Spin Foams for the Real, Complex Orthogonal Groups in 4D and the bivector scalar product reality constraint.

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March 24, 2022

Abstract

The Barrett-Crane model for the $SO(4, C)$ general relativity is systematically derived. This procedure makes rigorous the calculation of the Barrett-Crane intertwiners from the Barrett-Crane constraints of both real and complex Riemannian general relativity. The reality of the scalar products of the complex bivectors associated with the triangles of a flat four simplex is equivalent to the reality of the associated flat geometry. Spin foam models in 4D for the real and complex orthogonal gauge groups are discussed in a unified manner from the point of view of the bivector scalar product reality constraints. Many relevant issues are discussed and generalizations of the ideas are introduced. The asymptotic limit of the $SO(4, C)$ general relativity is discussed. The asymptotic limit is controlled by the $SO(4, C)$ Regge calculus which unifies the Regge calculus theories for all the real general relativity cases. The spin network functionals for the 3+1 formulation of the spin foams are discussed. The field theory over group formulation for the Barrett-Crane models is discussed briefly. I introduce the idea of a mixed Lorentzian Barrett-Crane model which mixes the intertwiners for the Lorentzian Barrett-Crane models. A mixed propagator is calculated. I also introduce a multi-signature spin foam model for real general relativity which is made by splicing together the four simplex amplitudes for the various signatures is defined. Further research that is to be done is listed and discussed.

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1 Introduction

The idea of spin foams [1, 2] briefly reviewed in this section is a proposal for background independent, non-perturbative and coordinate independent quantum general relativity\(^1\). Spin foams are essentially the path integral quantizations of general relativity and related theories on simplicial manifolds [1, 11, 17, 16]. Spin foams have various advantages. They are simply combinatorics and so they do not require a back-ground space-time to exist. Otherwise, a spin foam is a ‘thing-in-itself’ and so an ultimate object in terms of which reality could be understood. Spin foam models are connected to classical general relativity through Regge calculus [17]. Spin foam quantization is similar to lattice gauge\(^2\).

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\(^1\) refer the novice readers to the latest book by Rovelli [3] on background independent quantum gravity for startup.
theory. The later has been found useful in many issues in QCD \[8\]. A variation of Regge calculus called the dynamical triangulation has shown promising semiclassical limit \[9\].

1.1 Review of Spin Foams

Spin foams are dynamical generalization of the idea of spin networks \[10\] to higher dimensional space \[1\]. The essence of spin networks and spin foams is gauge invariance. An abstract closed spin foam in \(N\) dimensions for a group \(G\) is based on the following constructions\(^4\) (in simple terms):

- Consider a \(N\) dimensional closed oriented manifold triangulated using \(N\) dimensional simplices.
- With each \((N - 2)\)-simplex \(b\) associate an irreducible representation \(\rho_b\) of the group \(G\).
- With each \((N - 1)\)-simplex \(e\) associate an intertwiner \(i_e\) of the group that intertwines the representations associated with its \(N - 2\) simplices.

Spin foams are usually defined using a partition function. A typical definition of a spin foam partition function is

\[
Z(\Delta) = \sum_{\{\rho_b, i_e\}} A_{N-2}(\rho_b) A_{N-1}(\rho_b, i_e) A_N(\rho_b, i_e),
\]

where the \(\Delta\) denotes a triangulation. The sum is over all possible intertwiners and representations associated with the simplices. The \(A\)’s are the quantum amplitudes associated with the simplices of the various dimensions. The \(A_N(\rho_b, i_e)\) is the amplitude associated with a \(N\)-simplex as a function of the intertwiners and the representations associated with its lower dimensional simplices. Usually this amplitude is given by a spin network built using the intertwiners, the representations and the dual graph to the triangulation of the \(N\)-simplex. The \(A_{N-1}(\rho_b, i_e)\) is the amplitude of a \((N - 1)\)-simplex as a function of the intertwiner associated with it and the representations associated with its \((N - 2)\)-simplices. The \(A_{N-2}(\rho_b)\) is the amplitude of a \((N - 2)\)-simplex as a function of the representation associated with it.

The physics related to a spin foam is contained in the definition of its amplitudes. There are various possible spin foam models available based on the various definitions for the amplitudes \[9, 1\]. The amplitudes associated with these models can be derived by the path integral quantizations of discretized actions \[1, 5, 6, 4\].

An important class of spin foam models are those of the topological field theory called the BF theory\(^3\) \[11\]. A spin foam model for the four dimensional

\(^3\)A BF theory in \(n\) dimensions and for a group \(G\) refers to a field theory defined by the action \(S = \int B \wedge F\). Here the \(B\) is a \(n - 2\) form which takes values in the dual Lie algebra of \(G\). The \(F\) is a 2-form is the Cartan curvature of a \(G\)-connection \(A\). The free variables of the theory are the \(B\) and \(A\).

\(^4\)Please refer to Baez \[1\] for a more technical definition.
\( SO(4, R) \) BF theory was derived directly from its discretized action on simplicial manifolds by Ooguri [4]. The quantization of the discrete Riemannian BF theory results in a partition function which does not depend on the discretization of the four manifold [15]. This quantum model has only global degrees of freedom like the classical BF theory [1].

In case of the four dimensional Riemannian general relativity Barrett and Crane [12] proposed a systematic way to assign an amplitude to the four-simplex. They proposed a set of quantum constraints based on the classical properties of the bivectors associated with the triangles of a flat four-simplex. They also proposed a solution for these constraints [12]. The spin foam models constructed using the Barrett-Crane procedure are called the Barrett-Crane models.

The Barrett-Crane model can be considered as the quantization of discretized Plebanski formalism of general relativity [14] on a simplicial manifold. The Plebanski theory of general relativity is simply a four dimensional BF theory combined with certain constraint called the Plebanski constraint. This constraint enforces the \( B \) field to be a wedge product of a co-tetrad field with itself [14]. The co-tetrad field contains the metric information [14]. The Barrett-Crane constraints contain the information about the Plebanski constraint.

The Riemannian Barrett-Crane model can be formally derived starting from a discretized action on a simplicial four manifold [5]. It can be obtained by deriving the Ooguri model and imposing the Barrett-Crane constraints on it. Imposition of the Barrett-Crane constraints breaks the topological nature of the Ooguri model and the discretization independence of the theory. So the theory now acquires local excitations [2].

It is possible to rewrite the Riemannian Barrett-Crane four-simplex amplitude of a four-simplex in terms of certain propagators on the homogenous space \( S^3 = SO(4)/SO(3) \). Spin foam model of the Lorentzian general relativity were proposed by Barrett and Crane [16]. These models were constructed based on certain propagators on the homogenous spaces of the Lorentz group corresponding to the various subgroups of it in the Minkowski space, viz.

- The upper sheet of the double sheet hyperboloid: \( H^+ \approx SL(2, C)/SU(2) \)
- The single sheet hyperboloid with the antipodal points considered as a single point: \( H^- \approx SL(2, C)/U^(-) \) where \( U^(-) = SU(1, 1) \otimes \mathbb{Z}_2 \).
- The upper sheet of the null cone: \( N = SL(2, C)/E(2) \)

Rovelli and Perez proposed a way of deriving the first two models using the field theory over group formulation [17], [18].

1.2 Motivation for this article

There are two sets of issues at hand. The first set of issues relate to the construction of spin foam models starting from general physical and mathematical premises. Some of the issues involved here are:
1. How to understand the different spin foam models of general relativity from a general point of view?

2. Are there other models that exist for Lorentzian general relativity?

3. Even though the Barrett-Crane constraints appear to have solutions, it is not clear how to directly impose one of the constraints called the cross-simplicity constraint\(^4\).

4. The uniqueness of the Barrett-Crane model for the Riemannian general relativity has been argued by Reisenberger [19]. How to do this for the case of Lorentzian general relativity?

5. Is it possible to develop a unified understanding of the Barrett-Crane models for the various signatures and the \(SO(4,C)\) general relativity in four dimensions?

6. How to relate the ideas in the spin foams to canonical quantum general relativity and vice versa. For example, whether the reality condition of canonical quantum general relativity has any interpretation in spin foams?

The second set of issues is about how to extract physics from spin foam models. The two sets of issues are interlinked. This article is motivated by the first set of issues. The second set of issues is discussed as future research directions at the end of this article.

An attempt by me to rigorously develop and unify the various models for the Lorentzian general relativity was made in Ref. [20]. The attempt was made to derive the two models by directly solving the Barrett-Crane constraints. The Barrett-Crane cross-simplicity constraint operator was explicitly written using the Gelfand-Naimarck representation theory. But after numerous attempts I could not obtain any solution for the constraint. But the efforts in this research lead to the idea of the reality for spin foam models. It also led to the systematic quantization method for the Barrett Crane models of complex and real Riemannian general relativity.

Let us consider the Lorentzian Barrett-Crane models now. The Hilbert space of the unitary representations of the Lorentz group \(SL(2,C)\) is infinite dimensional [21]. A unitary representation of \(SL(2,C)\) is labeled by a complex number \(\chi = \frac{\rho}{2} + i \rho\) where \(\rho\) is a real number and \(n\) an integer. The idea of simplicity requires \(\rho n = 0\) [16]. Thus we are allowed to assign only one of either \(\chi = \rho\) or \(\chi = i \frac{n}{2}\) to each triangles. Now consider the eigen-values of the Casimir of \(SL(2,C)\) in the complex form [21],

\[
\chi^2 - 1 = -\rho^2 + \frac{n^2}{4} + i \rho n.
\]

\(^4\)It is known that the cross-simplicity condition implies that the internal representations of the Barrett-Crane intertwiners must be simple [12]. But the difficult part is the simultaneous imposition of all the cross-simplicity constraints on a general intertwiner in four dimensions.
The $\rho n$ is precisely the imaginary part of the Casimir. So if $\chi^2 - 1$ is interpreted as the square of the area of a triangle, then $\rho n = 0$ simply constrains the square of the area to be real. The situation is further clarified if I start from the $SO(4, C)$ general relativity theory as will be explained below.

The $SO(4, C)$ Barrett-Crane model need to be constructed based on unitary representation theory of the group $SO(4, C)$. The unitary representations of $SO(4, C)$ can be constructed using the relation

$$SO(4, C) \approx SL(2, C) \times SL(2, C) \mathbb{Z}^2.$$  \hspace{1cm} (1)

This is the complex analog of

$$SO(4, R) \approx SU(2, C) \times SU(2, C) \mathbb{Z}^2.$$  

So similar to the unitary representation theory of the Riemannian group, the unitary representations of $SO(4, C)$ can be labeled by two $\chi$'s: $(\chi_L = \rho_L + i n_L/2, \chi_R = \rho_R + i n_R/2)$, where each $\chi$ represents a unitary representation of $SL(2, C)$ \textsuperscript{[21]}, $n_L + n_R$ even number (Please see appendix B for details).

The $SO(4, C)$ Barrett-Crane simplicity constraint sets one of the $SO(4, C)$ Casimir’s eigen values $(\chi_L^2 - \chi_R^2)/2 = 0$, which in turn sets $\chi_L = \pm \chi_R$ (= $\chi$ say). Then the other Casimir’s eigen value is

$$\left(\chi_L^2 + \chi_R^2 - 2\right)/2 = \chi^2 - 1,$$

which corresponds to the square of area. By setting this eigenvalue to be real, we deduce the area quantum number that is to be assigned to a triangle of a Lorentzian spin foam. So from the point view of the $SO(4, C)$ Barrett-Crane model the simplicity condition of the Lorentzian general relativity is simply a reality condition.

The reality of the squares of areas can be imposed at the continuum classical level by imposing the condition that the area metric to be real. Since the area metric can be expressed as a function of a bivector field, this reality constraint can be naturally combined with the Plebanski theory for the $SO(4, C)$ general relativity. I have done this analysis in Ref: \textsuperscript{[27]}. There, I have shown that the area metric metric reality condition reduces a complex metric to a real or imaginary metric. An imaginary metric essentially describes a real geometry. I also have shown there that one can derive real general relativity by adding a Lagrange multiplier to the $SO(4, C)$ Plebanski action to impose the area metric reality constraint.

The idea of a Barrett-Crane intertwiner can be easily formalized. Then as will be discussed in this article the models for real general relativity theories for all signatures are related to that of the $SO(4, C)$ general relativity through the quantum version of the discretized area metric reality condition\textsuperscript{5}. In this

\textsuperscript{5}The Barrett-Crane model based on the propagators on the null-cone \textsuperscript{[10]} is an exception to this.
way we have a unified understanding of the Barrett-Crane models for the four dimensional real general relativity theories for all signatures (non-degenerate) and the $SO(4, C)$ general relativity. The discrete equivalent of the area metric reality condition in the context of Barrett-Crane theory is that the scalar products of bivectors associated with the triangles of a four- or three-simplex be real \[27\].

One of the new elements in the systematic derivation the $SO(4, C)$ Barrett-Crane model in this article is the rigorous imposition of the Barrett-Crane cross-simplicity constraint on the intertwiners initially defined as a function of many variables on the complex three sphere. This procedure is directly applicable to the Riemannian Barrett-Crane model. Also I calculate the asymptotic limit of the $SO(4, C)$ Barrett-Crane models and extract the bivectors that satisfy all the Barrett-Crane Constraints excluding the non-deneracy conditions.

In this article I discuss many ideas relating to the spin foams of the $SO(4, C)$ and real general relativity listed in the layout below. This article makes rigorous, unifies and generalizes the Barrett-Crane spin foam models of general relativity.

1.3 Article Layout

- Section One: I discuss the spin foam model for the $SO(4, C)$ BF theory based on Ooguri’s research \[4\].

- Section two: I briefly discuss the continuum $SO(4, C)$ model \[27\]. I call all the Barrett-Crane constraints excluding the non-degeneracy conditions as the essential conditions. I call the Barrett-Crane models obtained by quantizing these conditions as the essential Barrett-Crane models. I develop the essential $SO(4, C)$ Barrett-Crane model by solving the corresponding essential Barrett-Crane constraints. I explicitly solve the Barrett-Crane cross-simplicity constraint on the function. I describe the various properties of the propagators.

- In section three using the bivector scalar product reality constraint the Barrett-Crane models for the real general relativity for all signatures and $SO(4, C)$ general relativity are discussed in a unified manner.

- In section four I discuss various further developments.
  - I discuss the asymptotic limit \[32\] of the $SO(4, C)$ Barrett-Crane model.
  - To relate the canonical quantum general relativity to the spin foams I developed a $(N - 1) + 1$ model of the spin foams \[33\]. In this model the quantum partitions for general relativity and the BF theory can be written down as the sum over amplitudes for histories of spin networks functionals. This theory can be formally applied to various Barrett-Crane models.
– I also briefly discuss the field theory \cite{28}, \cite{29} over group version of the \(SO(4, C)\) Barrett-Crane model.

– I introduce two possible new quantum real general relativity models. One of them is a Lorentzian Barrett-Crane model and the other one is a multi-signature real spin foam model.

• In section five I briefly list the various new results in this article. I also list and discuss the possible future works need to be done.

2 Spin foam of the \(SO(4, C)\) BF model

Consider a four dimensional submanifold \(M\). Let \(A\) be a \(SO(4, C)\) connection 1-form and \(B_{ij}\) a complex bivector valued 2-form on \(M\). Let \(F\) be the curvature 2-form of the connection \(A\). Then I define a real continuum BF theory action, 

\[
S_{BF}(A, B_{ij}, \bar{A}, \bar{B}_{ij}) = \text{Re} \int_M B \wedge F, \tag{2}
\]

where \(A, B_{ij}\) and their complex conjugates are considered as independent free variables. This classical theory is a topological field theory. This property also holds on spin foam quantization as will be discussed below.

The Spin foam model for the \(SO(4, C)\) BF theory action can be derived from the discretized BF action by using the path integral quantization as illustrated in Ref:\cite{4} for compact groups. Let \(\Delta\) be a simplicial manifold obtained by a triangulation of \(M\). Let \(g_e \in SO(4, C)\) be the parallel propagators associated with the edges (three-simplices) representing the discretized connection. Let \(H_b = \prod_{c \supset b} g_c\) be the holonomies around the bones (two-simplices) in the four dimensional matrix representation of \(SO(4, C)\) representing the curvature. Let \(B_b\) be the \(4 \times 4\) antisymmetric complex matrices corresponding to the dual Lie algebra of \(SO(4, C)\) corresponding to the discrete analog of the \(B\) field. Then the discrete BF action is 

\[
S_d = \text{Re} \sum_{b \in M} \text{tr}(B_b \text{ln} H_b),
\]

which is considered as a function of the \(B_b\)’s and \(g_e\)’s. Here \(B_b\) the discrete analog of the \(B\) field are \(4 \times 4\) antisymmetric complex matrices corresponding to dual Lie algebra of \(SO(4, C)\). The \(\text{ln}\) maps from the group space to the Lie algebra space. The trace is taken over the Lie algebra indices. Then the quantum partition function can be calculated using the path integral formulation as,

\[
Z_{BF}(\Delta) = \int \prod_b dB_b d\bar{B}_b \exp(\text{i}S_d) \prod_c dg_c
\]

\[
= \int \prod_b \delta(H_b) \prod_c dg_c, \tag{3}
\]
where \( dg_e \) is the invariant measure on the group \( SO(4, C) \). The invariant measure can be defined as the product of the bi-invariant measures on the left and the right \( SL(2, C) \) matrix components. Please see appendix A and B for more details. Similar to the integral measure on the \( B \)'s an explicit expression for the \( dg_e \) involves product of conjugate measures of complex coordinates.

Now consider the identity

\[
\delta(g) = \frac{1}{64\pi^8} \int d\omega \text{tr}(T_\omega(g))d\omega, \tag{4}
\]

where the \( T_\omega(g) \) is a unitary representation of \( SO(4, C) \), where \( \omega = (\chi_L, \chi_R) \) such that \( n_L + n_R \) is even, \( d\omega = |\chi_L\chi_R|^2 \). The details of the representation theory is discussed in appendix B. The integration with respect to \( d\omega \) in the above equation is interpreted as the summation over the discrete \( n \)'s and the integration over the continuous \( \rho \)'s.

By substituting the harmonic expansion for \( \delta(g) \) into the equation (3) we can derive the spin foam partition of the \( SO(4, C) \) BF theory as explained in Ref: [1] or Ref: [4]. The partition function is defined using the \( SO(4, C) \) intertwiners and the \( \{15\omega\} \) symbols.

The relevant intertwiner for the BF spin foam is

The nodes where the three links meet are the Clebsch-Gordan coefficients of \( SO(4, C) \). The Clebsch-Gordan coefficients of \( SO(4, C) \) are just the product of the Clebsch-Gordan coefficients of the left and the right handed \( SL(2, C) \) components. The Clebsch-Gordan coefficients of \( SL(2, C) \) are discussed in the references [21] and [36].

The quantum amplitude associated with each simplex \( s \) is given below and can be referred to as the \( \{15\omega\} \) symbol,

The final partition function is

\[
Z_{BF}(\Delta) = \int_{\{\omega_b, \omega_e\}} \prod_b \frac{d\omega_b}{64\pi^8} \prod_s Z_{BF}(s) \prod_b \frac{d\omega_b}{64\pi^8} \prod_e d\omega_e, \tag{5}
\]
where the $Z_{BF}(s) = \{15\omega\}$ is the amplitude for a four-simplex $s$. The $d_{\omega_b} = |\chi_{LX}}|^2$ term is the quantum amplitude associated with the bone $b$. Here $\omega_c$ is the internal representation used to define the intertwiners. Usually $\omega_c$ is replaced by $i_\epsilon$ to indicate the intertwiner. The set $\{\omega_b, \omega_c\}$ of all $\omega_b$’s and $\omega_c$’s is usually called a coloring of the bones and the edges. This partition function may not be finite in general.

It is well known that the BF theories are topological field theories. A priori one cannot expect this to be true for the case of the BF spin foam models because of the discretization of the BF action. For the spin foam models of the BF theories for compact groups, it has been shown that the partition functions are triangulation independent up to a factor $[15]$. This analysis is purely based on spin foam diagrammatics and is independent of the group used as long the BF spin foam is defined formally by equation (3) and the harmonic expansion in equation (4) is formally valid. So one can apply the spin foam diagrammatics analysis directly to the $SO(4, C)$ BF spin foam and write down the triangulation independent partition function as

$$Z'_{BF}(\Delta) = \tau^{n_4-n_3}Z_{BF}(\Delta)$$

using the result from $[15]$. In the above equation $n_4, n_3$ is number of four bubbles and three bubbles in the triangulation $\Delta$ and

$$\tau = \delta_{SO(4, C)}(I)$$

$$= \frac{1}{64\pi^8} \int d_\omega^2 d\omega.$$ 

The above integral is divergent and so the partition functions need not be finite. The normalized partition function is to be considered as the proper partition function because the BF theory is supposed to be topological and so triangulation independent.

3 The $SO(4, C)$ Barrett-Crane Model

3.1 Classical $SO(4, C)$ General Relativity

Consider a four dimensional manifold $M$. Let $A$ be a $SO(4, C)$ connection 1-form and $B_{ij}$ be a complex bivector valued 2-form on $M$. I would like to restrict myself to the non-degenerate general relativity in this section by assuming $b = \frac{1}{2}e^{abcd}B_{ab} \wedge B_{cd} \neq 0$. The Plebanski action for the $SO(4, C)$ general relativity is obtained by adding a Lagrange multiplier term to impose the Plebanski constraint to the BF theory action given in equation (2). A simple way of writing the action $[22]$ is

$$S_C(A, B_{ij}, \bar{A}, \bar{B}_{ij}, \phi) = \text{Re} \left[ \int_M tr(B \wedge F) + \frac{b}{2}e^{abcd}B_{ab} \wedge B_{cd} \right], \quad (6)$$
where $\phi$ is a complex tensor with the symmetries of the Riemann curvature tensor such that $\phi^{abcd} e_{abcd} = 0$. The field equations corresponding to the extrema of the above action has been discussed by me in [27]. Two important results are

- The Plebanski constraint imposes the condition $B^{ij}_{ab} = \theta^i_a \theta^j_b$ where $\theta^i_a$ is a complex tetrad field [14], [27].
- The field equations correspond to the $SO(4, C)$ general relativity on the manifold $M$ [27].

3.1.1 Relation to Complex Geometry

Let $M$ be a real analytic manifold. Let $M_c$ be the complex analytic manifold which is obtained by analytically continuing the real coordinates on $M$. The analytical continuation of the field equations and their solutions on $M$ to complex $M_c$ can be used to define complex general relativity. Conversely, the field equations of complex general relativity or their solutions on $M_c$ when restricted to $M$ defines the $SO(4, C)$ general relativity. Because of these properties the action $S$ can also be considered as an action for complex general relativity.

Now consider the relation between the complex general relativity on $M_c$ and the $SO(4, C)$ general relativity on $M$. This relation critically depends on $M$ being a real analytic manifold. It also depends on the fields on it being analytic on some region may be except for some singularities. If the fields and the field equations are discretized we lose the relation to complex general relativity. Thus it is also not meaningful to relate a $SO(4, C)$ Barrett-Crane Model to complex general relativity. If the $SO(4, C)$ Barrett-Crane model has a semiclassical continuum general relativity limit then a relation to complex general relativity may be recovered.

3.2 The $SO(4, C)$ Barrett-Crane Constraints

My goal here is to systematically construct the Barrett-Crane model of the $SO(4, C)$ general relativity. In the previous section I discussed the $SO(4, C)$ BF spin foam model. The basic elements of the BF spin foams are spin networks built on graphs dual to the triangulations of the four simplices with arbitrary intertwiners and the principal unitary representations of $SO(4, C)$ discussed in appendix B. These closed spin networks can be considered as quantum states of four simplices in the BF theory and the essence of these spin networks is mainly gauge invariance. To construct a spin foam model of general relativity these spin networks need to be modified to include the Plebanski Constraints in the discrete form.

A quantization of a four-simplex for the Riemannian general relativity was proposed by Barrett and Crane [12]. The bivectors $B_i$ associated with the ten triangles of a four-simplex in a flat Riemannian space satisfy the following properties called the Barrett-Crane constraints$^6$.

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$^6$I would like to refer the readers to the original paper [12] for more details.
1. The bivector changes sign if the orientation of the triangle is changed.

2. Each bivector is simple.

3. If two triangles share a common edge, then the sum of the bivectors is also simple.

4. The sum of the bivectors corresponding to the edges of any tetrahedron is zero. This sum is calculated taking into account the orientations of the bivectors with respect to the tetrahedron.

5. The six bivectors of a four-simplex sharing the same vertex are linearly independent.

6. The volume of a tetrahedron calculated from the bivectors is real and non-zero.

The items two and three can be summarized as follows:

\[ B_i \wedge B_j = 0 \quad \forall i, j, \]

where \( A \wedge B = \varepsilon_{IJKL} A^{I}{}^{J} B^{K}{}^{L} \) and the \( i, j \) represents the triangles of a tetrahedron. If \( i = j \), it is referred to as the simplicity constraint. If \( i \neq j \) it is referred as the cross-simplicity constraints.

Barrett and Crane have shown that these constraints are sufficient to restrict a general set of ten bivectors \( E_b \) so that they correspond to the triangles of a geometric four-simplex up to translations and rotations in a four dimensional flat Riemannian space.

The Barrett-Crane constraints theory can be trivially extended to the \( SO(4, C) \) general relativity. In this case the bivectors are complex and so the volume calculated for the sixth constraint is complex. So we need to relax the condition of the reality of the volume.

A quantum four-simplex for Riemannian general relativity is defined by quantizing the Barrett-Crane constraints [12]. The bivectors \( B_i \) are promoted to the Lie operators \( \hat{B}_i \) on the representation space of the relevant group and the Barrett-Crane constraints are imposed at the quantum level. A four-simplex has been quantized and studied in the case of the Riemannian general relativity before [12]. All the first four constraints have been rigorously implemented in this case. The last two constraints are inequalities and they are difficult to impose. This could be related to the fact that the Riemannian Barrett-Crane model reveal the presence of degenerate sectors [31], [34] in the asymptotic limit [30] of the model. For these reasons here after I would like to refer to a spin foam model that satisfies only the first four constraints as an essential Barrett-Crane model, While a spin foam model that satisfies all the six constraints as a rigorous Barrett-Crane model.

Here I would like to derive the essential \( SO(4, C) \) Barrett-Crane model. For this one must deal with complex bivectors instead of real bivectors. The procedure that I would like to use to solve the constraints can be carried over
directly to the Riemannian Barrett-Crane model. This derivation essentially makes the derivation of the Barrett-Crane intertwiners for the real and the complex Riemannian general relativity more rigorous.

3.2.1 The Simplicity Constraint

The group \(SO(4, C)\) is locally isomorphic to \(\frac{SL(2, C) \times SL(2, C)}{Z_2}\). An element \(B\) of the Lie algebra space of \(SO(4, C)\) can be split into the left and the right handed \(SL(2, C)\) components,

\[
B = B_L + B_R.
\]

(7)

There are two Casimir operators for \(SO(4, C)\) which are \(\varepsilon_{IJKL} B^{IJ} B^{KL}\) and \(\eta_{IK} \eta_{JL} B^{IJ} B^{KL}\), where \(\eta_{IK}\) is the flat Euclidean metric. In terms of the left and right handed split I can expand the Casimir operators as

\[
\varepsilon_{IJKL} B^{IJ} B^{KL} = B_L \cdot B_L - B_R \cdot B_R
\]

and

\[
\eta_{IK} \eta_{JL} B^{IJ} B^{KL} = B_L \cdot B_L + B_R \cdot B_R,
\]

where the dot products are the trace in the \(SL(2, C)\) Lie algebra coordinates.

The bivectors are to be quantized by promoting the Lie algebra vectors to Lie operators on the unitary representation space of \(SO(4, C)\) \(\approx \frac{SL(2, C) \times SL(2, C)}{Z_2}\). The relevant unitary representations of \(SO(4, C)\) \(\approx SL(2, C) \otimes SL(2, C)/Z_2\) are labeled by a pair \((\chi_L, \chi_R)\) such that \(n_L + n_R\) is even (appendix B). The elements of the representation space \(D_{\chi_L} \otimes D_{\chi_R}\) are the eigen states of the Casimirs and on them the operators reduce to the following:

\[
\varepsilon_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 - \chi_R^2}{2} \hat{I}
\]

and

\[
\eta_{IK} \eta_{JL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 + \chi_R^2 - 2}{2} \hat{I}.
\]

(8)

(9)

The equation (8) implies that on \(D_{\chi_L} \otimes D_{\chi_R}\) the simplicity constraint \(B \wedge B = 0\) is equivalent to the condition \(\chi_L = \pm \chi_R\). I would like to find a representation space on which the representations of \(SO(4, C)\) are restricted precisely by \(\chi_L = \pm \chi_R\). Since a \(\chi\) representation is equivalent to \(-\chi\) representations [21], \(\chi_L = + \chi_R\) case is equivalent to \(\chi_L = - \chi_R\) [21].

Consider a square integrable function \(f(x)\) on the complex sphere \(CS^3\) defined by

\[
x \cdot x = 1, \forall x \in C^4.
\]

It can be Fourier expanded in the representation matrices of \(SL(2, C)\) using the isomorphism \(CS^3 \simeq SL(2, C)\),

\[
f(x) = \frac{1}{8\pi^4} \int Tr(F(\chi) T_\chi(g(x)^{-1}) \chi \bar{\chi} d\chi,
\]

(10)
where the isomorphism $g:CS^3 \rightarrow SL(2, C)$ is defined in the appendix A. The group action of $g = (g_L, g_R) \in SO(4, C)$ on $x \in CS^3$ is given by

$$g(x) = g_L^{-1}g(x)g_R.$$ (11)

Using equation (11) I can consider the $T_\chi(g(x))(z_1, z_2)$ as the basis functions of $L^2$ functions on $CS^3$. The matrix elements of the action of $g$ on $CS^3$ is given by (appendix B)

$$\int T_\chi(g(x))(\hat{z}_1, \hat{z}_2)T_\chi(g(x))(z_1, z_2)dx = T_{-\chi}(g_L)(\hat{z}_1, z_1)T_\chi(g_R)(\hat{z}_2, z_2)\delta(\chi - \chi).$$

I see that the representation matrices are precisely those of $SO(4, C)$ only restricted by the constraint $\chi_L = -\chi_R \approx \chi_R$. So the simplicity constraint effectively reduces the Hilbert space $H$ to the space of $L^2$ functions on $CS^3$. In Ref. [5] the analogous result has been shown for $SO(N, R)$ where the Hilbert space is reduced to the space of the $L^2$ functions on $S^{N-1}$.

### 3.2.2 The Cross-simplicity Constraints

Next let me quantize the cross-simplicity constraint part of the Barrett-Crane constraint. Consider the quantum state space associated with a pair of triangles 1 and 2 of a tetrahedron. A general quantum state that just satisfies the simplicity constraints $B_1 \wedge B_1 = 0$ and $B_2 \wedge B_2 = 0$ is of the form $f(x_1, x_2) \in L^2(CS^3 \ast CS^3)$, $x_1, x_2 \in CS^3$.

On the elements of $L^2(CS^3 \ast CS^3)$ the action $B_1 \wedge B_2$ is equivalent to the action of $(B_1 + B_2) \wedge (B_1 + B_2)^\dagger$. This implies that the cross-simplicity constraint $B_1 \wedge B_2 = 0$ requires the simultaneous rotation of $x_1, x_2$ involve only the $\chi_L = \pm \chi_R$ representations. The simultaneous action of $g = (g_L, g_R)$ on the arguments of $f(x_1, x_2)$ is

$$gf(x_1, x_2) = f(g_L^{-1}x_1g_R, g_L^{-1}x_2g_R).$$ (12)

The harmonic expansion of $f(x_1, x_2)$ in terms of the basis function $T_\chi(g(x))(z_1, z_2)$ is

$$f(x_1, x_2) = F^{\hat{z}_1 \hat{z}_2}_{\hat{z}_1 \hat{z}_2 \chi_1 \chi_2} T^{\hat{z}_1}_{\hat{z}_1 \chi_1}(g(x_1))T^{\hat{z}_2}_{\hat{z}_2 \chi_2}(g(x_2)),$$

where I have assumed all the repeated indices are either integrated or summed over for equation only. The rest of the calculations can be understood graphically. The last equation can be graphically written as follows:

\[\left(\hat{B}_1 + \hat{B}_2\right) \wedge \left(\hat{B}_1 + \hat{B}_2\right) = \hat{B}_1 \wedge \hat{B}_1 + \hat{B}_2 \wedge \hat{B}_2 + 2\hat{B}_1 \wedge \hat{B}_2.\]

But since $\hat{B}_1 \wedge \hat{B}_1 = \hat{B}_2 \wedge \hat{B}_2 = 0$ on $f(x_1, x_2)$ we have

$$\left(\hat{B}_1 + \hat{B}_2\right) \wedge \left(\hat{B}_1 + \hat{B}_2\right) f(x_1, x_2) = \hat{B}_1 \wedge \hat{B}_2 f(x_1, x_2).$$
\[ f(x_1, x_2) = \iiint_{\chi_1 \chi_2} F \ dx_1 dx_2, \]

where the box \( F \) represents the tensor \( F^{z_1 z_2}_{\chi_1 \chi_2} \). The action of \( g \in SO(4, C) \) on \( f \) is

\[ gf(x_1, x_2) = \iiint_{\chi_1 \chi_2} F \ dx_1 dx_2. \quad (13) \]

Now for any \( h \in SL(2, C) \),

\[ T_{a_1 \chi_1}^{b_1}(h) T_{a_2 \chi_2}^{b_2}(h) = C_{a_1 a_2 \chi_3}^{b_1 b_2 \chi_3} \tilde{C} \chi_1 \chi_2 \chi_3 T_{a_3 \chi_3}^{b_3}(h), \]

where \( C \)'s are the Clebsch-Gordan coefficients of \( SL(2, C) \) \[21], \[36\]. I have assumed all the repeated indices are either integrated or summed over for the previous and the next two equations. Using this I can rewrite the \( g_L \) and \( g_R \) parts of the result \[13\] as follows:

\[ T_{a_1 \chi_1}^{z_1}(g_L^{-1}) T_{a_2 \chi_2}^{z_2}(g_L^{-1}) = C_{z_1 z_2 \chi_3}^{z_1 \chi_1 \chi_3} \tilde{C} \chi_1 \chi_2 \chi_3 T_{a_3 \chi_3}^{z_3}(g_L^{-1}) \quad (14) \]

and

\[ T_{z_1 \chi_1}^{b_1}(g_R) T_{z_2 \chi_2}^{b_2}(g_R) = C_{z_1 z_2 \chi_3}^{b_1 \chi_1 \chi_3} \tilde{C} \chi_1 \chi_2 \chi_3 T_{z_3 \chi_3}^{b_3}(g_R). \quad (15) \]

Now we have

\[ gf(x_1, x_2) = \int \cdots \int_{\chi_1 \chi_2 \chi_L \chi_R} F \ dx_1 dx_2. \]

To satisfy the cross-simplicity constraint the expansion of \( gf(x_1, x_2) \) must have contribution only from the terms with \( \chi_L = \pm \chi_R \). In the expansion in equation \[14\] and equation \[15\] in the right hand side the terms are defined only up to a sign of \( \chi_L \) and \( \chi_R \). Let me remove all the terms which does not

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8 I derived this equation explicitly in the appendix of Ref: 20.
9 Please see appendix A for the explanation.
satisfy \( \chi_L = \pm \chi_R \) (say \( \pm \chi \)). Also let me set \( g = I \). Now we can deduce that the functions denoted by \( \tilde{f}(x_1, x_2) \) obtained by reducing \( f(x_1, x_2) \) using the cross-simplicity constraints must have the expansion \(^{10}\):

\[
f(x_1, x_2) = 2 \int \int \int_{\chi \chi \chi} c_{\chi} \chi d\chi_1 d\chi_2 d\chi,
\]

where the coefficients \( c_{\chi} \) are arbitrary. Now the Clebsch-Gordan coefficient terms in the expansion can be re-expressed using the following equation:

\[
C_{\chi \chi \chi}^{\chi \chi \chi} \tilde{C}_{\chi \chi \chi}^{\chi \chi \chi} = \frac{8\pi^4}{\lambda \chi} \int_{SL(2, C)} T_1^{\chi_1 \chi} (h) T_2^{\chi_2 \chi} (h) \tilde{T}_3^{\chi_3 \chi} (h) dh, \quad (16)
\]

where \( h, \tilde{h} \in SL(2, C) \) and \( dh \) the bi-invariant measure on \( SL(2, C) \). Using this in the middle two Clebsch-Gordan coefficients of \( \tilde{f}(x_1, x_2) \) we get

\[
\tilde{f}(x_1, x_2) = 2 \int \int \int_{SL(2, C)} \frac{8\pi^4 c_{\chi}}{\lambda \chi} \chi \tilde{\chi} \tilde{\chi} \tilde{\chi} d\chi_1 d\chi_2 d\chi.
\]

This result can be rewritten for clarity as

\[
\tilde{f}(x_1, x_2) = 2 \int \int \int_{SL(2, C)} \frac{8\pi^4 c_{\chi}}{\lambda \chi} \chi \tilde{\chi} \tilde{\chi} \tilde{\chi} d\chi_1 d\chi_2 d\chi.
\]

Once again applying equation (16) to the remaining two Clebsch-Gordan coefficients we get,

\[
\tilde{f}(x_1, x_2) = 2 \int \int \int_{SL(2, C) \times SL(2, C)} \frac{8\pi^4}{\lambda \chi} \chi \tilde{\chi} \tilde{\chi} \tilde{\chi} d\chi d\tilde{\chi} d\chi_1 d\chi_2 d\chi.
\]

\(^{10}\)The factor of 2 has been introduced to include the terms with \( \chi_L = -\chi_R \).
By rewriting the above expression, I deduce that a general function \( \hat{f}(x_1, x_2) \) that satisfies the cross-simplicity constraint must be of the form,

\[
\hat{f}(x_1, x_2) = \int \int_{SL(2, C)} c_{\chi_{x_1x_2}} F_{\chi_{x_1x_2}}(h) \text{tr}(T_{\chi_{x_1}}(g(x_1)h)\text{tr}(T_{\chi_{x_2}}(g(x_2)h)dhd\chi_{x_1}d\chi_{x_2},
\]

where \( F_{\chi_{x_1x_2}}(h) \) is arbitrary. Then if \( \Psi(x_1, x_2, x_3, x_4) \) is the quantum state of a tetrahedron that satisfies all of the simplicity constraints and the cross-simplicity constraints, it must be of the form,

\[
\Psi(x_1, x_2, x_3, x_4) = \int F_{\chi_{x_1x_2x_3x_4}}(h)\text{tr}(T_{\chi_{x_1}}(g(x_1)h)\text{tr}(T_{\chi_{x_2}}(g(x_2)h)\text{tr}(T_{\chi_{x_3}}(g(x_3)h)\text{tr}(T_{\chi_{x_4}}(g(x_4)h)dh \prod_i d\chi_i.
\]

This general form is deduced by requiring that for every pair of variables with the other two fixed, the function must be the form of the right hand side of equation (17).

### 3.2.3 The SO(4, C) Barrett-Crane Intertwiner

Now the quantization of the fourth Barrett-Crane constraint demands that \( \Psi \) is invariant under the simultaneous complex rotation of its variables. This is achieved if \( F_{\chi_{x_1x_2x_3x_4}}(h) \) is constant function of \( h \). Therefore the quantum state of a tetrahedron is spanned by

\[
\Psi(x_1, x_2, x_3, x_4) = \int_{n \in CS^3} \prod_i T_{\chi_i}(g(x_i)g(n))dn,
\]

where the measure \( dn \) on \( CS^3 \) is derived from the bi-invariant measure on \( SL(2, C) \). I would like to refer to the functions \( T_{\chi_i}(g(x_i) \) as the \( T-\)functions here after.

**Alternative forms** The quantum state can be diagrammatically represented as follows:

\[
\Psi(x_1, x_2, x_3, x_4) = \int \chi \cdot n \cdot \chi \cdot n \cdot d\chi \cdot d\chi.
\]
A unitary representation $T_\chi$ of $SL(2,C)$ can be considered as an element of $D_\chi \otimes D_\chi^*$ where $D_\chi^*$ is the dual representation of $D_\chi$. So using this the Barrett-Crane intertwiner can be written as an element $|\Psi\rangle \in \bigotimes_i D_\chi_i \otimes D_\chi_i^*$, as follows:

$$|\Psi\rangle = \int_{CS^3} \chi_1 n \chi_2 n \chi_3 n \chi_4 dn.$$  

Since $SL(2,C) \approx CS^3$, using the following graphical identity:

$$\int_{SL(2,C)} \chi_1 \chi_2 \chi_3 \chi_4 dg = \int \chi_1 \chi_2 \chi_3 \chi_4 8\pi^4 d\chi,$$

the Barrett-Crane solution can be rewritten as

$$|\Psi\rangle = \int \chi_1 \chi_2 \chi_3 \chi_4 8\pi^4 d\chi,$$

which emerges as an intertwiner in the familiar form in which Barrett and Crane proposed it for the Riemannian general relativity. It can be clearly seen that the simple representations for $SO(4,R)$ ($J_L = J_R$) has been replaced by the simple representation of $SO(4,C)$ ($\chi_L = \pm \chi_R$).

**Relation to the Riemannian Barrett-Crane model:** All the analysis done until for the $SO(4,C)$ Barrett-Crane theory can be directly applied to the Riemannian Barrett-Crane theory. The correspondences between the two models are listed in the following table:\footnote{BC stands for Barrett-Crane. For $\chi_L$ and $\chi_R$ we have $n_L + n_R = \text{even}$.}
### 3.2.4 The Spin Foam Model for the SO(4, C) General Relativity.

The SO(4, C) Barrett-Crane intertwiner derived in the previous section can be used to define a SO(4, C) Barrett-Crane spin foam model. The amplitude $Z_{BC}(s)$ of a four-simplex $s$ is given by the $\{10\chi\}_{SO(4,C)}$ symbol given below:

\[
\{10\chi\}_{SO(4,C)} = \chi_1 \chi_2 \chi_3 \chi_4,
\]

where the circles are the Barrett-Crane intertwiners. The integers represent the tetrahedra and the pairs of integers represent triangles. The intertwiners use the four $\chi$'s associated with the links that emerge from it for its definition in equation (19). In the next subsection, the propagators of this theory are defined and the $\{10\chi\}_{SO(4,C)}$ symbol is expressed in terms of the propagators in the subsubsection that follows it.

The SO(4, C) Barrett-Crane partition function of the spin foam associated with the four dimensional simplicial manifold with a triangulation $\Delta$ is

\[
Z(\Delta) = \sum_{\{\chi_b\}} \left( \prod_b \frac{d_{\chi_b}^2}{64 \pi^3} \right) \prod_s Z(s),
\]

where $Z(s)$ is the quantum amplitude associated with the 4-simplex $s$ and the $d_{\chi_b}$ adopted from the spin foam model of the BF theory can be interpreted as the quantum amplitude associated with the bone $b$.

### 3.2.5 The Features of the SO(4, C) Spin Foam

- **Areas:** The squares of the areas of the triangles (bones) of the triangulation are given by $\eta_{JK} \eta_{KL} B^I_J B^K_L$. The eigen values of the squares of the areas in the SO(4, C) Barrett-Crane model from equation (19) are given by

\[
\eta_{IK} \eta_{JL} \hat{B}_b^I_J \hat{B}_b^K_L = (\chi^2 - 1) \hat{I} = \left( \frac{n^2}{2} - \rho^2 - 1 + i\rho n \right) \hat{I}.
\]
One can clearly see that the area eigen values are complex. The $SO(4, C)$ Barrett-Crane model relates to the $SO(4, C)$ general relativity. Since in the $SO(4, C)$ general relativity the bivectors associated with any two dimensional flat object are complex, it is natural to expect that the areas defined in such a theory are complex too. This is a generalization of the concept of the space-like and the time-like areas for the real general relativity models: Area is imaginary if it is time-like and real if it is space-like.

- Propagators: Laurent and Freidel have investigated the idea of expressing simple spin networks as Feynman diagrams \[^{37}\]. Here we will apply this idea to the $SO(4, C)$ simple spin networks. Let $\Sigma$ be a triangulated three surface. Let $n_i \in CS^3$ be a vector associated with the $i^{th}$ tetrahedron of $\Sigma$. The propagator of the $SO(4, C)$ Barrett-Crane model associated with the triangle $ij$ is given by

$$G_{\chi_{ij}}(n_i, n_j) = Tr(T_{\chi_{ij}}(g(n_i))T_{\chi_{ij}}(g(n_j)))$$

$$= Tr(T_{\chi_{ij}}(g(n_i)g^{-1}(n_j))),$$

where $\chi_{ij}$ is a representation associated with the triangle common to the $i^{th}$ and the $j^{th}$ tetrahedron of $\Sigma$. If $X$ and $Y$ belong to $CS^3$ then

$$tr \left( (g(X)g(Y)^{-1}) \right) = 2X.Y,$$

where $X.Y$ is the Euclidean dot product and $tr$ is the matrix trace. If $\lambda = e^t$ and $\frac{1}{\lambda}$ are the eigen values of $g(X)g(Y)^{-1}$ then,

$$\lambda + \lambda^{-1} = 2X.Y$$

$$X.Y = \cosh(t).$$

From the expression for the trace of the $SL(2, C)$ unitary representations, (appendix A, \[^{21}\]) I have the propagator for the $SO(4, C)$ Barrett-Crane model calculated as

$$G_{\chi_{ij}}(n_i, n_j) = \frac{\cos(\rho_{ij} \eta_{ij} + n_{ij} \theta_{ij})}{|\sinh(\eta_{ij} + i\theta_{ij})|^2},$$

where $\eta_{ij} + i\theta_{ij}$ is defined by $n_i.n_j = \cosh(\eta_{ij} + i\theta_{ij})$. Two important properties of the propagators are listed below.

1. Using the expansion for the delta on $SL(2, C)$ I have

$$\delta_{CS^3}(X, Y) = \delta_{SL(2, C)}(g(X)g^{-1}(Y))$$

$$= \frac{1}{8\pi^4} \int \bar{\chi} \chi Tr(T_{\chi}(g(X)g^{-1}(Y)))d\chi,$$

where the suffix on the deltas indicate the space in which it is defined. Therefore

$$\int \bar{\chi} \chi G_{\chi}(X, Y) = 8\pi^4\delta_{CS^3}(X, Y).$$
2. Consider the orthonormality property of the principal unitary representations of $SL(2,\mathbb{C})$ given by

$$
\int_{CS^3} T_{z_1\chi_1}(g(X))T_{z_2\chi_2}(g(X))dX = \frac{8\pi^4}{\chi_1\chi_2} \delta(\chi_1 - \chi_2)\delta(z_1 - z_1)\delta(z_2 - z_2),
$$

where the delta on the $\chi$’s is defined up to a sign of them. From this I have

$$
\int_{CS^3} G_{\chi_1}(X,Y)G_{\chi_2}(Y,Z)dY = \frac{8\pi^4}{\chi_1\chi_2} \delta(\chi_1 - \chi_2)G_{\chi_1}(X,Z).
$$

- The $\{10\chi\}$ symbol can be defined using the propagators on the complex three sphere as follows:

$$
Z(s) = \int_{x_k \in CS^3} \prod_{i<j} T_{\chi_{ij}}(g(x_i)g(x_j)) \prod_k dx_k,
$$

and

$$
= \int_{x_k \in CS^3} \prod_{i<j} G_{\chi_{ij}}(x_i,x_j) \prod_k dx_k,
$$

where $i$ denotes a tetrahedron of the four-simplex. For each tetrahedron $k$, a free variable $x_k \in CS^3$ is associated. For each triangle $ij$ which is the intersection of the $i$’th and the $j$’th tetrahedron, a representation of $SL(2,\mathbb{C})$ denoted by $\chi_{ij}$ is associated.

- Discretization Dependence and Local Excitations: It is well known that the BF theory is discretization independent and is topological. The spin foam for the $SO(4,\mathbb{C})$ general relativity is got by imposing the Barrett-Crane constraints on the BF Spin foam. After the imposition of the Barrett-Crane constraints the theory loses the discretization independence and the topological nature. This can be seen in many ways.

  - The simplest reason is that the $SO(4,\mathbb{C})$ Barrett-Crane model corresponds to the quantization of the discrete $SO(4,\mathbb{C})$ general relativity which has local degrees of freedom.
  
  - After the restriction of the representations involved in BF spin foams to the simple representations and the intertwiners to the Barrett-Crane intertwiners, various important identities used in the spin foam diagrammatics and proof of the discretization independence of the BF theory spin foams in Ref. [15] are no longer available.
  
  - The BF partition function is simply gauge invariant measure of the volume of space of flat connections. Consider the following harmonic expansion of the delta function which was used in the derivation of the $SO(4,\mathbb{C})$ BF theory:

$$
\delta(g) = \frac{1}{8\pi^4} \int d\omega \text{tr}(T_\omega(g))d\omega.
$$
Imposition of the Barrett-Crane constraints on the BF theory spin foam, suppresses the terms corresponding to the non-simple representations. If only the simple representations are allowed in the right hand side, it is no longer peaked at the identity. This means that the partition function for the $SO(4, C)$ Barrett-Crane model involves contributions only from the non-flat connections which has local information.

- In the asymptotic limit study of the $SO(4, C)$ spin foams in section four the discrete version of the $SO(4, C)$ general relativity (Regge calculus) is obtained. The Regge calculus action is clearly discretization dependent and non-topological.

- The real Barrett-Crane models that are discussed in the next section are the restricted form of the $SO(4, C)$ Barrett-Crane model. The above reasoning can be applied to argue that they are also discretization dependent.

4 Spin Foams for Real General Relativity

4.1 The Formal Structure of Barrett-Crane Intertwiners

Let me briefly discuss the formal structure of the Barrett-Crane intertwiner of the $SO(4, C)$ general relativity for the purpose of the developing spin foam models for real general relativity theories. It has the following elements:

- A gauge group $G$,
- A homogenous space $X$ of $G$,
- A $G$ invariant measure on $X$ and,
- A complete orthonormal set of functions which call as $T$—functions which are maps from $X$ to the Hilbert spaces of a subset of unitary representations of $G$: $$T_{\rho} : X \rightarrow D_{\rho},$$

where $\rho$ is a representation of $G$. The $T$—functions correspond to the various unitary representations under the transformation of $X$ under $G$. The $T$—functions are complete in the sense that on the $L^2$ functions on $X$ they define invertible Fourier transforms.

Formally Barrett-Crane intertwiners are quantum states $\Psi$ associated to closed simplicial two surfaces defined as an integral of an outer product of $T$—functions on the space $X$:

$$\Psi = \int_{X} \prod_{\rho} T_{\rho}(x) d_{X} x \in \prod_{\rho} D_{\rho}.$$  

It can seen that $\Psi$ is gauge invariant under $G$ because of the invariance of the measure $d_{X} x$. 

22
4.2 The Real Barrett-Crane Models

In the case of non-degenerate general relativity the reality of the area metric is the necessary and sufficient condition for real geometry [27]. In the Plebanski formalism the area metric can be expressed in terms of the 2-form bivector variable \( B^{IJ}_{ab} dx^a \wedge dx^b \) as \( \eta_{IK} \eta_{JL} B^{IJ} \wedge B^{KL} \) where the \( \wedge \) represents the exterior product on the forms. It has been shown in case of Ref:[27] that by adding a Lagrange multiplier to the \( SO(4, C) \) Plebanski action, we can derive real general relativity.

On a simplicial manifold a bivector two form field can be discretized by associating bivectors to the triangles. The discrete equivalent of the area metric reality constraint is the bivector scalar product reality constraint [27]. Consider a four-simplex with complex bivectors \( B_i, i = 1 \) to 10 associated with its triangles. Then the bivector scalar product reality constraint requires

\[ \text{Im}(B_i \wedge B_j) = 0 \quad \forall i, j. \]

It can be shown that the necessary and sufficient condition for the reality of a flat four-simplex geometry is that the scalar products of the bivectors associated to the triangles be real [27].

I would like to formally reduce the Barrett-Crane models for real general relativity from that of the \( SO(4, C) \) Barrett-Crane model by using the bivector scalar products reality constraint [27]. Precisely I plan to use the following three ideas to reduce the Barrett-Crane models:

1. The formal structure of the reduced intertwiners should be the same as that of the \( SO(4, C) \) Barrett-Crane model,

2. The eigen value of the Casimir corresponding to the square of the area of any triangle must be real. I would like to refer to this as the self-reality constraint\(^{12}\),

3. The eigen values of the square of area Casimir corresponding to the representations associated with the internal links of the intertwiner must be real. I would like to refer to this as the cross-reality constraint.

The first idea sets a formal ansatz for the reduction process. The square of the area of a triangle is simply the scalar product of the bivector of a triangle with itself. Second condition is the quantum equivalent of the reality of the scalar product of a bivector associated with a triangle with itself. Once the second condition is imposed the third condition is the quantum equivalent of the reality of the scalar product of the two bivectors of any two triangle of a tetrahedron\(^{13}\).

\(^{12}\)I would like to mention that the areas being real necessarily does not mean that the bivectors must also be real.

\(^{13}\)We have ignored to impose reality of the scalar products of the bivectors associated to any two triangles of the same four simplex which intersect at only at one vertex. This is because these constraints appears not to be needed for a formal extraction of the Barrett-
My goal is to use the above principles to derive reduced Barrett-Crane models and later one can convince oneself by identifying and verifying that the Barrett-Crane constraints are satisfied for a subgroup of \( SO(4, C) \) for each of the reduced model.

In general by reducing a certain Hilbert space associated with the representations of a group \( G \) by some constraints, the resultant Hilbert space need not contain the states gauge invariant under \( G \). In that case one can look for gauge invariance states under subgroups of \( G \). In our case we will find that the suitable quantum states extracted by adhering to the above principles are gauge symmetry reduced versions of \( SO(4, C) \) Barrett-Crane states. They are gauge invariant only under the real subgroups of \( SO(4, C) \).

Let \( P \) be a formal projector which reduces the Hilbert space \( D_{\chi L} \otimes D_{\chi R} \) to a reduced Hilbert space such that the reality constraints are satisfied. Let me assume as an ansatz that now the complex three sphere is replaced by its subspace \( X \) due to projection. Now I expect, the projected \( SO(4, C) \) Barrett-Crane intertwiner is spanned by the following states for all \( \chi_i \) satisfying the reality constraint in equation (??):

\[
\Psi_X = \int_{x \in X} \prod_i P T_{\chi_i}(g(x)) \tilde{d}g(x),
\]

where \( \tilde{d}g(n) \) is the reduced measure of \( dg(n) \) on \( X \). The imposition of the constraint expressed at the quantum level sets \( \rho_i \) or \( n_i \) to be zero on each vertex of the \( SO(4, C) \) Barrett-Crane intertwiner. Let me rewrite the projected intertwiner as follows.

\[
\Psi_X = \int_{x,y \in X} \prod_{1,2} PT_{\chi_1}(g(x)) \delta_X(x,y) \prod_{3,4} PT_{\chi_1}(g(y)) d^X g(x) d^X g(y),
\]

where \( \delta_X(x,y) \) is the delta function on \( X \). Since \( X \) is a subspace of \( SL(2, C) \) a harmonic expansion can be derived for \( \delta(x,y) \) using the unitary representations of \( SL(2, C) \). Since the intertwiner must obey the cross reality constraint the harmonic expansion must only contain simple representations of \( SL(2, C) \) (\( \rho \) or \( n \) is zero).

For the Fourier transform defined by \( PT_{\chi}(g(x)) \) to be complete and orthonormal I must have

\[
\int_{\chi \in Q} \tilde{\chi} \chi PT_{\chi}(g(x)) PT_{\chi}(g(y)) d\chi = \delta_X(x,y),
\]

where \( Q \) is the set of all simple representations\(^{14}\) of \( SL(2, C) \) required for the expansion. Only the simple representations of \( SL(2, C) \) must be used to satisfy

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\(^{14}\)One could also call the simple representations of \( SL(2, C) \) as the real representations since it corresponds to the real areas and the real homogeneous spaces. But I will avoid this to avoid any possible confusion.
the cross-reality constraint. Thus, the number of reduced intertwiners deriv-
able is directly related to the possible solutions for this equation (subjected to
Barrett-Crane constraints).

The equation of a complex three sphere is

\[ x^2 + y^2 + z^2 + t^2 = 1. \]

There are four different topologically different maximally connected real sub-
spaces of \( CS^3 \) such that the harmonic (Fourier) expansions on these spaces use
the simple representations of \( SL(2, C) \) only. They are namely, the three sphere
\( S^3 \), the real hyperboloid \( H^+ \), the imaginary hyperboloid \( H^- \) and the Kleinien
hyperboloid\(^{15} K^3 \). Each of these subspace \( X \) are maximal real subspaces of \( CS^3 \).
They are all homogenous under the action of a maximal real subgroup\(^{16} G_X \) of
\( SO(4, C) \). There exists a \( G_X \) invariant measure \( d^X(x) \). The reduced bivectors
acting on the functions on \( X \) effectively take values in the Lie algebra of \( G_X \).
Since the measure \( d^X(n) \) is invariant, the reduced intertwiner is gauge invariant.
So the intertwiner \( \Psi_X \) must correspond to the quantum general relativity for
the group \( G_X \).

Let the coordinates of \( n = (x, y, z, t) \) be restricted to real values here after
in this section. Let me discuss the various reduced intertwiners:

1. \( \rho = 0 \) case: This uses only the \( \chi = (0, n) \) representations only. This
corresponds to \( X = S^3 \), satisfying

\[ x^2 + y^2 + z^2 + t^2 = 1, \]

which is invariant under \( SO(4, R) \). So this case corresponds to the Rie-
mannian general relativity. The appropriate projected \( T- \)functions are
the representation matrices of \( SU(2) \approx S^3 \) and the reduced measure is
the Haar measure of \( SU(2) \). The intertwiner I get is the Barrett-Crane
intertwiner for the Riemannian general relativity. Here the \( \chi' \)s has been
replaced by the \( J' \)s and the complex three sphere by the real three sphere.
The case of going from the \( SO(4, C) \) Barrett-Crane model to the Rieman-
nian Barrett-Crane model is intuitive. It is a simple process of going from
complex three sphere to its subspace the real three sphere.

2. \( n = 0 \) case: This uses \( \chi = (\rho, 0) \) representations only: This corresponds
to \( X \) as a space-like hyperboloid (only one sheet) with \( G_X = SO(3, 1, R) \):

\[ x^2 + y^2 + z^2 - t^2 = 1. \]

The intertwiner now corresponds to the Lorentzian general relativity. This
intertwiner was introduced in \[14\]. The unitary representations of the

\(^{15}\)By Kleinien hyperboloid I refer to the space described by \( x^2 + y^2 - z^2 - t^2 = 1 \) for real
\( x, y, z \) and \( t \).

\(^{16}\)The real group is maximal in the sense that there is no other real topologically connected
subgroup of \( SO(4, C) \) that is bigger.
Lorentz group on the real hyperboloid have been studied by Gelfand and Naimark [21], from which the $T-$functions are

$$T_{\rho}(x)[\xi] = [\xi, x]^{\frac{1}{2}}\rho^{-1},$$

where $\xi \in \text{null cone intersecting } t = 1 \text{ plane in the Minkowski space.}$

Here $\xi$ replaces $(z_1, z_2)$ in the $T-$function $T_{\chi}(g(x))(z_1, z_2)$ of the $SO(4, C)$ Barrett-Crane Model. An element $g \in SO(3, 1)$ acts as a shift operator as follows:

$$gT_{\rho}(x)[\xi] = T_{\rho}(gx)[\xi] = T_{\rho}(x)[g^{-1}\xi].$$

This intertwiner was first introduced in [16].

3. Combination of $(0, n)$ and$(\rho, 0)$ representations: There are two possible models corresponding to this case. One of them has $X$ as the Kleinien hyperboloid defined by

$$x^2 + y^2 - z^2 - t^2 = 1,$$

with $G_X = SO(2, 2, R)$. Here the $X$ is isomorphic to $SU(1, 1) \approx SL(2, R)$. The intertwiner now corresponds to Kleinien general relativity $(++--)$ signature). The $T-$functions are of the form $T_{\chi}(k(n))(z_1, z_2)$ where $z_1$ and $z_2$ takes real values only (please refer to appendix C), $\chi \neq 0$ and $k$ is an isomorphism from the Kleinien hyperboloid to $SU(1, 1)$ defined by

$$k(n) = \begin{bmatrix} x - iy & z - it \\ z + it & x + iy \end{bmatrix}.$$}

The representations corresponding to the $n = 0$ and $\rho = 0$ cases are qualitatively different. The representations corresponding to $\rho \neq 0$ are called the continuous representations and those to $n \neq 0$ are called the discrete representations. The action of $g \in SO(2, 2, R)$ on the $T-$functions is

$$gT_{\chi}(k(x)) = T_{\chi}(k(g(x))),$$

where $g(x)$ is the result of action of $g$ on $x \in X$.

4. The second model using both $(0, n)$ and$(\rho, 0)$ representations: This corresponds to the time-like hyperboloid with $G_X = SO(3, 1),

$$x^2 - y^2 - z^2 - t^2 = 1,$$

where two vectors that differ just by a sign are identified as a single point of the space $X$. The corresponding spin foam model has been introduced by Barrett and Crane [12]. It has been derived using a field theory over group formalism by Rovelli and Perez [18]. Similar to the previous case, I
have both continuous and discrete representations, with the $T$–functions given by

\[
T_\rho(x)[\xi] = [\xi, x]^{\frac{i}{\rho - 1}},
\]

\[
T_n(x)[l(a, \xi)] = \exp(-2in\theta)\delta(a, \xi),
\]

where the $l(a, \xi)$ is an isotropic line\footnote{A line on an imaginary hyperboloid \cite{21} is the intersection of a 2-plane of the Minkowski space with it. The line is called isotropic if the Lorentzian distance between any two points on it is zero. An isotropic line $l$ is described by the equation $x = s\xi + x_0$, $x$ is the variable point on $l$, $x_0$ is any fixed point on $l$, and $\xi$ is a null-vector. For more information please refer to \cite{21}.} on the imaginary hyperboloid along direction $\xi$ going through a point $a$ on the hyperboloid and the $\theta$ is the distance between $l(a, \xi)$ and $l(x, \xi)$ given by $\cos \theta = a.x$, where the dot is the Lorentzian scalar product. I have for $g \in SO(3, 1, R)$,

\[
gT_n(x)[l(a, \xi)] = T_n(x)[l(a, g\xi)] = T_n(g^{-1}x)[l(a, \xi)],
\]

and the action of $g$ on continuous representations are defined similar to equation (21). The corresponding spin foam model has been introduced and investigated before by Rovelli and Perez \cite{18}.

### 4.3 The Area Eigenvalues

Using the $T$–functions described above, the intertwiners for real general relativity can be constructed. Using these intertwiners, spin foam models (Barrett-Crane) for the real general relativity theories of the various different signatures can be constructed. The square of the area of a triangle of a four-simplex for all signatures associated with a representation $\chi$ is described by the same formula\footnote{Please refer to the end of appendix C regarding the differences between the Casimers of $SL(2, C)$ and $SU(1, 1)$.},

\[
\eta_{IK}\eta_{JL}\hat{B}^{IJ}\hat{B}^{KL} = (\chi^2 - 1) \hat{I} \equiv \left(\frac{n^2}{2} - \rho^2 - 1\right) \hat{I},
\]

where only of $n$ and $\rho$ is non-zero. The square of the area is negative or positive depending on whether $\rho$ or $n$ is non-zero. The negative (positive) sign corresponds to a time-like (space-like) area.

### 5 Further Considerations:

#### 5.1 A Mixed Lorentzian Quantum Model.

We have two intertwiners for the Lorentzian general relativity discussed in the previous section, one corresponding to the space-like hyperboloid $H^+$ \cite{16} and
another to the time hyperboloid $H^-$. We can consider a tetrahedron to be space-like (time-like) if it is associated with the intertwiner related to the space-like (time-like) hyperboloid. This is justified because in the semi-classical limit the tetrahedron becomes a space-like (time-like) hypersurface. I can construct quantum amplitudes for a general four-simplex with each tetrahedron of the 4-simplex either time-like or space-like. The intertwiners are straight forward to construct. This model is a more general form of the Lorentzian Barrett-Crane model. Let me next discuss the various propagators associated with this model.

The propagator from a space-like tetrahedron with an associated vector $t_1 \in H^+$ to another space-like tetrahedron in the same simplex with an associated vector $t_2 \in H^+$ is given by

$$
h^{++}_\rho(t_1, t_2) = \int (t_1, l)^{-1+i\rho}(t_2, l)^{-1-i\rho} dl = \frac{4\pi \sin(\eta \rho)}{i\rho \cosh \eta},
$$

where the unit vector $l$ is an element of the positive light cone intersecting $t = 1$ hypersurface in the Minkowski space-time, $dl$ is the measure on the intersection. This propagator has been introduced and discussed by Barrett and Crane. The propagators between two time-like tetrahedra were discussed by Rovelli-Perez. I refer the readers to Ref. [18] for the details.

One can define a propagator between a space-like and a time-like tetrahedra intersecting at a triangle associated with a continuous ($\rho \neq 0$) representation. The propagator from a time-like tetrahedron associated with a vector $t \in H^-$ to a tetrahedron associated with a space-like vector $s \in H^+$ is given by

$$
h^{+-}_\rho(t, s) = \int (t, l)^{-1+i\rho} |s, l|^{-1-i\rho} dl,
$$

where the unit vector $l$ is the element of the positive light cone with time component equal to 1 and the $dl$ the measure on it. An important difference between this propagator and the other two propagators discussed before is that there is no completion relation for this propagator, such as

$$\int h_{\chi}(x_1, x_2) d\chi = \delta_X(x_1 - x_2),$$

where a formal propagator between two elements $x_1, x_2$ of some space $X$ is summed and integrated over all possible representations.

To calculate this integral, using the Lorentz invariance of the integral, I can define the space time coordinates such that $t = (1, 0, 0, 0), s = (\sinh \eta, 0, 0, \cosh \eta), l = (1, n)$, where $n$ is a 3D unit vector expressed in terms of $\theta, \phi$ coordinates.
Then the integral is

\[ h_\rho(t, s) = \int (t.l)^{-1+i\rho} \overline{|s.l|}^{-1-i\rho} \, dl \]

\[ = \int \frac{|(\sinh \eta - \cos \theta \cosh \eta)|^{-1-i\rho}}{1-i\rho} \sin \theta \, d\theta \, d\phi \]

\[ = -2\pi \int_{-1}^{+1} |\sinh \eta - z \cosh \eta|^{-1-i\rho} \, dz, \]

where the \( \cos \theta \) has been replaced by a new variable \( z \). Let \( q = \sinh \eta - z \cosh \eta \). When \( z \) varies between \(-1\) and \(+1\), \( q \) varies between \( e^\eta \) and \( -e^{-\eta} \). In this range \( q \) is zero only once when \( z = \tanh(\eta) \). Rewriting the above integral using \( q \) as the variable of integration I get,

\[ h_\rho(t, s) = -2\pi \left\{ \int_{e^\eta}^{0} q^{-1-i\rho} \, dq + \int_{-e^{-\eta}}^{0} (-q)^{-1-i\rho} \, dq \right\} \]

\[ = -2\pi \left\{ \int_{0}^{e^\eta} q^{-1-i\rho} \, dq + \int_{0}^{-e^{-\eta}} q^{-1-i\rho} \, dq \right\}. \]

By setting \( q = e^x \), I get

\[ h_\rho(t, s) = -\frac{2\pi i \cos[\rho \eta]}{\rho \cosh \eta}. \]

At this point it is also not clear whether the new model has any physical significance. Further investigation is required.

5.2 A Multi-Signature Barrett-Crane Model

I formally deduced various intertwiners corresponding to the various signatures of real general relativity from the \( SO(4, C) \) Barrett-Crane intertwiner. By using each of these intertwiners I can construct a quantum four-simplex for each signature. By splicing the quantum four simplices of the various signatures on the tetrahedrons with common representations I can construct a spin foam model. This model could be considered as the most general Barrett-Crane model for real general relativity. The physical significance of this model is not clear and further study is required.

Putting together quantum general relativity models of various signatures has been considered before. For example, Hawking [38] has spliced together a

\[ ^{19}\text{It is not clear what is the physical meaning of the small positive imaginary part is.} \]
Euclidean geometry (imaginary time) universe in the initial stage of the universe to its Lorentzian future. But the Hawking theory is slightly different from mine. In the Hawking’s theory the Euclidean general relativity has an imaginary action and so it contributes magnitudes instead of phases to the path integral. In our theory the action that is used for the spin foam quantization is always real as is described in Ref. [27].

It has been suggested before that for quantum general relativity to be unitary it must involve all the signatures [39]. So the classical and quantum multi-signature real general relativity may be interesting new theories to look into and explore for new physics.

5.3 The Asymptotic Limit of the Barrett-Crane models.

The asymptotic limit of the real Barrett-Crane models has been studied before [31], [30], [32], [34] to a certain degree. Here I will discuss the asymptotic limit of the \( SO(4, \mathbb{C}) \) Barrett-Crane model. For the first time I show here that we can extract bivectors which satisfy the essential Barrett-Crane constraints from the asymptotic limit. Consider the amplitude of a four-simplex given by Eq. (19) with a real scale parameter \( \lambda \),

\[
Z_\lambda = \int_{n_k \in C S^3} \prod_{i<j} G_{\lambda \chi_{ij}}(n_i, n_j) \prod_k d n_k, \\
= \int_{n_k \in C S^3} \prod_{i<j} \frac{\cos(\lambda \rho_{ij} \eta_{ij} + \lambda n_{ij} \theta_{ij})}{\sinh(\lambda \eta_{ij} + i \lambda \theta_{ij})} \prod_k d x_k, \\
= \int_{n_k \in C S^3} \prod_{i<j} \frac{\exp(i \varepsilon_{ij} \lambda (\rho_{ij} \eta_{ij} + n_{ij} \theta_{ij}))}{2 \sinh(\lambda \eta_{ij} + i \lambda \theta_{ij})} \prod_k d x_k,
\]

where the \( \eta_{ij} + i \theta_{ij} \) is defined by \( n_i \cdot n_j = \cosh(\eta_{ij} + i \theta_{ij}) \). Here the \( \zeta_{ij} = \eta_{ij} + i \theta_{ij} \) is the complex angle between \( n_i \) and \( n_j \). The asymptotic limit of \( Z_\lambda(s) \) under \( \lambda \rightarrow \infty \) is controlled by

\[
S(\{n_i, \bar{n}_i\}, \{\chi_{ij}, \bar{\chi}_{ij}\}) \\
= \sum_{i<j} \varepsilon_{ij} (\rho_{ij} \eta_{ij} + n_{ij} \theta_{ij}) + \Re \left( \sum_i q_i (n_i, n_i - 1) \right) \\
= \Re \left( \sum_{i<j} \varepsilon_{ij} \bar{\chi}_{ij} \zeta_{ij} + \sum_i q_i (n_i, n_i - 1) \right),
\]

where the \( q_i \) are the Lagrange multipliers to impose \( n_i, n_i = 1, \forall i \). My goal now is to find stationary points for this action. The stationary points are determined by

\[
\sum_{i \neq j} \varepsilon_{ij} \bar{\chi}_{ij} \frac{\partial \zeta_{ij}}{\partial n_i} + q_j n_j = 0, \forall j, \quad (22a)
\]
and \( n_j, n_j = 1, \forall j \) where the \( j \) is a constant in the summation.

\[
\frac{\partial \zeta_{ij}}{\partial n_i} = \frac{n_j}{\sinh(\zeta_{ij})}.
\]  

(23)

Using equation (23) in equation (22a) and taking the wedge product of the equation with \( n_j \) we have,

\[
\sum_{i \neq j} \varepsilon_{ij} \chi_{ij} \frac{n_j \wedge n_j}{\sinh(\zeta_{ij})} = 0, \forall j.
\]

If

\[
\tilde{E}_{ij} = i \varepsilon_{ij} \chi_{ij} \frac{n_j \wedge n_j}{\sinh(\zeta_{ij})},
\]

then the last equation can be simplified to

\[
\sum_{i \neq j} E_{ij} = 0, \forall j.
\]  

(24)

We now consider the properties of \( E_{ij} \):

- Each \( i \) represents a tetrahedron. There are ten \( E_{ij} \)'s, each one of them is associated with one triangle of the four-simplex.

- The square of \( E_{ij} \):

\[
E_{ij} \cdot \tilde{E}_{ij} = \frac{-\chi_{ij}^2}{\sinh^2(\zeta_{ij})} \left(n_j^2 n_i^2 - (n_i \cdot n_j)^2\right)
= \frac{-\chi_{ij}^2}{\sinh^2(\zeta_{ij})} \left(1 - (\cosh(\zeta_{ij})^2\right)
= \chi_{ij}^2.
\]

- The wedge product of any two \( E_{ij} \) is zero if they are equal to each other or if their corresponding triangles belong to the same tetrahedron.

- Sum of all the \( E_{ij} \) belonging to the same tetrahedron are zero according to equation (24).

It is clear that these properties contain the first four Barrett-Crane constraints. So we have successfully extracted the bivectors corresponding to the triangles of a general flat four-simplex in \( SO(4, C) \) general relativity and the \( n_i \) are the normal vectors of the tetrahedra. The \( \chi_{ij} \) are the complex areas of the triangle as one would expect. Since we did not impose any non-degeneracy conditions, it is not guaranteed that the tetrahedra or the four-simplex have non-zero volumes.
The asymptotic limit of the partition function of the entire simplicial manifold with triangulation $\Delta$ is

$$S(\Delta, \{n_{is} \in CS^3, \chi_{ij}, \bar{\chi}_{ij}, \varepsilon_{ijs}\}) = \text{Re} \sum_{i < j, s} \varepsilon_{ijs} \bar{\chi}_{ij} \zeta_{ijs},$$

where I have assumed variable $s$ represents the four simplices of $\Delta$ and $i, j$ represents the tetrahedra. The $\varepsilon_{ijs}$ can be interpreted as the orientation of the triangles. Each triangle has a corresponding $\chi_{ij}$. The $n_{is}$ denote the unit complex vector associated with the side of the tetrahedron $i$ facing the inside of a simplex $s$. Now there is one bivector $E_{ijs}$ associated with each side facing inside of a simplex $s$ of a triangle $ij$ defined by

$$\bar{E}_{ijs} = i \varepsilon_{ijs} \bar{\chi}_{ij} \frac{n_{js} \wedge n_{js}}{\sinh(\zeta_{ijs})}.$$ 

If the $n_{is}$ are chosen such that they satisfy stationary conditions

$$\sum_{i \neq j} E_{ijs} = 0, \forall j, s,$$

and if

$$\theta_{ij} = \left( \sum_s \varepsilon_{ijs} \zeta_{ijs} \right),$$

then

$$S(\Delta, \{\chi_{ij}, \bar{\chi}_{ij}, \varepsilon_{ijs}\}) = \text{Re} \sum_{i < j, s} \varepsilon_{ijs} \bar{\chi}_{ij} \zeta_{ijs},$$

$$= \text{Re} \sum_{i < j} \bar{\chi}_{ij} \theta_{ij}$$

can be considered to describe the Regge calculus for the $SO(4, C)$ general relativity. The angle $\theta_{ij}$ are the deficit angles associated with the triangles and the $n_{is}$ are the complex vector normals associated with the tetrahedra. From the analysis that has been done in this section, it is easy see that the $SO(4, C)$ Regge calculus contains the Regge calculus theories for all the signatures. The Regge calculus for each signature can be obtained by restricting the $n_{is}$ and the $\chi_{ij}$ to the corresponding homogenous space and representations as described in the previous section. Also by the properly restricting the $n_{is}$ and the $\chi_{ij}$ we can derive the Regge calculus corresponding to the mixed Lorentzian and multi-signature Barrett-Crane models described in the previous subsections.

### 5.4 3+1 Formulation: Spin Networks Functionals.

A $(n - 1) + 1$ formulation was proposed in Ref:[33], with a motivation to relate spin foams to canonical quantum general relativity. I will briefly review the
basic ideas and discuss it in the context of the $SO(4, C)$ general relativity. For details, I refer to the original article Ref.[33]. The $nD$ simplicial manifold was foliated by a one parameter sequence of $(n − 1)D$ simplicial hypersurfaces. The parallel propagators associated with the edges in the foliating hypersurfaces can be thought of as the analog of the continuum connection in the (coordinate) time direction. It turns out that the integration of the Feynman weight $e^{iS}$ with respect to the parallel propagators associated with the edges of the hypersurfaces results in a product of spin network functionals shown in figure 1. These spin network functionals are defined on the parallel propagators associated with the edges that go between the hypersurfaces on the graphs that are dual to the triangulation of the foliating hypersurfaces. For the spin foam model of the BF theory, the BF intertwiners are used to intertwine the representations associated to the links of the graph. In the case of spin foam model of general relativity, the Barrett-Crane intertwiners are used. In this case the elements of the homogenous space on which the intertwiners are defined represent normal vectors to the simplicial hypersurfaces. The sum over the homogenous space vectors in the Barrett-Crane intertwiners can be interpreted as a sum over the normals. The spin foam partition functionals of the BF theories or general relativity (Barrett-Crane) can be reformulated using these spin network functionals.

It is straightforward to generalize the 3+1 theory to the $SO(4, C)$ general relativity. By using the Barrett-Crane intertwiners for $SO(4, C)$ and the $SO(4, C)$ parallel propagators, we can reconstruct the spin network functionals for the $SO(4, C)$ general relativity. In this case the normal vectors are complex. The $SO(4, C)$ 3+1 formulation essentially contains the 3 + 1 formulation of the real general relativity theories. It is also straightforward to see that spin network functional for the real general relativity models are: 1) restrictions of the complex normal vectors to the real normal vectors and 2) restrictions of representations as described in section three. These ideas are also applicable to the mixed Lorentzian and multi-signature Barrett-Crane models by using the appropriate intertwiners, representations and the parallel propagators.

5.5 Field Theory over Group and Homogenous Spaces.

One of the problems with the Barrett-Crane model for general relativity is its dependence on the discretization of the manifold. A discretization independent model can be defined by summing over all possible discretizations. With a proper choice of amplitudes for the lower dimensional simplices the BF spin foams can be reformulated as a field theory over a group (GFT) [4]. Similarly, the Barrett-Crane models can be reformulated as a field theory over the homogenous space of the group [29]. Let me briefly explain the GFT of the four dimensional spin foams for a compact group $G$. A field theory over a group is defined using an action. The action has two terms, namely the kinetic term and the potential terms. Consider a tetrahedron. Let a group element $g_i$ be associated with each triangle $i$ of the tetrahedron. Let a real field $\phi(g_1, g_2, g_3, g_4)$ invariant under the exchange of its arguments be associated with the tetrahedron. Let the field be invariant under the simultaneous (left or right) action of
Figure 1: In this diagram $\Sigma_i$ is the $i^{th}$ hypersurface of the foliation. $\Omega_i$ is the four dimensional slice between $\Sigma_i$ and $\Sigma_{i+1}$. $\Sigma_{i+1}$ is not shown. $b$ and $c$ are the triangles and tetrahedra on $\Sigma_i$. The links that go between the hypersurfaces are shown in blue. The $\hat{g}$'s are the parallel propagators associated to the tetrahedra that go between the hypersurfaces.
a group element $g$ on its variables. Then the kinetic term is defined as

$$K.E = \int \prod_{i=1}^{4} dg_i \phi^2.$$  

To define the potential term, consider a four-simplex. Let $g_i$, where $i = 1$ to 10 be the group elements associated with its ten triangles. With each tetrahedron $e$ of the four-simplex, associate a $\phi$ field which is a function of the group elements associated with its triangles. Denote it as $\phi_e$. Then the potential term is defined as

$$P.E = \frac{\lambda}{5!} \int \prod_{i=1}^{10} dg_i \prod_{e=1}^{5} \phi_e,$$

where $\lambda$ is an arbitrary constant. Now the action for a GFT can be defined as

$$S(\phi) = K.E + P.E = \int \prod_{i=1}^{4} dg_i \phi^2 + \frac{\lambda}{5!} \int \prod_{i=1}^{10} dg_i \prod_{e=1}^{5} \phi_e.$$

The Partition function of the GFT is

$$Z = \int D\phi e^{-S(\phi)}.$$  

Now, an analysis of this partition function yields the sum over spin foam partitions of the four dimensional BF theory for group $G$ for all possible triangulations. From the analysis of the GFT we can easily show that this result is valid for $G = SO(4, C)$ with the unitary representations defined in the appendix B.

Let us assume $\phi$ is invariant only under the simultaneous action of an element of a subgroup $H$ of $G$. Then, if $G = SO(4, R)$ and $H = SU(2)$ we get GFTs for the Barrett-Crane model\textsuperscript{20}. Similarly, if $G = SL(2, C)$ and $H = SU(2)$ or $SU(1, 1)$, we can define GFT for the Lorentzian general relativity \textsuperscript{18}, \textsuperscript{17}. The representation theories of $SO(4, C)$ and $SL(2, C)$ has similar structure to those of $SO(4, R)$ and $SU(2)$ respectively. So the GFT with $G = SO(4, C)$ and $H = SL(2, C)$ should yield the sum over triangulation formulation of the $SO(4, C)$ Barrett-Crane model. The details of this analysis and its variations will be presented elsewhere.

6 Summary

In this article I have comprehensively investigated various issues involved in the formulation of the spin foam models for general relativity. In this process many things has been accomplished. They can be listed as follows:\textsuperscript{20}

\textsuperscript{20}Depending on whether we are using the left or right action of $G$ on $\phi$, we get two different models that differ by amplitudes for the lower dimensional simplices.\textsuperscript{20}
• Formulated the spin foams for the $SO(4, C)$ BF theory.

• Systematically imposed the essential Barrett-Crane constraints for the $SO(4, C)$ general relativity:
  
  – Rigorously imposed the cross-simplicity constraints for the $SO(4, C)$ general relativity. This procedure can be directly applied to the Riemannian general relativity.
  
  – The Barrett-Crane intertwiner for the $SO(4, C)$ general relativity has been calculated.
  
  – The propagators of the $SO(4, C)$ Barrett-Crane model has been calculated and the four-simplex amplitude was formulated using them.

• Using the bivector scalar product reality condition the Barrett-Crane intertwiners for all non-degenerate signatures have been formally deduced.

• Discussed the asymptotic limit the $SO(4, C)$ general relativity which can be easily restricted to the real general relativity cases. Essentially the asymptotic limit is the $SO(4, C)$ Regge calculus which contains the Regge calculus theories of all the real general relativity cases.

• The $3 + 1$ formulations of the $SO(4, C)$ and the real Barrett-Crane models have been briefly discussed.

• Field theory over group for the $SO(4, C)$ Barrett-Crane model has been briefly introduced.

• We discussed the mixed Barrett-Crane models which mixes the intertwiners for the two Lorentzian Barrett-Crane models and calculate the mixed propagator.

• Proposed a multi-signature Barrett-Crane model which is obtained by coupling together the four-simplex amplitudes for the various different signatures.

6.1 Future directions

All of the above listed work focused on defining, deriving, and achieving general and unified understanding of the spin foam models of general relativity. This process is still incomplete. Also the process of extracting physics from the spin foams is incomplete and little explored. Let me list the future work that needs to be done in successfully formulating a coordinate and background independent quantum general relativity theory.

• Completing the Model:
– Fixing the degenerate contributions: A careful study of the asymptotic limit for the Riemannian Barrett-Crane model [31] revealed the existence of the degenerate contributions. These contributions are not only present in the model but they also dominate the asymptotic limit. This could be considered as a result of not having a nice way to impose the last two Barrett-Crane constraints which ensure the non-zero volumes. It is possible that the semi-classical limit may not be the same as the asymptotic limit. If the semi-classical limit is related to the asymptotic limit then one must find a physical explanation for the degenerate contributions or find a way to fix them. The semi-classical limit issues will be further discussed below.

– Sum over the spin foams and unitarity: Spin foam models are essentially the path integral quantization of the discretized gravitational actions. The spin foam amplitudes are the quantum transition amplitudes between spin networks. The spin foam transition amplitudes from one spin network associated with a hypersurface to another spin network associated with a similar hypersurface can be calculated. Assume these two spin networks are associated with graphs (dual triangulations) of different sizes. Then the Hilbert space associated with the two hypersurfaces need not be of the same size. In this case the spin foam transition amplitudes are clearly not unitary. Thus it becomes necessary that a sum over triangulation may need to be performed to realize unitarity, which directly leads to the field theory over group space formulations. But after performing the summation it is not clear and it is not yet shown that the spin foam transition amplitudes are unitary. So, this issue needs to be explored.

– Extracting Physics: An important area that needs to be explored is to understand the relation between classical physics and spin foam models. Clearly the spin foams are founded on the discretization and the quantization ideas of classical general relativity. But, extracting the classical physics from spin foam models needs to be done at various levels.

– Semiclassical and continuum limit issues: An important question is how to calculate rigorously the semiclassical limit for quantum general relativity. The calculation of the semiclassical limits is a less understood problem even in the conventional quantum mechanics itself [41]. In the case of spin foam models the asymptotic limit is usually considered to be the semi-classical limit of quantum general relativity. This has been motivated by the idea of semi-classical limits of the angular momentum [40]. But even though the mathematics involved in the spin foams is that of angular momentum calculus, the physics is not the same. So, the question is whether the asymptotic limit is the same as the semiclassical limit? If not, what is the semiclassical limit?
Convergence issues and amplitudes of lower dimensional simplices: The convergence issues involved in spin foam models are different from that involved in the usual perturbative quantum field theories like QED. In perturbative QED we are summing a series of terms (Feynman diagrams) corresponding to the various orders of the coupling constant. Calculating more elements of the series increases the precision of the quantity calculated. For the case of spin foam models this interpretation clearly does not hold. An important question is whether the partition functions of the spin foams are convergent? In general the partition functions spin foams do not converge [2]. Modified Barrett-Crane models for real general relativity based on GFTs were proposed [42]. It has been demonstrated that by choosing proper choices of quantum amplitudes for simplices in less than three dimensions a convergent spin foam model can be defined [42]. Before investigating convergence issues, the next important step might be the construction of a spin foam model with a physically motivated choice of amplitudes for lower dimensional simplices addressing all the issues discussed before and any other relevant issues.

• Unification with Matter: Unification with matter needs to investigated for the $SO(4, C)$ general relativity with the reality constraint for both the quantum and classical case. Currently there are various proposals and studies for the inclusion matter in case of the four dimensional and the three dimensional general relativity theories [43].

• Relationship with Canonical Quantum general relativity:

  – Relationship to canonical quantum general relativity needs to be investigated. My work on the $(n-1) + 1$ formulation of the spin foams [33] explicitly demonstrates how to relate the spin network functionals of canonical quantum general relativity to the spin foams of BF theory and General Relativity. But the important issues are in interpreting the diffeomorphism and Hamiltonian constraints in the context of the spin foams.

  – An important question that arises is whether there is a rigorous relationship of the reality constraint in the spin foams to that of the reality condition [23] in canonical quantum general relativity? Realization of the reality condition quantum mechanically is a non-trivial problem in canonical quantum general relativity. In fact many of the recent advances [24] in canonical quantum general relativity have been made by converting the complex formulation of the theory to a real formulation by transforming the configuration variable a complex $SL(2, C)$ connection to a real $SU(2)$ connection through a Legendre transformation [25].
7 Acknowledgement.

I thank Allen Janis, George Sparling and John Baez for correspondences.

A Unitary Representations of SL(2,C)

The Representation theory of $SL(2, C)$ was developed by Gelfand and Naimarck \[21\]. Representation theory of $SL(2, C)$ can be developed using functions on $C^2$ which are homogenous in their arguments. The space of functions $D_\chi$ is defined as functions $f(z_1, z_2)$ on $C^2$ whose homogeneity is described by

$$f(az_1, az_2) = a^{\chi_1-1}a^{\chi_2-1}f(z_1, z_2),$$

for all $a \neq 0$, where $\chi$ is a pair $(\chi_1, \chi_2)$. The linear action of $SL(2, C)$ on $C^2$ defines a representation of $SL(2, C)$ denoted by $T_\chi$. Because of the homogeneity of functions of $D_\chi$, the representations $T_\chi$ can be defined by its action on the functions $\phi(z)$ of one complex variable related to $f(z_1, z_2) \in D_\chi$ by

$$\phi(z) = f(z, 1).$$

There are two qualitatively different unitary representations of $SL(2, C)$: the principal series and the supplementary series, of which only the first one is relevant to quantum general relativity. The principal unitary irreducible representations of $SL(2, C)$ are the infinite dimensional. For these $\chi_1 = -\bar{\chi}_2 = \frac{n+i\rho}{2}$, where $n$ is an integer and $\rho$ is a real number. In this article I would like to label the representations by a single complex number $\chi = \frac{n}{2} + i\frac{\rho}{2}$, wherever necessary. The $T_\chi$ representations are equivalent to $T_{-\chi}$ representations \[21\].

Let $g$ be an element of $SL(2, C)$ given by

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where $\alpha, \beta, \gamma$ and $\delta$ are complex numbers such that $\alpha \delta - \beta \gamma = 1$. Then the $D_\chi$ representations are described by the action of a unitary operator $T_\chi(g)$ on the square integrable functions $\phi(z)$ of a complex variable $z$ as given below:

$$T_\chi(g)\phi(z) = (\beta z_1 + \delta)\chi^{-1}(\beta z_1 + \delta)\bar{\chi}^{-1}\phi(\frac{\alpha z + \gamma}{\beta z + \delta}). \quad (25)$$

This action on $\phi(z)$ is unitary under the inner product defined by

$$(\phi(z), \eta(z)) = \int \overline{\phi(z)}\eta(z)d^2z,$$

where $d^2z = \frac{i}{2}dz \wedge d\bar{z}$ and I would like to adopt this convention everywhere. Completing $D_\chi$ with the norm defined by the inner product makes it into a Hilbert space $H_\chi$.  

\[21\]These functions need not be holomorphic but infinitely differentiable may be except at the origin $(0, 0)$. 

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Equation (25) can also be written in kernel form [17],

\[ T_\chi(g)\phi(z_1) = \int T_\chi(g)(z_1, z_2)\phi(z_2)d^2z_2, \]

Here \( T_\chi(g)(z_1, z_2) \) is defined as

\[ T_\chi(g)(z_1, z_2) = (\beta z_1 + \delta)^{-1}(\bar{\beta}\bar{z}_1 + \bar{\delta})^{-1}\delta(z_2 - g(z_1)), \] (26)

where \( g(z_1) = \frac{az_1 + \gamma}{\bar{z}_1 + \bar{\gamma}} \). The Kernel \( T_\chi(g)(z_1, z_2) \) is the analog of the matrix representation of the finite dimensional unitary representations of compact groups.

An infinitesimal group element, \( a \), of \( SL(2, \mathbb{C}) \) can be parameterized by six real numbers \( \varepsilon_k \) and \( \eta_k \) as follows [44]:

\[ a \approx I + i\sum_{k=1}^{3} (\varepsilon_k \sigma_k + \eta_k i\sigma_k), \]

where the \( \sigma_k \) are the Pauli matrices. The corresponding six generators of the \( \chi \) representations are the \( H_k \) and the \( F_k \). The \( H_k \) correspond to rotations and the \( F_k \) correspond to boosts. The bi-invariant measure on \( SL(2, \mathbb{C}) \) is given by

\[ dg = \left( \frac{i}{2} \right)^3 \frac{d^2\beta d^2\gamma d^2\delta}{|\delta|^2} = \left( \frac{i}{2} \right)^3 \frac{d^2\alpha d^2\beta d^2\gamma}{|\alpha|^2}. \]

This measure is also invariant under inversion in \( SL(2, \mathbb{C}) \). The Casimir operators for \( SL(2, \mathbb{C}) \) are given by

\[ \hat{C} = \det \begin{bmatrix} \hat{X}_3 & \hat{X}_1 - i\hat{X}_2 \\ \hat{X}_1 + i\hat{X}_2 & -\hat{X}_3 \end{bmatrix} \]

and its complex conjugate \( \hat{C} \) where \( X_i = F_i + iH_i \). The action of \( C \ (\hat{C}) \) on the elements of \( D_\chi \) reduces to multiplication by \( \chi_1^2 - 1 \ (\chi_2^2 - 1) \). The real and imaginary parts of \( C \) are another way of writing the Casimirs. On \( D_\chi \) they reduce to the following

\[ \text{Re}(\hat{C}) = \left( -\rho^2 + \frac{n^2}{4} - 1 \right) \hat{I}, \]

\[ \text{Im}(\hat{C}) = \rho n \hat{I}. \]

The Fourier transform theory on \( SL(2, \mathbb{C}) \) was developed in Ref. [21]. If \( f(g) \) is a square integrable function on the group, it has a group Fourier transform defined by

\[ F(\chi) = \int f(g)T_\chi(g)dg, \] (27)

where is \( F(\chi) \) is linear operator defined by the kernel \( K_\chi(z_1, z_2) \) as follows:

\[ F(\chi)\phi(z) = \int K_\chi(z, \hat{z})\phi(\hat{z})d^2\hat{z}. \]
The associated inverse Fourier transform is

\[ f(g) = \frac{1}{8\pi^4} \int \text{Tr}(F(\chi)T_\chi(g^{-1}))\chi\bar{\chi}d\chi, \quad (28) \]

where the \( \int d\chi \) indicates the integration over \( \rho \) and the summation over \( n \).

From the expressions for the Fourier transforms, I can derive the orthonormality property of the \( T_\chi \) representations,

\[ \int_{SL(2, C)} T_{z_1}^{z_1}(g)T_{z_2}^{z_2}(g)dg = \frac{8\pi^4}{\chi_1\chi_1} \delta(\chi_1 - \chi_2)\delta(z_1 - \hat{z}_1)\delta(z_2 - \hat{z}_2), \]

where \( T_\chi^\dagger \) is the Hermitian conjugate of \( T_\chi \).

The Fourier analysis on \( SL(2, C) \) can be used to study the Fourier analysis on the complex three sphere \( CS^3 \). If \( x = (a, b, c, d) \in CS^3 \) then the isomorphism \( g: CS^3 \rightarrow SL(2, C) \) can be defined by the following:

\[ g(x) = \begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}. \]

Then, the Fourier expansion of \( f(x) \in L^2(CS^3) \) is given by

\[ f(x) = \frac{1}{8\pi^4} \int \text{Tr}(F(\chi)T_\chi(g(x)^{-1}))\chi\bar{\chi}d\chi \]

and its inverse is

\[ F(\chi) = \int f(g)T_\chi(g(x))dx, \]

where the \( dx \) is the measure on \( CS^3 \). The measure \( dx \) is equal to the bi-invariant measure on \( SL(2, C) \) under the isomorphism \( g \).

The expansion of the delta function on \( SL(2, C) \) from equation (28) is

\[ \delta(g) = \frac{1}{8\pi^4} \int \text{tr} [T_\chi(g)] \chi\bar{\chi}d\chi. \quad (29) \]

Let me calculate the trace \( \text{tr} [T_\chi(g)] \). If \( \lambda = e^{\rho + i\theta} \) and \( \frac{1}{\lambda} \) are the eigen values of \( g \) then

\[ \text{tr} [T_\chi(g)] = \frac{\lambda^{x_1}\bar{\lambda}^{x_2} + \lambda^{-x_1}\bar{\lambda}^{-x_2}}{|\lambda - \lambda^{-1}|^2}, \]

which is to be understood in the sense of distributions [21]. The trace can be explicitly calculated as

\[ \text{tr} [T_\chi(g)] = \frac{\cos(\eta\rho + n\theta)}{2|\sinh(\eta + i\theta)|^2}. \quad (30) \]

Therefore, the expression for the delta on \( SL(2, C) \) explicitly is

\[ \delta(g) = \frac{1}{8\pi^4} \sum_n \int d\rho(n^2 + \rho^2) \frac{\cos(\eta\rho + n\theta)}{|\sinh(\eta + i\theta)|^2}. \quad (31) \]
Let us consider the integrand in equation (28). Using equation (27) in it we have

\[ Tr(F(\chi)T_\chi(g^{-1}))\chi \tilde{\chi} = \chi \tilde{\chi} \int f(\hat{g}) Tr(T_\chi(\hat{g})T_\chi(g^{-1})) d\hat{g} \]

\[ = \chi \tilde{\chi} \int f(\hat{g}) Tr(T_\chi(\hat{g}g^{-1})) d\hat{g}. \quad (32) \]

But, since the trace is insensitive to an overall sign of \( \chi \), so are the terms of the Fourier expansion of the \( L^2 \) functions on \( SL(2, C) \) and \( CS^3 \).

**B  Unitary Representations of \( SO(4, C) \)**

The group \( SO(4, C) \) is related to its universal covering group \( SL(2, C) \times SL(2, C) \) by the relationship \( SO(4, C) \approx \frac{SL(2, C) \times SL(2, C)}{Z_2} \). The map from \( SO(4, C) \) to \( SL(2, C) \times SL(2, C) \) is given by the isomorphism between complex four vectors and \( GL(2, C) \) matrices. If \( X = (a, b, c, d) \) then \( G : C^4 \rightarrow GL(2, C) \) can be defined by the following:

\[ G(X) = \begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}. \]

It can be easily inferred that \( \det G(X) = a^2 + b^2 + c^2 + d^2 \) is the Euclidean norm of the vector \( X \). Then, in general a \( SO(4, C) \) rotation of a vector \( X \) to another vector \( Y \) is given in terms of two arbitrary \( SL(2, C) \) matrices \( g_L^A \), \( g_R^A \in SL(2, C) \) by

\[ G(Y)^{AA'} = g_L^A g_R^{A'} G^{AB}(X), \]

where \( G^{AB}(X) \) is the matrix elements of \( G(X) \). The above transformation does not differentiate between \( (L_B^A, R_B^{A'}) \) and \( (-L_B^A, -R_B^{A'}) \) which is responsible for the factor \( Z_2 \) in \( SO(4, C) \approx \frac{SL(2, C) \times SL(2, C)}{Z_2} \).

The unitary representation theory of the group \( SL(2, C) \times SL(2, C) \) is easily obtained by taking the tensor products of two Gelfand-Naimark representations of \( SL(2, C) \). The Fourier expansion for any function \( f(g_L, g_R) \) of the universal cover is given by

\[ f(g_L, g_R) = \frac{1}{64\pi^8} \int \chi_L \bar{\chi}_L \chi_R \bar{\chi}_R F(\chi_L, \chi_R) T_\chi(g_L^{-1}) T_\chi(g_R^{-1}) d\chi_L d\chi_R, \]

where \( \chi_L = \frac{n_L + i\omega}{2} \) and \( \chi_R = \frac{n_R + i\omega}{2} \). The Fourier expansion on \( SO(4, C) \) is given by reducing the above expansion such that \( f(g_L, g_R) = f(-g_L, -g_R) \).

From equation (20) I have

\[ tr [T_\chi(-g)] = (-1)^n tr [T_\chi(-g)], \]

where \( \chi = \frac{n + i\omega}{2} \). Therefore

\[ f(-g_L, -g_R) = \frac{1}{8\pi^4} \int \chi_L \bar{\chi}_L \chi_R \bar{\chi}_R F(\chi_L, \chi_R)(-1)^n T_\chi(g_L^{-1}) T_\chi(g_R^{-1}) d\chi_L d\chi_R. \]
This implies that for \( f(g_L, g_R) = f(-g_L, -g_R) \), I must have \((-1)^{n_L+n_R} = 1\). From this, I can infer that the representation theory of \( SO(4, C) \) is deduced from the representation theory of \( SL(2, C) \times SL(2, C) \) by restricting \( n_L + n_R \) to be even integers. This means that \( n_L \) and \( n_R \) should be either both odd numbers or even numbers. I would like to denote the pair \((\chi_L, \chi_R) (n_L + n_R \text{ even})\) by \( \omega \).

There are two Casimir operators available for \( SO(4, C) \), namely
\[
\varepsilon_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} \quad \text{and} \quad \eta_{IJKL} \hat{B}^{IJ} \hat{B}^{KL}.
\]
The elements of the representation space \( D_{\chi_L} \otimes D_{\chi_R} \) are the eigen states of the Casimirs. On them, the operators reduce to the following:
\[
\varepsilon_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 - \chi_R^2}{2} \quad \text{and} \quad (33)
\]
\[
\eta_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 + \chi_R^2 - 2}{2}.
\]

C  Unitary Representations of \( SU(1, 1) \)

The unitary representations of \( SU(1, 1) \approx SL(2, R) \), given in Ref. [45], is defined similar to that of \( SL(2, C) \). The main difference is that the \( D_\chi \) are now functions \( \phi(z) \) on \( C^1 \). The representations are indicated by a pair \( \chi = (\tau, \varepsilon) \), \( \varepsilon \) is the parity of the functions \((\varepsilon = 0 \text{ for even functions and } \frac{1}{2} \text{ for odd functions})\) and \( \tau \) is a complex number defining the homogeneity:
\[
\phi(az) = |a|^{2\tau} \text{sgn}(a)^{2\varepsilon} \phi(z),
\]
where \( a \) is a real number. Because of homogeneity the \( D_\chi \) functions can be related to the infinitely differentiable functions \( \phi(e^{i\theta}) \) on \( S^1 \) where \( \theta \) is the coordinate on \( S^1 \). The representations are defined by
\[
T_\chi(g)\phi(e^{i\theta}) = (\beta e^{i\theta} + \bar{\alpha})^{\tau+\varepsilon} (\bar{\beta} e^{-i\theta} + \alpha)^{\tau-\varepsilon} \phi(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}).
\]
(35)

There are two types of the unitary representations that are relevant for quantum general relativity: the continuous series and the discrete series. For the continuous series \( \chi = (i \rho - \frac{1}{2}, \varepsilon) \), where \( \rho \) is a non-zero real number. Let me denote the continuous series representations with suffix or prefix \( c \), for example \( T^c_\chi \).

There are two types of discrete series representations which are indicated by signs \( \pm \). They have their respective homogeneity as \( \chi_{\pm} = (l, \varepsilon^\pm_l) \) where \( \varepsilon^\pm_l = \pm 1 \) is defined by the condition \( l \pm \varepsilon^\pm_l \) is an integer. Let me denote the representations as \( T^+_l \) and \( T^-_l \). The \( T^+_l \) (\( T^-_l \)) representations can be re-expressed as linear operators on the functions \( \phi_{\pm}(z) \) on \( C^1 \) that are analytical inside (outside) the unit circle. The \( T^\pm_l (g) \) are defined as
\[
T^\pm_l(g)\phi_{\pm}(z) = |\beta z + \bar{\alpha}|^{2l} \phi_{\pm}(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}).
\]
The inner products are defined by
\[
(f_1, f_2)_c = \frac{1}{2\pi} \int_0^{2\pi} f_1(e^{i\theta}) \overline{f_2(e^{i\theta})} d\theta,
\]
\[
(f_1, f_2)_+^l = \frac{1}{\Gamma(-2l-1)} \int_{|z|<1} (1-|z|)^{-2l-2} f_1(z) f_2(z) \frac{dzd\bar{z}}{2\pi i},
\]
\[
(f_1, f_2)_-^l = \frac{1}{\Gamma(-2l-1)} \int_{|z|>1} (1-|z|)^{-2l-2} f_1(z) f_2(z) \frac{dzd\bar{z}}{2\pi i}.
\]

The Fourier transforms are defined for the unitary representations by
\[
F_c(\chi) = \int f(g) T^c_\chi(g) dg,
\]
\[
F_+(l) = \int f(g) T^+_l(g) dg, \quad \text{and}
\]
\[
F_-(l) = \int f(g) T^-_l(g) dg,
\]
where \(dg\) is the bi-invariant measure on the group.

The inverse Fourier transform is defined by
\[
f(g) = \frac{1}{4\pi^2} \left\{ \sum_{l \in \frac{1}{2}N_0} \sum_{\epsilon} \sum_{l} \frac{1}{\rho_0} \frac{1}{\rho} \frac{1}{\rho} \right\} + \frac{1}{2\pi} \int_0^{\infty} \frac{1}{\rho} \frac{1}{\rho} \right\} + \sum_{\epsilon} \int_0^{\infty} \rho \frac{1}{\rho} \frac{1}{\rho} \right\}.
\]

The \(T_{(\tau,\varepsilon)}\) is equivalent to \(T_{(-\tau-1,0)}\). The Casimir operator for the \(T_\chi\) representations (all) can be defined similar to \(SU(2)\) and its eigen values are
\[
C = \frac{\tau}{2} \left( \tau + 1 \right),
\]
where the \(\tau\) comes from \(\chi = (\tau, \varepsilon)\). The \(\tau\) in this section is related to the \(\chi\) in the representations of \(SL(2, \mathbb{C})\) by \(\chi = \frac{\tau}{2} + \frac{1}{2}\). The expressions for the Casimirs of the two groups differ by a factor of 4.

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