View-driven compressed sensing method of CT image reconstruction

Lijun Wu1, Fengrong Sun1,*, Jiangfei Yang2, Qianlei Yu1, Fangfang He1 and Guihua Yao3

1School of Information Science and Engineering, Shandong University, Qingdao, China
2Department of Medical Imaging, Shandong Provincial Hospital Affiliated to Shandong University, Jinan, China
3Department of Cardiology, Qilu Hospital of Shandong University, Jinan, China

*Corresponding author e-mail: sunfr_journal@163.com

Abstract. In order to significantly reducing the X-ray dose of CT examination, this study focuses on reconstructing the tomographic images from sparse projection data with sufficient accuracy. Aiming at two-dimensional (2D) image reconstruction of fan-beam projection under circular scanning, this paper proposes the view-driven system model and designs a CT iterative image reconstruction method by combining the iterative algorithm and the compressed sensing theory. The simulation results show that the algorithm has high numerical accuracy, low computational complexity, less memory overhead, and strong engineering practicability under the condition of ultra-sparse projection data (no more than 20 view angles for fan-beam scanning in the range of \([0, 2\pi]\)).

1. Introduction

Studies have shown that X-ray radiation from CT examinations may increase the risk of cancer in patients [1]. Clinically, the harm of X-rays to the human body can be weakened by reducing the number of view angles. Therefore, it is of great clinical significance to study how to reconstruct CT images from sparse projection data. There have been numerous iterative algorithms proposed to overcome incomplete projection data in CT image reconstruction. The TV algorithm [2] designed by Sidky et al, which uses the prior knowledge of sparseness of gradient images to improve the quality of reconstructed images, but the image smoothing effect may reduce the spatial resolution and contrast of the images. Subsequently, the prior image constrained compressed sensing (PICCS) algorithm [3] proposed by Dr. Chen et al, and it requires to select projection data sampled uniformly from redundant or complete projection data sets and uses filtered back-projection (FBP) algorithm to obtain prior images, which greatly limits its application range.

Synthesizing existing literature and communicating with CT technicians, we defined no more than 20 view angles for fan-beam scanning in the range of \([0, 2\pi]\) as ultra-sparse projection, and proposed a reconstruction method to reconstruct CT images from ultra-sparse projection data.
2. Theory and method

2.1. CT image reconstruction based on compressed sensing

Based on compressed sensing (CS) [4] theory, the CT iterative image reconstruction under the sparse projection data can be regarded as solving the ill-posed linear equation:

$$g = Wf$$  \hspace{1cm} (1)

where $g$ is the $M$-dimensional column vector of projection data, $f$ is the $N$-dimensional image vector to be reconstructed, usually $M < N$, and $W$ is the system matrix (i.e., the forward projection matrix mentioned below). According to CS theory, if the system matrix satisfies the strict isometry condition, Eq. (1) can be transformed into a constrained $l_1$ norm optimization problem:

$$\min_{f} \left\| \Psi f \right\|_1 \quad s.t. \ g = Wf$$  \hspace{1cm} (2)

where $\Psi f$ is the sparse representation of the image $f$ to be reconstructed.

2.2. View-driven system model and forward/backward projection matrix

2.2.1 View-driven system model. We propose a novel view-driven system model and describe the operation of the view-driven forward projection in Fig. 1. For a fixed view angle, the source is connected with the midpoints of pixel boundaries of a row and with the detector boundaries respectively, and then their respective intersections on the common axis (i.e., the $X$ axis in Fig. 1) are obtained; then we calculate the overlap lengths between two adjacent intersections, and use the normalized overlap length $s$ to represent the contribution of pixels to detectors. The operation of the view-driven backward projection runs in the same way. Finally, the view-driven forward/backward projection operation strategy in this paper can be summarized as: processing all pixels under one view angle in one iteration.

![Figure 1. Schematic diagram and detail diagram of two-dimensional fan-beam projection.](image)

In forward projection operation, the weighting coefficients are divided by the cosine value of the view angle $\alpha$ in order to simulate the line integral more accurately. The view-driven forward projection operation is expressed as:

$$d_{12} = \frac{1}{\cos \alpha} p_{12}$$

$$d_{23} = \frac{1}{\cos \alpha} \left(\frac{p_2 - d_2}{d_3 - d_2}\cdot p_{12} + \frac{1}{\cos \alpha} \frac{d_3 - p_2}{d_3 - d_2}\cdot p_{23}\right)$$

......
While for the backward projection we have:

\[
p_{12} = \frac{(d_2 - d_1) \cdot d_{12}}{p_2 - p_1} + \frac{(p_2 - d_2) \cdot d_{23}}{p_2 - p_1}
\]
\[
p_{23} = \frac{d_3 - p_2}{p_3 - p_2} \cdot d_{23} + \frac{d_4 - d_3}{p_3 - p_2} \cdot d_{34} + \frac{p_3 - d_4}{p_3 - p_2} \cdot d_{45}
\]
\[
\ldots
\]

(4)

2.2.2 Forward/backward projection matrix. The above view-driven system model determines the elements of the forward/backward projection matrix. The forward/backward projection operation shown in Eq. (5) and Eq. (6), where \( W \) and \( M \) are the forward/backward projection matrix respectively, the meaning of \( f \) and \( g \) have been mentioned above.

\[
g = Wf
\]  
(5)

\[
f = M^Tg
\]  
(6)

\( W \) and \( M \) have the same structure, but different matrix elements. In combination with Fig. 2, we further elaborate the meaning of each element of the forward/backward projection matrix.

The structure of forward projection matrix \( W \) is shown in Fig. 2. \( W_\theta \) is the sub-matrix of forward projection matrix \( W \) at the \( \theta \)th view angle. And \( W_\theta \{x\} \) represents the relative contribution of pixels in the \( x \)th row to all detector units at the \( \theta \)th view angle, matrix elements of which are the weighting coefficients in Eq. (3). Eq. (7) and Eq. (8) reveal the relationship among the above three matrices, where \( X \) is the number of pixel rows and \( Y \) is the number of pixel columns. Eq. (6) indicates that the transpose of the backward projection matrix \( M \) (i.e., \( M^T \)) participates in the backward projection operation directly, and the structure of matrix \( M^T \) is shown in Fig. 2. Similar to forward projection, the meaning of \( M^T_\theta \) and \( M^T_\theta \{x\} \) will not be repeated here. The relationship among the above three matrices can also be expressed by Eq. (9) and Eq. (10).

**Figure 2.** Schematic diagram of matrix elements of forward/backward projection.

\[
W = \begin{bmatrix}
W_1 & W_2 & \cdots & W_x \\
W_1 & W_2 & \cdots & W_x \\
& & & \\
& & & \\
& & & \\
W_{\text{ViewNum}} & W_{\text{ViewNum}} & \cdots & W_{\text{ViewNum}}
\end{bmatrix}
\]  
(7)
2.3. View-driven compressed sensing method of CT image reconstruction

Under the framework of compressed sensing theory, we use the view-driven system model mentioned above to design a CT iterative image reconstruction algorithm, which is called View-driven Compressed Sensing method of CT image reconstruction (VdCS) in this paper. VdCS algorithm mainly consists of rough image reconstruction and optimization calculation.

2.3.1 Rough image reconstruction. This module solves the ill-posed linear equation \( g = Mf \) iteratively to obtain solutions approximately satisfying the constraints given in Eq. (2). Eq. (11) shows that the projection data of each view angle is successively processed with 1) forward projection, 2) computing the difference, 3) backward projection, and 4) correction operations.

\[
W_\theta = \left[ W_\theta(1), W_\theta(2), \ldots, W_\theta(x), \ldots, W_\theta(X) \right] \\
M^T = \left[ M_\theta^T, M_\theta^T, \ldots, M_\theta^T \right] \\
M_\theta^T = \begin{bmatrix} M_\theta^T(1) \\ M_\theta^T(2) \\ \vdots \\ M_\theta^T(x) \\ \vdots \\ M_\theta^T(X) \end{bmatrix}
\]

\( W_\theta \) represents the image vector at the \( \theta \)th view angle in the \( r \)th rough reconstruction sub-iteration of the \( n \)th overall iteration. Adaptive correction factor \( \phi_\theta \) can be represented as:

\[
\phi_\theta = \frac{\lambda}{\omega_\theta \cdot r}
\]

where \( \lambda \) is the relax factor, \( \omega_\theta \) is an adaptive relaxation factor correction parameter, it is obtained by:

\[
\omega_\theta = \sum_{i,j,t} \| \omega[i, j, t, \theta] \|^2
\]

where \( \omega[i, j, t, \theta] \) (i.e., the value showed by the mark ‘*’ in Fig. 2) represents the forward projection coefficient of the pixel at position \((i, j)\) to the \( t \)th detector unit in the forward projection matrix \( W \) at the \( \theta \)th view angle.

2.3.2 Optimization calculation. This module optimizes the objective function \( \| \Psi f \|_1 \) of Eq. (2) using the gradient descent method to find the optimal solution of the rough image reconstruction. The optimization calculation here refers to the following calculations for each pixel value:

\[
V_{i,j}(n,k) = \frac{\partial \| \Psi f \|_1}{\partial f_{i,j}} |_{f_{i,j} = f^{(n,k)}} \\
f^{(n,k)} = f^{(n,k-1)} - \alpha \cdot d^{(n)} \cdot v^{(n,k-1)}
\]

where the image vector in the \( k \)th optimization sub-iteration of the \( n \)th overall iteration is \( f^{(n,k)} \), \( \alpha \) is the step size factor, and \( d^{(n)} \) represents the balance factor calculated in the \( n \)th overall iteration, which is calculated as:
\[ d^{(n)} = \frac{\| f^{(n,1,0)} - f^{(n)} \|}{\| f^{(n)} \|}, \text{ where } f^{(n)} = \begin{cases} f^{(n,R,\text{frame})} & n \neq 0 \\ f^{(n,R,\text{frame})}_{(0)} & n = 0 \end{cases} \]  

(16)

3. Simulation experiments and result analysis

In this paper, two sets of 2D CT image reconstruction of fan-beam projection under circular scanning are performed at 20 and 18 view angles. The reconstructed images are shown in Fig. 3 (b), (c). Fig. 3 (d) and (e) are the pixel values distribution curves in the horizontal direction between the Shepp-Logan model shown in Fig. 3 (a) and the reconstructed images at 20 and 18 view angles respectively. The results at 20 view angles are almost identical with the reference image; the reconstructed image at 18 view angles shows that partial edges are blurred and some areas have blocky artifacts, but each organizational structure can be clearly distinguished, and the pixel values distribution curve has an impulse in some places where the pixel value changes, while other parts of the curve almost coincide with the reference image’s.

In order to further demonstrate the performance of the VdCS algorithm, three indexes, normalized root mean squared error (NRMSE), peak signal to noise ratio (PSNR), and universal image quality index (UQI) [5] are used to objectively evaluate the reconstruction quality of the two algorithms (VdCS algorithm and FBP algorithm). The results are shown in Fig. 4 (the indexes at 36 view angles are also given). The VdCS algorithm is significantly better than the FBP algorithm by various indexes. With the number of view angles decreasing, the performance of the FBP algorithm deteriorates sharply, while the performance of the VdCS algorithm is stable; this indicates the VdCS algorithm is suitable for 2D fan-beam CT image reconstruction under ultra-sparse projection data.

![Figure 3. Shepp-Logan model and reconstructed images, as well as the comparison of pixel values distribution (center row of the image).](image)

(a) Shepp-Logan model    (b),(d)20 view angles    (c),(e)18 view angles

![Figure 4. Comparison of reconstructed images quality between FBP algorithm and VdCS algorithm.](image)

4. Discussion and conclusion

Theoretical analysis and experimental results demonstrate that the VdCS algorithm can reconstruct tomographic images with sufficient accuracy from ultra-sparse projection data. The good performance of the VdCS algorithm is mainly reflected in: (1) The VdCS algorithm processes one view angle in one iteration; and under one view angle, one iteration can get the contribution of one row of pixels to all detector units (here refers to the forward projection process), which reduces the computational complexity and avoids a lot of unnecessary traversal operations. (2) In CT imaging domain, the scale of conventional system matrices is usually large, and iterative reconstructions using such system matrices generally need to store the matrices. While image reconstruction based on the view-driven
system model does not require the storage of the system matrix so that the memory overhead is reduced significantly. In short, this paper provides a practical technical way for researchers in related fields to perform CT iterative image reconstruction under ultra-sparse projection data.

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