POSSIBILITY OF GEOMETRIC DESCRIPTION OF QUASIPARTICLES IN SOLIDS

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Abstract

New phenomenological approach for the description of elementary collective excitations is proposed. The crystal is considered to be an anisotropic space-time vacuum with a prescribed metric tensor in which the information on electromagnetic crystalline fields is included. The quasiparticles in this space are supposed to be described by the equations structurally similar to the relativistic wave equations for particles in empty space. The generalized Klein-Gordon-Fock equation and the generalized Dirac equation in external electromagnetic field are considered. The applicability of the proposed approach to the case of conduction electron in a crystal is discussed.

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1 Motivation

The concept of a quasiparticle as a quantum of collective excitation of a many-particle system is of extremely high importance in solid state physics. In many respects the quasiparticle behavior resembles that of usual particle in empty space, but the peculiarity of a crystal has an impact on the dynamics of its elementary excitations. "The properties of any quasiparticles reveal the properties of their areal – crystal, i.e. atoms, molecules, and ions, periodically distributed over space" 1.

Because a quasiparticle in its essence is a result of collective perturbation of all atoms the crystal consists of, the crystal symmetry can not but affect the quasiparticle structure and dynamics, and it seems to be natural to study all elementary collective excitations starting with the most general symmetry considerations.

A model in which a crystal is considered to be a continuous anisotropic medium turns out to be enough simple and fruitful. A general approaches making possible the study of long-wavelength collective excitations in the frame of this model have been developed for elastic waves (and, consequently, for the quanta of elastic excitations - phonons)2 and for spin waves (and their quanta – magnons)3,4.

Being restricted only to translational degrees of freedom, a conduction electron (and a hole) in a crystal with inversion can be described in the long-wave approximation in the frame of a model with effective dynamical mass tensor \( m_{\alpha\beta} \). This tensor involves the unknown internal forces exerting on a quasiparticle by the lattice. The wave function of such quasiparticle is so structured that its small-scale details are smeared due to the averaging: a crystal reveals itself only as some anisotropic physical space with metric tensor \( \gamma_{\alpha\beta} = m_{\alpha\beta}/m \), where \( m \) is a mass of free electron.

Below we shall mainly focus our attention to the case of spin 1/2 quasiparticles. Traditional approach takes into account the crystal anisotropy introducing a potential of periodic crystalline field in non-relativistic Hamiltonian of "bare" electron with relativistic terms required to meet spin and orbital motions. This means that the "dressing up" of an electron begins with the equation resulting from the Dirac equation with built-in spherical symmetry of empty space, perturbed by switching on of electromagnetic field. It is easy to understand that with such approach the breaking of "being on equal terms" between translational and spin motions is introduced \textit{ab initio}. In spite of the fact that for the most part of known effects related with conduction electrons the influence of spin motions is negligibly small, the role of the latters can considerably enhance for strong anisotropic quasi-two-dimensional and quasi-one-dimensional structures. So the problem is what to be understood under spin of a particle (and, naturally, quasiparticle).

1.1 What is Spin of a Quasiparticle?

From one hand we can consider spin to be some internal rotational motion of a point particle. In this case it is simple to imagine a picture of one- and two-dimensional motion
of an electron-quasiparticle, such that its spin magnetic moment equally reacts to the magnetic field in any direction of three-dimensional space. Such a picture is enough popular among physicists inspite that the rotation of a dimensionless point hardly can be understood from the viewpoint of common sense.

On the other hand, one can keep to the point of view proposed by Belinfante \(^7\,^8\) according to which spin of electron is completely similar to the angular momentum of classical circularly polarized wave. In this case the anisotropy of crystalline space has to be \textit{ab initio} taken into account in the equation both in translational and in spin motion of quasiparticle (the generalization of Belinfante’s theorem is given in Appendix). It is obvious that in this case any rotation in one-dimensional motion is out of question, i.e. spin magnetic moment of electron-quasiparticle in one-dimensional case must be equal zero, and in two-dimensional moment only component of spin magnetic moment perpendicular to the plane of motion remains finite.

Here we share the second point of view. We can mention the results of recent experimental works \(^9\,^10\) which count in its favor. They imply that in quasi-twodimensional crystalline systems (such as quantum well and layered crystal) only component of spin magnetic moment of quasiparticles, transversal to the plane of motion, reacts to the external magnetic field.

1.2 Geometrical Approach to Description of Quasiparticles

So we suppose that the influence of crystalline medium anisotropy must be taken into account both in translational and spin motion of quasiparticles. In order to carry out such procedure it is possible to make a natural assumption that the quasiparticle wave function in the above mentioned anisotropic space with metric tensor \(\gamma_{\alpha\beta}(\mathbf{x})\) (containing the unknown information on the crystalline field) satisfy the equation structurally similar to the Dirac equation. In other words, with such approach a \textit{quasiparticle is a particle in anisotropic space} – anisotropic vacuum, which role is fulfilled by crystal. The foundations of such approach were laid in Refs.\(^{11-13}\), where for the sake of simplicity a model of uniform anisotropic space was considered which applicability to crystal is restricted by long-wavelength states. Preliminary results for the case of nonuniform space were published in Ref.\(^{14}\). It should be noted that the phenomenological geometrical considerations were always attractive in physics (see, for example, Wheeler’s geometrodynamics \(^{15}\)).

The plan of the work is as follows. First, keeping in the mind methodical aims, we consider the generalized relativistic equation for quasiparticle with zero spin. Then we discuss the applicability of the effective mass method for space with metric tensor. In section 3 we study the generalized Dirac equation for quasiparticle with spin 1/2. The analysis given in the work and its consequences are discussed in conclusion.
2. Equation for Spin 0 Particle in Anisotropic Space

The Klein-Gordon-Fock equation for spin 0 particle in empty space is

\[
\left(\frac{\hbar}{c}\right)^2 \frac{\partial^2}{\partial t^2} \phi - \hbar^2 \Delta \phi = -m^2 c^2 \phi.
\] (1)

The generalized Klein-Gordon-Fock equation for the space with metric tensor \(g_{ij}(t, x)\) can be written as follows:

\[
\hbar^2 \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x_i} (g_{ij} \sqrt{-g} \frac{\partial}{\partial x_j}) \phi = -m^2 c^2 \phi.
\] (2)

Here \(g \equiv \det(g^{ij})\), and the summation rule over repeated indices \((i, j = 0, 3)\) is supposed. It is obvious that substituting \(g_{ij} = \eta_{ij} \equiv \text{diag}(1, -1, -1, -1)\) in (2), we obtain (1).

Further we shall restrict ourselves with the representation of metric tensor as

\[
g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma_{\alpha\beta} \end{pmatrix}, \quad \alpha, \beta = 1, 3,
\] (3)

where \(\gamma_{\alpha\beta}(x)\) is three-dimensional metric tensor, determining the geometrical properties of the space. Then (2) can be rewritten as follows

\[
\left(\frac{\hbar}{c}\right)^2 \frac{\partial^2}{\partial t^2} \phi - \hbar^2 \frac{\partial}{\sqrt{\gamma}} \left(\gamma_{\alpha\beta} \sqrt{\gamma} \frac{\partial}{\partial x_{\beta}}\right) \phi = -m^2 c^2 \phi,
\] (4)

where \(\gamma \equiv \det(\gamma^{\alpha\beta}) = -g\), and the summation is carried out over repeated indices.

The interaction between particle and external electromagnetic fields, described by potential \(A^i = (\varphi, A)\), is taken into account by changing of the momentum operator \(\hat{p}^i\) for generalized momentum operator \(\hat{P}^i \equiv \hat{p}^i + (e/c)A^i\), where \(e\) is the charge of particle.

Then we obtain from (4)

\[
\left(\frac{\hbar}{c}\right)^2 \left(\frac{\partial}{\partial t} + i e \varphi\right)^2 \phi - \hbar^2 \gamma \left(\gamma_{\alpha\beta} \sqrt{\gamma} \left(\frac{\partial}{\partial x_{\beta}} - i \frac{e}{\hbar c} A^\alpha\right)\right) \phi = -m^2 c^2 \phi.
\] (5)

2.1 Non-relativistic Limit

Let us substitute in (5) \(\phi = \exp(-imc^2t/\hbar)\phi'\) and suppose kinetic and interaction energies of a particle to be small in comparison with the rest energy. Disregarding the
corresponding terms, we obtain:

\[
\frac{i\hbar}{\partial t} \frac{\partial \phi'}{\partial t} = \frac{1}{2m\sqrt{\gamma}} (\dot{P}^\alpha - \frac{e}{c} A^\alpha) \left( \gamma_{\alpha\beta} \sqrt{\gamma} (\dot{P}^\beta - \frac{e}{c} A^\beta) \right) \phi' + e\varphi \phi'.
\]  

(6)

Let us emphasize that the derived equation has the aspect of the Schrödinger equation with Hamiltonian

\[
\mathcal{H} = -\frac{\hbar^2}{2m} \tilde{\Delta} + \frac{e}{mc} \gamma_{\alpha\beta} A^\alpha P^\beta - \frac{ie\hbar}{mc} (\tilde{\nabla}_\beta A^\beta) + \frac{e^2}{2mc^2} \gamma_{\alpha\beta} A^\alpha A^\beta + e\varphi,
\]

(7)

which is Hermitian operator in respect to scalar product \((\phi, \psi) = \int \phi^* \psi \sqrt{\gamma} \, d^3x\). Here we keep the following notations

\[
\tilde{\Delta} \equiv \tilde{\nabla}_\beta \tilde{\nabla}^\beta = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x_\alpha} (\gamma_{\alpha\beta} \sqrt{\gamma} \frac{\partial}{\partial x_\beta})
\]

and

\[
\tilde{\nabla}_\beta \equiv \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x_\alpha} (\gamma_{\alpha\beta} \sqrt{\gamma}).
\]

2.2 Translation-invariant Metrics

Consider the Hamiltonian (7) in the case when the space metric tensor is invariant in respect to some full crystalline group:

\[
\gamma_{\alpha\beta}(x + l) = \gamma_{\alpha\beta}(x),
\]

(8)

\[
A^\mu_\alpha A^\nu_\beta \gamma_{\mu\nu} (\hat{A}^{-1} x) = \gamma_{\alpha\beta}(x),
\]

(9)

where \(l\) is a vector from the translation group of the lattice, and \(\hat{A} = ||A^\alpha_\beta||\) is transformation matrix belonging to the point group of the lattice.

For zero external field one has

\[
\hat{\mathcal{H}}_0 = -\frac{\hbar^2}{2m} \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x_\alpha} (\gamma_{\alpha\beta} \sqrt{\gamma} \frac{\partial}{\partial x_\beta}).
\]

(10)

Due to properties of tensor \(\gamma_{\alpha\beta}(x)\), this Hamiltonian has the symmetry of crystalline lattice: it commutes with all operators from full crystalline symmetry group. Because the full group contains the group of discrete translations, the eigenfunctions of such Hamiltonian can be classified according to the irreducible representations of this group. Let \(\hat{T}_1\) is operator of translation by the lattice vector \(l\), \(\Psi(x)\) is eigenfunction of the Hamiltonian (10) then

\[
\hat{T}_1 \Psi(x) = \Psi(x + l), \quad \hat{\mathcal{H}}_0 \Psi(x) = E \Psi(x),
\]
and
\[ [\hat{T}_1, \hat{H}_0] \equiv \hat{T}_1\hat{H}_0 - \hat{H}_0\hat{T}_1 = 0. \]

It follows from the last equation that
\[ 0 = (\hat{T}_1\hat{H}_0 - \hat{H}_0\hat{T}_1)\psi(x) = \hat{T}_1\hat{H}_0\psi(x) - \hat{H}_0\hat{T}_1\psi(x) \]
i.e.
\[ \hat{H}_0(\hat{T}_1\psi(x)) = \mathcal{E}\hat{T}_1\psi(x). \]

It can be seen from here that \((\hat{T}_1\psi(x))\) in the same way as \(\psi(x)\) is eigenfunction of \(\hat{H}_0\), and the space of the eigenfunctions referring to a given eigenvalue \(\mathcal{E}\), can be expanded into the direct sum of the spaces-supports of irreducible representations of the group of discrete translations. Due to the fact that the translation group of the lattice is commutative, all its representations are one-dimensional, i.e. the eigenfunctions can be chosen in such way that for any \(\psi(x)\): \(\hat{H}_0\psi(x) = \mathcal{E}\psi(x)\) there exist complex numbers \(\xi_l, |\xi_l| = 1\), such that \(\hat{T}_1\psi(x) = \xi_l\psi(x)\). For \(\xi_l\) the relation \(\xi_l\xi_{l'} = \xi_{l+l'}\) is valid which follows from the chain of equations
\[ \hat{T}_1\hat{T}_{l'}\psi(x) = \hat{T}_{l+l'}\psi(x) = \xi_l\xi_{l'}\psi(x) = \xi_{l+l'}\psi(x). \]

Therefore, \(\xi_l = \exp(-i\mathbf{k}\mathbf{l})\), where \(\mathbf{k}\) is a vector determining the irreducible representation of the translation group, according to which \(\psi(x)\) transforms: \(\psi(x) \equiv \psi_k(x), \hat{T}_1\psi_k(x) = \exp(-i\mathbf{k}\mathbf{l})\psi_k(x)\). It is easy to see that \(\psi_k(x)\) can be written as
\[ \psi_k(x) = \exp(-i\mathbf{k}\mathbf{x})u_k(x), \quad (11) \]
where \(u_k(x)\) is periodic function with the lattice period, normalized to the condition
\[ \int |u_k|^2 \sqrt{\gamma} \, d^3x = 1, \quad (12) \]
where the integration is carried out over the volume of elementary cell.

The relations (11) and (12) extends the Bloch theorem to the case of the spaces with non-euclidean metrics.

### 2.3 Effective Mass Method

Let potential \(\varphi(x)\) slightly vary over the distances of the order of elementary cell. Consider the matrix elements of the Hamiltonian \(\hat{\mathcal{H}} = \hat{H}_0 + e\varphi\) in the Bloch wave representation
\[ \mathcal{H}_{kk'} = \int \psi_k^*(\hat{H}_0 + e\varphi)\psi_{k'}\sqrt{\gamma} \, d^3x = \mathcal{E}_k\delta_{kk'} + \int e^{-i(k-k')x}e\varphi(u_{k'}^*u_k\sqrt{\gamma}) \, d^3x. \]
Because \(| \mathbf{k} - \mathbf{k}' |\) must be small in comparison with the size of the reciprocal lattice cell, the rapidly oscillating factor \((u_{k'}^* u_k \sqrt{\gamma})\) can be changed for its average

\[
\mathcal{H}_{kk'} = \mathcal{E}_k \delta_{kk'} + \int u_{k'}^* u_k \sqrt{\gamma} \, d^3x \int e^{-i(k-k')^\mathbf{x}} \phi \, d^3x.
\]

(13)

Let us consider the integral

\[
\int u_{k'} u_k \sqrt{\gamma} \, d^3x = 1 + (k' - k) \int (\nabla_k u_k^*) u_k \sqrt{\gamma} \, d^3x + \ldots,
\]

where \(\nabla_k\) means differentiation by \(k\). In this expansion by powers of \(k' - k\) the terms of the higher than the first order, and due to (12) the value of integral for \(k' = k\) is equal to 1.

Let us show that if the Hamiltonian is invariant in respect to inversion, then

\[
\int u_k \nabla_k u_k^* \sqrt{\gamma} \, d^3x = 0 \quad \text{and} \quad \int u_{k'}^* u_k \sqrt{\gamma} \, d^3x = 1
\]

up to the terms of the second order in respect to \(k' - k\). Because the Hamiltonian \(\hat{\mathcal{H}}_0\) is real-valued, the following relations are valid

\[
\hat{\mathcal{H}}_0 \Psi_k^* (\mathbf{x}) = \mathcal{E}_k \Psi_k^* (\mathbf{x}), \quad \hat{\mathcal{H}}_0 \Psi_k^* (\mathbf{-x}) = \mathcal{E}_k \Psi_k^* (\mathbf{-x}).
\]

Let us determine now the wave quasimomentum \(\Psi_k^* (\mathbf{-x})\). To this aim let us act on it with the shift operator \(\hat{T}_1\):

\[
\hat{T}_1 \Psi_k^* (\mathbf{-x}) = \Psi_k^* (\mathbf{-x} - \mathbf{l}) = (\exp(i\mathbf{k}\mathbf{l}) \Psi_k (\mathbf{x}))^* = \exp(-i\mathbf{k}\mathbf{l}) \Psi_k^* (\mathbf{x}).
\]

Whence \(\Psi_k^* (\mathbf{-x}) = \Psi_k (\mathbf{x})\) and, therefore,

\[
u_k^* (\mathbf{-x}) = u_k (\mathbf{x}).
\]

(14)

Then

\[
u_k^* (\mathbf{-x}) \nabla_k u_k (\mathbf{-x}) = u_k (\mathbf{x}) \nabla_k u_k^* (\mathbf{x}),
\]

\[
\int u_k^* (\mathbf{-x}) \nabla_k u_k (\mathbf{-x}) \sqrt{\gamma} \, d^3x = \int u_k (\mathbf{x}) \nabla_k u_k^* (\mathbf{x}) \sqrt{\gamma} \, d^3x,
\]

\[
\int u_k (\mathbf{x}) \nabla_k u_k (\mathbf{x}) \sqrt{\gamma} \, d^3x = \int u_k (\mathbf{x}) \nabla_k u_k^* (\mathbf{x}) \sqrt{\gamma} \, d^3x
\]

\[
= \frac{1}{2} \int (u_k (\mathbf{x}) \nabla_k u_k^* (\mathbf{x}) + u_k^* (\mathbf{x}) \nabla_k u_k (\mathbf{x})) \sqrt{\gamma} \, d^3x
\]

\[
= \frac{1}{2} \nabla_k \left( \int u_k (\mathbf{x}) u_k^* (\mathbf{x}) \sqrt{\gamma} \, d^3x \right) = 0.
\]
Thus

\[ \mathcal{H}_{kk'} = \mathcal{E}_k \delta_{kk'} + \int e^{-i(k-k')x} e^{\varphi} \, d^3x \]  

(15)

describes the perturbed Hamiltonian up to the terms of the order \((k-k')^2\). It can be seen from (15) that matrix elements of the perturbed Hamiltonian \(\hat{\mathcal{H}}\) depend only on the Fourier-transform of the perturbing potential and spectrum of non-perturbed Hamiltonian. Therefore two unperturbed Hamiltonians with the same or close spectra \(\mathcal{E}_k\) have in slowly changing electric field the same eigenvalues and eigenfunctions in the Bloch wave representation. The same idea underlies the classical method of effective masses: the unperturbed Hamiltonian \(\hat{\mathcal{H}}_0\) with spectrum

\[ \mathcal{E}_k = \frac{\hbar^2}{2} \frac{\partial^2 \mathcal{E}_k}{\partial k_\alpha \partial k_\beta} k_\alpha k_\beta + \ldots, \]

is changed for the Hamiltonian with effective masses

\[ \hat{\mathcal{H}}' = \frac{1}{2} (m_{\mathrm{eff}}^{-1})^\alpha_\beta \hat{p}_\alpha \hat{p}_\beta, \quad (m_{\mathrm{eff}}^{-1})^\alpha_\beta \equiv \frac{1}{\hbar^2} \frac{\partial^2 \mathcal{E}_k}{\partial k_\alpha \partial k_\beta}. \]

3 Equation for Spin 1/2 Particle in Anisotropic Space

The search for the equation to be found must be obviously started from the well-known Dirac equation

\[ \left( i\hbar \gamma_i \frac{\partial}{\partial x_i} - mc \right) \psi = 0, \]  

(16)

describing particle with spin 1/2 in spherically symmetrical space-time (\(\gamma_i\) are Dirac’s matrices).

It is possible to assume that in order to generalize the equation to the case of the motion of a particle in a curved space-time it is sufficient to change partial derivatives \(\partial/\partial x_i\) for contravariant ones \(D^i\). However, here the difficulties ensue: the spinor quantities \(\psi\) are not tensor objects, which makes impossible the definition for them “good” operation of covariant differentiation.

The procedure of covariant differentiation for spinors was defined by Fock and Iwanenko \(^{17}\). These authors have written the generalized Dirac equation for the case of the particle motion in classical gravitational field. In this section the method developed in Ref.17 is employed for the description of the particle motion in the field of forces of non-gravitational origin. Recall that the action of electromagnetic forces is introduced via metric tensor depending on point of space.
The main requirement on the construction of covariant derivative of a spinor is that the observable tensor quantities bilinear in respect to $\psi$ could be translated as tensors in a curved space-time.

Let us introduce in each point of a space-time four orthonormalized vectors – tetrade $h_i^{(a)}(t, x)$, $a = 0, 3$, $i = 0, 3$, and conjugate tetrade $h_{(a)i}(t, x)$:

$$h_i^{(a)}h_{(a)j} = \delta_{ij} \cdot \text{diag}(1, -1, -1, -1).$$

The summation over repeated tetrade indices is supposed:

$$f^{(a)}g_{(a)} = \sum_{a=0}^{3} f^{(a)}g_{(a)}.$$

Shifting tetrade vectors from point $x_i$ to infinitesimally close point $x_i + \delta s_{(a)}h_i^{(a)}$, we obtain a set of quantities $\gamma^{bca}$, $\gamma^{bca} = -\gamma^{cba}$, called Ricci rotation coefficients. The antisymmetric matrices $\gamma^{bca}\delta s_{(a)}$ are matrices of infinitesimal rotation bringing into coincidence the tetrade in point $x_i + \delta s_{(a)}h_i^{(a)}$ with the tetrade shifted from $x_i$. The Ricci coefficients can be expressed in the terms of covariant derivatives of tetrade vectors $D_k h_i^{(a)} \equiv h_{i,k}^{(a)}$:

$$\gamma^{bca} = h_{i,k}^{(b)} h^{(c)i} h^{(a)k}.$$

Thus the operator of infinitesimal parallel shift of a spinor $\psi$ can be expressed in terms of $\gamma^{bca}$:

$$D^i \psi = \left( \frac{\partial}{\partial x_i} + \frac{1}{4} \gamma^{bca} \gamma^c_{\beta} h^{i}_i \right) \psi. \quad (17)$$

As a result, the general-covariant generalization of the Dirac equation can be written as follows $^{16,17}$:

$$\left( i \hbar \gamma^a h_k^{(a)} D^k - m \right) \psi = 0. \quad (18)$$

This equation depends explicitly on the tetrade $h_k^{(a)}$ and is invariant only in respect to local Lorentzian transformation of tetrade vectors.

Because further we shall restrict ourselves to the case where the metric tensor is of type (3), let us introduce tetrade vectors in such way that

$$h_k^{(0)} = (1, 0, 0, 0), \quad h_k^{(a)} = (0, h_1^{(a)}, h_2^{(a)}, h_3^{(a)}), \quad a = 1, 3.$$  

For such choice of vectors, Ricci coefficient $\gamma^{bca}$ vanishes if at least one of indices $a$, $b$ or $c$ is equal to zero. Therefore, the equation (18) can be written as

$$\left( \frac{i \hbar}{c} \gamma_0 \frac{\partial}{\partial t} + i \hbar \gamma^a h_{(a)\beta} D^\beta - m \right) \psi = 0. \quad (19)$$
Here and further the summation over repeated indices is supposed:

\[ \gamma_a h_{(a)} \equiv \sum_{a=1}^{3} \gamma_a h_{(a)}. \]

The lowering of index is carried out according to the following rule: \( f_{(a)} = -f^{(a)}. \) This rule implies that \( h_{(a)}^\alpha h_{(a)}^\beta = \gamma^{\alpha\beta} \) and \( h_{(a)}^\beta h_{(a)}^\beta = \delta_{ab}. \) The equation (19) can be written as

\[ i\hbar \frac{\partial \psi}{\partial t} = \left( i\hbar c \sigma_a h_{(a)}^\nu D^\nu + mc^2 \beta \right) \psi. \]  

(20)

where \( \alpha \equiv \gamma_0 \gamma_a, \) \( \beta \equiv \gamma_0. \)

Coming back to the operators \( D^\nu, \) we obtain

\[ D^\nu = \frac{\partial}{\partial x^\nu} - \frac{1}{4} \gamma_{beca} \gamma_b \gamma_c h_{(a)}^\nu = \frac{\partial}{\partial x^\nu} - \frac{1}{4} \gamma_{beca} h_{(a)}^\nu \left( \sigma_b \sigma_a \begin{array}{cc} 0 & 0 \\ \sigma_b \sigma_c \end{array} \right) = \frac{\partial}{\partial x^\nu} - \frac{i}{4} \epsilon_{bcd} \gamma_{beca} h_{(a)}^\nu \sigma_d. \]  

(21)

Here \( \sigma_d \) are Pauli matrices, \( \epsilon_{bcd} \) is completely antisymmetric tensor. In (21) the relation \( \sigma_a \sigma_b = \delta_{ab} + i \epsilon_{abc} \sigma_c, \) antisymmetricity of Ricci coefficients \( \gamma_{beca} = -\gamma_{cba} \) and the rule of index lowering have been used.

Introducing two two-component functions \( \phi \) and \( \chi : \psi \equiv \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \) one can bring the equation (20) to the form

\[ i\hbar \frac{\partial \phi}{\partial t} = -i\hbar c \sigma_a h_{(a)}^\nu \hat{\nabla}^\nu \chi + mc^2 \phi, \]  

(22)

\[ i\hbar \frac{\partial \chi}{\partial t} = -i\hbar c \sigma_a h_{(a)}^\nu \hat{\nabla}^\nu \phi - mc^2 \chi, \]

where \( \hat{\nabla}^\nu \equiv \frac{\partial}{\partial x^\nu} - \frac{i}{4} \epsilon_{bcd} \gamma_{beca} h_{(a)}^\nu \sigma_d. \)

Denoting

\[ \sigma_a h_{(a)}^\nu \hat{\nabla}^\nu \equiv (\sigma, \hat{\nabla}), \]

one obtains from (22)

\[ i\hbar \frac{\partial \phi}{\partial t} = -i\hbar c (\sigma, \hat{\nabla}) \chi + mc^2 \phi, \]  

(23)

\[ i\hbar \frac{\partial \chi}{\partial t} = -i\hbar c (\sigma, \hat{\nabla}) \phi - mc^2 \chi. \]
The interaction with external electromagnetic field can be taken into account by changing the derivatives $\partial/\partial x_i$ for $\partial/\partial x_i + (ie/\hbar c)A^i$. Then the equation (23) is transformed to

$$i\hbar \frac{\partial \phi}{\partial t} = -i\hbar c (\sigma, \hat{\nabla} + i\frac{e}{\hbar c}A)\chi + e\varphi \cdot \phi + mc^2 \phi,$$

(24)

$$i\hbar \frac{\partial \chi}{\partial t} = -i\hbar c (\sigma, \hat{\nabla} + i\frac{e}{\hbar c}A)\phi + e\varphi \cdot \chi - mc^2 \chi.$$

### 3.1 Non-relativistic Limit

Let us find now the analogue of the Pauli equation assuming that the rest energy $mc^2$ of the particle considerably exceeds its kinetic energy and the energy of interaction. To this aim let us make in (24) the following substitutions:

$$\phi = \exp \left(-\frac{mc^2}{\hbar}t\right) \phi' \quad \text{and} \quad \chi = \exp \left(-\frac{mc^2}{\hbar}t\right) \chi'.$$

Then

$$i\hbar \frac{\partial \phi'}{\partial t} = -i\hbar c (\sigma, \hat{\nabla} + i\frac{e}{\hbar c}A)\chi' + e\varphi \cdot \phi',$$

(25)

$$i\hbar \frac{\partial \chi'}{\partial t} = -i\hbar c (\sigma, \hat{\nabla} + i\frac{e}{\hbar c}A)\phi' + e\varphi \cdot \chi' - 2mc^2 \chi'.$$

(26)

It follows from (26) that

$$\chi' \simeq -\frac{i\hbar}{2mc}(\sigma, \hat{\nabla} + i\frac{e}{\hbar c}A)\phi'.$$

Substituting the latter expression in (25), we obtain

$$i\hbar \frac{\partial \phi'}{\partial t} = -\frac{\hbar^2}{2m}(\sigma, \hat{\nabla} + i\frac{e}{\hbar c}A)^2\phi' + e\varphi \cdot \phi'.$$

(27)

This equation has the aspect of the Schrödinger equation with Hamiltonian

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}(\sigma, \hat{\nabla} + i\frac{e}{\hbar c}A)^2\phi' + e\varphi \cdot \phi'.$$

(28)
Operator $\hat{H}$ is Hermitian in respect to the scalar product

$$(\phi, \psi) = \int \phi^+ \psi \sqrt{\gamma} \, d^3x,$$

where $\phi^+$ is the conjugate spinor.

Let us put the expression $(\sigma, \hat{\nabla} + i \frac{e}{\hbar c} \mathbf{A})^2$ in more detailed form:

$$(\sigma, \hat{\nabla} + i \frac{e}{\hbar c} \mathbf{A})^2 \phi = \sigma_e h_{(e)\nu} \mathcal{D}^\nu \sigma_f h_{(f)\mu} \mathcal{D}^\mu \phi$$

where $\mathcal{D}^3 = \partial / \partial x^3 + (i e / \hbar c) A^3$.

For the first summand in (29) it is valid that

$$\sigma_e h_{(e)\nu} \mathcal{D}^\nu \sigma_f h_{(f)\mu} \mathcal{D}^\mu \phi =$$

$$-\frac{i}{4} \left[ \sigma_e h_{(e)\nu} \epsilon_{bcd} \gamma_{bcd} h_{(a)\sigma} \right] \sigma_d \sigma_f h_{(f)\mu} \mathcal{D}^\mu \phi$$

$$+ \sigma_e h_{(e)\nu} \mathcal{D}^\nu \sigma_f h_{(f)\mu} \epsilon_{ghi} \gamma_{ghi} h_{(j)\sigma}^\mu \phi$$

$$- \frac{1}{16} \sigma_e h_{(e)\nu} \epsilon_{bcd} \gamma_{bcd} h_{(a)\sigma} \sigma_d \sigma_f h_{(f)\mu} \epsilon_{ghi} \gamma_{ghi} h_{(j)\sigma}^\mu \phi,$$

where $\mathcal{D}^3 = \partial / \partial x^3 + (i e / \hbar c) A^3$.

For the second summand it is valid that

$$-\frac{i}{4} \left( \sigma_a \sigma_d \sigma_f \epsilon_{bcd} \gamma_{bcd} h_{(f)\mu} \mathcal{D}^\mu + \sigma_e \sigma_f \sigma_i \epsilon_{ghi} h_{(e)\mu} \mathcal{D}^\mu \gamma_{ghf} \right) \phi =$$

$$-\frac{i}{4} \left[ \sigma_a \sigma_d \sigma_f \epsilon_{bcd} \gamma_{bcd} h_{(f)\mu} \mathcal{D}^\mu + \sigma_f \sigma_a \sigma_d \epsilon_{bcd} \gamma_{bcd} h_{(f)\mu} \mathcal{D}^\mu \right]$$

$$- \sigma_e \sigma_f \sigma_i \epsilon_{ghi} \gamma_{ghi} h_{(f)\mu} \mathcal{D}^\mu \phi =$$

$$-\frac{i}{4} \left( \sigma_a \sigma_d \sigma_f + \sigma_f \sigma_a \sigma_d \right) \left( \epsilon_{bcd} \gamma_{bcd} h_{(f)\mu} \mathcal{D}^\mu \right) \phi$$

$$- \frac{i}{4} \left( \sigma_e \delta f_i + i \epsilon_{f\mu} \sigma_a \sigma e \epsilon_{ghi} \gamma_{ghi} h_{f,e} \phi \right) =$$

$$-\frac{i}{4} \left( \delta_{ad} \sigma_f + 2 i \epsilon_{adf} \right) \left( \epsilon_{bcd} \gamma_{bcd} h_{(f)\mu} \mathcal{D}^\mu \right) \phi$$

$$- \frac{i}{4} \left( \sigma_e \delta f_i - \epsilon_{fie} \epsilon_{eeg} \sigma_g + i \epsilon_{fie} \epsilon_{ghi} \gamma_{ghi} h_{f,e} \phi \right) =$$

$$-\frac{i}{4} \left( \delta_{ad} \sigma_f + 2 i \epsilon_{adf} \right) \left( \epsilon_{bcd} \gamma_{bcd} h_{(f)\mu} \mathcal{D}^\mu \right) \phi.$$
And, at last, for the third summand in (29) it is valid that

\[ -\frac{1}{2} \epsilon_{bca} \gamma_{bca} h_{(f)\mu} D^\mu \phi + \frac{1}{2} (\delta_{f\mu}\delta_{ac} - \delta_{f\mu}\delta_{ab}) \gamma_{bca} h_{(f)\mu} D^\mu \phi \]

\[ -\frac{1}{4} (\sigma_{\delta_f - \sigma_f \delta_{ie} + \sigma_i \delta_{fe} + i \epsilon_{fie}) \epsilon_{ghf, e} \phi = \]

\[ -\frac{1}{2} \epsilon_{bca} \gamma_{bca} h_{(f)\mu} D^\mu \phi + \gamma_{bca} h_{(b)\mu} D^\mu \phi - \frac{1}{4} \sigma_e (\epsilon_{ghf} \gamma_{ghf, e} \]

\[ -\epsilon_{ghf} \gamma_{ghf, e} + \epsilon_{ghf} \gamma_{ghf, e} \phi + \frac{1}{2} \gamma_{eff, e} \phi. \]

Collecting all terms (30)–(32), we obtain Hamiltonian (28) in its explicit form:

\[
\hat{H} = -\frac{\hbar^2}{2m} \frac{1}{\sqrt{\gamma}} \left( \frac{\partial}{\partial x^\mu} + i \frac{\hbar}{\sqrt{\gamma}} A^\mu \right) \gamma_{\nu\mu} \sqrt{\gamma} \left( \frac{\partial}{\partial x^\nu} + i \frac{\hbar}{\sqrt{\gamma}} A^\nu \right) + i \frac{\hbar^2}{4m} \epsilon_{bca} \gamma_{bca} h_{(f)\mu} D^\mu
\]

\[ + i \frac{\hbar^2}{2m} \left( \frac{1}{16} \epsilon_{bca} \gamma_{bca} - \frac{1}{8} \gamma_{bca} \gamma_{bca} + \frac{1}{16} \epsilon_{bca} \gamma_{bca} \gamma_{ghf, e} - \frac{1}{2} \epsilon_{eff, e} \right) + \frac{\hbar^2}{8m} \sigma_c (\epsilon_{ghf} \gamma_{ghf, e} - \epsilon_{ghf} \gamma_{ghf, e} + \epsilon_{ghf} \gamma_{ghf, e} + \epsilon_{ghf} \gamma_{ghf, e}). \]

### 3.2 Long-wavelength Approximation

The Hamiltonian (33) can be considerably simplified if tensor $\gamma_{\alpha\beta}$ does not depend on the point of space. Such space is flat, and the metric tensor in this case can be brought to the diagonal form. The description of quasiparticles in solid state, corresponding in non-relativistic limit to the effective mass method, for the first time was proposed in Refs.12,13. In this case the Hamiltonian (33) is as follows

\[
\hat{H} = \sum_{i=1}^{3} \frac{1}{2m_i} (\hat{P}_i - \frac{e}{c} A_i)^2 + e \varphi + \frac{\hbar}{2c} \left( \frac{H_1 \sigma_1}{\sqrt{m_2 m_3}} + \frac{H_2 \sigma_2}{\sqrt{m_3 m_1}} + \frac{H_3 \sigma_3}{\sqrt{m_1 m_2}} \right).
\]
One can see that the frequencies of spin cyclotron resonance are expressed in terms of the components \( m_i \) of effective masses. Similar relation was derived in the frame of standard approach by Cohen and Blount\(^\text{18}\) in the limit of strong spin-orbital interaction.

Assuming that one of the effective mass turns into infinity, the motion becomes effectively two-dimensional. Only the field component perpendicular to the plane of motion contributes to the energy of interaction between spin and magnetic field. It should be once more mentioned that this theoretical result is in agreement with the magnetotransport experiment of Martin \textit{et al}\(^\text{9}\) on strained quantum wells and with the normal state magnetothermopower measurements of Jang \textit{et al}\(^\text{10}\) on high temperature superconducting \( Nd_{1.85}Ce_{0.15}CuO_4 \) crystal.

If two masses turn into infinity, say \( m_1 \) and \( m_2 \), then only the one-dimensional motion along \( x_3 \) axis survives, and the terms determining Zeeman interaction disappear. In the frame of such approximation one can not tell the electrons with opposite spins from each other, i.e. spin becomes a hidden coordinate. On this base in Refs.12,13 a hypothesis has been put forward that "one-dimensional" conduction electrons are governed by para-Fermi statistics of the rank \( p = 2 \) (the maximal occupation number). One can expect that this para-Fermi statistics is also a good approximation for quasi-one-dimensional case. This can follow from the continuity of physical parameters for a system of identical and quasi-identical particles.

Concept of quasiparticles with parastatistics and two models of superconducting systems with para-Fermi statistics (on the base of BCS model) were considered in Refs.19–21. The main conclusion is that the critical temperature increases in system with para-Fermi statistics. In Ref.22 the Fröhlich’s one-dimensional superconductor with para-Fermi statistics has been considered.

4 Conclusion

The given above analysis shows the self-consistency of the proposed geometrical approach and traditionally accepted language of solid state physics. The main aim of this work was to extend the analogy of "particle-quasiparticle" to the relativistic case. In principle this way allows to consider spin properties of elementary collective excitations. It is clear that such analysis enables to consider the question on quantum statistics of quasiparticles from a new viewpoint.

The question on spin (and spin relation to statistics) of such quasiparticles, for example, phonons, usually does not arise, because elementary excitations of oscillatory medium unavoidably are bosons. Essentially more complicated situation we meet under description of quasiparticles with half-integer spin. The matter is the extension of Pauli’s postulate on Fermi statistics for such quasiparticles is by no means so obvious as it may seem to be. According to general theorem by Govorkov\(^\text{23}\) on connection between spin of particle
and statistics, the realization of para-Fermi statistics for conduction electrons and holes can not be excluded \textit{a priori.}

To our opinion the construction of unified theory of elementary quasiparticles in the frame of only electromagnetic interaction is quite possible (the problem of the "Grand Unification" does not arise).

If the density of charged quasiparticles is not low, then the effects characteristic for quantum liquids come into play. In the frame of proposed approach it is necessary to take into account the change of metrics and as a consequence the change of the problem symmetry.

Another important feature of the proposed approach is that it enables to understand the interconnection between the excitations of "two-dimensional world" (of anyons and other planar formations) and real three-dimensional space.

The said above does not by any means exhaust the all possibilities of the proposed approach. For example, it allows to describe the excitation in fractal media, where due to strong anisotropy the description of quasiparticles drastically changed (instead of conventional differential equations the equations with fractional derivatives arise and instead of smooth solutions one meets fast oscillating functions and distributions) and to hope that the question on the classification of elementary excitations in fractal media can be settled as well.
APPENDIX

Generalized Belinfante’s Theorem

Symmetric energy-momentum tensor for a spinor field has the form

\[ T_{\mu\nu} = \frac{i}{4} (\bar{\psi} \gamma_\mu \nabla_\nu \psi + \bar{\psi} \gamma_\nu \nabla_\mu \psi) + \text{h.c.} \quad (A1) \]

Corresponding density of momentum is

\[ \pi_k = T_{0k} = \frac{i}{4} (\psi^+ \nabla_k \psi + \frac{1}{c} \psi^+ \alpha_k \frac{\partial \psi}{\partial t}) + \text{h.c.} \]

\[ = \frac{i}{4} (\psi^+ \nabla_k \psi) + \frac{i}{4} \psi^+ \alpha_k (\alpha^l \nabla_l + mc\beta) \psi + \text{h.c.} \quad (A2) \]

\[ = \frac{i}{4} (\psi^+ \nabla_k \psi) + \frac{i}{4} (\psi^+ \alpha_k \alpha^l \nabla_l \psi) + \text{h.c.} \]

From \( \alpha_k \alpha^l + \alpha^l \alpha_k = 2 \delta_k^l \) follows

\[ \frac{i}{4} (\psi^+ \alpha_k \alpha^l \nabla_l \psi) = \frac{i}{4} (\psi^+ \nabla_k \psi) + \frac{h}{4} \psi^+ \sigma_k^l (\nabla_l \psi), \quad (A3) \]

where \( \sigma_k^l = \frac{1}{2} [\alpha_k, \alpha^l] \). Thus the density of momentum (A2) can be expressed as

\[ \pi_k = \frac{i}{2} [\psi^+ (\nabla_k \psi) - (\nabla_k \psi^+) \psi] + \frac{h}{4} \nabla_l (\psi^+ \sigma_k^l \psi). \quad (A4) \]

Now we write the density of moment of momentum (as it was proposed by Belinfante\(^7\) for isotropic case) in the form

\[ j_m = \frac{1}{\sqrt{\gamma}} \epsilon_{mnk} x^n \pi^k = l_m + s_m, \quad (A5) \]

\[ l_m = \frac{1}{\sqrt{\gamma}} i \hbar \epsilon_{mnk} x^n [\psi^+ (\nabla_l \psi) - (\nabla_l \psi^+) \psi] \gamma^{lk}, \]

\[ s_m = \frac{1}{\sqrt{\gamma}} i \hbar \epsilon_{mnk} x^n \nabla_l (\psi^+ \sigma^{lk} \psi). \]

Here \( \gamma^{lk} = \text{diag} (m/m_1, m/m_2, m/m_3), \gamma = \det(\gamma^{lk}) \). One can see that \( l_m \) does not depend on the spinor state which is typical for the orbital moment and \( s_m \) are defined by the spinor state. The last term in (A5) can be rewritten as

\[ s_m = -\frac{1}{\sqrt{\gamma}} i \hbar \psi_m \sigma_m \psi = \frac{i}{\sqrt{\gamma}} \hbar \epsilon_{mkd} \alpha^k \alpha^l, \quad (A6) \]

where

\[ \sigma_m = \frac{1}{2} \epsilon_{mlk} \sigma^{lk} = \frac{i}{2} \epsilon_{mlk} \alpha^k \alpha^l. \]
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