ABSTRACT. We define and study proper permutations. Properness is a geometrically natural necessary criterion for a Schubert variety to be Levi-spherical. We prove the probability that a random permutation is proper goes to zero in the limit.

1. INTRODUCTION

Let $X$ denote the variety of complete flags $\langle 0 \rangle \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n$, where $F_i$ is a subspace of dimension $i$. The general linear group $GL_n$ of invertible $n \times n$ complex matrices acts naturally on $X$ by basis change. Let $B \subset GL_n$ be the Borel subgroup of upper triangular invertible matrices. $B$ acts on $X$ with finitely many orbits; these are the Schubert cells $X_w$ indexed by permutations $w$ in the symmetric group $S_n$ on $[n] := \{1, 2, \ldots, n\}$. Their closures $X_w := \overline{X_w}$ are the Schubert varieties; these objects are of significant interest in combinatorial algebraic geometry. A standard reference is [3] and we also point the reader to the expository papers [4, 6].

Now, $\dim X_w = \ell(w)$ where $\ell(w) = \# \{1 \leq i < j \leq n : w(i) > w(j) \}$ counts inversions of $w$. Also, let

$$J(w) = \{1 \leq i \leq n - 1 : \exists 1 \leq j < i, w(j) = i + 1\}$$

be the set of left descents of $w$. Assume $I \subseteq J(w)$ and let

$$D := [n-1] - I = \{d_1 < d_2 < \ldots < d_k\};$$

also, $d_0 := 0, d_{k+1} := n$. Let $L_I \subseteq GL_n$ be the Levi subgroup of invertible block diagonal matrices

$$L_I \cong GL_{d_1-d_0} \times GL_{d_2-d_1} \times \cdots \times GL_{d_k-d_{k-1}} \times GL_{d_{k+1}-d_k}.$$ 

As explained in, e.g., [5, Section 1.2], $L_I$ acts on $X_w$. Moreover, $X_w$ is said to be $L_I$-spherical if $X_w$ has a dense orbit of a Borel subgroup of $L_I$. If in addition, $I = J(w)$, we say $X_w$ is maximally spherical. We refer the reader to ibid., and the references therein, for background and motivation about this geometric condition on a Schubert variety.

Definition 1. Let $d(w) = \#J(w)$. $w \in S_n$ is proper if $\ell(w) - \binom{d(w)+1}{2} \leq n$.

Actually, for $1 \leq n \leq 10$, proper permutations are not rare; the enumeration is:

$$1, 2, 6, 24, 120, 684, 4348, 30549, 236394, 2006492, \ldots$$

Proposition 3.1 shows that if $X_w$ is $L_I$-spherical for some $I \subseteq J(w)$, then $w$ is proper. The proof explains the Lie-theoretic origins of the condition. In this paper, we study proper permutations using probabilistic considerations.

Theorem 1.1. If $w \in S_n$ is chosen uniformly at random, $\Pr[w \text{ is proper}] \to 0$, as $n \to \infty$. 

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Proposition 3.1 and Theorem 1.1 combined imply:

**Corollary 1.2.**

\[
\lim_{n \to \infty} \Pr[w \in S_n, X_w \text{ is } L_I\text{-spherical for some } I \subseteq J(w)] \to 0.
\]

In particular,

\[
\lim_{n \to \infty} \Pr[w \in S_n, X_w \text{ is maximally spherical}] \to 0.
\]

Theorem 1.1 resolves a conjecture from [5]. In *ibid.*, the second and third authors introduced the notion of permutation \(w \in S_n\) being *I*-spherical; in the case \(I = J(w)\) we call \(w \in S_n\) maximally spherical. This combinatorial definition (recapitulated in Section 3) conjecturally characterizes those \(w \in S_n\) such that \(X_w\) is \(L_I\)-spherical. Proposition 3.2 shows that if \(w \in S_n\) is *I*-spherical, then \(w\) is proper. That proposition, together with Theorem 1.1, confirms [5, Conjecture 3.7]:

**Corollary 1.3.**

\[
\lim_{n \to \infty} \Pr[w \in S_n, w \text{ is } I\text{-spherical for some } I \subseteq J(w)] \to 0.
\]

Therefore,

\[
\lim_{n \to \infty} \Pr[w \in S_n, w \text{ is maximally spherical}] \to 0.
\]

Since Corollary 1.3 is consistent with Corollary 1.2, the former provides additional evidence for the aforementioned conjectural characterization.

### 2. Proof of Theorem 1.1

For \(w \in S_n\), define

\[
\mathcal{E}_{ij} = \text{ the event } \{w^{-1}(i) > w^{-1}(j)\}.
\]

Let \(X_{ij}\) be the indicator for \(\mathcal{E}_{ij}\); that is, \(X_{ij} = 1\) if event \(\mathcal{E}_{ij}\) happens and \(X_{ij} = 0\) otherwise. Then if \(w\) is chosen from \(S_n\) uniformly at random, then:

\[
\mathbb{E}[X_{ij}] = \Pr[X_{ij} = 1] = \frac{1}{2!(1 - \delta_{i,j})} = 1 - \Pr[X_{ij} = 0].
\]

Since \(\ell(w) = \ell(w^{-1})\) and \(#J(w) = \#\{i : w^{-1}(i + 1) < w^{-1}(i)\}\), the random variable (r.v.)

\[
\ell(w) - \binom{d(w)+1}{2}
\]

can be modeled as the r.v.

\[
X := L - \binom{D + 1}{2},
\]

where:

\[
L = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij},
\]

and \(D = \sum_{i=1}^{n-1} X_{i,i+1}\).

Notice that if \(i_1, i_2, i_3, i_4 \in [n]\) are distinct, then \(X_{i_1,i_2}\) and \(X_{i_3,i_4}\) are independent.

**Lemma 2.1.** For \(n \geq 2\),

\[
\mathbb{E}[X] = \frac{3n^2 - 7n + 2}{24}.
\]
Proof. It is true that:

(a) \((X_{i,j})_{i<j}\) are identically distributed,
(b) \(\mathbb{E}[X_{i,i+1}X_{i,i+1}] = \mathbb{E}[X_{i,i+1}^2] = \mathbb{E}[X_{i,i+1}] = 1/2\) since \(X_{i,i+1}\) is an indicator r.v.,
(c) \(\mathbb{E}[X_{i,i+1}X_{i+1,i+2}] = \text{Pr}[w^{-1}(i) > w^{-1}(i + 1) > w^{-1}(i + 2)] = 1/3!\),
(d) \(X_{i,i+1}\) and \(X_{j,j+1}\) are independent if \(i + 1 < j\).

With this, the expression \(\mathbb{E}[L]\) can be expanded as:

\[
\mathbb{E}[L] = \mathbb{E}\left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}\right]
\]
\[
= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{E}[X_{ij}] \quad \text{lin. of expectation}
\]
\[
= \frac{1}{2} \left(\frac{n}{2}\right) \quad \text{identically distributed.}
\]

Similarly,
\[
\mathbb{E}[D] = \frac{n - 1}{2}.
\]

Next, the expression \(\mathbb{E}[D^2]\) can be expanded as:

\[
\mathbb{E}[D^2] = \mathbb{E}\left[\left(\sum_{i=1}^{n-1} X_{i,i+1}\right)^2\right]
\]
\[
= \mathbb{E}\left[\sum_{i=1}^{n-1} X_{i,i+1}^2 + \sum_{i=1}^{n-1} \sum_{j \neq i} X_{i,i+1}X_{j,j+1}\right]
\]
\[
= \sum_{i=1}^{n-1} \mathbb{E}[X_{i,i+1}^2] + \sum_{i=1}^{n-1} \sum_{j \neq i} \mathbb{E}[X_{i,i+1}X_{j,j+1}] \quad \text{lin. of expectation}
\]
\[
= \frac{n - 1}{2} + \sum_{i=1}^{n-1} \sum_{j \neq i} \mathbb{E}[X_{i,i+1}X_{j,j+1}] \quad \text{by (b)}
\]
\[
= \frac{n - 1}{2} + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} \mathbb{E}[X_{i,i+1}X_{j,j+1}]
\]
\[
= \frac{n - 1}{2} + 2 \left(\sum_{i=1}^{n-2} \mathbb{E}[X_{i,i+1}X_{i+1,i+2}] + \sum_{i=1}^{n-1} \sum_{j=i+2}^{n-1} \mathbb{E}[X_{i,i+1}X_{j,j+1}]\right)
\]
\[
= \frac{n - 1}{2} + 2 \left(\frac{n - 2}{3!} + \frac{1}{2^2}\left(\binom{n-1}{2} - (n-2)\right)\right) \quad \text{by (c) and (d))}
\]
\[
= \frac{n - 1}{2} + \frac{n - 2}{3} + \frac{1}{2}\left(\binom{n-1}{2} - (n-2)\right).
\]
Thus by linearity of expectation,

\[ \mathbb{E} \left[ \binom{D + 1}{2} \right] = \frac{1}{2} \mathbb{E} [D^2 + D] = \frac{n - 1}{2} + \frac{n - 2}{6} - \frac{n - 2}{4} + \frac{1}{4} \binom{n - 1}{2} \]

and

\[ \mathbb{E} [X] = \mathbb{E} \left[ L - \binom{D + 1}{2} \right] = \mathbb{E} [L] - \mathbb{E} \left[ \binom{D + 1}{2} \right] = \frac{3n^2 - 7n + 2}{24}. \quad \square \]

**Lemma 2.2.**

\[ \mathbb{E} [X^2] = \frac{n^4}{64} + o(n^4). \]

**Proof.** Notice that:

\[ \mathbb{E} [X^2] = \mathbb{E} [L^2] + \mathbb{E} \left[ \binom{D + 1}{2} \right]^2 - 2 \mathbb{E} \left[ L \binom{D + 1}{2} \right] = \mathbb{E} [L^2] + \frac{1}{4} (\mathbb{E} [D^4] + 2 \mathbb{E} [D^3] + \mathbb{E} [D^2]) - \mathbb{E} [LD^2] - \mathbb{E} [LD]. \]

Now, \( 0 \leq D^3, D^2, LD \leq n^3 \), so \( \mathbb{E} [D^3], \mathbb{E} [D^2], \mathbb{E} [LD] = o(n^4) \). Thus it suffices to study the asymptotics of \( \mathbb{E} [L^2], \mathbb{E} [D^4/4], \mathbb{E} [LD^2]. \)

We will repeatedly use the following observation. For a set \( S \) with \( |S| = o(f(n)) \):

\[ \sum_{(i_1,j_1,...,i_c,j_c)\in S} \mathbb{E} \left[ \prod_{k=1}^{c} X_{i_k,j_k} \right] \leq |S| = o(f(n)). \]  

(1)

Expanding \( \mathbb{E} [L^2] \) gives:

\[ \mathbb{E} [L^2] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{i'=1}^{n} \sum_{j'=i'+1}^{n} \mathbb{E} [X_{i,j} X_{i',j'}]. \]

There are \( \binom{n}{2}^2 = n^4/4 + o(n^4) \) many terms in this summation. Further, there are \( \binom{n}{2} \binom{n-2}{2} = n^4/4 + o(n^4) \) many terms in this summation such that \( i, j, i', j' \) are distinct. Therefore, there must be \( o(n^4) \) terms where \( i, j, i', j' \) are not distinct. Now,

\[ \sum_{\text{distinct } i,j,i',j' \in [n]} \mathbb{E} [X_{i,j} X_{i',j'}] = \sum_{\text{distinct } i,j,i',j' \in [n]} \mathbb{E} [X_{i,j}] \mathbb{E} [X_{i',j'}] \quad \text{(independence when indices are distinct)} \]

\[ = \left( \frac{1}{2} \right)^2 \binom{n}{2} \binom{n-2}{2} = \left( \frac{1}{2} \right)^2 (n^4/4 + o(n^4)). \]
Combining this with (1) gives

\[ \mathbb{E} [L^2] = \frac{1}{16} n^4 + o(n^4). \]

To expand \( \mathbb{E} [D^4/4]\), first we have

\[ \mathbb{E} [D^4] = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{i' = 1}^{n-1} \sum_{j' = 1}^{n-1} \mathbb{E} [X_{i,i+1}X_{j,j+1}X_{i',i'+1}X_{j',j'+1}] . \]

There are \((n-1)^4 = n^4 + o(n^4)\) many terms in this summation. Further, there are \(4!(n-4) = n^4 + o(n^4)\) many terms in this summation such that \(i, i + 1, j, j + 1, i', i' + 1, j', j' + 1\) are distinct. Here we have used the fact that there are \(\binom{n-4}{k}\) ways to choose \(k\) non-consecutive numbers from \([n-1]\). Therefore, there must be \(o(n^4)\) terms where \(i, i + 1, j, j + 1, i', i' + 1, j', j' + 1\) are not distinct. We compute

\[
\begin{align*}
\frac{1}{4} \cdot \sum_{i,j,i',j' \in [n]} & \mathbb{E} [X_{i,i+1}X_{j,j+1}X_{i',i'+1}X_{j',j'+1}] \\
& \text{where } i,i+1,j,j+1, i',i'+1, j',j'+1 \text{ are distinct} \\
= \frac{1}{4} \cdot \sum_{i,j,i',j' \in [n]} & \mathbb{E} [X_{i,i+1}] \mathbb{E} [X_{j,j+1}] \mathbb{E} [X_{i',i'+1}] \mathbb{E} [X_{j',j'+1}] \\
& \text{where } i,i+1,j,j+1, i',i'+1, j',j'+1 \text{ are distinct} \\
= \frac{1}{4} \cdot \left( \frac{1}{2} \right)^4 \cdot 4! \left( \frac{n-4}{4} \right) \\
= \frac{1}{4} \cdot \left( \frac{1}{2} \right)^4 (n^4 + o(n^4)).
\end{align*}
\]

Hence by (1),

\[ \mathbb{E} [D^4/4] = \frac{1}{64} n^4 + o(n^4). \]

Expanding \( \mathbb{E} [LD^2] \) gives:

\[ \mathbb{E} [LD^2] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{i' = 1}^{n-1} \sum_{j' = 1}^{n-1} \mathbb{E} [X_{i,j}X_{i',i'+1}X_{j',j'+1}] . \]

There are \(\binom{n}{2} (n-1)^2 = n^4/2 + o(n^4)\) many terms in this summation. Further, there are \(2! \binom{n-2}{2} \binom{n-4}{2} = n^4/2 + o(n^4)\) many terms such that \(i, j, i', i' + 1, j', j' + 1\) are distinct. This can be seen by first choosing \(i'\) and \(j'\), and then choosing the pair \((i, j)\) such that \(i < j\). Therefore, there must be \(o(n^4)\) terms where \(i, j, i', i' + 1, j', j' + 1\) are not distinct. We have:

\[
\begin{align*}
\sum_{i,j,i',j' \in [n]} & \mathbb{E} [X_{i,j}X_{i',i'+1}X_{j',j'+1}] \\
& \text{where } i,j,i',i'+1,j',j'+1 \text{ are distinct} \\
= \sum_{i,j,i',j' \in [n]} & \mathbb{E} [X_{i,j}] \mathbb{E} [X_{i',i'+1}] \mathbb{E} [X_{j',j'+1}] \\
& \text{where } i,j,i',i'+1,j',j'+1 \text{ are distinct}
\end{align*}
\]
\[
\left(\frac{1}{2}\right)^3 \cdot 2! \left(\begin{array}{c} n-2 \\ 2 \end{array}\right) \left(\begin{array}{c} n-4 \\ 2 \end{array}\right)
= \left(\frac{1}{2}\right)^3 \cdot (n^4/2 + o(n^4)).
\]

Therefore by (1),

(4) \[ \mathbb{E}[LD^2] = \frac{1}{16} n^4 + o(n^4). \]

Summarizing, we have shown that

\[ \mathbb{E}[X^2] = \mathbb{E}[L^2] + \mathbb{E}[D^4/4] - \mathbb{E}[LD^2] + o(n^4). \]

Now the result follows from (2), (3), (4). \( \square \)

Lemma 2.3. \( \lim_{n \to \infty} \Pr[X \leq n] \to 0. \)

Proof. The event \( \{X \leq n\} \) is contained in the event \( \{|X - \mathbb{E}[X]| \geq t\} \) when \( t = \mathbb{E}[X] - n \) because \( |X - \mathbb{E}[X]| \geq t \implies \text{either} \)

(A) \( X - \mathbb{E}[X] \geq t, \)
(B) \( \mathbb{E}[X] - X \geq t, \)

and the above choice of \( t \) causes inequality (B) to be \( X \leq n \). Now, we can apply Chebyshev’s Inequality to \( X \) and \( t = \mathbb{E}[X] - n \) to get:

\[ \Pr[X \leq n] \leq \Pr[|X - \mathbb{E}[X]| \geq \mathbb{E}[X] - n] \leq \frac{\text{Var}[X]}{(\mathbb{E}[X] - n)^2} \]
\[ = \frac{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}{(\mathbb{E}[X] - n)^2}. \]

The result follows from the fact that, by Lemma 2.2,

\[ \mathbb{E}[X^2] = \frac{n^4}{64} + o(n^4) \]

and by Lemma 2.1 both

\[ (\mathbb{E}[X])^2 = \frac{n^4}{64} + o(n^4) \quad \text{and} \quad (\mathbb{E}[X] - n)^2 = \Omega(n^4). \]

This completes the proof of Theorem 1.1. \( \square \)

3. PROPERNESS IS NECESSARY FOR SPHERICALITY; PROOF OF COROLLARIES 1.1 AND 1.3

Let \( T \) be the maximal torus of diagonal matrices in \( GL_n \). For \( I \subseteq J(w) \), define

\[ B_I = L_I \cap B. \]

Hence \( B_I \) is the Borel subgroup of upper triangular matrices in \( L_I \). For a positive integer \( j \), let \( U_j \) be the maximal unipotent subgroup of \( GL_j \) consisting of upper triangular matrices with 1’s on the diagonal. Then

(5) \[ \dim U_j = \binom{j}{2}. \]
Let $U_I$ be the maximal unipotent subgroup of $B_I$. It is basic (see, e.g., [1] Chapter IV) that
\[
U_I \cong U_{d_1-d_0} \times U_{d_2-d_1} \times \cdots \times U_{d_k-d_{k-1}} \times U_{d_{k+1}-d_k}.
\]

**Proposition 3.1.** If $X_w$ is $L_I$-spherical then $w$ is proper.

**Proof.** Since $L_I$ acts spherically on $X_w$, by definition, there is a Borel subgroup $K \subset L_I$ such that $K$ has a dense orbit $O$ in $X_w$. Thus
\[
\dim X_w = \dim O.
\]
Let $x \in O$. By [2, Proposition 1.11], $O = K \cdot x$ is a smooth, closed subvariety of $X_w$ of dimension $\dim K - \dim K_x$, where $K_x$ is the isotropy group of $x$. Hence
\[
\dim X_w = \dim O = \dim K - \dim K_x \leq \dim K.
\]
All Borel subgroups of a connected algebraic group are conjugate [1, §11.1], and so $\dim K = \dim B_I$. The fact that $L_I$ acts on $X_w$ implies $I \subseteq J(w)$, and hence $L_I \subseteq L_{J(w)}$ [5, Section 1.2]. This implies $B_I \subseteq B_{J(w)}$. By [1, Theorem 10.6.(4)], $B_I = T \rtimes U_I$. Combining all this we have
\[
\dim K = \dim B_I \leq \dim B_{J(w)} = \dim T + \dim U_{J(w)}.
\]
Let $D = [n - 1] - J(w) = \{d_1 < d_2 < \ldots < d_k\}$. It follows from (5) and (6) that
\[
\dim U_{J(w)} = \left(\frac{d_1 - d_0}{2}\right) + \left(\frac{d_2 - d_1}{2}\right) + \cdots + \left(\frac{d_{k+1} - d_k}{2}\right).
\]
The right hand side is maximized when there exists a $t$ such that $d_t - d_{t-1} = n - k$ and $d_j - d_{j-1} = 1$ for all $j \neq t$. Thus
\[
\dim U_{J(w)} \leq \left(\frac{n - k}{2}\right) = \left(\frac{n - ((n - 1) - d(w))}{2}\right) = \left(\frac{d(w) + 1}{2}\right).
\]
Combining this with (7), (8), and the fact that $\ell(w) = \dim X_w$, we see $\ell(w) \leq n + \left(\frac{d(w)+1}{2}\right)$, that is, $w$ is proper.

Next, we recall the definition of $I$-spherical permutations in $S_n$ [5]. Let $s_i = (i \ i + 1)$ denote the simple transposition interchanging $i$ and $i + 1$. An expression $w = s_{i_1}s_{i_2}\cdots s_{i_t}$ for $w \in S_n$ is reduced if $\ell = \ell(w)$. Let $\text{Red}(w)$ be the set of all reduced expressions for $w$.

**Definition 2** (Definition 3.1 of [5]). $w \in S_n$ is $I$-spherical if $R = s_{i_1}s_{i_2}\cdots s_{i_\ell(w)} \in \text{Red}(w)$ exists such that

(I) $s_i$ appears at most once in $R$

(II) $\# \{m : d_{t-1} < i_m < d_t\} \leq \left(\frac{d_t-d_{t-1}+1}{2}\right) - 1$ for $1 \leq t \leq k + 1$.

This is a combinatorial analogue of Proposition 3.1.

**Proposition 3.2.** Let $w \in S_n$ and $I \subseteq J(w)$. If $w$ is $I$-spherical then $w$ is proper.

**Proof.** First suppose $I = J(w)$. Consider a reduced word $R \in \text{Red}(w)$. By Definition 2(I), at most $n - 1 - d(w)$ of the factors of $R$ are of the form $s_x$ where $x \not\in J(w)$. Thus, at least
\( \ell(w) - (n - 1 - d(w)) \) factors are of the form \( s_x \) where \( x \in J(w) \). Clearly, if \( j_1, \ldots, j_k \) are positive integers then \( \sum_{i=1}^{k+1} \binom{j_i+1}{2} \leq \binom{j_1 + \ldots + j_{k+1} + 1}{2} \). Equivalently,

\[
\sum_{i=1}^{k+1} \binom{j_i + 2}{2} - 1 = \sum_{i=1}^{k+1} \binom{j_i + 1}{2} + j_i \leq \binom{j_1 + \ldots + j_{k+1} + 1}{2} + (j_1 + \ldots + j_{k+1}).
\]

Set \( j_i = d_i - d_i - 1 - 1 \) (for \( 1 \leq i \leq k + 1 \)). Then \( j_1 + \ldots + j_{k+1} = d_{k+1} - d_0 - (k + 1) = n - 1 - k = d(w) \). Thus, by Definition 1(II), at most \( \binom{d(w)+1}{2} + d(w) \) factors are of the form \( s_x \) where \( x \in J(w) \). Therefore,

\[
\binom{d(w)+1}{2} + d(w) \geq \ell(w) - (n - 1 - d(w)).
\]

Rearranging, \( \ell(w) \leq n - 1 - d(w) + \binom{d(w)+1}{2} + d(w) \iff \ell(w) < n + \binom{d(w)+1}{2} \). So, \( w \) is proper.

For \( I \neq J(w) \), we use that if \( w \) is \( I \)-spherical then \( w \) is \( J(w) \)-spherical [5, Proposition 2.12].

Conclusion of proof of Corollaries 1.2 and 1.3 These claims follow immediately from Theorem 1.1 combined with Proposition 3.1 and Proposition 3.2 respectively.

Although we chose not to pursue it, using similar techniques, it should be possible to prove analogues of our results for the other classical Lie types.

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