COMPLEXITY OF SIMPLE MODULES OVER THE LIE SUPERALGEBRA \( \mathfrak{osp}(k|2) \)

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Abstract. The complexity of a module is the rate of growth of the minimal projective resolution of the module while the \( z \)-complexity is the rate of growth of the number of indecomposable summands at each step in the resolution. Let \( \mathfrak{g} = \mathfrak{osp}(k|2) \) \((k > 2)\) be the type II orthosymplectic Lie superalgebra of types \( B \) or \( D \). In this paper, we compute the complexity and the \( z \)-complexity of the simple finite-dimensional \( \mathfrak{g} \)-supermodules. We then give these complexities certain geometric interpretations using support and associated varieties.

Keywords: Lie superalgebra; Complexity of module; \( z \)-Complexity; Support varieties; Associated varieties.

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1. Introduction

The complexity of a module over a finite group was first introduced by Alperin \cite{alperin}. The complexity of a module (cf. Subsection 2.2) is the rate of growth of a minimal projective resolution of the module. In \cite{carlson}, Carlson introduced support varieties of modules over group algebras which were then used to give a geometric interpretation of the complexity. In particular, the dimension of the support variety of the module is equal to the complexity.

This work was extended to Lie superalgebras in \cite{turkey} where the authors computed the complexity of the simple and Kac modules over the general linear Lie superalgebra \( \mathfrak{g} = \mathfrak{gl}(m|n) \) of type \( A \). Their work was carried in the category of finite-dimensional \( \mathfrak{g} \)-supermodules which are completely reducible over the even part \( \mathfrak{g}_0 \). Let \( \mathcal{F} \) denote this category (see \cite{fuchs} \cite{gelfand}). The authors in \cite{fuchs} showed that, for any basic classical Lie superalgebra \( \mathfrak{g} \), \( \mathcal{F} \) has enough projective modules and it satisfies: (i) it is a self-injective category and (ii) every module in this category admits a projective resolution which has a polynomial rate of growth. For a module \( M \in \mathcal{F} \), let \( c_\mathcal{F}(M) \) denote the complexity of \( M \). It was shown in \cite{fuchs} Theorem 2.5.1] that the complexity is always finite in classical Lie superalgebras. It is also important to mention that complexity detects projectivity. In particular, by \cite{fuchs} Corollary 2.7.1], \( c_\mathcal{F}(M) = 0 \) if and only if \( M \) is projective. We can look at complexity as a tool of measuring how far the module is from being projective.

In addition to computing the complexity of simple and Kac modules over \( \mathfrak{gl}(m|n) \) in \cite{turkey}, the authors interpreted their computations geometrically using two types of varieties. Explicitly, if \( \mathcal{X}_M \) denotes the associated variety defined by Duflo and Serganova \cite{duflo}, and \( \mathcal{V}_{(\mathfrak{g}_0)}(M) \) is the support variety as defined in \cite{fuchs}, then:

\[
c_\mathcal{F}(X(\lambda)) = \dim \mathcal{X}_X(\lambda) + \dim \mathcal{V}_{(\mathfrak{g}_0)}(X(\lambda)),
\]

\hspace{1cm} (1.0.1)
where $X(\lambda)$ is a Kac or a simple $\mathfrak{gl}(m|n)$-module.

In [10], the author computed the complexity of the simple and the Kac modules over the orthosymplectic Lie superalgebra $\mathfrak{osp}(2|2n)$ of type $C$. These computations were interpreted geometrically as in (1.0.1). It is important to note that both types $A$ and $C$ are type I Lie superalgebras and hence similar results were expected. However, it was shown in [10] that simple modules satisfy (1.0.1) for the type II superalgebras and hence similar results were expected. However, it was shown in [10] that simple modules satisfy (1.0.1) for the type $II$ superalgebras; $\mathfrak{osp}(3|2)$, and the three exceptional ones $D(2, 1; \alpha)$, $G(3)$, and $F(4)$.

In this paper we compute the complexity of the simple modules over the orthosymplectic Lie superalgebras $\mathfrak{osp}(k|2)$ ($k > 2$) of types $B$ and $D$. We show in Theorem 3.0.2 that the complexity of the atypical (cf. Subsection 2.6) simple modules is $k + 1$. Then we verify that (1.0.1) holds in both types.

Moreover, we compute the $z$-complexity of simple modules over $\mathfrak{osp}(k|2)$ ($k > 2$). The $z$-complexity of modules was introduced in [2, Section 9] and will be denoted by $z_F(-)$. In [2] the authors computed this categorical invariant for the simple and the Kac modules over $\mathfrak{gl}(m|n)$ and then used a detecting subsuperalgebra $\mathfrak{f}$ to interpret their computations geometrically. The same was done for the Lie superalgebras $\mathfrak{osp}(2|2n)$, $\mathfrak{osp}(3|2)$, $D(2, 1; \alpha)$, $G(3)$, and $F(4)$ in [10]. We carry these computations over $\mathfrak{osp}(k|2)$ and conclude in Theorem 4.0.2 that if $L(\lambda)$ is a simple module, we have:

$$z_F(L(\lambda)) = \dim \mathcal{V}_{(\mathfrak{f}, \mathfrak{h}_0)}(L(\lambda)). \quad (1.0.2)$$

The organization of the paper goes as follows. In Section 2 we give brief preliminaries for classical Lie superalgebras and their representations. We also provide brief definitions for the complexity, support and associated variety, and $z$-complexity of modules. In Section 3 we compute the complexity of the simple modules over $\mathfrak{osp}(k|2)$ for $k > 2$ and show that (1.0.1) holds. In Section 4 we compute the $z$-complexity of the same family of modules and show (1.0.2) holds.

2. Preliminaries

In this section we present some preliminary material on Lie superalgebras. We also give brief definitions for the complexity, $z$-complexity, support and associated varieties. Then we shift our focus to the Lie superalgebra $\mathfrak{osp}(k|2)$ and we provide a set of technical tools that will be used in our computations.

2.1. Lie superalgebras and representations. We will mainly use the same notations and conventions from [10] and [16]. We will work over the complex numbers $\mathbb{C}$ throughout this paper.

A Lie superalgebra $\mathfrak{g}$ is $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a bracket operation $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which preserves the $\mathbb{Z}_2$-grading and satisfies graded versions of the usual Lie bracket axioms. The subspace $\mathfrak{g}_0$ is a Lie algebra under the bracket and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. If $\mathfrak{g}$ has a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which is compatible with the $\mathbb{Z}_2$-grading, then $\mathfrak{g}$ is a type I Lie superalgebra. If $\mathfrak{g} = \bigoplus_{i=-2}^{\infty} \mathfrak{g}_i$ as a $\mathbb{Z}$-graded Lie superalgebra with $\mathfrak{g}_0 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, then it is of type II. If there is a connected reductive algebraic group $G_0$ such that Lie$(G_0) = \mathfrak{g}_0$, and an action of $G_0$ on $\mathfrak{g}_1$ which differentiates to the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$, then $\mathfrak{g}$ is called classical. Moreover, if it has a nondegenerate
invariant supersymmetric even bilinear form, then \( \mathfrak{g} \) is called \textit{basic}. For our purposes, the Lie superalgebra \( \mathfrak{osp}(k|2) \) \((k > 2)\) is a type II basic classical Lie superalgebra (See [12]).

The category of \( \mathfrak{g} \)-supermodules is described in [2] but we give a brief definition. Let \( U(\mathfrak{g}) \) be the universal enveloping superalgebra. The objects in the aforementioned category are all left \( U(\mathfrak{g}) \)-modules which are \( \mathbb{Z}_2 \)-graded vector spaces \( M = M_0 \oplus M_1 \) satisfying \( U(\mathfrak{g}), M_r \subseteq M_{r+s} \) for all \( r, s \in \mathbb{Z}_2 \). For \( \mathfrak{g} = \mathfrak{osp}(k|2) \) \((k > 2)\), we only consider the full subcategory, \( \mathcal{F} \) (as in [8, 4]), of all finite-dimensional \( \mathfrak{g} \)-supermodules. Note that these are completely reducible over \( \mathfrak{g}_0 \) since \( \mathfrak{g}_0 \) is semisimple and hence this aligns with the work done in [2] and [10]. For simplicity, we will from now on use the term “module” with the understanding that the prefix “super” is implicit.

2.2. Complexity. (See [2, Section 2.2].) Let \( \mathbf{V}_t = \{ V_t \mid t \in \mathbb{N} \} \) be a sequence of finite-dimensional \( \mathbb{C} \)-vector spaces. The rate of growth of \( \mathbf{V}_t \), \( r(\mathbf{V}_t) \), is the smallest nonnegative integer \( c \) such that there exists a constant \( C > 0 \) with \( \dim V_t \leq C \cdot t^{c-1} \) for all \( t \). If no such integer \( c \) exists, then \( \mathbf{V}_t \) is said to have an infinite rate of growth.

Let \( M \in \mathcal{F} \) and \( \mathbf{P}_t \rightarrow M \) be a minimal projective resolution of \( M \). The \textit{complexity} of \( M \) is defined to be \( c_{\mathcal{F}}(M) := r(\mathbf{P}_t) \). It was shown in [3, Theorem 2.5.1] that the complexity is always finite, in particular, \( c_{\mathcal{F}}(M) \leq \dim \mathfrak{g}_1 \). In addition, one can use the rate of growth of extension groups in \( \mathcal{F} \) to compute the complexity (See [3, Proposition 2.8.1]):

\[
c_{\mathcal{F}}(M) = r\left( \operatorname{Ext}^\bullet_{(\mathfrak{g}, \mathfrak{g}_0)}(M, \bigoplus S^{\dim P(S)}) \right),
\]

where the sum is over all the simple modules \( S \in \mathcal{F} \), and \( P(S) \) is the projective cover of \( S \). Here and elsewhere, we write \( \operatorname{Ext}^\bullet_{(\mathfrak{g}, \mathfrak{g}_0)}(M, N) \) for the relative cohomology for the pair \((\mathfrak{g}, \mathfrak{g}_0)\) as introduced in [4] Section 2.3. In this paper, this characterization of the complexity will not be used in computations. However, it shows that the complexity of a module is not a categorical invariant.

2.3. \( z \)-complexity. (See [2, Section 9].) The above problem of invariance can be fixed by removing the dimensions of the projective covers from (2.2.1). This gives the \textit{\( z \)-complexity} of \( M \in \mathcal{F} \):

\[
z_{\mathcal{F}}(M) := r\left( \operatorname{Ext}^\bullet_{(\mathfrak{g}, \mathfrak{g}_0)}(M, S) \right),
\]

where the direct sum runs over all simple modules \( S \in \mathcal{F} \). This shows that, unlike complexity, \( z_{\mathcal{F}}(\cdot) \) has the advantage of being invariant under category equivalences.

Once again, (2.3.1) will not be used for the purpose of computing the \( z \)-complexity. However, if \( \mathbf{P}_t \rightarrow M \) is a minimal projective resolution of \( M \), define \( s(\mathbf{P}_t) \) to be the rate of growth of the number of indecomposable summands at each step in the resolution. Then we can see that \( z_{\mathcal{F}}(M) = s(\mathbf{P}_t) \).

2.4. Support and associated varieties. (See [4, Section 6], [7, Section 2].) These varieties will be used to give geometric interpretations of the complexity and the \( z \)-complexity. We only provide brief definitions.

Let \( R = H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \) be the cohomology ring of \( \mathfrak{g} \) and let \( M \in \mathcal{F} \). According to [4, Theorem 2.7], \( \operatorname{Ext}^\bullet_{\mathcal{F}}(M, M) \) is a finitely generated \( R \)-module. Set \( J := \operatorname{Ann}_R(\operatorname{Ext}^\bullet_{\mathcal{F}}(M, M)) \).
The support variety of $M$ is defined by 
\[ \mathcal{V}(\rho_0)(M) := MaxSpec(R/J). \]

Now let $\mathcal{X}$ be the cone of odd self-commuting elements, that is, $\mathcal{X} = \{ x \in g_1 \mid [x, x] = 0 \}$. If $M \in \mathcal{F}$, then Duflo and Serganova [7] defined the associated variety of $M$ which is equivalent to:

\[ \mathcal{X}_M = \{ x \in \mathcal{X} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module} \cup \{0\}, \]

where $U(\langle x \rangle)$ denotes the enveloping algebra of the Lie superalgebra generated by $x$.

**2.5. Lie superalgebra $\mathfrak{osp}(k|2)$.** We denote by $\mathfrak{g}$ the orthosymplectic Lie superalgebra $\mathfrak{osp}(k|2)$ where $k > 2$. The matrix form of these Lie superalgebras is given in [12]. Note that when $k = 2$, $\mathfrak{g}$ is the type I Lie superalgebra $\mathfrak{osp}(2|2)$ of type C which was handled in [10]. The case $k = 3$ was also handled in [10]. Thus we can assume $k \geq 4$. If $k = 2m$, we have the Lie superalgebra $D(m,1)$ where $g_0 \cong \mathfrak{so}(k) \oplus \mathfrak{sl}_2$ is of type $D_m \oplus A_1$. If $k = 2m + 1$ we have the Lie superalgebra $B(m,1)$ where $g_0 \cong \mathfrak{so}(k) \oplus \mathfrak{sl}_2$ is of type $B_m \oplus A_1$. In both cases, $g_1$, as a $g_0$-module, to the outer tensor product of the natural representations of $\mathfrak{so}(k)$ and $\mathfrak{sl}_2$ and thus $\dim g_1 = 2k$.

The Cartan subalgebra $\mathfrak{h}$ will be chosen to be the set of diagonal matrices. Let $\mathfrak{h}^*$ denote the dual of the Cartan subalgebra with basis $\{ \delta, \varepsilon_i \mid 1 \leq i \leq m \}$. It has a bilinear form defined by

\[ (\delta, \delta) = -1, \quad (\delta, \varepsilon_i) = 0, \quad (\varepsilon_i, \varepsilon_j) = \delta_{i,j} \text{ for all } 1 \leq i, j \leq m. \] (2.5.1)

The set of simple roots is given by:

\[ \Delta = \begin{cases} 
\{ \delta - \varepsilon_1, \varepsilon_{m-1} + \varepsilon_m, \varepsilon_i - \varepsilon_{i+1}, \mid 1 \leq i \leq m - 1 \} & \text{if } k = 2m, \\
\{ \delta - \varepsilon_1, \varepsilon_m, \varepsilon_i - \varepsilon_{i+1}, \mid 1 \leq i \leq m - 1 \} & \text{if } k = 2m + 1.
\end{cases} \] (2.5.2)

Let $\Phi^+$ be the set of positive roots. The set of positive even and odd roots are given respectively:

\[ \Phi^+_0 = \{ 2 \delta, \varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq m \}, \quad \Phi^+_1 = \{ \delta \pm \varepsilon_i \mid 1 \leq i \leq m \} \text{ if } k = 2m, \] (2.5.3)

\[ \Phi^+_0 = \{ 2 \delta, \varepsilon_i, \varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq m \}, \quad \Phi^+_1 = \{ \delta, \delta \pm \varepsilon_i \mid 1 \leq i \leq m \} \text{ if } k = 2m + 1. \] (2.5.4)

For $\alpha \in \Phi^+$, we let $e_\alpha$ and $e_{-\alpha}$ to be respectively the positive and negative root vectors as defined in [10] Section 2. Let $g_{\pm 2} = \mathbb{C}e_{\pm 2\delta}$, $g_{\pm 1}$ be the $\mathbb{C}$-span of $\{ e_{\pm \alpha} \mid \alpha \in \Phi^+_1 \}$, and $g_0 = \mathfrak{so}(k) \oplus \mathbb{C}$ whose set of positive roots is $\Phi^+_0 = \Phi^+_0 \setminus \{ 2\delta \}$. We then have $g = \bigoplus_{i=-2}^{i=2} g_i$ as a $\mathbb{Z}_2$-consistent $\mathbb{Z}$-graded Lie superalgebra which shows that $g$ is a type II Lie superalgebra.

A weight $\lambda = \lambda_0 \delta + \sum_{i=1}^m \lambda_i \varepsilon_i \in \mathfrak{h}^*$ will be written as $(\lambda_0|\lambda_1, \ldots, \lambda_m)$ where $\lambda_i$ will be called the $i$-th coordinate of $\lambda$. Let $\rho = \rho_\bar{0} - \rho\bar{1}$ where $\rho_\bar{0}$ is half the sum of positive even roots and $\rho_\bar{1}$ is half the sum of positive odd roots. Let $s = 1$ if $k = 2m$ and $s = \frac{1}{2}$ if $k = 2m + 1$. Then by [10] equation 2.1, we have:

\[ \rho = (s - m|m - s, \ldots, 1 - s), \quad \rho\bar{1} = (m + 1 - s|0, \ldots, 0), \quad \rho_\bar{0} = (1|m - s, \ldots, 1 - s). \] (2.5.5)

In many occasions throughout this paper, we will use the following notation for a $\rho$-translated weight:

\[ \tilde{\lambda} = \lambda + \rho = (\tilde{\lambda}_0|\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m). \]
Let $W_0$ be the Weyl group of $\mathfrak{g}_0$. In particular, $W_0$ is the Weyl group of $\mathfrak{so}(m)$ and hence it is a semi-direct product of the symmetric group $S_m$ and $m$-copies of $\mathbb{Z}_2$ which acts on a weight $\lambda$ by permuting the coordinates and also changing their signs (the number of sign changes must be in $2s\mathbb{Z}_+$.). The Weyl group of $\mathfrak{g}$ is $W := W_0 \times \mathbb{Z}_2$ where the nontrivial element of $\mathbb{Z}_2$ changes the sign of the coordinate $\lambda_0$ when acting on a weight $\lambda$. The Weyl group acts on $\mathfrak{h}^*$ by the dot action: $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $w \in W$ and $\lambda \in \mathfrak{h}^*$. Since $\rho_1$ is $W_0$-invariant, then $w \cdot \lambda = w(\lambda + \rho_0) - \rho_0$ for $w \in W_0$.

Integral dominant weights are described in [16]. A weight $\lambda \in \mathfrak{h}^*$ is:

- **integral** if $\lambda_0 \in \mathbb{Z}$ and $\lambda_i \in \mathbb{Z}$ or $\lambda_i \in s + \mathbb{Z}$,
- **regular** if it is integral and $|\lambda_1|, \ldots, |\lambda_m|$ are distinct,
- **integral $\mathfrak{g}_0$-dominant** if it is integral and: $\lambda_1 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m|$ and further, $\lambda_m \geq 0$ if $k = 2m + 1$,
- **integral $\mathfrak{g}$-dominant** if it is integral $\mathfrak{g}_0$-dominant and: $l = \lambda_0 \in \mathbb{Z}_+$, and if $0 \leq l \leq m - 1$, then $\lambda_{l+1} = \lambda_{l+2} = \cdots = \lambda_m = 0$.

Denote by $P^{0+}$ and $P^+$ the sets of integral $\mathfrak{g}_0$-dominant and $\mathfrak{g}$-dominant weights respectively. For $\lambda \in P^{0+}$, the finite-dimensional irreducible $\mathfrak{g}_0$-module of highest weight $\lambda$ will be denoted by $L_\lambda^0$. We can then extend it to a $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$-module by making $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)L_\lambda^0 = 0$. Then the generalised Verma module $V_\lambda$ (as defined in [16]) is the induced module

$$V_\lambda = \text{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}^{\mathfrak{g}} L_\lambda^0 \cong U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \otimes_C L_\lambda^0.$$  

Denote by $L(\lambda)$ the unique irreducible quotient module of $V_\lambda$. It was shown in [12] that $\dim L(\lambda) < \infty$ if and only if $\lambda \in P^+$, thus $P^+$ indexes all simple modules in $\mathcal{F}$.

2.6. **Atypicality.** One of the important tools in studying representations of Lie superalgebras is atypicality. For a full definition, we refer the reader to [3 Subsection 7.2]. However, we give the definition in the case of $\mathfrak{osp}(k|2)$. A weight $\lambda \in P^{0+}$ is atypical with atypical root $\gamma = \delta \pm \varepsilon_i$ if $\lambda_0 = \pm \lambda_i$ for some $1 \leq i \leq m$. In this case, we define the atypicality of $\lambda$, $\text{atyp}(\lambda)$, to be one. Otherwise, it is called typical with $\text{atyp}(\lambda) = 0$.

We say a simple $\mathfrak{g}$-module $L(\lambda)$ of highest weight $\lambda$ is atypical (typical) if $\lambda$ is atypical (typical). Thus we define $\text{atyp}(L(\lambda)) := \text{atyp}(\lambda)$. It is known that the atypicality is the same for all simple modules in a given block. Hence it makes sense to refer to the atypicality of a block.

Moreover, if $P(\lambda)$ is the projective cover of $L(\lambda)$, then by [13 Theorem 1] we know that $P(\lambda) = L(\lambda)$ if $\text{atyp}(\lambda) = 0$, hence $L(\lambda)$ has zero complexity and $z$-complexity. This means that we only need to compute the complexity and $z$-complexity of atypical simple modules.

In the following, we make two important remarks on atypical weights:

**Remark 2.6.1.** In case $\lambda_0 = \lambda_1 = 0$ (this can only happen when $k = 2m$), both roots $\gamma_\pm = \delta \pm \varepsilon_i$ are atypical roots of $\lambda$. In this case, we always choose $\gamma = \delta - \varepsilon_i$. We also note that if $\lambda \in P^{0+}$ is atypical then $\lambda_i \in \mathbb{Z}$ for all $i$.

For atypical $\lambda \in P^{0+}$, define the atypical type of $\lambda$ by $\tilde{\lambda} = (|\lambda_1|, \ldots, |\lambda_{l-1}|, |\lambda_{l+1}|, \ldots, |\lambda_m|)$. We will also make use of the following set $S(\lambda) = \{ |\lambda_i| : i \neq 0, l \}$.

2.7. **Tools.** The description of the projective covers of the simple modules given in [6] utilizes the notation developed in [16]. In the following, we recall how this notation is
developed. First, we note that every regular weight $\lambda \in \mathfrak{h}^*$ is $W_0$-conjugate under the dot action to a unique integral $q_0$-dominant weight, which will be denoted by $\lambda^+$. For a weight $\lambda \in P^{0+}$ with atypical type $\check{\lambda}$ and atypical root $\gamma = \delta \pm \varepsilon_1$, the authors in [16] defined $\check{\lambda}$ and $\check{\lambda}$ using the following procedure. The first step is to find the smallest positive integers $a_\pm$ such that $|\check{\lambda}_i| 
eq S(\check{\lambda})$ if $\gamma = \delta + \varepsilon_1$ and $|\check{\lambda}_i| + a_\pm 
eq S(\check{\lambda})$ if $\gamma = \delta - \varepsilon_1$. Then

$$\check{\lambda} = (\lambda + a_+\gamma)^+ \quad \text{and} \quad \check{\lambda} = (\lambda - a_-\gamma)^+.$$  

(2.7.1)

A couple of examples are given in [16] to show how to compute $\check{\lambda}$ and $\check{\lambda}$. In the following, a few more notations are recalled from [16]. We assume that $\check{\lambda}$ is a fixed atypical type such that its coordinates $\check{\lambda}_1, \ldots, \check{\lambda}_m$ satisfy $|\check{\lambda}_1| > \cdots > |\check{\lambda}_{m-1}| > 0$.

**Definition 2.7.1.** (See [16] Definition 2.9)

Let $j$ be the smallest non-negative integer such that $a := j + 1 - s \neq S(\check{\lambda})$. Define $\lambda^{(0)}$ by

$$\check{\lambda}^{(0)} = \lambda^{(0)} + \rho = (-a|\check{\lambda}_1, \check{\lambda}_2, \ldots, \check{\lambda}_{m-j}, a, \check{\lambda}_{m+1-j}, \ldots, \check{\lambda}_m),$$

where we have re-labeled $\check{\lambda}_{m-j}, \ldots, \check{\lambda}_{m-j}$ by $\check{\lambda}_{m+1-j}, \ldots, \check{\lambda}_m$.

We will also utilize the weights $\lambda^i$ for $i \in \mathbb{Z}$ which are defined as follows:

**Definition 2.7.2.** (See [16] Definition 2.9)

If $k = 2m + 1$ or $k = 2m$ with $0 \in S(\lambda)$, define $\lambda^i (i \geq 1)$ inductively by:

$$\lambda^i = (\lambda^{(i-1)}), \quad \lambda^{-i} = (\lambda^{(i-1)})^\vee.$$

We compute these weights for the weight $\lambda = (0|0, \ldots, 0)$:

**Lemma 2.7.3.** Let $\lambda = (0|0, \ldots, 0)$, then $\lambda^{(0)} = \lambda$. For $i \geq 1$:

$$\lambda^i = (k + i - 3|i - 1, 0, \ldots, 0), \quad \lambda^{-i} = (-i|i, 0, \ldots, 0).$$

**Proof.** Consider the case $k = 2m$. The atypical type of $\lambda$ is $S(\check{\lambda}) = \{m - 2, m - 3, \ldots, 1\}$. To compute $\check{\lambda}$ we use $\gamma = \delta - \varepsilon_1, a_+ = 2m - 2$, and the Weyl group element, $\omega$, that changes the signs of the first and the last coordinates ($i = 1, m$). Then

$$\check{\lambda} = \omega \cdot (m - 1| - m + 1, m - 2, \ldots, 1, 0) - (1 - m|m - 1, m - 2, \ldots, 1, 0)$$

$$= (2m - 2|0, \ldots, 0).$$

Similarly, we use $a_- = 1$ to find $\check{\lambda} = (-1|1, 0, \ldots, 0)$. Since $a = j = m - 1$ (in definition [2.7.1]), we have $\lambda^{(0)} = \lambda$. Thus, $\check{\lambda} = \check{\lambda}$ and $\check{\lambda}^{-1} = \check{\lambda}$. An inductive approach can prove the result for $i > 1$ by using $a_+ = a_- = 1$. When $k = 2m + 1$, we use $j = m - 1$ and $a = m - 1/2$ to show that $\lambda^{(0)} = \lambda$. To find $\lambda^{(1)}$, we use $a_+ = 2m - 1$ and $a_- = 1$. For $\lambda^{(\pm 1)} (i > 1)$, we use $a_+ = 1$. \qed

2.8. **Blocks.** It was shown in [9] that every atypical block in $\mathcal{F}$ for $\mathfrak{osp}(k|2)$ is equivalent to an atypical block in $\mathfrak{osp}(3|2)$ if $k$ is odd, or $\mathfrak{osp}(4|2)$ or $\mathfrak{osp}(2|2)$ if $k$ is even. As stated in [6] Lemma 11.1, if $k = 2m + 1$ the quiver diagram of the block $\mathcal{F}_\chi$, for $\chi$ an atypical central character, is equal to the Dynkin diagram of type $D_\chi$. However, if $k = 2m$, the quiver diagram of $\mathcal{F}_\chi$ is either equal to the Dynkin diagram of type $D_\chi$ or of type $A_{2m}^\vee$. Moreover, by [6] Lemma 11.2, if the quiver diagram of $\mathcal{F}_\chi$ is of type $D_\chi$, the integral dominant weights corresponding to $\mathcal{F}_\chi$ are $\{\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \ldots\}$. However, if the quiver diagram of $\mathcal{F}_\chi$ is of
type $A_\infty^\infty$, the integral dominant weights corresponding to $F_\chi$ are $\{\lambda^{(0)} = \lambda_+^{(0)}, \lambda_-^{(0)}, \lambda_+^{(2)}, \ldots\}$. These weights are defined in [16] and will only be used slightly in this paper and hence we omit their definitions.

In the following we combined [6, Lemma 11.2] and [6, Proposition 11.3] to get a description of the radical structure of the projective covers.

**Lemma 2.8.1.** If $\lambda$ is in a block $F_\chi$ with a quiver diagram of type $D_\infty$, the projective indecomposable modules $P(\lambda^{(i)})$ ($i = 0, 1, 2$ and $i \geq 3$) have the following respective radical layer structure:

$$
\begin{array}{cccc}
L(\lambda^{(0)}) & L(\lambda^{(1)}) & L(\lambda^{(2)}) & L(\lambda^{(i)}) \\
L(\lambda^{(2)}) & L(\lambda^{(1)}) & L(\lambda^{(0)}) & L(\lambda^{(i)} - 1) \\
L(\lambda^{(1)}) & L(\lambda^{(2)}) & L(\lambda^{(1)}) & L(\lambda^{(i) + 1}) \\
L(\lambda^{(0)}) & L(\lambda^{(1)}) & L(\lambda^{(2)}) & L(\lambda^{(i)}) \\
\end{array}
$$

If $\lambda$ is in a block $F_\chi$ with a quiver diagram of type $A_\infty^\infty$, the projective indecomposable modules $P(\lambda_+^{(0)})$ and $P(\lambda_-^{(0)})$ ($i \geq 1$) have the following respective radical layer structure:

$$
\begin{array}{ccc}
L(\lambda_+^{(0)}) & L(\lambda_-^{(i)}) & L(\lambda_-^{(i + 1)}) \\
L(\lambda_-^{(1)}) & L(\lambda_-^{(0)}) & L(\lambda_-^{(i + 1)}) \\
L(\lambda_-^{(0)}) & L(\lambda_-^{(i) - 1}) & L(\lambda_-^{(i) + 1}) \\
\end{array}
$$

3. Computing the complexity

In this section we compute the complexity of atypical simple $\mathfrak{osp}(k|2)$-modules. Recall that typical simple modules over $\mathfrak{osp}(k|2)$ have zero complexity since they are projective. The key idea is to compute the complexity of the trivial module and utilize [13, Theorem 4.1.1] to find the complexity of other atypical simple modules. To perform these computations for the trivial module, we first find certain bounds on the dimension of the simple $\mathfrak{so}(k)$-module $L(r, 0, \ldots, 0)$ where $r$ is a positive integer (see $P^{\theta^+}$ in [2,5]).

**Lemma 3.0.1.** There are positive constants $C$ and $C'$ that depend only on $k$ such that

$$C r^{k-2} \leq \dim L(r, 0, \ldots, 0) \leq C' r^{k-2}.$$

**Proof.** Consider the case $k = 2m$. Let $\rho_{\mathfrak{so}(k)}$ be half the sum of the positive roots $\Phi_{\mathfrak{so}(k)}^+$ in $\mathfrak{so}(k)$, then $\rho_{\mathfrak{so}(k)} = (m - 1, m - 2, \ldots, 2, 1)$. By the Weyl-dimension formula ([11, Section 24.3]) we have:

$$\dim L(r, 0, \ldots, 0) = \frac{\prod_{j=2}^{m} (r + j - 1)(2m + r - j - 1) \prod_{1 \leq i < j \leq m} (j - i)(2m - i - j)}{\prod_{\alpha \in \Phi_{\mathfrak{so}(k)}^+} (\rho_{\mathfrak{so}(k)}, \alpha)},$$

thus

$$\dim L(r, 0, \ldots, 0) = C_1 \cdot \prod_{j=2}^{m} (r + j - 1)(2m + r - j - 1),$$

where $C_1$ is a constant that depends only on $m$ (and hence only on $k$). Then $\dim L(r, 0, \ldots, 0)$ is a polynomial in $r$ of degree $2m - 2$ with a positive leading coefficient. In fact, since
By using Lemma 2.7.3, a simple for some constants $C$ which can be rewritten as:

$$C \cdot r^{2m-2} \leq \dim L(r, 0, \ldots, 0) \leq C' \cdot r^{2m-2},$$

where $C$ and $C'$ are positive constants depending only on $k$. The proof is similar when $k = 2m + 1$. □

This will play a key role in finding the complexity of the atypical simple modules over $\mathfrak{osp}(k|2)$ when $k > 2$:

**Theorem 3.0.2.** For atypical $\lambda \in P^+$, $c_X(L(\lambda)) = k + 1$.

**Proof.** First, we will find the complexity of the trivial (atypical) module $C = L(0|0, \ldots, 0)$. The weight $\lambda = (0|0, \ldots, 0)$ is in the block with a quiver diagram of type $D_\infty$. The projective covers in Lemma 2.8.1 have the same structure as those over $\mathfrak{osp}(3|2)$ (see Theorem 2.1.1), hence we use the same minimal projective resolution of $L(0|0, \ldots, 0)$ from Theorem 6.5.1. It is given by:

$$\cdots \to P_d \to \cdots \to P_0 = P(\lambda(0)) \to C \to 0,$$

where the $d$th term, $d \geq 1$, in this resolution is given by:

$$P_d = \begin{cases} P(\lambda(d+1)) \oplus P(\lambda(d-1)) \oplus \cdots \oplus P(\lambda(2)) & \text{if } d \text{ is odd}, \\ P(\lambda(d+1)) \oplus P(\lambda(d-1)) \oplus \cdots \oplus P(\lambda(3)) \oplus P(\lambda(0)) & \text{if } d \equiv 0 \pmod{4}, \\ P(\lambda(d+1)) \oplus P(\lambda(d-1)) \oplus \cdots \oplus P(\lambda(3)) \oplus P(\lambda(1)) & \text{if } d \equiv 2 \pmod{4}. \end{cases}$$

By using Lemma 2.7.3 a simple $\mathfrak{g}_0$-module with weight $\lambda(i)$ is of the form: $V(k + i - 3) \boxtimes L(i - 1, 0, \ldots, 0)$ where the first module is the $\mathfrak{sl}_2$-module of dimension $k + i - 2$ and the second one is the $\mathfrak{so}(k)$-module whose dimension is bounded by Lemma 3.0.1. Thus for $i \geq 1$,

$$C(k + i - 2) \cdot (i - 1)^{k-2} \leq \dim V(k + i - 3) \boxtimes L(i - 1, 0, \ldots, 0) \leq C'(k + i - 2) \cdot (i - 1)^{k-2}.$$

As a $\mathfrak{g}_0$-module, the simple $\mathfrak{g}$-module $L(\mu)$ contains a simple $\mathfrak{g}_0$-module $L_0(\mu)$ as a composition factor. Using the discussion in Subsection 5.1, we have the following bounds:

$$\dim L_0(\mu) \leq \dim P(\mu) \leq 2^{\dim \mathfrak{g}_1} \cdot \dim L_0(\mu),$$

thus for $i \geq 1$,

$$C(k + i - 2)(i - 1)^{k-2} \leq \dim P(\lambda(i)) \leq C' \cdot 2^{\dim \mathfrak{g}_1} (k + i - 2)(i - 1)^{k-2}.$$
for some $K_2$ depending only on $k$. This upper bound can be established for $d = 1, 2$ by direct computations of dimensions. On the other hand, we note that $x = \frac{1}{2}(x + x) \geq \frac{1}{2}(x + x - 1)$ and if $t = \frac{d}{2}$ or $\frac{d+1}{2}$, then $t \geq \frac{d}{2}$. Thus

$$\dim P_d \geq C_1 \sum_{i \in T_d} \dim P^{(i)}(\lambda) \geq C_1 \frac{d+1}{2} \sum_{i=2}^{t} i^{k-1} \geq C_1 \frac{d+1}{2} \sum_{i=t}^{k} i^{k-1} \geq K_1 \cdot d^k,$$

for some $K_1$ depending only on $k$. Hence the complexity of the trivial module is $k + 1$. By [14 Theorem 4.1.1], all simple modules of the same atypicality have the same complexity. Thus the complexity of all atypical simple $\mathfrak{osp}(k|2)$-modules is $k + 1$. □

Next, we give the complexity a geometric interpretation as discussed earlier. We first recall the following result on support varieties:

**Proposition 3.0.3.** [14 Corollary 4.4.2] For $\lambda \in P^+$, dim $\mathcal{V}_{(\mathfrak{gl}_0)}(L(\lambda)) = \text{atyp}(\lambda)$.

In the following, we find the dimension of the associated variety of the simple modules:

**Proposition 3.0.4.** For $\lambda \in P^+$, dim $\mathcal{X}_{L(\lambda)} = \begin{cases} k & \text{if } \text{atyp}(\lambda) = 1, \\ 0 & \text{if } \text{atyp}(\lambda) = 0. \end{cases}$

**Proof.** When $\lambda$ is typical, $L(\lambda)$ is projective, hence dim $\mathcal{X}_{L(\lambda)} = 0$ by [7 Theorem 3.4]. For the rest of the proof, assume $\lambda$ is atypical. [15 Corollary 2.5] implies that $\mathcal{X}_{L(\lambda)} = \mathcal{X}$ when $\lambda$ is atypical. Now, dim $\mathcal{X}$ is equal to the dimension of an irreducible component which can be found by [7 Theorem 4.5]. By computing the dimension given in [7 Theorem 4.5] for both cases: $k = 2m$ and $k = 2m + 1$, we found dim $\mathcal{X} = k$, thus dim $\mathcal{X}_{L(\lambda)} = k$. □

Combining the computations from Theorem 3.0.2 and Propositions 3.0.3 and 3.0.4 we conclude that:

**Theorem 3.0.5.** For $\lambda \in P^+$, $c_\mathcal{F}(L(\lambda)) = \dim \mathcal{X}_{L(\lambda)} + \dim \mathcal{V}_{(\mathfrak{gl}_0)}(L(\lambda))$.

4. **Computing the $z$-complexity**

We compute the $z$-complexity (cf. Subsection 2.3) of the simple modules over $\mathfrak{osp}(k|2)$, $k > 2$. If $\lambda$ is typical, the simple module $L(\lambda)$ is projective and we can easily show that $z_\mathcal{F}(L(\lambda)) = 0$.

**Theorem 4.0.1.** If $\lambda$ is an atypical weight in $P^+$, then $z_\mathcal{F}(L(\lambda)) = 2$.

**Proof.** From Lemma 2.3.1 we observed that the projective covers of simple modules in the atypical block with quiver diagram of type $D_\infty$ have the same radical structure as those over $\mathfrak{osp}(3|2)$ (3 Theorem 2.1.1]). Since the $z$-complexity is a categorical invariant, we can use [10 Theorem 6.7.1] to show that the $z$-complexity of the simple modules in that block is 2.

On the other hand, the projective covers of simple modules in the atypical block with quiver diagram of type $A_\infty^\infty$ have the same radical structure (a diamond-shape) as those over $\mathfrak{osp}(2|2)$. Thus we can use [10 Theorem 5.1.1] to show that the $z$-complexity of the simple modules in that block is 2. □
Next, we give a geometric interpretation of the $z$-complexity using support varieties relative to a detecting Lie subsuperalgebra. As defined in [4], let $\mathfrak{f}_1 \subseteq \mathfrak{g}$ be the span of the root vectors $e_\alpha, f_\alpha$ where $\alpha = \delta - \varepsilon_1$. Set $\mathfrak{f}_0 = [\mathfrak{f}_0, \mathfrak{f}_1]$ spanned by $[e_\alpha, f_\alpha]$. We define a three-dimensional subalgebra of $\mathfrak{g}$ by
\[
\mathfrak{f} := \mathfrak{f}_0 \oplus \mathfrak{f}_1.
\]
The Lie superalgebra $\mathfrak{f}$ is classical and so has a support variety theory. Furthermore, as $[\mathfrak{f}_0, \mathfrak{f}_1] = 0$, it follows that these varieties admit a rank variety description and, in particular, can be identified as subvarieties of $\mathfrak{f}_1$, i.e.,
\[
\mathcal{V}_{(\mathfrak{f}_0 \mathfrak{f}_0)}(M) = \mathcal{V}_{\mathfrak{f}_1}^{\text{rank}}(M) = \{y \in \mathfrak{f}_1 \mid M \text{ is not projective as } U(y)-\text{module}\} \cup \{0\}.
\]
Using this detecting subsuperalgebra, we have the following geometric interpretation of the $z$-complexity:

**Theorem 4.0.2.** If $\lambda \in P^+$, then $\dim \mathcal{V}_{(\mathfrak{f}_0 \mathfrak{f}_0)}(L(\lambda)) = z_\mathcal{F}(L(\lambda))$.

**Proof.** By [14, Theorem 4.1.1], any atypical simple module $L(\lambda)$ satisfies $\mathcal{V}_{(\mathfrak{f}_0 \mathfrak{f}_0)}(L(\lambda)) = \mathcal{V}_{(\mathfrak{f}_0 \mathfrak{f}_0)}(\mathbb{C})$ since $L(\lambda)$ and $\mathbb{C}$ have the same atypicality. We have, however, $\mathcal{V}_{(\mathfrak{f}_0 \mathfrak{f}_0)}(\mathbb{C}) = \mathfrak{f}_1$ which gives $\dim \mathcal{V}_{(\mathfrak{f}_0 \mathfrak{f}_0)}(\mathbb{C}) = 2$. Thus,
\[
\dim \mathcal{V}_{(\mathfrak{f}_0 \mathfrak{f}_0)}(L(\lambda)) = z_\mathcal{F}(L(\lambda)) = 2.
\]
Now, if $L(\lambda)$ is typical, then it is projective and hence, $\dim \mathcal{V}_{(\mathfrak{f}_0 \mathfrak{f}_0)}(L(\lambda)) = z_\mathcal{F}(L(\lambda)) = 0$. \hfill \(\Box\)

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