Hamiltonian Equations of Reduced Conformal Geometrodynamics in Extrinsic Time

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1. INTRODUCTION

The Hamiltonian dynamics of the gravitational field is commonly formulated in redundant variables in an extended functional phase space as a consequence of the covariant description of Einstein’s theory. A time parameter should be conjugated to the Hamiltonian constraint. The Hamiltonian formulation of the theory makes it possible to reveal the physical meaning of geometrical variables. The problem of writing the vacuum Einstein equations in unconstrained variables for compact cosmological models is relevant. The reduced phase space is the cotangent bundle of the Teichmüller space of conformal structures on compact spacelike hypersurfaces [1]. The Hamiltonian is a volume functional. The Hamiltonian dynamics is constructed in York’s time [2]. The problem is that the Hamiltonian density as a volume functional is not expressed in an explicit form from the Hamiltonian constraint (the Lichnerowicz—York elliptic differential equation). This makes it difficult to obtain a Hamiltonian flow. In the present paper we obtain the Hamiltonian equations of motion. This is achieved by generalizing the theorem on implicit function derivative from mathematical analysis to functional analysis.

Let the spacetime $\mathcal{M} = \mathbb{R}^1 \times \Sigma_t$ be foliated into a family of spacelike hypersurfaces $\Sigma_t$, labeled by the time coordinate $t$ with just three spatial coordinates on each slice, $(x^1, x^2, x^3)$. The first quadratic form

$$\gamma := \gamma_{ik}(t, x) dx^i \otimes dx^k$$

defines the induced metric on every slice $\Sigma_t$. The Hamiltonian dynamics is built of the ADM variational functional

$$S_{ADM} = \int_0^{t_f} dt \int_{\Sigma_t} d^3 x \left( \pi^{ij} \frac{\partial \gamma_{ij}}{\partial t} \right.\ 
- N \mathcal{H}_\perp - N^i \mathcal{H}_i \left), \quad (2)\right.$$

where the ADM units, $c = 1$, $16\pi G = 1$ are used. Variation of the ADM action (2) in the lapse function $N$ leads to the Hamiltonian constraint expressed via components of the extrinsic curvature $K_{ij}$:

$$K_{ij} := \frac{1}{2N} \left( -\frac{\partial \gamma_{ij}}{\partial t} + \nabla_i N_j + \nabla_j N_i \right), \quad (3)$$

where the connection $\nabla_i$ is associated with the metric $\gamma_{ij}$, or in the components of the momentum densities $\pi^{ij}$

$$\mathcal{H}_\perp = \sqrt{\gamma} \left( K_{ij} K^{ij} - K^2 - R \right)$$

$$= \frac{1}{2\sqrt{\gamma}} (\gamma_{ik} \gamma_{jl} + \gamma_{il} \gamma_{jk} - \gamma_{ij} \gamma_{kl}) \pi^{ij} \pi^{kl} - \sqrt{\gamma} R.$$

Here, $\gamma$ is the determinant of the metric tensor,

$$K^{ij} := \gamma^{ik} \gamma^{jl} K_{kl}, \quad K := K_{ij} \gamma^{ij},$$

and $R$ is the Ricci scalar curvature. Varying the action (2) in the shift functions $N^i$, we get the momentum constraints

$$\mathcal{H}_i = 2\sqrt{\gamma} \left( \nabla_j K_{ij} - \nabla_i K \right) = -2\nabla_j \pi^j_i.$$

Variations of the action (2) with respect to the canonical variables $\pi^U(t, x)$ and $\gamma_{ij}(t, x)$ lead to the kinematic equations

$$\frac{\delta}{\delta \pi^U} S_{ADM} = \frac{\partial \gamma_{ij}}{\partial t} = -2NK_{ij} + \nabla_i N_j + \nabla_j N_i.$$
The Lie Equations (8) and (9) define to the dynamical equations

\[ \frac{\delta}{\delta \gamma} S_{ADM} = -\frac{\partial \pi^{ij}}{\partial t} = N \sqrt{\gamma} \left(R^{ij} - \frac{1}{2} \gamma^{ij} R \right) \]

\[ -\frac{N}{2 \sqrt{\gamma}} \gamma^{ij} \left( \pi^{mn} \pi_{mn} - \frac{1}{2} \pi^2 \right) + 2N \sqrt{\gamma} \left( \pi^{im} \pi_{im} - \frac{1}{2} \pi \pi^{ij} \right) - \sqrt{\gamma} \left( \nabla^i \nabla^j N - \gamma^{ij} \Delta N \right) - \nabla_m \left( \pi^{ij} N^m \right) + \nabla_m N^i \pi^{mj} + \nabla_m N^j \pi^{mi}, \]

where \( \Delta := \nabla_i \nabla^i \) is the Laplacian.

2. CONFORMAL DECOMPOSITION

The equations of motion (5), (6) contain the unknown Lagrange multipliers. To obtain the dynamic variables, conformal transformation are implemented [3]:

\[ \gamma_{ij} := \phi^4 \tilde{\gamma}_{ij}, \quad \phi^4 = \gamma^{1/3}. \]

To the conformal variables

\[ \tilde{\gamma}_{ij} := \frac{\gamma_{ij}}{\sqrt{\gamma}}, \quad \tilde{\pi}^{ij} := \sqrt{\gamma} \left( \pi^{ij} - \frac{1}{3} \pi \gamma^{ij} \right), \]

where \( \pi := \gamma_{ij} \pi^{ij} \), we add the canonical pair

\[ \tau := 2 \frac{\pi}{3 \sqrt{\gamma}} = \frac{4}{3} K, \quad \mathcal{H} := \sqrt{\gamma}. \]

Equations (8) and (9) define the Dirac’s mapping. The Lie-Poisson structure of the new variables in the extended phase space \( \Gamma_{\tau} \mathcal{H} \) is the following:

\[ \{ \tau(t, x), \mathcal{H}(t, x') \} = -\delta(x - x'), \]

\[ \{ \tilde{\gamma}_{ij}(t, x), \tilde{\pi}^{kl}(t, x') \} = \delta_{ij} \delta(x - x'), \]

\[ \{ \tilde{\pi}^{ij}(t, x), \tilde{\pi}^{kl}(t, x') \} = \frac{1}{3} \left( \tilde{\gamma}^{kl} \tilde{\pi}_{ij} - \tilde{\gamma}^{ij} \tilde{\pi}^{kl} \right) \delta(x - x'), \]

where

\[ \tilde{\delta}^{kl}_{ij} := (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k - \frac{1}{3} \delta_i^l \delta_j^k \tilde{\gamma}_{ij}). \]

Under the conformal transformation (7), its components transform according to

\[ A^{ij} = \phi^{-10} \tilde{A}^{ij}, \quad A_{ij} = \phi^{-2} \tilde{A}_{ij}. \]

They are connected with the components of the conformal momentum densities

\[ \tilde{A}^{ij} = -\tilde{\pi}^{ij}, \quad \tilde{A}_{ij} = -\tilde{\pi}_{ij}. \]

The conformal Ricci scalar \( \tilde{R} \) is expressed from the Ricci scalar \( R \) as

\[ R = \phi^{-4} \tilde{R} - 8 \phi^{-5} \Delta \phi, \]

where \( \tilde{\Delta} := \tilde{\nabla}_i \tilde{\nabla}^i \) is the conformal Laplacian, and \( \tilde{\nabla}_i \) is the conformal connection associated with the conformal metric \( \tilde{\gamma}_{ij} \). The conformal Hamiltonian constraint reads

\[ \tilde{H}_\perp := \tilde{\pi}_{ij} \tilde{\pi}^{ij} \phi^{-6} + \frac{8}{3} \phi \tilde{\Delta} \phi - \frac{3}{8} \phi^{-2} \phi^6, \]

and the conformal momentum constraints read

\[ \tilde{H}^i := -2 \phi^{-4} \tilde{\nabla}_j \tilde{\pi}^{ij} - \frac{4}{3} \phi^2 \tilde{\nabla}^i K. \]

We can introduce York’s global time and the canonically conjugated Hamiltonian:

\[ T := \frac{1}{3} \langle \tilde{\pi} \rangle = \frac{2}{3} \int \Sigma_t \frac{d^3 x \pi(x)}{\sqrt{\gamma}(x)}, \]

\[ H := \int \Sigma_t d^3 x \sqrt{\gamma}(x) = V_t, \]

where the Hamiltonian is the volume of the hypersurface \( V_t \), and the variables commute to minus one:

\[ \{ T, H \} = -1. \]

York proposed the constant curvature condition (CMC)[2]

\[ \tau = \frac{4}{3} K(t) = T = t \]

to fix the spacetime slicing. So, the local time \( \tau \) becomes the global time \( T \) (15). The last term in (14) is zero. This gauge allowed for decomposition of the conformal momentum densities into longitudinal \( \tilde{\pi}_{ij}^L \) and traceless-transverse \( (\tilde{\nabla}_i \tilde{\pi}_{ij}^T = 0) \) parts:

\[ \tilde{\pi}^{ij} = \tilde{\pi}^{ij}_L + \tilde{\pi}^{ij}_T. \]

The longitudinal part \( \tilde{\pi}_{ij}^L \) is the constrained part and is obtained as a solution of the linear elliptic differential equations (14).
3. CONFORMAL HAMILTONIAN
EQUATIONS OF MOTION

Expressing the conformal factor $\phi$ via the Hamiltonian density $\phi = \mathcal{H}^{1/6}$, we substitute it into the Hamiltonian constraint (13):

$$\tilde{\mathcal{H}}_{\perp} = \frac{1}{2} (\tilde{\gamma}_{ik} \tilde{\gamma}_{jl} + \tilde{\gamma}_{il} \tilde{\gamma}_{jk}) \tilde{\pi}^{ij} \tilde{\pi}^{kl} \mathcal{H}^{-1} + 8 \mathcal{H}^{1/6} \Delta \mathcal{H}^{1/6} - \tilde{R} \mathcal{H}^{1/3} - \frac{3}{8} T^2 \mathcal{H}. \tag{19}$$

The reduced ADM action (2) then reads

$$S_{\text{reduced}} = \int_{\Sigma_T} dT \int d^3 x \left( \tilde{\pi}^{ij} \frac{d\tilde{\gamma}_{ij}}{dT} - \mathcal{H}[\tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; T] - N^i \mathcal{H}_i[\tilde{\pi}^{ij}, \tilde{\gamma}_{ij}] \right).$$

The Hamiltonian density $\mathcal{H}$ is a functional of the variables $\tilde{\pi}^{ij}, \tilde{\gamma}_{ij}$ and a function of the time $T$; $\mathcal{H}_i$ are generators of changing of the coordinates in the hypersurface. The Hamiltonian

$$H := \int_{\Sigma_T} d^3 x \mathcal{H}[\tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; T] \tag{20}$$

generates the dynamics of the gravitational field. Unfortunately, we do not have its explicit form. Below, we can find the derivatives of the Hamiltonian (20) with respect to the conformal variables.

Variation of the functional of the conformal Hamiltonian constraint (19)

$$\delta \tilde{\mathcal{H}}_{\perp} := \int_{\Sigma_T} d^3 x \left( \delta \mathcal{H} \frac{\partial}{\partial \tilde{\pi}^{ij}} + \frac{\partial}{\partial \tilde{\gamma}_{ij}} \delta \mathcal{H} \right) \tag{21}$$

on a slice $T$ is zero, $\delta \tilde{\mathcal{H}}_{\perp} = 0$:

$$\int_{\Sigma_T} d^3 x \left( \delta \mathcal{H} \frac{\partial}{\partial \tilde{\pi}^{ij}} + \frac{\partial}{\partial \tilde{\gamma}_{ij}} \delta \mathcal{H} \right) \tag{22}$$

Variation of the Hamiltonian density can be presented as

$$\delta \mathcal{H} = \frac{\partial \mathcal{H}}{\partial \tilde{\pi}^{ij}} \delta \tilde{\pi}^{ij} + \frac{\partial \mathcal{H}}{\partial \tilde{\gamma}_{ij}} \delta \tilde{\gamma}_{ij}.$$

After substitution of $\delta \mathcal{H}$ into (22) one gets

$$\delta \tilde{\mathcal{H}}_{\perp} = \int_{\Sigma_T} d^3 x \left( \delta \mathcal{H} \frac{\partial}{\partial \tilde{\pi}^{ij}} + \frac{\partial \mathcal{H}}{\partial \tilde{\gamma}_{ij}} \frac{\partial}{\partial \tilde{\gamma}_{ij}} \right) \delta \tilde{\pi}^{ij} \tag{23}$$

Taking into account the independence of variations, we obtain the derivatives

$$\frac{\partial \mathcal{H}}{\partial \tilde{\pi}^{ij}} = - \frac{\delta \tilde{\mathcal{H}}_{\perp}}{\delta \tilde{\pi}^{ij}} \frac{\partial}{\partial \tilde{\gamma}_{ij}}, \quad \frac{\partial \mathcal{H}}{\partial \tilde{\gamma}_{ij}} = - \frac{\delta \tilde{\mathcal{H}}_{\perp}}{\delta \tilde{\gamma}_{ij}} \frac{\partial}{\partial \tilde{\gamma}_{ij}} \tag{24}$$

The Hamiltonian $H$ (20) generates a phase flow in the phase space $\Gamma[\tilde{\gamma}_{ij}, \tilde{\pi}^{ij}]$ on the Poisson brackets (11), (12)

$$\frac{d}{dT} \tilde{\gamma}_{ij}(x) = \{\tilde{\gamma}_{ij}(x), H\}$$

Let us calculate the functional derivative of (21) with respect to the Hamiltonian density:

$$\frac{\delta \tilde{\mathcal{H}}_{\perp}}{\delta \mathcal{H}(x)} = - \frac{1}{2} (\tilde{\gamma}_{ik} \tilde{\gamma}_{jl} + \tilde{\gamma}_{il} \tilde{\gamma}_{jk}) \tilde{\pi}^{ij} \tilde{\pi}^{kl} \mathcal{H}^{-2}(x) - \frac{8}{3} \mathcal{H}^{-2/3}(x) \tag{25}$$

The functional derivatives of (21) with respect to the conformal momentum densities are

$$\frac{\delta \tilde{\mathcal{H}}_{\perp}}{\delta \tilde{\pi}^{ij}(x)} = (\tilde{\gamma}_{ik} \tilde{\gamma}_{jl} + \tilde{\gamma}_{il} \tilde{\gamma}_{jk}) \tilde{\pi}^{kl} \mathcal{H}^{-1}(x), \tag{26}$$

and with respect to the conformal metric:

$$\frac{\delta \tilde{\mathcal{H}}_{\perp}}{\delta \tilde{\gamma}_{ij}(x)} = 2 \tilde{\gamma}_{kl} \tilde{\pi}^{ik} \tilde{\pi}^{jl} \mathcal{H}^{-1}$$

Substituting the functional derivatives (26), (27), (28) into (23), we get the partial derivatives desired. Then, taking into account the Lie–Poisson algebra (11), (12), we substitute these partial derivatives into (24), (25) and obtain the Hamiltonian equations

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of motion of the gravitational field. The differential equations
\[
\frac{d}{dT} \tilde{\gamma}_{ij}(x) = -\frac{4}{\delta H / \delta H} \tilde{\pi}^{ij}(x) \tag{29}
\]
present the kinematic equations for the conformal variables (compare with (5)). The following differential equations are the dynamical equations (compare with (6)):
\[
\frac{d}{dT} \tilde{\pi}^{ij}(x) = \frac{4}{\delta H / \delta H} \tilde{\gamma}_{kl} \tilde{\pi}^{ik} \tilde{\pi}^{jl} \\
- \frac{16}{\delta H / \delta H} \left( \tilde{\nabla}^{i} H^{1/6} \right) \left( \tilde{\nabla}^{j} H^{1/6} \right) \\
- \frac{8}{3} \frac{\delta H / \delta H}{\delta H / \delta H} \tilde{\gamma}_{ij} \left( \tilde{\nabla}_{k} H^{1/6} \right) \left( \tilde{\nabla}^{k} H^{1/6} \right) \\
- \frac{H^{1/3}}{\delta H / \delta H} \left( 2 \tilde{R}^{ij} - \frac{1}{3} \tilde{R} \tilde{\gamma}^{ij} \right). \tag{30}
\]

In the present paper, we did not make simplifications anywhere, so the form of the equations looks rather complicated. Their advantage in comparison with the ADM equations (5), (6) is that they do not contain Lagrange multipliers. They can be useful when considering model problems and perturbation theory, since their appearance should be simplified. We have considered vacuum general relativity. Our approach can be extended to the general case including conformal matter sources. In that case the Hamiltonian (20) would be a functional of the conformal matter variables additionally.

4. CONCLUSION AND DISCUSSION

The variables of the unconstrained system are the true degrees of freedom of the gravitational field. The redshift of a galactic spectrum and the modern Hubble diagram were interpreted \([4–6]\) within general relativity in conformal variables without introducing Dark Energy. The Ricci flow of three-manifolds was studied in \([7]\), and the conformal Ricci flow in \([8]\), which says about the importance of the subject. In the general case, the Hamiltonian constraint is an elliptic differential equation for the Hamiltonian density. For the systems with finite degrees of freedom it becomes an algebraic equation. For a homogeneous and isotropic minisuperspace model, the reduction was undertaken in \([9]\), and for an anisotropic model in \([10]\).

In a cosmological scenario \(\sqrt{\gamma} \sim a^{3}\), where \(a\) is the global scale factor, the extrinsic curvature (4) is
\[
K = -\frac{1}{2N} \tilde{\gamma}_{ij} \frac{\partial \tilde{\gamma}_{ij}}{\partial t} = -\frac{3}{N} \left( \frac{\dot{a}}{a} \right) = -\frac{3}{N} \left( a' \right),
\]
where the dot denotes differentiation with respect to coordinate time, and the prime differentiation with respect to conformal time. In York’s gauge (18)
\[
T = \frac{4}{3} K = -\frac{4}{N} \left( \frac{\dot{a}}{a} \right).
\]
Hence, York’s time is proportional to the Hubble parameter.

The global intrinsic time was constructed in \([11]\). It was achieved by averaging the geometric characteristics over hypersurfaces of constant coordinate time. The Hamiltonian equations of motion in the intrinsic time are written in \([12]\). In this case, the Hamiltonian and time exchange places. The advantage of this approach is that one can express the Hamiltonian from the Hamiltonian constraint explicitly. But in York’s gauge \(K = K(t)\) it is possible to split off the longitudinal components of the momentum densities.

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