POLYNOMIAL DECELERATION FOR A SYSTEM OF CUBIC NONLINEAR SCHRÖDINGER EQUATIONS IN ONE SPACE DIMENSION

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Abstract. In this paper, we consider the initial value problem of a specific system of cubic nonlinear Schrödinger equations. Our aim of this research is to specify the asymptotic profile of the solution in $L^\infty$ as $t \to \infty$. It is then revealed that the solution decays slower than a linear solution does. Further, the difference of the decay rate is a polynomial order. This deceleration of the decay is due to an amplification effect by the nonlinearity. This nonlinear amplification phenomena was previously known for several specific systems, however the deceleration of the decay in these results was by a logarithmic order. As far as we know, the system studied in this paper is the first model in that the deceleration in a polynomial order is justified.

1. Introduction

1.1. System of cubic NLS equations in one space dimension. In this paper, we consider the large time behavior of solutions to the Cauchy problem of the following system of cubic nonlinear Schrödinger (NLS) equations in one space dimension:

$$\begin{cases}
\mathcal{L}u_1 = 3\lambda_1 |u_1|^2 u_1, & t \in \mathbb{R}, x \in \mathbb{R}, \\
\mathcal{L}u_2 = \lambda_6 (2|u_1|^2 u_2 + u_1^2 \overline{u}_2), & t \in \mathbb{R}, x \in \mathbb{R},
\end{cases}$$

$$u_1(0, x) = u_{1,0}(x), \quad u_2(0, x) = u_{2,0}, \quad x \in \mathbb{R},$$

where $(u_1, u_2) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$ is a pair of unknown functions, $\mathcal{L} = i \partial_t + (1/2) \partial_x^2$, and $\lambda_1$ and $\lambda_6$ are real constants satisfying $(\lambda_1, \lambda_6) \neq (0, 0)$ and

$$\lambda_6 - \lambda_1 (\lambda_6 - 3\lambda_1) < 0.$$  

We give a precise assumption on the data $(u_{1,0}, u_{2,0})$ later. Throughout the paper, we use the notation $\eta = 3\lambda_1 - 2\lambda_6$ and $\mu = \sqrt{\lambda_6^2 - \eta^2}$. We remark that (1.2) reads as $\lambda_6^2 - \eta^2 > 0$, and hence it implies that $\mu$ is a real number. The system (1.1) is a special case of

$$\begin{cases}
\mathcal{L}u_1 = 3\lambda_1 |u_1|^2 u_1 + \lambda_2 (2|u_1|^2 u_2 + u_1^2 \overline{u}_2) \\
+ \lambda_3 (2u_1|u_2|^2 + \overline{u}_1 u_2^2) + 3\lambda_4 |u_2|^2 u_2, \\
\mathcal{L}u_2 = 3\lambda_5 |u_1|^2 u_1 + \lambda_6 (2|u_1|^2 u_2 + u_1^2 \overline{u}_2) \\
+ \lambda_7 (2u_1|u_2|^2 + \overline{u}_1 u_2^2) + 3\lambda_8 |u_2|^2 u_2.
\end{cases}$$

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Classification of systems of this form is introduced in [16, 18].

It is widely known that the cubic nonlinearity is critical in one space dimension in view of the large time behavior of solutions to several typical dispersive equations such as Schrödinger equation and Klein-Gordon equation. More precisely, the nonlinear effect can be seen in the large time behavior of solutions. The cubic nonlinearity is critical also for systems of dispersive equations. There is a variety of types on the large time behavior of solutions to systems, even if we restrict ourselves to the systems of the form (1.3).

To compare with, let us first recall that the asymptotic behavior of a solution to the linear Schrödinger equation \( \mathcal{L}u = 0 \) is given as
\[
t^\frac{1}{2} e^{i\frac{x^2}{2t} - i\frac{\pi}{4} F[u(0)](\frac{t}{T})}.
\]
In particular, the decay order of the linear solution in the \( L^\infty \)-topology is \( t^{-1/2} \).

Ozawa [22] showed that the asymptotic profile of a small good solution to the cubic equation \( \mathcal{L}u = \lambda |u|^2 u, \ \lambda \in \mathbb{R} \) (1.4) is given as
\[
t^\frac{1}{2} e^{i\frac{x^2}{2t} - i\frac{\pi}{4} (\hat{u}_+ e^{-i\lambda |\hat{u}_+|^2 \log t} + e^{-3i\lambda_1 |W_1|^2 \log t})(\frac{t}{T})},
\]
where \( u_+ \) is a suitable function (See also [3, 8]). Notice that, due to the presence of the gauge-invariant cubic nonlinear term, a logarithmic phase correction is involved in the asymptotic profile (See [15, 19, 28] for a similar result for other models). The first equation of the system (1.1) is nothing but the single cubic equation and hence its asymptotic behavior is given above.

It is also known that the cubic nonlinearity may change the decay of a solution. One such example is known as the so-called nonlinear dissipation phenomenon. The typical case is (1.4) with a complex coefficient \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

When \( \Im \lambda > 0 \) then (1.4) is dissipative for the positive time direction, and it has shown that the small solution decays faster than the free Schrödinger evolution (see [6, 7, 10, 14, 20, 21, 23, 24]). Note the logarithmic phase correction do not change the decay of solution.

Oppositely, there is a models in which the decay of a solution is decelerated due to the nonlinear effect. We call this phenomenon as the (weak) nonlinear amplification. A typical example is (1.1) with the choice \( \lambda_6 = \lambda_1 (\neq 0) \) or \( \lambda_6 = 3\lambda_1 (\neq 0) \). The second, third, and fourth authors show in [15] that there are two choices admits a solution of which asymptotic profile is the pair of
\[
t^\frac{1}{2} e^{i\frac{x^2}{2t} - i\frac{\pi}{4} (W_1 e^{-3i\lambda_1 |W_1|^2 \log t})(\frac{t}{T})},
\]
and
\[
t^\frac{1}{2} e^{i\frac{x^2}{2t} - i\frac{\pi}{4} ((W_2 + W \log t) e^{-3i\lambda_1 |W_1|^2 \log t})(\frac{t}{T})}
\]
by a suitable pair \((W_1, W_2)\) of functions, where \( W \) is given by
\[
W = \begin{cases} 
2\lambda_1 W_1 \Im [W_1 W_2] & \text{if } \lambda_6 = \lambda_1, \\
-6i\lambda_1 W_1 \Re [W_1 W_2] & \text{if } \lambda_6 = 3\lambda_1.
\end{cases}
\]
Besides the logarithmic phase correction, a \textit{logarithmic amplitude correction}, denoted by $W$, shows up in the asymptotic profile of the second component. One has
\[
\|u_2(t)\|_{L^\infty} \sim t^{-1/2}(1 + \|W\|_{L^\infty} \log t)
\]
as $t \to \infty$. We remark that the amplitude correction, which makes the decay of the solution slower, is due to the nonlinear effect between two components of unknowns. Indeed, $W_1 = 0$ implies $W = 0$, which tells us that, without the presence of the first component of the solution, the amplitude correction does not occur. For a generic non-trivial solution, we have $W \neq 0$.

As for (1.1), the case $(\lambda_6 - \lambda_1)(\lambda_6 - 3\lambda_1) = 0$ is a threshold. Indeed, it is proved in [18] that the nonlinear amplification do not take place in the case $(\lambda_6 - \lambda_1)(\lambda_6 - 3\lambda_1) > 0$. Further, it is revealed that the asymptotic behavior of $u_2$ in this case is a sum of two parts which oscillate in a different way. The oscillations of the two parts get close each other as $(\lambda_6 - \lambda_1)(\lambda_6 - 3\lambda_1) \downarrow 0$ and in the limit case, that is, at the threshold case, they become the same. As mentioned below, the asymptotic profile is given as a solution to a corresponding ODE system. At least in this ODE level, we see that the logarithmic amplitude correction is produced by the coincidence of the oscillations (cf. resonance phenomenon in the theory of ODEs).

Another example of a system in which the nonlinear amplification occurs is
\[
\begin{cases}
    Lu_1 = 0, & t \in \mathbb{R}, x \in \mathbb{R}, \\
    Lu_2 = 3|u_1|^2 u_1, & t \in \mathbb{R}, x \in \mathbb{R}.
\end{cases}
\tag{1.6}
\]
This system is also studied in [18] and a similar slowly-decaying solution is found. However, an ODE analysis shows that the mechanism of the appearance of the logarithmic amplitude correction is slightly different. We remark that a similar result is previously known for the corresponding system of Klein-Gordon equations, i.e., (1.6) with $L = \partial_t^2 - \partial_x^2 + 1$ (see [26]).

In this paper, we study the other side of the threshold, that is, the case (1.2), of the system (1.1). It will turn out that there is a stronger deceleration effect. Namely, the asymptotic profile of the second component involves a \textit{polynomial amplitude correction}. As far as we know, the deceleration of the time-decay by a polynomial order is not known before at least for the Schrödinger equations/systems and Klein-Gordon equation/systems.

One obstacle in justifying the existence of such a slowly-decaying solution comes from the fact that the criticalness of the cubic nonlinearity in one space dimension is related to the critical decay rate $O(t^{-1/2})$ in $L^\infty(\mathbb{R})$. To explain this respect in more detail, let us consider the power type nonlinear Schrödinger equation $Lu = |u|^{p-1}u$, where $p > 1$. By means of the propagator $U(t) = e^{i\frac{t}{2}\partial_x^2}$, it is rewritten as
\[
i\partial_t(U(-t)u(t)) = U(-t)(|u|^{p-1}u).
\]
By the conservation of mass, under the assumption that $u$ decays like $O(t^{-1/2})$ in $L^\infty(\mathbb{R})$ as $t \to \infty$, the right hand side becomes an integrable $L^2$-valued function of time on $\mathbb{R}_+$ if $p > 3$. Note that the integrability means that the solution $u(t)$ scatters (See [27]). The only-if part, i.e., the failure of scattering for $p \leq 3$ is well-known (See [1][23]). This is how the cubic nonlinearity appears as a critical index. The argument also suggests
that the cubic nonlinearity is not the critical index but a supercritical one when solutions decay at a slower rate than $O(t^{-1/2})$. From a technical point of view, the slower decay makes it difficult to obtain a closed estimate.

As for (1.1), the special structure of the system enables us to obtain a closed estimate in spite of the presence of the deceleration effect. Firstly, the first equation of the system is nothing but the single cubic equation (1.4). Hence, by applying the previous result on the single equation, we obtain the asymptotic behavior of the first component $u_1$. We remark that $u_1$ decays at the critical rate, the same rate as the linear solution does. Secondly, the equation for $u_2$ is linear with respect to $u_2$. Hence, the critical decay of $u_1$ is sufficient to obtain a global bound on $u_2$.

In many cases, the asymptotic profile of a solution is given by the corresponding ODE equation/system. Roughly speaking, the asymptotic profile of a good solution to a system of cubic Schrödinger equation

$$L u_\ell = N_\ell(u_1, u_2, \ldots, u_L), \quad (\ell = 1, 2, \ldots, L),$$

where $N_\ell$ is a homogeneous cubic nonlinearities, is described by a solution $(A_1, A_2, \ldots, A_L) = (A_1(\tau; \xi), A_2(\tau; \xi), \ldots, A_L(\tau; \xi))$ parametrized by $\xi \in \mathbb{R}$ to the system of ordinary equations

$$i A'_\ell = t^{-1} N_\ell(A_1, A_2, \ldots, A_L) \quad (\ell = 1, 2, \ldots, L). \quad (1.7)$$

Namely, in some sense,

$$u_\ell(t, x) \sim t^{-1/2} e^{i x^2/2 t - i \pi A_\ell(t; \xi)} \quad (\ell = 1, 2, \ldots, L)$$

as $t \to \infty$. As seen below, our main result ensures that this is true for (1.1) with (1.2) (See Remark 1.2). By the previous results, we see that this is also true for all systems introduced above, i.e., for the rest case of (1.1), and (1.4) and (1.6). Katayama and Sakoda [11] show that this is true for a class of systems of cubic Schrödinger equations.

There are two known systems for which the analysis of the corresponding ODE system suggests that a deceleration of the decay by a polynomial order takes place. One is

$$L u_1 = 3|u_1|^2 u_1 - 3(2u_1|u_2|^2 + \overline{u_1} u_2),$$

$$L u_2 = 3(2|u_1|^2 u_2 + u_1^2 \overline{u_2}) - 3|u_2|^2 u_2,$$

and the other is

$$L u_1 = 3|u_1|^2 u_1 - (2u_1|u_2|^2 + \overline{u_1} u_2^2),$$

$$L u_2 = (2|u_1|^2 u_2 + u_1^2 \overline{u_2}) - 3|u_2|^2 u_2.$$  

The corresponding ODE systems are studied in [17] and it is shown that the ODE systems admit solutions with polynomial deceleration. However, no PDE result is available on the large time behavior of solutions to these two systems. It is needless to say that these systems do not possess a good structure as (1.1) does.

Finally, let us briefly mention the strong nonlinear amplification phenomenon. In the case (1.1) with a complex coefficient $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the nonlinear dissipation occurs either forward or backward in time, as mentioned above. For this system, a different type of nonlinear amplification phenomenon takes place in the opposite time direction. The amplification is stronger.
than what we consider in this paper: At the ODE level, any nontrivial solution to the corresponding ODE equation $i A' = \lambda |A|^2 A$ ($\exists \lambda > 0$) blows up in finite time for positive time direction. At the PDE level, the first author [13] show that there exists an arbitrarily small (in $L^2$) initial datum which gives a solution blowing up in finite time, if $\exists \lambda > 0$ is sufficiently large compared with $|\Re \lambda|$. The existence of blow-up solutions is studied also in [3, 4, 12] without a smallness assumption on the data. It is also known that there exists a system of the form [13] (with real coefficients) for which strong nonlinear amplification phenomenon of this type occurs (see [15]).

1.2. Main result. To state our main result, let us introduce several notations. For $a \in \mathbb{R}$, $(a) := \sqrt{1+|a|^2}$. We denote $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$ holds and $A \sim B$ if $A \lesssim B \lesssim A$. It is also used with subscripts such as $A \lesssim_{a,b} B$ when we emphasize that the hidden constant $C$ depends on other quantities $a$ and $b$. For $1 \leq q \leq \infty$, let $L^q = L^q(\mathbb{R})$ denote the usual Lebesgue space with the norm

$$\|f\|_{L^q} = \begin{cases} \left( \int_{\mathbb{R}} |f(x)|^q \, dx \right)^{\frac{1}{q}} & 1 \leq q < \infty, \\ \text{ess.sup}_{x \in \mathbb{R}} |f(x)| & q = \infty. \end{cases}$$

$F$ and $F^{-1}$ stand for the Fourier and inverse Fourier transforms, respectively:

$$(F f)(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx,$$  

$$(F^{-1} f)(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) \, d\xi.$$  

The weighted Sobolev space $H^{s,m}$ is defined by

$$H^{s,m} = \{ f \in S' : \|f\|_{H^{s,m}} < \infty \},$$

$$\|f\|_{H^{s,m}} = \| (x)^m F^{-1} (\xi)^s F f \|_{L^2},$$

where $s, m \in \mathbb{R}$. For $t \neq 0$, we let $M(t) = e^{\frac{1}{2}t|x|^2}$ be a multiplication operator. We define the dilation operator by

$$(D(t)f)(x) = t^{-1/2} f(x/t) e^{-i\frac{t}{4} |x|^2}$$

for $t \neq 0$. It is well-known that the Schrödinger group $U(t) = \exp(it\partial_x^2/2)$ can be decomposed as

$$U(t) = M(t) D(t) F M(t).$$

Since $U(-t) = U(t)^{-1}$, we also have $U(-t) = M(t)^{-1} F^{-1} D(t)^{-1} M(t)^{-1}$. The standard generator of the Galilean transformation is given as

$$J(t) = U(t)xU(-t) = x + it \partial_x.$$

Note that the operators $J(t)$ and $\mathcal{L}$ commute. We often simply denote $J$ when the variable $t$ is clear from the context.

Recall that $\eta = 3\lambda_1 - 2\lambda_6$ and $\mu = \sqrt{\lambda_6^2 - \eta^2}$. Note that $|\mu \pm i\eta| = |\lambda_6| > 0$ holds. Our main result is the following.
Theorem 1.1. Let \( \varepsilon_j := \|u_{j,0}\|_{H^1} + \|u_{j,0}\|_{H^0,1} \) \((j = 1, 2)\). Then there exists \( \varepsilon_0 > 0 \) such that for any \( u_{j,0} \in H^1 \cap H^0,1 \) with \( \varepsilon_1 \leq \varepsilon_0 \) and without size-restriction on \( \varepsilon_2 \), there exists a unique global solution \((u_1, u_2) \in C(\mathbb{R}; H^1(\mathbb{R}) \cap H^0,1(\mathbb{R}))^2 \) to (1.1) satisfying
\[
\|u_1(t)\|_{H^1} + \|J u_1(t)\|_{L^2} \lesssim \varepsilon_1(t)^{C \varepsilon_2^2}, \quad \|u_1(t)\|_{L^\infty} \lesssim \varepsilon_1(t)^{-1/2},
\]
\[
\|u_2(t)\|_{H^1} + \|J u_2(t)\|_{L^2} \lesssim \varepsilon_2(t)^{C \varepsilon_2^2}, \quad \|u_2(t)\|_{L^\infty} \lesssim \varepsilon_2(t)^{-1/2 + C \varepsilon_2^2},
\]
for any \( t \in \mathbb{R} \). Furthermore, there exist two functions \( W_1, W_2 \in L^\infty \) such that
\[
u_1(t) = M(t) D(t) F_1(t) + O(t^{-\frac{3}{4} + C \varepsilon_2^2}),
\]
\[
u_2(t) = M(t) D(t) F_2(t) + O(t^{-\frac{3}{4} + C \varepsilon_2^2})
\]
in \( L^\infty \) as \( t \to \infty \), where
\[
F_1(t, \xi) = W_1(\xi) e^{-3 \lambda_1 |W_1(\xi)|^2 \log t},
\]
\[
F_2(t, \xi) = \tilde{W}_2(t, \xi) e^{-3 \lambda_1 |W_1(\xi)|^2 \log t}
\]
and \( \tilde{W}_2(t, \xi) \) is defined by \( \tilde{W}_2(t, \xi) := W_2(\xi) \) if \( W_1(\xi) = 0 \) and
\[
\tilde{W}_2(t, \xi) := \frac{1}{2 \mu} \left( (\mu - i \eta) W_2(\xi) + i \lambda_6 \frac{W_2(\xi)^2}{|W_1(\xi)|^2} \tilde{W}_2(\xi) \right) t^{-\mu|W_1(\xi)|^2}
\]
\[
+ \frac{1}{2 \mu} \left( (\mu + i \eta) W_2(\xi) - i \lambda_6 \frac{W_2(\xi)^2}{|W_1(\xi)|^2} \tilde{W}_2(\xi) \right) t^{\mu|W_1(\xi)|^2}
\]
if \( W_1(\xi) \neq 0 \).

Remark 1.1. If \( W_1 \) and \( W_2 \) are continuous then, for each fixed \( t > 0 \), \( \tilde{W}_2(t, \xi) \) is continuous in \( \xi \in \mathbb{R} \) including the zero points of \( W_1 \). Indeed, one sees this from \( \tilde{W}_2(t, \xi)^2 = W_2(\xi)^2(2 \mu \log t + o(1)) \to 0 \) as \( |W_1| \to 0 \).

Remark 1.2. The pair \((F_1, F_2)\) given by (1.13) and (1.14) is a unique solution to the following initial value problem of the system of ordinary differential equations:
\[
\begin{aligned}
i F_1' &= 3 \lambda_1 t^{-1} |F_1|^2 F_1, & t > 0 \\
i F_2' &= \lambda_6 t^{-1} (2|F_1|^2 F_2 + F_1^2 F_2), & t > 0, \\
F_1(1, \xi) &= W_1(\xi), & F_2(1, \xi) = W_2(\xi), & \xi \in \mathbb{R}.
\end{aligned}
\]
Notice that the system is the corresponding version of (1.7).

Since the function \( F_2 \) contains the factor \( t^{\mu|W_1|^2} \), the estimate (1.12) suggests that the second component \( u_2 \) decays in time slower than a linear solution does. However, in order to obtain a decay estimate from below, which makes the slow decay rigorous, we need to show that there exists a set \( \mathcal{O} \subset \mathbb{R} \) with positive measure such that \( W_1(\xi) \neq 0 \) and
\[
(\mu + i \eta) W_2(\xi) - i \lambda_6 \frac{W_2(\xi)^2}{|W_1(\xi)|^2} \tilde{W}_2(\xi) \neq 0
\]
hold on \( \mathcal{O} \). One may expect that a generic solution possesses this property and it would be actually possible to find a sufficient condition in terms of the initial data \((u_{1,0}, u_{2,0})\) which ensures the existence of the above set \( \mathcal{O} \).
We here take another way to prove the existence of a slowly-decaying solution. Namely, we consider the final state problem:

\[
\begin{cases}
L u_1 = 3\lambda_1 |u_1|^2 u_1, & t \in \mathbb{R}, x \in \mathbb{R}, \\
L u_2 = \lambda_6 (2|u_1|^2 u_2 + u_1^2 u_2), & t \in \mathbb{R}, x \in \mathbb{R}, \\
\|u_j(t) - u_{ap,j}(t)\|_{L^\infty} = o(\|u_{ap,j}(t)\|_{L^\infty}) & \text{as } t \to \infty, \ j = 1, 2,
\end{cases}
\]

where \(u_j : \mathbb{R} \times \mathbb{R} \to \mathbb{C} (j = 1, 2)\) are unknown functions, and \(\lambda_1\) and \(\lambda_6\) are real constants satisfying (1.2). Furthermore, \(u_{ap,j}\) are given functions defined by

\[
u_j = M(t) D(t) F_j(t, \cdot) = t^{\frac{1}{2}} e^{i \frac{\pi}{2} - i \frac{\pi}{2} t} F_j(t, \cdot), \quad j = 1, 2,
\]

where \(F_1\) and \(F_2\) are defined as in (1.13) and (1.14), respectively, from a pair of prescribed \(C\)-valued functions \(W_1\) and \(W_2\).

Our second result is as follows:

**Theorem 1.2.** Let \(W_1, W_2 \in H^2\). Fix \(\nu \in (1/2, 1)\) and \(\delta \in (0, \nu - 1/2)\). Then, if \(\|W_1\|_{H^2}\) is sufficiently small then there exist \(T = T(\nu, \delta, \|W_2\|_{H^2}) \geq 2\) and a solution \((u_1, u_2)\) in \(C([T, \infty); L^2(\mathbb{R}))^2\) such that \((J_{u_1}, J_{u_2}) \in C([T, \infty); L^2(\mathbb{R}))^2\) and

\[
\|u_1(t) - u_{ap,1}(t)\|_{L^2} \lesssim t^{-\nu}, \quad \|u_2(t) - u_{ap,2}(t)\|_{L^2} \lesssim t^{-\nu + \delta}
\]

in \(L^2(\mathbb{R})\) and

\[
\|u_1(t) - u_{ap,1}(t)\|_{L^\infty} \lesssim t^{-\nu - \frac{1}{2}}, \quad \|u_2(t) - u_{ap,2}(t)\|_{L^\infty} \lesssim t^{-\nu - \frac{1}{2} + \delta} \tag{1.19}
\]

in \(L^\infty(\mathbb{R})\) for \(t \geq T\). In particular, if \((W_1, W_2)\) is chosen so that \(W_1(\xi) \neq 0\) and

\[
(\mu + i \eta) W_2(\xi) - i \lambda_6 \frac{W_1(\xi)^2}{|W_1(\xi)|^2} \overline{W_2(\xi)} \neq 0 \tag{1.20}
\]

hold on \(\mathbb{R}\) in addition then

\[
\|u_2(t)\|_{L^\infty} \sim \|u_{ap,2}(t)\|_{L^\infty} \sim t^{-\frac{1}{2} + \mu} \|W_1\|_{L^\infty}.
\]

On the other hand, if \((W_1, W_2)\) is chosen so that

\[
(\mu + i \eta) W_2(\xi) - i \lambda_6 \frac{W_1(\xi)^2}{|W_1(\xi)|^2} \overline{W_2(\xi)} = 0 \tag{1.21}
\]

holds on \(\mathbb{R}\) then (1.19) implies

\[
u(t) = M(t) D(t) (t^{-\nu} |W_1|^2 W_2 e^{-i \lambda_1 |W_1|^2 \log t}) + O(t^{-\nu})
\]

as \(t \to \infty\).

**Remark 1.3.** A simple way to choose a pair of functions \((W_1, W_2)\) which satisfies the condition (1.20) is to take a pair of non-zero real-valued functions. For instance, the choice \((W_1, W_2) = (\varepsilon e^{-x^2}, e^{-x^2})\) works for small \(\varepsilon > 0\).

**Remark 1.4.** The upper bound of \(\nu\) can be compared with the asymptotic analysis for the linear equation. One has

\[
\|U(t)u_0 - t^{-\frac{1}{2}} e^{i \frac{\pi}{2} - i \frac{\pi}{2} t} F[u(0)](\cdot)\|_{L^p} \lesssim \|u_0\|_{H^{0.2}} e^{-t^{-\frac{1}{2} + \frac{1}{2p}}}.
\]
as \( t \to \infty \) for any \( p \in [2, \infty] \). The order in the right hand side is optimal. The estimates (1.18) and (1.19) almost reach to this order. They contain merely an epsilon loss.

The rest of the paper is organized as follows. We treat the Cauchy problem (1.1) in Section 2. We mainly prove the global existence and the asymptotic behavior of the second component \( u_2 \) to show Theorem 1.1. In section 3, we treat the final state problem and give the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

2.1. On the first component \( u_1 \). Note that \( u_1 \) satisfies the single equation

\[
\mathcal{L}u_1 = 3\lambda_1 |u_1|^2 u_1.
\]

As mentioned in the introduction, the asymptotic behavior is well-known.

Following the argument in Hayashi-Naumkin [8], one shows that if \( \varepsilon_1 > 0 \) is sufficiently small then (2.1) admits a unique time-global solution \( u_1(t) \) such that

\[
\| u_1(t) \|_{H^1} + \| Ju_1(t) \|_{L^2} \leq C\varepsilon_1(t)^{C_1} \varepsilon_1^2,
\]

\[
\| u_1(t) \|_{L^\infty} \leq C\varepsilon_1(t)^{-1/2}.
\]

Furthermore if we let \( v_1 := \mathcal{F}U(-t)u_1(t) \) then there exists some \( \alpha = \alpha(\xi) \in L^\infty \cap L^2 \) such that

\[
\| v_1(t) \|_{L^\infty} + \| \alpha \|_{L^\infty} \leq C_2 \varepsilon_1,
\]

\[
\| e^{3\lambda_1 \Phi_1(t)}v_1(t) - \alpha \|_{L^\infty} \leq C\varepsilon_1^{3/4} + 2C_1 \varepsilon_1^2
\]

for \( t \in [1, \infty) \), where \( \Phi_1 = \Phi_1(t, \xi) := \int_1^t \tau^{-1} |v_1(\tau)|^2 \, d\tau \).

Let us deduce (1.11). To this end, we introduce one lemma.

**Lemma 2.1.** One has \( \mathcal{F}M\mathcal{F}^{-1} = U(-1/t) \). Further, for any \( f \in H^1 \),

\[
\| (\mathcal{F}M\mathcal{F}^{-1} - 1)f \|_{L^\infty} \lesssim |t|^{-\frac{1}{2}} \| f \|_{H^1}
\]

for \( t \neq 0 \).

This is well-known. For reader’s convenience, we give a proof.

**Proof.** The identity is obvious. Let us prove (2.6). By Gagliardo-Nirenberg inequality,

\[
\| (\mathcal{F}M\mathcal{F}^{-1} - 1)f \|_{L^\infty} \lesssim \| (\mathcal{F}M\mathcal{F}^{-1} - 1)f \|_{L^2}^\frac{1}{2} \| (\mathcal{F}M\mathcal{F}^{-1} - 1)f \|_{H^1}^\frac{1}{2}
\]

Since \( U(-1/t) \) is unitary on \( \dot{H}^1 \), one has \( \| (\mathcal{F}M\mathcal{F}^{-1} - 1)f \|_{\dot{H}^1} \leq 2 \| f \|_{\dot{H}^1} \). Further, by using \( |M - 1| = |\sin(|x|^2/4t)| \leq (|x|^2/4|t|)^{1/2} \), one has

\[
\| (\mathcal{F}M\mathcal{F}^{-1} - 1)f \|_{L^2} \lesssim |t|^{-\frac{1}{2}} \| f \|_{H^1}.
\]

Hence, we obtain the result. \( \square \)
By Lemma 2.1, (2.2), (2.3) and the fact that \( \|v_1\|_{H^1} = \|Ju_1\|_{L^2} \), we see that
\[
u_1(t) = M(t)D(t)\mathcal{F}M(t)\mathcal{F}^{-1}v_1(t)
= M(t)D(t)v_1(t) + O(t^{-3/4+C_1\varepsilon_1^2})
= M(t)D(t)e^{-3i\lambda_1\Phi_1(t)\alpha} + O(t^{-3/4+2C_1\varepsilon_1^2}) \tag{2.7}
\]
in \( L^\infty \) as \( t \to \infty \). As for the phase correction, we write
\[
\Phi_1(t) = \int_1^t \tau^{-1}|v_1(\tau)|^2 \, d\tau
= |\alpha|^2 \log t + \int_1^t \tau^{-1}(|e^{3i\lambda_1 \Phi_1(\tau)}| - |\alpha|^2) \, d\tau.
\]
By virtue of (2.4) and (2.5), the above integral converges to some \( \theta_1 = \theta_1(\xi) \) in \( L^\infty \) as \( t \to \infty \), and hence we have
\[
\Phi_1(t) = |\alpha|^2 \log t + \theta_1 + O(t^{-1/4+C_1\varepsilon_1^2})
\]
in \( L^\infty \) as \( t \to \infty \). Plugging it into (2.7) and setting \( W_1 := e^{-3i\lambda_1\theta_1\alpha} \), we have
\[
u_1(t) = M(t)D(t)e^{-3i\lambda_1|\alpha|^2\log t}W_1 + O(t^{-3/4+C_1\varepsilon_1^2})
= M(t)D(t)e^{-3i\lambda_1|W_1|^2\log t}W_1 + O(t^{-3/4+C_1\varepsilon_1^2}),
\]
which is (1.11). Further, we see from (2.5) that
\[
\|e^{3i\lambda_1(\Phi_1-\theta_1)}v_1(t) - W_1\|_{L^\infty} \leq C\varepsilon_1 t^{-1/4+2C_1\varepsilon_1^2}. \tag{2.8}
\]

2.2. Global bound of the solution. Let us move on to the analysis of the second component \( u_2 \). Due to a standard local theory, one obtains a time-local solution
\[
u_2 \in C(I; H^1 \cap H^{0,1}), \quad 0 \in \exists I \subset \mathbb{R}.
\]
See [2], for instance. Notice that the second equation of the system (1.1) is a linear equation in \( u_2 \). Hence, it is obvious that if \( u_1 \) exists globally in time then so does \( u_2 \).

The main issue in this subsection is to obtain the bounds (1.10) on \( u_2 \). Recall that \( \varepsilon_2 := \|u_{2,0}\|_{H^2} + \|u_{2,0}\|_{H^{0,1}} \). For \( \delta \in (0, 1/5) \), we introduce
\[
T^* := \sup \left\{ T > 0 : \sup_{0 \leq t < T} \langle t \rangle^{-\delta} (\|u_2(t)\|_{H^1} + \|Ju_2(t)\|_{L^2}) < 4\varepsilon_2 \right\}. \tag{2.9}
\]
The main step of the proof is to show \( T^* = \infty \).

Before proving this, let us collect basic facts on \( u_2 \). Note that \( u_2 \) solves
\[
u_2(t) = U(t-t_0)u_2(t_0) - i\lambda_6 \int_{t_0}^t U(t-\tau)N_2(u_1(\tau), u_2(\tau)) \, d\tau, \tag{2.10}
\]
for any fixed \( t_0 \in \mathbb{R} \), where
\[
N_2(u_1, u_2) = 2|u_1|^2u_2 + u_1^2\overline{u_2}.
\]
One has
\[ J u_2(t) = U(t-t_0)J u_2(t_0) - i\lambda_6 \int_{t_0}^{t} U(t-\tau)J N_2(u_1, u_2)(\tau) \, d\tau. \] (2.11)

The following identity is useful:
\[
J N_2(u_1, u_2) = 2|u_1|^2 J u_2 - u_1^2 \overline{J u_2} + 2(\overline{\overline{u}_1 u_2} J u_1 - u_1 u_2 \overline{J u_1}) + 2 u_1 \overline{u}_2 J u_1.
\] (2.12)

Let us begin with the following lemma.

**Lemma 2.2.** For any \( T_0 > 1 \), there exists \( \varepsilon_1^* > 0 \) such that if \( \varepsilon_1 \in (0, \varepsilon_1^*) \), then
\[
\sup_{t \in [0, T_0]} (||u_2(t)||_{H^1} + ||J u_2(t)||_{L^2}) \leq 2\varepsilon_2.
\]

In particular, \( T^* > T_0 \) for such \( \varepsilon_1 \), where \( T^* \) is given in (2.9).

**Proof.** Plugging (2.2) and (2.3) to (2.10) and (2.11) with \( t_0 = 0 \), we have
\[
||u_2(t)||_{H^1} + ||J u_2(t)||_{L^2} \leq \varepsilon_2 + C \int_0^t (||u_1(\tau)||_{H^1} + ||J u_1(\tau)||_{L^2})^2 (||u_2(\tau)||_{H^1} + ||J u_2(\tau)||_{L^2}) \, d\tau
\]
\[
\leq \varepsilon_2 + C \varepsilon_1^2 \int_0^t \langle \tau \rangle^{2C_1 \varepsilon_1^2} (||u_2(\tau)||_{H^1} + ||J u_2(\tau)||_{L^2}) \, d\tau,
\]
where we have used (2.12) to estimate the nonlinearity. Hence, Gronwall’s inequality yields
\[
||u_2(t)||_{H^1} + ||J u_2(t)||_{L^2} \leq (1 + C \varepsilon_1^2 (T_0)^{1+2C_1 \varepsilon_1^2} e^{C \varepsilon_1^2 (T_0)^{1+2C_1 \varepsilon_1^2}}) \varepsilon_2 \leq 2\varepsilon_2
\] (2.13)
for \( t \in [0, T_0] \) if \( \varepsilon_1 \) is sufficiently small. Together with the definition of \( T^* \), one obtains \( T^* > T_0 \) as desired. \( \square \)

We introduce notation. Let \( v_2(t) := \mathcal{F} U(-t) u_2 \), or equivalently,
\[
u_2(t) = M(t) D(t) \mathcal{F} M(t) \mathcal{F}^{-1} v_2(t).
\] (2.14)
We further introduce
\[ w(t, \xi) := e^{3i\lambda_1(\Phi_1(t)-\theta_1)} v_2(t). \]

We have the following formula for \( w \):

**Lemma 2.3.** The following formula for \( w(t) \) is valid:
\[
\left( \frac{w(t)}{w(1)} \right) = PQ(t) P^{-1} \left( \frac{w(1)}{w(1)} \right) + PQ(t) \int_1^t Q(\tau)^{-1} P^{-1} \left( \frac{S_2 + e^{3i\lambda_1(\Phi_1-\theta_1)} R_2}{S_2 + e^{3i\lambda_1(\Phi_1-\theta_1)} R_2} \right) (\tau) \, d\tau
\] (2.15)
for $t \geq 1$, where $e^{i\theta} = W_1/|W_1|$, 

$$P = \begin{pmatrix} \lambda_6 e^{2i\theta} & \eta - i \mu \\
-\eta - i \mu & \lambda_6 e^{-2i\theta} \end{pmatrix},$$

$$P^{-1} = \begin{pmatrix} 1 & \frac{1}{2i\mu(\eta - i \mu)} \left( \lambda_6 e^{-2i\theta} \right) \\
-\frac{1}{\eta - i \mu} & \lambda_6 e^{2i\theta} \end{pmatrix},$$

$$Q(t) = \begin{pmatrix} t^{-\|W_1\|^2} & 0 \\
0 & \mu t\|W_1\|^2 \end{pmatrix},$$

for $\xi \in \mathbb{R}$ such that $W_1(\xi) \neq 0$, and

$$R_2(t) = -i\lambda_6 t^{-1} FM(t)^{-1} F^{-1} N_2(F M F^{-1} v_1, F M F^{-1} v_2)(t)$$
$$+ i\lambda_6 t^{-1} N_2(v_1, v_2)(t),$$

and

$$S_2(t) = i\eta t^{-1} \left( |e^{3i\lambda_1(\Phi_1 - \theta_1)} v_1(t)|^2 - |W_1|^2 \right) w(t)$$
$$- i\lambda_6 t^{-1} \left( |e^{3i\lambda_1(\Phi_1 - \theta_1)} v_1(t)|^2 - W_1^2 \right) w(t).$$

The identity is valid also on $\{W_1(\xi) = 0\}$ with the convention $P Q(t) P^{-1} = I_2$, where $I_2$ is the identity matrix diag(1, 1).

**Proof.** By definition of $R_2$, one sees that $v_2$ satisfies

$$i \partial_t v_2 = \lambda_6 t^{-1} FM(t)^{-1} F^{-1} N_2(F M(t) F^{-1} v_1, F M(t) F^{-1} v_2)$$
$$+ \lambda_6 t^{-1} N_2(v_1, v_2) + i R_2(s).$$

Hence, recalling the definition of $N_2$, one has

$$i \partial_t w = e^{3i\lambda_1(\Phi_1 - \theta_1)} i \partial_t v_2 - 3\lambda_1 t^{-1} |v_1|^2 w$$
$$= -\eta t^{-1} |v_1|^2 w + \lambda_6 t^{-1} (e^{3i\lambda_1(\Phi_1 - \theta_1)} v_1)^2 w + i e^{3i\lambda_1(\Phi_1 - \theta_1)} R_2(t)$$
$$= -\eta t^{-1} |W_1|^2 w + \lambda_6 t^{-1} |W_1|^2 w + i (S_2(t) + e^{3i\lambda_1(\Phi_1 - \theta_1)} R_2(t)),

(2.18)

where $\eta = 3\lambda_1 - 2\lambda_6$.

If $W_1(\xi_0) = 0$ then we have

$$w(t, \xi_0) = w(1, \xi_0) + \int_1^t (S_2 + e^{3i\lambda_1(\Phi_1 - \theta_1)} R_2)(\tau, \xi_0) d\tau,$$

which gives us (2.15) under the convention $P Q(t) P^{-1} = I_2$.

In the sequel we suppose that $W_1(\xi_0) \neq 0$. For simplicity, we omit $\xi_0$. We write (2.18) in a matrix form:

$$\partial_t \begin{pmatrix} w \\ \overline{w} \end{pmatrix} = it^{-1} |W_1|^2 \begin{pmatrix} \eta & -\lambda_6 e^{2i\theta} \\
-\lambda_6 e^{-2i\theta} & \eta \end{pmatrix} \begin{pmatrix} w \\ \overline{w} \end{pmatrix} + \begin{pmatrix} S_2 + e^{3i\lambda_1(\Phi_1 - \theta_1)} R_2 \\
S_2 + e^{3i\lambda_1(\Phi_1 - \theta_1)} R_2 \end{pmatrix},$$

where $e^{i\theta} = W_1/|W_1|$. The $2 \times 2$ matrix on the first term of the right hand side possesses the eigenvalues $\pm i \mu$. Further, the eigenvectors associated with the eigenvalue $\pm i \mu$ are $(\lambda_6 e^{2i\theta} \eta - i \mu)^T$ and $(\eta - i \mu \lambda_6 e^{-2i\theta})^T$, respectively. Then, from the diagonalization of the matrix, it follows that

$$\partial_t \begin{pmatrix} w \\ \overline{w} \end{pmatrix} = it^{-1} |W_1|^2 P \begin{pmatrix} i \mu & 0 \\
0 & -i \mu \end{pmatrix} P^{-1} \begin{pmatrix} w \\ \overline{w} \end{pmatrix} + \begin{pmatrix} S_2 + e^{3i\lambda_1(\Phi_1 - \theta_1)} R_2 \\
S_2 + e^{3i\lambda_1(\Phi_1 - \theta_1)} R_2 \end{pmatrix}.$$
We further rewrite it in such a way that
\[ \partial_t Q(t)^{-1} P^{-1} \left( \frac{w}{w} \right) = Q(t)^{-1} P^{-1} \left( \frac{S_2(t) + e^{3\lambda_1(\Phi_1 - \theta_1)} R_2(t)}{S_2(t) + e^{3\lambda_1(\Phi_1 - \theta_1)} R_2(t)} \right). \] (2.19)

An integration in time gives us the desired formula \[ \text{(2.15)}. \]

We now estimate the error terms \( R_2 \) and \( S_2 \).

**Lemma 2.4.** Let \( S_2 \) be defined in \( (2.17) \). Then there exists some positive constant \( C \) such that
\[ \| S_2(t) \|_{L^\infty} \leq C \varepsilon_1^2 t^{-\frac{\lambda}{t} + 2C_1} \| v_2(t) \|_{L^\infty} \] (2.20)
for any \( t \geq 1 \).

**Proof.** It is obvious by
\[ \| S_2(t) \|_{L^\infty} \lesssim t^{-1 \varepsilon_1^2} \| v_1 e^{3\lambda_1(\Phi_1 - \theta_1)} - W_1 \|_{L^\infty} (\| v_1(t) \|_{L^\infty} + \| W_1 \|_{L^\infty}) \| v_2(t) \|_{L^\infty} \]
and \( (2.4) \) and \( (2.8) \). \( \square \)

**Lemma 2.5.** Let \( \varepsilon_2 = \| u_{2,0} \|_{H^1} + \| u_{2,0} \|_{H^0.1} \) and \( R_2 \) be defined in \( (2.16) \). Then there exists some positive constant \( C \) such that
\[ \| R_2(t) \|_{L^\infty} \leq C \varepsilon_2^2 t^{-\frac{\lambda}{t} + 2C_1} \| v_2(t) \|_{H^1} \] (2.21)
for any \( t \geq 1 \).

**Proof.** We have
\[ R_2 = -i\lambda_6 t^{-1} F(M^{-1} - 1) F^{-1} N_2(FMF^{-1} v_1, FMF^{-1} v_2) \]
\[ - i\lambda_6 t^{-1} (N_2(FMF^{-1} v_1, FMF^{-1} v_2) - N_2(v_1, FMF^{-1} v_2)) \]
\[ - i\lambda_6 t^{-1} (N_2(v_1, FMF^{-1} v_2) - N_2(v_1, v_2)) \]
\[ =: R_{21} + R_{22} + R_{23}. \]
Combining \( (2.6) \), the fact that \( H^1(\mathbb{R}) \) is an algebra and that \( U(-1/t) \) is unitary on \( H^1(\mathbb{R}) \), we have
\[ \| R_{21} \|_{L^\infty} \lesssim t^{-\frac{\lambda}{t} \varepsilon_1^2} \| N_2(U(-1/t)v_1, U(-1/t)v_2) \|_{H^1} \]
\[ \lesssim t^{-\frac{\lambda}{t} \varepsilon_1^2} \| U(-1/t)v_1 \|_{H^1}^2 \| U(-1/t)v_2 \|_{H^1} = t^{-\frac{\lambda}{t} \varepsilon_1^2} \| v_1 \|_{H^1}^2 \| v_2 \|_{H^1} \]
for \( t \geq 1 \). Similarly, one has
\[ \| R_{22} \|_{L^\infty} \| R_{23} \|_{L^\infty} \lesssim t^{-\frac{\lambda}{t} \varepsilon_1^2} \| v_1 \|_{H^1}^2 \| v_2 \|_{H^1}. \]
Hence, using the fact that \( \| v_1 \|_{H^1} = \| u_1 \|_{L^2} + \| Ju_1 \|_{L^2} \) and the bound \( (2.2) \) for \( u_1 \), we obtain the result. \( \square \)

Now we are in the position to complete the proof of the global bound.

**Lemma 2.6.** \( T^* = \infty \) for sufficiently small \( \varepsilon_1 \), where \( T^* \) is given in \( (2.9) \).

**Proof.** We prove the lemma by contradiction. Fix \( \delta \in (0, 1/5) \) and assume that \( T^* \) is finite for any small \( \varepsilon_1 > 0 \). Then, we have a bound
\[ \| u_2(t) \|_{H^1} + \| Ju_2(t) \|_{L^2} < 4\varepsilon_2(t)^{\delta} \] (2.22)
for \( t \in (0, T^*) \). Further, by continuity of the solution, one has
\[ \| u_2(T^*) \|_{H^1} + \| Ju_2(T^*) \|_{L^2} = 4\varepsilon_2(T^*)^{\delta}. \]
To obtain a contradiction, we shall show (2.22) gives us a better bound

$$\|u_2(t)\|_{H^1} + \|J u_2(t)\|_{L^2} \leq 3\varepsilon_2(t)^{\delta}$$  \hspace{1cm} (2.23)

for $t \in (0, T^*)$ and sufficiently small $\varepsilon_1$. Once (2.23) is established, by letting $t \uparrow T^*$ in (2.22), we see that $4\varepsilon_2 \leq 3\varepsilon_2$. This is the contradiction, and we have $T^* = \infty$.

Let $T_0 > 1$ be a number to be chosen later. Note that if $\varepsilon_1 < \varepsilon_1^*(T_0)$ then $T^* > T_0$ and (2.14) gives us the desired bound for $t \in [0, T_0]$.

We obtain the estimates for large $t$. We deduce from (2.11) with $t_0 = 1$ that

$$\|J u_2(t)\|_{L^2} \leq \|J u_2(1)\|_{L^2} + C \int_1^t (\|u_1\|_{L^\infty}^2 \|J u_2\|_{L^2} + \|u_1\|_{L^\infty} \|u_2\|_{L^\infty} \|J u_1\|_{L^2}) \, d\tau. \quad (2.24)$$

Similarly,

$$\|u_2(t)\|_{H^1} \leq \|u_2(1)\|_{H^1} + C \int_1^t (\|u_1\|_{L^\infty}^2 \|u_2\|_{H^1} + \|u_1\|_{L^\infty} \|u_2\|_{L^\infty} \|u_1\|_{H^1}) \, d\tau. \quad (2.25)$$

Applying (2.24), (2.25), and (2.14) to (2.24) and (2.25), we see that

$$\|u_2(t)\|_{H^1} + \|J u_2(t)\|_{L^2} \leq 2\varepsilon_2 + C\varepsilon_1^2 \int_1^t \tau^{-1} (\|u_2(\tau)\|_{H^1} + \|J u_2(\tau)\|_{L^2}) \, d\tau + C\varepsilon_1^2 \int_1^t \tau^{-1/2+C\varepsilon_1^2} \|u_2(\tau)\|_{L^\infty} \, d\tau \quad (2.26)$$

for $t \geq 1$ if $\varepsilon_1 \leq \varepsilon_1^*(1)$.

We next claim that (2.22) implies that

$$\|u_2(t)\|_{L^\infty} \leq C\varepsilon_2 (1 + \varepsilon_1^2) t^{-1/2+\mu C_2^2 \varepsilon_1^2} \quad (2.27)$$

for $t \in [1, T^*)$. It follows from (2.22) that

$$\|u_2(t)\|_{L^\infty} = \|M(t)D(t)F M(t)F^{-1} v_2(t)\|_{L^\infty} \leq t^{-1/2} \|v_2(t)\|_{L^\infty} + t^{-1/2} \|F(M(t) - 1)F^{-1} v_2(t)\|_{L^\infty} \leq t^{-1/2} \|v_2(t)\|_{L^\infty} + C t^{-3/4} \|J u_2(t)\|_{L^2} \leq t^{-1/2} \|v_2(t)\|_{L^\infty} + C \varepsilon_2 t^{-3/4+\delta}. \quad (2.28)$$

on the interval $[1, T^*)$. The second term is acceptable since $\delta < 1/4$. We estimate $\|v_2(t)\|_{L^\infty} = \|w(t)\|_{L^\infty}$. By

$$Q(t) Q(\tau)^{-1} = Q(t \tau^{-1})$$

for $1 \leq \tau \leq t$, (2.13) and (2.24) yield

$$\|v_2(t)\|_{L^\infty} \leq C t^{\mu C_2^2 \varepsilon_1^2} \|w(1)\|_{L^\infty} + C t^{\mu C_2^2 \varepsilon_1^2} \int_1^t (\|S_2(\tau)\|_{L^\infty} + \|R_2(\tau)\|_{L^\infty}) \, d\tau.$$
Further, applying Lemmas 2.24 and 2.25, the embedding $H^1 \hookrightarrow L^\infty$, the identities $\|v_2(t)\|_{L^2} = \|u_2(t)\|_{L^2}$ and $\|v_2\|_{H^1} = \|Ju_2\|_{L^2}$, and (2.22), we have
\[
\|v_2(t)\|_{L^\infty} \leq \varepsilon_2 t^{C_2 \varepsilon_1^2} + \varepsilon_1^2 t^{C_2 \varepsilon_1^2} \int_1^t (\tau^{-5/4 + 2C_1 \varepsilon_1^2 + \delta} d\tau
\leq \varepsilon_2 t^{C_2 \varepsilon_1^2} + \varepsilon_1^2 t^{C_2 \varepsilon_1^2}
\]
for $t \in [1, T^*)$, if $\varepsilon_1 > 0$ is taken so that $2C_1 \varepsilon_1^2 \leq 1/40$. Note that the implicit constants are independent of $\varepsilon_1$ and $\delta$ since $2C_1 \varepsilon_1^2 + \delta \leq 1/40 + 1/5 < 1/4$. Plugging (2.24) to (2.25), we reach to the claim (2.27).

Applying the bound (2.27) to the third term on the right hand side of (2.24), and we have
\[
\|u_2(t)\|_{H^1} + \|Ju_2(t)\|_{L^2} \leq 2\varepsilon_2 + C_3\varepsilon_2 t(C_1 + C_2^2 \mu)\varepsilon_1^2 + C_4\varepsilon_1^2 \int_1^t \tau^{-1}(\|u_2(\tau)\|_{H^1} + \|Ju_2(\tau)\|_{L^2}) d\tau.
\]
Without loss of generality, one may suppose that $C_4 \geq 2(C_1 + C_2^2 \mu)$. Then, Gronwall’s inequality yields
\[
\|u_2(t)\|_{H^1} + \|Ju_2(t)\|_{L^2} \leq 2\varepsilon_2 + C_3\varepsilon_2 t(C_1 + C_2^2 \mu)\varepsilon_1^2 + C_4\varepsilon_1^2 \int_1^t \tau^{-1}(C_1 + C_2^2 \mu)\varepsilon_1^2 d\tau
\leq 2\varepsilon_2 + C_3\varepsilon_2 t(C_1 + C_2^2 \mu)\varepsilon_1^2 + C_4\varepsilon_1^2 \int_1^t \tau^{-1}(C_1 + C_2^2 \mu)\varepsilon_1^2 d\tau
\leq (2 + C_3/2)\varepsilon_2 t^{C_4 \varepsilon_1^2}
\]
for $t \geq 1$.

We first let $\varepsilon_1$ so small that $C_4\varepsilon_1^2 \leq \delta/2$ holds. Then, we next choose $T_0 > 1$ so that $(2 + C_3/2)T_0^{\delta/2} \leq 3T_0^{\delta}$. If $\varepsilon_1 < \varepsilon_1^*(T_0)$, we see from (2.13) that $T^* > T_0$ and
\[
\sup_{t \in [0, T_0]} (\|u_2(t)\|_{H^1} + \|Ju_2(t)\|_{L^2}) \leq 2\varepsilon_2.
\]
On the other hand, thanks to the choice of $T_0$, (2.30) gives us
\[
\sup_{t \in [T_0, T^*)} (t)^{-\delta}(\|u_2(t)\|_{H^1} + \|Ju_2(t)\|_{L^2}) \leq 3\varepsilon_2.
\]
Thus, we obtain (2.23). □

Let us complete the proof of the bound (1.10). If $\varepsilon_1$ is sufficiently small, we have $T^* = \infty$. Then, the claim (2.27) holds for $t \in [1, \infty)$. Further, arguing as in the proof of (2.30), we obtain
\[
\|u_2(t)\|_{H^1} + \|Ju_2(t)\|_{L^2} \leq \varepsilon_2 t^{C_4 \varepsilon_1^2}
\]
for $t \geq 1$. Combining this with (2.13), we obtain the desired estimate (1.10).
2.3. Asymptotic behavior. We complete the proof of Theorem 1.1 by establishing \((1.12)\).

Proof of Theorem \((1.12)\). Let \(\Phi_1 \in L^{\infty}\) and \(W_1 \in L^{\infty}\) be the functions defined in the proof of \((1.11)\).

If \(W_1(\xi_0) = 0\) at some \(\xi_0 \in \mathbb{R}\) then one sees from Lemmas \(2.4\) and \(2.5\) \((2.18)\) and \((2.31)\) that there exists \(\beta_0 = \beta_0(\xi_0)\) such that
\[
w(t, \xi_0) = \beta(\xi_0) + O(t^{-1/4+C_1\varepsilon_1^2}). \tag{2.32}
\]
Note that \(\beta_0\) and the second term of the right hand side are bounded uniformly in \(\xi_0 \in \mathbb{R}\).

We now consider the case \(W_1(\xi_0) \neq 0\). Plugging Lemmas \(2.4\) and \(2.5\) and \(2.31\) to \(2.19\), we see that there exist some \(\beta_1, \beta_2 \in L^{\infty}\) such that
\[
Q(t)^{-1}P^{-1} \left( \frac{w(\xi_0)}{w(\xi_0)} \right) = \left( \begin{array}{c} \beta_1(\xi_0) \\ \beta_2(\xi_0) \end{array} \right) + O(\varepsilon_1^2\varepsilon_2 t^{-1/4+C_1\varepsilon_1^2+2\mu C_2\varepsilon_1^2+C_3\varepsilon_1^2}) \tag{2.33}
\]
as \(t \to \infty\). The second term of the right hand side is bounded uniformly in \(\xi_0 \in \mathbb{R}\). Let us introduce
\[
Z := \left\{ \left( \begin{array}{c} z \\ \bar{z} \end{array} \right) \in \mathbb{C}^2 \mid z \in \mathbb{C} \right\},
\]
which is a closed subspace of \(\mathbb{C}^2\). It follows from \((2.33)\) that
\[
\lim_{t \to \infty} t^{-\mu|W_1|^2} P^{-1} \left( \frac{w(\xi_0)}{w(\xi_0)} \right) = \lim_{t \to \infty} \left( t^{-2\mu|W_1|^2} 0 1 \right) Q(t)^{-1} P^{-1} \left( \frac{w(\xi_0)}{w(\xi_0)} \right) = \left( \begin{array}{c} 0 \\ 0 \\ \beta_1(\xi_0) \\ \beta_2(\xi_0) \end{array} \right).
\]
This implies that
\[
P \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) = P \lim_{t \to \infty} t^{-\mu|W_1|^2} P^{-1} \left( \frac{w(\xi_0)}{w(\xi_0)} \right) = \lim_{t \to \infty} t^{-\mu|W_1|^2} \left( \frac{w(\xi_0)}{w(\xi_0)} \right) \in Z,
\]
since \(Z\) is closed. Further, if \(\varepsilon_1\) is sufficiently small then one has
\[
\lim_{t \to \infty} t^{\mu|W_1|^2} \left( P^{-1} \left( \frac{w(\xi_0)}{w(\xi_0)} \right) - \mu|W_1|^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right) = \lim_{t \to \infty} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) Q(t)^{-1} P^{-1} \left( \frac{w(\xi_0)}{w(\xi_0)} \right) + \lim_{t \to \infty} \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \left( Q(t)^{-1} P^{-1} \left( \frac{w(\xi_0)}{w(\xi_0)} \right) - \left( \begin{array}{c} \beta_1(\xi_0) \\ \beta_2(\xi_0) \end{array} \right) \right)
\]
by means of \((2.33)\). Hence one sees that
\[
P \left( \begin{array}{c} 1 \\ 0 \\ \beta_1(\xi_0) \\ \beta_2(\xi_0) \end{array} \right).
\]
takes value in $Z$ by arguing as above. Hence,
\[
P \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}
\]
also takes value in $Z$. Thus, there exists $W_2 \in L^\infty(\{W_1 \neq 0\})$ such that
\[
\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = P^{-1} \begin{pmatrix} W_2 \\ W_2 \end{pmatrix}.
\]
Then it follows that
\[
w = (1 \\ 0) \, PQ(t)P^{-1} \begin{pmatrix} W_2 \\ W_2 \end{pmatrix} + O(t^{-1/4+C_1\varepsilon_1^2+2\mu_1\varepsilon_1^2+C_4\varepsilon_1^2})
\]
in $L^\infty(\{W_1 \neq 0\})$. We remark that, in view of \ref{1.12}, the identity is valid also on the set \{\$W_1 = 0$\} by using the convention \$PQ(t)P^{-1} = I_2$ and extending $W_2$ to the set by \$W_2(\xi) := \beta_0(\xi)$\. Note that the extension $W_2$ belongs to $L^\infty(\mathbb{R})$. Let us now recall that \$w = e^{3i\lambda_1(\Phi_1-\theta_1)}v_2$\. Then, in $L^\infty(\mathbb{R})$, we have
\[
u_2(t) = M(t)D(t)FM(t)F^{-1}v_2(t)
\]
\[= M(t)D(t)v_2(t) + O(t^{-3/4+C_1\varepsilon_1^2}).
\]
Plugging \ref{2.31} and $e^{-3i\lambda_1(\Phi_1-\theta_1)} = e^{-3i\lambda_1|W_1|^2\log t + O(t^{-1/4+C_1\varepsilon_1^2})}$ in $L^\infty$ to the identity, we see that
\[
u_2(t) = M(t)D(t)e^{-3i\lambda_1|W_1|^2\log t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, PQ(t)P^{-1} \begin{pmatrix} W_2 \\ W_2 \end{pmatrix}
\]
\[+ O(t^{-3/4+C_1\varepsilon_1^2+2\mu_1\varepsilon_1^2+C_4\varepsilon_1^2}).
\]
One sees that the leading part is written as in \ref{1.11}. This completes the proof of Theorem \ref{1.1}.

\section{Proof of Theorem \ref{1.2}}

As mentioned in the introduction, we will construct the solution by solving the final state problem:
\[
\begin{aligned}
L \nu_1 &= 3\lambda_1|u_1|^2u_1, & t \in \mathbb{R}, x \in \mathbb{R}, \\
L \nu_2 &= \lambda_6(2|u_1|^2u_2 + u_1^2\nu_2), & t \in \mathbb{R}, x \in \mathbb{R}, \\
\|\nu_j(t) - \nu_{ap,j}(t)\|_{L^\infty} &= o(\|\nu_{ap,j}(t)\|_{L^\infty}), & j = 1, 2.
\end{aligned}
\]

\subsection{Reformulation as an integral equation}

Let us first reformulate \ref{1.15} as an integral equation employing the argument in \ref{9}. Introduce a modified approximate solution $\tilde{\nu}_{ap,j}(t)$ defined by
\[
\tilde{\nu}_{ap,j}(t) := U(t)F^{-1}[F_j(t, \xi)]
\]
with $F_j$ given by \ref{1.13} and \ref{1.14}. Let $v_j := \nu_j - \nu_{ap,j}$ $(j = 1, 2)$ and let
\[
\begin{aligned}
\tilde{N}_1(u_1, u_2) &= 3\lambda_1|u_1|^2u_1, \\
\tilde{N}_2(u_1, u_2) &= \lambda_6(2|u_1|^2u_2 + u_1^2\nu_2).
\end{aligned}
\]
Then, at least formally, we have for $j = 1, 2$,
\[
\mathcal{L}(u_j - \tilde{u}_{ap,j}) = \tilde{N}_j(u_1, u_2) - \mathcal{L}\tilde{u}_{ap,j}
\]
\[
= \tilde{N}_j(v_1 + u_{ap,1}, v_2 + u_{ap,2}) - \mathcal{L}\tilde{u}_{ap,j}
\]
\[
= \tilde{N}_j(v_1 + u_{ap,1}, v_2 + u_{ap,2}) - \tilde{N}_j(u_{ap,1}, u_{ap,2}) + \mathcal{E}_j,
\tag{3.1}
\]
where
\[
\mathcal{E}_j := -\mathcal{L}\tilde{u}_{ap,j} - \tilde{N}_j(u_{ap,1}, u_{ap,2}).
\tag{3.2}
\]
By the Duhamel principle, (3.1) is written as
\[
v_j(t) = i \int_0^\infty U(t - \tau) \left\{ \tilde{N}_j(v_1 + u_{ap,1}, v_2 + u_{ap,2}) - \tilde{N}_j(u_{ap,1}, u_{ap,2}) \right\}(\tau) d\tau
\]
\[
+ i \int_0^\infty U(t - \tau) \mathcal{E}_j(\tau)d\tau + R_j.
\tag{3.3}
\]
where
\[
R_j := \tilde{u}_{ap,j} - u_{ap,j}.
\tag{3.4}
\]
Note that the formula is chosen so that the integrands decay as $t \to \infty$.

To show the existence of $(v_1, v_2)$ satisfying (3.3), we shall prove that the map $\Phi = (\Phi_1, \Phi_2)$ defined by
\[
\Phi_j[(v_1, v_2)](t)
\]
\[
= i \int_0^\infty U(t - \tau) \left\{ \tilde{N}_j(v_1 + u_{ap,1}, v_2 + u_{ap,2}) - \tilde{N}_j(u_{ap,1}, u_{ap,2}) \right\}(\tau) d\tau
\]
\[
+ i \int_0^\infty U(t - \tau) \mathcal{E}_j(\tau)d\tau + R_j
\]
is a contraction on
\[
X_T = \{(v_1, v_2) \in C([T, \infty); L^2(\mathbb{R}); \|(v_1, v_2)||_{X_T} \leq 1)\},
\]
\[
\|(v_1, v_2)||_{X_T} = \sup_{t \geq T} \left( t^{\tilde{\nu} + \frac{1}{2}} v_1 ||v_1||_{L^2_x} + t^{\tilde{\nu}} ||Jv_1||_{L^2_x} + t^{\tilde{\nu} + \frac{1}{2} - \delta} ||v_2||_{L^2_x} + t^{\tilde{\nu} - \delta} ||Jv_2||_{L^2_x} \right)
\]
for some $T \geq 2$, where $\tilde{\nu} := \nu - 1/2 \in (0, 1/2)$ and $\delta \in (0, \tilde{\nu})$ are arbitrarily small numbers.

**Remark 3.1.** As seen below, the space $X_T$ is chosen so that decay of $||v_j||_{L^\infty}$ can be deduced.

**Remark 3.2.** Our definition of a solution to (1.16) is a pair of functions $(v_1 + u_{ap,1}, v_2 + u_{ap,2})$ such that $(v_1, v_2) \in X_T$ holds for some $T \geq 2$ and that $(v_1, v_2)$ satisfies (3.3) in $C([T, \infty), L^2)$ sense. Under a suitable regularity assumption on $u_{ap,j}$, one may see that $u_j := v_j + u_{ap,j}$ belongs to $C([T, \infty), H^1)$. Then, $(u_1, u_2)$ satisfies the differential equation (1.16) on $[T, \infty)$ in the $H^{-1}$ sense.

### 3.2. Estimates on approximate solutions.

Let us summarize decay properties of the given asymptotic profiles $u_{ap,j}$ and $\tilde{u}_{ap,j}$ and of the differences $R_j$ of those two. We also give an estimate of the error term $\mathcal{E}_j$. We assume $||W_1||_{H^2_x} \leq \varepsilon_1 \leq 1$ and $||W_2||_{H^2_x} \leq \varepsilon_2$. 

Lemma 3.1. Let \( u_{ap,1} \) be given by (1.17) and let \( \mathcal{R}_1 \) be defined by (3.4). It holds that

\[
\begin{align*}
\|u_{ap,1}\|_{L^\infty_x} &\lesssim \varepsilon_1 t^{-\frac{1}{2}}, \\
\|\mathcal{R}_1\|_{L^2_x} &\lesssim \varepsilon_1 t^{-1} (\log t)^2, \\
\|\mathcal{R}_1\|_{L^\infty_x} &\lesssim \varepsilon_1 t^{-\frac{3}{4}} (\log t)^2,
\end{align*}
\]  

(3.5)  

(3.6)  

(3.7)  

(3.8)  

(3.9)  

for any \( t \geq 2 \). In particular,

\[
\|\tilde{u}_{ap,1}\|_{L^\infty_x} \lesssim \varepsilon_1 t^{-\frac{1}{2}}
\]  

(3.10)  

for \( t \geq 2 \).

Proof. By definition, one immediately obtains (3.5). One has

\[
\mathcal{R}_1 = M(t) D(t) (\mathcal{F}M(t) \mathcal{F}^{-1} - 1) F_1(t).
\]

Taking \( L^p \) norm, we obtain

\[
\|\mathcal{R}_1\|_{L^p} = t^{-\frac{1}{2} + \frac{1}{p}} \|(\mathcal{F}M(t) \mathcal{F}^{-1} - 1) F_1(t)\|_{L^p}.
\]

When \( p = 2 \), we have

\[
\|(\mathcal{F}M(t) \mathcal{F}^{-1} - 1) F_1(t)\|_{L^2} \lesssim t^{-1} \|\partial_x^2 F_1\|_{L^2}
\]

\[
\lesssim t^{-1} \|W_1\|_{H^2} (\|W_1\|_{H^2})^4 (\log t)^2
\]

for \( t \geq 2 \). Thus, (3.6) follows. When \( p = \infty \), mimicking the proof of (2.6), one has

\[
\|(\mathcal{F}M \mathcal{F}^{-1} - 1) F_1(t)\|_{L^\infty} \lesssim \|(\mathcal{F}M \mathcal{F}^{-1} - 1) F_1(t)\|_{L^2}^{\frac{1}{2}} \|(\mathcal{F}M \mathcal{F}^{-1} - 1) F_1(t)\|_{H^1}^{\frac{1}{2}}
\]

\[
\lesssim (t^{-1} \|\partial_x^2 F_1\|_{L^2})^{\frac{1}{2}} (t^{-\frac{1}{2}} \|\partial_x |F_1|_{H^1})^{\frac{1}{2}}
\]

\[
\lesssim t^{-\frac{1}{4}} \|\partial_x^2 F_1\|_{L^2}
\]

\[
\lesssim t^{-\frac{3}{2}} \|W_1\|_{H^2} (\|W_1\|_{H^2})^4 (\log t)^2.
\]

We obtain (3.7). Since \( J(t) = M(t)(it\partial_x) M(-t) \), we have

\[
Ju_{ap,1} = M(t)(it\partial_x) D(t) F_1(t) = M(t) D(t) i\partial_x F_1(t).
\]

Then, (3.8) immediately follows. Further, arguing as in the proof of (3.6), one obtains (3.9). Finally, (3.10) is a consequence of (3.5) and (3.7). \( \square \)

Lemma 3.2. Let \( u_{ap,2} \) be given by (1.17) and let \( \mathcal{R}_2 \) be defined by (3.4). It holds that

\[
\begin{align*}
\|u_{ap,2}\|_{L^\infty_x} &\lesssim \varepsilon_2 t^{-\frac{1}{2} + C\varepsilon_1}, \\
\|\mathcal{R}_2\|_{L^2_x} &\lesssim \varepsilon_2 t^{-1 + C\varepsilon_1} (\log t)^2, \\
\|\mathcal{R}_2\|_{L^\infty_x} &\lesssim \varepsilon_2 t^{-\frac{3}{4} + C\varepsilon_1} (\log t)^2, \\
\|Ju_{ap,2}\|_{L^2_x} &\lesssim \varepsilon_2 t^{C\varepsilon_1^2} \log t, \\
\|J\mathcal{R}_2\|_{L^2_x} &\lesssim \varepsilon_2 t^{-\frac{1}{2} + C\varepsilon_1^2} (\log t)^2
\end{align*}
\]  

(3.11)  

(3.12)  

(3.13)  

(3.14)  

(3.15)
Lemma 3.3. Let $\bar{u}_{ap,2} \in L^p$, for $t \geq 2$. In particular,

$$\|\bar{u}_{ap,2}\|_{L^p} \lesssim \varepsilon_2 t^{-\frac{1}{2} + C\varepsilon_1^2}$$

(3.16)

for $t \geq 2$.

Proof. The proof of the estimate is similar to that for the corresponds estimate in the previous lemma. We only give estimates on $F_2(t)$. One has

$$\|F_2(t)\|_{L^p} \lesssim \|W_2\|_{L^p} t^\varepsilon \|W_1\|_\infty \lesssim \varepsilon_2 t^{C\varepsilon_1^2}$$

for $t \geq 2$. This yields (3.11) and (3.14). In order to estimate $\|F_2\|_{H^2}$, let us first check that $F_2 \in C^1(R)$. The continuity of the factor

$$F := \frac{W_2(t)^2}{W_1(t)} (t^\varepsilon |W_1(t)|^2 - t^{-\varepsilon} |W_1(t)|^2)$$

at the zero point of $W_1$ is only problematic. The continuity of $F$ at such points follows from the identity

$$F = W_1^2(2\tilde{\nu} \log t + o(1)) = O(W_1^2)$$

as $|W_1| \to 0$. The continuity of $\partial_x F$ at such points is also verified by

$$\partial_x F = 2W_1 \partial_x W_1(2\tilde{\nu} \log t + o(1)) + \frac{2W_1^2 \partial_x W_1}{|W_1|^2} o(1) = O(W_1)$$

as $|W_1| \to 0$. Integrability of $|F_2|^2 + |\partial_x F_2|^2$ in $\{|W_1| \leq 1\}$ also follows from these identities. Similarly, we have $|\partial_x^2 F| \lesssim |\partial_x^2 W_1|^2 + |W_1| |\partial_x^2 W_1|$ as $|W_1| \to 0$, which yields the integrability of $|\partial_x^2 F|^2$ in $\{|W_1| \leq 1\}$, together with $\partial_x W_1 \in H^1 \mapsto L^\infty$. Hence, we have (3.12), (3.13), and (3.15).

Finally, (3.16) follows from (3.11) and (3.13). □

Next we estimate the error term $\mathcal{E}_j$.

Lemma 3.3. Let $\mathcal{E}_j$ be given by (3.2). It holds that

$$\|\mathcal{E}_1(t)\|_{L^2} \lesssim \varepsilon_1^2 t^{-2}(\log t)^2,$$

(3.17)

$$\|\mathcal{J}\mathcal{E}_1(t)\|_{L^2} \lesssim \varepsilon_1^2 t^{-\frac{1}{2}}(\log t)^2.$$

(3.18)

$$\|\mathcal{E}_2(t)\|_{L^2} \lesssim \varepsilon_1^2 \varepsilon_2 t^{-2+C\varepsilon_1^2}(\log t)^2,$$

(3.19)

and

$$\|\mathcal{J}\mathcal{E}_2(t)\|_{L^2} \lesssim \varepsilon_1^2 \varepsilon_2 t^{-\frac{3}{2}+C\varepsilon_1^2}(\log t)^2$$

(3.20)

for $t \geq 2$.

Proof. We remark that $(F_1, F_2)$ is a solution to (1.15). Hence,

$$i\partial_t F_j = t^{-1}\tilde{\mathcal{N}}_j(F_1, F_2)$$

for $j = 1, 2$. Thanks to this identity, we have

$$\mathcal{L}\tilde{u}_{ap,j} = U(t)\mathcal{F}^{-1}(i\partial_t F_j) = U(t)\mathcal{F}^{-1}[t^{-1}\tilde{\mathcal{N}}_j(F_1, F_2)].$$

(3.21)

From (3.21) and $u_{ap,j} = M(t)D(t)F_j$, we obtain

$$\mathcal{E}_j = -U(t)\mathcal{F}^{-1}[t^{-1}\tilde{\mathcal{N}}_j(F_1, F_2)] + \tilde{\mathcal{N}}_j(M(t)D(t)F_1, M(t)D(t)F_2)$$

$$= -t^{-1}M(t)D(t)\mathcal{F}(M(t)\mathcal{F}^{-1}\tilde{\mathcal{N}}_j(F_1, F_2) + t^{-1}M(t)D(t)\tilde{\mathcal{N}}_j(F_1, F_2))$$

$$= -t^{-1}M(t)D(t)\mathcal{F}(M(t) - 1)\mathcal{F}^{-1}\tilde{\mathcal{N}}_j(F_1, F_2).$$
Then, one has $0 < \delta < \varepsilon_1$.

Hence, mimicking the proof of $(3.6)$, one has

$$
\|\mathcal{E}_1(t)\|_{L^2_x} \lesssim t^{-2}\|\mathcal{N}_1(F_1, F_2)\|_{H^2_x} \lesssim \varepsilon_1^2 t^{-2} (\log t)^2,
$$

and

$$
\|J\mathcal{E}_1(t)\|_{L^2_x} \lesssim t^{-\frac{4}{3}}\|\mathcal{N}_1(F_1, F_2)\|_{H^2_x} \lesssim \varepsilon_1^2 t^{-\frac{4}{3}} (\log t)^2.
$$

In a similar way, we obtain $(3.19)$ and $(3.20)$.

\[\square\]

### 3.3. Completion of the proof.

**Proof of Theorem 3.4.** Fix $\nu \in (1/2, 1)$ and $\delta \in (0, \nu - 1/2)$. Let $\bar{\nu} := \nu - 1/2$. Then, one has $0 < \delta < \bar{\nu} < 1/2$. We first show that $\Phi$ is a map onto $X_T$ for suitable $T \geq 2$ if $\varepsilon_1$ is sufficiently small.

Pick $(v_1, v_2) \in X_T$. We easily see that

$$
|\mathcal{N}_1(v_1 + u_{ap,1}, v_2 + u_{ap,2}) - \mathcal{N}_1(u_{ap,1}, u_{ap,2})| \lesssim (|v_1|^2 + |u_{ap,1}|^2)|v_1|,$$

$$
|\mathcal{N}_2(v_1 + u_{ap,1}, v_2 + u_{ap,2}) - \mathcal{N}_2(u_{ap,1}, u_{ap,2})| \lesssim |v_1|^2|v_2| + |u_{ap,2}|v_2|v_1| + |u_{ap,1}|v_1|v_2| + |u_{ap,1}|u_{ap,2}|v_1| + |u_{ap,1}|^2v_2|.
$$

Furthermore, noting that the nonlinear terms are gauge invariant, we have

$$
|J\mathcal{N}_1(v_1 + u_{ap,1}, v_2 + u_{ap,2}) - J\mathcal{N}_1(u_{ap,1}, u_{ap,2})| \lesssim |v_1|^2|Jv_1| + |Ju_{ap,1}|v_1|^2 + |u_{ap,1}|v_1|Jv_1| + |u_{ap,1}|Ju_{ap,1}|v_1| + |u_{ap,1}|^2v_1|Jv_1|,$$

$$
|J\mathcal{N}_2(v_1 + u_{ap,1}, v_2 + u_{ap,2}) - J\mathcal{N}_2(u_{ap,1}, u_{ap,2})| \lesssim |v_1|^2|Jv_1|v_2| + |v_1|^2|Jv_2| + |Ju_{ap,1}|v_1|^2 + |u_{ap,2}|v_1|Jv_1| + |Ju_{ap,1}|v_1|v_2| + |u_{ap,1}|Jv_1|v_2| + |u_{ap,1}|v_1|Jv_2| + |Ju_{ap,1}|u_{ap,2}|v_1| + |u_{ap,1}|Ju_{ap,2}|v_1| + |u_{ap,1}|u_{ap,2}|Jv_1| + |u_{ap,1}|Ju_{ap,1}|v_2| + |u_{ap,1}|^2v_2|Jv_2|.
$$

By plugging $\|v_1\|_{L_x^\infty} \lesssim t^{-1/2}\|v_1\|_{L^2_x}^{1/2}\|Jv_1\|_{L^2_x}^{1/2}$ and the estimates on $u_{ap,j}$ to those inequalities, we obtain

$$
\|\mathcal{N}_1(v_1 + u_{ap,1}, v_2 + u_{ap,2}) - \mathcal{N}_1(u_{ap,1}, u_{ap,2})\|_{L^2_x} \lesssim \|v_1\|_{L^2_x}^2\|v_1\|_{L^2_x} + \|u_{ap,1}\|_{L^2_x}^2\|v_1\|_{L^2_x} \lesssim t^{-1}\|v_1\|_{L^2_x}^2\|Jv_1\|_{L^2_x} + \|u_{ap,1}\|_{L^2_x}^2\|v_1\|_{L^2_x} \lesssim t^{-3\delta - 2} + t^{-\bar{\nu} - \frac{1}{3}\varepsilon_1^2},
$$

(3.22)
we obtain
\[ J \lesssim t \lesssim \| v \|_{L^2} \| Jv_1 \|_{L^2} + \| u_{a,1} \|_{L^2} + \| u_{a,2} \|_{L^2} + \| u_{a,1} \|_{L^\infty} \| v \|_{L^\infty} \| Jv_1 \|_{L^2} \]
+ \| u_{a,1} \|_{L^\infty} \| Jv_1 \|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2} + \| u_{a,1} \|_{L^\infty} \| v \|_{L^2} \| Jv_1 \|_{L^2} \]
+ \| u_{a,1} \|_{L^\infty} \| v \|_{L^2} \| Jv_1 \|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2}
\lesssim t^{-\frac{1}{2}} \| u_{a,1} \|_{L^\infty} \| v \|_{L^2} \| Jv_1 \|_{L^2} + t^{-\frac{1}{2}} \| u_{a,1} \|_{L^\infty} \| Jv_1 \|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2} \]
+ \| u_{a,1} \|_{L^\infty} \| Jv_1 \|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2}
\lesssim t^{-3\bar{d} - \frac{1}{2} + t^{-2\bar{d} - \frac{1}{2} + \delta} \epsilon_1 + t^{-2\bar{d} - \frac{1}{2} + C\delta_1^2 \epsilon_2} + t^{-\bar{d} - \frac{1}{2} + \delta} \epsilon_1 + t^{-\bar{d} - \frac{1}{2} + C\delta_1^2 \epsilon_1^2} \epsilon_2, (3.23)
\]

and
\[ |J|J\|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2} + \| u_{a,1} \|_{L^\infty} \| v \|_{L^2} \| Jv_1 \|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2} \]
+ \| u_{a,1} \|_{L^\infty} \| v \|_{L^2} \| Jv_1 \|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2} \]
+ \| u_{a,1} \|_{L^\infty} \| v \|_{L^2} \| Jv_1 \|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2} \]
+ \| u_{a,1} \|_{L^\infty} \| v \|_{L^2} \| Jv_1 \|_{L^2} \| v \|_{L^2} \| Jv_1 \|_{L^2}
\lesssim t^{-3\bar{d} - \frac{1}{2} + \delta} + t^{-2\bar{d} - \frac{1}{2} + \delta} \epsilon_1 + t^{-2\bar{d} - \frac{1}{2} + C\delta_1^2 \epsilon_2} + t^{-\bar{d} - \frac{1}{2} + \delta} \epsilon_1 + t^{-\bar{d} - \frac{1}{2} + C\delta_1^2 \epsilon_1^2} \epsilon_2, (3.24)
\]

for \( t \geq T \). Applying (3.23) and (3.25) to the inequalities
\[ \| \Phi_j(v, v_2)(t) \|_{L^2} \]
\[ \leq \| \tilde{N}_j(v, u_{a,1}, v_2 + u_{a,2}) - \tilde{N}_j(u_{a,1}, u_{a,2}) \|_{L^1 L^2(t, \infty)} \]
+ \| E_j \|_{L^1 L^2(t, \infty)} + \| R_j(t) \|_{L^2}
\]
and
\[ \| J\Phi_j(v, v_2)(t) \|_{L^2} \]
\[ \leq \| J(\tilde{N}_j(v, u_{a,1}, v_2 + u_{a,2}) - \tilde{N}_j(u_{a,1}, u_{a,2})) \|_{L^1 L^2(t, \infty)} \]
+ \| E_j \|_{L^1 L^2(t, \infty)} + \| J R_j(t) \|_{L^2},
\]
we obtain
\[ \Phi(v, v_2) \|_{X_T} \leq \tilde{C}(T^{-2\bar{d} - \frac{1}{2} + (\epsilon_1 + \epsilon_2)T^{-\bar{d} - \frac{1}{2} + \epsilon_1 \epsilon_2}T^{-\frac{1}{2}})
+ (\epsilon_1 + \epsilon_2)T^{\bar{d} - \frac{1}{2}(\log T)^2 + \epsilon_1^2}) (3.26)\]
if $C\varepsilon^2 \leq \delta/2$, where $\tilde{C} = \tilde{C}(\tilde{\nu}, \delta)$. Pick $\tilde{\varepsilon}_1 > 0$ so small that $\tilde{C}\varepsilon^2 \leq \frac{\delta}{2}$ and $\tilde{C}\varepsilon^2 \leq \delta/2$ are satisfied. Then, there exists $T = T(\tilde{\varepsilon}_1(\tilde{\nu}, \delta), \varepsilon_2, \tilde{\nu}, \delta) > 0$ such that if $\varepsilon_1 \leq \tilde{\varepsilon}_1$ then the right hand side of (3.26) is less than or equal to one. Then, $\Phi$ is a map onto $X_T$. In a similar way, one concludes that $\Phi$ is a contraction map on $X_T$ by letting $T$ even larger if necessary. We leave the details to the reader. Thus, by Banach fixed point theorem one infers that $\Phi$ has a unique fixed point in $X_T$ which is the solution to the final state problem (1.16).

Since $\| (v_1, v_2) \|_{X_T} \leq 1$, recalling that $\nu = \tilde{\nu} + 1/2$, we have

$$\| v_1(t) \|_{L^\infty} \lesssim t^{-\frac{1}{2}} \| Jv_1(t) \|_{L^2} \leq t^{-\nu - \frac{1}{4}}$$

and

$$\| v_2(t) \|_{L^\infty} \lesssim t^{-\frac{1}{2}} \| Jv_2(t) \|_{L^2} \leq t^{-\nu - \frac{1}{4} + \delta}$$

for $t \geq T$. This is (1.19). The estimate (1.18) immediately follows in a similar way. This completes the proof of Theorem 1.2. □

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