Semigraphs and its a-Domination

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Abstract. The domination concept having much role in graph theory. It gives a maximum space to the improvement of graphs and their applications. In this paper, a-domination number of different types of semigraphs are discussed.

1. INTRODUCTION

S. P. Subbiah [4] studied the concept of semigraphs and topologies. S.T. Hedetniemi et. Al [1] discussed some basic definitions of domination parameter in 1990. In 1998, W. Teresa et. Al [6] studied fundamentals of domination in graphs. N. Murugesan and D. Narmatha [2, 3] studied semigraphs properties and its associated graphs.

2. a-DOMINATIONS IN SEMIGRAPHS

2.1 Definition

Any subset D of a semigraph $S = (V, X)$ is a-dominating set, $\forall \ v \in V - D \ \exists \ u \in D \ \therefore u$ is adjacent to $v$. The a-domination number of $S$ is that the minimum cardinality of an a-dominating set is, it is noted as $\gamma_a(S)$.

2.2 Definition

If $u$ belongs to $D$ is said to be a-isolate, if $\exists \ no \ v \in D, \ \therefore u$ and $v$ are a-adjacent.

In the set $D = \{u_1, u_2, u_3\}$, $u_3$ is an a-isolate, but $u_1$ and $u_6$ are not a-isolates.
2.3 Lemma
An $a$-dominating set $D$ is contained in $V_a$ is minimum if and only if $\forall \ u \in D$ is an $a$ – isolate vertex.

2.4 Theorem [5]
The set $D \subseteq V$ of an $a$ – dominating set $S$ is minimal if and only if for every $u \in D$, the following satisfied. i.e., $u$ is an $a$ – isolate; $\exists$ another $v \in V \in N_a(v) \cap D$ is $\{u\}$.

2.5 Illustrations

In Fig 2.2 semigraph, example.2.2 (i), the set $\{w, x, y, z\}$ is the minimum $a$ – dominating set whose nodes satisfy the condition (i) of the theorem. In the semigraph given in example.(ii), $\{a, b, c, d\}$ is the set also is an $a$-dominating set, to which the vertices $a, c$ satisfy the condition (i), and the vertices $b, d$ satisfy the condition (ii) of the above theorem where the vertices $e$ and $f$ play the role of $V$.

2.6 Definition
Let $V$ is a minimal $a$ – dominating set in $S$. A node $v \in V$ is called strong $a$ – dominating vertex of $V$, if there is no $u$ in $V$ such that $(V \cup \{u\} – v)$ is a minimal a-dominating set, otherwise it is said to be weak a-dominated vertex of $V$. A minimum a- dominating set in which all vertices are strong a-dominating vertices, then the a- dominating set is said strong a- dominating set. Otherwise, a-domination is the weak a- dominating set.

For any 2 adjacent vertices $u$ and $v$ in a graph $G$, $u$ is called strongly dominates $v$ if $\text{deg}(u) \geq \text{deg}(v)$.
The weak dominating set is also defined analogously.

As an example let the semigraphs shown in fig. 2.3.

Fig 2.3. Semigraphs with strong and weak dominating sets

In the above graphs $u$ is weak $a$ – dominated vertex in (a), and $v$ is strong $a$ – dominated vertex in (b).
2.7 Lemma
\[ \gamma_a(S-k) = \gamma_a(S) + l, \text{ where } 0 \leq l \leq |k|-1. \]

2.8 Definition
A semigraph \( S \) is said to be connected if it has no \( a \)-isolates and simple if it has no loop (A loop is an edge which starts and ends with same end vertex). A semigraph \( S \) is called fundamental semigraph if \( d_a(v) \leq 2 \), for all \( u \in V \) of \( S \). An edge in a semigraph is said to principal edge, it is surrounded only by the vertices and not by the middle-end vertices.

![Fig. 2.4. The variants in semigraphs](image)

2.9 Definition
The set \( N_a(u) = \{ v \in V / u, v \text{ are a-adjacent} \} \) is “a-neighbourhood of \( u \) in \( S \)”. The set \( N_a[u] \) is \( \{u\} \cup N_a(u) \) is close neighbourhood of \( u \) in \( S \).

From definition it is noted that if \( D \) is a minimum \( a \)-dominating set in \( S \), \( \forall u \in V \), \( |N_a(u) \cap D| \) is 1.

2.10 Theorem
If \( S \) has no \( a \)-isolates, \( D \) is the minimum \( a \)-dominating set implies \( V-D \) is an \( a \)-dominating set.

Proof:
Every \( u \) in \( D \), either \( a \)-isolate in \( D \) or there is \( v \) in \( V \ni N_a(v) \cap D = \{u\} \). Because \( S \) has no \( a \)-isolates, \( u \) has a \( v \in V-D \ni u \& v \) are adjacent. If \( u \) is no an \( a \)-isolate in \( D \), then there exist \( v \) in \( D \ni u \& v \) both adjacent. Then there must be \( x \) and \( y \) in \( V \ni N_a(x) \cap D = \{u\} \), \( N_a(x) \cap N_a(y) \) is \( \phi \), \( N_a(y) \cap D = \{v\} \), \( N_a(y) \cap N_a(u) \) is \( \phi \). If there are no such \( x \) and \( y \), then \( u \) dominates all vertices which are dominated by \( v \) and vice versa. Hence removal of \( u \) or \( v \) in \( D \) is an \( a \)-dominating set. It is the contradiction.

2.11 Note
If a semigraph \( S \) has no \( a \)-isolates, then \( \gamma_a(S) \leq \frac{n}{2} \), where \( n = |V(S)| \).

2.12 Lemma
\[ \gamma_a(S) \leq \left\lfloor \frac{k}{2} \right\rfloor. \]
2.13 Illustrations
The semigraphs given in fig. 2.5 are examples of path and cycle semigraphs.

![Fig.2.5. The semigraphs $P_{5(4)}$ and $C_{3(5)}$.](image)

2.14 Example
Consider the following semigraphs, given in fig. 2.6.

![Fig.2.6. Some cycle and path semigraphs](image)

The $a$–domination number of the above cycle and path graphs are $\gamma_a(S_1)$ is 1, $\gamma_a(S_2) = \gamma_a(S_3)$ is 2, $\gamma_a(S_4)$ is 3.

2.15 Theorem
Minimal $a$-domination number is same as $a$-domination of $S$.

2.16 Corollary
From the above lemma, it is always a minimum dominating set containing end vertices in a semigraph.

The following observations provide an upper bound for $\gamma_a(S)$.

2.17 Observations
If $S$ has no $a$–isolates then $\gamma_a(S) \leq \frac{n}{2}$, where $n = \frac{|V(S)|}{2}$. In particular

i. $\gamma_a(S) \leq \frac{|V_a(S)|}{2}$
ii. If \( \deg(v) = |V_v(S)| \) is one for all vertices in end vertex set and hence \( \gamma_a(S) \) is one.

iii. \( \gamma_a(S) = \frac{|V_v(S)|}{2} \), when \( |V_v(S)| \) is even, and \( \deg(v) = 2 \), for all \( v \in V_v(S) \).

**2.18 Lemma**

For any semigraph \( S \) of size \( q \), \( \left\lceil \frac{q}{1 + \Delta_a(S)} \right\rceil \leq \gamma_a(S) \leq q \).

**Proof:**

If \( v \) belongs to \( S \), \( \exists v \) is a-adjacent to other vertices of \( S \), then \( v \) should be an end vertex to every edge in \( S \). Therefore the set \( \{v\} \) itself a minimum a – dominating set and \( d_a(v) = \Delta_a(S) \). Hence, \( \gamma_a(S) = \frac{q}{\Delta_a(S)} \). To get, better lower bound it can be written as \( \gamma_a(S) = \left\lceil \frac{q}{1 + \Delta_a(S)} \right\rceil \). The most possible case is that each edge in \( S \) is a component, and \( S \) has only isolated edges. Then \( \gamma_a(S) = q \).

For all other cases, \( \gamma_a(S) \) lies between \( \left\lceil \frac{q}{1 + \Delta_a(S)} \right\rceil \) and \( q \).

**2.19 Example**

Consider the semigraphs

![Semigraphs](Fig2.7)

**Fig. 2.7.** Connected and disconnected semigraphs

It can be seen that, \( \gamma_a(S_1) = 1 \) and \( \gamma_a(S_2) = 3 \).

**2.20 Definition**

Let \( V_E = \{v_1, v_2, \ldots, v_n\} \) in \( S \), and \( \deg(v_j) = d_j \neq 0 \). Then \( (d_1, d_2, \ldots, d_n) \), \( d_1 \geq d_2 \geq \ldots \geq d_n \) is called degree sequence of \( S \).

**2.21 Lemma**

If \( S \) has a degree sequence \( (d_1, d_2, \ldots, d_n) \) with \( d_1 \geq d_2 \geq \ldots \geq d_n \) & \( d_{j \rightarrow 1} = d_{j \rightarrow 2} = \ldots = d_n = 1 \Rightarrow \gamma_a(S) \leq f \).

**Proof:**

It is obvious that the vertex \( v \) in which \( \deg(v) \) is one, may not be a member of any \( a \) – dominating set.
3. Conclusion

The a-domination number of different types of semigraphs and its associated graphs have also been discussed in this paper.

4. References

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