A Simple Way to Incorporate Loss When Modelling Multimode Entangled State Generation

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Abstract: We prove that the light generated via spontaneous four-wave mixing in a set of \( M \) coupled, lossy cavities is an \( M \)-mode squeezed thermal state. The state generation and evolution is simply obtained by solving a set of \( 3M \) first-order differential equations, independent of the number of photons generated. © 2022 The Author(s)

1. Introduction
Multimode squeezed states can be generated in resonant structures, such as ring resonators (see Fig. 1a) or coupled-resonator optical waveguides (CROWs) in a photonic crystal (see Fig. 1b), via a nonlinear interaction. They are a source of continuous-variable (CV) entanglement, since the quadratures of the photons in the state are correlated with each other. CV entanglement has applications in boson sampling, quantum computing, and CV cluster states [1].

The theoretical generation of multimode squeezed states via spontaneous four-wave mixing (SFWM) or spontaneous parametric down conversion (SPDC) has been studied extensively for ring resonators, nonlinear waveguides, and CROWs. [2–4] Photon loss is a problem in these systems, since it can reduce the squeezing and inseparability of the state.

Loss in ring resonators and waveguides, due to photon scattering, can be handled by introducing reservoir modes that photons can couple to [2]. The waveguide-reservoir coupling parameters can be estimated with phenomenological values taken from experiment. In contrast, loss in CROWs can be handled intrinsically by calculating the complex frequencies of the CROW Bloch modes. In this approach, the evolution in these lossy systems is expressed as the non-unitary evolution of the reduced density operator of the generated light, obtained from the solution of the Lindblad master equation.

In this work, we prove that the density operator for entangled light generated in multiple lossy modes via SFWM or SPDC takes the form of a multimode squeezed thermal state (MSTS). This solution of the Lindblad master equation applies to the discrete lossy modes of a structure, where the loss can be handled either phenomenologically or intrinsically. For \( M \) modes, we derive a set of \( 3M \) coupled first-order differential equations that provide the complete evolution of the density operator. The size of the set of equations is independent of the number of generated photons and only increases linearly with \( M \). This semi-analytic method can greatly reduce the number of equations to solve, compared to current numerical methods that rely on Fock states [5], since in those methods the number of equations is \( (M+N)!/[M!N!)^{-1} \), where \( N \) is the total photon number.

In the limit that the modes become lossless, our formalism reproduces the results for a multimode squeezed vacuum state given by N. Quesada et al. [3]. Also, in the case of only one or two lossy modes, our multimode formalism reproduces the results of our previous work [6].

2. Theory
We consider a classical pump pulse with amplitude \( \alpha(t) \) that generates entangled light in multiple modes via SFWM. The nonlinear Hamiltonian in the undepleted pump approximation for the generated light is \( H_{NL} = \alpha^2(t) \sum_{m,l} S_{ml} b_m^\dagger b_l^\dagger + \text{H.c.} \), where \( b_m^\dagger \) creates a photon in the \( m \)th mode, and the nonlinear parameter \( S_{ml} \) is proportional to the overlap of the pump modes with the signal and idler modes [4]. We assume that \( S_{ml} \) is the same...
for every pump mode in the pump bandwidth and neglect its dependence on pump frequency. This is valid if the pump is in a single lossy mode or in a set of modes with similar overlap with the signal and idler modes. It is simple to alter $H_{NL}$ for SPDC, and all the results below still follow. We diagonalize $H_{NL}$ by performing a Takagi factorization of $S_m$ (i.e. $S_m = \sum_\mu \lambda_\mu U_{\mu m} U_{\mu m}^\dagger$), we obtain

$$H_{NL} = \alpha^2(t)\sum_\mu \lambda_\mu B_\mu^{\dagger 2} + \text{H.c.},$$

where we define $B_\mu^m \equiv \sum_m U_{\mu m} b_m^\dagger$ as the $\mu$th Schmidt operator, that has the standard commutator $[B_\mu^m, B_\mu^n] = \delta_{\mu\mu'}$, due to the orthogonality of the basis ($\sum_m U_{\mu m}^\dagger U_{\mu m}' = \delta_{\mu\mu'}$).

The reduced density operator for the generated light, $\rho(t)$, evolves according to the Lindblad master equation

$$\frac{d\rho}{dt} = -i\frac{\hbar}{\hbar} [H_0 + H_{NL}, \rho] + \sum_m \gamma_m \left(2b_m^\dagger \rho b_m^\dagger b_m - b_m b_m^\dagger \rho - \rho b_m^\dagger b_m \right),$$

where $H_0 = \hbar \sum_m \omega_m b_m^\dagger b_m$ is the free Hamiltonian for light generated at frequency $\omega_m$, $H_{NL}$ is defined in Eq. (1), and $\gamma_m$ is proportional to the rate of loss from the $m$th mode. In this work, we prove that the solution of Eq. (2) is a MSTS

$$\rho(t) = S(t) \rho_{th}(t) S(t)^\dagger,$$

where the thermal part of the state, $\rho_{th}(t)$, arises due to the photons leaking from the structure, and is a product of single-mode thermal states in each each mode $\rho_{th}(t) = \prod_m [1 + n_m(t)]^{-1/2}(n_m(t) + 1)^{-1/2} b_m^\dagger b_m$, where $n_m(t)$ is the average thermal photon number for the $m$th mode. Here $S(t)$ is a unitary multimode squeezing operator, which is written as a product of single-mode squeezing operators $S(t) = \prod_\mu \exp \left(\frac{i}{\hbar} r_\mu(t) b_\mu(t)^\dagger - \text{H.c.} \right)$, where $r_\mu(t)$ and $\phi_\mu(t)$ are the squeezing amplitude and squeezing phase of the $\mu$th Schmidt mode.

The evolution of the MSTS in Eq. (3) is entirely determined by the parameters $n_m(t)$, $r_\mu(t)$, and $\phi_\mu(t)$ defined above. Putting Eq. (3) into Eq. (2), we derive a set of coupled first-order differential equations that these parameters must satisfy in order for the MSTS to be a solution of the Lindblad master equation. For $M$ modes, there are only $3M$ equations that need to be solved to obtain the evolution of the MSTS.

Additionally, using the MSTS in Eq. (3) and the parameters $n_m(t)$, $r_\mu(t)$, and $\phi_\mu(t)$ obtained from solving the coupled-equations, the expectation value of a Schrödinger operator $O$ is easily calculated with $\langle O \rangle(t) = \text{Tr} [\rho(t) O]$. The expectation value of the operators $b_m^\dagger b_t$ and $b_m b_t$ are calculated by expanding them in terms of the Schmidt operators

$$\langle b_m^\dagger b_t \rangle(t) = \sum_{\mu, \mu'} U_{\mu m}^\dagger U_{\mu' t} \langle B_\mu^\dagger B_{\mu'} \rangle(t), \quad \langle b_m b_t \rangle(t) = \sum_{\mu, \mu'} U_{\mu m} U_{\mu' t} \langle B_\mu B_{\mu'} \rangle(t),$$

where the expectation values of the Schmidt operators in Eq. (4) only involve simple single-mode squeezing transformations. The expressions in Eq. (4) can be used to study the correlation variance between the quadratures of the photons in the MSTS, which is a measure of its inseparability.

3. Conclusion

We have proved that multimode entangled light generated in lossy cavities via SFWM or SPDC is a multimode squeezed thermal state. Using Schmidt operators we derived a set of coupled equations that need to be solved to obtain the evolution of the MSTS, where the set increases only linearly with the number of modes and is independent from the number of photons generated. We believe that our new solution to this important problem will make it more feasible to study multimode lossy structures and to optimize them for a wide variety of quantum information applications.

References

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