Design of optimal quantity discounts for multi-period bilevel production planning under uncertain demands

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Abstract
The analysis of the quantity discount of the decentralized supply chain has been studied only for single-period planning models. This article presents the design of an optimal quantity discounts for multi-period bilevel production planning for two-echelon supply chains under demand uncertainty. In order to derive an optimal contract for multi-period production planning, the cumulative order quantity and production quantity are introduced. From our proposed model, a Stackelberg equilibrium is analytically derived for the supply chain when the supplier is the leader and the retailer is the follower. An optimal discount contract is analytically designed through the optimal solution of the centralized problem. Computational results show the effectiveness of the proposed discount contract under demand uncertainty.

Keywords
Bilevel program, demand uncertainty, multi-period production planning, Stackelberg equilibrium, quantity discounts

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Introduction
The multi-period production planning under demand uncertainty over time periods is one of the important challenges in various business organizations. A variety of production planning models and theoretical developments have been developed for multi-period production planning under demand uncertainty.¹,² Among these studies, the coordination of multi-period newsvendor problem has received much attention from researchers. The classical newsvendor problem is the one in which the decision-maker is faced with the situation of placing an order of a perishable item to meet the uncertain demand for a single-period such that the expected profit is maximized. The optimal solution of the problem is the trade-off between the expected cost of overstocking and understocking of the product.

Several types of contracts such as quantity discounts, revenue sharing, and buyback have been employed to realize the entire optimization under uncertain demands. However, due to its complexity, the analysis of optimal contracts for multi-period production planning models has been one of the significant issues under uncertain demands, which has not been addressed before. This article presents an analytical model for the equilibrium analysis of multi-period

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planning for two-echelon supply chains under demand uncertainty.

The goal of the analysis is to design the optimal contract between the entities that allow the decentralized supply chains to perform as well as a centralized one in which all decisions are made by a single entity. The coordination of production planning problems for suppliers and retailers in decentralized supply chains can be treated as supply chain games. There are two types of games in supply chains. One is a cooperative game, and the other is a noncooperative game. One of the noncooperative games is a Stackelberg game which is conducted by the leader and the follower. In this case, supply chain contracts are introduced into a leader–follower relationship in order to maximize the total profit. Hohn studied some types of supply chain contracts in a Stackelberg game. Supply chain contracts include only wholesale price discount contracts, buyback contract and quantity discount contracts. Chen studied a buyback contract in a single-period newsvendor problem. Moreover, Tsai studied the optimization of the supply chain by introducing the quantity discount contract. Tsai’s model was developed only for manufacturer’s standpoint.

However, multi-period production planning problems have inventory balancing constraints, which makes it difficult to derive an optimal solution analytically. The multi-period supply chain models with some complex constraints are studied in Kogan and Tapiero. However, it is also difficult to analyze the solution of the multi-period bilevel production planning under demand uncertainty for the supplier and the retailer due to its complexity. Therefore, a multi-period production planning model is required when uncertain demands are considered. The customer’s demands have been regarded as uncertain in most studies related to supply chain planning models. Zhang et al. studied a multi-period production planning with quantity discounts. In their model, the retailer’s decision variable is defined as the total amount of order quantity, which makes it easier to solve the problem analytically. However, Zhang et al.’s model only considers the production planning problem from the retailer’s standpoint.

The problem that we study is motivated by the online retailer and manufacturing industries such as parts supplier and assembly manufacturer.

These companies require the analysis of the business environment for a period of time considering collaboration and cooperation between the supplier and manufacturer under demand uncertainty. To analyze the multi-period bilevel supply chain under demand uncertainty is one of the key challenges in real companies.

We address the following a general analytical model: there is a leader–follower relationship between a single supplier and a single retailer. The supplier determines wholesales price to the retailer and the order quantity to the supplier where the retailer faces a random demand according to stochastic demand functions. The bilevel programs and their solution methods have been studied in many years. The coordination in supply chains is achieved by introducing contracts in the Stackelberg game. Buyback contracts can be introduced in a newsvendor problem. In this contract, supplier has the promise to buy all unnecessary inventories if they are unsold to customers. However, quantity discounts have also been often used in the multi-period bilevel production planning models in supply chains. Under the contract, the supplier can reduce the wholesale price to induce the retailer to take more under risks under demand uncertainty.

In the conventional works, the multi-period supply chain planning problems under demand uncertainty have been addressed. However, the analysis of quantity discounts on the multi-period bilevel production planning under demand uncertainty has not been explored. To the best of our knowledge, this is the first investigation in the literature that addresses the theoretical analysis of optimal quantity discount on the multi-period bilevel supply chains under demand uncertainty. The novelty of this article is summarized as follows:

- A new multi-period bilevel supply chain model is proposed with the decision variable of the cumulative total order quantity. The model is developed for the supplier and the retailer. By considering the retailer’s decision variable as the total amount of order quantity, the model can be easily analyzed.
- A Stackelberg game is conducted by the supplier (leader) and the retailer (follower). By the assumption, the equilibrium solution between the supplier and the retailer is obtained in the multi-period production planning model.
- An optimal quantity discount contract is analyzed in order to maximize the total profit for the multi-period bilevel supply chain under demand uncertainty. By introducing the analytical solution of the optimal contract into the multi-period bilevel supply chain model, we show that the optimal profit is achieved in the decentralized model.

We introduce a novel formulation of the equilibrium analysis of the multi-period bilevel production planning in two cases. The first case is the coordination of a single supplier with a single-period model and a single retailer with a multi-period model under demand uncertainty. The second case is the coordination of a single supplier and a single retailer with multi-period models. The supplier and the retailer determine their own decision-making for production planning,
respectively, in a Stackelberg game. An optimal quantity discount contract is introduced into the decentralized supply chain. The optimal contract model which can maximize the total profit is studied. The solution of the decentralized problem is compared with that of the centralized problem from some computational examples.

The rest of the article is organized by the following sections. We introduce literature reviews on game theoretical approaches for production planning problems, Stackelberg game, bilevel programming, and contract decision in supply chains. Then, the multi-period bilevel planning model for a single supplier and a single retailer is defined. A novel reformulation of the problem is formulated. The Stackelberg equilibrium of the multi-period bilevel program is analyzed to obtain the optimal quantity discounts. Numerical examples are provided to show the effectiveness of introducing the optimal contract and their managerial insights. Finally, conclusion and future works are mentioned.

**Literature review**

Coordination of supply chains has been analyzed for the global marketplaces where there are multiple decision-makers in the supply chains. In the global production, the coordination of the different organizations is required to optimize the total supply chain. Nishi et al. introduced the coordination of supply chain planning for multiple companies. They proposed an augmented Lagrangian approach in order to optimize the distributed supply chain.

In the case where multiple decision-makers are involved in supply chains, there are some game theoretical situations for the decision-makers. Nagarajan and Sosse studied some types of game theoretical approaches. There are two types of supply chain games. One is the cooperative situations, the other is the noncooperative situations. One of the noncooperative situations is a Stackelberg game. The game is conducted by the leader and the follower in which the leader can know complete information about the follower’s decision-making in the future. Therefore, the Stackelberg game is advantageous for the leader because the leader can make the decision by considering the follower’s decision-making.

In the Stackelberg game, some types of contracts are required in order to increase the profit of the total supply chain. A supply chain contract was studied by Chen. In the model, buyback contract was introduced in a newsvendor problem. Hohn addressed the coordination of decision-makers in the supply chain. Some contracts such as buyback, revenue sharing, and quantity discount contracts were introduced in the Stackelberg game. Viswanathan and Wang addressed the discount pricing decision in a distribution channel. One of the supply chain contracts is a quantity discount contract which has also been studied. Tsai addressed an optimization approach to the supply chain coordination with quantity discounts. Yin et al. considered the optimal quantity discount contract for multiple suppliers and a single manufacturer. It can maximize the total profit in the supply chain. Yin and Nishi also addressed a solution approach for the supply chain which was formulated as the mixed integer nonlinear programming problem under demand uncertainty. Trejo et al. reported on the real-world attacker-defender security games where the solution was the Stackelberg equilibrium. They used an extra proximal method in order to compute the Stackelberg equilibrium experimentally. Chaudhary and Narahari analyzed the secure and efficient surveillance in the Stackelberg game, especially focusing on the defenders’ strategies. The model was analyzed under some scenarios and the model was reformulated as a linear programming problem. Ghotbi et al. proposed a sensitivity-based approach to finding Nash and Stackelberg solutions among some decision-makers. By applying their proposed method to the single-period model, their method was proven to be effective. Serin solved the newsvendor problems in a Nash game and in a Stackelberg game with an analytical method in the single-period model. Kim examined the radio spectrum sharing scheme in the Stackelberg game which was conducted by multi-leader and multi-follower in a single-period model. Choi et al. analyzed a supply chain between a single-supplier and retailer under the return policy. They analyzed the centralized model and the decentralized model both of which were single-period models using mean-variance analysis. Wan and Boyce examined the two-period Stackelberg model. They solved the equilibrium solution by the theoretical approach only in a two-period model. Julien addressed a Stackelberg game where there were some leaders and followers, respectively. They solved the Stackelberg equilibrium by a theoretical approach to the single-period model. Xiao et al. studied the dual-channel distribution structure in Stackelberg games where the retailer was the pricing leader and the manufacturer was the pricing follower under the fixed demand.

Stackelberg game is formulated as the bilevel programming problem. Yue and You proposed an optimization approach for a multi-echelon supply chain in a Stackelberg game using a piecewise linear approximation to the nonconvex functions. Alizadeh and Nishi addressed a dynamic p + q maximal hub location problem and an efficient decomposition method using the duality-based reformulation. Liu et al. studied a solution approach to a problem in the Stackelberg game. They solved a bilevel programming problem that
has too many solutions by integrating the gradient-based search with a genetic algorithm. Yeh et al.35 investigated an established timberlands system in terms of a new bio-refinery facility which was formulated as a bilevel programming problem. By formulation and analysis of the model, they compared a single-level and a bilevel programming problem representation.

However, the bilevel programming problem has also been studied in order to formulate a Stackelberg game. Bard26 introduced some optimization algorithms and applications for bilevel programming. Kalashnikov et al.27 reported a multi-period bilevel stochastic optimization in natural gas cash-out problem. They solved the problem in some scenarios. Therefore, the model was formulated as a linear programming problem. Wee et al.28 formulated a vendor–buyer Stackelberg game as the bilevel programming problem. The model was a time-dependent model, and the model was analyzed by a genetic algorithm. Du et al.29 studied the price-only contracts in supply chains in the Stackelberg game where one of the suppliers was as a leader and his manufacturer was as a follower. The model was a single-period model and was analyzed by an analytical method. Alizadeh et al.30 studied a two-stage stochastic bilevel pricing problem by reformulating as a single-stage problem. They especially studied theoretical analysis with a general model in a single-period model. Åksendal et al.31 analyzed the time-dependent newvendor models in a Stackelberg game. They offered an analytical method for the multi-period model in the Stackelberg game. However, this approach was defined only in terms of a coupled system of stochastic differential equations. Therefore, it is difficult to solve in terms of explicit expressions using the method. Zou et al.32 studied the two-period supply chain management with the contract in the decentralized assembly system. Zhang et al.7 addressed multi-period production planning with demand uncertainty. In the model, they proposed the theoretical method for the optimization of the production planning which includes the only one decision-maker with a quantity discount contract. The main difference between this article and the previous works is that we formulate the nonlinear bilevel programming model for the production planning which includes the single supplier and the single retailer under demand uncertainty.

Yoshida et al.33 addressed an analysis of the multi-period bilevel supply chain under demand uncertainty. The coordination between the supplier and the retailer is conducted by the cumulative order quantity only at the end of the time period. Nishi and Yoshida34 developed an algorithmic optimization approach for multi-period bilevel supply chains where an optimal response function cannot be derived analytically. The effects of the replacement of the leader and follower relationship are examined to maximize the total profit.35 Yue and You36 proposed an improved reformulation and decomposition method for supply chain design and operation. Chua et al.37 addressed a multi-period Stackelberg game formulation of production planning of make-to-order production system in supply chains. However, the optimal contract design between supplier and manufacturer is not considered in their study. The main similarities and differences between the previous works and this article are summarized in Table 1.

In this article, we propose a new analytical model for the general multi-period bilevel supply chain under demand uncertainty. The main feature of the proposed model is that the coordination is conducted for all time periods. We can derive an optimal quantity discount contract analytically from our proposed model. The proposed model is general in the sense that the cumulative order quantity and production quantity are introduced to analyze the model.

### Analytical model for multi-period planning problem

#### Problem definition

We consider a two-echelon supply chain consisting of a single supplier and a single retailer over multiple time periods. In the model, each supplier and each retailer makes an individual decision, respectively, in the decentralized supply chains. The relationship between the supplier and the retailer is shown in Figure 1. In the problem definition, we assume that the supplier is the leader and the retailer is the follower. We obtain the Stackelberg equilibrium when the supplier and the retailer make an individual decision regarding production planning.

The supplier has ample capacity. The length of a period is sufficiently longer than the supplier’s lead time, which implies that the supplier can deliver on time any quantity ordered by the retailer. The supplier sets a unit wholesale price and the retailer orders a quantity at the start of each period to the supplier during periods $1, 2, \ldots, t$ where demand is uncertain.

We consider two cases when the supplier sets a constant wholesale price during the planning horizon, and the supplier sets a wholesale price for each period. The supplier determines the wholesale price per unit products for the total time horizon in order to maximize the supplier’s profit. The supplier incurs a unit production cost and sells the product at a unit wholesale price. The supplier’s profit function is the total wholesale profit minus total production cost over the time horizon. All parameters are constant over the periods and uncertain demands of each product at each period are assumed to be independent. The demands follow a probability distribution according to historical data. The density and cumulative probability function, mean and
variance are known in advance. The remaining inventories from one period are stored for use in subsequent periods. Sales of products from the retailer to the customer are not lost if the demand exceeds the stock and the stock shortage is backlogged in subsequent periods. The retailer determines the order quantity to the supplier under uncertain demands incurring unit inventory holding cost and unit penalty of shortage cost. The retailer’s profit function is the expected value of the total sales minus the sum of the inventory holding costs, penalty costs of the stock shortage, and wholesale costs. To make the discussion easier, the number of product items is assumed to be one. We consider a noncooperative game situation between the single supplier and the single retailer in the supply chain. The sequence of the game is explained as follows. The supplier decides the wholesale price which is announced to the retailer following the supplier. The retailer decides the order quantity $q$ in order to maximize the retailer’s profit as the follower with the supplier’s wholesale price is given. By the process of the decision-making, the equilibrium $(w^*, q^*)$ which includes the optimal wholesale price $w^*$ and the optimal order quantity $q^*$ is obtained.

**Mathematical model of multi-period bilevel planning**

The production planning problems for the supplier and the retailer are formulated as the optimization problem. An algorithm to obtain the Stackelberg equilibrium
solution is also provided. In this case, a general model of the profit maximization problem between the supplier and the retailer is explained as follows.

**Parameters.** $c$ is the supplier’s production cost of one unit of product per period. $d_t$ is the random variable of the demand of the product in period $t$. $D_t$ is the cumulative random variable of the demand of the product until period $t$. $f_t(D_t)$ is the probability density function followed by the demand of the product until period $t$. $F_t(D_t)$ is the cumulative distribution followed by the demand of the product until period $t$. $g$ is the retailer’s penalty cost for stock shortage of one unit of product per period. $s$ is the retailer’s inventory holding cost of one unit of product per period. $T$ is the total time horizon. $w_{\text{min}}$, $w_{\text{max}}$ are the minimum and maximum values of the supplier’s wholesale price of one unit of product per period.

**Decision variables.** $Q_t$ is the cumulative order quantity offered by the supplier until period $t$. $w_t$ is the supplier’s wholesale price per unit product item in period $t$. $W_t$ is the supplier’s wholesale revenue in time period $t$.

**The supplier’s decision problem.** The supplier’s decision problem is to determine an optimal wholesale price over the time periods such that the total profit is maximized.

The objective function $J_s$ for the supplier is written by

$$\max J_s = \sum_{t=1}^{T} (w_t - c)q_t$$

(1)

The constraints on upper and lower values of the wholesale price are given by

$$w_{\text{min}} \leq w_t \leq w_{\text{max}}, \quad t = 1, \ldots, T$$

(2)

where $w_t$ is the wholesale price per unit product in period $t$, $c$ is the production cost per unit quantity of product, and $q_t$ is the order quantity offered by the retailer in period $t$.

**The retailer’s decision problem.** The retailer faces uncertain demand from customers. Therefore, we formulate the retailer’s decision problem as a stochastic model. A stochastic model under demand uncertainty is constructed for the formulation. The retailer’s decision problem is to determine an optimal order quantity over the time periods such that the total profit is maximized.

In order to explain the model easier, the model for a deterministic case is explained. Let $d_t$ denote the demand of the product during period $t$. If the demand is deterministic, the ordered products are sold only the deterministic quantity of the demand to the customer. The total profit function for the retailer $J_r$ is written by

$$\max J_r = \sum_{t=1}^{T} (pd_t - s(q_t - d_t)^+ - g(d_t - q_t)^+ - w_tq_t)$$

where $x^+ = \max(x, 0)$

(3)

The inventory balancing constraints for the retailer are represented by

$$x_t = x_{t-1} + q_t - d_t, \quad t = 1, \ldots, T$$

(4)

The nonnegative constraints of the order quantity are

$$q_t \geq 0, \quad t = 1, \ldots, T$$

(5)

Due to the inventory balancing constraints (4), it is difficult to analyze the optimal solution of the retailer’s problem. In that case, by considering the retailer’s inventory balancing constraints (4) in $t = 1, \ldots, T$

$$x_1 = x_0 + q_1 - d_1$$

$$x_2 = x_1 + q_2 - d_2$$

$$\vdots$$

$$x_T = x_{T-1} + q_T - d_T$$

By adding the equations

$$x_T = x_0 + \sum_{t=1}^{T} (q_t - d_t)$$

(7)

Then, the following equation is obtained

$$x_t = x_0 + Q_t - D_t$$

(8)

From equation (7), the inventory quantity in period $t$ can be represented by the sum of the initial quantity of inventory $x_0$, the cumulative production quantity $Q_t$, and the cumulative quantity of demand $D_t$ until period $t$. Then, the retailer’s decision variable of the order quantity $q_t$ in period $t$ can be changed into the cumulative order quantity $Q_t$ until period $t$. The retailer’s objective function $J_r$ (equation (3)) is reformulated as equation (9)

$$J_r = pD_t - s(Q_t - D_t)^+ - g(D_t - Q_t)^+ - w_t(Q_t - Q_{t-1})$$

(9)

Let $f_t(D_t)$ be the probability density function followed by the cumulative demand of the product until period $t$, and $F_t(D_t)$ be the cumulative distribution function followed by the cumulative demand of the product until period $t$. These functions can be represented by the sum of each probability density function at period...
\( t \), and the cumulative distribution function of demand at each period over time horizon \( T \).

Under demand uncertainty case, we formulate the expected value of the retailer’s objective function (equation (9)) as follows

\[
J_r = \int_0^{\infty} pD_T f_T(D_T) dD_T - \sum_{t=1}^T \int_0^{Q_t} s(Q_t - D_t)f_t(D_t) dD_t
\]

\[
+ \int_0^{\infty} pQ_T f_T(D_T) dD_T
\]

\[
- \sum_{t=1}^T \int_0^{\infty} g(D_t - Q_t)f_t(D_t) dD_t - \sum_{t=1}^T w_t(Q_t - Q_{t-1})
\]

The objective is to maximize the expected profit over the time horizon. The first and second terms of the objective function are the total expected revenue and cost for the overstocking situation, whereas the third and the fourth terms are the total expected revenue and cost for the understocking situation. The fifth term represents the procurement cost from the supplier.

**Constraints.** The constraints for the retailer’s problem are as follows.

\[
Q_{t-1} \leq Q_t, \quad t = 1, \ldots, T
\]

\[
Q_t \geq 0, \quad t = 1, \ldots, T
\]

The constraints (equation (11)) guarantee that there is no negative ordering quantity in all periods. Constraints (equation (12)) are the nonnegativity requirements.

**Formulation of multi-period bilevel supply chain with a constant wholesale price**

If the supplier sets a constant wholesale price during period 1 to \( T \), the supplier’s objective function is reformulated as equation (13)

\[
\max J_s = (w_T - c)Q_T
\]

where the constant wholesale price is expressed by \( w_T \).

Using the formulation, inventory balancing constraints of equation (4) can be eliminated in the retailer’s decision problem in equation (15). Then, the Stackelberg game of the multi-period model between the supplier and the retailer can be analyzed.

The Stackelberg model with a single supplier as the leader, and a single retailer as the follower, is formulated as follows

\[
\max J_s \quad s.t. \quad \text{equation (2)}
\]

\[
\max J_r \quad s.t. \quad \text{equations (11) and (12)}
\]

**Formulation of centralized problem**

If the multi-period planning for the supplier and the retailer can be optimized simultaneously, the centralized production planning can be formulated as the following equations

\[
\max J_c = \int_0^{\infty} pD_T f_T(D_T) dD_T + \int_0^{\infty} pQ_T f_T(D_T) dD_T - s(Q_T - D_T)f_T(D_T) dD_T
\]

\[
- \sum_{t=1}^T \int_0^{\infty} g(D_t - Q_t)f_t(D_t) dD_t - cQ_T
\]

s.t. equations (11) and (12)

**The multi-period bilevel supply chain planning model**

**Analysis of retailer’s objective function**

The retailer’s problem is analyzed as follows. By omitting the constraints (11) in the retailer’s problem, the retailer’s objective function (10) can be separated into the following two objective functions, \( J_{rT} \) and \( J_{rt} \) \( 0 \leq t \leq T - 1 \)

\[
\max J_r = J_{rT} - \sum_{t=1}^{T-1} J_{rt}
\]

where

\[
\max J_{rT} = \int_0^{\infty} pD_T f_T(D_T) dD_T
\]

\[
+ \int_0^{\infty} pQ_T f_T(D_T) dD_T
\]

\[
- \sum_{t=1}^T \int_0^{\infty} s(Q_T - D_T)f_T(D_T) dD_T
\]

\[
- \sum_{t=1}^T \int_0^{\infty} g(D_t - Q_T)f_t(D_t) dD_t - w_T Q_T
\]
min \( J_r = \int_{0}^{\infty} s(Q_t - D_t)f(D_t)dD_t + \int_{Q_t}^{\infty} g(D_t - Q_t)f(D_t)dD_t \)

(19)

The properties of \( J_r \) and \( J_n \) of equations (18) and (19) are studied, respectively. Proposition 1 for \( J_r \) is given as follows.\textsuperscript{33}

**Proposition 1.** The retailer’s objective function \( J_r \) is concave.\textsuperscript{33}

**Proof.** By the second differential of \( J_r \) with respect to \( Q_r \), the following inequality is obtained

\[
\frac{\partial^2 J_r}{\partial Q_r^2} = -(p + s + g)fr(Q_r) \leq 0
\]

Therefore, the objective function \( J_r \) is proven to be concave.

According to Proposition 1, the objective function \( J_r \) has a unique optimal solution \( Q_r^* \) that maximizes \( J_r \). In this case, \( Q_r^* \) is obtained by the following equation where the first-order differential of \( J_r \) with respect to \( Q_r \) is equal to 0

\[
Q_r^* = F_{t^{-1}} \left( \frac{p + g - w_r}{p + s + g} \right)
\]

We can show Proposition 2 for \( J_n \).

**Proposition 2.** The retailer’s objective function \( J_n \) is convex.\textsuperscript{33}

**Proof.** From the second differential of \( J_n \) with respect to \( Q_n \), the following inequality is derived

\[
\frac{\partial^2 J_n}{\partial Q_n^2} = (s + g)f_r(Q_n) \geq 0
\]

Therefore, the objective function \( J_n \) is proven to be convex.

From Proposition 2, the objective function \( J_n \) has a unique minimum solution \( Q_n^* \) that maximizes \( J_n \). In this case, \( Q_n^* \) is obtained by the following equation where the first-order differential of \( J_n \) with respect to \( Q_n \) is equal to 0

\[
Q_n^* = F_{t^{-1}} \left( \frac{g}{s + g} \right)
\]

**Feasibility algorithm for retailer’s decision problem**

From the analysis above, the optimal \( Q_r^* \) is obtained from equation (18), and the optimal \( Q_n^* \) is obtained from equation (19). However, if the constraints (11) are not satisfied, \( Q_r^* \) and \( Q_{r-1}^* \) should be modified in order to satisfy equation (11). The algorithm to obtain a feasible solution from equations (20) and (21) can be stated as follows.

**Proof:** The validity of the algorithm is shown in the following. Proposition 2 shows that the retailer’s objective function \( J_r \) is concave. Therefore, the equilibrium solution \( Q_r^* \), which is obtained by equation (21) minimizes \( J_r \) in each period \( t (1 \leq t \leq T - 1) \). It is assumed that there is another equilibrium solution \( Q_r' \), which is different from \( Q_r^* \) in the period \( t' (1 \leq t' \leq T - 1) \). If the assumption is considered, \( J_r(Q_r') \leq J_r(Q_r^*) \). Therefore, the optimal solution of the retailer’s cumulative order quantity is obtained by equation (21) in period \( t \neq t' \)

\[
\sum_{t=1}^{T} J_n = J_{r1} + J_{r2} + \cdots + J_{r(T-1)}
\]

\[
\geq J_{r1} + J_{r2} + \cdots + J_{r(T-1)}
\]

(22)

From equation (22), the cumulative order quantity which minimizes \( J_n \) is obtained by equation (21) in each period \( t (1 \leq t \leq T) \). Therefore, Algorithm 1 is proven to be valid.

**Analysis of equilibrium for supplier and retailer**

We derive the equilibrium between the supplier and the retailer for the multi-period bilevel production planning. The supplier and the retailer follow the Stackelberg game. In the Stackelberg game, the supplier determines the wholesale price \( w_T \) with the retailer’s response function in order to maximize own profit as the Stackelberg leader. However, the retailer can determine the optimal cumulative order quantity \( Q_T \) only with the wholesale price \( w_T \) in order to maximize own profit as the Stackelberg follower. The supplier announces \( w_T \) to the retailer under demand uncertainty. An equilibrium solution between the supplier and the retailer is obtained only at the end of period \( T \) where the supplier determines the wholesale price \( w_T \) considering the retailer’s response function

**Algorithm 1 (Feasibility algorithm)**

**Step 1** If \( Q_r^* \) satisfies \( Q_r^* \leq Q_{r+1} \) in period \( t = 1, \ldots, T \), every \( Q_r \) is an optimal solution in all periods. Then, the algorithm is completed. Otherwise, go to Step 2.

**Step 2** If there is \( Q_r^* \) such that \( Q_r^* > Q_{r+1} \) violating (11) and (12), \( Q_r = Q_{r+1} \). Then, return to Step 1. It is verified that this algorithm provides an optimal solution for the retailer’s problem with (10), (11) and (12).
and announces it to the retailer. Then, the retailer determines the cumulative order quantity $Q_t$ for each period $t$ ($1 \leq t \leq T$).

Let $(w^*_T, Q^*_T)$ be a Stackelberg equilibrium of the supplier and the retailer. The retailer’s response function $Q^*_T$ is written by

$$Q^*_T = F^{-1}_T \left( \frac{p + g - w^*_T}{p + s + g} \right) \tag{23}$$

However, by the first-order differential of $J_s$ in equation (13) with respect to $w_T$, the following equation is obtained

$$\frac{\partial J_s}{\partial w_T} = Q_T(w_T) + (w_T - c) \frac{\partial Q_T(w_T)}{\partial w_T} = 0 \tag{24}$$

We can readily verify that the supplier’s profit function is strictly concave because $\frac{\partial^2 J_s}{\partial w_T^2} < 0$, thus the supplier’s best response has unique. The differential of equation (20) leads to the following equation

$$\frac{\partial Q_T(w_T)}{\partial w_T} = -\frac{1}{(p + s + g) f'(Q_T(w_T))} \tag{25}$$

By substituting equation (25) into equation (24), the Stackelberg equilibrium can be derived

$$Q^*_T(w^*_T) - \frac{w^*_T - c}{(p + s + g) f'(Q^*_T(w^*_T))} = 0 \tag{26}$$

Then, the unique equilibrium solution is obtained by $(w^*_T, Q^*_T)$ which satisfies both equations (23) and (26).

**Optimal quantity discount contract**

The optimal quantity discount contract is analyzed in the setting of decentralized decision-making. The supplier can reduce the wholesale price if the retailer increases the cumulative order quantity by introducing quantity discounts into the decentralized production planning.

Therefore, the supplier’s wholesale price $w_T$ depends on the cumulative order quantity $Q_T$ indicating that the supplier’s revenue $W_T$ is represented by the function $Q_T$ as $W_T(Q_T)$. The quantity discount with the equilibrium solution in the bilevel production planning is analyzed as follows.

**Analysis of equilibrium with quantity discounts**

With the quantity discounts into the bilevel production planning, the supplier’s profit function is written by

$$\max J_s = W_T(Q_T) - cQ_T \tag{27}$$

Using equation (23), an equilibrium solution under the quantity discounts satisfies the following equation

$$Q^*_T = F^{-1}_T \left( \frac{p + g - aQ_T}{p + s + g} \right) \tag{28}$$

Then, the quantity discount, type 1, is assumed that the wholesale revenue is linearly dependent on $Q_T$ which is indicated by a linear function of $Q_T$. In this case, the supplier’s profit function and the retailer’s response function are represented by equations (29) and (30), respectively

$$\max J_s = W^0_T - aQ_T - cQ_T \tag{29}$$

$$Q^*_T = F^{-1}_T \left( \frac{p + g - a}{p + s + g} \right) \tag{30}$$

where $W^0_T$ is the initial wholesale revenue that is determined by the supplier and $a$ is a reduced price per unit of cumulative order quantity. Figure 2 shows the relationship between the supplier’s wholesale revenue $W_T$ and the retailer’s cumulative order quantity $Q_T$ with quantity discounts which is described by the linear function.

**Analysis of optimal quantity discounts**

The quantity discounts to maximize the total profit in the decentralized problem are analyzed when the optimal solution $Q^*_T$ is known in the centralized problem.

When the quantity discounts are introduced into the decentralized problem, the optimal $Q^*_T$ should satisfy equation (28). Moreover, the optimal solution $Q^*_T$ is obtained by the following equation in the centralized problem of equation (16)

$$Q^*_T = F^{-1}_T \left( \frac{p + g - a}{p + s + g} \right) \tag{31}$$

From equations (28) and (31), if $Q^*_T = Q^*_T$, the following equation should be satisfied.
\[
\frac{\partial W_T}{\partial Q_T} = c \tag{32}
\]

Then, the function \(w_T(Q_T)\) which is related to quantity discounts is assumed to satisfy the following inequalities

\[
\begin{align*}
\frac{\partial W_T(Q_T)}{\partial Q_T} &
\leq c, \\
\frac{\partial^2 W_T(Q_T)}{\partial Q_T^2} &
\geq 0 \quad (Q_T \leq Q_T^e) \\
\frac{\partial W_T(Q_T)}{\partial Q_T} &
\geq c \quad (Q_T > Q_T^e)
\end{align*}
\tag{33}
\]

If \(W_T(Q_T)\) satisfies equation (33), the retailer absolutely decides \(Q_T^e\) as the optimal solution.

**Proof**

(a) Case 1: the retailer’s optimal solution \(Q_T^e = Q_T\).

In this case, \(\frac{\partial W_T(Q_T^e)}{\partial Q_T} \leq c\) by equation (33) and \(Q_T^e = F_T^{-1}((p + s - \frac{\partial W_T(Q_T^e)}{\partial Q_T})/p + s + g)\) by equation (28) are satisfied, respectively. However, the cumulative distribution function \(F(x)\) is the monotone increasing function and \(Q_T^e > Q_T\) should be satisfied. Therefore, the assumption is not valid.

(b) Case 2: the retailer’s optimal solution \(Q_T^e \geq Q_T\).

In this case, \(\frac{\partial W_T(Q_T^e)}{\partial Q_T} \geq c\) by equation (33) and \(Q_T^e = F_T^{-1}((p + s - \frac{\partial W_T(Q_T^e)}{\partial Q_T})/p + s + g)\) by equation (28) are satisfied, respectively. However, in the same way as (a), \(Q_T^e < Q_T\) should be satisfied. Therefore, the assumption is not valid.

According to (a) and (b), the retailer’s optimal solution \(Q_T^e = Q_T\).

The quantity discount type 2 which satisfies equations (32) and (33) is shown with the wholesale price having the following function

\[
W_T(Q_T) = \begin{cases} 
A - bQ_T & (Q_T \leq Q_T^e) \\
A - bQ_T^e + d(Q_T^e - Q_T) & (Q_T > Q_T^e)
\end{cases} \tag{34}
\]

where \(A\) is the initial wholesale revenue which is determined by the supplier, and \(b\) and \(d\) are the slope parameters.

Figure 3 shows the linear relationship between the supplier’s wholesale revenue \(W_T\) and the retailer’s cumulative order quantity \(Q_T\) with quantity discounts.

If the supplier announces the wholesale price with equation (34), the retailer determines the cumulative order quantity, which is equal to the optimal solution \(Q_T^e\) in the centralized problem. Numerical results for the effects of quantity discounts for type 1 and type 2 are explained in the numerical example.

**Analysis of multi-period supply chain with a different wholesale price for each period**

The new supplier’s problem is formulated in order to obtain a Stackelberg equilibrium in the multi-period model. However, the formulation of the supplier’s decision problems of equations (2) and (13) is considered in the optimization only at the end of period \(T\). A Stackelberg game between the supplier and the retailer should be originally considered through the multi-period planning model. Therefore, a new formulation of the supplier’s problem is formulated as the multi-period model to obtain a Stackelberg equilibrium in each period. The multi-period problem of the supplier is formulated as follows

\[
\begin{align*}
\max_{s} & \quad J_s = \sum_{t=1}^{T} w_t(Q_t - Q_{t-1}) - cQ_T^e \\
\text{s.t.} & \quad c \leq w_t \leq w_t^{\max}, \forall t \\
& \quad \max J_s \text{ s.t. equations (11) and (12)}
\end{align*}
\tag{35}
\]

Then, the retailer’s decision problem of equations (10)–(12) is reformulated into

\[
\begin{align*}
\max J_r = & \int_{0}^{Q_T} \left[ p\zeta f_T(z_T)dz_T + \sum_{t=1}^{T} pQ_T f_T(z_T)dz_T \\
& - \sum_{t=1}^{T} \left( \int_{0}^{z_T} s(Q_t - z_t)f(z_t)dz_t + \int_{z_T}^{\infty} g(z_t - Q_t)f(z_t)dz_t \right) \\
& - \sum_{t=1}^{T} w_t(Q_t - Q_{t-1}) \right] \\
\text{s.t.} & \quad \text{equations (11) and (12)}
\end{align*}
\tag{37}
\]

The Stackelberg equilibrium is obtained using the response function of the multi-period planning of the supplier of equations (11), (12), and (35)–(37) and
the retailer of equations (11), (12), and (37). For example, if \( T = 2 \), the objective function of the supplier is written by

\[
\max J_s' = \sum_{t=1}^{T} w_t(Q_t - Q_{t-1} - cQ_t)
\]

(38)

\( Q_t \) is the function of \( w_t \); therefore, \( J_s' \) of equation (35) is the function of \( w_1 \) and \( w_2 \). By the differentiation of \( J_s' \) with respect to \( w_1 \) and \( w_2 \), the conditions for the equilibrium are obtained as follows

\[
\frac{\partial J_s'}{\partial w_1} = (Q_1 - Q_0) + (w_1 - w_2) \frac{\partial Q_1}{\partial w_1} = 0
\]

(39)

\[
\frac{\partial J_s'}{\partial w_2} = (Q_2 - Q_1) + (w_2 - c) \frac{\partial Q_2}{\partial w_2} = 0
\]

(40)

Then, the retailer’s objective function is rewritten by

\[
\max J_r = \int p_2 f_2(z) dz + \int p_2 f_2(z) dz
\]

\[
- \sum_{t=1}^{2} \left[ s(Q_t - z_t)f_t(z_t)dz_t + \int g(z_t - Q_t)f_t(z_t)dz_t \right]
\]

\[
- \sum_{t=1}^{2} w_t(Q_t - Q_{t-1})
\]

(41)

\( J_s' \) is the function of \( Q_t \), then \( J_s' \) is the dual function of \( Q_1 \) and \( Q_2 \). Therefore, the equilibrium solutions are obtained by the differentiation of \( J_s' \) with respect to \( Q_1 \) and \( Q_2 \)

\[
F_1(Q_1) = \frac{g + w_2 - w_1}{s + g}
\]

(42)

\[
F_2(Q_2) = \frac{p + g - w_2}{p + s + g}
\]

(43)

The demand function of customers follows the uniform function distribution where \( f_i(x) \) and \( F_i(x) \) are represented as

\[
f_i(z_t) = \begin{cases} \frac{1}{A} & (0 \leq z_t \leq A) \\ 0 & \text{(otherwise)} \end{cases}
\]

(44)

\[
F_i(z_t) = \begin{cases} \frac{z_t}{A} & (0 \leq z_t \leq A) \\ 0 & \text{(otherwise)} \end{cases}
\]

(44)

where \( A_t \) is the range of \( Q_t \) per period. Using equations (42)–(44), \( Q_t \) and \( Q_2 \) are written by

\[
Q_1 = \frac{g + w_2 - w_1}{s + g} A
\]

(45)

\[
Q_2 = \frac{p + g - w_2}{p + s + g} 2A
\]

(46)

The following conditions are obtained by substituting equations (45) and (46) into equations (39) and (40)

\[
\frac{2(p + g - w_2)A}{p + s + g} - \frac{(w_1 - w_2)A}{s + g} = 0
\]

(47)

\[
\frac{2(p + g - w_2)A}{p + s + g} - 2(g + w_2 - w_1) - \frac{2(w_2 - c)}{p + s + g} = 0
\]

(48)

The equilibrium wholesale prices \( w_1^* \) and \( w_2^* \) are obtained by solving equations (47) and (48)

\[
w_1^* = \frac{7g^2 + (9c + 3p + 7gs + (3c + 2c)p + 9cs}{11s + 11g + 3p}
\]

(49)

\[
w_2^* = \frac{3g^2 + (7c + 3p + 3gs + (3c + 4c)p + 7cs}{11s + 11g + 3p}
\]

(50)

Then, by substituting \( w_1^* \) and \( w_2^* \) into equations (45) and (46), \( Q_1 \) and \( Q_2 \) are obtained

\[
Q_1 = \frac{7g^2 + 3gp + 7gs + 2ps - 2cg - 2cs}{(11s + 11g + 3p)s + g}
\]

(51)

\[
Q_2 = \frac{(3s + 3g + p)(3g^2 + 3gp + 3gs + 4ps - 4cg - 4cs)}{(11s + 11g + 3p)(p + s + g)(s + g)} A
\]

(52)

From the above equations, the Stackelberg equilibrium is shown as

\[
(w_1^*, Q_1^*), (w_2^*, Q_2^*)
\]

(53)

The Stackelberg equilibrium can be represented by some parameters in the multi-period problem using equations (49)–(52).

However, it is obtained without the conditions in the decision problems of the supplier and the retailer. Therefore, the derived solution is adjusted to obtain the optimal solution which satisfies the conditions.

Then, we consider the supply chain contract in the Stackelberg game where the wholesale revenues \( W_1 \) and \( W_2 \) depend on the order quantities \( Q_1 \) and \( Q_2 \), respectively. In this case, the supplier’s wholesale revenue \( W_1 \) is represented by the function of \( Q_1 \) as \( W_1(Q_1) \) and \( W_2 \) is written by the function of \( Q_2 \) as \( W_2(Q_2) \). Therefore, using the supply chain contract, the retailer’s order quantity, \( Q_1 \) and \( Q_2 \) are rewritten by

\[
\frac{Q_1}{\frac{g + w_2 - w_1}{s + g} A}
\]

(54)

\[
\frac{Q_2}{\frac{p + g - w_2}{p + s + g} 2A}
\]

(55)
Table 2. Parameters for the numerical examples.

| Period (t) | $p_t$ | $c_t$ | $\mu_t$ | $\sigma^2_t$ | $a_t$ | $b_t$ |
|-----------|-------|-------|---------|-------------|------|------|
| 1         | 300   | 100   | 50      | 3           | 10   | 10   |
| 2         | 300   | 100   | 50      | 3           | 10   | 10   |
| 3         | 300   | 100   | 50      | 3           | 10   | 10   |
| 4         | 300   | 100   | 50      | 3           | 10   | 10   |
| 5         | 300   | 100   | 50      | 3           | 10   | 10   |

For example, for the quantity discount, type 1, depends on the wholesale price which is indicated by a linear function of $Q_t$, the supplier’s profit function is represented as equation (56) and the retailer’s response functions are represented as equations (57) and (58), respectively

$$\max J_s = \sum_{t=1}^{2} (W^0_t - a_t(Q_t - Q_{t-1}) - c(Q_t - Q_{t-1}))$$  \hspace{1cm} (56)

$$Q_1 = \frac{g + a_2 - a_1}{s + g} A$$ \hspace{1cm} (57)

$$Q_2 = \frac{p + g - a_2}{p + s + g} 2A$$ \hspace{1cm} (58)

where $W^0_t$ is the initial wholesale revenue which is determined by the supplier in period $t$. $a_t$ is a reduced price per unit in period $t$. The retailer’s order quantity equations (57) and (58) show that $a_1$ and $a_2$ should satisfy $Q_1 \geq 0$ and $Q_2 \geq 0$ because $Q_1$ and $Q_2$ are not negative.

**Numerical examples**

The Stackelberg equilibrium is compared with the optimal solution derived from the centralized problem. The decentralized problem is the equilibrium solution of the production planning between the single supplier and the single retailer. The test cases are randomly generated instances that are studied in Zhang et al.\(^7\)

**Description of test instances**

Since the benchmark instances are not available for the new model, the test instances are generated from the real-world problems which were studied in Zhang et al.\(^7\) The parameters for the test instances are randomly generated. The size of each instance is mainly defined by the length of the planning horizon. Table 2 shows the parameters for the test instances which were provided in Zhang et al.\(^7\) The total planning horizon was set to 5. The cumulative quantity of the customer’s demand until each period $t$ ($1 \leq t \leq T$) follows a normal distribution. $\mu$ indicates the mean of the cumulative quantity of the customer’s demand until period $t$, and $\sigma^2$ is the variance of the cumulative quantity of the customer’s demand until period $t$. If the mean of the customer’s demand is assumed to be equal to $\mu$ in each period $t$ ($1 \leq t \leq T$), the mean of the cumulative quantity of the customer’s demand until period $t$ is $t\mu$ and the variance is $t\sigma^2$.

**Effectiveness of supply chain contracts**

The effectiveness of supply chain contracts is investigated by some numerical experiments. The supplier’s decision problem is represented by equations (2) and (13). The retailer’s decision problem is formulated as equations (10)–(12). Figure 4 shows how to make decisions in order to derive the Stackelberg equilibrium between the supplier and the retailer. The equilibrium solution is obtained by differentiating $J_s$ and using the retailer’s response function in period $t = 5$. However, the equilibrium solution is obtained only with the retailer’s response function in period $t$ ($1 \leq t \leq 4$).

The total profit of the decentralized problem is the sum of the supplier’s profit and the retailer’s profit. The cumulative quantity of the customer’s demand until each period $t$ ($1 \leq t \leq T$) follows a normal distribution. $\mu$ indicates the mean of the cumulative quantity of the customer’s demand until period $t$, and $\sigma^2$ is the variance of the cumulative quantity of the customer’s demand until period $t$. If the mean of the customer’s demand is assumed to be equal to $\mu$ in each period $t$ ($1 \leq t \leq T$), the mean of the cumulative quantity of the customer’s demand until period $t$ becomes $t\mu$ and the variance becomes $t\sigma^2$. In the numerical experiments, the optimal type of quantity discounts is investigated by introducing some types of quantity discounts into the decentralized problem between the supplier and the retailer. The types of quantity discounts are indicated in section “Optimal quantity discount contract.” The computer program was coded by Intel® Core™ i5-3427U CPU 1.80 GHz under the environment of Microsoft Visual C+++2010 Express. The retailer’s response function (21) depends on the retailer’s penalty cost for stock shortage of one unit of product.
per period $g$ and the inventory holding cost of one unit of product per period $s$. Table 3 shows three cases of parameters for the problem instance: Case I: $s < g$, Case II: $s = g$, Case III: $s > g$.

Tables 4–6 show the comparison of the optimal solution and the total profits of the centralized problem, the decentralized problem without contract, the decentralized problem with contract type 1 and the decentralized problem with contract type 2 in Case I: $s_i < g_i$, Case II: $s_i = g_i$, and Case III: $s_i > g_i$, respectively.

The decentralized problem without a contract can obtain 94.41% of the total profit of the optimal solution. It is caused by the fact that the supplier and the manufacturer have their preferences trying to maximize their profits in the decentralized problem without a contract.

However, the decentralized problem with a contract can gain 99.56% of the total profit of the optimal solution. The total profit in the decentralized problem can be improved by introducing a contract type 1 into the problem. This is because the retailer can determine larger cumulative quantity than that the decentralized problem setting without a contract through quantity discounts of wholesale prices between the supplier and the retailer.

The total profit in the decentralized problem becomes close to the one in the centralized problem by introducing the contract type 2 in the problem. The result shows that the retailer can determine the cumulative order quantity which is very close to the optimal solution in the centralized problem.

The total profit in the decentralized problem is equal to the one in the centralized problem by introducing the contract type 2 in the

| Problem | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $s_1 < g_1$ | 50 | 50 | 50 | 50 | 50 | 125 | 125 | 125 | 125 | 125 |
| $s_1 = g_1$ | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 75 | 75 |
| $s_1 > g_1$ | 125 | 125 | 125 | 125 | 125 | 50 | 50 | 50 | 50 | 50 |

| Problem | $w_5$ | $Q_5$ | Profit | Percentage (%) |
|---------|-------|-------|--------|---------------|
| Centralized | — | 251.86 | 48,712.54 | 100.00 |
| Decentralized (no contract) | 423.25 | 239.62 | 45,991.18 | 94.41 |
| Decentralized (with type 1) | 185.83 | 255.33 | 45,496.18 | 99.56 |
| Decentralized (with type 2) | 198.52 | 251.86 | 48,712.54 | 100.00 |

| Problem | $w_5$ | $Q_5$ | Profit | Percentage (%) |
|---------|-------|-------|--------|---------------|
| Centralized | — | 251.10 | 48,694.80 | 100.00 |
| Decentralized (no contract) | 373.54 | 239.46 | 46,462.73 | 95.41 |
| Decentralized (with type 1) | 186.77 | 254.11 | 48,514.64 | 99.63 |
| Decentralized (with type 2) | 199.12 | 251.10 | 48,694.80 | 100.00 |

| Problem | $w_5$ | $Q_5$ | Profit | Percentage (%) |
|---------|-------|-------|--------|---------------|
| Centralized | — | 250.26 | 48,634.44 | 100.00 |
| Decentralized (no contract) | 348.69 | 239.25 | 46,677.74 | 95.98 |
| Decentralized (with type 1) | 185.83 | 252.71 | 48,949.04 | 99.71 |
| Decentralized (with type 2) | 199.79 | 250.26 | 48,634.44 | 100.00 |

Table 3. Parameters in each case.

Table 4. Comparison of the total profit ($s_1 < g_1$).

Table 5. Comparing the total profit ($s_1 = g_1$).

Table 6. Comparison of the total profit ($s_1 > g_1$).
problem. The result shows that the retailer can determine the cumulative order quantity which is close to the optimal solution in the centralized problem.

We also discuss the effects of cost parameters on the solution of centralized and decentralized problems. The profit and the cumulative order quantity are decreased both in the centralized and decentralized problems when the retailer’s penalty cost for the stock shortage \( s \) is set to a lower value than the retailer’s inventory holding cost \( g \). This is because the opportunity to obtain more sales revenue is decreased when the retailer’s penalty cost for the stock shortage is decreased. The utilizing contract type 2 is more effective when the inventory holding costs are lower than the penalty cost for the stock shortage to the retailer. However, in all cases, the total profit is improved by introducing the quantity discounts type 1 or type 2. Therefore, the quantity discounts are proven to be effective in order to improve the total profit of the decentralized problems in all cases.

Tables 5 and 6 show the results in cases \( s = g \) and \( s > g \), respectively. It is indicated that the profit and the cumulative order quantity are reduced both in the centralized and decentralized problems if the retailer’s penalty cost for the stock shortage \( g \) is set to a low value. However, in all cases, the total profit is improved by introducing the quantity discounts type 1 or type 2. Therefore, the quantity discounts are proven to be effective in order to improve the total profit of the decentralized problems in all cases.

Figures 5 and 6 show the effect of the contract type in case \( s < g \). Figure 5 shows that the total order quantity in the decentralized model with contract type 2 is equal to that in the centralized model. The total order quantity is largest in the decentralized model with contract type 1 because the supplier’s wholesale price is decreased by the linear function of the total order quantity. Figure 6 shows that the total profit increases with contract type 1 because the retailer can decide the larger \( Q_t \) than the model without contracts. The profit

Figure 5. Comparison of the order quantity \((s < g)\).

Figure 6. Comparison of the total profit \((s < g)\).

Figure 7. Decision-making in the multi-period planning with a different wholesale price for each period.

Numerical results for the multi-period supply chain with a different wholesale price for each period

The validity of the multi-period supply chain with a different wholesale price for each period is examined from numerical experiments in this section. Figure 7 shows how to make decisions in order to derive a Stackelberg equilibrium between the supplier and the retailer in the model with a different wholesale price for each period. In Figure 7, the supplier and the retailer decide a Stackelberg equilibrium in each period \( t \in \{1, 2\} \) which is solved for the optimization for each period.

Table 7 shows the result of the numerical experiments for the model of equations (49)–(52). According to the results of Table 7, the equilibrium is valid because their optimal values are larger than the optimal values obtained by each period, respectively. The effectiveness is confirmed for the proposed model.

Figures 8 and 9 show the comparison between the proposed model and the single-period model. Figure 8
shows that the comparison of the order quantity between the proposed model and the single-period model. The order quantity in period 2 is larger than that in period 1 because the supplier determines a higher wholesale price in period 2 than that in period 1 in both models. Figure 9 shows the result of the supplier’s total profits for the extended model and the single-period model. From the figure, the profit which is obtained by the multi-period optimization is larger than the profit obtained by each period optimization. Therefore, the proposed model is effective.

Managerial insights

In Table 8, the multi-period planning with a different wholesale price with the contract type 1 is compared with the model without a contract. Table 8 shows that the total profit of the proposed model with contract type 1 is larger than the total profit of the model without contract.

Figure 10 shows the effectiveness of the contract type 1 for the total profit in the proposed model. It shows that the total profit of the proposed model is approximately 2.16 times larger than the total profit of the proposed model with contract type 1. The total profit of the proposed model can be increased by the supply chain contract because the retailer can decide the larger order quantity due to introducing the

| Problem  | \(w_1\)  | \(w_2\)  | \(Q_1\)  | \(Q_2\)  | Supplier’s profit |
|----------|----------|----------|----------|----------|------------------|
| Whole period | 192.26   | 159.51   | 52.72    | 111.78   | 8378.79          |
| Each period   | 262.50   | 179.73   | 34.12    | 120.86   | 6909.35          |

| Problem  | \(w_1\)  | \(w_2\)  | \(Q_1\)  | \(Q_2\)  | \(J_s\) | \(J_r\) | Total profit |
|----------|----------|----------|----------|----------|--------|--------|-------------|
| No contract | 192.26   | 159.51   | 52.72    | 111.78   | 8378.79 | -1480.87 | 6897.93     |
| Contract type 1 | 200.0    | 207.28   | 71.43    | 72.32    | 14,901.58 | 2659.70 | 14,901.58   |

Figure 8. Comparison of the order quantity in the multi-period planning with a different wholesale price for each period.

Figure 9. Comparison of the supplier’s profit for the extended model and that for the single-period model.

Figure 10. Comparison of the total profit in the proposed model with no contract and with contract type 1.
contract. Moreover, the retailer’s objective value is positive using the supply chain contract. This result shows that the supply chain contract can relax the unfairness between the supplier and the retailer in a Stackelberg game. Therefore, the supply chain contract is verified to be effective for the proposed model.

From the experimental results, the equilibrium solution between the supplier and the retailer has an advantage to the supplier and the retailer’s objective value can be negative in a Stackelberg game where the leader is the supplier. Moreover, the supply chain contract can make the total profit larger in a Stackelberg game both in the single-period model and in the multi-period model. The supply chain contract can also relax unfairness in a leader–follower game. Therefore, the supply chain contract is valid to be introduced in a Stackelberg game in order to maximize the total profit. We have also confirmed that the same results are obtained when the length of the planning horizon is sufficiently large when the time horizon is greater than 5 and less than 10.

From the experimental results, the equilibrium solution between the supplier and the retailer has a disadvantage when the supplier and the retailer’s objective value can be negative in a Stackelberg game where the leader is the supplier. The supply chain contract can increase the total profit larger in a Stackelberg game both in the single-period model and in the multi-period model. The supply chain contract can also relax the unfairness in a leader–follower game. Therefore, the supply chain contract is valid to be introduced in a Stackelberg game in order to maximize the total profit. Utilizing the analytical model proposed in this article, we have obtained the following insights:

- The optimal contract type 2 derived from our analytical model is better than type 1. This is effective when the retailer’s inventory holding cost is less than the retailer’s penalty cost for the stock shortage. The total profit of centralized optimization can be achieved by contract type 2.
- The contract derived from the multi-period model has more profit than that derived from the single-period model.
- The contract type 1 can have more profit than no contract. It implies that these contracts type 1 and type 2 can increase the profit larger in a Stackelberg game both in single-period and multi-period models.

Conclusion
In this article, we have developed a new multi-period bilevel supply chain planning model for a single supplier and a single retailer under demand uncertainty. Utilizing the cumulative demand and the cumulative order quantity, the Stackelberg equilibrium between the supplier and the retailer is analytically derived in the decentralized problem. The analysis of the optimal quantity discounts is also conducted. The proposed model enhances the coordination between the single supplier and the single retailer for all time periods. Computational experiments are conducted to show the effectiveness of the multi-period planning model under demand uncertainty. The Stackelberg equilibrium with the derived quantity discount contract is compared with the profit in the centralized problem. The effectiveness of the quantity discounts is confirmed in all cases. In the future works, we will extend our analytical model to consider multiple product items and the multi-period bilevel supply chain problems are analyzed to obtain the optimal contracts for more general cases. The proposed multi-period model can be used to develop the decision-making model for optimal supply chain configuration and design problems, optimal pricing problems, optimal contract decision-making and optimal business decision-making linking to the tactical and strategic level of the supply chain optimization. The analytical approach will be enlarged for the dynamic supply chain planning problems under several uncertainties.

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