FROBENIUS AND SPHERICAL CODOMAINS AND NEIGHBOURHOODS

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Abstract. Given an exact functor between triangulated categories which admits both adjoints and whose cotwist is either zero or an autoequivalence, we show how to associate a unique full triangulated subcategory of the codomain on which the functor becomes either Frobenius or spherical, respectively. We illustrate our construction with examples coming from projective bundles and smooth blowups. This work generalises results about spherical subcategories obtained by Martin Kalck, David Ploog and the first author.

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Introduction

In this article we will study exact functors $F : \mathcal{A} \to \mathcal{B}$ between (suitably enhanced) triangulated categories which admit both a left adjoint $L$ and a right adjoint $R$. Using the unit $\eta$ and counit $\varepsilon$ of adjunction $F \dashv R$, one can associate two natural endofunctors to $F$, namely the cotwist $C$ and the twist $T$, which fit into the triangles:

$$C \to \text{id}_\mathcal{A} \xrightarrow{\eta} RF \quad \text{and} \quad FR \xrightarrow{\varepsilon} \text{id}_\mathcal{B} \to T.$$
These endofunctors are ubiquitous in nature because:

"adjoint functors arise everywhere".  
(Saunders Mac Lane)

In this paper, we will focus on the two most fundamental cases for the cotwist:

(i) \( C = 0 \), which is equivalent to \( F \) being fully faithful;
    we call a fully faithful functor with both adjoints **exceptional**;
(ii) \( C \) is an autoequivalence, in which case we call \( F \) **spherelike**.

At this point, we want to offer up an extension to Mac Lane’s famous slogan above with the following imperative, which will act as our guiding principle throughout:

"if a functor admits both adjoints then compare them!"

In particular, for the two fundamental cases described above, we have canonical natural transformations between \( R \) and \( L \), namely:

(i) \( \varphi : R \to RFL \overset{\sim}{\to} L \), if \( F \) is exceptional;
(ii) \( \varphi : R \to RFL \to CL[1] \), if \( F \) is spherelike.

Thus, a natural comparison question is whether \( \varphi \) is an isomorphism in either case? If \( \varphi \) is an isomorphism then we recover the well-established notions of \( F \) being:

(i) **exceptionally Frobenius** in the exceptional case;
(ii) **spherical** (or quasi-Frobenius) in the spherelike case.

However, if \( \varphi \) is not an isomorphism then one can complete \( \varphi \) to a triangle of functors and use the cocones to measure how far away \( F \) is from being (quasi-)Frobenius:

(i) if \( F \) is exceptional then we have a triangle \( P \to R \to L \),
    and we call \( \text{Frb}(F) := \ker P \subset B \) the Frobenius codomain of \( F \);
(ii) if \( F \) is spherelike then we have a triangle \( Q \to R \to CL[1] \),
    and we call \( \text{Sph}(F) := \ker Q \subset B \) the spherical codomain of \( F \).

**Theorem A.** Let \( F : A \to B \) be an exceptional or spherelike functor and let \( B_F \) be the Frobenius or spherical codomain, respectively. Then \( \text{im} F \subset B_F \) and the corestriction \( F|_{B_F} : A \to B_F \) is exceptionally Frobenius or spherical, respectively. Moreover, \( B_F \) is the maximal full triangulated subcategory of \( B \) with this property.

This theorem is the main result of Section 2.2 and Section 3.2, respectively. There is a local version of these codomains for objects \( FA \in B \), where \( A \in A \) is some object in the source category. For simplicity, we assume that \( A \) and \( B \) admit Serre functors \( S_A \) and \( S_B \), respectively. The local statements are as follows:

(i) if \( F \) is exceptional then our triangle becomes \( FS_A \to S_BF \to TS_BF \),
    and we call \( \text{Frb}(F, A) := \langle TS_BF \rangle \) the Frobenius neighbourhood of \( FA \in B \);
(ii) if \( F \) is spherelike then our triangle becomes \( FS_A \overset{C^{-1}[1]}{\to} S_BF \to Q^*S_A \),
    and we call \( \text{Sph}(F, A) := \langle Q^*S_A \rangle \) the spherical neighbourhood of \( FA \in B \).

Here we denote by \( Q^* \) the right adjoint of \( Q \).
Theorem B. Let $F: \mathcal{A} \to \mathcal{B}$ be an exceptional or spherelike functor and let $\mathcal{B}_{FA}$ be the Frobenius or spherical neighbourhood of $FA \in \mathcal{B}$, for some $A \in \mathcal{A}$. Then, inside $\mathcal{B}_{FA}$, the Serre dual of $FA$ is given by $FS_A A$ or $FS_A C^{-1}[-1]A$, respectively. Moreover, $\mathcal{B}_{FA}$ is the maximal full triangulated subcategory of $\mathcal{B}$ with this property.

This theorem is proven in Section 2.3 and Section 3.3. These neighbourhoods can be put into a set, which is ordered by inclusion, thus yielding the Frobenius or spherical poset of an exceptional or spherelike functor, respectively.

The symmetrical nature of $C$ and $T$ means that we could also consider the fundamental cases of when $T$ is zero or $T$ is an equivalence. The dual nature of these constructions might lead us to name the corresponding functors coexceptional and cospherelike, respectively, and it is easy to see how we would obtain analogous results to that of Theorem A and Theorem B.

We illustrate the theory by several examples. On the exceptional side, we study exceptional functors coming from projective bundles and blowups. We highlight Proposition 2.5.3 of blowing up a $\mathbb{P}^1$ on a threefold $\pi: \text{Bl}_{\mathbb{P}^1}(X) \to X$. There we can determine the Frobenius poset of the exceptional functor $\pi^*$: it encodes the poset of thick subcategories of $\mathbb{D}^b(\mathbb{P}^1)$. Additionally, we show that in case of hypersurfaces of degree $n$ in $\mathbb{P}^{2n-1}$, the linkage class appears actually as the triangle associated to an exceptional functor. On the spherelike side, we obtain a wealth of examples by Theorem 3.4.3: the composition of a spherical functor $F_1$ and an exceptional functor $F_2$ gives a spherelike functor $F_2F_1$ and the its spherical neighbourhoods can be expressed as Frobenius neighbourhoods of $F_1$. Currently, this is the only way we know how to build spherelike functors. It would be interesting to find examples which are not of this shape.

This article grew out of an attempt to generalise the notion of spherelike objects, as introduced in [HKP16, HKP19], to spherelike functors; see Section 3.5 for a detailed comparison. Whilst building up the theory, we realised that central statements and examples in loc. cit. are about embedding spherical objects by an exceptional functor, and thus they are actually statements about Frobenius neighbourhoods rather than spherical neighbourhoods; see Proposition 3.5.3 and the examples thereafter.

Conventions. Throughout, all categories will be triangulated and linear over an algebraically closed field $k$. In particular, all subcategories will be triangulated. Additionally, we will often implicitly assume that the triangulated categories admit an enhancement, in order to speak about triangles of functors. The shift functor will be denoted by $[1]$ and all triangles will be exact. We write $A \to B \to C$ for an (exact) triangle, suppressing the degree increasing map $C \to A[1]$. Finally, all functors will be exact. In particular, we will denote derived functors with the same symbol as its (non-exact) counterpart on the abelian level. For example, for a proper
morphism $\pi: X \to Y$, we write $\pi_*: \mathbb{D}^b(X) \to \mathbb{D}^b(Y)$ for the derived pushforward. Dualisation over $k$ is given by $(\_)^\vee := \text{Hom}(\_ , k)$ and we use $\text{Hom}^*(A, B)$ to mean the graded $k$-vector space $\bigoplus_i \text{Hom}^i(A, B[i][−i])$, which can also be considered as a complex with zero differential in $\mathbb{D}^b(k\text{-mod})$.

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1. **Preliminaries**

In this section, we collect some standard facts as well as detailing the terminology and notation that we will use throughout the article.

1.1. **Generating triangulated subcategories.** Recall that all categories are assumed to be triangulated, unless stated otherwise.

**Definition 1.1.1.** A subcategory $\mathcal{C}$ of $\mathcal{A}$ is called **thick** if it is full and closed under direct summands, i.e. if $C \oplus C' \in \mathcal{C}$ then $C, C' \in \mathcal{C}$ as well.

For an arbitrary family $\mathcal{F}$ of objects in $\mathcal{A}$, the **thick closure** of $\mathcal{F}$ is the smallest thick subcategory of $\mathcal{A}$ containing $\mathcal{F}$ and will be denoted by $\text{thick}(\mathcal{F})$.

**Definition 1.1.2.** Let $\mathcal{F}$ be an arbitrary family of objects in $\mathcal{A}$. Then the **right orthogonal** of $\mathcal{F}$ is 

$$\mathcal{F}^\perp := \{ A \in \mathcal{A} | \text{Hom}^*(F, A) = 0 \text{ for all } F \in \mathcal{F} \}.$$  
Likewise, the **left orthogonal** of $\mathcal{F}$ is 

$$\perp \mathcal{F} := \{ A \in \mathcal{A} | \text{Hom}^*(A, F) = 0 \text{ for all } F \in \mathcal{F} \}.$$  

**Remark 1.1.3.** The full subcategory of $\mathcal{A}$ with objects in $\mathcal{F}^\perp$ is automatically triangulated and thick. The same holds true for $\perp \mathcal{F}$. For this reason, we will in the following identify $\mathcal{F}^\perp$ and $\perp \mathcal{F}$ with the corresponding (full) subcategories of $\mathcal{A}$.

**Definition 1.1.4.** An object $A$ of $\mathcal{A}$ is said to be:

- a **weak generator** of $\mathcal{A}$ if $A^\perp = 0$;
- a **classical generator** of $\mathcal{A}$ if $\mathcal{A} = \text{thick}(A)$.

**Remark 1.1.5.** Note that if $A$ is a direct sum of exceptional objects, then both notions of weak and classical generator are equivalent. A classical generator is always a weak generator, but the converse implication does not hold in general.

**Example 1.1.6.** If $X$ is a smooth projective variety and $\mathcal{L}$ a very ample line bundle, then $A = \mathcal{O}_X \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^\otimes \text{dim}(X)$ is a classical generator of $\mathbb{D}^b(X)$, see [Orl09, Thm. 4].
Definition 1.1.7. A pair of full subcategories \((\mathcal{A}, \mathcal{B})\) of a triangulated category \(\mathcal{D}\) is said to be a semiorthogonal decomposition if

- \(\text{Hom}^*(B, A) = 0\) for all \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\);
- for all \(D \in \mathcal{D}\) there is an exact triangle
  \[ D_B \to D \to D_A \]
  with \(D_A \in \mathcal{A}\) and \(D_B \in \mathcal{B}\).

We denote a semiorthogonal decomposition by \(\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle\).

The following statements about semiorthogonal decompositions are standard and can be found, for example, in [Bon89] or [Kuz15].

Proposition 1.1.8. If \(\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle\) is a semiorthogonal decomposition then the assignments \(D \mapsto D_A\) and \(D \mapsto D_B\) are functorial in \(D\) and define left and right adjoints to the inclusions \(\mathcal{A} \to \mathcal{D}\) and \(\mathcal{B} \to \mathcal{D}\), respectively. Moreover, we have \(\mathcal{A} = \mathcal{B}^\perp\) and \(\mathcal{B} = \mathcal{A}^\perp\).

Definition 1.1.9. Let \(\mathcal{A}\) be a full subcategory of \(\mathcal{D}\). Then \(\mathcal{A}\) is called

- right admissible if the inclusion functor \(\mathcal{A} \hookrightarrow \mathcal{B}\) has a right adjoint;
- left admissible if the inclusion functor admits a left adjoint;
- admissible if it is both left and right admissible.

Proposition 1.1.10. Let \(\mathcal{A}\) be a left admissible subcategory of \(\mathcal{D}\). Then \(\mathcal{A}^\perp\) is right admissible and \(\mathcal{D} = \langle \mathcal{A}, \mathcal{A}^\perp \rangle\) is a semiorthogonal decomposition.

We can iterate the definition of semiorthogonal decompositions.

Definition 1.1.11. A sequence \((\mathcal{A}_1, \ldots, \mathcal{A}_n)\) of full subcategories in \(\mathcal{D}\) is called semiorthogonal decomposition if \(\mathcal{A}_n\) is right admissible in \(\mathcal{D}\) and \(\langle \mathcal{A}_1, \ldots, \mathcal{A}_{n-1} \rangle\) is a semiorthogonal decomposition of \(\mathcal{A}_n^\perp\). In this case, we write \(\mathcal{D} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle\).

Remark 1.1.12. By this definition, \(\mathcal{D}\) decomposes into a nested semiorthogonal decompositions:

\[ \mathcal{D} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle = \langle \ldots \langle \mathcal{A}_1, \mathcal{A}_2 \rangle, \ldots \rangle, \mathcal{A}_n \rangle. \]

Actually, one can check that the order of the nesting does not matter, since we have \(\mathcal{A}_i = \mathcal{A}_i^\perp \cap \mathcal{A}_{i+1}^\perp \cap \cdots \cap \mathcal{A}_n^\perp\). Moreover, note that \(\mathcal{A}_1\) is left admissible in \(\mathcal{D}\) (and \(\mathcal{A}_n\) right admissible), whereas for the terms in between we cannot make a general statement about left or right admissibility in \(\mathcal{D}\).

Remark 1.1.13. For a semiorthogonal decomposition \(\mathcal{D} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle\), it is often assumed in the literature that all \(\mathcal{A}_i\) are admissible in \(\mathcal{D}\), and the definition we gave above is sometimes called a weak semiorthogonal decomposition.

In the presence of a Serre functor of \(\mathcal{D}\), a left or right admissible subcategory of \(\mathcal{D}\) is automatically admissible. A particular consequence of this is that all terms of
a (weak) semiorthogonal decomposition become admissible. That is, if we have the luxury of Serre functors then there is no difference between the two notions.

1.2. Serre duality. We recall some basic facts about Serre duality, all of which can be found, for example, in [BK89] or [Huy06].

**Definition 1.2.1.** Let $A$ be an object in a triangulated category $\mathcal{A}$. An object $SA \in \mathcal{A}$ is called a Serre dual of $A$ if it represents the functor $\text{Hom}(A, \_)^\vee$. Moreover, $A$ is called $d$-Calabi-Yau if $A[d]$ is a Serre dual for $A$.

We say that $S : \mathcal{A} \to \mathcal{A}$ is a Serre functor of $\mathcal{A}$ if $S$ is an equivalence and $SA$ is a Serre dual for all $A \in \mathcal{A}$, i.e. there is an isomorphism

$$\text{Hom}(A, B) \cong \text{Hom}(B, SA)^\vee$$

which is natural in $A, B \in \mathcal{A}$. Finally, if $S = [d]$ for some $d \in \mathbb{Z}$, then we say that $\mathcal{A}$ is a $d$-Calabi-Yau category.

**Proposition 1.2.2.** Let $A$ be a right admissible subcategory of $\mathcal{B}$, and $SB$ be a Serre functor of $\mathcal{B}$. If $r : \mathcal{B} \to \mathcal{A}$ is the right adjoint of the inclusion $i : \mathcal{A} \to \mathcal{B}$ then $rSBi$ is a Serre functor of $\mathcal{A}$.

**Proposition 1.2.3.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between categories that admit Serre functors $SA$ and $SB$, respectively. If $L \dashv F$ then $F \dashv S_A L S_B^{-1}$. Similarly, if $F \dashv R$ then $S_A^{-1} R S_B \dashv F$.

1.3. Kernel, image and (co)restriction.

**Definition 1.3.1.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. The kernel of $F$ is the full subcategory:

$$\ker F = \{ A \in \mathcal{A} \mid F(A) = 0 \} \subset \mathcal{A}.$$ 

The (essential) image of $F$ is the subset:

$$\text{im} F = \{ B \in \mathcal{B} \mid B \cong F(A) \text{ for some } A \in \mathcal{A} \} \subset \mathcal{B}.$$ 

**Remark 1.3.2.** Note that $\ker F$ is automatically triangulated. Moreover, the kernel $\ker F$ is a thick subcategory of $\mathcal{A}$. Actually this generalises the notion of orthogonals, e.g. $A^\perp = \ker \text{Hom}^*(A, \_).$ On the other hand, if $F$ is fully faithful, then the full subcategory of $\mathcal{B}$ with objects $\text{im} F$ will be triangulated. For general $F$ this might not be true.

**Definition 1.3.3.** Let $F : \mathcal{A} \to \mathcal{B}$ a functor. If we have a full subcategory $\mathcal{A}' \subset \mathcal{A}$ then the restriction of $F$ to $\mathcal{A}'$ is the functor:

$$F|_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{B},$$

which does the same as $F$ on objects and morphisms.
Similarly, if we have a full subcategory \( B' \subset B \) such that \( \text{im} F \subset B' \) then the corestriction of \( F \) to \( B' \) is the functor:

\[
F|_{B'} : A \to B',
\]

which also does the same as \( F \) on objects and morphisms.

1.4. Functors with both adjoints.

**Definition 1.4.1.** If \( F : A \to B \) is an exact functor between triangulated categories with left adjoint \( L \) and right adjoint \( R \) then we can use Fourier–Mukai kernels, bimodules or dg-enhancements, to define the twist \( T \) and cotwist \( C \) of \( F \) by the following triangles:

\[
FR \overset{\varepsilon_R}{\longrightarrow} \text{id}_B \overset{\alpha_R}{\longrightarrow} T \overset{\beta_R}{\longrightarrow} FR[1] \quad \text{and} \quad C \overset{\delta_L}{\longrightarrow} \text{id}_A \overset{\eta_R}{\longrightarrow} RF \overset{\gamma_L}{\longrightarrow} C[1],
\]

where \( \eta_R \) and \( \varepsilon_R \) are the unit and counit of adjunction, respectively. Similarly, the dual twist \( T' \) and dual cotwist \( C' \) are defined by the adjoint triangles:

\[
T' \overset{\delta_L}{\longrightarrow} \text{id}_B \overset{\eta_L}{\longrightarrow} FL \overset{\gamma_L}{\longrightarrow} T'[1] \quad \text{and} \quad LF \overset{\varepsilon_L}{\longrightarrow} \text{id}_A \overset{\alpha_L}{\longrightarrow} C' \overset{\beta_L}{\longrightarrow} LF[1],
\]

where \( \eta_L \) and \( \varepsilon_L \) are again the unit and counit of adjunction, respectively.

**Remark 1.4.2.** For the construction of these triangles and the fact that they behave well under adjunction, we refer the reader to [CW10] or [AL17].

**Remark 1.4.3.** If we have more than one functor present in an argument, such as a composition \( F_2 \circ F_1 : A \to B \to C \), then we will use \( \eta_1 \) and \( \eta_2 \) for the unit morphisms associated to \( F_1 \) and \( F_2 \), respectively. In particular, \( \eta_1 \) will be used to denote either \( \eta_{R_1} : \text{id} \to R_1F_1 \) or \( \eta_{L_1} : \text{id} \to F_1L_1 \). Since these maps are taking place on different categories, this should not cause confusion.

**Lemma 1.4.4** ([Add16, §2.3] or [Mea16, Lem. 1.4]). We have natural isomorphisms:

\[
TF[-1] \simeq FC[1] \quad RT[-1] \simeq CR[1] \quad FC'[-1] \simeq T'[1] \quad C'L[-1] \simeq LT'[1].
\]

2. Exceptional functors

2.1. Definition and examples. We start with the central notion of this section.

**Definition 2.1.1.** We say that a functor \( F : A \to B \) is exceptional if it is fully faithful and admits both adjoints. If, in addition, there is an isomorphism \( R \simeq L \) between the adjoints of \( F \), then we say that \( F \) is exceptionally Frobenius.

**Remark 2.1.2.** Note that an exceptional functor is essentially the inclusion of an admissible subcategory.

Recall [Huy06, Cor. 1.23] that if \( L \dashv F \dashv R \) then \( F \) being fully faithful is equivalent to \( \eta_R : \text{id}_A \xrightarrow{\sim} RF \) and \( \varepsilon_L : LF \xrightarrow{\sim} \text{id}_A \) being isomorphisms.
Remark 2.1.3. A functor $F$ is called Frobenius, if there is an isomorphism $R \simeq L$ between the adjoints of $F$. Note that $F$ need not to be fully faithful. As an example consider $F = (\_ \otimes E) : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ with $E$ any object in $\mathcal{D}^b(X)$ where $X$ is smooth and projective. Then the adjoints of $F$ are $R = L = \mathcal{H}om(E, \_)$, but $F$ will not be fully faithful in general.

Lemma 2.1.4. If $F$ is exceptional then we have natural isomorphisms:

$$L\eta_L : L \xrightarrow{\sim} LFL \quad R\varepsilon_R : RFR \xrightarrow{\sim} R \quad \eta_L F : F \xrightarrow{\sim} FLF \quad \varepsilon_R F : FRF \xrightarrow{\sim} F.$$

Proof. Consider the triangle identity:

$$C'[L[-1]] \quad L \xrightarrow{L\eta_L} \quad LFL \xrightarrow{\varepsilon_L} \quad LT'[1].$$

Since $F$ is fully faithful we know that $\varepsilon_L : LF \xrightarrow{\sim} \text{id}_A$ and hence $C' = 0$. In particular, we have $LT'[1] \simeq C'[L[-1]] = 0$ which implies $L\eta_L : L \to LFL$ is an isomorphism. That is, even though $\eta_L : \text{id}_A \to FL$ is not an isomorphism, it becomes an isomorphism after applying $L$ on the left, or $F$ on the right. The other isomorphisms follow from similar arguments.

Remark 2.1.5. Note that as soon as $\text{id}_A$ and $RF$ are naturally isomorphic, then $\eta_R$ is already an isomorphism (and analogously for $\varepsilon_L$); see [Joh02, Lem. 1.1.1].

Lemma 2.1.6. Let $F : A \to B$ be an exceptional functor. Then the canonical maps:

$$\varphi : R \xrightarrow{R\eta_L} RFL \xrightarrow{\varepsilon_L^{-1}} L \quad \text{and} \quad \psi : R \xrightarrow{\varepsilon_L^{-1}R} LFR \xrightarrow{\text{L}\epsilon_R} L.$$

are equal.

Proof. The claim can be reformulated to show that the following diagram commutes:

$$\begin{array}{ccc}
LFR & \xrightarrow{L\epsilon_R} & L \\
\downarrow \varepsilon_L & & \downarrow \eta_L \\
R & \xrightarrow{R\eta_L} & RFL.
\end{array}$$

Since $F$ is fully faithful, the statement follows by the commutativity of the following diagram:

$$\begin{array}{ccc}
FLFR & \xrightarrow{FL\epsilon_R} & FL \\
F\varepsilon_L & \downarrow & F\eta_L \\
FR & \xrightarrow{FR\eta_L} & FRFL.
\end{array}$$
By Lemma 2.1.4, the maps $F \xrightarrow{\eta_R} FLF$ and $FLF \xrightarrow{\varepsilon_L} F$ are inverse to each other, and the same holds for $F \xrightarrow{F\eta_R} FRF$ and $FRF \xrightarrow{\varepsilon_R} F$. Extending the previous diagram by these isomorphisms we get:

The triangles on both sides commute by the remark above, whereas the bottom square commutes as the units and counits act on separate variables. To conclude that $(\ast)$ is commutative, we note that $\varepsilon_R F L$ is an isomorphism and

$$\varepsilon_R F L \circ F R \eta_L \circ F \eta_L R = F L \varepsilon_R \circ \eta_L F R \circ F \varepsilon_L R = \varepsilon_R F L \circ F \eta_R L \circ F L \varepsilon_R$$

which finishes the proof. For convenience of the reader we depict this chain:

Proposition 2.1.7. Let $F: A \rightarrow B$ be an exceptionally Frobenius functor. Then the canonical map

$$\varphi: R \xrightarrow{R \eta_R} RFL \xrightarrow{\eta_R^{-1} L} L$$

is an isomorphism.

Proof. Since $F$ is fully faithful, $\eta_R$ is an isomorphism and so it is sufficient to show that $R \eta_R: R \rightarrow RFL$ is an isomorphism. If we suppose the isomorphism between $R$ and $L$ is given by $\alpha: R \sim L$, then we can form the commutative diagram:

which commutes because the arrows act on separate variables. In particular, we have $R \eta_R = (\alpha F L)^{-1} \circ L \eta_R \circ \alpha$, which is an isomorphism by Lemma 2.1.4.

Example 2.1.8. Let $A \in A$ be an exceptional object, i.e. $\text{Hom}^\ast(A, A) \simeq \mathbb{k}$. Assume that $A$ admits an anti-Serre dual $S^{-1}A$ and $A$ is proper, i.e. $\text{Hom}^\ast(A, A')$ and $\text{Hom}^\ast(A', A)$ are finite-dimensional (graded) vector spaces for all $A' \in A$. Then the functor

$$F = F_A: \mathbb{D}^b(\text{k-mod}) \rightarrow A, V^\ast \mapsto V^\ast \otimes A$$
is exceptional. Its adjoints are $R = R_A = \text{Hom}^*(A, \_)$ and $L = L_A = \text{Hom}^*(S^{-1}A, \_) = \text{Hom}^*(\_, A)^\vee$.

**Example 2.1.9.** The inclusion of an admissible subcategory is, by definition, a fully faithful functor with both adjoints, hence exceptional. Moreover, any exceptional functor $F: A \to B$ factors into an equivalence $A \to \text{im } F$ and an inclusion of an admissible subcategory $\text{im } F \hookrightarrow B$.

As a special instance of this type, consider a cubic fourfold $Y \subset \mathbb{P}^5$. Then $A_Y = \langle 0, O(1), O(2) \rangle^\perp \subset D^b(Y)$ is called the Kuznetsov component, [Kuz10]. The category $A_Y$ 2-Calabi–Yau in the sense that it has a Serre functor given by $S_{A_Y} = [2]$ and, because of this, $A_Y$ is often referred to as a noncommutative K3 surface.

In Section 2.4 and Section 2.5 we will discuss in detail exceptional functors coming from projective bundles and smooth blowups.

**Proposition 2.1.10** (e.g. [Kuz15, Lem. 2.3]). Let $F: A \to B$ be an exceptional functor. Then there are semiorthogonal decompositions:

$$B = \langle \ker R, \text{im } F \rangle = \langle \text{im } F, \ker L \rangle$$

where the decompositions are given by twist and dual twist, respectively:

$$FR \to \text{id} \to T, \quad T' \to \text{id} \to FL.$$  

In particular, $T$ projects onto $\ker R$ and induces an equivalence $\ker L \to \ker R$, whereas $T'$ projects onto $\ker L$ and gives an equivalence $\ker R \to \ker L$.

**Remark 2.1.11.** We point out that the twist $T$ coincides with the left mutation functor $L_{\text{im } F}$ through $\text{im } F$. Similarly, the dual twist functor is the right mutation functor $R_{\text{im } F}$ through $\text{im } F$. See [Kuz07, §2.2] or [Bon89] for more details on this.

We note that even though $\text{im } F$ is admissible, $\ker L$ and $\ker R$ are in general only right and left admissible, respectively.

2.2. Frobenius codomains.

**Lemma 2.2.1.** If $F: A \to B$ is an exceptional functor then the cocone $P$ of the canonical map $\varphi: R \to L$ is isomorphic to $RT'$ and $LT[-1]$. In particular, we have triangles:

$$P \simeq RT' \to R \overset{\leq}{\to} L \quad \text{and} \quad R \overset{\leq}{\to} L \to LT \simeq P[1].$$  

**Proof.** Taking cones in Lemma 2.1.6 gives a commutative diagram of triangles:

$$
\begin{array}{ccc}
LFR & \xrightarrow{LFR} & L \\
\downarrow_{\varepsilon_R} & & \downarrow_{\eta_R} \\
R & \xrightarrow{R_{\eta_R}} & RFL \quad \xrightarrow{RT'[1]}
\end{array}
$$

from which the statements follow. \qed
**Definition 2.2.2.** Let $F : A \to B$ be an exceptional functor. Then we call $\text{Frob}(F) := \ker RT'$ the Frobenius codomain of $F$ and $F^\text{Frob}(F)$ the Frobenius corestriction of $F$. In particular, $F$ is exceptionally Frobenius if and only if $\text{Frob}(F) = B$.

**Theorem 2.2.3.** Let $F : A \to B$ be an exceptional functor. Then $\im F \subset \text{Frob}(F)$ and the corestriction $F^\text{Frob}(F) : A \to \text{Frob}(F)$ is exceptionally Frobenius. Furthermore, if $\mathcal{C}$ is a full subcategory of $B$ such that $\im F \subset \mathcal{C}$ and $F|^{\mathcal{C}} : A \to \mathcal{C}$ is exceptionally Frobenius, then $\mathcal{C} \subset \text{Frob}(F)$. That is, $\text{Frob}(F)$ is the maximal full subcategory on which $F$ becomes exceptionally Frobenius.

**Proof.** Since $F$ is fully faithful, the cotwist $C$ and its dual $C'$ are both zero. Therefore, by Lemma 1.4.4, we see that $PF := RT'F \simeq R\mathcal{C}'[-2] = 0$. In particular, we have $\im F \subset \ker P =: \text{Frob}(F)$ and the corestriction $F_1 := F|^{\text{Frob}(F)}$ makes sense.

Next we show that $F_1$ is Frobenius, that is, its adjoints are naturally isomorphic. If $F_2 : \ker P \to B$ denotes the inclusion then we have a natural isomorphism of functors $F \simeq F_2 F_1$ and the adjoints of $F_1$ are given by $R_1 \simeq R F_2$ and $L_1 \simeq L F_2$. We claim that we have a commutative diagram of triangles:

\[
\begin{array}{ccc}
RT'F_2 & \longrightarrow & RF_2 \\
\downarrow \wr & & \downarrow \wr \\
R_1T'_I & \longrightarrow & R_1F_1L_1.
\end{array}
\]

For commutativity of the right square, we apply $R$ to the compatibility condition:

\[
\begin{array}{ccc}
\Hom(LF_2, LF_2) & \sim & \Hom(F_2, FLF_2) \\
\downarrow & & \downarrow \\
\Hom(L_1, L_1) & \sim & \Hom(F_2, F_2F_1L_1) \\
\downarrow & & \downarrow \\
\Hom(id, F_1L_1) & \sim & \Hom(F_2, F_2F_1L_1).
\end{array}
\]

Therefore, we get an induced isomorphism $R_1T'_I \simeq RT'F_2 = 0$ as $F_2 : \ker RT' \to B$.

In particular, this yields an isomorphism $R_1\eta_1 : R_1 \simeq R_1F_1L_1$ and hence a composite isomorphism $R_1 \simeq R_1F_1L_1 \simeq L_1$, since $F_1$ is fully faithful, i.e. $\id_A \simeq R_1F_1$. So $F_1$ is exceptionally Frobenius.

For maximality, we let $F_1 : A \to \mathcal{C}$ be a corestriction of $F$ where $\mathcal{C}$ contains $\im F$. If $F_2 : \mathcal{C} \to B$ denotes the fully faithful embedding then a similar argument as above shows that we have

\[
\Hom(R\tilde{F}_2, RLF_2) \sim \Hom(R\tilde{F}_2, R_1F_1\tilde{L}_1), \quad R\tilde{F}_2 \rightarrow R_1\tilde{\eta}_1
\]

Moreover, if $\tilde{F}_1$ is exceptionally Frobenius then $R_1\tilde{\eta}_1$ is an isomorphism by Proposition 2.1.7, and hence $\im \tilde{F}_2$ is contained in $\ker P = \ker RT'$.

\[\Box\]
Actually, the structure of the Frobenius codomain is quite simple.

**Corollary 2.2.4.** Let $F: \mathcal{A} \to \mathcal{B}$ be an exceptional functor. Then the Frobenius codomain decomposes into

$$\text{Frb}(F) = \text{im } F \oplus (\ker R \cap \ker L).$$

**Proof.** Since $\ker R = \text{im } F$ and $\ker L = \bot \text{im } F$, we see that $\ker R \cap \ker L$ and $\text{im } F$ are mutually orthogonal. Hence $\text{im } F \oplus (\ker R \cap \ker L)$ is a subcategory of $\mathcal{B}$.

Now we check the inclusion “$\supseteq$”. We have checked already in the proof of Theorem 2.2.3 that $\text{im } F \subset \text{Frb}(F)$. Similarly, if $B \in \ker R \cap \ker L$ then the natural triangle $P \to R \to L$ shows that $B \in \ker P$, giving $\ker R \cap \ker L \subset \text{Frb}(F)$.

We turn to the converse inclusion “$\subseteq$”. If $B \in \text{Frb}(F) = \ker P \subset \mathcal{B}$, then we can use the semiorthogonal decomposition $\mathcal{B} = \langle \text{im } F, \ker L \rangle$ to break the object $B \in \mathcal{B}$ up via the triangle associated to the dual twist: $T'B \to B \to FLB$. Notice that $FLB \in \text{im } F \subset \ker P$ and $B \in \ker P$ together imply that $T'B \in \ker P$. Moreover, by Proposition 2.1.10 we have $T'B \in \ker L$ and so we see that $T'B \in \ker P \cap \ker L$. Finally, the triangle $P \to R \to L$ gives an equality $\ker P \cap \ker L = \ker R \cap \ker L$ and hence we see that $T'B \in \ker R \cap \ker L$, which completes the proof. \hfill \Box

**Remark 2.2.5.** The easiest example where the Frobenius codomain is strictly bigger than the image of $F$ is the inclusion of a direct summand $F: \mathcal{A} \hookrightarrow \mathcal{A} \oplus \mathcal{B}$. Here both adjoints are the same with kernel $\mathcal{B}$. In particular, $\text{Frb}(F) = \mathcal{A} \oplus \mathcal{B}$.

This behaviour is not pathological but rather the rule; see Section 2.4 and Section 2.5 for more details.

### 2.3. Frobenius neighbourhoods.

We can introduce a local analogue of the Frobenius codomain for objects.

**Definition 2.3.1.** Let $F: \mathcal{A} \to \mathcal{B}$ be an exceptional functor and $A \in \mathcal{A}$. The *Frobenius neighbourhood* of $FA \in \mathcal{B}$ is

$$\text{Frb}(F, A) := \{ B \in \mathcal{B} \mid \text{Hom}^\ast(A, RT'B) = 0 \}.$$

The Frobenius codomain is connected to the Frobenius neighbourhoods in the following way.

**Proposition 2.3.2.** Let $F: \mathcal{A} \to \mathcal{B}$ be an exceptional functor. Then

$$\text{Frb}(F) = \bigcap_{A \in \mathcal{A}} \text{Frb}(F, A).$$

**Proof.** We compute that

$$\text{Frb}(F) := \ker RT' = \{ B \in \mathcal{B} \mid RT'B = 0 \}$$

$$= \{ B \in \mathcal{B} \mid \text{Hom}^\ast(A, RT'B) = 0, \forall A \in \mathcal{A} \} \quad \text{(by Yoneda)}$$

$$= \bigcap_{A \in \mathcal{A}} \{ B \in \mathcal{B} \mid \text{Hom}^\ast(A, RT'B) = 0 \} = \bigcap_{A \in \mathcal{A}} \text{Frb}(F, A). \hfill \Box$$
**Proposition 2.3.3.** Let \( F : A \to B \) be an exceptional functor and \( A \in A \). Then \( \text{Frb}(F, A) \) is the maximal full subcategory of \( B \) such that

\[
\text{Hom}^*(A, \text{R}|_{\text{Frb}(F, A)}(\underline{\_})) \overset{\sim}{\to} \text{Hom}^*(A, \text{L}|_{\text{Frb}(F, A)}(\underline{\_})).
\]

**Proof.** First we check that \( FA \) lies inside \( \text{Frb}(F, A) \). Indeed, \( \text{Hom}^*(A, \text{RT}'FA) \) vanishes as \( \text{im} \mathbf{F} \subset \ker \text{RT}' \) by **Theorem 2.2.3**.

Applying \( \text{Hom}^*(A, \underline{\_}) \) to the triangle \( \text{RT}' \to \mathbf{R} \to \text{L} \) from **Lemma 2.2.1** yields the triangle

\[
\text{Hom}^*(A, \text{RT}'(\underline{\_})) \to \text{Hom}^*(A, \text{R}(\underline{\_})) \to \text{Hom}^*(A, \text{L}(\underline{\_})).
\]  

(2)

Plugging \( B \in \text{Frb}(F, A) \) into this triangle shows that

\[
\text{Hom}^*(A, \text{R}|_{\text{Frb}(F, A)}(\underline{\_})) \overset{\sim}{\to} \text{Hom}^*(A, \text{L}|_{\text{Frb}(F, A)}(\underline{\_})).
\]

Let \( \mathcal{C} \) be a full triangulated subcategory containing \( \text{im} \mathbf{F} \). We show that if

\[
\text{Hom}^*(A, \text{R}|_{\mathcal{C}}(\underline{\_})) \overset{\sim}{\to} \text{Hom}^*(A, \text{L}|_{\mathcal{C}}(\underline{\_}))
\]

then \( \mathcal{C} \subset \text{Frb}(F, A) \), which means that \( \text{Frb}(F, A) \) is maximal. Let \( C \in \mathcal{C} \) and plug it into (2). By assumption \( \text{Hom}^*(A, \text{R}(C)) \overset{\sim}{\to} \text{Hom}^*(A, \text{L}(C)) \), so \( \text{Hom}^*(A, \text{RT}'(C)) = 0 \). Consequently \( C \in \text{Frb}(F, A) \). \( \square \)

**Remark 2.3.4.** Note that this proposition fits nicely with **Theorem 2.2.3**: For \( B \in \bigcap_{A \in A} \text{Frb}(F, A) \) we get an isomorphism \( \text{Hom}^*(A, \text{R}B) \overset{\sim}{\to} \text{Hom}^*(A, \text{LB}) \) functorial in \( A \), which yields, by Yoneda, \( \text{RB} \overset{\sim}{\to} \text{LB} \) for \( B \in \bigcap_{A \in A} \text{Frb}(F, A) = \text{Frb}(F) \).

The following statement is our workhorse when computing the Frobenius codomains and neighbourhoods in examples.

**Theorem 2.3.5.** Let \( F : A \to B \) be an exceptional functor. Then for \( A \in A \),

\[
\text{Frb}(F, A) = (\text{im} \mathbf{F}, \ker \text{L} \cap \ker \text{Hom}^*(A, \text{R}(\underline{\_}))) = (\text{im} \mathbf{F}, \ker \text{L} \cap (\text{FA})^\perp).
\]

is a semiorthogonal decomposition.

**Proof.** Plugging \( B \in \mathcal{B} \) into the triangle (2) yields:

\[
\text{Hom}^*(A, \text{RT}'B) \to \text{Hom}^*(A, \text{R}B) \to \text{Hom}^*(A, \text{LB}).
\]

So for \( B \in (\text{im} \mathbf{F})^\perp \mathbf{L} \), we get \( \text{Hom}^*(A, \text{RT}'B) \cong \text{Hom}^*(A, \text{RB}) \). Therefore, by the definition of \( \text{Frb}(F, A) \) we get

\[
(\text{im} \mathbf{F})^\perp \mathbf{Frb}(F, A) = \ker \text{L} \cap \ker \text{Hom}^*(A, \text{RT}'(\underline{\_})) = \ker \text{L} \cap \ker \text{Hom}^*(A, \text{R}(\underline{\_})).
\]

Now let \( B \in \text{Frb}(F, A) \). As an object in \( \mathcal{B} = (\text{im} \mathbf{F}, (\text{im} \mathbf{F})^\perp \mathbf{F}) \), there is a the decomposition triangle \( \text{T}'B \to B \to \text{FLB} \). Since \( \text{FLB} \in \text{im} \mathbf{F} \subset \text{Frb}(F, A) \) by **Proposition 2.3.3**, \( \text{T}'B \in \text{Frb}(F, A) \) holds as well. So by the paragraph above \( \text{T}'B \in \ker \text{L} \cap \ker \text{Hom}^*(A, \text{R}(\underline{\_})) \), which completes the proof. \( \square \)
2.3.1. In presence of Serre functors. Even if both $A$ and $B$ admit Serre functors, the Frobenius neighbourhood $\text{Fr}(F, A)$ of an object will not have a Serre functor in general. Therefore we need the local notion of a Serre dual of an object, see Definition 1.2.1.

**Theorem 2.3.6.** Let $F : A \rightarrow B$ be an exceptional functor. If $A$ and $B$ admit Serre functors, then there is the natural triangle:

$$FS_A \rightarrow S_B F \rightarrow TS_B F.$$ 

In particular, we have $\text{Fr}(F) = \perp \text{im } TS_B F$ and $\text{Fr}(F, A) = \perp TS_B FA$ for $A \in A$.

**Proof.** This is a consequence of Lemma 2.2.1. Recall that $S_B^{-1} TS_B F \dashv T' \dashv T$. In particular, we can manipulate the first triangle there as follows:

$$\text{RT}' \rightarrow R \rightarrow L \iff \text{RT}' \rightarrow R \rightarrow S_A^{-1} RS_B \quad (\text{as } L \simeq S_A^{-1} RS_B)$$

$$\iff S_B^{-1} TS_B F \leftarrow F \leftarrow S_B^{-1} FS_A \quad (\text{taking left adjoints})$$

$$\iff TS_B F \leftarrow S_B F \leftarrow FS_A \quad (\text{applying } S_B).$$

From these manipulations we get that $\ker \text{RT}' = \text{im } (S_B^{-1} TS_B F) = \perp \text{im } TS_B F$, using Serre duality. The same reasoning for objects completes the proof:

$$\text{Hom}^*(A, \text{RT}'(\_)) = \text{Hom}^*(S_B^{-1} TS_B FA, \_ \_] = \text{Hom}^*(\_ \_ , TS_B FA)^\vee.$$ 

□

**Remark 2.3.7.** From the last triangle in the proof of Theorem 2.3.6 we get

$$\text{Hom}^*(A, \text{RT}' B) = \text{Hom}^*(B, TS_B FA)^\vee$$

which vanishes as soon as $B \in \text{Fr}(F, A)$. So we get from $FS_A A \rightarrow S_B FA \rightarrow TS_B FA$ that for $B \in \text{Fr}(F, A)$ holds functorially:

$$\text{Hom}^*_{\text{Fr}(F, A)}(B, FS_A A) = \text{Hom}^*_{\text{Fr}(F, A)}(B, FS_A A)$$

$$\cong \text{Hom}^*_{\text{Fr}(F, A)}(B, S_B FA)$$

$$\cong \text{Hom}^*_{\text{Fr}(F, A)}(FA, B)^\vee$$

$$= \text{Hom}^*_{\text{Fr}(F, A)}(FA, B)^\vee.$$ 

This means that $FS_A A$ is a Serre dual of $FA$ in $\text{Fr}(F, A)$.

**Corollary 2.3.8.** Let $F : A \rightarrow B$ be an exceptional functor and $A \in A$. Assume that $A$ and $B$ admit Serre functors. Then $\text{Fr}(F, A)$ is the maximal full subcategory of $B$ such that $FS_A A$ is a Serre dual of $FA$.

In particular, if $A$ is a $d$-Calabi-Yau object in $A$, then $\text{Fr}(F, A)$ is the maximal full subcategory of $B$ where $FA$ is $d$-Calabi-Yau. Therefore, we call in such a case $\text{Fr}(F, A)$ the Calabi-Yau neighbourhood of $FA$ in $B$.

**Proof.** This follows by combining Proposition 2.3.3 and Remark 2.3.7. □
Remark 2.3.9. In the situation of a Calabi-Yau object $A$ in Corollary 2.3.8, the Calabi-Yau neighbourhood $\text{Frb}(F, A)$ only depends on $A$ being a $d$-Calabi-Yau object somewhere. More precisely, if $\tilde{F}: \tilde{A} \to B$ is another exceptional functor and $\tilde{A} \in \tilde{A}$ a $d$-Calabi-Yau object such that $\tilde{F} \tilde{A} \cong FA$, then $\text{Frb}(\tilde{F}, \tilde{A}) = \text{Frb}(F, A)$.

To see this note that for $B := FS_A \cong FA[d] \cong FA[d] \cong FS_A \tilde{A}$, both $\text{Frb}(F, A)$ and $\text{Frb}(\tilde{F}, \tilde{A})$ are maximal with the property that $B$ is a Serre dual of $FA$.

2.3.2. Dual Frobenius neighbourhoods. For completeness, we mention that we could have started this subsection also using $LT$ instead of $RT'$. In this case, the key steps are

(i) The definition of a dual Frobenius neighbourhood of $A$ under $F$ is then

$$\text{Frb}^\lor(F, A) = \{ B \in \mathcal{B} \mid \text{Hom}^\ast(LTB, A) = 0 \}.$$

(ii) Proposition 2.3.3 can be extended by

$$\text{Hom}^\ast(R|_{\text{Frb}^\lor(F, A)}(\underline{\_}), A) \overset{\sim}{\longrightarrow} \text{Hom}^\ast(L|_{\text{Frb}^\lor(F, A)}(\underline{\_}), A).$$

(iii) In the presence of Serre functors, we get that $FS_A^{-1}A$ is an anti-Serre dual of $FA$ inside $\text{Frb}^\lor(F, A)$, i.e. corepresents $\text{Hom}^\ast(\underline{\_}, A)^\lor$. Moreover, one can check that $\text{Frb}^\lor(F, A) = \text{Frb}(F, S_A^{-1}A)$. In particular, if $A$ is a Calabi-Yau object, then $\text{Frb}(F, A) = \text{Frb}^\lor(F, A)$.

(iv) Finally, Theorem 2.3.5 can be extended by

$$\text{Frb}^\lor(F, A) = \langle \ker R \cap \ker \text{Hom}^\ast(L(\underline{\_}), A), \text{im } F \rangle = \langle \ker R \cap \bot FA, \text{im } F \rangle.$$

In particular, in presence of Serre functors, we arrive at

$$\text{Frb}(F, A) = \text{Frb}^\lor(F, S_A A) = \langle \ker R \cap \bot FS_A A, \text{im } F \rangle.$$

We leave the proofs as an exercise to the reader.

2.3.3. Frobenius poset. Inspired by the notion of a spherical poset of [HKP19, §2], we arrive at the following definition.

Definition 2.3.10. Let $F: \mathcal{A} \to \mathcal{B}$ be an exceptional functor. Then

$$\mathcal{P}(F) := \{ \text{Frb}(F, A) \mid A \in \mathcal{A} \}$$

is partially ordered by inclusion, which we call the Frobenius poset of $F$.

We collect here some general statements on the structure of such a poset.

Lemma 2.3.11. Let $F: \mathcal{A} \to \mathcal{B}$ be an exceptional functor. Then $\text{Frb}(F, 0) = \mathcal{B}$ is the maximal element of the Frobenius poset $\mathcal{P}(F)$.

Proof. Note that $\text{Frb}(F, 0) = \{ B \in \mathcal{B} \mid \text{Hom}^\ast(0, RT'B) = 0 \} = \mathcal{B}$. \qed
Remark 2.3.12. In many examples, \( \text{Frb}(F) \) is the minimal element of the Frobenius poset, see Section 2.4 and Section 2.5.

In general, if \( A \in \mathcal{A} \) is a weak generator, then \( \text{Frb}(F) = \text{Frb}(F, A) \). In particular, \( \text{Frb}(F) \) is the minimal element of the poset. To see this note that \( B \in \text{Frb}(F, A) \) if \( \text{Hom}^*(A, PB) = 0 \), which in turn implies that \( PB = 0 \) as \( A \) is a weak generator, hence \( B \in \ker P = \text{Frb}(F) \). Actually, in this argument it is only important that \( A \) is a weak generator for \( \text{im } P \).

Lemma 2.3.13. Let \( F: \mathcal{A} \rightarrow \mathcal{B} \) be an exceptional functor. Then for \( A, A' \in \mathcal{A} \) holds \( \text{Frb}(F, (A \oplus A')) = \text{Frb}(F, A) \cap \text{Frb}(F, A') \).

Proof. This follows easily from Theorem 2.3.5. \( \square \)

In examples, the following poset derived from the Frobenius poset will be useful.

Definition 2.3.14. Let \( F: \mathcal{A} \rightarrow \mathcal{B} \) be an exceptional functor. Then

\[
\hat{\mathcal{P}}(F) := \left\{ \bigcap_{A \in \mathcal{F}} \text{Frb}(F, A) \mid \mathcal{F} \text{ arbitrary family of objects in } \mathcal{A} \right\}
\]

is partially ordered by inclusion, which we call the completed Frobenius poset of \( F \).

We always have a natural inclusion \( \mathcal{P}(F) \subseteq \hat{\mathcal{P}}(F) \). Note that if \( \mathcal{A} \) admits arbitrary direct sums, then \( \mathcal{P}(F) = \hat{\mathcal{P}}(F) \) by Lemma 2.3.13.

2.4. Example: projective bundles. Let \( X \) be some projective variety and \( E \) a vector bundle on \( X \) of rank \( n+1 \). Consider the projective bundle \( q: \mathbb{P}(E) \rightarrow X \) and denote by \( \mathcal{O}_q(k) \) the relative twisting line bundles. By [Orl92, Lem. 2.5 & Thm. 2.6] the functor

\[
\Phi_k := q^* (\_ \otimes \mathcal{O}_q(k)): \text{D}^b(X) \rightarrow \text{D}^b(\mathbb{P}(E))
\]

is fully faithful for any \( k \in \mathbb{Z} \) and there is a semiorthogonal decomposition:

\[
\text{D}^b(\mathbb{P}(E)) = \langle \Phi_0(D^b(X)), \Phi_1(D^b(X)), \ldots, \Phi_n(D^b(X)) \rangle.
\]

In particular, we have that \( q^* = \Phi_0: D^b(X) \rightarrow D^b(\mathbb{P}(E)) \) is an exceptional functor.

The following is just the specialisation of Corollary 2.2.4 and Theorem 2.3.5 to the case of a projective bundle.

Proposition 2.4.1. Let \( q: \mathbb{P}(E) \rightarrow X \) be a \( \mathbb{P}^n \)-bundle. Then the Frobenius codomain of \( q^* \) is

\[
\text{Frb}(q^*) = q^* \text{D}^b(X) \oplus \ker q_* \cap \ker q!
\]

whereas the Frobenius neighbourhood for \( A \in \text{D}^b(X) \) is

\[
\text{Frb}(q^*, A) = \langle q^* \text{D}^b(X), \ker q_! \cap (q^* A)^\perp \rangle.
\]

For projective bundles of low rank we can say more.
Proposition 2.4.2. Let \( q: \mathbb{P}(\mathcal{E}) \to X \) be a \( \mathbb{P}^1 \)-bundle. Then we find that \( \text{Frb}(q^*) = q^*\mathcal{D}^b(X) \) and

\[
\text{Frb}(q^*, A) = q^*\mathcal{D}^b(X)
\]

for \( A \) a weak generator of \( \mathcal{D}^b(X) \).

Proof. The first part follows from the second one using Proposition 2.3.2:

\[
\text{Frb}(q^*) = \bigcap_{A \in \mathcal{D}^b(X)} \text{Frb}(q^*, A).
\]

Note that there is even a strong generator of \( \mathcal{D}^b(X) \) by Example 1.1.6.

For the second part, let \( A \) be a weak generator of \( \mathcal{D}^b(X) \), i.e. \( \text{Hom}^*(A, B) = 0 \) implies that \( B = 0 \). The Frobenius neighbourhood of \( A \) is

\[
\text{Frb}(q^*, A) = (q^*\mathcal{D}^b(X), (q^*\mathcal{D}^b(X) \otimes \mathcal{O}_q(1)) \cap (q^*A)^\perp).
\]

For \( B \in \mathcal{D}^b(X) \), we find that

\[
\text{Hom}^*(q^*A, q^*B \otimes \mathcal{O}_q(1)) = \text{Hom}^*(A, B \otimes q_*\mathcal{O}_q(1)) = \text{Hom}^*(A, B \otimes \mathcal{E}^\vee).
\]

In particular, if \( q^*B \otimes \mathcal{O}_q(1) \in q^*A^\perp \), then \( B \otimes \mathcal{E}^\vee = 0 \) using that \( A \) is a weak generator. For any closed point \( P \in X \), we therefore get for such \( B \)

\[
0 = \text{Hom}^*(\mathcal{O}_P, B \otimes \mathcal{E}^\vee) \cong \text{Hom}^*(\mathcal{O}_P \otimes \mathcal{E}, B) \cong \text{Hom}^*(\mathcal{O}_P^{\oplus 2}, B)
\]

As the skyscraper sheaves form a spanning class, we conclude that \( B = 0 \). \( \square \)

We consider the easiest examples of \( \mathbb{P}^1 \)-bundles: Hirzebruch surfaces. For this we recall some well-known facts about \( \mathcal{D}^b(\mathbb{P}^1) \). As the ground field \( k \) is algebraically closed, the indecomposable objects in \( \mathcal{D}^b(\mathbb{P}^1) \) are, up to shift, structure sheaves of (fat) closed points \( \mathcal{O}_{nP} \) and the line bundles \( \mathcal{O}(k) = \mathcal{O}_{p^1}(k) \). The thick subcategories of \( \mathcal{D}^b(\mathbb{P}^1) \) are \( \langle \mathcal{O}_{p^1}(k) \rangle, \mathcal{D}^b(\mathbb{P}^1) \) and \( \langle \mathcal{O}_P \mid P \in V \rangle \) with \( V \) any subset of closed points in \( \mathbb{P}^1 \). See for example [KS17, §4.1] for details, where \( k \) is not necessarily algebraically closed.

For a subset \( U \subset X \), we denote by \( \mathcal{D}^b_U(X) \) the full triangulated subcategory of \( \mathcal{D}^b(X) \) consisting of objects with support on \( U \). Hence in combination with Lemma 2.3.13, the following example gives a full description of a Frobenius poset.

Proposition 2.4.3. Let \( q: \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k)) \to \mathbb{P}^1 \). Then we find that

\[
\text{Frb}(q^*, \mathcal{O}_{nP}) = (q^*\mathcal{D}^b(\mathbb{P}^1), q^*\mathcal{D}^b_{\mathbb{P}^1 \setminus \{P\}}(\mathbb{P}^1) \otimes \mathcal{O}_q(1)),
\]

\[
\text{Frb}(q^*, \mathcal{O}(j)) = \begin{cases} q^*\mathcal{D}^b(\mathbb{P}^1) & \text{if } k \neq 0; \\ (q^*\mathcal{D}^b(\mathbb{P}^1), q^*\mathcal{O}(j-1) \otimes \mathcal{O}_q(1)) & \text{if } k = 0. \end{cases}
\]

In particular, \( \text{Frb}(q^*, \mathcal{O}_{nP}) \) is neither left nor right admissible.
Moreover, under the projection \( D^b(\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))) \to q^*D^b(\mathbb{P}^1) \otimes \mathcal{O}_q(1) \cong D^b(\mathbb{P}^1) \), the poset \( \widehat{\mathcal{P}}(q^*) \) and the poset of thick subcategories of \( D^b(\mathbb{P}^1) \) become isomorphic.

Proof. For the first part, note that for \( A \in D^b(\mathbb{P}^1) \)

\[
\text{Fr}b(q^*, A) = \langle q^*D^b(\mathbb{P}^1), (q^*D^b(\mathbb{P}^1) \otimes \mathcal{O}_q(1)) \cap q^*A^\perp \rangle
\]

For the intersection, we compute for \( B \in D^b(\mathbb{P}^1) \) similarly as in the proof above that

\[
\text{Hom}^\ast(q^*A, q^*B \otimes \mathcal{O}_q(1)) = \text{Hom}^\ast(A \oplus A \otimes \mathcal{O}(k), B).
\]

In particular for \( A = \mathcal{O}_{nP} \) we find that

\[
\text{Hom}^\ast(q^n\mathcal{O}_{nP}, q^nB \otimes \mathcal{O}_q(1)) = \text{Hom}^\ast(\mathcal{O}_{nP}, B)^\otimes 2 = 0
\]

if and only if \( B \in D^b_{\mathbb{P}^1 \setminus \{P\}}(\mathbb{P}^1) \) for support reasons. For \( A = \mathcal{O}(j) \) note that \( A \oplus A \otimes \mathcal{O}(k) \) is a weak generator of \( D^b(\mathbb{P}^1) \) if \( k \neq 0 \). Finally, in the case that \( k = 0 \), the right orthogonal of \( A \) inside \( D^b(\mathbb{P}^1) \) is generated by \( \mathcal{O}(j - 1) \).

To see the statement about the non-admissibility of \( \text{Fr}b(q^*, \mathcal{O}_{nP}) \), note that its (left or right) admissibility would be equivalent to the admissibility of \( D^b_{\mathbb{P}^1 \setminus \{P\}}(\mathbb{P}^1) \) inside \( D^b(\mathbb{P}^1) \). But the admissible subcategories of \( D^b(\mathbb{P}^1) \) are only 0, \( \langle \mathcal{O}(k) \rangle \) and \( D^b(\mathbb{P}^1) \).

Using the projection \( D^b(\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))) \to q^*D^b(\mathbb{P}^1) \otimes \mathcal{O}_q(1) \cong D^b(\mathbb{P}^1) \), we can embed the Frobenius poset \( \mathcal{P}(q^*) \) into the poset of thick subcategories of \( D^b(\mathbb{P}^1) \). By Lemma 2.3.13, under this embedding, the image of \( \mathcal{P}(q^*) \) consists of 0, \( \langle \mathcal{O}_{P1}(k) \rangle \), \( D^b(\mathbb{P}^1) \) and \( \langle \mathcal{O}_P \mid P \in V \rangle \) with \( V \) a cofinite subset of closed points in \( \mathbb{P}^1 \). Note that the completed \( \widehat{\mathcal{P}}(q^*) \), contains all subcategories \( \langle \mathcal{O}_P \mid P \in V \rangle \) with \( V \) an arbitrary subset of closed points. Hence we obtain the desired isomorphism of \( \widehat{\mathcal{P}}(q^*) \) and the poset of thick subcategories of \( D^b(\mathbb{P}^1) \). \( \square \)

Remark 2.4.4. The part about torsion sheaves in the previous proposition also works for \( \mathbb{P}^1 \)-bundles over arbitrary (smooth projective) curves.

We conclude this section with a qualitative statement about \( \mathbb{P}^2 \)-bundles.

Proposition 2.4.5. Let \( q : \mathbb{P}(\mathcal{E}) \to X \) be a \( \mathbb{P}^2 \)-bundle. Then the Frobenius codomain of \( q^* \) is neither left nor right admissible in \( D^b(\mathbb{P}(\mathcal{E})) \).

Proof. Recall that by [BvdB03] a left or right admissible subcategory in \( D^b(X) \) (with \( X \) smooth and projective) is automatically saturated, hence admissible. Therefore, it is sufficient to show that \( \text{Fr}b(q^*) \) is not admissible.
Assume the contrary, so there is a semiorthogonal decomposition $D^b(\mathbb{P}(\mathcal{E})) = \langle \text{Fr}(q^*), \text{Fr}(p^*) \rangle$. We can apply [Kuz11, Thm. 5.6] and get a semiorthogonal decomposition of a fibre, which turns out to be $D^b(\mathbb{P}^2) = \langle \text{Fr}(p^*), \text{Fr}(p^*) \rangle$

where $p : \mathbb{P}^2 \to \text{Spec}(k)$. Note that $\text{Fr}(p^*) = \langle 0 \rangle \oplus \frac{1}{2} 0 \cap \frac{1}{2} 0$. In particular, we conclude that $\frac{1}{2} 0 \cap \frac{1}{2} 0$ is admissible in $D^b(\mathbb{P}^2)$. But this contradicts [Bon13, §1.2].

**Proposition 2.4.6.** Let $q : \mathbb{P}(\mathcal{E}) \to X$ be a $\mathbb{P}^n$-bundle with $n \geq 2$. Then $\ker q \cap \ker q_*$ is non-zero.

**Proof.** We start with the relative Euler sequence:

$$0 \to \Omega_q \to q^* \mathcal{E}^\vee \otimes \mathcal{O}_q(-1) \to \mathcal{O}_q \to 0$$

taking its symmetric square and twisting by $\mathcal{O}_q(3)$ gives

$$0 \to \text{Sym}^2 \Omega_q(3) \to q^* (\text{Sym}^2 \mathcal{E}^\vee) \otimes \mathcal{O}_q(1) \to q^* \mathcal{E}^\vee \otimes \mathcal{O}_q(2) \to 0$$

From this it is obvious that $\text{Sym}^2 \Omega_q(3)$ lies in

$$\ker q_! = \langle q^* D^b(X) \otimes \mathcal{O}_q(1), \ldots, q^* D^b(X) \otimes \mathcal{O}_q(n) \rangle.$$ 

We claim that $\text{Sym}^2 \Omega_q(3)$ lies also in $\ker q_*$. We apply $q_*$ to the short exact sequence and get the triangle

$$q_* \text{Sym}^2 \Omega_q(3) \to \text{Sym}^2 \mathcal{E}^\vee \otimes_q \mathcal{O}_q(1) \xrightarrow{\varphi} \mathcal{E}^\vee \otimes_q \mathcal{O}_q(2)$$

using the projection formula. We claim that the map $\varphi$ is an isomorphism, and therefore $q_* \text{Sym}^2 \Omega_q(3) = 0$. First note that $R^i q_* \mathcal{O}_q(j) = 0$ for $i,j > 0$, so $\varphi$ is a morphism of vector bundles. Restricting to an arbitrary fibre $x \in X$, $\varphi \otimes k(x)$ becomes an isomorphism

$$\text{Sym}^2 \text{Hom}_{\mathbb{P}^n}(\mathcal{O}(1), \mathcal{O}(2)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \to \text{Hom}_{\mathbb{P}^n}(\mathcal{O}(1), \mathcal{O}(2)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(2)).$$

Hence $\varphi$ is an isomorphism of vector bundles (its kernel is a vector bundle of rank $\dim \ker (\varphi \otimes k(x)) = 0$; if its cokernel would be non-zero, we have $\text{coker}(\varphi \otimes k(x)) \neq 0$ for $x \in \text{Supp}(\text{coker}(\varphi))$, a contradiction to $\varphi \otimes k(x)$ being an isomorphism for all $x$). Therefore $\text{Sym}^2 \Omega_q(3) \in \ker q_! \cap \ker q_*$. □

**Remark 2.4.7.** We conjecture that for $q : \mathbb{P}^n \to \text{pt}$, the category $\ker q_! \cap \ker q_*$ is non-admissible in $D^b(\mathbb{P}^n)$ for all $n \geq 2$. Unfortunately, the result of [Bon13, §1.2] about non-admissibility of $\frac{1}{2} 0_{\mathbb{P}^2} \cap \frac{1}{2} 0_{\mathbb{P}^2}$ is based on tilting and the fact that $\text{End}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(n))$ is a hereditary algebra, which does not hold for $\text{End}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(n))$ as soon as $n > 2$. 
2.5. Example: blowups. Let \( \pi: \tilde{X} \to X \) be the blowup of a smooth projective variety \( X \) in a smooth closed subvariety \( Z \) of codimension \( c \geq 2 \), where the exceptional divisor \( E = \mathbb{P}(N_j) \) is the projectivisation of the rank \( c \) normal bundle \( N_j := N_{Z/X} \).

\[
E \xrightarrow{i} \tilde{X}
\]
\[
Z \xrightarrow{j} X.
\]

We will (sometimes tacitly) assume that \( \tilde{X} \) is again projective.

Recall that the canonical bundle of \( \tilde{X} \) is given by:
\[
\omega_{\tilde{X}} = \pi^*\omega_X \otimes \mathcal{O}_{\tilde{X}}((c - 1)E),
\]
and the restriction of the line bundle \( \mathcal{O}_{\tilde{X}}(E) \) is negative on the fibres of \( q \). That is, we have:
\[
\mathcal{O}_E(E) = i^*\mathcal{O}_{\tilde{X}}(E) = \mathcal{O}_q(-1).
\]

For all \( k \in \mathbb{Z} \), Orlov [Orl92, Ass. 4.2 & Thm. 4.3] shows that the functor:
\[
\Psi_k := i_*\left(q^*(\square) \otimes \mathcal{O}_q(k)\right): D^b(Z) \to D^b(\tilde{X})
\]
is fully faithful and we have a semiorthogonal decomposition:
\[
D^b(\tilde{X}) = \langle \pi^*D^b(X), \Psi_0(D^b(Z)), \Psi_1(D^b(Z)), \ldots, \Psi_{c-2}(D^b(Z)) \rangle.
\]

As in Section 2.4, we will not discuss the Frobenius codomains and neighbourhoods in general. We focus on cases where the center \( Z \) has low codimension.

**Proposition 2.5.1.** Let \( \pi: \tilde{X} \to X \) be the blowup in a smooth center \( Z \) of codimension 2. Then for \( A \in D^b(X) \), the Frobenius neighbourhood under \( \pi^* \) is
\[
\text{Frb}(\pi^*, A) = \langle \pi^*D^b(X), i_*q^*D^b(Z) \cap (\pi^*A)\perp \rangle
\]
In particular, we have two extremes:
\[
\text{Frb}(\pi^*, A) = \begin{cases} D^b(\tilde{X}) & \text{if and only if } j^*A = 0; \\ \pi^*D^b(X) & \text{if and only if } j^*A \text{ is a weak generator of } D^b(Z). \end{cases}
\]

Finally, the Frobenius codomain is \( \text{Frb}(\pi^*) = \pi^*D^b(X) \).

**Proof.** The Frobenius neighbourhood for a general \( A \) under \( \pi^* \) is of the stated shape by combining (3) with Theorem 2.3.5.

For general \( A \in D^b(X) \) and \( B \in D^b(Z) \) we compute
\[
\text{Hom}^*(\pi^*A, i_*q^*B) = \text{Hom}^*(A, \pi_*i_*q^*B) = \text{Hom}^*(A, j_*q_*q^*B) = \text{Hom}^*(j^*A, B) \tag{4}
\]
using adjunctions, fully faithfulness of \( q^* \) and \( \pi \circ i = j \circ q \).

If \( j^*A \) is a weak generator of \( D^b(Z) \), then the vanishing of (4) implies \( B = 0 \). So for such an \( A \), we get that \( i_*q^*D^b(Z) \cap \pi^*A\perp = 0 \) and hence \( \text{Frb}(\pi^*, A) = q^*D^b(X) \).
Whereas if \( j^*A = 0 \), then there is no restriction on \( B \in \mathcal{D}^b(Z) \) and we get \( \text{Frb}(\pi^*, A) = \mathcal{D}^b(\tilde{X}) \) in this case.

Note that if we choose a strong generator \( A \) of \( \mathcal{D}^b(X) \) as in Example 1.1.6 using a very ample line bundle, then \( j^*A \) will be a strong generator of \( \mathcal{D}^b(Z) \). So by Proposition 2.3.2, we obtain the statement about \( \text{Frb}(\pi^*) \).

**Example 2.5.2.** Let \( \pi: \tilde{X} \to X \) be the blowup in a point \( P \). Then the above proposition exhausts all possible cases and we find:

\[
\text{Frb}(\pi^*, A) = \begin{cases} 
\mathcal{D}^b(\tilde{X}) & \text{if } P \notin \text{Supp}(A); \\
\pi^*\mathcal{D}^b(X) & \text{if } P \in \text{Supp}(A).
\end{cases}
\]

Besides blowing up a point, also when blowing up a \( \mathbb{P}^1 \) on a threefold, we obtain a full description of the Frobenius poset.

**Proposition 2.5.3.** Let \( \pi: \tilde{X} \to X \) be the blowup of a threefold in a smooth rational curve \( C \). For \( A \in \mathcal{D}^b(X) \), its Frobenius neighbourhood \( \text{Frb}(\pi^*, A) \) is one of the following

\[
\begin{align*}
\mathcal{D}^b(\tilde{X}) &= \langle \pi^*\mathcal{D}^b(X), i_*q^*\mathcal{D}^b(C) \rangle & \text{if } j^*A = 0; \\
\langle \pi^*\mathcal{D}^b(X), i_*q^*\mathcal{D}^b_{\text{Supp}(j^*A)}(C) \rangle &= \bigoplus_i \mathcal{O}_n[\ell_i] & \text{if } j^*A \cong \bigoplus_i \mathcal{O}(\ell_i); \\
\langle \pi^*\mathcal{D}^b(X), i_*q^*\mathcal{O}_C(k - 1) \rangle &= \bigoplus_i \mathcal{O}(k)[\ell_i]; \\
\langle \pi^*\mathcal{D}^b(X) \rangle &= \bigoplus_i \mathcal{O}(k)[\ell_i].
\end{align*}
\]

Moreover, \( \mathcal{T}(\pi^*) \) is isomorphic to the poset of thick subcategories of \( \mathcal{D}^b(\mathbb{P}^1) \).

**Proof.** Proposition 2.5.1 yields already the minimal and maximal case. As \( C \cong \mathbb{P}^1 \), we recall that the indecomposable objects in \( \mathcal{D}^b(C) \) are line bundles \( \mathcal{O}_C(k) \) and fat (closed) points \( \mathcal{O}_nP \), see also the discussion before Proposition 2.4.3. Note that by [Orl09, Thm. 4], \( \mathcal{O}_C(k) \oplus \mathcal{O}_C(k') \) with \( k \neq k' \) is a weak generator of \( \mathcal{D}^b(C) \), and therefore also \( \mathcal{O}_C(k) \oplus \mathcal{O}_nP \). Therefore the non-minimal and non-maximal case only occurs for \( \pi^*A \) a direct sum of shifts of (several) skyscraper sheaves or of a single line bundle. Now using that \( \text{Frb}(\pi^*, A) = \langle \pi^*\mathcal{D}^b(X), i_*q^*\mathcal{D}^b(C) \cap (\pi^*A)\rangle \) and (4) gives the statement.

Almost all Frobenius neighbourhoods of the previous proposition do appear. The minimal and maximal element are no problem. To obtain the second one in the list, take \( A = \bigoplus \mathcal{O}_P \), for some collection of closed points in \( C \subset X \). Then \( j^*A \) is a direct sum of shifts of those skyscraper sheaves.

For the remaining case, let \( j^*A \) be the direct sum of shifts of a single line bundle \( \mathcal{O}_C(k) \). Note that \( \text{thick}(j^*A) = \langle \mathcal{O}_C(k) \rangle \) in this case, which leads to the claimed shape of the Frobenius neighbourhood. As \( j^* \) commutes with \( \otimes \) and \( \mathcal{F} \text{Kom} \) (and therefore \( j^*(A^\vee) = (j^*A)^\vee \)), there is a minimal positive integer \( k_0 \) such that we obtain all \( \mathcal{O}_C(mk_0) \) with \( m \in \mathbb{Z} \) by \( j^* \).
Using the projection $D^b(\tilde{X}) = (\pi^* D^b(X), i_* q^* D^b(C)) \rightarrow i_* q^* D^b(C) \cong D^b(\mathbb{P}^1)$, \(\bar{\mathcal{P}}(\pi^*)\) becomes in a natural way a subposet of all thick subcategories of $D^b(\mathbb{P}^1)$. It consists of the the thick subcategories 0, $\langle \mathcal{O}_P | P \in V \rangle$ where $m \in \mathbb{Z}$ and $V$ an arbitrary subset of closed points in $\mathbb{P}^1$. If $k_0 = 1$, then these two posets are isomorphic under this projection, but even for $k_0 > 1$, they are isomorphic as abstract posets.

**Remark 2.5.4.** In the proof of Proposition 2.5.3, it seems that we cannot expect that we can obtain all $\mathcal{O}_C(k)$ using pullbacks $j^* A$ with $A \in D^b(X)$. Therefore, $\mathcal{P}(\pi^*)$ encodes already the poset of thick subcategories of $D^b(C)$, and it might be that it also remembers something about the embedding $C \hookrightarrow X$.

**Example 2.5.5.** We consider the standard flip of $C_1 \cong \mathbb{P}^1$ inside a threefold $X$, see [Huy06, §11.3]:

\[
\begin{array}{c}
E \\
\downarrow q_1 \downarrow \pi_1 \\
\tilde{X} \\
\downarrow \pi_2 \\
X_1 \leftarrow j_1 C_1 \leftarrow j_2 X_2 \leftarrow C_2
\end{array}
\]

As $\tilde{X}$ is the blowup of $C_1 \hookrightarrow X_1$ and also of the flipped $C_2 \hookrightarrow X_2$, we have

\[
D^b(\tilde{X}) = (\pi_1^* D^b(X_1), i_* q_1^* D^b(C_1)) = (\pi_1^* D^b(X_1), \mathcal{O}_E(k, 0), \mathcal{O}_E(k + 1, 0))
\]

\[
= (\pi_2^* D^b(X_2), i_* q_2^* D^b(C_2)) = (\pi_2^* D^b(X_2), \mathcal{O}_E(0, l), \mathcal{O}_E(0, l + 1))
\]

where $k, l \in \mathbb{Z}$ arbitrary. Here we use the semiorthogonal decomposition coming from the blowup $\tilde{X} \rightarrow X$ and the standard exceptional sequence for $D^b(\mathbb{P}^1)$. Moreover, we can compare both $\mathcal{P}(\pi_1^*)$ and $\mathcal{P}(\pi_2^*)$, as they consist of thick subcategories of $D^b(\tilde{X})$. Using the list of Proposition 2.5.3, one can check that the only common element, besides $D^b(\tilde{X})$, is $0^E_E = (\pi_1^* D^b(X_1), \mathcal{O}_E(-1, 0)) = \text{Frb}(\pi_1^*, \mathcal{O}_{X_1})$

\[
= (\pi_2^* D^b(X_2), \mathcal{O}_E(0, -1)) = \text{Frb}(\pi_2^*, \mathcal{O}_{X_2}).
\]

Note that $0^E_E \in \mathcal{P}(\pi_1^*) \cap \mathcal{P}(\pi_2^*)$ is the minimal (geometric) category containing both $\pi_1^* D^b(X_1)$ and $\pi_2^* D^b(X_2)$. Hence we get that

\[
\mathcal{P}(\pi_1^*) \cap \mathcal{P}(\pi_2^*) = \{0^E_E \subset D^b(\tilde{X})\}.
\]

We conclude this section with a rough statement about the Frobenius codomain in case that the codimension of the center is bigger than 2.
**Proposition 2.5.6.** Let $\pi: \tilde{X} \to X$ be the blowup in a smooth center $Z$ of codimension $c > 2$. Then the Frobenius codomain of $\pi^*$ is $\text{Fr}(\pi^*) = \pi^* \text{D}^b(X) \oplus \ker \pi_s \cap \ker \pi_t$. Moreover, $\ker \pi_s \cap \ker \pi_t$ is non-zero.

**Proof.** The shape $\text{Fr}(\pi^*) = \pi^* \text{D}^b(X) \oplus \ker \pi_s \cap \ker \pi_t$ follows directly from Corollary 2.2.4. We claim that for $k = 1, \ldots, c - 2$ the objects $i_* \Omega^k_q(k)$ lie inside $\ker \pi_s \cap \ker \pi_t$.

First we have a closer look at $\Omega^k_q(k)$. Taking wedge powers of the relative Euler sequence:

$$0 \to \Omega_q \to q^* N^\vee \otimes \Omega_q(-1) \to \Omega_q \to 0,$$

and twisting by $\mathcal{O}_q(m)$, produces the short exact sequence:

$$0 \to \Omega^k_q(m) \to q^* \bigwedge^k N^\vee \otimes \Omega_q(m - k) \to \Omega^k_q(m) \to 0. \quad (5)$$

Now, pushing this forward along $q$, and using projection formula on the middle term, yields a triangle:

$$q_*(\Omega^k_q(m)) \to \bigwedge^k N^\vee \otimes q_*(\mathcal{O}_q(m - k)) \to q_*(\Omega^k_q(m - 1)).$$

In particular, for all $0 \leq m < k \leq c - 1$, which implies $1 - c \leq m - k < 0$, we have $q_*(\mathcal{O}_q(m - k)) = 0$ and so we see that

$$q_*(\Omega^k_q(m)) = 0 \quad \text{for all } 1 \leq m \leq k. \quad (6)$$

Indeed, if $k = c - 1$ then $\Omega^c_q(m) = \omega_q(m) = \mathcal{O}_q(m - c)$ and $q_*(\mathcal{O}_q(m - c)) = 0$ in the given range. The other cases follow by induction.

Next we apply $i_*$ to (5) which yields the triangle

$$i_* \Omega^k_q(m) \to i_* \left( q^* \bigwedge^k N^\vee \otimes \Omega_q(m - k) \right) \to i_* \Omega^k_q(m).$$

So by another induction, we conclude that

$$i_* \Omega^k_q(k) \in \ker \pi_t = \langle \Psi_0(\text{D}^b(Z)), \Psi_1(\text{D}^b(Z)), \ldots, \Psi_{c-2}(\text{D}^b(Z)) \rangle.$$ for $k = 1, \ldots, c - 2$, as $\Psi_k = i_*(q^*(_- \otimes \mathcal{O}_q(k)))$. Finally by (6) we find that

$$\pi_! i_* \Omega^k_q(k) = j_* q_! \Omega^k_q(k) = 0$$

so $i_* \Omega^k_q(k) \in \ker \pi_s$, as well. \qed

**Remark 2.5.7.** The objects $i_* \Omega^k_q(k)$ inside $\ker \pi_s \cap \ker \pi_t$ are not exceptional (one might be misled by the fact that in case of a projective bundle the $\Omega^k_q(k)$ form a full exceptional sequence).

Nevertheless, we conjecture that $i_* \Omega^k_q(k)$ with $k = 1, \ldots, c - 2$ generate $\ker \pi_s \cap \ker \pi_t$ and that $\ker \pi_s \cap \ker \pi_t$ is not admissible in $\text{D}^b(\tilde{X})$. 
2.6. **Example: linkage class.** Let $Y$ be a hypersurface of degree $n$ in $\mathbb{P} := \mathbb{P}^{2n-1}$ with $n \geq 3$, given by the inclusion $j: Y \hookrightarrow \mathbb{P}$. It is well-known that there is a semi-orthogonal decomposition:

$$\mathcal{D}^b(Y) = \langle A_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \ldots, \mathcal{O}_Y(n-1) \rangle,$$

see for example [Kuz04, §4] or [KMM10, Thm 2.13]. Moreover, $A_Y$ is a connected $(2n-4)$-Calabi-Yau category, i.e. the Serre functor $S_{A_Y}$ is just a shift by $2n-4$.

For more background in the case $n = 3$, see the article [Kuz10].

**Proposition 2.6.1** ([Huy06, Cor. 11.4], [KM09, §3], [KMM10, Rem. 5.2]). Let $B \in \mathcal{D}^b(Y)$ be an object. Then there is a morphism $e = e_B: B \to B \otimes \mathcal{O}_Y(-n)[2]$, called the linkage class of $B \in \mathcal{D}^b(Y)$, which fits functorially into the triangle

$$j^* j_* B \to B \xrightarrow{e_B} B \otimes \mathcal{O}_Y(-n)[2],$$

where the arrow $j^* j_* B \to B$ is the counit of adjunction.

Let now $i: A_Y \to \mathcal{D}^b(Y)$ be the inclusion coming from the semi-orthogonal decomposition. Note that for the exceptional functor $i$, the canonical triangle of Theorem 2.3.6 is

$$iS_{A_Y} \to S_Y iA \to TS_Y iA,$$

where $T$ denotes the twist functor associated to $i$. Using that $S_{A_Y} = [2n-4]$ and $S_Y = (-) \otimes \mathcal{O}_Y(-n)[2n-2]$, the triangle becomes (after shift and rotation):

$$T(iA \otimes \mathcal{O}_Y(-n))[1] \to iA \xrightarrow{w} iA \otimes \mathcal{O}_Y(-n)[2]. \quad (7)$$

**Proposition 2.6.2.** For $A \in A_Y$, the linkage class $e_{iA}$ and $w$ coincide. In particular, $j^* j_* iA \cong T(iA \otimes \mathcal{O}_Y(-n))[1]$ and so the Frobenius neighbourhood of $A$ in $\mathcal{D}^b(Y)$ is given by

$$\text{Frb}(i, A) = \perp (j^* j_* A).$$

**Proof.** By [KMM10, Prop. 5.8], $e_{iA}$ induces an isomorphism

$$\text{Hom}^*(iA', iA) \xrightarrow{\sim} \text{Hom}^*(iA', iA \otimes \mathcal{O}_Y(-n)[2])$$

which is functorial in $A' \in A_Y$ by [KM09, Prop. 3.1]. In particular for $A' = A$, we get that $\text{id}_{iA}$ is mapped to $e_{iA}$. As the same holds for $w$, they coincide.

The second part follows now directly from Theorem 2.3.6, noting that orthogonals are independent of shifts. \hfill $\square$

**Remark 2.6.3.** The linkage class is defined for all $B \in \mathcal{D}^b(Y)$. One can extend the definition of $w$ as in (7) to any $B \in \mathcal{D}^b(Y)$ by first projecting onto $A_Y$ using $i': \mathcal{D}^b(Y) \to A_Y$. 


**Question 2.6.4.** The linkage class exists in much greater generality, namely for any inclusion \( j: Y \rightarrow M \) as a locally complete intersection, see [KM09, §3]. Can the analogous triangle of Proposition 2.6.1 always be realised using some exceptional functor \( F: A_Y \rightarrow D^b(Y) \)?

By Proposition 2.6.2 and Corollary 2.2.4, \( A_Y \) has to be contained in \( \dim (j^*j_*) \).

### 3. Spherelike functors

#### 3.1. Definition and examples.

**Definition 3.1.1.** Let \( F: A \rightarrow B \) be a functor with both adjoints. If the cotwist \( C \) is an autoequivalence of \( A \) then we say that \( F \) is *spherelike*. If additionally, \( R \) and \( CL[1] \) are isomorphic, then we say that \( F \) is *spherical*.

Both conditions on a functor \( F \) to be spherical imply that \( R \) and \( L \) only differ by an autoequivalence. This property is also known as *quasi-Frobenius*. There is always a natural way to compare \( R \) and \( CL[1] \), namely by the canonical map

\[
\varphi := \gamma_R L \circ R \eta_L: R \rightarrow RFL \rightarrow CL[1]
\]

The dual version to \( \varphi \) is the canonical map

\[
\psi := \varepsilon_L R \circ L \beta_R: L [\dim -1] \rightarrow LFR \rightarrow R.
\]

**Proposition 3.1.2 ([Mea16, Prop. A.2]).** If \( F: A \rightarrow B \) is spherical, in particular there is some isomorphism \( R \simeq CL[1] \), then also the canonical map \( \varphi: R \rightarrow CL[1] \) is an isomorphism.

**Theorem 3.1.3 ([AL17, Thm. 1.1]).** Let \( F: A \rightarrow B \) be a functor with both adjoints. If \( F \) satisfies two of the following four conditions then \( F \) satisfies all four of them:

(i) the cotwist \( C \) is an autoequivalence of \( A \),

(ii) the canonical map \( \varphi: R \rightarrow CL[1] \) is an isomorphism,

(iii) the twist \( T \) is an autoequivalence of \( B \),

(iv) the canonical map \( \psi: LT[\dim -1] \rightarrow R \) is an isomorphism.

In particular, such an \( F \) is spherical.

The theorem above shows that one can define spherical functors in at least \( \binom{4}{2} \) different ways. However, we stick to the (classical) definition because in most applications, the spherical functor \( F: A \rightarrow B \) starts from a small source category with simple cotwist \( C \) and produces an interesting autoequivalence \( T \) of the target category.

**Theorem 3.1.4 ([Seg18, Thm. 2.10]).** Let \( T \) be an autoequivalence of \( B \). Then there is a category \( A \) and a spherical functor \( F: A \rightarrow B \) with twist \( T \).

**Example 3.1.5.** Let \( A \in A \) be an object. Then \( A \) is
• $d$-spherelike if $\text{Hom}^*(A, A) \cong k[t]/t^2$ with $\deg t = d$;
• $d$-Calabi-Yau if $A[d]$ is a Serre dual of $A$;
• $d$-spherical if $A$ is $d$-spherelike and $d$-Calabi-Yau.

If $A$ is spherelike, proper and admits an anti-Serre dual $S^{-1}A$ then the functor

$$F = F_A: D^b(k \text{-mod}) \to A, V^* \mapsto V^* \otimes A$$

is spherelike with adjoints $R = R_A = \text{Hom}^*(A, \_)$ and $L = L_A = \text{Hom}^*(S^{-1}A, \_)$.

To see this, by the triangle $C \to \text{id} \to RF$ we conclude that $C = [-d-1]$ is an autoequivalence. With this, one can check that an isomorphism $R \cong \text{CL}[1]$ translates into a $d$-Calabi-Yau property of $A$, in which case $A$ is spherical.

### 3.2. Spherical codomains.

Recall that if $F: A \to B$ is a functor with both adjoints then we have canonical maps $\varphi: R \to \text{CL}[1]$ and $\psi: LT[-1] \to R$. Using $\varphi$, we can measure the difference between $R$ and $\text{CL}[1]$ with the triangle:

$$Q \to R \xrightarrow{\varphi} CL[1].$$

and dually there is the triangle involving $\psi$:

$$LT[-1] \xrightarrow{\psi} R \to Q'.$$

**Definition 3.2.1.** If $F$ is spherelike then we call $\text{Sph}(F) := \ker Q$ the spherical codomain of $F$ and $F|_{\text{Sph}(F)}$ the spherical corestriction of $F$.

**Remark 3.2.2.** In particular, a spherelike functor $F$ is spherical if and only if $Q \simeq 0$, which is equivalent to $\ker Q = B$.

**Theorem 3.2.3.** Let $F: A \to B$ be a spherelike functor. Then $\text{im } F \subset \text{Sph}(F)$ and the corestriction $F|_{\text{Sph}(F)}: A \to \text{Sph}(F)$ is spherical. Furthermore, if $C$ is a full subcategory of $B$ such that $\text{im } F \subset C$ and the corestriction $F|_{C}: A \to C$ is spherical then $C \subset \text{Sph}(F)$. That is, $\text{Sph}(F)$ is the maximal full subcategory on which $F$ becomes spherical.

Note that in particular, as $F|_{\text{Sph}(F)}$ is spherical, its twist is an autoequivalence of $\text{Sph}(F)$.

**Proof.** First, we show that $\text{im } F \subset \text{Sph}(F)$. Precompose (8) with $F$ to get the triangle:

$$QF \to RF \xrightarrow{\varphi F} CLF[1].$$

Now, [Mea16, Lemma A.1] shows that the second map is an isomorphism which is equivalent to $QF \simeq 0$. Therefore, $\text{im } F \subset \ker Q =: \text{Sph}(F)$ and $F: A \to B$ naturally corestricts to a functor $F_1 := F|_{\text{Sph}(F)}: A \to \text{Sph}(F)$.

Next we show that $F_1$ is spherelike, that is, the cotwist $C_1$ is an autoequivalence. If $F_2: \ker Q \to B$ denotes the inclusion then we have a natural isomorphism of
functors $F \simeq F_2 F_1$ and the right adjoint of $F_1$ is given by $R_1 \simeq R F_2$. That is, we have natural isomorphisms $RF \simeq R F_2 F_1 \simeq R_1 F_1$ and the composition $RF \simeq R_1 F_1$ is compatible with both unit morphisms. Indeed, because $F_2 : \ker Q \to B$ is fully faithful, we have the following commutative diagram:

$$
\begin{array}{ccc}
\Hom(F, F) & \longrightarrow & \Hom(id_A, RF) \\
\downarrow & & \downarrow \\
\Hom(F_2 F_1, F_2 F_1) & \longrightarrow & \Hom(id_A, R_1 F_1) \\
\downarrow & & \downarrow \\
\Hom(F_1, F_1) & \longrightarrow & \Hom(id_A, R_1 F_1) \\
\downarrow & & \downarrow \\
C & \longrightarrow & \id_A \eta \longrightarrow \RF \\
\downarrow & & \downarrow \\
C_1 & \longrightarrow & \id_A \eta \longrightarrow \RF_1 F_1 \\
\end{array}
$$

Therefore, we have a commutative diagram of triangles:

$$
\begin{array}{ccc}
\Hom(F, F) & \longrightarrow & \Hom(id_A, RF) \\
\downarrow & & \downarrow \\
\Hom(F_2 F_1, F_2 F_1) & \longrightarrow & \Hom(id_A, R_1 F_1) \\
\downarrow & & \downarrow \\
\Hom(F_1, F_1) & \longrightarrow & \Hom(id_A, R_1 F_1) \\
\downarrow & & \downarrow \\
C & \longrightarrow & \id_A \eta \longrightarrow \RF \\
\downarrow & & \downarrow \\
C_1 & \longrightarrow & \id_A \eta \longrightarrow \RF_1 F_1 \\
\end{array}
$$

Since the second and third vertical maps are isomorphisms, we can conclude that the first vertical map is also an isomorphism. The cotwist of $F$ is an autoequivalence by assumption and so it follows that the cotwist of $F_1$ is an autoequivalence as well.

It remains to show that the canonical map $\varphi_1 : R_1 \to C_1 L_1[1]$ is an isomorphism. This also follows from the compatibility of units. Indeed, the same argument as above shows that we have natural isomorphisms $R_1 F_1 L_1 \simeq RF_2 F_1 L_2 \simeq RFLF_2$ which are compatible with the units:

$$
\begin{array}{ccc}
R_1 & \longrightarrow & R_1 F_1 L_1 \longrightarrow C_1 L_1[1] \\
\downarrow & & \downarrow \\
RF_2 & \longrightarrow & RFLF_2 \longrightarrow CLFF_2[1]. \\
\end{array}
$$

In particular, since $F_2 : \ker Q \to B$ is faithful, we see that $\varphi_1 : R_1 \to C_1 L_1[1]$ coincides with $\varphi : R \to CL[1]$ on the subcategory $\ker Q$, that is, $\varphi_1 = \varphi F_2$. Moreover, since $\varphi$ is an isomorphism on $\ker Q$ it follows that $\varphi_1$ is as well.

For maximality, we let $F_1 := F|_\mathcal{C} : A \to \mathcal{C}$ be a corestriction of $F$. If $\widetilde{F}_2 : \mathcal{C} \to B$ denotes the fully faithful embedding then a similar argument as above shows that we have $\tilde{\varphi}_1 = \varphi \widetilde{F}_2$. Moreover, if $F_1$ is spherical then $\tilde{\varphi}_1(B) = \varphi(\tilde{\mathcal{F}}_2(B))$ is an isomorphism for all $B \in \mathcal{C}$ which is equivalent to $Q(\tilde{\mathcal{F}}_2(B)) = 0$. Therefore, we see that $\mathcal{C} \subset \ker Q := \text{Sph}(F)$.

Remark 3.2.4. Instead of using the triangle $Q \to R \to \CL[1]$, we could have started this section also with the triangle $\LT[-1] \to R \to Q'$. By the same line of arguments as in Theorem 3.2.3, we arrive at the statement that the corestriction $F|_{\ker Q'}$ is...
spherical, since \( \psi: \text{LT}[-1] \to \text{R} \) becomes an isomorphism on \( \ker Q' \). Moreover, \( \ker Q' \) is maximal with this property. By Theorem 3.1.3, we also have an isomorphism \( \phi: \text{R} \to \text{CL}[1] \), so by the maximality property of both kernels we arrive at \( \ker Q' = \text{Sph}(F) = \ker Q \).

### 3.3. Spherical neighbourhoods

In close analogy to Section 2.3, we can also look at spherical neighbourhoods of objects under spherelike functors.

**Definition 3.3.1.** Let \( F: \mathcal{A} \to \mathcal{B} \) be a spherelike functor and \( A \in \mathcal{A} \). The **spherical neighbourhood** of \( A \) under \( F \) is

\[
\text{Sph}(F, A) := \{ B \in \mathcal{B} \mid \text{Hom}^\ast(A, QB) = 0 \}.
\]

**Remark 3.3.2.** To avoid confusion, we stress that in general \( FA \) will not be a spherical object inside its spherical neighbourhood \( \text{Frb}(F, A) \). In order that \( FA \) can be a spherical object inside \( \text{Frb}(F, A) \) it is necessary that \( FA \) is a spherelike object in \( \mathcal{B} \).

**Remark 3.3.3.** The spherical codomain of \( F \) is again the intersection of the spherical neighbourhoods of the objects in \( \mathcal{A} \) by Yoneda:

\[
\text{Sph}(F) = \ker Q = \bigcap_{A \in \mathcal{A}} \text{Sph}(F, A).
\]

If \( A \in \mathcal{A} \) is a weak generator, then we also find that \( \text{Sph}(F) = \text{Sph}(F, A) \). To see this note that \( B \in \text{Sph}(F, A) \) if \( \text{Hom}^\ast(A, QB) = 0 \), which in turn implies that \( QB = 0 \) as \( A \) is a weak generator, hence \( B \in \ker Q = \text{Sph}(F) \). Here we only use that \( A \) is a weak generator for \( \text{im} Q \).

**Proposition 3.3.4.** If \( F: \mathcal{A} \to \mathcal{B} \) is a spherelike functor and \( A \in \mathcal{A} \) then \( \text{Sph}(F, A) \) is the maximal full subcategory of \( \mathcal{B} \) such that

\[
\text{Hom}^\ast(A, \text{R}|_{\text{Sph}(F, A)}(\underline{\_})) \simeq \text{Hom}^\ast(A, \text{CL}|_{\text{Sph}(F, A)}(\underline{\_}))[\ast - 1].
\]

**Proof.** The proof of this statement is very similar to Proposition 2.3.3. Indeed, the triangle to use is:

\[
\text{Hom}^\ast(A, Q(\underline{\_})) \to \text{Hom}^\ast(A, \text{R}(\underline{\_})) \to \text{Hom}^\ast(A, \text{CL}(\underline{\_}))[-\ast].
\]

3.3.1. **In presence of Serre functors.** We specialise to the case that \( \mathcal{A} \) and \( \mathcal{B} \) admit Serre functors.

**Theorem 3.3.5.** Let \( F: \mathcal{A} \to \mathcal{B} \) be a spherelike functor. If \( \mathcal{A} \) and \( \mathcal{B} \) admit Serre functors, then there is a natural triangle

\[
\text{FS}_A C^{-1}[\ast - 1] \to S_B F \to Q^* S_A
\]

where \( Q^* \) is the right adjoint of \( Q \). In particular, we obtain \( \text{Sph}(F, A) = Q^* S_A A \) for \( A \in \mathcal{A} \) and that \( \text{FS}_A C^{-1} A[\ast - 1] \) is a Serre dual for \( FA \) inside \( \text{Sph}(F, A) \).
Proof. Taking right adjoints of $Q \to R \to \text{CL}[1]$ gives $\text{FC}^{-1}[-1] \to R^r \to Q^r$. Here we use that $R^r = S_B FS_A^{-1}$, in particular $Q^r$ also exists. We continue our calculation
\[ Q \to R \to \text{CL}[1] \iff \text{FC}^{-1}[-1] \to S_B FS_A^{-1} \to Q^r \quad \text{(taking right adjoints)} \]
\[ \iff FS_A C^{-1}[-1] \to S_B F \to Q^r S_A \quad \text{(precomposing with $S_A$)} \]
In the last step we used that Serre functors commute with autoequivalences. For $A \in \mathcal{A}$ we have
\[
\text{Hom}^*(A, Q(\_)) = \text{Hom}^*(Q(\_), S_A(A)) = \text{Hom}^*(\_, Q^r S_A(A))^\vee,
\]
so $\text{Sph}(F, A) = \ker \text{Hom}^*(A, Q(\_)) = \perp Q^r S_A(A)$. With the same reasoning as in Remark 2.3.7 we complete the proof. \qed

Remark 3.3.6. Let $F : \mathcal{A} \to \mathcal{B}$ be a spherelike functor. Note that if $S_A C^{-1} A = A[d]$ for some $d$, then $\text{Sph}(F, A)$ is the maximal full subcategory of $\mathcal{B}$ where $FA$ is $d$-Calabi-Yau. In such a case, we call $\text{Sph}(F, A)$ the Calabi-Yau neighbourhood of $FA$ in $\mathcal{B}$.

3.3.2. Dual spherical neighbourhoods. If we use the triangle $Q' \to \text{LT}[-1] \to R$ instead then we arrive at the following statements:
\begin{enumerate}
  \item[(i)] $\text{Sph}^v(F, A) = \{ B \in \mathcal{B} : \text{Hom}^v(Q', A)^\vee = 0 \}$,
  \item[(ii)] $\text{Hom}^v(\text{LT}_{\text{Sph}^v(F, A)}, A)^\vee[-1] \xrightarrow{\sim} \text{Hom}^v(\text{R}_{\text{Sph}^v(F, A)}, A)^\vee$,
  \item[(iii)] $\text{Sph}^v(F, A) = (Q^d S_A^{-1} A)\perp$ and the anti-Serre dual of $FA$ is $FS_A^{-1} CA[1]$.
\end{enumerate}

3.4. They go together. Most of our examples will be a composition of a spherical functor with an exceptional one.

Proposition 3.4.1. Suppose $F_1 : \mathcal{A} \to \mathcal{B}$ and $F_2 : \mathcal{B} \to \mathcal{C}$ are functors with both adjoints $L_1, R_1$ and $L_2, R_2$, as usual, and let $T_i$ and $C_i$ be the twist and cotwist associated to $F_i$ for $i = 1, 2$. If we consider the composition $F = F_2 \circ F_1 : \mathcal{A} \to \mathcal{C}$ together with its twist $T$ and cotwist $C$ then we have the following triangles:
\[
C_1 \to C \to R_1 C_2 F_1 \quad \text{and} \quad F_2 T_1 R_2 \to T \to T_2.
\]
In particular, if $F_2$ is exceptional, then there is an isomorphism $C_1 \simeq C$. So in this case, if $F_1$ is exceptional or spherelike then also $F$ is exceptional or spherelike, respectively.
Proof. Naturality of units and counits together with the octahedral axiom provides us with the following commutative diagrams of triangles:

\[
\begin{array}{ccc}
C_1 & \rightarrow & C_1 C_2 F_1 \\
\downarrow & & \downarrow \\
C_1 & \rightarrow & C_1 F_1 \\
\downarrow & & \downarrow \\
RF & \rightarrow & R_1 R_2 F_2 F_1 \\
\end{array}
\quad
\begin{array}{ccc}
C & \rightarrow & R_1 C_2 F_1 \\
\downarrow & & \downarrow \\
C & \rightarrow & R_1 F_1 \\
\downarrow & & \downarrow \\
FR & \rightarrow & T \\
\end{array}
\quad
\begin{array}{ccc}
F_1 & \rightarrow & F_1 R_2 \\
\downarrow & & \downarrow \\
F & \rightarrow & T \\
\end{array}
\quad
\begin{array}{ccc}
F_2 & \rightarrow & F_2 T R_2 \\
\downarrow & & \downarrow \\
F & \rightarrow & T \\
\end{array}
\]

Now observe that if \( F_2 \) is exceptional then \( C_2 = 0 \), and hence \( C \simeq C_1 \). \( \square \)

**Proposition 3.4.2.** Let \( F_1 : A \rightarrow B \) be a functor with both adjoints and \( F_2 : B \rightarrow C \) be an exceptional functor. Then there is the triangle

\[
R_1 P_2 \rightarrow Q \rightarrow Q_1 L_2.
\]

In particular, we get \( Q F_2 \simeq Q_1 \) and \( R_1 P_2 \simeq QT'_2 \), and consequently \( F_2(\ker Q_1) \subset \ker Q \).

**Proof.** We start with the following diagram of triangles, which compares \( Q = Q_F \) and \( Q_1 = Q_{F_1} \):

\[
\begin{array}{ccc}
Q & \rightarrow & R_1 R_2 = R \\
\downarrow & (\ast) & \downarrow c L[1] \\
Q_1 L_2 & \rightarrow & R_1 L_2 \\
\downarrow & & \downarrow \\
& & C_1 L_1 L_2[1]
\end{array}
\]

where \( c : C \rightarrow C_1 \) is the isomorphism of **Proposition 3.4.1** as \( F_2 \) is exceptional. We focus on the square (\( \ast \)), which we expand a bit:

\[
\begin{array}{ccc}
R_1 R_2 & \rightarrow & R_1 R_2 F_2 F_1 L_1 L_2 \\
\downarrow & \downarrow & \downarrow \\
R_1 L_2 & \rightarrow & R_1 F_1 L_1 L_2 \\
\downarrow & \downarrow & \downarrow \\
& & C_1 L_1 L_2[1]
\end{array}
\]

Here the left diagram commutes as it is the composition of adjoints, see [Mac71, Thm. IV.8.1]. The commutativity of the right diagram follows from the octahedron axiom as in the left diagram in the proof of **Proposition 3.4.1**. This shows that the square (\( \ast \)) commutes, so with another application of the octahedron axiom we
arrive at

\[
\begin{array}{ccc}
R_1P_2 & \rightarrow & R_1R_2T'_2 \\
\downarrow & & \downarrow \\
Q & \rightarrow & R_1R_2 = R \rightarrow CL[1] \\
\downarrow & & \downarrow \\
Q_1L_2 & \rightarrow & R_1L_2 \rightarrow C_1L_1L_2[1]
\end{array}
\]

Now precompose the obtained triangle with \(F_2\):

\[
R_1P_2F_2 \rightarrow QF_2 \rightarrow Q_1L_2F_2.
\]

As \(R_1P_2F_2 \simeq RT'_2F_2 \simeq RF_2C'_2[-2] = 0\), we get therefore \(QF_2 \simeq Q_1L_2F_2 \simeq Q_1\) as \(F_2\) is exceptional. Similarly precomposing with \(T'_2\) yields the triangle:

\[
R_1P_2T'_2 \rightarrow QT'_2 \rightarrow Q_1L_2T'_2
\]

As \(Q_1L_2T'_2 \simeq Q_1C'_2L_2[-2] = 0\), we hence get \(QT'_2 \simeq R_1P_2T'_2 \simeq RT'^2_2 \simeq RT'_2\).

Finally note that \(F_2(\ker Q_1) = \{F_2B \mid Q_1(B) = 0\}\), hence for such an \(F_2B\) holds \(QF_2B = Q_1B = 0\), as well. \(\square\)

**Theorem 3.4.3.** Let \(F_1: A \rightarrow B\) be a spherical functor and \(F_2: B \rightarrow C\) be an exceptional functor. Then the spherical codomain of the spherelike functor \(F = F_2F_1\) has the semiorthogonal decomposition

\[
\text{Sph}(F) = \langle \text{im } F_2, \ker L_2 \cap \ker R \rangle
\]

and for \(A\) in \(A\) its spherical neighbourhood is

\[
\text{Sph}(F, A) = \langle \text{im } F_2, \ker L_2 \cap FA^\perp \rangle = \text{Frb}(F_2, F_1A)
\]

**Proof.** By assumption \(F_1\) is spherical, so \(Q_1 = 0\). Therefore the triangle of Proposition 3.4.2 becomes an isomorphism \(R_1P_2 \sim Q\). In particular, we get

\[
\text{Sph}(F) = \ker Q = \ker R_1P_2
\]
and unraveling this with Yoneda and using Theorem 2.3.5
\[
\ker R_1 P_2 = \{ C \in \mathcal{C} \mid \forall A \in \mathcal{A} : \operatorname{Hom}^*(A, R_1 P_2 C) = 0 \}
\]
\[
= \{ C \in \mathcal{C} \mid \forall A \in \mathcal{A} : \operatorname{Hom}^*(F_1 A, P_2 C) = 0 \}
\]
\[
= \bigcap_{A \in \mathcal{A}} \{ C \in \mathcal{C} \mid \operatorname{Hom}^*(F_1 A, P_2 C) = 0 \}
\]
\[
= \bigcap_{B \in \operatorname{im} F_1} \{ C \in \mathcal{C} \mid \operatorname{Hom}^*(F_1 A, P_2 C) = 0 \}
\]
\[
= \bigcap_{B \in \operatorname{im} F_1} \operatorname{Fr}(F_2, B)
\]
\[
= \bigcap_{B \in \operatorname{im} F_1} \langle \operatorname{im} F_2, \ker L_2 \cap F_2 B \rangle
\]
\[
= \langle \operatorname{im} F_2, \ker L_2 \cap \bigcap_{B \in \operatorname{im} F_1} F_2 B \rangle
\]
\[
= \langle \operatorname{im} F_2, \ker L_2 \cap \operatorname{im} F \rangle
\]
\[
= \langle \operatorname{im} F_2, \ker L_2 \cap \ker R \rangle.
\]
Implicit in this chain of equalities we have
\[
\operatorname{Sph}(F, A) = \operatorname{Fr}(F_2, F_1 A) = \langle \operatorname{im} F_2, \ker L_2 \cap F A \rangle. \quad \Box
\]

Remark 3.4.4. Similar to the case of exceptional functors, we can also define the spherical poset \( \Omega(F) \) of a spherelike functor \( F \):
\[
\Omega(F) := \{ \operatorname{Sph}(F, A) \mid A \in \mathcal{A} \}.
\]
ordered by inclusion.

The proposition above shows that if \( F = F_2 F_1 \) with \( F_1 \) spherical and \( F_2 \) exceptional, then we have an inclusion of posets:
\[
\Omega(F) \subseteq \mathcal{P}(F_2).
\]

Example 3.4.5. Let \( \mathcal{C} = \langle \mathcal{B}, \mathcal{B}^\perp \rangle \) be a semiorthogonal decomposition and let \( T_1 : \mathcal{B} \rightarrow \mathcal{B} \) be an autoequivalence. Then by [Seg18], there is a spherical functor \( F_1 : \mathcal{A} \rightarrow \mathcal{B} \) with \( T_1 \) as its associated twist.

By Proposition 3.4.1, the composition \( F : \mathcal{A} \rightarrow \mathcal{C} \) of \( F_1 \) with \( F_2 : \mathcal{B} \rightarrow \mathcal{C} \) is a spherelike functor, whose twist \( T \) restricts to \( T_1 \) on \( \mathcal{B} \) and the identity on \( \mathcal{B}^\perp \).

3.5. Comparison to spherical subcategories. This article generalises results from [HKP16, HKP19, HP20] about spherical subcategories. In this section we show how these results fit into the language of exceptional and spherelike functors.

We first recall the central notions and results from [HKP16]. To simplify some arguments, we will assume that \( \mathcal{D} \) has a Serre functor. Given a \( d \)-spherelike object
A in a triangulated category $\mathcal{D}$, then there is a canonical map $A \to S_\mathcal{D} A[-d]$ which can be completed to the \textit{asphericity triangle}
$$A \to S_\mathcal{D} A[-d] \to Q_A.$$ \hfill (9)

The \textit{spherical subcategory} of $A$ in $\mathcal{D}$ is then
$$\mathcal{D}_A := \perp Q_A$$
and the main result is the following.

\textbf{Proposition 3.5.1} ([HKP16, Thm. 4.4 & 4.6]). The spherical subcategory $\mathcal{D}_A$ is the maximal full triangulated subcategory of $\mathcal{D}$ where $A$ is $d$-spherical.

To translate this result, note that a $d$-spherelike $A$ defines the spherelike functor $F_A : D^b(k\text{-mod}) \to \mathcal{D}$, see Example 3.1.5.

\textbf{Proposition 3.5.2.} The spherical subcategory $\mathcal{D}_A$ of $A$ and the spherical codomain $\text{Sph}(F_A)$ of $F_A$ coincide.

\textbf{Proof.} We set $\mathcal{A} := D^b(k\text{-mod})$ and $F := F_A : \mathcal{A} \to \mathcal{D}$. This proposition follows already from maximality, see [HKP16, Thm. 4.6] and Theorem 3.2.3. We show here a bit more, namely that the triangle
$$FS_A C^{-1}[−1] \to S_\mathcal{D} F \to Q^* S_A$$
of Theorem 3.3.5 is essentially the asphericity triangle (9). Note that $S_A = \text{id}_A$ and $C = [−d − 1]$, so the triangle simplifies to
$$F[d] \to S_\mathcal{D} F \to Q^*.$$ Now applying this triangle to the (strong) generator $k \in \mathcal{A}$, we get after shifting with $[−d]$: $$A \to S_\mathcal{D} A[−d] \to Q^* A[−d]$$since $Fk = A$. In particular, we conclude that $Q_A \cong Q^* A[−d]$. Hence we get that
$$\mathcal{D}_A = \perp Q_A = \perp Q^* A = \bigcap_{V \in \mathcal{A}} \perp Q^* FV = \bigcap_{V \in \mathcal{A}} \text{Sph}(F, V) = \text{Sph}(F). \quad \square$$

The next proposition is about comparing [HKP16, Thm. 4.7] and Theorem 3.4.3.

\textbf{Proposition 3.5.3.} Let $A \in \mathcal{C}$ be a spherical object, and $\iota : \mathcal{C} \to \mathcal{D}$ be an exceptional functor. Then
$$\mathcal{D}_{\iota A} = (\perp \mathcal{C}) \cap \perp \iota A, \iota \mathcal{C}) = \text{Fr}(\iota, A).$$

\textbf{Proof.} The first equality is just the statement of [HKP16, Thm. 4.7]. By Proposition 3.5.2, we obtain that $\mathcal{D}_{\iota A} = \text{Sph}(\iota F_A)$ where $F_A : D^b(k\text{-mod}) \to \mathcal{C}, k \mapsto A$. As $D^b(k\text{-mod})$ are just graded vector spaces, we get
$$\text{Sph}(\iota F_A) = \text{Sph}(\iota F_A, k) = \text{Fr}(\iota, F_A k) = \text{Fr}(\iota, A)$$
where we use Theorem 3.4.3 in the middle.

**Remark 3.5.4.** Most examples of spherelike objects in [HKP16, HKP19] are of the shape: spherical object $A \in \mathcal{C}$ embedded by an exceptional functor $\iota: \mathcal{C} \to \mathcal{D}$. So the spherical subcategory of $\iota A$ in $\mathcal{D}$ is actually the Frobenius neighbourhood of $A$ under $\iota$. In particular, the spherical subcategory of $\iota A$ becomes part of the Frobenius poset of $\iota$, which sometimes has a richer structure.

### 3.5.1 Geometric examples.

**Example 3.5.5 ([HKP16, §5.3]).** Let $\pi: X \to C$ be a ruled surface, where $C$ is a smooth, projective curve. There is a section $C_0 \subset X$, which allows to write us $X = \mathbb{P}_C(V)$ with $V := \pi_* O_X(C_0)$. Then for a spherical object $S \in D^b(C)$ we obtain

$$ D_{\pi^*S} = (\pi^* (\perp (S \otimes V))) \otimes O_X(C_0), \pi^* D^b(C). $$

In particular for the spherical $S = O_P$ with $P \in C$ a point, we get

$$ D_{\pi^* O_P} = (\pi^* D_U(C) \otimes O_X(C_0), \pi^* D^b(C)) $$

where $D^b_U(C)$ is the subcategory of objects of $D^b(C)$ supported on $U = C \setminus \{P\}$.

Since $D_{\pi^*S} = Frb(\pi^*, S)$ by Proposition 3.5.3, there is no need to restrict only to spherelike objects. Hence this example becomes a special case of the Frobenius neighbourhoods calculated in Section 2.4, see Proposition 2.4.3 and Remark 2.4.4 there.

**Example 3.5.6 ([HKP16, §5.2]).** Let $\pi: \tilde{X} \to X$ be the blowup of a smooth projective variety in a point $P$. [HKP16, Prop. 5.2] states that if $S \in D^b(X)$ is spherical with $P \in \text{Supp}(S)$ then $D^b(X)_{\pi^*S} = \pi^* D^b(X)$.

In light of the calculation in Section 2.5, this turns out to be wrong as soon as $\dim(X) > 2$: in this case,

$$ D^b(X)_{\pi^*S} = \text{Frb}(\pi^*, S) \supset \text{Frb}(\pi^*) = \pi^* D^b(X) \oplus \ker \pi_* \cap \ker \pi! $$

where $\ker \pi_* \cap \ker \pi!$ is non-zero for $\dim(X) > 2$, see Proposition 2.5.6. In the proof of [HKP16, Prop. 5.2], it was shown that the $O_E(-k)$ do not lie inside $D^b(X)_{\pi^*S}$ for $k = 1, \ldots, \text{codim}_X(Z) - 1$, where $E$ is the exceptional divisor. But this does not imply that the subcategory generated by these objects has non-zero intersection with $D^b(X)_{\pi^*S}$. Only in the case of a single exceptional object (that is, if $X$ is a surface) such a conclusion is true. For higher dimensional $X$, the proof of Proposition 2.5.6 shows that $i_* \Omega^k(k) \in D^b(X)_{\pi^*S}$ for $k = 1, \ldots, \text{codim}_X(Z) - 2$. Therefore, [HKP16, Prop. 5.2] is only valid for blowing up a point on a surface.

Unfortunately, the mistake in the proof has consequences for [HKP16, Cor. 5.3 & Prop. 5.5] about iterated blowups. It turns out that the statements there are
even wrong for iterated blowups on surfaces, the reason is again that the orthogonal of $\pi^*\mathcal{D}(X)$ is generated by more than one object. This problem appears already when blowing up twice, see [HP20, Prop. 5.5]. Again, even though the proposition there is about the pullback of a spherical object, it can be easily generalised to the following statement about Frobenius neighbourhoods.

**Example 3.5.7** (c.f. [HP20, Prop. 5.5]). Let $X$ be a smooth projective surface. Let $\pi: \tilde{X} \to X$ be the composition of a blowup in a point $P$ and a second blowup in a point on the exceptional divisor of the first blowup. Then the exceptional locus of $\pi$ consists of a $(-2)$-curve $C$ and a $(-1)$-curve $E$ which meet transversally in a point. For $A \in \mathcal{D}(X)$, the Frobenius neighbourhood under $\pi^*$ is then given by

$$
\text{Frb}(\pi^*, A) = \begin{cases} 
\mathcal{D}(\tilde{X}) & \text{if } P \not\in \text{Supp}(A); \\
\pi^*\mathcal{D}(X) \oplus \langle \mathcal{O}_C(-1) \rangle & \text{if } P \in \text{Supp}(A).
\end{cases}
$$

Note that $\langle \mathcal{O}_C(-1) \rangle \subset \mathcal{D}(\tilde{X})$ is not admissible, as $\mathcal{O}_C(-1)$ is spherical.

**Remark 3.5.8.** In [KPS18, §5.5], Calabi–Yau neighbourhoods are introduced as a generalisation of spherical subcategories. We believe that with a suitable exceptional functor, they can be written as Frobenius neighbourhoods. In particular, the Calabi–Yau property there does not seem necessary. For example, we think that in [KPS18, Prop. 5.15], $Y$ can be any projective variety with rational Gorenstein singularities and there is no need for a trivial canonical bundle.

3.5.2. **Algebraic examples.** In [HKP19], some examples from representation theory of finite dimensional algebras are treated. There, two constructions are presented – *insertion* and *tacking* – which attaches to an algebra $\Lambda$ a quiver $\Gamma$ without oriented cycles, yielding a new algebra $\Lambda'$ and an exceptional functor

$$
\j: \mathcal{D}(\Lambda-\text{mod}) \to \mathcal{D}(\Lambda'-\text{mod}).
$$

As in the geometric examples, the spherical subcategory of $\j A$ is computed in $\mathcal{D}(\Lambda'-\text{mod})$, where $A$ is spherical in $\mathcal{D}(\Lambda-\text{mod})$. Since the spherical subcategory is actually a Frobenius neighbourhood, we can consider arbitrary objects $A$.

**Example 3.5.9** (c.f. [HKP19, Thm. 3.12 & 3.18]). Let $\Lambda'$ be an algebra which is obtained from $\Lambda$ by tacking on or insertion of a quiver $\Gamma$ without oriented loops, yielding a new algebra $\Lambda'$ and an exceptional functor

$$
\j: \mathcal{D}(\Lambda-\text{mod}) \to \mathcal{D}(\Lambda'-\text{mod}).
$$

Then there is a simple module $S$ in $\Lambda'-\text{mod}$ such that the Frobenius neighbourhood of $A \in \mathcal{D}(\Lambda-\text{mod})$ under $\j$ is

$$
\text{Frb}(F, A) = \begin{cases} 
\mathcal{D}(\Lambda'-\text{mod}) & \text{if } \text{Hom}^*(S, \j A) = 0; \\
\j \mathcal{D}(\Lambda-\text{mod}) \oplus \mathcal{C} & \text{if } \text{Hom}^*(S, \j A) = 0,
\end{cases}
$$

where $\mathcal{C} \cong \mathcal{D}(k\Gamma'-\text{mod})$ and $\Gamma' \subset \Gamma$ is a subquiver where a single vertex (corresponding to $S$) is removed.
Remark 3.5.10. The problematic argument of Example 3.5.6 makes no problems here, as only a single exceptional object (namely $S$) is removed.

We want to highlight that as the simple module $S$ is exceptional, we obtain in particular that $\text{Fr}_{\mathcal{D}}(j) = \mathcal{D}^b(\Lambda \text{-mod}) \oplus \mathcal{C}$ is admissible in $\mathcal{D}^b(\Lambda' \text{-mod})$. This is in contrast to geometric examples, where the Frobenius codomain tends to be non-admissible.

3.5.3. Posets.

Remark 3.5.11. In [HKP19, §2], the notion of a spherical poset of $\mathcal{D}$ is introduced: it is defined as the poset

$$\{D_A \mid A \in \mathcal{D} \text{ spherelike}\}.$$ 

In contrast, we define the spherical poset in Remark 3.4.4 as the poset of spherical neighbourhoods under a fixed spherelike functor. So these two posets will be very different in general and we sincerely hope that this does not cause confusion.

We want to highlight the last remark by an example.

Example 3.5.12. By [Zub97, LNSZ19], there are exceptional line bundles $L_1, \ldots, L_{10}$ on a generic Enriques surface $X$, which are mutually orthogonal, that is $\text{Hom}^*(L_i, L_j) = 0$ for $i \neq j$. This induces a semiorthogonal decomposition

$$\mathcal{D}^b(X) = \langle A_X, L_1, \ldots, L_{10} \rangle.$$ 

By Serre duality, there is a morphism $L_i \to S_X L_i$ unique up to scalars, which we extend to a triangle

$$S_i \to L_i \to S_X L_i.$$ 

By [LNSZ19, Lem. 3.6 & Prop. 3.7], these $S_i$ are 3-spherical objects inside $A_X$ and any 3-spherical object inside $A_X$ is isomorphic to a shift of an $S_i$. Additionally, it was observed in the proof of [LNSZ19, Prop. 3.7] that $S_i$ fits into the triangle

$$S_i \to S_X S_i[-3] \to S_X L_i \oplus S_X L_i[-3]$$ 

which is the asphericity triangle of $S_i$. Therefore the spherical subcategory of $S_i$ is $\perp S_X L_i = L_i^\perp = \langle A_X, L_j \mid j \neq i \rangle$. In particular, the spherical poset in the sense of [HKP19] of $\mathcal{D}^b(X)$ contains

$$\{\langle A_X, L_j \mid j \neq i \rangle\}$$ 

where any two elements are not comparable.

For a spherelike object $S_i$ the spherical poset of the corresponding spherelike functor $F_i : \mathcal{D}^b(k) \rightarrow \mathcal{D}^b(X)$ consists of just two elements:

$$\{\langle A_X, L_j \mid j \neq i \rangle, \mathcal{D}^b(X)\},$$ 

where the maximal element is obtained by the zero object, and the minimal one by any non-zero object in $\mathcal{D}^b(k)$. 
The richest structure, we obtain by looking at the exceptional functor \( \iota: A_X \to D^b(X) \). The above discussion shows now that \( \text{Fr}(\iota, S_i) = \langle A_X, L_j \mid j \neq i \rangle \). Using that the \( L_i \) are mutually orthogonal, one can check that therefore the Frobenius poset is

\[
P(\iota) = \{ \langle A_X, L_j \in J \rangle \mid J \subset \{1, \ldots, n\} \}.
\]

### 3.6. Examples

In [KS15], a functor was called spherelike for the first time:

**Example 3.6.1.** Let \( X \) be an Enriques surface and \( \pi: \tilde{X} \to X \) its canonical cover, so \( \tilde{X} \) is an K3 surface. Note that \( \pi_*: D^b(\tilde{X}) \to D^b(X) \) is a spherical functor, whose cotwist is \( \tau^* \) with \( \tau \) the deck transformation. Let \( X^{[n]} \) be the Hilbert scheme of \( n \) points on \( X \). As \( O_X \) is exceptional, the Fourier-Mukai transform \( F: D^b(X) \to D^b(X^{[n]}) \) associated to the universal ideal sheaf is an exceptional functor.

It was observed in [KS15, Rem. 3.7], that the composition \( F \pi_* \) should be called spherelike functor. And indeed, by Proposition 3.4.1, \( F \pi_* \) is a spherelike functor as the composition of a spherical and an exceptional functor. By Theorem 3.4.3, we find that

\[
\text{Sph}(F \pi_*, A) = \text{Fr}(F, \pi_* A)
\]

In particular, as \( \pi_* \) is essentially surjective, we obtain that

\[
\text{Sph}(F \pi_*) = \text{Fr}(F) = \text{im} F \oplus (\ker R \oplus \ker L)
\]

where \( R \) and \( L \) are the adjoints of \( F \).

### Question 3.6.2.

Are there meaningful spherelike functors which are **not** the composition of a spherical and an exceptional functor?

Obviously, the answer to this question depends on the taste of the reader, as the following example shows.

**Example 3.6.3.** Let \( S \) be a bielliptic surface. Then its structure sheaf \( O_S \) is a (properly) 1-spherelike object in \( D^b(S) \). By [KO15, Prop. 4.1], \( D^b(S) \) admits no nontrivial semiorthogonal decomposition. In particular, the spherical subcategory of \( O_S \) is not admissible.

Note that a spherical object in the derived category of a \( d \)-dimensional variety is automatically \( d \)-Calabi–Yau. In contrast, \( O_S \) is a 1-spherelike object in the derived category of surface. It would be interesting to know, whether the cotwist of a spherical functor between categories of geometric origin is always of a specific shape.

We end with an example of spherelike objects from [HP20]. The first is still given by the inclusion of a spherical object via an exceptional functor into some bigger category. The second one is not of this kind, but to us, the second example seems rather a numerical accident than a meaningful example.
Example 3.6.4. Let $X$ be a surface containing three rational curves $B, E, C$ with the following dual intersection graph: \[\xymatrix{ 1 \ar@{-}[r] & 2 \ar@{-}[r] & 3} \], so $B^2 = -3$, $E^2 = -1$ and $C^2 = -2$. Then $\mathcal{O}_{B+E+C}$ is not the pullback of some spherical object using some birational morphism $\pi: X \to Y$. Still, $C$ is a $(-2)$-curve, so $\mathcal{O}_C(-1)$ is spherical, and actually $\mathcal{O}_{B+E+C} = T_{\mathcal{O}_C(-1)}(\mathcal{O}_B+E)$, see [HP20, Prop. 4.6]. So after applying this autoequivalence, $\mathcal{O}_{B+E}$ becomes contractible to a $(-2)$-curve. In particular, denoting by $\pi_E: X \to Y$ the contraction of $E$, we obtain an exceptional functor
\[
F : T_{\mathcal{O}_C(-1)}(\pi^*\mathcal{D}^b(Y)) \to \mathcal{D}^b(X) = \langle T_{\mathcal{O}_C(-1)}(\pi^*\mathcal{D}^b(Y)), T_{\mathcal{O}_C(-1)}(\pi^*\mathcal{D}^b(Y)) \rangle
\]
and $\mathcal{O}_{B+E+C}$ becomes the image of a spherical object under this $F$.

Example 3.6.5. Let $X$ be a surface containing five rational curves $B, C_1, C_2, E_1, E_2$ with the following dual intersection graph: \[\xymatrix{ 1 \ar@{-}[r] & 2 & 3 \ar@{-}[r] & 4 \ar@{-}[r] & 5} \], where $B^2 = -3$, $C_i^2 = -2$ and $E_i^2 = -1$. Consider the divisor $D = 2B + C_1 + C_2 + E_1 + E_2$. Then $\mathcal{O}_D$ is a spherelike divisor and it seems that it does not arise as the image of any spherical object under an exceptional functor. See [HP20, Ex. 5.11] for further discussion.

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