Infinitesimal symmetries of weakly pseudoconvex manifolds

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Abstract
We consider weakly pseudoconvex hypersurfaces with polynomial models in $\mathbb{C}^N$ and their symmetry algebras. In the most prominent case of special models, given by sums of squares of polynomials, we give their complete classification. In particular, we prove that such manifolds do not admit any nonlinear symmetries, depending only on complex tangential variables, nor do they admit real or nilpotent linear symmetries. This leads to a sharp 2-jet determination result for local automorphisms. We also give partial results in the general case and a more detailed description of the graded components in complex dimension three. The results also provide an important necessary step for solving the local equivalence problem on such manifolds.

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1 Introduction

Analysis and geometry in several complex variables lead inevitably to the study of boundaries of complex domains. Since the classical works of Poincaré, Levi, Cartan, Tanaka, Chern, Moser, Fefferman and many others, the study of invariants and symmetries of strongly pseudoconvex hypersurfaces played fundamental role in the development of CR geometry.

The study of weakly pseudoconvex manifolds was initiated in the seminal work of Kohn [15], which defined type of a point $p \in M \subseteq \mathbb{C}^2$ as the lowest order (integer valued) local CR invariant.

More refined rational local invariants of weakly pseudoconvex boundaries in $\mathbb{C}^N$ turned out crucial for characterizing subellipticity of the $\bar{\partial}$-Neumann problem, as follows from the work of Catlin [8,9].

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Recent work of D. Zaitsev and Sung-Yeon Kim, inspired by earlier work of Siu [24], aims to relate qualitative understanding of higher order CR invariants to effective termination of Kohn’s algorithm and explicit dependence of the order of subelliptic estimate on such invariants. In order to study boundary invariants, exact information on symmetries of the boundary is crucial (the symmetry group itself being a fundamental invariant). In relation to the work [13,14,29], Dmitri Zaitsev posed a question on possible symmetries of weakly pseudoconvex manifolds of finite Catlin multitype. Can the structure results of [17] be improved under the pseudoconvexity assumption?

In this paper we show that the structure of the Lie algebra of infinitesimal CR automorphisms $\mathfrak{g} = \text{aut}(M, p)$ for weakly pseudoconvex hypersurfaces given by sums of squares is indeed substantially simpler compared to the general case. In particular, there exist no nonlinear rigid vector fields on such manifolds, which leads to a sharp 2-jet determination result.

The ultimate motivation for the present work comes from the Poincaré problem on local equivalence of real hypersurfaces in $\mathbb{C}^N$. Let us briefly recall some of the history and some recent developments, which also motivated this work.

In the classical case of Levi nondegenerate hypersurfaces, the Poincaré problem was solved in the works of Cartan, Tanaka, Chern and Moser [4,6,7,10,14,23,27]. While Cartan, Tanaka and Chern applied differential geometric techniques, Moser developed a normal form approach, inspired by the normal form solution to the equivalence problem for singular analytic vector fields, originating also in the work of Poincaré.

The case of singular Levi form presents new challenges, which are often more of algebraic than of differential-geometric nature. In the first interesting case - finite type Levi-degenerate hypersurfaces in $\mathbb{C}^2$, the first attempt to extend Moser’s normal form approach is the work of Wong [28]. A complete normal form for this class of manifolds and the description of their symmetries, was given by the second author in [21]. In combination with a convergence result of Baouendi–Ebenfelt–Rothschild [1], this normal form solves the biholomorphic equivalence problem for this class.

In complex dimensions higher than two, local geometry of Levi degenerate hypersurfaces is substantially more complicated, even on the initial level. For pseudoconvex hypersurfaces, Catlin [8] introduced a notion of multitype. The entries of the Catlin multitype take rational values, but need not be integers. This approach provides a defining equation of the form

$$\text{Im } w = P(z, \bar{z}) + o_w(1),$$

where $P$ is a weighted homogeneous polynomial in the complex tangential variables $z = (z_1, \ldots, z_n)$ with respect to the multitype weights $(\mu_1, \ldots, \mu_n)$, $w$ is the normal variable and $o_w(1)$ denotes terms of weight bigger than one. Moreover, the multitype weights are the lexicographically smallest weights for which such a description is possible, thus providing a fundamental CR invariant.

We will denote by $M_P$ the corresponding polynomial model,

$$M_P := \{\text{Im } w = P(z, \bar{z})\}. \quad (1.2)$$

Kolář, Meylan and Zaitsev showed in [17] that hypersurfaces of finite Catlin multitype provide a natural class of manifolds for which a generalization of the Chern–Moser operator is well defined. Using this operator, they proved that the Lie algebra of infinitesimal automorphisms $\mathfrak{g} = \text{aut}(M_P, 0)$ admits the weighted grading given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \bigoplus_{j=1}^n \mathfrak{g}_{-\mu_j} \oplus \mathfrak{g}_0 \oplus \bigoplus_{\eta \in E} \mathfrak{g}_\eta \oplus \mathfrak{g}_1.$$
where $E$ is the set of integer combinations of the multitype weights, which lie between zero and one. As a consequence, they proved that the automorphisms of $M$ at $p$ are uniquely determined by their weighted 2-jets at $p$. See also [5,11,20,24] and references therein for related results on finite jet determination.

Since the kernel of the generalized Chern–Moser operator corresponds to the Lie algebra $\text{aut}(M_p, 0)$ of infinitesimal CR automorphisms of the model, this result gives a necessary tool for addressing the equivalence problem. However, full classification of such Lie algebras seems still out of reach. One of the main difficulties is the presence of a component (denoted below as $g_c$), containing nonlinear rigid vector fields, with arbitrarily high degree coefficients (only the weighted degree is controlled in general).

In this paper we show that in the most interesting case of pseudoconvex Levi degenerate manifolds defined by sums of squares, the structure of $\text{aut}(M_p, 0)$ is in fact much simpler, thus opening the possibility for a complete solution of the Poincaré equivalence problem for this important class.

Let us recall that by results of Stanton [25,26], finite dimensionality of $\text{aut}(M, p)$ is equivalent to holomorphic nondegeneracy of the hypersurface $M$ near $p$. Equivalently, holomorphically degenerate hypersurface can be characterized as a manifold whose defining equation at a generic point can be made independent of one of the variables. Hence, in this sense, the study of local invariants of $M$ should be reduced to one (or more) dimensions lower.

**Example 1.1** Let $M \subseteq \mathbb{C}^3$ be given by

$$\text{Im} \, w = |z_1^2 - z_2^3|^2. \quad (1.3)$$

It is easy to verify (using e.g. results of [16]) that $M$ is of finite Catlin multitype, with multitype weights $(\frac{1}{4}, \frac{1}{5})$. Notice that $M$ is of infinite D’Angelo type, containing a singular complex curve. The holomorphic vector field

$$Y = 3z_2^2 \partial_{z_1} + 2z_1 \partial_{z_2} \quad (1.4)$$

is complex tangent to $M$, i.e. $Y(\text{Im} \, w - P) = 0$. The same holds for every vector field of the form

$$Z = f(w, z_1, z_2)Y$$

for any holomorphic function $f$ in a neighbourhood of 0. Hence $M$ is holomorphically degenerate and $\dim g = \infty$.

As illustrated by this example, in our context holomorphic nondegeneracy is a natural and inevitable assumption.

We now formulate our results. Throughout the paper, $M$ will denote a hypersurface of finite Catlin multitype, described by (1.1).

**Theorem 1.2** Assume that $M$ is pseudoconvex in a neighbourhood of $p$ with a holomorphically nondegenerate model hypersurface $M_p$, given by (1.2), and nonvanishing $g_1$. Then the Lie algebra of infinitesimal CR automorphisms $g = \text{aut}(M_p, 0)$ of $M_p$ admits the weighted grading given by

$$g = g_{-1} \oplus \bigoplus_{j=1}^{n} g_{-\mu_j} \oplus g_0 \oplus g_2 \oplus g_1, \quad (1.5)$$

where $g_0$ is generated by the Euler field and a subalgebra of $u(n)$.
In order to formulate the next proposition, we recall the definition of a balanced polynomial.

**Definition 1.3** We say that a weighted homogeneous polynomial $P(z, \bar{z})$ is balanced if it can be written as

$$P(z, \bar{z}) = \sum_{|\alpha| = |\bar{\alpha}| = \frac{1}{2}} A_{\alpha, \bar{\alpha}} z^\alpha \bar{z}^{\bar{\alpha}},$$

for some nonzero n-tuple of real numbers $\Lambda = (\lambda_1, \ldots, \lambda_n)$, where

$$|\alpha|_{\Lambda} := \sum_{j=1}^{n} \lambda_j \alpha_j.$$

The associated hypersurface $M_P$ is then called a balanced hypersurface.

**Proposition 1.4** A pseudoconvex model hypersurface $M_P$, given by (1.2), has a nontrivial $g_1$ if and only if $P$ is a balanced polynomial.

$M_P$ has a nontrivial $g_2$ if and only if $P$ is of the form

$$P(z, \bar{z}) = |z_l|^2 + P_0(z, \bar{z}),$$

for some $l$, where $P_0$ is a balanced polynomial, independent of $z_l, \bar{z}_l$.

As a consequence, we obtain the following sharp jet determination result.

**Theorem 1.5** Assume that $M$ is pseudoconvex in a neighbourhood of $p$ with a holomorphically nondegenerate model hypersurface $M_P$, given by (1.2), and nonvanishing $g_1$. Then the automorphisms of $M$ at $p$ are uniquely determined by their jets of order 2.

Special domains have been used as the most prominent case in the context of the d-bar problem and effectivity of Kohn’s algorithm (see e.g. [2,8,24] and references therein).

The model corresponding to a special domain is a sum of squares model. If the model is holomorphically nondegenerate, then its symmetry algebra controls the symmetry algebra of the original manifold.

By $M_S$ we will denote a sum of squares model, given by

$$\text{Im} (w) = \sum_{j=1}^{k} |P_j(z)|^2,$$

where $P_j$ are weighted homogeneous holomorphic polynomials of weighted degree $\frac{1}{2}$ with respect to the multitype weights.

We will first consider the case of special models, given by sums of squares of homogeneous polynomials (hence necessarily of the same degree). We prove the following result.

**Theorem 1.6** Let $M_S$ be a sum of squares homogeneous polynomial model of degree $d > 2$, which is holomorphically nondegenerate. Then the Lie algebra of infinitesimal CR automorphisms $\mathfrak{g} = \text{aut}(M_S, 0)$ admits the weighted grading given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are of real dimension one and $\mathfrak{g}_0$ is generated by the Euler field and a subalgebra of $\mathfrak{u}(n)$.
We obtain an analogous result in the general case of a weighted homogeneous sum of squares model. Let us denote by $\kappa_M$ the number of multitype weights with $\mu_j = \frac{1}{2}$.

**Theorem 1.7** Let $M_S$ be a sum of squares weighted homogeneous polynomial model, which is holomorphically nondegenerate. Then the Lie algebra of infinitesimal CR automorphisms $\mathfrak{g} = \text{aut}(M_S, 0)$ admits the weighted grading given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1,$$

(1.10)

where $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are of real dimension one, $\mathfrak{g}_{-\frac{1}{2}}$ and $\mathfrak{g}_{\frac{1}{2}}$ are of real dimension $2\kappa_M$, and $\mathfrak{g}_0$ is generated by the Euler field and a subalgebra of $\mathfrak{u}(n)$.

In the lowest dimensional case of complex dimension three, we also describe explicitly the subalgebras of $\mathfrak{u}(2)$ which appear as constituents of $\mathfrak{g}_0$ (see Sect. 6). In particular, $\mathfrak{g}_0$ is non-abelian if and only if the model is given by

$$\text{Im } w = \left(\sum_{j=1}^{2} |z_j|^2\right)^l$$

(1.11)

for some integer $l \geq 1$.

The paper is organized as follows. In Sect. 2 we recall the needed definitions and notation, which is used in the sequel. In Sect. 3 we consider the $\mathfrak{g}_0$ component and prove that pseudoconvex models do not admit any real or nilpotent rotations. Section 4 considers generalized rotations. We show that such symmetries cannot occur on weakly pseudoconvex manifolds, under the assumption of nontriviality of $\mathfrak{g}_1$. In Sect. 5 we consider sum of squares models and prove Theorems 1.6 and 1.7. We illustrate the results on manifolds of infinite D’Angelo type, containing complex analytic sets. In Sect. 6, we consider the case of 5-dimensional real hypersurfaces in $\mathbb{C}^3$ and describe explicitly the form of $\mathfrak{g}_0$. Again we apply the results to compute all infinitesimal symmetries of hypersurfaces of infinite D’Angelo type, containing complex curves. In Sect. 7, the proofs of Theorem 1.2, Proposition 1.4 and Theorem 1.5 are given.

## 2 Preliminaries

In this section we recall the definitions and notation needed in the rest of the paper (for more details, see e.g. [17]).

**Definition 2.1** Let $n \in \mathbb{N}$ be an integer. A weight is an $n$-tuple of nonnegative rational numbers $\mu = (\mu_1, \ldots, \mu_n)$, where $0 \leq \mu_j \leq \frac{1}{2}$ and $\mu_j \geq \mu_{j+1}$ such that there exist an $n$-tuple of non-negative integers $(\alpha_1, \ldots, \alpha_n)$ satisfying $\alpha_j \neq 0$ if $\mu_j \neq 0$ for each $j$ and

$$\sum_j \alpha_j \mu_j = 1.$$  

(2.1)

For a given weight $\mu = (\mu_1, \ldots, \mu_n)$ and multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, the weighted length of $\alpha$ is

$$|\alpha|_\mu = \sum_{j=1}^{n} \alpha_j \mu_j.$$
Similarly, if \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n) \) are two multi-indices, the weighted length of \((\alpha, \hat{\alpha})\) is

\[
| (\alpha, \hat{\alpha}) |_\mu = \sum_{j=1}^{n} (\alpha_j + \hat{\alpha}_j) \mu_j.
\]

**Definition 2.2** For a given weight \( \mu = (\mu_1, \ldots, \mu_n) \), the weighted degree of a monomial \( A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^\hat{\alpha} \) is \( | (\alpha, \hat{\alpha}) |_\mu \), where \( A_{\alpha, \hat{\alpha}} \in \mathbb{C} \setminus \{0\} \) and \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \). A homogeneous polynomial \( P \) is called \( \mu \)-homogeneous of weighted degree \( l \), if it is a sum of monomials with weighted degree \( l \).

Next, consider coordinates \((z, w) = (z_1, \ldots, z_n, w)\) of \( \mathbb{C}^{n+1} \) with the weight \((\mu, 1) = (\mu_1, \ldots, \mu_n, 1)\). A holomorphic vector field \( Y \) is called \( \mu \)-homogeneous of weighted degree \( l \), if it is sum of holomorphic vector fields with polynomial coefficients, of the form

\[
f(z, w) \partial_{z_j} \quad \text{and} \quad g(z, w) \partial_w,
\]

where \( f(z, w) \) is \((\mu, 1)\)-homogeneous of weighted degree \( l + \mu_j \) and \( g(z, w) \) is a \((\mu, 1)\)-homogeneous of weighted degree \( l + 1 \).

Note that the weighted degrees of the vector fields \( \partial_{z_j} \) and \( \partial_{z_j} \) are \(-\mu_j \). The weighted degree of the vector field \( \partial_w \) is \(-1 \). Also, note that if \( P \) is a \( \mu \)-homogeneous polynomial of weighted degree \( 1 \), we can write \( P \) as

\[
P(z, \bar{z}) = \sum_{| (\alpha, \hat{\alpha}) |_\mu = 1} A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^\hat{\alpha}.
\]

**Definition 2.3** Let \( M \) be a hypersurface in \( \mathbb{C}^{n+1} \) and \( p \in M \) be a point. The weight \( \mu \) is called distinguished if there exist local holomorphic coordinates \((z, w)\) centered at \( p \in M \), where \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) and \( w = u + iv \), such that \( M \) is described as

\[
v = P(z, \bar{z}) + o_w(1)
\]

in a neighbourhood of \( p \), where \( P(z, \bar{z}) \) is a weighted homogeneous polynomial of degree \( 1 \) with respect to \( \mu \), without pluriharmonic terms and \( o_w(1) \) denotes terms of weighted degree bigger than one. For a distinguished weight \( \mu \), the local holomorphic coordinates are called \( \mu \)-adapted.

Let \( \mu_M = (\mu_1, \ldots, \mu_n) \) be the infimum of distinguished weights with respect to lexicographic ordering. The multitype of \( M \) at \( p \) is the \( n \)-tuple \((m_1, m_2, \ldots, m_n)\), where \( m_j = \frac{1}{\mu_j} \) if \( \mu_j \neq 0 \) or \( m_j = \infty \) if \( \mu_j = 0 \). If \( \mu_j \neq 0 \) for all \( j \), we say that \( M \) is of finite multitype at \( p \). The weight \( \mu_M \) is called the multitype weight and \( \mu_M \)-adapted coordinates are called multitype coordinates.

The model hypersurface \( M_P \) associated to \( M \) at \( p \) of the form \((2.4)\), is given by

\[
\Im(w) = P(z, \bar{z})
\]

in multitype coordinates \((z, w)\), where \( P(z, \bar{z}) \) is weighted homogeneous of degree \( 1 \) with respect to \( \mu_M \).

In the rest of the paper, we will assume that the hypersurface \( M \) is pseudoconvex. By a simple homogeneity argument, the same then holds for the corresponding model.
As an important example, let \( P_j \) be weighted homogeneous polynomials in \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) of weight \( \frac{1}{2} \) with respect to \( \mu_M \), and let \( M_S \) be the hypersurface given by

\[
\text{Im}(w) = \sum_{j=1}^{k} |P_j(z)|^2.
\]

Then \( M_S \) is automatically pseudoconvex and we call it a sum of squares model.

Let \( w = u + iv \) and let \( W \) be the vector field of degree \(-1\) given by

\[
W = \partial w = \partial u - i \partial v.
\]

Then we have \( \text{Re} \ W (\text{Im}(w) - P(z, \bar{z})) = 0 \), hence \( W \) is a symmetry.

Recall that, as was proved in [17], the Lie algebra \( g \) of infinitesimal automorphisms \( \text{aut}(M_P, 0) \) at \( 0 \in M_P \subset \mathbb{C}^{n+1} \) admits a weighted decomposition, which we rewrite now as

\[
g = g_{-1} \oplus \bigoplus_{j=1}^{n} g_{-\mu_j} \oplus g_0 \oplus g_c \oplus g_n \oplus g_1,
\]

where the vector fields in \( g_c \) commute with \( W \), the non-zero vector fields in \( g_n \) do not commute with \( W \) and their weights are between 0 and 1. In particular, \( W = \partial w \) is contained in \( g_{-1} \), which has real dimension one (for more details, see [17,19]).

Let \( E \) be the weighted Euler field, given by

\[
E = w \partial w + \sum_{j=1}^{n} \mu_j z_j \partial z_j.
\]

Then it is immediate that

\[
\text{Re} \ E (\text{Im}(w) - P(z, \bar{z})) = 0,
\]

which implies \( E \in g_0 \) and hence \( \text{dim} \ g_0 \geq 1 \) for an arbitrary model \( M_P \).

3 The structure of \( g_0 \)

Let \( X \) be a weight zero infinitesimal CR automorphism in \( g \). By results of [17], \( X \) is a linear vector field in suitable multitype coordinates. Its Jordan normal form can be decomposed into \( X^{Re} + X^{Im} + X^{Nil} \), where \( X^{Re} \) is the real diagonal of the Jordan normal form, \( X^{Im} \) is the imaginary diagonal and \( X^{Nil} \) is the nilpotent part. In particular, if we let \( \tilde{E} \) be the vector field given by

\[
i \sum_{j=1}^{n} \mu_j z_j \partial z_j,
\]

then \( \tilde{E} = \tilde{E}^{Im} \).

Lemma 3.1 Let the model hypersurface (2.5) be pseudoconvex and consider the expanded polynomial (2.3). There exist multitype coordinates in which for each \( j = 1, \ldots, n \) there exists a multiindex \( \alpha \) such that \( A_{\alpha, \alpha} \neq 0, \alpha_j \neq 0 \) and \( \alpha_l = 0 \) for all \( l > j \).
Proof By Lemma 3.1 of [16], there exist multitype coordinates such that for each \( j \) the \( j \)-th partial derivative of the polynomial
\[
P^j(z_1, \ldots, z_j) := P(z_1, \ldots, z_j, 0, 0, \ldots, 0)
\]
is not identically zero,
\[
\frac{\partial P^j}{\partial z_j} \neq 0.
\]
For any choice of the variables \( z_1, \ldots, z_{j-1} \) in a neighbourhood of 0, consider the restriction of \( P^j \) to the complex line \( z_1 = c_1, \ldots, z_{j-1} = c_{j-1} \) and denote this one variable polynomial by \( Q \). Since \( \Delta Q \) is nonnegative, it must contain a leading term of the form \( |z_j|^{2m} \), of the lowest degree, unless it is identically zero (which can happen on a thin set). We will consider the minimal such \( m \) over the choice of the complex lines \( z_1 = c_1, \ldots, z_{j-1} = c_{j-1} \). Then \( P^j \) contains the summand \( S(z_1, \ldots, z_{j-1}) |z_j|^{2m} \). \( S \) has to be a nonnegative polynomial, hence it contains a nonzero term with \( \alpha = \hat{\alpha} \) in \( z_1, \ldots, z_{j-1} \). That finishes the proof. \( \square \)

Let us remark that by Lemma 4.6. in [17], if \( X \) lies in \( \text{aut}(M_P, 0) \), then in Jordan normal form both its diagonal and nilpotent part lie in \( \text{aut}(M_P, 0) \). Moreover, both the real and imaginary parts of the diagonal component also lie in \( \text{aut}(M_P, 0) \).

Lemma 3.2 Let the model hypersurface (2.5) be pseudoconvex and let \( X \) be a rigid infinitesimal CR automorphism in \( g_0 \) and \( X \text{Re} \) be the real diagonal of the Jordan normal form of \( X \). Then \( X \text{Re} = 0 \).

Proof Let \( Y = Y \text{Re} \) be the vector field
\[
\sum_{j=1}^{n} \lambda_j z_j \partial z_j, \tag{3.2}
\]
where \( \lambda_j \in \mathbb{R} \), and let \( z^\alpha \bar{z}^{\hat{\alpha}} \) be a monomial, where \( z = (z_1, \ldots, z_n) \), \( \alpha = (\alpha_1, \ldots \alpha_n) \) and \( \hat{\alpha} = (\hat{\alpha}_1, \ldots \hat{\alpha}_n) \). We apply \( Y \) and \( \bar{Y} \) to \( z^\alpha \bar{z}^{\hat{\alpha}} \). Then
\[
Y(z^\alpha \bar{z}^{\hat{\alpha}}) = \sum_{j=1}^{n} \lambda_j \alpha_j z^\alpha \bar{z}^{\hat{\alpha}}, \quad \bar{Y}(z^\alpha \bar{z}^{\hat{\alpha}}) = \sum_{j=1}^{n} \lambda_j \hat{\alpha}_j z^\alpha \bar{z}^{\hat{\alpha}}
\]
and hence,
\[
0 = 2 \text{Re} Y(z^\alpha \bar{z}^{\hat{\alpha}}) = \sum_{j=1}^{n} \lambda_j (\alpha_j + \hat{\alpha}_j) z^\alpha \bar{z}^{\hat{\alpha}}
\]
if and only if
\[
\sum_{j=1}^{n} \lambda_j (\alpha_j + \hat{\alpha}_j) = 0. \tag{3.3}
\]
We denote this as \( \lambda \perp (\alpha + \hat{\alpha}) \).

Consider the decomposition of \( P \) based on the monomials
\[
P(z, \bar{z}) = \sum_{\alpha, \hat{\alpha}} A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^{\hat{\alpha}}. \tag{3.4}
\]
Then $YP(z, \bar{z})$ and $\bar{YP}(z, \bar{z})$ are given by

$$YP(z, \bar{z}) = \sum_{\alpha, \hat{\alpha}} \lambda_j \alpha_j A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^\hat{\alpha}$$

$$\bar{YP}(z, \bar{z}) = \sum_{\alpha, \hat{\alpha}} \lambda_j \hat{\alpha}_j A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^\hat{\alpha}.$$ 

Hence,

$$0 = 2 \text{Re}YP(z, \bar{z}) = \sum_{\alpha, \hat{\alpha}} \lambda_j (\alpha_j + \hat{\alpha}_j) A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^\hat{\alpha}$$

if and only if $\lambda \perp (\alpha + \hat{\alpha})$ for all $A_{\alpha, \hat{\alpha}} \neq 0$.

By Lemma 3.1, the subset $\{2\alpha \in \mathbb{C}^n | A_{\alpha, \hat{\alpha}} \neq 0\}$ of $\{\alpha + \hat{\alpha} \in \mathbb{C}^n | A_{\alpha, \hat{\alpha}} \neq 0\}$ spans $\mathbb{C}^n$. Hence, $\lambda = 0$. \hfill \Box

**Lemma 3.3** Let the model hypersurface (2.5) be pseudoconvex and let $X$ be a rigid infinitesimal CR automorphism in $\mathfrak{g}_0$ and $X^{Nil}$ be the nilpotent part of the Jordan normal form of $X$. Then $X^{Nil} = 0$.

**Proof** Let $Y = X^{Nil}$ be the nilpotent part of $X$,

$$\sum_{i=1}^{n-1} \lambda_i \bar{z}_i \partial z_{i+1}, \quad (3.5)$$

where $\lambda_i \in \{0, 1\}$.

Let

$$P(z, \bar{z}) = \sum_{\alpha, \hat{\alpha}} A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^\hat{\alpha}.$$ 

From the equation $YP(z, \bar{z}) = -\bar{YP}(z, \bar{z})$, we have

$$\sum_{\alpha, \hat{\alpha}, i=1}^{n} \lambda_i \alpha_{i+1} A_{\alpha, \hat{\alpha}} z^{\alpha(i)} \bar{z}^\hat{\alpha} = -\sum_{\alpha, \hat{\alpha}, i=1}^{n} \lambda_i \alpha_{i+1} A_{\alpha, \hat{\alpha}} z^{\hat{\alpha}(i)}, \quad (3.6)$$

where $\alpha(i) = (\alpha_1, \ldots, \alpha_i + 1, \alpha_{i+1} - 1, \ldots, \alpha_n)$ and $\hat{\alpha}(i) = (\hat{\alpha}_1, \ldots, \hat{\alpha}_i + 1, \alpha_{i+1} - 1, \ldots, \hat{\alpha}_n)$.

For the set $\{\alpha\} = \{\hat{\alpha}\}$, we give a partial ordering by $\alpha \succ \beta$ if $\alpha(i) = \beta$ for some $i \in \{1, \ldots, n-1\}$. We denote $\alpha \succeq \beta$ if $\alpha = \beta$, $\alpha \succ \beta$ or there is a subset $\{\gamma_i | i = 1, \ldots, t, \gamma_1 \succ 1 \ldots \succ 1 \gamma_t\}$ of $\{\alpha\}$ such that $\alpha \succ \gamma_1$ and $\gamma_1 \succ \beta$. Then $\{\alpha, \succeq\}$ is a partially ordered set.

Assume $Y \neq 0$. Let $i$ be the largest integer with $\lambda_i \neq 0$. By Lemma 3.1, there exists $\alpha$ such that $\alpha_{i+1} \neq 0$ and $A_{\alpha, \alpha} \neq 0$. In the Eq. (3.6), non-vanishing $\lambda_i \alpha_{i+1} A_{\alpha, \alpha}$ of the left-side implies there are $\beta$ and $\hat{\beta}$ such that $\lambda_k \beta_{k+1} A_{\beta, \hat{\beta}} \neq 0$ for some $k$ and

$$z^{\alpha(i)} \bar{z}^\alpha = z^{\beta(k)} \bar{z}^\hat{\beta(k)}$$

which gives us $\hat{\beta} \succ 1 \alpha$. Since $\lambda_k \beta_{k+1} \neq 0$, the equation (3.6) implies there are $\gamma$ and $\hat{\gamma}$ such that $\lambda_k \gamma_{l+1} A_{\beta, \hat{\beta}} \neq 0$ for some $l$ and

$$z^{\hat{\beta}(l)} \bar{z}^\hat{\beta} = z^{\gamma} \bar{z}^\hat{\gamma(l)}$$

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implies $\hat{\gamma}_1 > 1$ and $\hat{\beta}_1 > 1$. If we continue this, we get infinite-length chain, which is a contradiction, since the set $\{\alpha\}$ is finite. Hence, $Y = 0$. ∎

4 Vanishing of $g_c$

In this section we consider the $g_c$ component of the symmetry algebra.

Proposition 4.1 If $M_P$ given by (2.5) is pseudoconvex and $\dim g_1 = 1$, then $g_c = 0$.

Proof $Y \in g_1$ be a nonzero vector field. By Theorem 4.7 in [17], $Y$ has the form

$$Y = \sum_{j=1}^{n} \varphi_j(z) w \partial_{z_j} + \frac{1}{2} w^2 \partial_w$$

where the first term gives a complex reproducing field, i.e.,

$$2 \sum_{j=1}^{n} \varphi_j(z) P_{z_j}(z, \bar{z}) = P(z, \bar{z}).$$

It follows immediately that the Jordan normal form of the vector field $\sum_{j=1}^{n} \varphi_j(z) \partial_{z_j}$ is diagonal with real coefficients. Hence we consider multitype coordinates in which $Y$ becomes

$$\tilde{Y} = \sum_{j=1}^{n} \lambda_j w z_j \partial_{z_j} + \frac{1}{2} w^2 \partial_w.$$

Assume that $\dim g_c > 0$ and let $X \in g_c$. The commutator of $\tilde{Y}$ and $X$ has to vanish, since otherwise its weight is bigger than one. It follows that the vector field $Z = [W, \tilde{Y}]$, i.e.

$$Z = \sum_{j=1}^{n} \lambda_j z_j \partial_{z_j} + w \partial_w$$

is in $g_0$. Moreover, it is not a real multiple of $E$, since $[X, E] = \kappa X$ for a nonzero real constant $\kappa$, while $[X, Z] = 0$, where we have also used $X \in g_c$ and $[X, W] = 0$. It follows that the linear vector field

$$\sum_{j=1}^{n} (\lambda_j - \mu_j) \partial_{z_j}$$

is a real rotation, which is a contradiction, by Lemma 3.2. ∎

5 Sum of squares models

In this section we consider sum of squares models and give a description of their symmetry algebra.

Lemma 5.1 Let $M_S$ be a sum of squares weighted homogeneous polynomial model and let $\mu_1 < \frac{1}{2}$. Then the subspace $g_{-\mu_j}, j = 1, \ldots, n$ of the symmetry algebra vanishes.
Proof Since the vector field $\partial_{z_i}$ has the weight $-\mu_i$, it is equivalent to show that

$$\mathrm{Re} \partial_{z_i} \left( \mathrm{Im} (w) - \sum_{j=1}^{k} |P_j(z)|^2 \right)$$

is not pluriharmonic. Since $\partial_{z_i} \mathrm{Im} (w) = 0$, we need to show that $\mathrm{Re} \partial_{z_i} \sum_{j=1}^{k} |P_j(z)|^2$ is not pluriharmonic. Since $M_S$ is holomorphically non-degenerate, i.e., $\{\nabla P_j\}$ spans $\mathbb{C}^n$ at a generic point, we have $\max_j \{\deg_{z_i} P_j(z)\} > \max_j \{\deg_{z_i} \partial_{z_i} P_j(z)\} \geq 0$. It follows that, modulo pluriharmonic terms,

$$\sum_{j=1}^{k} (\partial_{z_i} P_j) P_j \neq -\sum_{j=1}^{k} (\partial_{z_i} P_j) P_j.$$  \text{(5.2)}

Let $X = X^{Im}$ be the vector field

$$\sum_{j=1}^{n} i\lambda_j z_j \partial_{z_j},$$

where $\lambda_j \in \mathbb{R}$, and let $z^\alpha \bar{z}^{\hat{\alpha}}$ be a monomial, where $z = (z_1, \ldots, z_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n)$. We have

$$X(z^\alpha \bar{z}^{\hat{\alpha}}) = \sum_{j=1}^{n} i\lambda_j \alpha_j z^\alpha \bar{z}^{\hat{\alpha}}, \quad \bar{X}(z^\alpha \bar{z}^{\hat{\alpha}}) = \sum_{j=1}^{n} -i\lambda_j \hat{\alpha}_j z^\alpha \bar{z}^{\hat{\alpha}}$$

and hence,

$$0 = 2\mathrm{Re} X P(z, \bar{z}) = \sum_{\alpha, \hat{\alpha}} \sum_{j=1}^{n} i\lambda_j (\alpha_j - \hat{\alpha}_j) A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^{\hat{\alpha}}$$

if and only if

$$\sum_{j=1}^{n} \lambda_j \alpha_j = \sum_{j=1}^{n} \lambda_j \hat{\alpha}_j.$$  \text{(5.4)}

Lemma 5.2 If $P(z, \bar{z}) = \sum_{j=1}^{k} |P_j(z)|^2$ is a sum of squares polynomial of weighted degree 1, then $\bar{E} \in g_0$, where

$$\bar{E} = i \sum_{j=1}^{n} \mu_j z_j \partial_{z_j}.$$  \text{(5.5)}

Proof If $P(z, \bar{z}) = \sum_{j=1}^{k} |P_j(z)|^2$ is a sum of squares, then

$$\sum_{j=1}^{n} \mu_j \alpha_j = \sum_{j=1}^{n} \mu_j \hat{\alpha}_j = \frac{1}{2},$$  \text{(5.6)}

for any monomial in the expansion of $P$, which implies the result. \hfill \Box
If \( P(z,\bar{z}) = \sum_{j=1}^k |P_j(z)|^2 \) is a sum of squares of degree \( l \), by Theorem 4.7 of [17] and by Lemma 5.2, the vector field

\[
\frac{1}{2} w^2 \partial_w + 2 \sum_{j=1}^n \mu_j w z_j \partial_{z_j}
\]

is contained in \( g_1 \). It provides the third symmetry in this case.

As a consequence, in combination with results of Sect. 3, we obtain the following precise description of the Lie algebra of infinitesimal automorphisms of \( M_S \) (Theorem 1.6 of the Sect. 1).

**Theorem 5.3** Let \( M_S \) be a sum of squares homogeneous polynomial model of degree \( k > 2 \). Then the Lie algebra of infinitesimal automorphisms \( \mathfrak{g} = \text{aut}(M_S, 0) \) of \( M_S \) admits the weighted grading given by

\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,
\]

where \( \mathfrak{g}_{-1} \) and \( \mathfrak{g}_1 \) are of real dimension one and \( \mathfrak{g}_0 \) is generated by the Euler field and a subalgebra of \( \mathfrak{u}(n) \).

Now we consider the general case of a weighted homogeneous polynomial model. We will denote by \( \kappa_M \) the number of multitype weights with \( \mu_j = \frac{1}{2} \). The following result is Theorem 1.7 of the Sect. 1.

**Theorem 5.4** Let \( M_S \) be the sum of squares weighted homogeneous polynomial model. Then the Lie algebra of infinitesimal automorphisms \( \mathfrak{g} = \text{aut}(M_S, 0) \) of \( M_S \) admits the weighted grading given by

\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1,
\]

where \( \mathfrak{g}_{-1} \) and \( \mathfrak{g}_1 \) are of real dimension one, \( \mathfrak{g}_{\frac{1}{2}} \) and \( \mathfrak{g}_{-\frac{1}{2}} \) are of real dimension \( 2\kappa_M \), and \( \mathfrak{g}_0 \) is generated by the Euler field and a subalgebra of \( \mathfrak{u}(n) \).

**Proof** It follows from the assumptions that in suitable multitype coordinates we can write

\[
P(z, \bar{z}) = \sum_{j=1}^{\kappa_M} |z_j|^2 + Q(z_{\kappa_M+1}, \ldots, z_n, \bar{z}_{\kappa_M+1}, \ldots, \bar{z}_n)
\]

where \( Q \) is weighted homogeneous and balanced (with respect to the multitype weights), not containing any quadratic terms. We verify that the vector fields

\[
a \partial_{z_j} + 2i\bar{a}z_j \partial_w
\]

for each \( j = 1, \ldots, \kappa_M \) and \( a \in \mathbb{C} \) lie in \( \mathfrak{g}_{-\frac{1}{2}} \), and can be integrated to vector fields

\[
a w \partial_{z_j} + 2i\bar{a}z_j \sum_{k=1}^n \mu_k z_k \partial_{z_k} + 2i\bar{a}z_j w \partial_w
\]

which lie in \( \mathfrak{g}_{\frac{1}{2}} \). By Lemma 5.1, there are no other elements of \( \mathfrak{g}_{-\frac{1}{2}} \), hence no other elements in \( \mathfrak{g}_{\frac{1}{2}} \), which gives the first part of the claim. By the results of Sect. 3, all elements of \( \mathfrak{g}_0 \) are purely imaginary in Jordan normal form, which proves the remaining part. \( \square \)
**Example 5.5** Let $M$ be a hypersurface in $\mathbb{C}^N$ given by $\text{Im } w = P(z, \bar{z})$, where

$$P(z, \bar{z}) = A_1|z_1|^{2k_{11}} + A_2|z_1|^{2k_{12}}|z_2|^{2k_{22}} + \cdots + A_n|z_1|^{2k_{1n}}|z_2|^{2k_{2n}} \cdots |z_n|^{2k_{nn}}.$$  

(5.13)

It has been proved in [2] that for any pseudoconvex model of finite multitype the corresponding defining polynomial in suitable coordinates contains such an expression, with $A_j > 0$ and $k_{jj} > 0$ for all $j = 1, \ldots, n$. As an illustration, consider the case when no other terms are present and $k_{jl} > 0$ for any $j, l$. Clearly, $M$ is weakly pseudoconvex (but not convex). Note that $M$ contains the $(n-1)$-dimensional complex analytic set

$$\{(w, z_1, \ldots, z_n) \in \mathbb{C}^N | w = z_1 = 0\}.$$  

(5.14)

Hence all components of D’Angelo multitype, except the first one, are infinite. On the other hand, $M$ is of finite Catlin multitype. The multitype weights are determined recursively by the numbers $k_{jl}$, as described in [2] or [16].

By Proposition 2.6 and 3.9 of [17], the elements of $g_0$ are linear in the given coordinates. Combining this with weighted homogeneity, any element of $g_0$ is of the form

$$X = \sum_{j=1}^n A_j z_j \partial z_j,$$  

(5.15)

where $A_j$ are arbitrary purely imaginary numbers. Since the model is given by sums of squares, by Theorem 1.7 we obtain

$$\dim \text{aut}(M, 0) = n + 3.$$

### 6 Symmetries in complex dimension three

In complex dimension three, we can obtain complete information also about the remaining class of symmetries, the imaginary rotations. The key result is the following lemma.

**Lemma 6.1** Assume that $\mu_1 = \mu_2$ and $P(z, \bar{z})$ is a weighted homogeneous polynomial of degree 1. Let $X = i\lambda_1 z_1 \partial z_1 + i\lambda_2 z_2 \partial z_2 \in g_0$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \neq \lambda_2$. Assume that

$$Y = az_1 \partial z_1 + bz_1 \partial z_2 + cz_2 \partial z_1 + dz_2 \partial z_2$$

also belongs to $g_0$, for some $a, b, c, d \in \mathbb{C}$, with $|b| + |c| > 0$. Then $P$ is of the form

$$P(z, \bar{z}) = (|z_1|^2 + |z_2|^2)^l$$

for some integer $l \in \mathbb{N}$. As a consequence, $\dim g_0 = 5$.

**Proof** We have

$$[X, Y] = i(\lambda_1 - \lambda_2)(bz_1 \partial z_2 - cz_2 \partial z_1) \in g_0$$

and

$$[X, [X, Y]] = -(\lambda_1 - \lambda_2)^2(bz_1 \partial z_2 + cz_2 \partial z_1) \in g_0.$$ 

Since the Jordan normal form of $[X, Y]$ only has purely imaginary part by Lemmas 3.2 and 3.3, the second-order equation $x^2 - (\lambda_1 - \lambda_2)^2bc = 0$ has to have purely imaginary solution, and hence, $bc < 0$ and $c = -\bar{b}$ up to positive scalar. After re-scaling the $z_2$-axis with positive
Let \( c = -\bar{b} \). Note that \( z_2 \partial_{z_2} \) is stable under the re-scaling of \( z_2 \)-axis and so is \( X \). Then

\[
[X, Y] = i(\lambda_1 - \lambda_2)(bz_1 \partial_{z_2} + \bar{b}z_2 \partial_{z_1})
\]

and

\[
[X, [X, Y]] = -(\lambda_1 - \lambda_2)^2(bz_1 \partial_{z_2} - \bar{b}z_2 \partial_{z_1})
\]

for some \( b \in \mathbb{C} \). Hence, \( Z_1 = z_1 \partial_{z_2} - z_2 \partial_{z_1} \in \mathfrak{g}_0 \) and \( Z_2 = iz_1 \partial_{z_2} + iz_2 \partial_{z_1} \in \mathfrak{g}_0 \).

It follows that away from the origin, \( M \) admits three linearly independent vector fields, \( Y, Z_1, Z_2 \), which are also tangent to the spherical model

\[
\text{Im } w = \sum_{j=1}^{2} |z_j|^2.
\]

(6.1)

It follows that the level sets of \( P(z, \bar{z}) \) are the same as those for the function \( \sum_{j=1}^{2} |z_j|^2 \). By homogeneity, we obtain

\[
P(z, \bar{z}) = \left( \sum_{j=1}^{2} |z_j|^2 \right)^l.
\]

(6.2)

for some integer \( l \in \mathbb{N} \).

\[ \square \]

**Corollary 6.2** Let \( M_P \subseteq \mathbb{C}^3 \) be a holomorphically nondegenerate model and assume that the Lie subalgebra \( \mathfrak{g}_0 \) is not abelian. Then \( M_P \) is biholomorphic to

\[
\text{Im } w = (|z_1|^2 + |z_2|^2)^l
\]

for some integer \( 2 \in \mathbb{N} \).

**Example 6.3** Let \( M \) be given by

\[
\text{Im } w = |z_1^2 - z_2^3|^2 + |z_1^5 - z_2^2|^2.
\]

(6.3)

Then \( M \) is not homogeneous with respect to any choice of weights. Using results of [16] it is easily verified that the multitype weights are \((\frac{1}{4}, \frac{1}{6})\). The model is given by

\[
\text{Im } w = |z_1^2 - z_2^3|^2 + |z_1|^4.
\]

(6.4)

It follows immediately from Neelon’s criterion [24] that the model is holomorphically non-degenerate.

By the previous lemma, \( \mathfrak{g}_0 \) is abelian, of real dimension two, spanned by the Euler field and the imaginary rotation \( X = \frac{i}{4} z_1 \partial_{z_1} + \frac{i}{6} z_2 \partial_{z_2} \). Using Theorem 1.7, we obtain

\[
\dim \text{aut}(M, 0) = 4.
\]

**Example 6.4** Let \( M \) be given by

\[
\text{Im } w = |(z_1^p - z_2^q)z_1^p|^2 + |(z_1^p - z_2^q)z_2^q|^2,
\]

(6.5)

for two integers \( p > q \). Again, using [16] we easily verify that the multitype weights are \((\frac{1}{p}, \frac{1}{q})\). Note that the complex analytic variety

\[
\{(w, z_1, z_2) \in \mathbb{C}^3 | w = (z_1^p - z_2^q) = 0\}
\]

(6.6)

is contained in \( M \). Hence \( M \) is of infinite D’Angelo type and the techniques of [3] are not applicable. By the same argument as in the previous example, we obtain \( \dim \text{aut}(M, 0) = 4 \).
7 Proof of the main results

Theorems 1.6 and 1.7 were proved in Sect. 5 (as Theorems 5.3 and 5.4). We give now the proof of Theorem 1.2.

Proof By Theorem 1.3 of [17] and Theorem 4.1, we obtain that \( g = \text{aut}(M_P, 0) \) admits the weighted grading given by

\[
g = g_{-1} \oplus \bigoplus_{j=1}^{n} g_{-\mu_j} \oplus g_0 \oplus g_n \oplus g_1.
\] (7.1)

It remains to prove that the weight of \( g_n \) is \( \frac{1}{2} \). We will use the following characterization of manifolds with nonvanishing \( g_n \).

Let \( M_P \) have nontrivial \( g_n \) and

\[
X = i \partial_{\bar{z}_l}
\] (7.2)

for some \( l \) be an infinitesimal symmetry of \( M_P \), which can be integrated, i.e. there exists \( Y \in \text{aut}(M_P, 0) \) such that \([Y, W] = X\). Let us write \( P \) as

\[
P(z, \bar{z}) = \sum_{j=0}^{m} (\text{Re} \, z_l)^j P_j(z', \bar{z}'),
\] (7.3)

for some weighted homogeneous polynomials \( P_j \) in \( z' = (z_1, \ldots, \hat{z}_l, \ldots, z_n) \), with \( P_m \neq 0 \). Then \( M_P \) has one of the following two forms. Either

\[
P(z, \bar{z}) = x_l^2 + x_l P_1(z', \bar{z}') + P_0(z', \bar{z}'),
\] (7.4)

or

\[
P(z, \bar{z}) = x_l P_1(z', \bar{z}') + P_0(z', \bar{z}').
\] (7.5)

This characterization was proved by Kolůř and Meylan in [20] in the \( \mathbb{C}^3 \) case, and in [19] in general. It is immediate to verify that in the second case, the manifold is not pseudoconvex. On the other hand, in the first case we have \( \mu_l = \frac{1}{2} \), which leads to the claim of Theorem 1.2.

\( \square \)

Proposition 1.4 now follows from Theorem 1.2, Theorem 4.7 in [17] and Proposition 4.3 in [19]. Theorem 1.5 is a direct consequence of Theorem 1.2 and Proposition 4.3 in [19].

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