A Ramsey characterisation of eventually periodic words

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Abstract
A factorisation $x = u_1 u_2 \cdots$ of an infinite word $x$ on alphabet $X$ is called ‘monochromatic’, for a given colouring of the finite words $X^*$ on alphabet $X$, if each $u_i$ is the same colour. Wojcik and Zamboni proved that the word $x$ is periodic if and only if for every finite colouring of $X^*$ there is a monochromatic factorisation of $x$. On the other hand, it follows from Ramsey’s theorem that, for any word $x$, for every finite colouring of $X^*$ there is a suffix of $x$ having a monochromatic factorisation. A factorisation $x = u_1 u_2 \cdots$ is called ‘super-monochromatic’ if each word $u_{k_1} u_{k_2} \cdots u_{k_n}$, where $k_1 < \cdots < k_n$, is the same colour. Our aim in this paper is to show that a word $x$ is eventually periodic if and only if for every finite colouring of $X^*$ there is a suffix of $x$ having a super-monochromatic factorisation. Our main tool is a Ramsey result about alternating sums that may be of independent interest.

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1 INTRODUCTION

Let $X$ be a non-empty finite or infinite set (called the alphabet) and let $X^*$ denote the set of all finite words $x_1 x_2 \cdots x_n$ with $n \geq 1$ and $x_i \in X$ for all $i$. Let $x = x_1 x_2 x_3 \cdots$ be an infinite word on $X$. Given

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a finite colouring of $X^*$, we say that a factorisation $x = u_1 u_2 u_3 \cdots$ (with $u_i \in X^*$) is monochromatic if all the $u_i$ have the same colour. When is it the case that $x$ always has a monochromatic factorisation, for any finite colouring of $X^*$?

This is certainly the case if $x$ is periodic. Indeed, if $x = uuu \cdots$, then for any colouring of $X^*$ that very factorisation is trivially monochromatic. In the other direction, Wojcik and Zamboni [12] proved that if $x$ is not periodic then there exists a finite colouring of $X^*$ for which $x$ does not have a monochromatic factorisation. Thus the above Ramsey condition actually characterises the periodic words.

We remark that if we are allowed to pass to a suffix of $x$, then this characterisation breaks down completely. Indeed, every word $x$ has the property that for every finite colouring of $X^*$ there is a suffix of $x$ having a monochromatic factorisation. This result is due to Schützenberger [8], and it follows from Ramsey’s theorem. To see this, let $x$ be the word $x_1 x_2 \cdots$. Given a finite colouring $\phi$ of $X^*$, we define a colouring of $\mathbb{N}/(2)$, the edge set of the complete graph on the natural numbers, by giving the pair $(i, j)$, where $i < j$, the colour $\phi(x_i \cdots x_{j-1})$. By Ramsey’s theorem, there is a monochromatic infinite set for this colouring, say $m_1 < m_2 < \cdots$. But now we note that the finite words $x_{m_i} x_{m_i+1} \cdots x_{m_{i+1}-1}$, for each $i$, are all assigned the same colour by $\phi$, and they form a factorisation of the suffix of $x$ starting at position $m_1$.

Actually, the above argument shows that more: it shows that, for any colouring, there is a suffix of $x$ having a factorisation $u_1 u_2 \cdots$ in which every word $u_i u_{i+1} \cdots u_j$, for $i \leq j$, has the same colour. It was shown by de Luca and Zamboni [1] that this strengthened form is actually equivalent to Ramsey’s theorem.

In light of these results, it is natural to ask if there is a Ramsey characterisation of the eventually periodic words over $X$, that is, infinite words of the form $uvuv \cdots$ with $u, v \in X^*$. We say that a factorisation $x = u_1 u_2 \cdots$ is super-monochromatic if each word $u_{k_1} u_{k_2} \cdots u_{k_n}$, where $k_1 < \cdots < k_n$, is the same colour. Our motivation for considering this notion comes from the following observation: if $x$ is eventually periodic, then for every finite colouring of $X^*$ there is a suffix of $x$ having a super-monochromatic factorisation.

Indeed, given a finite colouring $\phi$ of $X^*$, it suffices to take a suffix of $x$ that is periodic: say $y = uuu \cdots$. We induce a colouring of $\mathbb{N}$ by giving the number $n$ the colour $\phi(u^n)$. By Hindman’s theorem [3], there exists an infinite set $M \subset \mathbb{N}$, say $M = \{a_1, a_2, \ldots\}$, where $a_1 < a_2 < \cdots$, such that every (non-empty) finite sum of distinct elements of $M$ has the same colour. But now the factorisation $y = u^{a_1} u^{a_2} \cdots$ is super-monochromatic.

Our aim in this paper is to show that this condition actually characterises the eventually periodic words. In other words, we will show that if the word $x$ has the property that for every finite colouring of $X^*$ there is a suffix of $x$ having a super-monochromatic factorisation, then $x$ is eventually periodic.

**Theorem 1.** Let $x$ be an infinite word on alphabet $X$. Then $x$ is eventually periodic if and only if for every finite colouring of $X^*$ there is a suffix of $x$ having a super-monochromatic factorisation.

(Note that if we ‘swap the quantifiers’, we would have the statement that $x$ is eventually periodic if and only if there is a suffix of $x$ such that every finite colouring of this suffix has a super-monochromatic factorisation — which is true by the remarks above.)

This result has actually been around in the community as a conjecture for some time (see, for example, [10]). There have been some partial results, of which the strongest is perhaps the result of Wojcik [10], who showed that Theorem 1 holds for words $x$ that have at most finitely many distinct square factors, where by a square factor we mean a non-empty block of the form $uu$ which occurs...
in $x$. But the result was not even known for Sturmian words, which are regarded as the ‘simplest’ aperiodic words, that is, words that are not eventually periodic.

Let us also remark that if one is allowed to pass to the shift orbit closure then the situation is completely different. Recall that the shift orbit closure of an infinite word $x$ is the closure of the set of suffices of $x$ in the product topology: equivalently, it consists of all infinite words $y$ such that every factor of $y$ is a factor of $x$. Van Thé and Zamboni (see [11]) showed that, for any infinite word $x$ over a finite alphabet $X$, whenever $X^*$ is finitely coloured there is a word $y$ in the shift orbit closure of $x$ having a super-monochromatic factorisation.

Our proof is in two separate parts. In the first part, we reduce the problem to a problem that concerns not colourings of words, but colourings of $\mathbb{N}^{(2)}$. It will turn out from this reduction that Theorem 1 is implied by the following result, which concerns alternating sums and may be of independent interest.

**Theorem 2.** There exists a finite colouring of $\mathbb{N}^{(2)}$ such that there do not exist $x_1 < x_2 < \cdots$ for which the set of all pairs $(x_{k_1} - x_{k_2} + x_{k_3} - \cdots + x_{k_t}, x_{k_{t+1}})$, where $t$ is odd and $k_1 < k_2 < \cdots < k_{t+1}$, is monochromatic.

The second part of the proof thus consists of a proof of Theorem 2. What is interesting is the role played by the alternation. Indeed, if all the signs were plus-signs then the Ramsey statement would be in the affirmative. In other words, for any finite colouring of $\mathbb{N}^{(2)}$ there exist $x_1 < x_2 < \cdots$ for which the set of all pairs $(x_{k_1} + x_{k_2} + x_{k_3} + \cdots + x_{k_t}, x_{k_{t+1}})$, where $k_1 < k_2 < \cdots < k_{t+1}$, is monochromatic. This follows for example from the Milliken–Taylor theorem ([7, 9]). To see this, recall that the Milliken–Taylor theorem asserts that whenever the set of all pairs $(A, B)$, where $A$ and $B$ are (non-empty) finite subsets of $\mathbb{N}$ with $\max A < \min B$, is finitely coloured there exists a sequence $A_1, A_2, \ldots$ of finite subsets of $\mathbb{N}$, with $\max A_n < \min A_{n+1}$ for all $n$, such that all of the pairs $(S, T)$, where $S$ and $T$ are finite unions of the $A_n$ with $\max S < \min T$, are the same colour. So we just need to ‘transfer’ the colouring from numbers to finite sets: given a finite colouring $\vartheta$ of $\mathbb{N}^{(2)}$, we colour each pair $(A, B)$ as above with the colour $\vartheta(\sum_{i \in A} 2^i, \sum_{i \in B} 2^i)$. Given the sequence $A_1, A_2, \ldots$ as guaranteed by the Milliken–Taylor theorem, we set $x_n = \sum_{i \in A_n} 2^i$, and now we get that every pair $(x_{k_1} + x_{k_2} + x_{k_3} + \cdots + x_{k_t}, x_{k_{t+1}})$, and in fact even every pair $(x_{k_1} + x_{k_2} + x_{k_3} + \cdots + x_{k_1}, x_{k_{t+2}} + \cdots + x_{k_t})$, has the same colour. The interested reader is referred to [5] for a general discussion of the Milliken–Taylor theorem and many related results, although we stress that this paper is self-contained.

The colouring argument needed to establish Theorem 2 is rather complicated, and it is perhaps worthwhile to describe why this is the case. As we will see, it turns out to be useful to ‘change variables’ to some other variables, the $y_n$, that satisfy a related condition. However, this related condition is not preserved by passing to subsequences. This is in contrast to the usual situations when one is finding a ‘bad’ colouring (see, for example, [2, 4, 6]), where the first step is always to pass to a subsequence or sequence of sums in which the supports of the elements, when written say in binary, are disjoint, and even more are ordered in the sense that one variable’s support ends before the next one’s begins. Since this step is not available to us here, we have to deal with the situation when the $y_i$ do not have disjoint supports, and therefore we need to consider how the carry-digits behave when we add them to each other. This means that the colouring, and especially the proof that it works, is far more difficult than for other problems that superficially look similar.

The plan of the paper is as follows. In Section 2, we show that Theorem 1 is implied by Theorem 2, and then in Section 3 we prove Theorem 2. Section 4 is devoted to some related problems that we have been unable to solve.
2 | THE LINK BETWEEN THE TWO THEOREMS

In this section we will show that Theorem 2 implies Theorem 1. As explained above, we know that given any finite colouring of $X^*$, any eventually periodic word has a suffix that admits a super-monochromatic factorisation. Therefore, we only need to prove the reverse implication of Theorem 1: given an aperiodic word $x$ (that is, $x$ is not eventually periodic), we must construct a finite colouring of $X^*$ for which no suffix of $x$ has a super-monochromatic factorisation. This will be accomplished using the colouring given by Theorem 2.

**Theorem 3.** Theorem 2 implies Theorem 1.

**Proof.** By Theorem 2, there exists a finite colouring of $\mathbb{N}^{(2)}, \theta$, for which there is no increasing sequence $(x_k)_{k \geq 1}$ such that all edges of the form $(x_{k_1} - x_{k_2} + \cdots + x_{k_t}, x_{k_{t+1}})$ have the same colour, where $k_1 < k_2 < \cdots < k_{t+1}$ and $t$ is odd. Let $C$ be the set of colours of $\theta$.

Let $x$ be an aperiodic word. We will use $\theta$ to construct a finite colouring $\phi$ of $X^*$ for which no suffix of $x$ has a super-monochromatic factorisation. We denote by $x_i$ the $i$th letter of $x$.

For any factor $u$ of $x$, define $A_x(u) = \min\{n \in \mathbb{N} : u = x_n x_{n+1} \cdots x_{n + |u| - 1}\}$ and $B_x(u) = A_x(u) + |u|$, where $|u|$ is the length of $|u|$. In other words, $A_x(u)$ is the start position of the first occurrence of $u$ in $x$, while $B_x(u)$ is the first position after this first occurrence of $u$.

For an arbitrary factorisation $(u_i)_{i \geq 1}$ of $x$, we say $(w_i)_{i \geq 1}$ is a block subfactorisation of $(u_i)_{i \geq 1}$ if there exists a strictly increasing sequence of positive integers $(k_i)_{i \geq 1}$ such that $w_1 = u_1 u_2 \cdots u_{k_1}$ and $w_i = u_{k_{i-1}+1} \cdots u_{k_i}$ for each $i \geq 2$. Here by $(u_i)_{i \geq 1}$ being a factorisation of $x$ we mean $x = u_1 u_2 u_3 \cdots$. We immediately note that a block subfactorisation of a super-monochromatic factorisation is still super-monochromatic.

Now we are ready to define a colouring $\phi : X^* \to (C \times \{0, 1\}) \cup \{2\}$ as follows:

1. If $u$ is not a factor of $x$, then $\phi(u) = 2$.
2. If $u$ is a factor of $x$ and there exists a factorisation $u = vw$ such that $A_x(u) = A_x(v)$ and $B_x(u) = B_x(w)$ (in other words, the first occurrence of $v$ in $x$ is as the start of the first occurrence of $u$ in $x$ and also the first occurrence of $w$ in $x$ is as the end of the first occurrence of $u$ in $x$), then $\phi(u) = (\theta(A_x(u), B_x(u)), 0)$.
3. Otherwise $\phi(u) = (\theta(A_x(u), B_x(u)), 1)$.

We claim that for this colouring $\phi$ no suffix of $x$ has a super-monochromatic factorisation.

Suppose to the contrary that there is a suffix $y$ of $x$ having a super-monochromatic factorisation $y = u_1 u_2 \cdots$. Let $u_0$ be the (possibly empty) prefix of $x$ so that $x = u_0 y$. It is important to remember that each factor $u_i$ may occur in several places in $y$, not necessarily only in the place immediately following $u_1 u_2 \cdots u_{i-1}$. We call this place the standard position of $u_i$. Let the colour of all concatenations of the $u_i$ be $c \in (C \times \{0, 1\}) \cup \{2\}$. Since $\phi(u_1) = c$, and since $u_1$ is a factor of $x$, we have $c \neq 2$. Thus, $c = (a, b)$ where $a \in C$ and $b \in \{0, 1\}$.

**Claim 1.** By passing to a block subfactorisation, we may assume that for every $i \in \mathbb{N}$, the first occurrence of $u_i$ in $x$ is exactly the standard position of $u_i$.

Equivalently, this means $A_x(u_i) = |u_0| + |u_1| + \cdots + |u_{i-1}| + 1$ for all $i \geq 1$. 
Proof. We start by showing that we may assume $A_x(u_1) = |u_0| + 1$. If initially $A_x(u_1) < |u_0| + 1$, we consider all concatenations $u_1 u_2 \cdots u_k$. If $A_x(u_1 u_2 \cdots u_k) = |u_0| + 1$ for some $k$, we set our first factor to be $u_1 u_2 \cdots u_k$ and renumber the rest of them. Since concatenating consecutive factors does not change the super-monochromatic property, the new factorisation is still super-monochromatic and the first factor now has the desired property.

If on the other hand $A_x(u_1 u_2 \cdots u_k) < |u_0| + 1$ for all $k \geq 1$, then each concatenation $u_1 u_2 \cdots u_k$ first occurs in $x$ starting at some position in $u_0$. Since there are infinitely many of them and only finitely many positions in $u_0$, there exists a position $i$, with $i \leq |u_0|$, at which infinitely many $u_1 u_2 \cdots u_k$ start. This immediately implies that the suffix of $x$ starting at position $i$ is exactly $y$. But this means that $x$ has two suffixes equal to $y$, which implies that $x$ is eventually periodic. More precisely, we have $x_1 x_2 \cdots x_{i-1} y = x_1 x_2 \cdots x_{|u_0|} y$. Therefore, $y_k = x_{i-1+k}$ and $y_k = x_{|u_0|+k}$ for any $k$. It follows that $x_{i-1+k} = x_{|u_0|+k}$ for any $k$, thus $x$ is eventually periodic with period $|u_0| - i + 1$, contradicting our initial assumption.

Therefore we may assume $u_1$ has the desired property. We now move on to $u_2$ and repeat the same argument, looking at concatenations of the form $u_2 u_3 \cdots u_k$: so $u_2$ may be assumed to have the same property too. It follows inductively that we may assume that all $u_i$ have the property stated in the claim. □

We further observe that once we have the property that the first occurrence of each $u_i$ in $x$ is in the standard position, then any block subfactorisation has this property as well. For example, $u_1 u_2$ cannot appear earlier or else $u_1$ would. Therefore, we can further assume that $|u_{n+1}| \geq |u_1 u_2 \cdots u_n|$ for all $n \geq 1$.

We now look at $u_1 u_2$. Because of the above claim, we certainly have $A_x(u_1 u_2) = A_x(u_1)$ and $B_x(u_1 u_2) = B_x(u_2)$. This means that $u_1 u_2$ is a factor of $x$ that satisfies the factorisation condition specified in the colouring rule. Thus $\phi(u_1 u_2) = (a, b) = (\theta(A_x(u_1 u_2), B_x(u_1 u_2)), 0)$, and so the colour of the factorisation is $(a, 0)$ with $a \in C$.

Claim 2. The word $u_1 u_2 \cdots u_n$ is a suffix of $u_{n+1}$, for every $n \geq 1$.

Proof. Consider the concatenation $u_1 u_2 \cdots u_n u_{n+2}$. Because our factorisation $u_1 u_2 \cdots$ is super-monochromatic, we have that $\phi(u_1 u_2 \cdots u_n u_{n+2}) = (a, 0)$. This means that not only is $u_1 u_2 \cdots u_n u_{n+2}$ a factor of $x$, but also that $u_1 u_2 \cdots u_n u_{n+2} = vw$ for some $v, w$ with $A_x(u_1 u_2 \cdots u_n u_{n+2}) = A_x(v)$ and $B_x(u_1 u_2 \cdots u_n u_{n+2}) = B_x(w)$.

We now have two possibilities: either $v$ is a prefix of $u_1 u_2 \cdots u_n$ or $u_1 u_2 \cdots u_n$ is a prefix of $v$.

If $v$ is a prefix of $u_1 u_2 \cdots u_n$ then $u_{n+2}$ is a suffix of $w$. Therefore, we immediately have that $A_x(v) \leq A_x(u_1 u_2 \cdots u_n)$ and $B_x(w) \geq B_x(u_{n+2})$. It follows that

$$B_x(u_1 u_2 \cdots u_n u_{n+2}) = B_x(w) \geq B_x(u_{n+2}) = B_x(u_1 u_2 \cdots u_n u_{n+1} u_{n+2})$$

and

$$A_x(u_1 u_2 \cdots u_n u_{n+2}) = A_x(v) \leq A_x(u_1 u_2 \cdots u_n) = A_x(u_1 u_2 \cdots u_n u_{n+1} u_{n+2}),$$

where the last equalities in each line follow from the property that, for each $i$, the first occurrence of $u_i$ in $x$ is at its standard position. Consequently, any consecutive concatenation of such factors has the same property.
Putting these two inequalities together, we obtain
\[ B_x(u_1 u_2 \ldots u_n u_{n+2}) - A_x(u_1 u_2 \ldots u_n u_{n+2}) \geq B_x(u_1 u_2 \ldots u_n u_{n+1} u_{n+2}) - A_x(u_1 u_2 \ldots u_n u_{n+1} u_{n+2}). \]

This is equivalent to \(|u_1 u_2 \ldots u_n u_{n+2}| \geq |u_1 u_2 \ldots u_n u_{n+1} u_{n+2}|\), which is a contradiction.

Hence, \(u_1 u_2 \ldots u_n\) is a prefix of \(v\), and so \(w\) is a suffix of \(u_{n+2}\). This implies that \(B_x(w) \leq B_x(u_{n+2})\). Since \(u_{n+2}\) is a suffix of \(u_1 u_2 \ldots u_n u_{n+2}\), the same argument gives
\[ B_x(u_{n+2}) \leq B_x(u_1 u_2 \ldots u_n u_{n+2}) = B_x(w). \]

Therefore \(B_x(u_{n+2}) = B_x(u_1 u_2 \ldots u_n u_{n+2})\). By Claim 1, we also have \(B_x(u_{n+1} u_{n+2}) = B_x(u_{n+2})\). We conclude that \(B_x(u_{n+1} u_{n+2}) = B_x(u_1 u_2 \ldots u_n u_{n+2})\) which, combined with \(|u_{n+1}| > |u_1 u_2 \ldots u_n|\), gives that \(u_1 u_2 \ldots u_n\) is a suffix of \(u_{n+1}\).

**Claim 3.** The word \(u_k u_k \ldots u_k\) is a suffix of \(u\), for every \(k_1 < k_2 < \cdots < k_m < n\).

**Proof.** We prove the statement by induction on the number of factors.

From Claim 2, we get that \(u_t\) is a suffix of \(u_{t+1}\) for every \(t \geq 1\). Since ‘is a suffix of’ is a transitive property, we obtain that \(u_t\) is a suffix of \(u_s\) for every \(t < n\), thus the base case is proved.

Assume now that the result is true for all concatenations of at most \(s\) factors, and consider a concatenation \(u_k u_k \ldots u_k u_{k+1}\) with all \(k_i < n\). If the indices are consecutive numbers, Claim 2 guarantees that this is a suffix of \(u_{k+1}\), which is a suffix of \(u\). If that is not the case, then \(k_i + 1 < k_{i+1}\) for some \(i \leq s\). We take \(i\) to be the biggest such index and apply the induction hypothesis to obtain that \(u_k \ldots u_k\) is a suffix of \(u_{k+1}\), which is a suffix of \(u_{k+1}\). It then follows that \(u_{k_1} u_{k_2} \ldots u_{k_s} = u_{k_{s+1}}\) is a suffix of the consecutive concatenation of factors \(u_{k_{s+1}} u_{k_{s+1}} u_{k_{s+1}}\), which is a suffix of \(u_{k_{s+1}}\), thus a suffix of \(u\). This finishes the inductive step and hence proves the claim.

Combining Claims 1 and 3, we obtain that \(B_x(u_k u_k \ldots u_k) = B_x(u_k)\). This is because repeatedly applying Claim 3 tells us that \(u_{k_1} u_{k_2} \ldots u_{k_i}\) is a suffix of a consecutive concatenation of factors ending in \(u_{k_i}\). Note that, by construction, we also have \(A_x(u_{n+1}) = B_x(u_n)\).

We now return to our original colouring. By assumption, we have that
\[ \theta(A_x(u_k u_k \ldots u_k), B_x(u_k u_k \ldots u_k)) = a \]
for every \(k_1 < k_2 < \cdots < k_t\). We also know that
\[
A_x(u_{k_1} u_{k_2} \ldots u_{k_i}) = B_x(u_{k_1} u_{k_2} \ldots u_{k_i}) - |u_{k_1} u_{k_2} \ldots u_{k_i}|
= B_x(u_{k_1}) - |u_{k_1}| - |u_{k_2}| - \cdots - |u_{k_i}|
= A_x(u_{k_1}) + A_x(u_{k_1-1}) + \cdots + A_x(u_{k_1}) - B_x(u_{k_1-1}) - \cdots - B_x(u_{k_1}),
\]
where we used the fact that \(|u_{k_i}| = B_x(u_{k_i}) - A_x(u_{k_i})\).

Let \(m_i = B_x(u_i)\) for each \(i\). Clearly \((m_i)_{i \geq 1}\) is a strictly increasing sequence. We then have
\[ A_x(u_{k_1} u_{k_2} \ldots u_{k_i}) = m_{k_i-1} + m_{k_i-1} - m_{k_i-1} - \cdots - m_{k_1}. \]
It follows that for any choice of \( k_1 < k_2 < \cdots < k_l \), we have that
\[
\vartheta(m_{k_1-1} - m_{k_1} + m_{k_2-1} - m_{k_2} + \cdots + m_{k_{t-1}-1} - m_{k_{t-1}} + m_{k_t-1}, m_t) = a.
\]

By choosing the \( k_i \) appropriately, it follows that for any \( l \) odd and \( i_1 < i_2 < \cdots < i_l < i_{l+1} \), we have \( \vartheta(m_{i_1} - m_{i_2} + \cdots - m_{i_{l-1}} + m_{i_l}, m_{i_{l+1}}) = a \), which contradicts the choice of \( \vartheta \).

\[\square\]

### 3 Constructing the colouring \( \vartheta \)

In this section, we will construct a finite colouring of \( \mathbb{N}(2) \) with the property that for no infinite strictly increasing sequence \( (x_n)_{n \geq 1} \) do all pairs of the form \( (x_{k_1} - x_{k_2} + \cdots - x_{k_{t-1}} + x_{k_t}, x_{k_{t+1}}) \) have the same colour, where \( k_1 < k_2 < \cdots < k_{t+1} \) and \( t \) is odd.

We start with a simple observation. Let \( y_1 = x_1 \) and \( y_n = x_n - x_{n-1} \) for each \( n \geq 2 \). So \( x_n = y_n + y_{n-1} + \cdots + y_1 \).

Now let \( t \) be odd and \( k_1 < k_2 < \cdots < k_t \). We then have that \( x_{k_1} - x_{k_2} + \cdots - x_{k_{t-1}} + x_{k_t} = x_{k_1} + (x_{k_3} - x_{k_2}) + \cdots + (x_{k_t} - x_{k_{t-1}}) \). Thus \( x_{k_1} - x_{k_2} + \cdots - x_{k_{t-1}} + x_{k_t} = y_1 + y_2 + \cdots + y_{k_1} + (y_{k_3} + \cdots + y_{k_2}) + \cdots + (y_{k_{t-1}} + 1 + \cdots + y_{k_1}) \).

Let \( 1 < m_1 < \cdots < m_s \) be integers and set in the above expression \( k_1 = 1, k_2 + 1 = k_3 = m_1, \cdots, k_{2s} + 1 = k_{2s+1} = m_s \). Then we obtain that \( x_{k_1} - x_{k_2} + \cdots - x_{k_{t-1}} + x_{k_t} = y_1 + y_{m_1} + \cdots + y_{m_s} \).

This shows that Theorem 2 is equivalent to

**Theorem 4.** There exists a finite colouring of \( \mathbb{N}(2) \) such that there does not exist a sequence of natural numbers \( (y_n)_{n \geq 1} \) for which all pairs of the form \( (y_{i_1} + y_{k_1} + y_{k_2} + \cdots + y_{k_t}, y_{i_1} + y_2 + y_3 + \cdots + y_{k_{t+1}}) \) have the same colour, for all choices of \( 1 < k_1 < k_2 < \cdots < k_{t+1} \).

**Proof.** Our construction of the colouring will be in several stages. At each stage, we add more colours, meaning that we take the product colouring of the colouring we have so far with a new colouring. The conditions on a supposed sequence \( (y_n)_{n \geq 1} \) satisfying the conditions in Theorem 4 will thus become more and more stringent, eventually resulting in a contradiction.

As the colouring is rather complex, we give a brief overview of what each stage is supposed to achieve. We first need some notation. We work with natural numbers in their binary form, so strings of ‘0’ and ‘1’. The *position* of a digit is the power of 2 it represents. The *first* digit of \( n \) in binary is at position \( i \), where \( i \) is the greatest non-negative integer such that \( 2^i \) divides \( n \). The *last* digit of \( n \) in binary is at position \( j \), where \( 2^j \leq n < 2^{j+1} \). The *support* of \( n \) is the set of positions having the digit ‘1’ in its binary expansion. For example, let \( n = 2^7 + 2^6 + 2^3 \). Below \( n \) is shown in binary, where the first row represents the position of each digit. The support of \( n \) is \( \{3, 6, 7\} \).

| Position number | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|-----------------|---|---|---|---|---|---|---|---|
| Binary digit of \( n \) | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |

In Stage 1, we will ensure that the supports (in binary) of the \( y_n \) do roughly ‘go off to the left’. What we hope to achieve is a ‘staircase’ pattern. More precisely, writing \( n_i \) for the position of the first digit of \( y_i \) and \( m_i \) for the position of its last digit, we would like to ensure that the \( n_i \) and the
$m_i$ form strictly increasing sequences, with $y_{i+2}$ starting to the left of where $y_i$ ends for all $i$. This is the idea behind the definition of ‘Type A’ below. However, it will turn out that we cannot always achieve this, and so there is a residual case to deal with that we call ‘Type B’, which represents what happens when there is no way to pass to Type A. Of course, we cannot ignore this case, but somehow it has the feel of an annoying special case: the reader should perhaps view Type A as the ‘main’ case. Roughly speaking, the sequence is of Type B when the sequence, with $y_1$ removed, is of Type A, but also $y_1$ starts where $y_2$ starts, and $y_1 + y_2$ starts where $y_3$ starts, and $y_1 + y_2 + y_3$ starts where $y_4$ starts, and so on.

Then Stage 2 gives that the supports of the $y_i$, despite having the above staircase pattern, cannot be disjoint. And it then starts to deal with the unpleasant issues arising from ‘carry digits’, that arise when adding numbers whose supports are not disjoint. It will give that the carries must be short-range, in a certain sense. The fact that we forbid the carries to propagate arbitrarily far will actually show that Type B cannot occur. And finally, Stage 3 will eliminate these short-range carries as well.

We are now ready to turn to the proof itself. As stated above, we construct our colouring step by step. When colouring a pair $(a, b)$, $a < b$, we will often look at just $a$, just $b$, or just $b - a$. At other times, we will make full use of the fact that we are colouring pairs, not just numbers.

**Stage 1.** To start with, our colours are quadruples $(c_0, c_1, c_2, c_3)$ where $c_0, c_1, c_2 \in \{0, 1, 2\}$ and $c_3$ is one of the four possible bit-strings of length 3 having ‘1’ at their rightmost positions. We give the colour $(c_0, c_1, c_2, c_3)$ to the pair $(a, b)$, $a < b$, if the last digit of $b - a$ in binary is at position $c_0$ modulo 3, the first digit of $b - a$ in binary is at position $c_1$ modulo 3, the last digit of $a$ in binary is at position $c_2$ modulo 3, and the first 3 digits of $b - a$ form, from left to right, $c_3$.

Assume now $(y_n)_{n \geq 1}$ is a sequence that satisfies the conditions of Theorem 4 for the above colouring. The pairs that are all the same colour are the pairs of the form $(y_{k_1} + \cdots + y_{k_t} + y_{k_t +1} + y_{k_{t+1}} + \cdots + y_{k_{t+1} +1})$ for some $t \geq 0$ and $1 < k_1 < \cdots < k_t < k_{t+1}$. The differences $b - a$, where $(a, b)$ is a pair of the above form, are precisely the sums of the form $y_{k_1} + \cdots + y_{k_t}$ for some $t \geq 1$ and $1 < k_1 < \cdots < k_t$. It follows that any such finite sum must have the same first 3 digits, with the first digit at a fixed position modulo 3.

**Claim 1.** There do not exist $j > i > 1$ such that $y_j$ and $y_i$ have their first digit at the same position.

**Proof.** Assume that two such $y_i$ and $y_j$ exist. The position of their first digit is the same, say $n_0$, and by our colouring they have the same first 3 digits. On the other hand, the colouring also requires $y_i + y_j$ to have the first digit at a position congruent to $n_0$ modulo 3. However, adding two identical strings in binary shifts the support by exactly one to the left. Hence, when we add $y_i$ and $y_j$, their first ‘1’, which was at position $n_0$ for both of them, is moved to position $n_0 + 1 \not\equiv n_0$ modulo 3, a contradiction. $\square$

We now know that, except for possibly $y_1$, no two terms of the sequence start at the same position.

Let $(z_n)_{n \geq 1}$ be a sequence of natural numbers. We call $(w_n)_{n \geq 1}$ a **full block subsequence** or simply a **block subsequence** of $(z_n)_{n \geq 1}$ if there exists an increasing sequence of natural numbers $(k_n)_{n \geq 1}$ such that $w_1 = z_1 + \cdots + z_{k_1}$ and $w_n = z_{k_{n-1}+1} + \cdots + z_{k_n}$ for $n \geq 2$. We stress that there are no ‘gaps’: every $z_n$ appears as a summand in some $w_m$.

We observe that if the sequence $(y_n)_{n \geq 1}$ satisfies the conditions in Theorem 4 for a given colouring then so does any of its block subsequences. So, by passing to a block subsequence, we may assume that $(y_n)_{n \geq 1}$ is strictly increasing.
Let \((z_n)_{n \geq 1}\) be a sequence of natural numbers. Let \(n_i\) be the position of the first digit of \(z_i\) and \(m_i\) the position of the last digit of \(z_i\), for all \(i \geq 1\). We call the sequence \((z_n)_{n \geq 1}\) of Type A if for all \(i \geq 1\), \(n_i < n_{i+1}\), and \(m_i < m_{i+1}\), and \(m_i + 1 < n_{i+2}\). We call the sequence \((z_n)_{n \geq 1}\) of Type B if none of its block subsequences is of Type A, the sequence \((z_n)_{n \geq 2}\) is of Type A, and also \(n_1 = n_2\), and \(m_1 < m_2\), and \(m_1 + 1 < n_3\). We remark that this definition of ‘Type B’ is more abstract that the one informally described in the proof overview above: the reason is that we want this definition to capture the idea of ‘we cannot pass to Type A’.

Claim 2. By passing to a block subsequence, we may assume that \((y_n)_{n \geq 1}\) is of either Type A or Type B.

Proof. As above, let the positions of the first and last digits of \(y_i\) be \(n_i\) and \(m_i\) respectively.

Assume first that there is no \(k\) such that the first digit of \(y_k\) is at the same position as the first digit of \(y_1\). We will prove that we can find a block subsequence of Type A. We start with \(y_1\). By Claim 1, only finitely many terms have the position of their first digit at most the position of the first digit of \(y_1\). Let \(y_1\) be the last one of them. We replace \(y_1\) by the consecutive sum \(y_1 + y_2 + \cdots + y_l\) and relabel the sequence accordingly. Now we move on to the second term. All terms after \(y_1\) now have the position of their first digit greater than that of \(y_1\). Again, only finitely many terms have their first digit at a position at most one plus the position of the first digit of \(y_1\). Let \(y_1\) be the last one of them. We now replace \(y_2\) by \(y_2 + y_3 + \cdots + y_{l_2} + y_{l_2+1}\) and again relabel the sequence. Now all terms after \(y_2\) have their first digit at a position greater than one plus the position of the last digit of \(y_1\). Also, only finitely many have the position of their first digit at most one plus the position of the last digit of \(y_2\). Let \(y_{l_3}\) be the last one of them. We replace \(y_3\) by \(y_3 + y_4 + \cdots + y_{l_3+1}\). Now continue inductively. Hence, we obtain a block subsequence of Type A.

We now assume that \((y_n)_{n \geq 1}\) does not have any block subsequence of Type A. Therefore, there is a \(k\) such that the first digit of \(y_1\) is at the same position as the first digit of \(y_k\). We now construct a block subsequence of Type B.

First we note that we may assume that there is no \(i > 1\) such that \(n_i < n_1\): if such an \(i\) exists, then we replace \(y_i\) with \(y_1 + y_2 + \cdots + y_i\) and relabel. This new block subsequence has the property that the first digits of its terms are all on different positions. Therefore, by the argument presented at the beginning of the proof, we can construct a block subsequence of Type A, which is a contradiction.

We fix \(y_1\). We know that no \(y_i\) starts before it and only \(y_k\) starts at the same position. Only finitely many \(y_i\) have their first digit at a position at most one plus the position of the last digit of \(y_1\). Let \(y_1\) be the last of them and let \(t = \max(s + 1, k)\). We now replace \(y_2\) with \(y_2 + \cdots + y_t\). Note that in this block subsequence \(y_1\) and \(y_2\) start on the same position, and \(m_1 < m_2\). Since from now on the terms start on different positions, we repeat the inductive construction presented at the beginning of the proof and thus obtain the desired block subsequence of Type B. □

In what follows, a property that will play a crucial role is the fact that any block subsequence of \((y_n)_{n \geq 1}\) still satisfies Claim 2. More precisely, as we now show, Type A sequences are invariant under taking block subsequences, and the same holds for Type B sequences. That is a direct consequence of binary addition and our colouring so far.

Claim 3. If \((y_n)_{n \geq 1}\) satisfies the conditions in Theorem 4 for the above colouring and is of Type A, then the same also holds for each of its block subsequences, and similarly for Type B.
Proof. Consider a sum \( y_m + y_{m+1} + \cdots + y_n \), where \( 2 \leq m \leq n \). In any given position, at most two of the summands have a digit 1 and so the last digit of the sum is either at the same position as the last digit of \( y_n \), or one greater. Because of the colour \( c_0 \), we conclude that the last digit of \( y_m + y_{m+1} + \cdots + y_n \) is at the same position as the last digit of \( y_n \).

If \( n \geq 3 \), the sum \( y_1 + y_2 + \cdots + y_n \) has the last digit at the same position modulo 3 as \( y_1 + y_n \), by \( c_2 \). Since \( y_n \) and \( y_1 \) have disjoint supports, the last digit of \( y_1 + y_n \) is at the same position as \( y_n \). Similarly as above, the last digit of the sum \( y_1 + y_2 + \cdots + y_n \) is either at the same position as the last digit of \( y_n \), or one position greater. We conclude that the last digit of \( y_1 + y_2 + \cdots + y_n \) is at the same position as the last digit of \( y_n \), for \( n \geq 3 \).

Finally, we look at \( y_1 + y_2 \). Its last digit is either at the same position as the last digit of \( y_2 \), or one position greater. By \( c_2 \), the position of its last digit has to agree modulo 3 with the position of the last digit of \( y_1 + y_3 \), which is the position of the last digit of \( y_3 \), by disjointness. However, by \( c_0 \), \( y_2 \) and \( y_3 \) have the last digit at the same position modulo 3. We conclude that the last digit of \( y_1 + y_2 \) is at the same position as the last digit of \( y_2 \).

Thus, we have that for any \( 1 \leq m \leq n \), the position of the last digit of \( y_m + \cdots + y_n \) is the position of the last digit of \( y_n \).

If the sequence \( (y_n)_{n \geq 1} \) is of Type A, then the position of the first digit of \( y_m + \cdots + y_n \) is the position of the first digit of \( y_m \) for all \( 1 \leq m \leq n \). Combining these two observations we obtain that by passing to a block subsequence, we also obtain a Type A sequence.

If the sequence is of Type B, the first digit of \( y_m + \cdots + y_n \) is at the same position as the first digit of \( y_m \) if \( m > 1 \). If \( m = 1 \), then \( y_1 + \cdots + y_n \) has to start at the same position as some other term \( y_t + y_{t+1} + \cdots + y_{t+s} \), where \( t \geq n + 1 \). This is because \( (y_n)_{n \geq 1} \) is of Type B and thus cannot have any block subsequence of Type A, which can always be constructed from a sequence with terms starting at different positions. Since the last digit of this sum is at the same position as the last digit of \( y_n \), which is less than the position of the first digit of \( y_{n+2} \), we must have that the first digit of \( y_1 + \cdots + y_n \) is at the same position as the first digit of \( y_{n+1} \). This shows that any block subsequence is of Type B.

\( \square \)

We note that Claim 3 also implies that the last digit of any sum is at the same position as the last digit of its biggest term.

Stage 2. Let \( a, b \) with \( a < b \) be a pair of natural numbers. We write \( a \) and \( b \) in binary and we call a position \( i \) a ‘2’ if both \( a \) and \( b \) have at position \( i \) the digit 1. We call a position \( i \) a ‘1’ if exactly one of \( a \) and \( b \) has at position \( i \) the digit 1. We define the number of ‘2 to 1’-jumps of \( (a, b) \), denoted by \( J(a, b) \), to be the number of transitions from a ‘2’ to a ‘1’ as we traverse the positions in increasing order, ignoring the positions where both numbers have a ‘0’. For example, if \( a = 100000100 \) and \( b = 1101010111 \), then the positions labelled ‘1’ are 0, 1, 4, 6 and 9, the positions labelled ‘2’ are 2 and 8 and the positions ignored are 3, 5 and 7. Thus the number of ‘2 to 1’-jumps is 2, namely the jump from position 2 to position 4 and the jump from position 8 to position 9.

Let \( c \) be a natural number and \( c = l_p l_{p-1} \cdots l_1 \) its binary representation. We call a binary string not containing a ‘0’, \( a_s \cdots a_1 = 1 \cdots 1 \), an interval of \( c \) if there exits \( 1 \leq i \leq p - s + 1 \) such that \( l_{i+s-1} \cdots l_i = a_s \cdots a_1, l_{i-1} = 0 \) or \( i = 1 \), and \( l_{i+s} = 0 \) or \( i + s = p + 1 \). We denote by \( I(c) \) the number of intervals of \( c \), counted with multiplicity. For example, if \( c = 111011101011 \), then \( I(c) = 5 \) since \( c \) has two intervals of length 3 and three intervals of length 1.

We now incorporate this into the colouring: we define a new colouring by colouring \( (a, b) \) by \( (c_0, c_1, c_2, c_3, c_4, c_5) \) where \( c_0, c_1, c_2 \) and \( c_3 \) are defined above, and \( c_4 = J(a, b) \mod 2, c_5 = I(b - a) \mod 2 \), with \( c_4, c_5 = 0 \) or 1. So if \( (y_n)_{n \geq 1} \) satisfies the conditions in Theorem 4 for this new colouring, then it has all the properties we have already established, in addition to any new properties that may be forced by the new part of the colouring.
We say that two numbers $a$ and $b$, with $a < b$, have right to left disjoint supports if the last digit of $a$ is at a position smaller than the position of the first digit of $b$.

**Claim 4.** By passing to a block subsequence, we may assume that there is no $i \in \mathbb{N}$ such that both the pair $y_i, y_{i+1}$ and the pair $y_{i+1}, y_{i+2}$ have right to left disjoint supports.

**Proof.** Assume that such an $i$ exists. As the cases $i = 1$ and $i = 2$ are slightly different to the general case, we analyse them separately.

1. If $i = 1$, we look at the colour of the pairs $(y_1 + y_3, y_1 + y_2 + y_3 + y_4)$ and $(y_1 + y_2 + y_3, y_1 + y_2 + y_3 + y_4)$, which have to be the same colour. In particular, the value of $J \mod 2$ has to be the same. However, when we add $y_2$ to $y_1 + y_3$, we eliminate exactly one ‘2 to 1’-jump, namely the one where we moved from the last ‘2’ in the support of $y_1$ to the first ‘1’ in the support of $y_2$. By the disjointness of the supports, $y_2$ does not interact with $y_1$ or $y_3$, so indeed the value of $J$ changes by exactly 1, a contradiction.

2. If $i = 2$, we do not necessarily have that the supports of $y_1$ and $y_2$ are right to left disjoint. However, we observe that our colouring requires that the position of the last digit of $y_1 + y_2 + y_3$ is the position of the last digit of $y_3$. This tells us that $y_1 + y_2 + y_3$ and $y_4$ have right to left disjoint supports. By replacing $y_1$ with $y_1 + y_2 + y_3$ and relabelling the rest of the sequence, we may assume that $y_1$ and $y_2$ have right to left disjoint supports. With this assumption, if $y_2, y_3$ and $y_4$ still have right to left disjoint supports, we see that (with this choice of block subsequence) we are back in Case 1.

3. If $i > 2$, then we look at the colour of the pairs $(y_1 + y_2 + \cdots + y_i + y_{i+2}, y_1 + y_2 + \cdots + y_{i+3})$ and $(y_1 + y_2 + \cdots + y_{i+2}, y_1 + y_2 + \cdots + y_{i+3})$. As we argued above, the position of the last digit of $y_1 + y_2 + \cdots + y_i$ is the position of the last digit of $y_i$. This means that $y_1 + y_2 + \cdots + y_i, y_{i+1}$ and $y_{i+2}$ have right to left disjoint supports, thus this case is analogous to Case 1.

Therefore, we cannot have 3 consecutive terms with right to left disjoint supports.

**Claim 5.** By passing to a block subsequence, we may assume that the sequence $(y_n)_{n \geq 1}$ contains no two consecutive terms with right to left disjoint supports.

**Proof.** Using Claim 4, we construct the new block subsequence $(z_n)_{n \geq 1}$ inductively, with each term being either a $y_i$ or a sum of two consecutive $y_i$. If $y_1$ and $y_2$ do not have right to left disjoint supports, we do not change them. If they do, then $y_2$ and $y_3$ do not have right to left disjoint supports. We then replace $y_1$ by $y_1 + y_2$ and relabel the sequence. Thus now the first and the second terms do not have right to left disjoint supports.

Assume we have built our block subsequence up to the $i$th term: thus we have $(z_n)_{n=1}^i$, with $z_i$ being the sum of at most two consecutive terms of our original sequence. Thus $z_i = y_k + y_{k+1}$ or $z_i = y_{k+1}$, for some $k \in \mathbb{N}$. If $y_{k+1}$ and $y_{k+2}$ do not have right to left disjoint supports, then we let $z_{i+1} = y_{k+2}$. If on the other hand $y_{k+1}$ and $y_{k+2}$ do have right to left disjoint supports, then $y_{k+2}$ and $y_{k+3}$ do not have right to left disjoint supports, so we replace $z_i$ by $z_i + y_{k+2}$ and set $z_{i+1} = y_{k+3}$. We note that since $y_{k+1}$ and $y_{k+2}$ have right to left disjoint supports, by Claim 4 this implies that $y_k$ and $y_{k+1}$ do not have right to left disjoint supports, thus, by our inductive construction we must have $z_i = y_{k+1}$. Hence, when we perform the inductive step in this case, $z_i$ will be replaced by $y_{k+1} + y_{k+2}$, a sum of two consecutive terms of our original sequence. This gives us our block subsequence up to the $(i + 1)th$ term.
Note that Claim 5 is true for any block subsequence of \((y_n)_{n \geq 1}\) as well. This is an immediate consequence of Claim 2 and the fact that Claim 2 is preserved by passing to any block subsequence.

For a natural number \(n\), we denote the positions of its last and first digits by \(l_n\) and \(f_n\), respectively. Let \(a < b\) with \(f_a < f_b\), \(l_a < l_b\), and \(a\) and \(b\) having disjoint supports, but not right to left disjoint supports. We define a fragment of \(b\) in \(a\) to be a maximal binary string in \(a + b\) that appears in \(b\) at the same positions, has the digit 1 at its last position, and is situated between \(l_a\) and \(f_b\) inclusive. More formally, let \(a + b = r_k r_{k-1} \cdots r_1\), \(b = b_k b_{k-1} \cdots b_1\) and \(a = a_k a_{k-1} \cdots a_1\) be the binary representations of \(a + b\), \(b\) and \(a\). A binary string \(s_k s_{k-1} \cdots s_1\) is called a fragment of \(b\) in \(a\) if there exists a positive integer \(t\) such that \(k + t - 1 \leq l_a\), \(t \geq f_b\), \(r_{k+t-1} r_{k+t-2} \cdots r_1 = b_k b_{k-1} \cdots b_1\) and \(t = f_b\), or \(t \neq f_b\) and there exists no binary string \(w_d \cdots w_1\) with \(w_d = 1\) such that \(w_d \cdots w_1 = b_{k+t+d-1} \cdots b_{k+t} = r_{k+t+d-1} \cdots r_{k+t}\) and \(k + t + d - 1 \leq l_a\). We sometimes refer to these fragments as the right fragments of \(b\) in \(a\). Similarly, a binary string \(s_p s_{p-1} \cdots s_1\) is called a fragment of \(a\) in \(b\) if there exists a positive integer \(l\) such that \(p + l - 1 \leq l_b\), \(l \geq f_a\), \(r_p r_{p-1} r_{p-2} \cdots r_1 = a_p a_{p-1} \cdots a_1 = s_p s_{p-1} \cdots s_1\), \(r_{p+l} = a_{p+l-1} = 1\), \(r_{l-1} \neq a_{l-1}\) or \(l = f_b\), and there exists no binary string \(v_e \cdots v_1\) with \(v_e = 1\) such that \(v_e \cdots v_1 = a_{p+l+e-1} \cdots a_{p+l} = r_{p+l+e-1} \cdots r_{p+l}\) and \(p + l + e - 1 \leq l_a\). We sometimes refer to these fragments as the left fragments of \(a\) in \(b\). Note that there is always at least one left fragment of \(a\) in \(b\) and at least one right fragment of \(b\) in \(a\), because \(a\) and \(b\) have disjoint supports but not right to left disjoint supports. The picture below illustrates this definition in the case where there is only one fragment.

\[
\begin{align*}
\text{a} & \quad \text{left fragment of a in b} & \quad \text{b} & \quad \text{right fragment of b in a}
\end{align*}
\]

Now let \(a < b < c\) with the property that \(l_a < l_b < l_c\), \(f_a < f_b < f_c\), \(l_a + 1 < f_c\), and such that they have disjoint supports, but the pairs \((a, b)\) and \((b, c)\) do not have right to left disjoint supports. The fragments of \(b\) with respect to \(a\) and \(c\) are the fragments of \(b\) in \(a\) together with the fragments of \(b\) in \(c\). Whenever we count fragments, we count them with multiplicity — so for example, if the string 10110 occurs as a fragment in two different places, then we count this as two fragments. Note that fragments do not overlap by the maximality condition.

Let \(p < r < s\) be three natural numbers with \(f_p < f_r < f_s\), \(l_p < l_r < l_s\), \(l_p + 1 < f_s\). We define the centre of \(r\) with respect to \(p\) and \(s\) to be the binary string in \(r\) situated strictly between \(l_p\) and \(f_s\). We note that the centre of \(r\) cannot be the empty string, although, unlike a fragment in the disjoint case, it can certainly be a string of `0`'s.

The picture below illustrates the concepts we have just defined. The fragments are with respect to the three numbers \(a\), \(b\) and \(c\). For example, the centre of \(b\) is the centre of \(b\) with respect to \(a\) and \(c\). Here there is only one left fragment of \(a\) and only one right fragment of \(b\); in general, of course, there could be several, alternating from one to the other.
When working with a sequence \((y_n)_{n \geq 1}\), we consider the fragments or the centre of a term or of a consecutive sum of terms to be with respect to its neighbours. In other words, for any \(1 < i < j\), the fragments and centre of \(y_i + y_{i+1} + \cdots + y_{j-1}\) are implicitly understood to be with respect to \(y_{i-1}\) and \(y_j\).

Let \(m\) and \(n\) be two natural numbers such that \(l_m < l_n\) and \(f_m < f_n\). For each \(i\), let \(m_i, n_i\) and \((m + n)z\) be the digits of \(m, n\) and \(m + n\) at position \(i\), respectively. When adding \(m\) and \(n\) in binary, it is convenient to refer to the minimal interval in which all binary carrying occur as the carry region or just the carry. More precisely, the carry region starts at the least \(i\) for which \(m_i = n_i = 1\), and stops at position \(k\), where \(k\) is the maximum \(i\) such that \((m + n)_{i+1} \neq m_i + n_i\). For example, if \(m = 10011010010\) and \(n = 101001101100\), then the carry starts at position 4 and stops at position 9.

**Claim 6.** There exists no \(i \geq 1\) with the following property: \(y_i, y_{i+1}, y_{i+2}, y_{i+3}\) and \(y_{i+4}\) have pairwise disjoint supports and each of the centres of \(y_{i+1}, y_{i+2}, y_{i+3}, y_{i+4}, y_{i+1} + y_{i+2}\) and \(y_{i+2} + y_{i+3}\) are a string of ‘1’s.

**Proof.** Suppose for a contradiction that such an \(i\) exists. Let \(y_{i+1}\) have \(k_1\) intervals (that is, \(k_1\) disjoint strings of ‘1’s) between the position of the last digit of \(y_i\) and the position of its first digit inclusive, and \(k_2\) intervals between the position of its last digit and the position of the first digit of \(y_{i+2}\) inclusive. Because we assumed the centre of \(y_{i+1} + y_{i+2}\) is a string of ‘1’s, we get that \(y_{i+1}\) and \(y_{i+2}\) complement each other between the position of the first digit of \(y_{i+2}\) and the position of the last digit of \(y_{i+1}\) inclusive. Therefore, \(y_{i+2}\) has \(k_2\) intervals between these 2 positions too. Similarly, if \(y_{i+2}\) has \(k_3\) intervals between the position of its last digit and the position of the first digit of \(y_{i+3}\), then so does \(y_{i+3}\). Finally, let \(y_{i+3}\) have \(k_4\) intervals between the position of its last digit and the position of the first digit of \(y_{i+4}\) inclusive.

The reader might find the diagram below helpful, where the two dotted fragments are intervals as a result of \(y_{i+1}\) and \(y_{i+2}\) complementing each other in order to have an interval as the centre of \(y_{i+1} + y_{i+2}\). In the example below, we have \(k_2 = 1\), and only one right fragment of \(y_{i+1}\) in \(y_i\) that contains \(k_1\) fragments. The number \(k_1\) does not depend on the number of such fragments: it is the sum of the number of intervals in the fragments.

\[
\begin{align*}
 & y_i & y_{i+1} & y_{i+2} & y_{i+3} & y_{i+4} \\
 & \text{interval} & \text{interval (centre of } y_{i+1} ) & \text{interval} & \text{interval} & \text{interval} \text{here} \\
 & 11\ldots1 & 11\ldots1 & 11\ldots1 & 11\ldots1 & 11\ldots1 \\
 & y_{i+1} & y_{i+2} & y_{i+3} & y_{i+4} & y_{i+1} + y_{i+2} \\
 & 1 + k_1 + k_2 & 1 + k_2 + k_3 & 1 + k_3 + k_4 & 1 + k_1 + k_2 + k_3 + k_4 & 1 + k_1 + k_2 + k_3 + k_4 + 2 \text{ mod } 2
\end{align*}
\]

Since each centre is an interval and all numbers have disjoint supports, we get that \(y_{i+1}\) has \(1 + k_1 + k_2\) intervals, \(y_{i+2}\) has \(1 + k_2 + k_3\) intervals, \(y_{i+3}\) has \(1 + k_3 + k_4\) intervals, \(y_{i+1} + y_{i+2}\) has \(1 + k_1 + k_2\) intervals, and \(y_{i+1} + y_{i+3}\) has \(k_1 + k_2 + k_3 + k_4 + 2\) intervals since \(y_{i+1}\) and \(y_{i+3}\) have disjoint right to left supports that are at least one position apart.

By looking at the \(I\) value of these numbers, \(c_5\) tells us that

\[1 + k_1 + k_2 \equiv 1 + k_2 + k_3 \equiv 1 + k_3 + k_4 \equiv 1 + k_1 + k_3 \equiv k_1 + k_2 + k_3 + k_4 + 2 \mod 2.\]

The first four equations imply that \(k_1, k_2, k_3\) and \(k_4\) have the same parity. Hence \(k_1 + k_2 + k_3 + k_4 + 2\) is even, which implies that \(k_1 + k_2 + 1\) is even, a contradiction. \(\square\)
It is important to note that Claim 6 implies that our sequence \((y_n)_{n \geq 1}\), and thus any of its block subsequences, cannot be of Type B. Indeed, if the sequence \((y_n)_{n \geq 1}\) is of Type B, then so are all of its block subsequences, and so the first digit of \(y_1 + y_2 + \cdots + y_k\) is at the same position as the first digit of \(y_{k+1}\) for all \(k \geq 1\). If we first look at \(y_1, y_2\) and \(y_3\), we note that the above conditions imply that the centre of \(y_2\) has to be an interval, otherwise the carry in \(y_1 + y_2\) would stop before the position of the first digit of \(y_3\). Moreover, if we look at the block subsequence obtained by just replacing \(y_2\) with \(y_2 + y_3\), we must also have that the centre of \(y_2 + y_3\) is an interval. This immediately implies that \(y_2\) and \(y_3\) must have disjoint supports, otherwise the first position they both have a ‘1’ at will become a ‘0’ in \(y_2 + y_3\), as well as being part of the centre.

Recapping, we have shown that if \((y_n)_{n \geq 1}\) is of Type B, then the centre of \(y_2\) (with respect to \(y_1\) and \(y_3\)) is an interval, the centre of \(y_2 + y_3\) (with respect to \(y_1\) and \(y_4\)) is an interval, and \(y_2\) and \(y_3\) have disjoint supports. Passing to the block subsequence \(y_1 + y_2, y_3, y_4, \ldots\) and repeating the argument, we find that the centre of \(y_3\) (with respect to \(y_1 + y_2\) and \(y_4\)) is an interval, and \(y_1 + y_2\) and \(y_3\) have disjoint supports. By Claim 3, the position of the last digit of \(y_1 + y_2\) is the same as that of \(y_2\), so the centre of \(y_3\) with respect to \(y_1 + y_2\) and \(y_4\) is the same as the centre of \(y_3\) (with respect to \(y_2\) and \(y_4\)), and similarly for \(y_3 + y_4\). Continuing inductively, we obtain that for all \(n \geq 2\) the centres of \(y_n\) and \(y_n + y_{n+1}\) are intervals, and the terms \(y_n\) and \(y_{n+1}\) have disjoint supports, which contradicts Claim 6.

Therefore, we can guarantee that in what follows all sequences are of Type A.

Claim 7. There exists no \(i \in \mathbb{N}\) such that \(y_i, y_{i+1}, y_{i+2}, \ldots, y_{i+15}\) have pairwise disjoint supports.

Proof. Suppose for a contradiction that such an \(i\) exists. We will find a block subsequence of \((y_n)_{n \geq 1}\) that will not satisfy the conditions in Theorem 4, a contradiction. By Claim 2, we know that if three consecutive terms have disjoint supports, then the positions between the first and the last digit of their sum inclusive can be partitioned into fragments such that each fragment corresponds to exactly one term \(y_i\), as illustrated below.

\[
\begin{array}{cccc}
& & \text{the centre of } y_{i+1} & \\
y_i & \cdots & y_i & \\
y_{i+1} & y_{i+1} & \cdots & y_{i+1} \\
y_{i+2} & y_{i+2} & \cdots & y_{i+2} \\
\end{array}
\]

We immediately observe that every fragment in the picture, except for the centre of \(y_{i+1}\), has to contain the digit 1, by definition of fragments.

As we noted above, the centres can be strings of ‘0’. However, since the last digit of \(y_{i+1}\) is contained in the centre of \(y_{i+1} + y_{i+2}\) that sits between \(y_i\) and \(y_{i+3}\), we can replace \(y_{i+1}\) with \(y_{i+1} + y_{i+2}\), \(y_{i+2}\) with \(y_{i+3} + y_{i+4}\), \(y_{i+3}\) with \(y_{i+5} + y_{i+6}\), \(y_{i+4}\) with \(y_{i+7} + y_{i+8}\), and relabel the sequence. Thus, by passing to a block subsequence, we may assume that we can find nine consecutive terms, \(y_k, y_{k+1}, y_{k+2}, y_{k+3}, y_{k+4}, \ldots, y_{k+8}\), such that they have disjoint supports and the centre of \(y_{k+1}, y_{k+2}, \ldots, y_{k+7}\) all contain the digit 1.

The next step is to look at what happens with the sum \(y_1 + y_2 + \cdots + y_k\). We know that, by disjointness, at the position of the first digit of \(y_{k+1}\), \(y_k\) has a ‘0’. If the centre of \(y_k\) contains at least one ‘0’, or \(k = 1\), then the sum \(y_1 + y_2 + \cdots + y_k\) and \(y_{k+1}\) have the same fragment interaction as \(y_k\) and \(y_{k+1}\) (in other words, the fragments of \(y_k\) in \(y_{k+1}\) are the same as the fragments of \(y_1 + y_2 + \cdots + y_k\) in \(y_{k+1}\), and the fragments of \(y_{k+1}\) in \(y_k\) are the same as the fragments of \(y_{k+1}\) in \(y_{k+1}\).
in $y_1 + y_2 + \cdots + y_k$) since the carry stops before the fragments start, and when $k = 1$ there is no carry to consider as the above sum is just $y_1$. Here we used the fact that the last digit of $y_1 + y_2 + \cdots + y_{k-1}$ is at the same position as the last digit of $y_{k-1}$ for $k \geq 2$.

However, Claim 6 tells us that amongst five consecutive terms with disjoint supports, we can always find one or a sum of two consecutive terms that does not have the centre a string of ‘1’s (since Claim 6 is invariant under taking block subsequences). Therefore, by passing to a block subsequence or ignoring some previous terms, we can assume that the centre of $y_k$ is not an interval, or $k = 1$.

Finally, by passing to a block subsequence, we may assume that we can find 5 consecutive terms $y_{t+1}, y_{t+2}, y_{t+3}$ and $y_{t+4}$ such that the centres of $y_{t+1}, y_{t+2}$ and $y_{t+3}$ each contain at least one ‘1’, $y_1 + y_2 + \cdots + y_t$ interacts with the fragments of $y_{t+1}$ the same way $y_t$ does, and all 5 terms have pairwise disjoint supports.

We now look at the value of $J$ for the following pairs: $(y_1 + \cdots + y_{t+3}, y_1 + y_2 + \cdots + y_{t+4})$, $(y_1 + \cdots + y_t + y_{t+2} + y_{t+3}, y_1 + y_2 + \cdots + y_{t+4})$, $(y_1 + \cdots + y_t + y_{t+1} + y_{t+3}, y_1 + y_2 + \cdots + y_{t+4})$ and $(y_1 + \cdots + y_{t+3}, y_1 + \cdots + y_{t+4})$. Let $y_{t+1}$ have $l_{t+1}$ fragments on its left and $r_{t+1}$ fragments on its right. We define $r_{t+2}, r_{t+3}, l_{t+2}$ and $l_{t+3}$ similarly. We note that $y_{t+1} + y_{t+2}$ has $l_{t+2}$ fragments on its left and $r_{t+1}$ fragments on its right. We also note, by the definition of fragments, that $r_{t+2} = l_{t+1}$.

If we look at the first pair above, the term $y_{t+1} + y_{t+2}$ is missing from the first sum. So the non-zero digits in its fragments will all be labelled ‘1’. Therefore, its right fragments will give $r_{t+1} + l_{t+2} + 1$ jumps, while its left fragments will give $l_{t+2}$ jumps. Hence, the missing term gives $r_{t+1} + l_{t+2} + 1$ jumps. Similarly for the next two pairs, the missing terms give $r_{t+1} + l_{t+1} + 1$ and $r_{t+2} + l_{t+2} + 1$ jumps, respectively. For the last pair, there is no missing term, so the jumps come from the interaction between $y_{t+3}$ and $y_{t+4}$, which is identical for the other three pairs by disjointness. All the other digits in all four pairs remain unchanged.

The explanation above is summarised as follows:

\[
J(y_1 + \cdots + y_t + y_{t+3}, y_1 + \cdots + y_{t+4}) - J(y_1 + \cdots + y_{t+3}, y_1 + \cdots + y_{t+4}) = r_{t+1} + l_{t+2} + 1,
\]
\[
J(y_1 + \cdots + y_t + y_{t+2} + y_{t+3}, y_1 + \cdots + y_{t+4}) - J(y_1 + \cdots + y_{t+3}, y_1 + \cdots + y_{t+4}) = r_{t+1} + l_{t+1} + 1,
\]
\[
J(y_1 + \cdots + y_t + y_{t+1} + y_{t+3}, y_1 + \cdots + y_{t+4}) - J(y_1 + \cdots + y_{t+3}, y_1 + \cdots + y_{t+4}) = r_{t+2} + l_{t+2} + 1.
\]

Since our coloring asks for the $J$ values to have same parity, we need $0 \equiv r_{t+1} + l_{t+2} + 1 \equiv r_{t+1} + l_{t+1} + 1 \equiv r_{t+2} + l_{t+2} + 1 \pmod{2}$. Because $r_{t+2} = l_{t+1}$, the last equation tells us that $l_{t+2}$ and $l_{t+1}$ have different parities. However, by taking the difference of the first two equations, we must have that they have the same parity, a contradiction.

\[\square\]

Claim 8. By passing to a block subsequence, we may assume that the sequence $(y_n)_{n \geq 1}$ contains no two consecutive terms with disjoint supports.

Proof. The same as the proof of Claim 5. \[\square\]

Claim 9. By passing to a block subsequence, we may assume that for every $n \geq 1$ the carry in any sum where the biggest term is $y_n$, stops before the position of the first digit of $y_{n+1}$.

Proof. As in Claim 7, it is enough to show that for every $n \geq 2$, the centre of every $y_n$ contains at least one ‘0’. We will prove this by induction, replacing terms by consecutive sums and relabelling,
and also bearing in mind that our initial sequence does not have any two consecutive terms with disjoint supports. Assume we have built the sequence with the desired property up to the $n$th term. The terms $y_{n+1}$ and $y_{n+2}$ are consecutive terms of the original sequence, so their supports are not disjoint. If the centre of $y_{n+1}$ contains a ‘0’, then we have found the $(n+1)$th term. If it does not contain a ‘0’, then we take $y_{n+1} + y_{n+2}$ to be the $(n+1)$th term. To see that this satisfies the claim, we notice that since $y_{n+1}$ and $y_{n+2}$ do not have disjoint supports, the first position at which both have a ‘1’, becomes a ‘0’ in $y_{n+1} + y_{n+2}$. As the sequence $(y_n)_{n \geq 1}$ is of Type A, we see that that position is part of the centre of $y_{n+1} + y_{n+2}$. Note that the base case $n = 2$ is the same as the induction step. Thus the claim is proved.

Note that the condition in Claim 9 is invariant under passing to a block subsequence. We also note that the property that no two consecutive terms have disjoint supports is not necessarily preserved by passing to a block subsequence. We also observe that we have altered the sequence in Claim 8 that was assumed not to have two consecutive terms with disjoint supports, and obtained one such that the carry of any sum with biggest term $y_n$ stops before the support of $y_{n+1}$ begins. Further, this property is preserved by passing to a block subsequence. Therefore, starting with a sequence $(y_n)_{n \geq 1}$ with this property, we can repeat the process in Claims 7 and 8 again and assume that $(y_n)_{n \geq 1}$ has both the property that the binary carry of any sum stops before the support of the next term starts, and also the property that no two consecutive terms have disjoint supports. These two properties together are invariant under our standard operation of passing to a block subsequence (noting that the property of ‘consecutive terms do not have disjoint supports’ is preserved because the carry resulting from any earlier additions is guaranteed to stop before the supports overlap).

For a sequence $(z_n)_{n \geq 1}$ that is of Type A and has the two properties we have stated in the previous paragraph, we define $j_n$, for $n \geq 2$, to be the maximum of the position of where the carry of $z_n + z_{n-1}$ stops (or equivalently any finite sum of the $z_i$ with greatest terms $z_n$ and $z_{n-1}$) and the position of the last digit of $z_{n-1}$. For completeness, we set $j_1$ to be one less than the position of the first digit of $y_1$. We also define the middle of $z_n$ to be the (possible empty) binary string contained strictly between $j_n$ and the position of the first digit of $z_{n+1}$. We call the middle of $z_n$ proper if it is non-empty and it contains at least one non-zero digit. Finally, we define the overlapping zone of $z_n$ and $z_{n+1}$ to be the consecutive set of positions between the position of the first digit of $z_{n+1}$ and $j_{n+1}$ inclusive.

Claim 10. By passing to a block subsequence, we may assume that the middle of $y_n$ is proper for all $n \geq 2$.

Proof. We prove the claim by induction. Assume that all terms up to $y_{n-1}$, $n \geq 3$, have a proper middle. If $y_n$ has a proper middle, then we move on to the next term. If $y_n$ does not have a proper middle, then $y_n + y_{n+1}$ has a proper middle with respect to $y_{n-1}$ and $y_{n+2}$. This is because at position $j_{n+1}$ in the sum $y_n + y_{n+1}$ we find the digit 1 by definition. Note that by Claim 9 the ‘new $j_n$’ (corresponding to $y_n + y_{n+1}$) is equal to the ‘old $j_n$’ (corresponding to $y_n$). Also, $j_{n+1}$ is less than the position of the first digit of $y_{n+2}$ and, by construction, $j_{n+1} > j_n$. Thus $y_n + y_{n+1}$ does have a proper middle. Therefore we take the $n$th term to be $y_n + y_{n+1}$, and relabel the rest of the sequence, thus complete the induction step. We note that the same argument directly gives that the middle of $y_2$ can be assumed to be proper, which finishes the proof. □
Note that, given that a sequence satisfies the conditions of Claim 9, the conditions in Claim 10 are invariant under taking block subsequences. By earlier remarks, we may now therefore assume that our sequence satisfies Claims 8–10.

Stage 3. We now add a final piece of notation. For positive integers \(a\) and \(b\), that do not have disjoint supports, consider the positions where binary carries occur in the sum \(a + b\). Those positions form some intervals which we call the *carry intervals* of \(a\) and \(b\). For example, if \(a = 110 100 010\) and \(b = 10 100 111\), then the carry intervals are \(\{1, 2, 3\}\), \(\{5, 6\}\) and \(\{7, 8, 9\}\).

Let \(m < n\) be two positive integers such that \(m\) and \(n - m\) do not have disjoint supports. We label a position by ‘2’ if it is not part of any carry interval of \(m\) and \(n - m\), and both \(m\) and \(n\) have the digit 1 at that position. Also, we label a position by ‘1’ if it is not part of any carry interval of \(m\) and \(n - m\), and exactly one of \(m\) and \(n\) has a non-zero digit at that position. Let \(\tilde{J}(m, n)\) be the number of jumps from a position labelled ‘2’ to a position labelled ‘1’, as we read the labels from right to left (ignoring the positions that do not have labels).

Returning to our sequence, let \(y'_n\) be the number obtained from \(y_n\) by changing all the digits in the carry intervals of \(y_n\) and \(y_{n+1}\), and in the carry intervals of \(y_n\) and \(y_{n+1}\), to 0, for each \(n > 1\). Let also \(y'_1\) be the number obtained from \(y_1\) by changing all the digits in the carry interval of \(y_1\) and \(y_2\) to 0. Note that the new sequence \((y'_n)_{n \geq 1}\) is still increasing and of Type A as a consequence of Claim 10, and that its terms have pairwise disjoint supports.

With this in mind, our final colouring is: we colour \((a, b)\) by \((c_0, c_1, c_2, c_3, c_4, c_5, c_6)\), where \(c_0, c_1, c_2, c_3, c_4, c_5\) are defined above, and \(c_6 = J(a, b) \mod 2\), with \(c_6 = 0\) or \(1\), if \(a\) and \(b = a\) do not have disjoint supports, and \(c_6 = 3\) if \(a\) and \(b = a\) have disjoint supports.

Let \((y_1 + y_{k_1} + \cdots + y_{k_i}, y_1 + \cdots + y_{k_i+1})\) be any of the pairs that have the same colour. We first observe that, by Claims 8 and 9, \(y_1 + y_{k_1} + \cdots + y_{k_i}y_1 + \cdots + y_{k_i+1} - (y_1 + y_{k_1} + \cdots + y_{k_i}) = y_2 + \cdots + y_{k_i-1} + \cdots + y_{k_i+1} + \cdots + y_{k_i+1}\) never have disjoint supports — for example, \(y_{k_i}\) and \(y_{k_i+1}\) do not have disjoint supports and, in the above sums, they are unchanged in their overlapping zone. We therefore have that \(c_6 \neq 3\). Moreover, \(J(y_1 + y_{k_1} + \cdots + y_{k_i}, y_1 + \cdots + y_{k_i+1}) = J(y'_1 + y'_{k_1} + \cdots + y'_{k_i}, y'_1 + \cdots + y'_{k_i+1})\), and so the same argument as in Claim 7 gives us a contradiction. This completes the proof of Theorem 4.

\[\square\]

4 | OPEN PROBLEMS

The colouring of \(\mathbb{N}^{(2)}\) above, constructed in the previous section, involves colouring pairs. But can Theorem 4 be solved by a colouring that comes in a natural way just from a colouring of numbers? In particular, what happens if we promise that our colouring for Theorem 4 gives \((a, b)\) a colour that depends only on the value of \(a + b\)?

In this case, the sum \(a + b\), for a pair \((a, b)\) as in the statement of Theorem 4, is exactly a sum \(a_1y_1 + a_2y_2 + \cdots + a_ky_k\), where each \(a_i\) is 1 or 2 with \(a_k = 1\) and \(a_1 = 2\). Replacing \(y_1\) with \(2y_1\), this yields the following question.

**Question 5.** Is it true that whenever \(\mathbb{N}\) is finitely coloured, there exists a sequence \((y_n)_{n \geq 1}\) such that every sum \(a_1y_1 + a_2y_2 + \cdots + a_ky_k\), where each \(a_i\) is 1 or 2, with \(a_1 = a_k = 1\), has the same colour?

In general, such Ramsey-type statements, in which each coefficient can vary independently between some values, tend to be false. But here the fact that there are no ‘gaps’, in other words
that the $y_i$ in a given sum form an initial segment of the sequence $(y_n)_{n \geq 1}$, seems to perhaps make a difference.

We mention that if one allows $a_k$ to be 1 or 2, then the result is easily seen to be false, because one sum will be forced to be roughly double another, which can be ruled out by a suitable colouring. And if one instead allows $a_1$ to be 1 or 2, then the result is also false, by considering the 2-colouring given by the least significant non-zero digit in the base 3 expansion of a number. Finally, if one allows ‘gaps’, so that some of the $a_i$ are allowed to be zero, then it turns out that the result is again false, by using a colouring that examines the lengths of the jumps between successive elements of the support of a number: this is similar to the colourings considered in [2].

It is possible that Question 5 might be related to a problem considered by Hindman, Leader and Strauss [4]. They conjectured that whenever $\mathbb{N}$ is finitely coloured there exists a sequence $(y_n)_{n \geq 1}$ such that all finite sums of the $y_i$, and also all sums of the form $y_{n-1} + 2y_n + y_{n+1}$, are the same colour. In each of these problems, it is the fact that the terms must be consecutive (in each sum for Question 5, and for the sums $y_{n-1} + 2y_n + y_{n+1}$ in the conjecture of Hindman, Leader and Strauss) that causes the difficulty. We mention that if one attempts to strengthen the conjecture of Hindman, Leader and Strauss in almost any significant way, then the resulting statement turns out to be false: this is related to the ‘inconsistency’ of Milliken–Taylor systems (see [2] and the discussion in [4]).

Finally, returning to infinite words, what happens in Theorem 1 if we relax the condition that the factors $u_n$ form an actual factorisation of our word $x$: what if we allow some gaps between them? Could it be that we can actually allow gaps, as long as they are bounded, and still find a bad colouring? This is a natural question to ask, in light of some variants of Hindman’s theorem, such as [5, Theorem 5.23].

**Question 6.** Let $x$ be an infinite word on alphabet $X$ that is not eventually periodic. Must there exist a finite colouring of $X^*$ such that there does not exist a sequence $u_1, u_2, \ldots$ of factors of $x$, with $0 \leq A_x(u_{n+1}) - B_x(u_n) \leq C$ for all $n$ (for some $C$), such that all the words $u_{k_1}u_{k_2}\cdots u_{k_n}$, where $k_1 < k_2 \cdots < k_n$, have the same colour?

Note that if we insist that $C = 0$, then this is precisely Theorem 1.

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**REFERENCES**
1. A. de Luca and L. Q. Zamboni, *On some variations of coloring problems for infinite words*, J. Comb. Theory (Series A) 137 (2016), 166–178.
2. W. Deuber, N. Hindman, I. Leader, and H. Lefmann, *Infinite partition regular matrices*, Combinatorica 15 (1995), 333–355.
3. N. Hindman, *Finite sums from sequences within cells of a partition of \( \mathbb{N} \)*, J. Comb. Theory (Series A) **17** (1974), 1–11.
4. N. Hindman, I. Leader, and D. Strauss, *Extensions of infinite partition regular systems*, Electronic J. Combinatorics **22** (2015), 2–29.
5. N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, 2nd edn, Walter de Gruyter & Co., Berlin, 2012.
6. I. Leader, P. A. Russell, and M. J. Walters, *Transitive sets in Euclidean Ramsey theory*, J. Comb. Theory (Series A) **119** (2012), 382–396.
7. K. Milliken, *Ramsey’s theorem with sums or unions*, J. Comb. Theory (Series A) **18** (1975), 276–290.
8. M. P. Schützenberger, *Quelques problèmes combinatoires de la théorie des automates*, Cours professé à l’Institut de Programmation en 1966/67, notes by J.-F. Perrot, [http://igm.univ-mlv.fr/bers-tel/Mps/Cours/PolyRouge.pdf](http://igm.univ-mlv.fr/bers-tel/Mps/Cours/PolyRouge.pdf).
9. A. Taylor, *A canonical partition relation for finite subsets of \( \omega \)*, J. Comb. Theory (Series A) **21** (1976), 137–146.
10. C. Wojcik, *On a new conjecture about super-monochromatic factorisations and ultimate periodicity*, arXiv:1802.08670, 2018.
11. C. Wojcik and L. Q. Zamboni, *Coloring problems for infinite words, Sequences, groups and number theory*, Trends Math., Birkhauser/Springer (2018), pp. 213–231.
12. C. Wojcik and L. Q. Zamboni, *Monochromatic factorisations of words and periodicity*, Mathematika **64** (2018), 115–123.