Calculation of unsteady two-phase quasi-onedimensional channel flow based on the two-fluid model and the artificial-viscosity numerical scheme

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Abstract. An example of usage of the artificial viscosity technique for numerical simulation of unsteady one-dimensional two-phase flows based on the two-fluid approach is presented. The governing equations system is written under assumption of equal pressure in the fluids. No interfacial exchange is taken into account. Performance of the artificial viscosity technique suggested has been evaluated via numerical solution of a model problem of two-phase flow in a channel of variable cross-section. It has been shown that the used numerical scheme, implicit and first-order, involving the suggested artificial viscosity model, provides a stable process of getting a numerical solution and predicts a physically adequate flow dynamics without non-physical oscillations.

1. Introduction
In different industrial applications, calculations of unsteady flows of heterogeneous compressible media are carried out mostly with one-dimensional multi-velocity models that have been developed on the base of the interpenetrating continua concept. A leading role among the models of the sort, in sense of prevalence, belongs to the approach based on assumption of the pressure equilibrium between the “fluids” (phases) forming a heterogeneous medium. In particular, this approach is extensively employed in thermo-hydraulic codes developed for numerical simulation of hydrodynamic and thermal processes in cooling loops of nuclear power reactors (e.g. [1, 2]). Development of models that have, from one side, to provide closure of the system of one-dimensional differential governing equations, and, from the other side, to ensure stability of numerical algorithms in the framework of the pressure equilibrium assumption has a rather rich history. Initially, about half century ago, the so-called “basic model” was formed, where pressure was assumed same for all the fluids [3].
The positive effect of numerical viscosity, one gets the following system of governing equations describing unsteady one-dimensional flows of multiphase media

\[
\frac{\partial (\alpha_k \rho_k)}{\partial t} + \frac{\partial (\alpha_k \rho_k w_k)}{\partial x} = R_{e,k},
\]

\[
\frac{\partial (\alpha_k \rho_k w_k)}{\partial t} + \frac{\partial (\alpha_k \rho_k w_k^2 + \alpha_k p)}{\partial x} - p \frac{\partial \alpha_k}{\partial x} = R_{m,k},
\]

\[
\frac{\partial (\alpha_k \rho_k E_k)}{\partial t} + \frac{\partial (\alpha_k \rho_k w_k H_k)}{\partial x} = R_{e,k},
\]

\[
\sum \alpha_k = 1, \quad \rho_k = \rho_k \left( p, T_k \right), \quad e_k = e_k \left( p, T_k \right), \quad E_k = e_k + 0.5w_k^2, \quad H_k = E_k + \frac{P}{\rho_k}.
\]

Here \(\alpha_k\) is the volumetric fraction of \(k\)-fluid (\(k\)-phase), \(k = 1, \ldots, K\) is the number of fluids; \(\rho_k, T_k, w_k, p, e_k, E_k, H_k\) are density, temperature, velocity, pressure, specific internal energy, specific total energy, specific total enthalpy; \(R_{e,k}, R_{m,k}, R_{e,k}\) are terms representing the resultant action of interfacial-phase and phase-to-wall exchange on temporal and spatial variations of mass, momentum and energy. In the basic model, the components included into \(R_{e,k}, R_{m,k}, R_{e,k}\) do not involve derivatives of the sought variables. In this case, system (1) does not necessarily remain hyperbolic in all situations, since characteristics of the equation system can be complex-valued in the physically interesting region of state space [4, 5]. In other words, it means that the initial-value problem may be ill-posed for a large class of initial conditions.

There are a number of contributions covering various approaches to the basic system modification aimed at elimination or, at least, at a considerable reduction of the physical effects. The positive effect of numerical diffusion on calculation stability can be enforced when adding beforehand the artificial diffusivity terms to the transport equations (1). Note here that introducing an artificial viscosity was widely used at the early stage of development of numerical schemes aimed at calculation of compressible inviscid gas flows with shock-capturing methods [7, 8]. A recent application of the artificial viscosity technique was reported in [9] at consideration of the Riemann problem for multiphase flow described with the multi-fluid model. The present paper covers initial results obtained by the authors in application of the artificial viscosity approach for numerical computations of unsteady two-phase quasi-one-dimensional channel flows.

2. Mathematical model

According the above given considerations, the right-hand sides of governing equations (1) are added by the terms introducing effects of artificial viscosity. When dropping the terms associated with interfacial transfer, for sake of simplicity, one gets the following system of governing equations describing unsteady one-dimensional flows of multi-fluid media

\[
\frac{\partial (\alpha_k \rho_k)}{\partial t} + \frac{1}{A} \frac{\partial (\alpha_k \rho_k A)}{\partial x} = \frac{1}{A} \frac{\partial}{\partial x} \left( \nu_k \frac{\partial (\alpha_k \rho_k A)}{\partial x} \right),
\]

\[
\alpha_k \rho_k \frac{\partial w_k}{\partial t} + \alpha_k \rho_k w_k \frac{\partial w_k}{\partial x} + \alpha_k \frac{\partial p}{\partial x} = \nu_k \frac{\partial (\alpha_k \rho_k w_k)}{\partial x},
\]

\[
\frac{\partial (\alpha_k \rho_k h_k)}{\partial t} + \frac{1}{A} \frac{\partial (\alpha_k \rho_k h_k A)}{\partial x} - \alpha_k \left[ \frac{\partial p}{\partial t} + w_k \frac{\partial p}{\partial x} \right] = \frac{1}{A} \frac{\partial}{\partial x} \left( \nu_k \frac{\partial (\alpha_k \rho_k E_k A)}{\partial x} \right),
\]

\[
\sum \alpha_k = 1, \quad \rho_k = \rho_k \left( p, T_k \right), \quad h_k = h_k \left( p, T_k \right), \quad E_k = h_k - \frac{P}{\rho_k} + 0.5w_k^2.
\]
where $A$ is the cross-sectional area, $h_k$ is the specific enthalpy, $v_k$ is the artificial viscosity coefficient. Characteristic properties of system (2) are determined by higher-order derivatives. With respect to this system, the basic system (1) can be treated as a sub-system, and its properties become sub-characteristics. The standard procedure of getting and analyzing a characteristic determinant leads to conclusion that system (2) is parabolic in time. Consequently, the Cauchy problem for this multi-fluid flow model is well-posed.

In the present study, a model assuming identity of the artificial viscosity coefficient for all the flow components is used. At that, following [9], the speed of sound, $c$, is evaluated with the well-known Wood equation derived for multiphase mixtures. The model formulation is given by

$$v_s = \gamma \Delta \kappa, \quad \frac{1}{\rho c^2} = \sum \frac{\alpha_i}{\rho_i c_i^2}, \quad \rho = \sum \alpha_i \rho_i,$$

where $\gamma$ is a user-defined factor. All the simulation results presented below were obtained with $\gamma = 0.5$.

3. Numerical scheme

The numerical scheme used in the present study is a modification of the scheme implemented in the thermo-hydraulic code KORSAR [1]. The scheme is based on the control-volume method of first-order accuracy both in space and in time. The approximation of the conservation equations is done according to a partially implicit scheme. For the spatial coordinate, the “staggered” grid is used.

Discretized equations of mass and energy conservation for a current computational cell are given by

$$\begin{align*}
\frac{V_j}{\Delta t} \left[ (\alpha_i \rho_i)^{n+1} - (\alpha_i \rho_i)^n \right] &= \left[ (\alpha_i \rho_i)^n \right]_{j-1/2} \cdot w_{j+1/2}^{n+1} A_{j+1/2} - \left[ (\alpha_i \rho_i)^n \right]_{j+1/2} \cdot w_{j-1/2}^{n+1} A_{j-1/2} + V_j \cdot AVC_{i,j}^{n+1}, \\
\frac{V_j}{\Delta t} \left[ (\alpha_i \rho_i h_i)^{n+1} - (\alpha_i \rho_i h_i)^n \right] &= \left[ (\alpha_i \rho_i h_i)^n \right]_{j-1/2} \cdot w_{j+1/2}^{n+1} A_{j+1/2} - \left[ (\alpha_i \rho_i h_i)^n \right]_{j+1/2} \cdot w_{j-1/2}^{n+1} A_{j-1/2} + \\
&+ \left[ (p_j^n - p_{j-1/2}^n) \cdot \left( \alpha_i \rho_i h_i \right)^{n+1} \cdot \left( \alpha_i \rho_i h_i \right)^n \right]_{j-1/2} \cdot w_{j+1/2}^{n+1} A_{j+1/2} - \left[ (p_j^n - p_{j-1/2}^n) \cdot \left( \alpha_i \rho_i h_i \right)^n \right]_{j+1/2} \cdot w_{j-1/2}^{n+1} A_{j-1/2} + \\
&\quad + V_j \cdot AVM_{i,j}^{n+1},
\end{align*}$$

where $k = 1, 2$ is the fluid index, $j$ is the current cell index. The convective fluxes are evaluated at the cell faces marked by $j-1/2$ and $j+1/2$; $n, n+1$ are indices of time layers, $\Delta t$ is the time step, $V_j$ is the cell volume. Terms $AVC_{k,p}$, $AVE_{k,j}$ and $AVM_{k,j}$ in equation (10)) denote contribution of the artificial viscosity. They are approximated explicitly with the central difference scheme. Let’s $F$ stays for $\alpha_i, \rho_i, h_k, p$. The donor values, $E_{j-1/2}$, are defined by the upwind scheme as follows

$$E_{j-1/2} = 0.5 \left( F_{j-1/2} + F_{j-1} \right) \cdot \frac{w_{j-1/2}}{w_{j+1/2}} \left( F_{j+1} - F_{j-1} \right).$$

Solution of system (4) is sought introducing corrections, $\Delta F = F^{s+1} - F^s$, to the variable values on the current time layer and performing iterations ($s$ is the iteration number). With this, linearization of the unsteady terms is written as

$$\begin{align*}
\left[ (\alpha_i \rho_i)^{n+1} - (\alpha_i \rho_i)^n \right] \rightarrow \left[ (\alpha_i \rho_i)^{s+1} - (\alpha_i \rho_i)^n \right] = \left( \alpha_i \rho_i \right)^s + \alpha_i \Delta \rho + \rho_i \Delta \alpha_i - \left( \alpha_i \rho_i \right)^n, \\
\left[ (\alpha_i \rho_i h_i)^{n+1} - (\alpha_i \rho_i h_i)^n \right] \rightarrow \left( \alpha_i \rho_i h_i \right)^{s+1} + \alpha_i \rho_i \Delta h + \rho_i h_i \Delta \alpha_i + \alpha_i \rho_i h_i \Delta \rho - \left( \alpha_i \rho_i h_i \right)^n,
\end{align*}$$

where the phase densities are expressed through the flow parameters as

$$\Delta \rho = \left( \frac{\partial \rho}{\partial h} \right)^s \Delta h + \left( \frac{\partial \rho}{\partial \rho} \right)^s \Delta \rho.$$

The velocities and pressure in equation (2), calculated on the new time layer, can be rewritten as

$$w_{j+1/2}^{n+1} \rightarrow w_{j+1}^{n+1} = w_j^s + \Delta w_j, \quad \left( p^{s+1} - p^s \right) \rightarrow \left( p^{s+1} - p^s \right) = p^{s+1} + \Delta p - p^s.$$

In other quantities, the index $n$ is replaced by $s$.
Adding and subtracting the equations of mass conservation (the first equation in system (4)) and substituting in the resulting equations the expressions for the linearized unsteady terms (6) and densities (7), one obtains a system of four linear equations for each computational cell together with the equations of energy conservation (the second equation in system (4)):

\[ B_j^T X_j = g_j^T \Delta w_{j-1/2} - g_j^T \Delta w_{j+1/2} + I_j^T \Delta w_{j,1/2} - I_j^T \Delta w_{j,1/2} + b_j^T, \]

where \( X_j = (\Delta h_1, \Delta h_2, \Delta \alpha_1, \Delta \rho)_j \) is the vector of unknown scalar quantities; \( B_j \) is the matrix of coefficients; \( g_j, I_j, b_j \) are the vectors.

In a general form, the discretized momentum equations are written for the cell faces as

\[ \Delta x_{j+1/2} \left( \frac{g_j, b_j}{\Delta \tau} \right)^n = C_{j+1/2} \Delta x_{j+1/2} \left( P_j - P_{j+1} \right) - \Delta J_{x_{j+1/2}} + AVM_{x_{j+1/2}}, \]

where \( \Delta x \) is the distance between centers of neighboring cells, \( \Delta J \) is the convective flux approximated by the upwind scheme, \( F_{j+1/2} \) are the mean values of the corresponding quantities on the faces.

Implementation of the iterative algorithm is done according to expressions (8).

Equations (10) can be rewritten in the matrix form as

\[ W_{j+1/2} Y_{j+1/2} = D_{j+1/2} \Delta \rho_j + E_{j+1/2} \Delta \rho_{j+1} + C_{j+1/2}, \]

where \( Y_{j+1/2} = (\Delta w_1, \Delta w_2)_{j+1/2} \) is the vector of unknown velocities; \( W_{j+1/2} \) is the matrix of coefficients; \( D_{j+1/2}, E_{j+1/2}, C_{j+1/2} \) are vectors.

Multiplying equations (9) and (11) by the inverse matrices \((B_j)^{-1}\) and \((W_{j+1/2})^{-1}\) respectively, one obtains a system of linear relations given by

\[ X_j = \left( B_j^{-1} \right)^{-1} g_j^T \Delta w_{j-1/2} - \left( B_j^{-1} \right)^{-1} g_j^T \Delta w_{j+1/2} + \left( B_j^{-1} \right)^{-1} I_j^T \Delta w_{j,1/2} + \left( B_j^{-1} \right)^{-1} b_j^T, \]

\[ Y_{j+1/2} = \left( W_{j+1/2}^{-1} \right)^{-1} D_{j+1/2} \Delta \rho_j - \left( W_{j+1/2}^{-1} \right)^{-1} E_{j+1/2} \Delta \rho_{j+1} + \left( W_{j+1/2}^{-1} \right)^{-1} C_{j+1/2}. \]

From the last equation of system (12) one gets

\[ \Delta P_j = g_j^T \Delta w_{j-1/2} - g_j^T \Delta w_{j+1/2} + I_j^T \Delta w_{j,1/2} - I_j^T \Delta w_{j,1/2} + b_j^T, \]

Substituting expression (13) for the phase velocities into equation (14) results in a system of three-point equations for pressure corrections in cell centers

\[ A_j \Delta \rho_{j-1} + C_j \Delta \rho_j + B_j \Delta \rho_{j+1} = F_j. \]

After finding the pressure corrections, the enthalpies, \( \Delta h_k \), are calculated on the faces using relation (13). Substituting the obtained phase velocities into equation (12), corrections of the enthalpies, \( \Delta h_k \), and volume fractions, \( \Delta \alpha_1, \Delta \alpha_2 = -\Delta \alpha_1 \), of the phases are calculated. After finding the corrections, the transition to the next iteration is performed as \( F^{n+1} = F^n + \Delta F \).

4. Simulation results for a model problem
A model problem of two-phase flow in a channel of variable cross-section is considered. Computations have been carried out for a domain of \( 0 \leq x \leq 1 \) m. Areas of the inlet and outlet section: \( A_{in} = 1.256 \times 10^{-2} \) m², \( A_{out} = 0.3141 \times 10^{-2} \) m². The following relations (all dimensions are given in the SI system) define thermodynamic and caloric equations of state for the model medium components

Fluid 1 (perfect gas): \[ \rho_1 = \frac{P}{RT_1}, \quad h_1 = C_{p,1} \left( T_1 - 300 \right), \]

Fluid 2 (incompressible): \[ \rho_2 = 10^1, \quad h_2 = C_{p,2} \left( T_2 - 300 \right) + \frac{h_{2,form}}{M_2}, \]

where \( C_{p,1} = 1005 \) J/(kg·K), \( C_{p,2} = 4187 \) J/(kg·K) are the specific heat capacities of the fluids, \( R = 287 \) J/(kg·K) is the gas constant, \( h_{2,form} = -286 \times 10^3 \) J/(mol·K) is the standard enthalpy of formation, \( M_2 = 18 \times 10^{-3} \) kg/mol is the molar mass.
Initial values of the medium parameters are as follows: $p_{\text{init}} = 0.9 \cdot 10^5 \text{Pa}$, $T_{1,\text{init}} = T_{2,\text{init}} = 300 \text{K}$, $\alpha_{1,\text{init}} = \alpha_{2,\text{init}} = 0.5$, $w_{1,\text{init}} = 10 \text{ m/s}$, $w_{2,\text{init}}$ is varied from 1 m/s to 4 m/s. Boundary conditions are given by: $p_{\text{in}} = 10^5 \text{Pa}$, $T_{1,\text{in}} = T_{2,\text{in}} = 300 \text{K}$, $\alpha_{1,\text{in}} = \alpha_{2,\text{in}} = 0.5$, $p_{\text{out}} = 0.9 \cdot 10^5 \text{Pa}$. Values of the other variables at the outlet boundary are evaluated via extrapolation from inside the computational domain.

For different initial values of $w_{2,\text{init}}$, figure 1 shows pressure evolution at a point located in the channel center. The calculations were carried out with a grid of 30 cells using a time step of $10^{-5} \text{s}$ that corresponded to the Courant number of 0.1, if evaluated using the sound speed in the pure gas (Fluid 1). It is seen that in case of no artificial viscosity terms there are some sets of initial data (for instance, $w_{1,\text{init}} = 10 \text{ m/s}$, $w_{2,\text{init}} = 4 \text{ m/s}$) that result in occurrence of non-physical oscillations followed by complete destruction of the numerical solution. It means in particular that the numerical diffusivity produced by the first-order approximation used in the numerical scheme of the KORSAR algorithm [1] is not enough to suppress development of such oscillations.

Adding the artificial viscosity of the chosen level provides a stable process of obtaining numerical solution. It is illustrated by figure 2, where several sets of instant distributions of pressure and phase velocities are given. No non-physical oscillations are observed for any values of $w_{2,\text{init}}$ in the case of non-zero artificial viscosity.

![Figure 1](image1.png)

**Figure 1.** Pressure time variations at the channel center at different initial $w_{2,\text{init}}$-values: (red line) $w_{2,\text{init}} = 1 \text{ m/s}$, (green) 2 m/s, (blue) 4 m/s. Solid lines correspond to the case of zero artificial viscosity. The non-zero artificial viscosity data are shown by the dotted line (same for all $w_{2,\text{init}}$-values).

![Figure 2](image2.png)

**Figure 2.** Instant distributions of (left) pressure and velocities of (middle) fluid 1 and (right) fluid 2 calculated at (upper) zero and (lower) non-zero artificial viscosity: (red lines) $t = 0.05 \text{ s}$, (blue) 0.1 s, (black) 0.2 s. Data for $w_{2,\text{init}} = 1 \text{ m/s}$ and 4 m/s are shown by solid and dotted lines respectively.
Figure 3 illustrates influence of the artificial viscosity on the resulting steady-state solution. It can be seen that the artificial viscosity of the chosen level does not change the steady-state solution considerably.

**Figure 3.** Spatial distributions of (left) pressure and velocities of (middle) fluid 1 and (right) fluid 2 in the steady-state solution: (solid lines) zero artificial viscosity, (dotted) non-zero artificial viscosity.

5. Conclusions

Performance of the artificial viscosity technique suggested for numerical simulation of unsteady one-dimensional two-phase compressible flows with the two-fluid approach and the pressure equilibrium assumption has been evaluated via numerical solution of a test problem for two-phase flow in a channel of variable cross-section. It has been shown that in case of the basic model, leading to an ill-posed problem, some sets of initial data result in occurrence of intensive non-physical oscillations of the computed variables and finally in destruction of the numerical solution. Adding artificial viscosity provides a stable process of getting a numerical solution and predicts a physically adequate flow dynamics without non-physical oscillations.

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