LOCAL AND GLOBAL ANALYTICITY FOR A GENERALIZED CAMASSA-HOLM SYSTEM

HIDESHI YAMANE

Abstract. We solve the analytic Cauchy problem for the generalized two-component Camassa-Holm system introduced by R. M. Chen and Y. Liu. We show the existence of a unique local/global-in-time analytic solution under certain conditions.

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We consider the generalized two-component CH system
\[
\begin{align*}
\begin{cases}
m_t - \alpha u_x + \beta(2mu_x + um_x) + 3(1 - \beta)uu_x + \rho \rho_x = 0, \quad m = u - u_{xx}, \\
\rho_t + (\rho u)_x = 0.
\end{cases}
\end{align*}
\]
(0.1)

It was introduced in [6] as a model of shallow water waves and is equivalent to
\[
\begin{align*}
\begin{cases}
u_t - u_{txx} - \alpha u_x + 3uu_x - \beta(2u_xu_{xx} + uu_{xxx}) + \rho \rho_x = 0, \\\n\rho_t + (\rho u)_x = 0.
\end{cases}
\end{align*}
\]
(0.2)

Here it is assumed that \(u \to 0\) and \(\rho \to 1\) hold as \(|x| \to \infty\). It is natural to introduce \(v = \rho - 1\), which tends to 0 as \(|x| \to 0\). The system (0.1) with \(u \to 0, \rho \to 1\) is equivalent to
\[
\begin{align*}
\begin{cases}
(1 - \partial_x^2)u_t - \alpha u_x + 3uu_x - \beta(2u_xu_{xx} + uu_{xxx}) + (1 + v)v_x = 0, \\
v_t + u_x + (uv)_x = 0.
\end{cases}
\end{align*}
\]
(0.3)

with \(u \to 0, v \to 0\) as \(|x| \to \infty\). Applying \((1 - \partial_x^2)^{-1}\) to the first equation, we get, by using \(\partial^2_x(uu_x) = uu_{xxx} + 3u_xu_{xx}\),
\[
\begin{align*}
\begin{cases}
u_t + \beta uu_x + (1 - \partial_x^2)^{-1}\partial_x \left[ -\frac{3 - \beta}{2}u^2 + \frac{\beta}{2}u_x^2 + v + \frac{1}{2}v^2 \right] = 0, \\
v_t + u_x + (uv)_x = 0.
\end{cases}
\end{align*}
\]
(0.4)

In the sections below, we shall mainly consider (0.4) rather than (0.2) and (0.3).

We recall some background. The original Camassa-Holm equation
\[
u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}
\]
(0.5)
was first proposed in [9] in the context of integrability and was later studied in [8]. It has been found that (0.5) describes shallow water waves and other physical phenomena. See [6]. The single equation (0.5) admits many multi-component generalizations. A well-known one is
\[
\begin{align*}
\begin{cases}
m_t - Au_x + 2mu_x + um_x + \rho \rho_x = 0, \quad m = u - u_{xx}, \\
\rho_t + (\rho u)_x = 0,
\end{cases}
\end{align*}
\]
(0.6)
in [17] and it corresponds to the case of \(\beta = 1\) in (0.2).

The Camassa-Holm equation and its variants, including the ones above, are studied from several points of view. Some authors employ the inverse scattering technique (e.g. [4] [7]), while others apply Hirota’s bilinear method (e.g. [16]). In a significant number of articles, including [6], solutions have been obtained in the Sobolev or Besov spaces by using PDE techniques. Relatively fewer number of articles deal with real- or complex-analytic solutions. The analyticity of solutions is relevant since the equations model water waves. See [8] for some discussion about analyticity of water waves. In [1] [2] [3], the authors solved the Camassa-Holm and other equations in some spaces of analytic solutions locally or globally. Their methods were employed in [18] by the present author to solve the \(\mu\)-Camassa-Holm and similar equations. In the present paper, we use the same methods in the study of the two-component system (0.2). The difficulty lies in the fact that (0.2) is not symmetric in \(u\) and \(v\).
A global regularity result involving a higher-order inertia operator \((1 - \partial_x^2)^s\), \(s > 1\) can be found in [10], in which the authors studied the system

\[
\begin{align*}
  m_t - \alpha u_x + \beta (2m u_x + um_x) + 3(1 - \beta)uw_x + \rho p_x &= 0, \\
  \rho_t + (\rho u)_x &= 0, \\
  m &= (1 - \partial_x^2)^s u, s > 1
\end{align*}
\] (0.7)

and proved Gevrey regularity in \(x\) for any fixed \(t\).

The rest of the paper is organized as follows. In Section 1, we introduce some function spaces and prove their properties. Sections 2 and 3 are devoted to the local and global theories respectively. In the latter we need a lot of inequalities and their proofs are given in Sections 4 and 5.

1. Function spaces

In the present paper \(L^2 = L^2(\mathbb{R})\), \(C^\infty(\mathbb{R})\) and their subspaces consist of real-valued functions on \(\mathbb{R}\). We shall make frequent use of \(H^s = H^s(\mathbb{R}) \subset L^2\) with the norm \(\|f\|_s = \|(1 + \xi^2)^{s/2} \hat{f}(\xi)\|_{L^2}\) and the inner product \(\langle \cdot, \cdot \rangle_s\). We recall some known facts about \(H^s\) [3, 13].

**Lemma 1.** Set \(\Lambda = (1 - \partial_x^2)^{1/2}\).

(i) \(\|f\|_2^2 = \|\Lambda^2 f\|_0^2 = \|f\|_0^2 + 2\|f'\|_0^2 + \|f''\|_0^2\).

(ii) For \(f \in L^2\) and \(g \in H^s (s > 1/2)\), we have

\[
\|fg\|_0 \leq d(s)\|f\|_0\|g\|_s,
\]

where \(d(s) = \left[\int_0^1 (1 + \xi^2)^{-s} ds\right]^{1/2}\). In particular, we have

\[
\|fg\|_0 \leq \sqrt{\pi}\|f\|_0\|g\|_1 \leq 2\|f\|_0\|g\|_1,
\]

\[
\langle f, gh \rangle_0 \leq 2\|f\|_0\|g\|_0\|h\|_1.
\]

(iii) For \(f \in H^s (s \geq 0)\), we have \(\|\Lambda^{-2} f\|_{s+2} = \|f\|_s\).

(iv) For \(f \in H^{s+1} (s \geq 0)\), we have \(\|\partial_x f\|_s \leq \|f\|_{s+1}\).

(v) For \(f, g \in H^s (s \geq 1)\), we have

\[
\|fg\|_s \leq c_s (\|f\|_s\|g\|_1 + \|f\|_1\|g\|_s)
\]

for some constant \(c_s > 0\). In particular, for \(f, g \in H^s (s \geq 3)\), we have

\[
\|fg\|_2 \leq 8\|f\|_2\|g\|_1 + \|f\|_1\|g\|_2.
\]

(vi) For \(f, g \in H^s (s > 1/2)\), we have

\[
\|fg\|_s \leq c(s)\|f\|_s\|g\|_s,
\]

where \(c(s) = \left[(1 + 2^s) \int_0^1 (1 + \xi^2)^{-s} ds\right]^{1/2}\). In particular, we have

\[
\|fg\|_1 \leq 4\|f\|_1\|g\|_1, \quad \|fg\|_2 \leq 8\|f\|_2\|g\|_2.
\]

For \(r > 0\), set

\[
S(r) = \{ x + iy \in \mathbb{C} : |y| < r \},
\]

\[
A(r) = \{ f : \mathbb{R} \to \mathbb{R} : f(z) \text{ can be analytically continued to } S(r) \}
\]

\[
\cap \{ f \in L^2_{x,y}(S(r')) \text{ for all } 0 < r' < r \}.
\]

Notice that \(A(r)\) is a subspace of \(L^2\), the space of real-valued square-integrable functions on \(\mathbb{R}\).
Following \[12\], we set
\[
\|f\|_{\sigma,s}^2 = \sum_{j=0}^{\infty} \frac{e^{2j\sigma}}{j!^2} \|f^{(j)}\|_s^2, \; s \geq 0,
\]
for \( f \in H^\infty = \cap_{s \geq 0} H^s \), where \( f^{(j)} = \partial_j^j f(x), \partial_x = d/dx \).

**Lemma 2.** (\[12\]) The norms \( \| \cdot \|_{\sigma,s} \) have the following properties.
- (i) Assume \( s, s' \geq 0 \) and \( \sigma' < \sigma \). Then there exists a positive constant \( c > 0 \) such that
  \[
  \|f\|_{\sigma',s'} \leq c\|f\|_{\sigma,s}
  \]
  for any \( f \).
- (ii) If \( f \in A(r) \) and \( \sigma < \log r, s \geq 0 \), then \( \|f\|_{\sigma,s} < \infty \).
- (iii) Let \( s \geq 0 \) be fixed. If \( f \in H^\infty \) satisfies \( \|f\|_{\sigma,s} < \infty \) for some \( s \geq 0 \) and any \( \sigma \) with \( \sigma < \log r \), then \( f \in A(r) \).

**Proposition 3.** (\[12\]) When \( r > 0 \) is fixed, the following four families of norms determine the same topology of \( A(r) \) as a Fréchet space. With this topology, \( A(r) \) is continuously embedded in \( H^\infty \).
- (i) the \( L^2_{x,y}(S(r')) \) norms \( (0 < r' < r) \)
- (ii) \( \| \cdot \|_{\sigma,s} \) \( (s \geq 0, \sigma < \log r) \)
- (iii) \( \| \cdot \|_{\sigma,2} \) \( (\sigma < \log r) \)
- (iv) \( \| \cdot \|_{\sigma,0} \) \( (\sigma < \log r) \)

**Lemma 4.** (\[12\]) Let \( f_n \in A(r) \) be a sequence with \( \|f_n\|_{\sigma,2} \) bounded, where \( \sigma < \log r \). If \( f_n \to 0 \) in \( H^\infty \) as \( n \to \infty \), then \( \|f_n\|_{\sigma',2} \to 0 \) for each \( \sigma' < \sigma \).

Following \[3\] (with some generalization and a modified notation), we introduce
\[
\|f\|_{(\delta,s)} = \sup_{k \geq 0} \frac{\delta^k(k+1)!}{k!} |f^{(k)}|_s (0 < \delta \leq 1, s \geq 2).
\]

Do not confuse \( \| \cdot \|_{(\delta,s)} \) with \( \| \cdot \|_{\sigma,s} \). Moreover, notice that a different system of notation was employed in \[18\]. We introduce the Banach space \( E_{\delta,s} \) by
\[
E_{\delta,s} = \{ f \in C^\infty(\mathbb{R}); \|f\|_{(\delta,s)} < \infty \}.
\]

The property of \( E_{\delta,s} \) is similar to that of \( G^{\delta,s} \) used in \[1\,2\,3\]. We do without \( G^{\delta,s} \) in the present paper.

**Proposition 5.** (\[3\] Lemma 5.1) \( E_{\delta,s} \) is continuously embedded in \( A(\delta) \). Conversely, if \( \delta < r/e \) then \( A(r) \) is continuously embedded in \( E_{\delta,s} \).

**Proof.** In \[3\], this proposition is stated and proved only in the case \( s = 2m, m \in \mathbb{Z}_+ \). The same proof is valid in the general case. \( \square \)

**Proposition 6.** (i) If \( 0 < \delta' < \delta \leq 1 \) and \( 2 \leq s' < s \), then
\[
\|u\|_{(\delta',s)} \leq \|u\|_{(\delta,s)}, \quad \|u\|_{(\delta,s')} \leq \|u\|_{(\delta,s)}.
\]
(ii) If \( 0 < \delta' < \delta \leq 1, s \geq 2 \), then
\[
\|uv\|_{(\delta,s)} \leq C_s \|u\|_{(\delta,s)} \|v\|_{(\delta,s)}, \quad C_s = 18c_s.
\]
(iii) If $0 < \delta' < \delta \leq 1$, we have

$$
\|\partial_x u\|_{(\delta', s)} \leq \frac{1}{\delta - \delta'} \|u\|_{(\delta, s)},
$$

$$
\|\partial_x u\|_{(\delta, s)} \leq \|u\|_{(\delta, s+1)},
$$

$$
\|\Lambda^{-2} \partial_x^2 u\|_{(\delta, s)} \leq \|u\|_{(\delta, s)} (p = 0, 1, 2),
$$

$$
\|\Lambda^{-2} \partial_x u\|_{(\delta', s)} \leq \frac{\|u\|_{(\delta, s)}}{\delta - \delta'}.
$$

(iv) $\|\Lambda^{-2} u\|_{(\delta, s+2)} = \|u\|_{(\delta, s)} (p = 0, 1, 2)$,

(v) $\|\Lambda^{-2} \partial_x u\|_{(\delta', s+1)} \leq \|\Lambda^{-2} \partial_x u\|_{(\delta', s+2)} \leq \frac{1}{\delta - \delta'} \|u\|_{(\delta, s)}$.

Proof. The estimates in (i) are trivial. We prove (ii) and (iii) following [1]. By Lemma [1] (v), we have

$$
\|\partial_x (uv)\|_s \leq \sum_{\ell=0}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) \|u^{(k-\ell)}v^{(\ell)}\|_s
$$

$$
\leq c_s \sum_{\ell=0}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) \left( \|u^{(k-\ell)}\|_s \|v^{(\ell)}\|_1 + \|u^{(k-\ell)}\|_1 \|v^{(\ell)}\|_s \right)
$$

$$
= c_s \sum_{\ell=0}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) \left( \|u^{(k-\ell)}\|_s \|v^{(\ell)}\|_1 + \|u^{(\ell)}\|_1 \|v^{(k-\ell)}\|_s \right) .
$$

Since $s \geq 2$, we have $\|v\|_1 \leq \|v\|_s \leq \|v\|_{(\delta, s)}$ and $\|v^{(\ell)}\|_1 \leq \|v^{(\ell-1)}\|_s (\ell \geq 1)$. Hence

$$
\|\partial_x (uv)\|_s \leq c_s \left( \|u^{(k)}\|_s \|v^{(\ell)}\|_s + \sum_{\ell=1}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) \|u^{(k-\ell)}\|_s \|v^{(\ell-1)}\|_s \right)
$$

$$
+ c_s \left( \|u\|_{(\delta, s)} \|v^{(k)}\|_s + \sum_{\ell=1}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) \|u^{(\ell-1)}\|_s \|v^{(k-\ell)}\|_s \right) .
$$

(1.1)

What we want to prove is that $\delta^{k}(k+1)^2 \|\partial_x (uv)\|_s/k!$ is bounded by $18c_s$ times $\|u\|_{(\delta, s)} \|v\|_{(\delta, s)}$. In view of the symmetry in the right-hand side of (1.1), it is enough to prove

(a) $(k!)^{-1} \delta^{k}(k+1)^2 \|\partial_x (uv)\|_s/k!$ is bounded by $\|u\|_{(\delta, s)} \|v\|_{(\delta, s)}$,

(b) $(k!)^{-1} \delta^{k}(k+1)^2 \sum_{\ell=1}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) \|u^{(k-\ell)}\|_s \|v^{(\ell-1)}\|_s$ is bounded by $8\|u\|_{(\delta, s)} \|v\|_{(\delta, s)}$.

The estimate (a) is trivial. Now we show (b). We have

$$
\frac{\delta^{k}(k+1)^2}{k!} \sum_{\ell=1}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) \|u^{(k-\ell)}\|_s \|v^{(\ell-1)}\|_s
$$

$$
= \frac{\delta^{k}(k+1)^2}{k!} \sum_{\ell=1}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) (k-\ell)! (\ell-1)! \delta^{k-\ell}(k-\ell+1)^2 \|u^{(k-\ell)}\|_s \delta^{k-\ell+2} \|v^{(\ell-1)}\|_s
$$

$$
= \frac{(k+1)^2}{k!} \sum_{\ell=1}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) (k-\ell)! (\ell-1)! \|u\|_{(\delta, s)} \|v\|_{(\delta, s)}
$$

$$
\leq 8\|u\|_{(\delta, s)} \|v\|_{(\delta, s)} .
$$
Here we have used the fact \( \sum_{\ell=1}^{k} (k-\ell+1)^{-2} \ell^{-3}(k+1)^2 \leq 8 \) (II Lem 2.2). The proof of (ii) is complete.

Next we prove (iii). The first inequality follows from
\[
\|\partial_x u\|_{(\delta, s)} \leq \sup_{k \geq 0} \frac{\delta^k (k+1)^2 \|u^{(k+1)}\|_s}{k!} \\
= \sup_{k \geq 0} \frac{\delta^{k+1}(k+2)^2 \|u^{(k+1)}\|_s}{(k+1)!} \frac{\delta^k (k+1)^3}{\delta^{k+1}(k+2)^2} \\
\leq \|u\|_{(\delta, s)} \sup_{k \geq 0} \frac{\delta^k (k+1)^3}{\delta^{k+1}(k+2)^2} \leq \frac{\|u\|_{(\delta, s)}}{\delta - \delta'}.
\]
Here we employed \( \delta^k \delta^{-(k+1)} (k+1)^3(k+2)^{-2} \leq 1/(\delta - \delta') \) (II (2.7)). The second and third inequalities of (iii) follow from \( \|u^{(k+1)}\|_s \leq \|u^{(k)}\|_{s+1} \) and \( \|\Lambda^{-2}\partial_x u\|_s \leq \|u\|_s \) respectively. The fourth one follows from the first and the third.

The equality (iv) follows from \( \|\Lambda^{-2}\partial_x u\|_{s+2} = \|u\|_s \) and (v) follows from (iii) and (iv). \( \square \)

2. Local analyticity

2.1. General theory. We recall some basic facts about the autonomous Ovsyannikov theorem following [II (2)]. Let \( \{X_\delta, \|\cdot\|_\delta\}_{0<\delta \leq 1} \) be a (decreasing) scale of Banach spaces, i.e. each \( X_\delta \) is a Banach space and \( X_\delta \subset X_{\delta'} \), \( \|\cdot\|_{\delta'} \leq \|\cdot\|_\delta \) for any \( 0 < \delta' < \delta \leq 1 \). For example, \( \{E_{\delta,s}, \|\cdot\|_{\delta,s}\}_{0<\delta \leq 1} \) is a scale of Banach spaces. Assume that \( F: X_\delta \to X_{\delta'} \) is a mapping satisfying the following conditions.

(a) For any \( U_0 \in X_1 \) and \( R > 0 \), there exist \( L = L(U_0, R) > 0, M = M(U_0, R) > 0 \) such that we have
\[
\|F(U_0)\|_\delta \leq \frac{M}{1-\delta} \tag{2.1}
\]
if \( 0 < \delta < 1 \) and
\[
\|F(U) - F(V)\|_{\delta'} \leq \frac{L}{\delta - \delta'} \|U - V\|_\delta \tag{2.2}
\]
if \( 0 < \delta' < \delta \leq 1 \) and \( U, V \in X_\delta \) satisfies \( \|U - U_0\|_\delta < R, \|V - U_0\|_\delta < R \).

(b) If \( U(t) \) is a holomorphic function with values in \( X_\delta \) on the disk \( D(0, a(1-\delta)) = \{ t \in \mathbb{C} : |t| < a(1-\delta) \} \) for \( a > 0, 0 < \delta < 1 \) satisfying \( \sup_{|t| < a(1-\delta)} \|U(t) - U_0\|_\delta < R \), then the composite function \( F(U(t)) \) is a holomorphic function on \( D(0, a(1-\delta)) \) with values in \( X_{\delta'} \) for any \( 0 < \delta' < \delta \). The following autonomous Ovsyannikov theorem will be used in the next section.

**Theorem 7.** ([II]) Assume that the mapping \( F \) satisfies the conditions (a) and (b). For any \( U_0 \in X_1 \) and \( R > 0 \), set
\[
T = \frac{R}{16LR + 8M}. \tag{2.3}
\]
Then, for any \( \delta \in [0,1[ \), the Cauchy problem
\[
\frac{dU}{dt} = F(U), \quad U(0) = U_0 \tag{2.4}
\]
has a unique holomorphic solution \( U(t) \) in the disk \( D(0, T(1-\delta)) \) with values in \( X_\delta \) satisfying
\[
\sup_{|t| < T(1-\delta)} \|U(t) - U_0\|_\delta < R.
\]
2.2. Local analyticity for the Camassa-Holm system. We consider the analytic Cauchy problem for the generalized CH system (0.4), namely,

\[
\begin{cases}
    u_t + \beta uu_x + \Lambda^{-2}\partial_x \left[-\alpha u + \frac{3-\beta}{2} u^2 + \frac{\beta}{2} u_x^2 + v + \frac{1}{2} v^2\right] = 0, \\
v_t + u_x + (uv)_x = 0, \\
u(0, x) = u_0(x), \\
v(0, x) = v_0(x).
\end{cases}
\] (2.5)

**Theorem 8.** Let \( s \geq 2 \). If \( u_0, v_0 \in E_{1,s+1} \), then there exists a positive time \( T = T(u_0, v_0; s) \) such that for every \( \delta \in [0, 1] \), the Cauchy problem (2.5) has a unique solution which is a holomorphic function valued in \( \oplus^2 E_{\delta,s+1} \) in the disk \( D(0, T(1-\delta)) \). Furthermore, the analytic lifespan \( T \) satisfies

\[
T \approx \frac{\text{const.}}{\|(u_0, v_0)\|_{(1,s+1)}}
\]

for large initial values and

\[
T \approx \text{const.}
\]

for small initial values.

**Proof.** For \((u, v) \in \oplus^2 E_{\delta,s+1} \), set \( \|(u, v)\|_{(\delta,s+1)} = \|u\|_{(\delta,s+1)} + \|v\|_{(\delta,s+1)} \). We employ the same notation for the norms on \( E_{\delta,s+1} \) and \( \oplus^2 E_{\delta,s+1} \). Notice that \( \{\oplus^2 E_{\delta,s}\}_{0 \leq \delta \leq 1} \) is another scale of Banach spaces. Set

\[
F_1(u, v) = -\beta uu_x - \Lambda^{-2}\partial_x \left[-\alpha u + \frac{3-\beta}{2} u^2 + \frac{\beta}{2} u_x^2 + v + \frac{1}{2} v^2\right],
\]

\[
F_2(u, v) = -u_x - (uv)_x.
\]

We want to estimate \( F_j(u, v) - F_j(u', v') \) and \( F_j(u_0, v_0) \) by using Proposition 5 when \( u_0, v_0 \in E_{1,s+1} \) and

\[
\|(u, v) - (u_0, v_0)\|_{(\delta,s+1)} < R, \quad \|(u', v') - (u_0, v_0)\|_{(\delta,s+1)} < R.
\]

First we consider \( F_2 \). By using

\[
\|u\|_{(\delta,s+1)} \leq \|(u, v)\|_{(\delta,s+1)} \leq \|(u_0, v_0)\|_{(1,s+1)} + R
\]

and similar estimates on \( u', v \) and \( v' \), we get

\[
\|uv - u'v'\|_{(\delta,s+1)} \leq \|(u - u'v)\|_{(\delta,s+1)} + \|u'(v - v')\|_{(\delta,s+1)}
\]

\[
\leq C_{s+1}\|u - u'\|_{(\delta,s+1)}\|v\|_{(\delta,s+1)} + C_{s+1}\|u'\|_{(\delta,s+1)}\|v - v'\|_{(\delta,s+1)}
\]

\[
\leq C_{s+1} \left[\|(u_0, v_0)\|_{(1,s+1)} + R\right] \|(u', v') - (u_0, v_0)\|_{(\delta,s+1)}.
\]

Therefore

\[
\|(uv)_x - (u'v')_x\|_{(\delta',s+1)} \leq \frac{C_{s+1} \left[\|(u_0, v_0)\|_{(1,s+1)} + R\right] \|(u, v) - (u', v')\|_{(\delta,s+1)}}{\delta - \delta'}
\]

(2.10)

the other hand, we have

\[
\|u_x - u'_x\|_{(\delta',s+1)} \leq \frac{\|u - u'\|_{(\delta,s+1)}}{\delta - \delta'}.
\]

(2.11)
By using (2.10) and (2.11), we obtain
\[
\| F_2(u, v) - F_2(u', v') \|_{(\delta', s+1)} \\
\leq \frac{C_{s+1} \left[ \|(u_0, v_0)\|_{(1, s+1)} + R \right]}{\delta - \delta'} \| (u, v) - (u', v') \|_{(\delta, s+1)}.
\] (2.12)

Next, we consider \( F_1 \). By (2.9), we get
\[
\| u + u' \|_{(\delta, s+1)} \leq 2 \left[ \|(u_0, v_0)\|_{(1, s+1)} + R \right],
\] (2.13)
\[
\| u^2 - u'^2 \|_{(\delta, s+1)} \leq C_{s+1} \| u + u' \|_{(\delta, s+1)} \| u - u' \|_{(\delta, s+1)}
\] (2.14)
\[
\leq 2C_{s+1} \left[ \|(u_0, v_0)\|_{(1, s+1)} + R \right] \| u - u' \|_{(\delta, s+1)}.
\]

Since \( 2uu_x - 2u'u_x = (u^2 - u'^2)_x \), we get
\[
\| uu_x - u'u_x \|_{(\delta', s+1)} \leq \frac{C_{s+1} \left[ \|(u_0, v_0)\|_{(1, s+1)} + R \right]}{\delta - \delta'} \| u - u' \|_{(\delta, s+1)}.
\] (2.15)

On the other hand, Proposition 6 (iii) implies
\[
\| \Lambda^{-2} \partial_x (u - u') \|_{(\delta', s+1)} \leq \frac{1}{\delta - \delta'} \| u - u' \|_{(\delta, s+1)},
\] (2.16)
\[
\| \Lambda^{-2} \partial_x (v - v') \|_{(\delta', s+1)} \leq \frac{1}{\delta - \delta'} \| v - v' \|_{(\delta, s+1)}.
\] (2.17)

Similarly, the estimate (2.14) implies
\[
\| \Lambda^{-2} \partial_x (u^2 - u'^2) \|_{(\delta', s+1)} \leq \frac{1}{\delta - \delta'} \| u^2 - u'^2 \|_{(\delta, s+1)}
\] (2.18)
\[
\leq \frac{2C_{s+1} \left[ \|(u_0, v_0)\|_{(1, s+1)} + R \right]}{\delta - \delta'} \| u - u' \|_{(\delta, s+1)}
\]
and
\[
\| \Lambda^{-2} \partial_x (v^2 - v'^2) \|_{(\delta', s+1)} \leq \frac{2C_{s+1} \left[ \|(u_0, v_0)\|_{(1, s+1)} + R \right]}{\delta - \delta'} \| v - v' \|_{(\delta, s+1)}.
\] (2.19)

What remains is the estimate on \( \Lambda^{-2} \partial_x (u_x^2 - u'_x^2) \).

By using (2.13), Proposition 6 (v) and
\[
\| u_x^2 - u'_x^2 \|_{(\delta, s)} \leq C_s \|(u + u')_x\|_{(\delta, s)} \|(u - u')_x\|_{(\delta, s)}
\]
\[
\leq C_s \| u + u' \|_{(\delta, s+1)} \| u - u' \|_{(\delta, s+1)},
\]
we obtain
\[
\| \Lambda^{-2} \partial_x (u_x^2 - u'_x^2) \|_{(\delta', s+1)} \leq \frac{2C_s \left[ \|(u_0, v_0)\|_{(1, s+1)} + R \right]}{\delta - \delta'} \| u - u' \|_{(\delta, s+1)}.
\] (2.20)

Combining (2.10), (2.17), (2.18), (2.19) and (2.20), we obtain
\[
\| F_1(u, v) - F_1(u', v') \|_{(\delta', s+1)} \leq \frac{C_s (2|\beta| + |3 - \beta| + 1) \left[ \|(u_0, v_0)\|_{(1, s+1)} + R \right] + |\alpha| + 1}{\delta - \delta'}
\]
\[
\times \| (u, v) - (u', v') \|_{(\delta, s+1)},
\]
where \( C_s = \max \{ C_s, C_{s+1} \} \). The two estimates (2.21) and (2.12) mean that we have
\[
\| (F_1, F_2)(u, v) - (F_1, F_2)(u', v') \|_{(\delta', s)} \leq L \frac{\| (u, v) - (u', v') \|_{(\delta, s)}}{\delta - \delta'},
\] (2.22)
Finally by combining (2.24) and (2.25), we obtain

\[
L = C_\gamma (2|\beta| + |3 - \beta| + 2) \left[ \|(u_0, v_0)\|_{1,s+1} + R \right] + |\alpha| + 2, \tag{2.23}
\]

if \( \|(u, v) - (u_0, v_0)\|_{(\delta,s+1)} < R \) \( \|(u', v') - (u_0, v_0)\|_{(\delta,s+1)} < R \).

Next, we evaluate what corresponds to the constant \( M \) in the general theory. We have

\[
\|u_0(u_0)_x\|_{(\delta,s+1)} \leq \frac{(C_{s+1}/2)\|u_0\|^2_{1,s+1}}{1 - \delta},
\]

\[
\|\Lambda^{-2}\partial_x u_0\|_{(\delta,s+1)} \leq \frac{\|u_0\|_{1,s+1}}{1 - \delta},
\]

\[
\|\Lambda^{-2}\partial_x u_0^2\|_{(\delta,s+1)} \leq \frac{\|u_0^2\|_{1,s+1}}{1 - \delta} \leq \frac{C_{s+1}\|u_0\|^2_{1,s+1}}{1 - \delta},
\]

\[
\|\Lambda^{-2}\partial_x (\partial_x u_0)^2\|_{(\delta,s+1)} \leq \frac{\|\partial_x u_0^2\|_{1,s+1}}{1 - \delta} \leq \frac{C_{s+1}\|u_0\|^2_{1,s+1}}{1 - \delta},
\]

\[
\|\Lambda^{-2}\partial_x u_0^2\|_{(\delta,s+1)} \leq \frac{\|u_0^2\|_{1,s+1}}{1 - \delta} \leq \frac{C_{s+1}\|v_0\|^2_{1,s+1}}{1 - \delta}.
\]

These inequalities lead to

\[
\|F_1(u_0, v_0)\|_{(\delta,s+1)} \leq \frac{M_1}{1 - \delta}, \tag{2.24}
\]

\[
M_1 = \frac{C_{s+1}}{2} (2|\beta| + |3 - \beta|)\|u_0\|^2_{1,s+1} + |\alpha|\|u_0\|_{1,s+1}
\]

\[
+ \|v_0\|_{1,s+1} + \frac{C_{s+1}}{2} \|v_0\|^2_{1,s+1}.
\]

On the other hand, we have

\[
\|(u_0)_x\|_{(\delta,s+1)} \leq \frac{\|u_0\|_{1,s+1}}{1 - \delta},
\]

\[
\|(u_0v_0)_x\|_{(\delta,s+1)} \leq \frac{\|u_0v_0\|_{1,s+1}}{1 - \delta} \leq \frac{C_{s+1}\|u_0\|_{1,s+1}\|v_0\|_{1,s+1}}{1 - \delta}
\]

\[
= \frac{(C_{s+1}/2)(\|u_0\|_{1,s+1} + \|v_0\|_{1,s+1})^2}{1 - \delta}
\]

\[
= \frac{(C_{s+1}/2)(\|u_0, v_0\|_{1,s+1})^2}{1 - \delta}.
\]

Hence

\[
\|F_2(u_0, v_0)\|_{(\delta,s+1)} \leq \frac{M_2}{1 - \delta}, \tag{2.25}
\]

\[
M_2 = \|u_0\|_{1,s+1} + \frac{C_{s+1}}{2}\|(u_0, v_0)\|^2_{1,s+1}.
\]

Finally by combining (2.24) and (2.25), we obtain

\[
\|(F_1, F_2)(u_0, v_0)\|_{(\delta,s+1)} \leq \frac{M}{1 - \delta}, \tag{2.26}
\]

\[
M = \frac{C_{s+1}}{2} (2|\beta| + |3 - \beta| + 2)\|(u_0, v_0)\|_{1,s+1} + ((|\alpha| + 2)\|(u_0, v_0)\|_{1,s+1}. \tag{2.27}
\]
We can apply the general theory with $L, M$ as in (2.23) and (2.27). We set $R = \|(u_0, v_0)\|_{1,s+1}$. Then
\[
T = \frac{R}{16LR + 8M} = \frac{R}{(\gamma_1 R + \gamma_2)R} = \frac{1}{\gamma_1 R + \gamma_2}
\]
for some positive constants $\gamma_1, \gamma_2$ depending only on $\alpha, \beta, C_s, C_{s+1}$. We have $T \to 1/\gamma_2 \neq 0$ as $\|(u_0, v_0)\|_{1,s+1} = R \to 0$ and $T \approx 1/(\gamma_1 R)$ as $\|(u_0, v_0)\|_{1,s+1} = R \to \infty$.

In Theorem 9, we assumed the initial values $u_0$ and $v_0$ were in $E_{1,s+1}$. We can relax this assumption as in the following theorem.

**Theorem 9.** Let $0 < \Delta \leq 1, s > 2$. If $(u_0, v_0) \in \oplus^2 E_{\Delta,s+1}^1$, then there exists $T_\Delta > 0$ such that the Cauchy problem (2.3) has a unique solution which is a holomorphic function valued in $\oplus^2 E_{\Delta d,s+1}^1$ in the disk $D(0, T_\Delta(1-d))$ for every $d \in ]0,1[$.

**Proof.** Set $X_d = \oplus^2 E_{\Delta d,s+1}^1, ||(\Delta)_{(d,s+1)}|| = ||(\Delta)_{(d,s+1)}||_{(d,s+1)}$. Then $\{X_d, \cdot (\Delta)_{(d,s+1)}\}_{0<d\leq1}$ is a (decreasing) scale of of Banach spaces and $(u_0, v_0) \in X_1$. Assume $\|(u - u_0)\|_{(d,s+1)} < R, \|(v - v_0)\|_{(d,s+1)} < R$ and $0 < d' < d < 1$. Then by setting $\delta = \Delta d', \delta' = \Delta d'$, we obtain the following counterpart to (2.23):
\[
\|(F_1, F_2)(u, v) - (F_1, F_2)(u', v')\|_{(d',s)} \leq \frac{L_\Delta}{d - d'} \|(u, v) - (u', v')\|_{(d,s)}. \tag{2.28}
\]
Here
\[
L_\Delta = C_s \Delta^{-1} (2|\beta| + |3 - \beta| + 2) \left(\|(u_0, v_0)\|_{(1,s+1)} + R\right) + |\alpha| + 1, \tag{2.29}
\]
and $\|(u, v) - (u_0, v_0)\|_{(d,s+1)} < R, \|(u', v') - (u_0, v_0)\|_{(d,s+1)} < R$. On the other hand, we have
\[
\|(F_1, F_2)(u_0, v_0)\|_{(d,s+1)} \leq \frac{M_\Delta}{1 - d'}
\]
and
\[
M_\Delta = \frac{C_{s+1}}{2\Delta} (2|\beta| + |3 - \beta| + 2) \left(\|(u_0, v_0)\|_{(1,s+1)}\right)^2 + \frac{|\alpha| + 2}{\Delta} \|(u_0, v_0)\|_{(1,s+1)}.
\]

\[
\square
\]

### 3. Global analyticity

#### 3.1. Main result.

We recall a known result about global-in-time solutions to (2.5) in Sobolev spaces. We have

**Theorem 10.** ([6, Theorems 3.1, 5.1]) Assume $0 < \beta < 2$ and let $s > 3/2$. If $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ and $\inf_{x \in \mathbb{R}} v_0(x) > -1$, then (2.5) has a unique solution $(u, v)$ in the space $C([0, \infty), H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0, \infty), H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$.

Our main result is the following theorem, the proof of which shall be given in Propositions [15] and [18] below.

**Theorem 11.** Assume $0 < \beta < 2$. If $u_0, v_0 \in A(r_0)$ for some $r_0 > 0$ and $\inf_{x \in \mathbb{R}} v_0(x) > -1$, then the unique solution $(u, v)$ in Theorem [14] to the Cauchy problem (2.5) belongs to $\oplus^2 C^\infty([0, \infty), \mathbb{R}^2)$. More precisely, there exists a continuous function $\sigma(t)$ such that we have $u(\cdot, t), v(\cdot, t) \in A(e^{\sigma(T)})$ for $t \in [0, T]$. (The function $\sigma(t)$ shall be defined in Proposition [14].)
3.2. Kato-Masuda theory. In [12], the authors used their theory of Liapnov families to prove a regularity result about the KdV and other equations. Later, it was applied to a generalized Camassa-Holm equation in [3]. Here we recall the abstract theorem in [12] in a weaker, more concrete form. In applications, making a suitable choice of the subset $O$ of $Z$ is essential.

**Theorem 12.** Let $X$ and $Z$ be Banach spaces. Assume that $Z$ is a dense subspace of $X$. Let $O$ be an open subset of $Z$ and $F$ be a continuous mapping from $Z$ to $X$. Let $\{\Phi_s; -\infty < s < \bar{s} \leq \infty \}$ be a family of non-negative real-valued functions on $Z$ satisfying the conditions (a), (b) and (c) below.

(a) The Fréchet partial derivative of $\Phi_s(U)$ in $U \in Z$ exists not only in $L(Z; \mathbb{R})$ but also in $L(X; \mathbb{R})$. This statement makes sense because $L(X; \mathbb{R}) \subset L(Z; \mathbb{R})$ by the canonical identification. (a) follows from (b) below.

(b) The Fréchet derivative of $\Phi_s(U)$ in $(s, U)$ exists not only in $L(\mathbb{R} \times Z; \mathbb{R})$ but also in $L(\mathbb{R} \times X; \mathbb{R})$ and is continuous from $\mathbb{R} \times Z$ to $L(\mathbb{R} \times X; \mathbb{R})$. This statement makes sense because $L(\mathbb{R} \times X; \mathbb{R}) \subset L(\mathbb{R} \times Z; \mathbb{R})$ by the canonical identification.

(c) There exist positive constants $K, L$ and $M$ such that

$$|\langle F(U), D\Phi_s(U) \rangle| \leq K\Phi_s(U) + L\Phi_s(U)^{1/2}\partial_s\Phi_s(U) + M\partial_s\Phi_s(U)$$

(3.1) holds for any $U \in O$. Here $\langle \cdot, \cdot \rangle$ (no subscript) is the pairing of $X$ and $L(X; \mathbb{R})$.

Let $U \in C([0, T]; O) \cap C^1([0, T]; X)$ be the solution to the Cauchy problem

$$\begin{cases}
\frac{dU}{dt} = F(U), \\
U(0, x) = U_0(x).
\end{cases}$$

(3.2)

For a fixed constant $s_0 < \bar{s}$, set

$$r(t) = \Phi_{s_0}(U_0)e^{Kt},$$
$$s(t) = s_0 - \int_0^t \left( Lr(\tau)^{1/2} + M \right) d\tau = s_0 - \frac{2L\Phi_{s_0}(U_0)^{1/2}}{K}(e^{Kt/2} - 1) - Mt$$

for $t \in [0, T]$. Then we have

$$\Phi_{s(t)}(U(t)) \leq r(t), \ t \in [0, T].$$

Later we will use this theorem when $X = \oplus^2 H^{m+2}, Z = \oplus^2 H^{m+4}$ and $\Phi_s$ is related to the Sobolev norms. Roughly speaking, Theorem 12 means that the regularity of $U(t)$ for $t \in [0, T]$ follows from that of $U_0$.

3.3. Pairing and the main estimate. For $\sigma \in \mathbb{R}$, $(u, v) \in \oplus^2 H^\infty$ and a positive integer $m$, set
\[ \Phi_{\sigma,m}(u,v) = \Phi_{\sigma,m}^{(1)}(u) + \Phi_{\sigma,m}^{(2)}(v), \]

\[ \Phi_{\sigma,m}^{(1)}(u) = \frac{1}{2} \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \|u^{(j)}\|^2_2 \]

\[ = \frac{1}{2} \left( \|u^{(1)}\|^2_2 + \frac{e^{2\sigma}}{2!^2} \|u^{(2)}\|^2_2 + \cdots + \frac{e^{2m\sigma}}{(m+1)!^2} \|u^{(m+1)}\|^2_2 \right), \]

\[ \Phi_{\sigma,m}^{(2)}(v) = \frac{1}{2} \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2} \|v^{(j)}\|^2_2 \]

\[ = \frac{1}{2} \left( \|v\|^2_2 + \frac{e^{2\sigma}}{1!^2} \|v^{(1)}\|^2_2 + \cdots + \frac{e^{2m\sigma}}{m!^2} \|v^{(m)}\|^2_2 \right). \]

The asymmetry of the exponents and the ranges of \( j \) corresponds to that of \( \Phi_{\sigma,m} \): its second equation involves \( u \), while the first does not involve \( v \). We have

\[ \|u\|^2_2 + 2 \lim_{m \to \infty} e^{2\sigma} \Phi_{\sigma,m}^{(1)}(u) = \|u\|^2_{\sigma,2}, \quad \lim_{m \to \infty} 2 \Phi_{\sigma,m}^{(2)}(v) = \|v\|^2_{\sigma,2}. \]

It is trivial that \( \lim_{m \to \infty} \Phi_{\sigma,m}(u,v) < \infty \) if and only if \( \|u\|_{\sigma,2} < \infty, \|v\|_{\sigma,2} < \infty \). Later we will use

\[ \partial_x \Phi_{\sigma,m}(u,v) = \sum_{j=2}^{m+1} \frac{(j-1)e^{2(j-1)\sigma}}{j!^2} \|u^{(j)}\|^2_2 + \sum_{j=1}^{m} \frac{je^{2j\sigma}}{j!^2} \|v^{(j)}\|^2_2. \]

**Proposition 13.** Set \( F(u,v) = (F_1(u,v), F_2(u,v)) \), where \( F_1 \) and \( F_2 \) are as in \([2.0]\) and \([2.7]\). Then \( F \) is a continuous mapping from \( \oplus^2 H^{m+4} \) to \( \oplus^2 H^{m+2} \) and we have

\[ \langle F(u,v), D\Phi_{\sigma,m}(u,v) \rangle \]

\[ = \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \langle u^{(j)}, \partial_x F_1(u,v) \rangle_2 + \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2} \langle v^{(j)}, \partial_x F_2(u,v) \rangle_2, \]

where \( D\Phi_{\sigma,m} \) is the Fréchet derivative and the bracket on the left-hand side is the pairing of \( \oplus^2 H^{m+2} \) and its dual \( (\oplus^2 H^{m+2})^* \simeq \oplus^2 H^{m+2} \).

**Proof.** Set

\[ \Psi_j^{(1)}(u,v) = \Psi_j(u) = \frac{1}{2} \|u^{(j)}\|^2_2 = \frac{1}{2} \|\Lambda^2 u^{(j)}\|^2_0, \]

\[ \Psi_j^{(2)}(u,v) = \Psi_j(v) = \frac{1}{2} \|v^{(j)}\|^2_2 = \frac{1}{2} \|\Lambda^2 v^{(j)}\|^2_0. \]

Then

\[ \Phi_{\sigma,m}(u,v) = \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \Psi_j^{(1)}(u,v) + \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2} \Psi_j^{(2)}(u,v). \]

We have

\[ \langle (w_1, w_2), D\Psi_j^{(1)}(u,v) \rangle = D\Psi_j^{(1)}(u,v) ((w_1, w_2)) = \frac{d}{d\tau} \Psi_j^{(1)}((u,v) + \tau(w_1, w_2)) \bigg|_{\tau=0} \]

\[ = \frac{d}{d\tau} \Psi_j(u + \tau w_1) \bigg|_{\tau=0} = \langle u^{(j)}, w_1^{(j)} \rangle_2 \]
and similarly
\[ \langle (w_1, w_2), D\Psi_j^{(2)}(u, v) \rangle = \langle u^{(j)}, w_2^{(j)} \rangle_2. \]

The proposition follows immediately. \qed

**Proposition 14.** There exist positive constants \( K_1, K_2, L_1, L_2, M_1, M_2, M_3 \) independent of \( u, v \) and \( \sigma \) such that we have
\[
|\langle F(u, v), D\Phi_{\sigma,m}(u, v) \rangle| \leq \left[ K_1 + K_2 \| (u, v) \|_3 \right] \Phi_{\sigma,m}(u, v) \\
+ (L_1 + L_2 e^\sigma) \Phi_{\sigma,m}(u, v)^{1/2} \partial_\sigma \Phi_{\sigma,m}(u, v) \\
+ \left[ M_1 + (M_2 + M_3 e^{2\sigma}) \| (u, v) \|_3 \right] \partial_\sigma \Phi_{\sigma,m}(u, v).
\]
for \( (u, v) \in \oplus^2 H^{m+4} \), where \( D\Phi_{\sigma,m} \) is the Fréchet derivative of \( \Phi_{\sigma,m} \).

**Proof.** The estimate follows from (3.3) and the estimates (4.10), (4.11), (5.3), (5.4) below. \qed

3.4. **Analyticity in the space variable.** In this subsection, we prove a part of Theorem 11. We assume that \((u_0, v_0) \in \oplus^2 A(r_0)\) is as in Theorem 11. Then we can apply Theorem 11 with arbitrarily large \( s \). We will prove the analyticity of \( u(t) \) and \( v(t) \) in \( x \) for each fixed \( t \).

**Proposition 15.** Let \((u_0, v_0) \) and \((u, v)\) be as in Theorem 11. Fix \( \sigma_0 < \log r_0 \) and \( T > 0 \). Then there exists positive constants \( K, L, M \) such that if we set
\[
\Phi_{\sigma_0, \infty}(u_0, v_0) = \lim_{m \to \infty} \Phi_{\sigma_0,m}(u_0, v_0),
\]
\[
\sigma(t) = \sigma_0 - \frac{2L}{K} \Phi_{\sigma_0, \infty}(u_0, v_0)^{1/2}(e^{Kt/2} - 1) - Mt,
\]
then we have \((u(t), v(t)) \in \oplus^2 A(e^{\sigma(t)}) \subset \oplus^2 A(e^{\sigma(T)})\) for \( t \in [0, T] \).

**Proof.** Assume \( u_0, v_0 \in A(r_0) \) and set \( \bar{\sigma} = \log r_0 \). Theorem 11 implies \( u(t), v(t) \in H^\infty \). Set
\[
\mu_0 = 1 + \max \{ \| (u, v) \|_3; t \in [0, T] \} > 0,
\]
\[
\mathcal{O} = \{ (u, v) \in \oplus^2 H^{m+4}; \| (u, v) \|_3 < \mu_0 \},
\]
\[
K = K_1 + K_2 \mu_0,
\]
\[
L = L_1 + L_2 e^\sigma,
\]
\[
M = M_1 + (M_2 + M_3 e^{2\sigma}) \mu_0.
\]
Then \((u(t), v(t)) \in \mathcal{O}\) for \( t \in [0, T] \). Proposition 14 implies that
\[
|\langle F(u, v), D\Phi_{\sigma,m}(u, v) \rangle| \leq K \Phi_{\sigma,m}(u, v) + L \Phi_{\sigma,m}(u, v)^{1/2} \partial_\sigma \Phi_{\sigma,m}(u, v) + M \partial_\sigma \Phi_{\sigma,m}(u, v)
\]
holds for any \((u, v) \in \mathcal{O}, \sigma \leq \bar{\sigma}\). This corresponds to (4.11) in Theorem 12. Since \( u_0 \in A(r_0) \), we have \( \| u_0 \|_{\sigma_0, 2} < \infty \) by Proposition 3.

Set
\[
\rho_m(t) = \Phi_{\sigma_0,m}(u_0, v_0)e^{Kt},
\]
\[
\sigma_m(t) = \sigma_0 - \int_0^t \left( L \rho_m(\tau)^{1/2} + M \right) d\tau
\]
\[
= \sigma_0 - \frac{2L}{K} \Phi_{\sigma_0,m}(u_0, v_0)^{1/2}(e^{Kt/2} - 1) - Mt.
\]
These functions correspond to $r(t)$ and $s(t)$ in Theorem \[2\] We can apply Theorem \[2\] to obtain
\[ \Phi_{\sigma_m(t),m}(u(t),v(t)) \leq \rho_m(t), \quad t \in [0,T]. \] (3.6)
Set
\[ \rho(t) = \lim_{m \to \infty} \rho_m(t), \]
\[ \Phi_{\sigma_0,\infty}(u_0,v_0) = \lim_{m \to \infty} \Phi_{\sigma_{m,0}}(u_0,v_0) = \frac{1}{2} e^{-2\sigma_0} \left( \|u\|^2_{\sigma_0,2} - \|u\|^2_2 + \frac{1}{2} \|v\|^2_{\sigma_0,2} \right). \]
\[ \sigma(t) = \sigma_0 - \frac{2L}{K} \Phi_{\sigma_0,\infty}(u_0,v_0)^{1/2} (e^{Kt/2} - 1) - Mt, \]
\[ = \sigma_0 - \frac{L}{K} \left( e^{-2\sigma_0} \left( \|u\|^2_{\sigma_0,2} - \|u\|^2_2 + \|v\|^2_{\sigma_0,2} \right) \right) \left( e^{Kt/2} - 1 \right) - Mt. \]
Notice that $\sigma(t)$ is a decreasing function in $t$. We have $\rho_m(t) \leq \rho_{m+1}(t) \leq \rho(t)$ and $\sigma_m(t) \geq \sigma_{m+1}(t) \geq \sigma(t)$, $\sigma_m(t) \to \sigma(t)$ (as $m \to \infty$). By Fatou’s Lemma and (3.6), we get
\[ \|u(t)\|^2_{\sigma(t),2} + \|v(t)\|^2_{\sigma(t),2} \]
\[ = \sum_{j=0}^{\infty} \frac{1}{j!^2} e^{2j\sigma(t)} \left( \|u^{(j)}(t)\|^2_2 + \|v^{(j)}(t)\|^2_2 \right) \]
\[ \leq \liminf_{m \to \infty} \sum_{j=0}^{m} \frac{1}{j!^2} e^{2j\sigma_m(t)} \left( \|u^{(j)}(t)\|^2_2 + \|v^{(j)}(t)\|^2_2 \right) \]
\[ = \|u(t)\|^2_2 + 2 \liminf_{m \to \infty} \left( e^{2\sigma_m(t)} \Phi_{\sigma_m(t),m}^{(1)}(u(t)) + \Phi_{\sigma_m(t),m}^{(2)}(v(t)) \right) \]
\[ \leq \|u(t)\|^2_2 + 2 \liminf_{m \to \infty} \left( e^{2\sigma_m(t)} + 1 \right) \rho_m(t) \]
\[ = \|u(t)\|^2_2 + 2 \left( e^{2\sigma(t)} + 1 \right) \rho(t) < \infty. \]
Therefore $u(t), v(t) \in A(e^{\sigma(t)}) \subset A(e^{\sigma(T)})$ for $t \in [0,T]$. \[\square\]

**Proposition 16.** The mapping $[0,T] \to \oplus^2 A(e^{2\sigma(T)}), t \mapsto (u(t),v(t))$ is continuous.

**Proof.** Let \( \{t_n\} \subset [0,T] \) be a sequence converging to $t_\infty \in [0,T]$. We have $(u(t_n),v(t_n)) \to (u(t_\infty),v(t_\infty))$ in $\oplus^2 H^\infty$. In particular, $\|u(t_n)\|_2$ is bounded. On the other hand, $\|u(t_n)\|_{\sigma(T),2}, \|v(t_n)\|_{\sigma(T),2} (n \geq 0)$ are bounded since
\[ \|u(t_n)\|^2_{\sigma(T),2} + \|v(t_n)\|^2_{\sigma(T),2} \leq \|u(t_n)\|^2_{\sigma'_{\sigma(T)},2} + \|v(t_n)\|^2_{\sigma'_{\sigma(T)},2} \]
\[ \leq \|u(t)\|^2_2 + 2 \left( e^{2\sigma(t)} + 1 \right) \rho(t) \]
by the proof of Proposition 15. Lemma 4 implies that $(u(t_n),v(t_n))$ converges to $(u(t_\infty),v(t_\infty))$ with respect to $\| \cdot \|_{\sigma',2} (\sigma' < \sigma(T))$. By Proposition 3 this means convergence in $\oplus^2 A(e^{\sigma(T)})$. \[\square\]

### 3.5. Analyticity in the space and time variables

We continue the proof of Theorem 11. In the previous subsection, we have established the analyticity in the space variable. Here, we will prove the analyticity in the space and time variables. By convention, a real-analytic function on a closed interval is real-analytic on some open neighborhood of the closed interval.
Proposition 17. Under the condition of Theorem 17, for any $T > 0$ there exists $\delta_T > 0$ such that we have $(u, v) \in C^\omega([0, T], \oplus^2 A(\delta_T))$.

Proof. We have $(u_0, v_0) \in \oplus^2 E_{\Delta, s+1}$ for any $\Delta < r_0/e$. Let $s \geq 2$ and assume $\Delta \leq 1$. By Theorem 9 there exist $T_\Delta > 0$ such that the Cauchy problem (2.5) has a unique solution $(\hat{u}, \hat{v}) \in C^\omega(|t| \leq T_\Delta(1 - d), \oplus^2 E_{\Delta, d+1})$ for $0 < d < 1$. We have $(\hat{u}, \hat{v}) = (u, v)$ by the local uniqueness of [6] Theorem 3.1, where $(u, v)$ is the solution in Theorem 10. Set $d = 1/2$, $T_\Delta = T_{\Delta}/2$. Then $(\hat{u}, \hat{v}) = (u, v) \in C^\omega(|t| \leq T_\Delta, \oplus^2 E_{\Delta, s+1})$. By Proposition 15 a convergent series in $E_{\Delta, s+1}$ is convergent in $A(\Delta/2)$. We have $(u, v) \in C^\omega(|t| \leq T_\Delta, \oplus^2 A(\Delta/2))$.

We have shown that $(u, v)$ is analytic in $t$ at least locally. Our next step is to show that it is analytic in $t$ globally. Set

$$S = \{ T > 0; (u, v) \in C^\omega([0, T], \oplus^2 A(\delta_T)) \text{ for some } \delta_T > 0 \} \ni T_\Delta',$$

$$T^* = \sup S \geq T_\Delta'.$$

We prove $T^* = \infty$ by contradiction. Assume $T^* < \infty$. By Propositions 15 and 5, there exists $\delta^* \in [0, \Delta/2)$ such that

$$(u(T^*), v(T^*)) \in \oplus^2 E_{\delta', s+1} (0 < \delta' < \delta^*).$$

By Theorem 9 (with $t$ replaced with $T^* - t$), there exists $\varepsilon > 0$ and $(\hat{u}, \hat{v}) \in C^\omega([T^* - \varepsilon, T^* + \varepsilon], \oplus^2 E_{\delta', s+1})$ such that

$$\hat{u}_t + \beta \hat{u}_x + (1 - \hat{\partial}_l^2)^{-1} \hat{\partial}_x \left[ -\alpha \hat{u} + \frac{3 - \beta}{2} \hat{u}^2 + \frac{\beta}{2} \hat{v}^2 + \hat{v} + \frac{1}{2} \hat{v}^2 \right] = 0,$$

$$\hat{v}_t + \hat{u}_x + (\hat{u} \hat{v})_x = 0,$$

$$(\hat{u}, \hat{v})|_{t = T^*} = (u(T^*), v(T^*)).$$

By the local uniqueness, we have $(\hat{u}, \hat{v}) = (u, v)$. Namely, $(\hat{u}, \hat{v})$ is an extension of $(u, v)$ up to $t \leq T^* + \varepsilon$ (valued in $\oplus^2 E_{\delta', s+1} \subset \oplus^2 A(\delta'/2)$). Therefore $T^* + \varepsilon \in S$. This is a contradiction.

Finally we prove the analyticity in $(t, x)$.

\[ \square \]

Proposition 18. Under the condition of Theorem 17, the Cauchy problem (2.5) has a unique solution $(u, v) \in \oplus^2 C^\omega([0, \infty) \times \mathbb{R})$.

Proof. Let $T$ be fixed. For $r > 0$ sufficiently small, we have

$$\partial_t^k u(t) = \frac{k!}{2\pi i} \int_{|\tau - t| = r} \frac{u(t)}{(\tau - t)^{k+1}} d\tau, \ t \in [0, T].$$

The integral is performed in $A(\delta_T)$ and converges with respect to $|| \cdot ||_{(\sigma, 0)}$ ($\sigma < \log \delta_T$). By Cauchy’s estimate, there exists $C_0 > 1/r$ such that

$$|| \partial_t^k u(t) ||_{(\sigma, 0)} \leq C_0 k! r^{-k} < C_0^{k+1} k!.$$

Therefore we have

$$|| \partial_j^l \partial_t^k u(\cdot, t) ||_0 \leq C_0^{k+1} e^{-j \sigma} j! k!$$

and there exists $C > 0$ such that

$$|| \partial_j^l \partial_t^k u ||_{L^2(\mathbb{R} \times [0, T])} \leq \sqrt{T} C^{j+k+1} (j + k)!. \quad (3.7)$$

Set $\Delta = \partial^2/\partial x^2 + \partial^2/\partial t^2$. The binomial expansion of $\Delta^\ell (\ell = 0, 1, 2, \ldots)$ and (3.7) yield

$$\|\Delta^\ell u\|_{L^2} \leq \sqrt{T C^{2\ell+1}(2\ell)!} \sum_{p=0}^{\ell} \binom{\ell}{p}$$

This estimates implies the real-analyticity of $u$ due to [14] or [15]. □

4. Proof of Proposition [13]: Part 1

The goal of this section is to get an estimate of the first sum of the right-hand side of (4.3) in Proposition [13]. We have to estimate the six quantities

$$\langle u^{(j)}, \partial_x^2(uu_x) \rangle_2, \langle u^{(j)}, \Lambda^{-2}\partial_x^{j+1}u \rangle_2, \langle u^{(j)}, \Lambda^{-2}\partial_x^{j+1}u^2 \rangle_2,$$

$$\langle u^{(j)}, \Lambda^{-2}\partial_x^{j+1}u_x \rangle_2, \langle u^{(j)}, \Lambda^{-2}\partial_x^{j+1}u \rangle_2, \langle u^{(j)}, \Lambda^{-2}\partial_x^{j+1}u^2 \rangle_2$$

for $0 < j = 1, 2, \ldots, m + 1$.

4.1. Estimate 1. We have $\langle u^{(j)}, \partial_x^2(uu_x) \rangle_2 = P_j + Q_j$, where

$$P_j = \langle u^{(j)}, uu^{(j+1)} \rangle_2,$$

$$Q_j = \sum_{\ell=1}^{j}\binom{j}{\ell}\langle u^{(j)}, u^{(\ell)}u^{(j+\ell+1)} \rangle_2 (j \geq 1).$$

By [3] (6.5), we have

$$|P_j| \leq 2^{j}\|u\|_2\|u^{(j)}\|_2^2.$$

Therefore

$$\left| \sum_{j=1}^{m+1} e^{2(j-1)\sigma} j^{-2\ell} P_j \right| \leq 2^{\ell}\|u\|_2\Phi_{\sigma,m}^{(1)}(u) \leq 2^{\ell}\|u\|_2\Phi_{\sigma,m}(u,v). \quad (4.1)$$

On the other hand, by using Lemma [1] (v) and splitting the sum into the terms for $\ell = 1, j$ and those in-between, we get

$$|Q_j| \leq 8 \sum_{\ell=1}^{j}\binom{j}{\ell}\|u^{(j)}\|_2 \left(\|u^{(\ell)}\|_2\|u^{(j-\ell+1)}\|_1 + \|u^{(\ell)}\|_1\|u^{(j-\ell+1)}\|_2 \right) \quad (4.2)$$

$$\leq 16(j+1)\|u^{(1)}\|_2\|u^{(j)}\|_2^2 + q_j,$$

$$q_j = 8 \sum_{\ell=2}^{j-1}\binom{j}{\ell}\|u^{(J)}\|_2 \left(\|u^{(\ell)}\|_2\|u^{(j-\ell)}\|_2 + \|u^{(\ell-1)}\|_2\|u^{(j-\ell+1)}\|_2 \right). \quad (4.3)$$

We set $q_j = 0$ if $j \leq 2$. As for the first term in the right-hand side of (4.2), we have $16(j+1) = 32 + 16(j-1)$ and

$$\sum_{j=1}^{m+1} e^{2(j-1)\sigma} j^{-2\ell} \cdot 16(j+1)\|u^{(1)}\|_2\|u^{(j)}\|_2^2 \quad (4.4)$$

$$\leq 64\|u^{(1)}\|_2\Phi_{\sigma,m}^{(1)}(u) + 16\|u^{(1)}\|_2\partial_x\Phi_{\sigma,m}^{(1)}(u) \leq 64\|u\|_2\Phi_{\sigma,m}(u,v) + 16\|u\|_3\partial_x\Phi_{\sigma,m}(u,v).$$
Set \( a_k = e^{k \sigma} \| u^{(k)} \|^2 / k! \). Then, \((4.13), (5.19), (5.20)\) and Proposition \(20\) imply
\[
\sum_{j=3}^{m+1} \frac{e^{2(j-1)\sigma}}{j^2} q_j \leq 8e^{-2\sigma} \sum_{j=3}^{m+1} j \sum_{\ell=2}^{j-1} a_j a_{\ell-1} + 8e^{-2\sigma} \sum_{j=3}^{m+1} j \sum_{\ell=2}^{j-1} a_j a_{\ell-1} a_{j-\ell+1}
\]
\[
\leq \frac{32\pi}{\sqrt{3}} e^\sigma \sqrt{\Phi_{\sigma,m}(u) \partial_x \Phi_{\sigma,m}^1(u)} \leq \frac{32\pi}{\sqrt{3}} e^\sigma \sqrt{\Phi_{\sigma,m}(u, v) \partial_x \Phi_{\sigma,m}(u, v)}.
\]

By combining \((4.1), (4.2), (4.4)\) and \((4.5)\), we get
\[
\sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j^2} \langle u^{(j)}, \partial_x^j (u u_x) \rangle_2 \leq 96\| u \|_3 \Phi_{\sigma,m}(u, v) + \left(16\| u \|_3 + \frac{32\pi}{\sqrt{3}} e^\sigma \sqrt{\Phi_{\sigma,m}(u, v)}\right) \partial_x \Phi_{\sigma,m}(u, v).
\]

**4.2. Estimate 2.** Since
\[
\langle u^{(j)}, \Lambda^{-2} \partial_x^{j+1} u \rangle_2 \leq \| u^{(j)} \|_2 \| \Lambda^{-2} \partial_x^{j+1} u \|_2
\]
\[
\leq \| u^{(j)} \|_2 \| \partial_x^{j+1} u \|_0 \leq \| u^{(j)} \|_2^2,
\]
we have
\[
\sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j^2} \langle u^{(j)}, \partial_x^j u \rangle_2 \leq \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j^2} \| u^{(j)} \|_2^2 \leq 2\Phi_{\sigma,m}(u) \leq 2\Phi_{\sigma,m}(u, v).
\]

**4.3. Estimate 3.** In Subsection \[4.1\] we considered \( \langle u^{(j)}, \partial_x^j (u u_x) \rangle_2 \), which is \(1/2\) times \( \langle u^{(j)}, \partial_x^{j+1} u \rangle_2 \). In the present subsection we consider \( \langle u^{(j)}, \partial_x^{j+1} \Lambda^{-2} u \rangle_2 \). We just neglect \( \Lambda^{-2}: H^2 \to H^2 \) and follow \((4.6)\) to obtain
\[
\sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j^2} \langle u^{(j)}, \Lambda^{-2} \partial_x^j u \rangle_2 \leq 192\| u \|_3 \Phi_{\sigma,m}(u, v) + \left(32\| u \|_3 + \frac{64\pi}{\sqrt{3}} e^\sigma \sqrt{\Phi_{\sigma,m}(u, v)}\right) \partial_x \Phi_{\sigma,m}(u, v).
\]

**4.4. Estimate 4.** We have
\[
\langle u^{(j)}, \Lambda^{-2} \partial_x^{j+1} (u_x^2) \rangle_2 = \langle u^{(j)}, (\Lambda^{-2} \partial_x^2) \partial_x^{j-1} (u_x^2) \rangle_2
\]
\[
= \sum_{k=0}^{j-1} \binom{j-1}{k} \langle u^{(j)}, (\Lambda^{-2} \partial_x^2) (u^{(k+1)} u^{(j-k)}) \rangle_2
\]
\[
= \sum_{\ell=1}^j \binom{j-1}{\ell-1} \langle u^{(j)}, (\Lambda^{-2} \partial_x^2) (u^{(\ell)} u^{(j-\ell+1)}) \rangle_2.
\]
Since the norm of $\Lambda^{-2} \partial_x^2 : H^2 \to H^2$ is 1, this operator can be neglected in estimating $\langle u^{(j)}, (\Lambda^{-2} \partial_x^2)(u^{(j)}u^{(j-\ell+1)}) \rangle_2$. Moreover we have

$$\left( j - 1 \right) \left( \ell - 1 \right) \leq \left( j \right) \left( j \geq 1 \right).$$

Therefore, following the calculation about $Q_j$ in Subsection 4.1 we get

$$\left| \sum_{j=1}^{m+1} \frac{e^{2j\sigma}}{j!^2} \langle u^{(j)}', (\Lambda^{-2} \partial_x^2+1)u^{(j)} \rangle_2 \right| \leq 64\|u\|_3 \Phi_{\sigma,m}(u, v) + \left( 16\|u\|_3 + \frac{32\pi}{\sqrt{3}} \sqrt[\Phi_{\sigma,m}(u, v)} \right) \partial_\sigma \Phi_{\sigma,m}(u, v).$$

\textbf{4.5. Estimate 5.} Since

$$\left| \langle u^{(j)}, \Lambda^{-2} \partial_x^{j+1}v \rangle_2 \right| \leq \|u^{(j)}\|_2 \|\Lambda^{-2} \partial_x^{j+1}v\|_2 \leq 2\|u^{(j)}\|_2 \|v^{(j-1)}\|_2 \leq \frac{1}{2} \left( \|u^{(j)}\|_2^2 + \|v^{(j-1)}\|_2^2 \right),$$

we have

$$\left| \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \langle u^{(j)}, \Lambda^{-2} \partial_x^{j+1}v \rangle_2 \right| \leq \frac{1}{2} \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \left( \|u^{(j)}\|_2^2 + \|v^{(j-1)}\|_2^2 \right) \leq \frac{1}{2} \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \|u^{(j)}\|_2 + \frac{1}{2} \sum_{k=0}^{m} \frac{e^{2k\sigma}}{k!^2} \|v^{(k)}\|_2 = \Phi_{\sigma,m}(u, v).$$

\textbf{4.6. Estimate 6.} We consider $\langle u^{(j)}, \Lambda^{-2} \partial_x^{j+1}v^2 \rangle_2$. When $j = 1$, we have

$$\left| \langle u_x, \Lambda^{-2} \partial_x^2v^2 \rangle_2 \right| \leq \|u_x\|_2 \|\Lambda^{-2} \partial_x^2v^2\|_2 \leq \|u\|_3 \|v^2\|_2 \leq 8\|u\|_3 \|v\|_2^2.$$

We multiply it with $e^{2(1-1)\sigma}/1!^2 = 1$. The product is bounded by $16\|u\|_3 \Phi_{\sigma,m}^{(2)}(v)$. We estimate $\langle u^{(j)}, \Lambda^{-2} \partial_x^{j+1}v^2 \rangle_2$. Next assume $j \geq 2$. Since the norm of $\Lambda^{-2} \partial_x^2 : H^2 \to H^2$ is 1, we have

$$\left| \langle u^{(j)}, \Lambda^{-2} \partial_x^{j+1}v^2 \rangle_2 \right| \leq \|u^{(j)}\|_2 \|\partial_x^{j-1}v^2\|_2 \leq 8\|u^{(j)}\|_2 \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} \|v^{(\ell)}\|_2 \|v^{(j-\ell-1)}\|_2.$$

Separating the term for $\ell = 0$ from the other terms, we obtain

$$\left| \sum_{j=2}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \langle u^{(j)}, \Lambda^{-2} \partial_x^{j+1}v^2 \rangle_2 \right| \leq S_0 + S_1,$$

where

$$S_0 = 8\|v\|_2 \sum_{j=2}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \|u^{(j)}\|_2 \|v^{(j-1)}\|_2,$$

and

$$S_1 = 8\sum_{j=2}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \|u^{(j)}\|_2 \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \|v^{(\ell)}\|_2 \|v^{(j-\ell-1)}\|_2.$$
By using (5.21) and Proposition 20 imply
\[ S_1 = 8e^{-\sigma} \sum_{j=2}^{m+1} \sum_{\ell=1}^{j-1} a_j b_{\ell-j-1} \]
\[ \leq \frac{16\pi}{\sqrt{6}} \sqrt{\Phi_{\sigma,m}(v)} \sqrt{\partial_\sigma \Phi_{\sigma,m}(u) \cdot \partial_\sigma \Phi_{\sigma,m}(v)} \]
\[ \leq \frac{16\pi}{\sqrt{6}} \sqrt{\Phi_{\sigma,m}(u,v)} \partial_\sigma \Phi_{\sigma,m}(u,v). \]

Summing up, we obtain
\[ \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \langle u^{(j)}, \Lambda^{-2} \partial_x^j v^2 \rangle \]
\[ \leq (16\|u\|_3 + 8\|v\|_2) \Phi_{\sigma,m}(u,v) + \frac{16\pi}{\sqrt{6}} \sqrt{\Phi_{\sigma,m}(u,v)} \partial_\sigma \Phi_{\sigma,m}(u,v). \]  

5. PROOF OF PROPOSITION [13] PART 2

The goal of this section is to get an estimate of the second sum of the right-hand side of (5.3) in Proposition [13]. We have to estimate the two quantities
\[ \langle v^{(j)}, u^{(j+1)} \rangle_2, \langle v^{(j)}, (uv)^{(j+1)} \rangle_2 \]
for \( j = 0, 1, \ldots, m \). Notice that the range of \( j \) is different from that in the previous section.

5.1. \textbf{Estimate 7.} We have
\[ \langle v^{(j)}, u^{(j+1)} \rangle_2 = I_0 + 2I_1 + I_2, \quad I_i = \langle v^{(j+i)}, u^{(j+i+1)} \rangle_0, \]
where \( \langle \cdot, \cdot \rangle_0 \) is the inner product of \( H^0 = L^2 \). For \( i = 0, 1, 2 \), we have
\[ |I_i| \leq \|u^{(j+i+1)}\|_0 \|v^{(j+i)}\|_0 \leq \|u^{(j+1)}\|_2 \|v^{(j)}\|_2. \]

By using \( 2XY \leq X^2/(j+1) + (j+1)Y^2 \), we get
\[ |I_i| \leq \frac{1}{2} \left[ \frac{1}{j+1} \|u^{(j+1)}\|_2^2 + (j+1)\|v^{(j)}\|_2^2 \right]. \]  

(5.1)

for \( i = 0, 1, 2 \). Therefore
\[ \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2} \langle v^{(j)}, u^{(j+1)} \rangle_2 \leq 2 \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2} \left[ \frac{1}{j+1} \|u^{(j+1)}\|_2^2 + (1+j)\|v^{(j)}\|_2^2 \right]. \]  

(5.2)
Here the sum involving $u$ is
\[
\sum_{j=0}^{m} \frac{e^{2j\sigma}}{j^{12}} \|u^{(j+1)}\|_2^2 = \sum_{j=0}^{m} \frac{e^{2(j+1)\sigma}}{(j+1)^{12}} \|u^{(j+1)}\|_2^2 + \sum_{j=0}^{m} \frac{e^{2j\sigma}}{(j+1)^{12}} \|u^{(j+1)}\|_2^2
\]
\[
= \sum_{k=1}^{m+1} \frac{e^{2(k-1)\sigma}}{k^{12}} \|u^{(k)}\|_2^2 + \sum_{k=1}^{m+1} \frac{e^{2(k-1)\sigma}(k-1)}{k^{12}} \|u^{(k)}\|_2^2 = 2\Phi^{(1)}_{\sigma,m}(u) + \partial_{\sigma} \Phi^{(1)}_{\sigma,m}(u).
\]

The sum involving $v$ is
\[
\sum_{j=0}^{m} \frac{e^{2j\sigma}}{j^{12}} (1+j\|v^{(j)}\|_2^2) = 2\Phi^{(2)}_{\sigma,m}(v) + \partial_{\sigma} \Phi^{(2)}_{\sigma,m}(v).
\]

Hence we get
\[
\left| \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j^{12}} \langle v^{(j)}, u^{(j+1)} \rangle_2 \right| \leq 4\Phi^{(1)}_{\sigma,m}(u) + 2\partial_{\sigma} \Phi^{(1)}_{\sigma,m}(u) + 4\Phi^{(2)}_{\sigma,m}(v) + 2\partial_{\sigma} \Phi^{(2)}_{\sigma,m}(v)
\]
\[
= 4\Phi_{\sigma,m}(u,v) + 2\partial_{\sigma} \Phi_{\sigma,m}(u,v).
\]

5.2. Estimate 8. We shall prove
\[
\left| \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j^{12}} \langle v^{(j)}, (uv)^{(j+1)} \rangle_2 \right| \leq \left[ (66 + 16e^{2\sigma})\|u\|_3 + (18 + 8e^{2\sigma})\|v\|_3 \right] \Phi_{\sigma,m}(u,v)
\]
\[
+ \frac{16\pi}{\sqrt{3}} (1 + \sqrt{2})e^{\sigma} \sqrt{\Phi_{\sigma,m}(u,v)} \partial_{\sigma} \Phi_{\sigma,m}(u,v)
\]
\[
+ \left[ 8\|u\|_3 + (4e^{2\sigma} + 13)\|v\|_3 \right] \partial_{\sigma} \Phi_{\sigma,m}(u,v).
\]

This inequality is the result of (5.8), (5.9) and (5.18) below.

Since $(uv)^{(j+1)} = (uv_x + u_x v)^{(j)}$, we have $\langle v^{(j)}, (uv)^{(j+1)} \rangle_2 = J_1 + J_2 + J_3$, where
\[
J_1 = \langle v^{(j)}, u^{(j+1)}v \rangle_2,
\]
\[
J_2 = \langle v^{(j)}, uv^{(j+1)} \rangle_2,
\]
\[
J_3 = \sum_{\ell=1}^{j} \left( \frac{j}{\ell} \right) \langle v^{(j)} + u^{(j+1)}v^{(j-\ell+1)} + u^{(j-\ell+1)}v^{(j)} \rangle_2.
\]

Notice that $J_3 = 0$ if $j = 0$.

5.2.1. First term. We decompose $J_1$ into a sum of $L^2$ inner products by $J_1 = J_{10} + 2J_{11} + J_{12}$, where $J_{1i} = \langle v^{(j+1)}, \partial_{x}^{i}(u^{(j+1)}v) \rangle_0$. We have
\[
|J_{10}| \leq \|v^{(j)}\|_0 \|u^{(j+1)}v\|_0 \leq \|v^{(j)}\|_0 \cdot 2\|v\|_1 \|u^{(j+1)}\|_0 \leq \|v\|_1 \cdot 2\|u^{(j+1)}\|_2 \|v^{(j)}\|_2
\]
\[
\leq \|v\|_1 \left[ \frac{1}{j+1} \|u^{(j+1)}\|_2^2 + (1+j)\|v^{(j)}\|_2^2 \right].
\]
The right-hand side is similar to that of (5.1). As for $J_{11}$, we have
\begin{align}
|J_{11}| & \leq \|v^{(j+1)}\|_0 \left( \|u^{(j+1)}v_{x}\|_0 + \|u^{(j+2)}v\|_0 \right) \\
& \leq \|v^{(j+1)}\|_0 \left( 2\|u^{(j+1)}\|_0 \|v_{x}\|_1 + 2\|u^{(j+2)}\|_0 \|v\|_1 \right) \\
& \leq 4\|v^{(j)}\|_2 \|u^{(j+1)}\|_2 \|v\|_2 \\
& \leq 2\|v\|_2 \left[ \frac{1}{j + 1} \|u^{(j+1)}\|_2^2 + (1 + j) \|v^{(j)}\|_2^2 \right].
\end{align}

We further decompose $J_{12} = \langle u^{(j+2)}, \partial_x^2 (u^{(j+1)}v) \rangle_0$ by $J_{12} = J_{120} + 2J_{121} + J_{122}$, where
\[ J_{12i} = \langle v^{(j+2)}, u^{(j+3-i)} v^{(i)} \rangle_0. \]

We have
\begin{align}
|J_{12i}| & \leq \|v^{(j+2)}\|_0 \|u^{(j+3-i)} v^{(i)}\|_0 \leq 2\|v^{(j+2)}\|_0 \|u^{(j+3-i)}\|_0 \|v^{(i)}\|_1 \\
& = 2\|v^{(j)}\|_2 \|u^{(j+1)}\|_2 \|v\|_3 \\
& \leq \|v\|_3 \left[ \frac{1}{j + 1} \|u^{(j+1)}\|_2^2 + (1 + j) \|v^{(j)}\|_2^2 \right], \\
|J_{12}| & \leq 4\|v\|_3 \left[ \frac{1}{j + 1} \|u^{(j+1)}\|_2^2 + (1 + j) \|v^{(j)}\|_2^2 \right].
\end{align}

By combining (5.5), (5.6) and (5.7), we obtain
\[ |J_1| \leq 9\|v\|_3 \left[ \frac{1}{j + 1} \|u^{(j+1)}\|_2^2 + (1 + j) \|v^{(j)}\|_2^2 \right]. \]

Following (5.2) and (5.3), we get
\[ \left| \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2} J_1 \right| \leq 9\|v\|_3 \left[ 2\Phi_{\sigma,m}(u, v) + \partial_x \Phi_{\sigma,m}(u, v) \right]. \]

5.2.2. Second term. We decompose $J_2$ into a sum of $L^2$ inner products by $J_2 = J_{20} + 2J_{21} + J_{22}$, where $J_{2i} = \langle v^{(j+i)}, \partial_x^i (uv^{(j+1)}) \rangle_0$. We have
\begin{align}
|J_{20}| & \leq \|v^{(j)}\|_0 \|uv^{(j+1)}\|_0 \leq 2\|v^{(j)}\|_0 \|u\|_1 \|v^{(j+1)}\|_0 \\
& \leq 2\|u\|_1 \|v^{(j)}\|_2^2, \\
|J_{21}| & \leq \|v^{(j+1)}\|_0 \left( \|u_x v^{(j+1)}\|_0 + \|uv^{(j+2)}\|_0 \right) \\
& \leq \|v^{(j+1)}\|_0 \left( 2\|u_x\|_1 \|v^{(j+1)}\|_0 + 2\|u\|_1 \|v^{(j+2)}\|_0 \right) \leq 4\|u\|_2 \|v^{(j)}\|_2^2.
\end{align}

Therefore
\begin{align}
\left| \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2} J_{20} \right| & \leq 4\|u\|_1 \Phi_{\sigma,m}^{(2)}(v) \leq 4\|u\|_1 \Phi_{\sigma,m}(u, v), \\
\left| \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2} J_{21} \right| & \leq 8\|u\|_2 \Phi_{\sigma,m}^{(2)}(v) \leq 8\|u\|_2 \Phi_{\sigma,m}(u, v).
\end{align}
We further decompose $J_{22} = \langle v^{(j+2)}, \partial_x^2(u(v^{(j+1)})\rangle_0$ by $J_{22} = J_{220} + 2J_{221} + J_{222}$, where

$$J_{22i} = \langle v^{(j+2)}, u^{(2-i)}v^{(j+i+1)}\rangle_0.$$

For $i = 0, 1$, we have

$$|J_{22i}| \leq \|v^{(j+2)}\|_0\|v^{(2-i)}v^{(j+i+1)}\|_0 \leq 2\|v^{(j+2)}\|_0\|u^{(2-i)}\|_1\|v^{(j+i+1)}\|_0 \leq 2\|u\|_3\|v^{(j)}\|_2^2.$$

We get an alternative expression of $J_{222} = \langle v^{(j+2)}, uv^{(j+3)}\rangle_0$ by integration by parts. We have

$$J_{222} = \int_\mathbb{R} \frac{1}{2} u_\ell (v^{(j+2)})^2 dx = -\frac{1}{2} \int_\mathbb{R} u_\ell (v^{(j+2)})^2 dx = -\frac{1}{2} \langle u_\ell v^{(j+2)}, v^{(j+2)}\rangle_0.$$

Therefore

$$|J_{222}| \leq \frac{1}{2}\|u_\ell v^{(j+2)}\|_0\|v^{(j+2)}\|_0 \leq \|u_\ell\|_1\|v^{(j+2)}\|_0^2 \leq \|u\|_2\|v^{(j)}\|_2^2.$$

Summing up, we get

$$\left| \sum_{j=0}^m \frac{e^{2j\sigma}}{j^{12}} J_{222} \right| \leq \sum_{j=0}^m \frac{e^{2j\sigma}}{j^{12}} (|J_{220}| + 2|J_{221}| + |J_{222}|) \leq 14\|u\|_3\Phi^{(2)}_{\sigma,m}(v) \leq 14\|u\|_3\Phi_{\sigma,m}(u, v).$$

Finally we obtain

$$\left| \sum_{j=0}^m \frac{e^{2j\sigma}}{j^{12}} J_2 \right| \leq 34\|u\|_3\Phi_{\sigma,m}(u, v). \quad (5.9)$$

### 5.2.3. Third term

Recall

$$J_3 = \sum_{\ell=1}^j \binom{j}{\ell} \langle v^{(j)}, u^{(\ell)}v^{(j-\ell+1)} + u^{(j-\ell+1)}v^{(\ell)}\rangle_2 \quad (1 \leq j \leq m)$$

and that $J_3 = 0$ if $j = 0$. We assume $j \geq 1$. We have

$$J_3 = K_j + L_j, \quad K_j = K_{j1} + K_{j2}, \quad L_j = L_{j1} + L_{j2},$$

$$K_{j1} = j \langle v^{(j)}, u_xv^{(j)}\rangle_2, \quad K_{j2} = \sum_{\ell=2}^j \binom{j}{\ell} \langle u^{(j)}, u^{(\ell)}v^{(j-\ell+1)}\rangle_2,$$

$$L_{j1} = j \langle v^{(j)}, u^{(j)}v_x\rangle_2, \quad L_{j2} = \sum_{\ell=2}^j \binom{j}{\ell} \langle u^{(j)}, u^{(j-\ell+1)}v^{(\ell)}\rangle_2.$$

If $j = 1$, then $K_{j2} = L_{j2} = 0$.

For $1 \leq j \leq m$, we have

$$|K_{j1}| \leq 8j\|v^{(j)}\|_2\|u_x\|_2\|v^{(j)}\|_2 \leq 8\|u\|_3\cdot j\|v^{(j)}\|_2^2,$$

$$|L_{j1}| \leq 8j\|v^{(j)}\|_2\|u^{(j)}\|_2\|v_x\|_2 \leq 4\|v\|_3 \left( j\|u^{(j)}\|_2^2 + j\|v^{(j)}\|_2^2 \right),$$

and

$$\left| \sum_{j=1}^m \frac{e^{2\sigma j}}{j^{12}} K_{j1} \right| \leq 8\|u\|_3\|\partial_x\Phi^{(2)}_{\sigma,m}(v) \leq 8\|u\|_3\|\partial_x\Phi_{\sigma,m}(u, v), \quad (5.10)$$
\[
\left| \sum_{j=1}^{m} \frac{e^{2\sigma j}}{j!^2} L_{j1} \right| \leq 8e^{2\sigma} \|v\|_3 \Phi_{\sigma,m}^{(1)}(u) + 4e^{2\sigma} \|v\|_3 \partial_\sigma \Phi_{\sigma,m}^{(1)}(u) + 4\|v\|_3 \partial_\sigma \Phi_{\sigma,m}^{(2)}(v) \\
\leq 8e^{2\sigma} \|v\|_3 \Phi_{\sigma,m}(u,v) + (4e^{2\sigma} + 4)\|v\|_3 \partial_\sigma \Phi_{\sigma,m}(u,v).
\]

(5.11)

For \(j \geq 2\), we have
\[
|K_{j2}| \leq 8\|v^{(j)}\|_2 \sum_{\ell=2}^{j} \left( \frac{j}{\ell} \right) \left( \|u^{(\ell)}\|_2 \|v^{(j-\ell+1)}\|_1 + \|u^{(\ell)}\|_1 \|v^{(j-\ell+1)}\|_2 \right) \\
\leq 8 \sum_{\ell=2}^{j} \left( \frac{j}{\ell} \right) \|u^{(\ell)}\|_2 \|v^{(j)}\|_2 \|v^{(j-\ell)}\|_2 \\
+ 8 \sum_{\ell=2}^{j-1} \left( \frac{j}{\ell} \right) \|u^{(\ell-1)}\|_2 \|v^{(j)}\|_2 \|v^{(j-\ell+1)}\|_2.
\]

(5.12)

\[
|L_{j2}| \leq 8\|v^{(j)}\|_2 \sum_{\ell=2}^{j} \left( \frac{j}{\ell} \right) \left( \|u^{(j-\ell+1)}\|_1 \|v^{(j)}\|_2 + \|u^{(j-\ell+1)}\|_2 \|v^{(j)}\|_1 \right) \\
\leq 8\|u\|_2 \|v^{(j)}\|_2^2 + 8\|u^{(1)}\|_2 \|v^{(j)}\|_2 \|v^{(j-1)}\|_2 \\
+ 8 \sum_{\ell=2}^{j-1} \left( \frac{j}{\ell} \right) \|u^{(j-\ell)}\|_2 \|v^{(j)}\|_2 \|v^{(j)}\|_2 \\
+ 8 \sum_{\ell=2}^{j-1} \left( \frac{j}{\ell} \right) \|u^{(j-\ell+1)}\|_2 \|v^{(j)}\|_2 \|v^{(j-\ell)}\|_2.
\]

(5.13)

Notice that \(\sum_{\ell=2}^{j-1} = 0\) if \(j \leq 2\) in (5.13). As for the first term in the estimate (5.13) of \(L_{j2}\), we have
\[
\sum_{j=2}^{m} \frac{e^{2\sigma j}}{j!^2} \|u\|_2 \|v^{(j)}\|_2^2 \leq 2\|u\|_2 \Phi_{\sigma,m}^{(2)}(v).
\]

(5.14)

As for the second term, since
\[
\|u^{(1)}\|_2 \|v^{(j)}\|_2 \|v^{(j-1)}\|_2 \leq \frac{1}{2} \left( \|u^{(1)}\|_2^2 + \|v^{(j-1)}\|_2^2 \right),
\]
and
\[
\sum_{j=2}^{m} \frac{e^{2\sigma j}}{j!^2} \|u^{(1)}\|_2 \|v^{(j-1)}\|_2^2 \leq e^{2\sigma} \|u\|_3 \sum_{k=1}^{m-1} \frac{e^{2\sigma k}}{k!^2} \|v^{(k)}\|_2^2 \leq 2e^{2\sigma} \|u\|_3 \Phi_{\sigma,m}^{(2)}(v),
\]
we have
\[
\sum_{j=2}^{m} \frac{e^{2\sigma j}}{j!^2} \|u^{(1)}\|_2 \|v^{(j)}\|_2 \|v^{(j-1)}\|_2 \leq (1 + e^{2\sigma}) \|u\|_3 \Phi_{\sigma,m}^{(2)}(v).
\]

(5.15)

Now we estimate the four sums in (5.12) and (5.13). Set \(a_k = e^{k\sigma} \|v^{(k)}\|_2 / k! (k = 1, 2, \ldots, m + 1)\), \(b_k = e^{k\sigma} \|v^{(k)}\|_2 / k! (k = 0, 1, \ldots, m)\). Then by Proposition [19] we get
\[
\sum_{j=2}^{m} \frac{e^{2\sigma j}}{j!^2} \sum_{\ell=2}^{j} \left( \frac{j}{\ell} \right) \|u^{(\ell)}\|_2 \|v^{(j)}\|_2 \|v^{(j-\ell)}\|_2 = \sum_{j=2}^{m} \sum_{\ell=2}^{j} a_{j} b_{j-\ell} \leq \frac{\pi}{\sqrt{6}} \hat{A} \hat{B} \hat{B},
\]
Combining (5.10), (5.11), (5.16) and (5.17), we obtain

For non-negative numbers

Proposition 19.

Therefore by (5.12), (5.13), (5.14), (5.15) and Proposition 20

Combining (5.10), (5.11), (5.16) and (5.17), we obtain

APPENDIX

Proposition 19. For non-negative numbers \( a_j (j = 1, \ldots, m + 1) \), and \( b_j (j = 0, \ldots, m) \), set \( A = \left( \sum_{j=1}^{m+1} a_j^2 \right)^{1/2}, \hat{A} = \left( \sum_{j=2}^{m+1} j a_j^2 \right)^{1/2}, B = \left( \sum_{j=0}^{m} b_j^2 \right)^{1/2}, \hat{B} = \left( \sum_{j=1}^{m} j b_j^2 \right)^{1/2} \). Then we have

\[
\sum_{j=3}^{m} \sum_{\ell=2}^{j-1} \frac{j-\ell+1}{\ell} a_{\ell-1} b_{j-\ell+1} \leq \frac{\pi}{\sqrt{6}} A \hat{B},
\]

\[
\sum_{j=3}^{m} \sum_{\ell=2}^{j-1} \frac{j-\ell+1}{\ell} a_{\ell-1} b_{j-\ell+1} \leq \frac{\pi}{\sqrt{6}} A \hat{B}.
\]
Proof. We have

$$\sum_{j=2}^{m+1} \sum_{\ell=1}^{j-1} j \alpha_j b_\ell b_{j-\ell-1} \leq \frac{\pi}{\sqrt{6}} \tilde{A} \tilde{B},$$  \hspace{1cm} (5.21)$$

$$\sum_{j=2}^{m} j \alpha_j b_j \leq \frac{\pi}{\sqrt{6}} \tilde{A} \tilde{B},$$  \hspace{1cm} (5.22)$$

$$\sum_{j=2}^{m} \sum_{\ell=2}^{j-\ell+1} \alpha_{\ell-1} b_j b_{\ell-1} \leq \frac{\pi}{\sqrt{6}} \tilde{A} \tilde{B}^2,$$  \hspace{1cm} (5.23)$$

$$\sum_{j=3}^{m} \sum_{\ell=2}^{j-\ell} \alpha_{j-\ell} b_j b_{\ell} \leq \frac{\pi}{\sqrt{6}} \tilde{A} \tilde{B}^2,$$  \hspace{1cm} (5.24)$$

$$\sum_{j=3}^{m} \sum_{\ell=2}^{j-\ell+1} j \alpha_{j-\ell+1} b_j b_{\ell-1} \leq \frac{\pi}{\sqrt{6}} \tilde{A} \tilde{B}.$$  \hspace{1cm} (5.25)$$

\text{Camassa-Holm equations 25}
\[ \leq \frac{\pi}{\sqrt{6}} \tilde{A} \tilde{B}. \]

\[ \Box \]

**Proposition 20.** If \( a_k = e^{k\sigma} \| u^{(k)} \|_2 / k! \) \((k = 1, 2, \ldots, m + 1)\), \( b_k = e^{k\sigma} \| v^{(k)} \|_2 / k! \) \((k = 0, 1, \ldots, m)\), and \( A, \tilde{A}, B, \tilde{B} \) are defined as in Proposition 19, then

\[ e^{-2\sigma} A \tilde{A}^2 \leq 2\sqrt{2} e^{2\sigma} \sqrt{\Phi_{\sigma,m}^{(1)}(u)} \partial_\sigma \Phi_{\sigma,m}^{(1)}(u) \]

\[ \tilde{A} \tilde{B} \leq 2 e^{\sigma} \sqrt{\Phi_{\sigma,m}^{(2)}(v)} \sqrt{\partial_\sigma \Phi_{\sigma,m}^{(1)}(u) \cdot \partial_\sigma \Phi_{\sigma,m}^{(2)}(v)}, \]

\[ A \tilde{B}^2 = \sqrt{2} e^{\sigma} \sqrt{\Phi_{\sigma,m}^{(1)}(u)} \partial_\sigma \Phi_{\sigma,m}^{(2)}(v). \]

**Proof.** We have

\[ e^{-2\sigma} A \tilde{A}^2 = 2 \Phi_{\sigma,m}^{(1)}(u), \]

\[ e^{-2\sigma} A \tilde{A}^2 = \sum_{k=2}^{m+1} \frac{k e^{2(k-1)\sigma}}{k!^2} \| u^{(k)} \|_2^2 \leq \sum_{k=2}^{m+1} \frac{2(k-1)e^{2(k-1)\sigma}}{k!^2} \| u^{(k)} \|_2^2 = 2 \partial_\sigma \Phi_{\sigma,m}^{(1)}(u). \]

Moreover, we have

\[ B^2 = 2 \Phi_{\sigma,m}^{(2)}(v), \quad \tilde{B}^2 = \partial_\sigma \Phi_{\sigma,m}^{(2)}(v). \]

\[ \Box \]

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Current address: Department of Mathematical Sciences, Kwansei Gakuin University, Gakuen 2-1 Sanda, Hyogo 669-1337, Japan
E-mail address: yamane@kwansei.ac.jp