Finslerian metric function of totally anisotropic type.
Relativistic aspects

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Abstract

The work focuses upon the relativistic and geometric properties of the space–time endowed tentatively with the metric function of the Berwald–Moor type. The zero curvature of indicatrix is a remarkable property of the approach. We demonstrate how the associated geodesic equations can be solved in a transparent way, thereby obtaining possibility to introduce unambiguously the distance, angle, and scalar product. We find convenient indicatrix representation for the associated tetrads and, by attributing to them naturally the general meaning of the bases proper of inertial reference frames, elucidate respective fundamental kinematic relations, including the extensions of Lorentz transformations and velocity subtraction and composition laws. The invariance group for the metric tensor is found.
1. Introduction and Motivation

The pseudoeuclidean metric function suits the cases when the space-time is uniform in all directions. Alternatively, we may imagine a situation when there exist four geometrically distinguished directions and propose the fundamental metric function

\[ F(y) = \sqrt[4]{|y^1y^2y^3y^4|} \]  

(1.1)

\[ y^1 > 0, y^2 > 0, y^3 > 0, y^4 > 0. \]  

(1.5)

\[ y_A = \frac{F^2}{4y^A}, \quad g_{AB} = \frac{\partial y_A}{\partial y^B} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^A \partial y^B}. \]  

(1.3)

DEFINITION. The up-sector \( A_4^{(+)} \in A_4 \) is defined by the conditions:

\[ \{y^1, y^2, y^3, y^4\} \in A_4^{(+)} : y^1 > 0, y^2 > 0, y^3 > 0, y^4 > 0. \]  

(1.4)

We shall focus upon the 4-dimensional case \( N = 4 \); however, many relations and conclusions can be straightforwardly extended to any dimension \( N \geq 2 \), so that we shall retain in formulae a general \( N \) when they are applicable at arbitrary dimension.

Applying the rules (1.3) to (1.5) yields the explicit component values

\[ y_A = \frac{F^2}{4y^A} \]  

(1.6)

\[ g_{AB} = \frac{2y_Ay_B}{F^2} = \frac{F^2}{4y^Ay_B}\delta_{AB}; \]  

(1.7)

the contravariant version of the last tensor is \( \{g^{AB}\} \) with

\[ g^{AB} = \frac{2y^A y^B}{F^2} = \frac{4y^A y^B}{F^2}\delta^{AB}, \]  

(1.8)

DEFINITION. Given a centered vector space \( V_4 \) with some point “O” being the origin and with the members \( y \in V_4 \) issued from the point “O”. Let four directions \( \{e_A\}, A = 1, 2, 3, 4 \) be presupposed in \( V_4 \). We may decompose vectors \( y \) with respect to such a basis, obtaining the component representation \( y = \{y_A\} \). Under these conditions, we define the \( A_4\)-space:

\[ A_4 := \{V_4, e_A, F(y)\}. \]  

(1.2)

Applying the rules (1.3) to (1.5) yields the explicit component values

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\[ g^{AB} = \frac{2y^A y^B}{F^2} = \frac{4y^A y^B}{F^2}\delta^{AB}, \]  

(1.8)
so that $g^{AC}g_{BC} = \delta^A_B$; $\delta$ stands for the Kronecker symbol. The time–like nature of the metric tensor
$$\text{signature}\{g_{AB}\} = (+---)$$
(1.9)
and the constant determinant
$$\det(g_{AB}) = (-1)^{(N+1)}N^{-N}$$
(1.10)
(at any dimension number $N$) are remarkable properties of the space under study (cf. [1]). Owing to (1.9), the metric tensor may be represented as
$$g_{AB} = h^0_A h^0_B - h^1_A h^1_B - h^2_A h^2_B - h^3_A h^3_B$$
(1.11)
in terms of the associated tetrads $\{h^p_A\}$. The reciprocal representation reads
$$g^{AB} = h^0_A h^0_B - h^1_A h^1_B - h^2_A h^2_B - h^3_A h^3_B$$
(1.12)
subject to the reciprocity
$$h^p_A h^q_A = \delta^p_q$$
(1.13)
(the indices $p, q, \ldots$ will be specified over the range 0, 1, 2, 3 unless otherwise is stated explicitly). By comparing (1.10) with (1.11) we may conclude that
$$\det(h^A_p) = N^{-N/2}.$$  
(1.14)

**DEFINITION.** The indicatrix $I_{4}^{(+)} \in \mathcal{A}_{4}^{(+)}$ is defined as follows:

$$y \in I_{4}^{(+)} : \{y \in \mathcal{A}_{4}^{(+)}, F(y) = 1\}.$$  
(1.15)

Using the unit vectors
$$l^A \overset{\text{def}}{=} \frac{y^A}{F(y)}$$
(1.16)
(so that $F(l) = 1$) and choosing a convenient parameterization $\{u^a\}, a, b = 1, 2, 3$, over the indicatrix to have the representation

$$l^A = t^A(u^a),$$
(1.17)

we may construct the projection factors

$$t^A_a(u) \overset{\text{def}}{=} \frac{\partial t^A}{\partial u^a}$$
(1.18)
for the indicatrix to obtain the induced metric tensor over the indicatrix:

$$i_{ab}(u) \overset{\text{def}}{=} -t^A_a u^b g_{AB};$$
(1.19)
here, the minus in front of the right–hand side reflects the indefinite signature (1.9). As was demonstrated in [1], is convenient to treat the indicatrix in terms of the coordinate

$$z^0 \overset{\text{def}}{=} \ln F.$$  
(1.20)

Depending on the sign of the coordinate $z^0$, the space under study is broken into the unification

$$\mathcal{A}_{4}^{(+)} = \mathcal{A}_{4\{z^0>0\}}^{(+)} \cup \mathcal{A}_{4\{z^0=0\}}^{(+)} \cup \mathcal{A}_{4\{z^0<0\}}^{(+)}$$
(1.21)
of three following regions:

\[ \mathcal{A}_{4\{z^0>0\}}^{(+)} := \{ y \in \mathcal{A}_4^{(+)}, F(y) > 1 \}, \quad (1.22) \]

\[ \mathcal{A}_{4\{z^0=0\}}^{(+)} := \{ y \in \mathcal{A}_4^{(+)}, F(y) = 1 \}, \quad (1.23) \]

\[ \mathcal{A}_{4\{z^0<0\}}^{(+)} := \{ y \in \mathcal{A}_4^{(+)}, 1 > F(y) > 0 \}. \quad (1.24) \]

Notice that the sector (1.23) is the indicatrix:

\[ \mathcal{A}_{4\{z^0=0\}}^{(+)} = \mathcal{I}_{4}^{(+)}. \quad (1.25) \]

The known fact is that if we juxtapose (1.20) by an indicatrix coordinate set \( \{ u^a \} \) to obtain the four coordinates

\[ z^p = \{ z^0, z^a = u^a \}, \quad (1.26) \]

then the respective transformation of the Finslerian metric tensor would lead to the result

\[ g_{AB}(y) \frac{\partial y^A}{\partial z^p} \frac{\partial y^B}{\partial z^q} = e^{2z^0} g_{pq}^* \quad (1.27) \]

which is remarkable in that

\[ g_{00}^* = 1, \quad g_{0a}^* = 0, \quad g_{ab}^* = -i_{ab}, \quad (1.28) \]

where \( \{ i_{ab} \} \) is just the indicatrix metric tensor (1.19). Also, in case of the Finslerian metric function (1.5) the tensor \( \{ i_{ab} \} \) proves to be exactly euclidean, so that the conformal representation

\[ g_{pq}^* = e^{2z^0} r_{pq} \quad (1.29) \]

holds with \( \{ r_{pq} \} \) being the pseudoeuclidean metric tensor.

Therefore, it is attractive to introduce the associated conformally–pseudoeuclidean space \( C_4 \):

\[ C_4 := \{ V_4, z^p \in V_4, g_{pq}^* \} \quad (1.30) \]

to have the isometry

\[ \mathcal{A}_{4\{z^0>0\}}^{(+)} \Longleftrightarrow C_4 \quad (1.31) \]

with the decomposition

\[ C_4 = C_4^{(+)} \cup C_4^{(0)} \cup C_4^{(-)}, \quad (1.32) \]

where

\[ C_4^{(+)} := \{ z^p \in C_4^{(+)} : z^0 > 0 \}, \quad (1.33) \]

\[ C_4^{(0)} := \{ z^p \in C_4^{(0)} : z^0 = 0 \}, \quad (1.34) \]

\[ C_4^{(-)} := \{ z^p \in C_4^{(-)} : z^0 < 0 \}, \quad (1.35) \]

so that

\[ \mathcal{A}_{4\{z^0>0\}}^{(+)} \Longleftrightarrow C_4^{(+)}, \quad \mathcal{A}_{4\{z^0=0\}}^{(+)} \Longleftrightarrow C_4^{(0)}, \quad \mathcal{A}_{4\{z^0<0\}}^{(+)} \Longleftrightarrow C_4^{(-)}. \quad (1.36) \]
Now the question is what is the particular and convenient choice for the set \( \{u^a\} \) under which the tensor \( \{i_{ab}\} \) is exactly the diagonal unity, that is, when we get

\[
i_{ab} = \delta_{ab}.
\]  

(1.37)

Obviously, in the last case the fundamental length interval \( ds \) can be given merely by

\[
(ds)^2 = e^{2z_0} \left((dz)^2 - (dz_0)^2\right).
\]  

(1.38)

To anticipate true a due and possible answer to the question, it is useful to note that the choice

\[
l^A = \exp(C^A_a u^a)
\]  

(1.39)

with any constant \( C^A_a \) subjected to the condition

\[
\sum_{A=1}^{4} C^A_a = 0
\]  

(1.40)

would parameterize the indicatrix because of the product structure of the Finslerian metric function (1.5) under study. Also, if we subject the constants to the condition

\[
\sum_{A=1}^{4} C^A_a C^B_b = 4\delta_{ab},
\]  

(1.41)

then, because of the particular structure of the right-hand part in the metric tensor (1.7), we just obtain \( \delta_{ab} \) in the right-hand part of (1.19). When verifying this assertion, it is convenient to note that the projection coefficients (1.18) constructed on the basis of (1.39) bear the structure

\[
t^A_a = C^A_a \cdot l^A
\]  

(1.42)

at any value of the index \( A \). Owing to the exponential nature of the right-hand part in the representation (1.39), it is convenient to call the set \( \{u^a\} \) the \textit{indicatrix variables}.

It is convenient to supplement the constants by the members

\[
C_0^A = 1,
\]  

(1.43)

so that

\[
\sum_{A=1}^{4} C^A_p C^B_q = 4e_{pq},
\]  

(1.44)

where \( \{e_{pq}\} = \text{diagonal}(1, -1, -1, -1) \) is the pseudoeuclidean metric tensor. This entails

\[
\sum_{A=1}^{4} C^a_A = 0.
\]  

(1.45)

The inverse constants \( C^p_A \) obeying the relations

\[
C^p_A C^A_q = \delta^p_q
\]  

(1.46)

must show the properties

\[
C_0^A = \frac{1}{4}
\]  

(1.47)
and
\[ \sum_{A=1}^{4} C_A^a = 0. \tag{1.48} \]
Under these conditions, the representation (1.39) can be inverted to yield
\[ u^a = C_A^a \ln l^A \tag{1.49} \]
and
\[ z^p = C_A^p \ln l^A, \tag{1.50} \]
which in turn yields for the tetrads
\[ h^p_A = F z^p_A = C_A^p \frac{1}{l^A}, \tag{1.51} \]
where
\[ z_A^p = \frac{\partial z^p}{\partial y^A}. \tag{1.52} \]
From (1.51) it follows that
\[ g_{AB} = F^2 c_{AB} \tag{1.53} \]
with the tensor
\[ c_{AB} = z_A^p z_B^p \epsilon_{pq}, \tag{1.54} \]
which demonstrates that the Finslerian metric tensor associated with the metric function (1.1) is conformal to the pseudoeuclidean metric tensor. The conformal multiplier is the square \( F^2 \) of the metric function \( F \).

We shall frequently substitute variables \( \{a^A\} \) with \( \{y^A\} \):
\[ y^A = a^A, \tag{1.55} \]
thereafter the metric function (1.1) takes on the form
\[ F = \sqrt[4]{a^1 a^2 a^3 a^4}. \tag{1.56} \]

In Section 2 we deal with the geodesic equations of the space under study. It proves possible to find the adequate explicit solutions thereto in both the initial–value and fixed–edge forms. This opens up the straightforward way to obtain the angle
\[ \eta(a, b) = \frac{1}{2} \frac{F(b)}{F(a)} \sqrt{ \left( \ln \frac{a^1}{b^1} \right)^2 + \left( \ln \frac{a^2}{b^2} \right)^2 + \left( \ln \frac{a^3}{b^3} \right)^2 + \left( \ln \frac{a^4}{b^4} \right)^2 } \tag{1.57} \]
between two vectors by postulating the cosine theorem. The angle is actually defined by the unit vectors \( l^1(a) \) and \( l^1(b) \):
\[ \eta(a, b) = \frac{1}{2} \sqrt{ \left( \ln \frac{l^1(a)}{l^1(b)} \right)^2 + \left( \ln \frac{l^2(a)}{l^2(b)} \right)^2 + \left( \ln \frac{l^3(a)}{l^3(b)} \right)^2 + \left( \ln \frac{l^4(a)}{l^4(b)} \right)^2 } \tag{1.58} \]
The associated distance and scalar product are also found. The angle is additive when the vectors point to a fixed geodesic curve. In fact, this angle measures the euclidean length in the indicatrix.
In Section 3 we derive step-by-step the kinematic implications of the tetrad choice (1.51). The kinematic coefficients found are of the unit determinant (see (3.10)). Their structure (3.21)–(3.25) entail the following $A_4^{(+)}$-kinematic transformations:

\[ Y'^0 = \frac{Y^0 + s^1 Y^1 + s^2 Y^2 + s^3 Y^3}{\sqrt{(1 + s^1 + s^2 + s^3)(1 - s^1 + s^2 - s^3)(1 + s^1 - s^2 - s^3)(1 - s^1 - s^2 + s^3)}}, \]  

(1.59)

\[ Y'^1 = \frac{s^1 Y^0 + Y^1 + s^3 Y^2 + s^2 Y^3}{\sqrt{(1 + s^1 + s^2 + s^3)(1 - s^1 + s^2 - s^3)(1 + s^1 - s^2 - s^3)(1 - s^1 - s^2 + s^3)}}, \]

(1.60)

\[ Y'^2 = \frac{s^2 Y^0 + s^3 Y^1 + Y^2 + s^1 Y^3}{\sqrt{(1 + s^1 + s^2 + s^3)(1 - s^1 + s^2 - s^3)(1 + s^1 - s^2 - s^3)(1 - s^1 - s^2 + s^3)}}, \]

(1.61)

\[ Y'^3 = \frac{s^3 Y^0 + s^2 Y^1 + s^1 Y^2 + Y^3}{\sqrt{(1 + s^1 + s^2 + s^3)(1 - s^1 + s^2 - s^3)(1 + s^1 - s^2 - s^3)(1 - s^1 - s^2 + s^3)}}, \]

(1.62)

to extend the ordinary Lorentz transformations, where $s^1, s^2, s^3$ are components of three-dimensional motion velocity. For the velocity, we obtain the extended composition law as well the subtraction law in explicit forms. The kinematic $A_4^{(+)}$-length

\[ F(Y) = \sqrt{(Y^0 + Y^1 + Y^2 + Y^3)(Y^0 - Y^1 + Y^2 - Y^3)(Y^0 + Y^1 - Y^2 - Y^3)(Y^0 - Y^1 - Y^2 + Y^3)} \]

(1.63)

is appeared (see (3.30)) which fulfills the kinematic $A_4^{(+)}$-invariance

\[ F(Y') = F(Y) \]

(1.64)

(see (3.31)). The transformations (1.59)–(1.62) obviously extend the ordinary special-relativistic pseudoeuclidean Lorentz transformations

\[ Y'^0_{\text{special Lorentzian}} = \frac{Y^0 + s^1 Y^1}{\sqrt{(1 + s^1)(1 - s^1)}}, \quad Y'^1_{\text{special Lorentzian}} = \frac{s^1 Y^0 + Y^1}{\sqrt{(1 + s^1)(1 - s^1)}}. \]

(1.65)

accordingly the result (1.63) extends the function

\[ F(Y)_{\text{special Lorentzian}} = \sqrt{(Y^0 + Y^1)(Y^0 - Y^1)}. \]

(1.66)

In Section 4, we expose the transformations that leave invariant the Finslerian metric function as well as the Finslerian metric tensor. In the space under study, the transformations are found to be, in general, nonlinear. They realize euclidean rotations and translations in the indicatrix. That is to say, the group of such transformations is a nonlinear image of the euclidean invariance group. The translations in the euclidean indicatrix give rise to scale (product) transformations in the initial space, so that they form a linear (and abelian) subgroup. Detailed calculations are presented in Appendices A, B, and C.

The paper ends with a short Discussion of the key aspects of our approach.

2. Geodesics, distance, and angle in $A_N$–spaces
Given a conformally–pseudoeuclidean space $C_4$ (see (1.30)) with the metric tensor \( \{ g_{pq}^* \} \) prescribed by the conformal representation (1.29). Calculating the partial derivatives \( g_{pq,r}^* = \partial g_{pq}^*/\partial z^r \), we get \( g_{pq,a}^* = 0 \) and \( g_{pq,0}^* = 2g_{pq} \), so that for the components \( \Gamma_{pq} = \frac{1}{2}(g_{pq,r}^* + g_{qr,p}^* - g_{pq,r}^*) \) we shall have the values

\[
\begin{align*}
\Gamma_{000} &= g_{00}^*, \\
\Gamma_{a00} &= \Gamma_{0a0} = 0, \\
\Gamma_{ab0} &= -g_{ab}^*, \\
\Gamma_{abc} &= 0, \\
\Gamma_{pq0} &= g_{pq}^*.
\end{align*}
\]

The associated Christoffel symbols \( \Gamma_{prq} = g_{rs}^* \Gamma_{psq} \) are given by the components

\[
\begin{align*}
\Gamma_{000} &= 1, \\
\Gamma_{a00} &= \Gamma_{0a0} = 0, \\
\Gamma_{ab0} &= \delta_{ab}, \\
\Gamma_{a0b} &= r_{ab}, \\
\Gamma_{acb} &= 0, \\
\Gamma_{pq0} &= \delta_{pq}.
\end{align*}
\]

Let us consider a curve \( C(s) \) parameterized by the length parameter \( s \) (cf. (1.38)) and introduce the respective four-dimensional velocity

\[
U^p = \frac{dz^p}{ds},
\]

so that the velocity is unit:

\[
g_{pq}^*(z)U^pU^q = 1
\]

(the timelike case). The differential equation for the \( C(s) \) to be a geodesic curve

\[
\frac{dU^p}{ds} + \Gamma_{s,r}^p U^s U^r = 0
\]

proves to consist of two parts:

\[
\frac{dU^0}{ds} = -[(U^0)^2 + U^2] = -2(U^0)^2 + e^{-2z^0}
\]

and

\[
\frac{dU^a}{ds} = -2U^a U^0.
\]

The equation

\[
\frac{d^2z^0}{ds^2} + 2\left( \frac{dz^0}{ds} \right)^2 = e^{-2z^0}
\]

can readily be integrated, yielding

\[
z^0 = \ln(f(s))
\]

with

\[
f(s) = \sqrt{a^2 + 2bs + s^2},
\]

where \( a \) and \( b \) are integration constants.

Since \( z^0 = \ln F \) (see (1.20)), from (2.10) it follows that the Finslerian metric function varies along the geodesics according to the law

\[
F(s) = \sqrt{a^2 + 2bs + s^2}.
\]

Furthermore, using

\[
\frac{dz^0}{ds} = \frac{b + s}{(F(s))^2} = U^0
\]
in (2.7) enables us to readily find
\[ U^a = \frac{\sqrt{b^2 - a^2} n^a}{(F(s))^2}, \]  
where \( n^a \) is a set of constants. To fulfill (2.4), the set must be subjected to the unity length condition:
\[ \delta_{ab} n^a n^b = 1. \]  
Using \( U^a = dz^a/ds \) (see (2.3)) in (2.13) gives us a differential equation to find the functions \( z^a(s) \). The equation can readily be integrated to yield
\[ z^a(s) = m^a + n^a \frac{1}{2} \ln \frac{s + b - \sqrt{b^2 - a^2}}{s + b + \sqrt{b^2 - a^2}}, \]  
where \( m^a \) are new integration constants; we assume
\[ b^2 - a^2 \geq 0, \quad a > 0. \]  
Eqs. (2.11)–(2.14) upon the condition (2.16) fulfill (2.4).

In this way we obtain explicitly the following formulae:
\[ r_1^0 = \ln a, \quad r_2^0 = \ln (F(\Delta s)), \quad \sqrt{b^2 - a^2} = a^2 |v_1|, \quad b = a \sqrt{1 + a^2 |v_1|^2}, \]  
\[ r^0 = \frac{1}{2} \ln (a^2 + 2bs + s^2), \]  
\[ r(s) = r_1 + \frac{1}{2} v_1 \frac{a^2}{\sqrt{b^2 - a^2}} \ln (X(s)), \]  
where \( r^0 = z^0, \ r = \{z^a_1\}, \ v = \{v^a_1\}, \) and
\[ X(s) = \frac{s + b - \sqrt{b^2 - a^2} b + \sqrt{b^2 - a^2}}{s + b + \sqrt{b^2 - a^2} b - \sqrt{b^2 - a^2}}. \]  
The last function can also be represented in the forms
\[ X(s) = \frac{\left[ a^2 + (b + \sqrt{b^2 - a^2}) s \right]^2}{a^2 F^2(s)} = \frac{\left[ a^2 + bs + \sqrt{b^2 - a^2} s \right]^2}{a^2 F^2(s)}. \]  
Thus we have arrived at

**PROPOSITION.** The initial–value solution to the geodesic equations (2.5) under study can explicitly be given by Eqs. (2.17)–(2.20).

Also, it is possible to explicate the representation
\[ r(s) = r_1 + \frac{r_2 - r_1}{|r_2 - r_1|} \ln \sqrt{X(s)}, \]  
with
\[ |r_2 - r_1| = \ln \sqrt{X(\Delta s)}, \]
\[ b = \frac{\lvert \mathbf{r}_1 \rvert \lvert \mathbf{r}_2 \rvert \cosh \lvert \mathbf{r}_2 - \mathbf{r}_1 \rvert - \lvert \mathbf{r}_1 \rvert^2}{\Delta s}, \quad (2.24) \]

\[ \sqrt{b^2 - a^2} = \frac{\lvert \mathbf{r}_1 \rvert \lvert \mathbf{r}_2 \rvert \sinh \lvert \mathbf{r}_2 - \mathbf{r}_1 \rvert}{\Delta s}, \quad (2.25) \]

and

\[ (\Delta s)^2 = \lvert \mathbf{r}_1 \rvert^2 + \lvert \mathbf{r}_2 \rvert^2 - 2 \lvert \mathbf{r}_1 \rvert \lvert \mathbf{r}_2 \rvert \cosh \lvert \mathbf{r}_2 - \mathbf{r}_1 \rvert. \quad (2.26) \]

Thus, we have obtained

**PROPOSITION.** The fixed-edge solution to the geodesic equations (2.5) under study can explicitly be given by Eqs. (2.22)–(2.26).

![Geodesic curve](image.png)

Fig 1: A geodesic curve \( C \); the length of the curve from p. \( P_1 \) to p. \( P_2 \) is equal to \( \Delta s \), and from p. \( P_1 \) to p. \( P(s) \) is equal to \( s \).

The formula (2.26) can also be written as

\[ (\Delta s)^2 = \lvert \mathbf{r}_1 \rvert^2 + \lvert \mathbf{r}_2 \rvert^2 - 2 \lvert \mathbf{r}_1 \rvert \lvert \mathbf{r}_2 \rvert \cosh \left( \eta(\mathbf{r}_1, \mathbf{r}_2) \right) \]

with the following \( C_4 \)-angle:

\[ \eta(\mathbf{r}_1, \mathbf{r}_2) = \lvert \mathbf{r}_2 - \mathbf{r}_1 \rvert. \quad (2.28) \]

**PROPOSITION.** The \( C_4 \)-cosine theorem reads as (2.26) or (2.28).

In view of (2.21) and (2.24)–(2.25), we can write

\[ X(\Delta s) = e^{2\eta}. \quad (2.29) \]

By comparing (2.15) and (2.20) with the unit vector representation of the exponential type (1.39), we can readily conclude that the components of the unit vector \( l^A \) vary along geodesics in accordance with the law

\[ l^A(s) = l^A(0) \left( X(s) \right)^{\ln_{X(\Delta s)}(l^A(\Delta s)/l^A(0))}, \quad (2.30) \]

where

\[ \prod_{A=1}^{N} l^A(0) = 1, \quad \prod_{A=1}^{N} l^A(\Delta s) = 1, \quad \prod_{A=1}^{N} l^A(s) = 1.\quad (2.31) \]
This law is applicable at any dimension \( N \geq 2 \).

For the vector

\[
a^A(s) = F(s)l^A(s)
\]

we obtain from (2.30) the similar behaviour

\[
a^A(s) = \frac{F(s)a^A(0)}{F(0)} \left( X(s) \right)^{\ln X(\Delta s)(a^A(\Delta s)F(0)/a^A(0)F(\Delta s))},
\]

where \( F(0) = a \) (in view of (2.11)).

Thus we have arrived at

**PROPOSITION.** Given two vectors \( \{a^A_{\{1\}}\} \) and \( \{a^A_{\{2\}}\} \). Let \( C \) be a curve going from the end of the first vector to the end of the second vector. Put \( a^A(0) = \{a^A_{\{1\}}\} \) and \( a^A(\Delta s) = \{a^A_{\{2\}}\} \). Attribute the length values \( s = 0 \) and \( s = \Delta s \) to the vectors, where \( \Delta s \) is the length of the curve \( C \). If \( C \) is a geodesics, then the vector stretching to the geodesics point with a length value \( s \) is explicitly given by (2.33).

The result (2.11) entails the relation

\[
a^2F^2(\Delta s) = (a^2 + b\Delta s)^2 - (\sqrt{b^2 - a^2 \Delta s})^2,
\]

(2.34)
which can be used to introduce the angle \( \eta \) according to
\[
a^2 + b\Delta s = aF(\Delta s) \cosh \eta
\] (2.35)

and
\[
\sqrt{b^2 - a^2} \Delta s = aF(\Delta s) \sinh \eta,
\] (2.36)
or
\[
\frac{a^2 + b\Delta s}{\sqrt{b^2 - a^2} \Delta s} = \tanh \eta.
\] (2.37)

Applying (2.35) and (2.36) to (2.21), it follows that
\[
X(\Delta s) = (\cosh \eta + \sinh \eta)^2 = e^{2\eta},
\] (2.38)

so that the equality (2.29) has been reproduced. Therefore, we may write the laws (2.30) and (2.33) in the forms
\[
l^A(s) = l^A(0) \left( X(s) \right)^{\frac{1}{2\eta}} \ln\left( \frac{l^A(\Delta s)}{l^A(0)} \right)
\] (2.39)

and
\[
a^A(s) = \frac{F(s)a^A(0)}{F(0)} \left( X(s) \right)^{\frac{1}{2\eta}} \ln\left( \frac{a^A(\Delta s)F(0)}{a^A(0)F(\Delta s)} \right).
\] (2.40)

Since
\[
d\left( \frac{s + b - \sqrt{b^2 - a^2}}{s + b + \sqrt{b^2 - a^2}} \right) = \frac{2\sqrt{b^2 - a^2}}{F^2} \left( s + b + \sqrt{b^2 - a^2} \right),
\] (2.41)

from (2.16) and (2.40) we can conclude that
\[
F(s) \frac{d a^A}{ds} = \frac{dF(s)}{ds} a^A(s) + 2\sqrt{b^2 - a^2} a^A(s) \frac{1}{2\eta} \ln\left( \frac{l^A(\Delta s)}{l^A(0)} \right).
\] (2.42)

Using here
\[
g_{AB} = \frac{2a_A a_B}{F^2} - \frac{F^2}{Na^A a^B} \delta_{AB}, \quad a_A = \frac{F^2}{Na^A}
\] (2.43)

(see (1.6) and (1.7)), and noting that
\[
\prod_{A=1}^{N} \ln\left( \frac{l^A(\Delta s)}{l^A(0)} \right) = 0,
\] (2.44)

we find the equality
\[
g_{AB}(a^C) \frac{d a^A}{ds} \frac{d a^B}{ds} = \left( \frac{dF}{ds} \right)^2 - (b^2 - a^2) \frac{1}{F^2} \frac{1}{N\eta^2} \sum_{A=1}^{N} \left( \ln\left( \frac{l^A(\Delta s)}{l^A(0)} \right) \right)^2
\]

\[
= 1 + \frac{b^2 - a^2}{F^2} - (b^2 - a^2) \frac{1}{F^2} \frac{1}{N\eta^2} \sum_{A=1}^{N} \left( \ln\left( \frac{l^A(\Delta s)}{l^A(0)} \right) \right)^2.
\] (2.45)
The left-hand side here must be 1. Therefore, the angle \( \eta \) can be given by

\[
\eta = \sqrt{\frac{1}{N} \sum_{A=1}^{N} \left( \ln \frac{l^A(\Delta s)}{l^A(0)} \right)^2},
\]  

(2.46)

or equivalently,

\[
\eta = \sqrt{\frac{1}{N} \sum_{A=1}^{N} \left( \ln \frac{a^A(\Delta s)F(0)}{a^A(0)F(\Delta s)} \right)^2}.
\]  

(2.47)

If we merely consider two vectors \( \{a^A\} \) and \( \{b^A\} \), then (2.47) assigns for them the respective angle

\[
\eta(a, b) = \sqrt{\frac{1}{N} \sum_{A=1}^{N} \left( \ln \frac{a^A F(b)}{b^A F(a)} \right)^2}.
\]  

(2.48)

Thus, the following assertions are valid.

PROPOSITION. The angle between two vectors \( \{a^A\} \) and \( \{b^A\} \) is given by (2.48). The angle is symmetric

\[
\eta(a, b) = \eta(b, a).
\]  

(2.49)

Also, the angle is additive

\[
\eta(a, b) + \eta(b, c) = \eta(a, c),
\]  

(2.50)

when the vectors \( \{a^A, b^A, c^A\} \) point to a fixed geodesic curve.

Fig 4: The angle \( \eta(a_{(1)}, a_{(2)}) \)

Rewriting (2.34) as

\[
F^2(\Delta s) = (\Delta s)^2 - a^2 + 2(a^2 + b\Delta s)
\]  

(2.51)

and use (2.35), we get

the Finslerian \( \mathcal{A}_N^{(+)} \)-Cosine Theorem:

\[
(\Delta s)^2 = (F(a))^2 + (F(b))^2 - 2F(a)F(b) \cosh(\eta(a, b)).
\]  

(2.52)
Therefore, the Finslerian $A_{N}^{(+)}-$Distance $|b \ominus a|$ between end points of two given vectors is

$$|b \ominus a|^2 = (F(a))^2 + (F(b))^2 - 2F(a)F(b) \cosh(\eta(a, b)).$$  \hspace{1cm} (2.53)

The Finslerian $A_{N}^{(+)}-$Scalar Product

$$(ab) = F(a)F(b) \cosh(\eta(a, b))$$  \hspace{1cm} (2.54)

is obtained.

NOTE. In the dimension

$$N = 4$$  \hspace{1cm} (2.55)

we may use in the above expression (2.46) the indicatrix representation (1.39) and apply (1.40). On so doing, we obtain

$$\eta = \sqrt{(\Delta u^1)^2 + (\Delta u^2)^2 + (\Delta u^3)^2}.$$  \hspace{1cm} (2.56)

Since at the same time the variables $\{u^1, u^2, u^3\}$ are some euclidean coordinates on the indicatrix (see (1.37)), we may state the following result.

PROPOSITION. The Finslerian angle $\eta$ is tantamount to the indicatrix euclidean distance.

It may also be said that, to entire analogy to the euclidean geometry proper, the Finslerian angle $\eta$ found measures the geodesic lengths on the indicatrix. However, in the euclidean geometry the arcs are pieces of circles (the euclidean indicatrix is a unit sphere), while in our present case they are pieces of straightlines (since the indicatrix is a euclidean plane). It is useful to compare (2.56) with the representation (2.28) of the angle $\eta$.

NOTE. The two–dimensional case

$$N = 2$$  \hspace{1cm} (2.57)

is also comprised by the above formulae. Namely, the $\{N = 2\}$–dimensional precursor to the angle (2.48) reads

$$\eta^{(N=2)}(a, b) = \sqrt{\frac{1}{2} \sum_{A=1}^{2} \left( \ln \frac{a^A F(b)}{b^A F(a)} \right)^2} = \sqrt{\frac{1}{2} \left( \left( \ln \frac{a^1 F(b)}{b^1 F(a)} \right)^2 + \left( \ln \frac{a^2 F(b)}{b^2 F(a)} \right)^2 \right)}$$

(with $F(a) = \sqrt{a^1 a^2}$ and $F(b) = \sqrt{b^1 b^2}$), or

$$\eta^{(N=2)}(a, b) = \ln \frac{a^1 F(b)}{b^1 F(a)}.$$  \hspace{1cm} (2.58)

Therefore,

$$\cosh(\eta^{(N=2)}(a, b)) = \frac{1}{2} \left( \frac{a^1 F(b)}{b^1 F(a)} + \frac{b^1 F(a)}{a^1 F(b)} \right) = \frac{a^1 b^2 + b^1 a^2}{F(a)F(b)}.$$  \hspace{1cm} (2.59)

On taking

$$m^1 = \frac{a^1 + a^2}{2}, \quad m^2 = \frac{a^1 - a^2}{2}, \quad n^1 = \frac{b^1 + b^2}{2}, \quad n^2 = \frac{b^1 - b^2}{2},$$  \hspace{1cm} (2.60)
our explication(2.59) just reduces to the ordinary relativistic rule
\[
\cosh(\eta_{(N=2)}(a, b)) = \frac{m_1n_1 - m_2n_2}{\sqrt{(m_1)^2 - (m_2)^2}} \frac{\sqrt{(n_1)^2 - (n_2)^2}}{2} \tag{2.61}
\]

3. Explicated extension of Lorentzian relations

To treat the kinematic topics, we are to consider two (inertial) reference frames \(S\{a\}\) and \(S\{b\}\) moving in the four–dimensional directions of vectors \(a^A\) and \(b^A\). The tetrads \(h^p_A(a)\) and \(h^p_A(b)\) play naturally the roles of their reference systems proper. Let a signal move in the direction of a four–dimensional vector \(R^A\). Then with respect to the frames the components of the vector are
\[
R^p_{(a)} = h^p_A(a)R^A, \quad R^p_{(b)} = h^p_A(b)R^A, \tag{3.1}
\]
respectively. Therefore, the transformation law
\[
R^p_{(a)} = N^p_q(a, b)R^q_{(b)} \tag{3.2}
\]
from the reference frame \(S\{b\}\) into the reference frame \(S\{a\}\) is realized by means of the kinematic coefficients
\[
N^p_q(a, b) = h^p_A(a)h^A_q(b). \tag{3.3}
\]
If we apply here the representation (1.50)–(1.52) for the tetrads, we find that the very coefficients \(C^0_p\) disappear in the right–hand part of (3.3), the symmetry
\[
N^p_q = N^q_p, \tag{3.4}
\]
holds, and the components of (3.3) are given explicitly by means of the formulae
\[
N^1_2 = N^0_3, \quad N^1_3 = N^2_0, \quad N^2_3 = N^1_0, \tag{3.5}
\]
and
\[
N^0_0(a, b) = \frac{F(a)}{4F(b)} \left( \frac{b^1}{a^1} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + \frac{b^4}{a^4} \right), \tag{3.6}
\]
\[
N^1_1(a, b) = \frac{F(a)}{4F(b)} \left( \frac{b^1}{a^1} - \frac{b^2}{a^2} + \frac{b^3}{a^3} - \frac{b^4}{a^4} \right), \tag{3.7}
\]
\[
N^2_2(a, b) = \frac{F(a)}{4F(b)} \left( \frac{b^1}{a^1} + \frac{b^2}{a^2} - \frac{b^3}{a^3} - \frac{b^4}{a^4} \right), \tag{3.8}
\]
\[
N^3_3(a, b) = \frac{F(a)}{4F(b)} \left( \frac{b^1}{a^1} - \frac{b^2}{a^2} - \frac{b^3}{a^3} + \frac{b^4}{a^4} \right), \tag{3.9}
\]
from which it follows that the determinant is unit
\[
\text{det}(N^p_q) = 1, \tag{3.10}
\]
the identity
\[
(N^0_0 + N^1_1 + N^2_2 + N^3_3)(N^0_0 - N^1_1 + N^2_2 - N^3_3)(N^0_0 + N^1_1 - N^2_2 - N^3_3)(N^0_0 - N^1_1 - N^2_2 + N^3_3) = 1 \tag{3.11}
\]
PROPOSITION. The group property

\[ N_q^p(a, c) = N_r^p(a, b)N_q^r(b, c) \]  

holds.

To verify the identity (3.11) it is worth evaluating from (3.6)–(3.9) the equalities

\[ \frac{F(a) b}{F(b) a} = N_0^0(a, b) + N_0^1(a, b) + N_0^2(a, b) + N_0^3(a, b), \] (3.13)

\[ \frac{F(a) b^2}{F(b) a^2} = N_0^0(a, b) - N_0^1(a, b) + N_0^2(a, b) - N_0^3(a, b), \] (3.14)

\[ \frac{F(a) b^3}{F(b) a^3} = N_0^0(a, b) + N_0^1(a, b) - N_0^2(a, b) - N_0^3(a, b), \] (3.15)

\[ \frac{F(a) b^4}{F(b) a^4} = N_0^0(a, b) - N_0^1(a, b) - N_0^2(a, b) + N_0^3(a, b). \] (3.16)

The three-dimensional relative velocity

\[ s^a(a, b) = \frac{N_0^a(a, b)}{N_0^0(a, b)} = \frac{h_A^a(a)b^A}{h_B^a(a)b^B} \] (3.17)

of the reference frame \( S\{b\} \) with respect to the reference frame \( S\{a\} \), together with the adjoint velocity measure

\[ s_a(a, b) = \frac{N_0^a(a, b)}{N_0^0(a, b)}, \] (3.18)

can naturally be introduced. It proves that

\[ s^a = s_a. \] (3.19)

The kinematic coefficients can be written as functions of the velocity \( s^a \):

\[ N_q^p = N_q^p(s^a). \] (3.20)

Indeed, using (3.11) and introducing the relativistic \( A_4^{(+)} \)-dilatation factor

\[ A(s) = \sqrt{(1 + s^1 + s^2 + s^3)(1 - s^1 + s^2 - s^3)(1 + s^1 - s^2 - s^3)(1 - s^1 - s^2 + s^3)}, \] (3.21)

we can establish the equalities

\[ N_0^0 = N_1^1 = N_2^2 = N_3^3 = \frac{1}{A}, \] (3.22)

\[ N_0^a = \frac{1}{A}s^a, \] (3.23)
\[ N^0_a = \frac{1}{A} s_a, \quad (3.24) \]
\[ N_1^1 = N_2^2 = \frac{1}{A} s^3, \quad N_3^1 = N_3^3 = \frac{1}{A} s^2, \quad N_3^2 = \frac{1}{A} s^1. \quad (3.25) \]

Thus, the following result can be formulated.

**PROPOSITION.** The kinematic coefficients depend on the vectors \( a^A \) and \( b^A \) through only the relative velocity \( s^a \).

This assertion can be meant to claim the extended relativity principle.

For various purposes of calculations it occurs convenient to rewrite (3.13)–(3.16) as follows:

\[
\frac{F(a) b^1}{F(b) a^1} = \frac{1}{A(s)} (1 + s^1 + s^2 + s^3),
\quad (3.26)
\]
\[
\frac{F(a) b^2}{F(b) a^2} = \frac{1}{A(s)} (1 - s^1 + s^2 - s^3),
\quad (3.27)
\]
\[
\frac{F(a) b^3}{F(b) a^3} = \frac{1}{A(s)} (1 + s^1 - s^2 - s^3),
\quad (3.28)
\]
\[
\frac{F(a) b^4}{F(b) a^4} = \frac{1}{A(s)} (1 - s^1 - s^2 + s^3).
\quad (3.29)
\]

The above observations suggest the idea to introduce

**DEFINITION.** Given a four–dimensional vector \( Y = \{Y^p\} \) in an inertial reference frame. The *kinematic \( A^{(+)}_4 \)–length* \( \mathcal{F}(Y) \) of the vector reads

\[
\mathcal{F}(Y) = \sqrt{(Y^0 + Y^1 + Y^2 + Y^3)(Y^0 - Y^1 + Y^2 - Y^3)(Y^0 + Y^1 - Y^2 - Y^3)(Y^0 - Y^1 - Y^2 + Y^3)}. \quad (3.30)
\]

The definition entails the \( A^{(+)}_4 \)–invariance

\[
\mathcal{F}(N^p_q(s^a)Y^q) = \mathcal{F}(Y^p). \quad (3.31)
\]

There exists a simple way to verify the invariance. Namely, applying the coefficients (3.22)–(3.25) to the parentheses appeared under the root in the right–hand part of (3.30) yields

\[
(Y^0 + Y^1 + Y^2 + Y^3) \rightarrow \frac{1}{A(s)} (1 + s^1 + s^2 + s^3)(Y^0 + Y^1 + Y^2 + Y^3),
\quad (3.32)
\]
\[
(Y^0 - Y^1 + Y^2 - Y^3) \rightarrow \frac{1}{A(s)} (1 - s^1 + s^2 - s^3)(Y^0 - Y^1 + Y^2 - Y^3),
\quad (3.33)
\]
\[
(Y^0 + Y^1 - Y^2 - Y^3) \rightarrow \frac{1}{A(s)} (1 + s^1 - s^2 - s^3)(Y^0 + Y^1 - Y^2 - Y^3),
\quad (3.34)
\]
\[
(Y^0 - Y^1 - Y^2 + Y^3) \rightarrow \frac{1}{A(s)} (1 - s^1 - s^2 + s^3)(Y^0 - Y^1 - Y^2 + Y^3),
\quad (3.35)
\]

so that on taking into account the right–hand part in the expression (3.21) of \( A(s) \) we may just establish (3.31) (that is to say, the quantities \( s^a \) do disappear in the left–hand part of (3.31)).
Let us find the subtraction law

\[ s_{\{1\}} = s_{\{3\}} \ominus s_{\{2\}} \]  

(3.36)

for the velocities introduced. The law is an explication from the group law (3.12). Given three vectors \( a^A, b^A, c^A \), we may consider the velocities \( s_{\{1\}}^a = s_{\{1\}}^a(a, b), \ s_{\{2\}}^a = s_{\{2\}}^a(b, c), \ s_{\{3\}}^a = s_{\{3\}}^a(a, c) \) and find the components

\[
\begin{align*}
\frac{b^1}{a^1} - \frac{b^2}{a^2} + \frac{b^3}{a^3} - \frac{b^4}{a^4}, \ (3.37) \\
\frac{b^1}{a^1} + \frac{b^2}{a^2} - \frac{b^3}{a^3} - \frac{b^4}{a^4}, \ (3.38) \\
\frac{b^1}{a^1} - \frac{b^2}{a^2} - \frac{b^3}{a^3} + \frac{b^4}{a^4}, \ (3.39)
\end{align*}
\]

which can be transformed to read

\[
\begin{align*}
\frac{b^1 c^1}{c^1 a^1} - \frac{b^2 c^2}{c^2 a^2} + \frac{b^3 c^3}{c^3 a^3} - \frac{b^4 c^4}{c^4 a^4}, \ (3.40) \\
\frac{b^1 c^1}{c^1 a^1} + \frac{b^2 c^2}{c^2 a^2} - \frac{b^3 c^3}{c^3 a^3} - \frac{b^4 c^4}{c^4 a^4}, \ (3.41) \\
\frac{b^1 c^1}{c^1 a^1} - \frac{b^2 c^2}{c^2 a^2} - \frac{b^3 c^3}{c^3 a^3} + \frac{b^4 c^4}{c^4 a^4}. \ (3.42)
\end{align*}
\]

We may apply here (3.26)–(3.29) in terms of the quantities

\[
\begin{align*}
J_{11} &= 1 + s_{\{1\}}^1 + s_{\{1\}}^2 + s_{\{1\}}^3, \\
J_{21} &= 1 + s_{\{2\}}^1 + s_{\{2\}}^2 + s_{\{2\}}^3, \\
J_{31} &= 1 + s_{\{3\}}^1 + s_{\{3\}}^2 + s_{\{3\}}^3, \\
J_{12} &= 1 - s_{\{1\}}^1 + s_{\{1\}}^2 - s_{\{1\}}^3, \\
J_{22} &= 1 - s_{\{2\}}^1 + s_{\{2\}}^2 - s_{\{2\}}^3, \\
J_{32} &= 1 - s_{\{3\}}^1 + s_{\{3\}}^2 - s_{\{3\}}^3, \\
J_{13} &= 1 + s_{\{1\}}^1 - s_{\{1\}}^2 - s_{\{1\}}^3, \\
J_{23} &= 1 + s_{\{2\}}^1 - s_{\{2\}}^2 - s_{\{2\}}^3, \\
J_{33} &= 1 + s_{\{3\}}^1 - s_{\{3\}}^2 - s_{\{3\}}^3.
\end{align*}
\]  

(3.43)–(3.45)
\[ J_{14} = 1 - s_{(1)}^1 - s_{(1)}^2 + s_{(1)}^3, \quad J_{24} = 1 - s_{(2)}^1 - s_{(2)}^2 + s_{(2)}^3, \quad J_{34} = 1 - s_{(3)}^1 - s_{(3)}^2 + s_{(3)}^3, \]

obtaining

\[ s_{(3)}^1 = \frac{J_{31}/J_{21} - J_{32}/J_{22} + J_{33}/J_{23} - J_{34}/J_{24}}{J_{31}/J_{21} + J_{32}/J_{22} + J_{33}/J_{23} + J_{34}/J_{24}}, \quad (3.46) \]

\[ s_{(3)}^2 = \frac{J_{31}/J_{21} - J_{32}/J_{22} - J_{33}/J_{23} - J_{34}/J_{24}}{J_{31}/J_{21} + J_{32}/J_{22} + J_{33}/J_{23} + J_{34}/J_{24}}, \quad (3.47) \]

\[ s_{(3)}^3 = \frac{J_{31}/J_{21} - J_{32}/J_{22} - J_{33}/J_{23} + J_{34}/J_{24}}{J_{31}/J_{21} + J_{32}/J_{22} + J_{33}/J_{23} + J_{34}/J_{24}}, \quad (3.48) \]

Simplifying the right–hand parts eventually gives

\[ Hs_{(1)}^1 = \frac{1 + s_{(3)}^1 + s_{(3)}^2 + s_{(3)}^3}{1 + s_{(2)}^1 + s_{(2)}^2 + s_{(2)}^3} \left[ 1 - 1 - s_{(3)}^1 - s_{(3)}^2 - s_{(3)}^3 \right] + \frac{1 + s_{(3)}^1 - s_{(3)}^2 - s_{(3)}^3}{1 + s_{(2)}^1 - s_{(2)}^2 - s_{(2)}^3} \left[ 1 - s_{(3)}^2 + s_{(3)}^3 \right], \quad (3.50) \]

\[ Hs_{(1)}^2 = \frac{1 + s_{(3)}^1 + s_{(3)}^2 + s_{(3)}^3}{1 + s_{(2)}^1 + s_{(2)}^2 + s_{(2)}^3} \left[ 1 - 1 - s_{(3)}^1 - s_{(3)}^2 - s_{(3)}^3 \right] + \frac{1 + s_{(3)}^1 - s_{(3)}^2 - s_{(3)}^3}{1 + s_{(2)}^1 - s_{(2)}^2 - s_{(2)}^3} \left[ 1 - s_{(3)}^2 + s_{(3)}^3 \right], \quad (3.51) \]

\[ Hs_{(1)}^3 = \frac{1 + s_{(3)}^1 + s_{(3)}^2 + s_{(3)}^3}{1 + s_{(2)}^1 + s_{(2)}^2 + s_{(2)}^3} \left[ 1 - 1 - s_{(3)}^1 - s_{(3)}^2 - s_{(3)}^3 \right] + \frac{1 + s_{(3)}^1 - s_{(3)}^2 - s_{(3)}^3}{1 + s_{(2)}^1 - s_{(2)}^2 - s_{(2)}^3} \left[ 1 - s_{(3)}^2 + s_{(3)}^3 \right], \quad (3.52) \]

with

\[ H = \frac{1 + s_{(3)}^1 + s_{(3)}^2 + s_{(3)}^3}{1 + s_{(2)}^1 + s_{(2)}^2 + s_{(2)}^3} \left[ 1 - 1 - s_{(3)}^1 - s_{(3)}^2 - s_{(3)}^3 \right] + \frac{1 + s_{(3)}^1 - s_{(3)}^2 - s_{(3)}^3}{1 + s_{(2)}^1 - s_{(2)}^2 - s_{(2)}^3} \left[ 1 - s_{(3)}^2 + s_{(3)}^3 \right] + \frac{1 + s_{(3)}^1 - s_{(3)}^2 - s_{(3)}^3}{1 + s_{(2)}^1 - s_{(2)}^2 + s_{(2)}^3} \left[ 1 - s_{(2)}^1 - s_{(2)}^2 + s_{(2)}^3 \right]. \quad (3.53) \]

Thus we have arrived at the following result.

**PROPOSITION.** Under the Finslerian treatment of the \( A_4^{(+)} \)-space, the law of subtraction (3.36) for three-dimensional relativistic velocities is given by the explicit formulas (3.50)–(3.53).

The composition law

\[ s_{(3)} = s_{(1)} \oplus s_{(2)} \quad (3.54) \]

can be evaluated in a similar fashion. Namely, we have

\[ s_{(3)}^1 = \frac{c^1 - c^2 + c^3 - c^4}{a^1 + a^2 + a^3 + a^4}, \quad (3.55) \]
\begin{align}
 s_{\{3\}}^2 &= \frac{c^1 + c^2 - c^3 - c^4}{a^1 + a^2 + a^3 + a^4}, \quad \text{(3.56)} \\
 s_{\{3\}}^3 &= \frac{c^1 - c^2 - c^3 + c^4}{a^1 + a^2 + a^3 + a^4}, \quad \text{(3.57)}
\end{align}

and
\begin{align}
 s_{\{3\}}^1 &= \frac{c^1 b^1}{b^1 a^1} - \frac{c^2 b^2}{b^1 a^2} + \frac{c^3 b^3}{b^1 a^3} - \frac{c^4 b^4}{b^1 a^4}, \\
 s_{\{3\}}^2 &= \frac{c^1 b^1}{b^1 a^1} + \frac{c^2 b^2}{b^2 a^2} - \frac{c^3 b^3}{b^2 a^3} + \frac{c^4 b^4}{b^2 a^4}, \\
 s_{\{3\}}^3 &= \frac{c^1 b^1}{b^1 a^1} + \frac{c^2 b^2}{b^2 a^2} + \frac{c^3 b^3}{b^2 a^3} + \frac{c^4 b^4}{b^2 a^4},
\end{align}

\begin{align}
 s_{\{3\}}^1 &= \frac{J_{11} J_{21} - J_{12} J_{22} + J_{13} J_{23} - J_{14} J_{24}}{J_{11} J_{21} + J_{12} J_{22} + J_{13} J_{23} + J_{14} J_{24}}, \quad \text{(3.61)} \\
 s_{\{3\}}^2 &= \frac{J_{11} J_{21} + J_{12} J_{22} - J_{13} J_{23} - J_{14} J_{24}}{J_{11} J_{21} + J_{12} J_{22} + J_{13} J_{23} + J_{14} J_{24}}, \quad \text{(3.62)} \\
 s_{\{3\}}^3 &= \frac{J_{11} J_{21} - J_{12} J_{22} - J_{13} J_{23} + J_{14} J_{24}}{J_{11} J_{21} + J_{12} J_{22} + J_{13} J_{23} + J_{14} J_{24}}. \quad \text{(3.63)}
\end{align}

Due simplifications yield eventually

**PROPOSITION.** The $A_{4}^{(+)\text{-composition law}}$ (3.54) involves the explicit components
\begin{align}
 s_{\{3\}}^1 &= \frac{s_{\{1\}}^1 + s_{\{2\}}^1 + s_{\{1\}}^2 s_{\{2\}}^2 + s_{\{1\}}^3 s_{\{2\}}^3}{1 + s_{\{1\}}^1 s_{\{2\}}^2 + s_{\{1\}}^2 s_{\{2\}}^3 + s_{\{1\}}^3 s_{\{2\}}^1}, \quad \text{(3.64)} \\
 s_{\{3\}}^2 &= \frac{s_{\{1\}}^2 + s_{\{2\}}^2 + s_{\{1\}}^1 s_{\{2\}}^3 + s_{\{1\}}^3 s_{\{2\}}^1}{1 + s_{\{1\}}^1 s_{\{2\}}^2 + s_{\{1\}}^2 s_{\{2\}}^3 + s_{\{1\}}^3 s_{\{2\}}^1}, \quad \text{(3.65)} \\
 s_{\{3\}}^3 &= \frac{s_{\{1\}}^3 + s_{\{2\}}^3 + s_{\{1\}}^1 s_{\{2\}}^2 + s_{\{1\}}^2 s_{\{2\}}^1}{1 + s_{\{1\}}^1 s_{\{2\}}^2 + s_{\{1\}}^2 s_{\{2\}}^3 + s_{\{1\}}^3 s_{\{2\}}^1}. \quad \text{(3.66)}
\end{align}

An alternative convenient way to obtain the formulae (3.64)–(3.66) is to resolve the subtraction law set (3.50)–(3.53) with respect to the quantities $s_{\{3\}}^1, s_{\{3\}}^2, s_{\{3\}}^3$. 
Realizing the opposed way

$$R_{\{b\}}^p = N_{q}^p(b, a)R_{\{a\}}^q$$  \hspace{1cm} (3.67)

from the reference frame \(S\{a\}\) to the reference frame \(S\{b\}\) implies using the inverted coefficients

$$N_{q}^p(b, a) = h_{A}^{q}(b)h_{A}^{q}(a).$$  \hspace{1cm} (3.68)

We obtain

$$N_{0}^{0}(b, a) = \frac{F(b)}{4F(a)}\left(\frac{a^1}{b^1} + \frac{a^2}{b^2} + \frac{a^3}{b^3} + \frac{a^4}{b^4}\right),$$  \hspace{1cm} (3.69)

$$N_{1}^{0}(b, a) = \frac{F(b)}{4F(a)}\left(\frac{a^1}{b^1} - \frac{a^2}{b^2} - \frac{a^3}{b^3} + \frac{a^4}{b^4}\right),$$  \hspace{1cm} (3.70)

$$N_{2}^{0}(b, a) = \frac{F(b)}{4F(a)}\left(\frac{a^1}{b^1} + \frac{a^2}{b^2} - \frac{a^3}{b^3} - \frac{a^4}{b^4}\right),$$  \hspace{1cm} (3.71)

$$N_{3}^{0}(b, a) = \frac{F(b)}{4F(a)}\left(\frac{a^1}{b^1} - \frac{a^2}{b^2} + \frac{a^3}{b^3} + \frac{a^4}{b^4}\right),$$  \hspace{1cm} (3.72)

so that the coefficients \(N_{q}^p(b, a)\) are obtainable from the above coefficients \(N_{q}^p(a, b)\) merely by substituting \(a^A\) with \(b^A\), and \(b^A\) with \(a^A\).

The reciprocal relative velocity

$$s^{a}(b, a) = \frac{N_{0}^{a}(b, a)}{N_{0}^{0}(b, a)} = \frac{h_{A}^{a}(b)a^A}{h_{B}^{0}(b)a^B}$$  \hspace{1cm} (3.73)

(cf. (3.17)) bears naturally the meaning of the velocity which is inverse to the \(s^{a}(a, b)\):

$$s^{a}(b, a) = \ominus s^{a}(a, b).$$  \hspace{1cm} (3.74)

Given the velocity \(\{s^{a}\}\) with some values

$$s^{1} = s^{1}(a, b), \quad s^{2} = s^{2}(a, b), \quad s^{3} = s^{3}(a, b),$$  \hspace{1cm} (3.75)

the calculation of \(\{s^{a}(b, a)\}\) leads us to the components

$$\ominus s^{1} = \frac{1}{4}\left[\frac{1}{1 + s^{1} + s^{2} + s^{3}} - \frac{1}{1 - s^{1} + s^{2} - s^{3}} + \frac{1}{1 + s^{1} - s^{2} - s^{3}} - \frac{1}{1 - s^{1} - s^{2} + s^{3}}\right],$$  \hspace{1cm} (3.76)

$$\ominus s^{2} = \frac{1}{4}\left[\frac{1}{1 + s^{1} + s^{2} + s^{3}} + \frac{1}{1 - s^{1} + s^{2} - s^{3}} - \frac{1}{1 + s^{1} - s^{2} - s^{3}} - \frac{1}{1 - s^{1} - s^{2} + s^{3}}\right],$$  \hspace{1cm} (3.77)

$$\ominus s^{3} = \frac{1}{4}\left[\frac{1}{1 + s^{1} + s^{2} + s^{3}} - \frac{1}{1 - s^{1} + s^{2} - s^{3}} - \frac{1}{1 + s^{1} - s^{2} - s^{3}} + \frac{1}{1 - s^{1} - s^{2} + s^{3}}\right],$$  \hspace{1cm} (3.78)
or
\[
\ominus s^1 = - \frac{1}{(A(s))^4} \left[ s^1 - 2s^2 s^3 - (s^1)^3 + s^1(s^2 s^2 + s^3 s^3) \right], \tag{3.79}
\]
\[
\ominus s^2 = - \frac{1}{(A(s))^4} \left[ s^2 - 2s^1 s^3 - (s^2)^3 + s^2(s^1 s^1 + s^3 s^3) \right], \tag{3.80}
\]
\[
\ominus s^3 = - \frac{1}{(A(s))^4} \left[ s^3 - 2s^1 s^2 - (s^3)^3 + s^3(s^1 s^1 + s^2 s^2) \right]. \tag{3.81}
\]

**PROPOSITION.** The \(A_4^{(+)\text{--reciprocity for three-dimensional velocities}}\) reads as (3.79)–(3.81).

**NOTE.** From (3.50)–(3.66) the ordinary two–dimensional special–relativistic formulae ensue as follows:

\[
\{s^2_{(1)} = s^3_{(1)} = s^2_{(2)} = s^3_{(2)} = 0 \} \rightarrow s^1_{(1)} = \frac{s^1_{(3)} - s^1_{(2)}}{1 - s^2_{(2)} s^3_{(2)}} \quad \text{and} \quad s^3_{(3)} = \frac{s^1_{(1)} + s^1_{(2)}}{1 + s^3_{(1)} s^3_{(2)}}. \tag{3.82}
\]

In this case, the reciprocity of the ordinary “obvious” type
\[
\ominus s^a = -s^a \tag{3.83}
\]
is a true implication from the law (3.79)–(3.81).

In the small–velocity approximation up to O(5), we obtain
\[
A(s) \approx A_1(s) + A_2(s) \tag{3.84}
\]
with
\[
A_1(s) = 1 - \frac{1}{2} \left( (s^1)^2 + (s^2)^2 + (s^3)^2 \right) - \frac{1}{8} \left( (s^1)^4 + (s^2)^4 + (s^3)^4 \right) \tag{3.85}
\]
and
\[
A_2(s) = 2s^1 s^2 s^3 - \frac{5}{4} \left( (s^1)^2 (s^2)^2 + (s^2)^2 (s^3)^2 + (s^1)^2 (s^3)^2 \right). \tag{3.86}
\]

**4. Invariance in \(A_4^{(+)\text{--space}}\)**

Let us consider a non–singular, and non–linear in general, transformation
\[
y^A = F^A(y^B) \tag{4.1}
\]
under which the Finslerian metric function remains invariant, that is,
\[
F(y) = F(\tilde{y}). \tag{4.2}
\]
Let us construct from the coefficients \(F^A\) the derivatives
\[
F^A_B \overset{\text{def}}{=} \frac{\partial F^A}{\partial y^B} \tag{4.3}
\]
and
\[
F^A_{BC} \overset{\text{def}}{=} \frac{\partial F^A_B}{\partial y^C}. \tag{4.4}
\]
For our purposes it is worth assuming that the functions $F^A$ are sufficiently smooth and positively homogeneous of degree 1 with respect to $\tilde{y}$, so that
\[ F^A(k\tilde{y}) = kF^A(\tilde{y}), \quad k > 0, \] (4.5)
(for any admissible set of arguments). The last condition guarantees retaining the homogeneity property for the Finslerian metric function $F$ under the transformations (4.1) and allows rewriting them in the form
\[ y^A = F^A_B(\tilde{y})\tilde{y}^B \] (4.6)
(as this immediately follows from the Euler theorem for homogeneous functions).

Generally speaking, the second derivatives does not vanish identically:
\[ F^A_{BC} \neq 0. \] (4.7)

Differentiating (4.2) with respect to $\tilde{y}^C$ leads to new identity
\[ \tilde{y}^C = y_B F^B_C, \] (4.8)
which in turn can be differentiated with respect to $\tilde{y}^D$, which yields
\[ g_{CD}(\tilde{y}) = F^A_C(\tilde{y})F^B_D(\tilde{y})g_{AB}(F(\tilde{y})) + y_B F^B_{CD} \] (4.9)
(the definition (1.3) has been used).

If the transformation (4.1) fulfills also the condition
\[ y_B F^B_{CD} = 0, \] (4.10)
then we call it metric, keeping in mind that in such a case the transformation (4.9) leaves also invariant the Finslerian metric tensor:
\[ g_{CD}(\tilde{y}) = F^A_C(\tilde{y})F^B_D(\tilde{y})g_{AB}(F(\tilde{y})). \] (4.11)

Owing to (1.6), the metricity condition (4.10) can be written as
\[ F_B F^B_{CD} \equiv 0 \] (4.12)
with the functions
\[ F_B = 1/F^B. \] (4.13)

Obviously, the metric transformations comprise a group.

**DEFINITION.** Under the above conditions, the set of transformations (4.1) is called the group of Finslerian metric transformations.

In case of the particular Finslerian metric function (1.5), an attentive consideration of the role of the indicatrix variables \{u^a\} (see (1.39)) leads to the following conclusions.

**PROPOSITION.** The Euclidean rotations of the indicatrix variables \{u^a\} give rise to the nonlinear transformations of the vectors \{y^A\}, which leave the Finslerian metric function (1.5) invariant and simultaneously realize invariance transformation (4.11) for the associated Finslerian metric tensor.

The explicit form for the required coefficients $F^A$ will be evaluated below in Appendix A. Namely, under the rotation conditions (A.24)–(A.28), the nonlinear transformations
under our present concern prove to be given explicitly by means of the formulae (A.3)–(A.23). They involve three angles of rotations. For the transformations obtained the validity of the metricity condition (4.10) can be verified straightforwardly by applying the required Maple9-tools (see Appendix B below). The formulae are essentially got simplified in case of one-angle-rotations (see Appendix C below).

Additionally, the translations in the indicatrix:

\[ \tilde{u}^a = u^a + n^a \]  

induce obviously the unimodular dilatations

\[ \tilde{y}^A = y^A \cdot k^A, \quad k^1 k^2 k^3 k^4 = 1, \]

because of the exponential nature of the indicatrix representation (1.39) of unit vectors.

**Appendix A. Coefficients for three-angle rotations**

Let us start with an arbitrary linear nonsingular transformation of the indicatrix variables \( \{u^a\} \) entering (1.39). Specifying them for definiteness to fulfill \( \ln l^1 = \alpha + \beta + \gamma, \quad \ln l^2 = -\alpha + \beta - \gamma, \quad \ln l^3 = \alpha - \beta - \gamma, \quad \ln l^4 = -\alpha - \beta + \gamma \) with \( \{\alpha, \beta, \gamma\} = \{u^1, u^2, u^3\} \), we have

\[
\begin{align*}
\alpha &= l_1 \tilde{\alpha} + l_2 \tilde{\beta} + l_3 \tilde{\gamma}, \\
\beta &= m_1 \tilde{\alpha} + m_2 \tilde{\beta} + m_3 \tilde{\gamma}, \\
\gamma &= n_1 \tilde{\alpha} + n_2 \tilde{\beta} + n_3 \tilde{\gamma},
\end{align*}
\]

(A.1)

where

\[
\{m_1, m_2, m_3, n_1, n_2, n_3, l_1, l_2, l_3\}
\]

is a set of constants. This entails

\[
\ln a^1 = (l_1 \tilde{\alpha} + l_2 \tilde{\beta} + l_3 \tilde{\gamma}) + (m_1 \tilde{\alpha} + m_2 \tilde{\beta} + m_3 \tilde{\gamma}) + (n_1 \tilde{\alpha} + n_2 \tilde{\beta} + n_3 \tilde{\gamma}),
\]

\[
\ln a^2 = -(l_1 \tilde{\alpha} + l_2 \tilde{\beta} + l_3 \tilde{\gamma}) + (m_1 \tilde{\alpha} + m_2 \tilde{\beta} + m_3 \tilde{\gamma}) - (n_1 \tilde{\alpha} + n_2 \tilde{\beta} + n_3 \tilde{\gamma}),
\]

\[
\ln a^3 = (l_1 \tilde{\alpha} + l_2 \tilde{\beta} + l_3 \tilde{\gamma}) - (m_1 \tilde{\alpha} + m_2 \tilde{\beta} + m_3 \tilde{\gamma}) - (n_1 \tilde{\alpha} + n_2 \tilde{\beta} + n_3 \tilde{\gamma}),
\]

\[
\ln a^4 = -(l_1 \tilde{\alpha} + l_2 \tilde{\beta} + l_3 \tilde{\gamma}) - (m_1 \tilde{\alpha} + m_2 \tilde{\beta} + m_3 \tilde{\gamma}) + (n_1 \tilde{\alpha} + n_2 \tilde{\beta} + n_3 \tilde{\gamma}),
\]

or

\[
\begin{align*}
\ln a^1 &= (l_1 + m_1 + n_1) \tilde{\alpha} + (l_2 + m_2 + n_2) \tilde{\beta} + (l_3 + m_3 + n_3) \tilde{\gamma}, \\
\ln a^2 &= -(l_1 + m_1 - n_1) \tilde{\alpha} + (-l_2 + m_2 - n_2) \tilde{\beta} + (-l_3 + m_3 - n_3) \tilde{\gamma}, \\
\ln a^3 &= (l_1 - m_1 - n_1) \tilde{\alpha} + (l_2 - m_2 - n_2) \tilde{\beta} + (l_3 - m_3 - n_3) \tilde{\gamma}, \\
\ln a^4 &= -(l_1 - m_1 + n_1) \tilde{\alpha} + (-l_2 - m_2 + n_2) \tilde{\beta} + (-l_3 - m_3 + n_3) \tilde{\gamma},
\end{align*}
\]

from which it follows directly that

\[ 4 \ln a^1 = (l_1 + m_1 + n_1)(\ln \tilde{\alpha}^1 - \ln \tilde{\alpha}^2 + \ln \tilde{\alpha}^3 - \ln \tilde{\alpha}^4) \]
\[(l_2 + m_2 + n_2)(\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4)
\]
\[+(l_3 + m_3 + n_3)(\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4),\]
\[4 \ln a^2 = (-l_1 + m_1 - n_1)(\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4)
\]
\[+(-l_2 + m_2 - n_2)(\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4)
\]
\[+(-l_3 + m_3 - n_3)(\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4),\]
\[4 \ln a^3 = (l_1 - m_1 - n_1)(\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4)
\]
\[+(l_2 - m_2 - n_2)(\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4)
\]
\[+(l_3 - m_3 - n_3)(\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4),\]
\[4 \ln a^4 = (-l_1 - m_1 + n_1)(\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4)
\]
\[+(-l_2 - m_2 + n_2)(\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4)
\]
\[+(-l_3 - m_3 + n_3)(\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4),\]

or
\[4 \ln a^1 = (l_1 + m_1 + n_1 + l_2 + m_2 + n_2 + l_3 + m_3 + n_3) \ln \tilde{a}^1
\]
\[+(-l_1 - m_1 - n_1 + l_2 + m_2 + n_2 - l_3 - m_3 - n_3) \ln \tilde{a}^2
\]
\[+(l_1 + m_1 + n_1 - l_2 - m_2 - n_2 - l_3 - m_3 - n_3) \ln \tilde{a}^3
\]
\[+(-l_1 - m_1 - n_1 - l_2 - m_2 - n_2 + l_3 + m_3 + n_3) \ln \tilde{a}^4,
\]
\[4 \ln a^2 = (-l_1 + m_1 - l_2 + m_2 - n_2 - l_3 + m_3 - n_3) \ln \tilde{a}^1
\]
\[+(l_1 - m_1 - l_2 + m_2 - n_2 + l_3 - m_3 + n_3) \ln \tilde{a}^2
\]
Thus we can conclude that

\[ + ( -l_1 + m_1 - n_1 + l_2 - m_2 + n_2 + l_3 - m_3 + n_3 ) \ln \tilde{a}^3 \]

\[ + ( l_1 - m_1 + n_1 + l_2 - m_2 + n_2 - l_3 + m_3 - n_3 ) \ln \tilde{a}^4, \]

\[ 4 \ln a^3 = ( l_1 - m_1 - n_1 + l_2 - m_2 - n_2 + l_3 - m_3 - n_3 ) \ln \tilde{a}^1 \]

\[ + ( -l_1 + m_1 + n_1 + l_2 - m_2 - n_2 - l_3 + m_3 + n_3 ) \ln \tilde{a}^2 \]

\[ + ( l_1 - m_1 - n_1 - l_2 + m_2 + n_2 - l_3 + m_3 + n_3 ) \ln \tilde{a}^3 \]

\[ + ( -l_1 + m_1 + n_1 - l_2 + m_2 + n_2 + l_3 - m_3 - n_3 ) \ln \tilde{a}^4, \]

\[ 4 \ln a^4 = ( -l_1 - m_1 + n_1 - l_2 - m_2 + n_2 - l_3 - m_3 + n_3 ) \ln \tilde{a}^1 \]

\[ + ( l_1 + m_1 - n_1 - l_2 - m_2 + n_2 + l_3 + m_3 - n_3 ) \ln \tilde{a}^2 \]

\[ + ( -l_1 - m_1 + n_1 + l_2 + m_2 - n_2 + l_3 + m_3 - n_3 ) \ln \tilde{a}^3 \]

\[ + ( l_1 + m_1 - n_1 + l_2 + m_2 - n_2 - l_3 - m_3 + n_3 ) \ln \tilde{a}^4. \]

Thus we can conclude that

\[ (a^1)^4 = (\tilde{a}^1)(l_1+m_1+n_1+l_2+m_2+n_2+l_3+m_3+n_3) \]

\[ \cdot (\tilde{a}^2)(-l_1-m_1-n_1+l_2+m_2+n_2-l_3-m_3-n_3) \]

\[ \cdot (\tilde{a}^3)(l_1+m_1+n_1-l_2-m_2-n_2-l_3-m_3-n_3) \]

\[ \cdot (\tilde{a}^4)(-l_1-m_1+n_1-l_2-m_2+n_2+l_3+m_3+n_3), \]

\[ (a^2)^4 = (\tilde{a}^1)(-l_1+m_1-n_1-l_2+m_2-n_2-l_3+m_3-n_3) \]

\[ \cdot (\tilde{a}^2)(l_1-m_1+n_1-l_2+m_2+n_2+l_3-m_3+n_3) \]

\[ \cdot (\tilde{a}^3)(-l_1+m_1+n_1+l_2-m_2+n_2+l_3-m_3+n_3), \]
\[
\cdot (\tilde{a}^1) (l_1 - m_1 + n_1 + l_2 - m_2 + n_2 - l_3 + m_3 - n_3),
\]

\[ (a^3)^4 = (\tilde{a}^1)(l_1 - m_1 - n_1 + l_2 - m_2 - n_2 + l_3 - m_3 - n_3) \]

\[ \cdot (\tilde{a}^2)(-l_1 + m_1 + n_1 + l_2 - m_2 - n_2 - l_3 + m_3 + n_3) \]

\[ \cdot (\tilde{a}^3)(l_1 - m_1 - n_1 - l_2 + m_2 + n_2 - l_3 - m_3 + n_3) \]

\[ \cdot (\tilde{a}^4)(-l_1 + m_1 + n_1 - l_2 + m_2 + n_2 + l_3 - m_3 - n_3), \]

\[ (a^4)^4 = (\tilde{a}^1)(-l_1 + m_1 - n_1 - l_2 - m_2 + n_2 - l_3 - m_3 + n_3) \]

\[ \cdot (\tilde{a}^2)(l_1 + m_1 - n_1 - l_2 - m_2 + n_2 + l_3 + m_3 - n_3) \]

\[ \cdot (\tilde{a}^3)(-l_1 - m_1 + n_1 + l_2 + m_2 - n_2 + l_3 + m_3 - n_3) \]

\[ \cdot (\tilde{a}^4)(l_1 + m_1 - n_1 + l_2 + m_2 - n_2 - l_3 - m_3 + n_3). \]

Also, we obtain

\[ (a^1 a^2 a^3)^4 = (\tilde{a}^1)(l_1 + m_1 - n_1 + l_2 + m_2 - n_2 + l_3 + m_3 - n_3) \]

\[ \cdot (\tilde{a}^2)(-l_1 - m_1 + l_2 + m_2 - n_2 + l_3 - m_3 + n_3) \]

\[ \cdot (\tilde{a}^3)(l_1 + m_1 - n_1 - l_2 + m_2 + n_2 + l_3 + m_3 - n_3) \]

\[ \cdot (\tilde{a}^4)(-l_1 + m_1 + n_1 - l_2 + m_2 + n_2 + l_3 - m_3 - n_3), \]

\[ (a^1 a^2 a^4)^4 = (\tilde{a}^1)(-l_1 + m_1 + n_1 - l_2 + m_2 + n_2 - l_3 + m_3 + n_3) \]

\[ \cdot (\tilde{a}^2)(l_1 - m_1 - n_1 - l_2 - m_2 + n_2 - l_3 - m_3 - n_3) \]

\[ \cdot (\tilde{a}^3)(-l_1 + m_1 + n_1 + l_2 - m_2 - n_2 + l_3 - m_3 - n_3) \]

\[ \cdot (\tilde{a}^4)(l_1 - m_1 + n_1 + l_2 - m_2 - n_2 - l_3 + m_3 + n_3). \]
\[(a^1a^3a^4)^4 = (\tilde{a}^1)(l_1-m_1+n_1+l_2-m_2+n_2+l_3-m_3+n_3)\]

\[\cdot(\tilde{a}^2)(-l_1+m_1-n_1+l_2-m_2+n_2-l_3+m_3-n_3)\]

\[\cdot(\tilde{a}^3)(l_1-n_1-l_2+m_2-n_2-l_3+m_3-n_3)\]

\[\cdot(\tilde{a}^4)(l_1+m_1-n_1-l_2+m_2+n_2-l_3-m_3+n_3),\]

\[(a^2a^3a^4)^4 = (\tilde{a}^1)(-l_1-m_1-l_2-m_2-n_2-l_3-m_3-n_3)\]

\[\cdot(\tilde{a}^2)(l_1+m_1-n_1-l_2-m_2+n_2+l_3+m_3+n_3)\]

\[\cdot(\tilde{a}^3)(-l_1-n_1+l_2+m_2+n_2+l_3+m_3-n_3)\]

\[\cdot(\tilde{a}^4)(l_1+m_1+n_1+l_2+n_2-l_3-m_3-n_3),\]

This way we get the coefficients of the transformation

\[a^A = \mathcal{F}^A(\tilde{a}^B)\]  \hspace{1cm} \text{(A.3)}

to explicitly read

\[\mathcal{F}^1 = (\tilde{a}^1)^{f^{11}} (\tilde{a}^2)^{f^{12}} (\tilde{a}^3)^{f^{13}} (\tilde{a}^4)^{f^{14}},\]  \hspace{1cm} \text{(A.4)}

\[\mathcal{F}^2 = (\tilde{a}^1)^{f^{21}} (\tilde{a}^2)^{f^{22}} (\tilde{a}^3)^{f^{23}} (\tilde{a}^4)^{f^{24}},\]  \hspace{1cm} \text{(A.5)}

\[\mathcal{F}^3 = (\tilde{a}^1)^{f^{31}} (\tilde{a}^2)^{f^{32}} (\tilde{a}^3)^{f^{33}} (\tilde{a}^4)^{f^{34}},\]  \hspace{1cm} \text{(A.6)}

\[\mathcal{F}^4 = (\tilde{a}^1)^{f^{41}} (\tilde{a}^2)^{f^{42}} (\tilde{a}^3)^{f^{43}} (\tilde{a}^4)^{f^{44}},\]  \hspace{1cm} \text{(A.7)}

with

\[f^{11} = (l_1+m_1+n_1+l_2+m_2+n_2+l_3+m_3+n_3+1)/4,\]  \hspace{1cm} \text{(A.8)}

\[f^{12} = (-l_1-m_1-n_1+l_2+m_2+n_2-l_3-m_3-n_3+1)/4,\]  \hspace{1cm} \text{(A.9)}

\[f^{13} = (l_1+m_1+n_1-l_2-m_2-n_2-l_3-m_3-n_3+1)/4,\]  \hspace{1cm} \text{(A.10)}
\begin{align*}
f^{14} &= (-l_1 - m_1 - n_1 - l_2 - m_2 - n_2 + l_3 + m_3 + n_3 + 1)/4, \quad \text{(A.11)} \\
f^{21} &= (-l_1 + m_1 - n_1 - l_2 + m_2 - n_2 - l_3 + m_3 - n_3 + 1)/4, \quad \text{(A.12)} \\
f^{22} &= (l_1 - m_1 + n_1 - l_2 + m_2 - n_2 + l_3 - m_3 + n_3 + 1)/4, \quad \text{(A.13)} \\
f^{22} &= (-l_1 + m_1 - n_1 + l_2 - m_2 + n_2 - l_3 - m_3 + n_3 + 1)/4, \quad \text{(A.14)} \\
f^{24} &= (l_1 - m_1 + n_1 + l_2 - m_2 + n_2 - l_3 + m_3 - n_3 + 1)/4, \quad \text{(A.15)} \\
f^{31} &= (l_1 - m_1 + l_2 - m_2 - n_2 + l_3 - m_3 - n_3 + 1)/4, \quad \text{(A.16)} \\
f^{32} &= (-l_1 + m_1 + n_1 + l_2 - m_2 - n_2 - l_3 + m_3 + n_3 + 1)/4, \quad \text{(A.17)} \\
f^{33} &= (l_1 - m_1 - n_1 - l_2 + m_2 + n_2 - l_3 + m_3 + n_3 + 1)/4, \quad \text{(A.18)} \\
f^{34} &= (-l_1 + m_1 + n_1 - l_2 + m_2 + n_2 + l_3 - m_3 - n_3 + 1)/4, \quad \text{(A.19)} \\
f^{41} &= (-l_1 + m_1 - n_1 - l_2 - m_2 + n_2 - l_3 - m_3 + n_3 + 1)/4, \quad \text{(A.20)} \\
f^{42} &= (l_1 + m_1 - n_1 - l_2 - m_2 + n_2 + l_3 + m_3 - n_3 + 1)/4, \quad \text{(A.21)} \\
f^{43} &= (-l_1 - m_1 + n_1 + l_2 + m_2 - n_2 + l_3 + m_3 - n_3 + 1)/4, \quad \text{(A.22)} \\
f^{44} &= (l_1 + m_1 - n_1 + l_2 + m_2 - n_2 - l_3 - m_3 + n_3 + 1)/4. \quad \text{(A.23)}
\end{align*}

Finally, we are to subject the coefficients (A.2) to the condition that the transformation (A.1) realizes an euclidean rotation of the set \{\alpha, \beta, \gamma\}. To this end it is convenient to accept the (Euler) three angle choice:

\begin{align*}
c_1 &= \cos \theta, \quad c_2 = \cos \psi, \quad c_3 = \cos \phi, \quad \text{(A.24)} \\
s_1 &= \sin \theta, \quad s_2 = \sin \psi, \quad s_3 = \sin \phi \quad \text{(A.25)}
\end{align*}

to have

\begin{align*}
l_1 &= c_2 c_3 - c_1 s_2 s_3, \quad m_1 = s_2 c_3 + c_1 c_2 s_3, \quad n_1 = s_1 s_3, \quad \text{(A.26)}
\end{align*}
\[ l_2 = -c_2 s_3 - c_1 s_2 c_3, \quad m_2 = -s_2 s_3 + c_1 c_2 c_3, \quad n_2 = s_1 c_3, \quad (A.27) \]

\[ l_3 = s_1 s_2, \quad m_3 = -s_1 c_2, \quad n_3 = c_1. \quad (A.28) \]

Appendix B. Three-angle.mws(by Maple9)

The program presented below (created by means of Maple 9) does evaluate the metricity condition which fulfillment means that the transformation leaves the Finslerian metric tensor invariant.

\begin{verbatim}
> c1:=cos(theta); c2:=cos(psi); c3:=cos(phi);
> s1:=sin(theta); s2:=sin(psi); s3:=sin(phi);
> l1:=c2*c3-c1*s2*s3; l2:=-c2*s3-c1*s2*c3; l3:=s1*s2;
> m1:=s2*c3+c1*c2*s3; m2:=-s2*s3+c1*c2*c3; m3:=-s1*c2;
> n1:=s1*s3; n2:=s1*c3; n3:=c1;

\end{verbatim}
> a:=array(1..4,1..4):
> for i from 1 to 4
> do
>   for j from 1 to 4
>   do
>     a[i,j]:=diff(F||i,e||j);
>   end do:
> end do:

> b:=array(1..4,1..4):
> for i from 1 to 4
> do
>   for j from 1 to 4
>   do
>     b[i,j]:=simplify(add(1/F||k*diff(a[k,i],e||j),k=1..4),symbolic);
>   end do:
> end do:

> print(b);

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The result that all the entries of the matrix are zeroes means that the metricity condition holds true.

**Appendix C. One–angle rotation**

Let us take the particular case

\[
\alpha = \tilde{\alpha} \cos \eta + \tilde{\beta} \sin \eta, \quad \beta = -\tilde{\alpha} \sin \eta + \tilde{\beta} \cos \eta, \quad \gamma = \tilde{\gamma}
\]  

(C.1)

which represents the rotation by one angle, \(\eta\), in the \(\gamma\)-plane. We get

\[
\ln a^1 = \tilde{\alpha} \cos \eta + \tilde{\beta} \sin \eta - \tilde{\alpha} \sin \eta + \tilde{\beta} \cos \eta + \tilde{\gamma},
\]

\[
\ln a^2 = -\tilde{\alpha} \cos \eta - \tilde{\beta} \sin \eta - \tilde{\alpha} \sin \eta + \tilde{\beta} \cos \eta - \tilde{\gamma},
\]

\[
\ln a^3 = \tilde{\alpha} \cos \eta + \tilde{\beta} \sin \eta + \tilde{\alpha} \sin \eta - \tilde{\beta} \cos \eta - \tilde{\gamma},
\]

\[
\ln a^4 = -\tilde{\alpha} \cos \eta - \tilde{\beta} \sin \eta + \tilde{\alpha} \sin \eta - \tilde{\beta} \cos \eta + \tilde{\gamma},
\]

or

\[
\ln a^1 = \tilde{\alpha}(\cos \eta - \sin \eta) + \tilde{\beta}(\cos \eta + \sin \eta) + \tilde{\gamma},
\]
\[\ln a^2 = -\tilde{\alpha}(\cos \eta + \sin \eta) + \tilde{\beta}(\cos \eta - \sin \eta) - \tilde{\gamma},\]

\[\ln a^3 = \tilde{\alpha}(\cos \eta + \sin \eta) - \tilde{\beta}(\cos \eta - \sin \eta) - \tilde{\gamma},\]

\[\ln a^4 = -\tilde{\alpha}(\cos \eta - \sin \eta) - \tilde{\beta}(\cos \eta + \sin \eta) + \tilde{\gamma},\]

which entails

\[4\ln a^1 = (\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4)(\cos \eta - \sin \eta)\]

\[+ (\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4)(\cos \eta + \sin \eta)\]

\[+ (\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4),\]

\[4\ln a^2 = -(\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4)(\cos \eta + \sin \eta)\]

\[+ (\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4)(\cos \eta - \sin \eta)\]

\[- (\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4),\]

\[4\ln a^3 = (\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4)(\cos \eta + \sin \eta)\]

\[- (\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4)(\cos \eta - \sin \eta)\]

\[- (\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4),\]

\[4\ln a^4 = -(\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4)(\cos \eta - \sin \eta)\]

\[- (\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4)(\cos \eta + \sin \eta)\]

\[+ (\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4),\]

from which it follows that

\[4\ln a^1 = (\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4) \cos \eta\]
\[ + (\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4) \cos \eta \]

\[-(\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4) \sin \eta \]

\[ + (\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4) \sin \eta \]

so that

\[ 4 \ln a^1 = 2(\ln \tilde{a}^1 - \ln \tilde{a}^4) \cos \eta \]

\[ + 2(\ln \tilde{a}^2 - \ln \tilde{a}^3) \sin \eta \]

\[ + \ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4, \]

\[ 4 \ln a^2 = -(\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4) \cos \eta \]

\[ + (\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4) \cos \eta \]

\[-(\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4) \sin \eta \]

\[-(\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4) \sin \eta \]

so that

\[ 4 \ln a^2 = 2(\ln \tilde{a}^2 - \ln \tilde{a}^3) \cos \eta \]

\[-2(\ln \tilde{a}^1 - \ln \tilde{a}^4) \sin \eta \]

\[-(\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4), \]

\[ 4 \ln a^3 = (\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4) \cos \eta \]
\[-(\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4) \cos \eta \]

\[+ (\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4) \sin \eta \]

\[+ (\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4) \sin \eta \]

so that

\[4 \ln a^3 = -2(\ln \tilde{a}^2 - \ln \tilde{a}^3) \cos \eta \]

\[+ 2(\ln \tilde{a}^1 - \ln \tilde{a}^4) \sin \eta \]

\[-(\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4), \]

so that

\[4 \ln a^4 = -(\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4) \cos \eta \]

\[-(\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4) \cos \eta \]

\[+ (\ln \tilde{a}^1 - \ln \tilde{a}^2 + \ln \tilde{a}^3 - \ln \tilde{a}^4) \sin \eta \]

\[-(\ln \tilde{a}^1 + \ln \tilde{a}^2 - \ln \tilde{a}^3 - \ln \tilde{a}^4) \sin \eta \]

\[+(\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4), \]

so that

\[4 \ln a^4 = -2(\ln \tilde{a}^1 - \ln \tilde{a}^4) \cos \eta - 2(\ln \tilde{a}^2 - \ln \tilde{a}^3) \sin \eta \]

\[+(\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4). \]

Eventually we obtain

\[4 \ln a^1 = 2(\ln \tilde{a}^1 - \ln \tilde{a}^4) \cos \eta + 2(\ln \tilde{a}^2 - \ln \tilde{a}^3) \sin \eta + \ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4, \quad (C.2) \]

\[4 \ln a^2 = 2(\ln \tilde{a}^2 - \ln \tilde{a}^3) \cos \eta - 2(\ln \tilde{a}^1 - \ln \tilde{a}^4) \sin \eta - (\ln \tilde{a}^1 - \ln \tilde{a}^2 - \ln \tilde{a}^3 + \ln \tilde{a}^4), \quad (C.3) \]
4 \ln \alpha^3 = -2(\ln \alpha^2 - \ln \alpha^3) \cos \eta + 2(\ln \alpha^1 - \ln \alpha^4) \sin \eta - (\ln \alpha^1 - \ln \alpha^2 - \ln \alpha^3 + \ln \alpha^4), \tag{C.4}

4 \ln \alpha^4 = -2(\ln \alpha^1 - \ln \alpha^4) \cos \eta - 2(\ln \alpha^2 - \ln \alpha^3) \sin \eta + (\ln \alpha^1 - \ln \alpha^2 - \ln \alpha^3 + \ln \alpha^4). \tag{C.5}

The respective generalized rotation coefficients are given by the list:

\[ F^1 = \frac{(\ln \alpha^1)^{(2 \cos \eta + 1)/4}}{(\alpha^1)^{(2 \sin \eta - 1)/4}} \frac{(\ln \alpha^1)^{(-2 \sin \eta - 1)/4}}{(\alpha^1)^{(2 \sin \eta - 1)/4}}, \tag{C.6} \]

\[ F^2 = \frac{(\ln \alpha^2)^{(2 \cos \eta + 1)/4}}{(\alpha^2)^{(2 \sin \eta - 1)/4}} \frac{(\ln \alpha^1)^{(-2 \cos \eta + 1)/4}}{(\alpha^1)^{(2 \sin \eta - 1)/4}}, \tag{C.7} \]

\[ F^3 = \frac{(\ln \alpha^3)^{(2 \sin \eta - 1)/4}}{(\alpha^3)^{(2 \cos \eta + 1)/4}} \frac{(\ln \alpha^1)^{(-2 \cos \eta + 1)/4}}{(\alpha^1)^{(2 \sin \eta - 1)/4}}, \tag{C.8} \]

\[ F^4 = \frac{(\ln \alpha^4)^{(2 \cos \eta + 1)/4}}{(\alpha^4)^{(2 \sin \eta - 1)/4}} \frac{(\ln \alpha^1)^{(-2 \cos \eta + 1)/4}}{(\alpha^1)^{(2 \sin \eta - 1)/4}}, \tag{C.9} \]

**Discussion**

The \((N = 2)\)-dimensional precursor of the space \((1.2)\) under our study is the ordinary hyperbolic relativistic space \(A_2 := \{V_2, e_1, e_2, F^{(\text{two-dimensional})}(y)\} \) with \(F^{(\text{two-dimensional})} = \sqrt{|y^1 y^2|} \equiv |t^2 - x^2|\), where \(t = (y^1 + y^2)/2, \ x = (y^1 - y^2)/2\).

The anisotropic method exposed in the previous sections to increase the dimension \(N\) and arrive at the \(A_4^{(+)}\)-space differs drastically from the isotropic conventional pseudoeuclidean way. Generally, our methods of analysis were founded upon usage of the indicatrix geometry and indicatrix coordinates. The conformal nature of the associated Finslerian metric tensor (exhibited by \((1.53)\)) has played a crucial role in Section 2 in our getting explicit solutions to the \(A_4^{(+)}\)-geodesic equations. They are that solutions that entailed the distance, angle, and scalar product for the \(A_4^{(+)}\)-space.

In Section 3 the substantive items were concerned the \(A_4^{(+)}\)-kinematic aspects. Actually, all the obtained kinematic implications, included the \(A_4^{(+)}\)-extension \((1.59)-(1.62)\) of the Lorentz transformations and the \(A_4^{(+)}\)-extension \((1.63)\) of the pseudoeuclidean kinematic length of vector, as well the composition and subtraction laws found in Section 3, were direct implications of the particular \(A_4^{(+)}\)-tetrad \(h^p_A\) (introduced in Section 1 by means of \((1.51)\)). Since the tetrads present geometricaly the reference systems proper for the inertial reference frames (according to general kinematic principles; see, e.g., \([3-5]\)), and the vector components \(\{Y^p\}\) entering the representations \((1.59)-(1.63)\) are the tetradic components, the representations bear precisely physical space–time meaning; namely, \(\{Y^0\}\) and \(\{Y^1, Y^2, Y^3\}\) are respectively the time components and the spatial components of the vector \(\{Y^p\}\) as observed in the inertial reference frame. Similar Finslerian kinematic ideas were applied in context of the Finsleroid theory \([6-10]\).

Calculations involved the constants \(C^A_p\) coming from the fundamental indicatrix representation \((1.39)\) of the unit vector \(l^A\). When dealing with the four-dimensional case, the following suitable choice \(C^A_0 = \{1, 1, 1, 1\}, \ C^A_1 = \{-\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\}, \ C^A_2 = \{0, \sqrt{8}/3, -\sqrt{2}/3, -\sqrt{2}/3\}, \ C^A_3 = \{0, 0, -\sqrt{2}, \sqrt{2}\}\) was proposed in \([1]\); the possibility \(C^A_0 = \{1, 1, 1, 1\}, \ C^A_1 = \{1, -1, 1, -1\}, \ C^A_2 = \{1, 1, -1, -1\}, \ C^A_3 = \{1, -1, -1, 1\}\) which does not involve any roots was used in \([11]\). Obviously, these two sets of constants may
be expressed one through another by means of due euclidean rotations. However, many fundamental implications, e.g. the angle \((2.48)\), are independent of any such choice. With the latter choice of the constants, the tetradic components gain the particularly simple structure, namely,

\[
h^0_A = l_A = \frac{F}{4} \left( \frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4} \right), \quad h^1_A = \frac{F}{4} \left( \frac{1}{a^1}, -\frac{1}{a^2}, \frac{1}{a^3}, -\frac{1}{a^4} \right),
\]

\[
h^2_A = \frac{F}{4} \left( \frac{1}{a^1}, \frac{1}{a^2}, -\frac{1}{a^3}, -\frac{1}{a^4} \right), \quad h^3_A = \frac{F}{4} \left( \frac{1}{a^1}, -\frac{1}{a^2}, -\frac{1}{a^3}, \frac{1}{a^4} \right),
\]

together with their reciprocal components

\[
h^0_A = l^A = \frac{1}{F} (a^1, a^2, a^3, a^4), \quad h^1_A = \frac{1}{F} (a^1, -a^2, a^3, -a^4),
\]

\[
h^2_A = \frac{1}{F} (a^1, a^2, -a^3, -a^4), \quad h^3_A = \frac{1}{F} (a^1, -a^2, -a^3, a^4).
\]

The Finslerian metric tensor, which components can be presented by

\[
g_{AB} = h^0_A h^0_B - h^1_A h^1_B - h^2_A h^2_B - h^3_A h^3_B, \quad g^{AB} = h^0_A h^0_B - h^1_A h^1_B - h^2_A h^2_B - h^3_A h^3_B,
\]

describes the Finslerian aspects of geometry of the \(A_4^{(+)–}\)–space.

Study of the invariance properties of the \(A_4^{(+)–}\)–space faced us to conclude in Section 4 that the associated group of invariance is a nonlinear representation of the euclidean group of rotations and translations given rise to by the induced euclidean structure of the generalized \(A_4^{(+)–}\)–hyperboloid (which is the indicatrix of the space under study). At the same time, the \(A_4^{(+)–}\)–kinematic invariance transformations \((1.59)–(1.64)\) are of the essentially linear structure entailed by the tetrad projections.

Recently, a new impetus to development of geometrical and relativistic applications on the basis of the Berwald–Moor metric function was given in the work [11]. An intrinsic relationship with the hypercomplex numbers was emphasized, therefore the space was denoted by \(H_4\). The work was motivated by the desire to develop the theory prolonged “beyond square–root concepts”. It was argued that the ordinary physical motivation that only the Minkowskian framework with its well–accepted characteristics can be the basis proper to the nowadays theoretical and cosmological real world models is probably not exactly true. A novel feature was indicated that, while in the Minkowskian space the set of points equidistant from two static events (the space of relatively simultaneous events) forms a hyperplane, in the \(H_4\)–space the set probably forms a non-linear surface. Interesting list of Research Problems was set forth in [11] to investigate possibilities of finding adequate metric quantities, angles and scalar products included, the required conformal and congruent transformations as well as appropriate extended \(H_4\)–rotation transformations.

Generally, search for novel generalized physical aspects produced tentatively by the anisotropic structure of the space–time, in particular those referred to light behaviour, seems to be an urgent task for the new Finslerian framework outlined above. We hope, in particular, that the \(A_4^{(+)–}\)–extension \((1.59)–(1.62)\) of the Lorentz transformations may serve to propose the necessary kinematic grounds to think of respective light experiments.
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