The Polyakov loop dependence of bulk viscosity of QCD matter

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In this work, we show the dependence of bulk viscosity on Polyakov loop in 3+1 dimensional topologically massive model (TMM). This model contains equally massive non-Abelian gauge fields without spontaneous symmetry breaking. In earlier works, the bulk viscosity was found from the trace anomaly in massless $\phi^4$ model and Yang-Mills (YM) theory and its dependence on the quantum corrections was established. In TMM, the trace anomaly is absent due to the presence of kinetic term of a two-form field $B$ in the action. This model also provides the dependence of bulk viscosity on the mass of the gauge bosons. The mass of the gauge bosons in TMM acts as magnetic mass in the perturbative thermal field theory. This magnetic mass is gauge independent unlike what is found in massless YM theory. We also observe that the strong coupling constant has the same behaviour at high energy limit (i.e. asymptotic freedom) as that of massless YM theory at zero temperature.

PACS numbers: 05.60.Gg; 05.70.Fh; 05.70.Ln; 11.10.Wx; 11.25.Db; 12.38.Mh; 12.60.-i
Keywords: Topologically massive model; QCD; bulk viscosity; magnetic mass; Polyakov loop; heat-kernel method; asymptotic freedom

I. INTRODUCTION

In recent times, Schwarz-type topological field theory in 3+1 dimensions drew huge attentions due to some of its very important characteristics in gauge theory [1,3]. One of the interesting features of the model is that it contains the massive vector modes in spite of unbroken global symmetry i.e. without taking any recourse of Higgs mechanism [4,5]. TMM in 3+1 dimensions carries many advantages in the perturbative analysis in both zero and finite temperature field theories over the massless YM theory. The interest is increased manifold when the model was found to be unitary [6,7] and renormalizable [8]. In this model, YM fields become equally massive without leaving any extra degrees of freedom unlike the case of Higgs mechanism. Their masses are generated due to the presence of topological term $mB \wedge F$ which contains a quadratic mixing of a one-form YM field $A$ and a two-form field $B$. The massive one- and two-form fields have the same number of degrees of freedom [9,10]. Hence, an effective theory, constructed by integrating out YM field or $B$ field, becomes a massive theory of vector bosons. Unlike the massless YM theory, the complete propagator of YM field in TMM carries a non-zero pole, which is the coefficient of topological term $B \wedge F$ in the model. In TMM, the YM field acquires an additional physical longitudinal mode due to its mass. But, in spite of having longitudinal mode, the high energy behavior of scattering matrix maintains unitarity in the scattering process involving those modes. This is because of the fact that the model is invariant under Becchi-Rouet-Stora-Tyutin (BRST) symmetry transformations [6,7,11,12]. The unitarity is also maintained at every order of quantum corrections since it is renormalizable [8]. The massiveness of the gluon also plays a crucial role in maintaining the cluster decomposition principle i.e. causality [13,14] in quantum field theory. On the other hand, the non-zero pole of gluon propagator can explain gluon confinement in quantum chromodynamics (QCD) [15,17]. In the regime of strong interaction, we shall see that the TMM provides the same asymptotic behaviour of strong coupling at high energy limit (i.e. asymptotic freedom) as found in massless YM theory [18,20]. The asymptotic freedom is a very significant characteristic of strong sector in the Standard Model. Beside these important advantages of massive YM field at zero temperature, we consider its significant role in the perturbative thermal field theory (TFT). The masses of the vector fields put an infrared (IR) cut-off in the model, which behaves as magnetic mass and overcome the Linde infrared problem in TFT [21,22]. This assures the validity of perturbative analysis of the dynamics of YM field at finite temperature which is absent in massless case [22]. Moreover, it is interesting to point out that the magnetic mass in massless YM theory is not gauge independent [23,24], hence it is not a physical quantity. But the lattice gauge theories have showed the short range behaviour of chromomagnetic field [25,26] due to the presence of physical magnetic mass. In TMM, the mass of the gluon is gauge independent [5] and it plays the role of magnetic mass in the
It was also observed that massless gluons make the QCD vacuum unstable in the formation of bound state \(27, 28\). This problem can be cured in this model (without breaking global \(SU(N)\) symmetry) because of the massiveness of gluons. These essential characteristics motivate us to consider the TMM at finite temperature. We have already established the hard thermal loop effective action for TMM in \(29\).

We now consider the transport phenomenon in topologically massive gluonic fluid. This can be considered in the analysis of quark gluon plasma (QGP) formed in relativistic heavy ion collision. The QGP produced around mid-rapidity at top RHIC (Relativistic Heavy Ion Collider) and LHC (Large Hadron Collider) energies will contain a negligibly small number of net baryon (i.e. (number of quarks) - (number of antiquarks) \(\approx 0\)) (see Ref. \[30\] for a review). The bulk thermodynamic properties of such systems can be described by a single thermodynamic variable, temperature \(T\) as the corresponding baryonic chemical potential is negligibly small due to negligible small number of net baryons (quarks in this case). Therefore, the result of the present work can be applied to the system formed at top RHIC and LHC energies. Moreover, the tiny gluonic mass present in TMM may not cause a severe problem for such an application. However, the system formed at lower RHIC energies, at the upcoming Compressed Baryonic Matter (CBM) experiment at Facility for Anti-proton and Ion Research (FAIR) and Nuclotron based Ion Collider fAcility (NICA) at the Joint Institute for Nuclear Research (JINR) will have a large net baryons (i.e. the number of quarks-antiquarks is large at mid-rapidity). The system formed in these collisions will have a large baryonic chemical potential \(31\) and hence, both temperature and baryonic chemical potential are required to describe such systems.

Therefore, the results of the present work can not be applied to such systems.

QGP is created in a state slightly away from equilibrium characterized by various transport coefficients in TFT. The shear and bulk viscous coefficients of a fluid are useful quantities to characterize it. These quantities are required as inputs to solve the relativistic viscous hydrodynamical equations which have been used in the description of the space-time evolution of the strongly interacting QCD matter formed in nuclear collisions at relativistic energies. To understand the properties of QCD matter, it is important to reliably estimate both the shear and bulk viscous coefficients. Like shear viscosity \(32, 34, 44\), bulk viscosity also carries crucial physical significance which are discussed in many recent works \(35-41\). We consider the bulk viscosity in pure topologically massive gluodynamics. The lattice simulation finds a non-zero bulk viscosity \(42\) in pure gluodynamics. To make the analysis simpler, the linear response theory (LRT) \[23, 43, 45, 46\] which reflects the linear perturbation around equilibrium state of the fluid. As a consequence, the bulk viscosity \(\zeta_T\) can be calculated from the well-known Kubo formula \[32, 34, 44\]

\[
\zeta_T(\omega) = \frac{1}{18} \lim_{\omega \to 0} \frac{1}{\omega} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} \times \int d^3x \left\langle \Theta_T^i(x,t), \Theta_T^i(0,0) \right\rangle ,
\]

where \(\Theta_T^i\) is the energy momentum tensor density of field theory under consideration and \(\omega\) is the frequency appearing through the Fourier transformation of the correlation between the spatial trace of the energy momentum tensor (EMT) densities. We observe in Eq. (1) how the bulk viscosity \(\zeta_T\) depends on the trace of EMT densities of the quantum fields. Here, the Kubo formula is obtained from the linear response theory (LRT) \[23, 43, 15, 16\] which reflects the assumption that the system maintains local equilibrium. Since, energy \(\int d^3x \, \Theta^{00}(x)\) of the system is conserved, we can shift the spatial trace \(\Theta_T^i\) by the energy or any multiple of the energy. The bulk viscosity diverges near the critical point. The diverging nature can be taken into account phenomenologically by expressing \(\zeta_T\) in terms of the correlation length \(\xi\) as discussed below through Eq. (4).

The bulk viscosity was found in massless models like \(\phi^4\) and YM theories \[33, 31, 41\] where the conformal symmetry is obeyed classically \[47\]. A common characteristic of these models is that the classical conformal invariance breaks down due to the quantum correction in renormalization procedure (i.e. the spectral function corresponding to the correlation of energy momentum tensors depends on the breaking of conformal symmetry). The exceptional case is found in \(N = 4\) super YM theory where it is observed that \(\zeta_T = 0\) \[44\]. The trace anomaly causes problems when we consider the theories in curved spacetime. It was found that this anomaly causes violation of the fifth axiom in the construction of its uniqueness in curved spacetime which affects significantly in the semi-classical treatment of general theory of relativity \[48, 49\]. This carries a great importance in the consideration of de-Sitter spacetime. On the other hand, trace anomaly provides negative vacuum energy density from the perturbative regime \[50\] which contradicts with the cosmological observations \[51\]. The bulk viscosity may also lead to an alternative to the dark energy in cosmological scenario in de-Sitter spacetime \[52\].

From Eq. (1), we observe that the bulk viscosity depends on the correlation of energy momentum densities. We get the correlation from the low energy theorem (LET) at finite temperature \[33, 55\] as:
\[ \left( T \frac{\partial}{\partial T} - 4 \right)^n \langle \Theta^\mu \rangle = \int d\tau_n d^3x_n \cdots d\tau_1 d^3x_1 \langle \Theta^\mu_n(\tau_n, x_n) \cdots \Theta^\mu_1(\tau_1, x_1) \Theta^\mu_0(0,0) \rangle, \]  

(2)

where gluons degrees of freedom are relevant. We take background field method for the calculation of l.h.s. of Eq. (2). This calculation provides the vacuum expectation value of regularized trace of EMT density at finite temperature. With this purpose, we will construct the one-loop effective action using heat kernel method. This causes the dependence of \( \zeta \) of effective action. Hence, the modified heat kernel method, found in \cite{61, 62}, includes the contribution of \( L \) ordered product. It arises in the calculation due to the compactification of fourth Euclidean axis in thermal field theory. Hence, the modified heat kernel method, found in \cite{61, 62}, includes the contribution of \( L(x, \beta) \) in the construction of effective action. This cause the dependence of \( \zeta \) on \( L \) and we shall show it in the next section. Specifically, we consider the behavior of spectral function

\[ \rho(\omega) = \int_{-\infty}^{\infty} dt \ e^{-i\omega t} \int d^3x \ \langle \left\{ [\Theta^\mu(x, t), \Theta^\nu(0,0)] \right\} \rangle, \]  

(3)

from the model. In the massless YM theory, the bulk viscosity \( \zeta_T \) is calculated from quantum corrections of the trace of EMT density of the YM field (i.e. the leading contribution in the calculation comes from the conformal or trace anomaly). This causes the dependence of \( \zeta_T \) on the strong coupling in the perturbative computation. In the lattice QCD \cite{42}, the ratio \( \frac{\zeta}{\eta} \) was computed for pure gluodynamics where \( s \) is the entropy density. But, in the case of TMM, we shall observe how the leading order in the spectral function depends on the mass of gauge fields and expectation value of untraced Polyakov loop. The present investigation emphasizes the possibility of finding bulk viscosity in the perturbative regime of QCD with the same asymptotic freedom as found in the literature.

In this endeavor, we present an explicit calculation of the spectral function in Sec. II. Section III contains the discussion, conclusions and the future aspects of the model in the realm of thermal field theory. We take the signature of the Minkowski metric \( \eta_{\mu\nu} \) as \((+, -, -, -)\). We have also taken the convention: \( \hbar = k_B = c = 1 \) where \( k_B \) is Boltzmann’s constant.

\section{II. Calculation}

Within the scope of LRT, the hydrodynamical transport coefficients by using Green-Kubo formula can be written as follows:

\[ \eta(\omega) \left( \delta_{k(i} \delta_{j)l} - \frac{2}{3} \delta_{kl} \delta_{ij} \right) + \zeta_T(\omega) \delta_{kl} \delta_{lm} = \lim_{\omega \to 0} \frac{1}{\omega} \int d^3x \int_0^\infty dt \ e^{i(\omega t - k \cdot x)} \langle [\Theta_{ij}(t, x), \Theta_{kl}] \rangle, \]  

(4)

where \( \eta(\omega) \) is called shear viscosity and \( \delta_{m(i} \delta_{b)l} = \frac{1}{2} (\delta_{bm} \delta_{lm} + \delta_{bm} \delta_{un}) \). The bulk viscosity can be obtained from the above formula by contracting \( i, j \) and \( k, l \) as

\[ \zeta_T(\omega) = \lim_{\omega \to 0} \frac{1}{4\eta} \int d^3x \int_0^\infty dt \ e^{i(\omega t - k \cdot x)} \langle [\Theta_{ij}(t, x), \Theta_{kl}] \rangle. \]  

(5)

The energy momentum tensor is obtained from the following part of the action in Minkowski spacetime

\[ S_0 = \int d^4x \left( -\frac{1}{4} F^{\mu\nu} \Gamma^\alpha_{\mu\nu} + \frac{1}{12} H^{\alpha\mu\nu} H^\alpha_{\mu\nu} \right. \]  

\[ + \frac{m}{4} \epsilon^{\mu\nu\lambda} \partial_\mu B^\alpha_{\nu\lambda} \right), \]  

(6)

1 Here \( A_0 \) is the temporal component of quantum gauge field.
where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ and $H^a_{\mu\nu\lambda} = \partial_\mu B^a_\nu\lambda + g f^{abc} B^b_\mu B^c_\lambda - g f^{abc} F^b_{[\mu\nu} C^c_{\lambda]}$ are the field strength of YM field and tensor field, respectively, and $f^{abc}$ is the structure constant of $SU(N)$ group. The presence of an auxiliary field $C^a_\mu$ in the expression of $H^a_{\mu\nu\lambda}$ assures the invariance of the action under the following gauge transformations:

$$
A^a_\mu \to A^a_\mu, \quad B^a_{\mu\nu} \to B^a_{\mu\nu} + (D_{[\mu} \theta_{\nu]}^a)^a, \quad C^a_\mu \to C^a_\mu + \theta^a_\mu,
$$

(7)

where $\theta^a_\mu$ is a vector field in the adjoint representation of $SU(N)$. Including ghosts' sectors, we have the full action as given by

$$
S = S_0 + \int d^4 x \left[ h^a f^a + \frac{\xi}{2} h^a h^a + h^a_\mu (f^{a\mu} + \partial^\mu \tilde{a}^a) + \frac{\eta}{2} h^a_\mu h^a_\mu + \partial_\mu \tilde{a}^a \alpha^a + \tilde{a}^a \partial_\mu \omega^a \alpha^a + \tilde{a}^a \partial_\mu (D_\mu \omega^a)^a + \partial_\mu (g f^{abc} F_{\mu\nu} \omega^c)^a + \partial_\mu (D_\mu \eta^a)^a \right],
$$

(8)

where $S_0$ is the action given in Eq. (6) and $f^a = \partial^\mu A^a_\mu, f^a_\mu = \partial^\nu B^a_{\mu\nu}$. The parameters $\xi, \eta$ and $\tilde{\zeta}$ are the dimensionless gauge-fixing parameters. The auxiliary fields $h^a$ and $h^a_\mu$ play the role of Nakanishi-Lautrup type fields. Here $(\tilde{a}^a)\omega^a$ and $(\tilde{a}^a_\mu)\omega^a_\mu$ (with ghost number $(-1) + 1$) are the Fermionic scalar and vector (anti-)ghost fields for the vector gauge field $A^a_\mu$ and tensor field $B^a_{\mu\nu}$, respectively. The bosonic scalar fields $(\tilde{\beta}^a)^a$ (with ghost number $(-2) + 2$) are the (anti-)ghost fields for the Fermionic vector (anti-)ghost fields and $\tilde{n}^a$ is the bosonic scalar ghost field (with ghost number zero). These scalar ghost fields are required for the stage-one reducibility of the two-form field. Furthermore, $\alpha^a$ and $\tilde{\alpha}^a$ are the Grassmann valued auxiliary fields (with ghost number +1 and −1, respectively). This model contains massive non-Abelian gauge field and it was shown to be BRST invariant \[11, 12\]. In \[11, 12\], it is seen that the model is also invariant under the anti-BRST symmetry transformations. The $CP$ symmetry is not violated in this model.

The EMT density corresponding to the action [cf. Eq. (6)] in curved space time is given by

$$
\Theta_{\mu\nu} = \frac{2}{\sqrt{\bar{g}}} \frac{\delta S_0}{\delta g_{\mu\nu}},
$$

(9)

where $S_0 = \int \sqrt{-\bar{g}} \left( -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \frac{1}{12} H^a_{\mu\nu\lambda} H^a_{\mu\nu\lambda} + \frac{m}{4} \epsilon^{\mu\nu\alpha\beta} B^a_{\mu\nu} F^a_{\alpha\beta} \right) d^4 x$ and $g_{\mu\nu}$ is the metric in the curved spacetime. Here $\bar{g} = \text{Det}g_{\mu\nu}$. We find that the EMT density corresponding to the action of TMM classically as

$$
\Theta_{\mu\nu} = \Theta_{\mu\nu}^{YM} + \frac{1}{2} \left( H^a_{\mu\alpha\beta} H^a_{\nu\alpha\beta} - \frac{1}{6} g_{\mu\nu} H^a_{\mu\alpha\beta} H^a_{\nu\alpha\beta} \right),
$$

(10)

where $\Theta_{\mu\nu}^{YM} = -F^a_{\mu\alpha} F^a_{\nu\alpha} + \frac{1}{4} g_{\mu\nu} F^{a\alpha\beta} F^a_{\alpha\beta}$ is the standard EMT for the YM field $A^a_\mu$. Since the topological term is invariant under the variation of metric tensor, hence, it does not provide any contribution in TMT in Eq. (10). The trace of $\Theta^\mu_{\mu}$ is non-zero in 3+1 dimensional spacetime reads

$$
\Theta^\mu_{\mu} = \frac{1}{6} H^a_{\rho\mu\nu\lambda} H^a_{\rho\mu\nu\lambda},
$$

(11)

because $g^{\mu\nu} \Theta^{YM}_{\mu\nu} = 0$. Hence, it is clear from Eq. (11) and the action in Eq. (6) that the kinetic term of $B$ field is responsible for the absence of conformal symmetry at zero temperature. This implies that we can find the bulk viscosity for the topologically massive YM fluid at finite temperature. For our purpose, we use low energy theorem at finite temperature \[53, 55\]. We use the splitting

$$
A^a_\mu = A^a_\mu^\gamma + a^a_\mu^\gamma,
$$

(12)

$^2$ The quantum fields are designated by lowercase letters having Lorentz and gauge indices.
where \( \mathcal{A}_\mu^a \) is background and \( a_\mu^a \) is the quantum fields. Due to the splitting, the YM field strength becomes

\[
F_{\mu\nu}^a(A, a) = F_{\mu\nu}^a(A) + \left( D^A_\mu a_\nu \right)^c + g f^{cde} a_\mu^e a_\nu^d,
\]

(13)

where \( e \) is a gauge index. The covariant derivative \( D^A_\mu = \partial_\mu + g A_\mu \) is taken with respect to the background field \( A^a_\mu \). Here \( g \) is the gauge coupling constant. Our aim is to find an effective action at one-loop level from Eq. (8), which can provide regularized EMT density as shown in [10, 56, 57]. For this purpose, it is sufficient to find the terms in the action which contain quantum fields \( a_\mu^a \) quadratically. It is to be noted that the YM and \( B \) fields are coupled quadratically due to presence of the topological term \( B \wedge F \) in the Lagrangian density\(^3\) in Eq. (6). This mixing leads us to construct a matrix from the kinetic terms of YM and \( B \) fields and \( \frac{m}{2} B \wedge F \) term \(^4\)

\[
\Delta = \begin{pmatrix}
\Delta_A & \Delta_{AB} \\
\Delta_{BA} & \Delta_B
\end{pmatrix}.
\]

(14)

It is a block matrix whose determinant is given by [10]

\[
\det \Delta = (\det \Delta_B)^{-1} \det \left( \Delta_A - \Delta_{AB} \Delta_B^{-1} \Delta_{BA} \right)
= (\det \Delta_B)^{-1} \det \Delta_A \\
\times \det \left( 1 - \left( \Delta_A \right)^{-1} \Delta_{AB} \Delta_B^{-1} \Delta_{BA} \right).
\]

(15)

From the complete action of the model, we can clearly notice the presence of trilinear couplings among the ghost and gluon fields (due to the above splitting provides the quadratic terms in all quantum fields). The ghosts’ sector consists of Faddeev-Popov (FP) ghosts for \( a_\mu^a \), vector ghosts for \( B \) field, FP ghosts of the vector ghost fields and a scalar ghost. The general structure of the effective action, at \( T = 0 \), in one-loop level is given by [10, 64]

\[
W^1[A] \propto \int \sqrt{-} g \left( n_1 \text{Tr ln} \Delta + n_2 \text{Tr ln} \Delta_{gh}
+ n_3 \text{Tr ln} \Delta_{vecgh} + n_4 \text{Tr ln} \Delta_{ghvecgh}
+ n_5 \text{Tr ln} \Delta_{sclgh} \right) d^Dx,
\]

(16)

where \( n_i \)’s (with \( i = 1, 2, 3, 4, 5 \)) are the numerical factors which appear after integrating out the quantum fields in the partition functional. These numerical factors also depend on the spin-statistics of quantum fields [10, 64]; \( \Delta_{\Phi_i} \)'s are appearing from the action after integrating out the quantum fields from the partition functional [10, 64]

\[
Z[\Phi_{\text{field}}] = \int \prod_{i=1}^{n} \mathcal{D}\Phi_i \exp \left( \frac{i}{2} \int d^4x \sqrt{-} g \sum_{i=1}^{n} \Phi_i \Delta_{\Phi_i} \Phi_i \right),
\]

(17)

where \( \Phi_{\text{field}} \) is the background field of \( i \)-th type field; \( \Phi_i = \Phi_{\text{field}} + \tilde{\Phi}_i \); \( \tilde{\Phi}_i \) is the quantum part of \( \Phi_i \). We have suppressed the spin and gauge indices of the fields in Eq. (17) and \( \Delta_{\Phi_i} \)'s may be the function of background fields or not. For example, we note from the complete action of TMM that \( \Delta, \Delta_{gh}, \Delta_{vecgh} \) and \( \Delta_{ghvecgh} \) are the functions of covariant derivative with respect to the background YM field \( A \) but for the scalar ghost \( \Delta_{sclgh} \) does not contain any covariant derivative and \( A \). One can also check from the structure of the bulk matrix in Eq. (15) that \( \Delta \) contains \( \Delta_A, \Delta_{AB}, \) and \( \Delta_{BA} \). Here, \( \Delta_A \) and \( \Delta_B \) appear from the kinetic terms of \( A \) and \( B \) fields whereas other \( \Delta \)'s comes from \( B \wedge F \) term. Following the general rule [10], we can write the effective action at \( T = 0 \) as

\[
W^1 = -\frac{1}{2} \int \sqrt{-} g \left[ \text{Tr ln} \Delta - 2 \text{Tr ln} \Delta_{gh} + 2 \text{Tr ln} \Delta_{vecgh}
- 2 \text{Tr ln} \Delta_{ghvecgh} - \text{Tr ln} \Delta_{sclgh} \right] d^Dx.
\]

(18)

\(^3\) Due to Eq. (12), the following term \( \frac{m}{4} \epsilon_{\mu\nu\rho\lambda} B_{\mu\nu} F_{\rho\lambda} \) contributes \( \frac{m}{4} \epsilon_{\mu\nu\rho\lambda} b_{\mu\nu} \left( F_{\rho\lambda}(A) + 2 \left( D^A_\mu a_\mu \right) \right) \) in the quadratic part under consideration.

\(^4\) The \( \Delta \) is appeared by the re-expressing \( S_0 \) [cf. Eq. (46)] as \( S_0 = \int d^4x \Phi^T \Delta \Phi \) where \( \Phi = \left( \frac{\phi}{\rho} \right) \) and \( \Phi^T \) is the transpose of \( \Phi \).
For the computation of $\text{Tr} \ln \Delta_{\phi_i}$, we use the heat kernel method at finite temperature. The relation between the heat kernel coefficients at zero and finite temperature is given explicitly \[59 \, 61 \, 62\]. This calculation is done with covariant gauge-fixing condition $D^A \omega^\mu = D^A b^\mu = D^A \omega^\mu = 0$. We know from the detail of heat kernel method \[10 \, 58 \, 60\] how the trace of an operator $\Delta_{\phi_i}$ is calculated from the coincidence limit of a matrix with matrix element is

$$
\text{Tr} \ln \Delta_{\phi_i} = \int_0^\infty \frac{1}{(4\pi \tau)^{D/2}} H(x, x, \tau) \, \frac{d\tau}{\tau}, \quad (19)
$$

where $H(x, x, s) = \sum_{n=0}^\infty s^n \text{Tr}_n$ is called to be heat kernel corresponding to an operator $\Delta_{\phi_i}$, and $a_n$'s are the heat kernel coefficients in $D$-dimensional spacetime. Here $\text{Tr}_f(x, y) = \int \sqrt{-g} \, \text{tr} f(x, x) d^D x$ and 'tr' denotes the trace over the Lorentz and internal indices \[10 \, 58 \, 60\] of function $f(x, y)$. In the construction of the effective action at one-loop level, we consider the trace [cf. Eq. (18)] at finite temperature. The expression in Eq. (19) is modified at finite temperature as \[59\]

$$
H_{\beta}(x, y, \tau) = H(x, y, \tau) \left[ 1 + 2 \sum_{n=1}^\infty \kappa_n e^{-n^2 \tau^2} \right], \quad (20)
$$

where $\beta = \frac{1}{T}$ and $\kappa_n$ signifies the dependence of the expansion on the spin-statistics of the fields. In fact, $\kappa_n = (-1)^n$ for fermionic field and $\kappa_n = 1$ for bosonic field. The relation in Eq. (20) was found to be incomplete \[61 \, 62\]. This incompleteness occurs due to the exclusion of $L(x, \beta)$ as mentioned in the Sec. [4]. The heat kernel expansion of trace: $\text{Tr}(e^{-s(-D^\mu D_\mu + X)}) = \int ds \frac{1}{s} \left( \frac{4\pi s}{D/2} \right)^s \sum_{n=0}^\infty s^n \text{Tr}(a_n^T)$ is given with the following heat kernel coefficients (upto mass dimension 4) \[61 \, 62\]

$$
egin{align*}
a_0^T(x, x) &= \phi_0(L, s), \\
a_1^T(x, x) &= -\phi_0(L, s) X, \\
a_2^T(x, x) &= -\frac{1}{2} \phi_0(L, s) X^2 - \frac{1}{3} \phi_2(L, s) E_i^2 + \frac{1}{12} \phi_0(L, s) F_{ij}^2,
\end{align*}
$$

where $E_i = F_{0i}$, and

$$
\begin{align*}
\phi_0(L, s) &= \left[ 1 + 2 \sum_{n=1}^\infty L^n e^{-n^2 \tau^2} \right], \\
\phi_n(L) &= \frac{1}{\beta} \sqrt{4\pi s} \sum_{p>0} s^{n/2} Q_p e^{s Q_p^2}, \\
\phi_2 &= \phi_0 + 2\phi_2,
\end{align*}
$$

where $Q_r = i p_0 r - \frac{\tau}{2} \ln L \ [61 \, 62]$. Thus, $(\Theta_{\mu\nu})$, derived from the effective action, will depend on $L$ and as a consequence of LET [cf. Eq. (2)] we can see the correlation among the EMT densities become dependent on $L$, too. Its further consequence is very interesting which we shall show how $\omega_T$ depends on $L$.

According to the method outlined in \[10 \, 58 \, 60\], we need to identify $X$ in the Laplace-type operator appearing in the kinetic terms of quantum fields:

$$
\Delta_A \equiv -\frac{1}{2} \left( -D^\mu D_\mu + X \right), \quad (27)
$$

where covariant derivative is expressed as $D^A = \partial^A + \omega^\mu (A)$ and $\omega^\mu$ is the “connection” \[10 \, 58 \, 60\]. Generally, $X$ and $\omega^\mu$ are matrix valued functions in the non-Abelian gauge theory. Corresponding to the YM field, we have \[60\]

$$
\begin{align*}
(\omega^A)_{\lambda}^\rho &= -gf^{\rho\sigma} A_\sigma^\mu \delta^\nu_{\lambda}, \\
(X^A)_{\rho\lambda} &= 2gf^{\rho\sigma} F_{\rho\lambda}(A).
\end{align*}
$$
The expression of the $\tilde{\omega}_\mu$’s in Eq. (28) are same for all the quantum fields in Eq. (8) but X’s will be different. For example, we get $X_B$ corresponding to the $b_{\mu\nu}$ and vector ghost fields as:

$$
(X_B)_{\rho\lambda}^{cd} = 2g f_{\rho\lambda}^{\mu\nu}F_{\alpha\beta}(A)\eta^{\alpha|\mu}\delta^{\nu|\lambda}\delta^{\rho|\beta},
$$

(30)

$$
(X_{vegh})_{\mu\nu} = -g f_{\mu\nu}^{\rho\epsilon}F_{\rho\epsilon}(A).
$$

(31)

For the rest of the ghost fields, $X = 0$ which can be read-off from the action in Eq. (8). We also have the following explicit expressions

$$
(\Delta_{AB})^{\rho\sigma} = \frac{i}{2}\epsilon_{\rho\sigma}^{\alpha\beta}D_{\alpha},
$$

(32)

$$
(\Delta_{BA})^{\rho\sigma} = \frac{i}{2}\epsilon_{\rho\sigma}^{\beta\epsilon}D_{\beta},
$$

(33)

$$
(\Delta_B)_{\alpha\beta, \rho\sigma} = -\eta_{[\rho\sigma]}\eta_{[\beta\epsilon]D_\mu + (X)_{\alpha\beta, \rho\sigma}.}
$$

(34)

In the above, $\Delta_B$ can be expressed as

$$
(\Delta_B)_{\alpha\beta, \rho\sigma} = -\eta_{[\rho\sigma]}\eta_{[\beta\epsilon]\partial_\mu + \sigma_{\alpha\beta, \rho\sigma},}
$$

(35)

where $\sigma_{\alpha\beta, \rho\sigma} = \eta_{[\rho\sigma]}\eta_{[\beta\epsilon]}(2g A^\mu \partial_\mu + g \partial_\mu A^\nu + g^2 A^\mu A_\mu) + (E)_{\alpha\beta, \rho\sigma}$. We can safely neglect the contribution from $\Delta_{vegh}$ in the effective action [cf. Eq. (18)] because of the absence of the background field in kinetic term of the scalar ghost field of $b_{\mu\nu}$. Since, in the leading order

$$
(\Delta_{AB}\Delta_B^{-1}\Delta_{BA})_{x,y}^{\mu\nu} = m^2\delta^4(x-y)\eta^{\mu\nu} \left(1 - \frac{1}{D}\right) + \mathcal{J}_{xy}^{\mu\nu}(g^n, A), \ \ n \geq 1,
$$

(36)

we can re-express $\text{Tr} \ln \Delta$ as

$$
\text{Tr} \ln \Delta = \text{Tr} \ln \tilde{\Delta} - \text{Tr} \ln \Delta_B + \text{Tr} \ln \left(1 - \mathcal{J}(g^n, A)\tilde{\Delta}^{-1}\right),
$$

(37)

where $\tilde{\Delta} = \Delta_A + \tilde{m}^2$ and $\tilde{m}^2 = (1 - \frac{1}{D})m^2$. In Eq. (36), the matrix-valued operator $\mathcal{J}_{xy}^{\mu\nu}(g^n, A)$ designates the parts of $(\Delta_{BA}\Delta_B^{-1}\Delta_{AB})_{x,y}^{\mu\nu}$ which contains various non-zero powers of $g$ and background YM field $A$. The significance of the r.h.s. of Eq. (37) will be shown later in our analysis. Hence, we have now

$$
W^1 = -\frac{1}{2} \int \sqrt{-g} \left[\text{Tr} \ln \tilde{\Delta} - \text{Tr} \ln \Delta_B - 2\text{Tr} \ln \Delta_{gh} - \text{Tr} \ln \left(1 - \mathcal{J}(g^n, A)\tilde{\Delta}^{-1}\right)\right] d^D x.
$$

(38)

The last term is found from the series expansion

$$
\ln(1 - Y) = -\sum_{n=1}^{\infty} \frac{Y^n}{n}.
$$

(39)

It can be readily checked that the first term in the above expansion is: $-Y = \left(\mathcal{J}(g^n, A)\tilde{\Delta}^{-1}\right)$. The traces in the first three terms can be found from Eq. (19). Now we explain the significance of Eq. (37). Following Eq. (19), we can write

$$
\langle x | \ln \Delta | x \rangle = \int_0^\infty \frac{ds}{s} e^{-m^2 s} \tilde{H}(x, x, s),
$$

(40)

which shows the trace is convergent in the large-$s$ region. Further, $\tilde{H}(x, x, s)$ is different from $H(x, x, s)$ [cf. Eq. (19)] due to the rearrangement of the terms as shown in Eq. (36). To find the trace, we should note that the Laplace-type operator acts on the gluon and vector ghost fields in $(N^2 - 1)D$ dimensional internal space whereas it acts on the FP ghost fields in $(N^2 - 1)$ dimensional internal space. For the $b_{\mu\nu}$ field, the dimension of the internal space becomes
\( \frac{D(D-1)}{2}(N^2 - 1) \) where the operator acts. This leads to the following expression for the heat kernel expansion at finite temperature \([61, 62]\):

\[
W^1 = -\frac{1}{2} \int_0^\infty ds \frac{e^{-\tilde{m}^2 s}}{s^{D/2}} \sum_{n=0}^\infty \text{Tr} \ a_n^T(x,x) s^n,
\]

(41)

where

\[
a_0^T = \frac{7D - D^2 - 8}{2} \phi_0(L),
\]

(42)

\[
a_2^T = \left[ (2-D) + \frac{7D - D^2 - 8}{24} \phi_0(L)F^{\alpha\mu\nu}F_{\mu\nu}^{ab} \right]
+ \frac{7D - D^2 - 8}{12} E_i^a E_i^b N^{ab} \tilde{\phi}_2(L),
\]

(43)

and

\[
\phi_n(L) = \frac{1}{\beta} \sqrt{4\pi s} \sum_{p_{\nu\nu}} s^{n/2} Q^2_{\nu} e^{sQ^2_{\nu}}, \quad \tilde{\phi}_2 = \phi_0 + 2\phi_2,
\]

(44)

with \( Q_{\nu} = i \left( p_{\nu\nu} - \frac{\nu^2}{\beta} \ln L \right) \) \([61, 62]\) and \( N^{ab} = f^{ac} f^{bd} = N^{\delta_{ab}} \) in Eq. (43). Now we are going to get an explicit expression of the effective action at finite temperature using Eq. (20). From the general expression of heat kernel coefficients in \([61, 62]\), we can write the effective action for massless gluon field at finite temperature as

\[
W^1 = \int d^Dx \sqrt{-g} \left( -\frac{\pi^2}{45} T^4 (N^2 - 1)
+ \frac{2\pi^2}{3} T^4 \text{tr} \left[ \nu^2(1-\nu)^2 \right] + \mathcal{I}(A,T) \right), \quad 0 < \nu < 1,
\]

(45)

where \( \nu = \left( \ln \frac{L}{2\pi i} \right) \) and \( \mathcal{I}(A,T) \) designate the terms depending on both temperature and background YM field. In the massless case, it is already seen in \([62]\) that IR divergence exists in the large-\( s \) region. This causes a serious problem in perturbative TFT which we have pointed out in the introduction of the paper. In the TMM, the problem is settled due to the presence of the factor \( e^{-\tilde{m}^2 s} \) [cf. Eq. (40)]. We need the expression of \( \phi_0 \) [cf. Eq. (24)] to calculate the dimensionally regularized effective action as \([61, 62]\).

\[
W^1 = -\frac{1}{2} \int_0^\infty ds \frac{\mu^{2\epsilon}}{s^{(4\pi s)^{D/2}}} \sum_{n=0}^\infty \text{Tr} \ a_n^T(x,x) s^n,
\]

(46)

where \( \mu \) is called to be subtraction point and the regularization will be done in \( D = 4 - 2\epsilon \) dimensions. To calculate the terms in the leading order of heat kernel expansion, we need to work out the integration of the type \([61, 62]\):

\[
I_{1,n} = \int_0^\infty ds \frac{d^D \phi_n(\omega, s) e^{-\tilde{m}^2 s}}{s^{D/2}} \phi_n(\omega, s) e^{-\tilde{m}^2 s}, \quad |\omega| = 1,
\]

(47)

---

5 We have suppressed the terms involving the Riemann curvature, Ricci tensor and scalar and their derivatives because they do not contribute in the limit \( g_{\mu\nu} \rightarrow \eta_{\mu\nu} \).
which yields in the leading order for $l = -2$ and $n = 0$

\[
I_{-2,0} = \int_0^\infty \frac{ds}{s} e^{-\hat{m}^2 s} (4\pi \mu^2 s)^\epsilon s^{-2} \phi_0(\omega, s)
\]

\[
= (4\pi \mu^2)^\epsilon \int_0^\infty dss^{-2+\epsilon} e^{-\hat{m}^2 s} \sum_{k \in \mathbb{Z}} L^k e^{-k^2 \beta^2}
\]

\[
= (4\pi \mu^2)^\epsilon \left[ \Gamma(\epsilon - 2)m^2(2-\epsilon) \right]
\]

\[
+ 2 \sum_{k=1}^\infty L^k \left( \frac{k^2 \beta^2}{4\hat{m}^2} \right)^{-\frac{2+\epsilon}{\epsilon}} K_{-2+\epsilon}(\hat{m}k\beta),
\]

(48)

where we have used the formula \[65\]

\[
\int_0^\infty dx \ x^n e^{-\lambda x} = \frac{n!}{\lambda^{n+1}}, \quad [\text{Re } \lambda > 0],
\]

(49)

and the modified Bessel function of $K_n(x)$ in Eq. (48) appears through its integral representation:

\[
\int_0^\infty x^{n-1} e^{-\frac{x}{2} - \alpha x} dx = 2 \left( \frac{\beta}{\alpha} \right)^\frac{\beta}{2} K_n(2\sqrt{\alpha}x),
\]

\[
[\text{Re } \alpha > 0, \text{ Re } \beta > 0].
\]

(50)

The first term in Eq. (48) appears in any massive field theory at zero temperature. But, the next term is interesting because of its dependence on the untraced Polyakov loop ($L$) at finite temperature. The $L$-dependence of the EMT at high temperature will be discussed later. It is one of the main results of our present investigation. For the effective action at finite temperature, in the limit $\epsilon \to 0$, $\beta \to 0$, we have

\[
W_0^{1(T \neq 0)} = \int d^4x \sqrt{-g} \left[ -\frac{\pi^2}{45} T^4(N^2 - 1)
\]

\[
+ \mathcal{O}(L, \hat{m}\beta) \right],
\]

(51)

where the leading order term is matched with the expected result in Eq. (38). In getting the above expression, we have used $K_{-n}(x) = K_n(x)$ \[65\]. In the above, $\mathcal{O}(L, \hat{m}\beta)$ designates the terms which contain various non-zero power of $L$ and dimensionless quantity $\hat{m}\beta$ (with $\beta = T^{-1}$). The appearance of leading order terms can be understood from the behaviour of $K_n(x)$ for small argument ($x \to 0$) as \[65\]

\[
K_n(x) \sim \frac{1}{2} \Gamma(n) \left( \frac{x}{2} \right)^{-n},
\]

(52)

and the expansion of the time-ordered product is

\[
L^k(x, \beta) = \left( 1 - \int_{x_0}^{x_0+\beta} a_0(x', x) L(x', \beta) dx' \right)^k
\]

\[
= 1 - k \int_{x_0}^{x_0+\beta} a_0(x', x) L(x', \beta) \left. dx' \right. + \cdots,
\]

(53)

where $x' \equiv (x'_0, x)$.

We also obtain another important result which carries a great significance in a renormalizable massive non-Abelian gauge theory. From the dimensional regularization for the coefficient $a_2^{T=0}$, we obtain

\[
W_2^{1(T=0)} = \lim_{\epsilon \to 0} g^2 N \frac{1}{(4\pi)^2} \int d^4x \sqrt{-g} \frac{1}{12} \frac{11}{\epsilon} F_{\mu\nu}^a F_{\mu\nu}^a
\]

(54)

where $\frac{1}{\epsilon} = \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right)$ and $\gamma_E \approx 0.5772$ is called to be Euler’s constant. This result appears from the contribution of $a_2^{T=0}$. This contribution is same as found in the massless YM theory. Thus, we get the same asymptotic
behaviour (i.e. asymptotic freedom) of gauge coupling \( g \) in the non-Abelian TMM. Therefore, the applicability of the perturbative technique at high temperature is consistent.

Now we go back to the low energy theorem at finite temperature [cf. Eq. (2)]. For this purpose, we consider Eq. (2) for \( n = 1 \):

\[
\left( T \frac{\partial}{\partial T} - 4 \right) \left\langle \Theta_\mu^\mu \right\rangle = \int d\tau d^3x \left\langle \Theta_\mu^\mu(\tau, x) \Theta_\mu^\mu(0, 0) \right\rangle + \frac{(1 - 3c_s^2)}{c_s^2} h, \tag{55}
\]

where, we add a term in the right hand side of the above equation. This term is appeared from the consideration of right hydrodynamic limit to get the transport coefficients [66]. \( h \) is enthalpy density and \( c_s \), the speed of sound.

Now, putting the expression of \( a_T^0 \) [cf. Eq. (42)] in the above expression, we get in the high temperature limit

\[
\int d\tau d^3x \left\langle \Theta_\mu^\mu(\tau, x) \Theta_\nu^\nu(0, 0) \right\rangle = -\frac{g^2(N^2 - 1)}{(4\pi)^2} \sum_{n=1}^{\infty} \frac{4\tilde{m}^2}{n^2} \left[ -\frac{n\tilde{m}}{\beta} K_1(n\tilde{m}\beta) \langle \text{tr} \ L^n \rangle + \frac{2}{n\tilde{m}^2\beta^3} \langle \text{tr} (L'L^{n-1}) \rangle + \cdots \right] \tag{56}
\]

where \( L' = \frac{\partial L}{\partial \beta} \). In the last step of the above equation, we have used a series expansion of the modified Bessel function [65]:

\[
K_n(z) = \frac{1}{2} \left( \frac{1}{2} \right)^n \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left( -\frac{z^2}{4} \right)^k - (-1)^n \ln \left( \frac{1}{2} \right) I_n(z) \tag{57}
\]

and the following recursion relation [65] has also been utilized:

\[
\frac{dK_n(z)}{dz} = -K_{n-1} - \frac{n}{z} K_n(z). \tag{58}
\]

In the series expansion [cf. Eq. (57)], \( I_n(x) \) is the modified Bessel function [65]:

\[
I_n(x) = \left( \frac{x}{2} \right)^n \sum_{k=0}^{\infty} \frac{\left( \frac{x^2}{4} \right)^k}{k! \Gamma(n+k+1)} \tag{59}
\]

and \( \psi(n) \) is Euler’s \( \psi \) function or diagamma function defined as [65]

\[
\psi(x) = -\gamma_E - \sum_{k=0}^{\infty} \left( \frac{1}{x+k} - \frac{1}{k+1} \right). \tag{60}
\]

Introducing the spectral function \( \rho(\omega) \) as

\[
\int d\tau d^3x \left( \Theta_\mu^\mu(\tau, x) \Theta_\nu^\nu(0, 0) \right) = 2 \int_0^{\infty} \frac{\rho(\omega, 0)}{\omega} d\omega, \tag{61}
\]

and taking the expression of \( \rho(\omega, 0) \) for small frequencies [55]

\[
\rho(\omega, 0) = \frac{9\zeta_T}{\pi} \frac{\omega_0^2}{\omega^2 + \omega_0^2}, \tag{62}
\]
we get\(^\text{6}\) from Eq. 66,

\[
9\omega_0\zeta_T(\omega_0) = -\frac{g^2(N^2 - 1)}{(4\pi)^2} \sum_{n=1}^{\infty} \frac{8T^3}{n^2} \left[ \langle \text{tr} (L/L^{n-1}) \rangle \right] - \frac{\tilde{m}^2}{2T} \langle \text{tr} L^n \rangle + \cdots + \left( \frac{1 - 3c_s^2}{c_s^2} \right)h. \quad (63)
\]

The above equation shows the dependence of \(\zeta_T\) on the mass of the gluon and thermal average value of various power of untraced Polyakov loop in the deconfined phase. The bulk viscosity can be calculated by using Eq. 63 with known expression for the Polyakov loop. For simplicity, we use the analytical expression for the Polyakov loop of pure \(SU(3)\) field obtained by using gauge-string duality in [67] to estimate \(\zeta_T\). The expression for Polyakov loop given in Ref. [67] reproduces the lattice QCD results reasonably well. In Fig. 1, the variation of the ratio, \(\frac{9\zeta_0\omega_0}{sT}\) with \(T/T_c\) is displayed. The nature of variation is similar to that obtained in Ref. [55]. Here \(T_c\) is the critical temperature for quark-hadron transition and \(s\) is the entropy density which is estimated as follows. We have already observed from the expression of effective action in Eq. 51 that 'effectively' the transverse degrees of freedom (tDOF) of gluons participate in the leading term due to the combined contributions of ghost sectors corresponding to the one-form (\(A_\mu\)) and two-form (\(B_{\mu\nu}\)) fields (Refs. [10], [85], [86]). Consequently, the entropy of gluonic fluid is also constituted by the same contributions from the two tDOFs and eight colour degrees of freedom, that is, \(s = 4\frac{\pi^2}{90}gT^3\), \(g = 2 \times 8\) is the statistical degeneracy of the gluons.

\[
\frac{9\zeta_0\omega_0}{sT} = 1.5 \text{GeV}.
\]

FIG. 1: The variation of \(\frac{9\zeta_0\omega_0}{sT}\) with \(T/T_c\) with \(\omega_0 = 1.5\text{GeV}\).

The results obtained in this work can be applied to a system of pure gluonic matter only i.e. to a system which can be described by non-zero temperature and zero baryonic chemical potential (\(\mu_B\)). However, there are outstanding physics issues to be addressed for thermal QCD system at non-zero \(T\) and \(\mu_B\). One such issue is the existence and detection of the critical point in the QCD phase diagram at non-zero \(T\) and \(\mu_B\) [68]. The extension of the current formalism to the domain of non-zero chemical potential will help us to understand the behaviour of bulk viscosity near the critical point. In such case the variation of \(\langle \text{tr} L \rangle\), which is considered as an ordered parameter for confinement to deconfinement transition, with temperature and baryonic chemical potential will also govern the variation of \(\zeta_T\) near the transition point. Therefore, this relation will be useful to understand the variation of bulk viscosity with temperature and chemical potential to determine the value of critical exponent of bulk viscosity at the critical point of QCD-phase transition.

Commonly, the following procedure is used for estimating the bulk viscosity near the critical point. The LRT is used to calculate it away from the critical point. The bulk viscosity near the critical point \((\zeta_T(T,\mu_B))\) is obtained

\(\text{Retarded correlation between the traces of energy momentum tensors cannot be distinguished from the correlation involving commutator of the traces in linear response theory (see [62] for details).} \)

\(\text{Retarded correlation between the traces of energy momentum tensors cannot be distinguished from the correlation involving commutator of the traces in linear response theory (see [62] for details).} \)
then by using the following scaling behaviour ($\xi$) \cite{69}:

$$\zeta_T^2 = \zeta_T \left( \frac{\xi}{\xi_0} \right)^3$$ (64)

where $\zeta_T$ is the bulk viscosity away from the critical point, $\xi(T, \mu_B)$ is the correlation length which diverges near the critical point and $\xi_0$ is a constant, typically, $\xi_0 \sim 1.75$ fm \cite{69}.

It is worth mentioning that within the scope of the formalism adopted in the present work to estimate bulk viscosity, the effects of the critical point will infiltrate to bulk viscosity through the behaviour of the Polyakov loop near the critical point.

III. SUMMARY AND DISCUSSION

We have found the bulk viscosity $\zeta_T$ within the scope of TMM and its dependence on the thermally averaged untraced Polyakov loop as well as its various powers and derivative analytically in Eq. (63). We observe that $\zeta_T$ is positive in every order of quantum corrections. It is because of the BRST invariance of effective action in the quantum corrections which is a consequence of the renormalizibility of the model. This causes the maintenance of the convexity of effective potential \cite{64} in the corrections. The positivity of $\zeta_T$ is required to obey the second law of thermodynamics \cite{70}. The dependence on the various power of the thermal expectation value of $L$ [cf. Eq. (63)] appears from the LET where correlation among trace of energy momentum tensor densities is involved. In Eq. (63), the terms containing $\langle \text{tr} L^n \rangle$ and $\langle \text{tr} (L'L'^{-1}) \rangle$ are not invariant under $Z_N$ group\footnote{The elements of the $Z_N$ is $z = e^{\frac{2\pi i n}{N}}$, where $n = 0, 1, 2, \ldots, (N - 1)$; $1$ designates a $N \times N$ unit matrix.}, which is the centre of $SU(N)$ group. As a consequence, the contribution of $\zeta_T$ will also be significant in the study of QGP at heavy quark limit where the restoration of $Z_N$ symmetry implies the phase transition of QGP i.e. deconfined phase to confined phase.

Here we should make comments from our observations on a puzzle raised in \cite{34}. The authors in \cite{34} pointed out a mismatch of the power of gauge coupling in the sum rule that is given in \cite{55}. This issue was addressed in \cite{71} by considering an operator mixing in renormlization group approach. Generally, an operator product expansion is made in the deep ultraviolet region of Euclidean momentum space. The ultraviolet behaviour of various Green functions or correlators depend on their off-shell behaviour. But in thermal field theory, the real and imaginary time formalisms show that the off-shell nature of correlators, which causes the renormalization of the fields and couplings, is independent of temperature \cite{72,73}. Hence, the Callan-Symanzik renormalization group equation is always satisfied in a renormalizable gauge theory. For example, the $n$-point Green functions after quantum corrections generally takes the form in Lorentz covariant way \cite{74}:

$$\Gamma^{\mu\nu\ldots}_{R}(x_1, x_2, \ldots, x_n, T) = \Gamma^{\mu\nu\ldots}_{R}(x_1, x_2, \ldots, x_n, 0) + u^\mu u^\nu \ldots \Delta \Gamma((x_1, x_2, \ldots, x_n, T) + \cdots,$$ (65)

where the subscript $R$ designates the renormalized n-point function, $\Delta \Gamma$ represents the $i$-th order correction and $u^\mu$ is the four velocity of heat bath. It is interesting to note that the real and imaginary time formalisms in TFT provide the inequivalent 3-point functions \cite{76}. The quantum correction of 3-point vertex in pure YM theory at finite temperature in real time formalism leads us, logically, to the dependence of the gauge coupling $g$ on temperature \cite{77}. This dependence shows that $T \frac{d \beta}{dT}$ is not proportional to $g^6$ at the leading order even in the case of massless YM theory. Rather, $T \frac{d \beta}{dT} \propto g^4$ at leading order. The puzzle will also never arises in the model that is considered in our present work. It is because, the model does not provide any trace anomaly. Hence, at the leading order, the both side of the Eq. \cite{55} [cf. Eq. \cite{61}] are proportional to $g^2$.

The authors in \cite{66} have pointed out the domain of validity of the low energy theorem \cite{55} which is used in this work. However, the conformally Minkowski flat metric used in \cite{66} is not consistent with the phenomenology of general relativity in small scale\footnote{Here ‘small scale’ implies the scale which is much less than the cosmological scale.} \cite{80}. Besides this, there is no “physically meaningful” unique renormalized EMT in curved spacetime \cite{81} for massless fields \cite{82,83}. It is because the required Hadamard state has non-local singularity for massless field and there is no unique de-Sitter group invariant vacuum state of this field \cite{84}. Now, we are going to discuss on the results where we have reached in the last section. In arriving at Eq. (63), we have obtained two very significant results for QCD:
The leading terms in the expression of effective action [cf. Eq. (51)] match with the leading terms for the massless YM theory [cf. Eq. (45)]. This equality is due to the resultant null contribution from the kinetic terms of $B$, $\omega_{\mu}$, and $\tilde{\omega}_{\mu}$, $\beta$ and $\tilde{\beta}$, and $n$ (i.e. $B$ field sector)[cf. Eq. (2)]. It can be understood by counting the total degrees of freedom of the fields $\tilde{g}_{5}$ $\tilde{g}_{6}$ in the $B$ field sector, contributing in the effective action in 3+1 dimensions: $1 \times 6 = 2 \times 4 + 2 \times 1 = 0$. This resultant null contribution occurs because the kinetic term of $\tilde{n}$ does not contain any covariant derivative [cf. Eq. (3)].

(ii) The asymptotic freedom remains same as found in massless YM field theory. It (with the IR cut-off) assures the validity of the calculation of bulk viscosity in the perturbative regime at non-zero temperature. We also note that the resumation [23] are absent due to the presence of IR cut-off in the TMM. As a consequence, the terms originated with odd power of $g$ or fractional power of strong coupling $\alpha_{s} = \frac{g^{2}}{4\pi}$, like $O(g^{3}) \sim O(\alpha_{s}^{2})$ and the terms involving $\alpha_{s}^{2} \ln \alpha_{s}$, etc., in the expression of pressure in massless YM theory at high temperature [23], is absent in the case of TMM. The appearance of those terms in the analysis ensures the breakdown of analytic property of perturbative theory according to [24–27]. The absence of the terms $O(\alpha_{s}^{2})$ in the effective action shows that the particle number changing process is slower than the massless YM theories [33].

The present result has been obtained in the realm of perturbative approach. Even though, the hadronization is a non-perturbative process, we can make the following concluding remark. The variation of bulk viscosity near the transition point is governed by Polyakov loop. The enhancement of bulk viscosity near the transition point will reduce the effective pressure of the fluid which, in turn, will provide a smaller kick (as opposed the case when $\zeta_{T} = 0$) to the produced particles. This would be reflected in the experimentally measured value of average transverse momentum of the hadrons. Moreover, the reduced pressure will slow down the expansion resulting in production of more soft gluons enhancing the multiplicity of produced hadrons. Therefore, the present work indicates a possibility to measure Polyakov loop experimentally. It is worth mentioning here that calculations, based on the lattice QCD $[88, 89]$, indicate a sharp decrease of Polyakov loop with temperature near critical point. This causes a sharp rise of $\langle tr L \rangle = -\langle tr \frac{\partial L}{\partial \beta} \rangle$ appeared in Eq. (63). Hence, it implies a large increase of bulk viscosity near the critical point which is expected in phase transition. Such variation of bulk viscosity is consistent with the results obtained from calculations based on lattice QCD $[39]$ and phenomenological model $[90]$.

The present investigation may play an important role in the study of early universe and its evolution. In the Müller-Israel-Stewert theory of causal hydrodynamics $[91–94]$, it will be interesting to observe the importance of broken $Z_{N}$ symmetry through the dependence of entropy production rate on $\zeta_{T}$ at the time of QGP phase transition. We can also note that the contribution of $\zeta_{T}$ from TMM will be different from the case of massless YM theory in entropy production rate due to absence of resummation. Other transport coefficients are remained to be calculated from the TMM at finite temperature, whose behaviours at large-$N$ limit can be investigated. The significance of Eq. (63) can also be explored in the scenarios of bulk viscous cosmology $[95, 96]$ for the study of dark matter and dark energy.

Acknowledgments

DM is thankful to the Department of Atomic Energy, Government of India for financial support. RK would like to thank the University Grants Commission, Government of India, New Delhi, for financial support under the PDFSS scheme.

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