Solutions for dissipative quadratic open systems: part I - bosons

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This is a work in two parts in which we show how to solve a large class of Lindblad master equations for non-interacting particles on $L$ sites. In part I we concentrate on bosonic particles. We show how to reduce the problem to diagonalizing an $L \times L$ non-Hermitian matrix. In particular, for boundary dissipative driving of a uniform chain, the matrix is a tridiagonal bordered Toeplitz matrix which can be solved analytically for the normal master modes and their relaxation rates (rapidities). In the regimes in which an analytical solution cannot be found, our approach can still provide a speed-up in the numerical evaluation. We use this numerical method to study the relaxation gap at non-equilibrium phase transitions in a boundary driven bosonic ladder with synthetic gauge fields. We conclude by showing how to construct the non-equilibrium steady state.

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I. INTRODUCTION

Quantum systems in contact with an environment display a very rich physics including emergence of non-equilibrium phase transitions [11] which can be used to engineer particularly interesting non-equilibrium steady states (NESS). Moreover, also the relaxation towards an asymptotic or steady state can, for example, manifest a non-trivial dynamics, from power-law to stretched exponentials and even aging [12]. A particularly important class of open quantum system is that of boundary driven systems, in which a current may be induced by the coupling, only at the extremities, of the system to the environment. This class of systems is of particular relevance in the study of quantum transport.

The knowledge of analytical solutions for open quantum systems would allow to build a better intuition of the physics of these systems and to test numerical methods. The NESS of a boundary driven Heisenberg model can, in many regimes, be computed analytically using a Matrix Product Ansatz as shown in [8–13] (for a review see [14]). In [15] an exact solution for a diffusive XX chain is presented using a cleverly designed ansatz. For a boundary driven bosonic non-interacting system, in [16] the authors showed how to analytically compute the local densities and the current. In the seminal article [17], which is particularly relevant to our work, Prosen showed that for a quadratic open fermionic model with $L$ sites, solving for the relaxation rates of the quantum Lindblad equation, can be reduced to the diagonalization of a $4L \times 4L$ anti-symmetric matrix.

Here we build on this strategy while focusing on a boundary dissipative driven quadratic bosonic system. We are able to reduce the problem to diagonalizing a $L \times L$ matrix which can thus be studied numerically more effectively. In the second part of this work we will consider the case of fermions. Here we show that in many physically relevant cases the matrix to be diagonalized is a tridiagonal bordered Toeplitz matrix for which analytical expressions for the eigenvalues and eigenvectors are known. We also show an example for which we can explicitly write the relaxation rates (rapidities) of all the normal master modes of the Lindblad master equation. This can be used, for example, to compute the relaxation gap, that is the rapidity of the slowest decaying normal master mode. Last we use the framework developed to give an expression for the steady state.

This paper is organized as follows: In section II we introduce the quadratic bosonic model we study. In section III-D we show how to diagonalize the Lindblad master equation and obtain the normal master modes. In section IV we show how to solve analytically the boundary driven bosonic chain and study the scaling of the relaxation gap. In section V we introduce an efficient numerical algorithm to compute the quadratic observables such as currents and densities which we then use to study the relaxation gap of systems which present non-equilibrium phase transitions. In section VI we analytically construct a similarity transformation which maps the vacuum state to the steady state of the system and in section VII we draw our conclusions.

II. MODEL

We consider an open quantum systems of $L$ sites with bosonic particles. Its dynamics is described by the quantum Lindblad master equation [18, 19] with Lindbladian $\mathcal{L}$

$$\frac{d}{dt} \hat{\rho} = \mathcal{L}(\hat{\rho}) = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \mathcal{D}(\hat{\rho}). \quad (1)$$

Here $\hat{\rho}$ is the density operator of the system, $\hat{H}$ is the Hamiltonian, and $\mathcal{D}$, the dissipator, describes the dissipative part of the evolution. The Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \sum_{m,n=1}^{L} h_{m,n} \hat{a}_{m}^{\dagger} \hat{a}_{n}, \quad (2)$$

where $h$ is an $L \times L$ Hermitian matrix, and $\hat{a}_{m}^{\dagger} (\hat{a}_{m})$ creates (annihilates) one boson on site $m$. The dissipative
part is given by
\[ D(\hat{\rho}) = \sum_{i,j=1}^{L} \left[ \Lambda_{i,j}^+ (\hat{\alpha}_i^\dagger \hat{\rho} \hat{\alpha}_j^\dagger - \hat{\alpha}_j^\dagger \hat{\alpha}_i \hat{\rho}) + \Lambda_{i,j}^- \hat{\rho} \hat{\alpha}_i \hat{\alpha}_j^\dagger + \text{H.c.} \right], \]
where \( \Lambda^+ \) and \( \Lambda^- \) are \( L \times L \) Hermitian and non-negative matrices.

III. SOLVING THE MASTER EQUATION

A. Reshaping the density operator in a new representation

We perform a one-to-one mapping the density operator basis elements \( |n_1, n_2, \ldots, n_L, n'_1, n'_2, \ldots, n'_L \rangle \) to a state vector basis (with \( 2L \) sites) \( |n_1, \ldots, n_L, n'_1, \ldots, n'_L \rangle \) (see for example [20, 23]). From this, the operator \( \hat{\alpha}_i \) acting on site \( i \) on the left of the density matrix is mapped to \( \hat{\alpha}_i \) acting on the state vector on the \( i \)-th site, while the operator \( \hat{\alpha}_i \) on the right of the density matrix is mapped to \( \hat{\alpha}_i^\dagger \), acting on the state vector. We denote the density operator to be \( |\rho\rangle \) in the new representation.

B. The master equation in the new representation

\( \mathcal{L} \) in Eq.(1) can thus be written as
\[
\mathcal{L} = \left( \begin{array}{c} \text{a}_1^\dagger \text{l} \\ \text{a}_{L+1}^\dagger \text{2L} \end{array} \right)^t M \left( \begin{array}{c} \text{a}_1^\text{l} \\ \text{a}_{L+1}^{\text{2L}} \end{array} \right) + \left( \begin{array}{c} \text{a}_1 \text{l} \\ \text{a}_{L+1} \text{2L} \end{array} \right)^t M^\dagger \left( \begin{array}{c} \text{a}_1^{\dagger} \text{l} \\ \text{a}_{L+1}^{\dagger} \text{2L} \end{array} \right) + \text{tr}(\Lambda^\dagger - \Lambda)
\]
where \( M \) is a \( 2L \times 2L \) matrix,
\[
M = \left( \begin{array}{cc} K & A^\dagger \\ A & K \end{array} \right),
\]
Here \( K = (-i \hbar/\hbar) - \Lambda - \Lambda^{-1} / 2 \), where with \( A^t \) we indicate the transpose of the matrix \( A \). We have also used the notation \( \text{a}_1 \to \text{l} \) to denote the column vector made of operators \( \hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_L \) and \( \text{a}_1 \to \text{l} \) for the column vector made of \( \hat{\alpha}_1^\dagger, \hat{\alpha}_2^\dagger, \ldots, \hat{\alpha}_L^\dagger \), the same applies for \( \text{a}_{L+1} \to \text{2L} \) and \( \text{a}_{L+1}^\dagger \to \text{2L} \).

C. Normal master modes of the master equation

In general \( M \) is not Hermitian and it cannot always be diagonalized, however in the following we start from the assumption that we know a transformation which can diagonalize \( M \) and preserves bosonic commutation relations. This assumption is a posteriori verified in all the cases we considered. This transformation is given by the matrices \( W_1 \) and \( W_2 \) as follows
\[
\left( \begin{array}{c} \text{a}_{1 \to \text{l}} \\ \text{a}_{L+1 \to \text{2L}} \end{array} \right) = W_1 \left( \begin{array}{c} \text{b}_{1 \to \text{l}} \\ \text{b}_{L+1 \to \text{2L}} \end{array} \right) \]
\[
\left( \begin{array}{c} \text{a}_{1 \to \text{l}} \\ \text{a}_{L+1 \to \text{2L}} \end{array} \right) = W_2 \left( \begin{array}{c} \text{b}_{1 \to \text{l}} \\ \text{b}_{L+1 \to \text{2L}} \end{array} \right)
\]
where as for \( a_{1 \to \text{l}} \) and \( a_{1 \to \text{l}}^\dagger \), \( b_{1 \to \text{l}} \) means the column vector made of operators \( b_1, b_2, \ldots, b_L \) and \( b_{1 \to \text{l}}^\dagger \) means the column vector made of \( b_1^\dagger, b_2^\dagger, \ldots, b_L^\dagger \). Again the same notation applies for \( b_{L+1 \to \text{2L}} \) and \( b_{L+1 \to \text{2L}}^\dagger \).
Using this transformation we get
\[
\mathcal{L} = \left( \begin{array}{c} \text{b}_{1 \to \text{l}} \\ \text{b}_{L+1 \to \text{2L}} \end{array} \right)^t W_1 M W_1^{-1} \left( \begin{array}{c} \text{b}_{1 \to \text{l}} \\ \text{b}_{L+1 \to \text{2L}} \end{array} \right) + \text{tr}(\Lambda^\dagger - \Lambda)
\]
The bosonic commutation relation can be written as
\[
\left[ \left( \begin{array}{c} \text{a}_{1 \to \text{l}} \\ \text{a}_{L+1 \to \text{2L}} \end{array} \right), \left( \begin{array}{c} \text{a}_{1 \to \text{l}}^\dagger \\ \text{a}_{L+1 \to \text{2L}}^\dagger \end{array} \right) \right] = Z_L,
\]
and requiring for the bosonic commutation relation to apply also to the \( b \) we get
\[
\left[ \left( \begin{array}{c} \text{b}_{1 \to \text{l}} \\ \text{b}_{L+1 \to \text{2L}} \end{array} \right), \left( \begin{array}{c} \text{b}_{1 \to \text{l}}^\dagger \\ \text{b}_{L+1 \to \text{2L}}^\dagger \end{array} \right) \right] = Z_L
\]
and hence
\[
Z_L = W_1 Z_L W_1^t \iff W_2 = Z_L W_1^t Z_L^{-1}
\]
Here we have used
\[
Z_L = \left( \begin{array}{cc} 1_L & 0 \\ 0 & -1_L \end{array} \right)
\]
where we have denoted \( 1_L \) for an identity matrix of size \( L \). In the following we will also use the matrices
\[
X_L = \left( \begin{array}{cc} 0 & 1_L \\ 1_L & 0 \end{array} \right)
\]
and
\[
Y_L = -i \left( \begin{array}{cc} 0 & 1_L \\ -1_L & 0 \end{array} \right).
\]
Matrices in Eqs.(11) [12] [13], being given by a tensor product between Pauli matrices and identity, satisfy the relations
\[
Z_L^2 = 1_{2L}, \quad X_L^2 = 1_{2L}, \quad Y_L^2 = 1_{2L}, \quad Z_L X_L = -X_L Z_L = iY_L.
\]
It follows that
\[
\mathcal{L} = \left( \begin{bmatrix} b_{1 \to L} \\ b_{L+1 \to 2L} \end{bmatrix} \right)^t Z_L W_1^{-1} Z_L M W_1 \left( \begin{bmatrix} b_{1 \to L} \\ b_{L+1 \to 2L} \end{bmatrix} \right) \\
+ \left( \begin{bmatrix} b_{1 \to L} \\ b_{L+1 \to 2L} \end{bmatrix} \right)^t W_1^t M^t Z_L W_1^{-1} Z_L \left( \begin{bmatrix} b_{1 \to L} \\ b_{L+1 \to 2L} \end{bmatrix} \right) \\
+ \text{tr} (\Lambda^{-t} - \Lambda^+) \\
= \left( \begin{bmatrix} b_{1 \to L} \\ b_{L+1 \to 2L} \end{bmatrix} \right)^t Z_L (W_1^{-1} Z_L M W_1) \left( \begin{bmatrix} b_{1 \to L} \\ b_{L+1 \to 2L} \end{bmatrix} \right) \\
+ \left( \begin{bmatrix} b_{1 \to L} \\ b_{L+1 \to 2L} \end{bmatrix} \right)^t (W_1^{-1} Z_L M W_1)^t Z_L \left( \begin{bmatrix} b_{1 \to L} \\ b_{L+1 \to 2L} \end{bmatrix} \right) \\
+ \text{tr} (\Lambda^{-t} - \Lambda^+). \\
\tag{16}
\]
This implies that the problem of finding the normal modes of the system reduces to finding a $W_1$ such that $Z_L M$ can be diagonalized, that is,
\[
W_1^{-1} (Z_L M) W_1 = \text{diag}(\beta_1, \beta_2, \ldots, \beta_{2L}), \\
\tag{17}
\]
where $\text{diag}(\vec{\beta})$ is a diagonal matrix with the elements of the vector $\vec{\beta}$ on its diagonal. It would then be possible to write the following compact form for $\mathcal{L}$:
\[
\mathcal{L} = 2 \sum_{i=1}^{L} (\beta_i b_i^t - \beta_{L+i} b_{L+i}^t) \\
+ \sum_{i=1}^{L} (\beta_i - \beta_{L+i}) + \text{tr} (\Lambda^{-t} - \Lambda^+) \\
\tag{18}
\]

D. Diagonalizing $Z_L M$

Here we will explicitly construct the eigenvalues and eigenvectors of the matrix $Z_L M$. From Eq. (6), we note that the relation
\[
X_L M X_L = M^\dagger \\
\tag{19}
\]
we find that if $x = \begin{bmatrix} u \\ v \end{bmatrix}$ is a right eigenvector of $Z_L M$ with eigenvalue $\omega$, then $x^t Y_L$ is a left eigenvector of $Z_L M$ with eigenvalue $-\omega^*$. In fact, using Eq. (14)
\[
Z_L M x = \omega x \to x^t M^t Z_L = \omega^* x^t \\
\to x^t X_L X_L M^t X_L = \omega^* x^t Z_L X_L \\
\to x^t X_L M = \omega^* x^t Z_L X_L. \\
\]
This implies, using Eq. (15), that $x^t X_L Z_L Z_L M = x^t X_L M = \omega^* x^t Z_L X_L$, which means $x^t Y_L Z_L M = -\omega^* x^t Y_L$. Thus it follows that
\[
x^t Y_L (Z_L M) = -\omega^* x^t Y_L. \\
\tag{20}
\]
i.e. $x^t Y_L$ is a left eigenvector of $Z_L M$.

Moreover if $x_1$ is a right eigenvector of $Z_L M$ with eigenvalue $\omega_1$, and $x_2$ is a right eigenvector of $Z_L M$ with eigenvalue $\omega_2$, then if $\omega_1 + \omega_2^* \neq 0$ it follows that $x_1^t Y_L x_2 = 0$. In fact
\[
Z_L M x_1 = \omega_1 x_1; \\
Z_L M x_2 = \omega_2 x_2,
\]
then
\[
x_1^t Y_L Z_L M = -\omega_1^* x_1^t Y_L; \\
Z_L M x_2 = \omega_2 x_2; \\
\to (\omega_1^* + \omega_2) x_1^t Y_L x_2 = 0
\]
Since the eigenvalues of $Z_L M$ always appear in pairs, we could list the eigenvalues and the corresponding eigenvectors of $Z_L M$ as $\omega_1, \omega_2, \ldots, \omega_{L}, -\omega_{L}^*, \ldots, -\omega_1^*$, with the matrix $W_1$ composed in each column by the right eigenvectors $W_1 = (x_1, x_2, \ldots, x_{2L})$. Then following Eq. (20) we know that $x_{L+j}^t Y_L$ is the left eigenvector of $Z_L M$ corresponding to $\omega_j$, and $x_j^t Y_L$ is the left eigenvector corresponding to $-\omega_j^*$, for $1 \leq j \leq L$. Therefore the left eigenvectors of $Z_L M$ constitute the matrix $X_L W_1^t Y_L$.

We can now go to normalize the right eigenvectors as
\[
|x_L W_1^t Y_L W_1 = Z_L \iff Y_L W_1^t Y_L W_1 = -I_{2L} \\
\tag{21}
\]
so that we have
\[
W_1^{-1} = -Y_L W_1^t Y_L, \tag{22}
\]
\[
W_2 = -X_L W_1^t X_L. \tag{23}
\]
At this point we define a new $L \times L$ matrix $P$, which satisfies
\[
P = K + A^+ = (-i \hbar / \hbar + A^+ - A^{-1})/2, \tag{24}
\]
for which we assume to have the eigendecomposition
\[
P W_P = W_P \lambda_P, \tag{25}
\]
where $W_P$ and $\lambda_P$ are eigenvectors and eigenvalues. Then we find that the $2L \times L$ matrix formed by $W_P$, constitutes $L$ right eigenvectors of $Z_L M$, corresponding to $\lambda_P$, and the $L \times 2L$ matrix $(-W_P^t, -W_P^t)$ constitutes $L$ left eigenvectors of $Z_L M$, corresponding to $-\lambda_P^*$. This can be shown from
\[
Z_L M \left( \begin{bmatrix} W_P \\ W_P \end{bmatrix} \right) = \left( \begin{bmatrix} P W_P \\ P W_P \end{bmatrix} \right) = \left( \begin{bmatrix} W_P \\ W_P \end{bmatrix} \right) \lambda_P
\]
and
\[
(W_P^t, -W_P^t) Z_L M = (W_P^t, -W_P^t) \left( \begin{bmatrix} P W_P \\ P W_P \end{bmatrix} \right) = -\lambda_P^* (W_P^t, -W_P^t)
\]
Hence by denoting the remaining $L$ right eigenvectors of $Z_L M$ as $C D$, where $C, D$ are $L \times L$ matrices, we
know that they form the right eigenvectors with eigenvalues \(-\lambda_P\), which are paired with the left eigenvectors \((W_P^\dagger - W_Q^\dagger)\). Also \((-D^\dagger C^\dagger)\) will be the left eigenvectors corresponding the eigenvalues \(\lambda_P\), which are paired with the right eigenvectors \((W_P^\dagger)\).

Therefore \(W_1\) and \(W_2\) can be written more explicitly as

\[
W_1 = \begin{pmatrix} W_p & C \\ W_p & D \end{pmatrix}, \quad W_1^{-1} = \begin{pmatrix} -D^\dagger & C^\dagger \\ W_p^\dagger & -W_p^\dagger \end{pmatrix},
\]

\[
W_2 = -\begin{pmatrix} D^* W_p^\dagger & C^* W_p^\dagger \\ C^* W_p^\dagger & -C^\dagger D^\dagger \end{pmatrix}, \quad W_2^{-1} = \begin{pmatrix} W_p^\dagger & -W_p^\dagger \\ -C^\dagger & D^\dagger \end{pmatrix}
\]

which means

\[
\begin{align*}
\alpha_{1\rightarrow L} &= W_p b_{1\rightarrow L} + C b_{L+1\rightarrow 2L}; \\
\alpha_{1\rightarrow +1\rightarrow 2L}^\dagger &= W_p b_{1\rightarrow L} + D b_{L+1\rightarrow 2L}; \\
\alpha_{1\rightarrow L}^\dagger &= -D^\dagger b_{1\rightarrow L} - W_p^\dagger b_{L+1\rightarrow 2L}; \\
\alpha_{L+1\rightarrow 2L} &= -C^* b_{1\rightarrow L} - W_p^\dagger b_{L+1\rightarrow 2L}
\end{align*}
\]

and the inverse equation

\[
\begin{align*}
b_{1\rightarrow L} &= -D^\dagger \alpha_{1\rightarrow L} + C^* \alpha_{L+1\rightarrow 2L}; \\
b_{L+1\rightarrow 2L}^\dagger &= W_p^\dagger \alpha_{1\rightarrow L} - W_p^\dagger \alpha_{L+1\rightarrow 2L}; \\
b_{1\rightarrow L}^\dagger &= W_p^\dagger \alpha_{1\rightarrow L} - W_p^\dagger \alpha_{L+1\rightarrow 2L}; \\
b_{L+1\rightarrow 2L} &= -C^* \alpha_{1\rightarrow L} + D^\dagger \alpha_{L+1\rightarrow 2L}
\end{align*}
\]

Noticing that \(\sum \lambda_{P,i} = \text{tr}(P) = \left[-i \left(\frac{\hbar}{\tau} - \frac{1}{\Lambda^\dagger - \Lambda^-}\right)\right]/2\) and since the \((\lambda_{P,1}, \ldots, \lambda_{P,L}, -\lambda_{P,1}, \ldots, -\lambda_{P,L})\) correspond to \((\beta_1, \ldots, \beta_{2L})\), the eigenvalues of \(Z_L M\), we get the following identity

\[
\sum_{i=1}^{L} (\beta_i - \beta_{L+i}) = \sum_{i=1}^{L} (\lambda_{P,i} + \lambda_{P,i}^*) = \text{tr}(\Lambda^\dagger - \Lambda^-)
\]

which exactly cancels the last term in the expression of \(L\) in Eq.\[18\].

We can then write \(L\) as

\[
L = 2 \sum_{i=1}^{L} \lambda_{P,i} \hat{b}_i \hat{b}_i + 2 \sum_{i=1}^{L} \lambda_{P,i}^* \hat{b}_i^\dagger \hat{b}_{L+i}.
\]

The state \(|\rho_{ss}\rangle\) which annihilates all the operator \(b_{1\rightarrow 2L}\) is the steady state because \(L|\rho_{ss}\rangle = 0\). The \(\hat{b}_i\) are the normal master modes of the Lindblad master equation and the \(\lambda_{P,i}\) the rapidities.

### E. Non-positivity of the eigenvalues of the Lindbladian

To prove the eigenvalues of \(L\) are non-positive, it is sufficient to prove that all the eigenvalues of the matrix \(P\) are non-positive. The proof is similar as in \[22\]. Assuming \(P x = \omega x\), therefore \(x^\dagger P^\dagger = \omega x^\dagger\), then we have

\[
x^\dagger (P^\dagger + P) x = x^\dagger P^\dagger x + x^\dagger P x = 2 \mathcal{R}(\omega) x^\dagger x
\]

where \(\mathcal{R}(\omega)\) means the real part of \(\omega\). Moreover \(P^\dagger + P = \Lambda^\dagger - \Lambda^-\) hence all the eigenvalues of the matrix on the right-hand side have to be non-positive for the master equation to have a steady state. Hence we can conclude that

\[
\mathcal{R}(\omega) \leq 0,
\]

i.e. the real part of eigenvalues of the Lindbladian is non-positive.

### F. Computing the expectation value \(\langle \hat{a}_i^\dagger \hat{a}_j \rangle\)

We denote \(|1\rangle = \sum_{i_1, i_2, \ldots, i_L} |i_1, i_2, \ldots, i_L, i_1, i_2, \ldots, i_L\rangle\) to be the state vector resulting from the mapping of an identity operator, and \(|1\rangle\) to be its transpose. Computing the expectation value of observable \(\hat{O}\) on the steady state \(\rho_{ss}\), which is \(\text{tr}(\hat{O} \rho_{ss})\), is equivalent to the expression \(\langle 1 | \hat{O} | \rho_{ss} \rangle\) where we have simply re-written the trace of an operator times the density operator in the enlarged space. Of course the operator \(\hat{O}\) has also been mapped to the new space. In order to compute quadratic expectation values such as \(\text{tr}(\hat{a}_i^\dagger \hat{a}_j \rho_{ss}) = \langle 1 | \hat{a}_i^\dagger \hat{a}_j | \rho_{ss} \rangle\) it is convenient to rewrite the eigenvalue Eq.\[17\] in a different form.

#### 1. Equation for quadratic operators

To do so we start from \(W_1^{-1} W_1 = 1_{2L}\) and using Eq.\[26\], we have

\[
D = C - W_P^\dagger W_P^\dagger; \quad C = W_P^\dagger W_P^\dagger Q\mathbf{W}_P^\dagger
\]

where \(Q\) is a \(L \times L\) Hermitian matrix. Therefore \(W_1\) can also be written as

\[
W_1 = \begin{pmatrix} W_p & W_P^\dagger Q \\ W_p^\dagger & W_P^\dagger Q W_P - W_P^\dagger \end{pmatrix}
\]

From Eq.\[17\], which can be written now as

\[
Z_L M W_1 = W_1 \begin{pmatrix} \lambda_P & 0 \\ 0 & -\lambda_P^\dagger \end{pmatrix}
\]

and using Eq.\[36\], together with \(\Omega = W_P^\dagger Q W_P^\dagger\), we have

\[
P \Omega + \Omega P^\dagger = \Lambda^\dagger
\]

Solving this equation for \(\Omega\) will prove very useful in the following.
The observable matrix $O_{i,j} = \text{tr}(\hat{\rho} \hat{a}^\dagger_i \hat{a}_j) = \langle 1 | \hat{a}^\dagger_i \hat{a}_j | \rho_{ss} \rangle$ is then given by
\begin{equation}
O = -\Omega'.
\end{equation}

IV. EXACT SOLUTION OF A BOUNDARY DRIVEN BOSONIC CHAIN

Here we apply our method to directly obtain the spectrum of Eq. (5) for a class of linear chains (LC) which can then be solved analytically in the limit of a long chain. We consider a linear lattice of $L$ sites in which each site can have identical bosons and which is driven at boundaries (similar, for example, to [24]). The Lindbladian $\mathcal{L}_{\text{LC}}$ then becomes
\begin{equation}
\mathcal{L}_{\text{LC}}(\hat{\rho}) = \frac{1}{\hbar} [\hat{H}_{\text{LC}}, \hat{\rho}] + \mathcal{D}_{\text{LC}}(\hat{\rho})
\end{equation}
with
\begin{equation}
\hat{H}_{\text{LC}} = -J \sum_{l=1}^{L-1} (\hat{a}_l^\dagger \hat{a}_{l+1} + \hat{a}_{l+1}^\dagger \hat{a}_l) + \text{H.c.},
\end{equation}
and
\begin{equation}
\mathcal{D}_{\text{LC}}(\hat{\rho}) = \sum_{l=1}^{L} \left[ \Lambda_l^+ (\hat{a}_l^\dagger \hat{\rho} \hat{a}_l - \hat{\rho} \hat{a}_l^\dagger \hat{a}_l) + \Lambda_l^- (\hat{a}_l^\dagger \hat{\rho} \hat{a}_l - \hat{\rho} \hat{a}_l^\dagger \hat{a}_l) + \text{H.c.} \right],
\end{equation}
where $\Lambda_l^+$ and $\Lambda_l^-$ are respectively the raising and lowering rates at site $l$, while $J$ is the tunnelling amplitude.

In this case, the only non-zero elements of the matrix $\mathbf{h}$ are
\begin{equation}
h_{j,j+1} = h_{j+1,j} = -J.
\end{equation}
For the dissipation we rewrite the four coefficients $\Lambda_l^a$ with four new parameters
\begin{equation}
\Gamma_1 = \Lambda_1^- - \Lambda_1^+, \quad \bar{n}_1 = \frac{\Lambda_1^+}{\Gamma_1}
\end{equation}
\begin{equation}
\Gamma_L = \Lambda_L^+ - \Lambda_L^-, \quad \bar{n}_L = \frac{\Lambda_L^+}{\Gamma_L}
\end{equation}

Therefore, all the non-zero elements of matrix $\mathbf{P}$ are
\begin{equation}
P_{1,1} = -\frac{\Gamma_1}{2}, \quad P_{L,L} = -\frac{\Gamma_L}{2}
\end{equation}
\begin{equation}
P_{m,m+1} = P_{m+1,m} = \frac{iJ}{2\hbar}
\end{equation}
for $1 \leq m < L$. Note that $\bar{n}_1$ and $\bar{n}_L$ do not appear in the matrix $\mathbf{P}$, and hence will not affect the rapidities.

This results in the important fact that $\mathbf{P}$ is a tridiagonal matrix whose elements are constant along the diagonals (i.e. Toeplitz) except for top-left and bottom-right corners (i.e. bordered). The eigenvalues and eigenvectors
of this matrix can be analytically computed \[25\] (for the
eigendecomposition of more general tridiagonal matrices
see for example \[26,27\]). Assuming that \(\lambda\) is an
eigenvalue of \(\mathbf{P}\), and \(u\) is the corresponding right
eigenvector so that \(\mathbf{P}u = \lambda u\), then we find that \(\lambda\) and
\(u\) are given by
\[
\lambda = \frac{J}{\hbar} \cos (\theta)
\]
and the \(L\) elements of \(u\) are
\[
u_{j} = \frac{u_{1}}{\sin (\theta)} \left\{ \sin (j\theta) - \frac{\hbar \Gamma_{1}}{J} \sin ((j - 1)\theta) \right\}.
\]
In Eqs.\((53,54)\) \(\theta\) is a complex number which satisfies the equality
\[
- \frac{J^{2}}{\hbar^{2}} \sin [(L + 1)\theta] + \frac{J}{\hbar} (\Gamma_{1} + \Gamma_{L}) \sin (L\theta) + \Gamma_{1} \Gamma_{L} \sin ((L - 1)\theta) = 0
\]
extcept the trivial solutions \(\theta \neq m\pi\) with \(m \in \mathbb{Z}\).

Denoting \(\theta = \alpha + i\beta\), we transform the above equation into
two equations of real numbers
\[
\frac{J^{2}}{\hbar^{2}} \sin [(L + 1)\theta] \approx \sinh [(L - 1)\beta] \approx \sin [(L + 1)\beta]
\]
with Eqs.\((56,57)\) become
\[
\begin{align*}
\frac{J^{2}}{\hbar^{2}} \sin [(L + 1)\theta] & + \frac{J}{\hbar} (\Gamma_{1} + \Gamma_{L}) \cos (L\theta) - \Gamma_{1} \Gamma_{L} \sin ((L - 1)\theta) = 0, \\
\frac{J^{2}}{\hbar^{2}} \cos [(L + 1)\theta] & - \frac{J}{\hbar} (\Gamma_{1} + \Gamma_{L}) \sin (L\theta) - \Gamma_{1} \Gamma_{L} \cos ((L - 1)\theta) = 0.
\end{align*}
\]
In the following we solve the above equation approximately, in the limit \(L \to \infty\). All the solutions are found for \(\alpha \in [0, \pi]\) \[27\]. We then make the approximation that
\[
\begin{align*}
\sinh [(L + 1)\beta] & \approx \sinh [(L - 1)\beta] \approx \sin [(L + 1)\beta] \\
\cosh [(L + 1)\beta] & \approx \cosh [(L - 1)\beta] \approx \cosh [(L + 1)\beta]
\end{align*}
\]
which with Eqs.\((56,57)\) we get
\[
\begin{align*}
\frac{J^{2}}{\hbar^{2}} \sin [(L + 1)\theta] & + \frac{J}{\hbar} (\Gamma_{1} + \Gamma_{L}) \cos (L\theta) - \Gamma_{1} \Gamma_{L} \sin ((L - 1)\theta) = 0, \\
\frac{J^{2}}{\hbar^{2}} \cos [(L + 1)\theta] & - \frac{J}{\hbar} (\Gamma_{1} + \Gamma_{L}) \sin (L\theta) - \Gamma_{1} \Gamma_{L} \cos ((L - 1)\theta) = 0.
\end{align*}
\]
Combining Eqs.\((60,61)\) we get
\[
\begin{align*}
\frac{J^{2}}{\hbar^{2}} \sin [(L + 1)\alpha] & + \frac{J}{\hbar} (\Gamma_{1} + \Gamma_{L}) \cos (2\alpha) \tan \theta = 0, \\
\frac{J^{2}}{\hbar^{2}} \cos [(L + 1)\alpha] & - \frac{J}{\hbar} (\Gamma_{1} + \Gamma_{L}) \sin (2\alpha) \tan \theta = 0.
\end{align*}
\]
which can be rewritten as
\[
\begin{align*}
(\kappa_{1} + \kappa_{L}) \sin (2L\alpha) & + \sin [2(L - 1)\alpha] = 0, \\
\kappa_{1} \kappa_{L} \sin (2(L + 1)\alpha) & = 0.
\end{align*}
\]
where \(\kappa_{1} = J^{2}/(\hbar^{2}T_{0}^{2})\) and \(\kappa_{L} = J^{2}/(\hbar^{2}T_{L}^{2})\). Eq.\((63)\)
can be solved analytically when \(J^{2} = \hbar^{2}T_{1} \Gamma_{L}\). In fact
\(\kappa_{1} = \frac{1}{\kappa_{L}} = \kappa\) and Eq.\((63)\) reduces to
\[
\left( \kappa + \frac{1}{\kappa} \right) \sin (2L\alpha) + \sin [2(L - 1)\alpha] + \sin [2(L + 1)\alpha] = 0
\]
or equivalently \(\kappa + \frac{1}{\kappa} + 2 \cos (\alpha) \sin (2L\alpha) = 0\). Since
\(\kappa + 1/\kappa \geq 2\), the real solutions are
\[
\alpha = \frac{k\pi}{L},
\]
with \(1 \leq k < L\). Note that the solutions of Eq.\((64)\)
\(\alpha = k\pi/2L\) with \(k\) odd have been discarded because inconsistent with Eqs.\((60,61)\).

We then get for \(\beta\)
\[
\tanh (L\beta) = \frac{2\sqrt{\kappa}}{\kappa + 1} \sin \frac{k\pi}{L},
\]
which results in
\[
\beta = \frac{1}{2L} \ln \left( \frac{1 - 2\sqrt{\kappa} \sin \frac{k\pi}{L}}{1 + 2\sqrt{\kappa} \sin \frac{k\pi}{L}} \right).
\]
Using Eqs.\((53)\) and that \(\theta = \alpha + i\beta\) we can write
\[
\lambda = \frac{J}{\hbar} \sin (\alpha) \sinh (\beta) + i \frac{J}{\hbar} \cos (\alpha) \cosh (\beta).
\]
and inserting the various $\alpha$ and $\beta$ from Eqs. \ref{eq:12} \cite{65,67} we have an analytical solution for the eigenvalues. In Fig. \ref{fig:1} we compare the analytical solution Eq. \ref{eq:13}, blue diamonds, with numerical evaluation of the spectrum for Eq. \ref{eq:4}, red circles, for a system of length $L = 100$. The figure shows a remarkable match of the analytical and numerical spectra.

A natural consequence of knowing the spectrum, is that it is possible to compute the relaxation gap $\Delta$, which is given by the eigenvalues of the Lindbladian with the real part closest to zero. We thus get

$$\Delta = \frac{J^2}{\hbar} \sin \left( \frac{\pi}{L} \right) \sinh \left[ \frac{1}{2L} \ln \left( 1 + \frac{2\sqrt{\pi}}{\kappa+\pi} \sin \left( \frac{\pi}{L} \right) \right) \right]$$

(69)

$$\approx \frac{2\sqrt{\pi}J}{\hbar(\kappa+1)L} \sin^2 \left( \frac{\pi}{L} \right)$$

$$\approx \frac{2\pi^2 \sqrt{\kappa}}{\hbar(\kappa+1)L^3}$$

which thus scales as $1/L^3$, in agreement with the predictions in \cite{17,29}.

V. EFFICIENT ALGORITHM TO COMPUTE QUADRATIC OBSERVABLES

In this section we propose an efficient algorithm to solve Eq. \ref{eq:38}. We start by solving the eigenvalue decomposition problem of matrix $P$, to get the left eigenvector space $W_P^\text{f}$, the right eigenvector space $W_P$ and the diagonal matrix of eigenvalues $\lambda_P = \text{diag}(\lambda_{P,1}, \ldots, \lambda_{P,L})$. At this point it is already possible to figure whether the system has any dark mode; In fact this would be manifested by the existence of eigenvalues with zero real part. If the real part of all eigenvalues of $P$ is strictly smaller than 0, then the system has no dark modes, the steady state is unique and the following algorithm can be used. We can thus write $\Omega = W_P Q W_P^\text{f}$, which gives

$$P\Omega + \Omega P^\dagger = PW_P Q W_P^\text{f} + W_P Q W_P^\text{f} P^\dagger = W_P \lambda_P Q W_P^\text{f} + W_P \lambda_P^* W_P^\text{f}$$

$$= \Lambda^+$$

We can renormalize $W_P$ and $W_P^\text{f}$ so that $W_P^\text{f} = W_P^{-1}$. From Eq. \ref{eq:40} we get the elements of the Hermitian matrix $Q$ by

$$\lambda_P Q + Q \lambda_P^* = W_P^{-1} \Lambda^+ (W_P^{-1})^\dagger = W_P^\text{f} \Lambda^+ W_P^\dagger$$

$$\Leftrightarrow Q_{m,n} = \frac{(W_P^\dagger \Lambda^+ W_P)^{m,n}}{\lambda_{P,m} + \lambda_{P,n}^*}$$

(71)

The elements of the matrix corresponding to $\langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle$, i.e. $O_{i,j} = \text{tr} (\hat{\rho} \hat{\alpha}_j^\dagger \hat{\alpha}_j)$ are then given by, using Eq. \ref{eq:44}

$$O_{i,j} = -\Omega_{j,i} = - \sum_{m,n} W_{P,j,m} Q_{m,n} W_{P,i,n}^*$$

(72)

Note that with this approach we only need to solve an eigenvalue decomposition problem, plus a few matrix multiplications. All the matrices involved in these procedures are of size $L \times L$. The complexity of this algorithm is thus $O(L^3)$ which is the complexity of solving a $L \times L$ non-Hermitian eigendecomposition problem. Therefore this algorithm is faster than directly solving Eq. \ref{eq:38} as a linear equation for which the complexity would be $O(L^6)$ \cite{28}.

It should also be noted that because of the structure of Eq. \ref{eq:71}, this approach can become unstable if $P$ has eigenvalues whose real part is very close to 0.

A. Scaling of the relaxation time across the phase transition

With the algorithm introduced by Eqs. \ref{eq:71} \ref{eq:72}, we can more effectively explore larger systems allowing to easily study the scaling of the relaxation gap in various nonequilibrium phases. In the following we apply our method to study the boundary driven bosonic ladder when also a magnetic field is imposed on it. This system is depicted in Fig. \ref{fig:2}a. The Lindblad master equation for this bosonic ladder (BL) is

$$\mathcal{L}_{\text{BL}}(\hat{\rho}) = -\frac{i}{\hbar} \left[ \hat{H}_{\text{BL}}, \hat{\rho} \right] + D_{\text{BL}}(\hat{\rho}),$$

(73)

with Hamiltonian $\hat{H}_{\text{BL}}$ given by

$$\hat{H}_{\text{BL}} = - \left( \sum_{p,j} J_p e^{i(-1)^p \phi/2} \hat{\alpha}_{j,p}^\dagger \hat{\alpha}_{j+1,p} \right. + J^\perp \left. \sum_j \hat{\alpha}_{j+1}^\dagger \hat{\alpha}_{j,2} \right) + \text{H.c.}$$

(74)

Here $J^\parallel$ is the tunnelling constant in the legs, and $J^\perp$ for the rungs. $\hat{\alpha}_{j,p} (\hat{\alpha}_{j,p}^\dagger)$ annihilates (creates) a boson in the upper (for $p = 1$) or lower (for $p = 2$) chain at the $j$-th rung of the ladder. A particle tunneling around a plaquette would acquire a net phase of $\phi$. We consider a dissipative coupling $D_{\text{BL}}$ on the two edges modelled by

$$D_{\text{BL}}(\hat{\rho}) = \sum_{j=1,L} \Gamma \left[ \hat{n}_{j,1} \left( \hat{\alpha}_{j,1}^\dagger \hat{\alpha}_{j,1} - \hat{\alpha}_{j,1} \hat{\alpha}_{j,1}^\dagger \right) + \left( \hat{n}_{j,1} + 1 \right) \left( \hat{\alpha}_{j,1}^\dagger \hat{\alpha}_{j,1} - \hat{\alpha}_{j,1} \hat{\alpha}_{j,1}^\dagger \right) + \text{H.c.} \right]$$

(75)

where $\Gamma$ is the coupling constant of the bosons at sites $j = 1, L$, while $\hat{n}_{j,1}$ is the local particle density that the dissipator would impose to the bosonic site if the site was isolated. The dissipator is only coupled to the sites at the extremeties of the upper leg \cite{29}.

The unitary counterpart of this system is known to exhibit a quantum phase transition from the Meissner to the vortex phase. The transition between the two phases
where $\Delta$ is the tunnelling between the legs, while $J^\parallel$ is the tunnelling between sites in the legs. A gauge field imposes a phase $\phi$. (b) Chiral current $J_c$ as a function of $J^\perp$ and $\phi$ for $L = 500$. The white dashed and the white dot-dashed lines correspond respectively to the two phase transitions in Eqs. (80) and (81). The four lines correspond to the point $a$, $b$, $c$ and $d$ on this line. The four lines correspond to the point $a$, $b$, $c$ and $d$ in panel (b). The line marked with green diamonds (line c) shows the scaling at $\phi = 0.5398$ which corresponds to the non-equilibrium phase transition described by Eq. (81). The red dashed line is a linear fitting of line c, which has the exponent $-5$. The other three straight dashed lines are linear fittings of a, b and d, all with the same exponent $-3$.

is characterized by the chiral current $J_c$, defined as the difference of the current between the upper and the lower leg

$$J_c = \sum_j (J_{j,1} - J_{j,2})/L,$$  \hspace{1cm} (76)

where $J_{j,p} = \langle iJ^\parallel e^{i(-1)^{p+1}/2}\hat{\alpha}^\dagger_{j,p}\hat{\alpha}_{j+1,p} + H.c\rangle$ is the part-
opening of the gap affects the total and chiral currents, it does not change the scaling of the relaxation gap. At the intersection point between the two transition lines the scaling is indeed $L^{-5}$.

VI. COMPUTING THE STEADY STATE

From Eq. (31) we understand that the steady state of the system is the vacuum of the operators $\hat{b}_j$, that is $|\rho_{ss}\rangle = |0\rangle_b$. This is related to the vacuum of the $\hat{a}_j$, $|0\rangle_a$, by a linear transformation. We can then write

$$ |\rho_{ss}\rangle = \hat{S}^{-1}|0\rangle_a. \quad (82) $$

In the following we show how to compute $\hat{S}$ from $\hat{W}_1$. First we write $\hat{S} = e^{\hat{T}}$, where $\hat{T}$ is

$$ \hat{T} = \frac{1}{2} \left( \begin{array}{cc} a_{L+1-L}^\dagger t U & a_{L+1-L}^\dagger t J \\ a_{L+1-L}^\dagger t U^t & a_{L+1-L}^\dagger t J^t \end{array} \right) \left( \begin{array}{c} a_{L+1-L}^\dagger L \\ a_{L+1-L}^\dagger L \end{array} \right) + \frac{1}{2} \left( \begin{array}{cc} a_{L+1-L}^\dagger L t U & a_{L+1-L}^\dagger L t J \\ a_{L+1-L}^\dagger L t U^t & a_{L+1-L}^\dagger L t J^t \end{array} \right) \quad (83) $$

where $U, V, I, J$ are $L \times L$ matrix. Hereafter we will write

$$ \hat{W} = \left( \begin{array}{cc} U & V \\ I & J \end{array} \right) \quad (84) $$

To calculate $e^{\hat{T}} \hat{a}_j^\dagger e^{-\hat{T}}$ and $e^{\hat{T}} \hat{a}_{L+j} e^{-\hat{T}}$, we use the relations

$$ \hat{E} := e^{\hat{T}} \hat{a}_j^\dagger e^{-\hat{T}} = \sum_{m=1}^{\infty} \frac{1}{m!} \left[ \hat{T}, \hat{a}_j^\dagger \right]_m $$

$$ \hat{F} := e^{\hat{T}} \hat{a}_{L+j} e^{-\hat{T}} = \sum_{m=1}^{\infty} \frac{1}{m!} \left[ \hat{T}, \hat{a}_{L+j} \right]_m $$

where the nested commutator is defined recursively as $[A, B]_{m+1} \equiv [A, [A, B]_m]$ with $[A, B]_0 \equiv B$. After a little algebra it is possible to show that

$$ \hat{S} \left( \begin{array}{c} a_{L+1-L}^\dagger t a_{L+1-L}^\dagger L \\ a_{L+1-L}^\dagger t a_{L+1-L}^\dagger L \end{array} \right) \hat{S}^{-1} = \left( \begin{array}{c} a_{L+1-L}^\dagger L \end{array} \right) e^{\hat{W} \hat{Z}_L} \quad (85) $$

Similarly, we have

$$ \hat{S} \left( \begin{array}{c} a_{L+1-L}^\dagger L \\ a_{L+1-L}^\dagger a_{L+1-L}^\dagger L \end{array} \right) \hat{S}^{-1} = e^{-\hat{Z}_L \hat{W}} \left( \begin{array}{c} a_{L+1-L}^\dagger a_{L+1-L}^\dagger L \end{array} \right) \quad (86) $$

We now denote $\hat{W}_1 = e^{-\hat{Z}_L \hat{W}}$, $\hat{W}_2 = e^{\hat{W} \hat{Z}_L}$, and we see that

$$ \log \left( \hat{W}_1 \right) \hat{Z}_L + \hat{Z}_L \log \left( \hat{W}_2 \right) = 0 \quad (87) $$

which allows us to write

$$ \hat{W}_2 = \hat{Z}_L \hat{W}_1^{-1} \hat{Z}_L \quad (88) $$

Therefore we have that

$$ \hat{S} \hat{L} \hat{S}^{-1} = \left( \begin{array}{cc} a_{L+1-L}^\dagger L & a_{L+1-L}^\dagger L \\ a_{L+1-L}^\dagger a_{L+1-L}^\dagger L & a_{L+1-L}^\dagger a_{L+1-L}^\dagger L \end{array} \right) \hat{W}_2 \hat{M} \hat{W}_1 \left( \begin{array}{c} a_{L+1-L}^\dagger L \\ a_{L+1-L}^\dagger L \end{array} \right) $$

$$ + \text{tr}(\Lambda^{-t} - \Lambda^+) + \text{tr}(\Lambda^{-t} - \Lambda^+) $$

Thus we see that if we set $\hat{W}_1 = \hat{W}_1$, which means

$$ \hat{W} = -\hat{Z}_L \log \hat{W}_1, \quad (90) $$

then following from Eq. (18) and the explicit construction of $\hat{W}_1$ in Sec. III D, Eq. (89) can be simply written as

$$ \hat{S} \hat{L} \hat{S}^{-1} = 2 \sum_{i=1}^{L} \lambda_P \hat{a}_i^\dagger \hat{a}_i + 2 \sum_{i=1}^{L} \lambda_P \hat{a}_i^\dagger \hat{a}_i \hat{L}_i \quad (91) $$

It follows that the vacuum $|0\rangle_a$ is the steady state of $\hat{S} \hat{L} \hat{S}^{-1}$ which implies that the steady state of $\hat{L}$ is given by Eq. (82). Since $\hat{W}_1$ is given by Eq. (90), we can also reconstruct $\hat{T}$ from Eq. (83).

VII. CONCLUSIONS

We have shown how to map the problem of computing the relaxation rates and the normal master modes of a Lindblad master equation for dissipatively boundary driven uniform non-interacting bosons chain, to the diagonalization of a tridiagonal bordered Toeplitz matrix. This special structure of the matrix also allows to find explicit analytical solutions and we have shown an approximate solution for a large system. With the approach presented, for a system of size $L$, the matrix to be diagonalized is only of size $L \times L$ (when considering Hamiltonians which conserve the total number of particles, i.e. there are no terms of the type $\hat{a}_i^\dagger \hat{a}_j$ or $\hat{a}_i^\dagger \hat{a}_j^\dagger$). For more general Hamiltonians our approach can be readily extended, however the matrix to be diagonalized would be a $2L \times 2L$ block bordered Toeplitz matrix (for uniform bulk Hamiltonian with boundary dissipative driving) which cannot be diagonalized with the same analytical formulae. The method here presented can be useful to study both the time evolution (since it gives access to all the normal master modes and rapidities) and steady states for open bosonic systems far from equilibrium. Due to its simplicity, this method allows one to find more analytically solvable solutions.
We have also proposed a numerical algorithm which can efficiently compute observables of the type $\langle \hat{a}^\dagger_i \hat{a}_j \rangle$. We have then used this method to compute the relaxation gap of a boundary driven bosonic quadratic system which presents two different non-equilibrium phase transitions. A scaling analysis of the gap shows that the gap scale as $1/L^3$ in all the parameter space except at one of the two phase transitions, for which the relaxation gap scales as $1/L^5$. This is due to the different behavior of the spectrum of the Hamiltonian of the bulk of the system at the two transition points.

In the second part of our work, we are going to extend this approach to fermionic systems.

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