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**Stability of a parametric harmonic oscillator with dichotomic noise**

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(Dated: 22 June 2020)

The harmonic oscillator is a powerful model that can appear as a limit case when examining a nonlinear system. A well known fact is, that without driving, the inclusion of a friction term makes the origin of the phase space – which is a fixpoint of the system – linearly stable. In this work we include a telegraph process as perturbation of the oscillator’s frequency, for example to describe the motion of a particle with fluctuating charge gyrating in an external magnetic field. Increasing intensity of this coloured noise is capable of changing the quality of the fixed point. To determine the stability of the system, we use a common measure, the Lyapunov exponent and express it in a closed form. We expand the Lyapunov exponent for light friction and low noise intensity and compare both, the exact analytic solution and the expansion to numerical values. Our findings allow stability predictions for several suitable physical systems.

The harmonic oscillator with stochastic influences is a common “toy” model for the description of complex physical systems, e.g. particles with varying charge gyrating in a external magnetic field. It has been shown, that a white noise perturbation of the frequency of a harmonic oscillator induces a Hopf bifurcation, changing the stability of the system’s fixed point[1], while a coloured noise term generally defies an exact analytic treatment[2]. We focus on the case of a dichotomic parametric noise, i.e. a frequency term switching between two states, and show, that it leads to the same bifurcation as the aforementioned parametric white noise. The stability of the system can be described by the growth rate of the distance of the system’s current state from the phase space origin. We express this quantity in a closed analytic form. Our results can be used to predict the stability of several suitable physical systems.

I. INTRODUCTION

Studies of non-linear systems constitute a key-element of chaotic dynamics as an attempt to describe complex processes in nature and also society. A usual approach to serve this purpose is to investigate “toy” models which comprise and manifest typical properties of the underlying, more extensive systems. Despite their simplicity, these models often elude their analytical solution, as shown by the three body problem, for instance. A further simplification is delivered by the linearisation of the equations of motion in the vicinity of initial conditions, fixed points, periodic orbits or the like. The solutions now yield trends of the trajectory and allow to draw conclusions about the stability of the system. In this sense, the harmonic oscillator, naturally occurring by linearisation, is one of the most powerful models in physics.

Another difficulty for the analysis of physical models are stochastic terms in the equations of motion. These actions are often addressed by an additional stochastic force-like term, representing for example collisions with particles of the surrounding medium, that act as a heat bath. Still, there are cases were the random influences alter the system’s potential, e.g. via changes in mass or charge, and lastly modify the frequency of the linearised harmonic oscillator. These influences appear as a multiplicative noise in the equation of motion. A possible example is the gyration motion of charged particles immersed in a magnetic field, as they occur as erosion-remains of Saturn’s dense rings. These nanoparticles, freed by micro-meteoroid impacts, guided along the magnetic field lines into the planet’s atmosphere are subject to stochastic charging processes due to the planetary plasma environment[3]. In such systems, questions arise regarding the time evolution of the particle trajectory and the stability of the system.

To date, studies, that examined multiplicative noise for the harmonic oscillator used white and continuous coloured noise[4], while we aim to study noise terms that take discrete states, just as the charging of the nano-particles can no longer be approximated as continuous changes and has to be described as quantised steps. In this publication, we focus on the limit case of noisy switching between two discrete states and find an analytic description for the stability of the system. A similar system has already been focused on in a steady state setting, ensured by additional white noise forcing, that acted as a heat bath[5]. We omit this stabilisation by additional heating and use a common stability measure – the Lyapunov exponent – to calculate the average growth rate of an energy-like quantity to characterise the long-term behaviour of the parametric harmonic oscillator with a dichotomic noise.

II. CHARACTERISATION OF THE SYSTEM

We examine a harmonic oscillator with a multiplicative noise term that is described by the equation of motion

\[
\ddot{x} + \alpha \dot{x} + (1 + \xi(t))x = 0,
\]

(1)
given in a dimensionless form, where $x(t)$ is the displacement of the oscillator, $\alpha > 0$ is a friction parameter, and $\xi(t)$ is a dichotomous stochastic process that switches between the states $\xi_{1,2}$ with an average rate $\lambda_1$ from $\xi_1$ to $\xi_2$ and vice versa.

We make the simple assumption, that the transitions between states are independent events, which can be justified by turbulent environment and the molecular chaos of physical systems. To take account of this 'forgetfulness' of the system, the probability of an event in a small time interval $\Delta t$ only depends on its length and is simply given as $\lambda_1,2 \Delta t$ (the index is depending on the current state of $\xi(t)$).

For the probability $\omega(\Delta t)$ of an event after a certain time span $\Delta t$, we divide the intervals into $N$ small sub-intervals and calculate the probability of only a single event in the $N$-th sub-interval. In the limit $N \to \infty$ we obtain an exponential distribution of the transition times $\Delta t_{1,2}$

$$\omega_{\xi_{1,2}}(\Delta t) = \lambda_{1,2} e^{-\lambda_{1,2}\Delta t} .$$

In the stationary limit, the probabilities of the states $\xi_{1,2}$ are given by the balanced rate equation

$$0 = -\lambda_1 p_{\xi_1} + \lambda_2 p_{\xi_2}$$

with the normalisation $p_{\xi_1} + p_{\xi_2} = 1$. They are proportional to the average duration of stay of the according state, which is just the inverse rate, $p_{\xi_{1,2}} \propto \Delta t_{1,2} = \frac{1}{\lambda_{1,2}}$, normalised

$$p_{\xi_1} = \frac{\lambda_2}{\lambda_1 + \lambda_2} , \quad p_{\xi_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2} .$$

Without loss of generality, one can assume that $\lambda_1 \geq \lambda_2$ and the averaged value of $\xi(t)$ vanishes

$$\langle \xi(t) \rangle = \frac{\xi_1 \lambda_2 + \xi_2 \lambda_1}{\lambda_1 + \lambda_2} = 0$$

These assumptions allow reducing the parameter set from four to three by defining $k = \lambda_2/\lambda_1 \leq 1$ and simultaneously $k = -\xi_2/\xi_1$. Now, one can describe the process $\xi(t)$ by $\lambda := \lambda_1, \xi := \xi_1$ and $k$.

For the probability $p_T(n)$ of $n$ events in a fixed time interval $T$, we divide $T$ into sub-intervals and find the (binomial) distribution of $n$ events in the intervals, where we considered the probabilities of the system being in state $\xi_1$ or $\xi_2$. In the limit $N \to \infty$ we obtain

$$p_T(n) = e^{-\langle \lambda \rangle T} \left(\frac{\langle \lambda \rangle T}{n!}\right)^n$$

with $\langle \lambda \rangle = p_1 \lambda + p_2 k \lambda = 2 \lambda /(1+k)$. We use $p_T(n)$ to find the probability of an even or odd number of transitions in a time interval $T$ and average in the steady state limit to calculate the autocorrelation

$$\langle \xi(t) \xi(t+\tau) \rangle = k \xi_1^2 e^{-2\lambda \tau} .$$

In summary, $\xi(t)$ is a Poisson process, switching between its two states at exponentially distributed times. Its autocorrelation shows an exponential decay, thus it is a coloured noise. The continuous process with the same autocorrelation behaviour is the Ornstein-Uhlenbeck process, a stationary Gaussian process.

III. THE LYAPUNOV EXPONENT OF A PARAMETRIC OSCILLATOR

The energy of the normalised system is $E = \frac{x^2}{2} + \frac{\xi^2}{2}$, which is proportional to the squared length of the phase space vector $X = (x, \xi)$. The Lyapunov exponent $\Lambda$, which is defined as

$$\Lambda = \lim_{t \to \infty} \frac{1}{t} \ln E,$$ (7)

can therefore be seen as an averaged growth rate of $||X||$ and can describe the system’s tendency to diverge from the origin of the phase space or to approach it, depending on its sign. In that way, we will use the Lyapunov exponent to measure the stability of the system.

A different way for calculating $\Lambda$ for the considered system is finding the ensemble average of

$$z = \frac{x}{\xi} = \frac{d}{dx}(\ln x) .$$ (8)

in the stationary (or long time) limit $\lim_{t \to \infty} \Lambda = \langle z \rangle = \int z P(z) dz$ with the stationary probability density $P(z)$. One may illustrate that fact by pointing out, that Eq. $\langle \xi(t) \rangle$ gives just the asymptotic behaviour of $\frac{d}{dt} \ln ||X||$, while $||X||$ grows with $x$. To prove this conjecture, knowledge about $P(z)$ is required, which will be calculated in the next section. The proof itself is shown in Appendix A.

A. Solution of the Fokker-Planck equation

To find the desired probability density $P(z)$, we first change variables from $z \in \mathbb{R}$ to $v \in [-1, 1]$ to ease the numeric evaluation of occurring integrals. For that purpose, let $\phi \in [0, 2\pi]$,

$$\cot \phi := z ,$$ (9)

be a phase-like variable and $v = \cos \phi$. Then $v(z)$ has two branches $v = \pm(z^2 + 1)^{-1/2}$, where the positive sign represents $\phi \in [0, \pi)$ and the negative sign represents $\phi \in [\pi, 2\pi]$. In the following, we choose the second branch $v = -z(z^2 + 1)^{-1/2}$, as with this choice, $v$ increases with growing $z$ and the 'natural' sense of direction in the integrations will be preserved (see appendix B for more details). The equation of motion now translates to

$$\dot{v} = \sqrt{1 - v^2} - \alpha v (1 - v^2) + (1 - v^2)^{3/2} \xi(t)$$ (10)

$$=: f(v) + g(v) \xi(t).$$ (11)

Note, that both, the Itô integral and Stratonovich integral, lead us to the same result due to the exponential autocorrelation: The additional terms cancelled out the diffusive terms, indicating, that in this context, the Kramers-Moyal expansion is not sufficient and a more elaborate method, illustrated by Hors-lamke and Lefever needs to be considered.

We define the joint probability $p_i(v,t), i = 1,2$, of the processes $v(t)$ and $\xi(t)$ that is more exactly written as the transition probability

$$p_i(v,t) := p(v, \xi(t) = \xi_i, t | v(0) = v_0, \xi(0) = \xi_0) ,$$
where subscript 0 indicates the initial values of \( v(t) \) and \( \xi(t) \), respectively. The associated Fokker-Planck equation reads\(^{9,10}\)

\[
\begin{align*}
\dot{p}_1(v) &= -\frac{\partial}{\partial v} \left( f(v) + g(v) \xi(t) \right) p_1(v) - \lambda_1 p_1(v) + \lambda_2 p_2(v) \\
\dot{p}_2(v) &= -\frac{\partial}{\partial v} \left( f(v) + g(v) \xi(t) \right) p_2(v) + \lambda_1 p_1(v) - \lambda_2 p_2(v) \\
\end{align*}
\]

We introduce \( P(v) = p_1(v) + p_2(v) \), which is the desired probability density, and \( Q(v) = p_1(v) - p_2(v) \). In the stationary limit, i.e. for vanishing time derivatives, the equations read

\[
\begin{align*}
0 &= -\frac{\partial}{\partial v} \left( P \frac{2f + g\xi(1-k)}{2} + Qg\xi(1+k) \right) \\
0 &= -\frac{\partial}{\partial v} \left( Q \frac{2f + g\xi(1-k)}{2} + Pg\xi(1+k) \right) \\
&\quad - P\lambda(1-k) - Q\lambda(1+k) ,
\end{align*}
\]

where we use the reduced set of parameters \( \xi, k \) and \( \lambda \). A direct integration of Eq. \[12\] yields the stationary probability flux

\[
J = P \frac{2f + g\xi(1-k)}{2} + Qg\xi(1+k) = \text{const.} \quad \text{(14)}
\]

that allows us to eliminate \( Q \) from Eq. \[13\]. That way we end up with the final differential equation for \( P \)

\[
\frac{\partial}{\partial x} \left[ J(2f + g\xi(1-k)) - 2P(f - g\xi k)(f + g\xi) \right] \\
\quad = -2\lambda g (J - Pf) \quad \text{(15)}
\]

and the solution

\[
P = J \frac{f + g\xi(1-k)}{(f + g\xi)(f - g\xi k)} + \frac{g(1+k)}{(f - g\xi k)(f + g\xi)} G_{\text{part}} , \quad \text{(16)}
\]

with

\[
G_{\text{part}} = -J \exp \left\{ \phi(x) \right\} \int_{-\infty}^{x} \mathrm{d}x'^{n} \left( \frac{\lambda g(x') \xi^{2} k}{((f - g(x') \xi)(f + g(x') \xi))} \right) \times \exp \left\{ -\phi(x') \right\} \\
\phi(x) = -\lambda(1+k) \int_{-\infty}^{x} \mathrm{d}x' \left( \frac{f(x')}{(f - g(x') \xi)(f + g(x') \xi)} \right) .
\]

Note, that \( G_{\text{part}} \) vanishes in the noise-less case \( \xi(t) = \xi = 0 \) and \( P(v) = \frac{J}{f(v)} \propto v^{-1} \), as one would expect.

Now the Lyapunov exponent can be calculated by

\[
\Lambda = \left( \frac{v}{\sqrt{1 - v^{2}}} \right) , \quad \text{(17)}
\]

as \( P_{1}(z) \mathrm{d}z = P_{1}(v) \mathrm{d}v \) and thus \( \int z P_{1}(z) \mathrm{d}z = \int z(v) P_{1}(v) \mathrm{d}v \).

**IV. NUMERICAL EVALUATION**

Realisations of the process \( x(t) \) can be easily generated numerically, as each one is a piecewise compound of (known) solutions to the equation of motion (Eq. \[1\]) for \( \xi(t) = \xi_1 \) and \( \xi_2 \) with duration \( \Delta t_1 \) or \( \Delta t_2 \), respectively. These random transition times can be generated by library functions for exponential distributions and one only has to ensure the continuity of \( x(t) \) and \( \dot{x}(t) \) at the transition points. Fig. \[1\] illustrates this “concatenation” of the solutions, while example trajectories are given in Fig. \[2\].

The Lyapunov exponent can be found by averaging the value \( \frac{1}{t_{\text{max}}} \ln E(t_{\text{max}}) \) after a certain observation time \( t_{\text{max}} \) for several realisations of the process \( x(t) \) or by time averaging \( z(t) = x/t \) (which can be easily found for a sinusoid \( x(t) = A \cos(\sqrt{1 + \xi} t + \phi) \), where the amplitude \( A \) and the phase \( \phi \) are determined by the initial conditions and the continuity of \( x, \dot{x} \) at the transitions) for each interval \( \Delta t_{1,2} \) and a subsequent averaging of these values. We will compare the numerically determined Lyapunov exponent to the analytic solution in the next section, together with an expansion for small values of \( \xi \) and \( \alpha \).

**V. SMALL PARAMETER EXPANSION OF THE SOLUTION**

In the following we evaluate how \( \Lambda \) behaves for a small friction parameter \( \alpha \) and small noise strength \( \xi \) and find a critical curve \( \Lambda(\alpha, \xi) = 0 \), which separates the parameter space into a stable (\( \Lambda < 0 \)) and an unstable (\( \Lambda > 0 \)) domain.

As mentioned before \( P_{\xi=0} = \frac{J}{f(v)} \) and one finds easily, by linearising in \( \alpha \) and integrating, that

\[
\Lambda_{\xi=0} = -\frac{\alpha}{2} + O(\alpha^{3}) , \quad \text{(18)}
\]

which is simply the relaxation rate of an undriven harmonic oscillator with the influence of friction.
FIG. 2. Exemplary trajectories $x(t)$ for fixed parameters $\xi = 0.1$, $\lambda = 1$, $k = 1$ and different values of $\alpha$. (a) For a friction parameter $\alpha$ large enough, the amplitude decays due to friction losses and $x(t)$ approaches the resting point $x = 0$. b) For a decreasing $\alpha$, $\Lambda$ is close to zero and the growth or decay rate of $x(t)$ is small. c) For small $\alpha$ the noise $\xi(t)$ pumps energy into the system that cannot be compensated by friction. The amplitude of $x(t)$ grows exponentially in the asymptotic limit.

The time $t$ is given in units of the (averaged) period of the system, which is unity, as we considered a dimensionless equation of motion (Eq. 1). As this differential equation is homogeneous and linear in $x$, there is no distinct length scale and $x(t)$ can be given in arbitrary units.

As the first summand in Eq. 16 has no term linear in $\xi$ and $\Gamma_{\text{part}} \propto O(\xi^2)$ one can exclude mixed terms if expanding to second order and one only needs to find the coefficient $\gamma$ in

$$\Lambda = \frac{\alpha}{2} + \frac{\gamma}{2} \xi^2 + O(\alpha^n \xi^m) \quad , \quad m + n \geq 3 \quad .$$

(19)
Expanding Eq. [17] considering the $\xi$-dependency of $J$, we find
\[ \gamma = k \left( \frac{(1+k)\lambda}{2(4+(1+k)^2\lambda^2)} + 4k \frac{1-e^{-\pi(1+k)\lambda}}{\pi(4+(1+k)^2\lambda^2)^2} \right). \] (20)

Thus, the critical curve $\Lambda(\alpha, \xi) = 0$ is given by $\alpha_c = \gamma \xi^2$. For small values of the friction parameter, $\alpha < \alpha_c(\xi)$, $\Lambda > 0$ and the system is unstable as its energy grows; the origin is an unstable fixpoint. For larger values $\alpha > \alpha_c(\xi)$, the energy decays and the origin is a stable fixpoint. This transition behaviour of $\Lambda(\alpha, \xi)$ represents a Hopf bifurcation, similar to the one described by Mallick and Marcq[3] for a nonlinear oscillator with multiplicative white noise. In that case the relation of the critical friction coefficient $\alpha_c$ and the (quadratic) noise intensity $\Delta$ reads $\alpha_c = \Delta/4$ in the first order (Eq. 47 in[9]). Fig. 3 shows the analytic solution of Eq. [17] and the expansion $\Lambda = \frac{\xi^2}{2}$ compared to numerically values, calculated from $10^5$ trajectories. The quadratic expansion needs significantly less computation time than the exact solution, as no integrals need to be evaluated. The exact solution systematically exceeds the numeric values, as the latter only converge towards the real value of $\Lambda$ for an infinite observation time $t_{\text{max}} \to \infty$. In the considered range $\xi \in [0; 0.85]$ the quadratic expansion is even a better approximation to the numerical values and does not show a systematic deviation.

VI. CONCLUSION

We considered the basic physical model of a harmonic oscillator with a frequency switching between two values at random times. We found the stationary probability density of an associated phase-like variable and used it to calculate the Lyapunov exponent of the system. For a simple harmonic oscillator with friction, the origin of the phase space ($\chi = \hat{\chi} = 0$) is a globally stable fixed point. With increasing multiplicative dichotomic noise, this feature disappears and after a certain critical value the origin is the only stable initial condition, while for all other values the energy rises and the phase-space coordinates will grow above any given limit. In the future, we plan to use these results to characterise the stability of the trajectories of nanometer sized particles in the dust environment of Saturn’s rings. In this setting, the Lyapunov exponent will serve to give a time scale of the particles leaving their source region.

ACKNOWLEDGMENTS

The author would like to thank Prof. Frank Spahn for his supervision, support, and enriching discussions, as well as Dr. Holger Hoffmann and Prof. Ralph Metzler for constructive recommendations. This work was written within the scope of a project supported by the Deutsche Forschungsgemeinschaft (Sp384/33-1).

FIG. 3. Analytic solution of the Lyapunov exponent $\Lambda$ without friction (green line) and quadratic expansion with respect to the noise strength $\xi$ (red line), calculated according to Eq. [17] and [20] respectively. The values for the asymmetry and rate parameters are $k = 1$ and $\lambda = 1$. The exact solution systematically overestimates the numerically determinated value of $\Lambda$. The quadratic expansion does not follow this systematic behaviour und falls below the numeric trend for $\xi > 0.65$. In the evaluated range $0 \leq \xi \leq 0.85$, both analytic curves show a fair agreement and lie within a 15% band of the numerical results.

AIP PUBLISHING DATA SHARING POLICY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Appendix A

To show, that $\Lambda = \langle \cot \phi \rangle$, we find $\frac{\partial}{\partial \rho}$ for the radial variable $\rho = \frac{1}{2} \ln E$ and $\frac{\partial}{\partial \ln |\sin \phi|}$ by averaging $\rho$ and $\ln |\sin \phi|$ with respect to $P(\rho, \phi)$, which is generally given by the r.h.s. of Eq. [12].

After integration by parts and by using the boundary conditions $\lim_{\rho \to \pm \infty} P = 0$ and $P(\rho, 0) = P(\rho, 2\pi)$ we find
Note the dependencies inside the integrals $P = P(\rho, \phi)$ and $Q = Q(\rho, \phi)$, which are not written for brevity. Again following the arguments of Horsthemke and Lefever\textsuperscript{20} in the stationary limit $t \to \infty$ the term $\frac{d}{dt}\ln|\sin\phi|$ vanishes and we end up with $\Lambda = \lim_{t \to \infty} \frac{d}{dt}P = \langle \cot \phi \rangle$.

Appendix B

If expressing $z = \cot \phi$ by $v = \cos \phi$, one finds two branches of the solution:

$$\cot \phi = \begin{cases} \frac{\cos \phi}{\sqrt{1 - \cos^2 \phi}}, & \text{for } \phi \in [0, \pi] \\ \frac{-\cos \phi}{\sqrt{1 - \cos^2 \phi}}, & \text{for } \phi \in [\pi, 2\pi] \end{cases}$$

We take this to consideration in the following way (For simplicity, we only treat the friction-less case $\alpha = 0$. However, the general case follows in a similar way): Using $\dot{z} = -1 - z^2 - \xi(t)$ to find $\phi$, one ends up with $\phi = 1 + \xi(t)\sin^2 \phi > 0$, as long as $\xi, -k\xi > -1$.

Consequently

$$\dot{\nu} = \begin{cases} -\sqrt{1 - v^2} - (1 - v^2)^{3/2} \xi(t) < 0, & \text{for } \phi \in [0, \pi] \\ \sqrt{1 - v^2} + (1 - v^2)^{3/2} \xi(t) > 0, & \text{for } \phi \in [\pi, 2\pi] \end{cases}$$

and $\nu$ goes from 1 to $-1$ in the first branch and from $-1$ to 1 in the second branch. This sense of direction has to be considered for most integrals when calculating $P(\nu)$. For simplicity, we chose the second branch for the ‘natural’ direction of integration. However, the other branch would produce symmetric solutions with the same result. As soon as friction is included, the r.h.s. of $\dot{\phi}$ may get below zero. Although the detailed argumentation gets slightly more complicated in this case, the same arguments hold.

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