Polyhedral Voronoi Cells

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Abstract – Voronoi cells of a discrete set in Euclidean space are known as generalized polyhedra. We identify polyhedral cells of a discrete set through a direction cone. For an arbitrary set we distinguish polyhedral from non-polyhedral cells using inversion at a sphere and a theorem of semi-infinite linear programming.

Index Terms – Voronoi cell, polyhedron, discrete point set

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1 Introduction

The Voronoi diagram of a finite set in the $n$-dimensional Euclidean space $\mathbb{E}^n$ is a popular concept in Discrete and Computational Geometry, cf. Aichholzer and Aurenhammer \cite{1} or Okabe et al. \cite{5}, as well as in Minkowski Geometry, cf. Section 4 in Martini and Swanepoel \cite{4}.

A natural generalization from a finite set is the concept of a discrete set. By definition, a subset of $\mathbb{E}^n$ is discrete if its intersection with any bounded set of $\mathbb{E}^n$ is finite. Here $M \subset \mathbb{E}^n$ is bounded if $\sup_{x,y \in M} \|x - y\| < \infty$ with the Euclidean norm $\| \cdot \|$ based on the Euclidean scalar product $(\cdot, \cdot)$. Equivalently, a subset of $\mathbb{E}^n$ is discrete if it has no accumulation point.

We study the cardinality of half spaces needed to describe a Voronoi cell. A closed half space is defined for non-zero $u \in \mathbb{E}^n$ and $\lambda \in \mathbb{R}$ by

$$H^-(u, \lambda) := \{ x \in \mathbb{E}^n \mid (x, u) \leq \lambda \}.$$  

The Voronoi diagram of a non-empty generator $\mathcal{P} \subset \mathbb{E}^n$ is the tessellation of $\mathbb{E}^n$ consisting of the Voronoi cells

$$V(p) := \{ x \in \mathbb{E}^n \mid \|x - p\| \leq \|x - q\| \text{ for all } q \in \mathcal{P} \}, \quad p \in \mathcal{P}.$$  

By translational invariance we assume in this article that the origin $0_n$ of $\mathbb{E}^n$ belongs to $\mathcal{P}$ and we restrict to the cell $\mathcal{V} := V(0_n)$ at $0_n$. Notice the closed half space representation

$$2\mathcal{V} = \bigcap_{p \in \mathcal{P} \setminus \{0_n\}} H^-(p, \|p\|^2),$$  

the intersection over the empty index set being understood as $\mathbb{E}^n$.

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Figure 1: The Voronoi diagram of the generator $\mathcal{P} := \{(-1, z) : z \in \mathbb{Z}\} \cup \{0_2\}$ is depicted. The direction cone $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x < 0\} \cup \{0_2\}$ is not closed, hence it is not finitely generated. Theorem 1.2 concludes that the Voronoi cell $\mathcal{V}$ is not a polyhedron.

To catch the structure of a Voronoi cell of a discrete set we use the following definitions. Let a subset $M \subset \mathbb{R}^n$ be given. The affine hull $\text{aff}(M)$ resp. positive hull $\text{pos}(M)$ of $M$ consists of sums $\lambda_1 x_1 + \cdots + \lambda_N x_N$ such that for $i = 1, \ldots, N$ we have $\lambda_i \in \mathbb{R}$, $x_i \in M$ and $\lambda_1 + \cdots + \lambda_N = 1$ resp. $\lambda_i \geq 0$. Notice $\text{aff}(\emptyset) = \emptyset$ and $\text{pos}(\emptyset) = \{0_n\}$. The convex hull of $M$ is $\text{conv}(M) := \text{aff}(M) \cap \text{pos}(M)$. If $M = \text{conv}(M)$ then $M$ is convex. If for any $\lambda \geq 0$ and $\lambda M := \{\lambda m : m \in M\}$ we have $\lambda M \subset M$, then $M$ is a cone. A convex cone is finitely generated if it is the positive hull of a finite set. A polyhedron is the intersection of finitely many closed half spaces. A bounded polyhedron is a polytope. If $M$ is convex and if any intersection of $M$ with a polytope is a polytope, then $M$ is a generalized polyhedron.

Remark 1.1. It is clear from (1) that a Voronoi cell of a finite generator is a polyhedron. It is proved by Gruber [3], Chapter 32, that a Voronoi cell of a discrete generator is a generalized polyhedron.

The existence of a non-polyhedral Voronoi cell for a discrete generator is demonstrated in Figure 1. One of us has characterized a polyhedral Voronoi cell through the direction cone $\mathcal{D} := \text{pos}(\mathcal{P})$.

Theorem 1.2 (Ina Voigt [7]). If $\mathcal{P}$ is discrete then the Voronoi cell $\mathcal{V}$ is a polyhedron if and only if the direction cone $\mathcal{D}$ is finitely generated.

Further examples to apply this theorem are in Figure 2 and Figure 3. For subsequent discussions we recall that a convex cone is finitely generated if and only if it is a polyhedron, cf. Ziegler [8]. If $F$ is a convex subset of a

\[^3\text{In optimization the cone } \mathcal{D} \text{ is called the cone of feasible directions.}\]
convex subset $C \subset \mathbb{E}^n$ such that for $x, y \in C$ and $0 < \lambda < 1$ the inclusion $\lambda x + (1 - \lambda)y \in F$ always implies $x, y \in F$, then $F$ is a face of $C$. The zero dimensional faces are the extreme points and an extreme ray is a face which is a half line emanating from the origin. We notice that a polyhedron has at most finitely many faces, cf. §19 in Rockafellar [6].

In this article we recover Theorem 1.2 from results about a generator $P$ not necessarily discrete. As a tool we use the diffeomorphism of inversion at the unit sphere

$$\text{inv} : \mathbb{E}^n \setminus \{0_n\} \rightarrow \mathbb{E}^n \setminus \{0_n\}, \ x \mapsto \frac{x}{\|x\|^2},$$

the reciprocal $\mathcal{R} := \text{inv}(P \setminus \{0_n\})$ and the convex reciprocal

$$\mathcal{C} := \text{conv}(\mathcal{R} \cup \{0_n\}).$$

Illustrations of $\mathcal{C}$ are given in the following figures. We apply in Section 2 a theorem of semi-infinite linear programming by Goberna and López and prove that the Voronoi cell $\mathcal{V}$ is a polyhedron if and only if the closure $\overline{\mathcal{C}}$ is a polyhedron. The intuition is that $\mathcal{V}$ is completely surrounded by the Voronoi cells corresponding to the extreme points of $\overline{\mathcal{C}}$ while $\mathcal{V}$ has no extension in unbounded directions of $\overline{\mathcal{C}}$. The surrounding cells are finite in number only if $\overline{\mathcal{C}}$ is a polyhedron. In the case of a discrete generator $P$ the condition relaxes to the condition that $\mathcal{C}$ is a polytope.

Compared to the concept of convex reciprocal, the direction cone

$$\mathcal{D} = \text{pos}(P) = \text{pos}(\mathcal{R})$$

is nearer to the geometry of the generator $P$. We think of $\mathcal{D}$ as the area $0_n + \mathcal{D}$ occupied by $P$ from point of view of $0_n$. For arbitrary $P$ we find in Section 3 that a polyhedral closure $\overline{\mathcal{D}}$ is necessary for a polyhedral Voronoi cell $\mathcal{V}$, i.e. a finite number of extreme rays of $\overline{\mathcal{D}}$ is necessary. On the other hand, a polyhedral $\mathcal{D}$ is not sufficient, a simple example being Figure 4.

Stronger conditions apply to the case of a discrete generator $P$, we recover Theorem 1.2. The generator must be enclosed in a half space $H^-(u, 0)$ for non-zero $u \in \mathbb{E}^n$ to realize a non-polyhedral Voronoi cell $\mathcal{V}$. Then any asymptotic direction of the generator not in $\mathcal{D}$, i.e. any accumulation point of $\{\frac{x}{\|x\|} \mid x \in P \setminus \{0_n\}\}$ not in $\mathcal{D}$, makes a finite half space representation of $\mathcal{V}$ impossible, see Figure 1 and 2 for examples. (For the general generator there is no such condition, see Figure 5 as a counterexample.)

In Section 4 we discuss the polar $\mathcal{D}^*$ of the direction cone $\mathcal{D}$. This is the normal cone at $0_n$ of the convex hull $\text{conv}(P)$ of the generator,

$$\mathcal{D}^* = \{ x \in \mathbb{E}^n \mid \langle x, y \rangle \leq 0 \text{ for all } y \in \text{conv}(P) \}.$$ 

As we noted above, the cone $\mathcal{D} = \mathcal{D}^{**}$ is useless as a sufficient condition for a polyhedral Voronoi cell. However, it is useful to decide if a Voronoi cell is bounded. This problem is resolved in the literature for finite generators, see e.g. Okabe et al. [5].
2 The characteristic cone

We apply a theorem from semi-infinite linear programming to the special case of the Voronoi cell \( \mathcal{V} \) and obtain conditions on the cone of inequalities for the half spaces representation of \( \mathcal{V} \). The result is interpreted in terms of the convex reciprocal \( C \). The cone of inequalities is well-known from the lifting construction for Delaunay triangulations.

The starting point is the, possibly infinite, system of linear inequalities

\[
\sigma := \{ \langle p, x \rangle \leq \|p\|^2 \mid p \in \mathcal{P} \}
\]

satisfied by an unknown \( x \in \mathbb{R}^n \), if and only if \( x \) belongs to the doubly sized Voronoi cell \( 2\mathcal{V} \), see (1). The *characteristic cone* of \( \sigma \) is\(^4\)

\[
K := \text{pos}\left( \left\{ \left( \frac{p}{\|p\|^2} \right) \mid p \in \mathcal{P} \setminus \{0\}_n \right\} \cup \left\{ (0_n) \right\} \right) \subset \mathbb{R}^{n+1}.
\]

The trivial equation for \( 0_n \in \mathcal{P} \) is omitted. As a special case of Theorem 5.13 in Goberna and López [2] the following equivalence holds.

**Theorem 2.1** (Goberna and López). \( \mathcal{V} \) is a polyhedron if and only if the closure \( \overline{K} \) of the characteristic cone \( K \) is a polyhedron.

In place of \( K \subset \mathbb{R}^{n+1} \) we can study the convex reciprocal

\[
\mathcal{C} = \text{conv}\left( \left\{ x \|x\|^{-2} \mid x \in \mathcal{P} \setminus \{0\}_n \right\} \cup \{0_n\} \right) \subset \mathbb{R}^n.
\]

We have \( K = \bigcup_{\lambda \geq 0} \lambda \left( \mathcal{C} \times \{1\} \right) \) and \( \mathcal{C} \times \{1\} = K \cap \{ x_{n+1} = 1 \} \). We must be careful in a discussion of closures: if \( \mathcal{C} \) is unbounded then \( \overline{K} \) has non-zero points in the hyperplane \( x_{n+1} = 0 \) but \( K \) does not.

**Proposition 2.2.** The following statements are equivalent:

(i) \( \mathcal{V} \) is a polyhedron,

(ii) \( \overline{K} \) is a polyhedron,

(iii) \( \overline{\mathcal{C}}\) is a polyhedron.

**Proof.** We recall from Theorem 11.5 in Rockafellar [6] that a closed convex set is the intersection of the closed half spaces that contain the set. More specific assertions about closures of convex hulls and closures of positive hulls are Corollary 11.5.1 and Corollary 11.7.2 in the same reference. As a consequence we can write for the same (possibly empty) index set \( I \subset \mathbb{R}^n \setminus \{0_n\} \times \mathbb{R} \)

\[
\overline{\mathcal{C}} = \bigcap_{(u, \lambda) \in I} H^- (u, \lambda)
\]

\(^4\)In [2] the characteristic cone is used with the opposite sign (reflected at the origin) compared to our definition.
Proposition 2.2 is explained with non-discrete examples in Figure 4 and Figure 5. The statement of the proposition simplifies in the discrete case with the following remark. This is stated in Corollary 2.4 and explained with two discrete examples in Figure 2 and Figure 3.

Remark 2.3. If \( P \) is discrete then the convex reciprocal \( C \) is compact and the characteristic cone \( K \) is closed. Observe that the reciprocal \( R \) is bounded having \( 0_n \) as the only possible accumulation point. Then \( R \cup \{0_n\} \) is compact and from Carathéodory’s theorem follows that \( C = \text{conv}(R \cup \{0_n\}) \) is compact. Under the linear map \( \alpha : E^{n+1} \to E^{n+1}, (x, \lambda) \mapsto (\lambda x, \lambda) \), the closed cylinder \( \tilde{C} := C \times \{\lambda \geq 0\} \subset E^{n+1} \) is mapped to \( K \). The kernel \( \mathbb{E}^n \times \{0\} \) of \( \alpha \) does not contain the direction \( y = (0_n, 1) \) of recession of \( \tilde{C} \), i.e. a direction \( y \) with \( \tilde{C} + y \subset \tilde{C} \), so \( K \) is closed, cf. Theorem 9.1 in [6].

Corollary 2.4. If \( P \) is discrete then the following statements are equivalent:

(i) \( V \) is a polyhedron,

(ii) \( K \) is a polyhedron,

(iii) \( C \) is a polytope.
Figure 3: The Voronoi diagram of $\mathcal{P} := \{(t, f(t)) \mid t \in \frac{1}{2}\mathbb{Z}\} \cup \{0\}$ is depicted (left). The graph of the real function $f : t \mapsto \frac{1}{4}(t + 2)^2$ has tangents through $(-2, 0)$ and $(2, 4)$ meeting $0_2$. Thus, there is a steepest line through $0_2$ that meets a point of $\mathcal{P} \setminus \{0_2\}$ so the direction cone $\mathcal{D}$ is finitely generated. Then Theorem 1.2 proves that $\mathcal{V}$ is a polyhedron. A second argument that $\mathcal{V}$ is a polyhedron is Corollary 2.4. The two branches $\text{inv}(t, f(t))$ are concave near $0_2$ (for large $|t|$), so almost all points of $\mathcal{R}$ belong to the interior of $\mathcal{C}$, which therefore is a polytope (right).

**Remark 2.5** (Delaunay triangulations). A Delaunay diagram of a finite generator $\mathcal{P}$ is defined as a tessellation of $\text{conv}(\mathcal{P})$ where a circumsphere of a cell is an empty sphere. Thereby a circumsphere of a cell is an empty sphere (also empty circle) if the interior of the corresponding ball has an empty intersection with $\mathcal{P}$. A well known construction method for Delaunay diagrams is the lifting construction based on the map $L : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $x \mapsto (x, \|x\|^2)$. A Delaunay diagram of $\mathcal{P}$ is obtained as the orthogonal projection of the lower convex hull of $L(\mathcal{P})$ onto the $(x_1, \ldots, x_n)$-plane. That is, the edges of

$$\{(x, z) \in \text{conv}(\mathcal{P}) \times \mathbb{E} \mid \exists y \leq z \text{ such that } (x, y) \in \text{conv}(L(\mathcal{P}))\} \subset \mathbb{R}^{n+1},$$

projected orthogonally to the $(x_1, \ldots, x_n)$-plane, produce a Delaunay diagram for $\mathcal{P}$, see for example Okabe et al. [5]. In particular, the edges of $\text{pos}(L(\mathcal{P}))$ correspond to the edges emanating from $0_n$ in this diagram.

Heuristically, we consider the lifting construction for an infinite discrete generator $\mathcal{P}$. The positive hull of the lifted generator is

$$\text{pos}(L(\mathcal{P})) = \text{pos} \left( \left( \text{inv}(x) \right)_1, x \in \mathcal{P} \setminus \{0_n\} \right) = \bigcup_{\lambda \geq 0} \lambda \left[ \text{conv}(\mathcal{R}) \times \{1\} \right].$$

Similarly as in Remark 2.3 we have $\overline{\text{pos}(L(\mathcal{P}))} = K$, if $\mathcal{P}$ is unbounded (otherwise $\mathcal{P}$ is finite). If the Voronoi cell $\mathcal{V}$ is not a polyhedron then by Theorem 2.1 the characteristic cone $K$ is not a polyhedron, so $\text{pos}(L(\mathcal{P}))$ is not a polyhedron. This is in accordance with the Delaunay diagram having infinitely many edges emanating from the origin.
3 The direction cone

We compare the convex reciprocal $C$ to the direction cone $D$. While a polyhedral Voronoi cell $V$ was found equivalent to a polyhedral closure $\overline{C}$ in the last section, we will see in this section that a polyhedral Voronoi cell $V$ is a stronger condition compared to a polyhedral direction cone $D$. In the case of a discrete generator $P$ these conditions are equivalent in accordance with Theorem 1.2.

Let us study what consequences a polyhedral direction cone $D$ can have for the Voronoi cell $V$. We continue in Remark 3.1 with a proof sketch of a necessary assertion, omitting to explain the concepts needed for a proof.

Remark 3.1. If $G \subset \mathbb{E}^n$ is a polyhedron containing the origin $0_n$ and if $H = \text{pos}(G)$ then there exists an $\epsilon > 0$ such that for the open ball $B_\epsilon(0_n) := \{x \in \mathbb{E}^n \mid \|x\| < \epsilon\}$ we have

$$G \cap B_\epsilon(0_n) = H \cap B_\epsilon(0_n). \quad (2)$$

For a proof we can use the gauge of $G$ defined for $x \in \mathbb{E}^n$ by

$$\gamma(x) := \inf \{ \lambda \geq 0 \mid x \in \lambda C \}.$$  

For (2) to hold it is sufficient to find some $\epsilon > 0$ such that $\gamma(x) < \epsilon^{-1}$ holds for all $x$ in the unit sphere $S(H) := \{x \in H \mid \|x\| = 1\}$. The gauge $\gamma$ is a positively homogeneous function, whence it has finite values on $H$. On the other hand, with $0_n \in G$, the positive hull $H = \text{pos}(G)$ is a polyhedron and as such, is locally simplicial. These facts can be found in Rockafellar [6]. Theorem 10.2 in this reference concludes that $\gamma$ is upper semi-continuous on $H$. Since the unit sphere $S(H)$ is compact, $\gamma$ has a finite maximum there. This proves (2).

Asking for sharpness of (2), let $G := \{x \in \mathbb{E}^2 \mid \|x - (0,0.5)\| \leq 0.5\}$, this is a closed disk touching the origin. Here the gauge $\gamma$ still is upper semi-continuous on $H \setminus \{0_2\} = \{(x, y) \in \mathbb{E}^2 \mid x < 0\}$ but the unit sphere $S(H)$ is not compact and for each $\epsilon > 0$ we have $G \cap B_\epsilon(0_n) \subset H \cap B_\epsilon(0_n)$. A related example is the convex reciprocal for the example in Figure 1 with reciprocal $\mathcal{R} = \{(-1, t)(1 + t^2)^{-1} \mid t \in \mathbb{Z}\}$ included in the boundary of $G$.

Proposition 3.2. If the generator $P$ is discrete and if the direction cone $D$ is a polyhedron, then the convex reciprocal $C$ is a polytope.

Proof. If the cone $D = \text{pos}(P)$ is finitely generated, then we can assume that it is finitely generated by points of the generator $P$ or by points of the reciprocal $\mathcal{R}$, likewise: for $r_1, \ldots, r_k \in \mathcal{R}$ we have

$$D = \text{pos}(r_1, \ldots, r_k).$$
Figure 4: A finite sketch of the Voronoi diagram for unit sphere with center $P := \{(\cos(\varphi), \sin(\varphi)) \mid \varphi \in [0, 2\pi) \} \cup \{0_2\}$ is depicted (left). As the convex reciprocal $\overline{C} = C = \text{conv}(P)$ is not a polyhedron, the Voronoi cell $V$ is not a polyhedron, cf. Proposition 2.2. Another example is the Voronoi cell $V$ of $P := \{(e^{-x}, 1) \mid x = 1, 2, 3, \ldots \} \cup \{0, (0, 1)\}$ (right). Here the convex reciprocal $\overline{C}$, having the infinitely many extreme points $R$, is not a polyhedron (middle). Still, the direction cones are polyhedral in both examples.

Now we consider the polytope $\tilde{C} := \text{conv}(0, r_1, \ldots, r_k)$. Since $\mathcal{D} = \text{pos}(\tilde{C})$ we meet the assumptions of Remark 3.1 and can infer that $\mathcal{D} \setminus \text{inv}(\tilde{C})$ is bounded. So this set contains at most finitely many points of the discrete generator $\mathcal{P}$. Then all but finitely many points $s_1, \ldots, s_l$ of $R$ belong to $\tilde{C}$ and therefore the convex reciprocal $C = \text{conv}(0, r_1, \ldots, r_k, s_1, \ldots, s_l)$ is a polytope. \qed

**Corollary 3.3.** If the generator $\mathcal{P}$ is discrete and if the direction cone $\mathcal{D}$ is a polyhedron then the Voronoi cell $V$ is a polyhedron.

The above conclusion follows from Proposition 3.2 and Corollary 2.4. Figure 3 shows an application. Figure 4 demonstrates that a polyhedral direction cone is not sufficient for a polyhedral Voronoi cell without the assumption of the discrete generator.

Assuming the discrete generator, a polyhedral closure $\overline{\mathcal{D}}$ of the direction cone is not sufficient, we remember Figure 1 and Figure 2. Now we will see that a polyhedral closure $\overline{\mathcal{D}}$ of the direction cone is necessary regardless of the generator.

**Proposition 3.4.** If the convex reciprocal $C$ is a polyhedron then the direction cone $\mathcal{D}$ is a polyhedron. If the closure $\overline{C}$ is a polyhedron then the closure $\overline{\mathcal{D}}$ is a polyhedron.

**Proof.** We have the trivial chain of inclusions

$$\mathcal{D} = \text{pos}(C) \subset \text{pos}(\overline{C}) \subset \overline{\text{pos}(C)} = \overline{\mathcal{D}}.$$
Figure 5: We consider the generator \( P := \{0_2, p_{\alpha(1)}, p_{\alpha(2)}, p_{\alpha(3)}, \ldots\} \) with 
\[ p_{\alpha} := 2 \cos(\alpha)(-\sin(\alpha), \cos(\alpha)) \] 
and \( \alpha(t) := \frac{\pi}{4} e^{-t} \). The closed convex reciprocal \( \overline{C} \) is a triangle (middle), so the Voronoi cell \( V \) is a polyhedron by Proposition 2.2 (left). While the direction cone \( D = \text{pos}(\{\lambda, 1 \mid -1 \leq \lambda < 0\}) \) is not closed, the closure \( \overline{D} \) is a polyhedron by Corollary 3.5. The idea to the example (right): we narrow \( V \) below certain lines through \((0, 1)\).

With \( \overline{C} \) being a polyhedron containing the origin, Corollary 19.7.1 in [6] proves that the positive hull \( \text{pos}(\overline{C}) \) is a polyhedron. Thus, with \( \text{pos}(\overline{C}) \) being closed we obtain that \( \overline{D} = \text{pos}(\overline{C}) \) is a polyhedron. Assuming that \( C \) is a polyhedron, we can replace \( C \) by \( \overline{C} \) and obtain \( D = \text{pos}(\overline{C}) \). □

Corollary 3.5. If the Voronoi cell \( V \) is a polyhedron then the closure \( \overline{D} \) of the direction cone is a polyhedron.

Corollary 3.6. If the generator \( P \) is discrete and if the Voronoi cell \( V \) is a polyhedron then the direction cone \( D \) is a polyhedron.

The above conclusions follow from Proposition 3.4 together with Proposition 2.2 and Corollary 2.4 in this order. Figure 5 introduces a polyhedral Voronoi cell \( V \) where \( D \) is not closed. So, a polyhedral direction cone \( D \) is necessary for a polyhedral Voronoi cell \( V \) only in the case of a discrete generator.

4 Bounded cells

Using normal cones we prove a condition when the Voronoi cell \( V \) is bounded. For a discrete generator \( P \) this is a condition when \( V \) is a polytope.

The normal cone of a convex subset \( C \subset \mathbb{R}^n \) at \( x \in C \) is defined by

\[
N(C, x) := \{ u \in \mathbb{R}^n \mid \langle y - x, u \rangle \leq 0 \text{ for all } y \in C \}.
\]

This is the set of vectors \( u \in \mathbb{R}^n \) having no acute angle at \( x \) with any point \( y \in C \). An example is

\[
N(\text{conv}(P), 0_n) = N(C, 0_n) = \{ u \in \mathbb{R}^n \mid \langle y, u \rangle \leq 0 \text{ for all } y \in P \}, \quad (3)
\]
where the set equalities hold because for \( u \in E \) the inequality \( \langle y, u \rangle \leq 0 \) for all \( y \in P \) is equivalent to this inequality for all \( y \in R \) or for all \( y \) in the convex hull of one of these sets. The inequalities are even equivalent to these with \( y \) running through the positive hull pos(\( P \)) = \( D \), so the polar of the direction cone \( D^* := \{ u \in E \mid \langle y, u \rangle \leq 0 \text{ for all } y \in D \} \) satisfies

\[
D^* = N(\text{conv}(P), 0_n) = N(C, 0_n).
\] (4)

The recession cone of \( V \) describes unbounded directions of \( V \), it is

\[
\text{rec}(V) := \{ u \in E^n \mid x + \lambda u \in V \text{ for all } x \in V, \lambda \geq 0 \}.
\]

**Lemma 4.1.** The equality of cones \( N(\text{conv}(P), 0_n) = \text{rec}(V) \) holds.

**Proof.** We have the representation \( 2V = \bigcap_{p \in P \setminus \{0_n\}} H^-(p, \|p\|^2) \) by half spaces (1). With Corollary 8.3.3 in [6] the recession cone \( \text{rec}(V) = \text{rec}(2V) \) becomes

\[
\bigcap_{p \in P \setminus \{0_n\}} \text{rec}(H^-(p, \|p\|^2)) = \bigcap_{p \in P \setminus \{0_n\}} H^-(p, 0).
\]

The proof is completed by (3). \( \Box \)

As a consequence we can determine boundedness of the cell \( V \). For \( M \subset E^n \) let \( M^o \) denote the interior of \( M \) in the topology of \( E^n \).

**Proposition 4.2.** The Voronoi cell \( V \) is bounded if and only if \( 0_n \in C^o \) if and only if \( 0_n \in \text{conv}(P)^o \). If \( P \) is discrete, then \( V \) is a polytope if and only if one of these equivalent conditions holds.

**Proof.** By Theorem 8.4 in [6] the Voronoi cell \( V \) is bounded if and only if the recession cone \( \text{rec}(V) \) is zero. Using the equality \( N(\text{conv}(P), 0_n) = \text{rec}(V) \) in Lemma 4.1 this is equivalent to a zero normal cone \( N(\text{conv}(P), 0_n) \). Now by Theorem 13.1 in [6], the normal cone at \( 0_n \) is zero if and only if \( 0_n \in \text{conv}(P)^o \) holds. Using (3) we can argue with \( C \) in place of \( \text{conv}(P) \). In the discrete case, the cell \( V \) is a generalized polyhedron, see Remark 1.1. But a generalized polyhedron is bounded if and only if it is a polytope. \( \Box \)

The zero normal cone responsible for a bounded Voronoi cell in Proposition 4.2 is equivalent to the equality \( \overline{D} = E^n \) through polarity of closed cones, cf. (4) and Theorem 14.1 in [6]. Since a convex cone \( D \subseteq E^n \) is included in a half space, we have for arbitrary generator \( P \)

\[
V \text{ is bounded} \iff D = E^n \iff \overline{D} = E^n.
\]

Figure 6 shows an example of a discrete generator in \( E^2 \), which is bounded in \( y \)-direction and where nevertheless every Voronoi cell is a polytope.
Figure 6: All Voronoi cells of the generator $P := \{ \pm (n, 1 - \frac{1}{n}) \mid n \in \mathbb{N} \}$ are polytopes by Proposition 4.2 as $\text{conv}(P) = \{(x, y) \in \mathbb{R}^2 \mid |y| < 1\}$ is open.

5 Conclusion

By considering a Voronoi cell $V$ as a problem in linear semi-infinite programming, we obtain an equivalent condition when $V$ is a polyhedron. This is the condition that the closure $C$ of the convex reciprocal is a polyhedron and the condition simplifies in the case of a discrete generator $P$ to the condition that the convex reciprocal $C$ is a polytope.

While the polyhedral Voronoi cell $V$ implies the polyhedral closure $D$ of the direction cone, the polyhedral cone $D$ follows only in the discrete case. Conversely, the polyhedral direction cone $D$ implies a polyhedral Voronoi cell only in the discrete case.

The closure $D$ gives only advice if the Voronoi cell $V$ is bounded or not.

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