VIRTUAL MANIFOLDS AND LOCALIZATION

BOHUI CHEN AND GANG TIAN

Abstract. In this paper, we explore the virtual technique that is very useful in studying moduli problem from differential geometric point of view. We introduce a class of new objects "virtual manifolds/orbifolds", on which we develop the integration theory. In particular, the virtual localization formula is obtained.

1. Introductions

In this paper, we introduce a class of new objects, which we call them "virtual manifolds/orbifolds". As the terminology suggests, it is a generalization of manifold/orbifold. One of the main themes of this paper is to show that one can do most analysis on those objects as one does on usual manifolds, particularly, in we develop a modified integration theory on and show an analogue of the deRham theory for virtual manifolds/orbifolds. Furthermore, we study $G$-actions on virtual manifolds. We introduce a notion of $G$-virtual manifolds/orbifolds and develop a $G$-equivariant (integration) theory on them. One of the main results in this paper is the Atiyah-Bott type localization formula on $G$-virtual manifolds when $G$ is abelian. We call such a formula the "virtual localization formula" (Theorem 6.8).

Virtual manifolds/orbifolds provide a natural frame to study certain type of singular spaces that come from moduli problems in geometry. By "a moduli problem", we mean the construction of invariants on moduli spaces that are associated to Fredholm systems (cf. [13]). There are many famous moduli problems of this sort, e.g, the moduli space of anti-self-dual instantons in defining the Donaldson invariants, the moduli space in constructing the Seiberg-Witten invariants, the moduli space of $J$-holomorphic maps (from Riemann surfaces to symplectic manifolds) constructing the Gromov-Witten invariants. Let us take the moduli space of $J$-holomorphic maps as an example. The Gromov-Witten invariants were first constructed for semi-positive symplectic manifolds (cf. [13], [14], [15]). Since the involved moduli spaces may be singular, one needs to introduce the technique of “virtual cycles” in order to construct the Gromov-Witten invariants for general symplectic manifolds. In around 1996, several groups of people gave different constructions of virtual cycles. These groups include Fukaya-Ono([7]), Li-Tian([9], [10]), Liu-Tian([11]), Ruan([12] and etc.. In this paper, we explain that for any Fredholm system, we are able to construct a virtual manifold/orbifold associated to its corresponding moduli problem. The invariants then can be defined via the integration on this virtual object. Such a general construction can be applied to the Gromov-Witten theory to get these "virtual cocycles" in the symplectic case and therefore get the Gromov-Witten invariants. These are done in [3]. This approach by using integration follows the one used by Ruan in his construction of the Gromov-Witten invariants for general
symplectic manifolds \([12]\)). In some sense, one may also treat the theory of "virtual manifolds" as a dual to Fukaya-Ono’s construction of Kuranishi structures or Li-Tian’s construction of weakly smooth structures. Our construction can be also carried out for “weakly” Fredholm systems which are more general than Fredholm ones and require less smoothness. The problem of constructing the Gromov-Witten invariants is one of such systems.

We then go further to consider Fredholm systems with \(G\)-actions. The virtual-manifolds associated to moduli spaces then turn to be \(G\)-virtual manifolds. Therefore, we develop the abelian virtual localization formula for moduli problems when \(G\) is abelian. This is applied to derive the symplectic virtual localization formula for Gromov-Witten invariants in \([4]\). We should point out that such a formula was previously developed in the algebraic geometry category (\([8]\)).

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## 2. Virtual Orbifolds

In this section, we introduce a class of new objects ”virtual manifolds(orbifolds)”.

### 2.1. What is a virtual orbifold supposed to be?

An \(n\)-dimensional manifold/orbifold can be constructed by patching several pieces of \(n\)-dimensional manifolds/orbifolds together– note that this is not obvious for orbifolds. From this point of view, a virtual manifold(orbifold) is obtained by patching several pieces of possibly different dimensional manifolds (orbifolds) together properly.

We take a simplest example to explain what we mean by patching. Let \(A_1\) and \(A_2\) be two manifolds/orbifolds with dimension \(n\) and \(n+k\) respectively. Let \(U_i \subseteq A_i, i = 1, 2,\) be two open submanifolds/orbifolds of \(A_i\) and suppose that \(\pi : U_2 \to U_1\) is a rank \(k\) (orbifold) vector bundle. So \(U_1\) is identified with the 0-section in \(U_2\), say \(U_1'\). By patching \(A_1\) and \(A_2\) together, geometrically, we mean the quotient space

\[
A_1 \cup A_2/(U_1 \cong U_1').
\]

Such an object is a virtual manifold/orbifold.

To emphasize the role of the bundle structure \((U_2, U_1, \pi)\), we introduce a space \(A_1 \cup A_2/\sim\), where the equivalent relation is given by

\[x \in U_1 \sim y \in U_2 \iff \pi(y) = x.\]

This space is called the virtual space of the virtual manifold(orbifold) given above.

In the above example, we say that \(U_i \subseteq A_i\) are the overlapping areas in the sense of ”patching”. To summarize, **two different dimensional manifolds \(A_1\) and \(A_2\) are patched at \(U_i \subseteq A_i\) which are different up to a vector bundle structure.**

Technically, the formalism of such objects is not obvious. When more than two pieces are patched, certain compatibility is needed. For this purpose, we explain a useful, but rather obvious, principle of patching in next subsection.

### 2.2. A principle of patching.

Let \(N = \{1, \ldots, n\}\) and \(\mathcal{N} = 2^N\) be the set of all subsets of \(N\). Let

\[
\mathcal{X} = \{X_I | I \in \mathcal{N}\}
\]
be a collection of sets indexed by $\mathcal{N}$. For any $I \subseteq J$ there exist $X_{I,J} \subseteq X_I, X_{J,I} \subseteq X_J$ and a surjective map
\[ \phi_{J,I} : X_{J,I} \to X_{I,J}. \]
Set $\Phi = \{ \phi_{J,I} | I \subseteq J \}$. We always assume that $X_\emptyset \neq \emptyset$.

**Definition 2.1.** A pair $(X, \Phi)$ is called patchable if for any $I, J \in \mathcal{N}$ we have

1. $X_{I,J \cap I} = X_{I,J} \cap X_{I,J,J}$;
2. $X_{I \cap J, I,J} = X_{I,J} \cap X_{I,J,J}$;
3. $\phi_{I \cup J, I} \circ \phi_{I,J} = \phi_{I,J} \circ \phi_{I \cup J, I}$;
4. $\phi_{I \cup J, I} (X_{I,J, I,J}) = \phi_{I,J}^{-1}(X_{I,J,J,I})$;
5. $\phi_{I \cup J, I} (X_{I,J, I,J}) = \phi_{I,J}^{-1}(X_{I,J,I,J})$.

Set
\[
X_{I,J} = \phi_{I \cup J, I} (X_{I,J, I,J}) = \phi_{I,J}^{-1}(X_{I,J,J,I}),
\]
\[
X_{J,I} = \phi_{I \cup J, I} (X_{I,J, I,J}) = \phi_{I,J}^{-1}(X_{I,J,I,J}).
\]

In this paper, we always assume that $(X, \Phi)$ is patchable. We define a relation for points in $\bigcup X_I$.

**Definition 2.2.** For $x \in X_I$ and $y \in X_J$ we say that $x \sim y$ if and only if there exists a $K \subseteq I \cap J$ such that
\[ \phi_{I,K}(x) = \phi_{J,K}(y). \]

We claim that "$\sim$" is an equivalence relation. This follows from the next lemma.

**Lemma 2.3.** Let $x \in X_I, y \in X_J$ and $z \in X_K$. If $x \sim y$ and $y \sim z$, then $x \sim z$.

**Proof.** By assumptions, we have
\[ \phi_{I,K_1}(x) = \phi_{J,K_1}(y); \]
\[ \phi_{J,K_2}(y) = \phi_{K,K_2}(z); \]
for some $K_1 \subseteq I \cap J$ and $K_2 \subseteq J \cap K$. Then
\[ \phi_{I,K_1 \cap K_2}(x) = \phi_{J,K_1 \cap K_2}(y) = \phi_{K,K_1 \cap K_2}(z). \]
Clearly, $K_1 \cap K_2 \subseteq I \cap K$. Therefore $x \sim z$. q.e.d.

We "patch" $X_I$ together and get a set
\[ X = \bigcup_{I \in \mathcal{N}} X_I / \sim. \]

From a patchable $(X, \Phi)$ to $X$ is our so-called principle of patching in this paper.

### 2.3 Virtual manifolds/orbifolds

A virtual manifold is a patchable pair $(X, \Phi)$ with specified properties.

**Definition 2.4.** Let $(X, \Phi)$ be a patchable pair. Suppose that

- $X_I \in \mathcal{X}$ are smooth orbifolds;
- $X_{I,J}$ and $X_{J,I}$ are open suborbifolds in $X_I$ and $X_J$ respectively;
- $\Phi_{J,I} : X_{J,I} \to X_{I,J}$ is an orbifold vector bundle.
Then \((X, \Phi)\) is called a virtual orbifold if for any \(I\) and \(J\),

\[
\phi_{I\cap J} : X_{I,J} \to X_{I\cap J, I\cup J},
\]

\[
\phi_{J,I} : X_{J,I} \to X_{I\cap J, I\cup J},
\]

are orbifold vector bundles and

\[(1)\]

\[X_{I\cup J, I\cap J} = X_{I,J} \times_{X_{I\cap J, I\cup J}} X_{J,I} \].

We call \(X = \bigcup_{I \in \mathcal{I}} X_I/\sim\) the virtual space of \((X, \Phi)\). We denote the projection map \(X_I \to X\) by \(\phi_I\).

Let \(d_I\) be the dimension of \(X_I\). We call \(d_\emptyset\) the virtual dimension of \((X, \Phi)\).

For simplicity, from now on, we assume that \((X, \Phi)\) is a virtual manifold. The discussion is identical for virtual orbifolds.

A point in \(X\) is an equivalence class, denoted by \([x]\). Set \([X_I]\) = \(\phi_I(X_I)\). Then \([\{X_I\}]\) forms a cover of \(X\). For any \([x] \in X\), there exist some \(X_I\) such that \(\phi_I^{-1}([x])\) consists of only one single point \(x\). Furthermore, among them, there is a unique \(X_I\) such that \(i = |I|\) is smallest. For \(X_I\) and \(X_J\) are two such sets, so is \(A_{I\cap J}\). This contradicts to the assumption of smallest. We call such an \(X_I\) the support of \([x]\).

**Remark 2.5.** We can define the virtual manifolds with boundary by a slight modification of definition 2.4: (1), \(X_I\) are manifolds (possibly with boundaries); (2), if \(X_{I,J}\) contains boundary \(\partial X_{I,J} \subseteq \partial X_I\), we require that

\[
\partial X_{I,J} = \Psi_{I,J}^{-1}(\partial X_{I,J}).
\]

Such an object is called a virtual manifold(orbifold) with boundary. Set

\[
\partial X = \{\partial X_I | I \in \mathcal{I}\}
\]

and

\[
\partial \phi_{I,J} = \phi_{I,J}|_{\partial X_{I,J}}.
\]

Let \(\partial \Phi = \{\partial \phi_{I,J}\}\). Then \((\partial X, \partial \Phi)\) also forms a virtual manifold. We call it the boundary of \((X, \Phi)\). The induced virtual space \(\partial X\) is called the boundary of \(X\).

We say \([x]\) is an interior point if \(x\) is an interior point in its support. Let \(X^\circ\) denote the set of interior points of \(X\). We see that

\[
X^\circ = X - \partial X.
\]

If \(\partial X\) is empty, we say that \((X, \Phi)\), is boundary free. If \(X\) is compact, We say that \((X, \Phi)\) is compact.

We give examples of virtual manifolds. If \(X\) is a manifold. Itself is clearly a virtual manifold: let \(N = \emptyset, A_\emptyset = X\). However, we can construct a nontrivial virtual manifold out of \(X\). This is explained in the following example.

**Example 2.6.** Let \(X\) be a manifold. Let \(\{U_0, U_1, \ldots, U_n\}\) be an open cover of \(X\). Let \(U_i^\circ = \frac{3U_i}{4}, i \geq 1\). Here \(\frac{3U_i}{4}\) just means an open subset whose closure is in \(U_i\). We use \(\frac{3}{4}\) to make the notations more suggestive.
Let \( N = \{1, \ldots, n\} \) and \( I, J, K \) be as before. Define
\[
X_0 = U_0 - \bigcup_{i=1}^{n} U_i^o
\]
\[
X_I = \bigcap_{i \in I} U_i - \bigcup_{j \notin I} U_j^o.
\]

Let \( X = \{X_I|I \in N\} \). Define
\[
X_{I,J} = X_{J,I} = X_I \cap X_J.
\]

All possible \( \psi_{J,I} \) are taken to be identities and let \( \Phi = \{\phi_{J,I}\} \). Then \((X, \Phi)\) is a virtual manifold (cf. Proposition 2.6). Moreover, the virtual space \( X \) is \( X \).

**Proposition 2.7.** \((X, \Phi)\) given in Example 2.6 is a virtual manifold.

**Proof:** Let \( U_i^c = X - U_i^o \). By definition,
\[
X_I = \bigcap_{i \in I} U_i \bigcap_{j \notin I} U_j^c.
\]
If \( I \subseteq J \)
\[
X_{I,J} = X_{J,I} = \bigcap_{i \in I} U_i \bigcap_{j \notin J} U_j^c \bigcap_{k \in J - I} (U_k - U_k^o).
\]

Now for arbitrary \( I, J \), with computations
\[
X_{I\cup J,I} = \bigcap_{i \in I} U_i \bigcap_{j \notin I \cup J} U_j^c \bigcap_{k \in J - I} (U_k - U_k^o);
\]
\[
X_{I\cup J,J} = \bigcap_{k \in J} U_k \bigcap_{j \notin I \cup J} U_j^c \bigcap_{i \in I - J} (U_i - U_i^o);
\]
\[
X_{I,J\cup J} = \bigcap_{i \in I \cap J} U_i \bigcap_{j \notin J} U_j^c \bigcap_{k \in I - J} (U_k - U_k^o);
\]
\[
X_{J,I\cup J} = \bigcap_{i \in I \cap J} U_i \bigcap_{j \notin I} U_j^c \bigcap_{k \in J - I} (U_k - U_k^o),
\]
wwe have
\[
X_{I\cup J,I} \cap X_{I\cup J,J} = X_{I \cap J,I} \cap X_{I \cap J,J} = \bigcap_{i \in I \cap J} U_i \bigcap_{j \notin I \cup J} U_j^c \bigcap_{k \in I \cup J - I \cap J} (U_k - U_k^o).
\]
This says, by \( 2 \),
\[
X_{I\cup J,I} \cap X_{I\cup J,J} = X_{I \cap J,I} \cap X_{I \cap J,J} = X_{I \cup J,I \cup J}.
\]
It is also easy to check that they are same as \( X_I \cap X_J \). This coincides with the definition of \( X_{I,J} = X_{J,I} \). These imply (P1)-(P5) in Definition 2.1. q.e.d.

2.4. **Language of germs.** Let \((X, \Phi)\) be a virtual manifold and \( X \) be its virtual space. \( X \) admits a partition. For each \( I \in N \) we define
\[
X_I = [X_I] - \bigcup_{J \subseteq I} [X_J].
\]
Then clearly \( \{X_I\} \) is a partition of \( X \). Note that \( [x] \in X_I \) if and only if the support of \([x]\) is \( X_I \). For \([x] \in X_I \), let \( x \) be the corresponding point in \( X_I \). Let \( \mathcal{N}_x \) be the set of open neighborhoods of \( x \) in \( X_I \) given as following:

\[
\mathcal{N}_x = \{ \phi_I^{-1}(V) | V \text{ is neighborhood of } [x] \}.
\]

Suppose that \( U \) is an element in \( \mathcal{N}_x \). Then for any \( y \in U \) we have a unique element in \( \mathcal{N}_y \): if the support of \([y]\) is \( X_I \), we take the element to be \( U \); otherwise, suppose the support of \([y]\) is \( X_J, J \subset I \), we take the element to be \( \phi_{I,J}(U \cap X_{I,J}) \).

We wish to develop the theory on \( X \) via structure \((X, \Phi)\). Let

\[
\Gamma = \{ U | U \in \mathcal{N}_x \text{ for some } [x] \}.
\]

We say such a collection \( \Gamma \) is a complete collection if

- \( \{[U]|U \in \Gamma\} \) covers \( x \); and
- for any \( U = U_x \in \mathcal{N}_x \), the induced element \( U_y \) for \([y] \in [U] \) is also in \( \Gamma \).

Given a complete collection \( \Gamma \), we set

\[
X'_I = \bigcup_{U \subset X_I, U \in \Gamma} U.
\]

Then \( X'_I \subset X_I \). Set

\[
X'_{I,J} = X'_I \cap X_{I,J},
\]

and \( \phi'_{I,J} = \phi_{I,J}|_{X'_{I,J}} \). It is easy to see that \((X', \Phi')\) forms a virtual manifold. Moreover \( X' = X \). Let \( \mathcal{Z}(X, \Phi) \) be the collection of virtual manifolds constructed by this way. \( \mathcal{Z}(X, \Phi) \) admits a partial order: let \((X', \Phi')\) and \((X'', \Phi'')\) be two virtual manifolds in \( \mathcal{Z}(X, \Phi) \), we say that \((X', \Phi') \prec (X'', \Phi'')\) if and only if for any \( I \)

\[
X'_I \subset X''_I.
\]

Note that for any two virtual manifolds \((X_I, \Phi_I) \in \mathcal{Z}(X, \Phi), i = 1, 2\), there exists an \((X_3, \Phi_3) \in \mathcal{Z}(X, \Phi)\) such that

\[
(X_3, \Phi_3) \prec (X_I, \Phi_I), i = 1, 2.
\]

The germ of \((X, \Phi)\), denoted as \((X, \Phi)^{\text{germ}}\), is defined to be the direct limit of \( \mathcal{Z} \).

We propose the principle of theory on virtual manifolds:

**A theory on \((X, \Phi)\) is a theory on \((X, \Phi)^{\text{germ}}\).**

By the spirit of germs, a theory \( P \) on \((X, \Phi)^{\text{germ}}\) is constructed on some \((X', \Phi') \in \mathcal{Z}(X, \Phi)\).

**Remark 2.8.** Sometimes, we need a more subtle version. A theory \( P \) on \((X, \Phi)^{\text{germ}}\) means

- theory \( P_{[x]} \) on some \( U_{[x]} \) for each \([x]\);
- theory \( P_{[x]} \) is compatible with theory \( P_{[y]} \) for any \([x]\) and \([y]\).

The compatibility is equivalent to the following statement: let \( P_{[x]} \) be the theory on \( U_{[x]} \); for any \( y \in U_{[x]} \), \( U_{[x]} \) induces an element \( U' \in \mathcal{N}_{[y]} \); then we require that the theory \( P_{[y]} \) and \( P_{[x]} \) are compatible on \( U' \cap U_{[y]} \).

We remark that for any collection

\[
\Gamma' = \{ U_{[x]} | U_{[x]} \in \mathcal{N}_{[x]} \} \in \mathfrak{X}
\]

we can generate a complete collection \( \Gamma \) that contains \( \Gamma' \).
2.5. Category of virtual manifolds. We construct the category of virtual manifolds. The main task is to construct the maps between virtual manifolds.

Let $(\mathcal{X}, \Phi)$ and $(\mathcal{B}, \Psi)$ be two virtual manifolds, $\mathcal{X}$ and $\mathcal{B}$ be their virtual spaces. Let $f : \mathcal{X} \to \mathcal{B}$ be a continuous map. We define lifts of $f$ on $(\mathcal{X}, \Phi)^{\text{germ}} \to (\mathcal{B}, \Psi)^{\text{germ}}$ in terms of language of germs in the sense of Remark 2.8.

**Definition 2.9.** A collection of maps $\mathcal{F} = \{f_x : U_x \to V_{f(x)}, f_x \in N_x\}$, where $U_x \in N_x$ and $V_{f(x)} \in N_{f(x)}$, is a lift of $f$ if $f_x$ is compatible with $f_y$ for all $x$ and $y$, namely,

- **Case 1,** if $U_x$ and $U_y$ are in same $X_I$, $f_x = f_y$ on $U_x \cap U_y$;
- **Case 2,** otherwise, suppose $U_y \subset X_J, J \subset I$, then $f_x$ is a lifting of $f_y$ on the fibration

$$\phi_{I,J}^{-1}(\phi_I(U_x \cap X_I) \cap U_y) \to \phi_{I,J}(U_x \cap X_I) \cap U_y.$$ 

Two lifts $\mathcal{F}'$ and $\mathcal{F}''$ are equivalent if for all $x$, $f'_x$ and $f''_x$ are compatible. An equivalence class $[\mathcal{F}]$ is called a lift of $f$ on $(\mathcal{X}, \Phi)^{\text{germ}}$. $(f, [\mathcal{F}])$ is called a virtual map from $(\mathcal{X}, \Phi)$ to $(\mathcal{B}, \Psi)$.

Note that a collection of maps between $(\mathcal{X}, \Phi)$ and $(\mathcal{B}, \Psi)$ given by

$$f_1 : X_I \to B_I$$

satisfying that $f_J$ is a lifting of $f_I$ on the fibration

$$\Phi_{J,I} : X_{J,I} \to X_{I,J}, I \subset J$$

determines a virtual map.

**Proposition 2.10.** Let

$$(f_1, [\mathcal{F}_1]) : (\mathcal{X}_0, \Phi_0) \to (\mathcal{X}_1, \Phi_1),$$

$$(f_2, [\mathcal{F}_2]) : (\mathcal{X}_1, \Phi_1) \to (\mathcal{X}_2, \Phi_2);$$

be two virtual maps, then they are composed to a (unique) virtual map

$$(f_2 \circ f_1, [\mathcal{G}]) : (\mathcal{X}_0, \Phi_0) \to (\mathcal{X}_2, \Phi_2).$$

**Proof.** This is obvious via the definition of germs. q.e.d.

**Definition 2.11.** $(f, [\mathcal{F}])$ is called a smooth map if all lifts $f_x$ are smooth.

Then, by Proposition 2.10 we make the following definition.

**Definition 2.12.** The category of (smooth) virtual manifolds is denoted by $\mathcal{V}$. The objects of the category are virtual manifolds. The morphisms between two virtual manifolds are equivalence classes of smooth virtual maps.

### 3. Integration theory on virtual manifolds

We develop the integration theories on virtual manifolds. A similar theory holds for virtual orbifolds.
3.1. **Forms on virtual manifolds.** Let $(\mathcal{X}, \Phi)$ be a virtual manifold.

**Definition 3.1.** A pre-$k$-form on $(\mathcal{X}, \Phi)$ is

$$\alpha = \{ \alpha_I \in \Omega^k(X_I) | I \in \mathcal{N} \}$$

such that

$$\alpha_I = \phi^*_{I,J} \alpha_I$$
on $X_{I,J}$. Two pre-$k$-forms $\alpha'$ and $\alpha''$, on $(\mathcal{X}', \Phi')$ and $(\mathcal{X}'', \Phi'')$ respectively, are equivalent if they admit a smaller $(\mathcal{X}'''', \Phi''') \in \mathcal{Z}(\mathcal{X}, \Phi)$. Let $[\alpha]$ be the equivalence class. It is called a $k$-form on $(\mathcal{X}, \Phi)$.

Namely, a form on $(\mathcal{X}, \Phi)$ is a form on $(\mathcal{X}, \Phi)_{\text{perm}}$. Let $\Omega^k(\mathcal{X})$ be set of $k$-forms on $(\mathcal{X}, \Phi)$. Then $(\Omega^*(\mathcal{X}), d)$ is a complex. Define

$$H^*_d(\mathcal{X}) = H^*(\Omega^*(\mathcal{X}), d).$$

We next consider a very different type of forms on virtual manifolds. Let $\Theta_{I,J}$ be the Thom forms of the bundle $\Psi_{I,J} : X_{I,J} \to X_{I,J}$. To avoid the unnecessary complication caused by the degree of forms, we always assume that the degree of $\Theta_{I,J}$ is even.

**Definition 3.2.** A set of forms $\Theta = \{ \Theta_{I,J} \}_{I \subseteq J}$ is called a transition data of $\mathcal{X}$ if it satisfies the following compatibilities: for any $I$ and $J$,

$$\Theta_{I \cup J, I \cap J} = \Psi^*_{I \cup J, I \cap J} \Theta_{I, I \cap J} \wedge \Psi^*_{I \cup J, J} \Theta_{J, I \cap J}$$
on $X_{I \cup J, I \cap J}$.

**Definition 3.3.** A pre-virtual form on $(\mathcal{X}, \Phi)$ is

$$\tilde{z} = \{ z_I \in \Omega^*(X_I) | I \in \mathcal{N} \}$$

such that

$$z_I = \phi^*_{I,J} z_I \wedge \Theta_{I,J}$$
on $X_{I,J}$ for some transition data $\Theta$. $\tilde{z}$ is called a $\Theta$-form on $\mathcal{X}$. Two pre-$k$-forms $\tilde{z}'$ and $\tilde{z}''$, on $(\mathcal{X}', \Phi')$ and $(\mathcal{X}'', \Phi'')$ respectively, are equivalent if they admit a smaller $(\mathcal{X}'''', \Phi''') \in \mathcal{Z}$. Let $[\tilde{z}]$ be the equivalence class. It is called a virtual form on $(\mathcal{X}, \Phi)$. The virtual degree of $\tilde{z}$ is the degree of $z_\emptyset$.

Let $\Omega^*_v(\mathcal{X})$ be set of virtual forms on $(\mathcal{X}, \Phi)$. Then $(\Omega^*_v(\mathcal{X}), d)$ is a complex. Define

$$H^*_{v, dR}(\mathcal{X}) = H^*(\Omega^*_v(\mathcal{X}), d).$$

We may define the support of a form or a virtual-form. Let us take a form $[\alpha]$ as an example. Suppose $\alpha = (\alpha_I)$. For any $[x] \in \mathcal{X}$, let $X_I$ be its support, we say

$$[x] \in \text{supp}(\alpha_I) \iff x \in \text{supp}(\alpha_I).$$

If $[\alpha]$ is compact supported in $\mathcal{X}$ (or $\mathcal{X}^o$), we write $[\alpha] \in \Omega^*_v(\mathcal{X})$ (or $\alpha \in \Omega^*_v(\mathcal{X}^o)$). Similarly, we can define $\Omega^*_v(\mathcal{X}^c)$ and $\Omega^*_v(\mathcal{X}^c)$. For most of time, we are interested in forms $\Omega^*_v(\mathcal{X}^c)$.

If $\alpha \in \Omega^*_v(\mathcal{X})$ and $\tilde{z} \in \Omega^*_v(\mathcal{X})$, $\alpha \wedge \tilde{z}$ is in $\Omega^*_v(\mathcal{X})$. 

3.2. Integrations on virtual manifolds. We now describe how to define \( \int_X \mathfrak{z} \) for \( [\mathfrak{z}] \in \Omega_{\omega,c}(\mathcal{V}) \), where \( \deg \mathfrak{z} = \dim X \). Suppose \( \mathfrak{z} = (z_I) \) is a \( \Theta \)-form on \((\mathcal{X}, \Phi)\) representing \([\mathfrak{z}]\). The definition is almost obvious because of the following reason: suppose \([V] \subseteq [X_I] \cap [X_J]\), let \( V_I = \pi_I^{-1}([V]) \), \( V_J = \pi_J^{-1}([V]) \), \( V_{I \cap J} = \pi_{I \cap J}^{-1}([V]) \) and \( V_{I \cup J} = \pi_{I \cup J}^{-1}([V]) \), then

\[
\int_{V_I} z_I = \int_{V_{I \cup J}} z_{I \cup J} = \int_{V_J} z_J.
\]

The equalities of two ends are due to the Thom isomorphism. Similarly, the middle term can also be replaced by \( \int_{V_{I \cap J}} z_{I \cap J} \). Hence, \( \int_X \mathfrak{z} \) is well defined on \([X_I] \cap [X_J]\).

Now define

\[
\int_X \mathfrak{z} = \sum_I \int_{[X_I]} z_I - \sum_{I,J} \int_{[X_I] \cap [X_J]} z_I + \sum_{I,J,K} \int_{[X_I] \cap [X_J] \cap [X_K]} z_I - \cdots.
\]

It is not hard to see that \( \int_X [\mathfrak{z}] \) is independent of choice of representatives of \([\mathfrak{z}]\).

Let

\[
\Gamma = \{U | U \in \mathcal{N}_x \text{ for some } [x]\}
\]

be a collection of sets such that \( \{U | U \in \Gamma\} \) covers \( \mathcal{X} \). Let

\[
X_I' = \bigcup_{U \subseteq X_I} U.
\]

Then

\[
\int_X \mathfrak{z} = \sum_I \int_{[X_I']} z_I - \sum_{I,J} \int_{[X_I'] \cap [X_J']} z_I + \sum_{I,J,K} \int_{[X_I'] \cap [X_J'] \cap [X_K']} z_I - \cdots.
\]

Furthermore, we can construct a new virtual manifold out of \( \{X_I'\} \). Recall that \( \{[X_I']\} \) forms a cover of \( \mathcal{X} \). We now apply Example 2.6. To do this, we re-index the index set

\[
\iota : \mathcal{N} \rightarrow \mathcal{N}^{[1]} = \{0, 1, \ldots, 2^N - 1\}
\]

by requiring \( \iota(\emptyset) = 0 \). Rewrite the sets \( \{[X_I']\} \) as \( \{Y_i\} \), i.e., \( Y_i = [X_i'] \). Set \( \mathcal{N}^{[1]} = 2^{\mathcal{N}^{[1]}} \). Then as explained in Example 2.6, we can construct a patchable pair \((\mathcal{Y}, \Psi)\). We now construct a virtual manifold from this pair. For \( I^{[1]} = \{i_1, \ldots, i_k\} \in \mathcal{N}^{[1]} \), we have a set

\[
\Pi = \{I_{i_1}, \ldots, I_{i_k}\}, \text{ where } I_{i_j} = \iota^{-1}(i_j).
\]

Set

\[
I_{\max} = \bigcup_{j=1}^k I_{i_j} \in \mathcal{N}.
\]

We define

\[
Z_{I^{[1]}} = \pi_{I_{\max}}^{-1}(Y_{I^{[1]}}).
\]

Note that \( Y_{I^{[1]}} \subset \mathcal{X} \), in particular, in \([X_{I_{\max}}]\). Hence \( Z_{I^{[1]}} \) is in \( X_{I_{\max}} \). For \( I^{[1]} \subset J^{[1]} \),

\[
Z_{I^{[1]}, J^{[1]}} = \pi_{I_{\max}}^{-1}(Y_{I^{[1]}, J^{[1]}}), \quad Z_{J^{[1]}, I^{[1]}} = \pi_{J_{\max}}^{-1}(Y_{J^{[1]}, I^{[1]}}).
\]

and

\[
\Psi_{J^{[1]}, I^{[1]}} : Z_{I^{[1]}, J^{[1]}} \rightarrow Z_{J^{[1]}, I^{[1]}}
\]

is induced from \( \Phi_{J_{\max}, I_{\max}} \). Set

\[
Z_I = \{Z_{I^{[1]}}\}, \quad \Psi = \{\Psi_{J^{[1]}, I^{[1]}}\}.
\]

It is straightforward to prove that
Proposition 3.4. \((\mathcal{Z}_\Gamma, \Psi)\) is a virtual manifold and its virtual space \(\mathcal{Z}_\Gamma\) is same as \(X\).

Note that the integration on \(\mathcal{Z}_\Gamma\) is same as on \(X\). We call this new virtual manifold \(\mathcal{Z}_\Gamma\) to be a modification of \(X\).

To manipulate integrations, it is convenient to develop some type of theorems of partition of unity on virtual manifolds. These are discussed in the next sub-section.

3.3. Partition of unity. Let \((\mathcal{X}, \Phi)\) be a virtual manifold. For simplicity, we assume that \(\partial \mathcal{X} = \emptyset\). Also we assume \((??)\) holds.

Definition 3.5. Let \(W = (W_I)\) be pre-compact if it satisfies:

- a1. \([W]\) is compact in \(X\), where \([W] := \bigcup W_I\);
- a2. for each \(x \in [X_I]\), \(\pi^{-1}_I \cap W_I\) is compact;
- a3. \(\Psi_{J,I}(W_{J,I}) = W_{I,J}\), where \(W_{J,I} = X_{J,I} \cap W_J\).

Let \(W = (W_I)\) be pre-compact. Then we have

Lemma 3.6. There exists open subset \(Y_I \subset [X_I]\) for each \(I\) such that

- b1. \([W] \subset \bigcup Y_I\);
- b2. \(\overline{Y}_I \subset [X_I]\).

Proof. For any \([x] \in [W]\), let \(X_K\) be the support of \([x]\), i.e., there exists a unique \(x \in X_K\) such that \(\pi_K(x) = [x]\). Hence there exists a small neighborhood \(B_K(x) \subset X_K\) such that \(\bar{B}_K(x) \subset X_K\). Then \(\pi_K(\bar{B}_K(x)) \subset [X_K]\).

\[\{[B_K(x)], x \in [W]\}\]

covers \([W]\). Since the latter one is compact, there exists a finite cover, denoted by \(\{[B_K(x)], 1 \leq i \leq l\}\) for some \(l < \infty\). Set

\[Y_I = \bigcup_{K, i=1}^{l} [B_K(x_i)].\]

Clearly, the lemma follows. q.e.d.

Set \(Z_I = \pi_I^{-1}Y_I\), \(Z_I\) is an open subset of \(X_I\).

Definition 3.7. We say \(\{\eta_I\}\) is a smooth partition of unity with respect to \(W\) (i.e., for any \(x \in [W]\), \(\sum \eta_I(x) = 1\)) if

- c1. \(\text{supp}(\eta_I) \subset Y_I\);
- c2. \(\beta_I = \pi_I^* \eta_I\) is smooth on \(Z_I\).

It is not clear that a smooth partition of unity exists on a virtual manifold. But we can prove the existence on its modification. Let \(\mathcal{Z}_\Gamma\) be a modification of \(\mathcal{X}\) explained in the end of last subsection. Correspondingly, we have \(W\) on \(\mathcal{Z}_\Gamma\).

Lemma 3.8. For a proper choice of \(\Gamma\), there exists a partition of unity \(\{\eta_{I[I]}\}\) on \(\mathcal{Z}_\Gamma\) with respect to \(W\).
Proof. Step 1, by classic result, there is a continuous partition unity $\eta_I$ on $X$ with respect to $W$. We can first construct it on topological space $X$, then pull back functions to $X_I$. Set

$$[S_I] = \bigcup_{J \prec I} \partial[X_J]$$

and $S_I = \pi_I^{-1}([S_I])$. It is not hard to construct $\eta_I$ such that they are smooth away from $S_I$.

Step 2, by modifying functions $\eta_I$, we may have $\eta'_I$ defining on $X'_I = \pi_I^{-1}A_I$, where

$$A'_I \subset \overline{A'_I} \subset [X'_I]$$

and $\{A'_I\}$ covers $X$.

This can be done inductively. We start with $I = \emptyset$. Set $\eta'_I = \eta_I$. Next let $I = \{i\}, 1 \leq i \leq N$. We mollify $\eta_I$ at $S_I$ and get a smooth function $\eta'_I$ on $X'_I$.

We conclude that for $I \prec J$, $\eta'_I$ matches $\eta'_J$ on the overlapping area from $S_J$. Now we can choose proper $A'_I$ such that $\eta'_I$ matches $\eta'_J$ on the $A'_I \cap A'_J$. Moreover we can assume that $\{A'_I\}$ covers $X$.

Step 3, Set $\Gamma = \{A'_I\}$. Then we can construct a virtual manifold $Z_\Gamma$ with smooth functions $\eta'_I$ on $Z_{I[I]}$. Now on $Z\Gamma$, set

$$\eta_{I[I]} = \frac{\eta'_{I[I]}}{\sum_{J[I]} \eta'_{J[I]}}.$$ 

This gives a smooth partition of unity on $Z\Gamma$. q.e.d.

Suppose $X$ has a partition of unity. Another way to get $\int_X \xi$ is to use partition of unity. Set

$$W_I = (\text{supp}(z_I))^\circ$$

and $\mathcal{W} = \{W_I\}$. Let $\{\eta_I\}$ be a partition of unity with respect to $\mathcal{W}$. Then equivalently, we define the integral of $I$ to be

$$\int_X \xi = \sum_I \int_{X_I} \eta_I z_I.$$ 

3.4. The Stokes’ theorem on virtual manifolds. In order to define invariants, we focus on the following case: (1), $\mathfrak{z} \in \Omega^\circ_{\text{tr},c}(X^\circ)$ is close; (2), $a \in \Omega^{d-k}(X)$ is close. Define

$$\mu_z(a) = \int_X a \wedge \mathfrak{z}.$$ 

Then, Stokes’ theorem and the Thom isomorphism imply that

Lemma 3.9. Suppose $a$ and $b$ are exact,

$$\mu_z(a) = \mu_z(b).$$

Proof: Suppose $dc = a - b$, where $c = (c_I)$. Then

$$\mu_z(a) - \mu_z(b) = \sum_I \int_{X_I} dc_I \wedge \eta_I z_I = - \sum_I \int_{X_I} d(\eta_I) c_I \wedge z_I = 0.$$
The last equality follows from the Stokes’ theorem, the Thom isomorphism, and that \( \{ \eta_I \} \) is a partition of unity. q.e.d.

A similar argument implies the Stokes’ theorem for virtual manifolds. Suppose \( z \in \Omega_{\Theta,c}(\mathcal{L}) \). The restriction of \( z \) on \( \partial \mathcal{L} \), denoted by \( i^* z \), is a form in \( \Omega_{\Theta,c}(\partial \mathcal{L}) \). Here \( i : \partial \mathcal{L} \to \mathcal{L} \) is the standard embedding. Then

**Theorem 3.10 (Stokes' Theorem).** For \( z \in \Omega_{\Theta,c}(\mathcal{L}) \)

\[
\int_X d z = \int_{\partial X} i^* z.
\]

As a consequence,

**Corollary 3.11.** For any close form \( z \in \Omega_{\Theta,c}(\mathcal{L}) \)

\[
\int_{\partial X} i^* z = 0.
\]

Note that we just have a pairing

\[
\mu : H^*_{\text{v,c}}(\mathcal{L}^\circ) \times H^*(\mathcal{L}) \to \mathbb{R}.
\]

3.5. **Virtual bundles and the Euler classes.** Let \( (\mathcal{X}, \Phi) \) be a virtual manifold without boundary.

**Definition 3.12.** A virtual bundle over \( (\mathcal{X}, \Phi) \) is \( \mathcal{E} = (E_I) \), where \( E_I \to X_I \) is a vector bundles, such that for \( I \subset J \)

\[
E_J|_{X_I,J} = \phi^*_{IJ}(X_J,I \oplus E_I|_{X_I,J}).
\]

A section of the bundle is \( S = (S_I) \), where \( S_I : X_I \to E_I \) is a section of bundle, such that for \( I \subset J \)

\[
S_J(x, v) = (v, S_I(x)).
\]

Here \( (x, v) \in X_{I,J} \) is the local coordinate. The section is transverse to 0-section if each \( S_I \) is transverse.

Let \( \Theta = (\Theta_{IJ}) \) be a transition data. A \( \Theta \)-Thom form of \( \mathcal{E} \) is \( \Lambda = (\Lambda_I) \), where \( \Lambda_I \) is a Thom form of \( E_I \), such that for \( I \subset J \)

\[
\Lambda_J = \Theta_{IJ} \wedge \Lambda_I
\]
on \( E_I|_{X_{I,J}} \). \( S^* \Lambda \) defines a pre-\( \Theta \)-form. \( [S^* \Lambda] \) is called an Euler form of \( \mathcal{E} \).

The proofs of the following two statements are straightforward. We leave the proofs to readers.

**Proposition 3.13.** Let \( \mathcal{E} = (E_I) \) be a virtual bundle over a virtual manifold \( (\mathcal{X}, \Phi) \). Then there exists a sub-virtual manifold \( (\mathcal{X}', \Phi') \) in the sense of Remark ?? such that, over the virtual bundle given by \( \mathcal{E}' = (E_I|_{X_I}) \) we have

1. a section \( S \),
2. a \( \Theta \)-Thom form \( \Lambda \), and therefore
3. an Euler class \( S^* \Lambda \).

Let \( \mathcal{E} \to \mathcal{X} \) be a virtual bundle with a virtual transverse section \( S \). Then \( S^{-1}(0) \) is a virtual manifold.
**Proposition 3.14.** Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be two virtual bundles over $(X, \Phi)$. Let $\Lambda^i$ be their $\Theta$-Thom forms. Let $S^i, i = 1, 2$, be two transverse sections of $\mathcal{E}^i$. Then $S = S^1 \oplus S^2$ is a transverse section of $\mathcal{E}^1 \oplus \mathcal{E}^2$. Set $\lambda = \Lambda^1 \wedge \Lambda^2$. We have

\[ \int_X a \wedge (S^* \lambda) = \int_{(S^1)^{-1}(0)} a \wedge (S^2)^* (\Lambda^2) = \int_{(S^2)^{-1}(0)} a \wedge (S^1)^* (\Lambda^1). \]

4. **G-virtual manifolds and localization**

4.1. **G-virtual manifolds, equivariant forms and integration.** The discussion given in the previous section can be generalized to the equivariant case. Let $G$ be a compact Lie group.

**Definition 4.1.** By a $G$-virtual manifold $(X, \Phi)$, we mean that (a.) $(X, \Phi)$ is a virtual manifold, (b.) each $X_I$ is $G$-manifold and (c.) $\Psi_{J,I} : X_{J,I} \to X_{I,J}$ are $G$-equivariant bundles for any $I \subset J$.

To study the $G$-equivariant integration theory on $X$, we may consider $G$-equivariant transition data $\Theta_G = \{\Theta^G_{J,I}\}_{I \subseteq J}$. Then similarly, we may define: $G$-equivariant forms $\Omega^*_{\Theta_G}(X)$, $G$-equivariant $\Theta_G$ forms $\Omega_{\Theta_G,c}(X)$, $\Omega_{\Theta_G,c}(X)$ and $G$-invariant partition of unity, etc. For $\zeta = (\zeta_I) \in \Omega_{\Theta_G,c}(X)$, as before, we define

\[ \int^G_X \zeta = \sum_I \int_{X_I} \eta_I \zeta_I. \]

For a $G$-closed form $\zeta \in \Omega^*_{\Theta_G,c}(X)$ and $\alpha \in \Omega_G(X)$, define

\[ \mu_\zeta(\alpha) = \int^G_X \alpha \wedge \zeta. \]

For any space $Y$ with $G$ action, the notation $Y^G$ denotes the fixed loci of the action. Let $X^G = \{X^G_I\}$ Then

**Lemma 4.2.** $X^G$ is a virtual (sub-)orbifold (of $X$).

This follows directly from the definition. We skip the proof.

We will discuss the abelian localization formula of Atiyah-Bott type for the integration $\mu_\zeta(\alpha)$. For simplicity, we assume that $G = S^1$. We also assume that $X$ is compact, boundary free for the sake of Stokes’ theorem. Otherwise, the compact-supportedness of $\zeta$ would take care the issue.

4.2. **Virtual normal bundles and their Euler classes.** For each $X^G_I$, let $N_I$ denote its $G$-normal bundle in $X_I$. Now focus on $X_{J,I} \to X_{I,J}, I \subseteq J$. By restricting on $X^G_{I,J}$, this bundle splits as

\[ X_{J,I}|_{X^G_{I,J}} = X^G_{J,I} \oplus P \]

for some $G$-invariant bundle $P$. Therefore,

\[ N_J|_{X^G_{I,J}} = \Psi^*_{J,I}(N_I \oplus P). \]

This says that

**Lemma 4.3.** $\mathcal{N} = \{N_J\}$ forms a virtual bundle over $X^G$. 
Let \( e_{J,G} \) be the equivariant Euler forms of \( N_J \) over \( X_J^G \). We may arrange them such that
\[
e_{J,G}|_{X_J^G} = \Psi^*_{J,I} e_{I,G} \wedge \Theta_G(P),
\]
where \( \Theta_G(P) \) is the equivariant Thom form on \( P \). Denote \( \{e_{I,G}(X_J^G)\} \) by \( e_G(X_J^G) \).

On the other hand, Let \( \Theta_G = (\Theta^G_{J,I}) \) be the transition data. According to (7), we may assume that
\[
\Theta^G_{J,I} = \Theta_G(X_J^G) \wedge \Theta_G(P).
\]
over \( X_{J,I}|_{X_J^G} \). This induces a transition data
\[
\tilde{\Theta}_G = \{\Theta_G(X_{J,I}^G)\}
\]
on \( X^G \).

For any \( \Theta_G \)-form \( \zeta = (\zeta_I) \), we find that
\[
\frac{\zeta_J}{e_{J,G}}
\]
forms a \( \tilde{\Theta} \)-form on \( X^G \):
\[
\frac{\zeta_J}{e_{J,G}} = \Psi^*_{J,I} \frac{\zeta_I \wedge \Theta^G_{J,I}}{e_{I,G} \wedge \Theta_G(P)} = \Psi^*_{J,I} \frac{\zeta_I \wedge \tilde{\Theta}_G(X_{J,I}^G)}{e_{I,G}}
\]
Hence, we conclude that

**Proposition 4.4.** For a \( \Theta_G \) form \( \zeta, \{\zeta_I/e_{I,G}\} \) forms a \( \tilde{\Theta} \)-form on \( X^G \). The form is denoted by \( e_\zeta(X_G^G) \).

### 4.3. The Abelian localization formula

The standard localization technique implies that

**Theorem 4.5.** Let \( X \) be a finite dimensional virtual manifold with \( G = S^1 \) action. Let \( X \) be its virtual space. Let \( \zeta \in \Omega_{\Theta_G,c}(X^G) \) and \( \alpha \in \Omega_G^*(X) \), then
\[
\mu_\zeta(\alpha) = \int_{X^G} i^*_{X_G}(\alpha \wedge \zeta).
\]
The right hand side in the formula can be thought as an integration on virtual manifold \( X^G \):
\[
\mu_{e_\zeta(X_G^G)}(i^*_{X_G}(\alpha))
\]
**Proof:** Let \( \Omega_I, \Omega'_I \) be two equivariant Thom forms on \( N_I \). By identifying \( N_I \) with a neighborhood \( U(X_I^G) \) of \( X_I^G \), we require that they are equal in a smaller neighborhood \( U(X_I^G) \subseteq U(X_J^G) \) and \( i^*_{X_I} \Omega_I = e_{I,G}(X_J^G) \). Moreover, we require that: (1) the support of \( \Omega'_I \) is contained in that of \( \Omega_I \); (2) on the overlapping area \( N_J \cap X_{J,I} \to N_I \cap X_{I,J} \),
\[
\Omega_J = \Omega_I \wedge \Theta^G_{J,I}, \text{ and } \Omega'_J = \Omega'_I \wedge \Theta^G_{J,I}.
\]
It is not hard to see that such pairs always exist.

Then by the Thom isomorphism,
\[
\sum_I \int_{X_I} \eta_I \alpha_I \wedge \zeta_I \wedge \Omega'_I / \Omega_I = \sum_I \int_{X_I^G} i^*_{X_I^G}(\eta_I \alpha_I \wedge \zeta_I) / e_G(X_I^G).
\]
Note that the right hand side is same as the right hand side of the formula in the theorem.
On the other hand, let
\[ \tilde{\alpha}_I = \alpha_I - \alpha_I \wedge \frac{\Omega'_I}{\Omega_I}. \]
Then \( \tilde{\alpha} = (\tilde{\alpha}_I) \) becomes a \( G \)-equivariant form supported away from \( X^G \). It remains to prove that
\[ \mu_\zeta(\tilde{\alpha}) = 0. \]
The proof is standard. In fact, there exists a form \( \beta = (\beta_I) \) supported away from \( X^G \) such that \( \tilde{\alpha}_I = d_G \beta_I \). Then
\[ \mu_\zeta(\tilde{\alpha}) = \sum_I \int_{X^I} \eta_I d\beta_I = -\sum_I \int_{X^I} d(\eta_I)\beta_I = 0 \]
For the last equality, we use the fact that \( \sum_I \eta_I = 1 \). q.e.d.

5. Fredholm systems and Stabilizations

5.1. The Fredholm set-up. We start with the following set-up.

**Definition 5.1.** A Fredholm system consists of following data:

1. \( \pi: F \to B \) be a Banach orbifold bundle over a Banach orbifold \( B \);
2. \( S: B \to F \) be a proper smooth section. In particular, the properness implies that \( M = S^{-1}(0) \) is compact;
3. for any \( x \in M \), let \( L_x \) be the linearization of \( S \) at \( x \)
   \[ L_x: T_xB \to F_x. \]

We assume that \( L_x \) is a Fredholm operator. Let \( d \) be the index of the operator.

We refer the triple \((B, F, S)\) as a Fredholm system. \( M \) is called the moduli space of the system.

A core topic in studying moduli problems is to define invariants on such a system. This is based on the study of \( M \). It is well known that if \( L_x \) is surjective for all \( x \in M \), \( M \) is a compact smooth orbifold. Then \( M \) can be thought as a cycle in \( H_d(B) \) representing the Euler class of bundle \( F \to B \). Let \( a \in H^d(B, \mathbb{R}) \), define
\[ \Phi(a) = \int_M a. \]
The challenging problem is to define invariants when the surjectivity of \( L_x \) fails. In this case, the moduli may have dimension larger than expected. The virtual technique is introduced to deal with this bad situation. There are several different versions of this technique, however the main idea is the stabilization, which has become popular since 60’s. This section is a brief recollection of these constructions. We will construct a virtual orbifold which behaves well and replaces the moduli \( M \), then we will follow an approach used in [12] to define invariants by integration over such a virtual manifold.
5.2. Stabilization. Let \((\mathcal{B}, \mathcal{F}, S)\) be a Fredholm system as before. For simplicity, all orbifolds appeared in definition 5.1 are replaced by manifolds. A proper modification can be easily made when we consider orbifolds.

Let \(U\) be an open subset of \(\mathcal{B}\), let
\[
\sigma : O_U \to U
\]
be a rank-\(k\) vector bundle, let
\[
s : O_U \to \mathcal{F}_U
\]
be a bundle map. Define a map
\[
\hat{S} : O_U \to \mathcal{F}_U; \hat{S}(u, o) = (u, S(u) + s(o)),
\]
where the expression is given in the form of local coordinates and \(S(u) + s(o)\) is the sum on fibers. By abusing the notations, we usually use \(S + s\) for \(\hat{S}\) to emphasis that \(S\) is stabilized by \(s\).

Let \(\hat{L}_{(u, o)}\) be the linearization of \(\hat{S}\) as a map
\[
\hat{L}_{(u, o)} : T_{(u, o)}O_U \to \mathcal{F}_U.
\]
We say that the pair \((O_U, s)\) stabilizes the system \((\mathcal{B}, \mathcal{F}, S)\) at \(U\) if \(\hat{L}_{(u, o)}\) are surjective for all \((u, o) \in O_U\). Set
\[
V_U = \hat{S}^{-1}(0) \subseteq O_U.
\]
This is now a smooth manifold of dimension \(d + k\). Clearly, \(M \cap U \subseteq V_U\) and
\[(u, o) \in M \iff o = 0.
\]

We can restate this construction by using the concept of Fredholm system. Let \(\sigma^* \mathcal{F} \to O_U\) be the pull-back bundle over \(O_U\). \(\hat{S}\) then gives a canonical section of this bundle in an obvious way. For simplicity, we still denote the section by \(\hat{S}\). Therefore, we have a Fredholm system \((O_U, \sigma^* \mathcal{F}, \hat{S})\). If \((O_U, s)\) stabilizes the system at \(U\), we say that \((O_U, \sigma^* \mathcal{F}, \hat{S})\) stabilizes \((\mathcal{B}, \mathcal{F}, S)\) at \(U\). \(V_U \subseteq O_U\) is the moduli space of the new system.

We may construct a canonical bundle \(\sigma^* O_U \to V_U\), then there is a canonical section \(\sigma : V_U \to \sigma^* O_U\) given by \((u, o) \to (u, o, o)\) with respect to the local coordinates. Then \(M \cap U = \sigma^{-1}(0)\). This reduces the infinite dimensional system \((U, \mathcal{F}_U, S)\) to a finite dimensional system \((V_U, \sigma^* O_U, \sigma)\). We call \((V_U, \sigma^* O_U, \sigma)\), or simply \(V_U\), to be the virtual neighborhood of \(M\) at \(U\). Bundles \(O_U\) and \(\sigma^* O_U\) are called the obstruction bundles.

We now explain the existence of local stabilizations.

Suppose \(L_x\) is not surjective for some \(x \in M\). Let \(O^x\) be a finite dimensional subspace of \(\mathcal{F}_x\) such that
\[
\text{Image}(L_x) + O^x = \mathcal{F}_x.
\]
For example, we may take \(O^x\) to be the "cokernel" of \(L_x\).

Let \(U^x\) be a neighborhood of \(x\) in \(\mathcal{B}\). In order to make notations more suggestive, we assume that \(U^x = B_r(x)\) is the radius-\(r\) disk centered at \(x\) and \(cU^x = B_{cr}(x)\) for \(c \in \mathbb{R}^+\).

Suppose that \(\mathcal{F}_{U^x}\) is trivialized as \(\mathcal{F}_{U^x} = U^x \times \mathcal{F}_x\). We now describe the stabilization using the notations given above by setting \(U = U^x\):
(C1) the obstruction bundle is
\[ \mathcal{O}_{U^x} = U^x \times O^x; \]

(C2') the bundle map \( s = I^x : \mathcal{O}_{U^x} \to \mathcal{F}_{U^x} \) is the standard embedding via the trivialization of \( \mathcal{F}_{U^x} \) given above.

We may assume that the pair \((\mathcal{O}_{U^x}, I^x)\) stabilizes the system at \(U^x\) if \(U^x\) is chosen small. This explains the existence of local stabilization.

**Remark 5.2.** Following the constructions, we have a virtual neighborhood \( V_U \). One may use the projection map \( \sigma : V_U \to B \). Then \( \sigma(V_U) \) is taken as the virtual neighborhood of \( M \) at \( U \) in both [7] and [9].

The trivialization of \( \mathcal{F}_{U^x} \) prevents us to extend the construction outside \( U^x \). This is "taken care" by modifying the bundle map \( s \) as the following. Let \( \eta^x \) be a cut-off function on \( U^x \) such that \( \eta^x = 1 \) in \( \frac{U^x}{2} \) and = 0 outside \( \frac{3U^x}{4} \). (C2') is then replaced by

(C2) the bundle map is given by \( s^x = \eta^x I^x \).

Clearly, \((\mathcal{O}_{U^x}, s^x)\) stabilizes the system at \( \frac{U^x}{2} \). In this paper, we always use (C2) to construct virtual neighborhoods. It turns out that (C2) is the key towards the construction of virtual orbifolds from a Fredholm system.

Repeat the argument given earlier, we have a system \((\mathcal{O}_{U^x}, \sigma^* \mathcal{F}, S + s^x)\) that stabilizes \((B, \mathcal{F}, S)\) at \( \frac{U^x}{2} \). Let \( V_{U^x} = (S + s^x)^{-1}(0) \). Then \((V_{U^x}, \sigma^*(\mathcal{O}_{U^x}), \sigma)\) is a virtual neighborhood of \( M \) at \( \frac{U^x}{2} \).

The global stabilization does not exist in general. However, if \( B \) is a manifold and \( \mathcal{F} \) is a bundle, a global stabilization always exists. We now discuss the construction of global stabilizations explained in [12] and explain what the barrier from local stabilizations to a global one is.

By a global stabilization, we mean that \( U \) is \( B \) or, at least, an open neighborhood of \( M \) in \( B \). The construction of global stabilizations presented here is standard.

Since \( M \) is compact by our assumption, there exists finite points \( \{x_i\}_{i=1}^n \) in \( M \) such that
\[ M \subseteq \bigcup_{i=1}^n \frac{1}{2} U^{x_i} =: U, \]
where \( U^{x_i} \) are as above.

For simplicity, we set
\[ U_i = U^{x_i}, \mathcal{O}_i = \mathcal{O}_{U^{x_i}}, s_i = s^{x_i}. \]

We call the data \( \{(U_i, \mathcal{O}_i, s_i)\} \) a local stabilization system of \( U \).

Note that \( \mathcal{O}_i \) is only defined on \( U_i \). However, these (trivial !) bundles can be extended(!) over to \( U \) and so are \( s_i \)'s because of the cut-off functions.

With these preparation, we are able to define the pair \((\mathcal{O}_U, s)\) by setting
\[ \mathcal{O}_U = \mathcal{O}_1 \oplus \cdots \oplus \mathcal{O}_n; \]
\[ s = s_1 \oplus \cdots \oplus s_n. \]

Clearly, the pair \((\mathcal{O}_U, s)\) provides a global stabilization of the system \((B, \mathcal{F}, S)\). The stabilization system is \((\mathcal{O}_U, \sigma^* \mathcal{F}, S + s)\), and the virtual neighborhood is \((V_U, \sigma^* \mathcal{O}_U, \sigma)\).
Remark 5.3. One notices that the crucial step to construct a global stabilization is the extension of bundle $\mathcal{O}_i$ over $U_i$ to one over $U$. This is true in the current setting since $\mathcal{O}_i$ are trivial on $U_i$. However, it may fail in many general situations. For example, it fails when $\mathcal{B}$ is an orbifold and $x_i$ is a singular point.

5.3. Invariants via virtual neighborhoods: I. It is commonly believed that invariant $\Phi(a)$ is well defined if a global stabilization exists. We now explain this.

Suppose $(\mathcal{O}_U, s)$ is a global stabilization pair. $(V_U, s^*\mathcal{O}_U, \sigma)$ is the virtual neighborhood. $a \in H^d(\mathcal{B})$. Let $\Theta$ be a Thom form of $\mathcal{O}_U$ that is supported (arbitrary) near the 0-section. In particular, we may choose

\[ \Theta = \Theta_1 \wedge \cdots \wedge \Theta_n, \]

where $\Theta_i$ is a Thom form of $\mathcal{O}_i$. We then define

\[ \Phi(a) = \int_{V_U} s^*a \wedge \Theta. \]

Note that the expression in [12] is slightly different. But it is not hard to check that they are the same. It is standard to show that $\Phi(a)$ is well defined, i.e, it is independent of the choice of data in the construction of virtual neighborhoods, the choice of $\Theta$ and etc. (cf. [12]).

Our main goal of this paper is to explain that $\Phi$ can be defined without assuming the existence of global stabilizations. The method we introduce here differs from that in [7], [9]'s. We will construct a virtual orbifold out of a Fredholm system and then apply the integration theory to it.

To motivate the construction, we explain that (10) can be "reduced" to a formula that only involves local stabilizations. The process is not rigorous but very suggestive.

We introduce notations. Set $\eta_i = \eta^{x_i}$. Set $N = \{1, \ldots, n\}$. For any $I \subseteq \{1, \ldots, n\}$, define

\[ U_I = \{x \in \mathcal{B} | \eta_i(x) \neq 0, i \in I, \eta_j(x) = 0, j \notin I\}. \]

$\mathcal{B}$ is decomposed as a disjoint union of $U_I, I \subseteq N$.

Set $V_{U,I} = V_U \cap \sigma^{-1}(U_I)$. Then

\[ \Phi(a) = \int_{V_U} s^*a \wedge \Theta = \sum_I \int_{V_{U,I}} s^*a \wedge \Theta. \]

We explain how to simplify each integration on the right hand side.

We have following facts:

Fact 1: over $U_I$, set

\[ \mathcal{O}_I = \bigoplus_{i \in I} \mathcal{O}_i, s_I = \bigoplus_{i \in I} s_i; \]

then the pair $(\mathcal{O}_I, s_I)$ stabilizes the system $(\mathcal{B}, \mathcal{F}, S)$ at $U_I$. It then defines a virtual neighborhood denoted by $(V_I, s^*\mathcal{O}_I, \sigma)$.

Fact 2: over $U_I$, set

\[ \mathcal{O}_I^c = \bigoplus_{j \notin I} \mathcal{O}_j. \]

we claim that

\[ V_{U,I} = s_I^*\mathcal{O}_I^c \]
is a bundle over \( V_I \), where \( \sigma_I : V_I \to U_I \). To see this, suppose a point \( p \), whose local coordinate is given by \((u,o_1,\ldots,o_n)\), is in \( V_{U,I} \). Namely,
\[(S+s)(u,o_1,\ldots,o_n) = S(u) + s_1(o_1)\cdots + s_n(o_n) = 0.\]
Without the loss of generalities, we assume \( I = \{1,\ldots,m\}, m \leq n \). Note that \( s_j(o_j) = 0, j > m \). Hence
\[(u,o_1,\ldots,o_n) \in V_{U,I} \iff (u,o_1,\ldots,o_m) \in V_I.\]
Fact 3: over \( U_I \), by (9), we write the Thom form \( \Theta \) as \( \Theta_I \wedge \Theta^c_I \), where \( \Theta_I \) and \( \Theta^c_I \) are Thom forms of \( O_I \) and \( O^c_I \) respectively. Note that when restricting on \( V_{U,I} \), \( \Theta^c_I \) is the Thom form of the bundle \( V_{U,I} \to V_I \) explained in fact 2.

With these preparations, by the Thom isomorphism, we immediately have
\[\int_{V_{U,I}} \sigma^* a \wedge \Theta = \int_{V_I} \sigma^* a \wedge \Theta_I.\]
Note that the right hand side only needs local stabilizations.

In summary,
\[(11) \Phi(a) = \sum_I \int_{V_I} \sigma_I^*(a) \wedge \Theta_I.\]
Fact 2 above the key of this formula. Be precise, we detect the fact that \( V_{U,I} \to V_I \) has a natural bundle structure. Note that without the modified (C2), had we not have fact 2. Motivated by this procedure, we explain that we can associate a virtual orbifold to a local stabilization system.

6. FROM FREDHOLM SYSTEM TO VIRTUAL ORBIFOLDS

6.1. Virtual orbifolds associated to Fredholm systems. Let \((B,F,S)\) be a Fredholm system. Let \(\{(U_i,O_i,s_i)\}^{n}_{i=1}\) be one of its local stabilization system.

Set
\[U_0 = B - \bigcup_i \frac{1}{2}U_i.\]
Then \((U_0,U_1,\ldots,U_n)\) is a covering of \(B\).

Repeat the construction in example 2.6: let
\[U_i^0 = \frac{3}{4}U_i, 1 \leq i \leq n;\]
as in example 2.6 we construct \(X_I \subseteq B, I \subseteq N\).

Now note that over \(X_I\), the cut-off functions \(\eta_k = \eta^{2^k}\) are 0 if \(i \notin I\). By the same construction as in 5.3 (cf. Fact 1), over \(X_I\) we still have \((O_I,s_I)\) which stabilizes the system at \(X_I\). It then defines a virtual neighborhood \((W_I,\sigma^*_I O_I,\sigma)\). Here, we use \(W_I\) instead of \(V_I\) that are used earlier. We know that \(W_I\) are smooth. Since \(W_I \subseteq O_I\), we have map
\[\sigma_I : W_I \to X_I.\]
When \(I = \emptyset, O_I\) is trivial. Hence \(W_0 = M \cap X_\emptyset\), where \(M\) is the moduli space.

**Proposition 6.1.** \(W = \{W_I\}\) is a virtual orbifold.
Proof. To see this, we now describe how $W_I$ and $W_J$ intersect. First, suppose $I \subseteq J$. Define

$$W_{I,J} = o^{-1}(X_{I,J}) \subseteq W_I,$$

$$W_{J,I} = o^{-1}(X_{J,I}) \subseteq W_J.$$ 

Same as the argument in §5.3 (cf. Fact 2), we have that

$$\phi_{J,I} : W_{J,I} \rightarrow W_{I,J}$$

is a vector bundle. Be precise, let

$$O_{J-I} = \bigoplus_{j \in J-I} O_j$$

be the bundle over $X_{I,J} = X_{J,I}$. Then

$$W_{J,I} = o^* O_{J-I},$$

where $o : W_{I,J} \rightarrow X_{I,J}$.

Then using the property of $X_I$, it is straightforward to see that $(W, \Phi)$ is a virtual manifold. q.e.d.

**Proposition 6.2.** $O = \{o_i^* O_I\}$ is a virtual bundle over $W$. $\sigma$ is a section of $O$.

Proof. This follows from the construction of $O$. q.e.d.

6.2. Invariants via virtual neighborhoods:II. We now set up the integration theory for $W$.

Let $\Theta_i$ be the Thom form of $O_i$. Since the bundle $\Psi_{J,I} : W_{J,I} \rightarrow W_{I,J}, I \subseteq J$ is isomorphic to $O_{J-I}$, we take

$$\Theta_{J,I} = \bigwedge_{j \in J-I} \Theta_j.$$

Set $\Theta = \{\Theta_{J,I}\}_{I \subseteq J}$. From the definition of $\Theta_{J,I}$’s, we know that $\Theta$ is a transition data of $W$.

Take

$$\Theta_I = \bigwedge_{i \in I} \Theta_i$$

on $W_I \subseteq O_I$. Then $\theta = (\Theta_I)$ is a $\Theta$-form. We call $\theta$ the obstruction form and $\Theta_I$ the obstruction form on $W_I$.

We summarize what we have for the system $(B, \Phi, S)$.

**Proposition 6.3.** Let $(B, \Phi, S)$ be a Fredholm system.

1. there exists a local stabilization system $\{U_i, s_i, O_i\}$.
2. Let $X$ be the natural virtual manifold for $B$ generated by the covering $\{U_i\}$.

   Using the stabilization data given above, one is able to define a virtual manifold $W = \{W_I\}$, where $(W_I, o_i^* O_I, \sigma)$ is a virtual neighborhood over $U_I$. Let $W$ be the virtual space of $W$.

3. $O$ is a virtual bundle over $W$. $\sigma$ is a section of the bundle;
4. Let $\Theta_i$ be Thom form of $O_i$. All Thom forms $\Theta_I$ of $O_I$ restricting on $W_I$ form a $\Theta$-form. Denote the form by $\theta$. If the moduli space $M$ is compact, $\theta \in \Omega_{\Theta, \epsilon}(W)$, $\theta$ is an Euler class of $O$.
5. For any $a \in \Omega^*(B)$, let $a_I = \pi_I^* a$ on $W_I$. Then $(a_I)_{I \subseteq N} \in \Omega^*(V)$. To abuse the notations, we still denote the form by $a$. 
The proposition is directly followed from the construction.

By the proposition, we have \( \mu_\theta(a) \). Also we know that this is well defined not only on \( \Omega^*(B) \), but also on \( H^*(B) \). If a global stabilization as in \( \S 5.2 \) exists, it is easy to see that

\[
\Phi(a) = \mu_\theta(a).
\]

This leads to the following definition.

**Definition 6.4.** Let \((B,F,S)\) be a Fredholm system. Let \(\{(U_i,\mathcal{O}_i,s_i)\}\) be a local stabilization system constructed in \(\S ??\). Let \(W,\Theta\) be the virtual manifold and obstruction form given above. For \(a \in H^*(B)\), define the invariants \(\Phi(a)\) to be \(\mu_\theta(a)\).

The next subsections are to explain that

**Proposition 6.5.** \(\Phi(a)\) is well defined. Namely, it is independent of (1) the choice of \(\Theta_i\), (2) the choice of stabilizations.

6.3. The well-definedness of \(\Phi(a)\).

**Proposition 6.6.** \(\Phi(a)\) is independent of the choice of \(\Theta_i\)'s.

**Proof.** Without the loss of generality, we assume that \(\Theta_1\) is replaced by \(\Theta'_1\) and other \(\Theta_i, i \neq 1\), remains same. Suppose \(\Theta_1 = \Theta_1 \wedge \Theta'_I\). Then \(\Theta_I\) is replaced by

\[
\Theta'_I = (\Theta_1 + db) \wedge \Theta'_I.
\]

Denote \(\theta'\) to be the new obstruction form.

For closed forms \(a_I\)'s, we have

\[
\mu_\theta(a) - \mu_{\theta'}(a) = \sum_{I \in I} \int_{X_I} \eta_I a_I \wedge (\Theta_I - \Theta'_I)
\]

\[
= \sum_{I \in I} \int_{X_I} d(\eta_I a_I \wedge b \wedge \Theta'_I)
\]

\[
= - \sum_{I \in I} \int_{X_I} d\eta_I \wedge a_I \wedge b \wedge \Theta'_I
\]

\[
= 0
\]

q.e.d.

Now we discuss that \(\Phi(a)\) is independent of the local stabilizations. Suppose we have two different local stabilization systems for \((B,\mathcal{F},S)\). They are \(\{(U_i,\mathcal{O}_i,s_i)\}_{i=1}^n\) and \(\{(U'_j,\mathcal{O}'_j,s'_j)\}_{j=1}^{n'}\). They define two virtual manifolds \(W\) and \(W'\). Let \(\theta,\theta'\) be their obstruction forms respectively. Then

**Proposition 6.7.** \(\mu_{V,\theta}(a) = \mu_{V',\theta'}(a)\). Here we add virtual manifolds to the subindices of \(\mu\) for the obvious reason.

**Proof:** Set \(U_{n+j} = U'_j,\mathcal{O}_{n+j} = \mathcal{O}'_j, 1 \leq j \leq n'\) and \(s_{n+j} = s'_j\). Set \(m = n + n'\). Then \((U_i,\mathcal{O}_i,s_i), 1 \leq i \leq m\) is still a local stabilization system. It defines a virtual manifold, denoted by \(W'_2\), and obstruction form \(\theta_2\). It is easy to verify that
• the obstruction bundle $\mathcal{O}_k = \tilde{\mathcal{O}} \oplus \tilde{\mathcal{O}}'$;  
• the section $\sigma_k = \tilde{\sigma} \oplus \tilde{\sigma}'$, $\sigma$ and $\sigma'$ are transverse;  
• $\tilde{\sigma}^{-1}(0) = W'$, $\tilde{\mathcal{O}}'|_{W'} = \mathcal{O}'$, $\tilde{\sigma}'|_{W'} = \sigma'$, and $(\tilde{\sigma}')^{-1}(0) = W$, $\tilde{\mathcal{O}}|_{W} = \mathcal{O}$, $\tilde{\sigma}|_{W} = \sigma$.

Then by Proposition 3.14,

$$\mu_{\mathcal{W}, \theta}(a) = \mu_{\mathcal{W}', \theta}(a) = \mu_{\mathcal{W}', \varphi}(a).$$

q.e.d.

6.4. Virtual localization formula. We can extend the discussion to equivariant cases.

Suppose that $G$ acts on $\mathcal{B}$, $\mathcal{F}$ is a $G$-equivariant Hilbert bundle and $S$ is a $G$-equivariant section. We call such a system to be a $G$-Fredholm system.  

Let $U \subseteq \mathcal{B}$ be a $G$-invariant open subset. By a $G$-stabilization we mean a finite rank $G$-equivariant bundle $\mathcal{O}_U \to U$ and a $G$-equivariant bundle map $s : \mathcal{O}_U \to \mathcal{F}_U$ such that $S + s$ stabilizes $S$. Then $(V_U, \sigma^* \mathcal{O}_U, \sigma)$ is a virtual neighborhood with the $G$-equivariant section $\sigma : V_U \to \sigma^* \mathcal{O}_U$.

In order to apply the technique described in previous sections, we need local $G$-equivariant stabilizations. We repeat the construction given in §6.1. However, the construction of equivariant obstruction bundle $\mathcal{O}_{U,x}$ requires some extra work when $x$ is a fix point of the $G$-action. For this, we take $O^x$ to be orthogonal complementary to $\text{Image}(L_x)$. This is where we use the assumption that $\mathcal{F}$ is a Hilbert bundle.

With these preparations, we can construct a $G$-virtual manifold $V$ from a local $G$-stabilization system. Then we replace $\Theta_i$ by equivariant Thom forms $\Theta_i^G$. So we have $\Theta_i^G$'s and $\Theta_{i,j}^G$'s. Clearly, $\theta_G = \{\Theta_i^G\}_I$ is a $\Theta^G = \{\Theta_{i,j}^G\}$ form. For any $\alpha \in \Omega^*_G(\mathcal{B})$, define

$$\Phi_G(\alpha) = \mu_{\mathcal{V}, \Theta_G}(\alpha).$$

Now we can state the virtual localization formula for Fredholm systems. Again, let $G = S^1$. We consider the Fredholm system $(\mathcal{B}, \mathcal{F}, S)$ with $G$-action. Let $V$ be the virtual orbifold for the moduli space $M$. Let $V$ denote the virtual space. Then $V^G$ is the virtual orbifold for $M^G$ and its virtual space is $V^G$.

Now repeat the discussion in §4. We have

**Theorem 6.8.** Let $(\mathcal{B}, \mathcal{F}, S)$ be an $S^1$-Fredholm system. For $\alpha \in \Omega^*_G(\mathcal{B})$,

$$\Phi_G(\alpha) = \int_{V^G} i^*_G \alpha \wedge \theta_G = \mu_{\mathcal{V}^G, \Theta_G}(i^*_G \alpha).$$

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Department of Mathematics, Sichuan University, Chengdu, 610064, China
E-mail address: xfzhu1@yahoo.com

Department of Mathematics, Princeton University, tian@
E-mail address: tian@math.princeton.edu