COEFFICIENT ESTIMATES AND THE FEKETE-SZEGŐ PROBLEM FOR CERTAIN CLASSES OF POLYHARMONIC MAPPINGS

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Abstract. We give coefficient estimates for a class of close-to-convex harmonic mappings, and discuss the Fekete-Szegő problem of it. We also introduce two classes of polyharmonic mappings $\mathcal{H}S_p$ and $\mathcal{H}C_p$, consider the starlikeness and convexity of them, and obtain coefficient estimates on them. Finally, we give a necessary condition for a mapping $F$ to be in the class $\mathcal{H}C_p$.

1. Introduction

Let $D$ denote the unit disk $\{z : |z| < 1, z \in \mathbb{C}\}$, and let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

which are analytic in $D$. Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent. A continuous mapping $f = u + iv$ is a complex-valued harmonic mapping in a domain $D \subset \mathbb{C}$ if both $u$ and $v$ are real harmonic in $D$, i.e., $\Delta u = \Delta v = 0$, where $\Delta$ is the complex Laplacian operator

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For a simply connected domain $D$, we can write $f$ in the form $f = h + \bar{g}$, where $h$ and $g$ are analytic (see [14]). A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$ for all $z \in D$.

A continuous complex-valued mapping $F$ in $D$ is biharmonic if the Laplacian of $F$ is harmonic, i.e., $F$ satisfies the equation $\Delta(\Delta F) = 0$. It can be shown that in a simply connected domain $D$, every biharmonic mapping has the representation

$$F(z) = G_1(z) + |z|^2 G_2(z),$$

where both $G_1$ and $G_2$ are harmonic in $D$.

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More generally, a complex-valued mapping $F$ of a domain $D$ is called polyharmonic (or $p$-harmonic) if $F$ satisfies the equation $\Delta^p F = \Delta(\Delta^{p-1} F) = 0$ for some $p \in \mathbb{N}^+$. In a simply connected domain, a mapping $F$ is polyharmonic if and only if $F$ has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_k(z),$$

where each $G_k$ is harmonic, i.e., $\Delta G_k(z) = 0$ for $k \in \{1, \ldots, p\}$ (see [7, 8]). Obviously, when $p=1$ (resp. $p=2$), $F$ is harmonic (resp. biharmonic). The properties of biharmonic mappings have been investigated by many authors (see [2, 3, 4, 12, 18, 20, 23]). We refer to [7, 8, 9, 10, 11] for the discussions on polyharmonic mappings and [13, 14] for the basic properties of harmonic mappings.

We use $S_H$ to denote the class consisting of univalent harmonic mappings in $D$. Such mappings can be written in the form

$$(1.2) \quad f(z) = h(z) + g(z) = z + \sum_{j=2}^{\infty} a_j z^j + \sum_{j=1}^{\infty} b_j z^j,$$

with $|b_1| < 1$. Let $S_H^*$ and $C_H$ be the subclasses of $S_H$, where the images of $f(\mathbb{D})$ are starlike and convex, respectively. If $b_1 = 0$, then $S_H, S_H^*$ and $C_H$ reduce to the classes $S_H^0, S_H^{0,*}$ and $C_H^0$, respectively. See also [14].

A classical theorem of Fekete and Szegö [16] states that for $f \in S$ of the form (1.1), the functional $|a_3 - \lambda a_2^2|$ satisfies the following inequality:

$$|a_3 - \lambda a_2^2| \leq \begin{cases} 3 - 4\lambda, & \lambda \leq 0, \\ 1 + 2e^{-\frac{3\lambda}{2}}, & 0 \leq \lambda \leq 1, \\ 4\lambda - 3, & \lambda \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each real $\lambda$ there exists a function in $S$ such that equality holds (see [11, 21]). Thus the determination of sharp upper bounds for the nonlinear functional $|a_3 - \lambda a_2^2|$ for any compact family $\mathcal{F}$ of functions in $A$ is often called the Fekete-Szegő problem for $\mathcal{F}$. Many researchers have studied the Fekete-Szegő problem for analytic close-to-convex functions (see [19, 21, 22]). A natural question is whether we can get similar generalizations to harmonic close-to-convex mappings.

In [13], Clunie and Sheil-Small obtained the following result:

**Proposition 1.** ([13, Lemma 5.11]) If $f = h + g \in C_H$, then there exist angles $\alpha$ and $\beta$ such that

$$\text{Re} \left\{ (e^{i\alpha} h'(z) + e^{-i\alpha} g'(z)) (e^{i\beta} - e^{-i\beta} z^2) \right\} > 0$$

for all $z \in \mathbb{D}$.

Our purpose of this paper is twofold. In Section 3, we obtain coefficient estimates of a class of close-to-convex harmonic mappings and, as an application, show upper bounds for the Fekete-Szegő functionals $|a_3 - \lambda a_2^2|$ and $|b_3 - \lambda b_2^2|$. The main results in this section are Theorems 1 and 2. In Section 4, first we obtain two sufficient conditions for mappings $F \in \mathcal{H}_p$ to be starlike with respect to the origin and convex,
respectively, given in Theorems are \([\text{3}]\) and \([\text{4}]\). Then, we establish some coefficient estimates for two classes of polyharmonic mappings \(\mathcal{HS}_p\) and \(\mathcal{HC}_p\). Our main results are Theorems \([\text{5}]\) and \([\text{6}]\). Finally, we obtain a generalization of Proposition \([\text{1}]\) to the class \(\mathcal{HC}_p\), which is Theorem \([\text{7}]\).

2. Preliminaries

In \([\text{5}]\), Avci and Zlotkiewicz introduced the class \(\mathcal{HS}\) of univalent harmonic mappings \(f\) with the series expansion \((1.2)\) such that

\[
\sum_{j=2}^{\infty} j(|a_j| + |b_j|) \leq 1 - |b_1|, \quad (0 \leq |b_1| < 1),
\]

and the subclass \(\mathcal{HC}\) of \(\mathcal{HS}\), where

\[
\sum_{j=2}^{\infty} j^2(|a_j| + |b_j|) \leq 1 - |b_1|, \quad (0 \leq |b_1| < 1).
\]

These two classes constitute a harmonic counterparts of classes introduced by Goodman \([\text{17}]\). They are useful in studying questions of so-called \(\delta\)-neighborhoods originally considered by Ruscheweyh \([\text{20}]\) (see also \([\text{25}]\)) and in constructing explicit \(k\)-quasiconformal extensions of mappings (see Fait et al. \([\text{15}]\)).

We denote by \(\mathcal{H}_p\) the set of polyharmonic mappings \(F\) in \(\mathbb{D}\) with the form:

\[
(2.1) \quad F(z) = \sum_{k=1}^{p} z^{2(k-1)}(h_k(z) + g_k(z)) = \sum_{k=1}^{p} z^{2(k-1)} \sum_{j=1}^{\infty} (a_{k,j}z^j + b_{k,j}z^j),
\]

where \(a_{1,1} = 1, \ |b_{1,1}| < 1\). We say that a univalent polyharmonic mapping \(F\) with \(F(0) = 0\) is starlike with respect to the origin if the curve \(F(re^{i\theta})\) is starlike with respect to the origin for each \(r \in (0,1)\). The following result gives a convenient characterization of this property.

**Proposition 2.** (\([\text{24}]\)) If \(F\) is univalent, \(F(0) = 0\) and \(\frac{\partial}{\partial \theta}\arg F(re^{i\theta}) > 0\) for \(z = re^{i\theta} \neq 0\), then \(F\) is starlike with respect to the origin.

A univalent polyharmonic mapping \(F\) with \(F(0) = 0\) and \(\frac{\partial}{\partial \theta}F(re^{i\theta}) \neq 0\) whenever \(r \in (0,1)\), is said to be convex if the curve \(F(re^{i\theta})\) is convex for each \(r \in (0,1)\).

**Proposition 3.** (\([\text{24}]\)) If \(F\) is univalent, \(F(0) = 0\), \(\frac{\partial}{\partial \theta}F(re^{i\theta}) \neq 0\) whenever \(r \in (0,1)\), and \(\frac{\partial}{\partial \theta} \arg \left(\frac{\partial}{\partial \theta}F(re^{i\theta})\right) > 0\) for \(z = re^{i\theta} \neq 0\), then \(F\) is convex.

In \([\text{25}]\), J. Qiao and X. Wang introduced the class \(\mathcal{HS}_p\) of polyharmonic mappings \(F\) of the form \((2.1)\) satisfying the condition

\[
(2.2) \quad \begin{cases}
\sum_{k=1}^{p} \sum_{j=2}^{\infty} (2(k-1) + j)(|a_{k,j}| + |b_{k,j}|) \leq 1 - |b_{1,1}| - \sum_{k=2}^{p} (2k-1)(|a_{k,1}| + |b_{k,1}|), \\
0 \leq |b_{1,1}| + \sum_{k=2}^{p} (2k-1)(|a_{k,1}| + |b_{k,1}|) < 1,
\end{cases}
\]

\]
and the subclass $\mathcal{HC}_p$ of $\mathcal{HS}_p$, where
\begin{equation}
\left\{ \begin{array}{l}
\sum_{k=1}^{p} \sum_{j=2}^{\infty} (2(k-1) + j^2) \left( |a_{k,j}| + |b_{k,j}| \right) \leq 1 - |b_{1,1}| - \sum_{k=2}^{p} (2k-1)(|a_{k,1}| + |b_{k,1}|), \\
0 \leq |b_{1,1}| + \sum_{k=2}^{p} (2k-1)(|a_{k,1}| + |b_{k,1}|) < 1.
\end{array} \right.
\end{equation}

Obviously, for any $F \in \mathcal{HS}_p$, we have $|F(z)| < 2|z|$ for $z \in \mathbb{D}$. For $p = 1$, the classes $\mathcal{HS}_p$ and $\mathcal{HC}_p$ reduce to $\mathcal{HS}$ and $\mathcal{HC}$, respectively. An important property of these classes is given by the following result.

**Theorem A.** (\cite[Theorem 3.1]{25}) Suppose $F \in \mathcal{HS}_p$. Then $F$ is univalent and sense preserving in $\mathbb{D}$.

3. **Coefficient estimates for a class of close-to-convex harmonic mappings**

In this section, we will consider the coefficient estimates and the Fekete-Szegő problem of mappings of the class $\mathcal{F}$, defined as follows. In \cite{6}, Bharanedhar and Ponnusamy obtained the following result:

**Theorem B.** (\cite[Theorem 1]{6}) Let $f = h + \overline{g}$ be a harmonic mapping of $\mathbb{D}$, with $h'(0) \neq 0$, which satisfies
\begin{equation}
g'(z) = e^{i\theta}zh'(z) \quad \text{and} \quad \text{Re} \left( 1 + z \frac{h''(z)}{h'(z)} \right) > -\frac{1}{2}
\end{equation}

for all $z \in \mathbb{D}$. Then $f$ is a univalent close-to-convex mapping in $\mathbb{D}$.

We use $\mathcal{F}$ to denote the class of harmonic mapping $f$ in $\mathbb{D}$ of the form \cite{12}, satisfying 3.1. Let $\mathcal{H}$ and $\mathcal{G}$ be the subclasses of $\mathcal{F}$, where
\begin{equation}
\mathcal{H} = \{ F = h + \overline{g} : F \in \mathcal{F} \quad \text{and} \quad g \equiv 0 \}
\end{equation}

and
\begin{equation}
\mathcal{G} = \{ F = h + \overline{g} : F \in \mathcal{F} \quad \text{and} \quad h \equiv 0 \}.
\end{equation}

**Lemma 1.** The classes $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{F}$ are compact.

**Proof.** Suppose that $f_n = h_n + \overline{g_n} \in \mathcal{F}$ and that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. It follows from Hurwitz’s theorem that $f$ is harmonic, and therefore it has a canonical representation $f = h + \overline{g}$. It is easy to see that $h_n \to h$ and $g_n \to g$ locally uniformly, and that $h'(0) = 1$ and $g'(z) = e^{i\theta}zh'(z)$. Because
\begin{equation}
\text{Re} \left( 1 + z \frac{h''(z)}{h'(z)} \right) > -\frac{1}{2}
\end{equation}
in $\mathbb{D}$, it follows that when $n \to +\infty$, then we have
\begin{equation}
\text{Re} \left( \frac{3}{2} + z \frac{h''(z)}{h'(z)} \right) \geq 0
\end{equation}
for all $z \in \mathbb{D}$. It suffices to show that the equation in (3.2) cannot hold for any $z \in \mathbb{D}$. Obviously, the function $\text{Re} \left( \frac{3}{2} + z \frac{h''(z)}{h'(z)} \right)$ is harmonic in $\mathbb{D}$. By the maximum principle, if $\text{Re} \left( \frac{3}{2} + z_0 \frac{h''(z)}{h'(z)} \right) = 0$ for some $z_0 \in \mathbb{D}$, then $\text{Re} \left( \frac{3}{2} + z \frac{h''(z)}{h'(z)} \right) \equiv 0$, and hence

$$\frac{3}{2} + z \frac{h''(z)}{h'(z)} \equiv iC$$

for some real constant $C$. That is a contradiction. Hence,

$$\text{Re} \left( \frac{3}{2} + z \frac{h''(z)}{h'(z)} \right) > -\frac{1}{2}$$

for all $z \in \mathbb{D}$, and the proof is complete. \hfill \Box

**Theorem 1.** Let $f$ be of the form (1.2) satisfying (3.1). Then

$$(3.3) \quad |a_j| \leq \frac{j + 1}{2} \quad \text{and} \quad |b_j| \leq \frac{j - 1}{2}$$

for all $j = 1, 2, \ldots$.

**Proof.** Since $\text{Re} \left( \frac{3}{2} + z \frac{h''(z)}{h'(z)} \right) > 0$, then there exists an analytic function $p_1(z) = c_0 + c_1 z + c_2 z^2 + \ldots$, such that

$$(3.4) \quad z \frac{h''(z)}{h'(z)} = p_1(z) - \frac{3}{2} = c_1 z + c_2 z^2 + \ldots,$$

and $\text{Re}\{p_1(z)\} > 0$. Then (3.4) implies that

$$j(j+1)a_{j+1}z^j = (ja_j c_1 + (j-1)a_{j-1}c_2 + \ldots + a_1 c_j)z^j$$

for $j = 1, 2, \ldots$, and hence,

$$a_{j+1} = \frac{1}{j(j+1)} \sum_{\gamma=1}^{j} \gamma a_{\gamma} c_{j+1-\gamma}.$$ 

Because $p_1(z) = c_0 + c_1 z + c_2 z^2 + \ldots$, and $\text{Re} p_1(z) > 0$, then by [12, Lemma 1, p. 50], we have $|c_j| \leq 2\text{Re}\{c_0\} = 3$ for all $j = 1, 2, \ldots$. If we write $a_1 = 1$, it follows that

$$|a_j| \leq \frac{j + 1}{2} \quad \text{for all} \quad j = 1, 2, \ldots.$$

By (3.4), $g'(z) = e^{i\theta} h'(z)$, and therefore $\sum_{j=1}^{\infty} jb_j z^{j-1} = e^{i\theta} \sum_{j=1}^{\infty} ja_j z^j$. Thus, we have

$$b_1 = 0, \quad jb_j = e^{i\theta} (j - 1) a_{j-1} \quad \text{for all} \quad j = 2, 3, \ldots.$$

Then, we obtain

$$|b_j| = \frac{j - 1}{j} |a_{j-1}| \leq \frac{j - 1}{2} \quad \text{for all} \quad j = 1, 2, \ldots.$$ 

\hfill \Box
Now, we are ready to establish upper bounds for the Fekete-Szegő functionals
\(|a_3 - \lambda a_2^2|\) and \(|b_3 - \lambda b_2^2|\).

**Theorem 2.** Let \(f\) be of the form (1.2) and satisfy (3.1). Then

(3.5) \[|a_3 - \lambda a_2^2| \leq \max \left\{ \frac{1}{2} \frac{|8 - 9\lambda|}{4}, \frac{|8 - 9\lambda|}{4} \right\} \text{ and } |b_3 - \lambda b_2^2| \leq 1 + \frac{|\lambda|}{4}\]

for all \(\lambda \in \mathbb{R}\).

**Proof.** Let

\[p_2(z) = \frac{2}{3} \left( \frac{3}{2} + z \frac{h''(z)}{h'(z)} \right),\]

where \(h(z) = z + \sum_{j=1}^{\infty} a_j z^j\). Then by simple calculations, we obtain

(3.6) \[p_2(z) = 1 + 2 \left( 3 a_2 z + (6 a_3 - 4 a_2^2) z^2 + \ldots \right)\]

Write \(p_2(z) = 1 + u_1 z + u_2 z^2 + \ldots\). Because \(\text{Re} \, p_2(z) > 0\), then by [24, formula (10), p. 166], we have

\[|u_2 - \frac{u_1^2}{2}| \leq 2 - \frac{|u_1|^2}{2}\]

It follows from (3.6) that

\[a_2 = \frac{3}{4} u_1 \text{ and } a_3 = \frac{1}{4} u_2 + \frac{3}{8} u_1^2.\]

Hence,

(3.7) \[|a_3 - \lambda a_2^2| = \left| \frac{1}{4} u_2 + \frac{3}{8} u_1^2 - \frac{9}{16} \lambda u_1^2 \right| \leq \frac{1}{4} \left( 2 - \frac{1}{2} |u_1|^2 + \frac{|8 - 9\lambda|}{4} |u_1|^2 \right)\]

If \(\frac{|8 - 9\lambda|}{4} < \frac{1}{2}\), then (3.7) implies

\[|a_3 - \lambda a_2^2| \leq \frac{1}{2}.\]

Equality is attained if we choose \(a_2 = 0\) and \(a_3 = \pm \frac{1}{2}\).

If \(\frac{|8 - 9\lambda|}{4} \geq \frac{1}{2}\), then it follows from [14, Lemma 1, p. 50] that \(|u_1| \leq 2 \text{Re} \{p_2(0)\} = 2\) and (3.7) that

\[|a_3 - \lambda a_2^2| \leq \frac{1}{2} + \frac{1}{4} \left( \frac{|8 - 9\lambda|}{4} - \frac{1}{2} \right) |u_1|^2 \leq \frac{|8 - 9\lambda|}{4}.\]

Choosing \(a_2 = \pm \frac{3}{2}\) and \(a_3 = 2\) in (3.7) shows that the result is sharp.
Since \( g'(z) = e^{i\theta} z h'(z) \), we have
\[
\sum_{j=1}^{\infty} j b_j z^{j-1} = e^{i\theta} \sum_{j=1}^{\infty} j a_j z^j.
\]

Obviously, \( b_2 = \frac{e^{i\theta}}{2} a_1 = \frac{e^{i\theta}}{2} \) and \( b_3 = \frac{2 e^{i\theta}}{3} a_2 \). Hence, (3.3) implies
\[
|b_3 - \lambda b_2^2| = \left| \frac{2 e^{i\theta}}{3} a_2 - \frac{\lambda e^{2i\theta}}{4} \right| \leq \frac{2}{3} |a_2| + \left| \frac{\lambda}{4} \right| \leq 1 + \left| \frac{\lambda}{4} \right|.
\]

If \( \lambda \geq 0 \), then equality is attained when \( b_3 = -e^{2i\theta} \), i.e. \( a_2 = -\frac{3}{2} e^{i\theta} \). If \( \lambda < 0 \), then equality is attained when \( b_3 = e^{2i\theta} \), i.e. \( a_2 = \frac{3}{2} e^{i\theta} \).

**Remark 1.** Both equalities in (3.5) are attained when \( a_2 = \frac{3}{2} \) and \( b_3 = e^{i\theta} \) or \( a_2 = -\frac{3}{2} \) and \( b_3 = -e^{i\theta} \), but only in the case \(|8 - 9\lambda| \geq 2\) and \( \theta = 2k\pi \), where \( k \in \mathbb{Z} \).

4. Coefficient estimates for two classes of polyharmonic mappings

Let \( L \) denote the following differential operator defined on the class of complex-valued \( C^1 \) functions:
\[
L = z \frac{\partial}{\partial z} - \overline{z} \frac{\partial}{\partial \overline{z}}.
\]

An important property of the operator \( L[F] \) is that it behaves with respect to polyharmonic mappings much like the operator \( z f'(z) \) defined for analytic functions (see [4, Corollary 1(3)]).

**Theorem 3.** Each mapping \( F \in \mathcal{HS}_p \) is starlike with respect to the origin.

**Proof.** Let \( F \in \mathcal{HS}_p \) be of the form (2.1). It follows from Theorem A that
\[
J_F(0) = 1 - |b_{1,1}|^2 > 0.
\]

By computation, we have
\[
\begin{align*}
\frac{\partial}{\partial \theta} \left( \arg F(re^{i\theta}) \right) &= \frac{\partial}{\partial \theta} \left[ \operatorname{Im} \left( \log F(re^{i\theta}) \right) \right] \\
&= \operatorname{Im} \left[ \frac{\partial}{\partial \theta} \left( \log F(re^{i\theta}) \right) \right] \\
&= \operatorname{Im} \left[ \frac{iz F_z(z) - i\overline{z} F_{\overline{z}}(z)}{F(z)} \right] \\
&= \operatorname{Re} \left[ \frac{L[F(z)]}{F(z)} \right] \\
&= \operatorname{Re} \left[ \sum_{k=1}^{p} \left| z \right|^{2(k-1)} \sum_{j=1}^{\infty} \left( j a_{k,j} z^j - j b_{k,j} z^j \right) \right] \\
&= \operatorname{Re} \left[ \sum_{k=1}^{p} \left| z \right|^{2(k-1)} \sum_{j=1}^{\infty} \left( a_{k,j} z^j + b_{k,j} z^j \right) \right].
\end{align*}
\]
By (4.1) and (4.2), we obtain
\[
\lim_{z \to 0} \frac{\partial}{\partial \theta} \left( \arg F(re^{i\theta}) \right) = \lim_{z \to 0} \frac{1 - \overline{b_{1,1}}z}{1 + \overline{b_{1,1}}z} \left( 1 + \frac{b_{1,1}z}{\overline{b_{1,1}}} \right) = \lim_{z \to 0} \frac{\Re \left( \frac{1 - \overline{b_{1,1}}z}{1 + \overline{b_{1,1}}z} \right) \left( 1 + \frac{b_{1,1}z}{\overline{b_{1,1}}} \right)}{\left| 1 + \frac{b_{1,1}z}{\overline{b_{1,1}}} \right|^2}.
\]
(4.3)

Therefore, by (4.2), (4.3) and the continuity of \( F \), it follows from Theorem A that each mapping \( F \) is univalent in \( \mathbb{D} \). Hence, each mapping \( F \) is starlike with respect to the point \( z = 0 \).

It follows from Theorem A that each \( F \in \mathcal{HS}_p \) is univalent in \( \mathbb{D} \). Then, we have that \( F(z) \neq 0 \) for \( z \in \mathbb{D} \setminus \{0\} \), and the function \( \Re \frac{L[F(z)]}{F(z)} \) is continuous in \( \mathbb{D} \setminus \{0\} \). Therefore, by (4.2), (4.3) and the continuity of \( F \) in \( \mathbb{D} \setminus \{0\} \), we see the condition \( \frac{\partial}{\partial \theta} \left( \arg F(re^{i\theta}) \right) > 0 \) for all \( z \in \mathbb{D} \setminus \{0\} \) is equivalent to

\[
\frac{L[F(z)]}{F(z)} \neq \frac{\zeta - 1}{\zeta + 1}
\]

(4.4)

for all \( z \in \mathbb{D} \setminus \{0\} \) and all \( \zeta \in \mathbb{C} \) with \( |\zeta| = 1 \) and \( \zeta \neq -1 \). Hence, (4.4) holds if and only if

\[
\Phi(z) := (\zeta + 1)L[F(z)] - (\zeta - 1)F(z) \neq 0
\]

for all \( z \in \mathbb{D} \setminus \{0\} \) and all \( |\zeta| = 1 \). Calculations show that

\[
|\Phi(z)| = \left| (\zeta + 1) \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} j(a_{k,j}z^j - \overline{b_{k,j}z^j}) \right| - (\zeta - 1) \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left( a_{k,j}z^j + \overline{b_{k,j}z^j} \right)
\]

\[
= \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left( (j + 1 + \zeta(j - 1))a_{k,j}z^j - (j - 1 + \zeta(j + 1))b_{k,j}z^j \right). \]

If \( F \) is the identity, obviously, we have \( |\Phi(z)| = 2|z| \). If \( F(z) = z + \overline{b_{1,1}}z \), then

\[
|\Phi(z)| = 2 \left| z - \zeta \overline{b_{1,1}}z \right| \geq 2|z|(1 - |b_{1,1}|).
\]

If \( F \) is not an affine mapping, then

\[
|\Phi(z)| > |z| \left( 2 - 2|b_{1,1}| - 2 \sum_{k=1}^{p} \sum_{j=2}^{\infty} j(|a_{k,j}| + |b_{k,j}|) - 2 \sum_{k=2}^{p} (|a_{k,1}| + |b_{k,1}|) \right).
\]

Hence, each mapping \( F \in \mathcal{HS}_p \) is starlike with respect to the point \( z = 0 \). \( \square \)

**Theorem 4.** Each mapping \( F \in \mathcal{HC}_p \) is convex.
Proof. Let $F \in \mathcal{HC}_p$ be of the form (2.1). By (4.2), we have

$$
\frac{\partial}{\partial \theta} \left[ \arg \left( \frac{\partial}{\partial \theta} F(re^{i\theta}) \right) \right] = \text{Re} \frac{L \left[ \frac{\partial}{\partial \theta} F(re^{i\theta}) \right]}{L[F(z)]}
$$

(4.5)

Then by (4.1) and (4.5), we have

$$
\lim_{z \to 0} \frac{\partial}{\partial \theta} \left[ \arg \left( \frac{\partial}{\partial \theta} F(re^{i\theta}) \right) \right] = \lim_{z \to 0} \frac{1 + \frac{b_{1,1} \bar{z}}{z}}{1 - \frac{b_{1,1} \bar{z}}{z}}
$$

(4.6)

$$
= \lim_{z \to 0} \frac{1 - |b_{1,1}|^2}{\left| 1 - \frac{b_{1,1} \bar{z}}{z} \right|^2} \geq \frac{1 - |b_{1,1}|^2}{(1 + |b_{1,1}|)^2} > 0.
$$

If $F$ is the identity, obviously, we have $|L[F(z)]| = |z|$. If $F(z) = z + b_{1,1} z$, then

$$
|L[F(z)]| = |z - \bar{b}_{1,1} z| \geq |z|(1 - |b_{1,1}|).
$$

If $F$ is not an affine mapping, then

$$
|L[F(z)]| = \left| \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left( j a_{k,j} z^j - j b_{k,j} z^j \right) \right|
$$

$$
> |z| \left( 1 - |b_{1,1}| - \sum_{k=1}^{p} \sum_{j=2}^{\infty} j \left( |a_{k,j}| + |b_{k,j}| \right) - \sum_{k=2}^{p} j \left( |a_{k,1}| + |b_{k,1}| \right) \right).
$$

Therefore, $F \in \mathcal{HC}_p$ implies $L[F(z)] \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$, and hence the function $\text{Re} \frac{L[L[F(z)]]}{L[F(z)]}$ is continuous in $\mathbb{D} \setminus \{0\}$. Therefore, by (4.5), (4.6) and the continuity of $L[F(z)]$ in $\mathbb{D} \setminus \{0\}$, we see the condition $\frac{\partial}{\partial \theta} \left[ \arg \left( \frac{\partial}{\partial \theta} F(re^{i\theta}) \right) \right] > 0$ for all $z \in \mathbb{D} \setminus \{0\}$ is equivalent to

$$
\text{Re} \frac{L[L[F(z)]]}{L[F(z)]} \neq \frac{\zeta - 1}{\zeta + 1}
$$

(4.7)

for all $z \in \mathbb{D} \setminus \{0\}$ and all $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $\zeta \neq -1$. Hence, (4.7) holds if and only if

$$
\Psi(z) := (\zeta + 1)L[L[F(z)]] - (\zeta - 1)L[F(z)] \neq 0
$$
for all \( z \in \mathbb{D} \setminus \{0\} \) and all \( |\zeta| = 1 \). Calculations show that

\[
|\Psi(z)| = (\zeta + 1) \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} j^{2} |a_{k,j}z^{j} + \overline{b_{k,j}z^{j}}|
\]

\[
-(\zeta - 1) \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} j |a_{k,j}z^{j} - \overline{b_{k,j}z^{j}}|
\]

\[
= \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left( j^{2} + j + \zeta(j^{2} - j) \right) a_{k,j}z^{j} + \left( j^{2} - j + \zeta(j^{2} + j) \right) \overline{b_{k,j}z^{j}}.
\]

If \( F \) is the identity, obviously, we have \( |\Psi(z)| = 2|z| \). If \( F(z) = z + \overline{b_{1,1}z} \), then

\[
|\Psi(z)| = 2 \left| z + \zeta \overline{b_{1,1}z} \right| \geq 2|z|(1 - |b_{1,1}|).
\]

If \( F \) is not an affine mapping, then

\[
|\Psi(z)| > |z| \left( 2 - 2|b_{1,1}| - 2 \sum_{k=1}^{p} \sum_{j=2}^{\infty} j^{2} (|a_{k,j}| + |b_{k,j}|) - 2 \sum_{k=2}^{p} (|a_{k,1}| + |b_{k,1}|) \right).
\]

It follows that each mapping \( F \in \mathcal{HC}_{p} \) is convex. \( \square \)

**Example 1.** Let \( F_{1}(z) = z + \frac{1}{3}z + \frac{1}{6}z^{2}z \). Then \( F_{1} \) is convex. See also Figure 1.

It is well known that the coefficients of every starlike mapping \( f \in S_{H}^{*0} \) of the form (1.2) satisfy the sharp inequalities

\[
|a_{j}| \leq \frac{(2j + 1)(j + 1)}{6}, \quad |b_{j}| \leq \frac{(2j - 1)(j - 1)}{6}, \quad ||a_{j}| - |b_{j}|| \leq j
\]

for \( j = 2, 3, \ldots \) (see [27]). The coefficients of each mapping \( f \in C_{H}^{0} \) satisfy the sharp inequalities

\[
|a_{j}| \leq \frac{j + 1}{2}, \quad |b_{j}| \leq \frac{j - 1}{2}, \quad \text{and} \quad ||a_{j}| - |b_{j}|| \leq 1
\]

for \( j = 2, 3, \ldots \) (see [13]).

Next, we obtain similar results for mappings in \( \mathcal{HS}_{p} \) and \( \mathcal{HC}_{p} \).

**Theorem 5.** The coefficients of every mapping \( F \in \mathcal{HS}_{p} \) satisfy the sharp inequalities

\[
\sum_{k=1}^{p} (|a_{k,j}| + |b_{k,j}|) \leq \frac{1}{j}
\]

for all \( j = 2, 3, \ldots \).

**Proof.** Let \( F \in \mathcal{HS}_{p} \) be of the form (2.1). By (2.2), we have

\[
\sum_{k=1}^{p} j(|a_{k,j}| + |b_{k,j}|) \leq \sum_{k=1}^{p} \sum_{j=2}^{\infty} (2(k - 1) + j)(|a_{k,j}| + |b_{k,j}|) \leq 1.
\]
Figure 1. The images of $\mathbb{D}$ under the mappings $F_1(z) = z + \frac{4}{3} \pi + \frac{1}{6} |z|^2 \pi$ (left) and $F_2(z) = z + \frac{2}{3} e^{\pi i}$. It follows that
\[
\sum_{k=1}^{p}(|a_{k,j}| + |b_{k,j}|) \leq \frac{1}{j}
\]
for $j = 2, 3, \ldots$. \hfill \qed

**Example 2.** Let $F_2(z) = z + \frac{j}{j^2} e^{i\varphi}$ for all $j = 2, 3, \ldots$ and $\varphi \in \mathbb{R}$. Then $F_2 \in \mathcal{HS}$ is univalent, sense preserving and starlike with respect to the origin. Obviously, the coefficients of $F_2$ satisfy (4.8). See Figure 1 for the case where $j = 3$ and $\varphi = \pi/6$.

The above example shows that the coefficient estimate (4.8) is sharp for $p = 1$.

**Theorem 6.** The coefficients of each mapping $F \in \mathcal{HC}_p$ satisfy the sharp inequalities
\[
\sum_{k=1}^{p}(|a_{k,j}| + |b_{k,j}|) \leq \frac{1}{j^2}
\]
for $j = 2, 3, \ldots$.

**Proof.** The proof of Theorem 6 is similar to the proof of Theorem 5 and we will omit it. \hfill \qed

**Example 3.** Let $F_3(z) = z + \frac{2}{j} e^{j \varphi}$ for all $j = 2, 3, \ldots$ and $\varphi \in \mathbb{R}$. Then $F_3 \in \mathcal{HC}$ is a univalent, sense preserving and convex harmonic mapping. Obviously, the coefficients of $F_3$ satisfy (4.9). See Figure 2 for the case where $j = 3$ and $\varphi = \pi/6$.

This example shows that the coefficient estimate (4.9) is sharp for $p = 1$.

Now, we are ready to generalize Proposition 1 to the polyharmonic mappings of the class $\mathcal{HC}_p$. 
Obviously, the mapping $F_1(z) = z + \frac{4}{9} |z|^2 z = \mathcal{H}C_2$ (see Figure 1). Let $\alpha = \beta = 0$. Then $F_1$ satisfies the inequality (4.10).

However, the mapping $F_4(z) = z + \frac{4}{9} |z|^2 z + \frac{1}{4} \overline{z} = \mathcal{H}C_2$ (see Figure 2) also satisfies the inequality (4.10) for $\alpha = \beta = 0$ with $a_{2,1} = \frac{1}{9}$.

**Theorem 7.** If $F \in \mathcal{H}C_p$ and $a_{k,1} = 0$ for $k \in \{2, \ldots, p\}$, then there exist angles $\alpha$ and $\beta$ such that

$$\text{Re} \left\{ \left( e^{i\alpha} \sum_{k=1}^{p} |z|^{2(k-1)} h_k'(z) + e^{-i\alpha} \sum_{k=1}^{p} |z|^{2(k-1)} g_k'(z) \right) (e^{i\beta} - e^{-i\beta} z^2) \right\} > 0$$

for all $z \in \mathbb{D}$.

**Proof.** Let $F \in \mathcal{H}C_p$ be of the form (2.1), fix $r \in (0, 1)$, and let

$$F_r(z) = \sum_{k=1}^{p} r^{2(k-1)} \left( h_k(z) + g_k(z) \right) = \sum_{j=1}^{\infty} \sum_{k=1}^{p} \left( a_{k,j} r^{2(k-1)} z^j + b_{k,j} r^{2(k-1)} \overline{z}^j \right), \quad z \in \mathbb{D}.$$ 

Then $F_r$ is harmonic. By the hypothesis and (2.3), $F \in \mathcal{H}C_p$ implies

$$\sum_{j=2}^{\infty} \sum_{k=1}^{p} r^{2(k-1)} \left| a_{k,j} r^{2(k-1)} \right| + \sum_{j=2}^{\infty} \sum_{k=1}^{p} b_{k,j} r^{2(k-1)} \leq 1 - \sum_{k=1}^{p} \left| b_{k,j} r^{2(k-1)} \right|,$$

i.e., $F_r \in C_H$ (see [3]). Then Proposition 1 implies that there exist angles $\alpha$ and $\beta$ such that

$$\text{Re} \left\{ \left( e^{i\alpha} \sum_{k=1}^{p} r^{2(k-1)} h_k'(z) + e^{-i\alpha} \sum_{k=1}^{p} r^{2(k-1)} g_k'(z) \right) (e^{i\beta} - e^{-i\beta} z^2) \right\} > 0$$

for all $z \in \mathbb{D}$. Let $r = |z|$. The result is proved. \hfill $\square$

**Example 4.** Obviously, the mapping $F_3(z) = z + \frac{4}{9} |z|^2 z \in \mathcal{H}C_2$ (see Figure 1). Let $\alpha = \beta = 0$. Then $F_3$ satisfies the inequality (4.10).

However, the mapping $F_4(z) = z + \frac{4}{9} |z|^2 z + \frac{1}{4} \overline{z} = \mathcal{H}C_2$ (see Figure 2) also satisfies the inequality (4.10) for $\alpha = \beta = 0$ with $a_{2,1} = \frac{1}{9}$. 

![Figure 2](image-url)
Remark 2. The proof of Theorem 7 requires a somewhat unnatural additional assumption concerning the coefficients $a_{k,1}$. It is not obvious if the result holds without this assumption.

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