Third quantization of $f(R)$-type gravity II—general $f(R)$ case

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Abstract

In the previous paper, we examined the third quantization of the $f(R)$-type gravity and studied the Heisenberg uncertainty relation of the universe in the example of $f(R) = R^2$. In this work, the Heisenberg uncertainty relation of the universe is investigated in the general $f(R)$-type gravity where tachyonic states are avoided. It is shown that, at late times namely the scale factor of the universe is large, the spacetime becomes classical, and, at early times namely the scale factor of the universe is small, the quantum effects dominate.

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1. Introduction

The $f(R)$-type gravity is one of the kinds of the so-called generalized gravity theories. In the early stage, such theories were interested in because of their theoretical advantages. Within such theories, it is possible to construct a renormalizable theory of gravitons, in which the Lagrangian density depends on the scalar curvature and the Ricci tensor up to the quadratic terms, when the surface terms can be neglected [1—4]. It seems to be possible to avoid the initial singularity of the universe predicted by the theorem proved by Hawking [5, 6]. And inflationary model without introducing ad hoc inflaton field is possible [7]. Motivation for the $f(R)$-type gravity began after the discovery of the accelerated expansion of the universe [8—10], which could be explained using the dark energy whose density has been observed to be about 73% of the total energy density of the universe [10]. However, what is the dark energy is an open question, and explanation by the $f(R)$-type gravity is one of the hopeful candidates, so $f(R)$-type gravity theories have been attracting much attention [11—14].

Quantum mechanical aspects of the theory are mainly applied to cosmology, namely quantum cosmology [15—17], and black holes [18]. Shojai and Shojai started from the effective Lagrangian of $f(R)$-type gravity which is the same as equation (2.10) of [19], used Bohmian quantum mechanics and showed that the acceleration of the universe depends both on the form of $f(R)$ and on the quantum potential [15]. Vakili started also from the same effective Lagrangian and introduced the Noether symmetry, which restricted the form of $f(R) = R^{3/2}$,
in the case of the four-dimensional spacetime [16]. He derived the Wheeler–DeWitt equation (the WDW equation) [20] and another quantum equation which comes from the existence of the Noether symmetry, and he showed that, using the semiclassical approximation, in the late time the universe evolves with a power low accelerated expansion. Capozziello and Garattini started from the Hamiltonian of $f(R)$-type gravity which is similar to equation (3.144) of [21] and used the formulation by Buchbinder and Lyakhovich (BL) [22], which is one of the generalized Ostrogradski formulations used rather frequently in the recent literature [21]. Capozziello and Garattini found out the eigenvalues of the WDW equation with the meaning of vacuum states, i.e. cosmological constants [17]. Note that Deruelle et al showed that in $f(R)$-type gravity Jordan frame formulation, Einstein frame formulation and Ostrogradski formulation are related by canonical transformations [23]. However, there is a delicate issue in the canonical formalisms between Jordan and Einstein frames [24]. This arises primarily from the fact that two formalisms in the Jordan frame are used in the literature. One is the one by BL above, which modifies the Lagrangian density (2.1) below with the definitions of momenta and constraints by the Lagrange multiplier method, then by the Legendre transformation goes over to the canonical formalism. Another is a more straightforward generalization of the Ostrogradski transformation in terms of the Lie derivatives [25].

The fundamental equation describing the dynamics of the quantum universe is the WDW equation which is the differential equation for the wavefunction of the universe [20]. However, it is well known that, in general, the WDW equation has the problem that the probabilistic interpretation is difficult as in the case of the Klein–Gordon equation. One of the proposed ideas to solve this problem is the third quantization in analogy with the quantum field theory [18, 26–42]. Then the third-quantized universe theory describes a system of many universes. Third quantization is useful to describe bifurcating universes and merging universes, if an interacting term is introduced in the Lagrangian for the third quantization.

The quantum cosmology of the $f(R)$-type gravity using the WDW equation has already been studied [15–17]. As noted above the third quantized version is desirable, and the third quantization of $f(R)$-type gravity was also investigated in [18]. However, in this black holes were studied but cosmology was not treated. So in the previous work, we examined the third quantization of the $f(R)$-type gravity, using the explicit form of the action which yields the WDW equation of $f(R)$-type gravity, and we investigated the Heisenberg uncertainty relation of the universe in the example of $f(R) = R^2$ [43]. In this work, we investigate the uncertainty relation of the universe in the general $f(R)$-type gravity where tachyonic states are avoided.

We start from the effective theory of the $f(R)$-type gravity in a flat Friedmann–Lemaitre–Robertson–Walker metric. Then a suitable change of variable is performed and the WDW equation is written down. Quantizing this model once more, we obtain the third-quantized theory of $f(R)$-type gravity. The Heisenberg uncertainty relation is investigated in the general $f(R)$-type gravity where tachyonic states are avoided. It will be shown that, at late times namely the scale factor of the universe is large, the spacetime becomes classical, and, at early times namely the scale factor of the universe is small, the quantum effects dominate.

In section 2, the third quantization of the effective theory of $f(R)$-type gravity in the case of a flat Friedmann–Lemaitre–Robertson–Walker metric is summarized. In section 3, the uncertainty relation is studied in the general $f(R)$-type gravity where tachyonic states are avoided. Summary is given in section 4.

2. Third quantization of $f(R)$-type gravity

Generalized gravity of the $f(R)$-type is one of the higher curvature gravity in which the action is given by
The spacetime is taken to be four dimensional. Here \( g \equiv \det g_{\mu \nu} \) and \( R \) is the scalar curvature.

Let us consider the next action

\[
S = \int d^4x \sqrt{-g} f(R),
\]

(2.1)

where \( f'(\phi) = \frac{df(\phi)}{d\phi} \) and we assume \( f''(\phi) \neq 0 \). It is well known that from the field equation of this action obtained from the variation of \( \phi \), we have

\[
R = \phi, \tag{2.3}
\]

and from equation (2.3) and the field equations of this action obtained from the variation of \( g_{\mu \nu} \), we have the same field equations of action (2.1). If we substitute equation (2.3) into equation (2.2), we obtain equation (2.1) [12].

Considering the recent cosmological observation, we take the case of a flat Friedmann–Lemaître–Robertson–Walker metric,

\[
ds^2 = -dt^2 + a(t)^2 \sum_{k=1}^{3}(dx^k)^2.
\]

(2.4)

In order to make the kinematical part of the WDW equation simple, so that a feasible action for the third quantization is obtainable, let us make the change of a variable as follows [43]:

\[
\varphi = \varphi(\phi) \equiv \ln f'(\phi), \quad f'(\phi) = e^\varphi, \quad \phi = f^{-1}(e^\varphi).
\]

(2.5)

By this change of the variable, operator ordering problem is also avoided. Then as in [43], we obtain a WDW equation

\[
-\frac{1}{a} \frac{\partial^2 \psi}{\partial \tau^2} + \frac{\partial^2 \psi}{\partial a \partial \varphi} + 6a^5 f^{-1}(e^\psi) e^{2\varphi} \psi - 6a^5 e^{\varphi} f(f^{-1}(e^\varphi)) \psi = 0.
\]

(2.6)

Here, \( \psi(a, \varphi) \) is the wavefunction of the universe.

Now let us comment on the possibility of tachyonic states in this WDW equation. In order to examine equation (2.6) in the Klein–Gordon form, we make change of variables as

\[
\tau = a + \varphi + \ln a, \quad \sigma = a - \varphi - \ln a.
\]

(2.7)

Then, we obtain

\[
\frac{\partial^2 \psi}{\partial \tau^2} - \frac{\partial^2 \psi}{\partial \sigma^2} + U \psi = 0,
\]

(2.8)

where

\[
U = 6a^5 f^{-1}(e^\varphi) e^{2\varphi} - 6a^5 e^{\varphi} f(f^{-1}(e^\varphi)) = 6 \left( \frac{\tau + \sigma}{2} \right)^3 f^{-1} \left( \frac{2}{\tau + \sigma} e^{\frac{\tau + \sigma}{2}} \right) e^{\tau + \sigma} - 6 \left( \frac{\tau + \sigma}{2} \right)^4 f \left( f^{-1} \left( \frac{2}{\tau + \sigma} e^{\frac{\tau + \sigma}{2}} \right) \right) e^{\frac{\tau + \sigma}{2}}.
\]

(2.9)

From equation (2.7), we note that \( \tau \) can be considered as the time variable, since \( \tau \) is a monotonic increasing function of the scale factor \( a \). Therefore, in order to avoid tachyonic states, \( U \geq 0 \) is required, since \( U \) is the square of the effective mass [34]. The condition \( U \geq 0 \) means

\[
f'(R)(f'(R)R - f(R)) \geq 0,
\]

(2.10)
in the original variables. Note that this condition is satisfied, for example, when \( f(R) \) is a polynomial with positive coefficients without the cosmological constant and when \( R \gg 0 \).

The action for the third quantization to yield the WDW equation \((2.6)\) can be written as

\[
S_{3Q} = \int \, da \, d\varphi \frac{1}{2} \left[ \frac{1}{a} \left( \frac{\partial \psi}{\partial \varphi} \right)^2 - \frac{\partial \psi}{\partial a} \frac{\partial \psi}{\partial \varphi} + 6a^2 f^{-1}(e^{\varphi}) e^{2\varphi} \psi^2 - 6a^2 e^{\varphi} f(f^{-1}(e^{\varphi})) \psi^2 \right].
\]

\[
= \int \, da \, d\varphi \mathcal{L}_{3Q}.
\]

As discussed in [43], let us consider \( a \) to be the time coordinate, and in order to third quantize this system, we impose the equal time commutation relation

\[
[\hat{\psi}(a, \varphi), \hat{p}_\varphi(a, \varphi')] = i\delta(\varphi - \varphi').
\]

We use the Schrödinger picture, so we take the operator \( \hat{\psi}(a, \varphi) \) as the time independent \( c \)-number field \( \psi(\varphi) \), and we take the momentum operator as

\[
\hat{p}_\varphi(a, \varphi) \rightarrow -i \frac{\partial}{\partial \psi(\varphi)}.
\]

Then, we obtain the Schrödinger equation

\[
i \frac{\partial \Psi}{\partial a} = \hat{H}_{3Q} \Psi,
\]

explicitly

\[
i \frac{\partial \Psi}{\partial a} = \frac{2}{a} \left( \frac{\partial}{\partial \psi(\varphi)} \right)^2 - 3a^2 f^{-1}(e^{\varphi}) e^{2\varphi} \psi^2(\varphi) + 3a^2 e^{\varphi} f(f^{-1}(e^{\varphi})) \psi^2(\varphi) \right] \Psi,
\]

where \( \Psi \) is the third quantized wavefunction of universes [43].

Here, we comment on the Einstein limit of our theory. At the classical level, though we assumed \( f''(R) \neq 0 \) below equation \((2.2)\), the effective action \((2.9)\) of [43] holds in the Einstein–Lagrangian case. However, in our method, the Einstein limit of the quantum theory, \( f(R) = R + \epsilon R^2, \epsilon \rightarrow 0 \), is not identical to the Einstein case, i.e. \( f(R) = R \), although it is not desirable. Einstein limit can be different from each other on what quantization method is used [17].

3. Uncertainty relation

We calculate the Heisenberg uncertainty relation in the state described by the wavefunction \( \Psi(a, \varphi, \psi(\varphi)) \). It is given by the product of dispersions \( \Delta \psi(\varphi) \) and \( \Delta p_\varphi(\varphi) \), where \((\Delta \psi(\varphi))^2 \equiv \langle \psi^2(\varphi) \rangle - \langle \psi(\varphi) \rangle^2 \) and \((\Delta p_\varphi(\varphi))^2 \equiv \langle p_\varphi^2(\varphi) \rangle - \langle p_\varphi(\varphi) \rangle^2 \). The inner product of the two functions \( \Psi_1 \) and \( \Psi_2 \) is defined as

\[
\langle \Psi_1, \Psi_2 \rangle = \int \, d\psi(\varphi) \Psi_1^*(a, \varphi, \psi(\varphi)) \Psi_2(a, \varphi, \psi(\varphi)).
\]

In order to estimate this uncertainty, we assume the Gaussian ansatz for the wavefunction, corresponding to the coherent state, as is often done

\[
\Psi(a, \varphi, \psi(\varphi)) = C \exp \left\{ -\frac{1}{2} \left[ A(a, \varphi)[\psi(\varphi) - \eta(a, \varphi)]^2 + iB(a, \varphi)[\psi(\varphi) - \eta(a, \varphi)] \right] \right\}.
\]

where \( A(a, \varphi) = D(a, \varphi) + iB(a, \varphi) \) [33, 35, 39, 43, 44]. The real functions \( D(a, \varphi), I(a, \varphi), B(a, \varphi) \) and \( \eta(a, \varphi) \) should be determined from equation \((2.15)\). \( C \) is the normalization factor of the wavefunction. Then the Heisenberg uncertainty relation can be calculated as [43]

\[
(\Delta \psi(\varphi))^2(\Delta p_\varphi(\varphi))^2 = \frac{1}{4} \left( 1 + \frac{I^2(a, \varphi)}{D^2(a, \varphi)} \right).
\]
The key point of generalizing the previous work is finding new variables and functions as described in the following. Note that to evaluate (3.3), only $A(a, \varphi)$ is necessary. Substituting the ansatz (3.2) into the Schrödinger equation (2.15), we obtain

$$\left(-\frac{1}{2}\frac{\partial A(a, \varphi)}{\partial a}\right) = \frac{2}{a}A^2(a, \varphi) + 3a^2e^\varphi[f(f^{-1}(e^\varphi)) - e^\varphi f^{-1}(e^\varphi)].$$

Writing

$$\ln a = \frac{\alpha}{6},$$

we have

$$-3i\frac{\partial A(a, \varphi)}{\partial \alpha} = 2A^2(a, \varphi) + 3ae^\varphi[f(f^{-1}(e^\varphi)) - e^\varphi f^{-1}(e^\varphi)].$$

In order to solve this equation, let us make the following transformation of variables

$$A(a, \varphi) = \frac{3i}{2} a \ln u(a, \varphi),$$

where $u(a, \varphi)$ is a suitable function. Then $u(a, \varphi)$ must satisfy the equation,

$$\frac{\partial^2 u(a, \varphi)}{\partial \alpha^2} + k(\varphi)e^\varphi u(a, \varphi) = 0,$$

where

$$k(\varphi) = \frac{1}{3}e^\varphi[f(e^\varphi f^{-1}(e^\varphi)) - f(f^{-1}(e^\varphi))].$$

Now let us assume condition (2.10) to avoid tachyonic states, and let us introduce a new variable

$$z = 2\sqrt{k(\varphi)}e^\varphi,$$

which plays a role of time coordinate. Then, we have

$$\frac{\partial^2 u(z, \varphi)}{\partial z^2} + \frac{1}{z} \frac{\partial u(z, \varphi)}{\partial z} + u(z, \varphi) = 0.$$

As this equation can be regarded as the ordinary differential equation with respect to $z$ with a parameter $\varphi$, this equation is the case of the following Bessel’s equation for $\nu = 0$

$$\frac{d^2 u(z)}{dz^2} + \frac{1}{z} \frac{du(z)}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) u(z) = 0.$$

Therefore, we have the solution

$$u(z, \varphi) = c_J(\varphi)J_0(z) + c_Y(\varphi)Y_0(z),$$

where $J_0$, $Y_0$ are the Bessel functions of order 0 and $c_J$, $c_Y$ are arbitrary complex functions of $\varphi$.

From equations (2.3), (2.5), (3.5), (3.7), (3.9), (10.10) and (13.13), we obtain

$$z = 2\sqrt{\frac{2}{3} f'(R)[f'(R)R - f(R)]} a^3, \quad \varphi = \ln(f'(R))$$

and

$$A(z, \varphi) = -\frac{3}{4} c_J(\varphi)J_1(z) + c_Y(\varphi)Y_1(z),$$

where we have used $J_0'(z) = -J_1(z)$, $Y_0'(z) = -Y_1(z)$ [45].

Since $A(z, \varphi) = D(z, \varphi) + \tilde{I}(z, \varphi)$, we have

$$D(z, \varphi) = -\frac{3i}{4\pi |c_J(\varphi)J_0(z) + c_Y(\varphi)Y_0(z)|^2} [c_J(\varphi)c_Y(\varphi) - c_J'(\varphi)c_Y(\varphi)].$$
and
\[ I(z, \phi) = -\frac{3z}{8|c_J(\phi)J_0(z) + c_Y(\phi)Y_0(z)|^2} \left[ 2|c_J(\phi)|^2 J_0(z)J_1(z) + 2|c_Y(\phi)|^2 Y_0(z)Y_1(z) \right. \]
\[ + (c_J(\phi)c_Y(\phi) + c_J^*(\phi)c_Y^*(\phi))(J_0(z)Y_1(z) + J_1(z)Y_0(z)) \] \tag{3.17}

where we used \( J_0(z)Y_1(z) - J_1(z)Y_0(z) = -\frac{z}{2} \) \[45\].

So if we assume \( c_J(\phi)c_Y^*(\phi) - c_J^*(\phi)c_Y(\phi) \neq 0 \) (Note that in this case both of \( c_J(\phi), c_Y(\phi) \) are nonzero), we obtain
\[ P^2(z, \phi) = -\frac{\pi^2 z^2}{4|c_J(\phi)c_Y^*(\phi) - c_J^*(\phi)c_Y(\phi)|^2} \left[ 2|c_J(\phi)|^2 J_0(z)J_1(z) + 2|c_Y(\phi)|^2 Y_0(z)Y_1(z) \right. \]
\[ + (c_J(\phi)c_Y^*(\phi) + c_J^*(\phi)c_Y(\phi))(J_0(z)Y_1(z) + J_1(z)Y_0(z)) \] \tag{3.18}

Here \( c_J(\phi)c_Y^*(\phi) - c_J^*(\phi)c_Y(\phi) \) is a pure imaginary number, so its square is negative.

At late times namely \( a \to \infty \) i.e. \( z \to \infty \),
\[ J_0(z) \sim \sqrt{\frac{z}{\pi}} \cos \left( z - \frac{\pi}{4} \right), \quad J_1(z) \sim \sqrt{\frac{z}{\pi}} \sin \left( z - \frac{\pi}{4} \right), \]
\[ Y_0(z) \sim \frac{2}{\pi z} \sin \left( z - \frac{\pi}{4} \right), \quad Y_1(z) \sim -\frac{2}{\pi z} \cos \left( z - \frac{\pi}{4} \right). \]

\[45\] we have
\[ P^2(z, \phi) \sim -\frac{\pi^2 z^2}{[|c_J(\phi)|^2 - |c_Y(\phi)|^2] \cos(2z) + (c_J(\phi)c_Y^*(\phi) + c_J^*(\phi)c_Y(\phi)) \sin(2z)]^2 \]
\[ \sim O(1). \tag{3.19} \]

This and equation (3.3) mean that at late times namely \( a \to \infty \), it is plausible that the spacetime becomes classical in the sense that the quantum fluctuations become minimum.

On the other hand at early times namely \( a \to 0 \) i.e., \( z \to 0 \),
\[ J_0(z) \sim 1 - \frac{z^2}{4}, \quad J_1(z) \sim \frac{z}{2}, \]
\[ Y_0(z) \sim \frac{2}{\pi \ln z}, \quad Y_1(z) \sim -\frac{2}{\pi z}. \]

\[45\] we obtain
\[ P^2(z, \phi) \sim -\frac{16|c_Y(\phi)|^4}{\pi^2 |c_J(\phi)c_Y^*(\phi) - c_J^*(\phi)c_Y(\phi)|^2} (\ln z)^2 \sim \infty. \tag{3.20} \]

This and equation (3.3) mean that the fluctuation of the third quantized universe field becomes large at early times namely \( a \to 0 \). Therefore, the quantum effects dominate for the small values of the scale factor of the universe.

### 4. Summary

In this work, the third quantization of the \( f(R) \)-type gravity is investigated, when the metric is a flat Friedmann–Lemaître–Robertson–Walker one. The Heisenberg uncertainty relation of the universe is investigated in the general \( f(R) \)-type gravity where tachyonic states are avoided. It has been shown that, at late times namely the scale factor of the universe is large, the spacetime becomes classical, and, at early times namely the scale factor of the universe is small, the quantum effects dominate. This result is similar to \([34, 35, 39, 43]\) but not similar to \([33]\), where it was shown that quantum effects dominate also when the scale factor is large.
However, as pointed out in [35], this era corresponds to the classically forbidden region in that model [33].

As a future work, though our formulation started from the effective action (2.2), it will be interesting to quantize (2.1) directly as in [17, 18], since the quantization of the $f(R)$-type gravity would not be unique.

In addition there are many works on quantum cosmology which use various concrete forms of $f(R)$, e.g., $f(R) = R^2$ or $f(R) = R + R^2$ and make advantages of the characteristic nature of the models [22, 46–51]. It is interesting to apply our results for general $f(R)$-type and compare the results to the preceding ones, since the methods used would not be necessarily equivalent [23–25]. However, this issue is not treated in the context of the third quantized theory, so will be treated in a separate work.

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References

[1] Utiyama R and DeWitt B S 1962 J. Math. Phys. 3 608
[2] Stelle K 1977 Phys. Rev. D 16 953
[3] Buchbinder I L, Odintsov S D and Shapiro I L 1992 Effective Action in Quantum Gravity (Bristol: Institute of Physics Publishing)
[4] Narain G and Anishetty R 2012 arXiv:1210.0513 [hep-th]
[5] See for example Hawking S and Ellis G F R 1973 Large Scale Structure of Spacetime (London: Oxford University Press)
[6] Nariai H 1971 Prog. Theor. Phys. 46 433
Nariai H and Tomita K 1971 Prog. Theor. Phys. 46 776
[7] Starobinsky A A 1980 Phys. Lett. B 91 99
[8] Riess A G et al 1998 Astron. J. 116 1009
Perlmutter S et al 1999 Astrophys. J. 517 565
[9] Reid B A et al 2010 Mon. Not. R. Astron. Soc. 404 60
Finkbeiner P D et al 2009 Mon. Not. R. Astron. Soc. 400 1518
Hicken M et al 2009 Astrophys. J. 700 1097
Kessler R et al 2009 Astrophys. J. Suppl. 185 32
Vikhlinin A et al 2009 Astrophys. J. 692 1060
Mantz A et al 2010 Mon. Not. R. Astron. Soc. 406 1759
Riess A G et al 2009 Astrophys. J. 699 539
Suyu S H et al 2010 Astrophys. J. 711 201
Fadely R et al 2010 Astrophys. J. 711 246
Massey R et al 2007 Astrophys. J. Suppl. 172 239
[10] Komatsu E et al 2011 Astrophys. J. Suppl. 192 18
[11] Larson D et al 2011 Astrophys. J. Suppl. 192 16
Jarosik N et al 2011 Astrophys. J. Suppl. 192 14
Bennett C L et al 2011 Astrophys. J. Suppl. 192 17
[12] The pioneering work is Carroll S M, Duvvuri V, Trodden M and Turner M S 2004 Phys. Rev. D 70 043528
[13] Sotiriou T P and Faraoni V 2010 Rev. Mod. Phys. 82 451 (a good review)
[14] Nojiri S and Odintsov S D 2007 Int. J. Geom. Methods Mod. Phys. 4 115
[15] Shojai A and Shojai F 2008 Gen. Rel. Grav. 40 1967
[16] Vakili B 2010 Ann. Phys. 19 359
Vakili B 2008 Phys. Lett. B 669 206
