Small weights in Caccioppoli’s inequality and applications to Liouville-type theorems for non-standard problems

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Abstract

Using a variant of Caccioppoli’s inequality involving small weights, i.e. weights of the form \((1 + |\nabla u|^2)^{-\alpha/2}\) for some \(\alpha > 0\), we establish several Liouville-type theorems under general non-standard growth conditions.

Dedicated to the 70th birthday of Gregory Seregin

1 Introduction

Throughout this manuscript we always suppose that \(u: \mathbb{R}^2 \to \mathbb{R}, u \in C^2(\mathbb{R}^2)\), is a solution of the nonlinear equation

\[
\text{div} \left[ \nabla f(\nabla u) \right] = 0 \quad \text{on} \quad \mathbb{R}^2.
\]

Our main goal is the discussion of Liouville-type results under rather general hypotheses on the convex density \(f: \mathbb{R}^2 \to \mathbb{R}\) including non-standard growth conditions such as the case of linear growth or even allowing a certain degree of anisotropy in the superlinear case. For technical simplicity we restrict ourselves to the twodimensional case.

It is out of reach to give a complete overview on all the recent contributions on Liouville-type results. We refer to the beautiful survey of Farina [1] in the case of general elliptic problems including a lot of historical references. We also refer to Seregin’s discussion [2] of Liouville-type theorems in the case of the Navier-Stokes equations.

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Contributions in the case of linear or anisotropic growth are quite rarely found. We just mention the papers [3], [4], [5], [6] and the references quoted therein.

Before going into details we fix our main assumption which will be supplemented with appropriate hypotheses adapted to the applications of Section 4 and of Section 5.

**Assumption 1.1.** The convex energy density \( f : \mathbb{R}^2 \to \mathbb{R} \) is of class \( C^2(\mathbb{R}^2) \) and satisfies the non-uniform ellipticity condition

\[
c_1 (1 + |Z|^2)^{- \frac{\mu}{2}} |Y|^2 \leq D^2 f(Z)(Y,Y) \leq c_2 (1 + |Z|^2)^{- \frac{\mu}{2}} |Y|^2
\]

(1.2)

for all \( Z, Y \in \mathbb{R}^2 \) with exponents \( \mu > 1 \), \( \overline{\mu} \leq 1 \) and constants \( c_1, c_2 > 0 \).

Condition (1.2) also serves as one main assumption in the recent paper [6] on Liouville-type results in two dimensions for functionals satisfying a linear growth condition. These results are restricted to the case \( \overline{\mu} = 1 \), for instance we have:

**Theorem 1.1.** (Theorem 1.1, c), [6], the case \( N = 1 \) Let \( u \in C^2(\mathbb{R}^2) \) denote a solution of (1.1) with density \( f \) such that for some real number \( M > 0 \) we have

\[
|\nabla f(Z)| \leq M \quad \text{for all } Z \in \mathbb{R}^2
\]

and such (1.2) holds with the choice \( \overline{\mu} = 1 \).

If we have

\[
\limsup_{|x| \to \infty} \frac{|u(x)|}{|x|} < \infty,
\]

(1.3)

then \( u \) is affine.

Let us give some further explanations concerning the hypotheses of Theorem 1.1 from (1.2) and the boundedness of \( \nabla f \) it follows that \( f \) is of linear growth, and this actually holds for arbitrary exponents \( \overline{\mu} \leq 1 < \mu \). We refer to Lemma 2.1 in [6].

At the same time, if \( f \) is of linear growth satisfying inequality (1.2), then according to Lemma 1.1 of [7] we necessarily get the restriction \( \overline{\mu} \leq 1 \), whereas the bound \( \mu > 1 \) is an immediate consequence of the linear growth of \( f \).

To sum up, Theorem 1.1 addresses the case of energy densities with linear growth, but just covers the “limit case” for which \( \overline{\mu} = 1 \). So the first question arises, whether the result of Theorem 1.1 keeps valid, if we allow exponents \( \overline{\mu} < 1 \).
Closely related is the following setting in the case of superlinear growth. Suppose that we have \( \frac{1}{\mu} = 2 - q, q > 1 \), on the right-hand side. Then we are interested in the anisotropic case, which means that we do not narrow our discussion by assuming \( q \)-power growth of \( f \). We just impose the inequality
\[
a|Z|^s - b \leq f(Z) \quad \text{for all } Z \in \mathbb{R}^2,
\]
with exponent \( 1 < s \leq q \) and with constants \( a, A > 0, a, b \geq 0 \) as lower bound for the density \( f \) (see Section 5 for some further comments on the assumptions). A Liouville-type result in this setting is established in Section 5. We emphasize that we are not aware of similar Liouville-type theorems w.r.t. this general kind of anisotropic hypotheses.

Let us fix the notation.

**Notation.** We always abbreviate
\[
\Gamma := 1 + |\nabla u|^2, \quad \Gamma_Q := 1 + |\nabla u - Q|^2 \quad \text{for vectors } Q \in \mathbb{R}^2.
\]

For fixed radii \( r, R > 0 \) we consider open disks \( B_r \) and define
\[
T_R := B_{2R} - B_{R}, \quad \hat{T}_R := B_{5R/2} - B_{R/2},
\]
where the center \( x_0 \) is not indicated.

Moreover, for \( r > 0 \) we let
\[
(\nabla u) = (\nabla u)_r = \int_{B_r} \nabla u \, dx.
\]

Then we have

**Theorem 1.2.** Assume that Assumption \( 1.1 \) holds. Moreover, suppose that there are real numbers \( \gamma, \gamma_Q \geq 0 \) such that
\[
\gamma + \gamma_Q < \frac{1}{2} \quad \text{(1.4)}
\]
and that there exist \( Q = Q(R) \in \mathbb{R}^2 \) such that
\[
\liminf_{R \to \infty} \frac{1}{R^2} \Xi(R) := \liminf_{R \to \infty} \frac{1}{R^2} \int_{T_R} \Gamma^{\frac{-2q}{2q-2}} \Gamma_Q^{-\frac{2q}{2q-2}} |\nabla u - Q|^2 \, dx < \infty. \quad \text{(1.5)}
\]

Then \( u \) is an affine function.
We note that Theorem 1.2 just relies on condition (1.2) and it does not matter, whether the energy density $f$ is of linear growth or even shows a completely anisotropic behaviour. Moreover, the conclusion of the theorem is independent of the value of the exponent $\mu$.

Let us shortly comment on the main idea for the proof of Theorem 1.2 recalling the approach towards Theorem 1.1. This theorem is proved by first using a Caccioppoli-type inequality for the differentiated Euler equation. Since we have $\widehat{\mu} = 1$ as a hypothesis of Theorem 1.1, the right-hand side of this inequality can be measured in terms of the quantity $\nabla f(\nabla u) \cdot \nabla u$ which in turn occurs on the left-hand side of the weak form of (1.1) applied to a suitable testfunction.

In the case $\widehat{\mu} < 1$ a serious gap arises which cannot be closed by obvious arguments. Here, as the main new feature, we introduce small weights in Caccioppoli’s inequality such that both sides again fit together. The same arguments bridge the gap in the superlinear anisotropic case.

Theorem 1.2 immediately gives the following elementary corollary.

**Corollary 1.1.** Suppose that we have Assumption 1.1. Then $u$ is an affine function if one of the following conditions hold.

i) $\sup_{x \in \mathbb{R}^2} |\nabla u(x)| < \infty$.

ii) There exists some $\varepsilon > 0$ such that

$$\int_{I_R} \Gamma^\frac{4}{4+\varepsilon} dx < c$$

with a constant $c > 0$ not depending on $R$.

Before formulating more refined corollaries, we establish our Caccioppoli-type estimate in the next section as the main tool for proving Theorem 1.2 in Section 3.

Section 4 is devoted to applications in the linear growth setting while Section 5 concentrates on the main corollary in the superlinear case.

We finally note that the generality of Theorem 1.2 may be used to discuss a series of other applications which is left to the particular interest of the reader.
2 A Caccioppoli-type inequality

We start with a Caccioppoli-type inequality weighted with negative powers of $\Gamma$ and $\Gamma_Q$.

**Lemma 2.1.** Given Assumption 1.1 we fix $Q \in \mathbb{R}^2$, consider real numbers $s_Q > -1/4, s_1 > -1/4$ and let

$$c_Q := \begin{cases} 4|s_Q| & \text{if } s_Q \leq 0, \\ 0 & \text{if } s_Q > 0, \end{cases} \quad c_1 := \begin{cases} 4|s_1| & \text{if } s_1 \leq 0, \\ 0 & \text{if } s_1 > 0, \end{cases}$$

and suppose that $\eta \in C_0^\infty(\mathbb{R}^2)$, $0 \leq \eta \leq 1$.

If $c_Q + c_1 < 1$, then we have (summation w.r.t. $\alpha = 1, 2$)

$$\left[1 - c_Q - c_1\right] \int_{\mathbb{R}^2} D^2 f(\nabla u) \left(\nabla \partial_\alpha u, \nabla \partial_\alpha u\right) \Gamma_Q^{s_Q} \Gamma_Q^{s_1} \eta^2 \, dx$$

$$\leq c \left[ \int_{\text{spt} \nabla \eta} D^2 f(\nabla u) \left(\nabla \partial_\alpha u, \nabla \partial_\alpha u\right) \Gamma_Q^{s_Q} \Gamma_Q^{s_1} \eta^2 \, dx \right]^{\frac{1}{2}}$$

$$- \left[ \int_{\text{spt} \nabla \eta} D^2 f(\nabla u) \left(\nabla \eta, \nabla \eta\right) \nabla u \cdot \left(\nabla \partial_\alpha u, \nabla \partial_\alpha u\right) \Gamma_Q^{s_Q} \Gamma_Q^{s_1} \eta^2 \, dx \right]^{\frac{1}{2}}$$

where the constant $c$ is not depending on $\eta$. In particular we have

$$\int_{\mathbb{R}^2} D^2 f(\nabla u) \left(\nabla \partial_\alpha u, \nabla \partial_\alpha u\right) \Gamma_Q^{s_Q} \Gamma_Q^{s_1} \eta^2 \, dx$$

$$\leq c \int_{\text{spt} \nabla \eta} D^2 f(\nabla u) \left(\nabla \eta, \nabla \eta\right) \left(\partial_\alpha u - Q_\alpha\right)^2 \Gamma_Q^{s_Q} \Gamma_Q^{s_1} \, dx. \quad (2.1)$$

**Proof.** We first consider the case that $-1/4 < s_Q \leq 0$. For $\alpha = 1, 2$ and for all $\psi \in C_0^\infty(\mathbb{R}^2)$ equation (1.1) yields

$$0 = \int_{\mathbb{R}^2} D^2 f(\nabla u) \left(\nabla \partial_\alpha u, \nabla \psi\right) \, dx. \quad (2.3)$$

Inserting $\psi := \eta^2 \left(\partial_\alpha u - Q_\alpha\right) \Gamma_Q^{s_Q} \Gamma_Q^{s_1}$ in (2.3) we obtain for $\alpha = 1, 2$ and any testfunction $\eta$

$$\int_{\mathbb{R}^2} D^2 f(\nabla u) \left(\nabla \partial_\alpha u, \nabla \partial_\alpha u\right) \Gamma_Q^{s_Q} \Gamma_Q^{s_1} \eta^2 \, dx$$

$$= - \int_{\mathbb{R}^2} D^2 f(\nabla u) \left(\nabla \partial_\alpha u, (\partial_\alpha u - Q_\alpha) \Gamma_Q^{s_Q}\right) \Gamma_Q^{s_1} \eta^2 \, dx$$

$$- \int_{\mathbb{R}^2} D^2 f(\nabla u) \left(\nabla \partial_\alpha u, (\partial_\alpha u - Q_\alpha) \Gamma_Q^{s_1}\right) \Gamma_Q^{s_Q} \eta^2 \, dx$$

$$- 2 \int_{\mathbb{R}^2} D^2 f(\nabla u) \left(\nabla \partial_\alpha u, \nabla \eta\right) \left(\partial_\alpha u - Q_\alpha\right) \Gamma_Q^{s_Q} \Gamma_Q^{s_1} \eta \, dx. \quad (2.4)$$
We denote the bilinear form $D^2 f(\cdot, \cdot)$ by $\langle \cdot, \cdot \rangle$ and discuss the left-hand side of (2.4) by observing that

$$\sum_{\alpha=1}^{2} \left< \partial_{\alpha} \nabla u, \partial_{\alpha} \nabla u \right> \Gamma_{Q}^{sQ} \geq \sum_{\alpha=1}^{2} \left< \partial_{\alpha} u, \partial_{\alpha} u \right> \left[ (\partial_1 u - Q_1)^2 + (\partial_2 u - Q_2)^2 \right] \Gamma_{Q}^{sQ-1}$$

Moreover, for the first integral on the right-hand side of (2.4) we write

$$\sum_{\alpha=1}^{2} \left< \partial_{\alpha} \nabla u, \partial_{\alpha} u - Q_{\alpha} \right> \nabla \Gamma_{Q}^{sQ}$$

On account of

$$\left| \left< \partial_{1} u - Q_{1}, \partial_{1} \nabla u, \partial_{2} u - Q_{2}, \partial_{2} \nabla u \right> \right| \leq \frac{1}{2} \sum_{\alpha=1}^{2} \left< \partial_{\alpha} u - Q_{\alpha}, \partial_{\alpha} \nabla u \right>$$

we obtain from (2.6)

$$\left| \sum_{\alpha=1}^{2} \left< \partial_{\alpha} \nabla u, \partial_{\alpha} u - Q_{\alpha} \right> \nabla \Gamma_{Q}^{sQ} \right| \leq 4 |s_Q| \sum_{\alpha=1}^{2} \left< \partial_{\alpha} u - Q_{\alpha}, \partial_{\alpha} \nabla u \right> \Gamma_{Q}^{sQ-1}. \quad (2.7)$$

Combining (2.7) and (2.5) we get (where from now on we take the sum w.r.t. $\alpha = 1, 2$)

$$- \int_{\mathbb{R}^2} D^2 f(\nabla u) \left( \partial_{\alpha} \nabla u, (\partial_{\alpha} u - Q_{\alpha}) \nabla \Gamma_{Q}^{sQ} \right) \Gamma_{Q_{1}}^{s_1} \eta^2 \, dx \leq 4 |s_Q| \int_{\mathbb{R}^2} D^2 f(\nabla u) \left( \partial_{\alpha} \nabla u, \partial_{\gamma} \nabla u \right) \Gamma_{Q}^{sQ} \Gamma_{Q_{1}}^{s_1} \eta^2 \, dx. \quad (2.8)$$
In the case $0 < s_Q$ we just use the positive sign of
\[
\left< \partial_\alpha \nabla u, (\partial_\alpha u - Q_\alpha), \nabla \Gamma_{Q}^{s_Q} \right> = \frac{1}{2} \left< \nabla (\partial_\alpha u - Q_\alpha)^2, \nabla \Gamma_{Q}^{s_Q} \right>.
\tag{2.9}
\]
Having established (2.8) and (2.9) we recall $c_Q := 4|s_Q|$ if $-1/4 < s_Q \leq 0$ and $c_Q = 0$ if $s_Q > 0$. Then we summarize (2.8) and (2.9) by writing
\[
- \int_{\mathbb{R}^2} D^2 f(\nabla u) (\partial_\alpha \nabla u, (\partial_\alpha u - Q_\alpha) \nabla \Gamma_{Q}^{s_Q}) \Gamma_{Q}^{s_Q} \eta^2 \, dx 
\leq c_Q \int_{\mathbb{R}^2} D^2 f(\nabla u) (\partial_\alpha \nabla u, \partial_\alpha \nabla u) \Gamma_{Q}^{s_Q} \eta \, dx.
\tag{2.10}
\]
In the same way we recall $c_1 := 4|s_1|$ if $-1/4 < s_1 \leq 0$ and $c_1 = 0$ if $s_1 > 0$. With exactly the same reasoning as above we additionally obtain
\[
- \int_{\mathbb{R}^2} D^2 f(\nabla u) (\partial_\alpha \nabla u, (\partial_\alpha u - Q_\alpha) \nabla \Gamma_{s_1}^{s_Q}) \Gamma_{Q}^{s_Q} \eta^2 \, dx 
\leq c_1 \int_{\mathbb{R}^2} D^2 f(\nabla u) (\partial_\alpha \nabla u, \partial_\alpha \nabla u) \Gamma_{Q}^{s_Q} \eta \, dx.
\tag{2.11}
\]
Returning to (2.4) and using (2.10) and (2.11) we get
\[
[1 - c_e - c_1] \int_{\mathbb{R}^n} D^2 f(\nabla u) (\nabla \partial_\alpha u, \nabla \partial_\alpha u) \Gamma_{Q}^{s_Q} \eta^2 \, dx 
\leq -2 \int_{\mathbb{R}^n} D^2 f(\nabla u) (\eta \nabla \partial_\alpha u, (\partial_\alpha u - Q_\alpha) \nabla \eta) \Gamma_{Q}^{s_Q} \Gamma_{s_1}^{s_Q} \eta \, dx.
\tag{2.12}
\]
On the right-hand side of (2.12) we observe that the integration is performed w.r.t. the domain spt $\nabla \eta$ and an application of the Cauchy-Schwarz inequality completes the proof of Lemma 2.1. \qed

3 Proof of Theorem 1.2

For the proof of Theorem 1.2 we fix a disk $B_r \subset \mathbb{R}^2$. We apply the Sobolev-Poincaré inequality to the solution $u \in C^2(\mathbb{R}^2)$ under consideration and get
the inequality
\[
\int_{B_r} |\nabla u - (\nabla u)|^2 \, dx \leq c \left[ \int_{B_r} |\nabla^2 u| \, dx \right]^2
\]
\[
\leq c \left[ \int_{B_r} \Gamma^{-\frac{\gamma}{2}} |\nabla^2 u|^2 \Gamma^{-\frac{\gamma}{2}} \Gamma_Q^{-\frac{\gamma}{2}} \, dx \right]^{\frac{1}{2}} \left[ \int_{B_r} \Gamma^{\frac{\mu+\gamma}{2}} \Gamma_Q^{-\frac{\gamma}{2}} \, dx \right]^{\frac{1}{2}}
\]
\[
\leq c \int_{B_r} \Gamma^{-\frac{\gamma}{2}} |\nabla^2 u|^2 \Gamma^{-\frac{\gamma}{2}} \Gamma_Q^{-\frac{\gamma}{2}} \, dx ,
\]
where we used the fact that $|\nabla u|$ is bounded on the fixed ball $B_r$, however $c$ may depend on the radius $r$.

Now we choose $R \gg r$ and let $\eta \in C_0^\infty (B_{2R})$, $0 \leq \eta \leq 1$, such that $\eta \equiv 1$ on $B_R$, $|\nabla \eta| \leq c/R$. Then (2.1) of Lemma 2.1 gives recalling (1.4)
\[
\int_{B_r} \Gamma^{-\frac{\gamma}{2}} |\nabla^2 u|^2 \Gamma^{-\frac{\gamma}{2}} \Gamma_Q^{-\frac{\gamma}{2}} \, dx
\]
\[
\leq \int_{B_{2R}} \Gamma^{-\frac{\gamma}{2}} |\nabla^2 u|^2 \Gamma^{-\frac{\gamma}{2}} \Gamma_Q^{-\frac{\gamma}{2}} \eta^2 \, dx
\]
\[
\leq c \left[ \int_{\text{spt} \eta} D^2 f \left( \nabla \partial_\alpha u, \nabla \partial_\alpha u \right) \Gamma^{-\frac{\gamma}{2}} \Gamma_Q^{-\frac{\gamma}{2}} \eta^2 \, dx \right]^{\frac{1}{2}}
\]
\[
\cdot \left[ \frac{c}{R^2} \int_{T_R} \Gamma^{-\frac{\gamma}{2}} \Gamma_Q^{-\frac{\gamma}{2}} |\partial_\alpha u - Q_\alpha|^2 \, dx \right]^{\frac{1}{2}}
\]
\[
= \left( c I(R) \cdot \left[ \frac{1}{R^2} \Xi(R) \right] \right)^{\frac{1}{2}}.
\]

We observe that (2.2) implies (again recalling (1.4))
\[
\int_{B_r} D^2 f (\nabla u) \left( \nabla \partial_\alpha u, \nabla \partial_\alpha u \right) \Gamma^{-\frac{\gamma}{2}} \Gamma_Q^{-\frac{\gamma}{2}} \, dx
\]
\[
\leq \frac{c}{R^2} \int_{T_R} \Gamma^{-\frac{\gamma}{2}} \Gamma_Q^{-\frac{\gamma}{2}} |\partial_\alpha u - Q_\alpha|^2 \, dx ,
\]

8
hence we can make use of our assumption (1.5) by choosing a suitable sub-
sequence $R \to \infty$ and obtain
\[
\int_{\mathbb{R}^2} D^2 f(\nabla u)(\nabla \partial_\alpha u, \nabla \partial_\alpha u) \Gamma^{-\frac{\mu}{2}} \Gamma^{-\frac{\nu}{2}} dx < \infty,
\]
thus $I(R) \to 0$ as $R \to \infty$.

With this information we return to (3.1), (3.2) and obtain
\[
\int_{B_r} |\nabla u - (\nabla u)|^2 dx \leq I(R) \cdot \left[ \frac{1}{R^2} \Xi(R) \right]^{\frac{1}{2}}
\]
with $I(R) \to 0$ as $R \to \infty$. This proves the theorem with the help of hy-
pothesis (1.5). \hfill \Box

4 Applications to the linear growth case

Throughout this section we replace Assumption 1.1 by a suitable stronger
variant specifying the linear growth condition. More precisely, we require

**Assumption 4.1.** The convex energy density $f: \mathbb{R}^2 \to \mathbb{R}$ is of class $C^2(\mathbb{R}^2)$
and satisfies the non-uniform ellipticity condition (1.2) with exponents $\mu > 1,$
$\overline{\mu} \leq 1$.

Moreover we assume that there exists a constant $M > 0$ such that for all
$Z \in \mathbb{R}^2$
\[
|\nabla f(Z)| \leq M.
\]
As outlined in [6] (compare the discussion after inequality (1.3) in this ref-
ence), Assumption 4.1 implies with constants $a, A > 0, b, B \geq 0$ and for
all $Z \in \mathbb{R}^2$ the linear growth condition
\[
a|Z| - b \leq f(Z) \leq A|Z| + B.
\]
Before summarizing some corollaries of Theorem 1.2 in this particular setting,
we will show the following proposition which follows the line of the proof of
Theorem 1.1 of [6].

**Proposition 4.1.** Suppose that we have Assumption 4.1. Then
\[
\limsup_{|x| \to \infty} \frac{|u(x)|}{|x|} < \infty
\]
implies
\[
\int_{T_R} \Gamma^{\frac{1}{2}} dx \leq c [1 + R^2].
\]
Proof of Proposition 4.1. By the convexity of $f$ we have for all $Z \in \mathbb{R}^2$
\[ f(Z) \leq f(0) + \nabla f(Z) \cdot Z, \quad (4.5) \]
hence for any $\eta \in C^1_c(\mathbb{R}^2)$, $\eta \equiv 1$ on $T_R$, $0 \leq \eta \leq 1$, $|\nabla \eta| \leq c/R$, we obtain using (4.1), (4.2) and (4.5)
\[ \int_{T_R} \Gamma^2 \, dx \leq c \int_{\hat{T}_R} [1 + f(\nabla u)] \eta^2 \, dx \]
\[ \leq cR^2 + c \int_{\hat{T}_R} \nabla f(\nabla u) \cdot \nabla \eta^2 \, dx. \quad (4.6) \]
Now we use the weak form of equation (1.1) with testfunction $\psi = u\eta^2$, i.e.
\[ 0 = \int_{\mathbb{R}^2} \nabla f(\nabla u) \cdot \nabla [u\eta^2] \, dx \]
\[ = \int_{\hat{T}_R} \nabla f(\nabla u) \cdot \nabla u\eta^2 \, dx + \int_{\hat{T}_R} \nabla f(\nabla u) \cdot \nabla \eta^2 \eta u \, dx, \]
\[ \text{hence we have} \]
\[ \int_{\mathbb{R}^2} \nabla f(\nabla u) \cdot \nabla u\eta^2 \, dx \leq \frac{c}{R} \sup_{\hat{T}_R} |u| R^2. \quad (4.7) \]
Recalling our assumption (4.3) and combing (4.6) and (4.7) we find the claim (4.4) of the proposition. \[ \square \]
The first corollary to Theorem 1.2 immediately yields an extension of Theorem 1.1.

Corollary 4.1. Theorem 1.1 remains valid for $1/2 < \mu < 1$.

Proof of Corollary 4.1. We choose $\gamma_Q = 0$, $\gamma$ sufficiently close to $-1/2$ and let $Q = 0$, i.e.
\[ \Xi(R) = \int_{T_R} \Gamma^{-\frac{1}{2}} |\nabla u|^2 \, dx \leq \int_{T_R} |\nabla u| \, dx, \]
and apply Proposition 4.1 to obtain hypothesis (1.5) of Theorem 1.2. \[ \square \]

The next corollary gives an refinement of Corollary 4.1 by taking a measure for the relative oscillation into account (compare (4.8) and (4.9)).

Corollary 4.2. Suppose that we have Assumption 4.1 with $1/2 < \bar{\mu} < 1$.
For given $\gamma$, $\gamma_Q > 0$, $1 - \bar{\mu} < \gamma + \gamma_Q < 1/2$, we let
\[ p = \frac{1}{2 - \gamma_Q - \gamma - \bar{\mu}} > 1, \quad q = \frac{1}{\gamma_Q + \gamma + \bar{\mu} - 1} \]
and define
\[ \Theta_Q(x) := \left[ \frac{\Gamma_Q}{\Gamma} \right]^{2-\gamma_Q}, \quad \Theta(R) := \inf_{Q} \frac{1}{|T_R|} \int_{T_R} \Theta_Q(x) \, dx \leq 1. \] (4.8)

If we suppose that for all \( R \) sufficiently large
\[ \sup_{T_R} |u| \leq cR\Theta(R)^{-\frac{2}{q}} \] (4.9)
with a constant \( c > 0 \) not depending on \( R \), then \( u \) is an affine function.

**Proof of Corollary 4.2.** We estimate
\[ \int_{T_R} \Gamma^{-\gamma_Q-\gamma} \Gamma_Q^{-\gamma} |\nabla u - Q|^2 \, dx \leq \int_{T_R} \Theta_Q^{\frac{1}{2}} \Gamma_{2^{-\gamma_Q-\gamma}} \, dx \]
and obtain
\[ \Xi(R) \leq \int_{T_R} \Theta_Q^{\frac{1}{2}} \Gamma_{2^{-\gamma_Q-\gamma}} \, dx \]
\[ \leq \left[ \int_{T_R} \Theta_Q \, dx \right]^{\frac{1}{q}} \left[ \int_{T_R} \Gamma_0 \, dx \right]^{\frac{1}{p}}. \] (4.10)

We choose \( Q = Q(R) \) such that
\[ \frac{1}{|T_R|} \int_{T_R} \Theta_Q \, dx \leq 2\Theta(R). \]

Then (4.10) implies
\[ \Xi(R) \leq c\left[ \Theta(R)R^2 \right]^{\frac{1}{q}} \left[ \int_{T_R} \Gamma_0 \, dx \right]^{\frac{1}{p}}. \] (4.11)

Discussing the right-hand side of (4.11) we exactly follow the proof of Proposition 4.1 and just insert hypothesis (4.9) in (4.7). This shows
\[ \Xi(R) \leq c\left[ \Theta(R)R^2 \right]^{\frac{1}{q}} \cdot \left[ 1 + \Theta(R)^{-\frac{2}{q}} R^{4} \right]^{\frac{1}{p}}, \]
hence we obtain Corollary 4.2 by recalling \( \frac{1}{p} + \frac{1}{q} = 1 \).

A rather important class of energy densities with linear growth is of splitting-type, i.e. of the form
\[ f(Z) = f_1(Z_1) + f_2(Z_2) \]
with functions $f_1$, $f_2$ of linear growth. The particular features of splitting type energy densities with linear growth are discussed in [7] (compare also [8]). If $f_1$, $f_2$ satisfy (1.2) with exponents $\overline{\mu}_1$, $\overline{\mu}_2 \leq 0$, respectively, then we have $\overline{\mu} = 0$ for the energy density $f$.

Nevertheless we still can derive Liouville-type theorems from Theorem 1.2. In Corollary 4.3 we present an application, where in addition we make explicit use of the flexibility of the vector $Q \in \mathbb{R}^2$ by choosing $Q$ as a mean value. Then a smallness condition imposed on $|\nabla^2 u|$ provides the vanishing of the second derivatives.

**Corollary 4.3.** Suppose that we are given Assumption 4.1 with $\overline{\mu} > -1/2$. If we have for a finite constant $c$ that

$$\sup_{T_R} |\nabla^2 u| \leq c R^{-1}, \quad (4.12)$$

then $u$ is an affine function.

**Proof of Corollary (4.3).** We choose $\gamma = -\overline{\mu} < 1/2$, $\gamma_Q = 0$ such that (1.4) is satisfied. We have to show that (4.12) implies (1.5), i.e. we claim that in this case

$$\Xi(R) = \int_{T_R} |\nabla u - Q|^2 \, dx < c \left[ 1 + R^2 \right], \quad (4.13)$$

where we choose $Q = (\nabla u)_R$. In fact, we have by the Poincaré inequality (see [9], Theorem A.10, as the appropriate variant)

$$\int_{T_R} |\nabla u - Q|^2 \, dx \leq c R^2 \int_{T_R} |\nabla^2 u|^2 \, dx \leq c \left[ \sup_{T_R} |\nabla^2 u| \right]^2 R^4,$$

which proves the corollary on account of the hypothesis (4.12). \hfill $\Box$

## 5 Applications to the superlinear case

We adapt Assumption 4.1 to the case of superlinear growth, i.e. we now require

**Assumption 5.1.** Suppose that we are given numbers $\mu \in \mathbb{R}$, $q > 1$ such that $-\mu \leq q - 2$ and let $\overline{\mu} := 2 - q$ be the exponent on the right-hand side of (1.2), i.e. the convex energy density $f : \mathbb{R}^2 \to \mathbb{R}$ is of class $C^2(\mathbb{R}^2)$ and satisfies the non-uniform ellipticity condition

$$c_1 (1 + |Z|^2)^{-\frac{q}{2}} |Y|^2 \leq D^2 f(Z)(Y,Y) \leq c_2 (1 + |Z|^2)^{\frac{q-2}{2}} |Y|^2 \quad (5.1)$$
for all \( Z, Y \in \mathbb{R}^2 \) with exponents \( \mu > 1, q > 1 \) and constants \( c_1, c_2 > 0 \). Suppose that we have in addition
\[
a|Z|^s - b \leq f(Z) \quad \text{for some } 1 < s \leq q \tag{5.2}
\]
and with constants \( a > 0, b \geq 0 \).

Note that, as in the linear growth case, the auxiliary parameter \( \mu \) needs no further specification in our hypotheses.

Conditions (5.1) and (5.2) are introduced in [10] describing energy densities of \((s, \mu, q)\)-growth, we refer to [11], Section 3.2, for a more detailed discussion.

In particular (5.1) implies with some constant \( M > 0 \) and for all \( Z \in \mathbb{R}^2 \)
\[
|\nabla f(Z)| \leq M \left( 1 + |Z|^2 \right)^{\frac{q-1}{2}}. \tag{5.3}
\]
Examples are given, for instance, by
\[
f_1(Z) = (1 + |Z|^2)^{\frac{s}{2}} + |Z|^2 \quad 1 < s \leq 2,
\]
\[
f_2(Z) = (1 + |Z|^2)^{\frac{s}{2}} + (1 + |Z|^2)^{\frac{s}{2}} \quad 1 < s \leq q.
\]

The main result of this section is

**Corollary 5.1.** Given Assumption [5.1] we suppose in addition that
\[
s > q - \frac{1}{2}. \tag{5.4}
\]
If we have
\[
\limsup_{|x| \to \infty} \frac{|u(x)|}{|x|} < \infty, \tag{5.5}
\]
then \( u \) is an affine function.

**Proof of Corollary [5.7].** In order to apply Theorem [1.2] we let \( Q = 0, \gamma_Q = 0 \) and since we have (5.4) we can choose \( 0 < \gamma < 1/2 \) such that
\[
s > q - \frac{1}{2}. \tag{5.6}
\]
Then, in our main Theorem [1.2] we observe
\[
\Xi(R) \leq \int_{TR} \Gamma^{\frac{2s-\gamma}{2}} \, dx. \tag{5.7}
\]
We follow the proof of Proposition 4.1, where now (4.6) and (4.7) are replaced by (recalling (5.2), (5.3), (5.5) and choosing \( l \in \mathbb{N} \) sufficiently large).

\[
\int_{\hat{T}_R} \Gamma^{\frac{s}{2}} \eta^2 \, dx \leq c \int_{\hat{T}_R} [1 + f(\nabla u)] \eta^2 \, dx \\
\leq c R^2 + \frac{c}{R} \sup_{\hat{T}_R} |u| \int_{\hat{T}_R} |\nabla f(\nabla u)| \eta^{2l-1} \, dx \\
\leq c R^2 + c \int_{\hat{T}_R} \Gamma^{\frac{s-1}{2}} \eta^{2l-1} \, dx.
\] (5.8)

Since \( q - 1 < q - \gamma \) we find real numbers \( \hat{q}, \hat{p} > 1 \), \( \frac{1}{\hat{q}} + \frac{1}{\hat{p}} = 1 \) such that

\[
\int_{\hat{T}_R} \Gamma^{\frac{s-1}{2}} \eta^{2l-1} \, dx \leq c \left[ \int_{\hat{T}_R} \Gamma^{\frac{s-1}{2}} \eta^{2l} \, dx \right]^\frac{1}{\hat{q}} R^{\hat{p}}. 
\] (5.9)

From (5.6), (5.8) and (5.9) we obtain

\[
\int_{\hat{T}_R} \Gamma^{\frac{s-1}{2}} \eta^{2l} \, dx \leq c \int_{\hat{T}_R} \Gamma^{\frac{s}{2}} \eta^{2l} \, dx \\
\leq c R^2 + c \left[ \int_{\hat{T}_R} \Gamma^{\frac{s-1}{2}} \eta^{2l} \, dx \right]^\frac{1}{\hat{q}} R^{\hat{p}}. 
\] (5.10)

W.l.o.g. we suppose that

\[
R^2 \leq c \left[ \int_{\hat{T}_R} \Gamma^{\frac{s-1}{2}} \eta^{2l} \, dx \right]^\frac{1}{\hat{q}} R^{\hat{p}}
\]

and (5.10) yields

\[
\left[ \int_{\hat{T}_R} \Gamma^{\frac{s-1}{2}} \eta^{2l} \, dx \right]^{\frac{1-\frac{1}{\hat{q}}}{\frac{1}{\hat{q}}}} \leq c R^{\hat{p}}.
\] (5.11)

Recalling (5.7) and \( 1 - \frac{1}{\hat{q}} = \frac{1}{\hat{p}} \) we have proved (1.5) and this finally implies Corollary 5.1. \( \square \)

References

[1] Farina, A. Liouville-type theorems for elliptic problems. Handbook of differential equations: stationary partial differential equations. Vol. IV, Elsevier/North-Holland, Amsterdam, pages 61–116, 2007.
[2] Seregin, G. Remarks on Liouville type theorems for steady-state Navier-Stokes equations. *Algebra i Analiz*, 30(2):238–248, 2018.

[3] D’Ambrosio, L. Liouville theorems for anisotropic quasilinear inequalities. *Nonlinear Anal.*, 70(8):2855–2869, 2009.

[4] Adamowicz, T.; Górka, P. The Liouville theorems for elliptic equations with nonstandard growth. *Commun. Pure Appl. Anal.*, 14(6):2377–2392, 2015.

[5] Dudek, S. The Liouville-type theorem for problems with nonstandard growth derived by Caccioppoli-type estimate. *Monatsh. Math.*, 192(1):75–91, 2020.

[6] Bildhauer, M.; Fuchs, M. Liouville-type results in two dimensions for stationary points of functionals with linear growth. *arXiv:2101.00623*, 2021.

[7] Bildhauer, M.; Fuchs, M. Splitting type variational problems with linear growth conditions. *J. Math. Sci. (N.Y.), Problems in mathematical analysis. No. 105*, 250(2):45–58, 2020.

[8] Bildhauer, M.; Fuchs, M. Splitting-type variational problems with mixed linear- superlinear growth conditions. *To appear in J. Math. Anal. Appl.*, 2020.

[9] Struwe, M. *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*. Springer, Berlin, 1990.

[10] Bildhauer, M.; Fuchs, M. Partial regularity for variational integrals with (s,μ,q)-growth. *Calc. Var. Partial Differential Equations*, 13(4):537–560, 2001.

[11] Bildhauer, M. *Convex variational problems. Linear, nearly linear and anisotropic growth conditions*, volume 1818 of *Lecture Notes in Mathematics*. Springer, Berlin, 2003.