Two dimensional space-time symmetry in hyperbolic functions

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Summary.- An extension of the finite and infinite Lie groups properties of complex numbers and functions of complex variable is proposed. This extension is performed exploiting hypercomplex number systems that follow the elementary algebra rules. In particular the functions of such systems satisfy a set of partial differential equations that defines an infinite Lie group.

Emphasis is put on the functional transformations of a particular two dimensional hypercomplex number system, capable of maintaining the wave equation as invariant and then the speed of light invariant too. These functional transformations describe accelerated frames and can be considered as a generalisation of two dimensional Lorentz group of special relativity. As a first application the relativistic hyperbolic motion is obtained.

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1 Introduction

The wide use of complex numbers, far beyond their algebraic introduction as square root of negative numbers, stimulated in the past century the search for other systems of numbers and for a generalisation of the complex numbers properties. This led to the introduction of the so called hypercomplex numbers, whose theory was considered completed at the beginning of this century [1].

Today hypercomplex numbers are included as a part of abstract algebra, but a full understanding of their implications beyond the purely mathematical ones could contribute to new insights in many scientific fields.

In this paper we will use the relationship between the systems of hypercomplex numbers and the group theory, pointed out by S. Lie [2] and M. G. Scheffers [3].

In fact Lie showed that hypercomplex numbers define a finite group, while Scheffers demonstrated that for functions of commutative hypercomplex numbers the differential calculus holds as for functions of complex variable. This implies that for functions of hypercomplex variables a system of partial differential equations can be written as a generalisation of the Cauchy-Riemann conditions [4], and that these equations define an infinite Lie group [5]. This means that the transformations by functions of hypercomplex variables belong to a group of the same kind of conformal group, but
with specific properties (e.g. the invariants) depending on the particular number system. In this paper we will show that a two dimensional hypercomplex number system (hyperbolic numbers) is connected with the space-time group and we will give the physical meaning of its functional transformations, showing that these transformations can be considered as a generalisation of two dimensional Lorentz group of special relativity. Moreover, we will demonstrate that the exponential transformation (which, for functions of complex variable, gives the equipotentials of a field produced by a point charge) yields the constant acceleration motion in the case of hyperbolic functions.

2 Definition of hypercomplex numbers. Some properties of the two-dimensional systems

The hypercomplex numbers \[1, 2, 6\] are defined by the expression

\[ x = \sum_{\alpha=0}^{N-1} e_\alpha x^\alpha \]  

where \(x^\alpha\) are called components and \(e_\alpha\) units or versors, as in vector algebra. The expression of eq. \(1\) defines a hypercomplex number if the versors multiplication rule is given by a linear combination of versors:

\[ e_\alpha e_\beta = \sum_{\gamma=0}^{N-1} C_{\alpha\beta}^\gamma e_\gamma \]  

where \(C_{\alpha\beta}^\gamma\) are real constants, called structure constants, that define the characteristics of the system \[1, 2, 6\], as we will see in the two dimensional case considered in the following.

The versor product \(2\) defines also the product of hypercomplex numbers. This product definition makes the difference between vector algebra and hypercomplex systems and allows to relate the hypercomplex numbers to groups. In fact the vector product is not, in general, a vector while the product of hypercomplex numbers is still a hypercomplex number; the same for the division, that for vectors does not exist while for hypercomplex numbers, in general, does exist.

Let us now consider the two dimensional hypercomplex systems and their functions that have been recently treated in \[7\]. Here we will briefly recall the main properties. In \[7\] the generic system of two dimensional hypercomplex numbers is defined as:

\[ w = \{x + u y; \ u^2 = \alpha + u \beta \ \ x, y, \alpha, \beta \in \mathbb{R}\}. \]

For any value of \(\alpha\) and \(\beta\) only three systems exist having different properties. If we represent the structure constant \(\alpha\) and \(\beta\), in a Cartesian plane it can be shown \[7\] that the parabola \(\Delta \equiv \beta^2 + 4\alpha = 0\) divides the plane in two parts, and for \(\alpha\) and \(\beta\), on the parabola or on its left and right sides we have three equivalent systems that are called:

\(\Delta < 0\) \Rightarrow\text{Elliptic.}\n\(\Delta = 0\) \Rightarrow\text{Parabolic}\n\(\Delta > 0\) \Rightarrow\text{Hyperbolic}\n
Numbers of the same system can be related to each other by a linear transformation, \text{i.e.} they are equivalent. Due to this equivalence, special attention is paid to the simplest systems (Canonical systems) \[1, 2, 6, 7\] obtained with \(\beta = 0\) and:

\(\alpha = -1\) \Rightarrow\text{Elliptic, or ordinary complex numbers.}\n\(\alpha = 0\) \Rightarrow\text{Parabolic}\n\(\alpha = 1\) \Rightarrow\text{Hyperbolic}\n
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For complex numbers is, of course, \( u \equiv i \) whilst for hyperbolic numbers we put \( u \equiv h \).

It is known that the invariant of the group of complex numbers is \( \rho^2 = (x + iy)(x - iy) \equiv x^2 + y^2 \) i.e. the Euclidean distance. In the same way the invariant for canonical hyperbolic system is obtained as: \( \rho_h^2 = (x + hy)(x - hy) \equiv x^2 - y^2 \), i.e. the space-time invariant, that is not positive defined, and is equal to zero for \( x = \pm y \). In the following we will call the functions of hyperbolic numbers hyperbolic functions. The hyperbolic numbers are then related to complex numbers but, in spite of this, they have not so widely utilised as the complex numbers and their functions. One of the rare cases of application is reported in \([7]\) where the hyperbolic functions are used for the description of supersonic effects.

In this paper we will point out the properties which are common to complex and hyperbolic numbers and their differences. Moreover we will discuss the transformations by hyperbolic functions. In Appendix we recall some of the properties of the complex variable and their functions in a way that will allow for a direct comparison with hyperbolic variables and their functions.

3 Hyperbolic numbers and Lorentz group

For the hyperbolic variable one can follow an approach parallel to that discussed in Appendix for the complex variable. Let us start pointing out that the Lorentz transformation corresponds to the multiplicative group of the hyperbolic numbers. Let us write a space-time vector as a hyperbolic variable \( w = x + ht \) \([7]\) and a hyperbolic constant \( a = a_r + ha_h \) in the exponential form:\[a_r+ha_h \equiv \exp(\rho_h + h\theta_h) \equiv \exp \rho_h (\cosh \theta_h + h \sinh \theta_h) \text{ where } \rho_h = \ln (a_r^2 - a_h^2); \quad \theta_h = \tanh^{-1}(a_h/a_r).

Then the multiplicative group, \( w' \equiv x' + ht' = aw \) becomes:
\[
x' + ht' = \sqrt{(a_r^2 - a_h^2)} \left[ x \cosh \theta_h + t \sinh \theta_h + h (x \sinh \theta_h + t \cosh \theta_h) \right]
\]

In this equation by letting \((a_r^2 - a_h^2) = 1\) and considering as equal the coefficients of the versors “1” and “h”, as we make in complex analysis, we get the Lorentz transformation of special relativity \([9]\). It is interesting to note that the same result is normally achieved by following a number of “formal” steps \([10], p.94\), \([11], p.50\), i.e. by introducing an “imaginary” time \( t' = it \) which makes the Lorentz invariant \((x^2 - t^2)\) equivalent to the rotation invariant \((x^2 + y^2)\), and by introducing the hyperbolic functions through their equivalence with circular functions of an imaginary angle. Let us stress that this procedure is essentially formal, while the approach based on hyperbolic numbers leads to a direct description of the Lorentz transformation of special relativity explainable as a result of symmetry (or invariants) preservation: the Lorentz invariant (space-time “distance”) is the invariant of hyperbolic numbers. Therefore we can say that the hyperbolic numbers have the space-time symmetry, while the complex numbers have the symmetry of two spatial variables, represented in a Euclidean plane. Within the limits of our knowledge, the first algebraic description of Special Relativity, directly by these numbers (called “Perplex numbers”), has been introduced in \([8]\). With the exposed formalism we can see that the Lorentz transformation is equivalent to a “hyperbolic rotation”. In fact let us write in the Lorentz transformation, the hyperbolic variable \( x + ht \) in exponential form:

\[\text{We recall the definition of exponential and logarithmic hyperbolic functions. The former is given by \([7,8]\):}\]
\[
x + ht = \exp(X + hT) \equiv \exp X (\cosh T + h \sinh T)
\]
\[
\text{The logarithmic function can be obtained by inversion of exponential function \([7], p. 55\):}\]
\[
X + hT = \ln \sqrt{x^2 - t^2} + h \tanh^{-1}(t/x)
\]
\[
\text{In these transformations the complete } X, T \text{ plane is mapped in the region } x > 0 \text{ and } |x| > |t|. \text{ We note that the } x, t \text{ plane has the essential property of special relativity representative plane.}\]
\[
\text{For the constant } a = a_r + ha_h \text{ in exponential form we assume } |a_r| > |a_h|.
\]

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\[ x + ht = \exp \rho (\cosh \theta + h \sinh \theta) \] where \( \rho = \ln \sqrt{(x^2 - t^2)} \), \( \theta = \tanh^{-1}(t/x) \). Then:

\[ w' = aw = \exp \rho (\cosh(\theta h + \theta) + h \sinh(\theta h + \theta)) \] (4)

From this expression we see that the Lorentz transformation is equivalent to a “hyperbolic rotation” of the \( x + ht \) variable. Then the invariance under Lorentz transformation can be also expressed as an independence on the hyperbolic angle \( \theta \).

4 Physical meaning of transformations by hyperbolic functions

Let us now consider the functions of hyperbolic variable. As for the functions of complex variable, the functions \( f(w) = u(x, t) + hv(x, t) \) are said to be functions of the hyperbolic variable \( w = x + ht \) and consequently they are called hyperbolic functions if \( u, v \) satisfy the following system of partial differential equations [7]:

\[ u_{,x} = v_{,t} ; \quad u_{,t} = v_{,x} \] (5)

where the comma stands for derivation with respect to the variables which follow. From these equations, it follows that the relationship between the hyperbolic functions and the wave equation is the same as that existing between the functions of complex variable and the Laplace equation [11]. Accordingly, the wave equation is satisfied by hyperbolic functions and it is invariant for the transformations \( (x, t) \Rightarrow (u, v) \), with \( u, v \) satisfying the system of eq.s [8]. Therefore \( t \) can be referred to as a physical normalized (speed of light \( c=1 \)) time variable. The invariance of wave equation means that the speed of light does not change if the coordinate system is changed by these functional transformations, i.e. that the well known postulate of special relativity is also valid for all these coordinate systems. Consequently this infinite group of functional transformations can be considered as a generalisation of Lorentz-Poincaré two dimensional group.

Now let us emphasize the physical meaning of the functional transformations. It is known that the linear Lorentz transformation of special relativity represents a change of inertial frame. Relating the space and time variables to Cartesian coordinates, the inertial motion is represented by straight lines and, applying the linear Lorentz transformation, these lines remain straight lines. The functional transformations change the straight lines in one reference frame into curved lines in the other frame. From a physical point of view, a curved line represents a non inertial motion, i.e. a motion in a field. The question which arises is whether these functional transformations represent any physical fields. Now we show that extending to hyperbolic functions the procedure reported in Appendix for complex variable, we obtain the relativistic hyperbolic motion.

For the \( x, t \) variables the experimental evidence for symmetry is provided by the invariance under the Lorentz transformation, or by eq. 4 the independence on the hyperbolic angle \( \theta \). Thus let us find a solution \( U(x, t) \) of the wave equation \( U_{xx} - U_{,tt} = 0 \) independent of the hyperbolic angle. We can proceed as in Appendix, and write the hyperbolic variable \( x + ht \) as exponential function of the variable \( X + hT \). Then with the transformation:

\[ x = \exp X \cosh T, \quad t = \exp X \sinh T \quad \text{or} \quad X = \ln \sqrt{x^2 - t^2}, \quad T = \tanh^{-1}(t/x) \] we must solve the equation:

\[ U_{XX} - U_{,TT} = 0 \] (6)

The \( U \) invariance for “hyperbolic rotation” means independence on \( T \) variable. Therefore \( U_{,TT} \equiv 0 \), and \( U \) will depend only on the variable \( X \). The partial differential equation, eq. 6, becomes a normal differential equation \( d^2U/dX^2 = 0 \), with the elementary solution:

\[ U = AX + B \equiv A \ln \sqrt{(x^2 - t^2)} + B \] (7)
If we consider, as for the Laplace equation, the “equipotentials” \( U = \text{const} \), in the \( X, T \) plane these lines are straight lines, that can be expressed as \( X = \text{const} \equiv \ln g, \ T = g^{-1} \tau \). Going over to \( x, t \) variables these straight lines become the hyperbolas:

\[
x = g \cosh g^{-1} \tau; \ t = g \sinh g^{-1} \tau
\]

that represent the hyperbolic motion \([11, \text{p.}166]\), where \( \tau \), the time coordinate in a reference frame in which \( X = \text{const} \), is the proper time. This is the law of motion of a body in the field of a constant force, calculated by the relativistic Newton’s dynamic law. In this example the motion has been obtained as a transformation through the exponential function of a particular straight line. This transformation belongs to a group that preserves the symmetry in space-time, and, in so doing, it satisfies widely accepted concepts of modern physics \([12, \text{p.}48]\).

5 Conclusions

In summary, we can conclude that: the space-time symmetry is best treated by means of hyperbolic numbers and the space-time field is best described by functions of hyperbolic variable, following the fundamental principle according which a significant simplification in any physical problem can be obtained by using a mathematical approach with the same symmetry of the problem.

A Group properties of complex numbers and of functions of complex variable

The complex numbers can represent plane vectors and the related linear algebra \([10, \text{p.}73]\): \( z = x + iy \) is interpreted as a vector of components \( x \) and \( y \), and versors \( 1 \) and \( i \). In vector form we write \( \vec{z} = \vec{1}x + i\vec{y} \), where \( x \) and \( y \) are the current coordinates of the plane. If we consider the multiplication by a constant:

\[
z_1 = az \equiv (a_r + ia_i)(x + iy), \quad (8)
\]

the complex numbers play the role of both vector and operator (matrix) \([10, \text{p.}73]\) and eq. (8) is equivalent to the familiar expression of linear algebra:

\[
\begin{pmatrix}
x_1 \\
y_1
\end{pmatrix} = \begin{pmatrix}
a_r & -a_i \\
a_i & a_r
\end{pmatrix}\begin{pmatrix}
x \\
y
\end{pmatrix}.
\]

If we write the constant \( a \) in the exponential form:

\[
a \equiv (a_r + ia_i) = \exp \rho (\cos \phi + i \sin \phi), \quad \rho = \ln \sqrt{a_r^2 + a_i^2}, \quad \phi = \tan^{-1} a_i/a_r,
\]

we see that the constant \( a \) plays the role of an operator representing an orthogonal axis rotation with a homogeneous dilatation (homothety). If \( \rho = 0 \), and if we add another constant \( b = b_r + ib_i \), then \( z_1 = az+b \) gives the permissible vector transformations in a Cartesian plane. In group language this is the Euclidean group of roto-translations which depends on the three parameter \( \phi, \ b_r, \ b_i \). Then we can use complex numbers to describe plane vector algebra because the vectors are, usually, represented in an orthogonal coordinate system and the additive and multiplicative groups of complex numbers are related to Euclidean group. In mathematical physics, even more important is the conformal group deriving from functions of the complex variable.

According to Riemann \([4]\): a function \( w(z) = u(x,y) + iv(x,y) \) is said to be a function of the
complex variable \( z \) if its derivative is independent of direction. If this condition is verified, the partial derivatives of \( u, v \), satisfy the Cauchy-Riemann’s (C-R) equations:

\[
\begin{align*}
    u_x &= v_y; \\
    u_y &= -v_x
\end{align*}
\]  

where the comma stands for derivation with respect to the variables which follow. Thanks to eq. (9), \( u \) and \( v \) satisfy the Laplace equation \( U_{xx} + U_{yy} = 0 \), which is invariant under the transformations \( x, y \Rightarrow u, v \), where \( u, v \) are given by the real and imaginary part of the same function of complex variable \( z \). In the language of the classical Lie groups \([2, 5]\), the Euclidean group is said to be a finite group because it depends on three parameters, whilst the conformal group is said to be infinite because it depends on arbitrary functions. The Euclidean group is very important in itself. Moreover, if considered as addition and multiplication of complex constants and variables, it represents the simplest subgroup of the conformal group. Both groups derive from the symmetries related to the “operator” “\( i \)” of the complex variable.

A connection between these two groups can be found if one looks for a field that satisfies the Laplace equation and that is invariant for the rotation group. Having set these requirements the problem is equivalent to calculate the potential of a central field, that is a function only of the source distance. This problem is usually solved \([1, \text{p.341}]\) by means of a polar coordinate transformation \( (x, y \Rightarrow \rho, \phi) \), which transform the Laplace equation in: \( u_{\rho\rho} + \rho^{-1}u_{\rho} + \rho^{-2}u_{\phi\phi} = 0 \).

The use of a complex exponential transformation, that has the same symmetries of the polar one, leaves the Laplace equation invariant; then with the transformation:

\[
    x = \exp X \cos Y, \quad y = \exp X \sin Y \quad \text{or} \quad X = \ln \sqrt{x^2 + y^2}, \quad Y = \tan^{-1}(y/x)
\]

we must solve the equation:

\[
    U_{,XX} + U_{,YY} = 0 \quad (10)
\]

The \( U \) invariance for rotation means independence on the rotation angle, represented here by \( Y \). Therefore \( U_{,YY} \equiv 0 \), and \( U \) will depend only on the variable \( X \). The partial differential equation, eq. (10), becomes a normal differential equation \( d^2U/dX^2 = 0 \), with the elementary solution:

\[
    U = aX + b \equiv a \ln \sqrt{x^2 + y^2} + b
\]

which represents the potential of a point charge. Then in the \( X, Y \) plane the straight lines \( X = \text{const.} \) give the equipotential, and, as is better known, the circle \( x^2 + y^2 = \text{const.} \) are the equipotential in the \( x, y \) plane. We can note that from the “symmetry” of the finite rotation group, we have obtained the Green function for the partial differential Laplace equation.

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