Sub-leading conformal dimensions at the O(4) Wilson-Fisher fixed point

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In this work we focus on computing the conformal dimensions $D(j_L, j_R)$ of local fields that transform in an irreducible representation of $SU(2) \times SU(2)$ labeled by $(j_L, j_R)$ at the O(4) Wilson-Fisher fixed point using the Monte Carlo method. In the large charge expansion, among the sectors with a fixed large value of $j = \max(j_L, j_R)$, the leading sector has $|j_L - j_R| = 0$ and the sub-leading one has $|j_L - j_R| = 1$. Since Monte Carlo calculations at large $j$ become challenging in the traditional lattice formulation of the O(4) model, a qubit regularized O(4) lattice model was used recently to compute $D(j_L, j_R)$. Here we extend those calculations to the sub-leading sector. Our Monte Carlo results up to $j = 20$ fit well to the form $D(j_L, j_R - 1) - D(j_L, j_R) \sim \lambda_{1/2}/\sqrt{j} + \lambda_1/j + \lambda_{3/2}/j^{3/2}$, consistent with recent predictions of the large charge expansion. Taking into account systematic effects in our fitting procedures we estimate the two leading coefficients to be $\lambda_{1/2} = 2.08(5)$, $\lambda_1 = 2.2(3)$.

I. INTRODUCTION

There has been a resurgence of interest in conformal field theories in recent years, especially due to the success of the bootstrap approach in certain problems \cite{1,2}. It has also become clear that conformal field theories simplify in sectors with either large spin \cite{3} or large global charge \cite{4}. Due to these developments the field has seen a renaissance with several new results over the past few years \cite{5-15}. A recent review of the above progress can be found in Ref. \cite{16,17}.

An interesting quantity in a conformal field theory is the conformal dimension of local fields that transform according to some representation of the symmetries of the theory. In this work we focus on global symmetries and study CFTs that emerge in three-dimensional $O(N)$ models at the Wilson-Fisher fixed point. Recent work \cite{4} showed that in these theories the conformal dimension $D(Q)$ of local fields that transform under a representation with charge $Q$ satisfy a large charge expansion of the form

\[ D(Q) = \sqrt{Q^3 \over 4\pi} \left( c_{3/2} + 4\pi Q c_{1/2} + \mathcal{O}(1/Q^4) \right) + c_0, \]  

(1)

where $c_{3/2}$ and $c_{1/2}$ are low-energy constants that need to be determined non-perturbatively, while $c_0 \approx -0.094$ can be determined analytically \cite{18,19}. Non-perturbative Monte Carlo calculations confirming these predictions for the case of the O(2) model, where the global charges $Q$ are represented by integers, have also been performed, and it was discovered that in this case $c_{3/2} = 1.195(10)$ and $c_{1/2} = 0.075(10)$ \cite{20}. These calculations were later extended to the O(4) model where the local fields transform in some representation of the O(4) symmetry. We can classify them according to the irreducible representations of $SU_L(2) \times SU_R(2)$ labeled with charges $(j_L, j_R)$ where $j_L, j_R = 0, 1/2, 1, 3/2, \ldots$. In this case it is natural to define the charge as $Q = 2j$ where $j = \max(j_L, j_R)$. In the large charge expansion, the leading sector is given by $|j_L - j_R| = 0$ sector, for which Monte Carlo calculations give $c_{3/2} = 1.068(4)$ and $c_{1/2} = 0.083(3)$ \cite{21}. It was shown that the large charge expansion predicts the conformal dimensions, even at $Q = 1$ within a few percent.

How well does the large charge expansion do in predicting the conformal dimensions in the sub-leading sector, where $|j_L - j_R| = 1$? This is the question that motivates our research in this work. A natural quantity to measure in this case is $\tilde{D}(j) = D(j, j - 1) - D(j, j)$, which has the expansion of the form

\[ \tilde{D}(j) = \lambda_0 + \lambda_{1/2}/j^{1/2} + \lambda_1/j + \lambda_{3/2}/j^{3/2} + \mathcal{O}(1/j^2), \]  

(2)

valid for large values of $j$. In this expansion, the coefficients of fractional powers of $j$ like $\lambda_{1/2}$ and $\lambda_{3/2}$ are Wilson coefficients that cannot in principle be determined within the large charge effective field theory \cite{22}. Their origin is essentially classical with quantum corrections. On the other hand the coefficients of integer powers of $j$ like $\lambda_0$ and $\lambda_1$ arise from purely quantum mechanical effects and can in principle be calculable analytically, similar to $c_0$ introduced earlier. It is conjectured that $\lambda_0 = 0$ since it is difficult to imagine a calculation that would distinguish between $D(j, j)$ and $D(j, j - 1)$ in the large $j$ limit \cite{23}. However, the spin of the leading conformal field in the $(j, j)$ sector is different from the $(j, j - 1)$ sector. Such differences were first observed in \cite{24} and later clarified in the context of $O(4)$ in \cite{25,26}. For this reason $\lambda_0 \neq 0$ may still be possible. On the other hand $\lambda_1$ is most likely non-zero but remains undetermined until now \cite{27}.

In this work we design a new Monte Carlo method to compute $\tilde{D}(j)$ in order to explore the validity of Eq. (2). We present results in the range $1 \leq j \leq 20$. Within this range we study if the conjecture that $\lambda_0 = 0$ is consistent with our results and estimate the other three unknown coefficients.

II. QUBIT REGULARIZED O(4) MODEL

In order to construct a Monte Carlo method to compute $\tilde{D}(j)$ it is useful to understand how the $SU_L(2) \times SU_R(2)$ symmetry is manifest in our lattice model, which is the same as the one used in Ref. \cite{21}. For this purpose it is helpful to view our model as strongly coupled lattice Quantum Electrodynamics (QED) constructed with staggered fermions \cite{28}. When gauge fields are integrated out exactly, the microscopic
degrees of freedom are made up of bosons with fermionic constituents. These bosons naturally have a built-in hardcore interaction, and hence the Hilbert space on each lattice site is finite dimensional. Such bosonic lattice field models with a finite dimensional Hilbert space, that reproduce a continuum quantum field theory, can be referred to as a qubit regularized model of the continuum quantum field theory \[29\]. Other examples of qubit regularized models for studying continuum quantum field theories with \(O(N)\) symmetries, have been constructed recently \[30, 31\].

The lattice action of our qubit regularized \(O(4)\) model can be written using four Grassmann valued lattice fields \(\psi_{1,k}, \psi_{2,k}, \psi_{1,k}', \psi_{2,k}'\) at each lattice site \(k \equiv (r, \tau)\) on a cubic lattice, where we distinguish between the two-dimensional spatial coordinate \(r\) and the Euclidean temporal coordinate \(\tau\). The Euclidean action of our model is given by \[21\],

\[
S(\psi, \overline{\psi}) = -\sum_{(k,k')} \text{Tr}(M_k M_{k'}) - \frac{U}{2} \sum_k \text{Det}(M_k),
\]

where \((M_k)_{a,b} = \psi_{a,k} \overline{\psi}_{b,k}\) is a \(2 \times 2\) matrix defined at each lattice site \(k\). The symbol \((k, k')\) refers to neighboring sites \(k\) and \(k'\). The partition function is defined as usual through the Grassmann integral

\[
Z = \int \prod \text{Tr}(M_k) \ e^{-S(\psi, \overline{\psi})}.
\]

It is easy to verify that the action is invariant under the \(SU_L(2) \times SU_R(2)\) transformations given by \(M_k \rightarrow L M_k R_k\) when \(k \in \text{even sites}\), and \(M_k \rightarrow R_k M_k L_k\) when \(k \in \text{odd sites}\). Here we assume \(L\) and \(R\) are \(2 \times 2\) matrices, each of which are elements of the \(SU(2)\) group. This means \((\psi_{1,k}, \psi_{2,k})\) on even sites and \((-i\psi_{2,k}', i\psi_{1,k}')\) on odd sites transform as \(SU_L(2)\) doublets, while they are singlets of \(SU_R(2)\). The same fields on the opposite parity sites transform as \(SU_R(2)\) doublets and \(SU_L(2)\) singlets. When \(U = 0\), the theory has an additional \(U(1)\) symmetry: \(\psi_{a,x} \rightarrow e^{i\theta} \psi_{a,x}\) and \(\overline{\psi}_{a,x} \rightarrow e^{i\theta} \overline{\psi}_{a,x}\) for the odd sites, and \(\psi_{a,x} \rightarrow e^{-i\theta} \psi_{a,x}\) and \(\overline{\psi}_{a,x} \rightarrow e^{-i\theta} \overline{\psi}_{a,x}\) on the even sites. The \(U\)-term, therefore, mimics the anomalous axial symmetry of the action in Quantum Chromodynamics (QCD) \[28\]. For this reason the terms in the partition function that arise due to a non-zero value of \(U\) were referred to as instantons in the earlier work. It is known that instantons are known to break the anomalous axial symmetry in QCD. In the more recent viewpoint of qubit regularization, the instantons can be viewed as simply the local Fock vacuum states \[30\].

It is possible to perform the Grassmann integrations in Eq. \(4\) exactly and rewrite the partition as a sum over worldline configurations of pions and instantons \[28\]. One then obtains \(Z = \sum_{[\ell]} U_{\ell N_I}\), where \([\ell]\) is a configuration of worldlines, which is a collection of closed oriented loops, each of which can be in one of two colors, red or green. In addition, there are isolated sites which do not belong to the worldlines and are referred to as instantons. \(N_I\) is the total number of instantons in the configuration. An illustration of a worldline configuration on a two-dimensional lattice is shown in Fig. 1. In an earlier work, we solved our lattice model using this worldline approach and showed that at the critical coupling of \(U_c \approx 1.655394\) we can reproduce the critical scaling of the Wilson-Fisher fixed point at long distances \[21\].

**III. LEADING CHARGE SECTORS**

Our goal is to compute the conformal dimensions \(D(j_L, j_R)\) of field operators that transform in some irreducible representation \((j_L, j_R)\) of the \(SU_L(2) \times SU_R(2)\) group at the critical point. We can accomplish this by computing the correlation function

\[
C_{j_L, j_R} = \langle \widehat{O}_{j_L, j_R} \rangle = \frac{1}{Z} \int \prod \text{Tr}(M_k) \ e^{-S(\psi, \overline{\psi})} \overline{\widehat{O}}_{j_L, j_R} \widehat{O}_{j_L, j_R},
\]

where \(\widehat{O}_{j_L, j_R}\) and \(\overline{\widehat{O}}_{j_L, j_R}\) are the source and sink terms constructed with Grassmann valued fields that transform in the irreducible representation \((j_L, j_R)\). In our work, the source will be located at the temporal slice \(\tau = 0\), while the sink will be at the temporal slice \(\tau = L/2\). Thus, at the critical point, we expect

\[
C_{j_L, j_R} = A_{j_L, j_R} L^{-2D(j_L, j_R)}
\]

for sufficiently large values of \(L\).

In order to construct \(\widehat{O}_{j_L, j_R}\) and \(\overline{\widehat{O}}_{j_L, j_R}\) that transform under the irreducible representation \((j_L, j_R)\), let us denote \([j_L, m_L]; (j_R, m_R)\) as the \((2j_L + 1)(2j_R + 1)\) dimensional orthonormal basis that spans the irreducible representation space of \((j_L, j_R)\). Here \(-j_L \leq m_L \leq j_L \) and \(-j_R \leq m_R \leq j_R\). We can label the fields that transform according to this
irreducible representation as $O_{(j_L,m_L);(j_R,m_R)}$. For the sink terms, it is natural to choose fields that transform in the conjugate representation $\langle (j_L,m_L);(j_R,m_R) \rangle$. We can label these sink fields as $\overline{O}_{(j_L,m_L);(j_R,m_R)}$. While we can choose any of these fields as source and sink terms in the $(j_L,j_R)$ representation, we find that choosing $m_L = j_L$ and $m_R = j_R$ will be the most convenient choice for numerical work.

Based on the transformation property of $M_k$ it is easy to see that the four local fermion bilinear lattice fields $i\psi_1,k\overline{\psi}_1,k$, $-i\overline{\psi}_2,k\psi_2,k$, $-i\psi_1,k\overline{\psi}_2,k$, $-i\overline{\psi}_1,k\psi_1,k$ transform under the $(1/2, 1/2)$ (vector) representation of $O(4)$. However, the fields transform differently on even and odd sites. The exact mapping is given in the table in Table I. Given the source fields transform differently on even and odd sites. The green source is given on the other hand the sources and sinks of green worldlines depend on the parity of the sites. The green source is given by $G_k = i\psi_1,k\overline{\psi}_1,k$ on even sites, and $\overline{G}_k = -i\overline{\psi}_2,k\psi_2,k$ on odd sites. This is reversed for the sink of green lines. We have $\overline{G}_k = -i\overline{\psi}_2,k\psi_2,k$ on even sites, and $G_k = i\psi_1,k\overline{\psi}_1,k$ on odd sites.

In order to construct sources of more general irreducible representations, we use tensor product of the vector representation of local fields distributed over several spatial lattice sites on the time slice $\tau = 0$. The same sites on the time slice $\tau = L/2$ are used to construct the sinks. Let us first construct sources and sinks that transform in the representation $(j,j)$ which we refer to as the leading sector. We know we can construct the basis state $\langle (j,j);(j,j) \rangle$ as a tensor product of $2j$ states of the form $\langle (1/2,1/2);(1/2,1/2) \rangle$. The source fields we can construct the sink fields through the conjugate representation. This is given in Table III. Using the information in Table I and Table II we can identify each worldline in Fig. 1 as a vector particle carrying an appropriate charge. For example, we can identify the sources and sinks of red worldlines as $R_k = -i\overline{\psi}_1,k\psi_2,k$ and $\overline{R}_k = -i\overline{\psi}_2,k\psi_1,k$ on all sites. On the other hand the sources and sinks of green worldlines depend on the parity of the sites. The green source is given by $G_k = i\psi_1,k\overline{\psi}_1,k$ on even sites, and $\overline{G}_k = -i\overline{\psi}_2,k\psi_2,k$ on odd sites. This is reversed for the sink of green lines. We have $\overline{G}_k = -i\overline{\psi}_2,k\psi_2,k$ on even sites, and $G_k = i\psi_1,k\overline{\psi}_1,k$ on odd sites.

In Table II. Relationship between source and sink fields on a local site.

### Table II. Relationship between source and sink fields on a local site.

| local field | even site | odd site |
|-------------|-----------|----------|
| $O_{(1/2,1/2);(1/2,1/2)}$ | $-i\psi_1,k\overline{\psi}_1,k$ | $-i\overline{\psi}_1,k\psi_2,k$ |
| $O_{(1/2,-1/2);(1/2,-1/2)}$ | $i\overline{\psi}_2,k\psi_1,k$ | $i\psi_2,k\overline{\psi}_1,k$ |
| $O_{(1/2,1/2);(1/2,-1/2)}$ | $i\psi_1,k\overline{\psi}_1,k$ | $-i\overline{\psi}_1,k\psi_2,k$ |
| $O_{(1/2,-1/2);(1/2,1/2)}$ | $-i\psi_1,k\overline{\psi}_1,k$ | $i\overline{\psi}_1,k\psi_2,k$ |

These can be used to compute the correlation function

$C_{j,j} = \langle \overline{O}_{(j,j);(j,j)}O_{(j,j);(j,j)} \rangle$ (7)

and to obtain $D(j,j)$ one can use the relation $C_{j,j} \sim A_{j,j}L^{-2D(j,j)}$ for large values of $L$. In the actual worm algorithm, one in fact computes the ratio $R_j = C_{j,j}/C_{j-1,j-1}$ and fits to the form $(A_{j,j}/A_{j-1,j-1})L^{-2\Delta_j}$ to compute $\Delta_j = D(j,j) - D(j-1,j-1)$ for each value of $j$. From these differences and setting $D(0,0) = 1$ one can compute $D(j,j)$. Our results from [21] are tabulated in Table III for reference.

### Table III. Results for the conformal dimensions $D(j,j)$ up to $j = 5$ computed using worldline Monte-Carlo methods in [21].

| $j$ | $D(j,j)$ | $j$ | $D(j,j)$ |
|-----|----------|-----|----------|
| 1/2 | 0.515(3) | 1/2 | 1.185(4) |
| 3/2 | 1.989(5) | 3/2 | 2.915(6) |
| 5/2 | 3.945(6) | 5/2 | 5.069(7) |
| 7/2 | 6.284(8) | 7/2 | 7.575(9) |
| 9/2 | 8.949(10)| 9/2 | 10.386(11)|

FIG. 2. The figure shows the arrangement of local sources of red worldlines we used to create a source in the $(j,j)$ representation. The site labeled 0 is the origin is left empty. Depending on $j$ the sources are placed on the sites marked $k = 1, 2, \ldots, 2j$. The first 20 sites used in our calculations up to $j = 10$ are shown.
IV. SUBLEADING SECTOR

In this work we extend our earlier results in the leading sector and compute the conformal dimensions of the subleading sector, $D(j,j−1)$ for $j \geq 1$. For this we need to construct source and sink operators that transform in the representation $(j,j−1)$. We know we can construct the states $|j,j;j−1,1\rangle$ by applying the lowering operator $J^−_R$ to the $|j,j;j,j\rangle$ and then constructing orthogonal states in the tensor product space. This procedure naturally leads to $2j−1$ orthonormal states, which we can label with an additional index $M = 1, 2, \ldots, (2j−1)$. Translating this to the construction of sources, we now introduce the source $O_{\ell} = R_1 R_2 \ldots G_{\ell} \ldots R_{2j}$, where the red source on one lattice site $\ell$ is replaced by a green source where $\ell = 1, 2, \ldots, 2j$. Similarly, we introduce the corresponding sinks as $\overline{O}_{\ell} = R_1 R_2 \ldots G_{\ell} \ldots R_{2j}$. We can then argue that

$$O_{(j,j);(j,j−1)} = \frac{1}{\sqrt{2j}} \sum_{\ell=1}^{2j} O_{\ell},$$

(8)

where the right-hand side is a sum over the $2j$ source terms we introduced above. Note that there are $2j−1$ sources orthogonal to Eq. (8), which can label with $M = 1, 2, \ldots, 2j−1$ as before. These will naturally transform in the $(j,j−1)$ representation. Explicitly these sources are given by

$$O^M_{(j,j);(j,j−1)} = \frac{1}{\sqrt{2j}} \sum_{\ell=1}^{2j} e^{i2\pi(\ell−1)M/(2j)} O_{\ell}.$$  

(9)

We can similarly define the corresponding $2j−1$ sinks as

$$\overline{O}^M_{(j,j);(j,j−1)} = \frac{1}{\sqrt{2j}} \sum_{\ell=1}^{2j} e^{-i2\pi(\ell−1)M/(2j)} \overline{O}_{\ell}.$$  

(10)

Since all $2j−1$ sources and sinks transform under the same irreducible representation $(j,j−1)$ any combination of them can be used in Eq. (5) to extract $D(j,j−1)$. Let us define the correlation matrix

$$C^M_{j,j−1} = \langle \overline{O}^M_{(j,j);(j,j−1)} | O^M_{(j,j);(j,j−1)} \rangle$$

(11)

where we expect $C^M_{j,j−1} \sim A^M_{j,j−1} L^{-2D(j,j−1)}$ for large values of $L$. Practically it is more convenient to compute the average of the trace of this correlation matrix which can be simplified to the form

$$C_{j,j−1} = \sum_{\ell} \left\{ \frac{1}{2j} \langle \overline{O}_{\ell} O_{\ell} \rangle - \frac{1}{2j(2j−1)} \sum_{\ell′ \neq \ell} \langle \overline{O}_{\ell} O_{\ell′} \rangle \right\}.$$  

(12)

Note that we also expect $C_{j,j−1} \sim A_{j,j−1} L^{-2\Delta(j,j−1)}$.

As in the leading sector, in the worldline algorithm it much easier to compute the ratio $\tilde{R}_j = C_{j,j−1}/C_{j,j}$. For this one constructs a worldline Monte Carlo method to generate configurations with $2j$ red sources and sinks that contribute to $C_{j,j}$. In every configuration of this ensemble, we then imagine flipping each of the $2j$ sources located at the sites $\ell = 1, 2, \ldots, 2j$ to a green source. The worldline of the green source then naturally travels through the lattice to a sink at some location $\ell′$. Then we compute the contribution to $\tilde{R}_j$ from that configuration using Eq. (12), which means we add $1/2j$ if $\ell = \ell′$, or subtract the value $1/(2j(2j−1))$ if $\ell \neq \ell′$ for every value of $\ell$. Averaging this contribution over the

| $j$ | $L$-range | $A_{j,j−1}/A_{j,j}$ | $\Delta(j)$ | $\chi^2$ | DOF |
|-----|-----------|-----------------|-------------|---------|-----|
| 1   | 24 − 128  | 5.87(25)        | 0.813(6)    | 1.11    |     |
| 3/2 | 24 − 128  | 2.50(11)        | 0.750(6)    | 0.62    |     |
| 2   | 24 − 96   | 2.13(06)        | 0.722(4)    | 0.26    |     |
| 5/2 | 32 − 96   | 1.75(08)        | 0.685(6)    | 1.28    |     |
| 3   | 32 − 96   | 1.54(08)        | 0.659(7)    | 0.93    |     |
| 7/2 | 32 − 96   | 1.35(05)        | 0.633(5)    | 0.38    |     |
| 4   | 32 − 96   | 1.18(04)        | 0.607(4)    | 0.40    |     |
| 9/2 | 40 − 160  | 1.05(04)        | 0.586(5)    | 0.94    |     |
| 5   | 40 − 160  | 0.94(04)        | 0.566(5)    | 0.89    |     |
| 11/2| 48 − 160  | 0.90(03)        | 0.555(4)    | 0.66    |     |
| 6   | 48 − 160  | 0.83(03)        | 0.541(5)    | 1.40    |     |
| 13/2| 64 − 160  | 0.75(04)        | 0.525(7)    | 1.11    |     |
| 7   | 64 − 160  | 0.71(03)        | 0.513(5)    | 1.18    |     |
| 15/2| 64 − 160  | 0.69(04)        | 0.506(6)    | 1.45    |     |
| 8   | 64 − 160  | 0.60(03)        | 0.486(5)    | 0.77    |     |
| 17/2| 64 − 160  | 0.61(03)        | 0.484(5)    | 0.83    |     |
| 9   | 80 − 160  | 0.53(04)        | 0.467(8)    | 0.98    |     |
| 19/2| 80 − 160  | 0.53(03)        | 0.463(7)    | 0.47    |     |
| 10  | 80 − 160  | 0.50(02)        | 0.454(5)    | 0.63    |     |
| 20  | 96 − 256  | 0.28(01)        | 0.367(3)    | 0.65    |     |

TABLE IV. Results of the fit of $\tilde{R}_j$ shown in Fig. 3 to the form $(A_j/A_{j−1}) L^{-2\Delta(j)}$. The range of $L$ values used in the fit are given in the second column. We observe that as $j$ increases this range needs to involve larger lattice sizes for a good fit.
ensemble of configurations generated by the worldline algorithm gives us the ratio, $R_j$ which is expected to scale as $(A_j/A_{j-1})L^{-2\Delta_j}$ where $\Delta_j = D(j,j-1) - D(j,j)$. Using the values of $D(j,j)$ we compute $D(j,j-1)$.

| $j$ range | $\lambda_0$ | $\lambda_{1/2}$ | $\lambda_1$ | $\lambda_{3/2}$ | $\chi^2$ /DOF |
|------------|-------------|-----------------|-------------|-----------------|--------------|
| 1-20       | 0.07(2)     | 1.5(1)          | -1.0(2)     | 0.3(1)          | 0.6          |
| 1-20       | 0           | 1.42(1)         | 0           | -0.69(1)        | 48           |
| 1-20       | 0           | 1.96(2)         | -1.83(6)    | 0.69(5)         | 1.8          |
| 1-20       | 0.16(1)     | 0.98(2)         | 0           | -0.34(2)        | 2.3          |
| 1.5-20     | 0.13(1)     | 1.07(2)         | 0           | -0.47(3)        | 0.3          |
| 2-20       | 0           | 2.06(3)         | -2.27(14)   | 1.14(13)        | 1.1          |
| 8-20       | 0           | 1.81(2)         | 0           | -3.5(2)         | 1.1          |
| 8-20       | 0           | 2.09(4)         | -2.02(13)   | 0               | 0.6          |

TABLE V. Fits of $\tilde{\Delta}_j$ to the functional form given in Eq. (2). We first consider the whole range of $j$ in the first four rows. While including all four coefficients as fitting parameters gives an excellent fit, setting the two purely quantum mechanical terms $\lambda_0 = \lambda_1 = 0$ makes the fit quite bad. Including even one of them is sufficient to improve the fit considerably. Setting two of the fitting parameters to zero is only possible by shrinking the allowed region of $j$ considerably.

V. RESULTS

We have performed a series of Monte Carlo calculations for $1 \leq j \leq 10$ in increments of $1/2$, and for $j = 20$. For runs till $j = 10$, the lattice sizes ranged from $L = 32$ up to $L = 160$. For each value of $j$ we have always found that our data fits well to the expected form $C_{j,j-1}/C_{j,j} \sim L^{-2\tilde{\Delta}_j}$ for sufficiently large values of $L$. As examples, fits for $j = 1, 5, 10$, and 20 are illustrated in Fig. 3. From the figure, we observe that as $j$ increases the range of lattice sizes where a simple power law emerges changes to larger $L$ values. The range of $L$ values where we perform the fits and the obtained fit parameters for various values of $j$ are tabulated in Table V. At $j = 20$ we have extended our calculations up to lattice sizes of $L = 256$.

Having obtained the values of $\tilde{\Delta}_j$ for various values of $j$ we try to extract the constants $\lambda_i$’s in the large charge expansion based on Eq. (2), assuming we can neglect $O(1/j^2)$ terms. We try several fits to understand the role of the purely quantum terms $\lambda_0$ and $\lambda_1$. These fits are shown in Table VI. First, we note that our data for the entire range of $j$ values fits well to the form Eq. (2), if we assume all four coefficients are non-zero (first row in Table VI). On the other hand if we drop both quantum terms the fit becomes quite bad (second row in Table VI). The presence of either of the two quantum terms is sufficient to bring down the $\chi^2$/DOF considerably. For example setting $\lambda_1 = 0$, we can get an excellent fit if we just drop the $j = 1$ data (fifth row in Table VI). On the other hand setting $\lambda = 0$ and dropping both $j = 1, 1.5$ makes the fit acceptable (sixth row in Table VI). If we drop both quantum terms (i.e., set $\lambda_0 = \lambda_1 = 0$) we can only get a good fit in the range $j = 8 - 20$. We believe this is just an artifact of a small range of $j$ as can be seen in Fig. 4.

FIG. 4. The plot of $\tilde{\Delta}_j$ as a function of $j$ from Table VI. The lines shown are fits to the form given in Eq. (2) with parameter values given in the first (solid), fifth (dashes), sixth (dashes+dots) rows and seventh (dots) rows of Table VI.

Clearly, our data is consistent with the conjecture that $\lambda_0 = 0$ (see sixth row in Table VI). Assuming this, we can try to determine the leading two terms more reliably by fitting our data under the constraint $\lambda_{3/2} = 0$. In this case we can get a good fit only in the smaller range $j = 8 - 20$ (eighth row in Table VI). As expected, this changes $\lambda_{1/2}$ and $\lambda_1$ slightly. Taking such systematic fitting effects into account, we estimate that $\lambda_{1/2} = 2.08(5)$, $\lambda_1 = 2.2(3)$. Unfortunately, calculations at higher values of $j$ are difficult since we need larger lattice sizes, but they can be obtained with more computing resources and can help confirm the conjecture that $\lambda_0 = 0$.

VI. CONCLUSIONS

In this work we have constructed a Monte Carlo method to compute the sub-leading conformal dimensions in the large charge expansion at the $O(4)$ Wilson-Fisher fixed point. We used this method to compute $\tilde{\Delta}_j$ for several values of $j$ in the range $1 \leq j \leq 20$. While our results are consistent with the general predictions of the expansion given in Eq. (2), we cannot rule out the possibility that $\Delta_j$ approaches a non-zero constant $\lambda_0$ in the large $j$ limit. However, our data is consistent with the conjecture that $\lambda_0 = 0$. Assuming this to be true, we can estimate the leading two terms in the expansion to be $\lambda_{1/2} = 2.08(5)$, $\lambda_1 = 2.2(3)$. Since $\lambda_1$ must be a calculable number within the large charge effective field theory, we hope our work will motivate someone to calculate it in the future. Calculations at large $j$ may also help determine $\lambda_0$ reliably.
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