Parallel spinors and holonomy groups on pseudo-Riemannian spin manifolds

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Abstract

We describe the possible holonomy groups of simply connected irreducible non-locally symmetric pseudo-Riemannian spin manifolds which admit parallel spinors.

1 Introduction

In [Wa 89] McKenzie Y. Wang described the possible holonomy groups of complete simply connected irreducible non-flat Riemannian spin manifolds $M^n$ which admit parallel spinors. Parallel spinors occur exactly for the holonomy representations $SU(m)$, $n = 2m \geq 4$, $Sp(m)$, $n = 4m \geq 8$, $G_2$ and $Spin(7)$. A complete, simply connected locally symmetric Riemannian spin manifold with non-trivial parallel spinors is flat.

In the present paper we consider the same problem for pseudo-Riemannian spin manifolds. We describe the possible holonomy groups of simply connected irreducible non-locally symmetric pseudo-Riemannian spin manifolds $M^{r,s}$ of dimension $r+s$ and index $r$ and study the chiral type and the causal type of these parallel spinors with respect to the indefinite scalar product on the spinor bundle.

The method to attack this problem is based on the fact that the space of parallel spinor fields can be identified with the vector space

$$V_{\hat{H}_{\tilde{u}}} = \{ v \in \Delta_{r,s} \mid \rho(\hat{H}_{\tilde{u}})v = v \}$$

of all elements of the spinor module $\Delta_{r,s}$ which are invariant under the action of the holonomy group $\hat{H}_{\tilde{u}} \subset Spin(r,s)$ of the Levi-Civita connection $\hat{A}$ on the spin structure $Q$ of $M^{r,s}$ with respect to $\tilde{u} \in Q$. The first list of all possible holonomy groups $H_u$ of simply connected irreducible non-locally symmetric semi-Riemannian manifolds was established by M. Berger in [Be 55]. Later on there were made some corrections and additions to Berger’s original list by D. Alekseevskii [A 68], R. Brown and A. Gray [BG 72], R. McLean and by R. Bryant [Br 90]. Since $\lambda(\hat{H}_u) = H_{\lambda(\tilde{u})}$, where $\lambda : Spin(r,s) \rightarrow SO(r,s)$ is
the double covering of $\text{SO}(r, s)$ by $\text{Spin}(r, s)$, we have to check, for which of the holonomy groups of Berger’s list $V_{H_a}$ is non-zero.

We obtain

**Theorem.** Let $M^{r,s}$ be a simply connected, non-locally symmetric irreducible semi-Riemannian spin manifold of dimension $n = r + s$ and index $r$. Let $\mathfrak{P}$ be the space of parallel spinors of $M$. Then $\dim \mathfrak{P} = N > 0$ if and only if the holonomy representation $H$ of $M$ is (up to conjugacy in the full orthogonal group) one in the list given below. Furthermore, this list gives the chiral type and in the case of space- and time oriented spin manifolds the causal type of the parallel spinors.

1. $H = \text{SU}(p, q) \subset \text{SO}(2p, 2q)$, $n = 2(p + q)$, $r = 2p$.
   Then $N = 2$. There exists a basis $\{\varphi_1, \varphi_2\}$ of $\mathfrak{P}$ such that
   $\varphi_1, \varphi_2 \in \Gamma(S^+)$ or $\varphi_1, \varphi_2 \in \Gamma(S^-)$ if $p + q$ is even,
   $\varphi_1 \in \Gamma(S^+)$, $\varphi_2 \in \Gamma(S^-)$ if $p + q$ is odd,
   $\varphi_1, \varphi_2$ have the same causality if $p$ is even
   $\varphi_1, \varphi_2$ have different causality if $p$ is odd.

2. $H = \text{Sp}(p, q) \subset \text{SO}(4p, 4q)$, $n = 4(p + q)$, $r = 4p$.
   Then $N = p + q + 1$, $\mathfrak{P} \subset \Gamma(S^+)$ or $\mathfrak{P} \subset \Gamma(S^-)$. All non-trivial parallel spinors have the same causal type.

3. $H = G_2 \subset \text{SO}(7)$, $n = 7$, $r = 0$.
   Then $N = 1$.

4. $H = G_2^{*}(2) \subset \text{SO}(4, 3)$, $n = 7$, $r = 4$.
   Then $N = 1$. All non-trivial parallel spinors are non-isotropic.

5. $H = G_2^C \subset \text{SO}(7, 7)$, $n = 14$, $r = 7$.
   Then $N = 2$. There exists a basis $\{\varphi_1, \varphi_2\}$ of $\mathfrak{P}$ such that $\varphi_1 \in \Gamma(S^+)$,
   $\varphi_2 \in \Gamma(S^-)$, $\varphi_1, \varphi_2$ are isotropic but not orthogonal to each other.

6. $H = \text{Spin}(7) \subset \text{SO}(8)$, $n = 8$, $r = 0$.
   Then $N = 1$.

7. $H = \text{Spin}(4, 3) \subset \text{SO}(4, 4)$, $n = 8$, $r = 4$.
   Then $N = 1$. All non-trivial parallel spinors are non-isotropic.

8. $H = \text{Spin}(7)^C \subset \text{SO}(8, 8)$, $n = 16$, $r = 8$.
   Then $N = 1$. All non-trivial parallel spinors are non-isotropic.

This theorem shows that the possible holonomy group of a manifold with parallel spinors is uniquely determined (up to conjugacy) by the dimension and the index of $M$ and the number of linearly independent parallel spinor fields.
In the proof of the theorem we use the fact that the vector spaces \( \tilde{V}_{\tilde{H}_1} \) and \( \tilde{V}_{\tilde{H}_2} \) are isomorphic if the Lie algebras \( \tilde{\mathfrak{h}}_1 \) and \( \tilde{\mathfrak{h}}_2 \) of \( \tilde{H}_1 \) and \( \tilde{H}_2 \), respectively, are different real forms of the same complex Lie algebra. Therefore, the dimension of \( \tilde{V}_{\tilde{H}} \) is given by that of the space corresponding to the compact real form. The explicit description of \( \tilde{V}_{\tilde{H}} \) allows to determine the causal type of the parallel spinors in the pseudo-Riemannian situation. Another proof of the theorem by a straightforward direct calculation of the vector space \( \tilde{V}_{\tilde{H}} \) using standard formulas of the spinor calculus is given in [BK 97].

Section 2 of this paper contains some necessary notations from the spinor calculus. In Section 3 we recall the complete Berger list of the holonomy representations of irreducible simply connected non-locally symmetric semi-Riemannian manifolds, explain the relation between parallel spinors and these holonomy groups and fix some notation concerning the appearing groups. In Section 4 we calculate the spaces \( \tilde{V}_{\tilde{H}} \) for the holonomy groups of Berger’s list explicitly and determine the chiral and the causal type of the corresponding parallel spinors.

The authors would like to thank R. Bryant for helpfull remarks concerning the original Berger list.

### 2 Spinor representations

For a more detailed explanation of the following facts see e.g. [LM 89], [Ba 81], [F 97].

Let \( C_{r,s} \) denote the Clifford algebra of \( \mathbb{R}^{r,s} = (\mathbb{R}^{r+s}, \langle , \rangle_{r,s}) \), where \( \langle , \rangle_{r,s} \) is a scalar product of index \( r \). It is generated by an orthonormal basis \( e_1, \ldots, e_{r+s} \) of \( \mathbb{R}^{r,s} \) with relations \( e_i \cdot e_j + e_j \cdot e_i = -2\kappa_j \cdot \delta_{ij} \), where \( \kappa_j = \langle e_j, e_j \rangle_{r,s} = \pm 1 \).

If \( n = r+s \) is even, then the complexification \( C_{r,s}^c \) of \( C_{r,s} \) is isomorphic to the matrix algebra \( \mathbb{C}(2^n/2) \). If \( n = r + s \) is odd, then \( C_{r,s}^c \) is isomorphic to \( \mathbb{C}(2^{[n/2]}) \oplus \mathbb{C}(2^{[n/2]}) \).

We will use the following isomorphisms:

Let \( E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( U = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \), \( V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \) and

\[
\tau_k = \begin{cases} 
& i & \text{if} & \kappa_k = -1, \\
& 1 & \text{if} & \kappa_k = 1.
\end{cases}
\]

In case \( n = r + s \) even we define

\[
\Phi_{r,s} : C_{r,s}^c \rightarrow C(2^n/2)
\]

by

\[
\Phi_{r,s}(\epsilon_{2k-1}) = \tau_{2k-1} E \otimes \cdots \otimes E \otimes U \otimes T \otimes \cdots \otimes T \overset{k\text{-times}}{=} \quad \Phi_{r,s}(\epsilon_{2k}) = \tau_{2k} E \otimes \cdots \otimes E \otimes V \otimes T \otimes \cdots \otimes T ,
\]

(1)
$k = 1, \ldots, \frac{n}{2}$. The restriction of $\Phi_{r,s}$ to $\text{Spin}(r,s) \subset C_{r,s}$ yields the spinor representation $\Delta_{r,s}$ which we will use here.

In case $n = r + s$ odd

$$\Phi_{r,s} : C_{r,s}^C \rightarrow C(2\frac{n}{2}) \oplus C(2\frac{n}{2})$$

is given by

$$\Phi_{r,s}(e_k) = (\Phi_{r,s-1}(e_k), \Phi_{r,s-1}(e_k)), \ k = 1, \ldots, n-1$$

$$\Phi_{r,s}(e_n) = \tau_n (i T \otimes \cdots \otimes T, -i T \otimes \cdots \otimes T).$$

If we restrict $\Phi_{r,s}$ to $\text{Spin}(r,s) \subset C_{r,s}$ and project onto the first component we obtain the spinor representation $\Delta_{r,s}$ in the odd-dimensional case.

The Lie algebra $\text{spin}(r,s)$ of $\text{Spin}(r,s)$ is given by

$$\text{spin}(r,s) = \text{span}(e_k \cdot e_l | 1 \leq k < l \leq n) \subset C_{r,s}.$$

Let $D_{kl}$ be the $(n \times n)$-matrix whose $(k,l)$-entry is 1 and all of whose others are 0 and let $E_{kl}$ be the matrix

$$E_{kl} = -\kappa_l D_{kl} + \kappa_k D_{lk}.$$

The differential of the double covering $\lambda : \text{Spin}(r,s) \rightarrow \text{SO}(r,s)$ is given by

$$\lambda^*(e_k \cdot e_l) = 2E_{kl}.$$

Let $u(\varepsilon) \in C^2, \varepsilon = \pm 1$, be the vectors $u(\varepsilon) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -\varepsilon \cdot i \end{array} \right)$ of $C^2$. For calculations we will use the orthonormal basis

$$\{u(\varepsilon_1, \ldots, \varepsilon_1) := u(\varepsilon_m) \otimes u(\varepsilon_{m-1}) \otimes \cdots \otimes u(\varepsilon_1) | \varepsilon_j = \pm 1\}$$

of $\Delta_{r,s} = C^2 \otimes \cdots \otimes C^2 (m = \lfloor \frac{n}{2} \rfloor)$.

In case of even $n = r + s$ the splitting $\Delta_{r,s} = \Delta^+_{r,s} \oplus \Delta^-_{r,s}$ of $\Delta_{r,s}$ in positive and negative spinors is given by

$$\Delta^\pm_{r,s} = \left\{ u(\varepsilon_1, \ldots, \varepsilon_1) | \prod_{j=1}^m \varepsilon_j = \pm 1 \right\}.$$

If we are in the pseudo-Euclidean case ($0 < r < r+s$), there exists an indefinite scalar product $\langle , \rangle$ on $\Delta_{r,s}$ which is invariant under the action of the connected component $\text{Spin}_0(r,s)$ of $\text{Spin}(r,s)$. $\langle , \rangle$ is given by

$$\langle u, v \rangle = i^{\frac{r(r-1)}{2}} (e_i_1 \cdots e_i_r \cdot u, v)_{C^2m},$$

where $e_i_1, \ldots, e_i_r \in \mathbb{R}^{r,s}$ are the timelike vectors of the orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^{r,s}, i_1 < \cdots < i_r$. 

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In particular, if \( r \) and \( s \) are odd, then \( \Delta_{r,s}^+ \) and \( \Delta_{r,s}^- \) are totally isotropic. If \( r \) and \( s \) are even, \( \Delta_{r,s}^+ \) is \( \langle \cdot, \cdot \rangle \)-orthogonal to \( \Delta_{r,s}^- \).

In the case that
\[
\langle x, y \rangle_{r,s} = -x_1 y_1 - \cdots - x_r y_r + x_{r+1} y_{r+1} + \cdots + x_{r+s} y_{r+s}
\]
and that \( e_1, \ldots, e_{r+s} \) is the standard basis of \( \mathbb{R}^{r+s} \) we have for even index \( r = 2\hat{r} \)
\[
\langle u(\varepsilon_m, \ldots, \varepsilon_1), u(\delta_m, \ldots, \delta_1) \rangle
= \begin{cases} 
0 & \text{if } (\varepsilon_m, \ldots, \varepsilon_1) \neq (\delta_m, \ldots, \delta_1) \\
\varepsilon_1 \cdots \varepsilon_{\hat{r}} & \text{if } (\varepsilon_m, \ldots, \varepsilon_1) = (\delta_m, \ldots, \delta_1). 
\end{cases}
\]
(4)

If we consider the bilinear form
\[
\langle x, y \rangle_{r,r} = -\sum_{j=1}^{r} x_{2j-1} y_{2j-1} + \sum_{j=1}^{r} x_{2j} y_{2j}
\]
and the standard basis \( (e_1, \ldots, e_{2r}) \) of \( \mathbb{R}^{2r} \) we obtain
\[
\langle u(\varepsilon_r, \ldots, \varepsilon_1), u(\delta_r, \ldots, \delta_1) \rangle
= \begin{cases} 
0 & \text{if } (\varepsilon_r, \ldots, \varepsilon_1) \neq (-\delta_r, \ldots, -\delta_1) \\
(-i)^{\hat{r}} \varepsilon_1 \cdots \varepsilon_{r-1} & \text{if } r = 2\hat{r} \text{ and } (\varepsilon_r, \ldots, \varepsilon_1) = (-\delta_r, \ldots, -\delta_1) \\
-i^{\hat{r}} \varepsilon_2 \cdots \varepsilon_{r-1} & \text{if } r = 2\hat{r} + 1 \text{ and } (\varepsilon_r, \ldots, \varepsilon_1) = (-\delta_r, \ldots, -\delta_1). 
\end{cases}
\]
(5)

Let \( j : \mathcal{C}_{r,s} \hookrightarrow \mathcal{C}_n^\mathbb{C} \) be the embedding which sends \( e_k \) to \( e_k \) if \( \kappa_k = 1 \) and \( e_k \) to \( ie_k \) if \( \kappa = -1 \). Then \( j(\mathcal{C}_{r,s}) \) and \( j(\text{spin}(r,s)) \) are real forms of \( \mathcal{C}_n^\mathbb{C} \) and \( \text{spin}(n)^\mathbb{C} \), respectively. We have
\[
\Phi_{r,s} = \Phi_n \circ j.
\]
(6)

### 3 Some facts on holonomy groups of semi-Riemannian spin manifolds

A connected semi-Riemannian manifold \((M^{r,s}, g)\) of dimension \( r+s \) and index \( r \) is called \textit{irreducible}, if its holonomy representation is irreducible. It is called \textit{indecomposable} if its holonomy groups \( H_u \subset O(r,s) \) do not leave invariant any nondegenerate proper subspace of \( \mathbb{R}^{r,s} \). Of course, in the Riemannian case, indecomposable manifolds are irreducible. In the pseudo-Riemannian case there exist holonomy groups whose holonomy representations are not irreducible but
which are not decomposable into a direct sum of pseudo-Riemannian holonomy representations ([Wu 67], [BI 93]).

De Rham’s splitting theorem reduces the study of complete simply connected semi-Riemannian manifolds to indecomposable ones.

**Theorem 1** ([DR 52], [Wu 64]) Let \((M^{r,s}, g)\) be a simply connected complete semi-Riemannian manifold. Then \((M^{r,s}, g)\) is isometric to a product of simply connected complete indecomposable semi-Riemannian manifolds.

Up to now, there is no list of all indecomposable restricted holonomy groups in the pseudo-Riemannian setting, but the irreducible cases are known.

The irreducible pseudo-Riemannian symmetric spaces were classified by M. Berger in [Be 57]. The irreducible restricted holonomy groups of non-locally symmetric spaces are listed in the following (corrected) Berger list.

**Theorem 2** ([Be 55], [S 62], [A 68], [BG 72], [Br 96]) Let \((M^{r,s}, g)\) be a simply connected irreducible non-locally symmetric semi-Riemannian manifold of dimension \(n = r + s\) and index \(r\). Then its holonomy representation is (up to conjugacy in \(O(r, s)\)) one of the following:

- \(\text{SO}_0(p, q)\) \(n = p + q \geq 2, \quad r = p\)
- \(\text{U}(p, q) \subset \text{SO}(2p, 2q)\) \(n = 2(p + q) \geq 4, \quad r = 2p\)
- \(\text{SU}(p, q) \subset \text{SO}(2p, 2q)\) \(n = 2(p + q) \geq 4, \quad r = 2p\)
- \(\text{Sp}(p, q) \subset \text{SO}(4p, 4q)\) \(n = 4(p + q) \geq 8, \quad r = 4p\)
- \(\text{Sp}(p, q) \cdot \text{Sp}(1) \subset \text{SO}(4p, 4q)\) \(n = 4(p + q) \geq 8, \quad r = 4p\)
- \(\text{Sp}(p, R) \cdot \text{SL}(2, R) \subset \text{SO}(2p, 2p)\) \(n = 4p \geq 8, \quad r = 2p\)
- \(\text{Sp}(p, C) \cdot \text{SL}(2, C) \subset \text{SO}(4p, 4p)\) \(n = 8p \geq 16, \quad r = 4p\)
- \(\text{SO}(p, p) \subset \text{SO}(2p, 2p)\) \(n = 2p \geq 4, \quad r = p\)
- \(G_2 \subset \text{SO}(7)\) \(n = 7\)
- \(G_{2(2)}^\ast \subset \text{SO}(4, 3)\) \(n = 7\)
- \(G_2^\ast \subset \text{SO}(7, 7)\) \(n = 14\)
- \(\text{Spin}(7) \subset \text{SO}(8)\) \(n = 8\)
- \(\text{Spin}(4, 3) \subset \text{SO}(4, 4)\)
- \(\text{Spin}(7)^C \subset \text{SO}(8, 8)\) \(n = 16\)

It is known for all groups \(H\) of the list in Theorem 2 that there exists a non-symmetric semi-Riemannian manifold with holonomy \(H\) (see [Br 87], [Br 96]).

Now let \((M^{r,s}, g)\) be a semi-Riemannian spin manifold with spin structure \((Q,f)\) and spinor bundle \(S = Q \times_{\text{Spin}(r,s)} \Delta_{r,s}\).

A spinor field \(\varphi \in \Gamma(S)\) is called **parallel**, if \(\nabla^S \varphi = 0\), where \(\nabla^S\) is the spinor derivative associated to the Levi-Civita connection \(\tilde{\nabla}\) on \(Q\).
Let us denote by $\tau_\omega^\tilde{A} : Q_x \to Q_x$ the parallel transport in the spin structure $Q$ with respect to $\tilde{A}$ along a closed curve $\omega$ starting at $x \in M$. Then for $\tilde{u} \in Q_x$

$$\tilde{H}_{\tilde{u}} := \{ g \in \text{Spin}(r, s) \mid \text{There exists a closed curve } \omega \text{ starting at } x \text{ such that } \tilde{u} \cdot g = \tau_\omega^\tilde{A}(\tilde{u}) \}$$

denotes the holonomy group of $\tilde{A}$ with respect to $\tilde{u} \in Q_x$. If $\tilde{u}_1 \in Q_{x_1}$ is another point in $Q$ and $\sigma$ is a curve from $x$ to $x_1$, then

$$\tilde{H}_{\tilde{u}_1} = a^{-1} \cdot \tilde{H}_{\tilde{u}} \cdot a,$$

where $a \in \text{Spin}(r, s)$ is the element with $u_1 = \tau_\sigma^\tilde{A}(u) \cdot a$. Hence, all holonomy groups of $\tilde{A}$ are conjugated to each other.

Let

$$Q^{\tilde{A}}(\tilde{u}) := \{ \tilde{u} \in Q \mid \text{There exists an } \tilde{A} - \text{horizontal curve connecting } \tilde{u} \text{ with } \tilde{u} \}$$

be the holonomy bundle of $(Q, \tilde{A})$ with respect to $\tilde{u} \in Q$. According to the reduction theorem, $(Q, \tilde{A})$ reduces to the $\tilde{H}_{\tilde{u}}$-principal bundle $Q^{\tilde{A}}(\tilde{u})$. Hence, the spinor bundle $S$ of $M^{r,s}$ is given by

$$S = Q \times_{\text{Spin}(r,s), \rho} \Delta_{r,s} = Q^{\tilde{A}}(\tilde{u}) \times_{\tilde{H}_{\tilde{u}}, \rho} \Delta_{r,s}.$$ 

Then, we have a bijection between the space $\mathfrak{P}$ of all parallel spinor fields of the connected manifold $M^{r,s}$ and the space $V_{\tilde{H}_{\tilde{u}}}$ of all fixed spinors of $\Delta_{r,s}$ with respect to the holonomy group $\tilde{H}_{\tilde{u}}$ which is given by

$$V_{\tilde{H}_{\tilde{u}}} := \{ v \in \Delta_{r,s} \mid \rho(\tilde{H}_{\tilde{u}})v = v \} \mapsto \mathfrak{P}$$

$$\varphi_v \in \Gamma(S)$$

$$\varphi_v(y) = [\tau_\omega^\tilde{A}(\tilde{u}), v], \text{ where } \omega \text{ is a curve in } M \text{ from } x \text{ to } y.$$ 

Now, let us suppose that $M^{r,s}$ is simply connected. Then the holonomy groups $\tilde{H}_{\tilde{u}}$ coincide with the restricted holonomy groups, in particular, they are connected. Therefore, the vector space $V_{\tilde{H}_{\tilde{u}}}$ equals

$$V_{\tilde{h}_{\tilde{u}}} := \{ v \in \Delta_{r,s} \mid \rho_*(\tilde{h}_{\tilde{u}})v = 0 \},$$

where $\tilde{h}_{\tilde{u}}$ is the Lie algebra of $\tilde{H}_{\tilde{u}}$. If $H_u \subset \text{SO}(r, s)$ is the holonomy group of the Levi-Civita connection $A$ on the bundle of positively oriented frames of $M^{r,s}$ with respect to a frame $u$ in $x \in M$ and if $\tilde{u} \in Q_x$ denotes a lift of $u$ into the spin structure $Q$, then $\lambda(\tilde{H}_{\tilde{u}}) = H_u$, where $\lambda : \text{Spin}(r, s) \to \text{SO}(r, s)$ is the double
covering of \( \text{SO}(r, s) \) by \( \text{Spin}(r, s) \). Hence, we identify the Lie algebra \( \mathfrak{h}_u \) of \( H_u \) with \( \tilde{\mathfrak{h}} \) using \( \lambda^{-1} \).

In Berger’s list (Theorem 2) the holonomy groups are described up to conjugacy in the full orthogonal group \( \text{O}(r, s) \). Since we consider holonomy groups of oriented manifolds each of the conjugacy classes of the groups \( H \) of Theorem 2 gives rise to two different conjugacy classes in \( \text{SO}(r, s) \), generated by \( H \) and by \( H' = T_1 \cdot H \cdot T_1^{-1} \), where \( T_1 = \text{diag}(-1, 1, \ldots, 1) \). It depends on the chosen orientation of \( M \) which conjugacy class appears. For the Lie algebras \( \mathfrak{h}' \) and \( \mathfrak{h} \) of \( H' \) and \( H \) we have \( \mathfrak{h}' = \text{Ad}(T)\mathfrak{h} \). For the Lie algebras \( \tilde{\mathfrak{h}}' \) and \( \tilde{\mathfrak{h}} \) of the corresponding holonomy groups \( \tilde{H}' \) and \( \tilde{H} \) of the spin structure \( (Q, \tilde{A}) \) it follows

\[
\tilde{\mathfrak{h}}' = \text{Ad}(\tilde{T})\tilde{\mathfrak{h}},
\]

where \( \lambda(\tilde{T}) = T \). Since \( \tilde{T} = \pm e_1 \in \mathfrak{pin}(r, s) \subset C_{r,s} \) and \( \text{Ad}(\tilde{T})X = \tilde{T} \cdot X \cdot \tilde{T}^{-1} \in C_{r,s} \) for all \( X \in C_{r,s} \) it results \( \tilde{\mathfrak{h}}' = -\kappa_1 e_1 \cdot \tilde{\mathfrak{h}} \cdot e_1 \subset C_{r,s} \). Therefore we have

\[
V_{\tilde{H}'} = e_1 \cdot V_{\tilde{H}}.
\]

Hence, it is sufficient to calculate \( V_{\tilde{H}} \). By Clifford multiplication of \( v \in V_{\tilde{H}} \) with \( e_1 \) the chiral type of \( v \) is changed.

If we want to study the causal type of parallel spinors we have to restrict ourselves to space- and time-oriented pseudo-Riemannian manifolds in order to have an indefinite scalar product on the spinor bundle \( S \). In that case to each conjugacy class of the group \( H \) of Theorem 2 correspond four different conjugacy classes in \( \text{SO}_0(r, s) \) generated by

\[
H, \ H' = T_1 HT_1^{-1}, \ H'' = T_2 HT_2^{-1}, \ H''' = T_3 HT_3^{-1},
\]

where \( T_1 = \text{diag}(-1, 1, \ldots, 1) \), \( T_2 = \text{diag}(1, \ldots, 1, -1) \), \( T_3 = \text{diag}(-1, 1, \ldots, 1, -1) \). With analogous notations as above we obtain

\[
\begin{align*}
V_{H'} &= e_1 \cdot V_{H} \\
V_{H''} &= e_n \cdot V_{H} \\
V_{H'''} &= e_1 \cdot e_n \cdot V_{H}.
\end{align*}
\]

(Here \( e_1 \) is timelike and \( e_n \) is spacelike.) In the first two cases the chiral character of \( v \in V_{\tilde{H}} \) is changed, in the third case it remains the same. Since

\[
\langle e_j \cdot v, e_j \cdot v \rangle = (-1)^{r+1} \langle v, e_j \cdot e_j \cdot v \rangle = (-1)^{r} \kappa_j \langle v, v \rangle, \quad j = 1, \ldots, n, \ v \in \Delta_{r,s},
\]

in case of even (odd) index \( r \) the causal type of \( v \in V_{\tilde{H}} \) is changed (remains the same) by Clifford multiplication with \( e_1 \) or \( e_1 \cdot e_n \) and remains the same (is changed) by Clifford multiplication with \( e_n \).
Next we describe the groups which occur in Theorem 2 in more detail. We identify 
\[ H^n \cong C^{2n} \]

\[
\begin{pmatrix}
  a_1 \\
  \\
  a_n
\end{pmatrix}
\mapsto
\begin{pmatrix}
  z_1 \\
  \bar{w}_1 \\
  \vdots \\
  z_n \\
  \bar{w}_n
\end{pmatrix},
\]
where \( a_\nu = z_\nu + w_\nu \cdot j \) \( (\nu = 1, \ldots, n) \)

and

\[ C^n \cong R^{2n} \]

\[
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_n
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x_1 \\
  y_1 \\
  \vdots \\
  x_n \\
  y_n
\end{pmatrix},
\]
where \( z_\nu = x_\nu + iy_\nu \) \( (\nu = 1, \ldots, n) \).

We denote by \( J_R \) the matrix

\[
J_R = \begin{pmatrix}
  J & 0 & \cdots & 0 \\
  0 & J & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & J
\end{pmatrix} \in GL(2n, \mathbb{R}),
\]

where \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and by \( J_C \) the same matrix in \( GL(2n, \mathbb{C}) \). Let \( K(n) \) be the space of \((n \times n)\)-matrices with entries in the (skew-) field \( K \). We consider the embeddings \( i_C \) and \( i_R \) associated to the identifications \( H^n \cong C^{2n} \) and \( C^n \cong R^{2n} \):

\[
i_C : H(n) \hookrightarrow C(2n)
\]

\[
i_C ((a_{\mu\nu})_{\mu,\nu=1,...,n}) = \left( \begin{pmatrix}
  z_{\mu\nu} & -w_{\mu\nu} \\
  \bar{w}_{\mu\nu} & \bar{z}_{\mu\nu}
\end{pmatrix} \right)_{\mu,\nu=1,...,n},
\]

where \( a_{\mu\nu} = z_{\mu\nu} + w_{\mu\nu} \cdot j \),

\[
i_R : C(n) \hookrightarrow R(2n)
\]

\[
i_R ((z_{\mu\nu})_{\mu,\nu=1,...,n}) = \left( \begin{pmatrix}
  x_{\mu\nu} & -y_{\mu\nu} \\
  y_{\mu\nu} & x_{\mu\nu}
\end{pmatrix} \right)_{\mu,\nu=1,...,n},
\]

where \( z_{\mu\nu} = x_{\mu\nu} + iy_{\mu\nu} \) \( (\mu, \nu = 1, \ldots, n) \).
where \( z_{\mu \nu} = x_{\mu \nu} + iy_{\mu \nu} \).

Then we have

\[
i_{\mathbb{C}}(\mathbb{H}(n)) = \{ A \in \mathbb{C}(2n) \mid \bar{A}J_{\mathbb{C}} = J_{\mathbb{C}}A \} \]
\[
i_{\mathbb{R}}(\mathbb{C}(n)) = \{ A \in \mathbb{R}(2n) \mid AJ_{\mathbb{R}} = J_{\mathbb{R}}A \}.
\]

Furthermore, let \( I_{p,q}^\mathbb{K} \) be the matrix

\[
I_{p,q}^\mathbb{K} = \left( \begin{array}{cc} -E_p & 0 \\ 0 & E_q \end{array} \right) \in \text{GL}(p + q, \mathbb{K}),
\]

where \( E_r \) denotes the unity matrix in \( \mathbb{K}(r) \).

The special pseudo-orthogonal group

\[
SO(p, q) = \{ A \in \text{SL}(p + q, \mathbb{R}) \mid A^T I_{p,q}^\mathbb{R} A = I_{p,q}^\mathbb{R} \}
\]

(in its standard form) is the invariance group of the bilinear form

\[
\langle x, y \rangle = -x_1y_1 - \cdots - x_p y_p + x_{p+1}y_{p+1} + \cdots + x_{p+q}y_{p+q}
\]
on \( \mathbb{R}^{p+q} \).

We identify the pseudo-unitary group

\[
U(p, q) = \{ A \in \mathbb{C}(p + q) \mid \bar{A}^T I_{p,q}^\mathbb{C} A = I_{p,q}^\mathbb{C} \}
\]
with the subgroup \( i_{\mathbb{R}}(U(p, q)) = SO(2p, 2q) \cap i_{\mathbb{R}}\mathbb{C}(p + q) \) of \( SO(2p, 2q) \) and the symplectic group

\[
\text{Sp}(p, q) = \{ A \in \mathbb{H}(p + q) \mid \bar{A}^T I_{p,q}^\mathbb{H} A = I_{p,q}^\mathbb{H} \}
\]
with the subgroup \( i_{\mathbb{C}}\text{Sp}(p, q) = U(2p, 2q) \cap i_{\mathbb{C}}(\mathbb{H}(p + q)) \) of \( U(2p, 2q) \).

Now we are going to describe the subgroup \( \text{Sp}(p, q) \cdot \text{Sp}(1) \) of \( SO(4p, 4q) \). Each quaternion \( a \in \text{Sp}(1) \) defines an orthogonal map \( R_a : \mathbb{R}^{4p+4q} \to \mathbb{R}^{4p+4q} \) by right multiplication with \( a \) on \( \mathbb{R}^{p+q} = \mathbb{R}^{4p+4q} \). In particular, if \( a = x_0 + iy_0 + (x_1 + iy_1)j \) and

\[
r_a := \begin{pmatrix}
x_0 & -y_0 & -x_1 & y_1 \\
y_0 & x_0 & y_1 & x_1 \\
x_1 & -y_1 & x_0 & -y_0 \\
-y_1 & x_1 & -y_0 & x_0
\end{pmatrix}
\]

then we have

\[
R_a = \begin{pmatrix}
r_a & 0 & \cdots & 0 \\
0 & r_a \\
& & \ddots & \vdots \\
0 & \cdots & 0 & r_a
\end{pmatrix} \in SO(4p, 4q).
\]
The group $\text{Sp}(p, q) \cdot \text{Sp}(1)$ equals

$$
\text{Sp}(p, q) \cdot \text{Sp}(1) = \{ A \cdot R_a \mid A \in i_{\mathbb{R}}i_{\mathbb{C}}\text{Sp}(p, q), a \in \text{Sp}(1) \}.
$$

Let $\omega_K$ be a non-degenerate skew-symmetric bilinear form on $\mathbb{K}^{2p}$, $K = \mathbb{R}$, $\mathbb{C}$. By $\text{Sp}(p, K)$ we denote the group of all automorphisms of $(\mathbb{K}^{2p}, \omega_K)$. The bilinear form $g$ on $\mathbb{R}^{4p} = \mathbb{R}^{2p} \otimes \mathbb{R}^2$ defined by

$$
g(x \otimes a, y \otimes b) := \det(a, b) \cdot \omega_\mathbb{R}(x, y)
$$

is a metric of signature $(2p, 2p)$ on $\mathbb{R}^{4p}$. Hence, $\text{Sp}(p, \mathbb{R}) \cdot \text{SL}(2, \mathbb{R})$ can be considered as a subgroup of all automorphisms of $(\mathbb{R}^{4p}, g)$, where $A \cdot B \in \text{Sp}(p, \mathbb{R}) \cdot \text{SL}(2, \mathbb{R})$ acts on $\mathbb{R}^{4p} = \mathbb{R}^{2p} \otimes \mathbb{R}^2$ by

$$
A \cdot B(x \otimes a) = Ax \otimes Ba.
$$

In the same way, the bilinear form $h$ on $\mathbb{R}^{8p} = \mathbb{C}^{4p} = \mathbb{C}^{2p} \otimes \mathbb{C}^2$ defined by

$$
h(z \otimes a, w \otimes b) = \text{Re}(\det(a, b) \cdot \omega_\mathbb{C}(z, w))
$$

gives an embedding of $\text{Sp}(p, \mathbb{C}) \cdot \text{SL}(2, \mathbb{C})$ into $\text{SO}(4p, 4p)$.

The remaining classical group of the list in Theorem 2 is

$$
\text{SO}(n, \mathbb{C}) = \text{SL}(n, \mathbb{C}) \cap \{ A \in \mathbb{C}(n) \mid A^T A = E_n \}.
$$

Let us come to the exceptional cases. $G_2$, $G_2^*(2)$, $\text{Spin}(7)$, $\text{Spin}(4, 3)$ are subgroups of $\text{SO}(7)$, $\text{SO}(4, 3)$, $\text{SO}(8)$ and $\text{SO}(4, 4)$, respectively. They are defined as invariance groups of so-called nice 3-forms and 4-forms, respectively. Let $w_0, w_1 \in \Lambda^3(\mathbb{R}^7)$, $\sigma_0, \sigma_1 \in \Lambda^4(\mathbb{R}^8)$ be the forms

$$
w_0 = w^{127} + w^{135} - w^{146} - w^{236} - w^{245} + w^{347} + w^{567},
$$

$$
w_1 = -w^{127} - w^{135} + w^{146} + w^{236} + w^{245} - w^{347} + w^{567},
$$

$$
\sigma_0 = \sigma^{1234} + \sigma^{1256} - \sigma^{1278} + \sigma^{1357} + \sigma^{1368} + \sigma^{1458} - \sigma^{1467} - \sigma^{2358} + \sigma^{2367} + \sigma^{2457} + \sigma^{2468} - \sigma^{3456} + \sigma^{3478} + \sigma^{5678},
$$

$$
\sigma_1 = \sigma^{1234} - \sigma^{1256} + \sigma^{1278} - \sigma^{1357} - \sigma^{1368} + \sigma^{1458} + \sigma^{1467} - \sigma^{2358} + \sigma^{2367} - \sigma^{2457} - \sigma^{2468} + \sigma^{3456} - \sigma^{3478} + \sigma^{5678},
$$

where $w^{\alpha \beta \gamma} = w^\alpha \wedge w^\beta \wedge w^\gamma$ and $\sigma^{\alpha \beta \gamma \delta} = \sigma^\alpha \wedge \sigma^\beta \wedge \sigma^\gamma \wedge \sigma^\delta$ with respect to the dual bases $w^1, \ldots, w^7$ of $e_1, \ldots, e_7 \in \mathbb{R}^7$ and $\sigma^1, \ldots, \sigma^8$ of $e_1, \ldots, e_8 \in \mathbb{R}^8$.

Then

$$
G_2 = \{ A \in \text{SO}(7) \mid A^* w_0 = w_0 \}
$$

$$
G_2^*(2) = \{ A \in \text{SO}(4, 3) \mid A^* w_1 = w_1 \}
$$

$$
\text{Spin}(7) = \{ A \in \text{SO}(8) \mid A^* \sigma_0 = \sigma_0 \}
$$

$$
\text{Spin}(4, 3) = \{ A \in \text{SO}(4, 4) \mid A^* \sigma_1 = \sigma_1 \}
$$

(compare [Br 87]).

Finally, the groups $G_2^C \subset \text{SO}(7, \mathbb{C})$ and $\text{Spin}(7, \mathbb{C}) \subset \text{SO}(8, \mathbb{C})$ are the complexifications of $G_2 \subset \text{SO}(7)$ and $\text{Spin}(7) \subset \text{SO}(8)$, respectively.
4 The fixed spinors of the holonomy representation

Let \( M^{r,s} \) be a simply connected semi-Riemannian spin manifold of index \( r \) and dimension \( r+s = n \) with holonomy representation \( H \) and let \( \mathfrak{h} \) be the Lie algebra of \( H \). Then the parallel spinors are given by the kernel of the representation of \( \tilde{\mathfrak{h}} := \lambda^{-1}(\mathfrak{h}) \) on the spinor module \( \Delta_{r,s} \). Hence, we have to check the groups in the Berger-Simons list and to determine the kernel

\[
V_{\tilde{\mathfrak{h}}} = \{ v \in \Delta_{r,s} \mid \tilde{X} \cdot v = 0 \text{ for any } \tilde{X} \in \tilde{\mathfrak{h}} \}.
\]

We make use of the following obvious fact. If \( a \) is a subalgebra of \( \text{spin}(n) \) and \( a' \) a subalgebra of \( \text{spin}(r,s) \) such that \( j(a')^C = a'^C \subset \text{spin}(n)^C \) then we have by (6) and Weyl’s unitary trick \( V_a = V_{a'} \).

4.1  \( H = SO_0(p,q) \) \( (p = r, q = s) \)

The spinor representation is either irreducible or decomposes into two irreducible representations of the same dimension. Therefore, \( V_{\text{spin}(r,s)} = \{0\} \). Consequently, there is no parallel spinor on \( M^{p,q} \).

4.2  \( H = U(p,q) \) \( (r = 2p, s = 2q) \)

We consider \( \mathfrak{h} = i_R \ast u(p,q) \subset \text{so}(2p,2q) \) and \( \mathfrak{h}_0 = i_R \ast u\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) \subset \text{so}(n) \). Then \( j(\tilde{\mathfrak{h}})^C = \tilde{\mathfrak{h}}_0^C \) holds. It is known that \( V_{\text{u}(\begin{smallmatrix} n \\ 2 \end{smallmatrix})} = \{0\} \) (see [Wa 89]). Hence, \( V_{\text{u}(p,q)} = \{0\} \).

4.3  \( H = SU(p,q) \) \( (r = 2p, s = 2q) \)

Now we have \( \mathfrak{h} = i_R \ast \text{su}(p,q) \subset \text{so}(2p,2q) \) and \( \mathfrak{h}_0 = i_R \ast \text{su}(\begin{smallmatrix} n \\ 2 \end{smallmatrix}) \subset \text{so}(n) \). Again, \( j(\tilde{\mathfrak{h}})^C = \tilde{\mathfrak{h}}_0^C \) holds. By [Wa 89] the dimension of \( V_{\mathfrak{h}_0} \) equals 2, thus, \( \dim V_{\text{su}(p,q)} = 2 \).

The Lie subalgebra \( i_R \ast \text{su}(\begin{smallmatrix} n \\ 2 \end{smallmatrix}) \subset \text{so}(n) \) is spanned by \( X_{kl}, Y_{kl} \) and \( D_1 - D_k \ (2 \leq k \leq n/2) \), where

\[
X_{kl} := E_{2k-1,2l-1} + E_{2k,2l} \quad (1 \leq k < l \leq \frac{n}{2})
\]
\[
Y_{kl} := E_{2k-1,2l} - E_{2k,2l-1} \quad (1 \leq k < l \leq \frac{n}{2})
\]
\[
D_k := E_{2k-1,2k} \quad (1 \leq k \leq \frac{n}{2}).
\]

Using this and (1) we obtain

\[
V_{\text{su}(p,q)} = \text{span}\{u(1,1,\ldots,1), u(-1,-1,\ldots,-1)\}
\]

since both generators are annihilated by \( \text{su}(\begin{smallmatrix} n \\ 2 \end{smallmatrix}) \).

If \( p + q = \frac{1}{2} \dim M \) is even then \( V_{\text{su}(p,q)} \subset \Delta_{2p,2q}^+ \). If \( p + q = \frac{1}{2} \dim M \) is odd, we
have an 1-dimensional space of parallel spinors in $S^+$ as well as in $S^-$. According to (4) the quadratic length of $u(\varepsilon, \ldots, \varepsilon)$ is
\[ \langle u(\varepsilon, \ldots, \varepsilon), u(\varepsilon, \ldots, \varepsilon) \rangle = \varepsilon^p. \]

Hence $u(1, \ldots, 1)$ and $u(-1, \ldots, -1)$ are of the same causal type (spacelike) if $p$ is even and of opposite causal type (timelike) if $p$ is odd.

4.4 $H = \text{Sp}(p, q)$ ($r = 4p, s = 4q$)

In this case $\mathfrak{h}$ equals $i_{\mathbb{R}}i_{\mathbb{C}}\text{sp}(p, q) \subset \text{so}(4p, 4q)$ and $\mathfrak{h}_0$ is $i_{\mathbb{R}}i_{\mathbb{C}}\text{sp}(\frac{n}{4}) \subset \text{so}(n)$. Then $\tilde{j}(\mathfrak{h})^C = \tilde{h}_0^C$. According to [Wa 89] the dimension of $V_{\mathfrak{h}_0}$ equals $\frac{n}{4} + 1$. Hence, $\dim V_{\text{Sp}(p, q)} = p + q + 1$.

The subalgebra $\mathfrak{h} = i_{\mathbb{R}}i_{\mathbb{C}}\text{sp}(\frac{n}{4}) \subset \text{so}(n)$ is spanned by

\begin{align*}
X_{2k - 1} & \equiv X_{2k - 1} \equiv X_{2k - 1} - Y_{2k - 1}, \\
X_{2k - 1} & \equiv X_{2k - 1} \equiv X_{2k - 1} + Y_{2k - 1}, \quad (1 \leq k < l \leq \frac{n}{4}) \\
X_{2k - 1} & \equiv X_{2k - 1} \equiv Y_{2k - 1}, \quad (1 \leq k \leq \frac{n}{4}) \\
D_{2k - 1} & \equiv D_{2k} \quad (1 \leq k \leq \frac{n}{4}).
\end{align*}

Using this and (4) one proves that the spinors
\[ \varphi_k = \sum_{\varepsilon_i = -1 \text{ exactly } k \text{ times}} u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_1, \varepsilon_1) \]

$k = 0, 1, \ldots, p+q$, are annihilated by $\text{sp}(\frac{n}{4})$. Consequently, they constitute a basis of $V_{\text{sp}(p, q)}$. Obviously $V_{\text{sp}(p, q)} \subset \Delta^+$. Furthermore all $\varphi_k$ ($k = 0, \ldots, p+q$) have the same causal type since $\langle \varphi_k, \varphi_k \rangle = \binom{p+q}{k} > 0$.

4.5 $H = \text{Sp}(p, q) \cdot \text{Sp}(1)$ ($r = 4p, s = 4q$),
$H = \text{Sp}(p, \mathbb{R}) \cdot \text{SL}(2, \mathbb{R})$ ($r = 2p, s = 2p$)

The Lie algebras $\mathfrak{h}_1 = i_{\mathbb{R}}i_{\mathbb{C}}\text{sp}(p, q) \oplus \text{sp}(1) \subset \text{so}(4p, 4q)$, $\mathfrak{h}_2 = i_{\mathbb{R}}i_{\mathbb{C}}\text{sp}(\frac{n}{4}) \oplus \text{sp}(1) \subset \text{so}(n)$ and $\mathfrak{h}_3 = \text{sp}(p, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}) \subset \text{so}(2p, 2p)$ are real forms of the complex Lie algebra $\text{sp}(p, \mathbb{C}) \oplus \text{sl}(2, \mathbb{C})$. Therefore, the vector spaces $V_{\mathfrak{h}_1}, V_{\mathfrak{h}_2}$ and $V_{\mathfrak{h}_3}$ are isomorphic. From [Wa 89] we know that $V_{\text{sp}(\frac{n}{4}) \oplus \text{sp}(1)} = \{0\}$. Hence, there are no parallel spinors for $H = \text{Sp}(p, q) \cdot \text{Sp}(1)$ and $H = \text{Sp}(p, \mathbb{R}) \cdot \text{SL}(2, \mathbb{R})$. 

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4.6 \( H = G_2, \ G^*_2(2) \)

Here \( \mathfrak{h} = \mathfrak{g}_2(2) \subset \mathfrak{so}(4,3), \ \mathfrak{h}_0 = \mathfrak{g}_2 \subset \mathfrak{so}(7), \ j(\mathfrak{h})^C = \mathfrak{h}_0^C. \) It is well known that \( \dim \mathfrak{v}_{g_2} = 1 \) and, therefore, \( \dim \mathfrak{v}_{g_2(2)} = 1. \) The Lie algebra \( \mathfrak{g}_2 \subset \mathfrak{so}(7) \) is spanned by

\[
\begin{align*}
E_{12} - E_{34}, & \quad E_{12} - E_{56}, \quad E_{13} + E_{24}, \quad E_{13} - E_{67}, \quad E_{14} - E_{23}, \\
E_{14} - E_{57}, & \quad E_{15} + E_{26}, \quad E_{15} + E_{47}, \quad E_{16} - E_{25}, \quad E_{16} + E_{37}, \\
E_{17} - E_{36}, & \quad E_{17} - E_{45}, \quad E_{27} - E_{35}, \quad E_{27} + E_{46}.
\end{align*}
\]

Using this and (2) one checks that the spinor \( \varphi = u(1,1,1) + iu(-1,-1,-1) \) is annihilated by \( \mathfrak{g}_2. \) Consequently,

\[
V_{\mathfrak{g}_2} = \text{span}\{\varphi\} \subset \Delta_7 \quad V_{\mathfrak{g}_2(2)} = \text{span}\{\varphi\} \subset \Delta_{4,3}.
\]

From \( \langle \varphi, \varphi \rangle = 2 \) we see, that all non-zero spinors in \( V_{\mathfrak{g}_2(2)} \) are spacelike.

4.7 \( H = \text{Spin}(7), \ \text{Spin}(4,3) \)

Now, \( \mathfrak{h} = \mathfrak{spin}(4,3) \subset \mathfrak{so}(4,4), \ \mathfrak{h}_0 = \mathfrak{spin}(7) \subset \mathfrak{so}(8), \ j(\mathfrak{h})^C = \mathfrak{h}_0^C. \) It is well known that \( \dim \mathfrak{v}_{\mathfrak{spin}(7)} = 1. \) Consequently, \( \dim \mathfrak{v}_{\mathfrak{spin}(4,3)} = 1. \) The Lie subalgebra \( \mathfrak{spin}(7) \subset \mathfrak{so}(8) \) is generated by

\[
\begin{align*}
E_{12} + E_{34}, & \quad E_{13} - E_{24}, \quad E_{14} + E_{23}, \quad E_{56} + E_{78}, \quad -E_{57} + E_{68}, \quad E_{58} + E_{67}, \\
-E_{15} + E_{26}, & \quad E_{12} + E_{56}, \quad E_{16} + E_{25}, \quad E_{37} - E_{48}, \quad E_{34} + E_{78}, \quad E_{38} + E_{47}, \\
E_{12} - E_{78}, & \quad E_{17} + E_{28}, \quad E_{18} - E_{27}, \quad E_{34} - E_{56}, \quad E_{35} + E_{46}, \quad E_{36} - E_{45}, \\
E_{18} + E_{36}, & \quad E_{17} + E_{35}, \quad E_{26} - E_{48}, \quad E_{25} + E_{38}, \quad E_{23} + E_{67}, \quad E_{24} + E_{57}.
\end{align*}
\]

Hence by (11) the spinor \( \psi = u(1,-1,-1,1) - u(-1,1,1,-1) \) is annihilated by \( \mathfrak{spin}(7). \) This shows

\[
V_{\mathfrak{spin}(7)} = \text{span}\{\psi\} \subset \Delta^+_8, \quad V_{\mathfrak{spin}(4,3)} = \text{span}\{\psi\} \subset \Delta^+_{4,4}.
\]

Furthermore, in case of \( \mathfrak{spin}(4,3) \) the spinor \( \psi \) is timelike.

4.8 \( H = \text{SO}(p, \mathbb{C}) \quad (r = p, \ s = p) \)

Now let (in difference to the previous sections) \( \text{SO}(p,p) \subset \text{GL}(2p,\mathbb{R}) \) be the invariance group of the inner product \( \langle x, y \rangle_{p,p} = -x_1y_1 + x_2y_2 - x_3y_3 + \cdots + x_{2p}y_{2p}. \) According to that we have

\[
\kappa_k = \begin{cases} 
1 & \text{k even} \\
-1 & \text{k odd}
\end{cases} \quad \tau_k = \begin{cases} 
1 & \text{k even} \\
i & \text{k odd}.
\end{cases}
\]
Then \( i_\mathbb{R}(\text{SO}(p, \mathbb{C})) \) is contained in \( \text{SO}(p, p) \). The Lie algebra \( \mathfrak{so}(p) \oplus \mathfrak{so}(p) \subset \mathfrak{so}(2p) \) is the compact real form of \( (i_\mathbb{R}, \mathfrak{so}(p, \mathbb{C}))^\mathbb{C} \). Hence, we obtain \( \dim V_{\mathfrak{so}(p, \mathbb{C})} = \dim V_{\mathfrak{so}(p) \oplus \mathfrak{so}(p)} \).

Let \( p = 2k \). Then the restriction of the \( \mathfrak{so}(4k) \)-representation \( \Delta_{4k}^\pm \) to \( \mathfrak{so}(2k) \oplus \mathfrak{so}(2k) \) is equivalent to the sum \( \Delta_{2k}^\pm \otimes \Delta_{2k}^\pm \otimes \Delta_{2k}^\pm \otimes \Delta_{2k}^\pm \) of tensor products of the \( \mathfrak{so}(2k) \)-representations \( \Delta_{2k}^\pm \). Since \( V_{\mathfrak{so}(2k)} = \{0\} \) we obtain that \( V_{\mathfrak{so}(2k, \mathbb{C})} = \{0\} \).

In case \( p = 2k + 1 \) the restriction \( \Delta_{4k+2}^\pm(\mathfrak{so}(2k+1) \oplus \mathfrak{so}(2k+1)) \) is isomorphic to the tensor product \( \Delta_{2k+1}^\pm \otimes \Delta_{2k+1}^\pm \) of the \( \mathfrak{so}(2k + 1) \)-representation \( \Delta_{2k+1}^\pm \) by itself. Analogously, we deduce from \( V_{\mathfrak{so}(2k+1)} = \{0\} \) that \( V_{\mathfrak{so}(4k+2, \mathbb{C})} = \{0\} \).

Hence, there are no parallel spinors for \( H = \text{SO}(p, \mathbb{C}) \).

### 4.9 \( H = G_2^\mathbb{C} \) \( (r = 7, s = 7) \)

The compact real form of \( (i_\mathbb{R}, \mathfrak{g}_2^\mathbb{C})^\mathbb{C} \) is equal to \( \mathfrak{g}_2 \oplus \mathfrak{g}_2 \subset \mathfrak{so}(7) \oplus \mathfrak{so}(7) \). Hence, \( \dim V_{\mathfrak{g}_2^\mathbb{C}} = \dim V_{\mathfrak{g}_2 \oplus \mathfrak{g}_2} \). The representations \( \Delta_{14}^\pm |_{\mathfrak{g}_2} \oplus \mathfrak{g}_2 \) of \( \mathfrak{g}_2 \oplus \mathfrak{g}_2 \) are equivalent to \( \Delta_{7}^\pm |_{\mathfrak{g}_2} \otimes \Delta_{7}^\pm |_{\mathfrak{g}_2} \). Since \( \dim V_{\mathfrak{g}_2} = 1 \) we have \( \dim V_{\mathfrak{g}_2 \oplus \mathfrak{g}_2} = 2 \), where one parallel spinor lies in \( \Delta_{14}^+ \) and the other one in \( \Delta_{14}^- \). Consequently, \( \dim V_{\mathfrak{g}_2^\mathbb{C}} = 2 \).

The Lie algebra \( i_\mathbb{R} \mathfrak{g}_2^\mathbb{C} \subset i_\mathbb{R} \mathfrak{so}(7, \mathbb{C}) \subset \mathfrak{so}(7, 7) \) is spanned by

\[
\begin{align*}
\xi_{12} - \xi_{34}, & \quad \xi_{13} + \xi_{24}, \quad \xi_{14} - \xi_{23}, \quad \xi_{12} - \xi_{56}, \quad \xi_{16} - \xi_{25}, \quad \xi_{15} + \xi_{26}, \quad \xi_{36} - \xi_{45}, \\
\xi_{35} + \xi_{46}, & \quad -\xi_{13} + \xi_{67}, \quad -\xi_{14} + \xi_{57}, \quad \xi_{16} + \xi_{37}, \quad \xi_{15} + \xi_{47}, \quad \xi_{36} - \xi_{17}, \quad \xi_{35} - \xi_{27}, \\
\eta_{12} - \eta_{34}, & \quad \eta_{13} + \eta_{24}, \quad \eta_{14} - \eta_{23}, \quad \eta_{12} - \eta_{56}, \quad \eta_{16} - \eta_{25}, \quad \eta_{15} + \eta_{26}, \quad \eta_{36} - \eta_{45}, \\
\eta_{35} + \eta_{46}, & \quad -\eta_{13} + \eta_{67}, \quad -\eta_{14} + \eta_{57}, \quad \eta_{16} + \eta_{37}, \quad \eta_{15} + \eta_{47}, \quad \eta_{36} - \eta_{17}, \quad \eta_{35} - \eta_{27}.
\end{align*}
\]

Then a direct calculation shows that

\[
\begin{align*}
\psi_1 & = u(1, 1, 1, 1, 1, 1, 1) + u(1, 1, 1, -1, -1, 1, -1) \\
& \quad + u(1, -1, 1, -1, 1, -1, 1) + u(1, -1, 1, 1, 1, 1, -1) \\
& \quad + u(1, -1, 1, -1, 1, 1, -1) - u(-1, 1, 1, 1, -1, 1, 1) \\
& \quad - u(-1, 1, 1, 1, 1, 1, -1) - u(-1, -1, 1, 1, -1, 1, 1) \\
& \in \Delta_{7,7}^+ \\
\psi_2 & = u(-1, 1, 1, 1, 1, 1, -1) + u(-1, 1, 1, -1, 1, 1) \\
& \quad + u(-1, -1, 1, 1, 1, 1, 1) + u(-1, -1, -1, 1, 1, -1) \\
& \quad - u(1, -1, 1, 1, -1, 1, 1) + u(1, -1, 1, 1, 1, -1) \\
& \quad + u(1, -1, 1, 1, -1, 1, -1) + u(1, 1, 1, 1, -1, 1) \\
& \in \Delta_{7,7}^-
\end{align*}
\]

are generators of \( V_{\mathfrak{g}_2^\mathbb{C}} \).

From (3) one calculates \( \langle \psi_1, \psi_1 \rangle = \langle \psi_2, \psi_2 \rangle = 0 \) and \( \langle \psi_1, \psi_2 \rangle = 8i \).
4.10  \( H = \text{Spin}(7)^C \ (r = 8, \ s = 8) \)

The Lie algebra \((i_{SR} \cdot \text{spin}(7))^C \subset (i_{SR} \cdot \text{so}(8, C))^C\) has the compact real form \(\text{spin}(7) \oplus \text{spin}(7) \subset \text{so}(8) \oplus \text{so}(8) \subset \text{so}(16)\). Hence, \(\dim V_{\text{spin}(7)^C} = \dim V_{\text{spin}(7) \oplus \text{spin}(7)}\).

The representation \(\Delta^\pm|_{\text{spin}(7) \oplus \text{spin}(7)}\) is equivalent to \(\Delta^\pm|_{\text{spin}(7)} \otimes \Delta^\pm|_{\text{spin}(7)} \oplus \Delta^\mp|_{\text{spin}(7)} \otimes \Delta^\mp|_{\text{spin}(7)}\). Since \(\dim V_{\text{spin}(7)} = 1\) and \(V_{\text{spin}(7)} \subset \Delta^+_8\) we obtain \(\dim V_{\text{spin}(7) \oplus \text{spin}(7)} = 1\) and \(V_{\text{spin}(7) \oplus \text{spin}(7)} \subset \Delta^+_8\). In particular, \(\dim V_{\text{spin}(7)^C} = 1\).

The Lie algebra \((i_{SR} \cdot \text{spin}(7, C))^C \subset \text{so}(8, 8)\) is spanned by

\[
\begin{align*}
\xi_{12} + \xi_{34}, \ &\xi_{13} - \xi_{24}, \ &\xi_{14} + \xi_{23}, \ &\xi_{56} + \xi_{78}, \ &-\xi_{57} + \xi_{68}, \ &\xi_{58} + \xi_{67}, \ &-\xi_{15} + \xi_{26}, \\
\xi_{12} + \xi_{56}, \ &\xi_{16} + \xi_{25}, \ &\xi_{37} - \xi_{48}, \ &\xi_{38} + \xi_{47}, \ &\xi_{17} + \xi_{28}, \ &\xi_{18} - \xi_{27}, \ &\xi_{35} + \xi_{46}, \\
\xi_{36} - \xi_{45}, \ &\xi_{18} + \xi_{36}, \ &\xi_{17} + \xi_{35}, \ &\xi_{26} - \xi_{48}, \ &\xi_{25} + \xi_{38}, \ &\xi_{23} + \xi_{67}, \ &\xi_{24} + \xi_{57}, \\
\eta_{12} + \eta_{34}, \ &\eta_{13} - \eta_{24}, \ &\eta_{14} + \eta_{23}, \ &\eta_{56} + \eta_{78}, \ &-\eta_{57} + \eta_{68}, \ &\eta_{58} + \eta_{67}, \ &-\eta_{15} + \eta_{26}, \\
\eta_{12} + \eta_{56}, \ &\eta_{16} + \eta_{25}, \ &\eta_{37} - \eta_{48}, \ &\eta_{38} + \eta_{47}, \ &\eta_{17} + \eta_{28}, \ &\eta_{18} - \eta_{27}, \ &\eta_{35} + \eta_{46}, \\
\eta_{36} - \eta_{45}, \ &\eta_{18} + \eta_{36}, \ &\eta_{17} + \eta_{35}, \ &\eta_{26} - \eta_{48}, \ &\eta_{25} + \eta_{38}, \ &\eta_{23} + \eta_{67}, \ &\eta_{24} + \eta_{57}.
\end{align*}
\]

A direct calculation now shows \(V_{\text{spin}(7)^C}\) is generated by

\[
\begin{align*}
\eta = & u(1, 1, 1, 1, 1, 1, 1, 1) - u(1, 1, 1, 1, -1, -1, -1, -1) \\
& - u(-1, -1, -1, -1, 1, 1, 1, 1) + u(-1, -1, -1, -1, 1, 1, 1, 1) \\
& - u(1, 1, -1, 1, 1, -1, 1, 1) - u(-1, -1, 1, 1, -1, -1, -1, -1) \\
& + u(1, 1, 1, 1, -1, 1, 1, 1) + u(-1, -1, 1, 1, 1, 1, 1, -1) \\
& - u(-1, 1, -1, 1, 1, 1, 1, 1) - u(-1, -1, 1, 1, -1, -1, -1, -1) \\
& - u(-1, 1, -1, 1, 1, 1, 1, -1) - u(-1, 1, 1, 1, 1, -1, -1, -1) \\
& - u(1, -1, 1, 1, 1, 1, 1, 1) - u(-1, 1, 1, 1, 1, -1, -1, -1) \\
& - u(1, -1, 1, 1, 1, 1, -1, 1) - u(-1, 1, 1, 1, 1, 1, -1, 1) \\
& + u(1, -1, 1, 1, 1, 1, -1, 1) + u(-1, 1, 1, 1, 1, 1, -1, 1) \\
& \in \Delta^+_{8,8}.
\end{align*}
\]

According to (3) we obtain \(\langle \eta, \eta \rangle = 16\).

4.11  \( H = \text{Sp}(p, C) \cdot \text{SL}(2, C) \ (r = 4p, \ s = 4p) \)

The Lie algebra \(\text{sp}(p) \oplus \text{sp}(1)\) is the compact real form of \(\text{sp}(p, C) \oplus \text{sl}(2, C)\). Hence, \((\text{sp}(p) \oplus \text{sp}(1)) \oplus (\text{sp}(p) \oplus \text{sp}(1)) \subset \text{so}(4p) \oplus \text{so}(4p)\) is the compact real form of \((i_{SR} \cdot (\text{sp}(p, C) \oplus \text{sl}(2, C)))^C\) and we have

\[
\dim V_{\text{sp}(p,C) \oplus \text{sl}(2,C)} = \dim V_{(\text{sp}(p) \oplus \text{sp}(1)) \oplus (\text{sp}(p) \oplus \text{sp}(1))}.
\]

Because of

\[
\Delta^{sp}_{8p}((\text{sp}(p) \oplus \text{sp}(1)) \oplus (\text{sp}(p) \oplus \text{sp}(1))) \cong \Delta^{sp}_{4p}((\text{sp}(p) \oplus \text{sp}(1)) \oplus (\text{sp}(p) \oplus \text{sp}(1)) \oplus (\text{sp}(p) \oplus \text{sp}(1)) \oplus (\text{sp}(p) \oplus \text{sp}(1))}
\]

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and $V_{\text{sp}(\mu) \oplus \text{sp}(1)} = \{0\}$ we obtain $V_{\text{sp}(\mu, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})} = \{0\}$.

Summarizing the previous calculations we obtain the theorem formulated in the introduction.

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