ON THE ISOMORPHISM PROBLEM FOR NON-ERGODIC SYSTEMS WITH DISCRETE SPECTRUM

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Abstract. The article presents a new perspective on the isomorphism problem for non-ergodic measure-preserving dynamical systems with discrete spectrum which is based on the connection between ergodic theory and topological dynamics constituted by topological models. By first solving the isomorphism problem for a certain class of topological dynamical systems, it is shown that the measure-preserving case can in fact be deduced from the topological one via the construction of topological models. As a byproduct, a new characterization of mean ergodicity for topological dynamical systems is obtained.

The isomorphism problem is one of the most important problems in ergodic theory, first formulated by von Neumann in [vNe32, pp. 592–593], his seminal work on the Koopman operator method and dynamical systems with “pure point spectrum” (or “discrete spectrum”). Von Neumann, in particular, asked whether unitary equivalence of the associated Koopman operators (“spectral isomorphy”) implies the existence of a point isomorphism between two systems (“point isomorphy”). In [vNe32, Satz IV.5], he showed that two ergodic measure-preserving dynamical systems with discrete spectrum on standard probability spaces are point-isomorphic if and only if they are spectrally isomorphic. These first results on the isomorphism problem considerably shaped the ensuing development of ergodic theory. The next step in this direction was the Halmos-von Neumann paper [HN42] in which the authors gave a more complete solution to the isomorphism problem by addressing three different aspects:

- Uniqueness: For which class of dynamical systems is a given isomorphism invariant \( \Gamma \) complete, meaning that two systems \((X, \varphi)\) and \((Y, \psi)\) are isomorphic if and only if \( \Gamma(X, \varphi) = \Gamma(Y, \psi)\)?
- Representation: What are canonical representatives of isomorphy classes of dynamical systems?
- Realization: Given an isomorphism invariant \( \Gamma \), what is the precise class of objects that can be realized as \( \Gamma(X, \varphi) \) for a dynamical system \((X, \varphi)\)?

In addition to the uniqueness theorem from [vNe32], their representation theorem showed that for each isomorphy class of ergodic dynamical systems with discrete spectrum, there are canonical representatives given by ergodic rotations on compact groups. Moreover, their realization theorem established that every (countable) subgroup of \( \mathbb{T} \) can be realized as the point spectrum

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of the Koopman operator corresponding to an ergodic system with discrete spectrum. Hence, there is, up to isomorphy, a one-to-one correspondence between (countable) subgroups of $\mathbb{T}$ and (separable) systems with discrete spectrum.

In the following years, further efforts towards a solution of the isomorphism problem were made and we refer to [Wei72], [RW12] and [Wal75, Chapters 2 – 4] for a detailed account of these developments of which we only mention the following few: Nagel and Wolff generalized the Halmos-von Neumann theorem to an abstract, operator-theoretic statement in [NW72]. Mackey extended it to ergodic actions of separable, locally compact groups in [Mac64] and Zimmer took yet another approach, proving a version for extensions having relatively discrete spectrum in [Zim76]. However, all these results made use of ergodicity assumptions which was first justified by von Neumann in [vNe32, p. 624] by referring to the possibility of ergodic decomposition. Later, Choksi [Cho65] showed that the situation is not as simple as one might hope and it was only in 1981 that Kwiatkowski [Kwi81] solved the isomorphism problem for non-ergodic systems with discrete spectrum by using, as proposed by von Neumann, the ergodic decomposition as well as measure-theoretic methods. We also mention that recently, Austin generalized the Mackey-Zimmer theory to non-ergodic systems in [Aus10].

It is the purpose of this article to provide an alternative approach to the solution of the isomorphism problem for non-ergodic systems with discrete spectrum: The topological version of the Halmos-von Neumann theorem has an elegant and well-known proof using the Ellis (semi)group and Pontryagin duality for compact groups, see Theorem 3.1 and the introduction of Section 4. Knowing this, the measure-preserving version can be interpreted as a corollary of the topological result by constructing topological models. In this article, it is shown that this interplay of topological dynamics and ergodic theory extends to the non-ergodic case. More precisely, generalizing the proof sketched above, we first solve the isomorphism problem for non-minimal topological dynamical systems subject to a topological restriction and then obtain the analogous result for non-ergodic measure-preserving systems as a consequence by showing that this topological restriction can always be fulfilled when working with topological models. On the way, we obtain an interesting characterization of mean ergodicity for topological dynamical systems in [Theorem 2.14(b)] asserting that mean ergodicity, a global property, is equivalent to unique ergodicity of certain subsystems of a topological dynamical system, a local property.

The article is organized as follows: In Section 1 we fix our notation and recall basic results about operators and systems with discrete spectrum. In Section 2 we introduce the notion of a bundle of topological dynamical systems as well as group rotation bundles. The solution of the isomorphism problem is then broken down into the representation theorem in Section 3 and the uniqueness and realization results in Section 4. Our general philosophy for both sections, first solving the corresponding problems for topological systems and then obtain the same results for measure-preserving systems via topological models, was greatly inspired by [HM15].
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1. Preliminaries

1.1. Notation. Our notation and terminology are that of [EFHN15] in gen-
eral. We abbreviate a probability space \((X, \Sigma, \mu)\) by writing \(X := (X, \Sigma, \mu)\)
and if \(\varphi: X \to X\) is measurable and measure-preserving, we call \((X, \varphi)\)
a measure-preserving dynamical system. For such a system, we define the
Koopman operator \(T_\varphi: L^p(X) \to L^p(X), 1 \leq p \leq \infty, \text{ via } T_\varphi f := f \circ \varphi \)
for \(f \in L^p(X)\). With this definition, \(T_\varphi\) is a bounded operator and in fact
a Markov embedding, i.e., \(T_\varphi 1 = 1\), \(T_\varphi^* 1 = 1\) and \(\|T_\varphi f\| = \|T_\varphi f\|\)
for all \(f \in L^p(X)\). We say that two measure-preserving dynamical systems \((X, \varphi)\)
and \((Y, \psi)\) are point isomorphic if there exists an essentially invertible, mea-
surable, measure-preserving map \(\theta: X \to Y\) such that \(\theta \circ \varphi = \psi \circ \theta\). They
are Markov isomorphic if there is an invertible bi-Markov lattice homomor-
phism \(S: L^1(Y) \to L^1(X)\) such that \(T_\varphi S = ST_\psi\). If \(S\) is merely a bi-Markov
lattice homomorphism, we call \((Y, \psi)\) a Markov factor of \((X, \varphi)\).

If \(K\) is a compact space (i.e., \(K\) is quasi-compact and Hausdorff) and \(\varphi: K \to K\)
is continuous, we call \((K, \varphi)\) a topological dynamical system and again
define its Koopman operator \(T_\varphi: C(K) \to C(K)\) by \(T_\varphi f := f \circ \varphi\) for all
\(f \in C(K)\). We denote the space of regular Borel measures on \(K\) by \(M(K)\)
and identify it with the dual of \(C(K)\) via the Riesz-Markov-Kakutani represen-
tation theorem. We also let \(M_\varphi(K) := \{\mu \in M(K) \mid \mu\text{ is } \varphi\text{-invariant}\}\)
denote the subspace of \(\varphi\) -invariant measures and \(M^1_\varphi(K) := \{\mu \in M_\varphi(K) \mid \mu \geq 0, (1, \mu) = 1\}\) denote the subspace of \(\varphi\) -invariant probability mea-
ures.

If \(G\) is a compact topological group and \(a \in G\) we define \(\varphi_a: G \to G, \varphi_a(g) := ag\)
and call the dynamical system \((G, \varphi_a)\) the group rotation with \(a\). We may also abbreviate \((G, \varphi_a)\)
by writing \((G, a)\). Since the Haar measure \(m\) on \(G\) is invariant under rotation, the rotation can also be considered as a
measure-preserving dynamical system \((G, m; a)\).

If \(T\) is a linear operator on a vector space \(E\), we denote by
\[
A_n[T] := \frac{1}{n} \sum_{k=0}^{n-1} T^k
\]
its \(n\)th Cesàro mean and drop \(T\) from the notation if there is no room for
ambiguity. Furthermore, we call \(\text{fix}(T) := \{x \in E \mid Tx = x\}\) the fixed space
of \(T\). If \(F \subseteq E\) is a \(T\)-invariant subspace, we set \(\text{fix}_F(T) := \text{fix}(T|_F)\).
If \((K, \varphi)\) is a topological dynamical system, the fixed space \(\text{fix}(T_\varphi)\) of its
Koopman operator is a \(C^*\)-subalgebra of \(C(K)\). Similarly, if \((X, \varphi)\) is a measure-preserving dynamical system, \(\text{fix}_{L^\infty(X)}(T_\varphi)\) is a \(C^*\)-subalgebra of
\(L^\infty(X)\). By the Gelfand representation theorem (cf. [Tak79, Theorem I.4.4])
there is a compact space $L$ such that $\text{fix}_{L^\infty(X)}(T_\varphi) \cong C(L)$. The space $L$ is necessarily extremally disconnected: Since $\text{fix}(T_\varphi)$ is a closed sublattice of $L^1(X)$, the representation theorem for AL-spaces (see [Sch70, Theorem II.8.5]) shows that there is a compact space $M$ and a Borel probability measure $\mu_M$ on $M$ such that

$$C(L) \cong \text{fix}_{L^\infty(X)}(T_\varphi) \cong L^\infty(M, \mu_M).$$

But by [Sch70, Theorem II.9.3], $C(L)$ is isomorphic to a dual Banach lattice if and only if $L$ is hyperstonean. In particular, $L$ is extremally disconnected. This will be crucial for Theorem 3.10.

1.2. Operators with discrete spectrum. We start with a power-bounded operator $T$ on a Banach space $E$ and briefly recall the definition of discrete spectrum and the Jacobs semigroup generated by $T$ considered first by Konrad Jacobs in [Jac56, Definition III.1].

**Definition 1.1.** Let $E$ be a Banach space and $T \in \mathcal{L}(E)$ a power-bounded operator on $E$.

(i) The operator $T$ has **discrete spectrum** if its Kronecker space

$$\text{Kro}(T) := \bigcap_{|\lambda| = 1} \ker (\lambda I - T),$$

is all of $E$.

(ii) The **Jacobs semigroup** generated by $T$ is

$$J(T) := \left\{ T^n \mid n \in \mathbb{N} \right\}^{\text{wot}},$$

where the closure is taken with respect to the weak operator topology and the semigroup operation is the composition of operators.

The following characterization of an operator having discrete spectrum can be found in [EFHN15, Theorem, 16.36].

**Theorem 1.2.** The following assertions are equivalent.

(i) $T$ has discrete spectrum.

(ii) $J(T)$ is a weakly/strongly compact group of invertible operators.

(iii) The orbit $\{T^n x \mid n \geq 0\}$ is relatively compact and $\inf_{n \geq 0} ||T^n x|| > 0$ for all $0 \neq x \in E$.

**Remark 1.3.** If $T$ has discrete spectrum, it is mean ergodic and $J(T)$ is a compact abelian group on which the weak and strong operator topology coincide. It is metrizable if $E$ is.

1.3. Systems with discrete spectrum. Next, we consider Koopman operators corresponding to dynamical systems. See [EFHN15, Chapters 4, 7] for general information.

**Definition 1.4.** We say that a measure-preserving dynamical system $(X, \varphi)$ has discrete spectrum if its Koopman operator $T_\varphi$ has discrete spectrum on
Similarly, we say that a topological dynamical system \((K, \varphi)\) has discrete spectrum if \(T_\varphi\) has discrete spectrum as an operator on \(C(K)\).

**Example 1.5.** If \(B\) is a compact space, the trivial dynamical system \((B, \text{id}_B)\) has discrete spectrum. Also, if \(G\) is a compact group and \(a \in G\), the measure-preserving dynamical system \((G, m; a)\) has discrete spectrum and so does the topological dynamical system \((G, a)\).

If \((K, \varphi)\) is a topological dynamical system and \(T_\varphi \in \mathcal{L}(C(K))\) has discrete spectrum, the Jacobs semigroup \(J(T_\varphi)\) is related to the Ellis semigroup \(E(K, \varphi) \subseteq K^K\) defined as \(E(K, \varphi) := \{\varphi^n \mid n \in \mathbb{N}\}\), see [EFHN15, Section 19.3]. The following well-known result establishes this connection and gives a topological characterization of the operator theoretic notion of discrete spectrum.

**Proposition 1.6.** Let \((K, \varphi)\) be a topological dynamical system. For the Koopman operator \(T_\varphi\), the following assertions are equivalent.

(i) \(T_\varphi\) has discrete spectrum.

(ii) \(J(T_\varphi)\) is a group of Koopman operators.

(iii) \(E(K, \varphi)\) is a group of equicontinuous transformations on \(K\).

(iv) \((K, \varphi)\) is equicontinuous and invertible.

Moreover, if these conditions are fulfilled, the map
\[
\Phi: J(T_\varphi) \to E(K, \varphi), \quad T_\vartheta \mapsto \vartheta
\]
is an isomorphism of compact topological groups.

The equivalence of (i) and (ii) follows from Theorem 1.2 and [EFHN15, Theorem 4.13]. The equivalence of (ii) and (iii) follows via the canonical isomorphism \(\vartheta \mapsto T_\vartheta\) and for the equivalence of (iii) and (iv) see [Gla07, Proposition 2.5].

2. Bundles of dynamical systems

Bundles, e.g., in differential geometry or algebraic topology, allow to decompose an object into smaller objects such that the small parts fit together in a structured way. This perspective is of interest in our context when dealing with dynamical systems which are not “irreducible”, i.e., not minimal or ergodic. We therefore introduce the notion of a bundle of topological dynamical systems.

**Definition 2.1.** A triple \((K, B, p)\) is called a *bundle* if \(K\) and \(B\) are topological spaces, \(B\) is compact and \(p: K \to B\) is a continuous surjection. The subsets \(K_b := p^{-1}(b)\) are called the *fibers* of the bundle and if \(f: K \to S\) is a function into a set \(S\), we denote by \(f_b\) its restriction to \(K_b\). A bundle \((K, B, p)\) is called *compact* if \(K\) is compact. A tuple \((K, B, p; \varphi)\) is called a *bundle of topological dynamical systems* if \((K, B, p)\) is a compact bundle and \((K, \varphi)\) is a topological dynamical system such that each fiber \(K_b\) is \(\varphi\)-invariant. We denote by \(\varphi_b\) the restriction of \(\varphi\) to \(K_b\).
Remark 2.2. For a dynamical system \((K, \varphi)\) and a compact space \(B\), a tuple \((K, B, p; \varphi)\) is a bundle of dynamical systems if and only if \(p\) is a factor map from \((K, \varphi)\) to \((B, \text{id}_B)\).

Example 2.3. 
(1) Let \((K, \varphi)\) be a topological dynamical system, \(B\) a singleton and \(p: K \to B\) the unique map from \(K\) to \(B\). Then \((K, B, p; \varphi)\) is a compact bundle of dynamical systems. If \((K, \varphi)\) is minimal, this is the only possible choice of \(B\). However, the converse is not true as the system \(([0, 1], x \mapsto x^2)\) demonstrates.

(2) Let \(B = [0, 1], K = \mathbb{T} \times B, \alpha: B \to \mathbb{T}\) be continuous and \(\varphi_\alpha: K \to K, (z, t) \mapsto (\alpha(t)z, t)\) be the associated rotation on the cylinder \(K\). Then \((K, B, p_B; \varphi_\alpha)\) is a compact bundle of topological dynamical systems. If \(\alpha \equiv a\) for some \(a \in \mathbb{T}\), the system \((K, \varphi)\) is just the product of the torus rotation \((\mathbb{T}, \varphi_a)\) and the trivial system \((B, \text{id}_B)\).

(3) More generally, let \((M, \psi)\) be a topological dynamical system and \(B\) be a compact space. Then the product system \((M \times B, \psi \times \text{id}_B)\) can be viewed as the bundle \((M \times B, B, p_B; \psi \times \text{id}_B)\). Bundles of this form are called trivial.

Definition 2.4. A bundle morphism of bundles \((K_1, B_1, p_1)\) and \((K_2, B_2, p_2)\) is a pair \((\Theta, \vartheta)\) consisting of continuous functions \(\Theta: K_1 \to K_2\) and \(\vartheta: B_1 \to B_2\) such that the following diagram commutes:

\[
\begin{array}{ccc}
K_1 & \xrightarrow{\Theta} & K_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
B_1 & \xrightarrow{\vartheta} & B_2
\end{array}
\]

A morphism of compact bundles of topological dynamical systems \((K_1, B_1, p_1; \varphi_1)\) and \((K_2, B_2, p_2; \varphi_2)\) is a morphism \((\Theta, \vartheta)\) of the corresponding bundles such that \(\Theta\) is, in addition, a morphism of topological dynamical systems. If \(\Theta\) and \(\vartheta\) are homeomorphisms, we call \((\Theta, \vartheta)\) an isomorphism.

2.1. Sections. An important tool for capturing structure in bundles relative to the base space are sections. We recall the following definition.

Definition 2.5. Let \((K, B, p)\) be a bundle. A function \(s: B \to K\) is called a section of \((K, B, p)\) if \(s(b) \in K_b\) for each \(b \in B\).

Although the existence of sections is guaranteed by the axiom of choice, there may not exist continuous sections in general.

Example 2.6. Take \(K = B = \mathbb{T}\) and define \(p: K \to B\) by \(p(z) = z^2\). Then this bundle has no continuous section because sections are injective and an injective, continuous function \(s: \mathbb{T} \to \mathbb{T}\) is necessarily surjective and hence a homeomorphism.

Under additional topological conditions, there are positive results. The first result due to Ernest Michael involves zero-dimensional spaces. Since every totally disconnected compact Hausdorff space is zero-dimensional (see AT08).
Proposition 3.1.7), we only state the following special case. It can be found in [Mic56, Corollary 1.4].

**Proposition 2.7.** Let \((K, B, p)\) be a compact bundle such that \(K\) is metric, \(B\) is totally disconnected and \(p\) is open. Then there exists a continuous section.

For not necessarily metric \(K\), [Gle58, Theorem 2.5] yields the following.

**Theorem 2.8.** Let \((K, B, p)\) be a compact bundle such that \(B\) is extremally disconnected. Then there exists a continuous section.

In case that there is no continuous section, the so-called pullback construction allows to construct bigger bundles admitting sections. We repeat this standard construction in our setting.

**Construction 2.9** (Pullback bundle). Let \((K, B, p; \varphi)\) be a bundle of topological dynamical systems, \(M\) a compact space and \(r: M \to B\) a continuous surjection. We then define

\[ r^*K := \{(m, k) \in M \times K \mid r(m) = p(k)\} \]

and denote the restriction of the canonical projection \(p_M: M \times K \to M\) to \(r^*K\) by \(\pi_M\) and the restriction of \(id_M \times \varphi\) to \(r^*K\) by \(r^*\varphi\). Then \(r^*K, M, \pi_M; r^*\varphi\) is a bundle of topological dynamical systems and \((K, \varphi)\) is a factor of \((r^*K, r^*\varphi)\) with respect to the projection \(\pi_K\) onto the second component.

We obtain the following commutative diagram of dynamical systems:

\[
\begin{array}{ccc}
(r^*K, r^*\varphi) & \xrightarrow{\pi_K} & (K, \varphi) \\
\downarrow{\pi_M} & & \downarrow{p} \\
(M, id_M) & \xrightarrow{r} & (B, id_B)
\end{array}
\]

**Example 2.10.** Let \(K = B = \mathbb{T}\), define \(\varphi: K \to K\) by setting \(\varphi(z) = -z\) and \(p: K \to B\) by setting \(p(z) = z^2\). Set \(r: [0, 1] \to \mathbb{T}, t \mapsto e^{2\pi i t}\). Then the pullback bundle with respect to \(r\) is isomorphic to the bundle \(([0, 1] \times \{-1, 1\}, [0, 1], p|_{[0, 1]}; (t, x) \mapsto (t, -x))\).

**Remark 2.11.** Given a bundle \((K, B, p; \varphi)\), the pullback bundle \((p^*K, K, \pi; p^*\varphi)\) admits a continuous section: This pullback bundle is constructed by gluing to each point in \(K\) its fiber and so the map \(s: K \to p^*K, k \mapsto (k, k)\) is a canonical continuous section. In particular, every bundle of topological dynamical systems is a factor of a bundle admitting a continuous section.

**Remark 2.12.** Properties like minimality and unique ergodicity of each fiber as well as global properties such as equicontinuity, invertibility and mean ergodicity are preserved under forming pullback bundles. In particular, discrete spectrum is preserved by this construction.

### 2.2. Operator-theoretic aspects of bundles.

The following proposition shows that, up to isomorphism, there is a one-to-one correspondence between unital \(C^*\)-subalgebras of \(\text{fix}(T_\varphi)\) and trivial factors \((B, id_B)\) of the system \((K, \varphi)\).
Proposition 2.13. Let \((K_\varphi, \varphi)\) be a topological dynamical system and \(A\) a unital C*-subalgebra \(A\) of \(\text{fix}(T_\varphi)\). Then there is an (up to isomorphism) unique bundle \((K, B, p, \varphi)\) such that \(T_p(C(B)) = A\) where \(T_p\) denotes the Koopman operator of \(p\).

Proof. Let \(A\) be a unital C*-subalgebra of \(\text{fix}(T_\varphi)\). By the Gelfand-Naimark theorem, there is a compact space \(B\) such that \(A \cong C(B)\). The induced C*-embedding \(C(B) \ni C(K)\) is given by a Koopman operator \(T_p\) for a continuous map \(p: K \to B\). Because \(T_p\) is injective, \(p\) is surjective. Moreover, from one obtains from the commutativity of the two diagrams

\[
\begin{array}{ccc}
C(K) & \xrightarrow{T_\varphi} & C(K) \\
\downarrow{T_p} & & \downarrow{T_p} \\
C(B) & \xrightarrow{T_{id_B}} & C(B)
\end{array}
\]

\[
\begin{array}{ccc}
K & \xrightarrow{\varphi} & K \\
\downarrow{p} & & \downarrow{p} \\
B & \xrightarrow{id_B} & B
\end{array}
\]

that \(\varphi(K_\varphi) \subseteq K_\varphi\), so \((K, B, p; \varphi)\) is indeed a bundle of topological dynamical systems and \(T_p(C(B)) = A\) by construction.

Now take two such bundles \((K, B, p; \varphi)\) and \((K', B', p'; \varphi)\) of dynamical systems. Then \(C(B) \cong A \cong C(B')\) and this isomorphism is again given by a Koopman operator \(T_{\varphi}: C(B) \to C(B')\) corresponding to a homeomorphism \(\vartheta: B' \to B\). This yields that \((id, \vartheta)\) is an isomorphism between the two bundles.

Remark 2.14. Proposition 2.13 allows to order the bundles corresponding to a system \((K, \varphi)\) by saying that \((K, B_1, p_1; \varphi)\) is finer than \((K, B_2, p_2; \varphi)\) if \(T_{p_1}(C(B_1)) \supseteq T_{p_2}(C(B_2))\). The term finer is used here because the above inclusion induces a surjective map \(r: B_1 \to B_2\). In light of Proposition 2.13, there is a maximal trivial factor of \((K, \varphi)\) associated to the fixed space \(\text{fix}(T_\varphi)\). We denote this factor by \((L_\varphi, \text{id}_{L_\varphi})\) and the corresponding factor map \(q_\varphi: K \to L_\varphi\), but omit \(\varphi\) from the notation if the context leaves no room for ambiguity.

After these preparations, we characterize mean ergodicity via bundles, showing that the global notion of mean ergodicity is in fact equivalent to the “local” notion of fiberwise unique ergodicity. The elegant proof for implication (b) \(\Rightarrow\) (a) presented here was kindly provided by M. Haase.

Theorem 2.15. Let \((K, \varphi)\) be a topological dynamical system and \(q: K \to L\) the projection onto its maximal trivial factor. Then the following assertions are equivalent.

(a) The Koopman operator \(T_\varphi\) is mean ergodic on \(C(K)\).

(b) Each fiber \((K_l, \varphi_l)\) is uniquely ergodic.

(c) For each \(l \in L\) and \(f \in C(K)\) there is a \(c_l \in \mathbb{C}\) such that \(A_n, f(x) \to c_l\) for all \(x \in K_l\).

(d) The map \(M_\varphi(K) \to M(L), \mu \mapsto T'_{q^*\mu} = q_*\mu\) is an isomorphism.

Proof. Suppose that \(T_\varphi\) is mean ergodic. For each \(f \in \text{fix}(T_\varphi)\) there is a function \(\tilde{f} \in C(K)\) such that \(\tilde{f}|_{K_l} = f\). If we denote the mean ergodic
projection of $T_\varphi$ by $P$, then $P\hat{f} \in \text{fix}(T_\varphi) = T_q(C(L))$ and hence $P\hat{f}$ is constant on each fiber. Therefore, $f = P\hat{f}|_{K_i}$ is constant and $\text{fix}(T_{\varphi|_i})$ is one-dimensional. Thus, $T_{\varphi|_i}$ is mean ergodic since the Cesáro averages converge uniformly on $K$ and in particular on $K_i$. Hence, each fiber $(K_i, \varphi_i)$ is uniquely ergodic.

Now assume that each fiber $(K_i, \varphi_i)$ is uniquely ergodic and let $\mu_i$ denote the corresponding unique invariant probability measure. Using this and Lemma 2.16 below, we obtain that the graph of the map $l \mapsto \mu_i$ is closed. Since this map takes values in the compact set $M_1^1(K)$, applying the closed graph theorem for compact spaces (see [Dug67, Theorem XI.2.7]) we conclude that the map $l \mapsto \mu_i$ is weak*-continuous. Since each fiber is uniquely ergodic, we also have

$$\lim_{n \to \infty} A_n f(x) = \int_K f \, d\mu_q(x) = \langle f, \mu_q(x) \rangle$$

and this depends continuously on $x$, showing that $T_\varphi$ is mean ergodic. The equivalence of (b) and (c) is well-known for each fiber. Assertion (d) implies that $\text{fix}(T_\varphi)$ separates $\text{fix}(T_{\varphi|_i})$ and hence that $T_{\varphi|_i}$ is mean ergodic. Conversely, if $T_{\varphi|_i}$ is mean ergodic, a short calculation shows that the inverse of the map in (d) is given by $\nu \mapsto \int_{K_i} \mu_i \, d\nu$ where $\mu_i$ is the unique $\varphi$-invariant probability measure on $K_i$.

**Lemma 2.16.** Let $(K, \varphi)$ be a topological dynamical system and $\mu \in M(K)$ a probability measure. Then $\text{supp}(\mu) \subseteq K_i$ if and only if $T_q^i \mu = \delta_i$.

**Proof.** Assume that $\text{supp}(\mu) \subseteq K_i$. If $g \in C(L)$ satisfies $g(l) = 0$, then $T_q g$ is zero on $K_i$ and hence on $\text{supp}(\mu)$, meaning that

$$\langle g, T_q^i \mu \rangle = \langle T_q g, \mu \rangle = 0.$$

So $\text{supp}(T_q^i \mu) \subseteq \{l\}$ and since $T_q^i \mu$ is a probability measure, we conclude that $T_q^i \mu = \delta_i$.

Now suppose $T_q^i \mu = \delta_i$. If $f \in C(K)$ is positive and such that $\|f\| \leq 1$ and $\text{supp}(f) \cap K_i = \emptyset$, then $l \notin \text{supp}(f)$ and by Urysohn’s lemma there is a function $g \in C(L)$ equal to 1 on $q(\text{supp}(f))$ satisfying $g(l) = 0$. But then $f \leq T_q g$ and hence

$$0 \leq \langle f, \mu \rangle \leq \langle T_q g, \mu \rangle = \langle g, T_q^i \mu \rangle = \langle g, \delta_i \rangle = 0.$$

So $\langle f, \mu \rangle = 0$ and we conclude that $\text{supp}(\mu) \subseteq K_i$.

**Remark 2.17.** In the proof of the implication (b) $\Rightarrow$ (a) the information that we considered fibers with respect to the maximal trivial factor $L$ was not used. In fact, let $B$ be any trivial factor such that the corresponding fibers are uniquely ergodic. The existence of a continuous surjection $r : L \to B$ from Remark 2.14 then shows that each fiber $K_b$ is contained in a fiber $K_i$. But since each fiber $(K_i, \varphi_i)$ is also uniquely ergodic by Theorem 2.15, it cannot contain more than one of the sets $K_b$ and so $r$ has to be a homeomorphism. Therefore, any bundle of topological dynamical systems $(K, B, p, \varphi)$ with uniquely ergodic fibers is automatically isomorphic to the bundle $(K, L, q, \varphi)$ and we may hence assume that $B = L$ and $p = q$. 


2.3. **Group bundles.** We now introduce the main object of this paper: bundles of topological dynamical systems for which each fiber is a group rotation.

**Definition 2.18.** A bundle \((G, B, p)\) is called a *group bundle* if \((G, B, p)\) is a bundle and each fiber \(G_b\) carries a group structure such that

(i) the multiplication
\[
\{(g, g') \in G \times G \mid p(g) = p(g')\} \rightarrow G, \quad (g, g') \mapsto gg'
\]
is continuous,

(ii) the inverse \(G \rightarrow G, \ g \mapsto g^{-1}\) is continuous and

(iii) the neutral element \(e_b \in G_b\) depends continuously on \(b \in B\).

A bundle \((G, B, p; \varphi)\) is called a *group rotation bundle* if \((G, B, p)\) is a group bundle, \(\varphi: G \rightarrow G\) is continuous and

(iv) there is a continuous section \(\alpha: B \rightarrow G\) with \((G_b, \varphi_b) = (G_b, \varphi(\alpha(b)))\).

A morphism \((\Theta, \vartheta): (G_1, B_1, p_1) \rightarrow (G_2, B_2, p_2)\) of group bundles is a bundle morphism such that \(\Theta\) is a group homomorphism restricted to each fiber. It is called a *morphism of group rotation bundles* if, in addition, \(\Theta\) is a morphism of the corresponding dynamical systems. A group bundle \((G, B, p)\) is called **trivial** if there is a group \(G\) such that \((G, B, p) = (G \times B, B, \pi_B)\). We call \((G, B, p)\) **trivializable** if there is an isomorphism \(\iota: (G, B, p) \rightarrow (G \times B, B, \pi_B)\).

We call it **subtrivializable** and \(\iota\) a **\((G,\) subtrivialization** if \(\iota\) is merely an embedding. We say that two subtrivializations
\[
\iota_1: (G_1, B_1, p_1) \rightarrow (G \times B_1, B_1, \pi_{B_1}) \\
\iota_2: (G_2, B_2, p_2) \rightarrow (G \times B_2, B_2, \pi_{B_2})
\]
are **isomorphic** if there is an isomorphism \((\Theta, \vartheta): (G_1, B_1, p_1) \rightarrow (G_2, B_2, p_2)\) such that the diagram
\[
\begin{array}{ccc}
G \times B_1 & (g, b) \mapsto (g, \vartheta(b)) & G \times B_2 \\
\downarrow \iota_1 & & \downarrow \iota_2 \\
G_{\Theta, \vartheta} & \Theta & G_{\Theta, \vartheta}
\end{array}
\]
commutes.

**Example 2.19.** As an example of a bundle of topological dynamical systems for which each fiber is a group rotation, yet no continuous section \(\alpha: B \rightarrow K\) exists, recall the bundle from **Example 2.6** and equip it with the dynamic \(\varphi: K \rightarrow K, z \mapsto -z\). The fibers here may be interpreted as copies of \((\mathbb{Z}_2, n \mapsto n + 1)\) and it was seen in **Example 2.6** that this bundle does not admit continuous sections.

**Remark 2.20.** Products and pullbacks of group rotation bundles canonically are again group rotation bundles. However, when passing to factors, the existence of continuous sections may be lost, as seen in **Example 2.10**.
Remark 2.21. The notion of group bundles is not new: It has been considered as a special case of locally compact groupoids, in e.g., [Ren80, Chapter 1].

In order to decompose systems with discrete spectrum, we single out group rotation bundles for which each fiber is minimal. Recall the following characterization of minimal group rotations.

Theorem 2.22 ([EFHN15, Theorem 10.13]). Let \( G \) be a compact group, \( m \) the Haar measure on \( G \) and consider a group rotation \( (G, a) \). Then the following assertions are equivalent.

\begin{enumerate}[(a)]
\item \( (G, a) \) is minimal.
\item \( (G, a) \) is uniquely ergodic.
\item \( m \) is the only invariant probability measure for \( (G, a) \).
\item \( (G, m; a) \) is ergodic.
\end{enumerate}

Remark 2.23. Let \( (G, B, p; \varphi) \) be a group rotation bundle such that each fiber is minimal. Then by [Theorem 2.22] every fiber is uniquely ergodic, the unique \( \varphi \)-invariant probability measure being the Haar measure \( m_b \) on the group \( G_b \). [Remark 2.17] yields that we may therefore assume that \( B = L \) and \( p = q \) where \( q: G \to L \) is the projection onto the maximal trivial factor. Moreover, if \( m_l \) denotes the Haar measure on \( G_l \), the map \( l \mapsto m_l \) is weak* continuous. If \( \mu \) is a \( \varphi \)-invariant measure on \( G \), we define the pushforward measure \( \nu := q_* \mu \) on \( L \) and disintegrate \( \mu \) as in the proof of [Theorem 2.15] via

\[ \mu = \int_L m_l \, d\nu. \]

This will be important for [Theorem 3.10].

We now turn towards a generalization of the dual of a locally compact group. This will be needed for [Section 4]

Construction 2.24. Let \( (G, B, q) \) be a locally compact group bundle. Set

\[ G^* := \bigcup_{b \in B} (G_b)^* \]

where \( (G_b)^* \) is the dual group of \( G_b \) and denote by \( \pi_B: G^* \to B \) the canonical projection onto \( B \). Next, let \( h \in C_c(G) \), \( F \in C(G) \) and \( \varepsilon > 0 \). Set

\[ N(h, F, \varepsilon) := \left\{ \chi \in G^* \mid \| \chi h_{\pi_B(\chi)} - (F h)_{\pi_B(\chi)} \|_\infty < \varepsilon \right\}. \]

The family of these sets forms a subbasis for a topology which we call the topology of compact convergence on \( G^* \).

With this topology, the projection \( \pi_B \) is continuous as can be deduced from the continuity of the neutral element section \( e: B \to G \) by invoking Urysohn’s lemma and Tietze’s extension theorem to construct appropriate functions \( h \) and \( F \). Therefore, \( (G^*, B, \pi_B) \) is a bundle which we call the dual bundle
of \((G, B, q)\) and hence may also denote by \((G, B, q)^*\). If \((\Theta, \vartheta): (G, L, q) \to (H, L, p)\) is a morphism of group bundles such that \(\vartheta\) is bijective, define its **dual morphism** \((\Theta^*, \vartheta^{-1})\): \((H^*, B', q) \to (G^*, B, p)\) by setting \(\Theta^*: H^* \to G^*, \chi \mapsto (\Theta_{\pi_L(\chi)})^*\chi\).

For later reference and the convenience of the reader, we list some basic properties of dual bundles. To this end, we recall the following notions.

**Definition 2.25.** Let \(X\) and \(Y\) be topological spaces. A map \(F: X \to \mathcal{P}(Y)\) is called **lower-** (resp. **upper-**) **semicontinuous** in a point \(x \in X\) if for every open \(U \subseteq Y\) such that \(F(x) \cap U \neq \emptyset\) (resp. \(F(x) \subseteq U\)) there exists an open neighborhood \(V\) of \(x\) such that for all \(x' \in V\) one has \(F(x') \cap U \neq \emptyset\) (resp. \(F(x') \subseteq U\)). A bundle \((K, B, p)\) is called **lower-** (resp. **upper-**) **semicontinuous** if the map \(b \mapsto K_b\) is lower-semicontinuous in each point \(b \in B\) and **continuous** if it is both lower- and upper-semicontinuous.

**Remark 2.26.** A bundle is lower-semicontinuous if and only if the bundle projection is an open map.

**Definition 2.27.** Let \((X, B, p)\) and \((Y, B, q)\) be two bundles. Then their **sum** is defined as \((X \oplus Y, B, \pi_B)\) where

\[
X \oplus Y := \{ (x, y) \in X \times Y \mid p(x) = q(y) \}
\]

and \(\pi_B(x, y) := p(x)\).

**Proposition 2.28.** Let \((G, B, p)\) and \((H, B', p')\) be locally compact abelian group bundles and \((\Theta, \vartheta): (G, B, p) \to (H, B', p')\) a morphism of group bundles such that \(\vartheta\) is bijective.

(i) The **evaluation map** \(\text{ev}: G^* \oplus G \to \mathbb{C}, (\chi, g) \mapsto \chi(g)\) is continuous. In fact, a net \((\chi_i)_{i \in I}\) converges to \(\chi \in G^*\) if and only if \(\pi_B(\chi_i) \to \pi_B(\chi)\) and for every convergent net \((g_i)_{i \in I}\) with \(p(g_i) = \pi_B(\chi_i)\) and limit \(g \in G\) we have \(\chi_i(g_i) \to \chi(g)\).

(ii) For \(b \in B\), \(\text{id}_{(G_b)^*} : (G_b)^* \to (G^*)_b\) is an isomorphism of locally compact groups. In particular, the notation \(G_b^*\) is unambiguous.

(iii) If \(G\) is a locally compact group and \(L\) is a compact space, \((G \times L, L, \pi_L)^* = (G^* \times L, L, \pi_L)^*\).

(iv) The bundle \((G, B, p)\) is **lower-semicontinuous** if and only if \(G^*\) is a Hausdorff space.

(v) The **dual morphism** \(\Theta^*\) is continuous and

\[
(\Theta^*, \vartheta^{-1}): (H^*, B', \pi_{B'}) \to (G^*, B, \pi_B)
\]

is a morphism of group bundles.

(vi) If \(\Theta\) is proper and surjective, \(\Theta^*: H^* \to G^*\) is an embedding.

(vii) If \(G\) is compact and \(\Theta\) surjective, \(\Theta^*: H^* \to G^*\) is an embedding.

**Proof.** The first part of (i) follows from the definition of the topology on \(G^*\) using local compactness to invoke Urysohn’s lemma and Tietze’s extension theorem which provide appropriate functions \(h\) and \(F\). The second part of (iii) is a simple proof by contradiction. For part (iii) it suffices to show
that the two sets carry the same topology. This follows from (i) since it shows that the two topologies have the same convergent nets. By the same argument, (iv) follows directly from (i) and so does (v) since it suffices to show that $Θ^*$ is continuous. In (iv) we obtain the Hausdorff property from lower-semicontinuity and (i) showing that every convergent net in $G^*$ has a unique limit. For the converse implication in (iv) assume that $G^*$ is Hausdorff but $(G, B, p)$ is not lower-semicontinuous. Then there exist $b ∈ B$, an open subset $U ⊆ G$ with $U ∩ G_b ≠ ∅$ and a net $(b_i)i∈I$ such that $b_i → b$ and $G_{b_i} ∩ U = ∅$ for all $i ∈ I$. Consider

$$H := \left\{ g ∈ G_b : \text{There exists a net } (g_i)i∈I \text{ in } G \text{ with } g_i → g \text{ and } g_i ∈ G_{b_i} \text{ for all } i ∈ I \right\}.$$ 

This set forms a proper subgroup of $G_b$ which is not dense since $H ∩ U_b = ∅$. If we now denote by $χ_0(b_i)$ the trivial character on $G_{b_i}$, then $χ_0(b) → χ$ for any character $χ ∈ G^*$ such that $χ|_{H} = 1$. But since $H$ is a proper subgroup of $G_b$, there are at least two characters satisfying this. In particular, $G^*$ cannot be Hausdorff.

For part (vi) (which trivially implies (vii)), note that $Θ^*$ is injective because $Θ$ is surjective. Let $(χ_i)i∈I$ be a net in $H^*$ such that $Θ^*(χ_i) → η$ for $η ∈ G^*_b$. Then $η(g) = η(g')$ if $Θ(g) = Θ(g')$ and so $η = χ_0 Θ$ for a function $χ: H_b → G$. It is again multiplicative and continuous because $H_b$ carries the final topology with respect to $Θ_b$, so $χ ∈ H^*_b$. Let $N(U, h, F, ε)$ be an open neighborhood of $χ$. Then $N(Θ^{-1}(U), h ∘ Θ, F ∘ Θ, ε)$ is an open neighborhood of $χ_0 Θ$ and so $χ_i ∘ Θ \in N(Θ^{-1}(U), h ∘ Θ, F ∘ Θ, ε)$ for $i ≥ i_0$, implying $χ_i ∈ N(U, h, F, ε)$ for $i ≥ i_0$. Hence, $χ_i → χ$.

3. Representation

The classical examples for systems with discrete spectrum are group rotations $(G, a)$ and trivial systems $(B, id_B)$ as seen in Example 1.5. In Corollary 3.7 we show that, in fact, every system with discrete spectrum canonically is a factor of a product $(G, a) × (B, id_B)$ and therefore can be constructed from these two basic systems. We start with the topological case and derive the measure-preserving case from this via topological models. As a result, we generalize the Halmos-von Neumann representation theorem to not necessarily minimal or ergodic systems with discrete spectrum in Theorem 3.10.

We briefly recall the Halmos-von Neumann theorem for minimal topological systems $(K, φ)$ and, because the proof of Theorem 3.6 below is based on it, sketch a proof using the Ellis (semi)group $E(K, φ) := \{ φ_k : k ∈ N \} ⊆ K^K$ introduced by Ellis as the “enveloping semigroup”, see [Ell60].

**Theorem 3.1.** Let $(K, φ)$ be a minimal topological dynamical system with discrete spectrum. Then $(K, φ)$ is isomorphic to a minimal group rotation $(G, φ_a)$ on an abelian compact group $G$. More precisely, for each $x_0 ∈ K$ there is a unique isomorphism $δ_{x_0}: (E(K, φ), φ) → (K, φ)$ such that $δ_{x_0}(id_K) = x_0$. 
Proof. Pick a point \( x_0 \in K \) and consider the map
\[
\delta_{x_0} : E(K, \varphi) \to K, \quad \psi \mapsto \psi(x_0).
\]
Since \( K \) is minimal, \( \delta_{x_0} \) is injective. Moreover, \( \delta_{x_0}(E(K, \varphi)) \) is a closed, invariant subset of \( K \) which is not empty and hence \( \delta_{x_0}(E(K, \varphi)) = K \). It is not difficult to check that the system \( (E(K, \varphi), \varphi) \) is isomorphic to \( (K, \varphi) \) via \( \delta_{x_0} \).

Note that the isomorphism in Theorem 3.1 depends on the (non-canonical) choice of \( x_0 \in K \).

Definition 3.2. Let \((K, B, p; \varphi)\) be a bundle of topological dynamical systems. We then set
\[
E(K, B, p; \varphi) := \bigcup_{b \in B} E(K_b, \varphi_b),
\]
\[
\alpha : B \to E(K, B, p; \varphi), \quad b \mapsto \varphi_b,
\]
\[
\varphi_\alpha : (E(K, B, p; \varphi) \to E(K, B, p; \varphi), \quad \psi \mapsto \psi \circ \varphi_b
\]
and let \( \pi_B : E(K, B, p; \varphi) \to B \) denote the projection onto the second component. We equip \( E(K, B, p; \varphi) \) with the final topology induced by the map \( \rho : E(K, \varphi) \times B \to E(K, B, p; \varphi), (\psi, b) \mapsto \psi_b \) and call \( (E(K, B, p; \varphi), B, \pi_B; \alpha) \) the Ellis semigroup bundle of \((K, B, p; \varphi)\).

We abbreviate the Ellis semigroup bundle by \( E(K, B, p; \varphi) \) if the context leaves no room for ambiguity. We also note that it is a group rotation bundle if it is compact and \( E(K, \varphi) \) is a group, in which case we call it the Ellis group bundle. We now give a criterion for the space \( E(K, B, p; \varphi) \) to be compact.

Lemma 3.3. Let \( X \) be a locally compact space, \( Y \) a (Hausdorff) uniform space, \( B \) a compact space and \( F : B \to \mathcal{P}(X) \) a set-valued map. Define an equivalence relation \( \sim_F \) on \( C(X, Y) \times B \) via
\[
(f, b) \sim_F (g, b') \quad \text{if} \quad b = b' \text{ and } f|_{F(b)} = g|_{F(b)}
\]
and endow \( C(X, Y) \) with the topology of locally uniform convergence. Moreover, let \( A \subseteq C(X, Y) \) be a compact subset. If \( F \) is lower-semicontinuous, then the quotient \( A \times B/\sim_F \) is a compact space.

Proof. Since the quotient of a compact space by a closed equivalence relation is again compact (cf. [Bou95, Proposition 10.4.8]), it suffices to show that \( \sim_F \) is closed. So let \((f_i, b_i), (g_i, b_i)\) be a net in \( \sim_F \) with limit \((f, b), (g, b)\) \( \in (C(X, Y) \times B)^2 \). Pick \( x \in F(b) \). Since \( F \) is lower-semicontinuous and \( b_i \to b \), there is a net \((x_i)_{i \in I}\) such that \( x_i \in F(b_i) \) and \( x_i \to x \). But since \((f_i)_{i \in I}\) and \((g_i)_{i \in I}\) converge locally uniformly,
\[
f(x) = \lim_{i \to \infty} f_i(x_i) = \lim_{i \to \infty} g_i(x_i) = g(x).
\]
Since \( x \in F(b) \) was arbitrary, it follows that \( f|_{F(b)} = g|_{F(b)} \) and so \( \sim_F \) is closed.

Lemma 3.4. Let \((K, B, p; \varphi)\) be a bundle of topological dynamical systems such that \((K, \varphi)\) is equicontinuous. Then the following assertions are true.
(i) If $p$ is open, the space $E(K, B, p; \varphi)$ is compact.

(ii) If each fiber $(K_b, \varphi_b)$ is minimal, then $p$ is open.

Proof. Part (i) is a special case of Lemma 3.3 with $A = E(K, \varphi)$ and $F : B \to \mathcal{P}(K)$ with $F(b) = K_b$ since the topologies of pointwise and uniform convergence coincide on equicontinuous subsets of $C(K, K)$.

For (ii), assume that each fiber $(K_b, \varphi_b)$ is minimal. If $U \subseteq K_b$ is open in $K_b$, then

$$K_b = \bigcup_{k=0}^{\infty} \varphi_b^{-k}(U)$$

since $(K_b, \varphi_b)$ is minimal (cf. [EFHN13, Proposition 3.3]). Therefore, if $U \subseteq K$ is open, then

$$p^{-1}(p(U)) = \bigcup_{k=0}^{\infty} \varphi^{-k}(U)$$

is open in $K$ and hence $p(U)$ is open. □

Proposition 3.5. Let $(K, \varphi)$ be a topological dynamical system with discrete spectrum and $q : K \to L$ the canonical projection onto the maximal trivial factor. Then each fiber $(K_l, \varphi_l)$ is minimal and has discrete spectrum.

Proof. Each fiber $(K_l, \varphi_l)$ has discrete spectrum since $E(K_l, \varphi_l) = \{ \psi | K_l \}$, use Proposition 1.6. Moreover, for $x, y \in K_l$ one has $\text{orb}(x) = E(K_l, \varphi_l)x$ and $\text{orb}(y) = E(K_l, \varphi_l)y$. Since $E(K_l, \varphi_l)$ is a group, we conclude that either $\text{orb}(x) = \text{orb}(y)$ or $\text{orb}(x) \cap \text{orb}(y) = \emptyset$. However, by Remark 1.3 the system $(K, \varphi)$ is mean ergodic and hence $(K_l, \varphi_l)$ is uniquely ergodic by Theorem 2.15. We now conclude from the Krylov-Bogoljubov Theorem (cf. [EFHN13, Theorem 10.2]) that $K_l$ cannot contain two disjoint closed orbits. Consequently, $\text{orb}(x) = \text{orb}(y)$ for all $x, y \in K_l$ and hence $(K_l, \varphi_l)$ is minimal. □

Theorem 3.6. Let $(K, \varphi)$ be a topological dynamical system with discrete spectrum and assume that the canonical projection $q : K \to L$ onto the maximal trivial factor admits a continuous section. Then $(K, L, q; \varphi)$ is isomorphic to its Ellis group bundle.

Proof. Let $s : L \to K$ be a section for $q$. By Proposition 3.5 every fiber $(K_l, \varphi_l)$ is minimal and has discrete spectrum. By Theorem 3.1 we obtain an isomorphism $\Phi_l : (E(K_l, \varphi_l), \varphi_l) \to (K_l, \varphi_l)$ satisfying $\Phi_l(id_{K_l}) = s(l)$. This yields a bijection

$$\Phi : E(K, L, q; \varphi) \to K, \quad \psi_l \mapsto \psi_l(s(l)).$$

Because $(K, \varphi)$ has discrete spectrum, the map

$$E(K, \varphi) \times L \to K, \quad (\psi, l) \mapsto \psi(s(l))$$

is continuous, hence $\Phi$ is continuous and an isomorphism of topological dynamical systems. □

Example 2.19 shows that there are systems with discrete spectrum which are not isomorphic to a group rotation bundle. However, the following is still true.
Corollary 3.7. Let \((K, \varphi)\) be a topological dynamical system with discrete spectrum. Then \((K, \varphi)\) is a factor of a trivial group rotation bundle \((G, a) \times (B, \id_B)\) where the group rotation \((G, a)\) is minimal and can be taken as \((G, a) = (E(K, \varphi), \varphi)\).

Proof. Let \((K, \varphi)\) be a topological dynamical system with discrete spectrum and \(q: K \to L\) the projection onto its maximal trivial factor. As noted in Remark 2.12, the associated pullback system \((q^*K, K, \pi_K, q^*\varphi)\) also has discrete spectrum. Moreover, its fibers are uniquely ergodic and so Remark 2.17 shows that its maximal trivial factor is homeomorphic to \(K\). This, combined with Remark 2.11 yields that the canonical projection onto its maximal trivial factor admits a continuous section \(s: K \to q^*K\). By Theorem 3.6 we obtain that the bundle \((q^*K, K, \pi_K; q^*\varphi)\) is isomorphic to its Ellis group bundle which is, by construction, a factor of the trivial group rotation bundle \((E(q^*K, q^*\varphi), q^*\varphi) \times (K, \id_K)\). We now consider the following maps:

\[
Q: E(K, \varphi) \to E(q^*K, q^*\varphi), \quad \psi \mapsto q^*\psi,
\]

\[
P: E(q^*K, q^*\varphi) \to E(K, \varphi), \quad \psi \mapsto p_2 \circ \tilde{\psi} \circ s
\]

where \(p_2: q^*K \to K\) denotes the projection onto the second component. Both \(Q\) and \(P\) are continuous and satisfy \(Q(\varphi^k) = (q^*\varphi)^k\) and \(P((q^*\varphi)^k) = \varphi^k\) for all \(k \in \mathbb{N}\). Since \(\varphi\) and \(q^*\varphi\) generate their respective Ellis groups, \(P\) and \(Q\) are mutually inverse. Hence,

\[
(E(q^*K, q^*\varphi), q^*\varphi) \times (K, \id_K) \cong (E(K, \varphi), \varphi) \times (K, \id_K).
\]

\(\square\)

Remark 3.8. The group rotation \((E(K, \varphi), \varphi)\) is the smallest group rotation that can be taken as \((G, a)\) in Corollary 3.7 in the sense that any such group rotation \((G, a)\) admits an epimorphism \(\eta: (G, a) \to (E(K, \varphi), \varphi)\). This is true because a factor map \(\vartheta: (G, a) \times (B, \id_B) \to (K, \varphi)\) induces a continuous, surjective group homomorphism

\[
E(\vartheta): E((G, a) \times (B, \id_B)) \to E(K, \varphi)
\]

satisfying \(E(\vartheta)(a \times \id_B) = \varphi\) and

\[
(E((G, a) \times (B, \id_B)), a \times \id_B) \cong (E(G, a), \varphi_a) \cong (G, a).
\]

Remark 3.9. If \((K, \varphi)\) has discrete spectrum and the canonical projection \(q: K \to L\) admits a continuous section, the system is already isomorphic to its Ellis group bundle and hence, by definition of the latter, a factor of the system \((E(K, \varphi), \varphi) \times (L, \id_L)\). In this case, one can take \(B = L\) in Corollary 3.7.

3.1. The measure-preserving case. Since the problem of finding continuous sections can be solved for topological models of measure spaces as we will see below, we obtain a better result for measure-preserving systems. This is our generalization of the Halmos-von Neumann theorem to the non-ergodic case. It is proved by constructing a topological model and then applying Theorem 3.6. For background information on topological models, see [EFHN13, Chapter 12].
**Theorem 3.10.** Let \((X, \varphi)\) be a measure-preserving system with discrete spectrum. Then \((X, \varphi)\) is Markov-isomorphic to the rotation on a compact group rotation bundle. More precisely, there are a compact group rotation bundle \((\mathcal{G}, \mathcal{B}, \nu; \varphi_\alpha)\) with minimal fibers and a \(\varphi_\alpha\)-invariant measure \(\mu_\mathcal{G}\) on \(\mathcal{G}\) such that \((X, \varphi)\) and \((\mathcal{G}, \mu_\mathcal{G}; \varphi_\alpha)\) are Markov-isomorphic. Moreover, this group rotation bundle can be chosen such that the canonical map \(j: \text{Kro}_{\mathcal{G}}(T_{\varphi_\alpha}) \to \text{Kro}_{L^1(\mathcal{G}, \mu_\mathcal{G})}(T_{\varphi_\alpha})\) of Kronecker spaces is an isomorphism.

**Proof.** We define

\[
\mathcal{A} := \overline{\text{cl}}_{L^\infty} \left( \bigcup_{|\lambda|=1} \ker_{L^\infty}(\lambda I - T_\varphi) \right)
\]

and note that this is a \(T_\varphi\)-invariant, unital \(C^*\)-subalgebra of \(L^\infty(X)\) which is dense in \(L^1(X)\) by [EFHN15, Lemma 17.3] since \((X, \varphi)\) has discrete spectrum. The Gelfand representation theorem (cf. [Tak79, Theorem I.4.4]) yields that there is a compact space \(K\) and a \(C^*\)-isomorphism \(S: C(K) \to \mathcal{A}\). The Riesz-Markov-Kakutani representation theorem shows that there is a unique Borel probability measure \(\mu_K\) on \(K\) such that

\[
\int_K f \, d\mu_K = \int_X Sf \, d\mu_X \quad \text{for all } f \in C(K).
\]

Moreover, \(T := S^{-1} \circ T_\varphi \circ S: C(K) \to C(K)\) defines a \(C^*\)-homomorphism and so (cf. [EFHN15, Theorem 4.13]) there is a continuous map \(\psi: K \to K\) such that \(T = T_\psi\). The operator \(S\) is, by construction, an \(L^1\)-isometry and \(S[f] = |Sf|\) for all \(f \in C(K)\) by [EFHN15, Theorem 7.23]. Since \(\mathcal{A}\) is dense in \(L^1(X)\), we conclude that \(S\) extends to a bi-Markov lattice homomorphism \(S: L^1(K, \mu_K) \to L^1(X)\).

The (topological) system \((K, \psi)\) now has discrete spectrum by construction. Let \(L_\psi\) denote the maximal trivial factor of \((K, \psi)\). Then \(C(L_\psi) \cong \text{fix}(T_\psi) \cong \text{fix}_{L^\infty(X)}(T_\varphi)\) and so \(L_\psi\) is extremally disconnected as noted in Section 1. From Theorem 2.8 we therefore conclude that the canonical projection \(q: K \to L_\psi\) has a continuous section. Theorem 3.9 shows that there is an isomorphism \(\vartheta: (K, \psi) \to (\mathcal{G}, \alpha)\) where \((\mathcal{G}, \alpha)\) is the rotation on some compact group rotation bundle with minimal fibers. Equipping \((\mathcal{G}, \alpha)\) with the push-forward measure \(\mu_\mathcal{G} := \vartheta_* \mu_K\), we obtain that the system \((X, \varphi)\) is isomorphic to the system \((\mathcal{G}, \alpha; \mu_\mathcal{G})\).

**Corollary 3.11.** Let \((X, \varphi)\) be a measure-preserving dynamical system with discrete spectrum and \((L, \nu; \text{id}_L)\) a topological model for \(\text{fix}_{L^\infty(X)}(T_\varphi)\). Then \((X, \varphi)\) is a Markov factor of the trivial group rotation bundle \((J(T_\varphi), m; T_\varphi) \times (L, \nu; \text{id}_L)\).

**Proof.** This follows from Theorem 3.10 and Remark 3.9.

**Remark 3.12.** We can also interpret the Halmos-von Neumann theorem in the following way: If \((X, \varphi)\) is an ergodic, measure-preserving system with discrete spectrum, there is a compact, ergodic group rotation \((G, \alpha)\) and a
Markov isomorphism $S: L^1(X) \rightarrow L^1(G, m)$ such that the diagram

$$
\begin{array}{ccc}
L^1(X) & \xrightarrow{S} & L^1(G, m) \\
\downarrow T_\varphi & & \downarrow T_{\varphi_a} \\
L^1(X) & \xrightarrow{S} & L^1(G, m)
\end{array}
$$

commutes, i.e., $T_\varphi$ acts like an ergodic rotation on scalar-valued functions. If $(X, \varphi)$ is not ergodic, we can interpret Corollary 3.11 similarly: There is a compact, ergodic group rotation $(G, a)$, a compact probability space $(L, \nu)$ and a Markov embedding $S: L^1(X) \rightarrow L^1(G \times L, m \times \nu)$ such that $T_{\varphi_a \times \text{id}_L} S = S T_\varphi$. The rotation $\varphi_a$ induces a Koopman operator $T_{\varphi_a}$ on the vector-valued functions in $L^1(G, m; L^1(L, \nu))$. With the $\pi$-tensor product (see \cite[Théorème 2]{Gro52}), we obtain

$$
L^1(G, m; L^1(L, \nu)) \cong L^1(G, m) \otimes L^1(L, \nu) \cong L^1(G \times L, m \times \nu).
$$

Now, the diagram

$$
\begin{array}{ccc}
L^1(X; \mathbb{C}) & \xrightarrow{S} & L^1(G \times L, m \times \nu) \cong L^1(G, m; L^1(L, \nu)) \\
\downarrow T_\varphi & & \downarrow T_{\varphi_a \times \text{id}_L} \\
L^1(X; \mathbb{C}) & \xrightarrow{S} & L^1(G \times L, m \times \nu) \cong L^1(G, m; L^1(L, \nu))
\end{array}
$$

also commutes, i.e., $T_\varphi$ acts like an ergodic rotation on vector-valued functions. We can interpret the topological Halmos-von Neumann theorem Theorem 3.1 and Corollary 3.7 analogously.

4. Realization and Uniqueness

The topological Halmos-von Neumann theorem shows that every minimal dynamical system with discrete spectrum is isomorphic to a minimal group rotation $(G, a)$. Therefore, minimal group rotations can be seen as the canonical representatives of minimal systems with discrete spectrum. Moreover, the Pontryagin duality theorem shows that $(G, a)$ and $(G^*, \delta_a)$ are isomorphic which has two consequences: On the one hand, $G^* \cong G^*(a)$ via $\chi \mapsto \chi(a)$ and $G^*(a) = \sigma_p(T_{\varphi_a})$ where $T_{\varphi_a}$ denotes the Koopman operator of $\varphi_a$, see \cite[Propositions 14.22 and 14.24]{EFHN13}. In particular, $\sigma_p(T_{\varphi_a})$ is a subgroup of $\mathbb{T}$ and for the canonical inclusion $\iota: \sigma_p(T_{\varphi_a}) \hookrightarrow \mathbb{T}$

$$(G, a) \cong (G^*(a)^*, \iota) = (\sigma_p(T_{\varphi_a})^*, \iota)$$

if $\sigma_p(T_\varphi)$ is endowed with the discrete topology. Therefore, the point spectrum $\sigma_p(T_{\varphi_a})$ is a complete isomorphism invariant for the minimal group rotation $(G, a)$. Combined with the Halmos-von Neumann theorem, this shows that the point spectrum $\sigma_p(T_\varphi)$ is a complete isomorphism invariant for all minimal topological dynamical systems $(K, \varphi)$ with discrete spectrum. On the other hand, the Pontryagin duality theorem also implies that every subgroup of $\mathbb{T}$ can be realized as $\sigma_p(T_{\varphi_a})$ for some group rotation $(G, a)$. This completes the picture, showing that minimal systems with discrete spectrum are, up to isomorphism, in one-to-one correspondence with subgroups of $\mathbb{T}$. 
In order to generalize these results to the non-minimal setting, we need to adapt the Pontryagin duality theorem to group rotation bundles using the preparations from [Section 2.3] We start with the necessary terminology.

**Construction 4.1** (Dual bundles). If \((\mathcal{G}, L, q; \alpha)\) is a compact group rotation bundle with minimal fibers and discrete spectrum, the map
\[
\rho: E(\mathcal{G}, \varphi_\alpha) \times L \to \mathcal{G}, \quad (\psi, l) \mapsto \psi(e_l)
\]
yields a surjective morphism \((\rho, \text{id}_L)\) of group bundles which induces, by [Proposition 2.28], an embedding \(\rho^*: \mathcal{G}^* \to E(\mathcal{G}, \varphi_\alpha)^* \times L\). Since \(E(\mathcal{G}, \varphi_\alpha)\) is compact, its dual group is discrete and so we also have the embedding
\[
j: E(\mathcal{G}, \varphi_\alpha)^* \times L \to \mathbb{T} \times L, \quad (\chi, l) \mapsto (\chi(\varphi_\alpha), l)
\]
where \(\mathbb{T}\) carries the discrete topology. The composition \(\iota: \mathcal{G}^* \to \mathbb{T} \times L\) of these two maps is hence a subtrivialization of \(\mathcal{G}^*\) and we call \((\mathcal{G}^*, L, \pi_L; \iota)\) the dual bundle of \((\mathcal{G}, L, q; \alpha)\). (Note that \(\mathcal{G}^*\) is, in general, neither locally compact nor Hausdorff.) If, conversely, \((\mathcal{G}, L, q; \iota)\) is a group bundle with a \(\mathbb{T}\)-subtrivialization \(\iota\), we set \(\alpha_\iota: L \to \mathcal{G}^*, l \mapsto \iota_l\) and call \((\mathcal{G}^*, L, \pi_L; \alpha_\iota)\) the dual bundle of \((\mathcal{G}, L, q; \iota)\). We say that two group bundles with \(\mathbb{T}\)-subtrivializations \((\mathcal{G}, L, q; \iota)\) and \((\mathcal{G}', L', q'; \iota')\) are isomorphic if their respective subtrivializations are, i.e., if there is an isomorphism \((\Theta, \vartheta): (\mathcal{G}, L, q) \to (\mathcal{G}', L', q')\) such that the diagram
\[
\begin{array}{ccc}
\mathbb{T} \times L & \xrightarrow{\iota(z,l) \mapsto (z,\vartheta(l))} & \mathbb{T} \times L' \\
\downarrow \iota & & \downarrow \iota' \\
\mathcal{G}' & \xrightarrow{\Theta} & \mathcal{G}'
\end{array}
\]
commutes.

**Definition 4.2.** Let \((K, \varphi)\) be a topological dynamical system and \(q: K \to L\) the projection onto its maximal trivial factor \(L\). Then we define
\[
\Sigma_p(K, \varphi) := \bigcup_{l \in L} \sigma_p(T_{\varphi_l}) \times \{l\} \subseteq \mathbb{C} \times L.
\]
We denote the projection onto the second component by \(\pi_L\) and equip \(\Sigma_p(K, \varphi)\) with the subspace topology induced by \(\mathbb{C} \times L\) if \(\mathbb{C}\) carries the discrete topology. The bundle \((\Sigma_p(K, \varphi), L, \pi_L)\) is then called the point spectrum bundle of \((K, \varphi)\). We say that the point spectrum bundles of two systems are isomorphic if there is an isomorphism of their canonical subtrivializations. We say that the point spectrum bundles of two systems \((K, \varphi)\) and \((M, \psi)\) are isomorphic if there is a homeomorphism \(\eta: L_\varphi \to L_\psi\) such that
\[
H: \Sigma_p(K, \varphi) \to \Sigma_p(M, \psi), \quad (z, l) \mapsto (z, \eta(l))
\]
is a (well-defined) homeomorphism and call \((H, \eta)\) an isomorphism of the point spectrum bundles.

**Remark 4.3.** Let \((\mathcal{G}, L, \pi_L; \iota)\) be a group bundle with a \(\mathbb{T}\)-subtrivialization \(\iota: \mathcal{G} \to \mathbb{T} \times L\). Then \(\iota\) induces an isomorphism
\[
(\mathcal{G}, L, \pi_L; \iota) \cong (\iota(\mathcal{G}), L, \pi_L; \text{id}_L(\mathcal{G}))
\]
and hence
\((\mathcal{G}, L, \pi_L; \iota)^* \cong (\iota(\mathcal{G})^*, L, \pi_L; (\text{id}_{\iota(\mathcal{G})_l})_{l \in L}).\)

In particular, \(\mathcal{G}\) and hence its dual are completely determined by \(\iota(\mathcal{G})\). Now, if \((\mathcal{G}, L, \pi_L; \iota)\) is the dual of a compact group rotation bundle \((\mathcal{H}, L, \rho; \alpha)\) with minimal fibers and discrete spectrum, it follows from the introduction to this section that
\[(\iota(\mathcal{G})^*, L, \pi_L; (\text{id}_{\iota(\mathcal{G})_l})_{l \in L}) = (\mathcal{H}, L, \rho; (\text{id}_{\iota(\mathcal{H}_l)})_{l \in L}).\]

So we see that the dual bundle of a group rotation bundle with discrete spectrum and minimal fibers is completely determined by its point spectrum bundle.

**Lemma 4.4.** Let \((K, \varphi)\) be a topological dynamical system with discrete spectrum. Then its point spectrum bundle is lower-semicontinuous.

**Proof.** Suppose \((\lambda, l) \in \Sigma_p(T_\varphi)\) and let \(f \in C(K_l)\) be a corresponding eigenfunction. Since \(T_\varphi\) has discrete spectrum, \(\lambda T_\varphi\) is mean ergodic and since all the eigenvalues of \(T_\varphi\) are simple, \(\dim \text{fix}(\lambda T_\varphi) = 1\). So as in the proof of [Theorem 2.15](#), \(f\) can be extended to a global fixed function \(\tilde{f} \in C(K)\) of \(\lambda T_\varphi\). In particular, there is an open set \(U \subseteq L\) such that for each \(l \in U\), \(f_l \neq 0\) and \(T_\varphi(f_l) = \lambda f_l\). \(\square\)

**Proposition 4.5.** Let \((\mathcal{G}, L, q; \alpha)\) be a compact group rotation bundle with discrete spectrum and minimal fibers. Then it is isomorphic to its bi-dual bundle.

**Proof.** The following diagram commutes:

\[
\begin{array}{ccc}
(E(\mathcal{G}, \varphi_\alpha) \times L, \varphi_\alpha) & \xrightarrow{(\psi, l) \mapsto (\delta_{\psi}, l)} & (E(\mathcal{G}, \varphi_\alpha)^* \times L, \delta_{\varphi_\alpha}) \\
\downarrow \rho & & \downarrow \rho^* \\
(\mathcal{G}, \alpha) & \xrightarrow{g \mapsto \delta_g} & (\mathcal{G}^*, \delta_\alpha)
\end{array}
\]

Since \(\rho\) is a surjective, continuous map between compact spaces, \(\mathcal{G}\) carries the final topology with respect to \(\rho\). This shows that the map \(g \mapsto \delta_g\) is continuous and bijective. Combining [Remark 4.3](#) and [Lemma 4.4](#), we see that \((\mathcal{G}, L, q)^*\) is lower-semicontinuous and [Proposition 2.28](#) shows that \(\mathcal{G}^*\) embeds into \(E(\mathcal{G}, \varphi_\alpha) \times L\) and is therefore locally compact. By [Proposition 2.28](#) \(\mathcal{G}^{**}\) is thus Hausdorff. Since the map \(g \mapsto \delta_g\) is bijective, this implies \(\mathcal{G} \cong \mathcal{G}^{**}\) and the claim follows. \(\square\)

Here is now our final answer to the three aspects of the isomorphism problem presented in the introduction.

**Theorem 4.6.** Let \((K, \varphi)\) and \((M, \psi)\) be topological dynamical systems with discrete spectrum and continuous sections of the canonical projections onto their respective maximal trivial factor.

(a) (Representation) The system \((K, \varphi)\) is isomorphic to a compact group rotation bundle with minimal fibers.
(b) (Uniqueness) The systems $(K, \varphi)$ and $(M, \psi)$ are isomorphic if and only if their point spectrum bundles are.

(c) (Realization) The point spectrum bundle of $(K, \varphi)$ is lower-semicontinuous and if $L$ is any compact space, every lower-semicontinuous sub-group bundle of $(T \times L, L, \pi_L)$ can be realized as the point spectrum bundle of a topological dynamical system with discrete spectrum in the sense that the corresponding canonical subtrivializations are isomorphic.

Proof. The representation result is Theorem 3.6. Moreover, Remark 4.3 and Proposition 4.5 show that the point spectrum bundle is a complete isomorphism invariant for compact group rotation bundles with minimal fibers and discrete spectrum and the representation theorem allows to extend this to $(K, \varphi)$ and $(M, \psi)$. The last part follows, analogously to the minimal case, from Proposition 2.28(iv), Proposition 4.5 and Remark 4.3. \hfill \qed

Remark 4.7. Note that the statement of Theorem 4.6 is false if the continuous section assumption is removed. Indeed, one obtains a counterexample from Example 2.19.

In order to obtain a similar result for measure-preserving systems, we first need to define their point spectrum bundles. To motivate the definition, note that one could use the ergodic decomposition to do this for separable systems. However, to treat non-separable systems, we base our definition on topological models.

Definition 4.8. Let $(X, \varphi)$ be a measure-preserving dynamical system and take $(K, \mu; \psi)$ to be a topological model corresponding to the algebra

$$A := \text{Kro}L_\infty(T_\varphi) = \text{cl}_{L_\infty} \bigcup_{|\lambda|=1} \ker L_\infty(\lambda I - T_\varphi) \subseteq L_\infty(X).$$

Let $(\Sigma_p(X, \varphi), L, p)$ be the point spectrum bundle of $(K, \varphi)$ and set $\nu := p_* \mu_K$. We then call $(\Sigma(X, \varphi), L, p, \nu)$ the point spectrum bundle of $(X, \varphi)$. We say that the point spectrum bundles of two systems $(X, \varphi)$ and $(Y, \psi)$ are isomorphic if there is an isomorphism $(\Theta, \vartheta): (\Sigma_p(X, \varphi), L, p) \to (\Sigma_p(Y, \psi), L', p')$ such that $\vartheta$ is measure-preserving.

Remark 4.9. Let $(K, \varphi)$ be a topological dynamical system, $\mu$ a regular Borel measure on $K$, $q: K \to L$ the canonical projection onto the maximal trivial factor of $(K, \varphi)$ and $\nu := q_* \mu$. If the canonical map $j: \text{Kro}C(K)(T_\varphi) \to \text{Kro}L_\infty(K, \mu)(T_\varphi)$ is an isomorphism, then $\Sigma_p(K, \varphi) = \Sigma_p(K, \mu; \varphi)$. This is in particular the case for the group rotation bundles constructed in Theorem 3.10.

Recall that a regular Borel measure $\mu$ on a (hyper)stonean space $K$ is called normal if all rare sets are null-sets. If $\mu$ is a normal measure on $K$ with full support, then the canonical embedding $C(K) \to L_\infty(K, \mu)$ is an isomorphism, cf. [Tak79, Corollary III.1.16]. After this reminder, we can state the analogue of Theorem 4.6 for measure-preserving systems.

Theorem 4.10. Let $(X, \varphi)$ and $(Y, \psi)$ be measure-preserving dynamical systems with discrete spectrum.
(a) (Representation) The system \((X, \varphi)\) is Markov-isomorphic to a rotation \((G, \mu_G; \varphi_\alpha)\) on a compact group rotation bundle with minimal fibers.

(b) (Uniqueness) The systems \((X, \varphi)\) and \((Y, \psi)\) are Markov-isomorphic if and only if their point spectrum bundles are isomorphic.

(c) (Realization) The point spectrum bundle of \((X, \varphi)\) is continuous. Conversely, if \((L, \nu)\) is a hyperstonean compact probability space such that \(\nu\) is normal and \(\text{supp} \, \nu = L\) and \((\Sigma, L, p)\) is a continuous subgroup bundle of \((\mathbb{T} \times L, L, p)\) then \((\Sigma, L, p; \nu)\) can be realized as the point spectrum bundle of a measure-preserving dynamical system with discrete spectrum.

Proof. The representation result was proved in Theorem 3.10. Using it, the uniqueness can be reduced to the case of the special group rotation bundles from Theorem 3.10 and for these, it follows from Remark 4.9. Indeed, let \((G, \mu_G; \varphi_\alpha)\) and \((\mathcal{H}, \mu_\mathcal{H}; \varphi_\beta)\) be two such rotations and

\[
(\Theta, \vartheta): (\Sigma_p (G, \mu_G; \varphi_\alpha), L, p, \nu) \to (\Sigma_p (\mathcal{H}, \mu_\mathcal{H}; \varphi_\beta), M, q, \eta)
\]

an isomorphism of their point spectrum bundles. Then Remark 4.9 shows that \((\Theta, \vartheta)\) is, in particular, an isomorphism of their topological point spectrum bundles and thus induces a (topological) isomorphism \((\Theta^*, \vartheta^{-1})\) of the corresponding dual bundles. By Proposition 4.5 this yields an isomorphism

\[
(\Psi, \vartheta^{-1}): (\mathcal{H}, M, q; \varphi_\beta) \to (G, L, p; \varphi_\alpha).
\]

Using the disintegration formula from Remark 2.23 one quickly checks that \(\Psi_* \mu_\mathcal{H} = \mu_G\) because \(\vartheta^{-1} \eta = \nu\).

For part (c), let \((L, \nu)\) be a hyperstonean compact probability space such that \(\nu\) is normal and \(\text{supp} \, \nu = L\) and let \((\Sigma, L, p)\) be a continuous subgroup bundle of \((\mathbb{T} \times L, L, p)\). Let \((G, L, \pi_L, \varphi_\alpha)\) be its dual group rotation bundle endowed with the measure

\[
\mu_G := \int_L d\eta \, d\nu.
\]

To prove that \(\Sigma = \Sigma_p (G, \mu_G, \varphi_\alpha)\), it suffices to show that each eigenfunction \(f \in L^\infty(G, \mu_G)\) has a representative \(g \in C(G)\) since then

\[
\Sigma = \Sigma_p (G, \varphi_\alpha) = \Sigma_p (G, \mu_G, \varphi_\alpha)
\]

by Proposition 4.3 and Remark 4.9. So take \([f] \in L^\infty(G, \mu_G)\) with \(T_\nu [f] = \lambda [f]\). Then \(f_l = \lambda^l f_l\) for \(\nu\)-almost all \(l \in L\). Let \(U_\lambda \subseteq L\) be the open subset of \(l \in L\) such that \((\lambda, l) \in \Sigma\) and note that \(U_\lambda\) is also closed since \((\Sigma, L, p)\) is upper-semicontinuous.

Since \(G^*\) is isomorphic to the point spectrum bundle \(\Sigma_p (G, \varphi_\alpha)\) via an isomorphism \(\Phi\) by Proposition 4.5 the map \(\eta: U \to G^*, \lambda \mapsto \Phi^{-1}(\lambda, l)\) is continuous. Extend \(\eta\) to all of \(L\) by setting \(\eta(l)\) to the trivial character in \(G^*_l\) for \(l \in L \setminus U\) and note that \(\eta\) is continuous since \(U\) is open and closed.

Now, for \(l \in U\), each fiber \((G_l, \varphi_\alpha, l)\) of \((G, \varphi_\alpha)\) is a minimal group rotation and so the eigenspace of the Koopman operator \(T_{\varphi_\alpha, l}\) corresponding to \(\lambda\) is at most one-dimensional and therefore spanned by \(\eta(l) \in G^*_l\). So for \(\nu\)-almost
every \( l \in U \), there is a constant \( c_l \in \mathbb{C} \) such that \( f_l = c_l q(l) \) \( \mu_l \)-almost everywhere. If we extend \( c \) to \( L \) by 0, \( c \in L^\infty(L, \nu) \) since \( [f] \) is in \( L^\infty(\mathcal{G}, \mu) \).

But \( C(L) \cong L^\infty(L, \nu) \) via the canonical embedding and so we may assume that \( c \) is continuous. If \( g: \mathcal{G} \to L \) is the projection onto \( L \), using \([1]\) of Proposition 2.28 we see that the function \( \hat{f}: \mathcal{G} \to \mathbb{C}, x \mapsto e_q(x) q(x) \in \mathcal{G} \) and that \( f = \hat{f} \mu\)-almost everywhere by construction.

Now let \((X, \varphi)\) be a measure-preserving dynamical system with discrete spectrum. In order to show that its point spectrum bundle is upper-semicontinuous, we may switch to its representation \((\mathcal{G}, \mu_\mathcal{G}, \varphi_\alpha)\) on a compact group rotation bundle \((\mathcal{G}, L, p, \varphi_\alpha)\) constructed in Theorem 3.10. Take \( \lambda \in \mathbb{T} \). By Remark 4.9 and Lemma 4.4, the set

\[
U := \{ l \in L \mid (\lambda, l) \in \Sigma_p(\mathcal{G}, \mu_\mathcal{G}, \varphi_\alpha) \}
\]

is open. Via the isomorphism \( \Theta: \Sigma_p(\mathcal{G}, \varphi_\alpha) \cong \mathcal{G}^* \), we see that the function \( F: U \to \mathcal{G}^*, l \mapsto \Theta(\lambda, l) \) selecting the (unique) character on \( \mathcal{G} \) corresponding to the eigenvalue \( \lambda \) is continuous. By \([1]\) of Proposition 2.28 \( F \) defines a continuous function \( f: p^{-1}(U) \to \mathbb{C} \) and we may extend \( f \) to a measurable function on all of \( \mathcal{G} \) by 0. Then \( T_\varphi f = \lambda f \) and since the \( \mathcal{G}(\mu) \) and \( L^\infty(\mathcal{G}, \mu_\mathcal{G}) \)-Kronecker space for \( T_\varphi \) are canonically isomorphic, we can find a continuous representative \( g \in \mathcal{G}(\mathcal{G}) \) for \( f \).

Consider the following canonical isomorphisms:

\[
T_p: C(L) \to \text{fix}_{\mathcal{G}(\mathcal{G})}(T_{\varphi_\alpha}) \to \mathcal{G}(\mathcal{G}) \\
T_p: L^\infty(L, \nu) \to \text{fix}_{L^\infty}(T_{\varphi_\alpha}) \to L^\infty(\mathcal{G}, \mu_\mathcal{G})
\]

Since \( f \) is an eigenfunction \([f] \in \text{fix}_{L^\infty}(\mathcal{G}, \mu_\mathcal{G}) \) and in fact, \([f] = 1_{p^{-1}(U)}\). Therefore, \( T_p^{-1}([f]) = [1_U] \). Moreover, \( g \) is also an eigenfunction and so \([g] \in \text{fix}_{\mathcal{G}(\mathcal{G})}(T_{\varphi_\alpha}) \) and \( T_p^{-1}([g]) \in C(L) \).

But \([f] = [g] \mu\)-almost everywhere and hence \( T_p^{-1}([g]) = 1_U \) \( \nu\)-almost everywhere. But since \( C(L) \cong L^\infty(L, \nu) \) every equivalence class in \( L^\infty(L, \nu) \) contains precisely one continuous function, implying \( T_p^{-1}([g]) = 1_U \). In particular, \( g \) is an eigenfunction on \( \mathcal{G} \) satisfying \( g_l \neq 0 \) for each \( l \in \mathcal{U} \). Therefore, \( \mathcal{U} \subseteq U \) and hence \( U = \mathcal{U} \). This shows that the point spectrum bundle of \((\mathcal{X}, \varphi)\) is upper-semicontinuous.

\[\square\]

**Remark 4.11.** To conclude, let us briefly discuss how the different statements \([a]\) and \([b]\) in Theorem 4.10 can be improved in the special case that \( \mathcal{X} \) is a standard probability space:

(a) (Representation) It is not difficult to see that if the measure space \( \mathcal{X} \) is separable, the group rotation bundle can be chosen to be metrizable: Going back to the proof of Theorem 3.10 the algebra \( \mathcal{A} \) needs to be replaced by a separable subalgebra \( \mathcal{B} \) which is still dense in \( L^1(X) \). Using that \( T_\varphi \) is mean ergodic on \( \mathcal{A} \) and that there hence is a projection \( P: \mathcal{A} \to \text{fix}_\mathcal{A}(T_\varphi) \), this can be done in such a way that \( \text{fix}_\mathcal{A}(T_\varphi) \) is generated by its characteristic functions. Therefore, its Gelfand representation space is totally disconnected and using
Proposition 2.7 instead of Theorem 2.8, one can continue the proof of Theorem 3.10 analogously. By von Neumann’s theorem \cite{EFHN15, Theorem 7.20}, \((X, \varphi)\) is then not only Markov-isomorphic but point-isomorphic to the rotation on a compact group rotation bundle.

(b) (Uniqueness) By von Neumann’s theorem, Markov-isomorphy can be replaced by point-isomorphy.

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