Non-canonical isomorphisms

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Abstract

We give two examples of categorical axioms asserting that a canonically defined natural transformation is invertible where the invertibility of any natural transformation implies that the canonical one is invertible. The first example is distributive categories, the second (semi-)additive ones. We show that each follows from a general result about monoidal functors.

In any category \( \mathcal{D} \) with finite products and coproducts there is a natural family of maps

\[
X \times Y + X \times Z \xrightarrow{\delta_{X,Y,Z}} X \times (Y + Z)
\]

induced, via the universal property of the coproduct \( X \times Y + X \times Z \), by the morphisms \( X \times i \) and \( X \times j \), where \( i \) and \( j \) denote the coproduct injections of \( Y + Z \). Such a \( \mathcal{D} \) is said to be distributive \([2, 3]\) if the canonical maps are invertible; in other words, if the functor \( X \times - : \mathcal{D} \to \mathcal{D} \) preserves binary coproducts, for all objects \( X \). As observed by Cockett \([3]\), it follows that \( X \times 0 \cong 0 \), so that \( X \times - \) in fact preserves finite coproducts.

Claudio Pisani has asked whether the existence of any natural family of isomorphisms

\[
X \times Y + X \times Z \xrightarrow{\psi_{X,Y,Z}} X \times (Y + Z)
\]

might imply that \( \mathcal{D} \) is distributive. Such \( \psi \) are the non-canonical isomorphisms of the title. He suggested that this was probably not the case, and this was also my immediate reaction. But in fact it is true! This is the first result of the paper.

The second result is an analogue for semi-additive categories. Recall that a category is pointed when it has an initial object which is also terminal (\( 1 = 0 \)), and that for any any two objects \( Y \) and \( Z \) in a pointed category there is a unique morphism from \( Y \) to \( Z \) which factorizes through the zero object; this morphism is called \( 0_{Y,Z} \) or just \( 0 \). If the category has finite products and coproducts, then there is a natural family of morphisms

\[
Y + Z \xrightarrow{\alpha_{Y,Z}} Y \times Z
\]

induced by the identities on \( Y \) and \( Z \) and the zero morphisms \( 0 : Y \to Z \) and \( 0 : Z \to Y \). The category is semi-additive when these \( \alpha_{Y,Z} \) are invertible \([6, \text{VII.2}]\). A semi-additive category admits a canonical enrichment over commutative monoids; conversely, any category enriched over commutative monoids which has either finite products or finite coproducts is semi-additive. Our result for semi-additive categories asserts once again that the existence of any natural isomorphism \( Y + Z \cong Y \times Z \) implies that the category is semi-additive.

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We also show that the common part of the two arguments follows from a general result about monoidal functors; since the individual results are so easy to prove, however, we give them first, in Sections 1 and 2 respectively, before turning to the general result in Section 3.

1 Non-canonical distributivity isomorphisms

This section involves, as in the introduction, a category $\mathcal{D}$ with finite products and coproducts and a natural family of isomorphisms $X \times Y + X \times Z \cong X \times (Y + Z)$. First we show, in the following Lemma, that such a $\mathcal{D}$ will be distributive if $X \times 0 \cong 0$. Later on, we shall see that this Lemma follows from a more general result about coproduct-preserving functors due to Caccamo and Winskel; and that this in turn is a special case of a still more general result about monoidal functors: this is our Theorem 6 below.

**Lemma 1** Suppose that as above that we have natural isomorphisms

$$X \times Y + X \times Z \xrightarrow{\psi} X \times (Y + Z)$$

and that $X \times 0 \cong 0$. Then the category $\mathcal{D}$ is distributive.

**Proof:** If $X \times 0 \cong 0$, then $\varphi_{X,Y,0}$ gives an isomorphism $X \times Y + X \times 0 \cong X \times (Y + 0)$, which we can regard as simply being an isomorphism $X \times Y \cong X \times Y$. By naturality, the diagram

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\psi_{X,Y,0}} & X \times Y \\
\downarrow{\ i} & & \downarrow{\ X \times i} \\
X \times Y + X \times Z & \xrightarrow{\psi_{X,Y,Z}} & X \times (Y + Z)
\end{array}
$$

commutes, and similarly we have a commutative diagram

$$
\begin{array}{ccc}
X \times Z & \xrightarrow{\psi_{X,0,Z}} & X \times Y \\
\downarrow{\ i} & & \downarrow{\ X \times i} \\
X \times Y + X \times Z & \xrightarrow{\psi_{X,Y,Z}} & X \times (Y + Z)
\end{array}
$$

and now combining these we get a commutative diagram

$$
\begin{array}{ccc}
X \times Y + X \times Z & \xrightarrow{\psi_{X,Y,0} + \psi_{X,0,Z}} & X \times Y + X \times Z \\
\downarrow{\delta_{X,Y,Z}} & & \downarrow{\delta_{X,Y,Z}} \\
X \times Y + X \times Z & \xrightarrow{\psi_{X,Y,Z}} & X \times (Y + Z)
\end{array}
$$

In this last diagram, the $\psi$'s are all invertible, hence so is $\delta$. $\square$

Recall that an object $T$ is called *subterminal* if for any object $X$ there is at most one morphism from $X$ to $T$. (If, as here, a terminal object exists, this is equivalent to saying that the unique map $T \to 1$ is a monomorphism.)

Thus to prove our result about non-canonical distributivity isomorphisms, we must show that the assumption that $X \times 0 \cong 0$ made in the Lemma is unnecessary. The remainder of this section will be devoted to doing so.
**Proposition 2** The product $0 \times 0$ is initial, and so $0$ is subterminal.

**Proof:** For the first part, observe that $\psi_{0,0,1}$ gives an isomorphism $0 \times 0 + 0 \times 1 \cong 0 \times (0 + 1)$, and that $0 + 1 \cong 1$ and $0 \times 0 \cong 0$. For the second, we have an isomorphism $0 \cong 0 \times 0$, and since $0$ is initial, this can only be the diagonal $\Delta : 0 \to 0 \times 0$. Thus any morphism $X \to 0 \times 0$ factorizes through the diagonal, and so any two morphisms $X \to 0$ are equal. \qed

Next we consider the special case where $D$ is pointed ($0 = 1$); ultimately we shall reduce the general case to this. In a pointed category, every object has a (unique) morphism into $0$; but in a distributive category, any morphism into $0$ is invertible [2, Proposition 3.4]. It follows that any category which is pointed and distributive is equivalent to the terminal category $1$. Our next result shows that the same conclusion holds under the assumption of pointedness and a non-canonical distributivity isomorphism.

**Proposition 3** If $D$ is pointed then $D$ is equivalent to the terminal category $1$.

**Proof:** Taking $Y = Z = 1$ gives a natural family $\theta_X = \psi_{X,1,1} : X + X \cong X$. By naturality, the diagram

\[
\begin{array}{ccc}
X + X + X + X & \xrightarrow{\theta_{X+X}} & X + X \\
\downarrow \theta_X & & \downarrow \theta_X \\
X + X & \xrightarrow{\theta_X} & X
\end{array}
\]

commutes, and now since $\theta_X$ is invertible $\theta_X + \theta_X = \theta_{X+X}$. The diagram

\[
\begin{array}{ccc}
X + X & \xrightarrow{\theta_X} & X \\
\downarrow \nabla & & \downarrow \nabla \\
X + X + X + X & \xrightarrow{\theta_{X+X}} & X + X \\
\downarrow i & & \downarrow i \\
X + X & \xrightarrow{\theta_X} & X
\end{array}
\]

commutes, where $i_{X+X}$ denotes the injection of the first two copies of $X$ into $X + X + X + X$. But now $\theta_X = \nabla i \theta_X = \theta_X i \nabla$ is invertible, so $\nabla : X + X \to X$ is a monomorphism; since it also has a section $i$ (and $j$) it is invertible. This proves that any two maps $X \to Y$ must be equal. On the other hand, there is always at least one such map, since $D$ is pointed; thus there is exactly one, and so $X \cong 0$. Since $X$ was arbitrary, the result follows. \qed

**Theorem 4** If $D$ is a category with finite products and coproducts, and with a natural family

\[
\psi_{X,Y,Z} : X \times Y + X \times Z \cong X \times (Y + Z)
\]

of isomorphisms, then $D$ is distributive.

**Proof:** By Lemma 4 it will suffice to show that $X \times 0 \cong 0$. Since we have the projection $X \times 0 \to 0$, and the composite $0 \to X \times 0 \to 0$ is certainly the identity, we need only show that the other composite $e : X \times 0 \to 0 \to X \times 0$ is the identity. This is an endomorphism in the slice category $D/0$. So if $D/0$ is trivial, then this composite $e$ will be the identity, and $X \times 0$ will be isomorphic to $0$.

Since $0$ is subterminal, the projection $D/0 \to D$ is fully faithful, and preserves finite products as well as coproducts. Thus the isomorphisms $\psi_{X,Y,Z}$ restrict to $D/0$, thus equipping $D/0$ with non-standard distributivity isomorphisms. By Proposition 3 $D/0$ is trivial, and so $D$ is distributive. \qed
2 Non-canonical semi-additivity isomorphisms

We now give an analogous result for semi-additivity. An interesting feature is that this does not require us to assume that the category is pointed, although that will of course be a consequence.

**Theorem 5** If $\mathcal{A}$ is a category with finite products and coproducts and with a natural family 

$\psi_{Y,Z} : Y + Z \cong Y \times Z$

of isomorphisms, then $\mathcal{A}$ is semi-additive.

**Proof:** Taking $Y = 1$ and $Z = 0$ gives an isomorphism $\psi_{1,0} : 1 \cong 1 \times 0$; composing with the projection $1 \times 0 \to 0$ gives a morphism $1 \to 0$. By uniqueness of morphisms into 1 and out of 0, this is inverse to the unique map $0 \to 1$, and so $\mathcal{A}$ is pointed.

Taking one of $Y$ and $Z$ to be 0 gives natural isomorphisms $\psi_{Y,0} : Y \cong Y$ and $\psi_{0,Z} : Z \cong Z$. By naturality of the $\psi_{Y,Z}$, the diagrams

$$
\begin{array}{ccc}
Y & \xrightarrow{\psi_{Y,0}} & Y \\
\downarrow{i} & & \downarrow{(1)} \\
Y + Z & \xrightarrow{\psi_{Y,Z}} & Y \times Z
\end{array} \quad \begin{array}{ccc}
Z & \xrightarrow{\psi_{0,Z}} & Z \\
\downarrow{j} & & \downarrow{(2)} \\
Y + Z & \xrightarrow{\psi_{Y,Z}} & Y \times Z
\end{array}
$$

commute, and so also

$$
\begin{array}{ccc}
Y + Z & \xrightarrow{\psi_{Y,0} + \psi_{0,Z}} & Y + Z \\
\downarrow{\psi_{Y,Z}} & & \downarrow{\alpha_{Y,Z}} \\
Y + Z & \xrightarrow{\psi_{Y,Z}} & Y \times Z
\end{array}
$$

commutes. Just as in the proof of the lemma, $\psi_{Y,0} + \psi_{0,Z}$ and $\psi_{Y,Z}$ are invertible, hence so is $\alpha_{Y,Z}$. □

3 Non-canonical isomorphisms for monoidal functors

In this section we prove a general result on monoidal functors, which could be used in the proof of both of the other theorems. Recall that if $\mathcal{A}$ and $\mathcal{B}$ be monoidal categories, a monoidal functor $F : \mathcal{A} \to \mathcal{B}$ consists of a functor (also called $F$) equipped with maps $\varphi_{Y,Z} : FY \otimes FZ \to F(Y \otimes Z)$ and $\varphi_0 : I \to FI$ which need not be invertible, but which are natural and coherent [4]. The monoidal functor is said to be **strong** if $\varphi_{Y,Z}$ and $\varphi_0$ are invertible, and **normal** if $\varphi_0$ is invertible. Given such an $F$ and another monoidal functor $G : \mathcal{A} \to \mathcal{B}$ with structure maps $\psi_{X,Y}$ and $\psi_0$, a natural transformation $\alpha : F \to G$ is **monoidal** if the diagrams

$$
\begin{array}{ccc}
FY \otimes FZ & \xrightarrow{\varphi_{X,Y} \otimes \alpha_Y} & GY \otimes GZ \\
\downarrow{\varphi_{X,Y}} & & \downarrow{\psi_{X,Y}} \\
F(Y \otimes Z) & \xrightarrow{\alpha_{X \otimes Z}} & G(Y \otimes Z)
\end{array} \quad \begin{array}{ccc}
F & \xrightarrow{\psi_0} & FI \\
\downarrow{\varphi_0} & & \downarrow{\alpha I} \\
G & \xrightarrow{\psi_0} & GI
\end{array}
$$

commute. Recall further [5] that if $\mathcal{C}$ is braided monoidal, then the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is strong monoidal, with structure maps

$$
\begin{array}{ccc}
E \otimes X \otimes Y \otimes Z & \xrightarrow{W \otimes \gamma \otimes D} & W \otimes X \otimes Y \otimes Z \\
\downarrow{\alpha_X \otimes Y \otimes Z} & & \downarrow{\alpha X \otimes Y \otimes Z} \\
W \otimes X \otimes Y \otimes Z & \xrightarrow{I \otimes \gamma} & W \otimes X \otimes Y \otimes Z
\end{array}
$$

where $\gamma$ denotes the braiding and $\lambda$ the canonical isomorphism.
Theorem 6 Let $\mathcal{A}$ and $\mathcal{B}$ be braided monoidal categories, and $F = (F, \varphi, \varphi_0) : \mathcal{A} \to \mathcal{B}$ a normal monoidal functor (so that $\varphi_0$ is invertible). Suppose further that we have a monoidal isomorphism $A \times A \xrightarrow{\psi} B \times B \downarrow \varphi \downarrow \circ \circ$.

Then $\varphi$ is invertible, and so $F$ is strong monoidal.

Proof: The fact that $\psi$ is monoidal means in particular that the diagram

```
FW \otimes FX \otimes FY \otimes FZ \xrightarrow{\psi_{W,X} \otimes \psi_{Y,Z}} F(W \otimes X) \otimes F(Y \otimes Z)
```

commutes. Taking $X = Y = I$ and twice using the isomorphism $\varphi_0$ gives commutativity of

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FW \otimes FZ
```

in which all arrows except $\varphi_{W,Z}$ are invertible; thus $\varphi_{W,Z}$ too is invertible. □

Remark 7 In the proof of Theorem 6 we have used rather less than was assumed in the statement. For example, we do not use the nullary part of the assumption that the natural transformation is monoidal.

The following corollary appeared (in dual form) as [1, Theorem 3.3]:

Corollary 8 (Caccamo-Winskel) Let $\mathcal{A}$ and $\mathcal{B}$ be categories with finite coproducts, and $F : \mathcal{A} \to \mathcal{B}$ a functor which preserves the initial object. If there is a natural family of isomorphisms $X + Y \cong Y \times Z$ then $F$ preserves finite coproducts.

Proof: In this case $F$ has a unique monoidal structure, and $\psi$ is always monoidal. □

In particular if $\mathcal{D}$ has finite products and coproducts, we may apply the Corollary to the functor $X \times - : \mathcal{D} \to \mathcal{D}$ and recover Lemma 1.

Section 2 involves the case where the categories $\mathcal{A}$ and $\mathcal{B}$ are the same, but the monoidal structure on $\mathcal{A}$ is cartesian and that on $\mathcal{B}$ is cocartesian. The functor $F$ is the identity. One proves $0 \to 1$ is invertible, as in the proof of Theorem 5 and then the identity $1 : \mathcal{A} \to \mathcal{A}$ has a unique normal monoidal structure, with binary part precisely the canonical morphism $\alpha : Y + Z \to Y \times Z$. Furthermore, any natural isomorphism $\psi_{Y,Z} : Y + Z \cong Y \times Z$ is monoidal.
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