Optimality of minimum-error discrimination by the no-signalling condition

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In this work we relate the well-known no-go theorem that two non-orthogonal (mixed) quantum states cannot be perfectly discriminated, to the general principle in physics, the no-signalling condition. In fact, the minimum error in discrimination between two quantum states is derived from the no-signalling condition.

I. INTRODUCTION

Two non-orthogonal quantum states cannot be discriminated with certainty, while the discrimination error can be made smaller as their copies are provided. This leads to one of the well-known no-go theorems, that quantum states cannot be copied with certainty [1], although approximate quantum cloning is possible with the use of quantum operations and ancillary quantum systems [2]. Interestingly, the impossibility of perfect quantum cloning can be connected to the no-signalling principle in physics, which dictates that information cannot be sent faster than light. As a consequence, quantum communication that makes use of (non-local) quantum correlations cannot be performed faster than light.

In fact, the relation between the no-cloning theorem and the no-signalling constraint has been established in both qualitative and quantitative terms. For instance, it has been shown in Ref. [2] that any no-signalling theory predicting non-locality, i.e. violation of Bell inequalities, has a no-cloning theorem. This implies that quantum theory necessarily has the no-cloning theorem. In addition, the optimal fidelity of approximate quantum cloning has been derived by applying the no-signalling condition to the cloning process of quantum states [3], provided that corresponding quantum operations are positive [3].

On the other hand, the impossibility of perfect state discrimination can also be connected to the no-signalling condition in a qualitative way, at least through the existing relation that the no-cloning theorem is implied by the no-signalling constraint. Now the question naturally arises whether there is a quantitative connection as well. In fact, recent progress along this line has shown that the no-signalling condition would imply the optimality of state discrimination [13], constrained to those figures of merit such as minimum-error discrimination [3], unambiguous state discrimination [7], and maximum confidence measurement [5]. It is remarkable that the optimality of state discrimination in different figures of merit can be derived from the single condition, no-signalling constraint, despite the fact that quantum theory is not a maximally non-local theory [3].

In this work, we relate the no-signalling condition to minimum-error state discrimination of ensembles of quantum states i.e. mixed states. The novelty of this work is that two quantum states to be discriminated between are not purely quantum but mixtures of pure quantum states. We shall derive the minimum error for discriminating between two ensembles of quantum states from the no-signalling constraint. The proof is built on a communication scenario between two parties, Alice and Bob.

II. PRELIMINARIES

Before starting the main proof, for clarity we first fix notations to be used in the communication scenario and then briefly review some known facts about quantum states, minimum-error discrimination and ensemble decompositions. We also restrict our consideration to qubit states, so one can make use of the representation in which a single qubit state is fully characterised by its Bloch vector \( \vec{v} \), \( \rho(\vec{v}) = (\mathbb{I} + \vec{v} \cdot \vec{\sigma})/2 \) where \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \). Pure states have unit Bloch vectors \( \vec{v} \). The communication scenario to be considered assumes that, sharing entangled states with Bob, Alice measures her qubits to prepare the ensemble decomposition of Bob’s state, either one of two ensembles, \( \rho_B^{(0)} \) and \( \rho_B^{(1)} \), as follows,

\[
\rho_B^{(0)} = pp_0 + (1-p) |\delta\rangle \langle \delta|,
\rho_B^{(1)} = pp_1 + (1-p) |\delta\rangle \langle -\delta|,
\]

where \( p_0 \) and \( p_1 \) are two qubit states for which we are to derive the discrimination bound using the no-signalling constraint, and | \( \pm \delta \rangle \) are two orthogonal quantum states such that it holds \( \rho_B^{(0)} = \rho_B^{(1)} \), see Fig 1. It is convenient to express those quantum states in (1) in terms of Bloch vectors as follows,

\[
\rho_B^{(i)} = \rho_B^{(i)}(\vec{r}_B^{(i)}), \quad \rho_i = \rho_i(\vec{r}_i), \quad |\pm \delta\rangle \langle \pm \delta| = \rho(\vec{r}_\pm),
\]

where \( i = 0, 1 \). Then, the relation between quantum states in (1) can be translated to the following relations between Bloch vectors

\[
\vec{r}_B^{(0)} = pp_0 + (1-p)\vec{r}_\delta,
\vec{r}_B^{(1)} = pp_1 + (1-p)\vec{r}_{\mp \delta}.
\]

Recall that \( \rho_0(\vec{r}_0) \) and \( \rho_1(\vec{r}_1) \) in (1) are two quantum states that we wish to discriminate between. The minimum-error discrimination between two quantum
assuming that the \( a \text{ priori} \) tum state \( \rho \) is given, its actual decomposition cannot be known without knowing how the state was prepared. One can only deduce ensemble-averaged properties of the state, which does not depend on any particular decomposition.

Now let us assume that two quantum states in (4) are the same. Using the expression in (3), one can see that the vector of each ensemble of quantum states points to the same point in the Bloch sphere, see Fig. 1. That is, the two quantum states have different decompositions but have the same ensemble average. In addition, each \( \rho_j \), \( j = 0, 1 \), is again a mixture of pure quantum states, so finally quantum states of Bob in (4) can be expressed as an ensemble of pure quantum states. Recall that for any mixed state, there exists higher-dimensional pure quantum state such that partial reduction of the pure state would be the mixed state itself. In this case, two-dimensional ancillary systems suffice to purify those quantum states in (4). For two different decompositions of the same state in (4), the corresponding purifications are equivalent up to local unitary transformations. This simply depends on choice of the measurement basis on the ancillary systems.

If we assume that Alice holds the purification, her measurement will decide the decomposition of Bob’s quantum state. However, unless Alice announces what basis her measurement was made in, Bob will never know which decomposition he has. Performing quantum state tomography will only allow Bob to understand his quantum state as an ensemble average of other quantum states. This is in fact what prevents the two parties communicating faster than light.

III. STATE DISCRIMINATION BOUND BY THE NO-SIGNALLING CONSTRAINT

We are ready to prove that the no-signalling constraint implies that the Helstrom bound in (4) is optimal. Consider that Alice and Bob are separated in space so that local actions performed by one cannot affect the other. Suppose that they share copies of entangled states of \( C^2 \otimes C^2 \)

\[
|\psi\rangle_{AB} = \sum_j \sqrt{\lambda_j} |a_j\rangle_A |b_j\rangle_B,
\]

which is not written in orthonormal basis of Alice and Bob, so \( j \) may exceed the rank of systems belonging to two parties. As we have mentioned above, Alice chooses measurement basis so that after the measurement on her particle, the ensemble decomposition of Bob’s state is determined. In other words, Alice performs a measurement \( M_0 \) or \( M_1 \) (which are general positive-operator-valued-measures), after which decomposition of Bob’s state will be either \( \rho_B^{(0)} \) or \( \rho_B^{(1)} \) in (4) respectively. Alice actually can make use of the measurement \( M_0 \) and \( M_1 \) as an encoding such that a value \( j \) is encoded by applying \( M_j \). Bob then decodes the value by discriminating between two ensembles \( \rho_B^{(0)} \) and \( \rho_B^{(1)} \). Two ensembles are the same, i.e.,

\[
\rho_B^{(0)} = \rho_B^{(1)}
\]

and so their Bloch vectors are also the same, i.e. \( \vec{r}_B^{(0)} = \vec{r}_B^{(1)} \) from (3), and therefore the value \( p \) in (3) is uniquely determined,

\[
p = \frac{2}{\|\vec{r}_0 - \vec{r}_1\| + 2}.
\]
Let us now consider a detector that Bob applies, that works as follows. Once a qubit in a state $\rho$ being measured, the detector gives an outcome $i$ with some probability $P_i(\rho)$ where $i = 0, 1$, i.e. the detector can be thought of being in a black box from which we only know the input/output list. Therefore, it fulfills that

$$P_0(\rho) + P_1(\rho) = 1. \tag{8}$$

Note that we do not know all the properties of the detector, e.g. the internal structure, etc., but the probabilities $P_i(\rho)$ for $i = 0, 1$. For two states $\rho_0$ and $\rho_1$, we can assume that in general $P_0(\rho_0) \geq P_0(\rho_1)$, by which it also follows that $P_1(\rho_0) \leq P_1(\rho_1)$. While Alice not announcing which measurement is applied, Bob can make guess of it by designing his detector to discriminate the two states $\rho_0$ of $\rho_0^{(0)}$ and $\rho_1$ of $\rho_B^{(1)}$. If the detector discriminates between $\rho_0$ and $\rho_1$ and gives an output $i$ (either 0 or 1), Bob will then conclude that the ensemble is $\rho_B^{(j)}$. Note that Alice applies measurement $M_0$ or $M_1$ with equal probabilities, and then the error rate is

$$e = \frac{|P_1(\rho_0) + P_0(\rho_1)|}{2}. \tag{9}$$

Two ensembles in (11) consist of not only $\rho_i$ but also $|\pm \delta\rangle\langle \pm \delta|$. Although the detector being designed to discriminate between the two states $\rho_0$ and $\rho_1$ in the ensemble, the detector produces some outcomes for $|\delta\rangle\langle \delta|$ and $| - \delta\rangle\langle - \delta|$ with some probabilities $P_i(|\delta\rangle\langle \delta|)$ and $P_i(| - \delta\rangle\langle - \delta|)$ respectively. Assume first that Alice applies the measurement $M_0$ so that $\rho_B^{(0)}$ is prepared on the Bob’s side. Bob is then with the state $\rho_0$ with probability $p$, from which the value 0 is found with probability $P_0(\rho_0)$. Bob is also with the state $| - \delta\rangle\langle - \delta|$ with probability $1-p$, from which he can guess the value 0 with probability $P_0(\rho_0)$, a non-negative number. Finally, the overall probability that detector’s output says the value 0 when Alice prepares the ensemble $\rho_B^{(0)}$, denoted by $D_0^0$ \[^{[14]}\] and $D_0^0 = p P_0(\rho_0) + (1-p)P_0(| - \delta\rangle\langle - \delta|)$, is greater than or equal to $p P_0(\rho_0)$, i.e.,

$$D_0^0 \geq p P_0(\rho_0). \tag{10}$$

We also have $D_0^1$, the overall probability that detector’s output says the value 1 when the same ensemble $\rho_B^{(0)}$ is prepared. Clearly it holds that

$$D_0^0 + D_1^0 = 1. \tag{11}$$

Next, assume that Alice applies the measurement $M_1$ so that $\rho_B^{(1)}$ is prepared. In the same way we have done for the ensemble $\rho_B^{(0)}$, the overall probability that the detector says the value 1, denoted by $D_1^1$, is greater than or equal to $p P_1(\rho_1)$, i.e.,

$$D_1^1 \geq p P_1(\rho_1). \tag{12}$$

It holds again that

$$D_0^1 + D_1^1 = 1. \tag{13}$$

However, if the following holds

$$D_0^0 + D_1^1 > 1, \tag{14}$$

Bob can discriminate the two ensembles since equations in (11), (13), and (14) tell that

$$D_0^0 > D_0^1, \ D_1^1 < D_1^0. \tag{15}$$

The above (15) means that, once $M_j$ is applied so that $\rho_B^{(j)}$ is prepared, the measurement data show more frequency on the value $j$ than $j + 1 \mod 2$, through which the two ensemble can be discriminated. This implies that the no-signalling principle excludes (14) and therefore the following holds,

$$D_0^0 + D_1^1 \leq 1. \tag{16}$$

Combining (10), (12), and (16), we have

$$p P_0(\rho_0) + P_1(\rho_1)) \leq 1. \tag{17}$$

As well, from (8), (9), and (17) we obtain

$$e \geq 1 - \frac{1}{2p}. \tag{18}$$

Finally from (7) and (18), the following is the discrimination error limited by the non-signalling principle

$$e \geq \frac{1}{2} + \frac{1}{4} \| \bar{r}_0 - \bar{r}_1 \|, \tag{19}$$

which is already saturated by the Helstrom bound in (4).

IV. CONCLUSION

In conclusion, we relate two no-go theorems - the no-perfect state discrimination and the no-signalling - by showing that the optimal discrimination between two quantum states is derived from the no-signalling condition. It can thus be said that the impossibility of perfect state discrimination is a consequence of the no-signalling condition, not only qualitatively, but also quantitatively as is the case with the no-cloning theorem [4]. Furthermore, it was shown in Ref. [12] that the fidelity of optimal quantum cloning converges asymptotically to the fidelity obtained through optimal state estimation. Therefore, the quantitative relationship between three no-go theorems- no-signalling, no-cloning, and no-perfect state estimation- has been established at the most basic level i.e. two non-orthogonal quantum states. It would be interesting to investigate whether optimal values of all figures of merit in state discrimination, more generally state estimation, of quantum states could be derived solely from the no-signalling condition. Recently, it is shown that the optimal measurement for the maximum confidence is also implied by the no-signalling condition [8].
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[13] Note that, when no-signalling constraint is considered in state discrimination, the measurement postulate in quantum theory such as positive-operator-valued-measure is assumed.
[14] Let $D^i_j$ denote the probability that the detector says $i$ when Alice prepares $\rho_B^{(j)}$ for $i = 0, 1$ and $j = 0, 1$. 