The co-points are cut points of level sets for Busemann functions *†

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Abstract

We show that the corays to a ray in a complete non-compact Finsler manifold contain geodesic segments to level sets of Busemann functions. Moreover, we characterize the set of co-points to a ray as the cut locus of such set levels. The structure theorem of the co-points set on a surface, namely that is a local tree, and other properties follows immediately from the known results about cut locus. We point out that our Main Theorems 1.1 and 1.2, first statement, are new even for Riemannian manifolds.

1 Introduction

Roughly speaking, Busemann function is a function that measures the distance to a point at infinity on a complete non-compact Riemannian or Finsler manifold. Originally introduced by H. Busemann for constructing a theory of parallels for straight lines (see [Bu1], [Sh], [In1], [In2]), the function plays a fundamental role in the study of complete non-compact Riemannian or Finsler manifolds ([Oh], [Sh], [SST], etc).

In the present paper we study the differentiability of the Busemann function in terms of corays and co-points to a ray in the general case of a forward complete non-compact Finsler manifold. We show that the notions of geodesic segments to a closed subset and the cut locus of such sets can be extremely useful in the study of corays and co-points to a ray, that is points where Busemann function is not differentiable.

The originality of our research is two folded. Firstly, the detailed study of Busemann functions, corays and co-points on Finsler manifolds is new. Secondly, in the special case of Riemannian manifolds, our Main Theorems 1.1 and 1.2 first statement, are new and they lead to new elementary proofs of other results already known.

A Finslerian unit speed globally minimizing geodesic $\gamma : [0, \infty) \to M$ is called a (forward) ray. A ray $\gamma$ is called maximal if it is not a proper sub-ray of another ray, i.e.

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for any $\varepsilon > 0$ its extension to $[-\varepsilon, \infty)$ is not a ray anymore. If $(M, F)$ is bi-complete, i.e. forward and backward complete, then an indefinitely extensible geodesic at both ends, globally minimizing, $\gamma : \mathbb{R} \to M$ is called a straight line.

Let $\gamma : [0, \infty) \to M$ be a given forward ray and let $x$ be a point on a non-compact forward complete Finsler manifold $(M, F)$. Then a forward ray $\sigma : [0, \infty) \to M$ emanating from $x := \sigma(0)$ is called a forward coray (or a forward asymptotic ray) to $\gamma$ if there exists a sequence of minimal geodesics $\{\sigma_j\}$ from $q_j := \sigma_j(0)$ to $\sigma_j(l_j) := \gamma(t_j)$, for some divergent sequence of numbers $\{t_j\}$, such that $\lim_{j \to \infty} q_j = \sigma(0)$ and $\dot{\sigma}(0) = \lim_{j \to \infty} \dot{\sigma}_j(0)$.

An coray of $\gamma$ is called maximal if for any $\varepsilon > 0$ its extension to $[-\varepsilon, \infty)$ is not a coray anymore. The origin points of maximal corays of $\gamma$ are called the co-points to $\gamma$. Similarly one can define asymptotic straight lines to $\gamma$ (a slightly stronger definition can be found in [Oh]).

A unit speed (nonconstant) geodesic segment $\alpha : [0, a] \to M$ is called a forward $N$-segment if $d(\alpha(t), N) = d(\alpha(t), \alpha(a)) = a - t$ holds on $[0, a]$. The end point $\alpha(0)$ of the geodesic segment $\alpha$ is called a cut point of $N$ along the $N$-segment if any geodesic extension $\tilde{\alpha} : [-\varepsilon, a] \to M$ of $\alpha$ is not a forward $N$-segment anymore. The cut locus of $N$ is defined by the set of all cut points of $N$ along all nonconstant $N$-segments. It is well known that if a point admits two $N$-segments, then it is a cut point of $N$. Therefore, any interior point of $N$-segment is not a cut point of $N$.

We point out that in [IS], for a closed subset $N$ of a backward complete Finsler manifold $(M, F)$, a backward $N$-segment is defined analogously. The notions of forward and backward $N$-segments to a closed subset $N$ are equivalent. Indeed, if we consider the reverse Finsler metric $\tilde{F}$ on the manifold $M$ given by $\tilde{F}(x, y) := F(x, -y)$ for each $(x, y) \in TM$, a backward $N$-segment on $(M, F)$ is a forward $N$-segment on $(M, \tilde{F})$.

Notice that any geodesic segment on a compact interval admits forward and backward geodesic extensions even if the manifold $M$ is not forward nor backward complete. For more basics on Finsler manifolds see [BCS] or [S].

Here are the main results of our paper.

**Theorem 1.1** Let $(M, F)$ be a forward complete Finsler manifold and $\alpha : [0, a] \to M$ a unit-speed geodesic. The following three statements are equivalent.

1. $\alpha$ is a subarc of a coray of $\gamma$.
2. $\alpha$ satisfies
   \[ b_\gamma(\alpha(s)) = s + b_\gamma(\alpha(0)) \] (1.1)
   for all $s \in [0, a]$.
3. $\alpha$ is a forward $N^b$-segment, where $N^b := b_\gamma^{-1}[b, \infty)$ and $b = b_\gamma(\alpha(a))$.

From here it naturally follows the relation between the co-points to a forward ray and the cut points of a level set of Busemann function.

**Theorem 1.2** Let $(M, F)$ be a forward complete Finsler manifold.
1. A point $x$ of $M$ is a co-point of $\gamma$ if and only if for each $b$ with $b_\gamma(x) < b$, $x$ is a cut point of $N^b$. Thus $C_\gamma = \bigcup_{b > 0} C_{N^b}$, where $C_{N^b}$ denotes the cut locus of $N^b$.

2. The Busemann function $b_\gamma$ is differentiable at a point $x$ of $M$ if and only if $x$ admits a unique coray $\sigma$ to $\gamma$ emanating from $x = \sigma(0)$. In this case $\nabla b_\gamma(x) = \dot{\sigma}(0)$.

These two Main Theorems reduce the study of corays and co-points to the study of $N$-segments and cut points of a closed subset of $(M, F)$, respectively, making in this way possible to apply our previous results from [TS].

Seen in this light, the structure theorem for the co-points set on a Finsler surface, namely that is a local tree, it becomes trivial. It is also clear that the topology of $(C_\gamma, \delta)$, with the induced metric, coincides with the topology of the Finsler surface, as well as that $(C_\gamma, \delta)$ is forward complete (see Theorem 2.11). Other results are also straightforward from [TS] (see Theorem 2.12).

In the final Section we construct examples of Finsler manifolds whose co-points set $C_\gamma$ can be explicitly determined (Examples 4.1, 4.2). From here it follows that there exists forward complete Finsler metrics $(M, F)$ containing rays $\gamma$ whose set of co-points $C_\gamma$ is not closed.

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2 Busemann functions

Let $(M, F)$ be a forward complete non-compact Finsler manifold (see [BCS], [S] for details on the completeness of Finsler manifolds). In Riemannian geometry, the forward and backward completeness are equivalent, hence the words “forward” and “backward” are superfluous, but in Finsler geometry these are not equivalent anymore.

Definition 2.1 If $\gamma : [0, \infty) \to M$ is a ray in a forward complete non-compact Finsler manifold $(M, F)$, then the function

$$b_\gamma : M \to \mathbb{R}, \quad b_\gamma(x) := \lim_{t \to \infty} \{t - d(x, \gamma(t))\}$$ (2.1)

is called the Busemann function with respect to $\gamma$, where $d$ is the Finsler distance function.

The Busemann function for Finsler manifolds were introduced and partially studied by Egloff [Eg] and more recently by [Oh].

Remark 2.2 1. The limit in (2.1) always exists because the function $t \mapsto t - d(x, \gamma(t))$ is monotone nondecreasing and bounded above by $d(\gamma(0), x)$.

2. Obviously $b_\gamma(\gamma(t_0)) = t_0$, for all $t_0 \geq 0$. Moreover, if $\gamma_0$ is a sub-ray of the ray $\gamma$ such that there exists $t_0 \geq 0$, $\gamma_0(0) = \gamma(t_0)$, then $b_{\gamma_0}(p) = b_\gamma(p) - t_0$ for any point $p \notin \gamma$. 

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It follows that a point $x$ of $M$ is an element of $b^{-1}\gamma(a, \infty)$, for some real number $a$, if and only if $t - d(x, \gamma(t)) > a$ for some $t > 0$, and hence we get

**Lemma 2.3** For each $a \in \mathbb{R}$, $b^{-1}\gamma(a, \infty) = \bigcup_{t>0} B^{-}\gamma(t+a)$ holds, where $B^{-}\gamma(t+a) := \{x \in M \mid d(x, \gamma(t+a)) < t\}$ denotes the backward open ball centered at $\gamma(t+a)$ of radius $t$. In particular $b^{-1}\gamma(a, \infty)$ is arcwise connected for each $a \geq 0$.

Triangle inequality shows

**Lemma 2.4** The function $b_{\gamma}$ is locally Lipschitz, i.e.

\[ -d(x, y) \leq b_{\gamma}(x) - b_{\gamma}(y) \leq d(y, x) \quad (2.2) \]

for any two points $x, y \in M$.

The differentiability of Busemann function is fundamental for the study of corays. Some results are already known. Let us denote by $\nabla f(x)$ the Finslerian gradient of a smooth function $f : M \to \mathbb{R}$ (see [Oh], [S]).

**Theorem 2.5** [Oh] Let $\gamma$ be a forward ray in a non-compact forward complete Finsler manifold $(M, F)$.

1. For any $x \in M$, there exists at least an coray $\sigma$ of $\gamma$ such that $\sigma(0) = x$.

2. If the geodesic ray $\sigma$ is a coray of $\gamma$, then $b_{\gamma}(\sigma(s)) = s + b_{\gamma}(\sigma(0)), \forall s \geq 0$.

3. If $b_{\gamma}$ is differentiable at a point $x \in M$, then $\sigma(s) := \exp_{x}(s\nabla b_{\gamma}(x))$ is the unique coray to $\gamma$ emanating from $x$, where $\nabla b_{\gamma}(x)$ is the Finslerian gradient of $b_{\gamma}$ at $x$.

We point out that the converse of the statement 2 in Theorem 2.5 is also true, but we not need it here.

For any closed subset $N$ of $M$, we have defined $N$-segments in Introduction. From now on, any $N^{a}$-segment will mean forward $N^{a}$-segment, where $N^{a} := b_{\gamma}^{-1}[a, \infty)$.

**Proof.** (Proof of Theorem 1.1) 1 $\Rightarrow$ 2.

Suppose that the property 1 holds. Then, statement 2 follows immediately from Theorem 2.5 2.

2 $\Rightarrow$ 3.

Choose any $s \in [0, a]$ and any $x \in b_{\gamma}^{-1}[b, \infty)$. From Lemma 2.4 it follows

\[ b_{\gamma}(\alpha(a)) - b_{\gamma}(\alpha(s)) \leq b_{\gamma}(x) - b_{\gamma}(\alpha(s)) \leq d(\alpha(s), x). \] \[ (2.3) \]

On the other hand, hypothesis (1.1) implies

\[ d(\alpha(s), \alpha(a)) = a - s = b_{\gamma}(\alpha(a)) - b_{\gamma}(\alpha(s)). \] \[ (2.4) \]

From relations (2.3) and (2.4) it results $d(\alpha(s), \alpha(a)) = d(\alpha(s), N^{b})$ for any $s \in [0, a]$, and since the point $x$ is arbitrarily chosen from $N^{b}$ we obtain that $\alpha$ is an $N^{b}$-segment.

3 $\Rightarrow$ 1.
Choose any sufficiently small $\epsilon > 0$. Let $\sigma_\epsilon : [\epsilon, \infty) \to M$ denote a coray of $\gamma$ emanating from $\alpha(\epsilon)$. Since $\sigma_\epsilon$ satisfies (1.1) for all $s \geq \epsilon$, $\sigma_\epsilon|_{[\epsilon,a]}$ is also an $N^b$-segment emanating from $\alpha(\epsilon)$. Thus, the two geodesic segments $\alpha|_{[\epsilon,a]}$ and $\sigma_\epsilon|_{[\epsilon,a]}$ must coincide, since $\alpha(\epsilon)$ is an interior point of $\alpha$. Therefore, $\alpha$ is a subarc of the coray $\lim_{\epsilon \to 0} \sigma_\epsilon$.

By Theorem 1.1 we get

**Corollary 2.6** If a unit speed geodesic $\sigma : [0, \infty) \to M$ satisfies (1.1) for all $s > 0$ then $\sigma$ is a coray of $\gamma$.

**Corollary 2.7** For each $a \in \mathbb{R}$ with $b^{-1}\gamma(a) \neq \emptyset$, $d(x, N^a) = a - b^*\gamma(x)$ holds for all $x \in b^{-1}(-\infty, a]$. Hence, $b^*\gamma$ is differentiable at a point $x$ if and only if for each real number $a > b^*\gamma(x)$ the distance function $d(\cdot, N^a)$ is differentiable at $x$.

**Proof.** Choose any $x \in b^{-1}(-\infty, a]$, and denote by $\sigma : [0, \infty) \to M$ a coray of $\gamma$ emanating from $x = \sigma(0)$. By Theorem 1.1, $\sigma|_{[0, a - b^*\gamma(x)]}$ is an $N^a$-segment and noticing that $\sigma(a - b^*\gamma(x)) \in b^{-1}(-\infty, a]$ by (1.1), we obtain $d(x, N^a) = d(\sigma(0), \sigma(a - b^*\gamma(x))) = a - b^*\gamma(x)$.

**Proof.** (Proof of Theorem 1.2) 1. From Theorem 1.1 follows that for any maximal coray $\sigma : [0, \infty) \to M$, $\sigma|_{[0, a - b^*\gamma(x)]}$ is a maximal $N^a$-segment for each $b > b^*\gamma(\sigma(0))$ and conversely that for each maximal $N^b$-segment $\alpha$ emanating from a point $x$, there exists a maximal coray of $\gamma$ emanating from $x$ which contains $\alpha$ as a subarc, and 1 is proved.

2. Follows easily from Theorem A in [TS], Corollary 2.7 and Theorem 1.1.

**Corollary 2.8** If $x \in M$ is an interior point of a coray $\sigma$ of $\gamma$, then $b^*\gamma$ is differentiable at $x$.

**Proof.** Choose any point $\sigma(t_0), t_0 > 0$. By Theorem 1.1 the subray $\sigma|_{[t_0, \infty)}$ is a unique coray of $\gamma$ emanating from $\sigma(t_0)$. Thus, by Theorem 1.2 statement 2 it results that $b^*\gamma$ is differentiable at $\sigma(t_0)$.

Let us denote by $C_\gamma$ the set of co-points of the ray $\gamma$, that is the origin points of the maximal corays to $\gamma$. By Proposition 2.6 in [TS] and Theorem 1.1 we obtain,

**Corollary 2.9** Let $(M, F)$ be a forward complete Finsler manifold, $\gamma$ a forward ray in $M$ and $C_\gamma$ the set of co-points of $\gamma$.

Then the subset

$$C_\gamma^{(2)} := \{p \in C_\gamma : \text{there exist at least two maximal corays from } p \text{ to } \gamma \} \subset C_\gamma$$

is dense in $C_\gamma$.
Remark 2.10 Let $\mathcal{N}D(b_{\gamma}) \subset M$ be the set of non-differentiable points of the Busemann function $b_{\gamma}$. Then $\mathcal{N}D(b_{\gamma}) \subset C_{\gamma} \subset \overline{\mathcal{N}D(b_{\gamma})}$.

In the two dimensional case, the structure equations of the cut locus from [TS] can be easily extended. We recall that an injective continuous map from the open interval $(0, 1)$ (or closed interval $[0, 1]$) of $\mathbb{R}$ and from a circle $S^1$ into $M$ is called a Jordan arc and a Jordan curve, respectively.

A topological set $T$ is called a tree if any two points in $T$ can be joined by a unique Jordan arc in $T$. Likely, a topological set $C$ is called a local tree if for every point $x \in C$ and for any neighborhood $U$ of $x$, there exists a neighborhood $V \subset U$ of $x$ such that $C$ is a tree. A point of a local tree $C$ is called an endpoint of the local tree if there exists a unique sector at $x$.

A continuous curve $c : [a, b] \to M$ is called rectifiable if its length is finite.

By Theorem 1.2 and Theorem B in [TS] we obtain (compare with [L])

Theorem 2.11 Let $\gamma$ be a ray in a forward complete 2-dimensional Finsler manifold $(M, F)$. Then the set $C_{\gamma}$ of co-points of $\gamma$ satisfies the following three properties.

1. The set $C_{\gamma}$ is a local tree and any two copoints on the same connected component of $C_{\gamma}$ can be joined by a rectifiable curve in $C_{\gamma}$.

2. The topology of $C_{\gamma}$ induced from the intrinsic metric $\delta$ coincides with the induced topology of $C_{\gamma}$ from $(M, F)$.

3. The space $C_{\gamma}$ with the intrinsic metric $\delta$ is forward complete.

Indeed, by the first statement, any two copoints $y_1, y_2 \in C_{\gamma}$ can be joined by a rectifiable arc in $C_{\gamma}$ if $y_1$ and $y_2$ are in the same connected component. Therefore, the intrinsic metric $\delta$ on $C_{\gamma}$ defined as:

$$\delta(y_1, y_2) := \begin{cases} 
\inf \{l(c) \mid c \text{ is a rectifiable arc in } C_{\gamma} \text{ joining } y_1 \text{ and } y_2 \}, \\
+\infty, & \text{if } y_1, y_2 \in C_{\gamma} \text{ are in the same connected component,}
\end{cases}$$

is well defined.

By Theorem 1.2 and Theorem C in [TS] we have

Theorem 2.12 Let $\gamma$ be a ray in a forward complete 2-dimensional Finsler manifold $(M, F)$. Then there exists a set $E \subset [0, \infty)$ of measure zero with the following properties:

1. For each $t \in (0, \infty) \setminus E$, the set $b_{\gamma}^{-1}(t)$ consists of locally finitely many mutually disjoint arcs. In particular, if $b_{\gamma}^{-1}(a)$, is compact for some $a > t$, then $b_{\gamma}^{-1}(t)$ consists of finitely many mutually disjoint circles.

2. For each $t \in (0, \infty) \setminus E$, any point $q \in b_{\gamma}^{-1}(t)$ admits at most two maximal corays.
3 Implications of the differentiability of $b_\gamma$

Here are some results that follow from the previous section (compare with [In1]).

In [In1] it is proved for $G$-spaces that if $C_\gamma$ is compact, then $b_\gamma$ is an exhaustion function. We will give a more general result.

**Theorem 3.1** Let $(M, F)$ be a forward complete non-compact Finsler manifold and $\gamma$ a ray in $M$.

If there exists a divergent numerical sequence $\{c_i\}$ such that

1. $b_\gamma^{-1}(c_i)$ is connected,
2. $C_\gamma \cap b_\gamma((−∞, c_i])$ is compact,

then the Busemann function $b_\gamma$ is an exhaustion function, i.e. for any $a \in \mathbb{R}$, the set $b_\gamma^{-1}((−∞, a])$ is compact.

**Proof.** For each fixed $i$ and $c_i$, we define the set

$$S_i := \{q \in b_\gamma^{-1}(c_i) | q \text{ belongs to some coray to } \gamma \text{ emanating from a point in } C_\gamma\}.$$  

We claim that for any fixed $i$, we have $S_i = b_\gamma^{-1}(c_i)$.

We will prove this claim by showing that $S_i$ is both closed and open in $b_\gamma^{-1}(c_i)$. Indeed, firstly, we show that $S_i$ is closed. If we consider a convergent points sequence $\{q_j\} \subset S_i$, then we will show that $q := \lim_{j \to \infty} q_j$ belongs to $S_i$. If we denote by $\{\sigma_j\}$ the corresponding corays to $\gamma$ emanating from the initial points $\{x_j\} \subset C_\gamma$ and passing through $\{q_j\}$, respectively, then one can see that actually $\{x_j\} \subset C_\gamma \cap b_\gamma^{-1}((−∞, c_i])$. From hypothesis 2 it follows that $\{x_j\}$ must have a sub-sequence $\{x_{j_k}\}$ convergent to a point $x \in C_\gamma \cap b_\gamma^{-1}((−∞, c_i])$ and there exists an coray $\sigma$ from $x$ to $\gamma$ that intersects $b_\gamma^{-1}(c_i)$ in $q$, hence $S_i$ is closed (see Figure 1).

![Figure 1: $S_i$ is closed set.](image)

Next, we prove by contradiction that $S_i$ is open. Indeed, we assume contrary, that is, for $q \in S_i$, suppose there exists a points sequence $\{y_j\} \subset b_\gamma^{-1}(c_i) \setminus S_i$ such that
\[ q := \lim_{j \to \infty} y_j. \] We denote by \( \sigma_j \) and \( \sigma \) the corays sequence passing through \( y_j \) and \( q \), respectively. We denote by \( x \) the initial point on \( \sigma \), and by our assumption \( x \in C_\gamma \cap b_{\gamma}^{-1}(-\infty, c_i) \).

Consider now a scalar \( \delta > d(q, x) \) and the forward closed ball \( B_\delta^+(q) := \{ p \in M \mid d(q, p) \leq \delta \} \). Obviously \( B_\delta^+(q) \) is compact due to the forward complete hypothesis and Hopf-Rinow Theorem, and \( x \in B_\delta^+(q) \).

Let \( \sigma_j \) denote a coray to \( \gamma \) emanating from \( y_j = \sigma_j(0) \). Since \( B_\delta^+(q) \) is compact and \( y_j \notin S_i \), we can extend backward \( \sigma_j \) to some interval \([t_j, 0]\) with \( d(\sigma_j(t_j), q) = \delta \). Any limit geodesic of the sequence \( \{\sigma\}_j \) is a coray passing through \( q \) which contains \( x \) as an interior point, that is a contradiction (see Figure 2). It follows \( S_i \) must be open set.

![Figure 2: \( S_i \) is open set.](image)

Therefore, we have proved that \( S_i \) is closed and open in \( b_{\gamma}^{-1}(c_i) \), and hence the claim \( S_i = b_{\gamma}^{-1}(c_i) \) is proved. In other words, what we have proved up to this point is that for any point \( q \in b_{\gamma}^{-1}(c_i) \), there exists a maximal coray, i.e. a coray emanating from a point \( x \in C_\gamma \), passing through \( q \).

Using this we proceed to proving that \( b_{\gamma}^{-1}(-\infty, c_i] \) is compact. We assume the converse, that is there exists a divergent sequence \( \{x_j\} \) in \( b_{\gamma}^{-1}(-\infty, c_i] \). From our claim it follows that for each \( j \) there exists a coray \( \sigma_j \) from \( x_j \) that intersects \( b_{\gamma}^{-1}(c_i) \) in \( y_j \), and we extend \( \sigma_j \) up to the point \( z_j = \sigma_j(0) \in C_\gamma \).

From hypothesis 2 of the Theorem, there exists a subsequence \( z_{j_k} \) of \( z_j \) convergent to \( z \) and hence there exists a point \( y \in b_{\gamma}^{-1}(c_i) \) such that \( \lim_{j \to \infty} y_j = y \). Since \( x_j \) is interior point of the \( b_{\gamma}(c_i) \)-segment \( \sigma_j|[0, x_j] \), it follows that there exists a point \( x \) interior to the \( b_{\gamma}(c_i) \)-segment \( \sigma|[0, s] \), where \( y_j = \sigma_j(s_j) \) and \( y = \sigma(s) \). But this implies that the sequence \( \{x_j\} \) cannot be divergent, that is a contradiction. Therefore, \( b_{\gamma}^{-1}(-\infty, c_i] \) must be compact.

The following lemma shows that our Theorem 3.1 is a special case of Innami’s result in [In1].
Lemma 3.2 Let \((M, F)\) be a forward complete Finsler manifold and \(\gamma\) a ray in \(M\). If \(C_\gamma \neq \emptyset\) is compact, then for all sufficiently large \(a \in \mathbb{R}\), the level set \(b_\gamma^{-1}(a)\) is arcwise connected.

Proof. Since \(C_\gamma \neq \emptyset\) is compact we can choose a number \(a > \max b_\gamma(C_\gamma)\). Thus there does not exist a co-point of \(\gamma\) in \(b_\gamma^{-1}(a, \infty)\). Choose any two points \(x\) and \(y\) in \(b_\gamma^{-1}(a)\). By Lemma 2.3 there exists a continuous curve \(c\) in \(b_\gamma^{-1}(a, \infty)\) joining \(x\) to \(y\). Since \(C_\gamma \cap b_\gamma^{-1}(a, \infty) = \emptyset\), we can get a curve in \(b_\gamma^{-1}(a)\) joining \(x\) to \(y\) by deforming the curve \(c\) along the corays intersecting \(c\). Therefore, the level set is arcwise connected.

Moreover, we have

Theorem 3.3 Let \((M, F)\) be a bi-complete non-compact Finsler manifold and \(\gamma\) a ray in \(M\). If \(C_\gamma\) contains an isolated point \(p\), then

1. \(M\) is simply covered by maximal corays from \(p\).
2. \(C_\gamma = \{p\}\) only.
3. For any \(a > b_\gamma(p)\) the level sets \(b_\gamma^{-1}(a)\) coincide with the forward spheres \(S^+(p, a - b_\gamma(p))\).

Proof. 1. Since \(p \in C_\gamma\) is isolated it means we can choose a small enough \(\varepsilon > 0\) such that all corays with initial points in \(M \setminus S^+(p, \delta)\) do not intersect \(S^+(p, \varepsilon)\), for some \(\varepsilon < \delta\), and all maximal asymptotes straight lines (if any) do not intersect \(S^+(p, \varepsilon)\), where \(S^+(p, \varepsilon) := \{x \in M \mid d(p, x) = \varepsilon\}\) is the forward sphere in \((M, F)\).

This implies that every maximal coray containing a point in \(S^+(p, \varepsilon)\) must have \(p\) as initial point. Indeed, let us consider a convergent points sequence \(\{p_j\} \subset S^+(p, \varepsilon)\) such that \(\lim_{j \to \infty} p_j = p\), and denote by \(\sigma\) and \(\sigma_j\) the corays to \(\gamma\) from the points \(p\) and \(p_j\), respectively. By construction, the corays \(\sigma_j\) will extend backward to a maximal coray with initial point \(z_j\), or to an asymptotic straight line to \(\gamma\) (we do not need to assume here backward completeness, one of these two cases will happen anyway). In the case \(\sigma_j\) extends to a maximal coray, since \(p_j\) is convergent to \(p\), the coray \(\sigma_j\) is convergent to \(p\). But this means that \(\sigma\) extends backwards outside \(S^+(p, \varepsilon)\) to an initial point \(z := \lim_{j \to \infty} z_j\), contradiction with the fact that \(p \in C_\gamma\). The case of asymptotic straight lines is not possible either. It follows that \(S^+(p, \varepsilon)\) is simply covered by maximal corays with initial point \(p\) and therefore these simply covers \(M\).

2. We assume that there exists another point \(q \in C_\gamma\), except \(p\), that is \(q \in (M \setminus S^+(p, \delta)) \cap C_\gamma\).

From statement 1 proved above it follows that there exists a maximal coray through \(q\) with initial point \(p\), that is contradiction with the hypothesis \(q \in C_\gamma\). Obviously, maximal asymptotic straight lines are not allowed either.

3. Firstly, we show that for the forward sphere \(S^+(p, r) := \{q \in M \mid d(p, q) = r\}\), there exists a level set \(b_\gamma^{-1}(a)\) such that \(S^+(p, \delta) \subset b_\gamma^{-1}(a)\).

Indeed, for given radius \(r > 0\) we put \(a := b_\gamma(p) + r\). For any point \(q \in S^+(p, r)\), consider the maximal coray from \(p\) through \(q\). Then Theorem 1 implies \(b_\gamma(q) = b_\gamma(p) + r\),
where we take into account that \( r = d(p, q) = d(p, b^{-1}_\gamma(a)) \). Therefore \( q \in b^{-1}_\gamma(a) \) and the inclusion is proved.

Secondly, we will show that for each level set \( b^{-1}_\gamma(a) \), we have \( b^{-1}_\gamma(a) \subset S^+(p, r) \), where \( r = a - b_\gamma(p) \).

Indeed, let \( q \in b^{-1}_\gamma(a) \), that is \( b_\gamma(q) = a \), and let us again consider the maximal coray from \( p \) through \( q \). Then from Theorem 1.1 we have \( b_\gamma(q) = b_\gamma(p) + d(p, q) \), hence \( d(p, q) = b_\gamma(q) - b_\gamma(p) = a - b_\gamma(p) = r \), and therefore \( q \in S^+(p, \delta) \).

\[ \blacksquare \]

From Theorem 1.2 statement 2 we obtain

**Corollary 3.4** Let \((M, F)\) be a forward complete non-compact Finsler manifold. If from each point of \( M \) there exists a unique coray to a ray \( \gamma \), then \( C_\gamma = \emptyset \).

**Remark 3.5** It would be interesting to obtain some geometrical conditions on the Finsler manifold \((M, F)\) such that all Busemann functions are everywhere differentiable. Since this topic requires more elaboration, we leave it for a future research.

We recall that an *end* \( \varepsilon \) of a non-compact manifold \( X \) is an assignment to each compact set \( K \subset X \) a component \( \varepsilon(K) \) of \( X \setminus K \) such that \( \varepsilon(K_1) \supset \varepsilon(K_2) \) if \( K_1 \subset K_2 \). Every non-compact manifold has at least one end. For instance, \( \mathbb{R}^n \) has one end if \( n > 1 \) and two ends if \( n = 1 \). By definition one can see that a product \( \mathbb{R} \times N \) has one end if \( N \) is non-compact and two ends otherwise.

Here we prove

**Corollary 3.6** Let \((M, F)\) be a forward complete non-compact Finsler manifold.

1. If \( C_\gamma = \emptyset \), then \( M \) is homeomorphic to \( \mathbb{R} \times b^{-1}_\gamma(0) \).

2. If \( M \) has at least three ends, then there are no differentiable Busemann functions on \( M \).

**Proof.**

1. Since \( C_\gamma = \emptyset \), it follows that \( b_\gamma \) is smooth everywhere and hence from each point there is a unique coray to \( \gamma \). Thus, we can define the function \( \varphi : M \rightarrow \mathbb{R} \times b^{-1}_\gamma(0) \), \( p \mapsto \varphi(b_\gamma(p), h_1(p)) \), where \( h_1(p) \) is the intersection point of the coray from \( p \) with the level set \( b^{-1}_\gamma(0) \). Obviously this is a homeomorphism.

2. Due to statement 1 it follows that if \( b_\gamma \) is differentiable, then \( M \) have at most two ends. Statement 2 follows by logical negation. \( \blacksquare \)

## 4 Examples

**Example 4.1** We start by recalling here a Riemannian example from [N1]. Consider \( \mathbb{R}^3 \) to be the 3-dimensional Euclidean space with the canonical metric. In the \( xy \)-plane we consider the disk \( D \) of centre \((1, 1, 0)\) and radius \( 1/2 \). We can erect now a half cylinder

\[
(x - 1)^2 + (y - 1)^2 = \frac{1}{16}, \quad z \geq 0.
\] (4.1)
Moreover, we can now connect the points \((x_1, y_1, 0)\) on the circle \((x - 1)^2 + (y - 1)^2 = \frac{1}{4}\) to the points \((x_2, y_2, z_2)\) on the circle

\[(x - 1)^2 + (y - 1)^2 = \frac{1}{16}, \quad z = \frac{1}{4}\]

by a 4-th order algebraic curve such that we obtain a smooth surface made of the \(xy\)-plane and the cylinder attached to \((1, 1, 0)\) and smoothed out by the algebraic curve. Obviously this is a complete Riemannian surface.

Let the positive \(y\)-axis to be the ray \(\gamma\) and let us consider the curve \(c\) on \(S\) obtained by the intersection of the surface \(S\) with the plane \(x = 1, \quad y \in (-\infty, 1]\).

We consider a point \(P \in c\), a divergent sequence of numbers \(t_i\), and consider a sequence of geodesic segments from \(P\) to \(\gamma(t_i)\), for each \(i\). For \(i\) large enough there are two geodesic segments from \(P\) to \(\gamma_i\) denoted by \(\sigma_i^1\) and \(\sigma_i^2\) and by taking the limit \(i \to \infty\) we obtain two maximal corays \(\sigma^1\) and \(\sigma^2\) from \(P\) to \(\gamma\) given by \(\lim_{i \to \infty} \sigma_i^1\) and \(\lim_{i \to \infty} \sigma_i^2\), respectively. That is \(P \in C_\gamma\). One can see that any point on the curve \(c\) has this property and therefore one concludes that \(C_\gamma = c\).

Next, we will deform this Riemannian structure to a Randers metric on the same manifold. Denoting by \(h\) the restriction to the half cylinder (4.1) of the Riemannian metric constructed above, then we see that it is quite easy to construct a Finsler metric \(F = \alpha + \beta\) of Randers type on \(S\).

Indeed, we can regard the half cylinder (4.1) as a surface of revolution and consider the rotation around the straight line \(\{x = 1, y = 1\}\). Obviously this is a parallel straight line with the \(z\)-axis, piercing the \(xy\)-plane in the point \((1, 1, 0)\). More precisely, we consider
the vector field \( W := (-\frac{1}{4}y, \frac{1}{4}x, 0) \) whose flow is given by

\[
\phi(t; p) = p \cdot \begin{pmatrix}
\cos \frac{1}{4}t & \sin \frac{1}{4}t & 0 \\
-\cos \frac{1}{4}t & \sin \frac{1}{4}t & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \( p = (x, y, z) \) is a point of the half cylinder \((4.1)\). This is the rotation that leaves unchanged the parallels \( S \cap \{z = c\} \) and twist the meridians \( x - 1 = \frac{1}{4} \cos \theta, y - 1 = \frac{1}{4} \sin \theta, z \geq \frac{1}{4} \) along the half cylinder, here \((x, y)\) belongs to the circle of centre \((1, 1)\) and radius \( \frac{1}{4} \). Obviously the twisted meridians are geodesics of the Randers structure \( F = \alpha + \beta \) (on the half cylinder) obtained as solution of the Zermelo’s navigation problem for navigation data \((h, W)\).

We will extend this Randers metric to a smooth Finsler metric defined on all surface \( S \). Indeed, we consider

\[
\tilde{W}(p) := \begin{cases}
(-\frac{1}{4}y, \frac{1}{4}x, 0) , & \text{for } z \geq \frac{1}{4} \\
(-\phi(p)y, \phi(p)x, 0) , & \text{for } z \in (0, \frac{1}{4}) \\
(0, 0, 0) , & \text{for } z = 0,
\end{cases}
\]

on the smooth surface \( S \), where \( p = (x, y, z) \in S \). This is a smooth vector field on \( S \).

The Randers metric \( F = \alpha + \beta \) obtained as the solution of Zermelo’s navigation problem for navigation data \((\tilde{h}, \tilde{W})\), where \( \tilde{h} \) is the Riemannian metric on \( S \), is a Finsler metric on \( S \), that is Riemannian on \( S \cap \{z = 0\} \). It is easy to see that \( C_\gamma = c \) with respect to this Finsler metric.

It is known that the cut locus of a point in a Riemannian or Finsler manifold \( M \) is a closed subset of \( M \) (see [BCS]). Moreover, we have shown in [TS] that the cut locus of a closed subset in \( M \) is not closed in \( M \) anymore. A natural question is if the set of co-points \( C_\gamma \) is closed or not. The answer is given hereafter.

**Example 4.2** We start again with a Riemannian construction obtained by the iteration of the Riemannian construction in Example 4.1 (see [N1]). Indeed, consider again \( \mathbb{R}^3 \) and take a sequence of points \( D_n \) in the \( xy \) plane given by

\[
D_n = \left( \frac{2n + 1}{2n(n + 1)}, n, 0 \right)
\]

and denote by \( \gamma \) the \( y \)-axis.

Next consider the sequence of disks in the \( xy \) plane with center \( D_n \) and radius \( \frac{1}{2n(n+1)} \). As in the previous example we cut out this disk and smoothly connect the boundary of the disk with the half cylinder

\[
\left( x - \frac{2n + 1}{2n(n + 1)} \right)^2 + (y - n)^2 = \frac{1}{16n^2(n + 1)^2}, \quad z \geq 0.
\]

We obtain in this way a smooth surface \( S \) with countably many ends.

Let us denote by \( \gamma \) the positive \( y \)-axis. Then by the same arguments as in the the previous example, the co-points set \( C_\gamma \) contains the sequence of curves \( c_n \) obtained by intersecting the surface \( S \) with the planes \( x = \frac{2n + 1}{2n(n + 1)}, \quad y \in (-\infty, n] \).
We consider now the set of points \( q_n = \left( \frac{2n+1}{2^{n(n+1)}}, \lambda, 0 \right) \) for a fixed constant \( \lambda \in (0, \frac{3}{4}) \), for instance \( \lambda = \frac{3}{8} \) will do. Obviously \( \{q_n\} \) is a sequence of co-points of \( \gamma \), i.e. \( q_n \in C_\gamma \).

At limit, one can see that \( \lim_{n \to \infty} q_n = q = (0, \lambda, 0) \in \gamma \). But the ray from \( q \) to \( \gamma \) is subray of \( \gamma \) so it cannot be maximal. In other words, \( q \notin C_\gamma \). This means that \( C_\gamma \) is not closed. We can construct a Raders metric from this Riemannian construction as we did in Example 4.1. Using again the navigation data \((\check{h}, \check{W})\) and same construction from Example 4.1 we obtain a Finsler metric of Raders type whose set of co-points \( C_\gamma \) is not closed.

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