The zero temperature relaxational dynamics of a particle in a short range correlated random potential is addressed. We derive a set of “two-times” mean-field dynamical equations, accounting for a possible mean displacement of the particle when subject to an external force. We show first detailed results from the numerical integration of the above mention ed equations. We mainly pay attention to the exponentially decreasing spatial correlations case, for which simple analytical arguments provide valuable results about the hessian (or the “instantaneous normal modes” structure) of the energy landscape, and we propose a geometrical description of the “mean-field aging”. Our numerical results and further analytical arguments give access to the waiting-time dependence of the main characteristic time scales.

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I. INTRODUCTION

Understanding of the out-of-equilibrium dynamics of glassy systems, including spin glasses, structural glasses, supraconductor vortex glasses, etc, is a challenging problem. The need for exact, but non trivial results led to the introduction of ideal spin-glasses, like the celebrated Sherrington-Kirkpatrick model \[1\]. The spherical \(p\)-spin glass is such a model, where a closed set of equations exists for the time correlation functions of the dynamical variables, or soft spins \[2,3\]. In the thermodynamical limit, each spin becomes coupled to an infinity of other spins. Each variable experiences gaussian fluctuations in the effective environment created by all the other spins. The dynamics then simplifies drastically, and reduces to a set of correlation functions which have to be determined self-consistently. The result is the “dynamical mean-field solution” of the model.

The mean-field dynamics of spin-glasses has revealed extremely rich, the most striking feature being the existence of a non-trivial aging relaxation regime at low temperature \[4\]. For instance, the solution of the mean-field equations in this out-of-equilibrium regime demonstrate the existence of a generalised fluctuation-dissipation theorem (i.e. connecting correlation and response functions) whose validity seems now to extend to many realistic, non mean-field, models \[5\].

Then, mean-field solutions are valuable for explaining the experimental aging of disordered magnetic systems \[6\]. Finally, a sustained interest has followed the discovery of a deep formal analogy between the mode coupling description of structural glasses (supercooled liquids) and the mean field treatment of the spherical \(p\)-spin glass \[7\].

A crucial shortcoming of the mean-field description, however, is its inability to take into account properly thermally activated motion over energy barriers, leading to a sharp dynamical transition – divergence of an internal relaxational time scale – whereas the corresponding “finite dimensional” behaviour is only a strong but progressive slowing down of the dynamics.

Despite of this last point, mean-field dynamics remains a major issue in the study of out-of-equilibrium statistical physics of disordered systems, and any approach providing a physical insight on its aging mechanism is of interest. A major step in that direction was made by J.Kurchan and L.Laloux \[8\] who investigated the zero temperature relaxation of systems including ferromagnets and spin glasses. The zero temperature limit makes it possible to consider the energy landscape, rather than a ill-defined “free-energy” landscape, without reducing the dynamics to anything trivial.

In this work, we extend further their approach, and apply it to another system of interest: the mean-field dynamics of a particle in a short-range correlated random potential.

The out-of-equilibrium, aging dynamics of this model has been first studied in \[9\], and thoroughly investigated in \[10\]. Its glassy behaviour belongs to the same universality class than the spherical \(p\)-spin model. What makes this model interesting is its natural extension, when a finite and constant external force is applied to the particle. Then, it becomes a paradigm of “driven glassy system”, in which a non-linear response to the force as well as a significant violation of the fluctuation-dissipation theorem are expected, as shown by Horner \[11\].
In this paper, we present the dynamical mean-field equations in the presence of a constant force, allowing for an arbitrary mean displacement along this one. These equations are then numerically solved, in the zero temperature limit, for an exponentially decreasing correlator, in the absence, and in the presence of a weak external force. The corresponding numerical results are presented, restricting ourselves to the linear response regime. Then, we start our geometrical analysis of the zero temperature relaxation by a simple random matrix calculation that we believe to describe satisfactorily the hessian of the exponentially correlated gaussian potential. Next, we perform an “instantaneous normal mode” analysis of the relaxational motion. The key observable turns out to be the (intensive) energy difference between the energy $E(t)$ at a given time $t$, and its asymptotic value $\lim_{t \to \infty} E(t)$. We subsequently analyse the waiting time dependence of two characteristic time scales $t_1, t_2$, that we relate to $E(t) - E(\infty)$.

This work is preliminary to the study of the stationary driven situation in the presence of a finite force, which will be the subject of a forthcoming publication, and where the velocity-force characteristics, and the cross-over between linear and non-linear response will be exposed $[27]$.

II. THE OUT-OF-EQUILIBRIUM DYNAMICS IN THE MEAN-FIELD APPROXIMATION

We introduce in this section the mean-field dynamics equations and discuss the low temperature aging solution in the absence of force. Let $x(t)$ be the position of a particle, obeying a usual Langevin equation:

$$\dot{x}(t) = -\nabla V(x(t)) + F + \zeta(t), \quad (2.1)$$

where are introduced the random potential $V(x)$, the external force $F$, a white Langevin noise $\zeta(t)$ corresponding to a temperature $T$, and a friction coefficient equal to 1. Quantities $x, \nabla, F, \zeta$ are $N$-dimensional vectors. Three sub-cases of the dynamics defined by $\{[2.1]\}$ are of interest: 1/ the “isolated” dynamics, without force: $F = 0$; 2/ the driven relaxational dynamics, which is the zero temperature limit of $\{[2.1]\}$: $F \neq 0$ and $T \to 0$; 3/ the relaxational “isolated” dynamics: $F = 0, T \to 0$.

The potential $V(x)$ is a quenched disorder, chosen from a gaussian distribution. All the averages with respect to it will be denoted with an over-line $\overline{\cdot}$, while the average over the thermal noise (if any) $\zeta$ will be denoted by the brackets $\langle \cdot \rangle$. We suppose that the motion starts at $t = 0$ and $x(t=0) = 0$. After averaging over the quenched disorder, this choice becomes equivalent to start with a random, “infinite temperature” distribution of initial positions. We expect that the process $\{[2.1]\}$ is self-averaging with respect to $V(x)$ in the infinite dimensional limit. One introduces the correlator $f(y)$ of the gaussian disorder, explicitly dependent on the dimension $N$ of the configuration space $\{x\}$.

$$\overline{V(x) \cdot V(x')} = N \cdot f \left( \frac{\|x - x'\|^2}{N} \right); \overline{V(x)} = 0. \quad (2.2)$$

This form ensures a meaningful $N \to \infty$ limit, in which each coordinate $x_i(t)$, or gradient component $\partial_i V(x)$, remains of order 1, while the norms $\|x(t)\|$, $\|\nabla V\|$ scale like $N^{1/2}$. As a consequence, the external force must scale ($e_1$ being a unit vector) like:

$$F = N^{1/2} \cdot F \cdot e_1. \quad (2.3)$$

One expects a displacement $\langle x(t) \rangle = N^{1/2} \cdot u(t) \cdot e_1$, and possibly a mean velocity $\langle dx(t)/dt \rangle = N^{1/2} \cdot v \cdot e_1$. From now onwards, we arrange that $e_1$ coincides with the first coordinate axis $i = 1$.

In the present paper, we restrict ourselves to the exponentially correlated potential:

$$f(y) = \exp(-y). \quad (2.4)$$

This is a special case of short range correlated random potential, characterised by $\lim_{y \to \infty} f(y) < \infty$. The average difference $\overline{\|V(x) - V(x')\|^2}$ is bounded when $\|x - x'\|$ grows, and this ensures the existence of a normal diffusion regime at temperatures high enough. Another common choice is the power-law correlator: $f(y) = 2/(\gamma - 1) \cdot (1 + y)^{(1 - \gamma)/2}$; $\gamma > 1$ $[8, 11]$. Choice $\{[2.4]\}$ is also a particular case of $f(y) = U_p^2 \exp(-y/\xi^2)$, with a pinning energy $U_p$ and correlation length $\xi$ set to 1, thanks to a simple rescaling, without loss of generality.

The Langevin dynamics is handled with the help of a Martin-Siggia-Rose (MSR)-like functional integral, convenient for averaging over the gaussian disorder $[12, 13]$. The crucial point is that the limit $N \to \infty$ is taken first, before any other limit $T \to 0$ or $t \to \infty$. As a result, we obtain a general effective quadratic action $S[x_j(t), i\dot{x}_j(t)]$, involving the original field $x_j(t)$, and the MSR auxiliary field $i\dot{x}_j(t)$. Three among the four following correlation functions appear explicitly in the action $S[x_j(t), i\dot{x}_j(t)]:$
\[ u(t) = N^{-1/2} \langle x_1(t) \rangle; \]  
(2.5)

\[ r(t, t') = N^{-1} \sum_{j=1}^{N} (x_j(t) \cdot i\overline{x}_j(t')); \]  
(2.6)

\[ b(t, t') = N^{-1} \sum_{j=2}^{N} \langle (x_j(t) - x_j(t'))^2 \rangle; \]  
(2.7)

\[ d(t, t') = N^{-1} \sum_{j=1}^{N} \langle (x_j(t) - x_j(t'))^2 \rangle; \]  
\[ = b(t, t') + [u(t) - u(t')]^2. \]  
(2.8)

These are the displacement \( u(t) \), the response function \( r(t, t') \), and the correlation functions \( b(t, t') \) and \( d(t, t') \). The Dyson equations for \( r, b, d, u \) form a closed system of coupled integro-differential equations. For \( t \geq t' \) one has to solve:

\[ \partial_t r(t, t') = \delta(t - t') \]
\[ -\int_{0}^{t} ds \ 4f''(d(t, s)) \ r(t, s) \ [r(t, t') - r(s, t')]; \]  
(2.9)

\[ \partial_t b(t, t') = 2T - \int_{0}^{t} ds \ 4f'(d(t, s)) \ [r(t, s) - r(t', s)] \]
\[ -\int_{0}^{t} ds \ 4f''(d(t, s)) \ r(t, s) \ [b(t, s) + b(t, t') - b(s, t')]; \]  
(2.10)

\[ \partial_t u(t) = F - \int_{0}^{t} ds \ 4f''(d(t, s)) \ r(t, s) \cdot [u(t) - u(s)]. \]  
(2.11)

Equations (2.9-2.11) are original ones, and allow for a non uniform displacement \( u(t) \). The aging, isolated, situation corresponds to the limit \( u(t) = F = 0 \), and \( d(t, t') = b(t, t') \) in the above system. The stationary limit, investigated by Horner [11] amounts to write \( r(t, t') = R(t - t') \), \( b(t, t') = B(t - t') \), \( u(t) = v \cdot t \), and to reject the lower bound of the time integrals \( \int dt \int ds \) to \(-\infty \).

Three relevant observables: the energy \( \mathcal{E}(t) \), the curvature \( \mathcal{M}(t) \) and the pinning force \( F_p(t) \) can be expressed with the help of these correlation functions.

\[ \mathcal{E}(t) = N^{-1} \langle V(\mathbf{x}(t)) \rangle, \]
\[ = \int_{0}^{t} ds \ 2f'(d(t, s)) \ r(t, s); \]  
(2.12)

\[ \mathcal{M}(t) = N^{-1} \sum_{j=1}^{N} \langle \partial^2_{jj} V(\mathbf{x}(t)) \rangle, \]
\[ = \int_{0}^{t} ds \ 4f''(d(t, s)) \ r(t, s); \]  
(2.13)

\[ F_p(t) = N^{-1/2} \langle -\partial_t V(\mathbf{x}(t)) \rangle, \]
\[ = -\int_{0}^{t} ds \ 4f''(d(t, s)) \ r(t, s) \ [u(t) - u(s)]. \]  
(2.14)

The pinning force is such that \( \langle d\mathbf{x}(t)/dt \rangle = F_p(t) + F \) with \( F_p(t) = N^{1/2} : F_p(t) : \mathbf{e}_1 \). The pinning force \( F_p \) and the driving force \( F \) have opposite signs.

A proper study of the mean-field equilibrium phase diagram requires an extra quadratic confinement potential \( \mu \mathbf{x}^2 / 2 \). This ensures the existence of a true thermal equilibrium in the high temperature phase, while correlation functions reach their asymptotic values exponentially fast. Then, a transition line \( T_d(\mu) \), called dynamical temperature, separates the high temperature ergodic phase, from a low temperature, aging and non ergodic phase [12,13].

At high temperature, the system reaches a true stationary state, and the dynamics becomes time-translationally invariant (TTI), i.e the 2-times correlation functions depend only on the difference \( t - t' \), while the 1-time expectation values are constant. In this stationary situation, it is convenient to introduce the TTI correlation functions \( B(t - t') = \ldots \).
This implies that models can be solved thanks to the ansatz \([10,20]\):

\[
\lim_{t,t' \to \infty} b(t,t')|_{t-t' \text{ finite}} \Rightarrow R(t-t') = \lim_{t,t' \to \infty} r(t,t')|_{t-t' \text{ finite}}.
\]

The fluctuation-dissipation theorem (FDT) holds and reads:

\[
\frac{dB(t)}{dt} = 2T \cdot R(t).
\]  

(2.15)

As a consequence, the equal-time correlation functions coincide with their thermodynamical (canonical ensemble) counterparts. Taking the limit \(\mu \to 0\) does not lead to any singular result \([1]\). Provided the contribution from the harmonic potential has been subtracted off, the energy \(\mathcal{E}(t)\) behaves smoothly as \(\mu\) tends to 0. When \(\mu\) exactly equals 0, the system cannot be at equilibrium, and instead, one expects a long time behaviour corresponding to a normal diffusion situation, with a finite diffusivity \(D = \lim_{t \to \infty} \langle x^2(t) \rangle / (2Nt)\), a finite mobility \(\eta^{-1} = \lim_{t \to \infty} u(t)/(Ft)\), and the Einstein relation \(D = T \cdot \eta^{-1}\).

Kinzelbach and Horner described the dynamics in the stationary, high temperature phase \([16]\). They found that these correlation functions behave in the same way than those of the well known mode-coupling theories for supercooled liquids, as expected on general grounds \([7,8]\). The non-linearities of the self-consistent equations cause a dramatic slowing down of the dynamics as \(T_d\) is approached from above, leading to a sharp transition at \(T = T_d\).

For instance, the function \(B(t)\), after a fast increase at short times \(t \sim 1\), has a long plateau near a characteristic value \(B(t \sim t_f) \approx q\), before eventually reaching its asymptotic, long time regime \(B(t) = B(t/t_b)\). Both \(t_f\) and \(t_b\) diverge like power laws of the difference \(|T-T_d|\) \([10]\).

The low temperature region however corresponds to an out-of-equilibrium situation. In the absence of external force, this is meant by the loss of both time-translational invariance (TTI) and fluctuation-dissipation theorem (FDT). The 2-time correlation functions cannot be reduced any more to functions of the time differences \(t-t'\), and there is a domain in the \((t,t')\) plane, where the system ages \([8–10]\).

The addition of a weak, constant external force leads to a somewhat different picture. As proposed by Horner, the system is expected to reach a stationary state (TTI), but the FDT remains definitively lost \([1]\). It turns out, however (cf next section), that when the force is switched on at a time \(t = 0\), there is a finite time interval during which the dynamics can be successfully described as a perturbation around the aging isolated \((F=0)\) regime, with a linear response approach. The extent of this linear response regime is inversely related to the magnitude of the force.

The aging dynamics of the isolated particle has been exhaustively treated in \([10]\). The fluctuation-dissipation theorem is violated and must be replaced by:

\[
X(t,t') \propto \partial_t b(t,t') = r(t,t').
\]  

(2.16)

In the time sector \(t-t'\) finite; \(t' \to \infty\) of the \((t,t')\) plane, the behaviour is very similar to the one observed just above \(T_d\), and the value of \(X(t,t')\) is very close to its equilibrium value \(-1/(2T)\). When the time separation \(t-t'\) ceases to be small relatively to a characteristic time \(t_f(t')\) which has to be determined, \(X(t,t')\) departs from its equilibrium value, decreasing its magnitude \(|X|\).

The analytical study of the equations \([8]\) has only been possible in the asymptotic limit \(t,t' \to \infty\), by dropping out sub-leading terms presumably of order \(1/t, 1/t'\). In this limit, the authors have shown (this is the crucial point) that it was possible to parametrise the dynamics with the help of the correlation function \(b(t,t')\) of the system itself. This implies that \(X(t,t')\) becomes a one variable function \(X[b(t,t')]\), and it turns out that all short range correlated models can be solved thanks to the ansatz \([10,20]\):

\[
\begin{align*}
\quad b(t,t') < q \Rightarrow X[b] = -1/2T; \\
\quad b(t,t') > q \Rightarrow X[b] = \chi.
\end{align*}
\]  

(2.17)

This extension of the FDT is called “quasi-fluctuation-dissipation theorem” (QFDT). In this paper, we rather use the function \(\mathcal{T}(t,t') = -1/(2X(t,t'))\). In the aging regime, this effective temperature \(\mathcal{T} = -1/(2\chi)\) is higher than the thermostat temperature \(T\), and remains finite in the limit \(T \to 0\). The physical meaning of these or any other temperatures systems” is discussed in \([21]\).

In the case we are interested in, i.e. in the absence of confinement \((\mu \to 0)\), for a correlator \(f(y) = \exp(-y)\), \(\chi\) and \(q\) are for any temperature \(T < T_d\), solutions of the system \([10]\):

\[
\begin{align*}
\quad T &= q \sqrt{\mathcal{T}''(q)} = q \; e^{-q/2}; \\
\quad \chi &= \frac{\sqrt{\mathcal{T}''(q)}}{2\mathcal{T}'(q)} = -e^{q/2}/2;
\end{align*}
\]  

(2.18)

and in the low temperature limit:

\[
\begin{align*}
\quad q &\sim T + T^2/2 \ldots; \\
\quad \chi &\sim - \left(\frac{1}{2} + \frac{T}{4} + \frac{T^2}{8} \ldots\right)
\end{align*}
\]  

(2.19)
In the same way, given a triplet \( t_1 < t_2 < t_3 \), in the time domain where \( (t_1, t_2, t_3) \to \infty \) and \( t_1/t_2, t_2/t_3 \) finite, \( b(t_3, t_2) \) is uniquely determined by the knowledge of \( b(t_2, t_1) \) and \( b(t_3, t_1) \). Again, the explicit dependence can be carried out exactly when the correlator is exponential. The result is [10]:

\[
b(t_3, t_2) - q = b(t_3, t_1) - q - [b(t_2, t_1) - q].
\]  

(2.20)

A well known shortcoming of this approach, is that any reference to the original times \( t, t' \) is definitively lost. The asymptotic solution cannot distinguish between \( b(t, t') \) and \( b(h(t), h(t')) \) where \( t \mapsto h(t) \) can be any suitable reparametrization of the time variable. As a by-product, the previous analysis predicts only the more general form of the solution, in the aging regime \( t/t' \sim 1 \):

\[
b(t, t') = \tilde{B} \left[ \ln \left( \frac{h(t)}{h(t')} \right) \right] + q.
\]  

(2.21)

For exponentially correlated potentials, the master function is known [10], and without loss of generality:

\[
b(t, t') = \ln(h(t)) - \ln(h(t')) + q.
\]  

(2.22)

In [10] is made the conjecture \( h(t) = t^\delta \), compatible with the results found below. In what follows, we will refer to this solution as the time-reparametrization invariant (TRI) solution.

At the beginning of the aging regime, for \( t \) and \( t' \) such that \( (t - t')/t' \) is finite but small compared to 1, the scaling form (2.21), reads:

\[
b(t, t') = \tilde{B} \left( (t - t') \cdot t_0^{-1}(t') + \ldots \right) + q.
\]  

(2.23)

Here, \( t_0(t') \) is the characteristic time of the aging regime, defined by \( t_0(t') = h(t')/h'(t') \). This is the typical time needed by the particle for diffusing over a distance \( b(t, t') - q \sim 1 \). Non exponential correlators have a non analytic scaling function \( \tilde{B}(m) \) around \( m = 0 \) and the r.h.s of (2.23) is singular in \( t - t' \) [10,20].

The time-reparametrization invariant solution describes a situation where the time scales for the FDT regime \( (t_0 = 1) \) and for the aging regime \( (t_0(t')) \) are well separated \( i.e. \ t_0(t') \gg 1 \), which implicitly assumes \( t, t' \to \infty \). In order to go further, one needs to take into account the times derivatives \( \partial_t, \partial_{t'} \) neglected in the asymptotic regime of the TRI solution. It is enough, in principle, to fix up the reparametrization function \( h(t) \). Moreover, the TRI solution does not say how the parameter \( T \) goes from its FDT value \( b(t, t') < q \) to its QFDT value \( b(t, t') > q \). One defines for this purpose the new time scale \( t_f(t) \), such that, for instance, \( T(t, t - t_f(t)) \) takes a given value between \( T \) and \( -1/(2\chi) \). We shall see below that \( t_f(t) \) is much smaller than \( t_0(t) \).

III. RESULTS FROM THE NUMERICAL INTEGRATION OF MEAN-FIELD EQUATIONS

The mean-field equations, with \( F \) and \( u(t) \) equal to zero were first numerically integrated by Franz and Mézard [13]. The quadrature scheme is of order one in the time step \( h \), but reveals itself surprisingly robust as \( h \) is increased up to value as large as 0.3. The authors of [13] report being able to reach \( t \to 1000 \) at the best. Our investigations have shown that the quality of our solutions gets worst if \( h \) is increased above 0.2, and we present results up to \( t \sim 400 \).

For reasons detailed in the next section, we have only considered the exponential correlator case (2.4). We set \( T \) to 0 in (2.8) and took the initial value \( C(0,0) = 0 \). The information coming from the numerics may be pigeonholed in three categories.

1/ Results related to the TRI solution. First of all, we must check that the quasi-fluctuation dissipation relation (2.17) is true by plotting the integrated response versus the correlation function, on Figure (I). The observed value of \( \chi \) is close to 0.46, while the predicted value is 1/2. The TRI solution predicts also \( q \approx 0 \) and \( \lim_{t \to \infty} b(t, 0) = \infty \), in the absence of confinement. The measured asymptotic energy \( E(\infty) \) and mean curvature \( M(\infty) \) are found to be in excellent agreement with the predicted values \(-2 \) and \(+4 \) respectively.

2/ Beyond the TRI solution, without external force. This includes for instance the algebraic decay of the energy \( E(t) = -2 + c_1 \cdot t^{-\kappa} \). The exponent \( \kappa \) is determined by plotting \( \log(2 + E(t)) \) versus \( \log(t) \), and also by computing directly the logarithmic derivative, as shown on Figure (2). The exponent \( \kappa \) lies between 0.66 and 0.67 and our best estimate is \( c_1 = 1.08 \).

Also concerned are the characteristic times of the aging regime, and the precise nature of the cross-over from equilibrium to quasi-equilibrium fluctuation dissipation theorem. We are interested here in finding the characteristic time \( t_f(t) \) as a function of \( t \), defined by:

\[5\]
\[ \int_{t-t_f}^{t} ds \ 2f'(b(t,s)) \ r(t,s) = -1, \]  
\[ (3.1) \]

Equation (3.1) comes from the fact that the equilibrium \( X = -1/2T \) and aging \( X = \chi \) time sectors contribute for \(-1\) each to the energy. The value \( t_f \) which solves (3.1) separates the equilibrium regime \( (b(t,s) < b(t, t - t_f)) \) from the aging one \( (b(t,s) > b(t, t - t_f)) \). The equivalence between (3.1) and (3.2) is a straightforward consequence of (2.17).

One generalises (3.2) in:
\[ \int_{t-t_a}^{t} ds \ r(t,s) = a, \]  
\[ (3.3) \]

For \( a < 1/2 \), \( t_a \) must tend to a constant as \( t \to \infty \), while for \( a > 1/2 \), the asymptotic scaling (2.21) predicts:
\[ a - 1/2 = -\chi \cdot \beta \left[ \ln \left( \frac{h(t)}{h(t-t_a)} \right) \right]; \]  
\[ \Rightarrow -\chi \cdot \beta (t_a/t_b), \]  
\[ (3.4) \]
where terms \((t_a/t_b)^2\) have been neglected in the last expression, and \( t_b = h(t)/h'(t) \). If \( a \) is small enough, \( t_a \) is simply proportional to \( t_b \). Moreover, if \( h(t) \) is indeed \( t^\delta \), then \( a - 1/2 = -\chi \cdot \beta \left( 1 - t_a/t_b \right) \), and \( t_a/t_b \) is strictly constant. Our Figure (4) shows a plot of \( t_f, t_{a=0.55}, \) and \( t_{a=0.45} \). The characteristic time scale \( t_f \) tends asymptotically towards a power law \( c_2 t^a \), with \( c_2 \approx 0.51 \) and \( \alpha \approx 0.64 \) (according to our best estimate).

The correlation function is found to grow logarithmically with \( t \), and \( f(b(t,t')) = \exp(-b(t,t')) \) behaves as a power law of \( t \). Figure (5) presents \( \exp(-b(t,0)) \) and \( \exp(-b(t,t')) \) for a fixed \( t' \). An algebraic decay \( t^{-\delta} \) of \( \exp(-b(t,0)) \) is likely, while \( \exp(-b(t,t')) \) has not yet reached its asymptotic regime, but could tend to the same \( t^{-\delta} \) behaviour.

From the asymptotic form (2.21, 2.22) we note that \( \exp(-b(t,t')) \approx h(t')/h(t) ; \) \( t, t' \to \infty ; \) \( t' \) finite, and our Figure is consistent with \( h(t) = t^\delta , \delta \approx 1.10 \). Also shown is \( \exp(-b(t,t-t_f)) = t^{-\gamma} , \gamma \approx 0.42 \). Expanding \( \exp(-b(t,t_f)) \) as \((t/t_f)^{-\delta} \), and using \( t_f \sim t^\alpha \), one finds a relation \( \gamma = \delta(1 - \alpha) \) between exponents. The agreement between \( \delta(1 - \alpha) \approx 0.39 \) and \( \gamma \approx 0.42 \) is acceptable.

3/ The linear displacement regime, in the presence of a driving force. A small force \( F \) is applied and the displacement \( u(t) \) monitored. The linear response implies that \( u(t) \) must be proportional to \( F \), and it is indeed the case for time intervals not too large. Figure (5) presents \( u(t)/F \) for decreasing values of \( F \). The curve \( F = 0.05 \) is virtually indistinguishable from the integrated response \( R(t) = \int_0^t r(t,s) ds \), and this shows that \( \lim_{F \to 0} u_F(t)/F = R(t) \), i.e. the expected linear response behaviour. The other curves depart from the integrated response after a time \( t_f \) decreasing with \( F \). When starting from the isolated and aging situation \( F = 0 \), the linear response only holds during a finite time interval \( 0 < t \leq t_F \). What happens later is the onset of a stationary state, with a well defined velocity \( v \) and a non-linear dependence in the force as advocated by Horner [1]. A study of this regime is to be published soon [2].

In the linear response regime, Figure (6) is compatible with:
\[ u(t) = F \cdot (c_3 + c_4 \cdot \ln(t)), \]  
\[ (3.5) \]
\( c_3 \approx 0.71 \) and \( c_4 \approx 0.54 \).

IV. THE GEOMETRICAL APPROACH

In this section, we transpose to the particle in a random potential some of the ideas which have revealed fruitful when applied to the spherical \( p \)-spin model, namely the geometrical analysis of Kurchan and Laloux [7]. We expose first the main concepts of the method, and then propose a method for computing the limit value of the dynamical energy from generic properties of the potential \( V(x) \), working only for an exponential correlator \( f(y) = \exp(-y) \).

The \( p \)-spin model starts aging below a dynamical temperature \( T_d \) [2], and encounters a thermodynamical glassy transition at \( T_a < T_d \) [23]. Detailed investigations have brought an appealing picture of the complex free-energy landscape of the spherical \( p \)-spin model, accounting for many features of its thermodynamics [23, 24] and its dynamics [3, 24].
The phase space of the $p$-spin model can be investigated with the help of a “Thouless-Anderson-Palmer” free-energy $\Phi(m_i)$ of the magnetization $m_i, i = 1 \ldots N$. At low enough temperatures, the function $\Phi$ develops many extrema $m_i^{(\alpha)}$, the TAP solutions $\alpha$. Those extrema which are minima, i.e. the second derivative matrix $\partial^2 \Phi / \partial m_i \partial m_j$ is definite positive, are metastable states, as they are separated from each others by extensive free-energy barriers. A particular realization of the system, prepared in a given metastable state $\alpha$ remains for ever in this state in the thermodynamic limit.

The stability of a metastable state is related to the lowest eigenvalue $\lambda_{\min}$ of $\text{Spec}(\partial^2 \Phi / \partial m_i \partial m_j)$, spectrum of the hessian matrix. $\lambda_{\min}$ turns out to be a monotonically decreasing function of the free energy $\Phi(m_i^{(\alpha)})$ of the state itself. This defines the free-energy $\Phi_d$ of the marginal states as $\lambda_{\min} = 0$ for $\Phi = \Phi_d$. Magnetizations such that $\Phi(m_i) > \Phi_d$ represent regions of negative curvature which does not contribute to the thermodynamics but play a role in the dynamics [4].

The glassy dynamics of the $p$-spin model is observed when the stable metastable states $\Phi^{(\alpha)} < \Phi_d$ are populated, or equivalently, when the canonical Boltzmann measure is split into its metastable components $\alpha$. At a temperature lower than $T_d$, thermal equilibration requires the system to explore all the relevant metastable states $\Phi < \Phi_d$. Such an equilibration is impossible as going from one metastable state to the next one requires to go over an infinite barrier. What happens instead to a system quenched from high temperature, to $T < T_d$, is the onset of aging. The systems wander more and more slowly around the magnetization region $\Phi(m_i) \simeq \Phi_d$, i.e. around the marginal states.

The zero temperature relaxation dynamics is simpler because the free energy reduces to the hamiltonian $H$ of the spins $s_i$. At variance with the finite temperature case, the regions with negative curvature of $H$ are now well defined. Taking the limit $T \to 0$ in the mean-field equations does not lead to any singular behaviour. This somewhat counter-intuitive property is the consequence of sending $N \to \infty$ first, by keeping finite the times $t$ and $t'$. The dynamics is a pure gradient descent, but remains non trivial.

In order to perform a geometrical analysis of this relaxation dynamics, it is necessary to keep the dimension $N$ large but finite. Then, the relaxation process occurs till the particle falls into a true minimum of the Hamiltonian $H(s_i)$, and, at $T = 0$, remains stuck there indefinitely. According to the description advocated in [2], a system starting from a random configuration $\{s_i(0)\}$ will explore regions with smaller and smaller gradient $\|\nabla H(s_i)\|$, and a decreasing number of negative eigenvalues in the spectrum of $\mathcal{H}_{ij}$, hessian of $H(s_i)$. The typical time $t_{I \alpha}$ for reaching regions where $\mathcal{H}_{ij}$ has only $I$ negative eigenvalues, diverges as $N$ is sent to $\infty$ by keeping $I$ finite [3]. As a consequence, in the $N \to \infty$ limit, the system is unable to reach within a finite time $t$ a true minima, or even a saddle between two minima, and the difference $\mathcal{E} - \mathcal{E}_d$ remains positive.

To what extent does the above picture describe the particle in a random potential? Much less is known about the properties of the metastable states, and there is no obvious equivalent of the free energy functional $\Phi((x))$ of the mean particle’s position $\langle x \rangle^{(\alpha)}$ in the state $\alpha$. Nonetheless, we expect that the basic mechanism of the dynamical transition remain the same as for the spherical $p$-spin model, i.e. a slow relaxation toward a region of marginal states, $\lambda_{\min} \sim 0$. When considering the zero temperature limit, the dynamics reduces to a gradient descent $\dot{x}(t) = -\nabla V(x(t))$. The metastable states now correspond to local minima of the potential $V(x)$, and their stability will depend on the spectrum of the hessian $\mathcal{H}_{ij} = \partial^2_x V(x)$, at $\partial_i \text{ means } \partial / \partial x_i$ and $\partial^2_{ij} = \partial^2 / \partial x_i \partial x_j$.

The purpose of the geometrical approach, at zero temperature, is to relate the values provided by the more formal field-theoretical approach, to basic properties of the potential $V(x)$. For instance, one must be able to compute the asymptotic values of the energy $\mathcal{E}(t)$, curvature $\mathcal{M}(t)$, and the “plateau value” $\lim_{T \to 0} q/T$ of the correlation function. This is the first step, already outlined in [1]. One of the original contributions of this work concerns a second step, where we justify, with some geometrical arguments, many of the fine properties of the aging behaviour, beyond the time reparametrization invariant solution.

A random matrix computation of the spectrum of $\mathcal{H}$. A challenging problem in the study of the supercooled liquids dynamics, concerns the computation of the canonically averaged spectrum of the instantaneous normal modes $\langle \text{Spec}(\mathcal{H}) \rangle$. Here, $\mathcal{H}$ is the dynamical matrix, hessian of the potential energy $V$ of the interacting particles, function of the coordinates $r_i$. $\text{Spec}(\mathcal{H})$ is any representative characteristic function of the eigenvalues spectrum, e.g. the density of states.

$$\langle \text{Spec}(\mathcal{H}) \rangle = Z^{-1} \cdot \int_{\mathcal{D}} dx \left( e^{-\beta V} \cdot \text{Spec}(\mathcal{H}) \right). \tag{4.1}$$

$Z$ is the canonical partition function, and $\mathcal{D}$ is a bound domain eventually becoming infinite. In the mean field situation, as the energy $V$ is extensive, the canonical average is dominated by a saddle point value $V(\beta)$ of the potential $V(x)$. The analogous of [11] becomes:

$$\langle \text{Spec}(\mathcal{H}) \rangle = W_{\beta}^{-1} \int_{\mathcal{D}} dx \delta(V(x) - V(\beta)) \text{ Spec}(\mathcal{H})(x),$$
\[ W_\beta = \int_\mathcal{D} dx \, \delta(V(x) - V(\beta)). \quad (4.2) \]

In the \( p \)-spin case, the spectrum of \( \mathcal{H} \) is self-averaging with \( x \), \textit{i.e.} the spectrum of \( \mathcal{H} \) is a shifted semi-circle, by an amount controlled by \( V(\beta) \) only. The dynamical energy is found to be the highest value of \( V(\beta) \) such that \( \text{Spec}(\mathcal{H}) \geq 0 \) (marginality).

The averaged spectrum \( \langle \text{Spec}(\mathcal{H}) \rangle \) defined by equation (4.2) seems to be the natural quantity to consider when looking at the zero-temperature relaxational dynamics of our mean field model. We have found that as far as exponentially correlated potentials are concerned, \( \text{Spec}(\mathcal{H}) \) is, at the leading order, a non-fluctuating quantity determined by \( V_0 = V(\beta) \). More precisely, \( \text{Spec}(\mathcal{H}) \) has got a semi-circular distribution of radius \( \Lambda \), centred around \( \mathcal{D} \).

\[ \Lambda = 4\sqrt{f''(0)}; \quad (4.3) \]
\[ \mathcal{D} = \frac{2f'(0) V_0}{f(0)} \cdot N. \quad (4.4) \]

Let us outline our demonstration. We consider first the (\( \mathbf{r} \) independent) “annealed average”.

\[ \delta(V(\mathbf{r}) - V_0) \, \text{Spec}(\mathcal{H})(\mathbf{r}). \quad (4.5) \]

In order to compute (4.5), it is enough to enumerate the correlations of \( V(\mathbf{r}), \partial_i V(\mathbf{r}) \), where \( \mathbf{r} \) is an arbitrary point. All the \( \partial_i V(\mathbf{r}) \) are independent at the leading order \( N^{-1} \), whereas the \( N+1 \) remaining variables \( V(\mathbf{r}), \partial_i V(\mathbf{r}) \) are found to be correlated. One has:

\[ N \cdot \langle \partial_i V(\mathbf{r})^2 \rangle = 4f''(0) + O(N^{-1}); \]
\[ N \cdot \langle \partial_{ij} V(\mathbf{r}) \rangle^2 = 12f''(0) + O(N^{-1}); \]
\[ N \cdot \partial_i V(\mathbf{r}) \cdot \partial_j V(\mathbf{r}) = 4f''(0) + O(N^{-1}); \]
\[ N^{-1} \cdot \langle V(\mathbf{r}) \rangle^2 = f(0) + O(N^{-1}). \quad (4.6) \]

\( \mathcal{H} \) is split into a scalar part \( \mathcal{D} \delta_{ij} \) and a fluctuating part \( \mathcal{H}' \). The elements of \( \mathcal{H}' \) are independent and gaussian, and its eigenspectrum has, at the leading order, a semi-circular shape of radius \( 4\sqrt{f''(0)} \) centred around 0. If \( N \rightarrow \infty \) and \( V(\mathbf{r})/N \) finite, then \( \mathcal{D} \) is constant, up to fluctuations of order \( N^{-1/2} \) (cf appendix B).

\[ \mathcal{D} = \frac{2f'(0)}{f(0)} \cdot \frac{V(\mathbf{r})}{N} + O(N^{-1/2}). \quad (4.7) \]

The resulting spectrum is the one announced in equations (4.3, 4.4).

In order to bridge the gap between (4.3) and (4.4), we consider now the two-points annealed average:

\[ \delta(V(\mathbf{r}) - V_0) \, \delta(V(\mathbf{r}_1) - V_1) \, \text{Spec}(\mathcal{H})(\mathbf{r}). \quad (4.8) \]

The analysis involves now correlations between \( V(\mathbf{r}), \partial_i V(\mathbf{r}), V(\mathbf{r}_1), \partial_{ij} V(\mathbf{r}_1) \). One finds that for a generic correlator \( f(y) \), \( \text{Spec}(\mathcal{H})(\mathbf{r}) \) depends on both \( V(\mathbf{r}) \) and \( V(\mathbf{r}_1) \). However, if \( f(y) \) obeys \( f' - (f')^2 = 0 \), with \( f(y) = \exp(-y) \) as a particular case, the dependence in \( V(\mathbf{r}_1) \) disappears, and the result (4.4) holds.

Computing

\[ \delta(V(\mathbf{r}) - V_0) \, \delta(V(\mathbf{r}_1) - V_1) \ldots \]
\[ \times \delta(V(\mathbf{r}_n) - V_n) \text{Spec}(\mathcal{H})(\mathbf{r}), \quad (4.9) \]

becomes very difficult as \( n \geq 3 \), and we were not able to find a close expression for \( \text{Spec}(\mathcal{H})(\mathbf{r}) \) \((V_0, V_1, \ldots, V_n)\). However, if \( f' - (f')^2 = 0 \), again \( \text{Spec}(\mathcal{H})(\mathbf{r}) \) depends only on \( V_0 \), and (4.4) is valid. This shows that the spectrum of \( \mathcal{H} \) is a local quantity, independent of the environment of the particle.

Because \( \text{Spec}(\mathcal{H})(\mathbf{r}) \) is a function of \( V(\mathbf{r}) \) only, we conclude that the average (4.3) is described by (4.4) and that the self-averaging property of \( \text{Spec}(\mathcal{H})(\mathbf{r}) \) and its linear dependence in \( V(\mathbf{r}) \), which was true for the \( p \)-spin model, is still true for exponential correlators. The appendix B gives further details on the computation of (4.3) and (4.8).

Now, we suppose that the trajectory \( \mathbf{x}(t) \) explores representative regions of the potential (\textit{i.e.} non-exceptional points), for which the above mentioned results hold. The lowest eigenvalue \( -S \) of the hessian, defined by (4.4) becomes a time-dependent function:

\[ W_\beta = \int_\mathcal{D} dx \, \delta(V(x) - V(\beta)). \]
\[ S(t) = \Lambda - \frac{2f'(0)}{f(0)} E(t), \quad (4.10) \]

leading to the energy dependent (through \( S \)) density of eigenvalues of \( H_{ij} \). The number of eigenvalues between \( \lambda - S(t) \) and \( \lambda - S(t) + d\lambda \) is \( \rho(\lambda) d\lambda \) (time independent).

\[ \rho(\lambda) = 2(\pi\Lambda^2)^{-1} \sqrt{\lambda} (2\Lambda - \lambda). \quad (4.11) \]

The marginality condition, by definition, is \( S \equiv 0 \). Equation (4.10) yields the “geometrical energy”, necessary for \( H \) to be marginal:

\[ E_{\text{geom}} = 2 \sqrt{f''(0)} f(0), \quad (4.12) \]

and the curvature \( M_{\text{geom}} \):

\[ M_{\text{geom}} = \int d\lambda \rho(\lambda), \quad = 4 \sqrt{f''(0)}. \quad (4.13) \]

After a time \( t \) long enough, the particle evolves in a marginal region (\( S(t) \approx 0 \)) of the potential \( V(x) \), with a small gradient \( \| \nabla V(x) \| \). At low temperature, the potential may be developed up to the second order by means of local coordinates \( y_i \):

\[ V(y) = V(0) + y \cdot \nabla V(0) + \sum_{i=1,N} \lambda_i y_i^2 / 2. \]

The plateau value “\( q \)” of the correlation function \( b(t, t') \) is thus given by assuming that each direction of curvature \( \lambda_i \) is thermalized with \( \langle y_i^2 \rangle \approx T/\lambda_i \), and

\[ q_{\text{geom}} = T \sqrt{f''(0)}. \quad (4.14) \]

Let us compare now with the results from the dynamical mean-field theory, in the zero temperature limit [10].

\[ \lim_{t \to \infty} E(t) = \frac{f'(0)}{\sqrt{f''(0)}} + \frac{f(0) \sqrt{f'''(0)}}{f'(0)}, \quad (4.15) \]

\[ \lim_{t \to \infty} M(t) = 4 \sqrt{f''(0)}, \quad (4.16) \]

\[ q = \frac{T}{\sqrt{f'''(0)}}. \quad (4.17) \]

Agreement holds for the curvature and \( q \), whereas the geometrical and dynamical energy differ, unless \( f(0)f''(0) = f'(0)^2 \). We cannot conclude about the relevance of the geometrical approach for a generic correlator, e.g. power law, as (4.4) probably does not hold. However, the exponentially correlated toy-model turns out to be a very favourable model, for which the geometrical approach gives reasonable results. The following of this paper aims at demonstrating that many features of the zero temperature dynamics of this model (exponents, aging, driving with a force) can be explained with the help of geometrical arguments.

**V. THE DISTRIBUTION OF THE GRADIENT’S COORDINATES**

In this section, we define an orthonormal frame “attached” to the particle. The procedure used is reminiscent from the definition of the instantaneous normal modes in the study of supercooled liquids dynamics [15]. Then, we investigate the statistical properties of the components of \( \nabla V(x(t)) \) in this special frame. We find that these components are distributed according to a self-similar form, determined by the value of the exponent \( \kappa \) of the energy decay.

We develop up to the second order the potential around the actual position of the particle \( x(t) \).
\[
Q(x) = V(x(t)) + \sum_i (x_i - x_i(t)) \cdot \partial_i V(x(t)) \\
+ 1/2 \sum_{ij} \partial_i V(x(t)) \cdot (x_i - x_i(t)) \cdot (x_j - x_j(t)),
\] (5.1)

We define an orthonormal frame of eigendirections \( \{e_{\lambda_i}(t)\} \) in which the hessian \( H_{ij}(t) = \partial_i \partial_j V(x(t)) \) is diagonal. \( \lambda_i \) belongs to the -time independent– interval \([0, 2\Lambda]\), so that the corresponding eigenvalue of \( Q \) is just \( \lambda_i - S(t) \).

We follow “adiabatically” the eigenvectors \( \{e_{\lambda_i}(t)\} \) as the particle moves. A mild assumption is that the \( \{e_{\lambda_i}\} \) evolve smoothly, provided the levels \( \lambda_i \) are allowed to freely cross each other. This choice implies that any ordering of the \( \lambda_i \) lasts only for a short period of time. The \( \{e_{\lambda_i}\} \) define a comoving frame, in which the gradient \( \nabla V \), or equivalently the velocity, can be projected.

\[- \nabla V(x(t)) = \sum_i \gamma_i(t) \cdot e_{\lambda_i}(t), \quad \gamma_i(t) = \frac{\partial_{\lambda_i} S(t)}{S(t)}.
\] (5.2)

There are reasons to consider that the components \( \gamma_i(t) \) are randomly and evenly distributed, even in the deterministic zero temperature limit. First, this randomness reflects the average over the “white” initial conditions. Then, as the correlator \( \partial_{V} V(x) \partial_{\lambda_i} V(x') \) is exponentially short range correlated, one can suppose that the comoving frame is rotating on itself in a chaotic manner, as it does in the spherical \( (p = 3) \)-spin model \[6\]. So, during the particle’s motion, each component \( \gamma_i \) spreads continuously over the \( N - 1 \) others directions.

The sign of \( \gamma_i(t) \) itself is irrelevant, because of the arbitrary definition of the frame, invariant under the reflections \( e_{\lambda_i} \leftrightarrow -e_{\lambda_i} \). We claim that \( \gamma_i^2(t) \) has to be preferred to \( \gamma_i(t) \). On physical grounds, we propose to consider only the smoothed quantity \( \gamma_i^2(t) \), obtained by averaging locally over the few \( \sqrt{N} \) indices \( j \) such that \( \lambda_i - N^{-1/2} < \lambda_j < \lambda_i + N^{1/2} \). This is possible because the mean interspacing between the \( \lambda_i \) is \( \mathcal{O}(N^{-1}) \). As \( N \) goes to \( \infty \), one expects \( \gamma_i^2(t) \) to become a smooth function of \( \lambda_i \), varying only on the scale \( \delta \lambda \sim 1 \) (although, rigorously, the scale of variation is \( \delta \lambda \sim N^{-1/2} \)), making the dependence in the index \( i \) irrelevant.

The function:

\[ g(\lambda, t) = \left[ \gamma_i^2(t) \right], \] (5.3)

is the distribution of the gradient’s coordinates (or equivalently of the instantaneous velocity coordinates) and is a central object in the present study.

In this continuous limit, the two first time derivatives of \( \mathcal{E} \) can be expressed with the help of the density \( \rho(\lambda) \) and the distribution \( g(\lambda, t) \) as:

\[
\dot{\mathcal{E}}(t) = -\sum_i \partial_i V(x(t)) \cdot \partial_i V(x(t)), \\
= -\int d\lambda \rho(\lambda) \ g(\lambda, t),
\]

\[
\ddot{\mathcal{E}}(t) = \sum_{ij} \partial_i V(x(t)) \cdot \partial_{ij} V(x(t)) \cdot \partial_i V(x(t)), \\
= \int d\lambda \rho(\lambda) \ (\lambda - S(t)) \ g(\lambda, t).
\] (5.5)

In [7] was already noticed that, due to the algebraic decay of the energy \( \mathcal{E}(t) = -2 + 1.08 \cdot t^{-\kappa} \), the ratio \( \ddot{\mathcal{E}}(t)/\dot{\mathcal{E}}(t) \) was \( \sim 1/t \). From section [3], we know that \( S(t) \sim t^{-0.67} \), which implies \( \ddot{\mathcal{E}}(t) \ll S(t) \cdot \dot{\mathcal{E}}(t) \), and thus:

\[
\int d\lambda \rho(\lambda) \lambda \cdot g(\lambda, t) = \dot{\mathcal{E}}(t) + \mathcal{S}(t) \int d\lambda \rho(\lambda) g(\lambda, t),
\]

\[
\approx \mathcal{S}(t) \int d\lambda \rho(\lambda) g(\lambda, t). \] (5.6)

(5.7)

The first moment of \( g(\lambda, t) \rho(\lambda) \), is proportional to \( \mathcal{S}(t) \), suggesting a self-similar scaling form for \( g(\lambda, t) \), valid for \( t \to \infty \) and \( T = 0 \) (Figure [7]):

\[ g(\lambda, t) = \Gamma(t) \cdot \hat{G} \left( \frac{\lambda}{\mathcal{S}(t)} \right), \]

(5.8)
The knowledge of the other moments of $g$ would be useful to confirm equation 5.8, but unfortunately, they are very difficult to compute, and are no more given by the next derivatives of $\mathcal{E}$.

As $t$ increases, only the smaller $\lambda_i$ keep any relevance, and the density $\rho$ is well approximated by its $\lambda \sim 0$ equivalent $\pi^{-1}(2/\lambda)^{3/2}\sqrt{\lambda}$. In this limit, the loss of energy rate becomes, from (5.4) and (5.8):

$$\dot{S}(t) = -\frac{2f'(0)}{f(0)} \dot{S}(t) \propto -\Gamma(t) \cdot \mathcal{S}(t)^{3/2}. \tag{5.9}$$

The knowledge of the exponent $\kappa$ of $\mathcal{S}(t) \sim t^{-\kappa}$ (section VI) fixes the prefactor $\Gamma$ up to a constant, to:

$$\Gamma = \mathcal{S}^{(2-\kappa)/2\kappa}. \tag{5.10}$$

Our section II suggests $\kappa$ is very close to $2/3$, which would imply $\Gamma \propto \mathcal{S}$.

The next momentum of $g(\lambda, t) \rho(\lambda)$ provides information on the time correlations of the unit vector $\mathbf{w}(t)$ of the particle’s trajectory. One the one hand,

$$\left(\partial_t \mathbf{V}(\mathbf{x}(t))\right)^2 = \sum_i \left( \sum_j -\partial_j V(\mathbf{x}(t)) \partial_j V(\mathbf{x}(t)) \right),$$

$$\times \left( \sum_k -\partial_k V(\mathbf{x}(t)) \partial_k V(\mathbf{x}(t)) \right),$$

$$= \sum_{i,j,k} \partial_j V \cdot \partial_j V \cdot \partial_k V \cdot \partial_k V,$$

$$= \int d\lambda \rho(\lambda) (\lambda - \mathcal{S}(t))^2 g(\lambda, t). \tag{5.11}$$

With the scaling form for $g(\lambda, t)$, the right hand side is of order $\Gamma(t)\mathcal{S}(t)^{7/2}$. On the other hand, we perform a decomposition $-\nabla V(\mathbf{x}(t)) = M(t) \mathbf{w}(t)$. The norm $M(t)$ equals $(-\mathcal{E}(t))^{1/2}$, and $\mathbf{w}(t)$ is the unit vector, tangent to the trajectory. The following equality holds:

$$(\partial_t \mathbf{V})^2 = (\partial_t M)^2 + M^2 \cdot ||\partial_t \mathbf{w}||^2. \tag{5.12}$$

This sum is clearly dominated by $M^2 \cdot ||\partial_t \mathbf{w}||^2$, with $M^2 \approx \Gamma \cdot \mathcal{S}^{3/2}$. The unitary vector rotates, regardless to the actual value of $\Gamma(t)$, at a rate $||\partial_t \mathbf{w}|| \sim \mathcal{S}(t)$. One expects the “director” $\mathbf{w}(t)$ to have changed its orientation after a typical time $\mathcal{S}(t)^{-1}$, which looks like a “persistence time” for the trajectory of the particle. Consequently, the motion of $\mathbf{x}(t)$ crosses over from a “ballistic” regime $||\mathbf{x}(t+\delta t) - \mathbf{x}(t)|| \sim \mathcal{S}^{-1}$ to a diffusive regime $||\mathbf{x}(t+\delta t) - \mathbf{x}(t)|| \sim M^2 \cdot (\delta t)^2$; $\delta t \ll \mathcal{S}^{-1}$.

VI. A SHORT TIME, QUASI-STATIC APPROXIMATION

We investigate here the breakdown of the fluctuation dissipation theorem, in the zero temperature limit. The fluctuation-dissipation violation is measured by the function $T(t, t')$ (cf 2.17). We propose here a model for the short time evolution of $T(t, t')$, and show that its predictions are in good agreement with the findings of section II.

We approximate locally the potential $V(\mathbf{x})$ around $\mathbf{x}(t)$ by a quadratic function (5.1), which may be considered as constant provided we restrict ourselves to a time separation $t - t'$ small enough. One can always find a coordinate system $\{y_i\}$ such that this quadratic potential reads:

$$Q(\mathbf{y}) = Q(0) + 1/2 \sum_i (\lambda_i - \mathcal{S}) \cdot y_i^2, \tag{6.1}$$

where the coordinates $\mathbf{y}$ must not be confused with the original coordinates $\mathbf{x}$ of the relaxational motion.

This section aims at demonstrating that when a particle diffuses, or relaxes in such a parabolic potential, then a characteristic time $t_f$ scaling like $\mathcal{S}^{-1}$ arises, which turns out to be the time scale along which the function $T(t, t')$ departs from its equilibrium value 0, i.e. the fluctuation-dissipation violation characteristic time.
We consider a particle moving on the potential $V(x)$, starting at $t_0$, and define the time difference $\tau = t - t_0$. The intermediate steps of the calculation make use of $\tau, t_0$, while the final results are expressed in term of $t, t'$ in relation with the original out-of-equilibrium relaxation.

Let us consider the same local average $[y_i^2(\tau)]$ as in equation (5.3). The $y_i$ are related to the gradient’s coordinates by $\gamma_i(\tau) = -(\lambda_i - S) \cdot y_i(\tau)$:

$$[y_i^2(\tau)] = \frac{g(\lambda, t_0 + \tau)}{(\lambda - S)^2}. \quad (6.2)$$

The initial conditions $[y_i^2(\tau = 0)]$ are given by $g(\lambda, t_0)$. One computes the fluctuation dissipation violation $T(t, t')$, when the quadratic potential (6.1) does not evolve with time ($S$ fixed once for all), and with initial conditions arising from a realistic distribution $g(\lambda, t_0) = \Gamma \cdot \hat{G}(\lambda/S)$.

$$y_i(\tau) = y_i(0) \ e^{-(\lambda_i - S_i)(\tau)} \quad (6.3)$$

The distribution $g(\lambda, t_0 + \tau)$ evolves like $[y_i^2(\tau)] \cdot (\lambda - S)^2$. One has, for all $t > t_0$:

$$g(\lambda, t_0 + \tau) = g(\lambda, t_0) \ e^{-2(\lambda - S)(\tau)}; \quad \partial_t g(\lambda, t) = -2g(\lambda, t) \cdot (\lambda - S). \quad (6.4)$$

The usual response $r(t, t') = N^{-1} \sum_i \delta y_i(t)/\delta \xi_i(t')$, and correlation $b(t, t') = N^{-1} \sum_i (y_i(t) - y_i(t'))^2$ functions reexpress in terms of $g(\lambda, t')$:

$$r(t, t') = \int d\lambda \ \rho(\lambda) \ e^{-(\lambda - S)(t-t')}; \quad (6.5)$$

$$b(t, t') = \int d\lambda \ \rho(\lambda) [g(\lambda, t') \left(\frac{1 - e^{-(\lambda - S)(t-t')}}{\lambda - S}\right)^2]; \quad (6.6)$$

$$\partial_t b(t, t') = -2 \int d\lambda \ \rho(\lambda) \ g(\lambda, t') \left(\frac{1 - e^{-(\lambda - S)(t-t')}}{\lambda - S}\right); \quad (6.7)$$

By inserting $g(\lambda, t' = t_0) = \Gamma \hat{G}(\lambda/S)$ in (6.7), one deduces the short time, $\tau \ll S^{-1}$, value of $T(t, t')$,

$$T(t, t') = (t-t') \cdot \Gamma \cdot S^{3/2}, \quad (6.8)$$

and the intermediate time $\tau \sim S^{-1}$ one,

$$r(t, t') = S^{3/2} \Phi_0(S \cdot (t-t'))), \quad (6.9)$$

$$b(t, t') = -2\Gamma \cdot S^{1/2} \cdot \Phi_1(S \cdot (t-t')); \quad (6.9)$$

$$T(t, t') = \frac{\Gamma \Phi_0}{S \Phi_1}(S \cdot (t-t')).$$

$\Phi_0, \Phi_1$ are scaling function presented in appendix B. Equation (6.3) shows that $S^{-1}$ plays the role of a characteristic time for the onset of the effective temperature $T$.

Assuming now the very likely value $\kappa = 2/3$ and $\Gamma = S$, one finds that $T$ becomes an order one quantity after a time $t_f \sim S^{-1}$. One concludes that the characteristic time scale $t_f$ should scale like $t_f = t^\alpha \propto t^\kappa$, and $\alpha = \kappa = 2/3$.

Our numerics (Figure 4) lead to an estimated value $\alpha \approx 0.64$, while $\kappa \approx 0.67$. While we haven’t proved that $\kappa$ is actually $2/3$, we find the agreement satisfactory, and believe that the above picture describes correctly the first stage of the breaking of the “fluctuation-dissipation relation at zero temperature”.

How long can the quadratic approximation (6.1) accurately describe the original relaxation process? As $S(t)$ decreases algebraically, the necessary time $\delta t$ to have $|S(t + \delta t) - S(t)| \sim S(t)$ is $t$ itself. More seriously, we have seen in the previous section, that the unit vector of the trajectory $x(t)$ changes with the time scale $S^{-1} \sim \tilde{t}^{2/3}$. As this change is somewhat related to the frame’s chaotic motion, we deduce than $S^{-1}$ must be an upper limit of validity of the quasi-static approximation. Finally, the relaxation on the saddle becomes ill-defined when $S \cdot (t-t') \gg 1$, due to the exponential divergence of the functions $r(t, t')$ and $b(t, t')$, given by the equations (6.3, 6.6). We arrive to the conclusion that this quasi-static picture is not valid beyond times much greater than $S^{-1}$, but provides a
strong presumption in favour of \( t_f(t) \sim S^{-1}(t) \sim t^{2/3} \), in good agreement with our numerical findings (section IV and Figure 4).

Let us close this section by computing the typical distance covered during a time interval \( t - t' < S^{-1} \), with a gradient coordinates distribution \( g(\lambda, t') = \Gamma \cdot \hat{G}(\lambda/S) \).

\[
\begin{align*}
    b(t, t') &= \Gamma S^{3/2} \cdot (t - t')^2 \\
    &= S^{5/2} \cdot (t - t')^2 \quad \text{if } \kappa = 2/3.
\end{align*}
\]

For a time interval \( (t - t') \sim S^{-1} \), \( b(t, t') \sim \Gamma \cdot S^{-1/2} \ll 1 \), becoming \( b(t, t') \sim S^{1/2} \) for \( \kappa = 2/3 \). As \( S^{1/2} \) tends to zero, the characteristic time \( t_b \) of the evolution of \( b(t, t') \) in the aging regime, is necessarily much greater than \( t_f \sim S^{-1} \).

VII. A DYNAMICS RESTRICTED TO THE DOWNHILL DIRECTIONS

This section shows how the above approach describes the long time aging regime.

The equation (5.9) has a simple physical interpretation. The only non-vanishingly small components \( \gamma_i \) of \(-\nabla V\) are those corresponding to \( \lambda_i \leq S \). Only a number \( N \int _0 ^S d\lambda \rho(\lambda) \sim N \cdot S^{3/2} \) directions \( i \) are contributing to \(-\nabla V)^2 = -\sum _i \gamma_i^2 \). Each one of these \( \gamma_i \) has a magnitude of order \( \gamma_i^2 \sim \Gamma \). As a result, \( N \cdot \hat{E} \) scales like \( -\sum _i \gamma_i^2 = N \cdot S^{3/2} \cdot \Gamma \).

\[
\hat{E} \propto -(\hat{E}(t) - \hat{E}_d)^{3+1/\kappa}.
\]

The relaxation dynamics looks like if it was controlled by the difference \( \hat{E}(t) - \hat{E}_d \).

**The linear response regime.** A constant force \( F \) is now applied, uncorrelated to the potential \( V \). Each one of its (comoving) coordinate \( f_i \) is random, time-dependent as the frame rotates during the particle’s motion, and has a magnitude \( f_i \sim F \). We suppose \( F \) weak enough to be considered as a perturbation around the relaxational dynamics described in section IV.

At any time, there are “open”, or downhill directions, with \( \lambda_i \leq S \) and “close”, or uphill directions with \( \lambda_i \geq S \). The close directions behave as confining harmonic potentials which prevent the (weak) force \( f_i \) to drive the particle along this direction. The open directions are the one along with the external force drives efficiently the particle. The open directions are the one along with the external force drives efficiently the particle.

The force \( F \) induces a displacement \( \hat{x} \) whose components are \( \hat{x}_i \sim f_i \) along an open direction and \( \hat{x}_i \simeq 0 \) along a close direction. The average velocity \( \hat{x} \cdot \frac{F}{\|F\|} \) is given by (\( \theta \) Heaviside function):

\[
\begin{align*}
    \frac{\hat{x}(t) \cdot \frac{F}{\|F\|}}{\|F\|} &= \sum _i f_i^2 \theta(S - \lambda_i)/\|F\|; \\
    &= N \cdot S^{3/2} \cdot F^2 / (\sqrt{N} F); \\
    &= \sqrt{N} \ F \ S^{3/2};
\end{align*}
\]

that we identify to \( \sqrt{N} \ u(t) \). As a result, one finds a velocity proportional to the number of downhill directions:

\[
\hat{u} \propto F \cdot S^{3/2}.
\]

Inserting the likely value \( \kappa = 2/3 \), one finally gets a displacement \( u(t) - u(t') \propto F(\ln(t) - \ln(t')) \), well confirmed by the numerics (section IV and Figure 6). This pure relaxational motion is driven by the components \( \gamma_i \) of \(-\nabla V\) along the open directions, while the external force acts with \( f_i \) along the same open directions. One expects the linear response to hold if \( f_i^2 \ll \gamma_i^2 \), but to break down when \( f_i^2 \simeq \gamma_i^2 \). This leads to a predicted cross-over time scaling like \( \Gamma(t_F) = F^2 \), or \( t_F \sim F^{4/(\kappa-1)} \), to be investigated in a forthcoming publication [23].

**The diffusive regime.** The asymptotic behaviour predicted for \( b(t, t') \) is, from equations (2.21, 2.23):

\[
    b(t, t') = \frac{t - t'}{t_b} + O \left( \frac{t - t'}{t_b} \right)^2.
\]

One recognises a simple diffusive behaviour, with effective diffusivity \( t_b^{-1} \). From section IV, we know that the short-time motion \( (t - t') \leq S^{-1} \sim t_f \) is ballistic, and that the particles covers a distance of \( \Gamma \cdot S^{-1/2} \). On the other hand, our results from section IV show that the direction \( \mathbf{w} \) of the trajectory \( \mathbf{x}(t) \) uncorrelates itself after this same time.
This leads to the following predictions: decrease as a power law, and acknowledging the numerical estimate of $g$ in the comoving frame, reminiscent from the INM frame of a supercooled liquid, we derive an expression for the distribution of the random potential, and we were able to predict the correct value ($\kappa$) with an exponent $\kappa$ with an exponent $\sim (t-t')$. Asking for a power-law decay of the memory function $f(t, t') \sim (t-t')^{-\lambda}$, this makes $\exp(-b(t, t'))$ as well as $r(t, t')$ decaying as a power law. While we have no demonstration of that, we think that a power-law decay of the memory function $f(b(t, t'))$ is necessary for the “fine tuned” aging solution of the system (2.8). Asking for a power-law decay $f(b(t, t'))$ in turn fixes $\kappa$ to 2/3.

Finally, if $\kappa = 2/3$, $t_b = S^{-3/2}$, and the characteristic times for the linear response regime $u(t) \sim F/t_b$, and for the diffusion regime $b(t, t') \sim (t-t')/t_b$ are the same, which is consistent with the persistence of an “Einstein relation” at the beginning of the aging regime.

**VIII. CONCLUSION**

We have proposed a geometrical description of the mean-field relaxational dynamics of a particle, for a subclass of short-range correlated disorders. We have restricted ourselves to the isolated case, and to the driven case in the linear response regime.

A numerical integration of the mean-field equations gives evidence of a power-law decay of the dynamical energy with an exponent $\kappa$ numerically close to 2/3. We also found evidence of a logarithmic growth $b(t, t') \sim \ln t$ consistent with the conjecture $h(t) \sim t^\delta$ for the reparametrization function $h$.

The exponential correlator makes it possible to compute the density of eigenvalues of the hessian $H$ associated to the random potential, and we were able to predict the correct value (i.e. $-2$) of the dynamical energy $e_g$. Introducing a comoving frame, reminiscent from the INM frame of a supercooled liquid, we derive an expression for the distribution $g(\lambda, t)$ of the components of $\nabla V(x(t))$. This expression is $g = \Gamma G(\lambda/S)$, where $-S(t)$ is the (time dependent) lowest eigenvalue of $\text{Spec}(H)$.

For reasons exposed in section VII, namely the consistence with $h(t) \sim t^\delta$, the requirement that $f(b)$ is likely to be a free diffusion to the appearance of an effective temperature goes like $t_f \sim S^{-1} \sim t^{2/3}$. We conclude that the aging mechanism of this model comes from a simultaneous decrease of the number of downhill directions (going like $N \cdot S^{3/2} \sim Nt^{-1}$) and of the typical gradient component $|\gamma_i| \sim t^{-1/3}$.

We predict also that the effect of a constant force brings about a dramatic change in the dynamics after a time $t_F \sim F^{-3}$, reaching a out-of-equilibrium but stationary regime.
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APPENDIX A: THE MSR ACTION

The action leading to equations (2.3) is:

$$S[x, i\dot{x}] = \int_0^\infty dt \left\{ -T \sum_{j=1}^N (i\dot{x}_j)^2(t) + i\dot{x}_1(t) \cdot (\dot{x}_1(t) - \sqrt{N^{1/2}} \cdot F) + \sum_{j=2}^N i\dot{x}_j(t) \cdot \dot{x}_j(t) \right\}$$

$$+ \int_0^\infty dt \int ds \left\{ f'(d(t, s)) \sum_{j=1}^N i\dot{x}_j(t) \cdot i\dot{x}_j(s) + 4f''(d(t, s)) r(t, s) \sum_{j=1}^N i\dot{x}_j(t) \cdot (x_j(t) - x_j(s)) \right\}, \quad (A1)$$

and the expectation value of an observable $O[x(t), i\dot{x}(t)]$, averaged over the disorder, is given by:

$$\langle O \rangle = \int \mathcal{D}x[t] \mathcal{D}\dot{x}[t] O \exp(-S). \quad (A2)$$

APPENDIX B: THE SPECTRUM OF THE HESSIAN $\mathcal{H}$

We consider $\delta(V(r) - V_0) \text{Spec}(\mathcal{H})/r$ for any arbitrary potential. $V_0 = N\mathcal{E}$ is fixed, and the $H_{ij} = \partial^2_{ij} V(r)$ are $N(N+1)/2$ gaussian random variables. The correlations among the $H_{ij}$ are listed in equation (4.6). We define the self-averaging quantity $D = N^{-1}\sum_i \partial_i V(r)$ so that:

$$N(\partial_i V(r) - D)^2 = 8f''(0) - f''(0)/N,$$

$$N(\partial_i V(r) - D) V(r) = 0,$$

$$N(\partial_i V(r) - D) (\partial_j V(r) - D) = -8f''(0)/N,$$

$$= 0 + O(N^{-1}). \quad (B1)$$

The hessian is now $H_{ij} = D\delta_{ij} + H'_{ij}$. $H'$ is a matrix of independent gaussian centred random numbers. The diagonal elements are slightly correlated (of order $1/N^2$) and have a different variance than the off-diagonal elements, but this does not prevent the Wigner result to apply and the spectrum of $H'$ is a centred semi-circle of width $\Lambda = 4\sqrt{f''(0)}$.

The determination of $D$ follows from the fact that $\mathcal{E}$ and $D$ are gaussian distributed, with correlations:

$$N\mathcal{E}^2 = f(0);$$

$$N\mathcal{D}^2 = 4\left(\frac{N+2}{N}\right) f''(0);$$

$$N\mathcal{D} \cdot \mathcal{E} = 2f'(0). \quad (B2)$$

The joint probability distribution of $\mathcal{E}$ and $D$ is:

$$\mathcal{P}(D, \mathcal{E}) = \frac{N}{2\pi \sqrt{cf(0)}} \exp \left( -\frac{N}{2} \left[ \frac{\mathcal{E}^2}{f(0)} + \frac{(D - a\mathcal{E})^2}{c} \right] \right);$$

$$c = \frac{4}{f(0)} \cdot (f(0)f''(0) - f'(0)^2) + \frac{8}{N} f''(0);$$

$$a = \frac{2f'(0)}{f(0)}. \quad (B3)$$

Fluctuations of $D$ are of order $N^{1/2}$ around its saddle point value $2f'(0)/f(0) \times \mathcal{E}$. It follows that $\text{Spec}(\mathcal{H})$ is a semi-circle of radius $\Lambda$ shifted by an amount $D = 2f'(0)/f(0) \times \mathcal{E}$.
Let us consider now
\[ \delta(V(r) - V_0) \delta(V(r_1) - V_1) \text{Spec}(\mathcal{H})(r) \].
This simple average measures the non-locality of \text{Spec}(\mathcal{H}), i.e. its dependence on the values taken by the random potential \( V(r') \) around \( r \).

The rotational invariance of the above average is broken, and \( r - r' \) plays a special role. We relabel hereafter the direction 1 to coincide with \( r - r' \), and define \( b = \| r - r' \|^2/N \). Correlations are now, in addition to \( B4 \),
\[
\begin{align*}
\partial_{i_1} V(r)V(r') &= 2f'(b) \text{ if } i \geq 2; \\
\partial_{i_1} V(r)V(r') &= 2f'(b) + 4bf''(b); \\
\partial_{i_j} V(r)V(r') &= 0. 
\end{align*}
\]

\( D \) is again defined as \( N^{-1} \sum \partial_i V(r) \) and \( H'_{(i,j)} \geq 2 \) is equivalent to the above situation: independent, centred, gaussian random components, and the spectrum is a centred semi-circle. Adding one row and one column of random independents elements to \( H'_{(i,j) \geq 2} \) must not change the density profile of eigenvalues. This is because this eigenvalue distribution is a fixed point under the change \( N \rightarrow N+1 \), as argued in the cavity approach of the problem. A possible trouble come from the single component \( \partial_{i_1} V - D \) which does not average to 0, but this does not alter the final result more than by a single isolated eigenvalue.

It is possible to show, with the help of a formal field theoretical approach, that the correlations \( B4 \) indeed lead to the ordinary \( N \rightarrow \infty \) saddle point for \text{Spec}(\mathcal{H}'), i.e. a semi-circle law of radius \( \Lambda \).

The computation of \( D \) follows closely the lines of the previous paragraph. We found that if \( E = V(r)/N \), \( E' = V(r')/N \) and \( b = \| r - r' \|^2/N \), then:
\[
D = \left( \frac{f'(0) + f'(b)}{f(0) + f(b)} \cdot (E + E') + \frac{f'(0) - f'(b)}{f(0) - f(b)} \cdot (E - E') \right). 
\]

For a generic correlator, there is an explicit dependence on \( E' \) ("non locality") while for an exponential correlator \( f = \exp(-y) \), the above formula reduces to \( D = 2f'(0)/f(0) \times E \).

This suggests that the determination of \text{Spec}(\mathcal{H}) \text{ from } (4.2) \text{ is a complex problem and the simple behaviour (4.4) fails for a generic } f.

The exponential correlator, however, has a strong property. The average:
\[
N \left( \frac{2f'(0)}{f(0)} E - S \right)^2 = \frac{4}{f(0)} (ff'' - f^2) + \frac{8}{N} f''(0); \\
= \frac{8}{N} f''(0). 
\]
is 0 at the order \( N^{-1} \). This means that, while \( b \) is strictly positive, there is no possible fluctuations of \( D \) around \( 2f'(0)/f''(0) \times E \).

Repeating the argument for \( B5 \), \( n \) finite, shows that \text{Spec}(\mathcal{H}) \text{ depends only on } V(r) \text{ and not on its local environment. Thus, we argue that the average (4.2) is given by (4.3, 4.4), as announced.}

**APPENDIX C: THE QUASI-STATIC PICTURE**

From equation \( B6 \), we derive the expression for \( r(t' + \tau, t') \).
\[
r(t' + \tau, t') = \frac{e^{\Lambda \tau}}{\Lambda \tau} \frac{I_1(\Lambda \tau)}{e^{\Lambda \tau}} ,
\]
where \( I_1 \) is the first kind modified Bessel function. The short time expansion of \( B7 \) is:
\[
\partial_{\lambda} b(t' + \tau, t') = -2\tau \int d\lambda \rho(\lambda)g(\lambda, t'); \\
= -2\tau \left( \mathcal{E}(t') \right)^2; \\
= -2\tau \Gamma S^{3/2}. 
\]
In the intermediate time separation regime, \( \tau \) is of order \( S^{-1} \). The integral is dominated by \( \lambda \sim S \) and cut off by \( g(\lambda, t') \) for \( \lambda S \gg 1 \). \( \rho(\lambda) \) can be replaced by its \( \lambda \to 0 \) equivalent.

\[
\begin{align*}
    r(t' + \tau, t') &\approx \sqrt{2/\pi} e^{S \tau} (\Lambda \tau)^{-3/2}; \\
    &= S^{3/2} (\sqrt{2/\pi} \Lambda^{-3/2}) \cdot (S \tau)^{-3/2} e^{S \tau}; \\
    &= S^{3/2} \Phi_0(S \tau).
\end{align*}
\]  

\[\text{(C3)}\]

\[
\begin{align*}
    \partial_t b(t' + \tau, t') &= -2 \Gamma S^{1/2} \int_0^{2\Lambda/S \approx \infty} \frac{d u}{(2/\Lambda)^{3/2} \pi^{-1} \sqrt{u}} \\
    &\times \hat{G}(u) \left( \frac{1 - e^{-S \tau (u - 1)}}{u - 1} \right); \\
    &= -2 \Gamma S^{1/2} \Phi_1(S \tau).
\end{align*}
\]

\[\text{(C4)}\]

The effective temperature behaves as:

\[
\begin{align*}
    T(t' + \tau, t') &= \frac{\Gamma}{S} \frac{\Phi_1(S \tau)}{\Phi_0(S \tau)},
\end{align*}
\]

\[\text{(C5)}\]

reducing to

\[
\begin{align*}
    \overline{T}(t, t') &= \frac{\Phi_1}{\Phi_0}(S \cdot (t - t')), \quad \text{if } \kappa = 2/3 \text{ and } \Gamma = S.
\end{align*}
\]

\[\text{(C6)}\]
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FIGURE 1. Parametric plot of the integrated response \( b(t, t') \) vs \( R(t) = \int_0^t ds r(t, s) \) at zero temperature, for time steps \( h = 0.025, 0.05, 1.0, 2.0 \). The horizontal part corresponds to the short time regime, with \( T \to 0 \). Then, the aging regime is the straight line with a slope \( X^{-1} \simeq 0.21 \), to be compared with the theoretical value \( 1/X_{QFD} = 2 \). The inset shows the derivative \( X^{-1}(b) = dB/dR(t) \), stepping from 0 to 2.1.

FIGURE 2. Dynamical energy \( E(t) + 2 \) vs time, in log-log coordinate, for \( h = 0.025 \) and \( h = 0.2 \). The power-law decay is unambiguous, and a fit to \( \kappa = 0.67 \) has been done between the two vertical arrows.

FIGURE 3. Logarithmic derivative \( -\kappa \) of the energy \( E(t) + 2 \), for \( h = 0.025, 0.5, 1.0, 2.0 \). The curve is noisy as \( E(t) \to -2 \). The straight line stands for \( \kappa = 2/3 \) which we believe to be its exact value. Curves for \( h = 0.025, 0.05 \) seem to tend to 2/3 from above, while \( h = 0.2 \) seems to tend to 2/3 from below. \( \kappa \approx 2/3 \) is well realised for \( h = 1 \).

FIGURE 4. The characteristic times \( t_f(t) \propto t^a \) and \( t_u(t) \), \( a = 0.55 \) and \( a = 0.45 \) determined from equations (3.1), (3.2) and (3.3). Results are shown for the time steps \( h = 0.1 \) and \( h = 0.2 \), and the finiteness of \( h \) is visible at small \( t \). A numerical estimate of \( \alpha \) is 0.64 between the first and last vertical arrows. The exponent of \( t_u = 0.55 \) is close to 0.93 while we expect 1, and \( t_u = 0.45 \) should saturate to a constant.

FIGURE 5. The functions \( \exp[-b(t, 0)] \), \( \exp[-b(t, t-t_f(t))] \) vs \( t \), for \( h = 0.2 \) and \( h = 0.1 \), in logarithmic coordinates. The functions \( \exp[-b(t, 20)] \) vs \( t - 20 \) and \( \exp[-b(t, 40)] \) vs \( t - 40 \). Here, 20 and 40 are waiting times. The behaviour of \( b(t, 0) \) and \( b(t, t-t_f(t)) \) is doubtless logarithmic. The slopes of \( \exp[-b(t, 0)] \), \( \exp[-b(t, t-t_f(t))] \), on this figure are respectively -1.10 and -0.42. According to the predictions of [10], \( \exp[-b(t, t')] \) tends to \( h(t) - h(t') \) for \( t, t' \gg 1 \); \( t/t' \) finite. The curves \( \exp[-b(t, 20)] \) and \( \exp[-b(t, 40)] \) tend to imitate \( \exp[-b(t, 0)] \), with a delay.

FIGURE 6. A test of the linear response of the displacement \( u(t) \). We plot \( u(t)/F \), as a function of \( \ln(t) \), for \( F = 0.05, 0.1, 0.2 \) and 0.4; \( h = 0.2 \). As \( F \to 0 \), the curves are indistinguishable from the integrated response \( R(t) \). A departure from the straight line signals the breakdown of the linear response, as the particle acquires a finite velocity, dependent (non-linearly) on the force. The suggested behaviour of \( u(t) \) is thus: \( u(t) = F \cdot (F_3 + F_4 \ln(t)) \).

FIGURE 7. The density \( \rho(\lambda) \) of eigenvalues \( \lambda - S \) (on top). A sketch of the self-similar distribution \( g = \hat{S}G(\lambda/S) \), assuming \( \Gamma = S \) (bottom). The tail of \( \hat{G} \) goes to 0 as \( \lambda/S \to \infty \), in the asymptotic limit \( S \to 0 \).
FIG. 2.

FIG. 3.
$u(t) / F = R(t,t')$

**FIG. 6.**

$S = 0$

$\rho(\lambda)$

$\lambda - S = 0$

$g(\lambda, t)$

**FIG. 7.**