RANDOM ITERATIONS OF PARACONTRACTION MAPS AND APPLICATIONS TO FEASIBILITY PROBLEMS

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Abstract. In this paper, we consider the problem of finding an almost surely common fixed point of a family of paraccontraction maps indexed on a probability space, which we refer to as the stochastic feasibility problem. We show that a random iteration of paraccontraction maps driven by an ergodic stationary sequence converges, with probability one, to a solution of the stochastic feasibility problem, provided a solution exists. As applications, we obtain non-white noise randomized algorithms to solve the stochastic convex feasibility problem and the problem of finding an almost surely common zero of a collection of maximal monotone operators.

1. Introduction

A very common problem in several areas of mathematics is the convex feasibility problem, which consists of finding a point belonging to the intersection of a collection \{C_i, i \in I\} of convex closed subsets of \(\mathbb{R}^k\), where \(I\) is an index set. It arises in numerous applications including computerized tomography, image restoration, signal processing and many others [7, 14, 9].

One popular way of solving the convex feasibility problem is to use projection algorithms, originated in the works [17, 1, 20, 13], see also [10, 4]. If \(I = \{1, \ldots, m\}\), a classic version of these algorithms is the cyclic projection algorithm, which generates a sequence \((x_n)\) according to the recursion

\[
x_{n+1} = P(x_n),
\]

where \(P = P_{C_m} \circ \cdots \circ P_{C_1}\) and \(P_{C_i}\) is the projection map onto the set \(C_i\). The iteration above is well suited when \(m\) is not too large, however, in some applications this may not be the case.

To deal with the large dimensional case, one can perform the projections onto the sets \(C_i\) in a random order, rather than sequentially. That is, we can replace the iteration (1.1) by

\[
X_{n+1} = P_{\xi_n}(X_n),
\]

where \((\xi_n)\) is a white noise, i.e., \((\xi_n)\) is an independent and identically distributed (i.i.d.) sequence of random variables, taking values from the index set \(I\). Randomized projection algorithms driven by white noise like (1.2) have been widely studied in recent years, see for instance [21, 22, 24] and the references therein.

Furthermore, the random nature of (1.2) also allows to treat the case where the index set \(I\) is an infinite (possibly uncountable) set. However, for this case

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it is more suitable to consider a stochastic reformulation of the convex feasibility problem. Namely, if \( \xi \) is a random variable with values in \( I \), the **stochastic convex feasibility** problem seeks to find \( x^* \) such that
\[
x^* \in C_{\xi}
\]
almost surely.

It turns out that the stochastic convex feasibility problem can be formulated as finding an almost surely common fixed point for the family of projection maps \( P_{C_i} \). Therefore, it can be placed in the broader context of the random fixed point theory.

In this paper, we study the **stochastic feasibility** problem of finding an almost surely (a.s.) common fixed point of a family of maps. That is, if \( f_i : M \to M, i \in I \), is a family of maps defined on a metric space \( M \), the goal is to find \( x^* \) such that
\[
f_\xi(x^*) = x^* \quad \text{almost surely}.
\]
(1.3)

Note that for the case where \( f_i = P_{C_i}, i \in I \), the problem above is the stochastic convex feasibility problem. Another particular instance of (1.3) is the problem of finding an a.s. common zero of a family of maximal monotone operators, since finding a zero of a maximal monotone operator is equivalent to finding a fixed point of its resolvent mappings.

The aim of our paper is to analyze an iterative algorithm for solving (1.3) in the case \( \{f_i, i \in I\} \) is a family of paracontraction maps, see Section 2.1 for the definition of paracontraction. Specifically, we consider the random iteration
\[
X_{n+1} = f_{\xi_n}(X_n),
\]
(1.4)

where \( (\xi_n) \) is an ergodic stationary sequence of random variables taking values in \( I \). If the set \( C \) consisting of the points that satisfy (1.3) is non-empty and \( M \) is a separable metric space such that every bounded closed set is compact, we will show that for every initial point \( X_0 = x_0 \in M \), with probability 1, the sequence \( (X_n) \) generated by (1.4) converges to some point of \( C \), see Theorem 1.

We observe that this theorem was already obtained in [15] for the case where \( (\xi_n) \) is an i.i.d. sequence. However, in the last years it has been a growing interest in the implementation of randomized algorithms in the non-independent case, since they can be more suitable for applications, for instance in distributed optimization see [16]. Furthermore, the study of random iterations driven by a non-white noise like (1.4) is interesting in their own right, as they generalize the deterministic and white noise cases.

As applications of Theorem 1, we will obtain a randomized projection algorithm for solving the stochastic convex feasibility problem (see Section 3.1) as well as a **randomized proximal point algorithm** for finding an a.s. common zero of a family of maximal monotone operators in the non-independent case (see Section 3.2). In particular, this will allow us to obtain Markovian randomized algorithms for both problems.

Finally, let us make some comments regarding the proof of Theorem 1. As mentioned above, this theorem was proved in [15] for the case \( (\xi_n) \) is an i.i.d. sequence [15]. We note that their proof can not be extended to the case \( (\xi_n) \) is a general stationary sequence. Indeed, in [15], the authors explore the natural Markov structure provided by \( (\xi_n) \). That is, if \( (\xi_n) \) is an i.i.d. sequence, it is known that the sequence \( (X_n) \) is a homogeneous Markov chain and hence there is a well-defined notion of stationary measures. Using a “loss mass” argument, they show that every stationary measure is supported on \( C \). Then, the convergence is
obtained by manipulating standard convergence results on the space of probability measures endowed with the weak star topology. On the other hand, when \((\xi_n)\) is not an i.i.d. sequence, \((X_n)\) is no longer a homogeneous Markov chain (even in the case \((\xi_n)\) is a Markov chain).

To overcome this situation, we follow a dynamical system approach. Indeed, it turns out that the sequence \((X_n)\) can be seen as a random orbit of a discrete random dynamical system driven by a measure-preserving dynamical system. Thus, instead of exploring stationary measures, we consider invariant random measures of the associated random dynamical system. Using the Poincaré recurrence theorem we obtain the recurrence of \((X_n)\) in a certain sense, and we use this result to show that any invariant random measure is supported on \(C\). From this, the convergence is obtained using convergence results on the space of random measures endowed with the narrow topology (the weak star topology can not be used in general).

Organization of the paper. In Section 2 we precisely state the main definitions and results of this work. Section 3 is devoted to the applications of Theorem 1 to solve the stochastic convex feasibility problem and the problem of finding an a.s. common zero of a family of maximal monotone operators. Section 4 presents some preliminaries on the space of random measures and the narrow topology, and the proofs of the results.

2. Main results

2.1. Stochastic feasibility problem. Let \((I, \mathcal{F}, \nu)\) be a probability space and consider a collection of continuous maps \(f_i : M \to M, i \in I\), defined on a metric space \((M, d)\). The problem of interest consists of finding a point \(x^* \in M\) such that

\[
x^* \in C \overset{\text{def}}{=} \{ x \in M : f_i(x) = x \text{ for } \nu\text{-almost every } i \}.
\]  

We call this problem the stochastic feasibility problem. We study the following random algorithm for solving (2.1):

\[
X_{n+1} = f_{\xi_n}(X_n)
\]

starting from an initial constant random variable \(X_0 = x_0 \in M\), where \((\xi_n)\) is an ergodic stationary sequence of random variables with distribution \(\nu\).

Our goal is to show that the sequence \((X_k)\) generated by the random iteration 2.2 converges to a solution of the stochastic feasibility problem, under the assumption that the maps are paracontractions. Recall that a continuous map \(f : M \to M\) is called a paracontraction if for every fixed point \(x^* \in M\) it holds

\[
d(f(x), y) < d(x, y)
\]

for every \(y\) that is not a fixed point of \(f\).

In the space \(M = \mathbb{R}^k\), examples of paracontraction maps are the averaged mappings. Namely, \(f : \mathbb{R}^k \to \mathbb{R}^k\) is an averaged map if there is an \(\alpha \in (0, 1)\) such that

\[
\|f(x) - f(y)\|^2 + \frac{1-\alpha}{\alpha} \|x - f(x) - (y - f(y))\|^2 \leq \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^k.
\]

The projection map \(P_C\) onto a convex closed set \(C\) and, more generally, the resolvent mapping associated with a maximal monotone operator \(T : \mathbb{R}^k \to \mathcal{P}(\mathbb{R}^k)\) are averaged mappings in \(\mathbb{R}^k\) (see Section 3 and Lemma 4.7).
From now on, we assume that \((M, d)\) is a separable metric space such that every closed bounded subset is compact, and \(f_i : M \to M\) is a paracontraction map for every \(i \in I\).

**Theorem 1.** Assume that \(C\) in (2.1) is non-empty. Then, for every initial point \(X_0 = x_0 \in M\), the sequence \((X_n)\) of random variables generated by the random iteration (2.2) converges almost surely to a random variable \(X \in C\).

The theorem above was proved in [15] in the case where \((\xi_n)\) is an i.i.d sequence. Therein, was also obtained the convergence of \((X_n)\) for averaged mappings on separable Hilbert spaces.

**Remark 2.1.** We observe that in Theorem 1 we can not expect convergence of the sequence \((X_n)\) if the set \(C\) is empty. Indeed, in the case \((\xi_n)\) is an i.i.d sequence of random variables, \(I\) is finite and the maps \(f_i\) are contractions for every \(i \in I\), it is well-known that, with probability 1, the \(\omega\)-limit of \((X_n)\) is the Hutchinson attractor, which is a non-singleton compact set if \(C\) is empty, see [3].

2.2. Random iterations of paracontractions maps. In this section, we consider a reformulation of Theorem 1 which will allow us to use the ergodic theory of random maps for obtaining its proof. To this end, we recall some definitions.

Let \((\Omega, \mathcal{F}, \mathbb{P}, \theta)\) be an **ergodic measure-preserving dynamical system**, that is, \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space, \(\theta : \Omega \to \Omega\) is a measurable transformation such that \(\mathbb{P}\) is an invariant measure for \(\theta\), and for every \(A \in \mathcal{F}\) such that \(\theta^{-1}(A) \subset A\) we have either \(\mathbb{P}(A) = 0\) or \(\mathbb{P}(A) = 1\).

Consider a family \(\{f_\omega\}_{\omega \in \Omega}\) of continuous maps \(f_\omega : M \to M\) defined on the metric space \(M\). Assume that the map \((\omega, x) \mapsto f_\omega(x)\) is measurable (\(M\) is endowed with the Borel \(\sigma\)-algebra and \(\Omega \times M\) with the product \(\sigma\)-algebra). The map \(\varphi : \mathbb{N} \times \Omega \times M \to M\) defined by

\[
\varphi(n, \omega, x) \overset{\text{def}}{=} f_{\theta^{n-1}(\omega)} \circ \cdots \circ f_\omega(x) \quad \text{for} \quad n \geq 1,
\]

is called a **random iteration of maps** on \(M\) induced by the family \(\{f_\omega\}_{\omega \in \Omega}\) and the ergodic measure-preserving dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, \theta)\).

We observe that \((\theta^n)\) is an ergodic stationary sequence on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and hence the sequence \((\varphi(n, \omega, x))_{n \in \mathbb{N}}\) is a particular instance of the random iteration (2.2) for \(I = \Omega\) and \(\xi_n = \theta^n\).

However, for studying the convergence of the random iteration (2.2) it is enough to study the convergence of (2.3). Indeed, first consider \((W, \mathcal{A}, \mathbb{P})\) the sample space of \((\xi_n)\) and the measurable map \(\Xi : W \to I^\mathbb{N}\) given by

\[
\Xi(w) = (\xi_0(w), \xi_1(w), \ldots).
\]

Denote by \(\mathbb{P} = \Xi_* \mathbb{P}\) the **stochastic image** of \(\mathbb{P}\) by \(\Xi\), that is, \(\mathbb{P}(A) = \mathbb{P}(\Xi^{-1}(A))\). Then, the natural projections \(\Pi_n : I^\mathbb{N} \to I\) defined by

\[
\Pi_n(\omega) = \omega_n, \quad \omega = (\omega_i)_{i \in \mathbb{N}}
\]

constitute an ergodic stationary sequence on the probability space \((I^\mathbb{N}, \mathcal{A}, \mathbb{P})\), where \(\mathcal{A}\) is the product \(\sigma\)-algebra. Thus, the relation \(\mathbb{P} = \Xi_* \mathbb{P}\) says that to study the convergence of the random iteration in (2.2) it is sufficient to consider the case where \((\xi_n)\) is taken in its canonical form, i.e., as the natural projections on a product space.
Now, we observe that the random iteration (3.2) for the sequence \((\Pi_n)\) can be fit into the context of (2.3). Indeed, \(P\) is ergodic for the shift map \(\sigma: F^N \to F^N\) defined by

\[ \sigma((\omega_i)_{i \in \mathbb{N}}) = (\omega_{i+1})_{i \in \mathbb{N}} \]

and if we define the family of maps \(\{f_\omega\}_{\omega \in F^N}\) by \(f_\omega = f_{\omega_0}\), then we have that the random iteration of maps \(\varphi\) generated by this family and the ergodic measure-preserving dynamical system \((F^N, \mathcal{F}, P, \sigma)\) is given by

\[ \varphi(n, \omega, x) = f_{\sigma^{n-1}(\omega)} \circ \cdots \circ f_\omega(x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x), \]

which is exactly the random iteration (2.2) with \(\xi_n = \Pi_n\).

We now state the reformulation of Theorem 1. Given a random iteration of paracontraction maps \(\varphi\) on the metric space \(M\) over the ergodic measure-preserving dynamical system \((\Omega, \mathcal{F}, P, \sigma)\), we define

\[ C_\varphi \overset{\text{def}}{=} \{ x \in M : P(\omega \in \Omega : f_\omega(x) = x) = 1 \}. \]

**Theorem 2.** Consider a random iteration of paracontraction maps \(\varphi\) on the metric space \(M\) over the ergodic measure-preserving dynamical system \((\Omega, \mathcal{F}, P, \theta)\). Assume that \(C_\varphi \neq \emptyset\). Then, for every \(x \in M\), for \(P\)-almost every \(\omega\) the sequence \((\varphi(n, \omega, x))_{n \in \mathbb{N}}\) converges to some point of \(C_\varphi\) (depending on \(\omega\) and \(x\)).

From the discussion above, it is clear that Theorem 1 follows from Theorem 2.

### 3. Applications

In this section, we present non-white noise randomized algorithms to solve the stochastic convex feasibility problem and the problem of finding an almost surely common zero of a collection of maximal monotone operators.

#### 3.1. Convex feasibility problem.

Let \((I, \mathcal{I}, \nu)\) be a probability space, and consider a family of non-empty closed convex sets \(\{C_i, i \in I\}\) in \(\mathbb{R}^k\). The stochastic convex feasibility problem seeks to find a point \(x^*\) such that

\[ x^* \in \{ x \in \mathbb{R}^k : x \in C_i \text{ for } \nu\text{-almost every } i \} \overset{\text{def}}{=} C^*, \]

or equivalently, to find \(x^*\) such that there is a set \(I_0\) of \(\nu\)-full measure satisfying

\[ x^* \in \bigcap_{i \in I_0} C_i. \]

Here, we consider the following randomized projection algorithm for solving (3.1):

\[ X_{n+1} = X_n + \alpha_n(P_{C_{\xi_n}}(X_n) - X_n), \]

where \(X_0 = x_0 \in \mathbb{R}^k\) is an arbitrary initial point, \((\xi_n)\) is an ergodic stationary sequence of random variables with distribution \(\nu\) and \(\alpha \in (0, 2)\). The convergence of the algorithm above will be obtained as a consequence of Theorem 1. To the best of our knowledge, this is the first time that it is established convergence for randomized projection algorithms to solve (3.1) in the non-independent case, generalizing previous convergence results for this type of algorithms.

In particular, we observe that in [21] it was studied the random algorithm

\[ X_{n+1} = X_n + \alpha_n(P_{C_{\xi_n}}(X_n) - X_n), \]

where \(\alpha_n > 0\). For the algorithm above, it was proved convergence to a point in \(C^*\) assuming that \((\xi_n)\) is i.i.d and \((\alpha_n) \in \ell_2 \setminus \ell_1\). If we take \(\alpha_n \equiv \alpha \in (0, 2)\), then...
the iteration above falls within the framework of (3.2) and, consequently, we can establish convergence relaxing the assumption on the sequence of random variables \((\xi_n)\), as well as on the sequence of step sizes \((a_n)\).

**Theorem 3.** Assume that \(C^*\) in (3.1) is non-empty. Then, for every initial point \(X_0 = x_0 \in \mathbb{R}^k\), the sequence \((X_n)\) of random variables generated by (3.2) converges almost surely to a random variable \(X \in C^*\).

As a corollary of Theorem 3, we obtain a Markovian randomized projection algorithm, i.e., algorithm (3.2) with \((\xi_n)\) being an ergodic stationary Markov chain. Let us illustrate the Markovian randomized projection algorithm in the case \(I\) is finite.

**Example 1.** Let \(I = \{1, \ldots, m\}\) and consider an irreducible transition matrix \(P = (p_{ij}), 1 \leq i, j \leq m\). It follows from the Perron-Frobenius theorem that there is a unique stationary probability vector \(p = (p_1, \ldots, p_m)\). Moreover, \(p_i > 0\) for every \(i = 1, \ldots, m\). Let \((\xi_n)\) be a Markov chain with state space \(I\), having transition probability matrix \(P\) and initial distribution \(\nu\) given by

\[\nu = p_1\delta_1 + \cdots + p_m\delta_m,\]

where \(\delta_i\) is the Dirac measure at \(i\). It is well-known that \((\xi_n)\) is an ergodic stationary Markov chain. Then, it follows from Theorem 3 that the sequence \((X_n)\) in (3.2) converges to a point in the set \(C^*\) defined in (3.1). We observe that since \(I\) is the only set with \(\nu\)-full measure, the stochastic feasibility problem in (3.1) coincides with the convex feasibility problem. Thus, the sequence \((X_n)\) actually converges to a point in the intersection \(\cap_{i \in I} C_i\).

### 3.2. Randomized proximal point algorithm

In this section, we consider the problem of finding an almost surely common zero of a collection of maximal monotone operators. We observe that this problem can be seen as a stochastic feasibility problem. However, the structure of maximal monotone operators allows us to consider a randomized algorithm that generalizes the algorithm presented in Section 3.1. The problem of finding a common zero of a family of maximal monotone operators has been studied in several works, see for instance [8, 18] and references therein.

We first present some preliminaries on maximal monotone operators and their properties. An *operator* \(T\) on the space \(\mathbb{R}^k\) is a set-valued mapping \(T : \mathbb{R}^k \to \mathcal{P}(\mathbb{R}^k)\), where \(\mathcal{P}(\mathbb{R}^k)\) is the power set of \(\mathbb{R}^k\). The operator \(T\) can be equivalently identified with its graph, \(\text{Gr}(T) \overset{\text{def}}{=} \{(x, v) \in \mathbb{R}^k \times \mathbb{R}^k : v \in T(x)\}\). The inverse of the operator \(T\) is \(T^{-1} : \mathbb{R}^k \to \mathcal{P}(\mathbb{R}^k)\) defined as \(T^{-1}(v) \overset{\text{def}}{=} \{x : v \in T(x)\}\). We say that \(T\) is monotone if

\[\langle x' - x, v' - v \rangle \geq 0, \quad \forall (x, v), (x', v') \in \text{Gr}(T).\]

Further, a monotone operator \(T\) is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator. A typical example of a maximal monotone operator is the subdifferential \(\partial f\) of a proper lower semicontinuous convex function \(f : \mathbb{R}^k \to (-\infty, +\infty]\).

The set \(T^{-1}(0) = \{x : 0 \in T(x)\}\) is called the set of zeros of the operator \(T\), which is closed and convex if \(T\) is maximal monotone. A vast variety of problems such as convex or linear programming, monotone complementarity problems, variational inequalities, and equilibrium problems, can be regarded as finding a zero of...
a maximal monotone operator:
\[(3.4) \text{ find } x \in \mathbb{R}^k \text{ such that } 0 \in T(x).\]

More generally, we can consider the problem of finding an almost surely common zero of a collection of maximal monotone operators. Namely, considering a family of maximal monotone operators \(\{T_i, i \in I\}\) defined on \(\mathbb{R}^k\), the goal is to find \(x^* \in \mathbb{R}^k\) such that
\[(3.5) \quad 0 \in T_i(x^*) \quad \text{for } \nu\text{-almost every } i.\]

A well-known technique for solving (3.4) is the \textit{proximal point algorithm} \[23\]. This algorithm can be interpreted as a fixed point algorithmic framework using the \textit{resolvent} mappings associated with the operator \(T\), which we briefly discuss now.

For each \(u \in \mathbb{R}^k\) and \(\lambda > 0\), there is a unique \(x \in \mathbb{R}^k\) such that
\[u \in (\text{Id} + \lambda T)(x),\]

where \(\text{Id}\) is the identity mapping in \(\mathbb{R}^k\), see \[19\]. Hence, the function \(J_{\lambda T} \overset{\text{def}}{=} (\text{Id} + \lambda T)^{-1}\) is single-valued and is called the \textit{resolvent} mapping of \(T\) of proximal parameter \(\lambda\). The function \(J_{\lambda T} : \mathbb{R}^k \to \mathbb{R}^k\) is non-expansive and \(J_{\lambda T}(x) = x\) if and only if \(0 \in T(x)\). Therefore, the inclusion problem (3.4) can be interpreted as a fixed point problem for \(J_{\lambda T}\). This is the motivation for the proximal point algorithm, which generates a sequence \((x_n)\) by the rule
\[x_{n+1} = J_{\lambda T}(x_n),\]

starting from a given point \(x_0 \in \mathbb{R}^k\). Furthermore, in the iteration above the resolvent can be replaced by a suitable affine combination of \(J_{\lambda T}\) and the identity map. This method is known as the \textit{generalized proximal point algorithm} \[12\].

Inspired by this last method, we propose the following \textit{randomized proximal point algorithm} for solving problem (3.5):
\[(3.6) \quad X_{n+1} = X_n + \alpha(J_{\lambda T_{\xi_n}}(X_n) - X_n),\]

starting from a point \(X_0 = x_0 \in \mathbb{R}^k\), where \(\lambda > 0\) and \(\alpha \in (0, 2)\) are fixed, and \((\xi_n)\) is an ergodic stationary sequence of random variables.

The following result gives the convergence of iteration (3.6). Denote by \(Z\) the set of all points \(x^*\) satisfying relation (3.5).

\textbf{Theorem 4.} Suppose that \(Z\) is non-empty. Then, for every initial point \(X_0 = x_0 \in \mathbb{R}^k\), the sequence \((X_n)\) of random variables generated by (3.6) converges almost surely to a random variable \(X \in Z\).

In \[5\] is analyzed (in the case of Hilbert spaces) the following similar random iteration
\[X_{n+1} = J_{\lambda_n T_{\xi_n}}(X_n),\]

starting from an arbitrary initial point \(x_0\) and assuming that \((\xi_n)\) is an i.i.d. sequence. Besides the fact we consider a non-independent sequence \((\xi_n)\) in the design of the randomized proximal point algorithm, we also observe that the problem studied and the convergence results obtained in \[5\] are different from those we consider here. Indeed, the author proves that almost surely the sequence \((X_n)\) converges in average to a point within the set of zeroes of the \textit{mean operator} determined by

\[\text{Actually, the method in } [22] \text{ allows the parameter } \lambda \text{ to vary with } n.\]
the Aumann integral of the operators, provided that \((\lambda_n) \in \ell_2/\ell_1\) and the mean operator is maximal monotone.

4. Proofs

This section is devoted to proving Theorems 2, 3 and 4. Before presenting the proofs, we introduce some preliminary notions and results.

4.1. The space of random measures and the narrow topology. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and assume in this section that \((M, d)\) a compact metric space. A probability measure \(\mu\) on \(\Omega \times M\) such that the first marginal of \(\mu\) is \(\mathbb{P}\) (i.e., \(\mathbb{P}(A) = \mu(A \times M)\) for every measurable set \(A\)) is called a random measure. The space of random measures is denoted by \(\mathcal{P}_\mathbb{P}(\Omega \times M)\).

Following Crauel [11], we consider the narrow topology on the space of random measures, which we briefly recall now. A random continuous function is a map \(f: \Omega \times M \to \mathbb{R}\) such that

(i) for \(\mathbb{P}\)-almost every \(\omega\), the map \(x \mapsto f(\omega, x)\) is continuous;

(ii) for every \(x \in M\), the map \(\omega \to f(\omega, x)\) is measurable;

(iii) the map \(\omega \mapsto \sup\{|f(\omega, x)|: x \in M\}\) is integrable with respect to \(\mathbb{P}\).

It follows from items (i)-(iii) that every random continuous function is measurable and integrable with respect to any random measure \(\mu\). The smallest topology on \(\mathcal{P}_\mathbb{P}(\Omega \times M)\) such that the map \(\mu \mapsto \mu(f)\) is continuous for every random continuous function \(f\) is called the narrow topology. A sequence \(\mu_n\) converges to \(\mu\) in the narrow topology if and only if

\[
\int f \, d\mu_n \to \int f \, d\mu
\]

for every random continuous function \(f\). The space \(\mathcal{P}_\mathbb{P}(\Omega \times M)\) endowed with the narrow topology is sequentially compact, see Crauel [11].

A random iteration of maps \(\varphi\) on \(M\) (recall the definition in Section 2.2) induces a map \(F: \Omega \times M \to \Omega \times M\) defined by

\[
F(\omega, x) = (\theta(\omega), f_\omega(x)),
\]

which is called the skew product induced by \(\varphi\).

Definition 4.1 (\(\varphi\)-invariant measures). A probability measure \(\mu\) on \(\Omega \times M\) is called a \(\varphi\)-invariant measure if

1. \(\mu \in \mathcal{P}_\mathbb{P}(\Omega \times M)\);

2. \(\mu\) is a \(F\)-invariant measure.

We observe that the space of random measures is invariant under the action of \(F\). In particular, it holds that \(F_i^\ast \mu \in \mathcal{P}_\mathbb{P}(\Omega \times M)\) for every \(i \geq 0\) and \(\mu \in \mathcal{P}_\mathbb{P}(\Omega \times M)\), where \(F_i^\ast \mu\) is the stochastic image of \(\mu\) by \(F^i\). Next, we present the following version of Krylov-Bogoliubov theorem for the narrow topology.

Proposition 4.2 ([2], Theorem 1.5.8). For every probability measure \(\mu \in \mathcal{P}_\mathbb{P}(\Omega \times M)\), any accumulation point of the sequence

\[
\frac{1}{n} \sum_{i=0}^{n-1} F_i^\ast \mu
\]

is a \(\varphi\)-invariant measure.
4.2. Recurrence Poincaré theorem for random iterations of maps. Let \((Y, \mathcal{Y}, \eta)\) be a probability space and consider a measurable transformation \(G: Y \rightarrow Y\) preserving the probability measure \(\eta\). The classical recurrence Poincaré theorem says that, for every measurable set \(A\), there is a subset \(A_r \subset A\) (which may be empty if \(\eta(A) = 0\)) with \(\eta(A_r) = \eta(A)\) such that, for every \(y \in A_r\), we have

\[ G^n(y) \in A \quad \text{for infinitely many } n \geq 1. \]

A corollary of this result is the metric recurrence Poincaré theorem. Namely, if the space \(Y\) is a second-countable topological space, then the set of recurrent points has \(\eta\)-full measure. We recall that a point \(y \in Y\) is called recurrent if, for every neighborhood \(V\) of \(y\), there is a positive integer \(n\) such that \(G^n(y) \in V\).

In the proof of Theorem 2, we will need the following recurrence property for random iterations.

**Proposition 4.3.** Let \(\varphi\) be a random iteration of maps on \(M\). Denote by \(R_{\varphi}\) the set of points \((\omega, x)\) such that, for every open neighborhood \(V\) of \(x\),

\[ \varphi(n, \omega, x) \in V \quad \text{for infinitely many } n. \]

Then, if \(\mu\) is a \(\varphi\)-invariant measure, we have \(\mu(R_{\varphi}) = 1\).

The proposition above could be obtained from the metric recurrence Poincaré theorem applied to the skew product induced by \(\varphi\), if the space \(\Omega\) was a second-countable topological space. However, we are not assuming that \(\Omega\) is a topological space.

**Proof.** Since \(M\) is a separable metric space, it has a countable basis \((V_k)_{k \in \mathbb{N}}\) of open sets. For each \(k \in \mathbb{N}\), by the recurrence Poincaré theorem applied to the skew product \(F: \Omega \times M \rightarrow \Omega \times M\) induced by \(\varphi\) and the \(F\)-invariant measure \(\mu\), we have that there is a subset \(H_k \subset \Omega \times V_k\) such that \(\mu(H_k) = 0\), and for every \((\omega, x) \in \Omega \times V_k \setminus H_k\) the sequence \(F^n(\omega, x)\) returns infinitely many times to \(\Omega \times V_k\).

Hence, if we take

\[ H = \bigcup_{k=1}^{\infty} H_k, \]

we have that \(\mu(H) = 0\) and every point \((\omega, x) \in \Omega \times M \setminus H\) is in \(R_{\varphi}\). \(\square\)

**Remark 4.4.** It is clear that Proposition 4.3 can be proved under weaker assumptions. However, for the sake of clarity, we have preferred to present it for the context we are considering in Theorem 2.

4.3. Proof of Theorem 2. Let \(\varphi\) be a random iteration of paracontractions maps on \(M\) and \(F\) the skew product induced by \(\varphi\). Recall the definition of \(C_{\varphi}\) in Section 2.2. Throughout, for simplicity, we use the notation

\[ f^n_{\omega}(x) \overset{\text{def}}{=} \varphi(n, \omega, x). \]

We start by presenting the following lemmas.

**Lemma 4.5.** Let \(\{U_k\}_{k \in \mathbb{N}}\) be a countable basis for the topology on \(M\) and denote by \(A_{U_k}\) the measurable subset (possibly empty) of \(\Omega\) such that, for every \((\omega, x) \in A_{U_k} \times U_k\), we have \(f_{\omega}(x) \neq x\). Then, for every \(x \in M \setminus C_{\varphi}\), there is \(k\) such that \(x \in U_k\) and \(\mathbb{P}(A_{U_k}) > 0\).
Proof. By the definition of $C_\varphi$, for every $x \in M \setminus C_\varphi$ must exist a measurable subset $A_x \subset \Omega$ with $\mathbb{P}(A_x) > 0$ such that, for every $\omega \in A_x$, it holds that $f_\omega(x) \neq x$. For every $\omega \in A_x$, the set

$$U(\omega) \overset{\text{def}}{=} \{ y \in M : f_\omega(y) \neq y \}$$

contains $x$ and, by the continuity of $f_\omega$, it is open. Therefore, we have

$$A_x = \bigcup_{\ell=1}^{\infty} \{ \omega \in A_x : B \left( x, \frac{1}{\ell} \right) \subset U(\omega) \}.$$

In addition, since $\mathbb{P}(A_x) > 0$, there exists $\ell$ such that $\mathbb{P}(\{ \omega \in A_x : B \left( x, \frac{1}{\ell} \right) \subset U(\omega) \}) > 0$.

Now, by the assumption that $\{U_k\}_{k \in \mathbb{N}}$ is a basis for the topology of $M$, there is $U_k \ni x$ such that $U_k \subset B \left( x, \frac{1}{\ell} \right)$. Then,

$$\{ \omega \in A_x : B \left( x, \frac{1}{\ell} \right) \subset U(\omega) \} \subset A_{U_k},$$

which implies that $\mathbb{P}(A_{U_k}) > 0$. □

Lemma 4.6. Every $\varphi$-invariant measure is supported on $\Omega \times C_\varphi$.

Proof. Let $\mu$ be a $\varphi$-invariant measure. We need to show that $\mu(\Omega \times C_\varphi) = 1$. If $C_\varphi = M$, then there is nothing to do. So we assume that $M \setminus C_\varphi$ is non-empty.

The Birkhoff’s ergodic theorem implies that there exists a measurable set $\Lambda^*$ of $\mathbb{P}$-full measure such that for every $k$ with $\mathbb{P}(A_{U_k}) > 0$, we have

$$\theta^n(\omega) \in A_{U_k} \text{ for infinitely many } n,$$

for every $\omega \in \Lambda^*$.

Now, we observe that since $M$ is separable, there is a set $\Lambda \subset \Omega$ of $\mathbb{P}$-full measure such that

$$f_\omega(c) = c$$

for every $c \in C_\varphi$ and every $\omega \in \Lambda$. Taking $\Lambda' \overset{\text{def}}{=} \bigcap_{n=0}^{\infty} \theta^{-n}(\Lambda)$, by the definition of paracotructions, we have

$$d(f_\omega^{n+1}(x), c) \leq d(f_\omega^n(x), c)$$

for every $c \in C_\varphi$, $(\omega, x) \in \Lambda' \times M$ and $n \in \mathbb{N}$. Further, from the $\theta$-invariance of $\mathbb{P}$, it follows that $\mathbb{P}(\Lambda^') = 1$.

Next, we take an open set $V$ such that $V \cap C_\varphi = \emptyset$, which exists because $C_\varphi$ is closed. We claim that $\mu(\Omega \times V) = 0$. Indeed, assume by contradiction that $\mu(\Omega \times V) > 0$. Note that

$$\mu(\Lambda^* \cap \Lambda' \times M) = \mathbb{P}(\Lambda^* \cap \Lambda') = 1,$$

where in the first equality above we use that the first marginal of $\mu$ is $\mathbb{P}$. In particular, it follows from Proposition 1.3 and equation 4.3 that $\mu(R_\varphi \cap (\Lambda' \cap \Lambda^* \times M)) = 1$. Since $\mu(\Omega \times V) > 0$, we can find a point $(\bar{\omega}, \bar{x}) \in \Omega \times V$ such that $(\bar{\omega}, \bar{x}) \in R_\varphi \cap (\Lambda' \cap \Lambda^* \times V)$. 
Now, we fix \( c \in C_\varphi \) and observe that \( \bar{\omega} \in \Lambda^* \cap \Lambda' \) and \( \bar{x} \notin C_\varphi \). By the definition of \( \Lambda^* \) and Lemma 4.3, there is \( k \) such that \( \bar{x} \) is not a fixed point for the map \( f_{\bar{\nu}}(\bar{\omega}) \). Hence, since \( \bar{\omega} \in \Lambda' \), it follows from the definitions of \( \Lambda' \) and paracontractions that
\[
d(f_{\bar{\omega}}^{k+1}(\bar{x}), c) < d(\bar{x}, c),
\]
for every \( n \geq k \).

Define \( r \triangleq d(f_{\bar{\omega}}^{k+1}(\bar{x}), c) \geq 0 \) and denote by \( B[c, r] \) the closed ball of radius \( r \) centered at \( c \). By inequality \((4.2)\), it follows that
\[
d(f_{\bar{\omega}}^{n}(\bar{x}), c) \leq d(f_{\bar{\omega}}^{k+1}(\bar{x}), c)
\]
for every \( n \geq k + 1 \), which implies
\[
(4.4) \quad f_{\bar{\omega}}^{n}(\bar{x}) \in B[c, r]
\]
for every \( n \geq k + 1 \).

On the other hand, \( r < d(\bar{x}, c) \) implies that there is an open neighborhood \( U \) of \( \bar{x} \), such that \( U \cap B[c, r] = \emptyset \). However, the definition of \( R_\varphi \) yields an integer \( m \geq k + 1 \) such that
\[
f_{\bar{\omega}}^{m}(\bar{x}) \in U,
\]
which is a contradiction with \((4.4)\). Hence, we conclude that \( \mu(\Omega \times V) = 0 \).

Finally, to deduce that \( \mu(\Omega \times C_\varphi) = 1 \), we observe that the set \( \Omega \times (M \setminus C_\varphi) \) is an enumerable union of sets of the form \( \Omega \times V \) with \( V \) open and \( V \cap C_\varphi = \emptyset \).

**Proof of Theorem 2.** We first assume that \( M \) is a compact metric space. For every \( x \in M \), we will show that
\[
(4.5) \quad \lim_{n \to \infty} d(f_{\omega}^{n}(x), C_\varphi) = 0 \quad \text{for } \mathbb{P}\text{-almost every } \omega.
\]
To this end, recall the definition of \( \Lambda' \) in the proof of Lemma 4.6 and that \( \mathbb{P}(\Lambda') = 1 \). By equation \((4.2)\), we have that, for every \( \omega \in \Lambda' \),
\[
(4.6) \quad d(f_{\omega}^{n+1}(x), C_\varphi) = \inf_{c \in C_\varphi} d(f_{\omega}^{n+1}(x), c) \leq \inf_{c \in C_\varphi} d(f_{\omega}^{n}(x), c) = d(f_{\omega}^{n}(x), C_\varphi),
\]
which means that the sequence \( d(f_{\omega}^{n}(x), C_\varphi) \) is decreasing for \( \mathbb{P}\)-almost every \( \omega \). Therefore, there is a measurable map \( H: \Omega \to [0, \infty) \) such that
\[
\lim_{n \to \infty} d(f_{\omega}^{n}(x), C_\varphi) = H(\omega)
\]
for \( \mathbb{P}\)-almost every \( \omega \), and by the Dominated Convergence Theorem we have
\[
(4.7) \quad \lim_{n \to \infty} \int d(f_{\omega}^{n}(x), C_\varphi) \, d\mathbb{P}(\omega) = \int H(\omega) \, d\mathbb{P}(\omega).
\]
We now prove that the sequence on the left-hand side of the equation above has a subsequence converging to 0, which will imply that \( H(\omega) = 0 \) for \( \mathbb{P}\)-almost every \( \omega \). For this purpose, consider the probability measure \( \mathbb{P} \times \delta_x \in \mathcal{P}_\mathbb{P}(\Omega \times M) \). Since \( M \) is compact, the space \( \mathcal{P}_\mathbb{P}(\Omega \times M) \) is sequentially compact and, as a consequence, the sequence
\[
\frac{1}{m} \sum_{i=0}^{m-1} F^i_{\omega}(\mathbb{P} \times \delta_x)
\]
has an accumulation point $\mu$, which by Proposition 4.2 must be a $\varphi$-invariant measure. Let $(n_k)_{k \in \mathbb{N}}$ be a subsequence such that

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} F^i_\omega(\mathbb{P} \times \delta_x) \to \mu \quad (4.8)$$

in the narrow topology. Also, define the map $g: \Omega \times M \to \mathbb{R}$ by

$$g(\omega, y) = d(y, C_\varphi),$$

which clearly is a random continuous map. From equation $(4.8)$ and the property $(4.1)$ of the narrow topology applied to the map $g$, it follows

$$\lim_{k \to \infty} \int g \left( \frac{1}{n_k} \sum_{i=0}^{n_k-1} F^i_\omega(\mathbb{P} \times \delta_x) \right) = \int g \, d\mu. \quad (4.9)$$

Now, let us look more closely at the integrals on both sides of the equation above. First, we claim that $\int g \, d\mu = 0$. Indeed, by Lemma 4.6 we have that $\mu(\Omega \times C_\varphi) = 1$ and, therefore,

$$\int g \, d\mu = \int_{\Omega \times C_\varphi} d(y, C_\varphi) \, d\mu(\omega, y) = 0.$$

Next, for the integral on the left-hand side of $(4.9)$, we observe that

$$\int g \left( \frac{1}{n_k} \sum_{i=0}^{n_k-1} F^i_\omega(\mathbb{P} \times \delta_x) \right) = \frac{1}{n_k} \sum_{i=0}^{n_k-1} \int g \, dF^i_\omega(\mathbb{P} \times \delta_x),$$

and

$$\int g \, dF^i_\omega(\mathbb{P} \times \delta_x) = \int g \circ F^i \, d(\mathbb{P} \times \delta_x)$$

$$= \int \int g(F^i(\omega, y)) \, d\delta_x(y) \, d\mathbb{P}(\omega)$$

$$= \int g(F^i(\omega, x)) \, d\mathbb{P}(\omega)$$

$$= \int g(\theta^i(\omega), f^i_\omega(x)) \, d\mathbb{P}(\omega) = \int d(f^i_\omega(x), C_\varphi) \, d\mathbb{P}(\omega),$$

where the second equality above is due to Fubini’s theorem. Hence, combining the four equations above we conclude that

$$\lim_{k \to \infty} \int \frac{1}{n_k} \sum_{i=0}^{n_k-1} d(f^i_\omega(x), C_\varphi) \, d\mathbb{P}(\omega) = 0. \quad (4.10)$$

Furthermore, it follows from $(4.6)$ that

$$d(f^{n_k}_\omega(x), C_\varphi) \leq d(f^i_\omega(x), C_\varphi)$$

for all $0 \leq i \leq n_k$ and for $\mathbb{P}$-almost every $\omega$. Therefore,

$$\lim_{k \to \infty} \int d(f^{n_k}_\omega(x), C_\varphi) \, d\mathbb{P}(\omega) = \lim_{k \to \infty} \int \frac{1}{n_k} \sum_{i=0}^{n_k-1} d(f^i_\omega(x), C_\varphi) \, d\mathbb{P}(\omega)$$

$$\leq \lim_{k \to \infty} \int \frac{1}{n_k} \sum_{i=0}^{n_k-1} d(f^i_\omega(x), C_\varphi) \, d\mathbb{P}(\omega) = 0,$$
where the last equality follows from (4.10). Thus, the equation above, together with (4.7), yields that \( \int H(\omega) \, d\mathbb{P}(\omega) = 0 \), which implies that \( H(\omega) = 0 \) for \( \mathbb{P} \)-almost every \( \omega \).

To conclude the proof of the theorem in the case \( M \) is compact, let \( \Lambda'' \) be the set of \( \omega \) such that the sequence \( d(f^n_\omega(x), C_\varphi) \) is decreasing and converges to 0. Fix \( \omega \in \Lambda'' \) and let \( c \) be any accumulation point of \( f^n_\omega(x) \). By the choice of \( \omega \), it is clear that \( c \in C_\varphi \). Since \( d(f^n_\omega(x), c) \) is decreasing, we conclude that \( f^n_\omega(x) \) actually converges to \( c \in C_\varphi \). Finally, combing (4.5) with (4.6), we obtain that \( \mathbb{P}(\Lambda'') = 1 \).

Now, we assume that \( M \) is separable and every bounded closed subset is compact. Fix \( c \in C_\varphi \), let \( r > 0 \) and consider the compact metric space \( \hat{M} = B[c, r] \). For every \( \omega \in \Lambda' \) and every \( y \in \hat{M} \), by the definition of paracontraction we have \( f_\omega(y) \in \hat{M} \). Thus, this allows us to consider the family of paracontractions \( \{f_\omega\}_{\omega \in \Lambda'} \), given by \( \hat{f}_\omega : \hat{M} \to \hat{M}, \hat{f}_\omega(y) = f_\omega(y) \). Also, we consider the ergodic measure-preserving dynamical system \( (\Lambda', \mathcal{F}', \mathbb{P}', \hat{\theta}) \) where

\[
\mathcal{F}' = \{A \cap \Omega : A \in \mathcal{F}\}, \quad \mathbb{P}' = \mathbb{P}|_{\mathcal{F}'} \quad \text{and} \quad \hat{\theta} : \Lambda' \to \Lambda', \hat{\theta}(\omega) = \theta(\omega).
\]

We note that \( \mathbb{P}' \) is \( \hat{\theta} \)-invariant since \( \mathbb{P}(\Lambda') = 1 \). Hence, it follows from the compact case applied to the random iteration of maps induced by the family \( \{f_\omega\}_{\omega \in \Lambda'} \) and \( (\Lambda', \mathcal{F}', \mathbb{P}', \hat{\theta}) \) that, for every \( x \in M \), the sequence \( \hat{\varphi}(n, \omega, x) \) converges to some point of \( C_\varphi \subseteq C_\varphi \) for \( \mathbb{P}' \)-almost every \( \omega \). In particular, for every \( x \in M \), the sequence \( \varphi(n, \omega, x) \) converges to some point of \( C_\varphi \) for \( \mathbb{P} \)-almost every \( \omega \).

To conclude the proof of the theorem, we observe that every point of \( x \in M \) belongs to some closed ball centered at \( c \).

\[\square\]

### 4.4. Proofs of Theorems 3 and 4

We note that Theorem 3 can be placed in the context of Theorem 4. Thus, we first prove Theorem 4 and Theorem 3 will be obtained as a consequence. Nevertheless, we observe that Theorem 4 could be proved directly.

**Proof of Theorem 4.** We define \( Q_i(x) \overset{\text{def}}{=} x + \alpha(J_{\lambda T_i}(x) - x) \) and observe that the sequence \( (X_n) \) in (3.0) can be rewritten as

\[
X_{n+1} = Q_{i_n}(X_n),
\]

which is a particular instance of the random iteration (2.2). We also observe that

\[
Q_i(x) = x \quad \text{if and only if} \quad 0 \in T_i(x).
\]

Therefore, proving that \( (X_n) \) converges to a point of \( Z \) is equivalent to proving that it converges to a point in the set \( C \) given in (2.1) with \( f_i = Q_i \).

Hence, if we prove that \( Q_i \) is a paracontraction, the theorem would be a consequence of Theorem 1, since \( \mathbb{R}^k \) is a separable metric space such that every bounded closed subset is compact. The next lemma shows that \( Q_i \) is an averaged map and, in particular, a paracontraction map. This is a classical result that we present here for the sake of completeness.

**Lemma 4.7.** Let \( T \) be a maximal monotone operator, \( \lambda > 0 \), \( \alpha \in (0, 2) \) and consider the map \( Q : \mathbb{R}^k \to \mathbb{R}^k \), defined as

\[
Q(x) = x + \alpha(J_{\lambda T}(x) - x).
\]

Then, \( Q \) is an averaged mapping with parameter \( \alpha/2 \).
Proof. First, we consider the Yosida approximation of $T$, the operator $A_{\lambda T} \overset{\text{def}}{=} (I - J_{\lambda T})/\lambda$, and observe that for all $x \in \mathbb{R}^k$ we have the following

(i) $A_{\lambda T}(x) \in T(J_{\lambda T}(x))$;
(ii) $x = J_{\lambda T}(x) + \lambda A_{\lambda T}(x)$;
(iii) $Q(x) = x - \alpha \lambda A_{\lambda T}$.

Therefore, by (iii), for any $x, y \in \mathbb{R}^k$ it holds

$$
\|Q(x) - Q(y)\|^2 = \|x - \alpha \lambda A_{\lambda T}(x) - (y - \alpha \lambda A_{\lambda T}(y))\|^2
$$

$$
= \|x - y\|^2 - 2\alpha \lambda \langle x - y, A_{\lambda T}(x) - A_{\lambda T}(y) \rangle
$$

$$
+ \alpha^2 \lambda^2 \|A_{\lambda T}(x) - A_{\lambda T}(y)\|^2
$$

$$
= \|x - y\|^2 - 2\alpha \lambda \langle J_{\lambda T}(x) - J_{\lambda T}(y), A_{\lambda T}(x) - A_{\lambda T}(y) \rangle
$$

$$
- 2\alpha \lambda^2 \|A_{\lambda T}(x) - A_{\lambda T}(y)\|^2 + \alpha^2 \lambda^2 \|A_{\lambda T}(x) - A_{\lambda T}(y)\|^2,
$$

where the last equality above follows from (ii) and a simple manipulation. Now, combining (i) with the monotonicity property \[3.3\] and equation above, we obtain

$$
\|Q(x) - Q(y)\|^2 \leq \|x - y\|^2 - \alpha(2 - \alpha) \lambda^2 \|A_{\lambda T}(x) - A_{\lambda T}(y)\|^2
$$

$$
= \|x - y\|^2 - \alpha(2 - \alpha) \|A_{\lambda T}(x) - A_{\lambda T}(y)\|^2.
$$

Substituting (ii) in the second term of the right-hand side of the above equation we have

$$
\|Q(x) - Q(y)\|^2 \leq \|x - y\|^2 - \alpha(2 - \alpha) \|x - J_{\lambda T}(x) - (y - J_{\lambda T}(y))\|^2,
$$

and, after simple manipulations, we conclude. \hfill \Box

This completes the proof of the theorem. \hfill \Box

Proof of Theorem [3]. Consider the normal cone $N_{C_i}$ of the non-empty closed convex set $C_i$, which is defined as

$$
N_{C_i}(x) \overset{\text{def}}{=} \begin{cases} 
\{ v \in \mathbb{R}^k : \langle v, x' - x \rangle \leq 0 \ \forall x' \in C_i \} & \text{if } x \in C_i \\
\emptyset & \text{otherwise}
\end{cases}
$$

It is well-known that $N_{C_i}$ is a maximal monotone operator and that finding a point in $C_i$ is equivalent to finding a point in $N_{C_i}^{-1}(0)$. Therefore, the stochastic convex feasibility problem is a special case of the problem \[3.5\].

Hence, the theorem follows from Theorem \[3\] noting that $P_{C_i} = (\text{Id} + N_{C_i})^{-1}$. \hfill \Box

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