OPTIMAL SUBGROUPS AND APPLICATIONS TO NILPOTENT ELEMENTS

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ABSTRACT. Let $G$ be a reductive group acting on an affine variety $X$, let $x \in X$ be a point whose $G$-orbit is not closed, and let $S$ be a $G$-stable closed subvariety of $X$ which meets the closure of the $G$-orbit of $x$ but does not contain $x$. In this paper, we study G.R. Kempf’s optimal class $\Omega_G(x, S)$ of cocharacters of $G$ attached to the point $x$; in particular, we consider how this optimality transfers to subgroups of $G$.

Suppose $K$ is a $G$-completely reducible subgroup of $G$ which fixes $x$, and let $H = C_G(K)^0$. Our main result says that the $H$-orbit of $x$ is also not closed, and the optimal class $\Omega_H(x, S)$ for $H$ simply consists of the cocharacters in $\Omega_G(x, S)$ which evaluate in $H$. We apply this result in the case that $G$ acts on its Lie algebra via the adjoint representation to obtain some new information about cocharacters associated with nilpotent elements in good characteristic.

1. Introduction

Suppose $G$ is a connected reductive linear algebraic group acting on an affine variety $X$, and $x \in X$ is a point whose $G$-orbit is not closed in $X$. Let $G \cdot x$ denote the $G$-orbit of $x$, and $\overline{G \cdot x}$ denote its closure in $X$. A well known result in Geometric Invariant Theory, the Hilbert–Mumford Theorem [K, Thm. 1.4], states that there exists a cocharacter (one-parameter subgroup) $\lambda$ of $G$ which takes us from $G \cdot x$ to a point in $\overline{G \cdot x} \setminus G \cdot x$. In [K], G.R. Kempf strengthened this result to provide a so-called optimal class of such cocharacters which enjoys a number of useful properties; roughly speaking, these cocharacters are the ones which transport us as quickly as possible outside the orbit of $x$. The purpose of this paper is to study how these optimal classes behave under passing to subgroups of $G$; for a subgroup $H$ of $G$ and $x \in X$, we give some conditions under which the optimal class of cocharacters for $x$ in $H$ is the set of optimal cocharacters for $x$ in $G$ which evaluate in $H$ (this happens when at least one of the cocharacters evaluates in $H$, Proposition 4.2) and provide a class of subgroups $H$ for which these conditions are satisfied (Theorem 4.4).

The initial motivation for studying this problem came from the theory of associated cocharacters for nilpotent elements in the Lie algebra $\mathfrak{g}$ of $G$. This theory has been developed by, among others, Jantzen [J], Premet [P], and McNinch [M], as a way of replacing the $\mathfrak{sl}_2$-triples which help classify nilpotent orbits in characteristic zero. If we have a subgroup $H$ of $G$ and a nilpotent element $e \in \mathfrak{h} = \text{Lie}(H)$, a particular problem in this area is to identify whether or not the set of cocharacters of $H$ associated with $e$ is just the set of cocharacters of $G$ associated with $e$ which evaluate in $H$, see [J, §5.12], [FR]. In good characteristic, it has been shown that an associated cocharacter is one of Kempf’s optimal cocharacters, but not vice versa. However, using our general results on optimality, we show that understanding how the full class of optimal cocharacters behave with respect to subgroups is enough to also tackle the problem for associated cocharacters (Theorem 5.5), and provide a wide class...
of subgroups which satisfy the relevant properties. This gives a uniform approach to many results already in the literature, and also allows some new constructions of interest.

We now indicate the layout of the paper. In Section 2 we begin with some general notation and definitions for algebraic groups. In particular, we recall Serre’s notion of $G$-complete reducibility for subgroups of a reductive group, which plays a part in the ensuing discussion.

In Section 3 we recall the results we need from Kempf’s paper [K]. In doing this, we mostly try to use his notation and terminology, which hopefully makes cross-referencing easier for the interested reader. In particular, in introducing his results, we work in a scheme-theoretic context, as this is the way in which Kempf treats the topic.

In Section 4 we indicate how Kempf’s results can be used to transfer optimality to subgroups (Proposition 4.2), and how this ties in with $G$-complete reducibility (Theorem 4.1). Again, we work in quite a general way in this section, mirroring the set-up that Kempf has in his paper.

In Section 5 we apply our general results to the special case of the adjoint action of $G$ on its Lie algebra. Here we show that understanding the transfer of optimality to subgroups is enough to understand how associated cocharacters behave (Theorem 5.5), and apply this result to give a wide class of subgroups which behave well in this regard (Corollary 5.7).

In the final section, we indicate a few ways in which our results can be extended by considering non-connected groups à la [BMR], [BMR2]; this allows outer automorphisms of reductive groups to come into play.

2. Preliminaries

1. Basic Notation. Our basic reference for linear algebraic groups is [B]. Except for briefly in Section 6, $G$ is a connected reductive linear algebraic group defined over an algebraically closed field $k$ of characteristic $p \geq 0$. For a subgroup $H$ of $G$, we let $H^0$ denote the identity component of $H$, $DH$ the derived subgroup of $H$ and $R_u(H)$ the unipotent radical of $H$. The centralizer of $H$ in $G$ is denoted $C_G(H)$ and $N_G(H)$ is the normalizer of $H$ in $G$. The Lie algebra of $G$ (resp. $H$) is denoted $\mathfrak{g}$ (resp. $\mathfrak{h}$).

A $G$-scheme $X$ is a separated $k$-scheme of finite type on which $G$ acts morphically; a key example in this paper is the action of $G$ on $\mathfrak{g}$ via the adjoint representation $\text{Ad} : G \to \text{GL}(\mathfrak{g})$. A subscheme $S$ of $X$ is called a $G$-subscheme if $S$ is a $G$-scheme and the immersion $S \subseteq X$ is $G$-equivariant. In this paper, we are interested only in $k$-points of such $k$-schemes, and we use the notation $x \in X$ as shorthand for “$x$ is a $k$-point of $X$”. For $x \in X$, $C_G(x)$ denotes the stabilizer of $x$ in $G$, $G \cdot x$ denotes the $G$-orbit of $x$ in $X$, and $\overline{G \cdot x}$ denotes the closure of this orbit. For a subgroup $H$ of $G$, we let $X^H$ denote the fixed points of $H$ in $X$.

Let $\Psi = \Psi(G,T)$ denote the set of roots of $G$ with respect to a maximal torus $T$. Fix a Borel subgroup $B$ of $G$ containing $T$ and let $\Sigma = \Sigma(G,T)$ be the set of simple roots of $\Psi$ defined by $B$. Then $\Psi^+ = \Psi(B)$ is the set of positive roots of $G$. For $\beta \in \Psi^+$ write $\beta = \sum_{\alpha \in \Sigma} c_{\alpha \beta} \alpha$ with $c_{\alpha \beta} \in \mathbb{N}_0$. A prime $p$ is said to be good for $G$ if it does not divide $c_{\alpha \beta}$ for any $\alpha$ and $\beta$, and bad otherwise. A prime $p$ is good for $G$ if and only if it is good for every simple factor of $G$, [SS]; the bad primes for the simple groups are 2 for all groups except type $A_n$, 3 for the exceptional groups and 5 for type $E_8$. We say $p = \text{char} k$ is good for $G$ if $p = 0$ or $p$ is a good prime for $G$.

A linear algebraic group $\Gamma$ is called linearly reductive if all rational representations of $\Gamma$ are semisimple; a torus is linearly reductive in any characteristic.
2. Levi Decompositions. A linear algebraic group $H$ has a Levi decomposition if there exists a closed subgroup $L$ of $H$ such that $H = L \rtimes R_u(H)$. The subgroup $L$ is called a Levi subgroup of $H$. For example, parabolic subgroups of connected reductive groups always have Levi decompositions. The following result is an extension of a result of Richardson [R1 Prop. 6.1], and can be found in [FR, Prop. 2.3].

**Proposition 2.1.** Let $H$ be a linear algebraic group with a Levi decomposition such that $R_u(H)$ acts simply transitively on the set of Levi subgroups of $H$. Suppose $\Gamma$ is a linearly reductive algebraic group acting on $H$ by automorphisms. Then $H$ has a $\Gamma$-stable Levi subgroup.

3. $G$-Complete Reducibility. A subgroup $H$ of $G$ is said to be $G$-completely reducible ($G$-cr) if whenever $H$ is contained in a parabolic subgroup $P$ of $G$, there exists a Levi subgroup $L$ of $P$ containing $H$. This notion was introduced by Serre as a way of generalizing the notion of complete reducibility from representation theory, [S1, S2]. By [S1 Property 4], if $H$ is a $G$-completely reducible subgroup of $G$, then $R_u(H) = \{1\}$, i.e., $H$ is reductive. Another basic result that has bearing on this paper is that a linearly reductive subgroup of $G$ is always $G$-completely reducible. This follows from Proposition 2.1 see also [BMR, Lem. 3.12, Cor. 3.17]. The following result also plays an important part in what follows, it is [BMR Prop. 2.6].

**Proposition 2.2.** Let $H$ be a $G$-completely reducible subgroup of $G$. Then $C_G(H)$ is $G$-completely reducible. In particular, $C_G(H)$ is reductive, and $C_G(H)^0$ is connected reductive.

In [BMR], it is shown that the notion of $G$-complete reducibility has a geometric interpretation, obtained by considering the diagonal action of $G$ on $G^n$ for various $n \in \mathbb{N}$. Our results in Section 4 below show that the theory of $G$-complete reducibility has applications when one considers actions of $G$ on other affine $G$-schemes.

3. The Theory of Kempf–Rousseau–Hesselink

In this section we recall some of the main definitions and results from the paper of Kempf [K] about optimal cocharacters for actions of algebraic groups on affine varieties, see also Rousseau [Ro] and Hesselink [H]. We take our notation and terminology mainly from [K], and the reader should refer there for more detail; we recall only the parts of the exposition which are important for this paper. It is important to note, however, that some of the motivation for the results in this paper came from reading [H], so this is also a key reference.

1. Cocharacters and Length Functions. Let $\mathbb{G}_m$ denote the multiplicative group, whose $k$-points are isomorphic to the multiplicative group $k^*$ of $k$. A cocharacter of $G$ is a homomorphism of algebraic groups $\lambda : \mathbb{G}_m \rightarrow G$. We let $Y(G)$ denote the set of cocharacters of $G$. For every maximal torus $T$ of $G$, the subset $Y(T)$ is a free abelian group of rank equal to the dimension of $T$; $Y(G)$ is the union of these groups as $T$ ranges over the maximal tori of $G$. For any $n \in \mathbb{Z}$ and $\lambda \in Y(G)$, we define $n\lambda \in Y(G)$ by $(n\lambda)(t) = \lambda(t^n) = \lambda(t)^n$. A cocharacter $\lambda \in Y(G)$ is called indivisible if there is no $\mu \in Y(G)$ with $\lambda = n\mu$ for some $n \in \mathbb{N}$. There is a left action of $G$ on $Y(G)$; for $g \in G$, $\lambda \in Y(G)$, let $(g \cdot \lambda)(t) = g\lambda(t)g^{-1}$.

We need the concept of a length function $\| \cdot \|$ on $Y(G)$ defined on [K, p305].
Definition 3.1. A length function on $Y(G)$ is a function

$$\| \cdot \| : Y(G) \to \mathbb{R}$$

such that:
(a) $\|g \cdot \lambda\| = \|\lambda\|$ for all $g \in G$, $\lambda \in Y(G)$;
(b) for any maximal torus $T$ of $G$, there is a positive definite integer-valued bilinear form $(\ ,\ )$ on $Y(T)$ such that $\|\lambda\|^2 = (\lambda, \lambda)$ for all $\lambda \in Y(T)$.

To show length functions exist, one can start with any maximal torus of $G$ and any positive definite integer-valued bilinear form on $Y(T)$. Averaging this form over the Weyl group $W \simeq N_G(T)/T$, we obtain a $W$-invariant bilinear form $(\ ,\ )$, say. Now for each $\lambda \in Y(G)$, there exists $g \in G$ such that $g \cdot \lambda \in Y(T)$; we define $\| \cdot \|$ by $\|\lambda\|^2 = (g \cdot \lambda, g \cdot \lambda)$. This is well-defined because $(\ ,\ )$ is $W$-invariant.

2. Cocharacters and $G$-Actions. Let $X$ be an affine $G$-scheme. For each $\lambda \in Y(G)$ and each $x \in X$, there is a morphism $\phi_{\lambda,x} : \mathbb{A}^1 \setminus \{0\} \to X$ given by $\phi_{\lambda,x}(t) = \lambda(t) \cdot x$. If $\phi_{\lambda,x}$ extends to a morphism from all of $\mathbb{A}^1$ to $X$, then we say that $\lim_{t \to 0} \lambda(t) \cdot x$ exists.

Now fix $x \in X$ and let $S$ be a closed $G$-subscheme of $X$ with $x \not\in S$. Let $[X, x]$ denote the set of $\lambda \in Y(G)$ for which $\lim_{t \to 0} \lambda(t) \cdot x$ exists. Then for each $\lambda \in [X, x]$, we have a morphism $M(\lambda) : \mathbb{A}^1 \to X$ given by the unique extension of $\phi_{\lambda,x}$; i.e. $M(\lambda)(t) = \phi_{\lambda,x}(t)$ for all $t \in \mathbb{A}^1 \setminus \{0\}$, and $M(\lambda)(0) = \lim_{t \to 0} \lambda(t) \cdot x$. Following [K] p308, we define $\alpha_{S,x}(\lambda)$ to be the degree of the divisor $M(\lambda)^{-1}(S)$ on $\mathbb{A}^1$; this is a non-negative integer, which is positive if and only if $M(\lambda)(0) \in S$ ([K] Lem. 3.1), see also [H] Sec. 2).

3. Cocharacters and Parabolic Subgroups. To each $\lambda \in Y(G)$, we can associate a parabolic subgroup $P_\lambda$ of $G$, which consists of the points of $G$ for which $\lim_{t \to 0} \lambda(t) g \lambda(t)^{-1}$ exists. The unipotent radical of $P_\lambda$ consists of the points for which $\lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} = 1$, and the centralizer of the image of $\lambda$ in $G$ is a Levi subgroup of $P_\lambda$, denoted $L_\lambda$.

If $H$ is a reductive subgroup of $G$, and $\lambda \in Y(H)$, we can associate a parabolic subgroup of $G$ and of $H$ to $\lambda$. We reserve the notation $P_\lambda$ for parabolic subgroups of $G$, and let $P_\lambda(H)$ denote the corresponding subgroup of $H$. Note that $P_\lambda(H) = P_\lambda \cap H$ and $R_u(P_\lambda(H)) = R_u(P_\lambda) \cap H$.

4. Optimality. The following is [K] Thm 3.4).

Theorem 3.2. Let $X$ be an affine $G$-scheme, and let $x \in X$ be a point such that $G \cdot x$ is not closed. Let $S$ be a closed $G$-subscheme of $X$ such that $x \not\in S$ and $S$ meets the closure of $G \cdot x$. Then, for a fixed choice of length function $\| \cdot \|$ on $Y(G)$:
(a) $\alpha_{S,x}(\lambda)/\|\lambda\|$ has a maximum value on the set of non-trivial cocharacters in $[X, x]$;
(b) let $\Omega_G(x, S)$ denote the set of indivisible cocharacters which achieve the maximum value from (a). Then:
(i) $\Omega_G(x, S)$ is non-empty;
(ii) there is a parabolic subgroup $P(x, S)$ of $G$ such that $P_\lambda = P(x, S)$ for all $\lambda \in \Omega_G(x, S)$;
(iii) $R_u(P(x, S))$ acts simply transitively on $\Omega_G(x, S)$;
(iv) any maximal torus of $P(x, S)$ contains a unique member of $\Omega_G(x, S)$.
We call $\Omega_{G}(x, S)$ the optimal class of cocharacters for $x \in X$ (with respect to $S$); in what follows, we only consider optimal classes which are non-empty, as in Theorem 3.2. We refer to the parabolic $P(x, S)$ as the optimal parabolic subgroup. The optimal nature of $\Omega_{G}(x, S)$ allows the following corollary, [K] Cor. 3.5.

**Corollary 3.3.** Let the notation be as in Theorem 3.2. Then:

(a) for any $k$-point $g \in G$, $gP(x, S)g^{-1} = P(g \cdot x, S)$;
(b) $C_{G}(x) \subseteq P(x, S)$.

**Remark 3.4.** It is clear that the optimal class depends in general on the choice of length function $\| \cdot \|$ on $Y(G)$, as well as the particular subscheme $S$. However, Hesselink has shown that for certain $X$, $S$ and $x$, the optimal class is at least independent of the length function [H Thm. 7.2]. In particular, this happens for the special case we consider in Section 5, where $X = g, S = \{0\}$ and $x$ is a nilpotent element of $g$, cf. [H Ex. 7.1(b)].

We note here that the set-up in Hesselink’s paper is slightly different to Kempf’s. Firstly, he discusses a notion of uniform instability, where instead of a single point $x \in X$, one considers a whole family of points which are all moved together to the same distinguished point of $X$ (so $S$ is a single point, but $x$ is replaced with a subset of $X$). Secondly, he uses a norm on $Y(G)$ which is slightly different from Kempf’s length function, and derives an optimal class of so-called virtual cocharacters. However, his results are easily translated into our setting.

### 4. Transferring Optimality to Subgroups

In this section, we want to consider the following general idea: For a $G$-scheme $X$ and a subgroup $H$ of $G$, we get an obvious action of $H$ on $X$ inherited from $G$. Motivated by [H (4.4)], we would like to ask whether or not

\[(1) \quad \Omega_{H}(x, S) = \Omega_{G}(x, S) \cap Y(H), \]

for various $x \in X$, $S \subseteq X$. In order to make sense of this, we have to be slightly careful about the set-up; in particular, we need to know that $\Omega_{H}(x, S)$ is defined.

Let $X$ be an affine $G$-scheme, and $H$ a reductive subgroup of $G$. Suppose $x \in X$ is a point whose $G$- and $H$-orbits are not closed, and let $S$ be a closed $G$-subscheme of $X$ which does not contain $x$, but meets the closure of the $H$-orbit of $x$. These conditions ensure that $S$ is an $H$-subscheme, and $S$ also meets the closure of the $G$-orbit of $x$. Since $Y(H) \subseteq Y(G)$, it is clear that $|X, x|_{H} := |X, x| \cap Y(H) \subseteq |X, x|$. It is also clear that for $\lambda \in Y(H)$, the value of $\alpha(S, \lambda)$ does not depend on whether we consider $\lambda$ as a cocharacter of $H$ acting on the $H$-scheme $X$, or as a cocharacter of $G$ acting on the $G$-scheme $X$. The final easy observation that allows us to proceed is the following:

**Lemma 4.1.** Let $\| \cdot \|$ be a length function on $Y(G)$. Then the restriction of $\| \cdot \|$ to $Y(H)$ is a length function on $Y(H)$.

**Proof.** We need to check conditions (a) and (b) of Definition 3.1. Condition (a) is obvious; if $\| \cdot \|$ is $G$-invariant, then it is $H$-invariant. For (b), let $S$ be a maximal torus of $H$, and let $T$ be a maximal torus of $G$ containing $S$. Then $Y(S)$ is a subgroup of $Y(T)$, and it is clear that a positive definite integer-valued bilinear form on $Y(T)$ restricts to a positive definite integer valued bilinear form on $Y(S)$. \hfill $\square$
The preceding discussion shows that the optimal class $\Omega_H(x, S)$ exists and can be defined with respect to the same length function as $\Omega_G(x, S)$, so our question makes sense independently of the length function chosen on $Y(G)$. For the rest of this section, we fix a length function $\| \cdot \|$ on $Y(G)$, and use the restriction of this length function to $Y(H)$ for subgroups $H$ of $G$. Our next result shows that the desired equality holds if and only if $\Omega_G(x, S) \cap Y(H)$ is non-empty.

**Proposition 4.2.** Let $X$ be an affine $G$-scheme, let $x \in X$ be such that $G \cdot x$ is not closed, and let $S$ be a closed $G$-subscheme of $X$ which does not contain $x$, but meets $G \cdot x$. If $H$ is a reductive subgroup of $G$ such that $\Omega_G(x, S) \cap Y(H)$ is non-empty, then $H \cdot x$ is not closed, $S$ meets $H \cdot x$ and $\Omega_H(x, S) = \Omega_G(x, S) \cap Y(H)$.

**Proof.** Let $t \in \Omega_G(x, S) \cap Y(H)$. Then since $\lim_{t \to 0} \lambda(t) \cdot x \in S$ and $\lambda \in Y(H)$, we have $\lim_{t \to 0} \lambda(t) \cdot x \in S \cap H \cdot x$. Since $\lim_{t \to 0} \lambda(t) \cdot x$ is not in $G \cdot x$, it is not in $H \cdot x$. Thus $H \cdot x$ is not closed, and $S$ meets the closure of $H \cdot x$. In particular, it makes sense to define $\Omega_H(x, S)$ for $x, S$ and our fixed length.

Now $\lambda$ is an indivisible cocharacter of $G$, so $\lambda$ is an indivisible cocharacter of $H$. Since we use the same length function on $Y(G)$ and $Y(H)$, and $\alpha(S, \lambda)$ is independent of $G$ and $H$, it is clear that the maximal value of $\alpha(S, \lambda)/\|\lambda\|$ as $\lambda$ runs over $[X, x]_H$ is less than or equal to the maximum value as $\lambda$ runs over $[X, x]$. But since our choice of $\lambda$ attains this maximal value, there is equality here, and since $\lambda$ is an indivisible cocharacter of $H$, this means $\lambda = \Omega_{\frac{\lambda}{\|\lambda\|}}(x, S)$. Thus $\Omega_G(x, S) \cap Y(H) \subseteq \Omega_H(x, S)$.

Finally, consider $P_{\lambda}(H) = P(x, S) \cap H$; this is the optimal parabolic subgroup of $H$ given by $\Omega_H(x, S)$. Moreover, $R_u(P_{\lambda}(H)) = R_u(P(x, S)) \cap H$ acts transitively on $\Omega_H(x, S)$. Since $R_u(P(x, S))$ acts on $\Omega_G(x, S)$, we see that $\Omega_H(x, S) \subseteq \Omega_G(x, S) \cap Y(H)$, and we are done. \hfill \Box

Motivated by the previous result, we can now define our notion of optimality for subgroups.

**Definition 4.3.** Let $H$ be a reductive subgroup of $G$, and let $X$ be an affine $G$-scheme. Suppose $x \in X$ is a point whose $G$- and $H$-orbits are not closed, and let $S$ be a closed $G$-subscheme of $X$ which does not contain $x$, but meets the closure of the $H$-orbit of $x$. Then, following [H (4.4)], we call $H$ optimal for $x$, $S$ if $\Omega_H(x, S) = \Omega_G(x, S) \cap Y(H)$.

Our next result gives a wide class of optimal subgroups for a general action of $G$ on an affine variety $X$.

**Theorem 4.4.** Suppose $X$ is an affine $G$-scheme and that $K$ is a $G$-completely reducible subgroup of $G$. Let $x \in X^K$ be such that $G \cdot x$ is not closed, and let $S$ be any closed $G$-subscheme of $X$ which meets $G \cdot x$ and does not contain $x$. Set $H = C_G(K)^0$. Then:

(i) $H \cdot x$ is not closed;

(ii) $S$ meets $H \cdot x$, and hence $\Omega_H(x, S)$ is defined;

(iii) $H$ is optimal for $x$, $S$.

**Proof.** By Proposition 4.2, it is enough to show that $\Omega_G(x, S) \cap Y(H)$ is non-empty. Let $P(x, S)$ be the optimal parabolic subgroup of $G$ from Theorem 3.2. Then, by Corollary 3.3, $C_G(x) \subseteq P(x, S)$. Since $K \subseteq C_G(x)$, we also have $K \subseteq P(x, S)$. Now $K$ is $G$-cr, so there exists a Levi subgroup $L$ of $P(x, S)$ containing $K$. Let $\lambda \in \Omega_G(x, S)$; so $P(x, S) = P_{\lambda}$, and $L_{\lambda}$ is also Levi subgroup of $P(x, S)$. Then $L$ is $R_u(P(x, S))$-conjugate to $L_{\lambda}$, hence $L = L_{\mu}$ for
some $R_u(P(x, S))$-conjugate $\mu$ of $\lambda$. But $R_u(P(x, S))$ acts on $\Omega_G(x, S)$, hence $\mu \in \Omega_G(x, S)$. Finally, we have $K \subseteq L_\mu$, so $K$ centralizes the image of $\mu$, and $\mu \in Y(C_G(K)^0) = Y(H)$, as required.

Remarks 4.5. (i) Note that in Theorem 4.4 it is necessary to know that (i) and (ii) hold to even ask whether $H$ is optimal, as the definition of optimality for subgroups only applies to those $H$ for which (i) and (ii) hold.

(ii). In characteristic 0, a subgroup of $G$ is $G$-completely reducible if and only if it is reductive [BMR, Sec. 2.2]. In this case, part (i) of Theorem 4.4 can be deduced from [L, §3 Cor. 3.1], due to Luna, and parts (ii) and (iii) are similar to Kempf’s result [K, Cor. 4.5]. In fact, for (i) in characteristic 0, the stronger statement that the $G$-orbit of $x \in X^K$ is closed if and only if the $H$-orbit is closed follows from Luna’s result.

(iii). For any $x \in X$, there is a unique closed orbit $C$ in the closure of $G \cdot x$. Clearly, $C \subseteq S$ for any closed $G$-subscheme $S$ which meets $G \cdot x$. In applications, it usually suffices to just consider the case $S = C$ when applying Kempf’s results.

(iv). We can also interpret our results within the framework of the paper [H] of Hesselink (cf. Remark 3.3). In particular, Theorem 4.4 is a generalization of [H, Prop. 9.4], which gives the result when $K$ is a torus, so that $H$ is a Levi subgroup of some parabolic subgroup of $G$ (a critical subgroup in [H]).

We finish this section with an example which shows that the reverse direction of Theorem 4.4 (and hence also Proposition 4.2) does not work in general.

Example 4.6. In [BMR2, Ex. 5.1, 5.4], there are examples of commuting subgroups $A$ and $B$ of a reductive group $G$ such that $A$ and $B$ are $G$-cr, but $B$ is not $C_G(A)^0$-cr. We can find an $n$-tuple $x = (x_1, \ldots, x_n) \in G^n$ for some $n$ such that $B$ is the closure of the subgroup generated by $x_1, \ldots, x_n$.

In this situation, if we set $H = C_G(A)^0$, we know that $G \cdot x$ is closed in $G^n$, whereas $H \cdot x$ is not, by [BMR, Cor. 3.7]. Thus any closed $G$-subscheme $S$ which meets $G \cdot x$ actually contains all of $G \cdot x$, and hence all of $H \cdot x$, so we cannot define $\Omega_G(x, S)$ and $\Omega_H(x, S)$ for such subschemes. However, the unique closed $H$-orbit $C$ in $\overline{H \cdot x}$ is an $H$-subscheme which meets $H \cdot x$ and does not contain $x$, so we can talk about $\Omega_H(x, C)$. This example also shows why we have to be very careful in our set-up when trying to compare $G$- and $H$-actions, just to make sure that what we are saying even makes sense.

5. Cocharacters Associated with Nilpotent Elements

In this section, we consider the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$, and apply our results to the notion of a cocharacter associated with a nilpotent element of $\mathfrak{g}$. Throughout this section, $e$ denotes a nilpotent element in $\mathfrak{g}$, and our distinguished closed $G$-subscheme $S$ of $\mathfrak{g}$ is simply the set $S = \{0\}$, which is the unique closed $G$-orbit in $G \cdot e$. We keep our fixed length function, and drop $S$ from our notation; thus $\Omega_G(e)$ is the set of optimal cocharacters for $e$, and $P(e)$ is the corresponding parabolic subgroup of $G$. If $H$ is a reductive subgroup of $G$, and $e \in \mathfrak{h}$, then $\{0\}$ is also the unique closed $H$-orbit in $\overline{H \cdot e}$; thus $\Omega_H(e)$ is always defined in this situation.

Let $\lambda \in Y(G)$; then $\lambda$ gives a decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i, \lambda),$$
where \( g(i, \lambda) := \{ x \in g \mid \text{Ad}(\lambda(t))x = t^i x \text{ for all } t \in k^* \} \).

**Definition 5.1.** Let \( H \) be a closed subgroup of \( G \), with Lie algebra \( \mathfrak{h} \subseteq g \) and suppose \( e \in \mathfrak{h} \subseteq g \) is a nilpotent element. Then \( e \) is said to be distinguished in \( \mathfrak{h} \) if each torus in \( C_H(e) \) is contained in the centre of \( H \).

**Definition 5.2.** Let \( G \) be a connected reductive linear algebraic group, and let \( e \in g \) be nilpotent. A cocharacter \( \lambda \in Y(G) \) is called associated with \( e \) if

1. \( e \in \mathfrak{g}(2, \lambda) \);
2. there exists a Levi subgroup \( L \) of some parabolic subgroup of \( G \) such that
   - (a) \( e \) is distinguished in \( \text{Lie}(L) \);
   - (b) \( \text{Im}(\lambda) \subseteq DL \).

We let \( \Omega_G^a(e) \) denote the set of cocharacters of \( G \) associated with \( e \).

The following lemma gives a necessary condition for a Levi subgroup \( L \) to arise in part (ii) above, see [I, Rem. 4.7].

**Lemma 5.3.** Let \( e \) be a nilpotent element of \( \mathfrak{g} \), and suppose \( L \) is a Levi subgroup of some parabolic subgroup of \( G \). Then \( e \) is distinguished in \( \text{Lie}(L) \) if and only if \( L = C_G(S) \), where \( S \) is a maximal torus of \( C_G(e) \).

Premet [P] and McNinch [I] have shown that if the characteristic of \( k \) is good for \( G \), then there exist cocharacters associated with any nilpotent \( e \in g \). In fact, in good characteristic, we can say a lot about the set \( \Omega_G^a(e) \). The following is an amalgamation of several results, see [I, Lem. 5.3], [P] Thm. 2.3, Prop. 2.5, [FR]. Prop. 2.11, Prop. 2.15, Cor. 2.16, Rem. 2.17.

**Proposition 5.4.** Suppose \( p \) is good for \( G \). Let \( e \in \mathfrak{g} \) be nilpotent.

1. \( \Omega_G^a(e) \) is a non-empty subset of \( \Omega_G(e) \).
2. Let \( \lambda \in \Omega_G^a(e) \). Set \( R_e = R_u(P_\lambda) \cap C_G(e) \) and \( C_G(e, \lambda) = L_\lambda \cap C_G(e) \). Then
   - \( R_e = R_u(C_G(e)) \) and \( C_G(e) = R_e \rtimes C_G(e, \lambda) \) is a Levi decomposition of \( C_G(e) \).
3. Any two cocharacters of \( G \) associated with \( e \) are conjugate by an element of \( C_G(e)^0 \).
   - Conversely, \( C_G(e) \) acts on \( \Omega_G^a(e) \), and \( R_e \) acts simply transitively on \( \Omega_G^a(e) \).
4. The map \( \lambda \to C_G(e, \lambda) \) is a bijection between \( \Omega_G^a(e) \) and the set of Levi subgroups of \( C_G(e) \), and this map is compatible with the action of \( R_e \) on both these sets.

Note that since \( \Omega_G^a(e) \subseteq \Omega_G(e) \), the parabolic subgroups \( P_\lambda \) in part (ii) are all equal to the optimal parabolic \( P(e) \) given by Kempf’s Theorem 3.2 Building on the previous section, if \( H \) is a reductive subgroup of \( G \) and \( e \in \mathfrak{h} \) is nilpotent, it is interesting to know when

\[
\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H),
\]

see [I, §5.12], [FR]. Again, we have to be slightly careful to ensure that our question makes sense, which results in some characteristic restrictions in our results. The first result shows that Eqn. (2) holds if and only if Eqn. (1) holds.

**Theorem 5.5.** Let \( H \) be a reductive subgroup of \( G \) and \( e \in \mathfrak{h} \) a nilpotent element. If \( p \) is good for \( H \) and \( G \), then \( \Omega_H(e) = \Omega_G(e) \cap Y(H) \) if and only if \( \Omega_H^a(e) = \Omega_G^a(e) \cap Y(H) \).

**Proof.** First suppose that \( \Omega_H(e) = \Omega_G(e) \cap Y(H) \). Since \( \text{char} k \) is good for \( H \), we have \( \Omega_H^a(e) \) is non-empty and is a subset of \( \Omega_H(e) \). Let \( \lambda \in \Omega_H^a(e) \). Then \( \lambda \in \Omega_G(e) \). By [FR] Thm.
3.17], if we can show that \( \lambda \in \Omega^a_H(e) \), then we can conclude that \( \Omega^a_H(e) = \Omega^a_G(e) \cap Y(H) \), as required.

Since \( e \in \mathfrak{h}(2, \lambda) \subseteq \mathfrak{g}(2, \lambda) \), it is clear that \( \text{Im}(\lambda) \) normalizes \( C_G(e) \). Since, by our standing assumption, \( p \) is good for \( G \), \( C_G(e) \) has a Levi decomposition, and \( R_e \) acts simply transitively on the set of Levi subgroups of \( C_G(e) \), by Proposition 5.3. Moreover, the Levi subgroups of \( C_G(e) \) are in bijection with the elements of \( \Omega^a_G(e) \). By Proposition 27, there is a Levi subgroup \( C_G(e, \mu) \) of \( C_G(e) \) which is stable under \( \text{Im}(\lambda) \), where \( \mu \in \Omega^a_G(e) \).

Since \( \mu \) is associated with \( e \), \( e \in \mathfrak{g}(2, \mu) \). Also, by Lemma 5.3, there exists a maximal torus \( S \) of \( C_G(e) \) such that \( e \) is distinguished nilpotent in \( \text{Lie}(L) \), where \( L = C_G(S) \), and \( \text{Im}(\mu) \subseteq D_L \). Now for any \( x = \lambda(t) \in \text{Im}(\lambda) \), since \( \text{Im}(\lambda) \) normalizes \( C_G(e) \), \( xSx^{-1} \) is a maximal torus of \( C_G(e) \); thus \( e \) is distinguished nilpotent in \( \text{Lie}(xLx^{-1}) \). Moreover, since \( e \in \mathfrak{g}(2, \lambda) \) also, we have \( e \in \mathfrak{g}(2, x \cdot \mu) \). Finally, note that \( \mu \in Y(DL) \) implies \( x \cdot \mu \in Y(D(xLx^{-1})) \). These arguments suffice to show that for all \( x \in \text{Im}(\lambda) \), \( x \cdot \mu \in \Omega^a_G(e) \).

Now we have \( xC_G(e, \mu)x^{-1} = C_G(e, x \cdot \mu) = C_G(e, \mu) \) for all \( x \in \text{Im}(\lambda) \), so \( \text{Im}(\lambda) \) must fix \( \mu \). This means that \( \text{Im}(\lambda) \subseteq C_G(\text{Im}(\mu)) = L_\mu \). Thus there is a maximal torus \( T \) of \( L_\mu \) containing \( \text{Im}(\lambda) \) and \( \text{Im}(\mu) \); but \( \lambda, \mu \in \Omega^a_G(e) \), so \( \lambda = \mu \) by Theorem 3.2(b)(iv), and we are done.

For the reverse implication, suppose \( \Omega^a_H(e) = \Omega^a_G(e) \cap Y(H) \). Then, since the characteristic is good for \( G \) and \( H \), this set is non-empty. Let \( \lambda \in \Omega^a_H(e) \). Then \( \lambda \in \Omega^a_G(e) \subseteq \Omega^a_G(e) \), so \( \lambda \in \Omega^a_G(e) \cap Y(H) \) and Proposition 4.2 applies. \( \Box \) \( \Box \)

Remark 5.6. In characteristic 0, for any reductive subgroup \( H \) of \( G \) and any nilpotent element \( e \in \mathfrak{h} \), we always have \( \Omega^a_H(e) = \Omega^a_G(e) \cap Y(H) \), see [J 5.12]. Thus, by Theorem 5.5, we also always have \( \Omega_H(e) = \Omega_G(e) \cap Y(H) \). Here we are grateful to G. Röhrle for pointing out this application.

The following corollary is immediate from Theorem 4.4 and Theorem 5.5.

Corollary 5.7. Suppose \( K \) is a \( G \)-completely reducible subgroup of \( G \), set \( H = C_G(K)^0 \), and suppose that \( p \) is good for \( H \) and \( G \). Then \( \Omega^a_H(e) = \Omega^a_G(e) \cap Y(H) \) for all nilpotent elements \( e \in \mathfrak{h} \).

Remark 5.8. Corollary 5.7 covers in a uniform way many results already in the literature. For example, if \( K \) is a torus in \( G \), then \( C_G(K)^0 = C_G(K) \) is a Levi subgroup of some parabolic of \( G \). This gives [FR Cor. 3.22]. More generally, if \( s \in G \) is a semisimple element and \( K \) is the subgroup generated by \( s \), then \( K \) is linearly reductive, hence \( G \)-cr. In this case, the groups \( C_G(K)^0 = C_G(s)^0 \) are the pseudo-Levi subgroups of \( G \), which gives us [FR Cor. 3.27].

Corollary 5.9. Suppose \( H \) is a \( G \)-cr subgroup of \( G \) such that \( H = C_G(C_G(H)^0)^0 \), and \( p \) is good for \( H \) and \( G \). Then \( \Omega^a_H(e) = \Omega^a_G(e) \cap Y(H) \) for all nilpotent elements \( e \in \mathfrak{h} \).

Proof. This follows from Corollary 5.7, setting \( K = C_G(H)^0 \).

We finish this section with an example of how Corollaries 5.7 and 5.9 can be applied.

Example 5.10. In [LS] there are extensive tables of the subgroups of simple exceptional algebraic groups and their centralizers. These tables can be used to generate many cases. For example, looking at [LS] Table 8.1], if \( G = E_8 \) and \( p > 7 \), then there is a pair of subgroups \( X_1 = G_2 \) and \( X_2 = F_4 \) with \( C_G(X_1)^0 = X_2 \) and \( C_G(X_2)^0 = X_1 \). Both these subgroups are \( G \)-cr, by [LS] Thm. 1], so Corollary 5.7 applies. This is also an example of Corollary 5.9 since \( C_G(C_G(X_i)^0)^0 = X_i \) for \( i = 1, 2 \).
6. Extension to Non-Connected Groups

In [BMR, Sec. 6], and the later paper [BMR2], results about $G$-complete reducibility are proved in the more general case where $G$ is reductive, but not necessarily connected; we refer the reader to [BMR, Sec. 6] for the formalities. Kempf’s results can be immediately translated into this setting also, so the results in Section 4 are easily extended.

In this section we briefly indicate how to extend our results from Section 5. Since for any linear algebraic group $G$, $Y(G) = Y(G^0)$ and $\text{Lie}(G) = \text{Lie}(G^0)$, moving to non-connected $G$ is not difficult. To show what we mean, we give one possible extension of Corollary 5.7; the proof comes from obvious extensions of our earlier proofs.

**Proposition 6.1.** Let $G$ be a (possibly non-connected) reductive linear algebraic group, and let $K$ be a $G$-completely reducible subgroup of $G$. Set $H := C_G(K)$ and suppose that $p$ is good for $G^0$ and $H^0$. Then $\Omega_H^a(e) = \Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$ for all nilpotent $e \in \mathfrak{h}$.

**Remark 6.2.** Proposition 6.1 applies in particular in the case where $G = G_1 \times K$, where $G_1$ is a connected reductive group and $K$ is a reductive group acting on $G_1$ by automorphisms such that the image of $K$ in $G$ is $G$-cr. Note that, by [B, Cor. 14.11], $G$ is a reductive group in this situation. This allows one to consider outer automorphisms of a connected reductive group, for example graph automorphisms of simple groups in good characteristic, or automorphisms permuting the simple factors of a reductive group; we give a simple illustration in Example 6.3 below.

**Example 6.3.** Let $X$ be a connected reductive algebraic group such that $p$ is good for $X$. Let $G_1 = X \times \ldots \times X$ be the direct product of $r$ copies of $X$, and let $a$ be an automorphism of $G_1$ which acts as an $r$-cycle permuting the factors. Set $K := \langle a \rangle$ and $G = G_1 \times K$; then $C_G(K)^0$ is the diagonal embedding of $X$ in $G_1$. If $p$ divides $r$, then $K$ is not linearly reductive; however, the image of $K$ in $G$ is always $G$-cr. (In the language of [BMR, Sec. 6], any $R$-parabolic subgroup $P$ of $G$ which contains $K$ must be of the form $P = \langle Q \times \ldots \times Q \rangle \times K$, where $Q$ is a parabolic subgroup of $X$. Then for any Levi subgroup $M$ of $Q$, $K$ is contained in the $R$-Levi subgroup $L = \langle M \times \ldots \times M \rangle \times K$ of $P$.) Thus Proposition 6.1 says that the diagonal embedding of a cocharacter of $X$ associated with some $e \in \text{Lie}(X)$ is a cocharacter of $G$ (hence of $G_1$) associated with the diagonal embedding of $e$ in $\text{Lie}(G) = \text{Lie}(G_1)$.

We finish by noting that Corollary 5.9 holds for non-connected $G$ and $H$, with the weaker hypothesis that $H^0 = C_G(C_G(H))^0$, assuming $p$ is good for $G^0$ and $H^0$.

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