Minimax rate of estimation for invariant densities associated to continuous stochastic differential equations over anisotropic Hölder classes.

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Abstract

We study the problem of the nonparametric estimation for the density $\pi$ of the stationary distribution of a $d$-dimensional stochastic differential equation $(X_t)_{t \in [0, T]}$. From the continuous observation of the sampling path on $[0, T]$, we study the estimation of $\pi(x)$ as $T$ goes to infinity. For $d \geq 2$, we characterize the minimax rate for the $L^2$-risk in pointwise estimation over a class of anisotropic Hölder functions $\pi$ with regularity $\beta = (\beta_1, ..., \beta_d)$. For $d \geq 3$, our finding is that, having ordered the smoothness such that $\beta_1 \leq ... \leq \beta_d$, the minimax rate depends on whether $\beta_2 < \beta_3$ or $\beta_2 = \beta_3$. In the first case, this rate is $\left(\frac{\log T}{T}\right)^\gamma$, and in the second case, it is $\left(\frac{1}{T}\right)^\gamma$, where $\gamma$ is an explicit exponent dependent on the dimension and $\bar{\beta}_3$, the harmonic mean of smoothness over the $d$ directions after excluding $\beta_1$ and $\beta_2$, the smallest ones. We also demonstrate that kernel-based estimators achieve the optimal minimax rate. Furthermore, we propose an adaptive procedure for both $L^2$ integrated and pointwise risk. In the two-dimensional case, we show that kernel density estimators achieve the rate $\frac{\log T}{T}$, which is optimal in the minimax sense. Finally we illustrate the validity of our theoretical findings by proposing numerical results.

Non-parametric estimation, stationary measure, minimax rate, convergence rate, ergodic diffusion, anisotropic density estimation.

1 Introduction

In this work, we propose kernel density estimators for estimating the invariant density associated with a stochastic differential equation. Specifically, we investigate inference for the following $d$-dimensional process

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t a(X_s)dW_s, \quad t \in [0, T],$$

with $b : \mathbb{R}^d \to \mathbb{R}^d$, $a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $W = (W_t, t \geq 0)$ a $d$-dimensional Brownian motion.

Stochastic differential equations are very attractive for statisticians nowadays, as they model stochastic evolution as time evolves. These models have a variety of applications in many disciplines and emerge in a natural way in the study of many phenomena. For example, there are applications in astronomy [69], mechanics [49], physics [63], geology [28], ecology [39], biology [66] and epidemiology [7]. Some other examples are economics [13], mathematical finance [41], political analysis and social processes [19] as well as neurology [38], genetic analysis [52], cognitive psychology [72] and biomedical sciences [11].

Given the significance of stochastic differential equations, there has been extensive research into inference for such models. Authors have explored both parametric and non-parametric inference, beginning with continuous or discrete observations, and within various asymptotic frameworks. These frameworks include small diffusions asymptotics over a fixed time interval and long time...
intervals for ergodic models. Some landmarks in the area are the books of Kutoyants [48], Lacus [42], Kessler et al [43] and Höpfner [40]. In the meantime, a significant number of papers have been published on this topic. Among them, we mention Comte et al [21], Dalalyan and Reiss [23], Genon-Catalot [31], Gobet et al [33], Hoffmann [37], Larédou [52] and Yoshida [74].

The in-depth study on stochastic differential equations lead the way for statistical inference for more complicated models such as SDEs driven by Lévy processes (see for example [59]), diffusions with jumps ([65], [68]), stochastic partial differential equations ([2], [15]), diffusions with mixed effects ([65], [25]) and Hawkes processes ([8], [27]).

In this paper, we propose a kernel density estimator and aim to determine the convergence rates for pointwise $L^2$ estimation of the stationary measure $\pi$ associated with the process $X$ defined by (1). Additionally, we will assess the optimality of these rates.

The estimation of the invariant measure is a problem already widely faced in many different contexts by many authors, we quote [21], [23], [15], [75], [1], [18] and [12] among them. Nowadays, this problem attracts the attention of many statisticians due to the multitude of numerical methods associated with it, with the Markov Chain Monte Carlo method being the most prominent. For instance, in [51] and [62], one can find approximation algorithms for computing the invariant density, while the analysis of invariant distributions is employed to assess the stability of stochastic differential systems in [36] and [12]. It can also be used in order to estimate the drift coefficient in a non-parametric way (see [57] and [67]). However, there are some problems involving the estimation of the invariant density for SDEs in the anisotropic context which are still open, because of their challenging nature. One above all, the research of minimax convergence rate for the estimation of the invariant density associated to a multidimensional diffusion. The purpose of this work is to fill such a gap.

As it is easy to see, the estimation of the invariant measure presents itself as both a long-standing problem and an actively researched area. Kernel estimators, for various reasons, serve as powerful tools in this context. For instance, in [12] and [10], kernel estimators are employed to estimate the marginal density of a continuous-time process. In [5], [4], and [54], they find utility within a jump-diffusion framework, with the latter specifically applied for estimating volatility.

In a context closely related to ours, kernel density estimators have been employed in [23] and [70] to investigate the convergence rate for estimating the invariant density associated with a reversible diffusion process with unit diffusion. Notably, in [23], the authors establish convergence rates for pointwise estimation of the invariant density, as a by-product of their analysis, under isotropic Hölder smoothness constraints. The convergence rates they found are $\frac{\log T}{T}$ when the dimension $d$ is equal to 2 and $(\frac{1}{T})^{\frac{2\beta}{2\beta+d}}$ for $d \geq 3$, where $\beta$ is the common smoothness over the $d$ different directions. Strauch extended the work presented in [23] by developing adaptive kernel estimators for estimating the invariant density within anisotropic Hölder spaces. These estimators achieve identical convergence rates as those in [23], with the replacement of $\beta$ by $\beta_3$—the harmonic mean across the $d$ different directions. Given that the smoothness characteristics of function space elements can vary depending on the chosen direction in $\mathbb{R}^d$, the concept of anisotropy assumes a significant role.

In this work we aim at estimating the invariant density $\pi$ by means of the kernel estimator $\hat{\pi}_{h,T}$ assuming to have the continuous record of the process $X$ solution to (1) up to time $T$. We first of all prove the following upper bound for the mean squared error, for $d \geq 3$: there exist $T_0$ and $c > 0$ such that, for any $T \geq T_0$,

$$\sup_{(u,b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq \begin{cases} c \left(\frac{\log T}{T}\right)^{\frac{\beta_3}{\beta_3 + d}} & \text{for } \beta_2 < \beta_1, \\ c \left(\frac{1}{T}\right)^{\frac{\beta_3}{\beta_3 + d}} & \text{for } \beta_2 = \beta_1, \end{cases}$$

(2)

where $\Sigma$ is a class of coefficients for which the stationary density has some prescribed regularity, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_d$ and $\beta_3$ is the harmonic mean over the smoothness after having removed the two smallest. In particular, we have $\frac{1}{\beta_3} := \frac{1}{d-2} \sum_{k>2} \frac{1}{\beta_k}$ and so it clearly follows that $\beta_3 \geq \beta$. It is surprising that the presence of the logarithm in the convergence rate depends on whether $\beta_2 = \beta_3$ or not. This variation arises from different methods of bounding the variance, resulting in varying degrees of freedom for the optimal bandwidth $h = (h_1, \ldots, h_d)$. In some specific cases, as detailed in Remark 1, we can eliminate the logarithmic term.

We also introduce an adaptive procedure based on the Goldenshugler-Lepski method, which considers both the integrated and pointwise $L^2$ risk. This procedure selects a bandwidth that strikes an optimal balance between bias and variance based on the available data. The motivation be-
hind studying the data-selection procedure with respect to the integrated \( L^2 \) risk is to prevent the potential loss of a logarithm in our estimation, as discussed in \[17\] for example. This data-driven procedure is capable of automatically detecting the presence of a logarithmic term in the convergence rate.

It is important to note that while the convergence rate determined earlier, dependent on \( \bar{\beta}_3 \), may not appear immediately intuitive, it aligns with the findings in \[3\] for estimating the invariant density associated with a Lévy-driven SDE. In the mentioned paper, numerical results underscore the distinct role played by the two smallest levels of smoothness compared to the others. In our work, we delve into the specifics of this role, and with the result presented in \[2\], we can elucidate their impact on the convergence rate. Furthermore, we remove the boundedness condition on the drift coefficient. We also prove the following upper bound, in the bidimensional context: there exist \( T_0 \) and \( c > 0 \) such that, for any \( T \geq T_0 \),

\[
\sup_{(a,b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq c \frac{\log T}{T},
\]

The convergence rate described above is consistent with that in \[23\] and \[70\]. However, the question of whether this rate is optimal had remained unanswered until now. To our knowledge, the only existing results in this context are some lower bounds for the sup-norm risk, as demonstrated in \[70\] and analogous findings in the jump-framework discussed in \[5\].

In this paper we obtain a minimax lower bound on the \( L^2 \)-risk for the pointwise estimation with the same rate \( \frac{\log T}{T} \). To do that, we introduce the minimax risk

\[
\mathcal{R}_T := \inf_{\pi_T} \sup_{(a,b) \in \Sigma} \mathbb{E}[|\pi_T(x_0) - \pi_T(x_0)|^2],
\]

where the infimum is taken over all the possible estimators of the invariant density. Then, we show that there exist \( c > 0 \) and \( T_0 > 0 \) such that, for \( T \geq T_0 \),

\[
\mathcal{R}_T \geq c \frac{\log T}{T}.
\]

For \( d = 2 \), it is not possible to propose an estimator with a better convergence rate than the one we have found for the kernel density estimator we introduced. Regarding the case \( d \geq 3 \) we prove that, in general, the following lower bound holds true: there exists \( c > 0 \) and \( T_0 > 0 \) such that, for any \( T \geq T_0 \),

\[
\mathcal{R}_T \geq c \left( \frac{\log T}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}}.
\]

This result is not very surprising, as an analogous lower bound has been showed in \[3\] for the estimation of the invariant density associated to a Lévy driven SDE. What is instead quite surprising is that, when \( \beta_2 < \beta_3 \), it is possible to improve the result here above by showing there exist \( T_0 \) and \( c > 0 \) such that, for any \( T \geq T_0 \),

\[
\mathcal{R}_T \geq c \left( \frac{\log T}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}}.
\]

These results highlight how crucial the condition \( \beta_2 < \beta_3 \) is. They imply that, on a class of diffusions \( X \) whose invariant density has the prescribed regularity, it is not possible to find an estimator with rates of estimation faster than in \[3\]. It follows that the kernel density estimator achieves the best possible convergence rates.

It is worth remarking that the upper bound for the mean squared error relies on an upper bound for the transition density and the mixing properties of the process. Due to the latter, the constants involved are dependent on the coefficients, and it can be quite challenging, in general, to establish uniform control over these constants. However, relying on the theory of Lyapounov-Poincaré as introduced in \[9\], we prove a mixing inequality uniform on the class of coefficient \((a, b) \in \Sigma\) (see Proposition \[1\]). We remark that the Lyapounov-Poincaré method can be extended to get uniform mixing inequality also in more general frameworks, such as the jump context.

We therefore get a minimax upper bound on the risk of the estimator \( \hat{\pi}_{h,T} \) which is uniform over the class of coefficient \( \Sigma \) and we complement it by obtaining a minimax lower bound on the risk of the estimator \( \hat{\pi}_{h,T} \). Finally, we present numerical results to demonstrate the successful validation of our theoretical findings.
2 Model Assumptions

We want to estimate in a non-parametric way the invariant density associated to a $d$-dimensional diffusion process $X$. In the sequel we assume that a continuous record of the process $X^T = \{X_t, 0 \leq t \leq T\}$ up to time $T$ is available. The diffusion is a solution of the following stochastic differential equation:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t a(X_s)dW_s, \quad t \in [0, T], \quad (3)$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $a : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ and $W = (W_t, t \geq 0)$ is a $d$-dimensional Brownian motion. The initial condition $X_0$ and $W$ are independent. We denote $\tilde{a} := a \cdot a^T$.

We denote with $|.|$ and $<.,.>$ respectively the Euclidian norm and the scalar product in $\mathbb{R}^d$, and for a matrix in $\mathbb{R}^d \otimes \mathbb{R}^d$ we denote its operator norm by $|.|$.

**A1:** The functions $b(x)$ and $a(x)$ are globally Lipschitz functions of class $\mathcal{C}^1$, such that for all $x \in \mathbb{R}^d$,

$$|b(0)| \leq b_0, \quad |a(x)| \leq a_0, \quad |\frac{\partial}{\partial x_1} b(x)| \leq b_1, \quad |\frac{\partial}{\partial x_1} a(x)| \leq a_1, \quad \text{for } i \in \{1, \ldots, d\},$$

where $b_0 > 0$, $a_0 > 0$, $a_1 > 0$, $b_1 > 0$ are some constants. Moreover, for some $\alpha_{\min} > 0$,

$$a_{\min}^{-2}d \times d \leq \tilde{a}(x)$$

where $1_{d \times d}$ denotes the $d \times d$ identity matrix.

**A2 (Drift condition):**

There exist $\tilde{C} > 0$ and $\tilde{\rho} > 0$ such that $< x, b(x) > \leq -\tilde{C}|x|$, $\forall x : |x| \geq \tilde{\rho}$.

The assumptions here above involve the ergodicity of the process and they are needed in order to show the existence of a Lyapounov function (see for example [8]). Hence, the process $X$ admits a unique invariant distribution $\mu$ and the ergodic theorem holds (see Lemma 2 below). We suppose that the invariant probability measure $\mu$ of $X$ is absolutely continuous with respect to the Lebesgue measure and from now on we will denote its density as $\pi : d\mu = \pi dx$.

As in several cases the regularity of some function $g : \mathbb{R}^d \to \mathbb{R}$ depends on the direction in $\mathbb{R}^d$ chosen, we decide to work under anisotropic smoothness constraints. In particular, we assume that the density $\pi$ we want to estimate belongs to the anisotropic Hölder class $\mathcal{H}_d(\beta, \mathcal{L})$ defined below.

**Definition 1.** Let $\beta = (\beta_1, \ldots, \beta_d)$, $\beta_i \geq 0$, $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$, $\mathcal{L}_i > 0$. A function $g : \mathbb{R}^d \to \mathbb{R}$ is said to belong to the anisotropic Hölder class $\mathcal{H}_d(\beta, \mathcal{L})$ of functions if, for all $i \in \{1, \ldots, d\}$,

$$\left\|D^k_i g\right\|_{\infty} \leq \mathcal{L}_i^{\beta_i}, \quad \forall k = 0, 1, \ldots, \lfloor \beta_i \rfloor,$$

$$\left\|D^{|\beta_i|}_i g(., e_i) - D^{|\beta_i|}_i g(., e_i)\right\|_{\infty} \leq \mathcal{L}_i |t|^{|\beta_i| - |\beta_i|} \quad \forall t \in \mathbb{R},$$

for $D^k_i g$ denoting the $k$-th order partial derivative of $g$ with respect to the $i$-th component, $\lfloor \beta_i \rfloor$ denoting the largest integer strictly smaller than $\beta_i$ and $e_1, \ldots, e_d$ denoting the canonical basis in $\mathbb{R}^d$. 

The outline of the paper is the following. In Section 2 we introduce the model and its assumptions. In Section 3, we present the kernel estimator and provide upper bounds for the mean squared error. The Section 3.2 is devoted to the adaptive version of the estimator and associated upper bounds. Section 4 is dedicated to stating complementary lower bounds, complementing the results in the previous section. In Section 5, we present numerical results on the adaptive estimators. In Section 6, we prove the upper bounds stated in Section 5 while Section 7 contains proofs related to the adaptive version of the estimators. The Section 8 is devoted to the proof of the lower bounds. Some technical results are moreover showed in the appendix.
This leads us to consider a class of coefficients \((a, b)\) for which the stationary density \(\pi = \pi_{(a, b)}\) has some prescribed Hölder regularity.

**Definition 2.** Let \(\beta = (\beta_1, ..., \beta_d)\), \(\beta_i \geq 0\) and \(\mathcal{L} = (\mathcal{L}_1, ..., \mathcal{L}_d)\), \(\mathcal{L}_i > 0\), \(0 < a_{min} \leq a_0\) and \(b_0 > 0\), \(a_1 > 0\), \(b_1 > 0\), \(C > 0\), \(\rho > 0\).

We define \(\Sigma(\beta, \mathcal{L}, a_{min}, b_0, a_0, a_1, b_1, C, \rho)\) the set of couple of functions \((a, b)\) where \(a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d\) and \(b : \mathbb{R}^d \to \mathbb{R}^d\) are such that

- \(a\) and \(b\) satisfy Assumption A1 with the constants \((a_{min}, b_0, a_0, a_1, b_1)\),
- \(b\) satisfies Assumption A2 with the constants \((\bar{C}, \bar{\rho})\),
- the density \(\pi_{(a, b)}\) of the invariant measure associated to the stochastic differential equation \((\Sigma)\) belongs to \(\mathcal{H}_d(\beta, 2\mathcal{L})\).

In the sequel we will use repeatedly some known results about the transition density and the ergodicity of diffusion processes. The bound on the transition density in case of unbounded drift follows from \([60]\). As we will see, comparing it with the results known for the bound of the transition density in case of bounded drift, the main difference is that the departed point is now replaced by the flow of the departed point. For fixed \((s, x) \in \mathbb{R}^+ \times \mathbb{R}^d\), we denote by \(\theta_{t, s}(x)\) the deterministic flow solving

\[
\dot{\theta}_{t, s}(x) = b(\theta_{t, s}(x)), \quad t \geq 0, \quad \theta_{s, s}(x) = x.
\]

Then, the first point of Theorem 1.2 of \([60]\) ensures the following bounds:

**Lemma 1.** (Theorem 1.2 of \([60]\)) Under A1, for any \(T > 0\) \((s, t) \in [0, \infty)^2\), \(0 < t - s < T\) and \(x \in \mathbb{R}^d\), the unique weak solution of \((3)\) admits a density \(p_{t-s}(x, y)\) which is continuous in \(x, y \in \mathbb{R}^d\). Moreover, there exist \(\lambda_0 \in (0, 1]\), \(C_0 \geq 1\) depending on \(T, a_0, a_{min}, d\), and the Lipschitz constants in A1 such that, for any \((s, t)\) such that \(0 < t - s < T\) and \(x, y \in \mathbb{R}^d\) it is

\[
C_0^{-1}(t-s)^{-\frac{d}{2}} \exp \left( -\lambda_0^{-1}\frac{|\theta_{t, s}(x) - y|^2}{t-s} \right) \leq p_{t-s}(x, y) \leq C_0(t-s)^{-\frac{d}{2}} \exp \left( -\lambda_0\frac{|\theta_{t, s}(x) - y|^2}{t-s} \right). \tag{4}
\]

The lemma here above implies the following corollary, whose proof can be found in the Appendix.

**Corollary 1.** Under A1-A2, there exist \(\lambda_0 \in (0, 1]\), \(C_0, \bar{C}_1 \geq 1\) and \(c \geq 1\) such that, for any \((s, t)\) such that \(0 < t - s < T\) and \(x, y \in \mathbb{R}^d\) it is

\[
C_0(t-s)^{-\frac{d}{2}} e^{-\lambda_0^{-1}|x-y|^2} e^{-c(|x|^2+1)(t-s)} \leq p_{t-s}(x, y) \leq \bar{C}_1(t-s)^{-\frac{d}{2}} e^{-\lambda_0|x-y|^2} e^{c(|x|^2+1)(t-s)}.
\]

Moreover, the constants depend only on \(T, a_{min}, b_0, a_0, a_1, b_1\) and \(d\).

**Lemma 2.** Suppose that A1-A2 hold true. Then, the process \(X\) admits a unique invariant measure \(\pi\) it is exponentially \(\beta\)-mixing.

We will provide the proof of Corollary 1 in the appendix, while the ergodic property stated in Lemma 2 can be found in \([64]\).

### 3 Estimator and upper bounds

Given the observation \(X^T\) of a diffusion \(X\), solution of \((3)\), we propose to estimate the invariant density \(\pi \in \mathcal{H}_d(\beta, \mathcal{L})\) by means of a kernel estimator. We therefore introduce some kernel function \(K : \mathbb{R} \to \mathbb{R}\) satisfying

\[
\int_{\mathbb{R}} K(x)dx = 1, \quad \|K\|_{\infty} < \infty, \quad \text{supp}(K) \subset [-1, 1], \quad \int_{\mathbb{R}} K(x)x'dx = 0, \tag{6}
\]
for all \( l \in \{0, \ldots, M\} \) with \( M \geq \max_i \beta_i \).

For \( j \in \{1, \ldots, d\} \), we denote by \( X_j^l \) the \( j\)-th component of \( X_l \). A natural estimator of \( \pi \) at \( x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d \) in the anisotropic context is given by

\[
\hat{\pi}_{h,T}(x) = \frac{1}{T \prod_{i=1}^d h_i} \int_0^T \prod_{m=1}^d K(\frac{x_m - X_m^m}{h_m}) du =: \frac{1}{T} \int_0^T K_h(x - x_u) du. \tag{7}
\]

The multi-index \( h = (h_1, \ldots, h_d) \) is small. In particular, we assume \( h_i < 1 \) for any \( i \in \{1, \ldots, d\} \).

A simple reason why this estimator results appropriate for the estimation of the invariant density is that, thanks to Birkhoff’s ergodic theorem, \( \hat{\pi}_{h,T}(x) = \frac{1}{T} \int_0^T K_h(x - x_u) du \) converges almost surely, for \( T \to \infty \), to \( \mathbb{E}[K_h(x - X_0)] = (K_h \ast \pi)(x) \), which converges to \( \pi(x) \) for \( h \to 0 \).

In order to get minimax upper bound on the risk of the estimator \( \hat{\pi}_{h,T} \) we will need to apply a mixing inequality uniform on the class of coefficients \( (a, b) \in \Sigma(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, \beta, \tilde{C}, \rho) \). The following lemma will be useful in this sense. In the sequel, we will denote as \( \mathbb{E} \) the expectation with respect to the stationary process \( (X_t)_{t \geq 0} \), whose invariant probability density is \( \pi = \pi_{(a,b)} \).

Let us introduce the notation \( \pi(f) \) for the expectation of the function \( f \) with respect to the invariant density \( \pi \), i.e. \( \pi(f) := \mathbb{E}_\pi f(x) \). Then, the following bounds hold true:

**Lemma 3.** Let \( \beta = (\beta_1, \ldots, \beta_d) \), \( \beta_i \geq 0 \), \( \mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d) \), \( \mathcal{L}_i > 0 \), \( 0 < a_{\min} \leq a_0, b_0 > 0 \), \( a_1 > 0, b_1 > 0 \), \( \tilde{C} > 0 \), \( \rho > 0 \). There exists a constant \( \bar{C} > 0 \) independent of \( (a, b) \in \Sigma := \Sigma(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, \beta, \tilde{C}, \rho) \) such that, for all measurable function \( \varphi : \mathbb{R}^d \to \mathbb{R} \) bounded, we have

\[
|\mathbb{E}[\varphi(X_t) \varphi(X_0)] - \pi(\varphi)^2| \leq C e^{-t/\bar{C} \|\varphi\|_\infty^2}. \tag{8}
\]

As immediate consequence of the fact that \( K \) is compactly supported we get the proposition below, which ensures the uniformity of the constants involved in the mixing properties of the process.

**Proposition 1.** Let us consider the same notation as in Lemma 3. Then, there exist constants \( \rho_{\text{mix}} > 0 \) and \( C_{\text{mix}} > 0 \) such that \( \forall h = (h_1, \ldots, h_d) \) with \( h_i < 1 \),

\[
|\mathbb{E}[K_h(x - X_t) K_h(x - X_0)] - \pi(\mathbb{E}[K_h])^2| \leq C_{\text{mix}} e^{-\rho_{\text{mix}} t} \|\mathbb{E}[K_h]\|_\infty^2. \tag{9}
\]

Moreover, the constants \( C_{\text{mix}} \) and \( \rho_{\text{mix}} \) are uniform over the set of coefficients \((a, b) \in \Sigma\).

### 3.1 Upper bounds

In the sequel, we will denote as \( \hat{\beta}_3 \) the harmonic mean over the \( d - 2 \) largest components:

\[
\frac{1}{\hat{\beta}_3} := \frac{1}{d-2} \sum_{i \geq 3} \frac{1}{\beta_i}. \tag{10}
\]

Next theorem provides the convergence rate for the pointwise estimation of the invariant density for \( d \geq 3 \). Its proof can be found in Section 6.

**Theorem 1.** Suppose that \( d \geq 3 \). Consider \( 0 < a_{\min} \leq a_0 \) and \( b_0 > 0 \), \( a_1 > 0, b_1 > 0, \tilde{C} > 0 \), \( \rho > 0 \) and denote \( \Sigma := \Sigma(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, \beta, \tilde{C}, \rho) \). If \( \beta_1 \leq \beta_2 \leq \ldots \leq \beta_d \), then there exist \( c > 0 \) and \( T_0 > 0 \) such that, for \( T \geq T_0 \), the optimal choice for the multidimensional bandwidth \( h \) yields the following convergence rates.

- **If** \( \beta_2 < \beta_3 \), then

\[
\sup_{(a, b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq c \left( \frac{\log T}{T} \right)^{\frac{2\beta_3}{d + \beta_3}}.
\]

- **If** otherwise \( \beta_2 = \beta_3 \), then

\[
\sup_{(a, b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq c \left( \frac{1}{T} \right)^{\frac{2\beta_3}{d + \beta_3}}.
\]
In the isotropic context we have, in particular, $\beta_3 = \beta$, where $\beta$ is the common smoothness over the $d$ directions. The convergence rate we derive is therefore the same as in [23], being equal to $(\frac{1}{\beta})^{\frac{d}{d+2-\beta}}$.

One may argue that the fact that the result depends on the harmonic mean of the smoothness only on $d-2$ direction is not very intuitive. However, the exponent found in the convergence rate here above is not completely surprising, as it appears also in [3] for the estimation of the invariant density of a diffusion with jumps and it is proven to be optimal in this context, up to a logarithmic gap. What is surprising in this result is the fact that the role played by the logarithm in the proposition here above is not completely surprising, as it appears also in [3] for the estimation of the invariant density of a diffusion with jumps and it is proven to be optimal in this context, up to a logarithmic gap. What is surprising in this result is the fact that both propositions rely on the exponential $\beta$-mixing of the considered process and on a bound on the transition density. Comparing Lemma 1 with Lemma 1 of [4] it is possible to see that the upper bound for the transition density associated to a jump-diffusion consists in two terms: one derives from the gaussian component while the other is due to the presence of jumps. It is worth noting that in our computations the contribution of the second is always negligible compared to the one coming from the first.

We now study the behaviour of our estimator for $d = 2$.

Proposition 2. Suppose that $A1 - A2$ hold for some $a_{\min}, b_0, a_0, a_1, b_1, \tilde{C}$ and $\tilde{\rho}$; that $d \geq 3$ and that $\pi \in \mathcal{H}_d(\beta, 2\mathcal{L})$. Suppose moreover that $\beta_1 \leq \cdots \leq \beta_d$, and let $k_0 = \max\{i \in \{1, \ldots, d\} | \beta_1 = \cdots = \beta_i\}$. If $\hat{\pi}_{h,T}$ is the estimator given in (7), then there exist $c > 0$ and $T_0 > 0$ such that, for $T \geq T_0$, the following bounds hold true.

- If $k_0 = 1$ and $\beta_2 < \beta_3$ or $k_0 = 2$, then

$$\text{Var}(\hat{\pi}_{h,T}(x)) \leq \frac{c}{T} \frac{\sum_{j=1}^{d} |\log(h_j)|}{\prod_{l \geq 3} h_l}.$$  \hfill (11)

- If $k_0 \geq 3$, then

$$\text{Var}(\hat{\pi}_{h,T}(x)) \leq \frac{c}{T} \frac{1}{(\prod_{l=1}^{k_0} h_l)^{1-\frac{1}{\beta_2}} (\prod_{l \geq k_0+1} h_l)}.$$  \hfill (12)

- If otherwise $k_0 = 1$ and $\beta_2 = \beta_3$, then

$$\text{Var}(\hat{\pi}_{h,T}(x)) \leq \frac{c}{T} \frac{1}{\sqrt{h_2 h_3} \prod_{l \geq 4} h_l}.$$  

Moreover, the constants $c$ and $T_0$ are uniform over the set of coefficients

$$(a, b) \in \Sigma(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}).$$
Theorem 2. Suppose that $d = 2$. Consider $0 < a_{\min} \leq a_0$, $b_0 > 0$, $a_1 > 0$, $b_1 > 0$, $\bar{C}$, $\bar{\rho}$ and $\Sigma := \Sigma(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, b_1, \bar{C}, \bar{\rho})$. Then, there exist $c > 0$ and $T_0 > 0$ such that, for $T \geq T_0$, the optimal choice for the multidimensional bandwidth $h$ yields the following convergence rates.

$$\sup_{(a,b) \in \Sigma} E[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq c \frac{\log T}{T}.$$ 

To conclude the part regarding the upper bounds on the mean squared error associated to our estimator $\hat{\pi}$ when a continuous record of the process is available, we are left to discuss the monodimensional case. It is however known that, under our hypothesis, the proposed kernel estimator achieves the parametric rate $\frac{1}{T}$ and such a rate is optimal (see for example [47] or Theorem 1 in [46]).

3.2 Adaptive procedure

We observe that, in practice, the bandwidths $h_1, \ldots, h_d$ need to be selected from data. For this reason, it can be interesting to propose an adaptive procedure in the same spirit as the one firstly introduced by Goldenshluger and Lepski, in [34].

As explained in previous subsection, for $d = 1$ and $d = 2$ the bound on the variance does not depend on the unknown smoothness, and the same holds for the optimal choice for the bandwidth. Hence, there is no gain in implementing a data-driven adaptive procedure for $d < 3$. For $d \geq 3$, instead, the bandwidth choice depends on the smoothness $\beta$. We emphasize that, as we are in the anisotropic context, the smoothnesses over the different directions are different and, as a consequence, the bandwidth selection procedure has to be able to provide different choices for the bandwidth $h_1, h_2, \ldots, h_d$. To do that, the idea consists in providing a set of candidate bandwidths, associate to a set of potential estimators, and then in choosing the bandwidths such that the $L^2$ error is as small as possible. With this purpose in mind, we introduce a quantity that heuristically stands for the bias term and a penalty term whose size is the one found in the bound of the variance in previous subsection. The selected bandwidths will be the ones for which the sum of these two quantities achieves the minimum.

The adaptive procedure we propose encompasses both integrated and pointwise $L^2$ risk.

In the sequel, for $A \subset \mathbb{R}^d$ compact and for $g \in L^2(A)$, $\|g\|_A^2 := \int_A |g(x)|^2 dx$ denotes the $L^2$ norm with respect to Lebesgue on $A$. One might question why, in the adaptive procedure, we also analyze the integrated $L^2(A)$ norm, whereas in the previous section, we focused on the pointwise norm. It becomes evident from Theorem 1 and the constant $c$ being independent of $x$ that we can extend the upper bounds from pointwise estimation to estimation on $L^2(A)$, resulting in

$$E[\|\hat{\pi}_{h,T} - \pi\|^2_A] \leq \begin{cases} 2^{\frac{2\beta_3}{\beta_2 + d - 2}} & \text{if } \beta_2 < \beta_3 \\ \left(\frac{2^{\frac{2\beta_3}{\beta_2 + d - 2}}}{\log T}\right)^{\frac{2\beta_3}{\beta_2 + d - 2}} & \text{if } \beta_2 = \beta_3. \end{cases}$$

However, when transitioning from the data-driven procedure in $L^2(A)$ to pointwise estimation, a logarithmic factor is typically lost, as observed in prior works such as [47] for isotropic adaptive studies or [35] for anisotropic ones. This phenomenon also holds true in our context, as demonstrated in Theorems 3 and 4 below.

Furthermore, the analysis becomes more challenging when considering pointwise estimation. Indeed in this context, to claim that the rate-optimal choice for the bandwidth belongs to the set of candidate bandwidth, an additional condition involving $\beta$ emerges.

One of the main objectives of this work is to investigate the optimality of the logarithmic term in the convergence rate. To achieve this, we needed a procedure capable of automatically detecting the presence or absence of the logarithmic term. Consequently, the introduction of an additional logarithm due to the choice of a pointwise norm becomes a significant concern in this context. This is why we have opted to include the integrated $L^2$ risk in this section.

In order to introduce the quantities heuristically discussed at the beginning of this section we start by defining some auxiliary indexes:

$$k_1 := \arg \min_{l=1,\ldots,d} h_l, \quad k_2 := \arg \min_{l \neq k_1} h_l.$$
and, in an iterative way, we can introduce for any $j$,

$$k_j := \arg \min_{l \neq k_1, \ldots, k_{j-1}} h_l.$$

Then, we define the set of potential bandwidths for the $L^2(A)$ estimation $\mathcal{H}_T$ as

$$\mathcal{H}_T \subset \{ h \in (0,1]^d : \forall l = 1, \ldots, d \ \frac{1}{T}h^b \leq h_l \leq \left( \frac{1}{\log T} \right)^{1/a}; \min \left( \sum_{j=1}^d |\log h_j| h_{k_j} h_{k_j} a \right) h_l \geq \frac{c(log T)^{2a}}{\sqrt{T}} \}, \tag{13}$$

where $a > 1$, $b > 0$. For the pointwise estimation, instead, we introduce the analogous set $\mathcal{H}_T^P$ which is such that

$$\mathcal{H}_T^P \subset \{ h \in (0,1]^d : \forall l = 1, \ldots, d \ \frac{1}{T}h^b \leq h_l \leq \left( \frac{1}{\log T} \right)^{1/a}; \min \left( \sum_{j=1}^d |\log h_j| h_{k_j} h_{k_j} a \right) h_l \geq \frac{c(log T)^{2a}}{\sqrt{T}} \}, \tag{14}$$

where again $a > 1$ and $b > 0$. It is easy to check that the right hand side of (14) is included in the right hand side of (13).

Let us stress that it is possible to choose $a$ arbitrarily close to 1, and $b$ arbitrarily large. Moreover, we assume that there exists $c > 0$ for which $|\mathcal{H}_T| + |\mathcal{H}_T^P| \leq cT^e$, i.e. the growth of the sets $\mathcal{H}_T$ and $\mathcal{H}_T^P$ is at most polynomial.

According to the set of candidate bandwidths, we can introduce the set of candidate estimators:

$$\mathcal{F}(\mathcal{H}_T) := \left\{ \hat{\pi}_{h,T}(x) = \frac{1}{T} \int_0^T \mathbb{K}_h(X_u - x)du : \ x \in \mathbb{R}^d, \ h \in \mathcal{H}_T \right\},$$

$$\mathcal{F}(\mathcal{H}_T^P) := \left\{ \hat{\pi}_{h,T}(x) = \frac{1}{T} \int_0^T \mathbb{K}_h(X_u - x)du : \ x \in \mathbb{R}^d, \ h \in \mathcal{H}_T^P \right\}. \tag{15}$$

The goal of this section is to select two estimators from the families $\mathcal{F}(\mathcal{H}_T)$ and $\mathcal{F}(\mathcal{H}_T^P)$, respectively, in a completely data-driven way, based only on the observation of the continuous trajectory of the process $X$.

Following the idea in [34], our selection procedure relies on the introduction of auxiliary convolution estimators. According to our records, it was introduced in [35] as a device to circumvent the lack of ordering among a set of estimators in anisotropic case, where the increase of the variance of an estimator does not imply a decrease of its bias.

For any bandwidths $h = (h_1, \ldots, h_d)^T$, $\eta = (\eta_1, \ldots, \eta_d)^T \in \mathcal{H}_T \cup \mathcal{H}_T^P$ and $x \in \mathbb{R}^d$, we define

$$\mathbb{K}_h \ast \mathbb{K}_\eta(x) := \prod_{j=1}^d (K_{h_j} \ast K_{\eta_j})(x_j) = \prod_{j=1}^d \int_{\mathbb{R}} K_{h_j}(u - x_j)K_{\eta_j}(u)du.$$

We moreover define the kernel estimators

$$\hat{\pi}_{(h,\eta),T}(x) := \frac{1}{T} \int_0^T (\mathbb{K}_h \ast \mathbb{K}_\eta)(X_u - x)du, \ x \in \mathbb{R}^d.$$

As the convolution is commutative, we clearly have $\hat{\pi}_{(h,\eta),T} = \hat{\pi}_{(\eta,h),T}$. Then, the selection procedure we propose is based on a comparison of the differences $\hat{\pi}_{(h,\eta),T} - \hat{\pi}_{\eta,T}$. We remark that the bound on the variance in Proposition 2 when one does not know the order the regularities $\beta$ consists in

$$\frac{k}{T} \min \left( \frac{\sum_{j=1}^d |\log h_j|}{\prod_{l \neq k_1, k_2} h_l}, \ \frac{1}{\sqrt{h_{k_1} h_{k_2} \prod_{l \neq k_1, k_2} h_l}} \right) \min \left( \frac{1}{\prod_{l \neq k_1, \ldots, k_3} h_l} \right) \min \left( \frac{1}{\prod_{l \neq k_1, k_2, k_3} h_l} \right).$$

$$= \frac{k}{T} \min \left( \frac{\sum_{j=1}^d |\log h_j|}{\prod_{l \neq k_1, k_2} h_l}, \ \frac{1}{\sqrt{h_{k_1} h_{k_2} \prod_{l \neq k_1, k_2} h_l}} \right) \min \left( \frac{1}{\prod_{l \neq k_1, \ldots, k_3} h_l} \right).$$
the equivalence is a consequence of the definition of $h_{kj}$ for $j = 1, \ldots, d$. With this purpose in mind we introduce the penalty functions

$$V(h) := \frac{k}{T} \min\left(\frac{\sum_{j=1}^{d} |\log h_j|}{\prod_{l \neq k_1, k_2} h_l^{2}}, \frac{1}{\sqrt{h_{k_2}h_{k_3}}\prod_{l \neq k_1, k_2, k_3} h_l}\right) =: \frac{k}{T} \tilde{H}^2(h)$$

(16)

and

$$V^p(h) := (\log T)V(h) = \frac{kT}{T} \tilde{H}^2(h),$$

(17)

where $kT := k \log T$ and $k$ is a numerical constant which has to be taken large. Even though it is not explicit, it can be calibrated by simulations as done for example in Section 5 of [22] through the implementation of a method inspired by Lacour, Massart and Rivoirard in [50]. We present in Section 5 some numerical simulations which shows the impact of the choice of $k$ on the quality of estimation. Upon comparing the two penalty functions defined above, it becomes evident that opting for pointwise estimation instead of $L^2$ estimation results in the loss of a logarithmic term.

Then, we compare the differences $\hat{\pi}_{h, \eta} - \hat{\pi}_{\eta, T}$ in $A(h)$ and $A^p(h, x)$, defined as below:

$$A(h) := \sup_{\eta \in \mathcal{H}_T} (\|\hat{\pi}_{(h, \eta), T} - \hat{\pi}_{\eta, T}\|^2_A - V(\eta))_+,$$

(18)

$$A^p(h, x) := \sup_{\eta \in \mathcal{H}_T^p} (|\hat{\pi}_{(h, \eta), T}(x) - \hat{\pi}_{\eta, T}(x)|^2 - V^p(\eta))_+.$$

(19)

Heuristically, $A(h)$ and $A^p(h, x)$ are estimates of the squared bias and $V(h)$ and $V^p(h)$ of the variance bound. It is worth underlining that the penalty term which is used here comes from the three bounds obtained in Proposition 2 remarking that in this case the smoothness is unknown. The selection is done by setting

$$\hat{h} := \arg \min_{h \in \mathcal{H}_T} (A(h) + V(h)).$$

(20)

for the $L^2(A)$ estimation and

$$\hat{h}^p(x) := \arg \min_{h \in \mathcal{H}_T^p} (A^p(h, x) + V^p(h)).$$

(21)

for the pointwise one. Before proceeding with the main results of this subsection, let us introduce some notation. In the sequel, it will be useful to consider $\pi_h := \mathbb{K}_h \ast \pi$, which is the function that is estimated without bias by $\hat{\pi}_{h, \eta}$. It is indeed $\mathbb{E}[\hat{\pi}_{h, \eta,T}(x)] = \pi_h(x)$. Moreover, we define $\pi_{(h, \eta)} := \mathbb{K}_h \ast \mathbb{K}_\eta \ast \pi$. To conclude the notation paragraph, we introduce the bias for the pointwise estimation $B^p(h)(x)$ as $|(\pi_h(x) - \pi(x)|$, while for the procedure in the $L^2(A)$ we will employ the bias $B(h) := \|\pi_h - \pi\|^2_A$, where we have denoted as $\|\cdot\|_A$ the $L^2$-norm on $A$, a compact set in $\mathbb{R}^d$ which contains $A$. It is $A := \{\zeta \in \mathbb{R}^d : d(\zeta, A) \leq \sqrt{2}\}$, where $d(\zeta, A) := \min_{x \in A} |\zeta - x|$.

The following result holds for the adaptive procedure in integrated $L^2$ risk.

**Theorem 3.** Suppose that assumptions A1 - A2 hold and that $d \geq 3$. Then, we have that there exists $T_0 > 0$ such that, for any $T \geq T_0$,

$$\mathbb{E}[\|\hat{\pi}_{h, T} - \pi\|^2_A] \leq c_1 \inf_{h \in \mathcal{H}_T} (B(h) + V(h)) + c_1 e^{-c_2(\log T)^{c_3}},$$

for $c_1$, $c_2$ positive constants and $c_3 > 1$.

As the last term above is negligible, the bound stated above shows that the estimator leads to an automatic trade-off between the bias $\|\pi_h - \pi\|^2_A$ and the variance $V(h)$. The proof of Theorem 3 is postponed to Section 7.

An analogous result holds true when one considers the pointwise estimation, as stated in Theorem 4 below.
Theorem 4. Suppose that assumptions A1 - A2 hold and that \( d \geq 3 \). Then, we have that there exists \( T_0 > 0 \) such that, for any \( T \geq T_0 \),

\[
\mathbb{E}[|\hat{\pi}_{T}(x, T) - \pi(x)|^2] \leq c_1 \inf_{h \in \mathcal{H}^{c_2}_T} (B^{c_1}(h, x) + V^c(h)) + c_1 T^{-c_2},
\]

for a positive constant \( c_1 \) and \( c_2 \geq 1 \).

The proof of Theorem 4 is postponed to Section 7.

We want now to replace the rate optimal choice for \( h(T) \). With this purpose in mind we introduce the following sets of candidate bandwidths:

\[
\mathcal{H}_T := \left\{ h = (h_1, \ldots, h_d)^T \in (0, 1]^d : h_l = \frac{1}{z_l} \text{ with } z_l \in \{1, \ldots, [T] \} \text{ satisfying conditions in (13)} \right\},
\]

(22)

\[
\mathcal{H}^{c_2}_T := \left\{ h = (h_1, \ldots, h_d)^T \in (0, 1]^d : h_l = \frac{1}{z_l} \text{ with } z_l \in \{1, \ldots, [T] \} \text{ satisfying conditions in (14)} \right\}.
\]

(23)

As the ordering of the \( \beta \)'s is unknown, we adapt the formal definition of \( \bar{\beta}_3 \) which now is the harmonic mean of the \( \beta \)'s after having removed the smallest two:

\[
\frac{1}{\bar{\beta}_3} := \frac{1}{d-2} \sum_{l \neq k_1, k_2} \frac{1}{\beta_l}.
\]

Then, we obtain the following theorems:

Theorem 5. Suppose that assumptions A1 - A2 hold with \( \beta_{k_1} \geq 1 \) and let \( \mathcal{H}_T \) be defined by (22). Then, we have

\[
\mathbb{E}\left[\left|\hat{\pi}_{T} - \pi\right|^2 \right] \leq \begin{cases}
\left( \frac{\log T}{T} \right)^{2 \frac{3}{\beta_3} + \frac{3}{2}} + e^{-c_1 (\log T)^{\frac{1}{2}}} & \text{if } \beta_{k_2} < \beta_{k_3} \\
\left( \frac{\log T}{T} \right)^{2 \frac{3}{\beta_3} + \frac{3}{2}} + e^{-c_1 (\log T)^{\frac{1}{2}}} & \text{if } \beta_{k_2} = \beta_{k_3},
\end{cases}
\]

for a positive constant \( c_2 \geq 1 \).

Underlining once again that the exponential term above is negligible, we have that the risk estimates we get using the bandwidth provided by our selection procedure converges to zero at the same rate as in previous section, which we have proven being optimal in a minimax sense. It is important to highlight, then, that our procedure automatically selects in a data-driven way, having or not the logarithm in the convergence rate. Changes occur with the pointwise adaptive procedure. In fact, as emphasized in Theorem 5 below, this results in the addition of a logarithm in both cases.

Theorem 6. Suppose that assumptions A1 - A2 hold and let \( \mathcal{H}^{c_2}_T \) be defined by (23). Assume moreover that the smallest smoothness \( \beta_{k_1} \) is such that \( \beta_{k_1} > 2 \). Then, we have

\[
\mathbb{E}[|\hat{\pi}_{2c_1}(x, T) - \pi(x)|^2] \leq \begin{cases}
\left( \frac{\log T}{T} \right)^{2 \frac{3}{\beta_3} + \frac{3}{2}} + T^{-c_1} & \text{if } \beta_{k_2} < \beta_{k_3} \\
\left( \frac{\log T}{T} \right)^{2 \frac{3}{\beta_3} + \frac{3}{2}} + T^{-c_1} & \text{if } \beta_{k_2} = \beta_{k_3},
\end{cases}
\]

for \( c_1 \geq 1 \).

4 Lower bounds

One may wonder if the convergence rates found by using kernel density estimators are optimal or if it is possible to improve them by using other density estimators. We aim at showing that the convergence rates found in Theorems 1 and 2 are optimal. We will focus first on the case \( d \geq 3 \). We will start computing a lower bound in a general case. After that, we will show it is possible to
improve it, up to ask $\beta_2 < \beta_3$.

We can write down the expression of the minimax risk for the estimation, at some point $x_0$, of an invariant density $\pi$ belonging to the anisotropic Holder class $\mathcal{H}_d(\beta, 2\mathcal{L})$. Let $x_0 \in \mathbb{R}^d$ and $\Sigma(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, b_1, C, \hat{\rho})$ as in Definition here above. We define the minimax risk

$$R_T(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, b_1, \hat{C}, \hat{\rho}) := \inf_{\hat{\pi}_T} \sup_{(a,b) \in \Sigma(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, b_1, \hat{C}, \hat{\rho})} \mathbb{E}[(\hat{\pi}_T(x_0) - \pi_{(a,b)}(x_0))^2],$$

where the infimum is taken on all possible estimators of the invariant density, based on $X_t$ for $t \in [0, T]$. The following lower bound will be showed in Section 6.

**Theorem 7.** Let $\beta = (\beta_1, \ldots, \beta_d)$, $1 < \beta_1 \leq \cdots \leq \beta_d$, $\beta_2 > 2$, $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$, $\mathcal{L}_i > 0$. Consider $0 < a_{\min} \leq a_0, a_1 > 0, b_0 > 0, b_1 > 0$, then, there exist $C, \hat{\rho}, c > 0$ and $T_0 > 0$ such that, for $T \geq T_0$,

$$R_T(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, b_1, \hat{C}, \hat{\rho}) \geq c \left(\frac{1}{T}\right)^{\frac{2\beta_1}{2\beta_1 + d - 2}}.$$ 

**Theorem 7** implies that, on a class of diffusions $X$ whose invariant density belongs to $\mathcal{H}_d(\beta, 2\mathcal{L})$, it is not possible to find an estimator with a rate of estimation better than $T^{-\frac{2\beta_1}{2\beta_1 + d - 2}}$. Remark that $\hat{C}$ and $\hat{\rho}$ cannot be chosen freely in the statement of Theorem 7. Indeed, the theorem only ensures that the lower bound holds true for sufficiently small $\hat{C}$ and large $\hat{\rho}$. This is in contrast with the upper bound of Theorem 1 which holds true for all $\hat{C}, \hat{\rho}$. Actually, it is impossible to get a lower bound for any $\hat{C}, \hat{\rho}$. Indeed, for fixed values of the $\mathcal{L}_i$’s, $b_0$ and $b_1$, the set $\Sigma(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, b_1, \hat{C}, \hat{\rho})$ can be empty if $\hat{C}$ is too large and $\hat{\rho}$ too small. In such case, the lower bound cannot hold true.

Comparing the result here above with the second point of Theorem 1 we observe that the convergence rate we found in the lower bound and in the upper bound are the same, when $\beta_2 = \beta_3$. When $\beta_2 < \beta_3$, instead, it is possible to improve the lower bound previously obtained, as gathered in the following theorem. Its proof can be found in Section 6.

**Theorem 8.** Let $\beta = (\beta_1, \ldots, \beta_d)$, $0 < \beta_1 \leq \cdots \leq \beta_d$, $\beta_2 > 2$, and $\beta_2 < \beta_3$. Moreover, $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$, $\mathcal{L}_i > 0$. Consider $0 < a_{\min} \leq a_0, a_1 > 0, b_0 > 0, b_1 > 0$, then, there exist $\hat{C}, \hat{\rho}, c > 0$ and $T_0 > 0$ such that, for $T \geq T_0$,

$$R_T(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, b_1, \hat{C}, \hat{\rho}) \geq c \left(\frac{\log T}{T}\right)^{\frac{2\beta_1}{2\beta_1 + d - 2}}.$$ 

We will see that the condition $\beta_2 < \beta_3$ is crucial in order to recover a logarithm in the lower bound. Comparing the lower bounds in Theorems 7 and 8 with the upper bounds in Theorem 1 it follows that the kernel density estimator we proposed in Theorem 7 achieves the best possible convergence rate.

It is possible to ensure an analogous lower bound in the bi-dimensional case, which ensures the optimality of the convergence rate found in Theorem 2.

**Theorem 9.** Let $d = 2$, $\beta = (\beta_1, \beta_2)$, $0 < \beta_1 \leq \beta_2$, $\beta_2 > 2$, $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$, $\mathcal{L}_i > 0$. Consider $0 < a_{\min} \leq a_0, a_1 > 0, b_0 > 0, b_1 > 0$, then, there exist $\hat{C}, \hat{\rho}, c > 0$ and $T_0 > 0$ such that, for $T \geq T_0$,

$$R_T(\beta, \mathcal{L}, a_{\min}, b_0, a_0, a_1, b_1, \hat{C}, \hat{\rho}) \geq c \left(\frac{\log T}{T}\right).$$

We conclude that also in the two-dimensional case, the estimator proposed in Theorem 7 is minimax-optimal.
5 Numerical simulations

In this section, we study the numerical performance of the Goldenshluger-Lepski procedure on simulated data. In particular, we discuss the effect of the choice of the constant $k$ appearing in (17) on the quality of estimation. For simplicity, the diffusion is chosen to follow a stationary Ornstein-Uhlenbeck model, in dimension $d = 3$, solution to $dX_t = -\frac{X_t}{\tau}dt + dB_t$. In this situation, the density $\pi$ is the one of the Gaussian distribution $\mathcal{N}(0, \Id_d)$. We focus on the pointwise estimation of $\hat{\pi}(x)$. We use the family of estimators introduced in (15), where $\mathbb{K}_h(x-y) = \prod_{m=1}^d K\left(\frac{x_m-y_m}{\eta_m}\right)$, is a product of a one dimensional Gaussian kernel $K$, and $(h_1, h_2, h_3) \in \mathcal{H}_T^3 := \{2 \times 10^{-0.2k}, \ k \in \{0, \ldots, 11\}\}^3$. The computation of the quantity $A^p(h, x)$ defined in (19) appears difficult in practice. Indeed, the formula involves the computation of the estimators $\hat{\pi}(h, \eta; T)$ for $(h, \eta) \in (\mathcal{H}_T^p)^2$ which yields, in our situation, to the computation of $\text{card}(\mathcal{H}_T^p)^2 \approx 10^6$ different estimators. Thus, we slightly modify the definition (19) and use instead

$$\tilde{A}^p(h, x) = \sup_{\eta \in \mathcal{H}_T^p} \{||\hat{\pi}(h \vee \eta; T, x) - \hat{\pi}(\eta; T, x)||^2 - V^p(\eta)\} \tag{26}$$

where $h \vee \eta = (h_1 \vee \eta_1, h_2 \vee \eta_2, h_3 \vee \eta_3)$. Since $h \vee \eta \in \mathcal{H}_T^p$ for $(h, \eta) \in (\mathcal{H}_T^p)^2$, this modification reduces the computational task to the computation of the family of estimators $\hat{\pi}(h, T)$ for $h \in \mathcal{H}_T^p$.

Remark, that since we are using a Gaussian kernel, we have $\mathbb{K}_h \ast \mathbb{K}_{h'} = \mathbb{K}_{h+\eta}$, for which $\mathbb{K}_{h \vee \eta}$ can be used as proxy. This justifies the approximation of $A^p(h, x)$ by $\tilde{A}^p(h, x)$. Also, alternative definitions of the Goldenshluger-Lepski method directly relies on quantities analogous to (26) (e.g. see [54]).

The total time of observation is $T = 10^5$, the process $X$ is simulated using an Euler scheme with step $10^{-5}$, while the continuous time estimators (7) are approximated by discrete versions using the same grid as the Euler scheme. We use $m = 200$ replications and obtain a set of estimators $(\hat{\pi}^{(i)}(h, T))_{h \in \mathcal{H}_T^p}$ for each iteration $i \in \{1, \ldots, m\}$. From these $m$ independent realisations of the estimator, it is possible to compute the empirical variance and mean of $\hat{\pi}(h, T)$ for each $h \in \mathcal{H}_T^p$. Using the true value of the density $\pi(x)$, we can determine the bandwidth $h \in \mathcal{H}_T^p$ minimizing the empirical $L^2$ risk of $\hat{\pi}(h, T)$. For each $h \in \mathcal{H}_T^p$, we Monte-Carlo trials. Let us denote by $h_{\text{oracle}}$ the value of the bandwidth minimizing this empirical risk and by $\hat{h}_{\text{oracle}, T} = \hat{h}_{\text{oracle}}$. The associated estimator. As the determination of this optimal $h_{\text{oracle}}$ uses the knowledge of $\pi(x)$, it is unfeasible in practice. However, this oracle estimator will serve as a basis of comparison for the Goldenshluger-Lepski estimator. In Table 1 we show the empirical mean and standard deviation of the oracle estimator, together with the value $h_{\text{oracle}}$ written as $h_{\text{oracle}} = (2 \times 10^{-0.2k_1}, 2 \times 10^{-0.2k_2}, 2 \times 10^{-0.2k_3})$. The estimator works remarkably well. It is noticeable that when estimating $\pi(x)$ at the point $x = (0, 0, 0)$, around which the function $\pi$ is isotropic, the shape of the optimal kernel is isotropic as well, since $k_1 = k_2 = k_3 = 0$. On the other hand, for the estimation of $\pi(x)$ at $x = (0, 0, 1)$ the optimal $h_{\text{oracle}}$ is anisotropic.

In the definition of the Goldenshluger-Lepski estimator, we need to specify the value of the constant $k$ appearing in (17). Theoretically, it is sufficient to choose this constant above some threshold, which is related to the variance of the estimator. As a consequence, the exact value of this threshold is explicitly unknown. Since we are conducting a Monte Carlo experiment, it is possible to estimate the variance of the estimators $\hat{\pi}(h, T)$ and thus to deduce useful values for $k$. Indeed, we choose some arbitrary $h^* = (2 \times 10^{-0.2k_1}, 2 \times 10^{-0.2k_2}, 2 \times 10^{-0.2k_3})$ with $k_1 = k_2 = k_3 = 4$ and compute the empirical variance of the estimator $\hat{\pi}(h^*, T)$. Then, we define $k^*$ as the constant such that the penalty term $V(h^*)$ defined in (16) matches the empirical variance. This value $k^*$ gives a hint on the magnitude of the constant $k$ that could be chosen. To determine precisely the dependence of the Goldenshluger-Lepski estimator $\hat{\pi}(h^*(x), T)$ on the constant $k$, we now let $k$ range in $(0.01 \times k^*, 1.5 \times k^*)$ and compute the empirical $L^2$ risk of the corresponding estimators. It appears that the Goldenshluger-Lepski estimator works correctly on a large range of choices of $k$ in this interval. In Figures 1 we plot the ratio between the empirical $L^2$ risk of the Goldenshluger-Lepski estimator and the $L^2$ risk of the oracle estimator for different values of $k \in (0.01 \times k^*, 1.5 \times k^*)$. As expected, the oracle estimator exhibits a smaller risk than the Goldenshluger-Lepski's estimator for any values of $k$. However, we see that for $k \in (0.1 \times k^*, 0.75 \times k^*)$, the $L^2$ risk of the Goldenshluger-Lepski estimator remains less than twice that of the oracle estimator. Considering the strong performance of the oracle estimator, this indicates that the Goldenshluger-Lepski method provides a good estimation of the parameter for a wide range of constants $k$. 

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Figure 1: Ratio of the $L^2$ risk of $\hat{\pi}_{h^*, T}$ over the $L^2$ risk of $\hat{\pi}_{\text{oracle}, T}$ as a function of $k/k^*$.

(a) $x = (0, 0, 0)$  
(b) $x = (0, 0, 1)$

The procedure described here above to determine the value of $k^*$ is impossible in practice when only a single path of the process is observed, as the Monte Carlo estimation of the estimator’s variance is unfeasible. However, a rough estimation of the variance of the estimator could be achieved, for instance, by splitting one observed path of $X$ on $[0, T]$ into $N$ disjoint pieces, $(X_{kT/N + s})_{0 \leq s \leq T/N}$ for $k = 0, \ldots, N - 1$, such that $T/N$ remains large. We compute on each piece of the trajectory an estimator $\hat{\pi}_{h^*, T/N}(x)$. Taking the empirical variance of these $N$ estimators gives us a proxy for the variance of $\hat{\pi}_{h^*, T/N}(x)$. We can derive a data-driven value of $k^*$ by matching this empirical variance with the penalty term $V(h^*)$ defined in (16). Finally, we use the Goldenshluger-Lepski estimator with some constant $k$ chosen in $(0.1 \times k^*, 0.75 \times k^*)$ as suggested by the Figures 1. We have tested this method, with $N = 100$ and $k = k^*/4$, on $p = 200$ replications of the trajectory. Results can be found in Table 2 and show that this estimation procedure provides a sharp estimation of $\pi(x)$.

Table 1: Oracle estimator

| $x = (0, 0, 0)$ | $x = (0, 0, 1)$ |
|-----------------|-----------------|
| True value $\pi(x)$ | $6.349 \times 10^{-2}$ | $3.851 \times 10^{-2}$ |
| Mean $\hat{\pi}_{\text{oracle}, T}$ | $6.287 \times 10^{-2}$ | $3.829 \times 10^{-2}$ |
| Std $\hat{\pi}_{\text{oracle}, T}$ | $1.10 \times 10^{-3}$ | $6.93 \times 10^{-4}$ |
| $(k_1, k_2, k_3)$ | $(7, 7, 7)$ | $(7, 8, 4)$ |

Table 2: Goldenshluger-Lepski estimator with a data-driven choice of $k$

| $x = (0, 0, 0)$ | $x = (0, 0, 1)$ |
|-----------------|-----------------|
| True value $\pi(x)$ | $6.349 \times 10^{-2}$ | $3.851 \times 10^{-2}$ |
| Mean $\hat{\pi}_{h(x), T}$ | $6.229 \times 10^{-2}$ | $3.776 \times 10^{-2}$ |
| Std $\hat{\pi}_{h(x), T}$ | $1.25 \times 10^{-3}$ | $8.36 \times 10^{-4}$ |

6 Proofs upper bounds stated in Section 3

This section is devoted to the proof of all the results stated in Section 3, about the behaviour of the estimator proposed in (7), assuming that a continuous record of the process $X$ is available. As Theorem 1 is a consequence of Proposition 2 we start by proving Proposition 2.

6.1 Proof of Proposition 2

Proof. In the sequel, the constant $c$ may change from line to line and it is independent of $T$. From the definition (7) and the stationarity of the process we get

$$Var(\hat{\pi}_{h, T}(x)) = \frac{1}{T^2} \int_0^T \int_0^T k(t - s) dt ds,$$
where
\[ k(u) := \text{Cov}(\mathcal{K}_h(x - X_0), \mathcal{K}_h(x - X_u)). \]

We deduce that
\[ \text{Var}(\hat{s}_{h,T}(x)) \leq \frac{1}{T} \int_0^T |k(s)|ds. \]

We want to use different way to upper bound the variance, relying on the bound on the transition density gathered in Lemma 1 and on the mixing properties of the process as in Lemma 2. Hence, we split the time interval \([0, T]\) into 4 pieces:
\[ [0, T] = [0, \delta_1) \cup [\delta_1, \delta_2) \cup [\delta_2, D) \cup [D, T], \]
where \(\delta_1, \delta_2\) and \(D\) will be chosen later, to obtain an upper bound on the variance as sharp as possible. We will see that the bound on the variance will depend on \(k_0\) only on the interval \([\delta_1, \delta_2)\). For this reason we start considering what happens on \([0, \delta_1)\) without taking into account the fact that \(k_0\) can be larger or smaller than 3.

- For \(s \in [0, \delta_1)\), from Cauchy - Schwarz inequality and the stationarity of the process we get
\[ |k(s)| \leq \text{Var}(\mathcal{K}_h(x - X_0))^{\frac{1}{2}} \text{Var}(\mathcal{K}_h(x - X_s))^{\frac{1}{2}} = \text{Var}(\mathcal{K}_h(x - X_0)) \leq \int_{\mathbb{R}^d} (\mathcal{K}_h(x - y))^2 \pi(y)dy. \]

As \(\pi \in \mathcal{H}_d(\beta, \mathcal{L})\), its infinitive norm is bounded. Using also the definition of \(\mathcal{K}_h\) given in \(\mathcal{L}\) it follows
\[ |k(s)| \leq \frac{c}{\prod_{l=1}^d h_l} \] (27)
which implies
\[ \int_0^{\delta_1} |k(s)|ds \leq \frac{c \delta_1}{\prod_{l=1}^d h_l}. \] (28)

- For \(s \in [\delta_1, \delta_2)\), taking \(\delta_2 < 1\), we use the definition of transition density, for which
\[ |k(s)| \leq \int_{\mathbb{R}^d} |\mathcal{K}_h(x - y)| \int_{\mathbb{R}^d} |\mathcal{K}_h(x - y')| p_s(y, y')dy' \pi(y)dy. \]

We now act differently according on the value of \(k_0\). If \(k_0 = 1\) and \(\beta_2 < \beta_3\) or \(k_0 = 2\), then we introduce \(q_s(y_1, \ldots, y_d; y_1', y_2', y)\) as below:
\[ q_s(y_1, \ldots, y_d; y_1', y_2', y) := e^{-\lambda_0 |y_1 - y_1'|^2} \times e^{-\lambda_0 |y_2 - y_2'|^2} \times \frac{1}{\sqrt{8}} e^{-\lambda_0 |y_3 - y_3'|^2} \times \ldots \times \frac{1}{\sqrt{8}} e^{-\lambda_0 |y_d - y_d'|^2} e^{c(|y|^2 + 1)s}. \]

We remark that \(s < \delta_2 < 1\). From Lemma 1 we know it is
\[ p_s(y, y') \leq \frac{c_0}{s} q_s(y_1, \ldots, y_d; y_1', y_2', y). \]

Let us stress that
\[ \sup_{y_1', y_2' \in \mathbb{R}^d} \int_{\mathbb{R}^{d-2}} q_s(y_1, \ldots, y_d; y_1', y_2', y)dy_3 \ldots dy_d \leq ce^{c(|y|^2 + 1)s}. \] (29)

Then,
\[ |k(s)| \leq \frac{c_0}{s} \prod_{l=1}^d h_l \int_{\mathbb{R}^d} |\mathcal{K}_h(x - y)| \int_{\mathbb{R}^d} |\mathcal{K}_h(x - y')| q_s(y_1, \ldots, y_d; y_1', y_2', y)dy' \pi(y)dy. \] (30)

Using the definition of \(\mathcal{K}_h\) and (29) we obtain
\[
\begin{align*}
\int_{\mathbb{R}^d} |\mathcal{K}_h(x - y)| q_s(y_1, \ldots, y_d; y_1', y_2', y)dy' &
\leq \frac{c}{\prod_{l=1}^d h_l} \int_{\mathbb{R}^d} K\left(\frac{y_1' - x_1}{h_1}\right) \int_{\mathbb{R}^d} K\left(\frac{y_2' - x_2}{h_2}\right) \int_{\mathbb{R}^{d-2}} q_s(y_3, \ldots, y_d; y_1', y_2', y)dy_3 \ldots dy_d dy_1' \\
&\leq \frac{c}{\prod_{l=1}^d h_l} \int_{\mathbb{R}^d} K\left(\frac{y_1' - x_1}{h_1}\right) \int_{\mathbb{R}^d} K\left(\frac{y_2' - x_2}{h_2}\right) e^{c(|y'|^2 + 1)s} dy_2' dy_1' \\
&\leq \frac{c}{\prod_{l=1}^d h_l} e^{c(|y|^2 + 1)s}.
\end{align*}
\]

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Replacing this result in (30) we obtain

\[ |k(s)| \leq \frac{c_0}{s} \int_{\mathbb{R}^d} |K_k(x - y)| e^{c(|y|^2 + 1)^s} \pi(y) dy. \]

As the support of the kernel function is a compact set, the integral is bounded. It implies that, when \( k_0 = 1 \) and \( \beta_2 < \beta_1 \) or \( k_0 = 2 \), we get

\[ |k(s)| \leq \frac{c}{\prod_{j \geq 3} h_j s} \]  

and so

\[ \int_{\delta_1}^{\delta_2} |k(s)| ds \leq \frac{c \log(\delta_2) - \log(\delta_1)}{\prod_{j \geq 3} h_j}, \]

where the constant \( c \) does not depend on the coefficient \((a, b) \in \Sigma\). We now consider what happens on \([\delta_1, \delta_2]\) when \( k_0 \geq 3 \) or \( k_0 = 1 \) and \( \beta_2 = \beta_1 \). In analogy to what done before we introduce \( q_s(y_{k_0+1} | y_0 | y_1, \ldots, y_{k_0}, y) \) which is such that

\[ q_s(y_{k_0+1} | y_0 | y_1, \ldots, y_{k_0}, y) := e^{-\lambda_0 |y_{k_0+1} - y_0|^2} \times \cdots \times e^{-\lambda_0 |y_0 - y_{k_0}|^2} \times \frac{1}{\sqrt{\delta}} e^{-\lambda_0 |y_{k_0+1} - y_0|^2} \times \cdots \times \frac{1}{\sqrt{\delta}} e^{-\lambda_0 |y_0 - y_{k_0}|^2} e^{c(|y|^2 + 1)^s}. \]

Using again Lemma \[\right\] we can write

\[ p_s(y, y') \leq c_0 s^{-k_0} q_s(y_{k_0+1} | y_0 | y_1, \ldots, y_{k_0}, y) \]

and as before we have

\[ \sup_{y_1, \ldots, y_{k_0}} \int_{\mathbb{R}^d} q_s(y_{k_0+1} | y_0 | y_1, \ldots, y_{k_0}, y) dy_{k_0+1} \cdots dy_d \leq c e^{c(|y|^2 + 1)^s}. \]

Acting as in (30) and below it follows

\[ |k(s)| \leq \frac{c_0}{s} \int_{\mathbb{R}^d} |K_k(x - y)| \int_{\mathbb{R}^d} |K_k(x - y')| q_s(y_{k_0+1} | y_0 | y_1, \ldots, y_{k_0}, y) dy_{k_0+1} \cdots dy_d \pi(y) dy \]

\[ \leq \frac{c}{s} \frac{1}{\prod_{j \geq k_0+1} h_j}. \]

Hence, when \( k_0 \geq 3 \), we have

\[ \int_{\delta_1}^{\delta_2} |k(s)| ds \leq \frac{c}{\prod_{j \geq k_0+1} h_j} \int_{\delta_1}^{\delta_2} s^{-\frac{k_0}{2}} ds \leq \frac{c}{\prod_{j \geq k_0+1} h_j} \delta_1^{1 - \frac{k_0}{2}}. \]

We remark that, for \( k_0 = 1 \), the reasoning here above still applies. However, as \( 1 - \frac{k_0}{2} = \frac{1}{2} \) is now positive, we obtain

\[ \int_{\delta_1}^{\delta_2} |k(s)| ds \leq \frac{c}{\prod_{j \geq 2} h_j} \delta_1^{\frac{1}{2}}. \]

As the bound no longer depends on \( \delta_1 \), we can choose to take \( \delta_1 = 0 \).

\[ \bullet \] For \( s \in [\delta_2, D) \) we use the same estimation for any value of \( k_0 \), but for \( k_0 = 1 \), \( \beta_2 = \beta_3 \). We still consider the bound on the transition density gathered in Lemma \[\right\]. Such a bound is not uniform in \( t \) big. However, for \( t \geq 1 \), it is

\[ p_t(y, y') = \int_{\mathbb{R}^d} p_{\frac{1}{2}}(y, z) p_{\frac{1}{2}}(z, y') dz \]

\[ \leq c \int_{\mathbb{R}^d} e^{-2\lambda_0 |y - z|^2} e^{c(|y|^2 + 1)} p_{\frac{1}{2}}(z, y') dz \]

\[ \leq c e^{c(|y|^2 + 1)} \int_{\mathbb{R}^d} p_{\frac{1}{2}}(z, y') dz \leq c e^{c(|y|^2 + 1)}. \]

We deduce, for all \( t \),

\[ p_t(y, y') \leq c_0 t^{-\frac{d}{2}} e^{-\lambda_0 \frac{|y - y'|^2}{2}} + c e^{c(|y|^2 + 1)}. \]
It follows
\[ |k(s)| \leq c \int_{\mathbb{R}^d} |\mathbb{K}_h(x - y)| \int_{\mathbb{R}^d} |\mathbb{K}_h(x - y')|(s^{-\frac{d}{2}} + 1)e^{c(|y|^2 + 1)}dy'dy \]
\[ \leq c(s^{-\frac{d}{2}} + 1), \tag{37} \]
where we have also used that the support of the kernel function is a compact set. We therefore get
\[ \int_{\delta_i}^D |k(s)|ds \leq c(\delta_1^{1-\frac{d}{2}} + D). \tag{38} \]
When \(k_0 = 1\) and \(\beta_2 = \beta_3\), instead, we proceed analogously to how we did in the previous interval in order to obtain (33). We choose to remove, in particular, the contribution of the first three bandwidths. We obtain, \(\forall s \in [\delta_2, D),\)
\[ |k(s)| \leq c(s^{-\frac{d}{2}} \frac{1}{\prod_{l=4}^5 h_l} + 1), \tag{39} \]
where the final constant comes from (36), as now \(s\) can be larger than 1. It follows
\[ \int_{\delta_i}^D |k(s)|ds \leq c(\prod_{l=4}^5 h_l \frac{1}{\delta_2^{1-\frac{d}{2}}} + D). \tag{40} \]

• For \(s \in [D, T]\) we exploit the mixing properties of the process, as stated in Proposition 1. The following control on the covariance holds true:
\[ |k(s)| \leq c \|\mathbb{K}_h(x - \cdot)\|_2^2 \leq c(\prod_{j=1}^d h_j) e^{-\rho s}, \]
for \(\rho\) and \(c\) positive constants which are uniform over the set of coefficients \((a, b) \in \Sigma\). It entails
\[ \int_D^T |k(s)|ds \leq c(\prod_{j=1}^d h_j)^2 e^{-\rho D}. \tag{41} \]
We now put all the pieces together. For \(k_0 = 1\) and \(\beta_2 < \beta_3\) or \(k_0 = 2\) we collect together (28), (32), (38) and (41). We deduce
\[ \text{Var}(\hat{x}_{h,T}(x)) \leq c \prod_{j=4}^5 h_j \text{Var}(\hat{x}_{h,T}(x)) \leq c \sum_{j=1}^d \frac{|\log(h_j)|}{\prod_{j=3}^d h_j} + \frac{1}{\prod_{j=3}^d h_j} + \sum_{j=1}^d |\log(h_j)| + 1 \]
\[ \leq c \sum_{j=1}^d |\log(h_j)|. \]
If otherwise \(k_0 \geq 3\), we consider (34) instead of (32) which, together with (28), (38) and (41) provides
\[ \text{Var}(\hat{x}_{h,T}(x)) \leq c \prod_{j=1}^d h_j \frac{\delta_1}{\prod_{j=1}^d h_j} + \frac{\delta_1^{1-\frac{b_0}{T}}}{\prod_{j=1}^d h_j} + \frac{\delta_2^{1-\frac{d}{2}} + D + \frac{1}{\prod_{j=1}^d h_j} e^{-\rho D} \prod_{j=1}^d h_j}. \tag{43} \]
We observe that the balance between the first two terms is achieved for \(\delta_1 := (\prod_{j=1}^{k_0} h_j)^{\frac{T}{2}}\). Then, we can choose \(\delta_2 = 1\). We moreover take \(D := [\max(-\frac{2}{\rho} \log(\prod_{j=1}^d h_j), 1) \wedge T]\), as before. It yields
\[ \text{Var}(\hat{x}_{h,T}(x)) \leq c \frac{\prod_{j=1}^{k_0} h_j^{\frac{T}{2}}}{\prod_{j=1}^d h_j} + \frac{1}{\prod_{j=1}^d h_j} \frac{1}{\prod_{j=1}^{k_0} h_j^{1-\frac{d}{2}}} \frac{1}{\prod_{j=1}^{k_0} h_j^{1-\frac{d}{2}}} \]
\[ = \frac{1}{T} \prod_{j=1}^{k_0} h_j^{1-\frac{d}{2}} \frac{1}{\prod_{j=1}^{k_0} h_j}. \]
as we wanted.

To conclude, by (35) with \( \delta_1 = 0 \), (40) and (41) we have proven that when \( k_0 = 1 \) and \( \beta_2 = \beta_3 \) we have

\[
\text{Var} (\hat{\pi}_{h,T}(x)) \leq \frac{c}{T} \left( \frac{\delta_2^2}{\prod_{l \geq 2} h_l} + \frac{1}{\delta_2^2 \prod_{l \geq 4} h_l} + D + \frac{1}{\prod_{j=1}^d h_j} \right)
\]

where the last estimation follows by having chosen \( \delta_2 := h_2 \beta_3 \). The parameter \( D \) is moreover chosen as above. Remarking that all the constants are uniform over the set of coefficient \((a,b) \in \Sigma, \) the result is proven.

\[\square\]

### 6.2 Proof of Theorem 1

**Proof.** From the usual bias-variance decomposition it is

\[
\text{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] = |\text{E}[\hat{\pi}_{h,T}(x)] - \pi(x)|^2 + \text{Var}(\hat{\pi}_{h,T}(x)).
\]

An upper bound for the variance is gathered in Proposition 2. Regarding the bias, a standard computation (see for example the proof of Proposition 1.2 of [21] or Proposition 1 of [20]) provides

\[
|\text{E}[\hat{\pi}_{h,T}(x)] - \pi(x)|^2 \leq c \sum_{j=1}^d h_j^{\beta_j},
\]

with the constant \( c \) that does not depend on \( x \), nor on \((a,b) \in \Sigma.\)

The result is then obtained by looking for the balance between the two terms.

We start considering the case where \( k_0 = 1 \) and \( \beta_2 < \beta_3 \) or \( k_0 = 2 \). Thanks to Proposition 2 we know the bound (11) holds true. After simple computations (as in the proof of Theorem 1 of [21]) it is easy to see that the balance is achieved by choosing the rate optimal bandwidth \( h^* := \left( \frac{\log T}{T} \right)^{\beta_1}, \ldots, \left( \frac{\log T}{T} \right)^{\beta_d} \), with

\[
a_1 \geq \frac{\beta_3}{\beta_1 (2 \beta_3 + d - 2)}, \quad a_2 \geq \frac{\beta_3}{\beta_2 (2 \beta_3 + d - 2)}
\]

and

\[
a_l \left( \frac{\beta_3}{\beta_l (2 \beta_3 + d - 2)} \right) \quad \forall l \in \{3, ..., d\}.
\]

Replacing the value of \( h^* \) in the upper bounds of the variance and of the bias we get

\[
\text{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq c \left( \frac{\log T}{T} \right)^{2 \beta_3 + \alpha_2 - 2},
\]

with the same constant \( c \) for all \((a,b) \in \Sigma.\)

When \( k_0 = 1 \) and \( \beta_2 = \beta_3 \) the bias-variance decomposition consists in

\[
\text{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq c \sum_{j=1}^d h_j^{\beta_j} + \frac{c}{T} \frac{1}{\prod_{l \geq 4} h_l}.
\]

As \( \beta_2 = \beta_3 \) we choose \( h_2 = h_3 \), such that the upper bound on the variance gathered in Proposition 2 becomes simply \( \frac{c}{T} \prod_{l \geq 4} h_l \). Then, similar computations as above implies that the balance is achieved by choosing \( h^*(T) := \left( \frac{1}{T} \right)^{\alpha_1}, \ldots, \left( \frac{1}{T} \right)^{\alpha_d} \), with \( a_1 \) as in (45) and \( a_2 = a_3 = a_j \) as in (46), for \( j \geq 3 \). It yields

\[
\sup_{(a,b) \in \Sigma} \text{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq c \left( \frac{1}{T} \right)^{2 \beta_3 + \alpha_2 - 2}.
\]

Assuming now that \( k_0 \geq 3 \), we have

\[
\text{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq c \sum_{j=1}^d h_j^{\beta_j} + \frac{c}{T} \frac{1}{h_1^{\alpha_0} \prod_{l \geq k_0+1} h_l}.
\]

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where we have already chosen to take \( h_1 = h_2 = \ldots = h_{k_0} \). We now look for the rate optimal bandwidth by setting \( h_k^* := \left( \frac{1}{T} \right)^{h_k} \) and searching \( a_1, a_{k_0 + 1}, \ldots, a_d \) such that the bias and variance are balanced. It consists in solving the following system:

\[
\begin{aligned}
& a_l \beta_l = a_l \beta_l, \quad \forall l \in \{k_0 + 1, \ldots, d\} \\
& 1 - a_1(k_0 - 2) - \sum_{l \geq k_0 + 1} a_l = 2a_1 \beta_1. 
\end{aligned}
\] (47)

As a consequence of the first \( d - k_0 \) equations, we can write

\[
a_l = \frac{\beta_1}{\beta_l} a_1, \quad \forall l \in \{k_0 + 1, \ldots, d\}. 
\] (48)

Hence, the last equation becomes

\[
2\beta_1 a_1 = 1 - a_1(k_0 - 2) - \beta_1 a_1 \sum_{l \geq k_0 + 1} \frac{1}{\beta_l}
\]

\[
= 1 - a_1(k_0 - 2) - \beta_1 a_1 \frac{d - k_0}{\bar{\beta}_k},
\] (49)

where \( \bar{\beta}_k \) is the mean smoothness over \( \beta_{k_0 + 1}, \ldots, \beta_d \) and it is such that

\[
\frac{1}{\bar{\beta}_k} = \frac{1}{d - k_0} \sum_{l \geq k_0 + 1} \frac{1}{\beta_l}.
\]

We also observe that, as \( \frac{1}{\beta_3} = \frac{1}{d - k_0} \sum_{l \geq 3} \frac{1}{\beta_l} \), it is

\[
\frac{k_0 - 2}{\beta_1} + \frac{d - k_0}{\bar{\beta}_3} = \frac{d - 2}{\beta_3}
\] (50)

Hence, (49) can be seen as

\[
2\beta_1 a_1 = 1 - a_1 \beta_1 \frac{d - 2}{\beta_3},
\]

which leads to the choice

\[
a_1 = \frac{\bar{\beta}_3}{\beta_1(2\beta_3 + d - 2)}.
\]

Thanks to the first \( d - k_0 \) equations in the system it follows

\[
a_l = \frac{\bar{\beta}_3}{\beta_1(2\beta_3 + d - 2)} \quad \forall l \in \{k_0 + 1, \ldots, d\}.
\]

Plugging the rate optimal bandwidth \( h^* \) in the bound of the mean squared error and recalling that the constant \( c \) does not depend on \((a, b) \in \Sigma\), we obtain

\[
\sup_{(a,b) \in \Sigma} \mathbb{E}[\|\hat{\pi}_{h,T}(x) - \pi(x)\|^2] \leq c \left( \frac{1}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}},
\]

as we wanted.

\[\square\]

**Remark 1.** As the situation is complicated, it is worth highlighting how the optimal bandwidths depend on the smoothness.

- When \( \beta_2 < \beta_3 \), \( h_1 \) and \( h_2 \) are arbitrarily small while \( h_3, \ldots, h_d \) are fixed. In this case there is the logarithm in the rate.
- When \( \beta_2 = \beta_3 \) we have two possibilities for the bandwidths, depending on \( \beta_1 \). In both cases we remove the logarithm in the rate.
  - If \( \beta_1 < \beta_2 \), then \( h_1 \) can be arbitrarily small while \( h_2, \ldots, h_d \) are fixed.
  - If \( \beta_1 = \beta_2 \), then all the bandwidths \( h_1, \ldots, h_d \) are fixed.

It is interesting to see that when we have two degrees of freedom on the bandwidths it is impossible to remove the logarithm in the convergence rate, while having one degree of freedom on the bandwidths is enough to modify the convergence rate: it is the same it would have been without any degree of freedom on the bandwidths.
6.3 Proof of Theorem 2

Proof. The proof relies, as before, on the bias-variance decomposition. Concerning the bias, (44) still holds true. We need to provide an upper bound on the variance, based on Lemma 1 and Proposition 1 as before. The main difference compared to the proof of Proposition 2 is that we now split the integral over \([0, T]\) into three pieces:

\[
[0, T] = [0, \delta) \cup [\delta, D) \cup [D, T),
\]

where \(\delta\) and \(D\) will be chosen later. On the first and on the last piece we act as in the proof of Proposition 2 and so (28) and (41) keep holding. Regarding the interval \([\delta, D)\), we act on it as we did on \([\delta_2, D)\). From (37), recalling that now \(d = 2\), we obtain

\[
\int_\delta^D |k(s)|ds \leq c \int_\delta^D (s^{-1} + 1)ds \leq c(\log D + |\log \delta| + D).
\]

Putting all the pieces together we get

\[
\text{Var}(\hat{\pi}_{h,T}(x)) \leq \frac{c}{T} \left(\frac{\delta}{h_1 h_2} + |\log D| + |\log \delta| + D + \frac{1}{h_1 h_2} e^{-\rho D}\right)
\]

\[
\leq \frac{c}{T} \left(1 + \sum_{j=1}^2 |\log h_j| + \sum_{j=1}^2 |\log(|\log h_j|)|\right)
\]

\[
\leq \frac{c}{T} \sum_{j=1}^2 |\log h_j|,
\]

where we have opportuneely chosen \(\delta = h_1 h_2\) and \(D := [\max(-\frac{2}{\rho} \log(h_1 h_2), 1) \wedge T]\). It follows

\[
\mathbb{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq c h_1^{2\beta_1} + c h_2^{2\beta_2} + \frac{c \sum_{j=1}^2 |\log h_j|}{T}.
\]

To conclude it is enough to observe that the optimal choice for the bandwidth consists in taking \(h_l^* = \left(\frac{\log T}{T}\right)^{a_l}\), with \(a_l \geq \frac{1}{2\beta_l}\) for \(l = 1, 2\), that provides the wanted convergence rate. \(\square\)

7 Proof of the adaptive procedure stated in Section 3

7.1 Proof of Theorem 3

The adaptive procedure in Theorem 3 heavily relies on a bound on the expectation of \(A(h)\), as stated in following proposition.

Proposition 3. Suppose that assumptions A1 - A2 hold and that \(d \geq 3\). Then, \(\forall h \in H_T\), we have the following bound

\[
\mathbb{E}[A(h)] \leq c_1 B(h) + c_1 e^{-c_2 (\log T)^2}.
\]

The proof of the fact that Proposition 3 implies Theorem 3 is classical and it is therefore here omitted (see for example the proof of Theorem 1 in [4], relying on Proposition 5 therein).

The proof of Proposition 3 similarly as Proposition 5 in [4], relies on the use of Berbee’s coupling method as in Viennet [76] and on a version of Talagrand inequality given in Klein and Rio [44]. However, a different penalty function is here chosen, which results in some challenges when one wants to use Talagrand inequality.

7.1.1 Proof of Proposition 3

Proof. The proof of Proposition 3 is close in spirit to the proof of Proposition 5 of [4]. We want to highlight the main differences appearing in our context.

As in the proof of Proposition 5 of [4], the key point consists in Talagrand inequality, which is stated on independent random variables. Therefore, we start by introducing some blocks which
are mutually independent through Berbee’s coupling method as done in Viennet [76], Proposition 5.1 (p. 484).

Hence, we assume that \( T = 2pTqT \), with \( pT \) integer and \( qT \) a real to be chosen, and we split the initial process \( X = (X_t)_{t \in [0,T]} \) in \( 2pT \) processes of length \( qT \). For each \( j \in \{1, \ldots, pT\} \) we introduce \( X^{j,1} := (X_t)_{t \in [2(j-1)qT, (2j-1)qT]} \) and \( X^{j,2} := (X_t)_{t \in [2j-1)qT, 2jqT]} \) as in Proposition 5 of [4] and we construct the process \( (X^{j,*})_{j \in [0,T]} \) having the same distribution of \( X^{j,1} \) and \( X^{j,2} \) on the blocks but such that, for each \( k \in \{1,2, \ldots, pT\} \), \( X^{*,1,k}, \ldots, X^{*,pT,k} \) are independent (see points 1, 2, 3 in Proposition 5 of [4] for details).

All the quantities computed using \( X_t \) can be replaced by the same quantities computed through \( X^{*,1} \), we add in this case an \( * \) in the notation. In particular, we introduce in this way \( \hat{\pi}^{*,1}_{h,T} \) and \( \hat{\pi}^{*,2}_{h,T} = \frac{1}{2}(\hat{\pi}^{*,1}_{h,T} + \hat{\pi}^{*,2}_{h,T}) \), to separate the part coming from \( X^{*,1} \) (super-index \( 1 \)) and those coming from \( X^{*,2} \) (super-index \( 2 \)), having \( \hat{\pi}^{*,(1)}_{h,T} := \frac{1}{pqT} \sum_{j=1}^{pT} \int_{(2(j-1)qT, 2jqT]} K_h(X^*_t - x) du \).

In a natural way we define moreover \( \hat{\pi}^{*,(1)}_{(h,q),T} := \mathbb{K}_q \hat{\pi}^{*,1}_{h,T}, \) that can be written again as \( \frac{1}{2}(\hat{\pi}^{*,(1)}_{(h,q),T} + \hat{\pi}^{*,2}_{(h,q),T}) \), to separate the contribution of \( X^{*,1} \) and \( X^{*,2} \).

We recall that this notation is analogous to the one introduced in Proposition 5 of [4], where triangular inequality is used to obtain

\[
A(h) \leq \sup_{\eta \in \mathcal{H}_T} \left[ \left\| \hat{\pi}(h,q),T - \hat{\pi}(h,q),T \right\|_A^2 + \left\| \hat{\pi}^{*,(1)}_{(h,q),T} - \hat{\pi}^{*,(2)}_{(h,q),T} \right\|_A^2 \right] + \left\| \hat{\pi}^{*,(1)}_{(h,q),T} - \hat{\pi}^{*,(2)}_{(h,q),T} \right\|_A^2 \leq \sup_{\eta \in \mathcal{H}_T} \left[ \sum_{j=1}^{5} I_j^{(h,q)} \right],
\]

Then, the analysis of \( I_1, I_3 \) and \( I_5 \) is the same as in Proposition 5 of [4] and provides, respectively

\[
\sup_{\eta \in \mathcal{H}_T} I_3^{(h,q)} \leq c B(h)
\]

and

\[
\mathbb{E}\left[ \sup_{\eta \in \mathcal{H}_T} I_1^{(h,q)} \right] + \mathbb{E}\left[ \sup_{\eta \in \mathcal{H}_T} I_5^{(h,q)} \right] \leq \frac{c}{(\prod_{l=1}^d \eta_l)^2} \left( \frac{T}{(\log T)^2} e^{-\gamma (\log T)^2} \right),
\]

where we have also chosen \( qT \) as \( (\log T)^2 \). Then, the definition of the set of candidate bandwidths \( \mathcal{H}_T \) as in [13] implies

\[
\mathbb{E}\left[ \sup_{\eta \in \mathcal{H}_T} I_1^{(h,q)} \right] + \mathbb{E}\left[ \sup_{\eta \in \mathcal{H}_T} I_5^{(h,q)} \right] \leq c \left( \frac{T^{1+2d}}{(\log T)^2} e^{-\gamma (\log T)^2} \right).
\]

On \( I_2^{(h,q)} \) and \( I_4^{(h,q)} \) we want to use Talagrand inequality as formulated in Lemma 2 in [22]. It is a consequence of the Talagrand inequality given in Klein and Rio [14] (see also Lemma 5 in [4]):

**Lemma 4.** Let \( T_1, \ldots, T_p \) be independent random variables with values in some Polish space \( X, \mathcal{R} \), a countable class of measurable functions from \( X \) into \( \mathbb{R} \) and \( v_p(r) := \frac{1}{p} \sum_{j=1}^{p} (r(T_j) - \mathbb{E}[r(T_j)]) \). Then,

\[
\mathbb{E}\left[ (\sup_{r \in \mathcal{R}} |v_p(r)|^2 - 2H^2)_{+} \right] \leq c \left( \frac{\mathbb{E}[v_p(r)]^2}{p} + \frac{M^2}{p^2} e^{-c \gamma T} \right),
\]

with \( c \) a universal constant and where

\[
\sup_{r \in \mathcal{R}} |r|_{\infty} \leq M, \quad \mathbb{E}[\sup_{r \in \mathcal{R}} |v_p(r)|] \leq H, \quad \sum_{j=1}^{p} \frac{1}{p} \sum_{j=1}^{p} \operatorname{Var}(r(T_j)) \leq v.
\]

In order to use Lemma 4 we write that \( \left\| \hat{\pi}^{*,(1)}_{(h,q),T} - \hat{\pi}^{*,(2)}_{(h,q),T} \right\|_A \leq \sup_{r \in \mathcal{B}(1)} \left( \left\| \hat{\pi}^{*,(1)}_{(h,q),T} - r \right\|_A \right) \) where \( \mathcal{B}(1) \) is a countable dense set of the unit ball of \( L^2(A) \). For \( r \in \mathcal{B}(1) \), we set

\[
\mathbb{E}\left[ r(T)^{(2q-1)\epsilon r} \right] := \frac{1}{4T} \int_{(2(q-1)\epsilon r, 2q\epsilon r]} \mathbb{K}_q \ast r(X_t^{*,j-1}) ds
\]
which is such that \( < \hat{\pi}_{\eta_2}^{(1)}, r > = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} r_{\eta_j} (X^{*,j,1}) \), and \( < \pi_{\eta} - \hat{\pi}_{\eta_2}^{(1)}, r > = -v_p(r_\eta) \) where we use the notation \( v_p \) of Lemma 4. The application of Lemma 4 with the countable class \( \mathcal{R} = \{ r, r \in \mathbb{B}(1) \} \), now requires to compute the constants \( M, H \) and \( v \). To do so, one can closely follow the proof of Lemma 4 in [4]. Then, it is straightforward to see that

\[ M = c (\prod_{i=1}^{d} q_i)^{-\frac{1}{2}}. \]

Regarding the computation of \( H \), it comes directly from the computation of the variance. It is easy to check that, replacing the bound on the variance given by Proposition 1 in [4] by the one gathered in our Proposition 2, one obtains

\[ H^2 = \frac{c}{T} \min \left( \frac{\sum_{j=1}^{d} \left| \log \eta_j \right|}{\prod_{l \neq k_1, k_2} \eta_l}, \frac{1}{\sqrt{\prod_{l \neq k_1, k_2} \eta_l}} \right) \]

\[ = \frac{c}{T} \hat{H}^2(\eta). \]

To use Talagrand inequality, we moreover need to compute \( v \). Again, one can check the computations in Lemma 4 in [4] to easily obtain

\[ v := \frac{c}{qT} (1 + \log \left( \frac{1}{\prod_{i=1}^{d} \eta_i} \right)). \]

Then, we use Lemma [4] and it follows that there exists some \( k_0 > 0 \) such that, for any \( \hat{k} \geq k_0 \) we have

\[ \mathbb{E} \sup_{\eta \in \mathcal{H}_T} \left( \left\| \pi_{\eta} - \hat{\pi}_{\eta}^{(1)} \right\|^2_A - \frac{\hat{k}}{T} \hat{H}^2 \right) \leq \sum_{\eta \in \mathcal{H}_T} \mathbb{E} \left( \left\| \pi_{\eta} - \hat{\pi}_{\eta}^{(1)} \right\|^2_A - \frac{\hat{k}}{T} \hat{H}^2 \right) \]

\[ \leq c \sum_{\eta \in \mathcal{H}_T} \left( 1 + \log \left( \frac{1}{\prod_{i=1}^{d} \eta_i} \right) \right) e^{-c \frac{pT}{qT} \frac{\hat{H}^2(\eta)}{\prod_{i=1}^{d} \eta_i}} \left( \prod_{i=1}^{d} \eta_i \right)^{-\frac{1}{2}} e^{-c \frac{1}{\log T} \frac{1}{\prod_{i=1}^{d} \eta_i}} \right). \]

We recall that \( 2pTqT = T \), where \( qT \) is chosen as \( (\log T)^2 \); we can therefore upper bound the right hand side of the equation here above with

\[ \sum_{\eta \in \mathcal{H}_T} \left( 1 + \log \left( \frac{1}{\prod_{i=1}^{d} \eta_i} \right) \right) e^{-c \frac{pT}{qT} \frac{\hat{H}^2(\eta)}{\prod_{i=1}^{d} \eta_i}} \left( \prod_{i=1}^{d} \eta_i \right)^{-\frac{1}{2}} e^{-c \frac{1}{\log T} \frac{1}{\prod_{i=1}^{d} \eta_i}} \right) \]

\[ \leq \left( \prod_{i=1}^{d} \eta_i \right)^{\frac{1}{2}} \hat{H}(\eta) \]

\[ = \left( \prod_{i=1}^{d} \eta_i \right)^{\frac{1}{2}} \hat{H}(\eta) \]

\[ = \min \left( \left( \sum_{j=1}^{d} \left| \log \eta_j \right| \eta_{k_1} \eta_{k_2} \right)^{\frac{1}{2}}, \left( \eta_{k_2} \right)^{\frac{1}{2}} \left( \eta_{k_2} \right)^{\frac{1}{2}} \right). \]

Furthermore, it is \( (\eta_{k_1} \eta_{k_2})^{\frac{1}{2}} \leq (\eta_{k_2})^{\frac{1}{2}} (\eta_{k_2} \eta_{k_3})^{\frac{1}{2}} \) for \( \eta_{k_2} < \eta_{k_3} \), which implies that the minimum is achieved by the first in this context. Otherwise, for \( \eta_{k_2} = \eta_{k_3} \), because of the presence of the logarithm in the first term, the minimum is realised by the second. Then, by the definition we have given of \( \mathcal{H}_T \), \( \forall h \in \mathcal{H}_T \) we have \( \left( \prod_{i=1}^{d} \eta_i \right)^{\frac{1}{2}} \hat{H}(\eta) \geq \frac{(\log T)^{2+a}}{\sqrt{r}} \). Thus, \( (55) \) is upper bounded by

\[ \sum_{\eta \in \mathcal{H}_T} \frac{\log T}{T} e^{-c \frac{1}{\log T} \frac{1}{\prod_{i=1}^{d} \eta_i}} + T^{d+1-2} \left( \log T \right)^{d+1} e^{-c \frac{1}{\log T} \frac{1}{\prod_{i=1}^{d} \eta_i}} \]

\[ \leq T^{c-1} \log T e^{-c \frac{1}{\log T} \frac{1}{\prod_{i=1}^{d} \eta_i}} + T^{d+2} \left( \log T \right)^{d+1} e^{-c \frac{1}{\log T} \frac{1}{\prod_{i=1}^{d} \eta_i}}, \]
where in the last inequality we have used that, by the definition of $\mathcal{H}_T$, its cardinality $|\mathcal{H}_T|$ has polynomial growth in $T$ at most as $T^c$. Therefore, we have proven

$$
\mathbb{E}\left[ \sup_{\eta \in \mathcal{H}_T} \left\| \pi_{\eta} - \tilde{\pi}_{\eta,T}^{(1)} \right\|_A^2 \right] \leq T^{c-1} \log T e^{-c(\log T)^{\alpha(d-2)}} + T^{d b - 2 + c(\log T)^{\alpha}} \tag{57}
$$

from which it directly follows the wanted bound on $I_{4}^{h,\eta}$ since $a > 1$ and $d \geq 3$. Indeed, it is $\pi_{\eta} - \tilde{\pi}_{\eta,T}^{(1)} = \frac{1}{2}(\pi_{\eta} - \tilde{\pi}_{\eta,T}^{(1)} - \tilde{\pi}_{\eta,T}^{(2)})$. Hence, from triangular inequality and the definition of positive part function, we get

$$
I_{4}^{h,\eta} \leq c\left(\left\| \pi_{\eta} - \tilde{\pi}_{\eta,T}^{(1)} \right\|_A^2 - \frac{V(\eta)}{2} \right) + c\left(\left\| \pi_{\eta} - \tilde{\pi}_{\eta,T}^{(2)} \right\|_A^2 - \frac{V(\eta)}{2} \right). \tag{58}
$$

From (57), for a $k$ in the definition of $V(\eta)$ big enough, such that $\frac{c}{2} > k_0$, we get

$$
\mathbb{E}\left[ \sup_{\eta \in \mathcal{H}_T} I_{4}^{h,\eta} \right] \leq c_1 T^{c_2} e^{-c_3(\log T)^{\alpha}} \tag{59}
$$

with $c_4 > 1$. To end the proof, it remains to show a similar upper bound with $I_{2}^{h,\eta}$. From Lemma 3 of [4] and the definition of kernel function we have

$$
\left\| \pi_{(h,\eta)} - \tilde{\pi}_{(h,\eta),T}^{(1)} \right\|_A^2 \leq \left\| K_{h \ast} (\pi_{\eta} - \tilde{\pi}_{\eta,T}^{(1)}) \right\|_A^2 \leq \left\| K_{h} \right\|_A^2 \left\| \pi_{\eta} - \tilde{\pi}_{\eta,T}^{(1)} \right\|_A^2 \leq c \left\| \pi_{\eta} - \tilde{\pi}_{\eta,T}^{(1)} \right\|_A^2.
$$

Such a remark, together with (57) yields

$$
\mathbb{E}\left[ \sup_{\eta \in \mathcal{H}_T} \left\| \pi_{(h,\eta)} - \tilde{\pi}_{(h,\eta),T}^{(1)} \right\|_A^2 - \frac{\hat{k}}{T} H^2 \right] \leq T^{c-1} \log T e^{-c(\log T)^{\alpha(d-2)}} + T^{d b - 2 + c(\log T)^{\alpha}} e^{-c(\log T)^{\alpha}},
$$

which clearly implies

$$
\mathbb{E}\left[ \sup_{\eta \in \mathcal{H}_T} I_{2}^{h,\eta} \right] \leq c_1 T^{c_2} e^{-c_3(\log T)^{\alpha}}.
$$

The proof is then concluded. \qed

### 7.2 Proof of Theorem 4

**Proof.** The proof of the pointwise adaptive procedure closely follows the proof of Theorem 3 provided above. In particular, analogous to Proposition 3, the key point is to establish that, for any $h \in \mathcal{H}_T$,

$$
\mathbb{E}[A^p(h,x)] \leq c_1 B(h,x) + c_2 T^{-c_2} \tag{60}
$$

for some $c_1 > 0$, $c_2 \geq 1$. To establish (60), we can still employ Berbee’s coupling method, which, as seen in Equation (51), leads us to

$$
A^p(h,x) \leq \sup_{\eta \in \mathcal{H}_T} \left( |\tilde{\pi}_{(h,\eta),T}(x) - \tilde{\pi}_{(h,\eta),T}(x)|^2 + (|\tilde{\pi}_{(h,\eta),T}(x) - \pi_{(h,\eta)}(x)|^2 - \frac{V_p(\eta)}{2}) \right) + \left( |\pi_{(h,\eta)}(x) - \pi_{\eta}(x)|^2 + (|\pi_{(h,\eta)}(x) - \tilde{\pi}_{\eta,T}(x)|^2 - \frac{V_p(\eta)}{2}) \right)
$$

$$
=: \sup_{\eta \in \mathcal{H}_T} \left( \sum_{j=1}^{5} I_{j}^{h,\eta}(x) \right).
$$

The analysis of $I_{3}^{h,\eta}(x)$, $I_{4}^{h,\eta}(x)$ and $I_{5}^{h,\eta}(x)$ is the same as in (52) and (53), respectively. It yields

$$
\sup_{\eta \in \mathcal{H}_T} \left( \sum_{j=1}^{5} I_{j}^{h,\eta}(x) \right) \leq c B^p(h,x), \quad \mathbb{E}[\sup_{\eta \in \mathcal{H}_T} I_{j}^{h,\eta}(x)] \leq \mathbb{E}[\sup_{\eta \in \mathcal{H}_T} I_{j}^{h,\eta}(x)] \leq c T^{1+2d b} e^{-\gamma(\log T)^{2}}.
$$

On $I_{2}^{h,\eta}(x)$ and $I_{3}^{h,\eta}(x)$ we want to use Bernstein’s inequality. Let us start by considering $I_{3}^{h,\eta}(x)$. Notice that, similarly as in (58) it is

$$
I_{3}^{h,\eta}(x) \leq c\left( |\pi_{\eta}(x) - \tilde{\pi}_{\eta,T}(x)|^2 - \frac{V_p(\eta)}{2} \right) + c\left( |\pi_{\eta}(x) - \tilde{\pi}_{\eta,T}(x)|^2 - \frac{V_p(\eta)}{2} \right).
$$

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Then,

\[
\mathbb{E}\left(\sup_{\eta \in H_T^2} (|\pi_\eta(x) - \tilde{\pi}_{\eta,T}^{(1)}(x)|^2 - \frac{V_p(\eta)}{2})_+\right) \leq \sum_{\eta \in H_T^2} \int_0^\infty \mathbb{P}\left(|\pi_\eta(x) - \tilde{\pi}_{\eta,T}^{(1)}(x)|^2 - \frac{V_p(\eta)}{2} \geq t\right) dt
\]

\[
= \sum_{\eta \in H_T^2} \int_0^\infty \mathbb{P}\left(|\pi_\eta(x) - \tilde{\pi}_{\eta,T}^{(1)}(x)| \geq \sqrt{\frac{V_p(\eta)}{2} + t}\right) dt.
\]

We recall Bernstein’s inequality (see for example p.366 [14]): let \( T_1, \ldots, T_n \) independent variables and \( S_p(T) := \sum_{i=1}^n (T_i - \mathbb{E}[T_i]) \) such that \( \text{Var}(T_i) \leq v, |T_i|_\infty \leq M. \) Then, for any \( \gamma > 0, \)

\[
\mathbb{P}\left(|S_p(T) - \mathbb{E}[S_p(T)]| \geq p\gamma\right) \leq 2 \exp\left(-\frac{p^2\gamma^2}{2 M^2}\right) \leq 2 \max(\exp(-\frac{p\gamma^2}{4v}), \exp(-\frac{p\gamma}{4M})).
\]

Observe it is

\[
\pi_\eta(x) - \tilde{\pi}_{\eta,T}^{(1)}(x) = \frac{1}{p_T q_T} \sum_{l=1}^{p_T} \int_{2(l-1)q_T}^{2l q_T} (K_\eta(X_u^* - x) - \mathbb{E}[K_\eta(X_u^* - x)]) du
\]

\[
=: \frac{1}{p_T} \sum_{l=1}^{p_T} (T_i - \mathbb{E}[T_i]).
\]

The computation of \( v, \) in the same way as \( H \) appearing in Theorem [3] follows directly from Proposition [2]

\[
\text{Var}(T_i) = \text{Var}\left(\frac{1}{q_T} \int_{2(l-1)q_T}^{2l q_T} (K_\eta(X_u^* - x) - \mathbb{E}[K_\eta(X_u^* - x)]) du\right)
\]

\[
\leq \frac{c}{q_T} \tilde{H}^2(\eta) =: v,
\]

where \( \tilde{H}^2(\eta) \) is as in [16] and [17].

Regarding the computation of \( M, \) as \( T_i = \frac{1}{q_T} \int_{2(l-1)q_T}^{2l q_T} K_\eta(X_u^* - x) du, \) it is \( |T_i|_\infty \leq c(\prod_{l=1}^d \eta_l)^{-\frac{1}{2}} =: M. \) It is easy to see that it is worse compared to the integrated \( L^2 \) norm studied in previous theorem, where \( \|r\|_\infty \) was bounded by \( c(\prod_{l=1}^d \eta_l)^{-\frac{1}{2}}. \) Bernstein’s inequality gives, for any \( t \geq 0 \)

\[
\mathbb{P}\left(|\pi_\eta(x) - \tilde{\pi}_{\eta,T}^{(1)}(x)| \geq \sqrt{\frac{V_p(\eta)}{2} + t}\right) \leq 2 \max\left(\exp\left(-\frac{p_T V_p(\eta)}{4v}\right) + t\right),
\]

\[
\exp\left(-\frac{p_T}{8M} \sqrt{\frac{V_p(\eta)}{2}}\right) \exp\left(-\frac{p_T}{8M} \sqrt{t}\right),
\]

as \( \sqrt{\cdot} \) is a concave function. Then, using the definition of \( V_p(\eta), v \) and \( M \) one has, for some \( c_1, c_2, c_3 \) and \( c_4 > 0 \) the following identities: \( \frac{p_T V_p(\eta)}{4v} = c_1 k_T, \frac{p_T}{4v} = c_2 \frac{T}{H^2(\eta)}, \frac{p_T}{M} = c_3 \frac{T}{q_T} \prod_{l=1}^d \eta_l \) and \( \frac{p_T}{M} \sqrt{\frac{V_p(\eta)}{2}} = c_4 \frac{T}{q_T} \prod_{l=1}^d \eta_l \tilde{H}(\eta) \sqrt{\tilde{H}}, \) where we have also used that \( T = 2p_T q_T \) and we recall that
Let \( k_T = k \log T, \quad q_T = (\log T)^2 \). It follows, using also (61),

\[
\mathbb{E}[\sup_{\eta \in \mathcal{H}_T^p} (|\pi_\eta(x) - \hat{\pi}_\eta^{(1)}(x)|^2 - \frac{V_P(\eta)}{2})_+] \leq \sum_{\eta \in \mathcal{H}_T^p} \int_0^\infty 2 \max \left( \exp(-ck_T) \exp(-\frac{T}{H^2(\eta)}), \right. \\
\left. \exp(-c\sqrt{k_T} \frac{\sqrt{T}}{(\log T)^2} \prod_{i=1}^d \eta_i \hat{H}(\eta) \exp(-\frac{cT}{(\log T)^2} \prod_{i=1}^d \eta_i (1 - \alpha) \sqrt{t}) ) \right) dt \]

\[
\leq \sum_{\eta \in \mathcal{H}_T^p} \exp(-ck_T) \int_0^\infty \exp(-\frac{T}{H^2(\eta)}) dt + \sum_{\eta \in \mathcal{H}_T^p} \exp(-ck_T) \frac{\sqrt{T}}{(\log T)^2} \prod_{i=1}^d \eta_i \hat{H}(\eta) \alpha \frac{(\log T)^4}{(T \prod_{i=1}^d \eta_i (1 - \alpha)^2)}.
\]

Recall that, according to the definition of \( \mathcal{H}_T^p \) given in (14), it is \( \prod_{i=1}^d \eta_i \hat{H}(\eta) \geq (\frac{\log T}{\sqrt{T}})^{2+\delta} \). Moreover, by the definition of \( \hat{H} \) and the fact that for any \( \eta \in \mathcal{H}_T^p \) it is \( (\frac{1}{T})^\delta \leq \frac{1}{\eta_i} \leq (\frac{1}{\log T})^{2+\delta} \), one obtains \( \hat{H}(\eta) \leq (\frac{1}{\log T})^{\alpha(d-2)}. \) It follows

\[
\mathbb{E}[\sup_{\eta \in \mathcal{H}_T^p} (|\pi_\eta(x) - \hat{\pi}_\eta^{(1)}(x)|^2 - \frac{V_P(\eta)}{2})_+] \leq c \sum_{\eta \in \mathcal{H}_T^p} \left[ \exp(-ck_T) \frac{1}{T(\log T)^{\alpha(d-2)}} + \exp(-c\sqrt{k_T} (\log T)^{\alpha} \eta) \frac{(\log T)^4}{T^2(1 - \alpha)^2 (\log T)^{2+\delta}} \right] \\
\leq c_1 T^{-c_2} (e^{-ck_T \log T} + e^{-c(\log T)^{\alpha+\frac{1}{2}}}) \leq c_1 T^{-c_3},
\]

where we have also used that \( \|\mathcal{H}_T^p\| \) has polynomial growth in \( T \) at most as \( T^c \). Remark that the result above is in hold up to choose \( k \) large enough to guarantee that \( ck - c_2 \geq c_1 \geq 1 \). It yields the wanted bound on \( \mathbb{E}[\sup_{\eta \in \mathcal{H}_T^p} I_2^{p,h,\eta}(x)] \). To conclude, we study \( I_2^{p,h,\eta}(x) \). Observe that such term can be treated exactly as the previous one. Indeed, instead of \( T_i \) defined as above we have now to deal with \( \tilde{T}_i := \frac{1}{q_T} \int_{2(i-1)q_T}^{2i-1)q_T} \mathbb{E}_h \|\mathcal{K}_h \|_{1} \left( \prod_{i=1}^d \eta_i \right)^{-1} = M. \)

Moreover using Fubini Theorem and the fact that both \( \|\mathcal{K}_h\|_1 \) and \( \|\mathcal{K}_\eta\|_1 \) are bounded by a constant, it is easy to check the same for \( \|\mathcal{K}_h \|_{1} \). Furthermore, using that \( \mathbb{E}_h \|\mathcal{K}_h \|_{\infty} \) has support in \(-h, h\) and the improved upper bound \( \|\mathcal{K}_h \|_{\infty} \leq c(\prod_{i=1}^d (h_i + \eta_i))^{-1}, \) one can follow step by step the proof given in Proposition 2 to verify that it holds true with \( \mathbb{E}_h \|\mathcal{K}_h \|_{1} \) instead of \( \mathbb{E}_h \|\mathcal{K}_h \|_{\infty} \). It implies

\[
\text{Var}(\hat{T}_i) \leq \frac{c}{q_T} \hat{H}(\eta) = v.
\]

Then, applying Bernstein’s inequality in the same manner as demonstrated in the proof concerning \( I_2^{p,h,\eta}(x) \) directly yields

\[
\mathbb{E}[\sup_{\eta \in \mathcal{H}_T^p} I_2^{p,h,\eta}(x)] \leq c_1 T^{-c_2}
\]

for some \( c_2 \geq 1 \), as we wanted. 

\[\square\]

### 7.3 Proof of Theorem 5

**Proof.** We recall that, according to Theorem 1, the optimal choice for the bandwidth changes depending on how many \( \beta \)'s are coincident. This is equivalent to understand which term achieves
the minimum in the definition of the penalty term. In particular, when the \( \beta \)'s are not ordered we have for any bandwidth \( h \)

\[
\text{Var}(\hat{\pi}_h) \leq \frac{c}{T} \min\left( \sum_{j=1}^{d} |\log h_j| \frac{1}{\prod_{l \neq k_1, k_2} h_l} \right),
\]

with \( h_{k_1} \leq h_{k_2} \leq \ldots \leq h_{k_d} \) by construction. Then, if the minimum is realized by the first, it means that \( h_{k_3} > h_{k_2} \). In this case the rate optimal choice for the bandwidth consists in taking

\[
h_{k_j} := \left( \frac{\log T}{T} \right)^{\frac{\beta_j}{2} (\frac{1}{2} + d - 2)},
\]

(62)

for any \( j \geq 3 \) and \( h_{k_1} \) and \( h_{k_2} \) arbitrarily small. It provides the convergence rate \( \left( \frac{\log T}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}} \).

We want such a choice of bandwidth to be in \([22]\). Hence, we start by checking the conditions in \([13]\) are respected. One can observe that, for \( h_{k_3} > h_{k_2} \), it is

\[
\min(\sum_{j=1}^{d} |\log h_j|h_{k_1}h_{k_2})^{\frac{1}{2}}, (h_{k_1})^{\frac{1}{2}}(h_{k_2}h_{k_3})^{\frac{1}{2}}) = \left( \sum_{j=1}^{d} |\log h_j|h_{k_1}h_{k_2} \right)^{\frac{1}{2}}.
\]

Then, we need to choose \( h_{k_1} \) and \( h_{k_2} \) large enough to guarantee \( \left( \sum_{j=1}^{d} |\log h_j|h_{k_1}h_{k_2} \right)^{\frac{1}{2}} \geq \frac{c(\log T)^{2+a}}{\sqrt{T}} \) for some \( a > 0 \) arbitrarily small. It leads us to the choice \( h_{k_1} = h_{k_2} = \frac{(\log T)^{2+a}}{\sqrt{T}} \). It is easy to check that such a choice still implies the convergence rate to be \( \left( \frac{\log T}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}} \). Indeed, the bias term associated to \( h_{k_1} \) and \( h_{k_2} \) is

\[
h_{k_1}^{2\beta_1} + h_{k_2}^{2\beta_2} = \left( \frac{(\log T)^{2+a}}{\sqrt{T}} \right)^{2\beta_1} + \left( \frac{(\log T)^{2+a}}{\sqrt{T}} \right)^{2\beta_2} \leq \left( \frac{\log T}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}},
\]

being the last a consequence of the fact that both \( \beta_1 \) and \( \beta_2 \) are larger than 1, which is larger than \( \frac{2\beta_3}{2\beta_3 + d - 2} \). Clearly the optimal choice of the bandwidth with components \( h_{k_1} \) as above satisfies the conditions in \([13]\).

When the minimum of the variance is achieved by \( \frac{1}{h_{k_2}h_{k_1} \prod_{l \neq k_1, k_2} h_l} \), instead, it implies that \( h_{k_2} = h_{k_1} \). According to Theorem \([1]\) the rate optimal choice for the bandwidth is similar as in previous case, consisting in

\[
h_{k_j} := \left( \frac{1}{T} \right)^{\frac{\beta_j}{2} (\frac{1}{2} + d - 2)}, \quad \forall j \geq 2.
\]

(63)

Now the only bandwidth allowed to be arbitrarily small is \( h_{k_1} \). In this case the convergence rate is \( \left( \frac{\log T}{T} \right)^{\frac{2\beta_j}{2\beta_j + d - 2}} \).

Again, we need to check that conditions in \([13]\) are respected. One can observe that, as \( h_{k_3} = h_{k_2} \) (which coincides with \( \beta_1 = \beta_2 \) because of \([63]\)), it is

\[
\min(\sum_{j=1}^{d} |\log h_j|h_{k_1}h_{k_2})^{\frac{1}{2}}, (h_{k_1})^{\frac{1}{2}}(h_{k_2}h_{k_3})^{\frac{1}{2}}) = (h_{k_1})^{\frac{1}{2}}(h_{k_2}h_{k_3})^{\frac{1}{2}}
\]

(64)

(64)

Then, we need to choose \( h_{k_1} \) and \( h_{k_2} \) such that \( (h_{k_1}h_{k_2})^{\frac{1}{2}} \geq \frac{(\log T)^{2+a}}{\sqrt{T}} \) for some \( a > 0 \) arbitrarily small which, according to \([63]\), leads us to the condition

\[
h_{k_1}^{\frac{1}{2}} \geq \frac{(\log T)^{2+a}}{\sqrt{T}} T^{-\frac{\beta_3}{2\beta_3 + d - 2}}.
\]
This is equivalent to the constraint
\[
h_{k_1} \geq \frac{(\log T)^{4+2a}}{T} T^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)}.
\]

We choose \( h_{k_1} \) realizing the equivalence above, which implies the convergence rate to be \( \left( \frac{1}{T} \right)^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)} \), as the bias term associated to \( h_{k_1} \) is negligible. Indeed, it is
\[
h_{k_1}^{\frac{2}{3}} \left( \frac{1}{T} \right)^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)} \left( \frac{1}{T} \right)^{2 \beta_3 \left( \frac{1}{2} + \beta_3 \left( \frac{1}{2} + \frac{3}{d-2} \right) \right)}.
\]

This is negligible compared to \( \left( \frac{1}{T} \right)^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)} \) using that \( d \geq 3 \) and \( \beta_{k_1} \geq 1 \). We deduce the optimal choice of the bandwidth with components \( h_{k_1} \) as above satisfies the conditions in (13).

Then, in order to say that the chosen bandwidths belong to the set of candidate bandwidths \( \mathcal{H}_T \) proposed in (22) we should say they have the particular form \( h_l = \frac{1}{n_l} \) for \( z_l \in \{1, \ldots, [T] \} \), that in general is not the case. However, we can replace \( h_{k_1} \) in (62) with \( \bar{h}_{k_1} := \frac{1}{[T]^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)}} \),

which is asymptotically equivalent and which leads to the same convergence rate. Moreover, \( \bar{h}_{k_1}(T) \) has the wanted form, which entails \( \bar{h}_{k_1} \) belongs to \( \mathcal{H}_T \), defined as in (22).

We act similarly with \( h_{k_1} \) and \( h_{k_2} \) (or only on \( h_{k_1} \) when \( h_{k_2} = h_{k_1} \), to replace the rate optimal choice \( h(T) \) with \( \bar{h}(T) \), which belongs to the set of candidate bandwidth \( \mathcal{H}_T \), defined as in (22). From Theorem 1 it follows that the inf \( h \in \mathcal{H}_T \) of \( B(h) + V(h) \) is clearly realized by \( h_T \). Thus, the bound stated in Theorem 3 is actually
\[
\mathbb{E}\left[ \left\| \hat{p}_{\bar{h},T} - \pi \right\|_A^2 \right] \leq \left\{ \begin{array}{ll}
(\log T)^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)} + c_1 (\log T)^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)} & \text{if } \beta_{k_2} < \beta_{k_3} \\
(\frac{1}{T})^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)} + c_1 (\log T)^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)} & \text{if } \beta_{k_2} = \beta_{k_3},
\end{array} \right.
\]
as we wanted. \( \square \)

7.4 Proof of Theorem 6

Proof. The key point in this theorem consists in looking for the optimal choice of the bandwidth, that realizes the trade-off between the bias term \( \sum_{j=1}^d h_{k_j} \) and the variance
\[
\frac{\log T}{T} \min \left\{ \frac{\sum_{j=1}^d |\log h_j|}{\prod_{i \neq j} h_i}, \frac{1}{h_{k_1} h_{k_2} \prod_{i \neq j} h_i} \right\}.
\]
Such balance will be different depending on the fact that the min in the variance is achieved by the first term or the second.

- Case 1.
  - If \( \min \left\{ \frac{\sum_{j=1}^d |\log h_j|}{\prod_{i \neq j} h_i}, \frac{1}{h_{k_1} h_{k_2} \prod_{i \neq j} h_i} \right\} = \sum_{j=1}^d |\log h_j| \prod_{i \neq j} h_i \), then, as already said in previous theorem, it means that \( h_{k_3} > h_{k_1} \).
  - For \( l \neq k_1, k_2 \) let us introduce \( h_l(T) := (\log T)^{\frac{3}{\beta_3} \left( \frac{1}{2} + \frac{3}{d-2} \right)} \).
  - We now look for an optimal choice for \( a_1 \) and \( b_1 \), for which the following identities are true:
\[
(\log T)^{2 \cdot \sum_{i \neq k_1, k_2} a_i \left( \frac{1}{T} \right)^{2 \cdot k_3}}, (\frac{1}{T})^{2 \cdot k_3}, \beta_{k_3} = \ldots = (\log T)^{2 \cdot k_3}, \beta_{k_3} = \ldots = (\log T)^{2 \cdot k_3}, \beta_{k_3}.
\]

It implies we want \( a_{k_3}, b_{k_3}, \ldots, a_{k_d}, b_{k_d} \) satisfying the following system:
\[
2a_{k_3}, \beta_{k_3} = \ldots = 2a_{k_3}, \beta_{k_3} = 2 - \sum_{l \neq k_1, k_2} a_l \\
2b_{k_3}, \beta_{k_3} = \ldots = 2b_{k_3}, \beta_{k_3} = 1 - \sum_{l \neq k_1, k_2} b_l.
\]

It leads to \( a_l = \frac{2 \beta_3}{h_l(\beta_3) \left( \frac{1}{2} + \frac{3}{d-2} \right)} \) and \( b_l = \frac{\beta_3}{h_l(\beta_3) \left( \frac{1}{2} + \frac{3}{d-2} \right)} \) for any \( l \in \{k_3, \ldots, k_d\} \). Then, the rate optimal choice for the bandwidth consists in taking
\[
h_{k_j} := (\log^2 T)^{\frac{\beta_3}{h_j \left( \frac{1}{2} + \frac{3}{d-2} \right)}}
\]
for any \( j \geq 3 \) and \( h_{k_1}, h_{k_2} \) arbitrarily small. It yields the convergence rate \((\log^2 T)^{\frac{2\beta_{k_1}}{2\beta_{k_1} + d - 2}}\). Now we want to check, similarly as in the proof of Theorem 5, that such bandwidth belongs to \( H_T^p \) as defined in (23). We therefore want to ensure that the conditions in (14) are respected. It implies we have to choose \( h_{k_1}, h_{k_2} \) large enough to ensure that

\[
\left(\prod_{l=1}^{d} h_l\right)^{\frac{1}{2}} \min\left(\sum_{j=1}^{d} |\log h_j| h_{k_1}, h_{k_2}\right)^{\frac{1}{2}}, (h_{k_1})^{\frac{1}{2}} (h_{k_2} h_{k_3})^{\frac{1}{2}} \geq \frac{c(\log T)^{2+a}}{\sqrt{T}}
\]

for some \( a \) arbitrarily small. Remark that, as we are conducting our analysis in the case where \( h_{k_1} > h_{k_2} \), it is

\[
\min\left(\sum_{j=1}^{d} |\log h_j| h_{k_1}, h_{k_2}\right)^{\frac{1}{2}}, (h_{k_1})^{\frac{1}{2}} (h_{k_2} h_{k_3})^{\frac{1}{2}} = \left(\sum_{j=1}^{d} |\log h_j| h_{k_1}, h_{k_2}\right)^{\frac{1}{2}},
\]

Then, we have to guarantee that \((\prod_{l \neq k_1, k_2} h_l)^{\frac{1}{2}} h_{k_1} h_{k_2} (\sum_{j=1}^{d} |\log h_j|)^{\frac{1}{2}} \geq \frac{c(\log T)^{2+a}}{\sqrt{T}}\). According to the choice of \( h_{k_j} \) for \( j \geq 3 \) gathered in (65), it holds true if and only if \((\log^2 T)^{\frac{2\beta_{k_1}}{2\beta_{k_1} + d - 2}} \log T)^{\frac{1}{2}} h_{k_1} h_{k_2} \geq \frac{c(\log T)^{2+a}}{\sqrt{T}}\). It leads us to the choice

\[
h_{k_1} = h_{k_2} = (\log T)^{\frac{\beta_{k_1}}{\beta_{k_1} + d - 2}} + 1 + a \left(\frac{1}{T}\right)^{\frac{\beta_{k_1}}{\beta_{k_1} + d - 2}}.
\]

(66)

Observe such choice will still imply the convergence rate to be \((\log^2 T)^{\frac{2\beta_{k_1}}{2\beta_{k_1} + d - 2}}\) as we have assumed \( \beta_{k_2} \geq \beta_{k_1} > 2 \) and so it is

\[
h_{k_1}^{2\beta_{k_1}} + h_{k_2}^{2\beta_{k_2}} = (\log T)^{\frac{\beta_{k_1}}{\beta_{k_1} + d - 2} + 1 + a} + (\log T)^{\frac{\beta_{k_2}}{\beta_{k_2} + d - 2} + 1 + a} \leq (\log^2 T)^{\frac{2\beta_{k_1}}{\beta_{k_1} + d - 2}}.
\]

It implies the rate optimal choice as in (65), (66) satisfies the constraints in (14).

Case 2.

Assume now that \( \min\left(\sum_{j=1}^{d} |\log h_j| h_{k_1}, h_{k_2}\right)^{\frac{1}{2}}, (h_{k_1})^{\frac{1}{2}} (h_{k_2} h_{k_3})^{\frac{1}{2}} = \left(\sum_{j=1}^{d} |\log h_j| h_{k_1}, h_{k_2}\right)^{\frac{1}{2}}, \) which means that \( h_{k_2} = h_{k_3} \). Acting as in Case 1 it is easy to check that the rate optimal choice for the bandwidth consists in taking

\[
h_{k_1} := (\log T)^{\frac{\beta_{k_1}}{\beta_{k_1} + d - 2}}\]

(67)

for \( j \geq 2 \), leading to the convergence rate \((\log^2 T)^{\frac{2\beta_{k_1}}{2\beta_{k_1} + d - 2}}\).

Because of (64), we have to choose \( h_{k_1} \) such that \((\prod_{l=1}^{d} h_l h_{k_1} h_{k_2})^{\frac{1}{2}} \geq \frac{c(\log T)^{2+a}}{\sqrt{T}}\). Replacing for \( j \geq 2 h_{k_j} \) as in (67), the constraint here above becomes

\[
\left(\frac{\log T}{T}\right)^{\frac{\beta_{k_1}}{\beta_{k_1} + d - 2}} \left(\frac{\log T}{T}\right)^{\frac{\beta_{k_2}}{\beta_{k_2} + d - 2}} h_{k_1} \geq \frac{c(\log T)^{2+a}}{\sqrt{T}}.
\]

It leads us to the choice

\[
h_{k_1} = (\log T)^{\frac{2+a}{\beta_{k_1} + d - 2}} \left(\frac{1}{T}\right)^{\frac{\beta_{k_2}}{\beta_{k_2} + d - 2} \left(1 - \frac{1}{\beta_{k_2}}\right)}
\]

(68)

Then \( h_{k_1}^{2\beta_{k_1}} \) is negligible compared to \((\log^2 T)^{\frac{2\beta_{k_1}}{\beta_{k_1} + d - 2}}\) for \( \beta_{k_1} (1 - \frac{1}{\beta_{k_1}}) > 1 \), which holds true as \( \beta_{k_2} \geq \beta_{k_1} > 2 \) by hypothesis. Then, the rate optimal choice for the bandwidth as in (67), (68) satisfies the constraints in (14).

The concluding argument is then the same as in the proof of Theorem 5. Indeed, to say that the bandwidths belong to \( H_T^p \), they should have the form \( h_l = \frac{1}{z_l} \) for some \( z_l \in \{1, ..., |T|\} \), that is not true in general. However, up to replace \( h_{k_1} \) with \( h_{k_1} \), where the integer part of \( T \) has come into play, we obtain asymptotically equivalent bandwidth achieving the same convergence rate and belonging to the set of candidate bandwidth \( H_T^p \). The proof is then concluded. \( \square \)
8 Proof of the lower bounds stated in Section 4

This section is devoted to the proof of the lower bounds, as stated in Section 4

8.1 Proof of Theorem 7

The proof of Theorem 7 is based on the two hypothesis method, as explained for example in Section 2.3 of Tsybakov [71] and follows the standard scheme provided in Section 6 of [3]. We start by making explicit link between the drift and the stationary measure. Then, we provide two priors depending on some calibration parameters and, to conclude, we find some conditions on the calibration such that it is possible to prove a lower bound for the minimax risk introduced in (24).

First, remark that it is possible to restrict the class of coefficients \( \Sigma(\beta, L, a_{\text{min}}, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}) \) to a class where the drift coefficient is bounded, by setting

\[
\tilde{\Sigma} = \tilde{\Sigma}(\beta, L, a_{\text{min}}, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}) := \{(a, b) \in \Sigma(\beta, L, a_{\text{min}}, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}) \mid \|b\|_{\infty} \leq b_0\},
\]

and observing that it is sufficient to lower bound the minimax risk (24) with the class \( \tilde{\Sigma} \) to get the result on the larger class \( \Sigma(\beta, L, a_{\text{min}}, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}) \). Now, it is possible to check that the class of coefficients \( \tilde{\Sigma} \) is invariant by translation in the following way. For \( h \in \mathbb{R}^d \), if \((a, b) \in \tilde{\Sigma}(\beta, L, a_{\text{min}}, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho})\), then \((a(h + \cdot), b(h + \cdot)) \in \tilde{\Sigma}(\beta, L, a_{\text{min}}, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho})\) where \( \tilde{\rho} = \max(\tilde{\rho} + |h|, 2|h|\|\tilde{C}\|_{\infty}) \). As a consequence, it is sufficient to prove the theorem in the case where \( x_0 = (0, \ldots, 0) \), and the general case can be deduced by translation. A second remark, is that one can replace any \( X \) solution of (3) with the process \( t \to a_{\text{min}}^{-1}X_t \). Then, it is possible to assume without loss of generality that \( a_{\text{min}} = 1 \) and \( a_0 \geq 1 \), in the statement of Theorem 7.

In the proof, we will lower bound the risk on the subclass of model (3) given by the following simpler stochastic differential equation, for which the diffusion coefficient \( a = \mathbb{1}_{d \times d} \) is constant:

\[
dX_t = b(X_t)dt + dW_t,
\]

and where \( b \) is any drift function such that \((I_{d \times d}, b) \in \tilde{\Sigma}(\beta, L, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}) \) for some fixed constants \( b_0 > 0, a_0 > 1, a_1 > 0, b_1 > 0 \) and with \( \tilde{C}, \tilde{\rho} > 0 \) that will be specified later.

Proof. • Explicit link between the drift and the stationary measure.

We first of all need to introduce \( A \), the generator of the diffusion \( X \) solution of (69):

\[
Af(x) := \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{i=1}^{d} b^i(x) \frac{\partial}{\partial x_i} f(x).
\]

(70)

We now introduce a class of functions that will be useful in the sequel:

\[
\mathcal{C} := \left\{ f : \mathbb{R}^d \to \mathbb{R}, \ f \in C^2(\mathbb{R}^d) \text{ such that } \forall i \in \{1, \ldots, d\} \lim_{x_i \to \pm \infty} f(x) = 0, \right. \\
\left. \lim_{x_i \to \pm \infty} \frac{\partial}{\partial x_i} f(x) = 0 \text{ and } \int_{\mathbb{R}^d} f(x)dx < \infty \right\}.
\]

We denote furthermore as \( A^* \) the adjoint operator of \( A \) on \( L^2(\mathbb{R}^d, dx) \) which is such that, for \( f, g \in \mathcal{C}, \)

\[
\int_{\mathbb{R}^d} Af(x)g(x)dx = \int_{\mathbb{R}^d} f(x)A^*g(x)dx.
\]

The form of \( A^* \) is known (see for example Lemma 2 in [3] remarking that, in our case, the discrete part of the generator \( A_d \) is zero and so we do not have its adjoint in \( A^* \):

\[
A^* g(x) = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i} g(x) - \left( \sum_{i=1}^{d} \frac{\partial b^i}{\partial x_i} g(x) + b^i \frac{\partial g}{\partial x_i}(x) \right).
\]

If \( g : \mathbb{R}^d \to \mathbb{R} \) is a probability density of class \( C^2 \), solution of \( A^* g = 0 \), then it can be checked by Ito’s formula it is an invariant density for the process we are considering. When the stationary distribution \( \pi \) is unique, therefore, it can be computed as solution of the equation \( A^* \pi = 0 \). As proposed for example in [3] and in [26], we consider \( \pi \) as fixed and \( b \) as the unknown variable.
We therefore want to compute a function $b = b_\pi$ solution to $A^* g = 0$. For $g \in C$ and $g > 0$, we introduce for all $x \in \mathbb{R}^d$ and for all $i \in \{1, \ldots, d\}$,

$$b^i_g(x) = \frac{1}{g(x)} \frac{1}{2} \frac{\partial g}{\partial x_i}(x)$$

(71)

and $b_g(x) = (b^1_g(x), \ldots, b^d_g(x))$. It is enough to remark that, for $b^i_g(x)$ defined as above it is

$$\frac{\partial b^i_g}{\partial x_i}(x) = \frac{1}{g(x)} \frac{1}{2} \frac{\partial^2 g}{\partial x_i^2}(x) - \frac{b^i_g(x)}{g(x)} \frac{\partial g}{\partial x_i}(x)$$

(72)

to see that the function $b_g$ above introduced is actually solution of $A^* g(x) = 0$, for any $x \in \mathbb{R}^d$.

We know that $\pi$ is solution to $A^* \pi(x) = 0$ for $b = b_\pi$ and so it is a stationary measure for the process $X$ whose drift is $b_\pi$. However, if $b_\pi$ satisfies A1-A2 then, from Lemma 2, we know there exists a Lyapounov function and that the stationary measure of the equation with drift coefficient $b_\pi$ is unique. It follows it is equal to $\pi$.

Hence, we need $b_\pi$ to be a function satisfying A1-A2. We introduce some assumptions on $\pi$ for which the associated drift $b_\pi$ has the wanted properties.

A3: Let $\pi : \mathbb{R}^d \to \mathbb{R}$ a probability density with regularity $C^2$ such that, for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $\pi(x) = c_0 \prod_{j=1}^d \pi_j(x_j) > 0$, where $c_0$ is a normalization constant. We suppose moreover that the following holds true for each $j \in \{1, \ldots, d\}$:

1. $\lim_{y \to \pm \infty} \pi_j(y) = 0$ and $\lim_{y \to \pm \infty} \pi_j'(y) = 0$.

2. There exists $\tilde{\epsilon} > 0$ and $R_0 > 0$ such that, for any $y : |y| > \frac{R_0}{\sqrt{d}}$,

$$\frac{\pi_j'(y)}{\pi_j(y)} \leq -\tilde{\epsilon} \text{sgn}(y).$$

3. There exists a constant $c_1$ such that, for any $y \in \mathbb{R}$,

$$\left| \frac{\pi_j'(y)}{\pi_j(y)} \right| \leq c_1.$$

4. There exists a constant $c_2$ such that, for any $x \in \mathbb{R}^d$,

$$\left| \frac{\partial^2 \pi}{\partial x_i^2}(x) \right| \leq c_2 \pi(x).$$

The properties listed here above have been introduced in order to make the associated drift function satisfying A1-A2, and $b$ bounded, so that, up to know that $\pi_{(a,b)} \in H_d(\beta, 2\mathcal{L})$, it would follow $(I_{d\times d}, b) \in \Sigma$. It is easy to see from the definition of $b_\pi$ given in (71) that, having $\pi$ in a multiplicative form, we get

$$|b^i_\pi(x)| = \left| \frac{1}{2} \frac{\pi_j'(x)}{\pi_j(x)} \right| \leq \frac{c_1}{2} =: \tilde{b}_0,$$

where we have used the third point of A3. The drift function is also clearly bounded. Moreover from (72), the third and the fourth points of A3 and the just proven boundedness of $b_\pi$ we have

$$\left| \frac{\partial b^i_\pi}{\partial x_i}(x) \right| \leq \frac{c_2}{2} + c_1 \tilde{b}_0 =: \tilde{b}_1.$$

In order to show that also A2 holds true we need to investigate the behaviour of $x_i b_\pi^i(x)$. From the second point of A3, which holds true for any $x_i$ such that $|x_i| > \frac{R_0}{\sqrt{d}}$, it is

$$x_i b_\pi^i(x) = x_i \frac{\pi_j'(x_i)}{2 \pi_j(x_i)} \leq \frac{x_i}{2} \tilde{\epsilon} \text{sgn}(x_i) = -\frac{\tilde{\epsilon}}{2} |x_i|.$$
Using also the boundedness of $b'_\pi$ showed before, it follows

$$x \cdot b_\pi(x) = \sum_{i=1}^{d} x_i b'_\pi(x) = \sum_{x_i: |x_i| > \frac{\eta}{\sqrt{d}}} x_i b'_\pi(x) + \sum_{x_i: |x_i| \leq \frac{\eta}{\sqrt{d}}} x_i b'_\pi(x)$$

$$\leq -\tilde{\varepsilon} \sum_{x_i: |x_i| > \frac{\eta}{\sqrt{d}}} |x_i| + d \times \frac{R_0}{\sqrt{d}} \leq -\tilde{\varepsilon} |x| + \frac{R_0}{\sqrt{d}}$$

where the last inequality is a consequence of the fact that, for $|x| > R_0$, there has to be at least a component $x_i$ such that $|x_i| > \frac{\eta}{\sqrt{d}}$. Hence, we can use the $\sup$ norm and compare it with the euclidean one. Now, if $|x| \geq \tilde{\rho} := 2R_0d\tilde{b}_0/\tilde{\varepsilon}$, we have

$$x \cdot b_\pi(x) \leq -\frac{\tilde{\varepsilon}}{2\sqrt{d}} |x| + \frac{R_0}{\sqrt{d}} \leq -\frac{\tilde{\varepsilon}}{2\sqrt{d}} |x|.$$

The proposed drift $b_\pi$ is therefore a bounded lipschitz function that satisfies A2, up to know that on the linked invariant density the properties gathered in A3 hold true. In the next step we propose two priors with the prescribed properties.

- **Construction of the priors.**

  We want to provide two drift functions belonging to $\tilde{\Sigma}(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho})$ where $b_0 > 0$, $a_0 > 1$, $a_1 > 0$, $b_1 > 0$, and $\tilde{C}, \tilde{\rho}$ will be specified later. To do it, we introduce two probability densities defined on the purpose to make A3 hold true. We set $\pi^{(0)}(x) := c_\eta \prod_{k=1}^{d} \pi_{k,0}(x_k)$, where $c_\eta$ is the constant that makes $\pi^{(0)}$ a probability measure. For any $y \in \mathbb{R}$ we define $\pi_{k,0}(y) := f(\eta |y|)$, where

$$f(x) := \begin{cases} e^{-|x|} & \text{if } |x| \geq 1 \\ \in [1, e^{-1}] & \text{if } \frac{1}{2} < |x| < 1 \\ 1 & \text{if } |x| \leq \frac{1}{2} \end{cases}$$

and $\eta$ is a constant in $(0, \frac{1}{2})$ which plays the same role as $\tilde{\varepsilon}$ did in A3, as it can be chosen as small as we want. In particular we choose $\eta$ small enough to get $\pi^{(0)} \in \mathcal{H}_{d}(\beta, \mathcal{L})$. Moreover, we assume $f$ to be a $C^\infty$ function that satisfies $|f^{(k)}(x)| \leq 2e^{-|x|}$ for $k = 1, 2$. It is easy to see that $\pi_0$ satisfies A3, as it is clearly positive and in a multiplicative form. The first point holds true by construction. We observe then that, for $y$ such that $|y| > \frac{1}{\eta}$, we have $\pi_{j,0}'(y) = -\eta \text{sgn}(y)\pi_{j,0}(y)$ for any $j \in \{1, \ldots, d\}$. Therefore, point 2 of A3 is satisfied for $y$ such that $|y| > \frac{R_0}{\sqrt{d}}$, up to take $R_0 := \frac{\eta}{\sqrt{d}}$ and $\tilde{\varepsilon} = \eta$.

Regarding point 3 of A3, it clearly holds true for $y$ such that $|y| > \frac{1}{\eta}$ and $|y| < \frac{1}{2\eta}$ for what said before and as the derivative is zero, respectively. When $\frac{1}{2\eta} \leq |y| \leq \frac{1}{\eta}$, point 3 of A3 is satisfied thanks to the condition $|f'(x)| \leq 2e^{-|x|}$ with the possible choice $c_1 = 2\eta$. An analogous reasoning can be applied to ensure the validity of the fourth point of A3 with $c_2 = 2\eta^2$.

Choosing $\eta > 0$ small enough, $\pi^{(0)}$ belongs to $\mathcal{H}_{d}(\beta, \mathcal{L}) \subset \mathcal{H}_{d}(\beta, 2\mathcal{L})$ and is satisfying A3 with arbitrarily small constants $\tilde{\varepsilon}$, $c_1$, $c_2$. Thus, we can choose $\eta$ such that the associated coefficients satisfy $(I_{d \times d}, b^{(0)}) \in \tilde{\Sigma}(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho})$ with $b_0 < b_0$, $b_1 < b_1$ and $\tilde{C} = \eta/(2\sqrt{d})$, $\tilde{\rho} = \frac{2R_0d\tilde{b}_0}{\tilde{\varepsilon}} = \frac{2d^{3/2}b_0}{\eta^2}$ are some constants depending on $\eta$. We deduce that $(I_{d \times d}, b^{(0)}) \in \tilde{\Sigma} = \tilde{\Sigma}(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho})$. To provide the second hypothesis, we introduce the probability measure $\pi^{(1)}$. We are given it as $\pi^{(0)}$ to which we add a bump: let $K : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function with support on $[-1, 1]$ and such that

$$K(0) = 1, \quad \int_{-1}^{1} K(z)dz = 0. \quad (73)$$

We set

$$\pi^{(1)}(x) := \pi^{(0)}(x) + \frac{1}{MT} \prod_{l=1}^{d} K\left(\frac{x_l - x_0}{b_l(T)}\right), \quad (74)$$

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where $x_0 = (x_0^1, \ldots, x_0^d) \in \mathbb{R}^d$ is the point in which we are evaluating the minimax risk, as defined in [24], $M_T$ and $h_l(T)$ will be calibrated later and satisfy $M_T \to \infty$ and, $\forall l \in \{1, \ldots, d\}$, $h_l(T) \to 0$ as $T \to \infty$. From the properties of the kernel function given in (73) we obtain

$$\int_{\mathbb{R}^d} \pi^{(1)}(x)dx = \int_{\mathbb{R}^d} \pi^{(0)}(x)dx = 1.$$ 

Moreover, as $\pi^{(0)} > 0$, $K$ has compact support and $\frac{1}{M_T} \to 0$, for $T$ big enough we can say that $\pi^{(1)} > 0$ as well. The key point consists in calibrating $M_T$ and $h_l(T)$ such that both the densities $\pi^{(0)}$ and $\pi^{(1)}$ belong to the anisotropic Holder class $\mathcal{H}_d(\beta, 2L)$ (according with Definition 2) and the laws $\mathbb{P}^{(0)}$ and $\mathbb{P}^{(1)}$ are close.

To do that, we need to evaluate the difference between the two proposed drifts. We introduce the following set of $\mathbb{R}^d$:

$$K_T := [x_0^1 - h_1(T), x_0^1 + h_1(T)] \times \cdots \times [x_0^d - h_d(T), x_0^d + h_d(T)].$$

Then, for $T$ large enough,

1. For any $x \in K_T^c$ and $\forall i \in \{1, \ldots, d\}$: $b_i^{(1)}(x) = b_i^{(0)}(x)$.

2. For any $x \in K_T$ and $\forall i \in \{1, \ldots, d\}$: $|b_i^{(1)}(x) - b_i^{(0)}(x)| \leq \frac{c}{M_T} \sum_{j=1}^d \frac{1}{h_j(T)}$, where $c$ is a constant independent of $T$.

Before proceeding with the proof of these two points we introduce some notations:

$$I'[x](x) := \frac{1}{2} \frac{\partial \pi}{\partial x_i}(x), \quad d_T(x) := \pi^{(1)}(x) - \pi^{(0)}(x) = \frac{1}{M_T} \prod_{l=1}^d K(\frac{x_l - x_0^l}{h_l(T)}). \quad (75)$$

As the support of $K$ is in $[-1, 1]$, for any $x \in K_T^c$, both $d_T(x)$ and its derivatives are 0 and, in particular, $\pi^{(0)}(x) = \pi^{(1)}(x)$. We can therefore write, using the linearity of the operator $I'$,

$$b_i^{(1)}(x) = \frac{1}{\pi^{(0)}(x)} I'[\pi^{(0)}](x) + \frac{1}{\pi^{(1)}(x)} I'[d_T](x) = b_i^{(0)}(x) + \frac{1}{\pi^{(0)}(x)} I'[d_T](x) = b_i^{(0)}(x).$$

Regarding the second point here above, we observe that on $K_T$ we have

$$b_i^{(1)} - b_i^{(0)} = \frac{1}{\pi^{(1)}(x)} I'[\pi^{(0)}] + \frac{1}{\pi^{(1)}(x)} I'[d_T] = \frac{\pi^{(0)} - \pi^{(1)}}{\pi^{(1)}} I'[\pi^{(0)}] + \frac{1}{\pi^{(1)}(x)} I'[d_T] = \frac{d_T}{\pi^{(1)}(x)} b_i^{(0)} + \frac{1}{\pi^{(1)}(x)} I'[d_T].$$

For how we have defined $\pi^{(1)} = \pi^{(0)} + d_T$, we see first of all it is lower bounded away from 0. Moreover we know that $\pi^{(0)}$ satisfies Assumption A3 and so $b_i^{(0)}$ is bounded. Furthermore, we have the following controls on $d_T$:

$$\|d_T\|_{\infty} \leq \frac{c}{M_T}, \quad \left\| \frac{\partial d_T}{\partial x_j} \right\|_{\infty} \leq \frac{c}{M_T \ h_j(T)}.$$

It follows that, for any $x \in K_T$,

$$|b_i^{(1)} - b_i^{(0)}| \leq \frac{c}{M_T} \left(1 + \sum_{j=1}^d \frac{1}{h_j(T)}\right) \leq \frac{c}{M_T} \sum_{j=1}^d \frac{1}{h_j(T)},$$

where the last inequality is a consequence of the fact that, $\forall j \in \{1, \ldots, d\}$, $h_j(T) \to 0$ for $T \to \infty$ and so, if compared with the second term in the equation here above, all the other terms are negligible.

Then, it is possible to show that also $(I_{d \times d, b^{(1)}})$ belongs to $\tilde{\Sigma}$, up to calibrate properly $M_T$ and $h_i(T)$, for $i \in \{1, \ldots, d\}$. We recall that we already know that $b^{(0)}$ satisfies A1-A2. Due to points 1 and 2 above also $b^{(1)}$ is bounded, up to ask that $\sum_{j=1}^d \frac{1}{h_j(T)} = o(M_T)$. Remark also that
For all \( x \), \( x \leq -C|x| \) for \( |x| \geq \bar{r} \) for \( T \) large enough, using that \( b^{(1)} \) and \( b^{(0)} \) coincide on \( K^c_T \). Moreover, after some computations we have

\[
\left\| \nabla b^{(1)} - \nabla b^{(0)} \right\| \leq \frac{c}{M_T} \left[ \| d_T \|_{\infty} + \sum_{y \in \{x_1, \ldots, x_d\}} \| d_T \|_{\infty} \left\| \frac{\partial d_T}{\partial y} \right\|_{\infty} \right.
\]

\[
\left. + \frac{1}{M_T} \sum_{y \neq y' \in \{x_1, \ldots, x_d\}} \left[ \left\| \frac{\partial d_T}{\partial y} \right\|_{\infty} \left\| \frac{\partial d_T}{\partial y'} \right\|_{\infty} \right] \right] + \sum_{y \neq y' \in \{x_1, \ldots, x_d\}} \left[ \left\| \frac{\partial^2 d_T}{\partial y \partial y'} \right\|_{\infty} \right]
\]

\[
\leq \frac{1}{M_T} \sum_{i,j=1}^d \frac{c}{h_i h_j} \leq \frac{1}{M_T h_1}
\]

Hence, to get that \( \| \nabla b^{(1)} \| \leq b_1 \) it is sufficient that

\[
\frac{c}{M_T h_1} \to 0 \tag{76}
\]

for \( T \) going to \( \infty \). Furthermore, requiring that

\[
\frac{1}{M_T} \leq \epsilon h_i(T)^{\beta_i} \quad \forall i \in \{1, \ldots, d\}, \tag{77}
\]

it is easy to derive the Holder regularity of \( \pi^{(1)} \) starting from the regularity of \( \pi^{(0)} \), as proved for example in Lemma 3 of [3]. It follows that, under condition \( [77] \), which implies also \( \sum_{j=1}^d \frac{1}{\pi_j(T)} = o(M_T) \), as \( \beta_j > 1 \) for any \( j \) and \( \{76\} \), both \( (I_{d \times d}, b^{(0)}) \) and \( (I_{d \times d}, b^{(1)}) \) belong to \( \Sigma \).

- **Choice of the calibration**

Before we keep proceeding, we introduce some notations. We denote as \( P_0 \) (respectively \( P_1 \)) the law of a stationary solution \( (X_t)_{t \geq 0} \) of \( (69) \) whose drift coefficient is \( b^{(0)} \) (respectively \( b^{(1)} \)). Moreover we will note \( P_{0}^{(T)} \) (respectively \( P_{1}^{(T)} \)) the law of \( (X_t)_{t \in [0, T]} \), solution of the same stochastic differential equation as here above. The corresponding expectation will be denoted as \( E^{(T)}_{(I_{d \times d}, b^{(0)})} \) (respectively \( E^{(T)}_{(I_{d \times d}, b^{(1)})} \)).

To find a lower bound for the risk we will need to use that there exist \( C \) and \( \lambda > 0 \) such that, for all \( T \) large enough,

\[
P_0^{(T)}(Z^{(T)}) \geq \frac{1}{\lambda} \geq C, \tag{78}
\]

where we have introduced the notation \( Z^{(T)} := \frac{\delta P_{0}^{(T)}}{\delta P_{T}^{(0)}} \). To ensure its validity it is enough to remark that the proof of Lemma 4 in [3] is the same even in absence of jumps. Hence, we know \( [78] \) holds true if

\[
\sup_{T \geq 0} T \int_{\mathbb{R}^d} \left| b^{(1)}(x) - b^{(0)}(x) \right|^2 \pi^{(0)}(x) \, dx < \infty.
\]

From points 1 and 2 above, it is equivalent to ask

\[
\sup_{T \geq 0} T \frac{c}{M_T^2} \left( \sum_{j=1}^d h_j(T)^2 \right) |K_T| = \sup_{T \geq 0} T \frac{c}{M_T^2} \left( \prod_{j=1}^d h_j(T) \right) \left( \sum_{j=1}^d \frac{1}{h_j(T)} \right) < \infty. \tag{79}
\]

Then, as \( (I_{d \times d}, b^{(0)}) \) and \( (I_{d \times d}, b^{(1)}) \) belong to \( \Sigma \), we have

\[
R(\bar{\pi}_T(x_0)) \geq \frac{1}{2} E^{(T)}_{(I_{d \times d}, b^{(0)})}[(\bar{\pi}_T(x_0) - \pi^{(1)}(x_0))^2] + \frac{1}{2} E^{(T)}_{(I_{d \times d}, b^{(0)})}[(\bar{\pi}_T(x_0) - \pi^{(0)}(x_0))^2]
\]

\[
\geq \frac{1}{2} E^{(T)}_{(I_{d \times d}, b^{(0)})}[(\bar{\pi}_T(x_0) - \pi^{(1)}(x_0))^2] Z^{(T)} + \frac{1}{2} E^{(T)}_{(I_{d \times d}, b^{(0)})}[(\bar{\pi}_T(x_0) - \pi^{(0)}(x_0))^2]
\]

\[
\geq \frac{1}{2\lambda} E^{(T)}_{(I_{d \times d}, b^{(0)})}[(\bar{\pi}_T(x_0) - \pi^{(1)}(x_0))^2] 1\{z^{(T)} \geq \frac{1}{2}\} + \frac{1}{2} E^{(T)}_{(I_{d \times d}, b^{(0)})}[(\bar{\pi}_T(x_0) - \pi^{(0)}(x_0))^2] 1\{z^{(T)} \geq \frac{1}{2}\}
\]

\[
= \frac{1}{2\lambda} E^{(T)}_{(I_{d \times d}, b^{(0)})}[(\bar{\pi}_T(x_0) - \pi^{(1)}(x_0))^2] 1\{z^{(T)} \geq \frac{1}{2}\} + (\bar{\pi}_T(x_0) - \pi^{(0)}(x_0))^2 1\{z^{(T)} \geq \frac{1}{2}\},
\]

for all \( \lambda > 1 \). We remark it is

\[
(\bar{\pi}_T(x_0) - \pi^{(1)}(x_0))^2 + (\bar{\pi}_T(x_0) - \pi^{(0)}(x_0))^2 \geq \left( \frac{\pi^{(1)}(x_0) - \pi^{(0)}(x_0)}{2} \right)^2
\]

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and so we obtain

\[ R(\tilde{\pi}_T(x_0)) \geq \frac{1}{8\lambda} (\pi^{(1)}(x_0) - \pi^{(0)}(x_0))^2 y_0^{(T)}(Z^{(T)} \geq \frac{1}{\lambda}) \geq \frac{c}{M_T^2}, \]  

(80)

where we have used (78) and that, by construction, \( \pi^{(1)}(x_0) - \pi^{(0)}(x_0) = \frac{1}{M_T} \prod_{l=1}^{d} K(0) = \frac{1}{M_T} \). Hence, we have to find the largest choice for \( \frac{1}{M_T} \), subject to the constraints (77) and (79). To do that, we suppose at the beginning to saturate (77) for any \( j \in \{1, \ldots, d\} \). From the order of \( \beta \) we obtain

\[ h_1(T) = h_2(T) \frac{\beta_1}{\beta_2} \leq h_2(T) \leq \cdots \leq h_d(T). \]  

(81)

We plug it in (79) and we observe that the biggest term in the sum is \( \frac{\prod_{l=1}^{d} h_1(T)}{h_2(T)} \). In order to make it as small as possible, we decide to increment \( h_1(T) \) up to get \( h_1(T) = h_2(T) \), remarking that it is not an improvement to take \( h_1(T) \) also bigger than \( h_2(T) \) because otherwise \( \frac{\prod_{l=1}^{d} h_1(T)}{h_2(T)} \) would be the biggest term, and it would be larger than \( \frac{\prod_{l=1}^{d} h_1(T)}{h_1(T)} \) for \( h_1(T) = h_2(T) \). Therefore, we take \( h_1(T) = h_2(T) \) and \( h_l(T) = (\frac{1}{M_T})^{\beta_l} \) for \( l \geq 2 \). With this choice (76) is always satisfied as \( \beta_2 > 2 \). Moreover, we have

\[ \prod_{l=3}^{d} h_l(T) = (\frac{1}{M_T})^{\sum_{l=3}^{d} \beta_l} = (\frac{1}{M_T})^{\frac{2(\beta_2 - 1)}{\beta_2}}. \]

And so condition (79) turns out being

\[ \sup_T \frac{1}{M_T^2} \prod_{l=3}^{d} h_l(T) = \sup_T \frac{1}{M_T^2} (\frac{1}{M_T})^{\frac{2(\beta_2 - 1)}{\beta_2}} \leq c. \]

It leads us to the choice \( M_T = T^{\frac{2}{2\beta_2 - d - 2}} \). It implies

\[ R(\tilde{\pi}_T(x_0)) \geq (\frac{1}{T})^{\frac{2}{2\beta_2 - d - 2}}, \]

as we wanted.

\[ \square \]

8.2 Proof of Theorem 8

Proof. Following the same ideas as in the proof of Theorem 7, we construct a prior supported by two different models corresponding to two drift functions \( b^{(0)} \) and \( b^{(1)} \), and with associated stationary probabilities \( \pi^{(0)} \) and \( \pi^{(1)} \). However, the shape of the bump used in the construction of \( \pi^{(1)} \), given below by (85), is different from the one used in the proof of Theorem 7 and defined in (75).

In the construction of the prior, the first two components are treated differently than the other ones. For this reason, we introduce cylindrical coordinates \((r, \theta, x_3, \ldots, x_d)\) that translates into the Cartesian coordinates \((r \cos(\theta), r \sin(\theta), x_3, \ldots, x_d)\). We recall the expression of the Laplacian in cylindrical coordinates \( \Delta \pi = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial \pi}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 \pi}{\partial \theta^2} + \sum_{k=3}^{d} \frac{\partial^2 \pi}{\partial x_k^2} \) and of the divergence operator \( \nabla \cdot \xi = \frac{1}{r} \frac{\partial (r \xi_r)}{\partial r} + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \sum_{k=3}^{d} \frac{\partial \xi_k}{\partial x_k} \) where \( \xi \) is the vector field \( \xi = \xi_r \vec{e}_r + \xi_\theta \vec{e}_\theta + \sum_{k=3}^{d} \xi_k \vec{e}_k \) with \( \vec{e}_r = (\cos(\theta), \sin(\theta), 0, \ldots, 0)^T, \vec{e}_\theta = (-\sin(\theta), \cos(\theta), 0, \ldots, 0)^T \), and \( \vec{e}_k \) is the canonical Cartesian basis of \( \mathbb{R}^d \). For \( \pi \) smooth stationary probability of the diffusion \( dX_t = b(X_t) dt + dW_t \) the condition \( A_b^* \pi = 0 \) can be written as

\[ \frac{1}{2} \Delta \pi - \nabla \cdot (\pi b) = 0 \]  

(82)

which yields in cylindrical coordinates to

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \pi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \pi}{\partial \theta^2} + \sum_{k=3}^{d} \frac{\partial^2 \pi}{\partial x_k^2} = \frac{1}{r} \frac{\partial (r \pi b_r)}{\partial r} + \frac{1}{r} \frac{\partial (\pi b_\theta)}{\partial \theta} + \sum_{k=3}^{d} \frac{\partial (\pi b_k)}{\partial x_k}. \]  

(83)
Given a stationary probability \( \pi = \pi(r, x_3, \ldots, x_d) > 0 \) independent of \( \theta \), we see that the drift \( b \) solution of (83) is given by \( b = b_r \epsilon_r + b_0 \epsilon_0 + \sum_{k=3}^d b_k \epsilon_k \) with

\[
\begin{align*}
    b_r &= \frac{1}{\pi} \frac{\partial \pi}{\partial r}, \\
    b_0 &= 0, \\
    b_k &= \frac{1}{\pi} \frac{\partial \pi}{\partial x_k}, \quad \forall k \geq 3.
\end{align*}
\]  

(84)

- Construction of the priors. As in the proof of Theorem \([\text{7}]\) the prior is based on two points \((\pi^{(0)}, \pi^{(T)})\). First, we define \( \pi^{(0)} \). Let \( \psi : [0, \infty) \to [0, \infty) \) be a smooth function, vanishing on \([0,1/2]\) and satisfying \( \psi(x) = x \) for \( x \geq 1 \). We let

\[
\pi^{(0)}(r, x_3, \ldots, x_d) = c_\eta e^{-\eta \psi(r)} \prod_{k=3}^d e^{-\eta \psi(|x_k|)},
\]

where \( \eta > 0 \) and \( c_\eta \) is such that \( \int_{[0,\infty)} (|x_1|, \ldots, |x_d|) \frac{\partial \pi^{(0)}(r, x_3, \ldots, x_d)}{\partial r} dr d\theta dx_3 \ldots dx_d = 1 \). Remark that for \( \eta \to 0 \), we have \( c_\eta = O(\eta^d) \). Using that \( \psi \) has bounded derivatives and vanishes near 0 one can check that \( (x_1, \ldots, x_d) \mapsto \pi^{(0)}(\sqrt{x_1^2 + x_2^2}, x_3, \ldots, x_d) \) is a smooth function and that

\[
\left\| \frac{\partial^r \pi^{(0)}(\sqrt{x_1^2 + x_2^2}, x_3, \ldots, x_d)}{\partial |x_1|^r} \right\|_{\infty} = O(\eta^4) \text{ for all } 1 \leq k \leq d \text{ and } l \geq 0.
\]

Hence, we can choose \( \eta \) small enough such that \( \pi^{(0)} \in \mathcal{H}_d(\beta, \mathcal{L}) \). The drift function \( b^{(0)} \) associated to \( \pi^{(0)} \) given by (83) is such that \( b^{(0)}(r, x_3, \ldots, x_d) = -\eta \psi'(r) = -\eta \) if \( r > 1 \) and \( b^{(0)}(r, x_3, \ldots, x_d) = -\eta \) for \( |x_k| > 1 \). By computation analogous to the ones before, there exists \( \bar{C} > 0 \) and \( \bar{p} > 0 \) such that

\[
d > b^{(0)}(r, x, \bar{C}) \leq -\bar{C}|x| \text{ for } x \geq \bar{p}.
\]

Moreover, \( b^{(0)}(r)|_{\infty} + \|\nabla b^{(0)}\|_{\infty} = O(\eta) \). Hence, if \( \eta \) is chosen small enough, \( (x_3, \ldots, x_d) \in \mathcal{E}(\beta, \mathcal{L}) \). The construction of \( \pi^{(1)} \) is more elaborate. We add a bump centered at \( 0 \) to \( \pi^{(0)} \). Let \( K : \mathbb{R} \to \mathbb{R} \) be a smooth function with support on \([-1,1]\) and satisfying (72). We set \( \pi^{(1)} = \pi^{(0)} + \frac{1}{MT} d_T \) where

\[
d_T(r, x_3, \ldots, x_d) = J_{r_{\text{min}}(T), r_{\text{max}}(T)}(r) \times \prod_{i=3}^d K\left(\frac{x_i}{h_i(T)}\right),
\]

(85)

where \( 0 < r_{\text{min}}(T) < r_{\text{max}}(T)/4 < r_{\text{max}}(T) \leq h_3(T) \leq \cdots \leq h_d(T) \) will be calibrated later and goes to zero with a rate polynomial in \( 1/T \). In the following we will suppress the dependence on \( T \) of \( r_{\text{min}}, r_{\text{max}}, (h_j)_{3 \leq j \leq d} \) in order to lighten the notations. The function \( J_{r_{\text{min}}, r_{\text{max}}} : [0, \infty) \to \mathbb{R} \) is a smooth function with support on \([0, r_{\text{max}}]\) satisfying the following properties:

\[
J_{r_{\text{min}}, r_{\text{max}}} \text{ is constant on } [0, r_{\text{min}}/4] \text{ and decreasing on } [r_{\text{min}}/4, r_{\text{max}}] \text{ with } J_{r_{\text{min}}, r_{\text{max}}}(0) \geq 1
\]

(86)

\[
0 \leq J_{r_{\text{min}}, r_{\text{max}}}(r) \leq c \left( \frac{\ln(r_{\text{max}}/r)}{\ln(r_{\text{max}}/r_{\text{min}})} \wedge 1 \right),
\]

(87)

\[
\frac{\partial J_{r_{\text{min}}, r_{\text{max}}}(r)}{\partial r} \leq \frac{c}{(\ln(r_{\text{max}}/r_{\text{min}}))(r \wedge r_{\text{min}})},
\]

(88)

\[
\frac{\partial^k J_{r_{\text{min}}, r_{\text{max}}}(r)}{\partial r^k} \leq \frac{c(k)}{(\ln(r_{\text{max}}/r_{\text{min}}))^{r_{\text{min}}}}, \quad \forall k \geq 1,
\]

(89)

where the constants \( c \) and \( c(k) \) are independent of \( r_{\text{min}}, r_{\text{max}} \). The existence of such function \( J_{r_{\text{min}}, r_{\text{max}}} \) is postponed to Lemma \([\text{3}]\) and we give in Remark \([\text{2}]\) some hints underneath its construction. Remark that \( \pi^{(1)} \) is a probability measure for \( T \) large enough, as the mean of \( d_T \) is zero and \( \pi^{(1)} \) is positive if \( T \) is large enough, as we assume \( 1/MT \to 0 \).

We denote by \( b^{(1)} \) the drift function associated to \( \pi^{(1)} \) through the relations (84). We now prove the following lemma.

Lemma 5. We have,

\[
\int_{\mathbb{R}^d} \left| b^{(1)}(\sqrt{x_1^2 + x_2^2}, x_3, \ldots, x_d) - b^{(0)}(\sqrt{x_1^2 + x_2^2}, x_3, \ldots, x_d) \right|^2 \, dx_1 \ldots dx_d \leq c \left( \frac{\prod_{i=3}^d h_i}{M_T^2 \ln(r_{\text{max}})} \right).
\]  

(90)
\[
\|b^{(1)} - b^{(0)}\|_{\infty} \leq \frac{c}{M_T} \left[ \frac{1}{r_{\min} \ln(r_{\max}/r_{\min})} + \frac{1}{h_3} \right]; \tag{91}
\]
\[
\|\nabla b^{(1)} - \nabla b^{(0)}\|_{\infty} \leq \frac{c}{M_T} \left[ \frac{1}{r_{\min} \ln(r_{\max}/r_{\min})} + \frac{1}{h_3} \right]; \tag{92}
\]

**Proof.** We start with the proof of (90). Recalling \(\pi^{(1)} = \pi^{(0)} + \frac{dT}{M_T}\) and using (84), we deduce
\[
b^{(1)} - b^{(0)} = \frac{dT}{M_T \pi^{(1)}} h^{(0)} + \frac{1}{M_T \pi^{(1)}} \sum_{k=3}^{d} \frac{\partial dr_k}{\partial x_k} e_k + \frac{1}{M_T \pi^{(1)}} \frac{\partial dt}{\partial r} e_r. \tag{93}
\]

In the evaluation of the \(L^2\) norm of \(b^{(1)} - b^{(0)}\), we start by the contribution of the radial component. From (93), we have
\[
|b^{(1)}_r - b^{(0)}_r| = \left| \frac{dT}{M_T \pi^{(1)}} h^{(0)} + \frac{1}{M_T \pi^{(1)}} \frac{\partial dr}{\partial r} \right|.
\]

Remarking that \(\pi^{(1)}\) and \(\pi^{(0)}\) coincides out of the compact set \(K_T = \{r \leq r_{\max}\} \times [0, 2\pi] \times \prod_{i=3}^{d} [-h_i, h_i]\), we deduce that \(b^{(1)}\) and \(b^{(0)}\) are equal on \(K_T\). Thus,
\[
\|b^{(1)} - b^{(0)}\|_{2}^2 = \int_{K_T} |b^{(1)}_r(r, x_3, \ldots, x_d) - b^{(0)}_r(r, x_3, \ldots, x_d)|^2 r dr d\theta dx_3 \ldots dx_d
\]
\[
\leq \frac{c}{M_T^2} \int_{K_T} \left[ \left| \frac{dT}{M_T \pi^{(1)}} \right|^2 + \left| \frac{\partial dr}{\partial r} \right|^2 \right] r dr d\theta dx_3 \ldots dx_d
\]
\[
= \frac{c}{M_T^2} (I_1(T) + I_2(T)), \tag{94}
\]
where we used that \(\pi^{(1)}\) is lower bounded independently of \(T\) on the compact set \(K_T\), for \(T\) large enough, and the boundedness of \(b^{(0)}\). Using the definition of \(d_T\) given by (85), and (87), we have
\[
I_1(T) = \int_0^{r_{\max}} |J_{r_{\min}, r_{\max}}(r)|^2 r dr \int_{\prod_{i=3}^{d} [-h_i, h_i]} \prod_{i=3}^{d} K \left( \frac{x_i}{h_i} \right)^2 dx_3 \ldots dx_d
\]
\[
\leq c \|K\|_{\infty}^{2(d-3)} \int_0^{r_{\max}} \left( \ln(r_{\max}/r) + \frac{1}{\ln(r_{\max}/r_{\min})} \right)^2 r dr \left[ \prod_{i=3}^{d} h_i \right]
\]
\[
\leq c \|K\|_{\infty}^{2(d-3)} \left[ \prod_{i=3}^{d} h_i \right] \left[ \int_0^{r_{\max}/\sqrt{\ln(r_{\max}/r_{\min})}} r dr + \int_{r_{\max}/\sqrt{\ln(r_{\max}/r_{\min})}}^{r_{\max}} \left( \frac{\ln(r_{\max}/r) + 1}{\ln(r_{\max}/r_{\min})} \right)^2 r dr \right]
\]
\[
\leq c \|K\|_{\infty}^{2(d-3)} \left[ \prod_{i=3}^{d} h_i \right] \left[ \int_0^{r_{\max}/\sqrt{\ln(r_{\max}/r_{\min})}} \frac{r^2_{\max}}{\ln(r_{\max}/r_{\min})} + \int_{r_{\max}/\sqrt{\ln(r_{\max}/r_{\min})}}^{r_{\max}} \left( \frac{\ln(r_{\max}/r_{\min}) + 1}{\ln(r_{\max}/r_{\min})} \right)^2 r dr \right]
\]
\[
\leq c \|K\|_{\infty}^{2(d-3)} \left[ \prod_{i=3}^{d} h_i \right] \left[ \int_0^{r_{\max}/\sqrt{\ln(r_{\max}/r_{\min})}} \frac{r^2_{\max}}{\ln(r_{\max}/r_{\min})} + \int_{r_{\max}/\sqrt{\ln(r_{\max}/r_{\min})}}^{r_{\max}} \frac{c}{\ln(r_{\max}/r_{\min})} r dr \right]
\]
\[
\leq c \|K\|_{\infty}^{2(d-3)} \left[ \prod_{i=3}^{d} h_i \right] \frac{r^2_{\max}}{\ln(r_{\max}/r_{\min})}. \tag{95}
\]
Using now (88),
\[
I_2(T) = \int_0^{r_{\text{max}}} |\partial J_{\text{min},r_{\text{max}}}(r)|^2 r dr \int_{\Pi_{i=3}^{1}[-h_i,h_i]} \prod_{i=3}^{d} K(\frac{x_i}{h_i(T)})^2 dx_3 \ldots dx_d \\ \leq c\|K\|_{\infty}^{2(d-3)} \int_0^{r_{\text{max}}} \left( \frac{1}{(\ln(r_{\text{max}}/r_{\text{min}}))(r \wedge r_{\text{min}})} \right)^2 r dr \left( \prod_{i=3}^{d} h_i \right) \\ \leq c\|K\|_{\infty}^{2(d-3)} \left( \prod_{i=3}^{d} h_i \right) \frac{1}{(\ln(r_{\text{max}}/r_{\text{min}}))^2} \left[ \int_0^{r_{\text{min}}} \frac{1}{r_{\text{min}}} r dr + \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{r}{r^2} dr \right] \\ \leq c\|K\|_{\infty}^{2(d-3)} \left( \prod_{i=3}^{d} h_i \right) \frac{1}{(\ln(r_{\text{max}}/r_{\text{min}}))^2} \left[ 1 + \ln\left( \frac{r_{\text{max}}}{r_{\text{min}}} \right) \right].
\]

Collecting (94), (95) and (96), we deduce
\[
\|b_{r}^{(1)} - b_{r}^{(0)}\|_2^2 \leq c \frac{\prod_{i=3}^{d} h_i}{M_T^2 \ln(r_{\text{max}}/r_{\text{min}})}.
\]

We now compute the contribution of $b_k$ for $k \geq 3$ in the $L^2$ norms of $b^{(1)} - b^{(0)}$. Using (93), we have
\[
\|b_{k}^{(1)} - b_{k}^{(0)}\|_2 \leq \frac{c}{M_T^2} \int_{K_T} \left[ |d_T(r,x_3, \ldots, x_d)|^2 + \left| \frac{\partial d_T(r,x_3, \ldots, x_d)}{\partial x_k} \right| \right] rdrd\theta dx_3 \ldots dx_d \\ =: \frac{c}{M_T^2} (I_1(T) + I_2(T)).
\]

We have to upper bound $I_3(T)$ which is the new term. From the definition of $d_T$, we have
\[
I_3 = \int_0^{r_{\text{max}}} |J_{r_{\text{min}},r_{\text{max}}}(r)|^2 r dr \int_{\Pi_{i=3}^{1}[-h_i,h_i]} \left( \prod_{i=3}^{d} K(\frac{x_i}{h_i}) \right) |K'(\frac{x_i}{h_i})|^2 \frac{1}{h_k^2} dx_3 \ldots dx_d \\ \leq \|K\|_{\infty}^{2(d-4)} \|K'\|_{\infty} \int_0^{r_{\text{max}}} \left( \ln(r_{\text{max}}/r) + \frac{1}{\ln(r_{\text{max}}/r_{\text{min}}) \wedge 1} \right)^2 r dr \left( \prod_{i=3}^{d} h_i \right) h_k^2 \\ \leq \|K\|_{\infty}^{2(d-4)} \|K'\|_{\infty} \prod_{i=3}^{d} h_i \frac{r_{\text{max}}^2}{\ln(r_{\text{max}}/r_{\text{min}})}.
\]

Using that $r_{\text{max}} \leq h_k$, we deduce
\[
I_3 \leq c\|K\|_{\infty}^{2(d-4)} \|K'\|_{\infty} \prod_{i=3}^{d} h_i \frac{r_{\text{max}}^2}{\ln(r_{\text{max}}/r_{\text{min}})}.
\]

Collecting (98), (95) and (99) we get
\[
\|b_{k}^{(1)} - b_{k}^{(0)}\|_2^2 \leq c \frac{\prod_{i=3}^{d} h_i}{M_T^2 \ln(r_{\text{max}}/r_{\text{min}})}.
\]

Eventually, the upper bound (100) is a consequence of (89), (100) and $b_{0}^{(1)} = b_{0}^{(0)} = 0$.

• We now prove (91). Using again $\pi^{(1)} = \pi^{(0)} = 0$ and (94), we have
\[
\|b^{(1)} - b^{(0)}\|_\infty \leq c \frac{\|d_T\|_\infty}{M_T} + \left\| \frac{\partial d_T}{\partial x_k} \right\|_\infty + \sum_{k=3}^{d} \left\| \frac{\partial d_T}{\partial x_k} \right\|_\infty.
\]

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By the definition (85) of $d_T$ with (86), we have $\|d_T\|_\infty \leq c$ and $\left\|\frac{\partial d_T}{\partial y}\right\|_\infty \leq \frac{c}{\kappa_k}$ for $k \geq 3$. Using (89), we have $\left\|\frac{\partial d_T}{\partial x}\right\|_\infty \leq \frac{\ln(t_{\min})}{\min(T)}$. The upper bound (91) follows.

- Finally, we prove (92). By (84) and $\pi^{(1)} = \pi^{(0)} + \frac{d_T}{M_T}$, we have after some computations

$$\left\|\nabla b^{(1)} - \nabla b^{(0)}\right\|_\infty \leq \frac{c}{M_T} \left[\left\|d_T\right\|_\infty + \sum_{y \in \{r, s, x\}} \left\|d_T\right\|_\infty \right] + \frac{1}{M_T} \sum_{y, y' \in \{r, s, x\}} \left\|\frac{\partial d_T}{\partial y}\right\|_\infty \left\|\frac{\partial d_T}{\partial y'}\right\|_\infty + \sum_{y, y' \in \{r, s, x\}} \left\|\frac{\partial^2 d_T}{\partial y \partial y'}\right\|_\infty].$$

The upperbound (92) follows from $\left\|\frac{\partial^2 d_T}{\partial x\partial y}\right\|_\infty \leq c/(h_0 h_j)$, $\left\|\frac{\partial^2 d_T}{\partial x \partial \theta}\right\|_\infty \leq c/(h_i r_{\min} \ln(t_{\min}))$ and $\left\|\frac{\partial^2 d_T}{\partial \theta^2}\right\|_\infty \leq c/(r_{\min} \ln(t_{\max}^3)).$

We can now discuss the conditions ensuring that $(I_{d \times d}, b^{(1)}) \in \Sigma(\beta, \mathcal{L}, 1, a_0, a_1, b_0, b_1, \tilde{C}, \tilde{\rho})$. Using that $J_{\min, r_{\max}}$ is constant in a neighbourhood of zero, we deduce that $(x_1, \ldots, x_d) \mapsto \pi^{(1)}(\sqrt{x_1^2 + x_2^2}, x_3, \ldots, x_d)$ is a smooth function and by (85) with (89) we have

$$\left|\frac{\partial^k}{\partial x^k} \pi^{(1)}(\sqrt{x_1^2 + x_2^2}, x_3, \ldots, x_d)\right| + \left|\frac{\partial^k}{\partial x^k} \pi^{(1)}(\sqrt{x_1^2 + x_2^2}, x_3, \ldots, x_d)\right| \leq \frac{c(k)}{M_T \ln(t_{\max}^3) r_{\min}^k},$$

Then, the following condition is sufficient to ensure that $\pi^{(1)} \in \mathcal{H}(\beta, 2\mathcal{L})$:

$$\frac{1}{M_T} \leq \varepsilon_0 r_{\min}^3 \ln(r_{\max}/r_{\min}), \text{ for } l = 1, 2 \text{ and, } \frac{1}{M_T} \leq \varepsilon_0 h_j^l, \forall l \in \{3, \ldots, d\},$$

(101)

for some constant $\varepsilon_0 > 0$ small enough. Remark also that we have $b^{(1)}(x), x \geq \tilde{C}|x|$ for $|x| \geq \tilde{\rho}$ for $T$ large enough, using that $b^{(1)}$ and $b^{(0)}$ coincide on $K_T^1$. Moreover, to get that for $T$ large enough, $\|b^{(1)}\|_\infty \leq b_0$ and $\|\nabla b^{(1)}\| \leq b_1$ it is sufficient that

$$\frac{c}{M_T} \left[\frac{1}{r_{\min}^2 \ln(r_{\max}^3)} + \frac{1}{h_j^3}\right] \rightarrow \infty.$$

(102)

Thus, we have $(I_{d \times d}, b^{(1)}) \in \Sigma(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho})$ as soon as (101) and (102) are valid.

- Choice of the calibration and proof of (25). Repeating the proof of (80) we have

$$\mathcal{R}_T(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}) \geq \frac{1}{M_T^2},$$

as soon as $\sup_{T \geq 0} T \int_{\mathbb{R}^d} |b^{(1)}(x) - b^{(0)}(x)|^2 \pi^{(0)}(x) dx < \infty$. From (90) this condition is implied by

$$\sup_{T \geq 0} T \frac{1}{M_T^2} \prod_{l=1}^d h_l \left(\frac{r_{\min}}{r_{\max}}\right) < \infty.$$

(103)

Now, we search $1/M_T \rightarrow 0$ tending to zero as slow as possible subject to the existence of $0 < r_{\min} < r_{\max}/4 < r_{\max} \leq h_3 \leq \cdots \leq h_4$ satisfying (101)-(102) and (103). For $l \geq 3$, we set $h_l = \left(\frac{1}{M_T \pi_0}\right)^{1/b_{l}}$, $r_{\max} = h_3$ and $r_{\min} = \left(\frac{1}{M_T \pi_0}\right)^{1/2}$. We have $\log(r_{\max}/r_{\min}) \sim T \rightarrow \left(\frac{1}{b_{l}} - \frac{1}{b_4}\right) \ln(M_T) \rightarrow \infty$ using $\beta_2 < \beta_3$. Since $\beta_1 \leq \beta_2$, we deduce that the conditions $\frac{1}{M_T} \leq \varepsilon_0 r_{\min}^3 \ln(r_{\max}/r_{\min}),$ for $l = 1, 2$ hold true when $T$ is large enough. Hence, (101) is satisfied, and with these choices [102] becomes a consequence of $\beta_3 > \beta_2 > 2$. Replacing $h_i, r_{\min}, r_{\max}$ by their expression in function of $M_T$, the condition (103) writes

$$\sup_{T \geq 0} T \frac{1}{M_T^{2+3(d-2)/\beta_4} \ln(M_T)} < \infty.$$
and is satisfied for the choice $M_T = (T/\ln(T))^{2\gamma_2/(d-2)}$. This yields to

$$R_T(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{p}) \gtrsim (T/\ln(T))^{2\gamma_2/(2d-2)}.$$ 

To complete the proof Theorem 8, it remains to show the existence of the function $J_{r_{\text{min}}, r_{\text{max}}}$. This is done in the next lemma.

**Lemma 6.** There exists a smooth function $J_{r_{\text{min}}, r_{\text{max}}} : \mathbb{R} \to [0, \infty)$ with compact support on $[0, r_{\text{max}}]$ such that (86)–89 hold true.

**Proof.** We let $\varphi : [0, \infty) \to [0, 1]$ be a smooth function with $\varphi|_{[0, 1/4]} = 0$ and $\varphi|_{[1/2, \infty)} = 1$. We define piecewise the function $J_{r_{\text{min}}, r_{\text{max}}}$ as follows,

\[
J_{r_{\text{min}}, r_{\text{max}}}(r) = \begin{cases} 
0, & \text{for } r \geq r_{\text{max}}, \\
\frac{1}{\ln(r_{\text{max}}/r_{\text{min}})} \int_r^{r_{\text{max}}} \varphi(1-s/r_{\text{max}}) ds, & \text{for } r_{\text{max}}/2 \leq r \leq r_{\text{max}}, \\
J_{r_{\text{min}}, r_{\text{max}}}(r_{\text{max}}/2) + \frac{1}{\ln(r_{\text{max}}/r_{\text{min}})} \int_r^{r_{\text{max}}/2} ds, & \text{for } r_{\text{min}}/2 \leq r \leq r_{\text{max}}/2, \\
J_{r_{\text{min}}, r_{\text{max}}}(r_{\text{min}}/2) + \frac{1}{\ln(r_{\text{max}}/r_{\text{min}})} \int_r^{r_{\text{min}}/2} \varphi(s/r_{\text{min}}) ds, & \text{for } 0 \leq r \leq r_{\text{min}}/2.
\end{cases}
\]

Using the definition of $\varphi$, one can check that the derivative of the function $J$ is smooth and that $J$ is decreasing with $J'(r) = 0$ for $|r| \leq r_{\text{min}}/4$. By simple computation we have,

\[
J_{r_{\text{min}}, r_{\text{max}}}(r) = \frac{\Phi_1(r/r_{\text{max}})}{\ln(r_{\text{max}}/r_{\text{min}})}, \quad \text{for } r_{\text{max}}/2 \leq r \leq r_{\text{max}}, \tag{104}
\]

\[
J_{r_{\text{min}}, r_{\text{max}}}(r) = \frac{\Phi_1(1/2) + \ln(r_{\text{max}}/2r)}{\ln(r_{\text{max}}/r_{\text{min}})}, \quad \text{for } r_{\text{min}}/2 \leq r \leq r_{\text{max}}/2, \tag{105}
\]

\[
J_{r_{\text{min}}, r_{\text{max}}}(r) = 1 + \frac{\Phi_1(1/2) + \Phi_2(r/r_{\text{min}})}{\ln(r_{\text{max}}/r_{\text{min}})}, \quad \text{for } 0 \leq r \leq r_{\text{min}}/2, \tag{106}
\]

where $\Phi_1(u) = \int_0^u \varphi(1-s) ds$ and $\Phi_2(u) = J_{u}^{1/2} \varphi(s) ds$ are smooth functions on $[0, 1]$.

Thus, using monotonicity of $J_{r_{\text{min}}, r_{\text{max}}}$ and $\varphi \geq 0$, we have $J_{r_{\text{min}}, r_{\text{max}}}(0) \geq J_{r_{\text{min}}, r_{\text{max}}}(r_{\text{max}}/2) \geq 1$. We deduce that (86) is true.

Using (104)–(106) and that $\Phi_1$ and $\Phi_2$ are bounded on $[0, 1]$, we deduce (87). The condition $J_{r_{\text{min}}, r_{\text{max}}}$ follows again from (104)–(106) and the boundedness of $\Phi_1$ and $\Phi_2$. The condition (88) is shown by differentiating $\kappa$ times the representations (104)–(106) and using $r_{\text{min}} \leq r_{\text{max}}$. \hfill \square

**Remark 2.** The idea underneath the construction of the bump $(r, \theta) \mapsto J_{r_{\text{min}}, r_{\text{max}}}(r)$ in Lemma 6 is the following. Having in mind (82), we have constructed a smooth bump, with radius $r_{\text{max}}$, height greater than 1, and solution of $J_{r_{\text{min}}, r_{\text{max}}}(0) = 0$ on the torus $r \in [r_{\text{min}}/2, r_{\text{max}}/2]$.

### 8.3 Proof of Theorem 9

**Proof.** The proof of Theorem 9 heavily relies on the proof of Theorem 8. As done above, we prove the theorem in the case where $x_0 = (0, 0)$ and $a_{\text{min}} = 1$. We will lower bound the risk on the subclass of model (3) given by

\[
dX_t = b(X_t) dt + dW_t,
\]

where $b$ is any drift function such that $(I_{d \times d}, b) \in \tilde{\Sigma}(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{p})$. Following the proof of Theorem 8, we introduce cylindrical coordinates $(r, \theta)$ that translates into the Cartesian coordinates $(r \cos(\theta), r \sin(\theta))$. In two dimensions (83) is written without the sum over $k$ and the drift $b$ solution of (83) is given by $b = b_r \tilde{e}_r + b_\theta \tilde{e}_\theta$ with $\tilde{e}_r = (\cos(\theta), \sin(\theta))^T$, $\tilde{e}_\theta = (-\sin(\theta), \cos(\theta))^T$ and

\[
b_r = \frac{1}{\pi} \frac{\partial \pi}{\partial r}, \quad b_\theta = 0.
\]
Construction of the priors. Let \( \psi : [0, \infty) \rightarrow [0, \infty) \) be a smooth function, vanishing on \([0, 1/2]\) and satisfying \( \psi(x) = x \) for \( x \geq 1 \) as in the proof of Theorem 8. We let

\[
\pi^{(0)}(r) = c_0 e^{-\eta r},
\]

where \( \eta > 0 \) and \( c_0 \) is such that \( \int_{[0, \infty) \times [0, 2\pi]} \pi^{(0)}(r) r \, dr \, d\theta = 1 \). Again, for \( \eta \) small enough we have \( \pi^{(0)} \in \mathcal{H}_2(\beta, \mathcal{L}) \). Moreover, acting as in the proof of Theorem 8, it is easy to see that \( (b^{(0)}(d)) \in \Sigma(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1 / 2, \tilde{C}, \tilde{\rho}) \subset \Sigma(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}) \).

The construction of \( \pi^{(1)} \) is more elaborate, as before we add a bump centered at 0 to \( \pi^{(0)} \). We set

\[
\pi^{(1)} = \pi^{(0)} + \frac{1}{M_T} J_{r_{\min}(T), r_{\max}(T)}(r),
\]

where \( J_{r_{\min}(T), r_{\max}(T)} \) is the function introduced in the proof of Theorem 8 which satisfies (86), (87), (88) and (89) and whose existence has been proven in Lemma 6. We recall that 0 and 1.53.

\[
\frac{1}{M_T} \leq c_0 r_{\min} \ln r_{\max} / r_{\min}, \text{for } l = 1, 2 \quad (107)
\]

for some constant \( c_0 > 0 \) small enough. As before, if (107) holds true it is sufficient that

\[
\frac{c}{M_T r_{\min}^2 \ln r_{\max}} \rightarrow 0 \quad (108)
\]

to obtain \( (b^{(1)}(d)) \in \Sigma(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}) \).

Choice of the calibration and proof of (25).

Repeating the proof of (86) we clearly have

\[
\mathcal{R}_T(\beta, \mathcal{L}, 1, b_0, a_0, a_1, b_1, \tilde{C}, \tilde{\rho}) \geq \frac{1}{M_T},
\]

as soon as \( \sup_{T \geq 0} T \int_{R^2} |b^{(1)}(x) - b^{(0)}(x)|^2 \pi^{(0)}(x) dx < \infty \), which is implied by

\[
\sup_{T \geq 0} T \frac{1}{M_T^2 \ln r_{\max}} < \infty.
\]

We look for \( 1/M_T \) tending to zero as slow as possible subject to the existence of 0 < \( r_{\min} < r_{\max}/4 < r_{\max} \) satisfying (107), (108) and (109). We set \( r_{\min} = (1 / M_T^2) \) \( 1/\beta_1 \) and \( r_{\max} = (1 / M_T^2) \) \( 1/\beta_2 \) for any arbitrary \( \gamma > 0 \). It follows log(\( r_{\max}/r_{\min} \)) \(~ T \rightarrow \infty \) \( (1/\beta_2 - (1/\beta_2 - \gamma)) \) log\( (M_T) \) \( \rightarrow \infty \) as \( \gamma > 0 \). Since \( \beta_1 < \beta_2 \), we clearly have from the definition of \( r_{\min} \) and \( r_{\max} \) that the conditions

\[
\frac{1}{M_T} \leq c_0 r_{\min} \ln r_{\max} / r_{\min}, \text{for } l = 1, 2 \text{ hold true when } T \text{ is large enough. Hence, (107) is}
\]
satisfied. The same can be said about (108), recalling we have assumed $\beta_2 > 2$. Replacing $r_{\text{min}}$ and $r_{\text{max}}$ by their expression in function of $M_T$, condition (109) writes
\[
\sup_{T \geq 0} T^2 \frac{1}{M_T^2 \ln(M_T)} < \infty,
\]
and is satisfied for the choice $M_T = (\frac{T}{\log T})^{\frac{1}{2}}$. It provides
\[
R_T(\beta, \mathcal{L}, 1, b_0, a_0, b_1, \tilde{C}, \tilde{\rho}) \geq \frac{\log T}{T},
\]
as we wanted.

\[\square\]

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A Appendix: proof of technical results

This section is devoted to the proof of the results which are more technical and for which some preliminaries are needed.

A.1 Proof of Corollary 1

*Proof.* We start proving the upper bound. Clearly we have $|x - y|^2 \leq 2|\theta_t(x) - y|^2 + 2|\theta_t(x) - x|^2$ and thus
\[
-|\theta_t(x) - y|^2 \leq -\frac{1}{2}|x - y|^2 + |\theta_t(x) - x|^2.
\]

Then, the proof consists in finding a good bound for $|\theta_t(x) - x|$, together with the straightforward application of Lemma 1. We start by noticing that, when $|\theta_t(x)| > 1$, $|\theta_t(x)|$ is a decreasing function in $t$. Indeed, because of the definition of the flow, its derivative with respect to time is
\[
2 < \dot{\theta}_t(x), \theta_t(x) > = 2 < \theta_t(x), b(\theta_t(x)) > < 0,
\]
where the last is a consequence of A2. We remark that, if $|\theta_t(x)| > 1$, it is not possible to have $|x| \leq 1$, as in this case the flow could increase only until it reaches the value 1. Hence, for $|\theta_t(x)| > 1$ we have also $|x| > 1$ and it is $|\theta_t(x)| \leq |\theta_{s,s}(x)| = |x|$ for any $t \geq 0$ since $t \mapsto |\theta_t(x)|$ is decreasing until it possibly reaches the value 1. If instead $|\theta_t(x)| \leq 1$, then we have
\[
|\theta_t(x) - x| \leq 1 + |x|.
\]

The definition of the flow together with the mean value theorem provides
\[
|\theta_t(x) - x| = |\int_s^t b(\theta_{r,s}(x)) dr| \\
\leq \int_s^t |b(0)| dr + \int_s^t \|b'\|_{\infty} |\theta_{r,s}(x)| dr \\
\leq b_0|t - s| + b_1|t - s|(|x| + 1).
\]

It concludes the proof of the upper bound on the transition density. On the other side we have
\[
-|\theta_t(x) - y|^2 \geq -2|x - y|^2 - 2|\theta_t(x) - x|^2.
\]

Hence, we need to find a lower bound for $-|\theta_t(x) - x|^2$. What we have already showed, together with the application of Lemma 4, concludes the proof. \[\square\]
A.2 Proof of Lemma 3

Recalling Lemma 2, the main novelty is to prove that the control is uniform on the class of coefficients \((a, b) \in \Sigma\). To simplify the proof, we first strengthen the hypothesis on the coefficients of the diffusion by assuming that they are infinitely differentiable with bounded derivatives or any order, together with the condition \((a, b) \in \Sigma\).

The proof of relies on the theory of Lyapunov-Poincaré inequalities introduced in [9]. We recall some definitions related to these functional inequalities. First, if \(f \in C^2(\mathbb{R}^d, \mathbb{R})\) we define \(\Gamma(f) = A(f^2) - 2fA(f)\) where \(A\) is the generator of the diffusion, and we have \(\Gamma(f) = (\nabla f)^T \tilde{a} \nabla f = |a^T \nabla f|^2\). We let \(P_{t}^{\varphi, \pi}\) be the adjoint in \(L^2(\mathbb{R}^d, \pi)\) of the operator \(P_t\), which is given by the expression

$$P_{t}^{\varphi, \pi}(f)(x) = \frac{1}{\pi(x)} \int_{\mathbb{R}^d} f(y) \pi(y) p_t(y, x) dy,$$

and \(A^{\varphi, \pi}\) the generator of the semi group \((P_{t}^{\varphi, \pi})_{t \geq 0}\) (details can be found e.g. in [5]). If \(f\) is \(C^2\) then

$$A^{\varphi, \pi} f = \frac{1}{2} \sum_{1 \leq i, j \leq d} \tilde{a}_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial f}{\partial x_i}$$

with \(b_i = -b_i + \sum_{j=1}^d \tilde{a}_{i,j}^\varphi + \frac{1}{\pi} \tilde{a}_{i,j}^\varphi \tilde{a}_{j,i}^\varphi\) as soon as the expression in the right hand side of belongs to \(L^2(\pi)\).

Definition 3. (Lyapunov-Poincaré Inequality) Let \(W\) be a \(C^2(\mathbb{R}^d, \mathbb{R})\) function such that \(W \geq 1\), \(W \in L^2(\pi)\), and \(AW \in L^2(\pi)\). The probability \(\pi\) satisfies a \(W\)-Lyapunov-Poincaré inequality if there exists \(C_{L_P} > 0\) such that for all \(f \in C^2(\mathbb{R}^d, \mathbb{R})\), bounded function in the domain of \(A^{\varphi, \pi}\), with \(\Gamma W \in L^1(\pi)\) and \(\int_{\mathbb{R}^d} f(x) \pi(x) dx = 0\) we have

$$\int_{\mathbb{R}^d} f^2(x) W(x) \pi(x) dx \leq C_{L_P} \int_{\mathbb{R}^d} |W(x)\Gamma(f)(x) - f(x)^2 AW(x)| \pi(x) dx.$$  \(113\)

We can now state a result of [9] which shows that a \(W\)-Lyapunov-Poincaré inequality entails a mixing property of the semi-group. The mixing rate depends explicitly on the constant \(C_{L_P}\) and consequently controlling \(C_{L_P}\) is the central point to get the uniform mixing in Lemma 3. For the sake of completeness, we give a detailed proof adapted to our context of this result.

Lemma 7 (Bakry et al. [9]). Assume that the \(W\)-Lyapunov-Poincaré inequality holds true with some constant \(C_{L_P} > 0\). Then, for any \(\varphi: \mathbb{R}^d \to \mathbb{R}\) bounded function with \(\int_{\mathbb{R}^d} \varphi(x) \pi(x) dx = 0\), we have

$$\int_{\mathbb{R}^d} [P_t(\varphi)(x)]^2 \pi(x) dx \leq e^{-t/C_{L_P}} \int_{\mathbb{R}^d} \varphi^2(x) W(x) \pi(x) dx,$$

where \((P_t)_t\) is the semi-group associated to the process \(X\).

Proof. \(
\bullet\) First, we establish some properties on the stationary density \(\pi\). Using Theorem 1.2 in [20], we know that

$$\left| \frac{\partial^j}{\partial y^j} \pi_t(x, y) \right| \leq \frac{c}{t^{(d+j)/2}} e^{-\frac{|y-a^\varphi(x)-u|^2}{ct}}$$

for \(j \in \{0, 1, 2\}\). From the invariance of \(\pi\), \(\pi(y) = \int_{\mathbb{R}^d} \pi(x) p_t(x, y) dx\), we deduce that \(\pi \in C^2(\mathbb{R}^d, \mathbb{R})\), and that \(\pi\) is bounded, with bounded derivatives. We also deduce from Lemma [1] that \(\pi(x) > 0\) for all \(x \in \mathbb{R}^d\).

\(
\bullet\) We now prove \(114\). Remark that by a density argument we can assume that \(\varphi\) is \(C^2\) and compactly supported. We know that if \(f\) is in the domain of \(A^{\varphi, \pi}\) then \(t \to P_{t}^{\varphi, \pi}(f)\) is differentiable in \(L^2(\pi)\) and \(\frac{d}{dt} P_{t}^{\varphi, \pi}(f) = P_{t}^{\varphi, \pi} A^{\varphi, \pi}(f) = A^{\varphi, \pi} P_{t}^{\varphi, \pi}(f)\).

As in [9], we define \(I_t = \int_{\mathbb{R}^d} (P_{t}^{\varphi, \pi}(\varphi))^2 W(x) \pi(x) dx\). By formal differentiation, we get for \(t > 0\),

$$\frac{dI_t}{dt} = 2 \int_{\mathbb{R}^d} P_{t}^{\varphi, \pi}(\varphi(x) A^{\varphi, \pi} P_{t}^{\varphi, \pi}(\varphi(x) W(x) \pi(x) dx).$$

This differentiation can be justified by proving that \(\frac{d}{dt} (P_{t}^{\varphi, \pi}(\varphi))^2 = 2P_{t}^{\varphi, \pi} \varphi \times A^{\varphi, \pi} P_{t}^{\varphi, \pi} \varphi\) is dominated by a function in \(L^2(W \pi)\). Using \(111\) and that \(\pi\) is invariant we get \(|P_{t}^{\varphi, \pi}(\varphi)| \leq \|\varphi\|_{\infty}\). Also, \(|A^{\varphi, \pi} P_{t}^{\varphi, \pi} \varphi| = |P_{t}^{\varphi, \pi} A^{\varphi, \pi} \varphi|\) is bounded by \(\|A^{\varphi, \pi}\|_{\infty} < \infty\), using \(112\) and the fact that \(\varphi\) is
compactly supported. Since \( W \in L^1(\pi) \), we deduce (116). Then, using that \( A^{*,\pi} \) is a differential operator by (112), we have \( A^{*,\pi}(P_t^{*,\pi}(\varphi))^2 = 2P_t^{*,\pi}(\varphi)A^{*,\pi}(P_t^{*,\pi}(\varphi)) + \Gamma(P_t^{*,\pi}(\varphi)) \). Hence,
\[
\frac{\partial I_t}{\partial t} = \int_{\mathbb{R}^d} [A^{*,\pi}((P_t^{*,\pi}(\varphi))^2)(x) - \Gamma(P_t^{*,\pi}(\varphi))] W(x) \pi(x) dx.
\]

We now prove that \( \Gamma(P_t^{*,\pi}(\varphi))W \in L^1(\pi) \). Let \( \chi_M \) a sequence of \( C^2(\mathbb{R}, \mathbb{R}) \) functions bounded with bounded derivative and approximating \( x \mapsto x^2 \) in the sense that \( |\chi_M(x)| \leq x^2, |\chi_M'(x)| \leq 2|x|, \chi_M''(x) \geq 0, \chi_M''(x) \xrightarrow{u \to \infty} 2 \) for all \( x \in \mathbb{R} \). We have,
\[
A^{*,\pi}(\chi_M(P_t^{*,\pi}\varphi)) = \chi_M'(P_t^{*,\pi}\varphi)A^{*,\pi}(P_t^{*,\pi}\varphi) + \frac{1}{2} \chi_M''(P_t^{*,\pi}\varphi)\Gamma(P_t^{*,\pi}\varphi)
\]

Multiplying by \( W \) and integrating with respect to \( \pi \), it yields to
\[
\int_{\mathbb{R}^d} \frac{1}{2} \chi_M''(P_t^{*,\pi}\varphi(x)) \Gamma(P_t^{*,\pi}\varphi(x)) W(x) \pi(x) dx \leq \int_{\mathbb{R}^d} A^{*,\pi}(\chi_M(P_t^{*,\pi}\varphi)) W(x) \pi(x) dx
\]
\[
+ \int_{\mathbb{R}^d} \chi_M'(P_t^{*,\pi}\varphi(x)) A^{*,\pi}(P_t^{*,\pi}\varphi) W(x) \pi(x) dx
\]
\[
= \int_{\mathbb{R}^d} \chi_M(P_t^{*,\pi}\varphi(x)) AW(x) \pi(x) dx + \int_{\mathbb{R}^d} \chi_M'(P_t^{*,\pi}\varphi(x)) A^{*,\pi}(P_t^{*,\pi}\varphi) W(x) \pi(x) dx
\]
\[
\leq \|\chi_M''\|_{L^1(\mathbb{R})} \|AW\|_{L^1(\mathbb{R})} + \|\chi_M'\|_{L^1(\mathbb{R})} \|A^{*,\pi}(P_t^{*,\pi}\varphi)\|_{L^1(\mathbb{R})} \|W\|_{L^1(\mathbb{R})}.
\]

Letting \( M \to \infty \) and using Fatou’s lemma, we deduce \( \int_{\mathbb{R}^d} \Gamma(P_t^{*,\pi}\varphi(x)) W(x) \pi(x) dx < \infty \). Hence, we can write
\[
\frac{\partial I_t}{\partial t} = \int_{\mathbb{R}^d} A^{*,\pi}(\varphi^2) W(x) \pi(x) dx - \int_{\mathbb{R}^d} \Gamma(P_t^{*,\pi}\varphi) W(x) \pi(x) dx
\]
\[
+ \int_{\mathbb{R}^d} (P_t^{*,\pi}\varphi)^2 W(x) \pi(x) dx - \int_{\mathbb{R}^d} \Gamma(P_t^{*,\pi}\varphi) W(x) \pi(x) dx
\]
\[
= \int_{\mathbb{R}^d} [(P_t^{*,\pi}\varphi)^2 W(x) \pi(x) dx - \Gamma(P_t^{*,\pi}\varphi)(W(x) \pi(x) dx]
\]

Now we use the Lyapunov-Poincaré inequality with \( f = P_t^{*,\pi}(\varphi) \) and get \( \frac{\partial I_t}{\partial t} \leq -(1/C_{LP})I_t \). From Gronwall’s lemma it yields \( I_t \leq e^{-\frac{t}{C_{LP}}} I_0 \) and since \( W \geq 1 \),
\[
\int_{\mathbb{R}^d} \varphi^2(x) W(x) \pi(x) dx \leq I_t \leq e^{-\frac{t}{C_{LP}}} I_0 = e^{-\frac{t}{C_{LP}}} \int_{\mathbb{R}^d} \varphi^2(x) W(x) \pi(x) dx.
\]

By duality between \( P_t \) and \( P_t^{*,\pi} \) in \( L^2(\pi) \), we deduce (114).

\[\Box\]

**Lemma 8.** Assume that \( a \) and \( b \) are \( \mathcal{C}^2 \) with bounded derivatives and \( (a, b) \in \Sigma \). Then, for \( \varphi \)

bounded with \( \int_{\mathbb{R}^d} \varphi(x) \pi(x) dx = 0 \), and all \( t > 0 \),

\[
\int_{\mathbb{R}^d} [P_t(\varphi(x))]^2 \pi(x) dx \leq ce^{-t/c} \|\varphi\|_\infty^2,
\]

where the constant \( c > 0 \) is uniform over the class \( \Sigma \).

**Proof.** The main idea of the proof is that it is possible to show that for all \( (a, b) \in \Sigma \), the stationary probability \( \pi \) satisfies a \( W \)-Lyapunov-Poincaré inequality with the same function \( W \) and same constant \( C_{LP} \). Following [3], the existence of \( W \)-Lyapunov-Poincaré inequality is related to the existence of classical Lyapunov functions.

- First, we construct a Lyapunov function \( V \) independent of \( (a, b) \in \Sigma \). We let \( \chi \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}) \) such that \( 0 \leq \chi(x) \leq |x| \) and \( \chi(x) = |x| \) for \( |x| \geq 1 \) and we set \( V(x) = e^{x_0\chi(x)} \) for some \( \varepsilon_0 > 0 \) which will be calibrated later. We have for \( |x| \geq 1, \nabla V(x) = \varepsilon_0 \varphi(x) V(x) \) and thus,

\[
AV(x) = \frac{1}{2} \sum_{1 \leq i, j \leq d} \tilde{a}_{i, j}(x) \frac{\partial^2 V}{\partial x_i \partial x_j}(x) + \varepsilon_0 V(x) < \frac{x}{|x|}, b(x) >.
\]
Using that for $|x| \geq 1$, $|\frac{\partial^2 V}{\partial x_i \partial x_j}| = |e^{\varepsilon_0|x|} \{\varepsilon_0^2 2x_i x_j - \varepsilon_0 x_i x_j + \frac{\varepsilon_0}{|x|^2} (i=j)\}| \leq e^{\varepsilon_0|x|} \{\varepsilon_0^2 + \frac{2\varepsilon_0}{|x|^2}\}$, and that

\[< x, b(x) > \leq -\tilde{C}|x| \text{ for } |x| \geq \tilde{\rho} \text{ by Assumption A2}, \]

we get,

\[AV(x) \leq -\tilde{C}\varepsilon_0 V(x) + \frac{\varepsilon_0}{2} V(x) \sum_{i,j} |\tilde{a}_{i,j}(x)| \{\varepsilon_0 + \frac{2}{|x|}\},\]

for all $|x| \geq \tilde{\rho} \vee 1$. Using $|\tilde{a}_{i,j}(x)| = |(aa^T)_{i,j}(x)| \leq |a(x)|^2 \leq a_0^2$ by Assumption A1, we deduce

\[AV(x) \leq -\tilde{C}\varepsilon_0 V(x) \left(1 - \frac{d^2 a_0^2}{2C} \{\varepsilon_0 + \frac{2}{|x|}\}\right).\]

Now, we set $\varepsilon_0$ as any constant with $0 < \varepsilon_0 < \frac{\tilde{C}}{2d^2 a_0^2}$, and deduce for $|x| \geq \tilde{\rho} \vee 1 \vee \frac{4d^2 a_0^2}{C}$:

\[AV(x) \leq -\tilde{C}\varepsilon_0 V(x).\]

We then define $\alpha = \tilde{C}\varepsilon_0/2$, $R = \tilde{\rho} \vee 1 \vee \frac{4d^2 a_0^2}{C}$ and $\beta = e^{\varepsilon_0 R} \left[\frac{1}{2} d^2 a_0^2 \varepsilon_0 \|\nabla \chi\|_\infty^2 + \frac{1}{2} d^2 a_0^2 \varepsilon_0 \times \sup_{1 \leq i,j \leq d} \|\partial^2_{x_i x_j} \chi\|_\infty \right] + (b_0 + b_1 R) \varepsilon_0 \|\nabla \chi\|_\infty + \alpha$ where the notations $b_0$, $b_1$ are introduced in Assumption A1. Using that for $|x| \leq R$, $|b(x)| \leq b_0 + b_1 R$, we can check that $|AV(x) + \alpha V(x)| \leq \beta$ for $|x| \leq R$, and in turn $V$ is a Lyapunov function:

\[AV(x) \leq -\alpha V(x) + \beta 1_{|x| \leq R}, \quad \forall x \in \mathbb{R}^d. \quad (118)\]

- We now prove that (113) holds true with the same function $W$ and constant $C_{L,P}$ for all coefficients $(a, b) \in \Sigma$. Using Proposition 3.6 in [9] with (118), it is sufficient to find $R' > 0$ large enough such that

\[\{V \leq 2\beta/\alpha\} \subset B(0, R')\]

\[\pi(B(0, R')) > 1/2 \quad (119)\]

and the local Poincaré inequality is valid on $B(0, R')$ with some constant $\kappa_{R'}$: for all $f \in C^1$ with bounded derivative,

\[\int_{B(0, R')} f^2(x) \pi(x)\,dx \leq \kappa_{R'} \int_{\mathbb{R}^d} \Gamma(f) \pi(x)\,dx + \frac{1}{\pi(B(0, R'))} \left(\int_{B(0, R')} f(x) \pi(x)\,dx\right)^2. \quad (121)\]

If the conditions (119)–(121) are valid, then by Proposition 3.6 in [9] we deduce the Lyapunov-Poincaré Inequality (113) with $W = V + (\beta \kappa_{R'} - 1)_+$ and $1/C_{L,P} = 2\alpha (1 - \frac{\pi(B(0, R'))}{\pi(B(0, R'))}) \times (1 + (\beta \kappa_{R'} - 1)_+)^{-1}$.

To get (119), using the expression of $V$ it is sufficient to take $R' \geq (\frac{1}{\varepsilon_0} \ln(2\beta/\alpha)) \vee 1$. Considering (120), we take the expectation with respect to $\pi$ in (118) and obtain

\[0 = \int_{\mathbb{R}^d} AV(x) \pi(x)\,dx \leq -\alpha \int_{\mathbb{R}^d} V(x) \pi(x)\,dx + \beta \int_{\mathbb{R}^d} \pi(x)\,dx,\]

and thus

\[\int_{\mathbb{R}^d} V(x) \pi(x)\,dx \leq \frac{\beta}{\alpha}. \quad (122)\]

We deduce

\[\int_{|x| \geq 1} e^{\varepsilon_0 |x|} \pi(x)\,dx \leq \int_{\mathbb{R}^d} V(x) \pi(x)\,dx \leq \frac{\beta}{\alpha}. \quad (123)\]

Using the Markov inequality, this yields, for any $R' \geq 1$, $\pi(\{x \mid |x| \geq R'\}) \leq e^{-\varepsilon_0 R'} \frac{\beta}{\alpha}$. It entails that $R' > \frac{1}{\varepsilon_0} \ln(2\beta/\alpha)$ is sufficient for the condition (120) to hold true. We set $R' = (\frac{1}{\varepsilon_0} \ln(2\beta/\alpha)) \vee 1$, and now we have to check the condition (121). From the Poincaré inequality on the ball $B(0, R')$, endowed with the Lebesgue measure (see e.g. Theorem 4.9 in [9] and the Proposition 4.2.7 in [9]), we know that if

\[1/\varepsilon_\pi \leq \pi(x) \leq c_\pi, \quad \forall x \in B(0, R'), \quad (124)\]
for some $c_\pi > 0$ then,
\[
\int_{B(0,R')} f^2(x)\pi(x)dx \leq Cc_a R' \int_{B(0,R')} |\nabla f|^2 \pi(x)dx + \frac{1}{\pi(B(0,R'))} \left( \int_{B(0,R')} f(x)\pi(x)dx \right)^2,
\]
where $C$ is some universal constant. As the matrix $\tilde{a}$ is lower bounded by Assumption A1, we have $|\nabla f|^2 \leq \frac{1}{\min \Sigma \Gamma(f)}$, and we deduce that (121) holds true with $c_R = CR c_a^2$. Consequently, by Proposition 3.6 in [2], the Lyapunov-Poincaré inequality holds true. Moreover, the constant $C_{LP}$ in (113) is independent of $(a, b) \in \Sigma$, as soon as we can find a constant $c_{\pi}$ in (124) independent of $(a,b)$. Using the invariance of $\pi$ and [1] with $t = 1$ and $s = 0$, gives for $x \in B(0,R'),$
\[
C_0^{-1} \int_{B(0,R')} \pi(y)e^{-\lambda_\pi^{-1}|\theta_\pi(x)-y|^2} dy \leq \pi(x) \leq C_0 \int_{R^d} \pi(y)dy
\]
\[
C_0^{-1} \int_{B(0,R')} \pi(y)e^{-4\lambda_\pi^{-1} R'^2} dy \leq \pi(x) \leq C_0 \int_{R^d} \pi(y)dy
\]
\[
\frac{C_0^{-1}}{2} e^{-4\lambda_\pi^{-1} R'^2} \leq \pi(x) \leq C_0,
\]
where in the second line we used $|\theta_\pi(x)| \leq |x|$ for $|x| \geq 1$, and in the last line we used (120).

Hence, we have proved the W-Lyapunov-Poincaré inequality (113) with the same function $W$ and constant $C_{LP}$ for all $(a, b) \in \Sigma$. To see that we are in the scope of the Definition 3 we need to check that $W$ and $AW$ belong to $L^2(\pi)$. From (123) we know that the stationary measure integrates the exponential function $x \mapsto e^{c_\pi|x|}$ for any constant $c_\pi \in (0, \frac{C}{\min \Sigma \Gamma(f)}).$ As the constant $c_\pi$ can be chosen arbitrarily small, it suffices to choose $c_\pi < \frac{C}{\min \Sigma \Gamma(f)}$ to deduce that $W \in L^2(\pi)$.

- Eventually, we deduce (117) by applying Lemma 7 together with the upper bound
\[
\int_{R^d} W(x)\pi(x)dx \leq \int_{R^d} V(x)\pi(x)dx + (\beta \kappa_{R'} - 1)_+ \leq \beta/\alpha + (\beta \kappa_{R'} - 1)_+
\]
where in the last inequality we used (122).

**Remark 3.** Remark that the theory of W-Lyapunov-Poincaré inequality could be applied to more general processes than continuous diffusions. Indeed, following the proof of Lemma 8 we see that the two main ingredients for getting Lyapunov-Poincaré inequality are the existence of a classical Lyapunov function, and some lower bound on compact sets for the carré du champ of some operator $\Gamma(f)$ by $c|\nabla f|^2$. Both ingredients are possible to get, for instance, in the context of jump-diffusion processes. Adaptation of Lemma 7 seems also possible to the situation of jump-diffusion processes, taking into account that, in such a context, the carré du champ of the adjoint operator $A^{*\pi}$ is not necessary equal to the carré du champ of the operator $A$.

We now prove Lemma 3 as a consequence of Lemma 8.

**Proof.** First, remark that by a density argument, we can assume in the proof that $\varphi$ is a smooth function supported on $K$. The inequality (8) is a consequence of (117) in Lemma 3 but the latter requires that the coefficients of the S.D.E. are of class $C^2$. Hence, an approximation of the initial S.D.E. by one with smoother coefficients is required. For $(a,b) \in \Sigma = \Sigma(a_0, a_1, b_0, b_1, a_{min}, \tilde{C}_b, \tilde{p}_b)$, we introduce the following smooth approximations of $a$ and $b$. Let $\eta$ be a smooth function supported on the unit ball of $R^d$ such that $\int_{R^d} \eta(x)dx = 1$, and we set $a_n = a \ast \eta_n$, $b_n = b \ast \eta_n$, where $\ast$ is the convolution operator and $\eta_n(\cdot) = n\eta(n\cdot)$. As $(a, b) \in \Sigma$, we deduce that
\[
\|a - a_n\|_\infty + \|b - b_n\|_\infty \leq \frac{C}{n}.
\]
(126)

Moreover, for all $n$ large enough, $(a_n, b_n) \in \Sigma' = \Sigma(2a_0, 2a_1, 2b_0, 2b_1, a_{min}/2, \tilde{C}_b/2, 2\tilde{p}_b)$.

We denote by $X^n$ the solution of the S.D.E. (3) with the coefficients $(a_n, b_n)$ in place of $(a, b)$, and by $\pi_n$ the unique stationary distribution of $X^n$. In the sequel of the proof, we also emphasize the dependence on the initial condition of the process (3) by denoting as $E_x$ (resp. $E_+$) the
respect to \( \varepsilon \) in the last equation, we specify \( t \) where the constant \( c = c_{\Sigma} \) is independent of \( n \) and \( (a, b) \in \Sigma \). Using classical estimates for solutions of stochastic differential equations with \((128)\), it is possible to show

\[
\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ |X^n_t - X_t| \right] \leq \frac{c' e^{c't}}{n}, \forall t > 0, \forall n \geq 1, \tag{128}
\]

and with some constant \( c' > 0 \).

Now, we write

\[
\mathbb{E}_\pi [\varphi(X_t) \varphi(X_0)] = \int_{\mathbb{R}^d} P_t(\varphi)(x) \varphi(x) \pi(x) dx
\]

\[
= \int_{\mathbb{R}^d} \mathbb{E}_x[\varphi(X^n_t)] \varphi(x) \pi(x) dx + O\left(\frac{c' e^{c't}}{n} \|\varphi\|_\infty \|\varphi'\|_\infty\right),
\]

where we used \((128)\). On the compact \( K \) we can find a constant \( c_K \) such that \( c_K^{-1} \leq \pi \leq c_K \), \( c_K^{-1} \leq \pi_n \leq c_K \) (in the same way as we obtained \((125)\)), using \((127)\), we deduce

\[
\int_K |\mathbb{E}_x[\varphi(X^n_t)] - \pi_n(\varphi)|^2 \pi(x) dx \leq c_K^2 ce^{-t/c} \|\varphi\|_\infty^2. \tag{129}
\]

It follows

\[
\mathbb{E}_\pi [\varphi(X_t) \varphi(X_0)] = \int_K \pi_n(\varphi) \varphi(x) \pi(x) dx
\]

\[
+ O\left(\int_K |\mathbb{E}_x[\varphi(X^n_t)] - \pi_n(\varphi)| \varphi(x) \pi(x) dx\right) + O\left(\frac{c' e^{c't}}{n} \|\varphi\|_\infty \|\varphi'\|_\infty\right)
\]

\[
= \pi_n(\varphi) \pi(\varphi) + O(\|c_K \|\varphi\|_\infty^2 \|\varphi'\|_\infty) + O\left(\frac{c' e^{c't}}{n} \|\varphi\|_\infty \|\varphi'\|_\infty\right), \tag{130}
\]

where in the last line we used \((129)\). It remains to find some upper bound on \( |\pi_n(\varphi) - \pi(\varphi)| \). We write for \( t > 0 \)

\[
\pi_n(\varphi) - \pi(\varphi) = \pi_n(\varphi) - \mathbb{E}_x(\varphi(X^n_t)) + \mathbb{E}_x(\varphi(X^n_t)) - P_t(\varphi)(x) + P_t(\varphi)(x) - \pi(\varphi)
\]

and integrate on the compact set \( K \) with respect to the Lebesgue measure to find

\[
\text{vol}(K)|\pi_n(\varphi) - \pi(\varphi)| \leq c_K \|\pi_n(\varphi) - \mathbb{E}_x(\varphi(X^n_t))\|_{L^1(\pi_n)} +
\]

\[
\int_K |\mathbb{E}_x(\varphi(X^n_t)) - P_t(\varphi)(x)| dx + c_K \|P_t(\varphi)(x) - \pi(\varphi)\|_{L^1(\pi)},
\]

where we used that on \( K \), \( \pi \) and \( \pi_n \) are lower bounded by \( 1/c_K \). From \((128)\), we deduce

\[
\text{vol}(K)|\pi_n(\varphi) - \pi(\varphi)| \leq c_K \|\pi_n(\varphi) - \mathbb{E}_x(\varphi(X^n_t))\|_{L^1(\pi_n)} +
\]

\[
\text{vol}(K)\frac{c' e^{c't}}{n} + c_K \|P_t(\varphi)(x) - \pi(\varphi)\|_{L^1(\pi)},
\]

In the last equation, we specify \( t = \sqrt{\log(n)} \). The first term on the right hand side goes to zero by \((127)\), the second one goes to zero immediately, while the last one goes to zero by the mixing property of the process \( X \) (see Lemma \(8\)). We deduce that

\[
|\pi_n(\varphi) - \pi(\varphi)| \leq \varepsilon_n(\varphi, K, a, b), \tag{131}
\]

for some sequence \( \varepsilon_n(\varphi, K, a, b) \xrightarrow{n \to \infty} 0 \) (let us stress that this convergence is not uniform with respect to \( (a, b) \in \Sigma, K \), or the function \( \varphi \)).

Gathering \((130)\) and \((131)\), we have

\[
|\mathbb{E}_\pi [\varphi(X_t) \varphi(X_0)] - \pi(\varphi)|^2 \leq \frac{c^2}{K} \|\varphi\|_\infty^2 \sqrt{e^{-ct/2}} + \frac{c' e^{c't}}{n} \|\varphi\|_\infty \|\varphi'\|_\infty + \|\varphi\|_\infty \varepsilon_n(\varphi, K, a, b)
\]

where the constant \( c \) does not depend on \( n \). Letting \( n \to \infty \), we deduce \(8\).
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