On Semisymmetric Connection

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Abstract. Using the non-symmetry of a connection, it is possible to introduce four types of covariant derivatives. Based on these derivatives, several types of Ricci’s identities and twelve curvature tensors are obtained. Five of them are linearly independent but the other curvature tensors can be expressed as linear combinations of these five linearly independent curvature tensors and the curvature tensor of the corresponding associated symmetric space.

The semisymmetric connection is defined and the properties of two of the five independent curvature tensors are analyzed. In the same manner, the properties for three others curvature tensors may be derived.

1. Introduction

Although the notion of non-symmetric affine connection is used in several works before A. Einstein, for example in [2, 4], the use of non-symmetric connection became especially actual after appearance the works of Einstein, relating to create the Unified Field Theory (UFT).

Einstein was not satisfied with his General Theory of Relativity (GTR, 1916), and from 1923. to the end of his life (1955), he worked on various variants of UFT. This theory had the aim to unite the gravitation theory, to which is related GTR, and the theory of electromagnetism.

Let \( LN = (MN, L_{ij}^k \neq L_{ij}^k) \) be a non-symmetric affine connection space. Based on the non-symmetry of the connection, we can introduce four types of the covariant derivatives. For instance, for the tensor \( a_{ij} \), we have

\[
\begin{align*}
\alpha_{ij}^{(1)} &= a^{(1)}_{,ij} + L_{ij,}^k a_{k}, - L_{ij,}^k a_{k}^p - L_{ij,}^p a_{p} \\
\alpha_{ij}^{(2)} &= a^{(2)}_{,ij} + L_{ij,}^k a_{k}, - L_{ij,}^k a_{k}^p - L_{ij,}^p a_{p} \\
\alpha_{ij}^{(3)} &= a^{(3)}_{,ij} + L_{ij,}^k a_{k}, - L_{ij,}^k a_{k}^p - L_{ij,}^p a_{p} \\
\alpha_{ij}^{(4)} &= a^{(4)}_{,ij} + L_{ij,}^k a_{k}, - L_{ij,}^k a_{k}^p - L_{ij,}^p a_{p}
\end{align*}
\]

(1.1)

where by comma (, ) is denoted partial derivative. Based on the covariant derivatives, we can derive Ricci’s type identities and twelve curvature tensors, and among them five are linearly independent, while the rest of them can be expressed through this five curvature tensors [6, 8]. Linearly independent curvature tensors are:

\[
R_{mn}^i = L_{jm,}^i a_{mj} - L_{jm,}^i a_{mj} + L_{jm,}^i a_{mj}^p - L_{jm,}^i a_{mj}^p
\]

(1.3)
\[ R^{i}_{\ jmn} = L^{i}_{mj,n} - L^{i}_{nj,m} + L^{p}_{jm} T^{i}_{np} - L^{p}_{nj} T^{i}_{mp}, \]  
(1.4)

\[ R^{i}_{\ jmn} = L^{i}_{jm,n} - L^{i}_{nj,m} + L^{p}_{jm} L^{i}_{np} - L^{p}_{nj} L^{i}_{mp} + L^{p}_{mn} T^{i}_{pj}, \]  
(1.5)

\[ R^{i}_{\ jmn} = L^{i}_{jm,n} - L^{i}_{nj,m} + L^{p}_{jm} L^{i}_{np} - L^{p}_{nj} L^{i}_{mp} + L^{p}_{mn} T^{i}_{pj}, \]  
(1.6)

\[ R^{i}_{\ jmn} = \frac{1}{2}(L^{i}_{jm,n} - L^{i}_{nj,m} + L^{p}_{jm} L^{i}_{np} - L^{p}_{nj} L^{i}_{mp} + L^{p}_{mj} L^{i}_{np} - L^{p}_{nj} L^{i}_{pm}), \]  
(1.7)

The following theorem holds

**Theorem 1.1.** [6, 8] If the following notation are introduced (with an omission of the indices on the left side),

\[ A = \frac{1}{2} T^{ij}_{mn}; \]  
\[ A' = \frac{1}{2} T^{ij}_{jm}; \]  
(1.8)

\[ B = \frac{1}{4} T^{ji}_{mn} T^{ij}_{pm}; \]  
\[ B' = \frac{1}{4} T^{ji}_{jm} T^{ij}_{pm}; \]  
\[ C = \frac{1}{4} T^{pi}_{mn} T^{pi}_{pj}; \]  
(1.9)

where the covariant derivative with respect to the symmetric connection \( L^{i}_{jk} \) is denoted by semicolon, \( T^{i}_{jk} = L^{i}_{jk} - L^{i}_{kj} \) is the torsion tensor of the connection, then the equations (1.3) – (1.7) can be expressed as follows

\[ R^{1}_{i} = R + A - A' + B - B', \]  
(1.10)

\[ R^{2}_{i} = R - A + A' + B - B', \]  
(1.11)

\[ R^{3}_{i} = R + A + A' - B + B' - 2C, \]  
(1.12)

\[ R^{4}_{i} = R + A + A' - B + B' + 2C, \]  
(1.13)

\[ R^{5}_{i} = R + B + B', \]  
(1.14)

where \( R^{i}_{jmn} \) is the curvature tensor with respect to the symmetric connection \( L^{i}_{jk} \).

A lot of research papers and monographs [1] - [21] are dedicated to the theory of Riemannian spaces, affine connected ones and their generalizations.

### 2. Semisymmetric affine connection space

Particularly interesting cases can be observed at the non-symmetry of the connection. One such case is the semisymmetric affine connection.

**Definition 2.1.** Connection \( L^{i}_{jk} \) of the space \( L_{N} \) is **semisymmetric**, if

\[ L^{i}_{jk} = L^{i}_{jk} + \delta^{i}_{j} \tau^{k} - \delta^{i}_{k} \tau^{j}, \]  
(2.1)

where \( L^{i}_{jk} \) symmetrical part of \( L^{i}_{jk} \), and **torsion tensor**

\[ T^{i}_{jk} = L^{i}_{jk} - L^{i}_{kj} = 2(\delta^{i}_{j} \tau^{k} - \delta^{i}_{k} \tau^{j}), \]  
(2.2)

where \( \tau^{k} \) it is a vector.

Namely, it holds the following theorem
Theorem 2.1. If $T^i_{jk} = L^i_{jk} - L^i_{kj}$ is a torsion tensor of $L_N$, then
\[
\tau_i = \frac{1}{2(N-1)} T^p_{pi}.
\] (2.3)
for $\tau_i$ from the equation (2.1)

Proof. After contracting (2.1) with respect to the index $k$ and $i$, we get
\[
T^i_p = 2(\delta^i_j \tau_i - \delta^i_i \tau_j) = 2(\tau_j - N \tau_i) = 2(1-N) \tau_j,
\]
i.e. $-T^i_{ij} = 2(1-N) \tau_j$. Thus, $\tau_i = \frac{1}{2(N-1)} T^i_{ij}$. By substituting $i \to p$, $j \to i$, we get (2.1). \qed

Let us start from the equation (1.10), i.e. 
\[
R^i_{jmn} = R^i_{jmn} + \frac{1}{2} T^i_{jmn} - \frac{1}{2} T^i_{jm,n} + \frac{1}{4} T^p_{jm} R^i_{p,m} - \frac{1}{4} T^p_{jn} R^i_{p,m}.
\] (2.4)
If we replace the torsion tensor (2.2) in the previous equation, we obtain the curvature tensor $R$ of the semisymmetric affine connection
\[
R^i_{jmn} = R^i_{jmn} + (\delta^i_j \tau_m - \delta^i_m \tau_j)_n - (\delta^i_j \tau_n - \delta^i_n \tau_j)_m
+ (\delta^p_j \tau_m - \delta^p_m \tau_j)(\delta^i_n \tau_p - \delta^i_p \tau_n) - (\delta^p_j \tau_n - \delta^p_n \tau_j)(\delta^i_m \tau_p - \delta^i_p \tau_m).
\] (2.5)
After simple calculation, we get
\[
R^i_{jmn} = R^i_{jmn} + \delta^i_j (\tau_m,n - \tau_n,m) - \delta^i_m (\tau_j,n + \tau_j,n) + \delta^i_n (\tau_j,m + \tau_j,m).
\] (2.6)

Theorem 2.2. The curvature tensor $R^i_{jmn}$ of the semisymmetric affine space connection (2.1) is given by the equation (2.6), where $R^i_{jmn}$ is curvature tensor of the symmetric connection $L^i_{jk}$, $\tau_i$ is the vector expressed in the function of the torsion $T^i_{jk}$ by the equation (2.3), and covariant derivative ($\cdot$) is given by $L^i_{jk}$.

2.1. Properties of the curvature tensors of the semisymmetric connection

Based on the general case, and according to the equation (2.6) we get
\[
R^i_{jmn} = -R^i_{jnm}.
\] (2.7)
If we take into account that $\text{Cycl} R^i_{jmn} = 0$, the first Bianchi identity for $R^i_{jmn}$ is
\[
\text{Cycl} R^i_{jmn} = \text{Cycl} R^i_{jmn} + \delta^i_j (\tau_m,n - \tau_n,m) - \delta^i_m (\tau_j,n + \tau_j,n) + \delta^i_n (\tau_j,m + \tau_j,m),
\] (2.8)
i.e.
\[
\text{Cycl} R^i_{jmn} = 2\text{Cycl} \delta^i_j (\tau_m,n - \tau_n,m).
\] (2.9)
Therefore, the following theorem is valid

Theorem 2.3. The curvature tensor $R^i_{jmn}$ of the semisymmetric affine connection given by the equation (2.6) satisfies antisymmetry with respect to the last two lower indices (2.7), and the first Bianchi’s identity (2.9).
Let us consider the second Bianchi identity. In this case, we use (19) from [21].

\[
\text{Cycl} R^i_{jmn} = \text{Cycl} \left( \delta^p_n \tau_n - \delta^p_m \tau_m \right) R^i_{jpv} = 2 \text{Cycl} \left( \tau_n R^i_{jmv} - \tau_m R^i_{jnv} \right).
\] (2.10)

Since \( R \) is antisymmetric by \( n \) and \( v \), we obtain

\[
\text{Cycl} R^i_{jmn} = 2 \text{Cycl} \tau_n R^i_{jmv} + 2 \text{Cycl} \tau_m R^i_{jnv}.
\] (2.11)

For the addend \( \text{Cycl} \tau_n R^i_{jmv} \), we have

\[
2 \text{Cycl} \tau_n R^i_{jmv} = 2 \text{Cycl} \tau_m R^i_{jvn}.
\] (2.12)

By substituting (2.12) in the equation (2.11), we get

\[
\text{Cycl} R^i_{jmn} = 4 \text{Cycl} \tau_m R^i_{jmn}.
\] (2.13)

Therefore, we have proved the following theorem:

**Theorem 2.4.** The equation (2.13) is the second Bianchi identity for the curvature tensor \( R \) of the semisymmetric affine connection, where \( \tau_m \) is determined by (2.3).

In the same manner, we can derive the first and second Bianchi identities for the curvature tensors \( R, R^0, R^1, R^2 \). Let us consider, for example, the curvature tensor \( R^1 \) of the semisymmetric affine connection.

According to (1.14) and (1.9), (2.2) we have

\[
R^i_{jmn} = R^i_{jmn} + \frac{1}{4} (T^i_{jm} T^j_{mn} + T^i_{jm} T^j_{mn}),
\] (2.14)

wherefrom

\[
R^i_{jmn} = R^i_{jmn} + (\delta^p_j \tau_m - \delta^p_m \tau_j) (\delta^i_n \tau_n - \delta^i_n \tau_p) + (\delta^i_n \tau_n - \delta^i_n \tau_p) (\delta^j_m \tau_j - \delta^j_m \tau_p),
\] (2.15)

i.e.

\[
R^i_{jmn} = R^i_{jmn} + 2 \delta^i_j \tau_n \tau_p - \delta^i_m \tau_j \tau_p - \delta^i_n \tau_j \tau_m.
\] (2.16)

Thus, from the last equation, we can conclude that

\[
R^i_{jmn} = 2 \delta^i_j \tau_n \tau_p - \delta^i_m \tau_j \tau_p - \delta^i_n \tau_j \tau_m.
\] (2.17)

\[
R^i_{jmn} = R^i_{jmn}, \text{ because } R^i_{jmn} = -R^i_{jmn}.
\] (2.18)

Finally, it holds that

\[
R^i_{jmn} \neq R^i_{jmn}, \quad R^i_{jmn} \neq -R^i_{jmn}.
\] (2.19)

We are also checking the first Bianchi’s identity for the curvature tensor \( R \):
wherefrom
\[ \text{Cycl} R^i_\left(\mu_\nu\xi\eta\right) = 0. \] (2.20)

Regarding of the second Bianchi identity, all types of covariant derivatives should be examined. After some calculation, we conclude that
\[ R^i_\left(\mu_\nu\xi\eta\right) \neq 0 \quad \text{for} \quad \theta = 1, \ldots, 4. \]

So, the next theorem is valid:

**Theorem 2.5.** The curvature tensor \( R^i_\left(\mu_\nu\xi\eta\right) \) of the semisymmetric affine connection given by the equation (2.16) satisfies the following properties:

1. \( R^i_\left(\mu_\nu\xi\eta\right) \neq R^i_\left(\mu_\nu\xi\eta\right) \)
2. \( \text{Cycl} R^i_\left(\mu_\nu\xi\eta\right) = 0, \)
3. \( R^i_\left(\mu_\nu\xi\eta\right) \neq 0 \quad \text{for} \quad \theta = 1, \ldots, 4. \)

3. **Conclusion**

In this paper we consider one class of the space \( L_N \) with the non-symmetric connection. We obtained five linearly independent curvature tensors and find some their properties. This relationship will be of significance for the further development of semisymmetric affine connection spaces. Especially, these results are of interest for studying geodesic mappings of these spaces.

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