ON THE CAUCHY TRANSFORM OF THE BERGMAN SPACE

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Abstract. The range of the Bergman space $B_2(G)$ under the Cauchy transform $K$ is described for a large class of domains. For a quasidisk $G$ the relation $K(B_2^*(G)) = B_2^*(\mathbb{C} \setminus \overline{G})$ is proved.

1. Introduction

Let $G$ be a domain in the complex plane $\mathbb{C}$ bounded by a Jordan curve $\partial G$ with area($\partial G$)=0. We call these domains integrable domains. Consider the following classes of analytic functions:

$$B_2(G) = \left\{ g(z) \in \text{Hol}(G), \|g\|_{B_2(G)} = \left( \int_\mathbb{C} |g(z)|^2 \, dx \, dy \right)^{\frac{1}{2}} < \infty \right\};$$

$$H(\mathbb{C} \setminus \overline{G}) = \left\{ \gamma(\zeta) \in \text{Hol}(\mathbb{C} \setminus \overline{G}), \gamma(\infty) = 0 \right\};$$

$$B_2^*(\mathbb{C} \setminus \overline{G}) = \left\{ \gamma(\zeta) \in H(\mathbb{C} \setminus \overline{G}), \right.$$ 

$$\left. \|\gamma\|_{B_2^*(\mathbb{C} \setminus \overline{G})} = \left( \int_{\mathbb{C} \setminus \overline{G}} |\gamma'(\zeta)|^2 \, d\xi \, d\eta \right)^{\frac{1}{2}} < \infty \right\},$$

where $z = x + iy$, $\zeta = \xi + i\eta$; $\overline{G}$ is the closure of the domain $G$. The class $B_2(G)$ is called the Bergman space.

The transformation

$$(Kg)(\zeta) = \frac{1}{\pi} \int_{G} \frac{g(z)}{z - \zeta} \, dx \, dy,$$

where $g(z) \in B_2(G)$, $\zeta \notin \overline{G}$ is called the Cauchy transform of $B_2^*(G)$ which is dual to $B_2(G)$. Because the spaces $B_2(G)$ and $B_2^*(G)$ are isometric, we can think of $K$ as a transformation of $B_2(G)$.

The problem of describing the range of $X^*$ under the Cauchy transform for different spaces $X$ of analytic functions was investigated by many authors, see, for example, [6]-[7]. The motivation of the present

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work is the paper [1]. V.V.Napalkov(jr) and R.S.Yulmukhametov established that \( K(B^*_2(G)) = B^1_2(\mathbb{C} \setminus \overline{G}) \) for domains with sufficiently smooth boundary. We prove that this relation is valid for quasidisks, and also find \( K(B^*_2(G)) \) for a large class of domains.

It is obvious that the Cauchy transform converts a function \( g(z) \in B_2(G) \) into an analytic function \( \gamma(\zeta) \) on \( \mathbb{C} \setminus \overline{G} \) such that \( \gamma(\infty) = 0 \). Since polynomials are dense in \( B_2(G) \) [2, ch.1, 3] and the system \( \{1/(z - \zeta), \zeta \notin \overline{G}\} \) is dense in the space of functions holomorphic in \( \overline{G} \), the operator \( K \) is injective.

The operator
\[
(Tu)(\zeta) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int \int_{|z - \zeta| \geq \epsilon} \frac{u(z)}{(z - \zeta)^2} \, dx \, dy
\]
is an isometry on \( L_2(\mathbb{C}) \) [3, pp. 64-66]. Thus \( K : B^*_2(G) \to B^1_2(\mathbb{C} \setminus \overline{G}) \) is a continuous operator.

Throughout the paper we denote the unit disk by \( \mathbb{D} \) and its boundary by \( \partial \mathbb{D} \). The boundary of a domain \( G \) is denoted by \( \partial G \).

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2. General case

To study \( K(B^*_2(G)) \) we need the function space

\[
W(0, 2\pi) = \left\{ f(e^{i\theta}) \in L_1(0, 2\pi), f(e^{i\theta}) \sim \sum_{k=-\infty}^{\infty} f_k e^{ik\theta}, \right\}
\]

with the semi-norm \( \rho(f) = \left( \pi \sum_{k=1}^{\infty} k |f_k|^2 \right)^{1/2} < \infty \).

Functions of \( W(0, 2\pi) \) can be characterized as follows:

Lemma 1. Let \( f(t) \in L_1(\partial \mathbb{D}) \), i.e. \( f(e^{i\theta}) \in L_1(0, 2\pi) \), and \( F(\zeta) \) be the Cauchy-type integral corresponding to \( f(t) \):

\[
F(\zeta) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(t)}{t - \zeta} \, dt, \quad \zeta \in \mathbb{C} \setminus \overline{\mathbb{D}}.
\]

Then \( f \in W(0, 2\pi) \) if and only if \( F \in B^1_2(\mathbb{C} \setminus \overline{\mathbb{D}}) \), and

\[
\rho(f) = \|F\|_{B^1_2(\mathbb{C} \setminus \overline{\mathbb{D}})}.
\]

Proof It is obvious that \( F(\zeta) \in \text{Hol}(\mathbb{C} \setminus \overline{G}) \) and \( F(\infty) = 0 \).
Next we have

\[ F(\zeta) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(t)}{t - \zeta} \, dt = -\frac{1}{\zeta} \sum_{k=0}^{\infty} \frac{1}{\zeta^k} \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(t) t^k dt = -\sum_{k=1}^{\infty} \frac{\bar{f}_{-k}}{\zeta^k}. \]

The identity \( \| F \|_{B_2^1(\mathbb{C} \setminus \mathcal{G})} = \left( \pi \sum_{k=1}^{\infty} |F_k|^2 k \right)^{\frac{1}{2}} \), where \( \{F_k\}_1^\infty \) is the set of Taylor coefficients of \( F \), proves the lemma.

\[ \square \]

Let \( G \) be an integrable domain and let a sequence of Jordan domains \( \{G_n\}_1^\infty \) satisfies the conditions:

(i) \( \partial G_n \) is a smooth Jordan curve; (ii) \( \mathcal{G}_{n+1} \subset G_n, n = 1, 2, 3, \ldots \);
(iii) \( \cap_{n \geq 1} G_n = \overline{G} \). Let \( \varphi_n \) be a conformal map of \( \mathbb{D} \) onto \( G_n \).

**Theorem 1.** A function \( \gamma \) from \( B_2^1(\mathbb{C} \setminus \mathcal{G}) \) belongs to \( K(B_2^1(G)) \) if and only if

\[ \sup_{n \geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})) < \infty \]

for any sequence \( \{G_n\}_1^\infty \) with (i),(ii),(iii).

**Proof** First we show the implication

\[ \left\{ \gamma(\zeta) \in B_2^1(\mathbb{C} \setminus \mathcal{G}), \sup_{n \geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})) < \infty \right\} \subset K(B_2^1(G)). \]

Let \( \gamma(\zeta) \) belong to \( B_2^1(\mathbb{C} \setminus \mathcal{G}) \) and \( \sup_{n \geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})) < \infty \). We write \( h \in \text{Hol}(\mathcal{G}) \) if there exists an open set \( G_1 = G_1(h) \supset \mathcal{G} \) such that \( h \in \text{Hol}(G_1) \). For functions \( h \in \text{Hol}(\mathcal{G}) \) introduce the linear functional:

\[ \Phi(h) = \lim_{n \to \infty} \int_{\partial G_n} \gamma(\xi) h(\xi) \, d\xi. \]

If \( n_0 \) is such a number that \( h \) is holomorphic in \( G_{n_0} \), then the last integral is unaffected by \( n \geq n_0 \). Thus, \( F(h) \) is meaningful.

We show that \( \Phi \) is a bounded linear functional on the space \( \text{Hol}(\mathcal{G}) \) using the norm of the space \( B_2(G) \). Changing the variable by formula \( \xi = \varphi_n(e^{i\theta}) \), get

\[ \frac{1}{2\pi i} \int_{\partial G_n} \gamma(\xi) h(\xi) \, d\xi = \frac{1}{2\pi i} \int_0^{2\pi} \gamma(\varphi_n(e^{i\theta})) h(\varphi_n(e^{i\theta}))(\varphi_n)'(e^{i\theta}) \, d\theta. \]

The function \( h(\varphi_n(e^{i\theta}))(\varphi_n)'(e^{i\theta}) \) is the restriction to the unit circumference of the function \( h_n(z) = h(\varphi_n(z))(\varphi_n)'(z)z i [4, p.405] \). Changing the variable \( w = \varphi_n(z) \) we see that \( \| h_n \|_{B_2(\mathbb{D})} \leq \| h \|_{B_2(G_{n_0})} \). Since \( h(z) \) is continuous in \( \mathcal{G}_n \) for \( n \geq n_0 \) and \( \varphi_n(z) \) maps the unit disk onto the
domain $G_n$ bounded by a smooth Jordan curve, $\varphi_n(z)$ and $h_n(z)$ belong to $H_2(D)$ (Hardy space) [4, p.410]. If $\{c^n_k\}_{k=1}^\infty$ is the sequence of Taylor coefficients for the function $h_n(z)$, then an easy calculation shows

$$\|h_n\|_{B_2(D)} = \left( \pi \sum_{k=1}^\infty \frac{|c^n_k|^2}{k+1} \right)^{\frac{1}{2}} < \infty.$$  

Thus

$$\frac{1}{2\pi i} \int_{\partial G_n} \gamma(z) h(z) \, dz = \frac{1}{2\pi i} \int_0^{2\pi} \gamma(\varphi_n(e^{i\theta})) h_n(e^{i\theta}) \, d\theta = \frac{1}{i} \sum_{k=1}^\infty a^n_{-k} c^n_k,$$

where $\{a^n_k\}_{k=\infty}^{-\infty}$ is defined by the formula $\gamma(\varphi_n(e^{i\theta})) = \sum_{k=\infty}^{-\infty} a^n_k e^{ik\theta}$. Applying the Cauchy-Schwarz inequality, we get

$$|\frac{1}{2\pi i} \int_{\partial G_n} \gamma(z) h(z) \, dz| = \left| \sum_{k=1}^\infty a^n_{-k} c^n_k \right| \leq \left( \sum_{k=1}^\infty k |a^n_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^\infty \frac{|c^n_k|^2}{k} \right)^{\frac{1}{2}} = \frac{\sqrt{2}}{\pi} \rho(\gamma \circ \varphi_n(e^{i\theta})) \|h\|_{B_2(D)} \leq \frac{\sqrt{2}}{\pi} \rho(\gamma \circ \varphi_n(e^{i\theta})) \|h\|_{B_2(G_n)}.$$

Because the domain $G$ is integrable, $\|h\|_{B_2(G_n)} \to \|h\|_{B_2(G)}$ as $n \to \infty$. Hence

$$|\mathbb{F}(h)| \leq C \|h\|_{B_2(G)}, \quad \text{where} \quad C = \frac{\sqrt{2}}{\pi} \sup_{n \geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})).$$

Since the space $\text{Hol}(\overline{G})$ is dense in $B_2(G)$, the functional $\mathbb{F}$ can be uniquely extended to the linear continuous functional on $B_2(G)$ that we denote by $F$ also. It follows from the Riesz-Fisher representation theorem that there exists a function $g \in B_2(G)$ such that

$$\mathbb{F}(h) = \frac{1}{\pi} \iint_{G} h(z) g(z) \, dxdy, \quad h \in B_2(G).$$

Now calculate $\mathbb{F}(1/(z - \zeta))$ for $\zeta \notin \overline{G}$,

$$\mathbb{F}\left( \frac{1}{z - \zeta} \right) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\partial G_n} \frac{\gamma(z)}{z - \zeta} \, dz = -\gamma(\zeta).$$

We obtain that

$$\gamma(\zeta) = \frac{1}{\pi} \iint_{G} \frac{-g(z)}{z - \zeta} \, dxdy, \quad \zeta \notin \overline{G} \quad \text{and} \quad -g \in B_2(G).$$
The relation
\[ \left\{ \gamma(\zeta) \in B_2^1(\mathbb{C} \setminus \overline{G}), \sup_{n \geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})) < \infty \right\} \subset K(B_2(G)) \]
is proved.

To prove the relation
\[ K(B_2(G)) \subset \left\{ \gamma(\zeta) \in B_2^1(\mathbb{C} \setminus \overline{G}), \sup_{n \geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})) < \infty \right\} \]
we apply the lemma. It is sufficient to show that \( \sup_{n \geq 1} \|F_n\|_{B_2^1(\mathbb{C} \setminus \overline{D})} < \infty \), where
\[ F_n(\zeta) = \frac{1}{2\pi i} \int_{\partial \overline{D}} \frac{\gamma \circ \varphi_n(t)}{t - \zeta} d\zeta, \quad \gamma(\zeta) = \frac{1}{\pi} \iint_{\overline{G}} \frac{\overline{g(z)}}{z - \zeta} dxdy, \quad g \in B_2(G). \]

Putting the expression for \( \gamma(\zeta) \) in the formula for \( F_n(\zeta) \), have
\[ F_n(\zeta) = \frac{1}{2\pi i} \int_{\partial \overline{D}} \frac{1}{t - \zeta} \frac{1}{\pi} \iint_{\overline{G}} \frac{\overline{g(z)}}{z - \varphi_n(t)} dxdy dt. \]

Since \( \overline{g(z)}/((t - \zeta)(z - \varphi_n(t))) \in L_1(G \times \partial \overline{D}) \) for \( \zeta \in \mathbb{C} \setminus \overline{D} \), we can interchange the order of integration:
\[ F_n(\zeta) = \frac{1}{\pi} \iint_{\overline{G}} \frac{g(z)}{2\pi i} \int_{\partial \overline{D}} \frac{1}{t - \zeta} \frac{1}{z - \varphi_n(t)} dt dxdy. \]

Further, the residue theorem yields
\[ \frac{1}{2\pi i} \int_{\partial \overline{D}} \frac{1}{t - \zeta} \frac{1}{z - \varphi_n(t)} dt = -\frac{1}{(\varphi^{-1}_n(z) - \zeta)\varphi'_n(\varphi^{-1}_n(z))}, \]
where \( \varphi^{-1}_n \) is the inverse function of \( \varphi_n \). Let \( w = \varphi^{-1}_n(z) \) in the resulting integral, we then see that
\[ F_n(\zeta) = -\frac{1}{\pi} \iint_{\overline{D}_n} \frac{g(\varphi_n(w))\varphi'_n(w)}{w - \zeta} dudv, \]
where \( \overline{D}_n = \varphi^{-1}_n(G) \subset \overline{D} \). Hence in \( \mathbb{C} \setminus \overline{D} \)
\[ F'_n(\zeta) = \mathbb{T}(\overline{-g(\varphi_n(w))\varphi'_n(w)})(\zeta), \]
where the operator \( \mathbb{T} \) was introduced earlier. Since \( \mathbb{T} \) is isometric, we get
\[
\|F_n\|_{B_2^1(\mathbb{C} \setminus \overline{D})} \leq \|\mathbb{T}(\overline{-g(\varphi_n(w))\varphi'_n(w)})\|_{L_2(\mathbb{C} \setminus \overline{D})} \\
\leq \|g(\varphi_n(w))\varphi'_n(w)\|_{B_2(\overline{D}_n)} = \|g\|_{B_2(G)}.
\]
Thus
\[ \sup_{n \geq 1} \| F_n \|_{B^1_2(C \setminus \overline{D})} \leq \| g \|_{B^2_2(G)}, \]
Theorem 1 is proved. \(\square\)

3. THE CASE OF A QUASIDISK

As an application of Theorem 1 we prove a theorem concerning the Cauchy transform of the Bergman space on quasidisks.

We give some definitions [5, ch. 5].

**Definition 1.** A quasiconformal map of \( \mathbb{C} \) onto \( \mathbb{C} \) is a homeomorphism \( h \) such that:
1. \( h(x + iy) \) is absolutely continuous in \( x \) for almost all \( y \) and in \( y \) for almost all \( x \);
2. the partial derivatives are locally square integrable;
3. \( h(x + iy) \) satisfies the Beltrami differential equation
   \[ \frac{\partial h}{\partial \overline{z}} = \mu(z) \frac{\partial h}{\partial z} \]
   for almost all \( z \in \mathbb{C} \),
where \( \mu \) is a complex measurable function with \( |\mu(z)| \leq k < 1 \) for \( z \in \mathbb{C} \). In this case it is said \( h \) to be a \( k \)-quasiconformal map.

**Definition 2.** A quasicircle in \( \mathbb{C} \) is a Jordan curve \( J \) such that
\[ \text{diam } J(a, b) \leq M|a - b| \]
for \( a, b \in J \), where \( J(a, b) \) is the arc of the smaller diameter of \( J \) between \( a \) and \( b \). The domain interior to \( J \) is called a quasidisk.

**Remark 1.** An equivalent definition for \( J \) to be a quasicircle: \( J \) is the range of the circle under a quasiconformal map of \( \mathbb{C} \) onto \( \mathbb{C} \).

**Theorem 2.** Let \( G \) be a quasidisk, then
\[ K(B^2_1(G)) = B^1_2(C \setminus \overline{G}). \]

**Proof** Let \( \psi \) be a conformal map of \( \mathbb{C} \setminus \overline{D} \) onto \( \mathbb{C} \setminus \overline{G} \) with \( \psi(\infty) = \infty \). Denote the inner domain bounded by the curve \( \{ \psi(R_n e^{i\theta}) \mid \theta \in [0, 2\pi) \} \) by \( G_n \), where \( \{ R_n \}^\infty_1 \) be some sequence decreasing monotonically to 1. Let \( \varphi_n \) be a conformal map of \( \mathbb{D} \) onto \( G_n \).

Since \( K(B^2_1(G)) \subset B^1_2(C \setminus \overline{G}) \), we have only to show that for every \( \gamma(\zeta) \in B^2_2(C \setminus \overline{G}) \) the following holds true: \( \sup_{n \geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})) < \infty \). Then, in view of Theorem 1, we get Theorem 2.

To verify the inequality \( \sup_{n \geq 1} \rho(\gamma \circ \varphi_n(e^{i\theta})) < \infty \) apply the lemma. We have
\[ F_n(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{\gamma \circ \varphi_n(t)}{t - \zeta} dt, \quad \zeta \in \mathbb{C} \setminus \overline{D}. \]
It is clear that
\[ |ψ^{-1} \circ φ_n(t)| = R_n, \quad t ∈ ∂D, \quad n ≥ 1. \]

Hence \( γ \circ φ_n(t) = γ \circ ψ \left( \frac{R_n^2}{|ψ^{-1} \circ φ_n(t)|} \right), t ∈ ∂D. \)

Theorem 5.17 [5, p.114] states that any conformal map of the disk onto a quasidisk can be extended to a quasiconformal map of \( \mathbb{C} \) onto \( \mathbb{C} \). Evidently, the theorem remains true for a conformal map of \( \mathbb{C} \) onto \( \mathbb{C} \). Using isometricity of the operator \( \mathcal{T} \) defined above, we have:

\[ \frac{1}{\sqrt{1 - k^2}} \frac{1}{|z−γ|} \leq \frac{1}{\sqrt{1 - k^2}} \frac{1}{|z−γ|} \leq \frac{1}{\sqrt{1 - k^2}} \frac{1}{|z−γ|}. \]

where \( w = u + iv \). Since the operator \( ˜ψ : ˜ψ(γ)(ζ) = γ \circ ψ(ζ) \) is an isometry from \( B^1_2(\mathbb{C} \setminus \overline{G}) \) to \( B^1_2(\mathbb{C} \setminus \overline{D}) \), we have:

\[ \frac{1}{\sqrt{1 - k^2}} \frac{1}{|z−γ|} \leq \frac{1}{\sqrt{1 - k^2}} \frac{1}{|z−γ|}. \]

Now the Green formula gives:

\[ F_n(ζ) = \frac{1}{2πi} \int_{∂D} γ \circ ψ(f_n(t)) \frac{dt}{t−ζ} = \frac{1}{π} \int_{∂D} \frac{1}{z−ζ} \frac{∂}{∂z} γ \circ ψ(f_n(z)) dxdy \]

for \( ζ ∈ \mathbb{C} \setminus \overline{D} \). Using isometricity of the operator \( \mathcal{T} \) defined above, we get

\[ \|F_n\|_{B^1_2(\mathbb{C} \setminus \overline{D})} \leq \| \frac{∂}{∂z} γ \circ ψ(f_n(z)) \|_{L^2(\mathbb{D})} \leq \frac{1}{\sqrt{1 - k^2}} \| γ \|_{B^1_2(\mathbb{C} \setminus \overline{G})}. \]

Thus Theorem 2 is proved.
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