EQUIDISTRIBUTION AND INEQUALITIES FOR PARTITIONS INTO POWERS

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Abstract. In this paper, we study partitions into powers with an odd and with an even number of parts. We show that the two quantities are equidistributed, and that the one which is bigger alternates according to the parity of $n$. This extends a similar result established by the author (2020) for partitions into squares. By modifying a certain argument from the proof in the case of partitions into squares and by invoking a bound on Gauss sums found by Banks and Shparlinski (2015) using the work of Cohn and Elkies (2003) on lower bounds for the center density in the sphere packing problem, we generalize this result to partitions into higher powers.

1. Introduction and Statement of Results

A partition of $n \in \mathbb{N}$ is a non-increasing sequence (often written as a sum) of positive integers, called parts, adding up to $n$. By $p(n)$ we denote the number of partitions of $n$, and by convention we set $p(0) = 1$. For example, $p(4) = 5$ as the partitions of 4 are $4$, $3 + 1$, $2 + 2$, $2 + 1 + 1$, and $1 + 1 + 1 + 1$, this being the case of unrestricted partitions. One can consider, however, partitions with various conditions imposed on the parts, such as partitions with all their parts being in some set $S$ satisfying certain properties.

For $r \in \mathbb{N}$ we let $p_r(n)$ denote the number of partitions of $n$ into $r$th powers, $p_r(m, n)$ that of partitions of $n$ into $r$th powers with exactly $m$ parts, and $p_r(a, m, n)$ that of partitions of $n$ into $r$th powers with a number of parts that is congruent to $a$ modulo $m$.

Motivated by an interesting pattern noticed by Bringmann and Mahlburg and by their initial work [6] on the problem, the author [7] proved the following.

**Theorem 1** ([7]). For $n$ sufficiently large, we have

$$p_2(0, 2, n) \sim p_2(1, 2, n) \sim \frac{p_2(n)}{2}$$

and

$$\begin{cases} p_2(0, 2, n) > p_2(1, 2, n) & \text{if } n \text{ is even,} \\ p_2(0, 2, n) < p_2(1, 2, n) & \text{if } n \text{ is odd.} \end{cases}$$

The only analogous results of which the author is aware are due to Glaisher [10], who proved, using combinatorial arguments, that if $p_{\text{odd}}(n)$ denotes the number of partitions of $n$ into odd parts without repeated parts, then

$$p_1(0, 2, n) - p_1(1, 2, n) = (-1)^n p_{\text{odd}}(n), \quad (1)$$

and to Zhou [16], who has recently proved the equidistribution of partitions into parts that are certain polynomial functions. The identity established in (1) tells us that an even number $n$ has more partitions into an even number of parts than into an odd number of parts, and the other way around if $n$ is odd. The goal of this paper is to prove that Theorem 1 extends to partitions into any powers.

**Theorem 2.** For any $r \geq 2$ and $n$ sufficiently large, we have

$$p_r(0, 2, n) \sim p_r(1, 2, n) \sim \frac{p_r(n)}{2}$$

and

$$\begin{cases} p_r(0, 2, n) > p_r(1, 2, n) & \text{if } n \text{ is even,} \\ p_r(0, 2, n) < p_r(1, 2, n) & \text{if } n \text{ is odd.} \end{cases}$$

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In other words, we prove that the number of partitions of \( n \) into \( r \)th powers with an even number of parts is greater than that with an odd number of parts if \( n \) is even, and conversely if \( n \) is odd, and that the two quantities are asymptotically equal as \( n \to \infty \). The last claim will follow easily from our proof, but it is also a straightforward consequence of the aforementioned work of Zhou [16], who proved that

\[
p_f(a, k, n) \sim \frac{p_f(n)}{k}
\]

holds uniformly, as \( n \to \infty \), for all \( a, k, n \in \mathbb{N} \) with \( k^{2+2\deg(f)} \ll n \), where \( f \in \mathbb{Q}[x] \) is any non-constant polynomial such that the values \( f(n) \) are coprime positive integers, \( p_f(n) \) denotes the number of partitions of \( n \) with parts from the set \( \mathcal{S} = \{ f(n) : n \in \mathbb{N} \} \) and \( p_f(a, k, n) \) that of partitions of \( n \) with parts from \( \mathcal{S} \) having a number of parts congruent to \( a \) modulo \( k \). The asymptotic equidistribution stated in Theorem 2 is a consequence of this result for \( f(x) = x^r \). (As \( f \in \mathbb{Q}[x] \) is a non-constant polynomial, we believe there is no reason for confusion between this notation and \( p_r(n) \), used to denote partitions into \( r \)th powers.)

Before concluding this section, let us introduce some notation used in the sequel. By \( \zeta_n = e^{\frac{2\pi i}{n}} \) we will denote the standard primitive \( n \)th root of unity. For reasons of space, and with the hope that the reader will not consider this an inconsistency, we will sometimes use \( \exp(z) \) for \( e^z \). Whenever required to take logarithms or to extract roots of complex numbers, we will use the principal branch, and the principal branch of the (complex) logarithm will be denoted by \( \text{Log} \). The Vinogradov symbols \( o, O \) and \( \ll \) are used throughout with their standard meaning.

The paper is structured as follows. In Section 2 we explain the strategy of the proof, and the similarities and differences with the proof of the same result from [7] in the case \( r = 2 \). This will also be done, throughout the paper, in the form of commentaries at the end of each section. We consider this to be for the benefit of the reader interested in comparing the present paper with [7]. In Sections 3 and 4 we prove two estimates which, combined, will provide the proof of Theorem 2, given in Section 5.

2. Philosophy of the Proof

In view of what has already been mentioned, it is only of interest to us to prove the asymptotic inequalities from Theorem 2. In doing so, we will first reformulate the claim of our problem so that it becomes equivalent with proving that the coefficients of a certain generating function are positive.

2.1. A reformulation. It is well-known (see, for example, [8 Ch. 1]) that

\[
\prod_{n=1}^{\infty}(1 - q^n)^{-1} = \sum_{n=0}^{\infty} p_r(n) q^n,
\]

where, as usual, for \( \tau \in \mathbb{H} \) (the upper half-plane) we set \( q = e^{2\pi i \tau} \). Letting

\[
H_r(q) = \sum_{n=0}^{\infty} p_r(n) q^n,
\]

\[
H_r(w; q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_r(m, n) w^m q^n,
\]

\[
H_{r,a,m}(q) = \sum_{n=0}^{\infty} p_r(a, m, n) q^n,
\]

it is not difficult to see, by the orthogonality relations for roots of unity, that

\[
H_{r,a,m}(q) = \frac{1}{m} H_r(q) + \frac{1}{m} \sum_{j=1}^{m-1} \zeta_m^{-aj} H_r(\zeta_m^j; q),
\]

On noting now that

\[
H_{r,0,2}(-q) - H_{r,1,2}(-q) = \sum_{n=0}^{\infty} a_r(n) q^n,
\]
where

\[ a_r(n) = \begin{cases} p_r(0, 2, n) - p_r(1, 2, n) & \text{if } n \text{ is even}, \\ p_r(1, 2, n) - p_r(0, 2, n) & \text{if } n \text{ is odd}, \end{cases} \]

proving the asymptotic inequalities from Theorem 2 is equivalent to showing that \( a_r(n) > 0 \) as \( n \to \infty \).

Using, in turn, (2) and eq. (2.1.1) from [3] p. 16], we obtain

\[ H_{r,0,2}(q) - H_{r,1,2}(q) = H_r(-1; q) = \prod_{n=1}^{\infty} \frac{1}{1 + q^{nr}}. \]

Changing \( q \mapsto -q \) gives

\[ H_r(-1; -q) = \prod_{n=1}^{\infty} \frac{1}{1 + (-q)^{nr}} = \prod_{n=1}^{\infty} \frac{1}{(1 + q^{2n})/(1 - q^{2n+1})} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n+1})(1 - q^{nr})}, \]

from where, by setting

\[ G_r(q) = H_{r,0,2}(-q) - H_{r,1,2}(-q), \]

we get

\[ G_r(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n+1})(1 - q^{nr})} = \sum_{n=0}^{\infty} a_r(n)q^n. \quad (3) \]

In conclusion, what we need to prove now is that the coefficients \( a_r(n) \) are positive as \( n \to \infty \) and one natural way to verify this would be to compute asymptotics for them, which is what we are going to do.

2.2. A result by Meinardus. The reader familiar with asymptotics for infinite product generating functions might recognize at this point the similarity between the infinite product from [3] and that studied by Meinardus [11]. Writing \( q = e^{-\tau} \) with \( \text{Re}(\tau) > 0 \), the product in question is of the form

\[ F(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-a_n} = \sum_{n=0}^{\infty} r(n)q^n, \]

with \( a_n \geq 0 \) and, under certain assumptions on which we do not elaborate now, Meinardus found asymptotic formulas for the coefficients \( r(n) \). More precisely, if the Dirichlet series

\[ D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (s = \sigma + it) \]

converges for \( \sigma > \alpha > 0 \) and admits a meromorphic continuation to the region \( \sigma > -c_0 \ (0 < c_0 < 1) \), region in which \( D(s) \) is holomorphic everywhere except for a simple pole at \( s = \alpha \) with residue \( A \), then the following holds.

**Theorem 3** (Andrews [3] Ch. 6], cf. Meinardus [11]). As \( n \to \infty \), we have

\[ r(n) = cn^\kappa \exp \left( n^\frac{\alpha}{\alpha + 1} \left( 1 + \frac{1}{\alpha} \right) (A\Gamma(\alpha + 1)\zeta(\alpha + 1))^{\frac{1}{\alpha + 1}} \right) (1 + O(n^{-\kappa_1})), \]

where

\[ c = e^{D'(0)} (2\pi(\alpha + 1))^{-\frac{\alpha}{2}} (A\Gamma(\alpha + 1)\zeta(\alpha + 1))^{\frac{1 - 2D(0)}{2(\alpha + 1)}}, \]

\[ \kappa = \frac{2D(0) - 2 - \alpha}{2(\alpha + 1)}, \]

\[ \kappa_1 = \frac{\alpha}{\alpha + 1} \min \left\{ \frac{c_0}{\alpha} - \frac{\delta}{4} + \frac{1}{2} - \frac{\delta}{4} \right\}, \]

with \( \delta > 0 \) arbitrary.
Writing $\tau = y - 2\pi ix$, an application of Cauchy’s Theorem gives
\[
\tau(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(q)}{q^{n+1}} dq = e^{ny} \int_{-\frac{1}{2}}^{\frac{1}{2}} F(e^{-y+2\pi ix}) e^{-2\pi inx} dx,
\]
where $\mathcal{C}$ is the (positively oriented) circle of radius $e^{-y}$ around the origin. Meinardus found the estimate stated in Theorem 3 by splitting the integral from (4) into two integrals evaluated over $|x| \leq y^{\beta}$ and over $y^{\beta} < |x| \leq \frac{1}{2}$, for a certain choice of $\beta$ in terms of $\alpha$, and by showing that the former integral gives the main contribution for the coefficients $\tau(n)$, while the latter is only an error term.

The positivity condition $a_n \geq 0$ is, however, essential in Meinardus’ proof and, as one can readily note, this is not satisfied by the factors from the product in [3]. For this reason, we need to come up with a certain modification using the circle method and Wright’s modular transformations [15] for the function $G_r(q)$. This will be used to show that the integral over $y^{\beta} < |x| \leq \frac{1}{2}$ does not contribute.

On comparing with what was done for the case $r = 2$, the reader might notice that, up to this point, the strategy described here is analogous to that from [7]. The essential difference is that, in the case $r = 2$, a numerical check (Lemma 5) had to be carried out in order to prove a certain estimate (Lemma 6). This numerical check was rather technical and certainly cannot be carried out for all $r \geq 2$. In the present paper, we show how to avoid it by using a bound on Gauss sums found by Banks and Shparlinski [5] and by modifying an argument from [7]. It is precisely this step that allows for a significantly simpler proof and, at the same time, for a generalization of our results to any $r \geq 2$.

2.3. Two estimates. We keep the notation introduced in the previous subsection and write $q = e^{-\tau}$, with $\tau = y - 2\pi ix$ and $y > 0$. Recall that
\[
G_r(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n+1})(1 - q^n)}.
\]
Let $s = \sigma + it$ and
\[
D_r(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^s} - \frac{2}{2} \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = (1 + 2^{-1-s(r+1)} - 2^{-1-sr}) \zeta(rs),
\]
which is convergent for $\sigma > \frac{1}{r} = \alpha$, has a meromorphic continuation to $\mathcal{C}$ (thus we may choose $0 < c_0 < 1$ arbitrarily) and a simple pole at $s = \frac{1}{r}$ with residue $A = \frac{1}{r} \cdot 2^{-1-\frac{1}{r}}$.

If $\mathcal{C}$ is the (positively oriented) circle of radius $e^{-y}$ around the origin, Cauchy’s Theorem tells us that
\[
a_r(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{G_r(q)}{q^{n+1}} dq = e^{ny} \int_{-\frac{1}{2}}^{\frac{1}{2}} G_r(e^{-y+2\pi ix}) e^{-2\pi inx} dx,
\]
for $n > 0$. Set
\[
\beta = 1 + \frac{\alpha}{2} \left(1 - \frac{\delta}{2}\right), \quad \text{with} \quad 0 < \delta < \frac{2}{3},
\]
so that
\[
\frac{3r+1}{3r} < \beta < \frac{2r+1}{2r},
\]
and rewrite
\[
a_r(n) = I_r(n) + J_r(n),
\]
where
\[
I_r(n) = e^{ny} \int_{-y^{\beta}}^{y^{\beta}} G_r(e^{-y+2\pi ix}) e^{-2\pi inx} dx \quad \text{and} \quad J_r(n) = e^{ny} \int_{y^{\beta} \leq |x| \leq \frac{1}{2}} G_r(e^{-y+2\pi ix}) e^{-2\pi inx} dx.
\]

As already mentioned, the idea is that the main contribution for $a_r(n)$ is given by $I_r(n)$, and we will be able to prove this by using standard integration techniques. To show, however, that $J_r(n)$ is an error term will prove to be much more tricky.
In this section, we prove the following estimate.

**Lemma 1.** If $|x| \leq \frac{1}{2}$ and $|\text{Arg}(\tau)| \leq \frac{\pi}{4}$, then

$$G_r(e^{-\tau}) = 2 \cdot e^{-\frac{r+1}{2}} \exp \left( \Gamma \left( \frac{1}{r} \right) \zeta \left( 1 + \frac{1}{r} \right) \tau^{-\frac{1}{r}} + O(y^\omega) \right)$$

holds uniformly in $x$ as $y \to 0$, with $0 < c_0 < 1$.

**Proof.** By taking logarithms in (5), we obtain

$$\log(G_r(e^{-\tau})) = \sum_{k=1}^{\infty} \Gamma(s) \sum_{n=1}^{\infty} \left( e^{-kn^r \tau} + e^{-2r+1kn^r \tau} - 2e^{-2r^2kn^r \tau} \right).$$

Using the Mellin inversion formula (see, e.g., [2, p. 54]) we get

$$e^{-\tau} = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} e^{s \xi} ds$$

for $\text{Re}(\tau) > 0$ and $c_0 > 0$, thus

$$\log(G_r(e^{-\tau})) = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} \Gamma(s) \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} ((kn^r \tau)^{-s} + (2r+1kn^r \tau)^{-s} - 2(2r^2kn^r \tau)^{-s})$$

$$= \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} \Gamma(s) D_r(s) \zeta(s+1) \tau^{-s} ds. \quad (9)$$

By assumption,

$$|\tau^{-s}| = |\tau|^{-\sigma} e^{\epsilon \text{Arg}(\tau)} \leq |\tau|^{-\sigma} e^{\frac{\pi}{2}|t|}.$$

Classical results (see, e.g., [4, Ch. 1] and [14, Ch. 5]) tell us that the bounds

$$D_r(s) = O(|t|^{c_1}),$$

$$\zeta(s+1) = O(|t|^{c_2}),$$

$$\Gamma(s) = O(e^{-\frac{|t|}{2}})$$

hold uniformly in $-c_0 \leq \sigma \leq \frac{r+1}{r} = \alpha + 1$ as $|t| \to \infty$, for some $c_1, c_2$ and $c_3 > 0$. We may thus shift the path of integration from $\sigma = \alpha + 1$ to $\sigma = -c_0$. A quick computation gives $D_r(0) = 0$ and $D_r'(0) = -(r-1)\log 2$. The integrand in (9) has poles at $s = \frac{1}{2}$ and $s = 0$, with residues

$$\text{Res}_{s=\frac{1}{2}} A \left( \Gamma(s) D_r(s) \zeta(s+1) \tau^{-s} \right) = \Gamma \left( \frac{1}{r} \right) \zeta \left( 1 + \frac{1}{r} \right) \tau^{-\frac{1}{r}},$$

$$\text{Res}_{s=0} \left( \left( \frac{1}{s} + O(1) \right) \left( D_r(0)s + O(s^2) \right) \left( \frac{1}{s} + O(1) \right) \left( 1 + O(s) \right) \right) = D_r'(0) = -(r-1)\log 2,$$

whereas the remaining integral equals

$$\int_{-c_0 - i\infty}^{-c_0 + i\infty} \Gamma(s) D_r(s) \zeta(s+1) ds \ll |\tau|^{c_0} \int_0^\infty e^{\epsilon_1 + c_2 + c_3 - \frac{\pi}{2}it} dt \ll |\tau|^{c_0} \leq (\sqrt{2}y)^{c_0}$$

since, again by the assumption,

$$\frac{2\pi |x|}{y} = \tan(|\text{Arg}(\tau)|) \leq \tan \left( \frac{\pi}{4} \right) = 1.$$

In conclusion, integration along the shifted contour gives

$$\log(G_r(e^{-\tau})) = \left( \Gamma \left( \frac{1}{r} \right) \zeta \left( 1 + \frac{1}{r} \right) \tau^{-\frac{1}{r}} - \frac{(r-1)\log 2}{2} \right) + O(y^{c_0}). \quad \square$$

**Commentary.** This part is a straightforward generalization of the proof of [7, Lemma 1]. On replacing $r = 2$, the reader can easily trace back that argument.
4. The Error Term $J_r(n)$

This section is dedicated to proving that $J_r(n)$ does not contribute to the coefficients $a_r(n)$. More precisely, we prove the following estimate.

**Lemma 2.** There exists $\varepsilon > 0$ such that, as $y \to 0$,

$$G_r(e^{-\tau}) = O\left(\exp\left(\frac{1}{r} \zeta\left(1 + \frac{1}{r}\right) y^{-\frac{1}{r}} - cy^{-\varepsilon}\right)\right)$$

holds uniformly in $x$ with $y^\beta \leq |x| \leq \frac{1}{2}$, for some $c > 0$.

The proof is slightly more involved and will come in several steps. We start by describing the setup needed to apply the circle method.

4.1. Circle method. Inspired by Wright [15], we consider the Farey dissection of order $\lfloor y^{\frac{r}{r+1}} \rfloor$ of the circle $C$ over which we integrate in (6). We distinguish further two kinds of arcs:

(i) major arcs, denoted $M_{a,b}$, such that $b \leq y^{\frac{r}{r+1}}$;
(ii) minor arcs, denoted $m_{a,b}$, such that $y^{\frac{r}{r+1}} < b \leq y^{\frac{r}{r+1}}$.

In what follows, we express any $\tau \in M_{a,b} \cup m_{a,b}$ in the form

$$\tau = y - 2\pi ix = \tau' - 2\pi i \frac{a}{b},$$

with $\tau' = y - 2\pi ix'$. From basics of Farey theory it follows that

$$|x'| \leq \frac{y^{\frac{r}{r+1}}}{b}.$$  

For a neat introduction to Farey fractions and the circle method, the reader is referred to [1, Ch. 5.4].

4.2. Modular transformations. Recalling the definition of $H_r(q)$, we can rewrite (5) as

$$G_r(q) = \frac{H_r(q)H_r(q^{2r+1})}{H_r(q^{2r})^2}.$$  

In order to obtain more information about $G_r(q)$, we would next like to use Wright’s transformation law [15, Theorem 4] for the generating function $H_r(q)$ of partitions into $r$th powers.

Before doing so, we need to introduce a bit of notation. In what follows, $0 \leq a < b$ are assumed to be coprime positive integers, with $b_1$ the least positive integer such that $b \mid b_2$ and $b = b_1b_2$. First, set

$$j = j(r) = 0, \quad \omega_{a,b} = 1$$

if $r$ is even, and

$$j = j(r) = \frac{(-1)^{\frac{1}{2}(r+1)}}{(2\pi)^{r+1}} \Gamma(r+1)\zeta(r+1), \quad \omega_{a,b} = \exp\left(\pi \left(\frac{1}{b^2} \sum_{h=1}^{b} hd_h - \frac{1}{4}(b - b_2)\right)\right)$$

if $r$ is odd, where $0 \leq d_h < b$ is defined by the congruence

$$ah^2 \equiv d_h \pmod{b}$$

and

$$\mu_{h,s} = \begin{cases} \frac{d_h}{b} & \text{if } s \text{ is odd}, \\ \frac{b-d_h}{b} & \text{if } s \text{ is even}, \end{cases}$$

for $d_h \neq 0$. If $d_h = 0$, we set $\mu_{h,s} = 1$. Further, let

$$S_r(a,b) = \sum_{n=1}^{b} \exp\left(\frac{2\pi i an^r}{b}\right)$$

(14)
be the so-called Gauss sums (of order $r$), and

$$\Lambda_{a,b} = \frac{\Gamma \left( 1 + \frac{1}{r} \right)}{b} \sum_{m=1}^{\infty} \frac{S_r(ma,b)}{m^{1 + \frac{1}{r}}}.$$  \hfill (15)

Finally, put

$$C_{a,b} = \left( \frac{b_1}{2\pi} \right)^2 \omega_{a,b},$$

and

$$P_{a,b}(\tau') = \prod_{h=1}^{\frac{b}{r}} \prod_{\ell=1}^{r} \prod_{s=0}^{\infty} \left( 1 - g(h,\ell,s) \right)^{-1},$$

with

$$g(h,\ell,s) = \exp \left( \frac{(2\pi)^{\frac{r+1}{r}}}{r} \left( \ell + \mu_{h,s} \right)^{\frac{1}{r}} e^{\frac{2\pi}{b} \frac{r+1}{r}(2s+r+1)} - \frac{2\pi ih}{b} \right).$$

Having introduced all the required definitions, we can now state Wright’s modular transformation \cite{15, Theorem 4], which says, in our notation, that

$$H_r(q) = H_r \left( e^{\frac{2\pi i q}{b} - \tau'} \right) = C_{a,b} \sqrt{\tau'} e^{i\tau'} \exp \left( \frac{\Lambda_{a,b}}{\sqrt{\tau'}} \right) P_{a,b}(\tau').$$  \hfill (16)

On combining (13) and (16) we obtain, for some positive constant $C$ that can be made explicit if necessary,

$$G_r(q) = C e^{i\tau'} \exp \left( \frac{\Lambda_{a,b}}{\sqrt{\tau'}} \right) \frac{P_{a,b}(\tau') P'_{a,b}(2^{r+1} \tau')} {P''_{a,b}(2^{r} \tau')^{2}},$$  \hfill (17)

where

$$P'_{a,b} = P_{r+1, \frac{b}{2r+1}, \frac{b}{2r+1}} \quad \text{and} \quad P''_{a,b} = P_{2r+1, \frac{b}{2r+1}, \frac{b}{2r+1}}$$

and

$$\lambda_{a,b} = \Lambda_{a,b} + 2^{-\frac{r+1}{r}} \Lambda_{2r+1, \frac{b}{2r+1}, \frac{b}{2r+1}} - \Lambda_{2r+1, \frac{b}{2r+1}, \frac{b}{2r+1}} = 2^{-\frac{r+1}{r}} \Lambda_{a,b}.$$  \hfill (18)

4.3. A bound on Gauss sums. As we shall soon see, a crucial step in our proof is finding an upper bound for $\text{Re}(\lambda_{a,b})$ or, what is equivalent, a bound for $|\lambda_{a,b}|$. This is given by the following sharp estimate found by Banks and Shparlinski \cite{5} for the Gauss sums defined in (14).

**Theorem 2 ([5, Theorem 1]).** For any coprime positive integers $a, b$ with $b \geq 2$ and any $r \geq 2$, we have

$$|S_r(a, b)| \leq A b^{-\frac{1}{r}},$$  \hfill (19)

where $A = 4.709236 \ldots .$

The constant $A$ is known as Stechkin’s constant. Stechkin \cite{13} conjectured in 1975 that the quantity

$$A = \sup_{b,n \geq 2} \max_{(a,b) = 1} \frac{|S_r(a, b)|}{b^{-\frac{1}{r}}}$$

is finite, this being proven by Shparlinski \cite{12} in 1991. In the absence of any effective bounds on the sums $S_r(a, b)$, the precise value of $A$ remained a mystery until 2015 when, using the work of Cochrane and Pinner \cite{8} on Gauss sums with prime moduli and that of Cohn and Elkies \cite{9} on lower bounds for the center density in the sphere packing problem, Banks and Shparlinski \cite{5} were finally able to determine it.

Coming back to our problem, we can now prove the following estimate.

**Lemma 3.** If $0 \leq a < b$ are coprime integers with $b \geq 2$, we have

$$|\lambda_{a,b}| < 3A \cdot \Gamma \left( 1 + \frac{1}{r} \right) \zeta \left( 1 + \frac{1}{r} \right) b^{-\frac{1}{r}} \sum_{d|b} \frac{1}{d},$$

where $A$ is Stechkin’s constant.
Proof. Let us first give a bound for $|\Lambda_{a,b}|$. If we recall (15) and write $\Lambda_{a,b} = \Gamma \left( 1 + \frac{1}{r} \right) \Lambda_{a,b}^{*}$, we have, on using the fact that $S_r(ma,b) = dS_r \left( \frac{ma}{d}, \frac{b}{d} \right)$ to prove the second equality below, and on replacing $m \mapsto md$ and $d \mapsto \frac{b}{d}$ to prove the third and fourth respectively,

$$
\Lambda_{a,b}^{*} = \frac{1}{b} \sum_{m=1}^{\infty} \frac{S_r(ma,b)}{m^{1+\frac{1}{r}}} = \frac{1}{b} \sum_{db \geq m \geq 1 (m,b=d)} \frac{dS_r \left( \frac{ma}{d}, \frac{b}{d} \right)}{m^{1+\frac{1}{r}}} = \frac{1}{b} \sum_{db \geq m \geq 1 (m,b/d=1)} \frac{S_r(ma,b)}{(md)^{1+\frac{1}{r}}}
$$

$$
= \frac{1}{b} \sum_{db \geq m \geq 1 (m,b/d=1)} d^{-\frac{1}{r}} \sum_{m \geq 1} \frac{S_r(ma,b) m}{m^{1+\frac{1}{r}}} = \frac{1}{b} \sum_{db \geq m \geq 1 (m,d=1)} d^{-\frac{1}{r}} \sum_{m \geq 1} \frac{S_r(ma,b)}{m^{1+\frac{1}{r}}} = \frac{1}{b^{1+\frac{r}{2}}} \sum_{db \geq m \geq 1 (m,d=1)} d^{-\frac{1}{r}} \sum_{m \geq 1} \frac{S_r(ma,b)}{m^{1+\frac{1}{r}}}.
$$

On invoking (19), we obtain

$$
|\Lambda_{a,b}| \leq \frac{\Gamma \left( 1 + \frac{1}{r} \right)}{b^{1+\frac{r}{2}}} \sum_{db \geq m \geq 1 (m,d=1)} \frac{|S_r(ma,b)|}{m^{1+\frac{1}{r}}} \leq \frac{A \Gamma \left( 1 + \frac{1}{r} \right) \zeta \left( 1 + \frac{1}{r} \right)}{b^2} \sum_{db \geq m \geq 1 (m,d=1)} \frac{1}{d},
$$

from where the claim follows by applying this bound to the expression for $\lambda_{a,b}$ from (18).

\[ \square \]

4.4. Final estimates. We are now getting closer to our purpose and we only need a few last steps before giving the proof of Lemma 2. Let us begin by estimating the factors of the form $P_{a,b}$ appearing in (17).

Lemma 4. If $\tau \in \mathcal{M}_{a,b} \cup \mathfrak{m}_{a,b}$, then

$$
\log |P_{a,b}(\tau')| \ll b \quad \text{as} \quad y \to 0.
$$

Proof. Using (12) and letting $y \to 0$, we have

$$
|\tau'|^{1+\frac{1}{r}} = \left( y^2 + 4\pi^2 x^2 \right)^{\frac{1}{r+1}} \leq \left( y^2 + \frac{4\pi^2 y^{2r-1}}{b^2} \right)^{\frac{1}{r+1}} \leq \frac{c_4 y}{b^{\frac{r+2}{r+1}}} = \frac{c_4 \text{Re}(\tau')}{b^{\frac{r+2}{r+1}}},
$$

for some $c_4 > 0$. Thus, [15, Lemma 4] gives

$$
|g(h,\ell,s)| \leq e^{-c_5(\ell+1)^\frac{1}{r}},
$$

with $c_5 = \frac{4\sqrt{2\pi}}{cr_4}$, which in turn leads to

$$
|\log |P_{a,b}(\tau')|| \leq \sum_{h=1}^{b} \sum_{s=1}^{r} \sum_{\ell=1}^{\infty} |\log (1 - g(h,\ell,s))| \leq rb \sum_{\ell=1}^{\infty} \left| \log \left( 1 - e^{-c_5(\ell+1)^\frac{1}{r}} \right) \right| \ll b,
$$

concluding the proof. \[ \square \]

The next result gives a bound for $G_r(q)$ on the minor arcs. As it is an immediate consequence of replacing $a = \frac{1}{r}$, $b = \frac{1}{r+1}$, $c = 2r^{-1}$, $\gamma = \varepsilon$ and $N = y^{-1}$ in [15, Lemma 17], we omit its proof.

Lemma 5. If $\varepsilon > 0$ and $\tau \in \mathfrak{m}_{a,b}$, then

$$
|\text{Log}(G_r(q))| \ll \varepsilon \ y^{\frac{2r^{-1} - 1}{r+1} - \varepsilon}.
$$

Note that $r^{2r^{-1}} > r + 1$ for any $r \geq 2$, therefore the exponent of $y$ in Lemma 5 is positive for a small enough choice of $\varepsilon > 0$. At last, we need the following estimate, a modified version of [7, Lemma 6].

Lemma 6. If $0 \leq a < b$ are coprime integers with $b \geq 2$ and $x \notin \mathbb{Q}$, we have as $y \to 0$, for some $c > 0$,

$$
\text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{y}} \right) \leq \frac{\lambda_{0,1} - c}{\sqrt{y}}.
$$

(20)
Proof. Note that
\[ \lambda_{0,1} = 2^{-\frac{r+1}{r}} \Lambda_{0,1} = \frac{1}{2^{1+\frac{1}{r}}} \Gamma \left( 1 + \frac{1}{r} \right) \zeta \left( 1 + \frac{1}{r} \right) = \lambda g \left( \frac{1}{r} \right) \zeta \left( 1 + \frac{1}{r} \right). \]
Writing \( \tau' = y + it \) for some \( t \in \mathbb{R} \), we have
\[ \text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{\tau}} \right) = \frac{1}{\sqrt{y}} \text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{1+it}} \right) = \frac{1}{\sqrt{y}} \text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{1+t^2e^{\frac{\pi}{4}}}} \right), \]
and clearly \( f_r(t) \rightarrow 0 \) as \( t \rightarrow \infty \). Note now that the choice of \( x \) is independent from that of \( y \), and recall from (11) that \( \tau' = y - 2\pi ix' \), with \( x' = x - \frac{a}{b} \), hence \( t = -\frac{x}{2\pi y} \). The assumption \( x \notin \mathbb{Q} \) implies \( x' \neq 0 \), and so \( t \rightarrow \infty \) as \( y \rightarrow 0 \). Consequently, we have \( f_r(t) \rightarrow 0 \) as \( y \rightarrow 0 \). In combination with Lemma 3 and the well-known fact that \( d(n) = o(n^\varepsilon) \) for any \( \varepsilon > 0 \) (for a proof see, e.g., [1, p. 296]), where \( d(n) \) denotes the number of divisors of \( n \), this completes the proof. \( \square \)

4.5. Proof of the main lemma. We are now equipped with all the machinery needed for Lemma 2.

Proof of Lemma 2. If \( \tau \in \mathbb{M}_{a,b} \), then it suffices to apply Lemma 5 (because, as \( y \rightarrow 0 \), a negative power of \( y \) will dominate any positive power of \( y \); in particular, also the term \( jy \) coming from the factor \( e^{\tau y} \) in the case when \( r \) is odd), so let us assume that \( \tau \in \mathbb{M}_{a,b} \).

We first consider the behavior near 0, corresponding to \( a = 0, b = 1 \), \( \tau = \tau' = y - 2\pi ix \). Writing \( y^\varepsilon = y^{\frac{2r+1}{r}} + \varepsilon > 0 \) (here we use the second inequality from (8)), we have, on setting \( b = 1 \) in (12),
\[ y^{\frac{2r+1}{r}} + \varepsilon \leq |x| = |x'| \leq y^{\frac{r}{r+1}}. \]

By (17) we get
\[ G_r(q) = C e^{\tau y} \exp \left( \frac{\lambda_{0,1}}{\sqrt{\tau}} \right) \frac{P_{0,1}(\tau)P_{0,1}(2r+1+\tau)}{P_{0,1}(2r+\tau)^2} \]
for some \( C > 0 \) and thus, by Lemma 4,
\[ \log |G_r(q)| = \frac{\lambda_{0,1}}{\sqrt{|\tau|}} + jy + O(1). \]

On using (21) to prove the first inequality below and expanding into Taylor series to prove the second, we obtain, by letting \( y \rightarrow 0 \),
\[ \frac{1}{\sqrt{|\tau|}} = \frac{1}{\sqrt{y}} \frac{1}{\left( 1 + \frac{4\pi^2y^2}{1+r} \right)^{\frac{r}{2}}} \leq \frac{1}{\sqrt{y}} \frac{1}{\left( 1 + 4\pi^2y^{\frac{r}{r+1}} \right)^{\frac{r}{2}}} \leq \frac{1}{\sqrt{y}} \left( 1 - c_6 y^{\frac{1}{r+1}} \right) \]
for some \( c_6 > 0 \), and this concludes the proof in this case.

To finish the claim we assume \( \tau \in \mathbb{M}_{a,b} \), with \( 2 \leq b \leq \frac{1}{y^{\frac{1}{r+1}}} \). We distinguish two cases. First, let us deal with the case when \( x \notin \mathbb{Q} \). By (17) and Lemma 4 we obtain
\[ \log |G_r(q)| = \text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{\tau}} \right) + jy + O \left( y^{\frac{1}{r+1}} \right) = \text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{\tau}} \right) + O \left( y^{\frac{1}{r+1}} \right) \]
as \( y \rightarrow 0 \). Since by Lemma 6 there exists \( c_7 > 0 \) such that
\[ \text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{\tau}} \right) \leq \frac{\lambda_{0,1} - c_7}{\sqrt{y}}, \]
we infer from (23) that, as \( y \rightarrow 0 \), we have
\[ \log |G_r(q)| \leq \frac{\lambda_{0,1} - c_8}{\sqrt{y}} \]
for some \( c_8 > 0 \) and the proof is complete in this case.
Finally, assume that \( x = \frac{a}{y} \), that is, \( x' = 0 \) and \( \tau = y - 2\pi i \frac{a}{y} \). We claim that the estimate \( (10) \) is satisfied, with the same implied constant, say \( C_1 \). Suppose by sake of contradiction that this is not the case. Then there exist infinitely small values of \( y > 0 \) for which

\[
|G_r(e^{-\tau})| \geq C_2 \exp \left( \frac{\lambda_{a,1}}{\sqrt{y}} - cy^{-\varepsilon} \right),
\]

with \( C_2 > C_1 \). However, we can pick now \( x' \notin \mathbb{Q} \) infinitely small and set \( \tau_1 = y - 2\pi i \left( x' + \frac{a}{y} \right) \). For a fixed choice of \( y \), we have \( t \to 0 \) as \( x' \to 0 \); thus, by the same calculation done in the proof of Lemma \( 6 \) we obtain

\[
\text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{\tau_1}} \right) \to \text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{\tau}} \right) = \text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{\tau}} \right),
\]

since \( f_r(t) \to 1 \). On noting that \( \text{Re}(\tau_1^1) = \text{Re}(\tau) = y \), while clearly all factors of the form \( |P_{a,b}(k\tau')| \) tend to \( |P_{a,b}(k\tau')| \) as \( x' \to 0 \), we obtain a contradiction, in the sense that, on one hand, \( (22) \) and \( (24) \) yield

\[
|G_r(e^{-\tau_1})| \to |G_r(e^{-\tau})|
\]
as \( x' \to 0 \), whereas on the other, for a sufficiently small choice of \( y > 0 \), we have

\[
|G_r(e^{-\tau})| - |G_r(e^{-\tau_1})| \geq (C_2 - C_1) \exp \left( \frac{\lambda_{a,1}}{\sqrt{y}} - cy^{-\varepsilon} \right),
\]

quantity which gets arbitrarily large for sufficiently small choices of \( y > 0 \).

\( \square \)

Commentary. It is in this part where our proof differs substantially from that given in \[7\] in the case \( r = 2 \). More precisely, \[7\], Lemma 5 was used to prove the inequality \( (20) \) for all values of \( y \), inequality which was then used in the estimates made in the proof of \[7\], Lemma 2, the equivalent of Lemma \( 2 \) from the present paper. However, we are only interested in establishing the estimates from Lemma \( 2 \) on letting \( y \to 0 \), for which reason we only need the bound \( (20) \) to hold as \( y \to 0 \). The argument presented in Lemma \( 6 \) further tells us that, in order for this to happen, the estimate \( (19) \), obtained using the bound on Gauss sums found by Banks and Shparlinski \[5\], is enough. As a consequence, we can avoid the rather involved numerical check done in \[7\], Lemma 5, a check which we would, in fact, not be able to implement for all values \( r \geq 2 \). In particular, the present argument gives a simplified proof of the results from \[7\].

5. Proof of the Main Theorem

In this section we give the proof of Theorem \[2\]. Having already proven the two estimates from Lemma \[1\] and Lemma \[2\], the rest is only a matter of careful computations. The reader is reminded that, after the reformulation made in Section \[2.1\], what we are interested in is computing asymptotics for the coefficients

\[
a_r(n) = e^{ny} \int_{-\frac{1}{2}}^{\frac{1}{2}} G_r(e^{-y+2\pi i x})e^{-2\pi i n x} dx.
\]

5.1. Saddle-point method. Recall that, as defined in Section \[2.3\], we denote \( \alpha = \frac{1}{r} \) and \( A = \frac{1}{r} \cdot 2^{-\frac{r-1}{r}} \), notation which we keep, for simplicity, in what follows. Before delving into the proof, we make a particular choice for \( y \) as a function of \( n \). More precisely, let

\[
y = n^{-\frac{1}{r+1}} (A\Gamma(\alpha + 1)\zeta(\alpha + 1))^{-\frac{1}{r+1}} = n^{-\frac{1}{r+1}} \left( A\Gamma\left(\frac{1}{r}\right)\zeta\left(1 + \frac{1}{r}\right)\right)^{-\frac{1}{r+1}},
\]

and write \( m = ny \).

The reason for this choice of \( y \) is motivated by the saddle-point method and becomes apparent once the reader recognizes in \( (20) \) the quantity appearing in Lemmas \[1\] and \[2\]. As the maximum absolute value of the integrand from \( (25) \) occurs for \( x = 0 \), around which point Lemma \[1\] tells us that the integrand is well approximated by

\[
\exp(A\Gamma(\alpha)\zeta(\alpha + 1)y^{-\alpha} + ny),
\]
the saddle-point method suggests minimizing this expression, that is, finding the value of \( y \) for which
\[
\frac{d}{dy} (\exp(\Lambda y) \zeta(\alpha + 1) y^{-\alpha} + ny)) = 0.
\]

5.2. Proof of the main result. We have now all ingredients necessary to conclude the proof of Theorem 2 whose statement we repeat below for convenience. The proof merely consists of a skillful computation, which can be carried out in two ways. Since Lemma 1 and Lemma 2 are completely analogous to the two estimates found by Meinardus (combined in the Hilfssatz from [11, p. 390]), one way is to follow his approach and carry out the same computations done in [11, pp. 392–394]. The second way is slightly more explicit and is based entirely on the computation done in the proof of the case \( r = 2 \) from [7, pp. 139–141]. For sake of completeness and for comparison with the corresponding computation from [7], we will sketch in what follows the main steps of the argument, while leaving some details and technicalities as a check for the interested reader.

**Theorem 2.** For any \( r \geq 2 \) and \( n \) sufficiently large, we have
\[
p_r(0, 2, n) \sim p_r(1, 2, n)
\]
and
\[
\begin{cases} p_r(0, 2, n) > p_r(1, 2, n) & \text{if } n \text{ is even,} \\ p_r(0, 2, n) < p_r(1, 2, n) & \text{if } n \text{ is odd.}
\end{cases}
\]

**Proof.** As already explained, the interesting part is to prove the inequalities from (27), so let us begin by doing so. By Lemma 2 and (26) we have, as \( n \to 0, \)
\[
J_r(n) = e^{ny} \int_{y^\beta \leq |x| \leq \frac{1}{\beta}} G(e^{-y+2\pi i x}) e^{-2\pi i nx} dx = e^{ny} \int_{y^\beta \leq |x| \leq \frac{1}{\beta}} O(e^{-\alpha \Lambda y} (\zeta(\alpha + 1) - cy^{-n})) dx
\]
\[
e^{ny} \cdot O \left( e^{-\alpha \Lambda y} (\zeta(\alpha + 1) - cy^{-n}) \right) = O \left( e^{n \alpha \Lambda + (1 + \frac{1}{\alpha}) (\zeta(\alpha + 1)) \frac{1}{\beta} - C_1 n^\epsilon_1} \right),
\]
with \( \epsilon_1 = \frac{2 \pi}{r+1} > 0 \) and some \( C_1 > 0. \)

We now compute the main asymptotic contribution, which will be given by \( I_r(n) \). Let \( n \geq n_1 \) be large enough so that \( y^\beta - 1 \leq \frac{1}{2 \pi} \). This choice allows us to apply Lemma 1 as it ensures \( |x| \leq \frac{1}{r} \) and \( \arg(\tau) \leq \frac{\pi}{2} \). From Lemma 1 we obtain
\[
I_r(n) = e^{ny} \int_{-y^\beta}^{y^\beta} e^{\Lambda y (\zeta(\alpha + 1) - O(y^{\alpha} - 2\pi inx)) dx.}
\]
\[
(28)
\]

Writing
\[
\tau^{-\alpha} = \frac{1}{\sqrt{\tau}} = \frac{1}{\sqrt{y}} + \left( \frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{y}} \right),
\]

we can further express (28) as
\[
I_r(n) = e^{ny} \int_{-y^\beta}^{y^\beta} e^{\Lambda y (\zeta(\alpha + 1) - O(y^{\alpha} - 2\pi inx))} dx
\]
\[
e^{1+\frac{1}{\alpha}} (\Lambda y (\zeta(\alpha + 1) \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{\tau}}) e^{-2\pi inx + O(y^{\alpha})} dx
\]
\[
= e^{1+\frac{1}{\alpha}} (\Lambda y (\zeta(\alpha + 1) \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{\tau}}) e^{-2\pi inx + O(y^{\alpha})} dx
\]
\[
\frac{2^{-\beta-1} \Lambda y (\zeta(\alpha + 1) \alpha^{-1})} {2^{r-1} \Lambda y (\zeta(\alpha + 1) \alpha^{-1})} e^{-2\pi inx + O(y^{\alpha})} dx
\]

With \( u = -\frac{2 \pi x}{y} \), we obtain
\[
I_r(n) = \frac{ye^{1+\frac{1}{\alpha}} (\Lambda y (\zeta(\alpha + 1) \alpha^{-1})} {2^{r-1} \Lambda y (\zeta(\alpha + 1) \alpha^{-1})} e^{-2\pi inx + O(y^{\alpha})} dx.
\]
\[
(29)
\]

Set, for simplicity, \( B = \Lambda y (\zeta(\alpha + 1) \alpha^{-1}) \). We have the Taylor series expansion
\[
\frac{2^{-\beta-1} \Lambda y (\zeta(\alpha + 1) \alpha^{-1})} {2^{r-1} \Lambda y (\zeta(\alpha + 1) \alpha^{-1})} = 1 - \frac{iu}{r} - \frac{(r+1)u^2}{2r^2} + O(|u|^3),
\]
from where, on recalling that \( |u| \leq 2\pi y^{\beta-1} \) and using (20) to compute \( B = rny^{1 + \frac{\alpha}{2}} \), it follows that

\[
B \left( \frac{1}{\sqrt{y}} \left( \frac{1}{\sqrt{1 + ruu}} - 1 \right) + inuy = -\frac{Biu}{r \sqrt{y}} + inuy \right) - \frac{(r + 1)Bu^2}{2r^2 \sqrt{y}} + O \left( \frac{|u|^3}{\sqrt{y}} \right) = \frac{(r + 1)Bu^2}{2r^2 \sqrt{y}} + O \left( \frac{1}{n^{r+1}} \left( 1 + \frac{3(1-\beta)}{\alpha} \right) \right).
\]

For an appropriate constant \( C_2 \), we may then change the integral from the right-hand side of (29) into

\[
\int_{|u| \leq 2\pi y^{\beta-1}} e^{B \frac{1}{\sqrt{y}} \left( \frac{1}{\sqrt{1 + ruu}} - 1 \right) + inuy + O(y^\alpha)} du = \int_{|u| \leq C_2} e^{-\frac{(r + 1)Bu^2}{2r^2 \sqrt{y}}} \left( 1 + O \left( \frac{1}{n^{r+1}} \left( 1 + \frac{3(1-\beta)}{\alpha} \right) \right) \right) du.
\]

From the first inequality in (3), we see that \( 1 + 3r(1-\beta) < 0 \), and thus

\[
e^{-\left( n \frac{r \beta}{r+1} + n \frac{1 + 3r(1-\beta)}{r+1} \right)} - 1 = e^{O \left( n \frac{r \beta}{r+1} + n \frac{1 + 3r(1-\beta)}{r+1} \right)} - 1 = O(n^{-\kappa}),
\]

where \( \kappa = \frac{1}{r+1} \min \left\{ rC_0, \frac{1}{2}, \frac{3r}{4} \right\} \). We further get, on using (7) when changing the limits of integration,

\[
\int_{|u| \leq 2\pi y^{\beta-1}} e^{B \frac{1}{\sqrt{y}} \left( \frac{1}{\sqrt{1 + ruu}} - 1 \right) + inuy + O(y^\alpha)} du = \int_{|u| \leq C_2} e^{-\frac{(r + 1)Bu^2}{2r^2 \sqrt{y}}} (1 + O(n^{-\kappa})) du = c(n) \int_{|u| \leq C_3 \cdot n^{\frac{\delta}{r+1}}} e^{-\frac{u^2}{2}} (1 + O(n^{-\kappa})) du,
\]

where \( c(n) = \sqrt{\frac{2r}{r+1}} (\alpha Bn^\alpha)^{\frac{\alpha}{\alpha+1}} \) and \( C_3 > 0 \) is a constant. By letting \( n \to \infty \), and turning the integral from (30) into a Gauss integral, we obtain

\[
\int_{|u| \leq 2\pi y^{\beta-1}} e^{B \frac{1}{\sqrt{y}} \left( \frac{1}{\sqrt{1 + ruu}} - 1 \right) + inuy + O(y^\alpha)} du = c(n) \sqrt{\pi} (1 + O(n^{-\kappa_1})),
\]

where \( \kappa_1 = \frac{1}{r+1} \min \left\{ rC_0 - \frac{\delta}{2}, \frac{1}{2}, \frac{3r}{4} \right\} \). Putting together (29), (30) and (31) we see that, as predicted by Meinardus (Theorem 3), the main asymptotic contribution for our coefficients is given by

\[
a_r(n) \sim Cn^{\frac{\alpha + 2}{2(\alpha + 1)}} e^{\frac{n^{\alpha+1}}{2} (1 + \frac{1}{\beta}) (A\Gamma(\alpha + 1) \zeta(\alpha + 1))},
\]

where

\[
C = \frac{1}{\sqrt{2^r (\alpha + 1) \pi}} (A\Gamma(\alpha + 1) \zeta(\alpha + 1))^{\frac{1}{2(\alpha + 1)}}.
\]

This shows that the inequalities in (27) are true for \( n \to \infty \). The proof can be completed either by adding the estimate (32) for \( a_r(n) \) or by invoking the recent result of Zhou [16] Theorem 1.1. □

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