Darboux transformations for a Bogoyavlenskii equation in $2+1$ dimensions

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April 1, 2022

Abstract
We use the singular manifold method to obtain the Lax pair, Darboux transformations and soliton solutions for a (2+1) dimensional integrable equation.

1. Introduction

The equation
\[ u_{xt} + \frac{1}{4} u_{xxx} + u_{xxy} + \frac{1}{2} u_{xx} u_{y} + \frac{1}{4} \int u_{yyy} dx = 0 \] (1.1)
obtained by Bogoyavlenskii in [1], has been recently rederived as a (2+1) dimensional reduction of an equation which pretended to be a (3+1) dimensional generalization of the potential KP equation [12], [13], [14].

This equation is integrable in the sense of having the Painlevé property [12]. We can write equation (1) in a more appropriate way as the system
\[ 0 = u_{y} - m_{x} \]
\[ 0 = u_{xt} + \frac{1}{4} u_{xxx} + u_{xxy} + \frac{1}{2} u_{xx} u_{y} + \frac{1}{4} m_{yy} \] (1.2)

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2. Singular Manifold Method

2.1 Leading term analysis

The Painlevé property for this equation means that all solutions of (2) can be expanded as a generalized Laurent expansion in the neighbourhood of the manifold of movable singularities \( \chi(x, y, t) \) which is an arbitrary function. This expansion should be

\[
u(x, y, t) = \sum_{j=0}^{\infty} u_j(x, y, t)\chi(x, y, t)^{j-\alpha}, \quad m(x, y, t) = \sum_{j=0}^{\infty} m_j(x, y, t)\chi(x, y, t)^{j-\beta}
\]

(2.3)

where the \( u_j \) and \( m_j \) are analytical functions of \( x, y \) and \( t \) in the neighbourhood of \( \chi = 0 \). The leading term analysis yields:

\[
\alpha = \beta = 1 \quad u_0 = 2\chi_x \quad m_0 = 2\chi_y
\]

(2.4)

2.2 Truncated expansions. Bäcklund transformations

The singular manifold method requires the truncation of expansions (3) at the constant level \( j = 1 \). This implies that the manifold of movable singularities \( \chi \) is no longer an arbitrary function but it has to satisfy some equations, called the singular manifold equations, that we will see later on. Due to this fact, we denote this manifold as \( \phi \) and call it the singular manifold in the sense that it is singularized by the truncation condition. The truncated solutions are then

\[
u' = u + \frac{2\phi_x}{\phi}, \quad m' = m + \frac{2\phi_y}{\phi}
\]

(2.5)

Substitution of (5) in equation (2) provides us a polynomial in \( \phi \). Setting to zero the coefficients of this polynomial we find that \( u \) and \( m \) must be solutions of (2), which means that eqs. (5) can be considered a Bäcklund transformation between two solutions of (2).

2.3 Expression of the solutions in terms of the Singular Manifold

From the coefficients of the polynomial in \( \phi \) we obtain

\[
u_x = -\frac{1}{4}p_x^2 + \frac{1}{2}p_y - \frac{1}{2}v_x - \frac{1}{4}v^2 + h(y, t),
\]

\[
u_y = -2w - v_y - pxp_y - 2pxh(y, t)
\]

\[
m_y = -2pt - pxxv - \frac{1}{2}p_x^2 + pxp_y - \frac{1}{2}p_y^2 - pxxyv - pxpxv
\]

\[- \frac{1}{2}p_x^2v^2 + 2wp_x + +2p_x^2h(y, t) - 2p_yh(y, t)
\]

(2.8)

where \( h(y, t) \) arises from an integration in \( x \) and \( v, w \) and \( p \) are defined in terms of the singular manifold as

\[
v = \frac{\phi_{xx}}{\phi_x}, \quad w = \frac{\phi_t}{\phi_x}, \quad p_x = \frac{\phi_y}{\phi_x}
\]

(2.9)
2.4 Singular Manifold Equations

From the compatibility conditions between definitions (9) we obtain the following generic equations

\[
\begin{align*}
\phi_{xxx} &= \phi_{txx} \quad \Rightarrow \quad v_t = (w_x + vw)_x \\
\phi_{xy} &= \phi_{yxx} \quad \Rightarrow \quad v_y = (p_{xx} + vp_x)_x \\
\phi_{yt} &= \phi_{ty} \quad \Rightarrow \quad p_{xt} = w_y + w p_{xx} - p_x w_x
\end{align*}
\]  

(2.10)

Moreover, taking the cross derivatives of (6) and (7) \((u_{xy} = u_{yx})\) we have

\[
\begin{align*}
(p_{xxx} + 4w + 2p_x p_y + p_x \left(v_x - \frac{v^2}{2}\right) + 4p_x h(y,t) + \\
+ \left(p_y - \frac{1}{2} p_x^2 + 2h(y,t)\right) y + p_{xx} \left(v_x - \frac{v^2}{2}\right) = 0
\end{align*}
\]  

(2.11)

The set (10-11) constitutes the Singular Manifold Equations.

3. Lax Pair

3.1 Painlevé Analysis on the singular manifold equations

The singular manifold equations (10) and (11) can be considered as a system of non-linear coupled PDE’s in \(v, w, p\) and we can perform the Painlevé analysis over them. It’s not difficult to see, following the same procedure as in the previous section, that leading term analysis in the singular manifold equations yields the truncated expansions:

\[
\begin{align*}
v &= \frac{\psi^+_x}{\psi^+_y} + \frac{\psi^-_x}{\psi^-_y}, \quad p_x = \frac{\psi^+_x}{\psi^+_y} - \frac{\psi^-_x}{\psi^-_y}, \quad p_y = \frac{\psi^+_y}{\psi^+_y} - \frac{\psi^-_y}{\psi^-_y}, \quad p_t = \frac{\psi^+_t}{\psi^+_y} - \frac{\psi^-_t}{\psi^-_y}
\end{align*}
\]  

(3.12)

where we have two singular manifolds \(\psi^+\) and \(\psi^-\) because the Painlevé expansion has two branches [4], [3]. Taking the \(t\) and \(y\) derivatives of eqs. (12) and using (10) to integrate them in \(x\) we have

\[
\begin{align*}
w_x + vw &= \frac{\psi^+_y}{\psi^+_y} + \frac{\psi^-_y}{\psi^-_y}, \quad p_{xx} + vp_x = \frac{\psi^+_y}{\psi^+_y} + \frac{\psi^-_y}{\psi^-_y}
\end{align*}
\]  

(3.13)

Moreover, integration of (12) yields

\[
\begin{align*}
\phi_x &= \psi^+ \psi^- \quad \phi_y = \psi^+_x \psi^- - \psi^+ \psi^-_x
\end{align*}
\]  

(3.14)

These equations allow us to obtain the singular manifold \(\phi\) from the eigenfunctions \(\psi^+\) and \(\psi^-\).
3.2 Linearization of the singular manifold equations: Lax pair

From (12) and (13) we can easily obtain, with the help of (6)-(8), the spatial part of the Lax pair:

\[
\psi_{xx} + (u_x - h)\psi_y - \psi_{yy} = 0 \\
\psi_{xx}^+ + (u_x - h)\psi_y^+ - \psi_{yy}^+ = 0 \
\]

(3.15)

where \( h(y,t) \) is the spectral parameter.

The temporal part of the Lax pair can be obtained by combining equations (12)-(14) with (6)-(8) in order to eliminate \( w \) from them to obtain:

\[
\psi_t + \frac{1}{2}\psi_{yy} - \frac{1}{4}(u_{xy} - m_y - 2h_y)\psi_x + \frac{1}{2}u_y\psi^+_x + h\psi^+_y = 0 \\
\psi_t - \frac{1}{2}\psi_{yy} - \frac{1}{4}(u_{xy} + m_y - 2h_y)\psi^-_x + \frac{1}{2}u_y\psi^-_x + h\psi^-_y = 0 \\
\]

(3.16)

From the compatibility condition of the equations of the Lax pair among themselves and with equation (2) we have the condition

\[ h_t + hh_y = 0 \]

(3.17)

which means that the problem is non-isospectral.

4. Darboux Transformations

As far as \( u' \) and \( \omega' \) are also solutions of (2), we can define a new singular manifold \( \phi' \) for them by means of two eigenfunctions \( \psi'^+ \) and \( \psi'^- \) with eigenvalue \( h_2 \) as:

\[
\phi'_x = \psi'^+\psi'^- , \quad \phi'_y = \psi'^+_x\psi'^- - \psi'^+\psi'^-_x \\
\]

(4.18)

where \( \psi'^+ \) and \( \psi'^- \) must satisfy the Lax pair for \( u' \) with spectral parameter \( h_2 \). We can now consider the Lax pair itself as a system of coupled nonlinear equations in \( u' \), \( m' \), \( \psi'^+ \) and \( \psi'^- \) and hence we can expand the fields and eigenfunctions in the Painlevé series

\[
u' = u + \frac{2\phi'_{1x}}{\phi'_{1}}, \quad m' = m + \frac{2\phi'_{1y}}{\phi'_{1}} \\
\psi'^+ = \psi'^+_2 - \psi'^+_2\Omega'^+ , \quad \psi'^- = \psi'^-_2 - \psi'^-_2\Omega'^- \\
\phi' = \phi'_2 + \frac{\Delta}{\phi'_{1}} \\
\]

(4.19-4.21)

where \( \psi'^+_1 \) and \( \psi'^-_1 \) are eigenfunctions of \( u \) and \( m \) with eigenvalue \( h_1 \) and \( \psi'^+_2 \) and \( \psi'^-_2 \) are eigenfunctions with eigenvalue \( h_2 \) satisfying the Lax pair for \( u \).
Substitution of the truncated expansions in the Lax pair for $\psi'^+\,$ and $\psi'^-$ yields (we have used MapleV to handle the calculation): \begin{align*}
\Omega_x^+ &= \psi_x^+ \psi_1^- , \quad \Omega_y^+ = \Omega^+ (h_1 - h_2) + \psi_x^+ \psi_1^- - \psi_x^+ \psi_1^- \\
\Omega_x^- &= \psi_x^+ \psi_2^- , \quad \Omega_y^- = \Omega^- (h_2 - h_1) + \psi_x^+ \psi_2^- - \psi_x^+ \psi_2^- \tag{4.22}
\end{align*}
and from the substitution of (21) in (18) we have
\begin{equation}
\Delta = -\Omega^+ \Omega^- \tag{4.24}
\end{equation}
The set of equations (19)-(21) with $\Omega^+$ and $\Omega^-$ given by (22)-(23), constitutes a transformation of eigenfunctions and potentials that leaves invariant the Lax pair and hence it can be considered a Darboux transformation [9].

5. Hirota’s $\tau$-function

Hirota’s bilinear method is a tool used in many references to obtain multisolitonic solutions for nonlinear PDE’s. N-soliton solutions of (2) have been constructed with this method in [13]. In this section we shall build the $\tau$-functions of Hirota’s method through the iteration of the singular manifold. Using equation (21)
\begin{equation}
\phi' = \phi_2 - \frac{\Omega^+ \Omega^-}{\phi_1} \tag{5.25}
\end{equation}
which defines a singular manifold for $m'$, we can use such manifold in order to construct an iterated solution
\begin{equation}
\phi'' = \phi' + \frac{\phi'}{\phi'} \tag{5.26}
\end{equation}
Substituting equation (5) for $u'$ in (26) we have:
\begin{equation}
\phi'' = u + \frac{2\tau_x}{\tau} \tag{5.27}
\end{equation}
where
\begin{equation}
\tau = \phi' \phi_1 = \phi_1 \phi_2 - \Omega^+ \Omega^- \tag{5.28}
\end{equation}
is Hirota’s $\tau$-function [1].

6. Solutions

In this section we shall obtain the one and two soliton solutions of equation (2) using the results of the previous sections. We start from the seminal solutions
\begin{equation}
u = 0 \quad m = 0 \tag{6.29}
\end{equation}
In this case, and restricting ourselves to the case when \( h_1 \) and \( h_2 \) are constants, non-trivial simple solutions of the Lax pair are:

\[
\psi_i^+ = \exp \left[ \alpha_i^+ x + \beta_i^+ \left( y - \left( \alpha_i^{+2} - \frac{1}{2} \beta_i^+ \right) t \right) \right]
\]

\[
\psi_i^- = \exp \left[ \alpha_i^- x + \beta_i^- \left( y - \left( \alpha_i^{-2} + \frac{1}{2} \beta_i^- \right) t \right) \right]
\]

where \( i = 1, 2 \) and \( \alpha_i^+ \), \( \alpha_i^- \), \( \beta_i^+ \) and \( \beta_i^- \) are constants related with the spectral parameter and among themselves by

\[
h_i = \alpha_i^{-2} + \beta_i^- = \alpha_i^{+2} - \beta_i^+ \quad (6.32)
\]

Integration of (14) yields

\[
\phi_i = \frac{c_i}{\alpha_i^+ + \alpha_i^-} (1 + F_i) \quad (6.33)
\]

where

\[
F_i = \exp \left\{ (\alpha_i^+ + \alpha_i^-) x + (\beta_i^+ + \beta_i^-) y - \left( \beta_i^+ \alpha_i^{+2} + \beta_i^- \alpha_i^{-2} + \frac{\beta_i^{-2} - \beta_i^{+2}}{2} \right) t \right\} \quad (6.34)
\]

Integrating (22) and (23) we have:

\[
\Omega^+ = \frac{1}{\alpha_2^+ + \alpha_1^-} \left( d^+ + \psi_2^+ \psi_1^- \right), \quad \Omega^- = \frac{1}{\alpha_1^+ + \alpha_2^-} \left( d^- + \psi_1^+ \psi_2^- \right) \quad (6.35)
\]

where \( d^+ \) and \( d^- \) are arbitrary constants.

- **One soliton solution** The first iteration provides the solution

\[
u_x = 2 \partial_{xx} \left[ \ln \phi_1 \right] \quad (6.36)
\]

with \( \phi_1 \) given by (33). It corresponds with one line soliton.

- **Two soliton solution** From the second iteration we have:

\[
u'' = 2 \partial_{xx} \left[ \ln \tau \right] \quad (6.37)
\]

with

\[
\tau = \phi_1 \phi_2 - \Omega^+ \Omega^- = \frac{c_1 c_2}{(\alpha_1^+ + \alpha_1^-)(\alpha_2^+ + \alpha_2^-)} (1 + F_1 + F_2 + A_{12} F_1 F_2) \quad (6.38)
\]

\( F_i \) is given by (34) and

\[
A_{12} = \frac{(\alpha_2^+ - \alpha_1^+)(\alpha_2^- - \alpha_1^-)}{(\alpha_2^+ + \alpha_1^-)(\alpha_1^+ + \alpha_2^-)} \quad (6.39)
\]

Equation (38) represents the interaction of two line solitons. When \( \alpha_2^+ = \alpha_1^+ \) or \( \alpha_2^- = \alpha_1^- \), the interaction term \( A_{12} \) vanishes and this special case is termed the resonant state.
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