CONSTRUCTION AND RESEARCH OF FULL BALANCE ENERGY OF VARIATIONAL PROBLEM MOTION SURFACE AND GROUNDWATER FLOWS

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Abstract
Based on the laws of conservation of mass and momentum the basic equations of motion with unknown quantities velocity and piezometric pressure are written. These equations are supplemented with boundary and initial conditions describing the motion of compatible flows. Based on the laws of motion continuum, received conditions contact on the common border interaction of surface and groundwater flows. Variational problems formulated compatible flow. Energy norms of basic components of variational problem are analyzed. Correctness of constructing variational problem arising from construction of the energy system of equations that allow to investigate properties of the problem solution, its uniqueness, stability, dependence on initial data and more. Energy equation of motion of surface and groundwater flows are derived and investigated. It is shown that the total energy compatible flow depends on sources that are located inside the domain or on its border.

Keywords: surface flow, groundwater flow, watershed, incompressible fluid, velocity fluid and hydrostatic and piezometric pressure, energy equation, bilinear form, Initial and boundary conditions, interface conditions, coupling flow.

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1. Introduction
An important role in studying the water cycle plays hydrological system. In general, research integrity of the system, taking into account all impacts, are complex and not always feasible problem for the study because only investigated some of the area involved in the water cycle [1–3]. Highly likely part of the territory may be a watershed area (Fig. 1), which is characterized by similar climatic conditions and is influenced by such factors that affect the water movement.

![Fig. 1. Two-dimensional projection watershed on the plane XOX_2](image)

At the watershed may be an interaction between flow and located above and below water-bearing layers. Models of different dimensions are used in each layer to describe the water movement and their solutions are connected by boundary conditions [4–6]. We select in solid medium (liquid) moving surface layer \( F(t) \in \mathbb{R}^3 \) (Fig. 2) of such a structure

\[
\Omega_+(t) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3, \quad \eta(x) < x_3 < \nu(x, t) \quad \forall x = (x_1, x_2) \in \Omega(t) \right\}.
\]  

Let’s denote projection of its lower
\[ \Omega(t) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = \eta(x), \forall x = (x_1, x_2) \in \Omega(t) \right\} \] (2)

and upper
\[ \Lambda(t) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = \nu(x, t), \forall x = (x_1, x_2) \in \Omega(t) \right\} \] (3)

bases on the plane \( 0x_1x_2 \). The rest of the surface layer
\[ \Gamma_F(t) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3, \quad \eta(x) < x_3 < \nu(x, t), \forall x = (x_1, x_2) \in \Omega(t) \right\} \] (4)

will be called the lateral surface layer \( F(t) \).
Similarly denote part of fluid that moves in the soil, so
\[ \Omega_p(t) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3, \quad h(x) < x_3 < \eta(x), \forall x = (x_1, x_2) \in \Omega(t) \right\} \] (5)

the projection of the lower part will be written as
\[ \Lambda_p(t) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = h(x), \forall x = (x_1, x_2) \in \Omega(t) \right\} . \] (6)

Then, a layer of groundwater
\[ \Gamma_L(t) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3, \quad h(x) < x_3 < \eta(x) \quad \forall x \in \Gamma_L(t) \right\} . \] (7)

Fig. 2. General view of the model of flows and their cross-section

2. Materials and Methods

2.1. Initial boundary value problem of interaction of water flows

We formulate initial boundary problem of motion of surface and groundwater flows on the surface watershed considering boundary and initial conditions [7–9].

Find unknown quantities \( \{u, p, \phi\} \) such that satisfy the following system of equations:

\[ \frac{\partial}{\partial t} (\rho u_i) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left( \rho u_i u_k \right) - \rho f_i - \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} = 0, \] (8)

\[ \sigma_{ik} = -p \delta_{ik} + \tau_{ik}, \]

\[ \tau_{ik} = 2\mu \varepsilon_{ik}, \quad i, j = 1, 2, 3, \]

\[ \varepsilon_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]
\[ \frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \frac{\partial (\rho u_k)}{\partial x_k} = 0, \quad \text{in} \quad \Omega_\rho \times (0, T], \quad (9) \]

\[ m \frac{\partial \rho}{\partial t} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left( k \frac{\partial \rho}{\partial x_j} \right) + \epsilon \quad \text{in} \quad \Omega_p \times (0; T], \quad (10) \]

where \( \{u(x,t)\}_{i=1}^{3} \) and \( p_0 = p_0(x,t) \) – sought velocity vector of fluid and hydrostatic pressure, respectively; \( F = \{ f_i(x) \}_{i=1}^{3} \) – mass forces; \( \rho = \rho(x,t) > 0 \) – density of the mass water flow; \( \mu = \mu(x) > 0 \) – viscosity coefficient; \( \{ \sigma_{ij} \}_{i,j=1}^{3} \) – tensors of velocities of deformation and stress of the liquid at the point \( x \) in time \( t \); \( \delta_{ij} \) – Kronecker symbol; \( k = k(x,t) \) – filtration coefficient; \( m = m(x,t) \) – coefficient of specific water loss; \( \epsilon = \epsilon(x,t) \) – known function of sources of water influx;

\[ \varphi = x_i + \frac{p_0}{\rho g} \quad (11) \]

piezometric pressure;

\[ q = -k \nabla \varphi \quad (12) \]

flow (flow separation); \( \nu = \nu(x,t) \) – velocity vector of fluid in the ground; \( \nu = \frac{q}{\omega}, \omega \) – volume porosity; \( \mathbf{n}_i = -\mathbf{n}_p \) – vectors normal to the boundary area \( \Omega_\rho \) and \( \Omega_p \) in accordance;

\[ \Omega = \Omega_\rho \cup \Omega_p, \Omega_\rho \cap \Omega_p = \emptyset, \Omega_\rho \cap \Omega_p = \Gamma, \]

\[ \partial \Omega_p = \Gamma_p \cup \Lambda_p \cup \Gamma, \partial \Omega_\rho = \Gamma_\rho \cup \Lambda_\rho \cup \Gamma. \]

Boundary conditions [10, 11]:

\[ \bar{u}_i = 0 \quad \text{on} \quad \Gamma_p, \quad i = 1, 2, 3, \quad (13) \]

\[ \sigma_{st} = \sigma, \quad \text{on} \quad \Lambda_p, \quad (14) \]

\[ u_3 + R = \frac{\partial \nu}{\partial t} + u_1 \frac{\partial \nu}{\partial x_1} + u_2 \frac{\partial \nu}{\partial x_2} \quad \text{in} \quad \Omega_\rho \times (0, T], \quad (15) \]

where \( R \) – velocity of falling rain drops, \( u_1^0, u_2^0 \) – horizontal components of velocity on the free surface \( \nu(x,t) (\Lambda_\rho) \);

\[ \nu, \mathbf{n}_p = \vec{\nu} \quad \text{on} \quad \Gamma_p; \quad (16) \]

\[ \nu_l = u_2 = 0 \quad \text{on} \quad \Lambda_p, \quad (17) \]

\[ u_3 = -1 \quad \text{on} \quad \Lambda_p, \quad (18) \]

where \( I \) – known function that describes the velocity of fluid flow through the surface \( \Lambda_p \).

Initial conditions:

\[ u_{|t=0} = u_0, \quad \text{in} \quad \Omega, \quad (19) \]

Contact flow conditions on a common boundary \( \Gamma \) [4–6, 8]:

\[ \mathbf{Computer Sciences and Mathematics} \]
\[ \sigma_m(u,p_r) = p_r, \]
\[ \sigma_m = 0, \]  
\[ u_n = -u_n. \]  

2. Variational formulation of the problem of interaction of water flows

We introduce the following bilinear forms:

\[ M_v(r, w, q) = \sum_{i=1}^{3} r_i w_i q_i \, ds, \quad N_v(w; u, q) = \sum_{k=1}^{3} \sum_{j=1}^{3} \rho w_i \frac{\partial u_j}{\partial x_j} q_i \, ds, \]
\[ C_v(w, q) = \int_{\Omega} 2\mu \, e(w) : e(q) \, ds, \]
\[ A_v(w, q) = -\int_{\Omega} \text{div} q \, ds, \quad Y_v(w, q) = -\int_{\Omega} w q \, dy, \quad B_v(p, w) = -\int_{\Omega} p \, V w \, ds. \]

Introduce spaces:

\[ H_v := \{ \xi \in (H'(\Omega_v))^3 \mid \xi = 0 \text{ on } \Gamma \}, \]
\[ H_p := \{ \psi \in H'(\Omega_p) \mid \psi = 0 \text{ on } \Gamma \}, \quad W := H_v \times H_p, \quad \mathcal{S}_j : W \rightarrow R, \quad j = 1,3, \]
\[ \langle \mathcal{S}_1, \xi \rangle = \sum_{i=1}^{3} \int_{\Omega_v} \rho f_i \xi_i \, ds + \int_{\mathcal{L}_v} (\xi_p + \xi_r) \, d\gamma, \]
\[ \langle \mathcal{S}_2, \theta \rangle = -\int_{\Omega_p} \theta \, d\gamma, \quad \langle \mathcal{S}_3, \psi \rangle = \int_{\Omega_p} \frac{\varepsilon(x,1)\rho g \psi}{\omega} \, dp - \int_{\mathcal{L}_p} \psi \, \rho \, g \, d\gamma. \]

Let’s denote

\[ \tilde{\psi} = \psi\rho g, \quad m = \frac{m}{\omega}, \]

Then, let’s write the following variational problem [1–2, 10, 11]:

Find \( \{ u, p, \varphi \} \in V \times Q \times W, \)

\[ M_{\Omega_1} (p; u', \xi) + N_{\Omega_1} (u; u, \xi) + A_{\Omega_1} (p, \xi) + C_{\Omega_1} (u, \xi) + Y_{\Omega_1} (u, \xi) = \langle \mathcal{S}_1, \xi \rangle, \quad \forall \xi \in V, \]  
\[ B_{\Omega_1} (u, \theta) + Y_{\Omega_1} (\theta, u) = \langle \mathcal{S}_2, \theta \rangle, \quad \forall \theta \in Q, \]  
\[ M_{\Omega_2} (\tilde{m}; \varphi', \tilde{\psi}) + A_{\Omega_2} (\tilde{\psi}, \tilde{u}) + Y_{\Omega} (\tilde{\psi}, u) = \langle \mathcal{S}_3, \psi \rangle, \quad \forall \psi \in W \]  
with initial conditions

\[ M_{\Omega_1} (u''(0) - u_0, \xi) = 0, \]  
\[ B_{\Omega_1} (p(0) - p_0, \theta) = 0; \]  
\[ M_{\Omega_2} (\varphi'(0) - \varphi_0, \tilde{\psi}) = 0. \]
Let’s calculate, considering initial conditions (24)–(26) and boundary conditions (13)–(18), values of variables $u$ and $p$ with relations (21) and (22). Then on the basis of coupling flow conditions (interface conditions) (20) and boundary condition (11) the value of the variable $\varphi$ is calculated from (23).

### 2.3. The properties of the components and norms of variational problem interaction water flows.

It should be noted that trilinear form

$$ N_i(w;u,q) = \sum_{j,k} \rho w_j \frac{\partial u_i}{\partial x_j} q_k ds, $$

is continuous and bilinear form

$$ C_i(w,q) = \int 2\mu e(w) : e(q) ds $$

continuous and symmetrical.

It is a scalar product in the space $H_p$ and creates a norm

$$ \|w\|_{H_p} = \sqrt{C_i(w,w)}, \forall w \in H_p. $$

Then, let’s write the scalar function $\varphi$ bilinear forms

$$ D_i(\varphi, \psi) = \int k(x,t) \nabla \varphi \cdot \nabla \psi dp. $$

which is continuous and integral in the space of admissible functions $H_p$. It is also symmetrical and forms a semi-norm

$$ \|\varphi\|_{H_p} = \sqrt{D_i(\varphi, \varphi)}, \forall \varphi \in H'(\Omega_i). $$

Let’s consider the properties of bilinear forms

$$ A_i(w,q) = \int \text{div} w \text{div} q ds. $$

In space $H_p$, it is continuous, integral and symmetrical, and also forms the norm

$$ \|q\|_{H_p} = \sqrt{A_i(q,q)}, \forall q \in H_p. $$

### 3. Results of research

#### 3.1. Equation balance energy of coupling water flow

Let’s write variational equations for momentum

$$ M(u; p, u') + N(\sigma; u, u) + C(u; u, u) = \left(\varphi_1, u\right) - Y(\varphi_1, u; u) - A(u; u) = \left(\psi_1, u\right) - Y(u; u; u) - A(u; u). $$

Let’s write left side of the equation:

$$ \int_{\Omega_i} \sum_{k=1}^{3} \rho u'_i u_i ds + \int_{\Omega_i} \sum_{k=1}^{3} \rho u k \frac{\partial u_i}{\partial x_k} u_i ds - \int_{\Omega_i} \sum_{k=1}^{3} \sigma_{\alpha} \frac{\partial u_i}{\partial x_k} u_i ds = \int_{\Omega_i} \sum_{k=1}^{3} \rho u'_i u_i ds + $$

$$ + \int_{\Omega_i} \sum_{k=1}^{3} \rho u k \frac{\partial u_i}{\partial x_k} u_i ds + \int_{\Omega_i} \sum_{k=1}^{3} \sigma_{\alpha} \frac{\partial u_i}{\partial x_k} u_i ds - \int_{\Omega_i} u_i \sum_{k=1}^{3} \sigma_{\alpha} n_{i_k} d\gamma. $$

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Given that

\[ \sigma_{ik} = -p\delta_{ik} + 2\mu e_{ik} \]

rewrite (34) in the following form

\[
\int \sum_{k=1}^{3} \rho u_i u_i u_i d\gamma + \int \sum_{k=1}^{3} \sum_{j=1}^{3} \rho u_i \frac{\partial u_i}{\partial x_j} d\gamma - \int \rho u_i d\gamma + \\
+ \int_{\Omega_k} u \nabla d\gamma + \int 2\mu (e(u) : e(u)) d\gamma - \int u \sum_{k=1}^{3} \sigma_{ik} n_i d\gamma.
\] (35)

Let's write left side of variational equations for the law of conservation of mass flow

\[
\int u \cdot n_p d\gamma - \int_{\Omega_k} u \nabla d\gamma = 0.
\] (36)

Substituting (23) in place of \( \psi \) the function \( \varphi \) will be

\[
\int m \frac{\partial \varphi}{\partial t} \rho g dp - \int \kappa \frac{\partial \varphi}{\partial n_p} q d\gamma + \\
+ \int \sum_{k=1}^{3} \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_j} dp - \int c \varphi d\gamma = 0.
\] (37)

Let's multiply (37) on the expression \( \frac{\rho g}{\omega} \), then

\[
\frac{1}{2} \int \frac{\partial \varphi^2}{\partial t} \frac{\rho g}{\omega} dp - \int \frac{\partial \varphi}{\partial n_p} \rho g d\gamma + \\
+ \int \sum_{k=1}^{3} \frac{k(x,t)}{\omega} \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_j} dp - \int \frac{\rho g}{\omega} \varphi d\gamma = 0.
\] (38)

Let's estimate the term in (38) on the boundary \( \Omega_k \)

\[
- \int \frac{k}{\omega} \frac{\partial \varphi}{\partial n_p} \rho g d\gamma = - \int \frac{k}{\omega} \frac{\partial \varphi}{\partial n_p} \rho g d\gamma - \int \frac{k}{\omega} \frac{\partial \varphi}{\partial n_p} \rho g d\gamma - \\
- \int \frac{k}{\omega} \frac{\partial \varphi}{\partial n_p} \rho g d\gamma = - \int k \rho g \nabla \varphi n_p d\gamma - \int k \nabla \varphi n_p \rho g d\gamma = \\
= \int_{\gamma} p \cdot n u_i d\gamma - \int \nabla \varphi g d\gamma.
\] (39)

Simplifying a term on the border \( \Omega_k \) in the form (35), we obtain

\[
- \int \sum_{k=1}^{3} \sigma_{ik} n_i d\gamma = - \int \sum_{k=1}^{3} (u_n \sigma_{nk} + u_n \sigma_{ki}) d\gamma - \\
- \int \sum_{i=1}^{3} (u_n \sigma_{ni} + u_n \sigma_{ni}) d\gamma - \int \sum_{k=1}^{3} (u_n \sigma_{nk} + u_n \sigma_{nk}) d\gamma.
\] (40)
Rewriting the previous expression (41) in a more convenient form, including property incompressible environment and boundary conditions (13)–(18), we obtain

\[
\int \sum_{\xi_k} \rho \frac{\partial u_i}{\partial x_k} du_i dx + \int \sum_{\xi_k} \rho u_i \frac{\partial u_j}{\partial x_k} u_j dx - 
\int p \ u_n d\gamma - \int p \ u_s d\gamma - \int p \ u_n d\gamma + \int u \ V p \ ds + \int 2\mu \ e(u) : e(u) dx - 
\int (u_n p_n + u_s \sigma_n) d\gamma - \int (u_n \sigma_m + u_s \sigma_n) d\gamma + \int u_n p d\gamma - \int u \ V p d\gamma + 
\int \frac{1}{2} \rho \ \frac{\partial \rho}{\partial t} dx + \int \sum_{j=1}^{3} k(x,t) \left( \frac{\partial \rho}{\partial x_j} \right)^2 \rho g d\gamma - 
\int p \ u_n d\gamma + \int \bar{u} \ \rho g d\gamma = 0.
\] (42)

Let’s analyze the terms on joint border \( \Gamma \)

\[
\int (u_n p_n + u_s \sigma_n (u) - p_n \ u_n') d\gamma.
\]

Given the terms the coupling (20), integral to the common border \( \Gamma \) is zero.
From the expression (36) given kinematic condition (15) for equation of continuity will be

\[
\int u_n p d\gamma = \int u_n p d\gamma.
\] (43)

Thus, the energy balance equation of compatible motion of surface and groundwater flow is written:

\[
\int \sum_{\xi_k} \rho \frac{\partial u_i}{\partial x_k} du_i dx + \int \sum_{\xi_k} \rho u_i \frac{\partial u_j}{\partial x_k} u_j dx + \int 2\mu \ e(u) : e(u) dx - 
\int \frac{1}{2} \rho \ \frac{\partial \rho}{\partial t} dx + \int \sum_{j=1}^{3} k(x,t) \left( \frac{\partial \rho}{\partial x_j} \right)^2 \rho g d\gamma - 
\int p \ u_n d\gamma + \int \bar{u} \ \rho g d\gamma + \int \rho \ \frac{\partial \rho}{\partial t} dx.
\] (44)

Rewriting (44) through a total derivative, we have
As we see from (45), the total energy flow depends on the energy sources that are located within the region or within its boundaries.

4. Conclusions

On the basis of conservation laws basic equations and boundary and initial conditions are derived describing the compatible motion flow of surface and ground water with unknown values of velocity and piezometric pressure. Variational problems of compatible flow are formulated and the contact conditions on the common border are obtained based on the laws of motion continuum. Energy standards of basic components of variational problem are analyzed. Full energy equation of energy balance for coupling motion of surface and groundwater flows are constructed and studied that makes it possible to investigate the properties of solutions of the problem, such as stability, regularity, existence, convergence and so on.

References

[1] Shlychkov, V. A. (2007). Numerical simulation of currents and admixture transport in a multi–arm river channel. Bull. Nov. Comp. Center, Num. Model in Atmosph., etc., 11, 79–85.
[2] Kuchment, L. S., Gelfan, A. N. (2002). Estimation of Extreme Flood Characteristics Using Physically Based Models of Runoff Generation and Stochastic Meteorological Inputs. Water International, 27 (1), 77–86. doi: 10.1080/02508060208686980
[3] Panday, S., Huyakorn, P. S. (2004). A fully coupled physically-based spatially-distributed model for evaluating surface/subsurface flow. Advances in Water Resources, 27 (4), 361–382. doi:10.1016/j.advwatres.2004.02.016
[4] Lions, J. L., Temam, R., Wang, S. (1993). Models for the coupled atmosphere and ocean. (CAO I, II). Computational Mechanics, 1 (1), 120.
[5] Discacciati, M., Quarteroni, A., Valli, A. (2007). Robin-Robin Domain Decomposition Methods for the Stokes–Darcy Coupling. SIAM Journal on Numerical Analysis, 45 (3), 1246–1268. doi: 10.1137/06065091x
[6] Cesmelioglu, A., Chidyagwai, P., Riviere, B. (2013). Continuous and discontinuous finite element methods for coupled surface-subsurface flow and transport problems. Rice University, 23.
[7] Venherskyi, P. S. (2014). Numerical investigation mathematical models coupled flow of surface and ground water from the catchment area. Mathematical and computer modeling, 10, 33–42.
[8] Venherskyi, P. S. (2014). About the problem of coupled motion of surface and ground water from the catchment area. Bulletin of Lviv University. Series Applied Mathematics and Computer Science, 22, 41–53.
[9] Venherskyi, P. S., Demkovych, O. R. (2002). Mathematical modeling of ground water in the saturated zone. 9-th National Conference Modern Problems of Applied Mathematics and Informatics, 1–36.
[10] Temam, R. (1995). Navier-Stokes equations and nonlinear functional analysis. SIAM, 148. doi: 10.1137/1.9781611970050
[11] Trushhevskyi, V. M., Shynkarenko, H. A., Shcherbyna, N. M. (2014). Finite element method and artificial neural networks: theoretical aspects and application. Lviv: LNU Ivan Franko, 396.