Lusin-type Theorems for Cheeger Derivatives on Metric Measure Spaces

1 Introduction

A classical theorem of Lusin [19] states that for every Borel function \( f \) on \( \mathbb{R} \), there is a continuous function \( u \) on \( \mathbb{R} \) that is differentiable almost everywhere with derivative equal to \( f \).

In [1], Alberti gave a related result in higher dimensions. He proved the following theorem, in which \( |\cdot| \) denotes Lebesgue measure and \( Du \) denotes the standard Euclidean derivative of \( u \).

**Theorem 1.1** ([1], Theorem 1). Let \( \Omega \subset \mathbb{R}^k \) be open with \( |\Omega| < \infty \), and let \( f: \Omega \to \mathbb{R}^k \) be a Borel function. Then for every \( \varepsilon > 0 \), there exist an open set \( A \subset \Omega \) and a function \( u \in C^1_0(\Omega) \) such that

(a) \( |A| \leq \varepsilon |\Omega| \),
(b) \( f = Du \) on \( \Omega \setminus A \), and
(c) \( \|Du\|_p \leq C \varepsilon^{\frac{1}{p-1}} \|f\|_p \) for all \( p \in [1, \infty] \).

Here \( C > 0 \) is a constant that depends only on \( k \).

In other words, Alberti showed that it is possible to arbitrarily prescribe the gradient of a \( C^1_0 \) function \( u \) on \( \Omega \subset \mathbb{R}^k \) off of a set of arbitrarily small measure, with quantitative control on all \( L^p \) norms of \( Du \).

Moonens and Pfeffer [20] applied Alberti’s result to show a more direct analog of the Lusin theorem in higher dimensions:

**Theorem 1.2** ([20], Theorem 1.3). Let \( \Omega \subset \mathbb{R}^k \) be an open set and let \( f: \Omega \to \mathbb{R}^k \) be measurable. Then for any \( \varepsilon > 0 \), there is an almost everywhere differentiable function \( u \in C(\mathbb{R}^k) \) such that

(a) \( \|u\|_\infty \leq \varepsilon \) and \( \{u \neq 0\} \subset \Omega \),
(b) \( Du = f \) almost everywhere in \( \Omega \), and
(c) \( Df = 0 \) everywhere in \( \mathbb{R}^k \setminus \Omega \).
These "Lusin-type" results for derivatives in Euclidean space have applications to integral functionals on Sobolev spaces [1], to the construction of horizontal surfaces in the Heisenberg group ([2, 11]) and in the analysis of charges and normal currents [20]. In addition, we remark briefly that the results of Alberti and Moonens-Pfeffer have been generalized to higher order derivatives on Euclidean space in the work of Francos [10] and Hajłasz-Mirra [11], though we do not pursue those lines here.

The purpose of this note is to extend the results of Alberti and Moonens-Pfeffer, in a suitable sense, to a class of metric measure spaces on which differentiation is defined.

In his seminal 1999 paper, Cheeger [6] defined (without using this name) the notion of a "measurable differentiable structure" for a metric measure space. Cheeger showed that a large class of spaces, the so-called PI spaces, possess such a structure. A differentiable structure endows a metric measure space with a notion of differentiation and a version of Rademacher's theorem: every Lipschitz function is differentiable almost everywhere with respect to the structure.

The class of PI spaces includes Euclidean spaces, all Carnot groups (such as the Heisenberg group), and a host of more exotic examples like those of [21], [5], [18], [7], and [17].

We prove the following two analogs of the results of Alberti and Moonens-Pfeffer for PI spaces. All the definitions are given in Section 2 below.

**Theorem 1.3.** Let \((X, d, \mu)\) be a PI space and let \(\{(U_j, \phi_j : X \to \mathbb{R}^k)\}_{j \in J}\) be a measurable differentiable structure on \(X\). Then there are constants \(C, \eta > 0\) with the following property:

Let \(\Omega \subset X\) be open with \(\mu(\Omega) < \infty\) and let \(\{f_j : U_j \cap \Omega \to \mathbb{R}^k\}_{j \in J}\) be a collection of Borel functions. Then for every \(\epsilon > 0\) there is an open set \(A \subset \Omega\) and a Lipschitz function \(u \in C_0(\Omega)\) such that

\[\mu(A) \leq \epsilon \mu(\Omega),\]

for all \(j \in J\),

\[\|\text{Lip}_u\|_p \leq C \epsilon^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{j \in J} (\text{LIP}(\phi_j))^q\right)^{1/p} \int_{\Omega \cap U_j} |f_j|^p \right)^{1/p} \]

for all \(p \in [1, \infty)\), and

\[\|\text{Lip}_u\|_{\infty} \leq C \epsilon^{\frac{1}{q} - \frac{1}{p}} \sup_{j \in J} (\text{LIP}(\phi_j))\|f_j\|_{\infty}.\]

The constants \(C, \eta > 0\) depend only on the data of \(X\).

That the bounds (1.6) and (1.7) involve the chart functions \(\phi_j\) is in some sense inevitable, as one can easily discover by looking at the measurable differentiable structure \((\mathbb{R}, \phi(x) = 2x)\) on \(\mathbb{R}\). Note also that, unlike in the Euclidean setting of Theorem 1.1, the notion of \(C^1\) regularity is not defined in PI spaces. Thus, the natural regularity for our constructed function \(u\) in Theorem 1.3 is Lipschitz.

Our second result is the analog in PI spaces of Theorem 1.2, in which we prescribe the derivative almost everywhere, rather than off of a small set. We also include an innovation, originally due to Hajłasz-Mirra [11], which allows the constructed function to have any modulus of continuity worse than Lipschitz.

**Theorem 1.8.** Let \((X, d, \mu)\) be a PI space. Let \(\{(U_j, \phi_j : U_j \to \mathbb{R}^k)\}\) be a measurable differentiable structure on \(X\). Let \(\Omega \subset X\) be open, let \(\epsilon > 0\), and let \(\{f_j : U_j \cap \Omega \to \mathbb{R}^k\}_{j \in J}\) be a collection of Borel functions. Let \(v : [0, \infty) \to [0, \infty)\) be a continuous function with \(v(0) = 0\) and \(v(t) = O(t)\) as \(t \to \infty\).

Then there is a continuous function \(u\) on \(X\) that is differentiable almost everywhere and satisfies

\[\|u\|_{\infty} \leq \epsilon\text{ and }\{u \neq 0\} \subset \Omega,\]

\[d^i u = f_j \text{ a.e. in } U_j \cap \Omega\]

for each \(j \in J\),

\[\text{Lip}_u = 0 \text{ everywhere in } X \setminus \Omega,\]
and
\[ |u(x) - u(y)| \leq \frac{|x - y|}{\nu(|x - y|)} \]
for all \(x, y \in X\).

As mentioned above, equation (1.12) states that the function \(u\) in Theorem 1.8 can be taken to have any modulus of continuity worse than Lipschitz; e.g., it can be \(\lambda\)-Hölder for any \(\lambda \in (0, 1)\). This is sharp: even in Euclidean space it is impossible to construct Lipschitz functions with prescribed gradient almost everywhere (see the introduction of [11]). Part (1.12) of Theorem 1.8 is not present in the original theorem of [20], but was first done in [11] for derivatives of all orders in Euclidean space.

Theorems 1.3 and 1.8 give a unified method for prescribing derivatives on what is now a vast number of different known PI spaces. Indeed, these results are already interesting in the case where the measurable differentiable structure consists of a single, one-dimensional chart, as in the Laakso-type spaces of [18] and [7]. Each of these spaces \(G_\infty\) can be viewed as a Gromov-Hausdorff limit of a sequence of metric measure graphs \(G_n\), and is equipped with a natural 1-Lipschitz projection \(\pi : G_\infty \to \mathbb{R}\) which serves as the chart function for the measurable differentiable structure. In this case, it is already far from obvious that one can arbitrarily prescribe the derivative of a function \(u : G_\infty \to \mathbb{R}\) in a way which assigns different values to different points in the “slice” \(\pi^{-1}(t)\), since the function \(u\) cannot then be a simple lift \(u = \tilde{u} \circ \pi\) of a function \(\tilde{u} : \mathbb{R} \to \mathbb{R}\). Higher-dimensional versions of these examples also exist; see Section 11 of [7]. Generically, these examples admit no bi-Lipschitz embeddings into any \(\mathbb{R}^n\), and are purely unrectifiable and thus highly “non-Euclidean” in their infinitesimal structure.

Other examples of interesting PI spaces to which Theorems 1.3 and 1.8 apply are: some sub-Riemannian manifolds, topological manifolds with certain metric constraints [21], and fractal examples such as those of [5] and [17]. (For the specific case of Carnot groups, the problem of prescribed derivatives can be reduced to the Euclidean case of Theorems 1.1 and 1.2.)

2 Definitions and Preliminaries

We will work with metric measure spaces \((X, d, \mu)\) such that \((X, d)\) is complete and \(\mu\) is a Borel regular measure. If the metric and measure are understood, we will denote such a space simply by \(X\). An open ball in \(X\) with center \(x\) and radius \(r\) is denoted \(B(x, r)\). If \(B = B(x, r)\) is a ball in \(X\) and \(\lambda > 0\), we write \(\lambda B = B(x, \lambda r)\).

We generally use \(C\) and \(C'\) to denote positive constants that depend only on the quantitative data associated to the space \(X\) (see below); their values may change throughout the paper.

If \(\Omega \subset X\) is open, we let \(C_c(\Omega)\) denote the space of continuous functions with compact support in \(\Omega\). We also let \(C_0(\Omega)\) denote the completion of \(C_c(\Omega)\) in the supremum norm. Any function in \(C_0(\Omega)\) admits a natural extension by zero to a continuous function on all of \(X\).

Recall that a real-valued function \(u\) on a metric space \((X, d)\) is Lipschitz if there is a constant \(L \geq 0\) such that
\[ |u(x) - u(y)| \leq L d(x, y) \]
for all \(x, y \in X\). The infimum of all \(L \geq 0\) such that the above inequality holds is called the Lipschitz constant of \(u\) and is denoted \(\text{LIP}(u)\).

Given a real-valued (not necessarily Lipschitz) function \(u\) on \(X\), we also define its pointwise upper Lipschitz constant at points \(x \in X\) by
\[ \text{Lip}_u(x) = \limsup_{r \to 0} \frac{1}{r} \sup_{d(x, y) < r} |u(y) - u(x)|. \]

Two basic facts about Lip are easy to verify. First, for any two functions \(f\) and \(g\),
\[ \text{Lip}_{f+g}(x) \leq \text{Lip}_f(x) + \text{Lip}_g(x). \]
Second, if $f$ and $g$ are Lipschitz functions, then

$$\text{Lip}_{fg}(x) \leq f(x)(\text{Lip}_g(x)) + g(x)(\text{Lip}_f(x)).$$

(2.2)

A non-trivial Borel regular measure $\mu$ on a metric space $(X, d)$ is a doubling measure if there is a constant $C > 0$ such that $\mu(B(x, 2r)) \leq C \mu(B(x, r))$ for every ball $B(x, r)$ in $X$. The existence of a doubling measure $\mu$ on $(X, d)$ implies that $(X, d)$ is a doubling metric space, i.e. that every ball can be covered by at most $N$ balls of half the radius, for some fixed constant $N$. In particular, a complete metric space with a doubling measure is proper: every closed, bounded subset is compact.

**Definition 2.3.** A metric measure space $(X, d, \mu)$ is a PI space if $(X, d)$ is complete, $\mu$ is a doubling measure on $X$ and $(X, d, \mu)$ satisfies a “$(1, q)$-Poincaré inequality” for some $1 \leq q < \infty$; There is a constant $C > 0$ such that, for every compactly supported Lipschitz function $f : X \to \mathbb{R}$ and every open ball $B$ in $X$,

$$\int_B |f - f_B| d\mu \leq C(\text{diam } B) \left( \int_{CB} (\text{Lip}_f)^q d\mu \right)^{1/q}.$$  

(Here the notations $\int_B gd\mu$ and $g_B$ both denote the average value of the function $g$ on the set $E$, i.e., $\frac{1}{\mu(E)} \int_E gd\mu$.)

This definition can be found in [14]; it is equivalent to other versions of the Poincaré inequality in metric measure spaces, such as the original one of [12]. If $X$ is a PI space, then the collection of constants associated to the doubling property and Poincaré inequality on $X$ are known as the data of $X$. For a recent detailed treatment of PI spaces, see [13].

In addition to providing a differentiable structure (see below), the PI space property of $X$ will supply two other key facts for us, summarized in the following proposition.

**Proposition 2.4.** Let $(X, d, \mu)$ be a PI space. Then there is a constant $C > 0$, depending only on the data of $X$, such that the following two statements hold:

(a) $X$ is quasiconvex, meaning that any two points $x, y \in X$ can be joined by a rectifiable path of length at most $Cd(x, y)$.

(b) For any bounded Lipschitz function $u$ on $X$,

$$\text{LIP}(u) \leq C\|\text{Lip}_u\|_\infty.$$  

Proof. The first statement can be found in Theorem 17.1 of [6]. The second can be found (in greater generality than we need here) in [9], Theorem 4.7.

The following definition is due to Cheeger, in Section 4 of [6]. The form we state can also be found in Definition 2.1.1 of [15] (see also [4]), with one minor difference, explained below. The notation $\langle \cdot, \cdot \rangle$ denotes the standard inner product on Euclidean space of the appropriate dimension.

**Definition 2.5.** Let $(X, d, \mu)$ be a metric measure space. Let $\{U_j\}_{j \in J}$ be a countable collection of pairwise disjoint measurable sets such that $\mu(X \setminus \bigcup_j U_j) = 0$, let $\{k_j\}_{j \in J}$ be a collection of non-negative integers, and let $\{\phi_j : X \to \mathbb{R}^{k_j}\}_{j \in J}$ be a collection of Lipschitz functions.

We say that the collection $\{(U_j, \phi_j)\}$ forms a measurable differentiable structure for $X$ if the following holds: For every Lipschitz function $u$ on $X$ and every $j \in J$, there is a Borel measurable function $d^j u : U_j \to \mathbb{R}^{k_j}$ such that, for almost every $x \in U_j$,

$$\lim_{y \to x} \frac{|u(y) - u(x) - \langle d^j u(x), (\phi_j(y) - \phi_j(x))\rangle|}{d(y, x)} = 0.$$  

(2.6)

Furthermore, the function $d^j u$ should be unique (up to sets of measure zero).
We call each pair \((U_j, \phi_j)\) a chart for the differentiable structure on \(X\). For more background on differentiable structures (also called “strong measurable differentiable structures” and “Lipschitz differentiability spaces”) see [3, 15]. In Definition 2.6, we do not \textit{a priori} require that the integers \(k_j\) are uniformly bounded above, although this will follow whenever \(X\) is a PI space (see Theorem 2.7 below) and hence throughout this paper. Of course, we may assume without loss of generality that all the sets \(U_j\) in the measurable differentiable structure have positive measure.

Note that the defining property (2.6) for a measurable differentiable structure can be more succinctly rephrased as

\[
\text{Lip}_{\mathbb{R}^d} (\phi_j)(x) = 0.
\]

The link between PI spaces and measurable differentiable structures is given by the following theorem of Cheeger, one of the main results of [6]. (See also [3, 15, 16] for alternate approaches.)

**Theorem 2.7** ([6], Theorem 4.38). Every PI space \(X\) supports a measurable differentiable structure \(\{(U_j, \phi_j) : X \to \mathbb{R}^{k_j}\}\), and the dimensions \(k_j\) of the charts \(U_j\) are bounded by a uniform constant depending only on the constants associated to the doubling property and Poincaré inequality of \(X\).

If \(X\) supports a measurable differentiable structure, then it generally supports many other equivalent ones. For example, the sets \(U_j\) may be decomposed into measurable pieces or the functions \(\phi_j\) rescaled without altering the properties in Definition 2.5. At times, it will be helpful to assume certain extra properties of the charts.

**Definition 2.8.** A measurable differentiable structure \(\{(U_j, \phi_j) : X \to \mathbb{R}^{k_j}\}\) is normalized if, for each \(j \in J\), there exists \(c_j > 0\), such that

\[
\begin{align*}
U_j & \text{ is closed}, \quad (2.9) \\
\text{LIP}(\phi_j) & = 1, \quad (2.10) \\
|d^i u(x)| & \leq c_j \text{Lip}_u(x) \text{ whenever } u \text{ is differentiable at } x \in U_j. \quad (2.11)
\end{align*}
\]

The definition of a normalized chart is a minor modification of the notion of a “structured chart”, due to Bate ([3], Definition 3.6). The following lemma, essentially due to Bate, says that a given chart structure on \(X\) can always be normalized by rescaling and chopping.

**Lemma 2.12.** Let \(X\) be a PI space and let \(\{(U_j, \phi_j) : X \to \mathbb{R}^{k_j}\}\) be a measurable differentiable structure on \(X\). Then there exists a collection of sets \(\{U_{j,k}\}_{j \in J, k \in K_j}\) such that

- each set \(U_{j,k}\) is contained in \(U_j\) and
- \(\{(U_{j,k}, (\text{LIP}(\phi_j))^{-1} \phi_j)\}_{j \in J, k \in K_j}\) is a normalized measurable differentiable structure on \(X\).

**Proof.** By Lemma 3.4 of [3], we can decompose each chart \(U_j\) into charts \(U_{j,k}\) such that the measurable differentiable structure \(\{(U_{j,k}, \phi_j)\}\) satisfies (2.11).

As \((\text{LIP}(\phi_j))^{-1} \phi_j\) is just a rescaling of \(\phi_j\), the chart \((U_{j,k}, (\text{LIP}(\phi_j))^{-1} \phi_j)\) still possesses property (2.11) (with a different constant \(c_{j,k}\)).

As a final step, we decompose each \(U_{j,k}\) into closed sets, up to measure zero, while maintaining the same chart functions.

For technical reasons, it will be convenient in the proofs of Theorems 1.3 and 1.8 that the measurable differentiable structure is normalized. That this can be done without loss of generality is the content of the following simple lemma.

**Lemma 2.13.** To prove Theorems 1.3 and 1.8, we can assume without loss of generality that the measurable differentiable structure \(\{(U_j, \phi_j)\}\) is normalized.

**Proof.** Assume that we can prove Theorems 1.3 and 1.8 if the charts involved are normalized.
Suppose \((U_j, \phi_j)\) is an arbitrary (not necessarily normalized) measurable differentiable structure on \(X\). Let \(\Omega \subset X\) be open with \(\mu(\Omega) < \infty\), let \(\{f_j: U_j \cap \Omega \to \mathbb{R}^k\}_{j \in J}\) be a collection of Borel functions, and let \(c > 0\).

By Lemma 2.12, there is a normalized measurable differentiable structure
\[
\{(U_{j,k}, (\text{LIP}(\phi_j))^{-1} \phi_j)\}_{j \in J, k \in K_i}
\] (2.14)
on \(X\), where each \(U_{j,k}\) is contained in \(U_j\).

Let \(g_{j,k} = (\text{LIP}(\phi_j)f_j\). Apply Theorem 1.3 to the normalized measurable differentiable structure (2.14), with the functions \(g_{j,k}\) and the same parameter \(c\). We immediately obtain an open set \(A \subset \Omega\) and a Lipschitz function \(u \in C_0(\Omega)\) that satisfy all four requirements of Theorem 1.3.

A similar argument applies to reduce Theorem 1.8 to the normalized case.

The original arguments of [1] and [20] to prove Theorems 1.1 and 1.2 use the dyadic cube decomposition of Euclidean space. We will use the analogous decomposition in arbitrary doubling metric spaces provided by a result of Christ [8].

**Proposition 2.15** ([8], Theorem 11). Let \((X, d, \mu)\) be a doubling metric measure space. Then there exist constants \(c \in (0, 1), \eta > 0, a_0 > 0, a_1 > 0,\) and \(C_1 > 0\) such that for each \(k \in \mathbb{Z}\) there is a collection \(\Delta_k = \{Q^k_i: i \in I_k\}\) of disjoint open subsets of \(X\) with the following properties:

(i) For each \(k \in \mathbb{Z}\), \(\mu(X \setminus \bigcup_{i \in I_k} Q^k_i) = 0\).

(ii) For each \(k \in \mathbb{Z}\) and \(i \in I_k\), there is a point \(z^k_i \in Q^k_i\) such that
\[
B(z^k_i, a_0 c^k) \subset Q^k_i \subset B(z^k_i, a_1 c^k).
\]

(iii) For each \(k \in \mathbb{Z}\) and \(i \in I_k\), and for each \(t > 0\),
\[
\mu\left(\{x \in Q^k_i: \text{dist}(x, X \setminus Q^k_i) \leq tc^k\}\right) \leq C_1 t^\eta \mu(Q^k_i).
\]

(iv) If \(\ell \geq k\), \(Q \in \Delta_k\), and \(Q' \in \Delta_k\), then either \(Q \subset Q'\) or \(Q \cap Q' = \emptyset\).

(v) For each \(k \in \mathbb{Z}\), each \(i \in I_k\), and each integer \(\ell < k\) there is a unique \(j \in I_\ell\) such that \(Q^\ell_i \subset Q^k_j\).

We refer to the elements of any \(\Delta_k\) as cubes.

The next lemma is one of the primary differences between our proof of Theorem 1.3 and the proof of Theorem 1.1 from [1]. It allows us to replace a single-scale argument in [1] by an argument that uses multiple scales simultaneously, which will allow us to deal with the presence of multiple charts.

**Lemma 2.16.** Let \((X, d, \mu)\) be a complete doubling metric measure space. Suppose that \(\mu(X \setminus \bigcup_{j \in J} U_j) = 0\), where \(\{U_j\}_{j \in J}\) is a countable collection of pairwise disjoint measurable sets of positive measure. Fix \(\gamma > 0\) and positive numbers \(\{\delta_j\}_{j \in J}\). Then we can find a collection \(\mathcal{T}\) of pairwise disjoint cubes in \(X\) (of possibly different scales) such that the following conditions hold:

(i) \(\mu(X \setminus \bigcup_{T \in \mathcal{T}} T) = 0\).

(ii) There is a map \(j: \mathcal{T} \to J\) such that
\[
\mu(U_{j(T)} \cap T) \geq (1 - \gamma) \mu(T)
\] (2.17)

and
\[
\text{diam } T < \delta_{j(T)}
\] (2.18)

for each \(T \in \mathcal{T}\).

*Proof.* Let us call a cube \(T \in \Delta_k\) “good for \(j\)” if it satisfies (2.17) and (2.18) with \(j(T) = j\), and let us call \(T\) “good” if it is good for some \(j \in J\). Finally, let us call \(T\) “bad” if it is not good. Write \(\Delta_k^S\) for the sub-collection of \(\Delta_k\) consisting of good cubes.

We then define our collection of cubes \(\mathcal{T}\) to be
\[
\mathcal{T} = \bigcup_{k = 1}^{\infty} \{T \in \Delta_k^S: \text{for every } 1 \leq k' < k \text{ and every } Q \in \Delta_{k'} \text{ containing } T, Q \text{ is bad}\}.
\]
In other words, our collection consists of all cubes that are the first good cube among all their ancestors of scales below $1$. Note that any two distinct cubes in $\mathcal{T}$ are disjoint: if not, then one would contain the other, forcing the larger one to be bad.

For each cube $T$ in this collection $\mathcal{T}$, define $j(T)$ to be a choice of $j \in J$ such that $T$ is good for $j$. The collection $\mathcal{T}$ and the map $j: \mathcal{T} \to J$ then automatically satisfy condition (2.16) of the Lemma.

To verify (2.16), we show that almost every point $x \in X$ is contained in one of the cubes $T \in \mathcal{T}$. Let

$$Z = X \setminus \bigcup_{k \in \mathbb{Z}} \bigcup_{\ell \in I_k} Q^k$$

so that $\mu(Z) = 0$ by Proposition 2.15 (2.15).

Let $x \in X \setminus Z$ be a point of $\mu$-density of some $U_{j_0}$. We claim that $x \in T$ for some $T \in \mathcal{T}$. Suppose, to the contrary, that $x \not\in T$ for any $T \in \mathcal{T}$. Then $x$ lies in an infinite nested sequence of bad cubes. But this is impossible: if an infinite nested sequence of cubes satisfied $Q_1 \supset Q_2 \supset \cdots \supset x$, then eventually some $Q_i$ would be good for $j_0$, and the first such good cube would be in $\mathcal{T}$.

So $\bigcup_{T \in \mathcal{T}} T$ contains almost every point in $\left( \bigcup_{j \in J} U_j \right) \cap (X \setminus Z)$, which is almost every point of $X$. \hfill $\Box$

The following lemma will ensure that we obtain a Lipschitz function in our construction. Recall the definition of quasiconvexity from Proposition 2.4.

**Lemma 2.19.** Let $X$ be a complete and quasiconvex metric space and let $u: X \to \mathbb{R}$ be a function on $X$. Suppose that there are pairwise disjoint open sets $A_i \subset X$ ($i \in I$), and a constant $L \geq 0$ such that

$$\text{LIP}(u|_{A_i}) \leq L \text{ for each } i \in I$$

and

$$u = 0 \text{ on } B = X \setminus \bigcup_{i \in I} A_i.$$  

Then $u$ is $2CL$-Lipschitz on $X$, where $C$ is the quasiconvexity constant of $X$.

**Proof.** Without loss of generality, we may assume that $A_i \neq X$ for each $i \in I$, otherwise the lemma is trivial.

Fix points $x, y \in X$. We will show that

$$|u(x) - u(y)| \leq 2CLd(x, y),$$

(2.22)

where $C$ is the quasiconvexity constant of $X$.

Using the quasiconvexity of $X$, choose a rectifiable path $\gamma: [0, 1] \to X$ of length at most $Cd(x, y)$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Case 1: Suppose that, for some $i, j \in I$, we have $x \in A_i$ and $y \in A_j$. In this case, we may also suppose that $i \neq j$, otherwise (2.22) follows from the assumption (2.20). Let $t_0 = \inf \{ t: \gamma(t) \notin A_i \}$ and $t_1 = \sup \{ t: \gamma(t) \notin A_j \}$. By basic topology, $\gamma(t_0) \in \partial A_i \subset B$ and $\gamma(t_1) \in \partial A_j \subset B$. Thus, we have

$$|u(x) - u(y)| \leq |u(x) - u(\gamma(t_0))| + |u(\gamma(t_0)) - u(\gamma(t_1))| + |u(\gamma(t_1)) - u(y)|$$

$$\leq Ld(x, \gamma(t_0)) + 0 + Ld(y, \gamma(t_1))$$

$$\leq (2L)\text{length}(\gamma)$$

$$\leq 2CLd(x, y).$$

Case 2: Suppose that $x \in A_i$ for some $i \in I$ and that $y \in B$ (or vice versa). We then have that $u(y) = 0$ and

$$d(x, y) \geq \frac{1}{C}\text{length}(\gamma) \geq \frac{1}{C}\text{dist}(x, \partial A_i).$$

Thus,

$$|u(x) - u(y)| = |u(x)| \leq L\text{dist}(x, \partial A_i) \leq CLd(x, y).$$

Case 3: Suppose that $x \in B$ and $y \in B$. Then $u(x) = u(y) = 0$. \hfill $\Box$
Note that Lemma 2.19 is false without assumption (2.21), as the Cantor staircase function shows.

The following lemma is due to Francos ([10], Lemma 2.3). Although Francos stated it only for subsets of $\mathbb{R}^n$, the proof works equally well in our setting.

**Lemma 2.23.** Let $X$ be a locally compact metric space equipped with a locally finite Borel regular measure $\mu$. Let $f$ be a Borel function from an open set $\Omega \subset X$, with $\mu(\Omega) < \infty$, into some $\mathbb{R}^N$. Then, for any $\epsilon > 0$, there is a compact set $K \subset \Omega$ and a continuous function $g$ on $\Omega$ such that

- $\mu(\Omega \setminus K) < \epsilon$,
- $f = g$ on $K$,
- $\int_{\Omega} |g|^p \leq 2 \int_{\Omega} |f|^p$ for all $p \in [1, \infty)$, and
- $\|g\|_\infty \leq 2\|f\|_\infty$.

**3 Proof of Theorem 1.3**

Our main lemma is the analog of Lemma 7 of [1]:

**Lemma 3.1.** Let $(X, d, \mu)$ be a PL space and let $\{(U_j, \phi_j : X \to \mathbb{R}^k)\}_{j \in J}$ be a normalized measurable differentiable structure on $X$. Suppose that $\Omega \subset X$ is open with $\mu(\Omega) < \infty$ and $\Omega \neq X$, and that $\{f_j : \Omega \to \mathbb{R}^k\}$ is a uniformly bounded collection of continuous functions, i.e., that $\sup_{j \in J} \|f_j\|_\infty < \infty$. Fix $\alpha, \epsilon > 0$.

Then there exists a compact set $K \subset \Omega$ and a Lipschitz function $u \in C_c(\Omega)$ such that the following conditions hold:

$$\mu(\Omega \setminus K) \leq \epsilon \mu(\Omega).$$

$$\|f_j - d^j u\|_p \leq \alpha \ a.e. \ on \ U_j \cap K.$$  

$$\|\text{Lip}_u\|_p \leq C' \epsilon^{\frac{1}{p}} \left( \sum_{j \in J} \int_{U_j \cap \Omega} |f_j|^p \right)^{1/p} \text{ for all } p \in [1, \infty).}$$

$$\|\text{Lip}_u\|_\infty \leq C' \epsilon \sup_{j \in J} \|f_j\|_\infty.$$

The constants $\eta, C' > 0$ depend only on the data of $X$.

**Proof.** Without loss of generality, we assume that $\epsilon < 1$.

Fix a compact set $K' \subset \Omega$ such that $\mu(\Omega \setminus K') < \frac{\epsilon}{2} \mu(\Omega)$. For each $j \in J$, choose $\delta_j > 0$ small enough such that

if $|x - y| < \delta_j$ and $x \in K'$, then $|f_j(x) - f_j(y)| < \alpha/2$,

and

$$\delta_j < \text{dist}(K', X \setminus \Omega).$$

Using Lemma 2.16, we find a collection $J$ of pairwise disjoint cubes covering almost all of $X$, and a map $j : J \to J$ such that

$$\mu(U_{j(\mathcal{T})} \cap T) \geq \left(1 - \frac{\epsilon}{4}\right) \mu(T),$$

and

$$\text{diam} \ (T) < \delta_{j(T)}.$$

for each $T \in J$.

Consider the sub-collection consisting of all cubes $T \in J$ such that $T \cap K' \neq \emptyset$. Index these cubes $\{T_i\}_{i \in I}$, and write $j(i)$ for $j(T_i)$. By (3.7) and (3.9), each cube $T_i \ (i \in I)$ lies in $\Omega$.

For each $i \in I$, define $S_i \subset T_i$ as

$$S_i = \{x \in T_i : \text{dist}(x, X \setminus T_i) \geq tc^j\},$$

where $c_j$ is the constant from (3.9).
where \( k \) is such that \( T \subset A_k \) and

\[
t = (\varepsilon/4C_1)^{1/\eta}
\]

is fixed. This value of \( t \) was chosen to ensure (by Proposition 2.15 (2.15)) that

\[
\mu(T_i \setminus S_i) \leq C_1 t^q \mu(T_i) = \frac{\varepsilon}{4} \mu(T_i).
\]

Note that \( S_i \) is a compact subset of the open set \( T_i \). Let \( z_i \) be a “center” of \( T_i \) as in Proposition 2.15 (2.15), so that \( T_i \) both contains and is contained in a ball centered at \( z_i \) of radius approximately \( \text{diam}(S_i) \).

For each cube \( T_i \) in our collection, define \( a_i \in \mathbb{R}^{k,0} \) by

\[
a_i = \mu(U_{j_0} \cap T_i)^{-1} \int_{U_{j_0} \cap T_i} f_{j_0}. \]

Note that the collection \( \{|a_i| \in I \} \) is bounded, because the collection \( \{f_j\} \in J \) is uniformly bounded.

Let \( \psi_i : X \to \mathbb{R} \) be a Lipschitz function such that \( \psi_i = 1 \) on \( S_i \), \( \psi_i = 0 \) off \( T_i \), and \( \text{Lip}(\psi_i) \leq Ct\eta_k^{-1} \leq C(\text{diam}(T_i)^{-1}) \varepsilon^{-1/\eta} \). (Here \( C \) is some constant depending only on the data of \( X \).) By slightly widening the regions where \( \psi_i \) is constant, we can also easily arrange that \( \text{Lip}(\psi_i) = 0 \) everywhere in \( S_i \) and in \( X \setminus T_i \).

Define \( u : X \to \mathbb{R} \) by

\[
u(x) = \sum_{j \in J} \psi_i(x) \langle a_i, \phi_{j_0}(x) - \phi_{j_0}(z_i) \rangle.
\]

A simple calculation shows that, for each \( i \in I \),

\[
\text{Lip}(\nu_{T_i}) \leq C e^{-1/\eta} |a_i| \leq C e^{-1/\eta} \sup_i |a_i| < \infty.
\]

(Here we used the assumption that the measurable differentiable structure is normalized, and therefore \( \text{Lip}(\phi_j) \leq 1 \) for each \( j \in J \). Since \( \nu_{T_i} = \psi_i(x) \langle a_i, \phi_{j_0}(x) - \phi_{j_0}(z_i) \rangle \) is continuous on \( T_i \), it follows that)

\[
\text{Lip}(\nu_{T_i}) \leq C e^{-1/\eta} \sup_i |a_i| < \infty.
\]

Thus, as \( u = 0 \) outside \( \bigcup_{i \in J} T_i \), we see that \( u \) is Lipschitz on \( X \) by Lemma 2.19. In addition, \( u \in C_c(\Omega) \), with \( \text{supp } u \subset \bigcup_{i \in J} T_i \subset \Omega \), and \( d^j u = a_i \) a.e. on \( S_i \cap U_{j_0} \).

Let \( K_1 = \bigcup_{i \in I}(S_i \cap U_{j_0}) \), and let \( K \) be a compact subset of \( K_1 \) such that

\[
\mu(K_1 \setminus K) \leq \frac{\varepsilon}{4} \mu(\Omega).
\]

To verify (3.2), note that

\[
\mu(T_i \setminus (S_i \cap U_{j_0})) \leq \mu(T_i \setminus S_i) + \mu(T_i \setminus U_{j_0}) \leq \frac{\varepsilon}{4} \mu(T_i) + \frac{\varepsilon}{4} \mu(T_i) \leq \frac{3\varepsilon}{4} \mu(T_i)
\]

for each \( i \in I \). Therefore,

\[
\mu(\Omega \setminus K_1) \leq \mu(\Omega \setminus K') + \sum_i \mu(T_i \setminus (S_i \cap U_{j_0})) \leq \frac{3\varepsilon}{4} \mu(\Omega),
\]

and so

\[
\mu(\Omega \setminus K) \leq \mu(\Omega \setminus K_1) + \mu(K_1 \setminus K) \leq \varepsilon \mu(\Omega).
\]

Let us now verify (3.3). Suppose that \( x \in U_j \cap K \) for some \( j \in J \). Then \( x \in S_i \cap U_j \) for some \( i \in I \) such that \( j(i) = j \). Therefore, by (3.6), \( |f_j(x) - a_i| < \alpha \).

So if \( j \) is such that \( j(i) = j \), then \( d^j u = a_i \) almost everywhere in \( U_j \cap S_i \), we see that

\[
|f_j - d^j u| < \alpha
\]

almost everywhere in \( U_j \cap S_i \). This verifies (3.3), as \( K \subset \bigcup_{i}(S_i \cap U_{j_0}) \).
Finally, we must check (3.4) and (3.5). Observe that if \( T \) is a cube in \( \mathcal{T} \) and \( x \in T \), then the sum (3.10) defining \( u \) consists of at most one non-zero term. Therefore, for such \( x \), we have by (2.2) that

\[
\text{Lip}_x(u) \leq \sup_{i \in I} \left( \text{Lip}_{\phi_i}(x) \right) |a_i| \left( \phi_{j(0)}(x) \right) + |a_i| |\psi_i(0)|.
\]

Because almost every \( x \in X \) is contained in some \( T \in \mathcal{T} \), we have the bound (3.11) for almost every \( x \in \Omega \).

Recalling our normalization that \( \text{Lip}(\phi_j) \leq 1 \) for all \( j \in J \), we see from (3.11) that, for all \( 1 \leq p < \infty \),

\[
\|\text{Lip}_x\|_p \leq \left( \sum_{i \in I} \left( \text{Lip}_{\phi_i} \| \| a_i \| \| (\text{diam } T_i) \| \| \mu(T_i \setminus S_i) \| \right) \right)^{1/p} + \left( \sum_{i \in I} |a_i|^p \mu(T_i) \right)^{1/p}
\]

\[
\leq \left( \sum_{i \in I} \left( \frac{C e^{-\frac{2}{p}}}{e^{-\frac{2}{p}} + 1} \right) \left( \sum_{i \in I} |a_i|^p \mu(T_i) \right) \right)^{1/p} + \left( \sum_{i \in I} |a_i|^p \mu(T_i) \right)^{1/p}
\]

\[
\leq \left( \frac{C e^{-\frac{2}{p}}}{e^{-\frac{2}{p}} + 1} \right) \left( \sum_{i \in I} \frac{\mu(T_i)}{\mu(T_i \cap U_j)} \int_{T_i \setminus U_j} |f_{j(0)}|^p \right)^{1/p} + \left( \sum_{i \in I} |a_i|^p \mu(T_i) \right)^{1/p}
\]

\[
\leq 2 \left( \frac{C e^{-\frac{2}{p}}}{e^{-\frac{2}{p}} + 1} \right) \left( \sum_{j \in J} \int_{\Omega \setminus U_j} |f_j|^p \right)^{1/p}
\]

Note that in the last inequality we used (3.8).

The case \( p = \infty \), namely (3.5), follows from this by a limiting argument, or can alternatively be derived the same way. This completes the proof of Lemma 3.1. \( \square \)

Proof of Theorem 1.3. By Lemma 2.13, we may assume that the measurable differentiable structure is normalized.

It will also be convenient to assume that \( \Omega \) is a proper subset of \( X \), i.e., that \( \Omega \neq X \). We may assume this without loss of generality: If \( \Omega = X \), we replace \( \Omega \) by \( \Omega' = X \setminus \{x_0\} \) for some arbitrary \( x_0 \in X \). Proving Theorem 1.3 for \( \Omega' \) also proves it for \( \Omega \).

Finally, we may also assume that \( \epsilon < 1 \) and that \( \sup_{j \in J} \|f_j\|_\infty > 0 \). We also extend each \( f_j \) from \( U_j \cap \Omega \) to all of \( \Omega \) by setting \( f_j \) = \( 0 \) off of \( U_j \). The proof now proceeds in two steps.

**Step 1:** Assume that the functions \( f_j \) are uniformly bounded, i.e., that \( \sup_{j \in J} \|f_j\|_\infty < \infty \).

Let \( \{\alpha_n\} \) be a decreasing sequence of positive real numbers with \( \alpha_1 \leq \sup_{j \in J} \|f_j\|_\infty \), to be chosen later. For each integer \( n \geq 0 \), we will inductively build:

- a Lipschitz function \( u_n \in C_c(\Omega, \mathbb{R}) \),
- a compact set \( K_n \subset \Omega \), and
- a collection of continuous functions \( \{f_{n,j} : \Omega \to \mathbb{R}^k\} \) such that \( f_{n,j} = f_j \) on \( K_{0,j} \),

Let \( u_0 = 0 \). For each \( j \in J \), we apply Lemma 2.23, to find a compact set \( K_{0,j} \subset \Omega \) with \( \mu(\Omega \setminus K_{0,j}) \leq 2^{-1}2^{-1}\epsilon \mu(\Omega) \), and a continuous function \( f_{0,j} \) on \( \Omega \) such that \( f_{j} = f_j \) on \( K_{0,j} \),

\[
\int_{\Omega} |f_j|^p \leq \int_{\Omega \setminus U_j} |f_j|^p = 2 \int_{\Omega \setminus U_j} |f_j|^p
\]

for all \( 1 \leq p < \infty \), and

\[
\sup_{j \in J} \|f_j\|_\infty \leq 2 \|f_j\|_\infty.
\]

Let \( K_0 = \bigcap_{j \in J} K_{0,j} \). This completes stage \( n = 0 \) of the construction.
Suppose now that \( u_{n-1}, K_{n-1}, \{f^n_{j}\}_{j \in J} \) have been constructed. Apply Lemma 3.1 to get a compact set \( \tilde{K}_n \subset \Omega \) and a Lipschitz function \( u_n \in C_c(\Omega) \) such that
\[
\mu(\Omega \setminus \tilde{K}_n) \leq 2^{-(n+2)}e\mu(\Omega),
\]
\[
|f^n_{j}(x) - (d^i u_n)(x)| \leq \alpha_n/2
\]
for every \( j \in J \) and almost every \( x \in U_j \cap \tilde{K}_n \),
\[
\|\text{Lip}_{u_n}\|_p \leq C' \left(2^{-(n+2)}e^{p/2} \left( \sum_j \int_{U_j \cap \Omega} |f^n_{j}|^p \right)^{1/p} \right)
\]
for every \( p \in [1, \infty) \), and
\[
\|\text{Lip}_{u_n}\|_{\infty} \leq C'(2^{-(n+2)}e^{p/2})^{\frac{1}{p}} \sup_{j \in J} \|f^n_{j}\|_{\infty}.
\]
Given \( j \in J \), define \( \tilde{f}^n_{j}(x) : \Omega \rightarrow \mathbb{R}^k \) by
\[
\tilde{f}^n_{j}(x) = \begin{cases} 
  f^n_{j}(x) - d^i u_n(x) & \text{if } x \in U_j \cap \tilde{K}_n \\
  0 & \text{otherwise.}
\end{cases}
\]
(3.14)

For each \( j \in J \), apply Lemma 2.23 to find a compact set \( K_{n,j} \subset \Omega \) and a continuous map \( f^n_{j} : \Omega \rightarrow \mathbb{R}^k \) such that
\[
\mu(\Omega \setminus K_{n,j}) \leq 2^{-(n+2)}e2^{-j}\mu(\Omega),
\]
\[
f^n_{j} = \tilde{f}^n_{j} \text{ on } K_{n,j}, \text{ and}
\]
\[
\|f^n_{j}\|_{\infty} \leq 2\|\tilde{f}^n_{j}\|_{\infty} \leq \alpha_n.
\]
(3.16)

Let \( K_n = \tilde{K}_n \cap \left( \bigcap_{j \in J} K_{n,j} \right) \), so that
\[
\mu(\Omega \setminus K_n) \leq \mu(\tilde{K}_n) + \sum_{j \in J} \mu(\Omega \setminus K_{n,j}) \leq 2^{-(n+1)}e\mu(\Omega).
\]

This completes stage \( n \) of the inductive construction.
Now let
\[
A = \Omega \setminus \bigcap_{n=0}^{\infty} K_n = \bigcup_{n=0}^{\infty} \left( \Omega \setminus K_n \right)
\]
and
\[
u = \sum_{n=0}^{\infty} u_n.
\]
Note that
\[
\mu(A) \leq \sum_{n=0}^{\infty} \mu(\Omega \setminus K_n) \leq e\mu(\Omega),
\]
so (1.4) holds.

Purely for notational convenience, we now define a real-valued function
\[
F = \sum_{j \in J} \chi_{U_j} |f_j|,
\]
so that
\[
\|F\|_p = \left( \sum_j \int_{U_j \cap \Omega} |f_j|^p \right)^{1/p}
\]
for every $p \in [1, \infty)$ and

$$||F||_\infty = \sup_{j \in J} ||f_j||_\infty.$$  

Note that, if $p \in [1, \infty)$, we have, by (3.12) and (3.13), that

$$\left( \sum_{j \in J} \int_{U_j \cap \Omega} |f_j|^p \right)^{1/p} \leq 2 ||F||_p \text{ and } \sup_{j \in J} ||f_j||_\infty \leq 2 ||F||_\infty$$

In addition, $||F||_p$ is non-zero and finite for every $p \in [1, \infty)$, by our assumption that $0 < \sup_{j \in J} ||f_j||_\infty < \infty$.

We now calculate that, for $p \in [1, \infty)$,

$$\sum_{n=1}^{\infty} ||\text{Lip}_{u_n}||_p \leq \sum_{n=1}^{\infty} C' 2^{\frac{n+1}{p}} e^{\frac{1}{\lambda} \frac{1}{p}} \left( \sum_{j \in J} \int_{U_j \cap \Omega} |f_j|^{p-1} \right)^{1/p}$$

$$\leq 2C' e^{\frac{1}{\lambda} \frac{1}{p}} \left( ||F||_p + \sum_{n=1}^{\infty} 2^{\frac{n+1}{p}} \left( \sum_{j \in J} \int_{U_j \cap \Omega} |f_j|^p \right)^{1/p} \right)$$

$$\leq 2C' e^{\frac{1}{\lambda} \frac{1}{p}} \left( ||F||_p + \sum_{n=1}^{\infty} 2^{\frac{n+1}{p}} \left( \sum_{j \in J} ||f_j||_{L_p(U_j \cap \Omega)} \right)^{1/p} \right)$$

$$\leq 2C' e^{\frac{1}{\lambda} \frac{1}{p}} ||F||_p \left( 1 + \frac{\mu(\Omega)^{1/p}}{||F||_p} \sum_{n=1}^{\infty} 2^{\frac{n+1}{p}} a_n \right).$$

(3.18)

A similar calculation shows that (3.18) also holds if $p = \infty$.

The function $p \mapsto \frac{\mu(\Omega)^{1/p}}{||F||_p}$ is continuous for $p \in [1, \infty)$, and therefore has an upper bound $M > 0$. Choose our sequence $\{a_n\}$ to satisfy

$$\sum_{n=1}^{\infty} 2^{\frac{n+1}{p}} a_n \leq 1/M.$$  

Then the calculation (3.18) yields that

$$\sum_{n=1}^{\infty} ||\text{Lip}_{u_n}||_p \leq 4C' e^{\frac{1}{\lambda} \frac{1}{p}} ||F||_p < \infty$$

(3.19)

for any $p \in [1, \infty]$.

Proposition 2.4 says that, for each $n$,

$$\text{LIP}(u_n) \leq C ||\text{Lip}_{u_n}||_\infty,$$

and therefore (3.19) implies that

$$\sum_{n=1}^{\infty} \text{LIP}(u_n) < \infty.$$  

(3.20)

This, combined with the fact that each $u_n$ has compact support in $\Omega \neq X$, implies that the sum

$$u = \sum_{n=1}^{\infty} u_n$$

converges uniformly on compact sets to a Lipschitz function $u \in C_0(\Omega)$. It follows from (3.20), (3.19), and (2.1) that $u$ satisfies conditions (1.6) and (1.7) of Theorem 1.3.
To verify (1.5), fix \( j \in J \) and observe that, by (3.14) and (3.16),
\[
f_j - \sum_{n=1}^{m} (d^i u_n) = f_j^m,
\]
almost everywhere in \( U_j \cap (\Omega \setminus A) \), for each positive integer \( m \). Thus,
\[
\|f_j - d^i u\|_{L^\infty(U_j \cap (\Omega \setminus A))} \leq \|f_j^m - d^i u\|_{L^\infty(U_j \cap (\Omega \setminus A))} + \sum_{n=m+1}^{\infty} \|d^i u_n\|_{L^\infty(U_j \cap (\Omega \setminus A))}
\]
\[
\leq \alpha_m + \sum_{n=m+1}^{\infty} \|d^i u_n\|_{L^\infty(U_j \cap (\Omega \setminus A))}
\]
\[
\leq \alpha_m + c_j \sum_{n=m+1}^{\infty} \|\operatorname{Lip}_u\|_{L^\infty(U_j \cap (\Omega \setminus A))}
\]
and both of these tend to zero as \( m \) tends to infinity. In the last inequality, we used the fact that \((U_j, \phi_j)\) is a normalized chart, see Definition 2.8.

Thus,
\[
f_j = d^i u \text{ a.e. on } U_j \cap (\Omega \setminus A),
\]
so (1.5) holds. This completes Step 1.

**Step 2:** The functions \( \{f_j : U_j \cap \Omega \to \mathbb{R}^k\} \) are arbitrary Borel functions.

We first extend each \( f_j \) to be zero off of \( U_j \), so that each \( f_j \) is defined on all of \( \Omega \).

Fix \( \epsilon > 0 \). Choose \( r > 0 \) large so that
\[
B = \{x : |f_j(x)| > r \text{ for some } j \in J\}
\]
satisfies \( \mu(B) < \epsilon/2 \). Note that this is possible because, using the fact that \( f_j = 0 \text{ off } U_j \), we see that
\[
\mu\left(\Omega \setminus \bigcup_{\ell=1}^{\infty} \{x : |f_j(x)| \leq \ell \text{ for all } j \in J\}\right) = 0.
\]

For each \( j \in J \) let
\[
\tilde{f}_j(x) = \begin{cases} f_j(x) & \text{if } |f_j(x)| \leq r \\ r & \text{if } |f_j(x)| > r. \end{cases}
\]

Then \( \{\tilde{f}_j\} \) is a uniformly bounded collection of Borel functions on \( \Omega \) such that, for all \( j \in J \), \( |\tilde{f}_j| \leq |f_j| \) everywhere and \( \tilde{f}_j = f_j \) outside the set \( B \). Fix an open set \( A_1 \supset B \) such that \( \mu(A_1) < \epsilon/2 \). Then, for all \( j \in J \), \( \tilde{f}_j = f_j \) outside of \( A_1 \).

Now apply the result of Step 1 to the uniformly bounded collection \( \{\tilde{f}_j\} \). We obtain an open set \( A_2 \) with \( \mu(A_2) \leq \frac{\epsilon}{2} \mu(\Omega) \) and a Lipschitz function \( u \in C_0(\Omega) \) such that
\[
d^i u = \tilde{f}_j \text{ a.e. in } U_j \cap (\Omega \setminus A_2),
\]
\[
\|\operatorname{Lip}_u\|_p \leq 4C'((\epsilon/2)^{\frac{1}{p}} - \frac{1}{q}) \left( \sum_{j \in J} \int_{U_j \setminus A} |\tilde{f}_j|^q \right)^{1/p}
\]
for all \( p \in [1, \infty] \), and
\[
\|\operatorname{Lip}_u\|_\infty \leq 4C'((\epsilon/2)^{\frac{1}{p}} - \frac{1}{q}) \sup_{j \in J} |\tilde{f}_j|_\infty.
\]

Thus, for each \( j \in J \), \( f_j = d^i u \text{ a.e. in } U_j \cap (\Omega \setminus A) \), where \( A = A_1 \cup A_2 \) has \( \mu(A) \leq \epsilon \mu(\Omega) \). This verifies (1.4) and (1.5).

If \( p \in [1, \infty] \), we have
\[
\|\operatorname{Lip}_u\|_p \leq 4C'((\epsilon/2)^{\frac{1}{p}} - \frac{1}{q}) \left( \sum_{j \in J} \int_{U_j \setminus A} |\tilde{f}_j|^q \right)^{1/p} \leq 4C'2^{-1/2} e^{\frac{1}{p}} \left( \sum_{j \in J} \int_{U_j \setminus A} |f_j|^p \right)^{1/p},
\]
which verifies (1.6). A similar calculation verifies (1.7).

This completes the proof of Theorem 1.3. \( \square \)
4 Proof of Theorem 1.8

In this section, we give the proof of Theorem 1.8. Given our Theorem 1.3, we can now just closely follow the proof given by Moonens-Pfeffer in [20], with some modifications as in [11] to obtain (1.12). For the convenience of the reader, we give most of the details.

In our setting, the analog of Corollary 1.2 in [20] (with the additional constraints needed to prove (1.12)) is the following:

**Lemma 4.1.** Let $X$ be a PI space with a normalized differentiable structure $(U_j, \phi_j : U_j \rightarrow \mathbb{R}^k)$. Let $\Omega \subset X$ be a bounded open subset of $X$ and let $\{f_j : U_j \cap \Omega \rightarrow \mathbb{R}^k\}$ be a collection of Borel functions. Let $\epsilon, \sigma > 0$ and let $
u : [0, \infty) \rightarrow [0, \infty)$ be a continuous function with $\nu(0) = 0$ and $\nu(t) = O(t)$ as $t \rightarrow \infty$.

Then there exists a compact set $K \subset U$ and a Lipschitz function $u \in C_c(\Omega)$ such that

\[
\mu(\Omega \setminus K) < \epsilon, \quad (4.2)
\]
\[
d^j u = f_j \text{ a.e. in } U_j \cap K \quad (4.3)
\]

for each $j \in J$,

\[
|u(x)| \leq \epsilon \min\{1, \text{dist}^2(x, X \setminus \Omega)\} \quad (4.4)
\]

for all $x \in X$, and

\[
|u(x) - u(y)| \leq \sigma \frac{|x - y|}{\nu(|x - y|)} \quad (4.5)
\]

for all $x, y \in X$.

**Proof.** We can again assume without loss of generality that $\Omega \neq X$, otherwise we replace $\Omega = X$ by $X \setminus \{x_0\}$ for some $x_0 \in X$. Extend the functions $f_j$ to all of $X$ by letting $f_j = 0$ off of $U_j \cap \Omega$.

Fix an open set $\Omega'$ compactly contained in $\Omega$ with $\mu(\Omega \setminus \Omega') < \epsilon/2$. Let $\epsilon' = \epsilon/8\mu(\Omega')$.

As in Step 2 in the proof of Theorem 1.3, we can find a compact set $B \subset \Omega'$ such that $\mu(\Omega' \setminus B) < \epsilon/4$ and $\{f_j\}$ are uniformly bounded on $B$, i.e., $\sup_{j \in J} \|f_j\|_{L^\infty(B)} = M < \infty$.

Since $\nu$ is continuous, we can find $\delta \in (0, 1)$ such that

\[
\nu(t) \leq \frac{\sigma}{C(\epsilon')^{-1/\eta} M} \text{ for all } t \in [0, \delta],
\]

where $C$ is a constant depending only on the data of $X$, to be named below. We also set

\[
h = \min \left\{ \frac{\sigma}{2\epsilon \sup_{t \in [0, \delta]} \nu(t)/t}, 1 \right\},
\]

which is positive and finite, as $\nu(t) = O(t)$ as $t \rightarrow \infty$.

For each $j \in J$, let $g_j = f_j|_{\bar{B}}$, so the functions $g_j$ are uniformly bounded by the constant $M > 0$. Let

\[
\Delta = h \min\{1, \text{dist}^2(\Omega', X \setminus \Omega)\}
\]

and

\[
d = \epsilon^{1+\frac{2}{\eta}}(1 + (8\mu(\Omega'))^{\frac{1}{2}} CM),
\]

where $C$ and $\eta$ are the constants from Theorem 1.3.

Choose $k$ large so that there are cubes $Q_1, \ldots, Q_m \subset \Omega'$ in $\Lambda_k$, of diameter at most $d$, that satisfy

\[
\mu(\Omega' \setminus \bigcup_{i=1}^m Q_i) < \epsilon/4.
\]

(Note that the doubling property of $\mu$ and the boundedness of $\Omega$ implies that the collection $\{Q_1, \ldots, Q_m\}$ really is finite.)
For each $1 \leq i \leq m$, we now apply Theorem 1.3 to the collection \( \{ g_j \} \) in the cube \( Q_i \) with parameter \( \epsilon' \). For each $1 \leq i \leq m$, we obtain a compact set \( K_i \subset Q_i \) with \( \mu(Q_i \setminus K_i) \leq \epsilon' \mu(Q_i) \) and a Lipschitz function \( u_i \in \mathcal{C}(Q_i) \) such that, for each \( j \in J_1, d^1 u_j = g_j \) almost everywhere in \( U_{ij} \cap K_i \).

Furthermore, (1.7) (applied to \( u_i \)) and Proposition 2.4 (b) together imply that

\[
\text{Lip} u_i \leq C (\epsilon')^{-\frac{1}{2}} M. \tag{4.8}
\]

As \( u_i \in \mathcal{C}(Q_i) \), it follows that, for each $1 \leq i \leq m$, \( \| u_i \|_{\infty} \leq \text{diam } Q_i \text{Lip} (u_i) \leq d C (\epsilon')^{-\frac{1}{2}} M < \epsilon \Delta. \tag{4.9} \)

Let \( K = B \cap (\bigcup_{i=1}^{m} K_i) \), a compact subset of \( \Omega \). Our choices easily imply that

\[
\mu(\Omega \setminus K) < \epsilon,
\]

which verifies (4.2).

Let \( u = \sum_{i=1}^{m} u_i \). Then \( u \) is a Lipschitz function in \( \mathcal{C}(\Omega) \) that satisfies

\[
d^1 u = f_j \text{ almost everywhere in } U_{ij} \cap K,
\]

so (4.3) holds.

To verify (4.4), note that \( u \) is identically zero outside of \( \Omega' \), so (4.4) holds there automatically. For \( x \in \Omega' \), we have

\[
|u(x)| \leq \sup_i \| u_i \|_{\infty} < \epsilon \Delta \leq \epsilon \min \{ 1, \text{dist}^2 (x, X \setminus \Omega) \}.
\]

Thus, the condition (4.4) of Lemma 4.1 is verified.

Finally, we verify (4.5). First observe that (4.8), (4.9), and the fact that the functions \( u_i \) have disjoint supports imply that

\[
\text{Lip} u \leq C (\epsilon')^{-\frac{1}{2}} M \text{ and } \| u \|_{\infty} < \epsilon \Delta,
\]

where \( C \) depends only on the data of \( X \). (For this one can reuse 2.4 (b).)

Therefore, if \( x, y \in X \) have \( |x - y| < \delta \), we have by (4.6) that

\[
|u(x) - u(y)| \leq (\text{Lip} u) |x - y| \leq C (\epsilon')^{-\frac{1}{2}} M |x - y| \leq \sigma \frac{|x - y|}{v(x - y)},
\]

and if \( |x - y| \geq \delta \) we have by (4.7) that

\[
|u(x) - u(y)| \leq 2 \| u \|_{\infty} \leq 2 \epsilon \Delta \leq 2 \epsilon h \leq \sigma \frac{|x - y|}{v(x - y)}.
\]

This completes the proof of (4.5) and therefore of Lemma 4.1.

We now prove Theorem 1.8. (To avoid some cumbersome subscripts, we change notation slightly and write \( \text{Lip}(g)(x) \) instead of \( \text{Lip}_g(x) \).)

**Proof of Theorem 1.8.** We again closely follow [20], with some added ideas from [11].

By Lemma 2.13, we may assume that the measurable differentiable structure is normalized. Without loss of generality, we also assume that \( \epsilon < 1 \). Fix \( x_0 \in X \) and let \( B_i = B(x_0, i) \) for each \( i \in \mathbb{N} \).

We repeatedly apply Lemma 4.1. We inductively construct compact sets \( K_i \subset \Omega_i = \Omega \cap B_i \setminus \bigcup_{k=1}^{i-1} K_k \) and Lipschitz functions \( u_i \in \mathcal{C}(\Omega_i) \) such that, for each \( i \in \mathbb{N} \)

\[
\mu(\Omega \setminus K_i) < 2^{-i} \epsilon < 2^{-i}, \tag{4.10}
\]

\[
d^i u_i = f_j - \sum_{k=1}^{i-1} d^k u_i \text{ a.e. in } K_i, \tag{4.11}
\]

\[
|u_i(x)| \leq 2^{-i} \epsilon \min \{ 1, \text{dist}^2 (x, X \setminus \Omega_i) \}, \tag{4.12}
\]
for all $x \in X$, and
\[
|u_i(x) - u_i(y)| \leq 2^{-i} \frac{|x - y|}{v(|x - y|)}
\] (4.13)

for all $x, y \in X$.

Let $K = \bigcup_{i=1}^{\infty} K_i$ and let $u = \sum_{i=1}^{\infty} u_i$. Note that $u$ is a continuous function, because the bound $\|u_i\|_{\infty} \leq 2^{-i} \epsilon$ from (4.12) implies the uniform convergence of this sum. It also implies that $\|u\|_{\infty} \leq \epsilon$, verifying the first part of (1.9).

The second part of (1.9) also follows immediately, by observing that
\[
\{ u \neq 0 \} \subseteq \bigcup_{i=1}^{\infty} \{ u_i \neq 0 \} \subseteq \bigcup_{i=1}^{\infty} \Omega_i \subseteq \Omega.
\]

In addition, $\mu((\Omega \cap B_i) \setminus K) \leq \mu(\Omega \setminus K) \leq 2^{-k}$ whenever $k \geq i$, which implies that $\mu(\Omega \cap B_i \setminus K) = 0$ for each $i \in \mathbb{N}$ and thus that $\mu(\Omega \setminus K) = 0$.

Next, we verify (1.10).

We first claim that if $x \in K_i$ and $k > i$, then
\[
\text{Lip} \left( \sum_{k=i+1}^{\infty} u_k \right)(x) = 0.
\] (4.14)

Indeed, note that for $k > i$ and $x \in K_i$, we have $K_i \cap \Omega_k = \emptyset$ and so $u_k(x) = 0$. Fix any $y \in X$. If $y \notin \Omega_k$, then $u(y) = 0$ as well. If $y \in \Omega_k$, then
\[
|u_k(y) - u_k(x)| = |u_k(y)| \leq 2^{-i} \epsilon d(x, y)^2
\]
by (4.12). So, in either case, we have

\[
|u_k(y) - u_k(x)| \leq 2^{-i} \epsilon d(x, y)^2
\]

whenever $x \in K_i$, $y \in X$, and $k > i$. Summing this over all $k > i$ immediately proves (4.14).

Therefore, for almost every $x \in K_i \cap U_j$, we have the following:
\[
\text{Lip}(u - f_j(x) \cdot \phi_j)(x) \leq \text{Lip} \left( \sum_{k=1}^{i} u_k - f_j(x) \cdot \phi_j \right)(x)
\]
\[
\leq \text{Lip} \left( \sum_{k=1}^{i} u_k - \left( \sum_{k=1}^{i} d^i u_k(x) \right) \cdot \phi_j \right)(x)
\]
\[
= 0.
\]

It follows that at almost every point in $K_i \cap U_j$, the function $u$ is differentiable with $d^i u = f_j$. Because $\mu(\Omega \setminus \cup K_i) = 0$, it follows that $d^i u = f_j$ almost everywhere in $\Omega \cap U_j$. This proves (1.10).

Now we must show (1.11), that $\text{Lip}_u = 0$ everywhere in $X \setminus \Omega$. Fix $x \in X \setminus \Omega$. If $y \in X \setminus \Omega$, then $u(x) = u(y) = 0$. If $y \in \Omega$, then
\[
|u(y) - u(x)| = |u(y)| \leq \epsilon \text{dist}^2(x, X \setminus \Omega) \leq \epsilon d(x, y)^2.
\]

Thus, for any $x \in X \setminus \Omega$ and any $y \in X$, we have
\[
|u(y) - u(x)| \leq \epsilon d(x, y)^2,
\]
which immediately implies that $\text{Lip}_u(x) = 0$. This completes the proof of (1.11).

Finally, we must check (1.12). This, however, is immediate from equation (4.13) and the fact that $u = \sum u_i$.

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