Linear-Time Safe-Alternating DFS and SCCs

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Abstract

An alternating graph is a directed graph whose vertex set is partitioned into two colour classes, existential and universal.

This forms the basic arena for well-known models in formal verification, discrete optimal control, and many infinite duration two-player games where Player $\Box$ and his opponent Player $\Diamond$ alternate in a turn-based sliding of a pebble along the arcs they control.

We study alternating strongly-connectedness on alternating graphs as a generalization of strongly-connectedness in directed graphs, aiming at providing a linear-time decomposition and a sound structural graph characterization. For this a novel notion of alternating reachability is introduced: Player $\Box$ attempts to reach vertices without leaving a prescribed subset of the vertices while Player $\Diamond$ works against. This is named safe-alternating reachability. It is shown that every alternating graph uniquely decomposes into safe-alternating strongly-connected components, where Player $\Box$ can visit each vertex within a given component infinitely often without having to ever leave out the component itself.

Our main result is a linear-time algorithm for computing this alternating graph decomposition. Both the underlying graph structures and the algorithm generalize the classical decomposition of a directed graph into strongly-connected components. Indeed, the proposed algorithm builds on a non-trivial generalization to alternating graphs of the depth-first search and the strongly-connected components algorithm devised by Tarjan in 1972.

Our theory has direct applications e.g., in solving well-known infinite duration pebble games faster. Dinneen and Khoussainov showed in 1999 that deciding a given Update Game costs $O(mn)$ time, where $n$ is the number of vertices and $m$ is that of arcs. We solve that task in $\Theta(m + n)$ linear time. In turn the complexity of Explicit McNaughton-Müller Games improves from cubic to quadratic.

Keywords: Alternation, Infinite Pebble Games, Linear-Time Algorithm, McNaughton-Müller Games, Depth-First Search, Strongly-Connected Components, Update Games, Update Networks

1 Introduction

The alternating model of computation originated in [3, 4, 17] as a generalization of nondeterminism in which existential and universal quantifiers alternate along the course of the computation. Alternating Turing Machines were defined and the corresponding time and space complexity classes were characterized in terms of resource-bounded deterministic machines. In the complexity landscape, generalizing complete computational models to alternation leads more often than not to complexity blowups, e.g., alternating polynomial time equals deterministic polynomial space [3, 19].

Still alternation can be inquired by generalizing specific polynomial time computable problems. One of the classical P-complete problems is the Alternating Graph Accessibility Problem (AGAP) [3, 15]. We are given a finite directed graph $(V, A)$ whose vertex set $V = V_\Box \cup V_\Diamond$ is partitioned into two classes, existential $V_\Box$ and universal $V_\Diamond$ (i.e., an alternating graph), plus a source vertex $s$ and target vertex $t$. The task is to decide whether $t$ is alternating reachable from $s$, that is recursively defined as follows: either

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i) $s = t$, or ii) $s \in V_\square$ and for some outgoing arc $(s, s') \in A$ the target $t$ is alternating reachable from $s'$, or iii) $s \in V_\bigcirc$ and for every outgoing arc $(s, s') \in A$ the target $t$ is alternating reachable from $s'$. When restricted to only existential vertices, this is equivalent to the Directed Graph Accessibility Problem (GAP), also known as $(s,t)$-Connectivity (st-Con), which is complete for nondeterministic logarithmic space [19].

Both GAP and AGAP admit linear-time algorithms. In GAP, a depth-first search starting from $s$ works out. In AGAP a linear-time solution can be achieved by computing the $\square$-attractor set $T_\square$ of target vertex $t$, which is defined by: i) $t \in T_\square$; ii) if $x \in V_\square$ has an outgoing arc $(x, y)$ such that $y \in T_\square$, then $x \in T_\square$; iii) if $x \in V_\bigcirc$ has all outgoing arcs $(x, y)$ such that $y \in T_\square$, then $x \in T_\square$; iv) nothing else is in $T_\square$.

Algorithmic problems on alternating generalizations of graphs arise in the literature, e.g., in the field of formal probabilistic verification [5]. The input is a system that exhibits probabilistic behavior and a specification (set of desired behaviors), where the algorithmic problem is to answer whether the system satisfies the specification [8]. In probabilistic verification, systems are frequently modeled as a generalization of graphs called Markov decision processes (MDPs). The generalization is needed to model two different kind of behaviors. More specifically in MDPs there are two types of vertices, namely, the regular vertices $V_\square$ where the algorithm chooses which outgoing arc to follow, and the probabilistic vertices $V_P$ where the outgoing arc is chosen randomly according to some given probability distribution $\delta$; still, putting aside probabilistic behavior, in MDPs the underlying static graph arena is actually an alternating graph (think of it as if $V_\bigcirc = V_P$). Also notice: (i) a directed graph is a special case of an alternating graph with $V_\bigcirc = \emptyset$ (ii) similarly a directed graph is a special case of an MDP with $V_P = \emptyset$ and (iii) a Markov chain is a special case of an MDP with $V_\bigcirc = \emptyset$.

In the literature, algorithmic problems on alternating graphs have already been tackled by relying on classical graph algorithmics employed as interleaved subprocedures, such as the depth-first search (DFS) and the strongly-connected components (SCCs) algorithm of Tarjan [21]. One notable instance of that being the maximal end-component (MEC) decomposition that lies at the core of many algorithms in probabilistic verification, generalizing to alternating graphs the problem of decomposing a directed graph into SCCs [5]. The time complexity is nevertheless not known to be linear, the fastest algorithm which is currently known runs in time $O(\min(m^{3/2}, n^2))$ [5].

On the other hand recall that classical graph algorithmics shows that the DFS can be suitably adapted to decompose a finite directed graph into SCCs keeping the time complexity linear, e.g., the celebrated algorithm of Tarjan [21] finds SCCs in linear-time, also see [6,9,12,20].

In this work we introduce a novel notion of alternating strongly-connectedness on alternating graphs as a natural generalization of strongly-connectedness in directed graphs, ultimately aiming at providing a linear-time decomposition and a sound structural graph characterization. For this a novel notion of alternating reachability is also introduced, where Player $\square$ attempts to reach vertices without leaving a prescribed subset of the vertices, while Player $\bigcirc$ works against. This is named safe-alternating reachability. It is shown that every alternating graph uniquely decomposes into safe-alternating strongly-connected components where Player $\square$ can visit each vertex within a given component infinitely often, without having to ever leave out the component itself. Our main result is a linear-time algorithm for computing the corresponding alternating graph decomposition. A key technical ingredient will be to rely on the disjoint-sets union-find data structure [22], and the linearity of the so-called incremental-tree set-union problem [13] on RAM machines, for the fast computation of lowest common ancestors in certain search trees underlying our algorithm. This possibly departs away from the MEC decomposition, for which the existence of a sub-quadratic procedure remains an open question. Still our proposed algorithms and underlying graph structures do generalize the classical decomposition of a directed graph into SCCs.

Our theory has direct applications, e.g., for faster solving some infinite two-player pebble games that are well-known in the field of formal verification and automated synthesis. Infinite duration games can be applied in the construction of finite state reactive systems, like communication protocols or discrete dynamic control systems, where a central aim is to put the development of hardware and software on a mathematical basis which is both firm and practical. A characteristic feature of such systems is their perpetual interaction with the environment as well as their non-terminating behaviour. The theory of infinite duration games offers many appealing results under this prospect, see e.g., [14].
For instance consider the following communication network problem. Often one requirement is to share key information between all nodes of a network, suppose we have data stored on each node of a computer network and we want to continuously update all nodes with some consistent information. Imagine a data packet of current information continuously going through all nodes. Unfortunately not all routing choices are always under our control, some of them may be controlled by the network environment which could play against our benefits. Essentially this describes an infinite duration two-player game played on an alternating graph where Player □ wants to visit all vertices infinitely often, by keep moving the pebble everywhere around and forever, while Player ○ works against by trying to rule out at least one vertex from a certain time moment onwards. This model is named Update Game (UG) in [1,10,11]. Dinneen and Khoussainov [10] showed that deciding who’s the winner in a given UG costs \(O(mn)\) time, where \(n\) is the number of vertices and \(m\) is that of the arcs. Solving UGs turns out to be a foundamental subproblem when solving Explicit McNaughton-Müller Games in polynomial time as in the algorithm of Horn [16]. Now we can solve UGs in linear time. In turn the complexity of Explicit McNaughton-Müller Games improves from cubic to quadratic.

1.1 Results and Organization

To begin, Section 1.2 and 2 provide some background notions and formal notation.

In order to accomplish our tasks, a novel notion of alternating reachability is introduced in Section 2.1, namely, \textit{safe-alternating reachability}, where Player □ attempts to reach vertices without leaving a prescribed subset of the vertices, while Player ○ works against.

In Section 2.2, it is shown that every alternating graph uniquely decomposes into safe-alternating strongly-connected components where Player □ can visit infinitely often each vertex within a given component without having to ever leave out the component itself.

Our main result is a linear-time algorithm for computing this alternating graph decomposition.

**Main Result.** The DFS and SCCs Tarjan algorithm [21] admit a non-trivial generalization to alternating graphs w.r.t. safe-alternating reachability. The resulting algorithm can be implemented to run in linear-time on RAMs [13], and at least Ackermann-linear-time\(^1\) on pointer machines [22].

We leave open the existence of a linear-time solution for pointer machines, this is discussed in Section 5.

Both the underlying graph structures (which are analyzed in Section 3.2) and the algorithm (outlined and analyzed in Section 4) generalize the classical decomposition of a directed graph into strongly-connected components. The proposed linear-time decomposition is given in Section 3.

As a first direct application, we obtain the following neat result on Update Games [1,10,11]. The best previously known upper bound was \(O(|V||A|)\), as shown by Dinneen and Khoussainov in [10]. Section 2.3 offers formal definitions and more details on UGs.

**Corollary 1.** Deciding who wins a given Update Game \(A = (V, A, \langle V□, V○ \rangle)\) takes time \(Θ(|V| + |A|)\).

**Proof Sketch.** Player □ wins if and only if \(A\) has only one safe-alternating strongly-connected component; when there are at least two components the winner is Player ○. To decide this on input \(A\), it is sufficient one run of our proposed decomposition algorithm, \textit{safe-αSCC(A)} (Algorithm 2) given in Section 4.

Correctness and complexity will follow from that of Algorithm 2, see Section 4 and Appendix A.

McNaughton-Müller Games (MMGs) [16] also provide a useful model for the synthesis of controllers in reactive systems, but their complexity depends on the representation of the winning conditions. The most straightforward way to represent a Müller winning condition \(\mathcal{F} \subseteq 2^V\) is to provide an explicit list \(\mathcal{F} = \{F_i\}_{i=1}^\ell\) of subsets of vertices as in [16]. So-called Explicit MMGs can be solved in polynomial time, e.g., with Horn’s algorithm [16], by repeatedly deciding UGs as a basic subproblem. As aftermath of Corollary 1, the complexity of Explicit McNaughton-Müller Games also improves from cubic to quadratic.

This is established again in Section 2.3, where the formal definition of Explicit MMGs is recalled.

**Corollary 2.** Deciding who wins a given Explicit MMG \((A, F)\) takes time \(O(|F| \cdot (|A| + |F|))\).

\(^1\)i.e., \(O(n + \alpha(m, n))\), where \(n\) is the number of vertices, \(m\) that of the arcs, and \(\alpha()\) is the inverse Ackermann’s [22].
1.2 Notation and Preliminaries

An alternating graph (or graph) \( A = (V, A, \langle V_\square, V_\bigcirc \rangle) \) is a finite directed simple graph \( G_A = (V, A) \) (i.e., there are no loops nor parallel arcs) whose vertex set is split into the set of existential vertices \( V_\square \) owned by Player \( \square \), and the set of universal vertices \( V_\bigcirc \) owned by Player \( \bigcirc \). Notice \( G_A \) is not required to be a bipartite graph on colour classes \( V_\square \) and \( V_\bigcirc \). Also let \( [k] = \{1, \ldots, k\} \) for any \( k \in \mathbb{N} \).

The ingoing and outgoing neighbourhoods of any \( u \in V \) are denoted by \( N^\text{in}_A(u) = \{ v \in V \mid (v, u) \in A \} \) and \( N^\text{out}_A(u) = \{ v \in V \mid (u, v) \in A \} \), respectively.

An ograph can serve as an arena on which games can be played for infinitely many rounds by moving a pebble along the arcs from one vertex to an adjacent one. Initially the pebble is put on a starting position \( s \in V \). At each round, if the pebble is over \( v \in V_i \), for some \( i \in \{\square, \bigcirc\} \), Player \( i \) chooses an arc \((v, v') \in A\) and then the next round starts with the pebble on \( v' \).

A finite (or infinite) path in \( G_A \) is a sequence \( v_0v_1 \ldots v_n \in V^* \) (or \( V^\omega \)) such that \( \forall j \geq 0 \ (v_j, v_{j+1}) \in A \), where the length of \( v_0v_1 \ldots v_n \) is \( n \). A play path (or simply, a play) on \( A \) is any finite or infinite path in \( G_A \). A strategy for Player \( i \), where \( i \in \{\square, \bigcirc\} \), is a map \( \sigma_i : V^* \times V_i \to V \) such that for every finite path \( p'v \) in \( G_A \), where \( p' \in V^* \) and \( v \in V_i \), it holds that \( (v, \sigma_i(p', v)) \in A \). The set of all strategies of Player \( i \) in \( A \) is denoted by \( \Sigma_A^i \). A play \( v_0v_1 \ldots v_n \) is consistent with some \( \sigma \in \Sigma_A \) if \( v_{j+1} = \sigma(v_jv_1 \ldots v_j) \) whenever \( v_j \in V_i \). Given two strategies \( \sigma_\square, \sigma_\bigcirc \in \Sigma_A^\square \) and \( \sigma_\bigcirc, \sigma_\square \in \Sigma_A^\bigcirc \), and some \( s \in V \), the outcome play \( \rho_A(s, \sigma_\square, \sigma_\bigcirc) \) is the (unique) play that starts at \( s \) and is consistent with both \( \sigma_\square \) and \( \sigma_\bigcirc \). For any \( v \in V \), we denote by \( \rho_A(s, \sigma_\square, \sigma_\bigcirc, v) \) the (unique) prefix of \( \rho_A(s, \sigma_\square, \sigma_\bigcirc) \) which ends at the first occurrence of \( v \), if any; otherwise, \( \rho_A(s, \sigma_\square, \sigma_\bigcirc, v) \triangleq \rho_A(s, \sigma_\square, \sigma_\bigcirc) \). For any finite (or infinite) path \( p \in V^* \) (or \( p \in V^\omega \)), the alphabet \( \Gamma(p) \) is the set of vertices appearing in \( p \).

Let \( T = (V_T, A_T) \) be an inward directed tree rooted at \( r_T \in V_T \). We simply write \( u \in T \) for \( u \in V_T \). For each \( u \in T \), there is only one path \( p_u \) going from \( u \) to \( r_T \); the depth \( d(u) \) of \( u \) is the length of \( p_u \). An ancestor of \( u \in T \) is any \( v \in \Gamma(p_u) \); it is a proper ancestor if \( v \neq u \), it is the parent \( \pi_T(u) \) of \( u \) if \( (u, v) \in A_T \). The children of \( u \in T \) are all the \( v \in T \) such that \( \pi_T(v) = u \). A descendant of \( u \in T \) is any \( v \in T \) such that \( u \in \Gamma(p_v) \); it is a proper descendant if \( v \neq u \). A leaf of \( T \) is any \( u \in T \) having no children. The lowest common ancestor (LCA) \( \gamma_S \) of a subset of vertices \( S \subseteq T \) is:

\[
\gamma_S = \arg \max \{ d(\gamma) \in \mathbb{N} \mid \gamma \in T \text{ is an ancestor of each vertex in } S \}.
\]

The maximal subtree of \( T \) that is rooted at any \( u \in T \) is denoted by \( T_u \). Given a LIFO stack \( St \) containing some element \( v \in St \), then \( St(v) \) denotes the set of all elements \( u \in St \) going from the top of \( St \) down 'til the first occurrence of \( v \) extremes included.

2 Alternating Strongly-Connected Components

This section deepens alternating strongly-connectedness and its safe form. We shall see that both concepts can be built bottom-up (i.e., as a natural generalization of strongly-connectedness in directed graphs) and that they are sound and applicable (i.e., they enjoy a clear characterization in terms of quotient sets of reachability equivalence relations, and they can be directly applied for faster solving tasks concerning infinite pebble games on graphs).

Firstly, we consider alternating reachability and alternating strongly-connected components as the most natural notion in the neighborhood of possible definitions, already presenting some technical pitfalls compared to the traditional setting. Secondly, aiming at providing a linear-time decomposition algorithm and a sound structural graph characterization, we introduce safe-alternating reachability, a novel notion of alternating reachability on ographs that will form the backbone on which the forthcoming theory will sustain. Upon this, safe-alternating strongly-connectedness is introduced (in turn, a novel notion of alternating strongly-connectedness). It is offered a sound definition of safe-alternating strongly-connected components in terms of safe-alternating reachability quotient sets (i.e., equivalence classes).

safe-alternating reachability captures in a natural way the fundamental invariant property lying at the ground of both the forthcoming graph structures and linear-time decomposition algorithms – this is actually the reason why it seems necessary and not just interesting to introduce the safe form.
To conclude the section, we shall observe that both alternating strongly-connectedness and its safe form can be employed to solve Update Games, and thus Explicit McNaughton–Müller Games as shown in [16]. The algorithm of Section 4 will ultimately provide a faster solution to those two games. Let us start recalling alternating reachability.

**Definition 1** ([3, 15]). Let $A$ be an oagraph on vertex set $V$, and let $u, v \in V$ be any two vertices.

We say that $v$ is alternating reachable (reachable) from $u$ in $A$ if and only if there exists a strategy $\sigma_\square \in \Sigma^A_\square$ such that for every $\sigma_\bigcirc \in \Sigma^A_\bigcirc$ it holds that the target $v$ lies in the outcome play which starts at $u$ and proceeds consistently with the given strategies, i.e., if and only if

$$\exists \sigma_\square \in \Sigma^A_\square \forall \sigma_\bigcirc \in \Sigma^A_\bigcirc \exists v \in \Gamma(\rho_A(u, \sigma_\square, \sigma_\bigcirc)).$$

The reachability relation between $u$ and $v$ by strategy $\sigma_\square$ will be compactly denoted by $\sigma_\square : u \sim v$.

Then let’s consider a natural notion of alternating strongly-connectedness, also clarified in Example 1.

**Definition 2.** Let $A$ be an oagraph on vertex set $V$. We say $U \subseteq V$ is an alternating strongly-connected set (asc set) if and only if $\forall (u, v) \in U \times U$ $\exists \sigma_\square \in \Sigma^A_\square$ such that $\sigma_\square : u \sim v$.

Figure 1: An oagraph on vertex set $\{a, b, c\}$ and its $\alpha$SCCs $\gamma = \{a, c\}$ and $\beta = \{b\}$, as in Example 1.

**Definition 3.** $\sim_{\alpha}$ is a binary relation on $V$ defined as follows: $\sim_{\alpha} = \{(u, v) \in V \times V \mid \{u, v\} \text{ is } \alpha \text{-asc}\}$.

It is easy to check that $\sim_{\alpha}$ is an equivalence relation on $V$, where every equivalence class is asc.

**Definition 4.** Let $A$ be an oagraph on vertex set $V$. Let $C \subseteq V$ be a subset of the vertices and consider the relation $\sim_{\alpha_{\text{asc}}}$. We say that $C$ is an alternating strongly-connected component (ASC) of $A$ precisely when it is an equivalence class of $\sim_{\alpha_{\text{asc}}}$.

Otherwise stated, an ASC is any maximal (under set inclusion) asc set of vertices in the oagraph $A$.

**Example 1.** Consider the oagraph $A = (V, A, \langle V_\square, V_\bigcirc \rangle)$ such that $V = V_\square \cup V_\bigcirc$ where $V_\square = \{b, c\}$, $V_\bigcirc = \{a\}$, and $A = \{(a, c), (a, b), (b, c), (c, a)\}$. Fig. 1 highlights the corresponding decomposition into $\alpha$SCCs $\{\beta, \gamma\}$. Consider the component $\gamma = \{a, c\}$: if the play starts from $a$, of course the pebble will soon reach $c$ – but, in order to do this, it might escape out of $\gamma$, firstly reaching $b$; i.e., even though vertex $c$ is reachable from vertex $a$, Player $\square$ has no way for staying safe inside $\gamma$, because Player $\bigcirc$ can always push the pebble out at will to reach $\beta$.

It is easy to decompose an oagraph into its ASCs within time $O(|V| \cdot |A|)$. For each vertex $t \in V$, just compute the $\square$-attractor set $T_\square$ of vertex $t$ in time $O(|A|)$. Then any two vertices $u, v \in V$ lie in the same ASC if and only if each one is in the $\square$-attractor of the other. The existence of sub-quadratic time bounded solutions remains open.

These are some of the distinguishing characteristics of ASCs as compared to the classical counterpart notion in directed graphs. However, ASCs are well-posed and they coincide with equivalence classes of a natural equivalence binary relation defined on the vertex set.

We have decided to look at these simple properties explicitly because they will be taken back and transported to the safe notion of alternating strongly-connectedness. In order to formalize it, now we introduce safe-areachability.
2.1 Safe-Alternating Reachability

Instead of giving the definition directly in a flat manner, we wish to present safe-reachability by arguing that it emerges naturally as soon one tries gamifying the classical graph structures underlying depth-first search, which were pointed out by Tarjan [21], next recalled.

Recalling palm-trees and jungles. In [21] many fundamental properties and applications of the DFS were analyzed. Particularly, specific underlying graph structures were discussed, they were named palm-trees and jungles. This allowed the author to offer a renowned algorithm for computing SCCs in linear-time, also known as Tarjan's SCCs algorithm.

Following [7,21], the recursive strategy of the DFS is to search deeper in the graph whenever possible. Initially all vertices are unexplored. Start from some vertex \( u \) and choose an outgoing arc to follow. Recursively, the arcs are firstly explored out of the most recently discovered vertex \( v \) that still has unexplored arcs leaving it, by scanning the adjacency list of the already discovered vertex \( v \). When all of \( v \)'s arcs have been explored, the search backtracks one step back to explore the remaining arcs leaving that vertex from which \( v \) was discovered just before. This process continues until we have discovered all the vertices that are reachable from the original source vertex \( u \). If any undiscovered vertices remain, then one of them is picked as a brand new source and the search is repeated from that. The entire process is repeated until all vertices are discovered. Besides exploring the graph the DFS also timestamps each vertex twice, where each timestamp is a natural number: the first one, named \( \text{open}(v) : V \to \mathbb{N} \), records when \( v \in V \) is first discovered; the second timestamp \( \text{close}(v) \) records when the search finishes examining \( v \)'s adjacency list. These timestamps are used in many algorithms and are generally helpful in reasoning about the behavior of the DFS. Let us call it forward-DFS, for, at each step the chosen arc is outgoing.

Concerning palm-trees [21], observe in more detail what happens when DFS runs. The set of arcs \( A_\pi \) first leading to an unexplored vertex, when traversed during the search, forms a family of outward directed trees \( T \). The predecessor subgraph \( (V,A_\pi) \) of a DFS is thus a forest defined as:

\[
A_\pi = \{(\pi_v,v) \mid \pi_v,v \in V \text{ and } v \text{ is first discovered out of } \pi_v \text{ during the DFS}\}.
\]

All of the other arcs of the input graph \( G \) fall into four categories: (i) some arcs are running from ancestors to descendants in \( T \), these may well be ignored as (even if we remove them from the graph) they do not affect the strongly-connectedness of \( G \); still, (ii) some other arcs run from descendants to ancestors in \( T \), these are quite relevant to determine strongly-connectedness instead, and they are called fronds; (iii) other arcs run from one subtree to another within the same tree \( T \), these are also relevant and named internal cross-links; (iv) suppose to continue the DFS until all arcs are explored, the process creates a family of trees which contains all vertices, i.e., a spanning forest \( F = (V,A_\pi) \) of \( G \), plus sets of (fronds and) cross-links which may also connect two different trees in \( F \), and these would be called the external cross-links. Notice that any (internal or external) cross-link \((u,v)\) always has \( \text{open}(u) > \text{open}(v) \).

Any tree \( T \) of \( F \), comprising fronds and cross-links, is called palm-tree.

A directed graph consisting of a spanning forest, plus fronds and cross-links, is named jungle, i.e., a family of palm-trees plus external cross-links; this is a natural representation of the graph reachability relations of the input directed graph \( G \).

Reverse-DFS, palm-trees and jungles. As we are dealing with \( \alpha \)-graphs, it turns out we need to impose an opposite direction w.r.t. that in which the arcs are explored; i.e., at each step of the DFS, we actually choose an ingoing arc to follow instead of an outgoing one. This reversal is due to the fact that, on one side, Player \( \Box \) has no control on the arc choices of the opponent, and on the other side, we still aim at exploring the \( \alpha \)-graph in a depth-first manner but meanwhile preserving \( \alpha \)-reachability relations; we will see that we can achieve this but we have to reverse the direction of exploration so that to mimics the backward tread of computing a \( \Box \)-attractor. Let us call the corresponding search algorithm, reverse-DFS (think of it as if we had reversed the direction of each arc). A moment’s reflection reveals that, if run on a directed graph, all the basic properties of the DFS are still there (by symmetry). For instance, if the vertices are numbered in the order in which they are reached during the reverse-DFS, e.g., by \( \text{open}(u) : V \to \mathbb{N} \), now a cross-link \((u,v)\) always has \( \text{open}(u) < \text{open}(v) \). A forest of inward directed reverse-palm-trees is constructed during a reverse-DFS. Let us call reverse-jungle the underlying predecessor subgraph structure, that is a family of reverse-palm-trees comprising fronds and cross-links.
Also notice that, if run on an ograph having $V_\Box = \emptyset$, the reverse-palm-tree of a reverse-DFS is actually a $\Box$-attractor strategy. Since we will only deal with the reversed variants, from now on in the forthcoming sections we shall refer to them simply as “DFS”, “palm-trees” and “jungles.”

**Safe-$\alpha$Reachability.** Notice that graph reachability trivially holds in any palm-tree $T = (V_T, A_T)$: for any $u, v \in T$ such that $v$ is an ancestor of $u$ in $T$, there exists a simple path from $u$ to $v$ within $T$, i.e., $v$ is reachable from $u$ in $T$. With this in mind, let us now try to explore an ograph $A$ with a classical DFS. Let $J$ be the resulting jungle, and let $T$ be any palm-tree of $J$. An example is depicted in Fig. 2a and the corresponding palm-tree $T$ is in Fig. 2b, where above each vertex $v$ the opening and closing timestamps are depicted with the notation $\langle \text{open}[v] \rangle \mid \langle \text{close}[v] \rangle$. At this point, let us consider alternating reachability (instead of graph reachability), which is most relevant to 2-player pebble games. Observe that the palm-tree $T$, constructed as above, doesn’t respect reachability: e.g., consider the two vertices $F, B \in V_\Box$ in the palm-tree $T$ shown in Fig. 2b; starting from $F$, Player $\Box$ admits no strategy allowing him to reach $B$, even though $B$ is an ancestor of $F$ in $T$; indeed, any play starting from $F$ must first reach $D$, so at that point, if Player $\Box$ plays $(D, G)$ then Player $\bigcirc$ can go back to $F$ by playing $(G, F)$. Otherwise, if Player $\Box$ plays $(D, C)$, then Player $\bigcirc$ can reply $(C, H)$ thus reaching $H$ – and notice that once on $H$ the continuation of the play must reach $D$ back again. So, starting from $F$, Player $\bigcirc$ can prevent Player $\Box$ to reach $B$. We now aim at gamifying the classical DFS, as well as palm-trees and jungles, by generalizing them from directed graphs to ographs, in such a way as to preserve reachability within the (suitably adapted notion of) palm-trees. Particularly, a desirable “DFS on ographs” should maintain the following basic property: for any (suitably adapted) palm-tree $T$, if $u, v \in T$ and $v$ is an ancestor of $u$ in $T$, there exists $\sigma_\Box \in \Sigma_A^T$ which allows Player $\Box$ to eventually reach $v$ starting from $u$, without leaving $T$ at the same time, no matter which counter-strategy $\sigma_\bigcirc \in \Sigma_A^T$ is chosen by Player $\bigcirc$. The formal definition of safe-$\alpha$reachability follows next.

**Definition 5.** Given an ograph $A$ on vertex set $V$, let $U \subseteq V$ and $u, v \in U$. We say that vertex $v$ is $U$-safe-areachable from $u$ when there is $\sigma_\Box \in \Sigma_A^U$ such that for every $\sigma_\bigcirc \in \Sigma_A^U$:

- [reachability] $v$ is eventually reached by playing $\sigma_\Box$ starting from $u$, i.e., $v \in \Gamma[\rho_A(u, \sigma_\Box, \sigma_\bigcirc)]$;
- [safety] the pebble never leaves $U$, i.e., $\Gamma[\rho_A(u, \sigma_\Box, \sigma_\bigcirc, v)] \subseteq U$.

In that case denote $\sigma_\Box : u \sim v$, or $u \sim v$ when $\sigma_\Box$ is implicit; if $U = V$, denote $\sigma_\Box : u \sim v$ or $u \sim v$.

**Remark:** By convention, any $u \in U$ is $U$-safe-areachable from itself for every non-empty $U \subseteq V$. We are now in the position of revisiting alternating strongly-connectedness.

---

Figure 2: A reverse-palm-tree (b), generated by reverse-DFS (c) starting at $A$. 
2.2 Safe-Alternating Strongly-Connected Components

Definition 6. Let $A$ be an ograph on vertex set $V$. We say that $U \subseteq V$ is safe-alternating strongly-connected (safe-osc) if and only if, for every pair $(u, v) \in U \times U$, $v$ is $U$-safe-reachable from $u$ in $A$; i.e., if and only if there exists $\sigma_\Box \in \Sigma_\Box^A$ such that:

$$\sigma_\Box : u \sim_U v.$$ 

Notice that $\emptyset$ and $\{v\}$ are $\{v\}$-safe-osc for every $v \in V$. □

![Diagram](image)

Figure 3: An ograph on vertex set $\{a, b, c\}$, and its safe-$\alpha$SCCs $\alpha = \{a\}, \beta = \{b\}, \gamma = \{c\}$.

Next, let us observe the following property concerning safe-osc sets.

Lemma 1. Let $V_1, V_2 \subseteq V$ be two safe-osc sets. If $V_1 \cap V_2 \neq \emptyset$, then $V_1 \cup V_2$ is safe-osc.

Proof. Pick some $u, v \in V_1 \cup V_2$ and $z \in V_1 \cap V_2$, arbitrarily. Since $\{u, z\} \subseteq V_1$, and since $V_1$ is safe-osc, there exists some strategy $\sigma_\Box(u, z) \in \Sigma_\Box^A$ such that:

$$\sigma_\Box(u, z) : u \sim_{V_1} z,$$

similarly, there is some other strategy $\sigma_\Box(z, v) \in \Sigma_\Box^A$ such that:

$$\sigma_\Box(z, v) : z \sim_{V_2} v.$$ 

Then, consider the strategy $\sigma_\Box(u, v) \in \Sigma_\Box^A$ constructed by gluing $\sigma_\Box(u, z), \sigma_\Box(u, v)$ in sequence:

$$\sigma_\Box(u, v) \triangleq \begin{cases} (1) \text{ Starting from } u, \text{ play } \sigma_\Box(u, z) \text{ until } z \text{ is firstly reached;} \\
(2) \text{ once on } z, \text{ play } \sigma_\Box(z, v) \text{ until } v \text{ is finally reached.} \end{cases}$$

Clearly, $\sigma_\Box(u, v) : u \sim_{V_1 \cup V_2} v$. Since $u$ and $v$ were chosen arbitrarily, then $V_1 \cup V_2$ is safe-osc. □

Lemma 1 allows us to define and study the following binary relation on $V$.

Definition 7. The binary relation $\sim_{\text{safe}} \subseteq V \times V$ is defined as:

$$\sim_{\text{safe}} \triangleq \{ (u, v) \in V \times V \mid \exists U \subseteq V \text{ such that } U \text{ is safe-osc and } \{u, v\} \subseteq U \}.$$ 

Lemma 2. $\sim_{\text{safe}}$ is an equivalence relation on $V$.

Proof. To begin, (i) $\sim_{\text{safe}}$ is clearly reflexive: for any $u \in V$, let $U = \{u\}$; then, $u \sim_{\text{safe}} u$, so $U$ is safe-osc; this shows $u \sim_{\text{safe}} u$. (ii) $\sim_{\text{safe}}$ is symmetric, (actually, by definition): for any $u, v \in V$, assume $u \sim_{\text{safe}} v$; then, there exists some $U \subseteq V$ which is safe-osc and $u, v \in U$; so, the same set $U$ certifies $v \sim_{\text{safe}} u$. Finally, (iii) $\sim_{\text{safe}}$ is transitive: indeed, for any $a, b, c \in V$, assume $a \sim_{\text{safe}} b$ and $b \sim_{\text{safe}} c$. Since $a \sim_{\text{safe}} b$, there exists $V_1$ which is safe-osc and such that $a, b \in V_1$; similarly, there exists $V_2$ which is safe-osc and such that $b, c \in V_2$. Consider $U \triangleq V_1 \cup V_2$. Since $b \in V_1 \cap V_2$, and $V_1, V_2$ are both safe-osc, then $U$ is safe-osc by Lemma 1. Moreover, $a, c \in U$. So, $a \sim_{\text{safe}} c$.

Thus $\sim_{\text{safe}}$ is an equivalence relation. □
Let us point out some interesting properties of \(\sim_{\text{safe}}\) equivalence classes.

**Lemma 3.** Let \(\{C_i\}_{i=1}^k\) be all the distinct equivalence classes of \(\sim_{\text{safe}}\) on \(V\). Then, the following holds.

1. If \(U \subseteq V\) is safe-osc and \(U \cap C_i \neq \emptyset\) for some \(i \in [k]\), then \(U \subseteq C_i\);  
2. \(C_i\) is safe-osc for each \(i \in [k]\);  
3. Let \(U \subseteq V\) be safe-osc. Then, \(C_i \subseteq U\) for no \(i \in [k]\).

**Proof of (1).** Since \(U \cap C_i \neq \emptyset\), it's possible to pick \(z \in U \cap C_i\). Pick \(v \in U\), arbitrarily. Since \(U\) is safe-osc and \(z, v \in U\), then \(v \sim_{\text{safe}} z\). So, \(v \in C_i\) (because \(z \in C_i\), which is an equivalence class of \(\sim_{\text{safe}}\)).

**Proof of (2).** Let \(u, v \in C_i\), arbitrarily. Then, \(u \sim_{\text{safe}} v\). So, there exists some \(U \subseteq V\) which is safe-osc and such that \(u, v \in U\). Thus, \(u \sim_{U} v\). Notice, \(u, v \in U \cap C_i \neq \emptyset\). Then, by item 1 of Lemma 3, \(U \subseteq C_i\). Since \(u \sim_{U} v\) and \(U \subseteq C_i\), then \(u \sim_{C_i} v\). So, \(C_i\) is safe-osc.

**Proof of (3).** Assume that \(C_i \subseteq U\), for some \(i \in [k]\), and some \(U \subseteq V\) which is safe-osc. Then, since \(U \cap C_i = C_i \neq \emptyset\), by item 1 of Lemma 3 we have \(U \subseteq C_i\). So, \(C_i = U\).

**Definition 8.** Let \(A\) be an oagraph on vertex set \(V\). Let \(C \subseteq V\) be a subset of the vertices and consider the binary equivalence relation \(\sim_{\text{safe}}\) on \(V\). We say that \(C\) is an alternating strongly-connected component (\(\alpha\text{SCC}\)) of \(A\) precisely when it is an equivalence class of \(\sim_{\text{safe}}\).

Otherwise stated, by Lemma 3, an \(\alpha\text{SCC}\) is any maximal (under set inclusion) safe-osc vertex subset of the oagraph.

Moreover, since safe-osc is a more constrained form of osc, the former implies the latter (as below).

**Proposition 1.** The \(\sim_{\text{safe}}\) equivalence relation is finer than \(\sim_{\text{osc}}\).

**Proof.** It is enough to point out that every equivalence class of \(\sim_{\text{safe}}\) is a subset of an equivalence class of \(\sim_{\text{osc}}\) (and thus every equivalence class of the latter is a union of equivalence classes of the former).

This is clear as every safe-osc set is osc too.

### 2.3 Applications to Update Games and McNaughton-Müller Games

An **Update Game (UG)** \([1, 10, 11]\) is played on an oagraph \(A\) with vertex set \(V\) and arc set \(A\) for an infinite number of rounds. Here a **play** is an infinite path \(\rho = v_0v_1v_2\ldots \in V^{\omega}\) such that \((v_i, v_{i+1}) \in A\ \forall i \in \mathbb{N}\). Let \(\text{Inf}(\rho)\) be the set of all the vertices \(v \in V\) appearing infinitely often in \(\rho\); namely,

\[\text{Inf}(\rho) = \{v \in V \mid \exists j \in \mathbb{N} \exists k \in \mathbb{N}, k > j, \text{ such that } v = v_k\}\]

provided \(\rho = v_0v_1v_2\ldots v_k\ldots \in V^{\omega}\).

Player \(\Box\) wins the UG played on \(A\) if and only if there exists \(\sigma_\Box \in \Sigma_\Box\) such that, for every \(\sigma_\circ \in \Sigma_\circ\), every vertex is visited infinitely often in the unique play that is consistent with \(\sigma_\Box\) and \(\sigma_\circ\), independently w.r.t. the starting position \(s \in V\); namely, if and only if the following holds:

\[\exists \sigma_\Box \in \Sigma_\Box \forall \sigma_\circ \in \Sigma_\circ \forall s \in V \text{ Inf}(\rho_A(s, \sigma_\Box, \sigma_\circ)) = V;\]

otherwise, Player \(\bigcirc\) wins. When Player \(\Box\) wins an UG \(A\), then \(A\) is called **Update Network (UN)** \([1, 10, 11]\).

In order to decide who wins an UG, we can check whether the whole vertex set \(V\) is either safe-osc or simply osc (indifferently, as clearly the two conditions are equivalent for the whole vertex set, i.e., notice that the whole vertex set \(V\) is safe-osc if and only if it is osc).

**Proposition 2.** Let \(A\) be an UG on vertex set \(V\). Player \(\Box\) wins the UG played on \(A\) if and only if \(V\) is safe-osc; or equivalently (since \(V\) is the whole vertex set), if and only if \(V\) is osc.
\textbf{Proof.} If Player $\square$ wins the UG played on $\mathcal{A}$, then $V$ is safe-asc (it follows directly from definitions, as every vertex can be visited infinitely often then every vertex is reachable from any other one). Conversely, if $V$ is safe-asc, and $v_0, \ldots, v_{|V|-1}$ is a vertex ordering, for every $i$ there is $\sigma_{\square}(i) \in \Sigma_0^\omega$ such that $\sigma_{\square}(i) : v_i \sim v_{i'}$, where \(i' = (i + 1) \mod |V|\) for every $i \in \{0, \ldots, |V| - 1\}$. Starting from any $v_i$, Player $\square$ can visit infinitely often all vertices in $V$ by playing forever $\langle \sigma_{\square}(i), \sigma_{\square}(i'), \sigma_{\square}(i'') \rangle$ cascade. For the whole vertex set $V$, the same argument works if we consider osc instead of safe-asc. \hfill $\square$

The fact is that we are not currently aware of any sub-quadratic time algorithm for checking osc. Instead, our proposed solution for checking safe-osc runs in linear-time (as if we were computing all $|V|$ attractors in $O(|A|)$ aggregate time). Thus we employ safe-osc for solving UGs.

Let us consider alsoMcNaughton-Müller Games (MMGs) [16]. They provide a useful model for the synthesis of controllers in reactive systems, but their complexity depends on the representation of the winning conditions. The most straightforward way to represent a Müller winning condition $F \subseteq 2^V$ is to provide an explicit list of subsets of vertices as in [16], i.e., $F = \{F_i \subseteq V \mid 1 \leq i \leq \ell\}$ for some $\ell \in \mathbb{N}$.

A play $\rho \in V^\omega$ is winning for Player $\square$ if and only if $\text{Inf}[\rho] \in F$. So-called Explicit MMGs can be solved in polynomial time, e.g., with Horn’s algorithm [16]. Concerning time complexity, given an input ograph $\mathcal{A}$ and explicit winning condition $F$, there are at most $|F|$ loops in a run of that algorithm, and the most time consuming operation at each iteration is precisely to decide an UG of size at most $|A| + |F|$, see [16].

Thus deciding whether the whole vertex set of a game is safe-osc/osc is relevant to EMMGs too.

By Corollary 1, we can decide an UG in $\Theta(|A| + |F|)$ linear-time. As a consequence, the time complexity of Horn’s algorithm [16] improves by a factor $|A| + |F|$ (i.e., from cubic to quadratic).

In summary, from Corollary 1 and Horn’s algorithm [16], we obtain Corollary 2 (cfr Section 1.1).

\section{Safe-Alternating Depth-First Search}

This section offers a gamification of the DFS algorithm, as a depth-first reverse exploration of ographs.

On this route let us recall a basic vertex coloring induced by the depth-first search, as given in [7]. Imagine reverse-DFS runs on a directed graph, color the vertices during the search to indicate their state. Initially each vertex $v$ is white to mean unexplored, then $v$ becomes grey when it is first discovered (i.e., when open[$v$] is assigned), then black when the search backtracks (i.e., when close[$v$] is assigned). Each vertex changes color only twice, from white to grey and then blackened.

A fundamental underlying invariant property of \textit{Safe-Alternating Depth-First Search} (aDFS) goes as prescribed in the box below (this will be formally proved in Proposition 3). Recall that the exploration of the vertices goes backward like in reverse-DFS meanwhile building up a palm-tree $T$ in post-ordering; the task is precisely to decide \textit{which} particular post-ordering to follow, i.e., \textit{when} to explore any given vertex.

\begin{quote}
During the aDFS() exploration of an input ograph $\mathcal{A}$ on vertex set $V$, a new vertex $u \in V$ is visited and attached to the aDFS's palm-tree $T$ under formation (i.e., that one comprising at least one grey vertex) only when the \textit{T-safe-reachability} of its root $r_T$ becomes guaranteed starting from $u$ in such a way that any safe-reachability finite play path can only move through the non-white vertices of $T$.

This happens only after that a certain set of out-neighbours of $u$ becomes non-white in $T$: all of $u$’s out-neighbours must have been colored grey or black if $u \in V_\square$; and at least one if $u \in V_\open$.

(\text{So, safe-reachable is invariantly preserved in the palm-trees instead of just graph reachability})

Of course we will need additional (non-trivial) arguments to ensure the algorithm runs in linear-time. For instance, when a new vertex $u \in V_\open$ attaches to the aDFS’s palm-tree $T$ under formation, the parent of $u$ in $T$ must be chosen very carefully. The following rule stands out.

\begin{quote}
During the aDFS() exploration of the given input ograph $\mathcal{A}$ on vertex set $V$, assume that a new vertex $u \in V$ now attaches to the aDFS’s palm-tree $T$ under formation (i.e., that comprising at least
\end{quote}
one grey vertex). If $u \in V_\text{O}$, the parent of $u$ in $T$ can be any of the grey out-neighbours of $u$ in $T$; otherwise, if $u \in V_\text{V}$, the parent of $u$ in $T$ is precisely the Lowest Common Ancestor (LCA) (which is grey colored at that time) of all the out-neighbours of $u$ in $T$ (and all these must be non-white colored at that time).

So, safe-reachability is preserved from $u \in V_\text{O}$ to the LCA of its out-neighbours.

A detailed description of the algorithm comes next, where some additional technical machinery (e.g., counters, stacks, and disjoint-sets data structure) is employed precisely for running time efficiency.

As it starts to make sense, a major technical issue will be that to perform LCAs lookups efficiently.

### 3.1 Description of $\alpha$DFS

The main procedure is named $\alpha$DFS() (Algorithm 1), but the vertex visiting will be handled by subprocedure $\alpha$DFS-visit() (Proc. 1). See Algorithm 1 and Procedure 1 below for pseudocode.

The starting point for describing how everything works is recalling the reverse-DFS. In fact $\alpha$DFS($A$) (Algo. 1) can be viewed as a gamification of the latter, in the sense that, if $V_\text{O} = \emptyset$, it works like a reverse-DFS and the output forest $J_A$ is a jungle.

Indeed, given an ograph $A$ on vertex set $V = V_\text{V} \cup V_\text{O}$, a forest ograph $J_A$ can be built during the search process (like the traditional DFS constructs a jungle) and returned as output. So $J_A$ will comprise a forest of trees, each called alternating palm-tree ($\alpha$-palm-tree), having fronds and cross-links.

During the exploration, arcs $(u, v) \in A$ will be classified into four categories according to the state (color) of the tail vertex $u$ that is touched when the arc is first explored, namely, tree arcs $A_x$ (white), fronds $A_f$ (grey), stalk-arcs $A_s$ (white $u \in V_\text{V}$), and cross-links $A_c$ (black); at the end, their union $A'$ will be the whole arc set of what we call the alternating jungle (o-jungle) $J_A$.

An index, named open : $V \rightarrow \mathbb{N} \cup \{+\infty\}$, timestamps the vertices in the order in which they are firstly visited (i.e., the timestamp opens at the beginning of the visiting subprocedure); initially all vertices are unvisited, so $\forall u \in V$ open[$u$] $\leftarrow +\infty$. Another index, close : $V \rightarrow \mathbb{N} \cup \{+\infty\}$, timestamps the vertices in the order in which they are backtracked (i.e., the closing assignment happens at the end of the visiting subprocedure). In the pseudocode we assume open[], close[], rSt[], cnt[], time are all global variables.

We say vertex $u \in V$ is active (grey) if open[$u$] $<$ $+\infty$ and close[$u$] $=$ $+\infty$, say that $u$ has been visited (black) if open[$u$] $<$ $+\infty$ and close[$u$] $<$ $+\infty$, and that $u$ is unvisited (white) if open[$u$] $=$ close[$u$] $=$ $+\infty$.

Now, imagine that the search exploration proceeds by visiting and backtracking vertices like in a reverse-DFS. Any $u \in V_\text{V}$ is visited, and so it joins $J_A$, as soon as it is firstly discovered in the in-neighbourhood of some active vertex (i.e., precisely as in the reverse-DFS).

Let’s say by convention that any vertex $u \in V$ joins $J_A$ precisely when it becomes active and the tree arc $(u, v)$ is added to $A_x$ for some $v \in V$.

The $\odot$-rule (i.e., that allowing any $u \in V_\text{V}$ to be visited) is more involved: any $u \in V_\text{V}$ becomes active joining $J_A$, by attaching to some parent vertex $\pi_u$, only when all of $u$’s out-neighbours $v \in N_\text{A}^\text{out}(u)$ have already did it. So the visiting step of any circled $u$ has to be delayed w.r.t. the (possibly repeated) discovery of $u$ as an in-neighbour of (possibly many) active vertices $v$ such that $(u, v) \in A$; the exact moment being when the search backtracks, after the last visited out-neighbour $v \in N_\text{A}^\text{out}(u)$, up to the corresponding parent vertex $\pi_u$. And when $u \in V_\text{V}$ joins $J_A$ with parent $\pi_u$ (i.e., if $u \in V_\text{V}$ and $(u, \pi_u) \in A_x$ for some $\pi_u \in V$), then $\pi_u$ is prescribed by the $\odot$-rule to be the LCA $\gamma$ of $N_\text{A}^\text{out}(u)$ in the $\alpha$-palm-tree under formation; at that point all of the original outgoing arcs of $u$ are labeled stalk-arcs.

| Algorithm 1: Safe-Alternating DFS |
|----------------------------------|
| **Procedure $\alpha$DFS($A$)** |
| **input** : An ograph $A = (V, A, (V_\text{V}, V_\text{O}))$. |
| **output** : An ojungle $J_A$. |
| 1 | $A_x, A_f, A_s, A_c \leftarrow \emptyset$; |
| 2 | foreach $u \in V$ do |
| 3 | open[$u$] $\leftarrow +\infty$; |
| 4 | close[$u$] $\leftarrow +\infty$; |
| 5 | rSt[$u$] $\leftarrow \emptyset$; |
| 6 | if $u \in V_\text{V}$ then |
| 7 | cnt[$u$] $\leftarrow |N_\text{A}^\text{out}(u)|$; |
| 8 | time $\leftarrow 0$; // global time variable |
| 9 | foreach $u \in V_\text{O}$ do |
| 10 | if open[$u$] $=$ $+\infty$ then |
| 11 | $\alpha$DFS-visit($u$, $A$); |
| 12 | if open[$u$] $=$ $+\infty$ then |
| 13 | open[$u$] $\leftarrow$ time; |
| 14 | close[$u$] $\leftarrow$ time; |
| 15 | time $\leftarrow$ time + 1; |
| 16 | $A' \leftarrow A_x \cup A_f \cup A_s \cup A_c$; |
| 17 | return $J_A \leftarrow (V, (A', (V_\text{V}, V_\text{O})))$; |

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Proposition 3. Assume that $\alpha \in V_\circ$ belonged to the input directed graph. Notice that if $u \in V_\circ$ joins $A$ with parent $\pi_u$, and since $\pi_u$ is the LCA of $N^\text{out}_A(u)$, then the arc $(u, \pi_u) \in A_\pi$ may be a totally brand new arc, i.e., it might not have been in the original arc set $A$ of the input graph $A$ (in that case $A_\pi \not\subseteq A$ and $(u, \pi_u)$ is not labeled as a stalk-arc). The possibility that $A_\pi \not\subseteq A$ is a distinctive point with respect to reverse-DFS and Tarjan’s jungles, where all tree arcs belonged to the input directed graph.

In order to implement the $\circ$-rule efficiently, an additional counter of out-neighbour vertices $\text{cnt} : V_\circ \to N$ is employed, constantly checked and updated. The following invariant $\text{I}_{\text{cnt}}$ is kept maintained:

$$\forall u \in V_\circ \text{ cnt}[u] = |\{v \in N^\text{out}_A(u) \mid \text{open}[v] = +\infty\}|.$$ (I_{\text{cnt}})

Also, for each $v \in V$ it is employed a LIFO stack of vertices named $rSt[v]$ (named, the ready stack). Its role, during the $\alpha\text{DFS-visit()}$ subprocedure, is to memorize that a certain vertex $\pi_u \in V$ had been identified as the parent of some other vertex $u \in V_\circ$ (i.e., when $\text{cnt}[u] = 0$ and $\pi_u = \gamma$ is the LCA of $N^\text{out}_A(u)$); at that point $u$ would be promptly pushed to the ready stack $rSt[\pi_u]$. Then $u$ will have to join $A$ when visited by the search, this happens when the visit backtracks from $u$ up to his parent $\pi_u$.

By hooking to LCAs the $\circ$-rule allows us to preserve safe-arrachability, as shown in Proposition 3. A moment’s reflection reveals that, by construction, $(V, A_\pi)$ is really a forest graph, but more formal details on the graph structure of $A$ are postponed to Proposition 5. We also provide more details on the algorithm after the following Proposition 3, in which $\{P_i\}_{i=1}^k$ denotes the vertex disjoint family of all the palm-trees in $A$.

**Procedure 1:** Visit Procedure of Safe-Altering DFS

```
Procedure \alpha\text{DFS-visit}(v, A)
input : One vertex $v \in V$ of $A$.
1 open[v] ↷ (time ← time + 1);
2 foreach $u \in N^?_A(v)$ do
3 if open[u] = +\infty then
4 if $u \in V_\circ$ then
5 add $(u, v)$ to $A_\pi$;
6 $\alpha\text{DFS-visit}(u, A)$;
7 else $\text{cnt}[u] ↷ \text{cnt}[u] - 1$;
8 if $\text{cnt}[u] = 0$ and $\exists$ (LCA of $N^\text{out}_A(u)$ in $(V, A_\pi)$) then
9 $\gamma ↷$ the LCA of $N^\text{out}_A(u)$ in $(V, A_\pi)$;
10 $rSt[\gamma].\text{push}(u)$;
11 else if open[u] < +\infty and close[u] = +\infty then
12 add $(u, v)$ to $A_f$;
13 else add $(u, v)$ to $A_\circ$;
14 while $rSt[v] \neq \emptyset$ do
15 $u ↷ rSt[v].\text{pop}()$; // $u \in V_\circ$
16 add $(u, v)$ to $A_\circ$;
17 for each $t \in N^\text{out}_A(u)$ do add $(u, t)$ to $A_{\text{safe}}$;
18 $\alpha\text{DFS-visit}(u, A)$;
19 close[v] ↷ (time ← time + 1);
```

**Proposition 3.** Assume that $\alpha\text{DFS()}$ runs on a given input graph $A$. Consider the forest of palm-trees $\{P_i\}_{i=1}^k$ that are constructed during the visiting process; say that $P_i = (V_i, A_i, (V_{\circ_i}, V_{\bullet_i}))$ is the $i$-th palm-tree, on vertex set $V_i$ and arc set $A_i$ for each $i \in [k]$. For any two vertices $u, v \in V_i$, any $i \in [k]$, if $u$ is a descendant of $v$ in $P_i$, then $v$ is $V_i$-safe-arrachable from $u$ w.r.t. $A$. Particularly, this holds thanks to the following strategy $\sigma_\pi \in \Sigma_{A_\pi}$, where $\pi(u)$ denotes the parent of any $u \in V_{\bullet_i}$ in the forest $(V, A_\pi)$:

$$\forall u \in V_i \quad \sigma_\pi(u) \doteq \begin{cases} \pi(u), & \text{if } u \text{ is not the root of any palm-tree } P_i; \\ \text{any } u' \in N^\text{out}_A(u), & \text{if } u \text{ is the root of some palm-tree } P_i. \end{cases}$$

**Proof.** Assume $u, v \in V_i$ where $u$ is a descendant of $v$ in the palm-tree $P_i$, for some $i \in [k]$ fixed arbitrarily. Recall that during the $\alpha\text{DFS()}$ all vertices are given an index so that $\text{open}[v] < \text{open}[u]$ if
v is a proper ancestor of u in some palm-tree. Let us proceed arguing by induction on open[u]. Let 
\( z = \min_{x \in V} \text{open}[x] \) be the vertex with minimum index in \( P_i \). Assume \( \text{open}[u] = z \) as a base case. So, u is the root of \( P_i \). Then \( v = u \), so there is actually nothing to prove. Now, let \( \text{open}[u] > z \). Let w.l.o.g \( u \neq v \). Assume as induction hypothesis the thesis for every vertex \( x \in V_i \) such that \( \text{open}[x] < \text{open}[u] \).

Let us break the forthcoming analysis in two cases, according to whether \( u \in V_o \) or \( u \in V_o \).

- If \( u \in V_o \), since \( u \) is not the root of \( P_i \), then \( \sigma_{\square}(u) = \pi(u) \). By construction, \( \text{open} \pi(u) < \text{open}[u] \). Since \( \pi(u) \) is the parent of \( u \) in \( P_i \) and \( u \neq v \), then \( \pi(u) \) is still a descendant of \( v \) in \( P_i \) (possibly, \( \pi(u) = v \)); therefore, by induction hypothesis:

\[
\sigma_{\square} : \pi(u) \overset{V}{\sim} v.
\]

Since \( \sigma_{\square} : u \overset{V}{\sim} \pi(u) \) and \( \sigma_{\square} : \pi(u) \overset{V}{\sim} v \), therefore by composition \( \sigma_{\square} : u \overset{V}{\sim} v \).

- If \( u \in V_o \), recall that by definition of \( \alpha \text{DFS} \), \( \pi(u) \) is the LCA of the out-neighbours of \( u \) in \( A \), i.e., the LCA of \( N_{A}^{\text{out}}(u) = \{ u' \in V \mid (u, u') \in A_s \} \). Fix some \( u' \in N_{A}^{\text{out}}(u) \), arbitrarily. Notice that \( u' \) is still a descendant of \( \pi(u) \) in \( P_i \) (possibly, \( u' = \pi(u) \)), just because \( \pi(u) \) is the LCA of \( N_{A}^{\text{out}}(u) \) in \( P_i \). Thus, since \( \pi(u) \) is a descendant of \( v \) in \( P_i \) (possibly, \( \pi(u) = v \)), then by transitivity \( u' \) is also a descendant of \( v \) in \( P_i \) And, by definition of \( \alpha \text{DFS} \), it must be that \( \text{open}[u'] < \text{open}[u] \). Therefore, by induction hypothesis:

\[
\sigma_{\square} : u' \overset{V}{\sim} v.
\]

Since \( u' \) was chosen arbitrarily, the latter assertion holds for every \( u' \in N_{A}^{\text{out}}(u) \); so, \( \sigma_{\square} : u \overset{V}{\sim} v \).

This concludes the inductive step of the proof. So, anyway, \( \sigma_{\square} : u \overset{V}{\sim} v \).

It’s also clear at this point that, at anytime during the execution of \( \alpha \text{DFS}() \), any such safe-accessibility finite play path (that goes from descendants up to ancestors) can only move through the non-white vertices of its palm-tree.

\[\square\]

**More Details.** Let us further provide some lower-level implementation details of \( \alpha \text{DFS} \) (Algo. 1).

Concerning stacks and counters, \( rS\text{Fl}[u] \) is initialized to be empty for every \( u \in V \) and, for every \( u \in V_o \), it is initialized \( \text{cnt}[u] \leftarrow |N_{A}^{\text{out}}(u)| \) (see lines 5-7 of Algo. 1). Then \( \text{cnt}[u] \) is decremented whenever some out-neighbour \( v \) of \( u \) is visited during the search process. When \( \text{cnt}[u] = 0 \) (see line 9 of Proc. 1), all out-neighbours of \( u \) have already joined the jungle \( J_A \).

Notice, if any two out-neighbours of \( u \) belong to two distinct palm-trees in \( J_A \), there is no way to preserve safe-accessibility because Player \( \circ \) might choose to move from \( u \) to any of the two shafts at will, and the LCA \( \gamma \) of \( N_{A}^{\text{out}}(u) \) might not exist in \( (V, A_s) \); still, if all out-neighbours of \( u \) belong to the same palm-tree, the LCA \( \gamma \) does exist in \( (V, A_s) \). So, when \( \text{cnt}[u] = 0 \), firstly we seek for the LCA \( \gamma \) and if it exists we push \( u \) on top of \( rS\text{Fl}[\gamma] \) (cfr lines 9-11 of \( \alpha \text{DFS-visit}() \), Proc. 1).

In so doing, \( u \in V_o \) will join \( J_A \) only when \( \alpha \text{DFS-visit}() \) backtracks, from the last out-neighbour \( v \) of \( u \) that has been visited, up to \( \gamma \) (possibly \( \gamma = v \)). At that point (see lines 15-19), as \( rS\text{Fl}[\gamma] \) will be checked and \( u \) will be found therein, \( (u, \gamma) \) will be added to \( A_s \); and, for each \( t \in N_{A}^{\text{out}}(u) \) the arc \( (u, t) \) will be added to \( A_s \) (possibly, \( (u, \gamma) \in A_s \cap A_s \)). Finally \( \alpha \text{DFS-visit}(u, A) \) will be invoked for recursively visiting \( u \). In this way every vertex is visited exactly once.

During \( \alpha \text{DFS-visit}(v, A) \), when it is explored some in-neighbour \( u \) of \( v \) such that \( \text{open}[u] \neq +\infty \), if \( u \) is still active (grey) then \( (u, v) \) is added to the founds \( A_f \), otherwise \( u \) is inactive (black) and \( (u, v) \) goes to cross-links \( A_c \).

There’s still one detail which is worth mentioning as it helps keeping smooth the presentation. Firstly all \( u \in V_o \) are considered as roots of the palm-trees, i.e., no \( u \in V_o \) ever becomes a root of an palm-tree due to lines 9-11 of \( \alpha \text{DFS}() \) (Algo. 1). After the visiting is completed, for each \( u \in V_o \), which still remained unvisited, \( \text{open}[u] \) is assigned incrementally and the visiting process is not invoked anymore.

Indeed, w.l.o.g we can assume that for all \( v \in V \mid |N_{A}^{\text{out}}(v)| \geq 2 \). For this we just preprocess \( A \) as follows: for any \( v \in V \), if \( N_{A}^{\text{out}}(v) = \emptyset \), remove \( v \) from the agraph; if \( N_{A}^{\text{out}}(v) \neq \emptyset \) is a singleton, add \((u, v')\) to \( A \) for each \( u \in N_{A}^{\text{out}}(v) \) and then remove \( v \) from the agraph. So doing, observe that even if
\(\alpha\text{-DFS-visit}(v, A)\) would’ve been invoked for some \(v \in V_\bigodot\), say at line 15 of \(\alpha\text{DFS}()\), there would’ve been no actual \(\alpha\)-palm-tree to visit, i.e., no vertex \(u\) such that \((u, v) \in A_\alpha\). Of course all reachability relations are preserved after the preprocessing. So this self-reduction is fine, and it keeps simpler the presentation of the algorithm.

This ends the detailed description of \(\alpha\text{DFS}()\) (Algo. 1). Let us now begin to analyze its complexity.

**Proposition 4.** Assume that \(\alpha\text{DFS}()\) (Algo. 1) runs on a given input \(\alpha\text{graph} A\) on vertex set \(V\) and arc set \(A\). Each vertex \(v \in V\) is timestamped by \(\text{open}[v]\) exactly once, and the algorithm halts in time \(\Theta(|V| + |A| + \text{Time}[LCA])\), consuming space \(\Theta(|V| + |A| + \text{Space}[LCA])\), where \(\text{Time}[LCA]\) (Space[LCA]) is the aggregate total time (space) taken by all LCA computations that are done at lines 9-10 of \(\alpha\text{DFS-visit}()\) (Proc. 1).

**Proof.** The initialization phase takes \(\Theta(|V| + |A|)\) time (see lines 1-7 of Algo. 1). Recall Algo. 1 performs multiple calls to \(\alpha\text{DFS-visit}(v, A)\) (Proc. 1), each for some \(v \in V\). Any of these happens if and only if \(\text{open}[v] = +\infty\), and then \(\text{open}[v]\) is set to some non-zero value. Thus, the total number of invocations of \(\alpha\text{DFS-visit}()\) (Proc. 1) is at most \(|V|\). Actually, by lines 9 and 12 of \(\alpha\text{DFS}()\) (Algo. 1), it is exactly \(|V|\). So, each vertex \(v \in V\) is numbered by \(\text{open} : V \to \mathbb{N}\) exactly once.

Concerning time complexity, consider each of such visits independently from one another, where the in-neighbourhood \(N^{\text{in}}_A(v)\) is explored. For some \(u \in N^{\text{in}}_A(v) \cap V_\bigodot\), the LCA of \(N^{\text{out}}_A(u)\) might be computed, but notice that all the other operations about \(N^{\text{out}}_A(u)\) can be done in constant time per single \(u \in N^{\text{in}}_A(v)\). At the end of each visit the stack \(rSt[v]\) is emptied, still, due to the condition \(\text{cnt}[u] = 0\) any \(u \in V_\bigodot\) can be pushed on \(rSt[v]\) at most once and for at most one \(v \in V\). Therefore, the \(\Theta(|V| + |A| + \text{Time}[LCA])\) aggregate time bound holds.

Concerning space usage, a similar argument shows that the aggregate total space of storing \(rSt[v]\)_s \(v \in V\) is \(O(|V|)\). Also, the total size of \(\text{open}[]\), \(\text{close}[]\) and \(\text{cnt}[]\) is \(\Theta(|V|)\), and that of \(A' = \Theta(|A|)\).

Later on in [Section 3.3, Theorem 1], the aggregate total time and space of all LCA computations (i.e., \(\text{Time}[LCA]\) and \(\text{Space}[LCA]\)) will be bounded linearly. Before that, in the following Section 3.2, let us read out and carefully analyze the graph structure of the \(\alpha\text{jungle} J_A\).

### 3.2 Graph Structures

Let’s start by formalizing the structural properties of the \(\alpha\)-palm-trees. Examples are given in Fig. 4 and 5.

**Definition 9.** An alternating palm-tree (\(\alpha\)-palm-tree) is a triplet \((P, \text{open}[]\), \(\text{close}[]\)) where:

(i) \(P = (V, A, \{|V_\bigodot|, V_\bigodot\})\) is an \(\alpha\text{graph} on V = V_\bigodot \cup V_\bigodot\) and \(A = A_\alpha \cup A_f \cup A_s \cup A_c\), so the vertex set is split into squares and circles whereas the arc set into four categories.
(ii) open, close : V → N timestamp the vertex set V in pre and post order respectively;  
(iii) the following four main properties hold:

(αpt-1) T_P ⊆ (V, A_π) is an inward directed rooted tree such that:
(a) the root r_T_P of T_P is controlled by Player □, i.e., r_T_P ∈ V_□;  
(b) open[v] < open[u] < close[u] < close[v] whenever (u, v) ∈ A_π, i.e., if v = π(u) is the parent of u in T_P;

(αpt-2) Each frond-arc (u, v) ∈ A_f connects some u ∈ V_□ to one of its proper descendants v ∈ V in T_P;

(αpt-3) Each stalk-arc (u, v) ∈ A_s connects some u ∈ V_□ to one of the descendants v of its parent π(u) (i.e., possibly to π(u) itself); particularly, given any u ∈ V_□, the following three properties hold:
(a) {v ∈ V | (u, v) ∈ A_s} ∪ {π(u)} = N^out_T_P(u);  
(b) π(u) is the LCA of {v ∈ V | (u, v) ∈ A_s} in T_P;  
(c) open[v] < close[v] < open[u] < close[u] for every v ∈ N^out_T_P(u) \ {π(u)}.

Figure 5: An ograph (a), and the construction of a corresponding jungle (b-h).
(apt-4) Each cross-arc \((u, v) \in A_c\) connects some \(u \in V_{\square}\) to some \(v \in V\) such that:

(a) \(v\) is not a descendant of \(u\) in \(T_P\);

(b) either \(v\) is a proper ancestor of \(u\) in \(T_P\) (in that case \(open[v] < open[u] < close[u] < close[v]\)), or \(open[u] < close[u] < open[v] < close[v]\).

An \(\alpha\)jungle is formed by a disjoint union of \(\alpha\)palm-trees (see \(\alpha\)jn-1 and \(\alpha\)jn-2), possibly with external cross-arcs connecting two distinct \(\alpha\)palm-trees (see \(\alpha\)jn-3), plus a (possibly empty) set of circled vertices each having out-neighbours lying in at least two distinct \(\alpha\)palm-trees (see \(\alpha\)jn-4).

**Definition 10.** An alternating jungle (\(\alpha\)jungle) is an agraph \(J = (V, A, (V_{\square}, V_{\circ}))\) comprising a family of vertex-disjoint \(\alpha\)palm-trees \((\{P_i, open[i], close[i]\})_{i=1}^k\), whose vertices are timestamped, and these hold:

\(\alpha\)jn-1) \(\forall i \in [k]\) \(P_i = (V_i, A_i, (V_{\square}, V_{\circ}))\), where \(V_{\square} \subseteq V_i, V_{\circ} \subseteq V_i, A_i \subseteq A;\)

\(\alpha\)jn-2) \(\forall i, j \in [k]\) \(V_i \cap V_j = \emptyset\) if \(i \neq j;\)

\(\alpha\)jn-3) If \((u, v) \in A\) for some \(u \in V_i\) and \(v \in V_j\) such that \(i \neq j\), then \(u \in V_{\square}\) and \(i < j;\)

\(\alpha\)jn-4) If \(v \in V \backslash \bigcup_{i=1}^k V_i\), then \(v \in V_{\circ}\) and \(N^\text{out}_J(v) \subseteq V_i\) for no \(i \in [k]\).

Proposition 5 shows that \(a\)DFS() (Algo. 1) really constructs an \(\alpha\)jungle. It’s worth introducing a technical but conceptually simple notion, that of support for an \(\alpha\)jungle. The support of \(J\) is just the same agraph deprived of all the arcs in \(\{u, v\in A_n \backslash A_s \mid u \in V \circ\}\), i.e., those arcs that are added by \(a\)DFS() (Algo. 1) but that were not in the input agraph. More formal details below.

**Definition 11.** Given an \(\alpha\)palm-tree \((P, open[], close[])\), for \(P = (V, A, (V_{\square}, V_{\circ}))\), \(A = A_n \cup A_f \cup A_s \cup A_c\), the support of \(P\) is the agraph \(P_s = (V, A_s, (V_{\square}, V_{\circ}))\), where \(A_s = \{(u, v) \in A \mid u \in V_{\square}\}\) holds by (\(\alpha\)pt-3).

Given an \(\alpha\)jungle \(J\) with family of \(\alpha\)palm-trees \((P_i)_{i=1}^k\), let \(V := \bigcup_{i=1}^k V_i\) (where \(V_i\) is the vertex set of \(P_i\).) The support of \(J\) is the agraph \(J_s\) obtained from \(J\) by replacing each \(P_i\) with its support \((P_i)_{si}\), and by leaving intact all the vertices in \(V\) and all arcs \((u, v)\) of \(J\) such that: either, (i) \(u \in V_i\) and \(v \in V_j\) for some \(i \neq j\) (i.e., all external cross-arcs); or, (ii) \(u \in V\) or \(v \in V\) (possibly both).

Let us now argue more formally than an \(\alpha\)jungle really traces down the behaviour of \(a\)DFS() (Algo. 1).

**Proposition 5.** Let \(A = (V, A, (V_{\square}, V_{\circ}))\) be an agraph. The following two propositions hold.

1. Let \(J\) be the agraph constructed by executing \(a\)DFS(A) (Algo. 1). Then, \(J\) is an \(\alpha\)jungle.

2. Let \(J\) be an \(\alpha\)jungle with support \(J_s\). Then, \(a\)DFS\((J_s)\) (Algo. 1) reconstructs \(J\) itself, i.e., \(J_{J_s} = J\).

**Proof of (1).** Recall, \(a\)DFS(A) (Algo. 1) performs a sequence of invocations to \(a\)DFS-visit(\(\cdot, A\)) (Proc. 1). Let \(k\) be the total number of times that \(a\)DFS-visit(\(\cdot, A\)) is invoked only at line 11 of \(a\)DFS() (Algo. 1). For each \(i = 1, 2, \ldots, k\), let \(u_i \in V_{\square}\) be the vertex that is passed as a parameter to the \(i\)-th invocation, i.e., assume \(a\)DFS-visit(u\(_i\), A) is the \(i\)-th call; notice \(u_i \in V_{\square}\) by line 9 of \(a\)DFS() (Algo. 1). Let \(V_i \subseteq V\) be the set of all vertices timestamped by \(open[]\) during the \(i\)-th invocation (recursive calls included). Similarly, let \(A_i\) be the set of arcs that are explored during that invocation (recursive calls included), and consider the internal arcs i.e., \(A_{int} = \{(a, b) \in A_i \mid both a, b \in V_i\}\). Finally let \(P_i = (V_i, A_{int}, (V_{\square} \cap V_i, V_{\circ} \cap V_i))\). It is easy to check that \(P_i\) is an \(\alpha\)palm-tree since it satisfies all properties from (\(\alpha\)pt-1) to (\(\alpha\)pt-4). We also claim that \(J\) is an \(\alpha\)jungle with \(\alpha\)palm-tree family \((P_i)_{i=1}^k\). Clearly, we are given a family \((P_i)_{i=1}^k\) of vertex-disjoint \(\alpha\)palm-trees, so properties (\(\alpha\)jn-1) and (\(\alpha\)jn-2) hold. Concerning (\(\alpha\)jn-3), let \((u, v) \in A\) by any arc such that \(u \in V_i\) and \(v \in V_j\) for any \(i \neq j\); then \(u \in V_{\square}\) (we can argue this by exclusion: since \(P_i\) is an \(\alpha\)palm-tree, (\(\alpha\)pt-3) holds for \(V_o\), so the tail \(u\) of an external cross-link connecting two distinct \(\alpha\)palm-trees must be squared); also, \(i < j\) since otherwise \(u\) would’ve joined \(P_i\) instead of \(P_i\) (cfr lines 3-6 of \(a\)DFS-visit()). Concerning (\(\alpha\)jn-4), let \(v \in V \backslash \bigcup_{i=1}^k V_i\), then \(v \in V_{\circ}\) (cfr lines 9-16 of \(a\)DFS()); also, \(N^\text{out}_J(v) \subseteq V_i\) holds for no \(i \in [k]\), otherwise \(v\) would’ve joined \(P_i\) thanks to lines 9-11 and 15-19 of \(a\)DFS-visit(). All in, \(J\) is an \(\alpha\)jungle. □
Proof of (2). Recall that the support $J_*$ can be obtained from $J$ simply by removing from the opalm-trees of $J$ all the arcs $(u, v) \in A_\emptyset \setminus A_*$ such that $u \in V_\emptyset$. Consider the total ordering $<_\text{open}$ on the vertex set $V$ induced by the opening timestamp $\text{open}[]$ of $J$, i.e., $\forall_{a,b \in V\, a < \text{open}\, b \iff \text{open}[a] < \text{open}[b]}$. Encode an adjacency list of $J_*$ such that: (i) the main list of vertices is ordered according to $<_\text{open}$; (ii) for each $u \in V$, also the in-neighbourhood $N^\text{in}_A(u)$ is ordered according to $<_\text{open}$. Since $J$ satisfies all properties from (opt-1) to (opt-4) and their opalm-trees satisfy all properties from (opt-1) to (opt-4), it’s straightforward to check inductively that $\alpha\text{DFS}(J_*) = J$.

Still it remains to be seen how to perform efficiently, i.e., in linear-time, all the LCAs computations that are needed at lines 9-10 of $\alpha\text{DFS-visit()}$ (Proc. 1). In the next subsection, we suggest to adopt a disjoint-set forest data structure with a non-ranked union and a classical $\text{Find}$ primitive based on path-compression.

3.3 Computing LCAs by Disjoint-Set Forest

A disjoint-set forest (dsf) data structure [22], hereby denoted $D$, is a data structure that keeps track of a set of elements partitioned into a number of disjoint (non-overlapping) subsets, each of which is represented by a rooted tree. This is also known as union-find data structure or merge-find set.

The following three operations are supported: $D.\text{MakeSet}(\cdot)$, $D.\text{Union}(\cdot, \cdot)$ and $D.\text{Find}(\cdot)$, where:

(dsf-1) The representative element of each disjoint set is the root of that set’s tree;
(dsf-2) $\text{MakeSet}(v)$ initializes the parent of a vertex $v \in V$ to be $v$ itself, i.e., a singleton vertex tree;
(dsf-3) $\text{Union}(u, v)$ combines two trees, $T_1$ rooted at $u$ and $T_2$ rooted at $v$, into a new tree $T_3$ which is still rooted at $v$, i.e., $u$ simply becomes a child of $v$ (this is a non-ranked union).
(dsf-4) $\text{Find}(v)$, starting from $v$, traverses the ancestors of $v$ until the root $r$ of the tree containing $v$ is finally reached. While doing this, $\text{Find}(v)$ changes each ancestor’s parent reference to directly point to $r$ (this is path-compression); the resulting tree is much flatter, speeding up future operations, not only on these traversed elements but also on those referencing them from the downstairs of the tree.

Let us describe how to implement the LCAs computations at lines 9-10 of $\alpha\text{DFS-visit()}$ (Proc. 1). The resulting algorithm is named $\text{dsf-}\alpha\text{DFS}$, based on a global dsf data structure $\mathcal{D}$.

The main procedure of $\text{dsf-}\alpha\text{DFS}()$ is almost the same as $\alpha\text{DFS}()$ (Algo. 1), the only additions being:

(dsf-init-1) $D.\text{MakeSet}(v)$ is executed for each $v \in V$;
(dsf-init-2) For each $v \in V_\emptyset$, an array $\text{low\_ready}[] : V \to N \cup \{+\infty\}$ is initialized as $\text{low\_ready}[v] \leftarrow +\infty$.
Its role is tracking the $\text{open}[]$ timestamp of the unique out-neighbour of $v \in V_\emptyset$ which is visited firstly and before all other out-neighbours (i.e., the out-neighbour having minimum index). So, given $A$ in input, the following invariant property will be maintained:

$$\forall v \in V_\emptyset \quad \text{low\_ready}[v] = \min \{ \text{open}[u] \in N \cup \{+\infty\} \mid u \in N^\text{out}_A(v) \}.$$  \hfill (I_{\text{low}})

Lemma 4 shows that $\text{low\_ready}[v]$ can be used as a compass needle for making LCA lookups; indeed, because of the two forthcoming rules, the LCA that we need to find turns out to be the root of the disjoint set tree containing precisely the vertex indexed by $\text{low\_ready}[v]$.

Let us now describe in more detail the distinctive rules of the $\text{dsf-}\alpha\text{DFS}()$ algorithm. Let $v \in V$, then:

(dsf-visit-1) Whenever the visiting subprocedure, $\text{dsf-}\alpha\text{DFS-visit}(v, A)$ (Proc. 1), makes a recursive call on some ingoing neighbour $u \in N^\text{in}_A(v) \cup rSt[v]$ (see lines 6 and 19 of Proc. 1), soon after that, it is executed $D.\text{Union}(u, v)$. Doing so, as soon as the recursive call on $u$ returns, the disjoint set tree of the child $u$ is merged with that of its parent $v$; thus, parent-child ordering relations are preserved. This allows for fast lookup of the subtrees’ roots (i.e., the LCAs) that are needed in the (dsf-visit-2) rule coming next.

(dsf-visit-2) Suppose that $\text{dsf-}\alpha\text{DFS-visit}(v, A)$ is currently visiting some $v \in V$, and that it comes to consider some in-neighbour $u \in N^\text{in}_A(v) \cap V_\emptyset$ (at line 3 and 7). Then, assume at line 8, $\text{low\_ready}$ is updated as follows:

$$\text{low\_ready}[u] \leftarrow \min(\text{low\_ready}[u], \text{open}[v]);$$

this aims at satisfying the $I_{\text{low}}$ invariant. Next, $cnt[u]$ is decremented (cfr at line 8 of Proc. 1).
If the condition \( cnt[u] = 0 \) is met at line 9 of \( \text{dsf-} \alpha \text{DFS-visit}(v, A) \) (Proc. 1), the following is done:
(a) It is identified the unique \( x \in N_A^{\text{out}}(u) \) s.t. \( \text{open}[x] = \text{low-ready}[u] \), and it is assigned to \( \text{low}_v \leftarrow x \);
(b) Then, we lookup for the root \( \gamma \) of the corresponding disjoint set tree: \( \gamma \leftarrow \mathcal{D}.\text{Find}([\text{low}_v]) \);
(c) If \( \text{active}[\gamma] = \text{true} \), then \( \gamma \) is pushed to the ready stack \( \text{rSt}[\gamma] \); indeed, in that case, we can prove (see Lemma 4) that the LCA of \( N_A^{\text{out}}(u) \) in \((V, A_\pi)\) does exist and that it is really \( \gamma \) (i.e., the root of \( \text{low}_v \)).

The rest of \( \text{dsf-} \alpha \text{DFS-visit()} \) is the same as Proc. 1. This ends the description of \( \text{dsf-} \alpha \text{DFS()} \).

At this point we prove that the above mentioned claim concerning \( \gamma \) and LCAs really holds.

**Lemma 4.** Suppose \( \text{dsf-} \alpha \text{DFS-visit}(v, A) \) visits some \( v \in V \) and come considering an in-neighbour \( u \in N_A^{\text{out}}(v) \) s.t. \( V \cap V_\pi \). Assume that \( u \) is still unvisited, i.e., \( \text{open}[u] = +\infty \), and that \( v \) is the last out-neighbour of \( v \) that is being visited, i.e., that \( cnt[u] = 0 \). Let \( \gamma \) be the vertex returned by \( \mathcal{D}.\text{find}([\text{low}_v]) \), i.e., the root of the disjoint set tree of \( \text{low}_v \), where \( \text{low}_v \) is the unique \( x \in V \) such that \( \text{open}[x] = \text{low-ready}[u] \).

\( \text{If } \text{active}[\gamma] = \text{true } \) holds at that time, then the LCA of \( N_A^{\text{out}}(u) \) in \((V, A_\pi)\) is really \( \gamma \).

**Proof.** Notice that \((V, A_\pi)\) still grows as a forest during the execution of \( \text{dsf-} \alpha \text{DFS()} \). Indeed, if a new arc \((u, v)\) is added to \( A_\pi \) it still holds that \( \text{open}[u] = +\infty \) and \( \text{open}[v] < +\infty \); no cycle can be formed. Thus, assuming \( \alpha \text{DFS-visit}(v, A) \) is invoked for some \( v \in V \), we can consider the unique maximal tree \( T_v \) in \((V, A_\pi)\) containing \( v \) and comprising only non-white vertices – i.e., constructed until the time of that particular invocation. Let \( p_v \) be the path in \( T_v \) going from \( v \) up to the root \( r \) of \( T_v \). By properties \((\text{dsf-} \text{visit-1}, \text{dsf-} \text{visit-2})\) and by the definition of \( \text{low}_v \), and since \( \gamma = \mathcal{D}.\text{find}([\text{low}_v]) \) and \( \gamma \) is active by hypothesis, then \( \gamma \) lies on \( p_v \). Thus, \( \gamma \) must be the LCA of \( \text{low}_v \) and \( v \) in \( T_v \) (possibly \( \gamma = \text{low}_v \)). We argue that \( N_A^{\text{out}}(u) \subseteq T_v \), where \( T_v \) is the maximal subtree of \( T_v \) rooted at \( \gamma \). Indeed, by \((\text{dsf-} \text{visit-2})\), the \( I_{\text{low}} \) invariant holds:
\[
\text{open}([\text{low}_v]) = \min \{ \text{open}[x] \mid x \in N_A^{\text{out}}(u) \}.
\]
So, when \( \text{cnt}[u] = 0 \), and since \( \gamma \) is an ancestor of \( \text{low}_v \), then:
\[
\forall x \in N_A^{\text{out}}(u) \text{open}[\gamma] \leq \text{open}([\text{low}_v]) \leq \text{open}[x] < +\infty.
\]
Notice all vertices in \( T_v \) which are not descendants of \( \gamma \) still have a smaller opening timestamp than \( \gamma \) (i.e., they were all visited before \( \gamma \)), and all those which are proper descendants of \( \gamma \) have a greater opening timestamp than \( \gamma \). All these combined, it must be \( N_A^{\text{out}}(u) \subseteq T_v \). So, \( \gamma \) is a common ancestor of all out-neighbours of \( u \) in \( T_v \); but \( \gamma \) is also the LCA of \( \{\text{low}_v, v\} \subseteq N_A^{\text{out}}(u) \), this means that \( \gamma \) is the LCA of all \( N_A^{\text{out}}(u) \) in \( T_v \).  

By Lemma 4, Proposition 5 holds even for \( \text{dsf-} \alpha \text{DFS()} \), proving its correctness.

Concerning time complexity, by relying on technical results offered in [13], \( \text{dsf-} \alpha \text{DFS()} \) can be implemented so to run in linear-time on a RAM machine. Concretely, [13] showed that the incremental-tree set-union problem can be solved in linear-time on RAMs. The disjoint-sets union-tree \( T \) of \( \mathcal{D} \) is revealed one vertex at a time by attaching new singleton vertices to \( T \) incrementally and in interleaving with the \( \mathcal{D}.\text{Find}() \) operations (that can possibly be performed on those vertices that have already been revealed previously). The vertices \( u \) that are incrementally revealed and attached must be new singletons that were never attached before (i.e., there is only one underlying union-tree \( T \), that keeps growing, and many singleton vertices attached incrementally).

A moment’s reflection reveals that the incremental-tree set-union problem does encompass the way in which \( \text{dsf-} \alpha \text{DFS()} \) grows the union-tree. Indeed recall that our vertices are always attached incrementally to the union-tree during the backtracking and, thus, in a post-ordering. Concretely, recall that \( \text{dsf-} \alpha \text{DFS()} \) performs a depth-first search of the union-tree, and observe that once the docking points of the circled vertices have been decided and the post-order visiting of the vertices has been fixed, then one can also forget about the fact that the vertices belong to two players and reason about the union-tree downstream of that, as if it were a traditional dfs-tree of uncolored vertices. By this we mean that the distinction of the vertices into color classes, squares and circles, only affects the particular post-ordering that is being chosen (i.e., the particular order in which the \( \mathcal{D}.\text{Union}() \) operations are performed) but not the underlying fundamental graph structure. So [13] applies. Notice that we would need just a rather
special case of the incremental-tree set-union problem, i.e., the one in which the \texttt{D.Union()} operations are always done in a post-ordering simultaneously with the \texttt{dfs} backtracking. The following holds.

\textbf{Theorem 1.} Given an input \textit{α}-\textit{graph} \( \mathcal{A} \) on vertex set \( V \) and arc set \( A \), \texttt{dfα-DFS(\( \mathcal{A} \))} halts in \( \Theta(|V|+|A|) \) linear-time on a RAM machine, provided that the \texttt{dfs} data structure \( \mathcal{D} \) is implemented as proposed in [13].

If the \texttt{dfs} data structure \( \mathcal{D} \) is implemented more traditionally as proposed in [22], i.e., with ranked-unions and path-compressions, then \texttt{dfα-DFS()} runs in Ackermann-linear-time even on a pointer machine. As one would expect, due to it’s simplicity, this variant would also perform well in practice.

\textbf{Theorem 2.} Given an input \textit{α}-\textit{graph} \( \mathcal{A} \) on vertex set \( V \) and arc set \( A \), \texttt{dfα-DFS(\( \mathcal{A} \))} halts in \( O(|V|+|A||\alpha(|A|,|V|)) \) Ackermann-linear-time on a pointer machine, provided that the \texttt{dfs} data structure \( \mathcal{D} \) is implemented with ranked-unions and path-compressions as in [22].

We leave open the question of whether \texttt{dfα-DFS()} can be implemented so that to run in \( \Theta(|V|+|A|) \) linear-time on pointer machines, see Section 5 for further discussion.

\section{Linear-Time Algorithm for Safe-Alternating SCCs}

In order to offer a linear-time safe-\( \alpha \)-SCCs decomposition algorithm some more technical machinery is still needed, the catalyst being Definition 12 below.

It is shown that the problem of computing safe-\( \alpha \)-SCCs of a given \textit{α}-\textit{graph} \( \mathcal{A} \) can be tackled by finding the roots of the components’ subtrees in the \textit{α}-jungle \( \mathcal{J}_\mathcal{A} \), this is reminiscent to what happens in Tarjan’s algorithm for the classical problem of decomposing a directed graph into SCCs.

So we have identified an efficient procedure to decide whether a vertex is the root of a safe-\( \alpha \)-SCC subtree in \( \mathcal{J}_\mathcal{A} \). It is based on a \textit{lowlink} indexing, actually gamifying the \textit{lowlink} calculation performed in [21].

\textbf{Definition 12.} \textit{Let} \( \mathcal{J} \) \textit{be an \textit{α}-jungle constructed over an \textit{α}-\textit{graph} \( \mathcal{A} \) on vertex set \( V \). Let the vertices be timestamped by open[\( \cdot \)]: \( V \rightarrow \mathbb{N} \), and let \( \{\mathcal{P}_i\}_{i=1}^k \) \textit{be the \textit{α}-palms of} \( \mathcal{J} \) \textit{each having vertex set} \( V_i \) \textit{and arc set} \( A_i = A_{i\pi} \cup A_{i\text{ if}} \cup A_{i\text{ cf}} \cup A_{i\text{ s}} \).

\textit{Define} \( \alpha\text{lowlink}_\mathcal{J} : V \rightarrow \mathbb{N} \) \textit{as follows}: for \( v \in V \):

\[ \alpha\text{lowlink}_\mathcal{J}(v) = \min \{ \text{open}[v], \text{open}[u] \mid u \in V \setminus \{v\} \text{ and } \exists \bar{v} \in [k] \text{ such that the following two hold:} \]

\begin{enumerate}
  \item[(all-1)] \( \exists_{i \geq 1} [v_{i_1}, \ldots, v_{i_{t-1}}, (v_{i_t} = v)] \in (V_i)_\gamma \) such that:
    \begin{enumerate}
      \item[(a)] \( (u, v_{i_t}) \in A_{i\text{ if}} \cup A_{i\text{ ic}} \);
      \item[(b)] \( \text{if } t \geq 2, \forall j \in \{1, \ldots, t-1\} \text{ it holds } (v_{i_j}, v_{i_{j+1}}) \in A_{i\pi} \).
    \end{enumerate}
  \end{enumerate}

\begin{enumerate}
  \item[(all-2)] \( \exists \gamma \in V_i \) \text{ such that:} \]
    \begin{enumerate}
      \item[(a)] \( \gamma \text{ is a common ancestor of } u \text{ and } v \text{ in } (V_i, A_{i\text{ treed}}) \);
      \item[(b)] \( \gamma \text{ and } u \text{ are in the same safe-}\alpha\text{-SCC of } \mathcal{A} \text{, i.e., } \gamma \in C_u \).
    \end{enumerate}
\end{enumerate}

\textit{(where, for any } u \in V, C_u \text{ denotes the unique safe-}\alpha\text{-SCC of } \mathcal{A} \text{ which includes vertex } u) \)

However, in order to proceed on this route, at this point we must overcome some obstructions. Unfortunately it’s not generally true that, if \( C \subseteq V \) is a safe-\( \alpha \)-SCC of an \textit{α}-\textit{graph} \( \mathcal{A} \), then, \( C \) induces a subtree \( T_C \) in \( \mathcal{J}_\mathcal{A} \) – if \( \mathcal{J}_\mathcal{A} \) is the \textit{α}-\textit{jungle} constructed during an \texttt{αDFS()} as defined in Section 3. And even when it’s true, say by chance, still it is not generally true that a vertex \( v \) of \( \mathcal{A} \) is the root of some safe-\( \alpha \)-SCC if and only if \( \alpha\text{lowlink}_\mathcal{J}(v) = \text{open}[v] \) as it was in Tarjan’s SCCs algorithm.

Still, for all this to happen, we claim that a conceptually simple (but technically non-trivial) modification to the \texttt{αDFS()} (and, thus, to the structure of the \textit{α}-\textit{jungle}) can be introduced. To better illustrate the issue, let us first consider the following Example 2 and Example 3.
**Example 2.** Consider the α-graph $A_1 = (V, A, \langle V\Box, V\# \rangle)$ shown in Fig. 6a where $V = V\Box \cup V\#$ and $V\Box = \{a, b, c, d, e, f, h\}$ and $V\# = \{g\}$, where $V = V\Box \cup V\#$ and $A = \{(a, h), (b, a), (c, a), (d, b), (e, b), (f, b), (g, d), (g, e), (g, f), (h, c)\}$.

Fig. 6b shows the α-jungle $J_{A_1}$ tracing the execution of αDFS() on input $A_1$. Timestamps and α-lowlinks are shown above each vertex (denoted: $\langle \text{open}[v]\rangle | \langle \text{close}[v]\rangle | \langle \text{αlowlink}[v]\rangle$, for $v \in V$). Notice $(g, b)$ is an arc in $J_{A_1}$ but not in $A_1$. Concerning the safe-αSCCs of $A_1$, a moment’s reflection reveals that they are $C_1 = \{a, h, c\}$ and all of the remaining vertices are singleton safe-αSCCs. Notice that $b$ is always α-reachable from $C_1$, but Player $\bigcirc$ decides how to reach it by controlling $g$.

The main issue here is that $C_1$ doesn’t induce a subtree in $J_{A_1}$ because $(a, h)$ is a frond, $(h, c)$ is a cross arc, and $g \notin C_1$ act as a vertex in the middle between $h$ and $a$. The reason being that $g$ joined the α-jungle $J_{A_1}$ by attaching to parent $b$ (which is fine for deciding just safe-αreachability relations but it’s not for identifying safe-αSCCs).
Example 3. Consider the α-graph $A_2 = (V, A, \{V, V_\square\})$ of Fig. 7a: $V_\square = \{a, b, c, d, e, g, h\}$ and $V_\circ = \{f\}$, where $V = V_\square \cup V_\circ$ and $A = \{(a, h), (b, a), (c, a), (c, e), (d, a), (f, b), (f, e), (f, d), (g, f), (g, h), (h, g)\}$.

Fig. 7b shows the α-jungle $J_{A_2}$ tracing the execution of α-DFS( ) on input $A_2$. Timestamps and alowlinks are shown above each vertex (denoted: (open[v])|(close[v])|(alowlink[v])), for $v \in V$). Notice that the arc $(f, a)$ belongs to $J_{A_2}$ but not to $A_2$. Concerning the safe-αSCCs of $A_2$, a moment’s reflection reveals that they are $C_1 = \{c, e\}$, $C_2 = \{g, h\}$ and all of the remaining vertices are singleton safe-αSCCs. Notice that luckily enough both $C_1$ and $C_2$ induce a subtree in $J_{A_2}$. Notice c is the root of $C_1$ and $g$ is that of $C_2$.

But alowlink$J_{A}(g) = 1 \neq 11 = \text{open}[g]$, so g can’t be recognized as a root simply by testing the alowlink. The issue is still that f joined $J_{A_2}$ by attaching to parent a.

A revision of the α-DFS( ) is next provided in order to decompose a graph into safe-αSCC. Based on dfαDFS( ) (Algo. 1), but still, with three additional and distinctive rules for indentifying the components:

(r1) All vertices that have already been visited during the search, but whose safe-αSCC has not been identified yet, are stored on an auxiliary stack named cSt (i.e., the component stack);

(r2) cSt shrinks back when the condition alowlink(v) = open[v] is met at the end of the visiting subprocedure (see Propositions 6 and 9 below for correctness), at that point a brand new safe-αSCC $C$ is identified and detached.

(r3) The $V_\circ$-rule that allows circled vertices to join $J_A$ is revised by restriction. Now a circled vertex $u \in rSt[v]$ joins $J_A$ as a child of $v$ if and only if all of its out-neighbours are still found on the component stack cSt; otherwise, u is discarded.

Remark. The safe-αSCC algorithm doesn’t need to build the α-jungle $J_A$ explicitly (i.e., in principle there might be no real need to store it in memory; still, an α-jungle is defined implicitly just by following the trace of vertices that are visited and backtracked during the search. As it will be convenient to consider the α-jungle $J_A$ during the correctness proof, we shall continue refer to it anyways.

More details follow. The main procedure is now renamed safe-αSCC( ) (Algo. 2). Given an α-graph $\mathcal{A}$ in input, it aims at identifying and printing out all the safe-αSCC $C_1, \ldots, C_k$ of $\mathcal{A}$ without repetitions.

A subprocedure named safe-αSCC-visit( ) (Proc. 2) is also employed for visiting the vertices.

safe-αSCC( ) goes like dfαDFS( ), the major distinction being that now there is also an addional component stack cSt (which is initialized empty) and an additional flag vector on_stack : $V \to \{\text{true}, \text{false}\}$ (where all flags are initially false).
safe-αSCC-visit(v, $\mathcal{A}$) (Proc. 2) goes like dfαDFS-visit( ), but now there are some new features for computing the alowlinks and for keeping track of the components.

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**Algorithm 2: safe-αSCC**

**Procedure safe-αSCC(A)**

**input**: An α-graph $\mathcal{A} = (V, A, \{V_\square, V_\circ\})$.

**output**: The safe-αSCC of $\mathcal{A}$.

**foreach** $u \in V$ do

1. open[u] ← $+\infty$;
2. alowlink[u] ← $+\infty$;
3. on_stack[u] ← false;
4. $D$, make safe(u);
5. rSt[u] ← ∅;
6. if $u \in V_\circ$ then

7. if open[u] = $+\infty$ then

8. foreach $u \in V_\circ$ do

9. if open[u] = $+\infty$ then

10. time ← 1; cSt ← ∅;
11. foreach $u \in V_\circ$ do

12. if open[u] = $+\infty$ then

13. if open[u] = $+\infty$ then

14. foreach $u \in V_\circ$ do

15. if open[u] = $+\infty$ then

16. if open[u] = $+\infty$ then

17. if open[u] = $+\infty$ then

---

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The idea for computing the lowlinks being that to keep an eye just on the indices coming from active frond-arcs and cross-arcs, i.e., to pick the minimum alowlink that can be found in the following neighbours of the currently visited v:

\[ N^\text{in}_A(cSt)(v) = \{ u \in N^\text{in}_A(v) \mid u \in cSt \text{ when line } 21 \text{ of } \text{safe-}a\text{SCC-visit}(v,A) \text{ (Proc. 2) is executed} \}, \]

or from the recursive children of the currently visited vertex v, i.e., picking the minimum:

\[ \min \{ \text{alowlink}(c) \mid c \text{ is a child of } v \text{ in } J_A \}. \]

In order to identify the components, \( \text{safe-}a\text{SCC-visit}(\cdot, \cdot) \) tests whether \( \text{alowlink}[v] = \text{open}[v] \) (this is reminiscent to Tarjan’s algorithm for SCCs [21]). If that’s the case a brand new safe-\( a \)SCC \( C \) is identified; thus some vertices \( u \) will be repeatedly removed from \( cSt \) and added to \( C \), until \( u = v \) (\( v \) comprised).

However, in order for this test to be sound and complete, we have to overcome the issues observed before in Examples 2 and 3. As mentioned in (r3) above, the proposed solution is conceptually simple. Soon after that the whole in-neighbourhood of any \( v \in V \) has been visited by \( \text{safe-}a\text{SCC-visit}(v,A) \), a circled vertex \( u \in rSt[v] \) is visited with a recursive call (and thus attached to \( J_A \) as a child of \( v \)) if and only if all of its out-neighbours are still on the component stack \( cSt \) (see lines 21-26 of Proc. 2); otherwise, \( u \) is simply discarded and becomes a singleton component at the end of the search. Intuitively, this works because if some of the neighbours of \( u \) is no longer on the stack at that point, then (by reasoning
inductively) it has already been detached into another component that has been fully identified already, so it would not be possible to guarantee safe-reachability from $u \in V_0$ to the parent $\pi(u) = v$ within the safe-$\alpha$SCC of $v$ that is currently under formation. Along the lines of this intuitive observation, soundness and completeness is formally established in the proofs (see Appendix-A) of the forthcoming Propositions 6 and 9.

Remark. Notice that with (r3), the $\alpha$-jungle underlying safe-$\alpha$SCC (Algo. 2) might be different w.r.t. the $\alpha$-jungle $\mathcal{J}_A$ that would have been built by running $\alpha$DFS: like if some of the opalm-trees of $\mathcal{J}_A$ were pruned and partitioned into subtrees, where the cutting points are precisely those arcs $(u, v) \in A_\pi$ on circled vertices $u \in rSt[v] \cap V_0$ that can no longer join $\mathcal{J}_A$ because at that point $u' \notin rSt[v]$ for some $u' \in N_{A}^{-}(u)$. However, a moment’s reflection reveals that this is just a minor structural refinement of $\mathcal{J}_A$, the resulting graph structure still satisfies the fundamental properties of an $\alpha$-jungle given in Definitions 9 and 10. The only partial exception being property ($\alpha$-$\pi$-4), now there might be circled vertices $u$ that can no longer join $\mathcal{J}_A$ even if all out-neighbours belong to the same opalm-tree (cfr vertex $g$ in Example 2 and vertex $f$ in Example 3) – but this property would be still satisfied if only we imagine that, as soon as a safe-$\alpha$SCC is identified, the corresponding subtree detaches from the maximal tree to which it belongs. With this in mind the resulting graph structure is really an $\alpha$-jungle, so we will continue to denote it by $\mathcal{J}_A$ as the local context of safe-$\alpha$SCC() (Algo. 2) supersedes possible confusion.

Let us now provide some more implementation details of safe-$\alpha$SCC-visit($v,A$) (Proc. 2).

At the very beginning, the vertex $v$ which is currently being visited is pushed on top of the component stack $cSt$ and flagged on_stack[$v$] → true (see lines 3-4 of Proc. 2).

Then, whenever some in-neighbour $u \in N_{A}^{	ext{in}}(v)$ is visited, and as soon as the child recursive call safe-$\alpha$SCC-visit($u,A$) returns, the $\alpha$-lowlink is updated as follows:

\[ \text{alowlink}[v] \leftarrow \min(\text{alowlink}[v], \text{alowlink}[u]) \] 

(see lines 9 and 25 of Proc. 2)

besides executing a $D$.Union($u,v$) to update the disjoint-set forest as before in dsf-$\alpha$DFS-visit() .

Next, when exploring the in-neighbourhood $N_{A}^{	ext{in}}(v)$ aiming at visiting unexplored vertices: if an in-neighbour $u \in N_{A}^{	ext{in}}(v) \cap V_0$ is still unvisited (i.e., if open[$u$] = $+\infty$), and it happens that $\text{cnt}[u] = 0$, then $u$ is pushed to the ready stack $rSt[\gamma]$ if and only if on_stack[\gamma] = true (we now have the additional stack $cSt$ flagged by on_stack, and indeed we can use it to check whether $\gamma$ is still active): else, if $u \in N_{A}^{	ext{in}}(v)$ has been already visited (i.e., if open[$u$] $\neq +\infty$), and if on_stack[u] = true, then the $\alpha$-lowlink of $v$ is updated as follows:

\[ \text{alowlink}[v] \leftarrow \min(\text{alowlink}[v], \text{open}[u]) \] 

(see lines 19-20 of Proc. 2)

Soon after that the in-neighbourhood of $v$ has been visited (see lines 21-26 of Proc. 2), $rSt[v]$ is managed almost as it was in dsf-$\alpha$DFS-visit(); the only difference being that, as already mentioned, a circled vertex $u \in rSt[\gamma]$ is visited with a recursive call if and only if all of its out-neighbours are still on $cSt$. Of course when such an $u$ gets visited the disjoint-set forest is updated as usual by $D$.Union($u,v$), but now also the $\alpha$-lowlink of $v$ is updated by taking the minimum, i.e., $\text{alowlink}[v] \leftarrow \min(\text{alowlink}[v], \text{alowlink}[u])$.

This concludes the description of safe-$\alpha$SCC-visit() (Proc. 2) and that of Algorithm 2.

Let us revise Examples 2 and 3 to illustrate how safe-$\alpha$SCC (Algo. 2) runs on the ographs $A_1$ and $A_2$: the resulting $\alpha$-jungles are shown in Fig. 8a and Fig. 8b (respectively).

Concerning Example 2, Fig. 8a shows that all vertices in the safe-$\alpha$SCC $C_1 = \{a, h, c\}$ have $\alpha$-lowlink equal to open[$a$] = 1 and all other vertices are singletons. Now $C_1$ induces a subtree in the opalm-tree.

Similarly for Example 3, Fig. 8b shows that all vertices in the safe-$\alpha$SCC $C_1 = \{c, e\}$ have $\alpha$-lowlink equal to open[$c$] = 2, and all vertices in $C_2 = \{g, h\}$ have an $\alpha$-lowlink equal to open[g] = 6. All of the remaining vertices are singleton safe-$\alpha$SCCs. Again both $C_1 = \{c, e\}$ and $C_2 = \{g, h\}$ induce a subtree, rooted at $c$ and $g$ respectively.

In summary, safe-$\alpha$SCC() (Algo. 2) enjoys the following major properties (as proved in Appendix A).

**Proposition 6.** Assume safe-$\alpha$SCC() (Algo. 2) runs on a given input ograph $A$, and let $\mathcal{J}_A$ be the corresponding $\alpha$-jungle, then the $\alpha$-lowlink-$\mathcal{J}_A$ indexing is correctly computed as given in Definition 12.
Proposition 7. Let $\mathcal{J}_A$ be an αjungle constructed when safe-αSCC() (Algo. 2) runs on the αgraph $A$. If $C$ is a safe-αSCC of $A$, then $C$ induces a subtree in the forest of $\mathcal{J}_A$.

Since any safe-αSCC() $C$ induces a subtree in $\mathcal{J}_A$, we can identify the roots of the subtrees.

Proposition 8. Let $\mathcal{J}_A$ be an αjungle constructed when safe-αSCC() (Algo. 2) runs on the αgraph $A$. Let $\text{open}[\cdot] : V \to \mathbb{N}$ be the corresponding timestamp, and let $\text{αlowlink}_A : V \to \mathbb{N}$ be as in Definition 12. Any vertex $v \in V$ is the root of some safe-αSCC of $A$ if and only if $\text{αlowlink}_A(v) = \text{open}[v]$.

As a consequence safe-αSCC() (Algo. 2) is correct, the inductive proof is sketched in Appendix A. Concerning time complexity, Theorems 1 and 2 already imply that it is linear on RAMs and, at least, Ackermann-linear on pointer machines.

5 Related and Future Works

Firstly let us discuss about possible lines of investigation concerning the time complexity of dsf-αDFS() and safe-αSCC() on pointer machines. As already mentioned in Section 3.3, our current upper-bound is Ackermann-linear-time coming from the dsf union-find data structure. We observe that our proposed usage of the dsf union-find data structure falls within a rather special case of the incremental-tree set-union problem studied in [13], i.e., the special case in which the union operations are always done in a post-ordering, simultaneously with the backtracking of the depth-first search. This might be amenable e.g., to the techniques developed in [2], where a linear-time algorithm for the off-line LCA problem was offered. That algorithm does not seem to extend to the incremental-tree set-union problem in its full generality (i.e., where the union operations can arrive incrementally in any order), still, here one should investigate about the very special case in which all of the union operations arrive in a post-ordering.

In the neighbourhood of possibly related lines appearing in the literature, of course we find the αSCCs and the MECS [5] decompositions which may offer interesting directions of investigation for future works.

Let $A$ be an αgraph on vertex set $V$ and arc set $A$. As already mentioned, if $U \subseteq V$ is a safe-αSCC, then $U$ is an osc set, thus $U$ is included in some αSCC (though it may not correspond to the whole αSCC as it may lack maximality). On the other hand, if $U \subseteq V$ is an αSCC, then any two vertices $u, v \in U$ are strongly-connected in the original input directed graph $G_A$, thus $U$ is included in some SCC of $G_A$ (but it may lack maximality as well). Notice that the converse inclusions do not hold generally. Also
recall that the αSCC decomposition can be found in time $O(|V||A|)$ by computing $\Box$-attractors. One natural question at this point is whether our proposed theory can possibly help improving the latter time complexity upper bound. We leave open that question, and at the same time we observe what follows. As a simple variation of our safe-αSCC() algorithm, suppose just to drop the r3) rule: i.e., assume a circled vertex $u \in rSt[v]$ joins $J_A$ as a child of $v$ anyway if the LCA $\gamma$ exists, as it was for $\alpha$DFS(), without checking whether all of $u$ out-neighbours are still found on the component stack cSt. The conjecture may be this could be fine to find the $\alpha$SCC decomposition, unfortunately it’s not difficult to provide counterexamples that this is not enough on its own: it still seems necessary to run what is basically an attractor computation from a big fraction of the vertices, thus falling back in a $O(|V||A|)$ running time.

Figure 9: The four notions of strongly-connectedness ordered by set inclusion, each one showing the time complexity of the corresponding best currently known decomposition algorithms for $m = |A|, n = |V|$. Let us now consider the MECs [5] decomposition as a possibly related line of interest for future works. Given a directed graph $G = (V, A)$ with a finite set $V$ of vertices, a set $A \subseteq V \times V$ of directed arcs and a partition $(V_{\Box}, V_{\bigcirc})$ of the vertex set $V$, an end-component $U \subseteq V$ is a set of vertices such that: (i) the graph $(U, A \cap (U \times U))$ is strongly-connected; (ii) for all $u \in U \cap V_{\bigcirc}$ and all $(u, v) \in A$ we have $v \in U$; and (iii) either $|U| \geq 2$, or $U = \{v\}$ and there is a self-loop at $v$ (i.e., $(v, v) \in A$). Observe that if $U \subseteq V$ is a safe-αSCC and $|U| \geq 2$, then $U$ is an end-component according to the above definition. Of course the converse is not generally true, since, $U$ may well be an end-component (i.e., assume for every circled vertex $v \in U \cap V_{\bigcirc}$ and every arc $(u, v) \in A$ going out of $u$, it holds $v \in U$) and strongly-connected as a directed graph, but Player $\bigcirc$ may possibly prevent Player $\Box$ to visit one particular vertex from some moment in time onwards. On the other hand, every maximal end-component is included in some SCC of $G_A$ (again, it may lack maximality and the converse doesn’t hold generally). Finally, a moment’s reflection reveals that αSCCs and MECs are generally uncomparable (in the sense that no one implies the other). The relationship between the four notions is depicted in Fig. 9, as well as the best currently known time complexities.

We leave open whether our proposed theory can possibly help speeding up the efficient algorithms for MECs as devised in [5], at least by offering a novel approach for kernelization (pre-processing).

6 Conclusion

We expect that the proposed theory and the corresponding linear-time decomposition algorithm could possibly pave the way for speeding up computations in other problems concerning e.g., formal verification and infinite pebble games on graphs.

Future works will likely investigate further on this way.
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A Proof that Safe-$\alpha$SCC (Algo. 2) is Correct

For the sake of the argument let us recall that, during an execution of safe-$\alpha$SCC() (Algo. 2), any vertex $v$ is said active when safe-$\alpha$SCC-visit($v, A$) (Proc. 2) assigns open[$v$], remaining so until $v$ is finally backtracked (i.e., recursive calls included); let us say that $v$ is deactivated (i.e., blackened) as soon as the search backtracks $v$, i.e., when safe-$\alpha$SCC-visit($v, A$) assigns the closing timestamp close[$v$].

In order to prove Proposition 9, which basically asserts that safe-$\alpha$SCC (Algo. 2) is correct, let us dive into the following two technical lemmata.

**Lemma 5.** Assume that safe-$\alpha$SCC() (Algo. 2) runs on a given a graph $A$ on vertex set $V$. Let $u \in V$ be a descendant of $\gamma \in V$ in the forest of $J_{A}$ such that $u$ is still on the component stack $cSt(\gamma)$ when safe-$\alpha$SCC-visit() backtracks from $u$ up to $\gamma$ (i.e., let us say, at line 27 of safe-$\alpha$SCC-visit($\gamma, A$)) (Proc. 2).

Then, $\gamma$ lies in the same safe-$\alpha$SCC of $u$, i.e., $\gamma \in C_{u}$.

**Proof.** The proof goes by induction on the order in which vertices are deactivated during the execution of safe-$\alpha$SCC($A$), let it be $(v_{1}, \ldots, v_{i}, v_{i+1}, \ldots, v_{|V|})$. Also, w.l.o.g., let $u$ be a proper descendant of $\gamma \in V$.

**Base Case:** $u = v_{1}$. Notice the first deactivated vertex $u = v_{1}$ must be a leaf in the forest of $J_{A}$. Since $u$ is still on the component stack $cSt(\gamma)$ when safe-$\alpha$SCC-visit() backtracks up to $\gamma$, then,

\[ \alpha_{\text{lowlink}}[u] < \text{open}[u]; \quad (*) \]

actually, every ancestor of $u$ that is also a proper descendant of $\gamma$ must be still on $cSt(\gamma)$ (together with $u$) when safe-$\alpha$SCC-visit() backtracks up to $\gamma$, so $\alpha_{\text{lowlink}}[v] < \text{open}[v]$ for every such $v$. Since $\alpha_{\text{lowlink}}[u] < \text{open}[u]$, there is one vertex $\gamma' \neq u$ such that $\text{open}[\gamma'] = \alpha_{\text{lowlink}}[u]$. Since $u$ is the first deactivated vertex in the forest of $J_{A}$, then $\gamma'$ is a proper ancestor of $u$ and thus $(\gamma', u)$ is a frond arc. Let $x$ be any ancestor of $u$ that is also a descendant of $\gamma'$ (possibly $x = u$ or $x = \gamma'$, but not both).

We claim that $x$ can’t be a circled vertex, so it must be $x \in V_{C}$. Indeed, suppose $x \in V_{C}$ for the sake of contradiction, consider any out-neighbour $y \in N_{out}^{A}(x)$ which is not the parent of $x$ in $J_{A}$ (notice $y$ exists because w.l.o.g. $|N_{out}^{A}(x)| \geq 2$ if $x \in V_{C}$), so $(x, y)$ is just a stalk arc. By the $V_{C}$-rule, $y$ must have been deactivated before $x$: this is absurd as $u$ is the first deactivated vertex of $J_{A}$ and $u \neq y$. So $x \in V_{C}$.

Since any ancestor of $u$ that is also a descendant of $\gamma'$ lies in $V_{C}$, and since $(\gamma', u)$ is a frond arc, then $u$ and $\gamma'$ together with all the ancestors of $u$ that are also descendants of $\gamma'$, they form a safe-$\alpha$SCC set, so they all lie within the same safe-$\alpha$SCC of $A$. There are two cases now to analyze.

If $\text{open}[\gamma'] \leq \text{open}[\gamma]$, and since $u$ is the first deactivated vertex, then $\gamma$ must be an ancestor of $u$ and descendant of $\gamma'$, so $\gamma \in C_{u}$ and we are done.

Otherwise, if $\text{open}[\gamma'] > \text{open}[\gamma]$, and since $u$ is the first deactivated vertex, then $\gamma'$ must be a proper ancestor of $u$ and also a proper descendant of $\gamma$. Thus, as mentioned before at the beginning, also $\gamma'$ must be still on $cSt(\gamma)$ when safe-$\alpha$SCC-visit() backtracks up to $\gamma$. Therefore,

\[ \alpha_{\text{lowlink}}[\gamma'] < \text{open}[\gamma']. \]

At this point a moment’s reflection reveals that now we can reiterate the same argument that we have just applied on $u$ (cfr. inequality $(*)$ above), but this time to $\gamma'$. Even though $\gamma'$ is not the first deactivated
vertex, notice that all the same observations apply to \( \gamma' \) too. This happens because \( \gamma' \) is anyway an ancestor of the first deactivated vertex \( u \), it’s easy to check that this is enough to sustain the argument. After that, possibly, we may need to reiterate the argument along subsequent proper ancestors \( \hat{v} \) of \( \gamma' \) too, but at some point we must reach \( \gamma \), because at each iteration the corresponding \( \text{open}[\hat{v}] \) decreases by at least one unit. So, also in this case, \( \gamma \in C_u \).

**Inductive Step:** \( u = v_i \) for some \( i > 1 \). In this case, \( \text{allowlink}[v_i] \) can be assigned either at line 2, 9, 20, 25 of \( \text{safe-\alpha SCC-visit}(v_i, A) \) (Proc. 2). Since \( u \) is still on the component stack \( \text{cSt}(\gamma) \) when \( \text{safe-\alpha SCC-visit()} \) backtracks from \( u \) up to \( \gamma \), then,

\[
\text{allowlink}[u] < \text{open}[u];
\]

also, every ancestor of \( u \) that is a proper descendant of \( \gamma \) must be still on \( \text{cSt}(\gamma) \) (together with \( u \)) when \( \text{safe-\alpha SCC-visit()} \) backtracks from \( u \) up to \( \gamma \), so \( \text{allowlink}[\hat{v}] < \text{open}[\hat{v}] \) for every such \( \hat{v} \). Since \( \text{allowlink}[u] < \text{open}[u] \), there is one vertex \( u' \neq u \) such that \( \text{open}[u'] = \text{allowlink}[u] \). All in, \( \text{open}[u'] < \text{open}[u] \). A moment’s reflection reveals that there must be a descendant \( x \) of \( u \) (possibly, \( x = u \)) such that \( \text{allowlink}[x] = \text{open}[u' \gamma] \) and \( (u', x) \) is either a frond or a cross arc in \( J_A \). So, \( u' \) was still on \( \text{cSt} \) when \( \text{safe-\alpha SCC-visit()} \) backtracked on \( x \). Then let \( \gamma' \) be the LCA of \( u', x \) in the forest of \( J_A \) (possibly, \( \gamma' = u' \), but \( \gamma' \neq u' \)). Also, since \( u' \) was still on \( \text{cSt} \) when \( \text{safe-\alpha SCC-visit()} \) backtracked on \( x \), and since \( x \) is a descendant of \( u \), then the fact that \( \text{open}[u'] < \text{open}[u] \) implies that \( u' \) is still on \( \text{cSt}(\gamma') \) when \( \text{safe-\alpha SCC-visit()} \) backtracks up to \( \gamma' \) too.

We now claim that \( \gamma' \in C_{u'} \). If \( \gamma' = u' \), this is trivial. So, assume w.l.o.g. \( \gamma' \neq u' \). Then, since \( u' \) was still on \( \text{cSt} \) when \( \text{safe-\alpha SCC-visit()} \) backtracked on \( x \), and \( x \) is a descendant of \( u \), then \( u' \) must have been deactivated before \( u \). So the induction hypothesis applies to \( u' \) and \( \gamma' \), thus \( \gamma' \in C_{u'} \).

We now claim that \( \gamma' \in C_u \). Recall that \( \gamma' \in C_{u'} \) and \( (u', x) \) is either a frond or a cross arc in \( J_A \), and \( x \) is a descendant of \( u \), which is a descendant of \( \gamma' \). Let \( P \) be the set of all ancestors of \( x \) which are also descendants of \( \gamma' \). Proposition 3 implies that every vertex in \( P \) is \( T_{P_z}-\text{safe-oreachable from} \), with a strategy that simply goes up along the palm-tree \( P_z \) in which \( x \) resides, i.e., a strategy that goes from any \( c \in P \) to its parent \( \pi(c) \). To conclude the proof of \( \gamma' \in C_u \), it is now sufficient to show that: if \( c \in P \cap V_\gamma \), then for every out-neighbour \( c' \in N^\text{out}_A(c) \) such that \( c' \neq \pi(c) \) (i.e., \( (c, c') \) is just a stalk arc), it holds \( \pi(c) \in C_{c'} \). For this, observe that, by the \( V_\gamma \)-rule, \( c' \) must have been already deactivated when \( c \) joined \( J_A \). Thus, since \( x \) is a child of \( c, c' \) must have been deactivated before \( x \) was. Thus, since \( x \) is a child of \( u \), then \( c' \) must have been deactivated before \( u \) was. Therefore, the induction hypothesis applies to \( c' \) with parent \( \pi(c) \), i.e., \( \pi(c) \in C_{c'} \) as claimed. All in, \( C_u = C_p \) for every \( p \in P \). So, \( \gamma' \in C_u \).

In order to conclude the proof of the inductive step, there are two cases now to analyze.

If \( \text{open}[\gamma'] \leq \text{open}[\gamma] \), since \( \gamma, \gamma' \) are both ancestors of \( u \), then \( \gamma' \) must be an ancestor of \( \gamma \) (possibly, \( \gamma' = \gamma \)). In this case \( \gamma \in P \), and the argument above already proves \( \gamma \in C_u \).

Otherwise, if \( \text{open}[\gamma'] > \text{open}[\gamma] \), and since \( \gamma, \gamma' \) are both ancestors of \( u \), then \( \gamma \) must be a proper ancestor of \( \gamma' \). So, as mentioned before (at the beginning of the inductive step), also \( \gamma' \) must be still on \( \text{cSt}(\gamma) \) when \( \text{safe-\alpha SCC-visit()} \) backtracks up to \( \gamma \). Therefore,

\[
\text{allowlink}[\gamma'] < \text{open}[\gamma'] \text{.}
\]

At this point a moment’s reflection reveals that now we can reiterate the same argument that we have just applied on \( u \) (cfr. inequality \((*)\) above), but this time on \( \gamma' \). Even though \( \gamma' \) is deactivated after \( u \), notice that all the same observations apply to \( \gamma' \) too. This happens because \( \gamma' \) is anyway an ancestor of \( u \) in \( J_A \), and a moment’s reflection reveals that this is enough to sustain the argument even if the induction hypothesis still holds only with respect to those vertices that are deactivated before \( u \). Indeed, since \( \text{allowlink}[\gamma'] < \text{open}[\gamma'] \), there is still one vertex \( u'' \neq \gamma' \) such that \( \text{open}[u''] = \text{allowlink}[\gamma'] \). Since \( \text{open}[u''] < \text{open}[\gamma'] < \text{open}[u] \), so either \( u'' \) is an ancestor of \( \gamma' \) or must have been deactivated before \( u \) and thus the argument can proceed as before.

After that, possibly, we may need to reiterate the argument along subsequent proper ancestors \( \hat{v} \) of \( \gamma' \) too, but at some point we must reach \( \gamma \), because at each iteration of the argument the corresponding \( \text{open}[\hat{v}] \) decreases by at least one unit. So, also in this case, \( \gamma \in C_u \).
Lemma 6. Assume that safe-\(\alpha\)-SCC() (Algo. 2) runs on a given graph \(A\) on vertex set \(V\). Let \(u \in V\) be a descendant of \(\gamma \in V\) in the forest of \(J_A\) lying in the same safe-\(\alpha\)-SCC of \(u\), i.e., such that \(\gamma \in C_u\). Then, \(u\) is still on the component stack \(cSt(\gamma)\) when safe-\(\alpha\)-SCC-visit() backtracks from \(u\) up to \(\gamma\).

Proof. Firstly, assume w.l.o.g. that \(open[\gamma]\) is the smallest possible index in \(C_u\), i.e.,

\[ \gamma = \arg\min_{v \in C_u} open[v]. \]

Indeed, thanks to the structural connectivity properties of \(J_A\), cfr Definitions 9 and 10, particularly, properties \((\alpha pt\text{-}2, \alpha pt\text{-}4 and \alpha_{jn}\text{-}3)\), if \(\gamma \in C_u\) then \(\gamma\) belongs to the same palm-tree in which \(u\) resides, and any other possible ancestor \(\gamma' \neq \gamma\) of \(u\) such that \(open[\gamma']\) is minimum. So proving the thesis w.r.t. the smallest \(\gamma\) subsumes proving it for any other \(\gamma'\) satisfying the hypothesis.

The proof proceeds by induction on \(open[u]\), for \(open[u] \geq open[\gamma]\).

In the Base Case, \(u = \gamma\), so the thesis trivially holds.

Inductive Step: Since \(\gamma \in C_u\) because of the structural connectivity properties of \(J_A\), cfr Definitions 9 and 10, particularly, properties \((\alpha pt\text{-}2, \alpha pt\text{-}4 and \alpha_{jn}\text{-}3)\), a moment’s reflection reveals that along any of those paths that start at \(\gamma\) and reach \(u\) without ever leaving \(C_u\) (i.e., any of those paths thanks to which \(u\) is \(C_{u'}\)-safe-reachable from \(\gamma\)), at some point there must be a pair of vertices \(u'\), \(x \in C_u\) such that: \(open[u'] < open[u]\), \(x\) is a descendant of \(u\) in the forest of \(J_A\) (possibly \(x = u\), but \(x \neq u'\)), and \((u', x)\) is either a frond or a cross arc.

Now, notice that since \(open[\gamma]\) is minimum, then \(u'\) is still a descendant of \(\gamma\) (possibly, \(u' = \gamma\)). Also notice that, since \(u' \in C_u\), then \(C_{u'} = C_u\), so \(\gamma \in C_{u'}\).

Since \(open[u'] < open[u]\) and \(u'\) is a descendant of \(\gamma\) such that \(\gamma \in C_{u'}\), by induction hypothesis applied to \(u'\), then \(u'\) is still on the component stack \(cSt(\gamma)\) when safe-\(\alpha\)-SCC-visit() backtracks from \(u\) up to \(\gamma\).

Thus, since \(x\) is a descendant of \(u\) and \(open[u'] < open[u]\), \(u'\) is already on the component stack \(cSt(\gamma)\) when safe-\(\alpha\)-SCC-visit() backtracks on \(x\). Therefore, by line 20 of safe-\(\alpha\)-SCC-visit(), \(a\text{slowlink}[x] \leq open[u']\). So, \(x\) stays on \(cSt\) as long as \(u'\) stays there. Then also \(u\) stays on \(cSt\) as long as \(u'\) stays there. Therefore, since \(u'\) is on \(cSt(\gamma)\) when safe-\(\alpha\)-SCC-visit() backtracks from \(u\) up to \(\gamma\), also \(u \in cSt(\gamma)\) at that time.

We are finally in the position of proving correctness.

Proposition 9. If safe-\(\alpha\)-SCC() (Algo. 2) runs on a given graph \(A\), and safe-\(\alpha\)-SCC-visit() (Proc. 2) outputs some subset of vertices \(C \subseteq V\), then \(C\) is a safe-\(\alpha\)-SCC of \(A\).

Proof. Assume that safe-\(\alpha\)-SCC-visit(\(\gamma\), \(A\)) (Proc. 2) outputs some subset of vertices \(C \subseteq V\), for some \(\gamma \in C\), when \(a\text{slowlink}[\gamma] = open[\gamma]\) holds at line 26. So, \(C = cSt(\gamma)\). By Lemma 5, then \(\gamma \in C_u\) for every \(u \in cSt(\gamma)\). So, \(cSt(\gamma) \subseteq C_u\).

Now we claim that \(C_u \subseteq cSt(\gamma)\). Pick \(\gamma' \in C_u\). It is not possible for \(\gamma'\) to be an ancestor of \(\gamma\), because by Lemma 6 it would be \(\gamma \in cSt(\gamma')\), against \(a\text{slowlink}[\gamma] = open[\gamma]\). Thus, since there are no ancestors of \(\gamma\) in \(C_u\), it is not possible for \(\gamma'\) to be incomparable with \(\gamma\) (i.e., neither an ancestor nor a descendant), because this would contradict the structural connectivity properties of \(J_A\), cfr Definitions 9 and 10, particularly, properties \((\alpha pt\text{-}2, \alpha pt\text{-}4 and \alpha_{jn}\text{-}3)\). So, \(\gamma'\) must be a descendant of \(\gamma\) in the forest of \(J_A\). Then by Lemma 6 it holds \(\gamma' \in cSt(\gamma)\). All in, \(cSt(\gamma) = C_u\).

Next, for the sake of completeness, it is shown that safe-\(\alpha\)SCC (Algo. 2) correctly computes the slowlinks as prescribed by Definition 12. This follows from Lemma 5 and Lemma 6.

Proof of Proposition 6. For the sake of the argument, let us denote \(a\text{slowlink}[]\) (i.e., with squared brackets) the indices computed by safe-\(\alpha\)SCC (Algo. 2), whereas \(a\text{slowlink}()\) denotes the indices given by Definition 12. Thus we aim at showing that for every \(v \in V\), it holds \(a\text{slowlink}[v] = a\text{slowlink}(v)\).

The proof goes by induction on the order in which vertices are deactivated during the execution of safe-\(\alpha\)SCC(\(A\)), let it be \((v_1, \ldots, v_i, \ldots, v_{|V|})\).
For every \( v \in V \), for the sake of the argument, let us define the following in-neighbourhood by considering the state of the component stack \( cSt \) when line 21 of \( \text{safe-}\alpha\text{-SCC-visit}(v,A) \) is executed:

\[
N_A^\text{in}[cSt](v) = \left\{ u \in N_A^\text{in}(v) \mid u \in cSt \text{ when line 21 of } \text{safe-}\alpha\text{-SCC-visit}(v,A) \text{ (Proc. 2) is executed} \right\}.
\]

Also for the sake of the argument let us define for every \( v \in V \):

\[
N_A^\text{in}[\text{LCA}](v) = \{ u \in N_A^\text{in}(v) \cap \overline{V} \mid \text{the LCA } \gamma \text{ of } \{u,v\} \in J_A \text{ exists and } \gamma \in C_u \}.
\]

**Base Case:** \( i = 1 \). Notice that the first deactivated vertex \( v_1 \) must be a leaf in the forest of \( J_A \). In this case, \( \text{alowlink}[v_1] \) can be assigned only at line 20 of \( \text{safe-}\alpha\text{-SCC-visit}(v_1,A) \). So, the following holds:

\[
\text{alowlink}[v_1] = \min \{ \text{open}[v_1] \} \cup \{ \text{open}[u] \mid u \in N_A^\text{in}[\text{LCA}](v_1) \}. \tag{eq. 1}
\]

Since \( v_1 \) is the first deactivated leaf,

\[
N_A^\text{in}[cSt](v_1) = \{ u \in N_A^\text{in}(v_1) \mid u \text{ is an ancestor of } v_1 \text{ in } J_A \}. \tag{eq. 2}
\]

On the other hand, since \( v_1 \) is a leaf in \( J_A \) and by Definition 12, a moment’s reflection reveals:

\[
\text{alowlink}[v_1] = \min \{ \text{open}[v_1] \} \cup \{ \text{open}[u] \mid u \in N_A^\text{in}[\text{LCA}](v_1) \}. \tag{eq. 3}
\]

Since \( v_1 \) is the first deactivated leaf and by (eq. 2), \( N_A^\text{in}[\text{LCA}](v_1) = N_A^\text{in}[cSt](v_1) \).

Therefore, by (eq. 1) and (eq. 3), \( \text{alowlink}[v_1] = \text{alowlink}[v_1] \).

This concludes the proof of the base case.

**Inductive Step:** \( i > 1 \). In this case, \( \text{alowlink}[v_i] \) can be assigned either at line 2, 9, 20, 25 of \( \text{safe-}\alpha\text{-SCC-visit}(v_i,A) \) (Proc. 2). A moment’s reflection reveals that the following holds by construction:

\[
\text{alowlink}[v_i] = \min \{ \text{open}[v_i] \} \cup \{ \text{open}[u] \mid u \in N_A^\text{in}[\text{LCA}](v_i) \} \cup \{ \text{alowlink}[u] \mid u \text{ is a child of } v_i \text{ in } J_A \}.
\]

On the other side, by Definition 12, it is not difficult to see that by reasoning on the structure and connectivity properties of \( J_A \) (Definitions 9 and 10), and by the minimality of \( \text{alowlink} \), it holds:

\[
\text{alowlink}[v_i] = \min \{ \text{open}[v_i] \} \cup \{ \text{open}[u] \mid u \in N_A^\text{in}[\text{LCA}](v_i) \}
\]

\[
\cup \{ \text{alowlink}[u] \mid u \text{ is a child of } v_i \text{ in } J_A \}.
\]

If \( u \) is a child of \( v_i \) in \( J_A \), then \( u \) is deactivated before \( v_i \). By induction hypothesis, \( \text{alowlink}[u] = \text{alowlink}[u] \) for every child \( u \) of \( v_i \) in \( J_A \) that is considered either at line 9 or 25 of \( \text{safe-}\alpha\text{-SCC-visit}(v_i,A) \).

To finish the proof, it is sufficient to show that \( N_A^\text{in}[cSt](v_i) = N_A^\text{in}[\text{LCA}](v_i) \).

- Firstly, we claim \( N_A^\text{in}[cSt](v_i) \subseteq N_A^\text{in}[\text{LCA}](v_i) \).
  Let \( u \in N_A^\text{in}[cSt](v_i) \). Then, \( u \) and \( v_i \) lie within the same tree in \( J_A \): indeed, notice that \( cSt \) is completely emptied as soon as the root of an opalm-tree is deactivated, thus the stack \( cSt \) can’t contain vertices belonging to two distinct maximal opalm-tree. So, the LCA \( \gamma \) of \( \{u,v_i\} \) in \( J_A \) exists. Since \( u \in cSt \) when \( v_i \) is being visited, \( u \in cSt(\gamma) \) when \( \text{safe-}\alpha\text{-SCC-visit}() \) backtracks to \( \gamma \).
  By Lemma 5, \( \gamma \in C_u \). So, \( u \in N_A^\text{in}[\text{LCA}](v_i) \).

- Secondly, we claim \( N_A^\text{in}[\text{LCA}](v_i) \subseteq N_A^\text{in}[cSt](v_i) \).
  Let \( u \in N_A^\text{in}[\text{LCA}](v_i) \), and let \( \gamma \in C_u \) be the LCA of \( \{u,v_i\} \) in \( J_A \). By Lemma 6, since \( \gamma \in C_u \), then \( u \) is still on the component stack \( cSt(\gamma) \) when \( \text{safe-}\alpha\text{-SCC-visit}() \) backtracks to \( \gamma \). So, \( u \in N_A^\text{in}[cSt](v_i) \).

All in, \( N_A^\text{in}[cSt](v_i) = N_A^\text{in}[\text{LCA}](v_i) \). This concludes the inductive step. \( \square \)