Spanning trees and even integer eigenvalues of graphs

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Abstract
For a graph $G$, let $L(G)$ and $Q(G)$ be the Laplacian and signless Laplacian matrices of $G$, respectively, and $\tau(G)$ be the number of spanning trees of $G$. We prove that if $G$ has an odd number of vertices and $\tau(G)$ is not divisible by 4, then (i) $L(G)$ has no even integer eigenvalue, (ii) $Q(G)$ has no integer eigenvalue $\lambda \equiv 2 \pmod{4}$, and (iii) $Q(G)$ has at most one eigenvalue $\lambda \equiv 0 \pmod{4}$ and such an eigenvalue is simple. As a consequence, we extend previous results by Gutman and Sciriha and by Bapat on the nullity of adjacency matrices of the line graphs. We also show that if $\tau(G) = 2^t s$ with $s$ odd, then the multiplicity of any even integer eigenvalue of $Q(G)$ is at most $t+1$. Among other things, we prove that if $L(G)$ or $Q(G)$ has an even integer eigenvalue of multiplicity at least 2, then $\tau(G)$ is divisible by 4. As a very special case of this result, a conjecture by Zhou et al. [On the nullity of connected graphs with least eigenvalue at least $-2$, Appl. Anal. Discrete Math. (2013)] on the nullity of adjacency matrices of the line graphs of unicyclic graphs follows.

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1 Introduction

The graphs we consider are simple, that is, without loops or multiple edges. Let $G$ be a graph. The order of $G$ is the number of vertices of $G$. We denote by $A(G)$ the adjacency matrix, by $L(G)$ the line graph and by $\tau(G)$ the number of spanning trees of $G$.

The purpose of this paper is to study the interconnection between $\tau(G)$ and the multiplicities of even integer eigenvalues of $A(L(G))$. Our motivation comes partly from the previous works by
several authors on the connection between $\tau(G)$ and the multiplicity of zero eigenvalue, i.e. the nullity of $A(L(G))$. A brief review of the previous results is in order. Doob [7] proved that the binary rank (i.e. the rank over the two-element field) of $A(L(G))$ for any connected graph $G$ of order $n$ is $n - 1$ if $n$ is odd, and $n - 2$ if $n$ is even. This result was stated and proved in the context of Matroid Theory. Gutman and Sciriha [8] showed that the nullity (over reals) of $A(L(T))$ for any tree $T$ is at most 1 and if $A(L(T))$ is singular, then $T$ has an even order. Indeed, this is an immediate consequence of Doob’s result. Recently, Bapat [2] found an interesting generalization by proving that if $\tau(G)$ is odd, then $A(L(G))$ has nullity (over reals) at most 1. He also showed that a bipartite graph $G$ with odd $\tau(G)$ and with singular $A(L(G))$ must have even order. We extend these results to the following.

**Theorem 1.** Let $G$ be a connected graph and $\tau(G) = 2^t s$ with $s$ odd. Then the multiplicity of any even integer $\lambda \neq -2$ as an eigenvalue of $A(L(G))$ is at most $t + 1$.

**Theorem 2.** Suppose that $G$ is a graph with $\tau(G)$ not divisible by 4. If $\lambda \neq -2$ is an even integer eigenvalue of $A(L(G))$, then $\lambda \equiv 2 \pmod{4}$, $\lambda$ is a simple eigenvalue, and $A(L(G))$ has at most one such eigenvalue.

**Corollary 3.** If a graph $G$ has odd order and $\tau(G)$ is not divisible by 4, then $A(L(G))$ is nonsingular.

**Theorem 4.** If $A(L(G))$ has an even integer eigenvalue $\lambda \neq -2$ of multiplicity at least 2, then $\tau(G)$ is divisible by 4.

Since even integer eigenvalues of $A(L(G))$ and the signless Laplacian matrix $Q(G)$ are the same modulo a shift (see Section 2) it is enough to consider those of $Q(G)$ as we do in what follows. The rest of the paper is organized as follows. In Section 2 we recall some necessary preliminaries. In Section 3 we give a simple proof for Doob’s result which will be used later on. In Section 4, the proofs of Theorems [1, 2] and [3] in terms of $Q(G)$ are given along with some improvements and similar results for the eigenvalues of the Laplacian matrix $L(G)$.

## 2 Preliminaries

By $X = X(G)$ we denote the 0,1 vertex-edge incidence matrix of $G$. If we orient each edge of $G$, then $D = D(G)$ will denote the 0,±1 vertex-edge incidence matrix of the resulting graph. The Laplacian matrix of $G$ is $L = L(G) = DD^\top$ and the signless Laplacian matrix of $G$ is $Q = Q(G) = XX^\top$. Note that the Laplacian does not depend on the orientation. The matrices $L$
and $Q$ are positive semidefinite. It is easily seen that the incidence matrix of $G$ and the adjacency matrix of $\mathcal{L}(G)$ satisfy the following

$$A(\mathcal{L}(G)) + 2I = X^\top X. \quad (1)$$

Recall that for a matrix $M$, the matrices $MM^\top$ and $M^\top M$ have the same nonzero eigenvalues with the same multiplicities. This together with (1) imply that the matrices $A(\mathcal{L}(G)) + 2I$ and $Q(G)$ have the same nonzero eigenvalues with the same multiplicities. In particular, the multiplicity of eigenvalue 2 for $Q(G)$ is the same as the nullity of $A(\mathcal{L}(G))$. Therefore, studying even integer eigenvalues of $A(\mathcal{L}(G))$ and that of $Q(G)$ are equivalent.

We denote the vertex set and the edge set of $G$ by $V(G)$ and $E(G)$, respectively. If $S \subseteq E(G)$, then $(S)$ denotes the induced subgraph on $S$. For a matrix $M$ with $R, S$ being subsets of row and column indices of $M$, respectively, we denote the submatrix with row indices from $R$ and column indices from $S$ by $M(R, S)$.

The following two lemmas describe the invertible submatrices of $D$ and $X$. For the first one we refer to pp. 32 and 47 of [3] and for the second one to p. 30 of [1].

**Lemma 5.** Let $G$ be a graph and $R \subseteq V(G)$, $S \subseteq E(G)$ with $|R| = |S| \geq 1$. Let $V_0$ denote the vertex set of $(S)$. Then $D(R, S)$ is invertible if and only if the following conditions are satisfied:

(i) $R$ is a subset of $V_0$.

(ii) $(S)$ is a forest.

(iii) $V_0 \setminus R$ contains precisely one vertex from each connected component of $(S)$.

Moreover, if $D(R, S)$ is invertible, then $\det(D(R, S)) = \pm 1$.

**Lemma 6.** Let $G$ be a graph and $R \subseteq V(G)$, $S \subseteq E(G)$ with $|R| = |S| \geq 1$. Let $V_0$ denote the vertex set of $(S)$. Then $X(R, S)$ is invertible if and only if the following conditions are satisfied:

(i) $R$ is a subset of $V_0$.

(ii) each connected component of $(S)$ is either a tree or a unicyclic graph with odd cycle.

(iii) $V_0 \setminus R$ contains precisely one vertex from each tree in $(S)$.

Moreover, if $X(R, S)$ is invertible, then $\det(X(R, S)) = \pm 2^e$ where $e$ is the number of components of $(S)$ which are unicyclic with odd cycle.
The nullity of $L(G)$ and $Q(G)$ are respectively equal to the number of components and to the number of bipartite components of $G$. Let

$$p_Q(x) = x^n + q_1x^{n-1} + \cdots + q_n, \quad p_L(x) = x^n + \ell_1x^{n-1} + \cdots + \ell_{n-1}x$$

be the characteristic polynomials of $Q$ and $L$, respectively. A subgraph of $G$ whose components are trees or unicyclic graphs with odd cycles is called a $TU$-subgraph of $G$. Suppose that a $TU$-subgraph $H$ of $G$ contain $c$ unicyclic graphs and trees $T_1, T_2, \ldots, T_s$. Then the weight $W(H)$ of $H$ is defined by

$$W(H) = 4^c \prod_{i=1}^s (1 + e(T_i)),$$

where $e(T_i)$ denotes the number of edges of $T_i$. The weight of an acyclic subgraph, that is, a union of trees, is defined similarly with $c = 0$. We shall express the coefficients of $p_Q(x)$ and $p_L(x)$ in terms of the weights of $TU$-subgraphs and acyclic subgraphs of $G$.

By the Matrix-Tree Theorem, for any $1 \leq i, j \leq n$, $\tau(G)$ is equal to $(-1)^{i+j}$ times the determinant of the submatrix of $L(G)$ obtained by eliminating the $i$th row and $j$th column. It follows that $\ell_{n-1} = (-1)^{n-1}n\tau(G)$. The first part of the following theorem which is the generalization of the Matrix-Tree Theorem was appeared in [9] (see also [5, p. 90]). The second part was proved in [6] (see also [4]).

**Theorem 7.** The coefficients of $p_L(x)$ and $p_Q(x)$ are determined as follows.

(i) $\ell_j = (-1)^j \sum_{F_j} W(F_j)$, for $j = 1, \ldots, n - 1$, where the summation runs over all acyclic subgraphs $F_j$ of $G$ with $j$ edges.

(ii) $p_j = (-1)^j \sum_{H_j} W(H_j)$, for $j = 1, \ldots, n$, where the summation runs over all $TU$-subgraphs $H_j$ of $G$ with $j$ edges.

We close this section by stating the following well known lemma for later use.

**Lemma 8.** Any symmetric matrix of rank $r$ (over any field) has a principal $r \times r$ submatrix of full rank.

## 3 Binary rank of line graphs

In this section we give a simple proof of Doob’s result. For a matrix $M$, we use the notation $\text{rank}_2(M)$ to denote the binary rank of $M$.

**Theorem 9.** (Doob [7]) Let $G$ be a connected graph of order $n$ and $A = A(L(G))$. Then $\text{rank}_2(A)$ is equal to $n - 1$ if $n$ is odd, and $n - 2$ if $n$ is even.
Proof. If $S$ is the edge set of a spanning tree and $R$ any set of $n - 1$ vertices of $G$, then by Lemma 6, $\det(XX^\top (S, S)) = \pm 1$. Hence, $\text{rank}(X) \geq n - 1$. In fact we have equality since the rows of $X$ sum up to the all 2 vector. From 4, it follows that $\text{rank}(A) \leq \text{rank}(X) = n - 1$. Let $S \subseteq E(G)$ with $|S| = n - 1$. By the Binet–Cauchy Theorem and Lemma 6,

$$
\det(A(S, S)) \equiv \det((X^\top X)(S, S)) \pmod{2}
$$

$$
= \sum_{R \subseteq V(G), |R| = n - 1} \det(X(R, S))^2 = \begin{cases} 
n & \text{if } \langle S \rangle \text{ is a tree}, 
0 & \text{otherwise}.
\end{cases}
$$

This shows that $A$ has a principal submatrix of order $n - 1$ with full binary rank if $n$ is odd and has none if $n$ is even. This proves the theorem for odd $n$. Assume that $n$ is even. The above argument together with Lemma 8 show that $A$ has no submatrices of order $n - 1$ with full binary rank. Thus $\text{rank}(A) \leq n - 2$. Let $T$ be a subtree of $G$ with $n - 2$ edges. Then the adjacency matrix $B$ of $L(T)$ is a principal submatrix of $A$ and further $B$ has full binary rank by the above argument for odd $n$. This shows that $\text{rank}(A) = n - 2$. \hfill \Box

4 Even integer eigenvalues of Laplacian and signless Laplacian

In this section we demonstrate the interconnection between the number of spanning trees of a graph $G$ and the even integral eigenvalues of $L(G)$ and $Q(G)$. In view of the fact that the matrices $A(L(G)) + 2I$ and $Q(G)$ have the same nonzero eigenvalues, Theorems 1, 2, and 4 follow respectively from Theorems 10, 14, and 16 below.

Theorem 10. Let $G$ be a connected graph having $2^t$'s spanning trees with $s$ odd. Then the multiplicity of any even integer $\lambda$ as an eigenvalue of $Q(G)$ is at most $t + 1$.

Proof. It is well known that for a given integral matrix $A$ of rank $r$, there exist unimodular matrices (that is, integral matrices with determinant $\pm 1$) $U$ and $V$ such that

$$
UAV = \text{diag}(s_1, \ldots, s_r, 0, \ldots, 0)
$$

where $s_1, \ldots, s_r$ are positive integers with $s_1 s_2 \cdots s_i = d_i$ where $d_i$ is the greatest common divisor of all minors of $A$ of order $i$, $1 \leq i \leq r$. (The matrix $\text{diag}(s_1, \ldots, s_r, 0, \ldots, 0)$ is called the Smith form of $A$.)

Let $S = \text{diag}(s_1, \ldots, s_{n-1}, 0)$ be the Smith form of $L$. Note that $\text{rank}_2(L) = \text{rank}_2(S)$. By the Matrix-Tree Theorem, $\tau(G) = d_{n-1} = s_1 s_2 \cdots s_{n-1}$. It follows that at most $t$ of the $s_i$ are even. Therefore, $\text{rank}_2(L) \geq n - t - 1$ and so $\text{rank}_2(Q) \geq n - t - 1$. By Lemma 5, $Q$ has a principal
submatrix $B$ of order $n - t - 1$ with full binary rank. By interlacing, if an even integer $\lambda$ is an
eigenvalue of $Q$ with multiplicity at least $t + 2$, then any principal submatrix of $Q$ of order $n - t - 1$
has $\lambda$ as an eigenvalue. So $\lambda$ is an eigenvalue of $B$. This implies that det$(B)/\lambda$ is a rational
algebraic integer and thus an integer. Hence det$(B)$ is even, a contradiction. This completes the
proof.

Remark 11. The bound ‘$t + 1$’ of Theorem 10 on the multiplicity of even integer eigenvalues of
$Q$ is best possible. For, if we let $G$ to be the complete graph of order $n \equiv 2 \pmod{4}$, then by the
Cayley’s Formula, $\tau(G) = n^{n-2} = 2^{n-2} s$ for some odd $s$, and $Q(G)$ has the even integer $n - 2$ as
an eigenvalue of multiplicity $n - 1$.

Suppose that $G$ is a connected graph with $n$ vertices, $e(G)$ edges and $A = A(L(G))$. By
the same argument as the proof of Theorem 10 we see that the multiplicity of any even integer
eigenvalue $\lambda$ of $A$ is at most $e(G) - \text{rank}_2(A)$. Therefore, in view of Theorem 9 the multiplicity of
$\lambda$ is at most $e(G) - 2[n/2] + 2$. Combining this with Theorem 11 we come up with the following.

Theorem 12. Let $G$ be a connected graph with $n$ vertices, $e$ edges, and $2^i$’s spanning trees with
$s$ odd. Then the multiplicity of any even integer $\lambda \neq -2$ as an eigenvalue of $A(L(G))$ is at most
$\min\{t + 1, e(G) - 2[n/2] + 2\}$.

In the rest of the paper, we shall need a variation of Theorem 9 on the coefficients of the
characteristic polynomials of principal submatrices of order $n - 1$ of $L(G)$ and $Q(G)$. For simplicity
we denote by $L_1 = L_1(G)$ and $Q_1 = Q_1(G)$ the matrices obtained from $L(G)$ and $Q(G)$ by removing
the first row and the first column, respectively. Note that $L_1(G)$ and $Q_1(G)$ are not the same as
$L(G - v_1)$ and $Q(G - v_1)$ where $v_1$ is the vertex corresponding to the first rows of $L(G)$ and $Q(G)$.

A notion of ‘restricted weight’ with respect to $v_1$ is useful for describing the coefficients of
$p_{L_1}(x)$ and $p_{Q_1}(x)$. Let $U$ be a unicyclic subgraph of $G$ with odd cycle and $T$ be a tree subgraph
of $G$. We define

$$W_1(U) = \begin{cases} 0 & \text{if } U \text{ contains } v_1, \\ 4 & \text{otherwise}, \end{cases} \quad \text{and} \quad W_1(T) = \begin{cases} 1 & \text{if } T \text{ contains } v_1, \\ 1 + e(T) & \text{otherwise}. \end{cases}$$

We extend the domain of $W_1$ to all TU-subgraphs $H$ of $G$ by defining $W_1(H)$ to be the product
of the $W_1$’s of the connected components of $H$.

Lemma 13. Let $p_{L_1}(x) = x^{n-1} + \ell'_1 x^{n-2} + \cdots + \ell'_{n-1}$ and $p_{Q_1}(x) = x^{n-1} + p'_1 x^{n-2} + \cdots + p'_{n-1}$ be
the characteristic polynomials of $L_1$ and $Q_1$, respectively. Then their coefficients are determined
as follows.

(i) $\ell'_j = (-1)^j \sum_{F_j} W_1(F_j)$, for $j = 1, \ldots, n - 1$, where the summation runs over all spanning
forests $F_j$ of $G$ with $j$ edges.
(ii) \( p_j^\prime = (-1)^j \sum_{H_j} W_1(H_j) \), for \( j = 1, \ldots, n - 1 \), where the summation runs over all TU-subgraphs \( H_j \) of \( G \) with \( j \) edges.

**Proof.** Let \( E = E(G) \) and \( V_1 = V(G) \setminus \{v_1\} \).

(i) For \( j = 1, \ldots, n - 1 \), we have

\[
\ell_j^\prime = \sum_{R \subseteq V_1, |R| = j} \det(L(R, R)).
\]

From the Binet–Cauchy Theorem it follows that

\[
\det(L(R, R)) = \sum_{S \subseteq E, |S| = j} \det(D(R, S))^2.
\]

Thus,

\[
\ell_j^\prime = \sum \det(D(R, S))^2,
\]

where the summation is over \( R \subseteq V_1, S \subseteq E \) with \( |R| = |S| = j \). Now \( \det(D(R, S))^2 \) is either 0 or 1 by Lemma 5. Further, it takes the value 1 if and only if the three conditions of Lemma 5 hold. Hence, if \( \det(D(R, S))^2 = 1 \), then \( \langle S \rangle \) must be a union of some trees \( T_1, \ldots, T_r \). For such a \( S \), the contribution of \( \langle S \rangle \) in (2) is \((1 + e(T_2)) \cdots (1 + e(T_r))\) which is equal to \( W_1(\langle S \rangle) \).

(ii) The proof is similar to that of part (i). The only points different from part (i) are that here we use Lemma 6 instead of Lemma 5 and that if \( \langle S \rangle \) is a TU-subgraph and some unicyclic component of \( \langle S \rangle \) contains \( v_1 \), then for any \( R \subseteq V_1, \det(X(R, S)) = 0 \). Hence any TU-subgraph \( \langle S \rangle \) with nonzero contribution in \( p_j^\prime \) must have all of its unicyclic components included in \( V_1 \).

**Theorem 14.** Suppose that \( G \) is a connected graph with an odd order and \( \tau(G) \) is not divisible by 4. Then

(i) \( L(G) \) has no even integer eigenvalue;

(ii) \( Q(G) \) has no integer eigenvalue \( \lambda \equiv 2 \pmod{4} \);

(iii) \( Q(G) \) has at most one eigenvalue \( \lambda \equiv 0 \pmod{4} \) and such an eigenvalue is simple.

**Proof.** Let \( G \) be of order \( n \).

(i) We claim that the coefficient \( \ell_{n-2} \) of the characteristic polynomial \( p_L(x) = x^n + \ell_1 x^{n-1} + \cdots + \ell_{n-1} x \) of \( L(G) \) is even. By Theorem 7 we have \( \ell_{n-2} = (-1)^{n-2} \sum_{F_{n-2}} W(F_{n-2}) \) where the
Theorem 16. Improved.

Proof. Let $k$ be an even integer where $k = 2^s$ with $s$ odd and $t \geq 1$. Then, all the terms of $p_L(k)$ are divisible by $2^{t+2}$ except the last term, namely $\ell_{n-1}k = (-1)^{n-1}n\tau(G)k$ which is congruent to $2^t\tau(G)$ (mod $2^{t+2}$). Therefore, $p_L(k) \equiv 2^{t+1} \pmod{2^{t+2}}$ if $\tau(G) \equiv 2 \pmod{4}$ and $p_L(k) \equiv 2^t \pmod{2^{t+2}}$ if $\tau(G)$ is odd. This proves (i).

(ii) From Theorem 7 it follows that for some integers $s_1, \ldots, s_n$, we have $p_L = \ell_j + 4s_j, j = 1, \ldots, n - 1$, and $p_n = 4s_n$. This implies that $p_Q(x) = p_L(x) + 4f(x)$ where $f(x)$ is a polynomial with integer coefficients. First assume that $\tau(G)$ is odd. Hence $\ell_{n-1} = (-1)^{n-1}n\tau(G)$ is an odd integer. It follows that if $k \equiv 2 \pmod{4}$, then $p_Q(k) \equiv 2 \pmod{4}$, and we are done. Next assume that $\tau(G) \equiv 2 \pmod{4}$. We claim that $s_n$, the constant term of $f(x)$, is even. Let $U$ be the set of all spanning unicyclic subgraphs of $G$. By Theorem 7(ii), $s_n$ is the number of $U \in U$ such that the cycle of $U$ has an odd length. If $s_n = 0$, we are done. So assume that $s_n \geq 1$. Let $F$ be the set of all pairs $(T, U)$ such that $U \in U$ and $T$ is an spanning tree of $U$. For any fixed $U$, the number of pairs $(T, U) \in F$ is equal to the length of the cycle of $U$. Therefore, $|F|$ is congruent to $s_n \pmod{2}$.

On the other hand, for any spanning tree $T$ of $G$, there are exactly $e(G) - n + 1$ unicyclic graphs $U \in U$ containing $T$. Therefore, $|F| = \tau(G)(e(G) - n + 1) \equiv 0 \pmod{2}$. It follows that $s_n$ is even which in turn implies that $f(k)$ is even. On the other hand, from the proof of part (i) we see that $p_L(k) \equiv n\tau(G)k \equiv 4 \pmod{8}$. Therefore, $p_Q(k) = p_L(k) + 4f(k) \equiv 4 \pmod{8}$.

(iii) Suppose that $Q(G)$ has an even integer eigenvalue $\lambda$. By part (ii), $\lambda \equiv 0 \pmod{4}$. If the multiplicity of $\lambda$ is more than 1, then $\lambda$ is an eigenvalue of $Q_1$. Take $\theta$ as $\theta\lambda = \det(Q_1)$. Then $\theta$ is a rational algebraic integer, and so it is an integer. It follows that $\lambda$ divides $\det(Q_1)$ which means $\det(Q_1) \equiv 0 \pmod{4}$. On the other hand, from Theorem 8 it follows $p_{Q_1}(x) = p_{L_1}(x) + 4f(x)$ for some integer polynomial $f(x)$. This implies that $\det(Q_1) \equiv \det(L_1) \equiv \tau(G) \pmod{4}$ which is a contradiction. \hfill $\square$

Remark 15. Note that if $G$ is a $k$-regular graph, then $2k$ is the largest eigenvalue of $Q(G)$. Hence if $k$ is even, $Q(G)$ has an eigenvalue divisible by 4. This shows that Theorem 8(iii) cannot be improved.

Theorem 16. Let $G$ be a connected graph. If $L(G)$ or $Q(G)$ has an even integer eigenvalue of multiplicity at least 2, then $\tau(G)$ is divisible by 4.

Proof. Let $G$ be of order $n$. If $n$ is odd we are done by Theorem 8. So we may assume that $n$ is even.

First, let $\lambda$ be an even integer eigenvalue of $L(G)$ with multiplicity at least 2. By the interlacing property of eigenvalues of Hermitian matrices, $\lambda$ is also an eigenvalue of $L_1(G)$. Let $p_{L_1}(x) = x^{n-1} + \ell'_1x^{n-2} + \cdots + \ell'_{n-1}$ be the characteristic polynomials of $L_1$. By the Matrix-Tree Theorem,
\( \ell'_{n-1} = (-1)^{n-1} \tau(G) \). We show that \( \ell'_n \) is even. Any spanning forest of \( G \) with \( n-2 \) edges is a union of two trees \( T_1 \) and \( T_2 \) where we may assume that \( T_1 \) contains the vertex \( v_1 \), and hence by Lemma 13, \( \ell'_n = \sum_{T_1 \cup T_2} (1 + e(T_2)) \). We note that \( (1 + e(T_1))(1 + e(T_2)) \equiv (1 + e(T_2)) \pmod{2} \) for if \( e(T_2) \) is even, then \( e(T_1) \) is also even as \( e(T_1) + e(T_2) = n-2 \) so both sides are odd, and if \( e(T_2) \) is odd both sides are even. This implies that
\[
\ell'_n = \sum_{T_1 \cup T_2} (1 + e(T_2)) \equiv \sum_{T_1 \cup T_2} (1 + e(T_1))(1 + e(T_2)) = \ell_{n-2} \pmod{2}.
\]

Note that \( p_{L}(x) = (x - \lambda)^2 g(x) \) for some integer polynomial \( g(x) \). If \( ax^2 + bx \) are the last two terms of \( g(x) \), then \( \ell_{n-2} = \lambda^2 a - 2\lambda b \). It follows that \( \ell_{n-2} \) and so \( \ell'_n \) is even. Therefore,
\[
0 = p_{L_1}(\lambda) \equiv \ell'_n = (-1)^{n-1} \tau(G) \pmod{4}.
\]

Now let \( \lambda \) be an even integer eigenvalue of \( Q(G) \) with multiplicity at least 2. So \( \lambda \) is also an eigenvalue of \( Q_1(G) \). From Theorem 13, it follows \( p_{Q_1}(x) = p_{L_1}(x) + 4f(x) \) for some integer polynomial \( f(x) \). Therefore, \( 0 = p_{Q_1}(\lambda) \equiv p_{L_1}(\lambda) \equiv \tau(G) \pmod{4} \).

As a very special case of Theorem 16, we come up with the following which was conjectured in [10].

**Corollary 17.** Suppose that \( G \) is a unicyclic graph and the nullity of \( \text{A}(L(G)) \) is equal 2. Then the length of the unique cycle of \( G \) is divisible by 4.

More general assertions than Theorems 14 and 16 hold for Laplacian matrix. These are given below. We omit the proof which is essentially the same as the proofs of Theorems 14 and 16.

**Theorem 18.** Let \( G \) be a connected graph of order \( n \).

(i) If \( n \) is odd and \( \tau(G) = 2^t s \) with \( s \) odd, then \( L(G) \) has no eigenvalue \( \lambda \) with \( 2^{\max(1,t)}|\lambda| \).

(ii) If \( L(G) \) has an integer eigenvalue \( \lambda = 2^t s \) with \( t \geq 1, s \) odd and with multiplicity at least 2, then \( 2^{t+1}|\tau(G)| \).

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