Ramanujan Cayley graphs of Frobenius groups

Miki HIRANO, Kohei KATATA and Yoshinori YAMASAKI

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Abstract

In this paper, we determine the bound of the valency of Cayley graphs of Frobenius groups with respect to normal Cayley subsets which guarantees to be Ramanujan. We see that if the ratio between the orders of the Frobenius kernel and complement is not so small, then this bound coincides with the trivial one coming from the trivial estimate of the largest non-trivial eigenvalue of the graphs. Moreover, in the cases of the dihedral groups of order twice odd primes, which are special cases of the Frobenius groups, we determine the same bound for the Cayley graphs of the groups with respect to not only normal but also all Cayley subsets. As is the case of abelian groups which we have treated in the previous papers, such a bound is equal to the trivial one in the above sense or, as exceptional cases, exceeds one from it. We then clarify that the latter occurs if and only if the corresponding prime is represented by a quadratic polynomial in a finite family.

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1 Introduction

Let $X$ be a $k$-regular graph with standard assumptions, that is, finite, undirected, connected and simple. The graph $X$ is called Ramanujan if its largest non-trivial eigenvalue (in the sense of absolute value) is not greater than the Ramanujan bound $2\sqrt{k-1}$. The Ramanujan property of the graph means that the associated Ihara zeta function satisfies the “Riemann hypothesis”, which enables us to have a good estimate for the number of the prime cycles in it (see, e.g., [T]). See [La] for the other relations between this property and various mathematical objects.

We have considered the following problem in our previous papers [HKY] [K]. Let $G$ be a finite group and $S$ a set of Cayley subsets of $G$ which includes $G \setminus \{1\}$ with 1 being the identity element of $G$. Put

$$\mathcal{X} = \mathcal{X}_{G,S} = \{ X(S) \mid S \in S \},$$

where $X(S)$ is the Cayley graph of $G$ with respect to the Cayley subset $S \in S$. Letting $\mathcal{L} = \{ |G| - |S| \mid S \in S \}$ be the set of “covalencies” of graphs in $\mathcal{X}$, we write $\mathcal{X} = \bigsqcup_{l \in \mathcal{L}} \mathcal{X}_l$ where $\mathcal{X}_l = \{ X(S) \in \mathcal{X} \mid |G| - |S| = l \}$. Notice that $\mathcal{X}_1 = \{ K_{|G|} \}$ where $K_{|G|} = X(G \setminus \{1\})$ is the complete graph with $|G|$-vertices. According to [AR], some neighbors of $K_{|G|}$ are expected to be Ramanujan. We want to estimate them precisely, that is, to determine the number

$$\hat{l}_{G,S} = \max \{ l \in \mathcal{L} \mid X \in \mathcal{X}_k \text{ is Ramanujan for all } 1 \leq k \leq l \}$$

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of edge-removal preserving the Ramanujan property from the complete graph $K_{|G|} \in \mathcal{X}$.

Previously, we have discussed this problem when $G$ is abelian and $\mathcal{S} = \tilde{\mathcal{S}}$ is the set of all Cayley subsets of $G$. In this paper, we treat the cases when $G$ is a Frobenius group and $\mathcal{S} = \mathcal{S}_0$ is consist of all normal Cayley subsets of $G$. Here we call a Cayley subset normal if it is a union of conjugacy classes of $G$, and a Frobenius group is a non-abelian finite group such as the dihedral group $D_{2m}$ of order $2m$ with odd $m$, the non-abelian $pq$-group $F_{p,q}$ with odd primes $p$, $q$, etc (for the definition of the Frobenius groups, see [CR]). The main result is the following theorem.

**Theorem 1.1.** Let $G = N \rtimes H$ be a Frobenius group where $N$ and $H$ are the Frobenius kernel and complements, respectively, and $\mathcal{S}_0$ the set of all normal Cayley subsets of $G$. If $\frac{|N|-1}{|H|} \geq 4$, then it holds that

$$\hat{l}_{G, \mathcal{S}_0} = l_0,$$

where $l_0 = \max\{l \in \mathcal{L} \mid l \leq 2(\sqrt{|G|} - 1)\}$.

We remark that, when $G$ is non-abelian, the situation is completely different from whether $S$ is in $\mathcal{S}_0$ or not. Actually, when $S$ is normal, we have expressions of eigenvalues of $X(S)$ in terms of irreducible characters of $G$. Nevertheless, for the case of the dihedral group $G = D_{2m}$ of order $2m$ with odd $m$, one can calculate eigenvalues of $X(S)$ explicitly for any Cayley subset $S \in \tilde{\mathcal{S}}$ because the dimensions of irreducible representations of $D_{2m}$ are at most two. As a consequence, one can show that there exists $\varepsilon_m \in \{0, 1\}$ such that

$$\hat{l}_{D_{2m}, \tilde{S}} = \hat{l}_{D_{2m}, \mathcal{S}_0} + \varepsilon_m$$

for each $m$. For simplicity, we restrict to the case where $m = p$ is prime in this paper. When $\lfloor 2\sqrt{2p} \rfloor$ is even (here, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ represents for the largest integer not exceeding $x$), from Theorem [12] one immediately shows that $\varepsilon_p = 0$. On the other hand when $\lfloor 2\sqrt{2p} \rfloor$ is odd, we can see that most of $\varepsilon_p$’s are equal to 0. Now let us call $p$ exceptional if $\lfloor 2\sqrt{2p} \rfloor$ is odd and $\varepsilon_p = 1$ and ordinary otherwise. Similarly to the abelian results, we have the following theorem for $D_{2p}$.

**Theorem 1.2.** The odd prime $p$ is exceptional if and only if it is of the form of one of the following quadratic polynomials;

$$p = \begin{cases} 2k^2 + 4k - 3 & (k \geq 5), \\ 2k^2 + 4k - 1 & (k \geq 3), \\ 2k^2 + 4k + 1 & (k \geq 3) \end{cases}$$

or

$$p = \begin{cases} 2k^2 + 6k - 1 & (k \geq 7), \\ 2k^2 + 6k + 1 & (k \geq 3), \\ 2k^2 + 6k + 3 & (k \geq 3). \end{cases}$$

The classical Hardy-Littlewood conjecture asserts that every quadratic polynomials express infinitely many primes under some standard conditions. We finally remark that the above result implies that there exists infinitely many exceptional primes if and only if the Hardy-Littlewood conjecture is true for at least one of the above six quadratic polynomials.

Throughout of the present paper, we denote by $\mathbb{P}$ the set of all odd primes and $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ for $m \in \mathbb{Z}_{\geq 1}$.

# 2 Preliminary

## 2.1 Cayley graphs

Let $X$ be a $k$-regular graph with $m$-vertices ($m < \infty$) which is undirected, connected and simple. The adjacency matrix $A_X$ of $X$ is the symmetric matrix of size $m$ whose entry is 1 if the
corresponding pair of vertices are connected by an edge and 0 otherwise. We call the eigenvalues of \( A_X \) the eigenvalues of \( X \). The set \( \Lambda(X) \) of all eigenvalues of \( X \) is given as

\[
\Lambda(X) = \{ \lambda_i \mid k = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{m-1} \geq -k \}.
\]

Let \( \mu(X) \) be the largest non-trivial eigenvalue of \( X \) in the sense of absolute value, that is,

\[
\mu(X) = \max \{ |\lambda| \mid \lambda \in \Lambda(X), \ |\lambda| \neq k \}.
\]

Then, \( X \) is called Ramanujan if the inequality \( \mu(X) \leq 2\sqrt{k-1} \) holds. Here the constant \( 2\sqrt{k-1} \) is often called the Ramanujan bound for \( X \) and is denoted by \( RB(X) \).

Let \( G \) be a finite group with the identity element 1 and \( S \) a Cayley subset of \( G \), that is, it is a symmetric generating subset of \( G \) without 1. We denote by \( X(S) \) the Cayley graph of \( G \) with respect to the Cayley subset \( S \). This is undirected, connected and simple \( |S| \)-regular graph with the vertex set \( G \) and the edge set \( \{(x, y) \in G^2 \mid x^{-1}y \in S\} \). In what follows, for a Cayley subset \( S \), we write \( \Lambda(S) = \Lambda(X(S)) \), \( \mu(S) = \mu(X(S)) \), \( RB(S) = RB(X(S)) \), and so on. It is well known that if the Cayley subset \( S \) of \( G \) is normal, that is, it is a union of conjugacy classes of \( G \), then the eigenvalues of \( X(S) \) can be written by irreducible characters of \( G \).

**Lemma 2.1** (cf. [KS]). Let \( G \) be a finite group and \( \hat{G} \) the set of all irreducible characters of \( G \). For a normal Cayley subset \( S \) of \( G \), we have \( \Lambda(S) = \{ \lambda_{\chi} \mid \chi \in \hat{G} \} \) where

\[
\lambda_{\chi} = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)
\]

with the multiplicity \( \chi(1)^2 \).

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### 2.2 A problem for Ramanujan Cayley graphs

Let \( S \) be the set of all normal Cayley subsets of \( G \). For \( S \in \mathcal{S} \), we define \( l(S) = |G| - |S| = |G \setminus S| \) and call it a covalency of \( X(S) \). Letting \( \mathcal{L} = \{ l(S) \mid S \in \mathcal{S} \} \), we write \( \mathcal{S} = \bigsqcup_{l \in \mathcal{L}} \mathcal{S}_l \) where \( \mathcal{S}_l = \{ S \in \mathcal{S} \mid l(S) = l \} \). As we have explained in the introduction, our purpose is to determine

\[
\hat{l} = \hat{l}(G) = \max \{ l \in \mathcal{L} \mid X(S) \text{ is Ramanujan for all } S \in \mathcal{S}_k (1 \leq k \leq l) \}.
\]

The following lemma is fundamental.

**Lemma 2.2.** Let \( S \in \mathcal{S} \). If \( l(S) \leq 2(\sqrt{|G|} - 1) \), then \( X(S) \) is Ramanujan.

**Proof.** Take \( S \in \mathcal{S} \) with \( l(S) \leq \frac{|G|}{2} \). For any non-trivial irreducible character \( \chi \in \hat{G} \), we have

\[
\lambda_{\chi} = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s) = -\frac{1}{\chi(1)} \sum_{s \in G \setminus S} \chi(s)
\]

from the orthogonality. This together with \( |\chi(s)| \leq \chi(1) \) shows that \( |\lambda_{\chi}| \leq \min\{|s|, l(S)\} = l(S) \). Hence, if \( l(S) \leq RB(S) = 2\sqrt{|G|} - l(S) - 1 \), equivalently \( l(S) \leq 2(\sqrt{|G|} - 1) \), then \( X(S) \) is Ramanujan. Remark that \( 2(\sqrt{|G|} - 1) \leq \frac{|G|}{2} \) for any \( G \).

\[
\hat{l}_0 = l_0(G) = \max \{ l \in \mathcal{L} \mid l \leq 2(\sqrt{|G|} - 1) \}.
\]

From Lemma 2.2, we have \( \hat{l}_0 \leq \hat{l} \). We call \( \hat{l}_0 \) a trivial (lower) bound of \( \hat{l} \).
3 Ramanujan Frobenius graphs

In this section, we study Cayley graphs of Frobenius groups with respect to normal Cayley subsets. We call such graphs Frobenius graphs. From now on, for a group $G$ and $x, y \in G$, we write $x^y = y^{-1}xy$. Moreover, we denote by $\text{Conj}_G(x)$ the conjugacy class of $x \in G$ in $G$ and by $c(G)$ the number of conjugacy classes in $G$.

3.1 Character table of the Frobenius group

Let $G = N \rtimes H$ be a Frobenius group, where $N$ and $H$ are subgroups of $G$ called the Frobenius kernel and complement, respectively (see [CR] for details of the Frobenius groups and their characters). Notice that $r = |N| - 1 - |H|$ is a positive integer.

We first recall the character table of the Frobenius group $G$. It is known that a set of all representatives of the conjugacy classes of $G$ can be taken as $\{1\} \cup \{x_i\}_{i=1}^k \cup \{y_j\}_{j=1}^h$ where $x_i \in N \ (1 \leq i \leq k)$ and $y_j \in H \ (1 \leq j \leq h)$ with $k = \frac{c(N) - 1}{|H|}$ and $h = c(H) - 1$. Notice that $\text{Conj}_G(x_i) \subset N$ and $\text{Conj}_G(y_j) \subset G \setminus N$ with

$$|\text{Conj}_G(x_i)| = |\text{Conj}_N(x_i)||H|, \quad |\text{Conj}_G(y_j)| = |\text{Conj}_H(y_i)||N|.$$ 

The irreducible characters of $G$ are given as follows. Since $H \simeq G/N$, a non-trivial irreducible character of $H$ is corresponding to that of $G$ which has the kernel containing $N$. We write these as $\chi_\alpha$ ($1 \leq \alpha \leq h$). Moreover, for a non-trivial irreducible character $\psi_\beta$ of $N$, its induced character is again an irreducible character of $G$. We write these as $\phi_\beta = \text{Ind}(\psi_\beta)$ ($1 \leq \beta \leq k$). Notice that

$$\phi_\beta(x) = \frac{1}{|N|} \sum_{x^y \in N} \psi_\beta(x^y) = \sum_{z \in H} \psi_\beta(x^z).$$

It is known that these together with the trivial character 1 exhaust all irreducible characters of $G$. Now the character table of $G$ is given as follows:

|       | 1 | $x_i \ (1 \leq i \leq k)$ | $y_j \ (1 \leq j \leq h)$ |
|-------|---|----------------|----------------|
| 1     | 1 | 1              | 1              |
| $\chi_\alpha \ (1 \leq \alpha \leq h)$ | $\chi_\alpha (1)$ | $\chi_\alpha (1)$ | $\chi_\alpha (y_j)$ |
| $\phi_\beta \ (1 \leq \beta \leq k)$ | $|H|\psi_\beta (1)$ | $\sum_{z \in H} \psi_\beta (x^z)$ | 0 |

Table 1: The character table of the Frobenius group $G = N \rtimes H$.

3.2 Eigenvalues of Frobenius graphs

Now, let us calculate the eigenvalues of Frobenius graphs with respect to normal Cayley subsets. For subsets $X \subset \{x_i\}_{i=1}^k$ and $Y \subset \{y_j\}_{j=1}^h$, we put $S_{X,Y} = S_X \cup S_Y$ where

$$S_X = \bigcup_{x \in X} \text{Conj}_G(x), \quad S_Y = \bigcup_{y \in Y} \text{Conj}_G(y).$$
We say that $X \subset \{x_i\}_{i=1}^k$ (resp. $Y \subset \{y_j\}_{j=1}^h$) is symmetric if $S_X$ (resp. $S_Y$) is symmetric. It is clear that $S_{X,Y}$ is symmetric if and only if both $X$ and $Y$ are. We have

$$S \subset \left\{ S_{X,Y} \mid \text{both } X \subset \{x_i\}_{i=1}^k \text{ and } Y \subset \{y_j\}_{j=1}^h \text{ are symmetric} \right\}.$$

Notice that, if $S_{X,Y}$ in the right hand side satisfies $|S_{X,Y}| > \frac{|G|}{2}$, then it generates $G$ and hence is in $S$ and, moreover, $Y \neq \emptyset$ because otherwise $|S_{X,Y}| = |S_{X,\emptyset}| < |N| - 1$, which is contradict to $\frac{|G|}{2} \geq |N|$. This means that, in the determination of $l_0$ and $\hat{l}$, we may assume that $S \in S$ is always of the form of $S = S_{X,Y}$ for some symmetric subsets $X \subset \{x_i\}_{i=1}^k$ and $\emptyset \neq Y \subset \{y_j\}_{j=1}^h$.

From Lemma 3.1, we have

**Lemma 3.1.** For $X \subset \{x_i\}_{i=1}^k$ and $\emptyset \neq Y \subset \{y_j\}_{j=1}^h$, we have

$$\Lambda(S_{X,Y}) = \{\lambda_1\} \cup \{\lambda_{\alpha,1} \}_{\alpha=1}^{|S_X|} \cup \{\lambda_{\beta,1} \}_{\beta=1}^{|S_Y|},$$

where

$$(3.1) \quad \Lambda_1 = |S_{X,Y}| = |H| \sum_{x \in X} |\text{Conj}_X(x)| + |N| \sum_{y \in Y} |\text{Conj}_Y(y)|,$$

$$\lambda_{\alpha,1} = |H| \sum_{x \in X} |\text{Conj}_X(x)| + \frac{|N|}{\chi_{\alpha}(1)} \sum_{y \in Y} \chi_{\alpha}(y) |\text{Conj}_Y(y)|,$$

$$\lambda_{\beta,1} = \frac{1}{\psi_{\beta}(1)} \sum_{x \in X} \sum_{y \in Y} \psi_{\beta}(x)|\text{Conj}_X(x)|$$

with the multiplicities $1$, $\chi_{\alpha}(1)^2$ and $|H|^2 \psi_{\beta}(1)^2$, respectively.

**Proof.** These follow from the character table of $G$ obtained in the previous subsection. \qed

### 3.3 Main results

To determine $l_0$ and $\hat{l}$, we first describe the set $L = \{l(S) \mid S \in S\}$.

For symmetric subsets $X \subset \{x_i\}_{i=1}^k$ and $\emptyset \neq Y \subset \{y_j\}_{j=1}^h$, we respectively put

$$a_X = r - \sum_{x \in X} |\text{Conj}_X(x)|, \quad b_Y = |H| - 1 - \sum_{y \in Y} |\text{Conj}_Y(y)|.$$

We have $0 \leq a_X \leq r$, $0 \leq b_Y < |H| - 1$ and from (3.1)

$$(3.2) \quad l(S_{X,Y}) = |G| - |S_{X,Y}| = 1 + a_X |H| + b_Y |N|.$$

**Lemma 3.2.** Let $X, X' \subset \{x_i\}_{i=1}^k$ and $\emptyset \neq Y, Y' \subset \{y_j\}_{j=1}^h$ be symmetric subsets. Then, $l(S_{X,Y}) = l(S_{X',Y'})$ if and only if $(a_X, b_Y) = (a_{X'}, b_{Y'})$.

**Proof.** From (3.2), one sees that $l(S_{X,Y}) = l(S_{X',Y'})$ is equivalent to $(a_X - a_{X'})|H| + (b_Y - b_{Y'})|N| = 0$. Since $(a_X - a_{X'})|H| < |N|$, it has to hold that $a_X = a_{X'}$ and hence $b_Y = b_{Y'}$. \qed

Put $l(a, b) = 1 + a|H| + b|N|$. From Lemma 3.2, we write $S = \bigsqcup_{a \in A, b \in B} S_{(a,b)}$ where

$$A = \{a_X \mid X \subset \{x_i\}_{i=1}^k \text{ is symmetric} \} = \{a_1 < a_2 < \cdots < a_m\},$$

$$B = \{b_Y \mid \emptyset \neq Y \subset \{y_j\}_{j=1}^h \text{ is symmetric} \} = \{b_1 < b_2 < \cdots < b_n\}.$$
with \( m = |A| \) and \( n = |B| \). Remark that \( a_1 = 0 \) and \( a_m = r \), which respectively correspond to the cases \( X = \{x_i\}_{i=1}^k \) and \( X = \emptyset \). Similarly, \( b_1 = 0 \), which corresponds to the case \( Y = \{y_j\}_{j=1}^h \), and \( b_n < |H| - 1 \) because \( Y \neq \emptyset \). Moreover, when \( h \geq 2 \), since the center of \( H \) is not trivial, there exists \( y' \in \{y_j\}_{j=1}^h \) such that \( \text{Conj}_H(y') = \{y'\} \). This implies that \( b_2 = 1 \) if \( y' \) is not symmetric, that is, \( y'^2 \neq 1 \) and 2 otherwise. We below describe the relations among \( l(a,b) \) for \( a \in A \) and \( b \in B \):

\[
1 = l(a_1, b_1) < l(a_2, b_1) < \cdots < l(a_m, b_1) = (b_1 + 1)|N| = |N|
< b_2|N| + 1 = l(a_1, b_2) < l(a_2, b_2) < \cdots < l(a_m, b_2) = (b_2 + 1)|N|
< b_3|N| + 1 = l(a_1, b_3) < l(a_2, b_3) < \cdots < l(a_m, b_3) = (b_3 + 1)|N|
\vdots
< b_n|N| + 1 = l(a_1, b_n) < l(a_2, b_n) < \cdots < l(a_m, b_n) = (b_n + 1)|N|.
\]

The followings are our main results, which are involved to the determinations of \( l_0 \) and \( \hat{l} \) for Frobenius graphs with respect to normal Cayley subsets.

**Theorem 3.3.** Assume that \( r = \left| \frac{|N| - 1}{|H|} \right| \geq 4 \).

1. There exists \( 1 \leq i_0 < m \) such that \( l_0 = l(a_{i_0}, b_1) < |N| \).

2. It holds that \( \hat{l} = l_0 \).

**Proof.** Under the condition \( r \geq 4 \), we have \( |H| \leq \frac{|N| - 1}{r} < \frac{|N|}{r} \) and hence \( 2(\sqrt{|G|} - 1) < 2\sqrt{|N||H|} < 2\sqrt{|N|\frac{|N|}{r}} = |N| \). Therefore, the first assertion follows from the definition of \( l_0 \).

To prove the second one, it is sufficient to show that there exists \( S \in \mathcal{S}_{l(a_{i_0+1}, b_1)} \) such that \( X(S) \) is not Ramanujan. Actually, take any \( S = S_{X,Y} \in \mathcal{S}_{l(a_{i_0+1}, b_1)} \). Then, since \( Y = \{y_j\}_{j=1}^h \), we have from Lemma 3.1

\[
\lambda_{\chi_\alpha}(S_{X,Y}) = |H| \sum_{x \in X} |\text{Conj}_N(x)| + \frac{|N|}{\chi_\alpha(1)} \sum_{i=1}^h \chi_\alpha(y_j)|\text{Conj}_H(y_j)|
\]

\[
= |H| \left( |N| - 1 - a_{i_0+1} \right) + \frac{|N|}{\chi_\alpha(1)} (-\chi_\alpha(1))
\]

\[
= -(1 + a_{i_0+1})|H|
\]

\[
= -l(a_{i_0+1}, b_1)
\]

\[
= -l(S_{X,Y}).
\]

Here, the second equality follows from the orthogonality of characters together with the fact that \( \chi_\alpha \) is regarded as a non-trivial irreducible character of \( H \). This implies that \( |\lambda_{\chi_\alpha}(S_{X,Y})| = l(S_{X,Y}) \geq l_0 \) and hence \( |\lambda_{\chi_\alpha}(S_{X,Y})| > \text{RB}(S) \) by the definition of \( l_0 \). This ends the proof.

We remark that, since \( l(a_{i_0}, b_1) \leq 2(\sqrt{|G|} - 1) < l(a_{i_0+1}, b_1) \) with \( l(a_1, b_1) = 1 + a_1|H| \), it can be expressed as

\[
a_{i_0} = \max \left\{ a \in A \mid a \leq \frac{2\sqrt{|N||H|} - 3}{|H|} \right\}.
\]

Let us calculate \( \hat{l} \) for the following typical examples of the Frobenius groups.
First, for \( p \in \mathbb{P} \), consider the dihedral group
\[
D_{2p} = \mathbb{Z}_p \times \mathbb{Z}_2 = \langle x, y \mid x^p = y^2 = 1, \; y^{-1}xy = x^{-1} \rangle,
\]
which is a Frobenius group with \( r = \frac{p-1}{2} \). In this case, one can take representatives of the conjugacy classes of \( D_{2p} \) as \( \{1\} \cup \{x^v\}_{v=1}^{\frac{p-1}{2}} \cup \{y\} \). Since all the conjugacy classes \( \text{Conj}_{D_{2p}}(x^v) = \{x^v, x^{-v}\} \) for \( 1 \leq v \leq \frac{p-1}{2} \) and \( \text{Conj}_{D_{2p}}(y) = \{x^ay \mid 0 \leq a \leq p-1\} \) are symmetric, we have \( A = \{i \mid 0 \leq i \leq \frac{p-1}{2}\} \) and \( B = \{0\} \). Now, from Theorem 3.3 together with (3.3), we obtain the following result (notice that \( r \geq 4 \) if \( p \geq 11 \)).

**Corollary 3.4.** Let \( G = D_{2p} \) where \( p \in \mathbb{P} \). If \( p \geq 11 \), then we have
\[
\hat{l} = l_0 = 2\lfloor \sqrt{2p} - \frac{1}{2} \rfloor - 1.
\]

\[\square\]

We next consider the group
\[
F_{p,q} = \mathbb{Z}_p \times_u \mathbb{Z}_q = \langle x, y \mid x^p = y^q = 1, \; y^{-1}xy = x^v \rangle,
\]
where \( p, q \in \mathbb{P} \) with \( q \mid (p-1) \) and \( u \) an element of \( \mathbb{Z}_q^\times \) of order \( q \). It is known that \( F_{p,q} \) is also a Frobenius group with \( r = \frac{p-1}{q} \). Let \( S = \langle u \rangle = \{1, u, u^2, \ldots, u^{q-1}\} \) and \( V = \{v_1, \ldots, v_r\} \) a set of all representatives of \( \mathbb{Z}_q^\times / S \). Then, one can take representatives of the conjugacy classes of \( F_{p,q} \) as \( \{1\} \cup \{x^v\}_{v \in V} \cup \{y^b\}_{b=1}^{q-1} \). Because none the conjugacy classes \( \text{Conj}_{F_{p,q}}(x^v) = \{x^{vs} \mid s \in S\} \) for \( v \in V \) and \( \text{Conj}_{F_{p,q}}(y^b) = \{x^ay^b \mid 0 \leq a \leq p-1\} \) for \( 1 \leq b \leq q-1 \) are symmetric, we have \( A = \{2i \mid 0 \leq i \leq \frac{q-1}{2}\} \) and \( B = \{2j \mid 0 \leq j < \frac{q-1}{2}\} \). Similarly as above, one obtains the following result (notice that \( r \geq 4 \) if \( p \geq 4q + 1 \)).

**Corollary 3.5.** Let \( G = F_{p,q} \) where \( p, q \in \mathbb{P} \) with \( q \mid (p-1) \). If \( p \geq 4q + 1 \), then we have
\[
\hat{l} = l_0 = 2q\lfloor \sqrt{pq} - \frac{3}{2} \rfloor + 1.
\]

\[\square\]

**Remark 3.6.** There are several examples of the Frobenius groups with \( r = \frac{|N|-1}{|H|} \leq 3 \) for which the claims in Theorem 3.3 do not hold. Actually, consider the dihedral groups \( D_{2p} \) for \( p = 3, 5, 7 \), which are corresponding to the cases \( r = 1, 2, 3 \), respectively. In these cases, we have \( (m,n) = (\frac{p+1}{2},1) \) and can check that \( l_0 = l(a_{\frac{p+1}{2}}, b_1) = p-2 \). Moreover, it holds that \( \hat{l} = l(a_{\frac{p+1}{2}}, b_1) = p \) because the corresponding Cayley graph is \( X(S_{\emptyset, \{y\}}) \), which is Ramanujan because \( \Lambda(S_{\emptyset, \{y\}}) = \{\pm p, 0\} \).

## 4 Ramanujan dihedral graphs

In this section, we more precisely study Ramanujan Cayley graphs of the dihedral groups \( D_{2p} \) of order \( 2p \) with \( p \in \mathbb{P} \), which are special cases of the Frobenius groups. For simplicity, we call such graphs **dihedral graphs**.
4.1 An universal problem on Ramanujan Cayley graphs

We here consider our problem in more general situation. For a finite group $G$, let $\tilde{S}$ be the set of all Cayley subsets of $G$. Similarly to the case of $S$, we put $\tilde{L} = \{ l(S) \mid S \in \tilde{S} \}$, and write $\tilde{S} = \bigsqcup_{l \in \tilde{L}} \tilde{S}_l$ where $\tilde{S}_l = \{ S \in \tilde{S} \mid l(S) = l \}$. Now our interest is whether one can determine

$$\tilde{l} = \tilde{l}(G) = \max \left\{ l \in \tilde{L} \mid X(S) \text{ is Ramanujan for all } S \in \tilde{S}_k \ (1 \leq k \leq l) \right\}.$$ 

We remark that when $G$ is abelian, which we have studied in \cite{HKY}, $\tilde{l} = l$ because $\tilde{S} = S$.

The determination of $\tilde{l}$ is much more difficult rather than that of $l$ since one does not have explicit expressions of eigenvalues, such as \eqref{eigenvalue}, of general Cayley graphs. However, in the case of the dihedral group, we can manage this problem because of the fact that the dimensions of irreducible representations are at most two together with the relation

$$\tilde{l} < l_1 = \min \{ l \in \mathbb{L} \mid l > l_0 \},$$

which follows directly from the definition of $\tilde{l}$ and Theorem \ref{monodromy}.  

4.2 Initial results

Let us consider the dihedral graph $X(S)$ of $D_{2p}$ with respect to $S \in \tilde{S}$. Divide $D_{2p}$ into two parts as $D_{2p} = D_1 \sqcup D_2$ where $D_1 = \{ 1, x, x^2, \ldots, x^{p-1} \}$ and $D_2 = \{ y, xy, x^2y, \ldots, x^{p-1}y \}$. According to this decomposition, we write $S = S_1 \sqcup S_2$ and $l(S) = l_1(S) + l_2(S)$ where $S_1 = S \cap D_1$ and $l_i(S) = |D_i \setminus S_i| = p - |S_i|$ for $i = 1, 2$. Remark that, since any subset of $D_2$ is symmetric because the order of any element in $D_2$ is two, $S$ is symmetric if and only if $S_1$ is, which implies that $|S_1|$ is always even and hence $l_1(S)$ is odd. Define $z_j = z_j(S), w_j = w_j(S) \in \mathbb{C} \ (0 \leq j \leq p - 1)$ by $z_0 = |S_1| + |S_2|$, $w_0 = |S_1| - |S_2|$ and

$$z_j = \sum_{x^i \in S_1} \omega^{ja} = - \sum_{x^i \in D_1 \setminus S_1} \omega^{ja}, \quad w_j = \sum_{x^i y \in S_2} \omega^{ja} = - \sum_{x^i y \in D_2 \setminus S_2} \omega^{ja}$$

for $1 \leq j \leq p - 1$. Here, $\omega = e^{2\pi i/p}$. Note that $z_j \in \mathbb{R}$ because $S_1$ is symmetric. It is known that the eigenvalues of $X(S)$ are described by using $z_j$ and $w_j$.

**Lemma 4.1.**

(i) $\Lambda(S) = \{ \mu_j^{(+)} \mid 0 \leq j \leq p - 1 \}$ where $\mu_j^{(\pm)} = z_j \pm |w_j|$

(ii) Let $|\mu_j| = \max \{ |\mu_j^{(+)}|, |\mu_j^{(-)}| \}$. Then, we have $|\mu_j| = |z_j| + |w_j|.$

**Proof.** See, e.g., \cite{P} for the first assertion. The second one is direct.

One can obtain a lower bound of $\tilde{l}$ coming from the trivial estimate of the eigenvalues.

**Lemma 4.2.** For $p \geq 29$, we have $\tilde{l} \geq \lfloor 2\sqrt{2p} \rfloor - 2$.

**Proof.** We first remark that $\mu_0^{(+)} = |S_1| + |S_2| = |S|$ is the largest eigenvalue of $X(S)$ and hence can write $\mu(S) = \max \{ |\mu| \mid \mu \in \Lambda(S), \ |\mu| \neq \mu_0^{(+)} \} = \max \{ |\mu_0^{(-)}|, |\mu_1|, \ldots, |\mu_{p-1}| \}$.

Assume that $l(S) \leq \frac{p}{2}$, which implies that $l_i(S) \leq \frac{p}{2}$ for $i = 1, 2$. Then, from \eqref{proof}, for $1 \leq j \leq p - 1$, we see that $|\mu_j| = |z_j| + |w_j| \leq \min \{ |S_1|, l_1(S) \} + \min \{ |S_2|, l_2(S) \} \leq l_1(S) + l_2(S) = l(S)$. Moreover, it holds that $\mu_0^{(-)} = 2|S_1| - 2p + l(S) \leq l(S)$. These show that $\mu(S) \leq l(S)$. Therefore, if $l(S) \leq \text{RB}(S) = 2\sqrt{2p} - l(S) - 1$, equivalently $l(S) \leq \lfloor 2\sqrt{2p} \rfloor - 2$, then $X(S)$ is Ramanujan. Notice that $2\sqrt{2p} - 2 \leq \frac{p}{2}$ when $p \geq 29$. \hfill $\square$
By virtue of results on the Frobenius graphs obtained in the previous section, one gets an upper bound of $\bar{l}$. As a consequence, we can narrow the candidates of $\bar{l}$ down to at most two.

**Theorem 4.3.** Assume that $p \geq 29$.

(i) If $[2\sqrt{2p}]$ is even, then we have $\bar{l} = \hat{l} + 1$.

(ii) If $[2\sqrt{2p}]$ is odd, then we have $\bar{l} = \hat{l}$ or $\bar{l} = \hat{l} + 1$.

Here $\hat{l} = 2[\sqrt{2p} - \frac{1}{2}] - 1$ is obtained in Corollary 3.3.

**Proof.** We first remark that, for $\alpha \in \mathbb{R}$, it holds that

$$2|\alpha - \frac{1}{2}| - 1 = \begin{cases} |2\alpha| - 2 - 1 & (0 \leq \alpha - |\alpha| < \frac{1}{2} \text{ or } |2\alpha| \text{ is even}), \\ |2\alpha| - 2 & (\frac{1}{2} \leq \alpha - |\alpha| < 1 \text{ or } |2\alpha| \text{ is odd}). \end{cases}$$

Using this formula with $\alpha = \sqrt{2p}$, we see that $[2\sqrt{2p}] - 2$ coincides with $\hat{l} + 1$ (resp. $\hat{l}$) if $[2\sqrt{2p}]$ is even (resp. odd) and hence, from Lemma 1.12, $\bar{l} \geq \hat{l} + 1$ (resp. $\bar{l} \geq \hat{l}$). Now one obtains the results because $\bar{l} = l_1 = \hat{l} + 2$, which follows from (1.1).

From this theorem, it can be written as $\bar{l} = \hat{l} + \varepsilon$ for some $\varepsilon \in \{0, 1\}$. As is the case of the circulant graphs [HKY, K], we call $p$ exceptional if $[2\sqrt{2p}]$ is odd and $\varepsilon = 1$ and ordinary otherwise. Now our task is to clarify which $p \in \mathbb{P}$ is exceptional and whether such primes exist infinitely many.

**Remark 4.4.** The discussion in this section can be extended to that for $D_{2m}$ where $m$ is odd (not necessary prime), as we have done in the case of the circulant graphs in [HKY, K]. However, for simplicity, we only show results in the case where $m$ is odd prime.

### 4.3 A characterization of exceptional primes

In what follows, we assume that $[2\sqrt{2p}]$ is odd.

For $l \in \mathbb{N}$, let $\mu(l) = \max \{\mu(S) | S \subseteq S_l\}$ and $\text{RB}(l) = \text{RB}(S) = 2\sqrt{2p} - l - 1$ for $S \subseteq S_l$. From the definition, $p$ is exceptional if and only if $\mu(l + 1) \leq \text{RB}(l)$. To study this inequality, we at first construct $S \subseteq S_{l+1}$ such that $\mu(l + 1) = \mu(S)$.

For $l \in \mathbb{N}$, let $L(l) = \{(l_1, l_2) \in \mathbb{Z}_{\geq 0}^2 | l_1 + l_2 = l, l_1 \text{ is odd}\}$. Moreover, for $(l_1, l_2) \in L(l)$, define $S^{(l_1, l_2)} = S^{(l_1)} \cup S^{(l_2)} \in S_l$ by $S^{(l_1)} = D_1 \setminus \{1, x^{\pm 1}, x^{\pm 2}, \ldots, x^{\pm \frac{l_1-1}{2}}\}$ and $S^{(l_2)} = D_2 \setminus \{y, xy, x^{2}y, \ldots, x^{l_2-1}y\}$. One sees that $l_i(S^{(l_1, l_2)}) = l_i$ for $i = 1, 2$ and

$$z_j = \sum_{h=-\frac{l_1-1}{2}}^{\frac{l_1-1}{2}} \omega^{hj} = \frac{\sin \frac{\pi jl_2}{p}}{\sin \frac{\pi j}{2}}, \quad w_j = \sum_{h=0}^{\frac{l_2-1}{2}} \omega^{hj} = \frac{\sin \frac{\pi jl_2}{p} - \sin \frac{\pi j}{2}}{\sin \frac{\pi j}{2}},$$

whence $|\mu_j| = |\mu_j(l_1, l_2)|$ can be written as

$$|\mu_j| = |z_j| + |w_j| = \frac{\sin \frac{\pi jl_2}{p}}{\sin \frac{\pi j}{2}} + \frac{\sin \frac{\pi jl_2}{p}}{\sin \frac{\pi j}{2}} = 2 \frac{\sin \frac{\pi j}{p}}{\sin \frac{\pi j}{2}} \cos \frac{\pi j(l_1 - l_2)}{2p}.$$

Now let us write $\hat{l} = 2[\sqrt{2p} - \frac{1}{2}] - 1 = \lfloor 2\sqrt{2p} \rfloor - 2$ as

$$\hat{l} = 4k + r$$

for some $k \geq 0$ and $r \in \{1, 3\}$.
Lemma 4.5. Let \((\hat{l}_1, \hat{l}_2) = (\frac{i+1}{2}, \frac{i+1}{2})\) if \(r = 1\) and \((\frac{i+3}{2}, \frac{i-1}{2})\) otherwise. Then, we have
\[
\mu(\hat{l} + 1) = \mu(S(\hat{l}_1, \hat{l}_2)) = |\mu_1(\hat{l}_1, \hat{l}_2)|.
\]

Proof. Consider when \(|l_1-l_2|\) takes minimum under the condition that \(l_1 + l_2 = \hat{l} + 1\) and \(l_1\) is odd. Note that, since \(p\) is prime, \(|\mu_1(\hat{l}_1, \hat{l}_2)|\) takes maximum among \(|\mu_j(\hat{l}_1, \hat{l}_2)|\) for \(1 \leq j \leq p - 1\).

When \(\hat{l} = 4k + r\), we see that \(p \in I_{r,k} \cap \mathbb{P}\) where
\[
I_{r,k} = \left\{ t \in \mathbb{R} \mid [2\sqrt{2t} + 2 = 4k + r \right\} = \left[2k^2 + (r + 2)k + \frac{(r + 2)^2}{8}, 2k^2 + (r + 3)k + \frac{(r + 3)^2}{8}\right).
\]

This means that it is expressed as \(p = f_{r,c_r}(k)\) for some integers \(k \geq 3\) and \(c_r \in \mathbb{Z}\) with \(-k + 2 \leq c_1 \leq 1\) if \(r = 1\) and \(-k + 4 \leq c_3 \leq 4\) otherwise. Here,
\[
f_{r,c_r}(t) = 2t^2 + (r + 3)t + c_r.
\]

Let \(I_r = \bigcup_{k \geq 3} I_{r,k} \cap \mathbb{P}\) and \(C_r = \{r - 4, r - 2, r\}\). Moreover, for \(c_r \in C_r\), define an integer \(k_{r,c_r} \geq 3\) by \((k_{1,-3}, k_{1,-1}, k_{1,1}) = (5, 3, 3)\) and \((k_{3,-1}, k_{3,3}, k_{3,3}) = (7, 3, 3)\). We now obtain the following theorem, which gives a characterization for exceptional primes.

Theorem 4.6. A prime \(p \in I_r\) with \(p \geq 29\) is exceptional if and only if it is of the form of \(p = f_{r,c_r}(k)\) for some \(c_r \in C_r\) and \(k \geq k_{r,c_r}\).

Proof. To clarify when the inequality \(\mu(\hat{l} + 1) = |\mu_1(\hat{l}_1, \hat{l}_2)| \leq \text{RB}(\hat{l} + 1)\) holds, we introduce an interpolation function \(F_r(t)\) of the difference between \(|\mu_1(\hat{l}_1, \hat{l}_2)|\) and \(\text{RB}(\hat{l} + 1)\) on \(I_{r,k}\), that is,
\[
F_r(t) = 2\sin \frac{\pi(4k+r+1)}{2t} \cos \frac{\pi(r-1)}{2t} - 2\sqrt{2t} - 4k - r - 2.
\]

One can see that \(F_r(t)\) is monotone decreasing on \(I_{r,k}\) for sufficiently large \(k\). Moreover, at \(t = p = f_{r,c_r}(k) \in I_{r,k} \cap \mathbb{P}\), one has
\[
F_r(p) = \frac{3(r + 3)^2 - 24c_r - 16\pi^2}{24}k^{-1} + O(k^{-2})
\]
as \(k \to \infty\) because
\[
|\mu_1(\hat{l}_1, \hat{l}_2)| = 2\sin \frac{\pi(4k+r+1)}{2(2k^2 + (r+3)k + c_r)} \cos \frac{\pi(r-2)}{2(2k^2 + (r+3)k + c_r)}
\]
\[
= 4k + r + 1 - \frac{2\pi^2}{3}k + O(k^{-2}),
\]
\[
\text{RB}(\hat{l} + 1) = 2\sqrt{2(2k^2 + (r+3)k + c_r)} - 4k - r - 2
\]
\[
= 4k + r + 1 - \frac{(r+3)^2 - 8c_r}{8}k + O(k^{-2}).
\]

This shows that \(F_r(p) < 0\) for sufficiently large \(k\) if and only if \(3(r + 3)^2 - 24c_r - 16\pi^2 < 0\), that is, \(c_r \in C_r\) (note that \(c_r\) should be odd because \(p\) is). Actually, for each \(r \in \{1, 3\}\) and \(c_r \in C_r\), we can check that the inequality \(F_r(p) < 0\) with \(p = f_{r,c_r}(k)\) holds if and only if \(k \geq k_{r,c_r}\). This completes the proof of the theorem.
For $r \in \{1, 3\}$ and $c_r \in C_r$, let $J_{r,c_r} = \{ f_{r,c_r}(k) \in I_r \mid k \geq k_{r,c_r} \}$. Namely, $J_{r,c_r}$ is the set of exceptional primes $p$ of the form of $p = f_{r,c_r}(k)$. These are given as follows:

$J_{1,-3} = \{67, 93, 157, 283, 643, 877, 1453, 3037, 4603, 5197, \ldots \}$,

$J_{1,1} = \{29, 47, 197, 239, 389, 509, 719, 797, 2309, 2447, \ldots \}$,

$J_{1,1} = \{31, 71, 97, 127, 199, 241, 337, 449, 577, 647, \ldots \}$,

$J_{3,-1} = \{139, 307, 359, 607, 919, 1399, 1619, 1979, 2239, 2659, \ldots \}$,

$J_{3,1} = \{37, 109, 541, 757, 1009, 1297, 1621, 2377, 6841, 7561, \ldots \}$,

$J_{3,3} = \{59, 83, 179, 263, 311, 419, 479, 683, 839, 1103, \ldots \}$.

The classical Hardy-Littlewood conjecture [HL] asserts that if a quadratic polynomial $f(t) = at^2 + bt + c$ with $a, b, c \in \mathbb{Z}$ satisfies the conditions that $a > 0$, $(a, b, c) = 1$, $a + b$ and $c$ are not both even and $D = b^2 - 4ac$ is not a square, then there are infinitely many primes represented by $f(t)$ and, moreover, that $\pi(f; x) = \# \{ k \leq x \mid f(k) \in \mathbb{P} \}$ obeys the asymptotic behavior

$$\pi(f; x) \sim \frac{C(f)}{2} \frac{x}{\log x}, \quad C(f) = 2 \prod_{p \geq 3} \left(1 - \frac{f_p}{p-1}\right),$$

as $x \to \infty$. Here, $C(f)$ is called the Hardy-Littlewood constant with $(\frac{D}{p})$ being the Legendre symbol. From Theorem 4.6, one sees that the existence of infinitely many exceptional primes is related to this conjecture.

**Corollary 4.7.** There exists infinitely many exceptional primes if and only if the Hardy-Littlewood conjecture is true for at least one of $f_{r,c_r}(t)$ for $r \in \{1, 3\}$ and $c_r \in C_r$. \qed

**Remark 4.8.** From [HL3], we can expect that $\pi(f_{r,c_r}; x) \sim \frac{C(f_{r,c_r})}{2} \frac{x}{\log x}$ where

$$\frac{C(f_{r,c_r})}{2} = \prod_{p \geq 3} \left(1 - \frac{c'_r}{p-1}\right) = \begin{cases} 0.671043\ldots & (r = 1, \ c_1 = -3), \\ 1.03566\ldots & (r = 1, \ c_1 = -1), \\ 1.84998\ldots & (r = 1, \ c_1 = 1), \\ 1.14801\ldots & (r = 3, \ c_3 = -1), \\ 0.757353\ldots & (r = 3, \ c_3 = 1), \\ 1.38332\ldots & (r = 3, \ c_3 = 3), \end{cases}$$

with $c'_r = 4 - 2c_1$ and $c'_r = 9 - 2c_3$.

**Remark 4.9.** The existence of infinitely many ordinary primes are verified as follows. Let $J = \bigcup_{r \in \{1,3\}} \bigcup_{c_r \in C_r} J_{r,c_r}$. Moreover, for a positive integer $a$, let $J_{r,c_r}(a) = \{ n \in \mathbb{Z}_{\geq 0} \mid 0 \leq n \leq a - 1 \}$ and $n \equiv f_{r,c_r}(k) \pmod{a}$ for some $0 \leq k \leq a - 1$ and $n \equiv f_{r,c_r}(k) \pmod{a}$ for some $0 \leq k \leq a - 1$ and $n \equiv f_{r,c_r}(k) \pmod{a}$ for some $0 \leq k \leq a - 1$ and $n \equiv f_{r,c_r}(k) \pmod{a}$ for some $0 \leq k \leq a - 1$. If we can take $b \in \{0, 1, 2, \ldots, a - 1\} \setminus J(a)$ satisfying $(a, b) = 1$, then we have $\{at + b \mid t \in \mathbb{Z}\} \cap J = \emptyset$ and, from the Dirichlet theorem of arithmetic progression, can find infinitely many primes in $\{at + b \mid t \in \mathbb{Z}\}$. Now, these are ordinary from Theorem 4.6. To achieve such a purpose, one may take, for example, $(a, b) = (29, 4), (35, 8)$ or $(40, 33)$.

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MIKI HIRANO
Graduate School of Science and Engineering, Ehime University,
Bunkyo-cho, Matsuyama, 790-8577 JAPAN.
hirano@math.sci.ehime-u.ac.jp

KOHEI KATATA
Graduate School of Science and Engineering, Ehime University,
Bunkyo-cho, Matsuyama, 790-8577 JAPAN.
katata@math.sci.ehime-u.ac.jp

YOSHINORI YAMASAKI
Graduate School of Science and Engineering, Ehime University,
Bunkyo-cho, Matsuyama, 790-8577 JAPAN.
yamasaki@math.sci.ehime-u.ac.jp