IDEALS OF QUASI-SYMMETRIC FUNCTIONS AND SUPER-COVARIANT POLYNOMIALS FOR $S_n$

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ABSTRACT. The aim of this work is to study the quotient ring $R_n$ of the ring $\mathbb{Q}[x_1, \ldots, x_n]$ over the ideal $J_n$ generated by non-constant homogeneous quasi-symmetric functions. This article is a sequel to [2], in which is investigated the case of infinitely many variables. We prove here that the dimension of $R_n$ is given by $C_n$, the $n^{th}$ Catalan number. This is also the dimension of the space $SH_n$ of super-covariant polynomials, that is defined as the orthogonal complement of $J_n$ with respect to a given scalar product. We construct a basis for $R_n$ whose elements are naturally indexed by Dyck paths. This allows us to understand the Hilbert series of $SH_n$ in terms of number of Dyck paths with a given number of factors.

1. Introduction

We study, in this paper, a natural analog of the space $H_n$ of covariant polynomials of $S_n$. Letting $I_n$ denote the ideal generated by all symmetric polynomials with no constant term

$$I_n = \langle h_k, \ k > 0 \rangle,$$

where $h_k$ is the $k^{th}$ homogeneous symmetric polynomials (cf. [7]), the space $H_n$ is defined as the orthogonal complement, $I_n^\perp$, in $\mathbb{Q}[x_1, \ldots, x_n]$, of the ideal $I_n$, where the scalar product considered is

$$\langle P, Q \rangle = P(\partial)Q(X) |_{X=0},$$

where $X$ stands for the variables $x_1, \ldots, x_n$, $\partial$ stands for $\partial x_1, \ldots, \partial x_1$, and in the same spirit, $X = 0$ stands for $x_1 = \cdots = x_n = 0$.

Equivalently (cf. [7], Proposition I.2.3), covariant polynomials (also known as $S_n$-harmonic polynomials) can be defined as polynomials $P$ such that $Q(\partial)P = 0$, for any symmetric polynomial $Q$ with no constant term. Since, in particular, elements of $H_n$ satisfy the Laplace equation

$$(\partial x_1^2 + \cdots + \partial x_n^2) P = \Delta P = 0,$$

every covariant polynomial is also harmonic.

Classical results [1, 21] state that the space $H_n$ affords a graded $S_n$-module structure and is isomorphic (as a representation of $S_n$) to the left regular representation. Furthermore, as a graded $S_n$-module, $H_n$ is isomorphic to the quotient

$$Q_n = \mathbb{Q}[X]/I_n,$$

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with $X = (x_1, \ldots, x_n)$. The space $Q_n$ appears naturally in other contexts; for instance, as the cohomology ring of the variety of complete flags \cite{7}. In particular, this implies that
\begin{equation}
\dim H_n = n!.
\end{equation}
Part of the interesting results surrounding the study of $H_n$ involve the fact that it can also be described as the linear span of all partial derivatives of the Vandermonde determinant. This is just a special case of a more general result for finite groups generated by reflections \cite{21}.

By analogy, we consider here the space $SH_n = J_n^\perp$ of super-covariant polynomials, where $J_n$ is the ideal generated by quasi-symmetric polynomials with no constant term. Since the ring of symmetric polynomials is a subring of the ring of quasi-symmetric polynomials, we have $I_n \subseteq J_n$ hence $J_n^\perp \subseteq I_n^\perp$, thus $SH_n \subseteq H_n$,

which somewhat justifies the terminology. Quasi-symmetric polynomials where introduced by Gessel in 1984 \cite{13} and have since appeared as a crucial tool in many interesting algebraico-combinatorial contexts (cf. \cite{1, 2, 18, 19, 20}).

As in the corresponding symmetric setup, we have a graded isomorphism
\begin{equation}
SH_n \simeq R_n = \mathbb{Q}[X]/J_n
\end{equation}
and the approach used in the following work concentrates on this alternate description. We will construct a basis of $R_n$ by giving an explicit set of monomial representatives. As we will show, this set is naturally indexed by Dyck paths of length $n$, hence we obtain the following main theorem.

**Theorem 1.1.** The dimension of $SH_n$ is given by the well known Catalan numbers:
\begin{equation}
\dim SH_n = \dim R_n = C_n = \frac{1}{n+1} \binom{2n}{n}.
\end{equation}
In fact, taking into account the grading (with respect to degree), we have the Hilbert series
\begin{equation}
\sum_{k=0}^{n-1} \dim SH_n^{(k)} t^k = \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} t^k.
\end{equation}

The article is composed of five sections. In Section 2 we recall useful definitions and basic properties. In Section 3 we construct a family $G$ of generators for the ideal $J_n$ and state useful properties of this set. The Section 4 is devoted to the proof of the main Theorem \cite{12}. We construct an explicit basis for $R_n$ which allows us in Section 5 to obtain the Hilbert series of $SH_n$.

### 2. Basic definitions

A composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of a positive integer $d$ is an ordered list of positive integers ($> 0$) whose sum is $d$. We denote this by $\alpha \models d$ and also say that $\alpha$ is a composition of size $d$. The size of $\alpha$ is denoted $|\alpha|$. The integers $\alpha_i$ are the parts of $\alpha$, and the length $\ell(\alpha)$ is set to be the number of parts of $\alpha$. 
There is a natural one-to-one correspondence between compositions of \( d \) and subsets of \( \{1, 2, \ldots, d-1\} \). Let \( S = \{a_1, a_2, \ldots, a_k\} \) be such a subset, with \( a_1 < \cdots < a_k \), then the composition associated to \( S \) is \( \alpha_d(S) = (a_1 - a_0, a_2 - a_1, \ldots, a_k - a_{k-1}) \), where we set \( a_0 := 0 \) and \( a_{k+1} := d \). We denote \( D(\alpha) \) the set associated to \( \alpha \) by this correspondence. For compositions \( \alpha \) and \( \beta \), we say that \( \beta \) is a refinement of \( \alpha \), if \( D(\alpha) \subset D(\beta) \), and denote this by \( \beta \succeq \alpha \).

We will use vectorial notation for monomials. More precisely, for \( \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n \), we denote \( X^\nu \) the monomial \( x_1^{\nu_1}x_2^{\nu_2}\cdots x_n^{\nu_n} \). We further denote \([X^\nu]P(X)\) the coefficient of the monomial \( X^\nu \) in \( P(X) \).

For a vector \( \nu \in \mathbb{N}^n \), let \( c(\nu) \) the composition obtained by erasing zeros (if any) in \( \nu \). A polynomial \( P \in \mathbb{Q}[X] \) is said to be quasi-symmetric if and only if, for any \( \nu \) and \( \mu \) in \( \mathbb{N}^n \), we have \([X^\nu]P(X) = [X^\mu]P(X)\) whenever \( c(\nu) = c(\mu) \). The space of quasi-symmetric polynomials in \( n \) variables is denoted by \( \text{Qsym}_n \). The space \( Q_{sym}^{(d)} \) of homogeneous quasi-symmetric polynomials of degree \( d \) admits as linear basis the set of fundamental quasi-symmetric polynomials indexed by compositions of \( d \). More precisely, for each composition \( \alpha \) of \( d \) with at most \( n \) parts, we set

\[
M_\alpha = \sum_{c(\nu) = \alpha} X^\nu
\]

(2.1)

For the 0 composition, we set \( M_0 = 1 \). Another important linear basis is that of the fundamental quasi-symmetric polynomials (cf. [13]):

\[
F_\alpha = \sum_{\beta \succeq \alpha} M_\beta
\]

(2.2)

with \( \alpha \vdash n \) and \( \ell(\alpha) \leq n \). For example, with \( n = 4 \),

\[
F_{21}(x_1, x_2, x_3, x_4) = M_{21}(x_1, x_2, x_3, x_4) + M_{111}(x_1, x_2, x_3, x_4) = x_1^2 x_2 + x_1 x_2^2 + x_3 + x_1 x_2 x_3 + x_2 x_3 + x_2^2 x_4 + x_2 x_4 + x_3^2 + x_4 + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4.
\]

Part of the interest of fundamental quasi-symmetric functions comes from the following properties. The first is trivial, but very useful and the second comes from the theory of \( P \)-partitions [19, 20].

**Proposition 2.1.** For \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash d \),

\[
F_\alpha(X) = \begin{cases} 
  x_1 F_{(\alpha_1-1,\alpha_2,\ldots,\alpha_k)}(X) + F_\alpha(x_2, \ldots, x_n) & \text{if } \alpha_1 > 1, \\
  x_1 F_{(\alpha_2,\alpha_3,\ldots,\alpha_k)}(x_2, \ldots, x_n) + F_\alpha(x_2, \ldots, x_n) & \text{if } \alpha_1 = 1.
\end{cases}
\]

(2.3)

Let \( u = u_1 \cdots u_\ell \in S_\ell \) and \( v = v_1 \cdots v_m \in S_{\ell+1, \ell+m} \). Let \( u \omega v \) denote the set of shuffles of the words \( u \) and \( v \), i.e. \( u \omega v \) is the set of all permutations \( w \) of \( \ell + m \) such that \( u \) and \( v \) are subwords of \( w \). In particular \( u \omega v \) contains \( \binom{\ell+m}{m} \) permutations.
Let $D(u) = \{ i, u_i > u_{i+1} \}$ denote the descent set of $u$. If $\beta$ and $\gamma$ are the two compositions such that $D(\beta) = D(u)$ and $D(\gamma) = D(v)$, then

**Proposition 2.2** ([20], Exercise 7.93).

\begin{equation}
F_\beta F_\gamma = \sum_{w \in u \cup v} F_{\alpha_{\ell+\mu}(D(w))}.
\end{equation}

There is an evident bijection between elements $\nu$ of $\mathbb{N}^n$ and the corresponding monomial $X^\nu$. Elements of $\mathbb{N}^n$ are naturally called vectors. Just as for compositions, the size $\nu_1 + \cdots + \nu_n$ of $\nu$ is denoted $|\nu|$. It will also be convenient to denote $\ell(\nu)$ the position of its last non-zero component. As usual, $\nu + \mu$ is the componentwise addition of vectors.

To make for easier reading, we generally use $\alpha$, $\beta$, $\gamma$ to denote compositions, and $\mu$, $\nu$ to denote vectors. In general, $n$ the length of vectors (or number of variables) is fixed, and if $w$ is a word of integers (that is an element of $\mathbb{N}^k$ for $0 \leq k \leq n$) we denote by $w0^r = w0^{n-k}$ the vector whose first $k$ parts are the letters of $w$, to which are added $n-k$ 0’s at the end. If $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_m$ are words of integers, the word

$$uv := u_1 \cdots u_k v_1 \cdots v_m$$

is the concatenation of $u$ and $v$. We use the same symbol $\alpha$ for both the composition $(\alpha_1, \ldots, \alpha_\ell)$ and the word $\alpha_1, \ldots, \alpha_\ell$, likewise for vectors.

We next associate to any vector $\nu$ a path $\pi(\nu)$ in the $\mathbb{N} \times \mathbb{N}$ plane with steps going north or east as follows. If $\nu = (\nu_1, \ldots, \nu_n)$, the path $\pi(\nu)$ is

$$(0,0) \to (\nu_1,0) \to (\nu_1,1) \to (\nu_1 + \nu_2,1) \to (\nu_1 + \nu_2,2) \to \cdots$$

$$\to (\nu_1 + \cdots + \nu_n, n-1) \to (\nu_1 + \cdots + \nu_n, n).$$

For example the path associated to $\nu = (2,1,0,3,0,1)$ is

\begin{center}
\begin{tikzpicture}
\fill[gray!20] (0,0) rectangle (6,6);
\fill[gray!40] (1,0) rectangle (4,4);
\fill[gray!60] (2,1) rectangle (3,3);
\fill[gray!80] (2,2) rectangle (2,2);
\fill[gray!100] (3,3) rectangle (3,3);
\end{tikzpicture}
\end{center}

Observe that the height of the path is always $n$, whereas its width is $|\nu|$.

We distinguish two kinds of paths, thus two kinds of vectors, with respect to their “behavior” regarding the diagonal $y = x$. If the path remains above the diagonal, we call it a Dyck path, and say that the corresponding vector is Dyck. If not, we say that the path (or equivalently the associated vector) is transdiagonal. For example $\eta = (0,0,1,2,0,1)$ is Dyck and $\varepsilon = (0,3,1,1,0,2)$ is transdiagonal.
Observe that \( \nu = \nu_1 \cdots \nu_n \) is transdiagonal if and only if there exists \( 1 \leq \ell \leq n \) such that
\[
\ell < \nu_1 + \ldots + \nu_\ell.
\]
(2.5)

Recall that the classical lexicographic order, on monomials of same degree, is
\[
X^\nu \geq_{\text{lex}} X^\mu \text{ iff } \nu \geq_{\text{lex}} \mu,
\]
where we say that \( \nu \) is lexicographically larger than \( \mu \), \( \nu >_{\text{lex}} \mu \), if the first non-zero part of the vector \( \nu - \mu \) is positive. Thus
\[
x_1^3 >_{\text{lex}} x_1^2 x_2 >_{\text{lex}} x_1 x_2^2 >_{\text{lex}} x_2^3
\]
since
\[
(3, 0) >_{\text{lex}} (2, 1) >_{\text{lex}} (1, 2) >_{\text{lex}} (0, 3).
\]
We extend this order to all monomials (of possibly different degree) by setting
\[
X^\nu <_{\text{lex}} X^\mu \text{ whenever } |\nu| < |\mu|.
\]
This is known as the graded lex order, and it clearly makes sense for vectors.

3. The \( \mathcal{G} \) Basis

Following [2], we exploit relations (2.3) to construct a family
\[
\mathcal{G} = \{G_\varepsilon\} \subset \mathcal{J}_n
\]
indexed by vectors that are transdiagonal. For \( \alpha \) any composition of \( k \leq n \), the polynomial \( G_\varepsilon \), with \( \varepsilon := \alpha 0^* \), is defined to be
\[
G_\varepsilon := F_\alpha.
\]
(3.1)

When \( \alpha \neq 0 \), the vector \( \varepsilon = \alpha 0^* \) is clearly transdiagonal. For a general vector \( \varepsilon \) (not of the form \( \alpha 0^* \)), the polynomial \( G_\varepsilon \) is defined recursively in the following way. Let \( \varepsilon = w a \beta 0^* \) be the unique factorization of \( \varepsilon \) such that \( w \) is a word of \( k - 1 \) non-negative integers, \( a > 0 \) is a positive integer, and \( \beta \) is a composition (parts \( > 0 \)). Then we set
\[
G_\varepsilon = G_{wa \beta 0^*} - x_k G_{w(a-1) \beta 0^*}.
\]
(3.2)
Both terms on the right of (3.2) are well defined, and moreover we have
- \( \ell(wa \beta 0^*) = \ell(w(a-1) \beta 0^*) = \ell(\varepsilon) - 1 \);
- \( wa \beta 0^* \) and \( w(a-1) \beta 0^* \) are transdiagonal as soon as \( \varepsilon \) is transdiagonal.
In fact, let $\ell$ be the first ordinate where $\pi(\varepsilon)$ crosses the diagonal, this is to say that it is the smallest integer such that $\ell < \varepsilon_1 + \ldots + \varepsilon_\ell$. Then the second assertion follows from

$$\varphi_1 + \ldots + \varphi_\ell > \psi_1 + \ldots + \psi_\ell = \varepsilon_1 + \ldots + \varepsilon_\ell - 1 > \ell - 1,$$

where $\varphi = wa\beta 0^*$ and $\psi = w(a - 1)\beta 0^*$.

For example,

$$G_{1020} = G_{1200} - x_2 G_{1100}$$

$$= F_{12}(x_1, x_2, x_3, x_4) - x_2 F_{11}(x_1, x_2, x_3, x_4)$$

$$= x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 + x_2 x_3^2 + x_2 x_4^2 + x_3 x_4^2$$

$$+ x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4$$

$$- x_2 (x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4)$$

$$= x_1 x_3^2 + x_1 x_4^2 + x_2 x_3^2 + x_2 x_4^2 + x_3 x_4^2.
$$

Observe on this example that the leading monomial (in graded lex order) of $G_{1020}$ is $X_{1020} = x_1 x_2^2 x_3 x_4^0$. This holds in general for the $G$ family as stated in the following proposition, for which all technical details can be found in [2].

**Proposition 3.1** ([2], Corollary 3.4). The leading monomial $LM(G_\varepsilon)$ of $G_\varepsilon$ is $X^\varepsilon$.

**4. PROOF OF THE MAIN THEOREM**

We now prove our main Theorem [1], and more precisely obtain an explicit basis for the space $R_n$ naturally indexed by Dyck paths, thus of cardinality equal to $C_n$.

**Theorem 4.1.** The set of monomials

$$\mathcal{B}_n = \{X^\eta \mid \pi(\eta) \text{ is a Dyck path}\}$$

is a basis of the space $R_n$.

The proof will be achieved in several steps. We start with the following lemma.

**Lemma 4.2.** Any polynomial $P \in \mathbb{Q}[X]$ is in the linear span of $\mathcal{B}_n$ modulo $\mathcal{J}_n$, which is to say that

$$P(X) \equiv \sum_{X^\eta \in \mathcal{B}_n} c_\eta X^\eta \pmod{\mathcal{J}_n}.
$$

**Proof.** It clearly suffices to show that (4.2) holds for any monomial $X^\nu$, with $\nu$ trans-diagonal. We assume that there exists $X^\nu$ not reducible in the form (4.2) and we choose $X^\varepsilon$ to be the smallest amongst them with respect to the lexicographic order. Let us write

$$X^\varepsilon = LM(G_\varepsilon)$$

$$= (X^\varepsilon - G_\varepsilon) + G_\varepsilon$$

$$\equiv X^\varepsilon - G_\varepsilon \pmod{\mathcal{J}_n}.
$$

All monomial in $(X^\varepsilon - G_\varepsilon)$ are smaller than $X^\varepsilon$, thus they are reducible. This contradicts our assumption on $X^\varepsilon$ and completes our proof. $\square$
Thus $B_n$ spans the space $R_n$. We now prove its linear independence. This is equivalent to showing that the set $G$ is a Gröbner basis of the ideal $J_n$. A crucial lemma is the following one, which is the quasi-symmetric analogue of a classical result in the case of symmetric polynomials (\cite{11}, Theorem II.2.2).

**Lemma 4.3.** If we denote by $L[S]$ the linear span of a set $S$, then
\[((4.3)\ \ \ \ \ Q[X] = L[X^\alpha F_\alpha, \ X^\alpha \in B_n, \ \alpha \models r \geq 0].\]

**Proof.** We have already obtained
\[(X^\epsilon = \sum_{X^\eta \in B_n} c_\eta X^\eta \mod J_n),\]
which is equivalent to
\[(4.4) \ \ \ \ X^\epsilon = \sum_{X^\eta \in B_n} c_\eta X^\eta + \sum_{\alpha \models r \geq 1} Q_\alpha F_\alpha.\]

We then apply the reduction $(4.4)$ to each monomial of the $Q_\alpha$'s and use Proposition 2.2 to reduce products of fundamental quasi-symmetric functions. We obtain $(4.3)$ in a finite number of operations since degrees strictly decreases at each operation, because $\alpha \models r \geq 1$ implies $\deg F_\alpha \geq 1$.

The next lemma is the final step in our proof of the Theorem 4.1.

**Lemma 4.4.** The set $G$ is a linear basis of the ideal $J_n$, i.e.
\[(4.5) \ \ \ \ J_n = L[G_\epsilon | \epsilon \text{ transdiagonal}].\]

**Proof.** Let us denote by $A_n$ the set
\[(4.6) \ \ \ \ A_n = \{X^\xi | x_1^{\xi_1} x_2^{\xi_2-1} \cdots x_n^{\xi_n} \in B_n\} \]

Now the algebra endomorphism of $Q[X]$ that reverses the variables, that is
\[x_i \mapsto x_{n-i+1},\]
clearly fixes the subalgebra $Qsym$. In fact it maps $F_\alpha$ to $F_{\alpha'}$, where $\alpha'$ is the reverse composition.

It follows from Lemma 4.3 that:
\[(4.7) \ \ \ \ Q[X] = L[X^\xi F_\alpha | X^\xi \in A_n, \ \alpha \models r \geq 0].\]

Now to prove Lemma 4.4, we reduce the problem as follows. We first use $(4.7)$ and Proposition 2.2 to write
\[\mathcal{J}_n = \langle F_\alpha, \ \alpha \models s \geq 0 \rangle_{Q[X]} = L[X^\xi F_\alpha F_\beta | X^\xi \in A_n, \ \alpha \models s \geq 0, \ \beta \models t \geq 1] = L[X^\xi F_\gamma | X^\xi \in A_n, \ \gamma \models r \geq 1].\]

It is now sufficient to prove that for all $X^\xi \in A_n$ and all $\gamma \models r \geq 1$
\[(4.8) \ \ \ \ X^\xi F_\alpha \in L[G_\epsilon | \epsilon \text{ transdiagonal}].\]
But Lemma 4.2 implies that any monomial of degree greater than \( n \) is in \( J_n \). Hence to prove (4.8), we need only show it for \( \xi \) and \( \gamma \) such that \(|\xi| + |\gamma| \leq n\). To do that, we reduce the product

\[
x_n \eta_n (x_{n-1} \eta_{n-1} (x_{n-2} \eta_{n-2} (x_{n-3} \eta_{n-3} F_\alpha)))]\]

(recursively, using

\[
x_k G_{w b \beta} = G_{w(b+1)\beta} - G_{w0(b+1)\beta^*}
\]

or

\[
x_k G_{w0^*00^*} = G_{w0^*10^*} - G_{w0^*010^*}.
\]

Relations (4.10) and (4.11) are immediate consequences of the definition of the \( G \) basis (relation (3.2)).

We have to show that the vectors \( \varepsilon \) generated in this process are all transdiagonal and that the length \( \ell(\varepsilon) \) always remains at most equal to \( n \). Let us first check that the transdiagonal part. This is obvious in the case of relation (4.11). In the other case (relation (4.10)), it is sufficient to observe that, for \( \varphi = w b \beta \), if \( m \) is such that

\[
\varphi_1 + \ldots + \varphi_m > m
\]

with \( m > \ell(w) \) (if not, it is evident), then

\[
\varphi'_1 + \ldots + \varphi'_{m_1} > m + 1 > m \quad \text{and} \quad \varphi_1 + \ldots + \varphi_{m+1} > m + 1.
\]

where \( \varphi' = w(b+1)\beta \), and \( \varphi'' = w0(b+1)\beta^* \). We shall now prove that the length of the \( \varepsilon \)'s always remains at most equal to \( n \). For this we need to keep track of the term \( \varepsilon_{\ell(\varepsilon)} \). Two cases have to be considered.

- First case: \( \varepsilon_{\ell(\varepsilon)} \) comes from \( \alpha_{\ell(\alpha)} \) that has been shifted on the right by relation (4.11). It could have made at most \(|\xi|\) steps on the right, whence

\[
\ell(\varepsilon) \leq \ell(\alpha) + |\xi| \leq |\alpha| + |\xi| \leq n.
\]

- Second case: \( \varepsilon_{\ell(\varepsilon)} \) is a 1 generated by relation (4.11) that has been shifted on the right. If it is generated by multiplication by \( x_k \), we consider the vector

\[
\eta = \xi_n \xi_{n-1} \cdots \xi_k 0^*.
\]

Since \( X^\xi \in A_n \) implies \( \pi(\eta) \) is a Dyck path, we have

\[
|\eta| < \ell(\eta) = n - k + 1
\]

hence the 1 generated can be shifted at most in position

\[
k + |\eta| \leq k + n - k = n.
\]
The recursive process used to reduced a product of form (4.9) is illustrated in the following example, where $n = 5$.

$$x_1 x_3 F_{21} = x_3(x_1 F_{21})$$

$$= x_3(G_{31000} - G_{03100})$$

$$= x_3 G_{31000} - x_3 G_{03100}$$

$$= G_{31000} - G_{31010} - G_{03200} + G_{03020}.$$

*End of proof of Theorem 4.1.* By Lemma 4.2, the set $B_n$ spans the quotient $R_n$, and we are now in a position to prove its linear independence. Assume we have a linear dependence relation modulo $J_n$, i.e. there exists $P$

$$P = \sum_{\xi \in B_n} a_{\xi} X_{\xi} \in I_n.$$

By Lemma 4.4, $J_n$ is linearly spanned by the $G_{\varepsilon}$’s, thus

$$P = \sum_{\varepsilon \text{ transdiagonal}} b_{\varepsilon} G_{\varepsilon}.$$

This implies $LM(P) = X_{\varepsilon}$, with $\varepsilon$ transdiagonal, which is absurd. \qed

5. **Hilbert series**

Since Theorem 1.1 gives us an explicit basis for the quotient $R_n$, which is isomorphic to $\text{SH}_n$ as a graded vector space, we are able to refine Theorem 1.1 by giving the Hilbert series of the space of super-covariant polynomials. For $k \in \mathbb{N}$, let $\text{SH}_n^{(k)}$ and $R_n^{(k)}$ denote the projections

$$\text{SH}_n^{(k)} = \text{SH}_n \cap \mathbb{Q}^{(k)}[X] \simeq R_n \cap \mathbb{Q}^{(k)}[X] = R_n^{(k)}$$

(5.1)

where $\mathbb{Q}^{(k)}[X]$ is the vector space of homogeneous polynomials of degree $k$ together with zero. Here, we represent Dyck path horizontally, with $n$ rising steps $(1, 1)$ and $n$ falling steps $(1, -1)$. Let us denote by $D_n^{(k)}$ the number of Dyck paths of length $2n$ ending by exactly $k$ falling steps and by $C_n^{(k)}$ the number of Dyck paths of length $2n$ which have exactly $k$ factors, i.e. $k + 1$ points on the axis. The next figure gives an example of a Dyck path of length 28, ending with 4 falling steps and made of 3 factors.

It is well known that

$$D_n^{(k)} = C_n^{(k)} = \frac{k (2n - k - 1)!}{n! (n - k)!}.$$  

(5.2)
where the first equality is classical (cf. [22] for example for a bijective proof), and the second corresponds to [19], formula (7).

Let us denote by $F_n(t)$ the Hilbert series of $\text{SH}_n$, i.e.

$$F_n(t) = \sum_{k \geq 0} \dim \text{SH}_n^{(k)} t^k. \tag{5.3}$$

**Theorem 5.1.** For $0 \leq k \leq n-1$, the dimension of $\text{SH}_n^{(k)}$ is given by

$$\dim \text{SH}_n^{(k)} = \dim \text{R}_n^{(k)} = D_n^{(n-k)} = C_n^{(n-k)} = \frac{n-k}{n+k} \binom{n+k}{k}. \tag{5.4}$$

For $k \geq n$ the dimension of $\text{SH}_n^{(k)}$ is 0.

**Proof.** By Theorem 4.1, we know that the set

$$\mathcal{B}_n = \{X^\eta \mid \pi(\eta) \text{ is a Dyck path}\}$$

is a basis for $\text{R}_n$. It is then sufficient to observe that the path $\pi(\eta)$ associated to $\eta$ ends by exactly $n - |\eta|$ falling steps. $\square$

| $n$ | $F_n(t)$ |
|-----|----------|
| 1   | 1        |
| 2   | 1 + t    |
| 3   | 1 + 2t + 2t^2 |
| 4   | 1 + 3t + 5t^2 + 5t^3 |
| 5   | 1 + 4t + 9t^2 + 14t^3 + 14t^4 |
| 6   | 1 + 5t + 14t^2 + 28t^3 + 42t^4 + 42t^5 |
| 7   | 1 + 6t + 20t^2 + 48t^3 + 90t^4 + 132t^5 + 132t^6 |

This gives

$$F_n(t) = \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} t^k. \tag{5.5}$$

from which one easily deduces that the generating series for the $F_n(t)$’s is

$$\sum_n F_n(t) x^n = \frac{1 - \sqrt{1 - 4tx - 2t}}{2(t + x - 1)}. \tag{5.6}$$

**Remark 5.2.** The study of various filtrations of the space $\mathbb{Q}[X]$, with respect to family of ideals of quasi-symmetric polynomials, will be the object of a forthcoming paper [3].

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