The Head-On Collision of Two Equal Mass Black Holes: Numerical Methods

Peter Anninos(1), David Hobill(1,2), Edward Seidel(1,4), Larry Smarr(1,4) and Wai-Mo Suen(3)

(1) National Center for Supercomputing Applications
605 E. Springfield Ave., Champaign, IL 61820
(2) Department of Physics
University of Illinois, Urbana, IL 61801
(3) Department of Physics and Astronomy
University of Calgary, Calgary, Alberta, Canada T2N 1N4
(4) Department of Physics
Washington University, St. Louis, Missouri, 63130
(March 24, 2022)

The head-on collision of two nonrotating axisymmetric equal mass black holes is treated numerically. We take as initial data the single parameter family of time-symmetric solutions discovered by Misner which consists of two Einstein-Rosen bridges that can be placed arbitrarily distant from one another. A number of problems associated with previous attempts to evolve these data sets have been overcome. In this article, we discuss our choices for coordinate systems, gauges, and the numerical algorithms that we have developed to evolve this system.

PACS numbers: 04.30.+x, 95.30.Sf

I. INTRODUCTION

The coalescence of two black holes is considered to be one of the most promising sources of gravitational waves [1]. In a series of papers [2–4] we investigate numerically a special case of the black hole coalescence problem, namely the head-on collision of two equal mass black holes. Numerical computations are extremely difficult due to coordinate singularities, large gradients in the metric components, and numerical instabilities inherent in the two black hole spacetimes. Building on the work of Dewitt, Cadez, Smarr, and Epstein [5–9] (henceforth abbreviated as DCSE) and the more recent work involving distorted, single black holes [10–12], many of these numerical problems have been overcome in the present work.

Our work (like that of DCSE) is based on studying evolutions of the Misner initial data set [13], a single parameter family of time symmetric solutions that allows the initial separation between the two black holes and the total ADM mass of the system to be specified. Section II discusses this data set briefly. In section III we present the basic equations, formalism, coordinate system and gauge considerations. Section IV details some of the various numerical problems that we encountered along with the computational methods we developed to overcome them, drawing parallels between our methods to those of DCSE when appropriate. Section V reviews the general numerical methods we use to integrate the discrete Einstein equations. We also present convergence studies testing the reliability and robustness of our code.

Since this paper is devoted primarily to numerical methods and tests of our code, we refrain from presenting results such as extracted waveforms, energy radiated, horizon oscillations, etc., except where we discuss convergence results for our code. Instead we refer the reader to the series of related papers [2–4], for a detailed analysis of the physics of colliding black holes.

II. THE MISNER INITIAL DATA

There are a number of ways to construct initial data representing two black holes. One of the simplest to work with is the single parameter family of analytic data derived by Misner [13] for the Einstein-Rosen [14] model of two asymptotically flat sheets joined by two throats. Detailed studies of the Einstein-Rosen bridge construction and variations of it were discussed by Misner [15], Lindquist [16], Brill and Lindquist [17], and others. Data sets of this type were first investigated numerically by Hahn and Lindquist [18], and later by DCSE.

The spatial 3-metric for the Misner data is written in the following conformally flat form using cylindrical coordinates

$$dl^2 = \Psi_M^4 \left( d\rho^2 + dz^2 + \rho^2 d\phi^2 \right).$$  (1)
The conformal factor $\Psi_M$ defined by

$$\Psi_M = 1 + \sum_{n=1}^{\infty} \frac{1}{\sinh(n\mu)} \left( \frac{1}{1 + r_n} + \frac{1}{-r_n} \right),$$

(2)

and

$$\pm r_n = \sqrt{\rho^2 + [z \pm \coth(n\mu)]^2}$$

(3)

solves the Hamiltonian constraint (8) with the proper isometry imposed between the upper and lower sheets. This data set is both axisymmetric and time symmetric ($K_{ij} = 0$) and represents two equal mass black holes with zero rotation. The two black hole centers are aligned along the axis of symmetry (z-axis) so the physical interaction is a head-on collision along the axis. The free parameter $\mu$ is related to the physical parameters $M$ (half the total ADM mass and approximately the mass of a single black hole when the holes are “infinitely” separated)

$$M = M_{ADM}/2 = 2 \sum_{n=1}^{\infty} \frac{1}{\sinh(n\mu)},$$

(4)

and $L$ (the proper distance along the spacelike geodesic connecting the throats)

$$L = 2 \left[ 1 + 2\mu \sum_{n=1}^{\infty} \frac{n}{\sinh(n\mu)} \right].$$

(5)

The effect of increasing $\mu$ is to set the two black holes further away from one another and decrease the total mass of the system.

Comparing with the isotropic form of the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{m/2r}{1 + m/2r}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 (dx^2 + dy^2 + dz^2),$$

(6)

we can see how the Misner data (1) is similar to (6). If one associates $m_1 \sim \sum 2/\sinh n\mu$ as the mass of a single black hole when the two holes are separated by large distances ($\mu \to \infty$), then the Misner data resembles (6) if we identify $m = m_1$ for regions close to one of the throats and $m = 2m_1$ in the far field. (See also, for example, Brill and Lindquist [17]).

On the initial time-symmetric surface, the throats are minimal area surfaces. As the initial data parameter $\mu$ is varied, the shape of the initial apparent horizon varies. If the holes are close enough together (small $\mu$), a new minimal surface appears, surrounding both black holes on the initial slice. Cadez [19] has calculated the critical value $\mu_c = 1.362$ when this occurs. We know that event horizons lie outside of, or are coincident with the outermost trapped surface [20]. This implies that for values of $\mu$ less than or equal to $\mu_c$ the initial data is a single distorted black hole. As we discuss in [2,3,21] we have determined the $\mu$ required for a single connected event horizon by integrating photons through the spacetimes. We find that for values of $\mu$ greater than about 1.8 the black holes do not have a common event horizon on the initial timeslice.

### III. THEORETICAL FRAMEWORK

We use the 3+1 (or ADM) formalism [22] to write the general 4-metric as

$$ds^2 = -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j,$$

(7)

where $\alpha$ is the lapse function foliating the four dimensional spacetime with three dimensional spatial hypersurfaces $\gamma_{ij}$, and $\beta^i$ is the shift vector that specifies three dimensional coordinate transformations from time slice to time slice. Throughout this work we use Greek indices (ranging from 0 to 3) to label four dimensional coordinates, and Latin indices (ranging from 1 to 3) to label spatial coordinates. We use geometric units in which the gravitational constant and the speed of light are set to unity.

In the 3+1 formalism, the vacuum Einstein equations reduce to four constraint equations

$$R - K_{ij} K^{ij} + K^2 = 0,$$

(8)
\[ \nabla_j (K^{ij} - \gamma^{ij} K) = 0, \]  
(9)

and twelve evolution equations

\[
\begin{align*}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= -\nabla_i \nabla_j \alpha + \alpha \left( R_{ij} + K_{ij} K - 2 K_{ik} K^k \right) + \beta_i \nabla_k K_{ij} + K_{kj} \nabla_i \beta^k + K_{ik} \nabla_j \beta^k.
\end{align*}
\]
(10)
(11)

Here \( R \) is the Ricci scalar formed from the spatial 3-metric \( \gamma_{ij} \), \( K_{ij} \) is the extrinsic curvature, \( K \) is the trace of \( K_{ij} \) and \( \nabla_j \) is the covariant derivative with respect to \( \gamma_{ij} \).

A. The Coordinate Systems

Because the spacetimes we work with possess an axial Killing vector (which we set to be \( \partial/\partial x^3 \)), all variables are independent of \( x^3 \). Denoting the azimuthal angle by \( x^3 \equiv \phi \) and using the standard \((z,\rho,\phi)\) cylindrical coordinates, we write the 3-metric for a general axisymmetric spacetime as

\[
\gamma_{ij} = \Psi^4 \hat{\gamma}_{ij} = \Psi^4 \begin{pmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & \rho^2 d \end{pmatrix}
\]
(12)

in the coordinate order \( x^i = (z,\rho,\phi) \). The variables \( a, b, c \) and \( d \) are functions of the coordinates \( z, \rho \) and \( t \) and are assumed to be asymptotically flat. The conformal factor \( \Psi \) is a function of \( z \) and \( \rho \) only and does not evolve in time. It is determined on the initial time slice to satisfy the Hamiltonian constraint (8) and for the type of initial data that we consider, \( \Psi \) and all its derivatives are known analytically, given by equations (2) and (3). At the initial time slice, the 3-metric (12) takes the form of the time symmetric Misner data set (1) with \( a = b = d = 1, c = 0 \) and \( \Psi = \Psi_M \).

The Einstein equations are simplified when a conformal factor is introduced into the extrinsic curvature in a manner similar to the 3-metric (12). We define the following form for the extrinsic curvature

\[
K_{ij} = \Psi^4 \hat{K}_{ij} = \Psi^4 \begin{pmatrix} h_{a} & h_{c} & 0 \\ h_{c} & h_{b} & 0 \\ 0 & 0 & \rho^2 h_{d} \end{pmatrix}.
\]
(13)

The evolution equations (10) and (11) can now be formulated as evolution equations for the metric components \((a, b, c, d)\) and their corresponding curvature components \((h_{a}, h_{b}, h_{c}, h_{d})\).

As we discussed in section II, the initial data we consider consists of two throats connecting two isometric sheets. By construction the throats are spheres, centered on the axis along \( \rho = 0 \), on which boundary conditions relating the metric across the two sheets may be imposed. Since the natural boundaries (the throats and a sphere surrounding the system far from the throats) do not lie along constant \((z,\rho)\) coordinates, it is useful to introduce the “quasi-spherical” Cadez [5] coordinates \((\eta, \xi)\) with \( \eta \) being a logarithmic “radial” coordinate and \( \xi \) an “angular” coordinate. Cadez coordinates are related to cylindrical coordinates through the complex transformation

\[
\chi(\zeta) = \eta + i \xi = \frac{1}{2} \left[ \ln(\zeta + \zeta_0) + \ln(\zeta - \zeta_0) \right] + \sum_{n=1}^{\infty} C_n \left( \frac{1}{(\zeta_0 + \zeta)^n} + \frac{1}{(\zeta_0 - \zeta)^n} \right),
\]
(14)

where \( \zeta = z + i \rho \), and \( \zeta_0 = \coth \mu \) is the value of \( \zeta \) at the throat center. The constant \( \eta \) and \( \xi \) coordinate lines of (14) lie along the field and equipotential lines of two equally charged metallic cylinders located at the centers of the two throats \( z = \pm \coth \mu \). The coefficients \( C_n \) are determined by a least-squares method to set the throats (defined by \( \rho^2 + (z_\pm \coth \mu)^2 = 1/\sinh^2 \mu \)) to lie on an \( \eta = \eta_0 = \) constant coordinate line. Both \( \eta_0 \) and the different \( C_n \) are computed on a problem specific basis for different \( \mu \) using this least squares procedure. As the \( C_n \) are rapidly converging, the series (14) can be truncated to low order. In our simulations we typically keep only terms up to \( n \sim 15 \).

The constant Cadez coordinate lines as viewed in the cylindrical coordinate system are shown in Fig. 1a. The grid in the Cadez coordinates is shown in the accompanying Fig. 1b. The advantage afforded by this set of coordinates
is that they are spherical near the throats of the black holes and also far away in the wave zone, thus allowing us to deal with throat boundaries and asymptotic wave form extractions in a convenient way. Also, because the natural boundaries of the spacetime now lie on constant \((\eta, \xi)\) lines, our spacetimes are constructed and evolved in a manner similar to the single distorted black hole spacetimes in references \[10–12\]. The disadvantage is that the transformation \(14\) introduces a singular saddle point at the origin \((z = \rho = 0)\) that is not present in cylindrical coordinates. This creates certain numerical difficulties that we discuss in more detail in section IV.

The transformation \(14\) satisfies the following Cauchy-Riemann conditions

\[
\begin{align*}
\eta_{,z} &= \xi_{,\rho}, & \eta_{,zz} &= -\eta_{,\rho\rho} = \xi_{,z} \\
\eta_{,\rho} &= -\xi_{,z}, & \eta_{,z\rho} &= -\xi_{,zz} = \xi_{,\rho\rho}.
\end{align*}
\]

We note that these first and second derivatives of \(\eta\) and \(\xi\) can be computed “analytically” (to machine precision) from \(14\). The Jacobian of the two coordinate systems after the Cauchy-Riemann conditions have been applied becomes

\[
J = (\frac{\partial \eta}{\partial \rho})^2 + (\frac{\partial \eta}{\partial z})^2.
\]

In this set of coordinates, we define the analog of the cylindrical based 3-metric \((12)\) as

\[
\gamma_{ij} = \Psi^4 \hat{\gamma}_{ij} = \Psi^4 \begin{pmatrix} A & C & 0 \\ C & B & 0 \\ 0 & 0 & \sin^2 \xi D \end{pmatrix},
\]

with the corresponding extrinsic curvature

\[
K_{ij} = \Psi^4 \hat{K}_{ij} = \Psi^4 \begin{pmatrix} H_A & H_C & 0 \\ H_C & H_B & 0 \\ 0 & 0 & \sin^2 \xi H_D \end{pmatrix},
\]

in the order \([\eta, \xi, \phi]\). The Misner initial data \((1)\) can be written in the form of \((17)\) using Cadez coordinates as

\[
\text{ds}^2 = \Psi^4 \left( d\eta^2 + d\xi^2 + \sin^2 \xi \frac{J\rho^2}{\sin^2 \xi} d\phi^2 \right),
\]

and identifying \(A = B = 1, C = 0, D = J\rho^2 / \sin^2 \xi\) and \(\Psi = \Psi_C = \Psi_M / J^{1/4}\).

The success of our methodology depends critically on using both sets of coordinate systems (cylindrical and Cadez) to advantage as we discuss in section IV.

B. The Lapse and Shift

Kinematic conditions for the lapse function \(\alpha\) and shift vector \(\beta^i\) complete the set of Einstein equations \((8)\) through \((11)\). Even though the 3–metric is fixed on the initial slice by the Hamiltonian constraint, the lapse and shift can be chosen arbitrarily on the initial slice and thereafter.

We impose the maximal slicing condition

\[
K = \partial_t K = 0
\]

throughout the evolution. Taking the trace of Eq. \((11)\) and inserting Eq. \((20)\) results in the following elliptic equation for \(\alpha\)

\[
\nabla^i \nabla_i \alpha = \alpha K_{ij} K^{ij},
\]

where we have used the Hamiltonian constraint to replace \(R\) with \(K_{ij} K^{ij}\). We choose to solve the lapse equation in the nonsingular cylindrical coordinate system for improved accuracy over Cadez coordinates. Solutions to the lapse equation written in cylindrical coordinates tend to be smoother and better behaved near the saddle point than solutions obtained by solving the equation written in Cadez coordinates. Since the evolution equations for the extrinsic curvature components contain second derivatives of the lapse, smoothness in this sensitive region is very important.

For the initial lapse we have tried both \(\alpha = 1\), whereby observers are initially freely falling, and the solution of Cadez \([5]\)
which is a generalization of the standard static Schwarschild slicing for a single black hole, but here the system is not static. In the first case ($\alpha = 1$) the lapse is symmetric across the throat. In the second case it is antisymmetric and hence equal to zero on the throat. We find the antisymmetric lapse tends to work better as it “freezes” the evolution at the throat and slows it down in regions near and between the throats where calculations can be troublesome. For this reason, we work exclusively with antisymmetric boundary conditions for the lapse across the throat.

The shift vector $\beta^\i$ is defined by imposing

$$C = \partial_t C = 0$$

(23)

to maintain a diagonal 3-metric in the Cadez coordinates. This condition applied to Eq. (10) results in the differential equation

$$B \frac{\partial \beta^\xi}{\partial \eta} + A \frac{\partial \beta^\eta}{\partial \xi} = 2\alpha H C,$$

(24)

for the nonvanishing shift vector components $\beta^\eta$ and $\beta^\xi$. We can rewrite Eq. (24) by introducing a shift potential $\Omega$ that satisfies

$$\beta^\eta = \frac{\partial \Omega}{\partial \xi}, \quad \beta^\xi = \frac{\partial \Omega}{\partial \eta}$$

(25)

to get a single elliptic equation for $\Omega$

$$B \frac{\partial^2 \Omega}{\partial \eta^2} + A \frac{\partial^2 \Omega}{\partial \xi^2} = 2\alpha H C.$$

(26)

We emphasize that it is the off-diagonal Cadez metric component $C$ that is required to vanish and not the off-diagonal cylindrical component $c$. In general, the cylindrical metric is nondiagonal. This choice of gauge for the shift vector proved to be critical to the overall stability of the numerical evolution, particularly in suppressing the axis instability.

IV. “PATCHED” COORDINATES

We have investigated a number of different numerical schemes to solve the problem of colliding two black holes head-on. The basic idea that evolved from our progression of trials is to solve for the Cadez metric and extrinsic curvature components, defined by Eqs. (17) and (18), on the Cadez grid and set $C = \partial_t C = 0$. This approach has the advantage that the 3-metric is diagonal which helps to suppress the axis instability and simplifies the equations of evolution and the extraction of invariant gravitational waves in the far field.

This approach has the further advantage that it is possible to define variables for the two black hole system that obey the same evolution equations with similar boundary conditions, as the single distorted black hole code developed in previous work [10,11]. In fact, the two black hole code in its final incarnation evolved from the code we developed for distorted axisymmetric single black hole spacetimes and much of the discussion in [10,11] is directly applicable to the two black hole code. In this section we concentrate on those techniques developed and modified specifically for the two black hole problem and in particular the methods we use to overcome difficulties associated with the singular saddle point present in the Cadez coordinates. Further details relevant to both codes may be found in [10,11].

A. The Grid

Our spacetimes are equatorially plane symmetric, axisymmetric and isometric through a throat boundary. The computational grid is covered with Cadez coordinates and is bounded by the equator ($z = 0$), the axis ($\rho = 0$) and the isometry surface ($\eta = \eta_0$). The outer boundary is set at $\eta = \eta_{\text{max}}$ defined by $\eta_{\text{max}} - \eta_0 = 5.8$, which corresponds to an equivalent Schwarzschild radius ranging from $\sim 125M$ to $\sim 425M$ for $\mu = 1.2$ and $\mu = 3.25$ respectively. The outer boundary is sufficiently far from the two throats that we can safely impose static conformal flatness boundary conditions on the metric and extrinsic curvature components in the asymptotic far field.
Other boundary conditions are specified in an analogous way to the single black hole spacetimes described in references \[\text{[10][11]}\]. Specifically, we require for all the Cadez grid based variables (except $C$, $H_C$ and $\beta^\xi$) to be symmetric across the axis and equator. $C$, $H_C$ and $\beta^\xi$ are antisymmetric across both the axis and equator. The throat isometry takes the form

$$\partial_\eta A = \partial_\eta B = \partial_\eta D = \partial_\eta H_C = \partial_\eta \beta^\xi = 0$$

and

$$\alpha = C = H_A = H_B = H_D = \beta^\eta = 0$$

at $\eta = \eta_0$.

These choices result from establishing a consistent convention to satisfy the Einstein equations subject to the symmetries and isometries of the problem. The choice of lapse \[\text{(22)}\] for the initial data removes the freedom available to choose an isometry sign for the metric components and the boundary symmetric constraints on the shift help preserve the coordinate positions of the axis, equator and throat boundaries \[\text{[10][11]}\]. Boundary values for variables defined in the cylindrical coordinate basis are obtained from the tensor transformation of the corresponding Cadez grid variables.

We make one final comment regarding the grid: the grid is comoving with the black holes throughout the entire evolution. The dynamical history is carried solely by the metric components. As the black holes approach each other the grid cells do not track the black holes and squeeze together in the impact area between the two throats. Instead the conformal metric function $\hat{\gamma}_{\eta\eta} = A$, for example, goes to zero, signifying that the proper distance between grid lines is decreasing.

### B. The Saddle Point

The most difficult problem associated with the Cadez coordinates is the coordinate singularity at the origin ($z = \rho = 0$), clearly evident in Fig. 1h. The difficulty with this point, which is inside the computational domain, is two fold: (i) The transformation Jacobian \[\text{(16)}\] between the Cadez coordinates $(\eta, \xi)$ and cylindrical coordinates $(\rho, \theta)$ goes to zero at the saddle point as $(\rho^2 + \rho^2)K_\alpha$, where $K_\alpha$ is a constant defined in the neighborhood of the saddle point. Note that the Jacobian between cylindrical coordinates and Cartesian coordinates goes to zero just as $\sim \rho$ at the saddle point, so that Cadez coordinates are singular with respect to Cartesian coordinates at $z = \rho = 0$. (ii) The $\eta$ axis turns abruptly with respect to the local three geometry at the saddle point. The $\eta$ axis parallels the $z$ axis for $\eta < \eta_s$ while becoming the $\rho$ axis for $\eta > \eta_s$, where $\eta_s$ is the $\eta$ coordinate passing through the origin.

The problem (i) associated with the vanishing of the Jacobian (volume element) is a familiar one. Similar problems exist in spherical and cylindrical coordinates at the origin. The problem is often treated by rescaling variables, factoring out the Jacobian in the numerical evolution. The second problem (ii) is unfamiliar and more troublesome. As the spatial three geometry is symmetric to reflection about the $z = 0$ plane, and to rotation about the $\rho$ axis, the time development of the geometries along the $\rho$ axis and the $z$ axis are intrinsically different. The holes are falling towards each other in the $z$ direction. Although the initial data is smooth along the $\eta$ axis, as soon as the holes start falling towards each other, the discontinuity in the metric functions, say $\hat{\gamma}_{\eta\eta} = A$, will develop at $\eta = \eta_s$. For $\eta < \eta_s$, $\hat{\gamma}_{\eta\eta}$ decreases as the proper distance between grid points decreases. (In fact, there are two competing effects: the decrease in the proper distance between the holes, and the grid stretching effects whereby grid points along the $\eta$ axis with $\eta < \eta_s$ fall towards the “north” hole.) On the other hand, $\hat{\gamma}_{\eta\eta}$ increases for $\eta > \eta_s$ as grid points are falling towards the saddle point, again due to the grid stretching effect. The anisotropy in the $\rho$ and $z$ directions at the saddle point translates into the discontinuity along the $\eta$ axis at $\eta_s$. The functions $\hat{\gamma}_{\eta\eta}$ on the two sides of $\eta_s$ are really two different geometric objects. The spatial derivative $\partial_\eta \hat{\gamma}_{\eta\eta}$ is undefined analytically at $\eta = \eta_s$ on the $\eta$ axis, and is very large in the finite differencing approximation. Likewise the finite differencing $[\hat{\gamma}_{\eta\eta}(\eta = \eta_s + \Delta\eta, \xi \approx \xi_s)]/(2\Delta\eta)$ for grid points off the $\eta$ axis but near the saddle point will also be large.

To see this problem explicitly, we plot in Fig. 2 the extrinsic curvature function $\hat{K}_{\eta\eta} = H_A = -(\partial_\xi \hat{\gamma}_{\eta\eta})/\alpha$ evolved for just one time step. In this case, since the initial data and all spatial derivatives are known analytically, the extrinsic curvature can be computed analytically on the first time step (up to discretization in time), which can then be used to compute the metric function on the second time step without computing any numerical spatial derivatives. In this sense, the metric is known “analytically”, up to a finite difference in time. But the effect of the discontinuity at the origin ($\eta = \eta_s \sim 0.05$, plotted as a circle in Fig. 2) is clear immediately. It is easy to see that the same difficulty exists for other quantities $A, B, H_B, \ldots$ etc. Without special precautions or treatments, the evolution quickly contaminates with numerical noise. It is possible that certain gauge conditions, such as quasi-isotropic \[\text{[12]}\] or minimal distortion \[\text{[24]}\], could minimize the effect of these discontinuities, but we have not tried them. The method we use in dealing with this problem is discussed in the following section.
C. Numerical Issues

The most critical numerical issue is the treatment of the saddle point and the region near it. To help reduce this problem somewhat, we construct the Cadez coordinates in such a way as to avoid placing a grid point on the origin. Lines of constant $\xi$ are staggered across the axis of symmetry and the equator so that both axis and equator lie on the half zones. Lines of constant $\eta$ are set relative to the throat position $\eta = \eta_0$ at constant discrete spacing $\Delta \eta$ chosen to maintain a grid aspect ratio that is nearly unity. We do not enforce a constant $\eta$ line to intersect the saddle point. Hence, the origin is straddled by both the angular and radial grid lines (see Fig. 1). Although this procedure eliminates some spurious effects arising from the singular point, it does not resolve the problem completely. Gradients in this vicinity are extremely large, for the reasons discussed above.

The basic approach we use to evolve data near the saddle point is to take advantage of the fact that the cylindrical spacetime metric components \( g_{\rho \rho} \) are smooth everywhere, including the saddle point (although the volume element in cylindrical coordinates, or equivalently the Jacobian to Cartesian coordinates, is still zero at that point and along the $z$ axis). We can therefore define a cylindrical coordinate “patch” to cover regions near the saddle point in the singular Cadez system. The patch is constructed beginning at the angular coordinate value $\xi = \pi/2$ and extended along the angular direction towards $\xi = 0$ for a number of zones, depending on the angular resolution. The patch is also extended along the radial direction from the throat all the way out to the outer boundary to minimize distortions that might be suffered by radially propagating structures if they encounter patch boundaries “head-on”. In this patch region we evolve the cylindrical coordinate based metric \( g_{\rho \rho} \) and extrinsic curvature components \( \Gamma_{kl} \) on the Cadez grid and transform the solutions via the general tensor relations $T_{ij} = (\partial x^k/\partial x^i)(\partial x^l/\partial x^j)T_{kl}$ to reconstruct the Cadez components.

We stress that this scheme is not a coordinate patch in the formal sense. Grid lines are no where laid along the $(z, \rho)$ coordinates, and derivatives are not taken in $(z, \rho)$ coordinates. Rather, we are simply evolving two sets of components, Cadez and cylindrical, independently of each other (except for the coupling at the patch boundaries) on a single Cadez grid. The nonsingular cylindrical components are used to correct the singular Cadez components in the patched region. On the other hand, the Cadez components provide corrections to their cylindrical counterparts everywhere else, helping to suppress the axis instability that is inherently present in the cylindrical coordinate system possessing a nondiagonal metric. To help integrate the patched region into the rest of the spacetime for smooth evolutions, we construct a layer of buffer zones surrounding the patched region. Within this boundary of zones, both sets of components are evolved and a linear weighting scheme is used to blend all evolved variables to the values at the edges of this buffer domain.

In a typical calculation, the lapse collapses approximately “spherically” along constant $\eta$ lines, thereby freezing the fields interior to this domain. (This is the generic behavior of the maximal slicing condition or any singularity avoiding lapse function.) Hence, it is necessary to evolve with the patch in place only until the lapse drops below a critical value (typically $\sim 0.025$) at the origin to prevent any evolution from occurring there. Once the lapse falls below this value, the simulation is continued by evolving only the Cadez variable components over the entire Cadez grid, including the saddle area. The removal of the patch at relatively late times is necessary to maintain stable and accurate evolutions. Evolving the natural Cadez metric components on the Cadez grid does not suffer from the numerical instabilities inherited from applications of chain rule derivatives in regions of extreme gradients.

We end this section with a discussion of an alternative strategy that we tried for the numerical evolution. Although it was unsuccessful, it is instructive to point out why it failed. Given that the cylindrical metric variables are well behaved near the saddle point, one might consider evolving the entire system in those variables. It is well known that the axis instability can be suppressed by choosing the shift vector $(\beta^\rho, \beta^z, 0)$ such that $c = 0$. This strategy is effective at minimizing problems related to both the axis instability and the saddle point, but it introduces a new problem related to grid stretching. As noted above, very large gradients develop in the radial metric function surrounding the hole. This radial metric function is composed of the three cylindrical metric functions $\gamma_{zz}$, $\gamma_{pz}$, and $\gamma_{pp}$ that are actually being evolved. Using a shift that forces $c = 0$ in this coordinate system causes extreme angular gradients to develop near the transition between where $\gamma_{zz}$ is primarily radial (along the $z$-axis) and where $\gamma_{pp}$ is primarily radial (along the equator). Therefore, along a line of about 45 degrees between the axis and equator, instabilities develop as the grid stretching becomes severe and the metric functions $\gamma_{zz}$ and $\gamma_{pp}$ develop stepfunction-like discontinuities. A nonvanishing $\gamma_{pz}$ component serves to absorb some of this shear but introduces instabilities on the axis. For this reason, we adopted the hybrid scheme described above, whereby the Cadez metric variables are evolved over much of the grid, and the cylindrical metric variables are evolved over a smaller region covering the saddle. In this way we were able to benefit from the advantages afforded by each coordinate system, while minimizing the problems that each presents.
V. NUMERICAL METHODS AND CODE TESTS

A. Solving the Discrete Einstein Equations

The numerical integration of the evolution equations (10) and (11) is performed in an unconstrained manner because of practical computational time limitations. We do not enforce either the Hamiltonian (8) or the momentum constraints (9) during the course of evolution except at the initial time slice.

Eqs. (10) and (11) are solved using the standard time explicit second order accurate leap frog method whereby the extrinsic curvature components are staggered by a half step in time relative to the metric components. Schematically we have

\begin{equation}
\gamma_i^{n+1/2} = \gamma_i^{n-1/2} - \{2\alpha_i^n K_i^n - \nabla_i^n \} \Delta t,
\end{equation}

and

\begin{equation}
K_i^{n+1} = K_i^n + \alpha_i^{n+1/2} \left( K_i^{n+1/2} + K_i^{n+1/2} \right) \Delta t
+ \left( \beta_i^n \nabla_i^n K_i^n + K_i^n \nabla_i^n - \nabla_i \alpha_i^{n+1/2} \right) \Delta t,
\end{equation}

where subscripts \(i\) (superscripts \(n\)) refer to discrete spatial (temporal) positions. To maintain second order accuracy, variables are extrapolated to the proper time levels as needed using the second order formula

\begin{equation}
K_i^{n+1/2} = \frac{3}{2} K_i^n - \frac{1}{2} K_i^{n-1}.
\end{equation}

This method propagates gravitational waves with less dissipation and dispersion than other methods we have tried [11]. Explicit methods require stringent restrictions on the size of timesteps to maintain stability. A condition we found to provide a good balance between computational speed and accuracy is \(\Delta t = 4M\Delta \eta\), where \(M\) is half the ADM mass defined by (4) and \(\Delta \eta\) is the spacing interval in the radial direction.

As an added measure of stability, we introduce numerical diffusive terms to the discrete evolution equations. This effectively adds to the right-hand-side a term of the form \(k \nabla^2 \gamma_{ij}\) and \(k \nabla^2 K_{ij}\) to the evolution equations (10) and (11) respectively. The coefficient \(k\) is chosen as small as possible to minimize its effect on the accuracy of solutions while enhancing the stability of the time integration. We construct \(k\) dimensionally in the manner \(k \sim (\Delta x)^2/(2\Delta t)\) to scale appropriately with the grid parameters. The proportionality constant is typically chosen to be of order \(\leq 0.05\). We have verified that the addition of these diffusive terms has little effect on the extracted radiation waveforms for the dominant \(\ell = 2\) modes (but the higher frequency \(\ell = 4\) is more sensitive to this procedure, as discussed below).

Spatial derivatives appearing as source terms in the discrete time evolution equations are differenced using both standard second and fourth order center differences to approximate \(\partial_t\) and \(\partial_{\xi}\) on the uniform Cadez grid. We find that fourth order center differences provide more accurate solutions evident in calculations of apparent horizon masses and gravitational wave form extractions. However, because black hole simulations can develop large gradients, fourth order differences are generally less stable than using second order centered differences. (See also Refs. [11] for discussions of the effect of second and fourth order spatial derivatives.) The results presented in this paper and in the series of companion papers [12] were obtained by center differencing to fourth order all spatial Cadez derivatives appearing in the evolution Eqs. (10) and (11). Within the coordinate patch, it is necessary to compute spatial derivatives with respect to \(z\) and \(\rho\) of the cylindrical coordinate based variables. Finite differences of such derivatives (\(\partial_z a, \partial_\rho a, \ldots\)) are computed on the Cadez grid using the chain rule formulas (\(\partial_{\rho} = \eta_{\rho} \partial_\eta + \eta_z \partial_{\xi}, \ldots\)).

The elliptic equations (12) and (13) for the lapse and shift potential respectively are discretized using second order center differences. The resulting coupled algebraic equations give rise to large sparsely banded matrices which we solve using an iterative two dimensional multigrid algorithm developed by Steve Schaffer [26].

B. Convergence Studies

There are limitations to the grid resolutions that can be achieved. First, if the angular spacing \(\Delta \xi\) is too small, the zones poised next to the axis can trigger the axis instability during the evolution. Second, the phenomenon of “grid sucking” in which the black hole absorbs coordinate lines throughout the evolution is enhanced with resolution. As the resolution is increased, the peaks corresponding to the increased proper distance between nodes on the grid near the horizon become more pronounced and gradients grow ever sharper as one obtains more accurate solutions. These
sharp features are difficult to resolve numerically and are ultimately responsible for developing errors at late times, causing the code to crash. With our fixed mesh we cannot afford the computer time to add finer zones arbitrarily to accurately resolve the peaks with sufficient radial resolution to suppress numerical instabilities for the duration of a simulation. The convergence studies presented in this article include 100 (27), 200 (35), and 300 (55) radial (angular) zones on a uniformly spaced grid.

For each parameter run (different values of $\mu$) we extract both the $\ell = 2$ and $\ell = 4$ waveforms at radii of 30, 40, 50, 60, and 70$M$. (Coordinate positions corresponding to physical distances in units of $M$ are approximated from the initial data in the asymptotically spherical far field as $r \sim 1/\sqrt{\xi M / \Psi^2}$. We use results at each of these radii to check the propagation of waves and the consistency of our energy calculations. Table I shows the energy radiated across the gravitational field, ($\sqrt{\xi}$) the detectors closer in have a greater difficulty in separating gravitational wave signals from other parts of the field; ($\mu$) the detectors further out are more susceptible to numerical resolution problems and artificially induced diffusion, and ($\ell = 4$) the finite numerical run times allow for more of the gravitational wave train to pass through the inner detectors. The median deviation is $\sim 0.4\%$. We also find a general agreement among the different detectors at the same resolution. For example, the $\mu = 2.2$ deviation (taken over all detectors) from the average radiated energy is $\sim 5\%$ for the 200 radial zone evolution. A maximum deviation of $\sim 15\%$ occurs between the two detectors furthest from one another. Reasons for the larger discrepancies among the inner- and outermost detectors are the following: ($i$) the detectors closer in have a greater difficulty in separating gravitational wave signals from other parts of the gravitational field, ($ii$) the detectors further out are more susceptible to numerical resolution problems and artificially induced diffusion, and ($iii$) the finite numerical run times allow for more of the gravitational wave train to pass through the inner detectors.

In Fig. 3 we show the $\ell = 2$ waveform extracted at $r = 40M$ for the three different spatial resolutions of 100 (27), 200 (35), and 300 (55) radial (angular) zones for the case $\mu = 2.2$. Up until $t \sim 150M$ the results are quite similar in all three cases. After that time the low resolution run develops some difficulty due to the poorly resolved peak in the metric functions and becomes unstable. We note that with the higher resolution simulations one obtains a slightly better fit to the known quasinormal waveform, especially at late times. The higher resolved waveforms suffer less dispersion and damping attributable to numerical effects. Prior to $t \sim 125M$, the waveforms agree to within $\sim 3\%$ for all resolutions. Furthermore, the diffusion and patch parameters are different for each run, so the $\ell = 2$ waveforms are quite stable with respect changes in computational parameters. In fact for a fixed resolution, varying the patch parameters (such as the size of the patch, the time at which it is lifted and the numerical diffusion) the waveforms vary by no more than $\sim 10\%$, and typically by less than a few per cent.

Next we show results from the more difficult $\ell = 4$ extraction in Fig. 4 for the same $\mu = 2.2$ case. The $\ell = 4$ mode is clearly more sensitive to the computational parameters than the $\ell = 2$ extraction in both the signal preceding the strong quasinormal ringing (beginning at $t \sim 75M$) and in the amplitude of the ringing signal. Also for a fixed resolution, the amplitude of the $\ell = 4$ waveforms varies by about a factor of two with large changes in the computational parameters, with the largest effect coming from the patch parameters. As the energy output varies quadratically with the wave amplitude, the energy carried by the $\ell = 4$ modes is uncertain to about an order of magnitude. In spite of these effects, a fit of the two lowest $\ell = 4$ quasinormal modes to the high resolution run, shown in Refs. 2, 4, is quite good.

The reasons for the $\ell = 4$ extraction to be less accurate are clear. In the first place, the amplitude of the $\ell = 4$ component is much smaller than that of $\ell = 2$, hence harder to extract from the background noise level. Moreover the more complicated angular distribution of the $\ell = 4$ component needs more angular zones to be fully resolved than have been used in these runs. Even though we are unable to determine with great certainty the absolute amplitude of the $\ell = 4$ signal with our present code, it is clearly seen in the data and does match the expected quasinormal frequency. On the other hand, we are confident in the accuracy of the larger and more robust $\ell = 2$ signals, which are not sensitive to computational details as shown in Fig. 5. These figures show both the strengths and limitations of the present code.

**VI. SUMMARY**

In this paper we have presented the methods we developed to evolve the head-on collision of two equal mass black holes initially at rest. This problem is a difficult one that has required a number of numerical strategies designed to handle large gradients, to suppress instabilities, and to deal with singular points in the coordinate system. The result of our work is a code that can accurately evolve the black hole collision problem for a range of initial data sets. By considering simulations run with different numerical parameters and at different resolutions and we have demonstrated that the $\ell = 2$ waveforms and hence the total radiated energy calculations are accurate. The amplitude and early time behavior of the $\ell = 4$ waveforms are more sensitive to numerics, although the essential features (the
wavelength and damping time) of the quasinormal ringing are clearly present and can be accurately resolved. In a series of companion papers [2–4] we present detailed analyses of the physical results obtained using the methods outlined in this article.

Our work represents a step towards solving the more general and dynamic problem of coalescing binary black holes. It is our long term goal to develop a multi-purpose three dimensional code capable of simulating dynamic fully general relativistic spacetimes containing multiple black holes of arbitrary mass, rotation and impact parameters.

ACKNOWLEDGMENTS

We would like to thank David Bernstein for a number of helpful discussions. This work was supported by NCSA, NSF Grant 91-16682, and NSERC Grant No. OGP-121857, and calculations were performed at NCSA and the Pittsburgh Supercomputing Center.

[1] K. Thorne, in Proceedings of the Eight Nishinomiya-Yukawa Symposium on Relativistic Cosmology, edited by M. Sasaki (Universal Academy Press, Japan, 1994).
[2] P. Anninos, D. Hobill, E. Seidel, L. Smarr, and W.-M. Suen, Phys. Rev. Lett. 71, 2851 (1993).
[3] P. Anninos, D. Hobill, E. Seidel, L. Smarr, and W.-M. Suen, Physical Review D (1994), in preparation.
[4] P. Anninos, D. Bernstein, S. Brandt, D. Hobill, E. Seidel, and L. Smarr, Physical Review D (1994), in press.
[5] A. Čadež, Ph.D. thesis, University of North Carolina at Chapel Hill, 1971.
[6] L. Smarr, Ph.D. thesis, University of Texas, Austin, 1975.
[7] K. Eppley, Ph.D. thesis, Princeton University, 1975.
[8] L. Smarr, in Sources of Gravitational Radiation, edited by L. Smarr (Cambridge University Press, Cambridge, 1979), p. 245.
[9] L. Smarr, A. Čadež, B. DeWitt, and K. Eppley, Physical Review D 14, 2443 (1976).
[10] P. Anninos, D. Bernstein, D. Hobill, E. Seidel, L. Smarr, and J. Towns, in Computational Astrophysics: Gas Dynamics and Particle Methods, edited by W. Benz, J. Barnes, E. Muller, and M. Norman (Springer-Verlag, New York, 1994), to appear.
[11] D. Bernstein, D. Hobill, E. Seidel, L. Smarr, and J. Towns, Physical Review D (1994), in press.
[12] A. Abrahams, D. Bernstein, D. Hobill, E. Seidel, and L. Smarr, Physical Review D 45, 3544 (1992).
[13] C. Misner, Phys. Rev. 118, 1110 (1960).
[14] A. Einstein and N. Rosen, Phys. Rev. 48, 73 (1935).
[15] C. W. Misner, Ann. Phys. 24, 102 (1963).
[16] R. W. Lindquist, Jour. Math. Phys. 4, 938 (1963).
[17] D. S. Brill and R. W. Lindquist, Phys. Rev. 131, 471 (1963).
[18] S. G. Hahn and R. W. Lindquist, Ann. Phys. 29, 304 (1964).
[19] A. Čadež, Annals of Physics 83, 449 (1974).
[20] S. W. Hawking, in Black Holes, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973).
[21] P. Anninos, D. Bernstein, S. Brandt, J. Libson, J. Massó, É. Seidel, L. Smarr, W.-M. Suen, and P. Walker, Physical Review Letters (1994), submitted.
[22] R. Arnowitt, S. Deser, and C. W. Misner, in Gravitation: An Introduction to Current Research, edited by L. Witten (John Wiley, New York, 1962).
[23] C. Evans, in Dynamical Spacetimes and Numerical Relativity, edited by J. Centrella (Cambridge University Press, Cambridge, 1986).
[24] L. Smarr and J. York, Physical Review D 17, 2529 (1978).
[25] D. Bernstein, Ph.D. thesis, Dept. Of Physics, University of Illinois Urbana-Champaign, 1993.
[26] S. Schaffer, private communication.

FIG. 1. (a) The Cadez grid is constructed for the case \( \mu = 2.2 \) and displayed in a single quadrant with cylindrical coordinates. The throats are centered on the axis at \( z = \pm \coth \mu \). Lines of constant \( \eta \) concentrically surround the throat locally, and become spherical far from the holes. (b) The computational grid is shown in Cadez coordinates.
FIG. 2. The extrinsic curvature function \( \hat{K}_{\eta\eta} = H_{A} \) is plotted along a \( \xi = \) constant line (\( \sim \pi/2 \)) after a single time step in evolution. The geometric discontinuity in this function is evident.

FIG. 3. We show the \( \ell = 2 \) waveform at various resolutions of 100 (27), 200 (35), and 300 (55) radial (angular) zones for the case \( \mu = 2.2 \). Except for the low resolution at very late times, the waveforms agree to within less than 3% across all simulations.

FIG. 4. We show the \( \ell = 4 \) waveform at various resolutions of 100 (27), 200 (35), and 300 (55) radial (angular) zones for the case \( \mu = 2.2 \). The amplitude of the ringing modes for these simulations varies by about a factor of 2, depending upon the grid resolution and other computational parameters such as the size and duration of the numerical patch and the added numerical diffusion. In spite of the uncertainty in the amplitude of the signals, the \( \ell = 4 \) quasinormal mode is clearly present in all cases.

| detector | 100 radial zones | 200 radial zones | 300 radial zones |
|----------|------------------|------------------|------------------|
| 30M      | 7.032 \times 10^{-4} | 6.098 \times 10^{-4} | 6.068 \times 10^{-4} |
| 40M      | 6.052 \times 10^{-4} | 5.785 \times 10^{-4} | 5.773 \times 10^{-4} |
| 50M      | 5.710 \times 10^{-4} | 5.619 \times 10^{-4} | 5.606 \times 10^{-4} |
| 60M      | 5.346 \times 10^{-4} | 5.476 \times 10^{-4} | 5.461 \times 10^{-4} |
| 70M      | 5.069 \times 10^{-4} | 5.274 \times 10^{-4} | 5.313 \times 10^{-4} |

**TABLE I.** Convergence study of the total radiated energy for the case \( \mu = 2.2 \). The energies are computed at the five detector locations for three different spatial resolutions. The convergence rate is at least quadratic in the grid spacing for all detectors and deviations between the 200 and 300 radial zone simulations are on the order of a few percent.