Decompositions of tame profinite fundamental groups of non-archimedean curves using metrized complexes

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Abstract

In this paper, we study a natural covering functor from the category of tame étale coverings of a punctured curve over a complete algebraically closed non-archimedean field to the category of finite tame coverings of a metrized complex associated to the punctured curve. We enhance the latter category by adding a set of gluing data to every covering and we show that this yields an equivalence of categories. This enhanced category of so-called rigidified tame coverings then inherits the structure of a Galois category, yielding a natural notion of a profinite fundamental group for metrized complexes. Using this graph-theoretical interpretation, we define the (absolute) decomposition and inertia groups of the metrized complex in the fundamental group of a nonpunctured curve and we show that they define normal subgroups. We then show that the quotient of the fundamental group by the decomposition group is isomorphic to the profinite completion of the ordinary fundamental group of the underlying graph of the metrized complex. Furthermore, we prove that the extensions that arise from the abelianization of the decomposition and inertia quotients coincide with the extensions that arise from the toric and connected parts of the analytic Jacobian of the curve.

1 Introduction

Let $K$ be a complete, algebraically closed non-archimedean field with a nontrivial valuation. In this paper, we will study transitions from tame algebraic coverings to tame coverings of metrized complexes. To that end, we fix a punctured algebraic curve $(X,D)$ with $D \subset X(K)$ a finite subset and we consider tame étale coverings $X' \to X$. For these coverings $X' \to X$, we have that the inverse image of any semistable vertex set $V$ of $(X,D)$ gives a semistable vertex set $V'$ of $(X',D')$, where $D'$ is the inverse image of $D$. This seems to be well-known to experts (see [CTT16, Section 1.2.1]), but we include a proof in Section 3 since we couldn’t find an exact reference.

Using this fact, we can construct a natural functor for any fixed semistable vertex set $V$ of $(X,D)$ with skeleton $\Sigma(X,V \cup D)$ from the category of finite tame étale coverings of $(X,D)$ to the category of finite coverings of the metrized complex corresponding to $\Sigma(X,V \cup D)$. This already requires a small generalization with respect to the material in [ABBR15] in the form of disconnected coverings of metrized complexes. There is then a natural covering functor from $\text{Cov}(X,D)$ to $\text{Cov}(\Sigma(X))$ with respect to $V$ and we denote it by

$$F_\Sigma : \text{Cov}(X,D) \to \text{Cov}(\Sigma(X)).$$

From [ABBR15, Theorem 7.4], we see that this functor is essentially full. That is, every finite tame covering of $\Sigma$ comes from an algebraic covering $X' \to X$. The same theorem then classifies the possible liftings of these graph-theoretical coverings in terms of gluing data. It follows from this description that there can be multiple algebraic coverings reducing to the same covering of a metrized complex, so this covering functor does not induce an equivalence. An example of this phenomenon can be given using the unramified coverings that come from torsion subgroups of an elliptic curve, see [ABBR15, Example 7.8] or Examples 4.1 and 5.2 in this paper. To solve this noninjectivity, we consider the enhanced category of rigidified coverings of the metrized complex $\Sigma$. The objects in this category consist of a covering $\Sigma' \to \Sigma$ together with a set of gluing data $\mathcal{G} \in \mathcal{G}(\Sigma',X)$ and morphisms between these objects consist of commutative diagrams of coverings together with compatible diagrams of gluing data, see Definition 4.2. We then have the following
Theorem 4.1. Let $V$ be any semistable vertex set of $(X, D)$ and let $F_\Sigma$ be the corresponding covering functor from the category of finite tame étale coverings of $(X, D)$ to the category of finite tame rigidified coverings of $\Sigma := \Sigma(X, V \cup D)$. Then $F_\Sigma$ induces an equivalence of categories

$$\text{Cov}(X, D) \simeq \text{Cov}_G(\Sigma).$$

The proof of Theorem 4.1 is mostly based on Lemma 4.1. The additional gluing data in the covering category allows us to lift these morphisms of coverings of metrized complexes to morphisms of coverings of analytic spaces. Using a suitable version of GAGA (see [Ber12, Corollary 3.4.13]), we can then uniquely (up to $X$-isomorphism) lift these morphisms to algebraic morphisms. We then easily conclude that $F_\Sigma$ induces an equivalence of categories.

Since the category $\text{Cov}(X, D)$ of finite étale coverings of $(X, D)$ has the structure of a Galois category, we obtain an induced Galois category structure on $\text{Cov}_G(\Sigma(X))$. In other words, we have a profinite group $\pi(\Sigma(X, D))$ classifying the finite coverings of $\Sigma(X)$ and this group is isomorphic to the profinite fundamental group $\pi(X, D)$ of $(X, D)$. We call this group $\pi(\Sigma(X))$ the profinite fundamental group of $\Sigma(X, V \cup D)$.

We then define the absolute decomposition group $D(\Sigma^0)$ and the absolute inertia group $I(\Sigma^0)$ of a subcomplex $\Sigma^0$ of a metrized complex $\Sigma(X, V \cup D)$ in $\pi(X)$. We prove in Proposition 2 that these subgroups are in fact normal in $\pi(X)$. Their quotients $\pi_D(\Sigma^0(X)) := \pi(X)/D(\Sigma^0)$ and $\pi_I(\Sigma^0(X)) := \pi(X)/I(\Sigma^0)$ classify the finite connected coverings that are totally split (resp. étale) above $\Sigma^0$. We will be particularly interested in the case $\Sigma^0 = \Sigma$. The group $\pi_D(\Sigma(X))$ then classifies finite connected coverings of $\Sigma$ that split completely above $\Sigma$. Using this interpretation, the following theorem should come as no surprise:

Theorem 5.1. Let $D(\Sigma)$ be the decomposition group of $\Sigma$ in $\pi(X)$. Then $\pi_D(\Sigma(X)) := \pi(X)/D(\Sigma)$ is isomorphic to the profinite completion of the ordinary fundamental group of the underlying graph $\Gamma$ of a metrized complex $\Sigma$ corresponding to $X$.

We then turn to the abelianizations of these groups $\pi_D(\Sigma(X))$ and $\pi_I(\Sigma(X))$. The unramified abelian coverings of an algebraic curve $X$ are classified by the torsion points of the Jacobian $J$ of $X$ and the decomposition and inertia groups $\pi_D(\Sigma(X))$ and $\pi_I(\Sigma(D))$ naturally subdivide these abelian coverings. These subdivisions are then related to the natural decomposition of the Néron model $\mathcal{J}$ of the Jacobian over a discretely valued field

$$\mathcal{J}^0 \subseteq \mathcal{J} \subseteq \mathcal{J},$$

as the following theorem shows:

Theorem 5.2. Let $\pi_D(\Sigma(X))$ and $\pi_I(\Sigma(X))$ be the decomposition and inertia quotients of $\pi(X)$ respectively and let $T^{\text{an}}$ and $J^0$ be the toric and connected parts of $J^{\text{an}}$. Let $n$ be an integer such that $\gcd(n, \text{char}(\overline{k})) = 1$. Then

$$T^{\text{an}}[n] = \text{Hom}(\pi_D(\Sigma(X)), \mathbb{Z}/n\mathbb{Z})$$

and

$$J^0[n] = \text{Hom}(\pi_I(\Sigma(X)), \mathbb{Z}/n\mathbb{Z}).$$

The proof uses the analytic "Slope formula", see [BPR14, Theorem 5.15]. The main ideas behind this proof were already present in [Hell18, Propositions 8.2.1 and 8.3.1], where it was given in the discrete case.

The paper is organized as follows. We start in Section 2 by reviewing some notions from the theory of Berkovich spaces and metrized complexes (as introduced in [ABBR15]). As noted before, we expand upon the notions there slightly, in the sense that we allow disconnected coverings. We then review the theory of Galois categories in Section 2.6.

After this, we prove a simultaneous semistable reduction theorem for tame coverings in Section 3. We then study the notion of a rigidified morphism of metrized complexes in Section 4.1. This gives rise to a category and we prove Theorem 4.1. In Section 5, we reap the benefits of this equivalence, defining for instance the Galois closure of a finite rigidified covering of metrized complexes, see Definition 5.2. We then consider several subclasses of coverings in the category of finite rigidified coverings of $\Sigma$, giving rise to normal subgroups in $\pi(\Sigma(X))$. More explicitly, we study the set of coverings that are unramified above a subcomplex $\Sigma^0 \subseteq \Sigma$ and the coverings that are totally split above a subcomplex $\Sigma^0$. Taking $D = \emptyset$ and $\Sigma^0 = \Sigma$, we then reach our definition of the inertia and decomposition groups of $\pi(X)$. We then prove Theorem 5.1 and study the abelianization of $\pi(X)$ in Section 5.2.

2 Preliminaries

In this section, we will give a summary of the notions that we will be needing in this paper. We will give a short review of the building blocks of $K$-analytic spaces, the theory of tame coverings of metrized complexes.
as outlined in [ABBR15] and the theory of Galois categories. The section on metrized complexes will be slightly more general than in [ABBR15], since we will be needing the notion of disconnected coverings of a metrized complex.

We will use the following notation throughout this paper:

1. $K$ is a complete, algebraically closed non-archimedean field with nontrivial valuation $\text{val} : K \to \mathbb{R} \cup \{\infty\}$ and value group $\Lambda := \text{val}(K^*)$,
2. $R$ its valuation ring,
3. $\mathfrak{m}_R$ its maximal ideal,
4. $k = R/\mathfrak{m}_R$ its residue field and
5. $\varpi$ is any element in $K$ with $\text{val}(\varpi) > 0$.

We will assume that $K$ has characteristic zero. A curve $X$ over $K$ will be a smooth connected separated scheme of finite type over $K$ of dimension 1. We will sometimes refer to such a curve as an algebraic curve. We will denote its function field over $K$ by $K(X)$. Since $K$ has characteristic zero, this is a perfect field. We will denote the algebraic closure of $K(X)$ by $\overline{K(X)}$ and its profinite Galois group by $G_{K(X)}$ or $G_X$. Let $H$ be a commutative group scheme over $K$ and consider the corresponding group of $K$-points $H(K)$. Let $\ell$ be a prime number. We will assume that $\ell \neq \text{char}(k)$. We denote the $\ell$-adic Tate module of $G$ by

$$T_\ell(H) := \lim_{\longleftarrow} H(K)[\ell^n].$$

(6)

For the analytification $H^\text{an}$ of a commutative group scheme $H$ of finite type over $K$, we set $T_\ell(H^\text{an}) := T_\ell(H)$. Here, the analytification functor is the functor from the category of $K$-schemes to the category of analytic spaces as in [Ber12].

### 2.1 Analytic preliminaries

We will now quickly review the various analytic notions we will be needing in this paper. For any $K$-algebra $A$, we denote its Berkovich spectrum by $\mathcal{M}(A)$. That is, it is the set of all bounded multiplicative seminorms on $A$ and we give it the weakest topology such that all real-valued functions of the form $| | \mapsto |f|$ are continuous for all $f \in A$. An analytic curve will be a strictly $K$-analytic space $Y$ of dimension 1. For every algebraic curve $X$, there is a canonical analytic curve $X^\text{an}$ associated to it, which is also referred to as the analytification of $X$. For any finite map $\phi : X' \to X$ of algebraic curves over $K$, we then also obtain from the Berkovich analytification functor a finite map of analytic spaces $\phi^\text{an} : X'^\text{an} \to X^\text{an}$. We will say that $\phi$ is tame if $\phi^\text{an}$ is topologically tame in the sense of [CTT16, Section 3.2.3] and [Ber93, Section 6.3]. That is, $\phi^\text{an}$ is topologically tame if the local degrees $[\mathcal{H}(x') : \mathcal{H}(x)]$ are invertible in $k$ for all points $x' \in X'^\text{an}$ mapping to $x \in X^\text{an}$. Here $\mathcal{H}(x)$ is the completed residue field of a point in $X^\text{an}$.

Let $\mathbf{A}^n_1 := \mathcal{M}(K[T])$ be the analytic affine line. We define the tropicalization map $\text{trop} : \mathbf{A}^n_1 \to \mathbb{R} \cup \{\infty\}$ by

$$\text{trop}(|| \cdot ||) = -\log(||T||).$$

(7)

We will use this map to define the various analytic domains that we will be needing. We will be following [BPR14, Section 2.1].

- Let $G^\text{an}_m = \mathcal{M}(K[T, 1/T])$. Then $G^\text{an}_m = \text{trop}^{-1}(\mathbb{R})$.
- Let $a \in K^*$. The standard closed ball of radius $|a|$ is $B(a) = \text{trop}^{-1}([\text{val}(a), \infty])$. Its ring of analytic functions is then denoted by $K(a^{-1}t)$, with canonical reduction $k[\tau]$, where $\tau$ is the residue of $a^{-1}t$.
- Let $a \in K^*$. We define the standard open ball of radius $|a|$ to be $B(a)_+ = \text{trop}^{-1}((\text{val}(a), \infty))$. It can be written as an increasing union of standard closed balls.
- Let $a, b \in K^*$ with $|a| \leq |b|$. The standard closed annulus of inner radius $|a|$ and outer radius $|b|$ is $S(a, b) = \text{trop}^{-1}([\text{val}(a), \text{val}(b)])$. Its ring of analytic functions is denoted by $K(at^{-1}, b^{-1}t)$. Its canonical reduction is then given by $k[\sigma, \tau]/(\sigma \tau - a/b)$, where $\sigma$ (resp. $\tau$) is the residue of $at^{-1}$ (resp. $b^{-1}t$) and $a/b \in k$ is the residue of $a/b$. We define the length or modulus of $S(a, b)$ to be $\text{val}(a) - \text{val}(b)$.
- In the situation above, if $|a| \leq 1$ and $|b| = 1$, we will denote the annulus by $S(a) := S(a, 1)$.
- For $a, b \in K^*$ with $|a| < |b|$, we define the standard open annulus of inner radius $|a|$ and outer radius $|b|$ to be $S(a, b)_+ = \text{trop}^{-1}((\text{val}(b), \text{val}(a)))$. This open analytic domain can again be expressed as an increasing union of standard closed annuli, see Equation (26) for an explicit description.
• For $a \in K^*$, the standard punctured open ball of radius $|a|$ is $S(0, a)_+ = \text{trop}^{-1}((\text{val}(a), \infty))$. If $a = 1$, we will write $S(0)_+ := S(0, 1)_+$.

We now define the tropicalization map on a torus. Let $A$ be a lattice of rank $n$, i.e. $A \cong \mathbb{Z}^n$, let $B := \text{Hom}(A, \mathbb{Z})$ be the dual lattice and let $K[A]$ be the group ring corresponding to $A$. This is also called the character group of $A$. For $u \in A$, we write $\chi^u$ for the corresponding character in $K[A]$. We will write $A_\mathbb{R} := A \otimes \mathbb{R}$ and $B_\mathbb{R} := B \otimes \mathbb{R}$ for the corresponding real lattices. The torus with character lattice $A$ is then defined to be $T^a := \mathcal{M}(K[A])$, where $\mathcal{M}(\cdot)$ denotes the Berkovich spectrum as defined in [1]. We define the tropicalization map $\text{trop} : T^an \to B_\mathbb{R}$ by

$$\text{trop}(|| \cdot ||, u) = -\log(||\chi^u||).$$

We then define the affinoid torus to be $T_0 := \text{trop}^{-1}(0)$. We will see more of these tropicalizations of tori in Section 5.2.

2.2 Metrized complexes and harmonic morphisms

In this section, we slightly expand upon several notions from [ABBR15] to include disconnected metrized complexes and coverings thereof. The main reason we want to allow disconnected coverings is that we want our category to have coproducts, as in the algebraic case. The algebraic example we have in mind here is the étale covering $\text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{R})$, which splits into two disjoint schemes after taking the fiber product over the same morphism.

We now generalize the notions of metric graphs, metrized complexes and harmonic morphisms to the disconnected case.

Definition 2.1. (Disconnected metric graphs) A metric graph $\Gamma$ is a finite topological graph such that its connected components are metric graphs in the sense of [ABBR15, Definition 2.2]. We similarly define $\Lambda$-metric graphs and $\Lambda$-points. Here $\Lambda$ is the value group of the valuation on $K$.

A vertex set $V(\Gamma)$ is a finite subset of the $\Lambda$-points of $\Gamma$ containing all essential vertices, as defined in [ABBR15]. The edges of $\Gamma$ are the edges of the connected components $\Gamma_i$ of $\Gamma$. We will denote the induced vertex sets of the connected components $\Gamma_i$ by $V(\Gamma_i)$. The induced edge sets are denoted by $E(\Gamma_i)$.

We now define morphisms of $\Lambda$-metric graphs. Let $\phi : \Gamma^1 \to \Gamma$ be a surjective continuous map of $\Lambda$-metric graphs. Since the image of a connected subset is again connected, we have induced surjective maps

$$\phi_{i,j} : \Gamma^1_{i,j} \to \Gamma_i,$$

where the $\Gamma_i$ are the connected components of $\Gamma$ and the $\Gamma^1_{i,j}$ are the connected components of $\Gamma^1$ mapping to $\Gamma_i$. For the remainder of the paper, we will label the maps on the connected components as above.

Definition 2.2. (Harmonic morphisms of disconnected metric graphs) Fix vertex sets $V(\Gamma^1)$ and $V(\Gamma)$ for two (possibly disconnected) $\Lambda$-metric graphs $\Gamma^1$ and $\Gamma$ respectively and let $\phi : \Gamma^1 \to \Gamma$ be a surjective continuous map.

The map $\phi$ is called a $(V(\Gamma^1), V(\Gamma))$-morphism of (disconnected) $\Lambda$-metric graphs if the induced $\phi_{i,j}$ (as in Equation (9)) are $(V(\Gamma^1_{i,j}), V(\Gamma_i))$-morphisms of $\Lambda$-metric graphs as in [ABBR15, Definition 2.4] with dilation factors $d_{e'}(\phi) \in \mathbb{Z}_{\geq 0}$. The map $\phi$ is called a morphism of $\Lambda$-metric graphs if there exists a vertex set $V(\Gamma^1')$ of $\Gamma^1$ and a vertex set $V(\Gamma)$ of $\Gamma$ such that $\phi$ is a $(V(\Gamma^1'), V(\Gamma))$-morphism of $\Lambda$-metric graphs. We say that $\phi$ is finite if $d_{e'}(\phi) \neq 0$ for any edge $e' \in E(\Gamma^1)$.

Let $\phi : \Gamma^1 \to \Gamma$ be a morphism of $\Lambda$-metric graphs. We say that $\phi$ is harmonic at $p' \in \Gamma^1_{i,j} \subset \Gamma^1$ if the induced morphism $\phi_{i,j}$ from Equation (9) is harmonic at $p'$. The morphism $\phi$ is harmonic if it is harmonic at every $p' \in \Gamma^1$. The degree of $\phi$ is defined to be the sum of the degrees of the $\phi_{i,j}$.

Definition 2.3. (Augmented metrized complexes) An augmented $\Lambda$-metric graph is a $\Lambda$-metric graph $\Gamma$ as in Definition 2.1 with a function $g : \Gamma \to \mathbb{Z}_{\geq 0}$ such that $g(p) = 0$ for all points of $\Gamma$ except possibly for finitely many $\Lambda$-points $p \in \Gamma$. We now define a vertex set of $\Gamma$ to be a vertex set $V(\Gamma)$ as in Definition 2.1 of the underlying metric graph which contains all essential vertices of $\Gamma$. A harmonic morphism of augmented $\Lambda$-metric graphs $\phi : \Gamma^1 \to \Gamma$ is then defined as a map which is a harmonic morphism between the underlying metric graphs of $\Gamma^1$ and $\Gamma$, see Definition 2.2.

Definition 2.4. (Disconnected metrized complexes) A (disconnected) $\Lambda$-metrized complex of $k$-curves is an augmented $\Lambda$-metric graph $\Gamma$ (as in Definition 2.3) such that every connected component $\Gamma_i$ is a metrized complex as in [ABBR15, Definition 2.17].
**Definition 2.5.** (Harmonic morphisms of metrized complexes) A harmonic morphism of metrized complexes of curves as introduced in Definition 2.4 consists of the following data:

- A harmonic \((V(\Gamma'), V(\Gamma))\)-morphism \(\phi: \Gamma' \rightarrow \Gamma\) of augmented metric graphs as in Definition 2.3,
- For every finite vertex \(p'\) of \(\Gamma'\) with \(d_{p'} > 0\) a finite morphism of curves \(\phi_{p'}: C_{p'}' \rightarrow C_{\phi(p')}\) such that the \((V(\Gamma'_i, V(\Gamma_i)))\)-morphisms \(\phi_{i,j}\) of Equation (9) induce morphisms of metrized complexes of \(k\)-curves as introduced in [ABBR15, Definition 2.19].

We say that a morphism \(\phi: \Sigma' \rightarrow \Sigma\) of (disconnected) metrized complexes of curves is finite, tame or generically étale if the morphisms on the connected components in Equation (9) have the corresponding property as introduced in [ABBR15, Section 2.16]. We similarly define tame harmonic morphisms and tame coverings.

**Example 2.1.** We give an example of two tame coverings of metrized complexes such that the fiber product doesn’t exist in the original category of connected coverings of a single metrized complex. The fiber product turns out to be the disjoint union of two metrized complexes. The example was set-up to be analogous to the fiber product of two copies of the morphism \(\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{R})\).

Suppose that \(\text{char}(k) \neq 2\). Let \(\phi_i: C_{x_i}' \rightarrow C_{x_i}\) for \(i = 1, 2\) denote the covering of projective lines given locally by \(t \mapsto t^2\). We consider the finite tame covering of metrized complexes indicated in Figure 1. That is, the curves \(C_{x_i}\) and \(C_{x_i}'\) correspond to the finite vertices and the leaves correspond to the branch/ramification points of the \(\phi_i\). We also choose some identification of the inner edges with points on the curves \(C_{x_i}\) and \(C_{x_i}'\). The corresponding tame covering of metrized complexes will be denoted by \(\Sigma' \rightarrow \Sigma\). Note that there is an action of \(\mathbb{Z}/2\mathbb{Z}\) on \(\Sigma'\) such that \(\Sigma'/\mathbb{Z}/2\mathbb{Z} = \Sigma\). We will denote the nontrivial automorphism by \(\sigma\).

The fiber product of \(\Sigma' \rightarrow \Sigma\) over itself is then in fact a disjoint union of two copies of \(\Sigma' \rightarrow \Sigma\), albeit with different projection maps. These projection maps are given by \((\text{id}, \sigma)\) and \((\sigma, \text{id})\).

![Figure 1: The fiber product in the category of coverings of metrized complexes in Example 2.1.](image)

In Example 3.2, we will see that these coverings naturally arise from the degree two covering on an elliptic curve with multiplicative reduction.

**Remark 2.1.** Throughout this paper, we will assume that our metrized complexes \(\Sigma\) are loopless. That is, \(\Sigma\) contains no edges whose closure is homeomorphic to a circle.

### 2.3 Semistable vertex sets and triangulated punctured curves

We now review the notion of semistable vertex sets and triangulated punctured curves, where we again expand upon the definition of [ABBR15] by allowing disconnected algebraic curves.

Let \(X\) be a smooth, proper algebraic \(K\)-curve and let \(X_i\) be its irreducible (and connected) components. For every \(X_i\), let \(D_i \subseteq X_i(K)\) be a finite set of closed points. We naturally have

\[X^{an} = \bigcap X_i^{an} = \coprod (X_i)^{an},\]

so we can analytify every irreducible component separately and consider all the notions in [ABBR15] per connected component. We will write \(D := \coprod D_i\).
Definition 2.6. A semistable vertex set of \((X,D)\) is a collection of semistable vertex sets \(V_i\) for all the marked irreducible components \((X_i,D_i)\) of \(X\). That is, it is a finite set of type-2 points for every \(X^\an_i\) such that \(X^\an = \bigcup V_i\) is a disjoint union of open balls and finitely many open annuli, where the points of \(D_i\) are contained in distinct open ball connected components. We will write this collection of semistable vertex sets for \((X,D) = \bigsqcup (X_i,D_i)\) as \(V = \bigsqcup V_i\).

Definition 2.7. (Triangulated punctured curves) A triangulated punctured curve \((X,V \cup D)\) is a smooth proper algebraic curve together with a finite set \(D_i\) of punctures for every irreducible component \(X_i\) of \(X\) and a semistable vertex set \(V\) for \((X,D)\) as in Definition 2.6.

Definition 2.8. (Skeleton of a triangulated punctured curve) Let \((X,V \cup D)\) be a triangulated punctured curve. The skeleton \(\Sigma(X,V \cup D)\) of \((X,V \cup D)\) is the disjoint union of the skeletons of the connected triangulated punctured curves \((X_i,V_i \cup D_i)\), corresponding to the irreducible components \(X_i\) of \(X\).

Definition 2.9. (Finite morphisms of triangulated punctured curves) Let \((X,V \cup D)\) and \((X',V' \cup D')\) be triangulated punctured curves. A finite morphism \((X',V' \cup D') \to (X,V \cup D)\) of triangulated punctured curves is a finite morphism \(\phi : X' \to X\) such that \(\phi^{-1}(D) = D', \phi^{-1}(V) = V'\) and \(\phi^{-1}(\Sigma(X,V \cup D)) = \Sigma(X',V' \cup D')\) as sets.

For any semistable vertex set \(V\) of \((X,D)\), there is a natural retraction map \(\tau_V : X^\an \to \Sigma(X,V \cup D)\) by taking the retraction maps on the induced connected triangulated punctured curves \(\tau_{V_i} : X^\an_i \to \Sigma(X_i,V_i \cup D_i)\). See [BPR14, Definition 3.7] for the definition of retraction map in the connected case. If \(V\) is clear from context, we will just denote this retraction by \(\tau\).

2.4 Lifting tame coverings of disconnected metrized complexes

Let \(\phi : \Sigma' \to \Sigma\) be a tame covering of metrized complexes as in Definition 2.5, where we note that both \(\Sigma'\) and \(\Sigma\) can be disconnected. Let \(\Sigma'_{i,j}\) be the connected components of \(\Sigma'\) that map to the connected components \(\Sigma_i\). By definition, there are induced morphisms of connected metrized complexes

\[
\phi_{i,j} : \Sigma'_{i,j} \to \Sigma_i.
\]  

For every \(i\), let \((X_i,V_i \cup D_i)\) be a triangulated punctured curve with skeleton \(\Sigma_i\) and let \(\tau_i : X^\an_i \to \Sigma_i\) be the canonical retraction. We assume again that every \(\Sigma_i\) has no loop edges.

Definition 2.10. A lifting of \(\Sigma' \to \Sigma\) to a tame covering of \((X,V \cup D)\) is a set of liftings of \(\Sigma'_{i,j} \to \Sigma_i\) to triangulated punctured curves \((X'_{i,j},V'_{i,j} \cup D'_{i,j}) \to (X_i,V_i \cup D_i)\). Here liftings of connected curves are as defined in [ABBR15, Section 7].

We can now define the set of gluing data for a general covering of metrized complexes. For the original definition of the gluing data for connected coverings, we refer the reader to [ABBR15, Section 7.2].

Definition 2.11. Let \(G(\Sigma'_{i,j},X_i)\) be the set of gluing data for a lifting of \(\Sigma'_{i,j} \to \Sigma_i\) to a covering of \(X_i\). We define the set of gluing data for a lifting of \(\Sigma' \to \Sigma\) to be the product

\[
G(\Sigma',X) := \prod_{i} G(\Sigma'_{i,j},X_i).
\]

As in [ABBR15], we now define the conjugation action of \(\alpha\) on \(G(\Sigma',X)\). Let \(\Sigma'_{i,j}\) be a connected component of \(\Sigma'\) mapping to \(\Sigma_i \subset \Sigma\). Let \(\alpha \in \Aut_{\Sigma}(\Sigma')\), which is defined as the group of degree 1 harmonic morphisms of \(\Sigma'\) (as a metrized complex) that fix \(\Sigma\). Since \(\alpha\) fixes \(\Sigma\) and thus \(\Sigma_i\), we see that \(\alpha\) maps \(\Sigma'_{i,j}\) to another component above \(\Sigma_i\). We will denote this component by \(\Sigma'_{i',\alpha(j)}\). Let \(x' \in V(\Sigma'_{i,j})\) and let \(x'' = \alpha(x') \in V(\Sigma'_{i',\alpha(j)})\) and \(x = \phi(x') = \phi(x'')\). From [ABBR15, Theorem 6.18], we obtain a unique lift of the corresponding morphism of residue curves \(C_{x'} \to C_{\alpha(x')}\) to a \(Y(x)\)-isomorphism of star-shaped curves \(Y'(x') \to Y''(x'')\). Here a star-shaped curve is as defined in [ABBR15, Definition 6.2]. Let \(e' \in E_f(\Sigma'_{i,j})\) be an edge adjacent to \(x',\ e'' = \alpha(e')\) and let \(e = \phi(e') = \phi(e'')\). From the isomorphism of star-shaped curves, we obtain an isomorphism

\[
\alpha_{x'} : \tau_{x'}^{-1}(e'^o) \to \tau_{x'}^{-1}(e''o).
\]

We define the conjugation action of \(\Aut_{\Sigma}(\Sigma)\) by the rule

\[
\alpha \cdot (\Theta_{e'}) = (\alpha_{y'}^{-1} \circ \Theta_{\alpha(e')} \circ \alpha_{x'})
\]

where \(e' = \overrightarrow{x'y'}\). We now easily deduce the following theorem, which is the disconnected version of [ABBR15, Theorem 7.4].
Theorem 2.1. Let \((X,V \cup D)\) be a triangulated punctured curve with skeleton \(\Sigma\) and let \(\Sigma' \to \Sigma\) be a tame covering of (possibly disconnected) metrized complexes of curves. Then

1. There is a bijection between the set of liftings of \(\Sigma'\) to a tame covering of \(X\) as in Definition 2.10 (up to \(X\)-isomorphism preserving \(\Sigma'\)), and the set of gluing data \(G(\Sigma',X)\) as in Definition 2.11. Any such lifting has no nontrivial automorphisms which preserve \(\Sigma'\).

2. Two tuples of gluing data \(\Theta_1\) and \(\Theta_2\) determine \(X\)-isomorphic curves if and only if there exists an \(\alpha \in \text{Aut}_{\Sigma}(\Sigma')\) such that \(\alpha \cdot (\Theta_1) = \Theta_2\). Here the conjugation action is as defined in Equation (13). The stabilizer \(\text{Stab}(\Theta)\) in \(\text{Aut}_{\Sigma}(\Sigma')\) of an element \(\Theta\) of \(G(\Sigma',X)\) is canonically isomorphic to the \(X\)-automorphism group of the associated curve.

Proof. This follows as in [ABBR15, Theorem 7.4]. We leave the details to the reader. \(\square\)

2.5 Tropical Jacobians

We now define the Tropical Jacobian of a finite metric graph, which will be used in Section 5.2. There are multiple possible references for this material, to name a few: [BR14], [BBC17] and [Ale08, Page 203]. We will mostly be following [BR14, Section 3].

Let \(X\) be a smooth proper connected \(K\)-curve and let \(\Gamma\) be the intersection graph of a semistable formal \(R\)-model, considered as a metric graph. This graph can also be turned into a metrized complex, but we will only need the metric graph structure in this section. We let \(\Gamma(\Lambda)\) denote the set of all points of \(\Gamma\) whose distance from any vertex belongs to \(\Lambda\). Consider the canonical surjective retraction map

\[ \tau: X^{an} \to \Gamma. \] (14)

We then also obtain a map \(\tau_*: \text{Div}(X) \to \text{Div}_{\Lambda}(\Gamma)\) by extending \(\tau\) linearly, which induces a induced map on the divisors of degree zero \(\tau_*: \text{Div}^0(X) \to \text{Div}^0_{\Lambda}(\Gamma)\). We now define the notion of principal divisors on \(\Gamma\). A tropical meromorphic function on \(\Gamma\) is a continuous function \(F: \Gamma \to \mathbb{R}\) which is piecewise affine with integer slopes. It is \(\Lambda\)-rational provided that \(F(\Gamma(\Lambda)) \subset \Lambda\) and all points at which \(F\) is not differentiable are contained in \(\Gamma(\Lambda)\). We then define the divisor of \(F\) to be

\[ \text{div}(F) = \sum_{x \in \Gamma(\Lambda)} n_x(x), \] (15)

where \(n_x\) is the sum of the outgoing slopes of \(F\) at \(x\). The group of \((\Lambda\text{-rational})\) principal divisors on \(\Gamma\) is then the subgroup \(\text{Prin}_{\Lambda}(\Gamma) \subset \text{Div}^0_{\Lambda}(\Gamma)\) of divisors of \(\Lambda\)-rational tropical meromorphic functions.

For any rational function \(f \in K(X)\), we let \(F\) be the restriction of \(-\log(|f|)\) to \(\Gamma\). By the Poincaré-Lelong formula (See [BR14, Section 5] and [Thu05, Proposition 3.3.15]), we see that \(F\) is a \(\Lambda\)-rational tropical meromorphic function on \(\Gamma\) and that \(\tau_*(\text{div}(f)) = \text{div}(F)\). In the algebraic case of a strongly semistable regular model, this theorem follows from intersection theory, see [Hel18, Section 3]. At any rate, we can now define the following:

Definition 2.12. Let \(\Lambda = \mathbb{R}\). We define the Tropical Jacobian of \(\Gamma\) to be the group

\[ \text{Jac}(\Gamma) := \text{Div}^0_{\Lambda}(\Gamma)/\text{Prin}_{\Lambda}(\Gamma). \] (16)

One can alternatively define the Tropical Jacobian using differential forms as follows. Let \(H_1(\Gamma,\mathbb{Z})\) be the first singular homology group of \(\Gamma\). Using the integration pairing \(\gamma \mapsto \int_{\gamma}\), we obtain a map

\[ \mu : H_1(\Gamma,\mathbb{Z}) \to \text{Hom}(\Omega_{\Gamma(\Lambda)}(\Gamma),\Lambda) \] (17)

By [BR14, Proposition 3.8], we then have an isomorphism

\[ \text{Jac}(\Gamma) \cong \text{Hom}(\Omega_{\Gamma(\mathbb{R})}(\Gamma),\mathbb{R})/\mu(H_1(\Gamma,\mathbb{Z})). \] (18)

In particular, we obtain an isomorphism of the Tropical Jacobian with the real torus \(\mathbb{R}^g/\mathbb{Z}^g\), where \(g = \text{rank}_{\mathbb{Z}}(H_1(\Gamma,\mathbb{Z})) = \beta(\Gamma)\), the first Betti number of \(\Gamma\).
2.6 Galois categories

In this section we review the notion of Galois categories. We will use the definition found in [Sta18, Tag 0BMQ] and we refer the reader to [GR02], [Len08] and [Cad13] for more background on the subject. In this paper we will be mainly interested in the Galois category of finite tame coverings of an algebraic curve. We will show that the category of finite rigidified coverings of a metrized complex is equivalent to this category, giving an isomorphism of the corresponding profinite fundamental groups. This equivalence will be given in Section 4.2.

We will assume in this section that $\mathcal{C}$ is an essentially small category. Let $\mathcal{C}$ be such a category. For any $\phi \in \text{Ar}(\mathcal{C})$, we say that $\phi$ is a monomorphism if it is left cancellative. That is, $\phi : X \to Y$ is a monomorphism if for any two morphisms $g_1 : Z \to X$ in $\mathcal{C}$ such that $f \circ g_1 = f \circ g_2$, we have $g_1 = g_2$. A connected object of $\mathcal{C}$ is a noninitial object $X$ such that any monomorphism $Y \to X$ is initial or an isomorphism. Let $F$ be a functor $\mathcal{C} \to \mathcal{D}$. We say that $F$ is exact if it commutes with finite limits and finite colimits. We say that $F$ reflects isomorphisms if for any $f \in \text{Ar}(\mathcal{C})$, if $Ff$ is an isomorphism, then $f$ is an isomorphism.

**Definition 2.13.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to (\text{Sets})$ be a functor. The pair $(\mathcal{C}, F)$ is a Galois category if

1. $\mathcal{C}$ has finite limits and finite colimits,
2. every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects,
3. $F(X)$ is finite for every object $X \in \text{Ob}(\mathcal{C})$ and
4. $F$ reflects isomorphisms and is exact.

We refer to $F$ as the fundamental functor of the Galois category. The definition of a Galois category is completely categorical, so this structure is preserved under categorical equivalences. We can therefore define the notion of an induced Galois category structure.

**Definition 2.14.** (Induced Galois category structure) Let $(\mathcal{C}, F)$ be a Galois category and suppose that there is an equivalence of categories $G : \mathcal{D} \to \mathcal{C}$. Let $F'$ be the induced functor $FG : \mathcal{D} \to (\text{Sets})$ defined by $FG(X) = F(G(X))$. Then $(\mathcal{D}, F')$ is the induced Galois category structure on $\mathcal{D}$.

**Remark 2.2.** (An equivalent definition) Let $\mathcal{C}$ be a category and let $F$ be a functor to $(\text{Sets})$. In [Len08], [Cad13] and [GR02] a different definition for a Galois category is given. There, $F$ and $\mathcal{C}$ have to satisfy the following set of axioms:

1. $\mathcal{C}$ has a final object and and finite fiber products exist in $\mathcal{C}$.
2. Finite coproducts exist in $\mathcal{C}$ and categorical quotients by finite groups of automorphisms exist in $\mathcal{C}$.
3. Any morphism $u : Y \to X$ in $\mathcal{C}$ factors as $u'' \circ u' : Y \to X' \to X''$, where $u'$ is a strict epimorphism and $u''$ is a monomorphism which is an isomorphism onto a direct summand of $X$.
4. $F$ sends final objects to final objects and commutes with fiber products.
5. $F$ commutes with finite coproducts and quotients by finite groups and sends strict epimorphisms to strict epimorphisms.
6. $F$ reflects isomorphisms.

These definitions are in fact equivalent. To see this, let $\mathcal{C}$ be any category that is equivalent to $(\text{Finite-}G\text{-Sets})$, which is the category of finite sets with a continuous action by a profinite group $G$. Then $\mathcal{C}$ automatically satisfies the conditions in both definitions with respect to the composed functor $\mathcal{C} \to (\text{Finite-}G\text{-Sets}) \to (\text{Sets})$. By Theorem 2.2 and [Cad13, Theorem 2.8], we obtain an equivalence with $(\text{Finite-}G\text{-Sets})$ using either definition, so we see that the two definitions are equivalent.

**Example 2.2.** We will follow [Len08, Section 3.7] and [Cad13, Example 2.7]. Let $X$ be a topological space and consider the category $\text{Cov}(X)$ of finite coverings of $X$. Here a finite covering $\phi : X' \to X$ is a finite continuous surjective map such that for every $x \in X$, there exists a neighborhood $U$ of $x$ such that $\phi^{-1}(U)$ is homeomorphic to a disjoint union of $n$ copies of $U$ for some $n \in \mathbb{N}$. The fundamental functor, denoted by $F_\phi$ for any point $x$ in $X$ is then given as follows: one assigns to every covering $\phi : X' \to X$ the inverse image $\phi^{-1}(x)$ of $x$ under $\phi$. 

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Lemma 2.1. Let \( G \) be a connected scheme and consider the category \( \text{ECov}(X) \) of finite étale coverings \( X' \to X \). Let \( x : \text{Spec}(\Omega) \to X \) be any geometric point of \( X \). Here \( \Omega \) is some algebraically closed field. If \( Y \to X \) is finite étale, then so is \( Y \times_X \text{Spec}(\Omega) \). This gives a functor \( \text{ECov}(X) \to \text{ECov}(\text{Spec}(\Omega)) \). But the fundamental group of \( \text{Spec}(\Omega) \) is trivial, so we obtain an equivalence \( \text{ECov}(\text{Spec}(\Omega)) \simeq (\text{Finite Sets}) \). This gives the fundamental functor \( F_x \) for any geometric point \( x \). We denote the corresponding profinite group by \( \pi(X,x) \).

If \( X \) is a normal integral scheme, then there is a very concrete description of \( \pi(X,x) \) in terms of the absolute Galois group corresponding to the function field of \( X \). Let \( K(X) \) be that function field, \( \overline{K} \) an algebraic closure of \( K(X) \) and consider the composite \( M \) of all finite separable extensions \( L \subset \overline{K} \) such that \( X \) is unramified in \( L \). By unramified, we mean that the normalization of \( X \) in the field \( L \) is unramified over \( X \). The extension \( M/K \) is then Galois and we have an isomorphism \( \pi(X,x) \simeq \text{Gal}(M/K) \). See [Len08, Corollary 6.17] for the details. We note that by taking \( X := \text{Spec}(K) \) for some field \( K \), we obtain the usual Galois group \( \pi(X,x) \simeq \text{Gal}(K^{\text{sep}}/K) \) where \( x : \text{Spec}(K) \to \text{Spec}(K) \) is any morphism.

The theory of Galois categories can thus be seen as a generalization of usual Galois theory.

Lemma 2.1. Let \( (\mathcal{C}, F) \) be a Galois category. Then for any \( X \in \text{Ob}(\mathcal{C}) \), we have that
\[
|\text{Aut}(X)| \leq |F(X)|. \tag{19}
\]

Proof. See [Sta18, Lemma 0BN0]. \( \square \)

Definition 2.15. Let \( (\mathcal{C}, F) \) be a Galois category. We say that a connected object \( X \) is a Galois object if
\[
|\text{Aut}(X)| = |F(X)|. \tag{20}
\]

For any Galois object \( X \), we let \( X/\text{Aut}(X) \) be the coequalizer of the arrows \( \sigma : X \to X \), where \( \sigma \in \text{Aut}(X) \). This exists by condition (1) in the definition of a Galois category. We then have the following Lemma:

Lemma 2.2. Let \( X \) be a Galois object. Then the coequalizer \( X/\text{Aut}(X) \) is the terminal object of \( \mathcal{C} \).

Proof. By using the fact that \( F \) commutes with finite colimits and the fact that the action of \( \text{Aut}(X) \) is transitive on \( F(X) \), we easily see that \( F(X/\text{Aut}(X)) \) is the singleton set, implying that \( X/\text{Aut}(X) \) is the terminal object of \( \mathcal{C} \) by [Sta18, Lemma 0BN0, (5)] \( \square \)

Example 2.4. For a topological space \( X \), a Galois object in \( \text{Cov}(X) \) is a topological space \( X' \) with a continuous group action of a finite group \( G \) on it such that \( X'/G = X \). For a connected normal scheme \( X \) with function field \( K(X) \), every Galois object is obtained as follows. We take a finite Galois extension \( L \) of \( K(X) \) with Galois group \( G \) and consider the normalization \( X'/X \) in \( L \). There is then a natural action of \( G \) on \( X' \) and we have \( X'/G = X \).

Let \( F \) be any functor \( F : \mathcal{C} \to (\text{Sets}) \). An automorphism of functors is an invertible morphism of functors \( F \to F \). We then have a canonical injective map
\[
\text{Aut}(F) \to \prod_{X \in \text{Ob}(\mathcal{C})} \text{Aut}(F(X)). \tag{21}
\]

Assigning the discrete topology to every set on the righthand side of Equation (21) and the product topology to the product, we see that \( \text{Aut}(F) \) naturally has the structure of a profinite group for a Galois category. In particular, we see that \( \text{Aut}(F) \) is a topological group. We then also have natural continuous maps
\[
\text{Aut}(F) \times F(X) \to F(X) \tag{22}
\]
for every \( X \in \text{Ob}(\mathcal{C}) \). Letting \( G := \text{Aut}(F) \), we have that the functor \( F \) induces a functor
\[
\tilde{F} : \mathcal{C} \to (\text{Finite-G-Sets}), \tag{23}
\]
where \( (\text{Finite-G-Sets}) \) is the category of finite sets with a continuous \( G \)-action. We will often denote this functor by \( F \) again. We can now state the main theorem on Galois categories:

Theorem 2.2. Let \( (\mathcal{C}, F) \) be a Galois category and let \( \tilde{F} : \mathcal{C} \to (\text{Finite-G-Sets}) \) be the induced functor. We then have

1. \( \tilde{F} \) is an equivalence of categories.
2. If \( \pi \) is any profinite group such that the categories \( \mathcal{C} \) and \( (\pi\text{-Sets}) \) are equivalent by an equivalence that, composed with the forgetful functor \((\pi\text{-Sets}) \to (\text{Sets})\) yields the functor \( F \), then \( \pi \) and \( \text{Aut}(F) \) are canonically isomorphic.

3. If \( F \) and \( F' \) are two fundamental functors for \( \mathcal{C} \), then they are isomorphic.

4. If \( \pi \) is a profinite group such that \( \mathcal{C} \) and \((\pi\text{-Sets})\) are equivalent, then there is an isomorphism of profinite groups \( \text{Aut}(F) \cong \pi \) that is canonically determined up to an inner automorphism of \( \text{Aut}(F) \).

**Proof.** See [Sta18, Proposition 0BN4] or [Len08, Theorem 3.5].

**Remark 2.3.** Throughout this paper, we will often suppress the base-points in the fundamental functors \( F_x \) and the profinite fundamental groups \( \pi(X, x) \) since they are not of any significant relevance to us.

**Example 2.5.** We recall an important result by Grothendieck, which gives a finite representation of the tame profinite fundamental group of a punctured curve \( (X, \mathcal{D}) \), where \( X \) is now a smooth, proper, connected curve over a separably closed field \( k \) of characteristic \( p \). See [GR02, Définition 2.1.1] for the definition of the Galois category of finite étale coverings that are tamely ramified.

**Theorem 2.3.** Let \((X, \mathcal{D})\) be a smooth proper connected punctured curve over a separably closed field \( k \) of characteristic \( p \) (we allow \( p = 0 \)) of genus \( g := g(X) \) with \( d \) punctures \( P_i \in X(k) \). Then the profinite fundamental group \( \pi'(X, \mathcal{D}) \) corresponding to the Galois category of tamely ramified étale coverings of the punctured curve \((X, \mathcal{D})\) is the profinite completion of the group \( R = \langle x_i, y_i, z_i \rangle / I \) for \( i = 1, \ldots, g \) and \( j = 1, \ldots, d \), where \( I \) is the group generated by the single element

\[
\prod_{1 \leq i \leq g} x_i y_i x_i^{-1} y_i^{-1} \prod_{1 \leq i \leq d} z_i. \tag{24}
\]

Furthermore, every \( z_i \) generates an inertia group above the puncture \( P_i \).

**Proof.** See [GR02, Corollaire 2.12].

In other words, the tame profinite fundamental group coincides with the profinite completion of the fundamental group of any genus \( g \) curve \( X/C \) with \( d \) punctures. In fact, the proof of Theorem 2.5 uses this analytic result, along with a theorem on the specialization of the fundamental group.

### 3 A simultaneous semistable reduction theorem for tame coverings

In this section, we will give a proof of a simultaneous semistable reduction theorem for tame coverings of algebraic curves \( \phi : X' \to X \). The theorem seems to be well-known to experts (see [CTT16, Section 1.2.1]), but we included the proof since we couldn’t find an exact reference. The theorem will be used to go from the category of finite tame coverings of \((X, \mathcal{D})\) to the category of finite rigidified coverings of any metrized 

**Theorem 3.1.** Let \( \phi : X' \to X \) be a finite tame covering of smooth proper connected algebraic curves over \( K \) and let \( \mathcal{X} \) be any semistable model of \( X \) over \( R \) such that the closure \( \mathcal{B} \) of the branch locus \( B \subset X(K) \) in \( \mathcal{X} \) consists of disjoint smooth points. Let \( \mathcal{X}' \) be the normalization of \( \mathcal{X} \) in the function field \( K(X') \). Then \( \mathcal{X}' \) is semistable and \( \mathcal{X}' \to \mathcal{X} \) is a finite morphism of semistable curves over \( R \).

The earliest versions of this theorem appeared in [LL99] and [Liu06], where it was proved over a discretely valued ring \( R \) using techniques of birational geometry and a theorem by Grothendieck on the tame fundamental group of a local Henselian scheme with normal crossings. This theorem was consequently used by the author and Madeline Brandt to give explicit algorithms and descriptions of tropical moduli spaces in [Hel18] and [BH17]. The corresponding coverings were referred to as "disjointly branched coverings".

**Remark 3.1.** We note that there are at least two other equivalent ways of formulating the theorem using admissible formal \( R \)-schemes and Berkovich spaces:

1. Let \( \phi : X' \to X \) and \( \mathcal{X}' \to \mathcal{X} \) be as in the theorem and consider the corresponding induced finite morphism of admissible formal \( R \)-schemes \( \mathcal{X}'_{\text{form}} \to \mathcal{X}_{\text{form}} \). Then \( \mathcal{X}'_{\text{form}} \) is semistable over \( R \) and \( \mathcal{X}'_{\text{form}} \to \mathcal{X}_{\text{form}} \) is a finite morphism. This follows from [ABBR15, Lemma 5.1 and Theorem 5.13].
2. Let \( \phi : X' \to X \) be as in the theorem and let \( \phi^{an} : X'^{an} \to X^{an} \) be the corresponding analytification in the sense of Berkovich spaces. Let \( V \) be a semistable vertex set of \((X, D)\). Then \( (\phi^{an})^{-1}(V) \) is a semistable vertex set of \( X' \).

In our proof, we will use the last characterization.

We will start with some preliminary lemmas. Let \( X/K \) and \( X'/K \) be two proper, smooth connected curves and let \( \phi : X' \to X \) be a finite covering. Then the analytification functor gives a finite morphism \( \phi^{an} : X'^{an} \to X^{an} \) on the analytifications by [Ber12, Proposition 3.4.7]. We will denote this morphism by \( \phi \) again. For any affinoid domain \( U := M(A) \subset X^{an} \), we then have \( \phi^{-1}(U) = M(A') \) for some affinoid domain \( A' \), and \( A' \) is a finite \( A \)-module. Here we used [Ber12, 3.1.7.(i)]. For the upcoming lemma, we will use the following notation. For a \( K \)-analytic space \( Z \), we write \( Z^0 := \{ x \in Z : |H(x) : K| < \infty \} \).

**Lemma 3.1.** Suppose that \( \phi \) is étale above every point of \((X^{an})^0 \cap U \). Then \( \phi \) is étale above \( U \).

**Proof.** By [Ber93, 3.3.11], we have that the morphism of schemes \( \text{Spec}(A') \to \text{Spec}(A) \) is étale if and only if the morphism \( \phi^{-1}(U) \to U \) on the analytifications is étale. We will show that \( \text{Spec}(A') \to \text{Spec}(A) \) is étale. Let \( M := \text{Spec}(A'/A) \) be the module of Kähler differentials associated to the homomorphism \( A \to A' \). Recall that the maximal ideals of the ring \( A' \) correspond bijectively to type 1 points of \( X^{an} \cap \phi^{-1}(U) \), see [Ber93, Lemma 2.6.3]. By our assumption, the localization \( M_m \) of \( M \) at these points is zero. This implies that \( M = 0 \) by [Eis95, Chapter 2, Lemma 2.8]. This proves that \( A \to A' \) is unramified by [Liu06, Chapter 6, Corollary 6.2]. Using the fact that \( X' \) and \( X \) are nonsingular, we quickly obtain flatness using the same localization technique as above. Indeed, \( A \to A' \) is flat if and only if \( A_m' \) is flat over every \( A_m \) for every maximal ideal \( m' \) lying above a maximal ideal \( m \) of \( A \). But these are Dedekind domains, so we obtain flatness by [Liu06, Chapter 1, Corollary 2.14]. We conclude that \( \phi \) is flat and unramified, and thus étale as desired.

In other words, if the branch points of \( X' \to X \) don’t belong to some affinoid domain \( U \), then the corresponding covering \( \phi^{-1}(U) \to U \) is étale. We now recall the following important theorem by Berkovich.

**Theorem 3.2.** (Tame finite étale Galois coverings of open balls and annuli)

1. Let \( U \) be an open ball and let \( V \to U \) be a finite étale Galois covering. Then \( V \) is a disjoint union of open balls.

2. Let \( U \) be a generalized open annulus, let \( V \) be connected and let \( V \to U \) be a finite étale Galois covering. Then there exist isomorphisms \( S(a)_+ \simeq U \) and \( V \simeq S(a^{1/n}) \) for \( a \in K \) such that the composed map \( U \to \tilde{U} \) is given by \( t \mapsto t^n \).

**Proof.** The first part is exactly [Ber93, Theorem 6.3.2]. For the second part, we note that \( U \) can be written as a union of closed annuli \( S(a_1, a_2) \). To be explicit, let \( U \simeq S(a)_+ \) be an isomorphism for some \( a \in K^* \) and let \( c := \text{val}(a) \). Write \( T_m = [c/2m, c(1 - (1/2m))] \). Then

\[
S(a)_+ = \bigcup_{m=1}^{\infty} \text{trop}^{-1}(T_m).
\] (26)

The induced covering for these closed annuli can then be written as

\[
S(a^{1/n}_1, b^{1/n}_1) \to S(a_1, b_1)
\] (27)

for some \( n \) by [Ber93, Theorem 6.3.5]. Using Equation (26), we then obtain the isomorphism as described in the theorem.

That is, we know what happens to our covering on the \( U \) where \( \phi \) is étale. Let us give an example to illustrate this theorem.

**Example 3.1.** Let \( K := \mathbb{C}_p \) be the field of \( p \)-adic complex numbers for \( p \neq 2 \). Consider the curve \( X/\mathbb{C}_p \) defined by the affine equation

\[
y^2 = x(x - \omega)(x - \omega^2)(x + 1)(x + 2),
\]
where \( \text{val}(\varpi) > 0 \). The branch points are separated into three groups:

\[
S_1 := \{-1, -2, \infty\},
S_2 := \{\varpi\},
S_3 := \{0, \varpi^2\}.
\]

Consider the closed annuli \( V_1 = S(\varpi, 1) \) and \( V_2 = S(\varpi^2, \varpi) \) and we let \( U_1 \) and \( U_2 \) be the corresponding open annuli. Let \( y_1 \) and \( y_2 \) and be the type-2 points of \( V_1 \) and let \( y_3 \) be the remaining type-2 point of \( V_2 \). Here \( y_2 \) is the common type-2 point of \( V_1 \) and \( V_2 \). We let \( X \) be the corresponding semistable model, which has three irreducible components in the special fiber. The points of \( S_1 \) reduce to distinct points on the component corresponding to \( y_1 \), \( \varpi \) reduces to the component corresponding to \( y_2 \) and the points of \( S_3 \) reduce to distinct points of the third component corresponding to \( y_3 \). For every branch point \( P \), we let \( B_P \) be the punctured open ball corresponding to it with respect to \( X \). That is, we have \( B_P = \text{red} \overline{x}(\tilde{x}_P) \setminus \{P\} \), where \( \tilde{x}_P \) is the closed point on the special fiber of \( X \) corresponding to \( P \).

The induced coverings of the \( U_i \) and \( B_P \) of are finite tame and étale, with one connected component lying above \( U_1 \) and two components lying above \( U_2 \). The corresponding covering of metrized complexes is given in Figure 2.

![Figure 2: The covering of metrized complexes in Example 3.1 associated to a hyperelliptic covering \( X \to \mathbb{P}^1 \), where \( X \) has genus 2.](image)

Let us give an example of a covering of a closed subannulus of \( X^\text{an} \) that is not étale. Let us take the closed annulus \( A' = \tau^{-1}(\Sigma') \subset X^\text{an} \) where \( \Sigma' \) is the closed subset of \( \Sigma \) depicted in Figure 2 by the blue line. The corresponding induced covering is not étale. Indeed, the branch point \( \varpi^2 \) is contained in this annulus. We now easily see from Figure 2 that taking a semistable vertex that contains the endpoints of \( \Sigma' \) (and not any points in \( \Sigma' \)) would lead to a vertex set whose complement does not decompose into open annuli and open balls.

We are now ready to prove Theorem 3.1.

**Proof.** (Of Theorem 3.1) Let \( U \) be a generalized open annulus in \( X^\text{an} \setminus V \). By covering \( U \) by affinoid domains and using our assumption on the branch locus, we see from Lemma 3.1 that \( \phi^{-1}(U) \to U \) is étale. Let \( V \subseteq U \) be an affinoid domain with algebra \( A \) and let \( V' \) be a connected component in \( \phi^{-1}(V) \) with corresponding algebra \( A' \). We find by [Ber12, Corollary 3.3.21] that \( A' \) is an integral domain and we denote its field of fractions by \( K(A') \). Taking the Galois closure \( L \) of \( K(A') \supset K(A) \) and taking the normalization of \( A' \) in \( L \), we obtain an finite Galois covering \( A'' \to A \) which is étale. Indeed, by Lemma 3.1 we can prove this last claim using \( K \)-rational points. By some basic ramification theory, a covering is étale above a \( K \)-rational point if and only if the Galois closure is étale above that point. This proves that the Galois closure is again étale. By Theorem 3.2, part two, we now find that \( A'' \to A \) is Kummer. This then easily implies that the corresponding covering \( V' \to V \) is Kummer. Writing \( U \) as an increasing union of closed annuli, we thus see that \( \phi^{-1}(U) \) is a disjoint union of generalized open annuli, as desired. Similarly, let \( U \) be an open ball. As in the previous case, we can reduce to the Galois case and use Theorem 3.2, part one, to conclude that \( \phi^{-1}(U) \) is a disjoint union of open balls.
Now consider all type 2 points that reduce to type 2 points in the semistable vertex set $V$. By the previous considerations, we have that the complement consists of generalized open annuli and open balls. This means that it is a skeleton, as desired.

\[\square\]

Example 3.2. We now review Example 2.1 from the algebraic point of view. Let $E$ be the elliptic curve over $K := \mathbb{C}_p$ defined by the Weierstrass equation

\[y^2 = x(x - \varpi)(x + 1),\]

where $v(\varpi) > 0$ and $p \neq 2$. The branch points of the natural hyperelliptic covering $(x, y) \mapsto x$ are given by $0, \varpi, -1$ and $\infty$. A semistable model $\mathcal{X}$ that separates the branch locus is then easily found: we take the blow-up of $\mathbb{P}^1$ along the ideal $I = (x, \varpi)$. In terms of Berkovich spaces, we take the decomposition of $\mathbb{P}^1,\text{an}$ into the open annulus $\mathcal{S}(\varpi)_+$, four punctured open balls corresponding to the branch points and two type 2-points (which are the endpoints of $\mathcal{S}(\varpi)_+$). Using either an explicit calculation (see Example 5.2) or a general Kummer criterion for the splitting of edges in abelian extensions as in [Hel18, Proposition 7.2.1] and [BH17, Proposition 4.1], we see that the inverse image of this decomposition is then as given in Figure 1. Taking the fiber product of the covering $(x, y) \mapsto x$ over itself, we then see that we obtain the diagram of Example 2.1.

We will study this elliptic product of $E$ again in Example 5.2, where the focus will be on the 2-torsion of $E$ and the various coverings of metrized complexes arising from these points.

4 Rigidified coverings of metrized complexes

In this section we introduce the category of tame finite coverings of a metrized complex $\Sigma$ of $k$-curves, as introduced in [ABBR15]. The main difference between our approach and theirs is that we allow disconnected coverings. Furthermore, we enhance coverings with the additional datum of gluing data, leading to the definition of rigidified coverings of a metrized complex. These rigidified coverings then give rise to a category of rigidified coverings of a metrized complex and this category is in fact equivalent to the category of finite coverings of an algebraic curve, as we will see in Section 4.2. In Section 5, we will use this equivalence to endow the category of rigidified coverings with the structure of a Galois category. This in turn allows us to define fundamental groups for metrized complexes.

4.1 Rigidified tame coverings of metrized complexes

We now introduce the notion of a rigidified covering of a single metrized complex. The idea is to add a set of gluing data to a covering so that there is no possibility of multiple algebraic coverings giving the same covering of metrized complexes. We give a quick example with elliptic curves to show how the gluing data allows one to distinguish between various unramified coverings.

Definition 4.1. (Rigidified tame coverings) A rigidified tame covering of a metrized complex $\Sigma$ associated to a triangulated punctured curve $(X, V \cup D)$ is a pair $\phi := (\phi, g)$ consisting of

1. A tame covering $\phi : \Sigma' \to \Sigma$ of metrized complexes of $k$-curves, where $\Sigma$ is the skeleton of $(X, V \cup D)$.

2. An element $g$ of $\mathcal{G}((\Sigma', X))$, as in Definition 2.11.

We will also refer to such a pair as a rigidified covering of $\Sigma$.

By Theorem 2.1, we obtain a unique (up to isomorphism) lifting $(X', V' \cup D') \to (X, V \cup D)$ of a rigidified covering to an algebraic covering. We now introduce the notion of a morphism between two rigidified coverings of a single metrized complex $\Sigma$.

Definition 4.2. (Morphisms of rigidified coverings) Let $\phi_{1,g_1}$ and $\phi_{2,g_2}$ be two rigidified coverings of a metrized complex $\Sigma$. Here $\phi_i : \Sigma_i \to \Sigma$ are the morphisms on the metrized complexes and we write $\Theta_{1,e}$ for the gluing data. A morphism $\phi_{1,g_1} \to \phi_{2,g_2}$ is a morphism $\psi : \Sigma_1 \to \Sigma_2$ with the following two properties:

1. The diagram

\[
\begin{array}{ccc}
\Sigma_1 & \xrightarrow{\psi} & \Sigma_2 \\
\downarrow{\phi_1} & & \downarrow{\phi_2} \\
\Sigma & \xrightarrow{\psi} & \Sigma
\end{array}
\]
commutes.

2. The induced diagram

\[
\begin{array}{ccc}
\tau_{x}^{-1}(e_{1}^{1}) & \xrightarrow{\theta_{x},e_{1}} & \tau_{y}^{-1}(e_{2}^{2}) \\
\downarrow\psi & & \downarrow\psi \\
\tau_{\psi(x)}^{-1}(e_{2}^{2}) & \xrightarrow{\theta_{x},e_{2}} & \tau_{\psi(y)}^{-1}(e_{2}^{2}) \\
\downarrow\phi_{1} & & \downarrow\phi_{2} \\
\tau^{-1}(e_{1}) & \xrightarrow{\text{id}} & \tau^{-1}(e_{1}) \\
\end{array}
\]

commutes for every edge \( e_{1} \in \Sigma_{1} \) with images \( e_{2} \in \Sigma_{2} \) and \( e \in \Sigma \). Here we again write \( \psi \) and \( \phi_{2} \) for the induced unique maps (see [ABBR15, Theorem 6.18]) on the corresponding star-shaped curves (see [ABBR15, Definition 6.2]) and the restrictions of these maps to subannuli.

Let \( \phi : (X', V' \cup D') \to (X, V \cup D) \) be a covering of triangulated punctured curves, as in Definition 2.9, with a corresponding tame covering of metrized complexes \( \Sigma' \to \Sigma \). The analytic space \( X^{\text{an}} \) naturally gives rise to a set of gluing data by considering two simple neighborhoods of adjacent type-2 points in \( V_{f}(\Sigma') \). We thus have a natural rigidified covering of \( \Sigma \) associated to \( \phi \). Furthermore, if we have a commutative diagram

\[
\begin{array}{ccc}
(X_{1}, V_{1} \cup D_{1}) & \xrightarrow{\psi} & (X_{2}, V_{2} \cup D_{2}) \\
\downarrow\phi_{1} & & \downarrow\phi_{2} \\
(X, V \cup D) & & \\
\end{array}
\]

of triangulated punctured curves, then we easily see that this induces a morphism of rigidified coverings \( \phi_{1,\Sigma_{1}} \to \phi_{2,\Sigma_{2}} \). We would now like to go the other way, starting with a morphism of rigidified coverings of a metrized complex \( \Sigma \) of a triangulated punctured curve \((X, V \cup D)\). From this morphism of rigidified coverings, we would like to obtain a morphism of algebraic coverings.

**Lemma 4.1.** Let \( \phi_{1,\Sigma_{1}} \to \phi_{2,\Sigma_{2}} \) be a morphism of rigidified coverings of a metrized complex \( \Sigma \) associated to a triangulated punctured curve \((X, V \cup D)\). Then there is a unique lift (up to \( X \)-isomorphism) of the diagram

\[
\begin{array}{ccc}
\Sigma_{1} & \xrightarrow{\psi} & \Sigma_{2} \\
\downarrow\phi_{1} & & \downarrow\phi_{2} \\
\Sigma & & \\
\end{array}
\]

to a diagram of triangulated punctured curves

\[
\begin{array}{ccc}
(X_{1}, V_{1} \cup D_{1}) & \xrightarrow{\tilde{\psi}} & (X_{2}, V_{2} \cup D_{2}) \\
\downarrow\tilde{\phi}_{1} & & \downarrow\tilde{\phi}_{2} \\
(X, V \cup D) & & \\
\end{array}
\]

**Proof.** It suffices to prove the lemma for the connected case. Let \( C_{x'}, C_{y'} \) and \( C_{x} \) be curves corresponding to vertices \( x', x'' \) and \( x \) respectively. Here \( x' \) maps to \( x'' \) and \( x \) under \( \psi \) and \( \phi_{1} \) respectively. The commutativity of the \( \phi_{1} \) and \( \psi \) implies that there is a commutative diagram \( C_{x'} \to C_{x''} \to C_{x} \), and by [ABBR15, Theorem 6.18] this lifts uniquely to a morphism of star-shaped curves \( Y'(x') \to Y''(x'') \to Y(x) \) and we obtain a similar sequence of morphisms for vertices \( y', y'' \) and \( y \) adjacent to \( x', x'' \) and \( x \) respectively. The commutative diagram on the gluing data in the definition of morphisms of rigidified coverings is just another way of stating that we can define morphisms \( Y'(x') \cap Y'(y') \to Y''(x'') \cap Y''(y'') \to Y(x) \cap Y(y) \) using the induced isomorphism from either \( Y'(x') \) or \( Y'(y') \) and similarly for \( Y''(x'') \) and \( Y''(y'') \). In other words, these morphisms glue together to give a commutative diagram of algebraic curves, see [ABBR15, Theorem 7.4]. Furthermore, this diagram is unique up to \( X \)-isomorphism by the same theorem. This finishes the proof. \( \square \)

**Definition 4.3.** (Category of rigidified coverings) Let \( \Sigma \) be a metrized complex. We define the category \( \text{Cov}_{C}(\Sigma) \) of finite tame rigidified coverings of \( \Sigma \) to be the category consisting of
1. Rigidified tame coverings of $\Sigma$ as its objects (see Definition 4.1) and

2. Morphisms of rigidified coverings to be the morphisms between the objects (see Definition 4.2).

We will also call this the category of rigidified coverings of $\Sigma$.

Example 4.1. Let us examine [ABBR15, Example 7.8] from the viewpoint of rigidified coverings. The set-up is as follows. Assume that $\text{char}(k) \neq 2$. Consider the Tate curve $E$, given by $K^*/\langle q \rangle$ for some $q \in K^*$ with $v(q) > 0$ and let $\Sigma' \to \Sigma$ be the degree 2 covering of metrized complexes as in [ABBR15, Example 7.8]. The gluing data $G(\Sigma', E)$ of the covering $\Sigma' \to \Sigma$ is then easily seen to consist of four elements, corresponding to the choice of an automorphism per edge. The automorphism group $\text{Aut}_{\Sigma}(\Sigma')$ then also has order four. If we however choose a set of gluing data $\mathfrak{g}$ and consider the corresponding rigidified automorphism group $\text{Aut}_{\Sigma, \mathfrak{g}}(\Sigma')$, then it has order two.

Let us illustrate this fact in more detail. We will use the notation in [ABBR15, Example 7.8]. Write $y'$ and $z'$ for the two antipodal vertices of $\Sigma'$ mapping to $y$ and $z$ respectively. For $x' \in V'$, we fix isomorphisms $Y'(x') \simeq S(q^{1/2})$ such that the morphisms $Y'(x') \to Y(x)$ are given by $t \mapsto t^2$. A $\Sigma$-automorphism of metrized complexes then corresponds to a pair of automorphisms $(\psi_{y'}, \psi_{z'}) \in \text{Aut}_{C_{y'}}(C_{y'}) \times \text{Aut}_{C_{z'}}(C_{z'})$ that fix the two points $0, \infty \in \mathbb{P}_1^1$ corresponding to the edges.

For an automorphism $\alpha$ to respect the gluing data in the sense of Definition 4.1, we need to have $\alpha_{x'} \circ \Theta_x = \Theta_y \circ \alpha_{x'}$ for every $x'$ and $x' \in V'$. In other words, we need to have that $\alpha$ stabilizes the gluing data under the conjugation action. Since the stabilizer of every set of gluing data has order two, we see that the rigidified automorphism group $\text{Aut}_{\Sigma, \mathfrak{g}}(\Sigma')$ has order two for every set of gluing data $\mathfrak{g} = (\Theta_e)$.

4.2 Algebraic coverings and rigidified coverings of metrized complexes

We show in this section that the category $\text{Cov}_G(\Sigma)$ of rigidified coverings of $\Sigma$ is equivalent to the category of finite coverings of a punctured algebraic curve $(X, D)$. In Section 5, we will study the corresponding Galois category structure on this category and use this to define decompositions of the profinite fundamental group.

We first define the algebraic category of finite tame coverings of a single connected curve $(X, D)$. For the definition of a tame covering of algebraic curves, see Section 2.1.

Definition 4.4. Let $X$ be a smooth, proper connected curve with a finite set of punctures $D \subset X(K)$. The category $\text{Cov}(X, D)$ consists of

1. Finite tame coverings of $X$ that are étale outside $D$ as its objects and

2. Finite tame $X$-coverings $X_1 \to X_2$ as its morphisms.

Remark 4.1. We could also have defined the category by starting with the open subscheme $X \setminus D$ of $X$ and taking the category of finite tame étale coverings of $X \setminus D$. Since any such covering compactifies to a unique morphism $X' \to X$ of smooth curves, we easily obtain an equivalence of categories.

Now let $(X, D)$ be a punctured curve and consider the triangulated punctured curve $(X, V \cup D)$ induced by a fixed semistable vertex set. We denote its skeleton by $\Sigma(X)$ or $\Sigma(X, D)$. By Theorem 3.1, any finite tame covering of $(X, D)$ induces a finite tame covering of $(X, V \cup D)$ and thus of $\Sigma(X, D)$. This in turn induces a finite tame rigidified covering

$$\phi_{\mathfrak{g}} : \Sigma'(X', D') \to \Sigma(X, D).$$ (29)

Since this process is easily seen to be functorial, we can define the following

Definition 4.5. Let $(X, V \cup D)$ be a triangulated punctured curve associated to $(X, D)$ with skeleton $\Sigma(X) := \Sigma(X, D)$. We define the covering functor of $X$ with respect to $\Sigma$ to be the functor

$$\mathcal{F}_\Sigma : \text{Cov}(X, D) \to \text{Cov}_G(\Sigma(X))$$ (30)

assigning to every finite tame covering $X' \to X$ the canonical finite tame rigidified covering of metrized complexes $\Sigma'(X', D') \to \Sigma(X, D)$.

This covering functor allows us to pass from algebraic coverings to coverings of graphs in a natural way. We can ask ourselves if any information is lost in the procedure, for instance that two morphisms are considered to be the same in the graph-covering category or that two coverings are mapped to the same graph-theoretical covering. This is not the case, as we have an equivalence of categories.
**Theorem 4.1.** Let $F_{\Sigma}$ be the covering functor associated to a semistable vertex set $V$ of a punctured curve $(X, D)$ with skeleton $\Sigma(X) := \Sigma(X, V \cup D)$. Then $F_{\Sigma}$ induces an equivalence of categories
\[ \text{Cov}(X, D) \simeq \text{Cov}_G(\Sigma(X)). \] (31)

**Proof.** Let $\phi_g$ be a tame rigidified covering of $\Sigma(X, D)$. The gluing data then immediately gives rise to a finite covering of $(X, D)$, so we see that $F_{\Sigma}$ is essentially surjective. By Lemma 4.1, we see that $F_{\Sigma}$ is fully faithful. This finishes the proof. \[ \square \]

## 5 Profinite fundamental groups for metrized complexes

In this section, we will use the equivalence of categories in Theorem 4.1 and endow the category of rigidified coverings with the structure of a Galois category. Using this structure, we can define the *Galois closure* of a finite rigidified covering. Furthermore, we can use this structure to define normal subgroups of the tame fundamental group of a punctured curve, which we will call the decomposition group $I_{(\Sigma)}$. This ties in to the class field theory philosophy that the Jacobian classifies the unramified coverings of the curve and that decompositions of the Jacobian should give rise to a decomposition in the set of unramified coverings. The graph theory interpretation shows in what sense the corresponding coverings are different.

### 5.1 The Galois category structure on $\text{Cov}_G(\Sigma(X))$

For the rest of this chapter, we fix a semistable vertex $V$ with skeleton $\Sigma(X) := \Sigma(X, V \cup D)$ of a punctured curve $(X, D)$. By Theorem 4.1, the covering functor $F_{\Sigma}$ gives an equivalence of categories
\[ \text{Cov}(X, D) \simeq \text{Cov}_G(\Sigma(X)). \] (32)

Since $\text{Cov}(X, D)$ is a Galois category, we can give $\text{Cov}_G(\Sigma(X))$ the induced Galois category structure as in Definition 2.14. We will denote the fundamental functor of $\text{Cov}(X, D)$ by $F_X$.

**Definition 5.1.** Let $\text{Cov}_G(\Sigma(X))$ be the category of rigidified coverings of $\Sigma(X, D)$. We endow it with the Galois category structure induced by Theorem 4.1. We will denote its fundamental functor by $F_{\Sigma}$ and its automorphism group by $\pi(X, \Sigma)$. We will call this group the *profinite fundamental group* of the metrized complex $\Sigma(X, D)$.

By definition, we now have an isomorphism of profinite groups
\[ \pi(X, D) \simeq \pi(X, \Sigma). \] (33)

Let us state some facts about the interaction between the profinite group $\pi(X, \Sigma)$ and the category $\text{Cov}_G(\Sigma(X))$. All of these results directly follow from the theory of Galois categories.

**Proposition 1.** Let $\Sigma(X) := \Sigma(X, D)$ be a fixed metrized complex associated to the punctured curve $(X, D)$. Then the following are true.

1. There is an equivalence of categories
\[ \text{Cov}_G(\Sigma(X)) \to (\text{Finite } \pi(X, \Sigma)-\text{Sets}). \] (34)

2. The closed subgroups of finite index in $\pi(X, \Sigma)$ correspond bijectively to finite connected rigidified coverings $\Sigma' \to \Sigma$.

3. For every finite connected covering $\Sigma' \to \Sigma$, there is a unique minimal Galois object $\Sigma$ with Galois group $G$, finite morphisms $\Sigma \to \Sigma' \to \Sigma$ and a finite group $H \subseteq G$ such that $\Sigma/G = \Sigma$ and $\Sigma/H = \Sigma'$.

**Proof.** These follow directly from the theory of Galois categories. Note that for the last property, one can take the metrized complex arising from the Galois closure $\overline{X}$ of the corresponding connected algebraic morphism $X' \to X$. Note that there is also a purely categorical definition of the minimal Galois object, see [Cad13, Proposition 3.3]. \[ \square \]
Definition 5.2. (Galois closure) Let \( \phi : \Sigma' \to \Sigma \) be a finite connected rigidified covering of \( \Sigma = \Sigma(X,D) \). We define the object \( \Sigma \) from Proposition 1 to be the Galois closure of \( \phi \) and we define \( G \) to be its Galois group.

Example 5.1. (Example of a nontrivial Galois closure)
We will follow [Hel17, Section 7], where the details are given for some of the statements below. Let \( E \) be the elliptic curve given by the (slightly twisted) Weierstrass equation \( y^2 + x^3 + Ax + B = 0 \) and consider the 3 : 1 covering given by \( \phi : (x,y) \mapsto y \). For \( A \neq 0 \), this morphism is not Galois and we let \( \overline{E} \) be the Galois closure.

![Diagram](image)

Figure 3: The Galois closure of the covering of metrized complexes in Example 5.1 arising from the natural 3 : 1 mapping \( (x,y) \mapsto y \) for an elliptic curve in Weierstrass equation \( y^2 = x^3 + Ax + B \) with \( v(j) < 0 \).

We suppose now that \( v(j(E)) < 0 \), so that \( E \) has multiplicative reduction. Let \( S \) be a separating tree for \( \phi \). This has one nontrivial edge of length \( v(\Delta(E))/2 \), where \( \Delta(E) = 4A^3 + 27B^2 \) is the discriminant of \( E \). We let \( \Sigma' \) and \( \Sigma \) be the induced metrized complexes of \( E \) and \( \overline{E} \) respectively. It was shown in [Hel17, Theorem 4] that this induced Galois covering of metrized complexes \( \Sigma \to \Sigma' \to \Sigma \) is given as in Figure 3. In other words, \( \Sigma \to \Sigma \) is the Galois closure of \( \Sigma' \to \Sigma \) (as metrized complexes) and \( S_3 \) is its Galois group.

As an exercise, the reader might want to construct an action of \( S_3 \) on the top-left graph such that the other three emerge as quotients under this action.

We can now also think about obtaining Galois closures abstractly from metrized complexes only. That is to say, suppose we start with a tame rigidified covering \( \Sigma' \to \Sigma \) of metrized complexes. What is the Galois closure? The author does not know of a direct way of finding this closure other than obtaining it by algebraic/analytic methods, which is by no means algorithmic. This could be interesting research for the future.

Remark 5.1. (Function fields and algebraic coverings) Every algebraic covering \( X' \to X \) of smooth proper connected curves comes with a natural extension of function fields \( K(X) \to K(X') \) and this correspondence is a bijection by [Har77, Chapter 1, Corollary 6.12]. The fundamental group \( \pi(X,D) \) is then the Galois group of the composite of all field extensions that are unramified outside of \( D \), see Example 2.3. For every \( \sigma \in \pi(X,D) \) and connected covering \( \phi : X' \to X \), we then have an injection of function fields \( K(X) \to \sigma(K(X')) \), corresponding to a covering of curves \( \sigma(\phi) \) which we will denote by \( \sigma(X') \to X \). We will call this the action of \( \pi(X,D) \) on connected coverings of \( X \).

Using Theorem 4.1, we now transfer the action of \( \pi(X,D) \) on connected objects of \( \text{Cov}(X,D) \) to an action on connected objects of \( \text{Cov}_G(\Sigma(X)) \).

Definition 5.3. (Action of \( \pi(X,\Sigma) \) on connected coverings) Let \( \phi : \Sigma' \to \Sigma \) be a connected rigidified covering of metrized complexes and let \( \hat{\phi} : X' \to X \) be the corresponding algebraic covering. We define \( \sigma(\phi) : \Sigma(\Sigma') \to \Sigma \) to be the image of \( \sigma(\hat{\phi}) \) under the covering functor \( F \) of Definition 4.5.

Now let \( S \) be a set of connected coverings of \( (X,D) \) and consider the corresponding closed subgroup

\[
H_S = \{ \sigma \in \pi(X,D) : \sigma(\phi) = \phi \text{ for all } \phi \in S \}.
\]
If we denote the function field corresponding to $\phi$ by $K(X)^\phi$, then the fixed field of $H_S$ is the composite of all these fields $K(X)^\phi$. We would now like to study this subgroup $H_S$ for two special collections of connected coverings: the ones that are unramified or totally split above a subcomplex $\Sigma^0 \subseteq \Sigma(X, D)$. Let us first define what we mean by unramified and totally split.

**Definition 4.5.** *(Unramified and totally split coverings)* Let $\phi: \Sigma' \to \Sigma$ be a tame rigidified covering of metrized complexes and let $\Sigma^0 \subseteq \Sigma$ be a subcomplex. We say that an algebraic covering $\Sigma \phi$ is unramified over $\Sigma^0$ if for every edge $e \in E(\Sigma')$ mapping to $e \in E(\Sigma^0)$, we have that $d_{\Sigma'/\Sigma}(\phi) = 1$.

1. $\phi$ is unramified above $\Sigma^0$ if for every edge $e' \in E(\Sigma')$ mapping to $e \in E(\Sigma^0)$, we have that $d_{\Sigma'/\Sigma}(\phi) = 1$,

2. $\phi$ is totally split above $\Sigma^0$ if $\phi$ is unramified above $\Sigma^0$ and for every vertex $v \in V(\Sigma^0)$, we have that there are $\deg(\phi)$ vertices $v' \in V(\Sigma')$ such that $\phi(v') = v$.

We say that an algebraic covering $\phi: (X', D') \to (X, D)$ is unramified or totally split above a subcomplex $\Sigma^0 \subseteq \Sigma(X, D)$ if $\mathcal{F}_{\Sigma}(\phi)$ is unramified or totally split above $\Sigma^0$. Here $\mathcal{F}_{\Sigma}$ is the covering functor from Definition 4.5.

**Remark 5.2.** In terms of algebraic topology, we see that $\phi$ is totally split above $\Sigma^0$ if and only if $\phi^{-1}(\Sigma^0)$ is a covering space of $\Sigma^0$ in the sense of [Hat01, Section 1.3].

**Proposition 2.** Let $S_{\Sigma^0}$ be the set of all finite connected coverings of $(X, D)$ that are either unramified or totally split above a subcomplex $\Sigma^0 \subseteq \Sigma(X, D)$. Then $H_{S_{\Sigma^0}}$ is a normal subgroup of $\pi(X, D) = \pi(X, \Sigma)$.

**Proof.** The proof is similar to the one found in class field theory for the normality of the Hilbert class field. Let $\phi: X' \to X$ be a finite algebraic covering such that $F(\phi)$ is unramified or splits completely above $\Sigma^0$ and let $L$ be the subfield of $K(X)$ corresponding to the subgroup $H_{S_{\Sigma^0}}$. It then suffices to prove that for every $\sigma \in \pi(X, D)$, we have that $\sigma(K(X')) \subseteq L$. Note that the covering $\sigma(X') \to X$ satisfies the same conditions with respect to the edges and vertices as the covering $X' \to X$. Here one uses the fact that $\sigma$ defines isomorphisms between the local rings of $X'$ and $\sigma(X')$, where $X'$ is the canonical semistable model associated to the covering $\phi$. Since $L$ is the composite of these extensions, we find that $\sigma(K(X')) \subseteq L$, as desired.

**Definition 5.5.** *(Decomposition and inertia groups of a subcomplex)* Let $\Sigma^0 \subseteq \Sigma$ be a subcomplex and let $S_{\Sigma^0, 3}$ and $S_{\Sigma^0, 2}$ denote the coverings that are unramified and totally split above $\Sigma^0$, respectively. We denote the corresponding normal closed subgroups by $\mathcal{D}(\Sigma^0)$ and $\mathcal{D}(\Sigma^0)$. They are the (absolute) inertia and decomposition groups of $\Sigma^0$ in $\pi(X, D)$. We furthermore define $\pi_3(\Sigma^0(X, D)) := \pi(X, D)/\mathcal{D}(\Sigma^0)$ and $\pi_2(\Sigma^0(X, D)) := \pi(X, D)/\mathcal{D}(\Sigma^0)$. If $\Sigma^0 = \Sigma$ and $D = \emptyset$, we write $\pi_3(\Sigma(X))$ and $\pi_2(\Sigma(X))$ for the corresponding quotients.

**Remark 5.3.** We will see in Section 5.2 that there is a natural connection between the cyclic abelian extensions coming from $\pi_3(\Sigma(X))$ and $\pi_2(\Sigma(X))$ and the cyclic abelian extensions coming from the toric and connected parts of the analytic Jacobian of $X$. It is in this sense that we think of the groups $\pi_3(\Sigma(X))$ and $\pi_2(\Sigma(X))$ as natural non-abelian generalizations of the extensions coming from the toric and connected parts of the Jacobian in the tame case.

We will now connect the group $\pi_2(\Sigma(X))$ to the profinite completion of the ordinary fundamental group of the graph underlying $\Sigma$. Let $\Gamma$ be the finite connected graph underlying $\Sigma$. We then denote the category of finite coverings by $\text{Cov}(\Gamma)$, its profinite fundamental group by $\hat{\pi}(\Gamma)$ and its ordinary fundamental group by $\pi(\Gamma)$ (this is the only time we will use this notation for a nonprofinite group).

**Lemma 5.1.** Consider the full subcategory $\text{Cov}_2(X)$ of $\text{Cov}(X)$ consisting of all finite étale coverings of $X$ that split completely over $\Sigma$ in the sense of Definition 4.5. Then $\text{Cov}_2(X)$ is a Galois category with profinite fundamental group equal to $\pi_2(\Sigma(X))$ as defined in Definition 5.5.

**Proof.** The fundamental functor $\mathcal{F}_2$ of $\text{Cov}_2(X)$ is obtained from the following diagram:

\begin{equation}
\text{Cov}_2(X) \to \text{Cov}(X) \to \text{Sets},
\end{equation}

where the first arrow is the inclusion functor. We check the conditions as in Remark 2.2. As a general remark, note that a covering is totally split if and only if each of its connected components is totally split over $X$. This allows us to reduce many of the conditions to their counterparts for connected objects.

1. The final object is easily seen to be $X$. Let us show that finite fiber products exist in $\mathcal{C}$. By the above comment it suffices to consider connected objects. Then for finitely many connected objects $X_i$ with corresponding subgroups $H_i$, we have that the connected components of the coproduct correspond to the subgroup $\bigcap H_i$. Since the $H_i$ all contain $H_S$ (with $S$ the set of all connected totally split coverings, see
Equation 35), we must have that $\bigcap H_i$ also contains $H_S$, as desired. (2) By the remark above on connected objects, we see that finite coproducts exist in $\text{Cov}_2(X)$. Suppose now that $X' \to X$ is a totally split covering of $X$ with a finite group of automorphisms acting on it. Then the quotient $X'/G \to X$ is again totally split over $X$, since $G$ acts freely on the vertices and edges of the corresponding metrized complex $\Sigma(X')$. This shows that the quotient exists in the category of totally split coverings. (3) Any monomorphism is automatically totally split, so we see that $X' \to X''$ is totally split if and only if the epimorphism in the decomposition as in Remark 2.2 is totally split. We thus see that (3) holds.

For the remaining conditions (4)-(6), we argue as follows. Since the inclusion functor $i : \text{Cov}_2(X) \to \text{Cov}(X)$ turns $\text{Cov}_2(X)$ into a full subcategory of $\text{Cov}(X)$, we find that the conditions on $F_2$ in (4)–(6) hold because they hold for $F$. We conclude that $\text{Cov}_2(X)$ is a Galois category.

The corresponding profinite fundamental group $\pi_2$ of the Galois category $C_2 := \text{Cov}_2(X)$ is equal to the inverse limit of the automorphism groups of (connected) Galois objects, which we can easily describe:

$$\pi_2 = \lim_{X' \to X \in C_2} \text{Aut}(X'/X) = \lim_{H \supseteq \mathcal{D}(\Sigma)} \pi(X)/H = \lim_{H \supseteq \mathcal{D}(\Sigma)} (\pi(X)/\mathcal{D}(\Sigma))/(H/\mathcal{D}(\Sigma)).$$

(37)

Since $\pi(X)/\mathcal{D}(\Sigma) =: \pi_2(\Sigma(X))$, we obtain the desired result.

We now have the following

**Proposition 3.** Let $\text{Cov}_2(X)$ be as above and consider the composite $\mathcal{H}$ of the covering functor $F_2$ (see Definition 4.5 and Lemma 5.1) and the forgetful functor $\text{Cov}_2(\Sigma) \to \text{Cov}(\Gamma)$. Then $\mathcal{H}$ induces an equivalence of categories

$$\text{Cov}_2(X) \to \text{Cov}(\Gamma).$$

(38)

**Proof.** Let $\Gamma' \to \Gamma$ be a finite covering of finite graphs. By assigning the same length function on $\Gamma'$ as on $\Gamma$ (induced by $\Sigma$) and by assigning to every vertex of $\Gamma'$ a projective line with the right identifications of the edges, we easily obtain a finite tame covering of metrized complexes $\Sigma' \to \Sigma$. There is nontrivial gluing data (indeed, $\text{Aut}_{\Sigma}(e_0)(\pi^{-1}_{\Sigma}(e_0)) = (1)$ for every finite vertex $x'$ with image $x$ and adjacent edges $e'$ and $e$), so this also gives a canonical rigidified morphism. By lifting this morphism to a finite tame covering of metrized complexes, we easily see that $\text{Cov}_2(X)$ is a Galois category.

The corresponding profinite fundamental group $\pi_2$ of the Galois category $C_2 := \text{Cov}_2(X)$ is equal to the inverse limit of the automorphism groups of (connected) Galois objects, which we can easily describe:

$$\pi_2 = \lim_{X' \to X \in C_2} \text{Aut}(X'/X) = \lim_{H \supseteq \mathcal{D}(\Sigma)} \pi(X)/H = \lim_{H \supseteq \mathcal{D}(\Sigma)} (\pi(X)/\mathcal{D}(\Sigma))/(H/\mathcal{D}(\Sigma)).$$

(37)

Since $\pi(X)/\mathcal{D}(\Sigma) =: \pi_2(\Sigma(X))$, we obtain the desired result.

We now have the following

**Proposition 3.** Let $\text{Cov}_2(X)$ be as above and consider the composite $\mathcal{H}$ of the covering functor $F_2$ (see Definition 4.5 and Lemma 5.1) and the forgetful functor $\text{Cov}_2(\Sigma) \to \text{Cov}(\Gamma)$. Then $\mathcal{H}$ induces an equivalence of categories

$$\text{Cov}_2(X) \to \text{Cov}(\Gamma).$$

(38)

**Proof.** Let $\Gamma' \to \Gamma$ be a finite covering of finite graphs. By assigning the same length function on $\Gamma'$ as on $\Gamma$ (induced by $\Sigma$) and by assigning to every vertex of $\Gamma'$ a projective line with the right identifications of the edges, we easily obtain a finite tame covering of metrized complexes $\Sigma' \to \Sigma$. There is nontrivial gluing data (indeed, $\text{Aut}_{\Sigma}(e_0)(\pi^{-1}_{\Sigma}(e_0)) = (1)$ for every finite vertex $x'$ with image $x$ and adjacent edges $e'$ and $e$), so this also gives a canonical rigidified morphism. By lifting this morphism to a finite tame covering of metrized complexes, we easily see using Lemma 4.1 (or Theorem 4.1) that $\mathcal{H}$ is also fully faithful, yielding the desired equivalence of categories.

**Theorem 5.1.** Let $\mathcal{D}(\Sigma)$ be the decomposition group of $\Sigma$ in $\pi(X)$. Then $\pi_2(\Sigma(X)) := \pi(X)/\mathcal{D}(\Sigma)$ is isomorphic to the profinite completion of the ordinary fundamental group of the underlying graph $\Gamma$ of a metrized complex $\Sigma$ corresponding to $X$.

**Proof.** By [God97, Chapitre IX, numéro 6; 19, Chapter V] or a modification of [Hat01, Theorem 1.38], we have that the category of $\pi(\Gamma)$-sets is equivalent to the category of coverings of $\Gamma$. The profinite completion of $\pi(\Gamma)$ is then canonically equivalent to the category of finite coverings of $\Gamma$. By Proposition 3, this last category is equivalent to the category of finite $\pi_2(\Sigma(X))$-sets, which immediately yields the desired isomorphism by [Sta18, Tag 0BN5].

5.2 Decompositions of the abelianization of $\pi(X)$

In this section, we will study the abelianizations of the profinite groups $\pi(X), \pi_3(\Sigma(X))$ and $\pi_2(\Sigma(X))$ introduced in Definition 5.5. We will see that they are linked to the Jacobian of $X$, giving a decomposition akin to the one obtained using the Néron model $J$ of $X$ over a discretely valued field:

$$\mathcal{I}_0^0 \subseteq \mathcal{I}_0 \subseteq J$$

(39)

In a sense, the decomposition introduced in Definition 5.5 can be seen as a generalization of this decomposition for nonabelian unramified coverings. We will mostly follow [BR14, Sections 4-7] for the Berkovich approach to the analytification of Jacobians, see [BL84], [Bos14], [Sch16] and [FvdP04] for the rigid analytic approach.

Let $X$ be a smooth, proper, geometrically connected curve over $K$ and let $J := J(X)$ be its Jacobian. We will recall some facts from [BR14, Section 4, 5 and 6]. Let $X^{an}$ and $J^{an}$ be the $K$-analytic spaces associated to $X$ and $J$ respectively. Let $X'$ be a semistable model of $X$ with associated skeleton $\Gamma \subset X^{an}$ and let $\tau : X^{an} \to \Gamma$ be the retraction map. We let $\text{Jac}(\Gamma)$ be the Tropical Jacobian associated to $\Gamma$ as in Section 2.5. By [BR14, Proposition 5.3], there is a unique surjective homomorphism

$$\tau : J^{an} \to \text{Jac}(\Gamma)$$

(40)
that commutes with the maps induced by the retraction map \( \tau \) on the divisor groups. We furthermore have that its kernel is an analytic domain \( J^0 \), see [BR14, Corollary 6.8]. This analytic group \( J^0 \) is the analogue of the connected component of the identity \( J^0 \) of the Néron model of the Jacobian in the discretely valued case. \( J^0 \) then admits a decomposition

\[
(1) \to T^0 \to J^0 \to B^{an} \to (1)
\]

where \( T \) is a torus, \( T^0 = \text{trop}^{-1}(0) \) for the tropicalization map \( T^{an} \to N_\mathbb{R} \) and \( B \) is an abelian variety with good reduction. Note that if \( T = (G_{m,k})^n \), then the reduction \( \overline{T} \) of \( T^0 \) is isomorphic to \( (G_{m,k})^n \).

By [BL84, Theorem 5.1.c] and [BR14, Section 7.2], the reduction of \( J^0 \) is equal to the Jacobian of the special fiber \( \mathcal{X} \), which gives the exact sequence

\[
(1) \to \mathcal{T} \to \overline{J^0} \to \prod_{i=1}^n \text{Jac}(\Gamma_i) \to (1),
\]

where the \( \Gamma_i \) are the irreducible components of \( \mathcal{X} \) and \( \mathcal{T} \simeq (k^*)^t \) where \( t \) is the toric rank of \( J^0 \). This toric rank \( t \) is equal to the first Betti number of the intersection graph of \( \mathcal{X} \) by [Liu06, Chapter 7, Lemma 5.18]. We furthermore write \( a = \sum_{i=1}^n g(C_i) \) for the abelian rank of \( J^0 \). We then have the following result on the \( n \)-th torsion subgroup of \( J^0 \):

**Proposition 4.** Let \( n \) be any integer that is coprime to the characteristic of the residue field \( k \). Then

\[
T^{an}[n] = (\mathbb{Z}/n\mathbb{Z})^t,
\]

\[
B^{an}[n] = (\mathbb{Z}/n\mathbb{Z})^{2a},
\]

\[
J^0[n] = (\mathbb{Z}/n\mathbb{Z})^{t+2a}.
\]

**Proof.** This follows from [Liu06, Chapter 7, Corollary 4.41] and the exact sequence in Equation (42). \( \square \)

**Corollary 5.1.** Let \( \ell \) be a prime not equal to the characteristic of the residue field \( k \). On the level of Tate modules (see Equation (6)), we then have

\[
T_\ell(T^{an}) = (\mathbb{Z}_\ell)^t
\]

\[
T_\ell(J^0) = (\mathbb{Z}_\ell)^{t+2a}.
\]

We can now characterize the torsion points in this analytic domain \( J^0 \) using the groups \( \pi_D(\Sigma(X)) \) and \( \pi_3(\Sigma(X)) \). We first note that we have a canonical isomorphism

\[
J[n] = \text{Hom}(\pi(X), \mathbb{Z}/n\mathbb{Z})
\]

by [Mil08, Chapter III, Lemma 9.2]. That is, there is a natural bijection between cyclic unramified coverings of \( X \) and torsion points of \( J \). To be more explicit, let \( D \in J[n] \) and suppose that \( D \) has order \( n \). Then \( nD = \text{div}(f) \) for some \( f \in K(X) \) and we consider the covering on the level of function fields defined by

\[
z^n = f.
\]

The condition on the order of \( D \) ensures that this equation is irreducible and by local considerations, this covering is completely unramified. Conversely, every cyclic covering of \( X \) is given on the level of function fields by a covering of the form \( z^n = f \) by Kummer theory. In order for this covering to be unramified everywhere, the valuation of \( f \) at every point must be divisible by \( n \). This then quickly gives an \( n \)-torsion point in \( J \).

**Theorem 5.2.** Let \( \mathcal{D}(\Sigma(X)) \) and \( \mathcal{I}(\Sigma(X)) \) be the decomposition and inertia group of \( \Sigma \) in \( \pi(X) \) and let \( \pi_D(\Sigma(X)) \) and \( \pi_3(\Sigma(X)) \) be their corresponding quotients in \( \pi(X) \). Let \( n \) be an integer such that \( \gcd(n, \text{char}(k)) = 1 \). Then

\[
T^{an}[n] = \text{Hom}(\pi_D(\Sigma(X)), \mathbb{Z}/n\mathbb{Z})
\]

and

\[
J^0[n] = \text{Hom}(\pi_3(\Sigma(X)), \mathbb{Z}/n\mathbb{Z}).
\]

**Proof.** The proof is similar to the one in [Hel18, Proposition 8.2.1 and Proposition 8.3.1], where it is given in the discretely valued case. The idea is to study the Kummer coverings using the Slope formula (see [BPR14, Theorem 5.15]), which shows whether an edge ramifies or not in the induced extension. The splitting of the
vertices is then an easy condition on the level of function fields of the irreducible components of a semistable model $X$ of $X$.

Let $D \in J^0[n]$. Then $\tilde{\nu}(D) = 0$. Let $f \in K(C)$ be such that $\div(f) = nD$. Let $\phi$ be any piecewise-linear function on $\Sigma$ such that $\Delta(\phi_f) = \tau_n(\div(f))$ and let $\phi_f$ be a piecewise-linear function such that $\tau_n(D) = \phi_f$. Then $n \cdot \phi_f = \phi_f$ and thus the slope of $\phi_f$ on every edge is divisible by $n$. Let $v$ be a vertex in $\Sigma$ and scale $f$ such that $|f|_v = 1$. By the non-archimedean Slope formula (see [BPR14, Theorem 5.15]), we find that the order of the reduction of this scaled $f$ at the closed point in $C_v$ corresponding to an edge $e$ is divisible by $n$. By [BPR16, Definition 2.19, Condition 1], this implies that $\nu_e(\psi) = 1$ for every $e'$ lying above $e$, showing that $\psi$ is unramified above $e$.

Let $\psi : X' \to X$ be a cyclic unramified covering of degree $n$ and let $D$ be a divisor giving rise to the covering. That is, we have $nD = \div(f)$ for an $f \in K(X)$ and the covering is given by the extension of function fields induced by $z^n = f$. Suppose that the corresponding covering of metrized complexes is unramified above every edge $e \in \Sigma$. The Slope formula then again shows that the slope of any piecewise-linear function $\phi_f$ such that $\Delta(\phi_f) = \tilde{\nu}(\div(f))$ is divisible by $n$. We can thus write $\phi_f = n \cdot \phi'$. We then easily find that $\tau_n(D) = \phi'$ and thus $D \in J^0[n]$, as desired.

Suppose now that $D \in T^0[n]$. We have to check that the induced covering of metrized complexes splits completely. Since $T^0[n] \subseteq J^0[n]$, we already know that the covering splits on the edges. We thus only have to show that the covering splits on the vertices. Again, let $f$ be such that $nD = \div(f)$. By the exact sequence in Equation (42), we know that the reduction of $D$ at every component is principal, that is $\text{red}(D, \Gamma_i) = (f_i)$ for an $f_i \in k(\Gamma_i)$. Furthermore, we have that $f^n = \tilde{f}$ by the condition $nD = \div(f)$. It now suffices to show that for every component $\Gamma_i$ lying above $\Gamma_i$, the extension of function fields $k(\Gamma_{i,j}^n) \supset k(\Gamma_i)$ is an isomorphism. Note that the components above $\Gamma_i$ are given as the maximal ideals of the algebra

$$k(\Gamma_i)[z]/(z^n - \tilde{f}).$$

But in this algebra we have $\tilde{f} = f^n$, so the maximal ideals are given by $m_j = (z - \zeta_j f_i)$ for $j \in \{0, 1, ..., n-1\}$. Here $\zeta_n$ is a primitive $n$-th root of unity. We thus obtain the desired result.

Conversely, suppose that the extension induced by $D$ splits above every vertex. Then the spectrum of the algebra in Equation (50) consists of $n$ different maximal ideals and we obtain that $\tilde{f}$ is an $n$-th power. That is, for every component $\Gamma_i$, we can write $\tilde{f} = f^n$ for some $f_i \in k(\Gamma_i)$. We then see that $\text{red}(D, \Gamma_i) = (f_i)$ and thus the image of $D$ in $\text{Jac}(X_t)$ maps to zero in the Jacobian of every component $\Gamma_i$. In other words, $D \in J^0[n]$, as desired.

We now let consider the Tate modules of these varieties: $T_\ell(T^0)$ and $T_\ell(J^0)$. Using Theorem 5.2, we now easily obtain the following Corollary:

**Corollary 5.2.** Let $\pi_\Sigma(\Sigma(X))$ and $\pi_\Sigma(\Sigma(X))$ be the decomposition and inertia quotients of $\pi(X)$ and let $\ell$ be a prime not equal to the characteristic of the residue field $k$. Then

$$T_\ell(T^0) = \text{Hom}(\pi_\Sigma(\Sigma(X)), \mathbb{Z}_\ell)$$

$$T_\ell(J^0) = \text{Hom}(\pi_\Sigma(\Sigma(X)), \mathbb{Z}_\ell).$$

**Example 5.2.** Let $E$ be an elliptic curve with semistable reduction over $K$ and let $E \simeq \mathbb{G}_{m,K}/\langle q \rangle$ be the corresponding Tate uniformization. We will assume that $\text{char}(k) \neq 2$. An algebraic example to keep in mind here is $y^2 = x(x - \omega)(x + 1)$ for some $\omega \in K^*$ with $v(\omega) > 0$, which was also studied in Example 3.2.

The two-torsion subgroup $E[2] = \{P \in E(K) : [2]P = 0\}$ consists of four points, three of which are nontrivial points of order $2$. We thus obtain three nontrivial unramified extensions of degree $2$. In our algebraic example, these four points are given by $(0,0)$, $(\pi,0)$, $(-1,0)$ and $\infty$ and the three extensions are given by $y^2 = f$, for $f = x, x - \omega, x + 1$. The analytic points are given by $q^{1/2}, -q^{1/2}, -1$ and $1$ respectively. The decomposition in Theorem 5.2 then tells us that there is one $2$-torsion point that gives rise to a totally split extension. We easily see that this is the point $(-1,0)$. In fact, we can write down a semistable model using $xt = \omega$:

$$\left(\frac{y}{x}\right)^2 = (1 - t)(x + 1).$$

(51)

We then see that the reduction of $x + 1$ in both components corresponding to the prime ideals $p_1 = (x)$ and $p_2 = (t)$ of the algebra $R[x,t]/(xt - \omega)$ is a square. We then also see that the corresponding extension is totally split. Note that the corresponding analytic map is given by the natural map

$$\mathbb{G}_{m,K}/\langle q^2 \rangle \to \mathbb{G}_{m,K}/\langle q \rangle.$$  

(52)

The other two torsion points map to the nontrivial element of order two in the Tropical Jacobian. Explicitly, the corresponding two torsion point there is $D = \Gamma_1 - \Gamma_0$, where $\Gamma_1$ is the irreducible component
corresponding to \((x)\) and \(\Gamma_0\) to \((t)\). We then see that their extensions are as in Figure 4, where the corresponding fiber product extension corresponding to \(E[2]\) is also depicted.

Note that we see the necessity of considering rigidified coverings here: the two coverings of metrized complexes coming from \(\text{Jac}(\Gamma)[2] = (E_{\text{an}}/E_0)[2]\) are the same without gluing data, even though they come from nonisomorphic curves. Indeed, the \(q\)-expansions of these curves are different, so they cannot be isomorphic.

**Example 5.3.** Consider a genus two curve \(X\) with skeleton as in Figure 5. To be more explicit, we can take the curve defined by the equation

\[
y^2 = x(x - \wp)f(x),
\]

where \(f(x)\) is any polynomial in \(R[x]\) such that \(\overline{f}(x)\) has no multiple roots and no root at \(x = 0\). To see how to obtain the skeleton for these superelliptic curves, we refer the reader to [BH17].

We will denote the nontrivial component of genus one by \(\tilde{E}\). Let \(\ell\) be any prime not equal to the characteristic of \(k\). Then the Tate module has rank 4 over \(\mathbb{Z}_\ell\):

\[
T_\ell(J) = (\mathbb{Z}_\ell)^4.
\]

By Corollary 5.1, we have that

\[
T_\ell(J^0) = \mathbb{Z}_\ell
\]

\[
T_\ell(T_{\text{an}}) = (\mathbb{Z}_\ell)^3.
\]

By Corollary 5.2, the coverings coming from \(T_\ell(T_{\text{an}})\) correspond to the topological coverings of \(\Sigma\) of degree \(\ell^n\). The coverings in the quotient \(T_\ell(J^0)/T_\ell(T_{\text{an}})\) correspond to the coverings of degree \(\ell^n\) such that not all vertices split. The only choice for this vertex is the one with genus 1 and we thus obtain the identification

\[
T_\ell(J^0)/T_\ell(T_{\text{an}}) = \text{Hom}(\pi(\tilde{E}), \mathbb{Z}_\ell) = T_\ell(\tilde{E}).
\]
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