A COVARIANT METHOD FOR CALCULATING AMPLITUDES OF PROCESSES INVOLVING POLARIZED SPIN 1/2 PARTICLES. CALCULATION OF THE INTERFERENCE TERMS IN THEIR CROSS SECTIONS

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Abstract

A covariant method is proposed for calculating the amplitudes of processes involving polarized spin 1/2 particles. It is suitable for calculating the interference terms in the cross sections of such processes. As an illustration, expressions are given for the amplitudes of electron-electron scattering in the lowest order of perturbation theory and expressions for the electron current in the case of emission of two bremsstrahlung photons in the ultrarelativistic limit.

1 Introduction

In calculations of cross sections involving diagrams of higher orders (especially when allowance is made for the polarizations of the participating particles), the need to calculate
the traces of products of a large number of Dirac $\gamma$ matrices presents considerable difficulties. One of the ways of avoiding such difficulties is to calculate directly the amplitudes of the processes. In particular, expressions are given in [1] that were obtained by the multiplication of $\gamma$ matrices and bispinors expressed component by component in definite frames of reference. Because of computational difficulties, other authors too in later studies were forced to use such a device (see, for example, [2], [3]).

The obvious shortcomings of such an approach include the complexity of the calculations and the cumbersome and noncovariant nature of the results.

Various authors have attempted the covariant calculation of amplitudes (see [4] – [10]), but the expressions obtained in their studies could not be used to calculate the interference terms in the cross sections. We consider below a general scheme that embraces the results of the cited studies, and we propose a particularization of the scheme that permits the calculation of not only amplitudes but also interference terms in the cross sections.

2 General method of covariant calculation

In any reaction involving spin 1/2 particles in the initial and final states, there is an even number ($2N$) of fermions. Therefore, each diagram contains $N$ fermion lines that are not closed. In the amplitude of the process, two bispinors correspond to the ends of each line. For definiteness, we shall in what follows assume that both fermions are particles. However, the results also hold when the fermions are antiparticles or one fermion is a particle and the other an antiparticle.

In the amplitude of the process, there corresponds to each line an expression of the form

$$M_{12} = \bar{u}_2 Qu_1$$

where $u_1 = u(p_1, n_1)$ and $u_2 = u(p_2, n_2)$ are the bispinors for the free particles, $p_1$ and $p_2$ are the 4-momenta of the particles, $n_1$ and $n_2$ are the 4-vectors that specify the axes of the spin projections of the particles, $\bar{u}_2 = u_2^+ \gamma_4$ and $u^+ = \bar{u}^*$ (the asterisk denotes complex conjugation, the tilde matrix transposition), and $Q$ is the matrix operator that characterizes the interactions.

The operator $Q$ can be expressed as a linear combination of products of Dirac $\gamma$ matrices (or contractions of them with 4-vectors) and may have an arbitrary number of free Lorentz indices.

To calculate $M_{12}$, we use the scheme

$$M_{12} = \bar{u}_2 Qu_1 = (\bar{u}_2 Qu_1) \frac{\bar{u}_1 Zu_2}{\bar{u}_1 Zu_2} = \frac{\bar{u}_2 Qu_1 \bar{u}_1 Zu_2}{\bar{u}_1 Zu_2}$$

$$= \frac{(Qu_1 \bar{u}_1 Zu_2 \bar{u}_2)_t}{\bar{u}_1 Zu_2} \simeq \frac{(Qu_1 \bar{u}_1 Zu_2 \bar{u}_2)_t}{|\bar{u}_1 Zu_2|} = \frac{(Qu_1 \bar{u}_1 Zu_2 \bar{u}_2)_t}{|Zu_1 \bar{u}_1 Zu_2 \bar{u}_2|}^{1/2} = M_{12}$$

2
where $Z$ is an arbitrary $4 \times 4$ matrix,
\[
\tilde{Z} = \gamma_4 Z^+ \gamma_4, \quad Z^+ = \tilde{Z}^*.
\]
($t$ is the symbol of the matrix trace, and the symbol $\simeq$ means "up to a phase factor").

In place of $u\bar{u}$ in (2) projection operators are substituted. For a massive particle
\[
u(p,n)\bar{u}(p,n) = \frac{1}{4m} (m - i\hat{p})(1 + i\gamma_5 \hat{n}) = \mathcal{P},
\]
where $\hat{p} = p_\mu \gamma_\mu$, $p^2 = -m^2$, $n^2 = 1$, $\bar{u}u = 1$, $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ and $m$ is the particle mass [we use a metric in which $a_\mu = (\vec{a}, a_4 = i a_0)$, $a_\mu b_\nu = \vec{a} \vec{b} - a_0 b_0$].

For a massless particle, the projection operator has the form
\[
u_\pm(q)\bar{u}_\pm(q) = \frac{1}{4i\eta_0} (1 \mp \gamma_5) \hat{q} = \mathcal{P}_\pm
\]
where $q^2 = 0$ and $\bar{u}_\pm \gamma_4 u_\pm = 1$ (the signs $\pm$ of $\mathcal{P}$ correspond to the particle helicities).

In [4], [5], a choice proposed for $Z$ was $Z = 1$. Another proposal in [5] was $Z = \gamma_5$; in addition, the expressions obtained in [6] were given. The results of [7] – [9] reduce to $Z = 1 + \gamma_4$ and the results of [10] to $Z = m - i\hat{r}$ ($r$ is an arbitrary 4-momentum such that $r^2 = -m^2$); the 4-vectors used in [10] to specify the axes of the spin projections were
\[
n_1 = \frac{m^2 p_2 + (p_1 p_2) p_1}{m[(p_1 p_2)^2 - m^4]^{1/2}}, \quad n_2 = -\frac{m^2 p_1 + (p_1 p_2) p_2}{m[(p_1 p_2)^2 - m^4]^{1/2}}.
\]

However, as can be seen from (2), all the expressions obtained for $M_{12}$ are known up to a phase factor that depends on $\bar{u}_1 Z u_2$.

We note in passing that
\[
(M_{12})^* \simeq \frac{[(Q u_1 \bar{u}_1 Z u_2 \bar{u}_2)]_t^*}{[(Z u_1 \bar{u}_1 Z u_2 \bar{u}_2)]_t^{1/2}} = \frac{(\bar{u}_2 Q u_1 \bar{u}_1 Z u_2)^*}{[(Z u_1 \bar{u}_1 Z u_2 \bar{u}_2)]_t^{1/2}}
\]
\[
= \frac{\bar{u}_2 Z u_1 \bar{u}_1 Q u_2}{[(Z u_1 \bar{u}_1 Z u_2 \bar{u}_2)]_t^{1/2}} = \frac{Z u_1 \bar{u}_1 Q u_2 \bar{u}_2}{[(Z u_1 \bar{u}_1 Z u_2 \bar{u}_2)]_t^{1/2}} = (M_{12})^*.
\]

The presence of the unknown phase factors makes it impossible in the general case to use (2) and (3) to calculate amplitudes of processes that take place through several channels, since in this case errors in calculating the interference terms in the cross sections of such processes are possible.

### 3 Calculation of interference terms in the cross sections

We consider in general form the diagrams for a process that proceeds through two different channels (see Fig.4).
Let and to the second channels

where $A$ is a product of four currents.

To the first diagram there corresponds the expression

$$M = (\bar{u}_3 A u_1)(\bar{u}_4 B u_2) = M_{13} M_{24},$$

and to the second

$$M' = (\bar{u}_4 C u_1)(\bar{u}_3 D u_2) = M_{14} M_{23},$$

where $A, B, C, D$ are arbitrary matrix operators,

$$(M')^* = (\bar{u}_1 \bar{C} u_4)(\bar{u}_2 \bar{D} u_3).$$

Difficulties arise in the calculation of interference expressions of the form

$$M(M')^* = (\bar{u}_3 A u_1)(\bar{u}_4 B u_2)(\bar{u}_1 \bar{C} u_4)(\bar{u}_2 \bar{D} u_3)$$

$$= [Au_1 \bar{u}_1 \bar{C} u_4 B u_2 \bar{u}_2 \bar{D} u_3 u_3^3]_t. \tag{6}$$

To consider (6), we need the following identity:

$$[A_1 u_1 \bar{u}_1 A_2 u_3 \bar{u}_3^3]_t [Z_1 u_1 \bar{u}_1 Z_2 u_3 \bar{u}_3]_t \equiv [A_1 u_1 \bar{u}_1 Z_2 u_3 \bar{u}_3]_t [Z_1 u_1 \bar{u}_1 A_2 u_3 \bar{u}_3]_t, \tag{7}$$

where $A_1, A_2, Z_1, Z_2$ are arbitrary $4 \times 4$ matrices.

The validity of the identity (7) becomes obvious if each of its sides is rewritten as a product of four currents.

We apply the consequence of (7):

$$[A_1 u_1 \bar{u}_1 A_2 u_3 \bar{u}_3]_t = [A_1 u_1 \bar{u}_1 Z_2 u_3 \bar{u}_3]_t [Z_1 u_1 \bar{u}_1 A_2 u_3 \bar{u}_3]_t / [Z_1 u_1 \bar{u}_1 Z_2 u_3 \bar{u}_3]_t.$$

Let $A_1 = A$, $A_2 = [\bar{C} u_4 \bar{u}_4 B u_2 \bar{u}_2 \bar{D} ]$, $Z_2 = Z$, $Z_2 = \bar{Z}$. Then

$$M(M')^* = [Au_1 \bar{u}_1 Z u_3 \bar{u}_3]_t [\bar{Z} u_1 \bar{u}_1 \bar{C} u_4 \bar{u}_4 B u_2 \bar{u}_2 \bar{D} u_3 \bar{u}_3]_t$$

$$= [Au_1 \bar{u}_1 Z u_3 \bar{u}_3]_t / [\bar{Z} u_1 \bar{u}_1 Z u_3 \bar{u}_3]_t [\bar{C} u_4 \bar{u}_4 B u_2 \bar{u}_2 \bar{D} u_3 \bar{u}_3 \bar{Z} u_1 \bar{u}_1]_t. \tag{8}$$
Similarly, for the second factor in (8)

\[
[C\bar{u}_4\bar{u}_4Bu_2\bar{u}_2\bar{D}u_3\bar{u}_3\bar{Z}u_1\bar{u}_1]_t = \frac{[Cu_4\bar{u}_4\bar{X}u_1\bar{u}_1]_t [Xu_4\bar{u}_4Bu_2\bar{u}_2\bar{D}u_3\bar{u}_3\bar{Z}u_1\bar{u}_1]_t}{[Xu_4\bar{u}_4\bar{X}u_1\bar{u}_1]_t} \tag{9}
\]

and also

\[
[Bu_2\bar{u}_2\bar{D}u_3\bar{u}_3\bar{Z}u_1\bar{u}_1\bar{X}u_4\bar{u}_4]_t = \frac{[Bu_2\bar{u}_2\bar{Y}u_4\bar{u}_4]_t [Yu_2\bar{u}_2\bar{D}u_3\bar{u}_3\bar{Z}u_1\bar{u}_1\bar{X}u_4\bar{u}_4]_t}{[Yu_2\bar{u}_2\bar{Y}u_4\bar{u}_4]_t} \tag{10}
\]

Finally

\[
[\bar{D}u_3\bar{u}_3\bar{Z}u_1\bar{u}_1\bar{X}u_4\bar{u}_4\bar{Y}u_2\bar{u}_2]_t = \frac{[\bar{D}u_3\bar{u}_3\bar{V}u_2\bar{u}_2]_t [Vu_3\bar{u}_3\bar{Z}u_1\bar{u}_1\bar{X}u_4\bar{u}_4\bar{Y}u_2\bar{u}_2]_t}{[Vu_3\bar{u}_3\bar{V}u_2\bar{u}_2]_t} \tag{11}
\]

In obtaining (8) – (11), we have used when necessary cyclic permutations of matrices under the trace symbol; $X, Y, Z, V$ are as yet arbitrary $4 \times 4$ matrices.

Combining (3), (8) – (11), we obtain

\[
M(M')^* = \frac{[Au_1\bar{u}_1Zu_3\bar{u}_3]_t [Xu_1\bar{u}_1Cu_4\bar{u}_4]_t}{[Zu_1\bar{u}_1Zu_3\bar{u}_3]_t [Xu_1\bar{u}_1Xu_4\bar{u}_4]_t} \times \frac{[Bu_2\bar{u}_2\bar{Y}u_4\bar{u}_4]_t [Vu_3\bar{u}_3\bar{Z}u_1\bar{u}_1\bar{X}u_4\bar{u}_4\bar{Y}u_2\bar{u}_2]_t}{[Yu_2\bar{u}_2\bar{Y}u_4\bar{u}_4]_t [Vu_3\bar{u}_3\bar{V}u_2\bar{u}_2\bar{Y}u_2\bar{u}_2]_t} \times \frac{[\bar{D}u_3\bar{u}_3\bar{V}u_2\bar{u}_2]_t [Vu_3\bar{u}_3\bar{Z}u_1\bar{u}_1\bar{X}u_4\bar{u}_4\bar{Y}u_2\bar{u}_2]_t}{[Vu_3\bar{u}_3\bar{V}u_2\bar{u}_2\bar{V}u_3\bar{u}_3]_t} \tag{12}
\]

\[
= \frac{[Au_1\bar{u}_1Zu_3\bar{u}_3]_t [Xu_1\bar{u}_1Cu_4\bar{u}_4]_t}{([Zu_1\bar{u}_1Zu_3\bar{u}_3]_t)^{1/2} ([Xu_1\bar{u}_1Xu_4\bar{u}_4]_t)^{1/2}} \times \frac{[Bu_2\bar{u}_2\bar{Y}u_4\bar{u}_4]_t [Vu_3\bar{u}_3\bar{Z}u_1\bar{u}_1\bar{X}u_4\bar{u}_4\bar{Y}u_2\bar{u}_2]_t}{([Yu_2\bar{u}_2\bar{Y}u_4\bar{u}_4]_t)^{1/2} ([Vu_3\bar{u}_3\bar{V}u_2\bar{u}_2\bar{Y}u_2\bar{u}_2]_t)^{1/2}} \times \frac{[\bar{D}u_3\bar{u}_3\bar{V}u_2\bar{u}_2]_t [Vu_3\bar{u}_3\bar{Z}u_1\bar{u}_1\bar{X}u_4\bar{u}_4\bar{Y}u_2\bar{u}_2]_t}{([Vu_3\bar{u}_3\bar{V}u_2\bar{u}_2\bar{V}u_3\bar{u}_3]_t)^{1/2}}
\]

\[
= M_{13}M_{24}(M_{14})^*(M_{23})^*K,
\]

where $M_{13}, M_{24}, (M_{14})^*, (M_{23})^*$ are given by expressions analogous to (2) and (4), and the coefficient $K$ is given by

\[
K = \frac{[Zu_1\bar{u}_1Xu_4\bar{u}_4\bar{Y}u_2\bar{u}_2\bar{V}u_3\bar{u}_3]_t}{([Zu_1\bar{u}_1Zu_3\bar{u}_3]_t ([Xu_1\bar{u}_1Xu_4\bar{u}_4]_t ([Yu_2\bar{u}_2\bar{Y}u_4\bar{u}_4]_t ([Vu_3\bar{u}_3\bar{V}u_2\bar{u}_2\bar{V}u_3\bar{u}_3]_t)^{1/2}}. \tag{13}
\]
Obviously, for correct calculation of the interference contributions it is necessary to require $K \equiv 1$. This requirement is satisfied if we choose $Z = X = Y = V = \mathcal{P}$ [see (3)] or $Z = X = Y = V = \mathcal{P}_\pm$ [see (4)], since for the projection operators we have the identities

$$\mathcal{P}\mathcal{A}\mathcal{P} = [\mathcal{P}\mathcal{A}]_{\pm}\mathcal{P}, \quad \mathcal{P} = \mathcal{P}, \quad \mathcal{P}_\pm\mathcal{A}\mathcal{P}_\pm = [\mathcal{P}_\pm\mathcal{A}]_{\pm}\mathcal{P}_\pm, \quad \mathcal{P}_\pm = \mathcal{P}_\pm. \quad (14)$$

As an example, we give expressions for amplitudes of processes involving massless Dirac particles. In this case, the expression (2) becomes

$$\bar{u}_\pm(p_2)Qu_\pm(p_1) = \frac{i[Q\hat{p}_1\hat{q}_2(1 \pm \gamma_5)]_t}{8[(p_1)_0(p_2)_0(qp_1)(qp_2)]^{1/2}}, \quad (16)$$

where $Z = \frac{1}{4i\hat{q}_0}(1 \pm \gamma_5)\hat{q} = \mathcal{P}_\mp$, $q^2 = 0$.

The massless 4-vector $q$ can be arbitrary, but it must be the same for all the considered fermion lines that are not closed in the diagrams. Note that in (16) and similar expressions we shall in what follows use the equals sign instead of the symbol $\simeq$ since no problems can now arise with the phase factors.

Further,

$$\bar{u}_\pm(p_2)Qu_\pm(p_1) = -\frac{[Q\hat{p}_1(m + \gamma_5\hat{n})\hat{p}_2(1 \pm \gamma_5)]_t}{8\{(p_1)_0(p_2)_0[(pp_1) \pm m(np_1)][(pp_2) \mp m(np_2)]\}^{1/2}}, \quad (17)$$

where $Z = \frac{1}{4m}(m - i\hat{p})(1 + i\gamma_5\hat{n}) = \mathcal{P}$, $p^2 = -m^2$, $n^2 = 1$, $pn = 0$. With regard to the 4-vectors $p$ and $n$, the same remark holds as for the vector $q$ in (16).

In the last case, it is not possible to use as $Z$ the simpler operator $\mathcal{P}_\pm$ because both numerator and denominator would then be identically equal to zero.

If for certain values of $p_1$ and $p_2$ the denominators in (16) or (17) vanish in the numerical calculation, it is sufficient to change the values of the arbitrary vectors that appear in these expressions, namely, $q$ or $p$, $n$ (simultaneously for all considered lines of the diagrams).

As we have already noted, our method can be readily generalized to include antiparticles. For this, it is sufficient in (2) to replace the particle projection operators by the antiparticle analogs. For example, suppose we are interested in $\bar{v}_2Qu_1$, where $v_2$ is a free antiparticle bispinor. Then

$$\bar{v}_2Qu_1 = \frac{(Qu_1\bar{u}_1Zv_2\bar{v}_2)_t}{[(Zu_1\bar{u}_1Zv_2\bar{v}_2)_t]^{1/2}}$$

where

$$v(p, n)v_\mp(p, n) = -\frac{1}{4m}(m + i\hat{p})(1 + i\gamma_5\hat{n})$$
for a massive antiparticle or

$$v_\pm(q)\bar{v}_\pm(q) = \frac{1}{4i\hat{q}_0}(1 \pm \gamma_5)\hat{q}$$

for a massless antiparticle. For $Z$, we still use (3) or (4).

In conclusion, we note that the previously proposed direct methods of calculating the amplitudes of processes using $Z = 1, \gamma_5, 1 + \gamma_4, m - i\hat{r}$ do not permit correct calculation of the interference terms in the cross section; for when they are used, the coefficient $K$, which is determined by the expression (13) and takes into account the phase factors of all the fermion lines that are not closed in the diagrams, is not equal to 1 (this is readily demonstrated by considering, for example, processes involving massless particles).

**APPENDIX 1**

As an illustration of the application of the method to the calculation of amplitudes of processes involving massive Dirac particles, we consider electron-electron scattering in the lowest order of perturbation theory. To this process there correspond the two Feynman diagrams shown in Fig.2.

![Feynman diagrams](image)

Figure 2: The Feynman diagrams for electron-electron scattering in the lowest order of perturbation theory

For calculations of amplitudes of processes involving massive Dirac particles, formula (4) becomes

$$M_{12} = \bar{u}_2Qu_1 = \frac{(Qu_1\bar{u}_1P_\pm u_2\bar{u}_2)_t}{([P_\pm u_1\bar{u}_1P_\pm u_2\bar{u}_2]_t)^{1/2}},$$

and therefore to the electron current of the top line of the first diagram there will correspond the expression

$$(J_{13})_{\mu} = \bar{u}_3\gamma_\mu u_1 = \frac{[\gamma_\mu P_1P_\pm P_3]_t}{([P_\pm P_1P_\pm P_3]_t)^{1/2}},$$

$$= \frac{[\gamma_\mu \frac{1}{4m}(m - i\hat{p}_1)(1 + i\gamma_5\hat{n}_1)\frac{1}{4i\hat{q}_0}(1 - \gamma_5)\hat{q} \frac{1}{4m}(m - i\hat{p}_3)(1 + i\gamma_5\hat{n}_3)]_t}{([\frac{(1-\gamma_5\hat{q})(m-\hat{p}_1)(1+i\gamma_5\hat{n}_1)}{4m} \frac{(1-\gamma_5\hat{q})(m-\hat{p}_3)(1+i\gamma_5\hat{n}_3)}{4m}]_t)^{1/2}},$$

$$= \alpha^{-1}(aq_\mu + bp_{1\mu} + cp_{3\mu} + d_{n1\mu} + em_{3\mu} + f_\mu)$$
where
\[
\alpha = 4im[\frac{1}{2}\{-(qp_1) + m(qn_1)\} + \frac{1}{2}\{-(qp_2) + m(qn_3)\}],
\]
\[
a = [m^2 + (p_1 p_3)][1 + (n_1 n_3)] - m(p_1 n_3) - m(p_3 n_1) - (p_1 n_3)(p_3 n_1),
\]
\[
b = m(qn_1) + (q_3)[m + (p_1 n_3)] - (qp_2)[1 + (n_1 n_3)] - \varepsilon(q, p_3, n_1, n_3),
\]
\[
c = m(qn_3) + (q_1)[m + (p_1 n_3)] - (qp_1)[1 + (n_1 n_3)] - \varepsilon(q, p_1, n_1, n_3),
\]
\[
d = -m(qp_1) + (qp_3)[m + (p_1 n_3)] - (qp_3)[m^2 + (p_1 p_3)] - \varepsilon(q, p_1, p_3, n_3),
\]
\[
e = -m(qp_3) + (qp_1)[m + (p_3 n_1)] - (qn_1)[m^2 + (p_1 p_3)] - \varepsilon(q, p_1, p_3, n_1),
\]
\[
f_{\mu} = [m + (p_3 n_1)]\varepsilon(\mu, q, p_1, n_3) - [m + (p_1 n_3)]\varepsilon(\mu, q, p_3, n_1)
\]
\[
- [1 + (n_1 n_3)]\varepsilon(\mu, q, p_1, p_3) - [m^2 + (p_1 p_3)]\varepsilon(\mu, q, n_1, n_3)
\]
\[
- mz(\mu, q, p_3, n_3) + mz(\mu, q, p_1, n_1),
\]

in which \(\varepsilon(\mu, a, b, c) = \varepsilon_{\mu\nu\rho\sigma}a_\nu b_\rho c_\sigma\) is the contraction of the completely antisymmetric Levi-Civita tensor with the 4-vectors \(a, b, c\) and \(\vec{q}^2 = 0\).

The expression for \((J_{23})_\mu = \vec{u}_4 \gamma_\mu u_2\) is obtained from \((J_{13})_\mu\) by replacing the index 3 by 4 and 1 by 2; \((J_{14})_\nu = \vec{u}_4 \gamma_\nu u_1\) is obtained by \((J_{13})_\mu\) by replacing 3 by 4 and \(\mu\) by \(\nu\), and \((J_{23})_\nu = \vec{u}_3 \gamma_\nu u_2\) by replacing 1 by 2 and \(\mu\) by \(\nu\), respectively.

The expressions we have given are covariant and permit numerical calculations of amplitudes. The complex numbers obtained in the calculation serve for the calculation of the process cross section.

In this example, the calculation of the amplitude is as laborious as the calculation of the square of the modulus of the matrix element for one diagram but simpler than the calculation of the interference term. However, if the number of \(\gamma\) matrices in the operator \(Q\) is increased by \(N\) [see (18)], their number in the numerator of (18) increases only by \(N\) (at the same time, the denominator is unchanged), whereas in the construction \([Q u_1 \bar{u}_1 \bar{Q} u_2 \bar{u}_2]t\), which arises in calculations of the square of the modulus, the number of \(\gamma\) matrices is increased by \(2N\). Since the trace of product of \(2M\) \(\gamma\) matrices contains \(1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2M - 1)\) terms, we see that the more complicated the process the greater the gain from calculating it in the method of direct calculation of the amplitudes. This is also true for processes involving massless particles.

APPENDIX 2

We consider expressions for the electron current in the case of emission of two photons in the ultrarelativistic limit \((m_e = 0)\). The Feynman diagrams are given in Fig. 8.

We take the polarization vectors of a photon with helicity \(\lambda\) in the form (see (11))
\[
\hat{e}_\lambda(k_i) = 2N_i[k_i \hat{q} \omega_{-\lambda} - \hat{q} \hat{q} \hat{k}_i \omega_{\lambda}], \quad \omega_{\pm} = \frac{1}{2}(1 \pm \gamma_5),
\]
\[
N_i = [-16(qq')(qk_i)(q'k_i)]^{-1/2}, \quad i = 1, 2.
\]
In the considered approximation, the electrons being massless,
\[ \omega_{\pm} u_{\pm} = u_{\pm}, \quad \bar{u}_{\pm} \omega_{\pm} = \bar{u}_{\pm}, \quad \bar{u}_{\pm}(q')q' = 0, \quad \bar{q} u_{\pm}(q) = 0. \]

Using the above relations, we obtain
\[ \bar{u}_{\pm}(q')\hat{e}_{\pm}(k_{i}) = \bar{u}_{\pm}(q')\omega_{\pm} 2N_{i} [\hat{k}_{i}q'\hat{\omega}_{\pm} - \bar{q}'\hat{k}_{i}\omega_{\pm}] \]
\[ = 2N_{i} \bar{u}_{\pm}(q')\hat{k}_{i}\bar{q}' = 4N_{i}(q'k_{i})\bar{u}_{\pm}(q')\hat{q}, \quad \text{(19)} \]
\[ \hat{e}_{\mp}(k_{i})u_{\pm}(q) = 2N_{i}[\bar{k}_{i}q\hat{\omega}_{\pm} - q\hat{k}_{i}\omega_{\mp}]\omega_{\mp}u_{\pm}(q) \]
\[ = -2N_{i}\bar{k}'\hat{k}_{i}u_{\pm}(q) = -4N_{i}(qk_{i})q'u_{\pm}(q), \quad \text{(20)} \]
\[ \bar{u}_{\pm}(q')\hat{e}_{\mp}(k_{i}) = \hat{e}_{\pm}(k_{i})u_{\pm}(q) = 0. \]

Using Eqs. (19) and (19) – (21), we obtain
\[ J_{\mu}(\pm, \pm, \pm, \pm) = J_{5\mu}(\pm, \pm, \pm, \pm) + J_{6\mu}(\pm, \pm, \pm, \pm) \]
\[ = \bar{u}_{\pm}(q')\hat{e}_{\pm}(k_{2})q' + \hat{k}_{2}\bar{e}_{\pm}(k_{1})q' + \hat{k}_{1} + \hat{k}_{2}2(k_{1}k_{2} + q'k_{1} + q'k_{2})\gamma_{\mu}u_{\pm}(q) + (k_{1} \leftrightarrow k_{2}) \]
\[ = 8N_{1}N_{2}(q'q')\bar{u}_{\pm}(q')(1 \pm \gamma_{5})[\{q'q' + qk_{1} + qk_{2}\gamma_{\mu} - \mu(\hat{k}_{1} + \hat{k}_{2})\}]u_{\pm}(q) \]
\[ = 8N_{1}N_{2}(q'q')(\gamma_{\mu} - p_{\mu}(qq'))^{1/2}[\{q'q' - \mu(q'q')\}]q'q' + qk_{1} + qk_{2} \]
\[ + q_{\mu}[(q'q')(q'p + pk_{1} + pk_{2}) - (qp)(q'k_{1} + q'k_{2})] \mp (qp)\varepsilon(\mu, q, q', k_{1} + k_{2}) \]
\[ \mp (qp)\varepsilon(\mu, q, q' + k_{1} + k_{2}, p) \} . \]
Similarly

\[ J_\mu(\pm, \mp, \mp, \pm) = J_{1\mu}(\pm, \mp, \mp, \pm) + J_{2\mu}(\pm, \mp, \mp, \pm) \]

\[ = 8N_1N_2(qq') \frac{i}{[q_0q_0'qpp][qpp]} \{ [q_\mu(q'p) - p_\mu(q'q')](qq' - q'k_1 - q'k_2) \]

\[ + q'_\mu[(qq')(qp - pk_1 - pk_2) + (q'p)(qk_1 + qk_2)] \pm (q'p)\epsilon(\mu, q, q', k_1 + k_2) \]

\[ \mp (qq')\epsilon(\mu, q - k_1 - k_2, q', p) \} , \]

\[ J_\mu(\pm, \pm, \pm, \pm) = J_{2\mu}(\pm, \pm, \mp, \pm) + J_{4\mu}(\pm, \pm, \mp, \pm) + J_{6\mu}(\pm, \pm, \mp, \pm) , \]

\[ J_{2\mu}(\pm, \pm, \mp, \mp, \pm) = \frac{4N_1N_2(qq')}{k_1k_2 - qk_1 - qk_2 [q_0q_0'qpp][qpp]} \frac{i}{ \{ p_\mu[(q'k_2 - qq')(qk_1)(q'k_2) - (qq')(k_1k_2)] - (qk_2)(q'k_1)(q'k_2) - (qq')(qk_2)(q'k_1)] \}

\[ + q_\mu[(q'k_2 - qq')(qk_1)(pq_2) - (q'k_2)(p_k_2) + (q'p)(k_1k_2)] + 2(qk_2)(q'k_1)(q'p) \]

\[ + q'_\mu[(qq' + q'k_2)(qk_2)(pk_1) - (qk_1)(pk_2)(q'k_1) = (k_2)(q'k_2) - (qq')(pq_2)\mp \epsilon(q, q', p, k_1) \]

\[ + 2(qk_1)\{(q'k_2)(qp) \pm \epsilon(q, q', p, k_2)\} + 2(k_1k_2)\{(qq')(pk_1) - (q'p)(qk_2)\mp \epsilon(q, q', p, k_2)\}] \]

\[ + k_{1\mu}(q'k_2 - qq')(qk_1)(q'k_2) - (qq')(pq_2) + (qk_2)(q'p) \pm \epsilon(q, q', p, k_2) \]

\[ - k_{2\mu}(q'k_2 - qq')(pq_2) - (qq')(qk_1)(q'p) \pm \epsilon(q, q', p, k_1) \]

\[ \mp (q'k_2 - qq')(k_1k_2)\epsilon(\mu, q, q', p) + (qq')\epsilon(\mu, p, k_1, k_2) \]

\[ - (qp)\epsilon(\mu, q', k_1, k_2) - (q'p)\epsilon(\mu, q, k_1, k_2) \mp 2(qk_2)(q'k_1)\epsilon(\mu, q, q', p) \} , \]
\[ J_{4\mu}(\pm, \pm, \mp, \pm) = 4N_1N_2(qq') \frac{i}{q_0(q_0(qp)(qp')^{1/2}} \]
\[ \{ p_\mu[2(qq')(qk_1 - qk_2) + (qq')(k_1k_2) + (qk_1)(qk_2) - (qk_2)(q'k_1)] \]
\[ + q_\mu[2(qq')(q'p + pk_1) - 2(qp)(q'k_1) - 2(q'p)(q'k_2) + (qk_1)(pk_2) - (pk_1)(qk_2) - (q'p)(k_1k_2)] \]
\[ + q'_\mu[2(qq')(q'p - pk_2) + 2(qp)(qk_1) + 2(q'p)(qk_2) - (qk_1)(pk_2) - (pk_1)(qk_2) - (q'p)(k_1k_2)] \]
\[ + k_{1\mu}[(qp)(q'k_2) + (q'p)(k_2) - (qq')(p_{k_2})] \pm \varepsilon(q, q', p, k_2) \]
\[ + k_{2\mu}[(qp)(q'k_1) + (q'p)(q_1) - (qq')(p_{k_1})] \pm \varepsilon(q, q', p, k_1) \]
\[ \mp 2(qq')\{\varepsilon(\mu, q', p) - \varepsilon(\mu, q, p, k_1) - \varepsilon(\mu, q', p, k_2) \}
\[ \mp 2(qp)\varepsilon(\mu, q, q', k_1) \pm 2(q'p)\varepsilon(\mu, q, q', k_2) \pm (k_{1k_2})\varepsilon(\mu, q, q', p) \]
\[ \mp (qq')\varepsilon(\mu, p, k_1, k_2) \pm (qp)\varepsilon(\mu, q', k_1, k_2) \pm (q'p)\varepsilon(\mu, q, k_1, k_2) \} \]

\[ J_{0\mu}(\pm, \pm, \mp, \pm) = \frac{4N_1N_2(qq')}{k_1k_2 + q'k_1 + q'k_2 [q_0(q_0(qp)(qp'))^{1/2}} \]
\[ \{ p_\mu[(qq' + qk_1)((q_2)(k_1k_2) - (qk_1)(q'k_2)) + (qk_1)(qk_2)(q'k_1) - (qq')(qk_2)(q'k_1)] \]
\[ + q_\mu[(qq' - qk_1)((q'k_1)(p_{k_2}) - (q'k_2)(pk_1) - (q'p)(k_1k_2))] \mp 2(q'k_1)\varepsilon(q, q', p, k_2) \]
\[ + 2(q'k_2)((q_1)(q'p) \pm \varepsilon(q, q', p, k_1)) + 2(k_{1k_2})((qp)(q'k_1) - (qq')(p_{k_1}) \pm \varepsilon(q, q', p, k_1)) \}
\[ + q'_\mu[(qq' + qk_1)((q_1)(p_{k_2}) - (qk_2)(pk_1) - (qp)(k_1k_2)] + 2(q_{k_2})(qk_2)(q'k_1)(qp)] \]
\[ + k_{1\mu}(qq' + qk_1)((q'p)(q_{k_2}) - (qq')(p_{k_2}) + (qp)(q'k_2) \pm \varepsilon(q, q', p, k_2)] \]
\[ - k_{2\mu}(qq' + qk_1)((q'p)(q_{k_1}) - (qq')(p_{k_1}) + (qp)(q'k_1) \pm \varepsilon(q, q', p, k_1)] \]
\[ \pm (qq' + qk_1)((k_{1k_2})\varepsilon(\mu, q', q, p) + (q'q)\varepsilon(\mu, p, k_1, k_2) \]
\[ -(qp)\varepsilon(\mu, q', k_1, k_2) - (q'p)\varepsilon(\mu, k_1, k_2)] \mp 2(qk_2)(q'k_1)\varepsilon(\mu, q, q', p) \} . \]

The expression for
\[ J_{\mu}(\pm, \mp, \pm, \pm) = J_{1\mu}(\pm, \mp, \pm, \pm) + J_{3\mu}(\pm, \mp, \pm, \pm) + J_{5\mu}(\pm, \mp, \pm, \pm) \]
is obtained from the one for $J_{\mu}(\pm, \pm, \mp, \pm)$ by interchanging $k_1$ and $k_2$. In all expressions, $p$ is an arbitrary 4-momentum such that $p^2 = 0$.

Our expressions for leptonic currents are fairly compact. For comparison, we note that the leptonic tensors

\[ J_{\mu}(\pm, \mp, \pm, \pm) J_{\nu}^*(\pm, \mp, \pm, \pm) \quad \text{and} \quad J_{\mu}(\pm, \mp, \pm, \pm) J_{\nu}^*(\pm, \pm, \mp, \pm) \]

each contain 1416 terms when calculated by the classical method by means of the computer system SCHOONSCHIP.

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References

[1] R.P.Feynman, Quantum Electrodynamics, Benjamin (1961)
[2] F.A.Berends, P.De Causmaecker, R.Gastmans, R.Kleiss, W.Troost, T.T.Wu, Nucl.Phys.B239, p.395 (1984)
[3] F.A.Berends, P.De Causmaecker, R.Gastmans, R.Kleiss, W.Troost, T.T.Wu, Nucl.Phys.B264, p.243 (1986)
[4] E.Bellomo, Il Nuovo Cimento.Ser.X., v.21, p.730 (1961)
[5] H.W.Fearing, R.R.Silbar, Phys.Rev.D6, p.471 (1972)
[6] F.I.Fedorov, Izv.VUZ. Fiz., v.23, no.2, p.32 (1980) (in Russian)
   translated in: F.I.Fedorov, Sov.Phys.J., v.23, p.100 (1980)
[7] F.I.Fedorov, Teor.Mat.Fiz., v.18, p.329 (1974) (in Russian)
   translated in: F.I.Fedorov, Theor.Math.Phys., v.18, p.233 (1974)
[8] F.I.Fedorov, Vestsi Akad. Navuk BSSR. Ser. Fiz.-Mat. Navuk, no.2, p.58 (1974) (in Russian)
[9] F.I.Fedorov, Yad. Fiz., v.17, p.883 (1973) (in Russian)
   translated in: F.I.Fedorov, Sov.J.Nucl.Phys., v.17, p.883 (1973)
[10] S.M.Sikach, Vestsi Akad. Navuk BSSR. Ser. Fiz.-Mat. Navuk, no.2, p.84 (1984) (in Russian)
[11] P.De Causmaecker, R.Gastmans, W.Troost, T.T.Wu, Nucl.Phys.B206, p.53 (1982)