A REMARK ON A CURVATURE GAP FOR MINIMAL SURFACES IN THE BALL

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Abstract. We extend to higher codimension earlier characterization of the equatorial disk and the critical catenoid by a pinching condition on the length of their second fundamental form among free boundary minimal surfaces in the three dimensional Euclidean ball due to L. Ambrozio and I. Nunes.

1. Introduction

In this note we consider 2-dimensional free boundary minimal surfaces in the Euclidean ball $B^n$. The free boundary condition implies that these minimal surfaces meet the boundary of the ball orthogonally. Such surfaces arise as critical points of the area functional for relative cycles in the ball and as extremals for the Steklov problem [7, 8]. The simplest free boundary minimal surfaces in the ball are the equatorial disk and the critical catenoid. Recently, Ambrozio and Nunes [1] proved a characterization of the equatorial disk and the critical catenoid in the 3-dimensional ball by a pinching condition involving the length of the second fundamental form and the support function:

**Theorem 1.1 (Ambrozio-Nunes).** Let $\Sigma^2$ be a compact free boundary minimal surface in $B^3$. Assume that for all points $x \in \Sigma$,

\begin{equation}
|x^\perp|^2 |A(x)|^2 \leq 2
\end{equation}

where $x^\perp$ denotes the normal component of the point $x \in \Sigma$ and $A$ denotes the second fundamental form of $\Sigma$. Then

1. $|x^\perp|^2 |A(x)|^2 \equiv 0$ and $\Sigma$ is an equatorial disk;
2. $|x_0^\perp|^2 |A(x_0)|^2 = 2$ at some point $x_0 \in \Sigma$ and $\Sigma$ is a critical catenoid.

Theorem 1.1 share similarities with a classical theorem of Chern, do Carmo, and Kobayashi [4] (see also Lawson [10]) which characterizes the equatorial spheres and the Clifford hypersurfaces in $S^{n+1}$ and the Veronese surface in $S^4$ as the only minimal submanifolds of dimension $n$ in $S^{n+p}$ satisfying the inequality $|A|^2 \leq n / \left(2 - \frac{1}{p}\right)$. Despite the
analogy, the proof of Theorem 1.1 given in [1] is quite different. Besides working in dimension two, the codimension one is crucially used in some steps of the proof. The authors in [1] ask the question whether Theorem 1.1 can be generalized to higher ambient dimension and submanifold co-dimension. In this note we answer their question positively in the special case of 2-dimensional surfaces in the ball of any dimension, see Theorem 3.1 below.

Our proof follows closely the arguments in [1] and it is based on three ingredients: Fraser and Schoen’s Uniqueness Theorem for free boundary minimal disks, a standard dimension reduction argument, and an analysis of nodal sets for solutions of an elliptic system of partial differential equations.

We remark that Theorem 1.1 was recently generalized to geodesic balls in the 3-dimensional hyperbolic space and the hemisphere in [9]. Our proof also applies to these settings and their result can be extended in a similar way as discussed here. Finally, we mention that the pinching condition (1.1) also characterizes the plane and the catenoid among properly embedded minimal surfaces without boundary in $\mathbb{R}^3$ (see Remark 3.7 below). A version of this result was first proved by Meeks, Pérez, and Ros in [12, Section 7].

2. Preliminaries

The next two lemmas are standard, for the benefit of the reader we include their proofs.

**Lemma 2.1.** Let $\Sigma^k$ be a minimal submanifold in $\mathbb{R}^n$ given by the graph of the function $u : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ where $u(x) = (u_1(x), \ldots, u_{n-k}(x))$. Then for every $l = 1, \ldots, n-k$

\[
\frac{\alpha_{ij}(\nabla u_1, \ldots, \nabla u_{n-k})}{\sqrt{1 + |\nabla u_l|^2}} D_{ij} u_l = 0,
\]

for some smooth functions $\alpha_{ij}(\nabla u_1, \ldots, \nabla u_{n-k})$.

**Proof.** Parametrize $\Sigma$ as $\varphi(x) = (x, u_1(x), \ldots, u_{n-k}(x))$. The coordinate basis for $\Sigma$ is given by

\[
D_{x_i} \varphi = (0, \ldots, 1, \ldots, D_{x_i} u_1, \ldots, D_{x_i} u_{n-k}),
\]

for $i = 1, \ldots, k$. It follows that

\[
g_{ij} = \delta_{ij} + \sum_{l=1}^{n-k} D_{x_i} u_l D_{x_j} u_l \quad \text{and} \quad g^{ij} = \alpha_{ij}(\nabla u_1, \ldots, \nabla u_{n-k}).
\]
Now we consider for each \( l = 1, \ldots, n - k \) the unit normal vector
\[
N_l = \frac{1}{\sqrt{1 + |\nabla u_l|^2}} (-D_{x_1} u_l, \ldots, -D_{x_k} u_l, 0, \ldots, 1, \ldots, 0).
\]

A simple computation gives
\[
(N_l)_{x_i} = \left( \frac{1}{\sqrt{1 + |\nabla u_l|^2}} \right)_{x_i} \sqrt{1 + |\nabla u_l|^2} N_l + \frac{1}{\sqrt{1 + |\nabla u_l|^2}} (-D^2_{x_1 x_i} u_l, \ldots, -D^2_{x_k x_i} u_l, 0, \ldots, 0).
\]

Consequently,
\[
(A_N)_{ij} = \langle -dN_l(\varphi_{x_i}), \varphi_{x_j} \rangle = \frac{1}{\sqrt{1 + |\nabla u_l|^2}} D^2_{x_i x_j} u_l.
\]

Since \( \Sigma^k \) is minimal, \( 0 = g^{ij} (A_N)_{ij} \) and by (2.2) we obtain
\[
\frac{a_{ij}(\nabla u_1, \ldots, \nabla u_{n-k})}{\sqrt{1 + |\nabla u_l|^2}} D_{ij} u_l = 0.
\]

Lemma 2.2. If \( u, v : U \subset \mathbb{R}^k \to \mathbb{R}^p \) are smooth maps satisfying (2.1), then the difference \( \varphi = u - v \) satisfies, for each \( l = 1, \ldots, p \), the equation
\[
\frac{a_{ij}(\nabla u)}{\sqrt{1 + |\nabla u_l|^2}} D_{ij}(\varphi_l) + \sum_{m=1}^{p} b^m_{j}(\nabla u, \nabla v) D_{j}(\varphi_m) = 0,
\]
for some smooth functions \( a_{ij}(\nabla u) \) and \( b^m_{j}(\nabla u, \nabla v) \).

Proof. As \( u_l \) and \( v_l \) satisfy equation (2.1), therefore
\[
0 = \frac{a_{ij}(\nabla u)}{\sqrt{1 + |\nabla u_l|^2}} D_{ij} u_l - \frac{a_{ij}(\nabla v)}{\sqrt{1 + |\nabla v_l|^2}} D_{ij} v_l
\]
\[
= \frac{a_{ij}(\nabla u)}{\sqrt{1 + |\nabla u_l|^2}} D_{ij} u_l - \frac{a_{ij}(\nabla u)}{\sqrt{1 + |\nabla u_l|^2}} D_{ij} v_l
\]
\[
+ \frac{a_{ij}(\nabla u)}{\sqrt{1 + |\nabla u_l|^2}} D_{ij} v_l - \frac{a_{ij}(\nabla v)}{\sqrt{1 + |\nabla v_l|^2}} D_{ij} v_l
\]
\[
= \frac{a_{ij}(\nabla u_l)}{\sqrt{1 + |\nabla u_l|^2}} D_{ij}(\varphi_l) + \left( \frac{a_{ij}(\nabla u)}{\sqrt{1 + |\nabla u_l|^2}} - \frac{a_{ij}(\nabla v)}{\sqrt{1 + |\nabla v_l|^2}} \right) D_{ij} v_l.
\]

Now, let \( F_{ij} : \mathbb{R}^k \times \cdots \times \mathbb{R}^k \to \mathbb{R} \) be the function defined by
\[
F_{ij}(z_1, \ldots, z_p) = \frac{a_{ij}(z_1, \ldots, z_p)}{\sqrt{1 + |z_l|^2}}.
\]
By the Fundamental Theorem of Calculus we can write
\[ F_{ij}(\nabla u) - F_{ij}(\nabla v) = \left( \int_0^1 dF_{ij}(\nabla u + t(\nabla v - \nabla u))dt \right) \nabla (u - v). \]

The lemma follows by setting \( b^n_q \) to be
\[ b^n_q = \left( \int_0^1 dF_{ij}(\nabla u + t(\nabla v - \nabla u))dt \right)_{qm} D_{ij}v_l D_q(u - v)_m. \]

\[ \square \]

The next lemma, which is essentially contained in [9], concerns nodal sets for solutions of elliptic equations. We add the proof in order to include solutions of elliptic system of equations.

**Lemma 2.3** (Hardt-Simon [9]). Let \( u : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^p \) be a smooth map which satisfies for each \( k = 1, \ldots, p \) an elliptic equation of the form:
\begin{equation}
(2.3) \quad a_{ij}(x)D_{ij}u_k + \sum_{l=1}^p b^l_j(x)D_ju_l + \sum_{l=1}^p c_l(x)u_l = 0,
\end{equation}

where \( a_{ij}, b_j, \) and \( c_l \) are smooth functions. Let’s assume that \( a_{ij} \) is positive definite, and that \( |b_j| \leq C \) and \( |c_l| \leq C \) for some constant \( C > 0 \). If the order of vanishing of \( u_l \) at \( u^{-1}_l(0) \) is finite for each \( l \) and if \( x_0 \in u^{-1}(0) \cap |Du|^{-1}(0) \), then
\[ u^{-1}(0) \cap |Du|^{-1}(0) \cap B_r(x_0) \]
decomposes into a countable union of subsets of a pairwise disjoint collection of \( n - 2 \)-dimensional smooth submanifolds.

**Proof.** We define for each integer \( q = 1, 2, \ldots \) the set
\begin{equation}
(2.4) \quad S_q = \{ x : D^q u_l(x) = 0, \forall |\alpha| \leq q, \forall l \text{ and } D^{q+1}u_{l_0}(x) \neq 0 \text{ for some } l_0 \}. 
\end{equation}

We first note that if \( x \in u^{-1}(0) \cap |Du|^{-1}(0) \) and \( r > 0 \) is small enough, then
\begin{equation}
(2.5) \quad u^{-1}(0) \cap |Du|^{-1}(0) \cap B_r(x) = \bigcup_{q=1}^d S_q \cap B_r(x),
\end{equation}

where \( d - 1 \) is the order of vanishing of \( u \) at \( x \). Now for each \( x \in S_q \) we consider a multi-index \( \beta \) such that \( |\beta| = q - 1 \) and \( \text{Hess}(D^\beta u_{l_0})(x) \neq 0 \) for some \( l_0 \). Applying \( D^\beta \) to both sides of (2.3) with \( k = l_0 \) and recalling that \( D^\alpha u_l(x) = 0 \) for every multi-index \( \alpha \) such that \( |\alpha| \leq q \) we obtain
\[ a_{ij}(x)D_{ij}(D^\beta u_{l_0})(x) = 0. \]

Using that \( a_{ij} \) is positive definite and that \( \text{Hess}(D^\beta u_{l_0})(x) \neq 0 \) we conclude that \( \text{rank}(\text{Hess}(D^\beta u_{l_0})(x)) \geq 2 \). Thus there exist indexes \( i_1 \)
and $i_2$ for which $\text{grad}(D_i D^\beta u_0)(x)$ and $\text{grad}(D_i D^\beta u_0)(x)$ are linearly independent. This implies that for small $r > 0$ that

$$B_r(x) \cap \left(D_i D^\beta u_0\right)^{-1}(0) \cap \left(D_{i_2} D^\beta u_0\right)^{-1}(0)$$

is a $n-2$-dimensional submanifold $\Sigma_{x, r, \beta}$ which contains $B_r(x) \cap S_q$. In view of (2.5) we conclude that for each $x \in u^{-1}(0) \cap |Du|^{-1}(0)$ there exist $r > 0$ and smooth $n-2$-dimensional submanifolds $\Sigma_{x, r, q_1}, \ldots, \Sigma_{x, r, q_s}$ for which

$$B_r(x) \cap u^{-1}(0) \cap |Du|^{-1}(0) \subset \cup_{j=1}^s \Sigma_{x, r, q_j}.$$  

(2.6)

The Lemma follows from (2.6).

□

Lemma 2.4. If $\Sigma_1$ and $\Sigma_2$ are 2-dimensional minimal surfaces in $\mathbb{R}^n$ having a tangential intersection of infinite order at $x_0 \in \Sigma_1 \cap \Sigma_2$, then $\Sigma_1 = \Sigma_2$.

Proof. Let $v_k : \Omega \to \mathbb{R}^n$ be minimal map parameterizing a neighborhood of $\Sigma_k$ for each $k = 1, 2$ and assume that $v_k(0) = x_0$. We can assume that the coordinates $z = x + y^i$ in $\Omega$ are isothermal for both $v_1$ and $v_2$. As $v_k$ is minimal, each coordinate $v_k^i, i = 1, \ldots, n$, is harmonic, which implies by the conformal invariance of the Laplacian that $\partial^2_{z \bar{z}} v_k^i = 0$. Hence, if we define $v(z) = v_1(z) - v_2(z)$, then each component of $\partial_z v$ is holomorphic, i.e., $\partial^2_{z \bar{z}} v^i = 0$. Since $z = 0$ is an infinite order zero of $v$, the analytic continuation property for holomorphic functions implies that $v \equiv 0$. Therefore, $\Sigma_1 = \Sigma_2$. □

3. PROOF OF THEOREM

Theorem 3.1. Let $\Sigma^2$ be a compact free boundary minimal surface in $B^n$. Assume that for all points $x \in \Sigma$,

$$|x^\perp|^2 |A(x)|^2 \leq 2,$$

where $x^\perp$ denotes the normal component of $x$. Then

1. $|x^\perp|^2 |A(x)|^2 \equiv 0$ and $\Sigma^2$ is a flat equatorial disk.
2. $|x^\perp|^2 |A(x_0)|^2 = 2$ at some point $x_0 \in \Sigma^2$ and $\Sigma^2$ is the critical catenoid inside a 3-dimensional linear subspace.

Lemma 3.2. Let $\Sigma^2$ be a free boundary minimal surface in $B^n$ and $f$ be the function $f : \Sigma^2 \to \mathbb{R}$ defined by

$$f(x) = \frac{|x|^2}{2}, \ x \in \Sigma^2.$$  

Then $\nabla^\Sigma f = x^\perp$ for every $x \in \Sigma$ and

$$\text{Hess}_\Sigma f(x)(X, Y) = \langle X, Y \rangle + \langle A(X, Y), x^\perp \rangle.$$  

(3.2)
Proof. Given \( X \in \mathcal{X}(\Sigma) \), then
\[
X(f) = \frac{1}{2} X \langle \overrightarrow{x}, \overrightarrow{x} \rangle = \langle X, \overrightarrow{x} \rangle = \langle X, x^\top \rangle.
\]
Hence, \( \nabla^\Sigma f(x) = x^\top \). The hessian of \( f \) is then given by
\[
\text{Hess}_\Sigma f(X, Y) = \langle \nabla X \nabla f, Y \rangle = \langle \nabla X(x - x^\perp), Y \rangle = \langle X, Y \rangle + \langle x^\perp, \nabla X Y \rangle = \langle X, Y \rangle + \langle A(X, Y), \overrightarrow{x} \rangle,
\]
where \( X \) and \( Y \) are vector fields in \( \mathcal{X}(\Sigma) \).

Lemma 3.3. If \(|x^\perp|^2|A(x)|^2 \leq 2\), then \( \text{Hess}_\Sigma f \geq 0 \).

Proof. Let \( \{e_1, e_2\} \) be an orthonormal base of \( T\Sigma \) given by eigenvectors of \( \text{Hess}_\Sigma f \). The respective eigenvalues are \( \overline{\lambda}_i = 1 + \langle A(e_i, e_i), \overrightarrow{x} \rangle \). We want to prove that \( \overline{\lambda}_i \geq 0 \) for \( i = 1, 2 \).

\[
\overline{\lambda}_1 + \overline{\lambda}_2 = 2 + \sum_{i=1}^{2} \langle A(e_i, e_i), \overrightarrow{x} \rangle^2 \leq 2 + \sum_{i=1}^{2} |A(e_i, e_i)|^2 |x^\perp|^2 \leq 2 + |A|^2 |x^\perp|^2,
\]
where we used the Cauchy-Schwarz inequality in the first inequality. Since \((\overline{\lambda}_1 + \overline{\lambda}_2)^2 = 4\), we conclude that
\[
2 \overline{\lambda}_1 \overline{\lambda}_2 \geq 2 - |A|^2 |x^\perp|^2 \geq 0.
\]
Hence, \( \overline{\lambda}_1 \) and \( \overline{\lambda}_2 \) have the same sign. As \( \overline{\lambda}_1 + \overline{\lambda}_2 = 2 \), the lemma is proved.

Definition 3.4. Given a 2-dimensional free boundary minimal surface \( \Sigma^2 \) in \( B^n \) we define
\[
C(\Sigma) = \{ x \in \Sigma : f(x) = m_0 := \min_{\Sigma} f \}.
\]

The conormal vector of a free boundary minimal surface \( \Sigma \) being normal to the boundary of the ball implies that \( \partial \Sigma \) is convex on \( \Sigma \). Using this fact and that \( \text{Hess}_\Sigma f \geq 0 \), we obtain:

Lemma 3.5. If \( \Sigma \) is a free boundary minimal surface in \( B^n \) satisfying \( \text{Hess}_\Sigma f \geq 0 \), then the set \( C(\Sigma) \) is totally convex on \( \Sigma \), meaning that every geodesic segment with extremities in \( C(\Sigma) \) is in \( C(\Sigma) \).

Before we start proving Theorem 3.1, let us recall a simple fact from Riemannian Geometry that we will use later. Let \( c : [a, b] \to M \) be a curve in a Riemannian manifold \( M \) and \( P_s : T_{c(a)} M \to T_{c(s)} M \) the parallel transport map along \( c \). Let \( \Delta \) be a correspondence which associates for each \( s \in [a, b] \) a \( j \)-dimensional subspace \( \Delta(s) \subset T_{c(s)} M \).
The distribution $\Delta(s)$ is called parallel if $P_s(\Delta(a)) = \Delta(s)$ for every $s \in [a, b]$.

**Lemma 3.6** (Spivak [16]). If $\frac{DV}{ds}(s) \in \Delta(s)$ whenever $V$ is a vector field in $\Delta(s)$, then $\Delta(s)$ is parallel along $c$.

**Proof of Theorem 3.1.** By Lemma 3.3, the inequality (3.1) implies that $\text{Hess}_\Sigma f \geq 0$. Let us show this implies that $\Sigma$ is diffeomorphic to either a disk or an annulus.

If $\Sigma$ is simply connected, then $\Sigma$ is topologically a disk. Hence, we assume that $\pi_1(\Sigma, x) \neq \{0\}$, where $x$ is chosen to lie in $C(\Sigma)$. By minimizing the length in a nontrivial homotopy class $[\alpha] \in \pi_1(\Sigma, x)$ among closed loops passing through the fixed point $x \in C(\Sigma)$, we obtain a geodesic loop $\gamma : [0, 1] \to \Sigma$, where $\gamma(0) = \gamma(1) = x$; this follows from the fact that $\partial \Sigma$ is convex on $\Sigma$ due to the free boundary condition. We claim that $\gamma'(0) = \gamma'(1)$ and $C(\Sigma) = \gamma([0, 1])$. If either one of those properties are not true, then the total convexity of $C(\Sigma)$ guarantees that an open set $U$ of $\Sigma$ is contained in $C(\Sigma)$. In this case, $\text{Hess}_\Sigma f \equiv 0$ over $U$. Hence, $\langle A(X, Y), \vec{x}^2 \rangle = -\langle X, Y \rangle$.

Since $\vec{x}$ is a constant length normal vector to $\Sigma$ along $U$, we conclude that the mean curvature of $\Sigma$ in the direction of $\vec{x}$ is non-zero, a contradiction. Therefore, $C(\Sigma)$ is a smooth simple closed geodesic. Note that this implies that $\pi_1(\Sigma)$ is cyclic, from this we obtain that $\Sigma$ is an annulus.

If $\Sigma^2$ is a minimal disk, then Fraser and Schoen’s theorem in [6] implies that $\Sigma^2$ is an equatorial disk and $|x^\perp|^2 |A(x)|^2 \equiv 0$.

If $\Sigma^2$ is an annulus, then $C(\Sigma)$ is a smooth simple closed geodesic. This implies that $\lambda_1 = 0$ is an eigenvalue of $\text{Hess}_\Sigma f(x_0)$ for every $x_0 \in C(\Sigma)$ and $(\lambda_1^2 + \lambda_2^2) = (\lambda_1 + \lambda_2)^2$. On the other hand,

$$\sum_{i=1}^2 \lambda_i^2 = 2 + \langle A(e_i, e_i), \vec{x}^2 \rangle^2 \leq 2 + |x^\perp|^2 |A(x)|^2 \leq 4 = \left( \sum_{i=1}^2 \lambda_i \right)^2.$$

Hence, $\langle \sum_{i=1}^2 A(e_i, e_i), \vec{x}^2 \rangle^2 = |A(x)|^2 |x^\perp|^2 = 2$ for every $x \in C(\Sigma)$. It follows from the Cauchy-Schwarz inequality that

$$A(e_i, e_i) = \langle A(e_i, e_i), \frac{\vec{x}}{|x|} \frac{\vec{x}}{|x|} \rangle.$$

Consequently, if $e_1$ is tangent to $C(\Sigma)$, then

$$\nabla_{e_1} e_1 = \langle A(e_i, e_i), \frac{\vec{x}}{|x|} \frac{\vec{x}}{|x|} \rangle.$$
since $C(\Sigma)$ is a geodesic on $\Sigma$. Thus, $C(\Sigma)$ is also a geodesic in $\partial B_{2m_0}^{n+1}(0)$, i.e., a round circle. Now we consider the normal distribution $E$ along $C(\Sigma)$ defined by

$$E = \{\xi : \xi \in X^\perp(\Sigma)|_{C(\Sigma)} \text{ and } \langle \xi, \overrightarrow{x} \rangle = 0\}.$$  

It follows from (3.5) that for every $\xi \in E$ the following is true:

$$\nabla_{\gamma'(t)}\xi \in E.$$  

Lemma 3.6 implies that the distribution $E$ is parallel along $C(\Sigma)$. Hence, $E$ is a constant $(n - 2)$-dimensional plane through the origin. Therefore, there exists a critical catenoid $\Sigma_c$ which is tangent to $\Sigma$ along $C(\Sigma)$. Near $x_0 \in C(\Sigma)$ we write $\Sigma$ and $\Sigma_c$ locally as a graph over $T_{x_0}\Sigma$. Hence, $\Sigma_c = \text{graph}(f_c)$ and

$$\text{div}\left(\frac{\nabla f_c}{\sqrt{1 + |\nabla f_c|^2}}\right) = 0.$$  

Similarly, $\Sigma = \text{graph}(u)$, where $u : \mathbb{R}^2 = T_{x_0}\Sigma \to \mathbb{R}^{n-1}$, and by Lemma 2.1

$$\frac{a_{ij}(\nabla u_1, \ldots, \nabla u_{n-1})}{\sqrt{1 + |\nabla u|^2}}D_{ij}u_i = 0.$$  

for every $i = 1, \ldots, n - 1$. Lemma 2.2 implies that the difference $v = u - f_c$ satisfies a linear PDE of the following form:

$$\frac{a_{ij}(\nabla u)}{\sqrt{1 + |\nabla u|^2}}D_{ij}v_k + \sum_{l=1}^{n-1} b_{ij}^l(\nabla u, \nabla f_c)D_{lj}v_l = 0,$$

for each $k = 1, \ldots, n - 1$. Note that $v$ vanishes on $x_0$ and the order of vanishing is finite by Lemma 2.4. Therefore, $\mathcal{H}^1(v^{-1}(0) \cap |\nabla v|^{-1}(0)) = 0$ by Lemma 2.3. This is a contradiction since $\Sigma$ and $\Sigma_c$ are tangent along $C(\Sigma)$ and $\dim C(\Sigma) = 1$. We conclude that $v \equiv 0$ near $x_0$ and the theorem follows from standard analytic continuation property for minimal surfaces.

**Remark 3.7.** The same proof works for 2-dimensional minimal surfaces properly embedded in $\mathbb{R}^n$; the conclusion in this case is that such a surface $\Sigma^2$ satisfying (1.1) is either simply connected or the catenoid. In the special case $n = 3$, we can invoke the classification of properly embedded simply connected minimal surfaces by Meeks and Rosenberg [13] to conclude that $\Sigma$ is either the plane, the catenoid, or the helicoid. A simple computation shows that the helicoid does not satisfy (1.1).
A CURVATURE GAP FOR MINIMAL SURFACES IN THE BALL

References

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