INFINITESIMAL GLUING EQUATIONS AND THE
ADJOINT HYPERBOLIC REIDEMEISTER TORSION

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Abstract. We establish a link between the holomorphic derivatives of Thurston’s hyperbolic
 gluing equations on an ideally triangulated finite volume hyperbolic 3-manifold and the coho-
 mology of the sheaf of infinitesimal isometries. Moreover, we provide a geometric reformulation
 of the non-abelian Reidemeister torsion corresponding to the adjoint of the monodromy rep-
 resentation of the hyperbolic structure. These results are then applied to the study of the
 ‘1-loop Conjecture’ of Dimofte–Garoufalidis, which we generalize to arbitrary 1-cusped hy-
 perbolic 3-manifolds. We rigorously verify the generalized conjecture in the case of the sister
 manifold of the figure-eight knot complement.

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1. Introduction

This paper aims to establish a link between an infinitesimal version of Thurston’s hyperbolic gluing equations and the adjoint Reidemeister torsion of a finite volume hyperbolic 3–manifold. This goal is realized in two steps. Firstly, we explore the cohomological meaning of the derivatives of the edge consistency and completeness equations on a finite ideal triangulation \( T \) which admits a positively oriented geometric solution. The basic idea is to interpret the complex tangent space \( T_zC_{\text{im}}>0 \) as the space of infinitesimal deformations of the geometry of a hyperbolic ideal tetrahedron in \( \mathbb{H}^3 \) with the shape parameter \( z \in C_{\text{im}}>0 \). When several ideal tetrahedra are glued together to form a geometric triangulation \( T \), the deformations of the individual tetrahedra become infinitesimal deformations of the geometry of the resulting hyperbolic 3–manifold \( M \). These deformations are described as first cohomology classes with coefficients in the sheaf of infinitesimal isometries of \( M \) or a restriction thereof.

Secondly, since the bundle of infinitesimal isometries on a hyperbolic 3–manifold \( M \) is isomorphic to the flat, rank 3 complex vector bundle defined by the adjoint of the monodromy representation \( \pi_1(M) \to PSL_2\mathbb{C} \) of the hyperbolic structure, we can restate Porti’s construction [25] of the combinatorial adjoint hyperbolic torsion \( T_{\text{Ad}}(M) : H_1(\partial_\infty M; \mathbb{Z}) \to \mathbb{C}^*/\{\pm 1\} \) in terms of the sheaf of germs of Killing vector fields on \( M \). This provides a geometric interpretation of the adjoint torsion.

Finally, we show how these insights can be used to calculate the torsion invariant \( T_{\text{Ad}}(M) \) in terms of an ideal triangulation of \( M \). In particular, our method correctly reproduces the main factor of the ‘1–loop invariant’, a conjectural expression for the adjoint torsion given by Dimofte and Garoufalidis in [7].

1.1. Infinitesimal gluing equations. The hyperbolic gluing equations were first introduced by W. Thurston [28]. Neumann and Zagier [22] discovered a symplectic property of these equations which was further studied by Neumann in [20] and more recently reinterpreted by Dimofte and van der Veen [8] in terms of intersection theory on certain branched double covers. Within mathematical physics, gluing equations have been used to construct quantum Chern–Simons theories on ideal triangulations, with the symplectic structure serving as the starting point for geometric quantization; see in particular Dimofte [6] and Dimofte–Garoufalidis [7].

Suppose that \( T \) is an abstract ideal triangulation of a connected orientable open 3–manifold \( M \) with \( k \) ideal vertices, the links of which are all tori. Denote by \( N \) be the number of tetrahedra, and hence also of edges, of \( T \). Choi [3] reformulated the hyperbolic edge consistency equations in terms of a single map \( g : C_{\text{im}}>0 \to (\mathbb{C}^*)^N \), the domain of which is thought of as the space of shape parameters of \( N \) positively oriented ideal tetrahedra. By definition, \( g \) assigns to any \( N \)–tuple of shapes the \( N \)–tuple of their products about the edges of \( T \), so that the consistency equations read \( g(z) = 1 \). Consider also a collection \( \theta = (\theta_1, \ldots, \theta_k) \) of oriented, homotopically nontrivial curves in normal position with respect to \( T \), one in each vertex link. The log-parameters along the constituent curves of \( \theta \), defined by the way of [22], define a map \( u = u_\theta : C_{\text{im}}>0 \to \mathbb{C}^k \) so that the completeness condition becomes \( u(z) = 0 \). With these notations, the results of Neumann–Zagier [22] and Choi [3] imply the existence of the tangential exact sequence of ‘infinitesimal gluing equations’

\[
0 \to T_uU \xrightarrow{D_y} T_{y(u)}C_{\text{im}}>0 \xrightarrow{D_\theta} T_1(\mathbb{C}^*)^N \xrightarrow{D_p} \mathbb{C}^k \to 0,
\]

where \( y = y_\theta : U \to g^{-1}(1) \) is a local analytic inverse of \( u \) and \( p \) is a monomial map defined by the incidences of the edges of \( T \) to the ideal vertices; see Section 2.1 below for the details.

1.2. Infinitesimal hyperbolic isometries. Recall that a Killing field on a Riemannian manifold \( M \) is a vector field whose flows are local isometries of \( M \). We denote the Lie algebra of all Killing fields on \( M \) by \( \mathfrak{K}(M) \). The assignment of the space \( \mathfrak{K}(U) \) to any open set \( U \subset M \) defines a sheaf \( \mathfrak{K} \), called the sheaf of (germs of) Killing vector fields. Geometrically, the sheaf \( \mathfrak{K} \) can be viewed as the sheaf of local infinitesimal isometries of \( M \). The cohomology of \( \mathfrak{K} \) is therefore
closely related to the deformation theory of geometric structures, as explained in the case of hyperbolic 3–manifolds by Hodgson and Kerckhoff in [15]. We refer to [24, 18] for more information on Killing vector fields.

Suppose that the triangulation $T$ of $M$ is geometric, so that $M$ is endowed with a finite-volume hyperbolic structure, not necessarily complete. This in particular defines the sheaf $\mathcal{K} = \mathcal{K}_M$. Consider the closed subspace $M_0 \subset M$ resulting from the removal of disjoint open neighbourhoods of all edges of $T$ from $M$. Then $M_0 = M_0(T)$ produces the short exact sequence
\[
0 \to \mathcal{K}_{M,M_0} \to \mathcal{K} \to \mathcal{K}_{M_0} \to 0
\]
of sheaves on $M$, whose associated cohomology long exact sequence reduces to the four non-zero terms
\[
0 \to H^1(M;\mathcal{K}) \to H^1(M;\mathcal{K}_{M_0}) \to H^2(M;\mathcal{K}_{M,M_0}) \to H^2(M;\mathcal{K}) \to 0.
\]
The cohomological meaning of the gluing equations is then established by the following theorem.

**Theorem 1.1.** At a generic hyperbolic structure on $M$, with all ends incomplete, the acyclic complex (1.1) embeds as a subcomplex of (1.3).

The above theorem is stated in more detail as Theorem 3.3 below; cf. also Theorem 4.3.1 in [26].

Observe that the leftmost map $D\gamma$ of the exact sequence (1.1) depends on the chosen multicurve $\theta$. Given another multicurve $\theta$ with log-parameter map $\bar{u}$, the infinitesimal effect of the change of curves is described by the derivative $D(\bar{u} \circ \gamma): TU \to C^k$, which can be interpreted cohomologically as follows.

**Theorem 1.2.** The unique map $c$ which makes the diagram
\[
\begin{array}{ccc}
T_u U & \longrightarrow & H^1(M;\mathcal{K}) \\
\downarrow \begin{array}{c}
D(\bar{u} \circ \gamma) \end{array} & & \downarrow \begin{array}{c}
c \end{array} \\
\mathbb{C}^k & \longrightarrow & H^2(M;\mathcal{K})
\end{array}
\]
commutative is given by $c(x) = x - [\bar{\theta}]^*$, where $[\bar{\theta}]^* \in H^1(\partial_\infty M;\mathbb{Z})$ is the Poincaré dual of the homology class of $\bar{\theta}$.

In the diagram (1.4), the horizontal maps on the left side are the embeddings (in fact, isomorphisms) given by Theorem 1.1, and $\partial_\infty M$ denotes the toroidal boundary at infinity of $M$. We refer to Theorem 3.3 below for a precise statement and to [26] for an extended discussion.

### 1.3. Geometric approach to the adjoint hyperbolic torsion

The action of $PSL_2\mathbb{C}$ on $H^3$ by orientation-preserving hyperbolic isometries identifies the space $\mathcal{K}(\mathbb{H}^3)$ of global Killing fields with the Lie algebra $\mathfrak{sl}_2\mathbb{C}$. Similarly, the sheaf $\mathcal{K} = \mathcal{K}_M$ on an orientable hyperbolic 3–manifold $M$ is locally modeled after $\mathfrak{sl}_2\mathbb{C}$, which we consider here with the discrete topology. To understand this relationship geometrically, assume that $M$ is connected and consider a monodromy representation $\varphi: \pi_1(M) \to PSL_2\mathbb{C}$ of the hyperbolic structure on $M$. Let $E = E_{Ad,\varphi}$ be the rank 3 vector bundle on $M$ defined by
\[
E = \tilde{M} \times_{\pi_1(M)} \mathfrak{sl}_2\mathbb{C},
\]
where $\pi_1(M)$ acts on the universal covering space $\tilde{M}$ by deck transformations and on $\mathfrak{sl}_2\mathbb{C}$ via $Ad\varphi$. By a theorem of Matsushima–Murakami [15, Theorem 8.1], the sheaf $\mathcal{K}$ is isomorphic to the sheaf $\mathfrak{K}(E)$ of continuous sections of $E$. This isomorphism naturally endows $\mathcal{K} = \mathcal{K}_M$ with the structure of a locally constant sheaf of complex vector spaces. In particular, the monodromy of $E$ can be understood as analytic continuation of locally defined Killing fields, as studied by Nomizu [24].

The torsion invariant $T_{Ad}$ was constructed by Porti [25] as a combinatorial twisted Reidemeister torsion of $M$, where the twisting comes from the action of $\pi_1(M)$ on $\mathfrak{sl}_2\mathbb{C}$ via $Ad\varphi$. In general, adjoint torsion invariants can be interpreted as top degree differential forms on the regular locus of character varieties [25, 9] of special linear or projective groups. Using the
isomorphism $\mathcal{X} \cong \Gamma(E)$ we are able to express $T_{\text{Ad}}(M)$ in terms of cellular, simplicial or Čech cochains with coefficients in $\mathcal{X}$. In particular, the normalization of torsion introduced by Porti can be recovered from our geometric interpretation and from Theorem 1.2.

1.4. Application to the 1-loop Conjecture. Using the methods of mathematical physics, Dimofte and Garoufalidis [7] constructed a formal power series $Z_T(h)$ associated to a geometric ideal triangulation $T$ of a hyperbolic knot complement $M = S^3 \setminus K$. The coefficients in the power series $Z_T(h)$ are complex numbers defined as weighted sums over Feynman diagrams with an increasing number of loops, and consequently called ‘$n$–loop’ coefficients. They are determined by the combinatorics of $T$ and by the shape parameter solutions of the gluing equations; see [11] for an extended discussion. It is not known whether the $n$–loop coefficients are topological invariants of $M$ for $n > 0$.

A focal point of [7] is the conjectural ‘1–loop invariant’ $\tau_T \in \mathbb{C}/\{\pm 1\}$ determined by the 1–loop coefficient of $Z_T(h)$. If the power series $Z_T(h)$ recovers the asymptotic expansion of the Kashaev invariant [17] at least to the first order, then the Generalized Volume Conjecture [13] would predict that the 1–loop invariant coincides with the torsion $T_{\text{Ad}}(M, \mu)$ corresponding to the knot-theoretic meridian $\mu$. The equality $\tau_T = T_{\text{Ad}}(M, \mu)$ is the content of the 1-loop Conjecture [7] Conjecture 1.8, which we state as Conjecture 4.3 below.

If true, the 1–loop Conjecture would provide a particularly simple and explicit formula for $T_{\text{Ad}}$ in terms of a geometric ideal triangulation. We show that the main factor of this formula comes out naturally from the factorization of torsion induced by (1.2). This computation, presented in Section 5 below, implies in particular the non-vanishing of $\tau_T$.

Compared to the original statement in [7], our version of the 1–loop Conjecture is adapted to work with any system of homotopically non-trivial simple closed curves, not necessarily knot meridians. Hence, we can generalize the conjecture to all triangulated, orientable one-cusped hyperbolic 3–manifolds. Finally, we verify the generalized conjecture for the minimal triangulation of the figure-eight sister manifold ($m003$ in the SnapPea cusped census). This manifold is not a complement of a knot in an integral homology sphere.

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2. Hyperbolic gluing equations

2.1. Ideal triangulations and Thurston’s equations. Let $M$ be an orientable connected 3–manifold homeomorphic to the interior of a compact manifold $\overline{M}$ whose boundary is a union of $k > 0$ tori. Suppose that $T$ is an ideal triangulation of $M$ with $N$ tetrahedra. By Euler characteristic considerations, the number of edges of $T$ is also $N$. We fix an arbitrary numbering of the tetrahedra and of the edges of $T$ by integers $\{1, \ldots, N\}$. We also label the toroidal ends with integers $\{1, \ldots, k\}$. The hyperbolic gluing equations [28, 22] on $T$ can then be written as

$$
\prod_{j=1}^{N} z_j^{G_{ij}} z_j^{G''_{ij}} z_j^{G'''_{ij}} = 1 \quad \text{for } i \in \{1, \ldots, N\} = \{\text{edge indices}\},
$$

(2.1)

where $z_j, z'_j = 1/(1 - z_j)$ and $z''_j = 1 - 1/z_j$ are the three shape parameters associated to the $j$th tetrahedron (see Figure 2.1). We assemble the incidence numbers occurring as exponents in (2.1) into the integer matrices $G = [G_{ij}], G' = [G''_{ij}], G''' = [G'''_{ij}]$.

As discussed in [22], the equations expressing the completeness of a hyperbolic structure can be written in a similar form. Suppose that $\theta = \{\theta_i\}_{i=1}^k$ is a collection of homotopically non-trivial oriented simple closed peripheral curves, one at each end of $M$. Then the logarithmic form of
the completeness equations is
\begin{equation}
\sum_{j=1}^{N} C_{ij} \log z_j + C'_{ij} \log z'_j + C''_{ij} \log z''_j = 0, \text{ for } l \in \{1, \ldots, k\} = \{\text{indices of ends}\}.
\end{equation}

In the above equation, “log” denotes the standard branch of the logarithm on the upper halfplane \(\mathbb{C}_{\text{Im} > 0} = \{z \in \mathbb{C} : \text{Im} z > 0\}\) and the coefficients depend on the chosen representative of the free homotopy class of \(\theta\) in normal position with respect to \(\mathcal{T}\) by the way of \(\mathcal{G}\). As before, we assemble these coefficients into \(k \times N\) integer matrices \(C = [C_{ij}], C' = [C'_{ij}]\) and \(C'' = [C''_{ij}]\).

It is well known \(\mathcal{G}\) that Neumann–Zagier \([22]\) impose completeness conditions on a collection of oriented curves forming a \(\mathbb{Z}\)-basis of \(\mathcal{H}_1(\partial \mathcal{M}, \mathbb{Z})\). However, if the triangulation is positively oriented, then a result of Choi \([3]\, \text{Corollary 4.14}\) implies that it suffices to impose the completeness condition on only one nontrivial curve per end.

2.2. The tangential gluing complex. For the remainder of the section, we assume that \(\mathcal{T}\) admits a positively oriented solution \(z_* \in \mathbb{C}_{\text{Im} > 0}\) which recovers the unique complete hyperbolic structure on \(M\). We now summarize certain results concerning the derivatives of the gluing equations, due to Neumann–Zagier \([22]\) and Choi \([3]\).

Define the map \(g = g_{\mathcal{T}}\) by the left-hand sides of \((2.1)\):
\begin{equation}
g: \mathbb{C}^N_{\text{Im} > 0} \rightarrow (\mathbb{C}^*)^N, \quad g(z_1, \ldots, z_N) = \left(\prod_{j=1}^{N} z_j^{G_{ij}} z'_j^{G'_{ij}} z''_j^{G''_{ij}}\right)_{i=1}^{N}.
\end{equation}

The set \(\mathcal{V}_+^T := g^{-1}(1)\) is then called the positive gluing variety of the triangulation \(\mathcal{T}\). For every \(i \in \{1, \ldots, N\}\) and \(l \in \{1, \ldots, k\}\), denote by \(e_i\) the \(i\)th edge and by \(v_l\) the \(l\)th ideal vertex of \(\mathcal{T}\). Let \(K_{ij} \in \{0, 1, 2\}\) be the number of ends of \(e_i\) incident to \(v_l\), without any regard for orientations.

Following Choi \([3]\), we may define the monomial map \(p\) by
\begin{equation}
p: (\mathbb{C}^*)^N \rightarrow (\mathbb{C}^*)^k, \quad p(x_1, \ldots, x_N) = \left(\prod_{i=1}^{N} x_i^{K_{i}}\right)^{k}.
\end{equation}

Choi then proves \([3]\, \text{Theorem 3.4}\) that for any \(z \in \mathcal{V}_+^T\), there is an exact sequence
\begin{equation}
0 \rightarrow T_z \mathcal{V}_+^T \rightarrow T_z \mathbb{C}^N_{\text{Im} > 0} \xrightarrow{D_p} T_1(\mathbb{C}^*)^N \xrightarrow{D_p} \mathbb{C}^k \rightarrow 0
\end{equation}
given by the holomorphic derivatives of the maps defined above, where the last non-zero term \(T_1(\mathbb{C}^*)^k\) has been trivially identified with \(\mathbb{C}^k\). Denote by \(u_l = u_l(z)\) the log-parameter along the peripheral curve \(\theta_l\) for \(l \in \{1, \ldots, k\}\); explicitly, we have
\begin{equation}
u_l(z_1, \ldots, z_N) = \sum_{j=1}^{N} C_{ij} \log z_j + C'_ij \log z'_j + C''ij \log z''_j.
\end{equation}

Neumann–Zagier \([22]\, \text{§ 4}\) proved that there exists a neighbourhood of \(z\) in \(\mathcal{V}_+^T\) on which the log-parameters \(\{u_l\}_{l=1}^{k}\) form a holomorphic coordinate chart. Denote by \(y = y_0\) the inverse of this chart:
\begin{equation}
y: U \rightarrow \mathcal{V}_+^T, \quad y(u_1, \ldots, u_k) = (z_1(u_1, \ldots, u_k), \ldots, z_N(u_1, \ldots, u_k)),
\end{equation}
where \(U \subset \mathbb{C}^k\) is a sufficiently small neighbourhood of the origin; note that \(y(0) = z_*\) by definition. With these notations, we may replace \((2.5)\) with the exact sequence
\begin{equation}
0 \rightarrow T_u U \xrightarrow{D_y} T_{y(u)} \mathbb{C}^N_{\text{Im} > 0} \xrightarrow{D_y} T_1(\mathbb{C}^*)^N \xrightarrow{D_p} \mathbb{C}^k \rightarrow 0.
\end{equation}
2.3. Coordinates on character varieties via gluing equations. We now wish to summarize the tangential properties of the parametrizations of character varieties induced by the hyperbolic shapes. We fix the orientation of the tangential properties of the parametrizations of character varieties induced by the hyperbolic structure. For every edge e_i of T, 1 \leq i \leq N, we may choose an open tubular neighbourhood \nu_i of e_i in such a way that \nu_i \cap \nu_j = \emptyset for i \neq j.

**Definition 2.1.** We define M_0 = M_0(T) = M \setminus \bigcup_{i=1}^N \nu_i.

We remark that the space M_0 is called a ‘manifold with defects’ in [8]. In general, M_0 \subset M is a handlebody which deformation-retracts onto the union of ‘tetrapod’ graphs inscribed into the tetrahedra of T, as depicted on the right panel of Figure 2.1. A geometric version of this construction was used in [4] to study ideal triangulations.

Since the handlebody M_0 does not contain any edges of T, every collection of positively oriented shape parameters \( z = (z_1, \ldots, z_N) \in C_{im}^{N>0} \) determines a hyperbolic structure on M_0; this structure extends to M if and only if \( z \in V^+_T \). Let \( \varrho_z \in \text{Hom}(\pi_1(M_0), PSL_2\mathbb{C}) \) be a monodromy representation of the hyperbolic structure induced on M_0 by the shape parameters z. While \( \varrho_z \) itself is only defined up to conjugation, it makes sense to talk about the image of \( \varrho_z \) in the \( PSL_2\mathbb{C} \) character variety \( X(\pi_1(M_0), PSL_2\mathbb{C}) \). We refer the reader to [12, 13] for more information on character varieties.

**Definition 2.2.** We define

\[
\Sigma_T : C_{im}^{N>0} \to X(\pi_1(M_0), PSL_2\mathbb{C}), \quad \Sigma_T(z) = [\varrho_z],
\]

where \( X(\pi_1(M_0), PSL_2\mathbb{C}) = \text{Hom}(\pi_1(M_0), PSL_2\mathbb{C}) \# PSL_2\mathbb{C} \) is the \( PSL_2\mathbb{C} \)-character variety of \( \pi_1(M_0) \). We also define

\[
\Sigma_{T,0} : U \to X(\pi_1(M), PSL_2\mathbb{C}), \quad \Sigma_{T,0}(u) = \Sigma_T \circ y_0(u).
\]

**Lemma 2.3.** Let \( z_* \in C_{im}^{N>0} \) be a positively oriented solution of edge consistency and completeness equations on T. Then \( z_* \) has a neighbourhood \( V \subset C_{im}^N \) such that \( \Sigma_T(V) \) consists only of regular points. Moreover, \( \Sigma_T|_V \) is analytic.

**Proof.** As \( M_0 \) has the homotopy type of a graph, we see that \( \pi_1(M_0) \cong F_{N+1}, \) the free group of rank \( N+1 \). We assumed that \( M \) is orientable, so \( N \geq 2 \) and thus the rank of \( \pi_1(M_0) \) is at least three. By a result of Heusener–Porti [14] Proposition 5.8, a \( PSL_2\mathbb{C} \)-representation \( \varrho_z \)
maps to a regular point of $X(\pi_1(M_0), PSL_2\mathbb{C})$ if and only if the adjoint representation $\text{Ad}\varrho_\gamma$ is irreducible. Since $\text{Ad}\varrho_\gamma$ is irreducible and the GIT quotient map $\text{Hom}(\pi_1(M_0), PSL_2\mathbb{C}) \to X(\pi_1(M_0), PSL_2\mathbb{C})$ is an analytic submersion at regular points of the character variety, the result follows.

In particular, the above lemma implies that $\Sigma_{T,\theta}^\prime$ is an analytic map, provided that the Dehn surgery parameter space $U \subset \mathbb{C}^k$ is taken to be small enough. This parametrization was first studied by Neumann–Zagier [22, §4].

Lemma 2.4. If $u = (u_1, \ldots, u_k) \in U$ satisfies $0 < |u_l| < \pi$ for all $l$, then the derivative

$$D\Sigma_{T,\theta}^\prime: T_u U \to T_{[\theta]}X(\pi_1(M), PSL_2\mathbb{C}),$$

where $[\theta] = [\varrho_{\theta(u)}]$, is an isomorphism.

Proof. It is well known [22, §4] that the complex dimension of $X(\pi_1(M), PSL_2\mathbb{C})$ at the discrete faithful representation is equal to the number $k$ of cusps of $M$. In a small neighbourhood of the point $[\varrho_{\theta(u)}] \in X(\pi_1(M), PSL_2\mathbb{C})$, the chosen multicurve $\theta$ defines local holomorphic coordinates via squared traces $\text{tr}^2(\theta_l), 1 \leq l \leq k$; cf [13, 15]. In terms of $u$, we have $\text{tr}^2(\theta_l) = 4\cosh^2\frac{u_l}{\pi}$, showing that these coordinates diagonalize $D\Sigma_{T,\theta}^\prime$. We compute $\frac{\partial}{\partial u_l} \text{tr}^2(\theta_l) = 2\sinh(u_l)$, which is non-zero whenever $0 < |u_l| < \pi$. Hence, $D\Sigma_{T,\theta}^\prime$ is an isomorphism at these points. 

3. The cohomological content of the gluing equations

3.1. The long exact sequence associated to an ideal triangulation. Consider $M$ with a hyperbolic structure obtained from a positively oriented solution $z \in \mathcal{V}_T^+$ and denote by $\mathcal{K}$ the sheaf of germs of Killing vector fields on $M$. Since $M_0$ is a closed subspace of $M$, we obtain a short exact sequence of sheaves $0 \to \mathcal{K}_{M,M_0} \to \mathcal{K} \to \mathcal{K}_{M_0} \to 0$, whose associated long exact sequence in cohomology has the form

$$0 \longrightarrow H^0(M; \mathcal{K}_{M,M_0}) \longrightarrow H^0(M; \mathcal{K}) \longrightarrow H^0(M; \mathcal{K}_{M_0}) \longrightarrow H^1(M; \mathcal{K}_{M,M_0}) \longrightarrow H^1(M; \mathcal{K}) \longrightarrow H^1(M; \mathcal{K}_{M_0}) \longrightarrow H^2(M; \mathcal{K}_{M,M_0}) \longrightarrow H^2(M; \mathcal{K}) \longrightarrow H^2(M; \mathcal{K}_{M_0}) \longrightarrow 0. \tag{3.1}$$

Lemma 3.1 (Proposition 4.1.2 in [26]). We have

$$H^0(M; \mathcal{K}_{M,M_0}) = H^0(M; \mathcal{K}) = H^0(M; \mathcal{K}_{M_0}) = H^1(M; \mathcal{K}_{M,M_0}) = H^1(M; \mathcal{K}_{M_0}) = H^2(M; \mathcal{K}_{M,M_0}) = H^2(M; \mathcal{K}_{M_0}) = 0. \tag{3.2}$$

Proof. Since $M$ has a discrete group of isometries, we have $H^0(M; \mathcal{K}) = 0$ which also implies the vanishing of $H^0(M; \mathcal{K}_{M,M_0})$. $H^1(M; \mathcal{K}_{M,M_0})$ vanishes since the sheaf $\mathcal{K}_{M,M_0}$ is supported on the disjoint, contractible open sets $\nu_i$. The isomorphism $H^2(M; \mathcal{K}_{M_0}) = 0$ follows from the fact that $M_0$ has the homotopy type of a graph. Finally, $H^0(M; \mathcal{K}_{M_0})$ can be shown to vanish by calculating the algebraic Euler characteristic of (3.1). We refer the reader to [26] for more details.

Corollary 3.2. For every $z \in \mathcal{V}_T^+$ and the corresponding hyperbolic structure on $M$, we have an exact sequence

$$0 \to H^1(M; \mathcal{K}) \to H^1(M; \mathcal{K}_{M_0}) \xrightarrow{\Delta} H^2(M; \mathcal{K}_{M,M_0}) \to H^2(M; \mathcal{K}) \to 0. \tag{3.3}$$

The following theorem establishes a relationship between the infinitesimal gluing equations (2.1) and the exact sequence (3.3).

Theorem 3.3. Let $T$ be an ideal triangulation of an open manifold $M$ in which all links of the ideal vertices are tori and let $\theta = \{\theta_1, \ldots, \theta_k\}$ be a system of nontrivial oriented curves, one in each vertex link. Assume that the gluing equations (2.1) and (2.2) admit a positively oriented
solution. Then there exist maps $\alpha$, $\beta$ and a neighbourhood $U \subset \mathbb{C}^k$ of the origin such that for any log-parameter $u = (u_1, \ldots, u_k) \in U$ satisfying $0 < |u_l| < \pi$ for all $l$, the following diagram is commutative with exact rows and columns.

\[
\begin{array}{cccccccc}
0 & \longrightarrow & T_u U & \xrightarrow{D_y} & T_{y(u)} \mathbb{C}^N_{\text{Im} > 0} & \xrightarrow{D_g} & T_1(\mathbb{C}^*)^N & \xrightarrow{D_p} & \mathbb{C}^k & \longrightarrow & 0 \\
& \downarrow & D\Sigma_{T, \theta} & & \downarrow & D\Sigma_T & & \downarrow & \beta & & \\
0 & \longrightarrow & H^1(M; \mathcal{X}) & \xrightarrow{\Delta} & H^1(M; \mathcal{X}_{M_0}) & \longrightarrow & H^2(M; \mathcal{X}_{M,M_0}) & \longrightarrow & \mathbb{C}^k & \longrightarrow & 0 \\
& \downarrow & & & \downarrow & & & \downarrow & & & \\
0 & \longrightarrow & \text{Coker } D\Sigma_T & \xrightarrow{\alpha} & \text{Coker } \alpha & \longrightarrow & 0 & & & & \\
& & & & & & & & & & \\
& 0 & & & 0 & & & & & & \\
\end{array}
\]

Moreover, $\alpha$ and $\beta$ are unique.

The above theorem states in particular that we may view the derivative $Dg$ of the consistency equations (2.1) as the essential part of the connecting homomorphism $\Delta$ in the cohomology long exact sequence (3.3). We defer the proof until Section 3.4.

3.2. Cohomological meaning of complex lengths. In this section, we explain the cohomological meaning of the dependence of the completeness equations (2.2) on the chosen multicurve $\theta$. Assume that $\theta$ and $\tilde{\theta}$ are two non-trivial peripheral multicurves in normal position with respect to the triangulation $T$ and that $u = u(z)$ and $\tilde{u} = \tilde{u}(z)$ are their respective log-parameters. As before, we consider the local coordinates $y = y_\theta$ and $\tilde{y} = y_{\tilde{\theta}}$ on a neighbourhood of $z_* \in V_T^+$. Then $\tilde{u} \circ y: U \to \mathbb{C}^k$ expresses the log-parameter of $\tilde{\theta}$ in terms of the log-parameter of $\theta$, generalizing the situation described in Lemma 4.1 of [22]. Below, we provide a cohomological interpretation of the derivative of this map.

**Theorem 3.4.** For every $u = (u_1, \ldots, u_k) \in U$ satisfying $0 < |u_l| < \pi$ for all $l$, we have the commutative diagram

\[
\begin{array}{ccc}
T_u U & \xrightarrow{D(\tilde{u} \circ y)} & H^1(M; \mathcal{X}) \\
& D\Sigma_{T, \theta} & \downarrow \varpi \tilde{\theta}^* \\
& \mathbb{C}^k & \xrightarrow{\beta} & H^2(M; \mathcal{X}) \\
\end{array}
\]

where $\beta$ is the map of Theorem 3.3. In the above diagram, $[\tilde{\theta}]^* \in H^1(\partial M, \mathbb{Z})$ denotes the Poincaré dual of the homology class of $\tilde{\theta}$.

The above theorem is proved in Section 3.5.

3.3. Construction of the embedding. At present, we are going to construct the unique map $\alpha$ which makes Theorem 3.3 hold.

**Definition 3.5.**

(i) For any orientable manifold $M$, we denote by $\text{Or}(M)$ the set consisting of the two possible orientations of $M$.

(ii) Let $L$ be a simple geodesic in a symmetric Riemannian manifold $M$ and let $\nu(L) \subset M$ be an open tubular neighbourhood of $L$. For any orientation $\varepsilon \in \text{Or}(L)$ of $L$, we denote by $t(L, \varepsilon) \in \mathcal{X}(\nu(L))$ the unique local Killing vector field which acts as a unit-speed infinitesimal translation along $L$ in the direction of $\varepsilon$.

**Remark 3.6.** The existence and uniqueness of $t(L, \varepsilon)$ follow from the work of Nomizu [24, §2]. It is easy to see that $t(L, -\varepsilon) = -t(L, \varepsilon)$, where $-\varepsilon$ is the orientation opposite to $\varepsilon$. 


Example 3.7. Consider the geodesic $L \subset \mathbb{H}^3$ connecting the points $0, \infty \in CP^1 = \partial_\infty \mathbb{H}^3$ and let $\varepsilon$ be the orientation of $L$ from $0$ towards $\infty$. Although $t(L, \varepsilon)$ is defined a priori only on a neighbourhood of $L$, the fact that $\mathbb{H}^3$ is simply connected allows us to continue $t(L, \varepsilon)$, as a Killing field, unambiguously onto all of $\mathbb{H}^3$. Using the identification $\mathcal{K}(\mathbb{H}^3) = \mathfrak{sl}_2 \mathbb{C}$, we can write $t(L, \varepsilon)$ as a traceless $2 \times 2$ matrix with complex entries. To find this matrix, observe that for $s \in \mathbb{R}$, the Möbius transformation $z \mapsto e^sz$ acts on $L$ as a translation by $s$ units towards $\infty$. Hence,

\[(3.4) \quad t(L, \varepsilon) = \frac{d}{ds} \begin{bmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{bmatrix}_{s=0} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathfrak{sl}_2 \mathbb{C}.\]

For all edges $e_i$ of the ideal triangulation $T$, $1 \leq i \leq N$, consider the neighbourhoods $\nu_i$ of Definition 2.1. Since the sheaf $\mathcal{K}$ of Definition 2.1. Since the sheaf $(\mathcal{K}(M_0, M)) = \prod_{i=1}^N H^2(M; \mathcal{K}(M_0, M_i))$, where $M_i = M \setminus \nu_i$.

By excision, to calculate the cohomology groups on the right-hand side it suffices to consider, for each $i$, an embedded disc $D_i$ transverse to $e_i$ and satisfying $D_i \cap \nu_i = \text{int} D_i$. Using cellular cochains with int $D_i$ as a 2-cell, we immediately see that any element of $H^2(M; \mathcal{K}(M_0, M))$ is fully determined by its value on the oriented disc $D_i$. In other words, given an orientation $d_i \in \text{Or}(D_i)$, cohomology classes in $H^2(M; \mathcal{K}(M_0, M))$ are in a one-to-one correspondence with local sections of $\mathcal{K}$ on $\nu_i$. Observe that an edge orientation $\varepsilon_i \in \text{Or}(e_i)$ determines a dual orientation $d_i \in \text{Or}(D_i)$ by the requirement that $d_i \wedge \varepsilon_i$ agrees with the orientation of the ambient manifold $M$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N) \in \prod_i \text{Or}(e_i)$ be an arbitrary choice of orientations of the edges of $T$. Applying the foregoing reasoning to all edges of $T$, we obtain the isomorphism

\[(3.5) \quad \psi_\varepsilon : \prod_{i=1}^N \mathcal{K}(\nu_i) \xrightarrow{\cong} H^2(M; \mathcal{K}(M_0, M)).\]

We write $x_1, \ldots, x_N$ for the coordinates on $(\mathbb{C}^*)^N$; hence, the complex tangent space $T_1(\mathbb{C}^*)^N$ is spanned by the vectors $\partial/\partial x_1, \ldots, \partial/\partial x_N$. Define the map $\omega_\varepsilon$ by

\[(3.6) \quad \omega_\varepsilon : T_1(\mathbb{C}^*)^N \rightarrow \prod_{i=1}^N \mathcal{K}(\nu_i), \quad \omega_\varepsilon \left( \frac{\partial}{\partial x_i} \right) = t(e_i, \varepsilon_i) \text{ for all } 1 \leq i \leq N.\]

We set $\alpha = \psi_\varepsilon \circ \omega_\varepsilon$. By Remark 3.6, $\alpha$ does not depend on $\varepsilon$. We claim that this is the map needed in Theorem 3.3.

3.4. Proof of Theorem 3.3. We are now going to prove Theorem 3.3. We refer to [29] for more details and illustrations.

Proof of Theorem 3.3. Observe that the leftmost square in the top part of the diagram is commutative by the definition of $\Sigma_{T, \theta}$. We shall now prove the commutativity of the central square, i.e., the equality $\alpha \circ Dg = \Delta \circ D\Sigma_{T}$. We fix edge orientations $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N) \in \prod_i \text{Or}(e_i)$ arbitrarily. As discussed in the preceding section, $\varepsilon_i$ determines a dual orientation of a transverse disc $D_i$ for every $i$. Denote by $\gamma_i$ the oriented boundary of $D_i$. There exists a local orientation-preserving coordinate chart which identifies $e_i$ with the oriented infinite geodesic $L = (0, \infty) \subset \mathbb{H}^3$ of Example 3.7. Any such chart also identifies the space of Killing fields on $\nu_i$ with $\mathcal{K}(\mathbb{H}^3) = \mathfrak{sl}_2 \mathbb{C}$. In these coordinates, the monodromy of the hyperbolic structure along $\gamma_i$ is a homothety $H_s(w) = sw$ whose ratio $s = g_i(z)$ is the product of shape parameters of the tetrahedra incident to $e_i$; in particular, $s = 1$ when $z = z_0 \in \mathcal{V}_T$. We choose a local logarithm $\ell_i(z)$ so that $g_i(z) = \exp(\ell_i(z))$ and $\ell_i(z_0) = 0$ for a given $z_0 \in \mathcal{V}_T$. Then for any $z$ sufficiently close to $z_0$, the monodromy along $\gamma_i$ can be written in matrix form as

$$
\mu_i(z) = \pm \begin{bmatrix} \exp(\ell_i(z)/2) & 0 \\ 0 & \exp(-\ell_i(z)/2) \end{bmatrix} \in \text{PSL}_2 \mathbb{C}.
$$
Using Weil’s method \([30]\) and performing a calculation similar to (3.4), we see that the infinitesimal variation of the monodromy along \(\gamma_1\) with respect to \(z_j\) is given by

\[
\frac{\partial}{\partial z_j} \mu_i(z) \bigg|_{z=z_0} \mu_i^{-1}(z_0) = \frac{1}{2} \frac{\partial \ell_i(z)}{\partial z_j} \bigg|_{z=z_0} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{\partial g_i(z)}{\partial z_j} \bigg|_{z=z_0} t(e_i, e_i),
\]

where the last equality uses \(g_i(z_0) = 1\). Using the isomorphism (3.5) and the fact that \(\gamma_i = \partial D_i\), we see that the right-hand side of (3.7) is the \(i\)th component of \(\psi_{\varepsilon}^{-1}(\Delta(D \Sigma_T(\partial/\partial z_j)))\). On the other hand, using (3.6) we get

\[
\psi_{\varepsilon}^{-1}(\alpha(D g(\partial/\partial z_j))) = \omega_z \left( \sum_{i=1}^{N} \frac{\partial g_i(z)}{\partial z_j} \frac{\partial}{\partial x_i} \right) = \sum_{i=1}^{N} \frac{\partial g_i(z)}{\partial z_j} t(e_i, e_i).
\]

At \(z = z_0\), the \(i\)th term of the above sum is exactly the right-hand side of (3.7). Hence, the middle square commutes.

Using surjectivity of \(Dp\) and a standard diagram chase, we may now define \(\beta\) to be the unique map for which the right-hand side commutes. In this way, all squares in the top part have been shown commutative.

By Lemma 2.4, the map \(D \Sigma_T, \beta\) is an isomorphism. Moreover, \(\alpha\) is injective by definition. The Four Lemma now implies that \(D \Sigma_T\) is injective. Since both the first and second rows are exact, it follows that \(\beta\) is surjective and hence an isomorphism. \(\square\)

3.5. Complex lengths of peripheral curves. The boundary at infinity \(\partial M\) can be pushed into \(M\), yielding a disjoint union of embedded tori \(T_1 \cup \cdots \cup T_k\), numbered according to the chosen numbering of the ends of \(M\). When \(M\) is equipped with either the complete hyperbolic structure or a small deformation of it, we have the natural isomorphisms

\[
H^2(M; \mathcal{K}) \cong H^2(\partial M; \mathcal{K} \otimes \mathcal{P}) \cong \prod_{i=1}^{k} H^2(T_i; \mathcal{K}).
\]

It turns out that in the incomplete case, a basis of \(H^2(T_i; \mathcal{K})\) can be constructed geometrically.

**Lemma 3.8** (cf. [26 Proposition 4.4.1]). Let \(T\) be a torus about an incomplete end \(v\) of an oriented hyperbolic 3–manifold \(M\) and let \(R\) be a geodesic ray traveling into \(v\) with the orientation \(\varepsilon \in \text{Or}(R)\) pointing towards \(v\). Equip \(T\) with a cell decomposition containing a single 2–cell \(S\) and orient \(S\) positively using the orientation of \(M\). Then the 2–cochain mapping \(S\) to \(t(R, \varepsilon)\) defines a non-trivial free homotopy class \([t]\) \(\in H^2(T; \mathcal{K}_T)\). Moreover, \([t]\) does not depend on the choice of the ray \(R\).

**Proof.** Since the projective structure on \(T\) reduces to an affine structure [16 Lemma 2], the local system \(\mathcal{K}_T\) can be understood in terms of the adjoint of the monodromy representation \(\pi_1(T) \rightarrow \text{Aff}(\mathbb{C})\), where \(\text{Aff}(\mathbb{C})\) is embedded into \(\text{PSL}_2 \mathbb{C}\) as the upper-triangular Borel subgroup.

As in Example 3.7, \(t(R, \varepsilon)\) is then written as the traceless diagonal matrix \(\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \in \text{sl}_2 \mathbb{C}\). This reduces the proof to an elementary computation, which can be found in [26 pp. 94-95]. \(\square\)

Suppose that \(T\) is a torus about an incomplete end of \(M\) and that \(\gamma \subset T\) is an oriented simple closed curve representing a non-trivial free homotopy class in \(T\). Then the monodromy \(\mu(\gamma)\) of the hyperbolic structure along \(\gamma\) can be conjugated into the form

\[
\mu(\gamma) = \pm \begin{bmatrix} e^{\mathcal{L}/2} & 0 \\ 0 & e^{-\mathcal{L}/2} \end{bmatrix},
\]

where \(\mathcal{L}\) is called the complex length of \(\gamma\). Assuming \(\mathcal{L} \not\in 2\pi i \mathbb{Z}\), Bromberg [11 p. 25] defines an isomorphism

\[
B : H^1(\gamma; \mathcal{K}_\gamma) \xrightarrow{\cong} \mathbb{C}
\]

which sends a cohomology class to the corresponding infinitesimal variation of \(\mathcal{L}\). Observe that \(\mathcal{L}\) is not defined uniquely by \(\mu(\gamma)\) alone: even after choosing a branch of the logarithm, swapping the eigenvalues will replace \(\mathcal{L}\) with \(-\mathcal{L} + \text{const}\). Hence, the derivative of \(\mathcal{L}\) is defined \textit{a priori}
only up to sign. On the other hand, when \( \gamma \) is in normal position with respect to a positively oriented geometric triangulation \( \mathcal{T} \), the log-parameter \( u = u_\gamma \) provides a particular choice of \( \mathcal{L} \). Hence, by setting \( \mathcal{L} = u \) locally, we obtain a convenient choice of the isomorphism \( B \) which we characterize below.

**Lemma 3.9.** If \( \gamma \subset T \) is an oriented curve as above, we have a commutative diagram

\[
\begin{array}{ccc}
H^1(T; \mathcal{X}_T) & \overset{\gamma^*}{\longrightarrow} & H^2(T; \mathcal{X}_T) \\
\downarrow & & \downarrow \gamma \mapsto 1 \\
H^1(\gamma; \mathcal{X}_\gamma) & \overset{B}{\longrightarrow} & \mathbb{C},
\end{array}
\]

where \( \gamma^* \in H^1(T; \mathbb{Z}) \) is the Poincaré dual of the homology class of \( \gamma \) and \( \{t\} \) is the element constructed in Lemma 3.8.

**Proof.** We can choose a fundamental quadrilateral \( Q \subset C \) for \( T \) in such a way that one of the sides of \( Q \) is a lift of \( \gamma \) with the basepoint at one of the vertices. Moreover, we may place the basepoint vertex of \( Q \) at \( 0 \in C \). This conjugates the monodromy representation \( \mu: \pi_1(T) \to \text{Aff} \mathbb{C} \) into the form (3.8). If \( L \subset M \) is a hyperbolic geodesic intersecting \( T \) orthogonally at the basepoint, then it is clear that \( \mu(\gamma) \) acts on \( L \) by a translation of \( \text{Re} \mathcal{L} \) and rotation through the angle \( \text{Im} \mathcal{L} \). In other words, if \( L \) is taken to the line \((0, \infty) \subset \mathbb{H}^3 \) in the upper-halfspace model, \( \mu(\gamma) \) acts as the Möbius transformation \( z \mapsto e^{\mathcal{L}}z \).

Suppose now that a cohomology class \( h \in H^1(T; \mathcal{X}_T) \) is tangent to a holomorphic 1–parameter family of projective structures with monodromy \( \mu_s(\gamma) = (z \mapsto e^{\mathcal{L}(s)}z + c(s)) \), so that \( \mathcal{L} = \mathcal{L}(0) \). Denote by \( R \) the ray formed by points of \( L \) on the thin side of \( T \). Using Lemma 3.8 with this choice of \( R \), we see that \( h \sim [\gamma]^* = \frac{d\mathcal{L}(s)}{ds} \big|_{s=0} \{t\} \) and the result follows.

**Proof of Theorem 3.4.** Observe that Theorem 3.4 follows easily from Lemma 3.9 once we prove that the composition

\[
\mathbb{C}^k \overset{\beta}{\longrightarrow} H^2(M; \mathcal{X}) \to H^2(M; \mathcal{X}_{\partial M})
\]

sends the \( l \)th standard unit vector of \( \mathbb{C}^k \) to the cohomology class \( \{t_l\} \in H^2(T_l; \mathcal{X}_{T_l}) \) constructed in Lemma 3.8 on the \( l \)th boundary torus \( T_l \). To see that this is indeed the case, observe that the Jacobian matrix of the monomial map \( p \) of (2.4) is \( K = [K_{li}] \), where \( K_{li} \) was defined in Section 2.3 as the unsigned number of ends of the edge \( e_i \) incident to the \( l \)th end of \( M \). Hence, using the commutative diagram in Theorem 3.3, it suffices to show that for every \( i \), the basis vector \( \frac{\partial}{\partial x_i} \in T_1(\mathbb{C}^*)^N \) is sent by the composition

\[
r_i: T_1(\mathbb{C}^*)^N \overset{\partial}{\longrightarrow} H^2(M; \mathcal{X}_{M,M_i}) \to H^2(M; \mathcal{X}) \to H^2(T_l; \mathcal{X}_{T_l})
\]

to \( K_{li}[t_l] \). To compute \( r_i(\frac{\partial}{\partial x_i}) \), split the edge \( e_i \) into two rays \( R_1 \) and \( R_2 \) with orientations \( \varepsilon_1, \varepsilon_2 \) pointing outwards (towards infinity). Then \( \omega_{\varepsilon_n}(\frac{\partial}{\partial x_i}) = t(R_n, \varepsilon_n) \) for \( n = 1, 2 \). Hence, if \( t_l \) is the ideal vertex of \( \mathcal{T} \) whose link is \( T_l \), then

\[
r_i(\frac{\partial}{\partial x_i}) = \sum_{\text{ends of } e_i \text{ incident to } t_l} \{t_l\} = K_{li}[t_l],
\]

as desired.

\[\square\]

**4. Combinatorial Reidemeister torsion of 3-manifolds**

**4.1. Review of algebraic torsion.** In this section, we briefly summarize the definition of the algebraic torsion of a cochain complex, referring the reader to [3] [29] for more details.

Suppose that \( V \) is a vector space of finite dimension \( n \) over a field \( \mathbb{F} \). Any ordered basis \( b = (b_1, \ldots, b_n) \) of \( V \) determines a non-zero vector \( \text{vol}(b) := \wedge^n_{r=1} b_r \in \wedge^n V \) in the top-degree exterior power of \( V \). When the elements of \( b \) are not ordered, \( \text{vol}(b) \) is only defined up to sign. Similarly, when \( b' \subset V \) is another (unordered) basis, the ratio \( \text{vol}(b)/\text{vol}(b') \) is a scalar well
defined up to sign, which can be computed in practice as the determinant of a change-of-basis matrix from $b$ to $b'$.

Let $C^* = (0 \to C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{d-2}} C^{d-1} \xrightarrow{\delta^{d-1}} C^d \to 0)$ be a finite-dimensional cochain complex over $\mathbb{F}$ in which each cochain group $C^i$ is equipped with a preferred basis $\mathcal{C}^i$. Since we are working over a field, the short exact sequence

$$0 \to Z^i \xrightarrow{\partial^i} C^i \xrightarrow{\delta^i} B^{i+1} \to 0$$

always has a splitting $s^i: B^{i+1} \to C^i$. Denote by $h^i \subset H^i = Z^i/B^i$ an arbitrarily fixed basis of the $i$th cohomology group of $C^*$. Note that given any collection of bases $h^i \subset B^i$ for each $i$, we can form a new basis of $C^i$ defined as $h^i \cup \tilde{h}^i \cup s^i(b^{i+1})$, where $\tilde{h}^i$ consists of cocycles representing the cohomology classes of the elements of $h^i$. The combinatorial torsion of $C^*$ is then defined as

$$(4.1) \quad T(C^*, \mathcal{C}, h^*) = \pm \prod_{i \text{ even}} \frac{\text{vol}(h^i \cup \tilde{h}^i \cup s^i(b^{i+1}))}{\text{vol}(\mathcal{C}^i)} \prod_{i \text{ odd}} \frac{\text{vol}(\mathcal{C}^i)}{\text{vol}(h^i \cup \tilde{h}^i \cup s^i(b^{i+1})))} \in \mathbb{F}^*/\{\pm 1\},$$

and only depends on the choice of the bases $\mathcal{C}$ and $h^*$.

When the above construction is applied to a cellular cochain complex of a topological space, the resulting combinatorial invariant of cell complexes is called the combinatorial Reidemeister torsion. For further generalization to modules over non-commutative rings, see Milnor [19].

We shall also need the notion of compatible bases introduced in [19]. Suppose that

$$(4.2) \quad 0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$$

is a short exact sequence of finite-dimensional vector spaces. We say that the bases $\mathcal{A} \subset A$, $\mathcal{B} \subset B$, $\mathcal{C} \subset C$ are compatible if the torsion of (4.2) with respect to these bases equals $\pm 1$. More generally, if $A$, $B$ and $C$ are cochain complexes and $\iota$, $\pi$ cochain maps, we say that the bases $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are graded-compatible if their degree $d$ parts $\mathcal{A}^d$, $\mathcal{B}^d$, $\mathcal{C}^d$ are compatible for every $d$.

4.2. Definition of the adjoint torsion. The adjoint hyperbolic torsion $T_{Ad}$ was first defined by Porti in [25]. While Porti’s original treatment was in terms of homology groups, our approach using cohomology is equivalent [29].

Let $M$ be as in Section 2.1 and let $\varrho$ be a monodromy representation of the hyperbolic structure on $M$. We equip $M$ with an arbitrary finite CW-decomposition and consider the finite-dimensional cellular cochain complex $C^*(M; E) \cong C^*(M; \mathfrak{sl}_2 \mathbb{C}^{Ad} \varrho)$ with twisted $\mathfrak{sl}_2 \mathbb{C}$-coefficients. This complex can be constructed as $\text{Hom}_{\mathbb{Z}[\pi_1(M)]}(C^*(M; \mathbb{Z}), \mathfrak{sl}_2 \mathbb{C})$, where $\pi_1(M)$ acts on the universal covering space $\tilde{M}$ by deck transformations and on $\mathfrak{sl}_2 \mathbb{C}$ via $\text{Ad} \varrho$. As stated in [25, 26], the cohomology groups of this complex are $H^1(M; \mathfrak{sl}_2 \mathbb{C}^{Ad} \varrho) \cong H^2(M; \mathfrak{sl}_2 \mathbb{C}^{Ad} \varrho) \cong C^k$ and vanish in all other degrees.

In order to take the combinatorial torsion of $C^*(M; \mathfrak{sl}_2 \mathbb{C}^{Ad} \varrho)$, we equip it with a geometric basis which can be constructed as follows. Let $\mathcal{B} = \{b_1, b_2, b_3\} \subset \mathfrak{sl}_2 \mathbb{C}$ be an arbitrarily chosen basis. The cellular structure of $M$ determines a $\pi_1(M)$-invariant cell decomposition of $\tilde{M}$. For any oriented cell $s$ in $M$, choose a lift $\tilde{s}$ of $s$ to $\tilde{M}$ and form the three cochains $c_{s,r}$ ($r = 1, 2, 3$) defined by $c_{s,r}(\tilde{s}) = b_r$ and $c_{s,r}(f) = 0$ if the cell $f$ is not a lift of $s$. Then define

$$(4.3) \quad \mathcal{C}_{\text{com}}^* = \bigcup_{s: \text{a cell of } M} \{c_{s,1}, c_{s,2}, c_{s,3}\} \subset C^*(M; \mathfrak{sl}_2 \mathbb{C}^{Ad} \varrho),$$

which is easily seen to be a basis.

**Definition 4.1.** Let $\gamma = (\gamma_1, \ldots, \gamma_k) \subset \partial \overline{M}$ be a multicurve consisting of oriented, homotopically non-trivial simple closed curves, one in each torus component of $\partial \overline{M}$. 

(i) A cohomology basis $h^1 \cup h^2 \subset H^1(M; \mathfrak{s}l_2 \mathbb{C}^{Ad}) \oplus H^2(M; \mathfrak{s}l_2 \mathbb{C}^{Ad})$ is said to be balanced with respect to $\gamma$ if the images of $h^1$ and $h^2$ under the compositions

\[
\begin{array}{ccc}
h^1 \subset H^1(M; \mathfrak{s}l_2 \mathbb{C}^{Ad}) & \rightarrow & H^1(\partial M; \mathfrak{s}l_2 \mathbb{C}^{Ad}) \\
\downarrow \gamma^* & & \\
h^2 \subset H^2(M; \mathfrak{s}l_2 \mathbb{C}^{Ad}) & \rightarrow & H^2(\partial M; \mathfrak{s}l_2 \mathbb{C}^{Ad})
\end{array}
\]

coincide. In the above diagram, the horizontal maps are induced by restriction of local systems and the vertical map is given by the cup product with the Poincaré dual $[\gamma]^* \in H^1(\partial M; \mathbb{Z})$ of the homology class of $\gamma$.

(ii) The adjoint Reidemeister torsion of $(M, \gamma)$ is defined by

\[ \mathbb{T}_{Ad}(M, \gamma) := \mathbb{T}(C^\bullet(M; \mathfrak{s}l_2 \mathbb{C}^{Ad}), \mathfrak{c}^\bullet_{geom}; h^\bullet(\gamma)) \in \mathbb{C}^*/\{\pm 1\}, \]

where $\mathfrak{c}^\bullet_{geom}$ is any geometric basis constructed by the way of (4.3) and $h^\bullet(\gamma)$ is any cohomology basis balanced with respect to $\gamma$.

It is well known that the quantity $\mathbb{T}_{Ad}(M, \gamma)$ does not depend on the choice of a geometric basis [22]. Note that Part (i) of the above definition is an adaptation of Porti’s original construction to twisted cochain complexes.

4.3. Geometric construction of geometric bases. We shall now indicate how to reformulate Definition 4.1 in terms of the sheaf $\mathcal{K}$ of germs of Killing vector fields on $M$. As mentioned in the Introduction, the local system $\mathfrak{s}l_2 \mathbb{C}^{Ad}$ is isomorphic to the sheaf $\mathcal{K}$. Hence, Part (ii) of Definition 4.1 does not require any adaptations beyond replacing $\mathfrak{s}l_2 \mathbb{C}^{Ad}$ with “$\mathcal{K}$” throughout.

The definition of a geometric basis (4.3) can also be restated in a simple way which we now describe.

Given an ordered basis $b = (b_1, b_2, b_3) \subset \mathfrak{s}l_2 \mathbb{C}$. (1.5) implies that the non-zero element $\text{vol}(b) \in \Lambda^3 \mathfrak{s}l_2 \mathbb{C}$ defines a non-vanishing section of the bundle $\Lambda^3 E$. Since the adjoint action of $PSL_2 \mathbb{C}$ is unimodular, $\Lambda^3 E$ is trivial as a flat bundle and $\text{vol}(b)$ can be viewed as a global, constant section of $\Lambda^3 E$.

According to (4.3), a geometric basis contains, for every oriented cell $s$, three cochains $c_{s,r}$ ($r = 1, 2, 3$) such that $\Lambda_r c_{s,r}(\tilde{s}) = \text{vol}(b)$ as sections of $\Lambda^3 E$. Using Steenrod’s definition [27] of cellular cochains with coefficients in a local system, we can interpret this equality in terms of the cellular cochain complex $C^\bullet(M; \mathcal{K})$. For every oriented cell $s$, consider three germs $X_r \in \mathcal{K}_x$ ($r = 1, 2, 3$) such that $\bigwedge_r X_r = \text{vol}(b)_X$, where $x \in s$ is an arbitrarily chosen “representative point”. It is clear that replacing $c_{s,r}$ with cochains $c'_{s,r}$ mapping $s$ to $X_r$ for every $s$ produces a basis $\mathfrak{c}' \subset C^\bullet(M; \mathcal{K})$.

For every degree $d$, the identification of local systems $\mathfrak{s}l_2 \mathbb{C}^{Ad} \simeq \mathcal{K}$ induces an isomorphism of top exterior powers $\bigwedge C^d(M; \mathfrak{s}l_2 \mathbb{C}^{Ad}) \xrightarrow{\cong} \bigwedge C^d(M; \mathcal{K})$. It is easy to see that this isomorphism relates $\mathfrak{c}_{geom}^d$ to the degree $d$ part of $\mathfrak{c}_s'$. Hence, we have the equality of torsions $\mathbb{T}(C^\bullet(M; \mathcal{K}), \mathfrak{c}_s', b) = \mathbb{T}(C^\bullet(M; \mathfrak{s}l_2 \mathbb{C}^{Ad}), \mathfrak{c}^\bullet_{geom}, h)$. In order to make the construction of the germs $X_r$ more geometric, fix a cell $s$ of $M$ and a representative point $x \in s$. Let $U \subset M$ be a neighbourhood of $x$ and let $\varphi: U \rightarrow \mathbb{H}^3$ be an orientation-preserving geometric coordinate chart. Then $\varphi$ establishes an isomorphism $\mathcal{K}_x \cong \mathcal{K}_{\varphi(x)}$ of germ spaces and hence an isomorphism of their top exterior powers $\Lambda^3 \mathcal{K}_x \cong \Lambda^3 \mathcal{K}_{\varphi(x)}$. Since a germ of a Killing field at a point of $\mathbb{H}^3$ extends to a unique global Killing field, we obtain an isomorphism $\Phi: \bigwedge^3 \mathcal{K}_x \xrightarrow{\cong} \bigwedge^3 \mathfrak{s}l_2 \mathbb{C}$. By unimodularity of the adjoint action of $PSL_2 \mathbb{C} = \text{Isom}^+(\mathbb{H}^3)$, we see that $\Phi$ does not depend on the choice of the geometric chart $\varphi$. Moreover, once $\Phi$ is defined at an $x \in s$, it can be uniquely continued to any other point of $s$.

In conclusion, we arrive at the following equivalent definition of a geometric basis of the cellular cochain complex $C^\bullet(M; \mathcal{K})$. Fix a basis $b = (b_1, b_2, b_3)$ of $\mathfrak{s}l_2 \mathbb{C} \cong \mathcal{K}(\mathbb{H}^3)$. For any oriented cell $s$
of \( M \) and any representative point \( x \in s \), choose an arbitrary orientation-preserving geometric chart \( \varphi \) as above and define

\[
(4.4) \quad \mathbb{C}^\text{geom}_{s,\text{com}} = \bigcup_{s: \text{ a cell of } M} \bigcup_r \{ s \mapsto (\varphi^{-1})_*(\partial_r) \} \subset C^*(M; \mathcal{X}),
\]

where \( (\cdot)_* \) denotes the push-forward of vector fields. We remark that a similar coordinate-free interpretation of geometric bases exists for unimodular flat bundles of any rank \([26, 23]\).

4.4. Generalized 1-loop Conjecture. Assume for the entirety of this section that \( T \) is a geometric ideal triangulation of a hyperbolic 3–manifold \( M \) with one cusp \( (k = 1) \). Adopting notations of normal surface theory, we shall often write \( z_j \) for the unique of the three shape parameters \( z_j, z'_j, z''_j \) labeling the edges of the \( j \)th tetrahedron of \( T \) which are separated by the normal quadrilateral \( \square \). We extend this notation to all quantities corresponding bijectively to normal quadrilaterals.

**Definition 4.2.** A strong combinatorial flattening on \( T \) is a vector \((f, f', f'') \in (\mathbb{Z}^N)^3\) satisfying the equations

\[
\begin{align*}
(4.5) \quad Gf + G'f' + G''f'' & = (2, 2, \ldots, 2)^	op, \\
(4.6) \quad f + f' + f'' & = (1, 1, \ldots, 1)^	op, \\
(4.7) \quad Cf + C'f' + C''f'' & = 0,
\end{align*}
\]

whenever the rows of the matrices \( C_i \) contain the coefficients of the completeness equations \([2.2]\) along any nontrivial peripheral curves.

By linearity, it suffices to check condition \([4.7]\) on a \( Z \)-basis of \( H_1(\partial M; \mathbb{Z}) \). If \( M \) is a knot complement in \( S^3 \), one may use the knot-theoretic meridian–longitude pair, as is done in \([7]\).

The ‘1–loop Conjecture’ of Dimofte–Garoufalidis \([7, \text{Conjecture 1.8}]\) can be stated using the notations of Section 2.1 as follows. Let \( \theta \) be an oriented, homotopically nontrivial simple closed peripheral curve in normal position with respect to the triangulation \( T \). Define the \( N \times N \) integer matrices \( \hat{G}, \hat{G}', \hat{G}'' \) by

\[
(4.8) \quad \hat{G}_{ij}^C = \begin{cases} 
G_{ij}^C & \text{when } 1 \leq i \leq N - k \\
C_{i-N+k,j} & \text{when } i > N - k,
\end{cases}
\]

where \( C_i \) contains the coefficients of the completeness equation \([2.2]\) along \( \theta \). Since we assumed \( k = 1 \), the matrices \( \hat{G}_i \) differ from \( C_i \) only in their last rows but see Remark 4.4-(4) below. For any \( j \in \{1, \ldots, N\} \), we define the following rational functions of \( z \in \mathbb{C}^N_{\text{Im} > 0} \):

\[
(4.9) \quad \zeta_j(z) = \frac{d \log z_j}{dz_j} = \frac{1}{z_j}, \quad \zeta'_j(z) = \frac{d \log z'_j}{dz_j} = \frac{1}{1 - z_j}, \quad \zeta''_j(z) = \frac{d \log z''_j}{dz_j} = \frac{1}{z_j(z_j - 1)}.
\]

**Conjecture 4.3** (The 1-loop Conjecture). For any strong combinatorial flattening \((f, f', f'')\),

\[
(4.10) \quad T_{\text{Ad}}(M, \theta) = \pm \frac{\det(\hat{G} \text{ diag}(\zeta) + \hat{G}' \text{ diag}(\zeta') + \hat{G}'' \text{ diag}(\zeta'')))}{2^k \zeta^I \zeta'^I \zeta''^I}.
\]

In the formula \([4.10]\), the manifold \( M \) is considered with the hyperbolic structure defined by the shape parameters \( z = (z_1, \ldots, z_N) \in \mathcal{V}^T \) which enter the right-hand side via the functions \( \zeta_i(z) \) of \([4.9]\). The denominator uses multi-index notation.

**Remark 4.4.**

(1) We remark that our statement of the 1–loop Conjecture extends the original conjecture of Dimofte–Garoufalidis \([7]\) to the case of arbitrary 1–cusped hyperbolic manifolds and arbitrary nontrivial peripheral curves.

(2) The conjectural formula in \([7]\) is given in terms of the matrices \( A = \hat{G} - \hat{G}' \) and \( B = \hat{G}'' - \hat{G}' \), whence it can be easily transformed into the symmetric form presented above.
(3) Dimofte–Garoufalidis use a somewhat weaker concept of a combinatorial flattening and prove that the right-hand side of (4.10) does not depend on the choice of the flattening. The work of Neumann implies that a strong flattening in the sense of Definition 1.2 always exists, so there is no loss of generality in using strong flattenings in the statement.

(4) Since \( k = 1 \), the term \( 2^k \) in the denominator of (4.10) simply equals 2. Under certain additional assumptions on \( T \), a generalization of the 1-loop Conjecture to the case of multiple toroidal ends \( (k > 1) \) is discussed in [29]. This generalization involves the factor \( 2^k \) and the matrices \( \hat{G}^\square \) defined by (4.8) with a general \( k \).

5. Geometric computation of the torsion

5.1. Factorization of torsion with respect to an ideal triangulation. A geometric ideal triangulation \( T \) of a finite-volume hyperbolic 3–manifold \( M \) defines a finite cell complex \( X \) dual to \( T \). As usual, the 0–cells of \( X \) are in a bijective correspondence with the tetrahedra of \( T \). There is a 1–cell in \( X \) for every face of the triangulation and a 2–cell for every edge. In this way, the 2–dimensional CW–complex \( X \) represents the homotopy type of \( M \).

We shall use the cochain complex \( C^*(X; \mathcal{X}) \) to calculate the adjoint torsion \( T_{\text{Ad}}(M, \theta) \). Observe that the subspace \( \mathcal{M}_0 \subset M \) corresponds to the 1–skeleton \( X(1) \subset X \), so that the long exact sequence (5.3) can be constructed in cellular cohomology of \( \mathcal{X} \) as the long exact sequence of the pair \( (X, X(1)) \).

We fix the basis \( \{ \varepsilon, h, f \} \subset \mathfrak{sl}_2 \mathbb{C} \), where \( \varepsilon = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}] \), \( h = \frac{1}{2} [\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}] \), \( f = [\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] \). We are now going to construct a particularly convenient class of geometric bases \( \varepsilon_{\text{geom}} \) of the cellular cochain complex \( C^*(X; \mathcal{X}) \). Given a 2–cell \( s_i \) of \( X \) dual to an edge \( e_i \) of \( T \), choose a local coordinate chart \( \varphi_i \) which maps the edge \( e_i \) to the geodesic \( (0, \infty) \subset \mathbb{H}^3 \). We then define a geometric basis of \( C^2(X; \mathcal{X}) \) using (4.4) with \( \varphi = \varphi_i \) whenever \( \varepsilon = \varepsilon_i \), \( 1 \leq i \leq N \). Thanks to the factor of \( \frac{1}{2} \) in the definition of \( h \), equation (5.4) shows that the local Killing field \( (\varphi_i^{-1})_*(h) \) equals \( t(e_i, \varepsilon) \), where the orientation \( \varepsilon \in \text{Or}(e_i) \) is dual to the orientation of \( s_i \). From now on, we assume that the geometric basis \( \varepsilon_{\text{geom}} \) is constructed as above. We do not impose any special requirements on the charts \( \varphi \) for cells in dimensions zero and one.

Applying Milnor’s Multiplicativity Theorem [19, Theorem 3.2] to the short exact sequence of cochain complexes given by the pair \((X, X(1))\), we obtain the decomposition

\[
T_{\text{Ad}}(M, \theta) = T(C^*(X; \mathcal{X}), \varepsilon_{\text{geom}}, h^\bullet(\theta))
\]

\[
\cong T(C^*(X(1); \mathcal{X}), \varepsilon_{\text{geom}}^{(0)}, h_0^1(\theta)) \oplus T(C^*(X(1); \mathcal{X}), \varepsilon_{\text{geom}}^{(2)}, h_{rel}^1 \oplus h_{rel}^2, \theta),
\]

where \( h^\bullet(\theta) \) is any cohomology basis balanced with respect to \( \theta \), \( h_0^1 \) is any basis of \( H^1(X(1); \mathcal{X}) = H^1(M; \mathcal{X}_{M_*}) \) and \( h_{rel}^1 \) and \( h_{rel}^2 \) are any basis of \( H^2(X(1); \mathcal{X}) = H^2(M; \mathcal{X}_{M_*}) \). In the above formula, the symbol \( h^\bullet \) denotes the cohomology long exact sequence (3.3). Note that the bases \( h_0^1 \) and \( h_{rel}^1 \) occurring in (5.1) can be chosen arbitrarily, but we shall indicate a particularly good choice in Section 5.2.

Suppose that the log-parameter \( u = u_0 \) satisfies \( 0 < |u_0| < \pi \) for all \( l \), so that the incomplete hyperbolic structure on \( M \) is a small deformation of the unique complete structure. By Theorem 3.3 we can apply Milnor’s theorem to the last term of (5.1), obtaining

\[
T(H^\bullet, h^\bullet(\theta) \oplus h_0^1 \oplus h_{rel}^1, \theta) = T(G^\bullet, q^\bullet, \theta) \oplus T(\text{Coker } D\Sigma_T \to \text{Coker } \alpha, q^\bullet, \theta),
\]

where \( G^\bullet \) is the ‘gluing complex’ (2.7) and the bases \( q^\bullet, q^\bullet \) must be chosen so as to satisfy the graded compatibility assumptions of Milnor’s theorem. We construct such bases in the next section.

5.2. Construction of graded-compatible bases. We wish to equip the gluing complex \( G^\bullet \) of (2.7) with a basis \( q^\bullet \) given by the partial derivatives with respect to the standard coordinates. More precisely, take \( 1 \in \mathbb{C} \) as the basis vector of the last term \( \mathbb{C}^k = \mathbb{C} \) and pick the standard basis vectors \( \partial/\partial u \in T_uU, \partial/\partial z_1, \ldots, \partial/\partial z_N \in \mathbb{T}_0(\mathbb{C}^N) \). This defines the basis \( q^\bullet \) which we use in (5.2).
Observe that the maps $D_{\Sigma_T,\theta}$ and $\beta$ of Theorem 3.3 define cohomology basis vectors
\begin{equation}
(5.3)
D_{\Sigma_T,\theta}(\frac{\partial}{\partial \tau}) \in H^1(M; \mathcal{X}) \quad \text{and} \quad \beta(1) \in H^2(M; \mathcal{X}).
\end{equation}
An application of Theorem 3.3 with $\bar{\theta} = \theta$ immediately implies that the vectors \((5.3)\) form a cohomology basis balanced with respect to the curve $\theta$ in the sense of Definition 4.1. We therefore define $h^\bullet(\theta)$ to consist of the elements \((5.3)\).

Next, we define the basis $h^2_{\text{rel}} \subset H^2(M; \mathcal{X}_{\text{geom}})$ as the image of $\zeta_{\text{geom}}^{(2)}$ under the canonical isomorphism $C^2(X, X^{(1)}; \mathcal{X}) \cong H^2(M; \mathcal{X}_{\text{geom}})$. This choice of bases ensures that
\begin{equation}
(5.4)
T(C^\bullet(X, X^{(1)}; \mathcal{X}), \zeta_{\text{geom}}^{(2)}, h^2_{\text{rel}}) = \pm 1.
\end{equation}
By our assumption on the choice of the local charts $\varphi_i$, we see that the set $\{\alpha(\frac{\partial}{\partial \tau})\}_{i=1}^N$ is a subset of $h^2_{\text{rel}}$. The remaining part of $h^2_{\text{rel}}$ must therefore descend to a basis of $\text{Coker } \alpha$, which we denote by $q^\bullet$. In this way, the three bases $\{\alpha(\frac{\partial}{\partial \tau})\}$, $h^2_{\text{rel}}$ and $q^\bullet$ are compatible.

Observe that the cokernel complex $D_{\Sigma_T} \xrightarrow{[\Delta]} \text{Coker } \alpha$ consists of only two non-zero terms with the isomorphism $[\Delta]$ induced via Theorem 3.3 from the connecting homomorphism $\Delta$. Hence, it makes sense to set $q^1 = [\Delta]^{-1}(q^2)$. This choice of $q^\bullet$ ensures that
\begin{equation}
(5.5)
T(\text{Coker } D_{\Sigma_T} \xrightarrow{[\Delta]} \text{Coker } \alpha, q^\bullet, \emptyset) = \pm 1.
\end{equation}
Finally, we choose a collection $q_{\text{rel}}^3$ of lifts of the vectors of $q^1$ to $H^1(M; \mathcal{X}_{\text{geom}})$ and set
\begin{equation}
(5.6)
h^1_{\text{rel}} := q^1 \cup \{D_{\Sigma_T}(\frac{\partial}{\partial \tau})\}_{j=1}^N \subset H^1(M; \mathcal{X}_{\text{geom}}).
\end{equation}
This choice guarantees that $h^1_{\text{rel}}, \{\alpha(\frac{\partial}{\partial \tau})\}$ and $q^\bullet$ also satisfy the compatibility condition.

### 5.3. Reduction of the 1-loop Conjecture
Using the graded-compatible bases constructed in the preceding section, we can use the decompositions \((5.1)\) and \((5.2)\) to compute the adjoint hyperbolic torsion $T_{\text{Ad}}(M, \theta) \in \mathbb{C}^*/\{\pm 1\}$. Thanks to \((5.5)\) and \((5.4)\), we can write
\begin{equation}
T_{\text{Ad}}(M, \theta) = T(G^\star, q^\bullet, \emptyset) T(C^\bullet(X^{(1)}; \mathcal{X}), \zeta_{\text{geom}}^{(0,1)}, h^1_{\text{rel}}) =: T_1 T_2.
\end{equation}
We believe that it is possible to express the two factors $T_1$ and $T_2$ in closed form in terms of the so-called \textit{enhanced Neumann–Zagier datum} $(\tilde{G}, \tilde{G}', \tilde{G}'', z, f, f', f'')$. The lemma given below substantiates this belief in the case of $T_1$.

**Lemma 5.1.** $T_1 = \pm \frac{1}{2} \text{det } (\tilde{G} \text{ diag}(\zeta) + \tilde{G}' \text{ diag}(\zeta') + \tilde{G}'' \text{ diag}(\zeta''))$.

The proof can be found in Appendix B. Comparing the expression for $T_1$ given above with the 1–loop formula \((4.10)\), we obtain the following reduction of Conjecture 4.3

**Conjecture 5.2** (Reduced 1–loop Conjecture). Assume $M$ is a connected, orientable, finite-volume hyperbolic 3–manifold with any number $k > 0$ of toroidal ends equipped with a geometric, positively oriented triangulation $\mathcal{T}$. With notations of \((5.6)\), we have
\begin{equation}
(5.7)
T(C^\bullet(X^{(1)}; \mathcal{X}), \zeta_{\text{geom}}^{(0,1)}, h^1_{\text{rel}}) = \pm \zeta^{-f' \zeta'^{-1} - f'' \zeta''^{-1}}.
\end{equation}
Note that the decomposition \((5.7)\) holds \textit{a priori} for incomplete hyperbolic structures obtained as small deformations of the unique complete structure. However, it is shown in [7] that the shape parameters are rational functions on the geometric component $X_0$ of the $PSL_2 \mathbb{C}$–character variety $X(\pi_1(M), PSL_2 \mathbb{C})$. Hence, Lemma 5.1 implies that $T_1$ defines a rational function (up to sign) on a regular neighbourhood of the discrete faithful representation. By a result of Porti [25 Proposition 4.14], the adjoint torsion $T_{\text{Ad}}(M, \theta)$ is also a rational function on $X_0$. Hence, the equality \((5.7)\) guarantees that $T_2$ is rational as well. In particular, the decomposition \((5.7)\) extends to the discrete faithful representation.

**Theorem 5.3.** If $M$, $\mathcal{T}$ and $(f, f', f'')$ satisfy the assumptions of Conjecture 4.3, then
(i) Conjecture 5.2 implies the 1–loop Conjecture \((4.3)\) for all curves $\theta$;
(ii) The conjectural expression \((4.10)\) does not vanish.
Proof. Part (i) follows from the decomposition of torsion discussed above and from the fact that the Reduced Conjecture 5.2 does not involve the choice of a peripheral curve. Part (ii) follows from Lemma 5.1 and the non-vanishing of the monomial (5.8). □

Although it is clear that $T_2$ is a rational function of the shape parameters, it remains to be seen whether it can always be written in the form (5.8). We hope to address this problem in future work.

APPENDIX A. THE SISTER MANIFOLD OF THE FIGURE-EIGHT KNOT COMPLEMENT

We are going to verify the Reduced 1–loop Conjecture 5.2 for the minimal triangulation of the figure-eight sister manifold (SnapPea census designation $m003$). In this appendix, $T$ stands for the two-tetrahedron triangulation with Regina $[2]$ signature $cPcbb bdmx$. The complete hyperbolic structure on $M$ is recovered when both tetrahedra are regular, i.e., $z_1 = z_2 = \frac{1+i\sqrt{3}}{2}$.

We have used the computer algebra system Sage with the module sageRegina by M. Goerner. The calculation presented here uses only symbolic computation and works over rational numbers, which are represented by Sage exactly. Hence, all our results are rigorous.

With Regina’s default conventions, we find the gluing matrices of $T$ to be
\[ G = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad G' = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad G'' = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}. \]

Regina also finds two curves forming a basis of $H_1(\partial M; \mathbb{Z})$; the coefficients of the completeness equations along these curves form the rows of the matrices
\[ C = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \quad C' = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}, \quad C'' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]

Since one edge consistency equation is redundant, the gluing variety of $T$ is the zero locus of the single polynomial
\[ g_{\text{poly}}(z_1, z_2) = z_1^2 z_2^2 - 2z_1^2 z_2 + z_1^2 + z_1 z_2 - z_2. \]

The dual graph $X^{(1)}$ of the triangulation $T$ is depicted in Figure A.1. A neighbourhood of each vertex can be equipped with a geometric coordinate chart which identifies the stalks of $\mathcal{X}$ at these vertices with $\mathfrak{sl}_2 \mathbb{C}$; we choose to trivialize along the cell $F_3$, obtaining the following monodromies of $\mathcal{X}$ along the oriented cells:
\[ \mu_1 = \text{Ad}(H_{z_1^{-1}} C R H_{z_2^{-1}} C R), \quad \mu_2 = \text{Ad}(R C H_{z_2^{-1}} C R), \quad \mu_3 = \text{Id}, \quad \mu_4 = \text{Ad}(C H_{z_2}, R C H_{z_2} C), \]

where we follow $[26]$ in using the fundamental Möbius transformations
\[ C(z) = 1 - z, \quad R(z) = \frac{1}{z}, \quad H_s(z) = sz. \]

With respect to the basis $\{e, h, f\} \subset \mathfrak{sl}_2 \mathbb{C}$, we can write down $3 \times 3$ matrices $A_r$ representing $\mu_r$ for $1 \leq r \leq 4$; this is done automatically by our software.
Using the orientations of 1–cells as in Figure A.1 we see that the matrix of the zero coboundary operator $δ^0 : C^0(X^{(1)}; ℳ) \to C^1(X^{(1)}; ℳ)$ can be given the following block form with $3 × 3$ blocks:

$$δ^0 \sim \begin{bmatrix} -Id & A_1 \\ -Id & A_2 \\ -Id & A_3 \\ -Id & A_4 \end{bmatrix}. \tag{A.4}$$

It remains to find cocycles representing the requisite cohomology basis $h^1_j = \hat{g}^1_j \cup \{DΣ_j^T(\frac{∂}{∂z_j})\}^2_{j=1}$. We use Weil’s method to compute infinitesimal variation of the $PSL_2\mathbb{C}$ monodromies written as words in the generators $[A.3]$. For any shape parameter variable $z \in \{z_1, z_2\}$, we have

$$\frac{d}{ds}H_{z\square}H_{z\square}^{-1}\big|_{s=z} = \zeta^z \mathbf{h},$$

where the symbols $\zeta^z$ were defined in (4.9). In this way, we find representatives of the cohomology classes $DΣ_j^T(\frac{∂}{∂z_j})$ for $j = 1, 2$. Subsequently, we find the lifts $\hat{g}^1_j$ by performing Gauß-Jordan elimination over the coordinate ring $\mathbb{Q}[z_1, z_2]/(g_{\text{poly}})$. By adjoining the cohomology representatives thus found to the right of the matrix $δ^0$ of (A.4), we obtain a $12 × 12$ matrix $A$ over the transcendental extension $\mathbb{Q}(z_1, z_2)$ such that $(\det A)^{-1} = T(\mathcal{C}^\bullet(X^{(1)}; ℳ); \mathbf{h}^1, \mathbf{h}^1)$. We find that $\det A = n(z_1, z_2) |_{z_1, z_2}$, where

$$n(z_1, z_2) = z_1^6 z_2^5 - 5z_1^5 z_2^4 - z_1^4 z_2^3 + 4z_1^2 z_2^2 - 7z_1 z_2 - 12z_1^2 z_2^2 - z_1 z_2^2 + 2z_1^3 z_2^2 + 10z_1^2 z_2 + 2z_1 z_2^2 - 2z_1 z_2 + 2z_1 z_2 + z_2^2 - z_2$$

and

$$d(z_1, z_2) = z_1^2 z_2 - 3z_1 z_2^2 - z_1 z_2^3 + 2z_1^2 z_2^2 + 5z_1 z_2^2 - 6z_1^2 z_2 - z_1^2 z_2^2 + 2z_1^3 + 3z_1^2 z_2 - z_2^2 - 2z_1 z_2 + z_1 + z_2 - 1.$$ 

Sage quickly finds the strong combinatorial flattening $f = (0, 1)$, $f' = (1, 0)$, $f'' = (0, 0)$. Hence, the corresponding $\zeta$–monomial is given by

$$Π_\zeta := \prod_{j=1}^2 ζ_j f'_j ζ_j f''_j = \frac{1}{-z_1 z_2 + z_2}.$$

To test the Reduced 1–loop Conjecture, we check whether the ratio $(\det A)/Π_\zeta$ equals $±1$. Since this ratio is a rational function of the shape parameters $z_1$, $z_2$, it can be written as $\frac{a(z_1, z_2)}{b(z_1, z_2)}$ with $a, b \in \mathbb{Q}[z_1, z_2]$. We find that $a \equiv b$ on the gluing variety, since

$$a(z_1, z_2) - b(z_1, z_2) = -z_1 z_2^2 + 5z_1 z_2^2 - 9z_1 z_2 - 2z_1^2 z_2 + 7z_1^2 z_2 + 6z_1^2 z_2^2 + z_1^3 z_2 - 2z_1^5 - 5z_1^4 z_2^2 + z_1^3 z_2^2 + 5z_1^2 z_2^2 + z_1 z_2^3 - z_1^2 z_2 + 2z_1 z_2 + z_1 z_2^2.$$ 

Therefore, the Reduced 1–loop Conjecture holds on all of $V^+_T$. Using Theorem 5.3 we conclude that the 1–loop Conjecture holds on the entire gluing variety of $T$ for any choice of the peripheral curve $θ$.

APPENDIX B. TORSION OF THE INFINITESIMAL GLUING EQUATIONS

We are going to compute the torsion of the acyclic tangential gluing complex

$$0 \to \mathcal{T}_u U \xrightarrow{D_{\mathcal{T}_u}} \mathcal{T}_u U \mathfrak{C}_{\text{im}>0}^N \xrightarrow{D_{\mathcal{T}_u}} \mathcal{T}_u (\mathcal{C}^\ast)^N \xrightarrow{D_{\mathcal{T}_u}} \mathbb{C} \to 0,$$

with the basis $d^\ast$ consisting of

$$\{\frac{∂}{∂z_j}\} \subset \mathcal{T}_u U, \quad \{\frac{∂}{∂z_j}\}_j^N \subset \mathcal{T}_u U \mathfrak{C}_{\text{im}>0}^N, \quad \{\frac{∂}{∂z_j}\}_1^N \subset \mathcal{T}_u (\mathcal{C}^\ast)^N, \quad \{1\} \subset \mathbb{C}.$$

This calculation is presented in greater detail in [26], including the case of multiple ends.
For the last non-zero term $C$, we choose the basis $b^3 = c^3 = \{1\}$, so that the change-of-basis matrix equals $A_3 = [1]$.

Since $k = 1$, all edges of $T$ have both their ends incident to the only toroidal end of $M$, so the Jacobian matrix of the map $p$ of \( \mathbf{2.3} \) is $[2 \cdots 2]$. Hence, we may choose $\frac{1}{2} \cdot \frac{\partial}{\partial x_i} \in T_1(C^*)^N$ as a pre-image of $1 \in C$ under $Dp$. This vector can be completed to a basis by adjoining the vectors $\{ \frac{\partial}{\partial x_i} \}_i$. Expressing this basis in terms of the original basis $\{ \frac{\partial}{\partial x_i} \}_i$, we obtain

\[
A_2 = \begin{bmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & 0 & 0 \\
-1 & -1 & \ldots & -1 & \frac{1}{2}
\end{bmatrix}
\]

(B.1)

Note that $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} \in \text{Ker} \ Dp = \text{Im} \ Dg$ for every $i$. Hence, there exist vectors $w_i \in T_y(u)C_{\text{im} > 0}^N$ such that $Dg(w_i) = \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}$ for all $i$. By exactness, the set $\{w_1, \ldots, w_{N-1}\} \subset T_y(u)C_{\text{im} > 0}^N$ can be completed to a basis by adjoining the vector $w_N := Dg(\frac{\partial}{\partial m})$. Hence, the change-of-basis matrix $A_1$ has the form $A_1 = [w_1 \ w_2 \ \cdots \ \ w_N]$, where each column marked $w_i$ contains the coefficients of $w_i$ in the basis $\{ \frac{\partial}{\partial x_i} \}_i$. In order to compute the determinant of $A_1$, we need the following lemma.

**Lemma B.1.** The matrix $G \text{diag}(\zeta) + G' \text{diag}(\zeta') + G'' \text{diag}(\zeta'')$ is the Jacobian matrix of the map $g$ of \( \mathbf{2.3} \) at any point $z \in V_T^+$. Similarly, the matrix $\mathcal{C} \text{diag}(\zeta) + C' \text{diag}(\zeta') + C'' \text{diag}(\zeta'')$ is the Jacobian of the log-parameter map $u: C_{\text{im} > 0}^N \to C$.

**Proof.** We calculate the $(i, j)$th entry of the Jacobian of $g$; the proof for $u$ is analogous.

\[
\frac{\partial g_i(z)}{\partial z_j} = \left( G_{ij} z_j G_{ij}^{-1} z_j G_{ij} + \frac{G'_{ij} z_j G_{ij} z_j G_{ij}^{-1} z_j G_{ij} + G''_{ij} z_j G_{ij} z_j G_{ij}^{-1} z_j G_{ij} + G''_{ij} z_j G_{ij} z_j G_{ij}^{-1}}{(z_j - 1)^2} \right) 
\]

\[\times \prod_{m \neq j} z_m G_{im} z_m G_{im} z_m G_{im} G_{im} \]

\[= \left( \frac{G_{ij}}{z_j} + \frac{G'_{ij} z_j G_{ij} z_j G_{ij}^{-1}}{1 - z_j} \right) g_i(z) = G_{ij} \zeta_j + G'_{ij} \zeta_j + G''_{ij} \zeta_j. \]

Consider the matrices $\hat{G}^i$ of \( \mathbf{4.8} \) and define $\mathcal{G} := \hat{G} \text{diag}(\zeta) + \hat{G}' \text{diag}(\zeta') + \hat{G}'' \text{diag}(\zeta'')$. By Lemma B.1, the product $\mathcal{G} A_1$ has the block form

\[
\mathcal{G} A_1 = \begin{bmatrix}
\text{Id}_{(N-1) \times (N-1)} & \mathcal{G} w_N \\
* & \cdot
\end{bmatrix}
\]

(B.2)

Since the top $N - 1$ rows of $\mathcal{G}$ agree with those of the Jacobian of $g$ and $Dg \circ Dw = 0$, we find $\mathcal{G} w_N = [0, 0, \ldots, 0, 1]^T$. This implies that $\mathcal{G} A_1$ is a lower-triangular matrix with ones on the main diagonal, whence $\text{det} A_1 = 1/\text{det} \mathcal{G}$.

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The remaining change-of-basis matrix at $T_u U$ reduces to $A_0 = [1]$, because the preimage of $Dg(\frac{\partial}{\partial m})$ under $Dg$ is tautologically $\frac{\partial}{\partial m}$. Hence, the torsion of the complex \( \mathbf{2.7} \) equals

\[
\frac{\det A_0 \det A_2}{\det A_1 \det A_3} = \frac{1}{2} \text{det} \mathcal{G} = \frac{1}{2} \text{det} (\hat{G} \text{diag}(\zeta) + \hat{G}' \text{diag}(\zeta') + \hat{G}'' \text{diag}(\zeta'')).
\]

This concludes the proof of Lemma 5.1.
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