Abstract. Let \( p \equiv 1 \pmod{9} \) be a prime number and \( \zeta_3 \) be a primitive cube root of unity. Then \( k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3) \) is a pure metacyclic field with group \( \text{Gal}(k/\mathbb{Q}) \cong S_3 \). In the case that \( k \) possesses a 3-class group \( C_{k,3} \) of type \( (9, 3) \), the capitulation of 3-ideal classes of \( k \) in its unramified cyclic cubic extensions is determined, and conclusions concerning the maximal unramified pro-3-extension \( k_3^{(\infty)} \) of \( k \) are drawn.

1. Introduction

Let \( k \) be a number field, and \( L \) be an unramified abelian extension of \( k \). We say that an ideal \( I \) of \( k \) or its class capitulates in \( L \) if \( I \) becomes principal by extending it to an ideal of \( L \). Generally, it is difficult to determine the set of ideals of \( k \) which capitulate in \( L \). Among the first mathematicians who were concerned with this kind of problem is Kronecker, who in 1882 studied the capitulation problem for imaginary quadratic fields. In 1902, Hilbert [12] conjectured that “any ideal of a number field \( k \) capitulates in the Hilbert class field \( k^{(1)} \) of \( k \)”. This conjecture was demonstrated in 1930 by Furtwängler [7] after Artin had reduced it to a problem of group theory. Hilbert’s famous Theorem 94 states that “if \( L/k \) is an unramified cyclic extension of prime degree \( p \), then there exists an ideal of \( k \) of order \( p \) which capitulates in \( L \)”.

The number of ideal classes of \( k \) that capitulate in such an extension \( L \) is equal to \( [L : k][E_k : \mathcal{N}_{L/k}(E_L)] \), where \( E_k \) and \( E_L \) are respectively the groups of the units of \( k \) and \( L \), and \( \mathcal{N}_{L/k} \) is the relative norm of the extension \( L/k \). Around 1971, F. Terada, in his famous Tannaka-Terada theorem [24] asserted that “if \( k/k_0 \) is a cyclic extension, then all the ambiguous classes of \( k \), relative to \( k_0 \), capitulate in the genus field of \( k \)”. A generalization of Hilbert’s Theorem 94 has been conjectured by Miyake [20] as follows: “the degree of an unramified abelian extension \( L/k \) divides the number of classes of \( k \) that capitulate in \( L \)”. This conjecture was proved by Suzuki [21] in 1991, who also succeeded, after 7 years of research, in proving a theorem [22] which generalizes all previous results as follows: “if \( k/k_0 \) is a cyclic extension, and \( L \) is an unramified extension
of $k$ abelian over $k_0$, the degree $[L : k]$ divides the number of classes of $k$ invariant under $\text{Gal}(k/k_0)$.

Several mathematicians were interested in the study of the problem of capitulation for particular cases of number fields. We cite, for example, the study of capitulation in the intermediate extensions of $k^{(1)}/k$. When the $p$-class group of $k$ is of type $(p, p)$, this study was made by S. M. Chang and R. Foote. For other studies we cite the works [4], [5], [14], [15]. Here we contribute to this study with results on the problem of capitulation of the 3-ideal classes of the field $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ in the intermediate extensions of $k^{(1)}/k$, where $p$ is a prime such that $p \equiv 1 \pmod{9}$ and $k^{(1)}_3$ denotes the Hilbert 3-class field of $k$.

When the 3-class group $C_{k,3}$ is of type $(9, 3)$, the extension $k^{(1)}_3/k$ admits eight intermediate fields. Figure 1 schematizes the situation.

Figure 1. The unramified cubic and nonic sub-extensions of $k^{(1)}_3/k$

In section 2, we propose to determine the family $(K_{i,j})$ of all intermediate sub-fields of $k \subseteq K_{i,j} \subseteq k^{(1)}_3$, where $1 \leq i \leq 4$ and $j \in \{3, 9\}$, the object of Theorem 2.5, and we then study, in Theorem 3.1 of section 3, the capitulation of 3-ideal classes of $k$ in the fields $K_{i,j}$.

Section 4 is devoted to the identification of the maximal unramified pro-3-extension $k^{(\infty)}_3$ of $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, and to the proof that the dominating proportion (at least 94%) of the fields $k$ has a metabelian 3-class field tower $k^{(\infty)}_3$ with exactly two stages.

All our theoretical results are based on exhaustive computer results. The usual notation is as follows:
• The letter \( p \) designates a prime number congruent to 1 modulo 3;
• \( \Gamma = \mathbb{Q}(\sqrt[3]{d}) \): a pure cubic field, where \( d \geq 2 \) is a cube-free integer;
• \( k_0 = \mathbb{Q}(\zeta_3) \): the cyclotomic field, where \( \zeta_3 = e^{2\pi i/3} \);
• \( k = \Gamma(\zeta_3) \): the normal closure of \( \Gamma \);
• \( \Gamma' \) and \( \Gamma'' \): the two conjugate cubic fields of \( \Gamma \), contained in \( k \);
• \( u = [E_k : E_0] \): the index of the sub-group \( E_0 \) generated by the units of intermediate fields of the extension \( k/\mathbb{Q} \) in the group of units \( E_k \) of \( k \);
• \( \langle \tau \rangle = \text{Gal} (k/\Gamma) \), \( \tau^2 = id \), \( \tau(\zeta_3) = \zeta_3^2 \) and \( \tau(\sqrt[3]{d}) = \sqrt[3]{d} \);
• \( \langle \sigma \rangle = \text{Gal} (k/k_0) \), \( \sigma^3 = id \), \( \sigma(\zeta_3) = \zeta_3 \) and \( \sigma(\sqrt[3]{d}) = \zeta_3 \sqrt[3]{d} \);
• For an algebraic number field \( L \):
  - \( C_{L,3} \): the 3-class group of \( L \);
  - \( L_3^{(1)} \): the Hilbert 3-class field of \( L \);
  - \( [I] \): the class of a fractional ideal \( I \) in the class group of \( L \).

2. Unramified cubic and nonic sub-extensions of \( k_3^{(1)}/k \)

2.1. Preliminaries.

In his thesis [14], M. C. Ismaili established that the 3-class group \( C_{k,3} \) of \( k = \mathbb{Q}(\sqrt[3]{d}, \zeta_3) \) is of type \((3,3)\) if and only if 3 divides exactly the class number of \( \Gamma \) and \( u = 3 \), where \( u \) is the units index defined in the notations, and he determined all the integers \( d \) which satisfy this property by distinguishing three types of fields \( k \), namely type I, II, and III.

Here, we are interested in the case where the 3-class group \( C_{k,3} \) is of type \((3,9)\). Assuming that the number of classes of \( \Gamma \) is divisible exactly by 9, Theorem 2.1 describes the structure of \( C_{k,3} \) as follows:

**Theorem 2.1.** Let \( \Gamma \) be a pure cubic field, \( k \) be its normal closure, \( C_{k,3} \) (resp. \( C_{\Gamma,3} \)) the 3-class group of \( k \) (resp. \( \Gamma \)), and \( u \) be the units index defined in the notations, then:

\[
C_{k,3} \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \iff [C_{\Gamma,3} \cong \mathbb{Z}/9\mathbb{Z} \text{ and } u = 1].
\]

**Proof.** See [1, Lemma 2.5, p. 4]. \( \square \)

Further, we have classified all the integers \( d \) for which the 3-class group \( C_{k,3} \) is of type \((3,9)\) in the following Theorem:

**Theorem 2.2.** Let \( \Gamma = \mathbb{Q}(\sqrt[3]{d}) \) be a pure cubic field, where \( d \geq 2 \) is a cube free integer, and let \( k = \mathbb{Q}(\sqrt[3]{d}, \zeta_3) \) be its normal closure. Denote by \( u \) the index of the subgroup generated by the units of the intermediate fields of the extension \( k/\mathbb{Q} \) in the unit group of \( k \).

1) If the field \( k \) has a 3-class group of type \((9,3)\), then \( d = p^e \), where \( p \) is a prime congruent to 1 (mod 9) and \( e = 1 \) or 2.

2) Conversely, if \( p \) is a prime congruent to 1 (mod 9), and if 9 divides exactly the class number of \( \Gamma = \mathbb{Q}(\sqrt[3]{p}) \) and \( u = 1 \), then the 3-class group of \( k \) is of type \((3,9)\).

**Proof.** See [1, Theorem 1.1, p. 2]. \( \square \)
Theorem 2.3. Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, where $p$ is a prime such that $p \equiv 1 \pmod{9}$, $\Gamma = \mathbb{Q}(\sqrt[3]{p})$. The converse of the Calegari-Emerton result is shown by Frank Gerth III in [10] Thm. 1, p. 471. Assume that 9 divides exactly the class number of $\Gamma = \mathbb{Q}(\sqrt[3]{p})$, where $p \equiv 1 \pmod{9}$, then $C_{k,3}$ is of type (9, 3) if and only if $u = 1$. The generators of $C_{k,3}$ when $C_{k,3}$ is of type (9, 3) are determined as follows:

Theorem 2.3. Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, where $p$ is a prime such that $p \equiv 1 \pmod{9}$, $\Gamma = \mathbb{Q}(\sqrt[3]{p})$, $k_0 = \mathbb{Q}(\zeta_3)$, $k_{3,1}^{(1)}$ the 3-Hilbert class field of $k$, $C_{k,3}$ the 3-class group of $k$, we note also $C_{k,3}^+ = \{C \in C_{k,3} | C = C \}$, and $C_{k,3}^- = \{C \in C_{k,3} | C = C^{-1}\}$, where $\langle \tau \rangle = \text{Gal}(k/\Gamma)$.

Suppose that $C_{k,3}$ is of type (9, 3). Let $\langle A \rangle = C_{k,3}^+$, where $A \in C_{k,3}$ such that $A^0 = 1$ and $A^3 \neq 1$, $C_{k,3}^- = \langle B \rangle$, where $B \in C_{k,3}$ such that $B^3 = 1$ and $B \neq 1$, $C_{k,3}^{(\sigma)}$ is the ambiguous ideal class group of $k|k_0$, and $C_{k,3}^{1-\sigma} = \{A^{1-\sigma} | A \in C_{k,3}\}$ is the principal genus of $C_{k,3}$, where $\langle \sigma \rangle = \text{Gal}(k/k_0)$. Then:

1. $C_{k,3}^+ = C_{k,3}^- \times C_{k,3}^- = \langle A, B \rangle$.
2. The ambiguous class group $C_{k,3}^{(\sigma)}$ is a sub-group of $C_{k,3}^+$ of order 3, with $A \notin C_{k,3}^{(\sigma)}$.
3. $C_{k,3}^- = \langle (A^2)^{\sigma-1} \rangle$.
4. The principal genus $C_{k,3}^{1-\sigma} = C_{k,3}^{(\sigma)} \times C_{k,3}^- = \langle A^3, B \rangle$ is an elementary bi-cyclic 3-group of type (3, 3).

Proof. See [2] Proposition 3.4, pp. 9-10].

Theorem 2.4. Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, where $p$ is a prime such that $p \equiv 1 \pmod{9}$. The prime 3 decomposes in $k$ under form $3Q = P^2Q^3R^2$, where $P$, $Q$, and $R$ are prime ideals of $k$. Put $h = \frac{\text{class number of } k}{\text{class number of } k_0}$, where $h_k$ is the class number of $k$. Assume 9 divides exactly the class number of $\mathbb{Q}(\sqrt[3]{p})$ and $u = 1$. If 3 is not cubic residue modulo $p$, then:

1. the class $[R^h]$ generates $C_{k,3}$;
2. $C_{k,3}$ is generated by classes $[R^h]$ and $[R^h][P^2]$, and we have:

$$C_{k,3} = \langle [R^h] \rangle \times \langle [R^h][P^2]^2 \rangle$$

Proof. See [2] Theorem 3.5, pp. 10-11].
Figure 2. The group $G$ with $G/G'$ of type $(9, 3)$
1) Four unramified cyclic extensions of degree 3 denoted $K_{i,3}$, $1 \leq i \leq 4$, given by:
   - The field $K_{1,3}$ corresponds by class field theory to $C_{k,3}^{+} = \langle A \rangle$,
   - The field $K_{2,3}$ corresponds to $\langle AB \rangle = \langle A^{\sigma} \rangle$,
   - The field $K_{3,3}$ corresponds to $\langle AB^{2} \rangle = \langle A^{\sigma^{2}} \rangle$,
   - The field $K_{4,3}$ corresponds to the principal genus $C_{k,3}^{1-\sigma} = \langle A^{3}, B \rangle$.

   Furthermore, $K_{4,3} = k(\sqrt[3]{\pi_{1}^{2}}) = (k/k_{0})^{*} = k\Gamma^{*} = k(\Gamma^{\sigma})^{*} = k(\Gamma^{\sigma^{2}})^{*}$, where $(k/k_{0})^{*}$ is the relative genus field of $k/k_{0}$, $\langle \sigma \rangle = \text{Gal}(k/k_{0})$, $F^{*}$ for a number field $F$ is the absolute genus field of $F$, and $\pi_{1}, \pi_{2}$ are primes in $k_{0}$ such that $p = \pi_{1}\pi_{2}$.

2) Three unramified cyclic extensions of degree 9 denoted $K_{1,9}$, $2 \leq i \leq 4$, given by:
   - The field $K_{2,9}$ corresponds by class field theory to the sub-group $\langle A^{3}B \rangle$,
   - The field $K_{3,9}$ corresponds to the sub-group $\langle A^{3}B^{2} \rangle$,
   - The field $K_{1,9}$ corresponds to the sub-group $C_{k,3} = \langle B \rangle$.

   Furthermore, $K_{1,9} = k \cdot \Gamma_{3}^{(1)} = k \cdot (\Gamma^{\sigma})_{3}^{(1)} = k \cdot (\Gamma^{\sigma^{2}})_{3}^{(1)}$, where $F_{3}^{(1)}$ for a number field $F$ is the 3-Hilbert class field of $F$.

3) One bi-cyclic bi-cubic extension of degree 9, denoted $K_{4,9}$, and given by $K_{4,9} = K_{i,3} \cdot K_{j,3}$, $i \neq j$, which corresponds by class field theory to the ambiguous ideal class group $C_{k,3}^{(\sigma \tau)} = \langle A^{3} \rangle$ of the extension $k/k_{0}$.

Proof. Assume $C_{k,3}$ is of type $(3, 9)$. Let $\langle A \rangle = C_{k,3}^{+}$, where $A \in C_{k,3}$ such that $A^{9} = 1$ and $A^{3} \neq 1$, and let $C_{k,3}^{-} = \langle B \rangle$, where $B \in C_{k,3}$ such that $B^{3} = 1$ and $B \neq 1$.

The results of Theorem 2.3 follow immediately, according to the class field theory, from the fact that the 3-ideal class group $C_{k,3} = \langle A, B \rangle$ admits

- Four sub-groups $H_{i,3}$ cubic of order 9, where $1 \leq i \leq 4$, ordered as follows:
  - three of these subgroups are cyclic of order 9, given by:
    - $H_{1,3} = C_{k,3}^{+} = \langle A \rangle = \{ C \in C_{k,3} \mid C^{r} = C \}$,
    - $H_{2,3} = \langle AB \rangle = \langle A^{\sigma} \rangle = \{ C \in C_{k,3} \mid C^{r\sigma} = C \}$,
    - $H_{3,3} = \langle AB^{-1} \rangle = \langle AB^{2} \rangle = \langle A^{\sigma^{2}} \rangle = \{ C \in C_{k,3} \mid C^{-r\sigma^{2}} = C \}$,

Then, we have:

\[
\begin{align*}
C_{k,3}/H_{1,3} &= C_{k,3}/C_{k,3}^{+} = C_{k,3}/\langle A \rangle \simeq \mathbb{Z}/3\mathbb{Z}, \\
C_{k,3}/H_{2,3} &= C_{k,3}/\langle A^{\sigma} \rangle \simeq \mathbb{Z}/3\mathbb{Z}, \\
C_{k,3}/H_{3,3} &= C_{k,3}/\langle A^{\sigma^{2}} \rangle \simeq \mathbb{Z}/3\mathbb{Z},
\end{align*}
\]

then $\text{Gal}(K_{1,3}/k) \simeq \text{Gal}(K_{2,3}/k) \simeq \text{Gal}(K_{3,3}/k) \simeq \mathbb{Z}/3\mathbb{Z}$, which means that $K_{1,3}$, $K_{2,3}$, $K_{3,3}$ are unramified cyclic extensions of degree 3 on $k$ corresponds respectively to the subgroups $H_{1,3} = \langle A \rangle$, $H_{2,3} = \langle A^{\sigma} \rangle$, $H_{3,3} = \langle A^{\sigma^{2}} \rangle$ of $C_{k,3}$,

- the fourth subgroup of $C_{k,3}$ is exactly the principal genus $C_{k,3}^{1-\sigma}$, given by

\[
H_{4,3} = C_{k,3}^{1-\sigma} = C_{k,3}^{(\sigma)} \times C_{k,3}^{-} = \langle A^{3}, B \rangle = \prod_{i=1}^{4} H_{i,9},
\]
then

\[ C_{k,3}/H_{4,3} = C_{k,3}/C_{k,3}^{1-\sigma} \simeq \mathbb{Z}/3\mathbb{Z}, \]

and according to genus theory

\[ C_{k,3}/C_{k,3}^{1-\sigma} \simeq \text{Gal}((k/k_0)^*/k), \]

which is exactly the genus group, for more details see [8 §2, page 85]. Then, \( \text{Gal}(K_{4,3}/k) \simeq \text{Gal}((k/k_0)^*/k) \simeq \mathbb{Z}/3\mathbb{Z} \), which means that \( K_{4,3} = (k/k_0)^* \) is an unramified cyclic extension of degree 3 over \( k \) correspond to the sub-group \( H_{4,3} = \langle A^3, B \rangle \) of \( C_{k,3} \). For more details, see Theorem [2.3].

Furthermore, as the discriminant of the ring of integers of \( \Gamma \) is divisible by a single prime number \( p \) such that \( p \equiv 1 \pmod{3} \), then by [14 Corollary 2.1, p. 21], we have:

\[
\Gamma^* = M(p).\Gamma, \\
(\Gamma^\sigma)^* = M(p).\Gamma^\sigma, \\
(\Gamma^{\sigma^2})^* = M(p).\Gamma^{\sigma^2},
\]

where \( \Gamma^* \) (respectively \( (\Gamma^\sigma)^*, (\Gamma^{\sigma^2})^* \)) is the absolute genus field of \( \Gamma \) (respectively \( \Gamma^\sigma, \Gamma^{\sigma^2} \)), and \( M(p) \) is the unique cubic sub-field \( \mathbb{Q}(\zeta_p) \) of degree 3.

By switching to the composition we obtain

\[ k.\Gamma^* = k.(\Gamma^\sigma)^* = k.(\Gamma^{\sigma^2})^* = k.M(p). \]

The fact that \( p \equiv 1 \pmod{3} \) imply by [13 Chap. 9, Sec. 1, Proposition 9.1.4, p.110] that \( p = \pi_1\pi_2 \) with \( \pi_1 = \pi_2 \) and \( \pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_{k_0}} \), then by [9 § 3, Lemma 3.2, p. 56], we have \( (k/k_0)^* = k(\sqrt[3]{\pi_1\pi_2^2}). \)

We conclude that \( K_{4,3} = (k/k_0)^* = k\Gamma^* = k(\sqrt[3]{\pi_1\pi_2^2}). \)

- Four cyclic cubic subgroups \( H_{i,9} \) of order 3, where \( 1 \leq i \leq 4 \), ordered as follows:

  - \( H_{1,9} = C_{k,3}^{-} = \langle B \rangle = \{ C \in C_{k,3} \mid C^r = C^{-1} \} \),
  - \( H_{2,9} = \langle A^3B \rangle \),
  - \( H_{3,9} = \langle A^3B^{-1} \rangle = \langle A^3B^2 \rangle \),
  - \( H_{4,9} = C_{k,3}^{(\sigma)} = \langle A^3 \rangle \) is the ambiguous ideal class group of \( k/k_0 \).

Furthermore, \( H_{4,9} = \bigcap_{i=1}^{4} H_{i,9} \), and for each \( 1 \leq i \leq 3 \) we have \( H_{i,9} \) is contained only in \( H_{4,3} \).

On the one hand, we have

\[ C_{k,3}/H_{1,9} = C_{k,3}/C_{k,3}^{(\sigma)} = C_{k,3}/\langle A^3 \rangle \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \]

then \( \text{Gal}(K_{4,9}/k) \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \), which signifies that \( K_{4,9} \) is an unramified bicyclic bi-cubic extension of \( k \) corresponding to the sub-group \( H_{1,9} = C_{k,3}^{(\sigma)} \) of \( C_{k,3} \).

Also, we have

\[ C_{k,3}/H_{1,9} = C_{k,3}/C_{k,3}^{-} = C_{k,3}/\langle B \rangle \simeq \mathbb{Z}/9\mathbb{Z}, \]
Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, where $p$ is a prime number such that $p \equiv 1 \pmod{9}$, and $C_{k,3}$ the 3-class group of $k$. Suppose that $C_{k,3}$ is of type $(9,3)$. Let $(K_{i,j})$ be the family of all intermediate sub-fields of $k \subset K_{i,j} \subset k_{3}^{(1)}$, where $1 \leq i \leq 4$ and $j \in \{3, 9\}$. We denote by $\ker(T_{K_{i,j}/k})$ the kernel of the homomorphism $T_{K_{i,j}/k} : C_{k,3} \to C_{K_{i,j},3}$ induced by extension of ideals of $k$ in $K_{i,j}$, where $K_{i,j}$ is an unramified extension of $k$ included in $k_{3}^{(1)}$. We denote by $\kappa$ the quartet of Taussky’s conditions \[23].

**Definition 3.1.** Let $A_{i,j}$ be a generator of the sub-group $H_{i,j}$ of $C_{k,3}$, with $1 \leq i \leq 4$, $j \in \{3, 9\}$ corresponding to the field $K_{i,j}$. Let $l_i \in \{0, 1, 2, 3, 4\}$ with $1 \leq i \leq 4$.

We will say that the capitulation is of type $\{l_1, l_2, l_3, l_4\}$ to express the fact that when $l_i = n$ for one $n \in \{1, 2, 3, 4\}$, then only the class $A_{n,9}$ and its powers capitulate in $K_{i,3}$. If all classes capitulate in $K_{i,3}$ then we put $l_i = 0$. 
The main result of this paper is as follows:

**Theorem 3.1.** Let \( k = \mathbb{Q}(\sqrt[3]{3}, \zeta_3) \), where \( p \) is a prime number such that \( p \equiv 1 \) (mod 9), and \( C_{k,3} \) is the 3-class group of \( k \). Suppose that \( C_{k,3} \) is of type (9, 3). Put \( \langle A \rangle = C_{k,3}^+ \), where \( A \in C_{k,3} \) such that \( A^3 = 1 \) and \( A^3 \neq 1 \). Then:

1. \( K_{1,3}^\sigma = K_{2,3}, \quad K_{2,3}^\sigma = K_{3,3} \) and \( K_{3,3}^\sigma = K_{1,3} \) (\( \sigma \) permutes \( K_{1,3}, K_{2,3} \) and \( K_{3,3} \)).
2. \( K_{1,3}^\tau = K_{1,3}, \quad K_{2,3}^\tau = K_{3,3} \) and \( K_{3,3}^\tau = K_{2,3} \).
3. \( K_{2,9}^\tau = K_{4,9}, \quad K_{3,9}^\tau = K_{3,9} \) and \( K_{7,9}^\tau = K_{2,9} \),
   for all continuations of the automorphisms \( \sigma \) and \( \tau \).

2. The three classes \( A, A^\sigma \) and \( A^{\sigma^2} \) do not capitulate in \( K_{1,3} \), for \( 1 \leq i \leq 4 \).
3. Exactly the class \( A^3 \) and its powers capitulate in \( K_{4,3} \), and \( \ker(T_{K_{4,3}/k}) = \langle A^3 \rangle \),
   where \( T_{K_{4,3}/k} : C_{k,3} \to C_{K_{4,3},3} \) is the homomorphism induced by extension of ideals from \( k \) to \( K_{4,3} \).
4. The fields \( K_{2,3} \) and \( K_{3,3} \) have the same order of the capitulation kernel.
5. The three classes \( A, A^\sigma \) and \( A^{\sigma^2} \) capitulate in \( K_{1,9} \).
6. The fields \( K_{2,9} \) and \( K_{3,9} \) have the same order of the capitulation kernel.
7. Possible types of capitulation in \( K_{i,3} \), \( 1 \leq i \leq 4 \), are \((4, 4, 4, 4), (1, 2, 3; 4) \) and \((0, 0, 0, 4) \). Possible Taussky types in \( K_{i,3} \), \( 1 \leq i \leq 4 \), are \((AAA; A) \) or \((BBB; A) \).

**Proof.** Let \( C_{k,3}^{(\sigma)} \) be the 3-ambiguous class group of \( k/k_0 \) and \( C_{k,3}^{1-\sigma} = \{ A^{1-\sigma} \mid A \in C_{k,3} \} \) the principal genus of \( C_{k,3} \).

By Theorem 2.3 we have \( C_{k,3}^{(\sigma)} = \langle A^3 \rangle = \langle B^{1-\sigma} \rangle \), and \( C_{k,3}^{1-\sigma} = C_{k,3}^- \times C_{k,3}^{(\sigma)} = \langle B, A^3 \rangle \) is a 3-group of \( C_{k,3} \) of type \((3, 3) \), where \( B \in C_{k,3} \) such that \( C_{k,3}^- = \langle B \rangle = \langle (A^3)^{\sigma^{-1}} \rangle \).

1. We will agree that for all \( i, 1 \leq i \leq 4 \), \( j = 3 \) or 9, and for all \( \omega \in \text{Gal}(k|\mathbb{Q}) \), \( H_{i,j}^\omega = \{ C^\omega | C \in H_{i,j} \} \).
   a. According to Theorem 2.3, \( H_{1,3} = C_{k,3}^+ = \langle A \rangle \), \( H_{2,3} = \{ C \in C_{k,3} | C^{\sigma^2} = C \} = \langle A^\sigma \rangle \), and \( H_{3,3} = \{ C \in C_{k,3} | C^{\sigma^2} = C \} = \langle A^{\sigma^2} \rangle \). Then, \( H_{1,3}^\sigma = H_{2,3}, \quad H_{2,3}^\sigma = H_{3,3} \) and \( H_{3,3}^\sigma = H_{1,3} \) (\( \sigma \) permutes \( H_{1,3}, H_{2,3} \) and \( H_{3,3} \)).
   b. As \( H_{1,3} = C_{k,3}^+ = \{ C \in C_{k,3} | C^\tau = C \} \), then \( H_{1,3}^\tau = H_{1,3} \). We have \( H_{2,3}^\tau = \langle (A^{\tau})^\tau \rangle = \langle A^{\tau^2} \rangle \), and since \( A^{\tau^2} = A^{\sigma^2} = \langle A^{\sigma^2} \rangle \), \( H_{2,3}^\tau = H_{3,3} = \langle (A^{\sigma^2})^\tau \rangle = \langle (A^{\sigma^2}) \rangle = \langle (A^{\sigma^2})^\tau \rangle = \langle (A^{\sigma^2}) \rangle = \langle (A^{\tau})^\sigma \rangle \).
   c. We reason as in (2). \( H_{1,9}^\tau = H_{1,9} \) because \( H_{1,9} = C_{k,3}^- = \{ C \in C_{k,3} | C^\sigma = C^{-1} \} \).
   We have \( H_{2,9} = \langle A^3 B \rangle \), and \( H_{3,9} = \langle A^3 B^2 \rangle \), and since \( A^\tau = A \) and \( B^\tau = B^{-1} = B^2 \), then \( H_{2,9}^\tau = H_{3,9} \) and \( H_{3,9}^\tau = H_{2,9} \).
   The relations between the fields \( K_{i,j} \) in (1) are nothing else than the translations of the corresponding relations for the sub-groups \( H_{i,j} \) via class field theory.

2. For each \( 1 \leq i \leq 4 \), \( K_{i,3} \) is an unramified cyclic extension of degree 3 over \( k \). It is clear that for each class \( C \in C_{k,3} \) we have \( C^3 = (N_{K_{i,3}/k} \circ T_{K_{i,3}/k})(C) \). If the class \( C \) capitulates in \( K_{i,3} \), then \( T_{K_{i,3}/k}(C) = 1 \) and \( C^3 = 1 \). We conclude that the ideal classes which capitulate in \( K_{i,3} \) are of order 3. Since the classes \( A, A^\sigma \), and \( A^{\sigma^2} \) are of order 9, then these classes cannot capitulate in \( K_{i,3} \).
(3) By Theorem 2.5 we have $K_{4,3} = (k/k_0)^*$ is the relative genus field of $k/k_0$, and by Theorem 2.3 we have $\langle A^3 \rangle = \langle A^{3\sigma} \rangle = \langle A^{3\sigma^2} \rangle = C_{k,3}^{(\sigma)}$. We conclude according to Tannaka-Terada theorem [24], that all ambiguous ideal classes of $C_{k,3}$ capitulate in the relative genus field $(k/k_0)^*$. Thus, the class $A^3$ and its powers capitulate in $K_{4,3}$. We shall prove that the unique classes which capitulate in $K_{4,3}$ are only the ambiguous ideal classes. We have $C_{k,3}^- = \langle B \rangle = \langle (A^2)^{σ−1} \rangle$ and $C_{k,3}^0 = \langle A^3 \rangle = \langle B^{1−σ} \rangle$. On the one hand we have

$$A^{1+2σ} = A^{σ(1−σ)} = ((A^{−1})^{σ−1})^{σ} = ((A^2)^{σ−1})^{σ} = B^σ$$

because $(A^3)^{σ−1} = 1$. One the other hand, we have

$$B^{1−σ}B^2 = B^{3−σ} = B^{−σ}$$

then

$$(B^{1−σ}B^2)^{-1} = B^σ$$

So we get

$$B^σ \in \langle B^{1−σ}B^2 \rangle = \langle A^3B^2 \rangle$$

because $\langle A^3 \rangle = \langle B^{1−σ} \rangle$. Since $C_{k,3} = \langle A, B \rangle$ is of type $(9,3)$, then a class $χ \in C_{k,3}$ of order 3 capitulates in the cubic cyclic unramified extension $K_{4,3}/k$, if and only if the $B$ capitulates in the extension $K_{4,3}/k$, because a class $χ$ of order 3 is in one of the subgroup $\langle B \rangle, \langle A^3B \rangle, \langle A^3B^2 \rangle$ and that $B^σ \in \langle A^3B^2 \rangle$. If $B$ capitulates in the extension $K_{4,3}/k$, then $B^σ$ capitulates also in $K_{4,3}/k$. Since $A^{1+2σ} = B^σ$, then $A^{1+2σ}$ capitulates also in $K_{4,3}/k$, so $T_{K_{4,3}/k}(A^{1+2σ}) = 1$. Then

$$(T_{K_{4,3}/k}(A^σ))^2 = T_{K_{4,3}/k}(A^{−1}) = T_{K_{4,3}/k}(A^2)$$

because $T_{K_{4,3}/k}(A^3) = 1$, so

$$(T_{K_{4,3}/k}(A^σ))^2 = (T_{K_{4,3}/k}(A))^2$$

and then

$$T_{K_{4,3}/k}(A^σ) = T_{K_{4,3}/k}(A)$$

so we get $(Γ_3^{(1)}) = (Γ_3^{(1)})$, where $Γ_3^{(1)}$ (resp. $(Γ_3')^{(1)} = (Γ_3'')^{(1)}$) is the 3-Hilbert class field of $Γ$ (resp. $Γ'$), which is a contradiction.

Thus $B$ does not capitulate in $K_{4,3}/k$, and then only $A^3$ and its powers capitulate in $K_{4,3}/k$.

(4) $K_{2,3}$ and $K_{3,3}$ have the same order of the capitulation kernel, because $K_{2,3}$ and $K_{3,3}$ are isomorphic by (1)(b).

(5) Let $I$ be an ideal of $Γ$ whose class generates $C_{Γ,3}$. We know then that we can take $A = [T_{k/Γ}(I)]$, that the class of $I^σ$ (in $Γ^σ = Γ'$) generates $C_{Γ',3}$, that the class of $I^{σ^2}$ (in $Γ^{σ^2} = Γ''$) generates $C_{Γ'',3}$, and that $A^σ = [T_{k/Γ}(I^σ)] = [T_{k/Γ'}(I^σ)]$ and $A^{σ^2} = [T_{k/Γ}(I^{σ^2})] = [T_{k/Γ''}(I^{σ^2})]$.

Since $(Γ_3^{(1)})$ (resp. $(Γ_3')^{(1)}$ and $(Γ_3'')^{(1)}$) is the 3-Hilbert class field of $Γ$ (resp. $Γ'$ and $Γ''$), then $I$ (resp. $I^σ$ and $I^{σ^2}$) becomes principal in $Γ_3^{(1)}$ (resp. $(Γ_3')^{(1)}$ and $(Γ_3'')^{(1)}$).
Thus, when \( I \) (resp. \( I^\sigma \) and \( I^{\sigma^2} \)) is considered as an ideal of \( k.G_3^{(1)} \) (resp. \( k.(G_3')^{(1)} \) and \( k.(G''_3)^{(1)} \)), \( I \) (resp. \( I^\sigma \) and \( I^{\sigma^2} \)) becomes principal in \( k.G_3^{(1)} \) (resp. \( k.(G_3')^{(1)} \) and \( k.(G''_3)^{(1)} \)). So \( A \) (resp. \( A^\sigma \) and \( A^{\sigma^2} \)) capitulates in \( k.G_3^{(1)} \) (resp. \( k.(G_3')^{(1)} \) and \( k.(G''_3)^{(1)} \)). Since \( k.G_3^{(1)} = k.(G_3')^{(1)} = k.(G''_3)^{(1)} \) = \( K_{1,9} \), then the classes \( A, A^\sigma \) and \( A^{\sigma^2} \) capitulate in \( K_{1,9} \).

(6) This assertion follows from the fact that \( K_{2,9} \) and \( K_{3,9} \) are isomorphic by (1)(c).

(7) The possible types of capitulation in \( K_{i,3}, 1 \leq i \leq 4 \) are \( (4, 4, 4; 4) \), \( (1, 2, 3; 4) \) and \( (0, 0, 0; 4) \), and the possible Taussky types are (AAA;A) or (BBB;A). In fact, since exactly the class \( A^3 \) and its powers capitulate in \( K_{4,3} \), then:

(i) for each \( j \in \{1, 2, 3\} \), if all ideal classes \( C \) of \( C_{k,3} \) of order 3 capitulate in \( K_{j,3}/k \), then the type of capitulation is \( (0, 0, 0; 4) \), and the Taussky type is (AAA;A).

(ii) If exactly the class \( C \) of \( C_{k,3} \) of order 3 capitulates in the extension \( K_{1,3}/k \), then exactly one class of \( C_{k,3} \) of order 3 and its powers capitulate in the extensions \( K_{2,3}/k \) and \( K_{3,3}/k \), because by (1)(a) we have \( K_{1,3}^\sigma = K_{2,3}, K_{2,3}^\sigma = K_{3,3} \), and \( K_{3,3}^\sigma = K_{1,3} \). Then, there are two cases:

- If exactly the class \( A^3 \) capitulates in \( K_{1,3}/k \), then exactly the class \( (A^3)^\sigma \) capitulates in \( K_{2,3}/k \) and exactly the class \( (A^3)^{\sigma^2} \) capitulates in \( K_{3,3}/k \). In this case, the possible type of capitulation is \( (4, 4, 4; 4) \), and the Taussky type is (AAA;A).

- We have \( C_{k,3}^{-} = \langle B \rangle \), and \( C_{k,3}^{(\sigma)} = \langle A^3 \rangle = \langle B^{1-\sigma} \rangle \) by Theorem 2.3. Then

\[
\begin{align*}
B^{1-\sigma}B^2 &= B^{3-\sigma} \\
&= B^{-\sigma},
\end{align*}
\]

that is

\[
(B^{1-\sigma}B^2)^{-1} = B^\sigma.
\]

We get

\[
B^\sigma \in \langle B^{1-\sigma}B^2 \rangle = \langle A^3B^2 \rangle,
\]

because \( \langle A^3 \rangle = \langle B^{1-\sigma} \rangle \).

If exactly the class \( B \) capitulates in \( K_{1,3}/k \), then exactly the class \( B^\sigma \) capitulates in \( K_{2,3}/k \) and exactly the class \( B^{\sigma^2} \) capitulates in \( K_{3,3}/k \). Then, exactly the class \( A^3B^2 \) capitulates in \( K_{2,3}/k \) and exactly the class \( A^3B \) capitulates in \( K_{3,3}/k \).

In this case, the possible type of capitulation is \( (1, 2, 3; 4) \), and the Taussky type is (BBB;A).
4. 3-CLASS FIELD TOWER OF $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$

Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ be the normal closure of the pure cubic field $\Gamma = \mathbb{Q}(\sqrt[3]{p})$ with prime radicand $p \equiv 1 \pmod{9}$ of Dedekind’s second species. Then $k$ is a pure metacyclic field with absolute group $\text{Gal}(k/\mathbb{Q}) \simeq S_3$ the symmetric group of order six. Assume that $k$ possesses a 3-class group $C_{k,3} \simeq C_9 \times C_3$. Consequently, the 3-class group of $\Gamma$ is $C_{\Gamma,3} \simeq C_9$, according to Theorem 2.1 and $\Gamma$ is of principal factorization type $\alpha$, in the sense of [3].

In Theorem 3.1 we investigated the principalization of $k$ in its four unramified cyclic cubic extensions $K_{1,3}, \ldots, K_{4,3}$, i.e., we determined the three possibilities for the kernels $\ker(T_{K_{i,3}/k})$ of the transfer homomorphisms $T_{K_{i,3}/k} : C_{k,3} \to C_{K_{i,3},3}$, $aP_k \mapsto (aO_{K_{i,3}})P_{K_{i,3}}$.

Our numerical results for the 95 relevant cases $p < 20000$ in Table 1, computed with the aid of Magma [16], confirm the occurrence of precisely three situations for the punctured capitulation type $\varkappa(k) = \ker(T_{K_{i,3}/k})_{1 \leq i \leq 4}$, which we want to dub with succinct names in Definition 4.1. Recall that $H_{4,9} = \bigcap_{j=1}^{4} H_{j,3}$ in Figure 2 is the distinguished subgroup of $C_{k,3}$ which is generated by third powers of 3-ideal classes, i.e., the Frattini subgroup.

**Definition 4.1.** The punctured capitulation type, with puncture at the fourth component, for the subfield $K_{4,3}$ associated with the subgroup $H_{4,3} = \prod_{j=1}^{4} H_{j,3}$ in Figure 2 is called

1. **distinguished**, if $\varkappa(k) = (H_{4,9}, H_{4,9}, H_{4,9}; H_{4,9})$, briefly $(444; 4)$,
2. **harmonically balanced**, if $\varkappa(k) = (H_{1,9}, H_{2,9}, H_{3,9}; H_{4,9})$, briefly $(123; 4)$,
3. **total**, if $\varkappa(k) = (H_{1,3}, H_{4,3}, H_{4,3}; H_{4,9})$, briefly $(000; 4)$.

For the actual numerical determination of the (punctured) capitulation type $\varkappa$, we introduce the concept of Artin pattern of $k$.

**Definition 4.2.** Let $\tau(k) = \{\text{ATI}(C_{K_{i,3},3})\}_{1 \leq i \leq 4}$ be the family of abelian type invariants (ATI) (i.e., 3-primary type invariants) of the 3-class groups $C_{K_{i,3},3}$ of the four unramified cyclic cubic extensions of $k$. Then $\text{AP}(k) = (\varkappa(k), \tau(k))$ is called the Artin pattern of $k$.

It turns out that there is a bijective correspondence between $\varkappa$ and $\tau$ for the distinguished and total capitulation, whereas there are two variants of harmonically balanced capitulation.

Anyway, it is never required to perform the difficult computation of the capitulation type $\varkappa$. It is sufficient to determine the abelian type invariants $\tau$, which is computationally easier. Theorem 4.1 is a consequence of our results in Table 1.

**Theorem 4.1.** For a pure metacyclic field $k = \mathbb{Q}((\sqrt[3]{p}), \zeta_3)$ with prime radicand $p \equiv 1 \pmod{9}$, bounded by $p \leq 20000$, and 3-class group $C_{k,3} \simeq C_9 \times C_3$, the following statements determine $\varkappa(k)$ by means of $\tau(k)$:

1. $\varkappa(k) = (444; 4) \iff \tau(k) = [(9, 3)^3; (9, 3)].$
2. $\varkappa(k) = (000; 4) \iff \tau(k) = [(9, 3, 3)^3; (3, 3, 3, 3, 3)].$
3. $\varkappa(k) = (123; 4) \iff \tau(k) = \begin{cases} \text{either } [(27, 3)^3; (9, 3, 3)] & (1^{\text{st}} \text{variant}) \\ \text{or } [(27, 3)^3; (9, 9, 3)] & (2^{\text{nd}} \text{variant}). \end{cases}$

**Conjecture 4.1.** Theorem 4.1 is true for any prime $p \equiv 1 \pmod{9}$, not necessarily bounded from above by 20000.
We are now in the position to employ the strategy of pattern recognition via Artin transfers [19] in order to determine the 3-class field tower \( k_3^{(\infty)} \) of \( k \) by means of \( \text{AP}(k) = (\varpi(k), \tau(k)) \).

4.1. Relation rank and Galois action. Constraints arise from two issues, bounds for the relation rank of the tower group \( G = \text{Gal}(k_3^{(\infty)}/k) \), and the Galois action of \( \text{Gal}(k/Q) \) on \( C_{k,3} \simeq G/G' \). We denote by \( \langle o, i \rangle \) groups in the SmallGroups database of Magma [16].

**Theorem 4.2.** For any pure metacyclic field \( k = \mathbb{Q}(\zeta_3, \sqrt[d]{7}) \) with cube free radicand \( d \geq 2 \) and 3-class rank \( q = 2 \), the group \( G = \text{Gal}(k_3^{(\infty)}/k) \) of the 3-class field tower must satisfy the following conditions.

1. The relation rank \( d_2 \) of \( G \) must be bounded by \( 2 \leq d_2 \leq 5 \).
2. The automorphism group \( \text{Aut}(Q) \) of the Frattini quotient \( Q = G/\Phi(G) \) must contain a subgroup isomorphic to \( S_3 = \langle 6, 1 \rangle \). (This is true for any \( S_3 \)-field \( k \).)

**Proof.** According to the Burnside basis theorem, the generator rank \( d_1 \) of \( G \) coincides with the generator rank of the Frattini quotient \( Q = G/\Phi(G) = G/(G' \cdot G^3) \), resp. the derived quotient \( G/G' \simeq C_{k,3} \), that is the 3-class rank \( q \) of \( k \).

1. According to the Shafarevich Theorem [18, Thm. 5.1, p. 28], the relation rank \( d_2 \) of \( G \) is bounded by \( d_1 \leq d_2 \leq d_1 + r + \vartheta \), where the torsion free unit rank \( r = r_1 + r_2 - 1 \) of the totally complex field \( k \) with signature \( (r_1, r_2) = (0, 3) \) is \( r = 2 \), and \( \vartheta = 1 \), since \( k \) contains the primitive third roots of unity. Together with the generator rank \( d_1 = q = 2 \) this gives the bounds \( 2 \leq d_2 \leq 2 + 2 + 1 = 5 \). (For other complex, resp. real, \( S_3 \)-fields \( k \), the upper bound may be 4, resp. 7.)
2. The absolute Galois group \( \text{Gal}(k/Q) \simeq S_3 \) of \( k \) acts on the 3-class group \( C_{k,3} \simeq G/G' \) and thus also on the Frattini quotient \( Q = G/\Phi(G) = G/(G' \cdot G^3) \), whence \( \text{Aut}(Q) \) contains a subgroup isomorphic to \( S_3 = \langle 6, 1 \rangle \). \( \square \)

By the same proof as for item (2) of Theorem 4.2, with \( G/G' \simeq C_{k,3} \) replaced by \( G_n/G'_n \simeq \text{Gal}(k_3^{(n)}/k)/\text{Gal}(k_3^{(n)}/k_3^{(1)}) \simeq \text{Gal}(k_3^{(1)}/k) \simeq C_{k,3} \) we obtain:

**Corollary 4.1.** Let \( n \) be a positive integer, and denote by \( G_n = \text{Gal}(k_3^{(n)}/k) \) the Galois group of the \( n \)-th Hilbert 3-class field \( k_3^{(n)} \) of \( k \). The automorphism group \( \text{Aut}(Q) \) of the Frattini quotient \( Q = G_n/\Phi(G_n) \) must contain a subgroup isomorphic to \( S_3 = \langle 6, 1 \rangle \).

Furthermore, it will also be required to exploit data concerning the second layer of unramified abelian (three cyclic nonic and a single bicyclic bicubic) extensions.

**Definition 4.3.** Let \( \varpi_2(k) = (\text{ker}(T_{K_{1,9}/k}))_{1 \leq i \leq 4} \) be the punctured capitulation type, and \( \tau_2(k) = [\text{ATI}(C_{K_{1,9,3}})]_{1 \leq i \leq 4} \) be the family of abelian type invariants of the 3-class groups \( C_{K_{1,9,3}} \) of the four unramified abelian nonic extensions of \( k \), and \( \text{AP}_2(k) = (\varpi_2(k), \tau_2(k)) \).

According to item (5) of Theorem 3.1 we know that \( \text{ker}(T_{K_{1,9}/k}) = C_{k,3} \).
4.2. Distinguished capitulation.

**Proposition 4.1.** A power commutator presentation of the finite metabelian 3-group \(\langle 81, 4 \rangle\) with class 2 and coclass 2 in terms of the commutator \(s_2 = [y, x]\) is given by

\[
\langle x, y, s_2 \mid x^9 = 1, y^3 = s_2 \rangle
\]

**Proof.** Presentations of groups in the SmallGroups database are implemented in Magma [16].

**Theorem 4.3.** For a pure metacyclic field \(k = \mathbb{Q}(\zeta_3, \sqrt{p})\) with \(p \equiv 1 \pmod{9}\) having distinguished capitulation \(\varpi(k) = (444; 4)\), the Galois group \(G_2\) of the second Hilbert 3-class field \(k_3^{(2)}\) is unambiguously given by \(\text{Gal}(k_3^{(2)}/k) \simeq \langle 81, 4 \rangle\) with \(\varpi_2(k) = \mathbb{Z}_3 \times \mathbb{Z}_3\) and \(\tau_2(k) = [(9)^3; (3, 3)]\) (see Figure 3). The 3-class field tower of \(k\) must stop at the second stage, that is, \(k_3^{(2)} = k_3^{(\infty)}\) is the maximal unramified pro-3-extension of \(k\).

**Proof.** (Proof of Theorem 4.3) We use Theorem 4.1 in order to exploit the equivalence \(\varpi(P) \geq \varpi(D)\) and \(\tau(P) \leq \tau(D)\) of the components of the Artin pattern \((\varpi, \tau)\) with respect to (parent, descendant)-pairs \((P, D)\), where \(P\) is a quotient of \(D\), the abelian type invariants \(\tau = \mathbb{Z}_3 \times \mathbb{Z}_3\), which are common to the metabelian 3-groups \(\langle 81, 4 \rangle\) and \(\langle 243, 22 \rangle\), cannot occur for any other finite 3-group. Any other finite 3-group is descendant of the metabelian root \(R = \langle 81, 3 \rangle\) with pc-presentation \(\langle x, y, s_2 \mid x^9 = 1, y^3 = 1, s_2 = [y, x] \rangle\) and \(\tau(R) = \mathbb{Z}_3 \times \mathbb{Z}_3\). Consequently, at least one component of \(\tau(D)\) will always be of rank three, for any descendant \(D\) of the root \(R\). Furthermore, this argument also shows that there cannot be a non-metabelian 3-group \(G\) with second derived quotient \(G/G''\) isomorphic to either \(\langle 81, 4 \rangle\) or \(\langle 243, 22 \rangle\), since \(G\) would necessarily be required to have \(\tau(G) = \mathbb{Z}_3 \times \mathbb{Z}_3\), which is not compatible with being a descendant of \(R\). According to the Artin reciprocity law of class field theory, the 3-class field tower of \(k\) must therefore have precise length \(\ell_3(k) = 2\). Finally, both candidates for \(G_2 = G\) satisfy the inequalities \(2 \leq d_2 \leq 5\) for the relation rank in Theorem 4.2. Indeed, \(\langle 81, 4 \rangle\) has \(d_2 = 3\), and \(\langle 243, 22 \rangle\) has even the minimal value \(d_2 = 2\). However, for \(G = \langle 81, 4 \rangle\), the automorphism group \(\text{Aut}(Q)\) of the Frattini quotient \(Q = G/\Phi(G)\) contains a subgroup isomorphic to \(S_3 = \mathbb{S}_3\), whereas for \(G = \langle 243, 22 \rangle\), the corresponding \(\text{Aut}(Q)\) contains a subgroup \(C_2 = \mathbb{Z}_2\) only. Thus, \(G = \langle 81, 4 \rangle\) remains as unique candidate. See Figure 3.

We point out that the preceding proof is not really dependent on Theorem 4.1. A search for \(\varpi(k) = (444; 4)\) in the SmallGroups database yields \(\langle 81, 4 \rangle\), \(\langle 243, 22 \rangle\) and descendants of \(\langle 729, 10 \rangle\), \(\langle 729, 12 \rangle\). However, the latter two roots (and thus all of their descendants) do not have the required action by \(S_3\).

4.3. Harmonically balanced capitulation. We have seen that harmonically balanced capitulation \(\varpi = (123; 4)\) occurs in two variants with distinct fourth components \((9, 3, 3)\), resp. \((9, 9, 3)\), in the abelian type invariants \(\tau\). It turns out that the first variant leads to sporadic groups outside of coclass trees, and the second variant is connected with periodic groups on coclass trees.
Proposition 4.2. A power commutator presentation of the finite metabelian 3-group \((729, i)\) of class 3 in terms of the commutators \(s_2 = [y, x], s_3 = [s_2, x], t_3 = [s_2, y]\) is given by

\[
\begin{align*}
\{ & x, y, s_2, s_3, t_3 \mid x^9 = t_3, \quad y^3 = s_3 \} & \text{ if } i = 17, \\
\{ & x, y, s_2, s_3, t_3 \mid x^9 = t_3^2, \quad y^3 = s_3 \} & \text{ if } i = 20.
\end{align*}
\] (4.2)

Proof. Presentations of groups in the SmallGroups database are implemented in Magma [16]. The groups are sporadic of coclass 3. □

Theorem 4.4. For a pure metacyclic field \(k = \mathbb{Q}(\zeta_3, \sqrt[3]{p})\) with \(p \equiv 1 \pmod{9}\) having harmonically balanced capitulation \(\kappa(k) = (123; 4)\) and first variant of \(\tau = [(27, 3)^3; (9, 3, 3)]\), the sporadic Galois group \(G_2\) of \(k^{(2)}\), second Hilbert 3-class field, is given by \(\text{Gal}(k^{(2)}/k) \simeq \langle 729, \ell \rangle\) if \(\kappa(k) = ((C_{k,3})^3; H_{4,3}), \quad \tau_2(k) = [(9, 3)^3; (9, 3, 3)]\), (4.3)

\[
\begin{align*}
\{ & 2187, m \} & \text{ if } \kappa(k) = ((C_{k,3})^3; H_{4,3}), \quad \tau_2(k) = [(9, 9)^3; (9, 9, 3)]
\end{align*}
\]

where \(\ell \in \{17, 20\}, \quad m \in \{177, 178, 187, 188\}\) (see Figures 4 and 5).

Proof. (Proof of Theorem 4.4) All vertices of the entire descendant trees of the roots \(\langle 729, i \rangle\) with \(i \in \{17, 20\}\) share the required Artin pattern \((\kappa, \tau)\) with harmonically balanced capitulation \(\kappa = (123; 4)\) and the first variant of \(\tau = [(27, 3)^3; (9, 3, 3)]\). Since the trees are isomorphic as \textit{structured} graphs, we focus on \(\langle 729, 17 \rangle\), which gives rise to a finite “mainline”, standing out through an action by the direct product \(S_3 \times C_2 \simeq \langle 12, 4 \rangle\). The metabelian vertices of this finite mainline are \(\langle 729, 17 \rangle, \quad \langle 2187, 178 \rangle, \quad \langle 6561, 1733 \rangle, \quad \langle 6561, 1733 \rangle - \#1; 2\). The other two immediate descendants of the root \(\langle 729, 17 \rangle\) are \(\langle 2187, 1977 \rangle\) with action by \(S_3\) and \(\langle 2187, 179 \rangle\) with action by \(C_2\) only. There are exactly two further candidates for \(G_2\) with action by \(S_3\), namely the metabelian groups \(\langle 6561, 1731 \rangle\) and \(\langle 6561, 1733 \rangle - \#1; 3\). However, \(\langle 6561, n \rangle\) with \(n \in \{1731, 1733\}\) and \(\langle 6561, 1733 \rangle - \#1; s\) with \(s \in \{2, 3\}\) share the forbidden second layer \(\kappa_2 = (H_{1,3}, H_{2,3}, H_{3,3}; H_{4,3}), \quad \tau_2 = [(27, 9)^3; (9, 9, 3)]\). See Figures 4 and 5. □

Proposition 4.3. A power commutator presentation of the finite metabelian 3-group \((2187, i)\) in terms of the commutators \(s_2 = [y, x], s_3 = [s_2, x], s_4 = [s_3, x], t_3 = [s_2, y]\) is given by

\[
\begin{align*}
\{ & x, y, s_2, s_3, s_4, t_3 \mid x^9 = t_3, \quad y^3 = s_3, \quad s_2^3 = s_4^2 \} & \text{ if } i = 180, \\
\{ & x, y, s_2, s_3, s_4, t_3 \mid x^9 = t_3^2, \quad y^3 = s_3, \quad s_2^3 = s_4^2 \} & \text{ if } i = 190.
\end{align*}
\] (4.4)

Proof. Presentations of groups in the SmallGroups database are implemented in Magma [16]. The groups are periodic of class 4 and coclass 3. □

Theorem 4.5. For a pure metacyclic field \(k = \mathbb{Q}(\zeta_3, \sqrt[3]{p})\) with \(p \equiv 1 \pmod{9}\) having harmonically balanced capitulation \(\kappa(k) = (123; 4)\) and second variant of \(\tau = [(27, 3)^3; (9, 3, 3)]\), the periodic Galois group \(G_2\) of the second Hilbert 3-class field \(k^{(2)}\) is given by \(\text{Gal}(k^{(2)}/k) \simeq \langle 2187, m \rangle\) if \(\kappa(k) = ((C_{k,3})^3; H_{4,3}), \quad \tau_2(k) = [(9, 9)^3; (9, 9, 3)]\), (4.5)

\[
\begin{align*}
\{ & 2187, m \} & \text{ if } \kappa(k) = ((C_{k,3})^3; H_{4,3}), \quad \tau_2(k) = [(9, 9)^3; (9, 9, 3)]
\end{align*}
\]

where \(m \in \{180, 190\}\) and \(n \in \{1737, 1738, 1739, 1775, 1776, 1777\}\) (see Figures 6 and 7).
Proof. (Proof of Theorem 4.3) The required Artin pattern \((\kappa, \tau)\) with harmonically balanced capitulation \(\kappa = (123; 4)\) and second variant of \(\tau = [(27, 3)^3; (9, 9, 3)]\) cannot occur for descendants of the roots \(\langle 729, i \rangle\) with \(i \in \{17, 20\}\), because on the entire descendant trees of these sporadic roots \(\tau = [(27, 3)^3; (9, 9, 3)]\) remains stable.

The only possibility are vertices of the coclass trees with roots \(\langle 729, i \rangle\) for \(i \in \{18, 21\}\). Since the trees are isomorphic as structured graphs, we focus on \(\langle 729, 21 \rangle\), which has three immediate descendants, \(\langle 2187, 190 \rangle\) with \(\kappa = (123; 4)\), \(\tau = [(27, 3)^3; (9, 9, 3)]\), the mainline group \(\langle 2187, 191 \rangle\) with host type \(\kappa = (123; 0)\) like the parent \(\langle 729, 21 \rangle\), and \(\langle 2187, 192 \rangle\) with inadequate \(\kappa = (123; 2)\). Due to the antitony principle for the components of the Artin pattern \((\kappa, \tau)\), all descendants of \(\langle 2187, 191 \rangle\) can be eliminated, because they have \(\tau \geq [(27, 3)^3; (27, 9, 3)]\). The group \(\langle 2187, 190 \rangle\) has the required action by \(S_3 = \langle 6, 1 \rangle\), and this is also true for three of its immediate descendants \((6561, n)\) with \(1775 \leq n \leq 1777\) but not for \(n = 1778\) with action by \(C_3 = \langle 3, 1 \rangle\) only. Each of the three former has an immediate descendant \((6561, n) - \#1; 1\) with \(1775 \leq n \leq 1777\) and action by \(S_3\). The other descendant \((6561, n) - \#1; 1\) has action by \(C_3\), and three further descendants \((6561, n) - \#1; 1 - \#1; i\) with \(1 \leq i \leq 3\) have only an action by \(C_2 = \langle 2, 1 \rangle\). Further suitable candidates for \(G_2\) are impossible. Finally, the groups \((6561, n) - \#1; 1\) with \(n \in \{1775, 1776, 1777\}\) are discouraged by a wrong transfer kernel in the second layer with \(\kappa_2 = (H_{1,3}, H_{2,3}, H_{3,3}; H_{4,3}), \tau_2 = [(27, 27)^3; (9, 9, 9)]\). See Figures 6 and 7. \(\square\)

The proofs of the subsequent corollaries are based on the following fact. All the candidates for \(G_2\) in Theorem 4.4 and Theorem 4.5 satisfy the inequalities \(2 \leq d_2 \leq 5\) for the relation rank in Theorem 4.2, since they even satisfy the more severe estimates \(3 \leq d_2 \leq 4\). So there is no reason which precludes a metabelian tower with length \(\ell_3(k) = 2\).

**Corollary 4.2.** For the fields \(k\) with harmonically balanced capitulation \(\kappa = (123; 4)\) and first variant of \(\tau = [(27, 3)^3; (9, 9, 3)]\) (Theorem 4.4) the 3-class tower of \(k\) must stop at the second stage, that is, \(k_3^{(2)} = k_3^{(\infty)}\) is the maximal unramified pro-3-extension of \(k\).

**Proof.** (Proof of Corollary 4.2) The groups \(G_2\) in Theorem 4.4 are not second derived quotients \(G/G''\) of non-metabelian 3-groups \(G\). \(\square\)

**Remark 4.1.** We emphasize that the strict limitation \(\ell_3(k) = 2\) for the length of the 3-class field tower of \(k\) in Corollary 4.2 is only due to item (5) of Theorem 3.1, i.e. the requirement \(\ker(T_{K_{1,3}/k}) = C_{k,3}\) for the second layer.

Although we also had \(\ell_3(k) = 2\) if \(G_2 = \langle 6561, n \rangle, n \in \{1731, 1769\}\), were admissible, \(\ell_3(k) = 3\) would be enabled for \(r \in \{1733, 1771\}\), and \((6561, r) \simeq G/G''\) with \(G = \langle 6561, r \rangle - \#1; 4, \text{resp.} \langle 6561, r \rangle - \#1; s \simeq G/G''\) with \(G = \langle 2187, m \rangle - \#2; 1 - \#1; s, m \in \{178, 188\}, s \in \{2, 3\}\). The groups \(G\) are non-metabelian with action by \(S_3\), in contrast to \(\langle 6561, r \rangle\) with \(r \in \{1735, 1773\}\). On the other hand, it is known that the descendant trees of the roots \((729, i)\) with \(i \in \{17, 20\}\) contain vertices with unbounded derived length, whence any finite value \(\ell_3(k) \geq 4\) would also be possible. (Note that the tree continues at the non-metabelian vertex \(\langle 2187, m \rangle - \#2; 1 - \#1; 2\) with \(m = 178\), resp. \(m = 188\).)
Corollary 4.3. For the fields \( k \) with harmonically balanced capitulation \( \kappa = (123; 4) \) and second variant of \( \tau = [(27, 3)^3, (9, 9, 3)] \) (Theorem 4.5) the 3-class tower of \( k \) must stop at the second stage, that is, \( k_3^{(2)} = k_3^{(\infty)} \) is the maximal unramified pro-3-extension of \( k \).

Proof. (Proof of Corollary 4.3) According to the antity principle, there cannot exist non-metabelian 3-groups \( G \) whose metabelianization \( G/G'' \) is isomorphic to one of the 14 candidates for \( \text{Gal}(k_3^{(2)}/k) \) in Theorem 4.5. Therefore \( k_3^{(2)} = k_3^{(\infty)} \). ■

4.4. Total capitulation. Due to the wealth of metabelian groups \( M \) of low orders \( \#M \leq 3^8 \) in the descendant tree of the root \( \langle 729, 9 \rangle \), we restrict ourselves to immediate descendants of the root with action by \( S_3 \).

Proposition 4.4. A power commutator presentation of the finite metabelian 3-group \( \langle 729, 9 \rangle \) in terms of the commutators \( s_2 = [y, x], s_3 = [s_2, x], t_3 = [s_2, y] \) is given by

\[
\langle x, y, s_2, s_3, t_3 \mid x^9 = 1, y^3 = 1 \rangle.
\]

Proof. Presentations of groups in the SmallGroups database are implemented in Magma [16]. The group is periodic of class 3 and coclass 3. ■

Theorem 4.6. For a pure metacyclic field \( k = \mathbb{Q}(\zeta_3, \sqrt[p]{p}) \) with \( p \equiv 1 \pmod{9} \) having total capitulation \( \kappa(k) = (000; 4) \) and abelian type invariants \( \tau = [(9, 3, 3)^3; (3, 3, 3, 3)] \), the smallest possible Galois groups \( G_2 \) of the second Hilbert 3-class field \( k_3^{(2)} \) are given by

\[
\text{Gal}(k_3^{(2)}/k) \cong \begin{cases} 
\langle 729, 9 \rangle & \text{if } \tau_2(k) = [(3, 3, 3)^3; (3, 3, 3, 3)], \\
\langle 2187, 123 \rangle & \text{if } \tau_2(k) = [(3, 3, 3, 3)^3; (3, 3, 3, 3, 3)], \\
\langle 2187, 124 \rangle & \text{if } \tau_2(k) = [(9, 3, 3)^3; (3, 3, 3, 3)], \\
\langle 6561, i \rangle & \text{if } \tau_2(k) = [(3, 3, 3, 3)^3; (3, 3, 3, 3, 3)], \\
\langle 6561, 109 \rangle & \text{if } \tau_2(k) = [(3, 3, 3, 3)^3; (9, 3, 3, 3, 3)], \\
\langle 6561, j \rangle & \text{if } \tau_2(k) = [(9, 3, 3)^3; (9, 3, 3, 3, 3)],
\end{cases}
\]

with \( i \in \{103, 105\} \) and \( j \in \{110, 111\} \).

Corollary 4.4. For the fields \( k \) with total capitulation \( \kappa(k) = (000; 4) \) in Theorem 4.6, the length of the 3-class field tower \( k_3^{(\infty)} \) is given by

1. \( \ell_3(k) \geq 2 \), if \( G_2 \in \{\langle 729, 9 \rangle, \langle 2187, 123 \rangle, \langle 6561, 109 \rangle, \langle 6561, 110 \rangle, \langle 6561, 111 \rangle\} \),
2. \( \ell_3(k) \geq 3 \), if \( G_2 \in \{\langle 2187, 123 \rangle, \langle 6561, 109 \rangle, \langle 6561, 103 \rangle, \langle 6561, 105 \rangle\} \).

Proof. (Proof of Theorem 4.6 and Corollary 4.4) Since the root \( \langle 729, 9 \rangle \) has nuclear rank \( \nu = 3 \), it has descendants of step sizes \( s \in \{1, 2, 3\} \). The 37 children with \( s = 3 \) are of order \( 3^9 = 19683 \) and possess abelian quotient invariants beyond the threshold \( \tau \geq [(9, 9, 3)^3; (3, 3, 3, 3)] \). Among the 15, resp. 61, children with \( s = 1 \), resp. \( s = 2 \), and order \( 3^7 = 2187 \), resp. \( 3^8 = 6561 \), only the two, resp. five, mentioned possess an action by \( S_3 \). Concerning the length of 3-class field towers, the groups \( G_2 \) in item (1) of the corollary have relation ranks \( 4 \leq d_2 \leq 5 \), thus admitting a two-stage tower, whereas those in item (2) have \( 6 \leq d_2 \leq 7 \), which definitely excludes \( \ell_3(k) = 2 \). ■
The following Figures 3, 4, 5, 6 and 7 illustrate the location in descendant trees of all metabelian groups $M$ and certain non-metabelian groups $G$ which occur in section § 4.

5. Computational results

Table 1 has been computed with the aid of the computational algebra system MAGMA [16].

For each of the 95 pure metacyclic fields $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ with prime radicands $p \equiv 1 \pmod{9}$ in the range $0 < p < 20000$ and 3-class group of type $(9, 3)$, the capitulation kernels $\ker(T_{K_1,3/k})$ of the class extension homomorphisms $T_{K_1,3/k} : C_{k,3} \to C_{K_1,3,3}$ were computed and collected in the transfer kernel type $\kappa$.

An asterisk indicates the second variant of harmonically balanced capitulation $\kappa = (123; 4)$ with abelian type invariants $\tau = [(27, 3)^3; (9, 9, 3)]$.

Figure 3. Finite 3-groups $G$ with commutator quotient $G/G' \simeq C_9 \times C_3$
Figure 4. Descendant tree of \( \langle 729, 17 \rangle \) with stable \( \kappa = (123; 4), \tau = [(27, 3)^3; (9, 3, 3)] \)

![Descendant tree of \( \langle 729, 17 \rangle \) with stable \( \kappa = (123; 4), \tau = [(27, 3)^3; (9, 3, 3)] \)](image)

Legend: •... metabelian with action by \( S_3 \) □... other metabelian
○... other metabelian □... other non-metabelian

Figure 5. Descendant tree of \( \langle 729, 20 \rangle \) with stable \( \kappa = (132; 4), \tau = [(27, 3)^3; (9, 3, 3)] \)

![Descendant tree of \( \langle 729, 20 \rangle \) with stable \( \kappa = (132; 4), \tau = [(27, 3)^3; (9, 3, 3)] \)](image)

Legend: •... metabelian with action by \( S_3 \) □... non-metabelian with action by \( S_3 \)
○... other metabelian □... other non-metabelian
**Figure 6. Descendant tree of \langle 729, 18 \rangle**

Legend:
- \( \tau = [(27, 3), (27, 3), (27, 3); (9, 9, 3)] \)
- \( \kappa_0 = (132; 0) \)
- \( \kappa_1 = (132; 1) \)
- \( \kappa_2 = (132; 2) \)
- • ... metabelian with \( \kappa_4 \), \( \tau \) and action by \( S_3 \)
- ○ ... other metabelian
- □ ... non-metabelian

**Figure 7. Descendant tree of \langle 729, 21 \rangle**

Legend:
- \( \tau = [(27, 3), (27, 3), (27, 3); (9, 9, 3)] \)
- \( \kappa_0 = (123; 0) \)
- \( \kappa_1 = (123; 1) \)
- \( \kappa_2 = (123; 2) \)
- • ... metabelian with \( \kappa_4 \), \( \tau \) and action by \( S_3 \)
- ○ ... other metabelian
- □ ... non-metabelian
The following distribution of capitulation types \( \kappa \) arises in Table 1:

1. 61 (64\%) with \( \kappa = (444; 4) \) (distinguished capitulation),
2. 14 (15\%) with \( \kappa = (123; 4), \tau = [(27, 3), (27, 3), 9, 3, 3] \) (1\st\ variant),
3. 14 (15\%) with \( \kappa = (123; 4^*), \tau = [(27, 3), (27, 3), (27, 3); 9, 9, 3] \) (2\nd\ variant),
4. 6 (6\%) with \( \kappa = (000; 4) \) (total capitulation).

Table 1. Capitulation types \( \kappa \) of \( k = \mathbb{Q}(\sqrt[p]{p}, \zeta_3) \) in the range \( p < 20000 \)

| No. | \( p \) | \( \kappa \) | No. | \( p \) | \( \kappa \) | No. | \( p \) | \( \kappa \) | No. | \( p \) | \( \kappa \) |
|-----|--------|------|-----|--------|------|-----|--------|------|-----|--------|------|
| 1   | 199    | 4444 | 25  | 4951   | 4444 | 49  | 9829   | 4444 | 73  | 14293 | 4444 |
| 2   | 271    | 4444 | 26  | 5059   | 4444 | 50  | 10243 | 4444 | 74  | 14419 | 4444 |
| 3   | 487    | 4444 | 27  | 5077   | 1234*| 51  | 10459 | 0004 | 75  | 14563 | 4444 |
| 4   | 523    | 4444 | 28  | 5347   | 4444 | 52  | 10531 | 1234*| 76  | 14779 | 4444 |
| 5   | 1297   | 1234*| 29  | 5437   | 1234 | 53  | 10657 | 4444 | 77  | 14923 | 4444 |
| 6   | 1621   | 4444 | 30  | 5527   | 1234*| 54  | 10837 | 0004 | 78  | 15121 | 0004 |
| 7   | 1693   | 4444 | 31  | 5851   | 4444 | 55  | 10909 | 4444 | 79  | 15319 | 4444 |
| 8   | 1747   | 1234 | 32  | 6067   | 4444 | 56  | 11251 | 4444 | 80  | 15427 | 4444 |
| 9   | 1999   | 4444 | 33  | 6247   | 4444 | 57  | 11287 | 4444 | 81  | 16381 | 1234*|
| 10  | 2017   | 4444 | 34  | 6481   | 1234 | 58  | 11467 | 4444 | 82  | 16417 | 4444 |
| 11  | 2143   | 1234*| 35  | 6949   | 4444 | 59  | 11503 | 4444 | 83  | 16633 | 4444 |
| 12  | 2377   | 4444 | 36  | 7219   | 0004 | 60  | 11593 | 4444 | 84  | 16993 | 1234*|
| 13  | 2467   | 1234 | 37  | 7507   | 1234*| 61  | 11701 | 4444 | 85  | 17137 | 1234 |
| 14  | 2593   | 1234 | 38  | 7687   | 4444 | 62  | 11719 | 4444 | 86  | 17209 | 1234*|
| 15  | 2917   | 4444 | 39  | 8011   | 1234 | 63  | 12097 | 4444 | 87  | 17497 | 1234*|
| 16  | 3511   | 4444 | 40  | 8209   | 4444 | 64  | 12511 | 1234 | 88  | 17569 | 4444 |
| 17  | 3673   | 4444 | 41  | 8677   | 1234 | 65  | 12637 | 4444 | 89  | 18379 | 0004 |
| 18  | 3727   | 4444 | 42  | 8821   | 4444 | 66  | 12853 | 4444 | 90  | 18451 | 1234 |
| 19  | 3907   | 4444 | 43  | 9001   | 4444 | 67  | 12907 | 0004 | 91  | 18523 | 4444 |
| 20  | 4159   | 4444 | 44  | 9109   | 4444 | 68  | 13159 | 4444 | 92  | 18541 | 4444 |
| 21  | 4519   | 1234*| 45  | 9343   | 1234 | 69  | 13177 | 4444 | 93  | 19387 | 1234 |
| 22  | 4591   | 4444 | 46  | 9613   | 1234*| 70  | 13339 | 4444 | 94  | 19441 | 1234*|
| 23  | 4789   | 1234 | 47  | 9631   | 4444 | 71  | 13411 | 4444 | 95  | 19927 | 1234*|
| 24  | 4933   | 4444 | 48  | 9721   | 4444 | 72  | 13807 | 1234 |  |    |

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