Abstract

We prove sharp $L^p$ regularity results for a class of generalized Radon transforms for families of curves in a three-dimensional manifold associated with a canonical relation with fold and blowdown singularities. The proof relies on decoupling inequalities by Wolff and Bourgain–Demeter for plate decompositions of thin neighborhoods of cones and $L^2$ estimates for related oscillatory integrals.

Keywords

Regularity of integral operators · X-ray transforms · Radon transforms · Fourier integral operators

Mathematics Subject Classification

35S30 · 42B20 · 42B35 · 44A12 · 46E35

1 Introduction

Let $M$ be the family of all lines in $\mathbb{R}^3$. Given a function $f \in C_0^\infty(\mathbb{R}^3)$, its X-ray transform is a function defined on $M$ given by

$$Xf(l) = \int f, \quad l \in M$$

Since $M$ is a four-dimensional manifold, recovering $f$ from $Xf$ is an overdetermined problem. It is natural to ask for which three-dimensional submanifolds $\mathcal{F} \subset M$ the restriction $X\mathcal{F} f = Xf|_\mathcal{F}$ can be inverted. We study a class of these restricted X-ray transforms initially formulated in the complex setting by Gelfand and Graev [1] to give an essentially complete characterization of when inversion is possible.

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Definition 1 (Gelfand Admissibility) Given a three-dimensional line complex $\mathcal{F}$, let $\Gamma_{Q}$ be the cone of lines in $\mathcal{F}$ through the point $Q$. We say that $\mathcal{F}$ is Gelfand-admissible if $\Gamma_P$ is tangent to $\Gamma_Q$ along the line between the points $P$ and $Q$ for every $P$ in the cone $\Gamma_Q$.

This class of restricted X-ray transforms has been studied by many authors, including Greenleaf and Uhlmann who, in [2], showed that Gelfand admissibility, along with the condition that the cone of lines through each point is curved, is sufficient for the inversion of $X_\mathcal{F}$, extending the results of Gelfand–Graev to the real setting. Various estimates have been proven for this collection of restricted X-ray transforms. For instance, $L^2$-Sobolev estimates were proven by Greenleaf–Uhlmann in [3], and $L^p \to L^q$ estimates were proven by Greenleaf–Seeger in [4]. In this paper, we are interested in finding $L^p$-Sobolev estimates for $X_\mathcal{F}$ and similar operators. It is instructive to look at the following model case.

Let $I$ be a compact interval and suppose that $\gamma : I \to \mathbb{R}^2$ is a smooth regular curve with nonvanishing curvature (i.e., $\gamma'(s), \gamma''(s) \neq 0$). For a Schwartz function $f \in S(\mathbb{R}^3)$ and $\alpha \in I$, define

$$A f(x', \alpha) = \int_1^2 f(x' + s\gamma(\alpha), s) \chi_1(s) \chi_2(\alpha) \, ds,$$

where $\chi_1$ and $\chi_2$ are smooth real-valued functions supported in the interior of $[1, 2]$ and $I$, respectively. Pramanik and Seeger, in [5], proved that for sufficiently small $p$, the operator $A$ maps boundedly from $L^p(\mathbb{R}^3)$ into $L^p_{1-1/p}(\mathbb{R}^3)$, where $L^p_s(\mathbb{R}^3)$ is the standard Sobolev space on $\mathbb{R}^3$ with respect to Lebesgue measure. This result was proven by studying dyadic decompositions of the adjoint operator $A^*$ and using $\ell^p$-decoupling inequalities for the cone, originally proven by Wolff in [6] and later extended to the optimal range by Bourgain–Demeter [7]. Applying the Bourgain–Demeter decoupling result yields the boundedness of $A^*$ from $L^p(\mathbb{R}^3)$ into $L^p_{1/p}(\mathbb{R}^3)$ for $p > 4$ (and hence the boundedness of $A$ from $L^{p'}(\mathbb{R}^3)$ into $L^{p'}_{1/p}(\mathbb{R}^3)$ for the same range of $p$, where $p'$ is the dual exponent $p' = 1 - \frac{1}{p}$). This estimate is the best possible for the range of $p$, although it is unknown whether the range of $p$ can be extended to include $p = 4$.

A generalization of the main result of [5] is suggested by [8]. In this work, Pramanik and Seeger proved a gain of $1/p$ derivatives in $L^p$ for a class of integral operators in $\mathbb{R}^3$ with folding canonical relations, generalizing their previous result in [9], which considered averages over translations of curves in $\mathbb{R}^3$. We use similar techniques to Pramanik and Seeger to generalize the results of [5] to more general integral operators associated with fold and blowdown singularities. This class of integral operators includes the adjoints of generic Gelfand-admissible restricted X-ray transforms and also subsumes the main result of [10] for averaging operators over curves in the Heisenberg group.

To define our class of integral operators, we recall the double fibration formalism of Gelfand and Helgason [11, p. 4] (cf. [2,8]). Let $\Omega_L, \Omega_R$ be three-dimensional manifolds and consider families of curves $\mathcal{M}_x \subset \Omega_R$ parametrized by and smoothly
Let $d\sigma_x$ be the arclength measure on $\mathcal{M}_x$, and $\chi \in C_\infty_c(\mathbb{R}^3 \times \mathbb{R}^3)$. We define the generalized Radon transform operator $\mathcal{R} : C_\infty(\Omega_R) \rightarrow C_\infty(\Omega_L)$ by

$$\mathcal{R} f(x) = \int_{\mathcal{M}_x} f(y) \chi(x, y) \, d\sigma_x(y).$$

We assume that $\mathcal{M}_x$ are sections of a manifold $\mathcal{M} \subset \Omega_L \times \Omega_R$, so that the projections

$$\begin{array}{ccc}
\Omega_L & \xrightarrow{\pi_L} & \mathcal{M} \\
\mathcal{M} & \xrightarrow{\mathcal{C}} & \pi_R : \Omega_R
\end{array}$$

(1)

have surjective differentials; note this ensures that $\mathcal{R}$ is bounded on $L^1$ and $L^\infty$. The surjectivity assumption on the projections (1) also ensures that $\mathcal{M}_x$ and $\mathcal{M}_y = \{x \in \Omega_L : (x, y) \in \mathcal{M}\}$ are smooth immersed curves in $\Omega_R$ and $\Omega_L$, respectively.

The operator $\mathcal{R}$ can be realized as a Fourier integral operator of order $-1/2$ belonging to the Hörmander class $I^{-1/2}(\Omega_L, \Omega_R; (N^*\mathcal{M})')$, where

$$(N^*\mathcal{M})' = \{(x, \xi, y, \eta) : (x, \xi, y, -\xi) \in N^*\mathcal{M}\}$$

with $N^*\mathcal{M}$ the conormal bundle of $\mathcal{M}$. The assumptions on the projections (1) imply that

$$\mathcal{C} = (N^*\mathcal{M})' \subset (T^*\Omega_L \setminus 0_L) \times (T^*\Omega_R \setminus 0_R),$$

where $0_L$ and $0_R$ are the zero sections of the cotangent spaces $T^*\Omega_L$ and $T^*\Omega_R$, respectively. Moreover, $\mathcal{C}$ is a homogeneous canonical relation, i.e., if $\omega_L$ and $\omega_R$ are the canonical two-forms on $T^*\Omega_L$ and $T^*\Omega_R$, respectively, then $\mathcal{C}$ is Lagrangian with respect to $\omega_L - \omega_R$. As is known from the theory of Fourier integral operators (see [12,13]), the $L^2$-Sobolev regularity properties of $\mathcal{R}$ are governed by the geometry of the projections

$$\begin{array}{ccc}
T^*\Omega_L & \xrightarrow{\pi_L} & \mathcal{C} \\
\mathcal{C} & \xrightarrow{\pi_R} & T^*\Omega_R
\end{array}$$

(2)

This microlocal point of view is due to Guillemin–Sternberg [14]. Since $\mathcal{C}$ is Lagrangian, the ranks of the differentials $(D\pi_L)_P$ and $(D\pi_R)_P$ are equal; in particular, this implies that $(D\pi_L)_P$ is invertible if and only if $(D\pi_R)_P$ is invertible (see [12]). For averaging operators over curves in dimensions larger than 2, the projections $\pi_L$ and $\pi_R$ fail to be diffeomorphisms, meaning that for every point $(x, y) \in \mathcal{M}$, there is a $P = (x, \xi, y, \eta) \in (N^*\mathcal{M})'$ such that $(D\pi_L)_P$ and $(D\pi_R)_P$ are not invertible. However, we can restrict how singular the maps $\pi_L$ and $\pi_R$ are on $\mathcal{C}$. Following
the survey papers [15] and [16], we recall the definitions of a Whitney fold and a blowdown.

**Definition 2** Suppose \( g : X \to Y \) is a \( C^\infty \) map between \( C^\infty \) manifolds of corank \( \leq 1 \) such that \( d(\det(dg))_P \neq 0 \) for every \( P \in X \) such that \( \det(dg)_P = 0 \). By the implicit function theorem, the set \( \mathcal{L} = \{ P \in X : \det(dg)_P = 0 \} \) is thus an immersed hypersurface. We say \( V \), a nonzero smooth vector field on \( X \), is a kernel field of \( g \) if \( V|_P \in \ker(dg)_P \) for all \( P \in \mathcal{L} \).

We say \( g \) is a Whitney fold if for every kernel field \( V \) of \( g \) and every \( P \in \mathcal{L} \) we have \( V(\det dg)_P \neq 0 \) at \( P \).

We say \( g \) is a blowdown if every kernel field \( V \) of \( g \) restricted to \( \mathcal{L} \) is everywhere tangential to \( \mathcal{L} \). Note this implies that \( V^k(\det dg)|_P = 0 \) for all \( k \in \mathbb{N} \) and all \( P \in \mathcal{L} \).

In [8], Pramanik and Seeger proved \( \mathcal{R} \) maps \( L^p(\mathbb{R}^3) \) into \( L^p_{1/p}(\mathbb{R}^3) \) boundedly for \( p > 4 \) for a class of operators where the only singularities on \( \pi_L \) and \( \pi_R \) are Whitney folds. They conjectured that only the Whitney fold assumption on \( \pi_L \) is necessary for their result. In this paper, we consider a “worst” case, where \( \pi_R \) is instead a blowdown.

**Theorem 1** Let \( M \subset \Omega_L \times \Omega_R \) be a four-dimensional manifold such that the projections \( M \to \Omega_L \) and \( M \to \Omega_R \) are submersions. Assume that the only singularities on \( \pi_L : (N^*M)' \to T^*\Omega_L \) are Whitney folds, and that \( \pi_R : (N^*M)' \to T^*\Omega_R \) is a blowdown. Let \( \mathcal{L} \) be the conic submanifold on which \( d\pi_L \) and \( d\pi_R \) drop rank by one, and let \( \varpi \) be the projection of \((N^*M)'\) onto the base \( M \). Suppose that the restriction of \( \varpi \) to \( \mathcal{L} \),

\[ \varpi : \mathcal{L} \mapsto M \]

is a submersion. Then \( \mathcal{R} \) extends to a continuous operator

\[ \mathcal{R} : L^p_{\text{comp}}(\Omega_R) \to L^p_{1/p, \text{loc}}(\Omega_L), \quad 4 < p < \infty. \]

Theorem 1 generalizes the results of [10] and [5], and the sharpness examples in both papers show that the regularity index \( s = 1/p \) cannot be improved, and that the result fails for \( p < 4 \). Note that the assumption on the projection \( \varpi \) ensures a curvature condition on the fibers of \( \mathcal{L} \), first formulated in [4], and proven for \( \mathcal{R} \) in [8]. This curvature ensures that \( \ell^p\)-decoupling can be applied.

The layout of this paper is as follows. In Sect. 2, we introduce some example operators for which Theorem 1 applies. In Sects. 3 and 4, we begin the proof of Theorem 1 by relating it to an estimate of oscillatory integrals in Proposition 1. This is the main estimate of the paper, proven through the interpolation of a decoupling inequality and an \( L^2 \) estimate in Sects. 5 and 6, respectively. While the \( L^2 \) boundedness of \( \mathcal{R} \) has been established by the work of Greenleaf and Seeger in [4], these estimates rely on a Strichartz-type argument that does not yield the quantitative estimates that we need to interpolate with the \( \ell^p\)-decoupling estimates in Sect. 5. The work of Comech in [17] establishes these quantitative estimates if \( \pi_R \) is of finite type but does not cover the case when \( \pi_R \) is a blowdown, which is what we prove in Sect. 6 in a general
setting. Fortunately, it is not necessary to prove the endpoint $L^2$ estimate (see [18]) in order to interpolate. Finally in Sect. 7, we finish the proof of Theorem 1 with a Calderón–Zygmund-type estimate proven in [19].

2 Some Examples

Now we elaborate on some examples to which Theorem 1 applies. The notation in this section is self-contained.

2.1 Averages Along Curves in $\mathbb{H}^1$

Define the Heisenberg group $\mathbb{H}^1$ to be $\mathbb{R}^3$ with the group operation

$$x \odot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1)).$$

Let $\gamma : [0, 1] \to \mathbb{R}^3$ be a smooth regular curve whose tangent vector is nowhere parallel to $(0, 0, 1)$, so that without loss of generality, we can write $\gamma(t) = (t, \gamma_2(t), \gamma_3(t))$. Let $\mu$ be a smooth measure supported on $\gamma([0, 1])$, so for $f \in S(\mathbb{R}^3)$ we define

$$A f(x) = \int_0^1 f(\gamma(t)^{-1} \odot x) \, d\mu(t).$$

Secco, in [20], developed a group-invariant notion for higher derivatives of $\gamma$ and formulated two conditions which serve as right- and left-invariant analogues of non-vanishing curvature and torsion. These conditions are

\begin{align*}
\det \begin{pmatrix} \gamma''_2(t) & \gamma''_3(t) \\ \gamma''_2(t) & \gamma''_3(t) \end{pmatrix} + \frac{1}{2} (\gamma''_2(t))^2 &\neq 0 \\
\det \begin{pmatrix} \gamma''_2(t) & \gamma''_3(t) \\ \gamma''_2(t) & \gamma''_3(t) \end{pmatrix} - \frac{1}{2} (\gamma''_2(t))^2 &\neq 0.
\end{align*}

In [10], the author showed that if (4) holds for all $t \in \text{supp}(\chi)$ and (5) does not hold for any $t \in \text{supp}(\chi)$ then $A$ maps boundedly from $L^p_{\text{comp}}(\mathbb{R}^3)$ into $L^p_{1/p}(\mathbb{R}^3)$ for $p > 4$. Under this condition, the operator $A$ is a Fourier integral operator where $\pi_L$ is a fold and $\pi_R$ is a blowdown. An example of a curve satisfying this condition is $\gamma(t) = (t, t^2, \frac{1}{6} t^3)$.

We next check that $A$ satisfies the final condition of Theorem 1. The associated incidence manifold $\mathcal{M}$ is given by

$$\mathcal{M} = \{(x, y) : \Phi(x, y) = 0\},$$

where

$$\Phi(x, y) = \begin{pmatrix} \Phi_1(x, y) \\ \Phi_2(x, y) \end{pmatrix} = \begin{pmatrix} x_2 - y_2 - \gamma_2(x_1 - y_1) \\ x_3 - y_3 - \gamma_3(x_1 - y_1) + \frac{1}{2} x_1 y_2 (x_1 - y_1) - \frac{1}{2} x_2 (x_1 - y_1) \end{pmatrix}.$$
The twisted conormal bundle is given by

\[(N^\ast \mathcal{M})' = \{ (x, (\tau \cdot \Phi), y, -(\tau \cdot \Phi), y) : \Phi(x, y) = 0 \}\]

and \(\mathcal{L}\) is the submanifold of \((N^\ast \mathcal{M})'\) defined by

\[(\tau_2, \tau_3) \perp (\gamma_2''(x_1 - y_1), \gamma_3''(x_1 - y_1) - \frac{1}{2} x_1 \gamma_2''(x_1 - y_1)).\]

The condition that (4) holds for all \(t \in \text{supp}(\chi)\) and (5) does not hold for any \(t \in \text{supp}(\chi)\), hence the restriction of \((N^\ast \mathcal{M})'\) to \(\mathcal{L}\) amounts to a restriction of the \(\tau\) variables to a one-dimensional linear subspace for each \((x, y) \in \mathcal{M}\). Thus the projection \(\sigma\) defined in Theorem 1 is a submersion and we recover the result from [10] that \(A : L^p_{\text{comp}}(\mathbb{R}^3) \to L^{p/p}(\mathbb{R}^3)\) for \(p > 4\).

2.2 Restricted X-ray Transforms in \(\mathbb{R}^3\)

Let \(\mathcal{M}\) be the space of lines in \(\mathbb{R}^3\), let \(\mathcal{F} \subset \mathcal{M}\) be a three-dimensional line complex such that the cone of lines through each point is curved, and as defined in the introduction let

\[X_{\mathcal{F}} f(l) = \int_{l} f, \quad l \in \mathcal{F}.\]

We recall the parametrization of the Lagrangian of \(X_{\mathcal{F}}\) from a survey paper of Phong, [13] (see also [2]), to verify that \(X_{\mathcal{F}}\) satisfies the assumptions of Theorem 1. As shown in [2], the maps \(\pi_L\) and \(\pi_R\) are, respectively, a blowdown and a Whitney fold, so we only need to verify that the projection \(\sigma|_{\mathcal{L}}\) is a submersion in this case. We can view \(\mathcal{M}\) locally as a submanifold of \(T\mathbb{R}^3\), identifying each line \(l\) with a point \(P\) and a direction \(\gamma\). As a consequence, \(T^*\mathcal{M}\) (resp. its subspace \(T^*\mathcal{F}\)) can be identified with the restriction of \(T^*(T\mathbb{R}^3)\) to \(T\mathcal{M}\) (resp. \(T\mathcal{F}\)), viewed as functionals on \(T(T\mathbb{R}^3)\). The defining relation for \(X_{\mathcal{F}}\) is given by

\[Z = \{((P, \gamma), Q) : (P, \gamma) \in \mathcal{F}, \quad Q \in l\} = \{((P, \gamma), Q) : (Q - P) \wedge \gamma = 0\},\]

and its twisted conormal bundle, using the formalism above, is given by

\[N^* Z = \{(((P, \gamma); \gamma \wedge \tau, (Q - P) \wedge \tau); (Q; \tau \wedge \gamma))|_{\overline{\eta}_{\mathcal{F}}} : (Q - P) \wedge \gamma = 0\}.

At this point, we use Jacobi fields (see [21, Ch. 5] and [22]) to make a more concrete characterization of \(T_l \mathcal{F}\) and \(T_l \mathcal{M}\). Again, these calculations are based on the methods used in [2,13]. Fixing \(l = (P, \gamma) \in \mathcal{F}\), let \(e_0 = \gamma\) and pick \(e_1, e_2\) such that \(e_0, e_1, e_2\) form an orthonormal basis of vectors on \(\mathbb{R}^3\). With \(s\) parametrizing arclength on \(l\), the line \(l\) can be deformed to another line in \(\mathcal{M}\) by

\[P + s\gamma \mapsto P + s\gamma + (a_1 s + b_1) e_1 + (a_2 s + b_2) e_2,\]
where \( a_i, b_i \) are any constants. Thus the Jacobi fields \( e_1, se_1, e_2, and se_2 \) can be viewed as a basis for \( T_l M \). Given a Jacobi field \( X(s) = (a_1s + b_1)e_1 + (a_2s + b_2)e_2 \), we can view the deformation above using the identification \( l = (P, \gamma) \in T\mathbb{R}^3 \) as

\[
(P, \gamma) \mapsto (P + X(0), \gamma + X').
\]

Thus a tangent vector in \( T_l M \) can be identified as a pair \((X(0), X')\) lying in \( T^*(T\mathbb{R}^3) \).

The Gelfand admissibility condition (Definition 1) states that along the line \( l \), the normal space to \( F \) is proportional to a fixed vector. This implies we can pick a unit Jacobi field \( X_4(s) \) that is normal to \( T_l F \) and is proportional to a fixed vector perpendicular to \( \gamma \). Choose \( e_1 \) to be this vector, and choose \( a, b \in \mathbb{R} \) so that \( a^2 + b^2 = 1 \) and \( X_4(s) = (a - sb)e_1 \). Recall that \( M \) is a symplectic manifold with symplectic form given by

\[
\omega\left( \sum_{i=1}^{2} (a_is + b_i)e_i, \sum_{i=1}^{2} (c_is + d_i)e_i \right) = \sum_{i=1}^{2} b_ic_i - a_id_i.
\]

Then using \( X_4(s) \), we can form a symplectic basis for \( T_l M \), given by

\[
X_4(s) = (a - sb)e_1, \quad X_3(s) = (as + b)e_1, \quad X_2(s) = e_2, \quad X_1(s) = se_2
\]

Writing \( l = (P, \gamma) \) the basis for \( T_l F \) is given by \( \{(X_i(0), X'_i)\}_{i=1}^{3} \). Let \( \{\Phi_i\}_{i=1}^{3} \) be the dual basis in \( T_l^* F \). Parametrizing \( Q \) by its distance \( t \) from \( P \), i.e., \( Q = P + t\gamma \), we can rewrite \( \tau \wedge \gamma = \tau_1 e_1 + \tau_2 e_2 \) and \( \tau \wedge (Q - P) = t(\tau_1 e_1 + \tau_2 e_2) \). Then the twisted conormal bundle is given by

\[
N^* Z = \{(P, \gamma); -(at + b)\tau_1 \Phi_1 - \tau_2 \Phi_2 - t\tau_2 \Phi_3; P + t\gamma; \tau_1 e_1 + \tau_2 e_2 \}.
\]

We can parametrize \( (P, \gamma) = \sum_{i=1}^{3} \alpha_i(X_i(0), X'_i) \); thus we can parametrize \( N^* Z \) by \( t, \tau_1, \tau_2 \), and \( \alpha_i(X_i(0), X'_i), i = 1, 2, 3 \). Using this formalism we can describe \( d\pi_L \).

Using the above parametrization, we can identify \( \pi_L \) with the map

\[
(\{\alpha_i(X_i(0), X'_i)\}_{i=1}^{3}, t, \tau_1, \tau_2) \mapsto \\
\left( \sum_{i=3}^{\alpha_i(X_i(0), X'_i), -(at + b)\tau_1 \Phi_1 - \tau_2 \Phi_2 - t\tau_2 \Phi_3} \right),
\]

and thus analytically

\[
d\pi_L = \begin{pmatrix}
I & 0 \\
0 & B
\end{pmatrix},
\]
where

\[
\mathcal{B} = \begin{pmatrix}
-a \tau_1 & -(at + b) & 0 \\
0 & 0 & -1 \\
-\tau_2 & 0 & -t
\end{pmatrix}.
\]

The determinant of this matrix is \(\tau_2(at + b)\), so if we make the generic assumption that \(2at + b \neq 0\), \(\mathcal{L}\) is exactly the subvariety of \(N^*Z\) on which \(\tau_2 = 0\). The projection \(\varpi : N^*Z \to Z\) from Theorem 1 maps

\[
((P, \gamma); -(at + b)\tau_1 \Phi_1 - \tau_2 \Phi_2 - t \tau_2 \Phi_3; Q; \tau_1 e_1 + \tau_2 e_2) \mapsto ((P, \gamma); Q).
\]

Since \((P, \gamma; Q)\) is parametrized by only \(\alpha_i(X_i(0), X'_i)\) and \(t\) we see that \(\varpi|\mathcal{L}\) is a submersion. Thus again we see that Theorem 1 generalizes the results of [5] and applies to the adjoints of restricted X-ray transforms for Gelfand-admissible line complexes as long as the cones \(\Gamma_Q\) are curved. As Theorem 1 applies to the adjoints of these restricted X-ray transforms, we see that \(X_F\) maps boundedly from \(L^p_{\text{comp}}(\Omega_R) \to L^p_{1-1/p, \text{loc}}(\Omega_L)\) for \(1 \leq p < 4/3\), where \(\Omega_L, \Omega_R\) are sufficiently small coordinate patches in \(M\) and \(\mathbb{R}^3\), respectively.

3 Initial Setup

Using basic facts on generalized Radon transforms, we can simplify our operator \(\mathcal{R}\). By localization, we may assume that the Schwartz kernel of \(\mathcal{R}\) is supported in a small neighborhood of a base point \(P^\circ = (x^\circ, y^\circ) \in M\). On that neighborhood, the manifold \(M\) can be expressed locally by a defining function \(\Phi = (\Phi_1, \Phi_2)^T : \Omega_L \times \Omega_R \to \mathbb{R}^2\).

In other words, \(M = \{(x, y) : \Phi(x, y) = 0\}\) in a neighborhood of \(P^\circ\). Thus using the Fourier inversion formula, the Schwartz kernel of \(\mathcal{R}\) is given by an oscillatory integral distribution, formally written as

\[
\chi(x, y)\delta \circ \Phi(x, y) = (2\pi)^{-2} \iint e^{i\tau \cdot \Phi(x, y)} \chi(x, y) d\tau.
\]

Following the procedure found in [8], by local changes of variables and possible redefinition of \(\chi\), we can write \(\mathcal{R}\) locally as the oscillatory integral operator

\[
\mathcal{R}f(x) = \iint e^{i\tau \cdot (\Phi(x, y)-y')} \chi(x, y) f(y) d\tau dy.
\]

The twisted conormal bundle associated with \(\mathcal{R}\) is given by

\[
(N^*M)' = \{(x, \xi, y, \eta) : y_i = S^i(x, y_3), \; i = 1, 2, \; \xi = \tau_1 S^1(x, y_3) + \tau_2 S^2(x, y_3), \; \\
\eta = (\tau_1, \tau_2, -\tau_1 S^1_{y_3}(x, y_3) - \tau_2 S^2(x, y_3))\}.
\]
Thus parametrizing \((N^*\mathcal{M})'\) by the coordinates \((x_1, x_2, x_3, \tau_1, \tau_2, y_3)\), the projection \(\pi_L\) mapping \((N^*\mathcal{M})' \to T^*\Omega_L\) is identified with the map

\[
\tilde{\pi}_L : (x_1, x_2, x_3, \tau_1, \tau_2, y_3) \mapsto (x, \tau_1 S_x^1(x, y_3) + \tau_2 S_x^2(x, y_3)).
\]

Then we see

\[
D \tilde{\pi}_L = \begin{pmatrix}
I_{3 \times 3} & 0 \\
\partial x_i & \partial y_j (\tau \cdot S) B
\end{pmatrix},
\]

where \(B = (S_x^1, S_x^2, (\tau \cdot S)_{x,y_3})\). Thus we see

\[
\det D \tilde{\pi}_L = \det(S_x^1, S_x^2, \tau_1 S_{xy_3}^1 + \tau_2 S_{xy_3}^2) = \tau_1 \Delta^1 + \tau_2 \Delta^2,
\]

where

\[
\Delta^i (x, y_3) = \det(S_x^1, S_x^2, S_x^{i,y_3}) \bigg|_{x,y_3}, \quad i = 1, 2.
\]

We define \(\mathcal{L} = \{(x, \xi, y, \eta) \in \mathcal{C} : \det D \tilde{\pi}_L = 0\}\). Then \(\mathcal{L}\) is a conic submanifold of \((N^*\mathcal{M})'\) defined by

\[
\tau_1 \Delta^1(x, y_3) + \tau_2 \Delta^2(x, y_3) = 0.
\]

Similarly, we can identify \(\pi_R : (N^*\mathcal{M})' \to T^*\Omega_R\) with

\[
\tilde{\pi}_R : (x_1, x_2, x_3, \tau_1, \tau_2, y_3) \mapsto (S(x, y_3), y_3, \tau, -(\tau_1 S_{y_3}^1(x, y_3) + \tau_2 S_{y_3}^2(x, y_3))).
\]

Let \(N(x, y_3) = S_x^1(x, y_3) \wedge S_x^2(x, y_3)\). We see that a kernel field for \(\tilde{\pi}_R\) is given by

\[
V_R = \langle N(x, y_3), \nabla_x \rangle.
\]

Indeed, we see that \(\langle N(x, y_3), (\tau \cdot S_{y_3})_x \rangle = \tau \cdot \Delta\), and thus vanishes on \(\mathcal{L}\). Note this implies that \(-\Delta^2 S_{x,y_3}^1 + \Delta^1 S_{x,y_3}^2 \in \text{Span}(S_x^1, S_x^2)\). Since \(\pi_R\) is a blowdown, \(V_R\) is parallel to \(\mathcal{L}\), which implies \(V_R^k(\tau \cdot S_{y_3}) = 0\) on \(\mathcal{L}\) for all \(k \geq 1\).

Next, we will examine the fibers in \(T^*\Omega_L\) of \(\mathcal{L}\). Let \(\Sigma_x\) be the fibers of \(\pi_L(\mathcal{L})\), given by

\[
\Sigma_x = \{(\tau \cdot S)_x(x, y_3) : \tau \cdot \Delta(x, y_3) = 0\} = \{\pm \rho \mathcal{E}(x, y_3) : \rho > 0\},
\]

where

\[
\mathcal{E}(x, y_3) = -\Delta^2(x, y_3) S_x^1(x, y_3) + \Delta^1(x, y_3) S_x^2(x, y_3).
\]

Then we see two consequences, one related to our assumption on \(\sigma\).
Lemma 1 \[8, \S\ 3\] If \(\pi_L\) is a fold and \(\varpi\) is a submersion, then \(|\Delta| \neq 0\) near \(L\), and \(\Sigma_x\) is a two-dimensional cone that has one nonvanishing principal curvature given by

\[
\rho(\Sigma_{y_3}, N).
\]

Lemma 2 The direction normal to \(\Sigma_x\) at a point specified by \((y_3, \rho)\) is given by \(N(x, y_3)\).

Proof Let \(a \in \mathbb{R}^3\) be fixed. The tangent space of \(\Sigma_a\) at a point parametrized by \((y_3, \rho)\) is spanned by

\[
T_1(a, y_3) = \mathcal{E}(a, y_3)
\]

\[
T_2(a, y_3) = \mathcal{E}_{y_3}(a, y_3),
\]

so a normal vector at a point \((\rho, y_3)\) is given by

\[
T_1 \wedge T_2 = \mathcal{E} \wedge \mathcal{E}_{y_3}
\]

\[
= (\Delta^1 \mathcal{E}_{y_3}^1 - \Delta^2 \mathcal{E}_{y_3}^1)(S_x^1 \wedge S_x^2)
\]

\[
+ (\Delta^1 S_x^1 - \Delta^2 S_x^1) \wedge (\Delta^1 S_{xy_3}^1 - \Delta^2 S_{xy_3}^1).
\]

Since \(-\Delta^2 S_{xy_3}^1 + \Delta^1 S_{xy_3}^1 \in \text{Span}(S_x^1, S_x^2)\) for fixed \((x, y_3)\), the expression in the final line of the calculation of \(T_1 \wedge T_2\) is either 0 or a scalar multiple of the vector \(S_x^1 \wedge S_x^2 = N\), hence the sum is a multiple of \(N(a, y_3)\).

\(\square\)

4 Initial Decomposition

We localize in \(|\tau|\) then localize away from the singular variety \(L\), following the ideas of Phong and Stein in [23]. Let \(\chi_0 \in C^\infty_c(\mathbb{R})\) be equal to 1 on \([\frac{1}{2}, 2]\) and supported on \([\frac{1}{4}, 4]\) such that \(\sum_{k \in \mathbb{Z}} \chi_0(2^k) = 1\). For \(k \geq 1\) define \(\chi_k(|\tau|) = \chi_0(2^{-k}|\tau|)\). For \(\varepsilon > 0\) and \(0 \leq \ell \leq \ell_0 = \lfloor \frac{k}{2 + \varepsilon} \rfloor\) let

\[
ak, \ell, \pm(x, y_3, \tau) = \begin{cases} 
\chi_0(2^{\ell-k}(\pm \tau \cdot \Delta(x, y_3))) & \ell < \ell_0 \\
1 - \sum_{\ell < \ell_0} \chi_0(2^{\ell-k}(\pm \tau \cdot \Delta(x, y_3))) & \ell = \ell_0
\end{cases}
\]

and define

\[
\mathcal{R}_{k, \ell, \pm} f(x) = \chi(x_1) \int e^{i \tau \cdot \Phi(x, y)} \chi(y) f(y) \chi_k(|\tau|) a_{k, \ell, \pm}(x, y_3, \tau) \, dy \, d\tau.
\] (7)

We will suppress the dependence on \(\pm\). We prove the following estimate.
Proposition 1 For $p > 4$ and all $\varepsilon > 0$ there exists $\varepsilon_0(p) > 0$ such that for all $\ell \leq \ell_0 = \lfloor \frac{k}{2+\varepsilon} \rfloor$, 

$$\| R_{k,\ell} \|_{L^p \to L^p} \leq C_p 2^{-(k+\ell\varepsilon_0)/p}. $$

This proposition follows by interpolation with $L^2$ estimates, $L^\infty$ estimates, and a decoupling inequality. Let $I$ be a collection of intervals of length $2^{-\ell}$ with disjoint interiors intersecting a small neighborhood of 0. Then for a function $f : \mathbb{R}^3 \to \mathbb{R}$ supported in the unit cube and any $I \in \mathcal{I}$, let $f_I(y) := f(y) 1_I(y_3)$, so that $f = \sum_{I \in \mathcal{I}} f_I$ with almost disjoint supports in $y_3$. The necessary $L^2$ estimate is the following.

Proposition 2 Let $R_{k,\ell}$ be defined as above. For every $\varepsilon > 0$,

$$\| R_{k,\ell} \|_{L^2 \to L^2} \lesssim 2^{(\ell-k)/2+\ell\varepsilon}, \quad \ell \leq \ell_0 = \lfloor \frac{k}{2+\varepsilon} \rfloor. \quad (8)$$

Moreover, by almost disjoint supports of the functions $f_I$,

$$\left\| \sum_{I \in \mathcal{I}} R_{k,\ell} f_I \right\|_{L^2} \lesssim 2^{(\ell-k)/2+\ell\varepsilon} \left( \sum_{I \in \mathcal{I}} \| f_I \|_{L^2}^2 \right)^{1/2}, \quad \ell \leq \ell_0. \quad (9)$$

Proposition 2 will be proven in Sect. 6 following methods of almost-orthogonality found in the proof of the Calderón–Vaillancourt theorem (see [24], § 9.2), originally introduced into this context by Phong and Stein [23], Cuccagna [25], and Comech [15]. The main estimate in the proof of Theorem 1 is the decoupling inequality.

Proposition 3 For every $\varepsilon > 0$

$$\left\| \sum_{I \in \mathcal{I}} R_{k,\ell} f_I \right\|_{L^p} \lesssim_{\varepsilon} 2^{(1/2-1/p+\varepsilon)} \left( \sum_{I \in \mathcal{I}} \| R_{k,\ell} f_I \|_{L^p}^p \right)^{1/p} + 2^{-10k} \| f \|_{L^p}$$

for $2 \leq p \leq 6$.

Following a similar approach to [26] and [8], we prove Proposition 3 in Sect. 5 using an inductive argument, at each step combining $L^p$ decoupling with suitable changes of variables.

Proof that Propositions 2 and 3 imply Proposition 1 We begin by proving an $L^\infty$ estimate for $R_{k,\ell}$, namely that

$$\sup_{I \in \mathcal{I}} \| R_{k,\ell} f_I \|_{\infty} \lesssim 2^{-\ell} \sup_{I \in \mathcal{I}} \| f_I \|_{\infty} \quad (10)$$

$$\| R_{k,\ell} f \|_{\infty} \lesssim \| f \|_{\infty}. \quad (11)$$

To see (10), we estimate the Schwartz kernel of $R_{k,\ell}$ (call it $R_{k,\ell}(x, y)$) by integrating by parts in the $\tau$ variables, distinguishing the directions $(\Delta^1, \Delta^2)$ and $(-\Delta^1, \Delta^2)$. 

\begin{align*}
\end{align*}
This shows that \(|R_{k,\ell}(x, y)| \leq C_N U_1(x, y) U_2(x, y)|\), where

\[
U_1(x, y) = \frac{2^{k-\ell}}{(1 + 2^{k-\ell}|\Delta^1(y_1 - S_1^1) + \Delta^2(y_2 - S_2)|)^N}
\]
\[
U_2(x, y) = \frac{2^k}{(1 + 2^k| - \Delta^2(y_1 - S_1^1) + \Delta^2(y_2 - S_2)|)^N}
\]

We integrate in \(y'\) first, then in \(y_3\), which is supported in an interval of length \(2^{-\ell}\). To prove (11) the same argument holds, but we integrate over a larger interval in \(y_3\).

Interpolating (10) with (9) we obtain

\[
\left( \sum_{I \in I} \| \mathcal{R}_{k,\ell} f_I \|_p \right)^{1/p} \lesssim \varepsilon 2^{\ell(3/p-1+\varepsilon)/p} 2^{-k/p} \left( \sum_{I \in I} \| f_I \|_p \right)^{1/p}, \quad 2 \leq p \leq \infty. \tag{12}
\]

Combining this estimate with Proposition 3, we obtain

\[
\| \mathcal{R}_{k,\ell} f \|_p \lesssim \varepsilon 2^{\ell(p-1)/p} 2^{-k/p} \left( \sum_{I \in I} \| f_I \|_p \right)^{1/p} + 2^{-10k} \| f \|_p, \quad 2 \leq p \leq 6. \tag{13}
\]

Note that the power of \(2^\ell\) in (13) is negative if \(4 < p \leq 6\) and \(\varepsilon\) is sufficiently small. A further interpolation with the \(L^\infty\) estimate (11) yields Proposition 1 for \(p > 4\).

\[
\square
\]

5 Decoupling

We mirror the structure of the decoupling estimates in [8], working out a model case first then reducing the general case to the model case by changes of variables. In the model case, the functions \(S_i\) are replaced by \(S_i\) satisfying simplifying assumptions at the origin. Additionally, the blowdown condition in this model case implies some additional assumptions near the origin.

5.1 A Model Case

Consider \(C^\infty\) maps \((w, z_3) \mapsto \mathcal{G}^i(w, z_3)\) defined on a neighborhood of \([-r, r]^4\) for some \(r \in (0, 1)\). For \(n \in \mathbb{N}\) define \(M_n > 0\) such that

\[
M_n \geq 2 + \| \mathcal{G}^1 \|_{C^{n+5}([-r, r]^4)} + \| \mathcal{G}^2 \|_{C^{n+5}([-r, r]^4)}, \tag{14}
\]

where the \(C^n\) norm is the supremum of all derivatives orders 0 to \(n\). We assume that for \(w \in [-r, r]^3\),

\[
(\mathcal{G}^1, \mathcal{G}^2, \mathcal{G}^1_{z_3}) \big|_{(w, 0)} = (w_1, w_2, w_3); \tag{15}
\]
we also assume
\[ \mathcal{S}^2_{w,z_3}(0,0) = 0, \]  
and
\[ \mathcal{S}^2_{w_3 z_3}(0,0) = \kappa_0. \]

As the functions \( \mathcal{S}^1, \mathcal{S}^2 \) play the part of \( \mathcal{S}^1, \mathcal{S}^2 \) in our model case, we can analyze the geometry of the canonical relation associated with \( \mathcal{S}^1, \mathcal{S}^2 \). Define for \( i = 1, 2 \) the functions \( \Delta^i_{\mathcal{S}} = \det(\mathcal{S}^1_w, \mathcal{S}^2_w, \mathcal{S}^i_{w z_3}) \). In this model case, the singularity surface \( \mathcal{L}_{\mathcal{S}} \) is given by the restriction \( \mu_1 \Delta^1_{\mathcal{S}}(w, z_3) + \mu_2 \Delta^2_{\mathcal{S}}(w, z_3) = 0 \). We can define the analogue of the right projection \( \tilde{\pi}_R : (w, \mu, z_3) \mapsto (\mathcal{S}(w, z_3), z_3, \mu, -\mu_1 \mathcal{S}^1_{z_3}(w, z_3) + \mu_2 \mathcal{S}^2_{z_3}(w, z_3)) \), and a kernel field for this map at the point \( P \) parametrized by \( (w, z_3, \mu) \) is given by
\[ V_R(w, z_3) = \langle \mathcal{S}^1_w(w, z_3) \wedge \mathcal{S}^2_w(w, z_3), \nabla_w \rangle. \]

We assume a blowdown on \( \tilde{\pi}_R \), i.e., that \( V_R \) is parallel to \( \mathcal{L}_{\mathcal{S}} \), implying that
\[ V_R^N|_{(w, z_3), \mu \perp \mathcal{L}_{\mathcal{S}}(w, z_3)} = 0 \]
for all \( N > 0 \). Since \( \mathcal{S}^1_w(w, 0) = e_1 \) and \( \mathcal{S}^2_w(w, 0) = e_2 \), we see that \( V_R(w, 0) = \partial_{w_3} \).

The above conditions imply that
\[ \partial^N_{w_3} \mathcal{S}^2_{w z_3}(w, 0) = 0, \ \forall N \geq 1 \]  
\[ \partial^N_{w_3} \mathcal{S}^1_{w z_3}(w, 0) = 0, \ \forall N \geq 1. \]

Recall that the fibers of the singular manifold \( \mathcal{L}_{\mathcal{S}} \) are given for fixed \( w \) by
\[ \mathcal{\tilde{S}}_w = \{ \mu_1 \mathcal{S}^1_w(w, z_3) + \mu_2 \mathcal{S}^2_w(w, z_3) : \mu_1 \Delta^1_{\mathcal{S}}(w, z_3) + \Delta^2_{\mathcal{S}}(w, z_3) = 0 \} = \{\pm \rho \mathcal{S}_{\mathcal{S}}(w, z_3) : \rho > 0, |z_3| \leq r \}, \]
where \( \mathcal{S}_{\mathcal{S}}(w, z_3) \) is given by \( -\mathcal{S}^1_w(w, z_3) \Delta^2_{\mathcal{S}}(w, z_3) + \mathcal{S}^1_w(w, z_3) \Delta^1_{\mathcal{S}}(w, z_3) \). Thus \( \mathcal{\tilde{S}}_0 \) is a cone parametrized by \( (\rho, z_3) \) given by
\[ \{\pm \rho \mathcal{S}_{\mathcal{S}}(0, z_3) : \rho > 0, |z_3| \leq r \} =: \Sigma. \]

Recall from Sect. 3 that \( \mathcal{G}^1_w \wedge \mathcal{G}^2_w(0, b) =: N(b) \) is normal to \( \Sigma \) at the point \( P \) parametrized by \( (\rho', b) \). Thus \( T_P \Sigma \) has an orthogonal basis given by
\[ T_1(b) = \mathcal{S}_{\mathcal{S}}(0, b) \]  
\[ T_2(b) = T_1(b) \wedge N(b). \]
For \( A > 1 \) and \( \delta \ll 1 \) let \( \Pi_{A,b}(\delta) \) be set of \( \xi \in \mathbb{R}^3 \) such that
\[
A^{-1} \leq \left| \frac{T_1(b)}{|T_1(b)|} \xi \right| \leq A
\]
\[
\left| \frac{T_2(b)}{|T_2(b)|} \xi \right| \leq A\delta
\]
\[
\left| \frac{N(b)}{|N(b)|} \xi \right| \leq A\delta^2.
\]

The sets \( \Pi_{A,b}(\delta) \) are unions of \( A \times A\delta \times A\delta^2 \)-boxes with long, middle, and short sides parallel to \( T_1(b) \), \( T_2(b) \), and \( N(b) \), respectively. We will refer to \( \Pi_{A,b}(\delta) \) as a plate. Because the cone \( \tilde{\Sigma} \) is curved, we can apply decoupling to the plates \( \Pi_{A,b}(\delta) \).

**Theorem 2** [7] Let \( \varepsilon > 0 \) and \( A > 1 \). There exists a constant \( C(\varepsilon, A) \) such that the following holds for \( 0 < \delta_1 < \delta_0 < 1 \).

Let \( B = \{ b_v \}_{v=1}^M \) be a set of points in an interval \( J \subset [-1, 1] \) of length \( \delta_0 \) such that \( |b_v - b_{v'}| \geq \delta_1 \) for \( b_v, b_{v'} \in B, v \neq v' \). Let \( 2 \leq p \leq 6 \). Let \( f_v \in L^p(\mathbb{R}^3) \) such that the Fourier transform of \( f_v \) is supported in \( \Pi_{A,b}(\delta_1) \). Then
\[
\left\| \sum_v f_v \right\|_p \leq C(\varepsilon, A)(\delta_0/\delta_1)^{1/2-1/p+\varepsilon} \left( \sum_v \| f_v \|_p \right)^{1/p}.
\]

Let \( (w, z_3) \mapsto \alpha(w, z_3) \) be a \( C^\infty \) function satisfying for \( |(w, z_3)|_\infty < r \),
\[
M_0^{-1} \leq |\alpha(w, z_3)| \leq M_0 \quad (20)
\]
\[
|\nabla_w \alpha(w, z_3)| \leq M_0 \quad (21)
\]

Let \( (w, z, \mu) \mapsto \zeta(w, z, \mu) \) belong to a bounded family of \( C^\infty \) functions supported where \( |(w, z)|_\infty \leq r \) and \( 1/4 \leq |\mu| \leq 4 \).

Let \( T_{k,\ell} \) be an operator with Schwartz kernel
\[
2^{2k} \int e^{2i\xi \cdot (w, z_3)} \eta(2^\ell \alpha(w, z_3)(\mu_1 \Delta^1_\Theta(w, z_3) + \mu_2 \Delta^2_\Theta(w, z_3)) \zeta(w, z) \eta(|\mu|) \, d\mu.
\]

The operator \( T_{k,\ell} \) will play the role of \( R_{k,\ell} \) after a nonlinear change of variables, while \( \alpha(w, z_3) \) is introduced in the localization as a byproduct of those changes of variables.

**Proposition 4** Let \( 0 < \varepsilon \leq 1, k \gg 1, 0 \leq \ell \leq k/2 \),
\[
\delta_0 \in (2^{-\ell(1-\varepsilon)}, 2^{-\ell\varepsilon}),
\]
and \( \delta_0 > \delta_1 \geq \max\{2^{-\ell(1-\varepsilon)/2}, \delta_0 2^{-\ell\varepsilon/4} \} \). Define \( \varepsilon_1 = (\delta_1/\delta_0)^2 \). Let \( J \) be an interval of length \( \delta_0 \) containing \( 0 \), and \( I_J \) be a collection of intervals of length \( \delta_1 \) with disjoint interior and whose interiors all intersect \( J \). Let \( \sigma \in C^\infty_c(\mathbb{R}^3) \) be supported \((-1, 1)^3 \) and define \( \sigma_{\ell,\varepsilon_1}(w) = \sigma(2^\ell w_1, 2^\ell w_2, \varepsilon_1^{-1} w_3) \). Then for \( 2 \leq p \leq 6 \), \( g \in L^p(\mathbb{R}^3) \) with
$g_I(y) = g(y) \mathbb{1}_I(y_3)$, and any $N \in \mathbb{N}$,

$$\left\| \sigma_{\ell, \varepsilon} \sum_{I \in \mathcal{I}_J} T_{k, \ell} g_I \right\|_p \lesssim_{\varepsilon} \left( \delta_0 / \delta_1 \right)^{1/2-1/p+\varepsilon} \left( \sum_{I \in \mathcal{I}_J} \left\| \sigma_{\ell, \varepsilon} T_{k, \ell} g_I \right\|_p \right)^{1/p} + C(\varepsilon, N) 2^{-kN} 2^{-2\ell \varepsilon_1} \|g\|_p.$$  

The idea here is to show that the Fourier transforms of $\sigma_{\ell, \varepsilon} \sum_{I \in \mathcal{I}_J} T_{k, \ell} g_I$ are concentrated on the plates $\Pi_{A, b_I}(\delta_1)$ for some $b_I \in I$ and some large enough $A > 1$.

### 5.1.1 Derivatives of $\mathcal{G}$ and $\Delta$

Some approximations will be helpful to write down. For the rest of Sect. 5.1, we omit the subscript dependence on $\mathcal{G}$. Because of (15), we may conclude that for any multi-index $\beta$ of length at least 1,

$$\partial_\omega^\beta \mathcal{G}^1_\omega|_{(w,0)} = 0$$  

$$\partial_\omega^\beta \mathcal{G}^2_\omega|_{(w,0)} = 0$$  

$$\partial_\omega^\beta \mathcal{G}^3_{wz^3}|_{(w,0)} = 0.$$  

For $w \in [-r, r]^3$,

$$\Delta^1(w, 0) = 1$$  

$$\Delta^2(w, 0) = \mathcal{G}^2_{wz^3}(w, 0)$$  

$$\Delta^1_{z^3}(0, 0) = \mathcal{G}^1_{wz^3}(0, 0)$$  

$$\Delta^2_{z^3}(0, 0) = \mathcal{G}^2_{wz^3}(0, 0) = \kappa_0$$

and thus

$$\mathcal{E}(w, 0) = -\Delta^2(w, 0) \mathcal{G}^1_w(w, 0) + \Delta^1(w, 0) \mathcal{G}^2_w(w, 0) = e_2 - \mathcal{G}^2_{wz^3}(w, 0)e_1$$

$$\mathcal{E}_{wz^3}(0, 0) = -\mathcal{G}^2_{wz^3}(0, 0)e_1 = 0, \quad n \geq 1$$

$$\mathcal{E}_{z^3}(0, 0) = -\kappa_0 e_1 + \mathcal{G}^1_{wz^3}(0, 0)e_2.$$  

Using these,

$$T_1(b) = \mathcal{E}(0, b)$$

$$= \mathcal{E}(0, 0) + b \mathcal{E}_{z^3}(0, 0) + O(b^2)$$

$$= -\kappa_0 e_1 + (1 + b \mathcal{G}_{wz^3}(0, 0)) e_2 + O(b^2)$$
and

\[ N(b) = \mathcal{S}_w^1(0, b) \land \mathcal{S}_w^2(0, b) \]
\[ = (e_1 + be_3 + O(b^2)) \land (e_2 + O(b^2)) \]
\[ = -be_1 + e_3 + O(b^2). \]  \hspace{1cm} (36)

From these, we see that

\[ T_2(b) = T_1(b) \land N(b) = (1 + b\mathcal{S}_w^1_{w, z_3}^{0, b})e_1 + \kappa_0 be_2 + be_3 + O(b^2). \]  \hspace{1cm} (37)

Let \( \beta = (\beta_{w_1}, \beta_{w_2}, \beta_{w_3}, \beta_{z_3}) \) be a multi-index and let \( \partial^\beta_{(w, z_3)} \) denote a derivative of order \( |\beta| = \beta_{w_1} + \beta_{w_2} + \beta_{w_3} + \beta_{z_3} \) in the variables \( w, z_3 \). By using the upper bounds \( M_n \), trilinearity of determinants, and differentiation rules for products, we can estimate

\[ |\partial^\beta_{(w, z_3)} \Delta^i| \leq 3|\beta|M^3_{|\beta|}. \]  \hspace{1cm} (38)

Similarly, by differentiating products,

\[ |\partial^\beta_{(w, z_3)} \Xi| \leq 4|\beta|M^4_{|\beta|}. \]  \hspace{1cm} (39)

5.1.2 Plate Localization

**Lemma 3** Let \( \varepsilon > 0 \), and \( \delta_0, \delta_1, \varepsilon_1 \) be as in Proposition 4. Assume that \( 2^{-\ell} \ll r \), \( M_02^{-\ell} \leq 2^{-10}, \frac{1}{4} \leq |\mu| \leq 4, |w'| \leq 2^{-\ell}, |w_3| \leq \varepsilon_1, |b| \leq \delta_0, \) and \( |z_3 - b| \leq \delta_1. \)

If

\[ |\mu_1 \Delta^1(w, z_3) + \mu_2 \Delta^2(w, z_3)| \leq M_02^{-\ell}, \]  \hspace{1cm} (40)

then there exists \( A(\varepsilon) > 1 \) such that

\[ \mu_1 \mathcal{S}_w^1(w, z_3) + \mu_2 \mathcal{S}_w^2(w, z_3) \in \Pi_{A(\varepsilon), b}(\delta_1). \]

More specifically,

\[ A(\varepsilon)^{-1}|T_1(b)| \leq |\{T_1(b), \mu_1 \mathcal{S}_w^1(w, z_3) + \mu_2 \mathcal{S}_w^2(w, z_3)\}| \leq A(\varepsilon)|T_1(b)| \]  \hspace{1cm} (41)

\[ |\{T_2(b), \mu_1 \mathcal{S}_w^1(w, z_3) + \mu_2 \mathcal{S}_w^2(w, z_3)\}| \leq A(\varepsilon)|T_2(b)|\delta_1. \]  \hspace{1cm} (42)

\[ |\{N(b), \mu_1 \mathcal{S}_w^1(w, z_3) + \mu_2 \mathcal{S}_w^2(w, z_3)\}| \leq A(\varepsilon)|N(b)|\delta_1^2. \]  \hspace{1cm} (43)

**Proof** Throughout this proof, we use Taylor expansions with appropriate error remainders. Therefore, for any \( i = 1, 2, \ldots \), the function \( R_i(w, z_3) \) is \( C^\infty \) and uniformly bounded by 1.

\( \square \) Springer
The estimate in (41) is clearly true for some $A > 1$ independent of $\varepsilon$. We start with the proof of (43). Let $G = [3\varepsilon^{-1}]$. Employing a Taylor expansion about $(w, z_3) = (0, b)$, and reorganizing terms using that $2^{-\ell} \leq \delta_1$, $2^{-\ell} \delta_0 \leq \delta_1^2$, $\varepsilon_1^2 \delta_0 \leq \delta_1^2$, and $\varepsilon_1^2 \leq \delta_1^2$, we see that

$$
\langle N(b), \mu \cdot \mathcal{G}_w(w, z_3) \rangle = \sum_{n=0}^{G-n} \sum_{|w|=0} (N(b), \nabla_w \left( (\partial_{z_3})^n (\partial_w)^{\alpha} [\mu_1 \mathcal{G}^1 + \mu_2 \mathcal{G}^2] \right)(0, b))
$$

$$
+ M_G \delta_1^2 R_1(w, \mu, z_3)
= (z_3 - b) \langle N(b), \mu_1 \mathcal{G}^1_{wz_3} + \mu_2 \mathcal{G}^2_{wz_3}(0, b) \rangle
+ \left( \sum_{i=1}^{2} w_i \langle N(b), \mu_1 \mathcal{G}^1_{ww_i} + \mu_2 \mathcal{G}^2_{ww_i}(0, b) \rangle \right)
+ I + II + III + M_G \delta_1^2 R_2(w, \mu, z_3),
$$

where

$$
I = \sum_{n=1}^{G} \frac{w_3^n}{n!} \langle N(b), \mu_1 \mathcal{G}^1_{w3^n} + \mu_2 \mathcal{G}^2_{w3^n}(0, b) \rangle
$$

$$
II = \sum_{n=2}^{G} \frac{w_3^{n-1} (z_3 - b)}{n!} \langle N(b), \mu_1 \mathcal{G}^1_{w3^{n-1}z_3} + \mu_2 \mathcal{G}^2_{w3^{n-1}z_3}(0, b) \rangle
$$

$$
III = \sum_{n=2}^{G} \sum_{i=1}^{2} \frac{w_3^{n-1} w_i}{n!} \langle N(b), \mu_1 \mathcal{G}^1_{w3^{n-1}w_i} + \mu_2 \mathcal{G}^2_{w3^{n-1}w_i}(0, b) \rangle.
$$

Clearly the first term in (44) vanishes by the definition of $N(b)$ (see (36)). The second term in the expansion is

$$(z_3 - b) \langle N(b), \mu_1 \mathcal{G}^1_{wz_3} + \mu_2 \mathcal{G}^2_{wz_3}(0, b) \rangle = (z_3 - b)(\mu_1 \Delta^1(0, b) + \mu_2 \Delta^2(0, b)).$$

Now, since $|w'|, |z_3 - b| \leq \delta_1$, applying a Taylor expansion and using trilinearity of determinants, and differentiation of products, we get

$$
\mu_1 \Delta^1(0, b) + \mu_2 \Delta^2(0, b) = (\mu_1 \Delta^1(w, z_3) + \mu_2 \Delta^2(w, z_3))
+ \sum_{n=1}^{G} \frac{w_3^n}{n!} \left( \mu_1 \Delta^1_{w^n}(w, z_3) + \mu_2 \Delta^2_{w^n}(w, z_3) \right)
+ 3^G M_G^3 \delta_1 R_3(w, z_3).
$$
By (40) the first term is bounded by $M_0 2^{-\ell}$. For each $1 \leq n \leq G$, from (19) and (26) we have $\Delta^{i}_{w_3}(w, 0) = 0$ for $i = 1, 2$, and so by trilinearity of determinants, and differentiation of products, expanding about $z_3 = 0$, we get

$$|\Delta^{i}_{w_3}(w, z_3)| \leq |\Delta^{i}_{w_3}(w, 0) + 3^n M_n^3 z_3| \leq 3^n M_n^3 \delta_0.$$ 

Thus

$$|\mu_1 \Delta^{1}(0, b) + \mu_2 \Delta^{2}(0, b)| \leq M_0 2^{-\ell} + 3^G M_G^3 \varepsilon_1 \delta_0 + 3^G M_G^3 \delta_1 \leq 3^{G+1} M_G^3 \delta_1,$$

and the second term in (44) is bounded by $3^{G+1} M_G^3 \delta_1^2$.

Next we deal with the first-order $w'$ derivatives in (44). We approximate about $z_3 = 0$. For $i = 1, 2$, using the estimates (23) and (24), we get

$$|w_{i}(\mathcal{G}_w^1(0, b) \land \mathcal{G}_w^2(0, b), \mu_1 \mathcal{G}_w^1_{w_1}(0, b) + \mu_2 \mathcal{G}_w^2_{w_1}(0, b))| \leq |w_{i}||\langle \mathcal{G}_w^1(0, 0) \land \mathcal{G}_w^2(0, 0), \mu_1 \mathcal{G}_w^1_{w_1}(0, 0) \rangle + \mu_2 \mathcal{G}_w^2_{w_1}(0, 0)| + 3M_0^3 b R_4(0, b) \leq 2^{-\ell}(0 + 3M_0^3 \delta_0).$$

Note that the condition $\delta_1 \geq \max\{M_0^2 2^{20-\ell(1-\varepsilon/2)}, 2^{-\ell \varepsilon/4} \delta_0\}$ from Proposition 4 implies that $2^{-\ell} \delta_0 \leq \delta_1$.

Finally, we estimate $I$, $II$, and $III$. All rely on the blowdown condition at the origin.

First we estimate $I$. For all $n \geq 1$, we expand about the origin to obtain

$$(\mathcal{G}_w^1(0, b) \land \mathcal{G}_w^2(0, b), \mu_1 \mathcal{G}_w^1_{w_3}(0, b) + \mu_2 \mathcal{G}_w^2_{w_3}(0, b)) = (\mathcal{G}_w^1(0, 0) \land \mathcal{G}_w^2(0, 0), \mu_1 \mathcal{G}_w^1_{w_3}(0, 0) + \mu_2 \mathcal{G}_w^2_{w_3}(0, 0))$$

$$+ b \left[ \det(\mathcal{G}_w^1_{w_3} \mathcal{G}_w^2 \mu_1 \mathcal{G}_w^1_{w_3} + \mu_2 \mathcal{G}_w^2_{w_3}) \right]_{(0, 0)}$$

$$+ \det(\mathcal{G}_w^1 \mathcal{G}_w^2_{w_3} \mu_1 \mathcal{G}_w^1_{w_3} + \mu_2 \mathcal{G}_w^2_{w_3}) \right]_{(0, 0)}$$

$$+ \det(\mathcal{G}_w^1 \mathcal{G}_w^2_{w_3} \mu_1 \mathcal{G}_w^1_{w_3} + \mu_2 \mathcal{G}_w^2_{w_3}) \right]_{(0, 0)}$$

$$+ 3^2 M^n_3 b^2 R_5(0, b).$$

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Using the estimates (23), (24), (36), (25), and (18), we observe
\[
\langle S_w^1(0, 0) \wedge S_{w^3}^2(0, 0), \mu S_{w^3}^1(0, 0) + \mu_2 S_{w^3}^2(0, 0) \rangle = 0
\]
\[
\det(S_w^1 S_{w^3}^2, \mu S_{w^3}^1 + \mu_2 S_{w^3}^2)_{(0,0)} = 0
\]
\[
\det(S_w^1 S_{w^3}^2, \mu S_{w^3}^1 + \mu_2 S_{w^3}^2)_{(0,0)} = 0
\]
\[
\det(S_w^1 S_{w^3}^2, \mu S_{w^3}^1 + \mu_2 S_{w^3}^2)_{(0,0)} = 0.
\]
This implies
\[
|I| \leq 3^2 M_G^3 \sum_{n=1}^{\infty} \frac{\varepsilon^{n-1} \delta_0^2}{n!} \leq 3^3 M_G^3 \varepsilon \delta_0^2 \leq 3^3 M_G^3 \delta_1^2.
\]

Next we estimate \(II\). For \(n \geq 2\), we expand about the origin to obtain
\[
\langle S_w^1(0, b) \wedge S_{w^3}^2(0, b), \mu \cdot S_{w^3}^{n-1}(0, b) \rangle
\]
\[
= \det(S_w^1 S_{w^3}^2, \mu S_{w^3}^1 + \mu_2 S_{w^3}^2)_{(0,0)} + 3M_G^3 b R_6(0, b).
\]
Thus the calculation from \(I\) the determinant vanishes, and thus
\[
|II| \leq 3^3 M_G^3 \sum_{n=2}^{\infty} \frac{\varepsilon^{n-1} \delta_1^0}{n!} \leq 3^3 M_G^3 \varepsilon \delta_0 \leq 3M_G^3 \delta_1^2.
\]

Finally we estimate \(III\). Again using the calculations from \(I\), for \(n \geq 2\) and \(i = 1, 2\)
\[
\langle S_w^1(0, b) \wedge S_{w^3}^2(0, b), \mu \cdot S_{w^3}^{n-1}(0, b) \rangle
\]
\[
= \langle S_w^1(0, 0) \wedge S_{w^3}^2(0, 0), \mu \cdot S_{w^3}^{n-1}(0, 0) \rangle + 3M_G^3 b R_7(0, b)
\]
\[
= \mu \cdot S_{w^3}^1(0, 0) + 3M_G^3 b R_7(0, b)
\]
\[
= 3M_G^3 b R_7(0, b).
\]
This implies that
\[
|III| \leq 3^3 M_G^3 \sum_{n=2}^{\infty} \frac{\varepsilon^{n-1} \delta_0^0}{n!} \leq 3^3 M_G^3 \varepsilon 2^{-\ell} \delta_0 \leq 3M_G^3 \delta_1^2.
\]
Since \(|N(b)| \geq 1/2\) this proves (43) with any \(A(\varepsilon) \geq 3^{|3/\varepsilon|+2} M_G^3 |3/\varepsilon|\).
Having proven (43), we prove (42). Using (37), define

$$T_2^*(b) = (1 + b\mathcal{G}^1_{w3z}(0, 0))e_1 + \kappa_0 be_2 + be_3$$

and note that $|T_2(b) - T_2^*(b)| \leq M_0\delta_0^2$. Next, we will approximate $\mu$ by the projection of $\mu_1\Delta^1(w, z_3) + \mu_2\Delta^2(w, z_3)$ onto $\mathcal{L}_{\mathcal{G}}$. In particular, let

$$\mu^\circ = \pm \frac{\kappa}{|\Delta(w, z_3)|}(-\Delta^2(w, z_3), \Delta^1(w, z_3)),$$

so that $\mu^1\Delta^1(w, z_3) + \mu_2\Delta^2(w, z_3) = 0$, $|\mu| = |\mu^\circ|$, and where the sign is picked so that

$$|\mu - \mu^\circ| \leq 2|\mu|M_02^{-\ell}.$$

This is possible since $|\mu_1\Delta^1 + \mu_2\Delta^2| \leq M_02^{-\ell}$ and $|\Delta(w, z_3)| \neq 0$. Then

$$\mu_1\mathcal{G}^1_w(w, z_3) + \mu_2\mathcal{G}^2_w(w, z_3) = \frac{|\mu|}{|\Delta(w, z_3)|}\mathcal{E}(w, z_3),$$

and thus

$$|\mu_1\mathcal{G}^1_w(w, z_3) + \mu_2\mathcal{G}^2_w(w, z_3) - \frac{|\mu|}{|\Delta(w, z_3)|}\mathcal{E}(w, z_3)| \leq |\mu - \mu^\circ| |\mathcal{E}_w| \leq 8M_0^22^{-\ell}.$$ 

We approximate by a Taylor expansion about the origin, using the fact that $\varepsilon_1\delta_0 \leq \delta_1$, $|w'| \leq 2^{-\ell} \leq \delta_1$, $\delta_0^2 \leq \delta_1$, and $\varepsilon_1^G \leq \delta_1^2 \leq \delta_1$. Reorganizing, we obtain

$$\langle T_2^*(b), \mathcal{E}(w, z_3) \rangle = \sum_{n=0}^{G} \sum_{|\alpha|=0}^{G-n} \langle T_2^*(b), (\partial_{z_3})^\alpha(\partial_w)^\alpha \mathcal{E} \rangle \bigg|_{(0, 0)} + 4G^4M_0^4\delta_1 R_8(w, z_3)$$

$$= \langle T_2^*(b), \mathcal{E}(0, 0) \rangle + z_3 \langle T_2^*(b), \mathcal{E}_3(0, 0) \rangle$$

$$+ \sum_{n=1}^{G} \frac{w_3^n}{n!} \langle T^*(b), \mathcal{E}_{w_3^n}(0, 0) \rangle + 4G^4M_0^4\delta_1 R_9(w, z_3).$$

Using (30), (31), and (32)

$$\langle T_2^*(b), \mathcal{E}(0, 0) \rangle = \kappa_0 b$$

$$\langle T_2^*(b), \mathcal{E}_3(0, 0) \rangle = -\kappa_0(1 + b(\mathcal{G}^1_{w3z}(0, 0) - \mathcal{G}^1_{w3z}(0, b)))$$

$$\langle T_2^*(b), \mathcal{E}_{w_3^n}(0, 0) \rangle = 0, \quad n \geq 1.$$ 

Thus

$$|\langle T_2^*(b), \mathcal{E}(w, z_3) \rangle| \leq \kappa_0 \delta_1 + \kappa_0 M_0 \delta_0^2 + 4G^4M_0^4\delta_1$$

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and therefore we can estimate
\[
|\langle T_2(b), \mu_1 \mathcal{S}_w^1(w, z_3) + \mu_2 \mathcal{S}_w^2(w, z_3) \rangle| \\
\leq M_0 \delta_0^2 + 8 M_0^2 2^{-\ell} + \kappa_0 \delta_1 + \kappa_0 M_0 \delta_0^2 + 4^G M_0^4 \delta_1 \\
\leq \kappa_0 (1 + 4^{G+2} M_0^4) \delta_1.
\]

Thus picking
\[
A(\varepsilon) \geq \max \{3^{3/\varepsilon} + M_0^3, \kappa_0 (1 + 4^{3/\varepsilon} + 2 M_0^4)\} (46)
\]
the Lemma is proven.

5.2 Proof of Proposition 4

Fix an \( I \in \mathcal{I}_J \) and pick \( b_I \in I \). Let \( m_{A,b_I,\delta_1} \) be a multiplier equal to 1 on \( \Pi_{2I,b_I}(\delta_1) \) which vanishes on \( \Pi_{3I,b_I}(\delta_1) \). Let
\[
P_{k,A,b_I,\delta_1} f(\xi) = m_{A,b_I,\delta_1} (2^k \xi) \hat{f}(\xi).
\]
Then by Bourgain–Demeter decoupling on the cone,
\[
\left\| \sum_I P_{k,A,b_I,\delta_1} T_{k,\ell} f_I \right\|_p \leq C(\varepsilon, A)(\delta_0/\delta_1)^{1/2-1/p+\varepsilon} \left( \sum_I \left\| T_{k,\ell} f_I \right\|_p \right)^{1/p},
\]
for \( 2 \leq p \leq 6 \). The Schwartz kernel of the operator \( f \mapsto (I - P_{k,A,b_I,\delta_1}) T f \) is given by a sum of kernels \( \sum_{n=0}^{\infty} K_{n,k,\ell}(w, z) \), where
\[
K_{n,k,\ell,b_I}(w, z) = 2^{2k} \int \int \int e^{2\pi i \Psi(w, v, z, \mu, \xi)} \sigma_1(v, z, \mu) \sigma_{n,2}(\xi) dv d\xi d\mu,
\]
the phase function \( \Psi \) is given by
\[
\Psi(w, v, z, \mu, \xi) = \langle w - v, \xi \rangle + 2^k \mu \cdot (\mathcal{S}(v, z_3) - z')
\]
and the symbols \( \sigma_1, \sigma_{n,2} \) are given by
\[
\sigma_1(v, z, \mu) = \sigma_{l,\varepsilon_1}(v) \chi_1(2^l \alpha(v, z_3) \mu \cdot \Delta(v, z_3)) \xi(v, z, \mu),
\]
\[
\sigma_{n,2}(\xi) = (1 - m_{A(\varepsilon),b_I,\delta_1}(2^k \xi)) \chi_n(|\xi|)
\]
Note that the symbol of \( K_{n,k,\ell} \) is supported where \(|\xi| \sim 2^n \) for \( n \geq 1 \) (with obvious modifications for \( n = 0 \), \(|\mu| \sim 1 \), \(|v| + |z| \leq r \), and for a priori unbounded \( w \).

We prove the following lemma to reduce to the case when \(|\xi| \sim 2^k \).
Lemma 4 There exists a constant $C_1 > 0$ such that $|k - n| > C_1$ implies that for $N > 1$

$$|K_{n,k,\ell,b_I}(w,z)| \leq C_{N,e} \frac{1}{(1 + |w|)^4} 2^{-N(k+n)} 2^{[]_t r_r}(|z|).$$  \(47\)

If $|n - k| < C_1$, we can apply integration by parts using the fact that $2^{-k} \xi$ is bounded away from the plate $\Pi_{A(\varepsilon),b_I(\delta_1)}$ while $\mu \cdot \mathcal{S} w$ lies in $\Pi_{A(\varepsilon),b_I(\delta_1)}$ to obtain lower bounds on $|\Psi_v|$. In particular, we prove the following estimate.

Lemma 5 If $|n - k| < C_1$ then

$$|K_{n,k,\ell,b_I}(w,z)| \leq C_{\varepsilon} 2^{-11k} \frac{1}{(1 + |w|)^4} 2^{-N(k+n)} 2^{[]_t r_r}(|z|).$$

Together the estimates in Lemmas 4 and 5 along with the compact support of $K_{n,k,\ell,b_I}(w,z)$ in $z$ imply

$$\sup_{w} \int |K_{n,k,\ell,b_I}(w,z)| dw + \sup_{w} \int |K_{n,k,\ell,b_I}(w,z)| dz \leq C_{\varepsilon} 2^{-11k-n}.$$ 

Thus

$$\left\| \sum_{I \in \mathcal{I}_J} (\text{Id} - P_{k,A(\varepsilon),b_I,\delta_1}) [\sigma_{\ell,\ell_1} T_{k,\ell} g_I] \right\|_p = \sum_{I \in \mathcal{I}_J} \sum_{n \geq 0} \left\| \int K_{n,k,\ell,b_I}(\cdot,z) g_I(z) dz \right\|_p$$

and applying Young’s inequality and the almost disjoint support of $\{g_I\}_{I \in \mathcal{I}_J}$

$$\sum_{I \in \mathcal{I}_J} \sum_{n \geq 0} \left\| \int K_{n,k,\ell,b_I}(\cdot,z) g_I(z) dz \right\|_p \leq \sum_{I \in \mathcal{I}_J} \sum_{n \geq 0} C_{\varepsilon} 2^{-11k-n} \|g_I\|_p \leq C_{\varepsilon} 2^{-11k} \|g\|_p.$$ 

This will complete the proof of Proposition 4.

5.2.1 The Proof of Lemma 4

First, we integrate by parts in the $\xi$ variables with the differential operator

$$L_\xi = \langle \frac{w-v}{|w-v|^2}, \nabla_\xi \cdot \rangle,$$

which will give the desired decay in $w$. Note that $\nabla_\xi \Psi = w - v$, and

$$|\partial^\beta \sigma_{n,2}| \leq C_{|\beta|} \min\{A(\varepsilon)^{-1} \delta^2 \delta_1^2, 2^n\}^{-|\beta|} \leq C_{|\beta|} A(\varepsilon)^{|\beta|}$$
for any multi-index $\beta$ with $|\beta| \geq 1$. Thus applying integration by parts many times with the operator $L_\xi$ gives the bound

$$\left| (L_\xi^*)^N \sigma_{n,2}(\xi) \right| \leq \frac{CN A(\varepsilon)^N}{|w-v|^N}$$

for any $N > 0$. Since $\sigma_{n,2}$ is bounded and supported where $|\xi| \simeq 2^n$, we obtain an estimate

$$\left| \int e^{2\pi i \Psi(w,v,z,\mu,\xi)} \sigma_{n,2}(\xi) d\xi \right| \leq C_N \frac{2^{3n}}{(1 + A(\varepsilon)^{-1}|w-v|)^N}, \quad (48)$$

allowing us to later integrate in $w$.

By the implicit function theorem, there is a constant $C_1 > 0$ such that if $|n-k| > C_1$ then

$$|\nabla_v \Psi| = | - \xi + 2^k \nabla_v (\mu \cdot \mathcal{G}(v, z_3))| \geq ||\xi| - |2^k \nabla_v (\mu \cdot \mathcal{G}(v, z_3))|| \geq C_0 \max\{2^k, 2^n\}$$

for some $C_0 > 0$. We also see that $|\partial^\beta \psi| \leq A(\varepsilon)2^k$ for any multi-index $\beta$ with $|\beta| \geq 2$, and $|\partial^\beta \sigma| \leq C|\beta|2^{k|\beta|}$ for any multi-index $\beta$ with $|\beta| \geq 1$. Since $\ell < k/2$, integrating by parts in the $v$ variables with the differential operator $L_v = \langle \nabla_v \psi, \nabla_v \cdot \rangle$ gives the estimate

$$\left| (L_v^*)^N \sigma_1(v, z, \mu) \right| \leq C_N \frac{A(\varepsilon)2^\ell}{C_0 \max\{2^k, 2^n\}} \leq C_N A(\varepsilon) \max\{2^{k/2}, 2^{n/2}\} - N.$$ 

Combining this estimate with (48), we obtain

$$|K_{n,k,\ell,b_1}(w,z)| \leq \int \int \left| \int e^{2\pi i \Psi(w,v,z,\mu,\xi)} \sigma_{n,2}(\xi) d\xi \right| (L_v^*)^{2N} \sigma_1(v, z, \mu) d\mu d\nu \leq C_N \int \int \frac{A(\varepsilon)}{\max\{2^k, 2^n\}^N (1 + A(\varepsilon)^{-1}|w-v|)^N} d\mu d\nu.$$ 

As $\sigma_1(v, z, \mu)$ is supported where $|v| + |z| + |\mu| \leq 6$ by loss of a constant depending on $\varepsilon$, we can integrate in $v$ and $\mu$ to obtain (47).

### 5.2.2 The Proof of Lemma 5

Suppose that $|\langle T_2(b_1), \xi \rangle| \geq 3A(\varepsilon)2^k \delta_1$. Define $\partial T_2(b_1) = \langle T_2(b_1), \nabla_v \cdot \rangle$. Then by (42)

$$|\partial T_2(b_1) \Psi| \geq 2A(\varepsilon)2^k \delta_1.$$
We can also estimate for $j \geq 1$

$$|\partial_T^{j} T_{2}(b_I) \sigma_1| \leq C_j A(\varepsilon) 2^{\ell j},$$

and for $j \geq 2$

$$|\partial_T^{j} T_{2}(b_I) \Psi | \leq C_j A(\varepsilon) 2^{k} \leq C_j A(\varepsilon) 2^{(j-1)2^k} 2^k \delta_1.$$

Thus integrating by parts many times in the $T_2(b_I)$ direction and applying the estimate (48), we obtain

$$|K_{n,k,\ell,b_I}(w,z)| \leq C_N \int \int \frac{2^{3k}}{(1+A(\varepsilon)^{-1}|w-v|)^{4}} \frac{1}{(2^{k-\ell} \delta_1)^{N}} dv d\mu.$$  

Since $2^{k-\ell} \delta_1 \geq 2^{k\varepsilon/2}$, integrating by parts in the $T_2(b_I)$ direction $\approx 10/\varepsilon$ times and integrating over the compact support of $\sigma_1$ in $v, \mu$ gives the required estimate.

Next we assume that $|\langle N(b_I), \xi \rangle| \geq 3 A(\varepsilon) 2^k \delta_1^2$. Define $\partial N(b_I) = \langle N(b_I), \nabla v \rangle$.

Note that (43) implies $|\partial N(b_I) \Psi| \geq 2 A(\varepsilon) 2^k \delta_1^2$. We claim that $|\partial_{N(b_I)} \sigma_1(v,z,\mu)| \leq C_j A(\varepsilon) \max\{2^{\ell} \delta_0, \varepsilon_1^{-1}\}^j (50)$ for every $j \geq 1$. To see this, we use the approximation $N(b_I) = -b_I e_1 + e_3 + C(b_I) b_I^2$, where $|C(b_I)| < M_0$, from (36). From the definition of $\sigma_1$ we see for every $j \geq 1$ and every multi-index $\beta$ with $|\beta| \leq j$ that

$$|(b_I \partial_{\alpha_1})^{j-|\beta|} C(b_I)^{|\beta|} b_I^{2|\beta|} \partial_{v}^{|\beta|} \sigma_1(v, z, \mu)| \leq C_j (2^{\ell} \delta_0)^{j-|\beta|} (2^k \delta_0^2)^{|\beta|} \leq C_j 2^{\ell} \delta_0^j. \tag{51}$$

Thus it suffices to check that (50) holds for mixed derivatives of the form

$$|b_I^{|\beta|} \partial_{v_3}^{j-|\beta|} \partial_{v}^{|\beta|} \sigma_1(v, z, \mu),$$

where $v' = (v_1, v_2)$, and $\beta$ is a two-dimensional multi-index such that $|\beta| < j$. Note that

$$|b_I^{|\beta|} \partial_{v_3}^{j-|\beta|} \partial_{v}^{|\beta|} \sigma_1(v, z, \mu)| \leq C_j (2^\ell \delta_0)^{|\beta|} |\beta|^{j-|\beta|} \leq C_j 2^{\ell} \delta_0^j,$$

$$|b_I^{|\beta|} \partial_{v_3}^{j-|\beta|} \partial_{v}^{|\beta|} \xi(v, z, \mu)| \leq C_j \delta_0^{|\beta|},$$

so it suffices to estimate

$$|b_I^{|\beta|} \partial_{v_3}^{j-|\beta|} \partial_{v}^{|\beta|} \chi_1(2^{\ell} \alpha(v,z_3) \mu \cdot \Delta(v,z_3)).$$
Note that terms for which no derivative hits $\mu \cdot \Delta(v, z_3)$ will be negligible since $|\mu \cdot \Delta(v, z_3)| \simeq 2^{-\ell}$. Using (26), (19), and a Taylor expansion about $z_3 = 0$, we see that

$$
|b_I| |\beta| |\partial_{v_3}^{|\beta|-|\beta|} \partial_v^{|\beta|} \mu \cdot \Delta(v, z_3)| \leq \delta_0 |\beta| |\partial_v^{|\beta|} \mu \cdot \Delta_{v_3}^j(v, 0)| + A(\varepsilon)|\beta|^{\ell+1} = A(\varepsilon)\delta_0 |\beta|^{\ell+1}.
$$

Thus we see by differentiation of compositions and products

$$
|b_I| |\beta| |\partial_{v_3}^{|\beta|-|\beta|} \partial_v^{|\beta|} \sigma_1(v, z, \mu)| \leq C_j C_j A(\varepsilon) \max\{2^{\ell}\delta_0, \varepsilon_1^{-1}\}^j, \quad (52)
$$

and by combining (51) and (52) the claim (50) is proven.

To integrate by parts we also need to show that for $j \geq 2$

$$
|\partial_N^{j}(b_I) \Psi| \leq C_j A(\varepsilon) 2^k \max\{2^{\ell}\delta_0, \varepsilon_1^{-1}\}^{j-1} \delta_1^2.
$$

In fact, we claim that

$$
|\partial_N^{j}(b_I) \Psi| \leq C_j A(\varepsilon) 2^k \delta_0^2 \quad (53)
$$

for $j \geq 2$. We use (36) again to see that

$$
\partial_N^{j}(b_I) \Psi = (b_I e_1 + e_3 + C(b_I) b_I^2, \nabla_v)^j \Psi
$$

where again $|C(b_I)| \leq M_0$. Rearranging terms using the fact that $|\partial_v^\beta \Psi| \leq A(\varepsilon) 2^k$ for any multi-index $\beta$ with $|\beta| \geq 2$ we obtain

$$
\partial_N^{j}(b_I) \Psi = b_I \Psi_{v_1 v_3^{j-1}} + \Psi_{v_3^j} + A(\varepsilon) 2^k b_I^2 R_{10}(v, z_3).
$$

Next, we estimate via a Taylor expansion about $z_3 = 0$,

$$
\Psi_{v_1 v_3^{j-1}}(v, z_3) = \mu \cdot \mathcal{S}_{v_1 v_3^{j-1}}(v, 0) + 2M_j z_3^{11}(v, z_3)
$$

$$
\Psi_{v_3^j}(v, z_3) = \mu \cdot \mathcal{S}_{v_3^j}(v, 0) + z_3 \mu \cdot \mathcal{S}_{v_3 z_3}(v, 0) + 2M_j z_3^{2} R_{12}(v, z_3).
$$

From (23), (24), and (25) we see that

$$
\mu \cdot \mathcal{S}_{v_1 v_3^{j-1}}(v, 0) = 0,
$$

$$
\mu \cdot \mathcal{S}_{v_3^j}(v, 0) = 0,
$$

$$
\mathcal{S}_{v_3 z_3}^1(v, 0) = 0.
$$
Moreover, (18) ensures that

\[ \Theta^2_{v_j^3 z_3} (v, 0) = 0, \ j \geq 2. \]

Hence for \( j \geq 2 \)

\[ |\partial_j N_{(b_I)}^j \Psi | \leq C_j A(\varepsilon) \delta_0^j \leq C_j A(\varepsilon) A^{-j+1} \delta_1^2, \]

satisfying the claim (53).

Now that we have verified the conditions (49), (50), and (53), we integrate by parts \( M \) times in the \( N(b_I) \) direction and apply (48) to obtain

\[ |K_{n,k,\ell,b_I}(w,\varepsilon)| \leq \left( \frac{1}{\min\{2^{k-\ell} \delta_1^2 / \delta_0, 2^{k} \delta_1^2 \varepsilon_1 \}} \right)^M \int \int \frac{1}{(1 + A(\varepsilon) - 1 |w - v|)^{4}} dv d\mu. \]

Since \( \delta_1 \geq 2^{-\ell(1-\varepsilon/2)} \) and \( \delta_1 \geq 2^{-\ell \varepsilon/4} \delta_0 \), we have

\[ \frac{2^{k-\ell} \delta_1^2 / \delta_0}{\geq 2^{k-2\ell+\ell \varepsilon/4} \geq 2^{k \varepsilon/8}} \]
\[ \frac{2^{k} \delta_1^2 \varepsilon_1}{\geq 2^{k-2\ell+\ell \varepsilon/2} \geq 2^{k \varepsilon/4}}. \]

So if \( M \simeq 50 / \varepsilon \) and we integrate over the compact support of \( \sigma_1 \) in \( v \) and \( \mu \) we obtain the desired estimate.

### 5.3 Families of Changes of Variables

We use the family of changes of variables used in [8] to reduce the general case to the model case. Let \( P^\circ = (a^\circ, y^\circ) \in M \), with \( y^\circ = (S^1(a^\circ, b^\circ), S^2(a^\circ, b^\circ), b^\circ) \). For \( r > 0, q > 0 \) let

\[ Q(r) = \{(x_1, x_2, x_3) : |x - a^\circ| \leq r \} \]

and

\[ I(r) = \{y_3 : |y_3 - b^\circ| \leq r \}. \]

For \( i = 1, 2 \), let \( S^i \) be smooth functions in a neighborhood of \( Q(2r_0) \times I(2r_0) \), for some \( r_0 > 0 \). We assume that \( \Delta_1(x, y_3) = \det(S^1_x, S^2_x, S^1_{xy_3}) \neq 0 \) on \( Q(2r_0) \times I(2r_0) \). Choose \( M > 0 \) so that

\[ M > 2 + \|S\|_{C^5(Q(2r_0) \times I(2r_0))} + \max_{(x,y_3) \in Q(2r_0) \times I(2r_0)} \left| \frac{\Delta_1(x, y_3)}{\Delta_1(x, y_3)} \right|^{-1} \]
For $a \in Q(2r_0), b \in I(2r_0)$, let

$$
\Gamma_1(x, y_3) = \det(S^1_x, S^2_{xy}, S^1_{y_3y_3})
$$

$$
\Gamma_2(x, y_3) = \det(S^1_{xy}, S^1_x, S^2_{y_3y_3}),
$$

and let $\rho(a, b) \in \mathbb{R}^3$ be defined by

$$(\rho_1, \rho_2, \rho_3) := \frac{1}{\Delta_1(a, b)}(-\Gamma_2(a, b), \Gamma_1(a, b), \Delta_2(a, b)).$$

For $(x, y_3), (a, y_3) \in Q(r_0) \times I(2r_0)$, consider the map

$$(x, a, y_3) \mapsto \varpi(x, a, y_3) \in \mathbb{R}^3$$

given by

$$
\varpi_1(x, a, b) = S^1(x, b) - S^1(a, b)
$$

$$
\varpi_2(x, a, b) = S^2(x, b) - \rho_3(a, b)S^1(x, b) - S^2(a, b) + \rho_3(a, b)S^1(a, b)
$$

$$
\varpi_3(x, a, b) = S^1_{y_3}(x, b) - S^1_{y_3}(a, b).
$$

Then

$$
\det(D\varpi/Dx) = \det(S^1_x, S^2_x - \rho_3(a, b)S^1_x, S^1_{y_3y_3}, S^1_{y_3y_3}(x, b)) = \Delta^1(x, b) \neq 0.
$$

By the implicit function theorem, there exists $r_1 > 0$ with $r_1 < r_0$ such that $|w|_\infty < 2r_1, a \in Q(2r_1), b \in I(2r_1)$, the equation $\varpi(x, a, b) = w$ is solved by a unique $C^\infty$ function $x = \varphi(w, a, b)$ such that for $|x|_\infty < (50M^5)^{-1}r_1, \varphi(\varpi(x, a, b), a, b) = x$.

We also change variables in $y$. Define $\varphi : \mathbb{R}^2 \times Q(2r_0) \times I(2r_0)^2 \to \mathbb{R}^3$ by

$$
\varphi_1(y, a, b) = y_1 - S^1(a, y_3)
$$

$$
\varphi_2(y, a, b) = y_2 - \rho_3(a, b)y_1 - S^2(a, y_3)
$$

$$
+ \rho_3(a, b)S^1(a, y_3) - (y_3 - b) \sum_{i=1}^2 \rho_i(y_i - S^i(a, y_3))
$$

$$
\varphi_3(y, a, b) = y_3 - b.
$$

For $z_3, b \in I(r_3), a \in Q(2r_0)$, where $r_3 < \min\{r_1, (24M^4)^{-1}\}$, we define the inverse $z \mapsto \eta(z, a, b)$ by

$$
\eta_1(z, a, b) = z_1 + S^1(a, b + z_3)
$$

$$
\eta_2(z, a, b) = \frac{z_2 + z_1(\rho_3(a, b) + \rho_1(a, b)z_3) + (1 - z_3)S^2(a, b + z_3)}{1 - \rho_2(a, b)z_3}
$$

$$
\eta_3(z, a, b) = b + z_3.
$$
Lemma 6 [8] The function \( \tau, \eta \) defined above have the following properties.

1. \( \tau(0, a, b) = a, \eta(0, a, b) = (S^1(a, b), S^2(a, b), b) \), \( \eta_3(z, a, b) = b + z_3 \).

2. \( \det \left( \frac{D\tau(w, a, b)}{dw} \right) = -\frac{1}{\Delta^3(\tau(w, a, b), y_3)} \).

3. Let \( \rho = \rho(a, b) \), and let

\[
B(z_3, a, b) = \begin{pmatrix}
-\rho_3 - \rho_1 z_3 & 0 \\
1 - \rho_2 z_3 & 1
\end{pmatrix}.
\]

Then for \( |z_3| \leq r_3, a \in Q(2r_1, 2q_1), |w| \leq r_1 \)

\[
B(z_3, a, b) \left( S^1(\tau(w, a, b), b + z_3) - \eta_1(z, a, b) \right)
\]

\[
S^2(\tau(w, a, b), b + z_3) - \eta_2(z, a, b) \right) = \left( \mathcal{S}^1(w, z_3, a, b) - z_1 \right)
\]

\[
\mathcal{S}^2(w, z_3, a, b) - z_2
\]

where \( \mathcal{S}^i \) are \( C^\infty \) with

\[
(\mathcal{S}^1, \mathcal{S}^2, \mathcal{S}^1_{z_3})|_{(w, 0, a, b)} = w
\]

and \( \mathcal{S}^2_{w, z_3}(0, 0, a, b) = 0 \).

4. Let

\[
\Delta_S^i(x, y_3) = \det(\mathcal{S}^i_x, \mathcal{S}^i_x, \mathcal{S}^i_{xy_3})|_{(x, y_3)}
\]

\[
\Delta^i_\mathcal{S}(x, y_3, a, b) = \det(\mathcal{S}^1_w, \mathcal{S}^2_w, \mathcal{S}^i_{w, z_3})|_{(w, z_3, a, b)}
\]

Then for \( (\tau_1, \tau_2) = (\mu_1, \mu_2)B(z_3, a, b) \),

\[
\sum_{i=1}^{2} \tau_i \Delta_S^i(\tau(w, a, b), b + z_3) = \frac{\Delta_1^i(\tau(w, a, b), b)}{1 - \rho_2(a, b)z_3} \sum_{i=1}^{2} \mu_i \Delta^i_\mathcal{S}(w, z_3, a, b).
\]

5. Let

\[
\kappa(a, b) = \Gamma_2 \Delta^1_S - \Gamma_1 \Delta^2_S + \Delta^1_S \Delta^2_S, y_3 + \Delta^2_S \Delta^1_S, y_3|_{(a, b)}.
\]

Then

\[
\mathcal{S}^2_{w, z_3, z_3}(0, 0, a, b) = \frac{\kappa(a, b)}{\Delta_S^i(a, b)}.
\]

The proof is found in [8].

5.4 Decoupling in the General Case

Proposition 5 Let \( 0 < \varepsilon < \frac{1}{10}, k \geq 1, \ell \leq k/2 \). Let \( \delta_0 \in (2^{-\ell(1-\varepsilon)}, 2^{-\ell\varepsilon}) \), and define \( 0 < \delta_1 < \delta_0 \) such that

\[
\max\{2^{-\ell(1-\varepsilon)/2}, \delta_0 2^{-\ell\varepsilon/4}\} < \delta_1 < \delta_0.
\]
Define $\varepsilon_1 = (\delta_1/\delta_0)^2$. Let $J$ be an interval of length $\delta_0$ within $r_0$ of $b^0$, and let $I_J$ be a collection of intervals which have disjoint interior, intersecting $J$. For each $I \in I_J$, define $f_I(y) = f(y) 1_{J}(y_3)$. Then for $2 \leq p \leq 6$

$$ \left\| \sum_{I \in I_J} R_{k,\ell} f_I \right\|_p \leq C_p (\delta_0/\delta_1)^{1/2 - 1/p + \varepsilon} \left( \sum_{I \in I_J} \left\| R_{k,\ell} f_I \right\|_p^{p^*} \right)^{1/p} + C_{N,\varepsilon} 2^{-kN} \left\| f \right\|_p. $$

**Proof** Fix $b \in J$. Let $\sigma_0 \in C^\infty_c$ supported in $(-1, 1)$ such that $\sigma_0 \geq 0$ everywhere and $\sum_{n \in \mathbb{Z}} \sigma_0(-n) = 1$. For $n \in \mathbb{Z}$ define $\sigma_n(x) = \sigma_0(x - n)$ and for $a \in \mathbb{Z}^3$ let

$$ \varsigma_{\ell, \varepsilon_1}(x, a, b) = \sigma_{a_1}(2^\ell w_1(x, a, b)) \sigma_{a_2}(2^\ell w_2(x, a, b)) \sigma_{a_3}(\varepsilon_1^{-1} w_3(x, a, b)). $$

Then since $|x|, |a| < r_2/2$ and $|b| < r_3/2$, we have that

$$ \varsigma_{\ell, \varepsilon_1}(x(w, a, b), a, b) = \sigma_{a_1}(2^\ell w_1(x, a, b)) \sigma_{a_2}(2^\ell w_2(x, a, b)) \sigma_{a_3}(\varepsilon_1^{-1} w_3) =: \varsigma_{\ell, \varepsilon_1}(w, a), $$

and $\sum_{a \in \mathbb{Z}^3} \varsigma_{\ell, \varepsilon_1}(w, a) = 1$ with finite overlap for all $|w| < r_2/2$. Then by Hölder’s inequality

$$ \left\| \sum_{I \in I_J} R_{k,\ell} f_I \right\|_p \leq C_p \left( \sum_{a \in \mathbb{Z}^3} \left\| \sum_{I \in I_J} \varsigma_{\ell, \varepsilon_1, a, b} R_{k,\ell} f_I \right\|_p^{p^*} \right)^{1/p}. $$

Note that the terms vanish for $|a| > r_2$. Fix $a$. Write $g(z, a, b) = f(\eta(z, a, b))$. Apply changes of variables $y = \eta(z, a, b)$ and $\tau = B^T^{-1}(z_3, a, b)\mu$, noting that

$$ \det(D\eta/dz) \det B = 1 $$

so that

$$ R_{k,\ell} f(x) = 2^{2k} \int e^{i2k(\mu, \eta(x, a, b) - z')} \tilde{\chi}_{k,\ell}(x, z, \mu, a, b) g(z, a, b) d\mu dz, $$

with

$$ \tilde{\chi}_{k,\ell}(x, z, \mu, a, b) = \chi(x, \eta(z, a, b)) \eta_1(|B^T^{-1}(z_3, a, b)\mu|) \eta \left( 2^{\ell} \frac{\Delta_1^1(x)}{1 - \rho_3(a, b)z_3} \right) \times (\mu_1 \Delta_{\Theta}^1(\eta(x, a, b), z_3, a, b) + \mu_2 \Delta_{\Theta}^2(\eta(x, a, b), z_3, a, b)). $$

Thus we see that

$$ \varsigma_{\ell, \varepsilon_1, a, b}(\tau(w, a, b)) \sum_{I \in I_J} R_{k,\ell} f_I(\tau(w, a, b)) = \sigma_{\ell, \varepsilon_1, a, b}(w) \sum_{I \in I_J} T_{k,\ell, a, b} g_I(w), $$
where \( g_I(z, a, b) = g(z, a, b) \perp_{b+1} (z_3) \) and \( T_{k, \ell, a, b} = T_{k, \ell} \) from the model case. Define

\[
M_n(a, b) \geq 2 + \| \mathfrak{S}^1(\cdot, a, b) \|_{C^{n+5}([-\tau_0, \tau_0]^4)} + \| \mathfrak{S}^2(\cdot, a, b) \|_{C^{n+5}([-\tau_0, \tau_0]^4)}
\]

\[
\tilde{A}(\varepsilon) = \sup_{a, b \in [-\tau_0, \tau_0]^4} \max \{ 3^{[3/\varepsilon]} + 2 M_{[3/\varepsilon]}^4(a, b), \kappa_0(a, b)(1 + 4^{[3/\varepsilon]} + 2 M_{[3/\varepsilon]}^4(a, b)) \};
\]

these are the uniform versions of (14) and (46), respectively. We can then write

\[
\left\| \mathcal{S}_{\ell, 1, a, b} \sum_{I \in \mathcal{I}_J} R_{k, \ell} f_I \right\|_p
= \left( \int \left| \mathcal{S}_{\ell, 1, a, b}(w, a, b) \right| \sum_{I \in \mathcal{I}_J} R_{k, \ell} f_I(w, a, b) \right|_p | \det(D_{\varepsilon w}) | d w \right)^{1/p}
\]

by the uniform upper bound on \( | \det(D_{\varepsilon w}) | \). Then we can apply Proposition 4 with \( A(\varepsilon) = \tilde{A}(\varepsilon) \) to get

\[
\left\| \mathcal{S}_{\ell, 1, a, b} \sum_{I \in \mathcal{I}_J} T_{k, \ell} g_I \right\|_p \leq C_{\varepsilon} (\delta_0/\delta_1)^{1/2 - 1/p + \varepsilon} \left( \sum_{I \in \mathcal{I}_J} \left\| \mathcal{S}_{\ell, 1, a, b} T_{k, \ell} g_I \right\|_p \right)^{1/p}
+ C_{\varepsilon} 2^{-10k} 2^{-2\ell} \varepsilon_1 \| g \|_p.
\]

Then undoing the changes of variables above (and applying the uniform lower bounds on \( | \det(D_{\varepsilon w}) | \)), we may bound this by

\[
C'_{\varepsilon} (\delta_0/\delta_1)^{1/2 - 1/p + \varepsilon} \left( \sum_{I \in \mathcal{I}_J} \left\| \mathcal{S}_{\ell, 1, a, b} R_{k, \ell} f_I \right\|_p \right)^{1/p}
+ C_{\varepsilon} 2^{-10k} 2^{-2\ell} \varepsilon_1 \| f \|_p.
\]

Finally, we recombine our partition of unity in \( x \) using the fact that there are at most \( C 2^{2\ell} \varepsilon_1^{-1} \) many \( a \in \mathbb{Z}^3 \) for which \( \sigma_{\ell, 1, a, b} \) is nonzero, to get

\[
\left\| \sum_{I \in \mathcal{I}_J} R_{k, \ell} f_I \right\|_p \leq C_p \left( \sum_{a \in \mathbb{Z}^3} \left\| \sum_{I \in \mathcal{I}_J} \mathcal{S}_{\ell, 1, a, b} R_{k, \ell} f_I \right\|_p \right)^{1/p}
\leq C_p C_{\varepsilon} (\delta_0/\delta_1)^{1/2 - 1/p + \varepsilon} \left( \sum_{a \in \mathbb{Z}^3} \sum_{I \in \mathcal{I}_J} \left\| \mathcal{S}_{\ell, 1, a, b} R_{k, \ell} f_I \right\|_p \right)^{1/p}
+ \sum_{a \in \mathbb{Z}^3, |a|_\infty < r_2} C_{\varepsilon} 2^{-2\ell} \varepsilon_1 2^{-10k} \| f \|_p
\]
by applying Proposition 5 we get
\[
L^p_j \leq C_\varepsilon (\delta_0/\delta_1)^{1/2-1/p+\varepsilon} \left( \sum_{l \in \mathcal{I}_j} \| \mathcal{R}_{k,\ell} f_I \|_p \right)^{1/p} + C_\varepsilon 2^{-10^k} \| f \|_p.
\]

\[ \square \]

### 5.5 Iteration of the Decoupling Step

Let \( \delta_0 = 2^{-\ell \varepsilon} \), and define \( \delta_j = \delta_{j-1} 2^{-\ell \varepsilon/4} \) for \( j = 1, 2, \ldots \). Note that this implies \( \varepsilon_1 = (\delta_1/\delta_0)^2 = 2^{-\ell \varepsilon/2} \). We will iterate the estimate in Proposition 5 until \( \delta_j \leq 2^{-\ell (1-\varepsilon)} \). Let \( j^* \) be the smallest \( j \) such that \( \delta_j < 2^{-\ell (1-\varepsilon)} \). Clearly \( j^* \leq 1/\varepsilon \) and \( 2^{-\ell (1-\varepsilon)/2} \leq \delta_{j^*} \).

For \( j = 0, 1, 2, \ldots \) let \( I_j \) denote an interval of length \( \delta_j \) inside \([b^0 - r_0, b^0 + r_0]\), and let \( \mathcal{I}_j \) denote the collection of intervals \( I_{j+1} \) of length \( \delta_j \) intersecting \( I_j \) with disjoint interior. Finally, let \( J = [b^0 - r_0/2, b^0 + r_0/2] \) and let \( \mathcal{I}_{j,j} \) denote the collection of intervals \( I_j \) of length \( \delta_j \) intersecting \( J \) with disjoint interiors. Then since \( r_0 < 1 \) and \( \delta_0 = 2^{-\ell \varepsilon} \), using Hölder’s and Minkowski’s inequalities, we have

\[
\| \mathcal{R}_{k,\ell} f \|_p \lesssim 2^{\ell \varepsilon/p'} \left( \sum_{l_0 \in \mathcal{I}_j,0} \| \mathcal{R}_{k,\ell} f_{l_0} \|_p \right)^{1/p}.
\]

(54)

The function and operator \( \mathcal{R}_{k,\ell} f_{l_0} \) now satisfy the conditions of Proposition 5. We claim that for each \( 0 \leq j \leq j^* \),

\[
\| \mathcal{R}_{k,\ell} f \|_p \lesssim C(\varepsilon) j^{2\ell \varepsilon/(p')} (\delta_0/\delta_j)^{1/2-1/p+\varepsilon} \left( \sum_{l_j \in \mathcal{I}_j,j} \| \mathcal{R}_{k,\ell} f_{l_j} \|_p \right)^{1/p} + j^{2\ell} C(\varepsilon)^j 2^{-10^k} \| f \|_p.
\]

(55)

The case \( j = 0 \) follows immediately from (54). Assume (55) holds for some \( j \). Then by applying Proposition 5 we get

\[
\left( \sum_{l_j \in \mathcal{I}_j,j} \| \mathcal{R}_{k,\ell} f_{l_j} \|_p \right)^{1/p} \leq \left( \sum_{l_j \in \mathcal{I}_j,j} \left[ C(\varepsilon) \left( \frac{\delta_j}{\delta_{j+1}} \right)^{1/2-1/p+\varepsilon} \left( \sum_{l_{j+1} \in \mathcal{I}_j,j} \| \mathcal{R}_{k,\ell} f_{l_{j+1}} \|_p \right) \right]^{1/p} \right)^{1/p} + C(\varepsilon) 2^{-10^k} \| f \|_p \]

\[
\leq C(\varepsilon) \left( \frac{\delta_j}{\delta_{j+1}} \right)^{1/2-1/p+\varepsilon} \left( \sum_{l_{j+1} \in \mathcal{I}_j,j+1} \| \mathcal{R}_{k,\ell} f_{l_{j+1}} \|_p \right)^{1/p} + C(\varepsilon) 2^{-10^k} \| f \|_p.
\]

(56)
Plugging the above estimate into (55) gives us

\[
\|R_{k, \ell}f\|_p \leq C(\varepsilon)^{j+1} 2^{\ell \varepsilon / (p')} \left( \frac{\delta_0}{\delta_j} \right)^{1/2 - 1/p + \varepsilon} \left( \sum_{I_{j+1} \in \mathcal{I}_{j+1}} \|R_{k, \ell} f_{I_{j+1}}\|_p \right)^{1/p} \\
+ C(\varepsilon)^j 2^{\ell \varepsilon / (p')} \left( \frac{\delta_0}{\delta_j} \right)^{1/2 - 1/p + \varepsilon} C(\varepsilon)^{-1/p} 2^{-10k} \|f\|_p \\
+ j 2^{\ell} C(\varepsilon)^j 2^{-10k} \|f\|_p.
\]

Using the fact that \(\delta_0 = 2^{-\ell \varepsilon}\), \(\delta_j \geq 2^\ell (1 - \varepsilon/2)\) for \(j \leq j^*\), and \(2 \leq p \leq 6\), the last two terms of the above inequality are bounded by

\[(j + 1) C(\varepsilon)^{j+1} 2^{\ell} 2^{-10k} \|f\|_p,
\]

proving the claim.

We apply (55) for \(j = j^*\) and use the fact that \(j^* \leq 4/\varepsilon\) as well as the assertion

\[\frac{\varepsilon}{p'} - \frac{\varepsilon}{2} + \frac{\varepsilon}{p} - \varepsilon^2 - \frac{\varepsilon}{4} + \frac{\varepsilon}{2p} + \frac{\varepsilon^2}{2} \leq \varepsilon
\]

to deduce

\[
\|R_{k, \ell}f\|_p \leq C(\varepsilon)^{4/\varepsilon} 2^{\ell \varepsilon / p'} 2^{-\ell \varepsilon} \left( \frac{1}{p' + \varepsilon} \right) 2^\ell (1 - \frac{1}{p' + \varepsilon}) \left( \sum_{I_{j^*} \in \mathcal{I}_{j^*}} \|R_{k, \ell} f_{I_{j^*}}\|_p \right)^{1/p} \\
+ \frac{4}{\varepsilon} C(\varepsilon)^{4/\varepsilon} 2^{-10k + 2\ell} \|f\|_p \\
\lesssim_\varepsilon 2^{\ell (1/2 - 1/p + 2\varepsilon)} \left( \sum_{I_{j^*} \in \mathcal{I}_{j^*}} \|R_{k, \ell} f_{I_{j^*}}\|_p \right)^{1/p} + C(\varepsilon) 2^{-9k} \|f\|_p.
\]

Picking \(\varepsilon' = 2\varepsilon\) completes the proof.

### 6 \(L^2\) Estimates

The methods in this section draw from the work of Comech in [15,17], which was itself influenced by the estimates proven in [23]. While Comech proved \(L^2\) regularity estimates for fold and finite type conditions, here we prove \(L^2\) estimates for a general class of oscillatory integral operators associated with fold blowdown singularities in \(d\) dimensions. Let

\[
A_k f(x) = \int e^{i2k\phi(x,y)} f(y) \sigma(x, y) dy,
\]
where \( x, y \in \mathbb{R}^d, \phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d), \) and \( \sigma \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}^d) \). The canonical relation associated with this oscillatory integral operator is given by

\[
\{ (x, \phi_x) \times (y, \phi_y) : x \in \mathbb{R}^d, y \in \mathbb{R}^d \}
\]

We write the projections \( \pi_L : (x, y) \mapsto (x, \phi_x) \) and \( \pi_R : (x, y) \mapsto (y, \phi_y) \). The projections are degenerate on the variety \( \mathcal{L} \) where the determinant of the mixed Hessian of \( \phi \) vanishes. Let \( h(x, y) = \det \phi_{xy} \). We assume that \( \pi_L \) is a fold and \( \pi_R \) is a blowdown on \( \mathcal{L} \). We may choose the support of \( \sigma \) small enough and choose coordinates \( x = (x', x_d), y = (y', y_d) \) in \( \mathbb{R}^{d-1} \times \mathbb{R} \) vanishing at a reference point \( P^o = (x^o, y^o) \) so that

\[
\phi_{x'y'}(P^o) = I_{d-1}, \quad \phi_{x'dy'}(P^o) = 0, \quad \phi_{x'y'd}(P^o) = 0, \quad \phi_{x'dy'd}(P^o) = 0.
\]

Let \( \phi^{y'} = \phi^{-1}_{x'y'} \), and we can define the kernel fields

\[
V_R = \partial_{xd} - \phi_{x'dy'}(\phi^{y'}x') \partial_{x'} \\
V_L = \partial_{yd} - \phi_{x'y'd} \phi^{y'}x' \partial_{y'}.
\]

By the assumption on \( \pi_L \), in other words that \( \phi_{xy} \) has corank at most 1 and

\[
h(x, y) = 0 \implies |V_L h(x, y)| \geq c_L > 0.
\]

Since we assume that \( V_R \) is a blowdown, i.e., that \( V_R \) is tangent to the singularity surface \( \mathcal{L} \), we see that

\[
h(x, y) = 0 \implies V_R^j h(x, y) = 0 \forall j \geq 0.
\]

Assuming small enough support of \( \sigma \) we may assume that for \( (x, y) \in \text{supp}(\sigma) \)

\[
\max\{ |\phi_{y'd}(x, y)|, |\phi_{x'dy'}(x, y)| \} < \varepsilon.
\]

Note that this implies

\[
|(V_L - \partial_{yd}) h(x, y)| \leq \varepsilon \|\phi\|_{C^3}.
\]

We decompose first in distance to the singularity surface \( \mathcal{L} \); for \( \ell \leq \ell_0 = \left\lfloor \frac{k}{2 + \varepsilon} \right\rfloor < \frac{k}{2} \), we define

\[
A_{k, \ell} f(x) := \int e^{i2^k \phi(x,y)} f(y) \sigma(x,y) \chi(2^\ell h(x,y)) dy.
\]

**Theorem 3** For \( \varepsilon > 0 \) and \( \ell \leq \ell_0 \)

\[
\|A_{k, \ell} f\|_2 \leq C_\varepsilon 2^{(\ell - d k)/2 + \ell \varepsilon} \|f\|_2.
\]
To prove Proposition 2 for $R_{k,\ell}$, we apply a partial Fourier transform (in the $y'$ variables) then change variables $2^k\mu = \tau$, which satisfies the conditions for Theorem 3 with $d = 3$.

### 6.1 Proof of Theorem 3

First we note that if Theorem 3 holds for $\ell < \ell_0$, then the global estimate

$$\left\| \sum_{\ell \leq \ell_0} A_{k,\ell} f \right\|_2 \lesssim 2^{\frac{k}{4} - d^k/2} \| f \|_2.$$ 

from [4] implies the result for $\ell = \ell_0$.

Since $k = (2 + \varepsilon)\ell_0$, we see $\frac{k}{4} \leq \frac{\ell}{2} + \varepsilon \frac{\ell}{2}$, hence by triangle inequality

$$\| A_{k,\ell_0} f \|_2 \lesssim 2^{(\ell - d)k/2 + \ell\varepsilon} \| f \|_2.$$ 

For $\ell < \ell_0$, we decompose our operator further and use methods of the proof of the Calderon–Vaillancourt theorem, following the ideas of Comech in [15]. We decompose our operator along small boxes in $y$-space, by way of cutoffs

$$\chi_m(y) = \prod_{j=1}^{d} \chi(2^\ell y_j - m_j)).$$

We fix $k$, $\ell$ for now and let $A_m := A_{k,\ell}[\chi_m(y)]$. Then $A_m A_m^*$ has Schwartz kernel

$$K_{m,m}^* (x, w) = \int e^{i2^k(\phi(x,y) - \phi(w,y))} \sigma(x, w, k, \ell) \, dy,$$

where the amplitude is given by

$$\sigma(x, w, y) = \chi(2^\ell h(x, y))|\chi_m(y)|^2 \overline{\chi(2^\ell h(w, y))}.$$ 

Similarly, the Schwartz kernel for $A_m^* A_m$ is given by

$$K_{m,m}^* (y, z) = \int e^{i2^k(\phi(x,y) - \phi(x,z))} \overline{\sigma}(x, y, z) \, dx,$$

where

$$\overline{\sigma}(x, y, z) = \chi(2^\ell h(x, y))\chi_m(y)\chi(2^\ell h(x, z))\chi_{\bar{m}}(z).$$

By splitting our operator $A_{k,\ell}$ into a finite number of collections of $\{A_m\}$, we may assume that if $m_j \neq \bar{m}_j$ then $|m_j - \bar{m}_j| > \max\{\frac{15}{c_k}, 2\sqrt{d}\}$. We may also assume

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through the loss of a constant $2^{\ell\varepsilon}$ that the kernels of $A_m$ are supported where $|x_d| \leq 2^{\ell\varepsilon}$.

We first prove the following lemmas.

**Lemma 7** There exists a constant $C > 0$ such that

$$\|A_m\|_{L^2 \to L^2} \leq C 2^{\ell - dk}.$$ 

**Lemma 8** For any $N > 0$, and $\ell < \ell_0 = \lfloor \frac{k}{2} \rfloor$, the following estimates hold.

(a) If $m \neq \tilde{m}$ then

$$\|A_m A^*_\tilde{m}\| = 0.$$ 

(b) If $m \neq \tilde{m}$ and $|m' - \tilde{m}'| \leq \frac{c L}{10 \|\phi\|_{C^3}} |m_d - \tilde{m}_d|$ then

$$\|A^*_m A_{\tilde{m}}\| = 0.$$ 

(c) If $m \neq \tilde{m}$ and $|m' - \tilde{m}'| \geq \frac{c L}{10 \|\phi\|_{C^3}} |m_d - \tilde{m}_d|$ then

$$\|A^*_m A_{\tilde{m}}\| \lesssim N 2^{\ell - dk} (2^{k-2\ell} |m - \tilde{m}|)^{-N}.$$ 

A few remarks. First, the estimates in Lemma 8 do not rely on the blowdown assumption and essentially reprove the results of Comech in [15], albeit through a slightly different approach. Second, the separation of $\ell$ from $k/2$ is necessary for the proof of Lemma 8, but not Lemma 7.

### 6.1.1 Proof of Lemma 7

Since $|\phi_{x'y'}| > c > 0$, the set of equations $\nabla_{y'} (\phi(x, y) - \phi(w, y)) = 0$ is solved uniquely by $x' = \chi'(w, x_d, y)$. By the implicit function theorem, we can see that

$$\frac{1}{4} |x' - \chi'(w, x_d, y)| \leq |\phi_y(x, y) - \phi_y(w, y)| \leq 4 |x' - \chi'(w, x_d, y)|.$$ 

A further set of calculations reveals that

$$\phi_{y_d}(\chi'(w, x_d, y), x_d, y) - \phi_{y_d}(w, y) = \sum_{j=0}^{N} \frac{V_j^L (\det \phi_{xy}(\det \phi_{x'y'}))^{-1} (w, y) (x_d - w_d)^{j+1}}{(j+1)!} + b(w, y)(x_d - w_d)^{N+2},$$

where $b$ is uniformly bounded and $N$ is chosen such that $|x_d - w_d|^N \leq 2^{-\ell}$. Since $\pi_L$ is a fold and $V_L|_{(0,0)} = \delta_{y_d}$, we see that $h(x, y) = 0$ is solved uniquely by
\[ y_d = \eta_d(x, y') \] near 0. From this,
\[ \frac{1}{4}|y_d - \eta_d(x, y')| \leq |h(x, y)| \leq 4|y_d - \eta_d(x, y')|. \]

Because \( \pi_R \) is a blowdown and the bounds on \( h \), we see that
\[ |V_j^R h(x, y)| = |V_j^R h(x, y', \eta_d(x, y')) + (y_d - \eta_d(x, y'))\partial_{yd} V_j^R h(x, y', z_d)| \leq C 2^{-\ell} \]
implying by the properties of differentiation of products
\[ |\phi_{yd}(y(x, x_d, y), x_d, y) - \phi_{yd}(w, y)| \geq c 2^{-\ell} |x_d - w_d|. \]

Thus
\[ |\phi_{yd}(x, y) - \phi_{yd}(w, y)| \geq |\phi_{yd}(x, y) - \phi_{yd}(y'(w, x_d, y), x_d, y)| \]
\[ - |\phi_{yd}(w, y) - \phi_{yd}(y'(w, x_d, y), x_d, y)|, \]
and therefore,
\[ |\nabla_{yd}(\phi(x, y) - \phi(w, y)| \geq C \max\{2^{-\ell} |x_d - w_d|, |x' - y'(w, x_d, y)|\}. \]

With these estimates in place we integrate by parts in the \( y \) variables, noting that for a multi-index \( \alpha \)
\[ |\partial^\alpha_{yd} \phi| \leq C_{|\alpha|} 2^{\ell|\alpha|} \]
and for \( |\alpha| > 1 \),
\[ |\partial^\alpha_{yd} \phi| \leq C_{|\alpha|} |x - w|. \]

Thus, we get the estimate
\[ |K^{A_A}(x, w)| \lesssim_N \int \frac{1}{(1 + 2^{k-\ell}|x' - y'(w, x_d, y)|)^N} \frac{1}{(1 + 2^{k-2\ell}|x_d - w_d|)^N} dy. \]

Integrating in \( x \), we see that
\[ \int |K^{A_A}(x, w)| dx \lesssim_N \int \frac{1}{(1 + 2^{k-2\ell}|x_d - w_d|)^N} dx_d \]
\[ \times \sup_{x_d, y} 2^{-\ell} \int \frac{1}{(1 + 2^{k-\ell}|x' - y'(w, x_d, y)|)^N} dx' \]
\[ \lesssim 2^{2\ell-k} 2^{-d\ell} 2^{(d-1)(\ell-k)} \lesssim 2^{\ell-dk}. \]
6.1.2 Proof of Lemma 8

First, (a) follows immediately since it implies

$$\chi(2^\ell y - m)\chi(2^\ell y - \tilde{m}) = 0.$$  

The kernel $K_{m,\tilde{m}}^{A,\tilde{A}}$ vanishes under the assumption in (b) because $V_L \det d\pi_L$ is bounded away from 0 on $L$. Since $|\det \phi(x, y)|$ and $|\det \phi(x, z)|$ are both bounded above by $2^{-\ell+1}$, their sum is bounded by $2^{-\ell+3}$. Expanding the difference about $y = z$, we see

$$\det \phi_{xy}(x, y) - \det \phi_{xy}(x, z) = (y_d - z_d)\partial_{y_d} \det \phi_{xy}(x, z) + (y' - z') \cdot \nabla_{y'} \det \phi_{xy}(x, z) + O(|y - z|^2)$$

$$= (y_d - z_d)(\partial_{y_d} V_L) \det \phi_{xy}(x, z) + (y_d - z_d) V_L \det \phi_{xy}(x, z) + (y' - z') \cdot \nabla_{y'} \det \phi_{xy}(x, z) + O(|y - z|^2)$$

$$|\det \phi_{xy}(x, y) - \det \phi_{xy}(x, z)| \geq \frac{c_L}{3}|y_d - z_d| \geq 5(2^{-\ell}).$$

Thus, we see there are no $y, z$ that satisfy these conditions, hence

$$a_{k,\ell,\pm(x, y)}a_{k,\ell,\pm(x, z)} = 0.$$  

To prove (c), we split it into two cases: first, assume that $k \geq (2 + \varepsilon)\ell$. Then we use the following Taylor approximation of the derivative of the phase of $K_{m,\tilde{m}}^{A,\tilde{A}}$.  

$$\nabla_{x'}[\phi(x, y) - \phi(x, z)] = \phi_{x'y_d}(x, z)(y_d - z_d) + \phi_{x'z'}(y' - z') + O(|y - z|^2).$$

(59)

We know that $|\phi_{x'y}(x, z) \cdot (y' - z')| \geq C_d |y' - z'|$, and $|\phi_{x'z_d}(x, z)(y_d - z_d)| \leq \varepsilon |y_d - z_d|$. By assumption

$$|y' - z'| \geq \frac{c_L}{3\|\phi\|_{C^3}} |y_d - z_d| \geq \frac{10\varepsilon}{C_d} |y_d - z_d|.$$  

Thus

$$|\nabla_{x'}[\phi(x, y) - \phi(x, z)]| \geq c|y - z|$$

for some small constant $c > 0$. Define the operator

$$\mathcal{M}_{x'} = \frac{1}{i2^\ell} \frac{\nabla_{x'}[\phi(x, y) - \phi(x, z)]}{|\nabla_{x'}[\phi(x, y) - \phi(x, z)]|^2} \cdot \nabla_{x'}.$$
We apply $M_{x'}$ many times to $K_{m, \tilde{m}}^{A^* A}$, and by our lemma

$$|K_{m, \tilde{m}}^{A^* A}(y, z)| = \left| \int e^{i2k(\phi(x,y) - \phi(x,z))} \sigma \, dx \right|$$

$$= \left| \int e^{i2k(\varphi_Q(x,y) - \varphi_Q(x,z))} (M_{x'}^*)^N \sigma \, dx \right|$$

$$\lesssim N \int \frac{1}{(2k-\ell |y-z|)^N} |\tilde{\sigma}| \, dx$$

$$\lesssim N \frac{1}{(2k-\ell |y-z|)^N} \chi_m(y) \chi_{\tilde{m}}(z).$$

Since $|y - z| > 2^{-\ell}, k \geq 2\ell$, and $|y - z| \simeq 2^{-\ell} |m - \tilde{m}|$,

$$\frac{1}{2k-\ell |y-z|} \leq \min \left\{ \frac{C}{1 + 2k-\ell |y-z|}, \frac{C}{2^{k-2\ell} |m - \tilde{m}|} \right\}$$

Integrating in $y$ (or $z$)

$$\int |K_{m, \tilde{m}}^{A^* A}(y, z)| \, dy$$

$$\leq C_N.d \int \frac{1}{(1 + 2k-\ell |y-z|)^{d+1}} \frac{1}{(2k-2\ell |m - \tilde{m}|)^N} \chi_m(y) \chi_{\tilde{m}}(z) \, dy$$

$$\leq C_N.d 2^{d(\ell-k)} (2k-2\ell |m - \tilde{m}|)^{-N}.$$

Since $k - 2\ell \geq \ell \varepsilon$, if we let $N = d/\varepsilon$ then by Schur’s Lemma

$$\|A_m^* A_{\tilde{m}}\|_{2 \rightarrow 2} \leq C(\varepsilon, d) 2^{(\ell-dk)} |m - \tilde{m}|^{-N},$$

proving part (c) of Lemma 8.

### 7 $L^p$-Sobolev Estimate

As in [10], we prove Theorem 1 by applying a special case of Theorem 1.1 from [19] and a Littlewood–Paley estimate adapted from [27].

Let

$$R_{\ell} = \sum_{k \geq 2\ell} R_{k, \ell}.$$

We will prove for compactly supported $f$

$$\|R_{\ell} f\|_{L_{p,q}^p} \leq 2^{-\ell \varepsilon(p)} \|f\|_{B_{0,p}^{p,q}}, \quad 0 < q \leq 2 \leq 4 < p < \infty.$$
Lemma 2.1 in [27]). Note that the kernel of the operator depends on $C$ where $F^{p,q}_s$ and $B^p_{s'}$ are, respectively, the Triebel–Lizorkin space and Besov spaces (see [28]). Summing in $\ell$ with $q \geq 1$, we conclude that

\[ \mathcal{R} : B^p_{s,\text{comp}} \to F^{p,q}_{s+1/p}, \quad q \leq 2 < 4 < p < \infty. \]

Since $L^p_s = L^p_{s+2} \hookrightarrow B^p_{s,\text{comp}}$ for $p > 2$ and $F^{p,q}_{s+1/p} \hookrightarrow F^{p,q}_{s+1/p} = L^p_{s+1/p}$ for $q \leq 2$, this implies the asserted $L^p$-Sobolev bounds for $\mathcal{R}$.

Let $P_k$ be standard Littlewood–Paley multipliers on $\mathbb{R}^3$ for $k \in \mathbb{N}$ and $\tilde{\phi}_j(x, y) = S^j - y_j$ for $j = 1, 2$. Because $\nabla \tilde{\phi}_j(x, y)$ are linearly independent, as are $\nabla_y \tilde{\phi}_j(x, y)$, we can find $C_0 > 0$ such that

\[ 4C_0^{-1}|\tau| \leq |(\tau \cdot \tilde{\phi})_x|, |(\tau \cdot \tilde{\phi})_y| \leq C_0/|\tau| \]

This implies the following.

**Lemma 9** Suppose $k', k'' \in \mathbb{N}$, $k' \geq 2\ell$ and $\max\{|k - k'|, |k - k''|\} \geq C_1$, where $C_1$ depends on $C_0$. Then

\[ \|P_k \mathcal{R}_{k', \ell} P_{k''}\|_{L^p \to L^p} \leq C \min\{2^{-kN}, 2^{-k'N}, 2^{-k''N}\}. \]

**Proof of Lemma 9** This integration by parts argument is essentially due to Hörmander [12], based on the fact that the canonical relation stays away from zero sections (cf. Lemma 2.1 in [27]). Note that the kernel of the operator $P_k \mathcal{R}_{k', \ell} P_{k''}$ is given by

\[
\begin{aligned}
\int \int \int \int e^{i\left[(x-w, \eta) + \tau \cdot \tilde{\phi}(w, z) + (z-y, \xi)\right]} &\left(x_0(2^{-k} |\eta|) \chi_0(2^{-k'} |\tau|) \chi_0(2^{-k''} |\xi|) \right.
\times a_k, \ell, (z, \tau, \chi(|w|) \chi(|x|) \right) dw dz d\tau d\eta d\xi.
\end{aligned}
\]

Our assumption on $\Phi$ implies that if $\max\{|k - k'|, |k' - k''|\} > C_1$ we have

\[
\nabla_{(z, w)} \left[(x - w, \eta) + \tau \cdot \tilde{\phi}(w, z) + (z-y, \xi)\right] \geq c \max\{2^k, 2^{k'}, 2^{k''}\}.
\]

Thus we integrate by parts in the $(w, z)$ variables to get the above bound on the kernel, implying by Minkowski the desired bound on $L^p$.

Using the lemma above and an argument similar to a part of the proof of Lemma 2.1 in [27], we can reduce the proof of Theorem 1 to the estimate

\[
\left\| \left( \sum_{k \geq 2\ell} 2^{k/p} P_k \mathcal{R}_{k+s_1, \ell} P_{k+s_2} f \right)^q \right\|_{L^p}^{1/q} \leq C 2^{-\ell \ell_p} \left( \sum_{k \geq 0} |P_{k+s_2} f|^p \right)^{1/p} \left| L^p \right|.
\]

(60)

To prove (60), we apply the main result from [19].
Theorem 4 [19] Let $T_k$ be a family of operators defined for Schwartz functions by

$$T_k f(x) = \int K_k(x, y) f(y) dy.$$ 

Let $\phi \in S(\mathbb{R}^3)$, $\phi_k = 2^{3k} \phi(2^k \cdot)$, and $\Pi_k f = \phi_k \ast f$. Let $\epsilon > 0$ and $1 < p_0 < p < \infty$. Assume $T_k$ satisfies

$$\sup_{k > 0} 2^{k/p} \|T_k\|_{L^p \to L^p} \leq A \quad (61)$$

and

$$\sup_{k > 0} 2^{k/p_0} \|T_k\|_{L^{p_0} \to L^{p_0}} \leq B_0 \quad (62)$$

Further let $A_0 \geq 1$, and assume that for each cube $Q$, there is a measurable set $E_Q$ such that

$$|E_Q| \leq A_0 \max\{|Q|^{2/3}, |Q|\}. \quad (63)$$

and for every $k \in \mathbb{N}$ and every cube $Q$ with $2^k \text{diam}(Q) \geq 1$,

$$\sup_{x \in Q} \int_{\mathbb{R}^d \setminus E_Q} |K_k(x, y)| dy \leq B_1 \max \left\{ \left( 2^k \text{diam}(Q) \right)^{-\epsilon}, 2^{-k\epsilon} \right\}. \quad (64)$$

Let

$$B = B_0^{q/p} \left( A A_0^{1/p} + B_1 \right)^{1-q/p}.$$ 

Then for any $q > 0$, there is a $C$ depending on $\epsilon$, $p$, $p_0$, $q$ such that

$$\left\| \left( \sum_k 2^{kq/p} |P_k T_k f_k|^q \right)^{1/q} \right\|_p \leq C A \left[ \log \left( 3 + \frac{B}{A} \right) \right]^{1/q-1/p} \left( \sum_k \|f_k\|_p^p \right)^{1/p}. \quad (65)$$

We apply this theorem on the family of operators $T_k := R_{k, \ell}$ for $k \geq 2\ell$ (here $\ell$ is fixed). By Proposition 1, the assumptions (61) and (62) are satisfied with $A \lesssim 2^{-\ell \epsilon(p)}$ and $B_0 \lesssim 2^{-\ell \epsilon(p_0)}$. We next check the assumptions (63) and (64). For a given cube $Q$ with center $x_Q$ let

$$E_Q = \{ y : |S(x_Q, y_3) - y'| \leq C 2^\ell \text{diam}(Q) \}$$

if $\text{diam}(Q) < 1$, and a cube centered at $x_Q$ of diameter $C 2^\ell \text{diam}(Q)$ if $|Q| \geq 1$. By an integration by parts argument, we derive the bound

$$|K_k(x, y)| \lesssim N 2^{2k} \frac{2^{2k}}{(1 + 2^{k-\ell} |S(x_Q, y_3) - y'|)^N}.$$
Then clearly assumptions (63) and (64) are satisfied with $A_0 \lesssim 2^{3\ell}$ and $B_1 \lesssim 2^{2\ell}$, respectively. Theorem 4 then implies (60) with $\Pi_k = \mathcal{P}_{k+s_1}$ and $f_k = \mathcal{P}_{k+s_2}f$, finishing the proof of Theorem 1.

References

1. Gelfand, I.M., Graev, M.I.: Line complexes in the space $C^n$. Funkcional. Anal. i Priložen. 2(3), 39–52 (1968)
2. Greenleaf, A., Uhlmann, G.: Nonlocal inversion formulas for the X-ray transform. Duke Math. J. 58(1), 205–240 (1989). https://doi.org/10.1215/S0012-7094-89-05811-0
3. Greenleaf, A., Uhlmann, G.: Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms. Ann. Inst. Fourier (Grenoble) 40(2), 443–466 (1990)
4. Greenleaf, A., Seeger, A.: Fourier integral operators with fold singularities. J. Reine Angew. Math. 455, 35–56 (1994). https://doi.org/10.1515/crll.1994.455.35
5. Pramanik, M., Seeger, A.: $L^p$ Sobolev regularity of a restricted X-ray transform in mathbb R^3. In: Harmonic Analysis and Its Applications, pp. 47–64. Yokohama Publisher, Yokohama (2006)
6. Wolff, T.: Local smoothing type estimates on $L^p$ for large $p$. Geom. Funct. Anal. 10(5), 1237–1288 (2000). https://doi.org/10.1007/PL00001652
7. Bourgain, J., Demeter, C.: The proof of the $l^2$ decoupling conjecture. Ann. Math. 182(1), 351–389 (2015). https://doi.org/10.4007/annals.2015.182.1.9
8. Pramanik, M., Seeger, A.: $L^p$ Sobolev estimates for a class of integral operators with folding canonical relations. arXiv preprint arXiv:1909.04173. To appear in J. Geom. Anal. (2019)
9. Pramanik, M., Seeger, A.: $L^p$ regularity of averages over curves and bounds for associated maximal operators. Am. J. Math. 129(1), 61–103 (2007). https://doi.org/10.1353/ajm.2007.0003
10. Bentsen, G.: $L^p$ regularity for a class of averaging operators on the Heisenberg group. arXiv preprint arXiv:2002.01917. To appear in Indiana Univ. Math. J. (2020)
11. Helgason, S.: The Radon Transform. Progress in Mathematics, vol. 5, 2nd edn. Birkhäuser, Boston (1999)
12. Hörmander, L.: Fourier integral operators. I. Acta Math. 127(1–2), 79–183 (1971). https://doi.org/10.1007/BF02392052
13. Phong, D.H.: Singular integrals and Fourier integral operators. In: Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), Princeton Mathematical Series, vol. 42, pp. 286–320. Princeton University Press, Princeton (1995)
14. Guillemin, V., Sternberg, S.: Geometric Asymptotics. Mathematical Surveys, No. 14, American Mathematical Society, Providence (1977)
15. Comech, A.: Integral operators with singular canonical relations. In: Spectral Theory, Microlocal Analysis, Singular Manifolds. Mathematical Topic, vol. 14, pp. 200–248. Akademie, Berlin (1997)
16. Greenleaf, A., Seeger, A.: Oscillatory and Fourier integral operators with degenerate canonical relations. In: Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), pp. 93–141 (2002). https://doi.org/10.5565/PUBLMAT_Esco02_05
17. Comech, A.: Optimal regularity of Fourier integral operators with one-sided folds. Commun. Partial Differ. Equ. 24(7–8), 1263–1281 (1999). https://doi.org/10.1080/0360530090821465
18. Phong, D.H., Stein, E.M.: The Newton polyhedron and oscillatory integral operators. Acta Math. 179(1), 105–152 (1997). https://doi.org/10.1007/BF02392721
19. Pramanik, M., Rogers, K.M., Seeger, A.: A Calderón–Zygmund estimate with applications to general- lized Radon transforms and Fourier integral operators. Stud. Math. 202(1), 1–15 (2011). https://doi.org/10.4064/sm202-1-1
20. Secco, S.: $L^p$-improving properties of measures supported on curves on the Heisenberg group. Stud Math. 132(2), 179–201 (1999). https://doi.org/10.4064/sm-132-2-179-201
21. do Carmo, M.P.: Riemannian Geometry. Mathematics: Theory & Applications. Birkhäuser, Boston (1992). https://doi.org/10.1007/978-1-4757-2201-7. https://doi-org.ezproxy.library.wisc.edu/10.1007/978-1-4757-2201-7. Translated from the second Portuguese edition by Francis Flaherty
22. Klingenberg, W.: Riemannian Geometry. de Gruyter Studies in Mathematics, vol. 1. Walter de Gruyter & Co., Berlin (1982)
23. Phong, D.H., Stein, E.M.: Radon transforms and torsion. Int. Math. Res. Notices \textbf{4}, 49–60 (1991). https://doi.org/10.1155/S1073792891000077

24. Muscalu, C., Schlag, W.: Classical and multilinear harmonic analysis. In: Vol. I. (ed.) Cambridge Studies in Advanced Mathematics, vol. 137. Cambridge University Press, Cambridge (2013)

25. Cuccagna, S.: $L^2$ estimates for averaging operators along curves with two-sided $k$-fold singularities. Duke Math. J. \textbf{89}(2), 203–216 (1997). https://doi.org/10.1215/S0012-7094-97-08910-9

26. Anderson, T.C., Cladek, L., Pramanik, M., Seeger, A.: Spherical means on the Heisenberg group: stability of a maximal function estimate. arXiv preprint arXiv:1801.06981, To appear in J. Anal. Math. (2018)

27. Seeger, A.: Degenerate Fourier integral operators in the plane. Duke Math. J. \textbf{71}(3), 685–745 (1993). https://doi.org/10.1215/S0012-7094-93-07127-X

28. Triebel, H.: Theory of Function Spaces. Monographs in Mathematics, vol. 78. Birkhäuser, Basel (1983). https://doi.org/10.1007/978-3-0346-0416-1

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