RELATIVE PHANTOM MAPS

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Abstract. We define a map \( f : X \to Y \) to be a phantom map relative to a map \( \varphi : B \to Y \) if the restriction of \( f \) to any finite dimensional skeleton of \( X \) lifts to \( B \) through \( \varphi \), up to homotopy. There are two kinds of maps which are obviously relative phantom maps: (1) the composite of a map \( X \to B \) with \( \varphi \); (2) a usual phantom map \( X \to Y \). A relative phantom map of type (1) is called trivial, and a relative phantom map out of a suspension which is a sum of (1) and (2) is called relatively trivial. We study the (relative) triviality of relative phantom maps from a suspension, and in particular, we give rational homotopy conditions for the (relative) triviality. We also give a rational homotopy condition for the triviality of relative phantom maps from a non-suspension to a finite Postnikov section.

1. Introduction

Let \( X \) be a CW-complex of finite type. Recall that a map \( f : X \to Y \) is a phantom map if the restriction of \( f \) to any finite dimensional skeleton of \( X \) is null homotopic. Phantom maps are not detected by usual homotopy invariants such as homology and homotopy groups, so they are quite elusive in nature. But they certainly bear important parts of homotopy theory. We refer to [M1, S] for details. Let \( \text{Ph}(X, Y) \) be the set of pointed homotopy classes of phantom maps from \( X \) to \( Y \).

In this paper, we will study the following generalization of phantom maps: a map \( f : X \to Y \) is a phantom map relative to a map \( \varphi : B \to Y \) or a relative phantom map from \( X \) to \( \varphi : B \to Y \) if the restriction of \( f \) to any finite dimensional skeleton of \( X \) has a lift with respect to \( \varphi \), up to homotopy. To distinguish a usual phantom map from a relative phantom map, we call a usual phantom map an absolute phantom map. If \( B \) is a point and \( \varphi \) is the basepoint inclusion, then a phantom map relative to \( \varphi \) is an absolute phantom map. So one sees that our generalization of phantom maps is similar to sectional category for LS-category [J]. Let \( \text{Ph}(X, \varphi) \) be the set of pointed homotopy classes of phantom maps from \( X \) to \( Y \) relative to \( \varphi \).

We here note conventions on spaces involving relative phantom maps. As well as absolute phantom maps [M1, pp. 1239], we assume that the source space of a relative phantom map is always a connected CW-complex of finite type. When we deal with absolute phantom maps, it is usually assumed that the target space of an absolute phantom map is in the class \( F \) of connected CW-complexes whose \( n \)-th homotopy groups are finitely generated abelian groups for \( n \geq 2 \). We will later assume that for a target map \( \varphi : B \to Y \) of a relative phantom map, the spaces \( B, Y \) are in the class \( F \) as well.

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To start the study of relative phantom maps, we define a reasonable notion of the “triviality” of relative phantoms. Recall that an absolute phantom map is trivial if it is null homotopic. Then since the absolute phantom maps correspond to phantom maps relative to \( \varphi: B \to Y \) for \( B = \ast \), we define a phantom map \( f: X \to Y \) relative to \( \varphi \) if \( f \) itself has a lift with respect to \( \varphi \), up to homotopy. We call \( \text{Ph}(X, \varphi) \) trivial if it consists only of trivial relative phantom maps. There is certainly a non-trivial relative phantom map, which is not an absolute phantom map.

**Example 1.1** (Example 4.9). Let \( u: BS^3 \to K(\mathbb{Z}, 4) \) be a generator of \( H^4(BS^3; \mathbb{Z}) \cong \mathbb{Z} \), and extend it to a homotopy fibration sequence

\[
B \xrightarrow{\varphi} Y \to BS^3 \xrightarrow{u} K(\mathbb{Z}, 4).
\]

Then \( \text{Ph}(\Sigma CP^\infty, \varphi) \) is not trivial. On the other hand, there is no non-trivial absolute phantom maps from \( \Sigma CP^\infty \) to \( Y \) (see Corollary 2.4).

We will study the (non-)triviality of relative phantom maps out of a suspension and will give several conditions for the (non-)triviality, where the above example is produced by one of these results. For instance, we will generalize the fact that any absolute phantom map into a rationally contractible space is null homotopic.

**Theorem 1.2** (Proposition 4.6). Let \( B, Y \in \mathcal{F} \). Suppose that \( \varphi: B \to Y \) is an isomorphism in \( \pi_n \otimes \mathbb{Q} \) for \( n \geq 2 \). Then \( \text{Ph}(\Sigma X, \varphi) \) is trivial.

**Remark 1.3.** We will see in Corollary 6.14 below that Theorem 1.2 does not hold when the source space is not a suspension.

On the other hand, any absolute phantom map is always a relative phantom map, but this is not essential in studying relative phantom maps. If a source space is a suspension, we can sum up a trivial relative phantom map and an absolute phantom map, and this sum is, by definition, a relative phantom map. Since a relative phantom map of this form is inessential in studying relative phantom maps, we call it relatively trivial. We next consider the relative triviality of relative phantom maps out of a suspension space. We say that \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial if it consists only of relatively trivial relative phantom maps. Example 1.1 gives an example of a relative phantom map which is not relatively trivial. We look for a condition on \( X \) with respect to \( \varphi: B \to Y \) such that \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial. In [MR], it is given a condition on the rational homotopy type of a space \( A \) which is equivalent to that \( \text{Ph}(A, Y) = \ast \) for any space \( Y \). We are then by this result to consider a condition on a rational information of \( X \) which guarantees that \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial. For a map \( \varphi: B \to Y \), we put

\[
q(\varphi) = \{ n \geq 2 | \varphi_* \otimes \mathbb{Q}: \pi_n(B) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q} \text{ is not injective}\}.
\]

We will prove the following theorem which has several corollaries as we will see below.

**Theorem 1.4** (Theorem 5.9). Let \( B, Y \in \mathcal{F} \). If \( H_{n-1}(X; \mathbb{Q}) = 0 \) for \( n \in q(\varphi) \), then \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial.
Next we consider the triviality of relative phantom maps from a non-suspension space to the Postnikov section by extending the technique developed so far. Let \( s_n : B \to B_n \) be the \( n \)-th Postnikov section of a space \( B \). Put
\[
q(B) = \{ n \geq 2 \mid \pi_n(B) \otimes \mathbb{Q} \neq 0 \}.
\]
Then we will prove:

**Theorem 1.5** (Theorem 6.8). Suppose that \( B \in \mathcal{F} \) is nilpotent or has torsion annihilators (see Definition 6.6). If \( H_k(X; \mathbb{Q}) = 0 \) for \( k \in q(B) \), then \( \text{Ph}(X, s_n) \) is trivial.

We can apply this theorem to the case of the inclusion \( \mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty \) for odd \( n \).

\[\text{Corollary 1.6} \text{ (Corollary 6.9). If } n \text{ is odd and } H_n(X; \mathbb{Q}) = 0, \text{ then } \text{Ph}(X, i_n) \text{ is trivial.}\]

**Remark 1.7.**

1. The authors were originally interested in relative phantom maps to \( i_n : \mathbb{R}P^n \to \mathbb{R}P^\infty \) with a motivation from de Bruijn and Erdős theorem in combinatorics. This will be explained precisely in Section 6.

2. Corollary 1.6 will be shown to be optimal by Proposition 6.12 below.

The rest of this paper is organized as follows. In Section 2, we briefly review the description of \( \text{Ph}(X, Y) \) in terms of \( \text{lim}^1 \). In Section 3, we define relative phantom maps and give an exact sequence involving \( \text{Ph}(X, \varphi) \) and \( \text{Ph}(X, Y) \) which will be useful in studying \( \text{Ph}(X, \varphi) \) algebraically. In Section 4, we introduce the triviality of relative phantom maps and study conditions for the (non-)triviality of \( \text{Ph}(X, \varphi) \) including Theorem 1.2. In Section 5, we define the relative triviality of relative phantom maps out of a suspension. We then show conditions for the relative triviality of relative phantom maps including Theorem 1.4. In Section 6, by a way different from Section 5, we consider the triviality of relative phantom maps from a non-suspension to a finite Postnikov section and prove Theorem 1.5. We also give a non-trivial relative phantom map into the inclusion \( \mathbb{R}P^n \to \mathbb{R}P^\infty \) which shows that Corollary 1.6 is optimal.

2. Relative phantom maps and inverse limits

2.1. \( \text{lim} \) and \( \text{lim}^1 \) of groups. In this subsection, we recall the definition of \( \text{lim} \) and \( \text{lim}^1 \) of the inverse system of groups, not necessarily abelian. Let
\[
G_0 \xleftarrow{f_0} G_1 \xleftarrow{f_1} \ldots \xleftarrow{f_{n-1}} G_n \xleftarrow{f_n} \ldots
\]
be an inverse system of groups, and define the left action of \( \prod_{n=0}^\infty G_n \) on itself by
\[
(g_0, \ldots, g_n, \ldots) \cdot (x_0, \ldots, x_n, \ldots) = (g_0x_0f_0(g_1)^{-1}, \ldots, g_nx_nf_n(g_{n+1})^{-1}, \ldots).
\]
Then \( \text{lim} G_n \) and \( \text{lim}^1 G_n \) are defined by the isotropy subgroup of \( \prod_{n=0}^\infty G_n \) at \( (1, 1, \ldots) \in \prod_{n=0}^\infty G_n \) and the orbit space of this action, respectively. By definition, \( \text{lim} G_n \) is a group.
but $\lim^1 G_n$ is just a pointed set in general whose basepoint is the orbit containing $(1, 1, \cdots)$. However, if every $G_n$ is abelian, then $\lim^1 G_n$ has a natural abelian group structure.

Next we recall the 6-term exact sequence (Lemma 2.1) involving $\lim$ and $\lim^1$ which will be useful. For a basepoint preserving map $h \colon S \to T$ between pointed sets, we write $\text{Im} h$ and $\text{Ker} h$ to mean $h(S)$ and $h^{-1}(\ast)$, respectively. Recall that a sequence of pointed sets $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if $\text{Im} f = \text{Ker} g$. If $A, B, C$ are groups and $f, g$ are group homomorphisms, the exactness coincides with that of groups.

**Lemma 2.1.** Let $1 \to \{G_n\} \to \{H_n\} \to \{K_n\} \to 1$ be an exact sequence of inverse systems of groups. Then there is a natural exact sequence of pointed sets:

$$1 \to \lim^1 G_n \to \lim^1 H_n \to \lim^1 K_n \to \lim^1 G_n \to \lim^1 H_n \to \lim^1 K_n \to \ast$$

### 2.2 Absolute phantom maps.

Recall that a map $f \colon X \to Y$ is a phantom map if the restriction of $f$ to any finite dimensional skeleton of $X$ is null homotopic. Hereafter, we will always assume that the source space of a phantom map is a connected CW-complex of finite type. There is a different definition of phantom maps such that $f \colon X \to Y$ is a phantom map if for every map $u \colon K \to X$ from a finite complex $K$, the composite $f \circ u$ is null homotopic. Since $X$ is assumed to be a CW-complex of finite type, the two definitions above coincide. We will often call a usual phantom map an absolute phantom map to distinguish it from relative phantom maps. Let $\text{Ph}(X, Y)$ denote the set of homotopy classes of absolute phantom maps from $X$ to $Y$.

Let $X^n$ denote the $n$-skeleton of a CW-complex $X$. By the Milnor exact sequence (see [BK] or [Mi])

$$\ast \to \lim^1 [\Sigma X^n, Y] \to [X, Y] \xrightarrow{\pi_Y} \lim [X^n, Y] \to \ast$$

we have the following description of $\text{Ph}(X, Y)$ by $\lim^1$.

**Proposition 2.2.** There is an isomorphism of pointed sets

$$\text{Ph}(X, Y) \cong \lim^1 [\Sigma X^n, Y],$$

which is a group isomorphism whenever $X$ is a suspension.

We can dualize this proposition by considering the Postnikov tower of the target space, where the proof is omitted. Let $Y_1 \leftarrow Y_2 \leftarrow \cdots \leftarrow Y_n \leftarrow \cdots$ be the Postnikov tower of $Y$.

**Proposition 2.3.** There is an isomorphism of pointed sets

$$\text{Ph}(X, Y) \cong \lim^1 [X, \Omega Y_n],$$

which is a group isomorphism whenever $X$ is a suspension.

We record consequences of the two propositions above on the triviality of $\text{Ph}(X, Y)$. 
Corollary 2.4. (1) If $Y$ is a finite Postnikov piece, then $\text{Ph}(X, Y) = \ast$.
(2) If $Y \in \mathcal{F}$ satisfies that $\pi_\ast(Y) \otimes \mathbb{Q} = 0$ for $\ast \geq 2$, then $\text{Ph}(X, Y) = \ast$.

Proof. (1) is immediate from Proposition 2.3. (2) For any finite connected complex $A$, the homotopy set $[\Sigma A, Y]$ is a finite set by the assumption on $Y$, and the inverse system of finite groups satisfies the Mittag-Leffler condition (see [M1]). Then $\text{Ph}(X, Y) \cong \lim_{\leftarrow}^{1} [\Sigma X^n, Y] = \ast$. \hfill $\square$

3. Relative phantom maps

We first define relative phantom maps.

Definition 3.1. A map $f: X \rightarrow Y$ is a phantom map relative to a map $\varphi: B \rightarrow Y$ or a relative phantom map from $X$ to $\varphi: B \rightarrow Y$ if the restriction of $f$ to any finite dimensional skeleton of $X$ has a lift to $B$ through $\varphi$, up to homotopy.

Let $\text{Ph}(X, \varphi)$ denote the set of homotopy classes of phantom maps relative to $\varphi$.

If $\varphi$ is the basepoint inclusion (or more generally, $\varphi$ is null homotopic), then relative phantom maps from $X$ to $\varphi: B \rightarrow Y$ are exactly absolute phantom maps from $X$ to $Y$. So relative phantom maps are a generalization of absolute phantom maps.

As well as absolute phantom map, we will always assume that the source space of a relative phantom map is a connected CW-complex of finite type.

One can easily check that the above definition of relative phantom maps does not depend on the choice of CW-structures on $X$. As is the case of absolute phantom maps, we can consider another definition of relative phantom maps, using maps from finite complexes into $X$. However, these two definitions coincide since we are assuming that $X$ is a CW-complex of finite type.

As well as the absolute case in Proposition 2.3, let us dualize the definition of relative phantom maps. Let $Y_1 \leftarrow Y_2 \leftarrow \cdots \leftarrow Y_n \leftarrow \cdots$ be the Postnikov tower of $Y$ as in the previous section, and let $s_n: Y \rightarrow Y_n$ be the $n$-th Postnikov section of $Y$. By the naturality of Postnikov towers, a map $\varphi: B \rightarrow Y$ induces a map $\varphi_n: B_n \rightarrow Y_n$ between the Postnikov pieces satisfying $\varphi_n \circ s_n^B \simeq s_n^Y \circ \varphi$, where $s_n^B$ and $s_n^Y$ are the Postnikov sections of $B$ and $Y$, respectively.

Proposition 3.2. The following conditions on a map $f: X \rightarrow Y$ are equivalent:

1. $f$ is a phantom map relative to $\varphi$;
2. For any $n \geq 0$, $s_n \circ f: X \rightarrow Y_n$ has a lift with respect to $\varphi_n: B_n \rightarrow Y_n$, up to homotopy.

Proof. Suppose that $f$ is a phantom map relative to $\varphi$. We want to show that $s_n \circ f: X \rightarrow Y_n$ has a lift with respect to $\varphi_n$, up to homotopy, for any $n$. Since $f$ is a phantom map relative to $\varphi$, the map $f|_{X^{n+1}}: X^{n+1} \rightarrow Y$ has a lift $\tilde{f}: X^{n+1} \rightarrow B$ through $\varphi$, up to homotopy. Since the inclusion $X^{n+1} \rightarrow X$ induces an isomorphism $[X, B_n] \xrightarrow{\sim} [X^{n+1}, B_n]$ of pointed sets, there is a map $\tilde{f}: X \rightarrow B_n$ satisfying $\tilde{f}|_{X^{n+1}} \simeq s_n \circ \tilde{f}$. Now we have

$$\varphi_n \circ \tilde{f}|_{X^{n+1}} \simeq \varphi_n \circ s_n \circ \tilde{f} \simeq s_n \circ \varphi \circ \tilde{f} \simeq s_n \circ f|_{X^{n+1}}.$$
Since the inclusion $X^{n+1} \to X$ induces an isomorphism $[X, Y_n] \xrightarrow{\cong} [X^{n+1}, Y_n]$ as pointed sets, we obtain that $\varphi_n \circ \bar{f} \simeq s_n \circ f$. Thus $\bar{f}$ is a desired lift.

Suppose next that for any $n, s_{n+1} \circ f: X \to Y_{n+1}$ has a lift $g: X \to B_{n+1}$ with respect to $\varphi_{n+1}$, up to homotopy. We want to show that $f|_{X^n}: X^n \to Y$ has a lift with respect to $\varphi$, up to homotopy. Since there is an isomorphism $(s_{n+1})_*: [X^n, B] \xrightarrow{\cong} [X^n, B_{n+1}]$ of pointed sets, we have a map $\bar{g}: X^n \to B$ satisfying $s_{n+1} \circ \bar{g} \simeq g|_{X^n}$. Then we get

$$s_{n+1} \circ \varphi \circ \bar{g} \simeq \varphi_{n+1} \circ s_{n+1} \circ \bar{g} \simeq \varphi_{n+1} \circ g|_{X^n} \simeq s_{n+1} \circ f|_{X^n}.$$  

Since the map $(s_{n+1})_*: [X^n, Y] \to [X^n, Y_{n+1}]$ is also isomorphic, we get $\varphi \circ \bar{g} \simeq f|_{X^n}$ as required.

Next we give a description of $\text{Ph}(X, \varphi)$ by using $\text{Ph}(X, Y)$ which will be useful to deal with $\text{Ph}(X, \varphi)$ algebraically.

**Proposition 3.3.** There is an exact sequence of pointed sets

$$1 \to \text{Ph}(X, Y) \to \text{Ph}(X, \varphi) \xrightarrow{\pi_Y} \lim \varphi_* [X^n, B] \to 1$$

which is an exact sequence of groups whenever $X$ is a suspension.

**Proof.** Note that an element $f$ of $[X, Y]$ is a phantom map relative to $\varphi$ if and only if $\pi_Y(f) \in \lim [X^n, Y]$ is contained in $\lim \varphi_* [X^n, B]$. This means that the diagram

$$\begin{array}{ccc}
\text{Ph}(X, \varphi) & \xrightarrow{\pi_Y} & \lim \varphi_* [X^n, B] \\
\downarrow & & \downarrow \\
[X, Y] & \xrightarrow{\pi_Y} & \lim [X^n, Y]
\end{array}$$

is a pullback. By the Milnor exact sequence (2.1), the lower $\pi_Y$ is surjective, implying that the upper $\pi_Y$ is surjective too. By (2.1), we also have that the kernel of the lower $\pi_Y$ is $\lim^- [\Sigma X^n, Y]$. Thus the kernel of the upper $\pi_Y$ is isomorphic to $\lim^- [\Sigma X^n, Y]$ which is isomorphic with $\text{Ph}(X, Y)$ by Proposition 2.2, completing the proof. \qed

## 4. Triviality of Relative Phantom Maps Out of a Suspension

A phantom map $f: X \to Y$ relative to $\varphi: B \to Y$ is called **trivial** if the entire map $f$ has a lift with respect to $\varphi$, up to homotopy, and $\text{Ph}(X, \varphi)$ is called **trivial** if every element of $\text{Ph}(X, \varphi)$ is trivial. We consider the triviality of phantom maps relative to $\varphi: B \to Y$ when $\varphi$ is a fiber inclusion, that is, there is a homotopy fibration $B \xrightarrow{\varphi} Y \to W$. This case descends to relative phantom maps out of a suspension as follows. Given a map $\varphi: B \to Y$, there is a homotopy fibration $\Omega B \xrightarrow{\Omega \varphi} \Omega Y \to F$, where $F$ is the homotopy fiber of $\varphi$. Then $\Omega \varphi$ is a fiber inclusion and by the adjointness, we have

$$\text{Ph}(\Sigma X, \varphi) \cong \text{Ph}(X, \Omega \varphi).$$
The following proposition enables us to detect the (non-)triviality of relative phantom maps by that of related absolute phantom maps.

**Proposition 4.1.** Let $B \xrightarrow{\varphi} Y \xrightarrow{p} Z$ be a homotopy fibration. Then a map $f: X \to Y$ is a phantom map relative to $\varphi$ if and only if the composite $p \circ f: X \to Z$ is an absolute phantom map. Moreover, $f$ is a trivial relative phantom map if and only if $p \circ f$ is null homotopic.

**Proof.** For every $n$, $f|_{X^n}$ has a lift with respect to $\varphi$, up to homotopy, if and only if $p \circ f|_{X^n}$ is null homotopic. This implies that $f$ is a phantom map relative to $\varphi$ if and only if $p \circ f$ is an absolute phantom map. Similarly, $p \circ f$ is null homotopic if and only if $f$ has a lift with respect to $\varphi$, up to homotopy. Thus the proof is done. \hfill $\square$

We show two applications of Proposition 4.1. The first one is as follows. We denote the adjoint of a map $f: \Sigma X \to Y$ by $\text{ad}(f): X \to \Omega Y$.

**Corollary 4.2.** Let $\Omega Y \xrightarrow{\delta} F \to B \xrightarrow{\varphi} Y$ be a homotopy fibration sequence. A map $f: \Sigma X \to Y$ is a phantom map relative to $\varphi$ if and only if $\delta \circ \text{ad}(f)$ is an absolute phantom map. Moreover, $f$ is trivial if and only if $\delta \circ \text{ad}(f)$ is null homotopic.

**Proof.** Note that $f: \Sigma X \to Y$ is a (trivial) phantom map relative to $\varphi$ if and only if $\text{ad}(f): X \to \Omega Y$ is a (trivial) phantom map relative to $\Omega \varphi$. Then by applying Proposition 4.1 to the fibration sequence $\Omega B \xrightarrow{\Omega \varphi} \Omega Y \xrightarrow{\delta} F$, the proof is done. \hfill $\square$

**Corollary 4.3.** Suppose that we have a homotopy fibration $F \to B \xrightarrow{\varphi} Y$ such that the connecting map $\delta: \Omega Y \to F$ is null homotopic. Then $\text{Ph}(\Sigma X, \varphi)$ is trivial.

**Proof.** Since $\delta$ is null homotopic, so is $\delta \circ \text{ad}(f)$ for any $f \in \text{Ph}(\Sigma X, \varphi)$. Then by Corollary 4.2, $f$ is trivial, completing the proof. \hfill $\square$

**Example 4.4.** Let $F_n(Y) \to G_n(Y) \xrightarrow{p_n} Y$ be the $n$-th Ganea fibration. We shall show that $\text{Ph}(\Sigma X, p_n)$ is trivial for any space $X$. To this end, we apply Corollary 4.3 to the Ganea fibration, so we prove that the connecting map $\delta: \Omega Y \to F_n(Y)$ is null homotopic. Then it is sufficient to show that the map $\Omega p_n: \Omega G_n(Y) \to \Omega Y$ has a right homotopy inverse.

There is the natural map $i_n: G_1(Y) \to G_n(Y)$ such that the composite $G_1(Y) \xrightarrow{i_n} G_n(Y) \xrightarrow{p_n} Y$ is homotopic to $p_1$, and there is a homotopy equivalence $G_1(Y) \simeq \Sigma \Omega Y$ such that $p_1: G_1(Y) \to Y$ is homotopic to the adjoint of the identity map of $\Omega Y$. By the adjointness of $\Sigma$ and $\Omega$, $\Omega p_1$ has a right homotopy inverse, say $s: \Omega Y \to \Omega G_1(Y)$. Then for $t = \Omega i_n \circ s$, we have

$$\Omega p_n \circ t = \Omega p_n \circ \Omega i_n \circ s \simeq \Omega p_1 \circ s \simeq 1_{\Omega Y}.$$  

Thus $t$ is a right homotopy inverse of $\Omega p_n$.

Although we have seen that $\text{Ph}(\Sigma X, p_n)$ is trivial, we will see in Proposition 6.12 below that there is a non-suspension space $X(n)$ such that $\text{Ph}(X(n), p_n)$ is not trivial for $Y = \mathbb{R}P^\infty$ with $n > 2$. 

The next lemma is a variant of Corollary 4.2 and will be used to prove Proposition 4.6 below which is a generalization of Corollary 2.4 to the relative case.

**Lemma 4.5.** Let $F$ be the homotopy fiber of a map $\varphi: B \to Y$. Then $\text{Ph}(\Sigma X, \varphi)$ is trivial whenever $\text{Ph}(X, F) = \ast$.

**Proof.** Let $\delta: \Omega Y \to F$ be the connecting map of a homotopy fibration $F \to B \xrightarrow{\varphi} Y$. Since $\text{Ph}(X, F) = \ast$, $\delta \circ \text{ad}(f)$ is a trivial absolute phantom map for any $f \in \text{Ph}(\Sigma X, \varphi)$. Then $f$ is trivial by Corollary 4.2, completing the proof. □

As in Section 1, we will write by $\mathcal{F}$ the class of connected CW complexes each of which has finitely generated $\pi_n$ for $n \geq 2$.

**Proposition 4.6.** Let $B, Y \in \mathcal{F}$. Suppose that $\varphi: B \to Y$ is an isomorphism in $\pi_n \otimes \mathbb{Q}$ for $n \geq 2$. Then $\text{Ph}(\Sigma X, \varphi)$ is trivial.

**Proof.** By the assumption, the homotopy fiber $F$ of $\varphi$ satisfies the condition of Corollary 2.4, implying $\text{Ph}(X, F) = \ast$. Then we get the desired result by Lemma 4.5. □

Next we show the second application of Proposition 4.1.

**Proposition 4.7.** Suppose that there is a homotopy fibration sequence $B \xrightarrow{\beta} Y \xrightarrow{\alpha} V \xrightarrow{\beta} W$ such that either $\beta$ is null homotopic or $\text{Ph}(X, W) = \ast$. Then $\text{Ph}(X, \varphi)$ is trivial if and only if $\text{Ph}(X, F)$ is trivial.

**Proof.** Let $f: X \to V$ be an absolute phantom map. Then $\beta \circ f: X \to W$ is an absolute phantom map, so by the assumption, $\beta \circ f$ is null homotopic. Thus $f$ has a lift $\tilde{f}$ with respect to $\alpha$, up to homotopy. By Proposition 4.1, $\tilde{f}$ is a relative phantom map from $X$ to $\varphi$ which is trivial if and only if $f: X \to V$ is null homotopic. On the other hand, if there is a phantom map $f: X \to Y$ relative to $\varphi$, then by Proposition 4.1, $\alpha \circ f: Y \to V$ is an absolute phantom map which is null homotopic if and only if $f$ is trivial. Therefore the proof is completed. □

**Corollary 4.8.** Let $F \xrightarrow{j} B \xrightarrow{\varphi} Y$ be a homotopy fibration such that either $j$ is null homotopic or $\text{Ph}(X, B) = \ast$. Then $\text{Ph}(\Sigma X, \varphi)$ is trivial if and only if $\text{Ph}(X, F)$ is trivial.

**Proof.** Apply Proposition 4.7 to the homotopy fibration sequence $\Omega B \xrightarrow{\Omega \varphi} \Omega Y \xrightarrow{\alpha} F \xrightarrow{\beta} B$ together with the adjoint congruence (4.1). □

**Example 4.9.** We give an example of a space $X$ and a map $\varphi$ such that $\text{Ph}(\Sigma X, \varphi)$ is non-trivial although $\text{Ph}(\Sigma X, Y)$ is trivial. Let $u: BS^3 \to K(\mathbb{Z}, 4)$ be a generator of $H^4(BS^3; \mathbb{Z}) \cong \mathbb{Z}$, and extend it to a homotopy fibration sequence

$$S^3 \xrightarrow{\Omega u} K(\mathbb{Z}, 3) \to B \xrightarrow{\varphi} Y \to BS^3 \xrightarrow{u} K(\mathbb{Z}, 4).$$

By Corollary 2.4, we have $\text{Ph}(X, B) = \ast$ for any space $X$. So we can apply Corollary 4.8 to the homotopy fibration sequence $S^3 \xrightarrow{\Omega u} K(\mathbb{Z}, 3) = B \xrightarrow{\varphi} Y$. By [G], we have $\text{Ph}(CP\infty, S^3) \neq \ast$, and thus we obtain that $\text{Ph}(\Sigma CP\infty, \varphi)$ is not trivial. On the other hand, it follows from Corollary 2.4 that $\text{Ph}(\Sigma CP\infty, Y)$ is trivial.
5. Relative triviality of relative phantom maps out of a suspension

By definition, any absolute phantom map \( X \to Y \) is a phantom map relative to any map \( \varphi : B \to Y \), and this is not essential as well as trivial relative phantom maps in studying relative phantom maps. Then we define the following notion of relative triviality. We denote by + the multiplication of the homotopy set \([\Sigma X, Y]\) induced from the suspension comultiplication of \( \Sigma X \). A phantom map \( f : \Sigma X \to Y \) relative to \( \varphi : B \to Y \) is called relatively trivial if there are \( g \in [\Sigma X, B] \) and \( h \in \text{Ph}(\Sigma X, Y) \) such that

\[
 f \simeq \varphi^*(g) + h.
\]

We say that \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial if it consists only of relatively trivial relative phantom maps. Let \( \varphi : B \to Y \) be as in Example 4.9. By Corollary 2.4, we have \( \text{Ph}(\Sigma CP^\infty, Y) = * \), so Example 4.9 shows that there is certainly a relative phantom map which is not relatively trivial.

Let us observe a structure of the subset of relatively trivial phantom maps in \( \text{Ph}(\Sigma X, \varphi) \). Note that the set \( \text{Ph}(\Sigma X, \varphi) \) is a group.

**Proposition 5.1.** The set of relatively trivial relative phantom maps from \( \Sigma X \) to \( \varphi : B \to Y \) is a subgroup of \( \text{Ph}(\Sigma X, \varphi) \).

**Proof.** The map \( \pi_Y : \text{Ph}(\Sigma X, \varphi) \to \lim \left\langle \varphi^*[\Sigma X^n, B] \right\rangle \) in Proposition 3.3 is a group homomorphism whose kernel is \( \text{Ph}(\Sigma X, Y) \). In particular, \( \text{Ph}(\Sigma X, Y) \) is a normal subgroup of \( \text{Ph}(\Sigma X, \varphi) \). Then the set of relatively trivial relative phantom maps from \( X \) to \( \varphi \) is the subgroup \( \varphi^*[\Sigma X, B] + \text{Ph}(\Sigma X, Y) \) of \( \text{Ph}(\Sigma X, \varphi) \). Thus the proof is done. \( \square \)

We investigate conditions which guarantee that \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial.

**Lemma 5.2.** \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial if and only if the composite

\[
 [\Sigma X, B] \overset{\varphi^*}{\longrightarrow} \text{Ph}(\Sigma X, \varphi) \overset{\pi_Y}{\longrightarrow} \lim \varphi^*[\Sigma X^n, B]
\]

is surjective, where the map \( \pi_Y \) is as Proposition 3.3.

**Proof.** Suppose first that \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial. There is a commutative diagram of groups

\[
 (5.1) \begin{array}{ccc}
 [\Sigma X, B] & \overset{\pi_B}{\longrightarrow} & \lim [\Sigma X^n, B] \\
 \downarrow \varphi^* & & \downarrow \varphi^* \\
 \text{Ph}(\Sigma X, \varphi) & \overset{\pi_Y}{\longrightarrow} & \lim \varphi^*[\Sigma X^n, B]
\end{array}
\]

where \( \pi_B \) and \( \pi_Y \) denotes the natural projections as in (2.1) and Proposition 3.3. Then by Proposition 3.3, the bottom arrow \( \pi_Y \) of (5.1) is surjective, so for any \( f \in \lim \varphi^*[\Sigma X^n, B] \),
there is \( \tilde{f} \in \text{Ph}(\Sigma X, \varphi) \) satisfying \( \pi_Y(\tilde{f}) = f \). By the assumption, \( \tilde{f} \) is relatively trivial, so there are \( g \in [\Sigma X, B] \) and \( h \in \text{Ph}(\Sigma X, Y) \) such that \( \tilde{f} = \varphi_*(g) + h \). Now we have

\[
f = \pi_Y(\tilde{f}) = \pi_Y(\varphi_*(g) + h) = \pi_Y \circ \varphi_*(g) + \pi_Y(h)
\]

where \( \pi_Y \) is a group homomorphism. By definition, we have \( \pi_Y(h) = 0 \), and then we have proved that \( \pi_Y \circ \varphi_\ast \) is surjective.

Next suppose that \( \pi_Y \circ \varphi_\ast \) is surjective, and take any \( f \in \text{Ph}(\Sigma X, \varphi) \). Then there is \( g \in [\Sigma X, B] \) such that \( \pi_Y \circ \varphi_\ast(g) = \pi_Y(f) \), implying \( f - \varphi_\ast(g) \in \text{Ker} \pi_Y \). Since \( \text{Ker} \pi_Y = \text{Ph}(\Sigma X, Y) \) by Proposition 3.3, there is \( h \in \text{Ph}(\Sigma X, Y) \) satisfying \( f - \varphi_\ast(g) = h \), or equivalently, \( f = h + \varphi_\ast(g) \). Thus \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial. Therefore the proof is completed. \( \square \)

Let \( K_n \) be the kernel of the group homomorphism \( \varphi_\ast : [\Sigma X^n, B] \to [\Sigma X^n, Y] \). Then we have the following exact sequence of inverse systems of groups:

\[
1 \longrightarrow \{ K_n \} \longrightarrow \{ [\Sigma X^n, B] \} \longrightarrow \{ \varphi_\ast[\Sigma X^n, B] \} \longrightarrow 1
\]

**Proposition 5.3.** \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial if and only if the kernel of the map

\[
\lim_1 K_n \to \lim_1 [\Sigma X^n, B]
\]

is trivial.

**Proof.** Consider the commutative diagram (5.1). Since the top map \( \pi_B \) is surjective by the Milnor exact sequence (2.1), the map \( \varphi_\ast \circ \pi_B : [\Sigma X, B] \to \lim_1 \varphi_\ast[\Sigma X^n, B] \) is surjective if and only if so is \( \varphi_\ast : \lim_1 [\Sigma X^n, B] \to \lim_1 \varphi_\ast[\Sigma X^n, B] \). Applying Lemma 2.1 to the short exact sequence

\[
1 \to \{ K_n \} \to \{ [\Sigma X^n, B] \} \overset{\varphi_\ast}{\longrightarrow} \{ \varphi_\ast[\Sigma X^n, B] \} \to 1
\]

of inverse systems of groups, we get an exact sequence

\[
\lim_1 [\Sigma X^n, B] \overset{\varphi_\ast}{\longrightarrow} \lim_1 \varphi_\ast[\Sigma X^n, B] \to \lim_1 K_n \to \lim_1 [\Sigma X^n, B]
\]

of pointed sets. Thus the map \( \varphi_\ast : \lim_1 [\Sigma X^n, B] \to \lim_1 \varphi_\ast[\Sigma X^n, B] \) is surjective if and only if the kernel of the map \( \lim_1 K_n \to \lim_1 [\Sigma X^n, B] \) is trivial. This completes the proof. \( \square \)

The assumption of the following corollary trivially implies that of Proposition 5.3.

**Corollary 5.4.** \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial whenever \( \lim_1 K_n = * \).

We then consider practical conditions which guarantee \( \lim_1 K_n = * \). We first translate the condition \( \lim_1 K_n = * \) to that of absolute phantom maps.

**Lemma 5.5.** Let \( F \overset{\delta}{\to} B \overset{\zeta}{\to} Y \) be a homotopy fibration with the connecting map \( \delta : \Omega Y \to F \). For any space \( X \), \( \lim_1 K_n = * \) if and only if the map \( \delta_\ast : \text{Ph}(X, \Omega Y) \to \text{Ph}(X, F) \) is surjective.
Proof. Put \( L_n = \ker \{ j_n : [\Sigma X^n, F] \to [\Sigma X^n, B] \} \). By the exactness of the sequence
\[
[\Sigma X^n, F] \xrightarrow{j_n} [\Sigma X^n, B] \xrightarrow{\phi_n} [\Sigma X^n, Y],
\]
we have an exact sequence of inverse systems of groups
\[
1 \to \{ L_n \} \to \{ [\Sigma X^n, F] \} \to \{ K_n \} \to 1.
\]
Then by Lemma 2.1, we get an exact sequence of pointed sets
\[
\lim_\leftarrow \{ L_n \} \to \lim_\leftarrow \{ [\Sigma X^n, F] \} \to \lim_\leftarrow \{ K_n \} \to *.
\]
Thus \( \lim_\leftarrow \{ K_n \} = * \) if and only if the map \( \lim_\leftarrow \{ L_n \} \to \lim_\leftarrow \{ [\Sigma X^n, F] \} \) is surjective.

Next we put \( M_n = \ker \{ \delta_n : [\Sigma X^n, \Omega Y] \to [\Sigma X^n, F] \} \). Similarly to the above, from the exact sequence of groups
\[
[\Sigma X^n, \Omega Y] \xrightarrow{\delta_n} [\Sigma X^n, F] \xrightarrow{j_n} [\Sigma X^n, Y],
\]
we get an exact sequence of inverse systems of groups
\[
1 \to \{ M_n \} \to \{ [\Sigma X^n, \Omega Y] \} \to \{ L_n \} \to 1.
\]
Thus by Lemma 2.1, we have that \( \lim_\leftarrow \{ [\Sigma X^n, \Omega Y] \} \to \lim_\leftarrow \{ L_n \} \) is surjective. Then \( \lim_\leftarrow \{ K_n \} = * \) if and only if the composite \( \lim_\leftarrow \{ [\Sigma X^n, \Omega Y] \} \to \lim_\leftarrow \{ L_n \} \to \lim_\leftarrow \{ [\Sigma X^n, F] \} \) is surjective. By Proposition 2.2, this composite is identified with \( \delta : \text{Ph}(X, \Omega Y) \to \text{Ph}(X, F) \). Thus the proof is completed. \( \Box \)

As we have given a rational homotopy condition for the triviality of \( \text{Ph}(\Sigma X, \varphi) \) in Proposition 4.6, we expect to find a rational homotopy condition for the relative triviality of \( \text{Ph}(\Sigma X, \varphi) \). McGibbon and Roitberg [MR] gave a necessary and sufficient rational homotopy condition which guarantees that every phantom map \( X \to Y \) is null homotopic, and we are motivated by their result to consider a rational homotopy condition for the relative triviality of \( \text{Ph}(\Sigma X, \varphi) \).

We first recall the result of Roitberg and Touhey [RT].

**Theorem 5.6** (Roitberg and Touhey [RT]). For \( Y \in \mathcal{F} \), there is an isomorphism of pointed sets
\[
\text{Ph}(X, Y) \cong \prod_{n \geq 1} H^n(X; \pi_{n+1}(Y) \otimes \hat{\mathbb{Z}}/\mathbb{Z})/[X, \Omega \hat{Y}]
\]
which is natural with respect to \( X \) and \( Y \), where \( \hat{\mathbb{Z}} \) is the profinite completion of the integer ring \( \mathbb{Z} \) and \( \hat{Y} \) is the profinite completion of a space \( Y \) in the sense of Sullivan.

**Remark 5.7.** Although more conditions on \( Y \) are assumed in [RT], we may alter \( Y \) with its universal cover by Proposition 2.2 so that the conditions reduce to that \( Y \in \mathcal{F} \).

Next we apply Theorem 5.6 to the induced map between absolute phantom maps. For a map \( g : V \to W \), we put
\[
\hat{q}(g) = \{ n \geq 2 \mid g_n : \pi_n(V) \otimes \mathbb{Q} \to \pi_n(W) \otimes \mathbb{Q} \text{ is not surjective} \}.
\]
Lemma 5.8. Given a map \( g: V \to W \) for \( V, W \in \mathcal{F} \), suppose that \( H_{n-1}(X; \mathbb{Q}) = 0 \) for \( n \in \hat{q}(g) \). Then \( g_*: \text{Ph}(X, V) \to \text{Ph}(X, W) \) is surjective.

Proof. Since the isomorphism of Theorem (5.6) is natural with respect to \( Y \), the lemma immediately follows from the fact that \( \hat{Z}/\mathbb{Z} \) is a \( \mathbb{Q} \)-vector space. \qed

Put \( q(\varphi) = \{ n \geq 2 \mid \varphi_*: \pi_n(B) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q} \text{ is not injective} \} \).

Now we give a rational homotopy condition for the relative triviality of \( \text{Ph}(\Sigma X, \varphi) \).

Theorem 5.9. Let \( B, Y \in \mathcal{F} \). If \( H_{n-1}(X; \mathbb{Q}) = 0 \) for \( n \in q(\varphi) \), then \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial.

Proof. Let \( F \) be the homotopy fiber of \( \varphi: B \to Y \) and \( \delta: \Omega Y \to F \) be the corresponding connecting map. By the homotopy exact sequence, \( \pi_n(\Omega Y) \otimes \mathbb{Q} \to \pi_n(F) \otimes \mathbb{Q} \) is surjective if and only if \( \varphi_*: \pi_n(B) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q} \) is injective for \( n \geq 2 \). Then we have \( q(\varphi) = \hat{q}(\delta) \). Thus the proof is completed by Corollary 5.4 and Lemmas 5.5 and 5.8. \qed

We give three corollaries of this theorem.

Corollary 5.10. Let \( B, Y \in \mathcal{F} \). If \( \varphi_*: \pi_n(B) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q} \) is injective for \( n \geq 2 \), then \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial.

For a space \( A \), we put \( q(A) = \{ n \geq 2 \mid \pi_n(A) \otimes \mathbb{Q} \neq 0 \} \).

Corollary 5.11. Let \( B, Y \in \mathcal{F} \). If \( H_{n-1}(X; \mathbb{Q}) = 0 \) for \( n \in q(F) \), then \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial, where \( F \) is the homotopy fiber of \( \varphi: B \to Y \).

Proof. By the homotopy exact sequence of the homotopy fibration \( F \to Y \xrightarrow{\varphi} B \), we see that \( q(\varphi) \subset q(F) \). Thus the proof is done by Theorem 5.9. \qed

Corollary 5.12. Let \( B, Y \in \mathcal{F} \) and \( F \xrightarrow{j} B \xrightarrow{\varphi} Y \) be a homotopy fibration such that \( j \) is null homotopic. Then \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial.

We close this section by the following example.

Example 5.13. By definition, if \( \text{Ph}(\Sigma X, \varphi) \) is trivial, then it is relatively trivial. Here we give a space \( X \) and a map \( \varphi \) such that the converse of this implication does not hold, that is, \( \text{Ph}(\Sigma X, \varphi) \) is relatively trivial and is not trivial.

Let \( S^3 \to S^{4n+3} \xrightarrow{p_n} \mathbb{H}P^n \) be the Hopf fibration. Since the fiber inclusion \( S^3 \to S^{4n+3} \) is null homotopic, \( \text{Ph}(\Sigma X, p_n) \) is relatively trivial by Corollary 5.12. By Corollary 4.8, we also have that \( \text{Ph}(\Sigma X, p_n) \) is trivial if and only if \( \text{Ph}(X, S^3) = * \). Then since \( \text{Ph}(\mathbb{C}P^\infty, S^3) \neq * \) by [G], we get that \( \text{Ph}(\mathbb{C}P^\infty, p_n) \) is not trivial. Thus we have obtained that \( \text{Ph}(\Sigma \mathbb{C}P^\infty, p_n) \) is relatively trivial and is not trivial.
We first explain authors’ original motivation for introducing relative phantom maps. Recall the following de Bruijn and Erdős theorem in combinatorics.

**Theorem 6.1** (de Bruijn and Erdős [dBE]). Let $G$ be a simple graph, possibly infinite. If any finite subgraph of $G$ is $n$-colorable, then $G$ itself is $n$-colorable.

The minimum number of colors that we need to color a graph $G$ is called the chromatic number of $G$. Then as in [MZ], the chromatic number of $G$ is related with the index of a certain free $\mathbb{Z}/2$-complex associated with $G$, where the index of a free $\mathbb{Z}/2$-complex $K$ is the minimum integer $n$ such that there is a $\mathbb{Z}/2$-map $K \to S^n$. Then we can ask whether the index of a free $\mathbb{Z}/2$-complex has the same local-to-global property as the chromatic number in Theorem 6.1. Let us formulate this question. Subgraphs correspond to free $\mathbb{Z}/2$-subcomplexes. The index of a free $\mathbb{Z}/2$-complex $K$ is equivalent to the minimum integer $n$ such that the classifying map $K/(\mathbb{Z}/2) \to \mathbb{R}P^\infty$ of the free $\mathbb{Z}/2$-action can be compressed into $\mathbb{R}P^n$, up to homotopy. Then what we are asking is the following problem.

**Problem 6.2** (Topological de Bruijn-Erdős problem). Find whether or not there is a non-trivial phantom map relative to the inclusion $i_n: \mathbb{R}P^n \to \mathbb{R}P^\infty$.

Since the inclusion $\mathbb{R}P^n \to \mathbb{R}P^\infty$ is the first Postnikov section of $\mathbb{R}P^n$, the above problem is generalized to the following.

**Problem 6.3.** Find whether or not there is a non-trivial phantom map relative to the Postnikov section $s_n: B \to B_n$.

In this section, we consider Problem 6.3. By Proposition 2.4, we have $\text{Ph}(X, B_n) = \ast$, so the triviality and the relative triviality of phantom maps out of a suspension to $s_n: B \to B_n$ are the same. Then the case of relative phantom maps out of a suspension in Problem 6.3 has been studied in the last section. In particular, by Example 4.4, $\text{Ph}(\Sigma X, i_n)$ is trivial for the inclusion $i_n: \mathbb{R}P^n \to \mathbb{R}P^\infty$. Thus we consider relative phantom maps out of a non-suspension for Problem 6.3. When $X$ is not a suspension, the Puppe exact sequence associated with skeleta of $X$ is not an exact sequence of groups, so we cannot use Lemma 2.1 which has been fundamental in many places above. Instead, we will use the following lemma.

**Lemma 6.4** (cf. [RZ, Lemma 1.1.5]). Let $\{f_n\}: \{G_n\} \to \{H_n\}$ be a continuous map between inverse systems of compact Hausdorff topological spaces. Then the map $\lim f_n: \lim G_n \to \lim H_n$ is surjective whenever each $f_n: G_n \to H_n$ is so.

Let $V$ be a finite complex and $W$ be a torsion space, that is, $\tilde{H}_n(W; \mathbb{Q}) = 0$ for any $n$. Then it is well known that the homotopy set $[V, W]$ is finite. We generalize this fact in two cases. The first case is the following.
Lemma 6.5. If $B \in \mathcal{F}$ is nilpotent with finite $\pi_1$ and a finite complex $Z$ satisfies $H_k(Z; \mathbb{Q}) = 0$ for $k \in q(B)$, then $[Z, B]$ is finite.

Proof. Let $\cdots \xrightarrow{q_{k+1}} B(k+1) \xrightarrow{q_k} B(k) \xrightarrow{q_{k-1}} \cdots \xrightarrow{q_0} B(0) = *$ be a principal replacement of the Postnikov tower of $B$. Since $Z$ is a finite complex, we have $[Z, B] \cong [Z, B(k)]$ for large $k$. Then it suffices to show that $[Z, B(k)]$ is finite for any $k$. We prove this by induction on $k$.

Each arrow $q_k : B(k+1) \to B(k)$ is a principal fibration with fiber $K(A_k, m_k)$ such that $A_k$ is an abelian group. Then we have an exact sequence of pointed sets

$$H^{m_k}(Z; A_k) \to [Z, B(k)] \xrightarrow{(q_{k-1})_*} [Z, B(k-1)].$$

Since $q_{k-1} : B(k) \to B(k-1)$ is principal, we have $|q_{k-1}^{-1}(a)| \leq |H^{m_k}(Z; A_k)|$ for any $a \in [Z, B(k-1)]$. Moreover, by the assumption on $X$, $H^{m_k}(Z; A_k)$ is finite for any $k$. Then the proof is done by induction on $k$ starting with $[Z, B(0)] = *$ for $B(0) = *$. □

To consider the second case, we introduce:

Definition 6.6. We say that a space $Z$ has torsion annihilators if it has the following properties:

1. $\pi_1(Z)$ is an abelian group;
2. for any given integers $n, N$, there is a self-map $g : Z \to Z$ such that
   a. $g_* \otimes \mathbb{Q} : \pi_*(Z) \otimes \mathbb{Q} \to \pi_*(Z) \otimes \mathbb{Q}$ is an isomorphism, and
   b. for each $i \leq n$, the map $g_* : \pi_i(Z) \to \pi_i(Z)$ is the multiplication by an integer $m_i$ with $N | m_i$.

For example, $S^n \vee \mathbb{R}P^\infty$ is a space which has torsion annihilators but is not nilpotent.

Lemma 6.7. If $B \in \mathcal{F}$ has torsion annihilators and a finite complex $Z$ satisfies $H_k(Z; \mathbb{Q}) = 0$ for $k \in q(B)$, then $[Z, B]$ is finite.

Proof. Since $Z$ is a finite complex, we have $[Z, B] \cong [Z, B_n]$ for large $n$. Then it suffices to show that $[Z, B_n]$ is finite for any $n$. We prove this by induction on $n$.

Since $B_0 = *$, $[Z, B_0]$ is a singleton and there is a self-map $g : B \to B$ such that $g$ is an isomorphism in rational homotopy groups and $(g_0)_* : [Z, B_0] \to [Z, B_0]$ is the constant map. Suppose that $[Z, B_{n-1}]$ is finite and there is a self-map $h : B \to B$ such that $h$ is an isomorphism in rational homotopy groups and $(h_i)_* : [Z, B_i] \to [Z, B_i]$ is the constant map for $i < n$. By the naturality of Postnikov towers, we have the following homotopy commutative diagram.

$$
\begin{array}{ccc}
K(\pi_n(B), n) & \xrightarrow{h_*} & B_n \\
\downarrow{h_n} & & \downarrow{h_{n-1}} \\
K(\pi_n(B), n) & \xrightarrow{p_n} & B_{n-1}
\end{array}
$$

Then any map $f : Z \to B_n$ satisfies $p_n \circ h_n \circ f \simeq h_{n-1} \circ p_n \circ f \simeq *$, so $h_n \circ f$ has a lift $e : Z \to K(\pi_n(B), n)$, up to homotopy. By the assumption on $Z$, there is an integer $N$ such
that \(N \cdot H^n(Z; \pi_n(B)) = 0\), so \(Ne = 0\). Since \(B\) has torsion annihilators, there is a self-map \(\ell: B \to B\) such that \(\ell\) is an isomorphism in rational homotopy groups and the map \(\ell_*: \pi_n(B) \to \pi_n(B)\) is the multiplication by an integer \(M\) with \(N \mid M\). Then we see that \(\ell_n \circ h_n \circ f \simeq \ast\) for any \(f \in \{Z, B_n\}\). Let \(F\) be the homotopy fiber of \(\ell_n \circ h_n\). Then \(F\) is a torsion space and \([Z, F] \to [Z, B_n]\) is surjective. Since \(Z\) is a finite complex, \([Z, F]\) is a finite set, so \([Z, B_n]\) is too a finite set. This completes the proof. 

Now we give our answer to Problem 6.3.

**Theorem 6.8.** Suppose that \(B \in \mathcal{F}\) is nilpotent or has torsion annihilators. If \(H_k(X; \mathbb{Q}) = 0\) for \(k \in q(B)\), then \(\text{Ph}(X, s_n)\) is trivial for any \(n\).

**Proof.** Consider a map between the inverse systems of pointed sets \([X^k, B]\) \to \([X^k, B_n]\) induced by the Postnikov section \(s_n: B \to B_n\). There is a commutative diagram

\[
\begin{array}{ccc}
[X, B] & \xrightarrow{\pi_B} & \text{lim}[X^k, B] \\
\downarrow (s_n) & & \downarrow (s_n)_* \\
\text{Ph}(X, s_n) & \xrightarrow{\pi_{B_n}} & \text{lim} (s_n)_*[X^k, B],
\end{array}
\]

where the upper and the lower \(\pi_B\)'s are surjective by (2.1) and Proposition 3.3. Since \([X^k, B_n]\) \cong \([X, B_n]\) for \(k > n\), the map \(\pi_{B_n}: [X, B_n] \to \text{lim}[X^k, B_n]\) is injective. Then since \(\text{Ph}(X, s_n)\) is a subset of \([X, B_n]\) and the lower \(\pi_{B_n}\) is the restriction of \(\pi_{B_n}: [X, B_n] \to \text{lim}[X^k, B_n]\), the lower \(\pi_{B_n}\) is injective, so it is bijective. Then it follows that \(\text{Ph}(X, s_n)\) is trivial if and only if the right \((s_n)_*\) is surjective. Thus we shall show that the right \((s_n)_*\) is surjective.

Note that the map \((s_n)_*: [X^k, B] \to (s_n)_*[X^k, B]\) is surjective for any \(k\) and that by Lemmas 6.5 and 6.7, \([X^k, B]\) is a finite set for any \(k\). It follows from Lemma 6.4 that the right \((s_n)_*\) is surjective as desired. This completes the proof. 

Finally, we deal with the case that \(\varphi\) is the inclusion \(i_n: \mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty\). Since \(\mathbb{R}P^n\) is nilpotent for an odd \(n\), Theorem 6.8 deduces the following corollary:

**Corollary 6.9.** If \(n\) is odd and \(H_n(X; \mathbb{Q}) = 0\), then \(\text{Ph}(X, i_n)\) is trivial.

We finally show that Corollary 6.9 is optimal by giving an example of a space \(X\) such that \(H_n(X; \mathbb{Q}) \neq 0\) and there is a non-trivial relative phantom map from \(X\) to \(i_n: \mathbb{R}P^n \to \mathbb{R}P^\infty\). We will use the following simple lemma.

**Lemma 6.10.** Let \(\mathbb{Z}/2\) act on \(S^n\) by the antipodal map. For every odd integer \(k\), there is a \(\mathbb{Z}/2\)-map \(f: S^n \to S^n\) of degree \(k\).

**Proof.** The case \(n = 1\) is trivial, and for \(n > 1\), take the \((n - 1)\)-fold suspension of the \(\mathbb{Z}/2\)-map on \(S^1\). 

\[\square\]
Remark 6.11. Lemma 6.10 implies that there is a mistake in the calculation of the homotopy set $[\mathbb{R}P^n, \mathbb{R}P^m]$ for $n$ even due to McGibbon [M2]. It is calculated as follows. Consider a homotopy cofibration sequence

$$S^{n-1} \xrightarrow{p_{n-1}} \mathbb{R}P^{n-1} \xrightarrow{i_{n-1}} \mathbb{R}P^n \xrightarrow{q_n} S^n$$

where $p_{n-1}$ is the universal covering, $i_{n-1}$ is the inclusion, and $q_n$ is the pinch map to the top cell. Then for $n - k > 0$ and $k > 0$, there is an exact sequence of groups

$$\left[\Sigma^{k+1}\mathbb{R}P^{n-k-1}, \mathbb{R}P^n\right] \xrightarrow{(\Sigma^{k+1}p_{n-k-1})^*} \pi_n(\mathbb{R}P^n) \xrightarrow{(\Sigma^{k}q_{n-k})^*} \left[\Sigma^{k}\mathbb{R}P^{n-k}, \mathbb{R}P^n\right] \xrightarrow{(\Sigma^{k}i_{n-k-1})^*} \left[\Sigma^{k+1}\mathbb{R}P^{n-k-1}, \mathbb{R}P^n\right].$$

Since $\pi_n(\mathbb{R}P^n) = \mathbb{Z}\{p_n\}$, $q_k \circ p_k = 1 + (-1)^{k+1}$ and $\left[\Sigma^{k}\mathbb{R}P^{n-k-1}, \mathbb{R}P^n\right] = \ast$, we inductively get

$$\left[\Sigma^{k}\mathbb{R}P^{n-k}, \mathbb{R}P^n\right] \cong \begin{cases} \mathbb{Z} & n - k \text{ is odd} \\ \mathbb{Z}/2 & n - k \text{ is even} \end{cases}$$

where in both cases, $p_n \circ \Sigma^k q_{n-k}$ is a generator. We next consider an exact sequence of pointed sets

$$\left[\Sigma\mathbb{R}P^{n-1}, \mathbb{R}P^n\right] \xrightarrow{(\Sigma i_{n-1})^*} \pi_n(\mathbb{R}P^n) \xrightarrow{q_n^*} \left[\mathbb{R}P^n, \mathbb{R}P^n\right] \xrightarrow{i_n^*} \left[\mathbb{R}P^{n-1}, \mathbb{R}P^n\right] \xrightarrow{p_{n-1}^*} \pi_{n-1}(\mathbb{R}P^n)$$

where $\pi_{n-1}(\mathbb{R}P^n) = 0$ and $\left[\mathbb{R}P^{n-1}, \mathbb{R}P^n\right] = \{\ast, i_{n-1}\}$. Then by the above calculation, we have

$$(i_{n-1}^*)^{-1}(\ast) = \{\ast, p_n \circ q_n\}.$$ 

On the other hand, by considering the action of the top cell, we see that $(i_{n-1}^*)^{-1}(i_{n-1}) = \{h_{2j-1} | j \in \mathbb{Z}\}$, where $h_m$ is the self-map of $\mathbb{R}P^n$ which lifts to the degree $m$ self-map of $S^n$ as in Lemma 6.10. Thus we obtain that

$$\left[\mathbb{R}P^n, \mathbb{R}P^n\right] = \{\ast, p_n \circ q_n, h_{2j-1} (j \in \mathbb{Z})\}.$$ 

For $n > 2$, let $X(n)$ be the cofiber of the composite of maps

$$\bigvee_p S^{n+2p-3} \xrightarrow{\alpha_1} S^n \xrightarrow{\pi} \mathbb{R}P^n$$

where $p$ ranges over all odd primes and $\alpha_1|_{S^{n+2p-3}}$ is a generator $\alpha_1(p)$ of $\pi_{n+2p-3}(S^n) \cong \mathbb{Z}/p$ (see [T]). By definition, we have $H^1(X(n); \mathbb{Z}/2) \cong \mathbb{Z}/2$, and let $f: X(n) \to \mathbb{R}P^\infty$ be the generator of $H^1(X(n); \mathbb{Z}/2)$.

Proposition 6.12. The map $f: X(n) \to \mathbb{R}P^\infty$ is a non-trivial relative phantom map to the inclusion $i_n: \mathbb{R}P^n \to \mathbb{R}P^\infty$.

Proof. Suppose that $f$ is homotopic to a map $g: X(n) \to \mathbb{R}P^n$. Then since $g$ is an isomorphism in $\pi_1$, $g|_{\mathbb{R}P^n}$ lifts to a degree $k$ map of $S^n$ for some integer odd $k$. By definition, the composite $g|_{\mathbb{R}P^n} \circ \pi \circ \alpha_1(p)$ must be null homotopic for any odd prime $p$. Since $\alpha_1(p)$ is a co-H-map [BH], we have

$$g|_{\mathbb{R}P^n} \circ \pi \circ \alpha_1(p) \simeq \pi \circ k \circ \alpha_1(p) \simeq \pi \circ (k \alpha_1(p)).$$
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Then since \( \pi_*(S^n) \to \pi_*(\mathbb{R}P^n) \) is an isomorphism for \( * \geq 2 \), we get that \( k\alpha_1(p) \) is null homotopic. Thus \( k \) is divisible by any odd prime, which contradicts to that \( k \neq 0 \). Therefore \( f \) does not lift to \( \mathbb{R}P^n \) through the inclusion \( i_n: \mathbb{R}P^n \to \mathbb{R}P^\infty \), up to homotopy.

Fix an odd prime \( p \). By Lemma 6.10, for any given odd integer \( k \), there is a self-map \( h_k: \mathbb{R}P^n \to \mathbb{R}P^n \) which lifts to a degree \( k \) self-map of \( S^n \). Let \( p_1, \ldots, p_m \) be all odd primes less than or equal to an odd prime \( p \). Then by the above observation, we see that the map \( h_k: \mathbb{R}P^n \to \mathbb{R}P^n \) extends to a map \( \bar{h}_k: X(n) \to X(n) \), and by looking at \( \pi_1 \), we have

\[
f \simeq f \circ \bar{h}_{p_1} \circ \cdots \circ \bar{h}_{p_m}.
\]

Since \( \bar{h}_{p_i} \circ \pi \circ \alpha_1(p) \simeq \pi \circ (p_i \alpha_1(p)) \) as above, we see that the restriction of \( f \circ \bar{h}_{p_1} \circ \cdots \circ \bar{h}_{p_m} \) to \( X(n)^{n+2p-2} \) lifts to \( \mathbb{R}P^n \) through \( i_n \), up to homotopy. Since a prime \( p \) can be arbitrary large, \( f \) is a relative phantom map to the inclusion \( i_n: \mathbb{R}P^n \to \mathbb{R}P^\infty \). Therefore we obtain that \( f \) is a non-trivial relative phantom map to \( i_n: \mathbb{R}P^n \to \mathbb{R}P^\infty \), completing the proof. \( \square \)

Remark 6.13. Suppose that \( n \) is even. Then we have \( H_n(X(n); \mathbb{Q}) = 0 \) and \( H_k(X(n); \mathbb{Q}) \cong \mathbb{Q} \) for \( k = n + 2p - 2 \) for an odd prime \( p \). However, if \( X'(n) \) is a subcomplex of \( X(n) \) removed finitely many cells, then \( \text{Ph}(X(n), i_n) \) is not trivial by the same proof. Thus for even \( n \), we cannot derive a condition for triviality of \( \text{Ph}(X, i_n) \) in terms of the rational homology of \( X \).

We finally give an example showing that Proposition 4.6 does not hold if we consider a non-suspension source space.

**Corollary 6.14.** The map \( \star \times f: X(n) \to \mathbb{R}P^n \times \mathbb{R}P^\infty \) is a non-trivial phantom map to the map \( 1 \times i_n: \mathbb{R}P^n \to \mathbb{R}P^n \times \mathbb{R}P^\infty \) which is an isomorphism in \( \pi_\otimes \mathbb{Q} \) for \( n \geq 2 \).

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