A local systolic-diastolic inequality in contact and symplectic geometry

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Abstract

Let $\Sigma$ be a connected closed three-manifold, and let $t_\Sigma$ be the order of the torsion subgroup of $H_1(\Sigma; \mathbb{Z})$. For a contact form $\alpha$ on $\Sigma$, we denote by $\text{Volume}(\alpha)$ the contact volume of $\alpha$, and by $T_{\text{min}}(\alpha)$ and $T_{\text{max}}(\alpha)$ the minimal period (systole) and the maximal period (diastole) of prime periodic orbits of the Reeb flow of $\alpha$, respectively. We say that $\alpha$ is Zoll, if its Reeb flow generates a free $S^1$-action on $\Sigma$. We prove that every Zoll contact form $\alpha_*$ on $\Sigma$ admits a $C^3$-neighbourhood $\mathcal{U}$ in the space of contact forms such that

$$t_\Sigma T_{\text{min}}(\alpha)^2 \leq \text{Volume}(\alpha) \leq t_\Sigma T_{\text{max}}(\alpha)^2, \quad \forall \alpha \in \mathcal{U},$$

and the equality sign holds in any of the two inequalities if and only if $\alpha$ is Zoll.

We extend the above picture to odd-symplectic forms (i.e. maximally non-degenerate closed two-forms) $\Omega$ on $\Sigma$ in a given cohomology class, when $\Sigma$ is a connected oriented closed manifold of arbitrary odd dimension. We define the volume of $\Omega$, which generalises both the contact volume and the Calabi invariant of Hamiltonian functions. The action of closed characteristics of $\Omega$ is defined in such a way that it generalises both the period of periodic Reeb orbits and the action of fixed points of Hamiltonian diffeomorphisms. We say that $\Omega$ is Zoll if its characteristics are the orbits of a free $S^1$-action on $\Sigma$. We prove that the volume and the action of a Zoll odd-symplectic form satisfy a certain polynomial equation. This builds the equality case of a conjectural systolic-diastolic inequality for odd-symplectic forms close to a Zoll one. We establish the conjecture in some cases, e.g. when the $S^1$-action comes from a flat $S^1$-bundle or $\Omega$ is quasi-autonomous. This new inequality recovers the inequality between the minimal (or the maximal) action and the Calabi invariant of Hamiltonian isotopies $C^1$-close to the identity on a closed symplectic manifold, as well as the local contact systolic-diastolic inequality above.

Finally, we discuss applications to the study of curves with prescribed geodesic curvature, also known as magnetic geodesics, on a connected oriented closed surface endowed with a Riemannian metric. We establish a magnetic systolic-diastolic inequality, when the prescribed curvature is either close to a Zoll one or high enough.
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1 Introduction

1.1 Contact systolic geometry

Let $\Sigma$ be a connected closed manifold of dimension $2n + 1$ and $\xi$ a co-oriented hyperplane distribution on it. This means that $\xi \subset T\Sigma$ is a sub-bundle of codimension 1 and that an orientation on the quotient bundle $T\Sigma/\xi \to \Sigma$ has been fixed. We denote by $-\xi$ the same hyperplane distribution with opposite co-orientation. We say that $\xi$ is a co-oriented contact structure, if it admits a defining contact form $\alpha$. Namely, $\alpha$ is a one-form on $\Sigma$ such that

- $\xi = \ker \alpha$,
- $\alpha$ induces the co-orientation of $\xi$,
- $d\alpha|_{\xi}$ is non-degenerate.

Thus, $-\xi$ is also a co-oriented contact structure with defining form $-\alpha$. The form $\alpha$ induces an orientation $\sigma_\alpha$ on $\Sigma$ through the volume form $\alpha \wedge (d\alpha)^n$ and yields the Reeb vector field $R_\alpha$ on $\Sigma$ determined by the relations $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$.

Let $\xi$ be an isotopy class of co-oriented contact structures on $\Sigma$, and let $\text{Cont}(\xi)$ be the set of all contact one-forms $\alpha$ defining some element of $\xi$. If $\alpha \in \text{Cont}(\xi)$ and $\tau : \Sigma \to (0, \infty)$ is a function, then $\tau\alpha \in \text{Cont}(\xi)$. Conversely, by the Gray Stability Theorem [Gei08, Theorem 2.2.2], if $\xi' = \ker \alpha'$ is another contact structure in $\xi$, then there exists a diffeomorphism $\Psi : \Sigma \to \Sigma$ isotopic to the identity and a function $\tau : \Sigma \to (0, \infty)$ such that $\Psi^*\alpha' = \tau\alpha$. In particular, $\sigma_\xi := \sigma_\alpha$ does not depend on $\alpha$.

One of the main tasks of contact geometry is the classification of the isotopy classes $\xi$ on $\Sigma$. To this purpose, the study of the flow $\Phi^\alpha$, obtained integrating $R_\alpha$ for some $\alpha \in \text{Cont}(\xi)$, gives precious insights, as the periodic orbits of such a flow can be used to define a series of invariants of $\xi$. Partly because of this, the Weinstein conjecture, which asserts that every contact form possesses at least one periodic orbit [Wei79], has played a prominent role in the field. The conjecture has been established in many particular situations, most notably when $\Sigma$ is three-dimensional [Tau07]. In these cases a more refined question arises: What can be said about the period of the orbits that one finds? A natural problem is, namely, to give an explicit upper bound on $T_{\min}(\alpha)$, the minimal period of periodic orbits of $\Phi^\alpha$, in terms of some geometric quantity associated with $\alpha$. Following [APB14], we use here the contact volume (other choices are possible and can lead to different results, as in [AFM17])

$$\text{Volume}(\alpha) := \int_\Sigma \alpha \wedge (d\alpha)^n > 0, \quad (1.1)$$

where we integrate with respect to $\sigma_\alpha$, and consider the systolic ratio

$$\rho_{\text{sys}} : \text{Cont}(\xi) \to (0, +\infty], \quad \rho_{\text{sys}}(\alpha) := \frac{T_{\min}(\alpha)^{n+1}}{\text{Volume}(\alpha)}.$$

Abbondandolo, Bramham, Hryniewicz, and Salomão showed that if $\xi$ is an isomorphism class on a three-dimensional manifold, then $\sup \rho_{\text{sys}} = +\infty$ [ABHS17]. This indicates that the systolic ratio does not have a global upper bound on $\text{Cont}(\xi)$.

Such bound might hold, however, if one restricts the class of contact forms. For instance, a celebrated theorem of Viterbo [Vit00, Theorem 5.1] asserts that the systolic ratio is bounded from above on $\text{Convex}(n)$ (see also [AAMO08]). Here $\text{Convex}(n)$ denotes the set of contact forms, defining the standard contact structure on $S^{2n+1}$, that are obtained by restricting the Liouville form on $\mathbb{R}^{2(n+1)}$ to the boundary of some convex body containing the origin.
Another important class is given by the canonical contact forms on the unit tangent bundle of closed Riemannian or Finsler manifolds. This is the setting where systolic geometry originated and has been hitherto tremendously studied (see [Ber03, Chapter 7.2]).

In a similar vein, for a general Σ, one is led to study the local behaviour of \( \rho_{sys} \) around its critical set. This direction of inquiry was initiated in [APB14] by Álvarez-Paiva and Balacheff, who showed that \( \text{Crit} \rho_{sys} = Z(\xi) \), where \( Z(\xi) \) denotes the set of Zoll forms in \( \text{Cont}(\xi) \).

**Definition 1.1.** A contact form \( \alpha \) on a manifold \( \Sigma \) is called Zoll of period \( T(\alpha) > 0 \), if the flow \( \Phi^\alpha \) induces a free \( \mathbb{R}/T(\alpha)\mathbb{Z} \)-action (all orbits are periodic and have prime period \( T(\alpha) \)). We write \( Z(\Sigma) \) for the set of all Zoll forms on \( \Sigma \) and \( Z(\xi) \) for those contained in \( \text{Cont}(\xi) \).

Once a Zoll form \( \alpha_s \) is given, it is easy to deform it through a path \( s \mapsto \alpha_s \) of Zoll forms with \( \alpha_0 = \alpha_s \). We can just set \( \alpha_s := T_s \Psi_s^* \alpha_s \), where \( s \mapsto \Psi_s \) is any isotopy of \( \Sigma \) and \( s \mapsto T_s \) a path of positive numbers. By a theorem of Weinstein [Wei74], these represent all possible deformations of \( \alpha_s \) through Zoll contact forms. While the local structure of \( Z(\Sigma) \) is well understood, describing when \( Z(\Sigma) \) is non-empty and investigating its global structure are more subtle issues. In this regard, a celebrated construction by Boothby and Wang represents a useful tool [BW58]. They observed that if \( \alpha_s \) is Zoll of period 1, the quotient by the Reeb action yields an oriented \( S^1 \)-bundle \( p : \Sigma \to M \), where \( M \) is a closed manifold of dimension \( 2n \) and \( S^1 = \mathbb{R}/\mathbb{Z} \). The Zoll form \( \alpha_s \) becomes a connection form for \( p \), while the two-form \( d\alpha_s \) descends to a symplectic form \( \omega_s \) on \( M \), representing minus the Euler class of \( p \). Vice versa, given a symplectic manifold \((M,\omega_\ast)\) such that the cohomology class of \( \omega_\ast \) is integral, one can construct an oriented \( S^1 \)-bundle \( p : \Sigma \to M \) with a connection form \( \alpha_\ast \) satisfying \( d\alpha_\ast = p^* \omega_\ast \), so that \( \alpha_\ast \) is a Zoll form of period 1 on \( \Sigma \). The Boothby-Wang construction yields a complete description of Zoll contact forms on three-manifolds.

**Proposition 1.2.** Let \( \Sigma \) be a connected closed three-manifold, let \( b_\Sigma \) be the rank of \( H_1(\Sigma;\mathbb{Z}) \), and let \( t_\Sigma \) be the order of the torsion subgroup of \( H_1(\Sigma;\mathbb{Z}) \). There exists a Zoll contact form \( \alpha \) on \( \Sigma \) if and only if \( \Sigma \) is the total space of a non-trivial oriented \( S^1 \)-bundle \( p \) over a connected oriented closed surface \( M \). In this case, \( b_\Sigma \) is even, \( M \) has genus \( \frac{1}{2}b_\Sigma \) and minus the Euler number of \( p \) equals \( t_\Sigma \), so that the diffeomorphism-type of \( M \) and the isomorphism-type of \( p \) depend only on \( \Sigma \). Moreover,

\[
\rho_{sys}(\alpha) = \frac{1}{t_\Sigma}.
\]

If \( \alpha' \) is another Zoll contact form on \( \Sigma \), there is a diffeomorphism \( \Psi : \Sigma \to \Sigma \) and a positive constant \( T > 0 \) such that

\[
\Psi^* \alpha' = T \alpha.
\]

Finally, we have two cases:

1. If \( \Sigma \) is the three-sphere or the real projective space, then \( Z(\Sigma) \) has exactly two connected components \( Z(\xi_\ast) \) and \( Z(\xi_{\text{st}}) \). Here \( \xi_\ast \) denotes the isotopy class of the standard contact structure and \( \xi_{\text{st}} \) the class obtained from \( \xi_\ast \) applying an orientation reversing diffeomorphism of \( \Sigma \).

2. If \( \Sigma \) is neither the three-sphere nor the real projective three-space, then \( Z(\Sigma) \) has exactly two connected components \( Z(\xi_+) \) and \( Z(\xi_-) \). Here, \( \xi_+ \) and \( \xi_- \) are two distinct isotopy classes with \( \xi_- = -\xi_+ \). In particular, \( o_{\xi_+} = o_{\xi_-} \).
Actually, the results in [APB14] go beyond the characterisation of \( \text{Crit} \rho_{\text{sys}} \) and imply that if \( s \mapsto \alpha_s \) is a deformation of \( \alpha_* \in Z(\xi) \) with \( a_0 = \alpha_* \), then \( s \mapsto \rho_{\text{sys}}(\alpha_s) \) attains a strict maximum at 0, provided the deformation is not tangent in \( s = 0 \) to all orders to the space \( Z(\xi) \). On the other hand, by Weinstein’s theorem, if the deformation is contained in \( Z(\xi) \), then \( \rho_{\text{sys}}(\alpha_s) = \rho_{\text{sys}}(\alpha_s) \) for all \( s \). Based on such facts, one would hope to establish the following local upper bound.

**Conjecture 1** (Local contact systolic inequality). Let \( \alpha_* \) be a Zoll contact form on a connected closed manifold \( \Sigma \) of dimension \( 2n + 1 \). There is, for some \( k \geq 1 \), a \( C^k \)-neighbourhood \( U \) of \( \alpha_* \) in the set of contact forms on \( \Sigma \) such that

\[
\rho_{\text{sys}}(\alpha) \leq \rho_{\text{sys}}(\alpha_*), \quad \forall \alpha \in U
\]

and the equality holds if and only if \( \alpha \) is a Zoll form.

A big step toward this conjecture is taken in [ABHS17a]. On the one hand, Theorem 2 in [ABHS17a] shows that the local systolic inequality fails in the \( C^0 \)-topology for \( \Sigma = S^3 \). On the other hand, Theorem 1 in [ABHS17a] establishes the conjecture for \( k = 3 \) and \( \Sigma = S^3 \) (or more generally, by means of a simple covering argument, when the base \( M \) is the two-sphere).

In the present paper, building on the latter result, we prove the conjecture for every closed three-manifold (the base \( M \) is an arbitrary orientable closed surface), including a statement regarding the diastolic ratio

\[
\rho_{\text{dia}}(\alpha) := \frac{T_{\max}(\alpha)^{n+1}}{\text{Volume}(\alpha)},
\]

where \( T_{\max}(\alpha) \) is the maximal period of prime periodic orbits of \( \Phi^\alpha \). If \( \alpha \) is Zoll, then there holds \( T_{\min}(\alpha) = T(\alpha) = T_{\max}(\alpha) \) so that \( \rho_{\text{sys}}(\alpha) = \rho_{\text{dia}}(\alpha) \).

**Theorem 1.3.** Let \( \alpha_* \) be a Zoll contact form on a connected closed three-manifold \( \Sigma \). There exists a \( C^2 \)-neighbourhood \( U \) of \( d\alpha_* \) in the space of exact two-forms on \( \Sigma \) such that, for every contact form \( \alpha \) on \( \Sigma \) with \( d\alpha \in U \), we have

\[
\rho_{\text{sys}}(\alpha) \leq \frac{1}{t_\Sigma} \leq \rho_{\text{dia}}(\alpha)
\]

and any of the two equalities holds if and only if \( \alpha \) is Zoll. In particular, Zoll contact forms are strict local maximisers of the systolic ratio in the \( C^3 \)-topology.

**Sketch of proof of Theorem 1.3.** We can assume without loss of generality that all prime orbits of \( \alpha_* \) have period equal to 1, and that \( \alpha \) is \( C^2 \)-close to \( \alpha_* \) and \( d\alpha \) is \( C^2 \)-close to \( d\alpha_* \). We divide the proof of the theorem into two parts (corresponding to Section 3 and 4), which we now briefly outline. In the first part, we show that there exist real numbers \( T, T' \) with \( 1 < T' < T < 2 \) such that the set \( \mathcal{P}_\tau(\alpha) \) of prime periodic orbits \( \gamma \) of \( \Phi^\alpha \) with period \( T(\gamma) \leq T \) is not empty, and if \( \gamma \in \mathcal{P}_\tau(\alpha) \), then actually \( T(\gamma) \leq T' \). Moreover, given \( \gamma \in \mathcal{P}_\tau(\alpha) \), we will find a global surface of section \( N \to \Sigma \) for \( \Phi^\alpha \), which is diffeomorphic to \( M \) with an open disc removed and such that its boundary covers \( t_\Sigma \)-times the orbit \( \gamma \). If \( \lambda \) is the restriction of \( \alpha \) to \( N \), then \( d\lambda \) is symplectic in the interior \( \hat{N} \) and vanishes of order one at the boundary \( \partial N \). The first-return time, a priori only defined on \( \hat{N} \), extends to a function \( \tau : N \to (0, \infty) \), which is \( C^1 \)-close to the constant 1. The first-return map, a priori only defined on \( \hat{N} \), extends to a diffeomorphism \( \varphi : N \to N \), which is \( C^1 \)-close to \( \text{id}_N \). Moreover, there holds \( \varphi^* \lambda - \lambda = d\sigma \),
where \( \sigma := \tau - T(\gamma) \) is a \( C^1 \)-small function, called the action of \( \varphi \). The volume of \( \alpha \) is related to the Calabi invariant \( \text{CAL}(\varphi) := \frac{1}{2} \int_{\alpha} \sigma d\lambda \) of the map \( \varphi \) through the formula

\[
\text{Volume}(\alpha) = 2 \text{CAL}(\varphi) + t_{\Sigma} T(\gamma)^2.
\]

Furthermore, every fixed point \( q \in \tilde{N} \) of \( \varphi \) yields a prime periodic orbit \( \gamma_q \) of \( \Phi_\alpha \) with period

\[
T(\gamma_q) = \sigma(q) + T(\gamma).
\]

In particular, when \( \alpha \) is not Zoll, \( \varphi \neq \text{id}_N \) and the theorem is proven if we take \( \gamma \) to have minimal, respectively, maximal period among orbits in \( \mathcal{P}_T(\alpha) \), and are able to show that

\[
\begin{align*}
\varphi \neq \text{id}_N, \quad \text{CAL}(\varphi) \leq 0 & \quad \iff \quad \exists q_- \in \tilde{N} \cap \text{Fix}(\varphi), \quad \sigma(q_-) < 0, \\
\varphi \neq \text{id}_N, \quad \text{CAL}(\varphi) \geq 0 & \quad \iff \quad \exists q_+ \in \tilde{N} \cap \text{Fix}(\varphi), \quad \sigma(q_+) > 0.
\end{align*}
\]

Indeed, if we take \( \gamma \) to have \( T(\gamma) = T_{\min}(\alpha) \), then \( \rho_{\text{sys}}(\alpha) \geq \frac{1}{\Sigma} \) is equivalent to \( \text{CAL}(\varphi) \leq 0 \). But the first implication in (1.2) yields a periodic orbit \( \gamma_{q_-} \) with \( T(\gamma_{q_-}) < T(\gamma) = T_{\min}(\alpha) \). This contradiction proves \( \rho_{\text{sys}}(\alpha) < \frac{1}{\Sigma} \) for \( \alpha \) which is not Zoll. In the diastatic ratio case, one has to use that \( T' + \|\sigma\|_{C^0} \leq T \), which follows from the \( C^1 \)-smallness of \( \sigma \), to ensure that the periodic orbit \( \gamma_{q_+} \) produced from the second implication in (1.2) still belongs to \( \mathcal{P}_T(\alpha) \).

In the second part of the proof we establish implications (1.2) by proving that there exists a path \( t \mapsto \varphi_t \) of \( d\lambda \)-Hamiltonian diffeomorphisms of \( N \) such that \( \varphi_0 = \text{id}_N, \varphi_1 = \varphi \), and which is generated by a quasi-autonomous Hamiltonian \( H : N \times [0,1] \to \mathbb{R} \) normalised by \( H|_{\partial N \times [0,1]} = 0 \). We recall that, according to [BP94], a function \( H \) is quasi-autonomous if there exist \( x_{\min}, x_{\max} \in N \) such that

\[
\min_{q \in N} H(q,t) = H(q_{\min},t), \quad \max_{q \in N} H(q,t) = H(q_{\max},t), \quad \forall t \in [0,1].
\]

In particular, \( q_{\min} \) and \( q_{\max} \) are fixed points of \( \varphi \), if they lie in \( \tilde{N} \). In order to exhibit such a path, we construct a Weinstein neighbourhood of the diagonal in \((N \times (\mathbb{R} - d\lambda) \oplus d\lambda)) \). This yields a generating function \( G : N \to \mathbb{R} \) for \( \varphi \). Let \([0,\varepsilon) \times S^1 \subset N \) be a collar neighbourhood of the boundary with radial coordinate \( R \). At this point, crucially using that \( d\lambda \) vanishes at \( \partial N \) of order one in the radial direction, we can show that the generating function belongs to

\[
\mathcal{G} := \left\{ G : N \to \mathbb{R} \mid G = 0 \text{ on } \partial N, \ G \text{ is } C^2\text{-small on } N, \ \frac{1}{R} dG \text{ is } C^1\text{-small on } [0,\varepsilon) \times S^1 \right\}.
\]

Conversely, every \( G \in \mathcal{G} \) is the generating function of some diffeomorphism \( \varphi_G : N \to N \), which is \( C^1 \)-close to the identity. Therefore, since the set \( \mathcal{G} \) is star-shaped around the zero function, the Hamilton-Jacobi equation tells us that, for every \( \varphi = \varphi_G \), the Hamiltonian \( H : N \times [0,1] \to \mathbb{R} \) associated with the path \( t \mapsto \varphi_t G, t \in [0,1] \), is quasi-autonomous. Once the existence of a quasi-autonomous Hamiltonian is settled, implications (1.2) follow, as already observed in [ABHS17a] Remark 2.8. Indeed, we can rewrite the Calabi invariant of \( \varphi \) and the action of \( q_{\min} \) (and similarly of \( q_{\max} \)), provided it lies in \( \tilde{N} \), as

\[
\text{CAL}(\varphi) = \int_{N \times [0,1]} H \, d\lambda \wedge dt, \quad \sigma(q_{\min}) = \int_0^1 H(q_{\min},t) dt.
\]

This finishes the second part of the proof and the whole sketch.
Relations (1.3) suggest that one could interpret implications (1.2) as a local systolic-diastolic inequality for quasi-autonomous Hamiltonian systems. It is not difficult to see such an inequality for quasi-autonomous Hamiltonian systems on closed symplectic manifolds of arbitrary dimension. In order to make a precise statement, we introduce some notation. Let \((M, \omega)\) be a connected closed symplectic manifold of dimension \(2n\), and let \(\varphi : M \to M\) be a Hamiltonian diffeomorphism generated by a Hamiltonian \(H : M \times [0, 1] \to \mathbb{R}\). We orient \(M\) so that \(\omega^n\) is positive. Then, one can define the Hamiltonian action \(A_H : H_{\text{short}}(M) \to \mathbb{R}\) on the space of short one-periodic curves \(q : S^1 \to M\) (see Section 10 for a precise definition) by
\[
A_H(\gamma) := \int_{D^2} \hat{q}^* \omega + \int_0^1 H(q(t), t) dt,
\]
where \(\hat{q} : D^2 \to M\) is a small capping disc for \(q \in H_{\text{short}}(M)\). The minimal and maximal Hamiltonian actions of fixed points of \(\varphi\), whose associated curve is short, are given by
\[
\min A_H := \inf_{\gamma \in \text{Crit} A_H} A_H(\gamma), \quad \max A_H := \sup_{\gamma \in \text{Crit} A_H} A_H(\gamma).
\]
The Calabi invariant of \(H\) with respect to \(\omega\) is defined by
\[
\text{CAL}_\omega(H) := \int_{M \times [0, 1]} H \omega^n \wedge dt.
\]
Thus, we can give the following statement, which is one particular case of Proposition \[10.16\].

**Proposition 1.4.** Assume that a Hamiltonian diffeomorphism \(\varphi : (M, \omega) \to (M, \omega)\) is generated by a quasi-autonomous Hamiltonian \(H : M \times [0, 1] \to \mathbb{R}\) (for instance, \(\varphi\) is the time-one map of a Hamiltonian isotopy \(C^1\)-close to the identity). Then, there holds
\[
\min A_H \leq \frac{\text{CAL}(H)}{\int_M \omega^n} \leq \max A_H,
\]
and any of the two equalities holds if and only if \(H(q, t) = h(t)\) for all \((q, t) \in M \times [0, 1]\) and some function \(h : [0, 1] \to \mathbb{R}\). \(\square\)

**Remark 1.5.** The statement in Proposition \[1.4\] is false for general Hamiltonian diffeomorphisms. Indeed, adapting Proposition 2.28 in \[ABHS17a\], one can construct a time-one map \(\varphi\) of a Hamiltonian isotopy \(C^0\)-close to the identity not satisfying the inequality above. A proper way to go beyond the quasi-autonomous case might be to take into account not only the fixed points but all the periodic points of \(\varphi\) as explored by Hutchings in \[Hut16\] Theorem 1.2] in a special case.

### 1.2 Odd-symplectic systolic geometry

If \(M\) is a connected closed symplectic manifold, a Hamiltonian diffeomorphism \(\varphi : M \to M\) can be seen as the return map of a flow \(\Phi^X\) on the manifold \(M \times S^1\). If \(H : M \times S^1 \to \mathbb{R}\) is a Hamiltonian one-periodic in time generating \(\varphi\) as time one-map, then \(X(q, t) := \partial_t + X_{H_t}(q)\), for all \((q, t) \in M \times S^1\). Here, \(X_{H_t}\) is the Hamiltonian vector field tangent to \(M \times \{t\}\) associated with the function \(H_t := H(\cdot, t)\). Thus, we view Proposition \[1.4\] as giving a systolic-diastolic inequality for those flows on the trivial \(S^1\)-bundle \(p : M \times S^1 \to M\), which are obtained lifting
a Hamiltonian isotopy of $M$. This should be compared with Theorem 1.3 which yields a systolic-diastolic inequality for Reeb flows on the non-trivial $S^1$-bundles obtained from the Boothby-Wang construction. It turns out that these two types of flows are particular cases of the flows induced by odd-symplectic structures (also known as Hamiltonian structures, see [CM05]) on oriented odd-dimensional closed manifolds. Moreover, as we explain next, one can formulate a conjectural systolic-diastolic inequality in this setting recovering the contact and the Hamiltonian one.

We start by fixing some notation. Let $(\Sigma, o_\Sigma)$ be a connected oriented closed manifold of dimension $2n + 1$, and let $C \in H^2_{dR}(\Sigma)$ be a class in its de Rham cohomology. We write $\Omega^1(\Sigma)$ for the set of one-forms on $\Sigma$ and $\Xi^2_C(\Sigma)$ for the set of closed two-forms on $\Sigma$ representing the class $C$. We pick an auxiliary element $\Omega_0 \in \Xi^2_C(\Sigma)$ and get a surjective map $\Omega^1(\Sigma) \to \Xi^2_C(\Sigma)$, $\alpha \mapsto \Omega_\alpha := \Omega_0 + d\alpha,$ (1.6)
whose kernel is the space of closed one-forms on $\Sigma$. Using this map, we can associate to $\alpha \in \Omega^1(\Sigma)$ a real number $\text{Vol}(\alpha)$, which generalises both the contact volume (1.1) and the Calabi invariant (1.5). Namely, we define a functional $\text{Vol} : \Omega^1(\Sigma) \to \mathbb{R}$ by
\[\text{Vol}(0) = 0, \quad d\alpha \cdot \beta = \int_\Sigma \beta \wedge \Omega^n_\alpha, \quad \forall \beta \in \Omega^1(\Sigma).\]
For instance, when $n = 1$, we have
\[\text{Vol}(\alpha) = \frac{1}{2} \int_\Sigma \alpha \wedge d\alpha + \int_\Sigma \alpha \wedge \Omega_0.\]
We also remark that, when $C = 0$ and $\Omega_0 = 0$, the function $\text{Vol}$ recovers the Chern-Simons action for principal $S^1$-bundles over $\Sigma$ (see [CS74] and Remark 5.1).

If $C^n = 0$, then we can use the map (1.6) to push forward $\text{Vol}$ to a volume functional $\mathcal{Vol} : \Xi^2_C(\Sigma) \to \mathbb{R}$.

If $C^n \neq 0$, this procedure does not work, since there exists $\tau \in H^1_{dR}(\Sigma)$ such that $\tau \cup C^n \neq 0$. In this case, we say that $\alpha \in \Omega^1(\Sigma)$ is normalised, if $\text{Vol}(\alpha) = 0$. For every $\Omega \in \Xi^2_C(\Sigma)$, there exists $\alpha \in \Omega^1(\Sigma)$ normalised such that $\Omega = \Omega_\alpha$. Therefore, in this situation we can just work with normalised forms and declare $\mathcal{Vol} : \Xi^2_C(\Sigma) \to \mathbb{R}$ to be identically zero. In both cases, the volume functional is invariant under diffeomorphisms $\Psi : \Sigma \to \Sigma$ isotopic to the identity (see Proposition 5.8):
\[\mathcal{Vol}(\Psi^*\Omega) = \mathcal{Vol}(\Omega)\]

Having identified what the volume should be, we want to introduce the set $\mathcal{X}(\Omega)$ of closed characteristics of $\Omega$ generalising periodic Reeb and Hamiltonian orbits. To this purpose, we consider the possibly singularity distribution $\ker \Omega \to \Sigma$ called the characteristic distribution and define
\[\mathcal{X}(\Omega) := \left\{ \gamma : S^1 \to \Sigma \mid \dot{\gamma} \in \ker \Omega \right\} / \sim,\]
where $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1$ and $\gamma_2$ coincide up to an orientation-preserving reparametrisation of $S^1$. The distribution $\ker \Omega$ is co-oriented by $\Omega^n$ and we orient it using the given orientation $o_\Sigma$ on $\Sigma$.

We single out the forms in $\Xi^2_C(\Sigma)$, whose characteristic distribution is one-dimensional. They are of special importance, as their closed characteristics are the periodic orbits of a flow.
Definition 1.6. A two-form $\Omega$ on $\Sigma$ is said to be odd-symplectic, if it is closed and maximally non-degenerate, namely if its characteristic distribution is an (oriented) real line bundle over $\Sigma$. We write $\mathcal{S}_C(\Sigma)$ for the subset of all odd-symplectic forms in $\Xi^2(\Sigma)$.

In $\mathcal{S}_C(\Sigma)$ we hope to find elements whose characteristic distribution is as simple as possible. To this purpose we introduce the set of oriented $S^1$-bundles with total space $\Sigma$:

$$\Psi(\Sigma) := \{ p : \Sigma \to M \mid p \text{ is an oriented } S^1\text{-bundle} \}.$$ 

Definition 1.7. An odd-symplectic form $\Omega$ is said to be Zoll if the oriented leaves of its characteristic distribution are the fibres of some $p_\Omega : \Sigma \to M_\Omega$ in $\Psi(\Sigma)$. We write $Z_C(\Sigma)$ for the set of Zoll forms in $\mathcal{S}_C(\Sigma)$.

If an odd-symplectic form $\Omega$ has a primitive contact form $\alpha$, then $\Omega$ is Zoll if and only if $\alpha$ is Zoll. The classification of Zoll contact forms for (oriented) three-manifolds proved in Proposition 1.2 extends to Zoll odd-symplectic forms.

Proposition 1.8. Let $\Sigma$ be a connected oriented closed three-manifold. There is a Zoll odd-symplectic form on $\Sigma$ if and only if $\Sigma$ is the total space of an oriented $S^1$-bundle over a connected oriented closed surface $M$. When the bundle is non-trivial, then every Zoll odd-symplectic form is the differential of a Zoll contact form. When the bundle is trivial, then the group $H_1(\Sigma;\mathbb{Z})$ is free with rank equal to $2\text{genus}(M) + 1$, and if $\Omega$ and $\Omega'$ are Zoll odd-symplectic forms on $\Sigma$, there exists a real number $T > 0$ and a diffeomorphism $\Psi : \Sigma \to \Sigma$ such that $\Psi^*\Omega' = T\Omega$. Moreover,

- if $\Sigma = S^2 \times S^1$, the set $Z_C(\Sigma)$ has exactly two connected components, $\forall C \in H^2_{dR}(\Sigma) \setminus \{0\}$;
- if $\Sigma = \mathbb{T}^2 \times S^1$, the set $Z_C(\Sigma)$ is non-empty and connected, $\forall C \in H^2_{dR}(\Sigma) \setminus \{0\}$;
- if $\Sigma = M \times S^1$ with $\chi(M) < 0$, there is a one-dimensional subspace $L \subset H^2_{dR}(\Sigma)$ such that $Z_C(\Sigma)$ is non-empty if and only if $C \in L \setminus \{0\}$; in this case the set $Z_C(\Sigma)$ is connected.

Remark 1.9. The classification of Zoll odd-symplectic forms up to diffeomorphism on a three-manifold is equivalent to the classification of bundles in the set $\{ p_\Omega \mid \Omega \in \mathcal{Z}(\Sigma) \}$ up to isomorphism. Analogously, the connected components of $Z_C(\Sigma)$ on a three-manifold are in bijection with the connected components of $\{ p_\Omega \mid \Omega \in Z_C(\Sigma) \}$. This is due to the fact that the map $Z_C(\Sigma) \to \Psi(\Sigma)$, given by $\Omega \mapsto p_\Omega$, has contractible fibres.

Let us assume that $Z_C(\Sigma)$ is not empty and take $\Omega_* \in Z_C(\Sigma)$ with associated $S^1$-bundle $p_1 := p_{\Omega_*} : \Sigma \to M_1$.

This implies that there exists a positive symplectic form $\omega_*$ on $M_1$ such that $\Omega_* = p_1^*\omega_*$. We set $c_1 := [\omega_*] \in H^2_{dR}(M_1)$. We write $h \in [S^1, \Sigma]$ for the free-homotopy class of the oriented $p_1$-fibres and $e_1 \in H^2_{dR}(M_1)$ for minus the real Euler class of $p_1$. Let $\Psi^0(\Sigma)$ be the connected component of $p_1$ inside $\Psi(\Sigma)$. As we did for the volume, the action will be computed with respect to some reference object, which we now define.

Definition 1.10. A weakly Zoll pair is a couple $(p, c)$, where $p : \Sigma \to M$ is an element in $\Psi(\Sigma)$ and $c \in H^2_{dR}(M)$ is a cohomology class. We write $\mathcal{Z}(\Sigma)$ for the set of weakly Zoll pairs and $\mathcal{Z}_C(\Sigma)$ for the subset of those pairs $(p, c)$ such that $C = p^*c \in H^2_{dR}(\Sigma)$. We denote by $\mathcal{Z}_C^0(\Sigma)$ the set of those $(p, c) \in \mathcal{Z}_C(\Sigma)$ such that $p \in \Psi^0(\Sigma)$. 

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Remark 1.11. If $\Omega \in S_C(\Sigma)$ is Zoll, then $(p_0, [\omega_0]) \in Z_C(\Sigma)$ is a weakly Zoll pair, where $\omega_\Omega$ is the closed two-form on $M_\Omega$ such that $\Omega = p_0^*\omega_\Omega$. Conversely, if $(p, c) \in Z_C(\Sigma)$ is a weakly Zoll pair, we can consider any closed two-form $\omega$ on $M$ such that $c = [\omega]$ and build the two-form $p^*\omega \in Z^2_C(\Sigma)$, which is Zoll exactly when $\omega$ is symplectic.

Let us now fix a reference weakly Zoll pair

$$(p_0, c_0) \in Z^0_C(\Sigma), \quad p_0 : \Sigma \to M_0.$$  

We write $e_0 \in H^3_{dR}(M_0)$ for minus the real Euler class of $p_0$ and we have the equivalence

$$C^m \neq 0 \iff e_0 = 0 \iff e_1 = 0.$$

Let $\Lambda_b(\Sigma)$ be the space of one-periodic curves in the class $\mathfrak{h}$, and let $\tilde{\Lambda}_b(\Sigma)$ be the space of homotopies of paths $\{\gamma_r \mid r \in [0, 1]\}$ inside $\Lambda_b(\Sigma)$ such that $\gamma_0$ is some oriented $p_0$-fibre. The admissible homotopies are allowed to move $\gamma_0$ inside the set of oriented $p_0$-fibres but have to fix the periodic curve $\gamma_1$, so that the natural projection $\tilde{\Lambda}_b(\Sigma) \to \Lambda_b(\Sigma)$, $[\gamma_r] \mapsto \gamma_1$ is a covering map.

We pick a closed two-form $\omega_0 \in Z^2_{C}(M_0)$ and choose $\Omega_0 := p_0^*\omega_0 \in Z^2_{C}(\Sigma)$ as our auxiliary element for the computation of the volume. One sees that $\mathcal{Vol} : Z^2_{C}(\Sigma) \to \mathbb{R}$ depends only on $(p_0, c_0)$ and not on $\omega_0$. We associate to $\Omega = \Omega_0 + \alpha$ the action functional

$$\tilde{A}_\Omega : \tilde{\Lambda}_b(\Sigma) \to \mathbb{R}, \quad \tilde{A}_\Omega([\gamma_r]) := \int_{[0,1] \times S^1} \Gamma^*\Omega + \int_{S^1} \gamma_0^*\alpha,$$

where $\alpha$ is chosen to be normalised if $C^m \neq 0$, and $\Gamma : [0, 1] \times S^1$ is the cylinder traced by the path $\{\gamma_r\}$. Like the volume, the functional $\tilde{A}_\Omega$ depends only on $(p_0, c_0)$ and not on $\omega_0$. If $[\gamma_r]$ is a critical point of $\tilde{A}_\Omega$, then $\gamma_1 \in \mathcal{A}(\Omega)$, provided $\gamma_1$ is embedded. Furthermore, the action is invariant under isotopies $\{\Psi_r : \Sigma \to \Sigma\}$ starting at the identity (Proposition 9.10):

$$\tilde{A}_{\Psi_r^{-1} \circ \gamma_r}(\Psi_r^{-1}([\gamma_r])) = \tilde{A}_\Omega([\gamma_r]), \quad \forall [\gamma_r] \in \tilde{\Lambda}_b(\Sigma).$$

In general, $\tilde{A}_\Omega$ does not descend to a functional on $\Lambda_b(\Sigma)$. More precisely, by Lemma 9.3 this happens if and only if $c_0|_{\pi_2(M_0)} = ac_0|_{\pi_2(M_0)}$ for some $a \in \mathbb{R}$. However, as we see now, we can define an action on the set of Zoll forms $\Omega$ such that $(p_0, [\omega_0]) \in Z^0_C(\Sigma)$. In this case, the volume of $\Omega$ can be expressed as a polynomial function of the action.

Definition 1.12. The Zoll polynomial $P : \mathbb{R} \to \mathbb{R}$ associated with $(p_0, c_0)$ is given by

$$P(0) = 0, \quad \frac{dP}{dA}(A) = (\langle A c_0 + c_0 \rangle)^n, [M_0].$$

For instance, when $n = 1$, the polynomial reads

$$P(A) = \frac{1}{2} \langle c_0, [M_0] \rangle A^2 + \langle c_0, [M_0] \rangle A.$$

Theorem 1.13. There is a well-defined volume function

$$\mathcal{Vol} : Z_C(\Sigma) \to \mathbb{R}, \quad \mathcal{Vol}(p, c) := \mathcal{Vol}(p^*\omega)$$
and well-defined action functional
\[ A : \mathcal{Z}_C^0(\Sigma) \to \mathbb{R}, \quad A(p, c) := \tilde{A}_p^\omega([\delta_r]). \]

Here, \( \omega \) is any closed two-form on \( M \) in the class \( c \), \( \{ \delta_r \} \) is any path of periodic curves such that \( \delta_r \) is an oriented \( p_r \)-fibre, where \( \{ p_r \} \) is any path of oriented \( S^1 \)-bundles from \( p_0 \) to \( p_1 = p \). Moreover, there holds
\[ P(A(p, c)) = \mathcal{V}oI(p, c), \quad \forall (p, c) \in \mathcal{Z}_C^0(\Sigma). \]

If \( A_* := A(p_1, c_1) \), then \( \frac{dP}{dA}(A_*) > 0 \). In particular, the polynomial \( P \) is non-zero.

From this result, we can generalise the equality cases in Theorem 1.3 and Proposition 1.4.

**Corollary 1.14.** Let \( \Omega \in \mathcal{Z}_C(\Sigma) \) be a Zoll odd-symplectic form such that \( p_\Omega \in \mathcal{V}P^0(\Sigma) \). If we set \( A(\Omega) := A(p_\Omega, [\omega_\Omega]) \), then
\[ P(A(\Omega)) = \mathcal{V}oI(\Omega). \quad \square \]

In what follows, we describe a conjectural systolic-diastolic inequality for odd-symplectic forms close to \( \Omega_* \) and with class \( C \in H^2_{\text{dr}}(\Sigma) \). To this end, we fix a finite open covering \( \{ B_i \} \) of \( M_1 \) by balls so that all their pairwise intersections are also contractible. Let \( \Lambda(p_1) \) be the space of periodic curves \( \gamma \in \Lambda_{\delta}(\Sigma) \) with the property that \( p_1(\gamma) \) is contained in some \( B_i \) and there is a path \( \{ \gamma^\text{short}_r \} \rightarrow \Lambda_{\delta}(\Sigma) \) entirely contained in \( p_1^{-1}(B_i) \) with \( \gamma^\text{short}_1 = \gamma \). If \( \Omega \in \Xi_{\delta}^2(\Sigma) \), we set
\[ A_\Omega : \Lambda(p_1) \rightarrow \mathbb{R}, \quad A_\Omega(\gamma) := \tilde{A}_\Omega([\delta_r \# \gamma^\text{short}_r]), \quad \gamma^\text{short}_1 = \gamma. \]

Here, the symbol \( \# \) denotes the concatenation of paths, \( \{ \delta_r \} \) is any path of periodic curves such that \( \delta_1 = \gamma^\text{short}_0 \), and for every \( r \in [0, 1] \), \( \delta_r \) is an oriented \( p_r \)-fibre, where \( \{ p_r \} \) is a path of oriented \( S^1 \)-bundles connecting \( p_0 \) with \( p_1 = p_\Omega \). We define
\[ A_{\text{min}}(\Omega) := \inf_{\gamma \in \mathcal{X}(\Omega) \cap \Lambda(p_1)} A_\Omega(\gamma), \quad A_{\text{max}}(\Omega) := \sup_{\gamma \in \mathcal{X}(\Omega) \cap \Lambda(p_1)} A_\Omega(\gamma). \]

By [Gin87 Section III] or [APB14 Section 3.2], if \( \Omega \in \mathcal{S}_C(\Sigma) \) is \( C^1 \)-close to \( \Omega_* \), the set \( \mathcal{X}(\Omega) \cap \Lambda(p_1) \) is compact and non-empty. Furthermore, the numbers \( A_{\text{min}}(\Omega) \) and \( A_{\text{max}}(\Omega) \) are finite and vary \( C^1 \)-continuously with \( \Omega \).

**Conjecture 2** (Local systolic-diastolic inequality for odd-symplectic forms). Let \( \Omega_* \) be a Zoll odd-symplectic form with cohomology class \( C \in H^2_{\text{dr}}(\Sigma) \). There is a \( C^k-1 \)-neighbourhood \( U \) of \( \Omega_* \) in \( \mathcal{S}_C(\Sigma) \) with \( k \geq 2 \) such that
\[ P(A_{\text{min}}(\Omega)) \leq \mathcal{V}oI(\Omega) \leq P(A_{\text{max}}(\Omega)), \quad \forall \Omega \in U. \]

The equality holds in any of the two inequalities above, if and only if \( \Omega \) is Zoll.

**Remark 1.15.** If the real Euler class of the bundle associated with \( \Omega_* \) vanishes, then the inequality in the conjecture is equivalent to
\[ A_{\text{min}}(\Omega) \leq 0 \leq A_{\text{max}}(\Omega), \quad \forall \Omega \in U, \]
with equalities if and only if \( \Omega \) is Zoll.
Remark 1.16. The volume, the action and the Zoll polynomial depend on the choice of reference pair \((p_0, c_0) \in \mathcal{S}_C^1(\Sigma)\). However, thanks to Theorem 10.7 we will observe in Remark 10.7 that the functional

\[ (P \circ A_\Omega - \mathfrak{H} \Omega(\Omega)) : A(p_1) \to \mathbb{R} \]

is independent of such a choice. Hence, Conjecture 2 generalises Proposition 1.4, since if there exist \(\omega_0 \in \mathfrak{H} \Omega(\Omega)\), we take \(\omega = \omega_0\) for a contact form \(\alpha\), so that \(C = 0\). Up to multiplication by a positive constant, we can assume that \(T(\alpha) = 1\). We take \(\sigma_\Sigma = \sigma_\alpha\) and \(c_0 = 0\). There holds

\[ P(A) = \frac{1}{n + 1} (e_0^n, [M_0]) A^{n+1}, \]

where \((e_0^n, [M_0]) = \langle e_0^n, [M_0] \rangle > 0\). When \(n = 1\), we have \((e_0^n, [M_0]) = t_\Sigma\), as stated in Proposition 1.4. If \(\Omega = d\alpha\) is \(C^0\)-close to \(\Omega_*\), we can take \(\alpha\) to be a contact form.

Thus, there holds \(T_{\min}(\alpha) \leq A_{\min}(d\alpha) \leq A_{\max}(d\alpha) \leq T_{\max}(\alpha)\), and Conjecture 2 implies

\[ \rho_{\text{sys}}(\alpha) \leq \frac{1}{(e_0^n, [M_0])} \leq \rho_{\text{dia}}(\alpha) \]

with equality cases exactly when the contact form \(\alpha\) is Zoll.

In the Hamiltonian case, \(p_1\) is trivial so that \(\Sigma = M_1 \times S^1\) and we call \(t \in S^1\) the global angular coordinate. We take \(\Omega = \omega_* + d(H dt)\) for some function \(H : M_1 \times S^1 \to \mathbb{R}\). The characteristic distribution of \(\Omega\) is generated by the vector field \(\partial_t + X_H\), where \(X_H\) is the \(\omega_*\)-Hamiltonian vector field of \(H_t := H(\cdot, t)\) tangent to \(M_1 \times \{t\}\). Then, \(A_\Omega\) is the Hamiltonian action defined in 1.4, \(\langle e_0^n, [M_0] \rangle = \langle \omega^n_* - [M_0] \rangle > 0\), and

\[ P(A) = (e_0^n, [M_0]) A, \quad \text{Vol}(H dt) = \int_{M_1 \times S^1} (H dt) \wedge \omega^n_* = \text{CAL}_{\omega_*}(H). \]

Conjecture 2 generalises Proposition 1.4 since \(A_{\min}(\Omega) = \min A_H\) and \(A_{\max}(\Omega) = \max A_H\), if we take the Hamiltonian \(H\) to be normalised, namely \(\text{CAL}_{\omega_*}(H) = 0\).

Inspired by the Hamiltonian case, we will look first at a special class of \(\Omega \in \Xi_C^2(\Sigma)\), when studying Conjecture 2. Here, we assume without loss of generality that \(p_0 = p_1\). We fix an \(S^1\)-connection form \(\eta\) for the bundle \(p_0\). This means that \(\eta\) restricts to the angular form on each fibre and \(d\eta = p_0^* \kappa\) for some \(\kappa \in \Xi_0^2(M_0)\). We consequently choose the form \(\omega_0\) so that \(\omega_* = \omega_0 + A_* \kappa\), where \(A_* = A(\omega_*)\). We say that a form \(\Omega \in \Xi_C^2(\Sigma)\) is an \(H\)-form, if there exists a function \(H : \Sigma \to \mathbb{R}\) such that

\[ \Omega = \omega_0 + d(H \eta). \]

We call \(H\) a defining Hamiltonian for \(\Omega\). If \(H\) is \(C^0\)-close to \(A_*\), then \(\Omega\) is odd-symplectic, and if \(H\) is \(C^k\)-close to \(A_*\), then \(\Omega\) is \(C^{k-1}\)-close to \(\omega_*\). An \(H\)-form is called \textit{quasi-autonomous} if there exist \(q_{\min}, q_{\max} \in M_0\) and defining Hamiltonians \(H_{\min}\) and \(H_{\max}\) such that

\[ \min_{\Sigma} H_{\min} = H_{\min}(z), \quad \forall z \in p_0^{-1}(q_{\min}), \quad \max_{\Sigma} H_{\max} = H_{\max}(z), \quad \forall z \in p_0^{-1}(q_{\max}). \]
We say that $H$ is quasi-autonomous, if the corresponding $\Omega$ is quasi-autonomous. We establish Conjecture 2 for quasi-autonomous forms close to a Zoll odd-symplectic form.

**Proposition 1.18.** There exists a $C^2$-neighbourhood $\mathcal{H}$ of the constant $A_*$ in the space of quasi-autonomous functions over $\Sigma$ such that if $H \in \mathcal{H}$ and $\Omega = \Omega_* + d(H\eta)$ then

$$P(A_{\min}(\Omega)) \leq \text{vol}(\Omega) \leq P(A_{\max}(\Omega)),$$

and any of the two equalities holds if and only if $\Omega$ is Zoll. In the Zoll case, $H$ is invariant under the holonomy of $\eta$. In particular, it is constant if $e_0 \neq 0$.

Let $\eta_0 : M_0 \times S^1 \rightarrow M_0$ be the trivial bundle and take $\eta = dt$, where $t \in S^1$ is the angular coordinate. In this case, we show in Proposition 4.14 that if $\Omega \in \Xi^2_{C}(\Sigma)$ is $C^2$-close to $\Omega_*$, then $\Omega$ is an H-form with a $C^2$-small defining Hamiltonian $H$, after applying a diffeomorphism of $\Sigma$ isotopic to the identity. Therefore, in this setting the conjecture follows from Proposition 1.4 (or Proposition 1.18) and the invariance of the volume and the action under diffeomorphisms. From this result and the topological Lemma 7.6 Conjecture 2 for bundles with $e_0 = 0$ can readily be proven.

**Theorem 1.19.** The local systolic-diastolic inequality for odd-symplectic forms holds true in the $C^2$-topology, whose associated bundle has vanishing real Euler class.

If $\Omega_*$ is any Zoll odd-symplectic form on a closed three-manifold $\Sigma$, then either the Euler class of its associated bundle vanishes or $\Omega_* = d\alpha_*$ for some Zoll contact form $\alpha_*$. Hence, Theorem 1.19 and Theorem 1.3 establish Conjecture 2 in dimension three.

**Corollary 1.20.** The local systolic-diastolic inequality for odd-symplectic forms holds true in the $C^2$-topology on closed three-manifolds.

**Remark 1.21.** More generally, one could try to formulate a systolic-diastolic inequality in a neighbourhood of an odd-symplectic form $\Omega_*$, whose closed characteristics are tangent to a (not necessarily free) $S^1$-action on $\Sigma$, cf. [Tho76] and [BR94].

### 1.3 Applications to curves with prescribed geodesic curvature

The systolic-diastolic inequality established in Corollary 1.20 has applications to the study of immersed curves with prescribed geodesic curvature on a connected oriented closed surface $(M, \sigma_M)$ endowed with a Riemannian metric $g$. Let $f : M \rightarrow \mathbb{R}$ be a function and consider the curves $c : \mathbb{R} \rightarrow M$ with

$$|\dot{c}(t)| = 1, \quad \kappa_{c}(t) = -f(c(t)), \quad \forall t \in \mathbb{R},$$

where $\kappa_{c}$ is the geodesic curvature of $c$. We call the curves $c$ satisfying the condition above $f$-magnetic geodesics. The magnetic geodesics of $f$ and of $-f$ are in one-to-one correspondence through time reversal. This means that $t \mapsto c(t)$ is an $f$-magnetic geodesic if and only if $t \mapsto c(-t)$ is a $-f$-magnetic geodesic. It is a classical fact that the tangent lifts $(c, \dot{c})$ of $f$-magnetic geodesics are the integral curves of a vector field $X_f$ defined on the unit tangent bundle $T^1M$. The foot-point projection $p_\infty : T^1M \rightarrow M$ is an $S^1$-bundle, whose fibres we orient by the $\sigma_M$-negative direction. We write $h_\infty \in [S^1, T^1M]$ for the free-homotopy class of such fibres. We orient $T^1M$ by combining the orientation $\sigma_M$ on $M$ with the orientation of the
\( p_\infty \)-fibres given above. Actually, the vector field \( X_f \) is a non-zero section of the characteristic distribution of the odd-symplectic form

\[
\Omega_f := d\alpha_{\text{can}} + p_\infty^*(f\mu),
\]

where \( \mu \in \Omega^2(M) \) is the area form induced by \((g, o_M)\) and \( \alpha_{\text{can}} \) is the canonical one-form on \( T^1M \) given by \((\alpha_{\text{can}})_v \cdot Y = g(v, d_v p_\infty \cdot Y) \) for every \( Y \in T_v T^1M \).

**Definition 1.22.** We say that a function \( f : M \to \mathbb{R} \) is **Zoll** with respect to a given metric \( g \), if the corresponding odd-symplectic two-form \( \Omega_f \) is Zoll. In this case, we write \( p_f : T^1M \to M \) for the associated oriented \( S^1 \)-bundle and we have \( \Omega_f = p_f^*\omega_f \) for some symplectic form \( \omega_f \) on \( M \). We write \( e_f \) for minus the Euler class of \( p_f \), and \( h_f \) for the free-homotopy class of the \( p_f \)-fibres.

If we take \( f \equiv 0 \), we recover the notion of Zoll Riemannian metric and \( M \) must be the two-sphere. We refer the reader to [Bes78] for a thorough discussion of such metrics. A classical example of a Zoll function \( f^* : M \to \mathbb{R} \) can be given when \( g = g^* \) is a metric of constant Gaussian curvature \( K^* \). We just take \( f^* \) to be a constant function satisfying

\[
f^2 + K^* > 0.
\]

Then, the lifts of \( f^*-\text{magnetic geodesics} \) to the universal cover of \( M \) parametrise the boundary of geodesic balls of radius

\[
R = \begin{cases} 
\arctan \left( \frac{\sqrt{K^*}}{|f^*|} \right), & \text{if } K^* > 0; \\
\frac{1}{|f^*|} & \text{if } K^* = 0; \\
\arctanh \left( \frac{\sqrt{-K^*}}{|f^*|} \right) & \text{if } K^* < 0.
\end{cases}
\]

in the clockwise direction, if \( f^* \geq 0 \), or in the counter-clockwise direction, if \( f^* \leq 0 \). However, it is an open question to answer whether every Riemannian metric admits a Zoll function.

Let us go back and consider general functions \( f \). We associate two quantities to each of them. The former is the **average** of \( f \):

\[
f_{\text{avg}} := \frac{1}{\text{area}(M)} \int_M f\mu, \quad \text{area}(M) := \int_M \mu.
\]

The latter is the **average curvature** of \( f \), which generalises the left-hand side of (1.7):

\[
K_f := f_{\text{avg}}^2 + \frac{2\pi \chi(M)}{\text{area}(M)},
\]

where \( \chi(M) \) is the Euler characteristic of \( M \). This quantity is always positive for \( M = S^2 \). For \( M = T^2 \) it is always greater than or equal to 0 and equality holds exactly when \( f_{\text{avg}} = 0 \).

**Lemma 1.23.** If \( f : M \to \mathbb{R} \) is a Zoll function, then there is a path of oriented \( S^1 \)-bundles \( \{p_r\}_{r \in [0,1]} \) with \( p_0 = \pm p_\infty \) and \( p_1 = p_f \) (here \( -p_\infty \) is the bundle \( p_\infty \) with reversed orientation). Furthermore, there holds

\[
\langle e_f, [M_f] \rangle = \chi(M), \quad K_f > 0, \quad h_f = \pm h_\infty.
\]

If in addition \( M \) is the two-torus, then

\[
f_{\text{avg}} \neq 0, \quad \langle [\omega_f, [M_f]] \rangle = \text{area}(M)|f_{\text{avg}}|, \quad h_f = \text{sign}(f_{\text{avg}})h_\infty.
\]
In view of this lemma, given some function $f : M \to \mathbb{R}$, we are motivated to look for periodic $f$-magnetic geodesics, whose tangent lift belongs either to the free-homotopy class $\mathfrak{h}_\infty$ or to $-\mathfrak{h}_\infty$. Up to changing $f$ with $-f$, we focus henceforth on magnetic geodesics in the former class of curves, which we denote by $\Lambda(M; \mathfrak{h}_\infty)$, and suppose that $f_{\text{avg}} > 0$ when $M = T^2$. Moreover, from now on, when we talk about a Zoll function $f$, it will be tacitly assumed that all its prime periodic magnetic geodesics lie in $\Lambda(M; \mathfrak{h}_\infty)$.

**Definition 1.24.** Let $c$ be an element of $\Lambda(M; \mathfrak{h}_\infty)$. Namely, $c$ is a periodic immersed curve in $M$ parametrised by arc-length such that $(c, \dot{c}) \subset T^1M$ lies in the free-homotopy class $\mathfrak{h}_\infty$. In this case, there exists a cylinder $\Gamma : [0, 1] \times S^1 \to T^1M$ such that $\Gamma(0, \cdot)$ is an oriented $p_\infty$-fibre and $\Gamma(1, \cdot)$ coincides with $(c, \dot{c})$, up to reparametrisation. This projects to a disc $C : D^2 \to M$ bounding $c$. Any such a disc arising in this way is called an **admissible** capping disc for $c$.

Lemma 11.8 and Lemma 11.11 explain in more detail which curves belong to $\Lambda(M; \mathfrak{h}_\infty)$. Here, we call $\Lambda(f; \mathfrak{h}_\infty)$ the subset of periodic $f$-magnetic geodesics in $\Lambda(M; \mathfrak{h}_\infty)$. It is the critical set of the $f$-magnetic length functional

$$\ell_f : \Lambda(M; \mathfrak{h}_\infty) \to \mathbb{R}, \quad \ell_f(c) := \ell(c) + \int_{D^2} C^*(f\mu),$$

where $\ell(c)$ is the Riemannian length of $c$, and $C$ is an admissible capping disc for $c$. The systolic-diastolic inequality will give bounds for the quantities

$$\ell_{\min}(f) := \inf_{\substack{c \in \Lambda(f; \mathfrak{h}_\infty) \\text{c prime}}} \ell_f(c), \quad \ell_{\max}(f) := \sup_{\substack{c \in \Lambda(f; \mathfrak{h}_\infty) \\text{c prime}}} \ell_f(c),$$

in terms of the **average length** of $f$ which is defined as

$$\bar{\ell}(f) := \frac{2\pi}{f_{\text{avg}} + \sqrt{K_f}}.$$

**Definition 1.25.** We say that $f : M \to \mathbb{R}$ satisfies the **magnetic systolic-diastolic inequality**, if $K_f > 0$ and

$$\ell_{\min}(f) \leq \bar{\ell}(f) \leq \ell_{\max}(f),$$

with any of the two equalities holding if and only if $f$ is Zoll.

We prove the inequality in two cases. First, we show it for functions close to a Zoll one.

**Theorem 1.26.** Let $M$ be a connected oriented closed surface endowed with a Riemannian metric, and let $f_* : M \to \mathbb{R}$ be a Zoll function. There exists a $C^2$-neighbourhood $\mathcal{F}$ of $f_*$ in the space of functions such that every $f \in \mathcal{F}$ satisfies the magnetic systolic-diastolic inequality.

**Remark 1.27.** It is very reasonable to expect that the theorem is true also if we let the metric $g$ vary. To be precise, if $f_*$ is Zoll with respect to a metric $g_*$, then there should exist a $C^3$-neighbourhood $\mathcal{G}$ of $g_*$ and a $C^2$-neighbourhood $\mathcal{F}$ of $f_*$ such that if $(g, f) \in \mathcal{G} \times \mathcal{F}$, then $f$ satisfies the magnetic systolic-diastolic inequality with respect to $g$. Actually, in the purely Riemannian case (namely, when $f = 0$), the systolic-diastolic inequality holds true for metrics $g$ on $S^2$, whose curvature is suitably pinched [ABHS17c]. These represent a $C^2$-neighbourhood of the round metric $g_*$. See also [ABHS17a, Corollary 4].
Second, the magnetic systolic-diastolic inequality holds for functions \( f : M \to (0, \infty) \) with large average. To make this concept precise, let us define for every \( k \in \mathbb{N} \), the quantity

\[
\langle f \rangle_k := \frac{\| f \|_{C^k}}{\min f} \in [1, \infty).
\]

For a constant \( C > 0 \), we say that \( f : M \to \mathbb{R} \) is \( C \)-\textbf{strong}, if there holds

\[
f > 0, \quad f_{\text{avg}} > \left( \langle f \rangle_3^4 + \langle f \rangle_2^6 \right) e^{C \langle f \rangle_1^2}.
\]

**Theorem 1.28.** Let \( M \) be a connected oriented closed surface endowed with a Riemannian metric \( g \). There exists a constant \( C_g > 0 \) with the property that, if \( f : M \to \mathbb{R} \) is \( C_g \)-strong, then the function \( f \) satisfies the magnetic systolic-diastolic inequality.

**Remark 1.29.** A similar statement is expected to hold, if we let \( g \) vary in a \( C^3 \)-bounded set.

For \( s > 0 \), we have \( (sf)_{\text{avg}} = s(f_{\text{avg}}) \) and \( \langle sf \rangle_k = \langle f \rangle_k \), for all \( k \in \mathbb{N} \). Thus, Theorem 1.28 applies to large rescalings of any positive function.

**Corollary 1.30.** Let \( M \) be a connected oriented closed surface endowed with a Riemannian metric \( g \). For every \( f : M \to (0, \infty) \), there exists a positive number \( s(g, f) > 0 \) such that if \( s > s(g, f) \), then the function \( sf \) satisfies the magnetic systolic-diastolic inequality.

Theorems 1.26 and 1.28 will follow from Theorem 1.3, when \( M \) is different from the two-torus, as in this case the tangent lifts of magnetic geodesics are the trajectory of a Reeb flow on the unit tangent bundle, up to reparametrisation. If \( M \) is the two-torus, its unit tangent bundle is trivial, and therefore the theorems are consequences of Theorem 1.19.

### 1.4 Structure of the paper

The paper is divided into three parts, corresponding to the three subsections above. The first part deals with the local systolic-diastolic inequality for contact three-manifolds:

- Section 2 is entirely devoted to the proof of Proposition 1.12, which classifies Zoll contact forms in dimension three.
- In Section 3, we construct a global surface of section for contact forms close to a Zoll one on a given three-manifold. We collect properties of the associated return time and return map in Theorem 3.14. As a consequence, we identify in Corollary 3.15 the necessary conditions (1.2), which yield at once the proof of Theorem 1.3.
- The goal of Section 4 is to establish Corollary 4.21, which implies that the return map satisfies the conditions formulated in Corollary 3.15. In order to do so, we show the correspondence between \( C^1 \)-small Hamiltonian diffeomorphisms and generating functions on a general compact surface \( N \) endowed with a symplectic vanishing of order one at the boundary (Proposition 4.16). From this fact, the existence of quasi-autonomous Hamiltonian functions for such diffeomorphisms follows by the Hamilton-Jacobi equation (Proposition 4.20).

The second part aims at formulating the local systolic-diastolic inequality for odd-symplectic manifolds and to establish it in some cases.
• In Section 5 we introduce the volume of a closed two-form on an odd-dimensional oriented closed manifold and explore invariance properties of it.

• We define odd-symplectic forms and H-forms in Section 6. Under certain conditions, we prove a stability result for H-forms in the set of odd-symplectic forms.

• We devote Section 7 to review some basic facts about oriented $S^1$-bundles, which will be crucially used in the following discussion.

• In Section 8 we define weakly Zoll pairs and Zoll odd-symplectic forms. We also prove Proposition 1.8 which classifies Zoll odd-symplectic forms on oriented three-manifolds.

• In Section 9 we introduce the action of a closed two-form on an odd-dimensional manifold which is the total space of some oriented $S^1$-bundle. We prove Theorem 9.14 which is just a reformulation of Theorem 1.13 showing that the action and the volume of a weakly Zoll pair are related through the Zoll polynomial.

• We formulate the local systolic-diastolic inequality for odd-symplectic forms in Section 10. We prove it for quasi-autonomous H-forms (Proposition 10.16 corresponding to Proposition 1.18) or when the real Euler class of the bundle vanishes (end of Subsection 10.3 corresponding to Theorem 1.19). In the proof of the latter case, Proposition 1.4 is shown as an intermediate step.

The third part is devoted to applications to magnetic geodesics on closed oriented surfaces:

• In Section 11 we give the relevant definitions to the formulation of the magnetic systolic-diastolic inequality, and compute the volume and the action in this setting. Lemma 11.23 is a consequence of Lemma 11.5 and Corollary 11.22.

• In the two subsections making up Section 12 we establish the magnetic systolic-diastolic inequality for the cases considered in Theorem 1.26 and 1.28.

At the end of the paper, Appendix A is included, where we collect three classical regularity results for which we could not find a proper reference: the first about the pull-back of differential forms; the second about the primitive of exact differential forms; the third about estimating the Reeb vector field in terms of the contact form and of its exterior derivative.

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Conventions and List of Symbols

Throughout the article we make use of the following conventions.

- If a mathematical object is expressed by a certain symbol, we add a *prime* to it in order to denote another object of the same kind.
- The circle $S^1 = \mathbb{R}/\mathbb{Z}$ has period one. The set of natural number $\mathbb{N} = \{0, 1, 2, \cdots \}$ includes 0, and $\mathbb{N}^* = \{1, 2, \cdots \}$.
- If $\mathcal{M}$ is a manifold, $\partial \mathcal{M}$ denotes the boundary of $\mathcal{M}$ and $\mathcal{M}^\circ = \mathcal{M} \setminus \partial \mathcal{M}$ its interior part. If $\mathcal{F}$ is a map on $\mathcal{M}$, we write $\partial \mathcal{F}$ and $\mathcal{M}^\circ$ for its restriction to $\partial \mathcal{M}$ and $\mathcal{M}$ respectively.
- We use the notation $\text{id}_\mathcal{M}$ for the identity map on a set $\mathcal{M}$. If $\mathcal{S}$ is a subset of the domain of a map $\mathcal{F}$, we write $\mathcal{F}|\mathcal{S}$ for its restriction.
- We denote by $[S^1, \mathcal{M}]$ the set of free-homotopy classes of one-periodic curves on a manifold $\mathcal{M}$ and we write $\mathcal{h}$ for an element in it.
- We write $\Omega^h(\mathcal{M})$, for the space of differential $h$-forms on a manifold $\mathcal{M}$. We denote by $d$ the exterior derivative and by $i$ the interior derivative. We write $H^h_\text{dR}(\mathcal{M})$ for the de Rham cohomology groups.
- We write $\langle \cdot, \cdot \rangle$ for the Kronecker pairing between cohomology and homology and PD for the Poincaré duality.
- The square bracket $[\cdot]$ denotes (co)homology classes or homotopy classes of paths.
- Given an oriented $S^1$-bundle $p : \Sigma \to M$, we write $e_\mathbb{Z} \in H^2(M; \mathbb{Z})$ for minus the Euler class of $p$, $e \in H^2_\text{dR}(M) \cong H^2(M; \mathbb{R})$ for minus the real Euler class of $p$, and $p_* : \Omega^h(\Sigma) \to \Omega^{h-1}(M)$ for the integration along fibers.

Below, there is a list of symbols in alphabetical order. Special characters are at the beginning and Greek letters at the end.

**Symbols used in Part I**

- $\| \cdot \|_{C^2, +}$ norm on one-forms, 28
- $\| \cdot \|_{C^0_+}$ norm on $\mathbb{F}$, 53
- $\| \cdot \|_{V}$ norm on $V$, 50
- $A$ annulus $[0, a) \times S^1$
- $A'$ annulus $[0, a/2) \times S^1$
- $A''$ annulus $[0, a/4) \times S^1$
- $b$ function on $N$, 85
- $b_\Sigma$ rank of the group $H^1_\text{free}(\Sigma; \mathbb{Z})$
- $B$ closed ball of radius $a$ in $\mathbb{R}^2$
- $B'$ closed ball of radius $a/2$ in $\mathbb{R}^2$
- $B(\epsilon)$ open ball of one-forms, 28
- $B_\epsilon$ normalised one-forms in $B(\epsilon)$, 51
- $C_D$ bound for the norm of $\mathcal{D}_z$, 20
- $C'_D$ positive constant, 28
- $\text{dist}_{C^1}$ $C^1$-distance, 84
- $d_*$ constant in (DF2), 26
- $\delta_q$, $\delta_z$ Darboux covering, 26
- $\delta_*$, $\mathcal{D}_*$ special elements $\delta_\mathcal{D}_*$, 85
- $\mathcal{E}$ space of exact diffeomorphisms, 49
- $\mathcal{E}(\epsilon)$ open ball in $\mathcal{E}$, 49
- $\mathcal{E}$ inverse of $\mathcal{G}$, 56
- $\mathcal{F}$ functions on $\mathbb{A}$, 56
- $\mathcal{F}_0$ functions on $\mathbb{A}$ vanishing at $\partial \mathbb{A}$, 56
- $G$, $G_\varphi$ generating function, 52
- $G$ generating function map, 53
- $G_\varphi$ fixed point set of $\varphi$
- $G_\varphi$ generating function, 52
- $G$ generating function map, 53
- $g_{\text{st}}$ standard metric on $\mathbb{R}^2$
- $i$ canonical inclusion
- $i$ imaginary unit in $\mathbb{C}$
- $K$ function on $N$, 45
$k$ d$\lambda$-area of $N$, 44
$M$ closed orientable surface
$M$ $M \setminus \{q_\ast\}$, 33
$N$ compact oriented surface with one boundary component
$N$ neighbourhood of $\Delta_N$, 44
$O_N$ zero-section in $T^*N$
$\alpha_\ast$ orientation given by $\alpha \wedge (d\alpha)^n$
$P$ first return map, 39
$\mathcal{P}(\alpha)$ set of prime periodic orbits of $\alpha$
$\mathcal{P}_T(\alpha)$ set of prime periodic orbits of $\alpha$ with period at most $T$
$\mathcal{P}_p(\alpha)$ set of prime periodic orbits of $\alpha$ homologous to a $p$-fibre
$\mathcal{P}_p^k(\alpha)$ $\mathcal{P}_T(\alpha) \cap \mathcal{P}_p(\alpha)$
$p$ bundle associated with $\alpha_\ast$, 26
$p_{st}$ projection $B \times S^1 \to B$
$q$ point in $M$
$q_\ast$ reference point in $Q$, 33
$Q$ the set $p(Z)$, 26
$R_\alpha$ Reeb vector field of $\alpha$, 4
$R_{st}$ standard Reeb vector field, 26
$R_s$ Reeb vector field of $\alpha_s$, 26
$R_z$ Reeb vector field of $\alpha_z$, 28
$s$ coordinate on $N \times S^1$, 39
$S$ surface of section $N \to \Sigma$, 34
$t$ time coordinate
$t_\Sigma$ order of the group $H^1_{tor}(\Sigma; \mathbb{Z})$
$T(\gamma)$ period of $\gamma$
$T(\alpha)$ period of a Zoll contact form $\alpha$
$T_{\min}(\alpha)$ minimal period of $\alpha$, 27
$T_{\max}(\alpha)$ maximal prime period of $\alpha$, 27
$\mathcal{T}$ set $\mathcal{W}(\mathcal{N})$, 15
$\mathcal{W}$ Weinstein map, 15
$\mathcal{W}$ image of $\mathcal{W}_k$, 44
$\text{Volume}(\alpha)$ contact volume of $\alpha$, 4
$\mathcal{V}$ set of functions on $N$ vanishing at least of order two at $\partial N$, 50
$\mathcal{V}$ neighbourhood of $\Delta_k$, 44
$x$ point in $B$
$X$ vector field on $\Sigma$
$z$ point in $\Sigma$
$z_\ast$ reference point in $Z$, 33
$Z$ finite set of points in $\Sigma$, 26
$Z(\xi)$ Zoll contact forms in $\xi$, 5
$Z(\Sigma)$ Zoll contact forms on $\Sigma$, 5
$\alpha$ contact form
fixed Zoll contact form, 26
$\alpha_\ast$ one-form $\mathcal{D}^*\alpha$ on $B$, 28
$\alpha_z$ standard form on $B \times S^1$, 26
$\alpha_{st}$ $\frac{1}{k}\Psi^*\alpha$, 31
$\beta$ one-form $\Xi^*\alpha$ on $N \times S^1$, 37
$\Gamma_{\varphi}$ graph of $\varphi$, 50
$\gamma$ one-periodic curve $S^1 \to \Sigma$
$\gamma_\ast$ orbit of $\alpha_\ast$ with $\gamma(0) = \gamma(0)$, 27
$\gamma_{\text{rep}}$ one-periodic parametrisation of $\gamma$, 27
$\Delta_N$ diagonal of $N \times N$
$\epsilon_0$ constant, 28
$\epsilon_1$ constant, 29
$\epsilon_2, \epsilon_3$ constants, 32
$\epsilon_4, \epsilon_5$ constants, 35
$\epsilon_6, \epsilon_7$ constants, 37
$\epsilon_8, \epsilon_9$ constants, 39
$\epsilon_{10}, \epsilon_{11}$ constants, 41
$\epsilon_\ast$ constant, 50
$\epsilon_{**}$ constant, 50
$\zeta$ uniformisation map, 35
$\lambda$ one-form $S^*\alpha$ (Section 3, 35)
$\lambda_\ast$ fixed one-form on $N$ (Section 3, 41)
$\lambda_{st}$ standard one-form on $\Lambda$, 44
$\lambda_s$ standard Liouville form on $B$, 26
$\lambda_\ast(\alpha, q', q)$ one-form $S^*\alpha_\ast$, 34
$\lambda_\ast$-integral from $q$ to $q'$, 39
$\xi$ co-oriented contact structure
$\xi$ isotopy class of $\xi$
$\Xi$ open book, 36
$\nu, \nu_\varphi$ diffeomorphism of $N$, 50
$\pi_N$ projection $T^*N \to N$, 50
$\rho_{\text{sys}}$ systolic ratio, 4
$\rho_{\text{dia}}$ diastolic ratio, 6
$\sigma$ action of $\varphi$, 50
$\varsigma \frac{d\varphi}{dt}(0)$, 51
$\Sigma$ closed three-manifold
$\Sigma_{z}, \Sigma'_{z}$ $\mathcal{D}_z(B \times S^1), \mathcal{D}_z(B' \times S^1)$, 26
$\tau$ first return time, 39
$\Phi^\alpha$ Reeb flow of $\alpha$, 4
$\psi$ diffeomorphism $\Sigma \to \Sigma$
$\omega_{st}$ standard symplectic form on $B$, 26
$\omega_s$ form on $M$ with $p^*\omega_s = \alpha_s$, 26
$\Omega_{st}$ two-form $p_{st}^*\omega_{st}$ on $B \times S^1$, 26
$\Omega_s$ the two-form $\alpha_s$, 26
In the article a superscript \( ^{\vee} \) is added to objects pertaining to \( \Sigma^{\vee} \).

| Symbol | Description |
|--------|-------------|
| #      | concatenation of paths |
| \( A_* \) | action value of a Zoll form |
| \( A \) | action on \( \mathcal{S}_C(\Sigma) \) |
| \( \tilde{A}_\Omega \) | action on \( \Lambda_0(p_0) \) |
| \( \mathcal{A} \) | action over \( \Lambda(p_0) \) |
| \( \mathcal{A}_{min} \) | diastole over \( \Lambda(p_0) \) |
| \( \mathcal{A}_{max} \) | derivative of parameter in \( [0, q] \) |
| \( \tilde{A}_a \) | action on \( \Lambda_{0}(\Sigma) \) |
| \( \mathcal{A}_H \) | action induced from \( \tilde{A}_a \) |
| \( a, a(\Omega) \) | action one-form on \( \Lambda_{0}(\Sigma) \) |
| \( B_\Omega \) | map \( \Omega^{1}(\Sigma) \rightarrow \Xi^1_{c}(\Sigma) \) |
| \( B_{\Omega} \) | map \( \Omega^{1}(\Sigma)/\Xi^1(\Sigma) \rightarrow \Xi^1_{c}(\Sigma) \) |
| \( \{B_{i}\} \) | finite admissible cover of \( M_{1} \) |
| \( c \) | element of \( H^2_{\text{dr}}(M) \) |
| \( C \) | element in \( H^2_{\text{dr}}(\Sigma) \) |
| \( ev \) | evaluation map |
| \( H \) | Hamiltonian \( \Sigma \rightarrow \mathbb{R} \) |
| \( H_{\text{min}} \) | quasi-autonomous \( H \) |
| \( H_{\text{max}} \) | as above |
| \( I \) | maximal interval |
| \( J_\beta \) | \( p \)-fibre through a point in \( \Sigma \) |
| \( K(u) \) | connection forms for \( u \) |
| \( K(p) \) | connection forms for \( p \) |
| \( M \) | oriented closed 2n-manifold |
| \( P \) | Zoll polynomial |
| \( \mathcal{P} \) | integrand for \( P \) |
| \( \mathcal{P}(\Sigma) \) | oriented \( S^1 \)-bundle on \( \Sigma \) |
| \( \mathcal{P}_1 \) | bundle associated with \( \Omega_{\ast} \) |
| \( (p, c) \) | weakly Zoll pair |
| \( p^{-1}(pt) \) | arbitrary \( p \)-fibre |
| \( [p^{-1}(pt)]_{\mathcal{Z}} \) | class of a \( p \)-fibre in \( H_{1}(\Sigma; \mathcal{Z}) \) |
| \( [p^{-1}(pt)] \) | class of a \( p \)-fibre in \( H_{1}(\Sigma; \mathbb{R}) \) |
| \( \Psi(\Sigma) \) | oriented \( S^1 \)-bundles with \( \Sigma \) |
| \( \Psi(\Sigma) \) | as total space |
| \( \Psi^0(\Sigma) \) | component of \( \Psi(\Sigma) \) |
| \( \Psi \) | projection \( \mathcal{S}(\Sigma) \rightarrow \Psi(\Sigma) \) |
| \( \Psi^0(\Sigma) \) | projection \( \mathcal{S}(\Sigma) \rightarrow \Psi^0(\Sigma) \) |
| \( q_{min} \) | minimiser of \( H_{\text{min}} \) |
| \( q_{max} \) | maximiser of \( H_{\text{max}} \) |
| \( Q \) | derivative of \( P \) |
| \( Q \) | integrand for \( Q \) |
| \( r \) | parameter in \( [0, 1] \) |
| \( \mathcal{S}(\Sigma) \) | odd-symplectic forms |
| \( u \) | free \( S^1 \)-action on \( \Sigma \) |
| \( u_{\ast} \) | integration operator of \( u \) |
| \( \Omega(\Sigma) \) | free \( S^1 \)-actions on \( \Sigma \) |
| \( V \) | vector field of an \( S^1 \)-action |
| \( \nu \) | neighbourhood of \( \Omega_{\ast} \) |
| \( \text{vol} \) | volume of a one-form |
| \( \text{Vol} \) | volume of a two-form |
| \( \nu \) | volume of a weakly Zoll pair |
| \( \{\gamma_{r}^{\text{short}}\} \) | \( S^1 \)-periodic curve \( S^1 \rightarrow \Sigma \) |
| \( \gamma \) | short homotopy |
| \( \Gamma \) | map \( S^1 \times [0, 1] \rightarrow \Sigma \) |
| \( \delta_{r} \) | path of loops in \( \Sigma \) as in \( \mathbb{R}^3 \) |
| \( \eta \) | \( S^1 \)-connection one-form |
| \( \xi \) | one-form on \( M \) |
| \( \kappa \) | one-forms on \( \Sigma \) |
| \( \Lambda(\Sigma) \) | space of one-periodic curves in \( \Sigma \) |
| \( \Lambda(b)(\Sigma) \) | set of \( \gamma \in \Lambda(\Sigma) \) with class \( b \) |
| \( \Lambda(p_0) \) | covering space of \( \Lambda_{0}(\Sigma) \) |
| \( \Xi^{k}(\Sigma) \) | space of \( k \)-forms in \( \Xi(\Sigma) \) with class \( C \) |
| \( \Xi^{k}(M) \) | space of \( k \)-forms in \( \Xi(M) \) |
| \( \pi \) | quotient map \( \pi : \Sigma^{\vee} \rightarrow M \) |
| \( \sigma_0 \) | one-form on \( \Sigma \) |
| \( \sigma \) | \( \sigma \wedge \Omega^{n} \) volume form on \( \Sigma \) |
| \( \Sigma \) | oriented closed 2n-1-manifold |
| \( \Sigma^{\vee} \) | oriented closed 2n-1-manifold |
| \( \text{domain of } \Pi : \Sigma^{\vee} \rightarrow \Sigma \) |
Symbols used in Part III

- \# \quad \text{metric dual } T^*M \to TM
- * \quad \text{Hodge-star operator}
- \text{area}(M) \quad g\text{-area of } M, \text{100}
- A_{\min}(\Omega_f) \quad \text{minimal action, } \text{112}
- A_{\max}(\Omega_f) \quad \text{maximal action, } \text{112}
- A(\Omega_f) \quad \mathcal{A}(p_f, [\omega_f]), \text{110}
- B_{2,2} \quad \text{positive constant, } \text{116}
- c \quad \text{curve in } M
- \bar{c} \quad \text{lift of } c \text{ to } \bar{M}
- C \quad \text{capping disc of } c, \text{103}
- C_U \quad \text{positive constant, } \text{106}
- e_\infty \quad \text{minus the real Euler class of } p_\infty, \text{101}
- e_f \quad \text{minus the real Euler class of } p_f, \text{102}
- f \quad \text{real function on } M
- f_s \quad \text{Zoll function on } M, \text{112}
- f_{\text{avg}} \quad \text{average of } f, \text{100}
- f_{\text{norm}} \quad f/ f_{\text{avg}}, \text{106}
- (f)_k \quad \|f\|_k/ \text{min } f, \text{106}
- \mathcal{F} \quad \text{neighbourhood of } f, \text{112}
- g \quad \text{Riemannian metric on } M
- h_\infty \quad \text{class of } p_\infty\text{-fibres, } \text{101}
- h_f \quad \text{class of } p_f\text{-fibres, } \text{102}
- K \quad \text{Gaussian curvature of } g
- K_f \quad \text{average curvature of } f, \text{100}
- \ell_f \quad \text{magnetic length, } \text{104}
- \ell_{\text{min}}(f) \quad \text{magnetic systole, } \text{104}
- \ell_{\text{max}}(f) \quad \text{magnetic diastole, } \text{104}
- \bar{\ell}(f) \quad \text{average } f\text{-length, } \text{104}
- M \quad \text{orientable closed surface}
- M_f \quad \text{orbit space of } f \text{ Zoll, } \text{102}
- \bar{M} \quad \text{universal cover of } M
- \sigma_M \quad \text{orientation on } M
- \sigma_{T^1M} \quad \text{orientation } \sigma_M \oplus \sigma_V, \text{101}
- \sigma_V \quad \text{orientation on } p_\infty\text{-fibres, } \text{101}
- p_V \quad \text{projection } T^1M \to M, \text{101}
- p_f \quad \text{bundle } T^1M \to M_f, \text{102}
- p_\infty \oplus \mathcal{P}^0(T^1M) \quad \text{connected component of } p_\infty \text{ in } \mathcal{P}(T^1M), \text{108}
- T^1M \quad \text{unit tangent bundle of } M
- U \quad \text{neighbourhood of } \Omega_\infty, \text{106} \quad \text{114}
- V \quad \text{vector field on } T^1M \text{ rotating the fibres of } p_\infty, \text{101}
- C^1\text{-neighbourhood of } \Lambda(f; h_\infty), \text{112} \quad \text{113}
- X \quad \text{geodesic vector field, } \text{101}
- X_f \quad \text{magnetic vector field, } \text{101}
- \mathcal{X}^0_{[\Omega_\infty]}(T^1M) \quad \text{set of weakly Zoll pairs } (p, c) \text{ with } p \in \mathcal{P}^0(T^1M) \text{ and } p^*c = [\Omega_\infty], \text{108}
- \alpha_{\text{can}} \quad \text{canonical one-form on } T^1M, \text{101}
- \alpha_\infty \quad \text{primitive of } \Omega_\infty, \text{109}
- \alpha_f \quad \text{one-form associated to } \Omega_f, \text{109} \quad \text{111}
- \eta \quad \text{Levi-Civita one-form, } \text{101}
- \Lambda(M; h_\infty) \quad \text{periodic curves in } M \text{ admitting an admissible capping disk, } \text{103}
- \Lambda(f; h_\infty) \quad f\text{-magnetic geodesics in } \Lambda(M; h_\infty), \text{103}
- \kappa_c \quad \text{geodesic curvature of } c, \text{100}
- \mu \quad \text{area form on } M
- \phi \quad \text{angular coordinate on } S^1
- \chi(M) \quad \text{Euler characteristic of } M
- \omega \quad \text{symplectic form on } M
- \omega_f \quad \text{two-form } p_f^*\omega_f = \Omega_f, \text{102}
- \Omega \quad \Omega_f \quad \text{closed two-form on } \Sigma
- \Omega_f \quad \text{odd-symplectic form on } T^1M, \text{101}
- \Omega_\infty \quad \text{quotient } \Omega^1(\Sigma)/\Xi^1(\Sigma), \text{60}
- j_{\text{avg}} \quad \text{quotient } \Omega^1(\Sigma)/\Xi^1(\Sigma), \text{60}

\textbf{Symbols used in Part III}

- \{\Sigma_i\} \quad \{p_0^{-1}(B_i), \text{90}\}
- \tau \quad \text{closed one-form on } \Sigma
- \phi \quad \text{angular coordinate on } S^1
- \psi \quad \text{angular function, } \text{95}
- \Psi \quad \text{quotient map of } \Psi, \text{68}
- \Psi \quad \text{diffeomorphism } \Sigma \to \Sigma
- \Psi^* \quad \text{pull-back bundle, } \text{68}
- \omega \quad \text{symplectic form on } M
- \Omega \quad \text{closed two-form on } \Sigma
- \Omega_0 \quad \text{reference two-form on } \Sigma
- \Omega_s \quad \text{fixed Zoll two-form}
- \Omega_\alpha \quad \Omega_0 + \alpha\Omega
- \Omega_1(\Sigma) \quad \text{quotient } \Omega^1(\Sigma)/\Xi^1(\Sigma), \text{60}
Part I
A local systolic-diastolic inequality for contact forms

2 Classification of Zoll contact forms in dimension three

In this section Σ will denote a connected closed three-manifold. We use the notation

\[ b_\Sigma := \text{rank} \, H^1_\text{free}(\Sigma; \mathbb{Z}), \quad t_\Sigma := |H^1_\text{tor}(\Sigma; \mathbb{Z})|. \]

We divide the proof of Proposition 1.2 into three parts.

2.1 Existence

Following Boothby and Wang [BW58], we characterise Σ admitting a Zoll contact form.

**Lemma 2.1.** The manifold Σ admits a Zoll contact form if and only if it is a non-trivial orientable \( S^1 \)-bundle over a closed orientable surface \( M \). More precisely, for every Zoll contact form \( \alpha \) on Σ, we have a non-trivial orientable \( S^1 \)-bundle \( p : \Sigma \to M \) such that the Reeb orbits of \( \alpha \) are tangent to the fibres of \( p \) and there exists a symplectic form \( \omega \) on \( M \) satisfying \( p^* \omega = d\alpha \). In particular, the form \( \alpha \) induces an orientation on \( p \) and the form \( \omega \) on \( M \).

As a consequence, if \( e \in H^2_{\text{dR}}(M) \) denotes minus the real Euler class of the oriented bundle \( p \), then \( \langle e, [M] \rangle > 0 \) and we have an isomorphism

\[ H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^{2\text{genus}(M)} \oplus \frac{\mathbb{Z}}{\langle e, [M] \rangle \mathbb{Z}}, \tag{2.1} \]

which implies \( \text{genus}(M) = \frac{1}{2} b_\Sigma \), \( \langle e, [M] \rangle = t_\Sigma \). Moreover, the homology class of a \( p \)-fibre generates \( H^1_\text{tor}(\Sigma; \mathbb{Z}) \) and the homotopy class of a \( p \)-fibre has order \( t_\Sigma \), if \( M = S^2 \), and has infinite order, otherwise. Finally, if \( T(\alpha) > 0 \) is the period of \( \alpha \), then there holds

\[ \frac{T(\alpha)^2}{\text{Volume}(\alpha)} = \frac{1}{t_\Sigma}. \tag{2.2} \]

**Remark 2.2.** We refer to Section 7.1 for a thorough discussion about oriented \( S^1 \)-bundles and their real Euler class.

**Proof.** Let \( \alpha \) be a Zoll contact form on Σ with period \( T(\alpha) > 0 \). We endow Σ with the orientation induced by \( \alpha \). The Reeb flow of \( \frac{1}{T(\alpha)} \alpha \) yields a free \( S^1 \)-action on Σ, the orbit space \( M \) is a closed surface, and there is a quotient map \( p : \Sigma \to M \). The form \( \frac{1}{T(\alpha)} \alpha \) is a connection form for \( p \). Hence, \( d\omega \) descends to a closed form \( \omega \) on \( M \), so that \( p^* \omega = d\alpha \), and the Chern-Weil theory asserts that the cohomology class \( e := [\frac{1}{T(\alpha)} \omega] \in H^2_{\text{dR}}(M) \) is minus the real Euler class of \( p \). Since \( \alpha \) is a contact form, \( \omega \) is symplectic, and therefore, yields an orientation on \( M \). Thus, \( p \) is non-trivial since

\[ \langle e, [M] \rangle = \frac{1}{T(\alpha)} \int_M \omega > 0. \]
Moreover, if \( p_* : M \to \mathbb{R} \) denotes the integration of \( \alpha \) along the fibres, we have \( p_* \alpha \equiv T(\alpha) \) and by Fubini’s Theorem
\[
\text{Volume}(\alpha) = \int_{\Sigma} \alpha \wedge p^* \omega = \int_M (p_* \alpha) \cdot \omega = T(\alpha) \int_M \omega = T(\alpha)^2 \langle e, [M] \rangle. \tag{2.3}
\]

To prove the converse, we suppose that there exists a non-trivial orientable \( S^1 \)-bundle \( p : \Sigma \to M \), where \( M \) is a closed orientable surface. Choosing an orientation on the \( p \)-fibres yields a well-defined non-zero real Euler class \( -e \in H^2_{\text{dR}}(M) \) and we take the orientation on \( M \) by declaring \( \langle e, [M] \rangle > 0 \). Since \( M \) is an oriented surface, there exists a symplectic form \( \omega \) on \( M \) inducing the given orientation. Up to rescaling \( \omega \) by a positive factor, we can assume that \( \langle e, [M] \rangle = \int_M \omega \). As \( H^2_{\text{dR}}(M) \cong \mathbb{R} \), this implies that the cohomology class of \( \omega \) is equal to \( e \). Thus, by [KN96], there exists a connection form \( \alpha \) for \( p \) such that \( d\alpha = p^* \omega \). In particular \( \alpha \) is a contact form since \( \ker \alpha \) is transverse to the fibres of \( p \) and \( \omega \) is symplectic. This finishes the proof of the converse implication.

Equation (2.1) follows from either the Gysin sequence with integer coefficients or the Mayer-Vietoris sequence of the pair \((p^{-1}(D^2), p^{-1}(M \setminus \{q\}))\), where \( q \in M \) and \( D^2 \subset M \) is an open disc centred at \( q \). The statement about the homotopy class of a \( p \)-fibre can be easily verified using the homotopy long exact sequence. Finally, (2.2) is a consequence of (2.3) and the identity \( \langle e, [M] \rangle = t_{\Sigma} \), which follows from (2.1).

### 2.2 Uniqueness up to diffeomorphism

Having established when a Zoll contact form on \( \Sigma \) exists, we now prove that all of them are isomorphic. This should be compared with the fact that two Zoll metrics on \( S^2 \) are not necessarily isometric.

**Lemma 2.3.** Let \( \Sigma \) be a connected closed three-manifold. Let \( \alpha \) and \( \alpha' \) be two Zoll contact forms on \( \Sigma \) with unit period. There exists a diffeomorphism \( \Psi : \Sigma \to \Sigma \) such that
\[
\Psi^* \alpha' = \alpha.
\]

**Proof.** The Reeb flows of \( \alpha \) and \( \alpha' \) yield \( S^1 \)-actions on \( \Sigma \) and let \( p : \Sigma \to M \) and \( p' : \Sigma \to M' \) be the associated oriented \( S^1 \)-bundles. We write \( e \) and \( e' \) for minus the real Euler class of \( p \) and \( p' \). Let us orient \( M \) and \( M' \) through the forms \( \omega \) and \( \omega' \), where \( d\alpha = p^* \omega \) and \( d\alpha' = p'^* \omega' \). By Lemma 2.1, the surfaces \( M \) and \( M' \) have the same genus and \( \langle e, [M] \rangle = \langle e', [M'] \rangle \). As the Euler number \( \langle e, [M] \rangle \) is a complete invariant for principal \( S^1 \)-bundles over orientable surfaces, there exists an \( S^1 \)-equivariant diffeomorphism \( \Psi_1 : \Sigma \to \Sigma \) such that \( p' \circ \Psi_1 = \psi_1 \circ p \), for some orientation-preserving diffeomorphism \( \psi_1 : M \to M' \). As a result, if \( \alpha_1 := \Psi_1^* \alpha' \), then there exists a one-form \( \eta \) on \( M \) such that \( \alpha_1 = \alpha + p^* \eta \) and \( d\alpha_1 = p^* \omega, \) where \( \omega_1 := \psi_1^* \omega' \).

We construct now a diffeomorphism \( \Psi_2 : \Sigma \to \Sigma \) with the property \( \Psi_2^* \alpha_1 = \alpha \), so that \( \Psi := \Psi_2 \circ \Psi_1 \) will be the desired map. Using a stability argument, we look for an isotopy \( \Phi_u : \Sigma \to \Sigma \) generated by a vector field \( X_u \) such that
\[
\Phi_u^* \alpha_u = \alpha, \tag{2.4}
\]
where \( \alpha_u := \alpha + up^* \eta \), for all \( u \in [0,1] \). We will then set \( \Psi_2 := \Phi_1 \). We observe that \( \omega_u := (1-u)\omega + u \omega_1 \) is a path of symplectic forms on \( M \), as \( \Psi_1 \) preserves the orientation. Differentiating (2.4) with respect to \( u \), we see that (2.4) is satisfied once \( X_u \) is chosen as the vector field in \( \ker \alpha_u \) with the property that \( d\alpha(X_u) = X_u \), where \( X_u \) is the unique vector field on \( M \) satisfying the relation \( t_{X_u} \omega_u = -\eta \). \( \square \)
2.3 Isotopy classes

We now present the last step in the proof of Proposition 1.2. Recall that \( \mathcal{Z}(\Sigma) \) is the space of Zoll contact forms on \( \Sigma \) and \( \mathcal{Z}(\xi) \) is the set of Zoll contact forms supporting an isotopy class \( \xi \) of contact structures on \( \Sigma \).

**Lemma 2.4.** Let \( \Sigma \) denote the total space of a non-trivial orientable \( S^1 \)-bundle over a connected closed orientable surface.

1. If \( \Sigma \) is either \( S^3 \) or \( \mathbb{R}P^3 \), then \( \mathcal{Z}(\Sigma) \) has exactly two connected components \( \mathcal{Z}(\xi_{st}) \) and \( \mathcal{Z}(\xi_{st}) \). Here \( \xi_{st} \) is the isotopy class of the standard contact structure and \( \xi_{st} \) the isotopy class obtained from \( \xi_{st} \) by applying an orientation-reversing diffeomorphism.

2. If \( \Sigma \) is neither \( S^3 \) nor \( \mathbb{R}P^3 \), then \( \mathcal{Z}(\Sigma) \) has exactly two connected components \( \mathcal{Z}(\xi_+) \) and \( \mathcal{Z}(\xi_-) \). Here \( \xi_+ \) and \( \xi_- \) are two distinct isotopy classes with \( \xi_- = -\xi_+ \).

**Proof.** Let us fix a Zoll contact form \( \alpha \) on \( \Sigma \) with unit period and bundle map \( p : \Sigma \to M \). We consider any other Zoll form \( \alpha' \) with unit period on \( \Sigma \) and we distinguish two cases.

**Case 1:** \( M = S^2 \). By Lemma 2.1, the diffeomorphism type of \( \Sigma \) is determined by \( t_{\Sigma} \) so that \( \Sigma \) is the lens space \( L(t_{\Sigma},1) \). Lemma 2.2 yields a diffeomorphism \( \Psi : \Sigma \to \Sigma \) with the property \( \alpha' = \Psi^* \alpha \). Suppose that \( \Sigma \) is either \( S^3 \) or \( \mathbb{R}P^3 \) and let \( \Upsilon : \Sigma \to \Sigma \) be a diffeomorphism of \( \Sigma \) reversing the orientation. By Cerf’s Theorem (see [Cer68], and [Bon83 Théorème 3] or [HR85 Theorem 5.6]), \( \Psi \) is either isotopic to the identity or to \( \Upsilon \), thus showing that \( \alpha' \) is either homotopic to \( \alpha \) or to \( \Upsilon^* \alpha \) within \( \mathcal{Z}(\Sigma) \).

Suppose now that \( \Sigma \) is neither \( S^3 \) nor \( \mathbb{R}P^3 \). Then, a \( p \)-fibre is not homotopic to itself with reverse orientation. Therefore, \( \alpha \) and \( \alpha' = -\alpha \) are not homotopic in \( \mathcal{Z}(\Sigma) \). Moreover, by Lemma 2.3 there exists a diffeomorphism \( \Upsilon_- : \Sigma \to \Sigma \) such that \( \Upsilon_- \alpha = -\alpha \). In particular, \( \Upsilon_- \) changes the orientation of the fibres and is not isotopic to the identity. By [Bon83 Théorème 3] or [HR85 Theorem 5.6] again, the map \( \Psi \) is either isotopic to the identity or to \( \Upsilon_- \). Hence, \( \alpha' \) is either homotopic to \( \alpha \) or to \( \Upsilon_- \) within \( \mathcal{Z}(\Sigma) \).

**Case 2:** \( M \neq S^2 \). By Lemma 2.1 a \( p \)-fibre is not homotopic to itself. Therefore, \( \alpha \) and \( -\alpha \) are not homotopic within \( \mathcal{Z}(\Sigma) \). Moreover, [Wal67 Satz 5.5] implies that there exists a diffeomorphism \( \Psi : \Sigma \to \Sigma \) isotopic to the identity and such that \( \Psi^* \alpha' \) is an \( S^1 \)-connection for \( p \) or for \( p' \) with reversed orientation. The stability argument contained in the proof of Lemma 2.3 shows that \( \Psi^* \alpha' \) is homotopic to \( \alpha \) in the first case, or to \( -\alpha \) in the second case.

We finally observe that if \( \Sigma \) is not \( S^3 \) nor \( \mathbb{R}P^3 \), then \( \xi_+ \) and \( \xi_- \) are not isotopic. To this purpose, we use the last statement of Theorem D in [Mas08]. The fact that \( \Sigma \neq S^3, \mathbb{R}P^3 \) is equivalent to the fact that \( \langle e, [M] \rangle > \chi(M) \), and implies the hypothesis \( -b - r < 2g - 2 \) contained therein, where \( b = \langle e, [M] \rangle \), \( r = 0 \) and \( 2g - 2 = -\chi(M) \). Therefore, one only needs to check that the twisting number \( t(\xi_{\pm}) \) defined in [Mas08 p. 1730] equal to \(-1\). If we suppose that \( \alpha \) has period 1, then it is an \( S^1 \)-connection for \( p \) with \( d\alpha = p^* \omega \) and there exists a positively immersed disc \( D^2 \hookrightarrow M \), whose lift to the universal cover of \( M \) is embedded and such that \( \int_{D^2} \omega = 1 \). One readily sees that the horizontal lift of the boundary of \( D^2 \) traversed in the negative direction is a \( \xi_{\pm} \)-Legendrian curve in \( \Sigma \), which is isotopic to an oriented \( p \)-fibre and has twisting number \(-1\). \( \square \)
3 A global surface of section for contact forms near Zoll ones

Let $\Sigma$ be a connected closed three-manifold, and let $\alpha_s$ be a Zoll contact form on $\Sigma$ with unit period (see Definition 1.1). Let $R_s$ denote the Reeb vector field of $\alpha_s$. Since $\alpha_s$ is Zoll, $R_s$ induces a free $S^1$-action on $\Sigma$ and yields an oriented $S^1$-bundle $p : \Sigma \to M$, where $M$ is the quotient of $\Sigma$ by the action and $p$ is the canonical projection. Throughout this section, we fix auxiliary Riemannian metrics on $\Sigma$ and $M$, in order to compute the distance between points and between diffeomorphisms, and the norm of sections of vector bundles over these manifolds. The space $M$ is a connected closed surface having a symplectic form $\omega_s$ satisfying

\[ \Omega_s = p^* \omega_s, \quad \Omega_s := d\alpha_s. \]

We endow $M$ with the orientation induced by $\omega_s$.

Let $g_{st}$ and $i$ be the standard scalar product and complex structure on $\mathbb{R}^2 \cong \mathbb{C}$, respectively. If $a > 0$ is an arbitrary positive number, we denote by $B$ (respectively $B'$) the closed Euclidean ball in $\mathbb{R}^2$ of radius $a$ (respectively $a/2$). We write $x = (x_1, x_2)$ for a point in $B$ and let $\lambda_{st} = \frac{1}{2a}(x_1dx_2 - x_2dx_1)$ be the standard Liouville form (up to a constant) on $B$ with $\omega_{st} := d\lambda_{st}$. We consider the trivial bundle $p_{st} : B \times S^1 \to B$ and we write $\phi$ for the fibre coordinate. We set

\[ \alpha_{st} := d\phi + p_{st}^* \lambda_{st}, \quad \Omega_{st} := d\alpha_{st} = p_{st}^* \omega_{st}, \quad R_{st} := \partial_{\phi}. \]

We now define a finite Darboux covering for $M$. To this purpose, let $Z \subset \Sigma$ be a finite set of points and write $Q := p(Z)$. We consider $S^1$-equivariant embeddings

\[ D_z : B \times S^1 \to \Sigma, \quad D_z(0, 0) = z, \quad \forall z \in Z. \]

This means that there are corresponding embeddings

\[ d_q : B \to M, \quad d_q(0) = q, \quad \forall q \in Q \]

such that

\[ p \circ D_z = d_{p(z)} \circ p_{st}, \quad \forall z \in Z. \]

We write $\Sigma_z := D_z(B \times S^1)$, $\Sigma'_z := D_z(B' \times S^1)$, and $M_q := d_q(B)$, $M'_q := d_q(B')$. Finally, we denote by $(x_z, \phi_z) \in B \times S^1$ the coordinates given by $D_z$. By the compactness of $\Sigma$, we see that, if $a$ is small enough, the following four properties can be assumed to hold

\[ M = \bigcup_{q \in Q} M'_q, \quad (DF1) \]

\[ \exists d_s > 0, \quad \text{dist}(M'_q, M \setminus M_q) > d_s, \quad \forall q \in Q, \quad (DF2) \]

\[ \exists C_D > 0, \quad \|D_z\|_{C^2} \leq C_D, \quad \|d(D_z^{-1})\|_{C^2} \leq C_D, \quad \forall z \in Z, \quad (DF3) \]

\[ D_z^* \alpha_s = \alpha_{st}, \quad \forall z \in Z. \quad (DF4) \]

In this section, we define a neighbourhood of $\Omega_s$ in the space of exact two-forms on $\Sigma$ with special properties. The elements of the neighbourhood will be exterior differentials of contact forms, whose Reeb flow has a distinguished set of periodic Reeb orbits, which can be used to construct a global surface of section for the flow.
3.1 A distinguished class of periodic Reeb orbits

For any contact form $\alpha$ on $\Sigma$, let $R_\alpha$ be its Reeb vector field. Let $\mathcal{P}(\alpha)$ denote the set of prime periodic orbits of the Reeb flow $\Phi^\alpha$ of $\alpha$. We write $\mathcal{P}^p(\alpha)$ for the subset of $\mathcal{P}(\alpha)$, whose elements are homologous to an oriented $\mathfrak{p}$-fibre, and abbreviate $\mathcal{P}^p_T(\alpha) := \mathcal{P}^p(\alpha) \cap \mathcal{P}_T(\alpha)$. If $\gamma \in \mathcal{P}(\alpha)$, we write $T(\gamma)$ for the period of $\gamma$ and define the auxiliary one-periodic curves

$$\gamma_{\text{rep}}, \bar{\gamma}: S^1 \to \Sigma, \quad \gamma_{\text{rep}}(u) := \gamma(uT(\gamma)), \quad \bar{\gamma}(u) := \Phi^\alpha_u(\gamma(0)).$$

For all $T \in (0, \infty)$, we denote by $\mathcal{P}_T(\alpha)$ the subset of $\mathcal{P}(\alpha)$, whose elements have period less than or equal to $T$. We define

$$T_{\min}(\alpha) := \inf_{\gamma \in \mathcal{P}(\alpha)} T(\gamma), \quad T_{\max}(\alpha) := \sup_{\gamma \in \mathcal{P}(\alpha)} T(\gamma).$$

We now explore how much information of the Reeb dynamics is already encoded in the exterior differential of the contact form.

**Lemma 3.1.** Let $\alpha_1$ and $\alpha_2$ be contact forms such that $d\alpha_1 = d\alpha_2$. There holds $\mathfrak{o}_{\alpha_1} = \mathfrak{o}_{\alpha_2}$, $\text{Volume}(\alpha_1) = \text{Volume}(\alpha_2)$. Moreover, there is a bijection between $\mathcal{P}(\alpha_1)$ and $\mathcal{P}(\alpha_2)$ which preserves the oriented support of curves. The bijection is period-preserving when restricted to $\mathcal{P}^p(\alpha_1)$ and $\mathcal{P}^p(\alpha_2)$. If $\alpha_1$ is Zoll, then $\alpha_2$ is also Zoll, and $T(\alpha_1) = T(\alpha_2)$.

**Proof.** Since $d\alpha_1 = d\alpha_2$, we have $R_{\alpha_2} = \frac{1}{\alpha_2(R_{\alpha_1})} R_{\alpha_1}$ and $\alpha_2 = \alpha_1 + \eta$ for some closed one-form $\eta$. We fix an arbitrary orientation on $\Sigma$ and compute

$$\int_\Sigma \alpha_2 \wedge d\alpha_2 = \int_\Sigma \alpha_1 \wedge d\alpha_1 + \int_\Sigma \eta \wedge d\alpha_2 = \int_\Sigma \alpha_1 \wedge d\alpha_1.$$

Thus, the orientations induced by the contact forms $\alpha_1$ and $\alpha_2$ coincide. This implies that $\text{Volume}(\alpha_1) = \text{Volume}(\alpha_2)$ and that $\alpha_2(R_{\alpha_1}) > 0$. Therefore, for all $z \in \Sigma$, there holds

$$\Phi_{t_2(t_1, z)}^{\alpha_2}(z) = \Phi_{t_1}^{\alpha_1}(z), \quad t_2(t_1, z) := \int_0^{t_1} \left( t \mapsto \Phi_{t_1}^{\alpha_1}(z) \right)^* \alpha_2,$$

so that $t_1 \mapsto t_2(t_1, z)$ is strictly increasing. Hence, $\Phi^{\alpha_1}$ and $\Phi^{\alpha_2}$ have the same trajectories, up to an orientation-preserving reparametrisation, and we have a bijective correspondence between $\mathcal{P}(\alpha_1)$ and $\mathcal{P}(\alpha_2)$ preserving the oriented support of periodic orbits. If $\gamma_1 \in \mathcal{P}^p(\alpha_1)$, then $\gamma_2 \in \mathcal{P}^p(\alpha_2)$. Since, by Lemma 2.4, the homology class corresponding to $\gamma_1$ and $\gamma_2$ is torsion, the fact that $\eta$ is closed implies

$$T(\gamma_2) = \int_{\mathbb{R}/T(\gamma_2)\mathbb{Z}} \gamma_2^* \alpha_2 = \int_{\mathbb{R}/T(\gamma_1)\mathbb{Z}} \gamma_2^* \alpha_1 + \int_{\mathbb{R}/T(\gamma_2)\mathbb{Z}} \gamma_2^* \eta = T(\gamma_1) + 0.$$

Finally, if $\alpha_1$ is Zoll with period $T_1$, then $\alpha_2$ is also Zoll with period $T_2 := t_2(T_1, z)$ (independent of $z \in \Sigma$), as $t_1 \mapsto t_2(t_1, z)$ is monotone increasing. In this case, since every prime periodic orbit of $\Phi^{\alpha_1}$ is null-homologous by Lemma 2.4, we conclude as above that

$$T_2 = t_2(T_1, z) = \int_{\mathbb{R}/T_1\mathbb{Z}} \gamma_1^* \alpha_2 = \int_{\mathbb{R}/T_1\mathbb{Z}} \gamma_1^* \alpha_1 = T_1. \qed$$
On the space of one-forms \( \alpha \) on \( \Sigma \) we consider the norm \( \| \cdot \|_{C^2,+} \) defined by

\[
\| \alpha \|_{C^2,+} := \| \alpha \|_{C^2} + \| d\alpha \|_{C^2}.
\]

Considering also the \( C^{2,+} \)-norm for one-forms on \( B \), we see that Lemma A.1 and (D3) yield a constant \( C_D' \) depending only on \( C_D \) and \( \Sigma \) such that for all, one-forms \( \alpha \) on \( \Sigma \),

\[
\frac{1}{C_D'} \| \alpha \|_{C^2,+} \leq \| \mathcal{D}_1^* \alpha \|_{C^2,+} \leq C_D' \| \alpha \|_{C^2,+}, \quad \forall z \in Z. \tag{3.2}
\]

For every \( \epsilon > 0 \), we denote the \( C^{2,+} \)-ball with center \( \alpha_* \) and radius \( \epsilon \) by

\[
B(\epsilon) := \{ \alpha \text{ one-form on } \Sigma \mid \| \alpha - \alpha_* \|_{C^2,+} < \epsilon \}.
\]

The next result shows why it is natural to consider the \( C^{2,+} \)-norm for our purposes.

**Lemma 3.2.** There exists a constant \( C_0 > 0 \) such that for all one-forms \( \alpha' \) on \( \Sigma \), there is a one-form \( \alpha \) on \( \Sigma \) with the property that

- \( d\alpha = d\alpha' \),
- \( \forall \epsilon > 0, \quad \| d\alpha' - d\alpha_* \|_{C^2} < \epsilon \implies \alpha \in B(C_0\epsilon). \)

**Proof.** By Lemma A.3, there exists a constant \( C_0' > 0 \) such that for all exact two-forms \( \Omega \) on \( \Sigma \), we can find a one-form \( \eta_\Omega \) with

\[
d\eta_\Omega = \Omega - \Omega_* \quad \Rightarrow \quad \| \eta_\Omega \|_{C^2} \leq C_0' \| \Omega - \Omega_* \|_{C^2}.
\]

The form \( \alpha := \alpha_* + \eta_\Omega \) satisfies \( d\alpha = d\alpha' \) and the statement follows with \( C_0 := C_0' + 1 \), since

\[
\| \alpha - \alpha_* \|_{C^2} \leq C_0' \| d\alpha' - \Omega_* \|_{C^2}, \quad \| d\alpha - \Omega_* \|_{C^2} = \| d\alpha' - \Omega_* \|_{C^2}. \quad \square
\]

We can now proceed to study the Reeb dynamics for one-forms in the sets \( B(\epsilon) \).

**Lemma 3.3.** There exists a constant \( \epsilon_0 > 0 \) such that every element in \( B(\epsilon_0) \) is a contact form. There exists a constant \( C_1 > 0 \) with the following property: If \( \alpha \in B(\epsilon_0) \), \( z' \in \Sigma \), \( z \in Z \) and \( T \in (0, \infty) \) are such that the integral curve \( t \mapsto \Phi_t^\alpha(z') \) lies in \( \Sigma_z \) for all \( t \in [0,T] \), then the curve

\[
\gamma_z := (x_z(t), \phi_z(t)) = \mathcal{D}_z^{-1}(\Phi_t^\alpha(z))
\]

satisfies

\[
\| \gamma_z - R_{\text{st}} \|_{C^2} \leq C_1 \| \alpha - \alpha_* \|_{C^2,+}. \tag{3.3}
\]

Thus, if \( \hat{\phi}_z : [0,T] \to \mathbb{R} \) with \( \hat{\phi}_z(0) = 0 \) is a lift of \( \phi_z - \phi_z(0) \), then

\[
\| x_z(t) - x_z(0) \| \leq C_1 t \| \alpha - \alpha_* \|_{C^2,+}, \quad \| \hat{\phi}_z(t) - t \| \leq C_1 t \| \alpha - \alpha_* \|_{C^2,+}, \quad \forall t \in [0,T]. \tag{3.4}
\]

**Proof.** Let \( \alpha \in B(1) \) and set \( \alpha_* := \mathcal{D}_z^* \alpha \). From the estimates (3.2) we have

\[
\| \alpha_z - \alpha_* \|_{C^2,+} \leq C_D' \| \alpha - \alpha_* \|_{C^2,+}. \tag{3.5}
\]

We apply Lemma A.6 with \( k = 2 \), \( M = B \times S^1 \), \( \alpha_0 = \alpha_{\text{st}} \) and obtain corresponding constants \( \delta > 0 \) and \( A_2 > 0 \). We take

\[
\epsilon_0 := \frac{\delta}{C_D'}
\]
so that every \( \alpha \in \mathcal{B}(\epsilon_0) \) is a contact form, thanks to (3.5) and Lemma A.6. Moreover, if we write \( R_z \) for the Reeb vector field of \( \alpha_z \), using again (3.5), we have

\[
\|R_z - R_{st}\|_{C^2} \leq A_2 C^2 \|\alpha - \alpha_s\|_{C^2}.
\]

(3.6)

Therefore, we just need to estimate the left-hand side of (3.3) against \( \|R_z - R_{st}\|_{C^2} \). We know that \( \dot{\gamma}_z = R_z(\gamma_z) \). Therefore, \( \|\dot{\gamma}_z - R_{st}\|_{C^2} \leq \|R_z - R_{st}\|_{C^2} \). For the higher derivatives, we just observe that \( \|\dot{\gamma}_z\|_{C^0} \) is uniformly bounded by \( 1 + A_2 C^2 \) and

\[
\frac{d}{dt}(\dot{\gamma}_z - R_{st}) = \dot{\gamma}_z = d_{\gamma_z} R_z \cdot \dot{\gamma}_z = d_{\gamma_z}(R_z - R_{st}) \cdot \dot{\gamma}_z;
\]

\[
\frac{d^2}{dt^2}(\dot{\gamma}_z - R_{st}) = d^2_{\gamma_z} R_z(\gamma_z) + d_{\gamma_z} R_z \cdot \dot{\gamma}_z = d^2_{\gamma_z}(R_z - R_{st})(\dot{\gamma}_z, \dot{\gamma}_z) + d_{\gamma_z}(R_z - R_{st}) \frac{d}{dt}(\dot{\gamma}_z - R_{st}).
\]

This shows (3.3). Finally, integrating \( \dot{\phi}_z \) and \( \dot{x}_z \) and using (3.3), we obtain (3.4).

\[\square\]

**Proposition 3.4.** There exist \( C_2 > 0 \), and for all real numbers \( T \) in the interval \((1, 2)\), a radius \( \epsilon_1 = \epsilon_1(T) \in (0, \epsilon_0) \) such that for all \( \alpha \in \mathcal{B}(\epsilon_1) \) the following properties are true:

(i) A periodic orbit \( \gamma \) of \( \Phi^\alpha \) belongs to \( \mathcal{P}_T(\alpha) \) if and only if for all \( z \in \mathcal{Z} \) such that \( \gamma(0) \in \Sigma'_z \), then \( \gamma \) is contained in \( \Sigma_z \) and \( \gamma_{\text{rep}} \) is homotopic to \( \gamma \) within \( \Sigma_z \). In this case, if we set \( \gamma_z := D_z^{-1} \circ \gamma, \quad \gamma_z := D_z^{-1} \circ \gamma, \) there holds

\[
|T(\gamma) - 1| \leq C_2 \|\alpha - \alpha_s\|_{C^2}, \quad \|\gamma_{\text{rep}} - \gamma\|_{C^1} \leq C_2 \|\alpha - \alpha_s\|_{C^2}.
\]

(ii) If \( \alpha' \) is any other contact form such that \( d\alpha' = d\alpha \), then the bijection \( \mathcal{P}(\alpha) \to \mathcal{P}(\alpha') \) of Lemma 3.1 restricts to a period-preserving bijection \( \mathcal{P}_T(\alpha) \to \mathcal{P}_T(\alpha') \).

(iii) The set \( \mathcal{P}_T(\alpha) \) is compact and non-empty.

**Proof.** We claim that item (i) and (ii) hold with

\[
\epsilon_1 := \frac{1}{2C_1} \min\left\{ d_s, 2 - T, T - 1 \right\}, \quad C_2 := 10 \cdot C_1.
\]

Before proving (i), we observe that if a periodic curve \( \gamma \) is contained in \( \Sigma_z \), then \( \gamma = (x_\gamma, \phi_\gamma) \) in the coordinates \( D_z \). Moreover, if \( \hat{\phi}_\gamma : \mathbb{R} \to \mathbb{R} \) is the unique lift of \( \phi_\gamma - \phi_\gamma(0) \) such that \( \hat{\phi}_\gamma(0) = 0 \), then \( \hat{\phi}_\gamma(T(\gamma)) = 1 \) if and only if \( \gamma_{\text{rep}} \) is homotopic to \( \gamma \) within \( \Sigma_z \).

Let us now assume that \( \alpha \in \mathcal{B}(\epsilon_1) \) and that \( \gamma \in \mathcal{P}_T(\alpha) \). Let us take \( z \in \mathcal{Z} \) such that \( \gamma(0) \in \Sigma'_z \). Since \( T < 2 \), inequalities (3.4) and (DF2) imply that \( \gamma \) is contained in \( \Sigma_z \). By (3.3), we see that

\[
\hat{\phi}_\gamma \geq 1 - |1 - \hat{\phi}_\gamma| \geq 1 - \|\dot{\gamma}_z - R_{st}\|_{C^2} > 1 - C_1 \epsilon_1 \geq \frac{1}{2} > 0.
\]

Hence, \( \hat{\phi}_\gamma(T(\gamma)) > 0 \). On the other hand, using (3.4) and the fact that \( \epsilon_1 \leq \frac{2-T}{2C_1} \), we get

\[
\hat{\phi}_\gamma(T(\gamma)) < T(\gamma) + C_1 T \epsilon_1 \leq T + 2C_1 \epsilon_1 \leq 2.
\]

Since \( \hat{\phi}_\gamma(T(\gamma)) \) is an integer, we conclude that \( \hat{\phi}_\gamma(T(\gamma)) = 1 \).
Conversely, we assume that \( \gamma = (x_\gamma, \phi_\gamma) \subset \Sigma_2 \) and that \( \gamma_{\text{rep}} \) is homotopic to \( \tilde{\gamma} \) inside \( \Sigma_2 \) and prove that \( \gamma \in \mathcal{P}_T(\alpha) \). The curve \( \gamma \) is prime since \( \hat{\phi}_\gamma(T(\gamma)) = 1 \) has no non-trivial integer divisor. Substituting \( t = T(\gamma) \) in the second inequality in (3.3) yields
\[
|T(\gamma) - 1| \leq C_1 T(\gamma) \|\alpha - \alpha_*\|_{C^2+}. \tag{3.7}
\]
Using that \( \|\alpha - \alpha_*\|_{C^2+} < \epsilon_1 \), we solve for \( T(\gamma) \) and get \( T(\gamma) < 1/(1 - C_1 \epsilon_1)^{-1} \). This implies that \( T(\gamma) \leq T \) since
\[
1 - C_1 \epsilon_1 \geq 1 - \frac{T - 1}{2} \geq 1 - \frac{T - 1}{T} = \frac{1}{T}.
\]

We suppose that \( \gamma \in \mathcal{P}_T(\alpha) \) and prove the estimates in item (i). The first inequality comes from (3.7) using that \( T(\gamma) < 2 \) and \( C_2 \geq 2C_1 \). For the second inequality, exploiting (3.4) and (3.7) we have
\[
|\gamma_{z,\text{rep}}(s) - \hat{\gamma}_z(s)| \leq |x_\gamma(sT(\gamma))| + |\hat{\phi}_\gamma(sT(\gamma)) - s|
\leq C_1 \|\alpha - \alpha_*\|_{C^2+} T(\gamma) + |\hat{\phi}_\gamma(sT(\gamma)) - sT(\gamma)| + |T(\gamma) - 1|s
\leq C_1 \|\alpha - \alpha_*\|_{C^2+} T(\gamma) + C_1 \|\alpha - \alpha_*\|_{C^2+} T(\gamma) + |T(\gamma) - 1|
\leq 6C_1 \|\alpha - \alpha_*\|_{C^2+}.
\]

The higher derivatives can be bounded through (3.3) and (3.7):
\[
\left\|\frac{d\gamma_{z,\text{rep}}}{ds} - \frac{d\hat{\gamma}_z}{ds}\right\|_{C^2} \leq \left\|T(\gamma)(\dot{x}_\gamma, \dot{\phi}_\gamma)_{\text{rep}} - R_{st}\right\|_{C^2}
\leq T\left\|\left(\dot{x}_\gamma, \dot{\phi}_\gamma - 1\right)_{\text{rep}}\right\|_{C^2} + |T(\gamma) - 1|
\leq T \cdot T^2 \left\|\left(\dot{x}_\gamma, \dot{\phi}_\gamma - 1\right)\right\|_{C^2} + 2C_1 \|\alpha - \alpha_*\|_{C^2+}
\leq 3C_1 \|\alpha - \alpha_*\|_{C^2+} + 2C_1 \|\alpha - \alpha_*\|_{C^2+}.
\]

For every \( \gamma \in \mathcal{P}_T(\alpha) \), there exists \( z \in \mathbb{Z} \) such that \( \gamma(0) \in \Sigma_z \) due to (DF1). Therefore \( \mathcal{P}_T(\alpha) = \mathcal{P}_{T}(\alpha) \) by (i). Item (ii) now follows from Lemma 3.1.

Let us prove (iii). From [Gin87, Section III] or [APB14, Section 3.2], up to shrinking \( \epsilon_1 \), for every \( \alpha \in B(\epsilon_1) \), there exists a differentiable function \( S_{\alpha} : \Sigma \rightarrow \mathbb{R} \) with the following property.

The set \( \text{Crit} S_{\alpha} \) is the union of the supports of the orbits \( \gamma \in \mathcal{P}_T(\alpha) \). Therefore, \( \mathcal{P}_T(\alpha) \) is non-empty as \( \text{Crit} S_{\alpha} \) is non-empty. The set \( \mathcal{P}_T(\alpha) \) is also compact by the Arzelà-Ascoli theorem, as its elements have uniformly bounded period, and \( \Sigma \) is compact.

\[\square\]

3.2 Bringing the Reeb flow to normal form

In this subsection, we show that if \( \alpha \) lies in \( B(\epsilon_1) \) and \( \gamma \in \mathcal{P}_T(\alpha) \), we can suppose that \( \gamma \) is a given flow line of \( R_{st} \), up to rescaling \( \alpha \) and applying a diffeomorphism of \( \Sigma \).

Lemma 3.5. There is a constant \( C_3 > 0 \) with the following property. For all \( z_0, z_1 \in \Sigma \), there exists an \( S^1 \)-equivariant diffeomorphism \( \Psi_{z_0, z_1} : \Sigma \rightarrow \Sigma \) isotopic to the identity with
\[
\Psi_{z_0, z_1}(z_0) = z_1, \quad \Psi_{z_0, z_1}^* \alpha_* = \alpha_*, \quad \|d\Psi_{z_0, z_1}\|_{C^2} \leq C_3, \quad \|d(\Psi_{z_0, z_1}^{-1})\|_{C^2} \leq C_3.
\]

Proof. We start with a local construction. Let \( K_0 : B \rightarrow [0, 1] \) be a function which is equal to 1 in a neighbourhood of \( B' \) and whose support is contained in the interior of \( B \). For every \((x', \phi') \in B' \times S^1\), let \( \phi' \in [0, 1) \) be a lift of \( \phi' \). We define
\[
K_1 : B \rightarrow \mathbb{R}, \quad K_1(x) := \phi' + g_{st}(x, ix').
\]
We let $K_{x'} : B \to \mathbb{R}$ be the function $K_{x'} := K_0 K_1$ and $\Phi_t^X$ the flow on $B \times S^1$ generated by the unique vector field $X$ such that

$$\alpha_{st}(X) = K_{x'} \circ p_{st}, \quad t X d\alpha_{st} = -d(K_{x'} \circ p_{st}).$$

Namely, $K_{x'} \circ p_{st}$ is the contact Hamiltonian of $\Phi_t^X$ according to \cite[Section 2.3]{Gei08}. The vector field $X$ is compactly supported and an application of Moser’s trick shows that

$$(\Phi_t^X)^* \alpha_{st} = \alpha_{st}, \quad \forall t \in \mathbb{R}. \tag{3.8}$$

The flow $\Phi_t^X$ lifts the Hamiltonian flow of the function $K_{x'}$ with respect to $\omega_{st}$ on $B$. Moreover, since the curve $t \mapsto (tx', 0)$ is $\alpha_{st}$-Legendrian and $K_{x'}(tx') = \hat{\phi}'$, we see that

$$\Phi_t^X(0, 0) = (tx', t\hat{\phi'}) \in B' \times S^1, \quad \forall t \in [0, 1].$$

Then, the map $\Psi_{B', (x', \phi')} := \Phi_{x'}^X$ is a compactly supported diffeomorphism of $B \times S^1$ sending $(0, 0)$ to $(x', \phi')$ and there exists a positive constant $C'$, independent of $(x', \phi')$, such that

$$\|d\Psi_{B', (x', \phi')}\|_{C^2} \leq C', \quad \|d(\Psi_{B', (x', \phi')}^{-1})\|_{C^2} \leq C'.$$ \tag{3.9}

This completes the local construction. For the global argument, we observe that there exists $m \in \mathbb{N}^*$ independent of $z_0, z_1$ and a chain of points

$$(z_u) \subset \Sigma, \quad u \in U := \{ju_1 \mid j = 0, \ldots, m\}, \quad u_1 := 1/m$$

such that

$$\forall u \in U \setminus \{1\}, \ \exists y_u \in Z, \quad z_{u+1}, z_{u+u_1} \in \Sigma_{y_u}.$$

We construct $\Psi_{z_0, z_1}$ as the composition of $m$ maps $\Psi_{z_u, z_{u+u_1}} : \Sigma \to \Sigma, \ u \in U \setminus \{1\}$. Consider the trivialisation $\Omega_{y_u} : B \times S^1 \to \Sigma_{y_u}$ and define

$$\Psi_{z_u, z_{u+u_1}} : \Sigma \to \Sigma, \quad \Psi_{z_u, z_{u+u_1}} := \Omega_{y_u} \circ \left(\Psi_{B, \Omega_{y_u}^{-1}(z_{u+u_1})} \circ \Psi_{B, \Omega_{y_u}^{-1}(z_u)}^{-1}\right) \circ \Omega_{y_u}^{-1}.$$ 

The lemma follows from (4.8), (4.9), Lemma 3.2 and (DF3-DF4).

**Definition 3.6.** Let us fix a reference point $z_s \in Z$ with $q_s := p(z_s)$ and define $\gamma_s : S^1 \to \Sigma$ to be the prime periodic orbit of $R_s$ passing through $z_s$ at time 0. We say that a contact form $\alpha$ is **normalised**, if $\gamma_s \in \mathcal{P}(\alpha)$. For every $\epsilon \in (0, \epsilon_0]$, we define the set

$$B_{\epsilon} := \{\alpha \in B(\epsilon) \mid \alpha \text{ is normalised}\}.$$ 

**Definition 3.7.** Let $c$ be a positive number and $\Psi : \Sigma \to \Sigma$ a diffeomorphism. For every contact form $\alpha$ on $\Sigma$, we write $\alpha_{c, \Psi} := \frac{1}{c} \Psi^* \alpha$, so that

$$\text{Volume}(\alpha) = c^2 \text{Volume}(\alpha_{c, \Psi})$$

and we have a bijection

$$
\begin{align*}
P(\alpha) & \quad \longrightarrow \quad \mathcal{P}(\alpha_{c, \Psi}), \\
\gamma & \quad \longmapsto \quad \gamma_{c, \Psi},
\end{align*}
\begin{align*}
\gamma_{c, \Psi}(s) & := (\Psi^{-1} \circ \gamma)(cs), \quad \forall s \in \mathbb{R}, \\
T(\gamma_{c, \Psi}) & = \frac{1}{c} T(\gamma).
\end{align*}
$$

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Proposition 3.8. Let $T$ be a number in $(1, 2)$. For every $\epsilon_2 \in (0, \epsilon_0]$, there is $\epsilon_3 \in (0, \epsilon_0]$ (depending on $\epsilon_2$ and $T$) with the following properties. For all $\alpha \in B(\epsilon_3)$ and all $\gamma \in P_T(\alpha)$, there exists a diffeomorphism $\Psi : \Sigma \to \Sigma$ isotopic to the identity such that

$$aT(\gamma), \Psi \in B_*(\epsilon_2), \quad \gamma T(\gamma), \Psi = \gamma_*.$$  

Moreover, the bijection $P(\alpha) \to P(\alpha T(\gamma), \Psi)$ restricts to a bijection $P(\alpha) \to P(\alpha T(\gamma), \Psi)$.

Proof. Let $\alpha$ be an element of $B(\epsilon_3)$ for some $\epsilon_3 \leq \epsilon_1$ to be determined later on, and let $\gamma$ be a periodic orbit in $P_T(\alpha)$. Here the constant $\epsilon_1$ is given by Proposition 3.4. We apply Lemma 3.5 with $z_0 = z_\ast$ and $z_1 = \gamma(0)$ and get a diffeomorphism $\Psi_1 := \Psi_{z_\ast, \gamma(0)} : \Sigma \to \Sigma$. We abbreviate $\alpha_1 := \Psi_1 \alpha$. By Lemma A.1 with $\mathcal{M}_1 = \Sigma = \mathcal{M}_2$ we get some $C' \geq 1$ depending on $\Sigma$ and $C_3$ such that

$$\|\alpha_1 - \alpha_*\|_{C^{2,+}} \leq C'\|\alpha - \alpha_*\|_{C^{2,+}}. \tag{3.10}$$

The periodic curve $\gamma_1 := \Psi_1^{-1} \circ \gamma$ belongs to $P_T(\alpha)$ and has period $T(\gamma)$. As $\gamma_1(0) = z_\ast$, we have $\gamma_1 = \gamma_*$. If $\epsilon_3 \leq \frac{1}{2} \epsilon_1$, then $\alpha_1 \in B(\epsilon_1)$ and Proposition 3.4 implies that $\gamma_1 \in \Sigma_\ast$ and

$$\|(\gamma_1)_{z_\ast, \text{rep}} - (\gamma_*)_{z_\ast}\|_{C^3} \leq C''\|\alpha - \alpha_*\|_{C^{2,+}}, \quad C'' := C_2 C'.$$  

We write $(\gamma_1)_{z_\ast, \text{rep}} = (x_1, \phi_1)$ in the coordinates given by $\mathcal{D}_{z_\ast}$. If $\epsilon_3$ is small enough, from (3.11), we see that $\|x_1\|_{C_0} < 1/2$ and the map $\phi_1 : S^1 \to S^1$ is a diffeomorphism of degree 1. In particular, there exists a unique map $\Delta \phi_1 : S^1 \to \mathbb{R}$, which lifts $\phi_1 - \text{id}_{S^1}$ and it is still small. We define a diffeomorphism $\Psi_{2, z_*} : B \times S^1 \to B \times S^1$ by

$$\Psi_{2, z_*}(x, s) = \left(x + K(|x|) x_1(s), s + K(|x|) \Delta \phi_1(s)\right), \quad \forall (x, s) \in B \times S^1,$$

where $K : [0, 1] \to [0, 1]$ is a function which is equal to 1 on $[0, 1/2]$ and equal to 0 close to 1. By (3.11), we have

$$\|\Psi_{2, z_*} - \text{id}_{B \times S^1}\|_{C^3} \leq C''\|K\|_{C^1}\|\alpha - \alpha_*\|_{C^{2,+}},$$

which also implies

$$\|d\Psi_{2, z_*}\|_{C^3} \leq 1 + C''\|K\|_{C^1}\|\alpha - \alpha_*\|_{C^{2,+}}. \tag{3.12}$$

Since $\Psi_{2, z_*}$ is compactly supported in the interior of $B \times S^1$, we can define $\Psi_2 : \Sigma \to \Sigma$ as $\Psi_2 := \mathcal{D}_{z_*} \circ \Psi_{2, z_*} \circ \mathcal{D}_{z_*}^{-1}$ inside $\Sigma_\ast$, and as the identity in $\Sigma \setminus \Sigma_\ast$. We have $\Psi_2 \circ \gamma_* = \gamma_1, \text{rep}$, and thanks to Lemma A.2 and (3.12), we see that $\|d\Psi_2\|_{C^2}$ is bounded by a constant depending only on $\Sigma$, $C_\mathcal{D}$ and $C''\|K\|_{C^3} \frac{1}{\epsilon_1}$. Therefore, by Lemma A.1 there is also a constant $C''' > 0$ depending on the same quantities such that

$$\|\Psi_2(\alpha_1 - \alpha_*\|_{C^{2,+}} \leq C''\|\alpha_1 - \alpha_*\|_{C^{2,+}}. \tag{3.13}$$

We define

$$\Psi := \Psi_1 \circ \Psi_2 : \Sigma \to \Sigma, \quad \epsilon_2' := \min\left\{\epsilon_2, \epsilon_1, \frac{1}{C_2} \frac{T - 1}{T + 1}\right\},$$

and prove that $\alpha T(\gamma), \Psi$ belongs to $B_*(\epsilon_2')$, provided $\epsilon_3$ is suitably small. We take

$$\delta_0 := \frac{\epsilon_2'}{(T + 1)C_2\|\alpha_\ast\|_{C^4}}.$$
and let \( \delta_1 > 0 \) be the number associated with \( \delta_0 \) by Lemma A.4. We assume further that
\[
\varepsilon_3 \leq \min \left\{ \frac{\delta_1}{C''\|K\|C_3}, \frac{1}{C_2 T + 1} \right\}.
\]
This implies that \( \|\Psi_{2,z^*} - \id_{B \times S^1}\|_{C^3} \leq \delta_1 \) and we compute
\[
\|\alpha_{T(\gamma),\Psi} - \alpha_*\|_{C^{2,+}} \leq \frac{1}{T(\gamma)} - 1 \|\alpha_*\|_{C^{2,+}} + \frac{1}{T(\gamma)}\|\Psi^*\alpha - \alpha_*\|_{C^{2,+}}.
\]
For the first summand of the right-hand side, we first estimate \( T(\gamma)^{-1} \leq \frac{1}{2}(T + 1) \) and then
\[
\left| \frac{1}{T(\gamma)} - 1 \right| \|\alpha_*\|_{C^{2,+}} \leq \frac{T + 1}{2} C_2\varepsilon_3 \|\alpha_*\|_{C^{2,+}}.
\]
For the second summand, we estimate
\[
\|\Psi^*\alpha - \alpha_*\|_{C^{2,+}} \leq \|\Psi_2^*(\alpha_1 - \alpha_*\|_{C^{2,+}} + \|\Psi_2^*\alpha_* - \alpha_*\|_{C^2}
\leq C''\|\alpha_1 - \alpha_*\|_{C^{2,+}} + C'\|\Psi_{2,z^*}^*\alpha_{st} - \alpha_{st}\|_{C^{2,+}}
\leq C'C''\varepsilon_3 + C'\delta_0 \|\alpha_{st}\|_{C^4},
\]
where we used (5.10), (5.13), and Lemma A.4. Using the definition of \( \delta_0 \) and putting the computations together, we find that
\[
\|\alpha_{T(\gamma),\Psi} - \alpha_*\|_{C^{2,+}} \leq \frac{T + 1}{2} \left( C_2 \|\alpha_*\|_{C^{2,+}} + C'C''\varepsilon_3 \right) + \frac{\varepsilon_3}{2}.\]
The quantity on the right is smaller than \( \varepsilon_2 \leq \varepsilon_2 \), if \( \varepsilon_3 \) is small enough. Finally, we compute \( \gamma_{T(\gamma),\Psi} = \Psi^{-1} \circ \gamma_{\text{rep}} = \Psi_2^{-1} \circ \Psi_1^{-1} \circ \gamma_{\text{rep}} = \Psi_2^{-1} \circ \gamma_{1,\text{rep}} = \gamma_* \).

Let us now deal with the second part of the statement. Let \( \tilde{\gamma} \mapsto \tilde{\gamma}_{T(\gamma),\Psi} \) be the bijection between \( \mathcal{P}(\alpha) \) and \( \mathcal{P}(\alpha_{T(\gamma),\Psi}) \) introduced in Definition 3.7. Let us assume that \( T(\tilde{\gamma}) \leq T \). Since \( \varepsilon_3 \leq \varepsilon_1 \) we can use Proposition 3.4(i), and from \( C_2\varepsilon_3 \leq \frac{T - 1}{T + 1} \), we see that
\[
T(\tilde{\gamma}_{T(\gamma),\Psi}) \leq T(\tilde{\gamma}) \leq \frac{1 + C_2\varepsilon_3}{1 - C_2\varepsilon_3} \leq T.
\]
Assume, conversely, that \( T(\tilde{\gamma}_{T(\gamma),\Psi}) \leq T \). Since \( \varepsilon_2 \leq \varepsilon_1 \), we can use Proposition 3.4(i) and find that
\[
T(\tilde{\gamma}) = T(\tilde{\gamma}_{T(\gamma),\Psi})T(\gamma) \leq (1 + C_2\varepsilon_2')(1 + C_2\varepsilon_3) \leq \frac{2T}{T + 1} \frac{2T}{T + 1} = T \frac{4T}{(T + 1)^2} \leq T.
\]

### 3.3 Preparing the surface of section

Let \( z_* \) be a reference point on \( \Sigma \) with \( q_* := p(z_*) \), as in the previous subsection, and let us abbreviate \( d_* := d_{\Omega}, \mathcal{D}_* := \mathcal{D}_{z_*}, M := M \setminus \{q_*\} \). Let \( e \in H^2_{\text{dir}}(M) \) be minus the real Euler class of \( p \) and recall that
\[
t\Sigma = \langle e, [M]\rangle.
\]
where \( t_\Sigma = |H_1^{tor}(\Sigma; \mathbb{Z})| \) is a positive integer. We define the annuli
\[
\mathbb{A} := [0, a) \times S^1, \quad \mathbb{A}' := [0, a/2) \times S^1.
\]

We consider the open annulus \( \hat{\mathbb{A}} = (0, a) \times S^1 \) with inclusion \( i_1: \hat{\mathbb{A}} \to \mathbb{A} \) and the map
\[
i_2: \hat{\mathbb{A}} \to \hat{M}, \quad i_2(r, \theta) = d_* \left(r e^{2\pi i \theta}\right),
\]
where we identify the domain of \( d_* \) with a subset of the complex plane. We glue together \( \mathbb{A} \) and \( \hat{M} \) along the maps \( i_1 \) and \( i_2 \) to get a smooth compact surface \( \mathcal{N} \) with the same genus as \( M \) and one boundary component denoted by \( \partial \mathcal{N} \). Namely, we have the following commutative diagram
\[
\begin{array}{ccc}
\hat{\mathbb{A}} & \xrightarrow{i_2} & \hat{M} \\
i_1 & & \downarrow \\
\mathbb{A} & \xrightarrow{i_1} & \mathcal{N}
\end{array}
\]
so that \( \hat{M} \) is diffeomorphic to \( \hat{\mathcal{N}} = N \setminus \partial N \) and \( \mathbb{A} \) to a collar neighbourhood of \( \partial N \). On \( \hat{M} \) we have the orientation given by \( \omega_* \), while on \( \mathbb{A} \) the one given by \( dr \wedge d\theta \). These two orientations glue together to an orientation of \( \mathcal{N} \), since \( i_1 \) and \( i_2 \) are orientation preserving. Using the usual convention of putting the outward normal first, we see that the orientation induced on \( \partial \mathcal{N} \) is given by \( -d\theta \). As for \( M \) and \( \Sigma \), we fix on \( \mathcal{N} \) some auxiliary Riemannian metric to compute norms of sections, and distances between points and between diffeomorphisms. In particular, we write the \( C^1 \)-distance on the space of diffeomorphisms form \( \mathcal{N} \) to itself as
\[
dist_{C^1}: \text{Diff}(\mathcal{N}) \times \text{Diff}(\mathcal{N}) \to \mathbb{R}.
\]

Consider now the map
\[
S: \mathbb{A} \to \Sigma, \quad S_\theta = D_* (r e^{2\pi i \theta}, -t_\Sigma \theta)
\]
and observe that, for all \( \theta \in S^1 \), there holds \( S_\theta(0, \theta) = \gamma_* (-t_\Sigma \theta) \), so that
\[
d_\theta S_\theta \cdot \partial \theta = -t_\Sigma R_*.
\]

The map \( S_{\hat{M}} \circ i_2^{-1}: i_2(\hat{\mathbb{A}}) \to \Sigma \) is a local section of the bundle \( \mathfrak{p} \) with a singularity of order \( -t_\Sigma \) at \( q_* \). Since \( -t_\Sigma \) is the Euler number of \( \mathfrak{p} \), this section extends to a section on \( \hat{M} \) and yields a map \( S_{\hat{M}}: \hat{M} \to \Sigma \). By the commutativity of the diagram above, we get a map \( S: N \to \Sigma \) fitting into the diagram

Moreover, \( S_{\hat{M}}^* \Omega_* = (\mathfrak{p} \circ S_{\hat{M}})^* \omega_* = \omega_* \), and \( S_{\hat{M}}^* \Omega_* = r dr \wedge d\theta \). In particular, \( S^* \Omega_* \) is a two-form on \( N \), which is symplectic on the interior of \( N \) and vanishes of order 1 at the boundary of \( N \).

The one-form
\[
\lambda_* := S^* \alpha_*
\]
is a primitive for $S^*\Omega_s$ such that

$$\lambda_s|_{\mathbb{A}} = (\mathfrak{D}^{-1} \circ S_{\mathbb{A}})^*(d\phi + p_{st}\lambda_{st}) = (-t\Sigma + \frac{1}{2}r^2)d\theta.$$  

If $\alpha$ is a normalised form, so that $R_\alpha = R_s$ on $p^{-1}(q_s)$, we set

$$\lambda := S^*\alpha,$$

and by equation (3.14), we have

$$\lambda|_{\partial N} = \lambda_s|_{\partial N}, \quad \lambda = 0 \text{ at } \partial N.$$  

(3.15)

**Proposition 3.9.** For all $\epsilon_4 > 0$, there exists a number $\epsilon_5 \in (0, \epsilon_0]$ such that, if $\alpha \in \mathcal{B}_s(\epsilon_5)$, there exist a map $\zeta : N \to N$ isotopic to the identity and a function $b : N \to \mathbb{R}$ satisfying the following properties.

(i) **Triviality at the boundary:** $\zeta|_{\partial N} = \text{id}_{\partial N}, \quad b|_{\partial N} = 0,$

(ii) **$C^1$-smallness:** $\text{max} \{\text{dist}_{C^1}(\zeta, \text{id}_N), \|b\|_{C^1}\} < \epsilon_4,$

(iii) **Uniformisation:** $\zeta^*\lambda - \lambda_s = db.$

**Proof.** Let $\alpha \in \mathcal{B}_s(\epsilon_5)$, for some $\epsilon_5 \in (0, \epsilon_0]$ to be determined. For all $u \in [0, 1]$, we define $\lambda_u := \lambda_s + u(\lambda - \lambda_s)$. On $\mathbb{A}$, we get

$$\lambda - \lambda_s = c_1 dr + c_2 d\theta, \quad d\lambda_s = rdr \wedge d\theta, \quad d\lambda = fdr \wedge d\theta, \quad d\lambda_u = (r + u(f - r))dr \wedge d\theta,$$

for some functions $c_1, c_2, f : \mathbb{A} \to \mathbb{R}$. By (3.15), we have $c_2(0, \theta) = 0$ and $f(0, \theta) = 0$. Define the auxiliary function

$$c_3 : \mathbb{A} \to \mathbb{R}, \quad c_3(r, \theta) := c_2(r, \theta) - \int_0^r \partial_{r'} c_1(r', \theta)dr'.$$

From the definition of $c_1, c_2, c_3$ and $f$, we have the chain of identities

$$(\partial_r c_3)dr \wedge d\theta = (\partial_r c_2 - \partial_{r'} c_1)dr \wedge d\theta = d(\lambda - \lambda_s) = d\lambda - d\lambda_s = (f - r)dr \wedge d\theta,$$

which implies

$$\partial_r c_3 = f - r.$$

As a result, $c_3(0, \theta) = 0, \partial_r c_3(0, \theta) = 0$ and there exists a function $\hat{c}_3 : \mathbb{A} \to \mathbb{R}$ with $c_3 = r\hat{c}_3, \hat{c}_3|_{\partial N} = 0$, and a function $\hat{f} : \mathbb{A} \to \mathbb{R}$ with $f = rf$, defined by

$$\hat{c}_3(r, \theta) := \int_0^1 \partial_r c_3(vr, \theta)dv = \int_0^1 (f(vr, \theta) - vr)dv, \quad \hat{f}(r, \theta) := \int_0^1 \partial_r f(vr, \theta)dv.$$

In particular,

$$d\lambda_u = r(1 + u(\hat{f} - 1))dr \wedge d\theta$$

and we have the estimate

$$\text{max} \{\|\hat{c}_3\|_{C^2}, \|\hat{f} - 1\|_{C^1}\} \leq \|f - r\|_{C^2}.$$  

(3.17)
We now look for paths $u \mapsto \zeta_u$ and $u \mapsto b_u$ with $\zeta_0 = \text{id}_N$ and $b_0 = 0$ such that

$$
\zeta^*_u \lambda_u - dB_u = \lambda_u
$$

so that, for $u = 1$, we get a solution to item (iii) in the statement. Let $X_u$ denote the vector field generating $\zeta_u$ and set $a_u := \frac{d}{du} b_u$. By differentiating the equation above with respect to $u$, we find that such an equation can be solved for $\zeta_u$ and $b_u$ if and only if

$$(\lambda - \lambda_s) + \iota X_u d\lambda_u + d(\lambda_u(X_u)) - d(a_u \circ \zeta_u^{-1}) = 0.$$ 

Introducing an auxiliary function $h : N \to \mathbb{R}$, we see that $(X_u, a_u)$ is a solution if and only if

$$
\begin{align*}
\iota X_u d\lambda_u &= - (\lambda - \lambda_s) + dh, \\
a_u &= (\lambda_u(X_u) + h) \circ \zeta_u.
\end{align*}
$$

We choose $h := h_{\lambda} \cdot K$, where $K : N \to [0, 1]$ is a bump function which is equal to 1 on $\mathcal{A}'$ and its support is contained in $\mathcal{A}$, and $h_{\lambda} : \mathcal{A} \to \mathbb{R}$ is defined by

$$
h_{\lambda}(r, \theta) := \int_0^r c_1(r', \theta)dr'.
$$

This function has the crucial property that

$$
\lambda - \lambda_s - dh = c_1 dr + c_2 d\theta - c_1 dr - \left(\int_0^r \partial r c_1(r', \theta)dr'\right)d\theta = r \hat{c}_3 d\theta \quad \text{on } \mathcal{A}',
$$

which implies that the first equation in (3.18) admits a smooth solution $X_u$. Indeed, on the annulus $\mathcal{A}'$, we divide both sides of the equation by $r$ and using (3.16), (3.19), we get

$$
X_u = - \frac{\hat{c}_3}{1 + u(f - 1)} \partial r.
$$

On $N \setminus \mathcal{A}'$, $X_u$ is uniquely determined by the fact that $d\lambda_u|_{N \setminus \mathcal{A}'}$ is symplectic. The vector $X_u$ vanishes at $\partial N$, since $\hat{c}_3$ vanishes there, as observed before. This shows that $\zeta_u$ is the identity at the boundary. If we choose $\epsilon_5$ small, we see that the $C^2$-norm of $c_1$, $c_2$, and $f - r$ are small, and consequently, also the $C^2$-norm of $h$. By (3.17), we conclude that the $C^1$-norm of $X_u$ is small, as well. As a consequence, also $\text{dist}_{C^1}(\zeta_u, \text{id}_N)$ is small. Therefore, by defining $\zeta := \zeta_1$ and taking $\epsilon_5$ small enough, we get $\text{dist}_{C^1}(\zeta, \text{id}_N) < \epsilon_4$. We can now define $a_u$ through the second equation in (3.18). From the estimates on $\lambda, X_u, \zeta_u$ and $h$, we see that, if $\epsilon_5$ is small, $\|a_u\|_{C^1} < \epsilon_4$ and the same is true for $b := b_1$. Since $h$ and $X_u$ vanish at the boundary, we also have $a_u|_{\partial N} = 0$, and as $b_0|_{\partial N} = 0$, the function $b$ vanishes at the boundary, as well.

**3.4 The open book decomposition and the first return map**

Combining the map $S$ with the Reeb flow of $\alpha_s$, we get a rational open book for $\Sigma$:

$$
\Xi : N \times S^1 \longrightarrow \Sigma
$$

$$(q, s) \longmapsto \Phi_s^{\omega}(S(q)).$$

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If \( i_N : N \hookrightarrow N \times S^1 \) is the canonical embedding \( i_N(x) = (x, 0) \), then \( S = \Xi \circ i_N \). On the collar neighbourhood \( \mathcal{A} \times S^1 \) of \( \partial(N \times S^1) \), \( \Xi \) has the coordinate expression

\[
\Xi : \mathcal{A} \times S^1 \longrightarrow B \times S^1
\]

\[
((r, \theta), s) \longmapsto (re^{2\pi i \theta}, s - t\Sigma \theta).
\]  

(3.20)

The restricted map \( \hat{\Xi} : \tilde{N} \times S^1 \rightarrow p^{-1}(\tilde{M}) = \Sigma \setminus p^{-1}(q_s) \) is a diffeomorphism, and \( \hat{\Xi}^* \alpha_s \) is a contact form on \( \tilde{N} \times S^1 \) with Reeb vector field \( R_{\hat{\Xi}^* \alpha_s} = \partial_s \), which smoothly extends to the whole \( N \times S^1 \). We write \( i_{\partial N \times S^1} : \partial N \times S^1 \rightarrow N \times S^1 \) for the standard embedding of the boundary and set \( \Xi_{\beta} := \Xi \circ i_{\partial N \times S^1} : \partial N \times S^1 \rightarrow p^{-1}(q_s) \). This map has the coordinate expression \((\theta, s) \mapsto (s - t\Sigma \theta)\), so that

\[
d\Xi_{\beta} \cdot \partial_\theta = -t\Sigma R_s, \quad d\Xi_{\beta} \cdot \partial_s = R_s.
\]  

(3.21)

If \( \alpha \) is a normalised contact form, we define the pull-back forms

\[
\beta := \Xi^* \alpha, \quad \beta_{\theta} := \Xi^*_{\beta} \alpha.
\]

Proposition 3.10. If \( \alpha \) is a normalised contact form, then

(i) At every point on the boundary of \( N \times S^1 \), we have

\[
\beta_{\theta} = ds - t\Sigma d\theta, \quad d\beta = 0.
\]

(ii) The Reeb vector field \( R_{\hat{\Xi}^* \alpha} \) of \( \hat{\Xi}^* \alpha \) on \( \tilde{N} \times S^1 \) smoothly extends to a vector field \( R_{\beta} \) on the whole \( N \times S^1 \), so that, at every point in \( \partial(N \times S^1) \), \( R_{\beta} \) is tangent to \( \partial(N \times S^1) \).

(iii) If we denote by \( \Phi_{\beta}^t \) the flow of \( R_{\beta} \), we have

\[
\beta(R_{\beta}) = 1, \quad t_{R_{\beta}} d\beta = 0, \quad (\Phi_{\beta}^t)^* \beta = \beta, \quad \forall t \in \mathbb{R}.
\]

(iv) For every \( \epsilon > 0 \), there exists \( \epsilon_7 \in (0, \epsilon] \), independent of \( \alpha \), such that

\[
\alpha \in \mathcal{B}_s(\epsilon_7) \implies \|R_{\beta} - \partial_\beta\|_{C^1} < \epsilon_6.
\]

Proof. By (3.21), equation \( \alpha(R_{\alpha}) = 1 \), and the fact that \( R_{\alpha} = R_s \) on \( p^{-1}(q_s) \) as \( \alpha \) is normalised, we get the first equality in item (i). Since \( \partial N \times S^1 \) has co-dimension 1 in \( N \times S^1 \), to prove the second equality it is enough to show that for all vectors \( v \in T(\partial N \times S^1) \), we have \( t_v d\beta = 0 \). As \( v \) is a linear combination of \( \partial_\theta \) and \( \partial_s \), this follows again from (3.21) and the fact that \( R_{\alpha} \) annihilates \( \partial_\theta \).

Now we prove (ii). We set \( \alpha_{\beta} := \mathcal{D}_{\beta}^\alpha \alpha \), which is a contact form on \( B \times S^1 \) with corresponding Reeb vector field \( R_{\beta} \). Using coordinates \((x, \phi) \in B \times S^1 \), we have the splitting

\[
R_{\beta}(x, \phi) = R^x_{\beta}(x, \phi) + R^\phi_{\beta}(x, \phi) \partial_\phi.
\]

Since \( R_{\alpha} \) is tangent to \( p^{-1}(q_s) \), there holds \( R^x_{\alpha}(0, \phi) = 0 \), and therefore, there exists a matrix-valued function \( W_{\beta} \) such that

\[
R^x_{\beta}(x, \phi) = W_{\beta}(x, \phi) \cdot x, \quad \|W_{\beta}\|_{C^1} \leq \|R^x_{\beta}\|_{C^2}.
\]
by Lemma 4.12. We can then write $R^x_z$ in polar coordinates on $(B \setminus \{0\}) \times S^1$ as

$$R^x_z(re^{2\pi i \theta}, \phi) = g_{st}\left(W_{z+}(re^{2\pi i \theta}, \phi) \cdot r e^{2\pi i \theta} \right) \partial_r + g_{st}\left(W_{z+}(re^{2\pi i \theta}, \phi) \cdot r e^{2\pi i \theta}, \frac{ie^{2\pi i \theta}}{r} \right) \partial_\theta.$$ 

In particular, if we set

$$\begin{cases} R^r_z(r, \theta, \phi) := g_{st}\left(W_{z+}(re^{2\pi i \theta}, \phi) \cdot r e^{2\pi i \theta} \right), \\ R^\theta_z(r, \theta, \phi) := g_{st}\left(W_{z+}(re^{2\pi i \theta}, \phi) \cdot r e^{2\pi i \theta}, \frac{ie^{2\pi i \theta}}{r} \right), \end{cases}$$

then $R^r_z \circ \Xi_\kappa$ and $R^\theta_z \circ \Xi_\kappa$ are smooth functions on $\mathcal{A} \times S^1 \subset N \times S^1$ with

$$\max \left\{ \|R^r_z \circ \Xi_\kappa\|_{C^1}, \|R^\theta_z \circ \Xi_\kappa\|_{C^1} \right\} \leq (1 + \|d\Xi_\kappa\|_{C^0}) \|R^x_z\|_{C^2}. \quad (3.22)$$

Differentiating formula (3.20), we get

$$d\Xi_\kappa \cdot \partial_r = \partial_r, \quad d\Xi_\kappa \cdot \partial_\theta = \partial_\theta - t\Sigma \partial_\phi, \quad d\Xi_\kappa \cdot \partial_\theta = \partial_\phi.$$

Thus, we conclude that

$$R_\beta := (R^r_z \circ \Xi_\kappa) \partial_r + (R^\theta_z \circ \Xi_\kappa) \partial_\theta + (t\Sigma R^\theta_z \circ \Xi_\kappa + R^\phi_z \circ \Xi_\kappa) \partial_\phi$$

is the desired extension of $R^x_z\vert_\mathcal{O}$ in the collar neighbourhood $\mathcal{A} \times S^1$ of $\partial N \times S^1$. As $R^x_z \circ \Xi_\kappa$ vanishes at $r = 0$, the extended vector field is tangent to $\partial N \times S^1$ and (ii) is proven.

By the very definition of the Reeb vector field, $(\tilde{\Xi}^* \alpha)(R^x_z \alpha) = 1$ and $t_{R^x_z \alpha} d(\tilde{\Xi}^* \alpha) = 0$. By continuity of the extended vector field $R_\beta$, the first two relations in item (iii) follow. The third one is a consequence of the first two and Cartan’s formula. Point (iii) is established.

We assume that $\alpha \in B_\epsilon(\epsilon_7)$, for some $\epsilon_7$ to be determined independently of $\alpha$ and we prove (iv) by estimating $\|R_\beta - \partial_\theta\|_{C^1}$ separately on $N \setminus \mathcal{A} \times S^1$ and $\mathcal{A} \times S^1$. By Lemma 4.1 with $\mathcal{M} = (N \setminus \mathcal{A}) \times S^1$, there exists a constant $C' > 0$ depending on $\|d\tilde{\Xi}\|_{C^2}$ and $(N \setminus \mathcal{A}) \times S^1$ but not on $\alpha$ such that

$$\|\tilde{\Xi}^* \alpha - \tilde{\Xi}^* \alpha\|_{C^2, +} \leq C'\|\alpha - \alpha\|_{C^2, +}.$$ 

Therefore, by Lemma 4.6, $\|R_\beta - \partial_\theta\|_{C^2}$ is smaller than $\epsilon_6$ on $(N \setminus \mathcal{A}) \times S^1$, if $\epsilon_7$ is small enough. On $\mathcal{A} \times S^1$, there is some $C'' > 0$ for which we have the inequality

$$\|R_\beta - \partial_\theta\|_{C^1} \leq C'' \max \left\{ \|R^r_z \circ \Xi_\kappa\|_{C^1}, \|R^\theta_z \circ \Xi_\kappa\|_{C^1}, \|R^\phi_z \circ \Xi_\kappa - 1\|_{C^1} \right\} \leq C''(1 + \|d\Xi_\kappa\|_{C^0}) \|R_z - R_{st}\|_{C^2} \leq C''(1 + \|d\Xi_\kappa\|_{C^0}) A_2 C_{D}' \|\alpha - \alpha\|_{C^2, +},$$

where we have used (3.22), the equality $R_{st} = \partial_\phi$ and inequality (3.6). This proves that $\|R_\beta - \partial_\theta\|_{C^1}$ is smaller than $\epsilon_6$ on $\mathcal{A} \times S^1$, if $\epsilon_7$ is small enough.}

We can now show that $S$ is a global surface of section for $\Phi^\alpha$ with certain properties.

**Definition 3.11.** Let $\Phi$ be a flow on $\Sigma$ without rest points and $N_1$ a compact surface. A map $S_1 : N_1 \to \Sigma$ is a **global surface of section** for $\Phi$ if the following properties hold:
• The map $S_1|_{\tilde{N}_1}$ is an embedding and the map $S_1|_{\partial N_1}$ is a finite cover onto its image;

• The surface $S_1(\tilde{N}_1)$ is transverse to the flow $\Phi$ and $S_1(\partial N_1)$ is the support of a finite collection of periodic orbits of $\Phi$;

• For each $z \in \Sigma \setminus S_1(\partial N_1)$, there are $t_- < 0 < t_+$ such that $\Phi_{t_-}(z), \Phi_{t_+}(z)$ lie in $S_1(\tilde{N}_1)$.

Before stating the proposition, we need to introduce the following notation.

**Definition 3.12.** Let $q \in \partial N \cong S^1$ and denote by $-q \in \partial N$ its antipodal point. For every $q' \in \partial N \setminus \{-q\}$, we denote by

$$\lambda_*(q, q') := \int_q^{q'} \lambda_*$$

the integral of $\lambda_*$ over any path connecting $q$ and $q'$ within $\partial N \setminus \{-q\}$. Such a number does not depend on the choice of path.

**Proposition 3.13.** Let $T$ be a real number in the interval $(1, 2)$. For all $\epsilon_8 > 0$, there exists $\epsilon_9 \in (0, \epsilon_0]$ with the following properties. If $\alpha \in \mathcal{B}_+(\epsilon_9)$, then $S : N \to \Sigma$ is a global surface of section for $\Phi^\alpha$ with the first return time admitting an extension to the boundary

$$\tau : N \to \mathbb{R}, \quad \tau(q) := \inf \{t > 0 \mid \Phi^\alpha_t(q, 0) \in N \times \{0\}\}$$

and the first return map admitting an extension to the boundary

$$P : N \to N, \quad (P(q), 0) := \Phi^\beta_{\tau(q)}(q, 0).$$

Moreover, the following properties hold.

- (i) $C^1$-smallness: $\max \{\text{dist}_{C^1}(P, \text{id}_N), \|\tau - 1\|_{C^1} \} < \epsilon_8$,

- (ii) Normalisation: $\tau(q) = 1 + \lambda_* (q, P(q))$, $\forall q \in \partial N$,

- (iii) Exactness: $P^* \lambda = \lambda + d\tau$,

- (iv) Volume: $\text{Volume}(\alpha) = \int_N \tau \ d\lambda$,

- (v) Fixed points: $q \in \tilde{N} \implies \left[q \in \text{Fix}(P) \iff \gamma_q(t) := \Phi^\beta_t(S(q)) \in \mathcal{P}_T(\alpha)\right]$,

- (vi) Period: $q \in \tilde{N} \cap \text{Fix}(P) \implies T(\gamma_q) = \tau(q),$

- (vii) Zoll case: if $\alpha$ is Zoll, then $P = \text{id}_N$.

**Proof.** Let $\epsilon_6 \in (0, 1)$, which we will take small enough depending on $\epsilon_8$, and let $\epsilon_7$ be the number associated with $\epsilon_6$ in Proposition 3.10. If $\alpha \in \mathcal{B}_+(\epsilon_7)$, then $1 - \epsilon_6 < d\lambda(R_\beta) < 1 + \epsilon_6$. This implies at once that $S : N \to \Sigma$ is a global surface of section. In particular, if $q \in N$, there exists a smallest positive time $\tau(q)$ such that $\Phi^\beta_{\tau(q)}(q, 0)$ belongs to $N \times \{0\}$. We estimate the return time more precisely as $(1 + \epsilon_6)^{-1} < \tau(q) < (1 - \epsilon_6)^{-1}$. In particular, if $\epsilon_6$ is small enough, there holds

$$\max \tau < T < 2 \min \tau.$$  \hspace{1cm} (3.23)

Shrinking $\epsilon_6$ further, if necessary, we also get $\|\tau - 1\|_{C^1} < \epsilon_8$ from Proposition 3.10(iv).
We define the return point \( P(q) \) by the equation \((P(q), 0) = \Phi^\beta_\tau(q, 0)\). Again by Proposition 3.10(iv), we can achieve \( \text{dist}_{C^1}(P, \id_N) < \epsilon_\delta \), if \( \epsilon_\delta \) is small enough, so that item (i) is established. Let \( q \in \overline{N} \) and let us prove the statement in square brackets in item (v). If \( \gamma_q \) is prime, since intersects \( S(N) \) only once, and has period \( \tau(q) \). By 3.12, we have \( \tau(q) \leq T \). Suppose conversely, that \( \gamma_q \) has period \( T(\gamma_q) \leq T \). If \( q \neq P(q) \), then we would get the contradiction

\[ T(\gamma_q) \geq \tau(q) + \tau(P(q)) \geq 2 \min \tau T. \]

This establishes item (v) and (vi), at once. Let us assume that \( \alpha \) is Zoll. Since \( \gamma_* \in \mathcal{P}(\alpha) \), then, if \( q \in \overline{N} \), the orbit \( \gamma_q \) is prime and satisfies

\[ T(\gamma_q) = T(\gamma_*) = 1 < T. \]

By item (v), we conclude that \( q \in \text{Fix}(\overline{P}) \). This shows \( \overline{N} \subset \text{Fix}(\overline{P}) \), and by continuity \( \text{Fix}(\overline{P}) = N \). Namely, \( P = \id_N \) and item (vii) holds.

Let \( q \in \partial N \) and denote by \( \delta_q : [0, \tau(q)] \rightarrow \partial N \times S^1 \) the curve \( \delta_q(t) = \Phi^\beta_t q, 0 \). Using coordinates \((\theta, s)\) on \( \partial N \times S^1 \), we can write \( \delta_q(s) = (\theta_q(t), s_q(t)) \), so that \( \theta_q : [0, \tau(q)] \rightarrow \partial N \) is a path between \( \theta_q(0) = q \) and \( \theta_q(\tau(q)) = P(q) \), and \( s_q(0) = 0, s_q(\tau(q)) = 1 \). We compute

\[
\tau(q) = \int_0^{\tau(q)} dt = \int_0^{\tau(q)} \delta_q^* \beta_\theta = \int_0^{\tau(q)} \delta_q^* (ds - t \Sigma d\theta) = \int_0^{\tau(q)} (ds_q + \theta_q^*(-t \Sigma d\theta)) \\
= 1 + \int_0^{\tau(q)} \theta_q^* \lambda_*
\]

and the integral of \( \lambda_* \) over \( \theta_q \) is equal to \( \lambda_* (q, P(q)) \), as \( \theta_q \) is short if \( \epsilon_\delta \) is small enough. This establishes item (ii). Therefore, we can choose \( \epsilon_q := \epsilon_\tau \) in the statement of the corollary.

We prove now item (iii) and (iv) by considering the map

\[ Q : [0, 1] \times N \rightarrow N \times S^1, \quad Q(t, q) := \Phi^\beta_{t \tau(q)} (i_N(q)). \]

Its differential is given by

\[ d_{(t, q)} Q = d_{(t, q)} (t \tau) \otimes R_\beta(Q(t, q)) + d_{i_N(q)} \Phi^\beta_{t \tau(q)} \cdot d_q i_N. \]

Hence, using Proposition 3.10(iii) we compute

\[
Q^* \beta = \beta_Q \left( d(t \tau) \otimes R_\beta(Q) + d_{i_N} \Phi^\beta_{t \tau} \cdot d_i N \right) = d(t \tau) \beta(R_\beta) + (\Phi^\beta_{t \tau} \circ i_N)^* \beta \\
= d(t \tau) + i_N^* (\Phi^\beta_{t \tau})^* \beta \\
= d(t \tau) + i_N^* \beta \\
= d(t \tau) + \lambda.
\]

We define \( i_1 : N \rightarrow [0, 1] \times N \) by \( i_1(q) = (1, q) \) and observe that \( Q \circ i_1 = i_N \circ P \). Therefore,

\[ P^* \lambda = P^* i_N^* \beta = (i_N \circ P)^* \beta = (Q \circ i_1)^* \beta = i_1^* Q^* \beta = i_1^* (d(t \tau) + \lambda) = 1d \tau + \lambda. \]
This establishes item (iii). We calculate the volume of \( \alpha \) pulling back by \( \Xi \circ Q \):

\[
\text{Volume}(\alpha) = \int_{N \times S^1} \beta \wedge d\beta = \int_{[0,1] \times N} \left( d(t \tau) + \lambda \right) \wedge d\lambda = \int_{[0,1] \times N} d(t \tau) \wedge d\lambda = \int_{0,1 \times N} d(t \tau) d\lambda = \int_N 1 \tau d\lambda - \int_N 0 \tau d\lambda,
\]

which yields item (iv). \hfill \Box

### 3.5 Reduction to a two-dimensional problem

Putting together all the results of this section, we are able to translate the systolic-diastolic inequality into a statement for maps on \( N \). We recall the set-up. Let \( \alpha_* \) be a Zoll contact form on a closed three-manifold \( \Sigma \) with associated bundle number, which will be determined in the course of the proof depending on \( \lambda \). Section 3.3 and in Proposition 3.13. Let \( \lambda = S^* \alpha_* \) and remember that \( t_\Sigma = \langle e, [M] \rangle \).

**Theorem 3.14.** For any \( T \in (1,2) \) and \( \epsilon_{10} > 0 \), there is \( \epsilon_{11} > 0 \) such that for all contact forms \( \alpha' \) with \( ||d\alpha' - d\alpha_*||_{C^2} < \epsilon_{11} \), the set \( \mathcal{P}_T^p(\alpha') \) is compact and non-empty. Moreover for every \( \gamma \in \mathcal{P}_T^p(\alpha') \), there exist a diffeomorphism \( \varphi : N \to N \), and a function \( \sigma : N \to \mathbb{R} \) with the following properties.

1. **C^1-smallness:** \( d(\varphi, \text{id}_N)_{C^1} < \epsilon_{10} \),
2. **Normalisation:** \( \sigma(q) = \lambda_*(q, \varphi(q)) \), \( \forall q \in \partial N \).
3. **Exactness:** \( \varphi^* \lambda_* = \lambda_* + d\sigma \),
4. **Volume:** \( \text{Vol}(\alpha') - t_\Sigma T(\gamma)^2 = T(\gamma)^2 \int_N \sigma d\lambda_* \),
5. **Fixed points:** There is a map \( \tilde{N} \cap \text{Fix}(\varphi) \to \mathcal{P}_T^p(\alpha') \), \( q \mapsto \gamma_q \) such that \( T(\gamma_q) = T(\gamma)(1 + \sigma(q)) \).
6. **Zoll case:** if \( \alpha' \) is Zoll, then \( \varphi = \text{id}_N \).

**Proof.** Let \( C_0 \) be the constant given by Lemma 3.2 and let \( \epsilon_{11} \leq \frac{1}{C_0^9} \epsilon_1 \) be some positive real number, which will be determined in the course of the proof depending on \( T \) and \( \epsilon_{10} \). If \( \alpha' \) is a contact form with \( ||d\alpha' - d\alpha_*||_{C^2} < \epsilon_{11} \), then Lemma 3.2 yields a contact form \( \alpha \in \mathcal{B}(C_0 \epsilon_{11}) \) with \( d\alpha = d\alpha' \). Since \( C_0 \epsilon_{11} \leq \epsilon_1 \), by Proposition 3.4(ii), we have a period-preserving bijection \( \mathcal{P}_T^p(\alpha') \to \mathcal{P}_T^p(\alpha) \) and by Proposition 3.4(iii) the set \( \mathcal{P}_T^p(\alpha') \) is compact and non-empty.

We fix henceforth an element \( \gamma \in \mathcal{P}_T^p(\alpha') \). If \( \epsilon_2 \in (0, \epsilon_1] \) is an auxiliary number, we can find a corresponding \( \epsilon_3 \in (0, \epsilon_0] \) according to Proposition 3.3 so that, if \( \epsilon_{11} \leq \epsilon_3 \) there exists a diffeomorphism \( \Psi : \Sigma \to \Sigma \) such that \( \alpha T(\gamma), \Psi \in \mathcal{B}_\epsilon(\epsilon_2) \) and the map \( \gamma \mapsto \gamma T(\gamma), \Psi \) of Definition 3.7 restricts to a bijection \( \mathcal{P}_T(\alpha) \to \mathcal{P}_T(\alpha T(\gamma), \Psi) \). Thus, we get a bijection

\[
\mathcal{P}_T^p(\alpha') \to \mathcal{P}_T(\alpha, \Psi), \quad \gamma' \mapsto \gamma T(\gamma), \Psi \quad T(\gamma') = T(\gamma) T(\gamma T(\gamma), \Psi). \tag{3.24}
\]
We choose now an auxiliary $\epsilon_4 > 0$ and get a corresponding $\epsilon_5 \in (0, \epsilon_0]$ from Proposition 3.9 so that if $\epsilon_2 \leq \epsilon_5$, then there exist $\zeta : N \to N$ and $b : N \to \mathbb{R}$ associated with $\alpha_{T(\gamma), \varphi} \in \mathcal{B}_s(\epsilon_2)$ satisfying the properties contained therein. Finally, let $\epsilon_8 > 0$ be another auxiliary number and consider $\epsilon_9 \in (0, \epsilon_0]$, the number given by Proposition 3.13 so that, if $\epsilon_2 \leq \epsilon_9$, the statements contained therein hold for $\alpha_{T(\gamma), \varphi}$, the associated return time $\tau : N \to \mathbb{R}$ and return map $P : N \to N$.

Now we set

- $\varphi : N \to N$, $\varphi := \zeta^{-1} \circ P \circ \zeta$,
- $\sigma : N \to \mathbb{R}$, $\sigma := \tau \circ \zeta - b \circ \varphi + b - 1$.

First of all, we observe that by choosing $\epsilon_8$ and $\epsilon_4$ small enough, we obtain item (i). Then, we have $\varphi|_{\partial N} = P|_{\partial N}$ and $\sigma|_{\partial N} = \tau|_{\partial N} - 1$, so that item (ii) follows from Proposition 3.13(ii).

As far as item (iii) is concerned, we compute

$$\varphi^* \lambda_\epsilon = \zeta^* P^* \lambda - \varphi^* d\sigma = \zeta^*(\lambda + d\tau) - d(b \circ \varphi) = \lambda_\epsilon + d(b + \tau \circ \zeta - b \circ \varphi) = \lambda_\epsilon + d\sigma.$$ 

For item (iv), we recall from Lemma 3.11, Definition 3.7 and Proposition 3.13(iv) that

$$\text{Volume}(\alpha') = \text{Volume}(\alpha) = T(\gamma)^2 \text{Volume}(\alpha_{T(\gamma), \varphi}) = T(\gamma)^2 \int_N \tau d\lambda,$$

and we will show that

$$\int_N \tau d\lambda = \int_N \sigma d\lambda_\epsilon + t_\Sigma. \quad (3.25)$$

We can compute the integral of $\sigma d\lambda_\epsilon$ as

$$\int_N \sigma d\lambda_\epsilon = \int_N (\tau \circ \zeta) d\lambda_\epsilon - \int_N (b \circ \varphi) d\lambda_\epsilon + \int_N b d\lambda_\epsilon - \int_N d\lambda_\epsilon.$$

We deal with the first summand. The map $\zeta$ preserves the orientation on $N$, as it is isotopic to the identity, and satisfies $d\lambda_\epsilon = \zeta^*(d\lambda)$. Hence,

$$\int_N (\tau \circ \zeta) d\lambda_\epsilon = \int_N (\tau \circ \zeta) \zeta^*(d\lambda) = \int_N \zeta^*(\tau d\lambda) = \int_N \tau d\lambda.$$

The second and third summand cancel out. Indeed, as $\varphi$ preserves $d\lambda_\epsilon$, we get

$$\int_N (b \circ \varphi) d\lambda_\epsilon = \int_N (b \circ \varphi) \varphi^*(d\lambda_\epsilon) = \int_N \varphi^*(b d\lambda_\epsilon) = \int_N b d\lambda_\epsilon.$$

We deal with the last summand. By Stokes’ Theorem, the fact that $\lambda_\epsilon|_{\partial N} = -t_\Sigma d\theta$ and that the induced orientation on $\partial N$ is given by $-d\theta$, we get

$$\int_{\partial N} d\lambda_\epsilon = \int_{\partial N} \lambda_\epsilon = -\int_0^1 -t_\Sigma d\theta = t_\Sigma.$$

Plugging these last three identities in the computation above, we arrive at (3.25).

We move to item (v). We take $q \in N \cap \text{Fix}(\varphi)$ and observe that $\zeta(q) \in N \cap \text{Fix}(P)$. By Proposition 3.13(v), there exists a prime periodic orbit of $\Phi_{T(\gamma), \varphi}$ through $S(\zeta(q))$ with period $\tau(\zeta(q)) \leq T$. We denote by $\gamma_q \in \mathcal{D}_T(\alpha')$ the corresponding orbit through the bijection given in (3.24), so that $T(\gamma_q) = T(\gamma) \tau(\zeta(q))$. Finally, we observe that

$$\tau(\zeta(q)) = 1 + \sigma(q) + b(\varphi(q)) - b(q) = 1 + \sigma(q) + b(q) - b(q) = 1 + \sigma(q).$$

The implication in item (vi) follows at once, since $\alpha'$ is Zoll if and only if $\alpha_{T(\gamma), \varphi}$ is Zoll by Lemma 3.11 and moreover, $P = \text{id}_N$ if and only if $\varphi = \text{id}_N$.
**Corollary 3.15.** Suppose that we can choose \( \epsilon_{10} \) in Theorem 3.14 so that, with the corresponding \( \epsilon_{11} > 0 \), we have the following implications for a pair \((\varphi, \sigma)\) as above:

\[
\varphi \neq \text{id}_N, \quad \int_N \sigma \, d\lambda_s \leq 0 \quad \implies \quad \exists q_- \in \tilde{N} \cap \text{Fix} (\varphi), \quad \sigma(q_-) < 0, \tag{3.26}
\]

\[
\varphi \neq \text{id}_N, \quad \int_N \sigma \, d\lambda_s \geq 0 \quad \implies \quad \exists q_+ \in \tilde{N} \cap \text{Fix} (\varphi), \quad \sigma(q_+) > 0.
\]

Then, Theorem 1.3 holds taking \( U := \{ \Omega \text{ exact two-form on } \Sigma \mid \| \Omega - d\alpha_* \|_{C^2} < \epsilon_{11} \} \).

**Proof.** Let \( \alpha' \) be a contact form such that \( d\alpha' \in U \) as defined in the statement. If \( \alpha' \) is Zoll, the conclusion follows from Proposition 1.2. Thus, we assume that \( \alpha' \) is not Zoll and we want to prove that \( \rho_{\text{sys}}(\alpha) < \frac{1}{t_\Sigma} \). We first prove the inequality with the systolic ratio. Suppose by contradiction that \( \rho_{\text{sys}}(\alpha) \geq \frac{1}{t_\Sigma} \) and let \( \gamma \in P^p_T(\alpha') \) be such that

\[
T(\gamma) = \min_{\gamma' \in P^p_T(\alpha')} T(\gamma'), \tag{3.27}
\]

which exists by Theorem 3.14 and clearly satisfies \( T(\gamma) \geq T_{\text{min}}(\alpha') \). Thus, the assumption \( \rho_{\text{sys}}(\alpha) \geq \frac{1}{t_\Sigma} \) implies

\[
\text{Volume}(\alpha') - t_\Sigma T(\gamma)^2 \leq 0. \tag{3.28}
\]

Theorem 3.14 associates to \( \gamma \) the pair \((\varphi, \sigma)\) with the properties listed therein. In particular, by Theorem 3.14.(iv) and (3.28) above, we have that

\[
\int_N \sigma \, d\lambda_s \leq 0.
\]

As \( \alpha' \) is not Zoll, \( \varphi \neq \text{id}_N \) and we can use the first implication in (3.26) to produce a point \( q_- \in \tilde{N} \cap \text{Fix} (\varphi) \) with \( \sigma(q_-) < 0 \). By Theorem 3.14(v), this yields an element \( \gamma_{q_-} \in P^p_T(\alpha') \) with \( T(\gamma_{q_-}) < T(\gamma) \), which contradicts (3.27). This proves \( \rho_{\text{sys}}(\alpha) < \frac{1}{t_\Sigma} \).

The inequality with the diastolic ratio is analogously established. Suppose by contradiction that \( \rho_{\text{dia}}(\alpha) \leq \frac{1}{t_\Sigma} \). We take this time \( \gamma \in P^p_T(\alpha) \) to satisfy

\[
T(\gamma) = \max_{\gamma' \in P^p_T(\alpha')} T(\gamma'). \tag{3.29}
\]

If the pair \((\varphi, \sigma)\) is associated with \( \gamma \), the assumption \( \rho_{\text{dia}}(\alpha) \leq \frac{1}{t_\Sigma} \) implies

\[
\int_N \sigma \, d\lambda_s \geq 0.
\]

The second implication in (3.26) and Theorem 3.14(v) yield an orbit \( \gamma_{q_+} \in P^p_T(\alpha') \) with \( T(\gamma_{q_+}) > T(\gamma) \). This contradicts (3.29) and proves \( \rho_{\text{dia}}(\alpha) > \frac{1}{t_\Sigma} \).

In view of the last result, we only need to prove implications (3.26) above to establish Theorem 1.3. This will be done in the next section.
4 Surfaces with a symplectic form vanishing at the boundary

For \( a > 0 \), we recall the notation for the annuli \( A = [0, a) \times S^1 \) and \( A' = [0, a/2) \times S^1 \). In this section, \( N \) will denote a connected oriented compact surface with one boundary component. We fix a collar neighbourhood of the boundary \( i_A : A \to N \) with positively oriented coordinates \((r, \theta) \in A\), where \( r = 0 \) corresponds to \( \partial N \). Hence, we have the identification \( S^1 \cong \partial N \) and the orientation induced by \( N \) on \( \partial N \) is given by the one-form \(-d\theta\).

On \( N \), we consider a one-form \( \lambda \) such that \( d\lambda \) is a positive symplectic two-form on \( \hat{N} \) and \( \lambda_A := i^*_A \lambda = (-k + \frac{1}{2}r^2) d\theta \), where, by Stokes’ Theorem, \( k := \int_N d\lambda > 0 \). In particular, \( d\lambda_A = rdr \wedge d\theta \) vanishes of order 1 at \( r = 0 \). The pair \((N, \lambda)\) is an instance of an ideal Liouville domain, a notion due to Giroux (see [Gir17]). We will use the following notation if \( Q \) is a subset of \( N \) (respectively, of \( N \times N \) or \( T^*N \)): we denote by \( \hat{Q} \) the intersection of \( Q \) with \( \hat{N} \) (respectively with \( \hat{N} \times \hat{N} \) or \( T^*\hat{N} \)).

4.1 A neighbourhood theorem

In this subsection, we are interested to develop a version of the Weinstein neighbourhood theorem for the diagonal \( \Delta_N \subset (N \times N, (-d\lambda) \oplus d\lambda) \).

More precisely, we will consider the zero section \( O_N \subset (T^*N, d\lambda_{can}) \) in the standard cotangent bundle of \( N \) and look for an exact symplectic map \( W : N \to T^*N \) from a neighbourhood \( N \) of \( \Delta_N \) in \( N \times N \), so that \( W \circ i_{\Delta_N} = i_{O_N} \), where

\[
i_{\Delta_N} : N \hookrightarrow N \times N, \quad i_{\Delta_N}(q) = (q, q), \quad i_{O_N} : N \hookrightarrow T^*N, \quad i_{O_N}(q) = (q, 0)
\]

are the canonical inclusions of the diagonal and the zero section after the natural identifications of these sets with \( N \). We start by giving an explicit construction of \( W \) on \( A \times A \).

Let us endow the product \( A \times A \) with coordinates \((r, \theta, R, \Theta)\), so that the diagonal is \( \Delta_A := \{r = R, \theta = \Theta\} \). We make the identification \( T^*A = A \times \mathbb{R}^2 \) and let \((\rho, \vartheta, p_\rho, p_\vartheta)\) be the corresponding coordinates on \( T^*A \). We consider an open neighbourhood \( Y \) of \( \Delta_A \) defined as

\[
Y := \{(r, \theta, R, \Theta) \in A \times A \mid |\theta - \Theta| < \frac{1}{2}\}
\]

and define the auxiliary sets

\[
Y' := Y \cap (A' \times A'), \quad \partial Y := Y \cap (\partial A \times \partial A).
\]

We have a well-defined difference function

\[
Y \to (-\frac{1}{2}, \frac{1}{2}), \quad (r, \theta, R, \Theta) \mapsto \theta - \Theta.
\]

We consider the map \( W_A : Y \to T^*A \) given in coordinates by

\[
\begin{align*}
\rho &= R, \\
\vartheta &= \Theta, \\
p_\rho &= R(\theta - \Theta), \\
p_\vartheta &= \frac{1}{2}(R^2 - r^2),
\end{align*}
\]
so that $W_h \circ i_{\Delta_h} = i_{\Delta_h}$. We write $W \subset T^*A$ for the image of the map $W_h$. The restriction $W_h := W_h|_Y : Y \to W$ is a diffeomorphism with inverse given by

$$
\begin{align*}
\begin{cases}
  r = \sqrt{\rho^2 - 2p_\theta}, \\
  \theta = \vartheta, \\
  R = \rho, \\
  \Theta = \theta - \frac{p_\rho}{\rho}
\end{cases}
\end{align*}
$$

(4.2)

We also consider the restriction $W_{h'} := W_h|_{Y'} : Y' \to T^*A'$. Its image $W'$ has the following expression, which will be useful later on:

$$
W' = \left\{ (\rho, \vartheta, p_\rho, p_\theta) \in T^*A' \left| p_\rho \in \left( -\frac{1}{2}\rho, \frac{1}{2}\rho \right), \, p_\theta \in \left( \frac{1}{2}(\rho^2 - \frac{a^2}{4}), \frac{1}{2}\rho^2 \right) \right. \right\}.
$$

(4.3)

Finally, let us define the function

$$
K_h : Y \to \mathbb{R}, \quad K_h(r, \theta, R, \Theta) := (k - \frac{1}{2}R^2)(\theta - \Theta),
$$

(4.4)

and set $K_{h'} := K_h|_{Y'} : Y' \to \mathbb{R}$. There holds $K_h|_{\Delta_h} = 0$ and

$$
W_h^* \lambda_{\text{can}} = (\lambda_h) \oplus \lambda_h - dK_h.
$$

(4.5)

Indeed, we have

$$
W_h^* \lambda_{\text{can}} + \lambda_h \oplus (\lambda_h) = R(\theta - \Theta) dR + \frac{1}{2}(R^2 - r^2) d\theta + (-k + \frac{1}{2}r^2) d\theta - (-k + \frac{1}{2}R^2) d\Theta \\
= (\theta - \Theta) d\left( \frac{1}{2}R^2 \right) + \frac{1}{2}R^2 d(\theta - \Theta) - k d(\theta - \Theta) \\
= (\theta - \Theta) d\left( -k + \frac{1}{2}R^2 \right) + (-k + \frac{1}{2}R^2) d(\theta - \Theta) \\
= -dK_h.
$$

Since, for all $q \in \mathcal{A}$, $(\lambda_{\text{can}})_{(q,0)} = 0$, we also deduce

$$
d(\lambda_{\text{can}})_{(q,0)} = \left( (\lambda_h) \oplus \lambda_h \right)_{(q,0)}. 
$$

(4.6)

Finally, if $i_{\partial Y} : \partial Y \to Y$ is the natural inclusion, from $\textbf{[155]}$ we also conclude that

$$
i_{\partial Y}^* (\lambda_{\text{can}}) = d \left( K_h \circ i_{\partial Y} \right).
$$

(4.7)

Indeed, $(W_h \circ i_{\partial Y})^* \lambda_{\text{can}} = 0$ from the explicit formula for $W_h$ given in $\textbf{[411]}$ and the fact that both $r$ and $R$ vanish on $\partial Y$.

We can now state the neighbourhood theorem. The proof will be an adaptation of $\textbf{[MS98]}$ Theorem 3.33] (with different sign convention).

**Proposition 4.1.** There exist an open neighbourhood $\mathcal{N} \subset \mathcal{N} \times \mathcal{N}$ of the diagonal $\Delta_N$, a map $W : \mathcal{N} \to T^*\mathcal{N}$, and a function $K : \mathcal{N} \to \mathbb{R}$ with the following properties.

$(i)$ The set $\mathcal{N}$ contains $\mathcal{Y}$. If we write $T := W(\mathcal{N})$, then $T \subset T^*\mathcal{N}$ is an open neighbourhood of $\mathcal{O}_N$ and the restriction $W|_{\mathcal{N}} : \mathcal{N} \to T$ is a diffeomorphism.

$(ii)$ $W^* \lambda_{\text{can}} = (-\lambda) \oplus \lambda - dK$. 

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(iii) $\mathcal{W} \circ i_{\Delta N} = i_{\mathcal{O} N}$, $\mathcal{W}|_{\mathcal{Y}'} = \mathcal{W}_{\mathcal{A}'}$.

(iv) $K \circ i_{\Delta N} = 0$, $K|_{\mathcal{Y}'} = K_{\mathcal{A}'}$.

(v) If $\partial \mathcal{Y}' := \mathcal{Y}' \cap (\partial N \times \partial N)$ and $i_{\partial \mathcal{Y}'} : \partial \mathcal{Y}' \to N \times N$ is the inclusion, then

$$d(K \circ i_{\partial \mathcal{Y}'}) = i_{\partial \mathcal{Y}'}^*((-\lambda) \oplus \lambda).$$

Proof. Let us denote by $(q, p)$ the points in $T^*\mathcal{A} \cong \mathcal{A} \times \mathbb{R}^2$, where $q = (\rho, \vartheta)$ and $p = (p_\rho, p_\vartheta)$. Let $g_\mathcal{A}$ and $g_{T^*\mathcal{A}}$ be the standard metrics on $\mathcal{A}$ and $T^*\mathcal{A}$:

$$g_\mathcal{A} := d\rho^2 + d\vartheta^2, \quad g_{T^*\mathcal{A}} := d\rho^2 + d\vartheta^2 + dp_\rho^2 + dp_\vartheta^2.$$ 

Then, the metric $g_{T^*\mathcal{A}}$ is compatible with the canonical symplectic form $d\lambda_{\text{can}}$. Namely,

$$g_{T^*\mathcal{A}} = d\lambda_{\text{can}}(J_{T^*\mathcal{A}} \cdot , \cdot),$$

where $J_{T^*\mathcal{A}} : T(T^*\mathcal{A}) \to T(T^*\mathcal{A})$ is the standard complex structure given by

$$J_{T^*\mathcal{A}} \partial_\rho = \partial_{p_\rho}, \quad J_{T^*\mathcal{A}} \partial_\vartheta = \partial_{p_\vartheta}, \quad J_{T^*\mathcal{A}} \partial_{p_\rho} = -\partial_\rho, \quad J_{T^*\mathcal{A}} \partial_{p_\vartheta} = -\partial_\vartheta.$$

If $(q, 0) \in \mathcal{O}_\mathcal{A}$, then we have the horizontal and vertical embeddings

$$d_q i_{\mathcal{O}_\mathcal{A}} : T_q \mathcal{A} \to T_{(q, 0)}(T^*\mathcal{A}), \quad T_q^*\mathcal{A} \to T_{(q, 0)}(T^*\mathcal{A}), \quad p \mapsto p^*,$$

so that, if $\mathring{g} : T_q^*\mathcal{A} \to T_q \mathcal{A}$ is the metric duality given by $g_\mathcal{A}$, there holds

$$p^* = J_{T^*\mathcal{A}} \cdot d_q i_{\mathcal{O}_\mathcal{A}} \cdot p^*, \quad \forall p \in T_q^*\mathcal{A}.$$ 

We now combine this formula with the fact that, for every $(q, p) \in T^*\mathcal{A}$, the ray $t \mapsto (q, tp)$, $t \in [0, 1]$ is a geodesic for $g_{T^*\mathcal{A}}$ with initial velocity $p^*$. Thus, if $\exp_{T^*\mathcal{A}}$ denotes the exponential map of $g_{T^*\mathcal{A}}$, we arrive at

$$(q, p) = \exp_{T^*\mathcal{A}}(d_q i_{\mathcal{O}_\mathcal{A}} \cdot p^*). \quad (4.8)$$

We consider the pulled back objects $g_{\mathring{Y}} := \mathcal{W}_{\mathcal{A}}^* g_{T^*\mathcal{A}}$ and $J_{\mathring{Y}} := \mathcal{W}_{\mathcal{A}}^* J_{T^*\mathcal{A}}$. In particular, $\mathcal{W}_{\mathcal{A}}^*$ is a local isometry between $g_{\mathring{Y}}$ and $g_{T^*\mathcal{A}}$. Moreover, since $\mathcal{W}_{\mathcal{A}}^*(d\lambda_{\text{can}}) = ((-d\lambda_\mathcal{A}) \oplus d\lambda_\mathcal{A})$ by \ref{1.5}, we see that the structure $J_{\mathring{Y}}$ is compatible with $(-d\lambda) \oplus d\lambda$, since $J_{T^*\mathcal{A}}$ is compatible with $d\lambda_{\text{can}}$ and $\lambda_\mathcal{A} = i_{\mathcal{A}^*}^* \lambda$. Namely,

$$((-d\lambda) \oplus d\lambda)|_{\mathring{Y}} = g_{\mathring{Y}}(J_{\mathring{Y}} \cdot , \cdot).$$

Furthermore, using \ref{1.8}, we can compute the pre-image of a point $(q, p) \in \mathring{W}$ as

$$\mathcal{W}^{-1}_{\mathcal{A}}(q, p) = \mathcal{W}^{-1}_{\mathcal{A}}(\exp_{i_{\mathcal{O}_\mathcal{A}}(q)}^* (J_{T^*\mathcal{A}} \cdot d_q i_{\mathcal{O}_\mathcal{A}} \cdot p^*))$$

$$= \exp_{i_{\mathcal{O}_\mathcal{A}}(q)}^* (d_q i_{\mathcal{O}_\mathcal{A}}(q) \mathcal{W}^{-1}_{\mathcal{A}} \cdot J_{T^*\mathcal{A}} \cdot d_q i_{\mathcal{O}_\mathcal{A}} \cdot p^*)) = \exp_{i_{\mathcal{O}_\mathcal{A}}(q)}^* (J_{\mathring{Y}} \cdot d_q i_{\mathcal{O}_\mathcal{A}}(q) \mathcal{W}^{-1}_{\mathcal{A}} \cdot d_q i_{\mathcal{O}_\mathcal{A}} \cdot p^*))$$

$$= \exp_{i_{\mathcal{O}_\mathcal{A}}(q)}^* (J_{\mathring{Y}} \cdot d_q i_{\mathcal{O}_\mathcal{A}} \cdot p^*). \quad (4.9)$$
The space of almost complex structures, which are compatible with the symplectic form \((-d\lambda) \oplus d\lambda\)|\(\mathbb{N} \times \mathbb{N}\), is contractible. Therefore, we can find an almost complex structure \(J\) on \(\mathbb{N} \times \mathbb{N}\), which is compatible with \((-d\lambda) \oplus d\lambda\)|\(\mathbb{N} \times \mathbb{N}\) and such that

\[
J|_{\mathbb{N}} = J|_{\mathbb{N}}.
\] (4.10)

We denote by \(g\) the corresponding metric on \(\mathbb{N} \times \mathbb{N}\), which satisfies

\[
g|_{\mathbb{N}} = g|_{\mathbb{N}}.
\] (4.11)

We write \(g_N := i_{\Delta_N}^* g\) for the restricted metric on \(\mathbb{N}\). We observe that \(g_N|_{\mathbb{N}} = g|_{\mathbb{N}}\), and therefore, we denote the metric duality given by \(g_N\) also by \(\sharp : T^* \mathbb{N} \to T\mathbb{N}\). Let us consider the set \(\mathcal{T}_1\) made by all the points \((q, p) \in T^* \mathbb{N}\) with the property that the \(g\)-geodesic starting at time 0 from \(i_{\Delta_N}(q)\) with direction \(J \cdot d_q i_{\Delta_N} \cdot p^\ast\) is defined up to time 1. We claim that

\[
\mathcal{T}_1 \text{ is a fibre-wise star-shaped neighbourhood of } \mathcal{O}_N \text{ and it contains } \mathbb{W}'.
\] (4.12)

The second assertion follows from equation (4.9) and (4.10), (4.11). For the first one, we see from the homogeneity of the geodesic equation that \(\mathcal{T}_1\) contains \(\mathcal{O}_N\), and it is fibre-wise star-shaped around \(\mathcal{O}_N\). Finally, since \(\mathcal{T}_1 \setminus \mathbb{W}'\) is bounded away from \(\partial(T^* \mathbb{N})\), the set \(\mathcal{T}_1\) is a neighbourhood of \(\mathcal{O}_N\). We define the map

\[
\Upsilon : \mathcal{T}_1 \to \mathbb{N} \times \mathbb{N}, \quad \Upsilon(q, p) := \exp_{i_{\Delta_N}(q)}^g (J \cdot d_q i_{\Delta_N} \cdot p^\ast).
\]

It satisfies

\[
\Upsilon |_{\mathbb{W}'} = W^{-1}, \quad \Upsilon \circ i_{\mathcal{O}_N} = i_{\Delta_N}.
\] (4.13)

If \(q \in \mathbb{N}\), the differential of \(\Upsilon\) at \(i_{\mathcal{O}_N}(q)\) in the direction \(u = p^\ast + d_q i_{\mathcal{O}_N} \cdot v \in T_{i_{\mathcal{O}_N}(q)} T^* \mathbb{N}\), where \(p \in T_q^* \mathbb{N}\) and \(v \in T_q \mathbb{N}\), is given by

\[
d_{i_{\mathcal{O}_N}(q)} \Upsilon \cdot u = J \cdot d_q i_{\Delta_N} \cdot p^\ast + d_q i_{\Delta_N} \cdot v.
\]

If we abbreviate \(\Omega = (-d\lambda) \oplus d\lambda\), we claim that \((\Upsilon^* \Omega)_{i_{\mathcal{O}_N}(q)} = (d\lambda_{\text{can}})_{i_{\mathcal{O}_N}(q)}\), for all \(q \in \mathbb{N}\). For \(u_1, u_2 \in T_{i_{\mathcal{O}_N}(q)} T^* \mathbb{N}\), we compute

\[
\Upsilon^* \Omega(u_1, u_2) = \Omega(J \cdot d_q i_{\Delta_N} \cdot p^\ast_1 + d_q i_{\Delta_N} \cdot v_1, J \cdot d_q i_{\Delta_N} \cdot p^\ast_2 + d_q i_{\Delta_N} \cdot v_2)
\]

\[
= \Omega(J \cdot d_q i_{\Delta_N} \cdot p^\ast_1, d_q i_{\Delta_N} \cdot v_2) - \Omega(J \cdot d_q i_{\Delta_N} p^\ast_2, d_q i_{\Delta_N} \cdot v_1)
\]

\[
= g(d_q i_{\Delta_N} \cdot p^\ast_1, d_q i_{\Delta_N} \cdot v_2) - g(d_q i_{\Delta_N} \cdot p^\ast_2, d_q i_{\Delta_N} \cdot v_1)
\]

\[
= (i_{\Delta_N}^* g)(p^\ast_1, v_2) - (i_{\Delta_N}^* g)(p^\ast_2, v_1)
\]

\[
= g_N(p^\ast_1, v_2) - g_N(p^\ast_2, v_1)
\]

\[
= p_1(v_2) - p_2(v_1)
\]

\[
= d\lambda_{\text{can}}(u_1, u_2),
\] (4.14)

where in the second equality we used the fact that \(\Delta_N\) is Lagrangian and that \(J\) is a symplectic endomorphism.
We readily see that \( i_{\Delta_N} \Lambda = i_{\Delta_N}((-\lambda) \oplus \lambda) = -\lambda + \lambda = 0 \).

We consider any \( K_1 : \mathcal{T}_1 \to N \) satisfying, for all \( q \in \mathcal{N} \),
\[
K_1 \circ i_{\mathcal{O}_N}(q) = 0, \quad d_{i_{\mathcal{O}_N}(q)}K_1 = \Lambda_{i_{\mathcal{O}_N}(q)}. \tag{4.15}
\]

For example, we can set
\[
K_1(q,p) := \int_0^1 \Lambda_{(q,tp)}(p^*) dt.
\]

The first property in (4.15) is immediate and it implies that
\[
d_{i_{\mathcal{O}_N}(q)}K_1 \cdot d_q i_{\mathcal{O}_N} \cdot v = 0 = \Lambda_{i_{\mathcal{O}_N}(q)}(d_q i_{\mathcal{O}_N} \cdot v).
\]

Thus, we just need to check the second property on vertical tangent vectors \( p^* \in T_{(q,0)}(T^*N) \):
\[
d_{i_{\mathcal{O}_N}(q)}K_1 \cdot p^* = \lim_{s \to 0} \frac{K_1(q,sp) - K_1(q,0)}{s} = \lim_{s \to 0} \frac{1}{s} \int_0^1 \Lambda_{(q,tp)}(sp^*) dt = \lim_{s \to 0} \int_0^1 \Lambda_{(q,tp)}(p^*) dt
\]
\[= \Lambda_{i_{\mathcal{O}_N}(q)}(p^*).\]

At this point, we transfer the attention on \( N \times N \). First, we can shrink \( \mathcal{T}_1 \) in such a way that (4.12) still holds and that \( \Upsilon \) is a diffeomorphism onto its image \( \Upsilon(\mathcal{T}_1) \). We define the open neighbourhood \( \mathcal{O} \) of \( \Delta_N \) by
\[
\mathcal{O} := \Upsilon(\mathcal{T}_1) \cup \mathcal{N}'
\]

and the map \( \mathcal{W}_1 : \mathcal{N} \to T^*N \) obtained by gluing:
\[
\mathcal{W}_1|_{\Upsilon(\mathcal{T}_1)} = \Upsilon^{-1}, \quad \mathcal{W}_1|_{\mathcal{N}'} = \mathcal{W}_h.
\tag{4.16}
\]

Such a map is well-defined because of (4.13) and satisfies \( \mathcal{W}_1 \circ i_{\Delta_N} = i_{\mathcal{O}_N} \). Let \( \chi : \mathcal{N} \to [0,1] \) be a cut-off function which is equal to 0 in a neighbourhood of \( \mathcal{N}' \) and equal to 1 on \( \mathcal{N} \setminus \mathcal{N} \).

We set
\[
K_N : \mathcal{N} \to \mathbb{R}, \quad K_N = \chi \cdot (K_1 \circ \mathcal{W}_1) + (1 - \chi) \cdot K_h.
\]

We readily see that
\[
K_N \circ i_{\Delta_N} = 0, \quad K_N|_{\mathcal{N}'} = K_h|_{\mathcal{N}'}. \tag{4.17}
\]

Furthermore, for all \( q \in \mathcal{N} \), there holds
\[
d_{i_{\Delta_N}(q)}K_N = \chi \cdot \mathcal{W}_1^*(d_{i_{\mathcal{O}_N}(q)}K_1) + (1 - \chi) \cdot d_{i_{\Delta_N}(q)}K_h
\]
\[= \chi \cdot \mathcal{W}_1^*(\Lambda_{i_{\mathcal{O}_N}(q)}) + (1 - \chi) \cdot ((-\lambda) \oplus \lambda)_{i_{\Delta_N}(q)}
\]
\[= \chi \cdot ((-\lambda) \oplus \lambda)_{i_{\Delta_N}(q)} + (1 - \chi) \cdot ((-\lambda) \oplus \lambda)_{i_{\Delta_N}(q)},
\]
\[= ((-\lambda) \oplus \lambda)_{i_{\Delta_N}(q)},\]

\[1\]
where we used \( K_1 \circ W_1 \circ i_{\Delta_N}(q) = 0 = K_h \circ i_{\Delta_N}(q) \) in the first equality. While the second equality followed from (4.6) and (4.16). Since \( (\lambda_{\text{can}})_{i_{\Delta_N}(q)} = 0 \), we deduce

\[
(W_1^* \lambda_{\text{can}})_{i_{\Delta_N}(q)} = ((-\lambda) \oplus \lambda)_{i_{\Delta_N}(q)} - d_{i_{\Delta_N}(q)}K_N. 
\] (4.18)

The rest of the proof follows Moser's argument. We set

\[
\Lambda_t := t(W_1^* \lambda_{\text{can}} + dK_N) + (1 - t)((-\lambda) \oplus \lambda), \quad t \in [0, 1].
\]

By (4.15), (4.16), and (4.17), we have

\[
\Lambda_t = (-\lambda) \oplus \lambda \quad \text{on } \mathcal{Y}'.
\] (4.19)

Moreover, for all \( q \in \tilde{\mathcal{N}} \), by (4.14) and (4.18), we have

\[
(d\Lambda_t)_{i_{\Delta_N}(q)} = ((-d\lambda) \oplus d\lambda)_{i_{\Delta_N}(q)}, \quad (\Lambda_t)_{i_{\Delta_N}(q)} = ((-\lambda) \oplus \lambda)_{i_{\Delta_N}(q)}.
\] (4.20)

In particular \( d\Lambda_t \) is non-degenerate on \( \Delta_N \). Therefore, up to shrinking the neighbourhood away from \( \mathcal{Y}' \), we can assume that \( d\Lambda_t \) is non-degenerate on \( \mathcal{N} \). Let \( X_t \) be a time-dependent vector field and \( L_t \) a time-dependent function on \( \mathcal{N} \) defined by

\[
\iota_{X_t}d\Lambda_t = -\frac{d\Lambda_t}{dt}, \quad L_t := -\int_0^t \Lambda'_t(X_{t'}) \circ \Phi_{t'} \, dt'.
\]

where \( \Phi_t \) is the flow of \( X_t \). By (4.19), we see that \( X_t \) and \( L_t \) vanish on \( \mathcal{Y}' \) and we can extend them trivially to the whole \( \mathcal{N} \). Relations (4.20) imply that \( X_t \) and \( L_t \) vanish on \( \Delta_N \). In particular, \( \Phi_t \) is the identity map on \( \Delta_N \), and up to shrinking the neighbourhood \( \mathcal{N} \) away from \( \mathcal{Y}' \), we can suppose that \( \Phi_t \) is defined up to time 1. For all \( t \in [0, 1] \) we have

\[
\frac{d}{dt}(\Phi^*_t \Lambda_0 + dL_0) = \Phi^*_t \left( \iota_{X_t}d\Lambda_t + d(\Lambda_t(X_t)) + \frac{d\Lambda_t}{dt} \right) + d \left( \frac{dL_t}{dt} \right) = 0.
\]

Together with \( \Phi_0^* \Lambda_0 + dL_0 = \Lambda_0 \), this implies \( \Phi^*_t \Lambda_1 + dL_1 = \Lambda_0 \). Hence,

\[
\Phi^*_t W_1^* \lambda_{\text{can}} = (-\lambda) \oplus \lambda - d(L_1 + K_N \circ \Phi_1),
\]

and properties (i) and (ii) in the statement follow with

\[
\mathcal{W} := W_1 \circ \Phi_1, \quad K := L_1 + K_N \circ \Phi_1.
\]

Properties (iii) and (iv) hold as well, since they are satisfied by \( W_1 \) and \( K_N \) and we have shown that \( \Phi_1|_{\Delta_N} = \text{id}, \Phi_1|_{\mathcal{Y}} = \text{id} \) and \( L_1|_{\Delta_N} = 0, L_1|_{\mathcal{Y}} = 0 \). Property (v) follows from (iv) and equation (4.7). \( \square \)

### 4.2 Exact diffeomorphisms \( C^1 \)-close to the identity

We consider a special class of diffeomorphisms of \( \mathcal{N} \), which are compatible with \( \lambda \).

**Definition 4.2.** A diffeomorphism \( \varphi : \mathcal{N} \to \mathcal{N} \) is called **exact**, if \( \varphi^* \lambda - \lambda \) is an exact one-form.

Let \( \mathcal{E} \) denote the set of all exact diffeomorphisms, which we endow from now on with the **uniform \( C^1 \)-topology**, whose associated distance function we denote by \( \text{dist}_{C^1} \). For every \( \epsilon > 0 \), we consider the open ball around \( \text{id}_N \) of radius \( \epsilon \) in the space of exact diffeomorphisms

\[
\mathcal{E}(\epsilon) := \{ \varphi \in \mathcal{E} \mid \text{dist}_{C^1}(\varphi, \text{id}_N) < \epsilon \}
\]
If $\varphi \in \mathbb{E}$, we write $\Gamma_\varphi : N \to N \times N$ for its graph $\Gamma_\varphi(q) = (q, \varphi(q))$, and we have $\Gamma_\varphi(\partial N) \subset \partial N \times \partial N$. There is $\epsilon_* > 0$ such that all $\varphi \in \mathbb{E}(\epsilon_*)$ enjoy the following properties:

(a) $\Gamma_\varphi(N) \subset N$.

(b) If $\pi_N : T^*N \to N$ is the foot-point projection, then the map
$$\nu_\varphi : N \to N, \quad \nu_\varphi := \pi_N \circ W \circ \Gamma_\varphi$$
is a diffeomorphism. Indeed, $\nu_\varphi$ is $C^1$-close to $id_N$, if the same is true for $\varphi$. Henceforth, we write $\nu$ instead of $\nu_\varphi$, when the map $\varphi$ is clear from the context.

(c) We have the inclusions
$$\varphi(\mathbb{A}^\prime) \subset \mathbb{A}^\prime, \quad \nu^{-1}(\mathbb{A}^\prime) \subset \mathbb{A}^\prime$$
where $\mathbb{A}^\prime := [0, a/4] \times S^1$.

If $\varphi \in \mathbb{E}(\epsilon_*)$, then we can write its restriction to $\mathbb{A}^\prime$ as
$$\varphi(r, \theta) = (R_\varphi(r, \theta), \Theta_\varphi(r, \theta)).$$
By (4.1), the restriction of $\nu$ to $\mathbb{A}^\prime$ reads
$$\nu(r, \theta) = (R_\varphi(r, \theta), \theta), \quad (4.21)$$
which implies that
$$\nu_\varphi|_{\partial N} = id_{\partial N}.$$Let $i_{\partial N} : \partial N \to N$ be the inclusion and observe that $\Gamma_\varphi \circ i_{\partial N}$ takes values in $\partial \mathbb{Y}'$. Therefore, taking the pull-back by $\Gamma_\varphi \circ i_{\partial N}$ in Proposition 4.1(v), we get
$$d (K \circ \Gamma_\varphi \circ i_{\partial N}) = i_{\partial N}^*(\varphi^* \lambda - \lambda).$$
With this relation we can single out a special primitive of $\varphi^* \lambda - \lambda$.

**Definition 4.3.** The action of $\varphi \in \mathbb{E}(\epsilon_*)$ is the unique $C^1$-function $\sigma : N \to \mathbb{R}$ such that

(i) $\varphi^* \lambda - \lambda = d\sigma$, \hspace{1cm} (ii) $\sigma|_{\partial N} = K \circ \Gamma_\varphi|_{\partial N}$.

**Remark 4.4.** We observe that the normalisation of $\sigma$ at the boundary coincides with the one considered in Definition 3.12 and Theorem 3.14. Indeed, we have the explicit formulae $\lambda = -k d\theta$ on $\partial N$ and $K(0, \theta, 0, \Theta) = k(\theta - \Theta)$ on $\partial \mathbb{Y}'$, and for all $\theta_0 \in S^1 \cong \partial N$, there holds

$$K \circ \Gamma_\varphi(\theta_0) = -k(\Theta_\varphi(\theta_0) - \theta_0) = -k \int_{\theta_0}^{\varphi(\theta_0)} d\theta = \int_{\theta_0}^{\varphi(\theta_0)} \lambda = \lambda(\theta_0, \varphi(\theta_0)).$$

We describe the tangent space of $\mathbb{E}(\epsilon_*)$. To this purpose we introduce a space of functions.

**Definition 4.5.** We write $\mathbb{V}$ for the vector space of all smooth functions $f : N \to \mathbb{R}$ such that both $f$ and $df$ vanish at $\partial N$. We endow this space with the pre-Banach norm $\| \cdot \|_\mathbb{V}$ defined by

$$\| f \|_\mathbb{V} := \| f \|_{C^2} + \| d f \|_{C^1}, \quad \forall f \in \mathbb{V}.$$Choosing the restriction to a smaller annulus in the second term above yields an equivalent norm on $\mathbb{V}$. For all $\delta > 0$, we denote by $\mathbb{V}(\delta)$ the open ball of radius $\delta$ in $(\mathbb{V}, \| \cdot \|_\mathbb{V})$. 50
Let $\varphi$ denote some element in $\mathbb{E}(\epsilon_\ast)$ with action $\sigma$. First, we take any differentiable path $t \mapsto \varphi_t$ with values in $\mathbb{E}(\epsilon_\ast)$ such that $\varphi = \varphi_0$, and write $t \mapsto \sigma_t$ for the corresponding path of actions with $\sigma = \sigma_0$. Let $X$ be the $C^1$-vector field on $N$ and $\varsigma : N \to \mathbb{R}$ the $C^1$-function uniquely defined by
\[
\frac{d\varphi_t}{dt}|_{t=0} = X \circ \varphi, \quad \varsigma = \frac{d}{dt}|_{t=0} \sigma_t.
\]
We combine these two objects to get a function
\[
H : N \to \mathbb{R}, \quad H := \varsigma \circ \varphi^{-1} - \lambda(X).
\] (4.22)
Differentiating $\varphi_t^* \lambda = \lambda + d\sigma_t$ with respect to $t$ and evaluating at $t = 0$, we get
\[
\iota_X d\lambda = dH.
\]
From this last equation and the fact that $d\lambda = RdR \wedge d\Theta$ vanishes at $\partial N$, we see that $dH$ vanishes at $\partial N$. Hence, if we write $X = X^R \partial_R + X^\Theta \partial_\Theta$ on the annulus $\mathcal{A}$, we find
\[
X^R = \frac{1}{R} \partial_\Theta H, \quad X^\Theta = -\frac{1}{R} \partial_R H. \quad (4.23)
\]
Then, we observe that $H = 0$ at the boundary, since $\varsigma = \lambda(X) \circ \varphi$ there. Indeed, from Definition 4.3 and Proposition 4.11(v), we compute
\[
\varsigma = d\Gamma_{\varphi} K \cdot (0 \oplus X) = ((-\lambda) \oplus \lambda)(0 \oplus X) \big|_{\Gamma_{\varphi}} = \lambda(X) \circ \varphi.
\]
Therefore, we see that $H$ belongs to $\mathbb{V}$ and $\|H\|_\mathbb{V}$ is equivalent to $\|X\|_{C^1}$.

Conversely, let $H \in \mathbb{V}$ and take any path $t \to H_t$ with values in $\mathbb{V}$ and such that $H_0 = H$. We claim that there is a uniquely defined path $t \to X_t$ of $C^1$-vector fields with $\iota_{X_t} d\lambda = dH_t$. The vector fields are well defined away from $\partial N$, since $d\lambda$ is symplectic there. On $\mathcal{A}$, instead, they are well defined because of (4.23). Let $t \mapsto \varphi_t$ be the path of diffeomorphisms obtained integrating $X_t$ with the condition $\varphi_0 = \varphi$. Differentiating with respect to $t$, we get
\[
\frac{d}{dt}(\varphi_t^* \lambda) = \varphi_t^*(\iota_{X_t} d\lambda + d(\lambda(X_t))) = d((H_t + \lambda(X_t)) \circ \varphi_t)
\]
so that all the maps $\varphi_t$ are exact with some action $\sigma_t$. Relation (4.22) is also satisfied since $H$ and $\varsigma \circ \varphi^{-1} - \lambda(X)$ have the same differential and both vanish at $\partial N$.

We sum up the previous discussion in a lemma, which justifies the subsequent definition.

**Lemma 4.6.** There is an isomorphism between the pre-Banach spaces
\[
(T_{\varphi} \mathbb{E}(\epsilon_\ast), \| \cdot \|_{C^1}) \to (\mathbb{V}, \| \cdot \|_\mathbb{V})
\]
given by the map $X \mapsto H$, where $H$ is defined in (4.22). 

**Definition 4.7.** Let $I$ be an interval. If $t \mapsto \varphi_t$ with $t \in I$ is a differentiable path with values in $\mathbb{E}(\epsilon_\ast)$ with corresponding actions $t \mapsto \sigma_t$ and generating vector field $t \mapsto X_t$, then the path of functions $H_t : N \to \mathbb{R}$ in $\mathbb{V}$ defined by
\[
H_t := \frac{d\sigma_t}{dt} \circ \varphi_t^{-1} - \lambda(X_t)
\]
is called the Hamiltonian associated with the path. The Hamiltonian $H_t$ satisfies
\[
\iota_{X_t} d\lambda = dH_t,
\]
and for every $t_0, t_1 \in I$, we have
\[
\sigma_{t_1}(q) = \sigma_{t_0}(q) + \int_{t_0}^{t_1} \left[ H_t + \lambda(X_t) \right](\varphi_t(q)) \, dt, \quad \forall q \in N.
\]
4.3 Generating functions

In this subsection, we describe how to generalise the correspondence between $C^1$-small exact diffeomorphisms and generating functions to our setting. For a classical treatment, we refer the reader to [M89, Chapter 9]. Let $\varphi$ be an exact diffeomorphism in $E(\epsilon_*)$. There exists a one-form $\eta : N \to T^*N$ such that

\[W \circ \Gamma_\varphi = \eta \circ \nu.\]

Since $\lambda_{\text{can}}$ has the tautological property $\eta^* \lambda_{\text{can}} = \eta$, we have

\[\nu^* \eta = \nu^* \eta^* \lambda_{\text{can}} = \Gamma^*_\varphi W^* \lambda_{\text{can}} = \Gamma^*_\varphi ((-\lambda) \oplus \lambda - dK) = \varphi^* \lambda - \lambda - d(K \circ \Gamma_\varphi) = d(\sigma - K \circ \Gamma_\varphi).\]  \hspace{1cm} (4.24)

**Definition 4.8.** The generating function of $\varphi \in E(\epsilon_*)$ is defined as

\[G_\varphi : N \to \mathbb{R}, \quad G_\varphi := (\sigma - K \circ \Gamma_\varphi) \circ \nu^{-1}.\]

Henceforth, we will simply write $G$ instead of $G_\varphi$, when the map $\varphi$ is clear from the context. From the discussion above, and regarding the differential of $G$ as a map $dG : N \to T^*N$, we have the equality

\[W \circ \Gamma_\varphi = dG \circ \nu.\]  \hspace{1cm} (4.25)

We write the restriction of $\nu^{-1}$ to $\mathcal{H}''$ as $\nu^{-1}(\rho, \vartheta) = (r_\varphi(\rho, \vartheta), \vartheta)$, so that, for every $\theta = \vartheta$, the functions $R_\varphi(\cdot, \theta)$ and $r_\varphi(\cdot, \theta)$ are inverse of each other. Moreover, since $r_\varphi(0, \vartheta) = 0$, by Taylor’s Theorem with integral remainder, there exists a function $s_\varphi : \mathcal{H}'' \to \mathbb{R}$ such that

\[r_\varphi = \rho(1 + s_\varphi).\]

By (4.21), (4.25) and (4.1), we have

\[\begin{cases}
\partial_\rho G(\rho, \vartheta) = \rho(\vartheta - \Theta_\varphi(r_\varphi(\rho, \vartheta), \vartheta)), \\
\partial_\vartheta G(\rho, \vartheta) = \frac{1}{2}(\rho^2 - r_\varphi^2(\rho, \vartheta)) = -\rho^2\left(\frac{1}{2}s_\varphi^2(\rho, \vartheta) + s_\varphi(\rho, \vartheta)\right). 
\end{cases}\]  \hspace{1cm} (4.26)

**Proposition 4.9.** If $G : N \to \mathbb{R}$ is the generating function of $\varphi \in E(\epsilon_*)$, there holds

\[\tilde{N} \cap \text{Fix}(\varphi) = \tilde{N} \cap \text{Crit}G.\]

Moreover, if $z \in \tilde{N} \cap \text{Fix}(\varphi)$, then $\nu(z) = z$ and $\sigma(z) = G(z)$.

**Proof.** Let $z$ be a point in $\tilde{N}$. We suppose first that $\varphi(z) = z$. Then, $\Gamma_\varphi(z) \in \Delta_N$, and by (iii) in Proposition 4.8, we have $W(\Gamma_\varphi(z)) = i_{\sigma_N}(z)$, which implies that $\nu(z) = z$ and $d_G = 0$. Moreover, by Definition 4.8 and Proposition 4.1(iv), we have

\[G(z) = \sigma(\nu^{-1}(z)) - K(\Gamma_\varphi(\nu^{-1}(z))) = \sigma(z) - K \circ i_{\Delta_N}(z) = \sigma(z).\]

Conversely, suppose that $z$ is a critical point $G$. Then, by (4.25)

\[(z, z) = i_{\Delta_N}(z) = W^{-1}(dG(z)) = \Gamma_\varphi(\nu^{-1}(z)) = (\nu^{-1}(z), \varphi(\nu^{-1}(z))),\]

which implies $\nu^{-1}(z) = z$, and hence, $\varphi(z) = z$. \hfill \Box
Lemma 4.10. The generating function $G$ belongs to $\mathcal{V}$. Moreover, there holds
\[
\partial^2_{pp} G|_{\partial N} = \text{id}_{\partial N} - \Theta \circ i_{\partial N}.
\]
In particular, for every $z \in \partial N$, we have
\[
\varphi(z) = z \iff \partial^2_{pp} G(z) = 0, \iff \sigma(z) = 0.
\]

Proof. The vanishing of $G$ at the boundary follows from (ii) in Definition 4.3. To prove the vanishing of the differential of $G$ at the boundary, we just substitute $\rho = 0$ in (4.26). Moreover, dividing the first equation in (4.26) by $\rho$ and taking the limit for $\rho$ going to 0, we obtain the formula for $\partial^2_{pp} G$, which also implies the first equivalence above. The second one follows from (4.4) and Definition 4.3.

By the previous lemma we have a map
\[
\mathcal{G} : E(\epsilon_*) \to \mathcal{V}, \quad \mathcal{G}(\varphi) = G_{\varphi},
\]
whose properties we will study. To this aim, we need a definition and two lemmas about functions on $A$.

Definition 4.11. Fix a positive integer $m$. Let us denote by $\mathbb{F}$ the space of all smooth functions $\hat{f} : A \to \mathbb{R}^m$ and by $\| \cdot \|_{C^0_+}$ the norm on $\mathbb{F}$ defined by
\[
\| \hat{f} \|_{C^0_+} := \| \hat{f} \|_{C^0} + \| r \hat{d} \hat{f} \|_{C^0}, \quad \forall \hat{f} \in \mathbb{F}.
\]
Let $\mathbb{F}_0 \subset \mathbb{F}$ be the subspace of those functions $f : A \to \mathbb{R}^m$ such that $f(0, \theta) = 0$, for all $\theta \in S^1$. In this case, there exists a unique $\hat{f} \in \mathbb{F}$ such that
\[
f(r, \theta) = r \hat{f}(r, \theta), \quad \forall (r, \theta) \in A.
\]

Lemma 4.12. The following two statements hold.

(i) The map $(\mathbb{F}_0, \| \cdot \|_{C^1}) \to (\mathbb{F}, \| \cdot \|_{C^0_+})$, $f \mapsto \hat{f}$ is an isomorphism of pre-Banach spaces.

(ii) Let $U$ be an open set of $\mathbb{R}^m$, and let $A : U \to \mathbb{R}^m$ be a $C^2$-function with $\| A \|_{C^2} < \infty$. If $\mathbb{F}_U$ is the set of all functions $\hat{f} \in \mathbb{F}$ such that the image of $\hat{f}$ is a relatively compact subset of $U$, then the following map is continuous:
\[
(\mathbb{F}_U, \| \cdot \|_{C^0_+}) \to (\mathbb{F}, \| \cdot \|_{C^0}), \quad \hat{f} \mapsto A \circ \hat{f}.
\]

Proof. By Taylor’s Theorem with integral remainder, the function $\hat{f}$ is defined as
\[
\hat{f}(r, \theta) = \int_0^1 \partial_r f(ur, \theta) \, du.
\]
Moreover, differentiating the identity $f = r \hat{f}$, we deduce that
\[
df = r \hat{d} \hat{f} + \hat{f} \, dr.
\]
We see from (1.27) that the $C^0$-norm of $\hat{f}$ is controlled by the $C^1$-norm of $f$. Consequently, from (1.28), we conclude that the $C^0$-norm of $rd\hat{f} = df - \hat{f}dr$ is also controlled by the $C^1$-norm of $f$. On the other hand, we deduce from (1.28) that the $C^0$-norm of $df$ is controlled by the $C^0$-norm of $df$ and $\hat{f}$. As $f$ vanishes at $r = 0$, the $C^0$-norm of $f$ is controlled, as well.

Finally, we consider a map $A : U \to \mathbb{R}^m$ as in the statement. Let $\hat{f}_0 \in F_U$ be fixed and $\hat{f} \in F_U$ such that $\hat{f}_0 + r(\hat{f} - \hat{f}_0) \in F_U$, for all $r \in [0, 1]$. This happens if $\hat{f}$ is $C^0$-close to $\hat{f}_0$ since the images of $\hat{f}_0$ and $\hat{f}$ are relatively compact in $U$, by assumption. Then, we can estimate with the help of the mean value theorem:

$$\|A \circ \hat{f} - A \circ \hat{f}_0\|_{C^0} \leq \|A\|_{C^1} \|\hat{f} - \hat{f}_0\|_{C^0};$$

$$\|rd(A \circ \hat{f} - A \circ \hat{f}_0)\|_{C^0} = \|rd_A \hat{f} - rd_A \hat{f}_0\|_{C^0} + \|rd_A \hat{f}_0\|_{C^0} \leq \|d_A \hat{f}\|_{C^0} + \|A\|_{C^1} \|\hat{f} - \hat{f}_0\|_{C^0} \|rd\hat{f}_0\|_{C^0},$$

from which the continuity of the map $\hat{f} \mapsto A \circ \hat{f}$ at $\hat{f}_0$ follows.

**Lemma 4.13.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that, for all $\vartheta \in S^1$, we have $f(0, \vartheta) = 0$, $d_{(0, \vartheta)}f = 0$. Then, there exist functions $f_\rho, f_\vartheta : \mathbb{R} \to \mathbb{R}$ such that

$$\partial_\rho f = \rho f_\rho, \quad \partial_\vartheta f = \rho^2 f_\vartheta.$$  

Moreover, there exists a constant $C > 0$ (independent of $f$) such that

$$\frac{1}{C} \|\frac{1}{\rho} df\|_{C^1} \leq \|f_\rho\|_{C^1} + \|f_\vartheta\|_{C^0} \leq C \|\frac{1}{\rho} df\|_{C^1}.$$  

**Proof.** By Taylor’s Theorem with integral remainder, for all $(\rho, \vartheta) \in \mathbb{R}$, we can write

$$f(\rho, \vartheta) = \rho^2 \hat{f}(\rho, \vartheta),$$

for a function $\hat{f} : \mathbb{R} \to \mathbb{R}$, so that $f_\rho := 2\hat{f} + \rho \partial_\rho \hat{f}, f_\vartheta := \partial_\vartheta \hat{f}$ yield the desired functions. In order to prove the equivalence of the norms, we observe that $\frac{1}{\rho} df = f_\rho \rho + f_\vartheta \vartheta$. Thus, $\frac{1}{\rho} df$ is $C^1$-small if and only if $f_\rho$ is $C^1$-small and $\rho f_\vartheta$ is $C^1$-small. The conclusion now follows from Lemma 4.12(i).  

**Proposition 4.14.** The map $G : E(\epsilon_*) \to \mathbb{V}$ is continuous from the $C^1$-topology to the topology induced by $\|\cdot\|_{\mathbb{V}}$. Moreover, for every differentiable path $t \mapsto \varphi_t \subset E(\epsilon_*)$ with $\nu_t := \nu_{\varphi_t}$ and generated by some $t \mapsto H_t$, the corresponding path $t \mapsto G_t := G(\varphi_t) \subset \mathbb{V}$ has a smooth pointwise derivative $t \mapsto \frac{dG_t}{dt} \circ \nu_t \subset \mathbb{V}$, which satisfies the Hamilton-Jacobi equation:

$$\frac{dG_t}{dt} \circ \nu_t = H_t \circ \varphi_t.$$  

**Proof.** Since we can write $dG = W \circ \Gamma_{\varphi} \circ \nu^{-1}$, we readily see that the map $G$ is continuous from the $C^1$-topology to the topology induced by the $C^2$-norm. The lemma follows if we can establish the continuity from the $C^1$-topology to the topology induced by the semi-norm...
If $\pi_{S^1} : \mathbb{A}'' \to S^1$ is the standard projection, then, using equations (4.26), this amounts to showing that the map
\[
\varphi \mapsto \pi_{S^1} - \Theta_\varphi \circ \nu_\varphi^{-1}
\]
is continuous from the $C^1$-topology to the $C^1$-topology, and further employing Lemma 4.12(i), that the map
\[
\varphi \mapsto -\frac{1}{2}s_\varphi^2 - s_\varphi
\]
is continuous from the $C^1$-topology to the $C^0_+$-topology. The former map is continuous since $(f_1, f_2) \mapsto f_1 \circ f_2$ is continuous from the product $C^1$-topology into the $C^1$-topology and
\[
\varphi \mapsto \Theta_\varphi, \quad \varphi \mapsto \nu_\varphi^{-1} = (r_\varphi, \pi_{S^1})
\]
are continuous in the $C^1$-topology. The latter map is continuous since
\begin{itemize}
  \item[(a)] the map $\varphi \mapsto s_\varphi = \frac{1}{\rho}(r_\varphi - \rho)$ is continuous from the $C^1$-topology to the $C^0_+$-topology by Lemma 4.12(i);
  \item[(b)] the map $\hat{f} \mapsto A \circ \hat{f}$ with $A(x) = -\frac{1}{2}t^2 - x$ is continuous from the $C^0_+$-topology to the $C^0_-$-topology by Lemma 4.12(ii).
\end{itemize}

Putting everything together, we have shown that $\mathcal{G}$ is continuous.

We now deal with the second part of the statement. From the notation introduced above and Definition 4.8 we get
\[
G_t \circ \nu_t = \sigma_t - K \circ \Gamma_{\varphi_t},
\]
from which we see that $G_t$ has a pointwise derivative. Thus, we can differentiate both sides with respect to $t$, and using the definition of $H_t$, we obtain
\[
\frac{dG_t}{dt} \circ \nu_t + d_{\nu_t}G_t \left( \frac{d\nu_t}{dt} \right) = \frac{d\sigma_t}{dt} - dK \left( \frac{d}{dt} \Gamma_{\varphi_t} \right) = H_t \circ \varphi_t + \lambda(X_t) \circ \varphi_t - dK \left( \frac{d}{dt} \Gamma_{\varphi_t} \right).
\]
If we can show that
\[
d_{\nu_t}G_t \left( \frac{d\nu_t}{dt} \right) = \lambda(X_t) \circ \varphi_t - dK \left( \frac{d}{dt} \Gamma_{\varphi_t} \right), \tag{4.29}
\]
then the equation in the statement of the proposition follows. First, thanks to (4.25), we have the equation $\mathcal{W} \circ \Gamma_{\varphi_t} = dG_t \circ \nu_t$, which we differentiate with respect to $t$ to get
\[
d\mathcal{W} \cdot \frac{d}{dt} \Gamma_{\varphi_t} = \frac{d}{dt}(dG_t \circ \nu_t).
\]
Then, we combine this identity with property (iii) in Proposition 4.3 to get
\[
dK \left( \frac{d}{dt} \Gamma_{\varphi_t} \right) = (-\lambda) \oplus \lambda \left( \frac{d}{dt} \Gamma_{\varphi_t} \right) - \lambda_{\text{can}} \left( d\mathcal{W} \cdot \frac{d}{dt} \Gamma_{\varphi_t} \right)
\]
\[
= (-\lambda) \oplus \lambda(0, X_t(\varphi_t)) - \lambda_{\text{can}} \left( \frac{d}{dt}(dG_t \circ \nu_t) \right)
\]
\[
= \lambda(X_t) \circ \varphi_t - d_{\nu_t}G_t \left( \frac{d\nu_t}{dt} \right) - d_{\nu_t}G_t \left( \frac{d\nu_t}{dt} \right) \left[ \frac{d}{dt} \right](dG_t \circ \nu_t)
\]
\[
= \lambda(X_t) \circ \varphi_t - d_{\nu_t}G_t \left( \frac{d\nu_t}{dt} \right),
\]
where in the third equality, we used the tautological property of $\lambda_{\text{can}}$. Equation (4.29) is thus established. $\square$
Remark 4.15. If we endow $E(\epsilon_s)$ with the $C^2$-topology (instead of the coarser $C^1$-topology), then the map $G$ becomes of class $C^1$, and for all $\varphi \in E(\epsilon_s)$ and $H \in V \cong T_\varphi E(\epsilon_s)$, we can rephrase the equation in the statement of the proposition as

$$d_\varphi G \cdot H = H \circ (\varphi \circ \nu^{-1}).$$

Proposition 4.16. There are $\delta_*, \epsilon_** > 0$ and a continuous map $E : \mathcal{V}(\delta_*) \to \mathcal{E}(\epsilon_*)$ such that

(i) we have the inclusion $G(E(\epsilon_**)) \subset \mathcal{V}(\delta_*)$;

(ii) the map $E$ is the inverse of $G$, namely,

$$G(E(G)) = G, \quad \forall G \in \mathcal{V}(\delta_*), \quad E(G(\varphi)) = \varphi, \quad \forall \varphi \in E(\epsilon_**).$$

Proof. Let $\delta_*$ be a positive number. We first show that if $G \in \mathcal{V}(\delta_*)$, then $dG$ takes values into $T = \mathcal{V}(\mathcal{N})$, provided $\delta_*$ is small enough. Since $T$ is a neighbourhood of the zero section away from the boundary of $\mathcal{N}$, we see that $dG(N \setminus \mathcal{A}^\prime)$ is contained in $T$ if $\delta_*$ is small. On the other hand, since $T \supset \mathcal{W}(\mathcal{Y}) = \mathcal{W}'$ from Proposition 4.1(i), we just need to show that $dG(\mathcal{A}^\prime) \subset \mathcal{W}' \cap (T^* \mathcal{A}^\prime)$. Recall from (4.3) the description

$$\mathcal{W}' \cap (T^* \mathcal{A}^\prime) = \left\{ (\rho, \vartheta, p_\rho, p_\vartheta) \in T^* \mathcal{A}^\prime \mid p_\rho \in \left( -\frac{1}{2} \rho, \frac{1}{2} \rho \right), \quad p_\vartheta \in \left( \frac{1}{2} (\rho^2 - \frac{a^2}{2}), \frac{1}{2} \rho^2 \right) \right\},$$

so that the implication

$$0 \leq \rho < \frac{a}{2} \implies \frac{1}{2} (\rho^2 - \frac{a^2}{2}) > -\frac{3}{2} \rho^2,$$

yields the implication

$$(\rho, \vartheta, p_\rho, p_\vartheta) \in \mathcal{W}' \cap (T^* \mathcal{A}^\prime) \implies p_\vartheta \in \left( -\frac{3}{2} \rho^2, \frac{1}{2} \rho^2 \right].$$

By Lemma 4.13, we have the expressions $\partial_\rho G = \rho G_\rho$ and $\partial_\vartheta G = \rho^2 G_\vartheta$. Therefore, in order to have $dG(\mathcal{A}^\prime) \subset \mathcal{W}' \cap (T^* \mathcal{A}^\prime)$, we just need $\|G_\rho\|_{C^0(\mathcal{A}^\prime)} < \frac{1}{2}$ and $\|G_\vartheta\|_{C^0(\mathcal{A}^\prime)} < \frac{1}{2}$, which are true if $\delta_*$ is small, thanks to the inequality in Lemma 4.13 and the definition of $\|\cdot\|_V$.

Since $dG(N) \subset T$, we can consider the map

$$\hat{\mu} : \hat{\mathcal{N}} \to \hat{\mathcal{N}}, \quad \hat{\mu} := \pi_1 \circ \mathcal{W}^{-1} \circ dG|_{\hat{\mathcal{N}}}, \quad (4.30)$$

where $\pi_1 : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ is the projection on the first factor. On the annulus $\mathcal{A}^\prime$, we consider, furthermore, the map

$$\mu_\mathcal{A}^\prime : \mathcal{A}^\prime \to \mathcal{A}^\prime, \quad \mu_\mathcal{A}^\prime(\rho, \vartheta) = \left( \rho \sqrt{1 - 2G_\vartheta(\rho, \vartheta)}, \vartheta \right).$$

Thanks to (4.12), $\hat{\mu}$ and $\mu_\mathcal{A}^\prime$ glue together and yield a map $\mu_G : \mathcal{N} \to \mathcal{N}$. We claim that $G \mapsto \mu_G$ is continuous from the topology induced by $\|\cdot\|_V$ to the $C^1$-topology. We argue separately for $\hat{\mu}|_{\mathcal{N} \setminus \mathcal{A}^\prime}$ and $\mu_\mathcal{A}^\prime$. For the former map, the continuity is clear from the expression (4.12) and the fact that $\|G\|_{C^2} \leq \|G\|_V$. For the latter map, the continuity is clear in the second factor, and we only have to deal with the continuity of $G \mapsto \rho \sqrt{1 - 2G_\vartheta}$ by Lemma 4.12. This happens if and only if $G \mapsto \sqrt{1 - 2G_\vartheta}$ is continuous from the topology induced by $\|\cdot\|_V$ to the $C^0_\mathcal{L}$-topology. The latter map is the composition of $G \mapsto G_\vartheta$ with $f \mapsto A \circ f$, where $A$ is the...
where $A : (-\frac{1}{2}, +\frac{1}{2}) \to (0, \infty)$ is defined by $A(x) = \sqrt{1 - 2x}$. The map $G \mapsto G_\varphi$ is continuous from the $\| \cdot \|_V$-topology to the $C^0_{\mathcal{A}}$-topology by Lemma 4.13. The map $f \mapsto A \circ f$ is continuous from the $C^0_{\mathcal{A}}$-topology to the $C^0_{\mathcal{B}}$-topology by Lemma 4.12(ii). The claim is established.

Thus, taking $\delta_\ast$ small enough, we can assert that $\mu_G : N \to N$ is so $C^1$-close to the identity that is a diffeomorphism and we write $\nu_G : N \to N$ for its inverse, which satisfies

$$\nu_G(r, \theta) = (R_G(r, \theta), \theta), \quad \forall (r, \theta) \in \mathcal{A},$$

for some function $R_G : \mathcal{A} \to [0, a/2)$. The map $G \mapsto \nu_G$ is continuous in the $C^1$-topology.

We now construct a diffeomorphism $\varphi_G : N \to N$. Let $\pi_2 : N \times N \to N$ be the projection on the second factor and set

$$\hat{\varphi} : \hat{N} \to \hat{N}, \quad \hat{\varphi} := \pi_2 \circ W^{-1} \circ dG \circ \nu_G|_{\hat{N}}. \quad (4.31)$$

On the annulus $\mathcal{A}$, we set

$$\varphi_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}, \quad \varphi_{\mathcal{A}}(r, \theta) = (R_G(r, \theta), \theta - G_\rho(R_G(r, \theta), \theta)).$$

Thanks to (4.2), the maps $\hat{\varphi}$ and $\varphi_{\mathcal{A}}$ glue together to yield $\varphi_G : N \to N$. We claim that $\varphi$ is exact. Indeed, from (4.30) and (4.31), we get $W \circ \Gamma_{\hat{\varphi}} = dG \circ \nu$. Since $\nu_G$ and $\varphi_G$ are continuous up to the boundary, we deduce

$$W \circ \Gamma_{\varphi_G} = dG \circ \nu_G. \quad (4.32)$$

Repeating the computation as in (4.24), it follows that $\varphi_G$ is exact with action

$$\sigma_{\varphi_G} := G \circ \nu_G + K \circ \Gamma_{\varphi_G}. \quad (4.33)$$

Therefore, we have a map $E : V(\delta_\ast) \to \mathbb{E}$ defined by $E(G) = \varphi_G$. We claim that this map is continuous. As before, we argue separately for $\varphi|_{N \setminus \mathcal{A}}$ and $\varphi_{\mathcal{A}}$. For the former map, the continuity follows since we have a control on the $C^2$-norm of $G$. For the latter map, it follows from the continuity of $G \mapsto R_G$ from the $\| \cdot \|_V$-topology to the $C^1$-topology, which we have already established, the continuity of $G \mapsto G_\rho$ from the $\| \cdot \|_V$-topology to the $C^1$-topology, which follows from Lemma 4.13, and the continuity of $(f_0, f_1) \mapsto f_0 \circ f_1$ from the product $C^1$-topology into the $C^1$-topology. The claim is established. In particular, up to shrinking $\delta_\ast$, we can assume that $E(V(\delta_\ast)) \subset E(\epsilon_\ast)$. On the other hand, the existence of $\epsilon_\ast > 0$ with the property that $G(E(\epsilon_\ast)) \subset V(\delta_\ast)$ is a consequence of the continuity of $E$.

Next, we verify that $G(\varphi_G) = G$. First, recalling that $\pi_N : \Gamma^*N \to N$, we see that

$$\nu_{\varphi_G} = \pi_N \circ (W \circ \Gamma_{\varphi_G}) = \pi_N \circ (dG \circ \nu_G) = (\pi_N \circ dG) \circ \nu_G = \nu_G.$$

Therefore, comparing (4.33) with Definition 4.8, we get $G(\varphi_G) = G_{\varphi_G} = G$.

Finally, let $\varphi \in E(\epsilon_\ast)$. We show that $\varphi = \varphi(G_{\varphi})$. First, we get

$$\nu_{\varphi_{\mathcal{A}}^{-1}}|_{\hat{N}} = \pi_2 \circ W^{-1} \circ dG_{\varphi} |_{\hat{N}} = \pi_1 \circ \Gamma_{\varphi_{\mathcal{A}}} \circ \nu_{\varphi_{\mathcal{A}}}^{-1}|_{\hat{N}} = \nu_{\varphi_{\mathcal{A}}}^{-1}|_{\hat{N}}.$$

By continuity, this implies $\nu_\varphi = \nu_{G_\varphi}$, and we arrive at

$$\varphi|_{\hat{N}} = \pi_2 \circ W^{-1} \circ dG_{\varphi} \circ \nu_\varphi|_{\hat{N}} = \pi_2 \circ W^{-1} \circ dG_{\varphi} \circ \nu_{G_\varphi}|_{\hat{N}} = \pi_2 \circ \Gamma_{\varphi_{\mathcal{A}}} |_{\hat{N}} = \varphi_{\mathcal{A}} |_{\hat{N}}.$$

By continuity again, $\varphi = \varphi_{G_\varphi} = E(G_\varphi)$ as required, and the proof is completed. \qed
4.4 Quasi-autonomous diffeomorphisms

**Lemma 4.17.** Let $\varphi \in \mathbb{E}(\epsilon_*)$ be an exact diffeomorphism, and let $\sigma : N \to \mathbb{R}$ denote its action. Suppose that there exists a differentiable path $t \mapsto \varphi_t$ in $\mathbb{E}(\epsilon_*)$ with $\varphi_0 = \text{id}_N$ and $\varphi_1 = \varphi$. We write by $t \mapsto H_t \in \mathcal{V}$ the Hamiltonian associated with the path. There holds

$$
\sigma(q) = \int_0^1 \left[ H_t + \lambda(X_t) \right] (\varphi_t(q)) \, dt = \int_0^1 \left( t \mapsto \varphi_t(q) \right)^* \lambda + \int_0^1 H_t(\varphi_t(q)) \, dt, \quad \forall q \in N.
$$

As a consequence, we have

$$
\int_N \sigma \, d\lambda = 2 \int_0^1 \left( \int_N H_t \, d\lambda \right) \, dt.
$$

**Proof.** The first formula follows from Definition 4.7 taking $t_0 = 0$, $t_1 = 1$ and observing that $\sigma_0 = 0$ since $\varphi_0 = \text{id}_N$. The formula for the integral of $\sigma$ is classical (see [MS98, Lemma 10.27] and [ABHS17a, Proposition 2.6 & 2.7]), but we repeat the argument here for the convenience of the reader. Since the three-form $\lambda \wedge d\lambda$ vanishes as $N$ has dimension two, we have

$$
0 = \iota_{X_t}(\lambda \wedge d\lambda) = \lambda(X_t) \, d\lambda - \lambda \wedge (\iota_{X_t} \, d\lambda) = \lambda(X_t) \, d\lambda - \lambda \wedge dH_t,
$$

which in turn implies $\lambda(X_t) \, d\lambda = \lambda \wedge dH_t = H_t \, d\lambda - d(H_t \lambda)$. Using this identity, the formula for $\sigma$ we just proved, and the fact that $\varphi_t$ preserves $d\lambda$, we compute

$$
\int_N \sigma \, d\lambda = \int_N \left( \int_0^1 (H_t + \lambda(X_t)) \circ \varphi_t \, dt \right) \, d\lambda = \int_0^1 \left( \int_N (H_t + \lambda(X_t)) \circ \varphi_t \, d\lambda \right) \, dt
$$

$$
= \int_0^1 \left( \int_N (H_t + \lambda(X_t)) \, d\lambda \right) \, dt
$$

$$
= \int_0^1 \left( \int_N 2H_t \, d\lambda - d(H_t \lambda) \right) \, dt
$$

$$
= 2 \int_0^1 \left( \int_N H_t \, d\lambda \right) \, dt,
$$

where the last equality follows from Stokes’ Theorem together with $H_t|_{\partial N} \equiv 0$. $\square$

In [BP94], the following special class of Hamiltonian paths was introduced.

**Definition 4.18.** A Hamiltonian path $t \mapsto H_t \in \mathcal{V}$ parametrised in the interval $I$ is called **quasi-autonomous** if there exist a minimiser $q_{\min} \in N$ and a maximiser $q_{\max} \in N$ independent of time, i.e.

$$
\min_N H_t = H_t(q_{\min}), \quad \max_N H_t = H_t(q_{\max}), \quad \forall t \in I.
$$

A diffeomorphism $\varphi \in \mathbb{E}(\epsilon_*)$ is called quasi-autonomous, if there exists a differentiable path $t \mapsto \varphi_t \in \mathbb{E}(\epsilon_*)$ parametrised in $[0,1]$ with $\varphi_0 = \text{id}_N$, $\varphi_1 = \varphi$, whose associated Hamiltonian $t \mapsto H_t \in \mathcal{V}$ is quasi-autonomous.

**Lemma 4.19.** Let $\varphi \in \mathbb{E}(\epsilon_*)$ be quasi-autonomous with associated Hamiltonian $t \mapsto H_t$. The following implications hold:

$$
\exists t_- \in [0,1], \ H_{t_-}(q_{\min}) < 0, \quad \implies \quad q_{\min} \in \text{Fix} (\varphi) \cap \tilde{N}, \ \sigma(q_{\min}) < 0,
$$

$$
\exists t_+ \in [0,1], \ H_{t_+}(q_{\max}) < 0, \quad \implies \quad q_{\max} \in \text{Fix} (\varphi) \cap \tilde{N}, \ \sigma(q_{\max}) < 0.
$$
Proof. We show only the first implication. Since $H_{t_-}(q_{\min}) < 0$ and $H_{t_-}|_{\partial N} = 0$, we deduce that $q_{\min} \in \tilde{N}$. Moreover, since $q_{\min}$ minimises $H_t$ for all $t \in [0,1]$, we see that $d_{q_{\min}}H_t = 0$. Since $d\lambda$ is symplectic on $\tilde{N}$, by $\iota_Xd\lambda = dH_t$, we conclude that $X_t(q_{\min}) = 0$, which implies that $\varphi_t(q_{\min}) = q_{\min}$. We estimate the action of $q_{\min}$ using Lemma 4.17 and remembering that, for all $t \in [0,1]$, there holds $H_t(q_{\min}) \leq 0$, since $H_t$ vanishes on the boundary:

$$\sigma(q_{\min}) = \int_0^1 [H_t + \lambda(X_t)](\varphi_t(q_{\min}))dt = \int_0^1 H_t(q_{\min})dt < 0. \quad \square$$

**Proposition 4.20.** Every $\varphi \in \mathbb{E}(\epsilon_*)$ is quasi-autonomous.

*Proof.* By Proposition 4.16, the generating function $G$ of $\varphi$ belongs to $\mathbb{V}(\delta_*)$. Thus, for all $t \in [0,1]$, the function $tG$ belongs to $\mathbb{V}(\delta_*)$, and again by Proposition 4.16 we can consider the path $t \mapsto \varphi_t := \mathcal{E}(tG) \in \mathbb{E}(\epsilon_*)$. Let $t \mapsto H_t$ be the associated Hamiltonian. By Proposition 4.14 we deduce

$$G = \frac{d}{dt}(tG) = H_t \circ (\varphi_t \circ \nu_t^{-1}), \quad \forall t \in [0,1],$$

which implies

$$\min H_t = \min G, \quad \max H_t = \max G, \quad \forall t \in [0,1]. \quad (4.34)$$

Let $q_{\min}$ and $q_{\max}$ be the minimiser and the maximiser of $G$, respectively. We claim that

$$G(q_{\min}) = H_t(q_{\min}), \quad G(q_{\max}) = H_t(q_{\max}), \quad \forall t \in [0,1]. \quad (4.36)$$

We give only the argument for $q_{\min}$. If $q_{\min} \in \partial N$, we have $G(q_{\min}) = 0 = H_t(q_{\min})$, as $G$ and $H_t$ belong to $\mathbb{V}$. If $q_{\min} \in \tilde{N}$, then $q_{\min} \in \text{Crit} G$. We deduce that $\varphi_t(q_{\min}) = q_{\min} = \nu_t(q_{\min})$, as $\varphi_t$ and $\nu_t$ act as the identity on $\tilde{N} \cap \text{Crit}(tG) \supset \tilde{N} \cap \text{Crit} G$ by Proposition 4.9. The equality $G(q_{\min}) = H_t(q_{\min})$ follows then from (4.34). Now that the claim is established, relations (4.35) and (4.36) imply that $t \mapsto H_t$ is quasi-autonomous. \quad \square

We are now ready to prove implications (3.26) in Corollary 3.15, which are the last missing piece to establish the Main Theorem 1.3.

**Corollary 4.21.** Let $\varphi \in \mathbb{E}(\epsilon_*)$ be an exact diffeomorphism with action $\sigma : N \to \mathbb{R}$. If $\varphi \neq \text{id}_N$, the following implications hold:

- $\int_N \sigma d\lambda \leq 0 \implies \exists q_- \in \text{Fix}(\varphi) \cap \tilde{N}$ with $\sigma(q_-) < 0$,
- $\int_N \sigma d\lambda \geq 0 \implies \exists q_+ \in \text{Fix}(\varphi) \cap \tilde{N}$ with $\sigma(q_+) < 0$.

*Proof.* The implications follow with $q_- = q_{\min}$, $q_+ = q_{\max}$. We show only the former, the latter being analogous. Suppose that the integral of $\sigma$ is non-positive. By Proposition 4.20 $\varphi$ is quasi-autonomous, namely, there exists a quasi-autonomous $t \mapsto H_t$ generating $t \mapsto \varphi_t$ with $\varphi_0 = \text{id}_N$ and $\varphi_1 = \varphi$. By Lemma 4.17 the corollary is established, if we show that there exists $t_- \in [0,1]$ such that $H_{t_-}(q_{\min}) < 0$. Indeed, assume by contradiction that $H_t(q_{\min}) \geq 0$, for all $t \in [0,1]$. This means that $H_t \geq 0$. Furthermore, as $\varphi \neq \text{id}_N$, there exists $(s,w) \in [0,1] \times N$ with $H_s(w) > 0$, which, by Lemma 4.17 implies

$$0 < \int_0^1 \left( \int_N H_t \right) dt = \frac{1}{2} \int_N \sigma d\lambda.$$

From this contradiction we conclude the existence of a $t_-$ as above. \quad \square
Part II

A local systolic-diastolic inequality for odd-symplectic forms

In this second part, we formulate a generalisation of the contact systolic-diastolic inequality conjecture for odd-symplectic forms and establish it in some particular cases. In the following discussion \((\Sigma, o_{\Sigma})\) is a connected oriented closed manifold of dimension \(2n + 1\). As usual, we endow \(\Sigma\) with an auxiliary Riemannian metric in order to measure distances and norms of various objects defined on \(\Sigma\). Let us introduce some notation. In this part, \(r\) will always denote a real number in \([0, 1]\). We will often consider paths \(r \mapsto o_r\) with values in some target space \(O\). We use the short-hand \(\{o_r\} \subset O\) for such a path and \([o_r]\) for a homotopy class of paths with conveniently chosen boundary conditions. If \(\{o_r\}\) and \(\{o'_r\}\) are two paths with \(o_1 = o'_0\), we write \(\{o_r\} \# \{o'_r\}\) for the concatenated path.

5 The volume of a closed two-form

5.1 Definition of the volume

Let \(\Xi^k(\Sigma)\) be the set of closed \(k\)-forms on \(\Sigma\) and \(\Xi^2_\mathbb{C}(\Sigma)\) the set of elements in \(\Xi^2(\Sigma)\) representing the de Rham cohomology class \(C \in H^2_{dR}(\Sigma)\). The set \(\Xi^2_\mathbb{C}(\Sigma)\) is affine with underlying vector space

\[\hat{\Omega}^1(\Sigma) := \frac{\Omega^1(\Sigma)}{\Xi^1(\Sigma)}\).

Indeed, for every \(\Omega \in \Xi^2_\mathbb{C}(\Sigma)\), we have the surjective map

\[B_\Omega : \Omega^1(\Sigma) \rightarrow \Xi^2_\mathbb{C}(\Sigma), \quad \alpha \mapsto \Omega + d\alpha.\]

This map passes to the quotient under \(\Xi^1(\Sigma)\) and yields a bijection

\[B_\Omega : \hat{\Omega}^1(\Sigma) \rightarrow \Xi^2_\mathbb{C}(\Sigma)\]

so that we have an identification

\[T_\Omega \Xi^2_\mathbb{C}(\Sigma) = \hat{\Omega}^1(\Sigma).\]

We define now an exact one-form \(\text{vol} \in \Omega^1(\Omega^1(\Sigma))\). To this purpose, we fix a reference two-form \(\Omega_0 \in \Xi^2_\mathbb{C}(\Sigma)\), and for every \(\alpha \in \Omega^1(\Sigma)\), we use the short-hand

\[\Omega_\alpha := B_{\Omega_0}(\alpha) = \Omega_0 + d\alpha.\]

For every \(\beta \in \Omega^1(\Sigma) \cong T_\alpha \Omega^1(\Sigma)\), we set

\[\text{vol}_\alpha \cdot \beta := \int_{\Sigma} \beta \wedge \Omega^\alpha.\]

In the lemma below, we show that the one-form \(\text{vol}\) admits the primitive functional

\[\text{Vol} : \Omega^1(\Sigma) \rightarrow \mathbb{R}, \quad \text{Vol}(\alpha) := \int_0^1 \left(\int_{\Sigma} \alpha \wedge \Omega_{\alpha r}^\alpha\right) dr. \quad (5.1)\]
Remark 5.1. Exchanging the order of integration, we can rewrite
\[
\text{Vol}(\alpha) = \int_{\Sigma} \text{CS}(\alpha), \quad \text{CS}(\alpha) := \int_{0}^{1} \alpha \wedge \Omega_{ra}^{n} \, dr.
\]
When \( C = 0 \) and \( \Omega_{0} = 0 \), the form \((n + 1)\text{CS}(\alpha)\) reduces to the Chern-Simons form for principal \( S^1 \)-bundles with invariant polynomial \( P(x_1, \ldots, x_{n+1}) = x_1 \cdot \ldots \cdot x_{n+1} \) [CS74, equation (3.1)]. Furthermore, \((n + 1)\text{Vol}(\alpha)\) corresponds to the cohomology class in [CS74, Theorem 3.9, case 2] (where \( M \) therein is our \( \Sigma \) and \( l \) is our \( n + 1 \)).

Lemma 5.2. The functional \( \text{Vol} \) associated with \( \Omega_{0} \) is uniquely characterised by the properties
\[
\text{Vol}(0) = 0, \quad d\text{Vol} = \text{vol}.
\]
The following formula holds (see also [CS74, equation (3.5)]):
\[
\text{Vol}(\alpha) = \sum_{j=0}^{n} \frac{1}{j + 1} \binom{n}{j} \int_{\Sigma} \alpha \wedge (d\alpha)^{j} \wedge \Omega_{0}^{n-j}.
\]

Proof. Differentiating under the integral sign we write \( d_{\alpha}\text{Vol}(\beta) = \int_{0}^{1} V(r) dr \) and compute
\[
V(r) = \int_{\Sigma} \left( \beta \wedge \Omega_{ra}^{n} + \alpha \wedge \Omega_{ra}^{n-1} \wedge nr\beta \right)
\]
\[
= \int_{\Sigma} \left( \beta \wedge \Omega_{ra}^{n} + nr\Omega_{ra}^{n-1} \wedge (\beta \wedge d\alpha - d(\alpha \wedge \beta)) \right)
\]
\[
= \int_{\Sigma} \left( \beta \wedge \Omega_{ra}^{n} + r\beta \wedge \Omega_{ra}^{n-1} \wedge (n\alpha) \right)
\]
\[
= \frac{d}{dr} \left( \int_{\Sigma} r\beta \wedge \Omega_{ra}^{n} \right),
\]
where in the third equality we used Stokes’ Theorem. We conclude that
\[
d_{\alpha} \text{Vol} \cdot \beta = \int_{0}^{1} \frac{d}{dr} \left( \int_{\Sigma} r\beta \wedge \Omega_{ra}^{n} \right) dr = \int_{\Sigma} 1 \wedge \beta \wedge \Omega_{ra}^{n} - \int_{\Sigma} 0 \wedge \beta \wedge \Omega_{ra}^{n} = \text{vol}_{\alpha} \cdot \beta.
\]
The formula for \( \text{Vol} \) follows by expanding the binomial \( \Omega_{ra}^{n} = (\Omega_{0} + r\alpha)^{n} \) in \( \text{5.1} \) and integrating each term.

Remark 5.3. If \( \dim \Sigma = 3 \), namely \( n = 1 \), the formula for \( \text{Vol} \) turns into
\[
\text{Vol}(\alpha) = \int_{\Sigma} \alpha \wedge (\Omega_{0} + \frac{1}{2}d\alpha).
\]
Let us specify the dependence of \( \text{Vol} \) on the reference form \( \Omega_{0} \).

Lemma 5.4. Let \( \text{Vol}' : \Omega^{1}(\Sigma) \to \mathbb{R} \) be the volume functional associated with another reference two-form \( \Omega_{0}' \in \mathbb{E}^{2}(\Sigma) \). If \( \alpha' \in \Omega^{1}(\Sigma) \) is such that \( \Omega_{0}' = \Omega_{0} + d\alpha' \), then
\[
\text{Vol}(\alpha) = \text{Vol}(\alpha') + \text{Vol}'(\alpha - \alpha'), \quad \forall \alpha \in \Omega^{1}(\Sigma).
\]
Proof. For every $\alpha, \beta \in \Omega^1(\Sigma)$, we have
\[
\text{vol}'_{\alpha - \alpha_1} : \beta = \int_{\Sigma} \beta \wedge (\Omega'_0 + d\alpha - d\alpha')^n = \int_{\Sigma} \beta \wedge (\Omega_0 + d\alpha)^n = \text{vol}_{\alpha} \cdot \beta,
\]
where vol' := dVol'. Therefore, from the fundamental theorem of calculus, we get
\[
\text{Vol}(\alpha) - \text{Vol}(\alpha') = \int_{0}^{1} \text{vol}'_{r(\alpha - \alpha')} \cdot (\alpha - \alpha')dr = \int_{0}^{1} \text{vol}'_{r(\alpha - \alpha')} \cdot (\alpha - \alpha')dr
= \text{Vol}'(\alpha - \alpha').
\]

We now wish to use the map $B_{\Omega_0}$ to push the volume form and functional to the space of closed two-forms in the class $C$. To this purpose, we observe that Vol behaves linearly under the addition of closed one-forms. More precisely, for all $\alpha \in \Omega^1(\Sigma)$ we have
\[
\text{Vol}(\alpha + \tau) - \text{Vol}(\alpha) = \text{Vol}(\tau) = \langle [\tau] \cup C^n, [\Sigma] \rangle, \quad \forall \tau \in \Xi^1(\Sigma),
\]
and similarly,
\[
\text{vol}_{\alpha} \cdot \tau = \langle [\tau] \cup C^n, [\Sigma] \rangle, \quad \forall \tau \in \Xi^1(\Sigma).
\]
These formulae suggest to treat two separate cases.

Case 1: $C^n = 0$.

The volume functional $\text{Vol} : \Omega^1(\Sigma) \to \mathbb{R}$ passes to the quotient by $\Xi^1(\Sigma)$ according to (5.2):
\[
\text{Vol} : \Xi^2_C(\Sigma) \to \mathbb{R}, \quad \text{Vol}(\Omega_0) = 0.
\]

Following Lemma 5.4, if $\text{Vol}'$ denotes the functional obtained from $\Omega'_0$, then there holds
\[
\text{Vol}'(\Omega) = \text{Vol}'(\Omega'_0) + \text{Vol}'(\Omega), \quad \forall \Omega \in \Xi^2_C(\Sigma).
\]
This means that vol $\in \Omega^1(\Xi^2_C(\Sigma))$ descends to a one-form
\[
\text{vol} \in \Omega^1(\Xi^2_C(\Sigma)), \quad d\text{vol} = \text{vol},
\]
which is independent of the reference two-form $\Omega_0$.

Case 2: $C^n \neq 0$.

The functional Vol does not descend to $\Xi^2_C(\Sigma)$. Indeed, by Poincaré duality, there exists a form $\tau \in \Xi^1(\Sigma)$ such that
\[
[\tau] \cup C^n \neq 0.
\]
However, we can use the volume function to define a distinguished class of one-forms.

Definition 5.5. We say that $\alpha \in \Omega^1(\Sigma)$ is a normalised one-form, if the implication
\[
C^n \neq 0 \implies \text{Vol}(\alpha) = 0
\]
holds true (in particular, when $C^n = 0$ as in Case 1, all one-forms are normalised).
The inclusion \( \text{Vol}^{-1}(0) \subset \Omega^1(\Sigma) \) induces a bijection

\[
\text{Vol}^{-1}(0) \sim \rightarrow \Omega^1(\Sigma) \cong \Xi_C^2(\Sigma),
\]

where \( \alpha' \sim \alpha'' \) if and only if \( \alpha'' - \alpha' \in \Xi_1(\Sigma) \). Indeed, by (5.2) any class \( \bar{\alpha} := \alpha + \Xi_1(\Sigma) \in \bar{\Omega}_1(\Sigma) \) has a normalised representative \( \alpha' = \alpha + s \tau \), for some \( s \in \mathbb{R} \). We also observe that, if \( \alpha'' \) is another normalised representative of \( \bar{\alpha} \), then, again by (5.2), we have

\[
[\alpha'' - \alpha'] \cup C^n = 0.
\]

In view of (5.6), we define the volume form and functional on \( \Xi_C^2(\Sigma) \) trivially by setting

\[
\text{Vol} : \Xi_C^2(\Sigma) \rightarrow \mathbb{R}, \quad \text{Vol} := 0, \quad \text{vol} \in \Omega^1(\Xi_C^2(\Sigma)), \quad \text{vol} := 0.
\]

If \( \alpha \) is a normalised one-form representing \( \bar{\alpha} \in \bar{\Omega}_1(\Sigma) \), then the inclusion \( \ker \text{vol}_\alpha \subset \Omega^1(\Sigma) \) induces the surjection \( \ker \text{vol}_\alpha \rightarrow \Omega^1(\Sigma) \cong T_{\bar{\alpha}} \Omega^1(\Sigma) \) due to the existence of \( \tau \) satisfying (5.5), and in turn the isomorphism

\[
\ker \text{vol}_\alpha \cong \Xi_1(\Sigma) \rightarrow T_{\bar{\alpha}} \Omega^1(\Sigma) \cong T_{\bar{\alpha}} \Omega^1(\Sigma),
\]

thanks to (5.3) and (5.7).

**Remark 5.6.** For contact forms and Hamiltonian systems the volume functional recovers the following well-known formulae.

- **(Contact forms)** This is an instance of Case 1. Let \( \Omega_0 = 0 \) and \( \alpha \in \Omega^1(\Sigma) \) be a contact form. If we endow \( \Sigma \) with the orientation \( \phi_\alpha \), we recover the contact volume up to a constant factor

\[
\text{Vol}(d\alpha) = \text{Vol}(\alpha) = \frac{1}{n+1} \int_\Sigma \alpha \wedge (d\alpha)^n = \frac{1}{n+1} \text{Volume}(\alpha).
\]

- **(Hamiltonian systems)** This is an instance of Case 2. Let \( \Sigma = M \times S^1 \) and \( \Omega_0 = p^* \omega_0 \) for some symplectic form \( \omega_0 \) on \( M \), where \( p : M \times S^1 \rightarrow M \) is the projection on the first factor. Let \( \alpha = Hd\phi \), where \( H : M \times S^1 \rightarrow \mathbb{R} \) and \( \phi \) is the coordinate on \( S^1 \). Then, the volume functional recovers the **Calabi invariant** of \( H \)

\[
\text{Vol}(Hd\phi) = \int_{M \times S^1} (Hd\phi) \wedge \omega_0^n = \text{CAL}_{\omega_0}(H)
\]

and \( \alpha \) is normalised if and only if the Calabi invariant of \( H \) vanishes.

5.2 The volume is invariant under pull-back and isotopies

In this subsection, we prove two invariance results for the volume. Recall that \( \Omega_0 \) is the fixed reference form in \( \Xi_C^2(\Sigma) \) for \( C \in H^2_{\text{dR}}(\Sigma) \).
Proposition 5.7. Let $\Sigma$ and $\Sigma^\vee$ be two closed oriented manifolds of dimension $2n+1$ and $\Pi : \Sigma^\vee \to \Sigma$ a map of degree $\deg \Pi \in \mathbb{Z}$. Let $C^\vee := \Pi^* C \in H^2_{dR}(\Sigma^\vee)$ and $\Omega^\vee_0 := \Pi^* \Omega_0 \in \Xi^2_C(\Sigma^\vee)$. If $\text{Vol}$, $\text{Vol}^\vee$ are the volumes associated with $\Omega_0$ and $\text{Vol}^\vee$, $\text{Vol}^\vee$ the volumes associated with $\Omega^\vee_0$, there holds

$$\text{Vol}^\vee \circ \Pi^* = (\deg \Pi) \cdot \text{Vol}, \quad \text{Vol}_{\text{vol}}^\vee \circ \Pi^* = (\deg \Pi) \cdot \text{Vol}_{\text{vol}}.$$

Proof. The statement follows immediately from the definition of the volume and the observation that for all top dimensional forms $\mu$ on $\Sigma$ there holds

$$\int_{\Sigma^\vee} \Pi^* \mu = \deg \Pi \cdot \int_{\Sigma} \mu.$$ 

For the second result, we need a little preparation. Let $\Psi : \Sigma \to \Sigma$ be a diffeomorphism isotopic to the identity $\text{id}_\Sigma$. Since the pull-back $\Psi^*$ of $\Psi$ acts as the identity in cohomology, there is a normalised one-form $\theta \in \Omega^1(\Sigma)$ (determined up to a normalised closed one-form) such that

$$d\theta = \Psi^* \Omega_0 - \Omega_0.$$ (5.8)

We define

$$\check{\Psi}^*_\theta : \Omega^1(\Sigma) \to \Omega^1(\Sigma), \quad \check{\Psi}^*_\theta(\alpha) := \theta + \Psi^* \alpha,$$

so that the following diagram commutes:

$$\begin{array}{ccc}
\Omega^1(\Sigma) & \xrightarrow{\check{\Psi}^*_\theta} & \Omega^1(\Sigma) \\
\downarrow B_{\Omega_0} & & \downarrow B_{\Omega_0} \\
\Xi^2_C(\Sigma) & \xrightarrow{\Psi^*} & \Xi^2_C(\Sigma)
\end{array}$$ (5.9)

Let $\{\Psi_r\}$ be an isotopy with $\Psi_0 = \text{id}_\Sigma$ and $\Psi_1 = \Psi$. This gives rise to a smooth family of one-forms $\{\theta_r\}$ satisfying

$$\theta_0 = 0, \quad d\theta = \Psi^*_r \Omega_0 - \Omega_0, \quad \forall r \in [0, 1].$$

To construct such a family, we just take the time-dependent vector field $X_r$ generating the isotopy $\{\Psi_r\}$ and let $\{\theta_r\}$ be the unique path such that

$$\theta_0 = 0, \quad d\theta_r = \Psi^*_r (\iota_{X_r} \Omega_0).$$

To ease the notation, if $\alpha \in \Omega^1(\Sigma)$, we use the short-hand

$$\check{\Psi}^*_\theta(\alpha) := \check{\Psi}^*_\theta(\alpha) = \theta_r + \Psi^*_r \alpha.$$ (5.10)

Every $\theta_r$ is normalised, i.e. $\text{Vol}(\theta_r) = 0$. Indeed, $\text{Vol}(\theta_0) = \text{Vol}(0) = 0$ and

$$\frac{d}{dr} \text{Vol}(\theta_r) = \text{vol}_{\theta_r}(\dot{\theta}_r) = \int_{\Sigma} \Psi^*_r (\iota_{X_r} \Omega_0) \wedge (\Psi^*_r \Omega_0)^n = \int_{\Sigma} (\iota_{X_r} \Omega_0) \wedge \Omega^0_0 = 0.$$

In particular, the form $\tau_1 := \theta - \theta_0$ is closed and normalised and as observed in (5.7), there holds

$$[\tau_1] \cup C^n = 0$$

(note that this condition is automatically satisfied if $C^n = 0$).
Proposition 5.8. If $\Psi$ is a diffeomorphism isotopic to $\text{id}_\Sigma$ and $\theta$ is any normalised one-form satisfying (5.8), there holds
\[ \text{Vol} \circ \hat{\Psi}_\theta^* = \text{Vol}. \]
As a consequence, the set of normalised one-forms is $\hat{\Psi}_\theta^*$-invariant and we have
\[ \text{Vol} \circ \Psi^* = \text{Vol}. \]

Proof. For any $\alpha \in \Omega^1(\Sigma)$, we have
\[
\frac{d}{dr} \hat{\Psi}_r^*(\alpha) = \Psi_r^*(\iota_X, \Omega_0) + \Psi_r^* \left( \iota_X, d\alpha + d(\alpha(X_r)) \right) = \Psi_r^* \left( \iota_X, \Omega_\alpha + d(\alpha(X_r)) \right).
\]
Using this relation, we compute
\[
\frac{d}{dr} \text{Vol}(\hat{\Psi}_r^*(\alpha)) = \text{vol}_{\hat{\Psi}_r^*(\alpha)} \cdot \frac{d}{dr} \hat{\Psi}_r^*(\alpha) = \int_\Sigma \Psi_r^* \left( \iota_X, \Omega_\alpha + d(\alpha(X_r)) \right) \wedge (\Psi_r^*\Omega_\alpha)^n
\]
\[
= \int_\Sigma \iota_X \Omega_\alpha \wedge \Omega_\alpha^n + \int_\Sigma d(\alpha(X_r)) \wedge \Omega_\alpha^n
\]
\[= 0.\]
We conclude from (5.2) and the relation $[\tau_1] \cup C^n = 0$ that
\[ \text{Vol}(\hat{\Psi}_\theta^*(\alpha)) = \text{Vol}(\hat{\Psi}_1^*(\alpha) + \tau_1) = \text{Vol}(\hat{\Psi}_1^*(\alpha)) + \langle [\tau_1] \cup C^n, [\Sigma] \rangle = \text{Vol}(\hat{\Psi}_\theta^*(\alpha)) = \text{Vol}(\alpha). \]
This proves the first identity. The second identity follows from the first one and the commutation relation $B_{\Omega_0} \circ \hat{\Psi}_\theta^* = \Psi^* \circ B_{\Omega_0}$ in (5.9). \qed

6 Odd-symplectic forms

6.1 A couple of definitions

For a closed two-form $\Omega \in \Xi^2(\Sigma)$ on $\Sigma$, we consider the (possibly singular) distribution $\ker \Omega \to \Sigma$ defined by
\[ \ker \Omega := \{(z, u) \in T\Sigma \mid \Omega_z(u, v) = 0, \forall v \in T_z\Sigma\}. \]
The distribution $\ker \Omega$ is naturally co-oriented by $\Omega^n$ and we orient it combining the orientation on $\Sigma$ with such a co-orientation.

Definition 6.1. We say that $\Omega \in \Xi^2(\Sigma)$ is odd-symplectic if $\ker \Omega \to \Sigma$ is a one-dimensional distribution and denote the set of odd-symplectic forms by $\mathcal{S}(\Sigma) \subset \Xi^2(\Sigma)$. If $C \in H^2_{\text{dR}}(\Sigma)$, we write $\mathcal{S}_C(\Sigma) := \mathcal{S}(\Sigma) \cap \Xi^2_C(\Sigma)$.

Remark 6.2. Any of the following conditions gives an equivalent definition of $\Omega \in \Xi^2(\Sigma)$ being odd-symplectic:

- $\Omega^n$ is nowhere vanishing.
- There exists a one-form $\sigma$ on $\Sigma$ such that $\sigma \wedge \Omega^n$ is a volume form.
• There exists a nowhere vanishing one-form $\sigma$ such that $\Omega|_{\ker \sigma}$ is a non-degenerate bi-linear form on $\ker \sigma$.

**Definition 6.3.** Given a reference form $\Omega_0 \in \Xi_2^2(\Sigma)$ and a pair of one-forms $\alpha_0, \sigma_0 \in \Omega^1(\Sigma)$, we can build an affine map

$$C^\infty(\Sigma) \to \Xi_2^2(\Sigma),$$

$$H \mapsto \Omega_{\alpha_0 + H\sigma_0} = \Omega_0 + d(\alpha_0 + H\sigma_0).$$

We refer to the image of this map, as the set of $H$-forms (with respect to $\Omega_0$, $\alpha_0$, and $\sigma_0$).

### 6.2 A stability result

As we will see in Section 10, given $\Omega_0$, $\alpha_0$ and $\sigma_0$ as in Definition (6.3), it is extremely helpful to see, if an odd-symplectic form $\Omega_1 \in \mathcal{S}_C(\Sigma)$ admits a diffeomorphism $\Psi : \Sigma \to \Sigma$ isotopic to $\text{id}_\Sigma$ such that $\Psi^*\Omega_1$ is an $H$-form. We provide a criterion for such a diffeomorphism to exist.

**Proposition 6.4.** Let $\{\Omega_r = \Omega_0 + d\alpha_r\}$ be a path in $\mathcal{S}_C(\Sigma)$. Let $\{\sigma_r\}$ be a path in $\Omega^1(\Sigma)$ such that $\sigma_r \wedge \Omega^n_r$ is a volume form on $\Sigma$, which exists according to Remark 6.2. Let us denote by $I_r : (\ker \sigma_r)^* \to \ker \sigma_r$ the inverse of the map $v \mapsto \iota v \Omega_{\alpha_r}|_{\ker \sigma_r}$. If

$$\left(\dot{\sigma}_r - \iota_{I_r(\dot{\alpha}_r)}d\sigma_r\right)|_{\ker \sigma_r} = 0,$$

then there exist an isotopy $\{\Psi_r : \Sigma \to \Sigma\}$ with $\Psi_0 = \text{id}_\Sigma$ generated by a time-dependent vector field $X_r \in \ker \sigma_r$ and a path of functions $\{H_r : \Sigma \to \mathbb{R}\}$ such that, for every $r \in [0, 1]$,

$$\tilde{\Psi}_r^*\alpha_r = \alpha_0 + H_r\sigma_0 + dK_r, \quad K_r := \int_0^r \alpha_{r'}(X_{r'}) \circ \Psi_{r'} dr'.$$

Here $\tilde{\Psi}_r$ is the map in (5.10). In particular, we have

$$\tilde{\Psi}_1^*\Omega_1 = \Omega_0 + d(\alpha_0 + H_1\sigma_0), \quad \text{Vol}(\alpha_1) = \text{Vol}(\alpha_0 + H_1\sigma_0).$$

**Proof.** The argument is a refinement of the Gray stability theorem, see e.g. [Gei08, Theorem 2.2.2]. We introduce a vector field $V_r$ on $\Sigma$ uniquely determined by the relations

$$\iota_{V_r}\Omega_{\alpha_r} = 0, \quad \sigma_r(V_r) = 1.$$

We define a path of vector fields $r \mapsto X_r$ on $\Sigma$ satisfying the relations

$$X_r \in \ker \sigma_r, \quad \left(\iota_{X_r}\Omega_{\alpha_r} + \dot{\alpha}_r\right)|_{\ker \sigma_r} = 0,$$

which exists and is unique by (i). Applying (ii), we also have

$$\left(\iota_{X_r}d\sigma_r + \dot{\sigma}_r\right)|_{\ker \sigma_r} = 0.$$  

Let $r \mapsto \Psi_r$ be the isotopy on $\Sigma$ obtained by integrating $X_r$ and setting $\Psi_0 = \text{id}_\Sigma$. We construct an auxiliary path of functions $r \mapsto J_r$ through

$$J_0 = 0, \quad \dot{J}_r = \left(d\sigma_r(X_r, V_r) + \dot{\sigma}_r(V_r)\right) \circ \Psi_r.$$
Combining the last equation with (6.3), we arrive at

\[ \iota_{X_r}d\sigma_r + \dot{\sigma}_r = (\dot{J}_r \circ \Psi_r^{-1})\sigma_r. \]

With this piece of information, we can compute

\[ \frac{d}{dr}(\Psi_r^*\sigma_r) = \Psi_r^*(\iota_{X_r}d\sigma_r + \dot{\sigma}_r) = \Psi_r^*(\dot{J}_r \circ \Psi_r^{-1}\sigma_r) = \dot{J}_r(\Psi_r^*\sigma_r). \]

Since \( r \mapsto e^{J_r}\sigma_0 \) also satisfies the same ordinary differential equation and coincides with \( \Psi_r^*\sigma_r \) at \( r = 0 \), we see that

\[ \Psi_r^*\sigma_r = e^{J_r}\sigma_0. \tag{6.4} \]

Next we define the path of functions \( \{H_r : \Sigma \to \mathbb{R}\} \) by

\[ H_0 = 0, \quad \dot{H}_r = (\dot{\alpha}_r(V_r) \circ \Psi_r)e^{J_r}. \tag{6.5} \]

Conditions (6.2) and (6.5) together can be rephrased as

\[ \iota_{X_r}\Omega_{\alpha_r} + \dot{\alpha}_r = ((\dot{H}_re^{-J_r}) \circ \Psi_r^{-1})\sigma_r. \tag{6.6} \]

We are ready to prove the formula for \( \hat{\Psi}_r^*\alpha_r \) in the statement of the proposition. The identity clearly holds for \( r = 0 \) and we just need to show that the \( r \)-derivatives of both sides coincide for every \( r \):

\[
\frac{d}{dr}(\hat{\Psi}_r^*\alpha_r) = \Psi_r^*(\iota_{X_r}d\alpha_r + \dot{\alpha}_r) + \dot{\theta}_r = d(\alpha_r(X_r) \circ \Psi_r) + \Psi_r^*(\iota_{X_r}\Omega_{\alpha_r} + \dot{\alpha}_r)
= d\dot{K}_r + \dot{H}_re^{-J_r}\Psi_r^*\sigma_r
= d\dot{K}_r + \dot{H}_r\sigma_0,
\]

where we used that \( \dot{\theta}_r = \Psi_r^*\iota_{X_r}\Omega_0 \) and equations (6.4) and (6.6). Finally, Proposition 5.8 yields the identity \( \text{Vol}(\alpha_1) = \text{Vol}(\alpha_0 + H_1\sigma_0) \). \( \Box \)

**Remark 6.5.** The form \( \sigma_r \wedge \Omega_r^n \) is a volume form and condition (6.1) is satisfied in the following two extreme cases:

(a) The form \( \Omega_0 \) vanishes and \( \alpha_r = T_r\sigma_r \) is a contact form for every \( r \), where \( \{T_r\} \) is some path of real numbers. In this case, \( K_r \) and \( \theta_r \) vanish so that we have the usual Gray stability theorem

\[ \Psi_r^*\alpha_r = (1 + \frac{1}{T_0}H_r)\alpha_0. \]

(b) The form \( \sigma_0 \) is closed, there holds \( \sigma_r = \sigma_0 \) for all \( r \), and \( \Omega_r \) non-degenerate on \( \ker \sigma_0 \).

**Remark 6.6.** It would be interesting to find a condition not involving \( r \)-derivatives implying (6.1). Such a condition can indeed be found in cases (a) or (b) in the remark above, where the odd-symplectic forms \( \Omega_r \) are actually stable according to [CM05, Section 2.1].

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7 Oriented $S^1$-bundles

7.1 $S^1$-bundles and free $S^1$-actions

Let $\mathcal{P}(\Sigma)$ be the space of all oriented $S^1$-bundles having $\Sigma$ as total space, up to equivalence. Here we say that two bundles are equivalent if they have the same oriented fibres. If $p : \Sigma \to M$ is an element of $\mathcal{P}(\Sigma)$, then $M$ is a closed manifold of dimension $2n$. We write $\Xi^2(M)$ for the space of closed two-forms on $M$ and $\Xi^2_c(M)$ for the set of those $\omega \in \Xi^2(M)$ with $[\omega] = c$. We orient $M$ combining the orientation on $\Sigma$ with the orientation of the $p$-fibres. We denote by $-p$ the bundle obtained from $p$ by reversing the orientation.

**Definition 7.1.** If $p$ belongs to $\mathcal{P}(\Sigma)$, then $p^{-1}(pt) \subset \Sigma$ denotes an oriented $p$-fibre and $[p^{-1}(pt)]_\mathbb{Z} \in H_1(\Sigma; \mathbb{Z})$ its integral homology class.

**Definition 7.2.** A bundle map between oriented $S^1$-bundles $p : \Sigma \to M$ and $p^\vee : \Sigma^\vee \to M^\vee$ is a map $\Pi : \Sigma^\vee \to \Sigma$ diffeomorphically sending every oriented fibre of $p^\vee$ to an oriented fibre of $p$. In this case, we write $p^\vee = \Pi p$. A bundle map yields a quotient map $\pi : M^\vee \to M$ between the base manifolds so that $p \circ \Pi = \pi \circ p^\vee$. If $\Pi$ is also a diffeomorphism, we say that $\Pi$ is a bundle isomorphism. When $\Sigma^\vee = \Sigma$, we write $p^\vee = p'$, where $p' : \Sigma \to M'$, and denote by $\Psi : \Sigma \to \Sigma$ a bundle isomorphism between $p$ and $p'$ with quotient map $\psi : M' \to M$. We summarise the properties of $\Pi$ and $\Psi$ in two commutative diagrams.

\[
\begin{array}{ccc}
\Sigma^\vee & \xrightarrow{\Pi} & \Sigma \\
\downarrow{p^\vee} & & \downarrow{p} \\
M^\vee & \xrightarrow{\pi} & M \\
\end{array}
\quad
\begin{array}{ccc}
\Sigma & \xrightarrow{\Psi} & \Sigma \\
\downarrow{\sim} & & \downarrow{\sim} \\
\Psi : M' & \xrightarrow{\sim} & M' \\
\end{array}
\]

Let $\mathcal{U}(\Sigma)$ be the space of free $S^1$-actions on $\Sigma$. For $u \in \mathcal{U}(\Sigma)$, we denote by $V = V_u$ the fundamental vector field on $\Sigma$ generated by $u$. We have a natural map

\[
\mathcal{U}(\Sigma) \to \mathcal{P}(\Sigma),
\]

which associates to a free $S^1$-action the quotient map onto its orbit space. It is a classical fact that this map is surjective. Indeed, let us fix $p \in \mathcal{P}(\Sigma)$. Using a partition of unity, we find a vector field $X$ on $\Sigma$, positively tangent to the $p$-fibres at every point. If $T(z) > 0$ is the period of the periodic orbit of $X$ starting at $z \in \Sigma$, then the rescaled vector field $V := TX$ yields the desired $S^1$-action. The set $\mathcal{U}(\Sigma)$ carries a natural $C^1$-topology as a closed subset of the space $\text{Maps}(S^1 \times \Sigma, \Sigma)$ and we endow $\mathcal{P}(\Sigma)$ with the quotient topology brought by the map $[\Sigma]$. We write $\mathcal{U}(p)$ for the fibre above $p \in \mathcal{P}(\Sigma)$. This is a convex space in the following sense. If $V_0 = T_1 V_0$ are the fundamental vector fields of $u_0, u_1 \in \mathcal{U}(p)$, then

\[
V_r := \frac{V_1}{r + (1 - r)T_1}, \quad \forall r \in [0, 1]
\]

is the fundamental vector field of some $u_r \in \mathcal{U}(p)$.

Finally, if $\Lambda(\Sigma)$ denotes the space one-periodic curves in $\Sigma$, we have a map

\[
\mathcal{J}_p : \Sigma \to \Lambda(\Sigma)
\]

associating to a point the $u$-orbit through it for some $u \in \mathcal{U}(p)$. Up to an orientation-preserving change of parametrisation of the elements of $\Lambda(\Sigma)$, this map depends only on $p$. 

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Definition 7.3. If \( p : \Sigma \to M \) belongs to \( \Psi(\Sigma) \) and \( u \) belongs to \( \U(p) \), we write
\[
e_Z \in H^2(M; \mathbb{Z})
\]
for minus the Euler class of \( u \), as defined, for example in [Che77]. This class is independent of \( u \in \U(p) \), and therefore, we refer to it as minus the Euler class of \( p \).

Remark 7.4. The bundle \( p \) is trivial (namely admits a global section) if and only if \( e_Z = 0 \). If \( p^\vee : \Sigma^\vee \to M^\vee \) is another bundle with minus Euler class \( e_Z^\vee \in H^2(M^\vee; \mathbb{Z}) \), then
\[
p^\vee = \Pi^* p \implies e_Z^\vee = \pi^* e_Z,
\]
where \( \Pi \) is a bundle map as in (7.1).

The inclusion map \( \mathbb{Z} \hookrightarrow \mathbb{R} \) induces a map on the level of homology and cohomology and we write \( e \) and \( [p^{-1}(pt)] \) for the images of \( e_Z \) and \( [p^{-1}(pt)] \), respectively:
\[
H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R}) \cong H^2_{dR}(M), \quad e_Z \mapsto e, \quad H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{R}), \quad [p^{-1}(pt)] \mapsto [p^{-1}(pt)].
\]
From the piece of the Gysin exact sequence
\[
H^0_{dR}(M) \xrightarrow{[\cdot]} H^2_{dR}(M) \xrightarrow{p*} H^2_{dR}(\Sigma),
\]
we know that \( \mathbb{R} \cdot e = \ker \left( p^* : H^2_{dR}(M) \to H^2_{dR}(\Sigma) \right) \). (7.4)

Moreover, by the universal coefficient theorem, we get
\[
e_Z \text{ is torsion } \iff e = 0, \quad [p^{-1}(pt)] \text{ is torsion } \iff [p^{-1}(pt)] = 0.
\]

The Chern-Weil theory yields a convenient description of the real cohomology class \( e \). Let \( \mathcal{K}(u) \) denote the space of connection one-forms associated with \( u \in \U(p) \):
\[
\mathcal{K}(u) := \{ \eta \in \Omega^1(\Sigma) \mid \eta(V) = 1, \ d\eta = p^* \kappa_\eta \text{ for some } \kappa_\eta \in \Omega^2(M) \}.
\]
This space is non-empty, and for every \( \eta \in \mathcal{K}(u) \), the form \( \kappa_\eta \) is closed and we have
\[
e = [\kappa_\eta] \in H^2_{dR}(M).
\]
Conversely, for any \( \kappa \in \Omega^2(M) \) with \( e = [\kappa] \), there is \( \eta_\kappa \in \mathcal{K}(u) \) with \( \kappa_{\eta_\kappa} = \kappa \). We denote by
\[
\mathcal{K}(p) := \bigcup_{u \in \U(p)} \mathcal{K}(u)
\]
the set of connection one-forms for \( p \).

7.2 Flat \( S^1 \)-bundles and the local structure of \( \Psi(\Sigma) \)

This subsection is devoted to the proof of three lemmas. In the first two, we study flat bundles, i.e. with \( e = 0 \). In the last one, we prove a theorem of Weinstein [Wei74] showing that the space \( \Psi(\Sigma) \) is locally trivial.
Lemma 7.5. Let $p : \Sigma \to M$ be an oriented $S^1$-bundle. We have an equivalence

$$e = 0 \iff [p^{-1}(pt)] \neq 0.$$  \hspace{1cm} (7.5)

If $c \in H^2_{dR}(M)$ and we define $C := p^*c \in H^2_{dR}(\Sigma)$, there holds

$$\text{PD}(C^n) = \langle c^n, [M] \rangle : [p^{-1}(pt)] \in H_1(\Sigma; \mathbb{R}).$$  \hspace{1cm} (7.6)

In particular, if $c^n \neq 0$, then

$$\langle i \rangle \quad \text{ker} \left( H^1_{dR}(\Sigma) \xrightarrow{\langle \cdot, [p^{-1}(pt)] \rangle} \mathbb{R} \right) = \text{ker} \left( H^1_{dR}(\Sigma) \xrightarrow{\langle \cdot \rangle : C^n} H^{2n+1}_{dR}(\Sigma) \right),$$

$$\langle ii \rangle \quad C^n \neq 0 \iff e = 0.$$  \hspace{1cm} (7.7)

Proof. The proof of (7.5) relies on the Gysin sequence and of its Poincaré dual

\[
\begin{array}{ccccccc}
H^{2n-2}_{dR}(M) & \xrightarrow{\cap e} & H^{2n}_{dR}(M) & \xrightarrow{p^*} & H^{2n}_{dR}(\Sigma) & \xrightarrow{\text{PD}} & H^{2n-1}_{dR}(M) \\
\text{PD} & & \text{PD} & & \text{PD} & & \text{PD}
\end{array}
\]

where PD denotes Poincaré duality and $p_*$ stands for the integration along fibres. We claim that PD($p^*$) sends the class of a point $[pt]$ to $[p^{-1}(pt)]$. To this purpose, let $\mu \in \Xi^{2n}(M)$ be such that PD($[\mu]$) = $[pt]$, so that $\int_M \mu = 1$. Going around the central square in the diagram above, the claim follows, if we can show that PD($p^*[\mu]$) = $[p^{-1}(pt)]$. This last equality is true since for all $\tau \in \Xi^1(\Sigma)$ we have

$$\langle [\tau], [p^{-1}(pt)] \rangle = p_*[\tau] = \int_M e = \int_M p_*[\tau] \cdot \mu = \int_M p_*[\tau] \cdot \mu = \int_\Sigma [\tau] \cup p^*\mu = \langle [\tau] \cup p^*[\mu], [\Sigma] \rangle,$$

where in $(\ast)$ we have used Fubini’s Theorem. Therefore, $[p^{-1}(pt)] \neq 0$ if and only if the map $\cap e$ is equal to zero. This happens if and only if $\langle e, H_2(M; \mathbb{R}) \rangle = 0$ as one sees identifying $H_0(M; \mathbb{R})$ with $\mathbb{R}$. Finally, from the universal coefficient theorem, $\langle e, H_2(M; \mathbb{R}) \rangle = 0$ if and only if $e = 0$.

Let us show (7.6). If $\omega \in \Xi^2(M)$ and $\tau \in \Xi^1(\Sigma)$ is arbitrary, we have

$$\langle c^n, [M] \rangle : \langle [\tau], [p^{-1}(pt)] \rangle = \int_M p_*[\tau] \cdot \omega^n = \int_\Sigma [\tau] \cup p^*\omega^n = \langle [\tau] \cup C^n, [\Sigma] \rangle.$$  

Let us assume that $\langle c^n, [M] \rangle \neq 0$. Looking again at the last equation, we readily see that (i) in (7.7) holds. The equivalence in (ii) stems from a combination of (7.6) and (7.5)

$$C^n \neq 0 \iff \text{PD}(C^n) \neq 0 \iff [p^{-1}(pt)] \neq 0 \iff e = 0. \quad \Box$$

We now show that a flat bundle can always be pulled back to a trivial one.

Lemma 7.6. Let $p : \Sigma \to M$ be an oriented $S^1$-bundle. The class $e \in H^2_{dR}(M)$ vanishes if and only if there exists a finite cyclic cover $\pi : M' \to M$ such that the pull-back bundle $p'^* : \Sigma' \to M'$ of $p$ by $\pi$ is trivial. In this case, the order of $e_Z$ in $H^2(M; \mathbb{Z})$ equals the minimum degree of a cover with the properties above.
Proof. We preliminarily observe that if \( \pi : M^\vee \to M \) is a finite cover of degree \( k \) and the bundle \( p^\vee : \Sigma^\vee \to M^\vee \) is the pull-back of \( p \) through \( \pi \) with minus the Euler class \( e_{Z}^\vee \), then by Remark 7.4 there holds

\[
\pi_*e_{Z}^\vee = k \cdot e_{Z},
\]

where \( \pi_* : H^2(M^\vee; \mathbb{Z}) \to H^2(M; \mathbb{Z}) \) is the transfer map (see [Hat02, Chapter 3.G]). Therefore, if we suppose that \( p^\vee \) is trivial, we deduce that \( e_{Z}^\vee = 0 \), and hence, \( k \cdot e_{Z} = 0 \). In particular, the order of \( e_{Z} \) divides \( k \).

Conversely, let \( k \) be a positive integer such that \( k \cdot e_{Z} = 0 \) and take \( u \in \mathcal{U}(p) \). Let \( \widetilde{\mathcal{Z}}_{kZ} \to S^1 \) be the canonical homomorphism \( j \to j/k \), and we denote by \( u_k \) the free \( \mathcal{Z}_{kZ} \)-action on \( \Sigma \) obtained combining this homomorphism with \( u \). Let \( \Sigma_k = \Sigma/u_k \) be the quotient map by this action and denote by \( \Phi_k : \Sigma \to \Sigma_k \) the associated quotient map. The natural map \( p_k : \Sigma_k \to M \) is an oriented \( S^1 \)-bundle. Let \((e_{Z})_k \in H^2(M; \mathbb{Z})\) denote minus the Euler class of \( p_k \). Since \( S^1/\mathcal{Z}_{kZ} \cong S^1 \), through the map \( \phi \to k \cdot \phi \), we see that \((e_{Z})_k = k \cdot e_{Z} = 0\). Therefore, we have a section \( s_k : M \to \Sigma_k \) for \( p_k \) and we take a connected component \( M^\vee \subset \Sigma \) of \( \Phi_k^{-1}(s_k(M)) \). If \( i : M^\vee \to \Sigma \) is the inclusion map, we define \( \pi : M^\vee \to M \) as \( \pi := p \circ i \). The map \( \pi \) is a covering map, whose deck transformation group is given by the sub-action of \( u_k \) that leaves \( M^\vee \) invariant. The deck transformation group is isomorphic to \( \mathcal{Z}_j \), where \( j \) divides \( k \), since it is a subgroup of the cyclic group \( \mathcal{Z}_k \). Let \( \Pi : \Sigma^\vee \to \Sigma \) be a covering map lifting \( \pi \).

We sum up the construction above in the commutative diagram

\[
\begin{array}{ccc}
\Sigma^\vee & \xrightarrow{\Pi} & \Sigma_k \\
\downarrow{s^\vee} & & \downarrow{\Pi_k} \\
M^\vee & \xrightarrow{\pi} & M \\
\end{array}
\]

We construct a section \( s^\vee : M^\vee \to \Sigma^\vee \) tautologically, as follows. Let us consider an arbitrary \( q^\vee \in M^\vee \). By definition of pull-back bundle, the \( p^\vee \)-fibre over \( q^\vee \) is identified through \( \Pi \) with the \( p \)-fibre over \( \pi(q^\vee) \). Since \( i(q^\vee) \in \Sigma \) belongs to the \( p \)-fibre over \( \pi(q^\vee) \), we can define \( s^\vee(q^\vee) \) through the equation

\[
\Pi(s^\vee(q^\vee)) = i(q^\vee).
\]

This proves the existence of a cover as in the statement of the lemma and shows that the degree \( k^\vee \) is less than or equal to the order of \( e_{Z} \).

If \( p \in \Psi(\Sigma) \) and \( \{\Psi_r : \Sigma \to \Sigma\} \) is an isotopy with \( \Psi_0 = \text{id}_\Sigma \), then \( \{p_r := \Psi_r^*p\} \) yields a path in \( \Psi(\Sigma) \). The next lemma shows that all paths in \( \Psi(\Sigma) \) arise in this way.

**Lemma 7.7.** The following two statements hold:

(i) Every oriented \( S^1 \)-bundle \( p_0 \in \Psi(\Sigma) \) has a \( C^1 \)-neighbourhood \( W \) and a continuous map \((p \in W) \mapsto (\Psi_p \in \text{Diff}(\Sigma))\) such that

\[
\Psi_{p_0} = \text{id}_\Sigma, \quad \Psi_p^*p = p_0, \quad \forall p \in W.
\]

(ii) If \( \{p_r : \Sigma \to M_r\} \) is a path in \( \Psi(\Sigma) \), there exists an isotopy \( \{\Psi_r\} \) of \( \Sigma \) such that

\[
\Psi_0 = \text{id}_\Sigma, \quad \Psi_r^*p_r = p_0, \quad \forall r \in [0, 1].
\]
Proof. Pick an arbitrary connection $\eta \in \mathcal{K}(p_0)$ and an arbitrary Riemannian metric on $M$ with injectivity radius $\rho_{inj}$ and distance function $\text{dist} : M \times M \to [0, \infty)$. We define the open subset of $M \times \Sigma$

$$W := \left\{ (q, z) \in M \times \Sigma \mid \text{dist}(q, p_0(z)) < \rho_{inj}/2 \right\}$$

and let $q_M : W \to M$ and $\pi_\Sigma : W \to \Sigma$ be the projections on the two factors. We define $W$ as the space of all oriented $S^1$-bundles $\mathfrak{p}$ such that the following two properties hold:

- For every $z_1, z_2$ in $\Sigma$ belonging to the same $\mathfrak{p}$-fibre, we have $\text{dist}(p_0(z_1), p_0(z_2)) < \rho_{inj}/2$;
- For every $z \in \Sigma$, the disc $D_z \subset \Sigma$ is transverse to the orbits of $\mathfrak{p}$. Here, $D_z$ is the union of all the $\eta$-horizontal lifts through $z$ of the geodesic rays in $M$ emanating from $p_0(z)$ and with length $\rho_{inj}$.

For every $\mathfrak{p} \in W$, we construct a map of fibre bundles over $M$ given by

$$S : W \to TM, \quad S(q, z) := (q, v(q, z)), \quad v(q, z) := \int_{S^1} \exp_{q}^{-1} \circ p_0 \circ \gamma_z(t) \, dt,$$

where $\gamma_z : S^1 \to \Sigma$ is the oriented $\mathfrak{p}$-fibre passing through $z$. By definition, $S$ is constant along the $\mathfrak{p}$-fibres. Let $0_M \subset TM$ be the zero section and set $G := S^{-1}(0_M)$. We claim that

- $S$ is transverse to $0_M$,
- $\mathfrak{p}_M := q_M|_G : G \to M$ is an $S^1$-bundle map,
- $\Psi_\Sigma := \pi_\Sigma|_G : G \to \Sigma$ is a diffeomorphism.

This is clear if $\mathfrak{p} = p_0$, since in this case $S(q, z) = \exp_q^{-1}(p_0(z))$ and $G = \{(q, z) \mid p_0(z) = q\}$. Therefore, up to shrinking $W$, it also holds for $\mathfrak{p} \in W$, as $S$ depends continuously on $\mathfrak{p}$ and being transverse, or a submersion or a diffeomorphism is a $C^1$-open condition [Hir94, Chapter 2]. Thus, we get an $S^1$-bundle $\mathfrak{p}' := q_M \circ \Psi_\Sigma^{-1} : \Sigma \to M$, which is equivalent to $\mathfrak{p}$. We now define a map $\Phi_p : \Sigma \to \Sigma$ such that $\Phi^*_p p_0 = \mathfrak{p}$. For every $z \in \Sigma$, $\Phi_p(z)$ is the parallel transport with respect to $\eta$ along the radial geodesic on $M$ connecting $p_0(z)$ with $\mathfrak{p}'(z)$. The dependence of $\Phi_p$ from $\mathfrak{p}$ is continuous and clearly $\Phi^*_p p_0 = p_0$. As being a diffeomorphism is a $C^1$-open condition, we have $\Phi_p \in \text{Diff}(\Sigma)$ and (i) is proved by setting $\Psi_p = \Phi^{-1}_p$.

For part (ii), we break the given path into short paths and apply the first part. \hfill $\square$

8 Weakly Zoll pairs and Zoll odd-symplectic forms

8.1 Definitions and first properties

Definition 8.1. A couple $(p, c)$ is called a weakly Zoll pair if $p : \Sigma \to M$ is an element of $\mathfrak{B}(\Sigma)$ and $c \in H^2_{dR}(M)$. We say that $\Omega \in \Xi^2(\Sigma)$ is associated with $(p, c)$, if $\Omega = p^* \omega$ for some $\omega \in \Xi^2_c(M)$. We denote by $\mathfrak{Z}(\Sigma)$ the set of weakly Zoll pairs and by $\mathfrak{Z}_C(\Sigma)$ the subset of those $(p, c) \in \mathfrak{Z}(\Sigma)$ with $p^* c = C \in H^2_{dR}(\Sigma)$.

Let $(p_0, c_0) \in \mathfrak{Z}_C(\Sigma)$ with $p_0 : \Sigma \to M_0$ and take any $\omega_0 \in \Xi^2_{dR}(M_0)$. We consider the volume functionals $\text{Vol} : \Omega^1(\Sigma) \to \mathbb{R}$ and $\text{Vol}_p : \Xi^2_c(\Sigma) \to \mathbb{R}$ defined in Section 5 with respect to $\Omega_0 := p_0^* \omega_0 \in \Xi^2_c(M)$. We use $\text{Vol}_p$ to get a volume on $\mathfrak{Z}_C(\Sigma)$ denoted with the same name:

$$\text{Vol} : \mathfrak{Z}_C(\Sigma) \to \mathbb{R}, \quad \text{Vol}(p, c) := \text{Vol}(p^* \omega), \quad \omega \in \Xi^2_c(M).$$
Lemma 8.2. The following three statements hold.

(i) The volume functional \( \mathfrak{Vol} : \mathcal{Z}_C(\Sigma) \to \mathbb{R} \) is well-defined.

(ii) For all \( \zeta \in \Omega^1(M_0) \), \( p_0^*\zeta \) is \( p_0^*\omega_0 \)-normalised, namely

\[
\text{Vol}(p_0^*\zeta) = 0.
\]

(iii) The functions \( \mathfrak{Vol} : \Xi^2_C(\Sigma) \to \mathbb{R} \), \( \mathfrak{Vol} : S_C(\Sigma) \to \mathbb{R} \) depend only on the pair \((p_0, c_0)\) and not on the chosen \( \omega_0 \in \Xi^2_C(M_0) \).

Proof. All three items will stem out a preliminary result. Let \((p, c) \in S_C(\Sigma)\) with \( \omega \in \Xi^2_C(M) \) and pick \( \alpha_\omega \in \Omega^1(\Sigma) \) such that \( p_0^*\omega_0 + d\alpha_\omega = p^*\omega \). We claim that

\[
\text{Vol}(\alpha_\omega + p^*\zeta) = \text{Vol}(\alpha_\omega), \quad \forall \zeta \in \Omega^1(M).
\] (8.1)

Indeed, if we set \( \alpha_r := \alpha_\omega + rp^*\zeta \) so that \( p_0^*\omega_0 + d\alpha_r = p^*(\omega + rd\zeta) \) and \( \alpha_r = p^*\zeta \), then

\[
\text{Vol}(\alpha_\omega + p^*\zeta) - \text{Vol}(\alpha_\omega) = \int_0^1 \text{Vol}_{\alpha_r} \cdot \alpha_r \, dr = \int_0^1 \left( \int_\Sigma p^*\zeta \wedge p^*(\omega + rd\zeta)^n \right) \, dr = 0.
\]

If \( \omega' \) is another form in \( \Xi^2_C(M) \), then there is \( \zeta \in \Omega^1(M) \) with \( \omega' - \omega = d\zeta \) and (8.1) implies

\[
\mathfrak{Vol}(p^*\omega') = \mathfrak{Vol}(p^*\omega),
\]

which establishes item (i). Choosing \((p, c) = (p_0, c_0)\) and \( \alpha = 0 \) in (8.1), we get item (ii). For item (iii), we take another form \( \omega'_0 \in \Xi^2_{c_0}(M_0) \) and write \( \omega'_0 - \omega_0 = d\zeta_0 \) for some \( \zeta_0 \in \Omega^1(M_0) \). Applying Lemma 5.4 with \( \Omega_0 = p_0^*\omega_0 \) and \( \Omega'_0 = p_0^*\omega'_0 \) together with item (ii), we deduce

\[
\text{Vol}(\alpha) = \text{Vol}'(\alpha - p_0^*\zeta_0), \quad \forall \alpha \in \Omega^1(\Sigma),
\]

where Vol' is the volume functional associated with \( \Omega'_0 \). This implies \( \mathfrak{Vol}(\Omega) = \mathfrak{Vol}'(\Omega) \) for all \( \Omega \in \Xi^2_C(\Sigma) \).

We have a canonical projection

\[
\mathfrak{P} : S(\Sigma) \to \mathfrak{P}(\Sigma), \quad \mathfrak{P}(p, c) = p.
\]

Let us fix a class \( C \in H^2_{\text{dR}}(\Sigma) \) and a connected component \( \mathfrak{P}^0(\Sigma) \) of \( \mathfrak{P}(\Sigma) \). We define

\[
\mathfrak{Z}^0_C(\Sigma) := \mathfrak{P}^{-1}(\mathfrak{P}^0(\Sigma)) \cap \mathfrak{Z}_C(\Sigma).
\]

We consider the restriction of \( \mathfrak{P} \) on this set

\[
\mathfrak{P}^0_C : \mathfrak{Z}^0_C(\Sigma) \to \mathfrak{P}^0(\Sigma).
\]

By (7.4), for every \((p, c) \in \mathfrak{Z}^0_C(\Sigma)\), we have a surjective map

\[
\mathbb{R} \to (\mathfrak{P}^0_C)^{-1}(p), \quad A \mapsto (p, Ae + c),
\]

where \( e \) is minus the real Euler class of \( p \). Finally, we define the evaluation map

\[
ev : \mathfrak{Z}^0_C(\Sigma) \to \mathbb{R}, \quad \ev(p, c) := \langle e^n, [M] \rangle.
\]
Definition 8.3. We say that $\mathfrak{Z}^0_C(\Sigma)$ is non-degenerate, if the map $ev : \mathfrak{Z}^0_C(\Sigma) \to \mathbb{R}$ is non-zero, namely if there exists $(p, c) \in \mathfrak{Z}^0_C(\Sigma)$ with $ev(p, c) \neq 0$.

Lemma 8.4. Let $p_0 : \Sigma \to M_0$ be an element in $\mathfrak{P}^0(\Sigma)$, and let $e_0 \in H^2_{dR}(M_0)$ be minus the real Euler class of $p_0$. For every $p : \Sigma \to M$ in $\mathfrak{P}^0(\Sigma)$, there exists a diffeomorphism $\Psi : \Sigma \to \Sigma$ isotopic to $id_\Sigma$ such that $\Psi^*p = p_0$. If $\psi : M_0 \to M$ is the quotient map of $\Psi$ (see (7.1.)), then, for every $(p, c) \in (\mathfrak{P}^0_C)^{-1}(p)$ and $A \in \mathbb{R}$, there holds

$$(i) \quad \psi^*(Ae + c) = Ae_0 + \psi^*c,$$
$$(ii) \quad ev(p, Ae + c) = ev(p_0, Ae_0 + \psi^*c),$$
$$(iii) \quad \mathfrak{Vol}(p, Ae + c) = \mathfrak{Vol}(p_0, Ae_0 + \psi^*c).$$

Proof. Since $\mathfrak{P}^0(\Sigma)$ is connected, the existence of a diffeomorphism $\Psi$ as in the statement follows from Lemma 7.7 (ii). By Re, we see that $\psi^*c = e_0$, which immediately implies item (i). For item (ii), we observe that $[M] = \psi_*[M_0]$ since $\psi$ is an orientation-preserving diffeomorphism, and compute

$$ev(p, Ae + c) = \langle (Ae + c)^n, \psi_*[M_0] \rangle = \langle \psi^*(Ae + c)^n, [M_0] \rangle = ev(p_0, Ae_0 + \psi^*c).$$

For the last relation, we take any $\omega_A \in \Xi^2(M)$ such that $[\omega_A] = Ae + c$. Then $\Omega_A := p^*\omega_A$ is associated with $(p, Ae + c)$. The form $\Psi^*\Omega_A \in \Xi^2_C(\Sigma)$ is associated with $(p_0, Ae_0 + \psi^*c)$. Indeed, $\Psi^*\Omega_A = \Psi^*p^*\omega_A = p_0^*\psi^*\omega_A$ and $[\psi^*\omega_A] = \psi^*(Ae + c) = Ae_0 + \psi^*c$. Thus, from Proposition 8.8 we derive

$$\mathfrak{Vol}(p, Ae + c) = \mathfrak{Vol}(\Omega_A) = \mathfrak{Vol}(\Psi^*\Omega_A) = \mathfrak{Vol}(p_0, Ae_0 + \psi^*c).$$

Corollary 8.5. If $\mathfrak{Z}^0_C(\Sigma)$ is non-empty, the four statements below hold.

(i) The real Euler class of some bundle in $\mathfrak{P}^0(\Sigma)$ vanishes, if and only if the real Euler class of every element of $\mathfrak{P}^0(\Sigma)$ vanishes.

(ii) The set $\mathfrak{Z}^0_C(\Sigma)$ is non-degenerate, if and only if, for every $p \in \mathfrak{P}^0(\Sigma)$ there exists a pair $(p, c) \in \mathfrak{Z}^0_C(\Sigma)$ such that $ev(p, c) \neq 0$.

(iii) If the real Euler class of the bundles in $\mathfrak{P}^0(\Sigma)$ vanishes, then $\mathfrak{P}^0_C$ is a diffeomorphism and $ev : \mathfrak{Z}^0_C(\Sigma) \to \mathbb{R}$ is a constant map.

(iv) If the real Euler class of the bundles in $\mathfrak{P}^0(\Sigma)$ does not vanish, then $\mathfrak{P}^0_C$ has the structure of an affine $\mathbb{R}$-bundle. The $\mathbb{R}$-action on some $(p, c) \in \mathfrak{Z}^0_C(\Sigma)$ is given by

$$A \cdot (p, c) = (p, Ae + c), \quad \forall A \in \mathbb{R}.$$
\[ \Psi_p = \text{id}_\Sigma \text{ and } \Psi_p^* p = p_0. \] We define \( \Phi_p := \Psi_p^{-1} \) and let \( \phi_p : M \rightarrow M_0 \) be the quotient map. If \( (p_0, c_0) \in (\mathcal{P}_C)^{-1}(p_0) \), then the map
\[ \mathcal{W} \times \mathbb{R} \rightarrow (\mathcal{P}_C)^{-1}(\mathcal{W}), \quad (p, A) = (p, Ae + \phi_c^* c_0) \]
provides a local trivialisation around \( p_0 \).

**Definition 8.6.** A two-form \( \Omega \) on \( \Sigma \) is called \textbf{Zoll}, if it is odd-symplectic and there exists \( p_\Omega : \Sigma \rightarrow M_\Omega \) in \( \mathcal{P}(\Sigma) \) such that the oriented \( p_\Omega \)-fibres are positively tangent to ker \( \Omega \). In this case, we say that \( p_\Omega \), which is determined up to equivalence, is the bundle \textbf{associated} with \( \Omega \). We write \( \mathcal{Z}(\Sigma) \) for the set of Zoll (odd-symplectic) forms on \( \Sigma \) and \( \mathcal{Z}(\Sigma) \) for the subset of those forms with class \( C \in H^2_{\text{dR}}(\Sigma) \).

If \( \Omega \) is Zoll, then the form \( \Omega \) descends to the base manifold \( M_\Omega \), since \( d\Omega = 0 \). Namely, there exists \( \omega_\Omega \in \Omega^2(M_\Omega) \) such that
\[ \Omega = p_\Omega^* \omega_\Omega. \]
The form \( \omega_\Omega \) is closed as well, since \( p_\Omega^* \) is injective. Furthermore, \( \omega_\Omega \) is symplectic since \( \Omega \) is odd-symplectic and the orientations induced by \( \omega_\Omega \) and by \( p_\Omega \) on \( M_\Omega \) coincide. We have a natural inclusion
\[ \mathcal{Z}(\Sigma) \rightarrow \mathcal{J}(\Sigma), \quad \Omega \mapsto (p_\Omega, [\omega_\Omega]). \]
As \( \omega_\Omega \) is positive symplectic, we deduce that
\[ \text{ev}(p_\Omega, [\omega_\Omega]) > 0. \]

**Remark 8.7.** The inequality above implies that for \( \Omega \in \mathcal{Z}(\Sigma) \) the component \( \mathcal{J}^0_\Omega(\Sigma) \) which \((\pm p_\Omega, [\omega_\Omega])\) is belonging to, is non-degenerate.

**Remark 8.8.** By [BW58], Zoll forms with vanishing cohomology class are just the exterior differentials of Zoll contact forms. Indeed, if \( \Omega = p_\Omega^* \omega_\Omega \) is exact, then the cohomology class of \( \omega_\Omega \) is non-zero, as \( \omega_\Omega \) is symplectic, and lies in the kernel of the map \( p_\Omega^* \). In view of (7.4), this means that there exists \( T \in \mathbb{R} \setminus \{0\} \) such that \([T \omega_\Omega] = e\). Then, there exists a connection \( \eta \in \mathcal{K}(p_\Omega) \) with \( \kappa_\eta = \frac{1}{T} \omega_\Omega \) and \( \alpha := T \eta \) is a Zoll contact form with \( d\alpha = \Omega \). The orientations \( \sigma_\Sigma \) and \( \sigma_\alpha \) coincide exactly when \( T > 0 \).

**Example 8.9.** Let \( (N_1, \sigma_1) \) and \( (N_2, \sigma_2) \) be two connected closed symplectic manifolds and suppose that \([\sigma_1] \in H^2_{\text{dR}}(N_1)\) is an integral cohomology class. We build the product symplectic manifold \( (N_1 \times N_2, \sigma_1 \oplus \sigma_2) \) and consider the oriented \( S^1 \)-bundle \( p : \Sigma \rightarrow N_1 \times N_2 \) whose Euler class is \(-[\sigma_1 \oplus 0]\). Then, \( \Omega := p^*(\sigma_1 \oplus \sigma_2) \) is a Zoll form with cohomology class \( C := p^*[0 \oplus \sigma_2] \).

**Example 8.10.** Let \( (N, \sigma) \) be a connected closed symplectic manifold of dimension \( 2m \). Let \( p_{T^* N} : T^* N \rightarrow N \) be the cotangent bundle map and consider the twisted symplectic form
\[ d\lambda_{\text{can}} + p_{T^* N}^* \sigma \in \Omega^2(T^* N), \]
where \( \lambda_{\text{can}} \) is the canonical one-form on \( T^* N \). We fix a \( \sigma \)-compatible almost complex structure \( J \) on \( N \) with associated metric \( g_J \). The structure \( J \) turns \( p_{T^* N} \) into a complex vector bundle and we denote by \( p_J : \mathbb{P}(T^* N) \rightarrow N \) its projectivisation. For all Riemannian metrics \( g \) on \( N \) in the same conformal class of \( g_J \), let \( S^*_g N \) be the unit co-sphere bundle of \( g \), and write \( p_g : S^*_g N \rightarrow N \) and \( i_g : S^*_g N \rightarrow T^* N \) for the natural projection and inclusion. The
one-parameter group $t \mapsto e^{tJ}$ acts fibrewise on $T^*N$ and yields a free $S^1$-action on $S^*_gN$, since $J$ is $g$-orthogonal. The quotient is naturally identified with $\mathbb{P}_C(T^*N)$, so that we have an oriented $S^1$-bundle $p$ making the following diagram commute

$$
\begin{array}{ccc}
T^*N & \xleftarrow{i^*_g} & S^*_gN \\
\downarrow{p} & & \downarrow{p} \\
\mathbb{P}_C(T^*N) & \xrightarrow{p_J} & \mathbb{P}_C(T^*N).
\end{array}
$$

Therefore, $\Omega_{g,\sigma} := i^*_g(\lambda_{can} + p^*_T \cdot N \sigma)$ is an odd-symplectic two-form on $S^*_gN$ with cohomology class $C := p^*_g[\sigma]$. The class $C$ vanishes if and only if $N$ is a surface different from the two-torus. In general, it is well-known that there exists a Zoll form $p^*\omega_{\sigma}$ in the class $C$. More specifically, $\omega_{\sigma}$ is the symplectic form on $\mathbb{P}_C(T^*N)$ defined as

$$
\omega_{\sigma} := a \omega_{FS} + p^*_f \sigma,
$$

for some $a > 0$ small enough (see [Voi07, Proposition 3.18], when $(N, \sigma, J)$ is Kähler). Here, $p^*\omega_{FS} = d\eta$, where $\eta$ is a connection form for $p$, which, for all $x \in N$, restricts on the fibre $p^*_g(x) \cong S^{2m-1}$ to the standard contact form on the sphere. The $p$-fibres are almost tangent to the characteristic distribution of $\Omega_{g,\sigma}$ if $\sigma$ is very big. However, the form $\Omega_{g,\sigma}$ and $p^*\omega_{\sigma}$ are remarkably not close to each other, if $m > 1$.

The relevance of this example stems from the fact that the characteristics of $\Omega_{g,\sigma}$ are the tangent lifts of the magnetic geodesics of the pair $(g,\sigma)$. A curve $c : \mathbb{R} \to N$ is called a magnetic geodesic if $g(\dot{c}, \dot{c}) \equiv 1$ and there holds

$$
\nabla^g_{\dot{c}} \dot{c} = -fJ\dot{c},
$$

where $\nabla^g$ is the Levi-Civita connection of $g$ and $f : N \to (0, \infty)$ is the conformal factor given by $f \cdot g = g_J$. We will investigate the systolic inequality for magnetic geodesics when $N$ is a surface, namely $m = 1$, in Part [III].

Before moving further, it is worthwhile to briefly discuss stability properties of Zoll forms. Let $\{\Omega_r\}$ be a path in $\mathcal{Z}_C(\Sigma)$ with the corresponding path of associated bundles $\{p_r\}$. If the real Euler class of the bundles vanishes, we aim at finding an isotopy $\{\Psi_r\}$ of $\Sigma$, such that

$$
\Psi_r^* \Omega_r = \Omega_0.
$$

If the real Euler class of the bundles does not vanish, we aim at finding an isotopy $\{\Psi_r\}$ of $\Sigma$, a path of real numbers $\{A_r\}$ with $A_0 = 0$, and $\eta_0 \in \mathcal{K}(p_0)$ such that

$$
\Psi_r^* \Omega_r = \Omega_0 + A_r d\eta_0.
$$

In this last case, it seems unlikely that all paths admit such an expression. That this happens if $C = 0$ is a result of Weinstein [Wei74]. The stability for $e = 0$ is reminiscent of [Gin87].

**Proposition 8.11.** Let $\{\Omega_r\}$ be a path in $\mathcal{Z}_C(\Sigma)$ with the path of associated bundles $\{p_r\}$.

- If $C = 0$, there is $\eta_0 \in \mathcal{K}(p_0)$, an isotopy $\{\Psi_r\}$ of $\Sigma$ and real numbers $\{T_r\}$ such that

$$
\Psi_r^* \Omega_r = T_r d\eta_0 = \Omega_0 + (T_r - T_0) d\eta_0.
$$
If \( e = 0 \), there is an isotopy \( \{ \Psi_r \} \) of \( \Sigma \) such that \( \Psi_r^* \Omega_r = \Omega_0 \).

**Proof.** By Lemma 7.7, we can suppose in both cases that \( \{ \Omega_r = p_r^* \omega_r \} \), where \( \{ \omega_r \} \) is a path of symplectic forms on \( M \). If \( C = 0 \), [KN96] yield \( \{ \eta_r \} \subset K(p_0) \) and non-zero real numbers \( \{ T_r \} \) such that \( \Omega_r = d(T_r \eta_r) \) and \( \eta_r = \eta_0 + p^*_r \zeta_r \), where \( \{ \zeta_r \} \) are one-forms on \( M \). Setting \( \alpha_r := T_r \eta_r \) and applying the Gray stability theorem from Remark 6.5.(a), we see that there exists an isotopy \( \{ \Psi_r \} \) and a path \( \{ H_r : \Sigma \to \mathbb{R} \} \) with \( H_0 = 0 \) such that

\[
\Psi_r^* \alpha_r = (1 + \frac{1}{r!} H_r) \alpha_0.
\]

Since \( \dot{\alpha}_r = \dot{T}_r \eta_r + T_r p^* \zeta_r \), equation (6.5) implies \( \dot{H}_r = \dot{T}_r \) and hence \( H_r = T_r - T_0 \), which readily implies the desired formula.

If \( e = 0 \), then \( \omega_r = \omega_0 + d \zeta_r \) due to (7.4). By Remark 6.5(b) and Proposition 6.4 with \( \alpha_0 = 0 \), it follows that \( \Psi_r^* \Omega_r = \Omega_0 + d(H_r \eta_0) \), for some paths \( \{ \Psi_r \} \) and \( \{ H_r \} \) as above. Again by equation (6.5), we conclude that \( H_r = 0 \) for all \( r \).

We are now ready to classify Zoll odd-symplectic forms on a three-dimensional manifold, as promised in the introduction.

### 8.2 Classification of Zoll odd-symplectic forms in dimension three

**Proof of Proposition 1.8**

Let \( \Sigma \) have dimension three, let \( b_\Sigma \) be the rank of the free part of \( H^1(\Sigma; \mathbb{Z}) \), and let \( t_\Sigma \) denote the cardinality of its torsion subgroup. Let \( \Omega \in Z_C(\Sigma) \) be a Zoll form with cohomology class \( C \) and let \( p = p_1 \) its associated bundle. If \( C = 0 \), we know from Remark 8.8 that \( \Omega \) is the differential of a Zoll contact form. In particular, \( \Sigma \) is the total space of a non-trivial oriented \( S^1 \)-bundle, \( b_\Sigma \) is even and we already treated this case in Proposition 1.2.

Suppose that \( C \neq 0 \). In this case, \( e = 0 \) by equivalence (ii) in (4.27). Since \( M \) is a surface this implies that \( e_\Sigma = 0 \), and hence \( p \) is trivial. Therefore, \( \Sigma \cong M \times S^1 \) and we see that

\[
b_\Sigma = 1 + 2 \text{genus}(M), \quad t_\Sigma = 1. \tag{8.2}
\]

In particular, \( b_\Sigma \) is odd. Let \( \Omega' \in Z_{C'}(\Sigma) \) be another Zoll form with class \( C' \). Since \( b_\Sigma \) is odd, then \( C' \neq 0 \) and \( p' : \Sigma \to M' \) is the trivial bundle. We write \( \Omega = p^* \omega \) and \( \Omega' = p'^* \omega' \), where \( \omega \) and \( \omega' \) are symplectic forms on \( M \) and \( M' \) respectively. From (8.2), \( M \) and \( M' \) have the same genus, so that there is a diffeomorphism \( \psi : M \to M' \). Since \( \omega \) and \( \omega' \) are symplectic on \( M \) and \( M' \) and \( H^2_{\text{dR}}(M; \mathbb{R}) \cong \mathbb{R} \cong H^2_{\text{dR}}(M'; \mathbb{R}) \), by Moser’s trick, we can assume that

\[
\psi^* \omega' = T \omega, \quad \text{for some } T > 0.
\]

As both \( p \) and \( p' \) are trivial bundles, it is immediate to find a diffeomorphism

\[
\Psi : \Sigma \to \Sigma \tag{8.3}
\]

lifting \( \psi \) and such that \( \Psi^* p' = p \). Therefore, \( f \) preserves the orientation of \( \Sigma \) and \( \Psi^* \Omega' = T \Omega \).

We now want to describe the connected components of the space of Zoll forms on \( \Sigma \) with fixed cohomology class. In order to do so, we first determine the connected components of the space of oriented \( S^1 \)-bundles \( \mathfrak{P}(\Sigma) \) with the help of classical results in low-dimensional
topology. As a preliminary observation, we point out that if \( p \in \Psi(\Sigma) \), then the class \( [p^{-1}(pt)]_Z \in H_1(\Sigma; Z) \) is primitive, since its intersection number with a global section of \( p \) is equal to 1. We distinguish three cases: \( M = S^2, M = T^2 \), and genus(M) \( \geq 2 \).

**Case 1:** \( \Sigma \cong S^2 \times S^1 \). We regard \( S^2 \) as the unit sphere in \( \mathbb{R}^3 \). We recall that the mapping class group of orientation-preserving diffeomorphisms of \( S^2 \times S^1 \) is given by

\[
\text{MCG}(S^2 \times S^1) \cong \frac{Z}{2}[\Psi_1] \oplus \frac{Z}{2}[\Psi_2].
\]

Here, the generators \( \Psi_1, \Psi_2 : S^2 \times S^1 \to S^2 \times S^1 \) are given by

\[
\Psi_1(q, \phi) := (-q, -\phi), \quad \Psi_2(q, \phi) := (\rho_\phi(q), \phi), \quad \forall (q, \phi) \in S^2 \times S^1,
\]

where \( \rho_\phi : S^2 \to S^2 \) is the rotation of angle \( 2\pi \phi \) around the north pole. We consider the standard projection \( p_+ : S^1 \times S^2 \to S^2 \) along \( S^1 \) and we define \( p_- := \Psi_1^* p_+ \).

We claim that \( \Psi(S^2 \times S^1) \) has four connected components containing respectively \( p_+, \Psi_2^* p_+, p_- \) and \( \Psi_2^* p_- \). To this purpose, we observe that \( [p_+^{-1}(pt)]_Z = [(\Psi_2^* p_+)^{-1}(pt)]_Z \) and \( [p_-^{-1}(pt)]_Z = [(\Psi_2^* p_-)^{-1}(pt)]_Z \) are distinct and yield the two primitive homology classes in \( H_1(S^2 \times S^1; Z) \cong Z \). Therefore, we just need to show that \( p_+, \Psi_2^* p_+ \) are not in the same connected component. If, by contradiction, there were a path in \( \Psi(S^2 \times S^1) \) from \( p_+ \) to \( \Psi_2^* p_+ \), then by Lemma 7.7. (ii) there would exist a diffeomorphism \( \Psi_2' : S^2 \times S^1 \to S^2 \times S^1 \) isotopic to \( \Psi_2 \) such that \( p_+ = (\Psi_2')^* p_+ \). This forces \( \Psi_2' \) to be of the form

\[
\Psi_2'(q, \phi) = (\psi(q), \phi'(x, \phi))
\]

where \( \phi \mapsto \phi'(q, \phi) \) is an orientation-preserving diffeomorphism of \( S^1 \), for all \( q \in S^2 \), and \( \psi \) is an orientation-preserving diffeomorphism of \( S^2 \). However, every orientation-preserving diffeomorphisms of \( S^2 \) is isotopic to the identity and the set \( \text{Diff}_+(S^1) \) of orientation-preserving diffeomorphisms of \( S^1 \) is homotopy equivalent to \( S^1 \), so that \( \pi_2(\text{Diff}_+(S^1)) = 0 \). Thus, the map \( \Psi_2 \) would be isotopic to the identity, which is impossible, as it is isotopic to \( \Psi_2' \).

Let \( p \) be an arbitrary element in \( \Psi(S^2 \times S^1) \). We have either \( [p^{-1}(pt)]_Z = [p_+^{-1}(pt)]_Z \) or \( [p^{-1}(pt)]_Z = [p_-^{-1}(pt)]_Z \). We have seen in (8.3) that there exists an isomorphism of oriented \( S^1 \)-bundles \( \Phi : S^2 \times S^1 \to S^2 \times S^1 \) preserving the orientation and such that \( \Psi^* p_+ = p \). Therefore, if \( [p^{-1}(pt)]_Z = [p_+^{-1}(pt)]_Z \), then \( \Psi \) is either isotopic to \( \text{id}_{S^2 \times S^1} \) or to \( \Psi_2 \) and \( p \) is either homotopic to \( p_+ \) or to \( \Psi_2^* p_+ \); if \( [p^{-1}(pt)]_Z = [p_-^{-1}(pt)]_Z \), then \( \Psi \) is either isotopic to \( \Psi_1 \) or to \( \Psi_1 \circ \Psi_2 \) and \( p \) is either homotopic to \( p_- \) or to \( \Psi_2^* p_- \).

**Case 2:** \( \Sigma \cong T^2 \times S^1 = \mathbb{T}^3 \). To every orientation-preserving diffeomorphism \( \Psi \) on \( \mathbb{T}^3 \), we can associate the induced map in homology

\[
H_1(\Psi) : H_1(\mathbb{T}^3; Z) \to H_1(\mathbb{T}^3; Z).
\]

As \( H_1(\mathbb{T}^3; Z) \cong \mathbb{Z}^3 \), we can identify \( H_1(\Psi) \) with an element of \( \text{SL}(3; Z) \). Conversely, every \( A \in \text{SL}(3; Z) \) gives an orientation-preserving diffeomorphism \( \Psi_A : \mathbb{T}^3 \to \mathbb{T}^3 \) with \( H_1(\Psi_A) = A \). Moreover, the mapping class group is computed explicitly through the isomorphism

\[
\text{MCG}(\mathbb{T}^3) \to \text{SL}(3; Z), \quad \Psi \mapsto H_1(\Psi).
\]

(8.4)

Let \( p_0 : \mathbb{T}^3 \cong \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \) be the standard projection along \( S^1 \). For every primitive class \( h \in H_1(\mathbb{T}^3; Z) \), there exists \( A_h \in \text{SL}(3; Z) \) with \( A_h \cdot h = [p^{-1}(pt)]_Z \), so that the fibres of the oriented \( S^1 \)-bundle

\[
p_h := \Psi_A_h p_0
\]

are trivial.
lie in the homology class \( h \). The map \( A_h \) is not unique. However, if \( A'_h \) is another such map, then \( \Psi^* A_h p_0 \) and \( \Psi^* A_h p_0 \) are equivalent bundles, and up to equivalence, \( p_h \) depends only on \( h \).

On the other hand, if \( p \in \Omega(T^3) \), we claim that \( p_{[p^{-1}(pt)]2} \) and \( p \) are isotopic. Indeed, let \( \Psi : T^3 \rightarrow T^3 \) be an orientation-preserving diffeomorphism with \( \Psi^* p_0 = p \), as in \( \text{(8.3)} \). The equality \( H_1(\Psi)([p^{-1}(pt)]2) = [p_0^{-1}(pt)]2 \) implies that \( H_1(\Psi) = A_{[p^{-1}(pt)]2} \) so that there holds \( p_{[p^{-1}(pt)]2} = \Psi^* H_1(\Psi)p_0 \). Moreover, since \( \Psi \) and \( \Psi H_1(\Psi) \) are isotopic due to \( \text{(8.4)} \), \( \Psi^* H_1(\Psi)p_0 \) is homotopic to \( p \) and the claim is proven.

**Case 3:** \( \Sigma \cong M \times S^1 \) with genus(\( M \)) \( \geq 2 \). Let \( p_+ : M \times S^1 \rightarrow M \) be the oriented \( S^1 \)-bundles given by the standard projection and set \( p_- = -p_+ \). The bundles \( p_+ \) and \( p_- \) are not homotopic since the homotopy classes of their fibres are different. If \( p \) is any element of \( \Omega(M \times S^1) \), by \text{[Wal67] Satz (5.5)}], there exists a diffeomorphism \( \Psi : M \times S^1 \rightarrow M \times S^1 \) isotopic to the identity such that either \( \Psi^* p = p_+ \) or \( \Psi^* p = p_- \). Thus, \( p \) is either isotopic to \( p_- \) or to \( p_+ \).

This finishes the description of the connected components of \( \Omega(\Sigma) \) in the three cases. To determine the connected components of the space of Zoll forms with fixed cohomology class, we consider the map assigning to each Zoll odd-symplectic form its associated bundle

\[
\mathcal{P}_Z : Z(\Sigma) \rightarrow \Omega(\Sigma), \quad \Omega \mapsto p_\Omega
\]

If \( p = \mathcal{P}_Z(\Omega) \), then \( PD([\Omega]) \) is a positive multiple of \( [p^{-1}(pt)]2 \) by \( \text{(7.6)} \). Moreover, the map \( \mathcal{P}_Z \) is surjective, since the quotient manifold of any bundle is an orientable surface, and therefore, possesses a positive symplectic form. In particular, if \( p \in \Omega(\Sigma) \) and \( C \in H^2_{\text{DR}}(\Sigma) \) are such that \( PD(C) \) is a positive multiple of \( [p^{-1}(pt)]2 \), then there exists \( \Omega \in Z_C(\Sigma) \subset Z(\Sigma) \) such that \( \mathcal{P}_Z(\Omega) = p \). This shows that

- if \( M = S^2 \) or \( T^2 \), then \( Z_C(\Sigma) \) is not empty, for all \( C \neq 0 \);
- if \( M \) has higher genus, then \( Z_C(\Sigma) \) is not empty if and only if \( C \) is a non-zero element in \( p_+^* (H^2_{\text{DR}}(M)) \).

We fix now some \( C \in H^2_{\text{DR}}(\Sigma) \) for which \( Z_C(\Sigma) \) is non-empty and consider \( \mathcal{P}_{Z,C} := \mathcal{P}_Z|_{Z_C(\Sigma)} \). As its non-empty fibres are convex, the connected components of \( Z_C(\Sigma) \) correspond through \( \mathcal{P}_{Z,C} \) to the connected components of \( \Omega_{Z,C}(Z_C(\Sigma)) \). The latter set is the union of those connected components of \( \Omega(\Sigma) \), whose elements \( p \) satisfy \( PD(C) = a[p^{-1}(pt)]2 \) for some \( a > 0 \).

By the discussion above, this shows that:

- If \( M \) is the two-sphere, then the set \( \mathcal{P}_{Z,C}(Z_C(\Sigma)) \) has two connected components since \( [p_-^{-1}(pt)]2 = ([\Psi^*_2 p_+]^{-1}(pt)]2 \) and \( [p_+^{-1}(pt)]2 = ([\Psi^*_2 p_-]^{-1}(pt)]2 \);
- If \( M \) has positive genus, then the set \( \mathcal{P}_{Z,C} \) is connected.

The proof of the proposition is completed. \( \square \)

## 9 Action of closed two-forms

### 9.1 The action form

**Definition 9.1.** If \( \Omega \) is a closed two-form on \( \Sigma \), an embedded one-periodic curve \( \gamma : S^1 \rightarrow \Sigma \) with \( \dot{\gamma} \in \ker \Omega \) is said to be a closed characteristics of \( \Omega \). We write \( \mathcal{X}(\Omega) \) for the set of closed characteristics of \( \Omega \), up to orientation-preserving reparametrisations of \( S^1 \).
Fix a free homotopy class $\mathfrak{h} \in [S^1, \Sigma]$ and let $\Lambda_{\mathfrak{h}}(\Sigma)$ be the space of one-periodic curves in $\Sigma$ with class $\mathfrak{h}$. In what follows, given a pair $(C, \mathfrak{h})$, we shall study a variational principle on $\Lambda_{\mathfrak{h}}(\Sigma)$, which detects, for all $\Omega \in \mathbb{E}_2^2(\Sigma)$, the elements of $\mathcal{X}(\Omega)$ belonging to $\Lambda_{\mathfrak{h}}(\Sigma)$. To this purpose, we define the action form $a = a(\Omega) \in \Omega^1(\Lambda_{\mathfrak{h}}(\Sigma))$ by

$$a_\gamma(\xi) := \int_{S^1} \Omega(\xi(t), \dot{\gamma}(t)) \, dt, \quad \forall \gamma \in \Lambda_{\mathfrak{h}}(\Sigma), \, \forall \xi \in T\gamma \Lambda_{\mathfrak{h}}(\Sigma).$$

Any $C^1$-path $\{\gamma_r\} \subset \Lambda_{\mathfrak{h}}(\Sigma)$ can also be interpreted as a cylinder $\Gamma : [0, 1] \times S^1 \to \Sigma$ with $\Gamma(r, t) = \gamma_r(t)$, so that

$$\int_0^1 \{\gamma_r\}^* a(\Omega) = \int_{[0,1] \times S^1} \Gamma^* \Omega. \quad (9.1)$$

Furthermore, we have the following classical observation.

**Lemma 9.2.** The action form $a(\Omega)$ is closed. An embedded periodic curve $\gamma \in \Lambda_{\mathfrak{h}}(\Sigma)$ is a closed characteristic of $\Omega$ if and only if $a_\gamma(\Omega) = 0$.

If $\Omega$ and $\Omega'$ are forms in $\mathbb{E}_2^2(\Sigma)$ with $\Omega' - \Omega = d\alpha$, then there holds

$$a(\Omega') - a(\Omega) = dB_\alpha, \quad B_\alpha(\gamma) := \int_{S^1} \gamma^* \alpha.$$

Thus, it makes sense to define $a(C) := \{a(\Omega)\} \in H^1_{dR}(\Lambda_{\mathfrak{h}}(\Sigma))$. We have a homomorphism

$$H^2_{dR}(\Sigma) \to H^1_{dR}(\Lambda_{\mathfrak{h}}(\Sigma)), \quad C \mapsto a(C). \quad (9.2)$$

If $a(\Omega)$ admits a primitive functional, then the zeros of $a(\Omega)$ are critical points of the primitive. In the next lemma, we give a criterion ensuring that $a(\Omega)$ is exact on $\Lambda_{\mathfrak{h}}(\Sigma)$, in the case that there exists an oriented $S^1$-bundle with fibres in $\mathfrak{h}$. Below, we regard classes in $H^2_{dR}(M)$ as real homomorphisms on $\pi_2(M)$ through the canonical map $\pi_2(M) \to H_2(M; \mathbb{R})$.

**Lemma 9.3.** Let $C \in H^2_{dR}(\Sigma)$ and $(p, c) \in \mathcal{C}(\Sigma)$ such that $p : \Sigma \to M$ has minus the real Euler class $e \in H^2_{dR}(M)$ and the oriented $p$-fibres have class $\mathfrak{h} \in [S^1, \Sigma]$. There holds

$$a(C) = 0 \quad \text{in} \quad H^1_{dR}(\Lambda_{\mathfrak{h}}(\Sigma)) \quad \iff \quad \ker e|_{\pi_2(M)} \subseteq \ker c|_{\pi_2(M)}.$$

**Proof.** Let $\omega$ be any element in $\mathbb{E}_2^2(M)$ and define $\Omega := p^* \omega$. The cohomology class $a(C)$ is trivial if and only if its integral over any one-periodic curve $\Gamma : S^1 \to \Lambda_{\mathfrak{h}}(\Sigma)$ vanishes. Choosing any oriented fibre $\gamma : S^1 \to \Sigma$ such that $[\gamma] = \mathfrak{h}$, we may assume that $\Gamma(0) = \Gamma(1) = \gamma$, up to homotopy, since $a$ is closed and therefore the integral of $a$ depends only on the homotopy class of $\Gamma$. In view of (9.1), $a(C) \in H^1_{dR}(\Lambda_{\mathfrak{h}}(\Sigma))$ is trivial if and only if, for any such $\Gamma$,

$$\int_{[0,1] \times S^1} \Gamma^* \Omega = 0.$$

As $p \circ \gamma$ is a constant curve, we think of $\bar{\Gamma} := p \circ \Gamma$ as a map from $S^2$ into $M$ with homotopy class $[\bar{\Gamma}] \in \pi_2(M)$. We have

$$\int_{[0,1] \times S^1} \Gamma^* \Omega = \int_{S^2} \bar{\Gamma}^* \omega = \langle e, [\bar{\Gamma}] \rangle.$$
Hence, the lemma follows if we show that the map \( \Gamma \mapsto [\Gamma] \) is onto \( \ker e_{\pi_2(M)} \). To see that the image \([\Gamma]\) is indeed in \( \ker e_{\pi_2(M)} \), we compute
\[
\langle e, [\Gamma] \rangle = \int_{S^2} \bar{\Gamma}^* \kappa = \int_{[0,1] \times S^1} \Gamma^*(d\eta) = \int_{S^1} \gamma^* \eta - \int_{S^1} \gamma^* \eta = 0,
\]
where \( \eta \in K(p) \) is any connection for \( p \).

Then, we show that for any \( v : S^2 \to M \) with \( \langle e, [v] \rangle = 0 \), there is \( \Gamma : [0,1] \times S^1 \to \Sigma \) with \( \Gamma(0,\cdot) = \Gamma(1,\cdot) = \gamma \) such that \( \bar{\Gamma} = p \circ \Gamma = v \). By the naturality of the Euler class, the restriction of \( p \) over \( v \) admits a global section \( \Upsilon : S^2 \to \Sigma \). It satisfies \( p \circ \Upsilon = v \). Using the quotient map \( [0,1] \times S^1 \to S^2 \), we lift \( \Upsilon \) to a map \( \Upsilon' : [0,1] \times S^1 \to \Sigma \). Up to modifying \( v \) within its homotopy class, we can assume that \( \Upsilon'(0,\cdot) = \Upsilon'(1,\cdot) = \gamma(0) \). Finally, we define
\[
\Gamma : [0,1] \times S^1 \to \Sigma, \quad \Gamma(s,t) = u(t, \Upsilon'(s,t)),
\]
where \( u \in \Omega(\Upsilon) \). It follows that \( \Gamma(0,\cdot) = \Gamma(1,\cdot) = \gamma \) and \( \Gamma = \Upsilon' = \bar{\Upsilon} = v \), as required. \( \square \)

**Remark 9.4.** The condition \( a(C) = 0 \) in \( H^1_{dR}(\Lambda_h(\Sigma)) \) depends only on \( C \) and \( \mathfrak{h} \). Therefore, the condition \( \ker e_{\pi_2(M)} \subseteq \ker c_{\pi_2(M)} \) is also independent of the chosen pair \( (p,c) \in \mathcal{Z}_C(\Sigma) \) with the property that the \( p \)-fibres are in the class \( \mathfrak{h} \). This last statement can also be seen directly combining Remark 7.4, equation (7.4), and item (i) in Lemma 8.4.

### 9.2 The action functional on a covering space

In view of Lemma 9.3, we cannot expect the existence of a primitive functional of the action form \( a \) in general. One standard way to resolve this problem is to find a primitive functional on a suitable covering space of \( \Lambda_h(\Sigma) \). We define a natural covering space in (9.5) below. However, it will have the small disadvantage that in some cases the primitive functional depends on the choice of a one-form \( \alpha \) such that \( \Omega = \Omega_0 + d\alpha \), where \( \Omega_0 \) is a reference two-form in the same cohomology class of \( \Omega \). We overcome this nuisance through the notion of non-degeneracy, which we introduced in Definition 8.3.

As before, let \( C \) be a class in \( H^2_{dR}(\Sigma) \) and \( \mathfrak{P}^0(\Sigma) \) be a connected component of \( \mathfrak{P}(\Sigma) \). We write \( \mathfrak{h} \in [S^1, \Sigma] \) for the class of the oriented fibres of any bundle in \( \mathfrak{P}^0(\Sigma) \). For the rest of this section, we work under the assumption that
\[
\mathfrak{Z}_C^0(\Sigma) = \mathfrak{P}^{-1}(\mathfrak{P}_0(\Sigma)) \cap \mathfrak{Z}_C(\Sigma) \text{ is non-empty and non-degenerate}, \tag{9.3}
\]
where \( \mathfrak{P} : \mathfrak{Z}(\Sigma) \to \mathfrak{P}(\Sigma) \) is the standard projection. We fix a reference pair \( (p_0,c_0) \in \mathfrak{Z}_C^0(\Sigma) \) with \( p_0 : \Sigma \to M_0 \) and denote by \( c_0 \in H^2_{dR}(M_0) \) minus the real Euler class of \( p_0 \). From (7.3), we have an embedding
\[
j_{p_0} : \Sigma \to \Lambda_h(\Sigma), \tag{9.4}
\]
where \( j_{p_0}(z) \) is a parametrisation of the oriented \( p_0 \)- fibre passing through \( z \in \Sigma \). By definition, there holds \( j_{p_0}(z) \in \mathcal{X}(\Omega_0) \). We consider the following covering space of \( \Lambda_h(\Sigma) \):
\[
\tilde{\Lambda}_h(\Sigma) := \left\{ \{\gamma_r\} \mid \gamma_0 \in j_{p_0}(\Sigma), \gamma_r \in \Lambda_h(\Sigma), \forall r \in [0,1] \right\} / \sim, \tag{9.5}
\]
where \( \{\gamma_r^0\} \sim \{\gamma_r^1\} \), if there is a homotopy \( \{\gamma_r^s\}_{s \in [0,1]} \) such that
\[
\gamma_0^s \in j_{p_0}(\Sigma), \quad \gamma_1^0 = \gamma_1^1, \quad \forall s \in [0,1]. \tag{9.6}
\]
We denote the elements of $\overline{\Lambda}_h(\Sigma)$ as $[\gamma_r]$ so that the covering map is given by

$$\overline{\Lambda}_h(\Sigma) \to \Lambda_h(\Sigma), \quad [\gamma_r] \mapsto \gamma_1.$$ 

We further choose some $\omega_0 \in \Xi^2_{\alpha}(M_0)$ and set as reference form

$$\Omega_0 := p_0^*\omega_0 \in \Xi^2(\Sigma).$$

Let $\text{Vol} : \Omega^1(\Sigma) \to \mathbb{R}$ and $\Xi \mathfrak{ol} : \Xi^2_{\alpha}(\Sigma) \to \mathbb{R}$ be the volume functionals associated with $\Omega_0$. As observed in Lemma 5.2, $\Xi \mathfrak{ol}$ depends only on $(p_0, \omega_0)$ but not on $\omega_0$.

Let $\alpha \in \Omega^1(\Sigma)$ and recall the notation $\Omega_\alpha = \Omega_0 + d\alpha \in \Xi^2_{\alpha}(\Sigma)$. We define the action

$$\mathfrak{A}_\alpha : \overline{\Lambda}_h(\Sigma) \to \mathbb{R}, \quad \mathfrak{A}_\alpha([\gamma_r]) := \int_{S^1} \gamma_r^*\alpha + \int_{[0,1] \times S^1} \Gamma^*\Omega_\alpha,$$ 

where, as before, $\Gamma : [0,1] \times S^1 \to \Sigma$ is the cylinder associated with $\{\gamma_r\}$.

The action is well-defined as one sees by integrating $0 = d\Omega_\alpha$ over a homotopy satisfying (9.6) and then applying Stokes’ Theorem. Decomposing $\Omega_\alpha = \Omega_0 + d\alpha$ in the second integrand above and using Stokes’ Theorem, we can rewrite $\mathfrak{A}_\alpha$ as

$$\mathfrak{A}_\alpha([\gamma_r]) = \int_{S^1} \gamma_r^*\alpha + \int_{D^2} \Gamma^*\omega_0,$$ 

where $\tilde{\Gamma} = p_0 \circ \Gamma : D^2 \to M_0$. A straightforward computation shows that

$$d_{[\gamma_r]}\mathfrak{A}_\alpha([\xi_r]) = a_{\gamma_1}(\xi_1), \quad \forall [\gamma_r] \in \overline{\Lambda}_h(\Sigma), \quad \forall [\xi_r] \in T_{[\gamma_r]}\overline{\Lambda}_h(\Sigma),$$

where $a = a(\Omega_\alpha)$. Hence, Lemma 9.2 can be rephrased as follows.

**Corollary 9.5.** Let $[\gamma_r] \in \overline{\Lambda}_h(\Sigma)$ with $\gamma_1 \in \Lambda_h(\Sigma)$ embedded. Then $[\gamma_r]$ is a critical point of $\mathfrak{A}_\alpha$ if and only if $\gamma_1 \in \mathfrak{X}(\Omega_\alpha)$. \hfill \(\square\)

If $\alpha$ and $\alpha'$ are one-forms such that $\Omega_\alpha = \Omega = \Omega_{\alpha'}$, then $\tau := \alpha' - \alpha$ is closed and we have

$$\mathfrak{A}_{\alpha'} = \mathfrak{A}_\alpha + ([\tau], [p^{-1}(\text{pt})]).$$ 

(9.9)

**Case 1:** $e_0 \neq 0$.

By Lemma 7.5, we have $[p^{-1}(\text{pt})] = 0$. The action functional depends only on the two-form $\Omega \in \Xi^2_{\alpha}(\Sigma)$ due to (9.9). Therefore, for any $\alpha \in \Omega^1(\Sigma)$ with $\Omega = \Omega_\alpha$, we can set

$$\tilde{\Lambda}_\Omega := \mathfrak{A}_\alpha.$$

In this situation, the non-degeneracy (9.3) of $\Xi^2_{\alpha}(\Sigma)$ is not needed to associate the action functional with elements in $\Xi^2_{\alpha}(\Sigma)$, as opposed to the next case.

**Case 2:** $e_0 = 0$.

Here, the action functionals of $\alpha$ and $\alpha'$ might be different. Nevertheless, as $\Xi^2_{\alpha}(\Sigma)$ is non-degenerate, we have $\langle e_0^\alpha, [M_0] \rangle = \text{ev}(p_0, \alpha_0) \neq 0$. Thus, by (5.2) and (7.6), we can write

$$\langle [\tau], [p^{-1}(\text{pt})] \rangle = \frac{\langle [\tau] \cup C^n, [\Sigma] \rangle}{\langle e_0^\alpha, [M_0] \rangle} = \frac{\text{Vol}(\alpha') - \text{Vol}(\alpha)}{\langle e_0^\alpha, [M_0] \rangle},$$

so that if $\alpha$ and $\alpha'$ have the same volume, they also have the same action by (9.9). We set

$$\tilde{\Lambda}_\Omega := \mathfrak{A}_\alpha$$ 

(9.10)

for a normalised $\alpha \in \Omega^1(\Sigma)$, i.e. $\text{Vol}(\alpha) = 0$, with $\Omega = \Omega_\alpha$.
Proposition 9.7. For any associated covering space of $\Lambda_c$ and $\Lambda = \Omega$, then we denote by $\Omega = \Omega$ on the space of contractible curves with capping disc. Furthermore, the condition $\ker e_0|\pi_2(M_h) \subseteq \ker c_0|\pi_2(M_0)$ in Lemma 9.3 means that $\omega_0$ is symplectically aspherical. Next, we study the relation between the actions with respect to two different reference weakly Zoll pairs.

Remark 9.6. This remark is parallel to Remark 5.6. For contact forms and Hamiltonian systems, the action functional recovers the following well-known formulae.

- (Contact forms) Let $\Omega_0 = 0$ and $\alpha \in \Omega_1^c(\Sigma)$ be a (possibly contact) one-form. As $C = 0$, we have that $e_0 \neq 0$ by Lemma 7.5.(ii). Thus, this is a special instance of Case 1. In fact, $a(C) = 0$ due to (9.2), and the function $\mathcal{A}_\alpha : \Lambda_b(\Sigma) \to \mathbb{R}$, given by

$$\mathcal{A}_\alpha(\gamma) = \int_{S^1} \gamma^* \alpha,$$

is the unique primitive of $\alpha$ such that $\mathcal{A}_d(\gamma) = \mathcal{A}_\alpha(\gamma)$, for every $\gamma \in \tilde{\Lambda}_b(\Sigma)$.

- (Hamiltonian systems) Let $p_0$ be trivial, namely $\Sigma = M_0 \times S^1$, so that this is a special instance of Case 2. Assume $\Omega_0 = p_0^* \omega_0$, where $\omega_0$ is a symplectic form on $M_0$, and $\alpha = H dt$, where $t$ is the angular coordinate on $S^1$. If $\gamma_1(t) = (q_1(t), t)$, then we get the classical Hamiltonian action functional

$$\mathcal{A}_{H,l}(\gamma) := \int_{S^1} H(q_1(t), t) dt + \int_{D^2} \Gamma^* \omega_0$$

on the space of contractible curves with capping disc. Furthermore, the condition $\ker e_0|\pi_2(M_h) \subseteq \ker c_0|\pi_2(M_0)$ in Lemma 9.3 means that $\omega_0$ is symplectically aspherical.

Proposition 9.7. Let $(p_0', c_0')$ be another element in $\mathcal{Z}_0^c(\Sigma)$. We write $\tilde{\Lambda}_h'(\Sigma)$ for the associated covering space of $\Lambda_b(\Sigma)$ and $\tilde{\Lambda}_b(p_0, p_0') \subset \tilde{\Lambda}_h(\Sigma)$ for the set of elements $[\delta_r]$ such that $\delta_1 \in J_{p_0'}(\Sigma)$. We pick $\omega_0' \in \Xi_{p_0'}^c(M_0')$ and set $\Omega_0' := (p_0')^* \omega_0'$. We choose $\alpha' \in \Omega_1^c(\Sigma)$ such that

$$\Omega_0' = \Omega_0 + d\alpha', \quad \alpha' \text{ is } \Omega_0\text{-normalised}.$$

For any $\Omega \in \Xi_{p_0'}^c(\Sigma)$, we take $\alpha \in \Omega_1^c(\Sigma)$ such that

$$\Omega = \Omega_0 + d\alpha, \quad \alpha \text{ is } \Omega_0\text{-normalised}.$$

Then, there holds $\Omega = \Omega_0' + d(\alpha - \alpha')$ and the one-form $\alpha - \alpha'$ is $\Omega_0'$-normalised. Moreover, if we denote by $\mathcal{A}_{\Omega_0'}$ the action of $\Omega$ with respect to $\Omega_0'$, then

$$\mathcal{A}_{\Omega_0'}([\delta_r] \# \{\gamma_r\}) = \mathcal{A}_{\Omega_0'}([\delta_r]) + \mathcal{A}_{\Omega}([\gamma_r]), \quad (9.11)$$

for every $[\delta_r] \in \tilde{\Lambda}_b(p_0, p_0')$ and $[\gamma_r] \in \tilde{\Lambda}_h(\Sigma)$. Here, the concatenation is made by choosing any representative $\{\gamma_r\}$ of $\{\delta_r\}$ with $\gamma_0 = \delta_1$.

Proof. The one-form $\alpha - \alpha'$ is $\Omega_0'$-normalised thanks to Lemma 5.4. Let $\Gamma, \Delta : [0, 1] \times S^1 \to \Sigma$ be the cylinders traced by the paths $\{\gamma_r\}$ and $\{\delta_r\}$, respectively. Let us show equation (9.11) using (9.7) and (9.8):

$$\mathcal{A}_{\Omega_0'}([\delta_r]) + \mathcal{A}_{\Omega}([\gamma_r]) = \int_{S^1} \delta_r^* \alpha' + \int_{[0,1] \times S^1} \Delta^* \Omega_0 + \int_{S^1} \gamma_0^* (\alpha - \alpha') + \int_{[0,1] \times S^1} \Gamma^* \Omega$$

$$= \int_{[0,1] \times S^1} \Delta^* \Omega_0 + \int_{S^1} (\gamma_1)^* \alpha + \int_{[0,1] \times S^1} \Gamma^* \Omega_0 + \int_{[0,1] \times S^1} \Gamma^* (d\alpha)$$

$$= \int_{[0,1] \times S^1} (\Delta \# \Gamma)^* \Omega_0 + \int_{S^1} \gamma_0^* \alpha + \int_{S^1} \gamma_1^* \alpha - \int_{S^1} \gamma_0^* \alpha$$

$$= \mathcal{A}_{\Omega}([\delta_r] \# \{\gamma_r\}) \square$$
Corollary 9.8. The action $\tilde{A}_\Omega$ depends only on the fixed reference pair $(p_0, c_0) \in Z_C^0(\Sigma)$, and not on the specific choice of $\omega_0 \in \Xi^2_C(M_0)$.

Proof. Let $\omega'_0$ be another element in $\Xi^2_C(M_0)$. By (9.11) with $(p'_0, c'_0) = (p_0, c_0)$ and $[\delta] = [\gamma_0]$, it suffices to show that $\tilde{A}_{p'_0 \omega'_0}([\gamma_0]) = 0$. To this purpose, let $\zeta \in \Omega^1(M_0)$ be such that $\omega'_0 - \omega_0 = d\zeta$. By Lemma 8.2(ii), $p'_0 \zeta$ is $p'_0 \omega'_0$-normalised and we can use it to compute $\tilde{A}_{p'_0 \omega'_0}$:

$$\tilde{A}_{p'_0 \omega'_0}([\gamma_0]) = \int_{S^1} \gamma_0^*(p'_0 \zeta) = 0.$$ 

9.3 The action is invariant under pull-back and isotopies

In this subsection, we prove two invariance results for the action. For the first one, we consider an additional connected oriented closed manifold $\Sigma^\nu$ of dimension $2n + 1$. We suppose that there are a bundle $p^\nu_0 : \Sigma^\nu \to M'_0$ in $P^0(\Sigma^\nu)$ and a bundle map $\Pi : \Sigma^\nu \to \Sigma$ with $p^\nu_0 = \Pi^* p_0$. Let $P(\Sigma^\nu)$ be the connected component of $P(\Sigma^\nu)$ containing $p^\nu_0$. Let $\pi_0 : M^\nu_0 \to M_0$ be the map fitting into the first commutative diagram in (7.1). If we set

$$c_0^\nu := \pi_0^* c_0 \in H^2_{dR}(M'_0), \quad C^\nu := \Pi^* C \in H^2_{dR}(\Sigma^\nu),$$

then $(p_0^\nu, c_0^\nu) \in Z_C^0(\Sigma^\nu) := (P(\Sigma^\nu))^{-1}(Z_C^0(\Sigma^\nu)) \cap Z_C^0(\Sigma^\nu)$. In particular, $Z_C^0(\Sigma^\nu)$ is non-empty. We write $\mathfrak{F}^0 : Z_C^0(\Sigma^\nu) \to P^0(\Sigma^\nu)$ for the projection and $\mathfrak{ev}^\nu : Z_C^0(\Sigma^\nu) \to \mathbb{R}$ for the evaluation. Furthermore, we abbreviate

$$\omega^\nu := \pi_0^* \omega_0 \in \Xi^2_C(M'_0), \quad \Omega^\nu := (p_0^\nu)^* \omega^\nu = \Pi^* \Omega_0 \in \Xi^2_C(\Sigma^\nu),$$

so that $\Omega^\nu$ is associated with the weakly Zoll pair $(p_0^\nu, c_0^\nu)$. We note that

$$\pi_0^* c_0 = e_0^\nu, \quad \pi_0^* (A e_0 + c_0) = A e_0^\nu + c_0^\nu, \quad \forall A \in \mathbb{R}, \quad (9.12)$$

where $e_0^\nu$ is minus the real Euler class of $p_0^\nu$. If $h^\nu$ denotes the free homotopy class of oriented fibres of elements in $P^0(\Sigma^\nu)$, we write $\Lambda^\nu : \Lambda^\nu(\Sigma^\nu) \to \mathbb{R}$ for the action of $A^\nu \in \Omega^1(\Sigma^\nu)$ with respect to $\Omega^\nu$. The bundle map $\Pi$ yields a map between the spaces

$$\Pi_* : \Lambda^\nu(\Sigma^\nu) \to \Lambda^\nu(\Sigma), \quad \Pi_* \gamma^\nu := \Pi \circ \gamma^\nu$$

and between their covering spaces

$$\tilde{\Pi}_* : \tilde{\Lambda}_h^\nu(\Sigma^\nu) \to \tilde{\Lambda}_h(\Sigma), \quad \tilde{\Pi}_*[\gamma]^\nu := [\Pi \circ \gamma]^\nu.$$ 

Proposition 9.9. The set $Z_C^0(\Sigma^\nu)$ is non-degenerate if and only if $\deg \pi_0 \neq 0$. In this case, $\tilde{A}_\nu : \tilde{\Lambda}_h^\nu(\Sigma^\nu) \to \mathbb{R}$ is well-defined for all $\Omega^\nu \in \Xi^2_C(\Sigma^\nu)$ and there holds

$$\tilde{A}_{\Pi_* \Omega} = \tilde{A}_\Omega \circ \tilde{\Pi}_*, \quad \forall \Omega \in \Xi^2_C(\Sigma).$$

Proof. The map $(\mathfrak{F}^0)^{-1}(p_0) \to (\mathfrak{F}^0)^{-1}(p_0^\nu)$ given by $(p_0, c) \mapsto (p_0^\nu, \pi_0^* c)$ is well-defined. By (7.4) and (9.12), it is also surjective. Since $Z_C^0(\Sigma)$ is non-degenerate and there holds

$$\mathfrak{ev}^\nu(p_0^\nu, \pi_0^* c) = \deg \pi_0 \cdot \mathfrak{ev}(p_0, c), \quad \forall (p_0, c) \in (\mathfrak{F}^0)^{-1}(p_0), \quad (9.13)$$

we see that $Z_C^0(\Sigma^\nu)$ is non-degenerate exactly when $\deg \pi_0$ is non-zero.
Let $\Omega = \Omega_0 + d\alpha \in \Xi^2_C(\Sigma)$, where $\alpha \in \Omega^1(\Sigma)$ is $\Omega_0$-normalised. Then, $\Pi^*\Omega = \Omega_0^\ast + d(\Pi^*\alpha)$ and $\Pi^*\alpha$ is $\Omega_0^\ast$-normalised by Proposition 5.7. For all $[\gamma^\ast] \in \tilde{\Lambda}_h(\Sigma)$, we have

$$
\tilde{A}_{\Pi^*\Omega}([\gamma^\ast]) = \int_{S^1}(\gamma_0^\ast)^*\Pi^*\alpha + \int_{[0,1] \times S^1}(\Gamma^\ast)^*\Pi^*\Omega = \int_{S^1}(\Pi \circ \gamma_0^\ast)^*\alpha + \int_{[0,1] \times S^1}(\Pi \circ \Gamma^\ast)^*\Omega
= \tilde{A}_\Omega(\Pi_\ast[\gamma^\ast]).
$$

For the second invariance result, we consider a diffeomorphism $\Psi : \Sigma \to \Sigma$ isotopic to $\text{id}_\Sigma$. We define

$$
\Psi^\ast : \Lambda_h(\Sigma) \to \Lambda_h(\Sigma), \quad \Psi^\ast \gamma := (\Psi^{-1})_\ast \gamma = \Psi^{-1} \circ \gamma.
$$

Let $\{\Psi_r\}$ be an isotopy of diffeomorphisms of $\Sigma$ with $\Psi_0 = \text{id}_\Sigma$ and $\Psi_1 = \Psi$. We denote by $[\gamma_r]$ the homotopy class with fixed end-points of $\{\Psi_r\}$ in the space of isotopies. Given such a homotopy class, we define

$$
[\Psi_r]^\ast : \Lambda_h(\Sigma) \to \Lambda_h(\Sigma), \quad [\Psi_r]^\ast [\gamma_r] := [\Psi_r^{-1} \circ \gamma_r].
$$

**Proposition 9.10.** For every $\Omega \in \Xi^2_C(\Sigma)$ and every homotopy class $[\Psi_r]$ as above, there holds

$$
\tilde{A}_{\Psi_r^\ast \Omega} \circ [\Psi_r]^\ast = \tilde{A}_\Omega.
$$

**Proof.** We observe preliminarily that if $X_r$ and $Y_r$ are the time-dependent vector fields generating $\Psi_r$ and $\Psi_r^{-1}$, we have the relation

$$
-X_r = d\Psi_r(Y_r) \circ \Psi_r^{-1}.
$$

Let $\alpha$ be a normalised one-form such that $\Omega = \Omega_0 + d\alpha$ and let $\{\theta_r\}$ be the path of normalised one-forms introduced in Section 5.2 such that

$$
\Psi_r^\ast \Omega = \Omega_0 + d(\theta_r + \Psi_r^\ast \alpha), \quad \dot{\theta}_r = \Psi_r^\ast(i_{X_r} \Omega_0), \quad \theta_0 = 0.
$$

For $s \in [0, 1]$, we define the truncation $\{\Psi_r^s := \Psi_{rs}\}$ and we write $a^s := a(\Psi_r^s \Omega)$. For every $[\gamma_r] \in \tilde{\Lambda}_h(\Sigma)$, we compute

$$
\frac{d}{ds} \left( \tilde{A}_{\Psi_r^s \Omega}( [\Psi_r^s]^\ast [\gamma_r] ) \right) = \left( \frac{d}{ds} \tilde{A}_{\Psi_r^s \Omega} \right)( [\Psi_r^s]^\ast [\gamma_r] ) + \frac{d}{ds} [\Psi_r^s]^\ast \tilde{A}_{\Psi_r^s \Omega} \cdot \frac{d}{ds} [\Psi_r^s]^\ast [\gamma_r]
= \int_{S^1} (\Psi_r^s)^\ast \left( \frac{d}{ds} (\theta_s + \Psi_r^s \alpha) \right) dt + a_{\Psi_r^s \gamma_1} \cdot \frac{d}{ds} \Psi_r^s \gamma_1
= \int_{S^1} \gamma_1^\ast \left( i_{X_s} \Omega + d(\alpha(X_s)) \right) dt + \int_{S^1} (\Psi_r^s \Omega) \left( \frac{\partial}{\partial s} \Psi_r^s \gamma_1, \frac{\partial}{\partial t} \Psi_r^s \gamma_1 \right) dt
= \int_{S^1} \gamma_1^\ast \left( i_{X_s} \Omega \right) dt + \int_{S^1} \Omega \left( d\Psi_s \cdot \frac{\partial}{\partial s} (\Psi_r^{-1} \circ \gamma_1), d\Psi_s \cdot \frac{\partial}{\partial t} (\Psi_r^{-1} \circ \gamma_1) \right) dt
= \int_{S^1} \Omega (X_s \circ \gamma_1, \gamma_1) dt + \int_{S^1} \Omega (-X_s \circ \gamma_1, \gamma_1) dt
= 0,
$$

where in the second equality we used (5.8) to compute $\frac{d}{ds} \tilde{A}_{\Psi_r^s \Omega}$. Since $\tilde{A}_{\Psi_0^s \Omega} \circ [\Psi_0]^\ast = \tilde{A}_\Omega$, the proof is completed. \qed
9.4 The action of weakly Zoll pairs

Based on the action functional we have studied, we define the action functional on the space of weakly Zoll pairs:

$$A : \mathcal{Z}_C^0(\Sigma) \to \mathbb{R}, \quad A(p, c) := \tilde{A}_{p^*}(z_0)$$

where $z_0$ is a point in $\Sigma$, $\omega \in \Xi^2_c(M)$, and $\{p_r\}$ is a path in $\mathcal{P}^0(\Sigma)$ starting at the reference bundle $p_0$ and ending at $p_1 = p$. It will turn out that this action is well-defined without the need to pass to a covering space, even when the condition in Lemma 9.3 is not met. This fact is a striking consequence of the non-degeneracy (9.3) of $\mathcal{Z}_C^0(\Sigma)$. A key role will be played by the following polynomial.

**Definition 9.11.** Let $Q : \mathbb{R} \to \mathbb{R}$ be the auxiliary polynomial

$$Q(A) := \text{ev}(p_0, Ae_0 + c_0) = \langle (Ae_0 + c_0)^n, [M_0] \rangle, \quad \forall A \in \mathbb{R}.$$ 

The Zoll polynomial $P : \mathbb{R} \to \mathbb{R}$ of the pair $(p_0, c_0)$ is given through

$$P(0) = 0, \quad \frac{dP}{dA} = Q.$$

**Remark 9.12.** The Zoll polynomial is non-constant by Corollary 8.5.(ii) since we have assumed that $\mathcal{Z}_C^0(\Sigma)$ is non-degenerate. Furthermore, we have the explicit formula

$$P(A) = \langle e_0, [M_0] \rangle \frac{A^2}{2} + \langle c_0, [M_0] \rangle A, \quad \text{for } n = 1. \quad (9.15)$$

**Remark 9.13.** This remark is parallel to Remark 5.6 and 9.6. The Zoll polynomial has a simple form, when any of $c_0$ and $e_0$ vanishes.

- Let us assume that $C = 0$ and take $c_0 = 0$, which is relevant to the study of Zoll contact forms. We have

$$P(A) = \langle e_0^n, [M_0] \rangle \frac{A_{n+1}}{n+1}.$$ 

- Let us assume that $e_0 = 0$, which is relevant to the study of Hamiltonian systems on $M$ (in which case $p_0$ is trivial and $\omega_0$ is a symplectic form). We have

$$P(A) = \langle e_0^n, [M_0] \rangle A.$$ 

**Theorem 9.14.** The functional $A : \mathcal{Z}_C^0(\Sigma) \to \mathbb{R}$ does not depend on any choice involved. Moreover, for any $(p, c) \in \mathcal{Z}_C^0(\Sigma)$, there holds

$$A(p, c) = 0, \quad \text{if } e_0 = 0 \text{ or } (p, c) = (p_0, c_0), \quad (9.16)$$

and

$$\text{Vol}(p, c) = P(A(p, c)). \quad (9.17)$$

If $\Psi : \Sigma \to \Sigma$ is isotopic to $\text{id}_\Sigma$ satisfying $\Psi^*p = p_0$ (which exists by Lemma 7.7) and $\psi : M_0 \to M$ is its quotient map, then

$$\psi^*c = A(p, c)e_0 + c_0. \quad (9.18)$$
Proof. To ease the notation, we write \((p_1, c_1)\) instead of \((p, c)\) to denote an arbitrary element of \(\mathcal{Z}^0(C)(\Sigma)\) with \(p_1 : \Sigma \to M_1\) and \(c_1 \in H^2(\mathbb{R})\). Moreover, \(\{p_r\}\) will indicate any path connecting the reference bundle \(p_0\) with \(p_1\). We divide the proof in five steps.

Step 1. For any path \(\{p_r\}\), the action value \(\tilde{A}_{p_1^r}\omega_1([j_{p_r}(z)])\) does not depend on the choice of \(z_0 \in \Sigma\) and \(\omega_1 \in \mathcal{E}^2_c(M_1)\).

Let \(z_1\) be another point in \(\Sigma\), and let \(\{z_s\}\) be a path between \(z_0\) and \(z_1\) with \(s \in [0, 1]\). We claim that \(\tilde{A}_{p_1^r}\omega_1([j_{p_r}(z_s)])\) does not depend on \(s\). Indeed, since \(j_{p_1}(z_s) \in \mathcal{O}(p_1^s\omega_1)\) for all \(s \in [0, 1]\), we see that \(d_{j_{p_1}(z_s)}\tilde{A}_{p_1^r}\omega_1 = 0\), by Corollary 9.5 which in turn implies

\[
\frac{d}{ds}\tilde{A}_{p_1^r}\omega_1([j_{p_1}(z)]) = \frac{d}{ds}[j_{p_1}(z)]\tilde{A}_{p_1^r}\omega_1 = 0.
\]

This shows the independence of the action value from \(z_0\).

Next, we take another \(\omega_1' \in \mathcal{E}^2_c(M_1)\) such that \(\omega_1' = \omega_1 + d\zeta\) for some one-form \(\zeta\) on \(M_1\). By Lemma 8.2(ii), \(p_1^r\zeta\) is normalised with respect to \(p_1^r\omega_1\). Applying equation (9.11) with \(\Omega = p_1^r\omega_1'\), \(\Omega_0 = p_1^r\omega_1\), \(\{\delta_r\} = \{j_{p_1}(z_0)\}\) and \(\{\gamma_r\} = \{\gamma_0\}\) a constant path, we find

\[
\tilde{A}_{p_1^r}\omega_1([j_{p_1}(z_0)]) - \tilde{A}_{p_1^r}\omega_1([j_{p_1}(z)]) = \tilde{A}_{p_1^r}\omega_1([\gamma_0]) = \int_{S1} \gamma_0^*(p_1^r\zeta) = 0.
\]

Hence, the action depends on the cohomology class \(c_1\), not on the representative.

Step 2. Let \(\{p_r\}\) be a path and \(\{\Psi_r\}\) an isotopy of diffeomorphisms given by Lemma 9.7(ii) such that \(\Psi_r^*p_r = p_0\). If \(\psi_1 : M_0 \to M_1\) is the quotient map of \(\Psi_1\), then

\[
(i) \quad \psi_1^*c_1 = \tilde{A}_{p_1^r}\omega_1([j_{p_1}(z)]) \in \mathbb{E} \quad \text{for all } s \in [0, 1]; \quad (ii) \quad \tilde{A}_{p_1^r}\omega_1([j_{p_1}(z)]) = 0, \quad \text{if } e_0 = 0.
\]

Setting \(\gamma_1 := j_{p_0} \circ \psi_1^{-1}(z_0)\) and using Proposition 9.10 we compute

\[
\tilde{A}_{p_1^r}\omega_1([j_{p_1}(z)]) = \tilde{A}_{p_1^r}\omega_1([\psi_1^{-1}j_{p_0} \circ \psi_1^{-1}(z)]) = \tilde{A}_{p_1^r}\omega_1([j_{p_0} \circ \psi_1^{-1}(z)]) = \tilde{A}_{p_1^r}\omega_1([\gamma_1]) = 0
\]

where the last equality follows from \([j_{p_0} \circ \psi_1^{-1}(z_0)] = [\gamma_1]\) (see (9.5)). We observe that \(\psi_1^*p_1^r\omega_1 = p_0^*\psi_1^*\omega_1\) and thus \((p_0, \psi_1^*c_1) \in \mathcal{Z}^0_c(\Sigma)\). By items (iii) and (iv) in Corollary 8.5 there exists \(A_1 \in \mathbb{R}\) such that

\[
\psi_1^*c_1 = A_1 e_0 + c_0.
\]

If \(e_0 \neq 0\), \(A_1\) is uniquely defined by this property. If \(e_0 = 0\), we simply impose \(A_1 := 0\). Equation (9.20) implies that there are \(\eta \in \mathcal{K}(p_0)\) and \(\zeta \in \Omega^1(M_0)\) such that

\[
\Psi_1^*p_1^r\omega_1 = p_0^*\omega_0 + d(A_1 \eta + p_0^* \zeta), \quad p_0^* \zeta = d\eta
\]

for some \(\kappa \in \mathcal{E}^2_0(M_0)\). Since \(A_1 = 0\), if \(e_0 = 0\), the one-form \(A_1 \eta + p_0^* \zeta\) is \(p_0^*\omega_0\)-normalised by Lemma 8.2(ii) and we conclude with (9.19) that

\[
\tilde{A}_{p_1^r}\omega_1([j_{p_1}(z)]) = \tilde{A}_{\psi_1^*p_1^r}\omega_1([\gamma_1]) = \int_{S1} \gamma_1^*(A_1 \eta + p_0^* \zeta) = A_1.
\]

This proves both items in Step 2.
Step 3. For any path \( \{ p_r \} \), there holds \( \mathcal{V} \circ \mathcal{I}(p_1, c_1) = P(\tilde{A}_{p_1^* \omega_1}([j_{p_r}(z_0)])) \).

From item (iii) in Lemma 8.4 with \( A = 0 \), we have
\[
\mathcal{V} \circ \mathcal{I}(p_1, c_1) = \mathcal{V} \circ \mathcal{I}(p_0, \psi_1^* c_1) = \text{Vol}(A_1 \eta + p_0^* \zeta),
\]
where \( A_1 = \tilde{A}_{p_1^* \omega_1}([j_{p_r}(z_0)]) \), \( \eta \) and \( \zeta \) are from the previous step. Using the definition of the volume (5.1) and Fubini’s Theorem, we compute
\[
\text{Vol}(A_1 \eta + p_0^* \zeta) = \int_0^1 \left( \int_{M_0} (A_1 \eta + p_0^* \zeta) \wedge p_0^* (r A_1 \kappa + \omega_0)^n \right) dr
\]
\[
= \int_0^1 \left( \int_{M_0} ((p_0)_*(A_1 \eta + p_0^* \zeta)) \wedge (r A_1 \kappa + \omega_0)^n \right) dr
\]
\[
= \int_0^1 A_1 (r A_1 e_0 + c_0)^n [M_0] dr
\]
\[
= \int_0^1 A_1 Q(r A_1) dr
\]
\[
= P(A_1) - P(0)
\]
\[
= P(A_1).
\]

Step 4. If \( (p_1, c_1) = (p_0, c_0) \), then, for any path \( \{ p_r \} \), we have \( \tilde{A}_{p_1^* \omega_1}([j_{p_r}(z_0)]) = 0 \).

Let \( \psi_1 \) and \( A_1 = \tilde{A}_{p_1^* \omega_1}([j_{p_r}(z_0)]) \) as in Step 2. Since \( \psi_1^* e_0 = e_0 \), applying \( \psi_1 \) iteratively to item (i) in Step 2, we obtain
\[
(\psi_1^m)^* c_0 = mA_1 e_0 + c_0, \quad \forall m \in \mathbb{Z}.
\]
From item (ii) in Lemma 8.4 with \( A = 0 \), we deduce
\[
Q(0) = \text{ev}(p_0, c_0) = \text{ev}(p_0, (\psi_1^m)^* c_0) = \text{ev}(p_0, mA_1 e_0 + c_0) = Q(mA_1), \quad \forall m \in \mathbb{Z}.
\]
We assume by contradiction that \( A_1 \neq 0 \). Since \( Q \) is a polynomial, the above identity implies that \( Q(A) \equiv Q_0 \), where \( Q_0 \in \mathbb{R} \), so that \( P(A) = Q_0 A \) for all \( A \in \mathbb{R} \). Since \( \mathfrak{X}^0_C(\Sigma) \) is non-degenerate, the coefficient \( Q_0 \) is non-zero. This contradicts Step 3:
\[
0 = \mathcal{V} \circ \mathcal{I}(p_0, c_0) = P(A_1) = Q_0 A_1.
\]

Step 5. End of the proof.

To show that the functional \( \mathcal{A} : \mathfrak{X}^0_C(\Sigma) \to \mathbb{R} \) is well-defined, it remains to see that the action value \( \tilde{A}_{p_1^* \omega_1}([j_{p_r}(z_0)]) \) does not depend on the choice of path \( \{ p_r \} \). Indeed, if \( \{ p_r' \} \) is another path with \( p_0' = p_0 \) and \( p_1' = p_1 \), the concatenation \( \{ p_r' \} \# \{ p_{1-r} \} \) forms a loop based at \( p_0 \). Applying Step 4 and the definition of the action (9.8), we conclude
\[
\tilde{A}_{p_1^* \omega_1}([j_{p_r}(z_0)]) - \tilde{A}_{p_1^* \omega_1}([j_{p_r}(z_0)]) = \tilde{A}_{p_0^* \omega_0}([j_{p_r' \# p_{1-r}}(z_0)]) = 0.
\]

Finally, the identities claimed in the statement of the theorem follow directly from what we have proven. Equation (9.16) is a consequence of item (ii) in Step 2 and of Step 4. Equation (9.17) follows from Step 3. Item (i) in Step 2 implies (9.18).
Remark 9.15. If $\epsilon_0 = 0$, we could also have considered the space of weakly Zoll one-forms $\mathfrak{Z}^0_C(\Sigma)$, whose elements are pairs $(p, \alpha)$ with $(p, [\omega_\alpha]) \in \mathfrak{Z}^0_C(\Sigma)$, where $p^*\omega_\alpha = \Omega_0 + d\alpha$, and $\alpha$ not necessarily normalised. We have an action functional
\[ A : \mathfrak{Z}^0_C(\Sigma) \to \mathbb{R}, \quad A(p, \alpha) := \tilde{A}_\alpha(j_{p^*}(z_0)). \]
Since $\mathfrak{Z}^0_C(\Sigma)$ is non-degenerate, from Remark 9.12 and Theorem 9.14, we deduce
\[ \text{Vol}(\alpha) = \langle c^0_0, [M_0] \rangle \cdot A(p, \alpha), \quad \forall (p, \alpha) \in \mathfrak{Z}^0_C(\Sigma). \]

Remark 9.16. Let $\Omega \in Z_C(\Sigma)$ be a Zoll odd-symplectic form such that the associated bundle $p_{\Omega}$ belongs to $Z_C(\Sigma)$. If we define
\[ A(\Omega) := A(p_{\Omega}, [\omega_{\Omega}]), \quad \Omega = p_{\Omega}^*\omega_{\Omega}, \]
then Theorem 9.14 translates into
\[ \text{Vol}(\Omega) = P(A(\Omega)), \quad \forall \Omega \in Z_C(\Sigma). \]

As we did for the volume and the action, we study how the Zoll polynomial behaves under pull-back and change of pair $(p_0, c_0)$ within $\mathfrak{Z}^0_C(\Sigma)$.

Proposition 9.17. Let $p_0' : \Sigma^\gamma \to M_0'$ be an oriented $S^1$-bundle with maps $\Pi : \Sigma^\gamma \to \Sigma$ and $\pi_0 : M_0' \to M_0$ as described in (7.1). If $P'$ is the Zoll polynomial of the weakly Zoll pair $(\Pi^*p_0, \pi_0^*c_0) = (p_0', c_0')$, there holds
\[ P' = (\deg \pi_0) \cdot P. \]

Proof. From equations (9.12) and (9.13), we see that $Q' = (\deg \pi_0) \cdot Q$, where $Q'$ is the derivative of $P'$. Since $P(0) = 0 = P'(0)$, the statement follows. \hfill \Box

Proposition 9.18. Let $(p_0', c_0')$ be a pair in $\mathfrak{Z}^0_C(\Sigma)$ with $p_0' : \Sigma \to M_0'$. There holds
\[ \langle (c_0')^n, [M_0'] \rangle = \text{ev}(p_0', c_0') = \frac{dP}{dA}(A(p_0', c_0')). \]
If $P'$ is the Zoll polynomial associated with $(p_0', c_0')$, then
\[ P'(A) = P(A + A(p_0', c_0')) - P(A(p_0', c_0')), \quad \forall A \in \mathbb{R}. \]

Proof. If $Q'$ denotes the derivative of $P'$, by Lemma 8.4(ii) and (9.18), we have
\[ Q'(A) = Q(A + A(p_0', c_0')). \]
Setting $A = 0$, we get the first part of the statement. For the second part, we integrate the above identity and use the normalization $P'(0) = 0$:
\[ P'(A) = \int_0^A Q'(A')dA' = \int_0^A Q(A' + A(p_0', c_0'))dA' = P(A + A(p_0', c_0')) - P(A(p_0', c_0')). \]

Combining the result above with the transformation rules for the action and the volume under change of reference weakly Zoll pair, we conclude the following invariance property.
Corollary 9.19. Let $P'$, $\tilde{A}$, $\mathfrak{Vol}'$ be the Zoll polynomial, the action and the volume associated with another reference pair $(p_0', c_0') \in \mathfrak{Z}_C^0(\Sigma)$. For every $\Omega \in \Xi_C^2(\Sigma)$, there holds

$$P' \circ \tilde{A}_\Omega' - \mathfrak{Vol}'(\Omega) = P \circ \tilde{A}_\Omega - \mathfrak{Vol}(\Omega) \quad \text{on } \tilde{A}_h(\Sigma).$$

Proof. We preliminarily observe that

$$\tilde{A}_\Omega = A(p_0', c_0') + \tilde{A}_\Omega$$

thanks to Proposition 9.7. We also recall that Lemma 5.4 implies that

$$\mathfrak{Vol}(\Omega) = \mathfrak{Vol}(p_0', c_0') + \mathfrak{Vol}'(\Omega).$$

Using these identities together with Proposition 9.18 and Theorem 9.14, we readily compute

$$P(\tilde{A}_\Omega) - \mathfrak{Vol}(\Omega) = P(\tilde{A}_\Omega) - \mathfrak{Vol}(p_0', c_0') - \mathfrak{Vol}'(\Omega)$$

$$= P(\tilde{A}_\Omega + A(p_0', c_0')) - \mathfrak{Vol}(p_0', c_0') - \mathfrak{Vol}'(\Omega)$$

$$= P'(\tilde{A}_\Omega') + P(A(p_0', c_0')) - \mathfrak{Vol}(p_0', c_0') - \mathfrak{Vol}'(\Omega)$$

$$= P'(\tilde{A}_\Omega') - \mathfrak{Vol}'(\Omega).$$

\[\square\]

10 A conjectural local systolic-diastolic inequality

The following is our setting to study a systolic-diastolic inequality for odd-symplectic forms. We assume that there exists a Zoll odd-symplectic form $\Omega_\ast \in Z_C(\Sigma)$, for some $C \in H^2_{\text{dR}}(\Sigma)$. We denote the bundle associated with $\Omega_\ast$ by

$$p_1 := p_{\Omega_\ast} : \Sigma \to M_1$$

and let $\omega_\ast \in \Xi^2(M_\ast)$ be the symplectic form such that $\Omega_\ast = p_1^* \omega_\ast$. The space $\mathfrak{Z}_C^0(\Sigma)$ is the connected component of $p_1$ in $\mathfrak{Z}(\Sigma)$ and $\mathfrak{Z}_C^0(\Sigma)$ is the corresponding space of weakly Zoll pairs, which is non-empty and non-degenerate thanks to Remark 8.7. Namely, condition (9.3) is automatically satisfied and we can use all the results contained in Section 9.

We fix a reference weakly Zoll pair $(p_0, c_0) \in \mathfrak{Z}_C^0(\Sigma)$, where

$$p_0 : \Sigma \to M_0$$

and let $\omega_0$ be some form in $\Xi^2_{c_0}(M_0)$. Let $h \in [S^1, \Sigma]$ denote the free homotopy class of the $p_0$-fibres. We define the volume and the action functionals

$$\mathfrak{Vol} : \Xi_C^2(\Sigma) \to \mathbb{R}, \quad A : \mathfrak{Z}_C^0(\Sigma) \to \mathbb{R}, \quad \tilde{A}_\Omega : \tilde{A}_h(\Sigma) \to \mathbb{R}$$

with respect to $(p_0, c_0)$. Due to Proposition 9.18 we have

$$\frac{dP}{dA}(A_*) = ([\omega_*]^n, [M_1]) > 0, \quad A_* := A(p_1, \omega_*).$$

(10.1)

Definition 10.1. We say that a finite cover $\{B_i\}$ of $M_1$ is admissible, if all the sets $B_i$ and their intersections $B_i \cap B_j$ are homeomorphic to the open Euclidean ball, if non-empty. We write $\{\Sigma_i := p_1^{-1}(B_i)\}$ for the pull-back cover of $\Sigma$, so that $\Sigma_i \cap \Sigma_j$ retracts to a $p_1$-fibre, if non-empty. We readily see that admissible finite covers exist and we fix one of them throughout the present section.
We also consider another oriented $S^1$-bundle $p_0^\gamma : \Sigma^\gamma \to M_0^\gamma$ with bundle map $\Pi : \Sigma^\gamma \to \Sigma$ such that $p_0^\gamma = \Pi^* p_0$. We further assume that $\Pi$ is a finite covering map. This is equivalent to asking that the corresponding quotient map $\pi_0 : M_0^\gamma \to M_0$, which fits into the first commutative diagram in (7.1), is a finite covering map. Thus, we endow $M_0^\gamma$ and $\Sigma^\gamma$ with the orientations pulled back by $\pi_0$ and $\Pi$, respectively.

The pull-back form $\Omega^*_\gamma := \Pi^* \Omega^*$ is Zoll odd-symplectic on $\Sigma^\gamma$ and defines the oriented $S^1$-bundle $p^\gamma_1 := \Pi^* p_1 : \Sigma^\gamma \to M_1^\gamma$. Since $p_0$ and $p_1$ are connected by a path of oriented $S^1$-bundles, the quotient map $\pi_1 : M_1^\gamma \to M_1$ is also an orientation-preserving finite covering map, which fits into the first commutative diagram in (7.1). Moreover, $\pi_1$ evenly covers all the sets in $\{B_i\}$, as they are contractible. Therefore, the family $\{B_i^p\}$, whose elements are the connected components of the pre-images $\pi_1^{-1}(B_i)$, yields a finite admissible cover of $M_1^\gamma$.

10.1 A local primitive for the action form

We have seen that the action form does not admit a global primitive on $\Lambda_h(\Sigma)$ in general. However, we can always find a local primitive on the space of periodic curves that are close to $j_{p_1}(\Sigma)$. Let $\Lambda(p_1)$ be the subset of $\Lambda_h(\Sigma)$, whose elements $\gamma$ are contained in some $\Sigma_i$ and are homotopic within $\Sigma_i$ to some $p_1$-fibre. This means that $\gamma$ admits a short homotopy $\{\gamma^r_{\text{short}}\}$. Namely, such a homotopy has the properties

$$\gamma^r_{\text{short}} \subset \Sigma_i, \quad \forall r \in [0, 1], \quad \gamma^0_{\text{short}} = j_{p_1}(z_i), \quad \gamma^1_{\text{short}} = \gamma,$$

for some $z_i \in \Sigma_i$. We choose any path $\{p_r\}$ from $p_0$ and $p_1$ in $\mathcal{P}^0(\Sigma)$, and consider the map

$$\Lambda(p_1) \to \tilde{\Lambda}_h(\Sigma), \quad \gamma \mapsto \{j_{p_r}(z_i)\} \# \{\gamma^r_{\text{short}}\}$$

Since the intersection of two elements of the finite admissible cover is contractible, this map does not depend on the particular short homotopy nor on the set $\Sigma_i$. Composing this map with the functional $\tilde{\Lambda}_h$ on $\tilde{\Lambda}_h(\Sigma)$, we define the action functional on $\Lambda(p_1)$:

$$\mathcal{A}_\Omega : \Lambda(p_1) \to \mathbb{R}, \quad \mathcal{A}_\Omega(\gamma) := \tilde{\Lambda}_h(\{j_{p_r}(z_i)\} \# \{\gamma^r_{\text{short}}\}).$$  \hspace{1cm} (10.2)

We note that $d\mathcal{A}_\Omega = a(\Omega)$ on $\Lambda(p_1)$.

**Lemma 10.2.** The action value $\mathcal{A}_\Omega(\gamma)$ does not depend on the choice of path $\{p_r\}$.

**Proof.** If $\{p'_r\}$ is another path in $\mathcal{P}^0(\Sigma)$ with $p'_0 = p_0$ and $p'_1 = p_1$, then, by (9.8) and (9.16),

$$\tilde{\Lambda}_h(\{j_{p_r}(z_i)\} \# \{\gamma^r_{\text{short}}\}) - \tilde{\Lambda}_h(\{j_{p'_r}(z_i)\} \# \{\gamma^r_{\text{short}}\}) = \mathcal{A}(p_0, c_0) = 0.$$

We write $\mathcal{X}(\Omega; p_1) := \mathcal{X}(\Omega) \cap \Lambda(p_1)$ for the set of closed characteristics of $\Omega$ in $\Lambda(p_1)$. The **systole** over the set $\Lambda(p_1)$ is the functional

$$\mathcal{A}_{\min} : \Xi^2_{\mathcal{X}}(\Sigma) \to \mathbb{R} \cup \{-\infty, +\infty\}, \quad \mathcal{A}_{\min}(\Omega) := \inf_{\gamma \in \mathcal{X}(\Omega; p_1)} \mathcal{A}_\Omega(\gamma),$$  \hspace{1cm} (10.3)

while the **diastole** over the set $\Lambda(p_1)$ is the functional

$$\mathcal{A}_{\max} : \Xi^2_{\mathcal{X}}(\Sigma) \to \mathbb{R} \cup \{-\infty, +\infty\}, \quad \mathcal{A}_{\max}(\Omega) := \sup_{\gamma \in \mathcal{X}(\Omega; p_1)} \mathcal{A}_\Omega(\gamma).$$  \hspace{1cm} (10.4)
Proposition 10.3. Let $\Pi : \Sigma^\prime \to \Sigma$ be a bundle map with $p_0^\prime = \Pi^*p_0$, which is also a finite covering map. Let us consider the oriented $S^1$-bundle $p_1^\prime = \Pi^*p_1 : \Sigma^\prime \to M^\prime$. Then, $\Pi^\prime : (\gamma^\prime \in \Lambda(p_1^\prime)) \mapsto (\Pi \circ \gamma^\prime \in \Lambda(p_1))$ is a (surjective) finite covering map and there holds

$$A_{\Pi^*\Omega}^\prime = A_\Omega \circ \Pi^\prime, \quad \forall \Omega \in \Xi^2_1(\Sigma),$$

where $A_{\Pi^*\Omega}^\prime$ is the action functional on $\Lambda(p_1^\prime)$ with reference pair $(p_0^\prime, c_0^\prime)$. As a consequence, the restriction map $\Pi^\prime : \mathcal{X}(\Pi^*\Omega; p_1^\prime) \to \mathcal{X}(\Omega; p_1)$ is also a finite cover and we have

$$A_{\min}(\Pi^*\Omega) = A_{\min}(\Omega), \quad A_{\max}(\Pi^*\Omega) = A_{\max}(\Omega).$$

Proof. The statement follows from Proposition 9.9, since $\Pi^\prime$ lifts a path $\{p_t\}$ in $\mathcal{P}^0(\Sigma)$ to a path $\{p_t^\prime\}$ in $\mathcal{P}^0(\Sigma^\prime)$ and $\Pi$ maps short homotopies for $p_1^\prime$ to short homotopies for $p_1$. \qed

We recall some results from [Gin87], ensuring that the systole and the diastole of $\Omega$ are finite, if $\Omega$ is an odd-symplectic form sufficiently close to $\Omega_*$ in the same cohomology class. As usual, we measure distances and norms by fixing Riemannian metrics on the relevant spaces.

Proposition 10.4. There exists a $C^1$-neighbourhood $\mathcal{V}$ of $\Omega_*$ in $\mathcal{S}_C(\Sigma)$ such that, for all $\Omega \in \mathcal{V}$, the following properties hold:

(i) There is a constant $C > 0$ depending on $\mathcal{V}$ but not on $\Omega$ such that

$$\sup_{t \in \mathbb{R}} \left( \text{dist}(q_\gamma(0), q_\gamma(t)) \right) \leq C\|\Omega - \Omega_*\|_{C^0}, \quad q_\gamma := p_1 \circ \gamma, \quad \forall \gamma \in \mathcal{X}(\Omega; p_1).$$

(ii) The set $\mathcal{X}(\Omega; p_1)$ is compact and non-empty and the restrictions $A_{\min}|\mathcal{V} : \mathcal{V} \to \mathbb{R}$, $A_{\max}|\mathcal{V} : \mathcal{V} \to \mathbb{R}$ are Lipschitz-continuous real functions.

(iii) If $\Omega \in \mathcal{V}$ is a Zoll form, then all its closed characteristics (up to orientation) lie in $\Lambda(p_1)$, i.e. $\mathcal{X}(\Omega) = \mathcal{X}(\Omega; p_1)$, and, recalling the notation $A(\Omega) = A(p_1, [\omega_\Omega])$,

$$A_{\min}(\Omega) = A(\Omega) = A_{\max}(\Omega).$$

Proof. Let $\eta \in \mathcal{K}(p_1)$, and let $V$ be the vertical vector field with $\eta(V) = 1$. If $\Omega$ is odd-symplectic, the equation

$$\iota_{X_\Omega}(\eta \wedge \Omega^n) = \Omega^n$$

determines a nowhere vanishing section $X_\Omega \in \ker \Omega$. The periodic orbits of the flow $\Phi^{X_\Omega}$ yield elements in $\mathcal{X}(\Omega)$. Since $V = X_\Omega$, inequality (A.1) in the proof of Lemma A.6 gives

$$\|X_\Omega - V\|_{C^1} \leq C_1\|\Omega - \Omega_*\|_{C^1}.$$
is a one-to-one correspondence. In particular,
\[
\min S(\Omega) = A_{\min}(\Omega), \quad \max S(\Omega) = A_{\max}(\Omega).
\]
Since the functions \(\max, \min : C^1(\Sigma) \to \mathbb{R}\) are Lipschitz-continuous, this shows (ii).

To prove (iii), let \(\rho_{\text{Leb}} > 0\) be a Lebesgue number for the open cover \(\{B_i\}\) of \(M_1\) and let \(C\) be the constant given in (i). Up to shrinking \(\mathcal{V}\), we can assume that
\[
C\|\Omega - \Omega_*\|_{C^1} \leq \rho_{\text{Leb}}, \quad \forall \Omega \in \mathcal{V}.
\]
Let now \(\Omega \in \mathcal{V}\) be a Zoll form with associated bundle \(p_{\Omega}\), and consider the set
\[
\Sigma(\Omega; p_1) := \{z \in \Sigma \mid \{t \mapsto \Phi^X_t(z)\} \in \Lambda(p_1)\},
\]
which is non-empty since we proved that \(\mathcal{X}(\Omega; p_1)\) is non-empty. This set is open, as the sets \(B_i\) are open. Furthermore, from item (i) and the inequality \(C\|\Omega - \Omega_*\|_{C^1} \leq \rho_{\text{Leb}}\), it is also closed. Since \(\Sigma\) is connected, we deduce \(\Sigma(\Omega; p_1) = \Sigma\) as desired.

To prove the equality between the actions in (iii), we observe that, since \(\Xi(\Omega; p_1)\) is a one-to-one correspondence. In particular,
\[
\min S(\Omega) = A_{\min}(\Omega), \quad \max S(\Omega) = A_{\max}(\Omega).
\]

Corollary 10.5. If \(Y \subset S_C(\Sigma)\) is the \(C^1\)-neighbourhood of a Zoll odd-symplectic form \(\Omega_*\), given by Theorem 10.4, then there holds
\[
P(A_{\min}(\Omega)) = \mathfrak{Vol}(\Omega) = P(A_{\max}(\Omega)), \quad \forall \Omega \in Y \cap S_C(\Sigma).
\]

Remark 10.6. If the real Euler class vanishes, then the inequality in Conjecture 2 becomes
\[
A_{\min}(\Omega) \leq 0 \leq A_{\max}(\Omega), \quad \forall \Omega \in \mathcal{U}.
\]
Indeed, in this case \(\mathfrak{Vol} \equiv 0\), \(P(A) = \langle [\omega_*], M_1 \rangle A\) and \(\langle [\omega_*], M_1 \rangle > 0\) by (10.1).
**Remark 10.7.** The conjecture is independent of the chosen reference pair \((p_0, c_0) \in Z_C^0(\Sigma)\).
Indeed, if \((p_0', c_0') \in Z_C^0(\Sigma)\) is another pair with corresponding objects \(\mathcal{V}\), \(P\) and \(A_{\Omega}'\), then
\[
P' \circ A_{\Omega}' - \mathcal{V}(\Omega) = P \circ A_{\Omega} - \mathcal{V}(\Omega)
\]
by Corollary 9.19.

In the next subsection, we will see that \(P\) can be written as the integral over \(M_1\) of a function \(P : M_1 \times \mathbb{R} \to \mathbb{R}\) so that the monotonicity of \(P\) corresponds to the monotonicity of \(P\) in the second variable for an interval \(I\) containing \(A_s\). This will enable us to establish the conjecture in some simple cases. Before doing that, we end this subsection by observing that Conjecture 2 is also consistent with the pull-back operation.

**Proposition 10.8.** Let \(\Pi : \Sigma' \to \Sigma\) be a bundle map with \(p_0' = \Pi^* p_0\), which is also a finite covering map. If Conjecture 2 holds for \(\Omega' = \Pi^* \Omega \in S_C^C(\Sigma')\), then it holds for \(\Omega \in S_C(\Sigma)\).

**Proof.** Let us suppose that the conjecture is true for a neighbourhood \(U' \subset V'\) around \(\Omega'\), where \(V'\) is given by Proposition 10.4. Thus, for all \(\Omega \in U'\), there holds
\[
P'((\Omega')) \leq \mathcal{V}(\Omega') \leq P'((\Omega'))
\]
and any of the two equalities holds if and only if \(\Omega'\) is Zoll. We claim that the local systolic inequality holds in the neighbourhood \(U := (\Pi^*)^{-1}(U')\) of \(\Omega\). Indeed, let us take an arbitrary \(\Omega\) in \(U\). This means that \(\Pi^* \Omega\) belongs to \(U'\). Applying Proposition 10.3 and Proposition 9.17 it follows that
\[
P((\Omega)) = P((\Pi^* \Omega)) = \frac{1}{\deg \Pi} P((\Omega')) \leq \frac{1}{\deg \Pi} \mathcal{V}(\Omega) = \mathcal{V}(\Omega),
\]
where we used \(\deg \Pi = \deg \pi_0 > 0\) and (10.5). An analogous inequality holds for \(A_{\max}(\Omega)\).

Let us assume now, for instance, that \(P((\Omega)) = \mathcal{V}(\Omega)\). From the computation above, this implies \(P((\Omega')) = \mathcal{V}(\Omega')\). Since the conjecture is true for \(U'\), we see that \(\Pi^* \Omega\) is Zoll and we call \(p'\) the associated bundle. All the fibres of \(p'\) lie in some \(\Sigma_j\), by Proposition 10.4. Since \(\Pi\) is injective on such sets, it follows that there exists a bundle \(p : \Sigma \to M\) such that \(p' = \Pi^* p\), which implies that \(\Omega\) is a Zoll form with bundle \(p\). \(\square\)

### 10.2 A second look at \(H\)-forms

For the rest of this section, we assume that \(p_0 = p_1 = p_{\Omega_0}\), and take \(\{p_r\}\) as the constant path. This does not cause any loss in generality thanks to Remark 10.7. We fix a free \(S^1\)-action \(u \in \mathcal{U}(p_0)\) with generating vector field \(V\). Let us take \(\eta \in \mathcal{K}(u)\) with curvature \(\kappa \in \Xi^2(M_0)\).

Namely, we have \([\kappa] = e_0\) and \(p_0' \kappa = d \eta\). By Theorem 9.14 there is \(\omega_0 \in \Xi_{\omega_0}^2(M_0)\) such that
\[
\omega_s = A_s \kappa + \omega_0.
\]
Moreover, we can exploit our freedom in choosing \(\eta\) and \(c_0\) to get
- \(\omega_s = \omega_0\), if \(c_0 = 0\),
- \(\omega_s = A_s \kappa\), if \(C = 0\).

Indeed, if \(\omega_0 = 0\), we just take \(\eta\) to be a flat connection, i.e. \(\kappa = 0\). If \(C = 0\), we take \(c_0 = 0\) so that \(A_s \neq 0\) (as \([\omega_s] \neq 0\) and pick \(\eta \in \mathcal{K}(u)\) with the property that \(\kappa = \frac{1}{A_s} \omega_s\), which is equivalent to \(\omega_0 = 0\). We define the function
\[
Q : M_0 \times \mathbb{R} \to \mathbb{R}, \quad Q(\cdot, A) \omega_s^n = (A \kappa + \omega_0)^n, \quad \forall A \in \mathbb{R}.
\]
The function $\mathcal{P} : M_0 \times \mathbb{R} \to \mathbb{R}$ is determined by the properties

- $\mathcal{P}(q, 0) = 0$, \quad $\forall q \in M_0$
- $\frac{\partial \mathcal{P}}{\partial A}(q, A) = \mathcal{Q}(q, A)$, \quad $\forall (q, A) \in M_0 \times \mathbb{R}$.

There holds

$$Q(A) = \int_{M_0} Q(\cdot, A) \omega^p_0, \quad P(A) = \int_{M_0} \mathcal{P}(\cdot, A) \omega^a_0.$$ 

From the definition, $\mathcal{Q}(\cdot, A_s) \equiv 1$ and we define $\mathcal{I} \subset \mathbb{R}$ to be the maximal interval satisfying

- $A_s \in \mathcal{I}$,
- $\min_{q \in M_0} \mathcal{Q}(q, A) > 0$, \quad $\forall A \in \mathcal{I}$.

Therefore, $\mathcal{P}$ is strictly increasing in the second coordinate on $M_0 \times \mathcal{I}$. This fact will be crucially used in Proposition [10.16]

**Remark 10.9.** If $C = 0$, then $\mathcal{Q}(\cdot, A) = (A/A_s)^n$, $\mathcal{P}(\cdot, A) = \frac{n+1}{n+2}(A/A_s)^{n+1}$, which implies $\mathcal{I} = (0, +\infty)$ or $\mathcal{I} = (-\infty, 0)$ depending on the sign of $A_s$. If $e_0 = 0$, then $\mathcal{Q} \equiv 1$ and $\mathcal{P}(\cdot, A) \equiv A$, which implies $\mathcal{I} = \mathbb{R}$.

We now make use of the notion of H-form from Section 6.1. The H-forms we use below are associated with $\Omega_0$, $\alpha_0 = 0$ and $\sigma_0 = \eta$, the given $S^1$-connection. Namely, we deal with forms of the type

$$\Omega_{H\eta} = \Omega_0 + d(H\eta) \in \Xi^2_0(\Sigma), \quad H : \Sigma \to \mathbb{R}.$$ 

**Remark 10.10.** Let us describe the fibres of the map $H \mapsto \Omega_{H\eta}$. To this end, we compute

$$d(H\eta) = d^hH \wedge \eta + H p_0^* \kappa. \quad \text{(10.6)}$$

Here $d^hH$ is the horizontal part of $dH$:

$$d^hH := dH - dH(V)\eta.$$ 

Let us suppose now that $d(H\eta) = 0$. From [10.6], we deduce

$$d^hH = 0, \quad H p_0^* \kappa = 0.$$ 

The first relation tells us that $H$ is invariant along curves tangent to $\ker \eta$.

Thus, if $e_0 \neq 0$, then $H$ is constant, as the holonomy in this case is the whole $S^1$. Evaluating the second relation at a point $q \in M_0$, where $\kappa_q \neq 0$, we conclude that $H = 0$.

On the other hand, if $e_0 = 0$, then we can take the quotient $\Pi_k : \Sigma \to \Sigma_k$ by the holonomy of $\eta$, which is a finite subgroup of $S^1$ (see Lemma 7.6). The bundle $\Sigma_k \to M_0$ is trivial and admits an angular function $\phi_k : \Sigma_k \to S^1$. We define $\phi : \Sigma \to S^1$ as $\phi := \phi_k \circ \Pi_k$. We conclude that for each $H$ with $d^hH = 0$, there exists $\tilde{H} : S^1 \to \mathbb{R}$ such that $H = \tilde{H} \circ \phi$.

In our setting, the volume of an H-form can be computed in terms of the integration operator associated with the free $S^1$-action $u$:

$$u_* : C^0(\Sigma) \to C^0(M_0), \quad u_*(K)(q) := (p_0)_*(K\eta)(q) = \int_{S^1} K(\Phi^V_\tau(z)) \, dt, \quad \forall q \in M_0,$$

where $z$ is any point in $p^{-1}_0(q)$ and $(p_0)_*$ denotes the integration along the $p_0$-fibres.
Lemma 10.11. For a function $H : \Sigma \to \mathbb{R}$, there holds

$$\text{Vol}(H\eta) = \int_{M_0} u_* (\mathcal{P}(p_0, H)) \omega^n_*.$$ 

Proof. Recalling the definition of Vol from (5.1) and using Fubini’s Theorem, we compute

$$\text{Vol}(H\eta) = \int_0^1 \int_0^1 H\eta \wedge (p_0^*\omega_0 + r H p_0^*\kappa)^n = \int_0^1 \int_0^1 \mathcal{Q}(p_0, r H) (p_0^*\omega_0 + r H p_0^*\kappa)^n$$

$$= \int \left( \mathcal{P}(p_0, H) \eta \wedge (p_0^*\omega_0 + r H p_0^*\kappa)^n \right)$$

$$= \int M_0 (p_0)_* (\mathcal{P}(p_0, H) \eta) \wedge \omega_*^n$$

$$= \int M_0 u_* (\mathcal{P}(p_0, H)) \omega_*^n,$$

where in the second equality, we have used that $H\eta \wedge dH \wedge \eta = 0$.

Remark 10.12. When $e_0 = 0$, defining the Calabi invariant as the volume of $H\eta$

$$\text{CAL}_{\omega_*}(H) := \text{Vol}(H\eta) = \int_{M_0} u_* (H) \omega_*^n,$$

recovers the classical definition of the Calabi invariant for the trivial bundle $M_0 \times S^1 \to M_0$. We say that $H$ is normalised with respect to $\omega_*$ if its Calabi invariant $\text{CAL}_{\omega_*}(H)$ vanishes, which is equivalent to requiring that the one-form $H\eta$ is normalised according to Definition 5.5. When $H = h \circ \phi$, as in Remark 10.10, then $H$ is normalised if and only if the integral of $h$ is zero.

Lemma 10.13. If a function $H : \Sigma \to \mathbb{R}$ takes values in $\mathcal{I}$, the $H$-form $\Omega_{H\eta}$ is odd-symplectic.

Proof. To prove the statement, we show that $\Omega_{H\eta}$ is non-degenerate on the hyperplane distribution ker $\eta$. We compute preliminarily

$$\Omega_0 + d(H\eta) = \Omega_0 + d^h H \wedge \eta + H p_0^*\kappa = (p_0^*\omega_0 + H p_0^*\kappa) + d^h H \wedge \eta.$$

Let $z \in \Sigma$ be arbitrary, and set $q := p_0(z)$. We have an inverse $\mathcal{L}_z : T_q M_0 \to (\ker \eta)_z$ for the projection $dp_0 : (\ker \eta)_z \to T_q M_0$, which we can use to pull back the restriction of forms on ker $\eta$ to the tangent space of $M_0$:

$$\mathcal{L}_z^* \left( \Omega_0 + d(H\eta) \right)_z^n = \left( \mathcal{L}_z^* \Omega_0 + \mathcal{L}_z^* d(H\eta) \right)_z^n = \left( \mathcal{L}_z^* (p_0^*\omega_0 + H p_0^*\kappa) \right)_z^n$$

$$= \left( \omega_0 + H(z) \kappa \right)_q^n = \mathcal{Q}(q, H(z)) (\omega_*^n)_q,$$

where we used $\mathcal{L}_z^* (d^h H \wedge \eta) = 0$. The last form is non-degenerate since $\mathcal{Q}(q, H(z)) > 0$. 

Combining this lemma with the stability property proved in Proposition 6.4 we arrive at the following corollary.

**Corollary 10.14.** Suppose that \( p_0 : \Sigma \to M_0 \) is an oriented \( S^1 \)-bundle with \( e_0 = 0 \) and choose \( \eta \) to be a flat connection for \( p_0 \). Let \( \Omega_0 = p_0^* \omega_0 \) be given, where \( \omega_0 \) is some symplectic form on \( M_0 \). For every sufficiently small \( C^2 \)-neighbourhood \( U \) of \( \Omega_0 \) in \( \mathcal{S}_C(\Sigma) \), there exist a \( C^2 \)-neighbourhood \( \mathcal{D} \) of \( \text{id}_\Sigma \) in \( \text{Diff}(\Sigma) \) and a \( C^2 \)-neighbourhood \( \mathcal{H} \) of the zero function in the space of normalised functions on \( \Sigma \) with the following property:

\[
\forall \Omega \in U, \ \exists \Psi \in \mathcal{D}, \ \exists H \in \mathcal{H}, \quad \Psi^* \Omega = \Omega_0 + d(H \eta).
\]

**Proof.** Let \( \Omega \in \mathcal{S}_C(\Sigma) \) be \( C^2 \)-close to \( \Omega_0 \). By Lemma A.5 we have \( \Omega = \Omega_0 + d\alpha \) for a \( C^2 \)-small normalised one-form \( \alpha \). We apply Proposition 6.4 with \( \alpha_0 = 0 \), \( \sigma_r \equiv \eta \) and \( \alpha_r = r \alpha \) so that \( \Omega_r = \Omega_0 + r \alpha \). Indeed, the hypothesis in Remark 6.5(b) are met, as \( \Omega \) is \( C^2 \)-close to \( \Omega_0 \) and \( \sigma_r \equiv \eta \) is closed. Thus, we have the existence of a diffeomorphism \( \Psi : \Sigma \to \Sigma \) isotopic to \( \text{id}_\Sigma \) and of a normalised function \( H : \Sigma \to \mathbb{R} \) such that \( \Psi^* \Omega = \Omega_0 + d(H \eta) \). More explicitly, from (6.2) and (6.3), we see that we have paths \( \{ \Psi_r \} \) and \( \{ H_r \} \) with \( \Psi_0 = \text{id}_\Sigma \) and \( H_0 = 0 \) satisfying

\[
X_r \in \ker \eta, \quad (i X_r (\Omega_0 + r \alpha) + \alpha)|_{\ker \eta} = 0, \quad \dot{H}_r = \alpha(V) \circ \Psi_r,
\]

where \( X_r \) is the time-dependent vector field generating \( \Psi_r \) and \( V \) is the vertical vector field of \( \eta \). From these relations, we see that \( \Psi_r \) is \( C^2 \)-close to \( \text{id}_\Sigma \) and \( H_r \) is \( C^2 \)-small. \( \square \)

### 10.3 The local systolic-diastolic inequality for quasi-autonomous H-forms

We introduce H-forms of special type for which we can prove the systolic-diastolic inequality.

**Definition 10.15.** We say that an H-form \( \Omega_{H_\eta} \) is **quasi-autonomous**, if there are pairs \( q_{\text{min}}, q_{\text{max}} \in M_0 \) and \( H_{\text{min}}, H_{\text{max}} : \Sigma \to \mathbb{R} \) such that

\[
(i) \quad \Omega_{H_{\text{min}} \eta} = \Omega_{H_\eta} = \Omega_{H_{\text{max}} \eta} \quad \text{and} \quad \min_{\Sigma} H_{\text{min}} = H_{\text{min}}(z), \quad \forall \ z \in p_0^{-1}(q_{\text{min}}), \\
(ii) \quad \max_{\Sigma} H_{\text{max}} = H_{\text{max}}(z), \quad \forall \ z \in p_0^{-1}(q_{\text{max}}).
\]

We say that \( H : \Sigma \to \mathbb{R} \) is quasi-autonomous, if the form \( \Omega_{H \eta} \) is quasi-autonomous. Thanks to Remark 10.10 we have \( H_{\text{max}} = H = H_{\text{max}} \) if \( e_0 \neq 0 \). We recover the standard definition of quasi-autonomous Hamiltonians [HZ11, p. 186] if \( p_0 \) is trivial.

**Proposition 10.16.** Let \( \Omega_\ast \in \mathcal{Z}_C(\Sigma) \) be a Zoll odd-symplectic form with bundle \( p_{\Omega_\ast} = p_0 \). When \( e_0 = 0 \), the reference connection one-form \( \eta \) for \( p_0 \) is assumed to be flat. If \( H : \Sigma \to \mathbb{R} \) is quasi-autonomous with values in \( \mathcal{I} \) and satisfies \( \mathcal{A}_{\text{min}}(\Omega_{H \eta}) \in \mathcal{I} \), \( \mathcal{A}_{\text{max}}(\Omega_{H \eta}) \in \mathcal{I} \), then

\[
P(\mathcal{A}_{\text{min}}(\Omega_{H \eta})) \leq \mathfrak{Vol}(\Omega_{H \eta}) \leq P(\mathcal{A}_{\text{max}}(\Omega_{H \eta})).
\]

Moreover, let us consider the following three conditions:

(a) \( H \) is constant, when \( e_0 \neq 0 \), or factors through the map \( \phi \) of Remark 10.10, when \( e_0 = 0 \).

(b) The form \( \Omega_{H \eta} \) is Zoll.

(c) Any of the two inequalities above is an equality.
We have \((a) \Rightarrow (b)\) and \((a) \Leftrightarrow (c)\). The implication \((b) \Rightarrow (c)\) holds if \(\Omega_{H\eta}\) belongs to the neighbourhood \(V\) given in Proposition 10.4 or if

\[
A_{\text{max}}(\Omega_{H\eta}) - A_{\text{min}}(\Omega_{H\eta}) < \inf \left\{ a > 0 \mid a = \langle a(C), [\gamma_r] \rangle, \text{ for some } [\gamma_r] \in \pi_1(A_b(\Sigma)) \right\}.
\]

**Proof.** From \([10.6]\), we see that the oriented \(p_0\)-fibre over \(q_{\text{min}}\) is a closed characteristic for \(\Omega_{H\eta}\) with action \(\min H_{\text{min}}\). Therefore,

\[
A_{\text{min}}(\Omega_{H\eta}) \leq \min H_{\text{min}} \leq H_{\text{min}}. \tag{10.7}
\]

Since \(P\) is increasing in the second coordinate \(A\), when \(A \in \mathcal{I}\), we have by Lemma 10.11

\[
P(A_{\text{min}}(\Omega_{H\eta})) \leq P(\min H_{\text{min}}) = \int_{M_0} u_*(\mathcal{P}(p_0, \min H_{\text{min}})) \omega^n = \mathcal{M}(\Omega_{H\eta}). \tag{10.8}
\]

In the case of \(e_0 = 0\), this is true since \(\mathcal{I} = \mathbb{R}\) (see Remark 10.9) and we can assume that \(H_{\text{min}}\) is normalised up to adding a constant. The inequality for \(A_{\text{max}}\) is obtained similarly.

Next, we show the implications between \((a)\), \((b)\), and \((c)\). Assuming \((a)\), we have

\[
\Omega_{H\eta} = \begin{cases} p_0^*(\omega_0 + H\kappa), & \text{if } e_0 \neq 0 \\ p_0^*\omega, & \text{if } e_0 = 0, \end{cases}
\]

which is Zoll since the value of \(H\) is in \(\mathcal{I}\). This shows \((a) \Rightarrow (b)\). Moreover, if \(e_0 \neq 0\), then \(H = H_{\text{min}} = H_{\text{max}}\) is constant. Therefore, all the inequalities in \([10.7]\) and \([10.8]\) are actually equalities, and \((c)\) follows. If \(e_0 = 0\), then \(H = H_{\min} = H_{\max}\), and therefore, \(H\) factors through the map \(\phi\), as \(H_{\min}\) is constant and \(H - H_{\min}\) factors through \(\phi\) by Remark 10.10 since \(\Omega_{H\eta} = \Omega_{H_{\min}}\).

Finally, let us assume \((b)\), namely that \(\Omega_{H\eta}\) is Zoll. If \(\Omega_{H\eta} \in V\), we know from Proposition 10.4 that \(\mathcal{X}(\Omega_{H\eta}) \subset \Lambda(p_0)\). Therefore, \(A_{\text{min}}(\Omega_{H\eta}) = A_{\text{max}}(\Omega_{H\eta})\) due to Proposition 10.4 (iii) and \((c)\) follows. On the other hand, let

\[
\epsilon := A_{\text{max}}(\Omega_{H\eta}) - A_{\text{min}}(\Omega_{H\eta}) \geq 0
\]

and suppose that \(\epsilon\) is strictly less than the positive generator of \(\langle a(C), \pi_1(A_b(\Sigma)) \rangle\). Let \(p\) be the bundle associated with the Zoll form \(\Omega_{H\eta}\), and let \(z_0, z_1 \in \Sigma\) be two points such that \(j_p(z_0), j_p(z_1) \in \mathcal{X}(\Omega_{H\eta}; p_0)\) and

\[
A_{\Omega_{H\eta}}(j_p(z_0)) = A_{\text{min}}(\Omega_{H\eta}), \quad A_{\Omega_{H\eta}}(j_p(z_1)) = A_{\text{max}}(\Omega_{H\eta}).
\]
If \( s \mapsto z_s \) is any path connecting \( z_0 \) to \( z_1 \), we obtain an element \([\gamma_\tau] \in \pi_1(\Lambda_h(\Sigma))\) by concatenating a short homotopy for \( j_p(z_0) \), the path \( s \mapsto j_p(z_s) \), and a reverse short homotopy for \( j_p(z_1) \). This loop has the property that 
\[
\epsilon = \langle a(C), [\gamma_\tau] \rangle,
\]
which implies that \( \epsilon = 0 \) and condition (c) follows. \(\Box\)

**Remark 10.17.** By Proposition \([10.4]\) a quasi-autonomous function that is \(C^2\)-close to the constant \(A_* = A(\Omega_1)\) satisfies the hypotheses of Proposition \([10.16]\). Moreover for such a function, conditions (a), (b), and (c) above are equivalent since the corresponding odd-symplectic form belongs to \(\mathcal{V}\).

Combining the result we have just proven with the theory of generating functions on closed symplectic manifolds, we can give a proof of Theorem \([1.19]\) namely of the local systolic-diastolic inequality in the case of \(e_0 = 0\).

**Proof of Theorem \([1.19]\)**

Let \(\Omega_1\) be a Zoll-odd symplectic form such that the associated \(S^1\)-bundle \(p_{\Omega_1}\) has vanishing real Euler class. Due to Remark \([10.7]\) we may assume that \(p_{\Omega_1} = p_0 : \Sigma \to M_0\) and \(e_0 = 0\). Let \(\omega_\tau\) be the symplectic form on \(M_0\) such that \(p_0^* \omega_\tau = \Omega_1\). By Lemma \([7.6]\) there exists a trivial oriented \(S^1\)-bundle \(p_0^* : \Sigma^* \to M_0^*\) and a bundle map \(\Pi : \Sigma^* \to \Sigma\) such that \(p_0^* = \Pi^* p_0\).

Therefore, thanks to Proposition \([10.8]\) it is enough to prove Theorem \([1.19]\) for trivial bundles. Let \(p_0 : M_0 \times S^1 \to M_0\) with angular function \(\phi\). Assume that \(\Omega\) is \(C^2\)-close to \(\Omega_1\). Due to Corollary \([10.14]\) there is a diffeomorphism \(\Psi : \Sigma \to \Sigma\) such that \(\Psi = \Psi_1\), where \(\{\Psi_\tau\}\) is a \(C^1\)-small isotopy with \(\Psi_0 = \text{id}_{M_0 \times S^1}\), and
\[
\Psi^* \Omega = \Omega_{Hd\phi}
\]
for a \(C^2\)-small normalised Hamiltonian \(H : M_0 \times S^1 \to \mathbb{R}\). Thus, \(\text{Sol}(\Omega) = 0 = \text{Sol}(\Omega_{Hd\phi})\), namely \(\text{CAL}_{\omega_\tau}(H) = 0\). Moreover, \(\Omega\) and \(\Omega_{Hd\phi}\) have the same systole and diastole due to Proposition \([9.10]\) as \([\Psi_\tau]^*\) preserves the set of short homotopies. The closed characteristics in \(\mathcal{X}(\Omega_{Hd\phi}; p_0)\) are curves \(\gamma \in \Lambda(p_0)\) of the type \(\gamma(t) = (q(t), t)\), for some loop \(q : S^1 \to M\) with small capping disc \(\tilde{q} : D^2 \to M\). Thanks to \([9.3]\), the action of these curves recovers the classical Hamiltonian action
\[
A_{Hd\phi}(\gamma) = \int_{D^2} \tilde{q}^* \omega_\tau + \int_{S^1} H(q(t), t) dt
\]

Let \(\{\varphi_t\}_{t \in [0,1]}\) be the Hamiltonian isotopy on \((M_0, \omega_\tau)\) up to time one generated by \(H\). The maps \(\varphi_t\) are \(C^1\)-close to the identity, as \(H\) is \(C^2\)-small. From the theory of generating functions on arbitrary symplectic manifolds (see \([LM93]\) Proposition 5.11 and \([MS98]\) Proposition 9.31)), \(\{\varphi_t\}_{t \in [0,1]}\) is homotopic with fixed endpoints to a \(C^1\)-small Hamiltonian isotopy \(\{\varphi'_t\}_{t \in [0,1]}\) generated by a quasi-autonomous normalised Hamiltonian \(H' : M_0 \times S^1 \to \mathbb{R}\), namely \(\text{Sol}(\Omega_{H'd\phi}) = 0 = \text{CAL}_{\omega_\tau}(H')\). Adapting an argument in \([Sch00]\), we have an action-preserving bijection
\[
\mathcal{X}(\Omega_{Hd\phi}; p_0) \cong \text{Fix } \varphi_1' \cong \mathcal{X}(\Omega_{H'd\phi}; p_0).
\]

Therefore, \(\Omega_{Hd\phi}\) and \(\Omega_{H'd\phi}\) have the same systole and diastole. The local systolic-diastolic inequality holds for \(\Omega_{H'd\phi}\) due to Proposition \([10.16]\) and hence also for \(\Omega_{Hd\phi}\) and \(\Omega\). \(\Box\)
Part III

A systolic-diastolic inequality for magnetic geodesics

11 Curves with prescribed curvature on surfaces

In this last part, we apply the local systolic-diastolic inequality we have proven to the study of curves with prescribed geodesic curvature on a connected oriented closed surface \((M, o_M)\) endowed with a Riemannian metric \(g\). We will write \(M = S^2\) for the two-sphere and \(M = \mathbb{T}^2\) for the two-torus. The Riemannian metric \(g\) and the orientation \(o_M\) yield a well-defined way of measuring angles in each tangent space and an area form \(\mu \in \Omega^2(M)\). If \(c : I \to M\) is a curve parametrised by arc-length on some interval \(I\), we define its geodesic curvature \(\kappa_c : I \to \mathbb{R}\) to be the unique function satisfying the relation

\[
\nabla_{\dot{c}} \dot{c} = \kappa_c \dot{c}^\perp,
\]

where \(\nabla\) is the Levi-Civita connection, and \(\dot{c}^\perp\) is the unit vector with the property that the angle from \(\dot{c}\) to \(\dot{c}^\perp\) is \(\frac{\pi}{2}\).

Let \(f : M \to \mathbb{R}\) be a function. A curve \(c : \mathbb{R} \to M\) is said to be a magnetic geodesic, or an \(f\)-magnetic geodesics, when we want to mention the function \(f\) explicitly, if it is parametrised by arc-length and satisfies the equation

\[
\kappa_c(t) = -f(c(t)), \quad \forall t \in \mathbb{R}.
\]

The magnetic geodesics of \(-f\) are the same as those of \(f\) with reversed orientation.

**Example 11.1.** Let us suppose that \(g = g_*\) is a Riemannian metric of constant Gaussian curvature \(K_* \in \mathbb{R}\) and consider a constant function \(f_* \in \mathbb{R}\) such that

\[
f_*^2 + K_* > 0.
\]

Then, all \(f_*\)-magnetic geodesics are closed. If \(c\) is a prime periodic magnetic geodesic, its lift \(\tilde{c}\) to the universal cover \(\tilde{M}\) of \(M\) bounds a geodesic ball of radius

\[
R = \begin{cases} 
\arctan \left( \frac{\sqrt{|K_*|}}{|f_*|} \right), & \text{if } K_* > 0; \\
\frac{1}{|f_*|} & \text{if } K_* = 0; \\
\arctanh \left( \frac{\sqrt{|K_*|}}{|f_*|} \right) & \text{if } K_* < 0.
\end{cases}
\]

If \(f_* > 0\), then the curve \(\tilde{c}\) parametrises the boundary of the ball in the clockwise direction.

We end this introductory paragraph by associating two quantities to \(f\). The latter is a generalisation of the quantity on the left-hand side in (11.2).

**Definition 11.2.** The average of a function \(f : M \to \mathbb{R}\) is defined by

\[
f_{\text{avg}} := \frac{1}{\text{area}(M)} \int_M f \mu,
\]
where \( \text{area}(M) = \int_M \mu \) is the total area of \( M \) with respect to \( g \). The **average curvature** of \( f \) is the number

\[
K_f := (f_{\text{avg}})^2 + \frac{2\pi \chi(M)}{\text{area}(M)},
\]

where \( \chi(M) \) is the Euler characteristic of \( M \).

### 11.1 The unit tangent bundle

It is a classical fact that equation (11.1) defines a flow on the unit tangent bundle

\[
p_\infty : T^1M \to M
\]

of the metric \( g \), where \( T^1M \) are the vectors \((q,v) \in TM\) of unit norm and \( p_\infty \) is the foot-point projection \( p_\infty(q,v) = q \). This means that there exists a smooth vector field \( X_f \) on \( T^1M \) such that the trajectories of the associated flow \( \Phi^{X_f} \) are the tangent lifts \((c,\dot{c})\) of \( f \)-magnetic geodesics. The vector field can be explicitly written as

\[
X_f = X + \frac{1}{2\pi} (f \circ p_\infty)V,
\]

where \( X \) is the geodesic vector field of \( g \), and \( V \) is the vector field whose flow rotates the fibres of the map \( p_\infty \) in the \( o_{\partial M}\)-negative direction with constant angular speed \( \frac{1}{2\pi} \). Thus, the vector field \( V \) generates a free \( S^1 \)-action on \( T^1M \) (recall our convention \( S^1 = \mathbb{R}/\mathbb{Z} \)) and we denote by \( h_\infty \in [S^1,T^1M] \) the free homotopy class of the orbits of \( V \), namely of the oriented \( p_\infty \)-fibres. The Levi-Civita one-form \( \eta \in \Omega^1(T^1M) \) is the connection for \( p_\infty \) satisfying

\[
\eta(V) = 1, \quad d\eta = \frac{1}{2\pi} p_\infty^*(K\mu),
\]

where \( K \) is the Gaussian curvature of \( g \). This implies that, if \( e \in H^2_{\text{dR}}(M) \) is minus the real Euler class of \( p_\infty \), then \( e = \frac{1}{2\pi} [K\mu] \), and by the Gauß-Bonnet Theorem, we have

\[
\langle e, [M] \rangle = \chi(M).
\]

(11.3)

Finally, we endow \( T^1M \) with the orientation \( o_{T^1M} \) satisfying

\[
o_{T^1M} = o_M \oplus o_V
\]

so that

\[
\Omega_\infty := p_\infty^*\mu
\]

is a Zoll odd-symplectic form admitting \( p_\infty \) as an associated bundle.

Let \( \alpha_{\text{can}} \) be the canonical one-form on \( T^1M \) given by \((\alpha_{\text{can}})_v \cdot Y := g(v, d_y p_\infty \cdot Y)\) for \( Y \in T_y T^1M \). Then, \( \alpha_{\text{can}} \) is a contact form and the geodesic vector field \( X \) is its Reeb vector field. There holds (see [Ber65, V.2.5 and (5.2.12)])

\[
\alpha_{\text{can}} \wedge d\alpha_{\text{can}} = 2\pi \eta \wedge p_\infty^*\mu
\]

(11.4)

so that \( \alpha_{\text{can}} \wedge d\alpha_{\text{can}} \) is a positive form with respect to \( o_{T^1M} \).

The vector field \( X_f \) is a nowhere vanishing section of the characteristic distribution \( \ker \Omega_f \) of the odd-symplectic form

\[
\Omega_f := d\alpha_{\text{can}} + p_\infty^*(f\mu).
\]
Indeed, from the equation above, we have $\Omega_f = \iota_{X_f}(\alpha_{\text{can}} \wedge \d \alpha_{\text{can}})$. This also shows that
\begin{equation}
\mathfrak{o}_{T^1 M} = \mathfrak{o}_{\Omega_f} \oplus \mathfrak{o}_{X_f},
\end{equation}
where $\mathfrak{o}_{\Omega_f}$ is the co-orientation of the characteristic distribution of $\Omega_f$.

**Definition 11.3.** We say that a function $f : M \to \mathbb{R}$ is **Zoll**, if the associated odd-symplectic form $\Omega_f$ is Zoll. We denote by $p_f : T^1 M \to M_f$ the oriented $S^1$-bundle associated with $\Omega_f$. We write $\mathfrak{h}_f$ for the free homotopy class of oriented $p_f$-fibres, $e_f$ for minus the Euler class of $p_f$, and $\omega_f$ for the symplectic form on $M_f$ such that $\Omega_f = p_f^* \omega_f$. We put on $M_f$ the orientation induced by $T^1 M$ and $p_f$, or equivalently the orientation given by $\omega_f$.

**Remark 11.4.** If $f$ is Zoll, we have
\begin{equation}
\langle [\omega_f], [M_f] \rangle > 0.
\end{equation}
Moreover, by (11.5), the lifts $(c, \dot{c})$ of $f$-magnetic geodesics parametrise the $p_f$-fibres with the positive orientation.

**Lemma 11.5.** Let $f : M \to \mathbb{R}$ be a Zoll function. There exists a path of oriented $S^1$-bundles $\{p_r\}_{r \in [0,1]}$ such that
\begin{align*}
p_0 &= \pm p_\infty, \quad p_1 = p_f,
\end{align*}
where $-p_\infty$ denotes the bundle $p_\infty$ with opposite orientation. As a consequence, there holds
\begin{align*}
\mathfrak{h}_f &= \pm \mathfrak{h}_\infty, \quad \langle e_f, [M_f] \rangle = \chi(M).
\end{align*}
If $M$ is the two-torus $T^2$, then
\begin{equation}
f_{\text{avg}} \neq 0, \quad \langle [\omega_f], [T^2_f] \rangle = \text{area}(T^2) |f_{\text{avg}}|, \quad \mathfrak{h}_f = \text{sign}(f_{\text{avg}}) \mathfrak{h}_\infty.
\end{equation}

**Proof.** Let assume that $M \neq T^2$. The existence of a path $\{p_r\}$ is due to Proposition 1.2. It implies at once $\mathfrak{h}_f = \pm \mathfrak{h}_\infty$, and by (11.3) $\langle e_f, [M_f] \rangle = \chi(M)$. For the latter identity, notice that the Euler class of $-p_\infty$ is minus the Euler class of $p_\infty$ and that $-p_\infty$ induces the opposite orientation on $M$, so the two minus signs cancel out when computing the Euler number.

Let us assume that $M = T^2$. Since $p_\infty$ is the trivial bundle, Proposition 1.3 implies that $e_f = 0$ and that $\Omega_f$ is not exact. Applying identity (7.6) to the bundle $p_\infty$, we get
\begin{equation}
\text{PD}([\Omega_f]) = \text{PD}(p_\infty^*(f \mu)) = \text{area}(T^2) \cdot f_{\text{avg}} \cdot [p_\infty^{-1}(pt)] \in H_1(T^1 T^2; \mathbb{R}),
\end{equation}
which implies $f_{\text{avg}} \neq 0$. Applying the identity (7.6) to the bundle $p_f$, we get
\begin{equation}
\text{PD}([\Omega_f]) = \langle [\omega_f], [T^2_f] \rangle \cdot [p_f^{-1}(pt)].
\end{equation}
From Remark 11.4, we see that $\langle [\omega_f], [T^2_f] \rangle > 0$, and thus, $[p_f^{-1}(pt)] = \text{sign}(f_{\text{avg}}) \cdot [p_\infty^{-1}(pt)]$. Since $T^1 T^2 \cong T^2 \times S^1$ is the three-torus, we have an isomorphism between the set of free-homotopy classes and the set of first homology classes, so that $\mathfrak{h}_f = \text{sign}(f_{\text{avg}}) \mathfrak{h}_\infty$ holds, as well. Finally, as $[\Omega_f]$ is a multiple of $[\Omega_\infty]$, the existence of the path $\{p_r\}$ connecting $p_\infty$ with $p_f$ is a consequence of Proposition 1.8.

**Remark 11.6.** If $M = S^2$, $T^1 M \cong \mathbb{R} P^3$, and therefore, $\mathfrak{h}_\infty = -\mathfrak{h}_\infty$. If $M$ is a surface with $\chi(M) < 0$, we will show below that $K_f > 0$, so that $f_{\text{avg}} \neq 0$, as well. Therefore, we wonder if the relation $\mathfrak{h}_f = \text{sign}(f_{\text{avg}}) \mathfrak{h}_\infty$ holds in this case, as it happens for $T^2$. 

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In view of this lemma, henceforth we assume that, up to changing the sign of \( f \),

\[
(A) \quad f_{\text{avg}} > 0, \quad \text{for} \quad M = \mathbb{T}^2, \quad \quad (B) \quad f \text{ Zoll} \implies h_f = h_\infty. \quad (11.7)
\]

In the next subsection, we will formulate a variational principle and a systolic-diastolic inequality for periodic magnetic geodesics, whose tangent lifts have class \( h_\infty \).

### 11.2 A variational principle for magnetic geodesics

**Definition 11.7.** Let \( \Lambda(M; h_\infty) \) be the space of periodic curves \( c : \mathbb{R}/T\mathbb{Z} \to M \) parametrised by arc-length, where \( T > 0 \) can be any positive real number, such that their reparametrised tangent lifts \((t \in S^1) \mapsto ((c(tT), \dot{c}(tT)) \in T^1M) \) lie in \( \Lambda_{h_\infty}(T^1M) \), the space of 1-periodic curves in the class \( h_\infty \). Any cylinder \( \Gamma : [0, 1] \times S^1 \to T^1M \) with

\[
\Gamma(0, t) = \Phi^V_{a(t)}(z), \quad \Gamma(1, t) = (c(tT), \dot{c}(tT)), \quad \forall t \in S^1,
\]

yields a capping disc \( C := p_\infty \circ \Gamma : D^2 \to M \) for \( c \), i.e. \( C(e^{2\pi it}) = c(tT) \). Here \( z \) is an arbitrary point in \( T^1M \) and \( a : S^1 \to S^1 \) is an orientation-preserving diffeomorphism. We call capping discs arising in this way admissible. Given a function \( f : M \to \mathbb{R} \), we denote by \( \Lambda(f; h_\infty) \) the set of periodic \( f \)-magnetic geodesics in \( \Lambda(M; h_\infty) \).

The set \( \Lambda(M; h_\infty) \) is a \( C^1 \)-closed subspace of the space of contractible periodic curves. Before mentioning a classical characterisation of this space, we recall that the turning number of an immersed curve \( b : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^2 \) is the winding number of its velocity curve \( \dot{b} : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^2 \) with respect to \( 0 \in \mathbb{R}^2 \).

**Lemma 11.8.** Let \( c \) be an immersed periodic curve in \( M \) that is contractible. The curve \( c \) belongs to \( \Lambda(M; h_\infty) \) if and only if the following condition holds.

- **Case \( M = S^2 \).** The turning number of \( \psi \circ c \) is odd, where \( \psi : S^2 \setminus \{q\} \to \mathbb{R}^2 \) is a diffeomorphism and \( q \in M \) lies outside the support of \( c \).

- **Case \( M \neq S^2 \).** The turning number of \( \tilde{c} \) is equal to \(-1\), where \( \tilde{c} : \mathbb{R}/T\mathbb{Z} \to \tilde{M} \subset \mathbb{R}^2 \) is a lift of \( c \) to the universal cover of \( M \). In this case, the curve \( c \) is prime.

In particular, when \( M \neq S^2 \), the curves in \( \Lambda(M; h_\infty) \) are prime. \( \square \)

A somewhat more geometrical sufficient condition for a curve to be in \( \Lambda(M; h_\infty) \) is given by the notion of Alexandrov embeddedness.

**Definition 11.9.** A periodic and arc-length parametrised curve \( c \) in \( M \) is called negatively Alexandrov embedded if it admits a negatively immersed capping disc \( C : D^2 \to M \).

**Remark 11.10.** By the Schönflies Theorem, a periodic curve \( c \) in \( M \) is negatively Alexandrov embedded, if:

- \( M = S^2 \) and \( c \) is embedded;

- \( M \neq S^2 \) and the lift \( \tilde{c} \) to the universal cover \( \tilde{M} \) bounds a compact region in the clockwise direction.
Lemma 11.11. If a periodic curve $c$ in $M$ is negatively Alexandrov embedded, $c \in \Lambda(M; h_\infty)$ and any of its immersed capping discs is admissible. In particular, the curves from Remark 11.10 belong to $\Lambda(M; h_\infty)$.

Proof. Let $C : D^2 \to M$ be a negatively immersed capping disc for $c$. Then, we can define

$$(0, 1] \times S^1 \ni (s, t) \longmapsto \left( C(se^{2\pi it}), \frac{\partial_t C(se^{2\pi it})}{|\partial_t C(se^{2\pi it})|} \right) \in T^1 M.$$ 

Since $C$ is a local embedding around $0 \in D^2$, this map extends to $s = 0$ and yields a cylinder $\Gamma : [0, 1] \times S^1 \to T^1 M$ such that

(i) $p_\infty(\Gamma(s, t)) = C(se^{2\pi it}), \quad \forall (s, t) \in [0, 1] \times S^1$,

(ii) $\Gamma(0, t) = \Phi_{a(t)}^V(z), \quad \Gamma(1, t) = (c(tT), \hat{c}(tT)), \quad \forall t \in S^1$,

for some orientation-preserving diffeomorphism $a : S^1 \to S^1$ and element $z \in T^1 M$. This shows that $C$ is admissible. \qed

After having learned a few things on the space $\Lambda(M; h_\infty)$, we now see that the subset of periodic $f$-magnetic geodesics $\Lambda(f; h_\infty)$ is the critical set of the $f$-length functional

$$\ell_f : \Lambda(M; h_\infty) \to \mathbb{R}, \quad \ell_f(c) = \ell(c) + \int_{D^2} C^*(f \mu),$$

where $\ell(c)$ is the length of $c$ and $C$ is an admissible capping disc for $c$. We have

$$\ell(c) = \int_{\mathbb{R}/TZ} (c, \hat{c})^* \alpha_{\text{can}},$$

and if $\Gamma : [0, 1] \times S^1 \to T^1 M$ is the cylinder lifting $C$, then

$$\ell_f(c) = \int_{[0, 1] \times S^1} \Gamma^* \Omega_f,$$  \hspace{1cm} (11.9)

as one sees using $\int_0^1 \Gamma(0, \cdot)^* \alpha_{\text{can}} = 0$ and Stokes’ Theorem.

The value of $\ell_f(c)$ does not depend on the choice of admissible disc. If $M \neq S^2$, this is clear since $\pi_2(M)$ vanishes. If $M = S^2$, this is due to the fact that if $\Gamma'$ is another cylinder with $\Gamma'(1, t) = c(tT, \hat{c}(tT))$, then the cylinder $\Gamma''$ obtained concatenating $s \mapsto \Gamma'(s, \cdot)$ with the reversed cylinder $s \mapsto \Gamma(1 - s, \cdot)$ projects to a sphere $\sigma : S^2 \to M$ which is null-homologous. Indeed, this can be seen from

$$\langle e, [\sigma]\rangle = \int_{S^2} \sigma^*(\frac{1}{2\pi} K \mu) = \int_{[0, 1] \times S^1} (\Gamma'')^* \eta = \int_{S^1} (\gamma_0')^* \eta - \int_{S^1} \gamma_0^* \eta = 1 - 1 = 0.$$

Definition 11.12. The magnetic systole and diastole are given by

$$\ell_{\text{min}}(f) := \inf_{c \in \Lambda(f; h_\infty)} \ell_f(c), \quad \ell_{\text{max}}(f) := \sup_{c \prime \in \Lambda(f; h_\infty)} \ell_f(c).$$

We define the average $f$-length as

$$\bar{\ell}(f) := \frac{2\pi}{f_{\text{avg}} + \sqrt{K_f}}.$$
In Section 12, we will show a magnetic systolic-diastolic inequality for certain classes of functions involving the quantities we have just defined.

**Definition 11.13.** We say that a function $f : M \to \mathbb{R}$ satisfies the **magnetic systolic-diastolic inequality**, if $K_f > 0$ and

$$\ell_{\min}(f) \leq \bar{\ell}(f) \leq \ell_{\max}(f),$$

where any of the two equalities holds if and only if $f$ is Zoll.

**Remark 11.14.** One could define *mutatis mutandis* the analogous space $\Lambda(M; -h_\infty)$ and give a corresponding variational principle and a systolic-diastolic inequality for periodic $f$-magnetic geodesics contained therein. Such a formulation can be readily obtained from the one for $\Lambda(M; h_\infty)$ by substituting $f \leftrightarrow -f$ and inverting the orientations of the curves $c$ and of the capping discs $C$. In particular, $\ell_f$ has the same expression as in the definition of the $f$-length functional above, but the average $f$-length becomes $\bar{\ell}(f) = 2\pi (-f_{\text{avg}} + \sqrt{K_f})^{-1}$. In this case, on the two-torus one has to work with functions with negative average.

We finish this subsection by providing a partial answer to the following natural question. If all the $f$-magnetic geodesics are closed, is the odd-symplectic two-form $\Omega_f$ Zoll? We collect the result in a lemma, which is a magnetic counterpart of the Gromoll-Grove Theorem [GG82].

**Lemma 11.15.** Suppose that every $f$-magnetic geodesic is periodic. The form $\Omega_f$ is Zoll in the following two cases:

(i) There holds $M \neq S^2$ and all the prime geodesics lie in $\Lambda(M; h_\infty)$.

(ii) There holds $M = S^2$ and either all prime periodic magnetic geodesics are embedded or the function $f$ is positive and all prime periodic magnetic geodesics are negatively Alexandrov embedded.

**Proof.** A theorem of Epstein [Eps72] yields an $S^1$-action $\Phi_t : T^1 M \to T^1 M$, $t \in S^1$, whose orbits coincide with the tangent lifts of magnetic geodesics (up to reparametrisation) and such that the set

$$N := \{ z \in T^1 M \mid \Phi_t(z) \neq z, \forall t \in S^1 \setminus 0 \}$$

is non-empty. The lemma follows once we show that $N = T^1 M$. The set $N$ is open, so that, by the connectedness of $T^1 M$, we just have to prove that $N$ is also closed. Let $(z_m) \subset N$ be a sequence such that $z_m \to z \in T^1 M$. Let $(c_m)$ be the corresponding sequence of magnetic geodesics and $c$ the magnetic geodesic corresponding to $z$. Since $z_m \to z$, there exists $k \in \mathbb{N}^*$ such that $(c_m)$ converges in the $C^\infty$-topology to the $k$-th iteration of $c$. It suffices to show that $k = 1$. This would give that $z \in N$, and hence, that $N$ is closed.

Let us suppose that $M \neq S^2$. The lifts $(\tilde{c}_m)$ and $\tilde{c}$ to $\tilde{M}$ are such that $(\tilde{c}_m)$ converges to the $k$-th iteration of $\tilde{c}$. From Lemma 11.8, we conclude that $k$-times the turning number of $\tilde{c}$ is equal to $-1$, which forces $k = 1$.

Let us suppose that $M = S^2$. If all prime periodic magnetic geodesics are embedded, then all the curves $c_m$ are embedded. Since $S^2$ is an oriented surface, it follows that $c$ is also embedded, which forces $k = 1$. If $f$ is everywhere positive and the curves $c_m$ are negatively Alexandrov embedded, then by [Sch12] Lemma 3.2, $c$ is also negatively Alexandrov embedded. From [Sch12] Lemma 3.1, it follows that $c$ is prime, i.e. $k = 1$. 

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Remark 11.16. In the previous lemma, we had to put extra conditions when \( M = S^2 \), since there exists a sequence of prime Alexandrov embedded curves \((c_m)\) which converges in the \( C^\infty\)-topology to a curve \( c \), which is not prime. In particular, the set \( \{ c \in \Lambda(S^2; h_\infty) \mid c \text{ is prime} \} \) is not closed in the \( C^\infty\)-topology. Furthermore, there are examples of positive magnetic functions on the two-sphere all of whose magnetic geodesics are closed but their lifts to the unit tangent bundle are the orbits of a non-free \( S^1 \)-action [Ben16].

Remark 11.17. According to Lemma 11.15, the constant magnetic functions \( f_* \) considered in Example 11.1 are Zoll. If we assume \( f_* > 0 \) when \( M \neq S^2 \), then \( h_{f_*} = h_\infty \) and \( \bar{\ell}(f_*) > 0 \).

11.3 Strong magnetic functions

We have seen that for every \((M, g, \sigma_M)\), the odd-symplectic form \( \Omega_\infty = p_\infty^* \mu \) is Zoll. Even though this form does not come from a magnetic function, \( f\)-magnetic geodesics are close to the fibres of \( p_\infty \) when \( f : M \to \mathbb{R} \) is large. This observation, which we make precise in Lemma 11.20 below, will enable us to prove the magnetic systolic-diastolic inequality for such functions in Section 12.2.

Definition 11.18. Let \( f : M \to \mathbb{R} \) be a positive function, and let \( k \in \mathbb{N} \) be a natural number. We define the quantity

\[
\langle f \rangle_k := \frac{\|f\|_{C^k}}{\min f}
\]

and observe that it is invariant under rescaling. Namely, we have

\[
\langle sf \rangle_k = \langle f \rangle_k, \quad \forall s > 0.
\] (11.10)

Definition 11.19. For a constant \( C > 0 \), a function \( f : M \to \mathbb{R} \) is said to be \( C\)-strong, if

\[
f > 0, \quad f_{\text{avg}} > \left( \langle f \rangle_4^4 + \langle f \rangle_6^6 \right) e^{C\langle f \rangle_1^2}.
\]

The key point in the definition of \( C\)-strong functions is that their associated odd-symplectic forms can be brought in any given neighbourhood of \( \Omega_\infty \) after acting by a diffeomorphism isotopic to the identity \( \text{id}_{T^1 M} \), provided \( C \) is large enough.

Lemma 11.20. Let \( \mathcal{U} \) be a \( C^2\)-neighbourhood of \( \Omega_\infty \) in the space of two-forms on \( T^1 M \). There exists a constant \( C_\mathcal{U} > 0 \) with the following property: For every \( C_\mathcal{U}\)-strong \( f : M \to \mathbb{R} \), there is a diffeomorphism \( \Psi : T^1 M \to T^1 M \) isotopic to \( \text{id}_{T^1 M} \) such that \( \frac{1}{f_{\text{avg}}} \Psi^* \Omega_f \in \mathcal{U} \).

Proof. We define

\[
f_{\text{norm}} := \frac{f}{f_{\text{avg}}}
\]

and observe that there holds

\[
\min f_{\text{norm}} \leq 1 \leq \max f_{\text{norm}}.
\]

By Lemma A.5 there is \( \zeta \in \Omega^1(M) \) such that \( d\zeta = (f_{\text{norm}} - 1)\mu \) and

\[
\|\zeta\|_{C^k} \leq C_k \|f_{\text{norm}} - 1\|_{C^k} \leq C_k \|f_{\text{norm}}\|_{C^k}, \quad \forall k \in \mathbb{N}
\] (11.11)
for some constant $C_k > 0$ depending solely on $g$ and $k \in \mathbb{N}$. For $s \in [0, 1]$, let $\mu_s$ be the two-form given by $\mu_s := q(f, s)\mu$, where $q(f, s) := sf_{\text{norm}} + (1 - s)$, and $Y_s$ be the time-dependent vector field defined through

$$t_Y \mu_s = -\zeta.$$  

If $\psi : M \to M$ is the time-one map of $Y_s$, an application of Moser’s trick yields

$$\psi^*(f_{\text{norm}}\mu) = \mu. \quad (11.12)$$

If $\sharp : T^*M \to TM$ is the metric duality and $\ast : T^*M \to T^*M$ the Hodge star operator, we can write $Y_s$ explicitly as

$$Y_s = \frac{\sharp \ast \zeta}{q(f, s)}.$$  

Since $\ast$ and $\sharp$ are smooth bundle maps, we have (possibly with bigger $C_k > 0$)

$$\|Y_s\|_{C^k} \leq C_k \max_{s \in [0, 1]} \left\| \frac{\zeta}{q(f, s)} \right\|_{C^k}, \quad \forall k \in \mathbb{N}. \quad (11.13)$$

We claim that the following bound holds (possibly with bigger $C_k > 0$):

$$\max_{s \in [0, 1]} \left\| \frac{\zeta}{q(f, s)} \right\|_{C^k} \leq C_k (f_{\text{norm}})\|^{k+1}_k = C_k (f)\|^{k+1}_k, \quad \forall k \in \mathbb{N}. \quad (11.14)$$

where the last equality is due to $(11.10)$. We prove the claim by induction and observe preliminarily that $q(f, s) \geq \min f_{\text{norm}}$. For $k = 0$, the estimate follows directly from $(11.11)$. Suppose now that the estimate holds for all $k' \leq k - 1$. Since

$$\left\| \frac{\zeta}{q(f, s)} \right\|_{C^k} = \left\| \frac{\zeta}{q(f, s)} \right\|_{C^k-1} + \left\| \nabla^k \frac{\zeta}{q(f, s)} \right\|_{C^0},$$

we just have to bound the second term. We apply the Leibniz rule to the $k$-th derivative of the product $q(f, s) \cdot \frac{\zeta}{q(f, s)} = \zeta$ and obtain

$$\nabla^k \left( \frac{\zeta}{q(f, s)} \right) = \frac{1}{q(f, s)} \left[ \nabla^k \zeta - s \sum_{k' = 0}^{k-1} \binom{k}{k'} \nabla^{k-k'} f_{\text{norm}} \cdot \nabla^{k'} \frac{\zeta}{q(f, s)} \right],$$

where we have used that $\nabla^{k-k'} q(f, s) = s \nabla^{k-k'} f_{\text{norm}}$, since $k - k' \geq 1$. Consequently, we estimate using $(11.11)$ and $(11.14)$

$$\left\| \nabla^k \frac{\zeta}{q(f, s)} \right\|_{C^0} \leq \frac{1}{\min f_{\text{norm}}} \left[ C_k \|f_{\text{norm}}\|_{C^k} + \sum_{k' = 0}^{k-1} \binom{k}{k'} \|f_{\text{norm}}\|_{C^k} \|f_{\text{norm}}\|_{C^{k-1}} (f_{\text{norm}})_{k_k}^{k} \right]$$

$$\leq \frac{1}{\min f_{\text{norm}}} C_k' \left( \|f_{\text{norm}}\|_{C^k} + \|f_{\text{norm}}\|_{C^k}^{k+1} \right)$$

$$\leq C_k' \|f_{\text{norm}}\|_{C^k} + C_k' \left( \frac{\|f_{\text{norm}}\|_{C^k}}{\min f_{\text{norm}}} \right)^{k+1}$$

$$\leq (C_k' + 1) \left( \frac{\|f_{\text{norm}}\|_{C^k}}{\min f_{\text{norm}}} \right)^{k+1}$$

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with some $C'_k > 0$ depending only on $g$ and $k$. The claim is therefore established.

Using the Levi-Civita connection for $\omega_{\infty}$, we lift $Y_s$ horizontally to $Z_s$ on $T^1 M$, so that $d\rho_{\infty}(Z_s) = Y_s$. Since the lifting map $Y_s \mapsto Z_s$ is smooth and depends only on $g$, but not on $f$, there is a constant $C''_k > 0$ depending on $k$ and $g$ such that

$$\|Z_s\|_{C^k} \leq C''_k \|Y_s\|_{C^k}, \quad \forall k \in \mathbb{N}. \quad (11.15)$$

The time-one map $\Psi : T^1 M \to T^1 M$ of $Z_s$ lifts the time-one map $\psi$ of $Y_s$, so that

$$\Psi^*(\rho_{\infty}(f_{\text{norm}} \mu)) = \rho_{\infty}^* \mu,$$

by (11.12). Putting together (11.13), (11.14), (11.15), and Lemma A.3 we get

$$B_{2,2}(\|d\psi\|) \leq \left(\left\langle f \right\rangle_3 + \left\langle f \right\rangle_2^6\right)^{C_3} \left\langle f \right\rangle_2^2,$$

for a (possibly bigger) constant $C_3 > 0$. Hence, Lemma A.3 yields

$$\|\Psi^*(d\alpha_{\text{can}})\|_{C^2} \leq C'''_3 B_{2,2}(\|d\psi\|) \|d\alpha_{\text{can}}\|_{C^2} \leq \left(\left\langle f \right\rangle_3 + \left\langle f \right\rangle_2^6\right)^{C_3} \left\langle f \right\rangle_2^2,$$

where $C'''_3 > 0$ depends only on $g$ and where we take a bigger constant $C_3 > 0$ if necessary to incorporate $\|d\alpha_{\text{can}}\|_{C^2}$ and it is possible to bring the constant to the exponent since $\left\langle f \right\rangle_2^2 \geq 1$.

Let us suppose now that $f$ is $C$-strong for some positive number $C > 0$. We compute

$$\frac{1}{f_{\text{avg}}} \Psi^* \Omega_f - \Omega_{\infty} = \frac{1}{f_{\text{avg}}} \Psi^*(d\alpha_{\text{can}}) + \Psi^*(\rho_{\infty}(f_{\text{norm}} \mu)) - \rho_{\infty}^* \mu = \frac{1}{f_{\text{avg}}} \Psi^*(d\alpha_{\text{can}}).$$

Combining this identity with the bound for $\|\Psi^*(d\alpha)\|_{C^2}$ found above, we arrive at

$$\left\|\frac{1}{f_{\text{avg}}} \Psi^* \Omega_f - \Omega_{\infty}\right\|_{C^2} = \frac{1}{f_{\text{avg}}} \|\Psi^*(d\alpha_{\text{can}})\|_{C^2} \leq \frac{\left(\left\langle f \right\rangle_3 + \left\langle f \right\rangle_2^6\right)^{C_3} \left\langle f \right\rangle_2^2}{\left(\left\langle f \right\rangle_3 + \left\langle f \right\rangle_2^6\right)^{C_3} \left\langle f \right\rangle_2^2} = \left(1 - \frac{C - C}{\left\langle f \right\rangle_2^2}\right) \leq \left(1 - \frac{C - C}{\left\langle f \right\rangle_2^2}\right),$$

which can be made arbitrarily small, if $C$ is arbitrarily large. In particular, $\frac{1}{f_{\text{avg}}} \Psi^* \Omega_f$ belongs to the given $C^2$-neighbourhood $U$. \hfill \Box

### 11.4 Volume and action of magnetic functions

Let $\mathcal{P}(T^1 M)$ be the connected component of $\mathcal{P}(T^1 M)$ containing $\rho_{\infty} : T^1 M \to M$, where $\mathcal{P}(T^1 M)$ is the space of all oriented $S^1$-bundles with total space $T^1 M$. Let $\mathcal{Z}^0_{[\Omega_{\infty}]}(T^1 M)$ be the set of all weakly Zoll pairs $(p, [\omega])$ such that $p \in \mathcal{P}^0(T^1 M)$ and $[p^* \omega] = [\Omega_{\infty}] \in H^2_{\text{dR}}(T^1 M)$. Since $\Omega_{\infty} = \rho_{\infty}^* \mu$ is a Zoll odd-symplectic form, the set $\mathcal{Z}^0_{[\Omega_{\infty}]}(T^1 M)$ is non-empty and non-degenerate. We refer to Section 8 for generalities about weakly Zoll pairs. Below, we compute the volume (see Section 5), the action (see Section 9), and the Zoll polynomial (see Definition 9.11) with respect to some reference weakly Zoll pair in $\mathcal{Z}^0_{[\Omega_{\infty}]}(T^1 M)$. As usual, we need two different arguments depending on $M \neq T^2$ or $M = T^2$. 

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Case $M \neq T^2$

In this case, the two-forms $\Omega_\infty$ and $\Omega_f$, where $f : M \to \mathbb{R}$ is any function, are exact. Indeed, in the piece of the Gysin sequence

$$H^0_{dR}(M) \xrightarrow{\cup e} H^2_{dR}(M) \xrightarrow{p_*\infty} H^2_{dR}(T^1M)$$

the map $\cup e$ is an isomorphism and thus $p_*\infty$ vanishes. Explicit primitives are given by

$$\alpha_\infty := \frac{\text{area}(M)}{\chi(M)}(\eta + p_*\infty \zeta_\infty), \quad \alpha_f := \alpha_{\text{can}} + \frac{\text{area}(M)}{\chi(M)}(f_{\text{avg}} \eta + p_*\infty \zeta),$$

where $\zeta$ and $\zeta_\infty$ are one-forms on $M$ with differential

$$d\zeta_\infty = \left(\frac{\chi(M)}{\text{area}(M)} - \frac{K}{2\pi}\right)\mu, \quad d\zeta = \left(\frac{\chi(M)}{\text{area}(M)} f - \frac{f_{\text{avg}} \cdot K}{2\pi}\right)\mu.$$

We choose as a reference weakly Zoll pair $(p_\infty, 0) \in \mathcal{Z}^0_{\Omega_\infty}(T^1M)$.

From the formula in (9.15) and the identity in (11.3), we have the Zoll polynomial

$$P(A) = \frac{\chi(M)}{2} A^2. \quad (11.16)$$

Let $\Omega$ be an exact two-form on $T^1M$, and let $\alpha$ be an arbitrary primitive one-form of $\Omega$. As mentioned in Remark 5.6 the volume of $\Omega$ is given by

$$\mathcal{V}_\Omega(\alpha) = \frac{1}{2} \int_{T^1M} \alpha \wedge \Omega.$$

The action of $\Omega$ can be computed from Remark 9.6

$$\mathcal{A}_\Omega(\gamma) = \int_{S^1} \gamma^* \alpha, \quad \forall \gamma \in \Lambda_{h_\infty}(T^1M).$$

We note that the volume is 2-homogeneous while the action is 1-homogeneous. Namely,

$$\mathcal{V}_\Omega(s\Omega) = s^2 \mathcal{V}_\Omega(\Omega), \quad \mathcal{A}_\Omega(s\Omega) = s \mathcal{A}_\Omega, \quad \forall s \in \mathbb{R}. \quad (11.17)$$

**Lemma 11.21.** If $M$ is different from the two-torus and $f : M \to \mathbb{R}$ is a function, we have

$$\mathcal{V}_\Omega(\Omega_\infty) = \frac{\text{area}(M)^2}{2\chi(M)}, \quad \mathcal{V}_\Omega(\Omega_f) = \frac{\text{area}(M)^2}{2\chi(M)} f.$$

If $\gamma_0 : S^1 \to T^1M$ is an oriented fibre of $p_\infty$ and $c \in \Lambda(M; h_\infty)$, then

$$\mathcal{A}_{\Omega_\infty}(\gamma_0) = \frac{\text{area}(M)}{\chi(M)}, \quad \mathcal{A}_{\Omega_f}(c, \dot{c}) = f_{\text{avg}} \cdot \frac{\text{area}(M) \cdot f_{\text{avg}}}{\chi(M)}.$$
Proof. We compute the volume of \( \Omega_\infty \) as
\[
\mathfrak{Vol}(\Omega_\infty) = \frac{1}{2} \int_{T^1M} \alpha_\infty \wedge \Omega_\infty = \frac{\text{area}(M)}{2\chi(M)} \int_{T^1M} \eta \wedge p_\infty^* \mu = \frac{\text{area}(M)}{2\chi(M)} \int_M (\eta \circ \mu) \mu = \frac{\text{area}(M)^2}{2\chi(M)}.
\]
To determine the volume of \( \Omega_f \), we perform first the preliminary computation
\[
\alpha_f \wedge \Omega_f = \alpha_{\text{can}} \wedge d\alpha_{\text{can}} + \frac{\text{area}(M)}{\chi(M)} (f_{\text{avg}} \eta \wedge p_\infty^* (f \mu) + p_\infty^* \zeta \wedge d\alpha_{\text{can}}),
\]
using the fact that \( X \) annihilates \( \eta \wedge \alpha \) and \( V \) annihilates \( \alpha \wedge p_\infty^* (f \mu) \). Then,
\[
2 \mathfrak{Vol}(\Omega_f) = \int_{T^1M} \alpha_f \wedge \Omega_f
= \int_{T^1M} \alpha_{\text{can}} \wedge d\alpha_{\text{can}} + \frac{\text{area}(M)}{\chi(M)} \left[ \int_{T^1M} f_{\text{avg}} \eta \wedge p_\infty^* (f \mu) + \int_{T^1M} p_\infty^* \zeta \wedge d\alpha_{\text{can}} \right]
= 2\pi \int_{T^1M} \eta \wedge p_\infty^* \mu + \frac{\text{area}(M)}{\chi(M)} \left[ f_{\text{avg}} \int_M f \mu + \int_{T^1M} p_\infty^* (d\zeta) \wedge \alpha_{\text{can}} \right]
= 2\pi \cdot \text{area}(M) + \frac{(\text{area}(M) \cdot f_{\text{avg}})^2}{\chi(M)}
= \frac{\text{area}(M)^2}{\chi(M)} K_f,
\]
where we used (11.14) and the fact that \( V \) annihilates \( p_\infty^* (d\zeta) \wedge \alpha_{\text{can}} \).

Next we compute the actions. For the \( \Omega_\infty \)-action of \( \gamma_0 \) we find
\[
\mathcal{A}_{\Omega_\infty}(\gamma_0) = \int_{S^1} \gamma_0^* \alpha_\infty = \frac{\text{area}(M)}{\chi(M)} \left( \int_{S^1} \gamma_0^* (\eta + p_\infty^* \zeta_\infty) \right) = \frac{\text{area}(M)}{\chi(M)}.
\]
To compute the \( \Omega_f \)-action of \((c, \hat{c})\), let \( \Gamma : [0, 1] \times S^1 \rightarrow T^1M \) be a homotopy as in (11.8) and recall the formula for the magnetic length (11.9). Using Stokes’ theorem we compute
\[
\mathcal{A}_{\Omega_f}(c, \hat{c}) = \int_{\mathbb{R}/T\mathbb{Z}} (c, \hat{c})^* \alpha_f = \int_{[0, 1] \times S^1} \Gamma^* \Omega_f + \int_{S^1} \Gamma(0, \cdot)^* \alpha_f = \ell_f(c) + \frac{\text{area}(M) \cdot f_{\text{avg}}}{\chi(M)},
\]
where in the last passage we used that \( \int_{S^1} \Gamma(0, \cdot)^* \alpha_{\text{can}} = 0 \).

If \( f : M \rightarrow \mathbb{R} \) is a Zoll function, then \((p_f, [\omega_f])\) is the weakly Zoll pair associated with the Zoll odd-symplectic form \( \Omega_f \). By Lemma (11.5) and condition (B) in (11.7), there holds
\[
(p_f, [\omega_f]) \in \mathcal{Z}^0_{\Omega_\infty}(T^1M).
\]
Therefore, from Remark (11.4) Proposition (9.18) and Theorem (9.14) we have
\[
0 < ([\omega_f], [M_f]) = \left. \frac{dP}{dA}(\mathcal{A}(\Omega_f)) \right|_{P(\mathcal{A}(\Omega_f)) = \mathfrak{Vol}(\Omega_f)}, \quad P(\mathcal{A}(\Omega_f)) = \mathfrak{Vol}(\Omega_f), \quad (11.19)
\]
where \( \mathcal{A}(\Omega_f) := \mathcal{A}(p_f, [\omega_f]) \) is the action value defined in (9.14). In our case, it reads
\[
\mathcal{A}(\Omega_f) = \int_{S^1} (c_f, \hat{c}_f)^* \alpha_f,
\]
where \( c_f \) is a prime periodic \( f \)-magnetic geodesic.
Corollary 11.22. If $f : M \to \mathbb{R}$ is a Zoll function and $M \neq \mathbb{T}^2$, then

$$A(\Omega_f) = \frac{\langle [\omega_f], [M_f] \rangle}{\chi(M)}; \quad K_f = \left( \frac{\chi(M)A(\Omega_f)}{\text{area}(M)} \right)^2 = \left( \frac{\langle [\omega_f], [M_f] \rangle}{\text{area}(M)} \right)^2.$$

In particular, $A(\Omega_f)$ and $\mathfrak{Vol}(\Omega_f)$ have the same sign as $\chi(M)$, and $K_f$ is positive.

Proof. From (11.16) we get there holds $\chi(M)A(\Omega_f)$, which implies the formula for $K_f$ and the statements about the signs of $\mathfrak{Vol}(\Omega_f)$ and $K_f$. □

Case $M = \mathbb{T}^2$

As a reference weakly Zoll pair, we choose

$$(p_\infty, [\mu]) \in Z_{\Omega_\infty}(T^1(M)).$$

We note that the form $\Omega_\infty = p_\infty^*\mu$ is not exact as $p_\infty : T^1\mathbb{T}^2 \to \mathbb{T}^2$ is trivial. Let $f : \mathbb{T}^2 \to \mathbb{R}$ be an arbitrary function. As stated in (11.7), here we assume that $f_{\text{avg}} > 0$. In this case, there holds

$$\tilde{\ell}(f) = \frac{\pi}{f_{\text{avg}}} > 0.$$

We normalise $\Omega_f$ by

$$\bar{\Omega}_f := \frac{1}{f_{\text{avg}}}\Omega_f$$

so that $\bar{\Omega}_f$ and $\Omega_\infty$ are cohomologous. More precisely,

$$\bar{\Omega}_f = \Omega_\infty + d\left( \frac{1}{f_{\text{avg}}}\alpha_f \right), \quad \alpha_f := \alpha_{\text{can}} + p_\infty^*\zeta - \tilde{\ell}(f)d\phi,$$

where $\zeta$ is a one-form on $\mathbb{T}^2$ is such that $d\zeta = (f - f_{\text{avg}})\mu$ and $\phi : T^1\mathbb{T}^2 \to S^1$ is a global angular function for the bundle $p_\infty$, namely $d\phi(V) \equiv 1$. The term $-\tilde{\ell}(f)d\phi$ is added in order to normalise $\frac{1}{f_{\text{avg}}}\alpha_f$.

Lemma 11.23. Let $f : \mathbb{T}^2 \to \mathbb{R}$ be a function with $f_{\text{avg}} > 0$. Then the one-form $\frac{1}{f_{\text{avg}}}\alpha_f$ is normalised with respect to $\Omega_\infty$, i.e. $\text{Vol}(\frac{1}{f_{\text{avg}}}\alpha_f) = 0$.

Proof. We compute using (11.4) and the formula for the volume in Remark 5.3

$$(f_{\text{avg}})^2\text{Vol}(\frac{1}{f_{\text{avg}}}\alpha_f) = f_{\text{avg}}\int_{T^1\mathbb{T}^2} \alpha_f \wedge \left( \Omega_\infty + \frac{1}{2}d\left( \frac{1}{f_{\text{avg}}}\alpha_f \right) \right)$$

$$= \int_{T^1\mathbb{T}^2} \alpha_{\text{can}} + p_\infty^*\zeta - \tilde{\ell}(f)d\phi \wedge \left( f_{\text{avg}}p_\infty^*\mu + \frac{1}{2}(d\alpha_{\text{can}} + p_\infty^*(d\zeta)) \right)$$

$$= \frac{1}{2}\int_{T^1\mathbb{T}^2} \alpha_{\text{can}} \wedge d\alpha_{\text{can}} - \tilde{\ell}(f)f_{\text{avg}} \cdot \text{area}(\mathbb{T}^2)$$

$$= \pi \int_{T^1\mathbb{M}} \eta \wedge p_\infty^*\mu - \pi \cdot \text{area}(\mathbb{T}^2)$$

$$= 0.$$
Thanks to the lemma, we can use the one-form \( \frac{1}{f_{\text{avg}}} \alpha_f \) to compute the \( \Omega_f \)-action according to (9.10). Since the group \( \pi_2(T^2) \) vanishes, by Lemma 9.3 the action is well-defined as a functional \( A_{\Omega_f} : A_{\mathfrak{h}_\infty}(T^1T^2) \to \mathbb{R} \) given by

\[
A_{\Omega_f}(\gamma) = \frac{1}{f_{\text{avg}}} \int_{S^1} \Gamma(0, \cdot)^* \alpha_f + \frac{1}{f_{\text{avg}}} \int_{[0,1] \times S^1} \Gamma^* \Omega_f,
\]

where \( \Gamma : [0,1] \times S^1 \to T^1T^2 \) is a homotopy between an oriented \( p_\infty \)-fibre and \( \gamma \). This global action extends the one defined for curves close to the \( p_\infty \)-fibres given in (10.2).

**Lemma 11.24.** Let \( f : T^2 \to \mathbb{R} \) be a function with \( f_{\text{avg}} > 0 \). There holds

\[
A_{\Omega_f}(c, \dot{c}) = \frac{1}{f_{\text{avg}}} (\ell_f(c) - \bar{\ell}(f)), \quad \forall c \in \Lambda(T^2; \mathfrak{h}_\infty).
\]

**Proof.** The claim follows from substituting identity (11.9) in (11.20) and the computation

\[
\int_{S^1} \Gamma(0, \cdot)^* \alpha_f = \int_{S^1} \Gamma(0, \cdot)^* (\alpha + p_\infty^* \zeta - \bar{\ell}(f)d\phi) = -\bar{\ell}(f).
\]

Finally, we observe that if \( f : T^2 \to \mathbb{R} \) is a Zoll function and \( \bar{\omega}_f := \frac{1}{f_{\text{avg}}} \omega_f \), then \((p_f, [\bar{\omega}_f])\) is the weakly Zoll pair associated with the Zoll odd-symplectic form \( \Omega_f \). By Lemma 11.5 and condition (B) in (11.7), there holds

\[
(p_f, [\bar{\omega}_f]) \in \mathcal{Z}_{\Omega_\infty}(T^1M).
\]

**12 Establishing the magnetic systolic-diastolic inequality**

**12.1 The inequality in a neighbourhood of a Zoll magnetic function**

In this subsection we give a proof of Theorem 1.26 which states that the magnetic systolic-diastolic inequality holds in a \( C^2 \)-neighbourhood \( F \) of a Zoll function \( f_\ast : M \to \mathbb{R} \) (see Definition 11.3). This will be a consequence of Corollary 1.20 which says that Conjecture 2 is true with \( k = 3 \) for three-manifolds. As usual, we deal with the cases \( M \neq T^2 \) and \( M = T^2 \) separately.

**Proof of Theorem 1.26 for \( M \neq T^2 \)**

In view of (11.18) and Corollary 1.20 there exist a \( C^1 \)-neighbourhood \( \mathcal{W} \) of the set \( \Lambda(f_\ast; \mathfrak{h}_\infty) \) in \( \Lambda(M; \mathfrak{h}_\infty) \) and a \( C^2 \)-neighbourhood \( \mathcal{F} \) of the function \( f_\ast \) in the space of real functions on \( M \) such that, for all \( f \in \mathcal{F} \), we have

\[
P(A_{\min}(\Omega_f)) \leq \mathfrak{Vol}(\Omega_f) \leq P(A_{\max}(\Omega_f))
\]

with equality signs if and only if \( \Omega_f \) is Zoll. Here, following Section 10.1 we have defined

\[
A_{\min}(\Omega_f) := \min_{c \in \mathcal{W} \cap \Lambda(f_\ast; \mathfrak{h}_\infty)} A_{\Omega_f}(c, \dot{c}), \quad A_{\max}(\Omega_f) := \max_{c \in \mathcal{W} \cap \Lambda(f_\ast; \mathfrak{h}_\infty)} A_{\Omega_f}(c, \dot{c}).
\]
which vary continuously in $f \in \mathcal{F}$. Observe, indeed, that every critical point of $\mathcal{A}_{\Omega_f}$ is a tangent lift of a critical point of $\ell_f$, up to reparametrisation. Shrinking $\mathcal{F}$ if necessary, thanks to Corollary 11.22 we further assume that for all $f \in \mathcal{F}$:

$$K_f > 0, \quad \text{sign}(\mathcal{A}\min(\Omega_f)) = \text{sign}(\chi(M)) = \text{sign}(\mathcal{A}\max(\Omega_f)). \quad (12.2)$$

We claim that the magnetic systolic-diastolic inequality holds with $\mathcal{F}$. Let $f : M \to \mathbb{R}$ be a function in $\mathcal{F}$. According to Lemma 11.21 and equation (11.16), formula (12.1) becomes

$$\chi(M) \frac{\mathcal{A}\min(\Omega_f)^2}{2} \leq \frac{\text{area}(M)^2}{2\chi(M)} K_f \leq \chi(M) \frac{\mathcal{A}\max(\Omega_f)^2}{2}.$$

The identities in (12.2) simplify this inequality to

$$\mathcal{A}\min(\Omega_f) \leq \frac{\text{area}(M)}{\chi(M)} \sqrt{K_f} \leq \mathcal{A}\max(\Omega_f).$$

The formula for the action in Lemma 11.21 and the definition of $\ell_{\min}(f), \ell_{\max}(f)$ yield

$$\mathcal{A}\min(\Omega_f) \geq \ell_{\min}(f) + \frac{\text{area}(M) \cdot f_{\text{avg}}}{\chi(M)}, \quad \mathcal{A}\max(\Omega_f) \leq \ell_{\max}(f) + \frac{\text{area}(M) \cdot f_{\text{avg}}}{\chi(M)}$$

where the equalities hold when $f$ is Zoll. Combining the inequalities above, we get

$$\ell_{\min}(f) \leq \frac{\text{area}(M)}{\chi(M)} (\sqrt{K_f} - f_{\text{avg}}) \leq \ell_{\max}(f),$$

and using the definition of the average curvature, we rewrite the term in the middle as

$$\frac{\text{area}(M)}{\chi(M)} (\sqrt{K_f} - f_{\text{avg}}) = \frac{\text{area}(M) K_f - (f_{\text{avg}})^2}{\chi(M) \sqrt{K_f + f_{\text{avg}}}} = \frac{2\pi}{\sqrt{K_f + f_{\text{avg}}}} = \bar{\ell}(f).$$

This shows the magnetic systolic-diastolic inequality

$$\ell_{\min}(f) \leq \bar{\ell}(f) \leq \ell_{\max}(f).$$

Moreover if $f$ is Zoll, we actually have equalities. Conversely, if one of the two inequalities is an equality, we also have an equality in (12.1). This implies that $\Omega_f$, and thus $f$ is Zoll.

**Proof of Theorem 1.26 for $M = \mathbb{T}^2$**

Thanks to (11.21), Corollary 1.20 and Remark 10.6, there exists a $C^1$-neighbourhood $\mathcal{W}$ of $\Lambda(f_{*}; h_{\infty})$ inside $\Lambda(\mathbb{T}^2; h_{\infty})$ and a $C^2$-neighbourhood $\mathcal{F}$ of $f_{*}$ in the space of functions on $\mathbb{T}^2$ with the following properties. If $f \in \mathcal{F}$, then $f_{\text{avg}} > 0$ by (11.6) and (11.7), and accordingly $K_f > 0$. Moreover, for every $f \in \mathcal{F}$, there holds

$$\mathcal{A}\min(\bar{\Omega}_f) \leq 0 \leq \mathcal{A}\max(\bar{\Omega}_f), \quad (12.3)$$

and any of the two equalities holds if and only if $\bar{\Omega}_f$ is Zoll. Moreover, recall the definitions

$$\mathcal{A}\min(\bar{\Omega}_f) := \min_{c \in \mathcal{W} \cap \Lambda(f_{*}; h_{\infty})} \mathcal{A}_{\bar{\Omega}_f}(c, \dot{c}), \quad \mathcal{A}\max(\bar{\Omega}_f) := \max_{c \in \mathcal{W} \cap \Lambda(f_{*}; h_{\infty})} \mathcal{A}_{\bar{\Omega}_f}(c, \dot{c}).$$
From the definition of $\ell_{\text{min}}(f)$ and $\ell_{\text{max}}(f)$ and Lemma 11.24 we have

$$A_{\text{min}}(\bar{\Omega}_f) \geq \frac{1}{f_{\text{avg}}} (\ell_{\text{min}}(f) - \bar{\ell}(f)), \quad A_{\text{max}}(\bar{\Omega}_f) \leq \frac{1}{f_{\text{avg}}} (\ell_{\text{max}}(f) - \bar{\ell}(f))$$

where any of the two equalities holds, if $f$ is Zoll. Plugging these relations into (12.3), and using that $\frac{1}{f_{\text{avg}}} > 0$, we derive the desired inequality:

$$\ell_{\text{min}}(f) \leq \bar{\ell}(f) \leq \ell_{\text{max}}(f),$$

If any of the equalities holds, then there is an equality also in (12.3) and $f$ is Zoll. The converse is also true.

### 12.2 The inequality for strong magnetic functions

In this subsection we prove Theorem 1.28, which states that the magnetic systolic-diastolic inequality holds for $C_g$-strong functions (see Definition 11.19), where $C_g > 0$ is a constant depending only on $g$ that we will determine.

#### Proof of Theorem 1.28 for $M \neq T^2$

By Corollary 1.20 there exists a $C^0$-neighbourhood $\Lambda(p_{\infty}) \subset \Lambda_{\infty}(T^1M)$ of the $p_{\infty}$-fibres and a $C^2$-neighbourhood $\mathcal{U}$ of $\Omega_{\infty}$ in the space of exact odd-symplectic forms on $T^1M$ such that

$$P(A_{\text{min}}(\Omega)) \leq \mathfrak{Vol}(\Omega) \leq P(A_{\text{max}}(\Omega)) \quad \forall \Omega \in \mathcal{U} \tag{12.4}$$

with equality signs if and only if $\Omega$ is Zoll. Here, $A_{\text{min}}(\Omega)$ and $A_{\text{max}}(\Omega)$ are the minimal and maximal action among the closed characteristics in the set $\mathcal{X}(\Omega; p_{\infty})$ (see (10.3) and (10.4)). Since $\text{sign}(A_{\Omega_{\infty}}(\gamma)) = \text{sign}(\chi(M))$ by Lemma 11.21 and $A_{\text{min}}$, $A_{\text{max}}$ vary continuously by Proposition 10.4 in $\mathcal{U}$ (after shrinking if necessary), we have

$$\text{sign}(A_{\text{min}}(\Omega)) = \text{sign}(\chi(M)) = \text{sign}(A_{\text{max}}(\Omega)), \quad \forall \Omega \in \mathcal{U}.$$  

In particular, from (12.4) and the formula for $P$, we also have

$$\text{sign}(\mathfrak{Vol}(\Omega)) = \text{sign}(\chi(M)). \tag{12.5}$$

We prove the theorem with $C_g := C_{\mathcal{U}}$, the constant given by Lemma 11.20. Let $f : M \to \mathbb{R}$ be a $C_g$-strong function, and let $\Psi : T^1M \to T^1M$ be a diffeomorphism isotopic to $\text{id}_{T^1M}$ such that $\frac{1}{f_{\text{avg}}} \Psi^* \Omega_f \in \mathcal{U}$, whose existence is ensured by Lemma 11.20. From the homogeneity (11.17) of the volume and its invariance property in Proposition 5.8 we have

$$\mathfrak{Vol}(\frac{1}{f_{\text{avg}}} \Psi^* \Omega_f) = \left(\frac{1}{f_{\text{avg}}}\right)^2 \mathfrak{Vol}(\Omega_f).$$

From the formula for the volume in Lemma 11.21 and (12.5), we see that $K_f > 0$. Using also the homogeneity of the action and the formula (11.16) for $P$, we can rewrite (12.4) as

$$A_{\text{min}}(\Psi^* \Omega_f) \leq \frac{\text{area}(M)}{\chi(M)} \sqrt{K_f} \leq A_{\text{max}}(\Psi^* \Omega_f).$$
Since $\Psi$ is isotopic to $\text{id}_{T^1M}$, we also see that
\[ c_\gamma := p_\infty(\Psi(\gamma)) \in \Lambda(M; h_\infty), \quad \forall \gamma \in \mathcal{X}(\Psi^*\Omega_f; p_\infty). \]

From Lemma 11.21 and Proposition 9.10, we conclude that
\[ A_{\Psi^*\Omega_f}(\gamma) = A_{\Omega_f}(c_\gamma, \dot{c}_\gamma) = \ell_f(c_\gamma) + \frac{\text{area}(M) \cdot f_{\text{avg}}}{\chi(M)}, \quad \forall \gamma \in \mathcal{X}(\Psi^*\Omega_f; p_\infty). \]

From the definition of $\ell_{\text{min}}(f)$ and $\ell_{\text{max}}(f)$, we have
\[ \ell_{\text{min}}(f) + \frac{\text{area}(M) \cdot f_{\text{avg}}}{\chi(M)} \leq A_{\text{min}}(\Psi^*\Omega_f), \quad A_{\text{max}}(\Psi^*\Omega_f) \leq \ell_{\text{max}}(f) + \frac{\text{area}(M) \cdot f_{\text{avg}}}{\chi(M)} \]
and equalities hold if $f$ is Zoll. The rest of the proof goes along the same line as in the proof of Theorem 1.26 for $M \neq T^2$ in Section 12.1 above.

**Proof of Theorem 1.28** for $M = T^2$

Corollary 1.20 and Remark 10.6 yield a $C^0$-neighbourhood $\Lambda(p_\infty)$ of the $p_\infty$-fibres and a $C^2$-neighbourhood $U$ of $\Omega_\ast$ in the space of odd-symplectic forms cohomologous to $\Omega_\infty$ such that
\[ A_{\text{min}}(\Omega) \leq 0 \leq A_{\text{max}}(\Omega), \quad \forall \Omega \in U, \tag{12.6} \]
and any of the equalities holds if and only if $\Omega$ is Zoll. We prove the theorem with $C_g := C_\Omega$, the constant in Lemma 11.20. Let $f : T^2 \to \mathbb{R}$ be a $C_g$-strong function. In particular we have $f_{\text{avg}} > 0$ and $K_f > 0$. Let $\Psi$ be the diffeomorphism isotopic to $\text{id}_{T^1T^2}$ such that $\Psi^*\Omega_f \in U$ constructed in Lemma 11.20. Since $\Psi$ is isotopic to the identity, we see that $c_\gamma := p_\infty(\Psi(\gamma)) \in \Lambda(T^2; h_\infty)$ for all $\gamma \in \mathcal{X}(\Psi^*\Omega_f; p_\infty)$. From Proposition 9.10 and Lemma 11.24 we get
\[ A_{\Psi^*\Omega_f}(\gamma) = A_{\Omega_f}(c_\gamma, \dot{c}_\gamma) = \frac{1}{f_{\text{avg}}} (\ell_f(c_\gamma) - \bar{\ell}(f)), \quad \forall \gamma \in \mathcal{X}(\Psi^*\Omega_f; p_\infty). \]

This together with (12.6) yields
\[ \ell_{\text{min}}(f) - \bar{\ell}(f) \leq f_{\text{avg}} \cdot A_{\text{min}}(\Psi^*\bar{\Omega}_f) \leq 0 \leq f_{\text{avg}} \cdot A_{\text{max}}(\Psi^*\bar{\Omega}_f) \leq \ell_{\text{max}}(f) - \bar{\ell}(f) \]
which in turn implies
\[ \ell_{\text{min}}(f) \leq \bar{\ell}(f) \leq \ell_{\text{max}}(f). \]

If $f$ is Zoll, the equalities hold. Conversely if $\ell_{\text{min}}(f)$ or $\ell_{\text{max}}(f)$ are equal to $\bar{\ell}(f)$, then there is an equality also in (12.6), which yields that $\Psi^*\Omega_f$, and hence $f$ is Zoll.
A Some $C^k$-estimates on compact manifolds

Below, we discuss some results in abstract differential geometry, which we need in the paper.

A.1 The norm of the pull-back of a differential form

Let $\mathcal{M}$ be a connected manifold. We choose a Riemannian metric $g_\mathcal{M}$ and a connection $\nabla^\mathcal{M}$ on $T\mathcal{M}$, which we use to define uniform $C^k$-norms $\| \cdot \|_{C^k}$, for every natural number $k \in \mathbb{N}$, on the sections of the tensor algebra of $T\mathcal{M}$. Since $\mathcal{M}$ is compact, a different choice of metric and connection yields uniform norms that are Lipschitz-equivalent to the ones given by $g_\mathcal{M}$ and $\nabla^\mathcal{M}$, and therefore, the validity of the estimates below is independent of such a choice, up to adjusting constants suitably. If $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ is a map between compact manifolds as above, then $d\Psi : T\mathcal{M}_1 \to T\mathcal{M}_2$ is a section of the bundle $T^*\mathcal{M}_1 \otimes \Psi^*(T\mathcal{M}_2) \to \mathcal{M}_1$. On this bundle we have a metric $g_\Psi$ and a connection $\nabla^\Psi$ induced by $\Psi$, with respect to which we define the norms $\| d\Psi \|_{C^k}$, for every $k \in \mathbb{N}$. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are compact subsets of some Euclidean space, then there exists a constant $C > 0$ such that

\[
\| d\Psi \|_{C^k} \leq C \| \Psi \|_{C^{k+1}},
\]

where on the right-hand side, we have the standard uniform norm for maps in the Euclidean space. We collect below a couple of properties of $\| d\Psi \|_{C^k}$ that will be important for us. First, for every pair of natural numbers $h, k$, we define the following subset of multi-indices

\[
I_{h,k} := \left\{ a = (a_0, \cdots, a_k) \in \mathbb{N}^{k+1} \mid 0 < \sum_{j=0}^{k} (j+1)a_j \leq h + k \right\}
\]

and consider the polynomial

\[
B_{h,k} : \mathbb{R}^{k+1} \to \mathbb{R}, \quad B_{h,k}(x) = \sum_{a \in I_{h,k}} x^a,
\]

where $x = (x_0, \cdots, x_k)$ and $x^a := x_0^{a_0} \cdots x_k^{a_k}$. If $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ is a map, we use the short-hand

\[
B_{h,k}(\| d\Psi \|) = B_{h,k}(\| d\Psi \|_{C^0}, \cdots, \| d\Psi \|_{C^k}).
\]

**Lemma A.1.** Let $h, k$ be a pair of natural numbers, let $\mathcal{M}_1, \mathcal{M}_2$ be a pair of compact manifolds, and let $\mathcal{E}$ denote a vector bundle over $\mathcal{M}_2$. We denote by $\Omega^h(\mathcal{M}_2, \mathcal{E})$ the space of sections of the bundle $(\Lambda^h \mathcal{M}_2) \otimes \mathcal{E}$. There is a constant $C_k > 0$ (depending on $\mathcal{M}_1$ and $\mathcal{M}_2$) such that for all smooth maps $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$, there holds

\[
\Psi^* \eta \in \Omega^h(\mathcal{M}_1, \Psi^* \mathcal{E}), \quad \| \Psi^* \eta \|_{C^k} \leq C_k B_{h,k}(\| d\Psi \|) \| \eta \|_{C^k}, \quad \forall \eta \in \Omega^h(\mathcal{M}_2, \mathcal{E}). \quad \square
\]

As an immediate consequence, we can estimate the derivative of a composition of maps.

**Lemma A.2.** Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be a triple of compact manifolds, and $k \in \mathbb{N}$ a natural number. If $\Psi_1 : \mathcal{M}_1 \to \mathcal{M}_2$ and $\Psi_2 : \mathcal{M}_2 \to \mathcal{M}_3$ are smooth maps, then there is $C_k > 0$ (depending on $\mathcal{M}_1$ and $\mathcal{M}_2$) such that

\[
d(\Psi_2 \circ \Psi_1) = \Psi_1^*(d\Psi_2), \quad \| d(\Psi_2 \circ \Psi_1) \|_{C^k} \leq C_k B_{1,k}(\| d\Psi_1 \|) \| d\Psi_2 \|_{C^k}. \quad \square
\]

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Inductively using Gronwall’s Lemma, one can also give estimates for $B_{h,k}(\|d\Phi_1\|)$, when $\Phi_1$ is the time-one map of an isotopy obtained integrating a time-dependent vector field. Here, we give only the bound for $(h,k) = (2,2)$, since it is the one we need in Lemma [11.20]

**Lemma A.3.** For every compact manifold $M$, there exists a constant $C_k > 0$ with the following property. For every time-dependent vector field $X = \{X_s\}_{s \in [0,1]}$ on $M$ such that the corresponding flow $\{\Phi_s\}$ is defined up to time 1, there holds

$$B_{2,2}(\|d\Phi_1\|) \leq \left( (\nabla X)^2 + (\nabla X)^2 \right) e^{C(\nabla X)_{C^0}},$$

where we have set $\langle \nabla X \rangle_{C^k} := 1 + \max_{s \in [0,1]} \|\nabla X_s\|_{C^k}$, $\forall k \in \mathbb{N}$.

**Proof.** We preliminarily observe that if $V$ is a finite-dimensional vector space endowed with a norm coming from a scalar product, then for every $s \mapsto v(s) \in V$, there holds

$$\frac{d|v|}{ds} \leq \frac{dv}{ds}.$$ 

By the compactness of $M$, we just need to prove the lemma in local coordinates. Recalling that $\partial_s \Phi_s = X_s \circ \Phi_s$ by definition, we compute

$$\partial_s |d\Phi_s| \leq |d(X_s \circ \Phi_s)| \leq |\nabla X_s| \cdot |d\Phi_s|,$$

$$\partial_s |\nabla d\Phi_s| \leq |\nabla((\nabla d\Phi_s) \cdot X_s)_\Phi| \leq |\nabla^2 X_s| \cdot |d\Phi_s|^2 + |\nabla X_s| \cdot |\nabla d\Phi_s|,$$

$$\partial_s |\nabla^2 d\Phi_s| \leq |\nabla((\nabla d\Phi_s \cdot \nabla d\Phi_s) X_s + (\nabla \nabla d\Phi_s) X_s)_\Phi|$$

$$\leq |\nabla^3 X_s| \cdot |d\Phi_s|^3 + 3|\nabla^2 X_s| \cdot |\nabla d\Phi_s| \cdot |d\Phi_s| + |\nabla X_s| \cdot |\nabla^2 d\Phi_s|.$$

We now apply Gronwall’s Lemma [Gro19] and indicate with $C > 0$ a constant depending on $M$ but not on $X$. Below, we can always bring the constant to the exponent because, by definition, $\langle \nabla X \rangle_{C^0} \geq 1$. Thus, we find that

$$\| \max_s d\Phi_s \| \leq e^{C(\nabla X)_{C^0}},$$

$$\| \max_s \nabla d\Phi_s \| \leq \langle \nabla X \rangle_{C^1} \| \max_s d\Phi_s \|_{C^0}^2 e^{C(\nabla X)_{C^0}} \leq \langle \nabla X \rangle_{C^1} e^{C(\nabla X)_{C^0}},$$

$$\| \max_s \nabla^2 d\Phi_s \| \leq \left( \langle \nabla X \rangle_{C^2} \| \max_s d\Phi_s \|_{C^0}^3 + \langle \nabla X \rangle_{C^1} \| \max_s \nabla d\Phi_s \|_{C^0} \| \max_s d\Phi_s \|_{C^0} \right) e^{C(\nabla X)_{C^0}}$$

$$\leq \left( \langle \nabla X \rangle_{C^2} + \langle \nabla X \rangle_{C^1}^2 \right) e^{C(\nabla X)_{C^0}}.$$ 

Finally, from the definition of $B_{2,2}$, we get

$$B_{2,2}(\|d\Phi_1\|) \leq \left[ \sum_{a_1 + 2a_2 + 3a_3 \leq 4} \langle \nabla X \rangle_{C^0}^2 \left( \langle \nabla X \rangle_{C^2} + \langle \nabla X \rangle_{C^1}^2 \right)^{a_3} \right] e^{C(\nabla X)_{C^0}}$$

$$\leq \left( \langle \nabla X \rangle_{C^2} + \langle \nabla X \rangle_{C^1}^2 \right) e^{C(\nabla X)_{C^0}}. \quad \square$$

Finally, we have a lemma dealing with the dependence of the pull-back operation from the map $\Psi$. Here we consider only maps of the solid torus.

**Lemma A.4.** Let $B \subset \mathbb{R}^n$ be a closed ball. For every $k \in \mathbb{N}$ and $\delta_0 > 0$, there exists $\delta_1 = \delta_1(k,\delta_0) > 0$ such that, if $\Psi_1, \Psi_2 : B \times S^1 \to B \times S^1$ are smooth maps and $h \in \mathbb{N}$, then

$$\|\Psi_2 - \Psi_1\|_{C^{k+1}} < \delta_1 \quad \Rightarrow \quad \|\Psi_2^* \eta - \Psi_1^* \eta\|_{C^k} \leq \delta_0 \|\eta\|_{C^{k+1}}, \quad \forall \eta \in \Omega^h(B \times S^1). \quad \square$$
A.2 Primitives of differential forms via Hodge theory

In the next lemma, we see how the ellipticity of the Hodge-de Rham operator implies that an exact form $\Xi$ on a closed manifold $M$ admits a primitive $\xi$, whose norm is bounded in terms of that of $\Xi$. The proof is based on [Nic07, Chapter 10].

**Lemma A.5.** Let $M$ be a closed manifold and $k \in \mathbb{N}$. There exists a positive constant $C_k > 0$ (depending on $M$) with the property that if $\Xi \in \Omega^{h+1}(M)$ is an exact form for some $h \in \mathbb{N}$, there exists $\xi \in \Omega^h(M)$ such that

$$d\xi = \Xi, \quad \|\xi\|_{C^k} \leq C_k \|\Xi\|_{C^k}.$$

**Proof.** Let $m = \dim M$. By passing to a double cover, we assume that $M$ is orientable. We define the graded vector space (here and below we drop $M$ from the notation)

$$\Omega^L_2 := \bigoplus_{j=0}^{m} \Omega^j_L^2,$$

which is the completion of the space of smooth differential forms on $M$ with respect to the $L^2$-norm given by $g$. We consider the exterior differential $d : \Omega^L_2 \to \Omega^L_2$ and its formal adjoint operator $d^* : \Omega^L_2 \to \Omega^L_2$ as linear unbounded operators, whose dense domain is the Sobolev space $\Omega^{W_{1,2}}$. Let $H := d + d^* : \Omega^L_2 \to \Omega^L_2$ be the Hodge-de Rham operator. Its square equals the Laplace-Beltrami operator, $H^2 = dd^* + d^*d = \Delta$. Since $H$ is elliptic, Theorem 10.3.11 in [Nic07] implies that there exists a constant $E > 0$ such that

$$E \eta \in \Omega^W_{1,2}, H \eta \in \Omega^W_{k,p},$$

This implies that $\ker H$ is finite dimensional, the spaces $\text{Im} \, d$ and $\text{Im} \, d^*$ are $L^2$-closed, and we have the decompositions

$$\Omega^L_2 = \ker H \oplus \text{Im} \, H, \quad \text{Im} \, H = \ker d \oplus \text{Im} \, d^*.$$

In particular, since $\Xi \in \Omega^{h+1}_L \cap \ker \text{Im} \, d$ by assumption, there exists $\xi \in (\ker H)^\perp \cap \Omega^{W_{1,2}}$ such that $H \xi = \Xi$. Since $\Xi$ is already in the image of $d$, this implies that

$$d\xi = \Xi, \quad d^*\xi = 0, \quad \xi \in \Omega^h_W.$$

We claim that $\xi$ is the primitive of $\Xi$ we are looking for. To see this choose some $p > 2m$ and let $k$ be the natural number given in the statement. We take $\Omega^W_{k,p}$ to be the completion of the space of smooth differential forms with respect to the Sobolev norm $\| \cdot \|_{W^{k,p}}$. A further application of [Nic07, Theorem 10.3.11] yields a constant $E_{k,p} > 0$ with the following property: if $\eta \in \Omega^{W_{1,2}}$ and $H \eta \in \Omega^{W_{k,p}}$, then $\eta \in \Omega^{W_{k+1,p}}$ and

$$\|\eta\|_{W^{k+1,p}} \leq E_{k,p}(\|H \eta\|_{W^{k,p}} + \|\eta\|_L^p).$$

Translating Proposition 10.4.4, Lemma 10.4.8 and Lemma 10.4.9 in [Nic07] from exponent 2 to exponent $p$, we see that there exists a constant $E' > 0$ such that

$$\|\eta\|_L^p \leq E' \|H \eta\|_{L^p}, \quad \forall \eta \in (\ker H)^\perp \cap \Omega^{W_{1,p}}.$$
Moreover, by Morrey’s inequality \cite[Theorem 10.2.36.(d)]{Nic07}, there exists a positive constant \( E''_k > 0 \) such that if \( \eta \in \Omega_{W^{k+1,p}} \), then \( \eta \in \Omega_{C^k} \) and
\[
\| \eta \|_{C^k} \leq E''_k \| \eta \|_{W^{k+1,p}}.
\]
Finally, we readily see that there exists a constant \( E''_k > 0 \) such that
\[
\| \theta \|_{W^{k,p}} \leq E''_k \| \theta \|_{C^k}, \quad \forall \theta \in \Omega_{C^k}.
\]

Chaining the last four inequalities together, we conclude that there exists a positive constant \( C_k > 0 \) such that, if \( \eta \in (\ker H)^\perp \cap \Omega_{W^{1,2}} \) and \( H \eta \in \Omega_{W^{k,p}} \), then \( \eta \in \Omega_{C^k} \) and
\[
\| \eta \|_{C^k} \leq C_k \| H \eta \|_{C^k}.
\]

Taking \( \eta \) to be equal to the form \( \xi \in (\ker H)^\perp \cap \Omega^h_{W^{1,2}} \) with \( d\xi = H\xi = \Xi \), which we found before, we deduce that \( \xi \in \Omega^h_{C^k} \) and that the desired bound \( \| \xi \|_{C^k} \leq C_k \| \Xi \|_{C^k} \) holds. \( \square \)

### A.3 Estimating the size of the Reeb vector field

The result below transfers estimates on a contact form and its exterior differential to estimates on the Reeb vector field. If \( k \in \mathbb{N} \), we recall the definition of the \( C^{k,+} \)-norm of a one-form \( \alpha \):
\[
\| \alpha \|_{C^{k,+}} : = \| \alpha \|_{C^k} + \| d\alpha \|_{C^k}.
\]

**Lemma A.6.** Let \( k \in \mathbb{N} \), and let \( M \) be a compact manifold of dimension \( 2n+1 \) with contact form \( \alpha_0 \). There exists \( \delta \in (0,1) \) and a constant \( A_k > 0 \) such that, if \( \alpha \) is a one-form on \( M \) with \( \| \alpha - \alpha_0 \|_{C^{0,+}} < \delta \), then \( \alpha \) is a contact form and there holds
\[
\| R\alpha - R\alpha_0 \|_{C^k} \leq A_k \| \alpha - \alpha_0 \|_{C^{k,+}}.
\]

**Proof.** We assume that \( \| \alpha - \alpha_0 \|_{C^{0,+}} < \delta \) with \( \delta \in (0,1) \) to be determined during the proof. We consider the bundle maps \( I : \Lambda^2T^*M \to \Lambda^2\Lambda^2T^*M \) given by \( \Omega \mapsto \Omega^n \) and \( J : \Lambda^{2n}T^*M \to TM \) obtained as the inverse of the map \( X \mapsto LX(\alpha_0 \wedge (d\alpha_0)^n) \). The composition \( K := J \circ I \) induces a linear bijection between the sections of \( \Lambda^2T^*M \) and of \( TM \) which is locally Lipschitz in the \( C^k \)-norm. Namely, there exist \( B'_k > 0 \) such that if \( \Omega \) is a two-form on \( M \) with \( \| \Omega \|_{C^k} < 1 \), then \( \| K(\Omega) \|_{C^k} \leq B'_{k} \| \Omega \|_{C^k} \). In particular, we have
\[
\| K(\alpha) - R\alpha_0 \|_{C^k} = \| K(\alpha - d\alpha_0) \|_{C^k} \leq B'_{k} \| \alpha - d\alpha_0 \|_{C^k}.
\]

If \( R \) is the trivial \( \mathbb{R} \)-bundle over \( M \), the contraction bundle map \( T^*M \times TM \to \mathbb{R} \) is locally Lipschitz in the \( C^k \)-norm. At \( x \in M \), the contraction is defined as \( (\beta_x, Y_x) \mapsto \beta_x(Y_x) \). This yields a constant \( C'_k > 0 \) that can be used together with \((A.1)\) to estimate
\[
\| \alpha(K(\alpha)) - 1 \|_{C^k} = \| \alpha(K(\alpha)) - \alpha_0(R\alpha_0) \|_{C^k} \\
\leq C'_k \left( \| \alpha - \alpha_0 \|_{C^k} + \| K(\alpha) - R\alpha_0 \|_{C^k} \right) \\
\leq C'_k \left( \| \alpha - \alpha_0 \|_{C^k} + B'_k \| \alpha - d\alpha_0 \|_{C^k} \right) \\
\leq C'_k (B'_k + 1) \| \alpha - \alpha_0 \|_{C^{k,+}}.
\]

Let \( \epsilon \in (0,1) \) be an auxiliary number. If \( \delta \cdot C'_k (B'_k + 1) < 1 - \epsilon \), then \( \alpha \) is a contact form and we can write
\[
R\alpha = \frac{K(\alpha)}{\alpha(K(\alpha))}.
\]
If $(\epsilon, \infty)\mathcal{M}$ is the trivial $(\epsilon, \infty)$-bundle over $\mathcal{M}$, then the bundle map $(\epsilon, \infty)\mathcal{M} \times \mathcal{T}\mathcal{M} \to \mathcal{T}\mathcal{M}$ given by $(a_x, Y_x) \mapsto \frac{1}{a_x}Y_x$ is locally Lipschitz in the $C^k$-norm and the pair $(\alpha(K(d\alpha)), K(d\alpha))$ yields a section of $(\epsilon, \infty)\mathcal{M} \times \mathcal{T}\mathcal{M}$. Therefore, there exists a constant $D'_{k, \epsilon} > 0$ such that

$$\|R_{\alpha} - R_{\alpha_0}\|_{C^k} \leq D'_{k, \epsilon}(\|\alpha(K(d\alpha)) - 1\|_{C^k} + \|K(d\alpha) - R_{\alpha_0}\|_{C^k}).$$

The bound in the statement follows now by combining (A.1) with (A.2). \qed
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