Multilevel Code Construction for Compound Fading Channels

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Abstract—We consider explicit constructions of multi-level lattice codes that universally approach the capacity of the compound block-fading channel. Specifically, building on algebraic partitions of lattices, we show how to construct codes with negligible probability of error for any channel realization and normalized log-density approaching the Poltyrev limit. Capacity analyses and numerical results on the achievable rates for each partition level are provided. The proposed codes have several enjoyable properties such as constructiveness and good decoding complexity, as compared to random one-level codes. Numerical results for finite-dimensional multi-level lattices based on polar codes are exhibited.

I. INTRODUCTION

Compound channels are suitable models for open-loop communication scenarios, when a transmitter has limited knowledge of the channel state, but requires reliable transmission for all states. Another natural application is broadcasting a message to multiple receivers, where each transmission link is possibly in a different channel state. A code for a compound channel is universal, in the sense that the probability of error vanishes for any channel in a set, known as the compound set. The objective of this work is to study practical multi-level constructions of universal codes for block-fading channels.

It was previously shown in [3] that random algebraic lattice codes achieve the capacity of the compound block-fading (and MIMO) channels. However, as expected, random codes lack encoding and decoding efficiency. By employing multi-level algebraic constructions we can build on existing codes for regular finite-input channels to construct practical universal lattices. In particular, we show that chains of algebraic lattices (e.g. [7]) may be used for practical construction of efficient universal codes, with essentially the same complexity as codes for the Gaussian channel. Leveraging from the algebraic structure, we can identify the best and worst-channels in the compound set. As an illustration, we analyze the behavior of multi-level lattices tailored for the two extreme channels in each level as in [15], as well as a simple scheme based on a surrogate BEC channel in each level. Both schemes operate at within a fraction of dB from the limit with small probability of error for all channels and multi-stage decoding.

Related Works

Traditional random \( Z \)-lattice codes as in [16] can indeed achieve the compound capacity of the power constrained block-fading channels, with no extra algebraic structure. This is a consequence of a theorem in [5] along with a compactness argument (e.g. [3] Appx. A, [3] Thm. 2). However, algebraic random lattices possess several advantages, such as guaranteed full-diversity, better quantization of the channel space, and the possibility of decoupling equalization and decoding tasks. With respect to practical multi-level constructions, the advantages are even more evident. For instance, for integer partitions the achievable rates in each level varies significantly with the channel state (see, e.g., Fig. 2), implying a very poor compound capacity. Algebraic partitions, on the other hand, “absorb” part of channel realizations, making the capacity of each level vary smoothly enough in the channel space.

Regarding previous algebraic constructions, the work [2] studied one-level chains of lattices with full diversity and derived bounds on the alphabet size of the underlying codes. For independent fading channels, [10] showed that multi-level lattices constructed from nested polar codes via integer partitions \( \mathbb{Z}/2^{k}\mathbb{Z}\) are capacity achieving with essentially the same decoding and encoding complexity as the AWGN channel. In a recent contribution [1], the outage probability of two-dimensional multi-level lattices based on Reed-Muller codes was numerically evaluated. In a related discrete-model, a novel hierarchic polar coding scheme using two phases of polarization was proposed in [14] for binary-input block fading channels. Based on this scheme, a multi-level structure was then proposed for fading channels with additive exponential noise, using noise decomposition technique.

The rest of this work is organized as follows. In section II and we review the transmission model and Forney et al’s multi-level coset codes. In Section IV we provide an analysis of one-level partitions, revisiting the results in [3] via universal capacity-achieving codes for finite-input channels. In Section VII we show how to construct partitions with more level and analyze the compound capacity of each level. In Section VI numerical experiments with the constructed partitions are performed to show the performance of multi-level universal codes in finite block-lengths.

II. NOTATION AND INITIAL DEFINITIONS

Following [12], we define the diversity of a set as follows.

**Definition 1.** For two vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) define \( \mathcal{I}_{\mathbf{x},\mathbf{y}} = \{ i \in \{1, \ldots, n\} : x_i \neq y_i \} \). The diversity order of a set \( \Lambda \subset \mathbb{R}^n \) is given by \( l = \min_{\mathbf{x},\mathbf{y} \in \Lambda} |\mathcal{I}_{\mathbf{x},\mathbf{y}}| \). When \( l = n \), we say that \( \Lambda \)
has full-diversity.

Let $\Lambda$ be a lattice in $\mathbb{R}^n$. If $\Lambda$ has full diversity, it has an associated positive product-distance defined by $d_{\text{mod}}(\Lambda) = \min_{x \in \Lambda \setminus \{0\}} \prod_{i=1}^{n} ||x_i||$. A fundamental region $R_{\Lambda}(\Lambda)$ for $\Lambda$, is any interior-disjoint region that tiles $\mathbb{R}^n$. For any $x, y, z \in \mathbb{R}^n$ we say that $y = x \text{ (mod } \Lambda\text{)}$ if $y - x \in \Lambda$. By convention, we fix a fundamental region and denote by $y \text{ (mod } \Lambda\text{)}$ the unique representative $x \in R_{\Lambda}(\Lambda)$ such that $y = x \text{ (mod } \Lambda\text{)}$. A partition chain $\Lambda_1/\Lambda_2/\Lambda_3/\ldots/\Lambda_m$ is a lattice sequence such that $\Lambda_m \subset \Lambda_{m-1} \subset \ldots \subset \Lambda_1$.

**Definition 2.** The volume-to-noise ratio (VNR) of an $n$-dimensional lattice $\Lambda$ with respect to $\sigma > 0$ is defined as $\gamma_\Lambda(\sigma) = V(\Lambda)^{2/n}/\sigma^2$.

We write $M = \text{diag}(m_1, \ldots, m_n)$ for a diagonal matrix with diagonal elements $m_i$. We say that $M_1 \succeq M_2$ if the difference $M_1 - M_2$ is positive definite. We denote the $n \times n$ identity matrix by $I_n$ or simple $I$ when there is no ambiguity.

**A. Model Description**

In this paper we focus on infinite-dimensional constellations, for which the Poltyrev limit replaces the notion of capacity. Let $H$ be a diagonal matrix in $\mathbb{R}^{n \times n}$. For $x \in \Lambda$ transmitted over the block-fading channel, the received signal is given by

$$y = Hx + w,$$

where $w \sim \mathcal{N}(0, \sigma^2 I_n)$. The matrix $H$ is assumed to be constant in the whole transmission and known to the receiver.

For a real number $D \geq 1$, let

$$\mathbb{H}_\infty(D) = \left\{ H \in \text{diag}(\mathbb{R}^{n \times n}) : \det H^n H = D \right\}.$$ 

By convention, we will denote $\mathbb{H}_\infty(1)$ by $\mathbb{H}_\infty$. In the “infinite constellation setting”, [3] define universally good lattices as follows.

**Definition 3.** We say that a sequence of lattices $\Lambda_T$ of increasing dimension $n_T$ is universally good for the block-fading channel if for any VNR $\gamma_{\Lambda_T}(\sigma) > 2\pi e \sigma^{2/n}$ and all $H \in \mathbb{H}_\infty(D)$, $P_e(\Lambda_T, H) \to 0$.

For $H = I_n$, the condition is necessary and sufficient for vanishing error probability, and is known as the Poltyrev (see, e.g., [15 Ch. 7]). Sequences of lattices that achieve the Poltyrev limit are said to be AWGN-good. Notice that, if applied to a power-constrained scenario, achieving the capacity $\mathbb{H}_\infty$ is a more stringent condition than achieving the capacity of the compound block-fading channel. In particular, the definition forces $\Lambda_T$ to have diversity at least $n$. For multi-level constructions, this implies a non-trivial bound on the capacity of each level (Eq. (12)) not depending on the VNR.

**III. Multilevel Constructions**

In this section, we recall Forney et al.’s construction [6]. Let $\Lambda/\Lambda'$ be a lattice partition where $\Lambda$ and $\Lambda'$ have dimension $n$, e.g. $\Lambda = \mathbb{Z}$ and $\Lambda' = p\mathbb{Z}$, $p$ prime. To the quotient $\Lambda/\Lambda'$ is associated the $\Lambda/\Lambda'$-channel, whose input is drawn from $\Lambda$ and output the noise is calculated mod-$\Lambda$ (for more precise definitions see Section [V]). This channel is regular, and the optimum input distribution is uniform [6]. This channel has capacity

$$C(\Lambda/\Lambda', \sigma^2) = C(\Lambda', \sigma^2) - C(\Lambda, \sigma^2)$$

$$= h(\Lambda, \sigma^2) - h(\Lambda', \sigma^2) + \log V(\Lambda)/V(\Lambda)$$

where

$$C(\Lambda_i, \sigma^2) = \log V(\Lambda_i) - h(\Lambda_i, \sigma^2)$$

and $h(\Lambda_i, \sigma^2)$ is the differential entropy of the mod-$\Lambda_i$ Gaussian noise:

$$h(\Lambda_i, \sigma^2) = -\int_{V(\Lambda_i)} f_{\sigma, \Lambda_i}(x) \log f_{\sigma, \Lambda_i}(x) dx.$$ 

To construct an AWGN-good lattice for a partition chain $\Lambda_{m-1}/\Lambda_{m-2}/\Lambda_{m-3}/\ldots/\Lambda_m$, Forney et al. [6] choose $\Lambda$ and $\Lambda'$ as follows [6]:

1) $V(\Lambda_{m-1})$ is small enough that the mod-$\Lambda_{m-1}$ noise is almost uniform and $C(\Lambda_{m-1}, \sigma^2) \approx 0$.

2) $V(\Lambda_m)$ is large enough that the error probability $P_e(\Lambda_m, \sigma^2) \approx 0$. This means that the mod-$\Lambda'$ noise is almost unaliased and

$$C(\Lambda_m, \sigma^2) \approx \frac{n}{2} \log \frac{V(\Lambda_m)^{2/n}}{2\pi e \sigma^2}.$$ 

3) In each level a code $C_i$ is capacity achieving in the $\Lambda_i/\Lambda_{i+1}$ channel.

For a one-level partition $\Lambda/\Lambda'$, where $\Lambda$ and $\Lambda'$ have decreasing and increasing volume, the capacity of the $\Lambda/\Lambda'$ channel tends to the Poltyrev limit:

$$\lim_{V(\Lambda') \to \infty} \frac{C(\Lambda', \sigma)}{n/2 \log(\gamma_{\Lambda}(\sigma)/2\pi e)} = 1.$$ 

Indeed, in this case, increasing the volume $V(\Lambda')$ automatically decreases its probability of error.

As we move along a lattice partition chain, the capacity of each level increases. For our analysis, we need the more general fact that such channels are degraded. The following lemma is a generalization of [15 Lem. 3] (we omit the proof and precise definitions).

**Lemma 1** (Channel Degradation). If $|D| > 1$ is a diagonal matrix, the mod-$\Lambda$ channel is stochastically degraded with respect to the mod-$DA$ channel. Furthermore, if $\Lambda/DA/DA^2\Lambda$ is a lattice partition chain, the $\Lambda/DA$-channel is stochastically degraded with respect to the $DA/DA^2\Lambda$-channel.

The above lemma implies, for instance, that the mod-$DA$ has larger capacity than the mod-$\Lambda$ channel.

**IV. ONE-LEVEL UNIVERSAL LATTICES**

Typical proofs of the universality of lattices use one-level codes with increase alphabet size and follow from random arguments. In order to build partitions with more levels for the compound channel, we start by revisiting the one-level construction à la Forney et al. [6]. This construction has the advantage of being explicit and efficiently (two-stage) decodable, as compared to random lattices.
A. The mod-Λ block-fading channel

\[ x \in \mathcal{R}(\Lambda) \xrightarrow{\text{X}} \oplus \xrightarrow{\text{mod } H A} (Hx + w) \bmod(HA) \]

Fig. 1. Diagram of the mod HA channel

Given a vector \( x \in \mathcal{R}(\Lambda) \) the block-fading mod-\( \Lambda \) channel is a block-fading channel with the operation modulo \( HA \) performed at the receiver front-end, namely

\[ y = (Hx + w) \pmod{HA}. \]

where \( H \) is diagonal and belongs to \( \mathbb{H}_\infty \). The \( HA/HA' \)-channel is a modulo \( HA \) channel with inputs constrained to be in \( \Lambda \). Denote by \( C_H(\Lambda, \sigma^2) \) the capacity of the mod-\( HA \) channel and by \( C_H(\Lambda/\Lambda', \sigma^2) \) the capacity of the \( HA/HA' \) channel, given a realization \( H \). When \( H = I \) we use the shorthand notation \( C_I(\Lambda/\Lambda', \sigma^2) = C(\Lambda/\Lambda', \sigma^2) \). From [6] Thm. 5, we have

\[ C_H(\Lambda/\Lambda', \sigma^2) = C_H(\Lambda, \sigma^2) - C_H(\Lambda', \sigma^2). \]

The set of all \( HA/HA' \) channels, \( H \in \mathbb{H}_\infty \), is a compound channel with capacity

\[ C_{\mathbb{H}_\infty}(\Lambda/\Lambda', \sigma^2) = \inf_{H \in \mathbb{H}} C_H(\Lambda/\Lambda', \sigma^2). \] (7)

Achieving the compound capacity \( C_{\mathbb{H}_\infty}(\Lambda/\Lambda', \sigma^2) \) requires a universal code for \( p \)-ary symmetric channels.

B. Ideal lattices

Now we want to construct a universal lattice \( L \) which is good for \( \mathbb{H}_\infty \). To this purpose, we choose \( C \) as a universal code for the \( HA/HA_p \) channel and use Construction A. Upon receiving vector \( y = [y_1, y_2, \ldots, y_T] \) where \( y_i \in \mathbb{R}^n \), decoding consists of two stages:

- Decode \( y_i \) to coarse lattice \( HA_p \), and let

\[ \hat{y}_i = y_i \mod HA_p = y_i - \text{dec}(y_i); \]

- Decode \( \hat{y} \) to code \( C \).

In the first stage, we require \( P_e(HA_p, \sigma^2) \approx 0 \) for all \( H \) with fixed determinant \( D \). This can be satisfied by a bounded minimum Euclidean distance code, provided that \( HA_p \) has full diversity.

\[ \min_{H:|H|=D} \|Hx\|^2 = \min_{H:|H|=D} \min_{x \in \Lambda_p, x \neq 0} \|Hx\|^2 = \min_{x \in \Lambda_p, x \neq 0} nD^{2/n}(x_1 \cdots x_n)^{2/n} \] (8)

The second step follows from the inequality of arithmetic and geometric means; amazingly, the minimum distance is precisely known here, i.e., there exists a realization \( H \) such that the equality holds (this realization is \( h_i^2 = (D_{prod}(\Lambda_p))^{2/n} \) where \( x_1 \cdots x_n = d_{prod}(\Lambda_p) \); see [1] Prop. 3.6). Therefore, as long as the norm \( d_{prod} \) is sufficiently large, reliable decoding of \( HA_p \) is possible for all \( H \). More precisely, the worst-case probability can be bounded as follows:

**Proposition 1.** With notation as above,

\[ \sup_{H \in \mathbb{H}(D)} P_e(HA_p) \leq \left( \frac{\sqrt{2}Dd_{prod}(\Lambda_p)^{1/2}}{\sqrt{n}} \right)^{2/n} \to 0 \] (9)

as \( d_{prod}(\Lambda_p) \to \infty \).

The right-hand side of Eq. (9) can be further bounded using \( \Phi \)-function approximations in order to estimate the product-distance for a target probability of error (see [2] for more details). Notice that this strictly requires \( \Lambda \) to have full diversity. This can be seen geometrically, for instance, if \( H \in \mathbb{H}_\infty \) has some component arbitrarily close to zero. An explicit construction of full-diverse lattices is given by the generalized Construction A [3]. \( \Lambda_p \) is the embedding in \( \mathbb{R}^n \) of an ideal \( p \) of the ring of integers of a number field \( K \). In this case \( d_{prod}(\Lambda_p) = N(p) \) is the algebraic norm of the ideal. Due to space limitations, we refer to [12] for undefined algebraic terms.

In the second stage, the rate of the code \( C \) is bounded by the capacity \( C_H(\Lambda/\Lambda_p, \sigma^2) \) of the \( HA/HA_p \) channel. In general, we do not know whether \( C_H(\Lambda/\Lambda_p, \sigma^2) \) is fixed, since \( H \) is arbitrary. However if \( \Lambda \) has full diversity and we scale the bottom lattice \( \Lambda \) appropriately we can make \( C_H(\Lambda, \sigma^2) \approx 0 \) for any channel realization, so that in the limit [6],

\[ C_H(\Lambda/\Lambda_p, \sigma^2) \approx C_H(\Lambda_p, \sigma^2) \]

\[ \approx \frac{n}{2} \log \left( \frac{V(\Lambda_p)^{2/n}}{2\pi e \sigma^2} \right) = \frac{n}{2} \log \left( \frac{(DV(\Lambda_p)^{2/n}}{2\pi e \sigma^2} \right) \] (10)

which only depends on \( D \). So it is possible to design a universal good lattice, provided that the code for the \( HA/HA_p \) is universal. Examples include spatially-coupled LDPC codes or universal variants of polar codes [4], [8], [9].

V. Construction D

In order to reduce the alphabet-size and improve decoding complexity, multilevel constructions are employed. Assume a partition chain \( \Lambda_0 = \psi(K), \Lambda_1 = \psi(p_1), \ldots, \Lambda_m = \psi(p_r) \), where the \( p_i \)’s are nested ideals of a number field \( K \) of degree \( n \). For a fixed \( H \), from the capacity conservation rule [6],

\[ \sum_{i=1}^{n} C_H(\Lambda_i/\Lambda_{i+1}, \sigma^2) = C_H(\Lambda_0/\Lambda_r, \sigma^2), \] (11)

which is expected to be approximately the normalized log-density of the lattice. However we face two main problems

1) Rate allocation: what is the maximum rate \( R_i = C_H(\Lambda_i/\Lambda_{i+1}, \sigma^2) \) for each level?
2) Degradation: When can we assert that \( HA_{i-1}/HA_i \) is degraded with respect to \( HA_i/HA_{i+1} \)?
thus depending on the channel dimension the finite field will be very large. In case 2), we have $O_K = p^n$ and $O_K/p \cong F_p$. The algebraic partition chain is then $O_K/p^2/O_K/p^{n-1}/O_K$. If, in addition $p$ is a prime ideal, then each $p^i$ is a prime ideal, then each $p^i = \langle \alpha \rangle$ where $\alpha \in O_K$ is such that $\langle \alpha^n \rangle = pO_K$. Therefore, passing to the real case, we have $H_{\psi}(O_K/HD_a(\psi(O_K)/HD_a(\psi(O_K)$, where $D_a$ is a diagonal matrix with $[D_a]_{ii} = \sigma_i(\alpha)$. Building the the ideals to ensure $|[D_a]_{ii}| \times 1$, we can apply Lem. 1.

C. Worked Out Examples

Binary Partitions. Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ be the ring of integers of the field extension $\mathbb{Q}(\sqrt{2})$. A simple example of binary partition is given by $\mathbb{Z}[\sqrt{2}]/\sqrt{2}\mathbb{Z}[\sqrt{2}]/\mathbb{Z}[\sqrt{2}]/\ldots$ and its $n$-dimensional generalization, which have been thoroughly analyzed in [7], in terms of coding gain and product norm. Here we expand the analysis to the capacity of $\Lambda/\Lambda'$ channels. More precisely, let

$$\psi(a + b\sqrt{2}) = (a + b\sqrt{2}, a - b\sqrt{2}),$$

be the embedding of an algebraic number of the form $a + b\sqrt{2}$. Then we can construct the two-dimensional binary partition $\psi(\mathbb{Z}[\sqrt{2}])/\psi(\sqrt{2}\mathbb{Z}[\sqrt{2}]/\psi(\mathbb{Z}[\sqrt{2}]/\ldots$ We can see from Lem. 1 that the channels in each level are degraded with respect to the channel in the next level. For the rate allocation, we identify the worst channel. Let

$$C_{\Lambda}(\Lambda, \sigma^2) = h(\Lambda) - h(\Lambda') + 1$$

be the capacity of the first level, where $\Lambda = \psi(\mathbb{Z}[\sqrt{2}])$ and $\Lambda' = \psi(\sqrt{2}\mathbb{Z}[\sqrt{2}])$. For any $H \in \mathbb{H}_\infty$ denote by $C_h(\Lambda, \sigma^2)$ the capacity of the mod-$\Lambda$ fading channel for fading realization $(h, 1/h)$. Due to the invariance of the partition by units in $\mathbb{Z}[\sqrt{2}]$, $C_h(\Lambda, \sigma^2)$ is a multiplicatively periodic function, i.e. $C_h(\Lambda, \sigma^2) = C_h(\Lambda, \sigma)$. From this, and from the fact that $C_h(\Lambda, \sigma^2) = C_{1/(\sqrt{2})}(\Lambda, \sigma)$, we can conclude that $h = 1$ and $h = \sqrt{1 + \sqrt{2}}$ are extreme points of the capacity (see Fig. 3). Interestingly, [7] have shown that these are also the extreme points of Hermite parameter of $H$. The rate should be allocated according to the worst channel. We also observe a “phase-transition” in the worst case due to Fig. 3 from $h = \sqrt{1 + \sqrt{2}}$ to $h = 1$ as the level increases. This is due to the fact that the sums of the capacities of all relevant levels is approximately the Poltyrev limit.

VI. Numerical Results

In this section, we design explicit universal polar lattices for compound fading channels, using universal polar codes [8] and construction D. Let $\Lambda = \mathbb{Z}[\sqrt{2}]$. To simplify the design, we choose the binary partition chain $\Lambda/\sqrt{2}\Lambda/2\Lambda/\ldots$ and binary universal polar codes. The compound fading channel is also assumed to have only two extreme cases with $h_1 = 1$ and $h_2 = \sqrt{1 + \sqrt{2}}$. As can be seen from Fig. 3 the capacity $C_h(\mathbb{Z}[\sqrt{2}], \sigma)$ is negligible. Our aim is to design a universal polar code for each partition channel, with rate no more than $C_h(\Lambda/\sqrt{2}\Lambda, \sigma) = 0.2239$, $C_h(\sqrt{2}\Lambda/2\Lambda, \sigma) = 0.6516$, and $C_h(2\Lambda/2\sqrt{2}\Lambda, \sigma) = 0.9271$, respectively. Since there are two
for a surrogate BEC with erasure probability parameter of a partition channel, and construct a polar code approximation, i.e., to find an upper bound \( \epsilon \) convenient construction is to use binary erasure channel approximation for each level. However, the polarization rate is generalized to compound channels with finite states (see [8, Sec. V-C] for more details).

We have considered practical constructions and guidelines for the construction of universal multi-level lattice codes. Both theoretical bounds and numerical results were exhibited. Concretely, we illustrated the constructions with lattices from polar codes for the 2-dimensional compound-fading channel, showing that a small gap to the Poltyrev limit can be universally achieved in all channels. On a quantitative level, our analyses corroborate the fact that algebraic partitions can be greatly advantageous as compared to unstructured ones, even from an information-theoretic perspective.

ACKNOWLEDGMENTS

The work of L. Liu is supported by Huawei’s Shield Lab through the HIRP Flagship Program. The authors would like to thank J.-C. Belfiore for fruitful discussions.

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