General Formalism for the BRST Symmetry

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Abstract In this paper we will discuss Faddeev–Popov method for gauge theories with a general form of gauge symmetry in an abstract way. We will then develop a general formalism for dealing with the BRST symmetry. This formalism will make it possible to analyse the BRST symmetry for any theory.

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1 Introduction
It is not possible to directly quantize a field theory with gauge symmetry. In order to quantize such theories, we need to only sum over the physical field configurations and not the pure gauge ones. This can be achieved by Faddeev–Popov method. Analytic and numerical results concerning the confinement scenario in Coulomb gauge have been studied using a local renormalizable BRST-invariant action, for QCD in Coulomb gauge. The appearance of the GL(1) charged physical operators in the twistor string theory are also analyzed using BRST invariant symmetry. The nilpotency of the operators which describe the gauge dependence of the generating functionals of Green’s functions for the gauge theories with the soft breaking of BRST symmetry have been studied in the Batalin–Vilkovisky formalism. An explicit solution to classical master equations of the Sp(2)-symmetric Hamiltonian BRST quantization scheme has been studied in the case of irreducible gauge theories.

A gauge invariant formulation for massless and massive bosonic AdS fields of arbitrary symmetry type at the level of equations of motion has been studied using first order BRST formalism. An explicit formula for the BRST charge associated with Poisson superalgebras has been constructed. To this end, the master equation for the BRST charge has been split into a pair of equations such that one of them is equivalent to the original one. The linearly broken BRST symmetry directly translates into a set of Slavnov–Taylor identities and these identities guarantee the multiplicative renormalizability of both Gribov–Zwanziger and Refined Gribov–Zwanziger theories to all orders. The known property that only two renormalization factors are needed is recovered in this model. The non-renormalization theorem of the gluon-ghost-antighost vertex as well as the renormalization factor of the Gribov parameter are also derived within this linearly broken formulation.

Nonlocal BRST invariance of the Gribov–Zwanziger action has been studied by the introduction of additional fields. A modified action with a corresponding local BRST invariance has thus been used to show that the correlation functions of the original elementary fields do not change upon evaluation with the modified partition function. In SU(2) Yang–Mills theories a simple relation exists between lattice gluon propagators in Coulomb and Landau gauge. In particular, the realization of the Gribov–Zwanziger confinement mechanism in Coulomb gauge, linked to dual-superconductivity, implies that the standard BRST charge must be ill defined non-perturbatively. As a consequence, the Kugo–Ojima confinement criterion, which relies on BRST charge conservation beyond perturbation theory is not fulfilled. By giving physical interpretations to the components of detour complexes a large class of new gauge theories is obtained.

So, we will discuss Faddeev–Popov method for field theories with a gauge symmetry in an abstract way. This method gives rise to Faddeev–Popov ghosts. Ghosts occur in higher derivative theories and gauge theories. A way to deal with these ghosts in gauge theory is called the BRST symmetry. Recently, BRST symmetry has been studied in gravity and M-Theory. In BRST formalism the sum of the classical Lagrangian, the gauge fixing term and the ghost term is invariant the BRST transformations. This can be used to remove all the negative norm states associated with the Faddeev–Popov ghosts. In this paper we will study the BRST symmetry of in general formalism and thus this can be applied to any particular case.

2 Faddeev–Popov Method
Let $A^i$ denotes the field we are considering. Now the classical action $S$ of a theory is invariant under gauge transformations. Here $\Omega$ denotes spacetime as well as gauge indices. Suppose the gauge transformations are...
given by
\[
\mathcal{A}^t \rightarrow \mathcal{A}^t + g_{jk} A^j A^k ,
\] (1)
where \( g_{jk} A^j A^k \) is a general functional of the infinitesimal parameter \( \Lambda^t \). From now on we will suppress the indices "\( i \)". Thus we have
\[
\delta S_c = 0 ,
\] (2)
where
\[
\delta S_c = S_c [ A + g(\Lambda, \Lambda) ] - S_c [ A ] .
\] (3)
This will lead to counting the divergence of the functional integral
\[
Z = \int D \Lambda \exp i S_c, \quad (4)
\]
where
\[
Z \rightarrow \infty .
\] (5)
To fix this problem we want to restrict this path integral to \( F[\mathcal{A}] = 0 \). This condition is called a gauge fixing condition. This is achieved by inserting \( \delta (F[\mathcal{A}]) \) in the functional integral. First we note that
\[
Z = \int D \Lambda \delta (F[\mathcal{A}]) \exp i S_c .
\] (6)
Here we have used
\[
1 = \int D \Lambda \delta (F[\mathcal{A}]) \exp \left[ \frac{\delta F[\mathcal{A}]}{\delta \Lambda} \right] .
\] (7)
Then we define a function \( F'[\mathcal{A}] \) as
\[
F'[\mathcal{A}] = F[\mathcal{A}] - \phi .
\] (8)
Here \( \phi(x) \) is a scalar function. As \( \phi \) does not depend on \( \Lambda \), so we have
\[
\det \left[ \frac{\delta F'[\mathcal{A}]}{\delta \Lambda} \right] = \det \left[ \frac{\delta F'[\mathcal{A}]}{\delta \Lambda} \right] .
\] (9)
Now we can write
\[
Z = \int D \Lambda \delta (F'[\mathcal{A}]) \exp i S_c \] (10)
Now we change the variables from \( \mathcal{A} \) to \( \mathcal{A}' \) and this as a simple shift, we have \( D \Lambda = D \Lambda' \) and \( S_c[\mathcal{A}] = S_c[\mathcal{A}'] \). Now as \( \mathcal{A}' \) is a dummy variable we can rename it back to \( \mathcal{A} \) and obtain
\[
Z = \int D \mathcal{A} \delta (F'[\mathcal{A}]) \exp i S_c [ \mathcal{A} ] .
\] (11)
Now if \( c \) and \( \bar{c} \) are anticommuting fields then
\[
\det \left[ \frac{\delta F'[\mathcal{A}]}{\delta \Lambda} \right] = \int Dc D\bar{c} \exp \left( -i \int d^4x \sqrt{-g} \bar{c} \partial \phi \right) .
\] (12)
Now as this holds for any \( \phi(x) \), it will also hold for a normalized linear combination of \( \phi(x) \) involving different \( \phi(x) \). Now we integrate over all \( \phi(x) \) as follows
\[
\int D\phi \exp \left( -i \int d^4x \sqrt{-g} \phi^2(x) \right) Z
\]
\[
= N \int D \mathcal{A} Dc D\bar{c} \exp i S_t ,
\] (13)
where \( N \) is the normalization constant and \( S_t \) is given by
\[
S_t = S_c + S_g + S_{gh} .
\] (14)
Here \( S_c \) is the original classical action, \( S_g \) is the gauge fixing term, and \( S_{gh} \) is the ghost action.

3 General Formalism
We will now discuss the general formalism for BRST in an abstract way. To do so we write the gauge fixing term by adding an auxiliary field \( B \). The gauge fixing Lagrangian with an auxiliary field \( B \) is written as
\[
S_g = \int d^4x \sqrt{-g} \left[ -BF[\mathcal{A}] + \frac{g}{2} B^2 \right] .
\] (15)
\( B \) does not contain any derivatives and the functional integral over \( B \) can be done by completing the square and this way we will recover the gauge fixing term obtained by Faddeev–Popov method. Now if we take the gauge transformation of the gauge fixing condition,
\[
\delta F[\mathcal{A}] = \mathcal{G}[\mathcal{A}] ,
\] (16)
and replace \( \mathcal{A} \) by the ghosts \( c \) to get \( \mathcal{G}[c] \). Then the ghost action is given by
\[
S_{gh} = -i \int d^4x \sqrt{-g} [c \mathcal{G}[c]] .
\] (17)
The action \( S_t \) is invariant under a symmetry call the BRST symmetry. The BRST transformations are given by
\[
s\mathcal{A}^i = i g_{jk} c^j A^k , \quad sB^i = 0 ,
\]
\[
s c^i = B^i , \quad s c^i = -\frac{1}{2} f_{jk} c^j c^k .
\] (18)
Here the BRST transformation of \( \mathcal{A}^i \) is obtained by replacing the infinitesimal parameter \( \Lambda^t \) in the gauge transformations by the ghost field \( c^i \) and the BRST transformation of \( c^i \) is obtained by taking the commutator of two gauge transformations and then replacing all the infinitesimal parameters by \( c^i \). Thus the function \( f_{jk} \), which is usually a constant, is obtained by taking the commutator of the variation and then replacing \( \Lambda^t \) by \( c^t \). The BRST transformation of \( B^t \) vanishes and BRST transformation of \( c^t \) is \( B^t \). These BRST transformations are nilpotent
\[
s^2 \mathcal{A}^i = 0 , \quad s^2 c^i = 0 , \quad s^2 B^i = 0 .
\] (19)
This nilpotency is important is isolating the physical states of the theory.

4 Physical States
Now we will discuss general property of BRST charge. It is known that there is a conserved charge called Noether’s charge corresponding to each symmetry under which the action is invariant. The effective action which is formed by the sum of the original action, the gauge fixing term and the ghost term is invariant under the BRST transformation. The Noether’s charge corresponding to BRST transformation is the BRST charge \( Q \)
\[
Q = \int d^4x J^0 ,
\] (20)
where
\[ J^0 = \frac{\partial L_i}{\partial \dot{\phi}_i} \partial A + \frac{\partial L_i}{\partial \dot{\phi}_0 c} \partial c + \frac{\partial L_i}{\partial \dot{\phi}_0 \dot{c}} \partial \dot{c} + \frac{\partial L_i}{\partial \dot{\phi}_0 B} \partial B, \]
\[ S_i = \int d^4x \sqrt{-g} \bar{L}_i. \]

It is nilpotent as its action on any field \( |A| \) twice vanishes,
\[ Q^2 |A| = 0. \]

Physical states \(|P\rangle\) are annihilated by \( Q \)
\[ Q |P\rangle = 0. \]

Now physical states can be divided into two types: \(|Pt\rangle\)
which are those physical states that are obtained from
the action of \( Q \) on unphysical states \(|UP\rangle\)
\[ |Pt\rangle = Q |UP\rangle, \]
and \(|Pnt\rangle\) which are those physical states that are not
obtained from the action of \( Q \) on any state
\[ |Pnt\rangle \neq Q |UP\rangle. \]

It is obvious that any state that can be represented as
the action of the BRST charge on any other state is a physical
state, as it will be annihilated due to nilpotency of \( Q \)
\[ Q |Pt\rangle = Q^2 |UP\rangle = 0. \]

All such states are in fact orthogonal to all physical states
including themselves
\[ \langle P | Pt \rangle = \langle P | Q | UP \rangle = 0. \]

All physical amplitudes involving such null states vanish.
Two physical states that differ from each other by a null
state will be indistinguishable,
\[ |P\rangle = |P| + Q |A\rangle. \]

Thus the relevant physical states in the theory are those
physical states that are not obtained from the action of \( Q \)
on any other state i.e., \(|Pnt\rangle\). So we can identify the physical
Hilbert space as a set of equivalence classes. This is how
we factor out the physical state from the total Hilbert
space of states. It is interesting to note that the nilpotency
of these BRST transformations was crucial for isolating
the physical states of the theory. If this nilpotency was
broken then it would not be possible to isolate the physical
states of the theory.

5 Conclusion

In this paper we have analysed a general formalism
for the BRST symmetry. The nilpotency of these BRST
transformation can be used to isolate the physical state
of the theory. This formalism can be used to analyse
the BRST symmetry for gravity and M-theory. It will
be interesting to analyse the anti-BRST symmetry also
in this general way. It will also be interesting to per-
form this analyses in curved spacetime. To work out the
BRST transformations in anti-de Sitter will be trivial as
there are no IR divergences for the ghosts in anti-de Sitter
spacetime.\(^{[60-61]}\) However, it will be difficult to do it
in de Sitter spacetime as there are IR divergences for the
ghosts in anti-de Sitter spacetime.\(^{[62-64]}\) It will also be in-
teresting to analyse the third quantization\(^{[65-69]}\) of gravity
using this BRST charge in analogy with string field
theory.\(^{[70-73]}\)
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