Online Allocation and Pricing: Constant Regret via Bellman Inequalities

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We develop a framework for designing tractable heuristics for Markov Decision Processes (MDP), and use it to obtain constant regret policies for a variety of online allocation problems, including online packing, budget-constrained probing, dynamic pricing, and online contextual bandits with knapsacks. Our approach is based on adaptively constructing a benchmark for the value function, which we then use to select our actions. The centerpiece of our framework are the Bellman inequalities, which allow us to create benchmarks which both have access to future information, and also, can violate the one-step optimality equations (i.e., Bellman equations). The flexibility of balancing these allows us to get policies which are both tractable and have strong performance guarantees – in particular, our constant-regret policies only require solving an LP for selecting each action.

Key words: Stochastic Optimization, Approximate Dynamic Programming, Online Resource Allocation.

1. Introduction

Online decision-making under uncertainty is widely studied across a variety of fields, including operations research, control, and computer science. A canonical framework for such problems is that of Markov decision processes (MDP), with associated use of stochastic dynamic programming for designing policies. In complex settings, however, such approaches suffer from the known curse-of-dimensionality; moreover, they also fail to provide insights into structural properties of the problem: the performance of heuristics, dependence on distributional information, etc.

The above challenges have inspired an alternate approach to MDPs based on the use of benchmarks (variously known as offline/prophet/fluid relaxations). These are proxies for the value function that provide bounds for the optimal policy, and guide the design of heuristics. The performance of any policy can be quantified by its additive loss, or regret, relative to the benchmark; this consequently also bounds the additive optimality gap, i.e., performance against the optimal policy.

In this work, we develop new benchmark-driven policies for online resource allocation problems – settings where a finite set of resources are dynamically allocated to arriving requests, with associated constraints and rewards/costs. Our baseline setting is the online stochastic knapsack problem (henceforth OnlineKnapsack): a controller has an initial resource budget, and requests arrive sequentially over a horizon. Each arriving request has a random type corresponding to a resource requirement-reward pair. Item types are generated from some known stochastic process,
and are revealed upon arrival; the controller must then decide whether to accept/reject each request, in order to maximize collected rewards while satisfying budget constraints.

We consider three variants of this basic setting: (1) online probing, (2) online contextual bandits with knapsacks, and (3) dynamic pricing. The formal models for these settings are presented in Section 2; each setting augments the baseline \texttt{OnlineKnapsack} with additional constraints/controls, and the problems are widely-studied in their own right. Instead of solving each problem in an ad-hoc manner, however, our policies are derived from a single underlying framework. In particular, our results can be summarized as follows:

**Meta-theorem** Given an online resource allocation problem, we specify a simple online policy – based on solving a tractable optimization problem in each period – that achieves a constant regret compared to an offline benchmark; (and hence, bounded additive gap versus the optimal policy).

In more detail, our approach is based on adaptively constructing a benchmark that has privileged (but not necessarily full) information about future randomness. This can be viewed as an online primal-dual method, wherein our benchmark serves as a dual solution, which we then use to construct a feasible online policy. The centrepiece of our approach are the Bellman inequalities, which decompose the regret of an online policy into two distinct terms: The first, which we call the Bellman Loss, arises from computational considerations, specifically, from requiring that the benchmark is tractable (instead of a dynamic program, which is usually intractable); The second, which we call the Information Loss, accounts for the intrinsic randomness of the process. The flexibility of being able to balance between these losses allows us to get tractable policies with strong performance guarantees.

To understand why a flexible benchmark is important, consider two common benchmarks for the dynamic pricing problem, wherein each request has a random valuation for an item and we must post prices for our inventory. One common benchmark, known as the offline or prophet, is based on a controller that has full information of all the randomness; it is easy to show that no online policy can get better than \( \Omega(T) \) regret against this benchmark (where \( T \) is the decision horizon; see Example 1). An alternate benchmark, known as the ex ante or fluid, corresponds to replacing all random quantities with their expected values; again here, by the same argument as in \([2, 27]\), it can be shown that no online policy can get better than \( \Omega(\sqrt{T}) \) regret against this benchmark. Our approach allows us to build benchmarks which have \( O(1) \) regret for all our settings.

Prophet and fluid benchmarks are also widely used in adversarial models of online allocation, leading to algorithms with worst-case guarantees. In contrast, our work focuses on stochastic inputs, and consequently obtains much stronger guarantees. In particular, our guarantees are parametric, in the sense that they explicitly depend on the distribution and the primitives, i.e., constant parameters defining the instance.
2. Preliminaries and Overview

2.1. Problem Settings and Results

We illustrate our framework by developing low-regret algorithms for the following problems:

**Online Stochastic Knapsack.** This serves as a baseline for our other problems. The controller has an initial resource budget $B$, and items arrive sequentially over $T$ periods. Each item has a random type $j$ which corresponds to a known resource requirement (or ‘weight’) $w_j$ and a random reward $R_j$. In period $t = T, T-1, \ldots, 1$ (where $t$ denotes the periods to go), we assume the arriving type is drawn from a finite set $[n]$ from some known distribution $p[t] = (p_1[t], \ldots, p_n[t])$ is known. At the start of each period, the controller observes the type of the arriving item, and must decide to accept or reject the item. The expected reward from selecting a type-$j$ item is $E[R_j]$ so that maximizing the expected collected rewards is equivalent (in terms of optimal actions) to a setting where type-$j$ items bring a deterministic reward $r_j = E[R_j]$.

**Online Probing.** The controller – who knows the type of an arriving request, but not the realization of the reward $R_j$ – now has the option to probe each request to observe the realization, and then accept/reject the item based on the revealed reward. The controller can also choose to accept the item without probing; in this case, the decision must be based on the distribution of $R_j$ only. In addition to the resource budget $B$, the controller has an additional a probing budget $B_p$ that limits the number of arrivals the controller can probe. This introduces a trade-off between using the resource and probing budgets. We assume in this setting that $R_j$ has finite support $\{r_{jk}\}_{k \in [m]}$ of size $m$, and define $q_{jk} := P[R_j = r_{jk}]$ for $k \in [m]$.

**Dynamic Pricing.** The controller now has an inventory of $B$ identical copies of a product, and the arrival in each period $t$ corresponds to a customer who wants a single unit, and has a valuation $V_t$ drawn from some known distribution $F$. In period $t = T, T-1, \ldots, 1$, if there is inventory left, the controller first selects a price $f$ from a finite set of prices $\{f_1, \ldots, f_m\}$. The arriving customer then purchases iff $V_t > f$ (resulting in a reward of $f$), else leaves without purchasing.

**Online Contextual Bandits with Knapsacks.** We return to the OnlineKnapsack setting, with horizon $T$, capacity $B$, and finite item-types $[n]$ with item-type $j$ corresponding to weight $w_j$ and random reward $R_j$. Now however the controller is unaware of the distribution of $R_j$, and must learn $E[R_j]$ from observations. At every period $t$, the controller observes the type $j$ (i.e., context) of the arriving item and must decide to accept or reject the item based on the observations of rewards up to time $t$. We consider two feedback models: full feedback where the controller observes $R_j$ regardless of whether the item is accepted or rejected, and censored feedback where the controller only observes rewards of accepted items. For this setting, we also assume that the rewards $R_j$ have sub-Gaussian tails [6, Section 2.3].
Benchmarks and guarantees. The framework we develop, which we denote as RABBI (Re-solve and Act Based on the Bellman Inequalities, see Section 3.2) is based on comparing two ‘controllers’: Offline, who takes the optimal action given access to future information, and a non-anticipative controller Online who tries to follow Offline.

Both Online and Offline start in initial state $S^T$. We denote $v^{\text{off}}$ as the expected total reward collected by Offline acting optimally given its information structure; this is given by a Bellman equation that takes into consideration the extra information. In contrast, Online uses some non-anticipative policy $\pi$ that maps the current state to an action, resulting in a total expected reward $v^{\text{on}}_\pi$. We denote by $\pi_R$ the online policy produced by our RABBI framework. The expected regret relative to an offline benchmark is defined as

$$\mathbb{E}[\text{Regret}] := v^{\text{off}} - v^{\text{on}}_{\pi_R} \leq \max_{\pi} v^{\text{on}}_{\pi} - v^{\text{on}}_{\pi_R}$$

The last inequality, which follows from the fact that $v^{\text{on}}_{\pi} \leq v^{\text{off}}$ for any pair of benchmark and online policies, emphasizes that the regret is a bound on the additive gap w.r.t. the best online policy.

For all the above problems, we use the RABBI framework to obtain the following guarantees. For the OnlineKnapsack, our result is a re-statement of a theorem proved in [2, 27]; we recover it here to build intuition for our general framework.

**Theorem 1 (Baseline Setting [2, 27])**. For known reward distributions with finite mean, and compared to an offline benchmark with full knowledge of the types of future arrivals, RABBI obtains regret that depends only on the primitives $(n, p, r, w)$, but is independent of the horizon length $T$ and resource budget $B$.

In the case where rewards are deterministic, the above benchmark for OnlineKnapsack is the natural prophet benchmark. In the remaining settings, however, this turns out to be too loose to get constant regret policies. This is where our framework helps in guiding the choice of the right benchmark. In particular, in the remaining settings, we get the following results.

**Theorem 2 (Probing)**. For reward distributions with finite support of size $m$, compared to an offline benchmark which has full information of arriving types (but not the actual rewards), RABBI obtains regret that depends only on $(n, m, q, p, r)$, but is independent of the horizon length $T$, resource budget $B$ and probing budget $B_p$.

**Theorem 3 (Dynamic Pricing)**. For any reward distribution $F$ and prices $f$, compared to a benchmark which knows the number of future customers willing to buy at each price (but not the exact order of arrivals), RABBI obtains regret that depends only on $(m, f, F)$, but is independent of the horizon length $T$ and initial number of items $B$. 
For the bandit settings, we define a separation parameter $\delta = \min_{j \neq j'} |\mathbb{E}[R_j]/w_j - \mathbb{E}[R_{j'}]/w_{j'}|$; this is only for our bounds, and is not known to the algorithm.

**Theorem 4 (Contextual Bandits with Knapsacks).** Assuming the reward distributions are sub-Gaussian, compared to an offline benchmark which has full information of arriving types and the reward distributions (but not the actual rewards), in the full feedback setting, RABBI obtains regret that depends only on the primitives $(n, p, r, w, \delta)$ and is independent of the horizon length $T$ and knapsack capacity $B$.

The last result can also be used as a black-box for the censored feedback setting to get an $O(\log T)$ regret guarantee. The statement of the result (Corollary 1) requires some building blocks that we introduce in Section 6.4.

We provide explicit regret bounds for each of the above results in terms of the problem primitives. Note however that the algorithms in Theorems 2 to 4 enjoy constant-regret guarantees – in particular, they imply they are near optimal (up to additive constants) in any regime where $T, B$ scale, keeping other problem primitives fixed.

### 2.2. Overview of our Framework

We develop our framework in the full generality of MDPs in Section 3. To give an overview and gain insight into the general version, we first use the baseline OnlineKnapsack as a warm-up, using our framework to recover the results in [2, 27]. A schema for the framework is provided in Fig. 1.

In the OnlineKnapsack problem, at any time-to-go $t$, let $Z_j(t) \in \mathbb{N}$ denote the (random) number of type-$j$ arrivals in the remaining $t$ periods. Recall rewards of type-$j$ arrivals have expected value $r_j := \mathbb{E}[R_j]$. We define Offline to be a controller that knows $Z(T)$ in advance. The total reward collected by Offline can be written as an integer programming problem

$$V(t, b) = \max_{x_a, x_r \in \mathbb{N}} \{ r^t x : w^t x_a \leq b, x_a + x_r = Z(t) \} = \max_{x_a, x_r \in \mathbb{N}} \{ r^t x : w^t x_a \leq b, x_a + x_r = Z(t) \}.$$

For every $j$, the variables $x_{a,j}, x_{r,j}$ represent action summaries: the number of type-$j$ arrivals accepted and rejected, respectively. This function $V$ is Offline's value (see Fig. 1).

Offline's value and policy can also be represented via Bellman equations. Specifically, at time-to-go $t$, assuming the controller has budget $b$ at the start of the period, and the arriving type is $\xi$, the value function and optimal actions are given by the Bellman equation

$$V(t, b) = \max \{ r^t \xi + V(t - 1, b - w^t \xi) \mathbb{1}_{\{w^t \xi \leq b\}}, V(t - 1, b) \}, \quad \forall t, b, \xi^t.$$

Next consider the linear programming relaxation for $V(t, b)$

$$\varphi(t, b) := \max_{x_a, x_r \geq 0} \{ r^t x_a : w^t x_a \leq b, x_a + x_r = Z(t) \},$$
In our framework, we first define \( \text{Offline}'s \) value function by specifying its access to future information. Next, we develop a relaxation \( \varphi \) for \( \text{Offline}'s \) value under this same information structure (step I), such that \( \varphi \) approximates \( \text{Offline}'s \) policy in a tractable way. Next, we introduce a non-anticipative proxy \( \hat{\varphi} \) which serves as an estimate for \( \varphi \), and use it to design online controls (step II). Finally, the resulting online policy is evaluated against \( \text{Offline}'s \) value (step III).

Figure 1

It is clear that \( \varphi \) approximates \( V \) up to an integrality gap. However, \( \varphi \) does not obey a Bellman equation. To circumvent this, we introduce the notion of Bellman Inequalities, wherein we require that \( \varphi \) satisfies Bellman-like conditions for ‘most’ sample paths. Formally, for some random variables \( L_B \), we want \( \varphi \) to satisfy

\[
\varphi(t, b) \leq \max \left\{ (r_{t+1} + \varphi(t-1, b - w_{t+1})) \mathbb{1}_{\{w_{t+1} \leq b\}}, \varphi(t-1, b) \right\} + L_B(t, b).
\]

Note that, if \( \mathbb{E}[L_B(t, b)] \) is small, then \( \varphi \) ‘almost’ satisfies the Bellman equations. We henceforth refer to \( \varphi \) as a relaxed value for \( V \) and \( L_B \) the Bellman Loss.

The main advantage of \( \varphi \) is its tractability. On the other hand, using \( \varphi \) as a benchmark to design an online policy can induce an error because it is not \( \text{Offline}'s \) value function; this is quantified by the Bellman Loss, as it arises due to violations of the Bellman equations.

Establishing that actions derived from \( \varphi \) are nearly optimal for \( \text{Offline} \) accomplishes step (I) of our framework, as illustrated in Fig. 1. For step (II), we want to design an online policy that tries to guess \( \text{Offline}'s \) actions by estimating \( \varphi \) based on the current information. A natural estimate is obtained by taking expectations over future randomness to get

\[
\hat{\varphi}(t, b) := \max \left\{ r' y_a : w' y_a \leq b, y_a + y_r = \mathbb{E}[Z(t)] \right\}.
\]

It is known that \( \hat{\varphi} \) does not approximate \( V \) or \( \varphi \) up to constant additive error [27]; however, it can be used as a predictor for the action taken by \( \text{Offline} \). Specifically, at time \( t \) with current budget \( b \), in \text{RAABII} we first compute \( \hat{\varphi}(t, b) \) and then interpret the solution \( y \) as scores for each action (here, accept/reject). We show that taking the action with the highest score (i.e., action \( \arg\max_{a \in \{a,r\}} \{ y_{\xi_{t+1}, a} \} \)) leads to \( \text{Online} \) and \( \text{Offline} \) playing the same action with high probability.
Whenever Offline and Online play different actions, then we incur a loss; we refer to this as the information loss, as it quantifies how having less information impacts Online’s actions. This process of using \( \hat{\phi} \) to derive actions is represented as step III in Fig. 1.

The above three-step process forms a template for our general framework in Section 3.2. Moreover, for all the four problems introduced before, our approach gives similar policies, wherein we choose an Offline benchmark, establish a relaxed value \( \phi \) which takes the form of an optimization problem, and create an online policy based on an estimate \( \hat{\phi} \). Due to this structure, we refer to our framework as the Re-solve and Act Based on Bellman Inequalities (RABBI).

Moreover, as in the above discussion, in all the problems we consider, we decompose the regret into two distinct sources: the Bellman Loss arising from using the relaxation \( \phi \) instead of Offline’s value-function, and the Information Loss arising from estimating \( \phi \) using \( \hat{\phi} \). The former can be viewed as a loss arising from the need for computational tractability, while the latter is inherited from the randomness in the problem.

**Challenges for a general framework.** Constant regret policies for the baseline OnlineKnapsack were obtained in [2], and generalized to multiple dimensions (i.e., the network revenue management/online packing problem) in [27]; moreover, these builds on a long line of work [19, 23] demonstrating constant regret guarantees in restricted settings. In particular, [2, 27] propose a simple online algorithm based on re-solving a fluid LP, that attains constant regret.

The techniques developed in the above papers, however, are tailored to the baseline OnlineKnapsack, and have two fundamental shortcomings that prevent extension to general allocation problems. First, they obtain performance bounds based on the full information benchmark; however, for probing/pricing/learning settings, no algorithm can have constant regret compared to the full information benchmark (see Example 1). Secondly, the analysis in [27] requires an exact characterization of Offline’s value function, which may be intractable for other benchmarks.

In this regard, our work takes a significant step: we offer a framework whose generality allows us to obtain bounded regret algorithms (and guarantees) for several canonical resource allocation problems. In particular, our developments here expand those in [27] by (i) providing a flexible notion of offline information and (ii) allowing to bypass, via Bellman Inequalities, the direct analysis of Offline’s (typically intractable) value function.

### 2.3. Related Work

Our work uses benchmarks to bound the performance of the optimal online algorithm, and also for developing good heuristics. This has commonalities with two closely related approaches:
**Prophet Inequalities**: A well-known framework to obtain performance guarantees is to compare against a full information agent, or “prophet”. This line of work focuses on competitive-ratio bounds, see [12, 14, 20] for overviews of the area. In particular, [12] obtains a multiplicative guarantee for dynamic posted pricing with a single item under worst case distribution. In contrast, we obtain an additive guarantee for multiple items in a parametric setting.

**Information Relaxations.** MDP information-relaxations, in particular, the framework developed in [5, 7], are a fairly general way to create benchmarks, based on endowing Offline with additional information, while forcing him to ‘pay a penalty’ for using this information. [5, 7] use such relaxations in a dual-fitting approach, to construct performance bounds for standard greedy algorithms for a variety of problems. In contrast, our framework is similar to a primal-dual approach: we do not need to identify appropriate penalties, but rather, adaptively construct our relaxations, and derive controls directly from them. We compare the two approaches in more detail in Appendix C.

We test our framework on OnlineKnapsack variants which are widely studied in the literature.

**Probing.** This corresponds to a family of problems where the controller has the option to “pay for information”, leading to a trade-off between making a bad choice and obtaining better information. Approximation algorithms have been developed for offline probing problems, both under constraints on the probed items [16], as well as with costs for probing [25], with applications to Bayesian auction design and in kidney-exchange. Another line of work pursues tractable non-adaptive algorithms for this problem to achieve (multiplicative) constant bounds [17]. In terms of online adaptive algorithms, [11] introduces an algorithm that has bounded competitive ratio (hence linear regret) in an adversarial setting.

**Dynamic posted pricing.** This is a canonical problem in operations management, with a vast literature; we refer the reader to [26] for an overview. Much of this focusses on asymptotically optimal policies in regimes where the inventory $B$ and/or the horizon $T$ grow large. In the regime where both $B$ and $T$ are scaled together by a factor of $k$, there are known algorithms with regret that scales as $O(\sqrt{k})$ or $O(\log(k))$, depending on assumptions on the primitives (e.g., smoothness of the demand with price) [18]. There is also vast literature on pricing when the demand function is not known and has to be learned [10]. We assume that the demand distribution is known and prices are chosen from a discrete price menu. We make no further assumptions on the primitives.

**Knapsack with learning.** Multi-armed bandit problems have been well studied, we refer to [8, 9] for an overview. Bandit problems with combinatorial constraints on the arms that can be pulled are known as Bandits With Knapsacks [4]. The generalization wherein arms arrive online is known as Contextual Bandits With Knapsacks [1, 3]. Results in this bandit literature study worst-case distributions. We, in contrast, pursue parametric regret bounds, namely, bounds that
explicitly depend on the (unknown) discrete distribution. Closest to our work is [28], where an algorithm based on UCB is developed and shown to achieve $O(\sqrt{T})$ regret. The algorithm we develop strengthens their result to a logarithmic regret.

3. Approximate Control Policies via the Bellman Inequalities

In this section, we describe our general framework for benchmark-driven online algorithms. Before proceeding, we introduce some notation we use throughout: Given an optimization problem $(P)$, we denote its optimal value by $P$. We have an underlying probability space is $(\Omega, \Sigma, \mathbb{P})$. For an event $B \subseteq \Omega$, we denote by $B^c$ its complement. For a matrix $A$, we use interchangeably $A_{i,j} = A(i,j)$ to denote the coordinate $(i,j)$. We use boldface letters to indicate vector-valued variables (e.g. $p, w$, etc.), and capital letters to denote matrices and/or random variables.

3.1. Offline Benchmarks and Bellman Inequalities

We develop our framework in the context of finite-horizon MDP driven by some exogenous randomness. For the rest of this section, we consider a general online decision-making problem with state space $S$ and action space $U$, evolving over periods $t = T, T-1, \ldots, 1$; here $T$ denotes the horizon, and $t$ is the time to-go. In any period $t$, the controller first observes a random input $\xi_t \in \Xi$, following which it must choose an action $u \in U$. Given system-state $s \in S$ at the beginning of period $t$, a random input $\xi \in \Xi$ and action $u \in U$ result in a reward $R(s, \xi, u)$, and transition to the next state $T(s, \xi, u)$. Note that both reward and next state are random variables whose realizations are determined for every $u$ given $\xi$. The feasible actions for state $s$ and input $\xi$ correspond to the set $\{u \in U : R(s, \xi, u) > -\infty\}$. We assume that this feasible set is non-empty for all $s \in S, \xi \in \Xi$, and also, that the maximum reward is bounded, i.e., $\sup_{s \in S, \xi \in \Xi, u \in U} R(s, \xi, u) < \infty$.

The MDP described above induces a natural filtration $F$, with $F_t = \sigma(\{\xi^\tau : \tau \geq t\})$; a non-anticipative policy is one which is adapted to $F_t$. We allow Offline to use a richer information filtration $G$, where $G_t \supseteq F_t$. Note that since $t$ denotes the time-to-go, we have $G_{t-1} \supseteq G_t$. Henceforth, to keep track of the information structure, we use the notation $f(\cdot | G_t)$ to clarify that a function $f$ is measurable with respect to the sigma-algebra $G_t$.

Given any filtration $G$, Offline is assumed to play the optimal policy adapted to $G$, hence Offline’s value function is given by the following Bellman equation:

$$V(t, s, \xi^t | G_t) = \max_{u \in U} \{R(s, \xi^t, u) + \mathbb{E}[V(t-1, T(s, \xi^t, u), \xi^{t-1} | G_{t-1}) | G_t]\},$$

with the boundary condition $V(0, \cdot) = 0$. We denote the expected value as $v^{off} := \mathbb{E}[V(T, S^T, \xi^T | G_T)]$. Note that $v^{off}$ is an upper bound on the performance of the optimal non-anticipative policy.
Though our approach holds for any filtration, for concreteness, we present a specific class of filtrations (generated by augmenting the canonical filtration) that suffice for our applications (see Fig. 2 for an illustration of the definition).

**Definition 1 (Canonical augmented filtration).** Let $G_\Theta := (G_\theta : \theta \in \Theta)$ be a set of random variables. The canonical filtration w.r.t. $G_\Theta$ is

$$G_t = \sigma(\{\xi^l : l \geq t\} \cup G_\Theta) \supseteq F_t. \quad (2)$$

The richest augmented filtration is the full information filtration, wherein $G_t = F_1$ for all $t$; this however turns out to be too loose a benchmark in many settings as we show in Example 1. As $G_t$ gets coarser, the difference in performance between Offline and Online decreases. Indeed, when $G = F$, then Eq. (1) reduces to the Bellman equation corresponding to the value-function of an optimal non-anticipative policy:

$$V(t, s, \xi^t) = \max_{u \in U} \{R(s, \xi^t, u) + E[V(t - 1, T(s, \xi^t, u), \xi^{t-1})]\}, \quad V(0, \cdot, \cdot) = 0, \quad (3)$$

where the expectation is taken with respect to the next period’s input $\xi^{t-1}$. Since $\xi^t$ is included in all the filtrations we consider, we henceforth use the shorthand $V(t, s|G_t)$ for $V(t, s, \xi^t|G_t)$.

**Example 1 (Full Information is Too Loose).** Consider the dynamic pricing instance with prices $f = (1, 2)$. The valuation distribution is $P[V_t = 1 + \varepsilon] = p$ and $P[V_t = 2 + \varepsilon] = 1 - p$. In the case $B = T$, the optimal action is to always post the price that maximizes $f \cdot P[V_T > f]$. If we consider $p \geq 1/2$, then the optimal policy (DP) always posts price $f = 1$ and has an expected reward of $T$. On the other hand, full information extracts the realized valuation, i.e., it posts the exactly $V_t - \varepsilon$ at each time, hence $v^{off} = \sum_t E[V_t - \varepsilon] = T(2 - p)$. We conclude that the regret against full information must grow as $\Omega(T)$.

![Figure 2](image_url)

**Figure 2** Illustration of Definition 1. In online probing (see Section 4), arrivals first reveal their public type, then the controller chooses an action (accept/probe/reject), and then the private type (true reward) is revealed. Squares (resp. circles) represent public (resp. private) information. One possible filtration $G$ is to reveal all public types, i.e. $G_\Theta : \theta \in \Theta) = (\xi^\theta : \theta \in [T], \theta \text{ even})$. At time $t$, Offline knows all the information thus far (to the left and including $t$), plus the future squares. Given this filtration, Offline has to take expectation w.r.t. the future circles.

We are now ready to introduce the notion of relaxed value $\varphi$ and Bellman Inequalities. The main intuition behind our definition is that $\varphi$ is “almost” defined by a dynamic-programming recursion;
quantitatively, whenever \( \varphi \) does not satisfy the Bellman equation, we incur an additional loss \( L_B \), which we denote the Bellman loss.

**Definition 2 (Bellman Inequalities).** The family of r.v. \( \{ \varphi(t,s) \}_{t,s} \) satisfies the Bellman Inequalities w.r.t. filtration \( \mathcal{G} \) and r.v. \( \{ L_B(t,s) \}_{t,s} \) if \( \varphi(t,\cdot) \) and \( L_B(t,\cdot) \) are \( \mathcal{G}_t \)-measurable for all \( t \) and the following conditions hold:

1. Initial ordering: \( \mathbb{E}[V(T,S^T)|\mathcal{G}_T] \leq \mathbb{E}[\varphi(T,S^T)|\mathcal{G}_T] \).
2. Monotonicity: \( \forall s \in \mathcal{S}, t \in [T] \),
   \[
   \varphi(t,s|\mathcal{G}_t) \leq \max_{\omega \in \Omega} \{ \mathcal{R}(s,\xi^t,u) + \mathbb{E}[\varphi(t-1,T(s,\xi^t,u)|\mathcal{G}_{t-1})|\mathcal{G}_t] \} + L_B(t,s). \tag{4}
   \]
3. Terminal Condition: \( \varphi(0,s) = 0 \forall s \in \mathcal{S} \)

We refer to \( \varphi \) and \( L_B \) as the relaxed value and Bellman loss pair with respect to \( \mathcal{G} \).

Given any \( \varphi \), monotonicity holds trivially with \( L_B = \mathcal{R} \), but in this case, of course, the performance guarantee will be poor. On the other hand, if we require \( L_B = 0 \), then \( \varphi \) would be the value function, which is difficult to compute in general. A good choice of \( \varphi \) balances the loss and computational tractability. The crux of our approach is to identify relaxations \( \varphi \) with small Bellman Loss.

A special case we use in several of our results is when \( \varphi \) satisfies monotonicity with high probability, i.e., when \( \varphi(t,s|\mathcal{G}_t) \leq \max_{\omega \in \Omega} \{ \mathcal{R}(s,\xi^t,u) + \mathbb{E}[\varphi(t-1,T(s,\xi^t,u)|\mathcal{G}_{t-1})|\mathcal{G}_t] \} \) for all \( \omega \notin \mathcal{B}(t,s) \), where \( \mathcal{B}(t,s) \) is a set of small measure. We henceforth refer to \( \mathcal{B}(t,s) \subseteq \Omega \) as the exclusion sets.

To build intuition, we specify the Bellman Inequalities for our baseline OnlineKnapsack. For this end, we need the following LP lemma.

**Lemma 1.** Consider the standard-form LP \((P[\mathbf{d}])\): \( \max \{ r^t \mathbf{x} : \mathbf{Mx} = \mathbf{d}, \mathbf{x} \geq 0 \} \), where \( \mathbf{M} \in \mathbb{R}^{m \times n} \) is an arbitrary constraint matrix. If \( \tilde{x} \) solves \((P[\mathbf{d}])\) and \( \tilde{x}_j \geq 1 \) for some \( j \), then \( P[\mathbf{d}] = r_j + P[\mathbf{d} - \mathbf{M}_j] \).

**Proof.** By assumption, the optimal value of \((P[\mathbf{d}])\) remains unchanged if we add the inequality \( x_j \geq 1 \). Therefore we have \( P[\mathbf{d}] = \max \{ r^t (\mathbf{x} + \mathbf{e}_j) : \mathbf{M}(\mathbf{x} + \mathbf{e}_j) = \mathbf{d}, \mathbf{x} \geq 0 \} \).

Observe that Lemma 1 allows to divide \( P[\mathbf{d}] \) in two summands: the immediate reward \( r_j \) and the future reward \( P[\mathbf{d} - \mathbf{M}_j] \), which has the flavour of dynamic programming we need.

**Example 2 (Bellman Loss For Baseline Setting).** For the baseline OnlineKnapsack, discussed in Section 2.2, we chose \( \varphi(t,b|\mathcal{G}) := \max_{\mathbf{x} \geq 0} \{ r^t \mathbf{x} : \mathbf{w^t x} \leq b, \mathbf{x} + \mathbf{x_r} = Z(t) \} \). We define the exclusion sets as \( \mathcal{B}(t,b) = \{ \omega \in \Omega : \exists \mathbf{x} \text{ solving } \varphi(t,b) \text{ s.t. } x(a,\xi^t) \geq 1 \text{ or } x(r,\xi^t) \geq 1 \} \). In virtue of Lemma 1, outside the exclusion sets \( \mathcal{B}(t,b) \), monotonicity holds with zero Bellman Loss, i.e.,

\[
\varphi(t,s|\mathcal{G}_t) \leq \max_{\omega \in \Omega} \{ \mathcal{R}(s,\xi^t,u) + \mathbb{E}[\varphi(t-1,T(s,\xi^t,u)|\mathcal{G}_{t-1})|\mathcal{G}_t] \} \quad \forall \omega \notin \mathcal{B}(t,s).
\]
Let us define the maximum loss as
\[ r_\varphi = \max_{t,s,u:R(s,\xi^t,u) > -\infty} \{ \varphi(t,s|G_t) - (R(s,\xi^t,u) + \mathbb{E}[\varphi(t-1,T(s,\xi^t,u)|G_{t-1})|G_t]) \} \]

In words, the maximum violation of the Bellman equation is bounded by \( r_\varphi \). For our choice of \( \varphi \), since the optimal solution is to sort items by “bang for the buck” ratios \( r_j/w_j \), it is easily verified that \( r_\varphi \leq \max_j i \{ w_i r_j/w_j - r_i \} \), which depends on the primitives only. In conclusion, we can see that Definition 2 is satisfied with the Bellman Loss \( L_B(t,b) = r_\varphi 1 \).

Before proceeding to define online policies based \( \varphi \), we need another important definition. Note that though the RHS of Eq. (4) is defined in terms of an ‘optimal’ action, this action need not be unique, and indeed the inequality can be satisfied by multiple actions. For given \( \varphi \) and \( L_B \), we define the set of satisfying actions as follows.

**Definition 3 (Satisfying actions).** Given a filtration \( G \) and relaxed value \( \varphi \), we say that \( u \) is a satisfying action for state \( s \) at time \( t \) if
\[ \varphi(t,s|G_t) \leq R(s,\xi^t,u) + \mathbb{E}[\varphi(t-1,T(s,\xi^t,u)|G_{t-1})|G_t] + L_B(t,s). \] (5)

Note that, at any time \( t \) and state \( s \in S \), any action in \( \arg\max_u \{ R(s,\xi^t,u) + \mathbb{E}[\varphi(t-1,T(s,\xi^t,u)|G_{t-1})|G_t] \} \) is always a satisfying action; moreover, to identify a satisfying action, we need to know \( G_t \). We now have the following proposition.

**Proposition 1.** Consider a relaxation \( \varphi \) and Bellman loss \( L_B \) that satisfy the Bellman inequalities w.r.t. filtration \( G \). Let \( (S^t, t \in [T]) \) denote the state trajectory under a policy that, at time \( t \), takes any satisfying action \( U^t = U^t(S^t|G_t) \). Then,
\[ \mathbb{E}[V(T,S^T|G_T)] - \mathbb{E} \left[ \sum_{t=1}^{T} R(S^t,\xi^t,U^t) \right] \leq \mathbb{E} \left[ \sum_{t=1}^{T} L_B(t,S^t|G_t) \right]. \]

**Proof.** From the monotonicity condition in the Bellman inequalities (Definition 2), and the definition of a satisfying action (Definition 3), we have that for all time \( t \)
\[ \varphi(t,S^t|G_t) \leq \mathbb{E}[R(S^t,\xi^t,U^t)] + \varphi(t-1,S^{t-1}|G_{t-1}) + L_B(t,S^t|G_t)|G_t]. \]

Iterate the above inequality over \( t \), we obtain
\[ \varphi(T,S^T|G_T) \leq \sum_{t=1}^{T} \mathbb{E}[R(S^t,\xi^t,U^t)] + L_B(t,S^t|G_t)|G_t]. \]

Finally, the initial ordering condition gives us \( \mathbb{E}[V(T,S^T)|G_T] \leq \mathbb{E}[\varphi(T,S^T)|G_T] \), and hence taking expectations in the above equation, we get the result. \( \square \)
Proposition 1 shows that a policy that always plays a satisfying action \( U^t \) \textit{approximates} the performance of \textsc{Offline} up to an additive gap given by \( \mathbb{E} \left[ \sum_{t=1}^{T} L_B(t, S^t|G_t) \right] \), which we henceforth refer to as the \textit{total Bellman loss}. More importantly, the proposition suggests that a natural way to design an \textsc{Online} policy to track \textsc{Offline} is to ‘guess’ and play a satisfying action \( U^t \) in each period. We next illustrate how \textsc{Online} can generate such guesses.

Remark 1 (Slightly more general Bellman Inequalities). In the monotonicity requirement (4) we can, in fact, accommodate several generalizations. The loss \( L_B \) can be both random and dependent on the action \( u \). Additionally, rewards and transitions can be random, i.e., unknown at \( t \) given \( \xi^t \), the monotonicity requirement then becomes

\[
\varphi(t, s|G_t) \leq \max_{u \in U} \{ \mathbb{E} [\mathcal{R}(s, \xi^t, u) + \varphi(t-1, \mathcal{T}(s, \xi^t, u)|G_{t-1})|G_t] + \mathbb{E} [L_B(t, s, u)|G_t] \}. 
\]

This expansion of the definition does not change any of the results; we make use of it in our study of the pricing problem in Section 5.

3.2. From Relaxations to Online Policies

Suppose now we are given an augmented canonical filtration \( G_t = \sigma(\{\xi^i : l \geq t\} \cup G_\emptyset) \), and assume that the relaxed value \( \varphi \) can be represented as a function of the random variables \( \{\xi^i : l \geq t\} \cup G_\emptyset \) as \( \varphi(t, s|G_t) = \varphi(t, s; f_t(\xi^T, \ldots, \xi^t, G_\emptyset)) \). In particular, we henceforth focus on a special case where \( \varphi \) is expressed as the solution of an optimization problem:

\[
\varphi(t, s; f_t(\xi^T, \ldots, \xi^t, G_\emptyset)) = \max_{x \in \mathbb{R} \times U \times \Xi} \{ h_t(x; s, f_t(\xi^T, \ldots, \xi^t, G_\emptyset)) : g_t(x; s, f_t(\xi^T, \ldots, \xi^t, G_\emptyset)) \leq 0 \}. \tag{6}
\]

The decision variables give action summaries: for a given state \( s \) and time-to-go \( t \), \( x_{u, \xi} \) represents the number of times action \( u \) is taken for an input \( \xi \) in the remaining periods. We can also interpret \( x_{u, \xi} \) as a score for action \( u \) when the input \( \xi \) is presented.

To designing a non-anticipative policy, note that a natural ‘projection’ of \( \varphi(t, s|G_t) \) on the filtration \( \mathcal{F} \) is given via the following optimization problem

\[
\hat{\varphi}(t, s|\mathcal{F}_t) = \varphi(t, s; \mathbb{E} [f_t(\xi^T, \ldots, \xi^t, G_\emptyset)|\mathcal{F}_t]) = \max_{y \in \mathbb{R} \times U \times \Xi} \{ h_t(y; s, \mathbb{E} [f_t|\mathcal{F}_t]) : g_t(y; s, \mathbb{E} [f_t|\mathcal{F}_t]) \leq 0 \}. \tag{7}
\]

The solution of this optimization problem gives action summaries (or scores) \( y \); the main idea of the \textsc{RABBI} algorithm is to play the action with the highest score.

Before computing the regret of the \textsc{RABBI} policy, we need an additional definition

Definition 4 (Maximum Loss). For a given relaxation \( \varphi \), define the maximum loss as

\[
r_\varphi := \max_{t, s, u; \mathcal{R}(s, \xi^t, u) > -\infty} \{ \varphi(t, s|G_t) - (\mathcal{R}(s, \xi^t, u) + \mathbb{E} [\varphi(t-1, \mathcal{T}(s, \xi^t, u)|G_{t-1})|G_t]) \}.
\]
RABBI (Re-solve and Act Based on Bellman Inequalities)

Input: Access to functions $f_t$ such that $\varphi(t, s|G_t) = \varphi(t, s; f_t(\xi^T, \ldots, \xi^t, G_\Theta))$.

Output: Sequence of decisions $\hat{U}^t$ for ONLINE.

1: Set $S^T$ as the given initial state
2: for $t = T, \ldots, 1$ do
3: Compute $\hat{\varphi}(t, S^t) = \varphi(t, S^t; E[f_t(\xi^T, \ldots, \xi^t, G_\Theta)|F_t])$ with associated scores $y = \{y_{u, \xi}^t\}_{u\in U, \xi\in \Xi}$
4: Given input $\xi^t$, choose the action $\hat{U}^t$ with the highest score $y_{u, \xi}^t$
5: Collect reward $R(S^t, \xi^t, \hat{U}^t)$; update state $S^{t-1} \leftarrow \mathcal{T}(S^t, \xi^t, \hat{U}^t)$

Theorem 5. Let Offline be defined by the filtration $G_t$ as in Definition 1. Assume the relaxation $\varphi(t, s)$ satisfies the Bellman Inequalities with loss $L_B$. For $t \in [T], s \in S$, let $Q(t, s) \subseteq \Omega$ be the event where $\hat{U}^t$, as specified in RABBI, is not a satisfying action. If $(S^t, t \in [T])$ is the state trajectory when following the actions derived from RABBI, then

$$E[\text{Regret}] \leq E \left[ \sum_t (r_\varphi \mathbb{1}_{Q(t, S^t)} + 1_{Q(t, S^t)} L_B(t, S^t)) \right] \leq \sum_t (r_\varphi \mathbb{P}[Q(t, S^t)] + E[L_B(t, S^t)])$$

Remark 2 (Bellman and Information Loss). The bound in Theorem 5 has two distinct summands. The term $\sum_t \mathbb{P}[Q(t, S^t)]$ measures how often RABBI takes a non-satisfying action, hence we refer to it as information loss. On the other hand, $\sum_t E[L_B(t, S^t)]$ captures the fact that $\varphi$ “almost” has a DP representation, which defined before as the Bellman loss.

Remark 3 (More General Actions). In order to present a precise rule on how to select an action $\hat{U}^t$ in RABBI, we have assumed that $\varphi$ is based on an optimization problem. Nevertheless, Theorem 5 holds even if $\varphi$ has another representation, e.g., it is obtained through approximate DP techniques, but in this case the rule to select an action based on $\hat{\varphi}$ would be ad-hoc to the problem or special structure of $\varphi$.

Compensated Coupling: To proof of Theorem 5 is based on the compensated coupling approach introduced in [27]. The basic idea is as follows: Suppose we ‘simulate’ controllers Offline and Online in parallel with same random inputs $\xi^t$, with Online acting before Offline; moreover, suppose at some time $t$, both controllers are in the same state $s$. Recall that, for any given state $s$ at time $t$, an action $u$ is satisfying if Offline’s value does not decrease when playing $u$ (see Definition 3). If Online chooses to play a satisfying action, then we can make Offline play the same action, and consequently both move to the same state. On the other hand, if Online chooses an action that is not satisfying, then the two trajectories will separate; we can avoid this however by ‘compensating’ Offline so that he agrees to take the same action as Online. Specifically, if the maximum loss is bounded by $r_\varphi$, increasing the reward earned by Offline by $r_\varphi$ is enough
to make him choose Online’s action. As a consequence, we have that the (compensated) Offline and Online follow the same actions, and thus their state trajectories are coupled.

As an example, consider the baseline OnlineKnapsack with budget $B = 2$, unit weights ($w_j \equiv 1$), and horizon $T = 5$. Take a realization $\omega$ with rewards $(\xi^5, \xi^4, \xi^3, \xi^2, \xi^1) = (5, 7, 2, 7, 2)$. In particular, three different types arrived in this realization. The sequence of actions $(r, a, r, a, r)$ is optimal for Offline with a total reward of 14 (selecting the two 7-valued items). Suppose that Online, at period $t = 5$, wishes to accept the item (obtain a reward of 5, and lose one budget unit). Then, Offline is “willing” to follow this action if paid a compensation of 2 (in addition to the collected $\xi^5$); Offline and Online then start the next period $t = 4$ in the same state (same remaining budget), hence remain coupled.

**Proof of Theorem 5.** Denoting Offline’s state as $\bar{S}^t$, we have via Proposition 1 that $\forall t$:

$$\varphi(t, \bar{S}^t|G_t) \leq E[\mathcal{R}(\bar{S}^t, \xi^t, \bar{U}^t) + \varphi(t-1, \bar{S}^{t-1}|G_{t-1}) + L_B(t, \bar{S}^t)|G_t].$$

Let us assume as the induction hypothesis that $\bar{S}^t = S^t$. This holds for $t = T$ by definition. At any time $t$ and state $S^t$, if $\hat{U}^t$ is not a satisfying action for Offline, then we have from the definition of the maximum loss (Definition 4) that:

$$r_\varphi \geq \varphi(t, S^t|G_t) - \mathcal{R}(S^t, \xi^t, \hat{U}^t) + E[\varphi(t-1, S^{t-1}|G_{t-1})|G_t]) \text{ a.s..}$$

Now to make Offline take action $\hat{U}^t$ so as to have the same subsequent state as Online, it is sufficient to compensate Offline with an additional reward of $r_\varphi$. Specifically, we have

$$\varphi(t, S^t|G_t) \leq E[\mathcal{R}(S^t, \xi^t, \hat{U}^t) + \varphi(t-1, S^{t-1}|G_{t-1}) + r_\varphi 1_{Q(t, S^t)} + 1_{Q(t, S^t)^c} L_B(t, S^t)|G_t].$$

Finally, as in Proposition 1, we can iterate over $t$ to obtain

$$E[\varphi(T, S^T|G_T)] \leq E\left[\sum_t \mathcal{R}(S^t, \xi^t, \hat{U}^t) + \sum_t (r_\varphi 1_{Q(t, S^t)} + 1_{Q(t, S^t)^c} L_B(t, S^t))\right].$$

The first sum on the right-hand side corresponds exactly to Online’s total reward using the RABBI policy. By the initial ordering of Bellman Inequalities (Definition 2), $E[V(T, S^T)] \leq E[\varphi(T, S^T)]$ and we obtain the desired bound.

In the remaining sections we apply the general framework to each of the examples we introduced in Section 2.2 and prove the results stated there.
4. Partial Information and Probing

In this setting, each type \( j \) has an independent random reward \( R_j \) drawn from the set \( \{r_{jk} : k \in [m]\} \); the reward is \( r_{jk} \) with probability \( q_{jk} \). We assume without loss of generality that \( r_{j1} < r_{j2} < \ldots < r_{jm} \) and \( r_{jm} > 0 \). The parameters \( r \) and \( q \) are known. For ease of exposition, we assume that all items have weight 1; our analysis however extends to general weights \( w_j \). The controller has a resource budget (i.e., knapsack capacity) \( B_h \in \mathbb{N} \) and a probing budget \( B_p \in \mathbb{N} \). When an arrival is accepted (resp. probed), we reduce \( B_h \) (resp. \( B_p \)) by one. If an item of type \( j \) is probed, then \( R_j \) is revealed and the controller can decide whether to accept or reject the item. The controller can also accept an item of type \( j \) without probing, in which case the expected reward is \( \bar{r}_j := \sum_{k \in [m]} r_{jk} q_{jk} \).

Note that when either \( B_p \geq T \) or \( B_p = 0 \), this problem reduces to the baseline \textit{OnlineKnapsack}. In particular, when \( B_p \geq T \), the controller can probe at every single period, in which case we are back at the baseline \textit{OnlineKnapsack} with revealed rewards and \( mn \) types. If \( B_p = 0 \), we have the baseline \textit{OnlineKnapsack} with \( n \) types and reward equal to the expectation \( \bar{r}_j \) for type \( j \).

Dynamic programming formulation. To capture the two-stage nature of the decisions, it is useful to model each period as consisting of two stages. In this section, we assume each period \( t \in \{T, T - 1, \ldots, 1\} \) comprises of two stages \( \{t, t - 1/2\} \), driven by external inputs \( \xi^t \in [n] \) and \( \xi^{t-1/2} \in [n] \times [m] \). In the first stage \( t \), the controller observes the type of the arriving request \( \xi^t = j \); in the second stage \( t - 1/2 \), the realization of the reward is drawn according to probabilities \( q \). In particular, if the arrival is of type \( j \) (revealed in the first stage), then in the second stage the “subtype” \( \xi^{t-1/2} = (j, k) \), associated with reward \( r_{jk} \), is drawn with probability \( q_{jk} \). We augment the state space with a variable \( \diamond \) that captures the first stage decision (i.e., whether we accept/reject without probing or probe). The state space \( S \) of the controlled process is thus \( S = \{(b_h, b_p, \diamond) : b_h, b_p \in \mathbb{N}, \diamond \in \{a, p, r, \emptyset\}\} \), where \( b_h, b_p \) are the residual hiring and probing budgets. In first stages we set \( \diamond = \emptyset \), and only collect rewards in second stages. Fig. 3 displays the actions and state transitions under this reformulation.

4.1. Probing RABBI

Next, consider the following LP, parametrized by \( (b, z) \in \mathbb{N}^2 \times \mathbb{R}_0^n \),

\[
\begin{align*}
(P[b, z]) \quad \text{max} & \quad \sum_{j,k} r_{jk} x_{jka} + \sum_{j} \bar{r}_j x_{ja} \\
\text{s.t.} & \quad \sum_{j} x_{jka} + \sum_{j} x_{ja} \leq b_h \\
& \quad \sum_{j} x_{jp} \leq b_p \\
& \quad x_{ja} + x_{jp} + x_{j} = z_j \quad \forall j \in [n] \\
& \quad x_{jka} + x_{jk} = q_{jk} x_{jp} \quad \forall j \in [n], k \in [m] \\
& \quad x_{ja}, x_{jp}, x_{j}, x_{jka}, x_{jk} \geq 0 \quad \forall j \in [n], k \in [m].
\end{align*}
\]
Figure 3  

Actions/transitions in online probing in periods \( t, t - 1/2, \) and \( t - 1 \), with inputs \( \xi^t = j \) and \( \xi^{t-1/2} = R' \). Numbers below the arrows represent the reward of a transition. At \( t \), available actions are \( \{a, p, r\} \) (i.e., accept, probe, reject; from top to bottom); at \( t - 1/2 \), if we chose to probe in the first-stage (i.e., are in the middle state), then available actions are \( \{a, r\} \).

The decision variables \( x \in \mathbb{R}^{3n+2nm} \) have a natural interpretation as action summaries: \( x_{ja}, x_{jr}, x_{jp} \) are the total number of future type-\( j \) arrivals that are accepted without probing, rejected without probing, and probed respectively; \( x_{jka}, x_{jkr} \) are the number of probed future type-\( j \) arrivals that are revealed to have reward \( r_{jk} \), and then accepted/rejected respectively. The first constraint requires that the number of accepted items does not exceed the knapsack capacity; the second constraint guarantees that the number of items probed does not exceed the probing budget; the third is a “demand constraint” that guarantees the number of type-\( j \) items accepted, probed or rejected equals arrivals of that type. Finally, the last constraint guarantees that a \( q_{jk} \) fraction of probed type-\( j \) items have sub-type \( k \) (i.e., reward \( r_{jk} \)).

The LP \( P[b, Z] \) now plays the role of the proxy \( \hat{\varphi} \) for the Probing RABBI presented below in Algorithm 2 (with ties broken arbitrarily); we specify the relaxation \( \varphi \) later in Eq. (9).

4.2. Offline Information and Relaxation

We introduce an OFFLINE controller that knows the types of all arrivals in advance (i.e., it knows \( Z_j(t) \), the number of type-\( j \) items that will arrive in the last \( t \) periods), but does not know the realization of the rewards (sub-types). Formally, OFFLINE is endowed with the canonical filtration given by \( \Theta = [T] \) and \( \mathcal{G}_\emptyset = \xi^\emptyset \) (see Definition 1): with \( t \) steps to go, OFFLINE has the information filtration \( \mathcal{G}_t = \sigma(\{\xi^t : t \in [T]\} \cup \{\xi^\tau : \tau \geq t\}) \).

Solving for OFFLINE’s optimal actions may be non-trivial, as it corresponds to a dynamic program where, for each arrival, the controller must decide whether to accept/reject without probing, or probe, and then decide to accept/reject based on the realization of the reward. We circumvent this via an offline relaxation \( \varphi \) built using the LP \( P[b, Z] \) (8). Recall that a state is of the form \( s = (b_h, b_p, \diamond) = (b, \diamond) \) with \( \diamond \in \{a, p, r, \emptyset\} \). For period \( t \) (i.e., first stage, \( \diamond = \emptyset \)), we define 

\[
\varphi(t, s|\mathcal{G}_t) = P[b, Z]
\]
Algorithm 2 Probing RABBI

**Input:** Access to solutions of \((P|b, z])\)

**Output:** Sequence of decisions for ONLINE.

1. Set \((B_h^T, B_p^T) \leftarrow (B_h, B_p)\) as the given initial state
2. for period \(t \in \{T, T-1, \ldots, 1\}\) do
   3. Compute \(X^t\), an optimal solution to \((P|B^t, E[Z(t)])\)
   4. In the first stage:
      - Observe type \(\xi^t\); take action \(\hat{U}^t \in \arg\max_u \{X^t_{\xi^t, u}\}\).
   5. In the second stage (i.e., period \(t-1/2\)):
      - If \(\hat{U}^t = r\) or \(\hat{U}^t = a\), collect zero or random \(R_{\xi^t}\), respectively.
      - If \(\hat{U}^t = p\), observe sub-type \(\xi^{t-1/2}\); take action \(\arg\max_u \{X^t_{\xi^{t-1/2}, u}\}\)
   6. Update states \(B^{t-1}\) accordingly.

For \(t-1/2\) (i.e., second stage decisions), we modify \(\varphi\) to incorporate the action \((a, p, r)\) taken in the first stage. Overall, our relaxation is defined as follows

\[
\varphi(t-1/2, s|G_t) = \begin{cases} 
P[b, Z(t-1)] & \diamond = r \\
r_{\xi^{t-1/2}} + P[b, Z(t-1)] & \diamond = a \\
\max\{r_{\xi^{t-1/2}} + P[b - e_h, Z(t-1)], P[b, Z(t-1)]\} & \diamond = p
\end{cases}
\]

(9)

### 4.3. Bellman Inequalities and Loss

**Initial ordering.** Lemma 2 provides the first ingredient for the application of Definition 2.

**Lemma 2.** For any \(b_h, b_p \in \mathbb{N}\), and realization \(Z\) of arrivals, \(E[V(T, b|G_T)] \leq E[\varphi(T, (b, \emptyset)|G_T)]\).

**Proof.** The main idea is showing that any offline policy induces action summaries that satisfy the constraints defining \(\varphi\). Consider a policy for OFFLINE which determines when to probe, accept or reject. A policy is a mapping \(\pi : [T] \times S \rightarrow U\) such that \(\pi(t, s)\) is \(G_t\)-measurable for all \(t, s\).

The policy, once fixed, induces a random trajectory determined by the realization of the probed rewards. Arrivals, recall, are known to OFFLINE. Define the following random variables counting the number of times in which a type \(j\) was: probed, denoted \(X_{jp}\); accepted (rejected) without probing, \(X_{ja} (X_{jr})\); and was accepted (rejected) after the probe resulted in \((j, k)\), \(X_{jka} (X_{jkr})\).

Then, we can write \(E[V(T, b|G_T)] = E[\sum_j \bar{r}_j X_{ja} + \sum_{j, k} r_{jk} X_{jka}|G_T]\), where we use the fact that the policy is \(G_t\)-adapted for the term \(\bar{r}_j\), i.e., conditional on accepting without probing, the expected reward is \(\bar{r}_j\). We conclude,

\[
E[V(t, b|G_T)] = \sum_j \bar{r}_j E[X_{ja}|G_T] + \sum_{j, k} r_{jk} E[X_{jka}|G_T].
\]

We claim that the expectation of the random vector \(X\) yields a feasible solution to \((P[T, b, Z])\). In turn, \(P[T, b, G_T]\) can only be larger than the value. Indeed, with the exception of the constraint
\(x_{jka} + x_{jkr} = q_{jk} x_{jp}\), the random variables satisfy a.s. all the constraints of \((P[T, b, Z])\), hence their expectations do too. Furthermore, since Offline’s policy is adapted to \(G\), we obtain \(\mathbb{E}[X_{jka} + X_{jkr} | X_{jp}, G_T] = q_{jk} X_{jp}\), thus the expected values satisfy the desired constraint. To summarize, \(V(T, b | G_T)\) equals the value of the feasible solution given by the expectations. □

**Monotonicity and satisfying actions.** We will require the following LP lemma

**Lemma 3.** Consider the standard-form LP \((P[d])\): \(\max \{ r' x : M x = d, x \geq 0 \}\), where \(M \in \mathbb{R}^{m \times n}\) is an arbitrary constraint matrix and \(d \in \mathbb{R}^m\). The function \(d \mapsto P[d]\) is concave and therefore, if \(X\) is a random right-hand side, then \(\mathbb{E}[P[X]] \leq P[\mathbb{E}[X]]\).

**Proof.** The dual problem is \((D[d])\): \(\min \{ d' y : M' y \geq r \}\). The function \(d \mapsto D[d]\) is a minimum of linear functions, therefore concave. □

Before proving that \(\varphi\) satisfies the Bellman Inequalities, we recall the concept of a *satisfying action* in Definition 3 with our choice of \(L_B\). Given \(\varphi\), filtration \(G\), state \(s\) and time \(t\), action \(u\) is satisfying if \(\varphi(t, s | G_t) \leq \mathcal{R}(s, \xi^t, u) + \mathbb{E}[\varphi(t - 1, T(s, \xi^t, u) | G_{t-1}) | G_t]\).

**Lemma 4.** Let \(\bar{X}\) be a maximizer of \((P[b, Z(t)])\) for \(t\) a first stage and say \(\xi^t = i\). Then we have the following implications for satisfying actions

1. If \(\bar{X}_{is} \geq 1\) or \(\bar{X}_{ir} \geq 1\) (or \(\max \{\bar{X}_{is}, \bar{X}_{ir}\} \geq 1\)): Offline is satisfied (in the sense of Definition 3) accepting or rejecting, respectively, at time \(t\).

2. If \(\bar{X}_{ip} \geq 1\) and \(\xi^{t-1/2} = (i, k)\) is such that either \(\bar{X}_{ika} \geq 1\) or \(\bar{X}_{ikr} \geq 1\) (\(\bar{X}_{ip} \geq 1\) and \(\max \{\bar{X}_{ika}, \bar{X}_{ikr}\} \geq 1\)): Offline is (i) satisfied probing at time \(t\) and (ii) satisfied accepting if \(\bar{X}_{ika} \geq 1\) or rejecting if \(\bar{X}_{ikr} \geq 1\) at time \(t - 1/2\).

In conclusion, \(\varphi\) satisfies the Bellman Inequalities with exclusion sets

\[\mathcal{B}(t, b) = \{ \omega \in \Omega : \exists X \text{ solving } (P[b, Z(t)]) \text{ s.t. (1) or (2) hold} \} \].

**Proof.** We will establish the monotonicity condition (second requirement for Bellman Inequalities in Definition 2):

\[\varphi(t, s | G_t) \leq \max_{u \in U} \{ \mathcal{R}(s, \xi^t, u) + \mathbb{E}_{\xi^{t-1/2}}[\varphi(t - 1/2, T(s, \xi^t, u) | G_{t-1/2}) | G_t] \} \quad \forall \omega \notin \mathcal{B}(t, s)\]

Observe that, since \(t\) is a first stage, the instant reward \(\mathcal{R}(s, \xi^t, u)\) is zero. In case (1) of the lemma it is easy to verify the inequality by invoking Lemma 1.

For case (2) we need to introduce some notation. Let \(q_j \in \mathbb{R}^{n \times m}\) be a vector with value \(q_{jk}\) in components \((j, k), k \in [m]\), and zero otherwise (i.e. in components \((j', k)\) with \(j' \neq j\)). Similarly, let \(1_{(j,k)} \in \mathbb{R}^{n \times m}\) have value 1 in the single component \((j, k)\) and zero otherwise. The following LP is the same as in Eq. (8), but has the extra parameter \(y\) that facilitates the analysis; \(P[b, z] = \bar{P}[b, z, 0]\).
\( \max (\tilde{P}[b, z, y]) \quad \text{s.t.} \quad \sum_{j,k} r_{jk} x_{jka} + \sum_j \tilde{r}_j x_{ja} \)

\[
\begin{align*}
\sum_{j,k} x_{jka} + \sum_j x_{ja} & \leq b_h \\
\sum_j x_{jp} & \leq b_p \\
x_{ja} + x_{jp} + x_{kr} & = z_j \\
x_{jka} + x_{jkr} & = q_{jk} x_{jp} + y_{jk} \\
x & \geq 0.
\end{align*}
\]

In case (2), \( \bar{X}_{ip} \geq 1 \) and \( \xi^{t-1/2} = (i, k) \) is such that either \( \bar{X}_{ika} \geq 1 \) or \( \bar{X}_{ikr} \geq 1 \). By Lemma 1, we have the following breakdown (depending on the random \( \xi^{t-1/2} \))

\[ P[b, Z(t)] = \tilde{P}[b, Z(t), 0] = r_{\xi^{t-1/2}} I + \tilde{P}[b - e_p - e_h I, Z(t-1), q_{\xi^{t}} - 1_{\xi^{t-1/2}}], \quad \forall \omega \notin B(t, b), \]

where \( I := 1_{\{\bar{X}_{\xi^{t-1/2}, a} \geq 1\}} \) and the vectors \( q, 1 \) are evaluated in random components; since by assumption \( \bar{X}_{ip} \geq 1 \) under the optimal solution, the optimal value in the optimization problem is the same as the reward obtained “now” (\( r_{\xi^{t-1/2}} \)) and the residual value after discounting \( b_p \) by one.

Taking expectations \( E[\cdot | G_t] \) and using Lemma 3 we have

\[
\begin{align*}
P[b, Z(t)] &= E[r_{\xi^{t-1/2}} I + \tilde{P}[b - e_p - e_h I, Z(t-1), q_{\xi^{t}} - 1_{\xi^{t-1/2}}] | G_t] \\
&\leq E[r_{\xi^{t-1/2}} I + \tilde{P}[b - e_p - e_h I, Z(t-1), 0] | G_t] \\
&\leq E[\max\{r_{\xi^{t-1/2}} + \tilde{P}[b - e_p - e_h I, Z(t-1)], \tilde{P}[b - e_p, Z(t-1)]\] | G_t].
\end{align*}
\]

The last expression is obtained by considering the possible realizations of \( I \) and corresponds to the desired inequality. \( \square \)

4.4. Information Loss and Overall Performance Guarantee

Our relaxation \( \varphi \) in Eq. (9) is a function of the future arrivals at time \( t \), \( Z(t) \), and can be written in the form \( \varphi(t, s) = \varphi(t, s; Z(t)) \). This coincides with the general formulation in Section 3.2, specifically with Eq. (6) therein. The proxy is \( \tilde{\varphi}(t, s) = \varphi(t, s; E[Z(t)]) \), which is used to generate actions \( \tilde{\mathcal{U}} \) through the LP (8) that defines it.

Specifically, we use maximizers \( X^t \) of \( (P[B^t, \mu(t)]) \) as estimators for solutions of \( (P[B^t, Z(t)]) \), where \( \mu(t) := E[Z(t)] \). If the type of the arrival at time \( t \) is \( j \), then \( X^t(j, a), X^t(j, \mathcal{X}), X^t(j, \mathcal{P}) \) indicate how much of type \( j \) we want to accept, reject and probe, respectively. We choose the action with largest variable value. In case we decide to probe, we repeat the logic with variables \( X^t(j, k, a), X^t(j, k, \mathcal{X}) \).

**Proof of Theorem 2.** By Theorem 5, \( \text{Regret} \leq r_{\varphi} \sum_t (1_{B(t, s')} + 1_{Q(t, s')}) \). To bound this expression, we proceed in two steps: bounding the measure of the exclusion sets \( \mathcal{B} \) and the “disagreement” sets \( \mathcal{Q} \). We conclude using the fact that \( r_{\varphi} \leq \max_{j,k} r_{jk} \).
To bound the measure of the exclusion sets $\mathcal{B}$, as defined in Lemma 4, we exploit the following two restrictions of $(P[b, Z(t)])$: $x_{ja} + x_{jp} + x_{je} = Z_j(t) \forall j$ and $x_{jka} + x_{jkr} = q_{jk} x_{jp} \forall j, k$. If $Z_j(t) \geq 3$, then one of the variables $x_{ja}, x_{jp}, x_{je}$ must be at least 1. On the other hand, we need $q_{jk} x_{jp} \geq 2$ to guarantee that one of $x_{jka}, x_{jkr}$ is at least 1. Putting this argument together, in virtue of Lemma 4,

$$\mathbb{P}[B(t, b) | \xi_t^{1/2} = (j, k)] \leq \mathbb{P} \left[ Z_j(t) < \frac{6}{q_{jk}} \right] = \mathbb{P} \left[ Z_j(t) - \mu_j(t) < -\mu_j(t) \left( 1 - \frac{6}{\mu_j(t) q_{jk}} \right) \right]. \quad (11)$$

A standard Chernoff bound (see [6]) implies

$$\mathbb{P}[B(t, b) | \xi_t^{1/2} = (j, k)] \leq e^{-2(p_j/2)t} + \mathbb{1}_{\{t \geq 12/(p_j q_{jk})\}}.$$

Here, we applied the bound to the restricted range $\mu_j(t) \geq 12/q_{jk}$, which guarantees, in particular, that the the right-hand side of Eq. (11) is positive.

Finally,

$$\sum_t \mathbb{P}[B(t, B^t)] \leq \sum_t \sum_j p_j e^{-(p_j/2)t} + \sum_t \sum_{j,k} p_j q_{jk} \mathbb{1}_{\{t \geq 12/(p_j q_{jk})\}} \leq \sum_j 2/p_j + 12.$$

This finishes the first part of the proof, i.e., bounding the exclusion sets.

We turn to the second and last part of the proof, which is to bound the probabilities that our policy does not guess correctly a satisfying action. Recall that $Q(t, S^t) \subseteq \Omega$ is the event where $\hat{U}^t$ is not satisfying; the aim is to bound $\sum_t \mathbb{P}[Q(t, S^t)]$. Let $X^t$ be a solution to $(P[b, Z(t)])$, $t$ a first stage, and let $j = \xi^t$. We divide the analysis in two cases: if $\hat{U} \in \{a, r\}$ or if $\hat{U} = p$.

Assume $\hat{U}^t$ corresponds to accept or reject. According to Lemma 4, accepting or rejecting is satisfying whenever $\max\{X^t_a, X^t_r\} \geq 1$. Since $X^t(\xi^t, \hat{U}^t) = \max\{X^t(\xi^t, u) : u = a, p, r\}$, the error is given by

$$\mathbb{P}[X^t(j, \hat{U}^t) < 1 | X^t(j, \hat{U}^t) \geq \mu_j(t)/3] \leq \mathbb{P}[\|X^t - X^t\|_\infty \geq \mu_j(t)/3].$$

On the other hand, if $\hat{U} = p$, the error is bounded by

$$\mathbb{P}[X^t_{j,p} < 1 \text{ or } X^t_{\xi^t-1/2, u} < 1 | X^t_{j,p} \geq \mu_j(t)/3, X^t_{\xi^t-1/2, u} \geq q_{\xi^t-1/2} \mu_j(t)/6] \leq \mathbb{P}[\|X^t - X^t\|_\infty \geq q_{\xi^t-1/2} \mu_j(t)/6],$$

where $u$ is the action with largest value between the variables $X^t(\xi^t-1/2, a), X^t(\xi^t-1/2, r)$.

We conclude that, regardless of the action $\hat{U}^t$, the probability of choosing a non-satisfying action is bounded by $\mathbb{P}[\|X^t - X^t\|_\infty \geq \min_k q_{jk} \cdot \mu_j(t)/6]$. From the LP sensitivity result [21, Theorem 2.4], we deduce the existence of $\kappa$ that depends on $q, n, m$ only s.t. $\|X^t - X^t\|_\infty \leq \kappa \|Z(t) - \mu(t)\|_1$. Finally, the number of times that ONLINE chooses a non-satisfying action (measure of all sets $Q$) is bounded by

$$\sum_t \mathbb{P}[Q(t, S^t)] \leq \sum_t \mathbb{P}[\|Z(t) - \mu(t)\|_1 \geq \min_k q_{jk} \cdot \mu_j(t)/6] < \infty. \quad (12)$$

The summability follows from standard concentration bounds [27] and Theorem 2 now follows from Theorem 5. \qed
Remark 4 (non-i.i.d arrival processes). The key property of the arrival process that we used is a tail bound of the form \( P[|Z(t) - \mathbb{E}[Z(t)]| \geq c \mathbb{E}[Z(t)]] \leq g(t) \), see Eq. (12). The result thus holds for any arrival process that satisfies such deviation bounds.

Remark 5 (Probing cost). Suppose that the controller has infinite probing budget \((B_p = \infty)\), but incurs a penalty \(c_j\) when probing a type-\(j\) arrival. All else remains unchanged. The only change to results and proofs is the definition of \( P[b, Z] \) to one where the probing budget is dropped and a probing cost is added to the objective function:

\[
(P[b, z]) \max_{x} \sum_{j,k} r_{jk} x_{jka} + \sum_{j} \bar{r}_j x_{ja} - \sum_{j} c_j x_{jp} \\
\text{s.t.} \quad \sum_{j,k} x_{jka} + \sum_{j} x_{ja} \leq b_h \\
\quad x_{ja} + x_{jp} + x_{jrt} = z_j \quad \forall j \in [n] \\
\quad x_{jka} - x_{jkr} = q_{jk} x_{jp} \quad \forall j \in [n], k \in [m] \\
\quad x_{jp} - x_{jrt} \geq 0 \quad \forall j \in [n], k \in [m]
\]

5. Pricing

At each time \(t\), the controller chooses one posted price \(f_j\) among \((f_1, \ldots, f_m)\), then a valuation \(V_t \sim F\) is drawn and a purchase occurs iff \(V_t > f_j\). If the customer buys, \(f_j\) is collected and the inventory decreases by 1. On the other hand, if the customer does not buy, there is no reward and the inventory remains unchanged.

5.1. Pricing RABBI

The Pricing RABBI algorithm uses as proxy \(\hat{\varphi}\) the following LP (parameterized by \((t, b, q)\))

\[
(P[t, b, q]) \max \left\{ \sum_i f_i q_i x_i : \sum_i q_i x_i \leq b, \sum_i x_i \leq t, x_i \geq 0 \forall i \in [m] \right\}. \tag{13}
\]

If we take \(q_i = \bar{F}(f_i)\), i.e., the probability that price \(f_i\) is accepted, then the problem \((P[t, b, q])\) can be interpreted as follows: variable \(x_i\) represents the number of times that price \(f_i\) is offered, with \(\sum_i f_i q_i x_i\) the expected reward from the corresponding arrivals. Each time price \(f_i\) is offered, \(q_i\) units of inventory are consumed in expectation, and hence \(\sum_i q_i x_i\) is the total expected units sold. Finally, at most one price is offered per arrival, which is captured by \(\sum_i x_i \leq t\). The resulting online policy is presented in Algorithm 3. The relaxation \(\varphi\) is defined in the next subsection.

5.2. Offline Information and Relaxation

Assume, w.l.o.g., that the prices are ordered \(f_1 > f_2 > \ldots > f_m\). To formulate a suitable Offline, we introduce a sequence of independent random vectors \(\{Y_t : t = T, T-1, \ldots, 1\}\) defined by the relation \(Y_{it} := 1_{\{V_t > f_i\}}\) so that \(Y_{it} \sim \text{Bernoulli}(\bar{F}(f_i))\).
Algorithm 3 Pricing RABBI  
**Input:** Access to solutions of \((P[t, b, q])\)  
**Output:** Sequence of decisions for \textsc{Online}.  
1. Set \(B^T \leftarrow B\) as the given initial budget and \(q_i \leftarrow \bar{F}(f_i)\)  
2. for \(t = T, ..., 1\) do  
3. Compute \(X^t\), an optimal solution to \((P[t, B^t, q])\)  
4. Offer price \(f_j\) corresponding to any \(j \in \arg\max\{X^t_i : i \in [m]\}\)  
5. If \(V_t > f_j\), collect \(f_j\) and \(B^{t-1} \leftarrow B^t - 1\); else \(B^{t-1} \leftarrow B^t\)  

Define \(Q_i(t) := \frac{1}{t} \sum_{l=1}^{t} Y_{i,l}\) as the empirical average of the last \(t\) periods; \(Q_i(t)\) is the fraction of customers who would accept price \(f_i\). Observe that \(Q_i(t)\) is a martingale with \(\mathbb{E}[Q_i(t)] = \bar{F}(f_i)\) and \(Q_i(t) = \frac{t+1}{t} Q_i(t+1) - \frac{1}{t} Y_{i,t} + 1\).  

\textsc{Offline}’s information is given by the filtration \(\mathcal{G}_t = \sigma(\{Q_i(l) : l \geq t\})\), i.e., at every time \(t\), \textsc{Offline} knows the empirical averages \(Q_i(t)\), but does not know at what time which valuation arrived. This coincides with the canonical filtration (Definition 1) with variables \((Q_i(T) : i \in [m])\). The filtration \(\mathcal{G}\) is strictly coarser than the full information filtration, which would correspond to revealing all the variables \(Y_T, Y_{T-1}, ..., Y_1\) instead of their averages. We take the relaxation \(\varphi(t, b | \mathcal{G}_t) := P[t, b, Q(t)]\).

5.3. Bellman Inequalities and Loss

Proposition 2 shows that our choice of \(\varphi\) satisfies the Bellman Inequalities. The proof of the initial ordering is similar to that of Lemma 2, hence we omit it. We present the main ingredients to obtain the second part of Proposition 2 (monotonicity).

**Proposition 2.** Let \(\varphi\) and \(\mathcal{G}\) be as before.  
1. Let \(V(T, B | \mathcal{G}_T)\) be the value of \textsc{Offline}’s optimal policy, then \(\mathbb{E}[V(T, B | \mathcal{G}_T)] \leq \mathbb{E}[\varphi(T, B | \mathcal{G}_T)]\), hence \(\varphi\) satisfies the initial ordering condition.  
2. If the remaining inventory is \(b \geq 4\) and the remaining time is \(t \geq 2\), then monotonicity is satisfied with zero expected Bellman Loss, i.e., we can take \(\mathbb{E}[L_B(t, b)] \leq f_1 1_{\{t=1 \text{ or } b<4\}}\).  
3. If \(X\) solves \(\varphi(t, b | \mathcal{G}_t)\) and \(X_j \geq 1\), then posting \(f_j\) is a satisfying action.  

We have fixed the relaxation \(\varphi\) already, now we will find a Bellman Loss that satisfies the requirements. We start by recalling the monotonicity condition (Definition 2) we want to prove. Denote \(\mathbb{E}_t[:\mathcal{G}_t] = \mathbb{E}[:\mathcal{G}_t]\). If the inventory is \(b \geq 1\), the random reward of posting price \(j\) at \(t\) is \(f_j Y_{j,t}\) and the random new inventory is \(b - Y_{j,t}\), thus monotonicity is:

\[
\varphi(t + 1, b) \leq \max_j \{\mathbb{E}_{t+1}[f_j Y_{j,t+1} + \varphi(t, b - Y_{j,t+1})] + \mathbb{E}_{t+1}[L_B(t+1, b)]\}.
\]
Because $Q$ is a martingale, we have $E_t[Y_t] = Q(t)$ and we can further simplify the condition to
\[ \varphi(t + 1, b) \leq \max_j \{f_j Q_j(t + 1) + E_{t+1}[\varphi(t, b - Y_{j,t+1})]\} + E_{t+1}[L_B(t + 1, b)]. \] (14)

Let us define $L_B(t + 1, b, j) := \varphi(t + 1, b) - f_j Q_j(t + 1) - E_{t+1}[\varphi(t, b - Y_{j,t+1})]$, which corresponds to the loss in Eq. (14) when we assume a specific price $f_j$ is posted.

**Lemma 5.** If $X$ solves $P[t + 1, b, Q(t + 1)]$ and $X_j \geq 1$, then denoting $Y = Y_{t+1}$, we can write the loss in the Bellman inequality as
\[ L_B(t + 1, b, j) = P[t, b - Q_j(t + 1), Q(t + 1)] - E_{t+1}[P[t, b - Y_j, Q(t)]] \] (15)

**Proof.** By our definition,
\[ L_B(t + 1, b, j) = \varphi(t + 1, b) - f_j Q_j(t + 1) - E_{t+1}[\varphi(t, b - Y_{j,t+1})] \]
\[ = P[t + 1, b, Q(t + 1)] - f_j Q_j(t + 1) - E_{t+1}[P[t, b - Y_j, Q(t)]] . \]

Using Lemma 1, we have $P[t + 1, b, Q(t + 1)] = f_j Q_j(t + 1) + P[t, b - Q_j(t + 1), Q(t + 1)]$, which finishes the proof. \(\square\)

Observe that $L_B$ is characterized by a random LP that depends on $Y_{t+1}$ (which is unknown at time $t + 1$), see Eq. (15). To complete item (2) of Proposition 2, it remains to prove that $L_B(t, b, j)$ characterized in (15) satisfies $E_t[L_B(t, b, j) \leq f_q 1_{\{t = 1, b < 4\}}$. This is proved in Appendix B.

We can then conclude that, with $b \geq 4$ and for each $j$ with $X_j \geq 1$, $E_{t+1}[L_B(t, b, j)] \leq 0$ so that $\varphi(t + 1, b) \leq E_{t+1}[f_j Q_j(t + 1) + \varphi(t, b - Y_{j,t+1})]$, implying that posting price $f_j$ is a satisfying action, which is item (3) of Proposition 2. Notice that here we are using the more general definition of monotonicity; see Remark 1.

### 5.4. Information Loss and Overall Performance Guarantee

We have already established that if $X_j \geq 1$, then posting $f_j$ is a satisfying action. We will now study the information loss and specifically bound $P[Q(t, B^t)]$.

**Proposition 3.** Let $X$ be a solution of $(P[t, b, Q(t)])$. If $X_j \geq 1$, then posting $f_j$ is a satisfying action. Furthermore, the information loss is bounded by $P[Q(t, B^t)] \leq 1/t^2$ for all $t \geq c$, where $c$ depends only on $(f, F)$.

Recall that RABBI chooses $j$ as the maximum entry of the solution to $(P[t, b, E[Q(t)])$, which is a perturbed version of the object of interest. This information loss appears because we need to guess $j$ such that $X_j \geq 1$ without the knowledge of $Q(t)$. To build intuition, we now give a characterisation of the solution of $(P[t, b, q])$. See Fig. 4 for an illustration.
Figure 4  Solution to the pricing problem (13). If \( b/t \in (q_i, q_{i+1}] \), the prices used by the LP are \( f_i, f_{i+1} \) and the amount of time we offer each is piece-wise linear in the budget. For a perturbation \( \tilde{q} \) of \( q \), we superpose the solutions with these different parameters. We guess incorrectly when \( \tilde{x}_i \gg 1 \) and \( x_i < 1 \), which necessitates a substantial perturbation of \( q \).

**Lemma 6.** For given \((t, b, q)\), with \( f_1 > \ldots > f_m \) and \( q_1 < \ldots < q_m \), the solution of \( P[t, b, q] \) is as follows. (i) If \( b \leq t q_1 \), then \( x = (b/q_1, 0, \ldots, 0) \). (ii) If \( b > t q_m \), then \( x = (0, \ldots, 0, t) \).

The proof of Lemma 6 is straightforward. If \( b \) is neither too big nor too small, say \( b \in (t q_i, t q_{i+1}] \), then \( x_j = 0 \) for \( j \neq i, i+1 \), and

\[
\begin{align*}
  x_i &= t q_{i+1} - b \\
  q_{i+1} - q_i \\

  x_{i+1} &= b - t q_i \\
  q_{i+1} - q_i
\end{align*}
\]

As illustrated in Fig. 4, the intuition is that \( Q(t) \) and \( \mathbb{E}[Q(t)] \) must deviate considerably for Rabbi’s guess to be incorrect. The next concentration result is a direct application of the DKW inequality [22] and characterises the deviations of \( Q \) w.r.t. its mean.

**Lemma 7.** For any \( t \), \( \mathbb{P} \left[ \max_i |Q_i(t) - \mathbb{E}[Q_i(t)]| > \sqrt{\frac{\log(t)}{t}} \right] \leq \frac{2}{t^2} \).

**Proof.** Since \( Q_i(t) \) is the empirical distribution of \((V_s, s = 1, \ldots, t)\) at \( f_i \), the DKW inequality [22] states that

\[
\mathbb{P} \left[ \sqrt{t} \sup_i |Q_i(t) - \tilde{F}(f_i)| > \lambda \right] = \mathbb{P} \left[ \sqrt{t} \sup_i \left| \frac{1}{t} \sum_{i=1}^{t} \mathbb{1}_{\{V_i > f_i\}} - \tilde{F}(f_i) \right| > \lambda \right] \leq 2e^{-2\lambda^2}.
\]

We conclude by setting \( \lambda = \sqrt{\log(t)} \). \( \square \)

**Stability of Left-Hand Side Perturbations.** As stated in Algorithm 3, \( \text{Online} \) takes actions based on \( P[t, b, \mathbb{E}[Q(t)]] \), while \( \text{Offline} \) uses \( P[t, b, Q(t)] \). Therefore, for fixed \( (t, b) \), we need to compare solutions of \( P[t, b, q] \) to those of \( P[t, b, q + \Delta] \), where \( \Delta \) is the perturbation. For the remainder of this subsection, we set \( q = \mathbb{E}[Q(t)] \) and \( \Delta = Q(t) - \mathbb{E}[Q(t)] \).
Lemma 8 (Selection Program). Let $V_t = P[t, b, q + \Delta]$ and fix a component $j \in [m]$. Then posting price $f_j$ is satisfying if $P_S[V_t, q + \Delta] \geq 1$, where

$$(P_S[V_t, q + \Delta]) \max \{x_j: f \circ (q + \Delta)'x \geq V_t, (q + \Delta)'x \leq b, 1'x \leq t, x \geq 0\}.$$ 

In other words, $Q(t, b, j) = \{\omega \in \Omega : P_S[V_t, Q(t)] < 1\}$.

Proof. This problem selects, among all the solutions of $P[t, b, q + \Delta]$, the one with largest component $j$. From Proposition 2 we know that, if $X_j \geq 1$, then posting $f_j$ is a satisfying action. □

We turned the condition “there exists $X$ solving $P[t, b, q + \Delta]$ with $X_j \geq 1$” to an optimization program. Let $x$ be the solution to the proxy $P[t, b, q]$. Since the algorithm picks the price with the largest component, assume $x_j = \max_i x_i \gg 1$. In particular, $P_S[v_t, q] \gg 1$ for this fixed $j$. We want to show that $P_S[V_t, q + \Delta] \geq 1$ for that particular $j$. To that end, we need to bound the difference between $P_S[V_t, q + \Delta]$ and $P_S[v_t, q]$. This difference depends on (i) $v_t - V_t$, (ii) $\Delta$, and (iii) the dual variables of $(P_S[V_t, q + \Delta])$. Observe that the quantities (i)-(iii) are random. In Appendix B we establish the necessary bounds on $P_S[v_t, q] - P_S[V_t, q + \Delta]$.

5.5. Numerical Demonstration

We test our algorithm with a collection of ever larger instances. The $k$-th instance has budget $B = 6k$ and horizon $T = 20k$. For each scaling $k$, we run 100,000 simulations. We consider the following primitives: prices are (1, 2, 3) and the valuation $V_t$ has a an atomic distribution on (1, 2, 3) with probabilities (0.3, 0.4, 0.3). The instance is chosen such that it is dual degenerate for (13) (which is known to be hardest setting for heuristic pricing policies).

We consider instances small enough so that we can compute the optimal policy; this endeavour becomes infeasible already for moderate values of $k$ (RAIBI however scales gracefully with $k$ as it only requires re-solving an LP in each period). In Fig. 5 (LEFT) we display the gap between the optimal solution and both the RAIBI and OFFLINE’s value. We observe two important things: (i) the OFFLINE benchmark outperforms the optimal (as it should), but by a rather small margin, and (ii) RAIBI has a constant regret (i.e., independent of $k$) relative to OFFLINE, and hence constant optimality gap. In contrast, a full information benchmark would outperform the optimal by too much to be useful. We note that although in the figure, RAIBI occasionally outperforms the DP, this is a result of the simulation randomness.

In Fig. 5 (RIGHT), we compare RAIBI to the optimal static pricing policy [15], which is known to have regret $\Omega(\sqrt{k})$. In particular, we choose the static price to be the one that maximizes the revenue function $f \cdot D(f) = f \cdot T \cdot \bar{F}(f)$ subject to constraint that $D(f) \leq B$ (i.e., the better of
the two prices: (i) the market clearing price, i.e., that satisfies $D(f) = B$ or (ii) the monopoly price which maximizes $fD(f)$. We note though that when a continuum of prices are allowed, [18] propose an algorithm (that, like RABBI, is based on resolving an optimization problem in each period) which achieves a regret which is (asymptotically) logarithmic in $k$ under certain non-degeneracy assumptions on the optimization problem and differentiability assumptions on the valuation distribution. In contrast, our constant regret guarantees hold under a finite price menu.

6. Online Knapsack With Distribution Learning

We study first the full feedback setting, then in Section 6.4 turn to censored feedback. At each time $t$, the arrival is of type $j \in [n]$ with known probability $p_j$. Type $j$ has a random reward $R_j$, drawn from a distribution $F_j$ that is unknown to ONLINE, and a known weight $w_j$. The reward $R_j$ is revealed only after the decision of accept/reject has been made. At the end of each period, we observe the realization of both accepted and rejected items. In contrast, OFFLINE has access to the distribution $F_j$, but not to the realizations. Let $r_j := \mathbb{E}[R_j]$ and $R_j^t$ be the empirical average of the type-$j$ rewards observed with $t$ periods to go. We assume that, before the process starts, we are given one sample of each type. There is no probing in this setting. The variation relative to the baseline setting here is in terms of what is unknown to the decision maker.

Dynamic programming formulation. We again divide each period $t \in \{T, T-1, \ldots, 1\}$ into two stages, $t$ and $t-1/2$. In the first stage (i.e., period $t$) the input reveals the type $j \in [n]$, and in second stages (i.e., period $t-1/2$) the reward is revealed. The random inputs are given by $\xi^t \in [n]$ and $\xi^{t-1/2} \in \mathbb{R}$. The state space is $\mathcal{S} = \mathbb{R}_{\geq 0} \times \{\emptyset, a, r\}$, where the first component is the remaining
knapsack capacity (or hiring budget). At a first stage, the state is of the form \( s = (b, \emptyset) \), we then choose an action \( u \in \{a, r\} \), discount the capacity if \( u = a \), and transition to a second-stage state with \( \diamond = u \). At the second stage, the state is of the form \( s = (b, \diamond) \), and we collect the reward only if \( \diamond = a \). The only feasible action at a second stage is \( u = \emptyset \), which transitions to a state with the same budget. Formally, the rewards are \( R((b, a), \xi^{t-1/2}, \emptyset) = \xi^{t-1/2} \) and \( R((b, r), \xi^{t-1/2}, \emptyset) = 0 \).

### 6.1. Learning RABBI

The main object for the algorithm is the following LP parametrized by \((b, y, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}^n\)

\[
(P[b, y, z]) \quad \begin{array}{ll}
\text{max} & \sum_j y_j x_j \\
\text{s.t.} & \sum_j w_j x_{ja} \leq b \\
 & x_{ja} + x_{jr} = z_j \\
 & x_{ja}, x_{jr} \geq 0 \\
 & j \in [n].
\end{array}
\] (16)

If we knew the average rewards \( r \), then setting \( y = r \) we obtain the same LP we used for the baseline case. Hence, we interpret \( y \) as our best guess for the unknown rewards \( r \). Otherwise, the interpretation is the same as in Section 2.2. Recall that \( R^t \) are the empirical averages, hence the instantiation of RABBI in Algorithm 5 uses \( y = R^t \) as the natural estimator.

#### Algorithm 4 Learning RABBI

**Input:** Access to solutions of \((P[b, y, z])\)

**Output:** Sequence of decisions for ONLINE.

1. Set \( B^T \leftarrow B \) as the given initial state and \( R^T \) as the single sample of each \( j \).
2. for \( t \in \{T, T - 1, \ldots, 1\} \) do
3. Compute \( X^t \), an optimal solution to \((P[B^t, R^t, E[Z(t)])\).
4. Observe arrival-type \( \xi^t \) and take any action \( \hat{U}^t \in \arg\max_{u = a, r} \{X^t_{\xi^t, u}\} \)
5. if \( \hat{U}^t = a \), collect reward \( \xi^{t-1/2} \) and reduce the budget \( B^{t-1} \leftarrow B^t - w_{\xi^t} \). Else, \( B^{t-1} \leftarrow B^t \).
6. Update empirical averages \( R^{t-1} \) based on \( R^t \) and \( \xi^{t-1/2} \).

### 6.2. Offline Information, Relaxation, and Bellman Loss

We define **OFFLINE** through the filtration \( G_t = \sigma(\{\xi^t : t \in [T]\} \cup \{\xi^\tau : \tau \geq t\}) \). This is a canonical filtration (see Definition 1) with variables \( (G_{\theta} : \theta \in \Theta) = (\xi^t : t \in [T]) \). Observe that the future rewards, corresponding to times \( t - 1/2 \), are not revealed. The relaxation builds on the LP Eq. (16) and is defined as \( \varphi(t, s|G_t) = P[b, r, Z(t)] \) for first stages and

\[
\varphi(t - 1/2, s|G_t) = \begin{cases} 
P[b, r, Z(t - 1)] & \diamond = r \\
\xi^{t-1/2} + P[b, r, Z(t - 1)] & \diamond = a.
\end{cases}
\] (17)
Lemma 9. The relaxation \( \varphi \) defined in (17) satisfies the Bellman Inequalities with exclusion sets

\[ B(t, b) = \{ \omega \in \Omega : \exists X \text{ solving } (P[b, r, Z(t)]) \text{ s.t. } X_{t', r} \geq 1 \text{ or } X_{t', r} \geq 1 \} \]

Proof. The initial ordering in Definition 2 follows from an argument identical to that of Lemma 2. The monotonicity property follows from Proposition 4.

6.3. Information Loss and Overall Performance Guarantee

To complete the proof of Theorem 4, we first give an overview of the estimation process behind Algorithm 4. The relaxation relies on the knowledge of \( r \) (the true expectation) and \( Z(t) \). The natural estimators are the empirical averages \( R_t \) and expectation \( \mu(t) = \mathbb{E}[Z(t)] \), respectively. Specifically, we use maximizers \( X^t \) of \( (P[b, R^t, \mu(t)]) \) to “guess” those of \( (P[b, r, Z(t)]) \).

The overall regret bound is \( r^* \) (Regret_1 + Regret_2), where Regret_1 and Regret_2 are two specific sources of error. When the estimators \( R_t \) of \( r \) are accurate enough, the error is Regret_1 and is attributed to the incorrect “guess” of a satisfying action, i.e., Regret_1 is an algorithmic regret. The second term, Regret_2, is the error that arises from insufficient accuracy of \( R_t \), i.e., Regret_2 is the learning regret. The maximum loss satisfies \( r^* \leq \max_j \{ w_i r_j / w_j - r_i \} \) and we can show that

\[
\text{Regret}_1 \leq 2 \sum_j \frac{(w_{\max}/w_j)^2}{p_j} \quad \text{and} \quad \text{Regret}_2 \leq 16 \sum_j \frac{1}{p_j(w_j \hat{\delta})^2}.
\]

In sum, the regret is bounded by \((\max_j \{ w_i r_j / w_j - r_i \}) \cdot (2 \sum_j \frac{(w_{\max}/w_j)^2}{p_j}) + 16 \sum_j \frac{1}{p_j(w_j \hat{\delta})^2}\).

Proof of Theorem 4. To apply Theorem 5, we first bound the measure of the exclusion sets \( B \) and the “disagreement” sets \( Q \). Recall that \( B(t, b) \) is given in Lemma 9 and \( Q(t, b) \) is the event where \( \hat{U}^t \) is not a satisfying action.

Let \( \sigma : [n] \to [n] \) be an ordering of \( [n] \) w.r.t. the ratios \( \bar{r}_j := \frac{r_j}{w_j} \) such that \( \sigma_j = 1 \) if \( j \) has the highest ratio. Similarly, let \( \hat{\sigma}^t : [n] \to [n] \) be the ordering w.r.t. ratios \( \hat{R}_j^t := R_j^t / w_j \).

Call \( E^t \) the event \( B(t, B^t) \cup Q(t, B^t) \), then

\[
\mathbb{P}[E^t] = \mathbb{P}[E^t, \sigma = \hat{\sigma}^t] + \mathbb{P}[E^t, \sigma \neq \hat{\sigma}^t] \leq \mathbb{P}[E^t, \sigma = \hat{\sigma}^t] + \mathbb{P}[\sigma \neq \hat{\sigma}^t].
\]

Let \( N_j^t \) be the number of type-\( j \) samples observed by the beginning of period \( t \). By definition, since we are given a sample of each type before the process starts, we have \( N_j^t = Z_j(T) - Z_j(t) + 1 \). Since the reward distribution is sub-Gaussian, it satisfies the Chernoff bound [6]

\[
\mathbb{P}[R_j^t - r_j \geq x | N_j^t], \mathbb{P}[R_j^t - r_j \leq x | N_j^t] \leq e^{-N_j^t x^2 / 2} \quad \forall x \in \mathbb{R},
\]
A union bound relying on Eq. (18) gives that

\[ \mathbb{P}[\sigma \neq \hat{\sigma}^t | \mathcal{F}_t] \leq \mathbb{P}[\exists j \text{ s.t. } |\bar{r}_j - \bar{R}_j| \geq \delta/2 | \mathcal{F}_t] \leq 2 \sum_j e^{-N_j t (w_j \delta)^2 / 8}. \]

The variable \( N_j \), recall, is the number of type-\( j \) samples observed by the beginning of period \( t \), hence \( N_j - 1 \) is a Bin\((T - t, p_j)\) random variable. It a known fact that, given \( \theta > 0 \), \( \mathbb{E}[e^{-\theta \text{Bin}(p, m)}] = (1 - p + pe^{-\theta})^m \), thus

\[ \mathbb{P}[\sigma \neq \hat{\sigma}^t] = \mathbb{E}[\mathbb{P}[\sigma \neq \hat{\sigma}^t | \mathcal{F}_t]] \leq 2 \sum_j e^{-(w_j \delta)^2 / 8} (1 - p_j + p_j e^{-(w_j \delta)^2 / 8})^{T-t}. \]

Upper bounding by a geometric sum yields

\[ \text{Regret}_2 := \sum_t \mathbb{P}[\sigma \neq \hat{\sigma}^t] \leq 2 \sum_j \frac{1}{p_j (e^{(w_j \delta)^2 / 8} - 1)} \leq 2 \sum_j \frac{8}{p_j (w_j \delta)^2}. \quad (19) \]

We are left to bound \( \mathbb{P}[E^*, \sigma = \hat{\sigma}^t] \). Let us assume w.l.o.g. that the indexes are ordered so that \( \bar{r}_1 \geq \bar{r}_2 \geq \ldots \geq \bar{r}_n \). The optimal solution of \((P[B^t, r, Z(t)])\), i.e., Offline’s problem, is to sort the items and accept starting from \( j = 1 \), without exceeding the capacity \( B^t \) or the number of arrivals \( Z_j(t) \). Mathematically, the optimal solution \( X^{st} \) to \((P[B^t, r, Z(t)])\) is

\[ X_{ia}^{st} = \min \left\{ Z_i(t), \frac{B^t}{w_i} \right\}, \quad X_{ja}^{st} = \min \left\{ Z_j(t), \frac{B^t - \sum_{i<j} w_i X_{ia}^{st}}{w_j} \right\} \quad j = 2, \ldots, n. \]

For the proxy \((P[B^t, R^t, \mu(t)])\), the optimal solution has the same structure with \( Z_j(t) \) replaced everywhere by \( \mu_j(t) \).

Let \( \xi^t = j \) and \( U \) be any action in \( \text{argmax}\{X_{i,a}^t : u = a, r\} \). We study first the case \( U = a \). If \( X_{j,a}^{st} \geq 1 \) then \( U = a \) would be, by Lemma 9, a satisfying action. If it is not a satisfying action it must then be that \( X_{j,a}^{st} < 1 \) and since the algorithm chooses to accept it must be also that \( X_{j,a}^{st} \geq \mu_j(t)/2 \).

Thus we obtain the following two conditions

\[ X_{j,a}^{st} < 1 \Rightarrow \sum_{i<j} w_i Z_i(t) \geq b \quad \text{and} \quad X_{j,a}^{st} \geq \mu_j(t)/2 \Rightarrow \sum_{i<j} w_i \mu_i(t) + w_j \mu_j(t)/2 \leq b. \]

In the case \( U = r \), \( X_{j,r}^{st} < 1 \) and \( X_{j,r}^{st} \geq \mu_j(t)/2 \) imply

\[ \sum_{i\leq j} w_i Z_i(t) \leq b \quad \text{and} \quad \sum_{i<j} w_i \mu_i(t) + w_j \mu_j(t)/2 \geq b. \]

In conclusion,

\[ \mathbb{P}[E^*, \sigma = \hat{\sigma}^t] \leq \max \left\{ \mathbb{P} \left[ \sum_{i\leq j} w_i (Z_i(t) - \mu_i(t)) \geq \frac{w_j \mu_j(t)}{2} \right], \mathbb{P} \left[ \sum_{i\leq j} w_i (Z_i(t) - \mu_i(t)) \leq \frac{w_j \mu_j(t)}{2} \right] \right\}. \]
These probabilities are bounded symmetrically using the method of averaged bounded differences [13, Theorem 5.3]. Indeed, using the natural linear function \( f(\xi^1, \ldots, \xi^t) = \sum_i w_i \sum_{i=1}^t \mathbb{1}_{\{\xi^i = i\}} \), the differences are bounded by \( |\mathbb{E}[f|\mathcal{F}_l] - \mathbb{E}[f|\mathcal{F}_{l-1}]| \leq w_{\text{max}} \), hence

\[
\text{Regret}_1 := \sum_t \mathbb{P}[E^t, \sigma = \hat{\sigma}^t] \leq \sum_t \sum_j p_j \exp \left( -\frac{2(w_j \mu_j(t)/2)^2}{tw_{\text{max}}^2} \right) \leq 2 \sum_j \left( \frac{w_{\text{max}}/w_j}{p_j} \right)^2.
\]

Together with Eq. (19), we have the desired bound. \(\Box\)

**Remark 6 (Non-i.i.d Arrival Processes).** We used the i.i.d. arrival structure to bound two quantities in the proof of Theorem 4: (1) \( \mathbb{P}[[|Z(t) - \mathbb{E}[Z(t)]| \geq c\mathbb{E}[Z(t)]] \) and (2) \( \mathbb{E}[e^{-cN^j_t}] \), where, recall, \( N^j_t \) is the number of type \( j \) observations. The result holds for other arrival processes that admit these tail bounds.

### 6.4. Censored Feedback

We consider now the case where only accepted arrivals reveal their reward. We retain the assumption of Theorem 4 that there is a separation \( \delta > 0: |\bar{r}_j - \bar{r}_{j'}| \geq \delta \) for all \( j \neq j' \), where \( \bar{r}_j = \mathbb{E}[R_j]/w_j \).

In the absence of full feedback, we will introduce a unified approach to obtaining the optimal regret (up to constant factors), that takes the learning method is a plug-in. The learning algorithm will decide between explore or exploit actions. Examples of learning algorithms, that also give bounds that are explicit in \( t \), include modifications of UCB [28], \( \varepsilon \)-Greedy or simply to set apart some time for exploration (see Corollary 1 below).

Recall that \( \sigma: [n] \rightarrow [n] \) is the ordering of \( [n] \) w.r.t. the ratios \( \bar{r}_j = r_j/w_j \) and \( \hat{\sigma}^t: [n] \rightarrow [n] \) is the ordering w.r.t. ratios \( \bar{R}_j^t = R_j^t/w_j \). The discrepancy \( \mathbb{P}[\sigma \neq \hat{\sigma}^t] \) depends on the plug-in learning algorithm (henceforth BANDITS). BANDITS receives as inputs the current state \( S^t \) (remaining capacity), time, and the natural filtration \( \mathcal{F}_t \). The output of BANDITS is an action in \{explore, exploit\}. If the action is explore, we accept the current arrival in order to gather information, otherwise we call our algorithm to decide, as summarized in Algorithm 5. Note that \( \mathcal{F}_t \) has information only on the observed rewards, i.e., accepted items.

**Theorem 6.** Let \( \text{Regret}_1 \) be the regret of Algorithm 4, as given in Theorem 4. Define the indicators \( \text{explore}_t, \text{exploit}_t \), which denote the output of BANDITS at time \( t \). The regret of Algorithm 5 is at most \( r_{\varphi} M \), where

\[
M = \text{Regret}_1 + \mathbb{E} \left[ \sum_t \text{explore}_t \right] + \mathbb{E} \left[ \sum_t \mathbb{P}[\sigma \neq \hat{\sigma}^t] \text{exploit}_t \right] .
\]
### Algorithm 5: Bandits RABBI

**Input:** Access to Bandits and Algorithm 4.

**Output:** Sequence of decisions for Online.

1. Set $S^T$ as the given initial state
2. for $t = T, \ldots, 1$ do
   3. Observe input $\xi^t$ and let $U \leftarrow \text{Bandits}(T, t, S^t, F_t)$.
   4. If $U = \text{explore}$, accept the arrival
   5. If $U = \text{exploit}$, take the action given by Algorithm 4
   6. Update state $S^{t-1} \leftarrow S^t - w_{\xi^t}$ if accept or $S^{t-1} \leftarrow S^t$ if reject.

The expected regret of Algorithm 5 is thus bounded by the regret of Algorithm 4 in the full feedback setting, plus a quantity controlled by Bandits. In the periods where Bandits says explore (which, in particular, implies accepting the item), the decision might be the wrong one (i.e., different than Offline’s). We upper bound this by the number of exploration periods. This is the second term in $M$. The decision might also be wrong if Bandits says exploit (in which case we call Algorithm 4), but the (learned) ranking at time $t$, $\hat{\sigma}^t$, is different than $\sigma^t$. This is the last term in $M$. Finally, even if the learned ranking is correct, exploit can lead to the wrong “guess” by Algorithm 4 because the arrival process is uncertain. This is the first term in $M$.

Corollary 1 uses a naive Bandits which explores until obtaining $\Omega(\log T)$ samples and achieves the optimal (i.e., logarithmic) regret scaling. The constants may be better depending on the Bandits module we plug into our general algorithm. Any such algorithm has the guarantee given by Theorem 6. With the naive Bandits, the bound easily follows from a generalization of the coupon collector problem [24].

**Corollary 1.** If we first obtain $\frac{8}{(w_j^j)^2} \log T$ samples of every type $j$, then we can obtain $O(\log T)$ regret, which is optimal up to constant factors.

### 7. Concluding Remarks

We developed a framework that provides rigorous support to the use of simple optimization problems as a basis for online re-solving algorithms. The framework is based on comparing Online to a carefully chosen offline benchmark. The (often intuitive) optimization problem that guides the online algorithm is an outcome of two approximation steps: (i) an approximation for Offline’s value function and (ii) a projection thereof to Online’s smaller information set that produces the optimization problem guiding the online algorithm. The translation (or interpretation) of the solution to this optimization problem into actions is strongly grounded in Offline’s approximate value function.
The regret bounds follow from our use of Bellman Inequalities and a useful distinction between Bellman Loss and Information Loss. As is often the case in approximate dynamic programming, the identification of a function $\varphi$ satisfying the Bellman Inequalities requires some ad-hoc creativity but, as our example illustrate, is often rather intuitive. In Appendix A we provide sufficient conditions, applicable to cases where $\varphi$ has a natural linear representation, to verify the Bellman inequalities. These conditions are intuitive and likely to hold for a variety of resource allocation problems. Importantly, once such a function is identified, RABBI provides a way of obtaining online policies from $\varphi$ and mathematical results that produce upper bound on (i) the offline approximation gap and (ii) the online optimality gap.

The OnlineKnapsack with probing is an instance of the larger family of two-stage decision problems wherein there is an inherent trade-off between refined information and the cost of obtaining it. OnlineKnapsack with pricing is a well-studied problem and it is representative of settings where rewards and transitions are random. Our solution to the OnlineKnapsack with learning showcases a separation of the underlying combinatorial problem from the parameter estimation problem.

It is our hope that this structured framework will be useful in developing online algorithms for other problems, whether these are extensions of those we studied here or completely different. In our pricing model, for example, we focus on the one dimensional case (there is inventory of one product). We conjecture that the more general model where a purchase consumes units of multiple products (i.e. components) can be solved using our framework and generate similarly constant-regret guarantees.
Appendix

A. A Sufficient Condition for Bellman Inequalities

In this section, we provide one general sufficient condition for constructing an approximation \( \varphi \) to \textsc{Offline}'s problem. This serves to underscore some of the key elements in a problem's structure that allows one to construct low regret online policies. This guideline does not apply to all the examples we study here: in particular, it applies to the baseline and learning variants, but not to probing or pricing.

We study a particular case of canonical filtrations (see Definition 1), where the random variables \( G_\theta \) that we reveal are the inputs \( \xi^\theta \) for some fixed times \( \Theta \) (see Fig. 2 for an illustration).

Recall that we reveal some inputs to \textsc{Offline}, but not necessarily all of them; we call concealed inputs those not revealed to \textsc{Offline}. Informally speaking, we will show that \( \varphi \) satisfies the Bellman Inequalities if (i) \textsc{Offline}'s relaxed value \( \varphi \) can be computed with a linear program and (ii) the concealed inputs are in the objective function only (not in the constraints). Requirements (i) and (ii) are appealing because they are verifiable directly from the problem structure without any computation.

Recall that, with \( t \) periods to go, \textsc{Offline} knows the randomness \( \{\xi^T, \ldots, \xi^t, \xi_\Theta\} \), where we denote \( \xi_\Theta = (\xi^\theta : \theta \in \Theta) \). In other words, we reveal \( \{\xi^T, \ldots, \xi^t, \xi_\Theta\} \), while the inputs \( \{\xi^l : l < t, l \notin \Theta\} \) are concealed.

Suppose the relaxation is an LP with decision variables \( x \) (see Eq. (6)):

\[
\varphi(t, s|G_t) = \max_{x \in \mathbb{R}^{\Xi \times \Xi[1:T] \times \mathcal{U}}} \{ \mathbb{E}[h(x; \xi^1, \ldots, \xi^T)|G_t] : g(x; s, \xi^T, \ldots, \xi^t, \xi_\Theta) \geq 0 \},
\]

where \( \mathcal{U}, \Xi \) are the control and input spaces. For input \( \xi \), control \( u \), and time \( t \), we interpret \( x_{\xi,t,u} \) as a variable indicating if \textsc{Offline} uses \( u \) at time \( t \) when presented input \( \xi \).

**Proposition 4.** Let \( h, g \) be linear functions and let \( \varphi \) be given by (20). Assume further that the following holds for all \( s, t, u \)

(i) The function \( h \) captures rewards: \( \mathbb{E}[h(e_{\xi^t,t,u}; \xi^1, \ldots, \xi^T)|G_t] \leq \mathcal{R}(s, \xi^t, u) \) for actions \( u \) that are feasible in state \( s \).

(ii) The function \( g \) captures transitions: \( g(e_{\xi^t,t,u}; s, \xi^T, \ldots, \xi^t, \xi_\Theta) \leq g(0; \mathcal{T}(s, \xi^t, u), \xi^T, \ldots, \xi^{t-1}, \xi_\Theta) \).

Then, \( \varphi \) satisfies monotonicity with exclusion sets

\[
\mathcal{B}(t, s) = \{ \omega \in \Omega : \exists X[\omega] \text{ solving } \varphi(t, s|G_t) \text{ s.t. } X_{\xi^t,t,u} \geq 1 \text{ for some } u \in \mathcal{U} \}.
\]
It is natural to say that $h$ captures the reward if the incremental effect of taking the action $u$ given input $\xi^t$ is equal to the immediate reward $\mathbb{E}[h(e_{\xi^t};\xi^1,\ldots,\xi^T)|\mathcal{G}_t] = \mathcal{R}(s,\xi^t, u)$. It is similarly natural to say that $g$ captures transitions if it is stable under the one-step transition, namely, that $g(e_{\xi^t};s,\xi^T,\ldots,\xi^t,\xi_{t+1}) = g(0; \mathcal{T}(s,\xi^t, u),\xi^T,\ldots,\xi^{t-1},\xi_{t+1})$; in other words, this means that taking the action $u$ at time $t$, has the same effect as taking no action at the state $\mathcal{T}(s,\xi^t, u)$. This should hold in any reasonable resource consumption problem, e.g., consuming 1 with $B$ units of budget remaining is the same as not consuming anything with $B - 1$ units. In the result below we make the weaker assumption that these relationships hold as inequalities.

The baseline and learning variants are useful illustrations of Proposition 4.

**Example 3 (Baseline).** Let $\mathcal{G}$ be the full information filtration ($\Theta = [T]$). In Section 2.2 we introduced a linear relaxation for Offline. We start by writing a relaxation in the form of Proposition 4 and show how it subsequently simplifies to the final form in Section 2.2.

Recall that $a, r$ denote the actions accept and reject. A natural “expanded” linear program is

$$\max \left\{ \sum_j \sum_{l=1}^t x_{j,l,a}r_j : \sum_{j,l} w_j x_{j,l,a} \leq s, 0 \leq x_{j,l,a} \leq 1 \{\xi_l = j\} \right\}.$$

Defining the auxiliary variables $x_j := \sum_{l=1}^t x_{j,l,a}$, this is equivalent to $\varphi(t, s|\mathcal{G}) = \max \{ r'x : w'x \leq s, 0 \leq x \leq Z(t) \}$, where, recall $Z_j(t) = \sum_{i=1}^t 1\{\xi_i = j\}$ counts the number of type-$j$ arrivals in the last $t$ periods.

This $\varphi$ also has the form of Proposition 4, with the functions $h$ and $g$ given by (note that the action $r$ has zero objective coefficient)

$$h(x; \xi^1,\ldots,\xi^t) := \sum_j x_{j,a}r_j \quad \text{and} \quad g(x; s,\xi^T,\ldots,\xi^1) := \left( s - \sum_j x_{j,a} \right) \frac{1}{Z(t) - x}.$$

Conditions (i) and (ii) can be easily verified now. The objective $h$ is a linear function of the decision vector $x$ and the constraint function $g$ aggregates $\xi$ into the sums $Z(t)$.

In the learning setting, Offline is presented with a public type $j$ and must decide whether to accept or reject before seeing the private type, which is a reward $R_j$ drawn from an unknown distribution.

**Example 4 (Learning).** Let us model the problem with $2T$ time periods, where at even times the public type is revealed and at odd times the private (reward). In this model, the input $\xi^t$ is an index $j \in [n]$ at even times and it is a reward $R \in \mathbb{R}$ at odd times. Also let us model the random rewards by drawing i.i.d. copies $\{R_{jt}\}_t$ of $R_j$. 
Let us endow Offline with the information of all even times, i.e., Offline knows all the future arriving public types. Specifically, we set \( \Theta = \{ t \in [T] : t \text{ is even} \} \) (see Fig. 2 for a representation of \( \cal G \)). The realizations \( \{ R_{j,t} \}_{j,t} \), drawn at times \( t \notin \Theta \), are concealed. The expanded linear program is

\[
\max \left\{ \sum_{j} \sum_{l=1}^{t} x_{j,l,a} E[R_j] : \sum_{j,l} w_{j} x_{j,l,a} \leq s, 0 \leq x_{j,l,a} \leq 1 \right\}.
\]

As before, we can simplify this LP by aggregating variables, see Section 6 for the details. Here we prefer to study the expanded LP because it exemplifies the conditions in Proposition 4.

The objective function is \( h(x; \xi^1, \ldots, \xi^T) = \sum_{j,l} x_{j,l,a} R_{j,t} \). When we take expectations \( E[\cdot | \cal G_t] \) we arrive at the expression \( \sum_{j,l} x_{j,l,a} E[R_j] \). The constraint function \( g \) is given by the feasibility region of the LP. Conditions (i) and (ii) of Proposition 4 hold with equality. \( \square \)

**Proof of Proposition 4.** Let \( u \in \cal U \) be such that \( x_{\xi^t,l,a} \geq 1 \). Denote \( \theta_t := \{ l \in [T] : l \geq t \} \cup \Theta \), so all the inputs \( (\xi^l : l \in \Theta_t) \) are revealed at time \( t \) (the rest are concealed). By Lemma 1,

\[
\varphi(t,s|\cal G_t) = E[h(e_{\xi^t,l,a}; \xi^1, \ldots, \xi^T)|\cal G_t] + \max_{x} \{ E[h(x; \xi^1, \ldots, \xi^T)|\cal G_t] : g(x; e_{\xi^t,l,a}; s, (\xi^l : l \in \Theta_t)) \geq 0 \}.
\]

Using (i) and (ii) yields

\[
\varphi(t,s|\cal G_t) \leq \mathcal{R}(s, \xi^t, u) + \max_{x} \{ E[h(x; \xi^1, \ldots, \xi^T)|\cal G_t] : g(x; t, (\xi^l : l \in \Theta_{t-1})) \geq 0 \}.
\] (21)

Since \( \cal G_t \) is coarser than \( \cal G_{t-1} \), we know that \( E[\mathcal{E}[\cdot | \cal G_{t-1}]]|\cal G_t] = E[\cdot | \cal G_t] \). Using Eq. (21) and applying Jensen’s Inequality (recall that the maximum of linear functions is a convex function) we obtain

\[
\varphi(t,s|\cal G_t) \leq \mathcal{R}(s, \xi^t, u) + E\left[ \max_{x} \{ E[h(x; \xi^1, \ldots, \xi^T)|\cal G_{t-1}] : g(x; t, (\xi^l : l \in \Theta_{t-1})) \geq 0 \} \right]|\cal G_t].
\]

This corresponds to the required inequality in Definition 2. \( \square \)

The sufficient conditions in Proposition 4 are not necessary; they are not satisfied in the probing setting (Section 4) or in the pricing setting (Section 5). Nevertheless, we are still able to show monotonicity and draw the desired regret bounds.

**B. Proofs from Section 5**

**B.1. Proof of Proposition 2**

To complete the proof of the proposition, it remains to establish that, whenever \( b \geq 4 \) and \( X_j \leq 1 \), then \( E[L_B(t+1, b, j)|\cal G_t] \leq 0 \), where

\[
L_B(t+1, b, j) = P[t, b - Q_j(t+1), Q(t+1)] - E_{t+1}[P[t, b - Y_j, Q(t)]].
\]

**The Correction LP.** Let us fix \((t, b, q)\) and denote \( \tilde{x} \) the solution of \( P[t, b - q_j, q] \). To bound the loss, we must bound right-hand side of (15). That is, we wish to study the effect of perturbing the
budget from \( b - q_j \) to \( b - Y_j \) and the fractions from \( q \) to \( q + \Delta \), where \( \Delta \) is a random vector with zero mean.

Let us re-formulate \( P[t, b - Y_j, q + \Delta] \) based on how much we need to correct \( \bar{x} \):

\[
(P[b - Y_j, q + \Delta]) \max_z \{f \circ (q + \Delta)'(\bar{x} - z) : (q + \Delta)'(\bar{x} - z) \leq b - Y_j, 1'(\bar{x} - z) \leq t, \bar{x} - z \geq 0\}
\]

where we denote \( x \circ y := (x_1y_1, \ldots, x_ny_n) \) as the Hadamard product of two vectors \( x, y \). The new formulation uses decision variables \( z \), which may be negative:

Let us denote the slack variables of \( P[t, b - q_j, q] \) by \( s_1, s_2 \geq 0 \), i.e., \( q'\bar{x} = b - q_j - s_1 \) and \( 1'\bar{x} = t - s_2 \). Using the slack variables, the problem simplifies to

\[
P[b - Y_j, q + \Delta] = f \circ (q + \Delta)'\bar{x} - \min \{f \circ (q + \Delta)'z : (q + \Delta)'z \geq Y_j - q_j + \Delta'\bar{x} - s_1, 1'z \geq -s_2, z \leq \bar{x}\}
\]

(22)

Observe that, since \( \mathbb{E}[\Delta] = 0 \), the first term outside the maximization, namely \( f \circ (q + \Delta)'\bar{x} \), equals \( f \circ q'\bar{x} = P[b - q_j, q] \) in expectation. By Lemma 5, the expected value of the residual LP captures exactly the Bellman Loss. The following result readily proves Proposition 2.

**Lemma 10 (Correction LP).** If we denote \( q = Q(t + 1) \), then the Bellman Loss is bounded by \( \mathbb{E}[L_B(t + 1, b)] \leq \mathbb{E}[P_C[\bar{x}, q + \Delta]], \) where

\[
(P_C[\bar{x}, q + \Delta]) \min_z \{f \circ (q + \Delta)'z : (q + \Delta)'z \geq Y_j - q_j + \Delta'\bar{x} - s_1, 1'z \geq -s_2, z \leq \bar{x}\}
\]

(23)

Furthermore, if \( b \geq 4 \), then \( \mathbb{E}[L_B(t + 1, b)] \leq 0 \).

**Proof.** We already argued that \( \mathbb{E}[L_B(t + 1, b)] \leq \mathbb{E}[P_C[\bar{x}, q + \Delta]] \). To show the desired bound \( \mathbb{E}[L_B(t + 1, b)] \leq 0 \), we exhibit a \( \mathcal{G}_{t+1} \)-measurable feasible solution that achieves said objective value, hence the value of \( P_C \) can only be smaller.

First, if \( Y_j - q_j + \Delta'\bar{x} - s_1 \leq 0 \) (right-hand side of \( P_C \), we can set \( z = 0 \), hence \( P_C[\bar{x}, q + \Delta] \leq 0 \). Let us assume this is not the case. Take \( i \in \arg\max \{q_i \bar{x}_i : i \in [m] \} \) and set \( z = \frac{Y_j - q_j + \Delta'\bar{x} - s_1}{q_i + \Delta_i} \), i.e., only the \( i \)-th component is non-zero. This \( z \) satisfies the constraints \( (q + \Delta)'z \geq Y_j - q_j + \Delta'\bar{x} - s_1 \) and \( 1'z \geq 0 \). It satisfies \( z \leq \bar{x} \) if

\[
\frac{Y_j - q_j + \Delta'\bar{x} - s_1}{q_i + \Delta_i} \leq \bar{x}_i \iff Y_j - q_j + \Delta'\bar{x} - s_1 \leq \bar{x}_i (q_i + \Delta_i).
\]

At most two components of \( \bar{x} \) are non-zero (see Lemma 6), hence by our choice of \( i \) and the fact that \( q'\bar{x} = b - q_j - s_1 \), we get \( q_i \bar{x}_i \geq (b - q_j - s_1)/2 \). Since \( |\Delta| \leq 1/t \) and \( Y_j \in \{0, 1\} \), a simple algebraic check shows that \( b \geq 4 \) implies our desired condition \( z \leq \bar{x} \).

Finally, the objective value of our solution is \( f \circ (q + \Delta)'z = f_i(Y_j - q_j + \Delta'\bar{x} - s_1) \), which is bounded by zero in expectation. \( \square \)
B.2. Proof of Proposition 3

Recall that we wish to establish that, if \( \bar{\phi} \) (used by ONLINE) has a solution with \( x_j = \max_i x_i >> 1 \), then posting price \( f_j \) is a satisfying action. To establish this it remains to bound on the difference between the LP \( P_S[v_t, q] \) and its “perturbed” version \( P_S[V_t, q + \Delta] \). To that end, we first establish a bound on \( v_t - V_i \); see item (i) in the discussion following Lemma 8.

**Lemma 11.** For fixed \( b \), denote \( V_i = P[t, b, Q(t)] \) and \( v_i = P[t, b, \mathbb{E}[Q(t)]] \). If \( t \geq c \), then, with probability at least \( 1 - 2/t^2 \), we have \( v_i - V_i \geq -p_1 \sqrt{t \log(t)} \). The constant \( c \) depends on \( \bar{F}(p_1) \) only.

**Proof.** Set \( q = Q(t) \) and \( \Delta = \mathbb{E}[Q(t)] - Q(t) \). Take \( \bar{x} \) to be a solution of \( V_i \) and use the correction program in Eq. (23) without the terms \( Y \). Let \( z \) be the solution to the proxy \( V \) and \( \bar{v} \) be the solution to the proxy \( V \) and \( \bar{x} \). By Lemma 7, if \( q_i \geq \sqrt{\log(t)} \), we can guarantee the inequality with the desired probability. Finally, we have proved

\[
v_t = V_i + p \circ \Delta \bar{x} - \min \{ p \circ (q + \Delta) x : (q + \Delta) x \geq \Delta \bar{x} - s_1, 1 \bar{x} \geq -s_2, z \leq \bar{x} \}.
\]

We will argue an upper bound on the minimisation problem by exhibiting a feasible solution. Put

\[
z_i = \frac{\Delta_{i}^+}{q_i + \Delta_{i}^+} \bar{x}_i,
\]

where \( \Delta_i^+ = \Delta_{i} \mathbb{1}_{\{\Delta_i > 0\}} \). Clearly this choice of \( z \) satisfies the constraints \( (q + \Delta) z \geq \Delta \bar{x} - s_1 \) and \( 1 \bar{x} \geq -s_2 \). The constraint \( z \leq \bar{x} \) is equivalent to \( \Delta_i^+ \leq q_i + \Delta_i \), which holds whenever \( q_i \geq -\Delta_i \mathbb{1}_{\{\Delta_i < 0\}} \). By Lemma 7, if \( \sqrt{\log(t)} \geq q_i \), i.e., \( t \) large enough, we can guarantee the inequality with the desired probability. Finally, we have proved

\[
v_t \geq V_i + p \circ \Delta \bar{x} - \min \{ -q \circ (q + \Delta) x : \Delta \bar{x} - s_1, 1 \bar{x} \geq -s_2, z \leq \bar{x} \} = V_i + \sum_i p_i \bar{x}_i (\Delta_i - \Delta_i^+) \geq V_i - p_1 \sqrt{t \log(t)}.
\]

In the last inequality we used \( 1 \bar{x} \leq t \). \( \square \)

**Lemma 12.** Let \( x \) be the solution to the proxy \( P[t, b, q] \). If \( x_j \geq c \sqrt{t \log(t)} \), then posting price \( f_j \) is a satisfying action. The constant \( c \) depends only on the primitives \( (f, F) \).

**Proof.** Define \( D := \{ x : 1^2 \leq t, x \geq 0 \} \). Let \( L(x, \lambda, v_t, q) \) be the partial Lagrangean of \( (P_S[v_t, q]) \) in \( D \), where \( \lambda_1, \lambda_2 \geq 0 \)

\[
L(x, \lambda, v_t, q) = x_j + \lambda_1 (f \circ q' x - v_t) + \lambda_2 (b - q' x).
\]

Basic algebra shows that

\[
L(x, \lambda, V_t, q + \Delta) = L(x, \lambda, v_t, q) + \lambda_1 (V_t - V_i + f \circ \Delta' x) - \lambda_2 \Delta' x.
\]

From Lemma 7 and Lemma 11 we have \( |\Delta' x| \leq \sqrt{t \log(t)} \) and \( v_t - V_i \geq -f_1 \sqrt{t \log(t)} \). Let \( \lambda_1^*, \lambda_2^* \) be the optimal dual variables of \( (P_S[V_t, q + \Delta]) \) and assume bounds \( \lambda_i^* \leq \bar{\lambda}_i \), then

\[
L(x, \lambda^*, V_t, q + \Delta) \geq L(x, \lambda^*, v_t, q) - (2 \bar{\lambda}_1 f_1 + \bar{\lambda}_2) \sqrt{t \log(t)}.
\]
Let \( g(t) := (2\bar{\lambda}_1 f_1 + \bar{\lambda}_2) \sqrt{t \log(t)} \). We conclude,

\[
P_{S_t}[V_t, \mathbf{q} + \Delta] = \max_{\mathbf{x}, \lambda^*} L(\mathbf{x}, \lambda^*, \mathbf{q} + \Delta) \\
\geq \max_{\mathbf{x}, \lambda^*, v_t, \mathbf{q}} L(\mathbf{x}, \lambda^*, v_t, \mathbf{q}) - g(t) \\
\geq \min_{\lambda} \max_{\mathbf{x}, \lambda, v_t, \mathbf{q}} L(\mathbf{x}, \lambda, v_t, \mathbf{q}) - g(t) \geq \bar{x}_j - g(t).
\]

To bound the dual variables, we take the worst case change in the objective function \( x_j \) given right-hand side perturbations, hence we conclude \( \bar{\lambda}_1 \leq \frac{1}{f_1 \sqrt{Q_j(t)}} \) and \( \bar{\lambda}_2 \leq \frac{1}{Q_j(t)} \).

To finalize the proof of Proposition 3 let \( \mathbf{q} = \mathbb{E}[Q(t)] \) and \( \mathbf{x} \) be the solution to \( P[t, b, \mathbf{q}] \). Take \( j \in \arg\max\{x_i : i \in [m]\} \). If \( x_j \geq c \sqrt{t \log(t)} \), we can conclude the proof by invoking Lemma 12. Observe that, if \( \mathbf{1}' \mathbf{x} = t \), then \( x_j \geq t/m \), hence the condition holds for \( t \) large enough.

We are left with the case \( \mathbf{1}' \mathbf{x} < t \) and \( x_j < c \sqrt{t \log(t)} \). We will prove that, in this case, both \textsc{Online} and \textsc{Offline} post price \( f_1 \). By Lemma 6, if \( \mathbf{1}' \mathbf{x} < t \), then the solution is \( \mathbf{x} = (b/q_1, 0, \ldots, 0) \) and we are assuming \( b/q_1 < c \sqrt{t \log(t)} \). If \( Q_1(t) \geq b/t \), then, by Lemma 6, \( (P[t, b, Q(t)]) \) also posts price \( f_1 \) and there is no information loss. Finally, the loss is bounded by \( \mathbb{P}[Q_1(t) < b/t] \) when \( q_1 > \frac{b}{t} \sqrt{t \log(t)}/c \). An application of Lemma 7 concludes the proof.

\section*{C. Connections to Information Relaxations}

Our work is related to the information-relaxation framework developed in [5, 7]. The information-relaxation framework is a fairly general way to endow \textsc{Offline} with additional information, but at the same time forcing him to pay a penalty for using this information. The dualized problem (with the penalties) is an upper bound on the performance of the best online policy.

The main distinctions with our approach are:

1. Information Relaxation requires to identify \textsc{Offline}’s filtration and penalties to build a proxy for \textsc{Offline}’s value function. This proxy can then be used to assess the performance of specific online policies.

The proxy that is developed—as the true \textsc{Offline} value in our framework—may be difficult to compute. To overcome this difficulty, [5] proposes an approximation through which penalties can be computed and hence an upper bound can be obtained.

2. Our framework requires, as well, identifying a suitable information structure (a filtration) and a relaxation \( \varphi \). Because we allow for a Bellman Loss, we can develop \( \varphi \tilde{\varphi} \) that are computationally tractable. In most cases, a linear program. The framework explicitly then provides a mechanism, the RABBI algorithm, to derive a good online policy.

There is also an explicit mathematical connection. To state it, we first present a weaker version of our Bellman Inequalities, called thus because it is easier to find an object \( \varphi \) under this definition.
Recall that, for a given non-anticipatory policy $\pi$, we denote $v^*_\pi$ the expected value. Observe that the distinction with Definition 2 is in the initial ordering condition; we now require $\phi$ to upper bound the online value instead of the best offline.

**Definition 5 (Weak Bellman Inequalities).** The sequence of r.v. $\{\varphi(t,s)\}_{t \in \mathcal{T}, s \in \mathcal{S}}$ satisfies the Weak Bellman Inequalities w.r.t. filtration $\mathcal{G}$ and events $\mathcal{B}(t,s) \subseteq \Omega$ if $\varphi(t,s)$ is $\mathcal{G}_t$-measurable for all $t,s$ and the following holds:

1. Initial ordering: $\max_\pi v^*_\pi \leq \mathbb{E}[\varphi(T,S^T|\mathcal{G}_T)]$, where $S^T$ is the initial state.
2. Monotonicity: $\forall s \in \mathcal{S}, t \in [T], \omega \notin \mathcal{B}(t,s)$,

$$
\varphi(t,s|\mathcal{G}_t) \leq \max_{u \in \mathcal{U}} \{r(s, \xi^t, u) + \mathbb{E}[\varphi(t-1, T(s, \xi^t, u)|\mathcal{G}_{t-1})]|\mathcal{G}_t]\}.
$$

In Proposition 2.1 in [5] it is shown that if $\varphi$ is some function that satisfies the Bellman equation for Offline with the penalized immediate rewards function, then, in particular, it satisfies the initial ordering above. Since such $\varphi$ satisfies, by construction, the Bellman inequality the following is an immediate corollary.

**Proposition 5 (Proposition 2.1 in [5]).** Given feasible penalties $z_t$, the penalized value function satisfies Definition 2 with exclusion sets $\mathcal{B}(t,s) = \emptyset$.

Our framework is a structured approach for building a computationally tractable $\varphi$, and deriving an online policy is bounded regret, without pre-computing penalties.

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