(z \to 0) \).

In order to guarantee that the non-degeneracy conditions can persist at each KAM step, we estimate the derivative of each functions. Actually by

\[ \frac{dE_{\nu+1}}{d\epsilon} = \frac{dE_\nu}{d\epsilon} + \frac{d}{d\epsilon} \left( \frac{\partial Q_{\nu+1} (\theta)}{\partial z} \right) + \frac{Q_{\nu+1} (\theta)}{d\epsilon}, \]

we can get \( \frac{dE_{\nu+1}}{d\epsilon} > c - O(q^2) \). Thus, the choice of \( q \) should be small suitably (see (19) below). Different from this proof of Theorem 3.1, the choice of parameter \( q \) in the proof of Theorem 4.3 can be arbitrary value in (0, 1) because in Theorem 4.3 we do not need the persistence of non-degeneracy conditions.
3.1.2. The KAM Step. We will give an iterative lemma to prove Theorem 3.1. Let ∆ be an approximation function as defined in (11). Denote Λ(t) = t2∆12(t). We can find two constants c1 and C1 such that |λi| > 2c1 |λi − λj| > 2c1, 2c1 < |dλi/dε| < C1/2 and 2c1 < |d(λi − λj)/dε| < C1/2. We set m0 = m, s0 = s and r0 = r. Let Λ0 ≥ Λ(1) = ∆(1). Then, let τ0 := Λ−1(Λ0) be large enough such that
\[ 1/Λ0 < \min\{\frac{6\gamma^2}{64\Lambda_0^2}, \frac{4\gamma^2}{16\Lambda_0^2}\} \]
and
\[ 18 \int_{\tau_0}^{\infty} \frac{\ln \Lambda(t)}{t^2} dt < \min\{\frac{r_0}{2}, \frac{s_0}{2}, \frac{m_0}{2}\}. \]
We choose δ, a, b and q such that δ = q^\frac{1}{4}, 1 − a = q^3, b = q^3 and
\[ 1/Λ^3 < q < \min\{\frac{c_1\gamma^4}{64\Lambda_0^2}, \frac{c_2\gamma^4}{16\Lambda_0^2}\}. \]  
(19)
One can check that
\[ 2(4(1 − a) + b) < q = \delta^6 \leq \left( \frac{1}{32} \right)^6 \]
and
\[ \frac{\log(1 − a)}{\log(\delta^3 q^{\frac{1}{2}})} \int_{\tau_0}^{\infty} \frac{\ln \Lambda(t)}{t^2} dt = 18 \int_{\tau_0}^{\infty} \frac{\ln \Lambda(t)}{t^2} dt < \min\{\frac{r_0}{2}, \frac{s_0}{2}, \frac{m_0}{2}\}. \]
Finally, we choose ϵ0 = ϵ small sufficiently, where ϵ only relies on the constants mentioned above. Next, we define the sequences (εν)ν≥0, (Λν)ν≥0, (τν)ν≥0, (σν)ν≥0, (νν)ν≥0, (rν)ν≥0, (sν)ν≥0, (mν)ν≥0 and (γν)ν≥0 in the following manner:
\[ \begin{align*}
εν &= ϵ0ν^ν, \quad Λν = \left(\frac{δ}{q^{\frac{1}{2}}}\right)^ν Λ0, \quad τν = Λ−1(Λν), \quad 1 − a = e^{−τνσν}, \quad γν = γ0(\frac{1}{2})^ν, \\
νν+1 &= sν + σν, \quad rν+1 = rν − σν, \quad mν+1 = mν − σν.
\end{align*} \]
We will see that mν, rν and sν have a positive limit. Indeed, for N > τ0
\[ \int_{τ0}^{N} \frac{d\ln \Lambda(t)}{t} = \frac{\ln \Lambda(N)}{N} − \frac{\ln Λ0}{τ0} + \int_{τ0}^{N} \frac{\ln \Lambda(t)}{t^2} dt \]
\[ = \int_{N}^{∞} \frac{\ln \Lambda(N)}{t^2} dt − \frac{\ln Λ0}{τ0} + \int_{τ0}^{N} \frac{\ln \Lambda(t)}{t^2} dt \]
\[ \leq \int_{N}^{∞} \frac{\ln \Lambda(t)}{t^2} dt − \frac{\ln Λ0}{τ0} + \int_{τ0}^{N} \frac{\ln \Lambda(t)}{t^2} dt \]
\[ = −\frac{\ln Λ0}{τ0} + \int_{τ0}^{∞} \frac{\ln \Lambda(t)}{t^2} dt. \]
Making N → +∞, we obtain that
\[ \int_{τ0}^{∞} \frac{d\ln \Lambda(t)}{t} \leq −\frac{\ln Λ0}{τ0} + \int_{τ0}^{∞} \frac{\ln \Lambda(t)}{t^2} dt. \]
So, let t = Λ−1(Λ0(δ−1q^\frac{1}{4})−ν), it follows from Λ0 ≥ (δ−1q^\frac{1}{4})−1 that
\[ \sum_{ν≥0} \frac{1}{τ0} + \frac{1}{τ0} \int_{τ0}^{∞} \frac{dν}{Λ−1(Λ0(δ−1q^\frac{1}{4}))−ν} = \frac{1}{τ0} + \frac{1}{\log(δ−1q^\frac{1}{4})−1} \int_{τ0}^{∞} \frac{dΛ(t)}{tΛ(t)} \]
\[ \leq \frac{1}{τ0} + \frac{1}{\log(δ−1q^\frac{1}{4})−1} \left( −\frac{\ln Λ0}{τ0} + \int_{τ0}^{∞} \frac{\ln \Lambda(t)}{t^2} dt \right) . \]
\[
\frac{1}{\log(\delta^{-1} q^{1/2})} \int_{\tau_0}^{\infty} \frac{\ln A(t)}{t^2} dt.
\]

It follows that
\[
\sum_{\nu \geq 0} \sigma_{\nu} = \sum_{\nu \geq 0} \frac{\log(1 - a)^{-1}}{\tau_\nu} \leq \frac{\log(1 - a)}{\log(\delta^{-1} q^{1/2})} \int_{\tau_0}^{\infty} \frac{\log A(t)}{t^2} dt.
\]

Hence, we can achieve that \(\sum_{\nu \geq 0} \sigma_{\nu} \leq \min\{s_0/2, r_0/2, m_0/2\}\). Thus \(m_\nu \to m_* \geq m_0/2, s_\nu \to s_* \geq s_0/2\) and \(r_\nu \to r_* \geq r_0/2\).

We suppose that after \(\nu\) steps, the transformed system defined in the domain \(\Delta_{s_\nu, r_\nu}\) is of the form
\[
\dot{z} = (E_\nu(e) + Q_\nu(\theta, \epsilon))z + F_\nu(\theta, \epsilon) + G_\nu(\theta, z, \epsilon). \tag{Eq}_\nu
\]

**Lemma 3.2.** Suppose that there is a \(\gamma > 0\) for which the frequency vector \(\omega\) satisfies non-resonant condition (10). Let us consider the analytic almost-periodic system
\[
\begin{cases}
\dot{\theta} = \omega, \\
\dot{z} = (E_l(e) + Q_l(\theta, \epsilon))z + F_l(\theta, \epsilon) + G_l(\theta, z, \epsilon), \quad l = 0, \ldots, \nu, \quad (Eq)_l
\end{cases}
\]

where \(E_l(e), Q_l(\theta, \epsilon), F_l(\theta, \epsilon)\) and \(G_l(\theta, z, \epsilon)\) are \(C^1\)-smooth with respect to \(\epsilon\) and satisfy the following hypotheses:

1.1) \(E_l = \text{diag}(\lambda_{l,1}, \ldots, \lambda_{l,n})\) satisfies \(|E_l - E_{l-1}| \leq \epsilon_0 q^{1/2}\) and
\[
\begin{align*}
c_1 \leq |\lambda_{l,i}(\epsilon)| & \leq C_1, \quad \forall l \geq 0, \quad 0 \leq i \leq n, \\
c_1 \leq \frac{d\lambda_{l,i}(\epsilon)}{d\epsilon} & \leq C_1, \quad \forall l > 0, \quad 0 \leq i \leq n, \\
c_1 \leq |\lambda_{l,i}(\epsilon) - \lambda_{l,j}(\epsilon)| & \leq C_1, \quad \forall l \geq 0, \quad 0 \leq i \neq j \leq n, \\
c_1 \leq \frac{d(\lambda_{l,i}(\epsilon) - \lambda_{l,j}(\epsilon))}{d\epsilon} & \leq C_1, \quad \forall l > 0, \quad 0 \leq i \neq j \leq n;
\end{align*}
\]

1.2) \(Q_l(\theta, \epsilon) \in C^\omega_{m_1}(T_{s_l}^\mathbb{Z}, gl(n, \mathbb{C}))\), \(F_l(\theta, \epsilon) \in C^\omega_{m_1}(T_{s_l}^\mathbb{Z}, \mathbb{C}^n)\) and
\[
\begin{align*}
\|Q_l(\theta, \epsilon)\|_{s_l, m_1} & \leq \epsilon_0 q^{1/2}, \quad \|\frac{\partial Q_l(\theta, \epsilon)}{\partial \epsilon}\|_{s_l, m_1} \leq q^{1/2}, \\
\|F_l(\theta, \epsilon)\|_{s_l, m_1} & \leq \epsilon_1, \quad \|\frac{\partial F_l(\theta, \epsilon)}{\partial \epsilon}\|_{s_l, m_1} \leq \frac{\omega}{\epsilon_0};
\end{align*}
\]

1.3) \(G_l(z, \theta, \epsilon) \in C^\omega_{m_1}(\Delta_{s_l, r_l}, \mathbb{C}^n)\) and for any \(r \leq r_l\), it has
\[
\|G_l(z, \theta, \epsilon)\|_{s_l, r, m_1} \leq Cr^2, \quad \|\frac{\partial G_l(z, \theta, \epsilon)}{\partial \epsilon}\|_{s_l, r, m_1} \leq Cr^2.
\]

Then there exists a change of variables
\[
\Phi_{\nu} : \Delta_{s_{\nu+1}, r_{\nu+1}} \to \Delta_{s_\nu, r_\nu}
\]

of the form
\[
z = z^* + U_\nu(\theta, \epsilon)z^* + V_\nu(\theta, \epsilon),
\]

where \(z\) are ‘old’ variable and \(z^*\) are ‘new’ variable, \(U_\nu(\theta, \epsilon) \in C^\omega_{m_\nu}(T_{s_{\nu+1}}^\mathbb{Z}, gl(n, \mathbb{C}))\)
\(V_\nu(\theta) \in C^\omega_{m_\nu}(T_{s_{\nu+1}}^\mathbb{Z}, \mathbb{C}^n)\), such that the system \((Eq)_\nu\) is transformed into the system
\((Eq)_{\nu+1}\) and conditions (1.1) – (1.3) are fulfilled by replacing \(l\) by \(\nu+1\) and replacing \(z\) by \(z^*\), respectively.
Proof of Lemma 3.2. In this proof, we hide $\epsilon$ if there is no confusion. We divide the proof of Lemma 3.2 into the following several parts.

A. Truncation. At $\nu$-th KAM step, we have $(1 - a) = e^{-\tau_\nu \sigma_\nu}$. By (16) and (17), $P(\theta) \in \mathcal{C}^\infty_m(T^2_{s_{\nu}}, \mathbb{C})$ has the truncation $P(\theta) = \tilde{P}(\theta) + \hat{P}(\theta)$ with estimate

$$\|\tilde{P}\|_{s_{\nu+1}, s_{\nu}} \leq 2(1 - a)\|P\|_{s_{\nu}, s_{\nu}}, \|P\|_{s_{\nu}, s_{\nu}} \leq a^2\|P\|_{s_{\nu}, s_{\nu}}.$$

One can also check that

$$\|\frac{\partial \tilde{P}(\theta)}{\partial \epsilon}\|_{s_{\nu}, s_{\nu}} \leq a^2\|\frac{\partial P(\theta)}{\partial \epsilon}\|_{s_{\nu}, s_{\nu}} \text{ and } \|\frac{\partial \hat{P}(\theta)}{\partial \epsilon}\|_{s_{\nu+1}, s_{\nu+1}} \leq 2(1 - a)\|\frac{\partial P(\theta)}{\partial \epsilon}\|_{s_{\nu}, s_{\nu}}.$$

B. Construction of a transformation.

We define the transformation $\Phi_\nu : z^+ \rightarrow z$ by

$$z = z^+ + U_\nu(\theta)z^+ + V_\nu(\theta),$$

(20)

where $U_\nu(\theta) = (U^{ij}_\nu(\theta))_{1 \leq i, j \leq n}$ and $V_\nu(\theta) = (V^i_\nu(\theta), \cdots, V^n_\nu(\theta))^T$. And each $U^{ij}_\nu(\theta)$ and $V^i_\nu(\theta)$ admit spatial series expansion

$$U^{ij}_\nu(\theta) = \sum_{A \subseteq \mathbb{R}} \sum_{|A| \leq \nu} \sum_{\omega \in \omega(A)} U^{ij}_{\nu A}(\theta) e^{i(k, \theta)},$$

$$V^i_\nu(\theta) = \sum_{A \subseteq \mathbb{R}} \sum_{|A| \leq \nu} \sum_{\omega \in \omega(A)} V^i_{\nu A}(\theta) e^{i(k, \theta)}.$$

Inserting (20) into $(Eq)_\nu$, if the homological equations

$$\begin{cases}
\partial_\nu V_\nu(\theta) = E_\nu V_\nu(\theta) + \tilde{F}_\nu(\theta), \\
\partial_\nu U_\nu(\theta) = E_{\nu+1} U_\nu(\theta) - U_\nu(\theta)E_{\nu+1} + \mathcal{H}_\nu(\theta),
\end{cases}$$

(21)

where $\mathcal{H}_\nu(\theta) = [\tilde{Q}_\nu(\theta)] + [\frac{\partial \tilde{G}_\nu(\theta, V_\nu(\theta))}{\partial z}]$, are solved, then system $(Eq)_\nu$ becomes

$$z^+ = (E_{\nu+1} + Q_{\nu+1}(\theta))z^+ + F_{\nu+1}(\theta) + G_{\nu+1}(\theta, z^+),$$

where

$$E_{\nu+1} = E_\nu + \overline{Q_\nu(\theta)} + \frac{\partial \tilde{G}_\nu(\theta, V_\nu(\theta))}{\partial z},$$

$$Q_{\nu+1}(\theta) = (I + U_\nu(\theta))^{-1}\left[\frac{\partial \tilde{G}_\nu(\theta, V_\nu(\theta))}{\partial z} + \tilde{Q}_\nu(\theta) + \left([\frac{\partial \tilde{G}_\nu(\theta, V_\nu(\theta))}{\partial z}]ight)
+ \frac{\partial \tilde{G}_\nu(\theta, V_\nu(\theta))}{\partial z} + [\tilde{Q}_\nu(\theta)] + \tilde{Q}_\nu(\theta)\right]U_\nu(\theta)\right\},$$

$$F_{\nu+1}(\theta) = (I + U_\nu(\theta))^{-1}\left(Q_\nu(\theta)V_\nu(\theta) + \tilde{F}_\nu(\theta) + G_\nu(\theta, V_\nu(\theta))\right),$$

$$G_{\nu+1}(\theta, z^+) = (I + U_\nu(\theta))^{-1}\left(G_\nu(\theta, V_\nu(\theta) + (I + U_\nu(\theta))z^+)
- G_\nu(\theta, V_\nu(\theta)) - \frac{\partial G_\nu(\theta, V_\nu(\theta))}{\partial z}(I + U_\nu(\theta))z^+\right).$$

Here $\frac{\partial G_\nu}{\partial z}$ is the Jacobian matrix of $G_\nu$ with respect to $z$ and $\frac{\partial \tilde{G}_\nu}{\partial z} = \frac{\partial \tilde{G}_\nu}{\partial z} + \frac{\partial \tilde{G}_\nu}{\partial z}$

(See the definition of truncation).

C. Solving linear homological equations.
First of all, we solve the first equation of (21). For sufficiently small $\epsilon_*>0$, we define two sets by

$$D^1_\nu := \left\{ \epsilon \in (0, \bar{\epsilon}) : |i(k, \omega) - \lambda_{\nu,i}(\epsilon)| \geq \frac{\gamma_{\nu,i+1}}{\Delta^3(|k|)\Delta^3(|k|)} \right\}, \quad \forall k \in \mathbb{Z}_S^2, \quad 1 \leq i \leq n.$$ 

We solve the first equation of (21). Let $\vec{F}_\nu(\theta) = (\vec{F}_\nu^1(\theta), \ldots, \vec{F}_\nu^n(\theta))^T$ and

$$\vec{F}_\nu^i(\theta) = \sum_{A \in \mathcal{A}} \vec{F}_{\nu A}^i(\theta) = \sum_{|A| \leq \tau_\nu \wedge \supp k \subset A} \sum_{|k| \leq \bar{\epsilon}} \vec{F}_{\nu A,k}^i e^{i(k, \theta)}.$$ 

By the first equation (21), for each $A \in \mathcal{S}$, $|A| \leq \tau_\nu$, supp$k \subset A$ we have

$$i(k, \omega)V_{\nu A,k}^i = \lambda_{\nu,i}V_{\nu A,k}^i + \vec{F}_{\nu A,k}^i, \quad i = 1, \ldots, n.$$ 

For $A \in \mathcal{S}$, $|A| \leq \tau_\nu$, supp$k \subset A$, $k = 0$, we can choose $V_{\nu A,0}^i = - (\lambda_{\nu,i})^{-1} \vec{F}_{\nu A,0}^i$. For $A \in \mathcal{S}$, $|A| \leq \tau_\nu$, supp$k \subset A$, $0 < |k| \leq \tau_\nu$, we have

$$V_{\nu A,k}^i = - \frac{\vec{F}_{\nu A,k}^i}{i(k, \omega) - \lambda_{\nu,i}}.$$ 

In addition, considering the derivative of $V_{\nu A,k}^i$ with respect to $\epsilon$, we have

$$\frac{dV_{\nu A,0}^i}{d\epsilon} = -(\lambda_{\nu,i})^{-1} \frac{d\vec{F}_{\nu A,0}^i}{d\epsilon} + \frac{\vec{F}_{\nu A,k}^i}{(i(k, \omega) - \lambda_{\nu,i})^2}$$

and

$$\frac{dV_{\nu A,k}^i}{d\epsilon} = (i(k, \omega) - \lambda_{\nu,i})^{-1} \frac{d\vec{F}_{\nu A,k}^i}{d\epsilon} + \frac{\vec{F}_{\nu A,k}^i}{(i(k, \omega) - \lambda_{\nu,i})^2},$$

for all $A \in \mathcal{S}$, $|A| \leq \tau_\nu$, supp$k \subset A$, $0 < |k| \leq \tau_\nu$. Notice that

$$c_1 \leq |\lambda_{\nu,i}| \quad \text{and} \quad \left| \frac{d(\lambda_{\nu,i})}{d\epsilon} \right| \leq C_1, \quad \forall 1 \leq i \leq n.$$ 

For all $A \in \mathcal{S}$, $|A| \leq \tau_\nu$, supp$k \subset A$, $|k| = 0$, we have

$$|V_{\nu A,0}^i| \leq \frac{1}{c_1} |\vec{F}_{\nu A,0}^i|,$$

$$\left| \frac{dV_{\nu A,0}^i}{d\epsilon} \right| \leq C_1 \left( \frac{d\vec{F}_{\nu A,0}^i}{d\epsilon} + |\vec{F}_{\nu A,0}^i| \right).$$

If $\epsilon \in D^1_\nu$, then we have

$$|V_{\nu A,k}^i| \leq \gamma_{\nu+1}^{-1} \Delta^3([k]) \Delta^3(|k|) |\vec{F}_{\nu A,k}^i|$$

$$\leq \gamma_{\nu+1}^{-1} \Delta^3(|A|) \Delta^3(|k|) |\vec{F}_{\nu A,k}^i|$$

$$\leq \gamma_{\nu+1}^{-1} \Delta^6(\tau_\nu) |\vec{F}_{\nu A,k}^i|$$

and

$$\left| \frac{dV_{\nu A,k}^i}{d\epsilon} \right| \leq \gamma_{\nu+1}^{-1} \Delta^3([k]) \Delta^3(|k|) \left| \frac{d\vec{F}_{\nu A,k}^i}{d\epsilon} \right| + \gamma_{\nu+1}^{-2} \Delta^6([k]) \Delta^6(|k|) |\vec{F}_{\nu A,k}^i|$$

$$\leq \gamma_{\nu+1}^{-1} \Delta^3(|A|) \Delta^3(|k|) \left| \frac{d\vec{F}_{\nu A,k}^i}{d\epsilon} \right| + \gamma_{\nu+1}^{-2} \Delta^6(|A|) \Delta^6(|k|) |\vec{F}_{\nu A,k}^i|$$

$$\leq \gamma_{\nu+1}^{-1} \Delta^3(\tau_\nu) \left| \frac{d\vec{F}_{\nu A,k}^i}{d\epsilon} \right| + \gamma_{\nu+1}^{-2} \Delta^6(\tau_\nu) |\vec{F}_{\nu A,k}^i|.$$
for all $A \in \mathcal{S}$, $[A] \leq \tau_{\nu}$, supp$k \subset A$, $0 < |k| \leq \tau_{\nu}$. With $a = 1 - e^{-\tau_{\nu}\sigma_{\nu}} \leq \tau_{\nu}\sigma_{\nu}$, it follows that

$$
\|V_{\theta}^i(\theta)\|_{s_\nu,m_{\nu}} = \left\| \sum_{A \in \mathcal{S} \atop |A| \leq \tau_{\nu}} V_{i,A}^j(\theta) \right\|_{s_\nu,m_{\nu}} \\
\leq c_1^{-1} \sum_{A \in \mathcal{S} \atop |A| \leq \tau_{\nu}} \sum_{supp k \subset A \atop |k| \leq \tau_{\nu}} \gamma_{\nu+1}^{-1} \Delta^{6}(\tau_{\nu}) |\tilde{F}_{i,A,k}| \|k\|_{s_\nu,m_{\nu}} [A]_{m_{\nu}} \\
\leq c_1^{-1} \Delta^{6}(\tau_{\nu}) \gamma_{\nu+1} \|\tilde{F}_i(\theta)\|_{s_\nu,m_{\nu}} \leq c_1^{-1} \Delta^{6}(\tau_{\nu}) \gamma_{\nu+1}^2 a^2 \epsilon_{\nu} \\
\leq c_1^{-1} \Lambda_{\nu} \sigma_{\nu}^2 \epsilon_{\nu} \gamma_{\nu+1} \leq 2(c_1 \gamma)^{-1} \Lambda_{0} \epsilon_{0} (2\delta)^{\nu} \sigma_{\nu}^2 q^{\frac{2}{3}}\nu
$$

and

$$
\|\frac{\partial V_{\theta}^i(\theta)}{\partial \epsilon}\|_{s_\nu,m_{\nu}} = \left\| \sum_{A \in \mathcal{S} \atop |A| \leq \tau_{\nu}} \frac{\partial V_{i,A}^j(\theta)}{\partial \epsilon} \right\|_{s_\nu,m_{\nu}} \\
\leq \sum_{A \in \mathcal{S} \atop |A| \leq \tau_{\nu}} \sum_{supp k \subset A \atop |k| \leq \tau_{\nu}} (\gamma_{\nu+1}^{-1} \Delta^3(\tau_{\nu}) \frac{d\tilde{F}_{i,A,k}}{d\epsilon} | + \gamma_{\nu+1}^{-2} \Delta^6(\tau_{\nu}) |\tilde{F}_{i,A,k}|) \|k\|_{s_\nu,m_{\nu}} [A]_{m_{\nu}} \\
\leq \gamma_{\nu+1}^{-1} \|\tilde{F}_i(\theta)\|_{s_\nu,m_{\nu}} + \frac{C_1}{c_1^2} \gamma_{\nu+1} \frac{\partial \tilde{F}_i(\theta)}{\partial \epsilon} \|s_\nu,m_{\nu} \leq \frac{2C_1 \Delta^6(\tau_{\nu}) a^2 \epsilon_{\nu}}{c_1^2 \gamma_{\nu+1}^3 \epsilon_{0}} \\
\leq \frac{2C_1 \Lambda_{\nu} \sigma_{\nu}^2 \epsilon_{\nu}}{c_1^2 q^{\frac{2}{3}}\nu} \leq \frac{4C_1}{c_1^2 q^{\frac{2}{3}}\nu} \Lambda_{0} \epsilon_{0} q^{\frac{1}{2}}\nu.
$$

This implies that $V_{\theta}(\theta) \in C^{\infty}([s_{\nu}, C^n]$ is the solution of first equation of (21) and $C^{1}$-smooth with respect to $\epsilon$. Furthermore, we can get

$$
\|\frac{\partial \hat{G}_{\nu}}{\partial \epsilon}(\theta, V_{\nu}(\theta)) + \hat{Q}_{\nu}(\theta)\| \leq 4 \frac{\epsilon_{0}}{c_1 \gamma} \Lambda_{0} q^{\frac{1}{2}}\nu, \\
\|\frac{d}{d\epsilon} \left( \frac{\partial \hat{G}_{\nu}}{\partial \epsilon}(\theta, V_{\nu}(\theta)) + \hat{Q}_{\nu}(\theta)\right)\| \leq \frac{8C_1}{c_1^2 q^{\frac{1}{2}}\nu} \Lambda_{0} q^{\frac{1}{2}}\nu.
$$

Then in view of

$$
E_{\nu+1} = E_{\nu} + \frac{\partial \hat{G}_{\nu}}{\partial \epsilon}(\theta, V_{\nu}(\theta)) + \hat{Q}_{\nu}(\theta), \quad (22)
$$

we have $|E_{\nu+1} - E_{\nu}| \leq \frac{1}{2} q^{\frac{1}{2}}\nu$. Let $\frac{\partial \hat{G}_{\nu}}{\partial \epsilon}(\theta, V_{\nu}(\theta)) + \hat{Q}_{\nu}(\theta) := diag(h_{\nu,1}, \ldots, h_{\nu,n})$. Actually, one can check $\text{diag}(h_{0,1}, \ldots, h_{0,n}) = diag(\lambda_{1}(\epsilon), \ldots, \lambda_{n}(\epsilon))$. Thus we have

$$
|\lambda_{\nu+1,i}| \leq |\lambda_{0,i}| - \sum_{i=0}^{\nu} |h_{\nu,i}| \leq \frac{C_1}{2} + C \epsilon_{0} \leq C_1, \\
|\lambda_{\nu+1,i}| \geq |\lambda_{0,i}| - \sum_{i=0}^{\nu} |h_{\nu,i}| \geq 2c_1 - C \epsilon_{0} \geq c_1,
$$

$$
\frac{d}{d\epsilon} |\lambda_{\nu+1,i}| \leq \frac{\partial h_{0,i}}{\partial \epsilon} + \sum_{i=0}^{\nu} \frac{\partial h_{\nu,i}}{\partial \epsilon} \leq \frac{C_1}{2} + \frac{4C_1}{c_1^2 q^{\frac{1}{2}}\nu} \Lambda_{0} q^{\frac{1}{2}}\nu \leq C_1, \\
\frac{d}{d\epsilon} |\lambda_{\nu+1,i}| \geq \frac{\partial h_{0,i}}{\partial \epsilon} - \sum_{i=0}^{\nu} \frac{\partial h_{\nu,i}}{\partial \epsilon} \geq 2c_1 - \frac{4C_1}{c_1^2 q^{\frac{1}{2}}\nu} \Lambda_{0} q^{\frac{1}{2}}\nu > c_1,
$$

and

$$
\frac{d}{d\epsilon} |h_{\nu+1,i}| \leq |h_{\nu+1,i}| \leq |h_{\nu,i}| \leq \frac{C_1}{2} + C \epsilon_{0} \leq C_1, \\
\frac{d}{d\epsilon} |h_{\nu+1,i}| \geq |h_{\nu+1,i}| \geq 2c_1 - C \epsilon_{0} \geq c_1.
$$

and
for each $i$, if $q < \min\{\frac{3^4}{64M^2}, \frac{6^4}{16C^2M^2}\}$. In the same way, we can also get $c_1 < |\lambda_{\nu+1,i} - \lambda_{\nu+1,j}| < C_1$ and $c_1 < \left|\frac{d(\lambda_{\nu+1,i} - \lambda_{\nu+1,j})}{d\nu}\right| < C_1$. This proves assumption (1.1) holds true with $l = \nu + 1$.

Next, we solve the second equation of (21). Define set by

$$D^2_{\nu} := \{\epsilon \in (0, \tilde{\epsilon}) : |\langle k, \omega \rangle - \lambda_{v_i}(\epsilon) + \lambda_{v_j}(\epsilon)| \geq \frac{\gamma_{\nu+1}}{\sqrt{\nu}} ||\Delta(\nu)||^2, \forall k \in \mathbb{Z}_G \setminus \{0\}, 1 \leq i \neq j \leq n\}.$$

Let $H_{\nu}(\theta) = (H_{\nu}^{ij}(\theta))_{1 \leq i,j \leq n}$ and

$$H_{\nu}^{ij}(\theta) = \sum_{A \in \mathcal{S}} H_{\nu}^{ij}(\theta) = \sum_{A \in \mathcal{S}} H_{\nu}^{ij}(\theta) = \sum_{A \in \mathcal{S}} H_{\nu}^{ij}(\theta) = \sum_{A \in \mathcal{S}} H_{\nu}^{ij}(\theta).$$

By the second equation of (21), we can obtain

$$\begin{align*}
\partial_{\nu} U_{\nu}^{ij}(\theta) &= H_{\nu}^{ij}(\theta), \\
\partial_{\nu} U_{\nu}^{ij}(\theta) - (\lambda_{\nu+1,i} - \lambda_{\nu+1,j})U_{\nu}^{ij}(\theta) &= H_{\nu}^{ij}(\theta), \forall 1 \leq i \neq j \leq n, \\
\end{align*}

(23)

where $[H_{\nu}^{ij}(\theta)] = 0$. Thus, (23) is solvable. Similar to first equation of (21), if $\epsilon \in D^2_{\nu}$, we can get

$$\begin{align*}
\|U_{\nu}^{ij}(\theta)\|_{s_{\nu}, m_{\nu}} &\leq c_1^{-1}\gamma_{\nu+1}^{-1} \Delta^3(\nu)\|H_{\nu}^{ij}(\theta)\|_{s_{\nu}, m_{\nu}} \\leq \frac{2\Delta^9(\nu)\alpha^2q^{\frac{2}{2}}}{c_1^2\gamma_{\nu+1}^{-1}}, \\
\|\partial_{\nu} U_{\nu}^{ij}(\theta)\|_{s_{\nu}, m_{\nu}} &\leq \gamma_{\nu+1}^{-1} \Delta^3(\nu)\|\partial H_{\nu}^{ij}(\theta)\|_{s_{\nu}, m_{\nu}} + \gamma_{\nu+1}^{-2} \Delta^6(\nu)\|H_{\nu}^{ij}(\theta)\|_{s_{\nu}, m_{\nu}}, \\
&\leq \frac{4C^2\Delta^12(\nu)\alpha^2q^{\frac{2}{2}}}{c_1^4\gamma_{\nu+1}^{-1}},
\end{align*}

for $1 \leq i,j \leq n$. This implies that $U_{\nu}(\theta) \in C^\omega(\mathbb{T}_{s_{\nu}}, gl(2n, \mathbb{C}))$ is the solution of second equation of (21) and $C^1$-smooth with respect to $\epsilon$.

Finally, choosing $C_2 > \max\{\frac{2}{\alpha^2q}, \frac{4C_1}{\gamma_{\nu+1}^2}, \frac{8}{\gamma_{\nu+1}^2}, \frac{32C_2^2}{\gamma_{\nu+1}^2}\}$, we obtain that

$$\begin{align*}
\|V_{\nu}(\theta)\|_{s_{\nu}, m_{\nu}} &\leq C_2\Lambda_0\epsilon_0(2\delta)^\nu\sigma^2q^{\frac{3}{2}}\nu, \\
\|\frac{\partial V_{\nu}(\theta)}{\partial\epsilon}\|_{s_{\nu}, m_{\nu}} &\leq C_2\Lambda_0(\delta)^\nu\sigma^2q^{\frac{3}{2}}\nu, \\
\|U_{\nu}(\theta)\|_{s_{\nu}, m_{\nu}} &\leq C_2\Lambda_0(\delta)^\nu\sigma^2q^{\frac{3}{2}}\nu, \\
\|\frac{\partial U_{\nu}(\theta)}{\partial\epsilon}\|_{s_{\nu}, m_{\nu}} &\leq C_2\Lambda_0(\delta)^\nu\sigma^2q^{\frac{3}{2}}\nu.
\end{align*}

(24)

D. Estimates of perturbation terms.

By the third and fourth inequalities of (24) we have

$$\begin{align*}
\| (I + U_{\nu}(\theta))^{-1} \|_{s_{\nu+1}, m_{\nu+1}} + \| \frac{\partial}{\partial\epsilon} (I + U_{\nu}(\theta))^{-1} \|_{s_{\nu+1}, m_{\nu+1}} \\
\leq \sum_{j=0}^{\infty} \| U_{\nu}(\theta) \|_{s_{\nu+1}} + \sum_{j=1}^{\infty} \| U_{\nu}(\theta) \|_{s_{\nu+1}+1} \| \frac{\partial U_{\nu}(\theta)}{\partial\epsilon} \|_{s_{\nu+1}, m_{\nu+1}} \\
\leq \frac{1}{1 - (16\delta)^\nu} \leq 2
\end{align*}

(25)

as $\delta \leq 1/32$. Thus, according to (24) and (25) we have the following several estimates:

$$\begin{align*}
\| Q_{\nu+1}(\theta) \|_{s_{\nu+1}, m_{\nu+1}} \\
\leq \| (I + U_{\nu}(\theta))^{-1} \|_{s_{\nu+1}, m_{\nu+1}} \left\{ \| \frac{\partial G_{\nu}(\theta, V_{\nu}(\theta))}{\partial\epsilon} \|_{s_{\nu+1}, m_{\nu+1}} + \| Q_{\nu}(\theta) \|_{s_{\nu+1}, m_{\nu+1}} \\
+ \left( \| \frac{\partial G_{\nu}(\theta, V_{\nu}(\theta))}{\partial\epsilon} \|_{s_{\nu+1}, m_{\nu+1}} \right) + \| Q_{\nu}(\theta) \|_{s_{\nu+1}, m_{\nu+1}} \right\}
\end{align*}$$
\[\frac{\partial Q_{q+1}(\theta)}{\partial \epsilon} \leq \frac{2(1-a)\epsilon q 2^\nu + C_{\text{eq}} \epsilon q 2^\nu}{\epsilon_0} \leq 2(1-a) + \epsilon q 2^\nu \leq \frac{2(1-a) + \epsilon q 2^\nu}{\epsilon_0} \leq q_0 2^\nu,\]

\[\frac{\partial F_{q+1}(\theta)}{\partial \epsilon} \leq \frac{2(1-a) \epsilon q 2^\nu + C_{\text{eq}} \epsilon q 2^\nu}{\epsilon_0} \leq 2(1-a) + \epsilon q 2^\nu \leq q_0 2^\nu,\]
\[
\leq 2(1-a)\epsilon_\nu + C_0\epsilon_\nu + \frac{2(1-a)\epsilon_\nu + C_0\epsilon_\nu}{\epsilon_0}
\]
\[
\leq 2(1-a) + C_0 \frac{\epsilon_\nu}{\epsilon_0}
\]
\[
\leq 2(1-a) + b_0 \frac{\epsilon_\nu}{\epsilon_0} \leq q \frac{\epsilon_\nu}{\epsilon_0}
\]
\[
= \frac{\epsilon_{\nu+1}}{\epsilon_0}
\]
provided that \(\epsilon_0\) is small enough. This proves assumption (1.2) holds true with \(l = \nu + 1\).

Now we want to prove assumption (1.3) holds true with \(l = \nu + 1\). First of all, we note that there is a constant \(C_0 > 0\) such that
\[
\|\frac{\partial^2 G_0(\theta, z)}{\partial z^2}\|_{s_0, r_0, m_0} \leq C_0 \quad \text{and} \quad \|\frac{\partial^2 (\partial G_0(\theta, z))}{\partial \epsilon}\|_{s_0, r_0, m_0} \leq C_0.
\]
Assume by induction that
\[
\|\frac{\partial^2 G_l(\theta, z)}{\partial z^2}\|_{s_l, r_l, m_l} \leq C_0 \Gamma_{1,l} \quad \text{and} \quad \|\frac{\partial^2 (\partial G_l(\theta, z))}{\partial \epsilon}\|_{s_l, r_l, m_l} \leq C_0 \Gamma_{2,l}
\]
hold for \(l = 0, 1, \ldots, \nu\), where \(\Gamma_{1,0} = \Gamma_{2,0} = 1\),
\[
\Gamma_{1,l} = \prod_{i=1}^{l} \frac{(1 + (4\delta)^i)^2}{1 - (4\delta)^i} \quad \text{and} \quad \Gamma_{2,l} = \prod_{i=1}^{l} \frac{(1 + (4\delta)^i)^3}{1 - (4\delta)^i}.
\]
We need to show that
\[
\|\frac{\partial^2 G_{\nu+1}(\theta, z^+)}{\partial z^+^2}\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} \leq C_0 \Gamma_{1,\nu+1} \tag{26}
\]
and
\[
\|\frac{\partial^2 (\partial G_{\nu+1}(\theta, z^+))}{\partial \epsilon}\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} \leq C_0 \Gamma_{2,\nu+1} \tag{27}
\]
hold also true.

Now we show (26) and (27). First of all, by the definition of \(G_{\nu+1}(\theta, z^+)\) we may calculate that
\[
\frac{\partial^2 G_{\nu+1}(\theta, z^+)}{\partial z^+^2} = (I + U_\nu(\theta))^{-1} \frac{\partial^2 G_\nu(\theta, V_\nu(\theta)) + (I + U_\nu(\theta))z^+}{\partial z^2}(I + U_\nu(\theta))^2.
\]
Next, using the first and third inequalities of (24) one has that
\[
\|V_\nu(\theta) + (I + U_\nu(\theta))z^+\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} \leq c_\nu \sigma_\nu(2\delta)^\nu q^{2\nu} + \left(1 + c_\nu \sigma_\nu(4\delta)^\nu q^{2\nu}\right) r_{\nu+1}
\]
and one thus gets
\[
r_{\nu} - \|V_\nu(\theta) + (I + U_\nu(\theta))z^+\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} > \sigma_\nu(1 - c_\nu) > 0
\]
provided that \(c_\nu\) is small enough. This implies that \(V_\nu(\theta, \epsilon_0) + (I + U_\nu(\theta, \epsilon_0))z^+ \in B_r\) as \((\theta, z^+) \in \Delta_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}}\). Hence, we have
\[
\|\frac{\partial^2 G_{\nu+1}(\theta, z^+)}{\partial z^+^2}\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}}
\]
\[
= \|\frac{\partial^2 G_\nu(\theta, V_\nu(\theta)) + (I + U_\nu(\theta))z^+}{\partial z^2}(I + U_\nu(\theta))^2\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}}
\]
\[
\leq \frac{1}{1 - (4\delta)^{\nu+1}} C_0 \Gamma_{1,\nu}(1 + (4\delta)^{\nu+1})^2
\]
\[
= C_0 \Gamma_{1,\nu+1}.
\]
In the same way, we can obtain
\[ \left\| \frac{\partial^2}{\partial z^+^2} \left( \frac{G_{\nu+1}(\theta, z^+)}{\partial \epsilon} \right) \right\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} \leq C_0 \Gamma_{2, \nu+1}. \]

By Taylor’s formula, we see that for any \( r \leq r_{\nu+1} \), it has
\[ \left\| G_i(z, \theta) \right\|_{s_{\nu+1}, r, m_{\nu+1}} \leq C r^2, \quad \left\| \frac{\partial G_i(z, \theta)}{\partial \epsilon} \right\|_{s_{\nu+1}, r, m_{\nu+1}} \leq C r^2. \]

\[ \square \]

3.1.3. Convergence of KAM iteration. Let us take \( E_0 = E \), \( Q_0(\theta) = Q(\theta) \), \( F_0(\theta) = F(\theta) \), \( G_0(\theta, z) = G(\theta, z) \). It is easy to check that system (12) satisfies all hypotheses of Lemma 3.2 with \( l = 0 \). By induction, we can prove that for any \( \nu \geq 0 \) there is a sequence \( \Phi_\nu \) of transformations such that \( \Phi_\nu(\Delta_{s_{\nu+1}, r_{\nu+1}}) \subset \Delta_{s_{\nu}, r_{\nu}} \) and
\[
(Eq)_\nu \circ \Phi_\nu = (Eq)_{\nu+1}.
\]

It follows from (24) that
\[ \| \Phi_\nu - id \|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} \leq \| U_\nu(\theta) \|_{s_{\nu+1}, m_{\nu+1}} \| z^+ \| + \| V_\nu(\theta) \|_{s_{\nu+1}, m_{\nu+1}} \leq (4\delta)^\nu \]
and
\[ \| D\Phi_\nu - Id \|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} \leq \| U_\nu(\theta) \|_{s_{\nu+1}, m_{\nu+1}} \leq (4\delta)^\nu, \]
for sufficiently small \( \epsilon_0 \), where \( D \) denotes the Jacobian with respect to \( z^+ \). Let
\[ \Phi^\nu := \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : \Delta_{s_{\nu+1}, r_{\nu+1}} \rightarrow \Delta_{s_0, r_0}. \]

For sufficiently small \( \epsilon_0 > 0 \), it follows from the inequality (28) that
\[ \| D\Phi_j(\Phi_{j+1} \circ \cdots \circ \Phi_\nu) \| \leq 1 + (4\delta)^2, \quad j = 0, 1, \ldots, \nu - 1. \]

We infer
\[ \| D\Phi^{\nu-1} \|_{s_\nu, r_\nu, m_\nu} \leq \Pi_{\nu \geq 0} (1 + (4\delta)^\nu) \leq e^{\frac{1}{1 - \delta}} \]
and hence
\[ \| \Phi^\nu - \Phi^{\nu-1} \|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} = \| \Phi^{\nu-1}(\Phi_\nu) - \Phi^{\nu-1} \|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} \leq \| D\Phi^{\nu-1} \|_{s_{\nu}, r_\nu, m_\nu} \| \Phi_\nu - id \|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} \leq e^{\frac{1}{1 - \delta}} \delta^\nu. \]

The same inequality holds for \( \nu = 0 \) if we define \( \Phi^{-1} := id \). In view of \( s_\nu \to s_* \geq s/2, \ r_\nu \to r_* \geq r/2 \), as \( \nu \to \infty \), the mapping \( \Phi^\nu \):
\[ \Phi^\nu = \Phi^0 + \sum_{i=0}^{\nu} (\Phi^i - \Phi^{i-1}), \quad \Phi^{-1} := id \]
converges uniformly on
\[ \bigcap_{\nu \geq 0} \Delta_{s_{\nu+1}, r_{\nu+1}} = \Delta_{s_*, r_*}, \]
to a mapping \( \Phi^* \), which is analytic on \( \Delta_{s_*, r_*} \).

Similarly, in view of (22) we also show that \( E_\nu \) converges to a matrix \( E_* \). Then, \( (Eq)_0 \) is changed to
\[
(Eq)_0 \circ \Phi^* : \left\{ \begin{array}{l}
\dot{\theta} = \omega, \\
\dot{z} = E_* z + G_*(\theta, z),
\end{array} \right.
\]
(29)
where $G_\ast = \mathcal{O}(z^2)(z \to 0)$. It is obvious that (29) possesses an invariant torus
$$
\mathbb{T}^2 \times \{0\} : \theta = \omega t + \theta_0, \quad z = 0.
$$

It follows from (24), that
$$
\Phi^* = \lim_{\nu \to \infty} \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : \begin{cases} \theta = \theta, \\
z = z + U_\ast(\theta)z + V_\ast(\theta),
\end{cases}
$$

where
$$
U_\ast(\theta) = \mathcal{O}(\epsilon_0) \quad \text{and} \quad V_\ast(\theta) = \mathcal{O}(\epsilon_0).
$$

3.1.4. Measure estimate. In this section, we will estimate the measure of $E$ which will defined later. We first give two Lemmas which can be found in [15].

Lemma 3.3. For every given approximation function $\Theta$, there exists an approximation function $\Delta$ such that
$$
\sum_{A \in \mathcal{S},|A| = i} \frac{1}{\Delta(|A|)} \leq \frac{2N_0}{\Theta(t_i)}, \quad i \geq 1,
$$

where $N_0$, $t_i$ are given in Lemma 2.3.

Lemma 3.4. There is an approximation function $\Delta$ such that
$$
\sum_{l \in \mathbb{Z}\setminus\{0\}} \frac{1}{\Delta(|l|)} \leq M^{\log \log i}
$$

for all sufficiently large $i$ with some constant $M$, where $l = (l_1, l_2, \cdots, l_i)$ and $|l| = |l_1| + |l_2| + \cdots + |l_i|$.

Denote
$$
D_\nu := D_\nu^1 \cap D_\nu^2.
$$

For each $0 \neq \text{supp} k \subset A \in \mathcal{S}$ and $\nu \geq 0$, we let
$$
\begin{align*}
&f_{\nu,A,k}^{1,i}(\epsilon) = i\langle k, \omega \rangle - \lambda_{\nu,i}(\epsilon), \forall 1 \leq i \leq n, \\
&f_{\nu,A,k}^{2,ij}(\epsilon) = i\langle k, \omega \rangle - \lambda_{\nu,i}(\epsilon) + \lambda_{\nu,j}(\epsilon), \forall 1 \leq i \neq j \leq n.
\end{align*}
$$

Define sets
$$
\begin{align*}
&R_{\nu,A,k}^{1,i} = \{ \epsilon \in (0, \epsilon) : |f_{\nu,A,k}^{1,i}(\epsilon)| < \frac{\gamma_{\nu+1}}{\Delta(|k|)^3} \}, \forall 1 \leq i \leq n, \\
&R_{\nu,A,k}^{2,ij} = \{ \epsilon \in (0, \epsilon) : |f_{\nu,A,k}^{2,ij}(\epsilon)| < \frac{\gamma_{\nu+1}}{\Delta(|k|)^3} \}, \forall 1 \leq i \neq j \leq n.
\end{align*}
$$

In view of $|\lambda_{\nu,i}(\epsilon)| \leq C$, one can check that
$$
|f_{\nu,A,k}^{1,i}(\epsilon)| \geq |\langle k, \omega \rangle - \lambda_{0,i}| - |\lambda_{\nu,p}(\epsilon) - \lambda_{0,i}| \geq \frac{\gamma}{\Delta(|k|)\Delta(|k|)} - C\epsilon_0.
$$

Moreover, we have
$$
\left| \frac{df_{\nu,A,k}^{1,i}(\epsilon)}{d\epsilon} \right| \geq \left| \frac{d}{d\epsilon} \lambda_{\nu,i}(\epsilon) \right| \geq c
$$

for each $\epsilon \in (0, \epsilon)$. If $\frac{\gamma}{\Delta(|k|)\Delta(|k|)} \geq 2C\epsilon_0$, then
$$
|f_{\nu,A,k}^{1,i}(\epsilon)| \geq \frac{\gamma}{2\Delta(|k|)\Delta(|k|)} - C\epsilon_0 \geq \frac{\gamma}{2\Delta(|k|)\Delta(|k|)} \geq \frac{\gamma_{\nu+1}}{\Delta^3(|k|)\Delta^3(|k|)}.
$$
It implies that \( \text{meas} R_{\nu A, k}^{1, i} = 0 \). If \( \frac{\gamma}{\Delta(\varepsilon)} < \epsilon_0 \), then according to Lemma B.1 in [19] (for its proof, see [26]),

\[
\text{meas} R_{\nu A, k}^{1, i} \leq \frac{\gamma_{\nu + 1}}{\Delta(|\varepsilon|)} \leq \frac{4C^2\gamma_{\nu + 1}\epsilon_0^2}{\Delta(|\varepsilon|)} \leq C \frac{\gamma_{\nu + 1}}{\Delta(|\varepsilon|)} \epsilon_0^2,
\]

for all \( 1 \leq i \neq j \leq n \). Similarly, we can get

\[
\text{meas} R_{\nu A, k}^{2, ij} \leq C \frac{\gamma_{\nu + 1}}{\Delta(|\varepsilon|)} \epsilon_0^2,
\]

for all \( 1 \leq i \neq j \leq n \). Hence, we can get

\[
\text{meas} \left( \bigcup_{\nu \geq 0, k \in Z \setminus \{0\}} \bigcup_{1 \leq i \leq n} R_{\nu A, k}^{1, i} \right) \leq C \epsilon_0^2 \left( \sum_{A \subseteq \mathcal{S}} \frac{1}{\Delta(|\varepsilon|)} \sum_{A \subseteq \mathcal{S}} \frac{1}{\Delta(|\varepsilon|)} \right)
\]

\[
\leq C \epsilon_0^2 \sum_{A \subseteq \mathcal{S}} \left( \sum_{\supp(A) \subseteq \mathcal{S}} \frac{1}{\Delta(|\varepsilon|)} \sum_{k \neq 0} \frac{1}{\Delta(|\varepsilon|)} \right),
\]

From the proof of Lemma 3.3 and Lemma 3.4, there is a same approximation function \( \Delta \) such that Lemma 3.3 and Lemma 3.4 hold simultaneously. Applying Lemma 3.3 and Lemma 3.4, we can get

\[
\text{meas} \left( \bigcup_{\nu \geq 0, k \in Z \setminus \{0\}} \bigcup_{1 \leq i \leq n} R_{\nu A, k}^{1, i} \right) \leq C \epsilon_0^2 \left( C + C \sum_{i = i_0}^{+\infty} \frac{M^i \log \log i}{\Theta(t_i)} \right),
\]

where \( i_0 \) so large that \( t_j \geq i \log^{q - 1} i \) for \( i \geq i_0, q > 2 \) by hypotheses. Here, we can choose

\[
\Theta(t) = e^{\log t \log^{q - 1} \log t}, \quad t > e, \quad q > 2,
\]

which guarantees the infinite sum does converge. Similarly, we can get

\[
\text{meas} \left( \bigcup_{\nu \geq 0, k \in Z \setminus \{0\}} \bigcup_{1 \leq i \neq j \leq n} R_{\nu A, k}^{2, ij} \right) \leq C \epsilon_0^2 \left( C + C \sum_{i = i_0}^{+\infty} \frac{M^i \log \log i}{\Theta(t_i)} \right).
\]

Note that

\[
\mathcal{E} = \left( \bigcup_{\nu \geq 0} D_\nu = (0, \bar{\epsilon}) \setminus \left( \bigcup_{\nu \geq 0, k \in Z \setminus \{0\}} \bigcup_{1 \leq i \leq n} R_{\nu A, k}^{1, i} \right) \bigcup \left( \bigcup_{\nu \geq 0, k \in Z \setminus \{0\}} \bigcup_{1 \leq i \neq j \leq n} R_{\nu A, k}^{2, ij} \right) \right).
\]

Finally, we have

\[
\text{meas} \mathcal{E} \geq \bar{\epsilon} - \text{meas} \left( \bigcup_{\nu \geq 0, k \in Z \setminus \{0\}} \bigcup_{1 \leq i \leq n} R_{\nu A, k}^{1, i} \right) - \text{meas} \left( \bigcup_{\nu \geq 0, k \in Z \setminus \{0\}} \bigcup_{1 \leq i \neq j \leq n} R_{\nu A, k}^{2, ij} \right) \geq \bar{\epsilon} - C \epsilon_0^2 > \bar{\epsilon}(1 - C\epsilon).
\]
This means that $\mathcal{E}$ has relative positive measure. The proof of Theorem 3.1 is complete.

4. Degenerate case. In this section, we will consider the existence of a.p. response solutions for the perturbation system of a $n$-dimensional almost-periodically forced system with degenerate equilibrium point, the $n$-dimensional system (7). If we introduce the extended phase space $\mathbb{T}^n \times \mathbb{R}^n$, where writing $\theta = (\omega_1 t, \ldots, \omega_s t)$ for the toral variable yields the desired time-dependence, then the following system is equivalent to system (7)

$$\begin{cases}
\dot{\theta} = \omega, \\
\dot{x} = \phi(x) + h(\theta, x) + f(\theta, x, \epsilon), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,
\end{cases}$$

(30)

where $\phi(x) = (a_1 x_1^{l_1}, \ldots, a_{n-1} x_{n-1}^{l_{n-1}}, a_n x_n^{l_n})^T$ is a vector field that is comprised of $n$ power functions with $0 \neq a_i \in \mathbb{R}$, $l_i \in \mathbb{N}$, $l_i \geq 1$ and $h = (h_1, \ldots, h_n)^T$, with $h = O(x^{l_1+1})$, is a high order term and $f = (f_1, \ldots, f_n)^T$, with $f(\theta, x, 0) = 0$, is a low order perturbation term. If $\epsilon = 0$, then the unperturbed system (30) has the origin as a degenerate equilibrium and has the invariant torus

$$T_0 = \mathbb{T}^n \times \{0_n\},$$

here $0_n$ denotes $n$-dimensional zero vector, carrying a quasi-periodic flow $\theta(t) = \omega t + \theta_0$ with same frequencies $\omega$ as the forcing. Its normal space is described by the $x = (x_1, \ldots, x_n)$-coordinates. Obviously, the existence of response solutions of system (7) is equivalent to the persistence of such a degenerate response torus $T_0$ under sufficiently small perturbation $f$.

For system (30) with $n = 1$, we have the following theorem:

**Theorem 4.1.** For some $s, r > 0$. Consider system (30) in the domain $\Delta_{s,r}$. Suppose that $n = 1$, $a_1 = 1$, $l_1 > 1$ and

(i) $\omega$ satisfies the non-resonant condition

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{\Omega(|\langle k \rangle|)} \Omega(|k|), \quad \forall k \neq 0 \in \mathbb{Z}_S^n,$$

(31)

where $\Omega(t)$ is a positive function satisfying

$$\frac{\ln \Omega(t)}{t} \to 0, \quad \text{as } t \to \infty;$$

(ii) $h_1(\theta, x) \in C^m_n(\Delta_{s,r}, \mathbb{R})$, $h_1(\theta, x) = O(x^{l_1+1})(x \to 0)$;

(iii) $f_1(\theta, x, \epsilon) \in C^m_n(\Delta_{s,r}, \mathbb{R})$ and are analytic in $\epsilon$. Moreover, $f_1(\theta, x, \epsilon) = O(\epsilon)$ and

$$\left[ \frac{\partial f_1}{\partial \epsilon} \right]_{(0, 0)} < 0, \quad \text{as } l_1 \text{ is even,}$$

$$\neq 0, \quad \text{as } l_1 \text{ is odd;}$$

Then for sufficiently small positive constant $\epsilon$, system (30) has a degenerate response torus, i.e. the torus persists under small perturbations.

For system (30) with $n > 1$, we have the following two theorems:

**Theorem 4.2.** For some $s, r > 0$. Consider system (30) in the domain $\Delta_{s,r}$. Suppose that

(i) $h_i(\theta, x) \in C^m_n(\Delta_{s,r}, \mathbb{R})$, $h_i(\theta, x) = O(x^{l_i+1})(x \to 0)$, $i = 1, 2, \ldots, n;$

(ii) $f_i(\theta, x, \epsilon) \in C^m_n(\Delta_{s,r}, \mathbb{R})$ and are analytic in $\epsilon$. Moreover, $f_i(\theta, x, \epsilon) = O(\epsilon)$ and

$$\left[ \frac{\partial f_i}{\partial \epsilon} \right]_{(0, 0)} < 0, \quad \text{as } l_i \text{ is even,}$$

$$\neq 0, \quad \text{as } l_i \text{ is odd;}$$

Then for sufficiently small positive constant $\epsilon$, system (30) has a degenerate response torus, i.e. the torus persists under small perturbations.
(ii) \( f_i(\theta, x, \epsilon) \in C^m(\Delta_{x,r}, \mathbb{R}) \) and are analytic in \( \epsilon \) for \( i = 1, 2, \ldots, n \). Moreover, if \( l_i = 1 \), then \( f_i(\theta, x, \epsilon) = O(\epsilon^2) \); if \( l_i > 1 \), then
\[
a_i \left\{ \left[ \frac{\partial f_i(\theta, 0_{n+1})}{\partial \epsilon} \right] \right\} \begin{cases} 0, \text{ as } l_i \text{ are even}, \\
\neq 0, \text{ as } l_i \text{ are odd},
\end{cases}
\]
and
\[
f_i(\theta, x, \epsilon) = O(\epsilon), \quad f_i(\theta, x, \epsilon) - [\frac{\partial f_i(\theta, 0_{n+1})}{\partial \epsilon}] \epsilon = O(\epsilon^2),
\]
for \( i = 1, 2, \ldots, n \);

(iii) If \( l_i = l_j = 1 \), then \( a_i \neq a_j \), if \( l_i = l_j \neq 1 \), then
\[
a_i \left( -\frac{1}{a_i} \left[ \frac{\partial f_i(\theta, 0_{n+1})}{\partial \epsilon} \right] \right)^{\frac{i_j-1}{i_i}} \neq a_j \left( -\frac{1}{a_j} \left[ \frac{\partial f_j(\theta, 0_{n+1})}{\partial \epsilon} \right] \right)^{\frac{i_j-1}{i_j}},
\]
for \( 0 \leq i \neq j \leq n \).

Then for sufficiently small positive constant \( \epsilon \), system (30) has a degenerate response torus, i.e. the torus persists under small perturbations.

**Theorem 4.3.** For some \( s, r > 0 \). Consider system (30) in the domain \( \Delta_{s,r} \). Suppose that \( l_n = 2l + 1 > l, l \in \mathbb{N} \) and

(i) \( \omega \) satisfies the non-resonant condition (10);

(ii) \( h_i(\theta, x) \in C^m(\Delta_{s,r}, \mathbb{R}), \quad h_i(\theta, x) = O(x^{d+2})(x \to 0), \quad i = 1, 2, \ldots, n; \)

(iii) \( f_i(\theta, x, \epsilon) \in C^m(\Delta_{s,r}, \mathbb{R}) \) and are analytic in \( \epsilon \) for \( i = 1, 2, \ldots, n \). Moreover, if \( l_i = 1 \), then \( f_i(\theta, x, \epsilon) = O(\epsilon^2) \), if \( l_i > 1 \), then
\[
a_i \left\{ \frac{\partial f_i(\theta, 0_{n+1})}{\partial \epsilon} \right\} \begin{cases} 0, \text{ as } l_i \text{ are even}, \\
\neq 0, \text{ as } l_i \text{ are odd},
\end{cases}
\]
and
\[
f_i(\theta, x, \epsilon) = O(\epsilon), \quad f_i(\theta, x, \epsilon) - [\frac{\partial f_i(\theta, 0_{n+1})}{\partial \epsilon}] \epsilon = O(\epsilon^2)
\]
for \( i = 1, 2, \ldots, n-1 \);

(iv) \( f_n(\theta, x, \epsilon) = O(\epsilon^2) \);

(v) If \( l_i = l_j = 1 \), then \( a_i \neq a_j \), if \( l_i = l_j \neq 1 \), then
\[
a_i \left( -\frac{1}{a_i} \left[ \frac{\partial f_i(\theta, 0_{n+1})}{\partial \epsilon} \right] \right)^{\frac{i_j-1}{i_i}} \neq a_j \left( -\frac{1}{a_j} \left[ \frac{\partial f_j(\theta, 0_{n+1})}{\partial \epsilon} \right] \right)^{\frac{i_j-1}{i_j}},
\]
for \( 0 \leq i \neq j \leq n-1 \).

Then for sufficiently small positive constant \( \epsilon \), system (30) has a degenerate response torus, i.e. the torus persists under small perturbations.

**Remark 2.** Non-resonant condition (31) is weaker than the classical non-resonant condition (10). That is because if \( \Omega(t) \) is an approximation function in non-resonant condition (10), then for each \( x > 1 \), we have
\[
\int_x^{2x} \frac{\ln \Omega(t)}{t^2} dt > \int_x^{2x} \frac{\ln \Omega(x)}{t^2} dt > \frac{\ln \Omega(x)}{2x} > 0,
\]
which implies \( \lim_{t \to \infty} \frac{\ln \Omega(x)}{t^2} = \infty \) since \( \lim_{x \to \infty} \int_x^{2x} \frac{\ln \Omega(t)}{t^2} dt = 0 \). On the contrary, if we choose \( \Omega(t) = e^{\frac{t}{t^2}} \), then one can check that \( \lim_{t \to \infty} \frac{\ln \Omega(t)}{t^2} = \lim_{t \to \infty} \frac{1}{t^2} = 0 \) while \( \int_1^\infty \frac{\ln \Omega(t)}{t^2} dt \) is divergent.
Remark 3. Conditions (ii) and (iii) in Theorem 4.1, conditions (i) and (ii) in Theorem 4.6 and conditions (ii), (iii) and (iv) in Theorem 4.3 guarantee $l_2 > l_1$ and $l_3 > 2l_1$ hold still in system (9) after one KAM step. Condition (iii) in Theorem 4.6 and condition (v) in Theorem 4.3 guarantee that linear matrix of system (9) after each e KAM step has no multi-eigenvalues.

Remark 4. Theorem 4.2 not only generalizes Theorem 3.2 in [20] from quasi-periodic case to almost-periodic case but also from 1-dimensional system to $n$-dimensional system, and Theorem 4.3 is the direct generalization of Theorem 3.12 in [20]. Moreover, if we apply Theorem 4.3 to 1-dimensional system, then conditions (ii), (iii) and (v) vanish naturally. This shows that perturbation in Theorem 4.3, for 1-dimensional case, does not need any restriction which implies Theorem 4.3 also generalizes the work in [23] from quasi-periodic case to almost-periodic case.

4.1. The proofs of Theorems 4.1 and 4.2. In this subsection, we intend to give the proofs of theorems 4.1 and 4.2 which will divide the following two parts.

4.1.1. Normal form. We first reduce system (30) to a normal form for which KAM method can be applied. According to the different conditions of the two theorems, we represent the procedure of the normal forms of two theorems in two lemmas respectively. Actually, the main idea in these two lemmas is that the desired normal form transformation is achieved by first moving the equilibrium solution of the average system to the origin and then removing some lower order terms by means of a suitable normalizing transformation (only Lemma 4.4 use the nonresonance conditions on the frequency vector $\omega$ of the forcing). See the proofs of the following two Lemmas for more details.

Lemma 4.4. Under the conditions of Theorem 4.1, there exist $\bar{m} < m, \bar{r} < r, \bar{s} < s$ and the almost-periodic transformation

$$
\phi : \Delta_{\bar{s}, \bar{r}} \rightarrow \Delta_{s, r} \tag{32}
$$

of the form

$$
x = U(\theta, \eta)z + V(\theta, \eta), \quad \theta = \theta,
$$

with $\|U\|_{\bar{s}, \bar{r}, \bar{m}} \leq 1 + C\eta^{l_1}$ and $\|V\|_{\bar{s}, \bar{r}, \bar{m}} \leq C\eta$, such that system (30) is transformed into system

$$
\begin{cases}
\dot{\theta} = \omega, \\
\dot{z} = (e(\eta) + d(\theta, \eta))z + p(\theta, \eta) + q(\theta, z, \eta),
\end{cases} \tag{33}
$$

where $\eta = e^{\frac{1}{\tau}}$, $d(\theta, \eta), p(\theta, \eta) \in C^\omega_m(T^{\tau}_s, \mathbb{R}), q(\theta, z, \eta) \in C^\omega_m(\Delta_{\bar{s}, \bar{r}}, \mathbb{R})$ and the following estimates hold:

$$
c\eta^{l_1-1} < |e(\eta)| < C\eta^{l_1-1}, \quad \|d\|_{\bar{s}, \bar{m}, m} \leq C\eta^{2l_1}, \quad \|p\|_{\bar{s}, \bar{m}, m} \leq C\eta^{2l_1}, \quad q = O(z^2)(z \to 0). \tag{34}
$$

Proof of Lemma 4.4. First of all, we consider the averaged system of system (30):

$$
\begin{cases}
\dot{\theta} = \omega, \\
\dot{x} = x^{l_1} + [h_1(\theta, x, \epsilon)] + [f_1(\theta, x, \epsilon)].
\end{cases} \tag{35}
$$

We wish to investigate the solutions of system (30) near the equilibrium of the averaged system (35). To do this, we consider the implicit equation

$$
G(x, \epsilon) := x^{l_1} + [h_1(\theta, x, \epsilon)] + [f_1(\theta, x, \epsilon)] = 0. \tag{36}
$$
In view of \( f_1(\theta, x, \epsilon) = O(\epsilon) \), using a Taylor expansion, we have

\[
[f_1(\theta, x, \epsilon)] = \left( \frac{\partial f_1(\theta, 0, 0)}{\partial \epsilon} \right) \epsilon + O(\epsilon^2).
\]  

(37)

Putting \( x = \epsilon \tilde{x} \) in (36) and noting that \( h_1(\theta, x, \epsilon) = O(x^{l_1+1}) \) and (37), the implicit equation (36) becomes an implicit equation with regard to \( \tilde{x} \)

\[
H(\tilde{x}, \eta) := \frac{1}{\epsilon} G(\epsilon \tilde{x}, \epsilon) = \tilde{x}^{l_1} + \left[ \frac{\partial f_1(\theta, 0, 0)}{\partial \epsilon} \right] \epsilon + O(\epsilon) = 0
\]  

(38)

with \( \eta = \epsilon \tilde{x} \). From hypothesis (iii) of Theorem 4.1, (37) and (38) it is easy to see that

\[
H\left( \left[ -\frac{\partial f_1(\theta, 0, 0)}{\partial \epsilon} \right] \right) \neq 0.
\]

Then the implicit function theorem asserts that the equation (38) has a unique solution \( \tilde{x}(\eta) = \left[ -\frac{\partial f_1(\theta, 0, 0)}{\partial \epsilon} \right] \tilde{x}^{l_1} + O(\eta) \) in a neighborhood of \( \eta = 0 \), which implies the equation (36) has a unique solution \( x(\eta) = \eta \tilde{x}(\eta) \) in a neighborhood of \( \eta = 0 \). Made the transformation

\[
\phi_1 : x = y + x(\eta),
\]

system (30) becomes

\[
\begin{cases}
\dot{\theta} = \omega, \\
\dot{y} = l_1(\eta \tilde{x}(\eta))^{l_1-1} y + \left( \frac{\partial}{\partial \eta} h_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) + \frac{\partial}{\partial \eta} f_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) \right) y \\
+ h_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) - [h_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) + f_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) - f_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1})] \\
+ O(y^2), \quad y \to 0.
\end{cases}
\]  

(39)

System (39) can write as the following system

\[
\begin{cases}
\dot{\theta} = \omega, \\
\dot{y} = (\tilde{c}(\eta) + \tilde{d}(\theta, \eta)) y + \tilde{p}(\theta, \eta) + \tilde{q}(\theta, y, \eta),
\end{cases}
\]  

(40)

where

\[
\tilde{c}(\eta) = l(\eta \tilde{x}(\tau))^{l_1-1} + \left[ \frac{\partial}{\partial \eta} h_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) - \frac{\partial}{\partial x} f_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) \right],
\]

\[
\tilde{d}(\theta, \eta) = \frac{\partial}{\partial \eta} h_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) - \frac{\partial}{\partial x} f_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) - \frac{\partial}{\partial x} h_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1})
\]

\[
- \left[ \frac{\partial}{\partial x} f_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) \right],
\]

\[
\tilde{p}(\theta, \eta) = h_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) - [h_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1}) + f(\theta, \eta \tilde{x}(\eta), \eta^{l_1})
\]

\[
- [f_1(\theta, \eta \tilde{x}(\eta), \eta^{l_1})],
\]

\[
\tilde{q}(\theta, y, \eta) = O(y^2)(y \to 0).
\]

It is easy to check that there exist \( m_1 < m, s_1 < s \) and \( r_1 < r \) such that \( \tilde{d}(\theta, \eta) \in C^m_{m_1}(\mathbb{T}^d, \mathbb{R}) \) with \( |\tilde{d}(\theta)| = 0 \) and \( |\tilde{p}(\theta)| = 0 \), \( \tilde{q}(\theta, y, \eta) \in C^{m_1}_{s_1}(\Delta_{s_1, r_1}, \mathbb{R}) \) and \( C^{m_1-1} \leq |\tilde{c}(\eta)| \leq C^m_{m_1-1} \), \( \|\tilde{d}\|_{m_1, s_1} \leq C^s_{m_1}, \|\tilde{p}\|_{m_1, s_1} \leq C^s_{m_1}, \tilde{q} = O(y^2)(y \to 0) \).

Now we intend to find a transformation

\[
\phi_2 : y = z + u(\theta, \eta) z + v(\theta, \eta),
\]  

(41)
which changes system (40) into the following system
\[
\begin{cases}
\dot{\theta} = \omega, \\
\dot{z} = (e(\eta) + d(\theta, \eta))z + p(\theta, \eta) + q(\theta, z, \eta),
\end{cases}
\]
where
\[
e(\eta) = \tilde{e}(\eta) + \left[\frac{\partial \tilde{q}(\theta, v(\theta), \eta)}{\partial y}\right],
\]
\[
p(\theta, \eta) = (1 + u(\theta, \eta))^{-1}\left(\tilde{d}(\theta, \eta)v(\theta, \eta) + \tilde{q}(\theta, v(\theta), \eta)\right),
\]
\[
d(\theta, \eta) = (1 + u(\theta, \eta))^{-1}(\tilde{d}(\theta, \eta)u(\theta, \eta) + \left(\frac{\partial \tilde{q}(\theta, v(\theta), \eta)}{\partial y}\right)
- \left[\frac{\partial \tilde{q}(\theta, v(\theta), \eta)}{\partial y}\right]u(\theta, \eta)),
\]
\[
q(\theta, z, \eta) = (1 + u(\theta, \eta))^{-1}\left(q(\theta, (1 + u(\theta, \eta), \eta)z + v(\theta, \eta)) - \tilde{q}(\theta, v(\theta, \eta), \eta)
- \frac{\partial \tilde{q}(\theta, v(\theta), \eta), \eta)}{\partial y} (1 + u(\theta, \eta))z\right).
\]
The functions \(u\) and \(v\) in the transformation (41) are determined by the following equations
\[
\partial_\omega v(\theta, \eta) = \tilde{e}(\eta)v(\theta, \eta) + \tilde{p}(\theta, \eta),
\]
\[
\partial_\omega u(\theta, \eta) = \tilde{d}(\theta, \eta) + \left[\frac{\partial \tilde{q}(\theta, v(\theta), \eta)}{\partial y}\right] - \left[\frac{\partial \tilde{q}(\theta, v(\theta), \eta)}{\partial y}\right]u(\theta, \eta),
\]
(42) and (43).
First of all, we solve the equation (42). \(\tilde{p}(\theta, \eta)\) admits spatial series expansion
\[
\tilde{p}(\theta, \eta) = \sum_{\mathcal{A} \in \mathcal{S}} \tilde{p}_A(\theta, \eta) = \sum_{\mathcal{A} \in \mathcal{S}} \sum_{\text{supp} k \subseteq \mathcal{A}} \tilde{p}_{A,k}(\eta)e^{i(k, \theta)}.
\]
Let
\[
v(\theta, \eta) = \sum_{\mathcal{A} \in \mathcal{S}} v_A(\theta, \eta) = \sum_{\mathcal{A} \in \mathcal{S}} \sum_{\text{supp} k \subseteq \mathcal{A}} v_{A,k}(\eta)e^{i(k, \theta)}.
\]
According to equation (42), we have
\[
i(k, \omega)v_{A,k}(\eta) = \tilde{e}(\eta)v_{A,k}(\eta) + \tilde{p}_{A,k}(\eta), \text{ supp} k \subseteq \mathcal{A}, \mathcal{A} \in \mathcal{S}.
\]
Since \(\tilde{p}_{A,0}(\eta) = 0\), we can choose \(v_{A,0}(\eta) = 0\). For all \(\mathcal{A} \in \mathcal{S}, \text{ supp} k \subseteq \mathcal{A}\) and \(|k| > 0\), we have
\[
v_{A,k}(\eta) = \frac{\tilde{p}_{A,k}(\eta)}{i(k, \omega) - \tilde{e}(\eta)}.
\]
Since \(\tilde{e}(\eta)\) is a real number, we have
\[
|v_{A,k}(\eta)| \leq \frac{|\tilde{p}_{A,k}(\eta)|}{|i(k, \omega) - \tilde{e}(\eta)|} \leq \frac{|\tilde{p}_{A,k}(\eta)|}{|i(k, \omega)|} \leq \gamma^{-1}\Omega(|k|)\Omega(|k|)|\tilde{p}_{A,k}(\eta)|,
\]
for all \(|k| > 0\). Let \(\bar{m} < m_1, \bar{r} < r_1\) and \(\bar{s} < s_1\). It follows that
\[
\|v\|_{\bar{s}, \bar{m}} \leq \sum_{\mathcal{A} \in \mathcal{S}} \sum_{\text{supp} k \subseteq \mathcal{A}} |v_k(\eta)|e^{i(k, \bar{s})} e^{\bar{m}|A|} 
\leq \sum_{\mathcal{A} \in \mathcal{S}} \sum_{\text{supp} k \subseteq \mathcal{A}} \gamma^{-1}\Omega(|k|)\Omega(|k|)|p_k(\eta)|e^{i(k, \bar{s})} e^{\bar{m}|A|}.
since the sup
\[ \phi \]
transformation (32),
equation (43) is also solvable and \( \| C \) such that system (30) is transformed by the translation transformation following estimates hold:
\[ e \]
where \( z \in \epsilon \)
Making the transformation \( \alpha_i = \phi_1 \circ \phi_2 \) with \( U(\theta, \eta) = 1 + u(\theta, \eta) \) and \( V(\theta, \eta) = v(\theta, \eta) + x(\eta) \), and have the estimates
\[ ||U||_{\tilde{s}, \tilde{r}} \leq 1 + C \eta^{\tilde{l}} \text{ and } ||V||_{\tilde{s}, \tilde{r}} \leq C \eta. \]
In addition, it is easy to see the estimate (34).

**Lemma 4.5.** Under the conditions of Theorem 4.2, there exist \( m_1 < m, r_1 < r, s_1 < s \) such that system (30) is transformed by the translation transformation
\[ x_i = z_i + \left( -\frac{1}{a_i} \left[ \partial f_i(\theta, 0, 0) \right] \right) \epsilon \]
if \( l_i > 1 \),
\[ x_i = z_i \]
if \( l_i = 1 \),
into system
\[ \begin{cases} \dot{\theta} = \omega, \\ \dot{z} = (e(\epsilon) + d(\theta, \epsilon))z + p(\theta, \epsilon) + q(\theta, z, \epsilon), \end{cases} \]
where \( e(\epsilon) = \text{diag}(\lambda_1(\epsilon), \cdots, \lambda_n(\epsilon)) \) with \( \lambda_i(\epsilon) = a_i\epsilon_i\left( -\frac{1}{a_i} \left[ \partial f_i(\theta, 0, 0) \right] \right) \epsilon \)
\( d(\theta, \epsilon) \in \mathcal{C}_{m_1}^\omega(\mathbb{T}_{s_1}^2, \mathbb{R}^m) \) and \( q(\theta, z, \epsilon) \in \mathcal{C}_{m_1}^\omega(\Delta_{s_1, r_1}, \mathbb{R}) \), and the following estimates hold:
\[ \epsilon C e^{\frac{1}{1-n}} < \| \lambda_i(\epsilon) \| < C, \text{ for all } 1 \leq i \leq n, \]
\[ \| d \|_{s_1, m_1} \leq C e^{\frac{2\epsilon_{m_1}}{m_1}}, \| p \|_{s_1, m_1} \leq C \epsilon^2, \quad q(\theta, z, \epsilon) = \mathcal{O}(\epsilon^2)(z \to 0). \]

**Proof of Lemma 4.5.** Making the transformation \( x_i = z_i \) in (30), where \( b_i = \left( -\frac{1}{a_i} \left[ \partial f_i(\theta, 0, 0) \right] \right) \epsilon \) if \( l_i > 1 \) and \( b_i = 0 \) if \( l_i = 1 \), we get the following system
\[ \begin{cases} \dot{\theta} = \omega, \\ \dot{z} = (e(\epsilon) + d(\theta, \epsilon))z + p(\theta, \epsilon) + q(\theta, z, \epsilon), \end{cases} \]
where \( z = (z_1, \cdots, z_n) \), \( e(\epsilon) = \text{diag}(\lambda_1(\epsilon), \cdots, \lambda_n(\epsilon)) \) with \( \lambda_i(\epsilon) = a_i\epsilon_i\left( -\frac{1}{a_i} \left[ \partial f_i(\theta, 0, 0) \right] \right) \epsilon \)
\( p_i(\theta, \epsilon) = h_i(\theta, b_1, \cdots, b_n) + f_i(\theta, b_1, \cdots, b_n, \epsilon) - e \left( -\frac{1}{a_i} \left[ \partial f_i(\theta, 0, 0) \right] \right) \epsilon \), for \( l_i > 1 \),
\[ p_i(\theta, \epsilon) = h_i(\theta, b_1, \cdots, b_n) + f_i(\theta, b_1, \cdots, b_n, \epsilon), \quad \text{for } l_i = 1, \]
\[ d_{ij}(\theta, \epsilon) = \frac{\partial}{\partial x_j} h_i(\theta, b_1, \cdots, b_n) + \frac{\partial}{\partial x_j} f_i(\theta, b_1, \cdots, b_n, \epsilon). \]
In view of the conditions (i) and (ii) in Theorem 4.2, it is easy to see that there exist \( m_1 < m, s_1 < s \) and \( r_1 < r \) such that \( d(\theta, \epsilon) \in C^0_{m_1}(\mathbb{T}^Z_s, \mathbb{R}^n), p(\theta, \epsilon) \in C^0_{m_1}(\mathbb{T}^Z_s, \mathbb{R}^n), q(\theta, z, \epsilon) \in C^0_{m_1}(\Delta_{s_1, r_1}, \mathbb{R}^n) \) and the estimates (45) hold. \( \square \)

4.1.2. A KAM theorem. In the previous subsection, under the conditions in Theorems 4.1 and 4.2, system (30) can be transformed into system of the form

\[
\begin{aligned}
& V = (E(\theta, \kappa))z + F(\theta, \kappa) + G(\theta, z, \kappa),
\end{aligned}
\]

where

\[
(C1) \quad E(\kappa) = (\lambda_1(\kappa), \ldots, \lambda_n(\kappa)) \in \mathbb{R}^n, \quad Q(\theta, \kappa) \in C^0_m(\mathbb{T}^Z_s, \mathbb{R}^n), \quad F(\theta, \kappa) \in C^0_m(\mathbb{T}^Z_s, \mathbb{R}^n),
\]

\[
(G(\theta, z, \kappa) \in C^0_m(\Delta, \mathbb{R}^n), \quad \Phi(\cdot, \cdot, \kappa) \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^n)
\]

\[
(C2) \quad C^{\mu} < |\lambda_i(\kappa)| < C \text{ for all } 1 \leq i \leq n, \quad \|Q(\theta, \kappa)\|_{s, m} \leq \kappa^{\mu + 1}, \quad \|F(\theta, \kappa)\|_{s, m} \leq \kappa^{\mu + 4}, \quad G(\theta, z) = O(z^2) (z \to 0)
\]

In order to prove Theorems 4.1 and 4.2, we will give a KAM theorem for the more general system (46). Concretely, we have the following theorem:

**Theorem 4.6.** Consider system (46). Suppose that the assumptions (C1) and (C2) hold, then there is a sufficiently small \( \kappa^* \) such that for any positive \( \kappa < \kappa^* \) there exists a real analytic almost-periodic transformation

\[
\Phi(\cdot, \cdot, \kappa) : (\theta, z) \in \Delta_{s, r} \to (\vartheta, z) \in \Delta_{s, r}
\]

of the form

\[
z = z^* + V^*(\theta, \kappa), \quad \vartheta = \theta,
\]

where \( V^* \in C^\infty_{m_1}(\Delta, \mathbb{R}^n) \), \( m_1 = m/2, s_1 = s/2, r_1 = r/2 \) are positive constants and

\[
\|V^*(\theta, \kappa)\|_{s, m_1} \leq C\kappa^{\mu + 4},
\]

which transforms system (46) into system

\[
\begin{aligned}
& V = (E(\kappa) + Q_1(\theta, s_0))z + G_1(z, \kappa),
\end{aligned}
\]

where \( Q_1 \in C^\infty_{m_1}(\mathbb{T}^Z_s, \mathbb{R}^n) \), \( G_1 \in C^\infty_{m_1}(\Delta_{s_1, r_1}, \mathbb{R}^n) \) and \( G_1 = O((z^*)^2)(z^* \to 0) \).

The proof of Theorem 4.6 can divide into the several steps.

1. **KAM step** For \( m, r, s, \kappa > 0 \), we set \( m_0 = m, r_0 = r, s_0 = s \) and \( \kappa_0 = \kappa \).

Next, we define the sequences \( (\kappa_\nu)_{\nu \geq 0}, (\sigma_\nu)_{\nu \geq 0}, (s_\nu)_{\nu \geq 0}, (m_\nu)_{\nu \geq 0} \) and \( (r_\nu)_{\nu \geq 0} \) in the following manner:

\[
\begin{aligned}
& \kappa_{\nu+1} = \kappa_\nu^{\frac{\mu + 2}{\nu + 2}}, \quad \sigma_{\nu+1} = \sigma_\nu/2, \\
& s_{\nu+1} = s_\nu - \sigma_\nu, \quad r_{\nu+1} = r_\nu - \sigma_\nu, \quad m_{\nu+1} = m_\nu - \sigma_\nu.
\end{aligned}
\]

Choosing \( \sigma_0 < \min\{\frac{\nu_0}{2}, \frac{s_0}{2}\} \), we can achieve that \( m_\nu \to m_\nu^* > 0, s_\nu \to s_\nu^* > 0, r_\nu \to r_\nu^* > 0 \).

Now we can state the the following step Lemma.

**Lemma 4.7.** Let us consider the almost-periodic system

\[
\begin{aligned}
& \dot{\theta} = \omega, \\
& \dot{z} = (E(\kappa_0) + Q_l(\theta, s_0))z + Q_j^l(\theta, \kappa_0)z + F_l(\theta, \kappa_0) + G_l(\theta, z, \kappa_0), \quad (E \nu)
\end{aligned}
\]

where \( l = 0, 1, \ldots, \nu \), \( E(\kappa_0) = (\lambda_1(\kappa_0), \ldots, \lambda_n(\kappa_0))^T, \ c_0^\mu < |\lambda_i(\kappa_0)| < C \) for all \( 1 \leq i \leq n \). Assume that
The functions \( Q_l(\theta, \kappa_0) \), \( Q'_l(\theta, \kappa_0) \) \( \in C^\infty_m(\mathbb{T}^*_s, gl(n, \mathbb{R})) \), \( F_l(\theta, \kappa_0) \) \( \in C^\infty_m(\mathbb{T}^*_s, \mathbb{R}^n) \) and satisfy
\[
\|Q_l(\theta, \kappa_0)\|_{s,m} \leq C_{\mu_0}^{\mu_1}, \|Q'_l(\theta, \kappa_0)\|_{s,m} \leq C_{\mu_1}^{\mu_2}, \|F_l(\theta, \kappa_0)\|_{s,m} \leq C_{\mu_2}^{\mu_3};
\]
(1.2) The function \( G_l(\theta, z, \kappa_0) \) \( \in C^\infty_{m_1}(\Delta_{s_1, r_1}, \mathbb{R}^n) \) and
\[
G_l(\theta, z, \kappa_0) = O(z^2)(z \to 0).
\]
Then there exists a change of variables
\[
\Phi_{\nu} : \Delta_{s_{\nu+1}, r_{\nu+1}} \to \mathbb{D}_{s_{\nu}, r_{\nu}}
\]
of the form \( z = z^\nu + V_\nu(\theta, \kappa_0) \) where \( z \) are ‘old’ variable and \( z^\nu \) are ‘new’ variable,\n\( V_\nu = O(\kappa_\nu^{\nu+4}) \) \( \in C^\infty_m(\mathbb{T}^*_s, \mathbb{R}^n) \), such that system \( (Eq)_\nu \) is transformed into system \( (Eq)_{\nu+1} \) and conditions (1.1) and (1.2) are fulfilled by replacing \( l \) by \( \nu + 1 \) and replacing \( z \) by \( z^\nu \), respectively.

Proof of Lemma 4.7. In this proof, we hide parameter \( \kappa_0 \) for simplicity if there is no confusion. One can apply the transformation \( z = z^\nu + V_\nu(\theta) \) to system
\[
\dot{z} = (E + Q_\nu(\theta))z + Q'_\nu(\theta)z + F_\nu(\theta) + G_\nu(\theta, z).
\]
(47)
Once the homological equation
\[
\partial_z V_\nu(\theta) = (E + Q_\nu(\theta))V_\nu(\theta) + F_\nu(\theta)
\]
is solved, then system (47) becomes
\[
\dot{z}^\nu = (E + Q_{\nu+1}(\theta))z^\nu + Q'_{\nu+1}(\theta)z^\nu + F_{\nu+1}(\theta) + G_{\nu+1}(\theta, z^\nu),
\]
where
\[
Q_{\nu+1}(\theta) = Q_\nu(\theta) + Q'_\nu(\theta),
\]
\[
Q'_{\nu+1}(\theta) = \frac{\partial G_\nu(\theta, V_\nu(\theta))}{\partial z},
\]
\[
F_{\nu+1}(\theta) = G_\nu(\theta, V_\nu(\theta)) + Q'_\nu(\theta)V_\nu(\theta),
\]
\[
G_{\nu+1}(\theta, z^\nu) = G_\nu(\theta, V_\nu(\theta) + z^\nu) - G_\nu(\theta, V_\nu(\theta)) - \frac{\partial G_\nu(\theta, V_\nu(\theta))}{\partial z} z^\nu.
\]
Now, we solve equation (48). For each
\[
V(\theta) = \sum_{A \in \mathcal{S}} \sum_{\text{supp} \subseteq A} V_{A,k} e^{i(k, \omega)} \in C^\infty_m(\mathbb{T}^*_s, \mathbb{R}^n),
\]
we define an operator \( T : C^\infty_m(\mathbb{T}^*_s, \mathbb{R}^n) \to C^\infty_m(\mathbb{T}^*_s, \mathbb{R}^n) \) by
\[
T(V(\theta)) := \partial_z V(\theta) - EV(\theta) = \sum_{A \in \mathcal{S}} \sum_{\text{supp} \subseteq A} (i(k, \omega)I - E)V_{A,k} e^{i(k, \omega)},
\]
(49)
where \( I \) is identity matrix, and an operator \( W : C^\infty_m(\mathbb{T}^*_s, \mathbb{R}^n) \to C^\infty_m(\mathbb{T}^*_s, \mathbb{R}^n) \) by
\[
W(V(\theta)) = -Q_\nu(\theta)V(\theta).
\]
In fact, by (49) and \( E \) is a real matrix, we can get
\[
(i(-k, \omega)I - E)V_{A,-k} = \overline{(i(k, \omega)I - E)V_{A,k}}
\]
(50)
(here \( \overline{\cdot} \) denotes the complex conjugate of \( \cdot \)), and (50) implies \( T(V(\theta)) \) is a real analytic function. Then the homological equation (48) can be written as
\[
(T + W)V_\nu(\theta) = F_\nu(\theta),
\]
where
\[ Q_\nu(\theta) = \sum_{A \in S} \sum_{\suppk \subseteq A} Q^\nu_{A,k} e^{i(k,\omega)}, \]
\[ F_\nu(\theta) = \sum_{A \in S} \sum_{\suppk \subseteq A} F^\nu_{A,k} e^{i(k,\omega)} \in C^\omega_m(T^Z_{s_\nu}, \mathbb{R}). \]

By the definition of the operator \( T \), we can get \( T^{-1} : C^\omega_m(T^Z_{s_\nu}, \mathbb{R}) \to C^\omega_m(T^Z_{s_\nu}, \mathbb{R}) \), and
\[ T^{-1}(V(\theta)) = \sum_{A \in S} \sum_{\suppk \subseteq A} (i(k,\omega)I - E)^{-1} V_{A,k} e^{i(k,\omega)}, \]
where \( (i(k,\omega)I - E)^{-1} = \text{diag}((i(k,\omega) - \lambda_1)^{-1}, \cdots, (i(k,\omega) - \lambda_n)^{-1}) \). In view of \( |i(k,\omega) - \lambda_i| \geq c\kappa_0^\mu \) for all \( 1 \leq i \leq n \), one can check that
\[ \|T^{-1}\| = \sup_{0 \neq V(\theta) \in C^\omega_m(T^Z_{s_\nu}, \mathbb{R})} \frac{\|T^{-1}(V(\theta))\|_{s_\nu,m_\nu}}{\|V(\theta)\|_{s_\nu,m_\nu}} \leq C\kappa_0^{-\mu} \]
and
\[ \|W\| = \sup_{0 \neq V(\theta) \in C^\omega_m(T^Z_{s_\nu}, \mathbb{R})} \frac{\|W(V(\theta))\|_{s_\nu,m_\nu}}{\|V(\theta)\|_{s_\nu,m_\nu}} \leq C\kappa_0^{-\mu+1}. \]

Thus \( \|T^{-1}W\| \leq \|T^{-1}\||W\| \leq C\kappa_0 \), which implies the operator \( T + W \) is invertible, \( (T + W)^{-1} : C^\omega_m(T^Z_{s_\nu}, \mathbb{R}) \to C^\omega_m(T^Z_{s_\nu}, \mathbb{R}) \) and
\[ \|(T + W)^{-1}\| \leq \|(1 + T^{-1}W)^{-1}\||T^{-1}\| \leq C\kappa_0^{-\mu}. \]

Therefore, the equation (48) is solvable and \( V_\nu(\theta) = (T + W)^{-1}(F_\nu(\theta)) \). We have the estimation
\[ \|V_\nu(\theta)\|_{s_\nu,m_\nu} \leq C\kappa_0^{-\mu}\|F_\nu(\theta)\|_{s_\nu,m_\nu} \leq C\kappa_0^{-\mu} \kappa_0^{2\mu+4} \leq \kappa_0^{4\mu+4}. \]

Thus
\[ \|F_{\nu+1}(\theta)\|_{s_{\nu+1},m_{\nu+1}} \leq \|G_\nu(\theta, V_\nu(\theta))\|_{s_{\nu+1},m_{\nu+1}} + \|Q_\nu(\theta) V_\nu(\theta)\|_{s_{\nu+1},m_{\nu+1}} \leq \kappa_0^{2\mu+4}, \]
\[ \|Q_\nu^{\nu+1}(\theta)\|_{s_{\nu+1},m_{\nu+1}} \leq \frac{\partial G_\nu(\theta, V_\nu(\theta))}{\partial z} \|_{s_{\nu+1},m_{\nu+1}} \leq \kappa_0^{\mu+1}, \]
\[ \|Q_\nu^{\nu+1}(\theta)\|_{s_{\nu+1},m_{\nu+1}} \leq \|Q_0(\theta)\|_{s_0,m_0} + \sum_{l=0}^{\infty} \|Q^{(l)}(\theta)\|_{s_{l+1},m_{l+1}} \leq C\kappa_0^{\mu+1}. \]

One can check that \( r_{\nu+1} + \|V_\nu(\theta)\|_{s_{\nu+1},m_{\nu+1}} \leq r_{\nu} \), which implies
\[ \Phi_\nu(\Delta_{s_{\nu+1},r_{\nu+1}}) \subseteq \Delta_{s_{\nu},r_{\nu}}. \]

By the definition of \( G_{\nu+1}(\theta, z^+) \), one can also check
\[ \|\frac{\partial^2 G_{\nu+1}(\theta, z^+)}{\partial z^2}\|_{s_{\nu+1},r_{\nu+1},m_{\nu+1}} \leq \|\frac{\partial^2 G_\nu(\theta, z)}{\partial z^2}\|_{s_{\nu},r_{\nu},m_{\nu}}. \]
which implies that for each $0 < \tilde{r} \leq r_{\nu+1}$, 
\[ \|G_{\nu+1}(\theta, z^+)\|_{s_{\nu+1}, \tilde{r}, m_{\nu+1}} \leq C\tilde{r}^2, \]
for each $\nu$.

2. Iteration

Let us take $E_0 = E$, $Q_0 = Q$, $P_0 = 0$, $F_0 = F$, $G_0 = G$. So, it is easy to check that system (46) satisfies all hypotheses of Lemma 4.7 with $l = 0$. By induction, we can prove that for any $\nu \geq 0$ there is a sequence $\Phi_\nu$ of transformations such that $\Phi_\nu(\Delta_{s_{\nu+1}, r_{\nu+1}}) \subset \Delta_{s_{\nu}, r_{\nu}}$

\[ (Eq)_\nu \circ \Phi_\nu = (Eq)_{\nu+1}. \]

Let

\[ \Phi_\nu := \Phi_0 \circ \Phi_1 \circ \ldots \circ \Phi_\nu : \Delta_{s_{\nu+1}, r_{\nu+1}} \to \Delta_{s_0, r_0}. \]

Then, after the transformation $\Phi_\nu$, system (46) is changed to

\[ \dot{z} = (E + Q_\nu(\theta, \kappa_0))z + Q_\nu(\theta, \kappa_0)z + F_\nu(\theta, \kappa_0) + G_\nu(\theta, z, \kappa_0). \]

By the inductive assumptions of KAM iteration, for sufficiently small $\kappa_0$, we have

\[ \|Q_\nu\|_{s_\nu} \leq \kappa_0^{\mu+1}, \quad \|Q_\nu\|_{s_\nu} \leq \kappa_0^{\mu+1}, \quad \|F_\nu\|_{s_\nu} \leq \kappa_0^{2\mu+4}, \quad G_\nu = O(z^2)(z \to 0). \]

3. Convergence of KAM iteration

For sufficiently small $\kappa_0 > 0$, it follows easily that

\[ \|\Phi_\nu - \Phi_{\nu-1}\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}} \leq \kappa_0^{\mu+4}. \]

The same inequality holds for $\nu = 0$ if we define $\Phi^{-1} := id$. In view of $s_\nu \to s_* > 0$, $r_\nu \to r_* > 0$, $m_\nu \to m_* > 0$, as $\nu \to \infty$, the mapping $\Phi_\nu$:

\[ \Phi_\nu = \Phi_0 + \sum_{i=0}^\nu (\Phi_i - \Phi^{i-1}), \quad \Phi^{-1} := id \]

converges uniformly on

\[ \bigcap_{\nu \geq 0} \Delta_{s_{\nu+1}, r_{\nu+1}} = \Delta_{s_*, r_*}, \]

to a mapping $\Phi$, which is analytic on $\Delta_{s_*, r_*}$.

It is easy to see that $(Eq)_0$ is changed to, by

\[ \Phi = \lim_{\nu \to \infty} \Phi_0 \circ \Phi_1 \circ \ldots \circ \Phi_\nu : \begin{cases} \theta = \theta, \\ z = z^* + V^*(\theta) \end{cases} \]

with

\[ V^*(\theta) = O_{s_*, r_*}(\kappa_0^{\mu+4}), \]

\[ (Eq)_0 \circ \Phi : \begin{cases} \dot{\theta} = \omega, \\ \dot{z} = (E + Q_\ast(\theta))z + G_\ast(\theta, z^*), \end{cases} \]

where $Q_\ast \in C_{m_*}^\omega(T_\ast, gl(n, \mathbb{R}))$, $G_\ast \in C_{m_*}^\omega(\Delta_{s_*, r_*}, \mathbb{R}^n)$ and $G_\ast = O(z^2)(z^* \to 0)$. This completes the proof of Theorem 4.6.

As an application of Theorem 4.6, Theorems 4.1 and 4.2 are proved immediately. In fact, we can choose a positive integer $\hat{p}$ large enough such that $l_1 - 1 + 1/\hat{p} < l_1$. Note that $0 < \eta < 1$, the estimate (34) in Lemma 4.4 can write as

\[ c\eta^{l_1-1} < |e(\eta)| < C\eta^{l_1-1}, \|d\|_{s_1} \leq C\eta^{l_1-1+1/\hat{p}}, \|p\|_{s_1} \leq C\eta^{2(l_1-1)+1/\hat{p}}, q = O(z^2). \]
Taking $\mu = \hat{\rho}(l_1 - 1)$ and $\eta = \kappa \hat{\rho}$, the above estimates become (C2). Namely, system (33) satisfies all conditions of Theorem 4.6. Analogously, we can check that system (44) also satisfies all conditions of Theorem 4.6.

4.2. The proof of Theorem 4.3. In this subsection, we intend to give the proof of theorems 4.3 which will divide the following three parts.

4.2.1. The parameterization of system (30). The proof of Theorem 4.3 is based on a normal form theorem for an almost-periodic system with parameters. So we first introduce some parameters and change system (30) to a parameterized system. Let \( \xi = \Pi := [-1, 1] \) and \( y = (y_1, \ldots, y_n)^T \). Made the change of variables

\[
\begin{align*}
\dot{x}_i &= y_i + b_i, \quad i = 1, \ldots, n - 1, \\
\dot{x}_n &= y_n + \epsilon \frac{\partial}{\partial \xi} \xi,
\end{align*}
\]

where \( b_i = (-\frac{1}{n_i} [\hat{\rho} f_i (\theta, \phi, \omega, 0)] \epsilon) \) if \( i > 1 \) and \( b_i = 0 \) if \( i = 1 \), system (30) becomes

\[
\begin{align*}
\dot{\theta} &= \omega, \\
\dot{y}_i &= a_i b_i^{l_i} + a_i l_i b_i^{l_i - 1} y_i + h_i (\theta, y_1, \ldots, y_{n-1}, y_n, \epsilon \frac{\partial}{\partial \xi} \xi) \\
&\quad + f_i (\theta, y_1, \ldots, y_{n-1}, y_n, \epsilon \frac{\partial}{\partial \xi} \xi, \epsilon) + O(y_i^2)(y_i \to 0), \\
\dot{y}_n &= \epsilon \xi^{2l+1} + (2l+1) \epsilon \frac{\partial}{\partial \xi} \xi^{2l} y_n + h_n (\theta, y_1, \ldots, y_{n-1}, y_n, \epsilon \frac{\partial}{\partial \xi} \xi) \\
&\quad + f_2 (\theta, y_1, y_2, \ldots, y_{n-1}, y_n, \epsilon \frac{\partial}{\partial \xi} \xi) + O(y_n^2)(y_n \to 0).
\end{align*}
\]

Used a Taylor formula, system (51) can be rewritten as follows

\[
\begin{align*}
\dot{\theta} &= \omega, \\
\dot{y}_i &= a_i b_i^{l_i} y_i + Q_{i1} (\theta, \xi, \epsilon) y_i + Q_{i2} (\theta, \xi, \epsilon) y_2 + \ldots + Q_{in} (\theta, \xi, \epsilon) y_n \\
&\quad + F_i (\theta, \xi, \epsilon) + G_i (\theta, \xi, \epsilon), \quad i = 1, 2, \ldots, n - 1, \\
\dot{y}_n &= \epsilon \xi^{2l+1} + (2l+1) \epsilon \frac{\partial}{\partial \xi} \xi^{2l} y_n + \sum_{j=1}^{n} Q_{nj}(\theta, \xi, \epsilon) y_j + F_n (\theta, \xi, \epsilon) \\
&\quad + G_n (\theta, \xi, \epsilon),
\end{align*}
\]

where

\[
Q_{ij}(\theta, \xi, \epsilon) = \frac{\partial}{\partial x_j} h_i (\theta, b_1, \ldots, b_{n-1}, \epsilon \frac{\partial}{\partial \xi} \xi) + \frac{\partial}{\partial x_j} f_i (\theta, b_1, \ldots, b_{n-1}, \epsilon \frac{\partial}{\partial \xi} \xi, \epsilon), \\
F_i (\theta, \xi, \epsilon) = h_i (\theta, b_1, \ldots, b_{n-1}, \epsilon \frac{\partial}{\partial \xi} \xi) + f_i (\theta, b_1, \ldots, b_{n-1}, \epsilon \frac{\partial}{\partial \xi} \xi, \epsilon) - a_i b_i^{l_i}, \\
F_n (\theta, \xi, \epsilon) = h_n (\theta, b_1, \ldots, b_{n-1}, \epsilon \frac{\partial}{\partial \xi} \xi) + f_n (\theta, b_1, \ldots, b_{n-1}, \epsilon \frac{\partial}{\partial \xi} \xi), \\
G_j (\theta, y, \xi, \epsilon) = O(y^2)(y \to 0), \quad j = 1, 2, \ldots, n.
\]

Let

\[
\Pi_\rho = \{ \xi \in \mathbb{C} \mid \text{dist}(\xi, \Pi) \leq \rho \}
\]

and

\[
B(\Gamma, \rho) = \{ \lambda \in \mathbb{C} \mid \text{dist}(\lambda, \Gamma) \leq \rho \}
\]

be the complex \( \rho \)-neighborhood of \( \Pi \) and \( \Gamma \) in the complex space \( \mathbb{C} \), respectively. Note that conditions (ii), (iii) and (iv) hold in Theorem 4.3, it is easy to see that

\[
F_i = O(\epsilon^2) \quad \text{and} \quad Q_{i,j} = O(\epsilon), \quad i, j = 1, 2, \ldots, n.
\]
Now we introduce an artificial external parameter and consider the following system

\[
\begin{align*}
\dot{\theta} &= \omega, \\
\dot{y}_i &= a_i l_i b_i^{-1} y_i + Q_i(y_i, \xi, \lambda) + Q_i(\theta, \xi, \epsilon) y_i + \ldots + Q_{in}(\theta, \xi, \epsilon) y_n + F_i(\theta, \xi, \epsilon) + G_i(\theta, y, \xi, \epsilon), \\
\dot{y}_n &= N(\xi, \lambda, \epsilon) + E(\xi, \epsilon) y_n + \sum_{j=1}^{n} Q_{nj}(\theta, \xi, \epsilon) y_j + F_n(\theta, \xi, \epsilon) + G_n(\theta, y, \xi, \epsilon),
\end{align*}
\]

where

\[
N(\xi, \lambda, \epsilon) = -\lambda + \epsilon \xi^{l+1}, \\ E(\xi, \epsilon) = (2l + 1) \epsilon \xi^{2l}
\]

and \(\lambda \in \mathbb{C}\) is an artificial external parameter. System (52) corresponds to system (53) with \(\lambda = 0\). Let

\[
\beta = \max_{(\xi, \zeta) \in \Pi} |N(\xi, 0, \epsilon) - N(\zeta, 0, \epsilon)|
\]

and

\[
M = \Pi_{\rho} \times B(0, 2\beta + 1).
\]

For the sake of convenience, we rewrite system (53) as follows

\[
\begin{align*}
\dot{\theta} &= \omega, \\
\dot{y} &= N(\xi, \lambda, \epsilon) + (A(\xi, \lambda, \epsilon) + Q(\theta, \xi, \epsilon)) y + F(\theta, \xi, \epsilon) + G(\theta, y, \xi, \epsilon),
\end{align*}
\]

where

\[
\begin{align*}
y &= (y_1, \ldots, y_n)^T, \\
N &= (0_{n-1}, N)^T, \\
F &= (F_1, \ldots, F_n)^T, \\
G &= (G_1, \ldots, G_n)^T, \\
Q &= (Q_{ij})_{1 \leq i, j \leq n}, \\
A &= \text{diag} \left( a_1 l_1 b_1^{-1}, \ldots, a_n l_n b_n^{-1}, 1 ight)
\end{align*}
\]

and the following hypotheses hold

(H1) \(Q(\theta, \xi, \epsilon) \in C_{m}^{\infty}(\mathbb{T}^2 \times \Pi_{\rho} \times gl(n, \mathbb{R}))\), \(F(\theta, \xi, \epsilon) \in C_{m}^{\infty}(\mathbb{T}^2 \times \Pi_{\rho}, \mathbb{R}^n)\) and \(G(\theta, y, \xi, \lambda, \epsilon) \in C_{m}^{\infty}(\Delta_{s,r} \times M, \mathbb{R}^n)\). These functions all are analytic in \(\epsilon\);

(H2) \(Q(\theta, \xi, \epsilon) = \mathcal{O}(\epsilon), \ F(\theta, \xi, \epsilon) = \mathcal{O}(\epsilon^2), \ G(\theta, y, \xi, \lambda, \epsilon) = \mathcal{O}(y^2)(y \to 0)\).

Since \(b_i = 0\) as \(l_i = 1\), the place of \(a_i l_i b_i^{-1}\) actually is \(a_i\) if \(l_i = 1\). This is a simple case. Thus, in \(A\) and what follows, we assume \(l_i > 1\) for all \(1 \leq i \leq n\). We have the following KAM Theorem.

**Theorem 4.8.** Consider the real analytic almost-periodic system (54) in the domain \(\Delta_{s,r} \times M\). Suppose the frequency vector \(\omega\) satisfies the non-resonant condition (10) and the hypotheses (H1) and (H2) hold. Then for sufficiently small positive constant \(\epsilon\) there is a \(C^\infty\)-smooth curve

\[
\Gamma_\ast: \lambda = \lambda_\ast(\xi), \quad \xi \in \Pi,
\]

which is determined by the equation

\[
-\lambda + \epsilon \xi^{2l+1} + \tilde{N}_\ast(\xi, \lambda) = 0,
\]

where \(\tilde{N}_\ast(\xi, \lambda)\) is \(C^\infty\)-smooth function on \(M\) with

\[
|N_\ast(\xi, \lambda)| \leq C \epsilon^2, \quad \left| \frac{\partial}{\partial \xi} \tilde{N}_\ast(\xi, \lambda) \right| + \left| \frac{\partial}{\partial \lambda} \tilde{N}_\ast(\xi, \lambda) \right| \leq \frac{1}{2}.
\]

Moreover, there exists a parameterized family of almost-periodic transformation

\[
\Phi_\ast(\cdot, \cdot, \xi, \lambda): \Delta_{s,r} \times \mathbb{T}^2 \to \Delta_{s,r}, \quad (\xi, \lambda) \in \Gamma_\ast,
\]
where $\Phi^*$ is real analytic in $(\theta,x)$ on $\Delta_{\hat{z},\hat{z}}$ and $C^\infty$-smooth in $(\xi,\lambda)$ on $\Gamma_s$, such that for each $(\xi,\lambda) \in \Gamma_s$, the transformation $\Phi^*$ transforms system (54) into

$$
\begin{aligned}
\dot{\theta} &= \omega, \\
\dot{z} &= A_s(\xi,\lambda_s(\xi))z + G_s(\theta, z, \xi, \lambda_s(\xi)),
\end{aligned}
$$

where $G_s = O(z^2)(y \to 0)$. Hence, system (54) has an almost-periodic response torus $\Phi^*(\mathbb{T}^2, \{0_s\}, \xi, \lambda)$.

4.2.2. Iterative lemma. We will give an iterative lemma to prove Theorem 4.3. Let $\Delta$ be an approximation function as defined in (11) and let $\Lambda(t) = t^2\Delta^2(t)$. For $m, s, r, \epsilon > 0$, we set $m_0 = m$, $s_0 = s$, $r_0 = r$ and $\rho_0 = \epsilon_0$. Let $\Lambda_0 \geq \Lambda(1) = \Delta(1)$. We choose $0 < a, b < 1$ and $0 < \delta \leq 1/2$ such that

$$q^2 = 2(2(1 - a) + b) \leq \delta^2.$$  

Then, let $\Lambda_0 \geq (\delta^{-1}q^2t)^{-1}$ and $\tau_0 := \Lambda^{-1}(\Lambda_0)$ be large enough such that

$$\frac{\log(1 - a)}{\log(\delta^{-1}q^2)} \int_{\tau_0}^{\infty} \frac{\ln \Lambda(t)}{t^2} dt < \min\left\{\frac{r_0}{2}, \frac{s_0}{2}, \frac{m_0}{2}\right\}.$$  

Finally, we define $\epsilon_0 = \epsilon$ small sufficiently. Next, we define the sequences $(\epsilon_{\nu})_{\nu \geq 0}$, $(A_{\nu})_{\nu \geq 0}$, $(\tau_{\nu})_{\nu \geq 0}$, $(\sigma_{\nu})_{\nu \geq 0}$, $(s_{\nu})_{\nu \geq 0}$, $(r_{\nu})_{\nu \geq 0}$, $(d_{\nu})_{\nu \geq 0}$, $(m_{\nu})_{\nu \geq 0}$ and $(\rho_{\nu})_{\nu \geq 0}$ in the following manner:

$$
\begin{aligned}
\epsilon_{\nu} &= \epsilon_0 q^{\nu}, \\
A_{\nu} &= \left(\delta/q^2\right)^{1/\nu} \Lambda_0, \\
1 - a &= e^{-\tau_\nu \sigma_\nu}, \\
r_{\nu+1} &= r_{\nu} - \sigma_{\nu}, \\
m_{\nu+1} &= m_{\nu} - \sigma_{\nu}, \\
A_{\nu} &= \frac{d_{\nu}}{\nu} q^{-1/\nu}.
\end{aligned}
$$

Similar to the proof of Theorem 3.1, we will see that

$$\sum_{\nu \geq 0} \sigma_{\nu} = \sum_{\nu \geq 0} \frac{\log(1 - a)}{\tau_{\nu}} \leq \frac{\log(1 - a)}{\log(\delta^{-1}q^2)} \int_{\tau_0}^{\infty} \frac{\ln \Lambda(t)}{t^2} dt.$$  

Hence, we can achieve that $\sum_{\nu \geq 0} \sigma_{\nu} \leq \min\left\{s_0/2, r_0/2, m_0/2\right\}$. Thus $m_{\nu} \to m_s \geq s_0/2$, $\sigma_{\nu} \to \sigma_s \geq s_0/2$ and $r_{\nu} \to r_s \geq r_0/2$. We suppose that after $\nu$ steps, the transformed system defined in the domain $\Delta_{s_{\nu}, r_{\nu}} \times M_{\nu}$ is of the form

$$
\dot{y} = N_{\nu}(\xi, \lambda) + (A_{\nu}(\xi, \lambda) + Q_{\nu}(\theta, \xi, \lambda))g + F_{\nu}(\theta, \xi, \lambda) + G_{\nu}(\theta, y, \xi, \lambda), \quad \dot{\theta} = \omega, 
$$

(\nu, 1) \quad \|F_{\nu}\|_{s_{\nu}, m_{\nu}, M_{\nu}} \leq C_{\nu}^2, \quad \|Q_{\nu}\|_{s_{\nu}, m_{\nu}, M_{\nu}} \leq C_{\nu}, \text{ and } G_{\nu} = O(y^2)(y \to 0);$$

\textbf{Lemma 4.9.} Let us consider a family of the real analytic almost-periodic systems (\nu, 1) in the domain $\Delta_{s_{\nu}, r_{\nu}} \times M_{\nu}$. Suppose that the frequency vector $\omega$ satisfies non-resonant condition (10). Furthermore, we assume that

$$\text{(nu, 1): } \|F_{\nu}\|_{s_{\nu}, m_{\nu}, M_{\nu}} \leq C_{\nu}^2, \quad \|Q_{\nu}\|_{s_{\nu}, m_{\nu}, M_{\nu}} \leq C_{\nu}, \text{ and } G_{\nu} = O(y^2)(y \to 0);$$

We have the following lemma.
(ν.2) There exists a positive constant c such that \(|D^{i}_{\nu}| \geq c^{|i-1|}\), \(|D^{i}_{\nu} - D^{j}_{\nu}| \geq c^{|i-j|}\)
and \(|D^{i}_{\nu} - E_{\nu}| \geq c^{|i-1|}\), for \(\forall (\xi, \lambda) \in M_{\nu}\) and \(1 \leq i \neq j \leq n - 1\);
(ν.3) \(\tilde{N}_{\nu}\) satisfies
\[
|\frac{\partial}{\partial \xi} \tilde{N}_{\nu}(\xi, \lambda)| + |\frac{\partial}{\partial \lambda} \tilde{N}_{\nu}(\xi, \lambda)| \leq \frac{1}{2}
\]
for all \((\xi, \lambda) \in M_{\nu}\) and the equation
\[N_{\nu}(\xi, \lambda, \epsilon_{0}) = -\lambda + \epsilon \xi^{2l+1} + \tilde{N}_{\nu}(\xi, \lambda) = 0\]
defines implicitly an analytic curve
\[\Gamma_{\nu} : \lambda_{\nu} = \lambda_{\nu}(\xi), \xi \in \Pi_{\nu} \to \lambda_{\nu}(\xi) \in B(0, 2\beta + 1),\]
such that \(\Gamma_{\nu} = \{(\xi, \lambda_{\nu}(\xi)) | \xi \in \Pi_{\nu} \} \subset M_{\nu}\). We have
\[
\mathcal{U}(\Gamma_{\nu}, d_{\nu}) = \{(\xi, \lambda' \in \Pi_{\nu} \times \mathbb{C} | \xi \in \Pi_{\nu}, (\xi, \lambda' \in \Gamma_{\nu}, |\lambda' - \lambda_{\nu}| \leq d_{\nu}) \} \subset M_{\nu}.
\]
Then for sufficiently small \(\epsilon_{0} > 0\) there exist a set
\[M_{\nu+1} = \{\{\xi, \lambda' \in \Pi_{\nu+1} \times \mathbb{C} | \xi \in \Pi_{\nu+1}, (\xi, \lambda') \in \Gamma_{\nu}, |\lambda' - \lambda_{\nu}| \leq \frac{d_{\nu}}{2}\} \subset M_{\nu}
\]
and a transformation
\[\Phi_{\nu}(\cdot, \cdot; \xi, \lambda) : \Delta_{s_{\nu+1}} \times M_{\nu+1} \to \Delta_{s_{\nu}} \times M_{\nu}\]
with the following estimates:
\[
\|\Phi_{\nu} - id\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq C\epsilon_{0}^{\frac{1}{2}} \sigma_{\nu} \delta_{\nu} q^{\frac{1}{2} + \nu},
\]
\[
\|D \Phi_{\nu} - I\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq C\epsilon_{0}^{\frac{1}{2}} \sigma_{\nu} \delta_{\nu} q^{\frac{1}{2} + \nu},
\]
which changes system \((E_{\nu})_{\nu}\) to \((E_{\nu+1})_{\nu+1}\):
\[
\dot{y}_{\nu+1} = N_{\nu+1}(\xi, \lambda, \epsilon_{0}) + (A_{\nu+1}(\xi, \lambda, \epsilon_{0}) + Q_{\nu+1}(\theta, \xi, \lambda, \epsilon_{0}))y_{\nu+1} + F_{\nu+1}(\theta, \xi, \lambda, \epsilon_{0})
\]
\[+ G_{\nu+1}(\theta, y_{\nu+1}, \xi, \lambda, \epsilon_{0}),
\]
where
\[
A_{\nu+1} = \begin{pmatrix} D_{\nu+1} & 0_{n-1}^{T} - E_{\nu+1} \\ 0_{n-1} & 1_{n-1}^{T} \end{pmatrix} \quad \text{and} \quad N_{\nu+1} = \begin{pmatrix} 0_{n-1}^{T} \\ N_{\nu+1}^{T} \end{pmatrix}
\]
with \(A_{\nu+1} = A_{\nu} + \Delta A_{\nu}, N_{\nu+1} = N_{\nu} + \Delta N_{\nu}, \Delta A_{\nu} = \begin{pmatrix} \Delta D_{\nu} \\ 0_{n-1}^{T} - E_{\nu}^{T} \end{pmatrix} \quad \text{and} \quad \Delta N_{\nu} = \begin{pmatrix} 0_{n-1}^{T} - E_{\nu}^{T} \\ N_{\nu+1}^{T} \end{pmatrix}.
\]
Furthermore, the following conclusions hold:
(ν + 1.1) \(D_{\nu+1} + E_{\nu+1}\) satisfy
\[
|D^{i}_{\nu+1}| \geq c_{\nu+1}^{\frac{1}{1-i}}, \quad |D^{i}_{\nu} - D^{j}_{\nu}| \geq c_{\nu+1}^{\frac{1}{1-i}} \quad \text{and} \quad |D^{i}_{\nu+1} - E_{\nu+1}| \geq c_{\nu+1}^{\frac{1}{1-i}},
\]
\(1 \leq i \neq j \leq n - 1\), for all \((\xi, \lambda) \in M_{\nu}, (\nu + 1.2) \Delta N_{\nu}\) satisfies
\[
|\frac{\partial}{\partial \xi} \Delta N_{\nu}(\xi, \lambda)| + |\frac{\partial}{\partial \lambda} \Delta N_{\nu}(\xi, \lambda)| \leq C\epsilon_{0}^{\frac{1}{2}} q^{\frac{1}{2} + \nu}
\]
for all \((\xi, \lambda) \in M_{\nu+1}, \text{ and the equation}
\[N_{\nu+1}(\xi, \lambda) = -\lambda + \epsilon \xi^{2l+1} + \tilde{N}_{\nu}(\xi, \lambda) + \Delta N_{\nu}(\xi, \lambda) = 0\]
defines implicitly an analytic curve
\[ \Gamma_{\nu+1} : \lambda_{\nu+1} = \lambda_{\nu+1}(\xi) : \xi \in \Pi_{\rho_{\nu+1}} \to \lambda_{\nu+1}(\xi) \in \mathcal{B}(0, 2\beta + 1), \]
satisfying
\[ |\lambda_{\nu+1}(\xi) - \lambda_{\nu}(\xi)|_{\Pi_{\rho_{\nu+1}}} \leq \epsilon_{\nu} \leq \frac{d_{\nu}}{4} \]
and
\[ \Gamma_{\nu+1} : = \{(\xi, \lambda_{\nu+1}(\xi)) \mid \xi \in \Pi_{\rho_{\nu+1}} \} \subset M_{\nu+1}. \]
If
\[ d_{\nu+1} \leq \frac{d_{\nu}}{4}, \]
then \( U(\Gamma_{\nu+1}, d_{\nu+1}) \subset M_{\nu+1}. \)
\( (\nu + 1.3) \|F_{\nu+1}\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq C\epsilon_{\nu+1}^2, \|Q_{\nu+1}\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq C\epsilon_{\nu+1} \) and
\[ G_{\nu+1} = \mathcal{O}(y^2)(y^+ \to 0). \]

Proof of Lemma 4.9. We divide the proof of Lemma 4.9 into the following several parts, and in the following discussions we do not write parameters (\( \xi, \lambda \)) and \( \epsilon_0 \) explicitly if there is no confusion for simplicity.

A. Truncation. At \( \nu \)-th KAM step, we have \((1 - a) = e^{-\tau_{\nu}}\). By (16) and (17), \( P(\theta) \in C_{m_{\nu}}(\mathbb{T}_{s_{\nu}}, \mathbb{C}) \) has the truncation \( P(\theta) = \tilde{P}(\theta) + \tilde{\tilde{P}}(\theta) \) with estimate
\[ \|\tilde{P}\|_{s_{\nu+1}, m_{\nu+1}} \leq 2(1 - a)\|P\|_{s_{\nu}, m_{\nu}}, \|\tilde{\tilde{P}}\|_{s_{\nu}, m_{\nu}} \leq a^2\|P\|_{s_{\nu}, m_{\nu}}. \]

B. Constructing a transformation.

In the following, we will construct a transformation \( \Phi_\nu \) which changes system (Eq)\( \nu \) into (Eq)\( \nu+1 \). For any vector function \( F(\theta) = (F_1(\theta), \ldots, F_n(\theta))^T \), we also use the symbols \( \tau \) and \( | \cdot | \). We denote \( \bar{F}(\theta) := (0_{n-1}, [F_n(\theta)])^T \) and \([F(\theta)] = F(\theta) - \bar{F}(\theta)\).

We define the transformation \( \Phi_\nu : y^+ \to y \) by
\[ y = y^+ + U_\nu(\theta)y^+ + V_\nu(\theta), \]
where \( V_\nu(\theta) = (v^1_\nu(\theta), \ldots, v^n_\nu(\theta))^T \), \( U_\nu(\theta) = (u^{ij}_\nu(\theta))_{1 \leq i,j \leq n} \) and \( v^j_\nu(\theta) \) and \( u^{ij}_\nu(\theta) \) admit spatial series expansion
\[ u^{ij}_\nu(\theta) = \sum_{A \in \mathcal{S}} \sum_{|A| \leq \tau_\nu} u^{ij}_{\nu,A} e^{i(k, \theta)}, \]
\[ v^j_\nu(\theta) = \sum_{A \in \mathcal{S}} \sum_{|A| \leq \tau_\nu} v^j_{\nu,A} e^{i(k, \theta)}. \]

Inserting (60) into (Eq)\( \nu \), we get a new system:
\[ (I + U_\nu(\theta))y^+ = N_\nu + (A_\nu + Q_\nu(\theta)) \cdot (y^+ + U_\nu(\theta)y^+ + V_\nu(\theta)) + \bar{F}_\nu(\theta) \]
\[ + G_\nu(\theta, y^+ + U_\nu y^+ + V_\nu) - \partial_{\omega} U_\nu(\theta) \cdot y^+ - \partial_{\omega} V_\nu(\theta), \]
where \( \partial_{\omega} U_\nu = (\partial_{\omega} u^{ij}_\nu)_{1 \leq i,j \leq n} \) and \( \partial_{\omega} V_\nu = (\partial_{\omega} v^j_\nu, \ldots, \partial_{\omega} v^n_\nu)^T \). The functions \( U_\nu(\theta) \) and \( V_\nu(\theta) \) in the transformation (60) are determined by the following homological equations:
\[ \begin{align*}
\partial_{\omega} V_\nu(\theta) &= A_\nu V_\nu(\theta) + [\tilde{F}_\nu(\theta)], \\
\partial_{\omega} U_\nu(\theta) &= A_\nu U_\nu(\theta) - U_\nu(\theta)A_\nu + [\tilde{Q}_\nu(\theta)].
\end{align*} \]
Once these equations are solved, then (61) reads
\[ \dot{y}^+ = N_{\nu+1} + (A_{\nu+1} + Q_{\nu+1}(\theta))y^+ + F_{\nu+1}(\theta) + G_{\nu+1}(\theta, y^+), \]
where
\[ N_{\nu+1} = N_{\nu} + F_{\nu}(\theta), \]
\[ A_{\nu+1} = A_{\nu} + Q_{\nu}(\theta), \]
\[ F_{\nu+1}(\theta) = (I + U_{\nu}(\theta))^{-1}\left(-U_{\nu}(\theta)N_{\nu+1} + F_{\nu}(\theta) + Q_{\nu}(\theta)K_{\nu}(\theta) + G_{\nu}(\theta, V_{\nu}(\theta))\right), \]
\[ Q_{\nu+1}(\theta) = (I + U_{\nu}(\theta))^{-1}\left(Q_{\nu}(\theta) + (\partial Q_{\nu}(\theta))U_{\nu}(\theta) + \left(\frac{\partial G_{\nu}(\theta, V_{\nu}(\theta))}{\partial y}\right)(I + U_{\nu}(\theta))\right), \]
\[ G_{\nu+1}(\theta, y^+) = (I + U_{\nu}(\theta))^{-1}\left(G_{\nu}(\theta, V_{\nu}(\theta) + (I + U_{\nu}(\theta))y^+) - G_{\nu}(\theta, V_{\nu}(\theta))\right) \]
\[ -\left(\frac{\partial G_{\nu}(\theta, V_{\nu}(\theta))}{\partial y}\right)(I + U_{\nu}(\theta))y^+. \]

C. Solving linear homological equations.

In the following, we solve the linear equations (62). According to the first equation of (62), we have
\[ \partial_{\nu}v_j^\nu(\theta) = D^j_k v^\nu_k(\theta) + \tilde{F}_j^\nu(\theta), \quad 1 \leq j \leq n - 1, \]
\[ \partial_{\nu}v_0^\nu(\theta) = E_{\nu}v_0^\nu(\theta) + \tilde{F}_0^\nu(\theta) - [F^\nu_0(\theta)]. \]

Let
\[ \tilde{F}_j^\nu(\theta) = \sum_{A_1 \in S} \sum_{t_1 \leq \nu} \sum_{supp k \subseteq A_1 \subseteq \nu} \tilde{F}_{A,k}^j e^{i(k, \theta)}, \quad j = 1, \ldots, n. \]
Since \(|D^j_k| \geq c_{\nu}^{-\frac{1}{2}}\), we can choose \(v_{A,k}^\nu = \tilde{F}_{A,k}^j / (i(k, \omega) - D^j_k)\) for all \(supp k \subseteq A \in S, [A] \leq \nu_k, 0 \leq |k| \leq \nu_k\). One can check that \(D^j_k\) is a real number which implies
\[ |v_{A,k}^j| \leq \frac{|\tilde{F}_{A,k}^j|}{|i(k, \omega) - D^j_k|} \leq \frac{|\tilde{F}_{A,k}^j|}{|i(k, \omega)|} \leq \gamma^{-1}\Delta(|k|)|\tilde{F}_{A,k}^j| \]
for all \(supp k \subseteq A \in S, [A] \leq \nu_k, 0 \leq |k| \leq \nu_k\). With \(a = 1 - e^{-\tau_k \sigma_k} \leq \tau_k \sigma_k\), it follows that
\[ \|v_{A,k}^j\|_{s, m, \nu_k} \leq \sum_{A \subseteq S, [A] \leq \nu_k} \sum_{supp k \subseteq A \subseteq \nu_k} |v_{A,k}^j|e^{i(|k|s, m, [A]m, \nu_k)} \]
\[ \leq \sum_{A \subseteq S, [A] \leq \nu_k} \sum_{supp k \subseteq A \subseteq \nu_k} \gamma^{-1}\Delta(|k|)|\tilde{F}_{A,k}^j|e^{i(|k|s, m, [A]m, \nu_k)} \]
\[ \leq \sum_{A \subseteq S, [A] \leq \nu_k} |\tilde{F}_{A,0}^j|e^{i(|A|m, [A]m, \nu_k)} + \sum_{A \subseteq S, [A] \leq \nu_k} \sum_{supp k \subseteq A \subseteq \nu_k} \gamma^{-1}\Delta(|k|)|\tilde{F}_{A,k}^j|e^{i(|k|s, m, [A]m, \nu_k)} \]
\[ \leq C_0 \gamma^{-1}\Delta^2(\tau_k)\|\tilde{F}_{A,0}^j\|_{s, m, \nu_k} \]
\[ \leq C_{\nu_k} \gamma^{-1}\Delta^2(\tau_k)\|\tilde{F}_{A,0}^j\|_{s, m, \nu_k}. \]
We can choose \(v_{A,k}^j = 0\) and \(v_{A,k}^n = \tilde{F}_{A,k}^j / (i(k, \omega) - D^j_k)\) for all \(supp k \subseteq A \in S, [A] \leq \nu_k, 0 \leq |k| \leq \nu_k\). Similarly, we have
\[ \|v_{A,k}^n\|_{s, m, \nu_k} \leq C_0 \gamma^{-1}\Delta^2(\tau_k)\|\tilde{F}_{A,0}^j\|_{s, m, \nu_k}. \]
In view of $D^i_j(j = 1, \cdots, n)$ are all real number, we can get
\[ v^j_{\nu A, -k} = \overline{F^j_{\nu A, -k}} / (i \langle -k, \omega \rangle - D^j_{\nu}) = v^j_{\nu A, k} \]  
(63)  
(here $\overline{\cdot}$ denotes the complex conjugate of $\cdot$), for all $|k| > 0$. Thus by (63), $v^j_i(j = 1, \cdots, n)$ are all real analytic functions.

For sufficiently small $\epsilon_0$ we have the following estimates
\[
|D^j_{\nu+1}| \geq |D^j_0| - \sum_{l=0}^{\nu} ||Q^{ij}_l(\theta)|| \\
\geq c\epsilon_0^{l+1} - \sum_{l=0}^{\infty} \epsilon_l \geq c \frac{l+1}{2} \epsilon_0 \epsilon_0 (1 - \epsilon_0),
\]
\[
|D^j_{\nu+1} - D^j_{\nu}| \geq |D^j_0 - D^j_\nu| - \sum_{l=0}^{\nu} ||Q^{ij}_l(\theta)|| - |Q^{ij}_l(\theta)|| \\
\geq c\epsilon_0^{l+1} - \sum_{l=0}^{\infty} \epsilon_l \geq c \frac{l+1}{2} \epsilon_0 \epsilon_0 (1 - \epsilon_0)
\]
and
\[
|E_{\nu+1}| \leq |E_0| + \sum_{l=0}^{\nu} ||Q^{ij}_l(\theta)|| \leq C\epsilon_0^{\frac{2l+1}{2}}.
\]
We have $|E_{\nu+1} - D^j_{\nu+1}| \geq c\epsilon_0^{\frac{l}{2}} - C\epsilon_0^{\frac{2l+1}{2}} > c\epsilon_0^{\frac{l}{2}}$ since $2l + 1 \geq \tilde{l}$. This proves (62) holds true. In addition, it easily follows that
\[
|A_{\nu+1} - A_\nu| \leq C\epsilon_\nu.
\]
Now we solve the second equation of (62). Let
\[
Q^{ij}_l(\theta) = \sum_{A \in S} \sum_{|A| \leq \tau_\nu} \sum_{k, \omega} \overline{Q^{ij}_{\nu A, k}} e^{i(k, \omega)}, 1 \leq i, j \leq n.
\]
By the second equation of (62) we have $u^{ij}_{\nu A, 0} = 0$ and
\[
u^{ij}_{\nu A, k} = (i(k, \omega))^{-1} \overline{Q^{ij}_{\nu A, k}}, \forall \text{supp} k \subset A \in S, |A| \leq \tau_\nu, 0 < |k| \leq \tau_\nu,
\]
for $i = j$. Moreover, for $1 \leq i \neq j \leq n - 1$ we have
\[
u^{ij}_{\nu A, k} = (i(k, \omega) - D^i_{\nu+1} + D^j_{\nu+1})^{-1} \overline{Q^{ij}_{\nu A, k}},
\]
\[
u^{ij}_{\nu A, k} = (i(k, \omega) - D^i_{\nu+1} + E_{\nu+1})^{-1} \overline{Q^{ij}_{\nu A, k}},
\]
for all $\text{supp} k \subset A \in S, |A| \leq \tau_\nu, 0 \leq |k| \leq \tau_\nu$. Notice that both $D^i_{\nu+1}$ and $E_{\nu+1}$ are real numbers. With $\alpha = 1 - e^{-\tau_\nu \sigma_\nu} \leq \tau_\nu \sigma_\nu$, it follows that
\[
|u^{ij}_{\nu A, 0}|_{\nu, \nu \sigma_\nu, M_\nu} \leq \sum_{A \in S} |u^{ij}_{\nu A, 0}| e[|A|] M_\nu + \sum_{A \in S} \sum_{\text{supp} k \subset A} |u^{ij}_{\nu A, k}| e[|k|] e[|A|] e[\nu] M_\nu
\]
\[
\leq C\epsilon_0^{\frac{l+1}{2}} \gamma^{-1} \Delta^2(\tau_\nu) \left( \sum_{A \in S} \sum_{\text{supp} k \subset A} Q^{ij}_{\nu A, k} e[|k|] e[|A|] e[\nu] M_\nu \right)
\]
it follows that

where

Furthermore, we have

\[ \theta \]

\( \| \theta \|_{\nu, m, M} \leq C\epsilon_0^{-\frac{1}{\nu}} \sigma_0 \delta^\nu q^{-\frac{\nu-1}{\nu}}. \)

Similar to \( u^i_\nu(\theta) \), we can also get all \( u^i_\nu(\theta) \) are real analytic functions. Finally, we obtain that

\[ \| V_\nu(\theta) \|_{\nu, m, M} \leq C\epsilon_0^{-\frac{1}{\nu}} \sigma_0 \delta^\nu q^{-\frac{\nu-1}{\nu}} \text{ and } \| U_\nu(\theta) \|_{\nu, m, M} \leq C\epsilon_0^{-\frac{1}{\nu}} \sigma_0 \delta^\nu q^{-\frac{\nu-1}{\nu}}. \]

(64)

Noting that \( \theta = \theta \) and

\[ \| y \|_{\nu, r, M} = \| V_\nu(\theta) + (1 + U_\nu(\theta))y^+ \|_{\nu, r, M} \leq \| V_\nu(\theta) \|_{\nu, M} + (1 + \| U_\nu(\theta) \|_{\nu, M})r \]

\[ \leq r + C\epsilon_0^{-\frac{1}{\nu}} \sigma_0 \nu \leq r. \]

it follows that

\[ \Phi_\nu(\Delta_{\nu, M}) \subset \Delta_{\nu, M}. \]

Furthermore, we have

\[ \| \Phi_\nu - id \|_{\nu, M} \leq C\epsilon_0^{-\frac{1}{\nu}} \sigma_0 \delta^\nu q^{-\frac{\nu-1}{\nu}} \]

(65)

and

\[ \| D\Phi_\nu - I \|_{\nu, M} \leq C\epsilon_0^{-\frac{1}{\nu}} \sigma_0 \delta^\nu q^{-\frac{\nu-1}{\nu}}. \]

(66)

D. Estimates of perturbation terms.

For any \( (\xi, \lambda') \in \mathcal{U}(\Gamma_\nu, d_\nu) \), there exists \( (\xi, \lambda) \in \Gamma_\nu \) such that \( |\lambda' - \lambda| < d_\nu \). So it follows that

\[ |N_\nu(\xi, \lambda')| = |N_\nu(\xi, \lambda') - N_\nu(\xi, \lambda)| \leq \frac{3}{2} d_\nu. \]

Then we have

\[ \| U_\nu N_{\nu, 1} \|_{\nu, M} \leq C\epsilon_0^{-\frac{1}{\nu}} \sigma_0 \delta^\nu q^{-\frac{\nu-1}{\nu}}. \]

(67)

Let \( M_{\nu+1} \) be defined by (56), it follows easily that \( M_{\nu+1} \) is closed. Obviously, we have

\[ M_{\nu+1} \subset \mathcal{U}(\Gamma_\nu, d_\nu) \subset M_\nu \text{ and } \text{dist}(M_{\nu+1}, \partial M_\nu) \geq \frac{1}{2} d_\nu, \]

where \( \partial M_\nu \) is the boundary of \( M_\nu \). Applied Cauchy estimation, one has also that (57) holds true. Set \( \hat{N}_{\nu, 1} = \hat{N}_\nu + \Delta N_\nu. \)

In view of (57) and by the implicit function theorem, the equation

\[ N_{\nu+1}(\xi, \lambda) = -\lambda + \epsilon \xi^{2\nu+1} + \hat{N}_{\nu+1}(\xi, \lambda) = 0 \]

defines implicitly an analytic curve

\[ \Gamma_{\nu+1} : \lambda_{\nu+1} = \lambda_{\nu+1}(\xi) : \xi \in \Pi_{\nu+1} \rightarrow \lambda_{\nu+1}(\xi) \in B(0, 2\beta + 1). \]

By (55) it follows that \( \phi_{\nu, \nu+1}(\xi, \lambda) \leq 1/2 \) for all \( (\xi, \lambda) \in M_\nu \). Since \( \lambda_{\nu+1} \) and \( \lambda_\nu \) satisfy

\[ -\lambda_{\nu+1} + \epsilon \xi^{2\nu+1} + \hat{N}_{\nu+1}(\xi, \lambda_{\nu+1}) = -\lambda_\nu + \epsilon \xi^{2\nu+1} + \hat{N}_\nu(\xi, \lambda_\nu) = 0, \]

it is easy to see that

\[ |\lambda_{\nu+1} - \lambda_\nu| \leq |\hat{N}_{\nu+1}(\xi, \lambda_{\nu+1}) - \hat{N}_\nu(\xi, \lambda_\nu)| \]

\[ \leq |\hat{N}_\nu(\xi, \lambda_{\nu+1}) - \hat{N}_\nu(\xi, \lambda_\nu)| + |\Delta N_\nu(\xi, \lambda_{\nu+1})| \]

\[ \leq \frac{1}{2} |\lambda_{\nu+1}(\xi) - \lambda_\nu(\xi)| + C\epsilon_0^2. \]
Assume by induction that 
\[ |\lambda - \lambda_\nu| \leq |\lambda - \lambda_{\nu+1}| + |\lambda_{\nu+1} - \lambda_\nu| \leq d_{\nu+1} + C\varepsilon_\nu^2 \leq \frac{d_\nu}{2}, \]
which implies \( U(\Gamma_{\nu+1}, d_{\nu+1}) \subset M_{\nu+1} \). This proves the conclusion \((\nu + 1.2)\) holds true.

Now we estimate perturbation terms \( F_{\nu+1}(\theta) \) and \( Q_{\nu+1}(\theta) \). First of all, from (64) we have
\[ \|(I + U_\nu(\theta))^{-1}\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq 2 \]
as \( \delta \leq 1/2 \) and \( \epsilon_0 \) is small enough. Thus, by (64) and (67), we have
\[ \|F_{\nu+1}(\theta)\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq \|(I + U_\nu(\theta))^{-1}\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \left( \|U_\nu N_{\nu+1}\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \right. \]
\[ + \|F_{\nu}(\theta)\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} + \|Q_{\nu}(\theta)V_\nu(\theta)\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \]
\[ + \|G_{\nu}(\theta, V_\nu(\theta))\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \right) \]
\[ \leq 2(C\varepsilon_0^2 \varepsilon_\nu^2 + 2(1 - a)\varepsilon_\nu^2 + C\varepsilon_0^2 \varepsilon_\nu + C\varepsilon_0^2 \varepsilon_\nu^2) \]
\[ \leq 2(2(1 - a) + b)\varepsilon_\nu^2 \leq \epsilon_{\nu+1} \]
provided that \( \epsilon_0 \) is small enough. We also have that
\[ \|Q_{\nu+1}(\theta)\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq \|(I + U_\nu(\theta))^{-1}\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \left( \|\tilde{Q}_{\nu}(\theta)\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \right. \]
\[ + \|Q_{\nu}(\theta)V_\nu(\theta)\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \]
\[ + \|L_{\nu}(\theta, V_\nu(\theta))\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \left. \|I + U_\nu\|_{s_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \right) \]
\[ \leq 2\left( 2(1 - a)\varepsilon_\nu + C\varepsilon_0^2 \varepsilon_\nu + C\varepsilon_0^2 \varepsilon_\nu^2 \right) \]
\[ \leq 2(2(1 - a) + b)\varepsilon_\nu \leq \epsilon_{\nu+1} \]
provided that \( \epsilon_0 \) is small enough.

We note that there is a constant \( C_0 > 0 \) such that
\[ \|\frac{\partial^2 G_0(\theta, y)}{\partial y^2}\|_{s_0, r_0, m_0, M_0} \leq C_0. \]
Assume by induction that
\[ \|\frac{\partial^2 G_l(\theta, y)}{\partial y^2}\|_{s_l, r_l, m_l, M_l} \leq C_0 \Gamma_l \]
hold for \( l = 0, 1, \ldots, \nu \), where \( \Gamma_0 = 1 \),
\[ \Gamma_l = \prod_{i=1}^{l} \frac{(1 + \delta^i)^2}{1 - \delta^i}. \]
Similar to the proof of Theorem 3.1, we can get
\[ \|\frac{\partial^2 G_{\nu+1}(\theta, y^+)}{\partial y^2}\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq C_0 \Gamma_{\nu+1} \]
holds also true. It implies for any \( 0 < \bar{r} \leq r_{\nu+1} \),
\[ \|G_{\nu+1}(\theta, y^+)\|_{s_{\nu+1}, \bar{r}, m_{\nu+1}, M_{\nu+1}} \leq \frac{1}{2} C_0 \Gamma_{\nu+1} \bar{r}^2 \leq C\bar{r}^2. \]
Hence, \( G_{\nu+1} = \mathcal{O}(y^{+2})(y^+ \to 0) \). This proves the conclusion \((\nu + 1.3)\) holds true.
Let
\[ \Phi^\nu := \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : \Delta_{s_{\nu+1}, r_{\nu+1}} \times M_{\nu+1} \to \Delta_{s_0, r_0} \times M_\nu. \]
Then, after the transformation \( \Phi^\nu \), for each \( \nu \geq 0 \) system (54) is changed to
\[ \dot{y} = N_\nu + (A_\nu + Q_\nu(\theta))y + F_\nu(\theta) + G_\nu(y, \theta). \]
By the inductive assumptions of KAM iteration, for sufficiently small \( \epsilon_0 \), we have
\[ \|F_\nu\|_{s_0, m_\nu, M_\nu} \leq C\epsilon^2, \quad \|Q_\nu\|_{s_0, m_\nu, M_\nu} \leq C\epsilon, \quad G_\nu = O(y^3)(y \to 0). \]

2. Convergence of KAM iteration

Let us take \( N_0 = N, A_0 = A, Q_0 = Q, F_0 = F, G_0 = G \), and \( M_0 = \Pi_{\rho_0} \times \mathcal{B}(0, 2\beta + 1) \). So, it is easy to check that system (54) satisfies all hypotheses of Lemma 4.9 with \( \nu = 0 \).

For sufficiently small \( \epsilon_0 > 0 \), it follows from the inequality (65) and (66) that
\[ \|D\Phi_j(\Phi_{j+1} \circ \cdots \circ \Phi_\nu)(x)\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq 1 + \delta^j, \quad j = 0, 1, \ldots, \nu - 1. \]
We infer
\[ \|D\Phi_{\nu-1}(x)\|_{s_{\nu}, r_{\nu}, m_{\nu}, M_{\nu}} \leq \prod_{\nu \geq 0} (1 + \delta^\nu) \leq e^{1/\alpha}, \]
and hence
\[ \|\Phi^\nu - \Phi^{\nu-1}\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}, M_{\nu+1}} = \|\Phi^{\nu-1}(\Phi_\nu) - \Phi^{\nu-1}\|_{s_{\nu+1}, r_{\nu+1}, m_{\nu+1}, M_{\nu+1}} \leq e^{1/\alpha} \delta^\nu. \]
The same inequality holds for \( \nu = 0 \) if we define \( \Phi^{-1} := \text{id} \). Let \( M_\ast = \bigcap_{\nu \geq 0} M_\nu \) and \( \Delta_{s_\ast, r_\ast} = \bigcap_{\nu \geq 0} \Delta_{s_{\nu+1}, r_{\nu+1}} \). In view of \( s_\nu \to s_\ast \geq s/2, r_\nu \to r_\ast \geq r/2 \), as \( \nu \to \infty \), the mapping \( \Phi^\nu \):
\[ \Phi^\nu = \Phi^0 + \sum_{i=0}^\nu (\Phi^i - \Phi^{i-1}), \quad \Phi^{-1} := \text{id} \]
converges uniformly on \( \Delta_{s_\ast, r_\ast} \times M_\ast \) to a mapping \( \Phi^\ast \).

Now we show that the convergence of \( \hat{N}_\nu \). One has \( \|\hat{N}_{\nu+1} - \hat{N}_\nu\| \leq C\epsilon^2 \), which shows that \( \hat{N}_\nu \) is convergent on \( M_\ast \) to \( \hat{N}_\ast \), and \( \hat{N}_\ast = O(\epsilon_0^2) \). Let
\[ \rho_\ast = \rho_0 - \frac{1}{2} \sum_{j=0}^{\infty} d_j = \rho_0 - \frac{1}{2} \sum_{j=0}^{\infty} \frac{\epsilon_0^{2j+1}}{q - \frac{2j+1}{2}} = \rho_0 - \frac{\epsilon_0^{2j+1}}{2(1 - q^{-\frac{2j+1}{2}})}. \]
Then we have \( \rho_\ast > \frac{1}{2} \rho_0 \) provided that \( \epsilon \) is sufficiently small. Thus \( \Pi_{\rho_\ast} \subset \bigcap_{\nu \geq 0} \Pi_{\rho_\ast} \).

By (58), it is easy to see that \( \{\lambda_\nu(\xi)\} \) is convergent on \( \Pi_{\rho_\ast} \). Let
\[ \lambda_\ast(\xi) = \lim_{\nu \to \infty} \lambda_\nu(\xi), \quad \xi \in \Pi_{\rho_\ast} \]
Since \( \Gamma_\nu = \{(\xi, \lambda_\nu(\xi)) : \xi \in \Pi_{\rho_\ast}\} \subset M_\nu \) and \( \lambda_\nu \) are all analytic on \( \Pi_{\rho_\ast} \), so it is limit \( \lambda_\ast(\xi) \). By (58), we have
\[ |\lambda_j(\xi) - \lambda_\ast(\xi)| \leq \frac{d_\nu}{2} \]
provided that \( \epsilon_0 \) is sufficiently small. Hence, as \( j \to \infty \), we have
\[ |\lambda_\ast(\xi) - \lambda_\nu(\xi)| \leq \frac{d_\nu}{2}. \]
This implies that \( \Gamma_\ast = \{(\xi, \lambda_\ast(\xi)) : \xi \in \Pi_{\rho_\ast}\} \subset M_\ast \). So \( \Gamma_\ast \subset M_\ast \).
Thus, for all \((\xi, \lambda) \in \Gamma_\ast\), \((Eq)_0\) is changed to

\[
(Eq)_0 \circ \Phi^* : \begin{cases} 
\dot{\theta} = \omega, \\
\hat{y} = N_\ast(\xi, \lambda) + A_\ast(\xi, \lambda)y + \mathcal{G}_\ast(\theta, y, \xi, \lambda),
\end{cases}
\]

where \(\mathcal{G}_\ast = \mathcal{O}(y^2)(y \to 0)\), \(N_\ast(\xi, \lambda) = (0_{n-1}, N_\ast(\xi, \lambda))^T\) and

\[
A_\ast(\xi, \lambda) = \begin{pmatrix}
D_\ast(\xi, \lambda) & 0^T_{n-1} \\
0_{n-1} & E_\ast(\xi, \lambda)
\end{pmatrix}
\]

with

\[
N_\ast(\xi, \lambda) = -\lambda + \epsilon \xi^{2l+1} + \hat{N}_\ast(\xi, \lambda) = 0.
\]

Equation (69) determines a curve \(\Gamma_\ast : \{((\xi, \lambda_\ast)) \in \Pi_\theta \land \lambda_\ast = \lambda_\ast(\xi)\}\), on which we have

\[
\lambda_\ast(\xi) = \epsilon \xi^{2l+1} + \hat{N}_\ast(\xi, \lambda_\ast(\xi)).
\]

On curve \(\Gamma_\ast\), system (68) becomes

\[
(Eq)_0 \circ \hat{\Phi}^*(\cdot, \cdot, \xi, \lambda_\ast(\xi)) : \begin{cases} 
\dot{\theta} = \omega, \\
\hat{y} = A_\ast(\xi, \lambda_\ast(\xi))y + \mathcal{G}_\ast(\theta, y, \xi, \lambda_\ast(\xi)).
\end{cases}
\]

This completes the proof of Theorem 4.8. \(\square\)

4.2.3. Return to original system. By Theorem 4.8, we have proven that there exists a smooth curve

\[
\Gamma_\ast : \lambda_\ast = \lambda_\ast(\xi), \xi \in \Pi = [-1, 1]
\]

such that for \((\xi, \lambda_\ast) \in \Gamma_\ast\), system (54) can be reduced to (71). Now we want to show that there exists \(\xi_\ast \in \Pi\) such that \(\lambda_\ast(\xi_\ast) = 0\) and then come back to the original system (52) with \(\xi = \xi_\ast\). In fact, if \(\epsilon_0\) is sufficiently small, from \(\hat{N}_\ast = \mathcal{O}(\epsilon_0^2)\) and (70), we have

\[
\lambda_\ast(1) = \epsilon + \hat{N}_\ast(1, \lambda_\ast(1)) > 0,
\]

\[
\lambda_\ast(-1) = -\epsilon + \hat{N}_\ast(-1, \lambda_\ast(-1)) < 0,
\]

for sufficiently small \(\epsilon_0 > 0\), which implies there exists \(\xi_\ast \in \Pi\) such that \(\lambda_\ast(\xi_\ast) = 0\). By the transformation \(\Phi^*(\cdot, \cdot, \xi\ast, \lambda_\ast(\xi\ast)) = \Phi^*(\cdot, \cdot, \xi\ast, 0)\), system (54) is changed into

\[
(Eq)_0 \circ \hat{\Phi}^*(\cdot, \cdot, \xi\ast, 0) : \begin{cases} 
\dot{\theta} = \omega, \\
\dot{z} = A_\ast(\xi\ast, 0)z + \mathcal{G}_\ast(\theta, z, \xi\ast, 0).
\end{cases}
\]

Therefore, system (53) possesses a response torus at \(\lambda = \lambda_\ast(\xi\ast)\). That is, system (52) has a response solution. This proves Theorem 4.3.

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