Bosonic realization of algebras in the Calogero model

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Abstract

We study an $N$-body Calogero model in the $S_N$-symmetric subspace of the positive definite Fock space. We construct a new algebra of $S_N$-symmetric operators represented on the symmetric Fock space, and find a natural orthogonal basis by mapping the algebra onto the Heisenberg algebra. Our main result is the bosonic realization of nonlinear dynamical symmetry algebra describing the structure of degenerate levels of the Calogero model.

The Calogero model [1] describes a system of $N$ bosonic particles on a line interacting through the inverse square and harmonic potential. It is completely integrable in both the classical and quantum case, the spectrum is known and the wave functions are given implicitly. The model is connected with a host of physical problems, ranging from condensed matter physics to gravity and string theory, so there is considerable interest in finding the basis set of orthonormal eigenfunctions, and the structure of the dynamical algebra that characterizes the eigenstates of the system.

Investigations of the algebraic properties of the Calogero model in terms of the $S_N$-extended Heisenberg algebra [2] defined a basic algebraic setup for further research. For the Calogero model the orthonormal eigenfunctions in terms of (Hi-)Jack polynomials [3] were constructed [4]. The authors of Ref. [5] showed that the dynamical symmetry algebra of the two-body Calogero model was a polynomial generalization of the $SU(2)$ algebra. The three-body problem was also treated [6], and the algebra of polynomial type and the action of its generators on the orthonormal basis were obtained. It was shown that in the two-body case the polynomial $SU(2)$ algebra could be linearized, but an attempt to generalize this result to the $N$-body case led to $(N-1)$ linear $SU(2)$ subalgebras that operated only on subsets of the degenerate eigenspace [7]. The general construction of the dynamical symmetry algebra was given in Ref. [8]. It was shown that algebra is intrinsically polynomial, and examples for lower $N$ were provided, showing how everything could be computed in detail. However it would be interesting to determine the proper elements labelling the energy eigenstates and find corresponding ladder operators, for any $N$.

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In this letter we define the physical $S_N$-symmetric Fock space in terms of the minimal number of $S_N$-symmetric operators, and find the minimal set of relations which define the corresponding algebra $A_N$ of operators $\{A_k, A_k^\dagger\}$. We construct a mapping from ordinary Bose oscillators satisfying the Heisenberg algebra to operators $\{A_k, A_k^\dagger\}$, and vice versa, for $\nu > -1/N$. This mapping induces a natural orthogonal basis in $S_N$-symmetric Fock space, labelling the energy eigenstates in terms of free oscillators quantum numbers. Finally, we construct a symmetry algebra describing the structure of degenerate eigenspace, in terms of ladder operators build out of bosonic oscillators.

The Calogero model is defined by the following Hamiltonian:

$$H = -\frac{1}{2} \sum_{i=1}^{N} \partial_i^2 + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + \frac{\nu(\nu - 1)}{2} \sum_{i \neq j}^{N} \frac{1}{(x_i - x_j)^2}. \quad (1)$$

For simplicity, we have set $\hbar$, the mass of particles and the frequency of harmonic oscillators equal to one. The dimensionless constant $\nu$ is the coupling constant (and/or the statistical parameter) and $N$ is the number of particles. The ground-state wave function is, up to normalization,

$$\psi_0(x_1, \ldots, x_N) = \theta(x_1, \ldots, x_N) \exp\left(-\frac{1}{2} \sum_{i=1}^{N} x_i^2\right), \quad (2)$$

where

$$\theta(x_1, \ldots, x_N) = \prod_{i<j}^{N} |x_i - x_j|^\nu, \quad (3)$$

with the ground-state energy $E_0 = N[1 + (N - 1)\nu]/2$.

Let us introduce the following analogs of creation and annihilation operators $[2]$:

$$a_i^\dagger = \frac{1}{\sqrt{2}} \left(-\partial_i - \nu \sum_{j, j \neq i}^{N} \frac{1}{x_i - x_j} (1 - K_{ij}) + x_i\right),$$

$$a_i = \frac{1}{\sqrt{2}} \left(\partial_i + \nu \sum_{j, j \neq i}^{N} \frac{1}{x_i - x_j} (1 - K_{ij}) + x_i\right). \quad (4)$$

The elementary generators $K_{ij}$ of symmetry group $S_N$ exchange labels $i$ and $j$:

$$K_{ij}x_j = x_i K_{ij}, \quad K_{ij} = K_{ji}, \quad (K_{ij})^2 = 1,$$

$$K_{ij}K_{jl} = K_{jl}K_{ij} = K_{il}K_{ij}, \quad \text{for } i \neq j, i \neq l, j \neq l, \quad (5)$$

and we set $a_i|0\rangle = 0$ and $K_{ij}|0\rangle = |0\rangle$. One can easily check that the commutators of creation and annihilation operators $[4]$ are

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \left(1 + \nu \sum_{k=1}^{N} K_{ik}\right) \delta_{ij} - \nu K_{ij}. \quad (6)$$

After performing a similarity transformation on the Hamiltonian $[1]$, we obtain the reduced Hamiltonian

2
acting on the space of symmetric functions. We restrict the Fock space \( \{a_1^{\dagger n_1} \cdots a_N^{\dagger n_N} | 0 \} \) to the \( S_N \)-symmetric subspace \( F_{\text{symm}} \), where \( \mathcal{N} = \sum_{i=1}^{N} a_i^{\dagger} a_i \) acts as the total number operator. In the following we demand that all states have positive norm, i.e., \( \nu > -1/N \). Next, we introduce the collective \( S_N \)-symmetric operators \( I \).

The operator \( B_1 = \sum a_i \) represents the center-of-mass operator (up to a constant) and we can remove it from the further considerations, since it commutes with all \( A_i^{\dagger} \) operators. The symmetric Fock space \( F_{\text{symm}} \) is \( \{B_1^{\dagger n_1} A_2^{\dagger n_2} \cdots A_N^{\dagger n_N} | 0 \} \) and after removing \( B_1 \) it reduces to the Fock subspace \( \{A_2^{\dagger n_2} \cdots A_N^{\dagger n_N} | 0 \} \). The total number operator on \( F_{\text{symm}} \) splits into

\[
\mathcal{N} = \frac{1}{N} B_1^{\dagger} B_1 + \mathcal{N}^{\dagger}, \quad \mathcal{N} = \sum_{i=1}^{N} a_i^{\dagger} a_i,
\]

\[
\mathcal{N}^{\dagger} = \mathcal{N} \equiv \sum_{k=2}^{N} k \mathcal{N}_k.
\]

Note that \( \mathcal{N}_k \) are the number operators of \( A_k^{\dagger} \) but not of \( A_k \), \( \mathcal{N}_k^{\dagger} \neq \mathcal{N}_k \).

Let us discuss the \( \mathcal{A}_N \) algebra of the collective \( S_N \)-invariant operators \( A_k \) defined in Eq.\( (8) \) and acting on \( F_{\text{symm}} \), with additions of \( A_0 = N \cdot 1 \), \( A_1 = 0 \). It is obvious that

\[
[A_m, A_n] = [A_m^{\dagger}, A_n^{\dagger}] = 0, \quad \forall m, n,
\]

\[
[A_i, [A_j, X^{\dagger}]] = [A_j, [A_i, X^{\dagger}]], \quad \forall i, j, \quad \text{any } X^{\dagger}.
\]

The second relation in \( (10) \) is a consequence of the Jacobi identity. The commutator

\[
[A_m, A_n^{\dagger}] = \sum_{k=1}^{\min(m,n)} c_k(m,n) \left( \prod \pi_i^{\dagger} \right)^{n-k} \left( \prod \pi_i \right)^{m-k}
\]

is an \( S_N \)-symmetric, normally-ordered operator, where \( c_1 \sim mn \) is different from zero and does not depend on \( \nu \). The symbolical expression \( (\prod \mathcal{O})^k \) denotes a product of operators \( \mathcal{O}_i \) of the total order \( k \) in \( \pi_i^{\dagger}(\pi_i) \). Hence, the structure of the \( \mathcal{A}_N \) algebra is of the following type:

\[
[A_{i_1}, [A_{i_2}, \ldots, A_{i_j}^{\dagger}]] \cdots] = \sum (\prod A)^{I-j},
\]

where \( I = \sum_{a=1}^{j} i_a \geq 2j \). Generally, \( j \) successive commutators of \( A_{i_1}, \ldots, A_{i_j} \) with \( A_{i_j}^{\dagger} \), form a homogeneous polynomial \( \sum (\prod A)^{I-j} \) in \( a_i \) of order \( I - j \) with coefficients independent of \( \nu \). The term \( A_{I-j} \) on the r.h.s. of Eq.\( (12) \) appears with the coefficient \( (\prod \pi_i i_a)! \). There are \( \binom{2N-1}{N} - N \) linearly independent relations \( (12) \). Specially, we find

\[
[A_2, [A_2, A_3^{\dagger}] \ldots] = 2^n n! A_n.
\]
Note that any \( A_n \), \( n > N \) can be algebraically expressed in terms of \( A_m \), \( m \leq N \). For example, \( A_4 = 1/2A_2^2 \), \( A_5 = 5/6A_2A_3 \) for \( N \leq 3 \). The algebra \( \mathcal{A}_N \) can be expressed in terms of \( 2(N-1) \) algebraically independent operators.

For general \( N \), the algebraic relations (12) completely determine the action of \( A_k, k \leq N \) on any state:

\[
A_k A_i^{i_{n_2}} \cdots A_N^{i_{n_N}} |0\rangle = \sum \left( \prod A^i \right)^{N-k} |0\rangle,
\]

for \( k \leq \sum n_i \), reducing it to the states with \( k > \sum n_i \). For \( k > \sum n_i \), one calculates the finite set of relations from (8) and (6), directly. Hence, the \( \mathcal{A}_N \) algebra is closed in the sense of the successive commutation relation (12), finite and of polynomial type. Note that the action of \( A_i A_j^\dagger \) (and \( \mathcal{N}_i \)) on the symmetric Fock subspace can be written as an infinite, normally ordered expansion

\[
A_i A_j^\dagger = \sum_{k=0}^{\infty} \left( \prod A^i \right)^{k+j} \left( \prod A \right)^{k+i}, \forall i, j.
\]

Applied to a monomial state of the finite order \( N \) in \( F_{\text{symm}} \), only the finite number of terms in Eq. (13) will contribute.

The algebraic relations, Eq.(12), are independent of \( \nu \) and are common to all sets of operators \( \{A_k, A_k^\dagger\} \), with \( k = 2, 3, \ldots, N \), satisfying

\[
\left[ A_i, A_j \right] = \left[ A_i^\dagger, A_j^\dagger \right] = 0, \quad \left[ \mathcal{N}_i, A_j^\dagger \right] = \delta_{ij} A_j^\dagger.
\]

Two different sets of operators \( A(\nu) \) and \( A(\mu) \), satisfying the same algebra (12), differ only in the generalized vacuum conditions for \( k > \sum n_i \). Therefore, we denote the common algebra of operators \( \{A_k, A_k^\dagger\} \) by \( \mathcal{A}_N \), and its representation for a given \( \nu > -1/N \) by \( \mathcal{A}_N(\nu) \).

Let us illustrate this by the \( N = 3 \) case. The minimal set of relations which define the \( \mathcal{A}_3(\nu) \) algebra of operators \( \{A_2, A_3, A_2^\dagger, A_3^\dagger, \mathcal{N}\} \) is

\[
\begin{align*}
[A_i, A_2] &= 4iA_i, \\
[A_3, A_3^\dagger] &= 3A_2^2, \\
[A_i, [A_2, A_3^\dagger]] &= 6iA_4A_2, \quad i = 2, 3, \\
[A_2, [A_2, A_3^\dagger]] &= 48A_3, \\
[A_3, [A_3, A_3^\dagger]] &= 54A_3^2 - \frac{9}{2}A_2^2.
\end{align*}
\]

plus generalized vacuum conditions

\[
\begin{align*}
A_2|0\rangle &= A_3|0\rangle = A_2 A_3^\dagger |0\rangle = A_3 A_2^\dagger |0\rangle = A_3 A_2^\dagger |0\rangle = 0, \\
A_2 A_2^\dagger |0\rangle &= 4(1 + 3\nu)|0\rangle, \quad A_3 A_3^\dagger |0\rangle = 2(1 + 3\nu)(2 + 3\nu)|0\rangle, \\
A_3 A_2^\dagger A_3^\dagger |0\rangle &= 2(2 + 3\nu)(4 + 3\nu)A_2^2|0\rangle, \quad A_3 A_3^\dagger |0\rangle = 2(2 + 3\nu)(11 + 6\nu)A_3^\dagger |0\rangle.
\end{align*}
\]

This algebra Eq.(17) with the vacuum conditions (18) has a unique representation on \( F_{\text{symm}} \). Using Eqs.(17) and (18) one finds the action of operators \( A_2 \) and \( A_3 \) on any state in the Fock space:
$A_2|n_2, n_3\rangle = 3\binom{n_3}{2}|n_2 + 2, n_3 - 2\rangle + 4n_2(3n_3 + n_2 + 3\nu)|n_2 - 1, n_3\rangle$,

$A_3|n_2, n_3\rangle = 2\left(n_3(2 + 3\nu)(1 + 3\nu + 3n_2) + 9\binom{n_3}{2}(2 + 3\nu + n_2) + 27\binom{n_3}{3} + 6n_3\binom{n_2}{2}\right) \times$

$\times|n_2, n_3 - 1\rangle + 48\binom{n_2}{3}|n_2 - 3, n_3 + 1\rangle - \frac{81}{2}\binom{n_3}{3}|n_2 + 3, n_3 - 3\rangle$.  \hspace{1cm} (19)

The ket $|n_2, n_3\rangle$ denotes the state $A_2^{n_2}A_3^{n_3}|0\rangle$.

The general structure of the $A_N$ algebra can be viewed as a generalization of triple operator algebras \cite{12} to $(N + 1)$-tuple operator algebra. For $N = 2$, the $A_2$ algebra is just $[A_2, [A_2, A_3]] = 8A_2$ for a single oscillator, and $A_2 = (a_1 - a_2)^2/2$ describing the relative motion in the two-body Calogero model.

We point out that if the set of operators $\{A_k, A_k^\dagger\}$ satisfies relations \cite{10} there exists a mapping $A_k = f_k(b_i, b_i^\dagger)$ from ordinary Bose oscillators $\{b_i, b_i^\dagger\}$ to $\{A_k, A_k^\dagger\}$. Namely, the relations $[A_i^\dagger, A_j^\dagger] = 0$ imply that the monomial states $A_i^\dagger A_j^\dagger|0\rangle$ differing only in permutations of indices are all equal. The same property holds for the pure bosonic states. In order to construct the mapping, we first identify the vacua $|0\rangle_A = |0\rangle_b \equiv |0\rangle$ and the one-particle states $A_i^\dagger|0\rangle = b_i^\dagger|0\rangle \sqrt{\langle 0|A_i A_i^\dagger|0\rangle}$. Then we normally expand operators $A_i$ as an infinite series in terms of monomials of free bosonic oscillators

$$A_n = \sum_{k=0}^{\infty} \sum_{n_i, n_j} \left(\prod_i b_i^{n_i}\right) \left(\prod_j b_j^{n_j}\right), \quad \sum_i n_i = k, \quad \sum_j jn_j = k + n. \hspace{1cm} (20)$$

The coefficients in expansion (20) can be calculated from matrix elements of scalar products in $F_\text{symm}$ \cite{11} by applying expansion to one-particle matrix elements $\langle 0|A_i A_j^\dagger|0\rangle$, then to two-particle matrix elements $\langle 0|A_i A_j A_k^\dagger A_k^\dagger|0\rangle$, and so on. This mapping is invertible if there are no additional relations (null-norm states) in $F_\text{symm}$ when compared with the bosonic Fock space $F_b$. It is sufficient to demand that all Gram matrices of the scalar products have the same signature (structure of the signs of eigenvalues) in both Fock spaces. It has been found in Ref. [3] that the full Fock space $\{a_1^{n_1} \cdots a_N^{n_N}|0\rangle\}$ for $\nu > -1/N$ does not contain any additional null-norm states when compared with the bosonic Fock space $F_b$. As $F_\text{symm}$ is a subspace of full Fock space, it also does not contain any additional null-norm states, so $F_\text{symm}$ is isomorphic to $F_b$, and the mapping $f$ is invertible for $\nu > -1/N$.

The operators $a_i$, defined in Eq.(4), behave like free bosonic oscillators for $\nu = 0$ and this boundary condition uniquely determines all coefficients in the corresponding mapping from operators $a_i$ to the bosonic ones and vice versa. However, in the case of $A_i$ operators, there is no such boundary condition. Hence, we have a freedom of fixing the boundary condition, and this corresponds to freedom of choosing different orthogonal basis. Here, we choose a simple and natural way of fixing the boundary condition:

$A_2^{n_2}|0\rangle = \sqrt{\langle 0|A_2^{n_2} A_2^{n_2}|0\rangle} b_2^{n_2}|0\rangle$,

$A_3^{n_3}|0\rangle \sim \sum (\prod b_2^{i_{2}} b_3^{i_{3}}) |0\rangle$, \hspace{0.5cm} \hspace{0.5cm} 2i_2 + 3i_3 = 3n_3$
\[ A_N^{i,n}|0\rangle \sim \sum (\prod b_2^{i_2} \cdots b_N^{i_N}) |0\rangle, \sum_{k=2}^{N} k i_k = N n_N. \] (21)

Only after fixing the boundary condition one can determine the coefficients in Eq. (20) in the unique way. For the \( N=3 \) case, we present results for the first few coefficients in Eq. (20), up to \( k + n \leq 5 \), for the operators \( A_2 \) and \( A_3 \). First, we write general expression (20), up to \( k + n \leq 5 \):

\[
A_2 = f_2 b_2 + f_{22} b_2^2 b_2^2 + f_{23} b_2 b_3 + \cdots \\
A_3 = f_3 b_3 + f_{32} b_2^2 b_3 + \cdots,
\] (22)

with unknown coefficients \( f \). Next, we apply Eqs. (22) to matrix elements defined between the states in the symmetric Fock space \( (A_2^i|0\rangle, A_3^i|0\rangle, A_2^2|0\rangle, A_3^2|0\rangle, A_2^i A_3^i|0\rangle) \) and using relations (19), and the boundary conditions (21) we calculate unknown coefficients. Finally, we obtain

\[
A_2 = 2\sqrt{1 + 3\nu} b_2 + 2 \left( \sqrt{2 + 3\nu} - \sqrt{1 + 3\nu} \right) b_2^2 + 2 \left( \sqrt{4 + 3\nu} - \sqrt{1 + 3\nu} \right) b_2^2 b_3, \\
A_3 = \sqrt{2(1 + 3\nu)(2 + 3\nu)} b_3 + \left( \sqrt{2(4 + 3\nu)(2 + 3\nu)} - \sqrt{2(1 + 3\nu)(2 + 3\nu)} \right) b_2 b_3,
\] (23)

and similarly for hermitian conjugates. From this example is clear that the mapping is real if there are no negative-norm states. The inverse mapping exists for \( \nu > -1/3 \), and all relations (21), and (23) can be reversed:

\[
b_2 = \frac{A_2}{2\sqrt{1 + 3\nu}} + \frac{A_2^1 A_3^2}{8(1 + 3\nu)^{3/2}} \left( \frac{1 + 3\nu}{2 + 3\nu} - 1 \right) \\
+ \frac{A_3^1 A_2 A_3}{4(1 + 3\nu)^{3/2}(2 + 3\nu)} \left( \frac{1 + 3\nu}{4 + 3\nu} - 1 \right), \\
b_3 = \frac{A_3}{2\sqrt{2(1 + 3\nu)(2 + 3\nu)}} + \frac{A_3^1 A_2 A_3}{4\sqrt{2(1 + 3\nu)^3(2 + 3\nu)}} \left( \frac{1 + 3\nu}{4 + 3\nu} - 1 \right).\] (24)

Two different states \( A_2^{i_2} \cdots A_N^{i_N}|0\rangle \) and \( A_2^{i_2'} \cdots A_N^{i_N'}|0\rangle \) with the same energy \( (\sum i n_i = \sum i n_i') \) are not orthogonal. For example,

\[
\langle 0|A_2^2 A_3^{i_2}|0\rangle = \langle 0|A_2^3 A_3^{i_2}|0\rangle = \frac{144}{N}(N-1)(N-2)(1+\nu N)(2+\nu N).\] (25)

However, monomial states \( \prod b_i^{i_n}/\sqrt{n_i}|0\rangle \) in \( F_k \) are orthogonal, so when we express \( b_i = f_i^{-1}(A_k, A_k^\dagger) \), we obtain a natural orthogonal states in the symmetric Fock space, labelled by \( (n_2, \ldots, n_N) \), i.e., by free oscillators quantum numbers. Degenerate, orthogonal energy eigenstates of level \( \mathcal{N} \) are then defined by \( \mathcal{N} = \sum i n_i \). For example, some orthogonal states for \( N = 3 \) case are

\[
|0\rangle, \quad \frac{1}{2\sqrt{1 + 3\nu}} A_2^1|0\rangle, \quad \frac{1}{\sqrt{2(1 + 3\nu)(2 + 3\nu)}} A_3^1|0\rangle.
\]
Aν > B

Note that the generators \( J_\nu \) of the algebra (28) do not depend on \( N \). One can express the generators \( J_\nu \) in terms of \( A_1^\dagger \) and \( A_1 \), for \( N = 2 \), the degenerate states are \( B_1^\dagger |0\rangle \) and \( A_1^\dagger |0\rangle \); for \( N = 3 \), the degenerate states are \( B_1^\dagger A_1 |0\rangle \), \( B_1^\dagger A_2 |0\rangle \), and \( A_1^\dagger A_2 |0\rangle \), etc. The number of degenerate states of level \( N \) is given by partitions \( \mathcal{N}_1, \ldots, \mathcal{N}_k \) of \( N \) such that \( N = \sum_k k \mathcal{N}_k \). The generators of the algebra \( \mathcal{C}_N(\nu) \) can be chosen in different ways, and in the following we present a new and very simple bosonic realization of the dynamical algebra. The generators \( \{ \mathcal{J}_i^\pm, \mathcal{J}_i^0 \} \), \( i = 2, \ldots, N \) of the algebra represent a generalization of the linear \( SU(2) \) case discussed in Ref. [8]:

\[
\begin{align*}
\mathcal{J}_i^+ &= \frac{1}{\sqrt{i(\mathcal{N}_1 - 1)\cdots(\mathcal{N}_1 - i + 1)}} b_i^\dagger b_i, \\
\mathcal{J}_i^- &= b_i^\dagger b_i \frac{1}{\sqrt{i(\mathcal{N}_1 - 1)\cdots(\mathcal{N}_1 - i + 1)}} = (\mathcal{J}_i^+)\dagger, \\
\mathcal{J}_i^0 &= \frac{1}{2} \left( \frac{\mathcal{N}_1}{i} - b_i^\dagger b_i \right). \\
\end{align*}
\]

(27)

One can express the generators \( \mathcal{J}_i^{\pm,0} \) in terms of \( \{ A_k, A_k^\dagger \} \) using second relation in (20), for all \( \nu > -1/N \). The coefficients in expansion depend on the statistical parameter \( \nu \). The generators \( \{ \mathcal{J}_i^\pm, \mathcal{J}_i^0 \} \) satisfy the following new algebra for every \( i, j = 2, \ldots, N \):

\[
\begin{align*}
[\mathcal{J}_i^0, \mathcal{J}_j^\pm] &= \pm \frac{1}{2} \left( \delta_{ij} + \delta_{ij} \right) \mathcal{J}_j^\pm, \\
\mathcal{J}_i^+ \mathcal{J}_j^- - (\mathcal{J}_j^- \mathcal{J}_i^+) &= \delta_{ij} \left( \frac{\mathcal{N}_1}{i} \right), \\
\mathcal{J}_i^+ \mathcal{J}_j^+ - \sqrt{\frac{\mathcal{N}_1 - i}{\mathcal{N}_1 - j}} \mathcal{J}_j^+ \mathcal{J}_i^+ &= 0, \quad [\mathcal{J}_i^0, \mathcal{J}_j^0] = 0,
\end{align*}
\]

(28)

Note that the generators \( \mathcal{J}_i^+ \) and \( \mathcal{J}_i^- \) are hermitian conjugate to each other, and that relations (28) do not depend on \( \nu \). The \( \mathcal{C}_N \) algebra contains \( (N - 1) \) linear \( SU(2) \) subalgebras,
\[ i.e., \ [J_i^+, J_i^-] = 2J_i^0, \ [J_i^0, J_i^\pm] = \pm J_i^\pm. \] Different \( SU(2) \) subalgebras are connected in a nonlinear way, namely, commutation relations (28) have nonlinear (algebraic in \( N_1 \)) deformations.

The states with fixed energy \( E = \mathcal{N} \) are given by \( \prod_{i=1}^N b_i^{i n_i}/\sqrt{n_i!}0 \) with \( \sum_{i=1}^N i n_i = \mathcal{N} \). These states are orthonormal and build an irreducible representation (IRREP) of the symmetry algebra \( C_N \).

The structure of the IRREP on the states with fixed energy \( \mathcal{N} \) of the algebra \( C_N \) can be obtained recursively in \( N \):

\[
D(\mathcal{N}, N) = \sum_{i=0}^{\lfloor \mathcal{N}/N \rfloor} D(\mathcal{N} - N i, N - 1).
\]

For example, for \( N = 1 \), \( D(\mathcal{N}, 1) = 1 \), i.e., for a single oscillator, there is no degeneracy; for \( N = 2 \), \( D(\mathcal{N}, 2) = \lfloor \mathcal{N}/2 \rfloor + 1, \mathcal{N} \geq 0; \) for \( N = 3 \),

\[
D(\mathcal{N}, 3) = \sum_{i=0}^{\lfloor \mathcal{N}/3 \rfloor} \left[ \frac{\mathcal{N} - 3 i}{2} \right] + \left[ \frac{\mathcal{N}}{3} \right] + 1.
\]

More specifically, for \( \mathcal{N} = 6 n + i \), for some integer \( n \) and \( i = 0, \ldots, 5 \):

\[
D(6n + i, 3) = (3n + i)(n + 1), \quad \text{for } i = 1, \ldots, 5,
\]

\[
D(6n, 3) = 3n(n + 1) + 1.
\]

The structure of the IRREP’s of dynamical symmetry for \( N = 4 \) is

\[
\begin{align*}
& b_1^{i1}0 \ b_1^{i1}b_2^i0 \ b_1^2b_2^i0 \ b_2^{i1}0 \ b_1^i0 \\
& b_1^{i2}0 \ b_1^i0 \\
& b_1^{i3}0 \ b_1^i0 \\
& b_1^i0 \\
& b_1^i0 \\
& b_1^i0 \\
& b_1^i0 \\
& b_1^i0
\end{align*}
\]

(29)

Note that the dynamical symmetry algebra of the Hamiltonian \( H_1 = \sum_{k=1}^N b_k^i b_k \) is \( SU(N) \). The corresponding generators are \( b_i^i b_j \), and \( [H_1, b_i^i b_j] = 0 \) for all \( i \) and \( j \). The degenerate states are \( \prod_{i=1}^N b_i^{i n_i}/\sqrt{n_i!}0 \) with \( \sum_i n_i = \mathcal{N} \). They form a totally symmetric IRREP of \( SU(N) \) described by the Young diagram with \( \mathcal{N} \)-boxes \[ \begin{array}{c}
\mathcal{N}
\end{array} \] [13].

The states of the same Fock space \( \{ \prod_i b_i^{i n_i}/\sqrt{n_i!}0 \} \) are differently arranged in representations of \( SU(N) \) and \( C_N \) algebra. In respect to \( H_1 \), degenerate states are organized into the \( SU(N) \) symmetric IRREP’s. However, in respect to \( H_2 = H' - E_0 = \sum_{i=1}^N i \mathcal{N}_i = \sum_{i=1}^N i b_i^i b_i \), with \( \mathcal{N}_i \neq b_i^i b_i \), degenerate states build the \( \mathcal{N} \)-IRREP of the \( C_N \) algebra. In this sense, one can (formally) say that the nonlinear dynamical algebra \( C_N \) of the Hamiltonian \( H_2 \) is related to the boson realization of \( SU(N) \) algebra describing the dynamical symmetry of the Hamiltonian \( H_1 \).

We have studied the Calogero model in the harmonic potential for \( \mathcal{N} \) identical bosons. We have started with the Fock space of states with positive definite norms, and restricted
ourselves to the subspace of all symmetric states. After separating the center-of-mass coordinate, we have concentrated on the space \( \{ A_2^{n_2} \ldots A_N^{n_N} \} |0\rangle \) of operators \( \{ A_k, A_k^\dagger \} \). There exist number operators \( N_k \) that are not hermitian, since the monomial states in Fock space \( F_{\text{symm}} \) are not mutually orthogonal. The action of the operators \( A_k \) on the states in Fock space can be calculated using successive commutators defining the algebra \( \mathcal{A}_N \), and generalized vacuum conditions. We have shown that there exists a mapping from ordinary Bose oscillators to operators \( \{ A_k, A_k^\dagger \} \), and its inverse, provided that \( \nu > -1/N \). In that way we obtained a natural orthogonal basis for \( S_N \)–symmetric Fock space, an alternative to Jack polynomials. Finally, we constructed a dynamical symmetry algebra describing the structure of degenerate eigenspace, in terms of ladder operators build out of the bosonic oscillators. This bosonic realization of the symmetry algebra provides more insight on the nature of the degenerate energy eigenstates in the Calogero model. These results can also be applied to the finite Chern-Simons matrix model [14], since the observables, i.e., the \( SU(N) \) invariant operators \( \text{Tr} A^n \) defined in that model satisfy the same \( \mathcal{A}_N \) algebra.

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REFERENCES

[1] F. Calogero, J. Math. Phys. 10 (1971) 2191, 2197; J. Math. Phys. 12 (1971) 419.
[2] A.P. Polychronakos, Phys. Rev. Lett. 69 (1992) 703; L. Brink, T.H. Hansson, and M. Vasiliev, Phys. Lett. B 286 (1992) 109; L. Brink, T.H. Hansson, S.E. Konstein, and M. Vasiliev, Nucl. Phys. B 384 (1993) 591.
[3] R.P. Stanley, Adv. Math. 77 (1988) 76.
[4] H. Ujino and M. Wadati, J. Phys. Soc. Jnp. 65 (1996) 653; ibid. 2423 (1996); in ”Calogero-Moser-Sutherland Models”, Eds. J. F. van Diejen, L. Vinet, Springer-Verlag, New York, 2000, p. 522.
[5] R. Floreanini, L. Lapointe, and L. Vinet, Phys. Lett. B 389 (1996) 327.
[6] R. Floreanini, L. Lapointe, and L. Vinet, Phys. Lett. B 421 (1998) 229.
[7] N. Gurappa, P. K. Panigrahi, and V. Srinivasan, Mod. Phys. Lett. A 13 (1998) 339.
[8] L. Jonke, S. Meljanac, Phys. Lett. B 511 (2001) 276.
[9] S. Meljanac, M. Mileković, and M. Stojić, ”Permutation invariant algebras, a Fock space representation and the Calogero Model”, submitted to Eur. Phys. J., RBI-TH-01/08.
[10] A. D. de Veigy, Nucl. Phys. B 483 (1997) 580.
[11] S. Meljanac and M. Mileković, Int. Jour. Mod. Phys. A 11 (1996) 1391.
[12] Y. Ohnuki and S. Kamefuchi, ”Quantum Field Theory and Parastatistics”, University of Tokyo Press, 1982; S. Meljanac, M. Mileković, and M. Stojić, Mod. Phys. Lett. A 13 (1998) 995.
[13] R. Floreanini, L. Lapointe, and L. Vinet, J. Math. Phys. 39 (1998) 5739.
[14] A. P. Polychronakos, J. High Energy Phys. 04 (2001) 011.