Agnostic Learning of Halfspaces with Gradient Descent via Soft Margins

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Abstract

We analyze the properties of gradient descent on convex surrogates for the zero-one loss for the agnostic learning of linear halfspaces. If OPT is the best classification error achieved by a halfspace, by appealing to the notion of soft margins we are able to show that gradient descent finds halfspaces with classification error \( \tilde{O}(\OPT^{1/2}) + \varepsilon \) in time and sample complexity for a broad class of distributions that includes log-concave isotropic distributions as a subclass. Along the way we answer a question recently posed by Ji et al. (2020) on how the tail behavior of a loss function can affect sample complexity and runtime guarantees for gradient descent.

1 Introduction

We analyze the performance of gradient descent on a convex surrogate for the zero-one loss in the context of the agnostic learning of halfspaces. By a halfspace we mean a function \( x \mapsto \text{sgn}(w^\top x) \in \{\pm 1\} \) for some \( w \in \mathbb{R}^d \). Let \( \mathcal{D} \) be a joint distribution over \( (x, y) \), where the inputs \( x \in \mathbb{R}^d \) and the labels \( y \in \{\pm 1\} \), and denote by \( \mathcal{D}_x \) the marginal of \( \mathcal{D} \) over \( x \). We are interested in the performance of halfspaces found by gradient descent in comparison to the best-performing halfspace over \( \mathcal{D} \), so let us define, for \( w \in \mathbb{R}^d \),

\[
\text{err}_{\mathcal{D}}^{0-1}(w) := \mathbb{P}_{(x,y) \sim \mathcal{D}}(\text{sgn}(w^\top x) \neq y), \quad \OPT := \min_{\|w\|=1} \text{err}_{\mathcal{D}}^{0-1}(w).
\]

As the zero-one loss is non-convex, it is computationally infeasible to directly minimize this loss. The standard approach is to consider a convex surrogate loss \( \ell : \mathbb{R} \to \mathbb{R} \) for which \( 1(z < 0) \leq \ell(z) \) and to instead minimize the surrogate risk

\[
F_{\ell}(w) := \mathbb{E}_{(x,y) \sim \mathcal{D}}[\ell(yw^\top x)].
\]

Without access to the population risk itself, one can take samples \( \{(x_i, y_i)\}_{i=1}^{n} \overset{i.i.d.}{\sim} \mathcal{D} \) and optimize (1.1) by gradient descent on the empirical risk \( \hat{F}_{\ell}(w) \), defined by taking the expectation in (1.1) over the empirical distribution of the samples. By using standard tools from convex optimization and Rademacher complexity, such an approach is guaranteed to efficiently minimize the population

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surrogate risk up to optimization and statistical error. The question is then, given that we have found a halfspace $x \mapsto w^T x$ that minimizes the surrogate risk, how does this halfspace compare to the best halfspace as measured by the zero-one loss? And how does the choice of the surrogate loss affect this behavior?

We show below that the answer to these questions depend upon what we refer to as the soft margin function of the distribution at a given minimizer for the zero-one loss. (We note that in general, there may be multiple minimizers for the zero-one loss, and so we can only refer to a given minimizer.) For $\bar{v} \in \mathbb{R}^d$ satisfying $\|\bar{v}\| = 1$, we say that the halfspace $\bar{v}$ satisfies the $\phi_{\bar{v}}$-soft-margin property if for some function $\phi_{\bar{v}} : [0, 1] \rightarrow \mathbb{R}$, for all $\gamma \in [0, 1]$,

$$P_{D_x}(\|\bar{v}^T x\| \leq \gamma) \leq \phi_{\bar{v}}(\gamma).$$

To get a flavor for how this soft margin can be used to show that gradient descent finds approximately optimal linear halfspaces, for bounded distributions $D_x$, we show in Theorem 5.2 below that with high probability,

$$\text{err}^{0-1}_{\text{D}}(w_t) \leq \inf_{\gamma \in (0, 1)} \left\{ O(\gamma^{-1}\text{OPT}) + \phi_{\bar{v}}(\gamma) + O(\gamma^{-1}n^{-1/2}) + \varepsilon \right\},$$

where $\phi_{\bar{v}}$ is a soft margin function corresponding to a unit norm minimizer $\bar{v}$ of the population zero-one loss. Thus, by analyzing the properties of $\phi_{\bar{v}}$, one can immediately derive approximate agnostic learning results for the output of gradient descent. In particular, we are able to show the following guarantees for the output of gradient descent:

1. **Hard margin distributions.** If $\|x\| \leq B_X$ almost surely and there is $\bar{\gamma} > 0$ such that $\bar{v}^T x \geq \bar{\gamma}$ a.s., then $\text{err}^{0-1}_{\text{D}}(w_t) \leq \tilde{O}(\bar{\gamma}^{-1}\text{OPT}) + \varepsilon$.

2. **Sub-exponential distributions satisfying anti-concentration.** If random vectors from $D_x$ are sub-exponential and satisfy a simple anti-concentration inequality for projections onto one dimensional subspaces, then $\text{err}^{0-1}_{\text{D}}(w_t) \leq \tilde{O}(\text{OPT}^{1/2}) + \varepsilon$. This covers any log-concave isotropic distribution.

For each of our guarantees, the runtime and sample complexity are $\text{poly}(d, \varepsilon^{-1})$. The exact rates are given in Corollaries 5.3, 5.6 and 5.11. In Table 1 we compare our results with known lower bounds in the literature.

An additional consequence of our analysis is that loss functions with exponential tails have favorable sample complexity and runtime guarantees as compared to losses with fatter tails, answering a question recently posed by Ji et al. (2020). In particular, we show in Corollary 4.2 that if $\ell(z) = O(\exp(-z))$ for $z \geq 1$, then the sample complexity is $\tilde{O}(\gamma^{-2}\varepsilon^{-1})$ and runtime is $\tilde{O}(\gamma^{-2}\varepsilon^{-1})$ for linearly separable distributions with margin $\gamma > 0$, while for loss functions with tails $\ell(z) = O(z^{-p})$, the sample complexity and runtimes are each penalized by a factor of $O(\varepsilon^{-2/p})$.

2 Related Work

The problem of learning halfspaces is a classical problem in machine learning with a history almost as long as the history of machine learning itself, starting from the perceptron (Rosenblatt, 1958) and support vector machines (Boser et al., 1992) to today. Much of the early works on this problem focused on the realizable setting, i.e. where $\text{OPT} = 0$. In this setting, the Perceptron algorithm
Table 1: Comparison of our results with other upper and lower bounds in the literature.

| Algorithm                     | $D_x$                          | Population Risk | Known Lower Bound                                      |
|-------------------------------|--------------------------------|-----------------|--------------------------------------------------------|
| Non-convex G.D.               | Concentration, anti-concentration | $O(OPT)$       | N/A                                                   |
| *(Diakonikolas et al., 2020e)*|                                |                 |                                                        |
| Convex G.D.                   | Sub-exponential, anti-concentration | $\tilde{O}(OPT^{1/2})$ | $\Omega(OPT \log^\alpha(1/OPT))$                     |
| *(this paper)*                |                                |                 | *(Diakonikolas et al., 2020e)*                        |
| Convex G.D.                   | Hard margin                    | $\tilde{O}(\tilde{\gamma}^{-1}OPT)$ | $\Omega(\tilde{\gamma}^{-1}OPT)$                     |
| *(this paper)*                |                                |                 | *(Diakonikolas et al., 2020c)*                        |

or methods from linear programming can be used to efficiently find the optimal halfspace. In the setting of agnostic PAC learning *(Kearns et al., 1994)* where $OPT > 0$ in general, the question of which distributions can be learned up to classification error $OPT + \varepsilon$, and whether it is possible to do so in $\text{poly}(d, 1/\varepsilon)$ time (where $d$ is the input dimension), is significantly more difficult and is still an active area of research. It is known that without distributional assumptions, learning up to even $O(OPT) + \varepsilon$ is NP-hard *(Guruswami and Raghavendra, 2009; Daniely, 2016)*. Due to this difficulty, it is common to make a number of assumptions on either $D_x$ or to impose some type of structure to the learning problem.

A common structure imposed is that of structured noise: one can assume that there exists some underlying halfspace $y = \text{sgn}(v^\top x)$ that is corrupted with probability $\eta(x) \in [0, 1/2)$, possibly dependent on the features $x$. The simplest setting is that of random classification noise, where $\eta(x) \equiv \eta$, so that each label is flipped with the same probability *(Angluin and Laird, 1988)*; polynomial time algorithms for learning under this noise condition were shown by Blum et al. *(1998)*. The Massart noise model introduced by Massart et al. *(2006)* relaxes this assumption to $\eta(x) \leq \eta$ for some absolute constant $\eta < 1/2$. The Tsybakov noise model *(Tsybakov et al., 2004)* is a generalization of the Massart noise model that instead requires a tail bound on $\mathbb{P}(\eta(x) \geq 1/2 - t)$ for $t > 0$. Recently, Awasthi et al. *(2017)* showed that optimally learning halfspaces under Massart noise is possible for the uniform distribution on the unit sphere, and the recent landmark result of Diakonikolas et al. *(2019)* provided the first distribution-independent result for optimally learning halfspaces under Massart noise, answering a long-standing *(Sloan, 1988)* open problem in computational learning.

By contrast, in the agnostic PAC learning setting, one makes no assumptions on $\eta(x)$, so one can equivalently view agnostic PAC learning as an adversarial noise model in which an adversary can corrupt the label of a sample $x$ with any probability $\eta(x) < 1/2$. Recent work suggests that even when $D_x$ is the Gaussian, agnostically learning up to exactly $OPT + \varepsilon$ likely requires $\exp(1/\varepsilon)$ time *(Goel et al., 2020; Diakonikolas et al., 2020b)*. In terms of positive results in the agnostic setting, Awasthi et al. *(2017)* demonstrated that a localization-based algorithm can achieve $O(OPT) + \varepsilon$ under log-concave isotropic marginals. Diakonikolas et al. *(2020e)* showed that for a broad class of distributions, the output of projected SGD on a nonconvex surrogate for the zero-one loss produces a halfspace with risk $O(OPT) + \varepsilon$ in $\text{poly}(d, 1/\varepsilon)$ time.

Diakonikolas et al. *(2020e)* also showed that the minimizer of the surrogate risk of any convex surrogate for the zero-one loss is a halfspace with classification error $\omega(OPT)$. Ben-David et al. *(2012)* and Awasthi et al. *(2017)* showed similar lower bounds that together imply that empirical
risk minimization procedures for convex surrogates yield halfspaces with classification error $\Omega(\text{OPT})$. Given such lower bounds, we wish to emphasize that in this paper we are not making a claim about the optimality of gradient descent (on convex surrogates) for learning halfspaces. Rather, our main interest is the characterization of what are the strongest learning guarantees possible with what is perhaps the simplest learning algorithm possible. Given the success of gradient descent for the learning of deep neural networks, and the numerous questions that this success has brought to the theory of statistics and machine learning, we think it is important to develop a thorough understanding of what are the possibilities of vanilla gradient descent, especially in the simplest setting possible.

Recent work has shown that gradient descent finds approximate minimizers for the population risk of single neurons $x \mapsto \sigma(w^\top x)$ under the squared loss (Diakonikolas et al., 2020a; Frei et al., 2020), despite the computational intractability of finding the optimal single neuron (Goel et al., 2019). The main contribution of this paper is that despite the computational difficulties in exact agnostic learning, the standard gradient descent algorithm satisfies an approximate agnostic PAC learning guarantee, in line with the results found by Frei et al. (2020) for the single neuron.

2.1 Notation

We say that a differentiable loss function $\ell$ is $L$-Lipschitz if $|\ell'(z)| \leq L$ for all $z$ in its domain, and we say the loss is $H$-smooth if its derivative $\ell'$ is $H$-Lipschitz. We use the word “decreasing” interchangeably with “non-increasing”. We use the standard $O(\cdot), \Omega(\cdot)$ order notations to hide universal constants and $\tilde{O}(\cdot), \tilde{\Omega}(\cdot)$ to additionally suppress logarithmic factors. Throughout this paper, $\|x\|$ refers to the standard Euclidean norm on $\mathbb{R}^d$ induced by the inner product $x^\top x$. We will emphasize that a vector $v$ is of unit norm by writing $\bar{v}$. We assume $D$ is a probability distribution over $\mathbb{R}^d \times \{\pm 1\}$ with marginal distribution $D_x$ over $\mathbb{R}^d$.

3 Soft Margins

In this section we will formally introduce the soft margin function and describe some common distributions for which it takes a simple form.

**Definition 3.1.** Let $\bar{v} \in \mathbb{R}^d$ satisfy $\|\bar{v}\| = 1$. We say $\bar{v}$ satisfies the soft margin condition with respect to a function $\phi_{\bar{v}} : \mathbb{R} \to \mathbb{R}$ if for all $\gamma \in [0, 1]$, it holds that

$$
\mathbb{E}_{x \sim D_x} \left[ \mathbb{I} \left( |\bar{v}^\top x| \leq \gamma \right) \right] \leq \phi_{\bar{v}}(\gamma).
$$

We note that our definition of soft margin is essentially an unnormalized version of the soft margin function considered by Foster et al. (2018) in the context of learning GLMs, since they defined $\phi_{\bar{v}}(\gamma)$ as the probability that $|\bar{v}^\top x/\|x\|| \leq \gamma$. This concept was also considered by Balcan and Zhang (2017) for $s$-concave isotropic distributions under the name ‘probability of a band’.

We first show that (hard) margin distributions have simple soft margin functions.

**Example 3.2 (Hard margin distributions).** If $D_x$ is a hard margin distribution in the sense that $\bar{v}^\top x \geq \gamma^* > 0$ for some $\gamma^* > 0$ almost surely, then $\phi_{\bar{v}}(\gamma) = 0$ for $\gamma < \gamma^*$.

**Proof.** This follows immediately: $\mathbb{P}(|\bar{v}^\top x| \leq \gamma) = 0$ when $\gamma < \gamma^*$.
A typical example of a hard margin distribution is a linearly separable distribution where the labels are corrupted by some fixed probability $\eta$, such as in the random classification noise model, or by some structured noise function $\eta(x) : \mathbb{R}^d \rightarrow [0, 1/2)$, such as in the Massart or Tsybakov noise settings. Note that the soft margin function in Example 3.2 is specific to the vector $\bar{v}$, and does not necessarily hold for arbitrary unit vectors in $\mathbb{R}^d$. By contrast, for many distributions it is possible to derive bounds on soft margin functions that hold for any vector $\bar{v}$, which we shall see below is a key step for deriving approximate agnostic learning guarantees for the output of gradient descent.

The next example shows that provided the projections of $\mathcal{D}_x$ onto one dimensional subspaces satisfy an anti-concentration property, then all soft margins function for that distribution take a simple form. To do so we first introduce the following definition.

**Definition 3.3 (Anti-concentration).** For $\bar{v} \in \mathbb{R}^d$, denote by $p_{\bar{v}}(\cdot)$ the marginal distribution of $x \sim \mathcal{D}_x$ on the subspace spanned by $\bar{v}$. We say $\mathcal{D}_x$ satisfies $U$-anti-concentration if there is some $U > 0$ such that for all unit norm $\bar{v}$, $p_{\bar{v}}(z) \leq U$ for all $z \in \mathbb{R}$.

A similar assumption was used in Diakonikolas et al. (2020c,e,d) for learning halfspaces; in their setup, the anti-concentration assumption was for the projections of $\mathcal{D}_x$ onto two dimensional subspaces rather than the one dimensional version we consider here.

**Example 3.4 (Distributions satisfying anti-concentration).** If $\mathcal{D}_x$ satisfies $U$-anti-concentration, then for any unit norm $\bar{v}$, $\phi_{\bar{v}}(\gamma) \leq 2U\gamma$.

**Proof.** We can write $\mathbb{P}(|\bar{v}^\top x| \leq \gamma) = \int_{-\gamma}^{\gamma} p_{\bar{v}}(z)dz \leq 2\gamma U$. 

We will show below that log-concave isotropic distributions satisfy $U$-anti-concentration for $U = 1$. We first remind the reader of the definition of log-concave isotropic distributions.

**Definition 3.5.** We say that a distribution $\mathcal{D}_x$ over $x \in \mathbb{R}^d$ is log-concave if it has a density function $p(\cdot)$ such that $\log p(\cdot)$ is concave. We call $\mathcal{D}_x$ isotropic if its mean is the zero vector and its covariance matrix is the identity matrix.

Typical examples of log-concave isotropic distributions include the standard Gaussian and the uniform distribution over a convex set.

**Example 3.6 (Log-concave isotropic distributions).** If $\mathcal{D}_x$ is log-concave isotropic then it satisfies 1-anti-concentration, and thus for any $\bar{v}$ with $\|\bar{v}\| = 1$, $\phi_{\bar{v}}(\gamma) \leq 2\gamma$.

**Proof.** This was demonstrated in Balcan and Zhang (2017, Proof of Theorem 11).\(^1\)

## 4 Gradient Descent Finds Minimizers of the Surrogate Risk

We begin by demonstrating that gradient descent finds weights that achieve the best population-level surrogate risk. The following theorem is a standard result from stochastic optimization. For completeness, we present its proof in Appendix D.

\(^1\)The cited theorem implies a similar bound of the form $O(\gamma)$ holds for the more general set of $s$-concave isotropic distributions. We focus here on log-concave isotropic distributions for simplicity.
Theorem 4.1. Suppose $\|x\| \leq B_X$ a.s. Let $\ell$ be convex, $L$-Lipschitz, and $H$-smooth, with $\ell(0) \leq 1$. Let $v \in \mathbb{R}^d$ be arbitrary with $\|v\| \leq V$ for some $V > 1$, and suppose that the initialization $w_0$ satisfies $\|w_0\| \leq V$. For any $\varepsilon, \delta > 0$ and for any provided $\eta \leq (2/5)H^{-1}B_X^{-2}$, if gradient descent is run for $T = (4/3)\eta^{-1}\varepsilon^{-1} \|w_0 - v\|^2$, then with probability at least $1 - \delta$,

$$F_\ell(w_{T-1}) \leq F_\ell(v) + \frac{4B_XVL}{\sqrt{n}} + 8B_XV\sqrt{\frac{2\log(2/\delta)}{n}}.$$  

The above theorem shows that gradient descent learns halfspaces that have a population surrogate risk competitive with that of the best predictor with bounded norm for any norm threshold $V$. We can use this result to answer a recent question posed by Ji et al. (2020) concerning the properties of the loss function used for gradient descent on the linear halfspace problem:

*How does the loss function’s tail behavior affect time and sample complexity of gradient descent?*

For distributions that are linearly separable by some margin $\gamma > 0$, we show the below upper bounds on the sample complexity that suggest that exponentially tailed losses are preferable to polynomially tailed losses from both time and sample complexity perspectives.

Corollary 4.2 (Sample complexity for linearly separable data). Assume $\|x\| \leq B_X$ a.s. Suppose that for some $\bar{v} \in \mathbb{R}^d$, $\|\bar{v}\| = 1$, there is $\gamma > 0$ such that $y\bar{v}^\top x \geq \gamma$ a.s. If $\ell$ is convex, decreasing, $L$-Lipschitz, and $H$-smooth, and if we fix a step size of $\eta \leq (2/5)H^{-1}B_X^{-2}$, then

- Assume $\ell$ has polynomial tails, so that for some $C_0, p > 0$ and $\ell(z) \leq C_0 z^{-p}$ holds for all $z \geq 1$. Provided $n = \Omega(\gamma^{-2}\varepsilon^{-2-2/p})$, then running gradient descent for $T = \Omega(\gamma^{-2}\varepsilon^{-1-2/p})$ iterations guarantees that $\text{err}_D^{0-1}(w_T) \leq \varepsilon$.

- Assume $\ell$ has exponential tails, so that for some $C_0, C_1, p > 0$, $\ell(z) \leq C_0 \exp(-C_1 z^p)$ holds for all $z \geq 1$. Then $n = \tilde{\Omega}(\gamma^{-2}\varepsilon^{-2})$ and $T = \tilde{\Omega}(\gamma^{-2}\varepsilon^{-1})$ guarantees that $\text{err}_D^{0-1}(w_T) \leq \varepsilon$.

The proof for the above Corollary can be found in Appendix C. At a high level, the above result shows that if the tails of the loss function are heavier, one may need to run gradient descent for longer to drive the population surrogate risk, and hence the zero-one risk, to zero. In the subsequent sections, we shall see that this phenomenon persists beyond the linearly separable case to the more general agnostic learning setting.

Remark 4.3. The sample complexity in Theorem 4.1 can be improved from $O(\varepsilon^{-2})$ to $O(\varepsilon^{-1})$ if we use online stochastic gradient descent rather than vanilla gradient descent. The proof of this is somewhat more involved as it requires a technical workaround to the unboundedness of the loss function, and may be of independent interest. We present the full analysis of this in Appendix A.

5 Gradient Descent Finds Approximate Minimizers for the Zero-One Loss

We now show how we can use the soft margin function to develop bounds for the zero-one loss of the output of gradient descent.
5.1 Bounded Distributions

We first focus on the case when the marginal distribution \( \mathcal{D}_x \) is bounded almost surely.

By Theorem 4.1, since by Markov’s inequality we have that \( \text{err}_\mathcal{D}^0(w) \leq F_\ell(w) \), if we want to show that the zero-one population risk for the output of gradient descent is competitive with that of the optimal zero-one loss achieved by some halfspace \( v \in \mathbb{R}^d \), it suffices to bound \( F_\ell(v) \) by some function of \( \text{OPT} \). To do so we decompose the expectation for \( F_\ell(v) \) into a sum of three terms which incorporate \( \text{OPT} \), the soft margin function, and a term that drives the surrogate risk to zero by driving up the margin on those samples that are correctly classified.

**Lemma 5.1.** Let \( v \) be a unit norm population risk minimizer for the zero-one loss, and suppose \( v \) satisfies the soft margin condition with respect to some \( \phi : [0, 1] \to \mathbb{R} \). Assume that \( \|x\| \leq B_X \) a.s.

Let \( v = V \bar{v} \) for \( V > 0 \) be a scaled version of \( v \). If \( \ell \) is decreasing, \( L \)-Lipschitz and \( \ell(0) \leq 1 \), then

\[
F_\ell(v) = \mathbb{E}_{(x,y) \sim \mathcal{D}} [\ell(yv^\top x)] = \inf_{\gamma > 0} \left\{ (1 + LV B_X)\text{OPT} + \phi(\gamma) + \ell(V\gamma) \right\}.
\]

**Proof.** We begin by writing the expectation as a sum of three terms,

\[
\mathbb{E}[\ell(yv^\top x)] = \mathbb{E}\left[\ell(yv^\top x) \mathbb{1}(y\bar{v}^\top x \leq 0)\right] + \mathbb{E}\left[\ell(yv^\top x) \mathbb{1}(0 < y\bar{v}^\top x \leq \gamma)\right] + \mathbb{E}\left[\ell(y^\top x) \mathbb{1}(y\bar{v}^\top x > \gamma)\right]. \tag{5.1}
\]

For the first term, we use that \( \ell \) is \( L \)-Lipschitz and Cauchy–Schwarz to get

\[
\mathbb{E}[\ell(yv^\top x) \mathbb{1}(y\bar{v}^\top x \leq 0)] \leq \mathbb{E}\left[(1 + L|v^\top x|) \mathbb{1}(y\bar{v}^\top x \leq 0)\right] \leq (1 + LV B_X)\mathbb{E}[\mathbb{1}(y\bar{v}^\top x \leq 0)] = (1 + LV B_X)\text{OPT}.
\]

In the last inequality we use that \( \|x\| \leq B_X \) a.s. For the second term,

\[
\mathbb{E}\left[\ell(yv^\top x) \mathbb{1}(0 < y\bar{v}^\top x \leq \gamma)\right] \leq \ell(0)\mathbb{E}\left[\mathbb{1}(0 < y\bar{v}^\top x \leq \gamma)\right] \leq \phi(\gamma), \tag{5.2}
\]

where we have used that \( \ell \) is decreasing in the first inequality and Definition 3.1 in the second. Finally, for the last term, we can use that \( \ell \) is decreasing to get

\[
\mathbb{E}\left[\ell(yv^\top x) \mathbb{1}(y\bar{v}^\top x > \gamma)\right] = \mathbb{E}\left[\ell(yV\bar{v}^\top x) \mathbb{1}(yV\bar{v}^\top x > V\gamma)\right] \leq \ell(V\gamma). \tag{5.3}
\]

In order to concretize this bound, we want to take \( V \) large enough so that the \( \ell(V\gamma) \) term is driven to zero, but not so large so that the term in front of \( \text{OPT} \) grows too large. Theorem 4.1 is given in terms of an arbitrary \( v \in \mathbb{R}^d \), and so in particular holds for \( v = V\bar{v} \). We can view the results of Theorem 4.1 as stating an equivalence between running gradient descent for longer and for driving the norm \( \|v\| = V \) to be larger.

We formalize the above intuition into Theorem 5.2 below. Before doing so, we introduce the following notation. For general decreasing function \( \ell \), for which an inverse function may or may not exist, we overload the notation \( \ell^{-1} \) by denoting \( \ell^{-1}(t) := \inf\{z : \ell(z) \leq t\} \).

**Theorem 5.2.** Suppose \( \|x\| \leq B_X \) a.s. Let \( \ell \) be convex, decreasing, \( L \)-Lipschitz, and \( H \)-smooth, with \( \ell(0) \leq 1 \). Assume that a unit norm population risk minimizer of the zero-one loss, \( \bar{v} \), satisfies
the $\phi$-soft-margin condition for some increasing $\phi : \mathbb{R} \to \mathbb{R}$. Fix a step size $\eta \leq (2/5)H^{-1}B_{X}^{-2}$. Let $\varepsilon_{1}, \gamma > 0$ and $\varepsilon_{2} \geq 0$ be arbitrary. Denote by $w_{T}$ the output of gradient descent run for $T = (4/3)\eta^{-1}\varepsilon_{1}^{-1}\gamma^{-2}[\ell^{-1}(\varepsilon_{2})]^{-2}$ iterations after initialization at the origin. Then, with probability at least $1 - \delta$,

$$\text{err}_{D}^{0-1}(w_{T}) \leq (1 + LB_{X}\gamma^{-1}\ell^{-1}(\varepsilon_{2}))\text{OPT} + \phi(\gamma) + O(\gamma^{-1}\ell^{-1}(\varepsilon_{2})n^{-1/2}) + \varepsilon_{1} + \varepsilon_{2},$$

where $O(\cdot)$ hides absolute constants that depend on $L$, $H$, and $\log(1/\delta)$.

**Proof.** We take $v = \bar{V}v$ for a given unit-norm zero-one population risk minimizer $\bar{v}$ in Theorem 4.1 to get that for some universal constant $C > 0$ depending only on $L$ and $\log(1/\delta)$, with probability at least $1 - \delta$,

$$F_{\ell}(w_{T}) \leq F_{\ell}(v) + \varepsilon_{1}/2 + CVB_{X}n^{-1/2}. \quad (5.4)$$

By Lemma 5.1, for any $\gamma > 0$ it holds that

$$F_{\ell}(v) \leq (1 + LB_{X})\text{OPT} + \phi(\gamma) + \ell(V\gamma).$$

Let now $V = \gamma^{-1}\ell^{-1}(\varepsilon_{2})$. Then $\ell(V\gamma) = \varepsilon_{2}$, and putting this together with (5.4), we get

$$F_{\ell}(w_{T}) \leq (1 + L\gamma^{-1})\text{OPT} + \phi(\gamma) + O(\gamma^{-1}\ell^{-1}(\varepsilon_{2})n^{-1/2}) + \varepsilon_{1} + \varepsilon_{2}. \quad (5.5)$$

Finally, by Markov’s inequality,

$$\mathbb{P}(yw_{T}\top x < 0) \leq \frac{\mathbb{E}[\ell(yw_{T}\top x)]}{\ell(0)} = F_{\ell}(w_{T}). \quad (5.6)$$

Putting (5.5) together with (5.6) completes the proof. \hfill \Box

A few comments on the proof of the above theorem are in order. Note that the only place we use smoothness of the loss function is in showing that gradient descent minimizes the population risk in (5.4), and it is not difficult to remove the $H$-smoothness assumption to accommodate e.g., the hinge loss. On the other hand, that $\ell$ is $L$-Lipschitz is key to the proof of Lemma 5.1. Non-Lipschitz losses such as the exponential loss or squared hinge loss would incur additional factors of $\gamma^{-1}$ in front of $\text{OPT}$ in the final bound for Theorem 5.2.\footnote{This is because the first term in (5.1) would be bounded by $\text{OPT} \cdot \sup_{|x| \leq VB_{X}} \ell(x)$. For Lipschitz losses this incurs a term of order $O(V)$ while (for example) the exponential loss would have a term of order $O(\exp(V))$, and our proof requires $V = \Omega(\gamma^{-1})$.} We shall see below in the proof of Proposition 5.5 that this would yield worse guarantees for $\text{err}_{D}^{0-1}(w_{T})$.

Additionally, in concordance with the result from Corollary 4.2, we see that if the tail of $\ell$ is fatter, then $\ell^{-1}(\varepsilon_{2})$ will be larger and so our guarantees would be worse. In particular, for losses with exponential tails, $\ell^{-1}(\varepsilon_{2}) = O(\log(1/\varepsilon_{2}))$, and so by using such losses we incur only additional logarithmic factors in $1/\varepsilon_{2}$. For this reason, we will restrict our attention in the below results to the logistic loss—which is convex, decreasing, 1-Lipschitz and 1-smooth—although they apply equally to more general losses with different bounds that will depend on the tail behavior of the loss.

We now demonstrate how to convert the bounds given in Theorem 5.2 into bounds solely involving $\text{OPT}$ by substituting the forms of the soft margin functions given in Section 3.
Corollary 5.3 (Hard margin distributions). Suppose that $\|x\| \leq B_X$ a.s. and that a unit norm population risk minimizer $\hat{v}$ for the zero-one loss satisfies $|\hat{v}^\top x| \geq \hat{\gamma} > 0$ almost surely under $\mathcal{D}_x$ for some $\hat{\gamma} > 0$. For simplicity assume that $\ell(z) = \log(1 + \exp(-z))$ is the logistic loss. Then for any $\varepsilon, \delta > 0$, with probability at least $1 - \delta$, running gradient descent for $T = \tilde{O}(\eta^{-1} \varepsilon^{-1} \hat{\gamma}^{-2})$ is guaranteed to find a point $w_T$ such that

$$\text{err}_{\mathcal{D}}(w_T) \leq \text{OPT} + 2 B_X \bar{\gamma}^{-1} \text{OPT} \log(2/\text{OPT}) + \varepsilon,$$

provided $n = \tilde{\Omega}(\hat{\gamma}^{-2} B_X^2 \log(1/\delta) \varepsilon^{-2})$.

Proof. Since $|\hat{v}^\top x| \geq \gamma^* > 0$, $\phi(\gamma^*) = 0$. Note that the logistic loss is $\frac{1}{4}$-smooth and satisfies $\ell^{-1}(\varepsilon) \in [\log(1/(2\varepsilon)), \log(2/\varepsilon)]$. By taking $\varepsilon_2 = \text{OPT}$ the result follows by applying Theorem 5.2 with runtime $T = 4 \eta^{-1} \varepsilon^{-1} \hat{\gamma}^{-2} \log^2(1/2 \text{OPT})$. \hfill $\square$

Remark 5.4. The bound $\tilde{O}(\gamma^{-1} \text{OPT})$ in Corollary 5.3 is tight up to logarithmic factors$^3$ if one wishes to use gradient descent on a convex surrogate of the form $\ell(yw^\top x)$. Diakonikolas et al. (2019, Theorem 3.1) showed that for any convex and decreasing $\ell$, there exists a distribution over the unit ball with margin $\hat{\gamma} > 0$ such that a population risk minimizer $w^* := \text{argmin}_w \mathbb{E}[\ell(yw^\top x)]$ has zero-one population risk at least $\Omega(\hat{\gamma}^{-1} \kappa)$, where $\kappa$ is the upper bound for the Massart noise probability. The Massart noise case is more restrictive than the agnostic setting and satisfies $\text{OPT} \leq \kappa$. A similar matching lower bound was shown by Ben-David et al. (2012, Proposition 1).

In the below Proposition we demonstrate the utility of having soft margins. As we saw in the examples in Section 3, there any many distributions that satisfy $\phi(\gamma) = O(\gamma)$. We show below the types of bounds one can expect when $\phi(\gamma) = O(\gamma^p)$ for some $p > 0$.

Proposition 5.5 (Soft margin distributions). Suppose $\|x\| \leq B_X$ a.s. and that the soft margin function for a population risk minimizer of the zero-one loss satisfies $\phi(\gamma) \leq C_0 \gamma^p$ for some $p > 0$. For simplicity assume that $\ell$ is the logistic loss, and let $\eta \leq (2/5) B_X^2$. Assuming $\text{OPT} > 0$, then for any $\varepsilon, \delta > 0$, with probability at least $1 - \delta$, gradient descent run for $T = \tilde{\Omega}(\eta^{-1} \varepsilon^{-1} \text{OPT}^{-2/(1+p)})$ iterations with $n = \tilde{\Omega}(\text{OPT}^{-2/(1+p)} \log(1/\delta) \varepsilon^{-2})$ samples satisfies

$$\text{err}_{\mathcal{D}}(w_T) \leq \tilde{O} \left((C_0 + B_X) \text{OPT} \frac{\varepsilon^p}{\text{OPT}}\right) + \varepsilon,$$

Proof. By Theorem 5.2, we have

$$\text{err}_{\mathcal{D}}(w_T) \leq (1 + LB_X \gamma^{-1} \ell^{-1}(\varepsilon_2)) \text{OPT} + C_0 \gamma^p + O(\gamma^{-1} B_X \ell^{-1}(\varepsilon_2)n^{-1/2}) + \varepsilon_1 + \varepsilon_2.$$

For the logistic loss, $L = 1$ and $\ell^{-1}(\varepsilon) \in [\log(1/2\varepsilon), \log(2/\varepsilon)]$ and so we take $\varepsilon_2 = \text{OPT}$. Choosing $\gamma^p = \gamma^{-1} \text{OPT}$, we get $\gamma = \text{OPT}^{1/(1+p)}$ and hence

$$\text{err}_{\mathcal{D}}(w_T) \leq (2 + B_X \text{OPT}^{-1/(1+p)} \log(2/\text{OPT})) \text{OPT} + C_0 \text{OPT} \frac{1}{\text{OPT}} + \varepsilon_1,$$

provided $n = \Omega(\text{OPT}^{-2/(1+p)} \varepsilon^{-2} \log(1/\delta) \log^2(1/\text{OPT}))$ and $T = 4 \eta^{-1} \varepsilon^{-1} \text{OPT}^{-2/(1+p)} \log^2(1/2 \text{OPT}).$ \hfill $\square$

---

$^3$In fact, one can get rid of the logarithmic factors here and elsewhere in the paper by using the hinge loss rather than the logistic loss. In this case one needs to modify Lemma D.1 to accommodate non-smooth losses, which can be done with runtime $O(\varepsilon^{-2})$ rather than $O(\varepsilon^{-1})$ by e.g. Shalev-Shwartz and Ben-David (2014, Lemma 14.1). Then we use the fact that $\ell^{-1}(0) = 1$ for the hinge loss.
By applying Proposition 5.5 to Examples 3.4 and 3.6 we get the following approximate agnostic learning guarantees for the output of gradient descent for log-concave isotropic distributions and other distributions satisfying $U$-anti-concentration.

**Corollary 5.6.** Suppose that $\mathcal{D}_x$ satisfies $U$-anti-concentration and $\|x\| \leq B_X$ a.s. Then for any $\varepsilon, \delta > 0$, with probability at least $1 - \delta$, gradient descent on the logistic loss with step size $\eta \leq (2/5)B_X^2$ and run for $T = \tilde{O}(\eta^{-1}\varepsilon^{-1}\text{OPT}^{-1})$ iterations and $n = \tilde{O}(\text{OPT}^{-1}\log(1/\delta)\varepsilon^{-2})$ samples returns weights $w_T$ satisfying $\text{err}_D^T(w_T) \leq \tilde{O}(\text{OPT}^{1/2}) + \varepsilon$, where $\tilde{O}(\cdot), \Omega(\cdot)$ hide universal constant depending on $B_X, U, \log(1/\delta)$ and $\log(1/\text{OPT})$ only.

### 5.2 Unbounded Distributions

We show in this section that we can achieve essentially the same results from Section 5.1 if we relax the assumption that $\mathcal{D}_x$ is bounded almost surely to being sub-exponential.

**Definition 5.7** (Sub-exponential distributions). We say $\mathcal{D}_x$ is $C_m$-sub-exponential if every $x \sim \mathcal{D}_x$ is a sub-exponential random vector with sub-exponential norm at most $C_m$. In particular, for any $\bar{v}$ with $\|\bar{v}\| = 1$, $\mathbb{P}_{\mathcal{D}_x}(|\bar{v}^\top x| \geq t) \leq \exp(-t/C_m)$.

We show in the following example that any log-concave isotropic distribution is $C_m$-sub-exponential for an absolute constant $C_m$ independent of the dimension $d$.

**Example 5.8.** If $\mathcal{D}_x$ is log-concave isotropic, then $\mathcal{D}_x$ is $O(1)$-sub-exponential.

**Proof.** By Section 5.2.4 and Definition 5.22 of Vershynin (2010), it suffices to show that for any unit norm $\bar{v}$, we have $(\mathbb{E}[|\bar{v}^\top x|^p])^{1/p} \leq O(p)$. By Balcan and Zhang (2017, Theorem 3), if we define a coordinate system in which $\bar{v}$ is an axis, then $\bar{v}^\top x$ is equal to the first marginal of $\mathcal{D}_x$ and is a one dimensional log-concave isotropic distribution. By Lovász and Vempala (2007, Theorem 5.22), this implies

$$(\mathbb{E}[|\bar{v}^\top x|^p])^{1/p} \leq 2p\mathbb{E}[|\bar{v}^\top x|] \leq 2p\sqrt{\mathbb{E}[|\bar{v}^\top x|^2]} \leq 2p.$$ 

In the second inequality we use Jensen’s inequality and in the last inequality we have used that $\bar{v}^\top x = x_1$ is isotropic.

As was the case for bounded distributions, the key to the proof for unbounded distributions comes from bounding the surrogate risk at a minimizer for the zero-one loss by some function of the zero-one loss.

**Lemma 5.9.** Suppose $\mathcal{D}_x$ is $C_m$-sub-exponential. Denote by $\bar{v}$ as a unit norm population risk minimizer for the zero-one loss, and let $v = V\bar{v}$ for $V > 0$ be a scaled version of $\bar{v}$. If $\ell$ is decreasing, $L$-Lipschitz and $\ell(0) \leq 1$, then

$$\mathbb{E}_{(x,y) \sim \mathcal{D}} \ell(yv^\top x) \leq \inf_{\gamma > 0} \left\{ (1 + C_m + LVC_m \log(1/\text{OPT}))\text{OPT} + \phi(\gamma) + \ell(V\gamma) \right\}.$$ 

**Proof.** We again use the decomposition (5.1), with the only difference coming from the bound for the first term, which we show here. Fix $\xi > 0$ to be chosen later. We can write

$$\mathbb{E}[\ell(yv^\top x)\mathbb{1}(yv^\top x \leq 0)] \leq \mathbb{E}[(1 + LV|\bar{v}^\top x|)\mathbb{1}(y\bar{v}^\top x < 0)]$$

$$= \text{OPT} + LV\mathbb{E}[|\bar{v}^\top x|\mathbb{1}(y\bar{v}^\top x \leq 0, |\bar{v}^\top x| \leq \xi)].$$
Theorem 5.10. This bound is optimized when the value for \( \eta \) is chosen to be arbitrary, and fix a step size \( \eta \leq L - 2B_X^2 \varepsilon_1/4 \). By running online SGD for \( T = 2\eta^{-1}\varepsilon_1^{-1} \gamma^{-2} |\ell^{-1}(\varepsilon_2)|^{-2} \) iterations after initialization at the origin, SGD finds a point such that in expectation over the randomness of SGD, we get the desired bound with \( \gamma = \OPT \) in Theorem 5.10 to get

\[
\mathbb{E}[\text{err}_D^0(w_t)] \leq (2 + C_m + L\gamma m^{-1}\log(1/\OPT))\OPT + 2\gamma U + \varepsilon.
\]

This bound is optimized when \( U\gamma = C_m\gamma^{-1}\OPT \), i.e. when \( \gamma = U^{-1/2}C_m^{1/2}\OPT^{1/2} \). Substituting this value for \( \gamma \) we get the desired bound with \( T = 2\eta^{-1}\varepsilon_1^{-1}C_mU^{-1}\OPT^{-1}\log^2(1/2\OPT) \).

To derive an analogue of Theorem 5.2 for unbounded distributions, we need to extend the analysis for the generalization bound for the output of gradient descent we presented in Theorem 4.1 to unbounded distributions. Rather than using (full-batch) vanilla gradient descent, we instead use online stochastic gradient descent. The reason for this is that dealing with unbounded distributions is significantly simpler with online SGD due to the ability to work with expectations rather than high-probability bounds. It is straightforward to extend our results to vanilla gradient descent at the expense of a more involved proof by using methods from e.g. Zhang et al. (2019).

Below we present our result for unbounded distributions. Its proof is similar to that of Theorem 5.2 and can be found in Appendix B.

**Theorem 5.10.** Suppose \( D_x \) is \( C_m \)-sub-exponential, and let \( \mathbb{E}[\|x\|^2] \leq B_X^2 \). Let \( \ell \) be convex, \( L \)-Lipschitz, and decreasing with \( \ell(0) \leq 1 \). Let \( \varepsilon_1, \gamma > 0 \) and \( \varepsilon_2 \geq 0 \) be arbitrary, and fix a step size \( \eta \leq L^{-2}B_X^{-2}\varepsilon_1/4 \). By running online SGD for \( T = 2\eta^{-1}\varepsilon_1^{-1} \gamma^{-2} |\ell^{-1}(\varepsilon_2)|^{-2} \) iterations after initialization at the origin, SGD finds a point such that in expectation over the randomness of SGD, we get the desired bound with \( \gamma = \OPT \) in Theorem 5.10 to get

\[
\mathbb{E}[\text{err}_D^0(w_t)] \leq (1 + C_m + L\gamma m^{-1}\log(1/\OPT))\OPT + \phi(\gamma) + \varepsilon_1 + \varepsilon_2.
\]

The above theorem yields the following bound for sub-exponential distributions satisfying \( U \)-anti-concentration. Recall from Examples 3.6 and 5.8 that log-concave isotropic distributions are \( O(1) \)-sub-exponential and satisfy anti-concentration with \( U = 1 \).

**Corollary 5.11.** Suppose \( D_x \) is \( C_m \)-sub-exponential with \( \mathbb{E}[\|x\|^2] \leq B_X^2 \) and assume \( U \)-anti-concentration holds. Let \( \ell \) be the logistic loss and let \( \varepsilon > 0 \). Fix a step size \( \eta \leq B_X^{-2}\varepsilon/16 \). By running online SGD for \( T = \hat{O}(\eta^{-1}\varepsilon^{-1}C_mU^{-1}\OPT^{-1}) \) iterations, there exists a point \( w_t, t < T \), such that

\[
\mathbb{E}[\text{err}_D^0(w_t)] \leq \hat{O}\left((C_m/\varepsilon)^{1/2}\OPT^{1/2}\right) + \varepsilon.
\]

**Proof.** By Example 3.4, \( \phi(\gamma) \leq 2\gamma U \). Since \( \ell^{-1}(\varepsilon) \in [\log(1/2\varepsilon), \log(2/\varepsilon)] \), we can take \( \varepsilon_2 = \OPT \) in Theorem 5.10 to get

\[
\mathbb{E}[\text{err}_D^0(w_t)] \leq (2 + C_m + L\gamma m^{-1}\log^2(1/2\OPT))\OPT + 2\gamma U + \varepsilon.
\]
Remark 5.12. Diakonikolas et al. (2020e, Theorem 1.4) recently showed that if the marginal of $D$ over $x$ is the standard Gaussian in $d$ dimensions, for every convex, non-decreasing loss $\ell$, the minimizer $v = \arg\min_w F_\ell(w)$ satisfies $err_D^{\ell}(v) = \Omega(\text{OPT} \sqrt{\log(1/\text{OPT})})$. Thus, there is a large gap between our upper bound of $\tilde{O}(\text{OPT}^{1/2})$ and their corresponding lower bound. We think it is an interesting question if either the lower bound or the upper bound could be sharpened.

We also wish to note that Diakonikolas et al. (2020e) showed that by using gradient descent on a certain bounded and decreasing non-convex surrogate for the zero-one loss, it is possible to show that gradient descent finds a point with $err_D^{\ell}(w_T) \leq O(\text{OPT}) + \varepsilon$. In comparison with our result, this is perhaps not surprising: if one is able to show that gradient descent with a bounded and decreasing loss function can achieve population risk bounded by $O(\mathbb{E}[\ell(yv^T x)])$ for arbitrary $v \in \mathbb{R}^d$, then the same proof technique that yields Theorem 5.10 from Lemma 5.9 would demonstrate that $err_D^{\ell}(w_t) \leq O(\text{OPT})$. Since the only globally bounded convex function is constant, this approach would require working with a non-convex loss.

6 Conclusion and Future Work

We have showed a number of results related to the learning of halfspaces using convex surrogates for the zero-one loss. First, in the case of linearly separable datasets with large margin, we showed that the choice of the loss function can affect the sample complexity required to achieve a small zero-one loss: loss functions with exponential tails have the strongest guarantees, while losses with polynomial tails may, in general, have worse sample and runtime complexities, answering a recent question posed by Ji et al. (2020). Second, by utilizing the notion of a soft margin function, we showed that gradient descent on convex surrogates finds approximate minimizers for the zero-one loss for hard margin distributions and sub-exponential distributions satisfying a simple anti-concentration inequality that include log-concave isotropic distributions like the Gaussian. Our results match (up to logarithmic factors) lower bounds shown for agnostically learning hard margin distributions with convex surrogates for the zero-one loss.

For future work, we are interested in extending our analysis to more general distributions by analyzing the appropriate soft margin functions. We are also interested in seeing if one can show that gradient descent finds approximate minimizers for the zero-one loss in more complicated settings than the linear halfspace problem; we are particularly curious about whether or not such bounds are possible for neural networks.

Acknowledgements

We thank Peter Bartlett for helpful comments that led us to the result on fast rates for stochastic gradient descent.

A Fast Rates with Stochastic Gradient Descent

In Theorem 4.1, we showed that $F_\ell(w_T) \leq F_\ell(v) + O(1/\sqrt{n})$ given $n$ samples from $D$ by using vanilla (full-batch) gradient descent. In this section we demonstrate that by instead using stochastic gradient descent, one can achieve $F_\ell(w_T) \leq O(F_\ell(v)) + O(1/n)$ by appealing to a martingle Bernstein bound. We note that although the population risk guarantee degrades from $F_\ell(v)$ to $O(F_\ell(v))$, our
bounds for the zero-one risk in vanilla gradient descent already have constant-factor errors and so
the constant-factor error for $F_t(v)$ will not change the order of our final bounds.

The version of stochastic gradient descent that we study is the standard online SGD. Suppose
we sample $z_t = (x_t, y_t) \sim D$ for $t = 1, \ldots, T$, and let us denote the $\sigma$-algebra generated by the first
t samples as $\mathcal{G}_t = \sigma(z_1, \ldots, z_t)$. Define

$$\hat{F}_t(w) := \ell(y_t w^\top x_t), \quad \mathbb{E} [\hat{F}_t(w_t) | \mathcal{G}_{t-1}] = F(w_t) = \mathbb{E}_{(x,y) \sim D} \ell(yw_t^\top x).$$

The online stochastic gradient descent updates take the form

$$w_{t+1} := w_t - \eta \nabla \hat{F}_t(w_t).$$

We are able to show an improved rate of $O(\varepsilon^{-1})$ when using online SGD.

**Theorem A.1** (Fast rate for online SGD). Assume that $\ell(\cdot) \geq 0$ is convex, decreasing, $L$-Lipschitz
and $H$-smooth. Assume $\|x\| \leq B_X$ a.s. For simplicity assume that $w_0 = 0$. Let $v \in \mathbb{R}^d$ be arbitrary
with $\|v\| \leq V$. Let $\eta \leq (32HB_X^2)^{-1}$. Then for any $\varepsilon, \delta > 0$, by running online stochastic gradient
descent for $T = O(\varepsilon^{-1}V^2 \log(1/\delta))$ iterations, with probability at least $1 - \delta$ there exists a point
$w_{t^*}$, with $t^* < T$, such that

$$\text{err}_D^{0^{-1}}(w_{t^*}) \leq O(\mathbb{E}[\ell(yv^\top x)]) + \varepsilon,$$

where $O(\cdot)$ hides constant factors that depend on $L$, $H$ and $B_X$ only.

In this section we will sketch the proof for the above theorem. First, we note the following
guarantee for the empirical risk. This result is a standard result in online convex optimization (see,
e.g., Theorem 14.13 in Shalev-Shwartz and Ben-David (2014)).

**Lemma A.2.** Suppose that $\ell(\cdot) \geq 0$ is convex and $H$-smooth, and that $\|x\| \leq B_X$ a.s. Then for
any $\alpha \in (0,1)$, for fixed step size $\eta \leq \alpha/(8HB_X^2)$, and for any $T \geq 1$, it holds that

$$\frac{1}{T} \sum_{t=0}^{T-1} \hat{F}_t(w_t) \leq (1 + \alpha) \frac{1}{T} \sum_{t=0}^{T-1} \hat{F}_t(v) + \frac{\|w_0 - v\|^2}{\eta T}.$$

From here, one could take expectations and show that in expectation over the randomness of
SGD, the population risk found by gradient descent is at most $(1 + \alpha)F_t(v) + O(1/T)$, but we
are interested in developing a generalization bound that has the same fast rate but holds with
high probability, which requires significantly more work. Much of the literature for fast rates in
stochastic optimization require additional structure to achieve such results: Bartlett et al. (2006)
showed that the empirical risk minimizer converges at a fast rate to its expectation under a low-noise
assumption; Sridharan et al. (2009) achieved fast rates for the output of stochastic optimization
by using explicit regularization by a strongly convex regularizer; Srebro et al. (2010) shows that
projected online SGD achieves fast rates when $\min_v \mathbb{E}[\ell(yv^\top x)] = 0$. By contrast, we show below that
the standard online SGD algorithm achieves a constant-factor approximation to the best population
risk at a fast rate. We do so by appealing to the following martingale Bernstein inequality.

**Lemma A.3** (Beygelzimer et al. (2011), Theorem 1). Let $\{Y_t\}$ be a martingale adapted to the
filtration $\mathcal{G}_t$, and let $Y_0 = 0$. Let $\{D_t\}$ be the corresponding martingale difference sequence. Fix
T > 0, and define the sequence of conditional variance

\[ U_{T-1} := \sum_{t < T} \mathbb{E}[D_t^2 | G_{k-1}], \]

and assume that \( D_t \leq R \) almost surely. Then for any \( \delta \in (0, 1) \), with probability greater than \( 1 - \delta \),

\[ Y_{T-1} \leq R \log(1/\delta) + (e - 2)U_{T-1}/R. \]

We would like to take \( Y_t = \sum_{r < t} [F(w_t) - \hat{F}_r(w_t)] \), which has martingale difference sequence \( D_t = F(w_t) - \hat{F}_t(w_t) \). The difficulty here is showing that \( D_t \leq R \) almost surely for some absolute constant \( R \). The obvious fix would be to show that the weights \( w_t \) stay within a bounded region throughout gradient descent via early stopping. In the case of full-batch gradient descent, this is indeed possible: in Lemma D.1 we showed that \( \|w_t - v\| \leq \|w_0 - v\| \) throughout gradient descent, which would imply that \( \ell(yw_t^\top x) \) is uniformly bounded for all samples \( x \) throughout G.D., in which case \( D_t \leq F(w_t) \) would hold almost surely. But for online stochastic gradient descent, since we must continue to take draws from the distribution in order to reduce the optimization error, there isn’t a straightforward way to get a bound on \( \|w_t\| \) to hold almost surely throughout the gradient descent trajectory.

Our way around this is to realize that in the end, our end goal is to show something of the form

\[ \text{err}_D^{0-1}(w_t) \leq O(\mathbb{E}[\ell(yv^\top x)]) + O(1/T), \]

since then we could use a decomposition similar to Lemma 5.1 to bound the right hand side by terms involving OPT and a soft margin function. Since for a non-negative \( H \)-smooth loss \( [\ell'(z)]^2 \leq 4H\ell(z) \) holds, it actually suffices to show that the losses \( \{[\ell'(y_t w_t^\top x_t)]^2\}_1^T \) concentrate around their expectation at a fast rate. Roughly, this is because one would have

\[
\begin{align*}
\min_{t<T} \mathbb{E}_D \left( [\ell(y w_t^\top x)]^2 \right) &\leq \frac{1}{T} \sum_{t=0}^{T-1} [\ell'(y_t w_t^\top x_t]^2] + O(1/T) \\
&\leq 4H \frac{1}{T} \sum_{t=0}^{T-1} \ell(y_t w_t^\top x_t) + O(1/T) \\
&\leq 4H \frac{1}{T} \sum_{t=0}^{T-1} \ell(y_t v^\top x_t) + O(1/T). \tag{A.1}
\end{align*}
\]

To finish the proof we can then use the fact that \( v \) is a fixed vector of constant norm to show that the empirical risk on the last line of (A.1) concentrates around \( \mathbb{E}[\ell(yv^\top x)] \) at rate \( O(1/T) \). For decreasing and convex loss functions, \( \ell'(z)^2 \) is decreasing so the above provides a bound for \( \text{err}_D^{0-1}(w_t) \) by Markov’s inequality.

This shows that the key to the proof is to show that \( \{[\ell'(y_t w_t^\top x_t]^2\} \) concentrates at rate \( O(1/T) \). The reason this is easier than showing concentration of \( \{\ell(y_t w_t^\top x_t) \} \) is because for Lipschitz losses, \( \ell'(y_t w_t^\top x_t)^2 \) is uniformly bounded regardless of the norm of \( w_t \). This ensures that the almost sure condition needed for the martingale difference sequence in Lemma A.3 holds trivially. We note that a similar technique has been utilized before for the analysis of SGD (Ji and Telgarsky, 2020; Cao and Gu, 2020; Frei et al., 2019), although in these settings the authors used the concentration of
\{ \ell'(z_t) \} rather than \{ \ell'(z_t)^2 \} since they considered the logistic loss, for which \(|\ell'(z)| \leq \ell(z)\). Since not all smooth loss functions satisfy this inequality, we instead use concentration of \{ \ell'(z_t)^2 \}.

Below we formalize the above proof sketch. We first show that \{ \ell'(y_t w_t^\top x_t)^2 \} concentrates at rate \(O(1/T)\) for any fixed sequence of gradient descent iterates \(\{w_t\}\).

**Lemma A.4.** Let \(\ell\) be any differentiable \(L\)-Lipschitz function, and let \(z_t = (x_t, y_t)^{\frac{1}{1-d}} \sim D\). Denote \(G_t = \sigma(z_1, \ldots, z_t)\) the \(\sigma\)-algebra generated by the first \(t\) draws from \(D\), and let \(\{w_t\}\) be any sequence of random variables such that \(w_t\) is \(G_{t-1}\)-measurable for each \(t\). Then for any \(\delta > 0\), with probability at least \(1 - \delta\),

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{(x,y) \sim D} \left( \left[ \ell'(y_t w_t^\top x_t) \right]^2 \right) \leq \frac{4}{T} \sum_{t=0}^{T-1} \left[ \ell'(y_t w_t^\top x_t) \right]^2 + \frac{4L^2 \log(1/\delta)}{T}. \tag{A.2}
\]

**Proof.** For simplicity, let us denote

\[
J(w) := \mathbb{E}_{(x,y) \sim D} \left( \left[ \ell'(y w^\top x) \right]^2 \right), \quad \hat{J}_t(w) := \left[ \ell'(y_t w_t^\top x_t) \right]^2.
\]

We begin by showing the second inequality in (A.2). Define the random variable

\[
Y_t := \sum_{\tau < t} (J(w_{\tau}) - \hat{J}_t(w_{\tau})) \tag{A.3}
\]

Then \(Y_t\) is a martingale with respect to the filtration \(G_{t-1}\) with martingale difference sequence \(D_t := J(w_t) - \hat{J}_t(w_t)\). We need bounds on \(D_t\) and on \(\mathbb{E}[D_t^2|G_{t-1}]\) in order to apply Lemma A.3. Since \(\ell\) is \(L\)-Lipschitz,

\[
D_t \leq J(w_t) = \mathbb{E}_{(x,y) \sim D} \left( \left[ -\ell'(y v^\top x) \right]^2 \right) \leq L^2.
\]

Similarly,

\[
\mathbb{E}[\hat{J}_t(w_t)^2|G_{t-1}] = \mathbb{E} \left( \left[ \ell'(y_t w_t^\top x_t) \right]^4 | G_{t-1} \right)
\]

\[
\leq L^2 \mathbb{E} \left( \left[ \ell'(y_t w_t^\top x_t) \right]^2 | G_{t-1} \right)
\]

\[
= L^2 J(w_t). \tag{A.4}
\]

In the inequality we use that \(\ell\) is \(L\)-Lipschitz, so that \(|\ell'(\alpha)| \leq L\). We then can use (A.4) to bound the squared increments,

\[
\mathbb{E}[D_t^2|G_{t-1}] = J(w_t)^2 - 2J(w_t)\mathbb{E}[\hat{J}_t(w_t)|G_{t-1}] + \mathbb{E}[\hat{J}_t(w_t)^2|G_{t-1}]
\]

\[
\leq \mathbb{E}[\hat{J}_t(w_t)^2|G_{t-1}]
\]

\[
\leq L^2 J(w_t).
\]

This allows for us to bound

\[
U_{T-1} = \sum_{t=0}^{T-1} \mathbb{E}[D_t^2|G_{t-1}] \leq L^2 \sum_{t=0}^{T-1} J(w_t).
\]
Lemma A.3 thus implies that with probability at least $1 - \delta$, we have
\[
\sum_{t=0}^{T-1} (J(w_t) - \hat{J}_t(w_t)) \leq L^2 \log(1/\delta) + (\exp(1) - 2) \sum_{t=0}^{T-1} J(w_t).
\]

Using that $(1 - \exp(1) + 2)^{-1} \leq 4$, we divide each side by $T$ and get
\[
\frac{1}{T} \sum_{t=0}^{T-1} J(w_t) \leq \frac{4L^2 \log(1/\delta)}{T} + (\exp(1) - 2) \sum_{t=0}^{T-1} \hat{J}_t(w_t) + \frac{4}{T}.
\]

This completes the proof.

Next, we show that $\{(\ell(y_t v^\top x_t))\}$ concentrates around its expectation at rate $O(1/T)$.

**Lemma A.5.** Let $\ell$ be any $L$-Lipschitz function, and suppose that $\ell(0) \leq 1$ and $\|x\|_2 \leq B$ a.s. Let $v \in \mathbb{R}^d$ be arbitrary with $\|v\| \leq V$. For any $\delta > 0$, with probability at least $1 - \delta$,
\[
\frac{1}{T} \sum_{t=0}^{T-1} \hat{F}_t(v) \leq F(v) + \frac{2(1 + LBVX) \log(1/\delta)}{T}.
\]

**Proof.** Let $\mathcal{G}_t = \sigma(z_1, \ldots, z_t)$ be the $\sigma$-algebra generated by the first $t$ draws from $\mathcal{D}$. Then the random variable $Y_t := \sum_{\tau < t}(\hat{F}_\tau(v) - F(v))$ is a martingale with respect to the filtration $\mathcal{G}_{t-1}$ with martingale difference sequence $D_t := \hat{F}_t(v) - F(v)$. We need bounds on $D_t$ and on $\mathbb{E}[D_t^2|\mathcal{G}_{t-1}]$ in order to apply Lemma A.3. Since $\ell$ is $L$-Lipschitz and $\|x\| \leq B$ a.s., that $\|v\| \leq V$ implies that almost surely,
\[
D_t \leq \hat{F}_t(v) = \ell(y_t v^\top x_t) \leq (1 + LBVX).
\]

Similarly,
\[
\mathbb{E}[\hat{F}_t(v)^2|\mathcal{G}_{t-1}] = \mathbb{E}\left[\ell(y_t v^\top x_t)^2|\mathcal{G}_{t-1}\right] \
\leq (1 + LBVX)\mathbb{E}[\ell(y_t v^\top x_t)] \
= (1 + LBVX)F(v).
\]

In the inequality, we have used that $(x_t, y_t)$ is independent from $\mathcal{G}_{t-1}$ together with (A.5). We then can use (A.6) to bound the squared increments,
\[
\mathbb{E}[D_t^2|\mathcal{G}_{t-1}] = F(v)^2 - 2F(v)\mathbb{E}[\hat{F}_t(v)|\mathcal{G}_{t-1}] + \mathbb{E}[\hat{F}_t(v)^2|\mathcal{G}_{t-1}] \
\leq \mathbb{E}[\hat{F}_t(v)^2|\mathcal{G}_{t-1}] \leq (1 + LBVX)F(v).
\]

This allows for us to bound
\[
U_{T-1} := \sum_{t=0}^{T-1} \mathbb{E}[D_t^2|\mathcal{G}_{t-1}] \leq (1 + LBVX)TF(v).
\]
Lemma A.3 thus implies that with probability at least $1 - \delta$, we have
\[
\sum_{t=0}^{T-1} (\hat{F}_t(v) - F(v)) \leq (1 + LV B_X) \log(1/\delta) + (\exp(1) - 2)TF(v).
\]

Using that $\exp(1) - 2 \leq 1$, we divide each side by $T$ and get
\[
\frac{1}{T} \sum_{t=0}^{T-1} \hat{F}_t(v) \leq F(v) + \frac{2(1 + LV B_X) \log(1/\delta)}{T}.
\]

Finally, we put these ingredients together for the proof of Theorem A.1.

**Proof.** Since $\ell$ is convex and $H$-smooth, we can take $\alpha = 1/4$ in Lemma A.2 to get
\[
\frac{1}{T} \sum_{t=0}^{T-1} \hat{F}_t(w_t) \leq \frac{5}{4T} \sum_{t=0}^{T-1} \hat{F}_t(v) + \frac{V^2}{\eta T}.
\]

We can therefore bound
\[
\min_{t<T} \mathbb{E} \left( [\ell'(yw_t^\top x)]^2 \right) \leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{(x,y) \sim D} \left( [\ell'(yw_t^\top x)]^2 \right)
\]
\[
\leq \frac{4}{T} \sum_{t=0}^{T-1} [\ell'(y_t w_t^\top x_t)]^2 + \frac{4L^2 \log(2/\delta)}{T}
\]
\[
\leq \frac{16H}{T} \sum_{t=0}^{T-1} \hat{F}_t(w_t) + \frac{4L^2 \log(2/\delta)}{T}
\]
\[
\leq \frac{20H}{T} \sum_{t=0}^{T-1} \hat{F}_t(v) + \frac{5L^2 \log(2/\delta) + V^2}{\eta T}
\]
\[
\leq 20HF(v) + \frac{40H(1 + LV B_X)\eta \log(2/\delta) + 5L^2 \eta \log(2/\delta) + V^2}{\eta T}.
\]

The second inequality holds since $\ell$ is $L$-Lipschitz so that we can apply Lemma A.4. The third inequality uses that $\ell$ is non-negative and $H$-smooth, so that $[\ell'(z)]^2 \leq 4H\ell(z)$ (see Srebro et al. (2010, Lemma 2.1)). The fourth inequality uses (A.7), and the final inequality uses Lemma A.5.

Since $\ell$ is convex and decreasing, $\frac{d}{dz} \ell'(z)^2 = 2 \ell'(z)\ell''(z) \leq 0$, so $\ell'(z)^2$ is decreasing. By Markov’s inequality, this implies
\[
\mathbb{P}(yw_t^\top x < 0) = \mathbb{P} \left( [\ell'(yw_t^\top x)]^2 \geq \ell'(0)^2 \right) \leq [\ell'(0)]^{-2} \mathbb{E} \left( [\ell'(yw_t^\top x)]^2 \right).
\]

Substituting this into (A.8), this implies that with probability at least $1 - \delta$,
\[
err_{\mathcal{D}}^{0-1}(w_t) \leq O(F(v)) + O(V^2 \log(1/\delta)/T).
\]
We note that the above proof works for an arbitrary initialization \( w_0 \) such that \( \|w_0\| \) is bounded by an absolute constant with high probability, e.g. with the random initialization \( w_0 \overset{i.i.d.}{\sim} N(0, I_d/d) \). The only difference is that we need to replace \( V^2 \) with \( \|w_0 - v\|^2 \leq O(V^2) \) in (A.7) and the subsequent lines.

## B Proofs for Unbounded Distributions

In this section we prove Theorem 5.10.

### B.1 Empirical Risk

First, we derive an analogue of Lemma D.1 that holds for any distribution satisfying \( \mathbb{E}[\|x\|^2] \leq B_X^2 \) by appealing to online stochastic gradient descent. Note that any distribution over \( \mathbb{R}^d \) with sub-Gaussian coordinates satisfies \( \mathbb{E}[\|x\|^2] \leq B^2 \) for some \( B \in \mathbb{R} \).

We use the same notation from Section A, where we assume samples \( z_t = (x_t, y_t) \overset{i.i.d.}{\sim} \mathcal{D} \) for \( t = 1, \ldots, T \), and \( G_t := \sigma(z_1, \ldots, z_t) \), and denote

\[
\tilde{F}_t(w) := \ell(y_t w^\top x_t), \quad \mathbb{E}[\tilde{F}_t(w_t) | G_{t-1}] = F(w_t) = \mathbb{E}_{(x,y) \sim \mathcal{D}}(y w^\top x).
\]

The online stochastic gradient descent updates take the form

\[
w_{t+1} := w_t - \eta \nabla \tilde{F}_t(w_t).
\]

**Lemma B.1.** Suppose \( \mathbb{E}_\mathcal{D}_z[\|x\|^2] \leq B_X^2 \). Suppose that \( \ell \) is convex and \( L \)-Lipschitz. Let \( v \in \mathbb{R}^d \) and \( \epsilon, \alpha \in (0, 1) \) be arbitrary, and consider any initialization \( w_0 \in \mathbb{R}^d \). Provided \( \eta \leq L^{-2}B_x^{-2}\epsilon/2 \), then for any \( T \in \mathbb{N} \),

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} F(w_t) \leq F(v) + \frac{\|w_0 - v\|^2}{\eta T} + \epsilon.
\]

**Proof.** The proof is very similar to that of the proof of Lemma A.2 described in Appendix B.1, so we describe here the main modifications. The key difference comes from the gradient upper bound: for \( g_t = \ell'(y_t w_t^\top x_t) \), instead of getting an upper bound that holds a.s. in terms of the loss, we only show that its expectation is bounded by a constant:

\[
\mathbb{E}[\|g_t\|^2 | G_{t-1}] \leq \mathbb{E}[\ell'(y_t w_t x_t)^2 \|x_t\|^2 | G_{t-1}] \leq L^2 \mathbb{E}[\|x_t\|^2 | G_{t-1}] \leq L^2B_X^2.
\]

By convexity, \( \langle g_t, w_t - v \rangle \geq \tilde{F}_t(w_t) - \tilde{F}_t(v) \). Thus taking \( \eta = O(\epsilon) \), we get

\[
\|w_t - v\|^2 - \mathbb{E}[\|w_{t+1} - v\|^2 | G_{t-1}] \geq \mathbb{E}[2\eta(\tilde{F}_t(w_t) - \tilde{F}_t(v)) - \eta^2 \|g_t\|^2 | G_{t-1}] \\
\geq 2\eta(F(w_t) - F(v)) - \eta^2 L^2B_X^2 \\
\geq 2\eta(F(w_t) - F(v) - \epsilon).
\]

Taking expectations with respect to the randomness of SGD and summing from 0 to \( T - 1 \), we get

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} F(w_t) \leq F(v) + \frac{\|w_0 - v\|^2}{\eta T} + \epsilon.
\]

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We note that the above analysis is quite loose and we are aware of a number of ways to achieve faster rates by introducing various assumptions on $\ell$ and $D_x$; we chose the presentation above for simplicity.

With the above result in hand, we can prove Theorem 5.10.

Proof. Let $\varepsilon_1 > 0$. By taking $\eta \leq L^{-2}B_{x}^{-2}\varepsilon_1/8$ and $T = 2V^2\eta^{-1}\varepsilon_1^{-1}$, Lemma B.1 and Markov’s inequality, this implies that there exists some $t < T$ such that

$$
E[err_D^{0-1}(w_t)] \leq E[F(w_t)] \leq F(v) + \frac{V^2}{\eta T} + \varepsilon_1/2 \leq F(v) + \varepsilon_1.
$$

By Lemma 5.9, this implies that for any $\gamma > 0$,

$$
E[err_D^{0-1}(w_t)] \leq (1 + C_m + LV C_m \log(1/\text{OPT})) \cdot \text{OPT} + \phi(\gamma) + \ell(V \gamma) + \varepsilon_1.
$$

For $\varepsilon_2 \geq 0$, by taking $V = \gamma^{-1}\ell^{-1}(\varepsilon_2)$, this means that for any $\gamma > 0$, we have

$$
E[err_D^{0-1}(w_t)] \leq (1 + C_m + L C_m \ell^{-1}(\varepsilon_2) \gamma^{-1} \log(1/\text{OPT})) \cdot \text{OPT} + \phi(\gamma) + \varepsilon_1 + \varepsilon_2.
$$

For $V = \gamma^{-1}\ell^{-1}(\varepsilon_2)$, we need $T = 2\gamma^{-2}\eta^{-1}\varepsilon_1^{-1}[\ell^{-1}(\varepsilon_2)]^2$.

\[\square\]

C  Loss Functions and Sample Complexity for Separable Data

We present here the proof of Corollary 4.2.

Proof. Let $v = V\bar{v}$. By Theorem 4.1, for any $\varepsilon, \delta > 0$ and $V > 0$, running gradient descent for $T = 4[\ell(0)]^{-1}V^{-2}\varepsilon^{-1}$ iterations guarantees that $w = w_{T-1}$ satisfies

$$
F_{\ell}(w) \leq F_{\ell}(v) + \ell(0) \cdot \varepsilon/3 + CV n^{-1/2},
$$

for some absolute constant $C > 0$ depending only on $L$, $B_{x}$, and $\log(1/\delta)$. By Markov’s inequality, this implies

$$
\mathbb{P}(yw^\top x < 0) \leq \frac{1}{\ell(0)} F_{\ell}(w) \leq \frac{1}{\ell(0)} \left( F_{\ell}(v) + \frac{\ell(0)}{3} \varepsilon + CV n^{-1/2} \right).
$$

(C.1)

Since $y\bar{v}^\top x \geq \gamma$ a.s., we have

$$
F_{\ell}(v) = \mathbb{E}_{(x, y) \sim D}(y V \bar{v}^\top x) \leq \ell(V \gamma).
$$

If $\ell$ has polynomial tails, then by taking $V \geq \gamma^{-1}(6C_0[\ell(0)]^{-1} \varepsilon^{-1})^{1/p}$ we get $F_{\ell}(v) \leq C_0(\gamma V)^{-p} \leq \frac{\ell(0)}{6} \varepsilon$. Substituting this into (C.1), this implies

$$
\mathbb{P}(yw^\top x < 0) \leq \frac{\varepsilon}{2} + \frac{CV}{\ell(0)n^{1/2}}.
$$

(C.2)
Thus, provided \( n = \Omega(\gamma^{-2}\varepsilon^{-2-\frac{2}{\rho}}) \), if we run gradient descent for \( T = \tilde{\Omega}(\gamma^{-2}\varepsilon^{-1-\frac{2}{\rho}}) \) iterations, we have that \( \text{err}^{0\rightarrow1}(w) \leq \varepsilon \).

If \( \ell \) has exponential tails, then by taking \( V \geq \gamma^{-1}[C^{-1}_1 \log(6C_0\ell(0)\varepsilon^{-1})]^{1/\rho} \) we get \( F_\ell(v) \leq \frac{\ell(0)\varepsilon}{6} \), and so (C.2) holds in this case as well. This shows that for exponential tails, taking \( n = \Omega(\gamma^{-2}\varepsilon^{-2}) \) and \( T = \Omega(\gamma^{-2}\varepsilon^{-1}) \) suffices to achieve \( \text{err}^{0\rightarrow1}(w) \leq \varepsilon \).  

\[\square\]

## D Remaining Proofs

In this section we provide the proof of Theorem 4.1. We first will prove the following bound on the empirical risk.

**Lemma D.1.** Suppose that \( \ell \) is convex and \( H \)-smooth. Assume \( \|x\| \leq B_X \) a.s. Fix a step size \( \eta \leq (2/5)H^{-1}B_X^{-2} \), and let \( v \in \mathbb{R}^d \) be arbitrary. Then for any initialization \( w_0 \), and for any \( \varepsilon > 0 \), running gradient descent for \( T = (4/3)\varepsilon^{-1}\eta^{-1}\|w_0 - v\|^2 \) ensures that for all \( t < T \), \( \|w_t - v\| \leq \|w_0 - v\| \), and

\[
\hat{F}_\ell(w_{T-1}) \leq \frac{1}{T} \sum_{t=0}^{T-1} \hat{F}_\ell(w_t) \leq \hat{F}_\ell(v) + \varepsilon.
\]

To prove this, we first introduce the following upper bound for the norm of the gradient.

**Lemma D.2** *(Shamir (2020), Proof of Lemma 3).* Suppose that \( \ell \) is \( H \)-smooth. Then for any \( \rho \in (0, 1) \), provided \( \eta \leq 2\rho H^{-1}B_X^{-2} \), \( \hat{F}_\ell(w_t) \) is decreasing in \( t \). Moreover, if \( T \in \mathbb{N} \) is arbitrary and \( u \in \mathbb{R}^d \) is such that \( \hat{F}_\ell(u) \leq \hat{F}_\ell(w_T) \), then for any \( t < T \), we have the following gradient upper bound,

\[
\left\| \nabla \hat{F}_\ell(w_t) \right\|^2 \leq \frac{1}{\eta(1-\rho)} \left( \hat{F}_\ell(w_t) - \hat{F}_\ell(u) \right).
\]  

With this gradient upper bound, we can prove Lemma D.1.

**Proof.** Let \( \varepsilon > 0 \) be fixed and let \( T = (4/3)\varepsilon^{-1}\eta^{-1}\|w_0 - v\|^2 \) be as in the statement of the lemma. We are done if \( \hat{F}_\ell(w_T) < \hat{F}_\ell(v) \), so let us assume that \( \hat{F}_\ell(v) \leq \hat{F}_\ell(w_T) \). We proceed by providing the appropriate lower bounds for \( \|w_t - v\|^2 - \|w_{t+1} - v\|^2 = 2\eta \left\langle \hat{F}_\ell(w_t), w_t - v \right\rangle - \eta^2 \left\| \hat{F}_\ell(w_t) \right\|^2 \).

For any \( v \in \mathbb{R}^d \), by convexity of \( \ell \),

\[
\left\langle \nabla \hat{F}_\ell(w), w - v \right\rangle = \frac{1}{n} \sum_{i=1}^{n} \ell'(y_i w^\top x_i)(y_i w^\top x_i - y_i v^\top x_i)
\]

\[
\geq \frac{1}{n} \sum_{i=1}^{n} \left[ \ell(y_i w^\top x_i) - \ell(y_i v^\top x_i) \right]
\]

\[
= \hat{F}_\ell(w) - \hat{F}_\ell(v),
\]

by convexity of \( \ell \). On the other hand, since \( \hat{F}_\ell(v) \leq \hat{F}_\ell(w_T) \), by Lemma D.2, for any \( t < T \), (D.1)
holds, i.e.

\[
\left\| \nabla \hat{F}_\ell(w_t) \right\|^2 \leq \frac{1}{\eta(1-\rho)} \left( \hat{F}_\ell(w_t) - \hat{F}_\ell(v) \right).
\] (D.3)

Thus, for \( \eta \leq (2/5)H^{-1}B_X^{-2} \), putting eqs. (D.2) and (D.3) together yields

\[
\|w_t - v\|^2 - \|w_{t+1} - v\|^2 = 2\eta \langle \nabla \hat{F}_\ell(w_t), w_t - v \rangle - 2\eta^2 \left\| \nabla \hat{F}_\ell(w_t) \right\|^2 \\
\geq 2\eta \left( \hat{F}_\ell(w_t) - \hat{F}_\ell(v) \right) - \eta^2 \cdot \frac{1}{\eta(1-1/5)} \left( \hat{F}_\ell(w_t) - \hat{F}_\ell(v) \right) \\
= \frac{3}{4} \eta \left( \hat{F}_\ell(w_t) - \hat{F}_\ell(v) \right).
\]

Summing and telescoping over \( t < T \),

\[
\frac{1}{T} \sum_{t=0}^{T-1} \hat{F}_\ell(w_t) \leq \hat{F}_\ell(v) + \frac{4/3}{\eta T} \|w_0 - v\|^2 \leq \hat{F}_\ell(v) + \varepsilon.
\]

By Lemma D.2, \( \hat{F}_\ell(w_t) \) is decreasing in \( t \), and therefore

\[
\hat{F}_\ell(w_{T-1}) = \min_{t < T} \hat{F}_\ell(w_t) \leq T^{-1} \sum_{t < T} \hat{F}_\ell(w_t),
\]

completing the proof. \( \square \)

Lemma D.1 shows that throughout the trajectory of gradient descent, \( \|w_t\| \) stays bounded by the norm of the reference vector \( v \). We can thus use Rademacher complexity bounds to prove Theorem 4.1.

Proof. By Lemma D.1, it suffices to show that the gap between the empirical and population surrogate risk is small. To do so, we use a Rademacher complexity argument. Denote by \( \mathcal{G} \) the function class

\[
\mathcal{G}_V := \{ x \mapsto w^\top x : \|w\| \leq 3V \}.
\]

Since \( \ell \) is \( L \)-Lipschitz and \( \ell(0) \leq 1 \), it holds that \( \ell(yw^\top x) \leq 1 + 3LV \leq 4LV \). We therefore use standard results in Rademacher complexity (e.g. Theorem 26.12 of Shalev-Shwartz and Ben-David, 2014) to get that with probability at least \( 1 - \delta \), for any \( w \in \mathcal{G}_V \),

\[
F_\ell(w) \leq \hat{F}_\ell(w) + \frac{2B_XVL}{\sqrt{n}} + 4B_XV \sqrt{\frac{2\log(2/\delta)}{n}}.
\]

Since the output of gradient descent satisfies \( \|w_{T-1} - v\| \leq \|w_0 - v\| \leq 2V \), we see that \( w_{T-1} \in \mathcal{G}_V \). We can thus apply the Rademacher complexity bound to both \( w_{T-1} \in \mathcal{G}_V \) and \( v \in \mathcal{G}_V \), proving the theorem. \( \square \)

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