The Invariance of the Diffusion Coefficient with Iterative Operations of the Charged Particle Transport Equation

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Abstract

The spatial parallel diffusion coefficient (SPDC) is one of the important quantities describing energetic charged particle transport. There are three different definitions for the SPDC: the displacement variance definition \( \kappa_{zz}^{DV} = \lim_{\Delta t \to 0} \frac{\sigma_z^2}{2\Delta t} \), the Fick’s law definition \( \kappa_{zz}^{FL} = J/X \) with \( X = \partial F/\partial z \), and the Taylor–Green–Kubo (TGK) formula definition \( \kappa_{zz}^{TGK} = \int_0^\infty dt \langle \chi(t)\chi(0) \rangle \). For a constant mean magnetic field, the three different definitions of the SPDC give the same result. However, for a focusing field, it is demonstrated that the results of the different definitions are not the same. In this paper, from the Fokker–Planck equation, we find that different methods, e.g., the general Fourier expansion and iteration method, can give different equations of the isotropic distribution function (EIDFs). But it is shown that one EIDF can be transformed into another by some derivative iterative operations (DIOs). If one definition of the SPDC is invariant for the DIOs, it is clear that the definition is also invariant for different EIDFs; therefore, it is an invariant quantity for the different derivation methods of the EIDF. For the focusing field, we suggest that the TGK definition \( \kappa_{zz}^{TGK} \) is only an approximate formula, and the Fick’s law definition \( \kappa_{zz}^{FL} \) is not invariant to some DIOs. However, at least for the special condition, in this paper we show that the definition \( \kappa_{zz}^{DV} \) is an invariant quantity to the DIOs. Therefore, for a spatially varying field, the displacement variance definition \( \kappa_{zz}^{DV} \), rather than the Fick’s law definition \( \kappa_{zz}^{FL} \) and TGK formula definition \( \kappa_{zz}^{TGK} \), is the most appropriate definition of the SPDCs.

Unified Astronomy Thesaurus concepts: Interplanetary turbulence (830); Interplanetary magnetic fields (824); Particle astrophysics (96); Cosmic rays (329)

1. Introduction

The behavior of charged energetic particles in a turbulent magnetic field superposed on the mean magnetic field is an important long-standing problem in astrophysics, e.g., cosmic-ray physics, astrophysical plasmas, space weather research, and fusion plasma physics (Jokipii 1966; Schlickeiser 2002; Matthaeus et al. 2003; Shalchi & Schlickeiser 2005; Shalchi et al. 2006; Qin 2007; Hauff & Jenko 2008; Shalchi 2009, 2010; Qin & Zhang 2014; Wang et al. 2014; Qin & Wang 2015; Malkov & Sagdeev 2017). For the collisionless limit, the complicated interaction between energetic charged particles and turbulent magnetic fields, which takes the place of two-body Coulomb collisions as the principal scattering agent, leads to stochastic particle motion, so we have to use the statistical method to describe the complicated transport of charged particles (Earl 1974, 1976; Beeck & Wibberenz 1986; Schlickeiser et al. 2007; Shalchi 2011; Litvinenko 2012a, 2012b; Shalchi & Danos 2013; He & Schlickeiser 2014; Wang et al. 2017a, 2017b; Wang & Qin 2018).

The complete models describing the evolution of the phase-space density are based on the Fokker–Planck-type transport equation, which is deduced from the Master equation in phase space. For the analytical treatment, one needs to simplify the Fokker–Planck equation and obtain the spatial diffusion type of equations. Accordingly, specific forms of the spatial diffusion coefficient have to be defined in terms of different physical processes. So far, there are three different definitions for the spatial parallel diffusion coefficient (SPDC; the acronyms used in this paper is listed in Table 1), i.e., the Fick’s law definition \( \kappa_{zz}^{FL} \), the displacement variance definition \( \kappa_{zz}^{DV} \), and the TGK formula definition \( \kappa_{zz}^{TGK} \).

In response to the particle concentration gradient expressed as the change in concentration due to a change in position along the z direction, i.e., \( \partial F/\partial z \), with the particle concentration \( F(z, t) \), the local rule for the movement of flux \( J(z, t) \) is given by Fick’s first law:

\[
J(z, t) = -\kappa_{zz} \frac{\partial F}{\partial z},
\]

with the diffusion coefficient \( \kappa_{zz} \). Therefore, the spatial diffusion coefficient can be defined as the flux per unit area per unit time per concentration gradient:

\[
\kappa_{zz}^{FL} = \frac{J(z, t)}{\partial F/\partial z}.
\]

This is the Fick’s law definition of the SPDC in this paper. Accordingly, all the terms that can be written as \( \kappa \partial^2 F/\partial z^2 \) in the equations of the isotropic distribution function (EIDFs) are called the diffusion terms, and the corresponding parameters \( \kappa \) are the SPDCs from the Fick’s law definition.
For a nonvanishing mean motion of the charged energetic particles, the variance $\sigma^2$ of the particle random displacement \(\Delta z = z - z_0\) is written as

$$\sigma^2 = \langle (\Delta z)^2 \rangle - \langle (\Delta z) \rangle^2. \quad (3)$$

The parallel diffusion coefficient defined by the displacement variance is then given by

$$\kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to t_\infty} \frac{d\sigma^2}{dt}, \quad (4)$$

where $t_\infty \gg t_d$, in which the particles approach diffusive behavior with the characteristic timescale $t_d$. It should be emphasized that $t_\infty$ is not infinite because any physical process lasts for a finite time interval. In this paper, Equation (4) is called the displacement variance definition of the SPDC.

A useful tool to compute the diffusion coefficient is the so-called Taylor–Green–Kubo (TGK) formula (Taylor 1922; Green 1951; Kubo 1962), which is the time integral over the velocity correlation function,

$$\kappa_{zz}^{TGK} = \int_0^\infty dt \langle v_z(t) v_z(0) \rangle, \quad (5)$$

where $v_z$ is the $z$ component of the particle speed $v$. This is called the TGK definition of the SPDC.

When particles are transported in a constant mean magnetic field, the displacement variance definition $\kappa_{zz}^{DV}$ is equivalent to the TGK definition $\kappa_{zz}^{TGK}$ (Shalchi 2009). Wang & Qin (2019) also analytically demonstrated that the variance definition $\kappa_{zz}^{DV}$ is equal to the Fick’s law definition $\kappa_{zz}^{FL}$ for this scenario. Therefore, for a constant field, the three different definitions of the SPDC are equivalent.

For some scenarios, e.g., the position is close to the Sun or in the solar corona, we expect that the nonuniformity of the mean field has to be considered. The spatially varying background field has an impact on the transport of particles, i.e., the so-called adiabatic focusing of energetic charged particles appears. Many researchers have investigated the influence of this effect on the parallel and perpendicular diffusion of the charged particles (Roelof 1969; Earl 1976; Kunstmann 1979; Beeck & Wibberenz 1986; Bieber & Burger 1990; Kóta 2000; Schlickeiser & Shalchi 2008; Schalchi 2011; Litvinenko 2012a, 2012b; Danos et al. 2013; Shalchi & Danos 2013; He & Schlickeiser 2014; Wang & Qin 2016; Wang et al. 2017b; Wang & Qin 2018, 2019). For a focusing field, Wang & Qin (2019) found the relationship between the Fick’s law definition $\kappa_{zz}^{FL}$ and the displacement variance definition $\kappa_{zz}^{DV}$ as

$$\kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to t_\infty} \frac{d\sigma^2}{dt} = \kappa_{zz}^{FL} - \kappa_z \kappa_{iz}, \quad (6)$$

where $\kappa_z$ and $\kappa_{iz}$ are the coefficients of the convection term and the first-order spatial and temporal derivative term in the EIDF, respectively. Because $\kappa_z$ and $\kappa_{iz}$ are not equal to zero for a spatially varying mean magnetic field, Equation (6) demonstrates that the displacement variance definition $\kappa_{zz}^{DV}$ and the Fick’s law definition $\kappa_{zz}^{FL}$ give different results. In addition, it has been suggested that the TGK definition $\kappa_{zz}^{TGK}$ is not equivalent to the variance definition $\kappa_{zz}^{DV}$ for a focusing field (Danos et al. 2013; Litvinenko & Noble 2013; Lasuik et al. 2017). Thus, the three definitions of the SPDCs, i.e., the Fick’s law definition $\kappa_{zz}^{FL}$, the displacement variance definition $\kappa_{zz}^{DV}$, and the TGK formula definition $\kappa_{zz}^{TGK}$, are not equivalent for a focusing field.

Starting from the Fokker–Planck equation with the simple BGK collision operator or relaxation-time approximation (Gombosi et al. 1993; Zank et al. 2000), using the Legendre polynomial expansion method which belongs to the general Fourier expansion, Gombosi et al. (1993) provided a derivation method of the EIDF by obtaining the EIDF which contains infinite-order temporal derivative terms. The first- and second-order EIDFs were obtained according to the scaling analysis theory. It was shown that the first-order EIDF is the well-known diffusion equation, and the second-order EIDF belongs to the telegraph equation.

In this paper, we investigate whether the SPDC, i.e., the Fick’s law definition $\kappa_{zz}^{FL}$, the displacement variance definition $\kappa_{zz}^{DV}$, and the TGK formulation definition $\kappa_{zz}^{TGK}$, are changed by the iterative operations of the derivation. Because the modified Fokker–Planck equation (hereafter MFPE) conserves the number of particles while the standard Fokker–Planck equation (SFPE) does not, the MFPE

| Table 1 | Acronyms Used in This Paper |
| --- | --- |
| The Spatial Parallel Diffusion Coefficient | SPDC |
| The Equation of the Isotropic Distribution Function $F(z, t)$ | EIDF |
| The Derivative Iterative Operation | DIO |
| Taylor–Green–Kubo | TGK |
| The Derivation Method of the EIDF | DME |
| Malkov and Sagdeev (2015) | MS2015 |
| The Standard Fokker–Planck Equation | SFPE |
| The Modified Fokker–Planck Equation | MFPE |
rather than the SFPE is used in this paper (Kunstmann 1979; Shalchi & Danos 2013). The remainder of this paper is organized as follows: in Section 2, we demonstrate that the EIDFs with different forms derived using different methods can be transformed by derivative iterative operations (DIOs). In Section 3, starting from the MFPE with the adiabatic focusing effect, we derive the differential EIDF by using the iteration method. In Section 4, we demonstrate that the displacement variance definition $\kappa_{zz}^{DV}$ is invariant for the partial derivative over $z$ iterative (PzI) operation but the Fick’s law definition $\kappa_{zz}^{PL}$ is not. In Section 5, we explore the change of the displacement variance definition $\kappa_{zz}^{DV}$ for the partial derivative over $t$ and $z$ iterative (PtZ) operation. In Section 6, we investigate the change of the displacement variance definition $\kappa_{zz}^{DV}$ for the partial derivative over $t$ iterative operation. In Section 7, we show that the TGK formulation is an approximate formula for the focusing field. In Section 8, we evaluate the parallel diffusion coefficients $\kappa_{zz}^{DV}$. We conclude and summarize our results in Section 9.

2. The DIOs

2.1. The Derivation Methods of DIEF

There are different methods to derive EIDFs, indicated as the derivation methods of the EIDF (DMEs).

2.1.1. The First- and Second-order EIDFs Obtained by Gombosi et al. (1993)

In this subsection, we briefly introduce the DME provided by Gombosi et al. (1993) because it is the basis of this section. The propagation of energetic charged particles in magnetic turbulence superposed on a large-scale magnetic field is described by the Fokker–Planck equation with the simple BGK collision operator or relaxation-time approximation (Gombosi et al. 1993; Zank et al. 2000):

$$\frac{\partial f(z, \mu, t)}{\partial t} + v\mu \frac{\partial f(z, \mu, t)}{\partial z} = -\frac{f(z, \mu, t) - F(z, t)}{\tau},$$

which describes particle transport in phase space. Here, $f(z, \mu, t)$ is the distribution function of energetic charged particles, $t$ is time, $z$ is the distance along the background magnetic field, $F(z, t)$ is the isotropic part of the distribution function $f(z, \mu, t)$, $\tau$ is the pitch-angle scattering mean free time, $v$ is the particle velocity which is conserved, and $\mu$ is the pitch-angle cosine.

According to scale analysis theory, by setting the following relation of the terms,

$$\frac{\partial F}{\partial t} \sim \frac{\partial^2 F}{\partial z^2} \sim O(1),$$

Gombosi et al. (1993) obtained the first- and second-order EIDFs by employing the general Fourier expansion method

$$\frac{\partial F}{\partial t} - \frac{v\lambda}{3} \frac{\partial^2 F}{\partial z^2} = 0,$$

$$\frac{\partial F}{\partial t} + \frac{\tau}{5} \frac{\partial^2 F}{\partial t^2} - \frac{v\lambda}{3} \frac{\partial^2 F}{\partial z^2} = 0,$$

which are Equations (28) and (29) in Gombosi et al. (1993), respectively. Obviously, the first-order equation is the well-known diffusion equation, and the second-order equation belongs to the telegraph equation as it contains the second-order temporal derivative term $\partial^2 F / \partial t^2$.

2.1.2. The First- and Second-order EIDFs Obtained by the Method of Wang & Qin (2018)

Here, we also start from the Fokker–Planck equation (see Equation (7)) to provide a new DME by employing the method in Wang & Qin (2018), which belongs to the iteration method.

If the gyrotropic cosmic-ray phase-space density $f(z, \mu, t)$ due to dominating pitch-angle diffusion adjusts very quickly to a quasi-equilibrium through pitch-angle diffusion, the gyrotropic cosmic-ray phase-space density can be divided into its average $F(z, t)$ and the anisotropic part $g(z, \mu, t)$ (see, e.g., Schlickeiser et al. 2007; Schlickeiser & Shalchi 2008; He & Schlickeiser 2014; Wang et al. 2017b; Wang & Qin 2018, 2019):

$$f(z, \mu, t) = F(z, t) + g(z, \mu, t),$$

with

$$F(z, t) = \frac{1}{2} \int_{-1}^{1} d\mu f(z, \mu, t)$$

and

$$\int_{-1}^{1} d\mu g(z, \mu, t) = 0.$$

Inserting Equation (11) into Equation (7) gives

$$\frac{\partial F}{\partial t} + \frac{\partial g}{\partial t} + v\mu \frac{\partial F}{\partial z} + v\mu \frac{\partial g}{\partial z} = -\frac{g}{\tau}. $$

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From Equation (14), the formula of the anisotropic distribution function can be found:

\[ g = -\tau \left( \frac{\partial F}{\partial t} + \frac{\partial g}{\partial t} + \nu \frac{\partial F}{\partial \zeta} + \nu \mu \frac{\partial g}{\partial \zeta} \right). \]  

(15)

Because the anisotropic distribution function \( g(z, \mu, t) \) exists on the right-hand side of the latter equation, the formula of \( g(z, \mu, t) \) is an iterative function, actually.

Integrating Equation (7) over \( \mu \) from \(-1 \) to \( 1 \) gives

\[ \frac{\partial F}{\partial t} + \frac{\nu}{2} \frac{\partial g}{\partial t} \int_{-1}^{1} d\mu \mu g = 0. \]  

(16)

To proceed, the integral \( \nu \frac{\partial}{\partial \zeta} \int_{-1}^{1} d\mu \mu g \) has to be obtained. Using Equation (15), we find

\[ \frac{\nu}{2} \frac{\partial}{\partial \zeta} \int_{-1}^{1} d\mu \mu g = -\frac{\nu}{2} \tau \left( \int_{-1}^{1} d\mu \mu \frac{\partial^{2} g}{\partial \zeta \partial \zeta} + \frac{2\nu}{3} \frac{\partial^{2} F}{\partial \zeta^{2}} \right) \\
+ \nu \int_{-1}^{1} d\mu \mu \frac{\partial^{2} g}{\partial \zeta^{2}}. \]

(17)

Similarly, by defining the relation \( \lambda = \nu \tau \), we obtain the following integral formulas from Equation (15):

\[ \frac{\lambda}{2} \int_{-1}^{1} d\mu \mu \frac{\partial^{2} F}{\partial \zeta \partial \zeta} = -\frac{\lambda^{2}}{3} \frac{\partial^{3} F}{\partial t \partial \zeta^{2}} - \frac{\lambda \tau}{2} \int_{-1}^{1} d\mu \mu \frac{\partial^{3} g}{\partial t^{2} \partial \zeta} \]
\[ - \frac{\lambda^{2}}{2} \int_{-1}^{1} d\mu \mu^{2} \frac{\partial^{2} g}{\partial \zeta^{2}}, \]

(18)

\[ \frac{\nu \lambda}{2} \int_{-1}^{1} d\mu \mu^{2} \frac{\partial^{2} g}{\partial \zeta^{2}} = -\frac{\lambda^{2}}{3} \frac{\partial^{3} F}{\partial t \partial \zeta^{2}} - \frac{\lambda^{2}}{2} \int_{-1}^{1} d\mu \mu^{2} \frac{\partial^{3} g}{\partial t^{2} \partial \zeta} \]
\[ - \frac{\nu \lambda^{2}}{2} \int_{-1}^{1} d\mu \mu \frac{\partial^{3} g}{\partial \zeta^{2}}.
\]

(19)

\[ -\frac{\tau \lambda}{2} \int_{-1}^{1} d\mu \mu \frac{\partial^{3} g}{\partial t \partial \zeta^{2}} = \frac{\tau \lambda^{2}}{3} \frac{\partial^{4} F}{\partial t^{3} \partial \zeta} + \frac{\tau^{2} \lambda}{2} \int_{-1}^{1} d\mu \mu \frac{\partial^{4} g}{\partial t^{3} \partial \zeta} \]
\[ + \frac{\tau \lambda^{2}}{2} \int_{-1}^{1} d\mu \mu^{2} \frac{\partial^{4} g}{\partial t^{2} \partial \zeta^{2}}.
\]

(20)

\[ -\lambda \int_{-1}^{1} d\mu \mu \frac{\partial^{3} g}{\partial t \partial \zeta^{2}} = \frac{2 \tau \lambda^{2}}{3} \frac{\partial^{4} F}{\partial t^{2} \partial \zeta^{2}} + \lambda \int_{-1}^{1} d\mu \mu^{2} \frac{\partial^{4} g}{\partial t^{2} \partial \zeta^{2}} \]
\[ + \lambda \int_{-1}^{1} d\mu \mu \frac{\partial^{4} g}{\partial \zeta^{3}},
\]

(21)

\[ -\frac{\nu \lambda^{2}}{2} \int_{-1}^{1} d\mu \mu \frac{\partial^{3} g}{\partial \zeta^{2}} = \frac{\nu \lambda^{2}}{5} \frac{\partial^{4} F}{\partial \zeta^{4}} + \frac{\lambda^{3}}{2} \int_{-1}^{1} d\mu \mu^{3} \frac{\partial^{4} g}{\partial t \partial \zeta^{3}} \]
\[ + \frac{\nu \lambda^{3}}{2} \int_{-1}^{1} d\mu \mu \frac{\partial^{4} g}{\partial \zeta^{4}}.
\]

(22)

Combining Equations (16) and (17)–(22) yields

\[ \frac{\partial F}{\partial t} = \frac{\nu \lambda \frac{\partial^{2} F}{\partial \zeta^{2}}}{3} - \frac{2 \lambda \frac{\partial^{2} F}{\partial \zeta^{2}}}{3} + \frac{\lambda^{2}}{3} \frac{\partial^{4} F}{\partial t \partial \zeta^{2}} \]
\[ + \frac{\nu \lambda^{3} \frac{\partial^{4} F}{\partial \zeta^{4}}}{5} + H(g), \]

(23)
with

\[ H(g) = \frac{\lambda^3}{2} \int^1_{-1} d\mu \mu^3 \frac{\partial^4 g}{\partial \tau \partial z^4} + \frac{\nu \lambda^3}{2} \int^1_{-1} d\mu \mu^2 \frac{\partial^4 g}{\partial \tau^2 \partial z^2} + \frac{\lambda^3}{2} \int^1_{-1} d\mu \mu^3 \frac{\partial^4 g}{\partial \tau \partial z^3} \]

 Analogos to Equations (17)–(22), the integrals in the latter equation can also be derived by employing the anisotropic distribution function \( g(z, \mu, t) \) (see Equation (15)). By inserting Equations (15) into the right-hand side of Equation (15) again and again, we find that the anisotropic distribution function \( g(\mu, z, t) \) can be written as a series of derivative terms of the isotropic distribution function \( F(z, t) \):

\[ g(\mu, t) = \sum_{m,n} \Theta_{m,n}(\mu) \frac{\partial^{m+n} F}{\partial t^m \partial z^n}. \]

Here, \( \Theta_{m,n}(\mu) \) is the corresponding coefficient for the derivative term \( \frac{\partial^{m+n} F}{\partial t^m \partial z^n} \) with \( m, n = 0, 1, 2, 3, \ldots \), except for \( m = n = 0 \). Therefore, combining Equations (23) and (25) can give the EIDF

\[ \frac{\partial F}{\partial t} = \sum_{m,n} \Xi_{m,n} \frac{\partial^{m+n} F}{\partial t^m \partial z^n}, \]

where \( \Xi_{m,n} \) is the coefficient of the derivative term \( \frac{\partial^{m+n} F}{\partial t^m \partial z^n} \) with \( m, n = 0, 1, 2, 3, \ldots \), except for \( m = n = 0 \).

In order to obtain the first- and second-order EIDFs according to relation (8), the form of Equation (23) is enough and the terms in \( H(g) \) do not need to be expanded anymore as done in Equations (17)–(22). From Equation (23), considering relation (8), we easily find that the first-order equation is identical with Equation (9). The second-order equation can be obtained as

\[ \frac{\partial F}{\partial t} = \nu \lambda \frac{\partial^2 F}{\partial z^2} - \frac{2 \lambda^2}{3} \frac{\partial^3 F}{\partial \tau \partial z^3} + \frac{\nu \lambda^3}{5} \frac{\partial^4 F}{\partial z^4}, \]

which is different from the second-order equation in Gombosi et al. (1993) (see Equation (10) in Section 2.1.1). It is obvious that the higher-order governing equations of \( F(z, t) \) from the DME of Wang & Qin (2018) also have different forms from those obtained in Gombosi et al. (1993).

From the above investigation, we find that the different DMEs give different EIDFs, between which there may exist some relationship.

2.2. The Transformation between the EIDFs

2.2.1. The DIOs

Multiplying Equation (27) by the differential operator \( \frac{\partial^{m+n}}{\partial t^m \partial z^n} \) with \( n = 0, 1, 2, 3, \ldots, m = 0, 1, 2, 3, \ldots \), but \( (0, 0) \notin (n, m) \), we obtain the following equation:

\[ \frac{\partial^{n+m+1} F}{\partial t^{n+1} \partial z^m} = \frac{\nu \lambda}{3} \frac{\partial^{n+m+2} F}{\partial t \partial z^{n+m}} - \frac{2 \lambda^2}{3} \frac{\partial^{n+m+3} F}{\partial t \partial z^{n+m+2}} + \frac{\nu \lambda^3}{5} \frac{\partial^{n+m+4} F}{\partial t \partial z^{n+m+4}}. \]

By pulling out one term, e.g., \( \frac{\partial^{n+m+2} F}{\partial t \partial z^{n+m+2}} \), from Equation (28) and putting it on the left-hand side of the equal sign, and then leaving all the other terms on the right-hand side,

\[ \frac{\partial^{n+m} F}{\partial t \partial z^{n+m+2}} = \frac{3 \nu \lambda}{v \lambda} \frac{\partial^{n+m+1} F}{\partial t \partial z^m} + \frac{2 \lambda}{v} \frac{\partial^{n+m+3} F}{\partial t \partial z^{n+m+2}} - \frac{3 \nu \lambda^3}{v \lambda^3} \frac{\partial^{n+m+4} F}{\partial t \partial z^{n+m+4}}, \]

Of course, similarly, we also obtain the equations in which the term \( \frac{\partial^{n+m+3} F}{\partial t \partial z^{n+m+2}} \) or \( \frac{\partial^{n+m+4} F}{\partial t \partial z^{n+m+4}} \) is on the left-hand side and the rest are on the right-hand side. Inserting Equation (29) into Equation (26), we find another new EIDF. The above method shows the derivative operation with iteration indicated as the DIO, which is an equivalent substitution without any approximation or undetermined constants introduced, and is called the \((n + m)\)th-order PtZI operation in this paper. In Equation (28),
if \( n = 0 \) and \( m = 1,2,3, \cdots \), i.e.,

\[
\frac{\partial^{m+1}F}{\partial t \partial z^m} = \frac{\nu \lambda}{3} \frac{\partial^{m+2}F}{\partial z^{m+2}} - \frac{2 \lambda^2}{3} \frac{\partial^{m+3}F}{\partial t \partial z^{m+2}} + \frac{\nu \lambda^3}{5} \frac{\partial^{m+4}F}{\partial z^{m+4}},
\]

(30)

and operating the similar iterative method as done in the previous paragraphs, we also obtain the EIDFs with different forms. These manipulations are called the \( n \)-th-order PtI operations. Similarly, for \( n = 1, 2, 3, \cdots \) and \( m = 0 \), we obtain the \( n \)-th-order partial derivative over time \( t \) iterative (PtI) operation.

In the following subsection, we demonstrate that the DIOs can convert Equation (27) derived by the perturbation theory into Equation (10) obtained by the general Fourier expansion method.

### 2.2.2. The Transformation between the Second-order EIDFs with Different Forms

First, by employing the second-order PtI operation, i.e., taking the two-order derivative of Equation (27) over \( z \), we obtain

\[
\frac{\partial^2 F}{\partial t \partial z^2} = \frac{\nu \lambda}{3} \frac{\partial^3 F}{\partial z^3} - \frac{2 \lambda^2}{3} \frac{\partial^4 F}{\partial t \partial z^2} + \frac{\nu \lambda^3}{5} \frac{\partial^5 F}{\partial z^4},
\]

(31)

Replacing the term \( \partial^2 F/\partial z^2 \) in Equation (27) by the latter equation yields

\[
\frac{\partial F}{\partial t} = \frac{\nu \lambda}{3} \frac{\partial^2 F}{\partial z^2} - \frac{2 \lambda^2}{15} \frac{\partial^3 F}{\partial t \partial z^2}.
\]

(33)

Here, we only retain the first- and second-order terms, and ignore the higher-order ones in the latter equation.

To proceed, we have to employ the first-order PtI operation. Multiplying Equation (27) by the first-order differential operator \( \partial/\partial t \) gives

\[
\frac{\partial^2 F}{\partial t^2} = \frac{\nu \lambda}{3} \frac{\partial^3 F}{\partial t \partial z^2} - \frac{2 \lambda^2}{3} \frac{\partial^4 F}{\partial t^2 \partial z^2} + \frac{\nu \lambda^3}{5} \frac{\partial^5 F}{\partial t \partial z^4},
\]

(34)

The latter equation can be rewritten as

\[
\frac{\partial^3 F}{\partial t \partial z^2} = \frac{3}{\nu \lambda \partial z^2} \frac{\partial^2 F}{\partial t^2} + \frac{2 \lambda^2}{\nu \partial^2 t \partial z^2} \frac{\partial^4 F}{\partial t \partial z^2} - \frac{3 \lambda^2}{5} \frac{\partial^5 F}{\partial t \partial z^4}.
\]

(35)

By inserting the latter equation into Equation (33) and ignoring the third- and higher-order terms, we obtain

\[
\frac{\partial F}{\partial t} + \frac{\tau \partial F}{5 \partial t^2} - \frac{\nu \lambda}{3} \frac{\partial^2 F}{\partial z^2} = 0,
\]

(36)

which is identical with the second-order equation obtained in Gombosi et al. (1993). It is obvious that the transformation from Equation (10) into Equation (27) can also be found, but to save space, we ignore this content.

The above investigation demonstrates that the EIDFs derived from the DME with the perturbation theory method and from the DME with the general Fourier expansion method can be converted into each other by the DIOs. In fact, the EIDFs can give rise to all kinds of transformations by the DIOs. The purpose of this paper is to explore whether the SPDC has the same form for different DMEs which give many EIDFs, for the cases with adiabatic focusing effect and the small angle scattering. If the SPDC is not changed by the DIOs, it is also an invariant quantity for different EIDFs which can be transformed by DIOs; consequently, the SPDC is invariant for different DMEs which give many kinds of EIDFs.

### 3. The Differential Equation of the Isotropic Distribution Function

#### 3.1. The Differential Equation with Isotropic Distribution Function \( F(z, t) \) and Anisotropic Distribution Function \( g(z, \mu, t) \)

Because the background plasmas in the solar system and interstellar space are highly conducting, large-scale electric fields can be ignored. Throughout this paper, we only consider magnetic fluctuations that are superposed on the background magnetic field. When charged particles propagate close to the Sun or in the solar corona, we expect that the nonuniformity of the mean field is important. The spatially varying mean magnetic field gives rise to the so-called particle adiabatic focusing. For mathematical tractability, we ignore the perpendicular diffusion and only consider the parallel one in this paper. In these circumstances, the evolution of the two-dimensional distribution function \( f(z, \mu, t) \) of the energetic charged particles is described by the following two-dimensional MFPE with the effects of the along-field adiabatic focusing effect and the small pitch-angle scattering (Kunstmann 1979; Earl 1981; Rufolo 1995; Kóta 2000; Saiz et al. 2003; Schlickeiser & Jenko 2010; Litvinenko & Schlickeiser 2011; Schlickeiser 2011; Litvinenko 2012a, 2012b; Shalchi & Danos 2013; He & Schlickeiser 2014;
Wang & Qin 2016; Wang et al. 2017b; Wang & Qin 2018, 2019):

\[
\frac{\partial f}{\partial t} + v_D \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left[ D_{\mu}(\mu) \frac{\partial f}{\partial \mu} - \frac{v}{2L} (1 - \mu^2) f \right],
\]

(37)

which satisfies the conservation of particle number. Here, \( t \) is time, \( z \) is the distance along the background magnetic field, \( \mu = z/v \) is the pitch-angle cosine with particle speed \( v \) and its \( z \) component \( v_z \). \( D_{\mu}(\mu) \) is the pitch-angle diffusion coefficient which is only a function of the pitch-angle cosine \( \mu \) in this paper, \( f(z, \mu, t) = f_0(z, \mu, t)/B_0(z) \) is the modified distribution function of charged energetic particles with the distribution function of charged energetic particles \( f_0(z, \mu, t) \) and the background magnetic field \( B_0(z) \), and \( L(z) = -B_0(z)/ (dB_0(z)/ dz) \) is the focusing length of the large-scale magnetic field \( B_0(z) \). For simplification, in this paper the focusing length is assumed to be a constant. The terms related to source and momentum are ignored in Equation (37). In the following, we refer to \( f \) as the distribution function for the purpose of simplicity. The more complete form of the Fokker–Planck equation can be found in Schlickeiser (2002).

The distribution function \( f_0(z, \mu, t) \) of energetic charged particles satisfies the SFPE,

\[
\frac{\partial f_0}{\partial t} + v_D \frac{\partial f_0}{\partial z} = \frac{\partial}{\partial \mu} \left[ D_{\mu}(\mu) \frac{\partial f_0}{\partial \mu} - \frac{v}{2L} (1 - \mu^2) \frac{\partial f_0}{\partial \mu} \right].
\]

(38)

For the focusing field, the parameter \( \xi = \lambda/L \) with the mean free path of particles can be defined. Thus, there exists the relation \( \xi \ll 1 \) for the weak focusing limit. If the focusing length \( L \to \infty \), i.e., \( \xi \to 0 \), indicating that the field tends to constant, both the terms \( v(1 - \mu^2)/(2L) \partial f_0 / \partial \mu \) in Equation (38) and \( \partial / \partial \mu [v(1 - \mu^2)/(2L) \partial f / \partial \mu] \) in Equation (37) tend to zero. Therefore, Equations (37) and (38) are equal for \( L = \infty \) or \( \xi = 0 \).

The distribution function \( f(z, \mu, t) \) can be divided into the isotropic part and the anisotropic one as Equation (11). Formulas (12) and (13) are also satisfied. Integrating Equation (37) over \( \mu \) yields

\[
\frac{\partial F}{\partial t} + v \frac{\partial}{\partial z} \int_{-1}^{1} \mu g d\mu = 0.
\]

(39)

In the following, by integrating Equation (37) over \( \mu \) from \(-1\) to \( \mu \), we find

\[
\frac{\partial F}{\partial t} (\mu + 1) + \frac{\partial}{\partial t} \int_{-1}^{\mu} dv g + \frac{\partial}{\partial \mu} \left[ \int_{-1}^{\mu} dv g \right] + \frac{v(z^2 - 1)}{2} \frac{\partial F}{\partial z} + v \frac{\partial}{\partial \mu} \left[ \int_{-1}^{\mu} dv g \right] = D_{\mu}(\mu) \frac{\partial g}{\partial \mu} - \frac{v(1 - \mu^2)}{2L} F - \frac{v(1 - \mu^2)}{2L} g.
\]

(40)

Here, the regularity \( D_{\mu}(\mu = \pm 1) = 0 \) is used. Subtracting Equation (39) from Equation (40) gives

\[
\frac{\partial F}{\partial t} (\mu + 1) + \frac{\partial}{\partial t} \int_{-1}^{\mu} dv g + \frac{\partial}{\partial \mu} \left[ \int_{-1}^{\mu} dv g \right] + \frac{v(z^2 - 1)}{2} \frac{\partial F}{\partial z} + v \frac{\partial}{\partial \mu} \left[ \int_{-1}^{\mu} dv g \right] \\
- \frac{v}{2} \frac{\partial}{\partial \mu} \left[ \int_{-1}^{1} \mu g d\mu \right] = D_{\mu}(\mu) \frac{\partial g}{\partial \mu} - \frac{v(1 - \mu^2)}{2L} F - \frac{v(1 - \mu^2)}{2L} g.
\]

(41)

After a straightforward algebra, Equation (41) reduces to the following form as presented in Wang & Qin (2018):

\[
\frac{\partial}{\partial \mu} \left\{ \left[ g(\mu, t) - L \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) \right] e^{-M(\mu, t)} \right\} = e^{-M(\mu, t)} \Phi(\mu, t)
\]

(42)

with

\[
M(\mu) = \frac{v}{2L} \int_{-1}^{\mu} d\nu \frac{1 - \nu^2}{D_{\nu}(\nu)}
\]

(43)
and

\[
\Phi(\mu, t) = \frac{1}{D_{\mu}(\mu)} \left[ \frac{\partial F}{\partial t} + \frac{\partial}{\partial t} \int_{-1}^{\mu} g(\nu) d\nu \right] + \frac{\nu}{2} \frac{\partial}{\partial \zeta} \left( 2 \int_{-1}^{\mu} d\nu g(\nu, -) - \int_{-1}^{1} d\mu g(\mu, -) \right). \tag{44}
\]

As shown in Wang & Qin (2018), the anisotropic distribution function \(g(z, \mu, t)\) can be obtained from Equation (42):

\[
g(z, \mu, t) = L \left( \frac{\partial F}{\partial \zeta} - \frac{F}{L} \right) \left[ 1 - \frac{2e^M(\mu)}{\int_{-1}^{1} d\mu e^M(\mu)} \right] + e^M(\mu) \left[ R(\mu, t) - \frac{\int_{-1}^{1} d\mu e^M(\mu) R(\mu, t)}{\int_{-1}^{1} d\mu e^M(\mu)} \right] \tag{46}
\]

with

\[
R(\mu, t) = \int_{-1}^{\mu} d\nu e^{-M(\nu)} \Phi(\nu, t). \tag{47}
\]

By combining Equations (45)–(47), the iterated function of \(g(z, \mu, t)\) can be obtained as follows:

\[
g(z, \mu, t) = L \left( \frac{\partial F}{\partial \zeta} - \frac{F}{L} \right) \left[ 1 - \frac{2e^M(\mu)}{\int_{-1}^{1} d\mu e^M(\mu)} \right] + e^M(\mu) \left\{ \int_{-1}^{\mu} d\nu e^{-M(\nu)} \left( \frac{\partial F}{\partial \nu} + \frac{\partial}{\partial \nu} \int_{-1}^{\nu} g(\nu, t) d\rho \right) \right. \\
+ \frac{\nu}{2} \frac{\partial}{\partial \zeta} \left( 2 \int_{-1}^{\nu} d\rho g(\nu, \rho, t) - \int_{-1}^{1} d\mu g(\mu, \mu, t) \right) \bigg\} - \frac{1}{\int_{-1}^{1} d\mu e^M(\mu)} \int_{-1}^{1} d\mu e^M(\mu) \int_{-1}^{\nu} d\nu e^{-M(\nu)} \left( \frac{\partial F}{\partial \nu} + \frac{\partial}{\partial \nu} \int_{-1}^{\nu} g(\nu, t) d\rho \right) \\
+ \frac{\nu}{2} \frac{\partial}{\partial \zeta} \left( 2 \int_{-1}^{\nu} d\nu g(\nu, \nu, t) - \int_{-1}^{1} d\mu g(\mu, \mu, t) \right) \bigg\}. \tag{48}
\]

Using Equation (48), we obtain the following formula:

\[
\frac{\partial}{\partial \zeta} \int_{-1}^{1} d\mu g(\mu, \mu, t) = 2 \left( \frac{\partial F}{\partial \zeta} - \frac{L^2}{\partial \zeta^2} \right) \int_{-1}^{1} d\mu e^M(\mu) \\
+ \int_{-1}^{1} d\mu e^M(\mu) \int_{-1}^{\mu} d\nu e^{-M(\nu)} \left( \frac{\partial^2 F}{\partial \nu \partial \zeta} + \frac{\partial^2}{\partial \nu \partial \zeta} \int_{-1}^{\nu} g(\nu, t) d\rho \right) \\
+ \frac{\nu}{2} \frac{\partial^2}{\partial \zeta^2} \left( 2 \int_{-1}^{\nu} d\rho g(\nu, \rho, t) - \int_{-1}^{1} d\mu g(\mu, \mu, t) \right) \bigg\} - \frac{1}{\int_{-1}^{1} d\mu e^M(\mu)} \int_{-1}^{1} d\mu e^M(\mu) \int_{-1}^{\nu} d\nu e^{-M(\nu)} \left( \frac{\partial^2 F}{\partial \nu \partial \zeta} + \frac{\partial^2}{\partial \nu \partial \zeta} \int_{-1}^{\nu} g(\nu, t) d\rho \right) \\
+ \frac{\nu}{2} \frac{\partial^2}{\partial \zeta^2} \left( 2 \int_{-1}^{\nu} d\nu g(\nu, \nu, t) - \int_{-1}^{1} d\mu g(\mu, \mu, t) \right) \bigg\}. \tag{49}
\]
In order to find the differential equation with isotropic and anisotropic distribution functions, one inserts Equation (49) into Equation (39) to obtain

\[
\frac{\partial F}{\partial t} + \frac{\partial F}{\partial z} \int_{-1}^{1} d\mu \mu e^{M(\mu)} = \frac{\partial^2 F}{\partial z^2} v L \int_{-1}^{1} d\mu \mu e^{M(\mu)} \nonumber \
- \frac{v}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{1} d\nu v e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \left[ \left( \frac{\partial^2 F}{\partial t \partial z} + \frac{\partial^2 F}{\partial \nu \partial z} \right) \int_{-1}^{1} g(z, \nu, t) d\rho \right] \nonumber \
+ \frac{v}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{1} d\nu v e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \left[ \left( \frac{\partial^2 F}{\partial t \partial z} + \frac{\partial^2 F}{\partial \nu \partial z} \right) \int_{-1}^{1} g(z, \nu, t) d\rho \right] \nonumber \
+ \frac{v}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{1} d\nu v e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \left[ \left( \frac{\partial^2 F}{\partial t \partial z} + \frac{\partial^2 F}{\partial \nu \partial z} \right) \int_{-1}^{1} g(z, \nu, t) d\rho \right] \nonumber \
+ \frac{v}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{1} d\nu v e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \left[ \left( \frac{\partial^2 F}{\partial t \partial z} + \frac{\partial^2 F}{\partial \nu \partial z} \right) \int_{-1}^{1} g(z, \nu, t) d\rho \right]. \tag{50}
\]

The latter equation gives the relationship between the isotropic distribution function \( F(z, t) \) and the anisotropic distribution function \( g(z, \nu, t) \).

### 3.2. The Differential Equation of the Isotropic Distribution Function \( F(z, t) \)

From Equation (48), we find that the anisotropic distribution function \( g(\mu, t) \) is an iteration function. Thus, with the iteration operation, the anisotropic distribution function can be written as the function of the temporal and spatial derivatives of the isotropic distribution function \( F(z, t) \) (Wang & Qin 2018):

\[
g(\mu, t) = \sum_{m,n} \epsilon_{m,n}(\mu) \frac{\partial^{m+n} F}{\partial z^m \partial \nu^n}. \tag{51}
\]

Here, \( \epsilon_{m,n}(\mu) \) are the coefficients with \( m, n = 0, 1, 2, 3, \ldots \). By inserting Equation (51) into Equation (50), we obtain the EIDF as shown in the Appendix to Wang & Qin (2019):

\[
\frac{\partial F}{\partial t} = \left( -\kappa_{zz} \frac{\partial F}{\partial z} + \kappa_{zzz} \frac{\partial^3 F}{\partial z^3} + \kappa_{zzzz} \frac{\partial^4 F}{\partial z^4} + \cdots \right) + \left( \kappa_{zz} \frac{\partial^2 F}{\partial t \partial z} + \kappa_{zzz} \frac{\partial^3 F}{\partial t \partial z^2} + \kappa_{zzzz} \frac{\partial^4 F}{\partial t \partial z^3} + \cdots \right) + \left( \kappa_{zzz} \frac{\partial^3 F}{\partial t \partial z^2} + \kappa_{zzzz} \frac{\partial^4 F}{\partial t \partial z^3} + \cdots \right) + \cdots. \tag{52}
\]

This is a constant coefficient linear differential EIDF with an infinite number of derivative terms. From Equation (52), we find the SPDC with the Fick’s law definition,

\[
\kappa_{zz}^{FL} = \kappa_{zz}, \tag{53}
\]

and with the displacement variance definition (Wang & Qin 2019),

\[
\kappa_{zz}^{DV} = \lim_{\tau \to \infty} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_{z} \kappa_{zz}. \tag{54}
\]

The coefficients of the terms in Equation (52) can be obtained by using the method in the papers of Wang & Qin (2018, 2019). For example, with this method, we obtain the formula of the cross-term coefficient \( \kappa_{zz} \) as

\[
\kappa_{zz} = \frac{v}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{1} d\nu v e^{-M(\nu)} \left( 2 \int_{-1}^{1} d\mu e^{M(\mu)} - 1 \right) \nonumber \
- \frac{v}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{1} d\nu v e^{-M(\nu)} \left( 2 \int_{-1}^{1} d\mu e^{M(\mu)} - 1 \right) \nonumber \
+ \frac{v}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{1} d\nu v e^{-M(\nu)} \left( 2 \int_{-1}^{1} d\mu e^{M(\mu)} - 1 \right) \nonumber \
+ \frac{v}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{1} d\nu v e^{-M(\nu)} \left( 2 \int_{-1}^{1} d\mu e^{M(\mu)} - 1 \right) \nonumber \
+ \frac{v}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{1} d\nu v e^{-M(\nu)} \left( 2 \int_{-1}^{1} d\mu e^{M(\mu)} - 1 \right). \tag{55}
\]

with the details of the derivation shown in Appendix A. The coefficient \( \kappa_{zz} \) is also shown in Appendix A (see Equation (A15)).
4. Two Definitions for the PzI Operation

On the right-hand side of Equation (52), there are a lot of spatial derivative terms, e.g., \( \kappa_z \partial F/\partial z \), \( \kappa_{zz} \partial^2 F/\partial z^2 \), \( \kappa_{zzz} \partial^3 F/\partial z^3 \), \( \kappa_{zzzz} \partial^4 F/\partial z^4 \), and spatial and temporal cross-derivative terms (hereafter abbreviated as cross terms), e.g., \( \kappa_{zt} \partial^2 F/(\partial t \partial z) \), \( \kappa_{zzt} \partial^3 F/(\partial t^2 \partial z) \), \( \kappa_{zzzt} \partial^4 F/(\partial t^3 \partial z) \), \( \kappa_{zzzzt} \partial^5 F/(\partial t^4 \partial z) \), \( \kappa_{zzzzzt} \partial^6 F/(\partial t^5 \partial z) \), \( \kappa_{zzzzzz} \partial^7 F/(\partial t^6 \partial z) \), \( \kappa_{zzzzzzt} \partial^8 F/(\partial t^7 \partial z) \), \( \kappa_{zzzzzzzt} \partial^9 F/(\partial t^8 \partial z) \), and so on. As shown in Section 2, the DMEs from different methods, i.e., the general Fourier expansion and the perturbation theory, give different EIDFs, which, however, can be interconverted by the DIOs. Because EIDFs, no matter how different they are, actually describe the same transport process of particles, their corresponding physical quantities should be invariant. The SPDC has three different definitions, i.e., the Fick’s law definition \( \kappa^{FL}_{zz} \), the displacement variance definition \( \kappa^{DV}_{zz} \), and the TGK formula definition \( \kappa^{TK}_{zz} \). If one definition of the SPDC is invariant for the DIOs, it is more reasonable and should be used in the computer simulations, data analysis, and theoretical research over that which is changed.

4.1. The PzI Operation

We first multiply Equation (52) by the differential operator \( \partial^m/\partial z^m \) with \( m = 1, 2, 3, \ldots \), and find

\[
\frac{\partial^m+1 F}{\partial t \partial z^m} = \left( -\kappa_z \frac{\partial^m F}{\partial z^m} + \kappa_{zz} \frac{\partial^{m+1} F}{\partial z^{m+1}} + \kappa_{zzz} \frac{\partial^{m+2} F}{\partial z^{m+2}} + \kappa_{zzzz} \frac{\partial^{m+3} F}{\partial z^{m+3}} + \kappa_{zzzzz} \frac{\partial^{m+4} F}{\partial z^{m+4}} + \cdots \right) + \left( \kappa_{zt} \frac{\partial^{m+2} F}{\partial t \partial z^{m+1}} + \kappa_{zzt} \frac{\partial^{m+3} F}{\partial t^2 \partial z^{m+1}} + \kappa_{zzzt} \frac{\partial^{m+4} F}{\partial t^3 \partial z^{m+1}} + \cdots \right) + \left( \kappa_{zzt} \frac{\partial^{m+3} F}{\partial t \partial z^{m+2}} + \kappa_{zzzt} \frac{\partial^{m+4} F}{\partial t^2 \partial z^{m+2}} + \kappa_{zzzzt} \frac{\partial^{m+5} F}{\partial t^3 \partial z^{m+2}} + \cdots \right) + \cdots.
\]

After drawing any term on the right-hand side of Equation (56), and putting it on the left-hand side and the other terms on the right-hand side, we obtain a new equation. Thereafter, inserting this new equation into Equation (52), we obtain a new EIDF and call it Equation \( \mathcal{A} \). Mathematically speaking, Equation \( \mathcal{A} \) is equivalent to Equation (52).

The above manipulation is the \( m \)-th order Partial derivative over \( z \) iterative \( (m \text{th PzI}) \) operation. For \( m = 1 \), Equation (56) becomes

\[
\frac{\partial^2 F}{\partial t \partial z} = \left( -\kappa_z \frac{\partial^2 F}{\partial z^2} + \kappa_{zz} \frac{\partial^3 F}{\partial z^3} + \kappa_{zzz} \frac{\partial^4 F}{\partial z^4} + \kappa_{zzzz} \frac{\partial^5 F}{\partial z^5} + \cdots \right) + \left( \kappa_{zt} \frac{\partial^3 F}{\partial t \partial z^2} + \kappa_{zzt} \frac{\partial^4 F}{\partial t^2 \partial z^2} + \kappa_{zzzt} \frac{\partial^5 F}{\partial t^3 \partial z^2} + \cdots \right) + \cdots.
\]

which is the equation of the first-order PzI operation. In this subsection, we explore the Fick’s law definition \( \kappa^{FL}_{zz} \) and the displacement variance definitions \( \kappa^{DV}_{zz} \) for the first-order PzI operation.

4.1.1. \( R_z \) of the First-order PzI Operation

First, multiplying Equation (57) by the parameter \( \kappa_z \), we find

\[
\kappa_z \frac{\partial^2 F}{\partial t \partial z} = \left( -\kappa_z \kappa_z \frac{\partial^2 F}{\partial z^2} + \kappa_z \kappa_{zz} \frac{\partial^3 F}{\partial z^3} + \kappa_z \kappa_{zzz} \frac{\partial^4 F}{\partial z^4} + \kappa_z \kappa_{zzzz} \frac{\partial^5 F}{\partial z^5} + \cdots \right) + \left( \kappa_z \kappa_{zt} \frac{\partial^3 F}{\partial t \partial z^2} + \kappa_z \kappa_{zzt} \frac{\partial^4 F}{\partial t^2 \partial z^2} + \kappa_z \kappa_{zzzt} \frac{\partial^5 F}{\partial t^3 \partial z^2} + \cdots \right) + \cdots.
\]

Replacing \( \kappa_z \partial^2 F/(\partial t \partial z) \) in Equation (52) by the latter equation gives

\[
\frac{\partial F}{\partial t} = -\kappa_z \frac{\partial F}{\partial z} + \left( \kappa_z - \kappa_z \kappa_z \right) \frac{\partial^2 F}{\partial z^2} + \left( \kappa_{zz} + \kappa_z \kappa_z \right) \frac{\partial^3 F}{\partial z^3} + \left( \kappa_{zz} + \kappa_z \kappa_z \right) \frac{\partial^4 F}{\partial z^4} + \cdots + \left( \kappa_{zz} + \kappa_z \kappa_z \right) \frac{\partial^3 F}{\partial t^2 \partial z} + \left( \kappa_{zz} + \kappa_z \kappa_z \right) \frac{\partial^4 F}{\partial t^3 \partial z} + \cdots + \left( \kappa_{zz} + \kappa_z \kappa_z \right) \frac{\partial^3 F}{\partial t^2 \partial z} + \left( \kappa_{zz} + \kappa_z \kappa_z \right) \frac{\partial^4 F}{\partial t^3 \partial z} + \cdots.
\]

Here, the subscript \( 2\text{nd}z \) of \( \kappa_{2\text{nd}z} \) in the latter equation denotes that there are two letters \( t \) and three letters \( z \), i.e., \( \kappa_{zzzz} \). In the same way, the subscript \( n\text{th}z \) of \( \kappa_{n\text{th}z} \) in the latter equation presents \( n \) letters \( t \) and \( m \) letters \( z \). The other cases denote the same meaning. This notation is used throughout this paper. The above manipulation is called the \( R_z \) of the first PzI operation.

From Equation (59), we easily find that the SPDC of the Fick’s law definition is \( \kappa^{FL}_{zz} = \kappa_{zz} - \kappa_z \kappa_z \), which is different from Equation (53). Therefore, the Fick’s law definition \( \kappa^{FL}_{zz} \) is changed, or in other words, is not invariant, by the \( R_z \) of the first PzI operation.
By multiplying Equation (59) with \( \Delta z \) and integrating the result over \( z \), one can find

\[
\frac{d}{dt} \langle \Delta z \rangle = \int_{-\infty}^{\infty} d\zeta (\Delta z) \frac{\partial F}{\partial t} = \int_{-\infty}^{\infty} d\zeta (\Delta z) \left[ -\kappa \frac{\partial F}{\partial \zeta} + \left( \kappa_{zz} - \kappa_{z} \kappa_{zz} \right) \frac{\partial^2 F}{\partial \zeta^2} + \left( \kappa_{zz} + \kappa_{z} \kappa_{zz} \right) \frac{\partial^3 F}{\partial \zeta^3} + \left( \kappa_{zzz} + \kappa_{z} \kappa_{zzz} \right) \frac{\partial^4 F}{\partial \zeta^4} + \cdots \right]
\]

\[
+ \kappa_{mz} \frac{\partial^4 F}{\partial \zeta^2 \partial \zeta^2} + \cdots + \left( \kappa_{zz} + \kappa_{z} \kappa_{zz} \right) \frac{\partial^3 F}{\partial \zeta^2} + \left( \kappa_{mz} + \kappa_{z} \kappa_{mz} \right) \frac{\partial^4 F}{\partial \zeta^2} + \cdots
\]

\[
+ \left( \kappa_{zz} + \kappa_{z} \kappa_{zz} \right) \frac{\partial^3 F}{\partial \zeta^2} + \left( \kappa_{zzz} + \kappa_{z} \kappa_{zzz} \right) \frac{\partial^4 F}{\partial \zeta^2} + \cdots \right].
\]

The latter equation can be formally rewritten as

\[
\frac{d}{dt} \langle \Delta z \rangle = \sum_{i} \int_{-\infty}^{\infty} d\zeta (\Delta z) E_{i},
\]

where \( E_{i} \) denotes the terms on the right-hand side of Equation (59).

By using the following regularities,

\[
F(z = \pm \infty) = 0,
\]

\[
\frac{\partial^n F}{\partial \zeta^n} (z = \pm \infty) = 0 \quad n = 1, 2, 3, \ldots,
\]

and employing the following formulas:

\[
\int_{-\infty}^{\infty} d\zeta (\Delta z) \left( \kappa_{zz} + \kappa_{z} \kappa_{(n-1)z} \right) \frac{\partial^{n} F}{\partial \zeta^{n}} = (\kappa_{zz} + \kappa_{z} \kappa_{(n-1)z}) \int_{-\infty}^{\infty} d\zeta (\Delta z) \frac{\partial^{n} F}{\partial \zeta^{n}} = 0 \quad n = 2, 3, \ldots,
\]

\[
\int_{-\infty}^{\infty} d\zeta (\Delta z) \left( \kappa_{zz} + \kappa_{z} \kappa_{mz} \kappa_{(n-1)z} \right) \frac{\partial^{n+1} F}{\partial \zeta^{n+1}} = (\kappa_{zz} + \kappa_{z} \kappa_{mz} \kappa_{(n-1)z}) \int_{-\infty}^{\infty} d\zeta (\Delta z) \frac{\partial^{n+1} F}{\partial \zeta^{n+1}} = 0 \quad n = 1, 2, 3, \ldots,
\]

\[
\int_{-\infty}^{\infty} d\zeta (\Delta z) \left( \kappa_{zz} - \kappa_{z} \kappa_{zz} \right) \frac{\partial^{2} F}{\partial \zeta^{2}} = (\kappa_{zz} - \kappa_{z} \kappa_{zz}) \int_{-\infty}^{\infty} d\zeta (\Delta z) \frac{\partial^{2} F}{\partial \zeta^{2}} = 2(\kappa_{zz} - \kappa_{z} \kappa_{zz}),
\]

\[
\int_{-\infty}^{\infty} d\zeta (\Delta z) \left( \kappa_{zz} + \kappa_{z} \kappa_{(n-1)z} \right) \frac{\partial^{n+1} F}{\partial \zeta^{n}} = (\kappa_{zz} + \kappa_{z} \kappa_{(n-1)z}) \int_{-\infty}^{\infty} d\zeta (\Delta z) \frac{\partial^{n+1} F}{\partial \zeta^{n}} = 0 \quad n = 3, 4, 5, \ldots,
\]

\[
\int_{-\infty}^{\infty} d\zeta (\Delta z)^{2} \frac{\partial F}{\partial \zeta} = \kappa_{zz} \int_{-\infty}^{\infty} d\zeta (\Delta z) \frac{\partial F}{\partial \zeta} = 2\kappa_{zz} \langle \Delta z \rangle,
\]

\[
\int_{-\infty}^{\infty} d\zeta (\Delta z)^{2} \frac{\partial^{2} F}{\partial \zeta^{2}} = \kappa_{zz} \kappa_{zz} \int_{-\infty}^{\infty} d\zeta (\Delta z)^{2} \frac{\partial^{2} F}{\partial \zeta^{2}} = 2(\kappa_{zz} - \kappa_{z} \kappa_{zz}),
\]

\[
\int_{-\infty}^{\infty} d\zeta (\Delta z)^{2} \frac{\partial^{n+1} F}{\partial \zeta^{n}} = \kappa_{zz} \kappa_{zz} \int_{-\infty}^{\infty} d\zeta (\Delta z)^{2} \frac{\partial^{n+1} F}{\partial \zeta^{n}} = -2\kappa_{zz} \int_{-\infty}^{\infty} d\zeta (\Delta z)^{2} \frac{\partial^{n} F}{\partial \zeta^{n}} = 0 \quad n = 1, 2, 3, \ldots,
\]
The above manipulation, which is one of the DIOs, is called the Multiplying Equation (which is identical to Equation we easily find that only the term $\partial F/\partial z$ on the right-hand side of Equation (60) is left. That is,

$$
\frac{d}{dt} \langle (\Delta z) \rangle = \sum_i \int_{-\infty}^{\infty} dz (\Delta z) E_i = \int_{-\infty}^{\infty} dz (\Delta z) \left( -\kappa_z \frac{\partial F}{\partial z} \right).
$$

Using integration by parts, we rewrite the latter equation as

$$
\frac{d}{dt} \langle (\Delta z) \rangle = \kappa_z.
$$

Similarly, we find the following formula

$$
\frac{d}{dt} \langle (\Delta z)^2 \rangle = \sum_i \int_{-\infty}^{\infty} dz (\Delta z)^2 E_i = 2\kappa_z \langle (\Delta z) \rangle + 2\kappa_{zz} - 2\kappa_z \kappa_z.
$$

Replacing $\kappa_z$ on the right-hand side of the latter equation by Equation (73) and considering the relation

$$
\sigma^2 = \langle (\Delta z)^2 \rangle - \langle (\Delta z) \rangle^2,
$$

we easily find that the displacement variance definition of the SPDCs is as follows:

$$
\kappa_{zz}^{DV} = \lim_{t \to t_0} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_z \kappa_z,
$$

which is identical to Equation (54). Therefore, the displacement variance definition $\kappa_{zz}^{DV}$ is not changed by the $R_{zz}$ of the first PzI operation.

From the above investigation, we find the following conclusion: the Fick’s law definition $\kappa_{zz}^{FL}$ is changed by the $R_{zz}$ of the first-order PzI operation, which is one of the DIOs. However, the formula $\lim_{t \to t_0} d\sigma^2/(2dt) = \kappa_{zz} - \kappa_z \kappa_z$ is invariant for this manipulation. That is, the displacement variance definition $\kappa_{zz}^{DV} = \lim_{t \to t_0} d\sigma^2/(2dt)$ is invariant for the $R_{zz}$ of the first PzI operation.

4.1.2. $R_{zzz}$ of the First-order PzI Operation

By drawing $\partial^3 F/\partial z^3$ from Equation (57), we can rewrite the equation as

$$
\frac{\partial^3 F}{\partial z^3} = \frac{1}{\kappa_z} \frac{\partial^2 F}{\partial z^2} + \left( \frac{\kappa_z}{\kappa_{zz}} \frac{\partial^2 F}{\partial z^2} - \frac{\kappa_{zz} \kappa_z}{\kappa_z} \frac{\partial^3 F}{\partial z^3} - \frac{\kappa_{zz}}{\kappa_z} \frac{\partial^4 F}{\partial z^4} - \cdots \right) \left( \frac{\kappa_z}{\kappa_{zz}} \frac{\partial^3 F}{\partial z^3} + \frac{\kappa_z}{\kappa_{zz}} \frac{\partial^4 F}{\partial z^4} + \frac{\kappa_{zz}}{\kappa_z} \frac{\partial^5 F}{\partial z^5} + \cdots \right)
$$

Multiplying Equation (77) by $\kappa_{zzz}$ and inserting the result into Equation (52) gives

$$
\frac{\partial F}{\partial t} = \left[ -\kappa_z \frac{\partial F}{\partial z} + \left( \frac{\kappa_z}{\kappa_{zzz}} \frac{\partial^2 F}{\partial z^2} + \left( \frac{\kappa_z}{\kappa_{zzz}} - \frac{\kappa_{zzz}}{\kappa_z} \right) \frac{\partial^4 F}{\partial z^4} + \left( \frac{\kappa_{zzz}}{\kappa_z} \right) \frac{\partial^5 F}{\partial z^5} + \cdots \right] \frac{\partial F}{\partial z} + \left( \frac{\kappa_z}{\kappa_{zzz}} \right) \frac{\partial^3 F}{\partial z^3} + \left( \frac{\kappa_z}{\kappa_{zzz}} \right) \frac{\partial^4 F}{\partial z^4} + \left( \frac{\kappa_{zzz}}{\kappa_z} \right) \frac{\partial^5 F}{\partial z^5} + \cdots \right] \frac{\partial F}{\partial z}
$$

The above manipulation, which is one of the DIOs, is called the $R_{zzz}$ of the first PzI operation.

In Equation (78), the parallel diffusion coefficient, which is in front of $\partial^2 F/\partial z^2$, is $\kappa_{zz}^{FL} = \kappa_z + \kappa_{zz} \kappa_z / \kappa_{zz}$. However, it is $\kappa_{zz}^{FL} = \kappa_{zz}$ in Equations (52). Therefore, the Fick’s law definition $\kappa_{zz}^{FL}$ is variant for $R_{zzz}$ of the first PzI operation.
As done in Section 4.1.1, from Equation (78), the derivative of the first- and second-order moments of the parallel displacement over time can be obtained as

$$\frac{d}{dt}\langle (\Delta z) \rangle = \kappa_z,$$

(79)

$$\frac{d}{dt}\langle (\Delta z)^2 \rangle = 2\kappa_z \langle (\Delta z) \rangle + 2 \left( \kappa_{zz} + \frac{\kappa_{ijz} \kappa_z}{\kappa_{ij}} \right) \frac{d}{dt}\langle (\Delta z) \rangle - 2 \left( \kappa_{zz} + \kappa_z \right) \frac{1}{2} \frac{d}{dt}\langle (\Delta z)^2 \rangle.$$

(80)

Combining Equations (79) and (80) gives

$$\kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to t_c} \frac{d\sigma^2}{dt} = \frac{1}{2} \frac{d}{dt}\langle (\Delta z)^2 \rangle - \langle (\Delta z) \rangle \frac{d}{dt}\langle (\Delta z) \rangle$$

(81)

which is the same as Equation (54). Therefore, the displacement variance definition \( \kappa_{zz}^{DV} \) is invariant for the \( R_{zzz} \) of the first PzI operation.

4.1.3. \( R_{i;i+1} \) of the First-order PzI Operation

In this part, we explore the influence of other kinds of DIOs, i.e., \( R_{i;i+1} \) \((i, j) \in N^2 \) but \((0, 0) \notin N^2 \) of the first-order PzI operation, on the Fick’s law and the displacement variance definitions. Here, \( N^2 = N \times N \) denotes the ordered pairs of all the natural numbers. As with the manipulation in the previous parts, the EIDF corresponding to the \( R_{i;i+1} \) of the first-order PzI operation can be obtained

$$\frac{\partial F}{\partial t} = \left[ -\kappa_z \frac{\partial F}{\partial z} + \left( \kappa_{zz} + \frac{\kappa_{ijz} \kappa_z}{\kappa_{ij}} \right) \frac{\partial^2 F}{\partial z^2} + \left( \kappa_{zz} - \frac{\kappa_{ijz} \kappa_z}{\kappa_{ij}} \right) \frac{\partial^3 F}{\partial z^3} + \left( \kappa_{zz} - \frac{\kappa_{ijz} \kappa_z}{\kappa_{ij}} \right) \frac{\partial^4 F}{\partial z^4} + \cdots \right]$$

$$+ \left[ \left( \kappa_{zz} + \frac{\kappa_{ijz} \kappa_z}{\kappa_{ij}} \right) \frac{\partial^2 F}{\partial t \partial z} + \left( \kappa_{ijz} - \frac{\kappa_{ij} \kappa_z}{\kappa_{ij}} \right) \frac{\partial F}{\partial t} \frac{\partial z}{\partial z} \right]$$

(82)

Equation (82) shows that the Fick’s law definition of the SPDC is \( \kappa_{zz}^{PL} = \kappa_z + \frac{\kappa_{ijz} \kappa_z}{\kappa_{ij}} \), which is different from Equation (53), demonstrating that the Fick’s law definition \( \kappa_{zz}^{PL} \) is variant for the \( R_{i;i+1} \) of the first-order PzI operation.

The formulas of the first- and second-order moments of the parallel displacement can be obtained from Equation (82):

$$\frac{d}{dt}\langle (\Delta z) \rangle = \kappa_z = \text{constant},$$

(83)

$$\frac{d}{dt}\langle (\Delta z)^2 \rangle = 2\kappa_z \langle (\Delta z) \rangle + 2 \left( \kappa_{zz} + \frac{\kappa_{ijz} \kappa_z}{\kappa_{ij}} \right) \frac{d}{dt}\langle (\Delta z) \rangle - 2 \left( \kappa_{zz} + \frac{\kappa_{ijz} \kappa_z}{\kappa_{ij}} \right) \frac{1}{2} \frac{d}{dt}\langle (\Delta z)^2 \rangle.$$

(84)

To proceed, we easily derive the following formula from the latter two equations:

$$\kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to t_c} \frac{d\sigma^2}{dt} = \frac{1}{2} \frac{d}{dt}\langle (\Delta z)^2 \rangle - \langle (\Delta z) \rangle \frac{d}{dt}\langle (\Delta z) \rangle$$

(85)

The latter formula is identical to Equation (54) to show that the displacement variance definition \( \kappa_{zz}^{DV} \) is not changed by the \( R_{i;i+1} \) of the first-order PzI operation.
4.2. Second-order PzI Operation

For \( n = 2 \), Equation (56) becomes
\[
\frac{\partial^3 F}{\partial t \partial z^2} = \left(-\kappa_z \frac{\partial^3 F}{\partial z^3} + \kappa_{zz} \frac{\partial^4 F}{\partial z^4} + \kappa_{zzz} \frac{\partial^5 F}{\partial z^5} + \kappa_{zzzz} \frac{\partial^6 F}{\partial z^6} + \cdots \right) + \left(\kappa_{zz} \frac{\partial^4 F}{\partial t \partial z^3} + \kappa_{zzzz} \frac{\partial^5 F}{\partial t \partial z^4} + \kappa_{zzzzz} \frac{\partial^6 F}{\partial t \partial z^5} + \cdots \right) + \left(\kappa_{zzzz} \frac{\partial^5 F}{\partial t^2 \partial z^3} + \kappa_{zzzzzz} \frac{\partial^6 F}{\partial t^2 \partial z^4} + \cdots \right),
\]
which is the equation of the second-order PzI operation. Because there exist a limitless variety of terms in the latter equation, combining Equations (52) and (86) can give countless kinds of new EIDFs by numerous types of the DIOs, which are the replacement manipulations for the second-order PzI operation. In the following, we explore the influence of these manipulations on the Fick’s law and displacement variance definitions of the SPDC.

4.2.1. \( R_{zz} \) of the Second-order PzI Operation

Multiplying Equation (86) by \( \kappa_{zz} \) yields
\[
\kappa_{zz} \frac{\partial^3 F}{\partial t \partial z^2} = \left(-\kappa_z \kappa_{zz} \frac{\partial^3 F}{\partial z^3} + \kappa_{zz} \kappa_{zz} \frac{\partial^4 F}{\partial z^4} + \kappa_{zz} \kappa_{zzzz} \frac{\partial^5 F}{\partial z^5} + \kappa_{zz} \kappa_{zzzzz} \frac{\partial^6 F}{\partial z^6} + \cdots \right) + \left(\kappa_{zzzz} \kappa_{zz} \frac{\partial^4 F}{\partial t \partial z^3} + \kappa_{zz} \kappa_{zzzz} \frac{\partial^5 F}{\partial t \partial z^4} + \kappa_{zzzz} \kappa_{zzzz} \frac{\partial^6 F}{\partial t \partial z^5} + \cdots \right) + \left(\kappa_{zzzzz} \kappa_{zzzz} \frac{\partial^5 F}{\partial t^2 \partial z^3} + \kappa_{zzzz} \kappa_{zzzz} \frac{\partial^6 F}{\partial t^2 \partial z^4} + \cdots \right),
\]
Inserting the latter equation into Equation (52), we find
\[
\frac{\partial F}{\partial t} = -\kappa_z \frac{\partial F}{\partial z} + \kappa_{zz} \frac{\partial^2 F}{\partial z^2} + (\kappa_{3z} - \kappa_{zzz}) \frac{\partial^3 F}{\partial z^3} + (\kappa_{4z} + \kappa_{zzzz}) \frac{\partial^4 F}{\partial z^4} + (\kappa_{5z} + \kappa_{zzzzz}) \frac{\partial^5 F}{\partial z^5} + \cdots
\]
\[+ \left(\kappa_{zzzz} \frac{\partial^4 F}{\partial t \partial z^3} + \kappa_{zzzzzz} \frac{\partial^5 F}{\partial t \partial z^4} + \kappa_{zzzzzzzz} \frac{\partial^6 F}{\partial t \partial z^5} + \cdots \right)
\]
\[+ \left(\kappa_{zzzzzz} \frac{\partial^5 F}{\partial t^2 \partial z^3} + \kappa_{zzzzzzzz} \frac{\partial^6 F}{\partial t^2 \partial z^4} + \cdots \right) + \cdots.
\]
The manipulation in this subsection is called the \( R_{zz} \) of the second-order PzI operation, which is one of the DIOs.

From Equation (88), we find that the Fick’s law definition \( \kappa_{zz}^{FL} \) of the SPDC is equal to \( \kappa_{zz} \), which is identical to Equation (53). That is, the Fick’s law definition is not changed by the \( R_{zz} \) of the second-order PzI operation. In the following, we explore the displacement variance definition \( \kappa_{zz}^{DV} \) for the \( R_{zz} \) of the second-order PzI operation.

Using the method in Section 4.1.1, we obtain
\[
\frac{d}{dt} \langle (\Delta z) \rangle = \kappa_z,
\]
\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2 \kappa_z \langle (\Delta z) \rangle + 2 \kappa_{zz} - 2 \kappa_{zz} \frac{d}{dt} \langle \Delta z \rangle.
\] Replacing \( d \langle \Delta z \rangle / dt \) on the right-hand side of Equation (90) by Equation (89) yields
\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2 \kappa_z \langle (\Delta z) \rangle + 2 \kappa_{zz} - 2 \kappa_{zz} \kappa_z.
\]
Inserting Equation (89) into the latter equation gives
\[
\kappa_{zz}^{DV} = \frac{1}{2} \lim_{\alpha \to \kappa_z} \frac{d \sigma^2}{d \alpha} = \kappa_z - \kappa_{zz} \kappa_z.
\]
Here, the formula \( \sigma^2 = \langle (\Delta z)^2 \rangle - \langle (\Delta z) \rangle^2 \) is used. Obviously, Equation (92) is exactly identical to Equation (54). The investigation in this part demonstrates that the displacement variance definition \( \kappa_{zz}^{DV} \) is invariant for the \( R_{zz} \) of the second-order PzI operation.
4.2.2. \( R_{ijz} \) of the Second-order PzI Operation

Here, we derive the EIDF for another new DIO, i.e., \( R_{ijz} \) with \( i = 0, 1, 2, \ldots \) and \( j = 3, 4, 5, \ldots \) of the second-order PzI operation. Analogous to the previous parts, one can obtain

\[
\begin{align*}
\frac{\partial F}{\partial t} &= -\kappa_z \frac{\partial F}{\partial z} + \kappa_{zz} \frac{\partial^2 F}{\partial z^2} + \left( \kappa_{zzz} + \kappa_z \frac{\kappa_{ijz}}{\kappa_{ij(j-2)z}} \right) \frac{\partial^3 F}{\partial z^3} + \left( \kappa_{zzz} - \kappa_z \frac{\kappa_{ijz}}{\kappa_{ij(j-2)z}} \right) \frac{\partial^4 F}{\partial z^4} + \cdots + \kappa_z \frac{\partial^2 F}{\partial t \partial z} \\
&\quad + \kappa_{zz} \frac{\partial^3 F}{\partial t^2 \partial z} + \kappa_{zzz} \frac{\partial^4 F}{\partial t^3 \partial z} + \kappa_{zz} \frac{\partial^5 F}{\partial t^3 \partial z^2} + \cdots + \kappa_z \frac{\partial^4 F}{\partial t^4 \partial z^2} + \cdots \\
&\quad + \left( \kappa_{zzz} - \kappa_z \frac{\kappa_{ijz}}{\kappa_{ij(j-2)z}} \right) \frac{\partial^4 F}{\partial t \partial z^3} + \left( \kappa_{zzzz} - \kappa_z \frac{\kappa_{ijz}}{\kappa_{ij(j-2)z}} \right) \frac{\partial^5 F}{\partial t^2 \partial z^3} + \left( \kappa_{zzzz} - \kappa_z \frac{\kappa_{ijz}}{\kappa_{ij(j-2)z}} \right) \frac{\partial^6 F}{\partial t^3 \partial z^3} + \cdots \\
&\quad + \left( \kappa_{zzzz} - \kappa_z \frac{\kappa_{ijz}}{\kappa_{ij(j-2)z}} \right) \frac{\partial^5 F}{\partial t^2 \partial z^4} + \cdots + \left( \kappa_{zzzz} - \kappa_z \frac{\kappa_{ijz}}{\kappa_{ij(j-2)z}} \right) \frac{\partial^6 F}{\partial t^3 \partial z^4} + \cdots \\
&\quad + \left( \kappa_{zzzz} - \kappa_z \frac{\kappa_{ijz}}{\kappa_{ij(j-2)z}} \right) \frac{\partial^6 F}{\partial t^4 \partial z^4} + \cdots,
\end{align*}
\]

From Equation (93), we find that the Fick’s law definition \( \kappa^D_{zz} \) of the SPDC is equal to \( \kappa_{zz} \), which is identical to Equation (53). So the Fick’s law definition is invariant for the \( R_{ijz} \) of the second-order PzI operation. In what follows, we investigate the influence of the \( R_{ijz} \) of the second-order PzI operation on the displacement variance definition of the SPDC.

Analogous to the method in Section 4.1.1, we find

\[
\frac{d}{dt} \langle (\Delta z) \rangle = \kappa_z, 
\]

\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_z \langle (\Delta z) \rangle + 2\kappa_{zz} - 2\kappa_z \frac{d}{dt} \langle (\Delta z) \rangle.
\]

Similar to Section 4.2.1, we obviously obtain

\[
\kappa^D_{zz} = \frac{1}{2} \lim_{t \to t_0} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_z \kappa_{zz},
\]

which has the same form with Equation (54). Thus, the displacement variance definition is an invariant quantity for the \( R_{ijz} \) of the second-order PzI operation.

In fact, we easily find that there is no the second-order spatial derivative term in Equation (86). Thus, all the DIOs, i.e., any iteration operation by inserting the deformations of Equation (86) into Equation (52) cannot change the parallel diffusion coefficient in the results. Therefore, the Fick’s law definition \( \kappa^D_{zz} \) is invariant for the manipulations of the second-order PzI operation.

Every term in Equation (86) can be represented by \( E \), and the following formula

\[
\int_{-\infty}^{\infty} dz (\Delta z) E = 0
\]

holds. Therefore, the contributions of all terms in Equation (86) to the first moment is equal to zero. Thus, the first-order moment of the EIDFs for any EIDF derived by the DIOs of the second-order PzI operation only comes from Equation (52). Similarly, there is no contribution from any term in Equation (86) to the second-order moment:

\[
\int_{-\infty}^{\infty} dz (\Delta z)^2 E = 0.
\]

Consequently, the displacement variance definition, as a function of \( \langle (\Delta z) \rangle \) and \( \langle (\Delta z)^2 \rangle \), is only determined by Equation (52), from which Wang & Qin (2019) actually already obtained the following formula:

\[
\kappa^D_{zz} = \frac{1}{2} \lim_{t \to t_0} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_z \kappa_{zz}.
\]

The latter formula is identical to Equation (54). Therefore, the displacement variance definition is invariant for the \( R_{ijz} \) of the second-order PzI operation.

4.3. Two Definitions for the Third- and Higher-order PzI Operations

It is obvious that there is no the second-order spatial derivative term in Equation (56) for \( m \geq 3 \). Any manipulation of the third- and higher-order PzI operations cannot change the Fick’s law definition in the new EIDF or influence the first- and second-order
moments of the parallel displacement. Therefore, as with the discussion in Section 4.2.2, the Fick’s law definition \( \kappa_{zz}^{PL} \) and the displacement variance definition \( \kappa_{zz}^{DV} \) are all invariant quantities for the third- and higher-order PtzI operations.

5. Two Definitions for the PtzI Operation

In the above sections, we find that, for the SPDC, the displacement variance definition \( \kappa_{zz}^{DV} \) is invariant for the PtzI operation, while the Fick’s law definition \( \kappa_{zz}^{PL} \) is not. In what follows, we explore whether the Fick’s law definition \( \kappa_{zz}^{PL} \) and the displacement variance definition \( \kappa_{zz}^{DV} \) are invariant for the PtzI operation.

First, we multiply Equation (52) by the differential operator \( \partial^{n+m+1}/(\partial t^n \partial z^m) \) with \( n, m = 1, 2, 3, \ldots \), and find

\[
\frac{\partial^{n+m+1}F}{\partial t^{n} \partial z^{m}} = -\kappa_{zz} \frac{\partial^{n+2}F}{\partial t^{n+1} \partial z^{m+1}} + \kappa_{zzz} \frac{\partial^{n+3}F}{\partial t^{n+2} \partial z^{m+2}} + \kappa_{zzzz} \frac{\partial^{n+4}F}{\partial t^{n+3} \partial z^{m+3}} + \cdots \]

which is the equation of the \( n \)th-order temporal and \( m \)th-order spatial PtzI operation. For the lowest-order case, i.e., \( n = 1 \) and \( m = 1 \), the latter equation becomes

\[
\frac{\partial^3 F}{\partial t^2 \partial z} = -\kappa_{zz} \frac{\partial^3 F}{\partial t^2 \partial z^2} + \kappa_{zzz} \frac{\partial^3 F}{\partial t^3 \partial z^2} + \kappa_{zzzz} \frac{\partial^3 F}{\partial t^4 \partial z^2} + \cdots \]

which is the equation of the first-order temporal and first-order spatial PtzI operation.

Combining Equations (52) and (101), by employing the DIOs we obtain a lot of new EIDFs. As shown in Section 4.2.2, the contributions of Equation (101) to the formulas of the first- and second-order moments of the displacement are all equal to zero, and the displacement variance definitions \( \kappa_{zz}^{DV} \) are invariant for the lowest-order PtzI operation. At the same time, because there is no second-order spatial derivative term in Equation (101), the Fick’s law definition \( \kappa_{zz}^{PL} \) is not changed by any DIO, i.e., any replacement manipulation of the lowest-order PtzI operation. Similarly, the same results can be obtained for the manipulations of the higher-order PtzI operations. Therefore, the displacement variance definition \( \kappa_{zz}^{DV} \) and the Fick’s law definition \( \kappa_{zz}^{PL} \) are all invariant quantities for the DIOs of the PtzI operations.

6. Two Definitions for the Ptl Operations

6.1. The Ptl Operations

By multiplying Equation (52) by the partial differential operator \( \partial^\alpha/\partial t^\alpha \) with \( n, m = 1, 2, 3, \ldots \), we obtain the following equation:

\[
\frac{\partial^{n+1}F}{\partial t^{n+1}} = -\kappa_{zz} \frac{\partial^{n+2}F}{\partial t^{n+1} \partial z^{m+1}} + \kappa_{zzz} \frac{\partial^{n+3}F}{\partial t^{n+2} \partial z^{m+2}} + \kappa_{zzzz} \frac{\partial^{n+4}F}{\partial t^{n+3} \partial z^{m+3}} + \cdots \]

\[
+ \left( \kappa_{zz} \frac{\partial^{n+2}F}{\partial t^{n+1} \partial z^{m+1}} + \kappa_{zzz} \frac{\partial^{n+3}F}{\partial t^{n+2} \partial z^{m+2}} + \kappa_{zzzz} \frac{\partial^{n+4}F}{\partial t^{n+3} \partial z^{m+3}} + \cdots \right) \]

For \( n = 1 \), Equation (102) becomes

\[
\frac{\partial^2 F}{\partial t^2} = -\kappa_{zz} \frac{\partial^2 F}{\partial t^2 \partial z^2} + \kappa_{zzz} \frac{\partial^3 F}{\partial t^3 \partial z^2} + \kappa_{zzzz} \frac{\partial^4 F}{\partial t^4 \partial z^2} + \cdots \]

\[
+ \left( \kappa_{zz} \frac{\partial^2 F}{\partial t^2 \partial z^2} + \kappa_{zzz} \frac{\partial^3 F}{\partial t^3 \partial z^2} + \kappa_{zzzz} \frac{\partial^4 F}{\partial t^4 \partial z^2} + \cdots \right) \]

which is the governing equation of the first-order Ptl operation. After drawing one term from the latter equation, by putting it on the left-hand side of the equal sign and the other ones on the right-hand side, we obtain a new deformation equation. Obviously, we obtain a lot of deformation equations from Equation (103), one of which, replacing the corresponding term in Equation (52), as done
in the previous subsections, yields a new EIDF. In such way, a lot of new EIDFs can be obtained. All of the latter replacement manipulations, one of which is a DIO, are called the first-order PtI operation.

6.2.1. $R_{tt}$ of the First-order PtI Operation

Equation (103) can be rewritten as

$$
\frac{\partial^2 F}{\partial t \partial z} = \frac{1}{\kappa_z} \left[ \frac{\partial^2 F}{\partial t^2} + \left( \kappa_{zz} \frac{\partial^3 F}{\partial t \partial z^2} + \kappa_{zzz} \frac{\partial^4 F}{\partial t \partial z^4} + \kappa_{zzzz} \frac{\partial^5 F}{\partial t \partial z^6} + \cdots \right) + \left( \kappa_{tz} \frac{\partial^3 F}{\partial t^2 \partial z} + \kappa_{tzz} \frac{\partial^4 F}{\partial t^2 \partial z^3} + \kappa_{tzzz} \frac{\partial^5 F}{\partial t^2 \partial z^4} + \cdots \right) + \left( \kappa_{tz} \frac{\partial^3 F}{\partial t^2 \partial z} + \kappa_{tzz} \frac{\partial^4 F}{\partial t^2 \partial z^3} + \kappa_{tzzz} \frac{\partial^5 F}{\partial t^2 \partial z^4} + \cdots \right) \right].
$$

(104)

Multiplying the latter equation by the parameter $\kappa_{tz}$, we easily obtain

$$
\kappa_{tz} \frac{\partial^2 F}{\partial t \partial z} = -\kappa_z \frac{\partial F}{\partial z} + \left( \kappa_{zz} \frac{\partial^3 F}{\partial t \partial z^2} + \frac{\partial^3 F}{\partial t^2 \partial z} + \frac{\partial^4 F}{\partial t \partial z^3} + \frac{\partial^4 F}{\partial t^2 \partial z^2} + \frac{\partial^5 F}{\partial t \partial z^4} + \frac{\partial^5 F}{\partial t^2 \partial z^3} + \cdots \right) + \left( \kappa_{tzz} \frac{\partial^4 F}{\partial t \partial z^4} + \kappa_{tzzz} \frac{\partial^5 F}{\partial t \partial z^5} + \cdots \right)
$$

(105)

Replacing $\kappa_{tz} \frac{\partial^2 F}{(\partial t \partial z)}$ in Equation (52) by Equation (105) gives

$$
\frac{\partial F}{\partial t} = \left( -\kappa_z \frac{\partial F}{\partial z} + \kappa_{zz} \frac{\partial^3 F}{\partial z^3} + \kappa_{tzz} \frac{\partial^4 F}{\partial t \partial z^3} + \cdots \right) - \kappa_z \frac{\partial^2 F}{\partial z^2}
$$

$$
+ \left( \kappa_{tzz} \frac{\partial^3 F}{\partial t \partial z^2} + \kappa_{zzz} \frac{\partial^4 F}{\partial t \partial z^4} + \kappa_{tzzz} \frac{\partial^5 F}{\partial t \partial z^5} + \cdots \right)
$$

$$
+ \left( \kappa_{tzz} \frac{\partial^4 F}{\partial t \partial z^3} + \kappa_{tzzz} \frac{\partial^5 F}{\partial t \partial z^4} + \cdots \right)
$$

$$
+ \left( \kappa_{tzz} \frac{\partial^5 F}{\partial t \partial z^4} + \cdots \right)
$$

$$
+ \left( \kappa_{tzz} \frac{\partial^6 F}{\partial t \partial z^5} + \cdots \right) + \cdots
$$

(106)

The above replacement manipulation, which is one of the DIOs, is called the $R_{tt}$ of the first-order PtI operation in this paper.

From Equation (106), we find that the Fick’s law definition $\kappa_{tt}^{FL}$ is equal to $\kappa_{zz}$, $\kappa_{tzz}^{FL} = \kappa_{zz}$, which is identical to Equation (53). Thus, the DIOs, i.e., the $R_{tt}$ of the first-order PtI operation, cannot change the Fick’s law definition $\kappa_{tt}^{FL}$. In the following, we explore the invariance of the displacement variance definition $\kappa_{tt}^{DV}$ for the $R_{tt}$ of the first-order PtI operation.

Using the method given in Section 4.1.1, from Equation (106) we find

$$
\frac{d}{dt} \langle (\Delta z)^2 \rangle = \kappa_z \frac{\partial^2}{\partial t^2} \langle (\Delta z)^2 \rangle - \frac{\kappa_{tt}}{\kappa_z} \frac{\partial^2}{\partial t^2} \langle (\Delta z)^2 \rangle ,
$$

(107)

$$
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2 \kappa_z \langle (\Delta z)^2 \rangle + 2 \kappa_{zz} \frac{\partial^2}{\partial t^2} \langle (\Delta z)^2 \rangle - 2 T_{tt}.
$$

(108)

Here the term $T_{tt}$ is as follows:

$$
T_{tt} = \left( \kappa_{tt} \frac{\partial^2}{\partial t^2} \langle (\Delta z)^2 \rangle + \kappa_{tt} \frac{\partial^3}{\partial t^3} \langle (\Delta z)^3 \rangle + \kappa_{tt} \frac{\partial^4}{\partial t^4} \langle (\Delta z)^4 \rangle + \cdots \right)
$$

$$
+ \sum_{n=1}^{\infty} \left( \kappa_{tt} \frac{\partial^2}{\partial t^2} \langle (\Delta z)^2 \rangle + \kappa_{tt} \frac{\partial^3}{\partial t^3} \langle (\Delta z)^3 \rangle + \cdots \right) / \kappa_{tt} \frac{\partial^2}{\partial t^2} \langle (\Delta z)^2 \rangle
$$

(109)
Combining Equations (107) and (108) we obtain,
\[
\frac{1}{2} \frac{d\sigma^2}{dt} = \kappa_{ic} + \kappa_{zc} \left[ \left( \langle \Delta z \rangle \right) \frac{d^2}{dt^2} \langle \Delta z \rangle - \frac{1}{2} \frac{d^2}{dt^2} \langle (\Delta z)^2 \rangle \right] - T_{ic},
\] (110)

In order to simplify the latter equation, we need to investigate the term \( T_{ic} \). Applying the operator \( \partial^n/\partial t^n \) with \( n = 1, 2, 3, \ldots \) on Equation (107) gives
\[
\frac{d^{n+1}}{dt^{n+1}} \langle \Delta z \rangle = - \frac{\kappa_{zc}}{\kappa_{zc}} \frac{d^{n+2}}{dt^{n+2}} \langle \Delta z \rangle.
\] (111)

Employing the above equation, we find that Equation (109) can be simplified as
\[
T_{ic} = \lim_{n \to \infty} \left[ \left( \frac{\kappa_{zc}}{\kappa_{zc}} \right)^n \kappa_{zc} + \frac{1}{2D} \kappa_{zc} \frac{d^{n+1}}{dt^{n+1}} \langle \Delta z \rangle \right].
\] (112)

Here, the subscript \((n + 1)\) in the latter formula denotes there are \((n + 1)\) letters \( t \). That is, \( \kappa_{2zc} = \kappa_{zc}, \kappa_{3zc} = \kappa_{czc}, \kappa_{4zc} = \kappa_{3zc}, \) and so on. Using Equation (D20) in Appendix D, we can rewrite the latter equation as
\[
T_{ic} \approx \lim_{n \to \infty} \left[ \left( \frac{\kappa_{zc}}{\kappa_{zc}} \right)^n \kappa_{zc} + \frac{1}{2D} \kappa_{zc} \frac{d^{n+1}}{dt^{n+1}} \langle \Delta z \rangle \right].
\] (113)

Inserting \( \kappa_{zc}, \kappa_{zc} \), and \( \kappa_{2zc} \) (see Equation (B17), (B18), and (D7) in Appendix, respectively) into the above equation yields
\[
\frac{1}{v} T_{ic} \approx \lim_{n \to \infty} \left[ \left( \frac{2}{3} \right) \frac{2}{9} \xi + \left( \frac{1}{2} \right) \frac{9}{4} \xi^2 \right] \frac{d^{n+1}}{d(\partial v)^{n+1}} \langle \Delta z \rangle.
\] (114)

In order to evaluate the latter equation, we have to compute \( \frac{d^{n+1}}{d(\partial v)^{n+1}} \langle \Delta z \rangle \) in which the quantity \( \langle \Delta z \rangle \) is the solutions of Equations (107) as
\[
\langle \Delta z \rangle = c_1 + c_2 e^{-\kappa_{zc}/\kappa_{zc}} + c_2 t.
\] (115)

Here, \( c_1 \) and \( c_2 \) are the undetermined coefficients. From Equation (115) we obtain
\[
\frac{d^{n+1}}{dt^{n+1}} \langle \Delta z \rangle = c_2 (-1)^{n+1} \left( \frac{\kappa_{zc}}{\kappa_{zc}} \right)^{n+1} e^{-\kappa_{zc}/\kappa_{zc}}.
\] (116)

Using Equations (B17) and (B18) in Appendix B gives
\[
\frac{d^{n+1}}{dt^{n+1}} \langle \Delta z \rangle \approx c_2 (-1)^{n+1} \left( \frac{2}{3} \right)^{n+1} D e^{-3vD/2}.
\] (117)

Considering the latter equation and the limit \( t \to t_{ic} \), we find that Equation (114) becomes
\[
\frac{1}{v} \lim_{t \to t_{ic}} T_{ic} \approx \lim_{n \to \infty} \lim_{t \to t_{ic}} \left[ \left( \frac{2}{3} \right)^{n+1} \left( \frac{3}{4} \right)^{n-1} \left( -\frac{13}{108} \right) \right] (-1)^{n+2} c_2 e^{-3Dv/2} = 0,
\] (118)

where \( \xi \ll 1, \) and \( v, D \) are constants. The above equation can be rewritten as
\[
\lim_{t \to t_{ic}} T_{ic} = 0.
\] (119)

For the limit \( t \to t_{ic} \), Equation (110) becomes
\[
\frac{1}{2} \lim_{t \to t_{ic}} \frac{d\sigma^2}{dt} = \kappa_{zc} + \frac{\kappa_{zc}}{\kappa_{zc}} \left[ \left( \langle \Delta z \rangle \right) \frac{d^2}{dt^2} \langle \Delta z \rangle - \frac{1}{2} \frac{d^2}{dt^2} \langle (\Delta z)^2 \rangle \right].
\] (120)

where Equation (119) is used. Equations (108), (109), and (111) yield
\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2 \kappa_{zc} \langle \Delta z \rangle + 2 \kappa_{zc} - \frac{\kappa_{zc}}{\kappa_{zc}} \frac{d^2}{dt^2} \langle (\Delta z)^2 \rangle - T_{ic}
\] (121)

with
\[
T_{ic} = \lim_{n \to \infty} (-1)^{n-1} c_2 \frac{\kappa_{zc}}{\kappa_{zc}} e^{-\kappa_{zc}/\kappa_{zc}}.
\] (122)
The solution of Equation (121) can be obtained as
\[
\langle (\Delta z)^2 \rangle = c_1' + c_2' e^{-\kappa_z t/\kappa_z} + 2 (\kappa_{zz} + \kappa_z c_1) t + \kappa_z^2 t^2
- 2 \kappa_z \kappa_{zz} t - 2 \kappa_z c_2 t e^{-\kappa_z t/\kappa_z} + 2 c_2 \frac{\kappa_z}{\kappa_{zz}} \lim_{n \to \infty} (-1)^{n-1} t e^{-\kappa_z t/\kappa_z}
\] (123)
with the undetermined constants \(c_1'\) and \(c_2'.\) Inserting Equations (115) and (123) into Equation (120), we obtain
\[
\frac{1}{2} \frac{d\sigma^2}{dt} = \kappa_{zz} + (c_1 + c_2 e^{-\kappa_z t/\kappa_z} + \kappa_z t) \frac{\kappa_z}{\kappa_{zz}} c_2 e^{-\kappa_z t/\kappa_z} - \frac{1}{2} \left( \frac{\kappa_z}{\kappa_{zz}} c_2' e^{-\kappa_z t/\kappa_z} + 2 \kappa_z c_2 e^{-\kappa_z t/\kappa_z} - \frac{\kappa_z^2}{\kappa_{zz}} c_2 t e^{-\kappa_z t/\kappa_z} \right)
+ 2 c_2 \lim_{n \to \infty} (-1)^{n-1} \left[ \frac{\kappa_z}{\kappa_{zz}} c_2 - \left( \frac{\kappa_z}{\kappa_{zz}} \right)^2 t \right] e^{-\kappa_z t/\kappa_z}.
\] (124)
Employing Equations (B17) and (B18) in Appendix B gives
\[
\kappa_z / \kappa_{zz} = 3D/2 > 0.
\] (125)
Using the latter equation yields
\[
\lim_{t \to t_\infty} e^{-\kappa_z t/\kappa_z} \approx 0.
\] (126)
Inserting Equation (126) into Equation (124), for the limit \(t \to t_\infty\), we find
\[
\kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to t_\infty} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_z \kappa_{zz},
\] (127)
which is identical to Equation (54). Thus, we find that the displacement variance definition \(\kappa_{zz}^{DV}\) is an invariant quantity for the \(R_z\) of the first-order PtI operation.

6.2.2. Displacement Variance Definition \(\kappa_{zz}^{DV}\) for the \(R_z\) of the First-order PtI Operation under the Special Condition

The derivation process in the latter subsection is very lengthy and complicated. In fact, employing some special condition we give the same result through a simpler derivation.

From Equation (126), we find that the terms containing the exponent function \(e^{-\kappa_z t/\kappa_z}\) in Equations (115) and (123) tend to zero for the limit \(t \to t_\infty\). However, if using the special condition
\[
c_2 = c_2' = 0,
\] (128)
we also find that the same terms are all equal to zero. In what follows, we provide the derivation process of the displacement variance definition for the \(R_z\) of the first-order PtI operation under the special condition.

For the special condition Equation (128), Equation (115) becomes
\[
\langle (\Delta z) \rangle = c_1 + \kappa_z t,
\] (129)
employing which we find
\[
\lim_{n \to \infty} T_z = 0.
\] (130)
Thus, Equation (108) becomes
\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2 \kappa_z \langle (\Delta z) \rangle + 2 \kappa_{zz} - \frac{\kappa_{zz} \kappa_z d^2}{\kappa_z} \langle (\Delta z)^2 \rangle.
\] (131)
Similarly, for the special condition Equation (128), Equation (123) becomes
\[
\langle (\Delta z)^2 \rangle = c_1' + \kappa_z^2 t^2 + 2 (\kappa_z c_1 + \kappa_{zz} - \kappa_z \kappa_{zz}) t.
\] (132)
Thereafter, inserting Equations (129), (130), and (132) into Equation (110) yields, for the limit \(t \to t_\infty\),
\[
\kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to t_\infty} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_z \kappa_{zz}.
\] (133)
The latter equation is identical to Equation (54). Although the special condition used in this subsection is not necessary, it can simplify the derivation process significantly.
6.2.3. \( R_{\text{zz}} \) of the First-order PtI Operation

In this subsection, we introduce another DIO, i.e., the \( R_{\text{zz}} \) of the first-order PtI operation. Equation (103) can be rewritten as

\[
\frac{\partial^3 F}{\partial t \partial z^2} = \frac{\partial^3 F}{\partial t^3} - \left( -\kappa_z \frac{\partial^2 F}{\partial t \partial z} + \kappa_{zz} \frac{\partial^2 F}{\partial t \partial z^2} + \frac{\partial^3 F}{\partial t^2 \partial z^3} + \cdots \right)
\]

Inserting the latter equation into Equation (52) yields

\[
\frac{\partial F}{\partial t} = \left( -\kappa_z \frac{\partial F}{\partial z} + \kappa_{zz} \frac{\partial^2 F}{\partial z^2} + \kappa_{zzz} \frac{\partial^3 F}{\partial z^3} + \cdots \right) + \kappa_{zzz} \frac{\partial^2 F}{\partial t \partial z^2} + \left( \kappa_{zz} - \kappa_{zzz} \right) \frac{\partial^3 F}{\partial t \partial z^3} + \cdots.
\]

From Equation (135), we find that the Fick’s law definition \( \kappa_{\text{FL}}^{\text{zz}} \) is equal to \( \kappa_{zz} \), which is identical to Equation (53). Thus, \( \kappa_{\text{FL}}^{\text{zz}} \) is invariant for the \( R_{\text{zz}} \) of the first-order PtI operation. In order to explore the displacement variance definition \( \kappa_{\text{DV}}^{\text{zz}} \) for the \( R_{\text{zz}} \) of the first-order PtI operation, the following formulas have to be derived from Equation (135):

\[
\frac{d}{dt} \langle (\Delta z) \rangle = \kappa_z + \frac{\kappa_{zzz}}{\kappa_{zz}} d^2 \langle (\Delta z) \rangle dt^2,
\]

\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_z \langle (\Delta z) \rangle + 2\kappa_{zz} \frac{d^2}{dt^2} \langle (\Delta z)^2 \rangle - 2\kappa_{zzz} \frac{d^3}{dt^3} \langle (\Delta z) \rangle - 2T_{\text{zz}}
\]

with

\[
T_{\text{zz}} = \left( \kappa_{zz} - \kappa_{zzz} \right) \frac{d^2}{dt^2} \langle (\Delta z) \rangle + \left( \kappa_{zz} - \kappa_{zzz} \right) \frac{d^3}{dt^3} \langle (\Delta z) \rangle + \cdots.
\]

Combining Equations (136) and (137) gives

\[
\frac{1}{2} \frac{d\sigma^2}{dt} = \kappa_{zz} + \frac{1}{2} \kappa_{zzz} \frac{d^2}{dt^2} \langle (\Delta z)^2 \rangle - 2\langle (\Delta z) \rangle \frac{d^2}{dt^2} \langle (\Delta z) \rangle - 2T_{\text{zz}}.
\]

For the limit \( t \to t_\infty \), the latter equation can be written as

\[
\kappa_{\text{DV}}^{\text{zz}} = \frac{1}{2} \lim_{t \to t_\infty} \frac{d\sigma^2}{dt} = \lim_{t \to t_\infty} \left[ \kappa_z + \frac{1}{2} \kappa_{zzz} \frac{d^2}{dt^2} \langle (\Delta z)^2 \rangle - 2\langle (\Delta z) \rangle \frac{d^2}{dt^2} \langle (\Delta z) \rangle - \lim_{t \to t_\infty} T_{\text{zz}} \right].
\]

In what follows, we first deal with Equation (138), which is the function of the first-order moment \( \langle (\Delta z) \rangle \). From Equation (136), we obtain

\[
\langle (\Delta z) \rangle = c_1 + c_2 e^{\kappa_z t / \kappa_{zz}} + \kappa_z t,
\]
where $c_1$ and $c_2$ are undetermined constants. To proceed, taking the $n$th-order derivative of Equation (136) over time $t$ gives
\[
\frac{d^n}{dt^n} \langle \Delta z \rangle = \frac{\kappa_{zz}}{\kappa_{zz}} \frac{d^{n+1}}{dt^{n+1}} \langle \Delta z \rangle,
\] (142)
and the latter equation can be rewritten as
\[
\frac{d^{n+1}}{dt^{n+1}} \langle \Delta z \rangle = \frac{\kappa_{zz}}{\kappa_{zz}} \frac{d^n}{dt^n} \langle \Delta z \rangle.
\] (143)
Combining the above equation and Equation (138), we find
\[
T_{zz} = \lim_{n \to \infty} \left[ -\kappa_{zz} \frac{\kappa_{zz}}{\kappa_{zz}} + \kappa_{nt} \left( \frac{\kappa_{zz}}{\kappa_{zz}} \right)^n \right] \frac{d^2}{dt^2} \langle \Delta z \rangle.
\] (144)
Inserting Equation (141) into the latter equation yields
\[
T_{zz} = c_2 \lim_{n \to \infty} \left[ -\kappa_{zz} \frac{\kappa_{zz}}{\kappa_{zz}} + \kappa_{nt} \left( \frac{\kappa_{zz}}{\kappa_{zz}} \right)^n \right] \frac{d^2}{dt^2} \langle \Delta z \rangle.
\] (145)
The value of $T_{zz}$ in Equation (145) is determined by the constant $c_2$, the exponent function $e^{\kappa_{zz} t}$, $\kappa_{nt} (\kappa_{zz}/\kappa_{zz})^n$, and $\kappa_{nt} (\kappa_{zz}/\kappa_{zz})^n$. Because $\kappa_{zz} < 0$ as shown in Appendix C, we find $\lim_{n \to \infty} e^{\kappa_{zz} t} = 0$. However, the value of $\lim_{n \to \infty} (\kappa_{zz}/\kappa_{zz})^n$ is determined by $\kappa_{zz}$, the form of which is too complicated and hard to evaluate. For mathematical tractability, for Equation (141) we directly employ the special condition $c_2 = 0$ and obtain
\[
\langle \Delta z \rangle = c_1 + \kappa_z t.
\] (146)
Accordingly, we find
\[
\frac{d^n}{dt^n} \langle \Delta z \rangle = 0 \quad n = 2, 3, 4, \ldots,
\] (147)
so Equation (145) becomes
\[
T_{zz} = 0.
\] (148)
Inserting the latter equation into Equation (137) gives
\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_z \langle \Delta z \rangle + 2\kappa_{zz} \frac{d^2}{dt^2} \langle \Delta z \rangle^2 - 2\left( \kappa_{zz} + \frac{\kappa_{zz}}{\kappa_{zz}} \frac{d}{dt} \langle \Delta z \rangle \right).
\] (149)
Combining Equations (139) and (148), we find
\[
\frac{1}{2} \frac{d^2}{dt^2} \langle \Delta z \rangle = \kappa_{zz} \frac{d^2}{dt^2} \langle \Delta z \rangle^2 - 2\left( \kappa_{zz} \frac{d}{dt} \langle \Delta z \rangle \right).
\] (150)
Because of Equation (146), we obtain the following formula from Equation (149)
\[
\langle (\Delta z)^2 \rangle = c'_1 + c'_2 e^{\kappa_{zz}/\kappa_{zz}} + \left( 2\kappa_z \kappa_{zz} - 2\kappa_z c_1 - 2\kappa_{zz} + 2\kappa_z \frac{\kappa_{zz}}{\kappa_{zz}} \right) t + 2\kappa_z \frac{\kappa_{zz}}{\kappa_{zz}} t^2 + 2\kappa_z^2 t^2,
\] (151)
where $c'_1$ and $c'_2$ are all undetermined constants. For the special condition $c'_2 = 0$, the latter equation becomes
\[
\langle (\Delta z)^2 \rangle = c'_1 + \left( 2\kappa_z \kappa_{zz} - 2\kappa_z c_1 - 2\kappa_{zz} + 2\kappa_z^2 \frac{\kappa_{zz}}{\kappa_{zz}} \right) t + 2\kappa_z \frac{\kappa_{zz}}{\kappa_{zz}} t^2 + 2\kappa_z^2 t^2.
\] (152)
Inserting Equations (146) and (152) into Equation (150), we easily obtain
\[
\kappa_{zz} \frac{d^2}{dt^2} \langle \Delta z \rangle = \frac{1}{2} \lim_{t \to t_\infty} \frac{d^2}{dt^2} \langle \Delta z \rangle = \kappa_{zz} - \kappa_z \kappa_z,
\] (153)
which is identical to Equation (54). From the above deduction, we find that at least for the special condition \( c_2 = c'_2 = 0 \), the displacement variance definition \( \sigma_D^2 \) is invariant for the \( R_{ntmz} \) of the first-order PtI operation. Of course, it is possible that the special condition is unnecessary.

6.2.4. \( R_{ntmz} \) of the First-Order PtI Operation

To combine Equation (52) and the deformation of Equation (103), one can produce a lot of the EIDFs by the DIOs for the first-order PtI operation, i.e., \( R_{ntmz} \) with \( n = 2, 3, 4, \ldots, m = 1, 2, 3, \ldots \). Because Equation (103) does not contain the second-order spatial derivative terms, any replacement manipulation, i.e., the DIO, of the first-order PtI operation, cannot influence the Fick’s law definition. In the following, we explore the displacement variance definition for \( R_{ntmz} \) of the first-order PtI operation.

The equation of \( \langle (\Delta z)^2 \rangle \) can be derived as follows

\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = \kappa_z + \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} \frac{d^2}{dt^2} \langle (\Delta z) \rangle. \tag{154}
\]

The solution of the latter equation is found to be

\[
\langle (\Delta z)^2 \rangle = c_1 + c_2 e^{\kappa_{(n-1)tmz}/\kappa_{ntmz}} + \kappa_z t. \tag{155}
\]

Similarly, from the EIDF for the \( R_{ntmz} \) of the first-order PtI operation, the equation of \( d \langle (\Delta z)^2 \rangle / dt \) can be found:

\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_z \langle (\Delta z) \rangle + 2\kappa_{zz} + \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} \frac{d^2}{dt^2} \langle (\Delta z) \rangle
- 2 \left( \kappa_{zz} + \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} \kappa_z \right) \frac{d}{dt} \langle (\Delta z) \rangle - 2T_{ntmz} \tag{156}
\]

with

\[
T_{ntmz} = \left( \kappa_{zz} - \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} \kappa_z \right) \frac{d^2}{dt^2} \langle (\Delta z) \rangle
+ \left( \kappa_{zz} - \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} \kappa_z \right) \frac{d^3}{dt^3} \langle (\Delta z) \rangle + \cdots. \tag{157}
\]

From Equation (154), we obtain

\[
\frac{d^n}{dt^n} \langle (\Delta z)^2 \rangle = \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} \frac{d^{n+1}}{dt^{n+1}} \langle (\Delta z) \rangle, \tag{158}
\]

using which we find that Equation (157) becomes

\[
T_{ntmz} = c_2 \lim_{n \to \infty} \left[ - \kappa_z \frac{\kappa_{(n-1)tmz}}{\kappa_{ntmz}} + \kappa_{ntmz} \left( \frac{\kappa_{(n-1)tmz}}{\kappa_{ntmz}} \right)^n \right]
\times e^{\kappa_{(n-1)tmz}/\kappa_{ntmz}}. \tag{159}
\]

Because the forms of the parameters \( \kappa_{(n-1)tmz} \) and \( \kappa_{ntmz} \) are very complicated and hard to evaluate, for the simplification we only consider the case satisfying the special condition \( c_2 = 0 \) in this subsection. Thus, Equation (155) can be simplified as

\[
\langle (\Delta z)^2 \rangle = c_1 + \kappa_z t, \tag{160}
\]

consequently, Equation (159) becomes

\[
T_{ntmz} = 0. \tag{161}
\]

Inserting the latter equation into Equation (156) gives

\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_z \langle (\Delta z) \rangle + 2\kappa_{zz} + \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} \frac{d^2}{dt^2} \langle (\Delta z) \rangle
- 2 \left( \kappa_{zz} + \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} \kappa_z \right) \frac{d}{dt} \langle (\Delta z) \rangle, \tag{162}
\]

the solution of which can be obtained by employing Equation (155):

\[
\langle (\Delta z)^2 \rangle = c'_1 + c'_2 e^{\kappa_{(n-1)tmz}/\kappa_{ntmz}} + \left( 2\kappa_z \kappa_z - 2\kappa_z c_1 - 2\kappa_{zz}
+ 2\kappa_z^2 \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} \right) t + 2\kappa_z^2 \frac{\kappa_{ntmz}}{\kappa_{(n-1)tmz}} + 2\kappa_z^2 t^2. \tag{163}
\]
For the special condition \( c_1' = 0 \), the latter equation becomes
\[
\langle (\Delta z)^2 \rangle = c_1' + \left( 2\kappa_z \kappa_{tz} - 2\kappa_z c_1 - 2\kappa_z^2 + 2\kappa_z^2 \frac{\kappa_{ntmz}}{\kappa_{(n-1)ntmz}} \right) t \\
+ 2\kappa_z^2 \frac{\kappa_{ntmz}}{\kappa_{(n-1)ntmz}} + 2\kappa_z^2 t^2.
\] (164)

Considering Equations (154) and (162), we find
\[
\frac{1}{2} \frac{d\sigma^2}{dt} = \kappa_{zz} + \frac{1}{2} \frac{\kappa_{ntmz}}{\kappa_{(n-1)ntmz}} \left[ \frac{d^2}{dt^2} \langle (\Delta z)^2 \rangle - \frac{d^2}{dt^2} \langle (\Delta z) \rangle \right] - \left( \kappa_{zz} + \frac{\kappa_{ntmz}}{\kappa_{(n-1)ntmz}} \right) \frac{d}{dt} \langle (\Delta z) \rangle.
\] (165)

Inserting Equations (160) and (164) into the latter equation yields
\[
\kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to \infty} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_{zz} \kappa_z, 
\] (166)

which is identical to Equation (54).

From the investigation in this subsection, we find that for the \( R_{ntmz} \) of the first-order PtI operation, the Fick’s law definition \( \kappa_{zz}^{FL} \) is invariant, and the displacement variance definition \( \kappa_{zz}^{DV} \) is also invariant at least for the special condition.

### 6.3. The Second-order PtI Operation

For \( n = 2 \), i.e., the second-order PtI operation, Equation (102) becomes
\[
\frac{\partial^2 F}{\partial t^2} = \left( -\kappa_z \frac{\partial^2 F}{\partial t^2} + \kappa_{zzz} \frac{\partial^6 F}{\partial t^6} + \kappa_{zz} \frac{\partial^2 F}{\partial t^2} + \kappa_{zzz} \frac{\partial^4 F}{\partial t^4} + \kappa_{zz} \frac{\partial^2 F}{\partial t^2} + \kappa_{ntmz} \frac{\partial^4 F}{\partial t^4} + \cdots \right)
\] (167)

As shown in Section 6.2, rewriting the latter equation and inserting the results into Equation (52), we obtain the new EIDFs.

#### 6.3.1. \( R_{ntmz} \) of the Second-order PtI Operation

In this section, by pulling out the term \( \frac{\partial^3 F}{\partial t^2 \partial z} \) on the right-hand side of Equation (167), we can rewrite this equation and obtain
\[
\frac{\partial^3 F}{\partial t^2 \partial z} = \frac{1}{\kappa_z} \frac{\partial^3 F}{\partial t^3} + \left( \frac{1}{\kappa_z} \frac{\partial^3 F}{\partial t^2 \partial z} + \frac{1}{\kappa_z} \frac{\partial^5 F}{\partial t^5} + \frac{1}{\kappa_z} \frac{\partial^3 F}{\partial t^2 \partial z} + \frac{1}{\kappa_z} \frac{\partial^5 F}{\partial t^5} + \cdots \right)
\] (168)

After multiplying the latter equation by \( \kappa_{ntmz} \), we find
\[
\kappa_{ntmz} \frac{\partial^3 F}{\partial t^2 \partial z} = \frac{\kappa_{ntmz}}{\kappa_z} \frac{\partial^3 F}{\partial t^3} + \left( \frac{\kappa_{ntmz}}{\kappa_z} \frac{\partial^3 F}{\partial t^2 \partial z} + \frac{\kappa_{ntmz}}{\kappa_z} \frac{\partial^5 F}{\partial t^5} + \frac{\kappa_{ntmz}}{\kappa_z} \frac{\partial^3 F}{\partial t^2 \partial z} + \frac{\kappa_{ntmz}}{\kappa_z} \frac{\partial^5 F}{\partial t^5} + \cdots \right)
\] (169)
Replacing the term \( \partial^3 F / (\partial t^2 \partial z) \) in Equation (52) by the terms on the right-hand side of Equation (52) gives

\[
\frac{\partial F}{\partial t} = \left( -\kappa_{zz} \frac{\partial^2 F}{\partial z^2} + \kappa_{zzz} \frac{\partial^3 F}{\partial z^3} + \kappa_{zzzz} \frac{\partial^4 F}{\partial z^4} + \cdots \right) - \kappa_{zz} \frac{\partial^3 F}{\partial t^3} \\
+ \kappa_{zz} \frac{\partial^2 F}{\partial t \partial z} + \kappa_{zzz} \frac{\partial^3 F}{\partial t \partial z^2} + \kappa_{zzzz} \frac{\partial^4 F}{\partial t \partial z^3} + \kappa_{zzzzz} \frac{\partial^5 F}{\partial t \partial z^4} + \kappa_{zzzzzz} \frac{\partial^6 F}{\partial t \partial z^5} + \cdots \\
+ \left( \frac{\kappa_{zz}}{\kappa_{zt}} + \kappa_{zz} \right) \frac{\partial^4 F}{\partial t^2 \partial z^2} + \left( \frac{\kappa_{zz}}{\kappa_{zt}} \kappa_{zz} + \kappa_{zzz} \right) \frac{\partial^5 F}{\partial t^2 \partial z^3} + \cdots \\
+ \left( \frac{\kappa_{zz}}{\kappa_{zt}} \kappa_{zz} + \kappa_{zz} \right) \frac{\partial^5 F}{\partial t^3 \partial z^2} + \left( \frac{\kappa_{zz}}{\kappa_{zt}} \kappa_{zz} + \kappa_{zzz} \right) \frac{\partial^6 F}{\partial t^3 \partial z^3} + \cdots \\
+ \left( \frac{\kappa_{zz}}{\kappa_{zt}} \kappa_{zz} + \kappa_{zz} \right) \frac{\partial^6 F}{\partial t^4 \partial z^2} + \cdots.
\]

(170)

The manipulation in this part, which is one of the DIOs, is called the the \( \tilde{R}_{zt} \) of the second-order PtI operation.

As shown in Equation (170), the Fick’s law definition \( \kappa_{zt} \) is equal to \( \kappa_{zt} = \kappa_{zt} \), which is identical to Equation (53). Thus, the Fick’s law definition is not changed by the \( \tilde{R}_{zt} \) of the second-order PtI operation. In the following, we explore the displacement variance definition \( \kappa_{zt} \) for the new manipulation.

From Equation (170), we derive the following formulas:

\[
\frac{d}{dt} \langle (\Delta z) \rangle = \kappa_{zt} \frac{d^3}{dt^3} \langle (\Delta z) \rangle,
\]

(171)

\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_{zt} \langle (\Delta z) \rangle + 2\kappa_{zt} \kappa_{zt} \frac{d^3}{dt^3} \langle (\Delta z) \rangle \]

\[
- 2\kappa_{zt} \frac{d}{dt} \langle (\Delta z) \rangle - 2T_{zt},
\]

(172)

with

\[
T_{zt} = \left( \frac{\kappa_{zt}}{\kappa_{zt}} \kappa_{zt} + \kappa_{zt} \right) \frac{d^3}{dt^3} \langle (\Delta z) \rangle + \left( \frac{\kappa_{zt}}{\kappa_{zt}} \kappa_{zt} + \kappa_{zt} \right) \frac{d^4}{dt^4} \langle (\Delta z) \rangle + \cdots.
\]

(173)

Taking the \((n-1)\)th-order derivative of Equation (171) yields

\[
\frac{d^{n+2}}{dt^{n+2}} \langle (\Delta z) \rangle = - \kappa_{zt} \frac{d^n}{dt^n} \langle (\Delta z) \rangle.
\]

(174)

By inserting the latter equation into Equation (173), we obtain

\[
T_{zt} = \lim_{n \to \infty} \left( \frac{\kappa_{zt}}{\kappa_{zt}} \kappa_{zt} + \kappa_{zt} \right) \left[ \left( -\frac{\kappa_{zt}}{\kappa_{zt}} \right)^{n/2 - 2} \frac{d^3}{dt^3} \langle (\Delta z) \rangle \right]
\]

\[
+ \left( \frac{\kappa_{zt}}{\kappa_{zt}} \kappa_{zt} + \kappa_{zt} \right) \left[ \left( -\frac{\kappa_{zt}}{\kappa_{zt}} \right)^{n/2 - 2} \frac{d^4}{dt^4} \langle (\Delta z) \rangle \right],
\]

(175)

where the subscript \( ntz \) of \( \kappa_{ntz} \) denotes \( n \) letters \( t \) and one letter \( z \), e.g., \( \kappa_{ntz} \) can be rewritten as \( \kappa_{2zt} \), \( \kappa_{ntz} \) can be rewritten as \( \kappa_{3zt} \), and so on. The symbol \([n/2 - 2]\) indicates the smallest integer not smaller than \((n/2 - 2)\).

If considering the parameter \( \kappa_{ntz} \) (see Equation (D20) in Appendix D), we find

\[
\kappa_{ntz} \left( -\frac{\kappa_{zt}}{\kappa_{zt}} \right)^{n/2 - 2} \approx \left( \frac{1}{2D} \right)^{n-2} \kappa_{2zt} \left( \sqrt{-\frac{\kappa_{zt}}{\kappa_{zt}}} \right)^{n-4}
\]

\[
= \frac{1}{4} \left( \frac{1}{D} \right)^{n-2} \kappa_{2zt} \left( \frac{1}{2} \sqrt{-\frac{\kappa_{zt}}{\kappa_{zt}}} \right)^{n-4}.
\]

(176)
Combining the parameters $\kappa_{tz}$ and $\kappa_{ttz}$ (see Equations (B17) and (D7) in Appendix D), we find that Equation (176) becomes

$$\kappa_{ntz} - \frac{\kappa_{z}}{\kappa_{ntz}} \approx \frac{1}{4D^2} \left( 0.8320502943^\nu - \frac{13}{108} D^2 \xi \right)$$

(177)

Because $\lim_{n \to \infty} (0.8320502943)^n = 0$, Equation (177) becomes

$$\lim_{n \to \infty} \kappa_{ntz} \left( \frac{\kappa_{z}}{\kappa_{ntz}} \right)^{n/2-2} \approx \lim_{n \to \infty} \frac{1}{4D^2} \left( 0.8320502943^\nu - \frac{13}{108} D^2 \xi \right) = 0,$$

(178)

where $D$, $\xi$, and $\nu$ are all constants. Inserting the latter result into Equation (175) gives

$$T_{ntz} = \frac{\kappa_{ntz}}{\kappa_{z}} \frac{d^3}{dt^3} ((\Delta z)) + \frac{\kappa_{ntz}}{\kappa_{z}} \frac{d^4}{dt^4} ((\Delta z)).$$

(179)

Actually, Equation (179) can be simplified again. From Equation (174), we find

$$\frac{d^3}{dt^3} ((\Delta z)) = \kappa_{z} \frac{\kappa_{z}}{\kappa_{ntz}} - \frac{\kappa_{z} d}{\kappa_{ntz}} ((\Delta z)),
$$

(180)

$$\frac{d^4}{dt^4} ((\Delta z)) = -\frac{\kappa_{z} d^2}{\kappa_{ntz}} ((\Delta z)),$$

(181)

inserting which into Equation (179), one obtains

$$T_{ntz} = \kappa_{ntz} \kappa_{z} - \kappa_{ntz} \frac{d^2}{dt^2} ((\Delta z)) - \kappa_{z} \frac{d}{dt} ((\Delta z)).$$

(182)

Combining Equations (172) and (182) gives

$$\frac{d}{dt} ((\Delta z)^2) = 2\kappa_{zz} - 2\kappa_{tz} \kappa_{z} + 2\kappa_{zz} ((\Delta z))
- \frac{\kappa_{ntz}}{\kappa_{z}} \frac{d^3}{dt^3} ((\Delta z)^2) + 2\kappa_{ntz} \frac{d^2}{dt^2} ((\Delta z)),$$

(183)

which can be rewritten as

$$\frac{d^3}{dt^3} ((\Delta z)^2) + \frac{\kappa_{z}}{\kappa_{ntz}} \frac{d}{dt} ((\Delta z)^2) = 2\kappa_{zz} \kappa_{z} - 2\kappa_{tz} \kappa_{z} \kappa_{ntz} - 2\kappa_{zz} \kappa_{ntz} + 2\kappa_{zz} \frac{d^2}{dt^2} ((\Delta z)) + 2\kappa_{z} \frac{d^2}{dt^2} ((\Delta z)).$$

(184)

In order to derive the displacement variance

$$\kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to \infty} \frac{d\sigma^2}{dt} = \frac{1}{2} \lim_{t \to \infty} \frac{d}{dt} ((\Delta z)^2) - ((\Delta z)^2),$$

(185)

we have to obtain the formulas of the first- and second-order moments $((\Delta z))$ and $((\Delta z)^2)$, which are the solution of Equations (171) and (184), respectively. $((\Delta z))$ can be obtained as

$$((\Delta z)) = c_1 + c_2 e^{-\kappa_{ntz}/\kappa_{z}} + c_3 e^{\kappa_{ntz}/\kappa_{z}},$$

(186)

where $c_1$, $c_2$, and $c_3$ are the undetermined constants. After inserting Equation (186) into Equation (184), we find

$$((\Delta z)^2) = c_1' + c_2' e^{-\kappa_{ntz}/\kappa_{z}} + c_3' e^{\kappa_{ntz}/\kappa_{z}},
+ 2(\kappa_{zz} - \kappa_{tz} \kappa_{z} + \kappa_{ntz}) t + \kappa_{z} t^2$$

(187)

with the undetermined constants $c_1'$, $c_2'$, and $c_3'$.

Equation (171) can be rewritten as

$$\kappa_{z} = \frac{d}{dt} ((\Delta z)) + \frac{\kappa_{ntz}}{\kappa_{z}} \frac{d^3}{dt^3} ((\Delta z)).$$

(188)
Combining Equations (183) and (188) gives
\[ \frac{1}{2} \lim_{t \to t_\infty} \frac{d \sigma^2}{dt} = \kappa_{zz} - \kappa_{tz} \kappa_z + \kappa_{ttz} \frac{d^3}{dt^3} \langle (\Delta z)^2 \rangle + \kappa_{ttz} \frac{d^2}{dt^2} \langle (\Delta z) \rangle - \frac{1}{2} \frac{\kappa_{ttz}}{\kappa_z} \frac{d^3}{dt^3} \langle (\Delta z)^2 \rangle, \] (189)
where the formula \( \sigma^2 = \langle (\Delta z)^2 \rangle - \langle (\Delta z) \rangle^2 \) is used. Inserting Equations (186) and (187) into Equation (189) gives
\[ \frac{1}{2} \lim_{t \to t_\infty} \frac{d \sigma^2}{dt} = \kappa_{zz} - \kappa_{tz} \kappa_z + c_2^2 \sqrt{\frac{\kappa_z}{\kappa_{ttz}}} \lim_{t \to t_\infty} e^{-2t \sqrt{\frac{\kappa_z}{\kappa_{ttz}}}} - c_3^2 \sqrt{\frac{\kappa_z}{\kappa_{ttz}}} \lim_{t \to t_\infty} e^{2t \sqrt{\frac{\kappa_z}{\kappa_{ttz}}}} + \lim_{t \to t_\infty} \left( c_1 c_2 \sqrt{\frac{\kappa_z}{\kappa_{ttz}}} - c_2 \kappa_z - \frac{1}{2} c_2' \sqrt{\frac{\kappa_z}{\kappa_{ttz}}} + c_2 \kappa_z t \sqrt{\frac{\kappa_z}{\kappa_{ttz}}} \right) e^{-t \sqrt{\frac{\kappa_z}{\kappa_{ttz}}}} + \lim_{t \to t_\infty} \left( -c_1 c_3 \sqrt{\frac{\kappa_z}{\kappa_{ttz}}} - c_3 \kappa_z - \frac{1}{2} c_3' \sqrt{\frac{\kappa_z}{\kappa_{ttz}}} - c_3 \kappa_z t \sqrt{\frac{\kappa_z}{\kappa_{ttz}}} \right) e^{-t \sqrt{\frac{\kappa_z}{\kappa_{ttz}}}}. \] (190)
From Equations (B18) and (D7) in Appendices B and D, the relation \( \kappa_z / \kappa_{ttz} < 0 \) is found; therewith, one has
\[ \lim_{t \to t_\infty} e^{-t \sqrt{\frac{\kappa_z}{\kappa_{ttz}}}} = 0. \] (191)
Inserting Equation (191) into Equation (190) gives
\[ \frac{1}{2} \lim_{t \to t_\infty} \frac{d \sigma^2}{dt} = \kappa_{zz} - \kappa_{tz} \kappa_z. \] (192)
Because \( \sigma^2 \) is nonnegative, the third term on the right-hand side of the latter equation has to be zero. Thus, the condition \( c_3 = 0 \) needs to be satisfied, and Equation (192) becomes
\[ \frac{1}{2} \lim_{t \to t_\infty} \frac{d \sigma^2}{dt} = \kappa_{zz} - \kappa_{tz} \kappa_z, \] (193)
which is identical to Equation (54). Therefore, the displacement variance definition is invariant for the \( R_{ttz} \) of the second-order PtI operation.
Actually, if we adopt the special condition \( c_2 = c_3 = c_2' = c_3' = 0 \), we obtain Equation (193) easily. For the special condition, Equations (186) and (187) become
\[ \langle (\Delta z) \rangle = c_1 + \kappa_z t, \] (194)
\[ \langle (\Delta z)^2 \rangle = c_1^2 + 2(\kappa_{zz} - \kappa_{tz} \kappa_z + \kappa_z c_1) t + \kappa_z^2 t^2. \] (195)
Combining Equations (171) and (172) gives
\[ \frac{1}{2} \frac{d \sigma^2}{dt} = \kappa_{zz} + \kappa_{ttz} \frac{d^3}{dt^3} \langle (\Delta z) \rangle - \frac{1}{2} \frac{\kappa_{ttz}}{\kappa_z} \frac{d^3}{dt^3} \langle (\Delta z)^2 \rangle \]
\[ - \kappa_{ttz} \frac{d}{dt} \langle (\Delta z) \rangle - T_{ttz}, \] (196)
and inserting Equation (194) into Equation (173) gives
\[ T_{ttz} = 0. \] (197)
Thereafter, combining Equations (194)–(197) yields
\[ \frac{1}{2} \frac{d \sigma^2}{dt} = \kappa_{zz} - \kappa_{tz} \kappa_z, \] (198)
which also holds for the limit \( t \to t_\infty \)
\[ \kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to t_\infty} \frac{d \sigma^2}{dt} = \kappa_{zz} - \kappa_{tz} \kappa_z. \] (199)
6.3.2. $R_{zzz}$ of the Second-order PtI Operation

In this subsection, we introduce another DIO, i.e., the $R_{zzz}$ of the second-order PtI operation. As in the above subsection, we obtain the governing equation of $R_{zzz}$ to be

$$
\frac{\partial F}{\partial t} = \left( -\kappa_{zzz} \frac{\partial^2 F}{\partial z^2} + \kappa_{zzz} \frac{\partial^4 F}{\partial z^4} + \kappa_{zzz} \frac{\partial^6 F}{\partial t^6} + \ldots \right) + \kappa_{zzz} \frac{\partial^3 F}{\partial t^3} + \kappa_{zzz} \frac{\partial^5 F}{\partial t^5} + \ldots
$$

(200)

It is obvious that in Equation (200) the Fick’s law definition $\kappa_{zzz}^{\text{PL}}$ is equal to $\kappa_{zzz}$, i.e., $\kappa_{zzz}^{\text{PL}} = \kappa_{zzz}$, which is identical to Equation (53). From Equation (200), we obtain the following equations:

$$
\frac{d}{dt} \langle (\Delta z) \rangle = \kappa_z + \kappa_{zzz} \frac{d^3}{dt^3} \langle (\Delta z) \rangle ,
$$

(201)

$$
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_z \langle (\Delta z) \rangle + 2\kappa_{zz} \frac{d^3}{dt^3} \langle (\Delta z)^2 \rangle - 2\kappa_{zz} \frac{d}{dt} \langle (\Delta z) \rangle - 2T_{zzz}
$$

(202)

with

$$
T_{zzz} = \left( \kappa_{zzz} + \kappa_z \kappa_{zzz} \right) \frac{d^2}{dt^2} \langle (\Delta z) \rangle + \left( \kappa_{zzz} - \kappa_z \kappa_{zzz} \right) \frac{d^3}{dt^3} \langle (\Delta z) \rangle + \left( \kappa_{zz} - \kappa_z \kappa_{zzz} \right) \frac{d^4}{dt^4} \langle (\Delta z) \rangle + \ldots.
$$

(203)

The solution of Equation (201) can be obtained easily as

$$
\langle (\Delta z) \rangle = c_1 + c_2 e^{-t\sqrt{\kappa_z/\kappa_{zz}}/\kappa_{zzz}} + c_3 e^{t\sqrt{\kappa_z/\kappa_{zz}}/\kappa_{zzz}} + \kappa_z t.
$$

(204)

Because the formula of $\kappa_{zzz}$ is very complicated and hard to evaluate, in this subsection we only explore the displacement variance definition $\kappa_{zzz}^{\text{DV}}$ for the $R_{zzz}$ of the second-order PtI operation under the special condition, which requires the coefficients of the exponent functions in $\langle (\Delta z) \rangle$ and $\langle (\Delta z)^2 \rangle$ as zero. Thus, Equation (204) becomes

$$
\langle (\Delta z) \rangle = c_1 + \kappa_z t,
$$

(205)

inserting which into Equation (203), one finds

$$
T_{zzz} = 0.
$$

(206)

Inserting Equations (205) and (206) into Equation (202) yields

$$
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_z (c_1 + \kappa_z t) + 2\kappa_{zz} \frac{d^3}{dt^3} \langle (\Delta z)^2 \rangle - 2\kappa_{zz} \kappa_z.
$$

(207)

The solution of the latter equation can be easily obtained as follows:

$$
\langle (\Delta z)^2 \rangle = c'_1 + c'_2 e^{-t\sqrt{\kappa_z/\kappa_{zz}}/\kappa_{zzz}} + c'_3 e^{t\sqrt{\kappa_z/\kappa_{zz}}/\kappa_{zzz}} + 2(\kappa_{zz} \kappa_z - \kappa_{zz} - \kappa_z c_1) t + \kappa_z^2 t^2.
$$

(208)

For the special condition $c'_2 = c'_3 = 0$, the latter equation becomes

$$
\langle (\Delta z)^2 \rangle = c'_1 + 2(\kappa_{zz} \kappa_z - \kappa_{zz} - \kappa_z c_1) t + \kappa_z^2 t^2.
$$

(209)
By combining Equations (201) and (202), we find
\[
\frac{1}{2} \frac{d \sigma^2}{dt} = -\frac{\kappa_{zz}}{\kappa_{zz}} \left( (\Delta z) \frac{d^3}{dt^3} (\Delta z) \right) + \kappa_{zz}
\]
\[\quad + \frac{1}{2} \frac{\kappa_{\alpha\beta}}{\kappa_{zz}} \left( (\Delta z)^2 \right) - \kappa_{zz} \frac{d}{dt} ((\Delta z)) - T_{\alpha\beta}.
\]
(210)

Inserting Equations (205), (209), and (206) into the latter equation yields
\[
\frac{1}{2} \frac{d \sigma^2}{dt} = \kappa_{zz} - \kappa_{zz} \kappa_{zz},
\]
(211)

For the limit \( t \to t_{\infty} \), the latter equation also holds
\[
\kappa_{zz}^{\mathrm{VD}} = \frac{1}{2 \lim_{t \to t_{\infty}} \frac{d \sigma^2}{dt}} = \kappa_{zz} - \kappa_{zz} \kappa_{zz},
\]
(212)

which is identical to Equation (54). From the above investigation, we find that at least for the special condition, the displacement variance definition is invariant for the \( R_{\alpha\beta\gamma\delta} \) of the second-order PtI operation.

### 6.3.3. \( R_{\alpha\beta\gamma\delta} \) for the Second-order PtI Operation

Here, we give the general DIO, i.e., the \( R_{\alpha\beta\gamma\delta} \) of the second-order PtI operation. As done in Section 6.2.3, the equation corresponding to the \( R_{\alpha\beta\gamma\delta} \) of the second-order PtI operation can be obtained from Equation (103) as follows:

\[
\frac{\partial F}{\partial t} = \left( -\kappa_{\alpha\gamma} \frac{\partial F}{\partial \xi_{\gamma\delta}} + \kappa_{\alpha\gamma} \frac{\partial^2 F}{\partial \xi_{\gamma\delta}^2} + \kappa_{\alpha\gamma} \frac{\partial^3 F}{\partial \xi_{\gamma\delta}^3} + \kappa_{\alpha\gamma} \frac{\partial^4 F}{\partial \xi_{\gamma\delta}^4} + \ldots \right) + \kappa_{\alpha\gamma} \frac{\partial^4 F}{\partial \xi_{\gamma\delta}^4} + \ldots
\]

\[
+ \left( \kappa_{\alpha\gamma} - \kappa_{\alpha\gamma} \frac{\kappa_{\alpha\beta\gamma\delta}}{\kappa_{\alpha\beta\delta}} \frac{\partial^2 F}{\partial \xi_{\alpha\beta}^2} \right) + \left( \kappa_{\alpha\gamma} - \kappa_{\alpha\gamma} \frac{\kappa_{\alpha\beta\gamma\delta}}{\kappa_{\alpha\beta\delta}} \frac{\partial^2 F}{\partial \xi_{\alpha\beta}^2} \right) + \ldots
\]

\[
+ \left( \kappa_{\alpha\gamma} - \kappa_{\alpha\gamma} \frac{\kappa_{\alpha\beta\gamma\delta}}{\kappa_{\alpha\beta\delta}} \frac{\partial^2 F}{\partial \xi_{\alpha\beta}^2} \right) + \ldots
\]

\[
(213)
\]

From Equation (213), we find easily that the Fick’s law definition \( \kappa_{zz}^{\mathrm{FL}} \) is equal to \( \kappa_{zz} \), i.e., \( \kappa_{zz}^{\mathrm{FL}} = \kappa_{zz} \), which is the same as Equation (53). In what follows, we investigate the displacement variance definition for the \( R_{\alpha\beta\gamma\delta} \) of the second-order PtI operation.

As done in Section 6.2.4, from Equation (213) we obtain
\[
\frac{d}{dt} \left( (\Delta z) \right) = \kappa_{zz} + \frac{\kappa_{\alpha\beta\gamma\delta}}{\kappa_{\alpha\beta\delta}} \frac{d^2}{dt^2} (\Delta z).
\]
(214)

\[
\frac{d}{dt} \left( (\Delta z)^2 \right) = 2 \kappa_{zz} (\Delta z) + 2 \kappa_{zz} + \frac{\kappa_{\alpha\beta\gamma\delta}}{\kappa_{\alpha\beta\delta}} \frac{d^3}{dt^3} (\Delta z)
\]
\[\quad - 2 \kappa_{zz} \frac{d}{dt} (\Delta z) - 2 T_{\alpha\beta\gamma\delta}
\]
(215)

with
\[
T_{\alpha\beta\gamma\delta} = \left( \kappa_{\alpha\beta} - \frac{\kappa_{\alpha\beta\gamma\delta}}{\kappa_{\alpha\beta\delta}} \kappa_{\alpha\beta} \right) \frac{d^2}{dt^2} (\Delta z)
\]
\[\quad + \left( \kappa_{\alpha\beta} - \frac{\kappa_{\alpha\beta\gamma\delta}}{\kappa_{\alpha\beta\delta}} \kappa_{\alpha\beta} \right) \frac{d^3}{dt^3} (\Delta z) + \ldots
\]
(216)

Solving Equation (214) gives
\[
(\Delta z) = c_1 + c_2 e^{-t \sqrt{\kappa_{\alpha\beta\gamma\delta}/\kappa_{\alpha\beta\delta}}} + c_3 e^{t \sqrt{\kappa_{\alpha\beta\gamma\delta}/\kappa_{\alpha\beta\delta}}} + \kappa_{zz} t.
\]
(217)
which, for the special condition $c_2 = c_3 = 0$, becomes

$$\langle \Delta z \rangle = c_1 + \kappa_z t.$$  \hfill (218)

Thus, Equation (216) is simplified as

$$T_{nimz} = 0.$$  \hfill (219)

Inserting Equations (218) and (219) into Equation (215) yields

$$\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_z (c_1 + \kappa_z t) + 2\kappa_{zz} - 2\kappa_{zc} \kappa_z.$$  \hfill (220)

The solution of the above equation can be found easily:

$$\langle \Delta z \rangle = c'_1 + c'_2 e^{-\frac{\kappa_{nim} t}{\kappa_{(n-2)mz}}} + c'_3 e^{\frac{\kappa_{nim} t}{\kappa_{(n-2)mz}}} + 2(\kappa_{zz} - \kappa_{zc} \kappa_z + \kappa_z c_1) t + \kappa_z^2 t^2.$$  \hfill (221)

We only explore the case for the special condition, so the latter equation becomes

$$\langle \Delta z \rangle^2 = c'_1 + 2(\kappa_{zz} - \kappa_{zc} \kappa_z + \kappa_z c_1) t + \kappa_z^2 t^2.$$  \hfill (222)

Combining Equations (214) and (215), we obtain the equation of the displacement variance as follows:

$$\frac{1}{2} \frac{d\sigma_z^2}{dt} = -\frac{\kappa_{nimz}}{\kappa_{(n-2)mz}} \frac{d^3}{dt^3} \langle \Delta z \rangle + \kappa_{zz} \frac{d}{dt} \langle \Delta z \rangle.$$  \hfill (223)

Inserting Equations (218), (219), and (222) into the latter equation, we find, for the limit $t \to t_{\infty}$,

$$\kappa_{zz}^{VD} = \frac{1}{2t_{\infty}} \frac{d\sigma_z^2}{dt} = \kappa_{zz} - \kappa_{zc} \kappa_z,$$  \hfill (224)

which is identical to Equation (54). Therefore, the displacement variance definition $\kappa_{zz}^{VD}$ is invariant for the $R_{nimz}$ for the second-order PtI operation, at least for the special condition.

6.4. The Third-order PtI Operation

For $n = 3$, Equation (102) becomes

$$\frac{\partial F}{\partial t^2} = \left( -\kappa_{zz} \frac{\partial^2 F}{\partial t \partial z} + \kappa_{zz} \frac{\partial^3 F}{\partial t^2 \partial z} + \kappa_{zzz} \frac{\partial^4 F}{\partial t^3 \partial z} + \kappa_{zzzz} \frac{\partial^5 F}{\partial t^4 \partial z} + \cdots \right) + \left( \kappa_{zzz} \frac{\partial^5 F}{\partial t^4 \partial z} + \kappa_{zzz} \frac{\partial^6 F}{\partial t^5 \partial z} + \kappa_{zzzz} \frac{\partial^7 F}{\partial t^6 \partial z} + \cdots \right) \cdots,$$  \hfill (225)

which is the governing equation of the third-order PtI operation.

6.4.1. $R_{nimz}$ of the Third-order PtI Operation

Combining Equations (52) and (225), we find

$$\frac{\partial F}{\partial t} \left( \begin{array}{c} -\kappa_{zz} \frac{\partial F}{\partial z} + \kappa_{zz} \frac{\partial^2 F}{\partial z^2} + \kappa_{zzz} \frac{\partial^3 F}{\partial z^3} + \kappa_{zzzz} \frac{\partial^4 F}{\partial z^4} + \cdots \end{array} \right) - \frac{\kappa_{me}}{\kappa_z} \frac{\partial^4 F}{\partial z^4}$$

$$+ \kappa_{zz} \frac{\partial^4 F}{\partial t^4 \partial z} + \kappa_{zz} \frac{\partial^5 F}{\partial t^5 \partial z} + \kappa_{zzz} \frac{\partial^6 F}{\partial t^6 \partial z} + \kappa_{zzzz} \frac{\partial^7 F}{\partial t^7 \partial z} + \cdots$$

$$+ \kappa_{me} \frac{\partial^4 F}{\partial t^4 \partial z} + \kappa_{me} \frac{\partial^5 F}{\partial t^5 \partial z} + \kappa_{me} \frac{\partial^6 F}{\partial t^6 \partial z} + \kappa_{me} \frac{\partial^7 F}{\partial t^7 \partial z} + \cdots$$

$$+ \frac{\kappa_{me}}{\kappa_z} \frac{\partial^4 F}{\partial t^4 \partial z} + \frac{\kappa_{me}}{\kappa_z} \frac{\partial^5 F}{\partial t^5 \partial z} + \frac{\kappa_{me}}{\kappa_z} \frac{\partial^6 F}{\partial t^6 \partial z} + \cdots$$

$$+ \frac{\kappa_{me}}{\kappa_z} \frac{\partial^4 F}{\partial t^4 \partial z} + \frac{\kappa_{me}}{\kappa_z} \frac{\partial^5 F}{\partial t^5 \partial z} + \frac{\kappa_{me}}{\kappa_z} \frac{\partial^6 F}{\partial t^6 \partial z} + \cdots$$

$$+ \frac{\kappa_{me}}{\kappa_z} \frac{\partial^4 F}{\partial t^4 \partial z} + \frac{\kappa_{me}}{\kappa_z} \frac{\partial^5 F}{\partial t^5 \partial z} + \frac{\kappa_{me}}{\kappa_z} \frac{\partial^6 F}{\partial t^6 \partial z} + \cdots,$$  \hfill (226)

which is the EIDF corresponding to the $R_{nimz}$, one of the DIOs, of the third-order PtI operation.
From Equation (226), we easily find that the Fick’s law definition $k_{zz}^{FL}$ is equal to $\kappa_{zz}$, which is identical to Equation (53).

From Equation (226), we find

$$\frac{d}{dt}((\Delta z)) = \kappa_{zz} - \frac{\kappa_{3z}}{\kappa_{zz}} \frac{d^4}{dt^4}((\Delta z)),$$

(Eq. 227)

$$\frac{d}{dt}((\Delta z)^2) = 2\kappa_{zz}((\Delta z)) + 2\kappa_{zz} - \frac{\kappa_{3z}}{\kappa_{zz}} \frac{d^4}{dt^4}((\Delta z))$$

$$- 2\kappa_{zz} \frac{d}{dt}((\Delta z)) - 2\kappa_{3z} \frac{d^2}{dt^2}((\Delta z)) - 2T_{3z},$$

(Eq. 228)

with

$$T_{3z} = \left(\frac{\kappa_{3z}}{\kappa_{zz}} \kappa_{zz} + \kappa_{3z}\right) \frac{d^4}{dt^4}((\Delta z))$$

$$+ \left(\frac{\kappa_{3z}}{\kappa_{zz}} \kappa_{zz} + \kappa_{3z}\right) \frac{d^5}{dt^5}((\Delta z)) + \ldots,$$

(Eq. 229)

The solution of Equation (227) can be obtained as follows:

$$\langle (\Delta z) \rangle = c_0 + c_1 e^{\gamma t} + c_2 e^{\gamma t} + c_3 e^{\gamma t} + \kappa_{z} t$$

(Eq. 230)

with

$$r_1 = \left(\frac{\kappa_{zz}}{\kappa_{3z}}\right)^{1/3} e^{\gamma t/3},$$

(Eq. 231)

$$r_2 = -\left(\frac{\kappa_{zz}}{\kappa_{3z}}\right)^{1/3},$$

(Eq. 232)

$$r_3 = \left(\frac{\kappa_{zz}}{\kappa_{3z}}\right)^{1/3} e^{i\gamma t/3},$$

(Eq. 233)

where $r_1$, $r_2$, and $r_3$ satisfy the following formulas:

$$r_1 + r_2 + r_3 = 0,$$

(Eq. 234)

$$r_1^3 = r_2^3 = r_3^3 = -\frac{\kappa_{zz}}{\kappa_{3z}}.$$

(Eq. 235)

From Equation (214), we find

$$\frac{d^{n+3}}{dt^{n+3}}((\Delta z)) = -\frac{\kappa_{zz}}{\kappa_{3z}} \frac{d^n}{dt^n}((\Delta z)),$$

(Eq. 236)

inserting which into Equation (229), one obtains

$$T_{3z} = -\kappa_{zz} \frac{d^3}{dt^3}((\Delta z)) - \kappa_{3z} \frac{d^2}{dt^2}((\Delta z)) + \kappa_{zz} \kappa_{zz}$$

$$- \kappa_{zz} \frac{d}{dt}((\Delta z)).$$

(Eq. 237)

Replacing the term $T_{3z}$ in Equation (215) with the latter equation gives

$$\frac{1}{2} \frac{d}{dt}((\Delta z)^2) = \kappa_{zz}((\Delta z)) + \kappa_{zz} - \frac{1}{2} \frac{\kappa_{3z}}{\kappa_{zz}} \frac{d^4}{dt^4}((\Delta z))$$

$$- \kappa_{zz} \frac{d}{dt}((\Delta z)) - \kappa_{zz} \frac{d^2}{dt^2}((\Delta z))$$

$$+ \kappa_{3z} \frac{d^3}{dt^3}((\Delta z)) + \kappa_{zz} \frac{d^2}{dt^2}((\Delta z))$$

$$- \kappa_{zz} \kappa_{zz} + \kappa_{zz} \frac{d}{dt}((\Delta z)).$$

(Eq. 238)

Considering Equation (230), from Equation (238) we find

$$\langle (\Delta z)^2 \rangle = c_0' + c_1'e^{\gamma t} + c_2'e^{\gamma t} + c_3'e^{\gamma t}$$

$$+ 2(\kappa_{zz} - \kappa_{zz} \kappa_{zz} + \kappa_{zz} e_0') + \kappa_{zz}^2 r^2,$$

(Eq. 239)

where $r_1$, $r_2$, and $r_3$ are given in Equations (231)–(233).
Combining Equations (227), (228), and (237), we obtain

\[
\frac{1}{2} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_{zt} \kappa_z + \frac{K_{3t}}{\kappa_z} c_0 c_1 n_1^4 e_{r't'} + \frac{K_{3t}}{\kappa_z} c_0 c_2 r_2^4 e_{r''t'} + \frac{K_{3t}}{\kappa_z} c_0 c_3 r_3^4 e_{r'''t'}
\]

\[
+ \frac{K_{3t}}{\kappa_z} c_1 c_2 r_1^4 e^{(r_1 r_2)_t} + \frac{K_{3t}}{\kappa_z} c_1 c_2 r_2^4 e^{(r_1 r_2)_t} + \frac{K_{3t}}{\kappa_z} c_1 c_3 r_3^4 e^{(r_1 r_2)_t} + \frac{K_{3t}}{\kappa_z} c_1 c_3 r_3^4 e^{(r_1 r_2)_t} + \frac{K_{3t}}{\kappa_z} c_1 c_3 r_3^4 e^{(r_1 r_2)_t}
\]

\[
+ \frac{K_{3t}}{\kappa_z} c_1 c_2 r_1^3 e^{(r_2 r_3)_t} + \frac{K_{3t}}{\kappa_z} c_1 c_2 r_2^3 e^{(r_2 r_3)_t} + \frac{K_{3t}}{\kappa_z} c_1 c_2 r_2^3 e^{(r_2 r_3)_t} + \frac{K_{3t}}{\kappa_z} c_1 c_3 r_3^3 e^{(r_2 r_3)_t}
\]

\[
+ n_1^4 c_1 e_{r't'} + \frac{K_{3t}}{\kappa_z} c_0 c_2 e_{r't'} + \frac{K_{3t}}{\kappa_z} c_0 c_3 e_{r't'} + \frac{K_{3t}}{\kappa_z} c_0 c_3 e_{r't'} + \frac{K_{3t}}{\kappa_z} c_0 c_3 e_{r't'} + \frac{K_{3t}}{\kappa_z} c_0 c_3 e_{r't'} + \frac{K_{3t}}{\kappa_z} c_0 c_3 e_{r't'} + \frac{K_{3t}}{\kappa_z} c_0 c_3 e_{r't'}
\]

\[
- \frac{K_{3t}}{\kappa_z} n_1^4 c_1 e_{r't'} - \frac{K_{3t}}{\kappa_z} n_1^4 c_1 e_{r't'} - \frac{K_{3t}}{\kappa_z} n_1^4 c_1 e_{r't'} - \frac{K_{3t}}{\kappa_z} n_1^4 c_1 e_{r't'} - \frac{K_{3t}}{\kappa_z} n_1^4 c_1 e_{r't'} - \frac{K_{3t}}{\kappa_z} n_1^4 c_1 e_{r't'} - \frac{K_{3t}}{\kappa_z} n_1^4 c_1 e_{r't'} - \frac{K_{3t}}{\kappa_z} n_1^4 c_1 e_{r't'}.
\]

(240)

which, because of Equations (234) and (235), can be rewritten as

\[
\frac{1}{2} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_{zt} \kappa_z - c_0 c_1 r_1 e_{r't'} - \kappa_z c_1 e_{r't'} - c_0 c_2 r_2 e_{r't'} - \kappa_z c_2 e_{r't'} - c_0 c_3 r_3 e_{r't'} - \kappa_z c_3 e_{r't'}
\]

\[
- r_1 c_1^2 e_{r't'} - r_2 c_2^2 e_{r't'} - r_3 c_3^2 e_{r't'} + c_1 c_2 r_3 e_{r't'} - c_1 c_2 r_3 e_{r't'} + c_2 c_3 r_1 e_{r't'} + \frac{1}{2} n_1 c_1 e_{r't'} + \frac{1}{2} r_2 c_2 e_{r't'} + \frac{1}{2} r_3 c_3 e_{r't'}.
\]

(241)

In order to ensure that \(d\sigma^2/(2dt)\) is real, we have to ignore the terms containing \(c_1\) and \(c_3\) in the above equation, i.e., to employ the condition \(c_1 = c_1' = c_3 = c_3' = 0\), so Equation (241) becomes

\[
\frac{1}{2} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_{zt} \kappa_z - c_0 c_2 r_2 e_{r't'} - r_2 c_2^2 e_{r't'} - r_2 c_2^2 e_{r't'} - r_2 c_2^2 e_{r't'} - r_2 c_2^2 e_{r't'}
\]

\[
- \kappa_z r_2 c_2 e_{r't'} + \frac{1}{2} r_2 c_2 e_{r't'}.
\]

(242)

Because \(r_2 < 0\), for the limit \(t \to t_\infty\), the latter equation becomes

\[
\kappa_{zz}^{ND} = \frac{1}{2} \lim_{t \to t_\infty} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_{zt} \kappa_z,
\]

(243)

which is identical to Equation (54). Therefore, the displacement variance definition \(\kappa_{zz}^{ND}\) is an invariant quantity for the \(R_{nt}\) of the third-order PtI operation. From the above part, we find that the special condition \(c_1 = c_1' = c_2 = c_2' = c_3 = c_3' = 0\) is an inference from the derivation process.

As shown in Sections 6.2.4 and 6.3.3, if using the special condition, we also find that for the \(R_{nmc}\) of the third-order PtI operation, both the Fick’s law definition \(\kappa_{zz}^{NL}\) and the displacement variance definition \(\kappa_{zz}^{ND}\) are invariant quantities. For the simplification, in this subsection we do not give the detailed deduction process.

6.5. The \(i\)th-order PtI Operation

Setting \(n = i\) in Equation (102) yields

\[
\frac{\partial^{i+1} F}{\partial t^{i+1}} = \left( -\kappa_z \frac{\partial^{i+2} F}{\partial t^{i+2} \partial z} + \kappa_z \frac{\partial^{i+1} F}{\partial t^{i+1} \partial z} + \kappa_z \frac{\partial^{i+3} F}{\partial t^{i+1} \partial z} + \kappa_{nt} \frac{\partial^{i+4} F}{\partial t^{i+1} \partial z} + \cdots \right) + \left( \kappa_z \frac{\partial^{i+2} F}{\partial t^{i+1} \partial z} + \kappa_{nt} \frac{\partial^{i+3} F}{\partial t^{i+1} \partial z} + \kappa_{nt} \frac{\partial^{i+4} F}{\partial t^{i+1} \partial z} + \cdots \right)
\]

\[
+ \left( \kappa_z \frac{\partial^{i+1} F}{\partial t^{i+2} \partial z} + \kappa_{nt} \frac{\partial^{i+2} F}{\partial t^{i+2} \partial z} + \kappa_{nt} \frac{\partial^{i+3} F}{\partial t^{i+2} \partial z} + \cdots \right) + \cdots
\]

(244)

which is the equation of the \(i\)th-order PtI operation.
6.5.1. \( R_{i z} \) of the \( i \)th-order PtI Operation

Equation (244) can be rewritten as

\[
\frac{\kappa_{iz}}{t} \frac{\partial^{i+1}F}{\partial t^{i+1}} = -\frac{\kappa_{iz}}{t} \frac{\partial^{i+1}F}{\partial t^{i+1}} + \left( \frac{\kappa_{iz}}{t} \frac{\partial^{i+2}F}{\partial t^{i+2}} + \frac{\kappa_{iz}}{t} \frac{\partial^{i+3}F}{\partial t^{i+3}} + \frac{\kappa_{iz}}{t} \frac{\partial^{i+4}F}{\partial t^{i+4}} + \cdots \right) + \cdots 
\]

from which, considering Equation (52), we find

\[
\frac{\partial F}{\partial t} = \left( -\frac{\kappa_{iz}}{t} \frac{\partial^{i+1}F}{\partial t^{i+1}} + \right) + \cdots 
\]

The latter manipulation is a new DIO, i.e., the \( R_{i z} \) of the \( i \)th-order operation. It is obvious that the Fick’s law definition \( \kappa_{izd} \) is equal to \( \kappa_{izd} = \kappa_{iz} \), which is identical to Equation (53). Thus, \( \kappa_{izd} \) is an invariant quantity for \( R_{i z} \) of the \( i \)th-order operation.

From Equation (246), we obtain the first- and second-moment equations of the displacement as

\[
\frac{d}{dt} \langle \Delta z \rangle = \kappa_{iz} - \frac{\kappa_{iz}}{\kappa_{iz}} \frac{\partial^{i+1}F}{\partial t^{i+1}} \langle \Delta z \rangle, 
\]

\[
\frac{d}{dt} \langle (\Delta z)^2 \rangle = 2\kappa_{iz} \langle \Delta z \rangle + 2\kappa_{iz} \frac{\partial^{i+1}F}{\partial t^{i+1}} \langle (\Delta z)^2 \rangle - 2\kappa_{iz} \frac{d}{dt} \langle \Delta z \rangle - 2T_{iz} 
\]

with

\[
T_{iz} = \kappa_{iz} \frac{d^2}{dt^2} \langle \Delta z \rangle + \kappa_{iz} \frac{d^3}{dt^3} \langle \Delta z \rangle + \kappa_{iz} \frac{d^4}{dt^4} \langle \Delta z \rangle + \cdots + \kappa_{iz} \frac{d^i}{dt^i} \langle \Delta z \rangle + \kappa_{iz} \frac{d^{i+1}}{dt^{i+1}} \langle \Delta z \rangle + \kappa_{iz} \frac{d^{i+2}}{dt^{i+2}} \langle \Delta z \rangle + \cdots . 
\]

The solution of Equation (247) can be easily found:

\[
\langle \Delta z \rangle = c_1 + c_2 e^{r_1 t} + c_3 e^{r_2 t} + c_4 e^{r_3 t} + \cdots + c_{i+1} e^{r_{i+1} t} + \kappa_{iz} t, 
\]

where \( r_1, r_2, r_3, \ldots \), and \( r_i \) are the solution of the corresponding characteristic equation, and \( c_1, c_2, c_3, \ldots \), and \( c_{i+1} \) are undetermined constants. For the special condition \( c_2 = c_3 = \cdots = c_{i+1} = 0 \), Equation (250) becomes

\[
\langle \Delta z \rangle = c_1 + \kappa_{iz} t. 
\]

Inserting the latter formula into Equation (248) gives

\[
\frac{\kappa_{iz}}{t} \frac{\partial^{i+1}F}{\partial t^{i+1}} \langle (\Delta z)^2 \rangle + \frac{d}{dt} \langle (\Delta z)^2 \rangle = 2(\kappa_{iz} c_1 + \kappa_{iz} - \kappa_{iz} \kappa_{iz}) + 2\kappa_{iz}^2 t, 
\]

\[32\]
the solution of which can be found:

\[
(\Delta z)^2 = c_1' + c_1'e^{\mu t} + c_1'e^{\mu t} + c_1'e^{\mu t} + \cdots + c_1'e^{\mu (t-t_0)} + 2(\kappa_x c_1' + \kappa_z - \kappa_{zz} \kappa_z + \kappa_z^2 t^2).
\]  

Here, \(r_1', r_2', r_3', \ldots\), and \(r_i'\) are the solution of the characteristic equation corresponding to Equation (252), and \(c_1', c_2', c_3', \ldots\), and \(c_i'\) are undetermined constants. With the special condition \(c_1' = c_2' = \cdots = c_i' = 0\), Equation (253) becomes

\[
(\Delta z)^2 = c_1' + 2(\kappa_x c_1' + \kappa_z - \kappa_{zz} \kappa_z) t + \kappa_z^2 t^2.
\]  

Combining Equations (251) and (254) yields, for the limit \(t \rightarrow t_\infty\),

\[
\kappa_{zz} = \frac{1}{2} \lim_{t \rightarrow t_\infty} \frac{d\sigma^2}{dt} = \kappa_{zz} - \kappa_{zz} \kappa_z,
\]

which is identical to Equation (54). The above investigation shows that at least for the special condition, the displacement variance definition \(\kappa_{zz}^{VD}\) is invariant for the \(R_{zz}\) of the third-order PtI operation.

### 6.5.2. \(R_{zz}\) of the Ith-order PtI Operation and \(R_{zz}\) of the First-order PtI Operation

If considering Equations (106) and (246), that is, combining the \(R_{zz}\) of the \(i\)th-order PtI operation and the \(R_{zz}\) of the first-order PtI operation, which is another new DIO, we obtain the following equation:

\[
\begin{align*}
\frac{\partial F}{\partial t} &= \left( -\kappa_x \frac{\partial F}{\partial z} + \kappa_{zz} \frac{\partial^2 F}{\partial z^2} + \kappa_{zzz} \frac{\partial^3 F}{\partial z^3} + \cdots + \kappa_{zzzzz} \frac{\partial^5 F}{\partial z^5} + \cdots \right) - \kappa_{zz} \frac{\partial^2 F}{\partial z^2} - \kappa_{zzz} \frac{\partial^4 F}{\partial z^4} + \cdots \\
&\quad + \left( \kappa_{zz} + \frac{\kappa_{zz} \kappa_{zzz}}{\kappa_z} \right) \frac{\partial^3 F}{\partial z \partial \tau^2} + \left( \kappa_{zzz} + \frac{\kappa_{zz} \kappa_{zzzz}}{\kappa_z} \right) \frac{\partial^4 F}{\partial z \partial \tau^3} + \cdots \\
&\quad + \left( \kappa_{zzz} + \frac{\kappa_{zz} \kappa_{zzzz}}{\kappa_z} \right) \frac{\partial^4 F}{\partial \tau^2 \partial z^2} + \left( \kappa_{zzzz} + \frac{\kappa_{zz} \kappa_{zzzzz}}{\kappa_z} \right) \frac{\partial^5 F}{\partial \tau^3 \partial z^3} + \cdots \\
&\quad + \left( \kappa_{zzzz} + \frac{\kappa_{zz} \kappa_{zzzzz}}{\kappa_z} \right) \frac{\partial^5 F}{\partial \tau^3 \partial z^3} + \cdots \\
&\quad + \left( \kappa_{zzzzzz} + \frac{\kappa_{zz} \kappa_{zzzzzzz}}{\kappa_z} \right) \frac{\partial^6 F}{\partial \tau^4 \partial z^4} + \cdots \\
&\quad + \left( \kappa_{zzzzzzzz} + \frac{\kappa_{zz} \kappa_{zzzzzzzz}}{\kappa_z} \right) \frac{\partial^7 F}{\partial \tau^4 \partial z^4} + \cdots \\
&\quad + \left( \kappa_{zzzzzzzzzz} + \frac{\kappa_{zz} \kappa_{zzzzzzzzz}}{\kappa_z} \right) \frac{\partial^8 F}{\partial \tau^5 \partial z^5} + \cdots \\
&\quad + \left( \kappa_{zzzzzzzzzzz} + \frac{\kappa_{zz} \kappa_{zzzzzzzzzz}}{\kappa_z} \right) \frac{\partial^9 F}{\partial \tau^5 \partial z^5} + \cdots \\
&\quad + \left( \kappa_{zzzzzzzzzzzz} + \frac{\kappa_{zz} \kappa_{zzzzzzzzzz}}{\kappa_z} \right) \frac{\partial^{10} F}{\partial \tau^6 \partial z^6} + \cdots \\
&\quad + \left( \kappa_{zzzzzzzzzzzzzz} + \frac{\kappa_{zz} \kappa_{zzzzzzzzzzz}}{\kappa_z} \right) \frac{\partial^{11} F}{\partial \tau^6 \partial z^6} + \cdots \\
&\quad + \left( \kappa_{zzzzzzzzzzzzzzzz} + \frac{\kappa_{zz} \kappa_{zzzzzzzzzzzz}}{\kappa_z} \right) \frac{\partial^{12} F}{\partial \tau^7 \partial z^7} + \cdots.
\end{align*}
\]  

From the latter equation, we find

\[
\frac{d}{dt} (\Delta z)^2 = \frac{\kappa_x}{\kappa_z} \frac{d^2}{dt^2} (\Delta z) - \frac{\kappa_{zzz}}{\kappa_z} \frac{d^{i+1}}{dt^{i+1}} (\Delta z),
\]

\[
\frac{d}{dt} (\Delta z)^2 = 2\kappa_z (\Delta z) + 2\kappa_{zz} - \frac{\kappa_{zz}}{\kappa_z} \frac{d^2}{dt^2} (\Delta z)^2
\]

\[- \frac{\kappa_{zzz}}{\kappa_z} \frac{d^{i+1}}{dt^{i+1}} (\Delta z)^2 + T_{zzz}
\]
with

\[ T_{\text{it} + \text{Hz}} = \left( \kappa_{\text{id}} + \frac{\kappa_{\text{z}}^2}{\kappa_{\text{z}}} \right) \frac{d^2}{dt^2} \langle \Delta z \rangle + \left( \kappa_{\text{id}} + \frac{\kappa_{\text{z}}}{\kappa_{\text{z}}} \right) \frac{d^3}{dt^3} \langle \Delta z \rangle + \left( \kappa_{\text{id}} + \frac{\kappa_{\text{z}}}{\kappa_{\text{z}}} \right) \frac{d^4}{dt^4} \langle \Delta z \rangle + \cdots \]

\[ + \left( \kappa_{\text{id}} + \frac{\kappa_{\text{z}}^2}{\kappa_{\text{z}}} \right) \frac{d^i}{dt^i} \langle \Delta z \rangle + \left( \kappa_{\text{id}} + \frac{\kappa_{\text{z}}}{\kappa_{\text{z}}} \right) \frac{d^{i+1}}{dt^{i+1}} \langle \Delta z \rangle \]

\[ + \left( \kappa_{\text{id}} + \frac{\kappa_{\text{z}}}{\kappa_{\text{z}}} \right) \frac{d^{i+2}}{dt^{i+2}} \langle \Delta z \rangle + \cdots. \tag{259} \]

The characteristic equation corresponding to Equation (257) is found to be

\[ r^{i+1} + \frac{\kappa_{\text{z}}}{\kappa_{\text{id}}} r^2 + \frac{\kappa_{\text{z}}}{\kappa_{\text{id}}} r = 0, \tag{260} \]

which has \((i + 1)\) solutions. As in the above subsection, for the special condition we obtain the solution of Equation (257) to be

\[ \langle \Delta z \rangle = c_1 + \kappa_{\text{z}} t, \tag{261} \]

inserting which into Equations (258) and (259), one finds

\[ \frac{d}{dt} \langle \Delta z \rangle^2 = 2\kappa_{\text{z}} (c_1 + \kappa_{\text{z}} t) + 2\kappa_{\text{z}} - \frac{\kappa_{\text{z}}}{\kappa_{\text{id}}} \frac{d^2}{dt^2} \langle \Delta z \rangle^2 \]

\[ - \frac{\kappa_{\text{id}}}{\kappa_{\text{z}}} \frac{d^{i+1}}{dt^{i+1}} \langle \Delta z \rangle^2. \tag{262} \]

For the special condition, the solution of the latter equation is shown as

\[ \langle \Delta z \rangle^2 = c_1^2 + 2(\kappa_{\text{z}} c_1' + \kappa_{\text{z}} - \kappa_{\text{id}} \kappa_{\text{z}}) t + \kappa_{\text{z}}^2 t^2. \tag{263} \]

Combining Equations (257) and (262) gives

\[ \frac{1}{2} \frac{d\sigma^2}{dt} = \kappa_{\text{z}} + \frac{\kappa_{\text{z}}}{\kappa_{\text{id}}} \langle \Delta z \rangle \frac{d^2}{dt^2} \langle \Delta z \rangle + \frac{\kappa_{\text{id}}}{\kappa_{\text{z}}} \langle \Delta z \rangle \frac{d^{i+1}}{dt^{i+1}} \langle \Delta z \rangle - \frac{1}{2} \frac{\kappa_{\text{z}}}{\kappa_{\text{id}}} \frac{d^2}{dt^2} \langle \Delta z \rangle - \frac{1}{2} \frac{\kappa_{\text{id}}}{\kappa_{\text{z}}} \frac{d^{i+1}}{dt^{i+1}} \langle \Delta z \rangle. \tag{264} \]

Considering Equations (261), (263), and (264), for the limit \(t \to t_{\infty}\), we find

\[ \kappa_{\text{z}}^{\text{VD}} = \frac{1}{2} \lim_{t \to t_{\infty}} \frac{d\sigma^2}{dt} = \kappa_{\text{z}} - \kappa_{\text{id}} \kappa_{\text{z}}, \tag{265} \]

which is identical to Equation (54) and shows, at least for the special condition, that the displacement variance definition is an invariant quantity for the \(i\)th-order PtI operation and the \(R_{\text{e}}\) of the first-order PtI operation.

Because there is no second-order spatial derivative term in Equation (102) with \(n = 1, 2, 3, \ldots\), any manipulation to insert a deformation of Equation (102) into Equation (52) cannot change the Fick’s law definition of the SPDC. Analogous to the deduction in the previous part in this subsection, at least for the special condition, the displacement variance definition is invariant for the more complicated combination of the DIOs of the PtI operations.

### 7. TGK Definition for the Focusing Field

The TGK formulation is a useful tool to calculate diffusion coefficients (Taylor 1922; Green 1951; Kubo 1962); for the SPDC, it is given by

\[ \kappa_{\text{z}}^{\text{TGK}} = \int_0^\infty dt \langle v_z(t) v_z(0) \rangle. \tag{266} \]

Here, the \(z\) component of the energetic charged particle velocity \(v_z\) is equal to \(v \mu\) with the particle velocity \(v\) and the pitch-angle cosine \(\mu\), so Equation (266) can be rewritten as

\[ \kappa_{\text{z}}^{\text{TGK}} = v^2 \int_0^\infty dt \langle \mu(t) \mu(0) \rangle. \tag{267} \]

The integral in the latter equation is given by

\[ \int_0^\infty dt \langle \mu(t) \mu(0) \rangle = \frac{1}{4} \int_{-\infty}^\infty dz \int_0^\infty dt \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu_0 f(z, \mu, t), \tag{268} \]
where \( f(z, \mu, t) \) is the distribution function and satisfies the Fokker–Planck equation and \( \mu_0 \) is the initial pitch-angle cosine of the energetic particle. Thus, the TGK formulation becomes

\[
\kappa^\text{TGK}_{zz} = \frac{v^2}{4} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int_{-1}^{1} d\mu \mu \mu f(z, \mu, t),
\]

which, with Equations (11)–(13), can be expressed as

\[
\kappa^\text{TGK}_{zz} = \frac{v^2}{4} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int_{-1}^{1} d\mu \mu \mu g(z, \mu, t).
\]

Here, \( \int_{-1}^{1} d\mu \mu F(z, t) = 0 \) is used.

With the anisotropic distribution function \( g(z, \mu, t) \) (see Equation (46)), Equation (270) becomes

\[
\kappa^\text{TGK}_{zz} = \frac{v^2}{4} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int_{-1}^{1} d\mu \mu \mu \left\{ L \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) \left[ 1 - \frac{2e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \right] + e^{M(\mu)} \left[ R(\mu) - \frac{\int_{-1}^{1} d\mu e^{M(\mu)} R(\mu)}{\int_{-1}^{1} d\mu e^{M(\mu)}} \right] \right\}.
\]

Because the isotropic distribution function \( F(z, t) \) does not contain a variable \( \mu_0 \), the following integral can be obtained:

\[
\kappa^\text{TGK}_{zz} = \frac{v^2}{4} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu \mu \mu \left\{ L \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) \left[ 1 - \frac{2e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \right] \right\} = 0,
\]

from which Equation (271) becomes

\[
\kappa^\text{TGK}_{zz} = \frac{v^2}{4} (X_1 - X_2)
\]

with

\[
X_1 = \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int_{-1}^{1} d\mu e^{M(\mu)} R(\mu),
\]

\[
X_2 = \frac{\int_{-1}^{1} d\mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int_{-1}^{1} d\mu e^{M(\mu)} R(\mu).
\]

Inserting Equations (47) and (45) into Equations (274) and (275) yields

\[
X_1 = Y_1 + Y_2,
\]

\[
X_2 = Y_3 + Y_4
\]

with

\[
Y_1 = \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} D_{\nu}(\nu) \left( \frac{\partial F}{\partial t} + \int_{-1}^{\nu} \frac{\partial g}{\partial \nu} \right),
\]

\[
Y_2 = \frac{v}{2} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} D_{\nu}(\nu) \left( 2 \int_{-1}^{\nu} d\rho \frac{\partial g}{\partial \nu} - \int_{-1}^{1} d\mu \frac{\partial g}{\partial \nu} \right),
\]

\[
Y_3 = \frac{\int_{-1}^{1} d\mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} D_{\nu}(\nu) \left( \frac{\partial F}{\partial t} + \int_{-1}^{\nu} \frac{\partial g}{\partial \nu} \right),
\]

\[
Y_4 = \frac{\int_{-1}^{1} d\mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} D_{\nu}(\nu) \left( 2 \int_{-1}^{\nu} d\rho \frac{\partial g}{\partial \nu} - \int_{-1}^{1} d\mu \frac{\partial g}{\partial \nu} \right).
\]
By using the following formulas,

\[
\int_0^\infty dt \int_{-\infty}^\infty dz \frac{\partial g(z, t)}{\partial t} = \left[ \int_{-\infty}^\infty dz g(z) \right] (t = \infty) - \left[ \int_{-\infty}^\infty dz g(z) \right] (t = 0),
\]

(283)

\[
\int_0^\infty dz \frac{\partial g}{\partial z} = g(z = \infty) - g(z = -\infty)
\]

(284)

with

\[
\left[ \int_{-\infty}^\infty dz g(z) \right] (t = \infty) = 0,
\]

(285)

\[
\left[ \int_{-\infty}^\infty dz g(z) \right] (t = 0) = 2\delta(\mu - \mu_0),
\]

(286)

\[
g(z = \infty) = g(z = -\infty) = 0,
\]

(287)

we find

\[
Y_1 = -2 \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu \mu e^{M(\mu)} \int_{-1}^\mu d\nu \frac{e^{-M(\nu)}}{D_{\nu}(\nu)} \int_{-1}^\nu \delta(\nu - \nu_0) d\rho,
\]

(288)

\[
Y_2 = 0,
\]

(289)

\[
Y_3 = -2 \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu \mu e^{M(\mu)} \int_{-1}^\mu d\nu \frac{e^{-M(\nu)}}{D_{\nu}(\nu)} \int_{-1}^\nu \delta(\nu - \nu_0) d\rho,
\]

(290)

\[
Y_4 = 0.
\]

(291)

Equations (288)–(291) combined with Equations (276) and (277) become

\[
X_1 = -2 \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu \mu e^{M(\mu)} \int_{-1}^\mu d\nu \frac{e^{-M(\nu)}}{D_{\nu}(\nu)} \int_{-1}^\nu \delta(\nu - \nu_0) d\rho,
\]

(292)

\[
X_2 = -2 \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu \mu e^{M(\mu)} \int_{-1}^\mu d\nu \frac{e^{-M(\nu)}}{D_{\nu}(\nu)} \int_{-1}^\nu \delta(\nu - \nu_0) d\rho.
\]

(293)

With the formula

\[
\int_{-1}^1 d\nu \int_{-1}^\nu d\mu_0 \delta(\nu - \mu_0) = \frac{\mu^2 - 1}{2},
\]

(294)

we derive

\[
X_1 = \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu \mu e^{M(\mu)} \int_{-1}^\mu d\nu \frac{e^{-M(\nu)}(1 - \nu^2)}{D_{\nu}(\nu)},
\]

(295)

\[
X_2 = \int_{-1}^1 d\mu_0 \mu_0 \int_{-1}^1 d\mu \mu e^{M(\mu)} \int_{-1}^\mu d\nu \frac{e^{-M(\nu)}(1 - \nu^2)}{D_{\nu}(\nu)}.
\]

(296)

Combining Equations (295), (296), and (297) yields

\[
\kappa_{zz}^{\text{TGGK}} = \frac{v^2}{4} \int_{-1}^1 d\mu \mu_0 e^{M(\mu)} \int_{-1}^\mu d\nu e^{-M(\nu)}(1 - \nu^2) \frac{1}{D_{\nu}(\nu)},
\]

(297)

\[
= -\frac{v^2}{4} \int_{-1}^1 d\mu \mu_0 e^{M(\mu)} \int_{-1}^\mu d\nu e^{-M(\nu)}(1 - \nu^2) \frac{1}{D_{\nu}(\nu)}.
\]

The latter equation is identical to Equation (56) in the paper of Shalchi & Danos (2013).

From Equations (185) and (287), we find that the contribution of the terms containing \(\partial g/\partial z\) to the SPDC is equal to zero, i.e., ignoring the higher-order spatial derivative terms, which are used in the previous papers (Beeck & Wibberenz 1986; Litvinenko 2012b; He & Schlickeiser 2014). However, Wang & Qin (2018) showed that for the focusing field, the influence of the terms containing \(\partial g/\partial z\) on the SPDC cannot be ignored. Therefore, \(\kappa_{zz}^{\text{TGG}}\) only gives the approximate result in such condition. In order to prove this inference, in what follows we evaluate Equation (297) for the isotropic pitch-angle scattering model

\[
D_{\nu}(\mu) = D(1 - \mu^2)
\]

with the constant \(D\).
For this simple model, Equation (297) is easily simplified as

\[
\kappa_{zz}^{\text{TGK}} = \kappa_0^0(1 + S)
\]  

(298)

with

\[
S = -\frac{1}{15}\xi^2 + \frac{2}{315}\xi^4 + \ldots.
\]  

(299)

Here, \(\kappa_0^0\) is the SPDC for the constant mean magnetic field, and

\[
\xi = \frac{v}{2DL}
\]  

(300)

is the dimensionless parameter describing the focusing field with the adiabatic focusing characteristic length \(L\). Equations (298) and (299) show that the adiabatic focusing effect reduces the value of the SPDC regardless of the sign of the focusing length.

In fact, Equation (298) is identical to that derived by many researchers (Beeck & Wibberenz 1986; Bieber & Burger 1990; Kóta 2000; Litvinenko 2012a; Shalchi & Danos 2013; He & Schlickeiser 2014; hereafter, the result is denoted as \(\kappa_{zz}^{\text{BW}}\)); however, it is an approximate result (Wang & Qin 2018). In the above derivation, we find \(\kappa_{zz}^{\text{TGK}} = \kappa_{zz}^{\text{BW}}\); therefore, the TGK formula definition \(\kappa_{zz}^{\text{TGK}}\) is also an approximate result. Thus, we confirm the inference in the above part.

Considering the formula of \(m_{gz}\), (see Equation (48)) and Equation (270), we find that the TGK definition of the parallel diffusion \(\kappa_{zz}^{\text{TGK}}\) is determined by two factors, the initial condition

\[
\left[ \int_{-\infty}^{\infty} dz g(z) \right](t = 0) = 2\delta(\mu - \mu_0)
\]  

(301)

and the coefficients of the terms containing \(\partial g(z, \mu, t)/dt\) in the formula of \(g(z, \mu, t)\). The above factors are not influenced by the manipulations of the PzI, PtI, and PtzI operations, so the TGK definition \(\kappa_{zz}^{\text{TGK}}\) as well as the displacement variance definition \(\kappa_{zz}^{\text{DV}}\) is invariant for the PzI, PtI, and PtzI operations. However, for focusing field, the TGK definition is only an approximate result. Consequently, the displacement variance definition \(\kappa_{zz}^{\text{DV}}\) is more appropriate than the other ones.

### 8. Evaluating the Displacement Variance Definition

In this section, we evaluate the displacement variance definition through the formula

\[
\kappa_{zz}^{\text{VD}} = \kappa_{zz} - \kappa_{zz} \kappa_{\varepsilon}.
\]  

(302)

For the isotropic pitch-angle scattering model \(D_{\mu}(\mu) = D(1 - \mu^2)\) with the constant \(D\), the coefficient \(\kappa_{zz}\) becomes

\[
\kappa_{zz} \approx \frac{2\nu}{9D}\xi,
\]  

(303)

where the focusing parameter is \(\xi = v/(2DL)\). The detailed derivation is shown in Appendix A. The streaming coefficient \(\kappa_{\varepsilon}\) is also evaluated for the model \(D_{\mu}(\mu) = D(1 - \mu^2)\) as

\[
\kappa_{\varepsilon} \approx \frac{v}{3}\xi,
\]  

(304)

the evaluation process of which can be found in Appendix B.

Wang & Qin (2019) found the formula \(\kappa_{zz}^{\text{DV}} = \kappa_{zz} - \kappa_{\varepsilon}\kappa_{\varepsilon}\) from Equation (52), where \(\kappa_{zz}\) was obtained as

\[
\kappa_{zz} = \kappa_{zz}^{\text{BW}} + \eta_{0,2,0}
\]  

(305)

with

\[
\eta_{0,2,0} \approx \frac{1}{5}\xi^2\kappa_{0,0}.
\]  

(306)

Here, the quantity \(\kappa_{zz}^{\text{BW}}\) is shown as follows

\[
\kappa_{zz}^{\text{BW}} = \kappa_{0,0}(1 + S)
\]  

(307)

with

\[
S = -\frac{1}{15}\xi^2 + \frac{2}{315}\xi^4 + \ldots.
\]  

(308)
Equation (307) is an approximate formula of the SPDC (Beeck & Wirbenerz 1986; Litvinenko 2012b; Schalchi & Danos 2013; He & Schlickeiser 2014). Combining Equations (305)–(308), Wang & Qin (2018) found

\[ \kappa_{zz} \approx \kappa_{||}(1 + \frac{2}{15} \xi^2). \]  

(309)

Inserting Equations (303), (304), and (309) into Equation (302) yields

\[ \kappa_{zz}^{DV} = \kappa_{zz} - \kappa_z \kappa_{zz} \approx \kappa_{||}(1 - \frac{14}{45} \xi^2), \]  

(310)

which shows that the corrective action induced by the adiabatic focusing effect reduces the parallel diffusion coefficient regardless of the sign of the focusing parameter \( \xi \).

9. Summary and Conclusion

In previous years, much progress has been achieved in the theoretical description of energetic charged particle transport in a turbulent magnetic field superposed on a large-scale field. The SPDC is one of the key parameters for modeling particle transport and acceleration in the Galaxy and the solar system (Schlickeiser 2002; Schalchi 2009, 2020). In the past, people have found three different definitions of the SPDC, i.e., the displacement variance definition \( \kappa_{zz}^{i} = \lim_{r \to \infty} \langle \sigma^2 \rangle / (2d^2) \), the Fick’s Law definition \( \kappa_{zz}^{FL} = J/X \) with \( X = \partial F / \partial z \), and the TGK definition \( \kappa_{zz}^{TGK} = \int_0^\infty dt \langle (v(z) - v(0))^2 \rangle \). For a constant background magnetic field, the three different definitions of the SPDCs give the same result. However, some researchers (Danos et al. 2013; Litvinenko & Noble 2013; Schalchi & Danos 2013; Lasuiu et al. 2017) found that the displacement variance definition \( \kappa_{zz}^{DV} \) and the TGK definition \( \kappa_{zz}^{TGK} \) give different values for a spatially varying mean magnetic field and for which the Fick’s law definition \( \kappa_{zz}^{FL} \) is not equal to the displacement variance definition \( \kappa_{zz}^{DV} \) (Wang & Qin 2019). Thus, the three different definitions of the SPDC are not equal to one another for a focusing field.

In this paper, employing the iteration method, starting from the Fokker–Planck equation with the simple BGK collision operator, we obtain the EIDF which is different from the one derived through the Fourier expansion (Gombosi et al. 1993). In addition, with some DIOs, one can not only interconvert these EIDFs into each other, but also produce countless new EIDFs. Therefore, we get different equations to describe the same physical process. However, different EIDFs describing the same transport process should give the same SPDC. If one definition of the SPDC is invariant for the DIOs, it is also invariant for the different EIDFs; thus, it is an invariant quantity for different DMEs. Therefore, in the present paper we explore whether the EIDFs are invariant quantities for the DIOs.

Using the method of Wang & Qin (2018), through the DIOs belonging to the PzI, PtzI, and PtI operations, we obtain a limitless variety of EIDFs from the MFPE with the adiabatic focusing effect. The Fick’s law definition \( \kappa_{zz}^{i} \) is invariant with the DIOs of the PtzI and PtzI operations, but it is not with the PzI operation. The displacement variance definition \( \kappa_{zz}^{DV} \) is invariant not only for the PzI and PtzI operations, but also for the PtI operation at least under the special condition. The TGK definition \( \kappa_{zz}^{TGK} \) is the third kind of SPDC, which is invariant for the iteration operations. However, the TGK definition \( \kappa_{zz}^{TGK} \) ignores the effect of the higher-order spatial derivative terms, which actually have an influence on the parallel diffusion coefficient in a focusing field. Therefore, \( \kappa_{zz}^{TGK} \) only gives the approximate result in such condition. Consequently, the displacement variance definition \( \kappa_{zz}^{DV} \) is more appropriate than the others. Therefore, for data analysis and simulation, we should use \( \kappa_{zz}^{DV} \) rather than \( \kappa_{zz}^{FL} \) and \( \kappa_{zz}^{TGK} \) for a focusing field.

Wang & Qin (2018) derived the formula of \( \kappa_{zz} \) and evaluated \( \kappa_{zz} \approx \kappa_{||}(1 + 2\xi^2/15) \). In this paper, using the method of Wang & Qin (2018) we obtain \( \kappa_{zz} \approx 2\xi^2/(9D) \) and \( \kappa_{zz} \approx \xi^2/3 \) for the isotropic pitch-angle scattering model \( D_{ij}(\mu) = D(1 - \mu^2) \). Through the formula \( \kappa_{zz}^{DV} = \kappa_{zz} - \kappa_z \kappa_{zz} \), we find that the displacement variance definition \( \kappa_{zz}^{DV} \) is approximately equal to \( \kappa_{||}(1 - 14\xi^2/45) \), where the focusing parameter is \( \xi = 3\kappa_{||}/(vL) \). This result shows that the corrective factor \( 14\xi^2/45 \) induced by the adiabatic focusing effect reduces the parallel diffusion coefficient regardless of the sign of \( \xi \).

In this work, it is suggested that the displacement variance definition \( \kappa_{zz}^{DV} \) is invariant for the PzI and PtzI operations as well as for the PtI operation at least under the special condition. Here, the requirement of the special condition might not be necessary. In addition, the momentum diffusion with adiabatic focusing effect was also investigated.

Malkov & Sagdeev (2017; hereafter MS2015) used the Chapman and Enskog or multiple-timescale perturbation approach to deduce the EIDF. In MS2015, the starting point is the SFPE; however this paper is based on the MFPE. But because the SFPE and the MFPE are similar, it is possible that the form of EIDF derived from the SFPE through the multiple-timescale method in MS2015 and the form of the EIDF by employing the iteration method in this paper have certain similarity. Furthermore, although Equation (52) has infinite terms, it is easy to handle in this paper. Bian et al. (2017) obtained an integro-differential equation for the particle transport involving a nonlocal diffusive operator. But using an integro-differential equation similar to Bian et al. (2017) in this paper is too complicated. Maybe the infinite terms on the right-hand side of Equation (52) can be summed using some method, and a convenient EIDF with finite terms can be obtained. We may work on this task in the future.

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Appendix A

The Accurate Formula of \( \kappa_z \)

Equation (50) can be rewritten as

\[
\frac{\partial F}{\partial \tau} + \kappa_z^0 \frac{\partial F}{\partial \tau} = k_z^0 \frac{\partial^2 F}{\partial \tau^2} + (\alpha_1 + \alpha_2) \frac{\partial F}{\partial \tau} - \frac{\nu}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} dv e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \int_{-1}^{\nu} \frac{\partial^2 g}{\partial \tau^2} d\rho \\
+ \frac{\nu}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} dv e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \int_{-1}^{\nu} \frac{\partial^2 g}{\partial \tau^2} d\rho \\
- \frac{\nu^2}{4} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} dv e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \left(2 \int_{-1}^{\nu} d\mu \frac{\partial^2 g}{\partial \tau^2} - \int_{-1}^{1} d\mu \frac{\partial^2 g}{\partial \tau^2} \right) \\
+ \frac{\nu^2}{4} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} dv e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \left(2 \int_{-1}^{\nu} dv \frac{\partial^2 g}{\partial \tau^2} - \int_{-1}^{1} d\mu \frac{\partial^2 g}{\partial \tau^2} \right).
\]

(A1)

with

\[
k_z^0 = \nu \int_{-1}^{1} d\mu e^{M(\mu)} \\
k_z^0 = \nu \int_{-1}^{1} d\mu e^{M(\mu)} \\
\alpha_1 = -\frac{\nu}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} dv e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \nu \\
\alpha_2 = \frac{\nu}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} dv e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \nu
\]

(A2)

(A3)

(A4)

(A5)

It is obvious that in Equation (A1) there is no \( \partial^2 F / (\partial \tau \partial \tau) \) in the terms containing \( \partial^2 g / \partial \tau^2 \), but there might be \( \partial^2 F / (\partial \tau \partial \tau) \) in the terms containing \( \partial^2 g / (\partial \tau \partial \tau) \). Therefore, the correction to the coefficient of \( \partial^2 F / (\partial \tau \partial \tau) \) could only come from the fourth term,

\[
-\frac{\nu}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} dv e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \int_{-1}^{\nu} \frac{\partial^2 g}{\partial \tau \partial \tau} d\rho,
\]

(A6)

and the fifth term,

\[
+ \frac{\nu}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} dv e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \int_{-1}^{\nu} \frac{\partial^2 g}{\partial \tau \partial \tau} d\rho
\]

(A7)

on the right-hand side of Equation (A1). Operating with \( \partial^2 F / (\partial \tau \partial \tau) \) on Equation (48), we obtain

\[
\frac{\partial^2 g}{\partial \tau \partial \tau} = \left[L \frac{\partial^2 F}{\partial \tau^2} - \frac{\partial^2 F}{\partial \tau^2} \right] - \frac{2e^{M(\mu)}}{E_{\mu}(\mu)} \left[ \int_{-1}^{\mu} dv e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \left( \frac{\partial^2 F}{\partial \tau^2} + \int_{-1}^{\nu} \frac{\partial^2 g}{\partial \tau^2} d\rho \right) \right]
\]

(A8)
The latter equation can be rewritten as

\[
\frac{\partial^2 g}{\partial t \partial z} = \frac{\partial^2 F}{\partial t \partial z} \left[ \int_{-1}^{1} d\mu e^{M(\mu)} - 1 \right] + \frac{\partial^3 F}{\partial t^2 \partial z^2} \int_{-1}^{1} d\mu e^{M(\mu)} \left[ 1 - \frac{2e^{M(\mu)}}{D_{\nu}(\nu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} \left( \frac{\partial^3 F}{\partial \nu^2 \partial z^2} + \int_{-1}^{\mu} \frac{\partial^3 g}{\partial \nu^2 \partial z^2} d\nu \right) \right] + \frac{v}{2} \left[ 2 \int_{-1}^{\nu} d\mu e^{M(\mu)} \int_{-1}^{\nu} d\nu e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} \left( \frac{\partial^3 F}{\partial \nu^2 \partial z^2} + \int_{-1}^{\mu} \frac{\partial^3 g}{\partial \nu^2 \partial z^2} d\nu \right) \right] + \int_{-1}^{\nu} \frac{\partial^3 g}{\partial \nu^2 \partial z^2} d\nu \right].
\]  
(A9)

From Equation (A9), we find that the term containing \( \partial^2 F / (\partial t \partial z) \) is

\[
\frac{\partial^2 F}{\partial t \partial z} \left[ \int_{-1}^{1} d\mu e^{M(\mu)} - 1 \right],
\]  
(A10)

by substituting which for \( \partial^2 g / (\partial t \partial z) \) in Equations (A6) and (A7), we obtain \( \alpha_3 \partial^2 F / (\partial t \partial z) \) with

\[
\alpha_3 = -\frac{v}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\nu} d\nu e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} \left( \frac{\partial^3 F}{\partial \nu^2 \partial z^2} + \int_{-1}^{\mu} \frac{\partial^3 g}{\partial \nu^2 \partial z^2} d\nu \right) d\nu.
\]  
(A11)

and the fifth term \( \alpha_4 \partial^2 F / (\partial t \partial z) \) with

\[
\alpha_4 = \frac{v}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\nu} d\mu e^{M(\mu)} \int_{-1}^{\nu} d\nu e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} \left( \frac{\partial^3 F}{\partial \nu^2 \partial z^2} + \int_{-1}^{\mu} \frac{\partial^3 g}{\partial \nu^2 \partial z^2} d\nu \right) d\nu.
\]  
(A12)

Combining Equations (A1)–(A5) and expressions (A11)–(A12), we obtain the following formula:

\[
\kappa_{\mu} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \frac{v}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\nu} d\mu e^{M(\mu)} \int_{-1}^{\nu} d\nu e^{-M(\nu)} \frac{1}{D_{\nu}(\nu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} \left( \frac{\partial^3 F}{\partial \nu^2 \partial z^2} + \int_{-1}^{\mu} \frac{\partial^3 g}{\partial \nu^2 \partial z^2} d\nu \right) d\nu.
\]  
(A13)

In addition, we find that the coefficient of the convection term is

\[
\kappa_{\mu} = \kappa^0_{\mu} = \frac{v}{2} \int_{-1}^{1} d\mu e^{M(\mu)}.
\]  
(A14)

Similarly, by using the above method, the coefficient of the term \( \partial^3 F / (\partial t \partial z^2) \) can be obtained as

\[
\kappa_{\mu z} = \frac{v}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\nu} d\mu e^{M(\mu)} A - \frac{v}{2} \int_{-1}^{1} d\mu e^{M(\mu)} A.
\]  
(A15)

Here, the parameter \( A \) is

\[
A = \int_{-1}^{\nu} d\nu e^{-M(\nu)} \left( \int_{-1}^{\nu} d\rho B_1 + v \int_{-1}^{\nu} d\rho B_2 - \frac{v}{2} \int_{-1}^{1} d\mu B_1 \right).
\]  
(A16)
with

\[
B_1 = L \left( 1 - \frac{2e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \right) + \frac{\nu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \left[ \int_{-1}^{1} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} D_1 - J \int_{-1}^{1} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} \right]
- \frac{\nu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \int_{-1}^{1} d\mu e^{M(\mu)} \left[ \int_{-1}^{1} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} D_1 - J \int_{-1}^{1} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} \right].
\]

(A17)

\[
B_2 = e^{M(\mu)} \left[ \int_{-1}^{1} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} D_2 - \frac{1}{\int_{-1}^{1} d\mu e^{M(\mu)}} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{1} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} D_2 \right].
\]

(A18)

Here,

\[
J = \frac{\int_{-1}^{1} d\mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}}.
\]

(A19)

\[
D_1 = 2 \frac{\int_{-1}^{1} d\nu \nu e^{M(\nu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} - \frac{\nu^2 - 1}{2},
\]

(A20)

\[
D_2 = 2 \frac{\int_{-1}^{1} d\nu \nu e^{M(\nu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} - 1.
\]

(A21)

The other coefficients of Equation (52) can also be obtained through the same method. In general, the higher-order derivative terms have more complicated coefficients.

**Appendix B**

**Evaluating \(k_z\) for the Isotropic Pitch-angle Scattering**

For the isotropic pitch-angle scattering model \(D_{\mu\mu} = D(1 - \mu^2)\) with constant \(D\), He & Schlickeiser (2014) showed that Equation (43) can be simplified as

\[
M(\mu) = \xi(\mu + 1)
\]

(B1)

with

\[
\xi = \frac{\nu}{2DL}
\]

(B2)

By using Equation (B1) with Equation (300), we rewrite Equation (A13) as

\[
\kappa_z = \frac{\nu}{2} \frac{\int_{-1}^{1} d\mu e^{\xi_1}}{\int_{-1}^{1} d\mu e^{\xi_1}} V_1 - \frac{\nu}{2} V_2,
\]

(B3)

where

\[
V_1 = \int_{-1}^{1} d\mu e^{\xi_1} \int_{-1}^{\mu} d\nu W(\nu),
\]

(B4)

\[
V_2 = \int_{-1}^{1} d\mu e^{\xi_2} \int_{-1}^{\mu} d\nu W(\nu)
\]

(B5)

with

\[
W(\mu) = \frac{e^{-\xi_0}}{D_{\nu\nu}(\mu)} \left( \frac{\nu}{\int_{-1}^{1} d\mu e^{\xi_1}} - 1 \right).
\]

(B6)
Employing integration in parts for Equations (B4) and (B5), we obtain

\[
V_1 = \frac{e^\xi}{\xi} \int_{-1}^{1} d\mu W(\mu) - \frac{1}{\xi} \int_{-1}^{1} d\mu e^{\xi \mu} W(\mu),
\]

(B7)

\[
V_2 = \frac{e^\xi}{\xi} \int_{-1}^{1} d\mu W(\mu) - \frac{1}{\xi} \int_{-1}^{1} d\mu e^{\xi \mu} W(\mu) - \frac{V_1}{\xi},
\]

(B8)

\[
W(\mu) = \frac{e^{-\xi \mu}}{D_{\mu}(\mu)} \left( 2 e^{\xi \mu} - e^{\xi} - 1 \right).
\]

(B9)

From the formula

\[
\int_{-1}^{1} d\mu e^{M(\mu)} = \kappa_{\mu}(1 + S),
\]

(B10)

with

\[
S = -\frac{1}{15} \xi^2 + \frac{2}{315} \xi^4 + \ldots,
\]

(B11)

the following formula can be obtained:

\[
\frac{\int_{-1}^{1} d\mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} = \frac{\xi}{3}(1 + S).
\]

(B12)

Combining Equation (B3) with Equations (B7), (B8), and (B12) gives

\[
\kappa_{zz} = \frac{v}{2} \left[ \left( \frac{1}{3} + \frac{S}{3} + \frac{1}{\xi^2} - \frac{1}{\xi} \right) e^{\xi} \int_{-1}^{1} d\mu W(\mu) - \left( \frac{1}{3} + \frac{S}{3} + \frac{1}{\xi^2} \right) \int_{-1}^{1} d\mu e^{\xi \mu} W(\mu) + \frac{1}{\xi} \int_{-1}^{1} d\mu e^{\xi \mu} W(\mu) \right].
\]

(B13)

For \( \xi \ll 1 \), by employing the following formulas:

\[
e^{\xi} = 1 + \xi + \frac{1}{2} \xi^2 + \frac{1}{6} \xi^3 + \frac{1}{24} \xi^4 + \ldots,
\]

(B14)

\[
e^{\mu \xi} = 1 + \mu \xi + \frac{1}{2} (\mu \xi)^2 + \frac{1}{6} (\mu \xi)^3 + \frac{1}{24} (\mu \xi)^4 + \ldots,
\]

(B15)

we find that Equation (B13) becomes

\[
\kappa_{zz} \approx \frac{v}{2} \left[ \frac{1}{2} \int_{-1}^{1} d\mu (\mu^2 - 1) W(\mu) + \frac{\xi}{3} \int_{-1}^{1} d\mu (\mu^2 - 1) W(\mu) + \frac{\xi^2}{24} \int_{-1}^{1} d\mu (3 \mu^2 - 1) (\mu^2 - 1) W(\mu) + \frac{\xi^3}{96} \int_{-1}^{1} d\mu (3 \mu^2 - 2) (\mu^2 - 1) W(\mu) \right].
\]

(B16)

Inserting Equations (B14) and (B15) into Equation (B9), we obtain

\[
\kappa_{zz} \approx \frac{2v}{9D} \xi^2.
\]

(B17)

Similarly, we also obtain

\[
\kappa_{zz} \approx \frac{v}{3} \xi.
\]

(B18)

### Appendix C

**Determining the Sign of \( \kappa_{zz} \)**

In this paper, we only consider the very weak adiabatic focusing effect, for which the mean free path of the charged particles is much less than the characteristic length of the adiabatic focusing field, i.e., \( \xi = \lambda/L \ll 1 \) with \( \lambda = v/(2D) \). Therefore, the adiabatic focusing effect just has a very small correction function on the coefficients of Equation (52). Thus, if a certain coefficient is not equal zero, its sign cannot be changed by the very weak adiabatic focusing effect which is too weak. In order to judge the sign of one coefficient, one only needs to explore if it is negative or positive for the limit \( \xi \to 0 \). In this section, we only explore the sign of the coefficient \( \kappa_{zz} \) (see Equation (A15)).
For the isotropic model $D_{\mu
u} = D(1 - \mu^2)$ with the positive constant $D$ and the limit $\xi \to 0$, from Equation (B1) we obtain $M(\mu) \to 0$ and therewith the following results:

$$e^{M(\mu)} \to 1,$$

$$e^{-M(\mu)} \to 1.$$  \hfill (C1)

To proceed, using the latter relations, we find $J = 0$, $D_1 = 0$, and $D_2 = \mu$, inserting which into Equations (A16)–(A18), one obtains

$$A \to \int_{-1}^{1} d\nu \frac{1}{D_{\nu\nu}} \left( \int_{-1}^{1} d\rho B_1 + \nu \int_{-1}^{1} d\rho B_2 - \frac{v}{2} \int_{-1}^{1} d\mu B_2 \right),$$  \hfill (C3)

$$B_1 \to -\frac{v}{2D_\mu},$$  \hfill (C4)

$$B_2 \to \int_{-1}^{1} d\nu \frac{1}{D_{\nu\nu}} \nu - \frac{1}{2} \int_{-1}^{1} d\mu \frac{1 - \mu}{D_{\mu\mu}}.$$  \hfill (C5)

Combining Equations (A15) and (C3) yields

$$\kappa_{zz} \to -\frac{v}{2} \int_{-1}^{1} d\mu \frac{1}{D_{\nu\nu}} \left( \int_{-1}^{1} d\rho B_1 + \nu \int_{-1}^{1} d\rho B_2 - \frac{v}{2} \int_{-1}^{1} d\mu B_2 \right).$$  \hfill (C6)

Thereafter, using integrating by parts and employing the formula $D_{\mu\mu}(\mu) = D(1 - \mu^2)$, we obtain

$$\kappa_{zz} \to -\frac{v}{4D} \int_{-1}^{1} d\mu \left( \int_{-1}^{1} d\rho B_1 + \nu \int_{-1}^{1} d\nu B_2 - \frac{v}{2} \int_{-1}^{1} d\mu B_2 \right),$$  \hfill (C7)

considering which and Equations (C4) and (C5), we find

$$\kappa_{zz} \to -\frac{v}{4D} \int_{-1}^{1} d\mu \int_{-1}^{1} d\nu \left( \int_{-1}^{1} d\rho \frac{1}{D_{\rho\rho}} \nu - \frac{1}{2} \int_{-1}^{1} d\mu \frac{1 - \mu}{D_{\mu\mu}} \right).$$  \hfill (C8)

In order to simplify the latter equation, one can use integration by parts for the first terms to derive

$$\kappa_{zz} \to -\frac{v^2}{12D^2} - \frac{v^2}{4D} \int_{-1}^{1} d\mu \mu_\nu \left( \int_{-1}^{1} d\rho \frac{1}{D_{\rho\rho}} \nu - \frac{1}{2} \int_{-1}^{1} d\mu \frac{1 - \mu}{D_{\mu\mu}} \right)$$

$$+ \frac{v^2}{8D} \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu \mu_\mu \left( \int_{-1}^{1} d\nu \frac{1}{D_{\nu\nu}} - \frac{1}{2} \int_{-1}^{1} d\mu \frac{1 - \mu}{D_{\mu\mu}} \right).$$  \hfill (C9)

Integrating by parts again for the second term gives

$$\kappa_{zz} \to -\frac{v^2}{12D^2} - \frac{v^2}{12D} \int_{-1}^{1} d\mu \frac{1}{D_{\rho\rho}} \mu_\rho - \frac{v^2}{12D} \int_{-1}^{1} d\mu \frac{1 - \mu}{D_{\mu\mu}} \mu + \frac{v^2}{12D} \int_{-1}^{1} d\mu \frac{1 - \mu^2}{D_{\mu\mu}} \mu,$$  \hfill (C10)

therewith we obtain at last

$$\kappa_{zz} \to -\frac{v^2}{36D^2}.$$  \hfill (C11)

Because $v^2 > 0$ and $D^2 > 0$, the latter equation satisfies

$$\kappa_{zz} \to -\frac{v^2}{36D^2} < 0,$$  \hfill (C12)

from which we find that $\kappa_{zz}$ is negative for the limit $\xi \to 0$.

**Appendix D**

**The Formulas of $\kappa_{nn}$ with $n = 2, 3, 4, \cdots$**

Using the method in Wang & Qin (2018), we obtain the coefficients of the governing equation of $F(z, t)$, among which the formulas of $\kappa_{nn}$ with $n = 2, 3, \cdots$ is shown as

$$\kappa_{nn} = \frac{v}{2} \int_{-1}^{1} d\mu \epsilon^{M(\mu)} \int_{-1}^{1} d\rho \beta_{n-1}(\rho) - \frac{v}{2} \int_{-1}^{1} d\mu \epsilon^{M(\mu)} \int_{-1}^{1} d\rho \beta_{n-1}(\rho) + \frac{v}{2} \int_{-1}^{1} d\mu \epsilon^{M(\mu)} \int_{-1}^{1} d\rho \beta_{n-1}(\rho).$$  \hfill (D1)
Employing the evaluating method of Wang & Qin (2018), we find

\[ \kappa_{n+1} = -\frac{13}{108} \frac{\nu}{D^2} \xi, \]

\[ \kappa_{3n} = \frac{5}{81} \frac{\nu}{D^3} \xi, \]

\[ \kappa_{4n} = -\frac{121}{3888} \frac{\nu}{D^4} \xi, \]

\[ \kappa_{5n} = \frac{91}{5832} \frac{\nu}{D^5} \xi, \]

\[ \kappa_{6n} = -\frac{139968}{1093} \frac{\nu}{D^6} \xi, \]

\[ \kappa_{7n} = \frac{205}{52488} \frac{\nu}{D^7} \xi, \]

\[ \cdots \cdots \]

from which we obtain

\[ \kappa_{3n} D = -0.5125205128, \]

\[ \kappa_{4n} D = -0.504166667, \]

\[ \kappa_{5n} D = -0.5013774105, \]

\[ \kappa_{6n} D = -0.5004578755, \]

\[ \kappa_{7n} D = -0.5001524855, \cdots \cdots \]

Thus, we find the following formula:

\[ \lim_{n \to \infty} \frac{\kappa_{n+1}}{\kappa_{n+1}} D = -0.5, \]

and thereafter, the following formula can be obtained:

\[ \kappa_{n+1} \approx \left( \frac{1}{2D} \right)^{n-2} \kappa_{2n}. \]
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