ÉTALE AND MOTIVIC COHOMOLOGY AND ULTRAPRODUCTS OF SCHEMES

LARS BRÜNJES, CHRISTIAN SERPÉ

Abstract. This paper is a continuation of the authors article [BS07a]. We mainly study the behaviour of étale cohomology, algebraic cycles and motives under ultraproducts respectively enlargements. The main motivation for that is to find methods to transfer statements about étale cohomology and algebraic cycles from characteristic zero to positive characteristic and vice versa. We give one application to the independence of $l$ of Betti numbers in étale cohomology and applications to the complexity of algebraic cycles.

Contents

1. Introduction 1
2. Étale cohomology 3
3. Independence of $l$ of Betti numbers 13
4. Voevodsky motives and cycles 14
5. Complexity of cycles 20
Appendix A. Lengths and the Koszul complex 26
References 27

1. Introduction

Let $\{R_i\}_{i \in I}$ be a family of commutative rings and consider the usual product of rings $\prod_{i \in I} R_i$. Most properties of rings do not behave well under this construction. For example, even if all rings $R_i$ are fields, the product is full of zero divisors. The situation changes if we choose an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(I)$ and consider the ultraproduct

$$\prod_{i \in I, \mathcal{U}} R_i := \prod_{i \in I} R_i / \sim,$$

where $\sim$ is defined by $(r_i)_{i \in I} \sim (r'_i)_{i \in I} :\iff \{i \in I | r_i = r'_i\} \in \mathcal{U}$. For example, in this situation $\prod_{i \in I, \mathcal{U}} R_i$ is even a field if all $R_i$ are fields. In the case $I = \mathbb{N}$ and $R_i = \mathbb{R}$, Robinson used this methods to construct an enlargement $R := \prod_{\mathbb{N}, \mathcal{U}} R$ of $\mathbb{R}$ where one can do calculus with infinitesimals, leading to the area of mathematics known today as Nonstandard Analysis. In the case where $I = \mathbb{P}$ is the set of prime numbers and $R_p = \mathbb{F}_p$ are the finite prime fields, the ultraproduct $\prod_{p \in \mathbb{P}, \mathcal{U}} \mathbb{F}_p$ is an interesting field for algebraic
geometry. Namely, on the one hand, that field behaves like a finite field, because it is the ultraproduct of finite fields. But on the other hand, it is a field of characteristic zero. In order to use this ambiguity in algebraic geometry, we started in [BS07a] to investigate how schemes behave under ultraproducts. We constructed and explored the properties of a functor $N$ which turned a scheme over an ultraproduct of rings into an ultraproduct of schemes. We described the image of this functor and constructed a similar functor for coherent sheaves on schemes. Then we used the ambiguity mentioned above to give two applications to resolution of singularities and weak factorisation. For more motivational remarks we refer to the introduction of [BS07a].

In the present article we proceed with the investigation we started in [BS07a]. We study the behavior of étale cohomology and algebraic cycles under ultraproducts. We are mainly interested in a connection between the étale cohomology respectively cycle groups of an (in some sense limited) ultraproduct of schemes and the ultraproduct of the étale cohomology and various cycle groups. In short, we want to know whether étale cohomology and various cycle constructions commute with ultraproducts. Whereas in étale cohomology the results are quite convincing (cf. e.g. Proposition 2.15), the situation for the cycle groups is, not surprisingly, much more complicated.

In a forthcoming paper [BS07b], we apply our methods to the question whether a class in the $l$-adic cohomology of a smooth projective variety over $\mathbb{Q}$, which is algebraic over almost all finite fields, is also algebraic over $\mathbb{Q}$. We show that this can be expressed as a question about the uniform complexity of the cycles representing that class over the finite fields.

In the whole article, we do not work with ultraproducts, but with the enlargement of superstructures as in [BS07a]. Of course it would be possible to work throughout directly with ultraproducts. But from our point of view enlargements provide a conceptual way of handling ultraproduct constructions. As the structures like schemes, Chow groups and étale cohomology are quite involved we chose this more advanced viewpoint for all our considerations. For the use of enlargements in category theory, we refer to [BS05].

Now we describe the content of the paper in a little more detail.

In [BS07a], we constructed a functor $N$ which assigned to a scheme $X$ of finite type over an internal ring $R$ a *scheme $N(X)$ over $R$. In that situation, in the first section, we construct a functor $N$ from the category of constructible étale sheaves on $X$ to the category of constructible étale sheaves on $N(X)$. Then we study the relationship between the étale cohomology of a sheaf $F$ and the étale cohomology of $^*F$.

The independence of $l$ of the Betti numbers of $l$-adic cohomology is known in characteristic zero, but in general not in positive characteristic. In the second section, we use the results of the first section to give an application to the independence of $l$ of Betti numbers in $l$-adic cohomology. We show that in some sense the Betti numbers of the $l$-adic cohomology is independent of $l$ if the characteristic is large enough. How large depends on the complexity of the scheme and in some sense on $l$. For a precise statement see Theorem 3.4.

In the third section, we first construct a functor $N$ for the triangulated category of Voevodsky motives and then use this to give an appropriate map $N$ for motivic cohomology. We show that the maps defined in the first and third section are compatible with each other.
In section four we introduce a notion of complexity of algebraic cycles and use this to describe the image of the functor $N$ for cycles. For cycles of codimension one we show that $N$ is bijective for cycles with finite Hilbert polynomial. Using the result of Mumford, that Chow groups are not finite dimensional, we further show that in general $N$ is not injective. We also show how rational equivalence and the intersection product behave under our notion of complexity.

In the appendix, we give some lemmas about enlargements in commutative algebra, which we need in section three.

2. Étale cohomology

In [BS07a], we explained how a scheme over an ultraproduct of rings gives rise to an ultraproduct of schemes. Now we want to show that an étale sheaf on a scheme over an ultraproduct gives an ultraproduct of étale sheaves on the ultraproduct of schemes. This is, as in the case of schemes, only possible for étale sheaves which are compact in some sense. For that we generalize the construction of $N$ for schemes of finite presentation to algebraic spaces of finite presentation. After having achieved this, we explore how basic concepts of étale cohomology behave under this construction.

We consider the same basic setup as in [BS07a]. So let $\mathcal{R}$ be a small subcategory of all commutative rings with:

- $\mathbb{Z} \in \mathcal{R}$,
- for all $A \in \mathcal{R}$ and all $A$-algebras $B$ of finite presentation, we have $B \in \mathcal{R}$ (up to isomorphism),
- for all $A \in \mathcal{R}$ and all prime ideals $p \in \text{Spec}(A)$, the localisation $A_p$ is in $\mathcal{R}$ (up to isomorphism).

As in [BS07a], let $\text{sch}_\mathcal{X}^{fp}$ denote the fibred (over $\mathcal{R}$) category of schemes of finite presentation, and let $\text{AlgSp}_\mathcal{X}^{fp}$ denote the fibred category of algebraic spaces of finite presentation (also fibred over $\mathcal{R}$). For general facts about algebraic spaces we refer to [Knü71].

We choose a superstructure $\hat{\mathcal{M}}$ such that all our small categories are $\hat{\mathcal{M}}$-small, and let $*: \hat{\mathcal{M}} \to *\hat{\mathcal{M}}$ be an enlargement. As in the case of schemes, base change along $A \to *A$ for $A \in \mathcal{R}$ defines a functor

$$T: \text{AlgSp}_\mathcal{R}^{fp} \to \text{AlgSp}_{*\mathcal{R}}^{fp},$$

such that the diagram

$$\begin{align*}
\text{AlgSp}_\mathcal{R}^{fp} \ar[r]^T \ar[d] & \text{AlgSp}_{*\mathcal{R}}^{fp} \ar[d] \\
\mathcal{R}^{op} \ar[r]^* & *\mathcal{R}^{op}
\end{align*}$$

is commutative.
We are looking for a functor \( N : \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \to \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \) such that the diagram

\[
\begin{array}{ccc}
\text{AlgSpc}_{\mathcal{R}}^{\text{fp}} & \xrightarrow{T} & \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \\
\downarrow & & \downarrow \\
\text{AlgSpc}_{\mathcal{R}}^{\text{fp}} & \xrightarrow{N} & \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \\
\mathcal{R}^{\text{op}} & \xrightarrow{*} & \mathcal{R}^{\text{op}} \\
\downarrow & & \downarrow \\
\mathcal{R}^{\text{op}} & \xrightarrow{*} & \mathcal{R}^{\text{op}}
\end{array}
\]

is commutative.

2.1. Proposition/Definition. There is an essentially unique functor \( N : \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \to \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \) of fibrations over \( \mathcal{R} \) such that diagram (1) commutes.

2.2. Remark. The uniqueness of \( N \) from above can be made precise as follows: \( N : \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \to \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \) is the right Kan extension of \( * : \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \to \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \) along \( T : \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \to \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \) in the 2-category of fibrations.

The proof of the proposition relies mainly on the following lemma.

2.3. Lemma. Let \( A = \text{colim}_{\lambda \in L} A_\lambda \) be the colimit of a filtered system of rings \( (A_\lambda)_{\lambda \in L} \).

(i) Let \( \lambda_0 \in L \), and let \( X_{\lambda_0} \) and \( Y_{\lambda_0} \) be algebraic spaces over \( A_{\lambda_0} \). We assume that \( X_{\lambda_0} \) is quasi compact and that \( Y_{\lambda_0} \) is locally of finite presentation over \( A_{\lambda_0} \). Then the canonical map

\[
\text{colim}_{\lambda > \lambda_0} \text{Hom}_{A_{\lambda_0}}(X_{\lambda_0} \otimes_{A_{\lambda_0}} A_\lambda, Y_{\lambda_0} \otimes_{A_{\lambda_0}} A_\lambda) \to \text{Hom}_{A_{\lambda_0}}(X_{\lambda_0} \otimes_{A_{\lambda_0}} A Y_{\lambda_0} \otimes_{A_{\lambda_0}} A)
\]

is a bijection.

(ii) Let \( X \) be an algebraic space of finite presentation over \( A \). Then there is a \( \lambda_0 \in L \), an algebraic space \( X_0 \) of finite presentation over \( A_{\lambda_0} \) and an isomorphism

\[
X_0 \otimes_{A_{\lambda_0}} A \to X.
\]

Proof of Lemma 2.3 In [LMB00][Proposition 4.18], the analogous statements for algebraic stacks can be found, and from this, (i) follows immediately. The second part follows from the analogous statements for schemes, which can be found in [Gro66][8]. \( \text{q.e.d.} \)

Proof of Proposition 2.1 We give an explicit construction of \( N \) and leave the rest to the reader. For an \( A \in \mathcal{R} \) and \( X \in \text{AlgSpc}_{\mathcal{R}}^{\text{fp}} \) we can choose by lemma 2.3(ii) a subring \( A_0 \subset A \) of finite type over \( \mathbb{Z} \), a \( X_0 \in \text{AlgSpc}_{\mathcal{R}}^{\text{fp}}/A_0 \), and an isomorphism \( \varphi : X_0 \otimes_{A_0} A \to X \). Because \( A_0 \) is of finite type over \( \mathbb{Z} \), we have a canonical internal ring homomorphism \( *A_0 \to A \) (cf. [BS07A][Proposition/Definition 3.2]). With that we define

\[
N(X) := X_0 \otimes_{*A_0} A
\]
For a different choice \((A'_0, X'_0, \varphi')\) we have the isomorphism
\[
X_0 \otimes_{A_0} A \stackrel{\varphi'^{-1} \circ \varphi}{\longrightarrow} X'_0 \otimes_{A_0} A.
\]
By 2.3 (i) there is a \(B \subset A\) of finite type over \(\mathbb{Z}\) with \(A_0, A'_0 \subset B\) and an isomorphism
\[
\psi_0 : X_0 \otimes_{A_0} B \sim X'_0 \otimes_{A'_0} B
\]
such that
\[
\begin{array}{ccc}
X_0 \otimes_{A_0} A & \xrightarrow{\psi_0} & X'_0 \otimes_{A_0} A \\
\sim & & \sim \\
X_0 \otimes_{A_0} B_0 \otimes_{B_0} A & \xrightarrow{\psi_0 \otimes_{B_0} A} & X'_0 \otimes_{A'_0} B_0 \otimes_{B_0} A
\end{array}
\]
So this \(\psi_0\) defines an isomorphism
\[
*X_0 \otimes_{A_0} A \sim *X'_0 \otimes_{A_0} A.
\]
Again by 2.3 (i) it can be shown that this isomorphism is independent of the choices of \(B_0\) and \(\psi_0\). By similar argument as in [BS07a] [Theorem 3.4] for schemes, one shows that \(N\) is functorial. \textbf{q.e.d.}

2.4. \textbf{Remark.} In the construction it is important that \(A_0\) is not only in \(\mathcal{R}\) but even of finite type over \(\mathbb{Z}\). Otherwise there would be no canonical morphism \(*A_0 \to A.\)

For a scheme \(X\) we consider the \(\acute{\text{e}}\text{tale}\) topology on the category \(\text{Sch}^{\text{fp}}/X\) and denote the resulting site by \((\text{Sch}^{\text{fp}}/X)_{\acute{\text{e}}}\). We denote by \(\text{Shv}_{\acute{\text{e}}} := \text{Shv}((\text{Sch}^{\text{fp}}/X)_{\acute{\text{e}}} )\) the category of sheaves on \((\text{Sch}^{\text{fp}}/X)_{\acute{\text{e}}} ).\) For a *scheme\(X\) we use the notation \(*\text{Shv}_{\acute{\text{e}}} (X)\) for the internal category of *sheaves, and for \(B \in \mathcal{R}\) and \(X \in \text{Sch}^{\text{fp}}/B\) there is a canonical functor
\[
* : \text{Shv}_{\acute{\text{e}}} (X) \to *\text{Shv}_{\acute{\text{e}}} (*X).
\]
For more details about this we refer to our paper [BS04].

2.5. \textbf{Remark.} For a quasi compact \(X, \text{e.g.}\) if \(X\) is of finite presentation over an affine scheme, the restriction functor
\[
\text{Shv}((\text{Sch}/X)_{\acute{\text{e}}} ) \to \text{Shv}((\text{Sch}^{\text{fp}}/X)_{\acute{\text{e}}} )
\]
is an equivalence of categories. So in particular the cohomology on \(X\) in the site \((\text{Sch}^{\text{fp}}/X)_{\acute{\text{e}}}\) is the same as the usual \(\acute{\text{e}}\text{tale}\) cohomology.

Now let \(A \in *\mathcal{R}\) and \(X \in \text{Sch}^{\text{fp}}/A\), and consider the fully faithful Yoneda embedding
\[
\text{AlgSp}^{\text{fp}}/X \to \text{Shv}_{\acute{\text{e}}} (X).
\]
We denote by \(\text{Shv}_{\acute{\text{e}}}^{\text{fp}} (X)\) the essential image of the above functor. To define \(N\) on \(\text{Shv}_{\acute{\text{e}}}^{\text{fp}} (X)\), we choose for each \(\mathcal{F} \in \text{Shv}_{\acute{\text{e}}}^{\text{fp}} (X)\) a \(Y \in \text{AlgSp}^{\text{fp}}/X\) and an isomorphism \(h_Y \cong \mathcal{F}\) and define
\[
N(\mathcal{F}) := *h_{N(Y)},
\]
where \(*h\) denotes the *Yoneda embedding
\[
*h : *\text{AlgSp}^{\text{fp}}/N(X) \to *\text{Shv}_{\acute{\text{e}}} (N(X)).
\]
This defines a functor
\[ N : \text{Shv}^\text{fp}_{\text{ét}}(X) \to \text{*Shv}_{\text{ét}}(N(X)). \]

2.6. **Examples.** We are mainly interested in the following two cases:

(i) Let \( \mathcal{G} := G_{m,X} \). Then we have \( \mathcal{G} \in \text{Shv}^\text{fp}_{\text{ét}}(X) \), and there is an isomorphism
\[ N(\mathcal{G}) \cong *G_{m,N(X)}. \]

(ii) Let \( \mathcal{G} := \mu_n \). Then we have \( \mathcal{G} \in \text{Shv}^\text{fp}_{\text{ét}}(X) \), and there is an isomorphism
\[ N(\mu_n,X) \cong *\mu_n,N(X). \]

If \( \{U_i \to X\} \) is a finite étale covering of \( X \), then \( \{N(U_i) \to N(X)\} \) is an internal *étale covering of \( N(X) \). We denote by
\[ S : \text{*Shv}_{\text{ét}}(N(X)) \to \text{Shv}_{\text{ét}}(X) \]
the induced functor.

The morphisms
\[ \Gamma(U, \mathcal{G}) = \text{Hom}_{\text{AlgSpe}^\text{fp}/X}(U, Y) \xrightarrow{N} \text{Hom}_{\text{*AlgSpe}^\text{fp}/N(X)}(N(U), N(Y)) \]
define a natural transformation
\[ (2) \quad \varphi : h \to S \circ *h \circ N \]
from the Yoneda embedding
\[ h : \text{AlgSpe}^\text{fp}/X \to \text{Shv}_{\text{ét}}(X) \]
to
\[ S \circ *h \circ N : \text{AlgSpe}^\text{fp}/X \to \text{Shv}_{\text{ét}}(X). \]

For a *étale *sheaf \( \mathcal{G} \in \text{*Shv}_{\text{ét}}(N(X)) \), this gives a map
\[ (3) \quad \text{Hom}_{\text{*Shv}_{\text{ét}}(N(X))}(N(hY), \mathcal{G}) \xrightarrow{\varphi_Y} \text{Hom}_{\text{Shv}_{\text{ét}}(X)}([S*hN](Y), S(\mathcal{G})). \]

2.7. **Proposition.** For all \( Y \in \text{AlgSpe}^\text{fp}/X \) and all \( \mathcal{G} \in \text{*Shv}_{\text{ét}}(N(X)) \), map \( (3) \) is a bijection.

**Proof.** The restriction functors
\[ \text{Shv}((\text{AlgSpe}^\text{fp}/X)_{\text{ét}}) \to \text{Shv}((\text{Sch}^\text{fp}/X)_{\text{ét}}), \]
\[ \text{*Shv}((\text{*AlgSpe}^\text{fp}/X)_{\text{ét}}) \to \text{*Shv}((\text{*Sch}^\text{fp}/N(X)_{\text{ét}})) \]
are isomorphisms. So it is enough to see the bijection for algebraic spaces. But there we have the commutative diagram

\[ \text{Hom}_{\text{Shv}_{\text{et}}(\mathcal{A}_{\text{gp}Sp}/X)}(N(h_Y), \mathcal{G}_{\text{Yoneda}}) \sim \Gamma(N(Y), \mathcal{G}_{\text{Yoneda}}) \]

\[ \text{Hom}_{\text{Shv}_{\text{et}}(\mathcal{A}_{\text{gp}Sp}/X)}(h_Y, S(\mathcal{G})) \sim \Gamma(Y, S(\mathcal{G})) \rightarrow \Gamma(N(Y), \mathcal{G}_{\text{Yoneda}}) \]

q.e.d.

Next we want to study the behaviour of stalks under the functor $N$. For that let again $A \in \ast \mathcal{R}$, $X \in \mathcal{S}_{\text{gp}}/A$, and let $\mathcal{F} \in \text{Shv}_{\text{et}}(X)$ be an étale sheaf. If $K$ is a *artinian $A$-*algebra, there is by [BS07a][Theorem 4.13] a canonical bijection

\[ \text{Hom}_{\mathcal{S}_{\text{gp}}/A}(\ast \text{Spec } (K), N(X)) \rightarrow \text{Hom}_{\mathcal{S}_{\text{gp}}/A}(\text{Spec } (K), X). \]

Let now $K \in \ast \mathcal{R}$ be a separably closed field and

\[ \bar{x} : \text{Spec } (K) \rightarrow X \]

a geometric point of $X$. By abuse of notation, we denote by

\[ N(\bar{x}) : \ast \text{Spec } (K) \rightarrow N(X) \]

the corresponding *geometric point of $N(X)$. The stalk of $\mathcal{F}$ at $\bar{x}$ is by definition

\[ \mathcal{F}_{\bar{x}} = \text{colim}_U \Gamma(U, \mathcal{F}) \]

where $U$ runs through the inductive system of étale neighbourhoods of $\bar{x}$. If

\[ \begin{array}{ccc}
U & \rightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec } (K) & \rightarrow & X
\end{array} \]

is an étale neighbourhood of $\bar{x}$, then

\[ \begin{array}{ccc}
N(U) & \rightarrow & N(X) \\
\downarrow & & \downarrow \\
\ast \text{Spec } (K) & \rightarrow & N(\bar{x})
\end{array} \]

is a *étale neighbourhood of $N(\bar{x})$, and we have the canonical homomorphisms

\[ \Gamma(U, \mathcal{F}) \rightarrow \Gamma(N(U), N(\mathcal{F})). \]

These define a canonical homomorphism

\[ (4) \quad \mathcal{F}_{\bar{x}} \rightarrow N(\mathcal{F})_{N(\bar{x})}. \]

In general, this is not an isomorphism, but it is one for constructible sheaves. For that we recall:

2.8. **Definition.** An étale sheaf on a scheme $X$ is called constructible if it is representable by an algebraic space which is finite and étale over $X$. 
2.9. **Proposition.** In the above situation, if we assume that \( \mathcal{F} \) is a constructible sheaf, then the canonical morphism
\[
\mathcal{F}_{\bar{x}} \sim \rightarrow N(\mathcal{F})_{N(\bar{x})}
\]
is an isomorphism.

**Proof.** That \( \mathcal{F} \) is constructible means that it is representable by an algebraic space \( Y \to X \), finite and étale over \( X \). Then we have
\[
\mathcal{F}_{\bar{x}} = \text{Hom}_{\text{AlgSp}/X}(\text{Spec } (K), Y) = \text{Hom}_{\text{AlgSp}/K}(\text{Spec } (K), Y \otimes_X K)
\]
Now \( Y \otimes_X K \) is a scheme, and we have again by [BS07a][Theorem 4.13] the bijection
\[
\text{Hom}_{\text{Sch}/K}(\text{Spec } (K), Y \otimes_X K) \sim \rightarrow \text{Hom}_{\text{Sch}/K}(\ast \text{Spec } (K), N(Y \otimes_X K)),
\]
and \( N(Y \otimes_X K) = N(Y) \otimes_{N(X)} K \) as well as the identification
\[
N(\mathcal{F})_{N(\bar{x})} = \text{Hom}_{\text{Sch}/K}(\ast \text{Spec } (K), N(Y) \otimes_{N(X)} K).
\]

q.e.d.

2.10. **Remark.** We give an example showing that map (4) is not an isomorphism in general. For that let \( X = \text{Spec } (\ast Q) \), \( K = \ast \bar{Q} \), and let \( \bar{x} \) be given by the canonical \(*\)-embedding \( \ast Q \to \ast \bar{Q} \). Then \( G_{a,\bar{x}} \) is the algebraic closure of \( \ast Q \) in \( \ast \bar{Q} \), whereas \( \ast G_{a,N(\bar{x})} \) is \( \ast \bar{Q} \), which is surely different.

Next we want to remark that morphism (4) is compatible with specialisation morphisms. So let \( K, k \in \ast \mathbb{R} \) be separably closed fields, \( \bar{a} : \text{Spec } (K) \to X \) and \( \bar{s} : \text{Spec } (k) \to X \) two geometric points, and
\[
\varphi : \text{Spec } (O_{X,\bar{a}}) \to \text{Spec } (O_{X,\bar{s}})
\]
be a specialisation morphism, i.e. an \( X \)-morphism. Let
\[
\Psi : \ast \text{Spec } (O_{N(X),N(\bar{s})}) \to \ast \text{Spec } (O_{N(X),N(\bar{a})})
\]
be a \(*\)-specialisation morphism that prolongs \( \varphi \), i.e. the diagram
\[
\begin{array}{ccc}
O_{X,\bar{s}} & \longrightarrow & O_{N(X),N(\bar{s})} \\
\downarrow & & \downarrow \\
O_{X,\bar{a}} & \longrightarrow & O_{N(X),N(\bar{a})}
\end{array}
\]
commutes.

Then we have:

2.11. **Proposition.** In the situation described above, let \( \mathcal{F} \) be an étale sheaf on \( X \) which is representable by an algebraic space of finite presentation over \( X \). Then the induced diagram
\[
\begin{array}{ccc}
\mathcal{F}_{\bar{x}} & \longrightarrow & N(\mathcal{F})_{N(\bar{x})} \\
\downarrow & & \downarrow \\
\mathcal{F}_{\bar{a}} & \longrightarrow & N(\mathcal{F})_{N(\bar{a})}
\end{array}
\]
is commutative.
Proof. This follows directly from the explicit construction of the morphisms. q.e.d.

Now we want to see how cohomology and higher derived direct images behave under the functor $N$. First we consider the absolute case. For that let $A \in \mathcal{R}$ be an internal ring and $X \in \mathcal{S}_{\text{et}}^{\text{fp}}/A$.

2.12. Lemma. If $I \in \mathcal{S}_{\text{et}}^N(N(X))$ is a injective sheaf on $N(X)$, then $S(I)$ is a flabby sheaf on $X$.

Proof. Because all participating schemes are quasi compact, we only have to consider a finite covering $\{U_i \to U\}$. But then $\{N(U_i) \to N(U)\}$ is a covering, and we have

$$\check{H}^i(\{U_i \to U\}, S(I)) = \check{H}^i(\{N(U_i) \to N(U)\}, I) = 0$$

q.e.d.

We consider the following commutative diagram

$$\begin{array}{ccc}
\mathcal{S}_{\text{et}}(X) & \xrightarrow{S} & \mathcal{S}_{\text{et}}^N(N(X)) \\
\Gamma(X,-) & \downarrow & \downarrow \text{for } (N(X),-)
\end{array}$$

By Lemma 2.12 we get for a $\mathcal{G} \in \mathcal{S}_{\text{et}}(N(X))$ a spectral sequence

$$E_2^{p,q} = H^p_{\text{et}}(X, R^q S \mathcal{G}) \Rightarrow H^{p+q}_{\text{et}}(N(X), \mathcal{G})$$

and the edge homomorphism

$$H^p_{\text{et}}(X, S \mathcal{G}) \to H^p_{\text{et}}(N(X), \mathcal{G}).$$

For a sheaf $\mathcal{F} \in \mathcal{A}_{\text{et}}/X \subset \mathcal{S}_{\text{et}}(X)$ we compose the natural morphism induced by (2)

$$H^p_{\text{et}}(X, \mathcal{F}) \to H^p_{\text{et}}(X, S \circ N(\mathcal{F}))$$

with (7) and get

$$H^p_{\text{et}}(X, \mathcal{F}) \to H^p_{\text{et}}(N(X), N(\mathcal{F})).$$

2.13. Remark. We would like to give an alternative and more handy description of the above map. For given $A, X, \mathcal{F}$ we find a subring $A_0 \subset A$ of finite type over $\mathbb{Z}$, a scheme $X_0 \in \mathcal{S}_{\text{et}}^{\text{fp}}/A_0$ and an étale sheaf $\mathcal{F}_0 \in \mathcal{S}_{\text{et}}(X_0)$ with isomorphisms

$$X_0 \otimes_{A_0} A \simeq X$$

and

$$\mathcal{F} \simeq \pi_{A_0}^* \mathcal{F}_0,$$

where $\pi_{A_0}$ is the projection $\pi_{A_0} : X_0 \otimes_{A_0} A \to X_0$ and in (10) the identification (9) is used. We define the inductive system of subrings

$$L := \{B \subset A | B \text{ is of finite type over } \mathbb{Z} \text{ and } A_0 \subset B\}.$$
Then we have by \textbf{GVSD73}[VII.5.7] a canonical isomorphism
\begin{equation}
H^i_{\text{\acute{e}t}}(X, \mathcal{F}) \xrightarrow{\sim} \colim_{B \in L} H^i_{\text{\acute{e}t}}(X_0 \otimes_{A_0} B, \pi_B^* \mathcal{F}_0),
\end{equation}
where \( \pi_B \) denotes the projection \( \pi_B : X_0 \otimes_{A_0} B \to X_0 \). Now for each \( B \in L \) there is a canonical morphism
\begin{equation}
H^i_{\text{\acute{e}t}}(X_0 \otimes_{A_0} B, \pi_B^* \mathcal{F}_0) \to \ast H^i_{\text{\acute{e}t}}(N(X), N(\mathcal{F})),
\end{equation}
because \( N(X) \cong \ast(X_0 \otimes_{A_0} B) \otimes_{\ast B} A \). These morphisms induce a morphism
\begin{equation}
H^i_{\text{\acute{e}t}}(X, \mathcal{F}) \to \ast H^i_{\text{\acute{e}t}}(N(X), N(\mathcal{F})),
\end{equation}
which is identical to the morphism we have just constructed above.

For the relative case we consider an internal ring \( A \in \ast \mathcal{R} \), two schemes \( X, Y \in \text{Sch}^{\text{fp}}/A \) and a morphism of \( A \)-schemes \( f : X \to Y \).

We will construct a “base change” homomorphism in analogy to the usual base change homomorphism in étale cohomology. We consider the commutative diagram
\begin{equation}
\begin{array}{ccc}
\text{Shv}_{\text{\acute{e}t}}(X) & \xrightarrow{S} & \ast \text{Shv}_{\text{\acute{e}t}}(N(X)) \\
\downarrow f_* & & \downarrow N(f)_* \\
\text{Shv}_{\text{\acute{e}t}}(Y) & \xrightarrow{S} & \ast \text{Shv}_{\text{\acute{e}t}}(N(Y)).
\end{array}
\end{equation}

For a \(*\text{sheaf} \ G \in \ast \text{Shv}_{\text{\acute{e}t}}(N(X)) \) we have by Lemma 2.12 the spectral sequence
\begin{equation}
E_2^{p,q} = R^p f_* R^q S G \Rightarrow R^{p+q}(f_* \circ S) G
\end{equation}
and the edge homomorphisms
\begin{equation}
R^p f_*[S G] \to R^p(f_* \circ S) G.
\end{equation}
Because \( N(f)_* \) maps \(*\text{injectives to } \ast\text{injectives}, we further have the spectral sequence
\begin{equation}
E_2^{p,q} = R^p S R^q N(f)_* G \Rightarrow R^{p+q}(S \circ N(f)_*) G
\end{equation}
with edge homomorphisms
\begin{equation}
R^q(S \circ N(f)_*) G \to S R^q N(f)_* G.
\end{equation}
By the commutativity of (15) we can compose (16) and (17) to get a morphism
\begin{equation}
R^q f_*[S G] \to S R^q N(f)_* G.
\end{equation}
For an \( \mathcal{F} \in \text{Shv}_{\text{\acute{e}t}}^{f_p}(X) \) we get
\begin{equation}
R^q f_*[SN \mathcal{F}] \to S R^q N(f)_*[N \mathcal{F}] \tag{19}
\end{equation}
and then with (2)
\begin{equation}
R^q f_* \mathcal{F} \to S R^q N(f)_*[N \mathcal{F}] \tag{20}.
\end{equation}
2.14. **Remark.** As in 2.13, we can also give an easier description of this map. The construction of 2.13 gives a map of presheaves on $\mathcal{S}ch^{fp}/Y$ from the presheaf

$$(U \to Y) \mapsto H^i_{\text{et}}(X \times_Y U, \mathcal{F})$$

to the presheaf

$$(U \to Y) \mapsto H^i_{\text{et}}(N(X \times_Y U), N(\mathcal{F})).$$

Sheafification then induces morphism (20).

If we assume further that the sheaves $R^q f_* \mathcal{F}$ are constructible for all $q \geq 0$, we get by 2.7 the base change homomorphism

$$(21) \quad R^i f_* \mathcal{F} \to R^i N(f)_* [N \mathcal{F}].$$

For proper morphisms, we know that this base change homomorphism is actually an isomorphism:

2.15. **Proposition.** Let $f : X \to Y$ be a proper morphism with $X, Y \in \mathcal{S}ch^{fp}/A$, and $\mathcal{F}$ be a constructible sheaf on $X$. Then the base change homomorphism

$$(22) \quad R^i f_* \mathcal{F} \to R^i N(f)_* [N \mathcal{F}]$$

is an isomorphism.

**Proof.** By [Gro66][8.8.2] and lemma 2.3, we can choose $A_0 \subset A$ of finite type over $Z$, schemes $X_0, Y_0 \in \mathcal{S}ch^{fp}/A_0$ with $X_0 \otimes_{A_0} A \sim X$ and $Y_0 \otimes_{A_0} A \sim Y$, $\mathcal{F}_0 \in \mathcal{A}lg\mathcal{S}pc^{fp}/X_0 \subset \mathcal{S}hv_{\text{et}}/X_0$ with $\pi_X \mathcal{F}_0 \sim \mathcal{F}$ and a proper(!) morphism $f_0 : X_0 \to Y_0$, such that the diagram

$$\begin{array}{ccc}
X_0 \otimes_{A_0} A & \xrightarrow{f_0 \otimes id_A} & Y_0 \otimes_{A_0} A \\
\sim & & \sim \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes. By the base change theorem for proper morphisms for the cartesian square

$$\begin{array}{ccc}
X & \xrightarrow{\pi_X} & X_0 \\
\downarrow f & & \downarrow f_0 \\
Y & \xrightarrow{\pi_Y} & Y_0
\end{array}$$

we have

$$(23) \quad R^i f_* \mathcal{F} \simeq R^i f_* \pi_X^* \mathcal{F}_0 \simeq \pi_Y^* R^i f_0_* \mathcal{F}_0.$$ 

By construction we have

$$(24) \quad N(X) \simeq *X_0 \otimes_{A_0} A, N(Y) \simeq *Y_0 \otimes_{A_0} A, \text{ and } N(f) = *f_0 \otimes_{A_0} A.$$ 

So we have the cartesian square

$$\begin{array}{ccc}
N(X) & \xrightarrow{\pi_{N(X)}} & *X_0 \\
\downarrow N(f) & & \downarrow *f_0 \\
N(Y) & \xrightarrow{\pi_{N(Y)}} & *Y_0
\end{array}$$
and also by construction the identification

\[ N(\mathcal{F}) \simeq \pi_{N(X)}^* \mathcal{F}_0. \]

By the *base change theorem for the *proper morphism *\( f_0 \) in diagram (25), we have

\[ R^i N(f)_* N(\mathcal{F}) \simeq R^i N(f)_* \pi_{N(X)}^* \mathcal{F}_0 \simeq \pi_{N(Y)}^* R^i (\mathcal{F}_0)_* \mathcal{F}. \]

Because * is exact and maps injectives to injectives (cf [BS07a]), we have

\[ \ast(R^i f_0_* \mathcal{F}_0)) \simeq R^i (\ast f_0)_* \mathcal{F}_0. \]

Now we get what we want:

\[ N(R^i f_* \mathcal{F}) \simeq \pi_{N(Y)}^* (\ast(R^i f_0_* \mathcal{F}_0)) \simeq \pi_{N(Y)}^* (R^i (\ast f_0)_* \mathcal{F}_0) \simeq R^i N(f)_* N(\mathcal{F}). \]

q.e.d.

2.16. Corollary. Let \( K \in \ast \mathcal{R} \) be a separably closed field, \( f : X \to \text{Spec} (K) \) proper and \( \mathcal{F} \) a constructible étale sheaf on \( X \). Then the canonical morphism (8)

\[ H^1_{\text{ét}}(X, \mathcal{F}) \to \ast H^1_{\text{ét}}(N(X), N(\mathcal{F})) \]

is an isomorphism.

Proof. We just take the section of (22) on *Spec (\( K \)) and identify

\[ \Gamma(*\text{Spec} (K), N(R^i f_* \mathcal{F})) \cong \Gamma(\text{Spec} (K), R^i f_* \mathcal{F}) \]

by [BS07a] [Theorem 4.13] as in the proof of Proposition 2.9 q.e.d.

Now we want to prove a compatibility between the just defined morphism on cohomology and a morphism on the Picard group defined in [BS07a]:

2.17. Proposition. Let \( A \in \ast \mathcal{R} \) and \( X \in \text{Sch}^{\text{fp}}/A \). Then the diagram

\[ \begin{array}{ccc}
H^1_{\text{ét}}(X, G_m) & \xrightarrow{(i)} & \ast H^1_{\text{ét}}(N(X), G_m) \\
\downarrow & & \downarrow \\
\text{Pic}(X) & \xrightarrow{(ii)} & \ast \text{Pic}(N(X))
\end{array} \]

is commutative. Here (i) is defined by (8), and (ii) is defined by [BS07a] [Corollary 5.15].

Proof. Both horizontal maps are defined using a model \( X_0 \) of \( X \) which is defined over a subring \( A_0 \subset A \) of finite presentation over \( \mathbb{Z} \). Therefore we only have to show that the diagram

\[ \begin{array}{ccc}
H^1_{\text{ét}}(X_0, G_m) & \xrightarrow{(i)} & \ast H^1_{\text{ét}}(*X_0, \ast G_m) \\
\downarrow & & \downarrow \\
\text{Pic}(X_0) & \xrightarrow{(ii)} & \ast \text{Pic}(\ast X_0)
\end{array} \]

is commutative. But this is clear. q.e.d.
Now we give an application of the first section to the problem of the independence of \( l \) of Betti numbers for the étale cohomology of separated schemes of finite type over finite fields. The following is conjectured:

3.1. Conjecture. Let \( k \) be a field finite field, \( \bar{k} \) be an algebraic closure of \( k \), and \( X \) a separated scheme of finite type over \( k \). Then the dimension of the \( l \)-adic cohomology with compact support

\[
\dim_{\mathbb{Q}_l} H^i_{\text{ét}}(X \otimes_k \bar{k}, \mathbb{Q}_l)
\]

is independent of \( l \). (see for example [Kat91])

It is well known that the corresponding statement is true when the ground field \( k \) is of characteristic zero (cf. [GAD73][exp. XVI, 4]). Furthermore, it is generally believed that a theorem which is true for fields of characteristic zero is also true for fields of large positive characteristic. The aim of this section is to turn this belief into a precise statement in the case of the independence of \( l \) of Betti numbers.

First we prove the following general result about the dimension of \( l \)-adic cohomology.

3.2. Theorem. Let \( B \in \mathcal{R} \) be of finite type over \( \mathbb{Z} \), \( X \to \text{Spec}(B) \) a proper morphism and \((\mathcal{G}_n)_{n \in \mathbb{N}}\) an AR-\( l \)-adic system of constructible étale sheaves on \( X \). Let \( K \in \ast \mathcal{R} \) be an algebraically closed field and \( *B \to K \) an internal homomorphism. Then we have

\[
\dim_{\mathbb{Q}_l}(\lim_{n \in \mathbb{N}} H^i_{\text{ét}}(X_K, \mathcal{G}_n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) = \dim_{\mathbb{Q}_l}(\ast \lim_{n \in \ast \mathbb{N}} H^i_{\text{ét}}(*X_K, *\mathcal{G}_n) \otimes_{\ast \mathbb{Z}_l} \ast \mathbb{Q}_l)
\]

Proof. We have \( N(X_K) = *X_K \) and \( N(\mathcal{G}_n) = *\mathcal{G}_n \) on \( *X_K \). Therefore by (2.16) the canonical morphism

\[
H^i_{\text{ét}}(X_K, \mathcal{G}_n) \to *H^i_{\text{ét}}(*X_K, *\mathcal{G}_n)
\]

is an isomorphism for all \( n \in \mathbb{N} \). If both sides were \( l \)-adic respectively \( *l \)-adic systems, the claim would follow. But by applying the next proposition to the AR-\( l \)-adic system \((R^if_*\mathcal{G}_n)_{n \in \mathbb{N}}\) and using the fact that

\[
*R^if_*\mathcal{G}_n \sim R^i(*f)^*\mathcal{G}_n
\]

is an isomorphism, we see that we only need a finite number of terms to calculate a term of \( l \)-adic respectively \( *l \)-adic systems which are AR-isomorphic to the systems above. \textbf{q.e.d.}

3.3. Proposition. Let \( X \) be a noetherian scheme and \( \mathcal{G} = (\mathcal{G}_n)_{n \in \mathbb{N}} \) be an AR-\( l \)-adic system of constructible étale sheaves on \( X \). Then there are constants \( n_0, n_1, n_2 \in \mathbb{N} \) with the following property: If we define

\[
\mathcal{H}_n := \mathcal{G}_n / \ker(\mathcal{G}_n \to \mathcal{G}_n) \text{ and } \mathcal{F}_n := \text{im}(\mathcal{H}_{n_1+n_2+n} \to \mathcal{H}_{n_2+n})/l^{n+1},
\]

the system \((\mathcal{F}_n)_{n \in \mathbb{N}}\) is a torsion free \( l \)-adic system which is up to torsion AR-\( l \)-isomorphic to \( \mathcal{G} \).

Proof. The category of AR-\( l \)-adic systems of constructible étale sheaves is by [FK88][Prop. 12.12] noetherian. Therefore there is an \( n_0 \) so that for all \( m > n_0 \) the inclusion

\[
\ker(\mathcal{G} \to \mathcal{G}) \to \ker(\mathcal{G} \to \mathcal{G})
\]
is an AR-isomorphism. So
\[ G / \ker(G \xrightarrow{\iota_{0}} G) =: \mathcal{H} \]
is an AR-torsion free AR-\(l\)-adic system. Now for each AR-\(l\)-adic system \(\mathcal{H}\) there are integers \(n_1, n_2 \in \mathbb{N}\) such that
\[ J_n := \text{im}(\mathcal{H}_{n_1+n_2+n} \to \mathcal{H}_{n_2+n})/l^{n+1} \]
is an \(l\)-adic sheaf which is AR-isomorphic to \(\mathcal{H}\). \textbf{q.e.d.}

Now we restrict ourselves to projective varieties to have an easier notion of complexity. For natural numbers \(n, d \in \mathbb{N}\) and a field \(k\) we define \(H(n, d, k)\) as the set of all closed subschemes of \(\mathbb{P}^n_k\) of degree \(d\).

Now we consider the function \(B_i := B_{i,d,n} : \mathbb{P} \to \mathbb{N} \cup \{\infty\}\) on the set of prime numbers \(\mathbb{P}\) which is defined by
\[ p \mapsto \max \left\{ m \in \mathbb{N} \mid \text{for all finite fields } k \text{ with char}(k) = p \text{ and all } X \in H(n, d, k) \text{ and all primes } l_1, l_2 < m \text{ we have } \dim_{Q_{l_1}} H^{i}_{\text{ét}}(\bar{X}, Q_{l_1}) = \dim_{Q_{l_2}} H^{i}_{\text{ét}}(\bar{X}, Q_{l_2}) \right\} \]

Note that conjecture 3.1 says \(B_{i,d,n} \equiv \infty\). From the fact that the Conjecture holds in characteristic zero, we can deduce the following with our methods:

3.4. \textbf{Theorem.} With the above notations we have:
\[ \lim_{p \to \infty} B_{i,d,n}(p) = \infty \]

\textit{Proof.} Let us assume the statement is not true. Then by transfer there are an infinite prime \(P \in \mathbb{P} \setminus \mathbb{P}\), a *finite field \(k\) of internal characteristic \(P\), a *scheme \(X \hookrightarrow \mathbb{P}^n_k\) of *degree \(d\) and two standard primes \(l_1, l_2 \in \mathbb{P}\) such that
\[ \dim_{Q_{l_1}} H^{i}_{\text{ét}}(\bar{X}, *Q_{l_1}) = \dim_{Q_{l_2}} H^{i}_{\text{ét}}(\bar{X}, *Q_{l_2}) \]

But \(X\) is of *degree \(d \in \mathbb{N}\) in *\(\mathbb{P}^n_k\), so that by [BS07a][6.21], there is a \(X' \in \mathcal{S} \mathcal{C} h^{\text{fp}}/\mathbb{P}\) with \(\mathcal{N}(X') = \bar{X}\). Then by 3.2 we have for all standard primes \(l \in \mathbb{P}\)
\[ \dim_{Q_l} H^{i}_{\text{ét}}(X', Q_l) = \dim_{Q_l} H^{i}_{\text{ét}}(\bar{X}, *Q_l) \]
Therefore we have a contradiction to the independence of \(l\) for fields of characteristic zero. \textbf{q.e.d.}

4. \textbf{VOEVOVSKY MOTIVES AND CYCLES}

The aim of this section is to construct a functor \(N\) for the motivic cohomology of schemes. For that, we use the geometric construction of the triangulated category of mixed motives by Voevodsky from [Voe00]. The advantage of this way is that we only have to deal with finite correspondences and proper intersections.

For the convenience of the reader, we shortly recall the construction of Voevodsky’s triangulated category of geometrical motives. For details we refer to [Voe00] and [MVW06].
After that, we discuss enlargements of these motives and — most importantly — the existence of a functor $N$ for them.

For a field $k$, we denote by $\text{Sm}_k$ the category of smooth schemes of finite type over $k$. For $X \in \text{Sch}^{fp}_k$, we denote by $Z(X)$ the group of algebraic cycles. Let $V \subseteq X$ be a closed subscheme. Recall that $[V]$, the cycle associated to $V$, is defined as

$$\sum_x l_{O_{X,x}} (O_{V,x}) \in Z(X),$$

where the sum is taken over the generic points $x$ of the irreducible components of $V$ and $l_{O_{X,x}} (\cdot)$ denotes length of an $O_{X,x}$-module.

For two schemes $X,Y \in \text{Sch}^{fp}_k$, we denote by $c(X,Y)$ the subgroup of $Z(X \times Y)$ generated by cycles of $X \times Y$ which are finite and surjective over an irreducible component of $X$ via the projection $X \times Y \to X$. Elements of $c(X,Y)$ are called finite correspondences.

Now let $X,Y,Z \in \text{Sm}_k$, $W_1 \hookrightarrow X \times Y$ an irreducible closed subscheme, finite and surjective over a component of $X$, and $W_2 \hookrightarrow Y \times Z$ an irreducible closed subscheme, finite and surjective over a component of $Y$. One point of using finite correspondences is that $W_1 \times Z$ and $X \times W_2$ intersect properly on $X \times Y \times Z$. So we can define the intersection product $[W_1 \times Z] \cdot [X \times W_2]$, and it is a sum of prime cycles which are finite over $X$. Therefore one can define $W_1 \circ W_2 := p_{13*}([W_1 \times Z] \cdot [X \times W_2]) \in c(X,Z)$. In particular, we do not have to work with rational equivalence, and we do not have to use a moving lemma. This extends to a composition of finite correspondences. With this composition we get an additive category $\text{SmCor}(k)$ of smooth correspondences where the objects are smooth schemes of finite type over $k$ and where the morphisms are finite correspondences.

The graph $\Gamma_f$ of a usual morphism $f : X \to Y$ gives us a covariant functor $[-] : \text{Sm}_k \to \text{SmCor}(k)$.

Now we consider the homotopy category $\mathcal{K}^b(\text{SmCor}(k))$ of bounded complexes in $\text{SmCor}(k)$, and we let $T$ be the smallest thick subcategory of the triangulated category $\mathcal{K}^b(\text{SmCor}(k))$ which contains the following types of complexes:

1. $[X \times \mathbb{A}^1] \xrightarrow{[pr_1]} [X]$ for all $X \in \text{Sm}_k$
2. $[U \cap V] \to [U] \oplus [V] \to [X]$ for all $X \in \text{Sm}_k$ and Zariski open coverings $X = U \cup V$ of $X$ with the obvious morphisms.

Then let $DM_{gm}^{eff}(k)$ be the pseudo abelian hull of $\mathcal{K}^b(\text{SmCor}(k))/T$.

It turns out that $DM_{gm}^{eff}(k)$ is again a triangulated category, that the product $[X] \otimes [Y] = [X \times Y]$ for $X,Y \in \text{Sm}_k$ defines a tensor triangulated structure on $DM_{gm}^{eff}(k)$ and that
we have a canonical functor
\[ M_{gm} : \mathcal{S}m/k \to DM_{gm}^{eff}(k). \]

We denote by \( \mathbb{Z} \) the object \( M_{gm}(\text{Spec}(K)) \). It is given by the complex
\[ \cdots \to 0 \to [\text{Spec}(K)] \to 0 \to 0 \cdots. \]

We denote by \( \mathbb{Z}/n \) the object in \( DM_{gm}^{eff} \) which is given by the complex
\[ \cdots \to 0 \to [\text{Spec}(K)] \overset{n}{\to} [\text{Spec}(K)] \to 0 \to \cdots \]
living in degree \(-1\) and \(0\). So we have the exact triangle
\[ \mathbb{Z} \overset{n}{\to} \mathbb{Z} \to \mathbb{Z}/n \to \mathbb{Z}[1] \]
in \( DM_{gm}^{eff} \).

The Tate object \( \mathbb{Z}(1) \in DM_{gm}^{eff}(k) \) is defined to be the image of the complex \( [\mathbb{P}^1] \to [\text{Spec}(k)] \), where \( [\mathbb{P}^1] \) sits in degree \(2\). For an \( n \in \mathbb{N} \) we define \( \mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n} \), and for an object \( A \in DM_{gm}^{eff}(k) \) we set \( A(n) := A \otimes \mathbb{Z}(n) \).

Finally we get \( DM_{gm}(k) \) by inverting \( \mathbb{Z}(1) \), and it can be shown that the tensor structure lifts from \( DM_{gm}^{eff}(k) \) to \( DM_{gm}(k) \).

For varying fields \( k \in \mathcal{R} \), we get for each construction step a fibration of categories over the categories of fields in \( \mathcal{R} \) and if we choose an appropriate superstructure we get for each internal field \( K \in \mathcal{R} \) internal categories \( *\mathcal{S}m/K, *\mathcal{S}mCor(K), *\mathcal{X}^{h}(\mathcal{S}mCor(K))/T, *DM_{gm}^{eff}(K) \) and \( *DM_{gm}(K) \). For the tensor product in \( *DM_{gm}(K) \) we write again \( \otimes \) instead of \( \otimes \), and we again have the Tate object \( *\mathbb{Z}(1) \in *DM_{gm}(K) \). For an \( n \in \mathbb{N} \) we define \( *\mathbb{Z}(n) := *\mathbb{Z}(1)^{\otimes n} \), and for an object \( A \in *DM_{gm}(K) \) we set \( A(n) := A \otimes *\mathbb{Z}(n) \).

For each standard field \( k \) we get a functor of \( \otimes \)-triangulated categories
\[ * : DM_{gm}(k) \to *DM_{gm}(k). \]

with \( *(\mathbb{Z}(1)) = *\mathbb{Z}(1) \).

For an internal field \( K \in \mathcal{R} \), we want to define a functor
\[ N : DM_{gm}(K) \to *DM_{gm}(K). \]

First we have functors
\[ N : \mathcal{S}m/K \to *\mathcal{S}m/K \]
and
\[ N : \mathcal{S}ch^{fp}/K \to *\mathcal{S}ch^{fp}/K \]
which are constructed and analysed in [BS07a] [section 4].

Since \( N : \mathcal{S}ch^{fp}/K \to *\mathcal{S}ch^{fp}/K \) maps prime cycles to \( * \)prime \( * \)cycles by [BS07a] 6.4], we get an induced group homomorphism
\[ N : Z(X) \to *Z(N X), \quad \sum_{j=1}^{n} \alpha_j \cdot Z_j \mapsto \sum_{j=1}^{n} \alpha_j \cdot N Z_j \]
for each $X \in \text{Sch}^{fp}/K$.

4.1. **Proposition.** $N$ commutes with taking the associated cycle, i.e.

$$N[V] = ^*[N V] \in ^*Z^i(N X)$$

for every closed subscheme $V$ of $X$ of codimension $i$.

**Proof.** Let $V$ be a closed subscheme of $X$ of codimension $i$. Without loss of generality, we can assume that $V$ is irreducible with generic point $x$. Furthermore, we can assume that $X = \text{Spec}(A)$ is affine, so that $V$ corresponds to an ideal $a$ of $A$ and $x$ corresponds to a prime ideal $p$ which is the unique minimal prime ideal above $a$. Then [A.3] applied to $M := A/a$, shows

$$l_{0X,x} (\mathcal{O}_{V,x}) = l_{Ap} (A_p/A_p) = *l_{[NA]_{Np}} ([N A]_{Np}/a[N A]_{Np}) = *l_{0N_x,x'} (\mathcal{O}_{N V, x'})$$

where $x'$, the point given by the prime ideal $N_p = p[N A]$, is the generic point of $N V$ by [vdDS84, 2.7]. By definition of “associated cycle” this finishes the proof. q.e.d.

4.2. **Proposition.** $N$ commutes with push forward of cycles along proper morphisms.

**Proof.** Let $f : X \to Y$ be a proper morphism with $X,Y \in \text{Sch}^{fp}/K$, and let $W \subset X$ be a closed integral subscheme. By definition we have

$$f_*[W] = \begin{cases} [\kappa(W) : \kappa(f(W))]: [f(W)] & \text{if } \dim(W) = \dim(f(W)) \\ 0 & \text{otherwise,} \end{cases}$$

and the proposition follows from [BS07a][Lemma 6.27] and [BS07a][Theorem 6.4]. q.e.d.

Now for $X,Y \in \text{Sm}/K$, let $W \hookrightarrow X \times Y$ be a cycle which is finite and surjective over a connected component of $X$. Then again by [BS07a][section 4],

$$N(W) \hookrightarrow N(X \times Y) = N(X) \times N(Y)$$

is a cycle, *finite and surjective over $N(X)$, i.e. we get a morphism

$$c(X, Y) \to ^*c(N(X), N(Y))$$

With that we get the following theorem:

4.3. **Proposition.** The above construction defines a functor

$$N : \text{SmCor}(K) \to ^*\text{SmCor}(K).$$

**Proof.** We have to show that the construction is compatible with composition in the categories. By Proposition 4.1 and 4.2 it is enough to show that $N$ commutes with the intersection product of two cycles which intersect properly on a smooth scheme. By reduction to the diagonal, the occurring multiplicities are multiplicities of a Koszul complex. Because $N$ is exact on modules (cf [BS07a][Theorem 6.4]), it is enough to show the compatibility of $N$ with the Koszul complex and with the notion of length. But this is done in lemma [A.3] and lemma [A.4] q.e.d.

The functor $N : \text{SmCor}(K) \to ^*\text{SmCor}(K)$ induces a functor $\mathcal{K}^b(\text{SmCor}(K)) \to \mathcal{K}^b(^*\text{SmCor}(K))$. We compose this with the canonical functor

$$\mathcal{K}^b(^*\text{SmCor}(K)) \to ^*\mathcal{K}^b(^*\text{SmCor}(K))$$
which was studied in [BS05][Section 6] to get the functor
\[ N : K^b(SmCor(K)) \to *K^b(*SmCor)(K). \]

Again by [BS07a][Section 4] we have
\[ \begin{align*}
&\bullet N([X \times \mathbb{A}^1] \xrightarrow{[pr_1]} [X]) = [N(X) \times *\mathbb{A}^1] \xrightarrow{[pr_1]} [N(X)] \text{ for all } X \in Sm/K \\
&\bullet N([U \cap V]) \to [U] \oplus [V] \to [X]) = [N(U) \cap N(V)] \to [N(U)] \oplus [N(V)] \to [N(X)] \\
\end{align*} \]
for all \( X \in Sm/k \)
and if \( U \cup V = X \) is an open covering of \( X \), then \( N(U) \cup N(V) = N(X) \) is an open covering of \( N(X) \). Hence we have \( N(\mathcal{T}) \subset *\mathcal{T} \). Therefore we get a functor
\[ N : K^b(SmCor(K))/\mathcal{T} \to *K^b(*SmCor)(K)/*\mathcal{T}. \]

By the universal property of the pseudo abelian hull this further induces a functor
\[ N : DM_{gm}^{eff}(K) \to *DM_{gm}^{eff}(K) \]
and then again by a universal property a functor
\[ N : DM_{gm}(K) \to *DM_{gm}(K). \]

Furthermore, by [BS07a][Section 4] we have
\[ N(X \times Y) = N(X) \times N(Y) \]
and
\[ N([\mathbb{P}^1_K] \to [\text{Spec}(K)]) = [\mathbb{P}^1_K] \to [*\text{Spec}(K)], \]
and therefore \( N : DM_{gm}(K) \to *DM_{gm}(K) \) is compatible with the tensor structure on both sides, and we have \( N(\mathbb{Z}(n)) = *\mathbb{Z}(n) \). To summarise this we formulate the next

4.4. **Proposition.** Let \( K \) be an internal field. The functor \( N : SmCor(K) \to *SmCor(K) \) of Proposition 4.3 induces a natural functor of tensor triangulated categories
\[ N : DM_{gm}(K) \to *DM_{gm}(K), \]
and the diagram
\[ \begin{array}{ccc}
Sm/K & \xrightarrow{N} & *Sm/K \\
M \downarrow & & \downarrow *M \\
DM_{gm}(K) & \xrightarrow{N} & *DM_{gm}(K)
\end{array} \]
is commutative.

Now we want to show how this functor \( N \) induces a morphism for motivic cohomology. First we recall the definition of motivic cohomology in terms of Voevodsky’s triangulated category of motives.

Let \( X \) be a smooth scheme of finite type over a field \( k \).
4.5. **Definition** (motivic cohomology). The motivic cohomology of $X$ is defined as

$$H^i_{\text{M}}(X, \mathbb{Z}(j)) := \text{Hom}_{DM_{gm}(k)}(M_{gm}(X), \mathbb{Z}(j)[i]).$$

The $\otimes$-triangulated structure and the pullback along the diagonal define on $\bigoplus_{i,j \in \mathbb{N}} H^i_{\text{M}}(X, \mathbb{Z}(j))$ a graded ring structure. In the same way we define motivic cohomology with finite coefficients

$$H^i_{\text{M}}(X, \mathbb{Z}/n(j)) := \text{Hom}_{DM_{gm}(k)}(M_{gm}(X), \mathbb{Z}/n(j)[i]).$$

For an internal field $K$ and an $X \in \ast Sm/K$ we define

4.6. **Definition.** The *motivic cohomology of $X$ is defined as

$$\ast H^i_{\text{M}}(X, \ast \mathbb{Z}(j)) := \text{Hom}_{\ast DM_{gm}(k)}(\ast M_{gm}(X), \ast \mathbb{Z}(j)[i])$$

resp.

$$\ast H^i_{\text{M}}(X, \ast \mathbb{Z}/n(j)) := \text{Hom}_{\ast DM_{gm}(k)}(\ast M_{gm}(X), \ast \mathbb{Z}/n(j)[i])$$

for $i,j,n \in \ast \mathbb{N}$. In the same way as above

$$\ast \bigoplus_{i,j \in \ast \mathbb{N}} \ast H^i_{\text{M}}(X, \ast \mathbb{Z}(j))$$

is a *graded ring.

4.7. **Remark.** An alternative way to define *motivic cohomology is the following: For each field $k$, motivic cohomology is a functor

$$H^i(-, \mathbb{Z}(\cdot)) : (Sm/k)^{op} \rightarrow \text{graded rings}.$$ 

By transfer, for details see [BS05], we get for each internal field $K$ an (internal) functor

$$\ast H^i(-, \mathbb{Z}(\cdot)) : (\ast Sm/K)^{op} \rightarrow (\text{internal}) \text{ graded rings}.$$ 

Obviously this agrees with the above definition.

4.8. **Remark.** For a smooth equidimensional scheme $X$ we have by [Voe02] the following identification

$$H^i_{\text{M}}(X, \mathbb{Z}(j)) = CH^j(X, 2j - i),$$

where $CH^*(X, *)$ denotes the higher Chow groups of Bloch. In particular we have

$$H^0_{\text{M}}(X, \mathbb{Z}(i)) = CH^i(X),$$

where $CH^i(X)$ denotes the usual Chow groups, i.e. cycles of codimension $i$ on $X$ modulo rational equivalence.

Now let $K$ be an internal field.

4.9. **Proposition/ Definition.** The functor $N : DM_{gm} \rightarrow \ast DM_{gm}(K)$ of Proposition 4.4 induces for all $i,j \in \mathbb{N}_0$ and all $X \in Sm/K$ a natural morphism

$$N : H^i_{\text{M}}(X, \mathbb{Z}(j)) \rightarrow \ast H^i_{\text{M}}(N(X), \ast \mathbb{Z}(j))$$

and then a morphism of graded rings

$$N : \bigoplus_{i,j \in \mathbb{N}_0} H^i_{\text{M}}(X, \mathbb{Z}(j)) \rightarrow \ast \bigoplus_{i,j \in \ast \mathbb{N}_0} \ast H^i_{\text{M}}(N(X), \ast \mathbb{Z}(j)).$$
4.10. **Remark.** With remark 4.8 we see that we get a morphism

\[ N : CH^i(X) \to ^*CH^i(N(X)) \]

, and by the construction it is easy to see that for a prime cycle \([Y]\), we have

\[ N([Y]) = [N(Y)]. \]

For a smooth scheme \(X\) over a field \(K\) and an \(n \in \mathbb{N}\), prime to \(\text{char}(K)\), there is the cycle class map

\[ cl_n : CH^i(X) \to H^{2i}_{\text{ét}}(X, \mu_n^\otimes). \]

By transfer, for a *smooth schemes \(X\) over an internal field \(K\) and an \(n \in ^*\mathbb{N}\), *prime to \(*\text{char}(K)\). we have the induced map

\[ *cl_n : ^*CH^i(X) \to ^*H^{2i}_{\text{ét}}(X, ^*\mu_n^\otimes). \]

These are compatible with \(N\):

4.11. **Proposition.** Let \(K\) be an internal field, \(X\) be a smooth scheme of finite type over \(K\), and \(n \in \mathbb{N}\) prime to \(\text{char}(K)\). Then the diagram

\[
\begin{array}{ccc}
CH^i(X) & \xrightarrow{N} & ^*CH^i(N(X)) \\
\downarrow{cl} & & \downarrow{*cl} \\
H^{2i}_{\text{ét}}(X, \mu_n^\otimes) & \xrightarrow{N} & ^*H^{2i}(N(X), ^*\mu_n^\otimes)
\end{array}
\]

is commutative.

**Proof.** For \(i = 1\) the map \(cl_n\) is defined as follows. One first identifies

\[ CH^1(X) \cong Pic(X) \cong H^1_{\text{ét}}(X, G_m) \]

and then uses the connection homomorphism

\[ H^1_{\text{ét}}(X, G_m) \xrightarrow{\delta} H^2_{\text{ét}}(X, \mu_n) \]

of the short exact sequence

\[ 1 \to \mu_n \to G_m \xrightarrow{n} G_m \to 1. \]

The same is true for \(*cl_n\) and therefore in this case the claim follows from Proposition 2.17. By the compatibility of \(N\) with the intersection product 4.4 the diagram is commutative for cycles which are products of divisors. Then the cohomological methods of [Del77][Cycles, sect. 2.2] reduce the general case to this case. **q.e.d.**

5. **Complexity of cycles**

In this section we give a notion of complexity of a cycle and show how that can be used to describe the image of \(N : CH^i(X) \to ^*CH^i(N(X))\). We show that for divisors, the image of \(N\) can be describe much easier, and that in this case \(N\) is injective. Then we show that Mumford’s result, that Chow groups are not finite dimensional, implies that in general the morphism \(N : CH^i(X) \to ^*CH^i(N(X))\) is not injective. After that we show
that our notion of complexity behaves well under the intersection product of cycles, and
we show that rational equivalence behaves somehow bad under this notion of complexity.

Let $K$ be a field and $X \hookrightarrow \mathbb{P}^n_K$ be a closed immersion. Then we define a notion of
complexity on the Chow ring of $X$ as follows:

5.1. **Definition.** An element $x \in CH^i(X)$ has complexity less than $c \in \mathbb{N}$ if we can write
$x = \sum_{i=1}^{n} \alpha_i [X_i]$ where $|\alpha_i| < c$, $n < c$ and the $X_i \hookrightarrow X$ are integral subschemes of degree $< c$. We
define the notion of *complexity less than $c \in \ast \mathbb{N}$ for *cycles on *projective varieties in
the obvious analogue way.

The next lemma shows that this notion of complexity is quite natural, if we want to
understand the image of $N$ for Chow groups.

5.2. **Lemma.** Let $K$ be an internal field and $X \hookrightarrow \mathbb{P}^n_K$ a closed immersion. Then a cycle
$x' \in \ast CH^i(N(X))$ is in the image of the morphism
$$N : CH^i(X) \to \ast CH^i(N(X))$$
if and only if the *complexity of $x'$ is less than $d$ for a $d \in \mathbb{N} \subset \ast \mathbb{N}$.

**Proof.** An element $x = \sum \alpha_i [X_i]$ is mapped to $N(x) = \sum \alpha_i [N(X_i)]$. By [BS07a][Cor.
6.18] the *degree of $N(X_i)$ is equal to the degree of $X_i$. Furthermore, by [BS07a][Corollary
6.21], a prime cycle $Y_i \hookrightarrow N(X)$ is of the form $N(X_i)$ for an $X_i \hookrightarrow X$ if and only if its
*degree is in $\mathbb{N} \subset \ast \mathbb{N}$. q.e.d.

For cycles of codimension one the situation is much easier, and we can simply use the
Hilbert polynomial instead of the notion of complexity.

5.3. **Theorem.** Let $K$ be an internal field, $X$ a projective $K$-scheme with integral geo-
metric fiber and $\phi \in \mathbb{Q}[t]$ a rational polynomial. Then the morphism
$$Pic^\phi(X) \to \ast Pic^\phi(N(X))$$
is bijective.

**Proof.** We find a subring $A_0 \subset K$ of finite type over $\mathbb{Z}$ and a projective $A_0$-scheme $\mathfrak{x}$
with geometrically integral fibers such that $\mathfrak{x}_0 \otimes_{A_0} K = X$. We denote by $Pic^\phi_{\mathfrak{x}_0/A_0}$
the relative Picard-functor. By [GBI71][XIII, 3.2 (iii) and 2.11] $Pic^\phi_{\mathfrak{x}_0/A_0}$ is representable by
a scheme $Pic^\phi_{\mathfrak{x}_0/A_0}$ of finite type over $A_0$. Then $Pic^\phi_{\mathfrak{x}_0/A_0}$ is represented by $Pic^\phi_{\mathfrak{x}_0/A_0} \otimes_{A_0} K$,
and $\ast Pic^\phi_{N(X)/k}$ is represented by
$$\ast Pic^\phi_{\mathfrak{x}_0/A_0} \otimes_{A_0} K = N(Pic^\phi_{\mathfrak{x}_0/A_0} \otimes_{A_0} K),$$
so the theorem follows from [BS07a][theorem 4.14]. q.e.d.
5.4. **Corollary.** Let $X$ be as in the above theorem. Then the morphism

$$N : CH^1(X) \to *CH^1(N(X))$$

is injective, and the image consists of those $*$divisors whose Hilbert polynomial is in $\mathbb{Q}[t] \subset *\mathbb{Q}[t]$.

Lemma 5.2 describes the image of $N : CH^i(X) \to *CH^i(N(X))$. Now we want to show that in general $N$ fails to be injective.

For that we consider a smooth projective irreducible surface $X$ over $\mathbb{C}$ with $H^2(X, O_X) \neq 0$. For such a surface Mumford showed:

5.5. **Theorem.** Let $X$ be as above. Then $CH^2(X)_0 := \{0$-cycles of degree zero\} is not finite dimensional, i.e. for all $n \in \mathbb{N}$ the natural map

$$S^n X(\mathbb{C}) \times S^n X(\mathbb{C}) \to CH^2(X)_0$$

is not surjective.

*Proof.* That is the main result of [Mum68]. q.e.d.

We have further the following characterisation of finite dimensionality:

5.6. **Proposition.** Let $X$ be a smooth, projective, geometrically irreducible variety of dimension $d$ over a field $k$, and let $\Omega \supseteq k$ be an algebraically closed field. Consider the following statements

(i) There is an $n \in \mathbb{N}$ such that

$$S^n X(\Omega) \times S^n X(\Omega) \to CH^d(X_\Omega)_0$$

is surjective

(ii) If $B \subseteq X_\Omega$ is a smooth linear space section of dimension one, then

$$CH^d(X_\Omega \setminus B) = 0$$

(iii) The canonical map

$$CH^d(X_\Omega)_0 \to Alb(X)(\Omega)$$

is an isomorphism, where $Alb(X)$ is the albanese variety of $X$.

Then $(iii) \Rightarrow (ii) \Rightarrow (i)$. If $\Omega$ is uncountable, all three statements are equivalent.

*Proof.* See [Jan94][1.6]. q.e.d.

Furthermore, For the algebraic closure of a finite field we have the following result:

5.7. **Theorem.** Let $X$ be as in the above proposition, where $k$ is now a finite field. Then the morphism

$$CH^d(\bar{X})_0 \to Alb(\bar{X})(\bar{k})$$

is an isomorphism. Here $Alb(X)$ denotes the albanese variety of a scheme $X$. 
Proof. See [KS83][9] q.e.d.

Now let \( P \in \{ \mathbb{P} \setminus P \} \) be an infinite prime with \( \bar{Q} \subset \mathbb{F}_P \), and let \( k \) be an *algebraic closure of \( \mathbb{F}_P \). Then \( k \) is in particular an algebraically closed field, and \( \mathbb{C} \) can be embedded in \( k \).

We can then use Theorem 5.5 to prove the following theorem.

5.8. Theorem. Let \( X \) be as above. Then the morphism

\[
CH^2(X_k) \to ^*CH^2(N(X_k))
\]

is not injective.

Proof. From 5.5 and 5.6 it follows that for a smooth linear space section \( B \hookrightarrow X_k \) of dimension one we have

\[
CH^2(X_k \setminus B) \neq 0.
\]

But \( k \) is internally the *algebraic closure of a *finite field. Therefore by transfer of 5.7 the map

\[
^*CH^2(N(X_k))_0 \to ^*Alb(N(X))(k)
\]

is an isomorphism. Then by 5.6 again we have

\[
^*CH^2(N(X_k) \setminus N(B)) = 0.
\]

Now we consider the commutative diagram

\[
\begin{array}{ccc}
CH^1(B) & \longrightarrow & ^*CH^1(N(B)) \\
\downarrow _{i_*} & & \downarrow _{N(i)_*} \\
CH^2(X) & \longrightarrow & ^*CH^2(N(X)) \\
\downarrow _{j^*} & & \downarrow _{N(j)^*} \\
CH^2(X \setminus B) & \longrightarrow & ^*CH^2(N(X) \setminus N(B)) \\
\downarrow & & \downarrow _{\sim} \\
0 & & 0
\end{array}
\]

which has exact columns and where \( j : (X \setminus B) \hookrightarrow X \) denotes the open immersion. Now let \( x \in CH^2(X) \) with \( j^*x \neq 0 \). Then by diagram chasing there is a \( y \in ^*CH^1(N(B)) \) such that \( N(i_*)y = N(x) \). Now we have \( \deg(y) = \deg(N(x)) = \deg(x) \). Therefore by 5.3 there is an \( \tilde{y} \in CH^1(B) \) with \( N(\tilde{y}) = y \). But then we have \( x - i_*\tilde{y} \neq 0 \) but \( N(x - i_*\tilde{y}) = 0 \). q.e.d.

On the other hand, the above cited results can also be used to show surjectivity of \( N \) in another situation:

5.9. Theorem. Let \( k \) be the *algebraic closure of a *finite field, and let \( X/k \) a smooth, projective and geometrically irreducible scheme of dimension \( d \). Then the morphism

\[
N : CH^d(X)_0 \to ^*CH^d(N(X))_0
\]

is surjective.
Proof. By transfer of \[5.7\] and \[5.6\] we have the following diagram
\[
\begin{array}{c}
\ast CH^1(NB)_0 \quad \xrightarrow{(N_\ast)} \ast CH^d(N(X))_0 \quad \xrightarrow{\ast CH^d(N(X) \setminus N(B)) = 0} \\
CH^1(B)_0 \quad \xrightarrow{} \quad CH^d(X)
\end{array}
\]
with exact first line. But the vertical map on the left is an isomorphism by \[5.4\] and so the claim follows. q.e.d.

In contrast to that we have the following result for surfaces over \(*C*:\)

5.10. Proposition. Let \(X\) be a smooth, projective, and irreducible surface over \(C\) with \(H^2(X,O_X) \neq 0\). Then the map
\[
CH_0(X \cdot C)_0 \to \ast CH_0(N(X \cdot C))_0
\]
is not surjective.

Proof. By Theorem \[5.5\] and its transfer we have:
\[
\forall n \in \mathbb{N} : S^nX \cdot C(\ast C) \times S^nX \cdot C(\ast C) \to CH_0(X \cdot C)_0 \text{ is not surjective.}
\]
and
\[
\forall n \in \ast \mathbb{N} : S^nN(X \cdot C)(\ast C) \times S^nN(X \cdot C)(\ast C) \to \ast CH_0(N(X \cdot C))_0 \text{ is not surjective,}
\]
and we have the following commutative diagram:
\[
\begin{array}{c}
S^nX \cdot C(\ast C) \times S^nX \cdot C(\ast C) \quad \xrightarrow{} \quad CH_0(X \cdot C)_0 \\
S^nN(X \cdot C)(\ast C) \times S^nN(X \cdot C)(\ast C) \quad \xrightarrow{} \quad \ast CH_0(N(X \cdot C))_0
\end{array}
\]
Now \(\bigcup_{n \in \mathbb{R}} \text{im}(S^nX \cdot C(\ast C) \times S^nX \cdot C(\ast C) \to CH_0(X \cdot C)_0) = CH_0(X \cdot C)_0\), and if \(CH_0(X \cdot C)_0 \to \ast CH_0(N(X \cdot C))_0\) was surjective, then for all \(n \in \ast \mathbb{N} - \mathbb{N}\) the map
\[
S^nN(X \cdot C) \times S^nN(X \cdot C) \to \ast CH_0(N(X))_0
\]
would be surjective. q.e.d.

Now we want to show that the non-injectivity of the morphism \(N\) somehow says that our notion of complexity does not go along very well with rational equivalence. For that let us remind you first another, similar situation.

Let \(f_1, f_2\) and \(g\) be polynomials over a field \(k\) such that \(g \in (f_1, f_2)\). That means that there are polynomials \(a_1\) and \(a_2\) with
\[
(29) \quad g = a_1 \cdot f_1 + a_2 \cdot f_2.
\]
Now one can ask for a bound of the minimal degree of \(a_1\) and \(a_2\) in terms of the degrees of \(f_1, f_2\) and \(g\) such that \[29\] holds. And in fact such a bound exists, and it only depend on the degrees of \(f_1, f_2\) and \(g\), but not on their coefficients or the field \(k\) (!). This follows for example from the nonstandard fact that the ring homomorphism
\[
k[x_1, \cdots, x_n] \to k^*[x_1, \cdots, x_n]
\]
is flat for an internal field \( k \) (cf. \[\text{vdDS84}\]).

Now we want to answer a similar question for the rational equivalence of cycles. For that let \( x, y \in Z_k(X) \) be two cycles, which are rational equivalent. That means that there is a cycle \( z \in Z_{k+1}(X \times \mathbb{P}^1) \) such that \( x - y = z|_{X \times \{0\}} - z|_{X \times \{1\}} \). Now a natural question is whether we can bind the complexity of \( z \) by the complexities of \( x \) and \( y \). One could hope that the following was true:

For all \( d, n \in \mathbb{N} \) there is a constant \( C(n, d) \in \mathbb{N} \) such that for all fields \( K \), all closed subschemes \( X \hookrightarrow \mathbb{P}^n_K \) of degree less than \( d \) and all rational equivalent cycles \( x, y \in Z_k(X) \) with complexity less than \( d \), there is a cycle \( z \in Z_{k+1}(X \times \mathbb{P}^1_K) \) with complexity less than \( C(n, d) \), such that

\[
x - y = z|_{X \times \{0\}} - z|_{X \times \{1\}}.
\]

But this can not be true:

5.11. **Theorem.** The above statement is false.

**Proof.** If the statement were true, then \( N : CH_k(X) \to CH_k(N(X)) \) would be injective by Lemma 5.2. q.e.d.

Next we want to see how our notion of complexity of a cycles behaves under intersection products. If we consider two cycles \( x \in CH^i(X) \) and \( y \in CH^j(X) \), both of complexity less than \( d \), it is natural to ask about the complexity of their intersection product \( x \cdot y \in CH^{i+j}(X) \). If we could write \( x = \sum_{i=1}^n \alpha_i[X_i] \) and \( y = \sum_{j=1}^m \beta_j[Y_j] \) with \( \alpha_i, \beta_j, n, m, \deg(X_i), \deg(Y_j) < d \) such that all \( X_i \) and \( Y_j \) intersect properly, the complexity of \( x \cdot y \) would be \( < d^4 \). But in general one has to move the cycles in their rational equivalence class to get proper intersections, and it is hard to control the complexities during this process (cf. 5.11). But using the previous result, we can at least prove the existence of a uniform bound for the complexity of the product:

5.12. **Theorem.** For all \( d, n \in \mathbb{N} \) there is a constant \( C(d, n) \) with the following property: For all fields \( k \), all closed subschemes \( X \hookrightarrow \mathbb{P}^n_k \) of degree less than \( d \), and cycles \( x \in CH^i(X) \) and \( y \in CH^j(X) \), both of complexity less than \( d \), the product \( x \cdot y \in CH^{i+j}(X) \) is of complexity less than \( C(d, n) \).

**Proof.** All statements are about schemes or subschemes which are of finite presentation over a field, and the intersection product behaves well under field extension. Therefore it is enough to consider all fields which are finitely generated over their prime fields, and we can choose a category of rings \( \mathcal{R} \) which contains all such fields. Now we assume that the statement is false. Then by transfer there are an internal field \( K \in \mathcal{R} \), a *closed subscheme \( X' \hookrightarrow \mathbb{P}^n_K \) of degree less than \( d \) and *cycles \( x' \in *CH^i(X) \) and \( y' \in *CH^j(X) \), both of complexity less than \( d \), such that the product \( x' \cdot y' \in *CH^{i+j}(X') \) is not of complexity less than \( n \) for all \( n \in \mathbb{N} \). But because of our assumption about the degree of \( X' \) and the complexities of \( x \) and \( y \), there are a closed subscheme \( X \in \mathbb{P}^n_K \) and cycles \( x \in CH^i(X) \) and \( y \in CH^j(Y) \) with

\[
N(X) = X', N(x) = x' \text{ and } N(y) = y'.
\]
Furthermore, by 4.9 we have $N(x \cdot y) = x' \cdot y'$. But the complexity of $x \cdot y$ is less than $n_0$ for an $n_0 \in \mathbb{N}$, and then the complexity of $N(x \cdot y)$ is also less than $n_0 \in \mathbb{N}$, a contradiction.

q.e.d.

5.13. Remark. This theorem corresponds to the fact that the intersection product is constructible, proven in [Mac00] by careful analysis of the construction given in [Ful84].

5.14. Remark. With the same argument, a similar result can be shown for higher Chow groups.

APPENDIX A. LENGTHS AND THE KOSZUL COMPLEX

Let $k$ be an internal field, let $A$ be a $k$-algebra of finite type, let $p \subseteq A$ be a prime ideal, and let $M$ be a finitely generated $A$-module.

A.1. Lemma. $M_p = 0 \implies [N M]_{N p} = 0$.

Proof.

$M_p = 0 \implies \exists f \in A \setminus p : f M = 0 \implies f [N M] = 0 \implies [N M]_{N p} = 0$.

q.e.d.

A.2. Lemma. Let $\varphi : M' \to M$ be a morphism of finitely generated $A$-modules such that Ker $(\varphi_p) = 0$. Then

(i) Ker $([N \varphi]_{N p}) = 0$ and

(ii) $\kappa(p) \cong \text{Coker } (\varphi_p) \implies \ast \kappa(p) \cong \text{Coker } ([N \varphi]_{N p})$.

Proof.

Ker $(\varphi_p) = 0 \implies \exists f \in A \setminus p : \text{Ker } (\varphi_f) = 0 \implies [N \varphi]_f = 0 \implies \text{Ker } ([N \varphi]_{N p}) = 0$,

which proves (i). For (ii), assume $\kappa(p) \cong \text{Coker } (\varphi_p)$. Then

$\exists f \in A \setminus p : A_f / p A_f \cong \text{Coker } (\varphi_f) \implies [N A]_f / p [N A]_f \cong \text{Coker } ([N \varphi]_f) \implies \ast \kappa(p) \cong \text{Coker } ([N \varphi]_{N p})$,

where we use $N p = p[N A]$ (valDS84 2.5). q.e.d.

A.3. Lemma. $l_{A_p} (M_p) = \ast l_{[N A]_{N p}} ([N M]_{N p}) \in \mathbb{N}_0 \cup \{\infty\}$.
Proof. Put \( l := l_{A_p}(M_p) \) and \(*l := *l_{[N A]N_p}([N M]N_p)\). First consider the case \( l < \infty\). Then by the definition of “length”, there exists a chain

\[
0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_{l-1} \subseteq M_l = M_p
\]

of (finitely generated) \( A_p\)-modules satisfying \( \kappa(p) \cong M_i/M_{i-1} \) for \( i = 1, \ldots, l \). Since the \( M_i \) are finitely generated, there exist an \( f \in A \setminus p \) and a tower

\[
0 = M_0' \xrightarrow{\varphi_0} M_1' \xrightarrow{\varphi_1} M_2' \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_{l-2}} M_{l-1}' \xrightarrow{\varphi_{l-1}} M_l' = M_f
\]

of (finitely generated) \( A_f\)-modules with \( (M_i')_p \cong M_i \) for \( i = 0, \ldots, l \). Applying A.1 and A.2 to \( A_f \) instead of \( A \), it follows that we have a chain

\[
0 = \tilde{M}_0 \subseteq \tilde{M}_1 \subseteq \tilde{M}_2 \subseteq \ldots \subseteq \tilde{M}_{l-1} \subseteq \tilde{M}_l = M_p
\]

of *finitely generated \([N A]N_p\)-modules, where \( \tilde{M}_i := [N M_i']N_p \) for \( i = 0, \ldots, l \). Furthermore, \( *\kappa(N p) \cong \tilde{M}_i/\tilde{M}_{i-1} \) for \( i = 1, \ldots, l \), which proves \(*l = l\).

Now let \( M_p \) have infinite length, and choose an arbitrary \( n \in \mathbb{N}_+ \). Since \( M_p \) has infinite length, there exists an \( A_p\)-submodule \( M' \) of \( M_p \) of length \( n \), and \( M' \cong M''_p \) for a suitable \( f \in A \setminus p \) and an \( A_f\)-module \( M'' \). By what has already been proven (taking \( A_f \) instead of \( A \)), \( *l_{[N A]N_p}([N M'']N_p) = n \), so \([N M]N_p\) contains a *submodule of *length \( n \). Since \( n \) was chosen arbitrarily, \(*l \geq n\) for all \( n \in \mathbb{N}_+ \), so \(*l \) lies in \((\mathbb{N}_0 \setminus \mathbb{N}_0) \cup \{\infty\}\) and is hence infinite. q.e.d.

Now we consider an element \( m \in M \) and the Koszul complex

\[
\mathcal{K}(M,m) := 0 \to A \to M \to \Lambda^2 M \to \ldots
\]

In fact this is a bounded complex of finitely generated \( A \)-modules.

A.4. Lemma.

\[
N(\mathcal{K}(M,m)) = *\mathcal{K}(N(M),N(m))
\]

Proof. By transfer we have

\[
*\mathcal{K}(N(M),N(m)) := 0 \to N(A) \to N(M) \to \Lambda^2 N(M) \to \ldots
\]

In [BS07a] 5.14 it is shown that \( N \) for modules commutes with tensor products. Similarly one can show that \( N \) commutes with alternating products. So the claim follows. q.e.d.

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E-mail address: lbrunjes@gmx.de

CHRISTIAN SERPÉ, WESTFÄLISCHE WILHELM-S-UNIVERSITÄT MÜNSTER, MATHEMATISCHES INSTITUT, SONDERFORSCHUNGSBEREICH 478 “GEOMETRISCHE STRUKTUREN IN DER MATHEMATIK”, HITTORFSTR. 27, D-48149 MÜNSTER, GERMANY

E-mail address: serpe@uni-muenster.de