Can parity-time-symmetric potentials support continuous families of non-parity-time-symmetric solitons?

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Abstract
For the one-dimensional nonlinear Schrödinger equations with parity-time (PT) symmetric potentials, it is shown that when a real symmetric potential is perturbed by weak PT-symmetric perturbations, continuous families of asymmetric solitary waves in the real potential are destroyed. It is also shown that in the same model with a general PT-symmetric potential, symmetry breaking of PT-symmetric solitary waves do not occur. Based on these findings, it is conjectured that one-dimensional PT-symmetric potentials cannot support continuous families of non-PT-symmetric solitary waves.

1 Introduction
Solitary waves play an important role in the dynamics of nonlinear wave equations (Kivshar & Agrawal 2003, Yang 2010). In conservative systems, solitary waves generally exist as continuous families. Familiar examples include the nonlinear Schrödinger equation with or without external real potentials. In dissipative systems, solitary waves generally exist as isolated solutions, with the Ginzburg-Landau equation as one of the best known examples (Akhmediev & Ankiewicz 2005). However, a recent discovery is that, in dissipative but parity-time (PT) symmetric systems (Bender & Boettcher 1998), solitary waves can still exist as continuous families, parameterized by their propagation constants. This exact balance of continually deformed wave profiles in the presence of gain and loss is very remarkable. So far, various PT-symmetric wave systems have been investigated. Examples include the NLS equations with linear and/or nonlinear PT-symmetric potentials (Musslimani et al. 2008, Wang & Wang 2011, Lu & Zhang 2011, Abdullaev et al. 2011, He et al. 2012, Nixon et al. 2012, Yang 2012a, Zezyulin & Konotop 2012a, Huang et al. 2013), vector NLS equations with PT-symmetric potentials (Kartashov 2013), PT-symmetric couplers (Driben & Malomed 2011, Alexeeva et al. 2012), the $\chi^{(2)}$ system with PT-symmetric potentials (Moreira et al. 2012), discrete NLS-PT lattices (Konotop et al. 2012, Kevrekidis et al. 2013), and so on. In these systems, continuous families of PT-symmetric solitary waves (or solitons in short) have been reported. Other PT systems, such as finite-dimensional PT systems, have also been explored (Li & Kevrekidis 2011, Zezyulin & Konotop 2012b, Zezyulin & Konotop 2013). In certain finite-dimensional PT systems (such as the quadrimer model), it was found that continuous families of PT-symmetric solutions could coexist with isolated solutions. Experimentally, PT-symmetric potentials have been fabricated in optical settings (Guo et al. 2009, Rüter et al. 2010).
Since solitons in PT systems could exist for continuous ranges of propagation constants, bifurcations of such solitons become an important issue. In conservative wave systems, various archetypical bifurcations of solitons have been reported, including fold bifurcations (also known as saddle-node or saddle-center bifurcations), symmetry-breaking bifurcations (also known as pitchfork bifurcations), and transcritical bifurcations, see Yang (2012b) and the references therein. In PT systems, fold bifurcations have been found (Yang 2012a, Zezyulin & Konotop 2012a, Konotop et al. 2012, Kevrekidis et al. 2013), but the other types of bifurcations are still unknown.

Of all bifurcations, symmetry-breaking bifurcations are particularly interesting, since such bifurcations create solitary waves that do not obey the symmetry of the original system. One of the most familiar symmetry-breaking bifurcations is in the NLS equation with a real symmetric potential, where families of asymmetric solitons bifurcate out from symmetric solitons at certain propagation-constant values (Jackson & Weinstein 2004, Kirr et al. 2008, Sacchetti 2009, Kirr et al. 2011, Akylas et al. 2012, Pelinovsky & Phan 2012, Yang 2013). These asymmetric solitons are often more stable than their symmetric counterparts. PT-symmetric systems have been shown to admit families of PT-symmetric solitons. Then a natural question is, can symmetry breaking occur for these PT-symmetric solitons? If so, continuous families of non-PT-symmetric solitons would appear in a PT-symmetric system.

Symmetry breaking is a dominant way for the creation of families of asymmetric solitons, but it may not be the only way. Thus, a more general question is, can PT-symmetric systems admit continuous families of non-PT-symmetric solitons?

In this article, we investigate the existence of families of non-PT-symmetric solitons in a familiar PT system — the one-dimensional NLS equation with a linear PT-symmetric potential. This PT system governs paraxial nonlinear light propagation in a medium with symmetric refractive index and anti-symmetric gain and loss (Musslimani et al. 2008, Yang 2010), as well as Bose-Einstein condensates in a symmetric potential with balanced gain and loss (Pitaevskii & Stringari 2003). For this PT system, we show that continuous families of asymmetric solitons in a real symmetric potential are destroyed when this real potential is perturbed by weak PT-symmetric perturbations. We further show that in a general one-dimensional PT-symmetric potential, symmetry breaking of PT-symmetric solitons cannot occur. Based on these findings, we conjecture that one-dimensional PT-symmetric potentials cannot support continuous families of non-PT-symmetric solitons. In other words, continuous families of solitons in one-dimensional PT-symmetric potentials must be PT-symmetric.

2 Preliminaries

Our study of solitary waves in PT-symmetric systems is based on the following one-dimensional nonlinear Schrödinger (1D NLS) equation with a linear PT-symmetric potential

\[ iU_t + U_{xx} - V(x)\Psi + \sigma|\Psi|^2\Psi = 0, \]  

(2.1)

where \( V(x) \) is a complex-valued (non-real) PT-symmetric potential

\[ V^*(x) = V(-x), \]  

(2.2)

with the asterisk representing complex conjugation, and \( \sigma = \pm 1 \) is the sign of nonlinearity (\( \sigma = 1 \) for self-focusing and \( \sigma = -1 \) for self-defocusing). Here, the PT-symmetry (2.2) means that the real
part of the potential $V(x)$ is symmetric in $x$, and the imaginary part of $V(x)$ is antisymmetric in $x$. The equation (2.1) is the appropriate mathematical model for paraxial light transmission in PT-symmetric media (where the refractive index is symmetric and gain-loss profile anti-symmetric). It also governs the dynamics of Bose-Einstein condensates in a symmetric potential with spatially-balanced gain and loss (in this community, Eq. (2.1) is called the Gross-Pitaevskii equation). In the model (2.1), the nonlinearity is only cubic. But extension of our analysis to an arbitrary form of nonlinearity is straightforward without much more effort (see Yang 2012b).

Solitary waves in Eq. (2.1) are sought of the form
\[ U(x,t) = e^{i\mu t} u(x), \]  
(2.3)
where $u(x)$ is a complex-valued localized function which satisfies the equation
\[ u_{xx} - \mu u - V(x)u + \sigma |u|^2 u = 0, \]  
(2.4)
and $\mu$ is a real-valued propagation constant. Even though Eq. (2.1) is dissipative due to the complex potential $V(x)$, a remarkable phenomenon is that it can support continuous families of solitary waves (2.3), parameterized by the propagation constant $\mu$ — just like in real potentials (Wang & Wang 2011, Lu & Zhang 2011, He et al. 2012, Nixon et al. 2012, Yang 2012a, Zezyulin & Konotop 2012a, Huang et al. 2013). Then under certain conditions, these solitary waves may undergo bifurcations at special values of $\mu$.

If the potential $V(x)$ were strictly real, then the PT symmetry condition (2.2) would become $V(-x) = V(x)$, i.e., this potential would be symmetric. It is well known that in real symmetric potentials, symmetry breaking of solitary waves often occurs. Specifically, in addition to continuous families of symmetric and anti-symmetric solitons, continuous families of asymmetric solitons can also bifurcate out from those symmetric and anti-symmetric soliton branches. This symmetry breaking is most familiar in double-well potentials (Jackson & Weinstein 2004, Sacchetti 2009), but it can occur in other symmetric potentials (such as periodic potentials) as well (Kerr et al. 2008, Akylas et al. 2012). Due to this symmetry breaking, continuous families of asymmetric solitons appear in a real symmetric potential.

When the potential $V(x)$ is complex but PT-symmetric, there exist continuous families of solitary waves $u(x;\mu)$ with the same PT-symmetry
\[ u^*(x;\mu) = u(-x;\mu), \]  
(2.5)
see Wang & Wang (2011), Lu & Zhang (2011), Nixon et al. (2012), Zezyulin & Konotop (2012a). Then, the question is, can PT-symmetry breaking occur for these PT-symmetric solitons? More generally, can continuous families of non-PT-symmetric solitons exist in a one-dimensional PT-symmetric potential? These are the questions we will address in this article.

Remark 1 It is noted that Eq. (2.1) is phase-invariant. That is, if $u(x)$ is a solitary wave, then so is $u(x)e^{i\alpha}$, where $\alpha$ is any real constant. For a solitary wave $u(x)$, if there exists a real constant $\alpha$ so that $u(x)e^{i\alpha}$ is PT-symmetric, then we say $u(x)$ is reducible to PT-symmetric. For instance, a complex solitary wave $u(x)$ with anti-PT-symmetry $u^*(x) = -u(-x)$, i.e., with an anti-symmetric real part and symmetric imaginary part, is reducible to PT-symmetric by multiplying it by $i$. In general, a simple way to determine whether a complex-valued solitary wave $u(x)$ is reducible to PT-symmetric or not is to examine the function $u(x)e^{-i\theta}$, where $\theta$ is the phase of $u(0)$. If $u(x)e^{-i\theta}$ is PT-symmetric, then $u(x)$ is reducible to PT-symmetric; and vise versa. Graphically, a simple
way to decide whether a solitary wave $u(x)$ is reducible to PT-symmetric is to plot the amplitude $|u(x)|$ of the function. If this amplitude is not symmetric in $x$, then $u(x)$ is not reducible to PT-symmetric. In this article, when we say non-PT-symmetric solitary waves or solitons, we mean solitary waves that are not reducible to PT-symmetric.

3 Disappearance of families of asymmetric solitons under weak PT-potential perturbations

To explore the existence of continuous families of non-PT-symmetric solitons in Eq. (2.1) with a PT-symmetric potential, we first investigate what happens to families of asymmetric solitons of a real symmetric potential when this real potential is weakly perturbed by an imaginary anti-symmetric term (which makes the perturbed potential non-real but PT-symmetric). Can these families of asymmetric solitons survive and turn into families of non-PT-symmetric solitons in the resulting PT potential? The answer is negative. We will show that these families of asymmetric solitons of real potentials disappear under weak PT-potential perturbations.

When a real symmetric potential is perturbed by an imaginary anti-symmetric term, the model equation (2.4) can be written as

$$u_{xx} - \mu u - V_r(x)u + \sigma |u|^2 u = i\epsilon W(x)u,$$

where $V_r(x)$ is a real symmetric potential, $W(x)$ is a real anti-symmetric function,

$$V_r(x) = V_r(-x), \quad W(x) = -W(-x),$$

and $\epsilon$ is a small real parameter. Note that the combined potential in Eq. (3.1), $V = V_r + i\epsilon W$, is complex and PT-symmetric.

Suppose we seek a continuous family of solitons in the perturbed potential (with $0 < \epsilon \ll 1$), near a continuous family of asymmetric solitons in the unperturbed real potential (with $\epsilon = 0$). Since the soliton family in the perturbed potential should exist for a continuous range of $\mu$ values, each soliton in that family with a fixed $\mu$ value should converge to the asymmetric soliton of $\epsilon = 0$ with the same $\mu$ value when $\epsilon \to 0$. Because of this, in our perturbation expansion we can fix $\mu$ and expand the soliton at this $\mu$ value into a perturbation series of $\epsilon$,

$$u(x; \epsilon) = u_r(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \ldots,$$

where $u_r(x)$ is an asymmetric (i.e. neither symmetric nor anti-symmetric) real soliton in the real potential $V_r$, i.e.,

$$\frac{d^2 u_r}{dx^2} - \mu u_r - V_r(x)u_r + \sigma u_r^3 = 0,$$

and

$$u_r(-x) \neq \pm u_r(x).$$

We will show below that, in order for this perturbation series to be constructed, an infinite number of non-trivial conditions would have to be satisfied, which is impossible in practice. This conclusion will also be corroborated by several specific examples.
We start by substituting the expansion \((3.3)\) into Eq. \((3.1)\). The \(O(1)\) equation is satisfied automatically due to Eq. \((3.4)\). At \(O(\epsilon)\), the equation for \(u_1\) is

\[
L_r \left( \begin{array}{c} u_1 \\ u_1^* \end{array} \right) = \left( \begin{array}{c} iW u_r \\ -iW u_r \end{array} \right),
\]

where

\[
L_r = \begin{bmatrix}
\frac{d^2}{dx^2} - V_r(x) - \mu + 2\sigma u_r^2 \\
\sigma u_r^2 \\
\frac{d^2}{dx^2} - V_r(x) - \mu + 2\sigma u_r^2
\end{bmatrix}
\]

(3.7) is a real and self-adjoint operator. The kernel of \(L_r\) contains an eigenfunction \([u_r, -u_r]^T\), where the superscript ‘T’ represents the transpose of a vector, due to Eq. \((3.4)\). Thus,

\[
L_r \left( \begin{array}{c} u_r \\ -u_r \end{array} \right) = 0.
\]

(3.8)

Let us assume that the kernel of \(L_r\) does not contain any additional eigenfunctions, which is true for generic values of \(\mu\). Then the solvability condition for Eq. \((3.6)\) is that its right hand side be orthogonal to \([u_r, -u_r]^T\), which reduces to

\[
Q_1(\mu) \equiv \langle u_r, W u_r \rangle = 0,
\]

(3.9)

where the inner product is defined as

\[
\langle f, g \rangle = \int_{-\infty}^{\infty} f^*(x) g(x) \, dx.
\]

(3.10)

Since \(W\) is anti-symmetric and \(u_r\) asymmetric [see \((3.5)\)], Eq. \((3.9)\) is then a non-trivial condition which must be satisfied in order for the asymmetric soliton \(u_r(x)\) to persist under weak PT-potential perturbations.

It turns out that Eq. \((3.9)\) is only the first of infinitely many conditions which must be satisfied in order for the perturbation-series solution \((3.3)\) to be constructed. Indeed, at each higher odd order, a new condition would appear. For instance, if condition \((3.9)\) is met, then the \(u_1\) equation \((3.6)\) can be solved. This solution can be written as

\[
u_1 = i\hat{u}_1 + id_1 u_r,
\]

(3.11)

where \(\hat{u}_1\) is a real and localized function solving the equation

\[
L_r \left( \begin{array}{c} \hat{u}_1 \\ -\hat{u}_1 \end{array} \right) = \left( \begin{array}{c} W u_r \\ -W u_r \end{array} \right),
\]

(3.12)

and \(d_1\) is a real constant. This \(id_1 u_r\) term in \(u_1\), when combined with the first term \(u_r\) in the expansion \((3.3)\), only amounts to a constant phase shift to the solution \(u(x; \epsilon)\), thus it can be set to be zero without any loss of generality (see Remark 1). Thus,

\[
u_1 = i\hat{u}_1.
\]

(3.13)

At \(O(\epsilon^2)\), the \(u_2\) equation is

\[
L_r \left( \begin{array}{c} u_2 \\ u_2^* \end{array} \right) = \left( \begin{array}{c} h_2 \\ h_2 \end{array} \right),
\]

(3.14)
where
\[ h_2 = -\sigma u_r \hat{u}_1^2 - W \hat{u}_1 \] (3.15)
is a real function. The solvability condition for this equation is satisfied automatically, thus \( u_2 \) has a solution
\[ u_2 = \hat{u}_2, \] (3.16)
where \( \hat{u}_2 \) is a real and localized function. As before, we have excluded the homogeneous term (proportional to \( iu_r \)) in the \( u_2 \) solution without loss of generality.

Now we proceed to \( O(x^3) \), where the \( u_3 \) equation is
\[ L_r \left( \begin{array}{c} u_3 \\ u_3^* \end{array} \right) = \left( \begin{array}{c} i h_3 \\ -i h_3 \end{array} \right), \] (3.17)
and
\[ h_3 = -\sigma (\hat{u}_1^3 + 2u_r \hat{u}_1 \hat{u}_2) + W \hat{u}_2 \] (3.18)
is a real function. The solvability condition of this \( u_3 \) equation is that
\[ Q_2(\mu) \equiv \langle u_r, h_3 \rangle = 0. \] (3.19)
Since \( u_r, \hat{u}_1 \) and \( \hat{u}_2 \) are all asymmetric functions, so is \( h_3 \). Then, Eq. (3.19) is the second non-trivial condition which has to be met. Following similar calculations to higher orders, infinitely more conditions will appear.

The fundamental reason for this infinite number of conditions for the perturbation series solution (3.3) is that, due to phase invariance of the solitons in Remark 1, each \( u_n \) solution does not contain any non-reducible free constants. But for each odd \( n \), the \( u_n \) equation is of the form
\[ L_r [u_n, u_n]^T = [ih_n, -ih_n]^T, \] where \( h_n \) is a certain real function. In order for this \( u_n \) equation to be solvable, the solvability condition \( \langle u_r, h_n \rangle = 0 \) must be satisfied. All these solvability conditions then constitute an infinite number of conditions for the perturbation series solution (3.3).

In one spatial dimension, neither the perturbed PT-symmetric potential nor the underlying asymmetric solitons possesses additional spatial symmetries. Because of that, each of these infinitely many conditions is non-trivial and is generally not satisfied for generic values of \( \mu \). The requirement of them all satisfied simultaneously is practically impossible. This means that continuous families of asymmetric solitons in the real potential would disappear under weak PT-potential perturbations.

Now we use three specific examples to corroborate the above statement.

**Example 1.** In this first example, we take \( V_r \) to be a symmetric double-well potential
\[ V_r(x) = -3 \left[ \text{sech}^2(x + 1.5) + \text{sech}^2(x - 1.5) \right], \] (3.20)
and \( W \) to be an anti-symmetric function
\[ W(x) = \text{sech}^2(x + 1.5) - \text{sech}^2(x - 1.5). \] (3.21)
Both functions are displayed in Fig. 1(a). In addition, we take \( \sigma = 1 \), i.e., self-focusing nonlinearity.

In this double-well potential \( V_r \), a branch of real symmetric solitons exist, whose power curve is shown in Fig. 1(b) (solid blue line). In addition, symmetry breaking occurs at \( \mu_0 \approx 2.1153 \), where a branch of asymmetric solitons appear for \( \mu > \mu_0 \). An example of such asymmetric solitons (with \( \mu = 2.3 \)) is illustrated in Fig. 1(c). For these asymmetric solitons, we have numerically
calculated the function $Q_1(\mu)$ as defined in Eq. \ref{eq:3.9}, and this function is plotted in Fig. 1(d). We can see that this function is non-zero for all $\mu > \mu_0$, thus the first condition \ref{eq:3.9} is never satisfied, let alone all the other conditions such as \ref{eq:3.19}. Thus we conclude that in this example, the continuous family of asymmetric solitons in the real symmetric potential \ref{eq:3.20} are destroyed under weak PT-potential perturbations \ref{eq:3.21}.

**Example 2.** In the second example, we keep the real potential $V_r$ of Example 1, but choose a different anti-symmetric function for $W$ as

$$W(x) = \text{sech}^2(x+1.5) \tanh(x+1.5) + \text{sech}^2(x-1.5) \tanh(x-1.5). \quad \tag{3.22}$$

This new function $W$ is displayed in Fig. 2(a). The significance of this new $W$ function is that it is proportional to $V'_r(x)$. In this case, multiplying Eq. \ref{eq:3.11} by $u'_r(x)$ and integrating from $-\infty$ to $+\infty$, we find that

$$\langle u_r, V'_r u_r \rangle = 0.$$

Since $W \propto V'_r$, $Q_1(\mu)$ is then always zero, thus the first condition \ref{eq:3.9} is satisfied automatically.
for all $\mu > \mu_0$. However, for this $W$ function, the second condition (3.19) is never satisfied. Indeed, we have numerically computed the function $Q_2$ in this condition and plotted it in Fig. 2(b); one can see that it is never zero for $\mu > \mu_0$. Since this second condition is not met, this family of asymmetric solitons cannot persist and have to disappear under PT-potential perturbations (3.22) as well.

**Example 3.** In the third example, we keep the real potential $V_r$ of Examples 1 and 2, but choose yet another anti-symmetric function for $W$ as

$$W(x) = \text{sech}^2(x + 1.5) - \text{sech}^2(x - 1.5) - 1.15 \left[ \text{sech}^2(x + 2) - \text{sech}^2(x - 2) \right].$$

(3.23)

This $W$ function is plotted in Fig. 3(a). For this choice of $W$, the function $Q_1(\mu)$ in condition (3.9) is displayed in Fig. 3(b). We see that this $Q_1$ is zero only at a special $\mu$ value of $\mu_c \approx 2.5343$, which is marked as a red dot in Fig. 3(b). Because of that, under this $W(x)$ perturbation, the continuous family of asymmetric solitons (with $\mu \neq \mu_c$) in the real potential (3.20) are all destroyed.

What about families of symmetric and anti-symmetric solitons of real potentials under weak PT perturbations? In that case, repeating the above perturbation calculations, we can easily show that all conditions, such as (3.9) and (3.19), are automatically satisfied due to symmetries of the involved functions. As a consequence, perturbation series (3.3) for solitary waves $u(x; \epsilon)$ can be constructed to all orders. In addition, the constructed solutions $u(x; \epsilon)$ are PT-symmetric or reducible to PT-symmetric. This indicates that, families of symmetric and anti-symmetric solitons of real potentials, under PT perturbations, persist and turn into families of PT-symmetric solitons.

## 4 No symmetry breaking in PT-symmetric potentials

In this section, we turn our attention to general PT-symmetric potentials whose imaginary parts are not necessarily small. In such a general PT potential, if Eq. (2.4) admits a branch of PT-symmetric solitons (which is often the case), we ask whether a PT-symmetry-breaking bifurcation can occur, where new branches of non-PT-symmetric solitons bifurcate out from this PT-symmetric branch.
If the potential were real symmetric, symmetry breaking of solitary waves would occur frequently (Jackson & Weinstein 2004, Kirr et al. 2008, Sacchetti 2009, Akylas et al. 2012). However, when the potential becomes complex and PT-symmetric, we will show that PT-symmetry breaking cannot occur in Eq. (2.4).

Before the analysis, we first introduce some notations and make some basic observations.

### 4.1 Notations and simple observations

The linearization operator of the solitary-wave equation (2.4) plays an important role in the bifurcation analysis. Since the solitary wave $u(x)$ is complex-valued due to the complex potential, this linearization operator is vector rather than scalar and can be written as

$$L = \begin{bmatrix} \frac{d^2}{dx^2} - V(x) - \mu + 2\sigma|u|^2 & \sigma u^2 \\ \sigma u^2 & \frac{d^2}{dx^2} - V^*(x) - \mu + 2\sigma|u|^2 \end{bmatrix}. \quad (4.1)$$

This operator is non-Hermitian. Under the standard inner product (3.10), the adjoint operator of $L$ is then

$$L^A = \begin{bmatrix} \frac{d^2}{dx^2} - V^*(x) - \mu + 2\sigma|u|^2 & \sigma u^2 \\ \sigma u^2 & \frac{d^2}{dx^2} - V(x) - \mu + 2\sigma|u|^2 \end{bmatrix}. \quad (4.2)$$

The kernel of the linearization operator $L$ is clearly not empty. Indeed, it is easy to see that

$$L \begin{bmatrix} u \\ -u^* \end{bmatrix} = 0$$

for all $\mu$ values in view of Eq. (2.1), thus the dimension of the kernel of $L$ is at least one.

For any eigenfunction $[f, g]^T$ in the kernel of $L$, it is easy to see that $[g^*, f^*]^T$ is also in this kernel. By adding these two eigenfunctions, we get an eigenfunction in the form of $[w, w^*]^T$, or equivalently $[\hat{w}, -\hat{w}^*]^T$ if one sets $w = i\hat{w}$. Eigenfunctions in these special forms will be chosen as

![Figure 3](image_url)
the basis to span the kernel of \( L \). Similar statements go to the kernel of the adjoint operator \( L^A \) as well.

Our basic observation on solitary-wave bifurcations in Eq. (2.4) is that, if a bifurcation occurs at \( \mu = \mu_0 \), by denoting the corresponding solitary wave and the linearization operator as

\[

u_0(x) \equiv u(x; \mu_0), \quad L_0 \equiv L|_{\mu=\mu_0, u=u_0},

\]

then dimension of the kernel of \( L_0 \) should be at least two, i.e., \( \dim(\ker(L_0)) \geq 2 \). This means that the kernel of \( L_0 \) should contain at least another localized eigenfunction in addition to \( [u_0, -u_0^*]^T \). Using the language of multiplicity of eigenvalues, this means that zero should be a discrete eigenvalue of \( L_0 \) with geometric multiplicity at least two. This is a necessary (but not sufficient) condition for bifurcations.

The above necessary condition for bifurcations can be made even more explicit. Since \( L_0 \) is a fourth-order ordinary differential operator, and the Wronskian of fundamental solutions in its kernel is a non-zero constant (see Remark 2 below), this kernel then cannot contain more than two linearly independent localized eigenfunctions. In other words, \( \dim(\ker(L_0)) \leq 2 \). Then, combined with the above observation, we see that a necessary condition for solitary-wave bifurcations in Eq. (2.4) is that

\[

\dim(\ker(L_0)) = 2.

\]

Due to this condition, if a bifurcation occurs at \( \mu = \mu_0 \), then the kernel of \( L_0 \) would contain exactly one additional eigenfunction, which can be denoted as \( [\psi, \psi^*]^T \). Thus,

\[

L_0 \begin{bmatrix} u_0 \\ -u_0^* \end{bmatrix} = L_0 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} = 0.

\]

Since operator \( L_0 \) for PT potentials is non-self-adjoint, the kernel of the adjoint operator \( L_0^A \) will also play an important role in the bifurcation analysis. When \( \dim(\ker(L_0)) = 2 \), we will show in Remark 2 below that

\[

\dim(\ker(L_0^A)) = 2

\]

as well. Thus the kernel of \( L_0^A \) also contains exactly two linearly independent localized eigenfunctions: one can be denoted as \( [\phi_1, -\phi_1^*]^T \), and the other denoted as \( [\phi_2, \phi_2^*]^T \), i.e.,

\[

L_0^A \begin{bmatrix} \phi_1 \\ -\phi_1^* \end{bmatrix} = L_0^A \begin{bmatrix} \phi_2 \\ \phi_2^* \end{bmatrix} = 0.

\]

**Remark 2** Here we show that if \( \dim(\ker(L_0)) = 2 \), then \( \dim(\ker(L_0^A)) = 2 \). To show this, notice that both ordinary differential operators \( L_0 \) and \( L_0^A \) are four-dimensional. Let us rewrite the operator equations \( L_0 Y = 0 \) and \( L_0^A Y^A = 0 \) as systems of four first-order equations, \( Z_x = QZ \) and \( -Z_x^A = Q^T Z^A \), where \( Z = [Y_1, Y_{1x}, Y_2, Y_{2x}]^T \), \( Z^A = [Y_1^A, Y_{1x}^A, Y_2^A, Y_{2x}^A]^T \), and the superscript ‘\( \dagger \)’ represents Hermitian (i.e., transpose conjugation). Then it is easy to see that \( \text{tr}(Q) = 0 \), thus Wronskians of fundamental matrices for these two first-order systems are both non-zero constants. It is also known that if the fundamental matrix of the system \( Z_x = QZ \) is \( Z = M \), then the fundamental matrix of the adjoint system \( -Z_x^A = Q^T Z^A \) would be \( Z^A = (M^{-1})^\dagger \). Under the assumption of \( \dim(\ker(L_0)) = 2 \), two columns of the fundamental matrix \( M \) are localized functions. Since the determinant of \( M \) is a non-zero constant, using the \( M^{-1} \) formula in terms of cofactors, the other two columns in the adjoint fundamental matrix \((M^{-1})^\dagger\) then are localized functions. Hence, dimension of the kernel of \( L_0^A \) is also two.
\[ \psi^*(x) = \pm \psi(-x), \quad \phi_1^*(x) = \phi_1(-x), \quad \phi_2^*(x) = \pm \phi_2(-x). \] (4.9)

To prove this, we notice that since \( V(x) \) and \( u_0(x) \) are both PT-symmetric, by taking the complex conjugate of the second equation in (4.6) and switching \( x \) to \(-x\), then \( L_0 \) is invariant, and \([\psi^*(-x), \psi(-x)]^T\) is also in the kernel of \( L_0 \). Since the kernel of \( L_0 \) has dimension two, \([\psi^*(-x), \psi(-x)]^T\) then should be a linear combination of \([u_0(x), -u_0^*(x)]^T\) and \([\psi(x), \psi^*(x)]^T\), i.e.,

\[ \psi^*(-x) = c_1 \psi(x) + c_2 u_0(x), \] (4.10)

and

\[ \psi(-x) = c_1 \psi^*(x) - c_2 u_0^*(x), \] (4.11)

where \( c_1, c_2 \) are certain complex constants. Switching \( x \) to \(-x\) in (4.11) and using the PT symmetry of \( u_0 \), we get

\[ \psi(x) = c_1 \psi^*(-x) - c_2 u_0(x). \] (4.12)

Then adding (4.10) and (4.12), we get

\[ (1 - c_1) [\psi(x) + \psi^*(-x)] = 0. \] (4.13)

Thus, if \( c_1 \neq 1 \), then \( \psi^*(x) = -\psi(-x) \), i.e., \( \psi(x) \) is anti-PT-symmetric. If \( c_1 = 1 \), by taking the complex conjugate of Eq. (4.11) and then subtracting it from Eq. (4.10), we get \( c_2 = -c_2 \), i.e., \( c_2 \) is purely imaginary. Denoting \( c_2 = i\beta \), where \( \beta \) is a real parameter, Eq. (4.12) can be rewritten as

\[ \hat{\psi}^*(x) = \hat{\psi}(-x), \] (4.14)

where \( \hat{\psi} \equiv \psi + \frac{1}{2} i\beta u_0 \). It is easy to see that \([\hat{\psi}, \hat{\psi}^*]^T\) is a linear combination of \([\psi(x), \psi^*(x)]^T\) and \([u_0(x), -u_0^*(x)]^T\), hence it is also in the kernel of \( L_0 \). Then, instead of \([\psi(x), \psi^*(x)]^T\), we can choose \([\hat{\psi}, \hat{\psi}^*]^T\) in the eigenvalue equation (4.11); and now \( \hat{\psi} \) is PT-symmetric in view of Eq. (4.14).

\section{Nonexistence of PT-symmetry breaking}

The observations and remarks in the previous subsection apply to all bifurcations in the PT system (4.4). Now we focus on the particular type of bifurcation: symmetry-breaking bifurcation.

Suppose \( u_s(x; \mu) \) is a base branch of PT-symmetric solitons. If a symmetry-breaking bifurcation occurs at \( \mu = \mu_0 \) of this base branch, with \( u_0(x) \equiv u_s(x; \mu_0) \), then eigenfunctions (4.6) and (4.8) in the kernels of \( L_0 \) and \( L_0^A \) should have the following symmetries

\[ u_0^*(x) = u_0(-x), \quad \psi^*(x) = -\psi(-x), \] (4.15)

\[ \phi_1^*(x) = \phi_1(-x), \quad \phi_2^*(x) = -\phi_2(-x), \] (4.16)

i.e., \( u_0, \phi_1 \) are PT-symmetric, and \( \psi, \phi_2 \) anti-PT-symmetric (see Remark 3). In addition, since the two functions in the kernel of \( L_0 \) should be linearly independent, \( \psi \neq iu_0 \). For a similar reason, \( \phi_2 \neq i\phi_1 \).
Below we will show that, in a general PT-symmetric potential, the kernel of \( L_0 \) generically cannot contain the second eigenfunction \([\psi, \psi^*]^T\) with anti-PT-symmetry (4.15). Thus the necessary condition for symmetry breaking is not met. We will also show that even if such a second eigenfunction \([\psi, \psi^*]^T\) appears in the kernel of \( L_0 \), symmetry breaking still cannot occur.

First, we show that in a general PT-symmetric potential, the kernel of \( L_0 \) generically cannot contain the second eigenfunction \([\psi, \psi^*]^T\) with anti-PT-symmetry (4.15). Using the language of multiplicity of eigenvalues, we will show that when the zero eigenvalue of \( L_0 \) has algebraic multiplicity higher than one, its geometric multiplicity generically cannot be higher than one with an anti-PT-symmetric second eigenfunction \([\psi, \psi^*]^T\).

Suppose when \( \mu = \mu_0 \), the zero eigenvalue of \( L_0 \) has algebraic multiplicity higher than one and geometric multiplicity two, and the second eigenfunction \([\psi, \psi^*]^T\) of this zero eigenvalue is anti-PT-symmetric. When \( \mu \neq \mu_0 \), the zero eigenvalue of this \([\psi, \psi^*]^T\) eigenmode would move out of the origin. Let us calculate this eigenvalue of \( L \) for \(|\mu - \mu_0| \ll 1\) by perturbation methods. The eigenvalue equation is

\[
L \begin{bmatrix} w \\ w^* \end{bmatrix} = \lambda \begin{bmatrix} w \\ w^* \end{bmatrix}.
\]

When \(|\mu - \mu_0| \ll 1\), we can expand the eigenvalue \( \lambda \) and the eigenfunction \([w, w^*]^T\) into a perturbation series,

\[
\lambda = \lambda_1 (\mu - \mu_0) + \lambda_2 (\mu - \mu_0)^2 + \ldots,
\]

\[
\begin{bmatrix} w \\ w^* \end{bmatrix} = \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} + (\mu - \mu_0) \begin{bmatrix} w_1 \\ w_1^* \end{bmatrix} + (\mu - \mu_0)^2 \begin{bmatrix} w_2 \\ w_2^* \end{bmatrix} + \ldots.
\]

Similarly, we also expand the operator \( L \) into a perturbation series,

\[
L = L_0 + (\mu - \mu_0) L_1 + (\mu - \mu_0) L_2 + \ldots.
\]

When this eigenmode \([w, w^*]^T\) moves out of the origin, using similar arguments as in Remark 3, we can show that \( w \) can be made PT-symmetric or anti-PT-symmetric. Since \( w \to \psi \) as \( \mu \to \mu_0 \) and \( \psi \) is anti-PT-symmetric, \( w \) then should be anti-PT-symmetric. As a consequence, the other functions \( w_1, w_2, \ldots \) in the \( w \) expansion are also anti-PT-symmetric.

Substituting the above expansions into Eq. (4.17), the \( O(1) \) equation is satisfied automatically since \([\psi, \psi^*]^T\) is in the kernel of \( L_0 \) by assumption. At \( O(\mu - \mu_0) \), we get

\[
L_0 \begin{bmatrix} w_1 \\ w_1^* \end{bmatrix} = \lambda_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} - \begin{bmatrix} g_1 \\ g_1^* \end{bmatrix},
\]

where

\[
\begin{bmatrix} g_1 \\ g_1^* \end{bmatrix} \equiv L_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix}.
\]

The function \( g_1 \) is anti-PT-symmetric in view that \( L_1 \) is PT-symmetric and \( \psi \) anti-PT-symmetric.

Since the kernel of \( L_0 \) has dimension two under the current assumption, the kernel of \( L_0^A \) has dimension two as well (see Remark 2), and the two linearly independent eigenfunctions in the kernel of \( L_0^A \) are denoted in Eq. (4.18) with symmetries (4.16). Then in order for the \( w_1 \) equation (4.21) to be solvable, the solvability condition is that the right side of (4.21) be orthogonal to the two eigenfunctions in the kernel of \( L_0^A \), i.e.,

\[
\left\langle \begin{bmatrix} \phi_1 \\ -\phi_1^* \end{bmatrix}, \lambda_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} - \begin{bmatrix} g_1 \\ g_1^* \end{bmatrix} \right\rangle = 0,
\]

\[
\left(4.23\right)
\]
\[
\left\langle \begin{bmatrix} \phi_2 \\ \phi_2^* \end{bmatrix}, \lambda_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} - \begin{bmatrix} g_1 \\ g_1^* \end{bmatrix} \right\rangle = 0.
\] (4.24)

These two conditions give two different expressions for the same eigenvalue coefficient \(\lambda_1\). In order for these two formulae to be consistent, the following compatibility condition must be satisfied,

\[
\frac{\text{Im}\langle \phi_1, g_1 \rangle}{\text{Im}\langle \phi_1, \psi \rangle} = \frac{\text{Re}\langle \phi_2, g_1 \rangle}{\text{Re}\langle \phi_2, \psi \rangle}.
\] (4.25)

Here ‘Re’ and ‘Im’ represent the real and imaginary parts of a complex number. Recalling the PT symmetry of \(\phi_1\) and anti-PT-symmetries of \(\phi_2, \psi\) and \(g_1\), \(\langle \phi_1, g_1 \rangle\) and \(\langle \phi_1, \psi \rangle\) are purely imaginary, and \(\langle \phi_2, g_1 \rangle, \langle \phi_2, \psi \rangle\) are strictly real. In addition, recalling that \(\phi_2 \neq i\phi_1\), Eq. (4.25) then is a non-trivial compatibility condition for the existence of a second eigenfunction \([\psi, \psi^*]^T\) in the kernel of \(L_0\). Since this compatibility condition is not satisfied generically, the necessary condition for symmetry breaking is then not met.

Next, we show that even if such a second eigenfunction \([\psi, \psi^*]^T\) appears in the kernel of \(L_0\), symmetry breaking still cannot occur, because such a bifurcation further requires an infinite number of additional non-trivial conditions to be satisfied simultaneously, which is impossible in practice.

Suppose at a propagation constant \(\mu = \mu_0\), the kernels of \(L_0\) and \(L_0^A\) have dimension two, and their eigenfunctions are given in Eqs. (4.6) and (4.8) with symmetries (4.15) and (4.16). If a symmetry-breaking bifurcation occurs at this point, then two new branches of non-PT-symmetric solitons would bifurcate out from \(u_0(x)\) on only one side of \(\mu = \mu_0\). Let us seek such non-PT-symmetric solitons near \(\mu = \mu_0\) by perturbation methods.

Suppose these new solitons bifurcate to the right side of \(\mu_0\), then their perturbation series can be written as

\[
u_a(x; \mu) = \sum_{k=0}^{\infty} (\mu - \mu_0)^{k/2} u_k(x).
\] (4.26)

Substituting this perturbation series into Eq. (2.4), the \(O(1)\) equation is satisfied automatically since \(u_0\) is a solitary wave at \(\mu = \mu_0\). At \(O[(\mu - \mu_0)^{1/2}]\), the equation for \(u_1\) is

\[
L_0 \begin{bmatrix} u_1 \\ u_1^* \end{bmatrix} = 0.
\] (4.27)

In view of the kernel structure of \(L_0\) in Eq. (4.6), we see that

\[
\begin{bmatrix} u_1 \\ u_1^* \end{bmatrix} = c_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} + d_1 \begin{bmatrix} u_0 \\ -u_0^* \end{bmatrix},
\] (4.28)

where \(c_1, d_1\) are constants. In order for the resulting \(u_1^*\) formula to be complex conjugate of the \(u_1\) formula, \(c_1\) must be strictly real, and \(d_1\) purely imaginary. Then the \(d_1 u_0\) term in \(u_1\), when combined with the leading-order term \(u_0\) in the expansion (4.26), only amounts to a phase shift to \(u_a(x; \mu)\), which is insignificant in view of Remark 1. Thus we can set \(d_1 = 0\) without loss of generality. Then the \(u_1\) solution becomes

\[
u_1 = c_1 \psi,
\] (4.29)

where \(c_1\) is a real constant.
For symmetry-breaking bifurcation to occur, \( c_1 \) should be non-zero. In this case, the first-two-term solution of (4.26),

\[
    u_0 + (\mu - \mu_0)^{1/2} c_1 \psi,
\]

is not PT-symmetric, nor is it reducible to PT-symmetric, because \( u_0 \) is PT-symmetric but \( \psi \) is anti-PT-symmetric and \( \psi \neq \mu u_0 \). This non-PT-symmetry will not be affected by higher-order terms of (4.26), thus the resulting solution \( u_a(x; \mu) \) in (4.26) would be non-PT-symmetric.

At \( O(\mu - \mu_0) \), the equation for \( u_2 \) is

\[
    L_0 \begin{pmatrix} u_2 \\ u_2^* \end{pmatrix} = \begin{pmatrix} g_2 \\ g_2^* \end{pmatrix},
\]

where

\[
    g_2 = u_0 - \sigma c_1^2 (2u_0|\psi|^2 + u_0^* \psi^2).
\]

Here the \( u_1 \) solution (4.29) has been utilized. Since \( u_0 \) is PT-symmetric and \( \psi \) anti-PT-symmetric, \( g_2 \) is PT-symmetric.

The solvability conditions of Eq. (4.30) are that its right hand side be orthogonal to the kernels of \( L_0^2 \) in Eq. (4.38), i.e.,

\[
    \text{Im}(\phi_1, g_2) = \text{Re}(\phi_2, g_2) = 0.
\]

Recalling the symmetries of \( \phi_1 \) and \( \phi_2 \) in (4.16) as well as the PT-symmetry of \( g_2 \), we see that \( \langle \phi_1, g_2 \rangle \) is strictly real, and \( \langle \phi_2, g_2 \rangle \) is purely imaginary, thus both solvability conditions in Eq. (4.32) are automatically satisfied. As a result, a localized particular solution \( \hat{u}_2 \) can be found. This particular solution can be split into two parts, corresponding to the two terms of \( g_2 \) in (4.31):

\[
    \hat{u}_2 = \hat{u}_{21} + c_1^2 \hat{u}_{22}.
\]

Here, \( \hat{u}_{21} \) solves

\[
    L_0 \begin{pmatrix} \hat{u}_{21} \\ \hat{u}_{21}^* \end{pmatrix} = \begin{pmatrix} u_0 \\ u_0^* \end{pmatrix},
\]

and \( \hat{u}_{22} \) solves

\[
    L_0 \begin{pmatrix} \hat{u}_{22} \\ \hat{u}_{22}^* \end{pmatrix} = \begin{pmatrix} -\sigma(2u_0|\psi|^2 + u_0^* \psi^2) \\ -\sigma(2u_0^*|\psi|^2 + u_0 \psi^2)^* \end{pmatrix}.
\]

Since both terms of \( g_2 \) are PT-symmetric, \( \hat{u}_{21} \) and \( \hat{u}_{22} \) can be made PT-symmetric as well. The general solution of \( u_2 \) is then this particular solution plus the homogeneous solutions. Similar to the \( u_1 \) solution case, we can exclude the homogeneous \( u_0 \) term and set

\[
    u_2 = \hat{u}_{21} + c_1^2 \hat{u}_{22} + c_2 \psi
\]

without loss of generality. Here \( c_2 \) is another real constant to be determined.

The calculations so far have been benign. However, from the next order, we will start to get an infinite number of additional conditions which have to be satisfied in order for the perturbation series (4.26) to be constructed. Let us begin with the \( u_3 \) equation, which is

\[
    L_0 \begin{pmatrix} u_3 \\ u_3^* \end{pmatrix} = \begin{pmatrix} g_3 \\ g_3^* \end{pmatrix},
\]
where 
\[ g_3 = u_1 - \sigma \left( 2u_0^2u_1 u_2 + 2u_0 u_1^*u_2 + 2u_0 u_1 u_2^* + |u_1|^2 u_1 \right). \] (4.38)

Substituting the \( u_1 \) and \( u_2 \) solutions (4.29) and (4.36) into \( g_3 \), we get

\[ g_3 = c_1 \left( g_{31} + c_1^2 g_{32} + c_2 g_{33} \right), \] (4.39)

where

\[ g_{31} = \psi - 2\sigma \left( u_0^* \psi \hat{u}_{21} + u_0 \psi^* \hat{u}_{21} + u_0 \psi \hat{u}_{21}^* \right), \]
\[ g_{32} = -2\sigma \left( u_0^* \psi \hat{u}_{22} + u_0 \psi^* \hat{u}_{22} + u_0 \psi \hat{u}_{22}^* \right) - \sigma |\psi|^2 \psi, \]

and

\[ g_{33} = -2\sigma (2u_0 |\psi|^2 + u_0^* \psi^2). \]

Notice that both \( g_{31} \) and \( g_{32} \) are anti-PT-symmetric, and \( g_{33} \) is PT-symmetric.

The solvability conditions of Eq. (4.37) are

\[ \text{Im} \langle \phi_1, g_3 \rangle = \text{Re} \langle \phi_2, g_3 \rangle = 0. \] (4.40)

Using the \( g_3 \) formula (4.39) and the symmetry properties of the involved functions, these solvability conditions then yield a condition for symmetry-breaking bifurcations as

\[ \frac{\text{Im} \langle \phi_1, g_{31} \rangle}{\text{Im} \langle \phi_1, g_{32} \rangle} = \frac{\text{Re} \langle \phi_2, g_{31} \rangle}{\text{Re} \langle \phi_2, g_{32} \rangle}. \] (4.41)

Due to the PT symmetry of \( \phi_1 \) and anti-PT-symmetries of \( \phi_2 \) and \( g_{31} \) and \( g_{32} \), \( \langle \phi_1, g_{31} \rangle \) and \( \langle \phi_1, g_{32} \rangle \) are purely imaginary, and \( \langle \phi_2, g_{31} \rangle \), \( \langle \phi_2, g_{32} \rangle \) are strictly real. In addition, \( \phi_2 \neq i\phi_1 \). Thus Eq. (4.25) is a non-trivial condition for symmetry-breaking bifurcations.

When we pursue this perturbation expansion to higher orders, infinitely more non-trivial conditions will also appear (since these calculations are straightforward, details are omitted here for brevity). The fundamental reason for this infinite number of conditions is that, due to the phase invariance of solitary waves, when we solve the inhomogeneous \( u_n \) equation, we can only introduce one real parameter into the \( u_n \) solution, which is the coefficient of the \( \psi \) term. But each \( u_n \) equation has two solvability conditions (since the kernel of \( L_0^A \) has dimension two), and neither solvability condition can be satisfied automatically from symmetry considerations (for \( n \geq 3 \)). This means that we have twice as many solvability conditions as real parameters. Because of this, we have an over-determined system for real parameters, which results in an infinite number of non-trivial conditions for symmetry-breaking bifurcations. In one spatial dimension, Eq. (2.4) does not admit any additional spatial symmetries (except the PT symmetry). Due to this lack of additional symmetries, it is practically impossible for these infinite conditions to be satisfied simultaneously.

From the above analysis, we see that in a PT-symmetric potential, the necessary condition for symmetry breaking, i.e., \( \text{dim} |\ker(L_0)|=2 \) with eigenfunction symmetries (4.15), is generically not satisfied. Even if that necessary condition is met, symmetry breaking still requires infinitely more conditions to be satisfied simultaneously, which is practically impossible for the 1D system (2.4). Thus, we conclude that symmetry breaking cannot occur in the PT-symmetric system (2.4).

Regarding the three specific examples (3.20)–(3.23) in the PT system (3.1) for various values of \( \epsilon \) (not necessarily small), we have found numerically that the kernel of \( L \) never contains a second eigenfunction \( [\psi, \psi^*]^T \) with anti-PT-symmetry at any \( \mu \) value, thus the necessary condition for symmetry breaking is not satisfied. This numerical finding corroborates our analytical result that this necessary condition for symmetry breaking is generically not met for a PT-symmetric potential.
5 Summary and discussion

In this article, we have investigated the possibility of continuous families of non-PT-symmetric solitons in one-dimensional PT-symmetric potentials. We have shown that families of asymmetric solitons in a real symmetric potential are destroyed when this real potential is perturbed by weak PT-symmetric perturbations. We have also shown that in a general one-dimensional PT-symmetric potential, symmetry breaking of PT-symmetric solitons cannot occur. This contrasts real symmetric potentials where symmetry breaking of solitary waves often takes place.

Based on these findings and Remark 1, we make the following conjecture:

The one-dimensional NLS equation (2.1) with a complex PT-symmetric potential cannot admit continuous families of non-PT-symmetric solitary waves.

Equivalently, this conjecture says that all continuous families of solitary waves in a one-dimensional PT-symmetric potential must be PT-symmetric.

The absence of continuous families of non-PT-symmetric solitons in 1D PT-symmetric potentials is an interesting phenomenon, since it contrasts real symmetric potentials, where families of asymmetric solitons often exist. This means that, even though PT-symmetric potentials can support continuous families of solitons, which makes such dissipative potentials analogous to conservative real potentials, the types of soliton families allowed by PT-symmetric potentials are nonetheless limited. So the dissipative nature of a PT-symmetric potential does leave its signature on the structure of its solitary waves, and this signature distinguishes PT-symmetric potentials from real symmetric ones.

We would like to point out that the above conjecture does not exclude the possibility of 1D PT-symmetric potentials supporting isolated non-PT-symmetric solitons (i.e., non-PT-symmetric solitons existing at isolated propagation-constant values). In a certain finite-dimensional PT-symmetric system (the quadrimer model), isolated non-PT-symmetric solutions have been reported (Li & Kevrekidis 2011). In the 1D NLS equation (2.1) with a PT-symmetric potential, such isolated non-PT-symmetric solitons can also exist, as our preliminary numerics has shown. These isolated solitons are reminiscent of dissipative solitons in the Ginzburg-Landau and other dissipative equations (Akhmediev & Ankiewicz 2005), and they can coexist with continuous families of solitons in a PT-symmetric potential.

The analytical results in this article can be extended to a large class of higher-dimensional NLS equations with PT-symmetric potentials, but not to all of them. In higher spatial dimensions, the PT symmetry of a potential is compatible with certain other spatial symmetries, such as $x$-symmetry or $y$-symmetry. For instance, we can easily construct two-dimensional complex potentials $V(x, y)$ with the following PT-symmetry as well as $x$-symmetry,

$$V^*(x, y) = V(-x, -y), \quad V(x, y) = V(-x, y),$$

i.e., the real part of the potential is symmetric in both $x$ and $y$, but the imaginary part of the potential is symmetric in $x$ and anti-symmetric in $y$. Due to this additional $x$-symmetry, continuous families of non-PT-symmetric solitons can exist in this 2D PT-symmetric potential, and symmetry breaking of PT-symmetric solitons can occur. Putting this 2D problem in the framework of the earlier analysis in this article, the reason for the existence of PT-symmetry breaking and families of non-PT-symmetric solitons in this 2D PT potential is that, due to the additional $x$-symmetry of the potential and its ramifications for the symmetries of the underlying solitons and eigenfunctions in the kernels of the linearization operators, those infinite conditions in our earlier analysis can
now be all satisfied. Details on this 2D problem will be reported elsewhere. However, if this
2D PT-symmetric potential $V(x,y)$ does not admit those additional spatial symmetries (such as
$x$-symmetry and $y$-symmetry), then the analysis in this article would still apply, and families of
non-PT-symmetric solitons still cannot be expected. Thus, absence of continuous families of non-
PT-symmetric solitons in PT-symmetric potentials is not restricted to one spatial dimension, but
holds for most higher-dimensional PT-symmetric potentials as well.

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References

[1] F. K. Abdullaev, Y. V. Kartashov, V. V. Konotop & D. A. Zezyulin 2011 Solitons in PT-
symmetric nonlinear lattices. Phys. Rev. A 83, 041805.

[2] N. Akhmediev & A. Ankiewicz (Editors) 2005 Dissipative Solitons (Springer, Berlin).

[3] T.R. Akylas, G. Hwang & J. Yang 2012 From nonlocal gap solitary waves to bound states in
periodic media, Proc. Roy. Soc. A 468, 116-135.

[4] N. V. Alexeeva, I. V. Barashenkov, A. A. Sukhorukov & Yu. S. Kivshar 2012 Optical solitons
in PT-symmetric nonlinear couplers with gain and loss, Phys. Rev. A 85, 063837.

[5] C.M. Bender & S. Boettcher 1998 Real spectra in non-Hermitian Hamiltonians having PT
symmetry, Phys. Rev. Lett. 80, 5243–5246.

[6] R. Driben & B. A. Malomed 2011 Stability of solitons in parity-time-symmetric couplers, Opt.
Lett. 36, 4323–4325.

[7] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A.
Siviloglou & D. N. Christodoulides 2009 Observation of PT-Symmetry Breaking in Complex
Optical Potentials, Phys. Rev. Lett. 103, 093902 (2009).

[8] Y. He, X. Zhu, D. Mihalache, J. Liu & Z. Chen 2012 Lattice solitons in PT-symmetric mixed
linear-nonlinear optical lattices, Phys. Rev. A 85, 013831.

[9] C. Huang, C. Li, & L. Dong 2013 Stabilization of multipole-mode solitons in mixed linear-
nonlinear lattices with a PT symmetry, Opt. Exp. 21, 3917–3925.

[10] R.K. Jackson & M.I. Weinstein 2004 Geometric analysis of bifurcation and symmetry breaking
in a GrossPitaevskii equation, J. Stat. Phys. 116, 881-905.

[11] Y.V. Kartashov 2013, Vector solitons in parity-time-symmetric lattices, Kartashov, Opt. Lett.
38, 2600–2603.

[12] P. G. Kevrekidis, D. E. Pelinovsky & D. Y. Tyugin 2013 Nonlinear stationary states in PT-
symmetric lattices, arXiv:1303.3298 [nlin.PS].
[13] E.W. Kirr, P.G. Kevrekidis, E. Shlizerman & M.I. Weinstein 2008 Symmetry-breaking bifurcation in nonlinear Schrödinger/GrossPitaevskii equations, SIAM J. Math. Anal. 40, 56-604.
[14] E. W. Kirr, P. G. Kevrekidis & D. E. Pelinovsky 2011 Symmetry-breaking bifurcation in the nonlinear Schrodinger equation with symmetric potentials, Comm. Math. Phys. 308, 795-844.
[15] Y. S. Kivshar & G. P. Agrawal 2003 Optical Solitons: From Fibers to Photonic Crystals (Academic Press, San Diego).
[16] V. V. Konotop, D. E. Pelinovsky & D. A. Zezyulin 2012 Discrete solitons in PT-symmetric lattices, Euro. Phys. Lett. 100, 56006.
[17] K. Li & P. G. Kevrekidis 2011 PT-symmetric oligomers: Analytical solutions, linear stability, and nonlinear dynamics, Phys. Rev. E 83, 066608 (2011).
[18] Z. Lu & Z. Zhang 2011 Defect solitons in parity-time symmetric superlattices, Opt. Express 19, 11457–11462.
[19] F. C. Moreira, F. Kh. Abdullaev, V. V. Konotop & A. V. Yulin 2012 Localized modes in $\chi^{(2)}$ media with PT-symmetric localized potential, Phys. Rev. A 86, 053815.
[20] Z.H. Musslimani, K.G. Makris, R. El-Ganainy & D.N. Christodoulides 2008 Optical solitons in PT periodic potentials, Phys. Rev. Lett. 100, 030402.
[21] S. Nixon, L. Ge & J. Yang 2012 Stability analysis for solitons in PT-symmetric optical lattices, Phys. Rev. A 85, 023822.
[22] D. E. Pelinovsky & T. Phan 2012 Normal form for the symmetry-breaking bifurcation in the nonlinear Schrodinger equation, J. Differential Equations 253, 2796-2824.
[23] L.P. Pitaevskii & S. Stringari 2003 BoseEinstein Condensation (Oxford University Press, Oxford).
[24] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev & D. Kip 2010 Observation of parity-time symmetry in optics, Nature Physics 6, 192195.
[25] A. Sacchetti 2009 Universal critical power for nonlinear Schrodinger equations with symmetric double well potential, Phys. Rev. Lett. 103, 194101.
[26] H. Wang & J. Wang 2011 Defect solitons in parity-time periodic potentials. Opt. Express 19, 4030–4035.
[27] J. Yang 2010 Nonlinear Waves in Integrable and Nonintegrable Systems (SIAM, Philadelphia).
[28] J. Yang 2012a No Stability Switching at Saddle-Node Bifurcations of Solitary Waves in Generalized Nonlinear Schrodinger Equations, Phys. Rev. E 85, 037602.
[29] J. Yang 2012b Classification of solitary wave bifurcations in generalized nonlinear Schrodinger equations, Stud. Appl. Math. 129, 133–162.
[30] J. Yang 2013 Stability analysis for pitchfork bifurcations of solitary waves in generalized nonlinear Schrodinger equations, Physica D 244, 50–67.
[31] D. A. Zezyulin & V.V. Konotop 2012a Nonlinear modes in the harmonic PT-symmetric potential, Phys. Rev. A 85, 043840.

[32] D. A. Zezyulin & V.V. Konotop 2012b Nonlinear Modes in Finite-Dimensional PT-Symmetric Systems, Phys. Rev. Lett. 108, 213906.

[33] D. A. Zezyulin & V.V. Konotop 2013 Stationary modes and integrals of motion in nonlinear lattices with PT-symmetric linear part, [arXiv:1306.5286](http://arxiv.org/abs/1306.5286) [nlin.PS].