Framings of $W_{g,1}$

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Abstract. We compute the set of framings of $W_{g,1} = D^{2n} \# gS^n \times S^n$, up to homotopy and diffeomorphism relative to the boundary.

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1. Introduction

Closed surfaces do not admit a framing unless they have genus 1, but surfaces of any genus with non-empty boundary do admit framings and there has been recent interest in understanding the set of such framings up to homotopy and diffeomorphism and, relatedly, the stabilisers of framings with respect to the action of the mapping class group [RW14, Kaw18, CS20b, CS20a, PCS20].

The analogues in higher dimensions of genus $g$ surfaces with one boundary component are the $2n$-manifolds $W_{g,1} := D^{2n} \# gS^n \times S^n$, which play a distinguished role in the study of diffeomorphism groups of $2n$-manifolds via homological stability [GRW18, GRW17]. These manifolds also admit framings, and all framings of $W_{g,1}$ induce the same homotopy class of framing of $TW_{g,1}|_{\partial W_{g,1}}$. Fixing once and for all a framing $\ell_\partial$ of $TW_{g,1}|_{\partial W_{g,1}}$ in this homotopy class, in our work on Torelli groups and diffeomorphism groups of disks [KRW19, KRW20] we have needed to study the moduli spaces of framed manifolds diffeomorphic to $W_{g,1}$ and with boundary condition $\ell_\partial$. The set of path components of this space is the set $\text{Str}^\ell_\partial(W_{g,1})$ of homotopy classes of framings of $W_{g,1}$ extending $\ell_\partial$, modulo the action of the diffeomorphism group. In that work we could get away with qualitative information about this set of path components: that it is finite. In this note we precisely determine it.

Theorem A. Let $g \geq 2$ and $n \geq 1$. The action of the mapping class group $\pi_0(\text{Diff}_\partial(W_{g,1}))$ on the set $\text{Str}^\ell_\partial(W_{g,1})$ of homotopy classes of framings extending $\ell_\partial$ has

(i) two orbits if $n = 1, 3, 7$ or $n \equiv 0 \pmod{4}$;
(ii) one orbit if $n \neq 1, 3, 7$ and $n \not\equiv 0 \pmod{4}$.

If $n > 1$ then in fact these hold for $g \geq 1$.

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While Theorem A is quite simple to formulate, our main interest is not so much in this statement but rather in related results concerning the stabilisers of framings with respect to the action of the mapping class group $\pi_0(\text{Diff}_\partial(W_{g,1}))$, especially in dimensions $2n \geq 6$. These results require substantial background to formulate, and we leave them to the body of the text: the main results in this direction are Propositions 3.3 and 3.5, and Corollary 5.2. In Section 6 we explain how highly-connected tangential structures can be analysed similarly.

Remark 1.1. The exceptions in Theorem A are covered by the following:

(i) The cases $g = 0$ are as in Table 2. (This uses that framings of $D^d$ up to homotopy and diffeomorphism, relative to a fixed boundary condition, are in bijection with $\pi_d(O(d))$, and that the diffeomorphisms of the disc act trivially on them by Lemma 2.2.)

(ii) The case $n = 1$ and $g = 1$ is given in [Kaw18, Theorem 3.12]; it is rather complicated.

The case $n = 1$ and $g \geq 2$ is contained in [RW14, Theorem 2.9]. We shall nonetheless continue discussing this case, to draw parallels between it and the high-dimensional case which is our main focus.

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2. Recollections and generalities

In this section we recall some notation and results from [KRW19], in particular Section 8 of that paper, specialized to framings.

2.1. The mapping class group. The mapping class group of $W_{g,1}$ is defined as

$$\Gamma_g := \pi_0(\text{Diff}_\partial(W_{g,1})) = \pi_1(B\text{Diff}_\partial(W_{g,1})),$$

where $\text{Diff}_\partial(W_{g,1})$ is the topological group of diffeomorphisms fixing a neighborhood of the boundary pointwise, in the $C^\infty$-topology. Let us write

$$H_n := H_n(W_{g,1}; \mathbb{Z}),$$

which has an action of the mapping class group. In high dimensions we will use an analysis of $\Gamma_g$ due to Kreck [Kre79] (see that paper for details about the definitions of the homomorphisms between the terms):

**Theorem 2.1** (Kreck). For $2n \geq 6$, the mapping class group $\Gamma_g := \pi_0(\text{Diff}_\partial(W_{g,1}))$ is described by the pair of extensions

$$1 \rightarrow I_g \rightarrow \Gamma_g \rightarrow G'_g \rightarrow 1,$$

$$1 \rightarrow \Theta_{2n+1} \rightarrow I_g \rightarrow \text{Hom}(H_n, S\pi_n(SO(n))) \rightarrow 1,$$

with $S\pi_n(SO(n))$ as in Table 1 and

$$G'_g := \begin{cases} 
\text{Sp}_{2g}(\mathbb{Z}) & \text{if } n \text{ is 3 or 7}, \\
\text{Sp}_{2g}^0(\mathbb{Z}) & \text{if } n \text{ is odd but not 3 or 7}, \\
O_{2g}(\mathbb{Z}) & \text{if } n \text{ is even},
\end{cases}$$

where $\text{Sp}_{2g}^0(\mathbb{Z}) \leq \text{Sp}_{2g}(\mathbb{Z})$ denotes the proper subgroup of symplectic matrices which preserve the standard quadratic refinement $\mu_0$. 
Table 1. The abelian groups $S\pi_n(SO(n))$ for $n \geq 1$, with the exceptions that $S\pi_1(SO(1)) = 0$, $S\pi_2(SO(2)) = 0$ and $S\pi_6(SO(6)) = 0$.

| $n \pmod{8}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------------|---|---|---|---|---|---|---|---|
| $S\pi_n(SO(n))$ | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}$ |

In particular, the homomorphism $\Gamma_g : G'_g \to GL_2(\mathbb{Z})$ is given by sending a diffeomorphism to the induced automorphism of $H_2$. Part of Kreck’s theorem is the analysis of precisely which automorphisms of $H_2(W_{g,1}; \mathbb{Z})$ arise from diffeomorphisms: they must preserve the intersection form, and for $n \neq 3, 7$ odd they must also preserve a further quadratic refinement. We will discuss this in more detail in Section 2.5.2.

For $n = 1, 2$ acting on middle homology, and preserving the intersection form, gives homomorphisms

\[ \Gamma_g \mapsto G'_g := \begin{cases} \text{Sp}_{2g}(\mathbb{Z}) & \text{if } n \text{ is } 1, \\ O_{g,g}(\mathbb{Z}) & \text{if } n \text{ is } 2, \end{cases} \]

and these are still surjective: the first is folklore, the second is [Wal64, Theorem 2].

The subgroup $\Theta_{2n+1} \subseteq \Gamma_g$ corresponds to $\pi_0(\text{Diff}_\beta(D^{2n}))$. We will make use of the following well-known fact about diffeomorphisms of the disc.

**Lemma 2.2.** For $2n \geq 6$ the derivative map

\[ \Theta_{2n+1} = \pi_0(\text{Diff}_\beta(D^{2n})) \mapsto \pi_0(\text{Bun}_0(TD^{2n})) = \pi_{2n}(SO(2n)). \]

is trivial.

**Proof.** This is equivalent to the well-known statement that the tangent bundle of a homotopy sphere is isomorphic to the tangent bundle of the standard sphere, see e.g. [RP80, §1]. Another argument is given in [KRW19, Lemma 8.15] (take $g = 0$), using [BL74]. \qed

### 2.2. Framings

Tangential structure on $2n$-dimensional manifolds can be equivalently described as $GL_{2n}(\mathbb{R})$-spaces, or as fibrations over $BO(2n)$ [GRW19, Section 4.5]. In particular, framings can be described by the $GL_{2n}(\mathbb{R})$-space given by $\Theta_{fr} = GL_{2n}(\mathbb{R})$ with action given by right multiplication, or by the fibration $\theta^\ell : EO(2n) \to BO(2n)$. Though we used the description in terms of the fibration $\theta^\ell$ in [KRW19, KRW20], here we find the former description more convenient.

A **framing** on $W_{g,1}$ is a map of $GL_{2n}(\mathbb{R})$-spaces $\ell : \text{Fr}(TW_{g,1}) \to \Theta_{fr}$, with $\text{Fr}(TW_{g,1})$ the frame bundle of the tangent bundle of $W_{g,1}$, which is a principal $GL_{2n}(\mathbb{R})$-bundle. We shall fix a boundary condition $\ell_\partial : \text{Fr}(TW_{g,1}|_{\partial W_{g,1}}) \to \Theta_{fr}$ and only consider those $\theta$-structures which extend this boundary condition: the **space of framings of $W_{g,1}$ extending $\ell_\partial$** is defined to be the space

\[ \text{Bun}_0(\text{Fr}(TW_{g,1}), \Theta_{fr}; \ell_\partial) \]

of $GL_{2n}(\mathbb{R})$-equivariant maps $\text{Fr}(TW_{g,1}) \to \Theta_{fr}$ extending $\ell_\partial$. We write

\[ \text{Str}^\ell_{fr}(W_{g,1}) := \pi_0(\text{Bun}_0(\text{Fr}(TW_{g,1}), \Theta_{fr}; \ell_\partial)) \]

for its set of path components.

The manifold $W_{g,1}$ indeed admits a framing: viewing it as the boundary connect-sum of $g$ copies of the plumbing of $S^n \times D^n$ with itself, it is enough to note that $S^n \times D^n$ may be framed. The following is a special case of [KRW19, Lemma 8.5]:

**Lemma 2.3.**

(i) Up to homotopy there is a unique orientation preserving boundary condition $\ell_\partial$ which extends to a framing $\ell$ on all of $W_{g,1}$.
(ii) For such a boundary condition there is a homeomorphism

$$\text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_\partial) \simeq \text{map}_\partial(W_{g,1}, \text{SO}(2n)),$$

depending on a choice of reference framing $\tau$ satisfying this boundary condition.

A choice of reference framing $\tau$ therefore induces, by Lemma 2.3 (ii), a bijection

$$\text{Str}^\ell_\partial(W_{g,1}) \overset{\sim}{\longrightarrow} \pi_0(\text{map}_\partial(W_{g,1}, \text{SO}(2n))),$$

though one must use it carefully as it depends on the choice of framing $\tau$.

The advantage of using $\text{GL}_{2n}(\mathbb{R})$-spaces as a model for tangential structures, rather than fibrations over $BO(2n)$, is that the homotopy equivalence in Lemma 2.3 (ii) can be upgraded to a homeomorphism

$$\text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_\partial) \overset{\sim}{\longrightarrow} \text{map}_\partial(W_{g,1}, \text{GL}_{2n}(\mathbb{R})).$$

The group $\text{Diff}_\partial(W_{g,1})$ acts on the space of framings $\text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_\partial)$ by taking derivatives. In particular it acts through the topological monoid $\text{Bun}_\partial(\text{Fr}(TW_{g,1}))$ of $\text{GL}_{2n}(\mathbb{R})$-maps $\text{Fr}(TW_{g,1}) \to \text{Fr}(TW_{g,1})$ that are the identity on the boundary. Using the reference framing $\tau$, the argument in [KRW19, Section 4] provides a homeomorphism of topological monoids

$$\text{Bun}_\partial(TW_{g,1}) \overset{\sim}{\longrightarrow} \text{map}_\partial(W_{g,1}, W_{g,1} \times \text{GL}_{2n}(\mathbb{R}))$$

under which composition of bundle maps corresponds to the operation

$$(f, \lambda) \circ (g, \rho) := (f \circ g, (\lambda \circ g) \cdot \rho),$$

with $\circ$ denoting composition of maps and $\cdot$ denoting pointwise multiplication. The right action of $\text{Bun}_\partial(\text{Fr}(TW_{g,1}))$ on $\text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_\partial)$ by precomposition then corresponds to

$$\text{map}_\partial(W_{g,1}, \text{GL}_{2n}(\mathbb{R})) \times \text{map}_\partial(W_{g,1}, W_{g,1} \times \text{GL}_{2n}(\mathbb{R})) \longrightarrow \text{map}_\partial(W_{g,1}, \text{GL}_{2n}(\mathbb{R}))$$

$$(h, (f, \lambda)) \longmapsto (h \circ f) \cdot \lambda,$$

under the homeomorphisms (1) and (2), where here $\cdot$ denotes the multiplication in $\text{GL}_{2n}(\mathbb{R})$. We write $h \oplus (f, \lambda)$ for this operation.

Since $\partial W_{g,1} \to W_{g,1}$ is 0-connected, all maps in the right hand side of (2) have image in $W_{g,1} \times \text{GL}_{2n}^+(\mathbb{R})$, with $\text{GL}_{2n}^+(\mathbb{R}) \leq \text{GL}_{2n}(\mathbb{R})$ the path-component of orientation-preserving invertible matrices. As the inclusion $\text{SO}(2n) \hookrightarrow \text{GL}_{2n}^+(\mathbb{R})$ is a homotopy equivalence, we phrase later computations in terms of the homotopy groups of $\text{SO}(2n)$ rather than $\text{GL}_{2n}^+(\mathbb{R})$.

### 2.3. The moduli space of framed manifolds

The moduli space of framed manifolds diffeomorphic to $W_{g,1}$ relative to the boundary, mentioned in the introduction, is defined to be the homotopy quotient

$$BDiff^\ell_\partial(W_{g,1}; \ell_\partial) := \text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_\partial) \sslash \text{Diff}_\partial(W_{g,1}).$$

There is a bijection

$$\pi_0(BDiff^\ell_\partial(W_{g,1}; \ell_\partial)) \overset{\sim}{\longrightarrow} \text{Str}^\ell_\partial(W_{g,1})/\Gamma_g,$$

and this is the set which Theorem A proposes to describe.

For $[\ell] \in \text{Str}^\ell_\partial(W_{g,1})$ we shall write

$$\Gamma^\ell_g([\ell]) := \text{Stab}_{\partial}([\ell])$$

for its stabiliser.

The framed mapping class group of a framing $\ell \in \text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_\partial)$ is defined as

$$\Gamma^\ell_g := \pi_1(BDiff^\ell_\partial(W_{g,1}; \ell_\partial), \ell),$$
and the long exact sequence for the homotopy orbits (3) gives a surjection \( \tilde{\Gamma}_g^{fr/} \to \Gamma_g^{fr,[\ell]} \).

We write

\[
G_g^{fr,[\ell]} := \text{im}(\Gamma_g^{fr,[\ell]} \to \Gamma_g \to G_g').
\]

2.4. Relaxing the boundary condition. It will be helpful to relax the condition that framings agree with \( \ell_g \) on all of the boundary and only ask that they agree at a point.

Fix a point \( * \in \partial W_{g,1} \). Then we let \( \text{Str}_{g}^{fr}(W_{g,1}) \) denote the homotopy classes of framings on \( W_{g,1} \) which agree with \( \ell_g \) at \( * \in \partial W_{g,1} \). Using the vanishing of Whitehead brackets in \( SO(2n) \), as in [KRW19, Section 8.2.2] we obtain a short exact sequence

\[
0 \longrightarrow \text{Str}_{g}^{fr}(D^{2n}) \longrightarrow \text{Str}_{g}^{fr}(W_{g,1}) \longrightarrow \text{Str}_{g}^{fr}(W_{g,1}) \longrightarrow 0
\]

Here the vertical isomorphisms are induced by \( \tau \), and this is in fact a short exact sequence of groups using the group structure coming from pointwise multiplication in \( SO(2n) \). The groups \( \pi_n(SO(2n)) \) are well-known by Bott periodicity, and the groups \( \pi_{2n}(SO(2n)) \) were computed by Kervaire [Ker60] (see Table 2).

**Table 2.** The groups \( \pi_{2n}(SO(2n)) \) for \( n \geq 1 \), with the exceptions that \( \pi_2(SO(2)) = 0 \) and \( \pi_6(SO(6)) = 0 \).

| \( n \) (mod 4) | 0 | 1 | 2 | 3 |
|----------------|---|---|---|---|
| \( \pi_{2n}(SO(2n)) \) | \( \mathbb{Z}/2^3 \) | \( \mathbb{Z}/4 \) | \( \mathbb{Z}/2^2 \) | \( \mathbb{Z}/4 \) |

The top short exact sequence is evidently equivariant for the right action of the mapping class group \( \Gamma_g \). The induced right actions on the bottom short exact sequence are as follows. It acts on the middle term via the derivative map \( \Gamma_g \to \pi_0(\text{Bun}_0(\text{Fr}(TW_{g,1}))) \), the identification \( \text{Bun}_0(\text{Fr}(TW_{g,1})) \cong \pi_0(\text{map}_0(W_{g,1}, W_{g,1} \times GL_{2n}(\mathbb{R}))) \) from (2) given by \( \tau \), and the action \( \circ \) described above. It acts on the right-hand term via the derivative map composed with the map \( \text{Bun}_0(\text{Fr}(TW_{g,1})) \to \text{Bun}_{1/2}(\text{Fr}(TW_{g,1})) \) which relaxes the boundary composition, followed by the analogue

\[
\text{Bun}_{1/2}(\text{Fr}(TW_{g,1})) \longrightarrow \text{map}_{1/2}(W_{g,1}, W_{g,1} \times GL_{2n}(\mathbb{R}))
\]

of (2) induced by \( \tau \), followed by the formula \( \alpha \circ (B, \beta) = \alpha \circ B + \beta \) written in terms of \( \pi_0(\text{map}_{1/2}(W_{g,1}, W_{g,1} \times GL_{2n}(\mathbb{R}))) \cong GL(H_n) \rtimes \text{Hom}(H_n, \pi_n(SO(2n))) \).

The \( \Gamma_g \)- and \( \text{Str}_{g}^{fr}(D^{2n}) \)-actions on \( \text{Str}_{g}^{fr}(W_{g,1}) \) commute because the \( \text{Str}_{g}^{fr}(D^{2n}) \)-action is by changing the framings in a small disc near the boundary, and diffeomorphisms in \( \Gamma_g \) can be changed by an isotopy so that they fix such a disc.

We write \( [[\ell]] \in \text{Str}_g^{fr}(W_{g,1}) \) for the class of a framing \( \ell \), and let

\[
\Gamma_g^{fr,[\ell]} := \text{Stab}_g(\Gamma_g(\ell))
\]

We define

\[
G_g^{fr,[\ell]} := \text{im}(\Gamma_g^{fr,[\ell]} \to \Gamma_g \to G_g').
\]

**Remark 2.4.** In [KRW19] we defined \( G_g^{fr,[\ell]} \) as the image of the stabiliser \( A_g^{fr,[\ell]} := \text{Stab}_g(\Gamma_g(\ell)) \) in \( G_g' \), where \( A_g := \Gamma_g/\Theta_{2n+1} \). The group \( A_g \) acts on \( \text{Str}_g^{fr}(W_{g,1}) \) because \( \Theta_{2n+1} \) consist of diffeomorphisms supported in a small disc near the boundary, and when the boundary condition has been relaxed near this ball the derivatives of such diffeomorphisms are homotopic to the identity (in fact, by Lemma 2.2) the subgroup \( A_g^{fr,[\ell]} \) already acts trivially on \( \text{Str}_g^{fr}(W_{g,1}) \). Then \( A_g^{fr,[\ell]} = G_g^{fr,[\ell]}/\Theta_{2n+1} \), so the images
of \( \Lambda^{fr}_{g}([ℓ]) \) and \( I^g_{fr}([ℓ]) \) in \( G'_1 \) are equal. The group \( \Lambda^{fr}_{g}([ℓ]) \) can also be interpreted in terms of self-embeddings, c.f. [KRW19, Section 8.5.2].

2.5. **Quadratic refinements.** The group \( H_n = H_n(W_{g,1}; \mathbb{Z}) \) is equipped with the intersection form \( λ: H_n \otimes H_n \to \mathbb{Z} \), which is \((-1)^n\)-symmetric. A function

\[
μ: H_n \to \begin{cases} 
\mathbb{Z} & \text{if } n \text{ is even} \\
\mathbb{Z}/2 & \text{if } n \text{ is odd}
\end{cases}
\]

is called a *quadratic refinement of \((H_n, λ)\)* if it satisfies

\[
μ(a ⋅ x) = a^2 μ(x)
μ(x + y) = μ(x) + μ(y) + λ(x, y),
\]

for \( a \in \mathbb{Z} \) and \( x, y \in H_n \), where in the latter \( λ(x, y) \) is reduced modulo 2 if \( n \) is odd. If \( n \) is even then these properties imply that \( μ(x) = \frac{1}{2}λ(x, x) \), so \((H_n, λ)\) has a unique quadratic refinement and it carries no further information: we shall therefore now suppose that \( n \) is odd, and write \( Quad(H_n, λ) \) for the set of quadratic refinements.

If \( μ \) and \( μ' \) are quadratic refinements of \((H_n, λ)\) then \( μ' - μ: H_n \to \mathbb{Z}/2 \) is linear, so \( Quad(H_n, λ) \) forms a \( \text{Hom}(H_n, \mathbb{Z}/2) \)-torsor. Equivalently, choosing a symplectic basis \( e_1, f_1, e_2, f_2, \ldots, e_g, f_g \) for \( H_n \), a quadratic refinement \( μ \) is uniquely and freely determined by the quadratic property (4) and the values \( μ(e_1), \ldots, μ(f_g) \in \mathbb{Z}/2 \).

2.5.1. **Classification of quadratic refinements.** The action of \( \text{Sp}_{2g}(\mathbb{Z}) \) on \( Quad(H_n, λ) \) is well-known [Arf41] to have two orbits, distinguished by the *Arf invariant*

\[
\text{Arf}(μ) = \sum_{i=1}^{g} μ(e_i)μ(f_i) \in \mathbb{Z}/2,
\]

where \( e_1, f_1, e_2, f_2, \ldots, e_g, f_g \in H_n \) is a symplectic basis.

Let us introduce some further notation for specific quadratic forms. Let \( H(0) \) denote the module \( \mathbb{Z}\{e, f\} \) with anti-symmetric form determined by \( λ(e, f) = 1 \) and quadratic refinement determined by \( μ(e) = 0 = μ(f) \). Let \( H(1) \) denote the module \( \mathbb{Z}\{e, f\} \) with the same anti-symmetric form but quadratic refinement determined by \( μ(e) = 1 = μ(f) \). These quadratic forms have Arf invariant 0 and 1 respectively. Since the Arf-invariant is additive under orthogonal sum, is valued in \( \mathbb{Z}/2 \), and is a complete invariant of quadratic forms, there is an isomorphism \( H(1)^{\otimes 2} \cong H(0)^{\otimes 2} \) of quadratic forms. (In the proof of Lemma 4.2 we will make an explicit choice of such an isomorphism.)

**Example 2.5.** There are \( 2^{2g-1} + 2^{g-1} \) quadratic refinements of Arf invariant 0. The *standard quadratic refinement* \( μ_0 \) is that of \( H(0)^{\otimes g} \), determined by

\[
μ_0(e_1) = μ_0(f_1) = \cdots = μ_0(e_g) = μ_0(f_g) = 0.
\]

The group \( \text{Sp}_{2g}(\mathbb{Z}) \) in the statement of Theorem 2.1 is the stabiliser of \( μ_0 \) for the action of \( \text{Sp}_{2g}(\mathbb{Z}) \) on the set of quadratic refinements.

**Example 2.6.** Similarly, there are \( 2^{2g-1} - 2^{g-1} \) quadratic refinements of Arf invariant 1. For concreteness, we take \( H(1) \oplus H(0)^{\otimes g-1} \) as the standard choice, and write \( \text{Sp}_{2g}(\mathbb{Z}) \leq \text{Sp}_{2g}(\mathbb{Z}) \) for its stabiliser.

2.5.2. **A quadratic form.** Suppose that \( n > 1 \). The group \( π_n(\text{Fr}(TW_{g,1})) \) may be interpreted via Hirsch–Smale theory as the set \( I^g_n(W_{g,1}) \) of regular homotopy classes of immersions \( j: S^n × D^n \to W_{g,1} \). Using the map \( π_n(\text{Fr}(TW_{g,1})) \to π_n(W_{g,1}) = H_n \) and the intersection form \( λ \), this has a (degenerate) \((-1)^n\)-symmetric bilinear form \( λ^{fr} \). This has a quadratic refinement \( μ^{fr} \) given by

\[
μ^{fr}(\{j\}) = \#\{\text{self-intersections of } j|_{S^n \times \{0\}}\} \pmod{2}.
\]
This construction is due to Wall [Wal99, Theorem 5.2]; see [GRW18, Definition 5.2] for a discussion specific to the manifolds $W_{g,1}$.

As the manifolds $W_{g,1}$ admit a framing there is a splittable short exact sequence

$$0 \rightarrow \pi_n(\SO(2n)) \xrightarrow{i} \pi_n(\Fr(TW_{g,1})) \rightarrow \pi_n(W_{g,1}) \rightarrow 0.$$  

Although the function $\mu^\ell$ is quadratic, the composition $\mu^\ell \circ i : \pi_n(\SO(2n)) \rightarrow \Z/2$ is linear (because $i(\pi_n(\SO(2n)))$ is the radical of the bilinear form $\lambda^\ell$).

**Lemma 2.7.** The map $\mu^\ell \circ i : \pi_n(\SO(2n)) \rightarrow \Z/2$ is onto if and only if $n = 3, 7$.

*Proof.* Recall the Whitney “figure eight” immersion $S^n \hookrightarrow D^{2n}$, which has one double point and has normal bundle isomorphic to $TS^n$.

If $n = 3, 7$ then $TS^n$ is trivial, so the Whitney immersion may be extended to an immersion $j : S^n \times D^n \hookrightarrow D^{2n} \subset W_{g,1}$ which satisfies $\mu([j]) = 1$, and so the composition

$$\pi_n(\SO(2n)) = \pi_n(\Fr(D^{2n})) \xrightarrow{i} \pi_n(\Fr(TW_{g,1})) \xrightarrow{\mu^\ell} \Z/2$$  

is surjective.

If $n \neq 3, 7$, suppose for a contradiction that $j : S^n \times D^n \hookrightarrow D^{2n} \subset W_{g,1}$ has $\mu^\ell([j]) = 1$. We may form the ambient connect-sum inside $D^{2n}$ of the immersion $j|_{S^n \times \{0\}}$ with a disjoint copy of the Whitney immersion, giving an immersion $j' : S^n \hookrightarrow D^{2n}$ having an even number of double points and having normal bundle $TS^n$. Using the Whitney trick we can eliminate the double points to obtain an embedding $j''$, still having normal bundle $TS^n$: as $TS^n$ is a non-trivial bundle for $n \neq 3, 7$ this is a contradiction, by [Ker59].

It follows that for $n \neq 3, 7$ the function $\mu^\ell$ descends to a function $\mu : H_n \rightarrow \Z/2$, a quadratic refinement of $(H_n, \lambda)$. As the standard symplectic basis of $H_n$ may be represented by embedded normally-framed spheres, in the notation of the previous section this gives the quadratic form $H(0)^{\oplus 2g}$. Diffeomorphisms of $W_{g,1}$ must also preserve this quadratic refinement: this accounts for the fact that $G'_g = \Sp_{2g}^2(\Z)$ in these dimensions in Theorem 2.1.

2.5.3. Quadratic refinements from framings. For $n$ odd a framing $\ell : W_{g,1} \rightarrow \Fr(TW_{g,1})$ may be used to define

$$\mu : H_n = \pi_n(W_{g,1}) \xrightarrow{\ell^*} \pi_n(\Fr(TW_{g,1})) \xrightarrow{\mu^\ell} \Z/2,$$

which is a quadratic refinement of $(H_n, \lambda)$. This construction provides a $\Gamma_g$-equivariant function

$$\Phi : \St_{2g}^1(W_{g,1}) \rightarrow \Quad(H_n, \lambda),$$

where the action of $\Gamma_g$ on $\Quad(H_n, \lambda)$ is via $G'_g$.

**Lemma 2.8.** For $n = 3, 7$ the function $\Phi$ is surjective.

*Proof.* Choose a framing $\ell$, and let $\mu' \in \Quad(H_n, \lambda)$. The function $\mu' - \mu_\ell$ is a homomorphism $L : H_n \rightarrow \Z/2$, and as $H_n$ is a free $\Z$-module and $\mu^\ell \circ i$ is surjective by Lemma 2.7, we may choose a homomorphism $\delta : H_n \rightarrow \pi_n(\SO(2n))$ such that $\mu^\ell \circ i \circ \delta = L$. But then if the framing $\ell$ is changed using $\delta$ to get a new framing $\delta \cdot \ell$, we have $\mu_{\delta \cdot \ell} = \mu_\ell + L$, so $\mu' = \mu_{\delta \cdot \ell}$. Thus $\Phi$ is a surjection. □
2.6. Orbits and stabilisers. Our proof of Theorem A will be in terms of the sequence
\[ 0 \to \Gamma_g^{\text{fr.}[\ell]} \to \Gamma_g^{\text{fr.}[\ell]} \xrightarrow{f_g} \text{Str}^\text{fr}_g(D^{2n}) \xrightarrow{\ell[^g]} \text{Str}_g^\text{fr}(W_{g,1})/\Gamma_g \to \text{Str}_g^\text{fr}(W_{g,1})/\Gamma_g \to \{e\} \] (5)
which is exact in the sense of groups and pointed sets. In particular, \( f_g \) is a group homomorphism. This sequence comes from the long exact sequence on homotopy groups for the principal \( \text{Str}_g^\text{fr}(D^{2n}) \)-bundle
\[ \text{Str}_g^\text{fr}(D^{2n}) \to \text{Str}_g^\text{fr}(W_{g,1})/\Gamma_g \to \text{Str}_g^\text{fr}(W_{g,1})/\Gamma_g, \]
and the orbit-stabiliser theorem.

The strategy of our argument will be as follows. We will estimate the size of \( \text{im}(f_g) = I_g^{\text{fr.}[\ell]} / \Gamma_g^{\text{fr.}[\ell]} \) from below using the surjection
\[ I_g^{\text{fr.}[\ell]} / \Gamma_g^{\text{fr.}[\ell]} \to G_g^{\text{fr.}[\ell]} / G_g^{\text{fr.}[\ell]}. \]
We will identify the group \( G_g^{\text{fr.}[\ell]} \) as the automorphism group of a certain quadratic form (in fact we will have \( G_g^{\text{fr.}[\ell]} = G_g^{\text{fr}} \) for \( n \neq 1, 3, 7 \), and a mild variant for \( n = 1, 3, 7 \)), whose abelianisation is known to have order 4. We will then show by geometric considerations that \( G_g^{\text{fr.}[\ell]} \) has trivial abelianisation, so that \( G_g^{\text{fr.}[\ell]} / G_g^{\text{fr}[\ell]} \) has order at least 4. We then use \( \text{Str}_g^\text{fr}(D^{2n}) = \pi_{2n}(\text{SO}(2n)) \) and consult Table 2.

For \( n \neq 0 \) (mod 4) the group \( \pi_{2n}(\text{SO}(2n)) \) has order 4, so it follows that the map \( f_g \) is a surjection, and so \( I_g^{\text{fr.}[\ell]} \) has index precisely 4 in \( \Gamma_g^{\text{fr.}[\ell]} \). Thus \( G_g^{\text{fr.}[\ell]} / G_g^{\text{fr}[\ell]} \) has order precisely 4, from which we will deduce that \( G_g^{\text{fr}[\ell]} \) is precisely the commutator subgroup of \( G_g^{\text{fr.}[\ell]} \).

For \( n \equiv 0 \) (mod 4) the group \( \pi_{2n}(\text{SO}(2n)) \) has order 8, and the argument is a little more complicated. We will show that the image of \( f_g \) has index 2 in \( \text{Str}_g^\text{fr}(D^{2n}) = \pi_{2n}(\text{SO}(2n)) = (\mathbb{Z}/2)^3 \).

3. Counting framings relative to a point

In this section we determine \( \text{Str}_g^\text{fr}(W_{g,1})/\Gamma_g \).

Proposition 3.1. Suppose \( n \geq 1 \) and \( g \geq 2 \), then

- If \( n \neq 1, 3, 7 \), then \( \text{Str}_g^\text{fr}(W_{g,1})/\Gamma_g \) consists of a single element.
- If \( n = 1, 3, 7 \), then \( \text{Str}_g^\text{fr}(W_{g,1})/\Gamma_g \) consists of two elements.

If \( n > 1 \) then these in fact hold for \( g \geq 1 \).

3.1. The cases \( n \geq 3 \). In this case Theorem 2.1 is available, and we will study the action of \( I_g \leq \Gamma_g \) on \( \text{Str}_g^\text{fr}(W_{g,1}) \). Derivatives of elements of \( \Theta_{2n+1} \leq I_g \) are bundle maps supported in a small disc which can be taken to be near the boundary: as in Remark 2.4 these act trivially on \( \text{Str}_g^\text{fr}(W_{g,1}) \). Thus the action of \( I_g \) on \( \text{Hom}(H_n, \pi_n(\text{SO}(2n))) \) factors over \( \text{Hom}(H_n, \pi_n(\text{SO}(2n))) \) to the target. This homomorphism was studied by Levine; Theorem 1.4 of [Lev85] and Table 1 imply:

Lemma 3.2. For \( 2n \geq 6 \) the stabilisation \( S\pi_n(\text{SO}(n)) \to \pi_n(\text{SO}(2n)) \) is:

(i) surjective with kernel \( \mathbb{Z}/2 \) when \( n \) is even,
(ii) an isomorphism when \( n \) is odd but not 3 or 7,
(iii) injective with cokernel \( \mathbb{Z}/2 \) when \( n = 3, 7 \).

We conclude that:

Proposition 3.3. Suppose \( 2n \geq 6 \) and consider the action of \( I_g \) on \( \text{Str}_g^\text{fr}(W_{g,1}) \).
(i) When \( n \neq 3, 7 \), this action has a single orbit.

(ii) When \( n = 3, 7 \), the set of orbits is in bijection with \( \text{Hom}(H_n, \mathbb{Z}/2) \).

In either case the stabiliser \( I_g^{fr, [[\ell]]} \) of any \( [[\ell]] \in \text{Str}^g_{fr}(W_{g,1}) \) satisfies

\[
I_g^{fr, [[\ell]]}/\Theta_{2n+1} \cong \begin{cases} 
0 & \text{if } n \text{ is odd} \\
\text{Hom}(H_n, \mathbb{Z}/2) & \text{if } n \text{ is even}.
\end{cases}
\]

**Proof.** When \( n \neq 3, 7 \), Lemma 3.2 says that \( I_g \) surjects on to \( \text{Hom}(H_n, \pi_n(SO(2n))) \). The action on \( \text{Hom}(H_n, \pi_n(SO(2n))) \) is through addition, which is therefore transitive. When \( n = 3, 7 \) the lemma shows that \( I_g \) maps to \( \text{Hom}(H_n, \pi_n(SO(2n))) \) with cokernel \( \text{Hom}(H_n, \mathbb{Z}/2) \), so the orbits are in bijection with this set. For the stabiliser, Lemma 3.2 shows that the kernel of \( S\pi_n(O(n)) \to \pi_n(SO(2n)) \) if 0 if \( n \) is odd and \( \mathbb{Z}/2 \) if \( n \) is even.

This proves Proposition 3.1 when \( n \geq 3 \) and \( n \neq 3, 7 \), as \( \text{Str}^g_{fr}(W_{g,1})/I_g \) already consists of a single point, so \( \text{Str}^g_{fr}(W_{g,1})/\Gamma_g \) does too.

To finish the argument in the cases \( n = 3, 7 \), we use quadratic refinements to give a more invariant description of \( \text{Str}^g_{fr}(W_{g,1})/I_g \). Recall that in Section 2.5.3 we described a \( \Gamma_g \)-equivariant function \( \Phi: \text{Str}^g_{fr}(W_{g,1}) \to \text{Quad}(H_n, \lambda) \).

**Lemma 3.4.** For \( n = 3, 7 \) the induced function

\[ \text{Str}^g_{fr}(W_{g,1}) / I_g \longrightarrow \text{Quad}(H_n, \lambda) \]

is a bijection.

**Proof.** By Lemma 2.8 it is surjective. The target has \( 2^{2g} \) elements as it is a \( \text{Hom}(H_n, \mathbb{Z}/2) \)-torsor, and the domain has \( 2^{2g} \) elements by Proposition 3.3 (ii), so it is a bijection.

Referring to the discussion in Section 2.5.1, it follows from the theorem of Arf that \( \text{Str}^g_{fr}(W_{g,1})/\Gamma_g \) consists of two elements, distinguished by the Arf invariants of their associated quadratic forms. This completes the proof of Proposition 3.1 in the cases \( n = 3, 7 \).

3.2. The case \( n = 2 \). When \( n = 2 \) we have \( \pi_n(SO(2n)) = 0 \) so \( \text{Str}^g_{fr}(W_{g,1}) \) consists of a single point, so has a single \( \Gamma_g \)-orbit too.

3.3. The case \( n = 1 \). As \( \text{Str}^g_{fr}(D^2) = \pi_2(SO(2)) = 0 \), the orbit-stabiliser sequence gives a bijection \( \text{Str}^g_{fr}(W_{g,1})/\Gamma_g \cong \text{Str}^g_{fr}(W_{g,1})/I_g \). For \( g \geq 2 \) the latter is described in [RW14, Theorem 2.9] (for framings one takes \( r = 0 \)) as having two elements, which proves Proposition 3.1 in this case.

In fact the analogue of Lemma 3.4 also holds in this case. A framing of \( W_{g,1} \) determines a Spin structure, which via a construction of Johnson [Job80] gives a quadratic refinement of the symplectic form \( (H_1(W_{g,1}; \mathbb{Z}), \lambda) \). This construction yields a surjective map

\[ \text{Str}^g_{fr}(W_{g,1}) \longrightarrow \text{Quad}(H_1, \lambda) \]

which as long as \( g \geq 2 \) becomes, just as in Lemma 3.4, a bijection after dividing out the action of the Torelli group \( I_g \). This has been shown in [CS20b, Proposition 5.1]. It may also be seen using the methods of [RW14, Sections 2.3, 2.4] and the fact that the Torelli group is generated by Dehn twists along separating curves and bounding pairs of curves.
3.4. The group $G^\text{fr.}[[\ell]]$. The results of the previous sections combine to give the following complete description of the group $G^\text{fr.}[[\ell]]$.

**Proposition 3.5.** For $n \geq 1$, and $g \geq 2$ if $n = 1$, we have

$$G^\text{fr.}[[\ell]] = \begin{cases} \text{Sp}^0_{2g} \text{ or } \text{Sp}^0_{2g}(\mathbb{Z}) & \text{if } n = 1, 3 \text{ or } 7, \text{ and } \ell \text{ has Arf invariant } 0 \text{ or } 1, \\ \text{Sp}^0_{2g}(\mathbb{Z}) & \text{if } n \text{ is odd but not } 1, 3 \text{ or } 7, \\ \text{O}_{g,g}(\mathbb{Z}) & \text{if } n \text{ is even.} \end{cases}$$

**Proof.** If $n = 2$ then as in Section 3.2 there is only one framing relative to a point, so $\Gamma^\text{fr.}[[\ell]] = \Gamma_g$ and hence $G^\text{fr.}[[\ell]] = G_g^\text{fr}$.

For $n \neq 2$ the group $G^\text{fr.}[[\ell]]$ is the stabiliser of the class of $\ell$ in $\text{Str}_{n,1}^g(W_{g,1})/I_g$ under the residual $\Gamma_g/I_g = G^\text{g}_g$-action. If $n \neq 1, 3, 7$ then this set has a single element and so the stabiliser is $G^\text{fr}_g$ itself. If $n = 1, 3, 7$ then this set is identified with $\text{Quad}(H_n, \lambda)$ and so $G^\text{fr.}[[\ell]]$ is the stabiliser of the quadratic form determined by $\ell$. As there are two orbits of quadratic forms, distinguished by their Arf invariant, this stabiliser is conjugate to $\text{Sp}^0_{2g}(\mathbb{Z})$ if the Arf invariant is 0, and to $\text{Sp}^0_{2g}(\mathbb{Z})$ if the Arf invariant is 1.

For $2n \geq 6$ there is by definition there is an extension

$$0 \longrightarrow I^\text{fr.}[[\ell]] \longrightarrow \Gamma^\text{fr.}[[\ell]] \longrightarrow G^\text{fr.}[[\ell]] \longrightarrow 0$$

and in Proposition 3.3 we have shown that there is an extension

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow I^\text{fr.}[[\ell]] \longrightarrow \begin{cases} 0 & \text{if } n \text{ is odd} \\ \text{Hom}(H_n, \mathbb{Z}/2) & \text{if } n \text{ is even} \end{cases} \longrightarrow 0.$$

With Proposition 3.5 this gives a description of $\Gamma^\text{fr.}[[\ell]]$ analogous to the theorem of Kreck (Theorem 2.1).

4. Arithmetic groups

4.1. Abelianisations of some arithmetic groups. By stabilising by direct sum with a hyperbolic form, of Arf invariant 0 in the case of quadratic structures, we have stable groups

$$\text{Sp}_\infty(\mathbb{Z}), \ \text{Sp}^0_\infty(\mathbb{Z}), \ \text{Sp}^0_\infty(\mathbb{Z}), \ \text{O}_\infty(\mathbb{Z}).$$

Furthermore, as $H(0)\text{^{ab2}} \cong H(1)^{ab2}$ there is a direct system of groups containing both $\{\text{Sp}^0_{2g}(\mathbb{Z})\}_{g \geq 0}$ and $\{\text{Sp}^0_{2g}(\mathbb{Z})\}_{g \geq 0}$ cofinally, so $\text{Sp}^0_\infty(\mathbb{Z}) \cong \text{Sp}^0_{\infty}(\mathbb{Z})$. The abelianisations of these stable groups are well-known:

$$H_1(\text{Sp}_{\infty}(\mathbb{Z})) = 0,$$

$$H_1(\text{Sp}^0_{\infty}(\mathbb{Z})) = H_1(\text{Sp}^0_{\infty}(\mathbb{Z})) = \mathbb{Z}/4,$$

$$H_1(\text{O}_\infty(\mathbb{Z})) = (\mathbb{Z}/2)^2.$$  

These are collected from the literature in [GRW16, Proposition 2.2]. Such automorphism groups of quadratic forms over $\mathbb{Z}$ enjoy homological stability: in the generality needed here this may be found in [Fri17, Theorem 3.25], but for some of these groups it was known much earlier. We shall only need to know that the abelianisation of the $g = 1$ group surjects onto to the abelianisation of the $g = \infty$ one, but we give complete information about their abelianisations for all $g$ in Table 3.

**Remark 4.1.** It may be helpful to alert the reader that [Kre79, page 645] states incorrectly that $\text{O}_{1,1}(\mathbb{Z}) = \mathbb{Z}/4$, a mistake going back to [Sat69].
4.1.1. The (quadratic) symplectic groups. The first homology group of $\text{Sp}_2^q(\mathbb{Z})$ and $\text{Sp}_4^q(\mathbb{Z})$ in low genus is tabulated in [Kra19, Lemma A.1], and that of $\text{Sp}_2^q(\mathbb{Z})$ has recently been calculated by Sierra [Sic] and will appear in his forthcoming Cambridge PhD thesis. They are as shown in Table 3.

**Lemma 4.2.** Each of the stabilisation maps $H_1(\text{Sp}_2(\mathbb{Z})) \rightarrow H_1(\text{Sp}_q^0(\mathbb{Z}))$, $H_1(\text{Sp}_2^q(\mathbb{Z})) \rightarrow H_1(\text{Sp}_q^0(\mathbb{Z}))$, and $H_1(\text{Sp}_2^q(\mathbb{Z})) \rightarrow H_1(\text{Sp}_q(\mathbb{Z}))$ is surjective.

**Proof.** In the first case there is nothing to show as the stable homology is trivial. The second case is [Kra19, Lemma A.1 (iii)].

In the third case observe that as in Example 2.6 the number of quadratic refinements of $(H_n(W_{1,1};\mathbb{Z}),\lambda)$ having Arf invariant 1 is $2^{2-1} - 2^{1-1} = 1$, so we have $\text{Sp}_q^0(\mathbb{Z}) = \text{Sp}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})$. The first homology of this group is $\mathbb{Z}/12$ and an element of order 4 is represented by the matrix

$$S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. $$

Stabilising $H(1)$ by taking the direct sum with another copy of $H(1)$, we get a stabilisation map $\text{Sp}_2^q(\mathbb{Z}) \rightarrow \text{Sp}_q^0(\mathbb{Z})$. We shall compute its image under the stable abelianisation map $\lambda: \text{Sp}_2^q(\mathbb{Z}) \rightarrow \mathbb{Z}/4$ using a formula of Johnson and Millson [JM90] for this map.

To do so, we need to make explicit the isomorphism $H(1) \oplus H(1) \cong H(0) \oplus H(0)$. Let $e_1, f_1, e_2, f_2$ be the standard “hyperbolic” basis of $H(1) \oplus H(1)$. Making explicit this isomorphism amounts to finding a basis $\tilde{e}_1, \tilde{f}_1, \tilde{e}_2, \tilde{f}_2$ of $H(1) \oplus H(1)$ satisfying

\[
\begin{align*}
\lambda(\tilde{e}_i, \tilde{e}_j) &= 0 = \lambda(\tilde{f}_i, \tilde{f}_j), \\
\lambda(\tilde{e}_i, \tilde{f}_j) &= \delta_{ij}, \text{ and} \\
\mu(\tilde{e}_i) &= \mu(\tilde{f}_i) = 0.
\end{align*}
\]

The choice

$\tilde{e}_1 = e_1 + e_2$, $\tilde{f}_1 = f_1 + e_1 + e_2$, $\tilde{e}_2 = e_2 - f_1 + f_2$, $\tilde{f}_2 = -f_1 + f_2$

will do. Writing $S \oplus [1 0] \oplus [0 1]$ in terms of the basis $\{f_1, \tilde{e}_2, \tilde{f}_1, \tilde{f}_2\}$ gives the matrix

$$\tilde{S} = \begin{bmatrix}
1 & 1 & -1 & -1 \\
-2 & 2 & 1 & -1 \\
2 & 0 & -1 & 0 \\
-2 & 1 & 1 & 0
\end{bmatrix}. $$

Johnson and Millson give an explicit formula for the homomorphism $\lambda: \text{Sp}_2^q(\mathbb{Z}) \rightarrow \mathbb{Z}/4 = \{1, -1, i, -i\}$ [JM90, pages 147-148] (their conventions are the reason for reordering our basis). Evaluated on $\tilde{S}$ we are in the case “2 (ii)”, and $\lambda(\tilde{S}) = i^{-1}e(1:1 - (-2):1) = i$, which has order 4.

4.1.2. The orthogonal groups. We provide the analogue of the results of the last section for the groups $\text{O}_{g,\rho}(\mathbb{Z})$.

**Lemma 4.3.** For $g \neq 2$ the first homology groups of $\text{O}_{g,\rho}(\mathbb{Z})$ are as in Table 3. The stabilisation map $H_1(\text{O}_{1,1}(\mathbb{Z})) \rightarrow H_1(\text{O}_{\infty,\infty}(\mathbb{Z}))$ is a surjection.
Proof. We shall use results of Hahn–O’Meara: [HO89, Theorem 9.2.8] says the kernel of the map $O_{g,2}(Z) \to (Z/2)^2$ given by the determinant and spinor norm is equal to the subgroup generated by elementary matrices for $g \geq 2$, and [HO89, 5.3.8] says that this subgroup is perfect for $g \geq 3$. It follows that $(Z/2)^2$ is the abelianisation for $g \geq 3$.

For $g = 1$ it is easy to verify that the determinant and spinor norm map $O_{1,1}(Z) \to (Z/2)^2$ is an isomorphism of groups, and the claim about the stabilisation map follows from this.

It remains to describe the case $g = 2$.

Lemma 4.4. $H_1(O_{2,2}(Z)) \cong (Z/2)^3$.

Proof. We claim that $O_{2,2}^+(Z) := \text{ker}(\det \oplus \text{spin}: O_{2,2}(Z) \to (Z/2)^2)$ is isomorphic to the group $\text{SL}_2(Z) \times \text{SL}_2(Z)/\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$.

By [HO89, Theorem 7.2.21] there is an extension $1 \to Z^\times \to \text{Spin}_{2,2}(Z) \to O_{2,2}^+(Z) \to 1$, and by [HO89, p. 434] there is an exceptional isomorphism $\text{Spin}_{2,2}(Z) \cong \text{SL}_2(Z) \times \text{SL}_2(Z)$.

We can make this explicit as follows. Consider the set $M_{2,2}(Z)$ of $(2 \times 2)$-matrices over $Z$, equipped with the bilinear form given by $\langle X, Y \rangle = \text{tr}(X \Omega Y \Omega^t)$, $\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Explicitly it is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = ad' - bc' - cb' + da',$$

so is symmetric and even. This formula also makes it clear that $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $f_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$ provides a hyperbolic basis for $(M_{2,2}(Z), \langle \cdot, \cdot \rangle)$. There is a left action of $\text{SL}_2(Z) \times \text{SL}_2(Z)$ on $M_{2,2}(Z)$ by $(A, B) \cdot X = AXB^{-1}$, and one may check that this action preserves the form $\langle \cdot, \cdot \rangle$, using that $A^{-1} = \Omega A^t \Omega$ for $A \in \text{SL}_2(Z)$. This describes the composition $\text{SL}_2(Z) \times \text{SL}_2(Z) \cong \text{Spin}_{2,2}(Z) \to O_{2,2}(Z)$.

We see that $Z^\times \to \text{Spin}_{2,2}(Z)$ is given by $( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix})$, which establishes the claim, and also that the elements $(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ and $(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ in $\text{SL}_2(Z) \times \text{SL}_2(Z)$ map to

$$T_1 := \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

respectively, with respect to the basis $(e_1, f_1, e_2, f_2)$.

We now calculate as follows. It is well known that $H_1(\text{SL}_2(Z)) = Z/12$ is generated by $T := \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and minus the identity matrix represents the element of order 2 in this group, so $H_1(O_{2,2}(Z))$ has a presentation as an abelian group by $(T_1, T_2 | 12T_1, 12T_2, 6(T_1 + T_2))$.

The group $O_{1,1}(Z) = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ may be included in $O_{2,2}(Z)$ by stabilisation, and is mapped isomorphically to $(Z/2)^2$ by the determinant and spinor norm. Thus the outer action of $(Z/2)^2$ on $O_{2,2}(Z)$ may be described by conjugating by (the stabilisations of) these two matrices. Conjugating by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acts as

$$T_1 \mapsto T_1^{-1}, \quad T_2 \mapsto T_2^{-1}.$$
The homotopy groups which is an isomorphism in the limit simplifies to give vanishing result does not hold for \( n \) a quotient of the framed mapping class group \( \tilde{\Gamma} \), Lemma 4.5. Considerations of framed fibre bundles. It will follow by combining Corollary 5.2, Proposition 3.5, and Table 3 that this vanishing result does not hold for \( n \neq 1, 3 \).

**Proof.** Recall that we write \( \tilde{\Gamma} \) := \( \pi_1(B\text{Diff}_g^r(W_{g,1}; \ell_\partial), \ell) \), so that \( \Gamma_{\tilde{g}}^r \) is the image of the forgetful map \( \tilde{\Gamma} \rightarrow \Gamma \). The composition
\[
\tilde{\Gamma} = H_1(\Gamma_{\tilde{g}_{\ell}}) \rightarrow H_1(\tilde{\Gamma}_{\tilde{g}_{\ell}}) \rightarrow H_1(\Gamma_{\tilde{g}_{\ell}}) \rightarrow H_1(\Gamma_{\tilde{g}_{\ell}})
\]
is zero, as diffeomorphisms supported inside a disc act trivially on the middle homology of \( W_{g,1} \). The two rightmost maps are surjective, as the underlying maps of groups are surjective.

By an application of [GRW17, Theorem 1.5] there is a map
\[
H_1(\tilde{\Gamma}_{\tilde{g}_{\ell}}) = H_1(B\text{Diff}_g^r(W_{g,1}; \ell_\partial), \ell) \rightarrow H_1(\Omega_2^s S^{2n}) = \pi_{2n+1} = \pi_{2n+1}(G)
\]
which is an isomorphism in the limit \( g \rightarrow \infty \) (this formulation is valid even if \( 2n = 4 \), as it does not rely on homological stability). Considering the target as framed cobordism, this map is given by the Pontrjagin–Thom construction.

If \( n = 2 \) then we use that \( \pi_2^s = 0 \), so that \( \text{colim}_{g \rightarrow \infty} H_1(\tilde{\Gamma}_{\tilde{g}_{\ell}}) = 0 \) surjects onto \( \text{colim}_{g \rightarrow \infty} H_1(\Gamma_{\tilde{g}_{\ell}}) \).

For \( n > 2 \) will show that \( H_1(\tilde{\Gamma}_{\tilde{g}_{\ell}}) \rightarrow \text{colim}_{g \rightarrow \infty} H_1(\tilde{\Gamma}_{\tilde{g}_{\ell}}) \) is surjective, with which the fact that the composition (6) is zero gives the result. To do so we use smoothing theory to obtain an identification \( \tilde{\Gamma}_{\tilde{g}_{\ell}} \) := \( \pi_{2n+1}(\text{Top}(2n)) \), and it is a matter of interpreting the Pontrjagin–Thom construction to see that \( \tilde{\Gamma}_{\tilde{g}_{\ell}} \rightarrow \text{colim}_{g \rightarrow \infty} H_1(\tilde{\Gamma}_{\tilde{g}_{\ell}}) = \pi_{2n+1} = \pi_{2n+1}(G) \) agrees with the natural composition
\[
\pi_{2n+1}(\text{Top}(2n)) \rightarrow \pi_{2n+1}(\text{Top}) \rightarrow \pi_{2n+1}(G).
\]
The homotopy groups \( \pi_i(G/\text{Top}) \) are identified with the simply-connected surgery obstruction groups \( L_i(\mathbb{Z}) \) [KS77, p. 274], which vanish in odd degrees so follows that the right map is surjective; thus we shall be done if we show that the left map is surjective.

To show this we will instead show that the connecting map
\[
\partial: \pi_{2n+1}(\text{Top}, \text{Top}(2n)) \rightarrow \pi_2(\text{Top}(2n))
\]
is injective. From [KS77, page 246], it follows that the map
\[
\pi_{2n+1}(O, O(2n)) \rightarrow \pi_{2n+1}(\text{Top}, \text{Top}(2n))
\]
is an isomorphism, and as in [GRW16, Lemma 5.2] by work of Paechter [Pae56] we have

\[ \pi_{2n+1}(O, O(2n)) = \begin{cases} \mathbb{Z}/4 & n \text{ odd} \\ (\mathbb{Z}/2)^2 & n \text{ even} \end{cases} \]

We therefore consider the diagram

\[
\begin{array}{c}
\pi_{2n+1}(\text{Top}(2n)/O(2n)) \xrightarrow{\cong} \pi_0(\text{Diff}_D(D^{2n})) \\
\downarrow \\
\pi_{2n+1}(O, O(2n)) \to \pi_{2n}(O(2n)) \to \pi_{2n}(O)
\end{array}
\]

The top vertical map is zero: it corresponds to the map sending a diffeomorphism to its derivative, which is trivial by Lemma 2.2. Thus the bottom middle vertical map is injective. On the other hand we know the groups \( \pi_{2n}(O(2n)) \) from Table 2, and the groups \( \pi_{2n}(O) \) from Bott periodicity, so we may simply check that \( \pi_{2n+1}(O, O(2n)) \to \pi_{2n}(O(2n)) \) must be injective for \( n > 3 \). It follows from commutativity of the square that the bottom map is injective, as required. \( \square \)

5. Comparing stabilisers and the proof of Theorem A

In this section we wish to analyse the exact sequence

\[ 0 \to \Gamma^\fr_g,[[\ell]] \to \Gamma^\fr_g,[[\ell]] \xrightarrow{f_\ell} \text{Str}_{\ell}^\fr_0(D^{2n}) \cong \pi_{2n}(\text{SO}(2n)), \]  

coming from (5), as follows.

**Proposition 5.1.** For any \( n > 0 \) the map \( f_\ell \) factors as

\[ \Gamma^\fr_g,[[\ell]] \to G^\fr_g,[[\ell]] \to H_1(G^\fr_g,[[\ell]]) \to H_1(\text{SO}(2n)) \]

for some \( h_\ell \), where the first three maps are the natural quotient, abelianisation, and stabilisation maps.

(i) If \( n = 1, 3 \) then \( \pi_{2n}(\text{SO}(2n)) = 0 \) (so the map \( f_\ell \) is surjective).

(ii) If \( n \not\equiv 0 \pmod{4} \) and \( n \neq 1, 3 \) then \( h_\ell \) is an isomorphism.

(iii) If \( n \equiv 0 \pmod{4} \) then \( h_\ell \) is injective with image of index 2 in \( \pi_{2n}(\text{SO}(2n)) \cong (\mathbb{Z}/2)^3 \).

In Remark 6.4 we will determine more precisely the index 2 subgroup in part (iii).

**Proof.** If \( n = 1, 3 \) then \( \pi_{2n}(\text{SO}(2n)) = 0 \) so the claims are vacuous.

By naturality of the sequence (7) under boundary connect-sum, the connecting map \( f_\ell \) factors over (the abelianisation of) its stabilisation, i.e. as

\[ \Gamma^\fr_g,[[\ell]] \to \Gamma^\fr_g,[[\ell]] \to H_1(\Gamma^\fr_g,[[\ell]]) \xrightarrow{g_\ell} \pi_{2n}(\text{SO}(2n)). \]

To prove the first part we must show that this map \( g_\ell \) factors as

\[ H_1(\Gamma^\fr_\infty,[[\ell]]) \to H_1(G^\fr_\infty,[[\ell]]) \to \pi_{2n}(\text{SO}(2n)) \]

for some (unique, as the first map is surjective) map \( h_\ell \).

If \( n = 2 \) then by [Kre79, Theorem 1] the map \( \Gamma^\fr_g,[[\ell]] \to G^\fr_g,[[\ell]] \) has kernel consisting of isotopy classes of diffeomorphisms which are pseudoisotopic to the identity, and by [Qui86, Theorem 1.4] such a diffeomorphism becomes isotopic to the identity after sufficiently-many stabilisations by \( S^2 \times S^2 \). Thus these maps become isomorphisms of groups in the colimit, and so in particular \( H_1(\Gamma^\fr_\infty,[[\ell]]) \to H_1(G^\fr_\infty,[[\ell]]) \) is an isomorphism, so \( g_\ell \) factors as desired.
Suppose now that \( n > 3 \). By stabilising if necessary we may suppose that \( g \) is large. By Lemma 2.2 the subgroup \( \Theta_{2n+1} \leq \Gamma_g \) acts trivially on the set of framings relative to the boundary, so \( \Theta_{2n+1} \leq \Gamma_g^{fr}[\ell] \leq \Gamma_g^{fr}[\ell] \) and is therefore annihilated by \( f_\ell \). Now \( f_\ell \) is a homomorphism to an abelian group, so factors over \( H_1(\Gamma_g^{fr}[\ell]/\Theta_{2n+1}) \). To calculate the latter group, we use the extension

\[
0 \longrightarrow \frac{\text{Hom}(H_n, \mathbb{Z}/2)}{n \text{ odd}} \quad 0 \text{ odd} \quad \Gamma_g^{fr}[\ell]/\Theta_{2n+1} \longrightarrow G_g^{fr}[\ell] \longrightarrow 0
\]

from Section 3.4. The Serre spectral sequence for this extension gives an exact sequence

\[
\cdots \longrightarrow \frac{\text{Hom}(H_n, \mathbb{Z}/2)}{n \text{ odd}} \quad 0 \text{ odd} \quad \Gamma_g^{fr}[\ell]/\Theta_{2n+1} \longrightarrow H_1(\Gamma_g^{fr}[\ell]/\Theta_{2n+1}) \longrightarrow H_1(G_g^{fr}[\ell]) \longrightarrow 0
\]

and it follows from [Kra19, Lemma A.2] that the leftmost term is zero. Thus we have \( H_1(\Gamma_g^{fr}[\ell]/\Theta_{2n+1}) \cong H_1(G_g^{fr}[\ell]) \) from which the factorisation of \( g_\ell \) over some \( h_\ell \) follows.

Before proving (ii) and (iii), we first show that for large enough \( g \) the group \( \Gamma_g^{fr}[\ell]/G_g^{fr}[\ell] \) has order at least 4. To see this we use the quotient map to \( G_g^{fr}[\ell]/G_g^{fr}[\ell] \). By Lemma 4.5, for \( n \neq 1,3 \) the group \( G_g^{fr}[\ell] \) has trivial abelianisation, so there is an induced surjection

\[
G_g^{fr}[\ell]/G_g^{fr}[\ell] \longrightarrow H_1(G_g^{fr}[\ell]) = \begin{cases} \mathbb{Z}/4 & n \text{ odd} \\ (\mathbb{Z}/2)^2 & n \text{ even.} \end{cases}
\]

Thus \( \Gamma_g^{fr}[\ell]/G_g^{fr}[\ell] = \ker(f_\ell) = \ker(h_\ell) \) indeed has order at least 4.

On the other hand \( H_1(G_g^{fr}[\ell]) \) has order precisely 4, so \( h_\ell \) must be injective. If \( n \neq 0 \) (mod 4) and \( n \neq 3 \) then \( \pi_{2n}(SO(2n)) \) has order 4, so \( h_\ell \) must be an isomorphism. If \( n \equiv 0 \) (mod 4) then \( \pi_{2n}(SO(2n)) \) has order 8, so \( h_\ell \) must be injective onto an index 2 subgroup. This proves parts (ii) and (iii).

This discussion allows us to describe the subgroups \( G_g^{fr}[\ell] \leq G_g^{fr}[\ell] \), where they have the following interesting intrinsic descriptions.

**Corollary 5.2.** If \( n \neq 1,3 \) then \( G_g^{fr}[\ell] = \ker(G_g^{fr}[\ell] \to H_1(G_g^{fr}[\ell])) \).

If \( n = 1,3 \) then \( G_g^{fr}[\ell] = G_g^{fr}[\ell] \).

**Proof.** For \( n \neq 1,3 \) the kernel of the composition

\[
\Gamma_g^{fr}[\ell] \longrightarrow G_g^{fr}[\ell] \longrightarrow H_1(G_g^{fr}[\ell])
\]

is the same as the kernel \( \Gamma_g^{fr}[\ell] \) of \( f_\ell \) by Proposition 5.1. Thus \( G_g^{fr}[\ell] \to H_1(G_g^{fr}[\ell]) \) has kernel \( G_g^{fr}[\ell] \).

If \( n = 1,3 \) then \( \pi_{2n}(SO(2n)) = 0 \) so \( \Gamma_g^{fr}[\ell] = \Gamma_g^{fr}[\ell] \) and hence their images in \( G_g^{fr}[\ell] \) are equal too.

**Proof of Theorem A.** The argument will be by analysing the sequence (5). Suppose that \( n \geq 2 \) and \( g \geq 1 \). By the calculations in Section 4.1 the composition

\[
\Gamma_g^{fr}[\ell] \longrightarrow G_g^{fr}[\ell] \longrightarrow H_1(G_g^{fr}[\ell]) \longrightarrow H_1(G_g^{fr}[\ell])
\]

is surjective as long as \( g \geq 1 \). By Proposition 5.1 if \( n \neq 0 \) (mod 4) then the map \( f_\ell \) is surjective; if \( n \equiv 0 \) (mod 4) then it has cokernel \( \mathbb{Z}/2 \). By Proposition 3.1 the set \( \text{Str}^*_n(W_{g,1})/\Gamma_g \) has a single element, unless \( n = 3,7 \) in which case it has two. It follows by combining these that \( \text{Str}^*_n(W_{g,1})/\Gamma_g \) has a single element unless \( n = 3,7 \) or \( n \equiv 0 \) (mod 4) in which case it has two elements.

The case \( n = 1 \) and \( g \geq 2 \) is [RW14, Theorem 2.9].
6. $\theta$-STRUCTURES ON $W_{g,1}$

In [KRW19, Section 8] we more generally considered tangential structures whose $GL_{2n}(\mathbb{R})$-space $\Theta$ has the property that the homotopy quotient $B := \Theta \sslash GL_{2n}(\mathbb{R})$ is $n$-connected (in terms of the associated fibration $\theta : B \to BO(2n)$ this means that $B$ is $n$-connected). Our results about framings can be used to also classify such $\theta$-structures on $W_{g,1}$, up to homotopy and diffeomorphisms. We will assume that $n \geq 2$.

Given a boundary condition $\ell_\partial : \text{Fr}(TW_{g,1} | \partial W_{g,1}) \to \Theta$ we let $\text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta; \ell_\partial)$ be the space of $GL_{2n}(\mathbb{R})$-maps $\text{Fr}(TW_{g,1}) \to \Theta$ extending $\ell_\partial$. Its set of path components is denoted $\text{Str}_\partial^\theta(W_{g,1})$. The moduli space of $W_{g,1}$'s with $\theta$-structures is defined as

$$BDiff_\partial^\theta(W_{g,1}) := \text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta; \ell_\partial) \sslash \text{Diff}_\partial(W_{g,1}).$$

Its set of path components is the set of orbits $\text{Str}_\partial^\theta(W_{g,1})/\Gamma_g$. This is what we shall compute in this section, but we first make some definitions.

The boundary condition $\ell_\partial$ singles out a path component $\Theta^+$ of $\Theta$. Since $\Theta \sslash GL_{2n}(\mathbb{R})$ is $n$-connected, the map $\pi_n(SO(2n)) \to \pi_n(\Theta^+)$ is surjective. By Lemma 3.2 the map $S\pi_n(SO(n)) \to \pi_n(SO(2n))$ is surjective unless $n = 3, 7$, in which case it has cokernel $\mathbb{Z}/2$. This leads to two cases when $n = 3, 7$:

1. $S\pi_n(SO(n)) \to \pi_n(SO(2n)) \to \pi_n(\Theta^+)$ is not surjective (and thus has index 2),
2. $S\pi_n(SO(n)) \to \pi_n(SO(2n)) \to \pi_n(\Theta^+)$ is surjective.

We also define

$$C\pi_{2n}(\Theta^+) := \text{coker} \left( H_1(G_{\infty}^\theta, [[\tau]] \to \pi_{2n}(SO(2n)) \to \pi_{2n}(\Theta^+) \right),$$

which seems at first sight to depend on the orbit of homotopy class $[[\tau]]$ of reference framing $\tau$ relative to $\ast$, but is in fact independent of this choice: by Proposition 5.1 the map $h_\tau$ is surjective unless $n \equiv 0 \pmod{4}$, but in that case the orbit of $[[\tau]]$ is unique by Proposition 3.1.

**Theorem 6.1.** Suppose that $\Theta \sslash GL_{2n}(\mathbb{R})$ is $n$-connected. Let $g \geq 1$ and $n \geq 2$. The action of the mapping class group $\Gamma_g$ on the set $\text{Str}_\partial^\theta(W_{g,1})$ of homotopy classes of $\theta$-structure extending $\ell_\partial$ is in bijection with

1. $C\pi_{2n}(\Theta^+)$ if $n \neq 3, 7$;
2. $C\pi_{2n}(\Theta^+) \times \mathbb{Z}/2$ if $n = 3, 7$ and we are in case (A);
3. $\pi_{2n}(\Theta^+)$ if $n = 3, 7$ and we are in case (B).

We will explain the proof of this theorem in parallel with that of Theorem A. The definitions and results of Sections 2.2, 2.3, and 2.4 go through for $\theta$-structures. By [KRW19, Lemma 8.5], up to homotopy there is a unique orientation preserving boundary condition $\ell_\partial$ which extends to a $\theta$-structure on all of $W_{g,1}$, which we may take to be $\ell_\partial^\theta$ coming from a reference framing $\tau$. The reference framing induces a homeomorphism

$$\text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta; \ell_\partial) \cong \text{map}_\partial(W_{g,1}, \Theta).$$

Its path components will be denoted $\text{Str}_\partial^\theta(W_{g,1})$. We can relax boundary conditions to get an exact sequence

$$0 \to \text{Str}_\partial^\theta(D^{2n}) \to \oplus \to \text{Str}_\partial^\theta(W_{g,1}) \to \text{Str}_\partial^\theta(W_{g,1}) \to 0$$

The arguments for Proposition 3.1 go through for $\theta$-structures, giving the following:

**Proposition 6.2.** Suppose $n \geq 2$ and $g \geq 1$, then

1. If $n \neq 3, 7$ or we are in case (B), then $\text{Str}_\partial^\theta(W_{g,1})/\Gamma_g$ consists of a single element.
\begin{itemize}
  \item If \( n = 3, 7 \) and we are in case (A), then \( \text{Str}_g^\theta(W_{g,1})/\Gamma_g \) consists of two elements.
\end{itemize}

These arguments also give information about the stabiliser \( \text{I}_g^\theta \) of \( [[\ell]] \in \text{Str}_g^\theta(W_{g,1}) \), as well as its image \( G_g^\theta\cdot[[\ell]] \) in \( G_g^\theta \). The map
\[
\text{Str}_g^\theta(W_{g,1}) \rightarrow \text{Str}_g^\theta(W_{g,1})
\]
associating to a framing the induced \( \theta \)-structure is surjective, as by assumption the map \( \pi_n(\text{SO}(2n)) \rightarrow \pi_n(\Theta^+) \) is. If \( n = 2 \) it follows that \( \text{Str}_g^\theta(W_{g,1}) \) is a single point; if \( n \neq 3, 7 \) it follows that \( \text{Str}_g^\theta(W_{g,1})/I_g \) is a single point; if \( n = 3, 7 \), it follows as in the proof of Proposition 3.3 that \( \text{Str}_g^\theta(W_{g,1})/I_g \) is in bijection with \( \text{Quad}(H_n, \lambda) \) in case (A) and is a single point in case (B). Thus for \( n \geq 2 \) and \( g \geq 1 \), we have
\[
G_g^\theta\cdot[[\ell]] = \begin{cases} 
\text{Sp}_{2g}^\theta(Z) & \text{if } n = 3, 7, \text{ and } \ell \text{ has Arf invariant 0 or 1}, \\
\text{Sp}_{2g}(Z) & \text{if } n = 3, 7 \text{ and we are in case (B)}, \\
\text{O}_{g,g}(Z) & \text{if } n \text{ is odd but not 3 or 7}, \\
\text{O}_{g,g}(Z) & \text{if } n \text{ is even}.
\end{cases}
\]
Furthermore, if \( n \neq 3, 7 \) or \( n = 3, 7 \) and we are in case (A), it follows that the natural inclusion of stabilisers \( \text{I}_g^\theta\cdot[[\ell]] \rightarrow \text{I}_g^\theta\cdot[[\ell]] \) is an isomorphism.

The analogue of the fundamental sequence \((5)\) for \( \theta \)-structures gives, for any framing \( \ell^\theta \), a commutative diagram
\[
\begin{array}{ccccccc}
\text{I}_g^\theta\cdot[[\ell]] & \longrightarrow & \Gamma_g^\theta\cdot[[\ell]] & \longrightarrow & \text{Str}_g^\theta(D^2n) & \longrightarrow & \text{Str}_g^\theta(W_{g,1})/\Gamma_g & \longrightarrow & \text{Str}_g^\theta(W_{g,1})/\Gamma_g \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma_g^\theta\cdot[[\ell]] & \longrightarrow & \Gamma_g^\theta\cdot[[\ell]] & \longrightarrow & \text{Str}_g^\theta(D^2n) & \longrightarrow & \text{Str}_g^\theta(W_{g,1})/\Gamma_g & \longrightarrow & \text{Str}_g^\theta(W_{g,1})/\Gamma_g,
\end{array}
\]

\textbf{Proof of Theorem 6.1.} We proceed in three cases.

\textbf{The cases } \( n \neq 3, 7 \). In this case the map indicated by \((*)\) is an isomorphism and \( f_\ell \) is determined by \( f_{\ell^\theta} \). Using Proposition 5.1, we then identify the cokernel of \( f_\ell \) with \( C\pi_{2n}(\Theta^+) \). As \( \text{Str}_g^\theta(W_{g,1})/\Gamma_g \) consists of a single element, we conclude that \( \text{Str}_g^\theta(W_{g,1})/\Gamma_g \cong C\pi_{2n}(\Theta^+) \).

\textbf{Case (A).} In this case \( n = 3 \) or \( 7 \) and \( S\pi_n(\text{SO}(n)) \rightarrow \pi_n(\text{SO}(2n)) \rightarrow \pi_n(\Theta) \) not surjective. Then \( \text{Str}_g^\theta(W_{g,1})/\Gamma_g \) consists of two elements, distinguished by an Arf invariant. Choosing framings \( \ell^\theta \) with Arf invariant 0 and 1 respectively, we get two commutative diagrams as above. In both cases, the map indicated by \((*)\) is an isomorphism, and as above we identify the cokernel of \( f_\ell \) with \( C\pi_{2n}(\Theta^+) \). Thus we get a collection of orbits with Arf invariant 0 and another collection of orbits with Arf invariant 1, each in bijection with \( C\pi_{2n}(\Theta^+) \).

\textbf{Case (B).} In this case \( n = 3 \) or \( 7 \) and \( S\pi_n(\text{SO}(n)) \rightarrow \pi_n(\text{SO}(2n)) \rightarrow \pi_n(\Theta) \) surjective. The proof of Proposition 5.1 gives a factorisation
\[
\begin{array}{ccccccc}
\text{I}_g^\theta\cdot[[\ell]] & \longrightarrow & f_\ell & \longrightarrow & \pi_{2n}(\Theta) & \longrightarrow & H_1(G_g^\theta\cdot[[\ell]]) \\
\downarrow & & h_\ell & & & & H_1(G_g^\theta\cdot[[\ell]]) \\
G_g^\theta\cdot[[\ell]] & \longrightarrow & H_1(G_g^\theta\cdot[[\ell]]) & \longrightarrow & H_1(G_g^\theta\cdot[[\ell]])
\end{array}
\]
with left map the quotient map, and bottom maps abelianisation followed by stabilisation. As \( G_g^\theta\cdot[[\ell]] = \text{Sp}_{2g}(Z) \), the bottom-right term vanishes by the computations in
Section 4.1, so $f_\ell = 0$. Combining this with the fact that $\text{Str}_{g}^0(W_{g,1})/\Gamma_g$ is a single point in this case gives the claimed result. \qed

6.1. Example: stable framings. Stable framings are trivialisations of the stable tangent bundle. In this case we take $\Theta = \text{GL}_\infty(\mathbb{R})$, made into a $\text{GL}_{2n}(\mathbb{R})$-space by stabilisation. Then $\Theta^+ = \text{GL}_+^\infty(\mathbb{R})$, which deformation retracts onto $\text{SO}$. The map $\pi_n(\text{SO}(2n)) \to \pi_n(\text{SO})$ induced by stabilisation is an isomorphism as long as $n \geq 3$, so when $n = 3, 7$ we are in case (A).

Lemma 6.3. The map $f_\ell: \Gamma_{g}^{\text{str.}}([\ell]) \to \pi_{2n}(\text{SO})$ is zero.

Proof. The group $\pi_{2n}(\text{SO})$ vanishes unless $n \equiv 0 \pmod{4}$ in which case it is given by $\mathbb{Z}/2$. In this case we claim that $h_\ell: H_1(\Gamma_{g}^{\text{str.}}([\ell]); \mathbb{Z}) \to \pi_{2n}(\text{SO})$ is zero. To see this note that the composition

$$H_1(\Gamma_{g}^{\text{str.}}; \mathbb{Z}) \to H_1(G_{\infty}^{\text{str.}}([\ell]; \mathbb{Z}) \to H_1(G_{\infty}^{\text{str.}}([\ell]; \mathbb{Z}) \to \pi_{2n}(\text{SO})$$

is zero by the analogue for stable framings of the exact sequence (7). But $G_{g}^{\text{str.}}([\ell]) = G_{g}^{\text{str.}}([\ell]) = \text{O}_{g,\ell}(\mathbb{Z})$ for $n$ even as we have discussed above, and by Section 5.2 of [GRW16] the composition

$$\pi_1(Y_{SO}^g/\text{SO}(2n)) \cong H_1(\Gamma_{g}^{\text{str.}}; \mathbb{Z}) \to \pi_1(M\text{Th}_g) \cong H_1(\Gamma_{g}; \mathbb{Z}) \to H_1(\text{O}_{\infty}(\mathbb{Z}); \mathbb{Z})$$

is surjective. \qed

We conclude that there are, up to homotopy and diffeomorphism, two stable framings on $W_{g,1}$ when $n = 3, 7$ or $n \equiv 0 \pmod{4}$, and a unique one otherwise. That is, the classification of stable framings is the same as that of framings: up to homotopy and diffeomorphism, every stable framing of $W_{g,1}$ arises from a unique framing.

Remark 6.4. This identifies the index 2 subgroup hit by $h_\ell$ in Proposition 5.1 (iii): it is the kernel of the stabilisation map $\pi_{2n}(\text{SO}(2n)) \to \pi_{2n}(\text{SO})$.

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FRAMINGS OF $W_{g,1}$

[258x748]FRAMINGS OF

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