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Adiabatic approximation for the motion of Ginzburg-Landau vortex filaments

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Abstract

In this paper, we consider the concentration property of solutions to the dispersive Ginzburg-Landau (or Gross-Pitaevskii) equation in three dimensions. On a spatial domain, it has long been conjectured that such a solution concentrates near some curve evolving according to the binormal curvature flow, and conversely, that a curve moving this way can be realized in a suitable sense by some solution to the dispersive Ginzburg-Landau equation. Some partial results are known with rather strong symmetry assumptions.

Our main theorems here provide affirmative answer to both conjectures under certain small curvature assumption. The results are valid for small but fixed material parameter in the equation, in contrast to the general practice to take this parameter to its zero limit. The advantage is that we can retain precise description of the vortex filament structure. The results hold on a long but finite time interval, depending on the curvature assumption.

1 Introduction

Consider the dispersive Ginzburg-Landau (or Gross-Pitaevskii) equation on a spatial domain $\Omega \subset \mathbb{R}^3$,

$$
\frac{i}{\epsilon} \frac{\partial \psi}{\partial t} = -\Delta \psi + \frac{1}{\epsilon^2}(|\psi|^2 - 1)\psi.
$$

This equation has its origin from condensed matter physics. The complex scalar field $\psi : \Omega \to \mathbb{C}$ describes the Bose-Einstein condensate in superfluidity, or, ignoring the effect of magnetic field, the electronic condensate in superconductivity. Here $\epsilon > 0$ is a material parameter measuring the characteristic length scale of $|\psi|$.

Realistically, the coherent length $\epsilon$ is very small compared to the size of the domain $\Omega$. Thus in what follows we take

$$
\Omega := \{(x, z) : x \in \omega \subset \mathbb{R}^2, z \in I := [0, 1]\},
$$

where $\omega$ is a bounded domain in $\mathbb{R}^2$, whose size is large compared to the scale $\epsilon$. The choice $I = [0, 1]$ can be replaced by any interval $[a, b]$ with
\[ b - a \gg \epsilon, \] which amounts to a change of coordinate along z-direction. The reason for taking a bounded domain is to circumvent certain undesirable decay properties that are not essential to our argument (see Sec. 2.1 for details). In addition, for technical reason we also assume \( \omega \) is star-shaped around the origin, and the boundary \( \partial \omega \) is smooth.

On this cylindrical domain \( \Omega \), we impose the boundary conditions \( |\psi| \to 1 \) as \( x \) approaches \( \partial \omega \) horizontally, and \( \psi(x,0) = \psi(x,1) \) for every \( x \in \omega \). Note that through the transform \( \psi = e^{i \epsilon t} u \), equation (1) becomes the cubic defocusing subcritical nonlinear Schrödinger equation. Global well-posedness for such equations has been known since the eighties [13]. Using standard blow-up arguments, one can show that for every initial configuration \( u_0 \in H^1(\Omega) \) satisfying the boundary conditions above, there exists a unique global solution \( u_t \in H^1(\Omega) \) to (1) generated by \( u_0 \). Therefore in this paper we will mainly work with Sobolev spaces \( H^k, k \geq 1 \).

1.1 The geometric structure of (1)

The dispersive Ginzburg-Landau equation (1) can be viewed as a Hamiltonian system as follows. Let \( X \subset L^2(\Omega, \mathbb{C}) \) be a suitable configuration space for (1). If we identify the space \( L^2(\Omega, \mathbb{C}) = L^2(\Omega, \mathbb{R}) \times L^2(\Omega, \mathbb{R}) \) through \( u \mapsto (\Re u, \Im u) \), then we can regard \( X \) as a real vector space, equipped with the inner product \( \langle (\Re u, \Im u), (\Re v, \Im v) \rangle = \int_{\Omega} (\Re u \Re v + \Im u \Im v) \).

Under this identification, the operator \( J : \psi \mapsto -i \psi \) can be represented by the symplectic matrix
\[
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Thus we say \( J \) is a symplectic operator, in the sense that it induces a symplectic form \( \tau = \langle J \cdot, \cdot \rangle \) on \( X \), satisfying \( \tau(w,w') = -\tau(w',w) \) for every \( w,w' \in X \).

Using the operator \( J \), we can write (1) as the Hamiltonian system
\[
\frac{\partial \psi}{\partial t} = JE'(\psi), \quad E(\psi) = E_0(\psi) := \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 + \frac{1}{4\epsilon^2} (|\psi|^2 - 1)^2. \tag{4}
\]

Here \( E \) is the Ginzburg-Landau energy, measuring the difference of Helmholtz free energy between a transition phase \( \psi \) and the normal phase. \( E'(\psi) \) denotes the \( L^2 \)-gradient of the energy functional w.r.t. the real inner product given above. This gradient \( E'(\psi) \) is given explicitly as the r.h.s. of (1).

It is well-known by now that as the material parameter \( \epsilon \to 0 \) in (1), the energy \( E_0(\psi) \) of a finite-energy configuration \( \psi \) concentrates near some lower dimensional manifold \( \gamma \). This concentration phenomenon can be made precise in measure theoretic terms. For instance, see [2,3,17]. This leads one to expect, at least at the heuristic level, that as \( \epsilon \to 0 \), (4) reduces, in an appropriate manner, to a Hamiltonian system in the space of curves,
\[
\frac{\partial \gamma}{\partial t} = JL'(\gamma), \tag{5}
\]
where \( \gamma = \gamma_t \) is a \( C^1 \) path of curves in \( \Omega \), and \( J \) is a suitable (symplectic) operator.
1.2 Outline of the main results

Suppose $\gamma$ is parametrized by arclength, $L(\gamma) = \int |\partial_\gamma \gamma|^2$, $L'(\gamma) = -\partial_\gamma \gamma + \mathcal{J}$, where $\mathcal{J} := -\partial_\gamma \gamma \times$. Then (1) becomes the binormal curvature flow (11). The purpose of this paper is to formulate rigorously the connection between the dispersive Ginzburg-Landau equation (1) and the binormal curvature flow. To this end, we first make precise the notion of concentration set in Lemma 2 for a class of low energy configurations. Then our first main result, Theorem 1, states that if $u_t$ is a solution to (1) generated by $u_0$, and $u_0$ has concentration set $\gamma_0$, then the flow $u_t$ has concentration set given in the leading order by $\gamma_t$, the flow generated by $\gamma_0$ under the binormal curvature flow (11). As a corollary, we derive the second main result, Theorem 3, which states the converse: If $\gamma_t$ is the flow generated by $\gamma_0$, then we can find some solution $u_t$ to (1) such that the concentration set associated to $u_t$ is given in the leading order by $\gamma_t$. Both main results hold on some long but possibly finite time interval depending on the initial configuration.

The precise statements of the main results of this paper are as follows:

Theorem 1. 1. For any $\epsilon > 0$, there are $\delta_1, \delta_2 < \epsilon$ such that the following holds: Let $M$ be the manifold of approximate vortex filaments as in Definition 1. Let $u_0 \in X^1 + \psi_0$ be an initial configuration in the energy space $(X^k$ and $\psi_0$ are defined in Sec. 2.3), such that $\text{dist}(u_0, M) < \delta_2$. Let $u_t$ be the flow generated by $u_0$ under (11).

Then there is some $T > 0$ independent of $\epsilon$ and $\delta_1, \delta_2$, such that for $t \leq T$, there exists a flow of moduli $\sigma_t$ associated to $u_t$ as in Lemma 2 with

$$||u_t - f(\sigma_t)||_{X^2} = o(\sqrt{\delta_1}),$$

where $f : \Sigma_{\delta_1} \rightarrow M$ is the parametrization defined in (22).

Moreover, for $t \leq T$, the flow of moduli moduli $\sigma_t$ evolves according to

$$\partial_t \sigma = J_{\sigma}^{-1} d_J E(f(\sigma)) + o_1 \|v_0\|_{\sigma}(\delta_1),$$

where the operator $J_{\sigma}$ is defined in (20).

2. For any $\beta > 0$, there exist $\delta, \epsilon_0 > 0$ such that the following holds: Let $\epsilon < \epsilon_0$ in (11). Let $\sigma_0 = (\lambda_0, \gamma_0) \in \Sigma_{\delta}$ be given, where $\Sigma_{\delta}$ is defined in (24). Let $\gamma_t$ be the flow generated by $(\gamma_0(z), z)$ under the binormal curvature flow (11).

Then there exist a solution $u_t$ to (11), and some $T > 0$ independent of $\epsilon$ and $\delta$, such that for all $t \leq T$, $X \in C^1([R^3, R^3])$ and $\phi \in C^1([R^3, R])$ with $||X||_{C^1}, ||\phi||_{C^1} = O(\delta^{-1/4})$, we have

$$\left| \int_{\Omega} X \times J u - \pi \int_{\gamma_t} X \right| \leq \beta,$$

$$\left| \int_{\Omega} \frac{e(u)}{|\log \epsilon|} \phi - \pi \int_{\gamma_t} \phi dH^1 \right| \leq \beta,$$

where for $\psi = \psi^1 + i \psi^2$,

$$e(\psi) = \frac{1}{2} |\nabla \psi|^2 + \frac{1}{4 \epsilon^2} (|\psi|^2 - 1)^2, \quad J \psi = \nabla \psi^1 \times \nabla \psi^2.$$
Remark 1. In the formulation of Theorem 1, we do not adopt the convention of taking $\epsilon \to 0$. Instead, for small but fixed $\epsilon$, the main assumption is that the concentration sets $\gamma_t$, $t \geq 0$ have uniformly small curvature. (See the definitions of the various spaces in Theorem 1 in Sec. 2.2.) This allows us to retain precise description of the vortex structure of the evolving configurations.

We obtain the results in Theorem 1 by developing an adiabatic approximation scheme for (1). This allows us to decompose a flow $u_t$ solving (1) to a slowly evolving main part $v_t$ and a uniformly small remainder $w_t := u_t - v_t$. The flow $v_t$ consists of some low energy configurations whose motion is explicitly governed by their concentration sets $\gamma_t$. The small curvature assumption on $\gamma_t$ is used here to ensure $v_t$ has low energy, which is essential for the validity of adiabatic approximation. (If the curvature is large at a point on the concentration curve, then the speed of the vortex filament at this point is large, and consequently the adiabatic approximation breaks down.)

The main technical ingredients include 1. to show that the concentration sets of $v_t$ indeed evolve according to the binormal curvature flow (41) to the leading order, assuming the the remainder $w_t$ is small and 2. to get apriori estimate on the remainder $w_t$. The second part is analogous to proving dynamical stability of solitons, except for that the configurations $v_t$ need not to be stationary.

1.3 Historical remark

Let us briefly comment on the existing literature on related problems. If the evolution (1) is instead the energy dissipative dynamics (so that (1) becomes a cubic nonlinear heat equation), rigorous result that formulates the kind of connection in which we are interested is known [3]. This is one example of a vast literature in the $\epsilon \to 0$ scheme. See an excellent review [16] of works along this line. Note our method is completely different from the measure theoretic schemes for the $\epsilon \to 0$ limit. One can also ask similar questions about the hyperbolic and gauged analogues of (1). The method we develop in the present paper is applicable to the latter cases, which we will treat elsewhere.

The problem we study here is motivated by a conjecture of R.L. Jerrard [16, Conj. 7.1]. The method we use here is inspired by a series of papers by I.M. Sigal with several co-authors [5, 11, 12, 14, 19, 20]. Other results along this line which we have referred to include [6–10, 15, 23, 26–28]. Most of these papers consider planar problems (possibly after some symmetry reduction), or higher dimensional problems with concentration sets given by finite collections of points. In this regard, the geometry of our problem is more involved.

1.4 Arrangement

The structure of this paper is as follows: In Section 2, we construct the manifold $M$ consisting of what we call approximate vortex filaments. These are low energy configurations with explicit concentration sets. We find a path of appropriate approximate filaments $v_t$ as the “adiabatic
part" of a given solution \( u_t \) to (1), provided \( u_t \) stays uniformly close to \( M \). In Section 3, we show that the path \( v_t \) as above is governed by the bi-normal curvature flow (41). We prove then two main theorems by finding an apriori estimate for \( w_t := u_t - v_t \), valid on a long but finite interval. The proofs rely on the adiabaticity of the approximate filaments, as well as the energy conservation for (1). In Section 4, we discuss the properties of certain linearized operators involved in the preceding analysis. In Appendix, we recall some basic concepts of Fréchet derivatives that we use repeatedly, and collect various technical estimates for operators used earlier.

Notations

Throughout the paper, when no confusion arises, we shall drop the time dependence \( t \) in subscripts. A domain \( \Omega \subset \mathbb{R}^d \) is an open connected subset. The symbols \( L^p(\Omega), H^s(\Omega), 0 < p \leq \infty, s \in \mathbb{R} \) denote respectively the Lebesgue space of order \( p \), and the Sobolev space of order \( s \), consisting of functions from a domain \( \Omega \) into \( \mathbb{C} \).

2 Approximate vortex filaments

In this section, we construct the manifold of approximate (vortex) filaments and discuss their key properties.

2.1 The planar 1-vortex

On the entire plane \( \mathbb{R}^2 \), it is well-known that (1) has smooth, stationary, radially symmetric solutions \( \psi^{(n)} : \mathbb{R}^2 \to \mathbb{C} \), where \( n \in \mathbb{Z} \) labels the winding number

\[
\deg \psi|_{|x|=R} = n
\]

for large \( R \gg 0 \). The characteristic feature of \( \psi^{(n)} \) is that they concentrate near the origin \( r = 0 \). Among these vortices, the simple ones with \( |n| = 1 \) are stable, and the higher order ones with \( |n| > 1 \) are unstable [21].

For this reason, in what follows we will only be concerned with the 1-vortex.

Write \( \psi^{(1)} = \varphi(r) e^{i\theta} \) in polar coordinate. Then \( \varphi \in C^\infty, \varphi' > 0 \) for \( r > 0 \), and the following asymptotics hold [22, Sec. 3.1]:

\[
\varphi \sim 1 - \frac{\epsilon^2}{2i^2} (r \to \infty), \quad \varphi \sim \frac{r}{\epsilon} - \frac{r^3}{8\epsilon} (r \to 0).
\] (10)

The planar stationary equation

\[-\Delta \psi + \frac{1}{\epsilon^2} (|\psi|^2 - 1) \psi = 0, \quad \psi : \mathbb{R}^2 \to \mathbb{C}\]

is called the (ungauged) Ginzburg-Landau equation. It has rotation, translation, and global gauge symmetries. Among these the, latter two classes are broken by \( \psi^{(1)} \). Consequently, if \( L^1_{\epsilon^2} : H^2(\omega) \to L^2(\omega) \) is the linearized operator at \( \psi^{(1)} \), then the vectors

\[
\frac{\partial \psi^{(1)}}{\partial x_j}, i\psi^{(1)}, \quad j = 1, 2
\]
are in the kernel of $L^1_\omega$. We call these vectors the symmetry zero modes. Note that these modes are not in $L^2(\mathbb{R}^2)$.

On a bounded star-shaped domain $\omega \subset \mathbb{R}^2$ around the origin, the following estimates are well-known [4, Chap. III, X]:

\begin{align}
E_\omega(\psi^{(1)}) &\leq \pi |\log \epsilon| + C(\omega), \\
\|\psi^{(1)}\|_{L^\infty(\omega)} &\leq 1, \\
\|\nabla \psi^{(1)}\|_{L^\infty(\omega)} &\leq \frac{C(\omega)}{\epsilon}, \\
\|\nabla \psi^{(1)}\|_{L^2(\omega)} &\leq C(\omega) |\log \epsilon|^{1/2}, \\
\frac{1}{\epsilon^2} \int_\omega \left( |\psi^{(1)}|^2 - 1 \right)^2 &\leq C(\omega).
\end{align}

(11) - (15)

The finite energy property (11) fails if $\omega$ is unbounded. Consequently, if $\omega$ is unbounded, then in general (say $\omega = \mathbb{R}^2$), even the simple planar vortex $\psi^{(1)}$ has infinite energy. Moreover, the translation zero modes are not $L^2$.

One can study the problem with $\omega = \mathbb{R}^2$ by either using some kind of renormalized energy [21], or posing the problem in a weighted Sobolev space [22]. Though it will incur significantly more involved estimates, we believe the arguments below can extend to the non-compact case using one of these methods.

2.2 Construction of approximate vortex filaments

To construct the manifold of approximate vortex filaments, we first define some configuration spaces. Put

\begin{align}
X^s &:= \{ \psi \in H^s(\Omega, \mathbb{C}) : \psi|_{\partial \omega \times I} = 0, \ \psi(x,0) = \psi(x,1) \text{ for every } x \in \omega \}, \\
Y^k &:= \mathbb{R} \times C^k(I, \mathbb{R}^2) \quad (s \in \mathbb{R}, \ k \in \mathbb{N}).
\end{align}

(16) - (17)

We write elements in $Y^k$ as $\sigma = (\lambda, \gamma)$. $X^s$, $Y^k$ are real Hilbert spaces with the inner products given respectively by

\begin{align}
\langle \psi, \psi' \rangle_X &= \int_\Omega \mathbb{R}(\overline{\psi} \psi') \quad (\psi, \psi' \in X^s), \\
\langle \sigma, \sigma' \rangle_Y &= \int_I \gamma \cdot \gamma' + \mu \mu' \quad (\sigma, \sigma' \in Y^k).
\end{align}

(18) - (19)

The norm on $Y^k$ is $\| (\lambda, \gamma) \|_{Y^k} := \| \gamma \|_{C^k} + |\lambda|$. The Ginzburg-Landau energy $E$ defined in (4) is smooth on $1 + X^k$ with $k \geq 1$, which we call the energy space.

Write $C^k_{\text{per}} := \{ \gamma \in C^k(I, \omega) : \gamma(0) = \gamma(1) \} \quad (k \in \mathbb{N})$.

Here we require periodic boundary condition for $\gamma$, so as to match the periodic boundary condition along the $z$-axis for (1). Indeed, we impose
such boundary conditions so that the arguments below can be naturally extended to an infinitely long cylindrical domain with $I = \mathbb{R}$. In general, one can take $I = [0, a]$ for any $a \gg \epsilon$, and impose other appropriate boundary conditions. Notice that if one varies the vertical boundary condition for (1), then the definition of $C^k_{\text{per}}$ must be changed accordingly.

Define

$$\Sigma := \mathbb{R} \times \{ \gamma \in C^\infty_{\text{per}} : z \mapsto (\gamma(z), z) \text{ is an embedding of } I \text{ into } \Omega \} ,$$

$$\Sigma_\delta := \{ \sigma \in \Sigma : \| \sigma \|_{Y^2} < \delta \} .$$

We view $\Sigma$ as a manifold. Each fibre $T_\sigma \Sigma$ can be trivialized as a subspace of $Y^0$. We view $\Sigma_\delta$ as an open submanifold of $\Sigma$.

**Definition 1** (Manifold of approximate vortex filaments). For $\gamma \in C^k_{\text{per}}$ and a function $\psi : \Omega \to \mathbb{C}$, let $\psi_\gamma(x, z) := \psi^{(1)}(x - \gamma(z))$, where $\psi^{(1)}$ is the simple planar vortex in Sec. 2.1.

Define a map

$$f : \Sigma_\delta \to X^0 + \psi_0$$

where $\psi_0 : \Omega \to \mathbb{C}$ is the lift of the planar vortex $\psi^{(1)}$ to $X^0$. The map $f$ is $C^1$, as we show in Appendix. $f$ parametrizes a submanifold

$$M := f(\Sigma_\delta) \subset X^0 + \psi_0.$$

The tangent space to $M$ at $f(\sigma)$ is $T_{f(\sigma)} M = df(\sigma)(T_\sigma \Sigma_\delta)$, where $df(\sigma)$ denotes the Fréchet derivative of $f$ at $\sigma$, given explicitly in Appendix. We call the elements in $M$ the approximate vortex filaments.

**Remark 2.** The construction of $M$ is motivated by the broken symmetries by $\psi^{(1)}$. We use the term approximate (vortex) filaments because the configurations in $M$ concentrate near some curves around $\{0\} \times I$ by construction. We can trivialize $T_{f(\sigma)} M$ as a subspace of $X^0$. The spaces $M, \Sigma$ are Riemannian manifolds w.r.t. the inner products given in (18)-(19).

### 2.3 Properties of approximate filaments

In what follows, we always assume the material parameter $\epsilon \ll 1$ in (1).

A key observation is that since $u := |\psi^{(1)}|$ (resp. $v := |
abla \psi^{(1)}|$) is strictly increasing (resp. decreasing) sufficiently away from $r = 0$, using the asymptotics in (10), we have control over the oscillation of $u$ and $v$ as

$$|u(r) - u(s)| \leq C \frac{\epsilon^2}{R^2}, \quad |v(r) - v(s)| \leq C \frac{\epsilon}{R} \quad (r > s \geq R \gg 0),$$

where $C$ is independent of $\epsilon$. 


Let $\alpha > 0$ be given such that the planar domain $\omega$ contains the ball of radius $1 + \epsilon^\alpha$. Then it is not hard to see that for $\sigma \in \Sigma_{\alpha}$,

$$\|f(\sigma)\|_{X^0}^2 = \int_{\Omega} |\psi_1|^2 \leq \int_{\Omega} \int_{\omega} |\psi_1|^2 (x, z) \, dx \, dz \leq \int_1 \int_{\omega} \left( \left| \psi_1 \right|^2 (x) \, dx + \int_{\omega} \left| \nabla_x \psi_1 \nabla z (x, z) - \left| \psi_1 \right|^2 (x) \, dx \right) \, dz \leq \int_1 \int_{\omega} \left| \psi_1 \right|^2 (x) \, dx + \|\sigma\|_{L^2} \text{diam}(\omega) \sup_{r>\gamma_{0}} |u(r) - u(s)|^2 \, dz \leq \left( \left| \psi_1 \right|^2 \right)_{L^2} + C(\omega) \epsilon^{1+\alpha}. $$

Therefore we have

$$\|\psi_1\|_{X^0} = \|\psi_1\|_{L^2(\omega)} + \Theta(\epsilon^{1+\alpha/2}). $$

Similarly, one can show using (23) that

$$\|\nabla_x \psi_1\|_{X^0} = \|\nabla_x \psi_1\|_{L^2(\omega)} + \Theta(\epsilon^{1+\alpha/2}). $$

In what follows, for a $C^4$-curve $\gamma : I \to \mathbb{R}^d$, we write $\gamma_s = \partial_t \gamma_s$, etc. (Not to be confused with the subscript $t$ in time-parametrized families.)

**Lemma 1** (approximate critical point). Let $\alpha > 0$. If $\sigma \in \Sigma_{\alpha}$, then

$$\|E'(f(\sigma))\|_{X^0} \leq C\epsilon^{\alpha} |\log \epsilon|^{1/2} \text{ where } C \text{ is independent of } \sigma. $$

**Proof.** Using the fact that $\psi^{(1)}$ is stationary solution to (1), one can compute

$$E'(f(\sigma)) = \epsilon^{i\lambda} (\nabla_x \psi_1 \cdot \gamma_{zz} - \nabla_x^2 \psi_1 \gamma_{zz} \cdot \gamma_z).$$

For $\|\gamma\|_{C^2} \ll 1$, the leading order term in this expression is $\nabla_x \psi_1 \cdot \gamma_{zz}$. One can estimate this as

$$\|\nabla_x \psi_1 \cdot \gamma_{zz}\|_{X^0} \leq 2 \|\nabla_x \psi_1\|_{X^0} \|\gamma_{zz}\|_{C^0} \leq 2 \left( \left\| \nabla_x \psi_1 \right\|_{L^2(\omega)} + C(\omega) \epsilon^{1+\alpha/2} \right) \|\gamma\|_{C^2} \leq C(\omega) \epsilon^{\alpha} |\log \epsilon|^{1/2},$$

where in the last step one uses $\sigma \in \Sigma_{\alpha}$ and the estimates (13), (23). \hfill $\Box$

To simplify notation, write $g_\sigma : Y^0 \to X^0$ for the action of $df(\sigma) : T_{\gamma} \Sigma_{\delta} \to T_{\gamma(\sigma)} M$ on each fibre. Let $g_{\sigma}^*$ be the adjoint to $g_\sigma$ w.r.t. the inner products defined in (13)-(14). In Appendix, we calculate these operators explicitly in (S7)-(S8).

**Remark 3.** Note here that 1. $g_\sigma$ is injective, and therefore $f$ is an immersion; 2. Using the regularity of $\psi^{(1)}$ and Sobolev embedding, we have

$$g_\sigma : Y^0 \to X^s, \quad g_{\sigma}^* : X^r \to Y^0 \quad (s \in \mathbb{R}, r \geq 2).$$
Let $J : X^0 \to X^0$ be the symplectic operator sending $\psi$ to $-i\psi$. We show in Appendix that the map
\[
J_{\sigma} : Y^0 \rightarrow Y^0, \quad \xi \mapsto g_{\sigma} J^{-1} g_{\sigma} \xi
\]
defines a symplectic operator, in the sense that $J_{\sigma}$ induces a symplectic form on the tangent bundle $T\Sigma$. Moreover, $J_{\sigma}$ is invertible, and satisfies the uniform estimate \[\|J_{\sigma}\|_{Y^k \to Y^k} \leq C(\Omega) |\log \epsilon|^{-2} \] for any $k \in \mathbb{N}$.

Geometrically, notice that $J_{\sigma}$ is the pullback of $J$ by the parameterization $f$. Recall in Sec. 1.1, we have defined the bilinear map induced by $J$ as
\[
\tau : (u,v) \mapsto \langle Ju, v \rangle_{X} \quad (u,v \in X^s).
\]
This $\tau$ is a non-degenerate symplectic form on $X^s$. Thus through the immersion $f$, the manifold of approximate filaments $M$ also inherits a symplectic structure, with a non-degenerate symplectic form induced by $J_{\sigma}$.

### 2.4 The adiabatic decomposition

In this section we derive a key result that is essential for the development in Section 3. The point is that on a tubular neighbourhood with definite volume around the manifold $M$ of approximate filaments, one can define a nonlinear projection into $\Sigma$. This way we can make precise the notion of concentration set for low energy configurations that are uniformly close to the approximate filaments.

Recall the symplectic form $\tau$ is given in (27).

**Lemma 2** (concentration set). Let $M := f(\Sigma_{\alpha})$ be the manifold of approximate vortex filaments. Then there is $\delta > 0$ such that for every $u \in X^1 + \psi_0$ with $\text{dist}_{X^2}(u,M) < \delta$, there exists a unique $\sigma \in \Sigma_{\alpha}$ such that
\[
\tau(u - f(\sigma), \phi) = 0 \quad (\phi \in T_{f(\sigma)}M).
\]

**Remark 4.** We call $\sigma = (\lambda, \gamma)$ associated to $u$ the moduli of the latter. This terminology is taken from e.g. [26, 27] with similar context.

We call the curve $\gamma$ the concentration set of $u$. The real number $\lambda$ is the global gauge parameter and is not physically meaningful. One can view (28) as an orthogonality condition w.r.t. the symplectic form (27), and the association $u \mapsto \sigma$ is optimal in this sense.

**Proof.** 1. First, for $\sigma \in \Sigma_{\alpha}$, define the linear projection $Q_{\sigma}$ by
\[
Q_{\sigma} : X^0 \rightarrow X^0, \quad \phi \mapsto g_{\sigma} J_{\sigma}^{-1} g_{\sigma}^{-1} \phi.
\]
This map $Q_{\sigma}$ is the skewed (i.e. $Q_{\sigma}^* = J^* Q_{\sigma} J$) projection onto the tangent space $T_{f(\sigma)}M$. In the definition of $Q_{\sigma}$, each factor is uniformly bounded, as we show in Appendix [67-68] and [72]. So we get the uniform estimate $\|Q_{\sigma}\|_{X^0 \to X^0} \leq C$. One can check $Q_{\sigma} \phi = \phi$ for every $\phi \in T_{f(\sigma)}M$ by writing $\phi = g_{\sigma} \xi$ for some $\xi \in Y^0$, since $Q_{\sigma} g_{\sigma} \xi = g_{\sigma} J_{\sigma}^{-1} (g_{\sigma} J^{-1} g_{\sigma}) \phi = g_{\sigma} (J_{\sigma}^{-1} J_{\sigma}) \xi = g_{\sigma} \xi = \phi$. 


Next, we find the concentration set \( \sigma \) using Implicit Function Theorem. Consider the map

\[
F : X^2 \times \Sigma \longrightarrow Y^0 \times \mathbb{R} \\
(\phi, \sigma) \longmapsto g^* J^{-1}(\phi - f(\sigma))
\]

Condition (28) is satisfied if \( Q(\mathcal{U} - f(\sigma)) = 0 \). To see this, one uses the property \( Q^* = -JQJ \), which implies \( \tau(Q\phi, \phi') = \tau(\phi, Q\phi') \). Thus, if \( F(\phi, \sigma) = 0 \), then (28) is satisfied.

By construction, we have the following expression for the partial Fréchet derivative

\[
\partial_\sigma F|_{(f(\sigma), \sigma)} = -J_{\sigma}.
\]

It is invertible since \( J_{\sigma} \) is invertible, see Appendix. The equation \( F(\phi, \sigma) = 0 \) has the trivial solution \( (f(\sigma), \sigma) \). It follows that for any fixed \( \sigma \in \Sigma \), there is \( \delta = \delta(\sigma, \epsilon) > 0 \) and a map \( S_\sigma : B_\delta(f(\sigma)) \to \Sigma \) such that \( F(\phi, S(\phi)) = 0 \) for \( \phi \in B_\delta(f(\sigma)) \).

2. It remains to show that in fact \( \delta \) can be made independent of \( \sigma \). This is important because we must retain a definite volume for the projecton neighbourhood, so that later on a flow can fluctuate within this neighbourhood.

Write

\[
A_\phi := \partial_\sigma F|_{(\phi + f(\sigma), \sigma)}, \quad V_\phi := A_\phi - A_0.
\]

Then

\[
A_0 = -J_{\sigma}, \quad V_\phi = (d_\sigma g^*_\sigma)(.)|_{J^{-1}B_\phi}.
\]

The size of \( \delta \) is determined by the condition that for every \( \phi \in B_\delta(f(\sigma)) \),

\[
A_\phi \text{ is invertible,} \quad (30)
\]

\[
\|A_{\phi}^{-1}\|_{Y^0 \rightarrow Y^0} \leq \frac{1}{4\|J_{\sigma}^{-1}\|_{Y^0 \rightarrow Y^0}}, \quad (31)
\]

\[
\|F(\phi + f(\sigma), \sigma)\|_{Y^0} \leq \frac{\delta_0}{4\|J_{\sigma}^{-1}\|_{Y^0 \rightarrow Y^0}}, \quad (32)
\]

where \( \delta_0 \) is determined that for every \( \xi \in B_{\delta_0}(\sigma) \) and \( \phi \in B_\delta(f(\sigma)) \),

\[
\|R(\phi, \xi)\|_{Y^0} \leq \frac{\delta_0}{4\|J_{\sigma}^{-1}\|_{Y^0 \rightarrow Y^0}}, \quad (33)
\]

\[
R(\phi, \xi) := F(\phi + f(\sigma), \sigma + \xi) - F(\phi + f(\sigma), \sigma) - \partial_\sigma F(\phi + f(\sigma), \sigma)\xi,
\]

\[
\|J_{\sigma + \xi} - J_{\sigma}\|_{Y^0 \rightarrow Y^0} \leq \frac{1}{4\|J_{\sigma}^{-1}\|_{Y^0 \rightarrow Y^0}}, \quad (34)
\]

See for instance [1 Sec. 2]. Note that the r.h.s. of (31) - (34) are independent of \( \sigma \) by the uniform estimate for \( J_{\sigma}^{-1} \).

Conditions (33) - (34) are satisfied for some \( \delta_0 = C(\Omega)\epsilon \). To get (33), one uses (29) and the fact that \( \|R(\phi, \xi)\|_{Y^0} = o(\|\xi\|_{Y^0}) \), since it is the super-linear remainder of the expansion of \( F \) in \( \sigma \). To get (34), one uses the continuity of the map \( \sigma \mapsto J_{\sigma} \in L(Y^0, Y^0) \).

The claim now is that (31) - (32) are satisfied for \( \delta = O(\epsilon^3) \). Indeed, plugging \( \delta_0 = C\epsilon \) to (29) and using the uniform estimate for \( g^*_\sigma \) and \( J_{\sigma}^{-1} \), one sees that (32) is satisfied so long as \( \delta = O(\epsilon \log \epsilon^{-1/2}) \).
By elementary perturbation theory, since $A_0$ is invertible, and the partial Fréchet derivative $A_\phi$ is continuous in $\phi$ as a map from $X^2 \rightarrow L^2(Y^0, Y^0)$, it follows that condition (30) is satisfied provided $\|V_\phi\|_{Y^0 \rightarrow Y^0} < \|A_0^{-1}\|_{Y^0 \rightarrow Y^0}^{-1} \leq C|\log \epsilon|^2$. By the uniform estimate on $d_\sigma g_\sigma$, we can arrange this with $\delta = O(\epsilon^3)$.

Lastly, referring to the Neumann series for the inverse $A^{-1}_\phi = \sum_{n=0}^{\infty} A^{-1}_0 (-V_\phi A^{-1}_0)^n$, one can see that $A^{-1}_\phi$ is also continuous in $\phi$, and $\|A^{-1}_\phi\|_{Y^0 \rightarrow Y^0} = O(\log \epsilon)^{-2}$ so long as $\|V_\phi\|_{Y^0 \rightarrow Y^0} = o(\log \epsilon)^2$. The latter holds with $\delta = O(\epsilon^3)$. This completes the proof.

Lemma 1 gives a unique decomposition for every configuration $u$ sufficiently close to $M$ as $u = v + w$, where $v$ is in $M$, and $\tau(v, w) = 0$. This orthogonality ensures the decomposition we find is optimal. In turn, $v$ is characterized by the moduli $\sigma = (\lambda, \gamma)$, which are, through the projection lemma above, functions of $u$. We call $v$ the adiabatic part of $u$.

In the remaining sections, we use Lemma 1 to decompose an entire flow $u_t$ starting near $M$ under (1) into an adiabatic flow $v_t$, of which we have explicit information, and a uniformly small remainder. Then we check that the concentration set $\gamma_t$ associated to $u_t$ indeed evolves according to the binormal curvature flow.

### 3 Effective dynamics

#### 3.1 The connection of (1) to the binormal curvature flow

The following lemma translates the r.h.s. of (1) restricted to $M$ to an expression on the tangent bundle $T\Sigma_{\epsilon^\alpha}$. Afterwards, we show this defines an evolution on $C^0_k$ that agrees with the binormal curvature flow in the leading order.

In this subsection, the operator $J$ denotes multiplication by the standard symplectic matrix $J$.

**Lemma 3.** Let $\alpha > 2$. If $\sigma \in \Sigma_{\epsilon^\alpha}$, then

$$J^{-1}_\sigma d_\sigma E(f(\sigma)) = \left( o(\epsilon^\alpha), J\gamma_{zz} + \sigma ||_{C^0}(\epsilon^\alpha) \right).$$

**Proof.** Recall $\sigma = (\lambda, \gamma)$ consists of a $U(1)$-gauge parameter $\lambda$ and a concentration curve $\gamma$. The $\lambda$-component is therefore not physically relevant, and we are interested in the $\gamma$ component. For $\sigma \in Y^k$, we write $\sigma =: ([\sigma]_\lambda, [\sigma]_\gamma)$.

The partial-Fréchet derivative of the energy $E(f(\sigma))$ w.r.t. $\gamma$ is given by

$$\partial_\gamma E(f(\sigma)) = \int_x \nabla_x \psi_\gamma \cdot \gamma_{zz} + \nabla_x^2 \psi_\gamma \cdot \gamma_{zz},$$

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where we write $\gamma_s = \partial_s \gamma$, etc.. Using this and the expression for $J_\sigma$ given in (39), we find that the first term in the r.h.s. of (36)

$$[J_\sigma^{-1} \int_\Omega (\nabla \psi_\gamma \cdot \gamma_{ss}, \nabla \psi_\gamma)]_\gamma = J \gamma_{ss} + o(\epsilon^\alpha).$$

It suffices then to show the second term

$$\left\| [J_\sigma^{-1} \int_\Omega (\nabla^2 \psi_\gamma \cdot \gamma_{ss}, \nabla \psi_\gamma)]_\gamma \right\|_{C^0} = o(\epsilon^\alpha), \quad (37)$$

i.e. this term can be absorbed into the remainder.

First, we use (11) and (25) to get

$$\left\| \int_\Omega (\nabla^2 \psi_\gamma \cdot \gamma_{ss}, \nabla \psi_\gamma) \right\|_{C^0} \leq C(\Omega) \| \nabla \psi_\gamma \|_{C^0} \| \nabla^2 \psi_\gamma \|_{L^\infty(\Omega)} \leq C(\Omega) \epsilon^{2\alpha} |\log \epsilon|^{1/2} \| \nabla^2 \psi_\gamma \|_{L^\infty(\Omega)}. \quad (38)$$

Next, note the function $\psi_\gamma$ satisfies the linear second order equation

$$\Delta \psi + \nabla^2 \psi_\gamma \cdot \gamma_{ss} + \nabla \psi_\gamma \cdot \gamma_{ss} + \frac{1}{\epsilon^2} (1 - |\psi_\gamma|^2) \psi = 0 \quad \text{on } \Omega.$$ 

This equation is elliptic for $\|\gamma_s\|_{C^0} < 1$, so the Schauder estimate implies for all $\epsilon < \epsilon_0 \ll 1$ and $0 < \mu < 1$,

$$\|\psi_\gamma\|_{C^{2,\mu}(\Omega)} \leq C(\Omega, \epsilon_0) \left( \|\psi_\gamma\|_{L^\infty(\Omega)} + \left\| \frac{1}{\epsilon^2} (1 - |\psi_\gamma|^2) \right\|_{C^{0,\mu}(\Omega)} \right) \quad (39)$$

Here we have used the estimates on the uniform norm $\|\psi_\gamma\|_{L^\infty(\Omega)} \leq 1$ and Hölder seminorm $\|\psi_\gamma\|_{C^{2,\mu}(\Omega)} \leq C(\Omega) \epsilon^{1-\mu}$. These follow from (13) and (24).

Plugging (39) to (38), we see that

$$\left\| \int_\Omega (\nabla^2 \psi_\gamma \cdot \gamma_{ss}, \nabla \psi_\gamma) \right\|_{C^0} \leq C(\Omega, \epsilon_0) |\log \epsilon|^{1/2} \epsilon^{2(\alpha - 1 + \mu)}. \quad (40)$$

In Appendix we show $\|J_\sigma^{-1} \|_{L^\infty(\Omega)} \leq C(\omega) |\log \epsilon|^{-2}$. This, the assumption $\alpha > 2$, and the above estimate together show that if we choose $\mu < \alpha/2 - 1$ then (39) holds. The proof is complete. \[\square\]

We now discuss the geometric meaning of the r.h.s. of (35). A family of curves $\gamma_t \in C^2(I, \Omega)$ satisfies the binormal curvature flow if parametrized by arclength, the curves satisfy

$$\partial_t \vec{\gamma} = \vec{\gamma}_s \times \vec{\gamma}_{ss} \quad (|\vec{\gamma}_s| \equiv 1). \quad (41)$$

Here we write derivatives w.r.t. to the arclength parameter as $\vec{\gamma}_s \equiv \partial_s \vec{\gamma}$, etc.

Now reparametrize $\gamma_t$ as $\vec{\gamma}_t(z) = (\gamma_t(z), z)$ with $\gamma \in C^2(I, \omega)$. Then (41) becomes

$$\partial_t \vec{\gamma} = \vec{\gamma}_s z_s \times (\vec{\gamma}_{ss} z_s^2 + \vec{\gamma}_s z_{ss}). \quad (42)$$
The arclength parameter $s$ is given in terms of the new parameter $z$ by

$$s(z) = \int_0^z |\gamma'|^2 = z + \int_0^z |\gamma_z|^2.$$ 

Differentiating this expression, for $\|\gamma\|_{C^2} = O(\delta)$, we get

$$z_s = \frac{1}{|\gamma_z|^2 + 1} = 1 + O(\delta^2), \quad z_{ss} = -\frac{\gamma_z \cdot \gamma_{zz}}{(1 + |\gamma_z|^2)^2} = O(\delta^2).$$

Geometrically, $z$ and $s$ are close because the curve parametrized by $\tilde{\gamma}$ with $\|\tilde{\gamma}\|_{C^2} \ll 1$ is approximately a vertical straight line, in which case $z = s$.

Thus (42) in the leading order reads

$$\partial_t \tilde{\gamma} = \gamma_z \times \gamma_{zz} + O(\delta^2) = (J\gamma_{zz}, \gamma_z^2 \gamma_{zz} - \gamma_z^2 \gamma_{zz}) + O(\delta^2). \quad (43)$$

In the r.h.s. of (43), $\gamma_z^2 \gamma_{zz} - \gamma_z^2 \gamma_{zz}$ is also $O(\delta^2)$. In conclusion, combing with Lemma 3 we can say

$$\partial_t \sigma = (\partial_t \lambda, \partial_t \gamma) = J^{-1}_{\sigma} d_{\sigma} E(f(\sigma)) + o(\delta) \implies \tilde{\gamma}_t \text{ solves (1)} \text{ up to } o(\delta).$$

In the next subsection, we show that this equation for $\sigma$ is indeed the effective dynamics of vortex filaments.

### 3.2 Proof of the first main theorem

Suppose $u_t$ is a solution to (1) such that $\text{dist}(u_t, M) < \delta \ll 1$ for $t \leq T$. So far, using Lemma 2, this allows us to define an adiabatic flow $v_t = f(\sigma_t) \in M$ consisting of the approximate filaments associated to $u_t$. At time $t \leq T$, the filament $v_t$ is characterized by the moduli $\sigma_t$, and the curve $\gamma_t = [\sigma_t]_v$ defines the concentration set of $u_t$ (see Remark 3).

In what follows, we show that the velocity $\partial_t \sigma$ governing the motion of the adiabatic flow is given by $J^{-1}_{\sigma} d_{\sigma} E(f(\sigma))$ uniformly up to the leading order. We then find an a priori estimate for the remainder, so that as long as $u_0$ is close to $M$, the full flow $u_t$ remains uniformly close to $M$ up to a large time. In this sense one can view $M$ as an invariant manifold for (1).

**Theorem 2** (effective dynamics). For any $\epsilon > 0$, there are $\delta_1, \delta_2 < \epsilon$ such that the following holds: Let $\Sigma_{\delta_1} \to M$ be the manifold of approximate vortex filaments as in Section 2. Let $u_0$ be an initial configuration such that $\text{dist}_{X^2}(u_0, M) < \delta_2$.

Then there is some $T > 0$ independent of $\epsilon$ and $\delta_1, \delta_2$, such that for all $\epsilon t \leq T$, there exists moduli $\sigma_t$ associated to $u_t$ as in Lemma 3 and

$$\|u_t - f(\sigma_t)\|_{X^2} = o(\sqrt{\delta_1}). \quad (44)$$

Moreover, for all $\epsilon t \leq T$, the moduli $\sigma_t$ evolves according to

$$\partial_t \sigma = J^{-1}_{\sigma} d_{\sigma} E(f(\sigma)) + o_2(\delta_1). \quad (45)$$

**Proof.** 1. To begin with, choose $\delta_2 \ll \sqrt{\delta_1}$. We show $w_0 := u_0 - f(\sigma_0) = u_0 - v_0$ satisfies $\|w_0\|_{X^2} \ll \sqrt{\delta_1}$.

Suppose the $X^2$-closest approximate vortex filament to $u_0$ is $v_* = f(\sigma_*) \in M$. Then $w_0 = (u_0 - v_*) + (v_* - v_0)$. The first difference on the
r.h.s. has size $\delta_2$. The second difference can be bounded by $\|v_\ast - v_0\|_X = \|f(\sigma_\ast) - f(\sigma_0)\|_X \leq C \|\sigma_\ast - \sigma_0\|_X \leq C\delta_2$. It follows that $\|w_0\|_X \leq C\delta_2$, and by the choice of $\delta_2 < \delta_1$, that $\|w_0\|_X \ll \delta_1$. Therefore, by the continuity of the evolution \(14\), 14 holds at least locally.

2. So long as the decomposition $u_t = u_t + u_1$ in Lemma 14 is valid, we can expand (14) as

$$\partial_t u + \partial_t w = J(E'(v) + L_\sigma w + N_\sigma (w)),$$

where $J : \psi \mapsto -i\psi$,

$$L_\sigma \phi := -\Delta \phi + \frac{1}{\epsilon^2}(|\psi_\gamma|^2 - 1)\phi + \frac{2\epsilon^2 \cos \lambda}{\epsilon^2} \psi_\gamma \langle \psi_\gamma, \phi \rangle$$

is the linearized operator at $f(\sigma) \equiv v$, and

$$N_\sigma (\phi) := E'(\psi_\gamma + \phi) - E'(\psi_\gamma) - L_\sigma \phi$$

is the nonlinearity.

For the moduli $\sigma = \sigma_\ast$ associated to $u_1$ given in Lemma 14 let $Q_\sigma : X^0 \to X^0$ be the fibrewise projection onto $T^*_f(\sigma)m$, given in (29). Applying $Q_\sigma$ to both sides of (52), we have

$$\partial_t v - Q_\sigma J E'(v) = Q_\sigma (J L_\sigma w - \partial_t w + J N_\sigma (w)).$$

Consider the identity

$$\mathcal{J}_\sigma^{-1} g_\ast^{-1} (\partial_t v - Q_\sigma J E'(v)) = \partial_t \sigma - \mathcal{J}_\sigma^{-1} d_\sigma E(f(\sigma)).$$

To verify this, one uses two facts that follow readily from the chain rule:

$$\partial_t v = g_\ast \partial_t \sigma, \quad g_\ast E'(f(\sigma)) = d_\sigma E(f(\sigma)).$$

Thus by the uniform estimates on $g_\ast$ and $\mathcal{J}_\sigma^{-1}$ in (88), (92), we have

$$\|\partial_t \sigma - \mathcal{J}_\sigma^{-1} d_\sigma E(f(\sigma))\|_{Y_0} \leq C \|\log \epsilon\|^{-3/2} \|\partial_t v - Q_\sigma J E'(v)\|_X,$$

for some $C$ independent of $\sigma$. This shows that the claim (45) would follow if we have uniform control over (18).

3. We now derive the a priori estimate for the remainder $w = w_1$, the fluctuation of $u_t$ around the adiabatic part $v_\ast$.

Consider the r.h.s. of (18). These three terms can be bounded respectively as follows:

$$\|Q_\sigma J L_\sigma w\|_{X^0} \leq C \|\log \epsilon\|^{-1} \delta_{1/2} \|w\|_X,$$

$$\|Q_\sigma \partial_t w\|_{X^0} \leq C \|\log \epsilon\|^{-1} \|\partial_t \sigma\|_{Y_0} \|w\|_X,$$

$$\|Q_\sigma J N_\sigma (w)\|_{X^0} \leq C \|\log \epsilon\|^{-1} \epsilon^{-2} \|w\|_X^2.$$

Here $C$ is independent of $\sigma$. In all these three inequalities we use the uniform bound $\|Q_\sigma\|_{X^0} \leq C(\Omega) \|\log \epsilon\|^{-1}$, which follows its definition (29) and the estimates for each of its factors.

First, we show (50) using the identity

$$|\langle Q_\sigma J L_\sigma w, w' \rangle| = |\langle w, L_\sigma Q_\sigma J w' \rangle|, \quad (w, w' \in X^2).$$
To get (53), one uses the relation $Q_\sigma J = J Q_\sigma^*$, which follows from the definition of $Q_\sigma$ (see also the first step in the proof of Lemma 2). Plugging $w' = Q_\sigma J_\sigma L w$ into (53), and using estimate (72), we find
\[
\|Q_\sigma J_\sigma w\|_{X^0} \leq C(\delta_1^{1/2} \|w\|_{X^2}).
\]
This gives (50).

Next, differentiating $Q_\sigma w$ with respect to $t$, we have
\[
0 = \partial_t (Q_\sigma w) = (\partial_t Q_\sigma) w + Q_\sigma \partial_t w = (d_\sigma Q_\sigma \partial_t \sigma) w + Q_\sigma \partial_t w. 
\] (54)
Here $d_\sigma Q_\sigma$ is an operator from $Y^0$ to the space of linear operators $L(X^1, X^0)$. Since $Q_\sigma$ is the projection onto $T_\sigma \mathcal{M}$, and $\sigma$ is slowly varying, we get the uniform estimate $\|d_\sigma Q_\sigma\|_{Y^0 \to L(X^1, X^0)} \leq C$. Plugging this to (54) gives (51). Lastly, we have the following explicit expression for the nonlinearity:
\[
N_\sigma(w) = \frac{1}{\epsilon^2} (2w(v, w) + |w|^2 (v + w)).
\]
Using (12) we can bound this as (52).

Plugging (50)-(52) into (48), and then using (49), one can see that if one can bound $\|w\|_{X^2}$ uniformly in time by a sufficiently small number compared to $\epsilon$, it suffices then to show (44) with $\delta_1 \ll \epsilon$.

4. Consider the expansion
\[
E(v + w) = E(v) + \langle E'(v), w \rangle + \frac{1}{2} \langle L_\sigma w, w \rangle + R_\sigma(w), 
\] (55)
where $R_\sigma(w)$ is the super-quadratic remainder defined by this expression.

We want to apply the coercivity of $L_\sigma$ shown in Lemma 5 to $w$, so that we can rearrange (55) to get
\[
\|w\|_{X^2}^2 \leq C_\epsilon (E(v + w) - E(v) - \langle E'(v), w \rangle - R_\sigma(w)).
\] (56)
The constant $C_\epsilon$ will eventually be eliminated by the choice of $\delta$. This requires the uniform estimates
\[
\|w\|_{X^2} \leq C_\epsilon \|w\|_{X^0}. 
\] (57)
For this, one uses the fact that $w$ satisfies the elliptic equation
\[
\Delta w + \frac{1}{\epsilon^2} (1 - |v|^2) w = -i \partial_t u + \nabla_x v \cdot \gamma_z - \nabla_z^2 v_z \cdot \gamma_z + (|u|^2 - |v|^2) u.
\]
The r.h.s. of this equation is $O(\delta_1)$. Thus standard elliptic regularity gives $\|w\|_{X^2} \leq C(\Omega, \epsilon) \|w\|_{X^0}$, with $C(\Omega, \epsilon) = O(\epsilon^{-2})$.

Since $u = v + w$ solves (1), the energy $E(u)$ is conserved for all $t$. Consequently,
\[
E(v + w) = E(v_0 + w_0) = E(v_0) + \langle E'(v_0), w_0 \rangle + \frac{1}{2} \langle L_\sigma w_0, w_0 \rangle + R_\sigma(w_0).
\]
Plugging this into (56), and using (57), one gets
\[
\|w\|_{X^2}^2 \leq C_\epsilon \alpha^{-1} (E(v_0) - E(v)) + \langle E'(v_0), w_0 \rangle - \langle E'(v), w \rangle + \frac{1}{2} \langle L_\sigma w_0, w_0 \rangle - R_\sigma(w_0) + R_\sigma(w).
\] (58)
Here $\alpha = \alpha(\epsilon) = O(|\log \epsilon|^{-1})$ as in Lemma 5.

Similar to the nonlinear estimate on $N_\sigma(w)$, it follows from the regularity of $E$ on the energy space that $R_\sigma(w) \leq C \epsilon^{-2} \|w\|^2_{X^2}$ where $C$ is independent of $\sigma$. Combining this with Lemma 4, we can bound

\[
\langle E'(v_0), w_0 \rangle - \langle E'(v), w \rangle + \frac{1}{2} \langle L_\sigma w_0, w_0 \rangle - R_\sigma(w_0) + R_\sigma(w)
\]

by

\[
C \left( \|E'(v_0)\|_{X^1} \|w_0\|_{X^0} + \epsilon^{-1} \|w_0\|^2_{X^1} + \epsilon^{-2} \|w_0\|^3_{X^2} \right) + \|E'(v)\|_{X^1} \|w\|_{X^0} + \epsilon^{-2} \|w\|^3_{X^2},
\]

(59)

Recall in Step 1 we have shown $\|w_0\|_{X^2} \leq C \text{dist}_{X^2}(u_0, M) = C \delta_2$. So (59) becomes

\[
\langle E'(v_0), w_0 \rangle - \langle E'(v), w \rangle + \frac{1}{2} \langle L_\sigma w_0, w_0 \rangle - R_\sigma(w_0) + R_\sigma(w)
\]

by

\[
C \left( \|E'(v_0)\|_{X^1} \delta_2 + \epsilon^{-1} \delta_2^2 + \epsilon^{-2} \delta_2^3 + \|E'(v)\|_{X^1} \|w\|_{X^0} + \epsilon^{-2} \|w\|^3_{X^2} \right).
\]

This gives control over the last five terms in the r.h.s. of (60).

5. We now exploit energy conservation to control the first two terms in the r.h.s. of (60), as this difference is the energy fluctuation of the approximate filaments. Differentiate the energy $E(t) = E(f(\sigma_t))$ and using (48), we have

\[
\frac{dE}{dt} = \langle E'(v), \partial_t v \rangle = \langle E'(v), Q\sigma J E'(v) \rangle + \langle E'(v), Q\sigma (J L_\sigma w - \partial_t w) \rangle + \langle E'(v), Q\sigma J N_\sigma(w) \rangle.
\]

(61)

We now bound the three inner products respectively.

Using the relation $Q\sigma J = J Q_\sigma^*$, the property of projection $Q_\sigma^2 = Q_\sigma$, and the fact that $J$ is symplectic, we find

\[
\langle E'(v), Q\sigma J E'(v) \rangle = \langle E'(v), Q_\sigma^2 J E'(v) \rangle = \langle Q_\sigma E'(v), Q_\sigma J E'(v) \rangle = \langle (J^{-1} J) Q_\sigma E'(v), J Q_\sigma E'(v) \rangle = -(J Q_\sigma J E'(v), Q_\sigma J E'(v)) = 0.
\]

Thus the first term in (61) vanishes.

Using (60) - (61), the second inner product can be bounded as

\[
\left| \langle E'(v), Q\sigma (J L_\sigma w - \partial_t w) \rangle \right| \leq \|E'(v)\|_{X^1} C \delta_1^{1/2} + \|\partial_t \sigma\|_{Y^0} \|w\|^2_{X^2}.
\]

By (59), (48) - (52), so long as $\|w\|_{X^1} < 1/2$ we have

\[
\|\partial_t \sigma\|_{Y^0} \leq C (\delta_1 + \delta_1^{1/2} \|w\|_{X^2} + \epsilon^{-2} \|w\|^2_{X^2}).
\]

Plugging this back to the previous estimate, we have

\[
\left| \langle E'(v), Q\sigma (J L_\sigma w - \partial_t w) \rangle \right| \leq \|E'(v)\|_{X^1} \left( C_1 \delta_1^{1/2} + C_2 (\delta_1 + \delta_1^{1/2} \|w\|_{X^2} + \epsilon^{-2} \|w\|^2_{X^2}) \right) \|w\|_{X^2}.
\]

(62)
Lastly, by the nonlinear estimate \( \text{(52)} \), the third inner product in \( \text{(61)} \) can be bounded as
\[
|\langle E''(v), Q_\sigma J_N^\sigma (w) \rangle| \leq C \varepsilon^{-2} \| E'(v) \|_{X_0} \| w \|_{X_0}^2. \tag{63}
\]

6. Combining \( \text{(61)} \) and \( \text{(63)} \) and integrating from 0 to \( t \), we have
\[
|E(v_t) - E(v(0))| \leq \| E'(v) \|_{X_0} t (C_1 \delta_1^{1/2} + C_2 (\delta_1 + \delta_2 M(t) + \varepsilon^{-2} M(t) + \varepsilon^{-2} M(t)^2)) M(t), \tag{64}
\]
where \( M(t) := \sup_{t' \leq t} \| u(t') \|_{X_0} \). Plugging \( \text{(60)} \) and \( \text{(64)} \) into \( \text{(58)} \), we have
\[
M(t) \leq \log \varepsilon |C_t| \| E'(v) \|_{X_0} t \left( C_1 \delta_1^{1/2} + C_2 (\delta_1 + \delta_2 M(t) + \varepsilon^{-2} M(t) + \varepsilon^{-2} M(t)^2) \right) + C_3 \left( \| E'(v) \|_{X_0} (1 + \varepsilon^{-2} M(t)^2) + \| E'(v_0) \|_{X_0} \delta_2 + \varepsilon^{-1} \delta_2^2 + \varepsilon^{-2} \delta_2^3 \right). \tag{65}
\]
If we now choose \( \delta_2 = \min(\| \log \varepsilon \|^{-1/2}, \varepsilon_1^{1/2}) \) and the manifold of approximate filaments \( \Sigma_{\delta_1} \to M \) to be sufficiently small, say with
\[
\delta_1 = O(\| \log \varepsilon \|^{-2} C^{-1} \varepsilon^{1/2}) \tag{66}
\]
for some \( \mu > 0 \), then \( \text{(66)} \) and Lemma 1 implies that for all \( t \leq T = C \varepsilon^{-1} \), we have \( M(t) \leq C \varepsilon^{1/2} \), as claimed in \( \text{(13)} \). (Actually, if we further shrink \( \delta_1 \) (i.e. the class of initial configurations), we can ensure \( \text{(13)} \) on a longer time interval.)

Plugging \( \text{(14)} \) into \( \text{(18) - (22)} \), we have
\[
\| \partial_t \sigma - J_{\sigma^{-1}}^\tau d_\sigma E(f(\sigma)) \|_{y_0} \leq C \| \log \varepsilon \|^{-5/2} \varepsilon^{-2} \delta_1.
\]
Shrinking \( \delta_1 \), say with \( \mu > 1 \) in \( \text{(66)} \), we can ensure this expression is still \( o(\delta_1) \). The proof is complete. \( \square \)

3.3 Theorem 2 relates to Jerrard’s conjecture

The following theorem refers to \( \text{[14] Conjecture 7.1} \). For small but fixed \( \varepsilon \), Theorem 3 below, in particular \( \text{(67) - (68)} \), provides an affirmative answer for a certain class of initial conditions with uniformly small curvature.

**Theorem 3.** For any \( \beta > 0 \), there exist \( \delta, \varepsilon_0 > 0 \) such that the following holds: Let \( \varepsilon < \varepsilon_0 \) in \( \text{(1)} \). Let \( \sigma_0 \in \Sigma_\delta \) be given, and \( \gamma_0 = [\sigma_0]_{\gamma} \). Let \( \gamma_t \) be the flow generated by \( \langle \gamma_0(z), z \rangle \) under \( \text{(11)} \).

Then there exist a solution \( u_t \) to \( \text{(1)} \), and some \( T > 0 \) independent of \( \varepsilon \) and \( \delta \), such that for all \( t \leq T \), \( X \in C^1_t(\mathbb{R}^3, \mathbb{R}^3) \) and \( \phi \in C^1_t(\mathbb{R}^3, \mathbb{R}) \) with \( \| X \|_{C^1}, \| \phi \|_{C^1} = O(\delta^{-1/4}) \), we have
\[
\left| \int_\Omega X \times J u - \pi \int_{\Gamma_t} X \right| \leq \beta, \tag{67}
\]
\[
\left| \int_\Omega \frac{e(u)}{|\log \varepsilon|} \phi - \pi \int_{\Gamma_t} \phi d\mathcal{H}^1 \right| \leq \beta, \tag{68}
\]
where for \( \psi = \psi_1 + i \psi_2 \),
\[
e(u) = \frac{1}{2} |\nabla \psi|^2 + \frac{1}{4\varepsilon^2} (|\psi|^2 - 1)^2, \quad J \psi = \nabla \psi_1 \times \nabla \psi_2.
\]

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Proof. Let \( u_t \) be the flow generated by \( u_0 := f(\sigma_0) \) under (1). Then for sufficiently small \( \delta > 0 \), Theorem \( 2 \) applies with \( \delta_1 = \delta, \delta_2 = 0 \). Let \( \tilde{\sigma}_t \) be the flow generated by \( \sigma_0 \) under the effective dynamics \( (1) \). It follows that \( u_t = v_t + O_1(X^2(\sqrt{\delta})) \), where \( v_t = f(\tilde{\sigma}_t) \).

Given the explicit construction, one can check using classical concentration properties of the planar vortex \( \psi^{(1)} \) (for instance \( [17] \)) that, for sufficiently small \( \epsilon_0 = \epsilon_0(\beta) > 0 \) and all \( 0 < \epsilon \leq \epsilon_0 \), the flow \( v_t \) satisfies \( (67)-(68) \) with \( \sigma_t \) in place of \( \sigma_t \). For example, we can compute using (29) and the assumption \( \| \phi \|_{C^0} = O(\delta^{-1/4}) \) that

\[
\int_{\Omega} \frac{e(\psi)}{|\log \epsilon|} \phi = \int_{\tilde{\gamma}_t} \left( \int_{\Omega} \frac{e(\psi_{u_0})}{|\log \epsilon|} \phi \, dx \right) \, dH^1 \\
\leq \int_{\tilde{\gamma}_t} \left( \int_{\Omega} \frac{e(\psi^{(1)})}{|\log \epsilon|} \phi \, dx \right) \\
\leq \pi \int_{\tilde{\gamma}_t} \phi \, dH^1 + C \left( \beta + \frac{\epsilon^2 \delta^{3/4}}{|\log \epsilon|} \right).
\]

Here \( \tilde{\gamma} = \tilde{\sigma}_t \), and \( \tilde{\gamma}(z) = (\tilde{\gamma}(z), z) \).

Using (10) with \( \alpha = 3 \) and \( \mu = 1/4 \), we can get the uniform estimate \( \tilde{\sigma}_t = \sigma_t + O(\delta^{3/2}) \). If \( \| X \|_{C^1} = O(\delta^{-1/4}) \), then by the mean value theorem,

\[
\left| \int_{\tilde{\gamma}} X - \int_{\tilde{\gamma}} X \right| \leq \| \gamma - \gamma \|_{L^\infty (\tilde{\gamma})} \| X \|_{C^1} = O(\delta^{5/4}).
\]

Similarly one can show

\[
\left| \int_{\tilde{\gamma}} \phi \, dH^1 - \int_{\tilde{\gamma}} \phi \, dH^1 \right| = O(\delta^{5/4}).
\]

It follows that for all sufficiently small \( \epsilon_0 \) and all \( \epsilon < \epsilon_0 \),

\[
\left| \int_{\Omega} X \times Jv - \pi \int_{\tilde{\gamma}_t} X \right| \leq C \left( \beta + \frac{\epsilon^2 \delta^{3/4}}{|\log \epsilon|} + \delta^{5/4} \right), \quad (69)
\]

\[
\left| \int_{\Omega} \frac{e(v)}{|\log \epsilon|} \phi - \pi \int_{\tilde{\gamma}_t} \phi \, dH^1 \right| \leq C \left( \beta + \frac{\epsilon^2 \delta^{3/4}}{|\log \epsilon|} + \delta^{5/4} \right), \quad (70)
\]

On the other hand, using (44) and the continuity of \( e(\psi), J\psi \) on \( X^2 \), we have

\[
\left| \int_{\Omega} X \times Ju - \int_{\Omega} X \times Jv \right| \leq C \| X \|_{C^0 (\Omega)} \| u - v \|_{X^2} \leq C \delta^{1/4},
\]

and similarly

\[
\left| \int_{\Omega} \frac{e(u)}{|\log \epsilon|} \phi \, dH^1 - \int_{\Omega} \frac{e(v)}{|\log \epsilon|} \phi \, dH^1 \right| \leq C |\log \epsilon|^{-1} \delta^{1/4}.
\]

Plugging these into (69)–(70), we get that \( u_t \) satisfies (67)–(68) for sufficiently small \( \delta = \delta(\beta, \epsilon_0) > 0 \). \( \square \)
4 Properties of the linearized operators

In this section we consider various estimates for the linearized operator \( L_\sigma = E'(f(\sigma)) \) defined in (17). Recall that so far we have always suppressed the dependence of the various functions on the material parameter \( \epsilon \ll 1 \). Throughout this section we assume \( \|\sigma\|_{L^2} < \delta \ll \epsilon \). Without specification, various inner products are as in (18).

Lemma 4 (uniform bound of \( L_\sigma \)). There exists \( 0 < C < \infty \) independent of \( \sigma \) and \( \epsilon \) such that

\[
(L_\sigma \phi, \phi) \leq C \epsilon^{-1} \|\phi\|_{X^0}^2 \quad (\phi \in X^1),
\]

\[
\|L_\sigma Q_\sigma \phi\|_{X^0} \leq C \delta^{1/2} \|\phi\|_{X^2} \quad (\phi \in X^2).
\]

Here, and \( Q_\sigma \) is the projection onto the tangent space \( T_f(\sigma)M \) defined in (29).

Proof. We write the Schrödinger operator \( L_\sigma \) defined in (17) as \( L_\sigma = -\Delta + V \). By Poincaré’s inequality, the kinetic part of (71) is bounded by \( \langle -\Delta \phi, \phi \rangle \leq C \|\phi\|_{X^0} \). To bound the potential part of (71), consider

\[
\|V(\phi)\|_{X^0} \leq \epsilon^{-2} \left( \|\phi\|_{X^0}^2 - 1 \right) + 2 \epsilon \lambda \cos \lambda \|\phi\|_{X^0} + \epsilon \|\phi\|_{X^0}^2 \leq \epsilon^{-2} \left( \|\phi\|_{X^0}^2 - 1 \right) + \epsilon \|\phi\|_{X^0}^2 \leq \epsilon^{-2} \left( \|\phi\|_{X^0}^2 - 1 \right) + \epsilon \|\phi\|_{X^0}^2 \leq \epsilon^{-2} \left( \|\phi\|_{X^0}^2 - 1 \right) + \epsilon \|\phi\|_{X^0}^2 
\]

Recall \( \omega \subset \mathbb{R}^2 \) is the cross section of the domain \( \Omega \). In the last step we use (32), (15), and (21). This implies \( \langle V(\phi), \phi \rangle \leq C \epsilon^{-1} \|\phi\|_{X^0}^2 \), and therefore (71) follows.

Next, we show (72). Since \( Q_\sigma \) is the projection onto the tangent space \( T_f(\sigma)M \), using the formula (87) for the trivialization, every \( \phi \in \mathcal{R} Q_\sigma \) can be written as

\[
\phi = e^{\lambda x} (i \mu \psi_{\gamma} - \nabla_x \psi_{\gamma} \cdot \xi)
\]

for some \( \xi \in \mathcal{C}_0(\mu) \). Using this representation, the assumption \( \lambda < \delta \) (which implies \( e^{\lambda x} = 1 + iO(\delta) \)), together with the formula (17) for \( L_\sigma \) (in which only the last term is not real-linear), we can write

\[
L_\sigma Q_\sigma \phi = L_\sigma (i \mu \psi_{\gamma} - \nabla_x \psi_{\gamma} \cdot \xi) + O(\delta \epsilon^{-2} \phi).
\]

For fixed \( \gamma \in \mathcal{C}_0^1 \), let \( L_{x,z} : H^2(\omega) \to L^2(\omega) \) be the planar linearized operator at \( \psi_{\gamma,z} := f(\sigma)(x, z) \), given explicitly as

\[
L_{x,z} \phi := -\Delta_x \phi + \frac{1}{\epsilon^2} \|\psi_{\gamma,z}\|^2 - \Delta_x \psi_{\gamma}(x, z) \cdot \xi(z), \quad (\phi : \omega \subset \mathbb{R}^2 \to \mathbb{C}).
\]

Consider the inner product \( \langle L_\sigma Q_\sigma \phi, \phi' \rangle \) for \( \phi, \phi' \in X^2 \). Using (74)-(75), we can split this into three parts,

\[
\langle L_\sigma Q_\sigma \phi, \phi' \rangle = \int_{\omega} \left( \int \left( \langle L_{x,z} (i \mu \psi_{\gamma}(x, z) - \nabla_x \psi_{\gamma}(x, z) \cdot \xi(z), \phi' (x, z))^2 dx \right) \right. \\
+ \langle \partial_{x,z} (i \mu \psi_{\gamma} - \nabla_x \psi_{\gamma} \cdot \xi), \phi' \rangle + O(\delta \epsilon^{-2} \|\phi\|_{X^2} \|\phi'\|_{X^2}.
\]
The operator \( L_{z,x} \) is obtained by translating the planar linearized operator \( L^{(1)}_z \) at the simple vortex \( \psi^{(1)} \) by \( x \mapsto x - \gamma(z) \). Consequently, for each fixed \( z \), the three vectors \( \partial_x \psi_{\gamma}(x,z) \), \( j = 1, 2 \), and \( i \psi_{\gamma}(x,z) \) are in the kernel of \( L_{z,x} \). (See the discussion in Sec. 2.1 about symmetry zero modes). Because of this fact, the first term in the r.h.s. of (76) vanishes.

To estimate the second term, we compute

\[
\partial_{zz} (i \mu \gamma \nabla \gamma \xi) = -i \mu \nabla \gamma \gamma_{zz} + \nabla^2 \gamma \gamma_{zz} \xi + O(||\gamma||^2_{C^2}) \sum_{i,j=1,2} \partial^2_{z,z,j} \phi. 
\]

(77)

By the assumption \( ||\gamma||_{C^2} = O(\delta) \), (77) implies that the second term in the r.h.s. of (76) is \( O(\delta) ||\phi||_{X^2} ||\phi'||_{X^2} \). Lastly, setting \( \phi' = L_0 Q \phi \) in (76), we obtain (72) so long as \( \delta = O(\epsilon^4) \).

\[ \text{Proof.} \]

Recall that \( L^{(1)}_z \) is the planar linearized operator at the simple vortex \( \psi^{(1)} \) as in Sec. 2.1. The classical stability result for planar vortices states that there is \( \alpha > 0 \) such that for every \( \eta \in L^2(\omega) \) orthogonal to the symmetry zero modes \( G := i \psi^{(1)}, T_j := \partial_{z,j} \psi^{(1)} \),

\[
\langle L^{(1)}_z \eta(), \eta() \rangle_{L^2(\omega)} \geq \beta \| \eta \|^2_{L^2(\omega)}. 
\]

(79)

This \( \beta \) measures the spectral gap at 0. See [21, Secs. 7-8] for a discussion in the same setting. Moreover, in [18] it is shown that \( \beta = O(||\log \epsilon||^{-1}) \).

Let \( L_0 : X^k \rightarrow X^{k-2} \) be the linearized operator at the lift of \( \psi^{(1)} \) to \( X^k \). Integrating (78) along \( z \)-direction, and use periodicity to drop the term \( \langle \partial_{zz} \eta, \eta \rangle \geq 0 \), we find

\[
\langle L_0 \eta, \eta \rangle \geq \beta \| \eta \|^2_{X^0} \quad \text{if} \quad \eta(.,z) \text{ is orthogonal to } T_j \text{ and } G. 
\]

(80)

The orthogonality condition for (80) holds trivially if \( \eta \) is orthogonal to the lift of \( T_j \) and \( G \) to \( X^k \).

2. Put \( \hat{Q} = 1 - Q \), then we can rewrite (78) as

\[
\langle L \eta, \eta \rangle = \langle L \hat{Q} \eta, \eta \rangle + \langle L \hat{Q} \eta, \eta \rangle. 
\]

(81)

The first term is \( O(\delta^{1/2}) \| \eta \|^2_{X^2} \) by the approximate zero mode property (72). The second term further splits as \( \langle L \hat{Q} \eta, \hat{Q} \eta \rangle + \langle L \hat{Q} \eta, \hat{Q} \eta \rangle = \langle L \hat{Q} \eta, \hat{Q} \eta \rangle + \langle \hat{Q} \eta, L \hat{Q} \eta \rangle = \langle L \hat{Q} \eta, \hat{Q} \eta \rangle + O(\delta^{1/2}) \| \eta \|^2_{X^2} \), again by (72).

Thus it suffices to control \( \langle L \hat{Q} \eta, \hat{Q} \eta \rangle \).

Write \( \hat{\eta} = Q \eta \). We claim

\[
\langle L \hat{\eta}, \hat{\eta} \rangle \geq \frac{\beta}{4} \| \eta \|^2_{X^0}, 
\]

(82)
which, so long as $\delta \ll \beta$, implies (78) with $\alpha = \beta/8$.

3. Choose a partition of unity $\chi_1, \chi_2$ on $\Omega$, such that $\chi_j \geq 0$, $\chi_1^2 + \chi_2^2 = 1$ and $\text{supp} \chi_1 \subset \{(x,z) : x - \gamma(z) < \rho\}$ for $\gamma = [\sigma]_\gamma$ and some $\rho$ with $\delta < \rho < \text{diam}(\omega)$ to be determined. These cut-off functions $\chi_1, \chi_2$ separate $\Omega$ into an inner region, where the vortex filament $\psi_\gamma$ is small, and an outer region, where $|\psi_\gamma|$ is close to 1.

We use the localization formula [24, Eqn. (1.1)],

$$ L = \sum \chi_j L \chi_j - \sum |\nabla \chi_j|^2. $$

If we choose $|\nabla \chi| \leq \rho^{-1}$, then this formula allows us to write the l.h.s. of (82) as

$$ \langle \mathcal{L} \bar{\eta}, \bar{\eta} \rangle \geq \langle \mathcal{L} \chi_1 \bar{\eta}, \chi_1 \bar{\eta} \rangle + \langle \mathcal{L} \chi_2 \bar{\eta}, \chi_2 \bar{\eta} \rangle - C_1 \rho^{-2} \|\eta\|^2_{X^0}. $$

(83)

Since $\chi_2$ is supported away from the vortex filament, using the far-off asymptotics in (10), the second term in the r.h.s. can be bounded from below as

$$ \langle \mathcal{L} \chi_2 \bar{\eta}, \chi_2 \bar{\eta} \rangle \geq (1 - C_2 \rho^{-2}) \|\chi_2 \eta\|^2_{X^0}. $$

(84)

Recall $Q$ is the projection onto the approximate zero modes satisfying (72). This, the fact that $\bar{\eta} \in \text{ker} Q_\sigma$, and the lower bound (80) together give

$$ \langle \mathcal{L} \chi_1 \bar{\eta}, \chi_1 \bar{\eta} \rangle \geq (\beta - C_3 (\delta^{1/2} + \rho^{-1})) \|\chi_1 \eta\|^2_{X^0}. $$

(85)

Plugging (85) to (84), and choosing $\rho = 2C_3 \beta^{-1}$, we find

$$ \langle \mathcal{L} \bar{\eta}, \bar{\eta} \rangle \geq \left( \min \left( \frac{\beta}{2} - C_3 \delta^{1/2}, 1 - 4C_4 C_5 \beta^2 \right) - (4C_1 C_5 \beta^2 + C_4 \delta \rho^{-1}) \right) \|\eta\|^2_{X^0}. $$

(86)

Since $\beta = O(|\log \epsilon|^{-1})$, for $\delta = \epsilon^{1+s}$, $s > 0$ and all $\epsilon \ll 1$, we get the claim (82) from (86).

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Declarations

- Competing interests: The Author has no conflicts of interest to declare that are relevant to the content of this article.

A Properties of the Fréchet derivatives

In this section we consider the Fréchet derivatives of the immersion $f$ defined in (22).
A.1 Basic variational calculus

First, we recall some basic elements of variational derivatives that we used repeatedly. For details, see for instance [25 Appendix C]

Fréchet derivative. Let $X, Y$ be two Banach spaces, $U$ be an open set in $X$. For a map $g : U \subset X \to Y$ and a vector $u \in U$, the Fréchet derivative $dg(u)$ is a linear map from $X \to Y$ such that $g(u + v) - g(u) - dg(u)v = o(\|v\|_X)$ for every $v \in X$. If $dg(u)$ exists at $u$, then it is unique. If $dg(u)$ exists for every $u \in U$, and the map $u \mapsto dg(u)$ is continuous from $U$ to the space of linear operators $L(X, Y)$, then we say $g$ is $C^1$ on $U$. In this case, $dg(u)$ is uniquely given by

$$v \mapsto \frac{dg(u + tv)}{dt}|_{t=0} \quad (v \in X).$$

Iteratively, we can define higher order derivatives this way.

Gradient and Hessian. If $X$ is a Hilbert space over scalar field $Y$, then by Riesz representation, we can identify $dg(u)$ as an element in $X$, denoted $g'(u)$. The vector $g'(u)$ is called the $X$-gradient of $g$. Similarly, we denote $g''(u)$ the second-order Fréchet derivative $d^2g(u)$. If $g$ is $C^2$, then $g''$ can be identified as a symmetric linear operator uniquely determined by the relation

$$\langle g''(u)v, w \rangle = \frac{d^2g(u + tv + tw)}{dt^2}|_{t=1} = o(\|v\| Y).$$

Expansion. Let $X$ be a Hilbert space over scalar field $Y$. Suppose $g$ is $C^2$ on $U \subset X$. Define a scalar function $\phi(t) := g(u + tv)$ for vectors $v, w$ such that $v + tw \in U$ for every $0 \leq t \leq 1$. Then the elementary Taylor expansion at $\phi(1)$ gives

$$g(v + w) = g(v) + \langle g'(v), w \rangle + \frac{1}{2} \langle g''(v)w, w \rangle + o(\|w\|^2).$$

Here we have used the definition of $g'$ and $g''$ from the last paragraph.

Composition. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Fix $r > d/2$, $f \in C^{r+1}(\mathbb{R}^n)$. For $u : \Omega \to \mathbb{R}^n$, define a map $g : u \mapsto f \circ u$. Then $g : H^r(\Omega) \to H^r(\Omega)$, and is $C^1$. The Fréchet derivative is given by $v \mapsto \nabla f \cdot v$.

A.2 Various uniform estimates

In this section we assume $\epsilon \ll 1$ in (1). For two complex numbers, we use the real inner product $\langle u, v \rangle = \Re(u\bar{v})$.

Fix some $\alpha > 0$. Using the definition of the immersion $f$ in Section 2.2 for $\sigma = (\lambda, \gamma) \in \Sigma_{\epsilon, \alpha}$ and $(\mu, \xi) \in Y^k$, we compute the Fréchet derivative of $f$ as

$$df(\sigma)(\mu, \xi) = e^{i\lambda} (i\mu\psi_\gamma - \nabla_x \psi_\gamma \cdot \xi).$$

This is uniformly bounded in $\sigma$ as an operator from $Y^0 \to X^0$, since

$$\|df(\sigma)(\mu, \xi)\|_{X^0} \leq \mu \|\psi_\gamma\|_{X^0} + \|\nabla_x \psi_\gamma \cdot \xi\|_{X^0} \leq \mu \left(\|\psi_\gamma^{(1)}\|_{L^2(\omega)} + O(\epsilon^{2+\alpha/2})\right) + \left(\|\nabla_x \psi_\gamma^{(1)}\|_{L^2(\omega)} + O(\epsilon^{1+\alpha/2})\right)
\|\xi\|_{C^0} \leq C(\Omega) \log \epsilon^{1/2} \|\sigma\|_{Y^0}.$$
Here we have used \([14]\) and \([25]\). Using \([37]\) and \([39]\), one can get a uniform estimate for \(d_\sigma df(\sigma) : Y^0 \to L(Y^0, X^0)\).

Write an element in \(Y^k\) as \(\sigma = ([\sigma], [\sigma]_\xi)\). The adjoint operator \(df(\sigma)^*\) is determined by the relation

\[
\left< df(\sigma)(\mu, \xi), \phi \right> = \int_x \int_z \left< e^{i\lambda}(i\mu \psi_\gamma - \nabla_x \psi_\gamma) \cdot \xi, \phi \right>
= \mu \left[ df(\sigma)^* \phi \right]_\lambda + \int_z \xi \cdot \left[ df(\sigma)^* \phi \right]_\gamma \quad (\phi \in X^0).
\]

Here and in the remaining of this section, it is understood that various integrals are taken over \((x, z) \in \omega \times I = \Omega\).

By Fubini’s Theorem and the identity \(* v \cdot w, u \rangle = \langle u, v \rangle \cdot w\) for real vector \(w\), the above relation implies

\[
df(\sigma)^* \phi = \left( \int_x \int_z \phi, ie^{i\lambda} \psi_\gamma \right), - \int_x \int_z \langle e^{i\lambda} \nabla_x \psi_\gamma(x, \cdot), \phi(x, \cdot) \rangle\right).
\]

This adjoint operator is also uniformly bounded in \(\sigma\) with \(\|df(\sigma)^*\|_{X^0 \to Y^0} \leq C (\log \epsilon)^{1/2}\). Moreover, by Sobolev embedding, \(df(\sigma)^*\) maps \(X^2\) into \(Y^0\). Using \([88]\) and \([39]\), one can get a uniform estimate for \(d_\sigma df(\sigma)^* : Y^0 \to L(X^0, Y^0)\).

The operator \(\mathcal{J}_\sigma\) is defined in \([26]\). It induces a symplectic form w.r.t. the inner product \([19]\) on the tangent bundle \(T\Sigma\), since

\[
\langle \mathcal{J}_\sigma \chi, \chi \rangle = \langle g^*_\sigma g^{-1}_\sigma \chi, \chi \rangle = \langle J^{-1} g^*_\sigma g_\sigma \chi, g_\sigma \chi \rangle = 0 \quad (\chi \in Y^0).
\]

Using \([37]\)-\([38]\), we can compute \(\mathcal{J}_\sigma\) explicitly as

\[
[J_\sigma(\mu, \xi)]_\lambda = - \int_x \int_z \langle \nabla_x \psi_\gamma \cdot \xi, \psi_\gamma \rangle,

[J_\sigma(\mu, \xi)]_\gamma = \mu \int_x \int_z \langle \nabla_x \psi_\gamma(x, \cdot), \psi_\gamma(x, \cdot) \rangle + \int_x \langle \nabla_x \psi_\gamma(x, \cdot) \cdot J_\xi(\cdot), \nabla_x \psi_\gamma(x, \cdot) \rangle.
\]

Here we have used the identity that for any complex-valued \(C^1\) function \(\phi\) and vector field \(\chi\) in \(\mathbb{R}^2\), by the Cauchy-Riemann equation,

\[
-i \nabla \phi \cdot \chi = \nabla \phi \cdot J \chi.
\]

One can also write \(\mathcal{J}_\sigma = \mathcal{J}_\sigma(\mu, \xi(z))\) as the multiplication operator by
the matrix $B_{ij}$, where

\begin{align*}
B_{ii} &= 0 \quad (i = 1, 2, 3), \\
B_{12} &= -\int_x \int_z \langle \frac{\partial \psi}{\partial x_1}(x, z), \psi_\gamma(x, z) \rangle, \\
B_{13} &= \int_x \int_z \langle \frac{\partial \psi}{\partial x_1}(x, z), \frac{\partial \psi}{\partial x_2}(x, z) \rangle, \\
B_{21} &= \int_x \int_z \langle \frac{\partial \psi}{\partial x_1}(x, z), \psi_\gamma(x, z) \rangle, \\
B_{23} &= \int_x \int_z \langle \frac{\partial \psi}{\partial x_2}(x, z), \psi_\gamma(x, z) \rangle, \\
B_{31} &= \int_x \int_z \langle \frac{\partial \psi}{\partial x_1}(x, z), \psi_\gamma(x, z) \rangle, \\
B_{32} &= -\int_x \langle \frac{\partial \psi}{\partial x_1}(x, z), \frac{\partial \psi}{\partial x_2}(x, z) \rangle.
\end{align*}

(90)

Here we have used the Cauchy-Riemann equation to eliminate certain cross-terms of the form $\langle \frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2} \rangle$.

Using (90), the fact $\|\nabla \psi^{(1)}\|_{L^2(\omega)} \sim C(\omega) |\log \epsilon|^{1/2}$ (see [14] and [4] Chap. V.1), and the asymptotics (10) for $\psi^{(1)}$, one can check that $J_\sigma$ is invertible and satisfies the uniform estimates

\begin{align*}
\|J_\sigma\|_{Y_k \to Y_k} &\leq C(\omega) |\log \epsilon|^2, \\
\|J_\sigma^{-1}\|_{Y_k \to Y_k} &\leq C(\omega) |\log \epsilon|^{-2}
\end{align*}

(91) (92)

for every $k \in \mathbb{N}$.

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