Lagrangian and Hamiltonian Dynamics on Para-Kählerian Space Form

Mehmet Tekkoyun *
Department of Mathematics, Pamukkale University,
20070 Denizli, Turkey
February 26, 2009

Abstract

In this study, we introduce Euler-Lagrange and Hamiltonian equations on \((\mathbb{R}^2, g, J)\) being a model of para-Kählerian Space Forms. Finally, some geometrical and physical results on the related mechanic systems have been discussed.

Keywords: Para-Kählerian Manifolds, Para-Kählerian Space Forms, Lagrangian and Hamiltonian Systems.

MSC (2000): 53C, 37F.

*tekkoyun@pamukkale.edu.tr
1 Introduction

Modern Differential Geometry is a suitable frame for studying Lagrangian and Hamiltonian formalisms of Classical Mechanics. To show this, it is possible to find many articles and books in the relevant fields. It is well-known that the dynamics of Lagrangian and Hamiltonian systems is characterized by a convenient vector field $X$ defined on the tangent and cotangent bundles which are phase-spaces of velocities and momentum of a given configuration manifold. If $Q$ is an $m$-dimensional configuration manifold and $L : TQ \to \mathbb{R}$ is a regular Lagrangian function, then there is a unique vector field $X_L$ on $TQ$ such that

$$i_{X_L} \omega_L = dE_L,$$

where $\omega_L$ is the symplectic form and $E_L$ is energy associated to $L$. The so-called Euler-Lagrange vector field $X$ is a semispray (or second order differential equation) on $Q$ since its integral curves are the solutions of the Euler-Lagrange equations. The triple, either $(TQ, \omega_L, \xi)$ or $(TQ, \omega_L, L)$, is called Lagrangian system on the tangent bundle $TQ$. If $H : T^*Q \to \mathbb{R}$ is a regular Hamiltonian function then there is a unique vector field $X_H$ on $T^*Q$ such that

$$i_{X_H} \omega = dH,$$

where $\omega$ is the symplectic form and $H$ stands for Hamiltonian function. The paths of the so-called Hamiltonian vector field $X_H$ are the solutions of the Hamiltonian equations. The triple, either $(T^*Q, \omega, Z_H)$ or $(T^*Q, \omega, H)$, is called Hamiltonian system on the cotangent bundle $T^*Q$ fixed with symplectic form $\omega$.

From the before some studies given in [1-2, 3, 4, 5, 6, 7]; we know that time-dependent or not real, complex and paracomplex analogues of the Euler-Lagrange and Hamiltonian equations have detailed been introduced. But, we see that is not mentioned about Lagrangian and Hamiltonian dynamics on para-Kählerian space forms. Therefore, in this paper we present the Euler-Lagrange equations and Hamiltonian equations on a model of para-Kählerian space forms and to derive geometrical and physical results on related dynamics systems.

In this study, all the manifolds and geometric objects are $C^\infty$ and the Einstein summation convention is in use. Also, $\mathbb{R}$, $\mathcal{F}(M)$, $\chi(M)$ and $\Lambda^1(M)$ denote the set of real numbers, the set of functions on $M$, the set of vector fields on $M$ and the set of 1-forms on $M$, respectively.

2 Para-Kählerian Space Forms

Definition 1 [8,9]: Let a manifold $M$ be endowed with an almost product structure $J \neq \pm Id$; which is a $(1; 1)$-tensor field such that $J^2 = Id$: We say that $(M, J)$ (resp.$(M, J, g)$) is an almost
product (resp. almost Hermitian) manifold, where \( g \) is a semi-Riemannian metric on \( M \) with respect to which \( J \) is skew-symmetric, that is

\[
g(JX, Y) + g(X, JY) = 0, \forall X, Y \in \chi(M)
\]  

(3)

Then \((M, J, g)\) is para-Kählerian if \( J \) is parallel with respect to the Levi-Civita connection.

Let \((M, J, g)\) be a para-Kählerian manifold and let denote the curvature \((0, 4)\)-tensor field by

\[
R(X, Y, Z, V) = g(R(X, Y)Z, V); \forall X, Y, Z, V \in \chi(M)
\]  

(4)

where the Riemannian curvature \((1, 3)\)-tensor field associated to the Levi-Civita connection \( \nabla \) is given by

\[
R = [\nabla, \nabla] - \nabla[\ , \ , ].
\]

Then

\[
R(X, Y, Z, V) = -R(Y, X, Z, V) = -R(X, Y, V, Z) = R(JX, JY, Z, V)
\]  

and 

\[
\sum_{\sigma} R(X, Y, Z, V) = 0,
\]  

(5)

where \( \sigma \) denotes the sum over all cyclic permutations. We know that the following \((0,4)\)-tensor field is defined by

\[
R_0(X, Y, Z, V) = \frac{1}{4} \begin{Bmatrix}
   g(X, Z)g(Y, V) - g(X, V)g(Y, Z) - g(X, JZ)g(Y, JV) \\
   +g(X, JV)g(Y, JZ) - 2g(X, JY)g(Z, JV)
\end{Bmatrix},
\]  

(6)

where \( \forall X, Y, Z, V \in \chi(M) \). For any \( p \in M \), a subspace \( S \subset T_pM \) is called non-degenerate if \( g \) restricted to \( S \) is non-degenerate. If \( \{u, v\} \) is a basis of a plane \( \sigma \subset T_pM \), then \( \sigma \) is non-degenerate iff \( g(u, u)g(v, v) - [g(u, v)]^2 \neq 0 \). In this case the sectional curvature of \( \sigma = \text{span}\{u, v\} \) is

\[
k(\sigma) = \frac{R(u, v, u, v)}{g(u, u)g(v, v) - [g(u, v)]^2}
\]  

(7)

From (3) it follows that \( X \) and \( JX \) are orthogonal for any \( X \in \chi(M) \). By a \( J \)-plane we mean a plane which is invariant by \( J \). For any \( p \in M \), a vector \( u \in T_pM \) is isotropic provided \( g(u, u) = 0 \). If \( u \in T_pM \) is not isotropic, then the sectional curvature \( H(u) \) of the \( J \)-plane \( \text{span}\{u, Jv\} \) is called the \( J \)-sectional curvature defined by \( u \). When \( H(u) \) is constant,

then \((M, J, g)\) is called of constant \( J \)-sectional curvature, or a para-Kählerian space form.

**Theorem 1:** Let \((M, J, g)\) be a para-Kählerian manifold such that for each \( p \in M \), there exists \( c_p \in R \) satisfying \( H(u) = c_p \) for \( u \in T_pM \) such that \( g(u, u)g(Ju, Ju) \neq 0 \). Then the Riemann-Christoffel tensor \( R \) satisfies \( R = cR_0 \), where \( c \) is the function defined by \( p \to c_p \). And conversely.
**Definition 2:** A para-Kählerian manifold \((M, J, g)\) is said to be of constant paraholomorphic sectional curvature \(c\) if it satisfies the conditions of **Theorem 1**.

**Theorem 2:** Let \((M, J, g)\) be a para-Kählerian manifold with \(\text{dim} M > 2\). Then the following properties are equivalent:

1) \(M\) is a space of constant paraholomorphic sectional curvature \(c\)

2) The Riemann-Christoffel tensor curvature tensor \(R\) has the expression

\[
R(X, Y, Z, V) = \frac{c}{4} \left\{ g(X, Z)g(Y, V) - g(X, V)g(Y, Z) - g(X, JZ)g(Y, JV) + g(X, JV)g(Y, JZ) - 2g(X, JY)g(Z, JV) \right\},
\]

where \(\forall X, Y, Z, V \in \chi(M)\). Let \((x, y)\) be a real coordinate system on a neighborhood \(U\) of any point \(p\) of \(\mathbb{R}^2\), and \(\{(\frac{\partial}{\partial x})_p, (\frac{\partial}{\partial y})_p\}\) and \(\{(dx)_p, (dy)_p\}\) natural bases over \(\mathbb{R}\) of the tangent space \(T_p(\mathbb{R}^2)\) and the cotangent space \(T^*_p(\mathbb{R}^2)\) of \(\mathbb{R}^2\), respectively.

The space \((\mathbb{R}^2, g, J)\), is the model of the para-Kählerian space forms of dimension 2 and paraholomorphic sectional curvature \(c \neq 0\), where \(g\) is the metric

\[
g = \frac{4}{c} \left( \cosh^2 2y dx \otimes dx - dy \otimes dy \right), 0 \neq c \in \mathbb{R},
\]

and \(J\) the almost product structure

\[
J = -\frac{1}{\cosh 2y} \frac{\partial}{\partial x} \otimes dy - \cosh 2y \frac{\partial}{\partial y} \otimes dx.
\]

Then we have

\[
J(\frac{\partial}{\partial x}) = -\cosh 2y \frac{\partial}{\partial y}, J(\frac{\partial}{\partial y}) = -\frac{1}{\cosh 2y} \frac{\partial}{\partial x}.
\]

The dual endomorphism \(J^*\) of the cotangent space \(T^*_p(\mathbb{R}^2)\) at any point \(p\) of manifold \(\mathbb{R}^2\) satisfies \(J^{*2} = \text{Id}\) and is defined by

\[
J^*(dx) = -\cosh 2yd y, J^*(dy) = -\frac{1}{\cosh 2y} dx.
\]

### 3 Lagrangian Dynamics

Here, we find Euler-Lagrange equations for Classical Mechanics constructed on para-Kählerian space form \((\mathbb{R}^2, g, J)\).

Denote by \(J\) almost product structure and by \((x, y)\) the coordinates of \(\mathbb{R}^2\). Assume that semispray be a vector field as follows:

\[
\xi = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y}, X = \dot{x} = y, Y = \dot{y}.
\]
By *Liouville vector field* on para-Kählerian space form \((\mathbb{R}^2, g, J)\), we call the vector field determined by \(V = J\xi\) and calculated by

\[
J\xi = -\frac{1}{\cosh 2y}Y \frac{\partial}{\partial x} - \cosh 2y.X \frac{\partial}{\partial y},
\]

(14)

Given \(T\) by the kinetic energy and \(P\) by the potential energy of mechanics system on para-Kählerian space form. Then we write by \(L = T - P\) Lagrangian function and by \(E_L = V(L) - L\) the energy function associated \(L\).

Operator \(i_J\) defined by

\[
i_J : \wedge^2 \mathbb{R}^2 \rightarrow \wedge^1 \mathbb{R}^2, \quad i_J(\omega)(X) = \omega(X, JX)
\]

(15)
is called the *interior product* with \(J\), or sometimes the *insertion operator*, or *contraction* by \(J\), where \(\omega \in \wedge^2 \mathbb{R}^2\), \(X \in \chi(\mathbb{R}^2)\). The exterior vertical derivation \(d_J\) is defined by

\[
d_J = [i_J, d] = i_J d - d i_J,
\]

(16)

where \(d\) is the usual exterior derivation. For almost product structure \(J\) determined by (11), the closed para-Kählerian form is the closed 2-form given by \(\Phi_L = -dd_J L\) such that

\[
d_J = -\cosh 2y \frac{\partial^2 L}{\partial a \partial y} dx + \cosh 2y \frac{\partial^2 L}{\partial b \partial y} db \wedge dx
\]

\[
+ \frac{1}{\cosh 2y} \frac{\partial^2 L}{\partial a \partial x} da \wedge dy + \frac{1}{\cosh 2y} \frac{\partial^2 L}{\partial b \partial x} db \wedge dy.
\]

(17)

Thus we get

\[
\Phi_L = \cosh 2y.X \frac{\partial L}{\partial a} \delta_a^x dx - \cosh 2y.X \frac{\partial L}{\partial b} \delta_b^x dx + \cosh 2y.Y \frac{\partial L}{\partial a} \delta_a y db + \cosh 2y.Y \frac{\partial L}{\partial b} \delta_b y db
\]

\[
+ \frac{1}{\cosh 2y} Y \frac{\partial^2 L}{\partial a \partial x} \delta_a^y dy - \frac{1}{\cosh 2y} Y \frac{\partial^2 L}{\partial b \partial x} \delta_b^y db.
\]

(18)

Since the closed para-Kählerian form \(\Phi_L\) on para-Kählerian space form \((\mathbb{R}^2, g, J)\) is para-symplectic structure, one may find

\[
E_L = -\frac{1}{\cosh 2y} Y \frac{\partial L}{\partial x} + \cosh 2y.X \frac{\partial L}{\partial y} - L,
\]

(20)

and thus

\[
dE_L = -\frac{1}{\cosh 2y} Y \frac{\partial L}{\partial a} da - \cosh 2y.X \frac{\partial L}{\partial b} da - \frac{\partial L}{\partial a} da
\]

\[
- \frac{1}{\cosh 2y} Y \frac{\partial^2 L}{\partial b \partial x} db - \cosh 2y.X \frac{\partial^2 L}{\partial a \partial x} db - \frac{\partial L}{\partial b} db.
\]

(21)
Considering \( i_\xi \Phi_L = dE_L \), we calculate
\[
\cosh 2y.X \frac{\partial^2 L}{\partial a \partial y} dx + \cosh 2y.Y \frac{\partial^2 L}{\partial b \partial y} dx \\
+ \frac{1}{\cosh 2y} X \frac{\partial^2 L}{\partial a \partial x} dy + \frac{1}{\cosh 2y} Y \frac{\partial^2 L}{\partial b \partial x} dy + \frac{\partial L}{\partial x} dx + \frac{\partial L}{\partial y} dy = 0.
\]
(22)

If the curve \( \alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^2 \) be integral curve of \( \xi \), which satisfies
\[
\cosh 2y \left[ X \frac{\partial^2 L}{\partial a \partial y} + Y \frac{\partial^2 L}{\partial b \partial y} \right] dx + \frac{\partial L}{\partial x} dx \\
+ \frac{1}{\cosh 2y} \left[ X \frac{\partial^2 L}{\partial a \partial x} + Y \frac{\partial^2 L}{\partial b \partial x} \right] dy + \frac{\partial L}{\partial y} dy = 0,
\]
(23)
we get equations
\[
\cosh 2y \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial y} \right) + \frac{\partial L}{\partial x} = 0, \quad \frac{1}{\cosh 2y} \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x} \right) + \frac{\partial L}{\partial y} = 0
\]
(24)
so-called Euler-Lagrange equations whose solutions are the paths of the semispray \( \xi \) on para-Kählerian space form \((\mathbb{R}^2, g, J)\). Finally one may say that the triple \((\mathbb{R}^2, \Phi_L, \xi)\) is mechanical system on para-Kählerian space form \((\mathbb{R}^2, g, J)\). Therefore we say the following:

**Proposition 1:** Let \( J^* \) almost product structure on para-Kählerian space form \((\mathbb{R}^2, g, J)\). Also let \((f_1, f_2)\) be natural bases of \( \mathbb{R}^2 \). Then it follows
\[
\cosh 2y.J(f_2) + f_1 = 0 \iff \cosh 2y.J_1 + f_2 = 0,
\]
where \( f_1 = \frac{\partial L}{\partial x}, \ f_2 = \frac{\partial L}{\partial y}, \ \dot{f}_1 = \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x} \right), \ \dot{f}_2 = \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial y} \right) \).

## 4 Hamiltonian Dynamics

Now, we conclude Hamiltonian equations for Classical Mechanics structured on para-Kählerian space form \((\mathbb{R}^2, g, J)\).

Let \( J^* \) be an almost product structure defined by \((12)\) and \( \lambda \) Liouville form determined by \( J^* (\omega) = -x \cosh 2y dy - y \frac{1}{\cosh 2y} dx \) such that \( \omega = x dx + y dy \) 1-form on \( \mathbb{R}^2 \). If \( \Phi = -d\lambda \) is closed para-Kählerian form, then it is also a para-symplectic structure on \( \mathbb{R}^2 \).

Let \((\mathbb{R}^2, g, J)\) be para-Kählerian space form fixed with closed para-Kählerian form \( \Phi \). Suppose that Hamiltonian vector field \( Z_H \) associated to Hamiltonian energy \( H \) is given by
\[
Z_H = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y}.
\]
(25)

For the closed para-Kählerian form \( \Phi \) on \( \mathbb{R}^2 \), we have
\[
\Phi = -d\lambda = -d(-x \cosh 2y dy - y \frac{1}{\cosh 2y} dx) = \cosh 2y - \frac{1}{\cosh 2y} dx \wedge dy.
\]
(26)
Then it follows
\[ i_{Z_H} \Phi = i_{Z_H}(-d\lambda) = -\cosh^2 2y - 1 \frac{Y}{\cosh 2y} dx + \cosh^2 2y - 1 \frac{X}{\cosh 2y} dy. \] (27)

Otherwise, we find the differential of Hamiltonian energy the following as
\[ dH = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial y} dy. \] (28)

From (27) and (28) with respect to \( i_{Z_H} \Phi = dH \), we find para-Hamiltonian vector field on para-Kählerian space form to be
\[ Z_H = \cosh^2 2y \frac{\partial H}{\cosh^2 2y - 1} \frac{\partial}{\partial x} - \cosh^2 2y \frac{\partial H}{\cosh^2 2y - 1} \frac{\partial}{\partial y}. \] (29)

Assume that the curve
\[ \alpha : I \subset \mathbb{R} \to \mathbb{R}^2 \] (30)
be an integral curve of Hamiltonian vector field \( Z_H \), i.e.,
\[ Z_H(\alpha(t)) = \dot{\alpha}, \quad t \in I. \] (31)

In the local coordinates we get
\[ \alpha(t) = (x(t), y(t)), \] (32)
and
\[ \dot{\alpha}(t) = \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y}. \] (33)

Now, by means of (31), from (29) and (33), we deduce the equations so-called para-Hamiltonian equations
\[ \frac{dx}{dt} = \cosh 2y \frac{\partial H}{\cosh^2 2y - 1} \frac{dy}{dt} = -\cosh 2y \frac{\partial H}{\cosh^2 2y - 1} \frac{dx}{dt}. \] (34)

In the end, we may say to be para-mechanical system \((\mathbb{R}^2, \Phi, Z_H)\) triple on para-Kählerian space form \((\mathbb{R}^2, g, J)\).

5 Discussion

From above, we understand that Lagrangian and Hamiltonian formalisms in generalized Classical Mechanics and field theory can be intrinsically characterized on \((\mathbb{R}^2, g, J)\) being a model of para-Kählerian space forms. So, the paths of semispray \( \xi \) on \( \mathbb{R}^2 \) are the solutions of the Euler-Lagrange equations given by (24) on the mechanical system \((\mathbb{R}^2, \Phi_L, \xi)\). Also, the solutions of the Hamiltonian equations determined by (34) on the mechanical system \((\mathbb{R}^2, \Phi, Z_H)\) are the paths of vector field \( Z_H \) on \( \mathbb{R}^2 \).
References

[1] Crampin M., On the differential Geometry of Euler-Lagrange Equations, and the inverse problem of Lagrangian dynamics, J. Phys. A-Math. and Gen., Vol: 14, Issue: 10, (1981) 2567-2575.

[2] De Leon M., Rodrigues P.R., Methods of Differential Geometry in Analytical Mechanics, North-Hol. Math. St.,152, Elsevier Sc. Pub. Com., Inc., Amsterdam, 1989.

[3] Crampin M., Lagrangian Submanifolds and the Euler-Lagrange Equations in the Higher-Order Mechanics, Letters in Mathematical Physics, Vol.: 19, Issue: 1, (1990)53-58.

[4] Tekkoyun, M., ” On Para-Euler Lagrange and para- Hamiltonian equations”. Physics Letters A, Vol. 340, (2005) 7-11.

[5] Tekkoyun, M., Görgülü A., Higher Order Complex Lagrangian and Hamiltonian Mechanics Systems”, Physics Letters A, Vol.357, (2006) 261-269.

[6] Tekkoyun, M., ”A Note On Constrained Complex Hamiltonian Mechanics” Differential Geometry-Dynamical Systems (DGDS), Vol.8, No.1 , (2006) 262-267.

[7] Tekkoyun M., Cabar G., Complex Lagrangians and Hamiltonians, Journal of Arts and Sciences, Çankaya Üniv., Fen-Ed.Fak., Issue 8/December 2007.

[8] Bejan, C. L, Ferrara, M.,Para-Kähler Manifolds of Quasi-Constant-P Sectional Curvature.

[9] Cruceanu V., Gadea P.M., Muñoz Masqué J., Para-Hermitian and Para- Kähler Manifolds, Supported by the commission of the European Communities’ Action for Cooperation in Sciences and Technology with Central Eastern European Countries n. ERB3510PL920841.