OPTIMALITY OF THE FINAL MODEL FOUND VIA STOCHASTIC GRADIENT DESCENT

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ABSTRACT. We study convergence properties of Stochastic Gradient Descent (SGD) for convex objectives without assumptions on smoothness or strict convexity. We consider the question of establishing that with high probability the objective evaluated at the candidate minimizer returned by SGD is close to the minimal value of the objective. We compare this result concerning the final candidate minimizer (i.e. the final model parameters learned after all gradient steps) to the online learning techniques of [Zin03] that take a rolling average of the model parameters at the different steps of SGD.

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1. INTRODUCTION

1.1. Motivation. Stochastic Gradient Descent (SGD) is a popular approach to build machine learning models by learning parameters that single out an “(approximately) optimal” hypothesis in a given hypothesis space. Main reasons for the popularity are the simplicity of the algorithm and the ability to deal with real-life large datasets. Moreover, SGD can be used also to learn and optimize in real-time (e.g. online learning) and the gradient update rules can be refined (e.g. using algorithms like Adagrad, Adam or FTRL) to improve convergence, especially in problems where the hypothesis space is more complex either due to the large number of parameters (e.g. regressions with categorical features having high-cardinality) or the complexity of the hypothesis (e.g. neural networks).

In this note we only consider convex problems, a case where theoretical guarantees are well-understood [Bot03, Nes98, Haz16]. Let us start with the classical mathematical setting of minimizing a convex function $f : C \rightarrow \mathbb{R}$ where $C$ is a convex compact subset of some Euclidean space $\mathbb{R}^N$. We want to find a $u \in C$ that minimizes $f$ and for this Gradient Descent (GD) uses steps in the direction of the subgradient $\partial f$ to improve on an initial guess on the minimizer. A common hypothesis in this case is that there is a uniform bound on the norm $||\partial f||$ and a minor complication is to keep the constraint $u \in C$ during the minimization process.

In the typical supervised learning setting the situation is more complex as the objective function $f$ is not directly known, while one can sample objective functions
$f_t$ from a distribution $D$ with the guarantees $E[f_t] = f$ and $E[\partial f_t] = \partial f^1$. Concretely, one often has the case that $f_t(u) = F(x_t, u)$ where $F$ is known but $x_t$ is sampled from a distribution (e.g. $x_t$ is the training example consisting of the predictive features and the target variable(s)). We will call the variable $u$ parameters (e.g. the parameters/weights of a linear model). In this problem there are two main complications:

1. gradient steps are in the direction of $\partial f$ only on average.
2. a probabilistic approach, e.g. PAC-learning (see [MRT12], [Haz16, Ch. 9]) is needed to evaluate the goodness of the final hypothesis. In particular, the sequence of gradient updates generates a sequence of parameters $(u_t)_{t}$ which is no longer deterministic but a stochastic process.

This work was motivated by the following question:

**Q1:** Assume that we run SGD for $T$ iterations; how good of an approximate minimizer of $f$ is the final parameter $u_{T+1}$ returned by SGD? More precisely, can we claim that if $T$ is sufficiently large then $f(u_{T+1})$ is close to the minimal value of $f$ with high probability?

Despite the amount of literature on SGD we were not able to find an answer to Q1 that we found satisfactory\(^2\). We are aware of results on the expectation either of $E[f(u_{T+1})]$ or $E[u_{T+1}]$ (for example the recent [NNP+18]) or results about $u_{T+1}$ under additional assumptions on the stochastic process generated by SGD (for example [ZWSL10]). In particular, in [ZWSL10] the authors are in a sufficiently smooth and regularized setting so that the gradient updates result in a contraction in the parameter space. Using Wasserstein distances, they can then guarantee probabilistic results on $u_{T+1}$.

To study Q1 we consider two strategies. The first one uses the connection between online learning and convex optimization of [Zin03] and replaces $u_{T+1}$ by a running average of the parameter weights. Our understanding is that the averaging reduces the uncertainties in the gradient steps. However, this approach answers Q1 only partially. In the second approach we work directly with $u_{T+1}$ but we need to overcome 3 technical issues:

1. the subgradients $\partial f_t$ are not uniformly Lipschitz, this breaks some arguments in gradient descent.
2. martingales arguments in the PAC-framework do not work well when the projection onto the convex set $C$ is not linear.
3. we need a slightly improvement of Hoeffding’s concentration inequality [Hoe63] that uses one of Doob’s maximal inequalities.

### 1.2. Warm-up: the deterministic case.

As a warm-up case let us consider the case in which the sampling distribution $D$ is concentrated at the objective $f$. This is the case where the objective $f$ is known and available ([Nes98] for overview of results on convex optimization). We assume that the set $C \subset \mathbb{R}^N$ is compact and convex and let $\pi_C : \mathbb{R}^N \rightarrow C$ denote the projection onto $C$ which is known to be $1$-Lipschitz. Let $u_{opt} \in C$ denote a point where $f$ attains the minimum in $C$. The first strategy is to choose a sequence of learning rates $\{\varepsilon_t\}_{t=1}^T$, take gradient steps, project back to $C$ and then take the average of the parameters at different steps (which lies in $C$ by convexity). We call this algorithm Running Average Projected

\(^1\)formally this involves saying that $E[\partial f_t]$ is a subgradient of $f$ when $f$ is not sufficiently smooth

\(^2\)Overlooks here are to blame on me!
Gradient Descent, abbrev. RAPGD and pseudocode 1. The running average is because one can do the computation of the final average by keeping running sums ($v, \rho$ in the pseudocode) over the parameters and the learning rates. RAPGD is analyzed in Theorem 2.1 using the analysis of [Zin03]. In particular, under the assumption that the norm of $\partial f$ is uniformly bounded on $\mathcal{C}$ one shows that:

$$f(u_{\text{end}}) - f(u_{\text{opt}}) = O \left( \sum_{t=1}^{T} \frac{\varepsilon_t^2}{\sum_{t=1}^{T} \varepsilon_t} \right);$$

(1.1)

taking for example $\varepsilon_t = \frac{1}{\sqrt{t}}$ one gets a bound:

$$f(u_{\text{end}}) - f(u_{\text{opt}}) = O \left( \frac{\log T}{\sqrt{T}} \right).$$

(1.2)

The second strategy is to take gradient steps followed by projection. We call this algorithm Plain Projected Gradient Descent, abbrev. PPGD, pseudocode 2. The analysis of this approach is done in Theorem 2.35 and Remark 2.42. Under the
assumption of a uniform bound $L$ on the Lipschitz constant of $\partial f$ we obtain a bound:

\begin{equation}
 f(u_{\text{end}}) - f(u_{\text{opt}}) = O\left(\frac{L}{\sqrt{T}}\right);
\end{equation}

as remarked in Remark 2.42 a more careful analysis following [Nes98, Corollary 2.1.2] would give

\begin{equation}
 f(u_{\text{end}}) - f(u_{\text{opt}}) = O\left(\frac{L}{T}\right).
\end{equation}

Here the role of the Lipschitz hypothesis is to guarantee that the objective goes down after every iteration, not just on average after some iterations. Moreover, the nonlinearity of the projection $\pi_C$ introduces some complications. In particular, we introduce a notion of local norm (Definition 2.16) for vectors with respect to the convex set $C$; in this setting, even though $\partial f$ does not vanish at a minimizer $u_{\text{opt}} \in C$, one is guaranteed that $\|\partial f(u_{\text{opt}})\|_C = 0$.

1.3. The stochastic setting. In the general case we do not have direct access to $f$ but we base our gradient steps on sampling from a distribution $\mathcal{D}$ of convex functions. In Running Average Projected Stochastic Gradient Descent, aabb. RAPSGD and pseudocode 3 we proceed similarly to RAPGD. At each step we take a sample from $\mathcal{D}$ independent of what happens at the previous steps, update the gradient and project back to $C$. The theoretical guarantee that one proves in Theorem 3.78

\begin{algorithm}
\begin{algorithmic}
  \STATE \textbf{input}: convex objective $f$, compact convex set $C$ with projection $\pi_C$, sequence of learning rates $\{\epsilon_t\}_{t=1}^T$, initial point $u_1 \in C$, distribution of convex functions $\mathcal{D}$ whose average is $f$.
  \STATE \textbf{output}: approximate minimizer $u_{\text{end}}$.
  \STATE /* Note that the sequence of iterations gives rise to a filtration $\{\mathcal{F}_t\}_t$. */
  \STATE begin
  \STATE \hspace{1em} $\rho \leftarrow 0$;
  \STATE \hspace{1em} $v \leftarrow 0 \in \mathbb{R}^N$;
  \STATE \hspace{1em} for $t \leftarrow 1$ to $T$ do
  \STATE \hspace{2em} Sample $f_{t+1/2}$ from $\mathcal{D}$ independently of $\mathcal{F}_{t-1}$;
  \STATE \hspace{2em} $u_{t+1/2} \leftarrow u_t - \epsilon_t \partial f_{t+1/2}(u_t)$;
  \STATE \hspace{2em} $u_{t+1} \leftarrow \pi_C(u_{t+1/2})$;
  \STATE \hspace{2em} $\rho \leftarrow \rho + \epsilon_t$;
  \STATE \hspace{2em} $v \leftarrow v + \epsilon_t u_t$;
  \STATE \hspace{1em} end
  \STATE $u_{\text{end}} \leftarrow \frac{v}{\rho}$
  \STATE end
\end{algorithmic}
\end{algorithm}

is that, for an appropriate choice of the learning rates, with a number of steps $T = O(\epsilon^{-2})$ with probability $1 - O(\epsilon^2)$ one has:

\begin{equation}
 f(u_{\text{end}}) - f(u_{\text{opt}}) \lesssim 2\epsilon \log \frac{1}{\epsilon}.
\end{equation}
Note that the notation $\preceq$ is an asymptotic notation. For an error parameter $\varepsilon$ writing $a(\varepsilon) \preceq b(\varepsilon)$ means that $\lim_{\varepsilon \downarrow 0} \frac{a(\varepsilon)}{b(\varepsilon)} \leq 1$, while for the number of steps parameters $T$ writing $a(T) \preceq b(T)$ means that $\lim_{T \to \infty} \frac{a(T)}{b(T)} \leq 1$. The proof of Theorem 3.78 relies on that of Theorem 2.1 and the probability bound is obtained via Hoeffding’s inequality. As we obtain $u_{\text{end}}$ as an average, the nonlinearity of $\pi_C$ is dealt with by using Jensen’s inequality.

The second approach allows to get a theoretical guarantee for the final parameters returned by the algorithm. We call this algorithm *Smoothed Stochastic Gradient Descent*, abb. SSGD and pseudocode 4. SSGD differs from RASPGD in two respects:

1. the constraint $u_t \in C$ is not strongly enforced at each step, but only at the final step taking a projection (line 7). At the general step one uses a penalization function $\psi_C$ introduced in Lemma 3.9. This idea is not too different from that of relaxing a constraint by adding a penalization to the objective function via Lagrange multipliers.

2. a perturbation term (see [Haz16, Sec. 2.3.2, Algorithm 4]) is sampled (line 4) to smooth out the gradient updates (line 5).

In Theorem 3.34 we prove that, for a particular choice of the learning rates, with a number of steps $T = O(\varepsilon^{-6})$ with probability $1 - O(\varepsilon)$ one has that:

\[ f(u_{\text{end}}) - f(u_{\text{opt}}) \preceq 128G^2 \varepsilon \log \frac{1}{\varepsilon}. \]

On the other hand, if $\partial f$ was known to be Lipschitz, one would require a number of steps $T = O(\varepsilon^{-4})$. The ability to bound the properties of the final parameter $u_{T+1}$ (note that in Theorem 3.34 we show that the distance between $u_{T+1}$ and $u_{\text{end}} = \pi_C(u_{T+1})$ is $O(\varepsilon)$ so one can use $u_{T+1}$ and $u_{\text{end}}$ interchangeably even though only
the latter is guaranteed to lie in $\mathcal{C}$) come at the cost of a slower convergence (though we conjecture that with an adaptative selection of the learning rate depending on the size of the current gradient one might improve to $O(\varepsilon^{-2} \log \frac{1}{\varepsilon})$). SSGD handles the issues we mentioned above as follows:

**I1:** the smoothing (lines 4 and 5) slightly perturbs the objective $f$ to one which has Lipschitz subgradient. This idea is essentially a use of mollifications in real analysis.

**I2:** the penalization $\psi_C$ allows to avoid taking projections; thus we can use linearity in taking expectations.

**I3:** in deriving (3.47) we use a stronger version of Hoeffding’s concentration inequality; as observed by Hoeffding in [Hoe63, equation 2.17] this comes almost for free combining his proof with one of Doob’s maximal inequalities for martingales.

### 1.4. Summary.

We address the question concerning whether the final parameters $u_{T+1}$ returned by SGD is minimizing the convex objective $f$ up to a small error:

- In Theorem 3.34 we show that $f(u_{T+1})$ is within distance $O(\varepsilon \log 1/\varepsilon)$ from the minimum with probability $1 - O(\varepsilon)$ if the number of gradient descent steps $T$ is $O(\varepsilon^{-6})$.
- In Theorem 3.78 we show that the same holds if instead of $u_{T+1}$ one considers a rolling average of the parameters with $T$ now $O(\varepsilon^{-2})$.

The paper is organized in two sections. In the first one we deal with the deterministic case in which each gradient step works directly with $f$. This is mainly for illustrative purposes. In the second section we deal with the case in which at each step we sample from a distribution of convex functions whose mean is $f$.

### 2. The determinstic case

In this section we analyze RAPGD and PPGD. In order to analyze PPGD we need to introduce a notion of local norm and prove a geometric result, Theorem 2.23.

#### 2.1. Analysis of RAPGD.

Following [Zin03] we prove:

**Theorem 2.1 (Analysis of RAPGD).** If $u_{\text{opt}} \in \mathcal{C}$ is a minimizer of $f$ in $\mathcal{C}$ and if $\sup_{u \in \mathcal{C}} \|\partial f(u)\| < \infty$, letting

$$\|\partial f\|_{\mathcal{C},\infty} = \sup_{u \in \mathcal{C}} \|\partial f(u)\|,$$

then

$$f(u_{\text{end}}) - f(u_{\text{opt}}) \leq \frac{\|u_1 - u_{\text{opt}}\|^2 + \|\partial f\|^2_{\mathcal{C},\infty} \sum_{t=1}^{T} \varepsilon_t^2}{2 \sum_{t=1}^{T} \varepsilon_t}.$$

*Proof.* Using convexity of $f$ and the definition of the subgradient we obtain

$$\varepsilon_t (f(u_t) - f(u_{\text{opt}})) \leq \varepsilon_t \langle \partial f(u_t), u_t - u_{\text{opt}} \rangle$$

$$= \langle u_t - u_{t+1/2}, u_t - u_{\text{opt}} \rangle$$

$$= \langle (u_t - u_{\text{opt}}) - (u_{t+1/2} - u_{\text{opt}}), u_t - u_{\text{opt}} \rangle.$$
Using that projection onto \( C \) is 1-Lipschitz and expanding the Hilbert norm into products we get:

\[
\|u_{t+1} - u_{\text{opt}}\|^2 \leq \|u_{t+1/2} - u_{\text{opt}}\|^2 = \|u_{t+1/2} - u_t + u_t - u_{\text{opt}}\|^2 \\
= \varepsilon_t^2 \|\partial f(u_t)\|^2 + \|u_t - u_{\text{opt}}\|^2 + 2\langle u_{t+1/2} - u_t, u_t - u_{\text{opt}} \rangle.
\]

Substituting (2.5) into (2.4) we get:

\[
\varepsilon_t(f(u_t) - f(u_{\text{opt}})) \leq \frac{\|u_1 - u_{\text{opt}}\|^2 - \|u_{t+1} - u_{\text{opt}}\|^2 + \varepsilon_t^2 \|\partial f(u_t)\|^2}{2};
\]

summing in \( t \) we get:

\[
\sum_{t=1}^{T} \varepsilon_t(f(u_t) - f(u_{\text{opt}})) \leq \frac{\|u_1 - u_{\text{opt}}\|^2 + \sum_{t=1}^{T} \varepsilon_t^2 \|\partial f\|^2_{C,\infty}}{2},
\]

and application of Jensen’s inequality finally yields

\[
\sum_{t=1}^{T} \varepsilon_t(f(u_{\text{end}}) - f(u_{\text{opt}})) \leq \frac{\|u_1 - u_{\text{opt}}\|^2 + \sum_{t=1}^{T} \varepsilon_t^2 \|\partial f\|^2_{C,\infty}}{2}.
\]

\( \square \)

2.2. Preliminary analysis of PPGD. A preliminary analysis of PPGD is based on the following Lemma which analyzes the effect of a single gradient step. Note that the term \(-\varepsilon_t L\) in (2.10) might be improved to \(-\varepsilon L/2\) (compare [Nes98, Lemma 1.2.3]).

**Lemma 2.9.** Let \( \partial f \) be \( L \)-Lipschitz; then in plain projected gradient descent one has

\[
f(u_{t+1}) - f(u_t) \leq -(1 - \varepsilon_t L)\langle \partial f(u_t), u_t - u_{t+1} \rangle.
\]

Moreover,

\[
\langle \partial f(u_t), u_t - u_{t+1} \rangle \geq 0,
\]

and thus if \( \varepsilon_t \leq \frac{1}{L} \) then \( \{f(u_t)\}_{t=1}^{T+1} \) is non-increasing.

**Proof.** Applying first convexity of \( f \) and the definition of the subgradient \( \partial f \) and the Lipschitz condition on \( \partial f \) we obtain:

\[
f(u_{t+1}) - f(u_t) \leq \langle \partial f(u_t), u_{t+1} - u_t \rangle \\
= \langle \partial f(u_{t+1}) - \partial f(u_t), u_{t+1} - u_t \rangle + \langle \partial f(u_t), u_{t+1} - u_t \rangle \\
\leq L \|u_{t+1} - u_t\|^2 + \langle \partial f(u_t), u_{t+1} - u_t \rangle.
\]

Rewriting \( \|u_{t+1} - u_t\|^2 \) as

\[
\|u_{t+1} - u_t\|^2 = \langle u_{t+1} - u_t, u_{t+1} - u_{t+1/2} \rangle + \langle u_{t+1} - u_t, u_{t+1/2} - u_t \rangle,
\]

and recalling that \( \langle u_{t+1} - u_t, u_{t+1} - u_{t+1/2} \rangle \leq 0 \) as \( u_{t+1} \) is the point of \( C \) closest to \( u_{t+1/2} \) and \( u_t \in C \), we obtain

\[
\|u_{t+1} - u_t\|^2 \leq \langle u_{t+1} - u_t, -\varepsilon_t \partial f(u_t) \rangle;
\]

substitution into (2.12) yields (2.10). Finally, for (2.11) we assume that \( \partial f(u_t) \neq 0 \); then from the decomposition

\[
u_{t+1} = u_t + R \partial f(u_t) + v^t,
\]
where \( v^+ \) is orthogonal to \( \partial f(u_t) \), we observe that if (2.11) did not hold, \( R \) would be less than 0 and hence \( u_t \) closer to \( u_{t+1/2} \) than \( u_{t+1} \), contradicting that \( u_{t+1} \) is the point of \( C \) closest to \( u_{t+1/2} \).

**2.3. Geometric results.** As we use the non-linear projection \( \pi_C \) onto \( C \) we introduce a notion of local norm to measure, having fixed a vector \( u \) and a \( p \in C \), how big is the effective norm of \( u \) when we consider only steps that start from \( p \) and hit the sphere centered at \( p \) of radius \( r \) in some point of \( C \). We then prove a geometric result that we need when we want to relate gradient steps to the local norm. As in the stochastic case we use a penalization function \( \psi \) to weakly enforce the constraint \( u_t \in C \), we might have skipped a discussion of local norms, but we think it is useful to get an idea of the complications that can arise because of the non-linearity of projections.

**Definition 2.16** (Local norm). Given \( p \in C \), a vector \( u \) and \( r > 0 \), we define the local norm of \( u \) at \( p \) at scale \( r \) relatively to \( C \) as:

\[
\|u\|_C(p; r) = \begin{cases} 
\sup_{v \in C, \|v-p\| = r} \max \left( \frac{\langle u, v-p \rangle}{r}, 0 \right) & \text{if there is a } v \in C: \|v-p\| = r \\
0 & \text{otherwise.}
\end{cases}
\]

**Lemma 2.18.** The map \( r \mapsto \|u\|_C(p; r) \) is non-increasing and \( \limsup_{r \searrow 0} \|u\|_C(p; r) \leq \|u\| \).

**Proof.** Fix \( \varepsilon > 0 \) and \( r_1 \geq r_0 > 0 \). Choose \( v_1 \in C \) with \( \|v_1-p\| = r_1 \) such that:

\[
\max \left( \frac{\langle u, v_1-p \rangle}{r_1}, 0 \right) \geq \|u\|_C(p; r_1) - \varepsilon;
\]

writing \( v_1 = p + w_1 \) we have \( \|w_1\| = r_1 \); as \( p \in C \) and \( C \) is convex, the point \( v_0 = p + r_0 w_1 / r_1 \) lies also in \( C \) and we have:

\[
\max \left( \frac{\langle u, v_0-p \rangle}{r_0}, 0 \right) = \max \left( \frac{\langle u, v_1-p \rangle}{r_1}, 0 \right);
\]

we thus conclude that

\[
\|u\|_C(p; r_0) \geq \|u\|_C(p; r_1).
\]

Note however, that for any \( r > 0 \) we have

\[
\max \left( \frac{\langle u, v-p \rangle}{r}, 0 \right) \leq \|u\|
\]

whenever \( \|v-p\| = r \); thus \( \limsup_{r \searrow 0} \|u\|_C(p; r) \leq \|u\| \). \( \square \)

**Theorem 2.23.** Whenever \( u \) is a unit-norm vector, i.e. \( \|u\| = 1 \), for the local norm we have the fundamental inequality linking it to projections, where the implied constants are universal (also in the underlying \( \mathbb{R}^N \)'s dimension):

\[
\langle u, \pi_C(p + ru) - p \rangle \geq \frac{r}{2} \|u\|^2_C(p; r).
\]

**Proof.** Step 1: a weak bound. Without loss of generality we can assume that \( p \) is the origin 0. Note that if \( \pi_C(ru) \neq 0 \) then the left hand side of (2.24) would have to be \( \geq 0 \), otherwise \( 0 \in C \) would be closer to \( ru \) than \( \pi_C(ru) \). We conclude that the left hand side of (2.24) is always nonnegative.
If there were no \( v_{\text{cmp}} \in C \) such that \( \|v_{\text{cmp}}\| = r \) the right side of (2.24) would be 0. If \( \pi_C(ru) = 0 \) then, as the angle between \( ru \) and \( v_{\text{cmp}} \) would have to be \( \geq \pi/2 \), the right hand side of (2.24) would be 0 too.

We thus focus on the case in which \( v_{\text{pj}} = \pi_C(ru) \neq 0 \), and there is a \( v_{\text{cmp}} \in C \) such that \( \|v_{\text{cmp}}\| = r \). Let us define the angle \( \alpha = \angle(u, 0, v_{\text{pj}}) \); we know that \( \alpha \in [0, \pi/2] \) from the above discussion. We now have a weak bound:

\[
\angle(u, 0, v_{\text{cmp}}) \geq \alpha,
\]

otherwise the point \( \frac{\|v_{\text{pj}}\|}{\|v_{\text{cmp}}\|} v_{\text{cmp}} \), which belongs to \( C \) by convexity, would be closer to \( ru \) than \( v_{\text{pj}} \).

**Step 2: planar reduction.** Let

\[
v_{\text{cmp}} = v_{\text{pj}} + w_P + \frac{1}{2} w_P,
\]

be an orthogonal decomposition of \( v_{\text{cmp}} - v_{\text{pj}} \) with respect to the plane \( P \) spanned by \( u \) and \( v_{\text{pj}} \). By the properties of projections onto convex sets:

\[
\langle ru - v_{\text{pj}}, v_{\text{cmp}} - v_{\text{pj}} \rangle \leq 0;
\]

thus \( \langle ru - v_{\text{pj}}, w_P \rangle \leq 0 \). If we let \( \tilde{v}_{\text{cmp}} = v_{\text{pj}} + w_P \) we have that:

\[
\langle u, v_{\text{cmp}} \rangle = \langle u, \tilde{v}_{\text{cmp}} \rangle.
\]

To establish (2.24) we can thus replace \( v_{\text{cmp}} \) by \( \tilde{v}_{\text{cmp}} \) even though, in general, \( \tilde{v}_{\text{cmp}} \) does not belong to \( C \). By expanding \( \|ru - \tilde{v}_{\text{cmp}}\|^2 \) we find:

\[
\langle ru, v_{\text{cmp}} \rangle = \frac{r^2 + \|\tilde{v}_{\text{cmp}}\|^2 - \|ru - \tilde{v}_{\text{cmp}}\|^2}{2} \leq \frac{r^2 + \|v_{\text{cmp}}\|^2 - \|ru - v_{\text{pj}}\|^2 - \|w_P\|^2}{2} = \frac{r^2 + \|v_{\text{pj}}\|^2 - \|ru - v_{\text{pj}}\|^2}{2} + \langle w_P, v_{\text{pj}} \rangle;
\]

as also \( \angle(u, v_{\text{pj}}) \geq \pi/2 \) the \( w_P \) that maximizes the right hand side of (2.29) would have to be orthogonal to \( ru - v_{\text{pj}} \). Let \( \hat{v}_{\text{cmp}} \) be obtained from \( v_{\text{cmp}} \) by replacing \( w_P \) with the vector of the same norm and orthogonal to \( ru - v_{\text{pj}} \) so to maximize the right hand side of (2.29).

**Step 3: trigonometric inequalities.** Let \( \beta = \angle(ru, v_{\text{pj}}, 0) \) so that \( \beta \in [\alpha, \pi/2] \); applying the law of sines to the triangle \( \triangle(ru, v_{\text{pj}}, 0) \) we find:

\[
\|v_{\text{pj}}\| = \frac{r \sin(\alpha + \beta)}{\sin \beta}.
\]

Letting \( \delta = \angle(v_{\text{pj}}, \hat{v}_{\text{cmp}}, 0) \) and applying the law of sines to \( \triangle(v_{\text{pj}}, \hat{v}_{\text{cmp}}, 0) \) one has:

\[
\sin \delta = \frac{\|v_{\text{pj}}\|}{\|\hat{v}_{\text{cmp}}\|} \sin(3\pi/2 - \beta).
\]

Thus we obtain

\[
\frac{\langle v_{\text{pj}}, u \rangle}{\|\hat{v}_{\text{cmp}}\|} = \frac{r \sin(\alpha + \beta)}{\sin \beta} \cos(\alpha + \beta - \pi/2 - \delta).
\]
We now claim that the right hand side of (2.32) is at least \( \cos \frac{\alpha}{2} \). Indeed setting that right hand side to be \( \geq \cos \frac{\alpha}{2} \) we get:

\[
2 \frac{r}{\|\hat{v}_{\text{cmp}}\|} \sin(\alpha + \beta) \sin(\beta) \geq \sin \beta \cos \delta \sin(\alpha + \beta - \pi/2) \sin \delta \\
= \sin \beta \cos \delta \sin(\alpha + \beta) - \cos(\alpha + \beta) \sin(\alpha + \beta) \frac{r}{\|\hat{v}_{\text{cmp}}\|} \sin(3\pi/2 - \beta);
\]

in this form we see that (2.33) holds. We conclude observing that:

\[
\langle \nu_{pj}, u \rangle \geq \frac{\cos \alpha}{2} \frac{\langle \hat{v}_{\text{cmp}}, u \rangle}{r} \geq \frac{\cos \alpha}{2} \frac{\langle \hat{v}_{\text{cmp}}, u \rangle}{r} \geq \frac{1}{2} \left( \frac{\langle \hat{v}_{\text{cmp}}, u \rangle}{r} \right)^2.
\]

\( \square \)

2.4. Analysis of PPGD. We can now prove Theorem 2.35. In this case the best choice of learning rates is constant in \( t \), see Remark 2.42.

Theorem 2.35. If \( u_{opt} \in C \) is a minimizer of \( f \) in \( C \), if \( \partial f \) is \( L \)-Lipschitz and if for each \( t \in \{1, \ldots, T\} \) one has:

\[
1 - \varepsilon_t L \geq \gamma > 0;
\]

then one either has:

\[
f(u_{\text{end}}) - f(u_{opt}) \leq \varepsilon_{\text{err}},
\]

or for some \( t^* \in \{1, \ldots, T\} \) one has:

\[
\|\partial f(u_t^*)\|_{C} (u_t^*; \varepsilon_t, \|\partial f(u_t^*)\|) \leq \left( \frac{2f(u_1) - f(u_{opt}) - \varepsilon_{\text{err}}}{\gamma \sum_{t=1}^{T} \varepsilon_t} \right)^{1/2}.
\]

Proof. From Lemma 2.9 we get:

\[
f(u_{\text{end}}) - f(u_1) \leq -\gamma \sum_{t=1}^{T} \langle \partial f(u_t), u_t - u_{t+1} \rangle;
\]

we claim that (2.39) implies:

\[
f(u_{\text{end}}) - f(u_1) \leq -\frac{\gamma}{2} \sum_{t=1}^{T} \varepsilon_t \|\partial f(u_t)\|_{C}^2 (u_t; \varepsilon_t, \|\partial f(u_t)\|);
\]

indeed, if some \( \partial f(u_t) \) is 0 there is nothing to prove, otherwise we apply Theorem 2.23 to the unit vector \( \partial f(u_t)/\|\partial f(u_t)\| \). Adding \( f(u_1) - f(u_{opt}) \) to (2.40) we obtain:

\[
f(u_{\text{end}}) - f(u_{opt}) \leq f(u_1) - f(u_{opt}) - \frac{\gamma}{2} \sum_{t=1}^{T} \varepsilon_t \|\partial f(u_t)\|_{C}^2 (u_t; \varepsilon_t, \|\partial f(u_t)\|);
\]

thus if (2.37) is violated, for some \( t^* \) (explicitly a \( t \) for which \( \|\partial f(u_t)\|_{C}^2 (u_t; \varepsilon_t, \|\partial f(u_t)\|) \) is minimal), we have that (2.38) must hold. \( \square \)
Remark 2.42. Note that Theorem 2.35 gives, choosing $\varepsilon_t = 1/(2L)$ for each $t$ an order of iterations of $O(L/\varepsilon_{err}^2)$ to achieve an error $\varepsilon_{err}$ in the minimization condition (2.39). This is not optimal however, as for $f$ $L$-Lipschitz one can achieve a better bound $O(L/\varepsilon)$ as shown in [Nes98, Corollary 2.1.2]. However, we were not able to adapt that argument to the next stochastic case.

3. The stochastic case

In this section we analyze RASPGD and SSGD. We first deal with SSGD, whose proof is more involved.

3.1. An extension result. The following result is added for completeness. It shows that if $f$ is just defined on $C$ we can extend it to all of $\mathbb{R}^N$ while keeping the gradients bounded. This extension property is needed in the smoothing step.

Lemma 3.1. Let $C \subset \mathbb{R}^N$ a convex compact subset and $f : C \to \mathbb{R}$ be a continuous convex function such that there is a choice $\partial f$ of the subgradient of $f$ such that:

\begin{equation}
\sup_{x \in C} \|\partial f(x)\| \leq G < \infty.
\end{equation}

Then there is a convex extension $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ of $f$ such that there is a choice of the subgradient $\partial \tilde{f}$ such that:

\begin{equation}
\sup_{x \in \mathbb{R}^N} \|\partial \tilde{f}(x)\| \leq G < \infty.
\end{equation}

Proof. For $x \in C$ define the affine function:

\begin{equation}
a_x(y) = f(x) + \langle \partial f(x), y - x \rangle;
\end{equation}

then we set

\begin{equation}
\tilde{f}(y) = \sup_{x \in C} a_x(y);
\end{equation}

then $\tilde{f}$ is convex being the pointwise sup of affine (and hence convex) functions; fix $y$ and take a maximizing sequence $\{x_n\}$ for the definition of $\tilde{f}(y)$; by compactness of $C$ and of the closed ball of radius $G$ in $\mathbb{R}^N$ we can find $z_y \in C$ and $v_y \in \mathbb{R}^N$ with $\|v_y\| \leq G$ such that:

\begin{equation}
\tilde{f}(y) = f(z_y) + \langle v_y, y - x \rangle.
\end{equation}

Now fix $y \in C$; evaluating the sup at $y$ gives $\tilde{f}(y) \geq f(y)$; on the other hand, for any other $z \in C$ the very definition of convexity and subgradient imply that $a_z(y) \leq f(y)$ and hence $\tilde{f}(y) = f(y)$. Finally, a bounded choice of the subgradient $\partial \tilde{f}(y)$ is obtained by choosing $v_y$. $\square$

3.2. Weakly enforcing constraints via penalization. We show how to construct a penalization term $\psi_C$ to constrain the membership of the parameters to $C$.

Definition 3.7 (Support vectors). Let $C$ be a convex set; then for each $x \in \partial C$ let $S(x)$ denote the set of unit vectors such that

\begin{equation}
\sup_{y \in C} \langle v, y - x \rangle \leq 0;
\end{equation}

from convex analysis we know that $S(x)$ is nonempty and its elements are the support vectors of $C$ at $x$. 

Lemma 3.9 (Penalization function). Let \( C \) be a compact convex set; for \( x \in \partial C \) and \( v \in S(x) \cup \{0\} \) define the affine function:
\[
(3.10) \quad a_{x,v}(y) = \langle v, y - x \rangle;
\]
then we define the gauge:
\[
(3.11) \quad \psi_C(y) = \sup_{x \in \partial C, v \in S(x) \cup \{0\}} a_{x,v}(y).
\]
Then:
- \( \psi_C \) is convex;
- there is a choice of the subgradient such that \( \|\partial \psi_C\| \leq 1 \);
- \( \psi_C = 0 \) on \( C \);
- for \( x \not\in C \) we have
\[
(3.12) \quad \psi_C(x) \geq \|x - \pi_C(x)\|.
\]
Proof. The function \( \psi_C \) is the pointwise sup of a family of affine functions, hence is convex. As the zero affine function is in the family, \( \psi_C \geq 0 \) everywhere. But for each \( y \in C \), from the definition of supporting vector, we also have \( a_{x,v}(y) \leq 0 \) so \( \psi_C \) vanishes on \( C \). For any \( x \in \mathbb{R}^N \) a compactness argument gives an \( x, v \in S(x) \cup \{0\} \) (hence a \( v \) of norm at most 1) and such that:
\[
(3.13) \quad \psi_C(y) = a_{x,v}(y).
\]
From the following equations we see that we can use \( y \mapsto v \) as a subgradient at \( y \):
\[
\psi(z) + \langle v, y - z \rangle = \langle v, z - x \rangle + \langle v, y - z \rangle
\]
\[
\leq \psi_C(y).
\]
Let \( x \not\in C \); then \( \pi_C(x) \in \partial C \) and let \( v \) be the unit vector in the direction of \( x - \pi_C(x) \); then by the minimizing properties of \( \pi_C(x) \) for any \( y \in C \) we have
\[
(3.15) \quad \langle v, y - x \rangle \leq 0,
\]
which implies \( v \in S(\pi_C(x)) \); but then we get (3.12). \( \square \)

Lemma 3.16 (Constraint via penalization). Let \( f : \mathbb{R}^N \to \mathbb{R} \) be (continuous) convex with \( \|\partial f\| \leq G \) (for some choice of the subgradient); let \( C \subset \mathbb{R}^N \) be compact convex and \( x_{\text{opt}} \in C \) be a minimizer of the restriction of \( f \) to \( C \). Then \( f \) restricted on \( C \) and \( f + 2G\psi_C \) (on the whole \( \mathbb{R}^N \)) have the same minimizer; if for some \( x \in \mathbb{R}^N \) we have:
\[
(3.17) \quad f(x) + 2G\psi_C(x) - f(x_{\text{opt}}) \leq \varepsilon;
\]
then
\[
(3.18) \quad \|x - \pi_C(x)\| \leq \frac{\varepsilon}{G}.
\]
Proof. To show that \( f \) (restricted on \( C \)) and \( f + 2G\psi_C \) (on the whole \( \mathbb{R}^N \)) have the same minimizer we argue by contradiction, assuming for some \( u \not\in C \) we have
\[
(3.19) \quad f(\pi_C(u)) > f(u) + 2G\psi_C(u);
\]
then as \( u \neq \pi_C(u) \) we get the contradiction:
\[
(3.20) \quad G\|u - \pi_C(u)\| > 2G\|u - \pi_C(u)\|.
\]
The proof of (3.18) is immediate from Lemma 3.9:
\[ G \|x - \pi_C(x)\| \leq f(x) + 2G\psi_C(x) - f(\pi_C(x)) \]
(3.21)
\[ \leq f(x) + 2G\psi_C(x) - f(x_{\text{opt}}) \]
\[ \leq \varepsilon. \]
\[ (3.21) \]
\[ \square \]

Lemma 3.22 (Approximate optimization). Let \( f, \tilde{f} \) be real-valued functions on \( \mathbb{R}^N \) with \( |f - \tilde{f}| \leq \varepsilon \); let \( C \subset \mathbb{R}^N \) (not necessarily convex) and assume that \( x_{\text{opt}} \) is a minimizer of \( f \) on \( C \) and \( \tilde{x}_{\text{opt}} \) is a minimizer of \( \tilde{f} \) on \( C \). Then if \( x \) is a good candidate minimizer for \( \tilde{f} \) it is so also for \( f \):
(3.23)
\[ f(x) - f(x_{\text{opt}}) \leq 2\varepsilon + \tilde{f}(x) - \tilde{f}(\tilde{x}_{\text{opt}}). \]
Proof. We apply two times the uniform closeness of \( \tilde{f} \) and \( f \) and the definition of minimizers:
(3.24)
\[ f(x) - f(x_{\text{opt}}) \leq 2\varepsilon + \tilde{f}(x) - \tilde{f}(\tilde{x}_{\text{opt}}). \]
\[ (3.24) \]
\[ \square \]

3.3. Smoothing. As \( \partial f \) is not in general Lipschitz we resort to a perturbation argument to regularize \( f \) while staying close to \( f \). This technique is standard in real analysis, for a machine learning reference see [Haz16, Sec. 2.3.2, Algorithm 4].

Definition 3.25 (Mollifications). Let
(3.26)
\[ \eta_\varepsilon = \frac{1}{\varepsilon^N \text{Vol}(B(0,1))} \chi_{B(0,1)}, \]
so that \( \eta_\varepsilon \) is a probability distribution with mass absolutely continuous with respect to the Lebesgue measure. Moreover, \( \eta_\varepsilon \) is a function of bounded variation and Stokes’ Theorem shows that its gradient is:
(3.27)
\[ D\eta_\varepsilon = -\frac{1}{\varepsilon^N \text{Vol}(B(0,1))} \chi_{B(0,1)} \tilde{S}(0, \varepsilon), \]
where \( \tilde{S}(0, \varepsilon) \) is the signed measure on the boundary \( \partial B(0, \varepsilon) \) where the positive direction is that of the outward normal. Comparing the surface area of \( \partial B(0, \varepsilon) \) with the volume of \( B(0, \varepsilon) \) we obtain the total mass of \( D\eta_\varepsilon \):
(3.28)
\[ \|D\eta_\varepsilon\| (\mathbb{R}^N) = \frac{N}{\varepsilon}. \]
Given a function \( f : \mathbb{R}^N \to \mathbb{R}^M \) we can define the smoothing \( f_\varepsilon \) as the expectation:
(3.29)
\[ f_\varepsilon(x) = E_{\eta_\varepsilon} f(x - \cdot) = \int_{\mathbb{R}^N} f(x - v) \eta_\varepsilon(v) \, dv. \]
If \( f \) is convex then \( f_\varepsilon \) is convex as we take an expectation of a family of convex functions. Similarly, if \( \|f\| \) is bounded by a constant \( G \) so is \( \|f_\varepsilon\| \). For the gradient \( \partial f_\varepsilon \) we have the formulas:
(3.30)
\[ \partial f_\varepsilon(x) = \int \partial f(x - v) \eta_\varepsilon(v) \, dv = -\int f(x - v) \, dD\eta_\varepsilon(v), \]
where the second integral is with respect to the measure $D\eta$.

Using (3.30) we see that if $f$ is $G$-Lipschitz then $\partial f_\varepsilon$ is $\frac{GN}{\varepsilon}$-Lipschitz:

\begin{equation}
(3.31)
\partial f_\varepsilon(x) - \partial f_\varepsilon(y) = -\int (f(x - v) - f(y - v)) \, dD\eta_\varepsilon(v),
\end{equation}

from which

\begin{equation}
(3.32)
\|\partial f_\varepsilon(x) - \partial f_\varepsilon(y)\| \leq G \|x - y\| \|D\eta_\varepsilon\| (\mathbb{R}^N).
\end{equation}

Finally we also have the bound:

\begin{equation}
(3.33)
|f_\varepsilon(x) - f(x)| \leq \int_{\mathbb{R}^N} G \|v\| \eta_\varepsilon(v) \, dv \leq G\varepsilon.
\end{equation}

### 3.4. Analysis of SSGD.

We now analyze the convergence of SSGD. The main ideas of the proof are the use of $\psi_C$, using martingale bounds combined with the analysis of gradient descent on the expected smoothed objective $g$, and a case by case analysis (Step 5: this is the place of the argument that should be improved to speed up convergence).

**Theorem 3.34** (Analysis of SSGD). Assume a uniform bound $G$ on the norms of $\partial f_{t+1/2}$, $\partial f$; let $u_{opt}$ be a minimizer of $f$ on $C$; in the asymptotic regime where $\varepsilon \searrow 0$ one has that if

\begin{equation}
(3.35)
T \geq \left[ \frac{3G \text{diam} C}{2\varepsilon^3} + 1 \right]^2 - 1 = O(\varepsilon^{-6}),
\end{equation}

with probability $\geq 1 - 2\varepsilon$ the algorithm SSGD returns a final point $u_{end}$ which minimizes $f$ up to an error $O(\varepsilon \log(1/\varepsilon))$:

\begin{equation}
(3.36)
f(u_{end}) - f(u_{opt}) \leq 128G^2 \varepsilon \log(1/\varepsilon).
\end{equation}

On the other hand, in the case in which $\partial f$ is $L$-Lipschitz, one can achieve (3.36) for

\begin{equation}
(3.37)
T \geq \left[ \frac{3G \text{diam} C}{2\varepsilon^2} + 1 \right]^2 - 1 = O(\varepsilon^{-4}).
\end{equation}

**Proof.** Step 1: A gradient descent bound. The function that SSGD is effectively trying to minimize is the smoothing:

\begin{equation}
(3.38)
g = E_{\eta_{sm}}[f + 2G\psi_C]
\end{equation}

whose subgradient $\partial g$ satisfies the bound:

\begin{equation}
(3.39)
\|\partial g\| \leq 3G
\end{equation}

and is $\frac{2GN}{\varepsilon_{sm}}$-Lipschitz by (3.32); from now on we will use several times the bound (3.33) which implies that $|g - (f + 2G\psi_C)| \leq 3G\varepsilon_{sm}$. Moreover at time $t + 1/2$ we are using the function:

\begin{equation}
(3.40)
g_{t+1/2} = f_{t+1/2}(\cdot - v) + 2G\psi_C(\cdot - v),
\end{equation}

to decide the gradient step where $v$ is the point sampled from $\eta_{sm}$. Using the argument in Lemma 2.9 we get

\begin{equation}
(3.41)
g(u_{t+1}) - g(u_t) \leq \frac{27G^3 N}{\varepsilon_{sm}^2} \varepsilon_t^2 - \varepsilon_t(\partial g(u_t), \partial g_{t+1/2}(u_t));
\end{equation}
then for $m \geq 1$ we get:

\begin{equation}
(3.42) \quad g(u_{t+m}) - g(u_t) \leq \frac{27G^3 N}{\varepsilon_{\text{sm}}} \sum_{s=t}^{t+m-1} \varepsilon_s^2 - \sum_{s=t}^{t+m-1} \varepsilon_s (\partial g(u_s), \partial g_{s+1/2}(u_s)).
\end{equation}

**Step 2: A martingale bound.** Let $F_t$ denote the filtration at time $t$ (which can be integer or integer plus 1/2); SS GD gives rise to a random variable

\begin{equation}
(3.43) \quad X_T = -\sum_{s=1}^{T} \varepsilon_s [(\partial g(u_s), \partial g_{s+1/2}(u_s)) - \|\partial g(u_s)\|^2];
\end{equation}

then for $t \leq T$ define

\begin{equation}
(3.44) \quad X_t = E[X_T|F_{t+1}],
\end{equation}

so that $\{X_t\}_t$ defines a martingale. As $g_{s+1/2}$ is sampled independent of the filtration $F_s$ we find:

\begin{equation}
(3.45) \quad X_t = -\sum_{s=1}^{t} \varepsilon_s [(\partial g(u_s), \partial g_{s+1/2}(u_s)) - \|\partial g(u_s)\|^2];
\end{equation}

in particular we have the bound:

\begin{equation}
(3.46) \quad |X_t - X_{t-1}| \leq 18G^2 \varepsilon_t.
\end{equation}

We can then use Hoeffding’s inequality in the form explained in [Hoe63, equation 2.17] (this gives a slightly better bound and it uses one of Doob’s maximal inequalities) to obtain:

\begin{equation}
(3.47) \quad P(\max_t |X_t| \geq \varepsilon_{\text{prob}}) \leq 2 \exp \left( -\frac{\varepsilon_{\text{prob}}^2}{648G^4 \sum_{t=1}^{T} \varepsilon_t^2} \right).
\end{equation}

**Step 3: Combining the martingale and the gradient descent bounds.** We now let $u_{\text{opt}}^g \in \mathbb{R}^N$ be an optimal point for $g$; by Lemma 3.22 a minimizer $u_{\text{opt}} \in \mathcal{C}$ of $f + 2G\psi_{\mathcal{C}}$ (and of $f$ restricted on $\mathcal{C}$ by Lemma 3.16) is also almost a minimizer of the smoothing $g$:

\begin{equation}
(3.48) \quad g(u_{\text{opt}}) \leq 6G\varepsilon_{\text{sm}} + g(u_{\text{opt}}^g).
\end{equation}

Let $\Omega$ denote the set of events where one has:

\begin{equation}
(3.49) \quad \max_t |X_t| \leq \varepsilon_{\text{prob}}.
\end{equation}

Combining (3.42) with (3.48)–(3.49) we get:

\begin{equation}
(3.50) \quad g(u_{T+1}) - g(u_{\text{opt}}^g) \leq \frac{27G^3 N}{\varepsilon_{\text{sm}}} \sum_{s=1}^{T} \varepsilon_s^2 - \sum_{s=1}^{T} \varepsilon_s \|\partial g(u_s)\|^2 + g(u_1) - g(u_{\text{opt}}) + 6G\varepsilon_{\text{sm}} + \varepsilon_{\text{prob}}.
\end{equation}
Step 4: Bounding the distance of the $u_t$’s from the convex set. We will now prove a bound on the distance of any $u_t$ from the set $C$:

$$\|u_t - \pi_C(u_t)\| \leq (f + 2G\psi_C(u_t)) - (\pi_C(u_t)) \leq 6G\varepsilon_{sm} + g(u_t) - g(\pi_C(u_t))$$

$$\leq 6G\varepsilon_{sm} + g(u_t) - g(\pi_C(u_t)) + g(\pi_C(u_t)) - g(u_{opt})$$

$$\leq 12G\varepsilon_{sm} + g(u_t) - g(u_{opt})$$

$$\leq \frac{27G^3N}{\varepsilon_{sm}} \sum_{s=1}^{T} \varepsilon_s^2 - \sum_{s=1}^{T} \varepsilon_s \|\partial g(u_s)\|^2 + \varepsilon_{\text{prob}} + 12G\varepsilon_{sm} + g(u_1) - g(u_{opt})$$

$$\leq \frac{27G^3N}{\varepsilon_{sm}} \sum_{s=1}^{T} \varepsilon_s^2 + \varepsilon_{\text{prob}} + 12G\varepsilon_{sm} + 3G \|u_1 - u_{opt}\|;$$

from which we get:

$$\text{dist}(u_t, C) \leq \frac{27G^2N}{\varepsilon_{sm}} \sum_{s=1}^{T} \varepsilon_s^2 + \frac{\varepsilon_{\text{prob}} + 12G\varepsilon_{sm}}{G} + 3 \text{diam } C.$$

Step 5: Bounding the algorithm in different cases. We now need to make an analysis in different cases; in case C1 we have:

$$\frac{27G^3N}{\varepsilon_{sm}} \sum_{s=1}^{T} \varepsilon_s^2 - \sum_{s=1}^{T} \varepsilon_s \|\partial g(u_s)\|^2 + g(u_1) - g(u_{opt}) + 6G\varepsilon_{sm} + \varepsilon_{\text{prob}} \leq \varepsilon_{\text{err}};$$

this implies

$$g(u_{T+1}) - g(u_{opt}) \leq \varepsilon_{\text{err}}.$$ 

In case C1 is violated we have:

$$\sum_{s=1}^{T} \varepsilon_s \|\partial g(u_s)\|^2 \leq \frac{27G^3N}{\varepsilon_{sm}} \sum_{s=1}^{T} \varepsilon_s^2 + g(u_1) - g(u_{opt}) + 6G\varepsilon_{sm} + \varepsilon_{\text{prob}} - \varepsilon_{\text{err}};$$

in this case there must be values of $t$ for which one has:

$$\|\partial g(u_t)\| \leq \left( \frac{27G^3N}{\varepsilon_{sm}} \sum_{s=1}^{T} \varepsilon_s^2 + g(u_1) - g(u_{opt}) + 6G\varepsilon_{sm} + \varepsilon_{\text{prob}} - \varepsilon_{\text{err}} \right)^{1/2};$$

let $t^*$ be the maximal such $t$ satisfying (3.56). Combining (3.42) with the fact that on $\Omega$ (3.49) holds we get:

$$g(u_{T+1}) - g(u_{t^*}) \leq \frac{27G^3N}{\varepsilon_{sm}} \sum_{s=1}^{T} \varepsilon_s^2 + 2\varepsilon_{\text{prob}} - \sum_{s=t^*}^{T} \varepsilon_s \|\partial g(u_s)\|^2.$$

In case C2 we have:

$$\frac{27G^3N}{\varepsilon_{sm}} \sum_{s=t^*}^{T} \varepsilon_s^2 - \sum_{s=t^*}^{T} \varepsilon_s \|\partial g(u_s)\|^2 \leq 0;$$
in this case we combine (3.48), (3.52), (3.56) and (3.57) to obtain:

\begin{equation}
(3.59) \quad g(u_{T+1}) - g(u^*_{opt}) \leq g(u_{T+1}) - g(u_{opt}) + g(u_{opt}) - g(u^*_{opt}) + 2\varepsilon_{\text{prob}}
\leq BG1 \times (d_\varepsilon + \text{diam } C) + 2\varepsilon_{\text{prob}} + 6G\varepsilon_{sm}.
\end{equation}

If case \textbf{C2} fails we are in case \textbf{C3} in which we see that the set of those \( s \geq t^* \) such that:

\begin{equation}
(3.60) \quad \|\partial g(u_s)\|^2 \leq \frac{27G^3N\sum_{t \geq s+1} \varepsilon_s^2}{\varepsilon_{sm} \sum_{t \geq s} \varepsilon_t}
\end{equation}

is not empty. Let \( t^{**} \) be a maximal \( s \in \{t^*, \ldots, T\} \) satisfying (3.60). Let:

\begin{equation}
(3.61) \quad BG2 = \left( \frac{27G^3N\sum_{s \geq t^{**}} \varepsilon_s^2}{\sum_{s \geq t^{**}} \varepsilon_s} \right)^{1/2}.
\end{equation}

Then arguing as we obtained (3.57) we get:

\begin{equation}
(3.62) \quad g(u_{T+1}) - g(u_{t^{**}+1}) \leq \frac{27G^3N}{\varepsilon_{sm}} \sum_{s = t^{**}+1}^T \varepsilon_s^2 + 2\varepsilon_{\text{prob}} - \sum_{s = t^{**}+1}^T \varepsilon_s \|\partial g(u_s)\|^2.
\end{equation}

By maximality of \( t^{**} \) we have

\begin{equation}
(3.63) \quad \frac{27G^3N}{\varepsilon_{sm}} \sum_{s = t^{**}+1}^T \varepsilon_s^2 - \sum_{s = t^{**}+1}^T \varepsilon_s \|\partial g(u_s)\|^2 \leq 0;
\end{equation}

now combining (3.48), (3.51), (3.60) and (3.61) we get:

\begin{equation}
(3.64) \quad g(u_{T+1}) - g(u^*_{opt}) \leq g(u_{T+1}) - g(u_{t^{**}+1}) + g(u_{t^{**}+1}) - g(u_{t^{**}})
+ g(u_{t^{**}}) - g(u_{opt}) + g(u_{opt}) - g(u^*_{opt})
\leq 2\varepsilon_{\text{prob}} + 3G\varepsilon_{t^{**}} + BG2 \times (d_\varepsilon + \text{diam } C)
+ 6G\varepsilon_{sm}.
\end{equation}

**Step 6:** Choice of the learning rate sequence and asymptotic bounds. We start by requiring \( \varepsilon_{\text{prob}} = \varepsilon_{\text{err}} \) and setting

\begin{equation}
(3.65) \quad \varepsilon_s = \varepsilon_{sm} \frac{1}{\sqrt{s}};
\end{equation}

from the bounds

\begin{equation}
(3.66) \quad \frac{1}{\sqrt{t}} \chi_{|t_1, t_2+1]}(t) \leq \sum_{s = t_1}^{t_2} \frac{1}{\sqrt{s}} \chi_{|s, s+1]}(t) \leq \frac{1}{\sqrt{t-1}} \chi_{|\max(t_1, 2), t_2+1]} + \chi_{t_1=1}\chi_{|1, 2]};
\end{equation}

we derive

\begin{equation}
(3.67) \quad \varepsilon_{sm}^2 (\log(t_2 + 1) - \log(t_1)) \leq \sum_{s = t_1}^{t_2} \varepsilon_s^2
\leq \varepsilon_{sm}^2 (\chi_{t_1=1} + \log(t_2) - \log(\max(t_1, 2) - 1)),
\end{equation}

and

\begin{equation}
(3.68) \quad 2\varepsilon_{sm}(\sqrt{t_2 + 1} - \sqrt{t_1}) \leq \sum_{s = t_1}^{t_2} \varepsilon_s
\leq 2\varepsilon_{sm}(\chi_{t_1=1} + \sqrt{t_2} - \sqrt{\max(t_1, 2) - 1}).
\end{equation}
Substitution of these bounds in (3.56) yields:

\[
(3.69) \quad \text{BG}_1 \leq \left( \frac{27G^3N(1 + \log T) + 6G}{2(\sqrt{T} + 1 - 1)} + \frac{3G \text{diam} \mathcal{C}}{2\varepsilon_{\text{sm}}(\sqrt{T} + 1 - 1)} \right)^{1/2};
\]

making the choice of the \( T(\varepsilon_{\text{sm}}) \) as:

\[
(3.70) \quad T = \left[ \frac{3G \text{diam} \mathcal{C}}{2\varepsilon_{\text{sm}}^3} + 1 \right]^2 - 1 = O(\varepsilon_{\text{sm}}^{-6})
\]

we get that in the asymptotic case \( \varepsilon_{\text{sm}} \searrow 0 \) one has:

\[
(3.71) \quad \text{BG}_1 \lessgtr \varepsilon_{\text{sm}}.
\]

Using the bounds in (3.61) we get:

\[
(3.72) \quad \text{BG}_2 \leq \left( \frac{27G^3N(\log T + \chi_{t^{**} = 1} - \log(\max(t^{**}, 2) - 1))}{2(\sqrt{T} + 1 - \sqrt{t^{**}})} \right)^{1/2}
\]

\[
\leq \left( \frac{27G^3N \log T - \log(T - 1)}{2\sqrt{T} + 1 - \sqrt{T}} \right)^{1/2}
\]

\[
\leq \left( \frac{27G^3N}{T - 1} \right)^{1/2};
\]

from which we get the asymptotic bound

\[
(3.73) \quad \text{BG}_2 \lessgtr \left( \frac{18G^2N}{\text{diam} \mathcal{C}} \right)^{1/2} \varepsilon_{\text{sm}}^{3/2}.
\]

We now turn to (3.47), making the choice

\[
(3.74) \quad \varepsilon_{\text{prob}} = 64G^2\varepsilon_{\text{sm}} \log(1/\varepsilon_{\text{sm}})
\]

we get:

\[
(3.75) \quad P(\max_i |X_i| \geq 64G^2\varepsilon_{\text{sm}} \log(1/\varepsilon_{\text{sm}})) \leq 2 \times \exp \left( -\frac{\varepsilon_{\text{sm}}^2 (\log(1/\varepsilon_{\text{sm}}))^2 (64G)^2}{648G^4\varepsilon_{\text{sm}}^2 \log\left( \left( \frac{3G \text{diam} \mathcal{C}}{2\varepsilon_{\text{sm}}^3} + 1 \right)^2 - 1 \right) + 1} \right)
\]

\[
\leq 2 \exp(-\log(1/\varepsilon_{\text{sm}})) = 2\varepsilon_{\text{sm}}.
\]

Now in the asymptotic regime when \( \varepsilon_{\text{sm}} \searrow 0 \), the quantity \( \varepsilon_{\text{prob}} \) dominates in the bounds (3.54) (case C1), (3.59) (case C2), (3.64) (case C3); i.e. on \( \Omega \) one has that

\[
(3.76) \quad g(u_{T+1}) - g(u_{\text{opt}}) \lessgtr 2\varepsilon_{\text{prob}} = 128G^2\varepsilon_{\text{sm}} \log(1/\varepsilon_{\text{sm}}).
\]
We can now obtain an asymptotic bound for \( u_{\text{end}} \) being an approximate minimizer of \( f \) on \( C \):

\[
 f(u_{\text{end}}) - f(u_{\text{opt}}) \leq f(u_{\text{end}}) - (f(u_{T+1}) + 2G\psi_C(u_{T+1})) \\
\leq 0 \text{ by (3.12)} \\
+ (f(u_{T+1}) + 2G\psi_C(u_{T+1})) - f(u_{\text{opt}}) \\
\leq 2\varepsilon_{\text{sm}} + g(u_{T+1}) - g(u_{\text{opt}}) \\
\leq 2\varepsilon_{\text{sm}} + g(u_{T+1}) - g(u_{\text{opt}}) \\
\leq 0 \text{ by (3.12)} \\
\leq 2\varepsilon_{\text{prob}}.
\]

Now (3.36) follows by setting \( \varepsilon_{\text{sm}} = \varepsilon \).

**Step 7: Proof of (3.37).** In this case a minor variation is required, we leave most details to the reader. The point is that no smoothing is required and thus one can directly work with \( f + 2G\psi_C \) and then set \( \varepsilon_s = \frac{1}{\sqrt{s}} \).

\[\square\]

### 3.5. Analysis of RASPGD

In this section we analyze RASPGD. The argument is close to that of RAPGD, the main difference is the use of a concentration inequality argument.

**Theorem 3.78 (Analysis of RASPGD).** Assume a uniform bound \( G \) on the norms of \( \partial f_{t+1/2}, \partial f \); let \( u_{\text{opt}} \) be a minimizer of \( f \) on \( C \); in the asymptotic regime where \( \varepsilon \to 0 \) one has that if

\[
 T \geq \left\lceil \frac{2G \text{diam} C + G^2}{4\varepsilon} \right\rceil^2 = O(\varepsilon^{-2})
\]

with probability at least

\[
 1 - \frac{32\varepsilon^2}{(2G \text{diam} C + G^2)^2} = 1 - O(\varepsilon^2)
\]

the algorithm RASPGD returns a final point \( u_{\text{end}} \) which minimizes \( f \) up to an error \( O(\varepsilon \log(1/\varepsilon)) \):

\[
 f(u_{\text{end}}) - f(u_{\text{opt}}) \leq 2\varepsilon \log(1/\varepsilon).
\]

**Proof.**

**Step 1:** Generalize the bounds in Theorem 2.1. Equation (2.6) has been established at the level of each iteration, so in RASPGD we get:

\[
 \varepsilon_t(f_{t+1/2}(u_t) - f_{t+1/2}(u_{\text{opt}})) \leq \frac{\|u_t - u_{\text{opt}}\|^2 - \|u_{t+1} - u_{\text{opt}}\|^2 + G^2\varepsilon_t^2}{2},
\]

thus we can also generalize (2.7):

\[
 \sum_{t=1}^{T} \varepsilon_t(f_{t+1/2}(u_t) - f_{t+1/2}(u_{\text{opt}})) \leq \frac{\|u_1 - u_{\text{opt}}\|^2 + G^2\sum_{t=1}^{T} \varepsilon_t^2}{2}.
\]

**Step 2:** A martingale bound. Let \( F_t \) denote the filtration at time \( t \) (which can be integer or integer plus 1/2); RASPGD gives rise to a random variable

\[
 X_T = \sum_{s=1}^{T} \varepsilon_s [(f_{s+1/2}(u_s) - f_{s+1/2}(u_{\text{opt}})) - (f(u_s) - f(u_{\text{opt}}))];
\]

then for \( t \leq T \) define

\[
 X_t = E[X_T | F_{t+1}],
\]
so that \( \{X_t\}_t \) defines a martingale. As \( f_{s+1/2} \) is sampled independent of the filtration \( \mathcal{F}_s \) we find:

\[
X_t = \sum_{s=1}^{t} \varepsilon_s [(f_{s+1/2}(u_s) - f_{s+1/2}(u_{\text{opt}})) - (f(u_s) - f(u_{\text{opt}}))];
\]

in particular we have the bound:

\[
|X_t - X_{t-1}| \leq 2G \text{diam} C \varepsilon_t.
\]

We can then use Hoeffding’s inequality [Hoe63, equation 2.17] (this gives a slightly better bound and it uses one of Doob’s maximal inequalities) to obtain:

\[
P(\max_t |X_t| \geq \varepsilon_{\text{prob}}) \leq 2 \exp\left(-\frac{\varepsilon_{\text{prob}}^2}{4G^2(\text{diam} C)^2 \sum_{t=1}^{T} \varepsilon_t^2}\right).
\]

Let \( \Omega \) denote the set of event where one has:

\[
(3.89) \quad |X_T| \leq \varepsilon_{\text{prob}}.
\]

**Step 3: Bounding the Algorithm on \( \Omega \).** Using (3.83) and the definitions of \( X_T \) and \( \Omega \) we conclude that on \( \Omega \):

\[
\sum_{t=1}^{T} \varepsilon_t (f(u_t) - f(u_{\text{opt}})) \leq \frac{(\text{diam} C)^2 + G^2 \sum_{t=1}^{T} \varepsilon_t^2}{2} + \varepsilon_{\text{prob}};
\]

application of Jensen’s inequality finally yields:

\[
f(u_{\text{end}}) - f(u_{\text{opt}}) \leq \frac{(\text{diam} C)^2 + G^2 \sum_{t=1}^{T} \varepsilon_t^2 + 2\varepsilon_{\text{prob}}}{2 \sum_{t=1}^{T} \varepsilon_t}.
\]

**Step 4: Choice of the sequence \( \varepsilon_t \).** We set

\[
\varepsilon_t = \frac{1}{\sqrt{t}}
\]

and

\[
\varepsilon_{\text{prob}} = 2G \text{diam} C (\log T + 1);
\]

in analogy with (3.66)–(3.68) we get:

\[
P(\Omega^c) \leq \frac{2}{T}
\]

and

\[
f(u_{\text{end}}) - f(u_{\text{opt}}) \leq \frac{(\text{diam} C)^2 + G^2(\log T + 1) + 2G \text{diam} C (\log T + 1)}{4(\sqrt{T} - 1)}
\]

\[
\leq \frac{2G \text{diam} C + G^2 \log T}{4 \sqrt{T}}.
\]

If we choose

\[
T \geq \left\lceil \frac{2G \text{diam} C + G^2}{4\varepsilon} \right\rceil^2 = O(\varepsilon^{-2})
\]

we then get

\[
P(\Omega^c) \lesssim \frac{32\varepsilon^2}{(2G \text{diam} C + G^2)^2}
\]
and

\[(3.98) \quad f(\text{u}_{\text{end}}) - f(\text{u}_{\text{opt}}) \lesssim 2\varepsilon \log(1/\varepsilon).\]

\[\square\]

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