A HOMOTOPY THEORY FOR ENRICHMENT IN SIMPLICIAL MODULES

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Abstract. We put a Quillen model structure on the category of small categories enriched in simplicial $k$-modules and non-negatively graded chain complexes of $k$-modules, where $k$ is a commutative ring. The model structure is obtained by transfer from the model structure on simplicial categories due to J. Bergner.

1. Introduction: DK-equivalences and DK-fibrations

1.1. Let $\textbf{Cat}$ the category of small categories. It has a natural model structure in which a cofibration is a functor monic on objects, a weak equivalence is an equivalence of categories and a fibration is an isofibration [5]. The fibration weak equivalences are the equivalences surjective on objects.

Let $\mathcal{V}$ be a monoidal model category [8] with unit $I$. We denote by $\mathcal{W}$ the class of weak equivalences of $\mathcal{V}$, by $\mathcal{Fib}$ the class of fibrations and by $\mathcal{Cof}$ the class of cofibrations.

The small $\mathcal{V}$-categories together with the $\mathcal{V}$-functors between them form a category written $\mathcal{V} \textbf{Cat}$. Let $\mathcal{M}$ be a class of maps of $\mathcal{V}$. We say that a $\mathcal{V}$-functor $f: A \to B$ is locally in $\mathcal{M}$ if for each pair $x, y \in A$ of objects, the map $f_{x,y}: A \to B$ is in $\mathcal{M}$.

We have a functor $\gamma: \mathcal{V} \textbf{Cat} \to \textbf{Cat}$ obtained by change of base along the (symmetric monoidal) composite functor

$$\gamma: \mathcal{V} \to \text{Ho}(\mathcal{V}) \xrightarrow{\text{Hom}_{\text{Ho}(\mathcal{V})}(I, -)} \text{Set}.$$  

**Definition 1.1.** Let $f: A \to B$ be a morphism in $\mathcal{V} \textbf{Cat}$.

1. The morphism $f$ is homotopy essentially surjective if the induced functor $[f]_\mathcal{V}: [A]_\mathcal{V} \to [B]_\mathcal{V}$ is essentially surjective.

2. The morphism $f$ is a DK− equivalence if it is homotopy essentially surjective and locally in $\mathcal{M}$.

3. The morphism $f$ is a DK− fibration if it satisfies the following two conditions.

(a) $f$ is locally in $\mathcal{Fib}$.

(b) For any $x \in A$, and any isomorphism $v: [f]_\mathcal{V}(x) \to y'$ in $[B]_\mathcal{V}$, there exists an isomorphism $u: x \to y$ in $[A]_\mathcal{V}$ such that $[f]_\mathcal{V}(u) = v$. That is, if $[f]_\mathcal{V}$ is an isofibration.

One can easily see that a morphism $f$ is a DK-equivalence and a DK-fibration iff $f$ is surjective on objects and locally in $\mathcal{W} \cap \mathcal{Fib}$. The class of maps having the left lifting property with respect to the $\mathcal{V}$-functors surjective on object and locally in $\mathcal{W} \cap \mathcal{Fib}$ is generated by the map $u: \emptyset \to I$, where $I$ is the $\mathcal{V}$-category with a single object $*$ and $I(*, *) = I$, together with the maps

$$\varphi_i: \bar{2}_A \to \bar{2}_B,$$

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where $i$ is a generating cofibration of $\mathcal{V}$. Here the $\mathcal{V}$-category $\bar{2}_A$ has objects 0 and 1, with $\bar{2}_A(0, 0) = 2_A(1, 1) = I$, $2_A(0, 1) = A$ and $2_A(1, 0) = \emptyset$.

1.2. Let $k$ be a commutative ring. We denote by $\text{SMod}_k$ the category of simplicial $k$-modules and by $\text{Ch}^+(k)$ the category of non-negatively graded chain complexes of $k$-modules. The purpose of this note is to prove the following theorem.

**Theorem 1.2.** Let $\mathcal{V}$ be one of the categories $\text{SMod}_k$ or $\text{Ch}^+(k)$. Then $\mathcal{V}\text{Cat}$ admits a model structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations.

To prove this result we use the (similar) model structure on simplicial categories \cite{2} and Quillen’s path object argument (\cite{7}, Lemma 2.3(2) and \cite{1}, 2.6). An explicit description of a cofibration of $\mathcal{V}\text{Cat}$ can be given \cite{9}.

1.3. The proof of theorem 1.2 relies decisively on the construction of path objects for dg-categories due to G. Tabuada (\cite{11}, 4.1). In fact, our attempt to understand his construction led us to the proof of our result.

1.4. In \cite{12}, B. To{"e}n characterised the maps in the homotopy category of dg-categories, where the category of dg-categories has a model structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations. One can show that his results (loc. cit., Thm. 4.2 and 6.1) hold for $\mathcal{V}\text{Cat}$, where $\mathcal{V}$ is $\text{SMod}_k$ or $\text{Ch}^+(k)$.

**Note added in proof.** After the completion of this work we learned about the existence of a paper by G. Tabuada \cite{10}, which treats the same subject matter, and more, but differently. One can see that the model structure proposed in theorem 1.2 coincides with the one in \cite{10}, although the classes of fibrations and cofibrations are not explicitly identified in loc. cit. On the other hand, Tabuada shows that the model structures on $\text{SMod}_k\text{Cat}$ and $\text{Ch}^+(k)\text{Cat}$ are Quillen equivalent, an issue that we have initially neglected. One can easily give a proof of this fact, adapted to our context, using section 2.2 below and the general results of \cite{9}.

2. Categories enriched in $\text{SMod}_k$ and $\text{Ch}^+(k)$

2.1. The category $\text{SMod}_k$ is a closed symmetric monoidal category with tensor product defined pointwise and unit $ck$, where $(ck)_n = k$ for all $n \geq 0$. A model structure on $\text{SMod}_k$ is obtained by transfer from the category $\text{S}$ of simplicial sets, regarded as having the classical model structure, via the free-forgetful adjunction $k : \text{S} \rightleftarrows \text{SMod}_k : U$.

All objects are fibrant and the model structure is simplicial. The functor $k$ is strong symmetric monoidal (and it preserves the unit), hence $\text{SMod}_k$ is a monoidal model category. The adjunction $(k, U)$ induces an adjunction $k' : \text{SCat} \rightleftarrows \text{SMod}_k\text{Cat} : U'$.

We claim that a map $f$ of $\text{SMod}_k\text{Cat}$ is a DK-equivalence (resp. DK-fibration) iff $U'(f)$ is a weak equivalence (resp. fibration) in the Bergner model structure on $\text{SCat}$ \cite{2}. Clearly, $f$ is locally in $\text{S}$ (resp. $\text{Sib}$) iff $U'(f)$ is locally in $\text{S}$ (resp. $\text{Sib}$).

In the induced adjoint pair

$$Lk : Ho(\text{S}) \rightleftarrows Ho(\text{SMod}_k) : RU,$$

the functor $Lk$ is strong symmetric monoidal and preserves the unit object, hence one has a natural isomorphism of functors

$$\eta : [\cdot, \text{SMod}_k] \cong [\cdot, \text{S}U'] : \text{SMod}_k\text{Cat} \rightarrow \text{Cat}$$

such that for all $A \in \text{SMod}_k\text{Cat}$, $\eta_A$ is the identity on objects. The rest of the claim follows from this observation.
2.2. Consider the normalized chain complex functor \( N : \text{SMod}_k \to Ch^+(k) \). It was shown in ([8], 4.3) that \( N \) is part of a weak monoidal Quillen equivalence

\[
N : \text{SMod}_k \rightleftarrows Ch^+(k) : \Gamma
\]
in which both functors preserve the unit objects. Therefore the composite adjunction

\[
Nk : S \rightleftarrows Ch^+(k) : \Gamma
\]
is a weak monoidal Quillen pair with \( Nk \) preserving the unit object. The functor \( \Gamma \) induces a functor \((\Gamma')' : Ch^+(k)\text{Cat} \to \text{SCat}\) which has a left adjoint \( F \) defined "fibrewise". We claim that a map \( f \) in \( Ch^+(k)\text{Cat} \) is a DK-equivalence (resp. DK-fibration) iff \((\Gamma')'(f)\) is a weak equivalence (resp. fibration) in the Bergner model structure on \( \text{SCat} \). For this, it is enough to remark that in the induced adjunction

\[
L(Nk) : Ho(S) \rightleftarrows Ho(Ch^+(k)) : R(\Gamma),
\]
the functor \( L(Nk) \) is strong monoidal and preserves the unit object, and then conclude as in 2.1.

2.3. In order to prove theorem 1.2, it suffices to apply Quillen’s path object argument to the adjunctions \((k', U')\) and \((F, (\Gamma')')\). This will be achieved in the next section.

3. Cocomategory object structure on the interval

3.1. Let \( V \) be a monoidal model category with cofibrant unit \( I \) and all objects fibrant. We write \( Y^X \) for the internal hom of two objects \( X, Y \) of \( V \). We say that \( V \) has a cocomcategory interval if there is a cocomategory object structure

\[
I \xrightarrow{d_0} I[1] \xrightarrow{i_0} I[2]
\]
such that

\[
\begin{CD}
I \sqcup I @<d_0\sqcup d_1<< I[1] \Rightarrow @>i_1>> I[2] @>c>> I
\end{CD}
\]
is a cylinder object for \( I \). The map \( c \) denotes the cocomposition.

Examples. (a) The standard example is when \( V = \text{Cat} \) as in 1.1. Here \( I[1] \) is the "free-living" isomorphism and \( I[2] \) is the groupoid with three objects and one isomorphism between any two objects. We leave to the reader the task to identify all the maps involved.

(b) The case which interests us is when \( V = Ch^+(k) \). The interval \( I[1] \) is well known to be \( ... \to 0 \to ke \xrightarrow{\partial} ka \oplus kb \), where \( \partial(e) = b - a \). The maps \( d_0 \) and \( d_1 \) are the inclusions, and the map \( p \) is \( a, b \mapsto 1 \). The object \( I[2] \) is

\[
... \to 0 \to ke_1 \oplus ke_2 \xrightarrow{\partial} ka_0 \oplus ka_1 \oplus ka_2,
\]
where \( \partial(e_1) = a_1 - a_0 \) and \( \partial(e_2) = a_2 - a_1 \). The cocomposition \( c \) is given by \( e \mapsto e_1 + e_2 \), \( a \mapsto a_0 \) and \( b \mapsto a_2 \). The map \( i_0 \) is given by \( e \mapsto e_1 \), \( a \mapsto a_0 \) and \( b \mapsto a_1 \); the map \( i_1 \) is given by \( e \mapsto e_2 \), \( a \mapsto a_1 \) and \( b \mapsto a_2 \).

(c) Since the functor \( \Gamma \) from 2.2 preserves the unit object and is an equivalence of categories, we obtain that \( \text{SMod}_k \) has a cocomcategory interval.

(d) Let \( k \) be a field and let \( H \) be a finite dimensional cocommutative Hopf algebra over \( k \). We let \( V = H \text{Mod} \), the category of left \( H \)-modules, and we view \( V \) as having
the stable model structure \([4]\). Let \(u\) (resp. \(e\)) be the unit (resp. counit) of \(H\). A cylinder object for \(k\) is

\[
\begin{array}{c}
\begin{tikzcd}
k \oplus k & k \\
& k \\
\end{tikzcd}
\end{array}
\]

The maps \(d_0\) and \(d_1\) are given by \(d_0(1) = (1, 0)\) and \(d_1(1) = (0, 1)\). We set \(I[2] = k \oplus k \oplus H\) and \(c(a, h) = (\alpha, 0, h)\). The map \(i_0\) is \((\alpha, h) \mapsto (\alpha, \epsilon(h), 0)\) and the map \(i_1\) is \((\alpha, h) \mapsto (0, \alpha, h)\). One can check that the resulting gadget is a cocategory interval with \(I[1] = k \oplus H\). It is easy to see that \(I[1]\) is an “interval with a coassociative and cocommutative comultiplication” in the sense of ([1], page 813).

3.2. We shall now construct (DK)-path objects for \(\mathcal{V}\)-categories, where \(\mathcal{V}\) is as in 3.1. In the case of dg-categories, the construction is due to G. Tabuada ([11], 4.1). Let \(\mathcal{A} \in \mathcal{V}Cat\). We first construct a factorisation

\[
\begin{array}{c}
\begin{tikzcd}
\mathcal{A} & \mathcal{A} \times \mathcal{A} \\
&(s,t) \\
\end{tikzcd}
\end{array}
\]

such that \(i_0\) is locally in \(\mathfrak{M}\) and \((s, t)\) is locally in \(\mathfrak{N}\). An object of \(P_0\mathcal{A}\) is a map \(f: a \to b\) of \([\mathcal{A}]_\mathcal{V}\). If \(f_0 : a_0 \to b_0\) and \(f_1 : a_1 \to b_1\) are two objects of \(P_0\mathcal{A}\), we define \(P_0\mathcal{A}(f_0, f_1)\) to be the limit of the diagram

\[
\begin{array}{ccc}
\mathcal{A}(a_0, a_1) & \mathcal{A}(a_0, b_1)^{I[1]} & \mathcal{A}(b_0, b_1) \\
\downarrow f_1^* & \downarrow \alpha & \downarrow f_0^* \\
\mathcal{A}(a_0, b_1) & \mathcal{A}(a_0, b_1) & \mathcal{A}(b_0, b_1) \\
\end{array}
\]

The unit of \(P_0\mathcal{A}(f, f)\) is induced by the adjoint transpose of \(I[1] \overset{p_i}{\to} I \overset{i}{\to} \mathcal{A}(a, b)\). Let \(f_i : a_i \to b_i\) \((i = 0, 2)\) be three objects of \(P_0\mathcal{A}\) and let \(A_i = P_0\mathcal{A}(f_i, f_{i+1})\) \((i = 0, 1)\). We denote by \(p_i\) (resp. \(q_i\)) the canonical map \(A_i \to \mathcal{A}(a_i, a_{i+1})\) (resp. \(A_i \to \mathcal{A}(b_i, b_{i+1})\)) \((i = 0, 1)\). The pair \((p_i, q_i)\) gives rise to a commutative diagram

\[
\begin{array}{ccc}
A_i & \mathcal{A}(a_i, a_{i+1}) & A_i \\
\downarrow j_i & \downarrow f_i^* & \downarrow j_{i+1} \\
I[1] \otimes A_i & \mathcal{A}(a_i, b_{i+1}) & A_i \\
\downarrow H_i & \downarrow f_i^* & \downarrow j_{i+1} \\
A_i & \mathcal{A}(b_i, b_{i+1}) & A_i \\
\end{array}
\]

where \(j_{k,i} = d_k \otimes A_i\) \((k = 0, 1)\). Observe that in order to define a map \(A_0 \otimes A_1 \to P_0\mathcal{A}(f_0, f_2)\) it suffices to find a map \(G : A_0 \otimes A_1 \to \mathcal{A}(a_0, b_2)^{I[1]}\) which makes
Then \( G \) as the composite such that the diagram \[ \begin{array}{ccc} A_0 \otimes A_1 & \xrightarrow{G} & A(a_0, b_2)^{I[1]} \\ f_0^*(q_0 \otimes q_1) & \downarrow s & \uparrow t \\ f_2^*(p_0 \otimes p_1) & \end{array} \] commutes, where \( \overline{\text{commutative the diagram}} \)

We define the map \( G_1 \) as the composite

\[ I[1] \otimes A_0 \otimes A_1 \xrightarrow{H_0 \otimes A_1} A(a_0, b_2) \otimes A_1 \xrightarrow{id \otimes q_1} A(a_0, b_2) \otimes A(b_1, b_2) \rightarrow A(a_0, b_2). \]

Then \( G_1 \) is a "homotopy" between \( f_0^*(q_0 \otimes q_1) \) and \( p_0 \otimes q_1 \). We define the map \( G_2 \) as the composite

\[ I[1] \otimes A_0 \otimes A_1 \xrightarrow{A_0 \otimes H_1} A_0 \otimes A(a_1, b_2) \xrightarrow{p_0 \otimes id} A(a_0, b_2) \otimes A(a_1, b_2) \rightarrow A(a_0, b_2). \]

Then \( G_2 \) is a "homotopy" between \( p_0 \otimes q_1 \) and \( f_2^*(p_0 \otimes p_1) \). The two homotopies induce a map

\[ A_0 \otimes A_1 \rightarrow A(a_0, b_2)^{I[1]} \times A(a_0, b_2) A(a_0, b_2)^{I[1]} \]

such that the diagram

\[ \begin{array}{ccc} A_0 \otimes A_1 & \xrightarrow{G_1} & A(a_0, b_2) \\ & \searrow G_2 \downarrow & \searrow \downarrow \\ & A(a_0, b_2) \\ \end{array} \]

commutes, where \( \overline{G_1} \) is the adjoint transpose of \( G_1 \) (\( i = 0, 1 \)). Since \( A(a_0, b_2)^{I[1]} \) is a category object in \( V \), we have then a map

\[ A_0 \otimes A_1 \rightarrow A(a_0, b_2)^{I[1]} \times A(a_0, b_2) A(a_0, b_2)^{I[1]} \xrightarrow{m} A(a_0, b_2)^{I[1]} \]

which is the required map \( G \).

In this way \( P_0 \mathcal{A} \) becomes a \( V \)-category. The association \( i_0: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(P_0 \mathcal{A}), a \mapsto (id_a : a \rightarrow a) \), \( (i_0)_{a,b} = \mathcal{A}(a,b) \), is a \( V \)-functor \( \mathcal{A} \rightarrow P_0 \mathcal{A} \). By construction, the maps \( s,t : P_0 \mathcal{A} \rightarrow \mathcal{A}, s(f_0 : a_0 \rightarrow b_0) = a_0, t(f_0 : a_0 \rightarrow b_0) = b_0 \), \( s_{f_0,f_1} = p_0 \) and \( t_{f_0,f_1} = g_0 \), are \( V \)-functors. One clearly has \( (s,t)i_0 = \Delta \). Moreover, \( (s,t) \) is locally in \( \mathfrak{ib} \) since \( (s,t)f_0,f_1 : A_0 \rightarrow \mathcal{A}(a_0,a_1) \times \mathcal{A}(b_0,b_1) \) is the pullback

\[ \mathcal{A}(a_0, a_1) \times \mathcal{A}(b_0, b_1) \]

Next, let \( PA \) be the full sub-\( V \)-category of \( P_0 \mathcal{A} \) whose objects consist of isomorphisms \( f : a \rightarrow b \) of \( [\mathcal{A}]_V \). Then \( i_0 \) factors through \( P_0 \mathcal{A} \). The resulting factorisation

\[ \begin{array}{ccc} \Delta & \xrightarrow{(s,t)} & \mathcal{A} \\ i \downarrow & \searrow & \downarrow \\ PA & \xrightarrow{(s,t)} & \mathcal{A} \times \mathcal{A} \end{array} \]
is the desired (DK-)path object: a lengthy but straightforward computation shows that \( i \) is homotopy essentially surjective and that \( \{(s, t)\}_V \) is an isofibration.

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