Rapidly rotating superfluid neutron stars in Newtonian
dynamics

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ABSTRACT

We develop a formulation for constructing and examining rapidly rotating Newtonian
neutron star models that contain two superfluids, taking account of the effect of the
rotation velocity difference between two superfluids. We assume neutron stars to be
composed of the superfluid neutrons and the mixture of the superfluid protons and
the normal fluid electrons. To describe Newtonian dynamics of the two superfluids,
the Newtonian version of the so-called two-fluid formalism is employed. The effect
of the rotation velocity difference on the structure of equilibrium state is treated as
a small perturbation to rapidly rotating superfluid stars whose angular velocities of
two superfluids are assumed to be exactly the same. We derive basic equations for
the perturbed structures of rapidly rotating superfluid stars due to the rotation ve-
clocity difference between two superfluids. Assuming the superfluids to obey a simple
analytical equation of state proposed by Prix, Comer, and Andersson, we obtain nu-
merical solutions for the perturbations and find that the density distributions of the
superfluids are strongly dependent on the parameter \(\sigma\) which appears in the analyti-
cal equation of state and characterizes the so-called symmetry energy. It is also found
that if Prix et al.’s analytical equation of state is assumed, the perturbations can be
represented in terms of the universal functions that are independent of the parameters
of the equation of state.

Key words: stars: neutron – stars: rotation – hydrodynamics

1 INTRODUCTION

To investigate properties of equilibrium configurations of rotating neutron stars, so far, most neutron star models have been
obtained by assuming neutron star matter to be a one-constituent perfect fluid (for a review, see, e.g. Stergioulas 2003). This
treatment of equilibrium states of neutron stars seems to be quite reasonable as a first approximation for examining global
properties of neutron stars such as the gravitational mass, the radius, or the maximum rotation frequency. However, it has long
been suggested that neutrons in the inner crust and neutrons and protons in the core of neutron stars are in superfluid states
when the interior temperatures cool down below \(T_c \sim 10^9\)K (e.g., Shapiro & Teukolsky 1983). Since the interior temperature
of neutron stars is believed to cool down quickly via the neutrino emission (e.g., Baym & Pethick 1979), it is likely that
each of many observable neutron stars, except newly born ones, has a core containing superfluids. Although superfluidity
in the interior might be one of the important ingredients that affect the structures of neutron stars, the superfluidity has
been neglected in most studies on equilibrium configurations of rotating neutron stars. Thus, it is necessary to examine how
superfluidity affects fundamental properties of rotating neutron stars.

The superfluidity in neutron stars has been mainly argued in connection with the glitch phenomenon and the post-
glitch relaxation observed in pulsars. The glitch is a sudden decrease of an observed pulse period \(T_p\) of a pulsar, whose largest

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magnitude is $\Delta T_p/T_p \sim -10^{-6}$, that is followed by a period of continuous relaxation. In most explanations of glitch phenomena, a two-component model has been employed to describe rotational dynamics of neutron stars containing superfluid neutrons. In those treatments, one of the components is a mixture of the charged particles such as protons (or ions) and electrons, which are assumed to co-rotate because the charged components are strongly coupled each other due to short-range electromagnetic interactions (Easson 1979, Alpar et al. 1984), and the other the superfluid neutrons, which are supposed to rotate separately from the charged components because the superfluid neutrons are inviscous and loosely couple to the charged components. In this paper, we call the mixture of the charged components “protons” for brevity. It has been shown that those two-component superfluid models have succeeded in explaining observed features of the glitches and the post-glitch relaxations qualitatively (Anderson & Itoh 1975; Alpar et al. 1984; Sedrakian et al. 1995, Link & Epstein 1996). Those results mean that the existence of superfluid components inside neutron stars is supported not only from the theory of nuclear physics but also from observations of pulsars.

If they co-rotate, as mentioned in Prix & Rieutord (2002), two constituents in a rotating superfluid star behave as one-constituent ordinary fluid, because the so-called entrainment effect, due to which the momentum of one superfluid constituent is dependent on the mass current of all superfluid constituents, does not operate and two constituents are in a chemical equilibrium state. On the other hand, specific characteristics of neutron stars due to the superfluidity are considered to be observed when two constituents rotate at different rotation rates. To study the effect of rotation velocity difference between two constituents on the global structures of rotating neutron stars with superfluidity, Prix (1999) derived basic equations for the equilibrium configurations and obtained analytical solutions for slowly rotating superfluid stars within the framework of Newtonian dynamics, generalizing the so-called Chandrasekhar-Milne expansion for ordinary fluid stars (see, e.g. Tassoul 1978), in which rotational effects are treated as perturbations to non-rotating spherical stars. To describe dynamics of superfluids, he made use of a variant of the so-called two-fluid formalism, which had been mainly developed by Carter, Langlois, and their co-workers (Carter 1989; Carter & Langlois 1998; Langlois, Sedrakian, & Carter 1998; Prix 2002). In order to obtain analytical solutions, Prix (1999) employed an equation of state whose functional form is $P \propto \rho^2$, where $P$ and $\rho$ are the pressure and density, respectively. More recently, Prix, Comer, & Andersson (2002a) extended the Prix formalism so as to include the entrainment effect between two superfluids in order to investigate its effect on the stellar structures. In the same paper, Prix et al. (2002a) also argued the effect of the so-called symmetry energy, which can be included as a parameter in the equation of state, and found that the effect of symmetry energy is significant in determination of the density distributions of the neutrons and the protons. As for relativistic rotating superfluid stars, Andersson & Comer (2001) calculated equilibrium configurations of slowly rotating superfluid neutron stars, extending the slow rotation approximation formalism devised by Hartle (1967) (see, also Hartle & Thorne 1968) for ordinary fluid stars to relativistic two-constituent superfluid stars. Very recently, Prix, Novak, & Comer (2002b) have obtained preliminary results for rapidly rotating superfluid stars, without the slow rotation approximation.

The purpose of this paper is to improve our understanding of properties of equilibrium configurations of rapidly rotating neutron stars with superfluids. We calculate equilibrium configurations of rapidly rotating neutron stars with neutron and proton superfluids. In this paper, we are concerned with rotating neutron stars of two superfluids whose angular velocities are different. We however assume that the rotation velocity difference between two superfluids is very small in comparison with the angular velocity of the star. This assumption could be reasonable for models of superfluid neutron stars because observations of pulsar glitches show that the rotation velocity of the protons may differ from that of the neutrons, but the amount of the rotation velocity difference is not very large. We can therefore treat the effect of the rotation velocity difference between two superfluids on the structures of rapidly rotating neutron stars as a small perturbation to equilibrium configurations of rapidly rotating neutron stars in which two superfluids co-rotate. For the dynamics of superfluids inside stars, we employ a Newtonian version of the two-constituent formalism developed by Prix (2002). In order to obtain basic equations for determining the effect of the rotation velocity difference between two superfluids, equations of superfluid hydrostatic equilibrium and the Poisson equation are expanded in terms of the rotation velocity difference between two superfluids. To obtain the numerical solutions of equilibrium configurations, we make use of a variant of the so-called Self Consistent Field method (SCF) devised by Ostriker & Mark (1968). The equation of state for the superfluid we use is the analytical one employed by Prix et al. (2002a), which is a natural extension of an $N = 1$ polytropic equation of state for a barotropic ordinary fluid to the case of two superfluids. In §2 we present the basic equations employed in this paper for the rapidly rotating superfluid stars. Numerical results are given in §3, and §4 is devoted for summary and discussion.

2 FORMULATION

2.1 Two-constituent formalism for Newtonian superfluid dynamics

In this paper, a neutron star is assumed to be composed of superfluid neutrons and a mixture of superfluid protons and normal fluid electrons, because the interior temperatures of old neutron stars are much lower than the transition temperature to neutron and proton superfluids, which is considered to be $T_c \sim 10^9$ K (see, e.g., Epstein 1988). We further assume that
this mixture of charged particles is perfectly in the state of charge neutrality because of strong coupling between the protons and the electrons by electromagnetic interactions. The electrons therefore co-move with the protons, and their motions can be described with one fluid velocity $v_p\alpha$. Through this paper, we call the mixture of the the protons and the electrons “protons” for brevity as mentioned before. To describe the dynamics of the superfluids, we make use of the two-constituent formalism for Newtonian superfluid dynamics, which has been recently developed by Prix (2002).

We take account of no “transfusion” between the neutrons and the protons. The particle numbers of the neutron and the proton must be therefore conserved separately. The particle number conservation equations for the neutron and the proton are then given by

$$\nabla_a(n_n v_n^a) = 0, \quad \nabla_a(n_p v_p^a) = 0,$$

where $n_n$ and $n_p$ are the number densities of the neutron and the proton, and $v_n^a$ and $v_p^a$ denote the fluid velocities of the neutron and the proton, respectively. Here, $\nabla_a$ means the covariant derivative in the three dimensional flat space. The mass densities of two superfluids can be written as

$$\rho_n = m n_n, \quad \rho_p = m n_p,$$

where $m$ is the mass of the neutron, and we assume the mass of the proton to be equal to that of the neutron. The fundamental quantity of the two-constituent formalism for the Newtonian superfluid dynamics is the internal energy density $E$, which is a function of $\rho_n, \rho_p$, and $\Delta$, where $\Delta^2 = \Delta^a \Delta_a$ and $\Delta^n = v_n^a - v_n^a$. Note that we have neglected the entropy $s$ carried by the normal fluid of the electron for simplicity. This treatment is justified for neutron stars whose internal temperatures are sufficiently low. This function $E$ defines several basic thermodynamical quantities which describe the dynamics of two superfluids in terms of the total difference as follows

$$dE = \tilde{\mu}_n d\rho_n + \tilde{\mu}_p d\rho_p + \alpha d\Delta^2,$$

where $\tilde{\mu}_n$ and $\tilde{\mu}_p$ denote the specific chemical potentials of the neutron and the proton, respectively. The function $\alpha$ represents the strength of the entrainment effect. By using a variant of the Gibbs-Duhem relation, the generalized pressure of two fluids is defined by

$$dP = \rho_n d\tilde{\mu}_n + \rho_p d\tilde{\mu}_p - \alpha d\Delta^2,$$

and we can obtain the relation,

$$P = \rho_n \tilde{\mu}_n + \rho_p \tilde{\mu}_p - E.$$

According to Prix (2002), the Euler equations for two superfluids can be given by

$$(\partial_t + \mathcal{L}_{v_n})(v_{n,a} + \varepsilon_n \Delta_a) + \nabla_a \left(\tilde{\mu}_n + \Phi - \frac{1}{2} |v_n|^2\right) = 0,$$

$$(\partial_t + \mathcal{L}_{v_p})(v_{p,a} + \varepsilon_p \Delta_a) + \nabla_a \left(\tilde{\mu}_p + \Phi - \frac{1}{2} |v_p|^2\right) = 0,$$

where $\varepsilon_n = 2\alpha/\rho_n$ and $\varepsilon_p = 2\alpha/\rho_p$. Here, $\mathcal{L}_u$ means the Lie derivative along the vector field $u^a$. Note that we have considered no direct interaction force between two superfluids such as the mutual friction or the Magnus-type force. In equations (6) and (7), the function $\Phi$ is the gravitational potential, which is determined with the total mass density $\rho$, given by $\rho = \rho_n + \rho_p$, through the Poisson equation

$$\nabla_a \nabla^a \Phi = 4 \pi G \rho,$$

where $G$ is the gravitational constant. Considering the exterior differentiation of equations (6) and (7), we can obtain the vorticity equations, given by

$$\partial_t \omega_{n,ab} = 0, \quad (\partial_t + \mathcal{L}_{v_p}) \omega_{p,ab} = 0,$$

where $\omega_{n,ab}$ and $\omega_{p,ab}$ are, respectively, the exterior derivatives of the one-forms $v_{n,a} + \varepsilon_n \Delta_a$ and $v_{p,a} - \varepsilon_p \Delta_a$, which are defined by

$$\omega_{n,ab} = \partial_a (v_{n,b} + \varepsilon_n \Delta_b) - \partial_b (v_{n,a} + \varepsilon_n \Delta_a), \quad \omega_{p,ab} = \partial_a (v_{p,b} - \varepsilon_p \Delta_b) - \partial_b (v_{p,a} - \varepsilon_p \Delta_a).$$

### 2.2 Stationary and axisymmetric equilibrium configurations

We consider stationary and axisymmetric equilibrium configurations. Thus, the time and azimuthal derivatives of physical quantities must vanish because a star is assumed to be in a stationary and axisymmetric state. In this paper, we further assume that the neutrons and the protons, respectively, rotate with the angular velocities $\Omega_n$ and $\Omega_p$ around the axis of rotation, and that no meridional circulation is present in a superfluid star. The fluid velocities for two fluids are then given by
where \( \varphi^a \) is the rotational Killing vector. In this paper, we employ the spherical polar coordinates \((r, \theta, \varphi)\). The components of \( \varphi^a \) can be written as \( \varphi^a = \delta^a_\varphi \) in the spherical polar coordinates. First of all, let us consider the vorticity conservation for the flow given by equation (11). Substituting equation (11) into equation (9), we obtain

\[
(\mathcal{L}_v \omega_n)_{ab} dx^a \wedge dx^b = d[\varpi^2 \{(1 - \varepsilon_n)\Omega_n + \varepsilon_n \omega_n\}] \wedge d\Omega_n = 0,
\]

\[
(\mathcal{L}_p \omega_p)_{ab} dx^a \wedge dx^b = d[\varpi^2 \{(1 - \varepsilon_p)\Omega_p + \varepsilon_p \omega_p\}] \wedge d\Omega_p = 0,
\]

where \( \varpi \) is the distance from the rotation axis, defined by \( \varpi = r \cos \theta \). Thus, these vorticity conservation equations are automatically satisfied if the rotation laws for two superfluids are given as follows:

\[
\varpi^2 \{(1 - \varepsilon_n)\Omega_n + \varepsilon_n \omega_n\} = j_n(\Omega_n), \quad \varpi^2 \{(1 - \varepsilon_p)\Omega_p + \varepsilon_p \omega_p\} = j_p(\Omega_p),
\]

where \( j_n(\Omega_n) \) and \( j_p(\Omega_p) \) are arbitrary functions of \( \Omega_n \) and \( \Omega_p \), respectively. In other words, the rotation laws of superfluid stars must be strongly restricted because the rotation velocities \( \Omega_n \) and \( \Omega_p \) may depend on the entrainment functions \( \varepsilon_n \) and \( \varepsilon_p \), whose functional forms are in general determined with the matter distributions of stars. When the effect of the entrainment between two superfluids does not operate, that is, the case of \( \alpha = 0 \) or of \( \Omega_n = \Omega_p \), the vorticity conservation equations become relatively simple as follows

\[
d\varpi \wedge d\Omega_n = 0, \quad d\varpi \wedge d\Omega_p = 0.
\]

The functions \( \Omega_n \) and \( \Omega_p \) must not therefore depend on \( z = r \sin \theta \) in this situation. In other words, we can choose the rotation velocities \( \Omega_n \) and \( \Omega_p \) freely as long as \( \Omega_n = \Omega_n(\varpi) \) and \( \Omega_p = \Omega_p(\varpi) \) are fulfilled. Note that this condition is the same as that of barotropic ordinary fluid stars. For simplicity, in this paper, we assume that both the neutrons and the protons are uniformly rotating, i.e. \( \Omega_n = \text{const} \) and \( \Omega_p = \text{const} \), because vorticity conservations are automatically satisfied for uniformly rotating configurations. Assuming the uniform rotations and integrating the Euler equations (9) and (7), we can obtain integrated equations for hydrostatic equilibrium states, given by

\[
\hat{\mu}_n + \Phi - \frac{\varpi^2}{2} \Omega_n^2 = C_n, \quad \hat{\mu}_p + \Phi - \frac{\varpi^2}{2} \Omega_p^2 = C_p,
\]

where \( C_n \) and \( C_p \) are integral constants. Here, we have used the relationships

\[
\mathcal{L}_v(\varpi + \varepsilon_n \Delta) dx^a = \varpi^2 \{(1 - \varepsilon_n)\Omega_n + \varepsilon_n \omega_n\} d\Omega_n = 0,
\]

\[
\mathcal{L}_p(\varpi - \varepsilon_p \Delta) dx^a = \varpi^2 \{(1 - \varepsilon_p)\Omega_p + \varepsilon_p \omega_p\} d\Omega_p = 0.
\]

We are interested in configurations in which two superfluids are rotating rapidly with different angular velocities, \( \Omega_n \neq \Omega_p \). We do not therefore make use of any slow-rotation approximation such as the Chandrasekhar-Milne expansion. We however assume the difference between rotation velocities of two superfluids to be very small. We can then treat the effect of the angular velocity difference \( \Omega_p - \Omega_n \) on the stellar structure as a perturbation to rapidly rotating stars whose rotation rates of two superfluids are exactly the same. For superfluid neutron stars, although the angular velocity \( \Omega_n \) may differ from the angular velocity \( \Omega_p \) because the interaction between two superfluids may be very weak due to the superfluidity, it is believed that this difference between two angular velocities is small in comparison with the averaged angular velocity of the star. Thus, this treatment could be appropriate for neutron star models with superfluidity.

We expand physical quantities appeared in equations (8) and (16) up to the first order in \( (\Omega_n - \Omega_p)/(|\Omega_n| + |\Omega_p|) \) as follows:

\[
\Omega_n = \Omega(1 + \delta \Omega_n), \quad \Omega_p = \Omega(1 + \delta \Omega_p),
\]

\[
\rho_n = \rho_n(1 + \delta \rho_n), \quad \rho_p = \rho_p(1 + \delta \rho_p),
\]

\[
\tilde{\mu}_n = \tilde{\mu}_n(1 + \delta \tilde{\mu}_n), \quad \tilde{\mu}_p = \tilde{\mu}_p(1 + \delta \tilde{\mu}_p),
\]

\[
\Phi = \Phi(1 + \delta \Phi), \quad C_n = C_n(1 + \delta C_n), \quad C_p = C_p(1 + \delta C_p),
\]

where \( \tilde{\mu}_n, \tilde{\mu}_p, \delta \tilde{\mu}_n, \) and \( \delta \tilde{\mu}_p \) are given in terms of partial derivatives of the internal energy density \( \mathcal{E} \), \( \delta \rho_n \), and \( \delta \rho_p \) by

\[
\tilde{\mu}_n = \frac{\partial \mathcal{E}}{\partial \rho_n}, \quad \tilde{\mu}_p = \frac{\partial \mathcal{E}}{\partial \rho_p},
\]

\[
\delta \tilde{\mu}_n = \frac{\partial^2 \mathcal{E}}{\partial \rho_n^2} \delta \rho_n + \frac{\partial^2 \mathcal{E}}{\partial \rho_n \partial \rho_p} \delta \rho_p, \quad \delta \tilde{\mu}_p = \frac{\partial^2 \mathcal{E}}{\partial \rho_p^2} \delta \rho_p + \frac{\partial^2 \mathcal{E}}{\partial \rho_p \partial \rho_n} \delta \rho_n.
\]
In order to obtain equilibrium configurations, the internal energy density $\mathcal{E}$ must be specified. In this paper, we employ the same analytical internal energy density as that used in Prix et al. (2002a), which is given by

$$\mathcal{E} = \frac{1}{2k\rho_p(1 - x_p(\sigma + 1))} \left[ x_p\rho_p^2 + (1 - x_p + \sigma(1 - 2x_p))\rho_p^2 - 2\sigma x_p\rho_n\rho_p \right],$$  

(33)

where $x_p$, $\sigma$, and $k$ are constants. Note that the entrainment that is included in the equation of state of Prix et al. (2002a) is neglected because the entrainment appears in second order in $\delta\Omega_n$ or $\delta\Omega_p$. This internal energy density is a natural generalization of an $N = 1$ polytropic equation of state for a barotropic ordinary fluid into the two superfluids because it is written in the general quadratic form of $\rho_n$ and $\rho_p$, given by

$$\mathcal{E} = \kappa_{nn}\rho_n^2 + 2\kappa_{np}\rho_n\rho_p + \kappa_{pp}\rho_p^2,$$

(34)

where $\kappa_{nn}$, $\kappa_{np}$, and $\kappa_{pp}$ are constants. The chemical potentials for the internal energy density $\mathcal{E}$ are then given by

$$\tilde{\mu}_n = \frac{1}{k\rho_p(1 - x_p(\sigma + 1))}(\rho_n - \sigma p),$$

$$\tilde{\mu}_p = \frac{1}{k\rho_p(1 - x_p(\sigma + 1))}\left[ (1 - x_p + \sigma(1 - 2x_p))\rho_p - \sigma x_p\rho_n \right].$$

(35)

On the other hand, $\rho_n$ and $\rho_p$ can be written in terms of $\tilde{\mu}_n$ and $\tilde{\mu}_p$ as

$$\rho_n = \frac{k}{\sigma + 1}\left[ (1 - x_p + \sigma(1 - 2x_p))\tilde{\mu}_n + \sigma x_p\tilde{\mu}_p \right],$$

$$\rho_p = \frac{kx_p}{\sigma + 1}\left[ (\sigma\tilde{\mu}_n + \tilde{\mu}_p) \right].$$

(36)

which lead to

$$\rho = \rho_n + \rho_p = k\left[ (1 - x_p)\tilde{\mu}_n + x_p\tilde{\mu}_p \right].$$

(37)

Let us consider the situation where the neutrons and the protons are in chemical equilibrium, which is given by the condition $\tilde{\mu}_n = \tilde{\mu}_p$. From equation (36), we can obtain

$$x_p\rho_n = (1 - x_p)\rho_p.$$

(38)
This equation means that \( \rho_n/\rho_p = \text{constant} \) inside the star and that \( x_p \) represents the proton fraction when chemical equilibrium between the neutrons and the protons is achieved. On the other hand, the parameter \( \sigma \) of the internal energy density is interpreted as the so-called “symmetry energy” term (Prix et al. 2002a; Prakash, Lattimer, & Ainsworth 1988). The appropriate range for \( \sigma \) might be \( \sigma \in (-1, 1) \) and we consider only three values for \( \sigma \), i.e. \( \sigma = -0.5, 0, 0.5 \), in this paper (Prix et al. 2002a).

### 2.4 Unperturbed state: rapidly rotating stars

For numerical calculations, it is convenient to introduce non-dimensional physical quantities as follows:

\[
\begin{align*}
  r &= r_0 \hat{r}, \quad \rho_n = \rho_0 \hat{\rho}_n, \quad \rho_p = \rho_0 \hat{\rho}_p, \quad \hat{\mu}_n = \mu_0 \hat{\mu}_n, \quad \hat{\mu}_p = \mu_0 \hat{\mu}_p, \\
  \Omega_n &= \sqrt{4\pi G \rho_0 \hat{\Omega}_n}, \quad \Omega_p = \sqrt{4\pi G \rho_0 \hat{\Omega}_p}, \quad \Phi = 4\pi G \rho_0 r_0^2 \hat{\Phi}, \quad k = k_0 \hat{k},
\end{align*}
\]

where quantities with hat are non-dimensional ones, and \( r_0 \) and \( \mu_0 \) are defined by

\[
r_0 = \sqrt{1/(4\pi G \rho_0)}, \quad \mu_0 = \rho_0/k_0,
\]

where \( \hat{k} \) is determined so as to be \( \hat{r}_{\text{max}} = 1 \) for the unperturbed star, and \( \hat{r}_{\text{max}} \) is the largest distance from the stellar surface to the center of the unperturbed star. Here, \( r_0 \) and \( \rho_0 \) can be given freely because of the polytropic equation of state.

Since, for unperturbed states, chemical equilibrium between two superfluids are assumed, the chemical potential \( \hat{\mu}_{n0} \) can be written as \( \hat{\mu}_{n0} = \hat{k}^{-1}(1 - x_p)^{-1} \hat{\rho}_{n0} = \hat{k}^{-1} \hat{\rho}_0 \) by virtue of the analytical equation of state, where \( \hat{\rho}_0 \) is the total mass density. Then the master equations for unperturbed stars are reduced to

\[
\hat{k}^{-1} \hat{\rho}_0 + \hat{\Phi}_0 - \frac{\hat{\pi}}{2} \hat{r}^2 = \hat{C}_{n0},
\]

\[
\hat{\Phi}_0 = -\frac{1}{4\pi} \int \hat{\rho}_0(\hat{r}') ||\hat{r} - \hat{r}'||^{-2} \hat{r}' d\hat{r}',
\]

\[
= -\sum_{n=0}^{\infty} P_{2n}(\cos \theta) \int_0^{\hat{r}_{\text{max}}} \hat{r}'^2 d\hat{r}' f_{2n}(\hat{r}, \hat{r}') \int_0^{\pi/2} \sin \theta' d\theta' P_{2n}(\cos \theta') \hat{\rho}_0(\hat{r}', \theta'),
\]

where

\[
f_{2n}(\hat{r}, \hat{r}') = \begin{cases} 
\frac{1}{2} \left( \frac{\hat{r}}{\hat{r}'} \right)^{2n} & \text{for } \hat{r}' < \hat{r}, \\
\frac{1}{2} \left( \frac{\hat{r}'}{\hat{r}} \right)^{2n} & \text{for } \hat{r}' \geq \hat{r}.
\end{cases}
\]

and \( P_{2n}(\cos \theta) \) are the Legendre polynomials. Here, the gravitational potential \( \hat{\Phi}_0 \) has been written in the integral representation, in which a proper Green function is employed to include physically appropriate boundary conditions both at \( \hat{r} = 0 \) and at spatial infinity \( \hat{r}_{\text{max}} \). In this integral representation, equatorial symmetry of the matter distribution has been assumed because we are concerned about stars having this equatorial symmetry. Note that these equations are exactly the same as those of polytropic ordinary fluid stars with polytrope index \( N = 1 \). Thus, solutions to equations (40 and 41) can be obtained by the so-called Hachisu’s Self Consistent Field scheme (HSCF), once the axis ratio, \( r_p \), is given (Hachisu 1986). Here, \( r_p \) is defined by

\[
r_p = \hat{r}_{\text{min}}/\hat{r}_{\text{max}} = \hat{r}_{\text{min}},
\]

where \( \hat{r}_{\text{min}} \) is the minimum distance from the stellar surface to the center of the star. Note that solutions \( \hat{\rho}_0, \hat{\Phi}_0, \) and \( \hat{C}_{n0} \) are universal functions in the sense that those are independent of the proton fraction \( x_p \). After getting a solution \( \hat{\rho}_0 \), and giving the proton fraction \( x_p \), we can obtain the mass densities and chemical potentials of two superfluids through the formulas

\[
\hat{\rho}_{n0} = (1 - x_p) \hat{\rho}_0, \quad \hat{\rho}_{p0} = x_p \hat{\rho}_0, \quad \hat{\mu}_{n0} = \hat{\mu}_{p0} = \hat{\rho}_0 \hat{k}^{-1}.
\]

The surface of the star \( R(\theta) \) for unperturbed stars is defined by

\[
P(R(\theta), \theta) = 0,
\]

where \( P \) is the generalized pressure, which is given, for our internal energy density, by

\[
P = \frac{\mu_0 \rho_0}{2k x_p [1 - x_p (\sigma + 1)]} \left[ x_p \rho_n^2 + (1 - x_p + \sigma (1 - 2x_p)) \rho_p^2 - 2\rho_p \rho_n \rho_p \right].
\]

Pressure \( P_0 \) for the unperturbed state can be explicitly written as

\[
P_0 = \frac{\mu_0 \rho_0}{2k (1 - x_p)^2} \rho_n = \frac{\mu_0 \rho_0}{2k} \rho_n^2.
\]
The non-dimensional stellar surface $\hat{R}_0(\theta)$ for an unperturbed star is then given by a solution of the algebraic equation $\hat{\rho}_0(\hat{R}_0(\theta), \theta) = 0$.

2.5 Effect of the rotation velocity difference on the stellar structure

In this paper, we consider perturbed states whose densities of two superfluids $\rho_n$ and $\rho_p$ at the center of the star have the same values as those of unperturbed states. Because of our equations of state \[, this requirement is equivalent to the assumptions of $\delta\mu_n = \delta\mu_p = 0$ at the center of the star. Thus, the integral constants in equations \[ and \[ may be determined so as to be

$$ \delta C_n = \delta C_p = \delta \Phi(r = 0, \theta). $$

(49)

For the analytical internal energy density \[, perturbations of the total mass density $\delta\rho$ is given in terms of perturbed chemical potentials $\delta\mu_n$ and $\delta\mu_p$ as

$$ \delta\rho = \delta\mu_n + \delta\rho_p = \hat{k} \{ (1 - x_p)\delta\mu_n + x_p\delta\mu_p \}. $$

(50)

From equations \[ and \[, thus, we obtain

$$ \hat{k}^{-1}\delta\rho + \delta\Phi - 2\hat{\Omega}^2 \{ (1 - x_p)\delta\Omega_n + x_p\delta\Omega_p \} = \delta\hat{C}_n, $$

(51)

where the relations of the integral constants \[ have been assumed. Note that the perturbed total density is dependent on the proton fraction $x_p$ but not on the symmetry energy parameter $\sigma$. From equation \[, it is found that perturbations $\delta\rho$, $\delta\Phi$, and $\delta\hat{C}_n$ can be represented in terms of three functions independent of $\delta\Omega_n$, $\delta\Omega_p$, and $x_p$ as follows:

$$ \delta\rho = \{ (1 - x_p)\delta\Omega_n + x_p\delta\Omega_p \} \delta\rho, \quad \delta\Phi = \{ (1 - x_p)\delta\Omega_n + x_p\delta\Omega_p \} \delta\Phi, $$

$$ \delta\hat{C}_n = \{ (1 - x_p)\delta\Omega_n + x_p\delta\Omega_p \} \delta\hat{C}_n, $$

(52)

where the three functions $\delta\rho$, $\delta\Phi$, and $\delta\hat{C}_n$ are the solutions of the equations

$$ \hat{k}^{-1}\delta\rho + \delta\Phi - 2\hat{\Omega}^2 = \delta\hat{C}_n, $$

(53)

$$ \nabla_n \nabla^\sigma \delta\Phi = \delta\rho. $$

(54)

Note that three quantities $\delta\mu$, $\delta\Phi$, and $\delta\hat{C}_n$ are universal functions in the sense that those are independent of the parameters $x_p$ and $\sigma$, and that those are dependent only on the structure of the unperturbed star. With equations \[ and \[, the chemical potentials of two superfluids are given by

$$ \delta\mu_n = \{ x_p(\delta\Phi - \delta\hat{C}_n) + \hat{k}^{-1}\delta\rho \} \delta\Omega_n - x_p(\delta\Phi - \delta\hat{C}_n) \delta\Omega_p, $$

$$ \delta\mu_p = \{ (1 - x_p)(\delta\hat{C}_n - \delta\Phi) \} \delta\Omega_n + \{ (1 - x_p)(\delta\Phi - \delta\hat{C}_n) + \hat{k}^{-1}\delta\rho \} \delta\Omega_p. $$

(55)

Perturbations of the mass densities, on the other hand, can be written in terms of $\delta\mu_n$ and $\delta\mu_p$ as

$$ \delta\hat{\rho}_n = \frac{\hat{k}}{\sigma + 1} \{ [1 - x_p + \sigma(1 - 2x_p)] \delta\mu_n + \sigma x_p\delta\mu_p \}, \quad \delta\hat{\rho}_p = \frac{\hat{k} x_p}{\sigma + 1} (\sigma \delta\mu_n + \delta\mu_p), $$

(56)

Note that, for our internal energy density, the chemical potentials do not depend on the symmetry energy parameter $\sigma$, while the mass densities do. The perturbations of the generalized pressure $\delta\hat{P}$ is given by

$$ \delta\hat{P} = \rho_0 \rho_0 \hat{k}^{-1} \rho_0 \{ (1 - x_p)\delta\mu_n + x_p\delta\mu_p \}. $$

(57)

If we write the stellar surface with accuracy up to the first order of the rotation velocity difference as

$$ \hat{R}(\theta) = \hat{R}_0(\theta) + \delta\hat{R}(\theta), $$

(58)

the first order corrections for the stellar surface $\delta\hat{R}(\theta)$ can be due to equation \[, written in terms of $\delta\mu_n$ and $\delta\mu_p$ as

$$ \delta\hat{R}(\theta) = (1 - x_p)\delta\hat{R}_n + x_p\delta\hat{R}_p, $$

(59)

where $\delta\hat{R}_n$ and $\delta\hat{R}_p$ are defined by

$$ \delta\hat{R}_n = \frac{\delta\mu_n}{\hat{\rho}_0} (\hat{r} = \hat{R}_0(\theta), \theta), \quad \delta\hat{R}_p = -\frac{\delta\mu_p}{\hat{\rho}_0} (\hat{r} = \hat{R}_0(\theta), \theta). $$

(60)

Because $\mu_n(\hat{R}_0 + \delta\hat{R}_n) = 0$ and $\mu_p(\hat{R}_0 + \delta\hat{R}_p) = 0$ are satisfied, two surfaces of $\hat{r} = \hat{R}_0 + \delta\hat{R}_n$ and $\hat{r} = \hat{R}_0 + \delta\hat{R}_p$ can be interpreted as the surfaces of the neutron and proton superfluids, respectively. Although we can define the respective fluid surfaces of two superfluids as the zero-density surfaces (Prix 1999; Prix et al. 2002a), we make use of the definition with the chemical potential for the surfaces because the definition with the chemical potentials is considered to be natural.
generalization of the surface definition for ordinary fluid stars, for which the surface is defined as a zero-pressure surface not as a zero-density surface. Note that as we can see from equation (35), in general, “zero-chemical potential surfaces” do not coincide with “zero-density surfaces”. In other words, the densities do not necessarily vanish on the zero-chemical potential surfaces because the equi-potential surfaces in general incline to the equi-density surfaces in the two-fluid model.

In our numerical procedure of solving equations (33) and (41), in order for the Poisson equation (41) to satisfy the boundary condition (32) explicitly, equation (54) is converted into the integral representation, given by

\[
\delta \Phi = - \sum_{n=0}^{\infty} P_{2n}(\cos \theta) \int_0^{\pi/2} \sin \theta' d\theta' \int_0^{R_i(\theta')} \hat{r}'^2 d\hat{r}' f_{2n}(\hat{r}, \hat{r}') P_{2n}(\cos \theta') \delta \hat{\rho}(\hat{r}', \theta').
\]  

(61)

In this paper, equations (33) and (41) are numerically solved with a variant of the so-called Self Consistent Field scheme (Ostriker & Mark 1968). To obtain solutions, we follow the following steps: i) By assuming an initial guess for \(\delta \hat{\rho}\), compute the gravitational potential through the two-dimensional integration (41). ii) Determine the value of parameter \(\delta \tilde{C}_n\) from equation (40). iii) By using the obtained \(\delta \Phi\) and \(\delta \tilde{C}_n\), solve equation (33) for the perturbed density \(\delta \hat{\rho}\). iv) Compare the obtained perturbed density with the one used in obtaining the perturbed potential. If the relative changes of \(\delta \hat{\rho}\) are less than \(10^{-8}\) at all grid points, then the obtained perturbed density distribution is considered to be a converged solution. If the condition for the relative changes is not satisfied, go back to step ii) and there the obtained perturbed density is treated as a new initial guess for \(\delta \hat{\rho}\) for the next iteration cycle.

In actual computations, we make use of equidistantly spaced discrete meshes in the radial direction (0 \(\leq \hat{r} \leq \hat{r}_{\text{max}} = 1\)). In order to calculate integrations in \(\hat{r}\) direction, we employ a classical trapezoidal rule. As for the angular variable \(\theta\), we take the angular grid points at \(\mu_i = \cos \theta_i\), where \(\mu_i\)'s are zeros of the Legendre polynomial of order \(2L - 1\), i.e. \(P_{2L-1}(\mu_i) = 0\), and there are \(L\) grid points for \(0 \leq \theta \leq \pi/2\). We employ the Gaussian quadrature (see, e.g., Abramowitz & Stegun 1964) to evaluate integrations in the \(\theta\) direction. In the present investigation, we take the number of mesh points to be \(500 \times 25\) (\(r \times \theta\)). The Legendre polynomials in equations (40) and (41) are added up to \(P_{40}(\mu)\).

3 NUMERICAL RESULTS

In this paper, we assume the proton fraction to be \(x_p = 0.1\) because a typical proton fraction in the cores of old neutron stars is expected to be around \(x_p = 0.1\). For the parameter of the symmetry energy, \(\sigma\), since we have little information about the range of \(\sigma\), we investigate three cases, i.e. \(\sigma = -0.5, 0, 0.5\), in order to examine the effect of the symmetry energy on the stellar structures. Note that the total density of superfluids and the chemical potentials are dependent on the proton fraction but not on the symmetry energy. We exhibit the results for two cases of \((\delta \Omega_n, \delta \Omega_p) = (1, 0)\) and \((\delta \Omega_n, \delta \Omega_p) = (0, 1)\) because solutions for other values of \((\delta \Omega_n, \delta \Omega_p)\) are expressed in terms of linear superpositions of those two solutions.

First, let us consider the unperturbed stars. In our unperturbed states, two superfluids are assumed to be in the same rotational motion and in chemical equilibrium. As mentioned in the last section, the structures of unperturbed stars are, therefore, almost the same as those of uniformly rotating \(N = 1\) polytropic stars. Thus, there are no new features for the unperturbed stars. For reader’s convenience, however, we exhibit some results for the unperturbed stars. In Figures 1 through 4, we show the non-dimensional fundamental quantities of the unperturbed stars, the axis ratios, \(r_p\), the total masses, \(M\), the total moments of inertia, \(I\), and the ratios of the rotational energy to the absolute value of the gravitational energy, \(T/|W|\), as functions of the angular velocity, \(\Omega\), where \(\Omega\) is the non-dimensional angular velocity of the star, defined by \(\Omega = \Omega/(GM/R^3)^{1/2}\), and \(R\) denotes the stellar radius on the equatorial plane. Here, the total mass, \(M\) is defined by

\[
M = \int \rho_0 d^3r = \frac{4\pi}{3} \rho_0 r_0^3 M = M_0 \hat{M}.
\]  

(62)

The total moments of inertia, \(I\), is given by

\[
I = \int \rho_0 \hat{r}^2 d^3r = 4\pi \rho_0 r_0^5 I = I_0 \hat{I}.
\]  

(63)

The rotational energy, \(T\), and the gravitational energy, \(W\), are, respectively, defined by

\[
T = \frac{1}{2} \int \rho_0 \hat{r}^2 \hat{\Omega}^2 d^3r, \quad W = - \frac{1}{2} \int \rho_0 \Phi_0 d^3r.
\]  

(64)

Since \(\rho_n/\rho_p = \text{const}\) for the unperturbed stars, we can write the masses, \(M_n, M_p\), and the moments of inertia, \(I_n, I_p\), for the neutrons and the protons as

\[
M_n = (1 - x_p) M, \quad M_p = x_p M, \quad I_n = (1 - x_p) I, \quad I_p = x_p I.
\]  

(65)

In order to check our numerical code, we have calculated a solution for a slowly rotating star with the axis ratio \(r_p = 0.992\), whose angular velocity is \(\Omega = 0.10298\), and compared them with the analytical solution obtained by Prix et al. (2002a), in
which the structure of a slowly rotating superfluid star was examined analytically with the slow rotation approximation. The perturbed densities obtained from two methods are shown as functions of $\hat{r}$ in Figure 5. The solid lines show the analytical result for $\delta \rho_n$ with the slow rotation approximation, and the solid squares our numerical result without assuming the slow rotation approximation. The parameters of the model shown in Figure 5 have been chosen to be $\sigma = -0.5$, $0$, $0.5$. For the models shown in Figures 6 through 11, the axis ratio $r_p$ for the unperturbed star has been taken to be $r_p = 0.668$, and the corresponding angular velocity is given by $\Omega = 0.8037$. Figures 6 through 8 show the density distributions for $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$, and Figures 9 through 11 for $(\delta \Omega_n, \delta \Omega_p) = (0, 1)$. In those figures, the perturbed densities are shown versus the radial coordinate $\hat{r}$ for six different values of $\theta_i$, in which the longest curve corresponds to the result on the equator, $\theta = \pi/2$, and $\theta_i$'s decrease for each successively shorter curve toward the value $\theta = 0$ on the symmetry axis. Comparing Figure 6 (left panel) with Figure 5, in which distributions of $\delta \rho_n$ are shown for the same parameter as those in Figure 6 except for the axis ratio $r_p$ or for the rotation velocity $\Omega$, we observe that the basic qualitative properties of the density perturbations for two superfluids do not depend on the value of $\Omega$ so much. We do not therefore show the density distributions for other rotation rates $\Omega$ in this paper. From Figures 6 through 11, we can see that the amplitudes of $\delta \rho_n$ are much larger than those of $\delta \rho_p$ for the solutions of $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$, while the amplitudes of $\delta \rho_p$ is smaller than those of $\delta \rho_p$ for the case of $(\delta \Omega_n, \delta \Omega_p) = (0, 1)$. As shown in Figures 9 through 11, however, for the solutions of $(\delta \Omega_n, \delta \Omega_p) = (0, 1)$, difference of the amplitude between $\delta \rho_n$ and $\delta \rho_p$ is not so large. This is because the proton fraction is not very large in the interior. Similar behaviors were found by Prix et al. (2002a). It is also found from Figures 6 through 11 that $\delta \rho_p$ is strongly dependent on the values of $\sigma$ for the $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$ case, while $\delta \rho_n$ depends strongly on $\sigma$ for the $(\delta \Omega_n, \delta \Omega_p) = (0, 1)$ case. This means that the significant parameters to model a superfluid neutron star are appropriate combinations of the symmetry energy parameter $\sigma$ and the velocity difference between two superfluids.

In Figures 12 and 13, the perturbed axis ratios are plotted as functions of the angular velocity $\Omega$. Here, the functions, $\delta r_p$, $\delta r_{p,n}$, and $\delta r_{p,p}$ can be interpreted as the perturbations of the axis ratios of the whole star, the neutron superfluid, and the proton superfluid, respectively, and are defined by

\[
\delta r_p = \delta \hat{R}(\theta = 0) - r_p \delta \hat{R}(\theta = \pi/2), \quad \delta r_{p,n} = \delta \hat{R}_n(\theta = 0) - r_p \delta \hat{R}_n(\theta = \pi/2), \quad \delta r_{p,p} = \delta \hat{R}_p(\theta = 0) - r_p \delta \hat{R}_p(\theta = \pi/2).
\]

The results for $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$ and $(\delta \Omega_n, \delta \Omega_p) = (0, 1)$ are shown in Figures 12 and 13, respectively. Note that in these figures, the perturbed axis ratios are shown for $0 \leq \bar{\Omega} \leq 0.9$ because they diverge as $\bar{\Omega}$ goes to its maximum value. These figures illustrate how the surface of the star is deformed due to the effect of the rotation velocity difference between two superfluids. It is found that the change of the axis ratio of the rotating component with the slower angular velocity becomes positive when the angular velocity $\bar{\Omega}$ becomes larger than some critical value. In other words, the rotating component with the slower angular velocity tends to become prolate when a star rotates very rapidly.

In Figure 14, the perturbed mass is shown as a function of $\bar{\Omega}$. Here, the perturbed total mass $\delta M$ is defined as

\[
\delta M = \frac{3}{4\pi M} \int d^3 \mathbf{r} \delta \rho d^3 \mathbf{r}.
\]

The solid curve and the dashed curve show $\delta M$ for the solutions with $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$ and $(\delta \Omega_n, \delta \Omega_p) = (0, 1)$, respectively. In this paper, we consider the perturbed states whose central density is the same as that of the unperturbed states. Thus, the mass of the perturbed stars can change. In Figures 15 and 16, we plot the perturbed moments of inertia for the neutron and the proton, $\delta I_n$ and $\delta I_p$, as functions of the angular velocity $\bar{\Omega}$. Here, the perturbed moments of inertia for the neutron and the proton, $\delta I_n$ and $\delta I_p$, are defined as

\[
\delta I_n = \frac{1}{4\pi \rho_n(1 - x_p)} \int d^3 \mathbf{r} \rho_n \varpi^2 d^3 \mathbf{r}, \quad \delta I_p = \frac{1}{4\pi \rho_p(1 - x_p)} \int d^3 \mathbf{r} \rho_p \varpi^2 d^3 \mathbf{r}.
\]

Figures 15 and 16 show the perturbations of the moments of inertia for $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$ and $(\delta \Omega_n, \delta \Omega_p) = (0, 1)$, respectively. In these figures, results for three different values of $\sigma$, $\sigma = -0.5$, $0$, $0.5$ are displayed. Figures 15 and 16 show that the perturbed moments of inertia for the protons are strongly dependent on the symmetry energy parameter $\sigma$ for $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$ case, while the perturbed moments of inertia for the neutrons strongly depend on $\sigma$ for $(\delta \Omega_n, \delta \Omega_p) = (0, 1)$ case. This is consistent with that of the properties of the density distributions for the neutrons and the protons. It is noted that amplitudes of $\delta I_n$ and $\delta I_p$ become large as the angular velocity $\bar{\Omega}$ is increased because the perturbations are roughly proportional to $\bar{\Omega}^2$ due to equation (6).
4 SUMMARY AND DISCUSSION

In this paper we have developed a formulation for constructing and examining rapidly rotating neutron stars that contain two superfluids, taking account of the effect of the rotation velocity difference between two superfluids. We assumed neutron stars to be composed of the superfluid neutrons and the mixture of the superfluid protons and the normal fluid electrons. To describe Newtonian dynamics of the two superfluids, the Newtonian version of the two-fluid formalism developed by Prix (2002) was employed. In this paper, we considered the situation where two superfluids rapidly rotate around the same rotation axis, but the rotation velocity difference between two superfluids is very small. We then treated the effect of the rotation velocity difference on the equilibrium configurations as a small perturbation to rapidly rotating superfluid stars whose rotation velocities of two superfluids are the same. We derived basic equations for perturbations on the structure of rapidly rotating superfluid stars due to the rotation velocity difference between two superfluids. Assuming the superfluids to obey a simple analytical equation of state used by Prix et al. (2002a), we obtained numerical solutions for the perturbed quantities and found that the density distributions of the superfluids are strongly dependent of the symmetry energy parameter $\sigma$, which appears in the analytical equation of state. Similar properties were found in Prix et al. (2002a). It was also found that if Prix et al. (2002a)'s analytical equation of state is assumed, the perturbations can be represented in terms of the universal functions that are independent of the parameters of the equation of state.

Although we only considered one special equation of state for superfluids in the present investigation, the formalism we derived in this paper is straightforwardly applicable to general equations of state. We treated the superfluids inside neutron stars in the framework of Newtonian dynamics. The effect of general relativity must not be however neglected for the structures of neutron stars because neutron stars are quite compact in a sense that a general relativistic effect can be expressed by the factor $GM/c^2R$ whose typical value is $\sim 0.2$ for neutron stars, where $c$ is the speed of light. It is straightforward to extend the present formulation to general relativistic configurations and we will do it in the future. Recently, Prix et al. (2002b) have obtained rapidly rotating relativistic superfluid stars whose rotation velocity of the neutrons differs from that of the protons. They did not assume that the rotation velocity difference between two fluids is very small. In other words, their method can be applied to equilibrium configurations with any rotation velocities. Yet, our perturbation method developed in this investigation has an advantage in studying structures of real neutron stars because it would give results with higher accuracy by taking account of the smallness of the velocity difference directly, as long as the rotation velocity difference between two fluids in real neutron stars is very small. A solid crust is believed to exist near the surface of cold and old neutron stars, and can have a significant influence on the structure of the equilibrium states if the solid crust is not in a strain-free state in the equilibrium states of rotating neutron stars. Therefore, an investigation of the effect of the solid crust on the equilibrium states remains as a challenging problem in the future too.

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Rapidly rotating superfluid neutron stars in Newtonian dynamics

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Figure 1. Axis ratio $r_p$ for the unperturbed stars is given as a function of $\Omega$.

Figure 2. Same as Figure 1 but for the mass $\dot{M}$. 
Figure 3. Same as Figure 1 but for the moment of inertia $\tilde{I}$

Figure 4. Same as Figure 1 but for the ratios of the rotational energy to the absolute value of the gravitational energy $T/|W|$
Figure 5. Perturbed densities for the neutron $\delta \rho_n$ are given as functions of $\hat{r}$. The solid curves and solid squares are used to indicate $\delta \rho_n$ obtained from Prix et al. (2002a)'s analytical formula and our numerical scheme, respectively. The model parameters are $\tilde{\Omega} = 0.10298$, $\sigma = -0.5$, and $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$.

Figure 6. Perturbed densities for the neutron (left) and the proton (right) are given as functions of $\hat{r}$ for six different values of $\theta$. The model parameters are $\tilde{\Omega} = 0.8032$, $\sigma = -0.5$, and $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$.
Figure 7. Same as Figure 6 but for $\sigma = 0$.

Figure 8. Same as Figure 7 but for $\sigma = 0.5$. 
Figure 9. Perturbed densities for the neutron (left) and the proton (right) are given as functions of $\hat{r}$ for six different values of $\theta$. The model parameters are $\Omega = 0.8032$, $\sigma = -0.5$, and $(\delta\Omega_n, \delta\Omega_p) = (0, 1)$.

Figure 10. Same as Figure 9 but for $\sigma = 0$. 
Figure 11. Same as Figure 10 but for $\sigma = 0.5$.

Figure 12. Perturbed axis ratios $\delta r_p$, $\delta r_{p,n}$, and $\delta r_{p,p}$ are given as functions of $\bar{\Omega}$. The solid, dashed, and dotted curves denote $\delta r_p$, $\delta r_{p,n}$, and $\delta r_{p,p}$, respectively. The model parameter is $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$. 

Figure 13. Same as Figure 12 but for \((\delta \Omega_n, \delta \Omega_p) = (0, 1)\).

Figure 14. Perturbed mass \(\delta M\) is given as a function of \(\varpi\). The solid and dashed curves denote results for \((\delta \Omega_n, \delta \Omega_p) = (1, 0)\) and \((\delta \Omega_n, \delta \Omega_p) = (0, 1)\), respectively.
Figure 15. Perturbed moments of inertia for the neutron and the proton $\delta I_n, \delta I_p$ are given as functions of $\bar{\Omega}$. The model parameter is $(\delta \Omega_n, \delta \Omega_p) = (1, 0)$.

Figure 16. Same as Figure 15 but for $(\delta \Omega_n, \delta \Omega_p) = (0, 1)$. 