Searching for BPS Vortex with Nonzero Internal Pressure in Generalized Born-Infeld-Higgs Model

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ABSTRACT: In this article we show that the BPS equations of vortex with nonzero internal pressure derived in [1] for the generalized Maxwell-Higgs model can also be obtained using the BPS Lagrangian method developed in [2]. Two additional terms are added into the original BPS Lagrangian $L_{BPS} = \int dQ$, with $Q$ is the BPS energy function, which are proportional to quadratic of first-derivative of the effective scalar field and a function of the effective scalar field. These additional terms produce additional constraint equations coming from Euler-Lagrange equations of the BPS Lagrangian. We repeat the procedure for generalized Born-Infeld-Higgs model and show that the total energy is finite for the BPS equations if the scalar potential is bounded from above $V \leq 2b^2$, with $b$ is the Born-Infeld parameter. We also compute the energy-momentum tensor and show that the pressure densities in radial and angular directions are nonzero. We argue that the BPS Lagrangian method could be used to find BPS equations for other solitons with nonzero internal pressure.

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1 Introduction

Vortex is a soliton in two-dimensions, or in general it is solitonic object with the number of co-dimension is two [3, 4]. In field theory with three-dimensional spacetime it is a point-like object while in four-dimensional it becomes a string-like object, i.e. vortex strings. In order for vortex to have finite energy, the field theory must be equipped with additional gauge field due to the Derrick’s theorem [5], and so vortex is featured with electromagnetic charges. Vortex finds its applications in many branches of physics. As an example, magnetic vortex of the standard Maxwell-Higgs model (sMH), obtained by Nielsen and Olesen [6], correspond to the Type-II superconductor identified by Abrikosov [7]. Some other applications of vortex are in Bose-Einstein condensates [8], in quantum Hall effect [9], including cosmic strings in the early formation of the Universe [10], and many more.

The dynamics of vortex is given by the Euler-Lagrange equations which is a second-order differential equations. These equations in general are non-linear and finding their solutions could be very difficult. In some cases, we may take some limit to the parameters of field theory that could make the Euler-Lagrange equations simpler and easy to solve as shown by Prasad and Sommerfield in the case of monopole and dyon [11]. It turns out that these solutions are also solutions to the first-order differential equations that also solve the Euler-Lagrange equations as shown by Bogomolnyi [12]. One could obtain these first-order differential equations directly from the static energy density using Bogomolnyi’s trick by squaring the energy density, in which the first-order differential equations are sometimes called Bogomolnyi-Prasad-Sommerfield (BPS) equations. This trick has been used for many solitonic objects including the vortices found in the standard Maxwell-Higgs model [6].
The existence of these BPS equations does not only simplify the problem of solving the second-order equations but they also have intimate relation with the supersymmetry extension of the theory \[4\]. Furthermore, in most of the cases, the static energy is bounded from below which is determined fully by topological charge of the theory. The BPS solitons, which are solutions to the BPS equations, saturate this bound of energy thus easy to prove their stability property. However, extracting these BPS equations from the static energy density for general cases is somewhat difficult and tricky. An additional dummy term in the Lagrangian might be needed to execute the Bogomolnyi’s trick, with the cost of producing a constraint equation to eliminate this term at the end \[13, 14\]. An attempt was made in \[15\] by imposing a pressureless condition on the energy-momentum tensor of two-dimensional scalar field theory, with general kinetic term. The kink solutions then can be extracted from this condition. However, in general cases, extracting the BPS equations is also a bit tricky for the higher-dimensional theory. A question was aroused if there is a more rigorous way to obtain these BPS equations. A proposal was given not along ago and it is know as the \textit{On-Shell} method \[16\]. The main idea of this method is by introducing auxiliary fields into the (second-order) Euler-Lagrange equations. These auxiliary fields, which would generate the (first-order) BPS equations that are solutions to the Euler-Lagrange equations. For details procedure see \[1, 16\]. The advantage of this method is that since one works in the Euler-Lagrange equations, whatever Bogomolnyi equations obtained are always solutions to the Euler-Lagrange equations. However, the procedure is a bit tedious and involves solving additional constraint equations though do not depend on coordinate explicitly. Later on, a BPS Lagrangian method was developed in \[2\] by introducing a BPS Lagrangian, equating this BPS Lagrangian with effective Lagrangian of the model, and then solving it as quadratic equation of first-derivative of all effective fields. We will describe the procedure in more details below. Another a method called First-Order Euler-Lagrange(FOEL) formalism was developed in \[17\] which is generalization of Bogomolnyi decomposition using a concept of strong necessary condition \[18\].

An interestingly new BPS equations of vortex, with \(C_0 \neq 0\), was found in generalized Maxwell-Higgs model using the \textit{On-Shell} method \[16\]. It was shown that these are BPS equations of vortex that has nonzero internal pressure. One might ask if we could find BPS equations of vortex with nonzero internal pressure in other models. In this article, we will consider the Born-Infeld extension of the generalized Maxwell-Higgs model, which is generalization of the Born-Infeld-Higgs model \[19\], that is given by Casana et al. in \[13\]. However, applying the \textit{On-Shell} method to this model will be very complicated since the Lagrangian has terms inside a square-root. Therefore in this article we will use the BPS Lagrangian method as shown in \[2\] to be more simple in deriving BPS equations of vortex in Born-Infeld type of actions. In the section 2, we explain in detail about the BPS Lagrangian method and how to use it to find BPS equations of vortex in the standard Maxwell-Higgs model. Next section 3, we use the BPS Lagrangian method for the case of generalized Maxwell-Higgs model with the form of BPS Lagrangian that is motivated by the form of static energy in \[1\]. In section 4, we then use this form of BPS Lagrangian for the case of generalized Born-Infeld-Higgs model and obtain BPS equations of vortex. In section 5, we compute the energy-momentum tensor and show that its radial-angular and angular-angular
components are nonzero for the BPS equations. In the last section 6, we give conclusions and remarks.

2 BPS Lagrangian Method

At first let us write the ansatz for scalar and $U(1)$ gauge fields in polar coordinates as follows, respectively,

$$\phi = f(r)e^{in\theta}, \hspace{1cm} A_\theta = a(r) - n,$$

(2.1)

where $n = \pm 1, \pm 2, \ldots$ is the winding number and we have set the gauge coupling to be unity for simplification. Here we consider static configuration with temporal and radial gauge, $A_t = A_r = 0$. Although the ansatz (2.1) depends on angle coordinate $\theta$, the effective Lagrangian $L_{\text{eff}}$ will be eventually independent of $\theta$, and $n$ as well. As an example the (standard) kinetic terms are

$$|D_\mu \phi|^2 = -f'^2 - \left( \frac{af'}{r} \right)^2,$$

(2.2)

and

$$B^2 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \left( \frac{a'}{r} \right)^2,$$

(2.3)

with $D_\mu \phi = \partial_\mu \phi + iA_\mu \phi$ and the signature of the metric is taken to be $(+,-,-)$. We use $' \equiv \frac{\partial}{\partial r}$ unless there is an explicit argument in the function then it means taking derivative over the argument. In this case the effective Lagrangian will be a function $L_{\text{eff}}(a', f', a, f; r)$.

In this case the BPS Lagrangian is simply written as [2]

$$L_{\text{BPS}} = \int dQ = \int d^2x \left( \frac{\partial Q}{\partial a} a' + \frac{\partial Q}{\partial f} f' \right) = \int d^2x L_{\text{BPS}},$$

(2.4)

where $Q$ is called BPS energy function and it is assumed to be function of $a$ and $f$, but not of $r$ explicitly. Equating $L_{\text{eff}}$ with $L_{\text{BPS}}, L_{\text{eff}} = L_{\text{BPS}}$, we may consider it as a quadratic equation for $a'$ or $f'$. One can try to split this equation into two quadratic equations for $a'$ and $f'$ separately. However in general splitting this equation may a bit tricky. We can consider it first as a quadratic equation for $a'$ and solve it using the quadratic formula that will give us two solutions, $a'_\pm$. We must set these two solutions to be equal, $a'_+ = a'_-$, as a requirement that $L_{\text{eff}}$ can be rewritten in a complete square form. This can be done by setting the square-root term in the quadratic formula to be zero, which then becomes a quadratic equation of $f'$. Solving it and following the similar steps as before for $a'$, we obtain at the end an equation that does not contain $a'$ and $f'$. This last equation must be valid for all values of $r$ and so we can solve it order by order of power $r$. The BPS equations then are given by $a' = a'_\pm$ and $f' = f'_\pm$ with additional constraint equations that might come from solving the last equation. This last equation should be the same even if we consider the equation $L_{\text{eff}} = L_{\text{BPS}}$ as a quadratic equation for $f'$ first.
As an example let us consider the standard Maxwell-Higgs model with the following Lagrangian
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - V(|\phi|). \] (2.5)

Using the ansatz (2.1), its effective Lagrangian is given by
\[ \mathcal{L}_{\text{eff}} = -\frac{1}{2} \left( \frac{a'}{r} \right)^2 - \left( f'^2 + \left( \frac{af}{r} \right)^2 \right) - V(f). \] (2.6)

Now, equating this effective Lagrangian with the BPS Lagrangian, \( \mathcal{L}_{\text{BPS}} \) in (2.4), give us an equation
\[ -\frac{1}{2} \left( \frac{a'}{r} \right)^2 - \left( f'^2 + \left( \frac{af}{r} \right)^2 \right) - V(f) = \frac{\partial Q}{\partial a} \frac{a'}{r} + \frac{\partial Q}{\partial f} f', \] (2.7)
where we have rescaled \( Q \rightarrow 2\pi Q \) for simplification. Consider it as quadratic equation for \( a' \) first, the two solutions are
\[ a'_\pm = -\frac{\partial Q}{\partial a} r \pm \sqrt{-2r^2 f'^2 - 2r \frac{\partial Q}{\partial f} f' + \left( \frac{\partial Q}{\partial a} \right)^2 r^2 - 2r^2 V - 2a^2 f'^2}. \] (2.8)

The two solutions will be equal if
\[ -2r^2 f'^2 - 2r \frac{\partial Q}{\partial f} f' + \left( \frac{\partial Q}{\partial a} \right)^2 r^2 - 2r^2 V - 2a^2 f'^2 = 0, \] (2.9)
which is a quadratic equation for \( f' \). Solutions to this equations are
\[ f'_\pm = -\frac{1}{2r} \frac{\partial Q}{\partial f} \pm \frac{1}{2} \sqrt{2r^2 \left( -4a^2 f'^2 + \left( \frac{\partial Q}{\partial f} \right)^2 + 2r^2 \left( \frac{\partial Q}{\partial a} \right)^2 - 2V \right)} \] (2.10)
thus give us the last equation, upon equation these solutions,
\[ -4a^2 f'^2 + \left( \frac{\partial Q}{\partial f} \right)^2 + 2r^2 \left( \frac{\partial Q}{\partial a} \right)^2 - 2V = 0. \] (2.11)

Solving the last equations order by power \( r \) yields two equations
\[ \frac{\partial Q}{\partial a} = \pm \sqrt{2V}, \] (2.12)
\[ \frac{\partial Q}{\partial f} = \pm 2af, \] (2.13)
which have solution \( Q = \pm a (f^2 - 1) \) and \( V = \frac{1}{2} (f^2 - 1)^2 \). The BPS equations are then given by
\[ \frac{a'}{r} = \pm \left( 1 - f^2 \right), \] (2.14)
\[ f' = \mp \frac{af}{r}. \] (2.15)
3 Generalized Maxwell-Higgs Model

The generalized Maxwell-Higgs model described by the following Lagrangian
\begin{align}
L_{GenMH} = \frac{G(|\phi|)}{4} F_{\mu\nu} F^{\mu\nu} + w(|\phi|) |D_\mu \phi|^2 - V(|\phi|),
\end{align}
and the effective Lagrangian, using the ansatz (2.1), is given by
\begin{align}
L_{GenMH} = \frac{G(f)}{2} \left( \frac{a'}{r} \right)^2 - w(f) \left( f'^2 + \left( \frac{af}{r} \right)^2 \right) - V(f),
\end{align}
with \( G(f), w(f) > 0 \) and \( V(f) \geq 0 \). Taking the BPS energy function to be \( Q = 2\pi F(f) A(a) \), let us write the BPS Lagrangian to be
\begin{align}
L_{BPS} = -F'(f) \frac{A'}{r} f' - A'(a) F(f) \frac{a'}{r} - X_2(a, f, r) f'^2 - X_0(a, f, r).
\end{align}
This BPS Lagrangian is different from the resulting BPS energy density of vortices, with non-zero pressure, computed in [2]. We will show using this BPS Lagrangian we can reproduce the BPS vortex, with non-zero internal pressure, as shown in [1]. Equating both Langragians and collecting all terms that contains derivative of \( f \), we then have
\begin{align}
w(f) \left( f'^2 + \left( \frac{af}{r} \right)^2 \right) = X_2(a, f, r) f'^2 + F'(f) \frac{A'}{r} f'.
\end{align}
Solution to this equation is
\begin{align}
(w - X_2) \left( f' \mp \frac{af}{r} \sqrt{w - X_2} \right)^2 = 0,
\end{align}
with a constraint equation
\begin{align}
F'(f) A = \pm 2 a f \sqrt{w(w - X_2)}.
\end{align}
We may conclude from this constraint equation that \( A = a \) and \( X_2 \equiv X_2(f) \). So, we obtain a BPS equation
\begin{align}
f' = \pm \frac{af}{r} \sqrt{\frac{w}{w - X_2}}
\end{align}
and the constraint equation
\begin{align}
F'(f) = \pm 2 f \sqrt{w(w - X_2)}.
\end{align}
The remaining terms from equating both Lagragians, we have
\begin{align}
\frac{G(f)}{2} \left( \frac{a'}{r} \right)^2 + V(f) = A'(a) F(f) \frac{a'}{r} + X_0(a, f, r).
\end{align}
Solution to this equation is
\begin{align}
\frac{G}{2} \left( \frac{a'}{r} \mp \sqrt{\frac{2(V - X_0)}{G}} \right)^2 = 0,
\end{align}
with a constraint equation
\[ F = \pm \sqrt{2G(V - X_0)}. \]  
(3.11)

This constraint equation also implies \( X_0 \equiv X_0(f) \). So, we have another BPS equation
\[ \frac{a'}{r} = \pm \sqrt{\frac{2(V - X_0)}{G}} \]  
(3.12)

along with the other constraint equation (3.11).

Now we still have two unknown functions \( X_2 \) and \( X_0 \). Using the two constraint equations (3.8) and (3.11), we reduce the unknown functions to only for \( X_2 \) such that
\[ \frac{\partial}{\partial f} \sqrt{2G(V - X_0)} = 2f \sqrt{w(w - X_2)}. \]  
(3.13)

As shown in [1], finite energy condition requires that
\[ V - \frac{G}{2} \left( \frac{a'}{r} \right)^2 = f_{C_0} \]  
(3.14)

and
\[ \int rf_{C_0} dr = 0, \]  
(3.15)

with \( f_{C_0} = 0 \), for \( C_0 = 0 \), where \( C_0 \) is a constant. Substituting the BPS equation (3.12) into the finite energy condition implies \( X_0 = f_{C_0} \).

Having BPS equations (3.7) and (3.12), we can rewrite the effective Lagragian (3.2) to become
\[ L_{eff} = w \left( f' \mp \frac{af}{r} \sqrt{\frac{w}{w - X_2}} \right)^2 - \left( \frac{af}{r} \right)^2 \frac{wX_2}{w - X_2} \pm 2w \frac{af'}{r} \sqrt{\frac{w}{w - X_2}} + G \left( \frac{a'}{r} \mp \sqrt{\frac{2(V - X_0)}{G}} \right)^2 + X_0 \pm \frac{a'}{r} \sqrt{2G(V - X_0)}. \]  
(3.16)

One can check that its Euler-Lagrange equation for \( a \) is simply just the equation (3.13) after substituting the BPS equations (3.7) and (3.12). Its Euler-Lagrange equation for \( f \) is given by
\[ \pm \frac{\partial}{\partial r} \left( 2waf \sqrt{\frac{w}{w - X_2}} \right) = \frac{a^2}{r} \frac{\partial}{\partial f} \left( \frac{wX_2^2}{w - X_2} \right) \pm 2af' \frac{\partial}{\partial f} \left( waf \sqrt{\frac{w}{w - X_2}} \right) + rX_0'(f) \pm \frac{a'}{r} \sqrt{2G(V - X_0)}. \]  
(3.17)

Substituting the BPS equations (3.7) and (3.12), and the constraint equations (3.8) and (3.11), the Euler-Lagrange equation for \( f \) can be simplified to
\[ \frac{a^2}{r^2} \frac{\partial}{\partial f} \left( \frac{wX_2^2}{w - X_2} \right) = X_0'(f) - 2fX_2 \sqrt{\frac{2w(V - X_0)}{G(w - X_2)}}. \]  
(3.18)

The left hand side of equation, which contain function \( a \) and coordinate \( r \) explicitly, should be zero since all terms on the right hand side of equation are only functions of \( f \). Another
reason is because the equation (3.18) can be solved order by order of power \( r \) thus implies the left hand side of equation to be zero. This would give

\[
\frac{w f^2 X_2}{w - X_2} = C_0 \quad \rightarrow \quad X_2 = \frac{w C_0}{f^2 w + C_0},
\]

where \( C_0 \) is the similar constant mentioned previously in the finite energy condition. Now, suppose that \( X_0 = V - \frac{G}{2} R^2 \), with \( R \equiv (a'/r)^2 \) is a function of \( f \). Then the equation (3.18) can be written as

\[
\frac{\partial}{\partial f} \left( V - \frac{G}{2} R^2 \right) = 2w C_0 \frac{R}{f^2 w + C_0},
\]

which is the same constraint equation in [1]. The solution for \( R \) can be derived from the equation (3.13) as such

\[
R = \frac{1}{G} \left( 2 \int df \frac{f^2 w^{3/2}}{\sqrt{f^2 w + C_0}} + C_1 \right),
\]

with \( C_1 \) is an integration constant.

### 3.1 Euler-Lagrange equations of BPS Lagrangian

Another way to obtain the additional constraint (3.20) is by deriving Euler-Lagrange equation of the BPS Lagrangian (3.3). The first two terms of the BPS Lagrangian are

\[
\propto \int dr \left( \frac{\partial Q}{\partial a} a' + \frac{\partial Q}{\partial f} f' \right)
\]

and its Euler-Lagrange equations are

\[
\frac{d}{dr} \left( \frac{\partial Q}{\partial a} \right) = \frac{\partial^2 Q}{\partial a^2} a' + \frac{\partial^2 Q}{\partial a \partial f} f'
\]

and

\[
\frac{d}{dr} \left( \frac{\partial Q}{\partial f} \right) = \frac{\partial^2 Q}{\partial f^2} f + \frac{\partial^2 Q}{\partial a \partial f} a'.
\]

The left hand side of both equations above is equal to its right hand side and so they are trivially zero. The remaining two terms in the BPS Lagrangian are

\[
\propto \int dr \ R \left( X_0(f) + X_2(f) f'^2 \right)
\]

and its Euler-Lagrange equation is\(^1\)

\[
2 \frac{d}{dr} \left( X_2 r f' \right) = r \left( \frac{\partial X_2}{\partial f} f'^2 + \frac{\partial X_0}{\partial f} \right).
\]

It can be simplified further to

\[
2X_2 \frac{d}{dr} \left( r f' \right) + r \frac{\partial X_2}{\partial f} f'^2 = r \frac{\partial X_0}{\partial f}.
\]

In a BPS limit, in which the BPS equations (3.7) and (3.12) are satisfied, this equation is equal to the equation (3.18) and hence later gives us the constraint equation (3.20). Therefore it is not necessary to write down the effective Lagrangian in a complete square-forms (3.16), but instead we could use the BPS Lagrangian (3.3) and derive its Euler-Lagrange equations as additional constraint equations, with the BPS limit in our disposal.

\(^1\)Since the remaining terms do not depends on \( a \), there is only Euler-Lagrange equation for \( f \).
3.2 General BPS Lagrangian

One may ask what if we add term which is proportional to $a'^2$ in the BPS Lagrangian (3.3) do we get the same result as previously or not. Let us now consider a more general BPS Lagrangian as follows

$$\mathcal{L}_{BPS} = -F'(f) \frac{A}{r} f' - A'(a) F(f) \frac{a'}{r} - X_0(a, f, r) - X_1(a, f, r) a'^2 - X_2(a, f, r) f'^2. \quad (3.28)$$

Equating with the effective Lagrangian (3.2), $\mathcal{L}_{GenMH} = \mathcal{L}_{BPS}$, as discussed in the previous section we may consider it as a quadratic equation for $a'$, which has solutions

$$a'_{\pm} = \frac{A'(a) F(r) \pm \sqrt{S_a}}{G - 2r^2 X_1}, \quad (3.29)$$

$$S_a = A'(a)^2 F^2 r^2 - 2 \left( G - 2r^2 X_1 \right) \left( -A F'(f) f' r + a'^2 f^2 w + r^2 \left( V - X_0 + f'^2 (w - X_2) \right) \right). \quad (3.30)$$

The two solutions will be equal if $S_a = 0$. The later equation can be considered as a quadratic equation for $f'$, which has solutions

$$f' = \frac{A F'(f) r \left( G - 2r^2 X_1 \right) \pm \sqrt{S_f}}{2r^2 \left( G - 2r^2 X_1 \right) \left( w - X_2 \right)}, \quad (3.31)$$

$$S_f = r^2 \left( G - 2r^2 X_1 \right) \left( 2 \left( w - X_2 \right) \left( A'(a)^2 F^2 r^2 - 2 \left( G - 2r^2 X_1 \right) \left( a'^2 f^2 w + r^2 \left( V - X_0 \right) \right) \right) \right) + A'^2 F'(f)^2 \left( G - 2r^2 X_1 \right). \quad (3.32)$$

Similarly two solutions will be equal if $S_f = 0$. Now $S_f$ does not contain any first-derivative of the effective fields and so we can solve it algebraically. But at first notice that in order for BPS equations (3.29) and (3.31) to be well-defined we must have $(G - 2r^2 X_1) \neq 0$ and $(w - X_2) \neq 0$. The equation $S_f = 0$ can be rewritten explicitly in terms of power of $r$ as follows

$$0 = \frac{Q'(f)^2}{4\pi^2} G - 4a^2 f^2 G \left( w - X_2 \right) + 2 \left( w - X_2 \right) \left( \frac{Q'(a)^2}{4\pi^2} + 4a'^2 f^2 w X_1 - 2G \left( V - X_0 \right) \right) - \frac{Q'(f)^2}{4\pi^2} X_1 \left( w - X_2 \right) r^4, \quad (3.33)$$

with $Q = 2\pi A(a) F(f)$. The above equation must be solved for all values of $r$ and thus we could solve it order by order in power of $r$. Here we have three equations of $r^0$, $r^2$, and $r^4$-orders. Along with these equations we will have at least additional two equations from Euler-Lagrange equations of the BPS Lagrangian (3.28) as shown in the previous section. On the other hand we have four unknown functions $Q, X_0, X_1, \text{and } X_2$. So in total there are four (additional) unknown functions and five equations which means we have overdetermined set of equations. This is even before we consider the functions $X_0, X_1, \text{and } X_2$ to depend explicitly on coordinate $r$. Should one of these functions depends explicitly on coordinate $r$ the number equations extracted from (3.33) might be more than three. To reduce this problem, we will take $X_0, X_1, \text{and } X_2$ to not depend explicitly on coordinate $r$. 

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There are few possibilities to equating the number of unknown functions and of equations. In doing so we can take functions \(X_0, X_1, \) and \(X_2\) to depend only on field \(a\) or \(f\). This would reduce the number of Euler-Lagrange equations from the BPS Lagrangian. From the \(r^4\)-order equation of (3.33), we can take \(V = X_0\) and so those functions must depend only on field \(f\). Then from the \(r^0\)-order equation we obtain

\[
\frac{Q'(f)^2}{4\pi^2} = 4a^2 f^2 w (w - X_2).
\] (3.34)

Thus \(Q \propto a^2\) and \(w > X_2\). This equation is similar to the constraint equation (3.8). However, substituting these to the \(r^2\)-order equation forces us to take \(w = X_2\) which contradict with previous condition that the BPS equation (3.31) must be well-defined. Instead of taking \(V = X_0\) in the \(r^4\)-order equation, we could take \(X_1 = 0\) which also means we reduce the number of unknown function by one. This also reduces the number of equations extracted from the equation (3.33) by one; explicitly there is no \(r^4\)-order equation. The \(r^0\)-order equation is still given by equation (3.34) and we get the same result as mentioned previously. Now the \(r^2\)-order equation becomes

\[
\frac{Q'(a)^2}{4\pi^2} - 2G (V - X_0) = 0,
\] (3.35)

where it does not depend on field \(a\) explicitly and it is similar to the constraint equation (3.11). Therefore we may consistently take \(X_1 = 0\), as in BPS Lagrangian (3.3), and functions \(X_0\) and \(X_2\) to depend only on field \(f\). Here now the number of constraint equations is actually four while the number of unknown functions are three \((Q, X_0, X_2)\). However this does not mean the set of equations are overdetermined because we still have four unknown functions from the effective Lagrangian (3.2), which are \(w, G,\) and \(V\). This would mean one of these functions will be fixed by the constraint equations.

4 Generalized Born-Infeld-Higgs Model

Let us now extend the generalized Maxwell-Higgs model (3.2) into the Born-Infeld type of action \([19, 21]\) in which the effective action is given by \([13]\)

\[
\mathcal{L}_{GenBIH} = -b^2 \left( \sqrt{1 + \frac{G(|\phi|)}{2b^2} F_{\mu\nu} F^{\mu\nu} - 1} \right) + w(|\phi|) |D_\mu \phi|^2 - V(|\phi|),
\] (4.1)

with \(G, w > 0, V \geq 0,\) and \(b\) is the Born-Infeld parameter. Using the ansatz (2.1), we obtain an effective Lagrangian

\[
\mathcal{L}_{GenBIH} = -b^2 \left( \sqrt{1 + \frac{1}{b^2} G(f) \frac{a'^2}{r^2} - 1} \right) - w(f) \left( f'^2 + \frac{a'^2 f^2}{r^2} \right) - V(f).
\] (4.2)

We will be using the following BPS Lagrangian

\[
\mathcal{L}_{BPS} = -\frac{Q'(f)}{r} f' - \frac{Q'(a)}{r} a' - X_0(a,f,r) - X_1(a,f,r)a' - X_2(a,f,r)f'^2,
\] (4.3)
with \( Q = Q(a, f) \), in which the fourth term on right hand side is proportional to \( a' \) instead of \( a'^2 \) as in (3.28). The reason for not taking the fourth term to be proportional to \( a'^2 \) is because higher power of \( a' \) will not make equation \( \mathcal{L}_{\text{GenBIH}} = \mathcal{L}_{\text{BPS}} \) becoming a quadratic equation for \( a' \). To be precise, the proportionality to \( a'^2 \) will make equation \( \mathcal{L}_{\text{GenBIH}} = \mathcal{L}_{\text{BPS}} \) becoming quartic equation for \( a' \) that could lead to more complicated analysis.

Equating both Lagrangians (4.2) and (4.3), and considering it first as quadratic equations for \( f' \) we obtain solutions

\[
f'_\pm = \frac{Q'(f)}{2r(w - X_2)} \pm \sqrt{\frac{S_f}{2r(w - X_2)}},
\]

\[
S_f = Q'(f)^2 + 4(w - X_2) \left( a'r(Q'(a) + rX_1) - r^2(V - X_0) - a^2f^2w \right) - b^2r^2 \left( \sqrt{1 + \frac{1}{b^2G} \frac{a'^2w}{r^2}} - 1 \right).
\]

Two solutions will be equal if \( S_f = 0 \) then this will be considered as quadratic equation for \( a' \) with solutions

\[
a'_\pm = \frac{(Q'(a) + rX_1)(4(w - X_2)(r^2(V - X_0 - b^2) + a^2f^2w) - Q'(f)^2) \pm \sqrt{S_a}}{4r(w - X_2)(\{(Q'(a) + rX_1)^2 - b^2G\}}),
\]

\[
S_a = b^2r^2(w - X - 2)^2 \left( G(Q'(f)^2 - 4(w - X - 2)(a^2f^2w + r^2(V - X_0)) \right)
\]

\[
+ 16b^2r^2(4(w - X - 2)(r^2(2b^2 - V + X_0) - a^2f^2w) + Q'(f)^2)
\]

\[
+ 16b^2r^4(w - X_2)^2(Q'(a) + rX_1)^2.
\]

The solutions will be equal if \( S_a = 0 \) and as previous procedure we solve it order by order in power of \( r \) as such we obtain several equations

\[
r^2 : b^2G(w - X_2)^2(4a^2f^2w(X_2 - w) + Q'(f)^2)^2 = 0,
\]

\[
r^4 : 8b^2G(b^2 - V + X_0)(w - X_2)(Q'(f)^2 + 4a^2f^2w(-w + X_2)) = 0,
\]

\[
r^6 : 16b^2G(V - X_0)^2 + b^2(Q'(a)^2 + 2G(-V + X_0))(w - X_2)^4 = 0,
\]

\[
r^7 : 32b^4Q'(a)X_1(w - X_2)^4 = 0,
\]

\[
r^8 : 16b^4X_1(w - X_2)^4 = 0.
\]

There are five equations with four unknown functions in which we have taken \( X_0, X_1 \), and \( X_2 \) to be explicitly independent of \( r \). From equation (4.12), we conclude that \( X_1 = 0 \) since \( w \neq X_2 \) and \( (Q'(a) + rX_1)^2 \neq b^2G \) are implied by equations (4.4) and (4.6). This will remove the equations (4.11) and (4.12). As mentioned previously we can further reduce the number of constraint equations from Euler-Lagrange equations of (4.3) by assuming \( X_0 \) and \( X_2 \) are only functions of \( f \). Notice that from equation (4.8), the equation is solved by \( Q'(f)^2 = 4a^2f^2w(w - X_2) \) which is also solution to equation (4.9). Therefore equations (4.8) and (4.9) are mutually solved by

\[
Q'(f)^2 = 4a^2f^2w(w - X_2).
\]
Similar to the previous case of generalized Maxwell-Higgs model total number of constraint equations are four with three unknown functions, which also mean one of the unknown functions in the effective Lagrangian (4.2) is fixed.

The equation (4.13) implies

\[ Q = \pm 2a \int f \sqrt{w(w - X_2)} \, df, \quad (4.14) \]

while the equation (4.10) give us

\[ Q^2 = \frac{a^2}{b^2} G (V - X_0) (2b^2 - (V - X_0)), \quad (4.15) \]

with \((V - X_0) (2b^2 - (V - X_0)) > 0\). Using the previous results, we obtain the BPS equations

\[ \frac{a'}{r} = \pm \sqrt{b^2 G (V - X_0) (2b^2 - V + X_0)} \frac{G (b^2 - V + X_0)}{G (b^2 - V + X_0)}, \quad (4.16) \]

\[ f' = \pm \frac{af}{r} \sqrt{\frac{w}{w - X_2}}, \quad (4.17) \]

with additional constraint equation (3.27). Substituting the above BPS equations into the constraint equation (3.27) and solving it order by order of power \(r\) yield

\[ r^{-1} : \frac{\partial}{\partial f} \left( \frac{f^2 X_2 w}{w - X_2} \right) = 0, \quad (4.18) \]

\[ r^1 : 2X_2 f \sqrt{\frac{w}{w - X_2}} = \frac{G (b^2 - V + X_0)}{\sqrt{b^2 G (V - X_0) (2b^2 - V + X_0)}} \frac{\partial X_0}{\partial f}. \quad (4.19) \]

The equation (4.18) has solution equal to (3.19) in the case of generalized Maxwell-Higgs model. Substituting this solution and using BPS equations (4.16) and (4.17) into equation (4.19) yields

\[ X_0' = C_0 \frac{(a^2)^f}{r^2}. \quad (4.20) \]

Here the solution for \(X_0\) have the same form as in the case of generalized Maxwell-Higgs model differs only by explicit form of \(a'\), or the BPS equation for \(a'\).

5 Energy-Momentum Tensor

The energy-momentum tensor for generalized Born-Infeld-Higgs model is give by [13]

\[ T_{\mu \nu} = -\frac{G F^a_{\mu \nu} F_{\alpha \nu}}{\sqrt{1 + \frac{1}{2b^2} GF_{\mu \nu} F^\mu_{\nu}}} + w \left( D_\mu \phi D_\nu \phi + D_\mu \phi D_\nu \phi \right) - \eta_{\mu \nu} \mathcal{L}. \quad (5.1) \]
Substituting the ansatz (2.1) and writing in polar coordinate, the non-zero components of the energy-momentum tensor are

\[
T_{tt} = b^2 \left( \sqrt{1 + \frac{G a^2}{b^2 r^2}} - 1 \right) + w \left( f'^2 + \frac{a^2 f'^2}{r^2} \right) + V, \tag{5.2}
\]

\[
T_{rr} = \frac{b^2}{\sqrt{1 + \frac{G a^2}{b^2 r^2}}} \left( \sqrt{1 + \frac{G a^2}{b^2 r^2}} - 1 \right) + w \left( f'(r)^2 - \frac{a(r)^2 f'(r)^2}{r^2} \right) - V, \tag{5.3}
\]

\[
T_{\theta\theta} = \frac{b^2}{\sqrt{1 + \frac{G a^2}{b^2 r^2}}} \left( \sqrt{1 + \frac{G a^2}{b^2 r^2}} - 1 \right) - w \left( f'(r)^2 - \frac{a(r)^2 f'(r)^2}{r^2} \right) - r^2 V. \tag{5.4}
\]

Using the BPS equations, (4.16) and (4.17), and solution for \(X_2\), (3.19), they are simplified further to

\[
T_{tt} = b^4 b^2 - V + X_0 - (b^2 - V) + (2 f^2 w + C_0^2) \frac{a^2}{r^2}, \tag{5.5}
\]

\[
T_{rr} = \frac{a^2}{r^2} C_0^2 - X_0, \tag{5.6}
\]

\[
T_{\theta\theta} = -r^2 \left( \frac{a^2}{r^2} C_0^2 + X_0 \right). \tag{5.7}
\]

5.1 Finiteness of energy

Let us consider one-parameter family of solutions by rescaling the space with a constant \(0 < \lambda < \infty\), \(x \to \frac{x}{\lambda}\), and now total static energy is given by a function of \(\lambda\),

\[
E(\lambda) = \int d^2 x \left[ \frac{b^2}{\lambda^2} \left( \sqrt{1 + \frac{\lambda^4 G}{2b^2} F^2_{ij}} - 1 \right) + w |D_i \phi|^2 + \frac{V}{\lambda^2} \right]. \tag{5.8}
\]

Due to Derrick's theorem [5], the total static energy can be finite if its extremum over \(\lambda\) is given by a finite and positive value of \(\lambda\). The extremum of total static energy is given by the following equation

\[
\frac{dE}{d\lambda} = \int d^2 x \left[ \frac{2b^2}{\lambda^3 \sqrt{1 + \frac{\lambda^4 G}{2b^2} F^2_{ij}}} \left( \sqrt{1 + \frac{\lambda^4 G}{2b^2} F^2_{ij}} - 1 \right) - \frac{2}{\lambda^3} V \right] = 0. \tag{5.9}
\]

Now, it is rather difficult to prove that solution to this equation is a positive and finite \(\lambda\) because there is \(\lambda\) inside the square root. However we can prove it by considering the equation locally as such

\[
\frac{b^2}{\sqrt{1 + \frac{\lambda^4 G}{2b^2} F^2_{ij}}} \left( \sqrt{1 + \frac{\lambda^4 G}{2b^2} F^2_{ij}} - 1 \right) = V, \tag{5.10}
\]

which then give us

\[
\lambda^4 = \frac{2 b^2 b^4 - (b^2 - V)^2}{(b^2 - V)^2 F^2_{ij}} > 0 \tag{5.11}
\]
if $0 < V \leq 2b^2$ and $F_{ij}^2 > 0$. If the positive and finite value of $\lambda$ extremizes the equation pointwise then it extremizes in the whole space and so the total energy can be finite. Setting $\lambda = 1$, we obtain a virial identity

$$\frac{b^2}{\sqrt{1 + \frac{G}{2b^2} F_{ij}^2}} \left( \sqrt{1 + \frac{G}{2b^2} F_{ij}^2} - 1 \right) = V.$$  \hspace{1cm} (5.12)

Using the ansatz (2.1), it becomes

$$\frac{G}{b^2} \left( \frac{a'}{r} \right)^2 = \frac{V(2b^2 - V)}{(b^2 - V)^2},$$  \hspace{1cm} (5.13)

which is equal to the BPS equation (4.16) if we shift $V \rightarrow V + X_0$. However, we could consider the virial identity globally by means of taking its integral over the whole space as such

$$\int d^2x \left[ \frac{b^2}{\sqrt{1 + \frac{G}{2b^2} F_{ij}^2}} \left( \sqrt{1 + \frac{G}{2b^2} F_{ij}^2} - 1 \right) - V \right] = \int d^2x Y(f) = 0,$$  \hspace{1cm} (5.14)

where $Y$ is a non-trivial function and it has been shown to be exist in polar coordinate in [1]. Substituting the ansatz (2.1) and the BPS equation (4.16) into the above integran, we find that $Y = -X_0$. Therefore solutions to the BPS equations (4.16) and (4.17) could give a finite total energy if $X_0$ satisfies equation (4.20) and its integration over whole space is

$$\int_0^\infty dr r X_0(f(r)) = 0.$$  \hspace{1cm} (5.15)

6 Conclusions and Remarks

In this article we have shown the BPS Lagrangian method can be used to derive BPS equations for vortex with internal pressure in the generalized Maxwell-Higgs and the generalized Born-Infeld-Higgs models; (3.2) and (4.2) respectively. Additional terms are needed in the BPS Lagrangian besides the usual total derivative terms. However, these additional terms imply additional constraint equations coming from Euler-Lagrange equations of the BPS Lagrangian and so these additional terms must be choosen carefully as such number of the constraint equations are (at least) equal to number of the unknown functions in the BPS Lagrangian particulary. We found that the additional terms in both models, (3.2) and (4.2), are similarly given by

$$L_{BPS} = -\frac{Q'(f)}{r} f' - \frac{Q'(a)}{r} a' - X_0(f) - X_2(f) f'^2.$$  \hspace{1cm} (6.1)

Here $Q \equiv Q(a, f)$. Furthermore in both models solutions for $X_2$ are equally given by (3.19) and solutions for $X_0$ are in the same form of (4.20), with their integral over whole space is (5.15). The pressure densities in radial and angular directions, $T_{rr}$ and $T_{\theta\theta}$, are in the same forms of (5.3) and (5.4), respectively, for both models. Either $T_{rr}$ or $T_{\theta\theta}$ can not
be zero otherwise it will lead to constant value of $a$ by the equation (4.20) and thus all pressure densities are nonzero. The total energy of the BPS equations of both models is finite. However, for the generalized Born-Infeld-Higgs model the scalar potential must be bounded from above $V \leq 2b^2$. Furthermore, reality condition in the BPS equations require $V - X_0 \geq 0$ and $0 \leq V - X_0 \leq 2b^2$ respectively for the generalized Maxwell-Higgs and the generalized Born-Infeld-Higgs models.

From the two models considered in this article, we may learn that BPS equations for vortex without internal pressure are due to BPS Lagrangian having only terms in the form of total derivative as such

$$L_{BPS} = \int d^2 x \ L_{BPS} = \int dQ \propto \int d^2 x \left( -\frac{Q'(f)}{r} f' - \frac{Q'(a)}{r} a' \right). \quad (6.2)$$

The BPS equations for vortex with internal pressure can be constructed from the corresponding BPS vortex without internal pressure, or zero internal pressure, by adding two additional terms in the BPS Lagrangian as shown previously. Hence in this way we would expect that we could find BPS equations for other type of solitons with (non-zero) internal pressure. However this may not the only way to get BPS equations for vortex with internal pressure. As an example one could consider BPS Lagrangian for the case of generalized Maxwell-Higgs model as follows

$$L_{BPS} = X_0(a, f, r) + X_{a1}(a, f, r)a' + X_{f1}(a, f, r)f'. \quad (6.3)$$

Following the previous procedures, the BPS equations are given by

$$a' = -\frac{r^2}{G} X_{a1}, \quad (6.4)$$

$$f' = -\frac{1}{2w} X_{f1}, \quad (6.5)$$

which imply $X_{a1} \neq 0$ and $X_{f1} \neq 0$. There are three constraint equations which are

$$\left( \frac{r^2 X_{a1}^2}{2G} - V - X_0 \right) + w \left( \frac{X_{f1}^2}{4w^2} - \frac{a^2 f^2}{r^2} \right) = 0, \quad (6.6)$$

and the other two are from the Euler-Lagrangian of its BPS Lagrangian,

$$\frac{X_{a1}}{r} = \frac{\partial X_0}{\partial a} - \frac{\partial X_{a1}}{\partial r} - \left( \frac{\partial X_{f1}}{\partial a} - \frac{\partial X_{a1}}{\partial f} \right) \frac{X_{f1}}{2w} \quad (6.7)$$

and

$$\frac{X_{f1}}{r} = \frac{\partial X_0}{\partial f} - \frac{\partial X_{f1}}{\partial r} - \left( \frac{\partial X_{a1}}{\partial f} - \frac{\partial X_{f1}}{\partial a} \right) \frac{r^2 X_{a1}}{G}. \quad (6.8)$$

The pressure densities in radial and angular directions are, respectively,

$$T_{rr} = \frac{r^2 X_{a1}^2}{2G} + w \left( \frac{X_{f1}^2}{4w^2} - \frac{a^2 f^2}{r^2} \right) - V, \quad (6.9)$$

$$T_{\theta\theta} = r \left( \frac{r^2 X_{a1}^2}{2G} - w \left( \frac{X_{f1}^2}{4w^2} - \frac{a^2 f^2}{r^2} \right) - V \right), \quad (6.10)$$
where we have taken the BPS limit. Pressureless condition, $T_{rr} = T_{\theta\theta} = 0$, gives us

$$X_{f_1} = \pm 2w \frac{a f}{r}, \quad X_{a_1} = \pm \frac{\sqrt{2G}}{r}V,$$

which implies $X_0 = 0$ using the constraint equation (6.6). Notes that the constrain equation (6.6) is actually $T_{rr} = X_0$. Furthermore using either the constraint equation (6.7) or (6.8), we obtain $\frac{\partial X_{a_1}}{\partial f} = \frac{\partial X_{f_1}}{\partial a}$, which implies that these two terms are total derivative term. The BPS equations for vortex with internal pressure would correspond to $X_0 \neq 0$ and since the number of unknown functions in the BPS Lagrangian is equal to the number of constraint equations, which is three, there could be other BPS equations for vortex with internal pressure besides the ones discussed in this article. Further simplification by substituting $X_0$ from equation (6.6) into equations (6.7) and (6.8) yields two constraint equations:

$$\frac{dX_{a_1}}{dr} + \frac{X_{a_1}}{r} = -\frac{2}{r}awf^2$$

and

$$\frac{dX_{f_1}}{dr} + \frac{X_{f_1}}{r} = -\left(\frac{r^2X_{a_1}^2}{2G^2} \frac{\partial G}{\partial f} + \frac{X_{f_1}^2}{4w^2} \frac{\partial w}{\partial r}\right).$$

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