THE DERIVED CATEGORY OF THE PROJECTIVE LINE

HENNING KRAUSE AND GREG STEVENSON

Abstract. We examine the localizing subcategories of the derived category of quasi-coherent sheaves on the projective line over a field. We provide a complete classification of all such subcategories which arise as the kernel of a cohomological functor to a Grothendieck category.

1. Introduction

Ostensibly, this article is about the projective line over a field, but secretly it is an invitation to a discussion of some open questions in the study of derived categories. More specifically, we are thinking of localizing subcategories and to what extent one can hope for a complete classification. The case of affine schemes is by now quite well understood, having been settled by Neeman in his celebrated chromatic tower paper [15]. However, surprisingly little is known in the simplest non-affine case, namely the projective line over a field. We seek to begin to rectify this state of affairs and to advertise this and similar problems.

Let us start by recalling what is known. We write QCoh\(\mathbb{P}^1_k\) for the category of quasi-coherent sheaves on the projective line \(\mathbb{P}^1_k\) over a field \(k\), and Coh\(\mathbb{P}^1_k\) denotes the full subcategory of coherent sheaves. There is a complete description of the objects of Coh\(\mathbb{P}^1_k\), due to Grothendieck [6]. The localizing subcategories of QCoh\(\mathbb{P}^1_k\) are known by work of Gabriel [5], and are parametrized by specialization closed collections of points of \(\mathbb{P}^1_k\). When one passes to the derived category \(D(QCoh\mathbb{P}^1)\), the situation becomes considerably more complicated. Several new localizations appear as a result of the fact that one can no longer non-trivially talk about sub-objects and it remains a challenge to provide a complete classification of localizing subcategories.

An enticing aspect of this problem is that it not only represents the first stumbling block for those coming from algebraic geometry, but also for the representation theorists. There is an equivalence of triangulated categories

\[ D(QCoh\mathbb{P}^1) \sim \rightarrow D(Mod\mathcal{A}) \]

where Mod\(\mathcal{A}\) denotes the module category of

\[ \mathcal{A} = \left[ \begin{array}{cc} k & k^2 \\ 0 & k \end{array} \right]. \]

The algebra \(\mathcal{A}\) is isomorphic to the path algebra of the Kronecker quiver \(\cdot \rightarrow \cdot\cdot\cdot\), and is known to be of tame representation type. The ring \(\mathcal{A}\) is one of the simplest non-representation finite algebras and so understanding its derived category is also a key question from the point of view of representation theory. Of particular note, it is known by work of Ringel [16, 17] that Mod\(\mathcal{A}\), the category of all representations, is wild and so it is very natural to ask if, as in the case of commutative noetherian rings, localizations can nonetheless be classified.

In this article we make a contribution toward this challenge in two different ways. First of all, one of the main points of this work is to highlight this problem, provide some appropriate background, and set out what is known. To this end the first part of the article discusses the various types of localization one might consider...
in a compactly generated triangulated category and sketches the localizations of \( D(\text{QCoh}\, \mathbb{P}^1) \) which are known.

Our second contribution is to provide new perspective and new tools. The main new result is that the subcategories we understand admit a natural intrinsic characterization: it is shown in Theorem 4.4.9 that they are precisely the cohomological ones. In the final section we provide a discussion of the various restrictions that would have to be met by a non-cohomological localizing subcategory. Here our main results are that such subcategories come in \( \mathbb{Z} \)-families and consist of objects with full support on \( \mathbb{P}^1 \).

2. Preliminaries

This section contains some background on localizations, localizing subcategories, purity, and the projective line. Also it serves to fix notation and may be safely skipped, especially by experts, and referred back to as needed.

2.1. Localizing subcategories and localizations. Let \( T \) be a triangulated category with all small coproducts and products. The case we have in mind is that \( T \) is either well-generated or compactly generated.

**Definition 2.1.1.** A full subcategory \( L \) of \( T \) is **localizing** if it is closed under suspensions, cones, and coproducts. This is equivalent to saying that \( L \) is a coproduct closed triangulated subcategory of \( T \).

**Remark 2.1.2.** It is a consequence of closure under (countable) coproducts that \( L \) is closed under direct summands and hence **thick** (which means closed under finite sums, summands, suspensions, and cones).

Given a collection of objects \( X \) of \( T \) we denote by \( \text{Loc}(X) \) the localizing subcategory generated by \( X \), i.e. the smallest localizing subcategory of \( T \) containing \( X \). The collection of localizing subcategories is partially ordered by inclusion, and forms a lattice (with the caveat it might not be a set) with meet given by intersection.

We next present the most basic reasonableness condition a localizing subcategory can satisfy.

**Definition 2.1.3.** A localizing subcategory \( L \) of \( T \) is said to be **strictly localizing** if the inclusion \( i^*: L \to T \) admits a right adjoint \( i_! \), i.e. if \( L \) is coreflective.

Some remarks on this are in order. First of all, it follows that \( i_! \) is a Verdier quotient, and that there is a localization sequence

\[
\begin{array}{c}
L \xrightarrow{i_*} T \xrightarrow{j^*} T/L
\end{array}
\]

inducing a canonical equivalence

\[
L^\perp := \{ X \in T \mid \text{Hom}(L, X) = 0 \} \xrightarrow{\sim} T/L.
\]

Next we note that in nature localizing subcategories tend to be strictly localizing. This is, almost uniformly, a consequence of Brown representability; if \( T \) is well-generated and \( L \) has a generating set of objects then \( L \) is strictly localizing.

Now let us return to the localization sequence above. From it we obtain two endofunctors of \( T \), namely

\[
i_*i^* \quad \text{and} \quad j_*j^*
\]

which we refer to as the associated **acyclization** and **localization** respectively. They come together with a counit and a unit which endow them with the structure of an idempotent comonoid and monoid respectively. The localization (or acyclization) is equivalent information to \( L \). One can give an abstract definition of a localization
functor on $T$ (or in fact any category) and then work backward from such a functor to a strictly localizing subcategory. Further details can be found in [12]. We will use the language of (strictly) localizing subcategories and localizations interchangeably.

2.2. **Purity.** Let $T$ be a compactly generated triangulated category and let $T^c$ denote the thick subcategory of compact objects. We denote by $\text{Mod}^c T$ the Grothendieck category of modules over $T^c$, i.e. the category of contravariant additive functors $T^c \to \text{Ab}$. There is a restricted Yoneda functor

$$(2.1) \quad H : T \to \text{Mod}^c T \text{ defined by } HX = \text{Hom}(\cdot, X)|_{T^c},$$

which is cohomological, conservative, and preserves both products and coproducts.

**Definition 2.2.1.** A morphism $f : X \to Y$ in $T$ is a pure-monomorphism (resp. pure-epimorphism) if $Hf$ is a monomorphism (resp. epimorphism).

An object $I \in T$ is pure-injective if every pure-monomorphism $I \to X$ is split, i.e. it is injective with respect to pure-monomorphisms.

It is clear from the definition that if $I \in T$ with $HI$ injective then $I$ is pure-injective. It turns out that the converse is true and so $I$ is pure-injective if and only if $HI$ is injective. Moreover, Brown representability allows one to lift any injective object of $\text{Mod}^c T$ uniquely to $T$ and thus one obtains an equivalence of categories

$$\{\text{pure-injectives in } T\} \sim \{\text{injectives in } \text{Mod}^c T\}.$$

Further details on purity, together with proofs and references for the above facts can be found, for instance, in [11].

2.3. **The projective line.** Throughout we will work over a fixed base field $k$ which will be suppressed from the notation. For instance, $\mathbb{P}^1$ denotes the projective line $\mathbb{P}^1_k$ over $k$. We will denote by $\eta$ the generic point of $\mathbb{P}^1$. The points of $\mathbb{P}^1$ that are different from $\eta$ are closed. A subset $V \subseteq \mathbb{P}^1$ is specialization closed if it is the union of the closures of its points. In our situation this just says that $V$ is specialization closed if $\eta \in V$ implies $V = \mathbb{P}^1$.

As usual $\text{QCoh} \mathbb{P}^1$ is the Grothendieck category of quasi-coherent sheaves on $\mathbb{P}^1$ and $\text{Coh} \mathbb{P}^1$ is the full abelian subcategory of coherent sheaves.

We use standard notation for the usual ‘distinguished’ objects of $\text{QCoh} \mathbb{P}^1$. The $i$th twisting sheaf is denoted $\mathcal{O}(i)$ and for a point $x \in \mathbb{P}^1$ we let $k(x)$ denote the residue field at $x$. In particular, $k(\eta)$ is the sheaf of rational functions on $\mathbb{P}^1$. For an object $X \in D(\text{QCoh} \mathbb{P}^1)$ or a localizing subcategory $L$ we will often write $X(i)$ and $L(i)$ for $X \otimes \mathcal{O}(i)$ and $L \otimes \mathcal{O}(i)$ respectively.

All functors, unless explicitly mentioned otherwise, are derived. In particular, $\otimes$ denotes the left derived tensor product of quasi-coherent sheaves and $\mathcal{H}\text{om}$ the right derived functor of the internal hom in $\text{QCoh} \mathbb{P}^1$.

For an object $X \in D(\text{QCoh} \mathbb{P}^1)$ we set

$$\text{supp } X = \{x \in \mathbb{P}^1 \mid k(x) \otimes X \neq 0\}.$$ 

This agrees with the notion of support one gets as in [3] by allowing $D(\text{QCoh} \mathbb{P}^1)$ to act on itself; the localizing subcategories generated by $k(x)$ and $\Gamma_x \mathcal{O}$ coincide.

**Remark 2.3.1.** Let $A$ be a hereditary abelian category, for example $\text{QCoh} \mathbb{P}^1$. Then $\text{Ext}^n(X, Y)$ vanishes for all $n > 1$ and therefore every object of the derived category $D(A)$ decomposes into complexes that are concentrated in a single degree. It follows that the functor $H^0 : D(A) \to A$ induces a bijection between the localizing subcategories of $D(A)$ and the full subcategories of $A$ that are closed under kernels, cokernels, extensions, and coproducts.
3. Types of localization

In this section we give a further review of the notions of localization, or equivalently localizing subcategory, that naturally arise and that we treat in this article. These come in various strengths and what is known in general varies accordingly. We take advantage of this review to give a whirlwind tour of certain aspects of the subject and to expose some technical results that are absent from the literature.

Unless otherwise specified we will denote by $\mathcal{T}$ a compactly generated triangulated category. One also can, and should, consider the well-generated case and it arises naturally even when one starts with a compactly generated category. However, our focus will, eventually, be on those categories controlled by pure-injectives which more or less binds us to the compactly generated case.

3.1. Smashing localizations. In this section we make some brief recollections on the most well understood class of localizing subcategories.

**Definition 3.1.1.** A localizing subcategory $L$ of $\mathcal{T}$ is *smashing* if it is strictly localizing and satisfies one, and hence all, of the following equivalent conditions:

- the subcategory $L^\perp$ is localizing;
- the corresponding localization functor preserves coproducts, i.e. the right adjoint to $\mathcal{T} \to \mathcal{T}/L$ preserves coproducts;
- the quotient functor $\mathcal{T} \to \mathcal{T}/L$ preserves compactness;
- the corresponding acyclization functor preserves coproducts, i.e. the right adjoint to $L \to \mathcal{T}$ preserves coproducts.

The smashing subcategories always form a set. Amongst the smashing subcategories there is a potentially smaller distinguished set of localizing subcategories. Unfortunately, there is not a standard way to refer to such categories; the snappy nomenclature only exists for the corresponding localizations.

**Definition 3.1.2.** A localization is *finite* if its kernel is generated by objects of $\mathcal{T}^c$, i.e. the corresponding localizing subcategory is generated by objects which are compact in $\mathcal{T}$.

If $L$ is the kernel of a finite localization then it is smashing. It is also compactly generated, although there are in general many localizing subcategories of $\mathcal{T}$ which are, as abstract triangulated categories, compactly generated but are not generated by objects compact in $\mathcal{T}$.

The *smashing conjecture* for $\mathcal{T}$ asserts that every smashing localization is a finite localization. This is true in many situations, for instance it holds for the derived category $D(\text{Mod } A)$ of a ring $A$ when it is commutative noetherian [15] or hereditary [13]. On the other hand it is known to fail for certain rings (see for instance [10]) and is open in many cases of interest, for example the stable homotopy category.

3.2. Cohomological localizations. We now come to the next types of localizing subcategories in our hierarchy, which are defined by certain orthogonality conditions. This gives a significantly weaker hierarchy of notions than being smashing.

First a couple of reminders. An abelian category $A$ is said to be (AB5) if it is cocomplete and filtered colimits are exact. If in addition $A$ has a generator then it is a Grothendieck category. An additive functor $H: \mathcal{T} \to A$ is *cohomological* if it sends triangles to long exact sequences i.e. given a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

the sequence

$$\cdots \longrightarrow H(\Sigma^{-1} Z) \xrightarrow{H(\Sigma^{-1} h)} H(X) \xrightarrow{H(g)} H(Y) \xrightarrow{H(f)} H(Z) \xrightarrow{H(h)} H(\Sigma X) \longrightarrow \cdots$$
is exact in $A$.

**Definition 3.2.1.** A localizing subcategory $L \subseteq T$ is **cohomological** if there exists a cohomological functor $H : T \to A$ into an (AB5) abelian category such that $H$ preserves all coproducts and

$$L = \{X \in T \mid H(\Sigma^n X) = 0 \text{ for all } n \in \mathbb{Z}\},$$

that is $L$ is the kernel of $H^*$.

We can extend this definition to an analogue for arbitrary regular cardinals, with Definition 3.2.1 being the $\aleph_0$ or ‘base’ case. The idea is to relax the exactness condition on the target abelian category. This requires a little terminological preparation.

Let $J$ be a small category and $\alpha$ a regular cardinal. We say that $J$ is $\alpha$-filtered if for every category $I$ with $|I| < \alpha$, i.e. $I$ has fewer than $\alpha$ arrows, every functor $F : I \to J$ has a cocone. For instance, this implies that any collection of fewer than $\alpha$ objects of $J$ has an upper bound and any collection of fewer than $\alpha$ parallel arrows has a weak coequalizer. If $\alpha = \aleph_0$ we just get the usual notion of a filtered category.

Let $A$ be an abelian category. We say it satisfies (AB5$\alpha$) if it is cocomplete and has exact $\alpha$-filtered colimits.

**Definition 3.2.2.** A localizing subcategory $L \subseteq T$ is $\alpha$-cohomological if there exists an (AB5$\alpha$) abelian category $A$ and a coproduct preserving cohomological functor $H : T \to A$ such that

$$L = \{X \in T \mid H(\Sigma^n X) = 0 \text{ for all } n \in \mathbb{Z}\},$$

that is $L$ is the kernel of $H^*$.

If $L$ is $\alpha$-cohomological then it is clearly $\beta$-cohomological for all $\beta \geq \alpha$.

**Remark 3.2.3.** An $\aleph_0$-cohomological localizing subcategory is just a cohomological localizing subcategory. We will usually stick to the shorter terminology for the sake of brevity and to avoid a proliferation of $\aleph$’s.

We now make a few observations on $\alpha$-cohomological localizing subcategories and then make some further remarks on the case $\alpha = \aleph_0$.

**Lemma 3.2.4.** Smashing subcategories are cohomological.

**Proof.** Suppose $L$ is smashing. Then $T/L$ is compactly generated and for $H$ we can take the composite

$$T \to T/L \to \text{Mod}(T/L)^c$$

where the latter functor is the restricted Yoneda functor. \(\square\)

**Theorem 3.2.5.** Let $L$ be an $\alpha$-cohomological localizing subcategory. Then $L$ is generated by a set of objects and so it is, in particular, strictly localizing.

**Proof.** This follows by applying \[12\,\text{Theorem 7.1.1}\] and then \[12\,\text{Theorem 7.4.1}\]. \(\square\)

**Corollary 3.2.6.** A localizing subcategory $L$ is generated by a set of objects of $T$ if and only if there exists an $\alpha$ such that $L$ is $\alpha$-cohomological.

**Proof.** We have just seen that an $\alpha$-cohomological localizing subcategory has a generating set. On the other hand if $L$ is generated by a set of objects then $L$ is well-generated, and so is strictly localizing, and the quotient $T/L$ is also well-generated (see \[12\,\text{Theorem 7.2.1}\]). One can then compose the quotient $T \to T/L$ with the universal functor from $T/L$ to an (AB5$\alpha$) abelian category, for a sufficiently large $\alpha$, to get the required cohomological functor. \(\square\)
Let us now restrict to cohomological localizations and make the connection to purity in triangulated categories.

Proposition 3.2.7. A localizing subcategory \( L \subseteq T \) is cohomological if and only if there is a suspension stable collection of pure-injective objects \((Y_i)_{i \in I}\) in \( T \) such that \( L = \{ X \in T \mid \text{Hom}(X, Y_i) = 0 \text{ for all } i \in I \} \).

Proof. Recall from (2.1) the restricted Yoneda functor which we denote by \( H_T \), for clarity, for the duration of the proof. This functor identifies the full subcategory of pure-injective objects in \( T \) with the full subcategory of injective objects in \( \text{Mod} T^c \) as noted earlier (see [11, Corollary 1.9] for details).

A cohomological functor \( H : T \to A \) that preserves coproducts admits a factorisation \( H = \bar{H} \circ H_T \) such that \( \bar{H} : \text{Mod} T^c \to A \) is exact and preserves coproducts; see [11, Proposition 2.3]. The full subcategory \( \text{Ker} \bar{H} = \{ M \in \text{Mod} T^c \mid \bar{H}(M) = 0 \} \) is a localizing subcategory, so of the form \( \{ M \in \text{Mod} T^c \mid \text{Hom}(M, N_i) = 0 \text{ for all } i \in I \} \) for a collection of injective objects \((N_i)_{i \in I}\) in \( \text{Mod} T^c \). Now choose pure-injective objects \((Y_i)_{i \in I}\) in \( T \) such that \( H_T(Y_i) \cong N_i \) for all \( i \in I \). \( \square \)

3.3. When things are sets. As has been alluded to in the previous sections, it is a significant subtlety that one does not usually know the class of all localizing subcategories forms a set. In fact there is no example where one knows that there are a set of localizing subcategories by ‘abstract means’; all of the examples come from classification results.

If one does know there are a set of localizing subcategories then life is much easier. The purpose of this section is to give some indication of this, and record some other simple observations. Everything here should be known to experts, but these observations have not yet found a home in the literature.

Let \( T \) be a well-generated triangulated category.

Lemma 3.3.1. If the localising subcategories of \( T \) form a set then every localizing subcategory is generated by a set of objects (and hence by a single object).

Proof. Suppose, for a contradiction, that \( L \) is a localizing subcategory of \( T \) which is not generated by a set of objects. We define a proper chain of proper localizing subcategories

\[
L_0 \subset L_1 \subset \cdots \subset L_\alpha \subset L_{\alpha+1} \subset \cdots \subset L,
\]

each of which is generated by a set of objects, by transfinite induction. For the base case pick any object \( X_0 \) of \( L \) and set \( L_0 = \text{Loc}(X_0) \). This is evidently generated by a set of objects, namely \( \{X_0\} \). By assumption \( L \) is not generated by a set of objects so \( L_0 \subset L \). Suppose we have defined a proper localizing subcategory \( L_\alpha \) of \( L \) which is generated by a set of objects. Since \( L_\alpha \) is proper we may pick an object \( X_{\alpha+1} \) in \( L \) but not in \( L_\alpha \) and set

\[
L_{\alpha+1} = \text{Loc}(L_\alpha, X_{\alpha+1}) \supseteq L_\alpha.
\]

This is clearly still generated by a set of objects and hence is still a proper subcategory of \( L \). For a limit ordinal \( \lambda \) we set

\[
L_\lambda = \text{Loc}(L_\kappa \mid \kappa < \lambda).
\]

Again this is generated by a set of objects and hence is still a proper subcategory of \( L \). This gives an ordinal indexed chain of distinct localizing subcategories of \( T \). However, this is absurd since the collection of ordinals is not a set and so cannot be embedded into the set of all localising subcategories of \( T \). Hence \( L \) must have a generating set (i.e. the above construction must terminate). \( \square \)

Remark 3.3.2. The above argument does not use that \( T \) is well-generated.
One then deduces that all localizations are cohomological for an appropriate cardinal.

**Lemma 3.3.3.** If the localising subcategories of $T$ form a set then every localizing subcategory of $T$ is $\alpha$-cohomological for some regular cardinal $\alpha$.

**Proof.** By the previous lemma the hypothesis imply that every localizing subcategory of $T$ is generated by a set of objects. It then follows from Corollary 3.2.6 that they are all cohomological. \qed

One can, to some extent, also work in the other direction.

**Lemma 3.3.4.** If the collection $\bigcup_{\alpha \in \text{Card}} \{ L \mid L \text{ is } \alpha\text{-cohomological} \}$ forms a set then the collection of all localising subcategories of $T$ also forms a set.

**Proof.** We have seen in Corollary 3.2.6 that being $\alpha$-cohomological for some $\alpha$ is the same as being generated by a set of objects. Thus the hypothesis asserts that there are a set of localizing subcategories which have generating sets. From this perspective it is clear we can pick a regular cardinal $\kappa$ such that every localizing subcategory of $T$ which is generated by a set is generated by $\kappa$-compact objects. Moreover, since the union in the statement of the lemma is both a set and indexed by a class, we conclude that the chain stabilises and so, taking $\kappa$ larger if necessary, we may also assume every $\alpha$-cohomological localizing subcategory of $T$ is $\kappa$-cohomological.

If the localising subcategories of $T$ do not form a set then, as there are a set of $\kappa$-cohomological localizing subcategories, there must be a localizing subcategory $L$ which is not generated by a set of objects. In particular

$$L' = \text{Loc}(L \cap T^\kappa) \subseteq L.$$  

But this is nonsense. Since $L'$ is a proper localizing subcategory of $L$ we can find some object $X$ in $L$ but not in $L'$ and consider $L'' = \text{Loc}(L', X)$. Clearly $L''$ is still contained in $L$, it properly contains $L'$, it is generated by a set and hence $\kappa$-cohomological, and it contains the $\kappa$-compact objects of $L$. These are not compatible statements: we have assumed $\kappa$ large enough so that $L''$ must be generated by the $\kappa$-compact objects it contains but this contradicts $L' \subseteq L''$. \qed

### 4. Cohomological localizations for the projective line

We now turn to the example we have in mind, namely $\mathcal{D}(\text{QCoh } \mathbb{P}^1)$ the unbounded derived category of quasi-coherent sheaves on $\mathbb{P}^1$. We first describe the thick subcategories of $\mathcal{D}^b(\text{Coh } \mathbb{P}^1)$, the bounded derived category of coherent sheaves on $\mathbb{P}^1$. We then recall the classifications of smashing subcategories and of tensor ideals in $\mathcal{D}(\text{QCoh } \mathbb{P}^1)$. Finally, we classify the ($\aleph_0$-)cohomological localizing subcategories—there are no surprises and they are exactly the ones which have been understood for some time.

It is of course possible that there are $\alpha$-cohomological localizing subcategories for $\alpha > \aleph_0$ which we are not aware of. It is in some sense tempting to guess that this is not the case, i.e. that our list is already a complete list of localizing subcategories, but there is no real evidence for this. We close by making some remarks on the hurdles that such an ‘exotic’ localization would have to clear.

Before getting on with this let us recapitulate the connection with representation theory. By a result of Beilinson [4] there is a tilting object $T \in \text{Coh } \mathbb{P}^1$ which induces an exact equivalence

$$R\text{Hom}(T, -) : \mathcal{D}(\text{QCoh } \mathbb{P}^1) \to \mathcal{D}(\text{Mod } A)$$
where Mod $A$ denotes the module category of 

$$A = \text{End}(T) \cong \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}.$$ 

Note that $A$ is isomorphic to the path algebra of the Kronecker quiver $\cdot \to \cdot \to \cdot$, and this algebra is known to be of tame representation type. In fact, the representation theory of this algebra amounts to the classification of pairs of $k$-linear maps, up to simultaneous conjugation. The finite dimensional representations were already known to Kronecker [14].

4.1. Thick subcategories of the bounded derived category. The structure of the lattice of thick subcategories of $D^b(\text{Coh } P^1)$, which we recall in this section, has been known for some time; it can be computed by hand using the fact that $\text{Coh } P^1$ is tame and hereditary.

The structure of the coherent sheaves on $P^1$ is well known: there is a $\mathbb{Z}$-indexed family of indecomposable vector bundles and a 1-parameter family of torsion sheaves for each point on $P^1$.

For each $i \in \mathbb{Z}$ one has a thick subcategory

$$\text{Thick}(\mathcal{O}(i)) = \text{add}(\Sigma^j \mathcal{O}(i) \mid j \in \mathbb{Z}) \cong D^b(k)$$

where the identifications follow from the computation of the cohomology of $P^1$. These are the only proper non-trivial thick subcategories which are generated by vector bundles and are also the only thick subcategories which are not tensor ideals. Thus we have a lattice isomorphism

$$\{\text{thick subcategories of } D^b(\text{Coh } P^1) \text{ generated by vector bundles} \} \cong \mathbb{Z}$$

where $\mathbb{Z}$ denotes the lattice given by the following Hasse diagram:

\[\cdot\cdot\cdot \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \cdot\cdot\cdot \]

This is a special case of a general result because the indecomposable vector bundles are precisely the exceptional objects of $D^b(\text{Coh } P^1)$. For any hereditary artin algebra $A$ the thick subcategories of $D^b(\text{mod } A)$ that are generated by exceptional objects form a poset which is isomorphic to the poset of non-crossing partitions given by the Weyl group $W(A)$; see [7, 8]. Note that $W(A)$ is an affine Coxeter group of type $\tilde{A}_1$ for the Kronecker algebra $A = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$, keeping in mind the derived equivalence

$$D^b(\text{Coh } P^1) \cong D^b(\text{mod } A).$$

The thick tensor ideals are classified by $\text{Spc } D^b(\text{Coh } P^1) \cong P^1$, where the space $\text{Spc } D^b(\text{Coh } P^1)$ is meant in the sense of Balmer [2], and its computation is a special case of a general result of Thomason [20]. What all this boils down to is that for any set of closed points $V$ of $P^1$ there is a thick tensor ideal

$$D^b_V(\text{Coh } P^1) := \{E \mid \text{supp } E \subseteq V\} = \text{Thick}(k(x) \mid x \in V)$$

consisting of complexes of torsion sheaves supported on $V$. Moreover, together with 0 and $D^b(\text{Coh } P^1)$ this is a complete list of thick tensor ideals. One can make this uniform by considering subsets of $P^1$ which are specialization closed. In this language, by extending the above notation to allow $D^b_0(\text{Coh } P^1) = 0$ and $D^b_{P^1}(\text{Coh } P^1) = D^b(\text{Coh } P^1)$, we have a lattice isomorphism

$$\{\text{thick tensor ideals of } D^b(\text{Coh } P^1)\} \cong \{\text{spc subsets of } P^1\},$$
where ‘spc’ is an abbreviation for ‘specialization closed’, which is given by
\[ l \mapsto \text{supp}(l) = \bigcup_{E \in l} \text{supp}(E) \quad \text{and} \quad V \mapsto D^b_V(\text{Coh P}^1) \]
for \( l \) a thick tensor ideal and \( V \) a specialization closed subset.

We know every object of \( D^b(\text{Coh P}^1) \) is a direct sum of shifts of line bundles and torsion sheaves and so one can readily combine these classifications to obtain a lattice isomorphism.\(^1\)

\[ \{\text{thick subcategories of } D^b(\text{Coh P}^1)\} \xrightarrow{\sim} \{\text{spc subsets of } \mathbb{P}^1\} \sqcup \mathbb{Z}. \]

The verification that the evident bijection is indeed a lattice map as claimed is elementary: the twisting sheaves are supported everywhere so are not contained in any proper ideal, and any twisting sheaf and a torsion sheaf, or any pair of distinct twisting sheaves, generate the category. Thus for \( i \neq j \) and \( V \) proper non-empty and specialization closed in \( \mathbb{P}^1 \) we have
\[ D^b_V(\text{Coh P}^1) \vee \text{Thick}(\mathcal{O}(i)) = D^b(\text{Coh P}^1) = \text{Thick}(\mathcal{O}(i)) \vee \text{Thick}(\mathcal{O}(j)) \]
and
\[ D^b_V(\text{Coh P}^1) \wedge \text{Thick}(\mathcal{O}(i)) = 0 = \text{Thick}(\mathcal{O}(i)) \wedge \text{Thick}(\mathcal{O}(j)). \]

### 4.2. Ideals and smashing subcategories

We now describe the localizing subcategories that one easily constructs from our understanding of the compact objects \( D^b(\text{Coh P}^1) \) in \( D(\text{QCoh P}^1) \).

By [13] the smashing conjecture holds for \( D(\text{QCoh P}^1) \) (our computations will also essentially give a direct proof of this fact). Thus the finite localizations one obtains by inflating the thick subcategories listed above exhaust the smashing localizations i.e.

\[ \{\text{thick subcategories of } D^b(\text{Coh P}^1)\} \xrightarrow{\sim} \{\text{smashing subcategories of } D(\text{QCoh P}^1)\} \]

The localizing ideals are also understood. Again this is known more generally (there is such a classification for any locally noetherian scheme, see [11]) but can easily be computed by hand for \( \mathbb{P}^1 \). The precise statement is that there is a lattice isomorphism

\[ \{\text{localizing tensor ideals of } D(\text{QCoh P}^1)\} \xrightarrow{\sim} 2^{\mathbb{P}^1} \]
where \( 2^{\mathbb{P}^1} \) denotes the powerset of \( \mathbb{P}^1 \) with the obvious lattice structure. The bijection is given by the assignments

\[ L \mapsto \{x \in \mathbb{P}^1 \mid k(x) \otimes L \neq 0\} \]
for a localizing ideal \( L \) and

\[ V \mapsto \Gamma_V D(\text{QCoh P}^1) := \{X \in D(\text{QCoh P}^1) \mid X \otimes k(y) \cong 0 \text{ for } y \notin V\} \]
for a subset \( V \) of points on \( \mathbb{P}^1 \).

This agrees with the classification of smashing subcategories in the sense that the smashing ideals are precisely those inflated from the compacts, i.e. those corresponding to specialization closed subsets of points. Since \( \mathbb{P}^1 \) is 1-dimensional the only new localizing ideals that occur are obtained by throwing the residue field of the generic point, \( k(\eta) \), into a finite localization.

Thus we have identified a sublattice consisting of a copy of \( \mathbb{Z} \) and the powerset of \( \mathbb{P}^1 \) inside the lattice of localizing subcategories of \( D(\text{QCoh P}^1) \). The lattice structure extends that of the lattice of thick subcategories of \( D^b(\text{Coh P}^1) \) in the expected way. The naive guess is that this is, in fact, the whole lattice. While we do not know

\(^1\)Let \( L', L'' \) be a pair of lattices with smallest elements \( 0', 0'' \) and greatest elements \( 1', 1'' \). Then \( L' \sqcup L'' \) denotes the new lattice which is obtained from the disjoint union \( L' \cup L'' \) (viewed as sum of posets) by identifying \( 0' = 0'' \) and \( 1' = 1'' \).
if this is the case, we can give an intrinsic definition of the localizations we have stumbled into so far. This description is the goal of the next two subsections.

4.3. An aside on continuous pure-injectives. In order to describe the localizations we have listed so far a word on continuous pure-injectives is required.

Definition 4.3.1. A pure-injective object $I$ is continuous (or superdecomposable) if it has no indecomposable direct summands.

We say that $T$ has no continuous pure-injective objects if every non-zero pure-injective object has an indecomposable direct summand or, in other words, if there are no continuous pure-injectives. An equivalent condition is that every pure-injective object is the pure-injective envelope of a coproduct of indecomposable pure-injective objects.

Proposition 4.3.2. The category $D(QCoh \mathbb{P}^1)$ has no continuous pure-injective objects.

Proof. Let $A = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$ denote the Kronecker algebra. We use the derived equivalence $D(QCoh \mathbb{P}^1) \xrightarrow{\sim} D(Mod \ A)$. Let $X$ be a pure-injective object in $D(Mod \ A)$. Observe that $X$ decomposes into a coproduct $X = \coprod_{n \in \mathbb{Z}} X_n$ of complexes with cohomology concentrated in a single degree, since $A$ is a hereditary algebra. Thus we may assume that $X$ is concentrated in degree zero and identifies with a pure-injective $A$-module. Now the assertion follows from the description of the pure-injective $A$-modules in [9, Theorem 8.58].

Corollary 4.3.3. A localizing subcategory $L \subseteq D(QCoh \mathbb{P}^1)$ is cohomological if and only if there is a collection of indecomposable pure-injective objects $(Y_i)_{i \in I}$ in $QCoh \mathbb{P}^1$ such that $L = \{ X \in D(QCoh \mathbb{P}^1) \mid \text{Hom}(X, \Sigma^j Y_i) = 0 \text{ for all } i \in I, j \in \mathbb{Z} \}$.

Proof. By Proposition 4.3.2 a being cohomological is equivalent to being the left perpendicular of a collection of pure-injective objects. By the last proposition $D(QCoh \mathbb{P}^1)$ has no continuous pure-injectives and so we may replace such a collection of pure-injectives with the collection of its indecomposable summands without changing the left perpendicular. These are all honest sheaves since $QCoh \mathbb{P}^1$ is hereditary.

4.4. Classifying cohomological localizations. In this section we give a classification of the cohomological localizing subcategories of $D(QCoh \mathbb{P}^1)$. As we will show in Theorem 4.4.8 they are precisely the subcategories described in Section 4.2. Our strategy is to use Corollary 4.3.3 and the classification of indecomposable pure-injectives for $D(QCoh \mathbb{P}^1)$ to compute everything explicitly; we can compute the set of indecomposable pure-injectives associated to each of the localizing subcategories described in Section 4.2 and show any suspension stable set of pure-injectives has the same left perpendicular as one of these.

To this end we first recall the description of the indecomposable pure-injective objects of $QCoh \mathbb{P}^1$. Let us set up a little notation: given a closed point $x \in \mathbb{P}^1$ we can consider the corresponding map of schemes

$$i_x : \text{Spec } \mathcal{O}_{\mathbb{P}^1, x} \longrightarrow \mathbb{P}^1.$$  

We denote the maximal ideal of $\mathcal{O}_{\mathbb{P}^1, x}$ by $\mathfrak{m}_x$ and the residue field $\mathcal{O}_{\mathbb{P}^1, x}/\mathfrak{m}_x$ by $k(x)$. Let $E(x)$ be the injective envelope of the residue field $k(x)$, and $A(x)$ the $\mathfrak{m}_x$-adic completion of $\mathcal{O}_{\mathbb{P}^1, x}$, which is the Matlis dual of $E(x)$. Pushing these forward along $i_x$ gives objects in $QCoh \mathbb{P}^1$ which we denote by

$$E(x) = i_x \ast E(x) \quad \text{and} \quad A(x) = i_x \ast A(x).$$
Proposition 4.4.1. The indecomposable pure-injective quasi-coherent sheaves are given by the following disjoint classes of sheaves:

- the indecomposable coherent sheaves;
- the Prüfer sheaves $\mathcal{E}(x)$ for $x \in \mathbb{P}^1$ closed;
- the adic sheaves $\mathcal{A}(x)$ for $x \in \mathbb{P}^1$ closed;
- the sheaf of rational functions $k(\eta)$.

Proof. The indecomposable pure-injective quasi-coherent sheaves correspond to the indecomposable pure-injective modules over the Kronecker algebra via the derived equivalence $\mathcal{D}(\text{QCoh}\mathbb{P}^1) \to \mathcal{D}(\text{Mod} A)$. The latter have been classified in [9, Theorem 8.58].

Remark 4.4.2. The Prüfer sheaves and $k(\eta)$ are precisely the indecomposable injective quasi-coherent sheaves.

Having recalled the indecomposable pure-injective sheaves we now determine how they interact with the localizations described in Section 4.2. Let us begin by recording their supports.

Lemma 4.4.3. We have

\[ \text{supp } \mathcal{E}(x) = \{x\}, \text{ supp } k(\eta) = \{\eta\}, \text{ and supp } \mathcal{A}(x) = \{x, \eta\}. \]

Proof. All of these sheaves are pushforwards along the inclusions of the spectra of local rings at points, and so their supports are contained in the relevant subset $\text{Spec } \mathcal{O}_{\mathbb{P}^1, x}$. Having reduced to computing the support over $\mathcal{O}_{\mathbb{P}^1, x}$ this is then a standard computation.

As one would expect the localizations $\text{Loc}(\mathcal{O}(i))$ are particularly simple.

Lemma 4.4.4. The only indecomposable pure-injective quasi-coherent sheaf in $\text{Loc}(\mathcal{O}(i))^\perp$ is $\mathcal{O}(i - 1)$.

Proof. There is a localization sequence for the compacts

\[
\begin{array}{cccccc}
\text{Thick}(\mathcal{O}(i)) & \hookrightarrow & \mathcal{D}^b(\text{Coh } \mathbb{P}^1) & \hookrightarrow & \text{Thick}(\mathcal{O}(i - 1)) \\
| & & | & & | \\
\mathcal{D}^b(k) & \to & \mathcal{D}^b(k) & \to & \mathcal{D}^b(k)
\end{array}
\]

where the adjoints exist since $\mathcal{O}(i)$ is exceptional and the computation of the right orthogonal follows from the computation of the cohomology of $\mathbb{P}^1$. Applying Thomason’s localization theorem shows that

\[ \text{Loc}(\mathcal{O}(i))^\perp = \text{Loc}(\mathcal{O}(i - 1)) = \text{Add}(\Sigma^j \mathcal{O}(i - 1) \mid j \in \mathbb{Z}) \]

and the claim is then immediate.

We next compute the pure-injectives lying in the right perpendicular of the localizing ideals.

Lemma 4.4.5. Let $\mathcal{V}$ be a set of closed points with complement $\mathcal{U}$. Then the indecomposable pure-injective sheaves in $\Gamma_{\mathcal{V}} \mathcal{D}(\text{QCoh } \mathbb{P}^1)^\perp$ are:

- the indecomposable coherent sheaves supported at closed points in $\mathcal{U}$;
- the Prüfer sheaves $\mathcal{E}(x)$ for $x \in \mathcal{U}$;
- the adic sheaves $\mathcal{A}(x)$ for $x \in \mathcal{U}$;
- the sheaf of rational functions $k(\eta)$.
Corollary 4.4.8.  

Proof. By the classification of localizing ideals of $\text{D}(\text{QCoh} \mathbb{P}^1)$ we know that the category $\Gamma \text{D}(\text{QCoh} \mathbb{P}^1)^\perp$ consists of precisely those objects supported on $\mathcal{U}$. Since $\mathcal{V}$ consists of closed points we know $\mathcal{U}$ contains the generic point $q$. The list is then an immediate consequence of Lemma 4.4.3. □

Lemma 4.4.6. Let $\mathcal{V}$ be a subset of $\mathbb{P}^1$ containing the generic point and denote its complement by $\mathcal{U}$. Then the indecomposable pure-injective sheaves in $\Gamma \text{D}(\text{QCoh} \mathbb{P}^1)^\perp$ are:

- the indecomposable coherent sheaves supported at closed points in $\mathcal{U}$;
- the adic sheaves $A(x)$ for $x \in \mathcal{U}$.

Proof. The sheaf of rational functions $k(\eta)$ has a map to every indecomposable injective sheaf and so no $\mathcal{E}(x)$ is contained in the right perpendicular category (and clearly $k(\eta)$ is not). The category $\Gamma \text{D}(\text{QCoh} \mathbb{P}^1)$ contains the torsion and adic sheaves for points in $\mathcal{V}$ so the only indecomposable pure-injective sheaves which could lie in the right perpendicular are those indicated; it remains to check they really don’t receive maps from objects of $\Gamma \text{D}(\text{QCoh} \mathbb{P}^1)$.

This is clear for the residue fields $k(x)$ for $x \in \mathcal{U}$, as they cannot receive a map from any of the residue fields generating $\Gamma \text{D}(\text{QCoh} \mathbb{P}^1)$. Since the right perpendicular is thick it thus contains all the indecomposable coherent sheaves supported on $\mathcal{U}$. Moreover, the right perpendicular is closed under homotopy limits and so contains the corresponding adic sheaves $A(x)$. □

We now know which subsets of indecomposable pure-injectives occur in the right perpendiculars of the localizing subcategories we understand. It’s natural to ask for the minimal set giving rise to one of these categories, as in Corollary 4.3.3. Let us make the convention that for an object $E \in \text{D}(\text{QCoh} \mathbb{P}^1)$

$$\mathcal{V}E = \{ F \in \text{D}(\text{QCoh} \mathbb{P}^1) \mid \text{Hom}(F, \Sigma^j E) = 0 \ \forall j \in \mathbb{Z} \}.$$  

We can, without too much difficulty, compute all of the left perpendiculars of the indecomposable pure-injectives.

Lemma 4.4.7. The left perpendicular categories to the suspension closures of the indecomposable pure-injectives are as follows:

1. $\perp F = \Gamma_{\mathbb{P}^1 \setminus \{x\}} \text{D}(\text{QCoh} \mathbb{P}^1)$ for any $F \in \text{Coh} \mathbb{P}^1$ supported at $x \in \mathbb{P}^1$;
2. $\perp 0(i) = \text{Loc}(0(i + 1))$;
3. $\perp \mathcal{E}(x) = \Gamma_{\mathbb{P}^1 \setminus \{x, \eta\}} \text{D}(\text{QCoh} \mathbb{P}^1)$;
4. $\perp A(x) = \Gamma_{\mathbb{P}^1 \setminus \{x\}} \text{D}(\text{QCoh} \mathbb{P}^1)$;
5. $\perp k(\eta) = \Gamma_{\mathbb{P}^1 \setminus \{\eta\}} \text{D}(\text{QCoh} \mathbb{P}^1)$.

Proof. These are all (more or less) straightforward computations. □

Knowing this it is not hard to write down minimal sets of pure-injectives determining the ideals.

Corollary 4.4.8. Let $\mathcal{V}$ be a subset of $\mathbb{P}^1$. Then we have

$$\Gamma \text{D}(\text{QCoh} \mathbb{P}^1) = \{ F \mid x \notin \mathcal{V} \}.$$  

We also now have enough information to confirm that we have a complete list of the cohomological localizing subcategories.

Theorem 4.4.9. There is a lattice isomorphism

$$\{ \text{cohomological localizing subcategories of } \text{D}(\text{QCoh} \mathbb{P}^1) \} \xrightarrow{\sim} 2^{2\mathbb{P}^1} \amalg \mathbb{Z},$$

where $2^{2\mathbb{P}^1}$ is the powerset of $\mathbb{P}^1$, with inverse defined by

$$\mathcal{V} \mapsto \Gamma \text{D}(\text{QCoh} \mathbb{P}^1) \text{ and } i \mapsto \text{Loc}(0(i)).$$
That is, the cohomological localizing subcategories are precisely the localizing ideals and the \( \text{Loc}(\mathcal{O}(i)) \) for \( i \in \mathbb{Z} \).

**Proof.** By Corollary 4.3.3 the cohomological localizing subcategories are precisely the localizing subcategories which are left perpendicular to a set of indecomposable pure-injectives. Taking the left perpendicular of a set of pure-injectives corresponds to intersecting the corresponding left perpendiculars. By Lemma 4.4.7 we thus see that any such localizing subcategory is of the form claimed. \( \square \)

**Remark 4.4.10.** Denote by \( \text{Ind} \mathbb{P}^1 \) the set of isomorphism classes of indecomposable pure-injective sheaves. The subsets of the form \( L^\perp \cap \text{Ind} \mathbb{P}^1 \) for some cohomological localizing subcategory \( L \) are listed in Lemmas 4.4.4, 4.4.5, and 4.4.6.

5. **Exotic localizations**

As noted in Section 4.2 we have a classification both of ideals and of smashing localizations for \( \mathcal{D}(\text{QCoh} \mathbb{P}^1) \). Moreover, we have just shown in Theorem 4.4.9 that together these are precisely the cohomological localizations. It is obvious to ask if there are non-cohomological localizations; we do not know the answer to this question and don’t hazard a guess.

In this section we at least provide some foundation for future work in this direction by presenting some criteria to guarantee a localizing subcategory is an ideal. This is relevant as any non-cohomological localization could not be an ideal—we proved that all ideals are cohomological. As we shall see this dramatically restricts the possible form of a potential ‘exotic’ localizing subcategory.

5.1. **A restriction on supports.** We begin by analysing support theoretic conditions that ensure a localizing subcategory is an ideal. Since \( \mathbb{P}^1 \) is 1-dimensional the consequences we obtain are quite strong. However, the ideas present in the arguments should be of more general interest.

The first observation is that if the support of an object does not contain some closed point then that object generates an ideal.

**Lemma 5.1.1.** Let \( y \) be a closed point of \( \mathbb{P}^1 \) and let \( X \in \mathcal{D}(\text{QCoh} \mathbb{P}^1) \) be such that \( y \notin \text{supp} \ X \). Then \( L = \text{Loc}(X) \) is an ideal.

**Proof.** By definition we have \( k(y) \otimes X \cong 0 \). Since \( y \) is a closed point the torsion sheaf \( k(y) \) is compact, and hence rigid, so we deduce that

\[
\mathcal{H}om(k(y), X) \cong 0.
\]

In particular, \( X \in \text{Loc}(k(y))^\perp \cong \mathcal{D}(\text{QCoh} \mathbb{A}^1) \), where we have made an identification of \( \mathbb{P}^1 \setminus \{y\} \) with the affine line. Since \( k(y) \) is compact the subcategory \( \text{Loc}(k(y))^\perp \) is localizing and so

\[
L \subseteq \text{Loc}(k(y))^\perp \cong \mathcal{D}(\text{QCoh} \mathbb{A}^1).
\]

It just remains to note that every localizing subcategory of \( \mathcal{D}(\text{QCoh} \mathbb{A}^1) \) is an ideal and that \( \text{Loc}(k(y))^\perp \) is itself an ideal, from which it is immediate that \( L \) is an ideal in \( \mathcal{D}(\text{QCoh} \mathbb{P}^1) \). \( \square \)

Let \( V = \mathbb{P}^1 \setminus \{y\} \) denote the set of closed points of \( \mathbb{P}^1 \). Corresponding to this Thomason subset there is a smashing subcategory \( \Gamma_V \mathcal{D}(\text{QCoh} \mathbb{P}^1) \) which comes with a natural action of \( \mathcal{D}(\text{QCoh} \mathbb{P}^1) \), in the sense of [18], via the corresponding acyclization functor. Moreover, \( \Gamma_V \mathcal{D}(\text{QCoh} \mathbb{P}^1) \) is a tensor triangulated category in its own right, with tensor unit \( \Gamma_V \mathcal{O} \) (which is, however, not compact).
**Lemma 5.1.2.** The category $\Gamma_V D(QCoh \mathbb{P}^1)$ is generated by its tensor unit and hence every localizing subcategory contained in it is an ideal in it, and thus a submodule for the $D(QCoh \mathbb{P}^1)$ action. In particular, every localizing subcategory of $D(QCoh \mathbb{P}^1)$ contained in $\Gamma_V D(QCoh \mathbb{P}^1)$ is an ideal of $D(QCoh \mathbb{P}^1)$.

*Proof.* The subset $V$ is discrete, in the sense that there are no specialization relations between any distinct pair of points in it. It follows from [19] that $\Gamma_V D(QCoh \mathbb{P}^1)$ decomposes as

$$\Gamma_V D(QCoh \mathbb{P}^1) \cong \prod_{x \in V} \Gamma_x D(QCoh \mathbb{P}^1).$$

With respect to this decomposition the monoidal unit $\Gamma_V O$ is just $\bigoplus_{x \in V} \Gamma_x O$, which clearly generates. It follows that every localizing subcategory of $\Gamma_V D(QCoh \mathbb{P}^1)$ is an ideal (see for instance [18, Lemma 3.13]) and from this the remaining statements are clear. \hfill \Box

As a particular consequence we get the following statement, which is more in the spirit of Lemma 5.1.1.

**Lemma 5.1.3.** Let $X \in D(QCoh \mathbb{P}^1)$ be an object such that $\eta \notin \text{supp} X$. Then $\text{Loc}(X)$ is an ideal.

*Proof.* Since $\eta \notin \text{supp} X$ we have $X \in \Gamma_V D(QCoh \mathbb{P}^1)$. Thus $\text{Loc}(X)$ is contained in $\Gamma_V D(QCoh \mathbb{P}^1)$ and therefore an ideal by the previous lemma. \hfill \Box

We have shown that for any object $X$ with proper support the category $\text{Loc}(X)$ is an ideal. Next we will show that any localizing subcategory containing such an object is automatically an ideal. This requires the following technical lemma.

**Lemma 5.1.4.** If $L$ is a non-zero localizing ideal of $D(QCoh \mathbb{P}^1)$ then the quotient $T = D(QCoh \mathbb{P}^1)/L$ is generated by the tensor unit.

*Proof.* Since the property of being generated by the tensor unit is preserved under taking quotients it is enough to verify the statement when $L$ has support a single point. If $L$ is a closed point then we can identify $T$ with the derived category of the open complement, which is isomorphic to $\mathbb{A}^1$. Having made this observation the conclusion follows immediately.

It remains to verify the lemma in the case that $\text{supp} L = \{\eta\}$. In this situation there is a recollement

$$\Gamma_V D(QCoh \mathbb{P}^1) \longleftarrow D(QCoh \mathbb{P}^1) \longrightarrow L$$

where, as above, $V$ denotes the set of closed points of $\mathbb{P}^1$. The bottom four arrows identify $\Gamma_V D(QCoh \mathbb{P}^1)$ with the quotient $T$ and the desired conclusion is given by Lemma 5.1.2. \hfill \Box

Combining all of this we arrive at the following proposition.

**Proposition 5.1.5.** If $L$ is a localizing subcategory of $D(QCoh \mathbb{P}^1)$ such that there is a non-zero $X \in L$ with $\text{supp} X \subseteq \mathbb{P}^1$ then $L$ is an ideal.

*Proof.* Let $X \in L$ as in the statement of the proposition. The object $X$ generates a non-zero localizing subcategory $\text{Loc}(X) \subseteq L$. Since the support of $X$ is proper and non-empty it fails to contain some point of $\mathbb{P}^1$ and so, by one of Lemma 5.1.1 or 5.1.3 it is an ideal. We thus have a monoidal quotient functor $D(QCoh \mathbb{P}^1) \rightarrow D(QCoh \mathbb{P}^1)/\text{Loc}(X)$ and an induced localizing subcategory $L/\text{Loc}(X)$ in the quotient. By Lemma 5.1.3 the quotient $D(QCoh \mathbb{P}^1)/\text{Loc}(X)$ is generated by the tensor unit and so $L/\text{Loc}(X)$ is a tensor ideal in it. But then $L = \pi^{-1}(L/\text{Loc}(X))$ is also an ideal, which completes the proof. \hfill \Box
Example 5.1.6. The non-ideals we know, namely the $\text{Loc}(\mathcal{O}(i))$, are of course compatible with the proposition: every object of $\text{Loc}(\mathcal{O}(i))$ is just a sum of suspensions of copies of $\mathcal{O}(i)$, and these are all supported everywhere.

The following interpretation is the most striking in our context.

Corollary 5.1.7. If $L$ is a localizing subcategory which is not cohomological then every non-zero object of $L$ is supported everywhere.

5.2. Twisting sheaves and avoiding compacts. We next make a few comments concerning the interactions between localizing subcategories and the twisting sheaves.

Lemma 5.2.1. If $L$ is a localizing subcategory which is not an ideal then $L \cap L(i) = 0$ for all $i \in \mathbb{Z} \setminus \{0\}$.

Proof. Without loss of generality we may assume $i > 0$. Suppose, for a contradiction, that $X \in L \cap L(i)$ is non-zero. Pick a closed point $y$ and consider a triangle

$$\mathcal{O}(-i) \rightarrow \mathcal{O} \rightarrow Z(y) \rightarrow \Sigma \mathcal{O}(-i)$$

where $Z(y)$ is the cyclic torsion sheaf of length $i$ supported at $y$. We can tensor with $X$ to get a new triangle

$$X(-i) \rightarrow X \rightarrow X \otimes Z(y) \rightarrow \Sigma X(-i),$$

where both $X$ and $X(-i)$ lie in $L$ by hypothesis. Thus, since $L$ is localizing, we see that $X \otimes Z(y)$ lies in $L$. By Proposition 5.1.5 we know that $X$ is supported everywhere and so $X \otimes Z(y) \neq 0$. But on the other hand, $X \otimes Z(y)$ is supported only at $y$ which, by the same Proposition, implies that $L$ is an ideal yielding a contradiction. \qed

Remark 5.2.2. The lemma implies that non-cohomological localizing subcategories would have to come in $\mathbb{Z}$-indexed families.

Lemma 5.2.3. If $L$ is a localizing subcategory such that

$$\text{Loc}(\mathcal{O}(i)) \subseteq L$$

for some $i \in \mathbb{Z}$ then $L = D(\text{QCoh } \mathbb{P}^1)$.

Proof. Localizing subcategories containing $\text{Loc}(\mathcal{O}(i))$ are in bijection with localizing subcategories of $D(\text{QCoh } \mathbb{P}^1)/\text{Loc}(\mathcal{O}(i))$. This quotient is just $D(k)$ and so, since we have asked for a proper containment, the result follows. \qed

We can now conclude that any non-cohomological localizing subcategory must intersect the compact objects trivially.

Proposition 5.2.4. If $L$ is a localizing subcategory which is not cohomological then $L$ contains no non-zero compact object.

Proof. The indecomposable compact objects are just the indecomposable torsion sheaves at each point and the twisting sheaves. By Lemma 5.1.3 we know $L$ cannot contain a torsion sheaf and by the last lemma it cannot contain a twisting sheaf. \qed
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Henning Krause, Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany.
E-mail address: hkrause@math.uni-bielefeld.de

Greg Stevenson, School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow G12 8SQ, Scotland U.K.
E-mail address: gregory.stevenson@glasgow.ac.uk