Characteristic function and operator approach to M-indeterminate probability densities

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Abstract: Based on a quantum mechanical approach, we investigate moment- (or M-) indeterminate probability densities by way of the characteristic function and self-adjoint operators. The approach leads to new methods to construct classes of M-indeterminate probability densities.

Keywords: M-indeterminate probability density; moments; characteristic function; self-adjoint operators; Stieltjes

1 Introduction

In a well-known series of papers, Aharonov et al. [1, 2, 3] developed a quantum mechanical state function that resulted in a probability density that depends on a parameter while the moments do not. An analogous situation arises in probability theory. Specifically, while many of the common probability densities are uniquely determined by their moments, some are not; such densities are said to be “moment-indeterminate,” or “M-indeterminate” [7, 10, 22]. Yet, the approach of Aharonov et al. and the way in which probability densities are obtained in quantum mechanics is very different compared to standard probability theory. Nevertheless, one does not need to understand quantum theory in order to apply that approach to the problem of generating M-indeterminate densities in standard probability, as we do here. In particular, we show that using the characteristic function as defined in probability theory together with operators that are standard in quantum mechanics allows one to readily generate an infinite number of M-indeterminate densities.

Consider the generally complex function $g(x)$, which we shall take to be normalizable to one,

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = 1$$

Although not essential to our developments here, in quantum mechanical language $g(x)$ is called a state function and $x$ may be the position of a particle. What is important for our considerations is that the probability that the particle resides within some interval $x_1 < x < x_2$ is given by

$$\Pr\{x_1 < x < x_2\} = \int_{x_1}^{x_2} |g(x)|^2 dx$$

Hence, the probability density is $f_X(x) = |g(x)|^2$.

To obtain the probability density of other variables, $g(x)$ is expanded in a complete set of functions $u(r, x)$ [13],

$$g(x) = \int_{-\infty}^{\infty} G(r)u(r, x) dr$$ (1.1)
where
\[ G(r) = \int_{-\infty}^{\infty} g(x)u^*(r, x) \, dx \quad (1.2) \]
The functions \( u(r, x) \) are eigenfunctions that are obtained by solving the eigenvalue problem
\[ \mathcal{A} u(r, x) = r u(r, x) \]
where the operator \( \mathcal{A} \) is self-adjoint. (In writing these equations, we have assumed the eigenvalues are continuous.) The function \( G(r) \) is the representation of \( g(x) \) in the \( r \)-domain. Note that because \( g(x) \) is normalized, so is \( G(r) \), namely
\[ \int_{-\infty}^{\infty} |G(r)|^2 \, dr = \int_{-\infty}^{\infty} |g(x)|^2 \, dx = 1 \]
The important point is that the probability density for \( r \) is
\[ f_R(r) = |G(r)|^2 \]
Hence, we see that \( g(x) \) and \( G(r) \) are a Fourier transform pair. Now consider the moments
\[ \mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n |g(x)|^2 \, dx \]
By virtue of the Fourier theory, these may be calculated from \( G \) by way of
\[ \mathbb{E}[X^n] = \int_{-\infty}^{\infty} G^*(r) \left( -\frac{1}{i} \frac{d}{dr} \right)^n G(r) \, dr \]
Similarly, one has
\[ \mathbb{E}[R^n] = \int_{-\infty}^{\infty} r^n |G(r)|^2 \, dr = \int_{-\infty}^{\infty} g^*(x) \left( \frac{1}{i} \frac{d}{dx} \right)^n g(x) \, dx \]
We build on these relationships and their generalization for other self-adjoint operators to develop a characteristic function approach to M-indeterminate densities. Before that, however, we briefly discuss the function \( g(x) \) considered by Aharonov and colleagues [1, 2, 3].

1.1 The approach of Aharonov et al.

Consider the function
\[ g(x) = \frac{1}{\sqrt{2}} \left( g_1(x) + e^{i\theta} g_2(x) \right) \]
where $\beta$ is a real parameter and where each of $g_1(x)$ and $g_2(x)$ is normalized to one. In addition, $g_1(x)$ and $g_2(x)$ are taken to be finite extent and, importantly, to have disjoint support such that $g_1(x)g_2(x) = 0$. The probability density is then

$$f_X(x) = |g(x)|^2 = \frac{1}{2} |g_1(x) + e^{i\beta}g_2(x)|^2 = \frac{1}{2} \left(|g_1(x)|^2 + |g_2(x)|^2\right)$$

and the moments are given by

$$E[X^n] = \frac{1}{2} \int_{-\infty}^{\infty} x^n \left(|g_1(x)|^2 + |g_2(x)|^2\right) dx$$

Since the probability density is independent of $\beta$, naturally so are the moments.

Now consider the $r$-domain density obtained by expanding $g(x)$ in terms of the eigenfunctions $u(r, x)$ for the momentum operator considered above; doing so, one obtains the density

$$f_R(r) = |G(r)|^2 = \frac{1}{2} \left(|G_1(r)|^2 + |G_2(r)|^2 + e^{i\beta}G_1^*(r)G_2(r) + e^{-i\beta}G_1(r)G_2^*(r)\right)$$

where

$$G_{1,2}(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_{1,2}(x)e^{-i\beta x} dx$$

Note that, although $g_1(x)g_2(x) = 0$, here we have $G_1(r)G_2(r) \neq 0$ because $G_1(r)$ and $G_2(r)$ are Fourier transforms of finite extent functions and therefore they extend over all $r$ [19, 11]. Hence, unlike the case with $f_X(x)$, the density $f_R(r)$ does depend on $\beta$. However, as Aharonov et al. showed, the moments $E[R^n]$ are independent of $\beta$.

Specifically, by virtue of the Fourier relations given above for calculating moments, one has

$$E[R^n] = \frac{1}{2} \int_{-\infty}^{\infty} \left(g_1^*(x) + e^{-i\beta}g_2^*(x)\right) \left(\frac{1}{i} \frac{d}{dx}\right)^n \left(g_1(x) + e^{i\beta}g_2(x)\right) dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} g_1^*(x) \left(\frac{1}{i} \frac{d}{dx}\right)^n g_1(x) dx + \frac{1}{2} \int_{-\infty}^{\infty} g_2^*(x) \left(\frac{1}{i} \frac{d}{dx}\right)^n g_2(x) dx$$

where the final expression follows since $g_1(x)g_2(x) = 0$. Hence, the density $f_R(r)$ is in general M-indeterminate. Historically, M-indeterminate densities were defined as those for which the moments do not uniquely determine the density, and for which all moments exist. In their original considerations, Aharonov et al. did not address this latter criterion, but it has been addressed in the quantum context [16, 17, 14, 15] and we do so herein as well.

We generalize these ideas by considering other self-adjoint operators to obtain formulations for M-indeterminate densities. A key aspect of M-indeterminate densities that we take advantage of here is that, while M-indeterminate densities do not have a moment generating function, the characteristic function always exists.

2 Characteristic function and self-adjoint operators

Let $R$ be a continuous real random variable with continuous probability density function $f_R(r)$ ($r \in \mathbb{R}^1$). Then the characteristic function of $f(r)$ is defined as [7, 10]

$$M(t) = \int_{-\infty}^{\infty} f_R(r) e^{irt} dr$$
Given the characteristic function, one may obtain the probability density by Fourier inversion of the previous equation, namely
\[ f_R(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M(t) e^{-itr} dt \]

**Theorem 2.1.** If \( f_R(r) = |G(r)|^2 \) and \( G(r) \) and \( g(x) \) are related by Eqs. (1.1) and (1.2), then the characteristic function may be obtained directly in terms of \( g(x) \) and the operator \( A \) as
\[ M(t) = \int_{-\infty}^{\infty} g^*(x) e^{itA} g(x) \, dx \]  \hspace{1cm} (2.1)

**Proof.** See [5, 6] and references therein. □

**Corollary 2.1** For the particular case of the self-adjoint operator
\[ A = \frac{1}{i} \frac{d}{dx} \]
we have that
\[ M(t) = \int_{-\infty}^{\infty} g^*(x) e^{it\frac{d}{dx}} g(x) \, dx = \int_{-\infty}^{\infty} g^*(x) g(x + t) \, dx \]  \hspace{1cm} (2.2)

**Proof.** The result follows readily by a Taylor series expansion of \( e^{it\frac{d}{dx}} \). □

**Remark 2.1:** The right-hand side of Eq. (2.2) is a well-known condition, first given by Khinchin [9, 5].

### 3 A Stieltjes class characteristic function approach

One general formulation for generating so-called “Stieltjes classes” of M-indeterminate densities is [20, 21]
\[ f_\varepsilon(x) = f_0(x) \left[ 1 + \varepsilon h(x) \right] \]  \hspace{1cm} (3.1)
where \(-1 \leq \varepsilon \leq 1\) is a real parameter, and \( h(x) \neq 0 \) is a real, continuous, bounded (\(|h(x)| \leq 1\)) function that satisfies the constraint
\[ \int_{-\infty}^{\infty} x^n f_0(x) h(x) \, dx = 0, \quad n \in \mathbb{N}^+ \]  \hspace{1cm} (3.2)
by which it follows that the densities \( f_\varepsilon(x) \) all have the same moments.

#### 3.1 Characteristic function constraint

For the M-indeterminate densities of Eq. (3.1), we express the characteristic function as
\[ M_\varepsilon(t) = \int_{-\infty}^{\infty} f_\varepsilon(x) e^{itx} \, dx = \int_{-\infty}^{\infty} f_0(x) e^{itx} \, dx + \varepsilon \int_{-\infty}^{\infty} h(x) f_0(x) e^{itx} \, dx = M_0(t) + \varepsilon Q(t) \]
where
\[ M_0(t) = \int_{-\infty}^{\infty} f_0(x) e^{itx} \, dx \quad ; \quad Q(t) = \int_{-\infty}^{\infty} f_0(x) h(x) e^{itx} \, dx \]
Further, let $H(t)$ be the Fourier transform of $h(x)$,

$$H(t) = \int_{-\infty}^{\infty} h(x) e^{-itx} dx$$

**Theorem 3.1.** Let $M_0(t)$ and $H(t)$ and their derivatives $M^{(k)}_0(t) = \frac{d^k}{dt^k} M_0(t)$ and $H^{(k)} = \frac{d^k}{dt^k} H(t)$ vanish at $\pm \infty$. Then for the probability densities $f_\varepsilon(x)$ of Eq. (3.1), the moments will be independent of $\varepsilon$ if

$$\int_{-\infty}^{\infty} M_0^{(k)}(t) H^{(n-k)}(t) dt = 0, \quad k = 0, 1, \ldots, n \quad (3.3)$$

for all $n \in I^+$ and any value of $k$ as indicated.

**Proof.** The moments of $f_\varepsilon(x)$ are given by

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_\varepsilon(x) dx = \int_{-\infty}^{\infty} x^n f_0(x) dx + \varepsilon \int_{-\infty}^{\infty} x^n f_0(x) h(x) dx$$

The moments can be equivalently obtained from the characteristic function by

$$E[X^n] = \left\{ \frac{1}{i^n} \frac{d^n}{dt^n} M_0(t) \right\}_{t=0} + \varepsilon \left\{ \frac{1}{i^n} \frac{d^n}{dt^n} Q(t) \right\}_{t=0}$$

In order for the moments to be independent of $\varepsilon$, we require

$$\left\{ \frac{d^n}{dt^n} Q(t) \right\}_{t=0} = 0$$

Substituting in for $Q(t)$ and evaluating this expression, one can confirm that this condition is precisely that given in Eq. (3.2). In terms of $M_0(t)$ and $H(t)$, straightforward evaluation yields

$$\left\{ \frac{d^n}{dt^n} Q(t) \right\}_{t=0} = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} M_0^{(n)}(t') H(t') dt' = 0$$

by which Eq. (3.3) follows via integration by parts as the characteristic function domain condition for the Stieltjes-classes of M-indeterminate densities of Eq. (3.1).

3.2 Generating Stieltjes class densities via the characteristic function

The characteristic function constraint for M-indeterminate densities leads to the following results:

**Theorem 3.2.** Let

1. The characteristic function $M_0(t)$ be of finite extent: $M_0(t) = 0$ for $|t| > L > 0$

2. The function $H(t) = 0$ for $|t| < L$, and is normalized such that $\frac{1}{2\pi} \int_{-\infty}^{\infty} H(t) e^{itx} dt = |h(x)| \leq 1$

Then the resulting densities $f_\varepsilon(x)$ of Eq. (3.1) are independent of $\varepsilon$.

**Corollary 3.1:** If all derivatives $\frac{d^n M_0(t)}{dt^n}$ are finite at $t = 0$, then the densities $f_\varepsilon(x)$ are M-indeterminate.
Proof. The normalization on \( H(t) \) insures that \( f_\varepsilon(x) \geq 0 \). Then, imposing the finite support constraint on \( M_0(t) \) yields, via Eq. (3.3) with \( k = 0 \),
\[
\int_{-\infty}^{\infty} M_0(t') H^{(n)}(t') \, dt' = \int_{-L}^{L} M_0(t') H^{(n)}(t') \, dt'
\]
Since \( H(t') = 0 \) over the limits of integration, the integral equals zero; hence, the moments are independent of \( \varepsilon \) but the densities \( f_\varepsilon(x) \) are not. Finally, the moments may be obtained by
\[
\left\{ \int_{-\infty}^{\infty} \frac{d^n}{dt^n} M_0(t) \right\}|_{t=0} = \left\{ \frac{1}{i^n} \frac{d^n}{dt^n} M_0(t) \right\}|_{t=0}.
\]
By virtue of the corollary, all moments are finite. Hence, the densities are M-indeterminate. □

These results point to a simple procedure for generating an unlimited number of Stieltjes classes of M-indeterminate densities:

Choose any infinitely differentiable, finite-extent function \( g(x) \), normalized to 1; namely,
\[
\frac{d^n g(x)}{dx^n} < \infty, \quad n = 0, 1, 2, ... \quad g(x) = 0, \quad |x| > \frac{L}{2} > 0 \quad \int_{-\infty}^{\infty} |g(x)|^2 \, dx = 1
\]
Examples of such functions are “bump functions” \[23\]. Then calculate
\[
M_0(t) = \int_{-\infty}^{\infty} g^*(x) g(x + t) \, dx
\]
The density function is
\[
f_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_0(t) e^{-itx} \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^*(y) g(y + t) \, dy \, e^{-itx} \, dt = \left\{ \int_{-\infty}^{\infty} g(y) e^{-ixy} \, dy \right\}^2
\]
Note that because \( g(x) \) is normalized, \( f_0(x) \) is likewise normalized, i.e., \( \int_{-\infty}^{\infty} f_0(x) \, dx = \int_{-\infty}^{\infty} |g(x)|^2 \, dx = 1 \).
Note further that, by virtue of the finite extent of \( g(x) \), the characteristic function \( M_0(t) \) is also of finite extent and in particular it equals zero for \( |t| > L \). Hence the density \( f_0(x) \) always exists.

Then, form the Stieltjes classes of M-indeterminate densities
\[
f_\varepsilon(x; \lambda, \phi) = f_0(x) \left[ 1 + \varepsilon \cos (\lambda x + \phi) \right], \quad \lambda > L, \quad -1 \leq \varepsilon \leq 1, \quad -\pi \leq \phi \leq \pi \quad (3.5)
\]
Because \( \lambda > L \) and the characteristic function equals zero for \( |t| > L \), it follows that \( \int_{-\infty}^{\infty} x^n f_0(x) \cos (\lambda x + \phi) \, dx = 0 \) (which is more readily evaluated in the characteristic function domain via Eq. (3.4)). Hence, \( f_0(x) \) and the densities \( f_\varepsilon(x; \lambda, \phi) \) all have identical moments \( \text{E}[X^n] \), consistent with Theorem 3.2.
Remark 3.1: Because \( g(x) \) is infinitely differentiable, it follows that all derivatives \( \frac{d^r M(t)}{dt^r} \) are finite at \( t = 0 \), and hence all moments exist, which is a necessary condition for a density to be classified as M-indeterminate.

Remark 3.2: This form of the characteristic function corresponds to the special case considered in Corollary 2.1.

4 Generalized characteristic function and operator approach

We now extend the characteristic function approach by making use of Eq. (2.1) with general self-adjoint operators \( \mathcal{A} \) and taking the (normalized) function to be

\[
g(x; \beta) = g_1(x) + e^{i\beta}g_2(x), \quad \text{with} \quad g_1(x)g_2(x) = 0, \quad \int_{-\infty}^{\infty} |g_1(x)|^2 dx = \int_{-\infty}^{\infty} |g_2(x)|^2 dx = \frac{1}{2} \tag{4.1}
\]

where we have made the dependence on the real parameter \( \beta \) explicit in our notation. For clarity, we shall denote here the characteristic function of Eq. (2.1) by \( M_{\mathcal{A}}(t; \beta) \), with corresponding probability density \( f_{\mathcal{A}}(r; \beta) \) for the continuous real random variable \( R \).

Theorem 4.1. Let \( g(x; \beta) \) be given by Eq. (4.1) and let the characteristic function \( M_{\mathcal{A}}(t; \beta) \) be given by Eq. (2.1). Further, let the operator \( \mathcal{A} \) be such that the support of \( \mathcal{A}^n g(x; \beta) \) is the same as that of \( g(x; \beta) \). Then, the moments \( E[R^n] \) of the family of densities

\[
f_{\mathcal{A}}(r; \beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_{\mathcal{A}}(t; \beta) e^{-itr} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^*(x; \beta)e^{it\mathcal{A}} g(x; \beta) dx e^{-itr} dt \tag{4.2}
\]

are independent of the parameter \( \beta \).

Proof. The moments are

\[
E[R^n] = \int_{-\infty}^{\infty} r^n f_{\mathcal{A}}(r; \beta) dr = \left\{ \left( \frac{d}{dt} \right)^n M_{\mathcal{A}}(t; \beta) \right\}_{t=0}
\]

\[
= \int_{-\infty}^{\infty} g_1^n(x) \mathcal{A}^n g_1(x) dx + \int_{-\infty}^{\infty} g_2^n(x) \mathcal{A}^n g_2(x) dx + e^{i\beta} \int_{-\infty}^{\infty} g_1^n(x) \mathcal{A}^n g_2(x) dx + e^{-i\beta} \int_{-\infty}^{\infty} g_2^n(x) \mathcal{A}^n g_1(x) dx
\]

But since \( g_1(x)g_2(x) = 0 \) and \( \mathcal{A}^n g_1(x) \) has the same support as \( g_1(x) \), and likewise for \( g_2(x) \), it follows that

\[
e^{i\beta} \int_{-\infty}^{\infty} g_1^n(x) \mathcal{A}^n g_2(x) dx + e^{-i\beta} \int_{-\infty}^{\infty} g_2^n(x) \mathcal{A}^n g_1(x) dx = 0
\]

Accordingly, the moments are independent of \( \beta \). \( \square \)

Corollary 4.1. In addition to the conditions of Theorem 4.1, let

\[
m_1^{(n)} = \int_{-\infty}^{\infty} g_1^n(x) \mathcal{A}^n g_1(x) dx < \infty
\]

\[
m_2^{(n)} = \int_{-\infty}^{\infty} g_2^n(x) \mathcal{A}^n g_2(x) dx < \infty
\]
The operator

\[ \mathcal{A} = \frac{d}{dx} + c(n + 1)x^n \]

where \( c \in \mathbb{R} \) and \( n \in I^+ \), satisfies the conditions of Theorem 4.1 and Corollary 4.1; namely, it is self-adjoint, and the support of \( \mathcal{A}^n g(x) \) equals that of \( g(x) \), whereas the support of \( e^{it\mathcal{A}} g(x) \) differs from the support of \( g(x) \).
For \( g(x) \) as in Eq. (4.1), we have by Eq. (2.1) that the characteristic function is

\[
M_A(t; \beta) = \int_{-\infty}^{\infty} (g_1^*(x) + e^{-i\beta}g_2^*(x)) e^{itA} (g_1(x) + e^{i\beta}f_2(x)) \, dx 
\]

\[
= M_{11}(t) + M_{22}(t) + e^{i\beta}M_{12}(t) + e^{-i\beta}M_{21}(t) 
\]

where

\[
M_{lm}(t) = \int_{-\infty}^{\infty} g_l^*(x)e^{itA}g_m(x)\,dx 
\]

Note that the characteristic function, and hence the density, depends on \( \beta \). For the moments to be independent of \( \beta \), we require that

\[
\left. \left\{ e^{i\beta}M_{12}^{(n)}(t) + e^{-i\beta}M_{21}^{(n)}(t) \right\} \right|_{t=0} = 0 
\]

where

\[
M_{lm}^{(n)}(t) = \frac{d^n}{dt^n}M_{lm}(t) 
\]

Evaluating this expression, we have

\[
\left\{ e^{i\beta}M_{12}^{(n)}(t) + e^{-i\beta}M_{21}^{(n)}(t) \right\}_{t=0} = \left\{ e^{i\beta} \int_{-\infty}^{\infty} g_1^*(x)i^nA^n e^{itA}g_2(x)\,dx + e^{-i\beta} \int_{-\infty}^{\infty} g_2^*(x)i^nA^n e^{itA}g_1(x)\,dx \right\}_{t=0} 
\]

\[
= e^{i\beta}i^n \int_{-\infty}^{\infty} g_1^*(x)A^n g_2(x)\,dx + e^{-i\beta}i^n \int_{-\infty}^{\infty} g_2^*(x)A^n g_1(x)\,dx 
\]

\[
= 0 
\]

where the last step follows since \( g_1(x)g_2(x) = 0 \) and the support of \( A^n g_1(x) \) equals that of \( g_1(x) \), and similarly for \( A^n g_2(x) \).

**Remark 4.3:** Special cases of this operator are the creation and annihilation operators for the quantum harmonic oscillator.

### 5 Closing remarks

We have developed new methods to construct M-indeterminate probability densities via the characteristic function together with operator methods used in quantum mechanics. We now briefly remark on applications as appropriate for this journal. Generally speaking, the M-indeterminate moment problem has been formulated with the necessary condition that all moments exist. Of course, there are important densities for which all moments do not exist. The procedures we have given for generating an unlimited number of densities that depend on a parameter, whereas the moments do not, is applicable in general, regardless of whether all moments exist (as of course is the approach of Eq. (3.1)).

Since the density and the characteristic function are Fourier transform pairs, the M-indeterminate condition that all moments exist implies that the characteristic function is infinitely differentiable at the origin. Using so-called “bump” functions, as noted in section 3.2 ensures that this condition is met. The existence of all moments
coupled with the Fourier relation between the density and the characteristic function also has implications on the tails of the density. Practical issues involving the number of moments actually known and possible errors in the moments are important questions. Also, the general issue of the differentiability of a function from a practical point of view, and relatedly the estimation of the tails of a density as well as the influence of the tails on physical properties, has been addressed by a number of authors (e.g. [8, 18, 19]).

References

[1] Aharonov, Y., Pendleton, H. and Petersen, A., 1969. Modular Variables in Quantum Theory. Int. J. Th. Phys. 2(3), 213-230.
[2] Aharonov, Y., Pendleton, H. and Petersen, A., 1970. Deterministic Quantum Interference Experiments. Int. J. Th. Phys. 3(6), 443-442.
[3] Aharonov, Y. and Rohrlich, D., 2005. Quantum Paradoxes. Wiley-VCH.
[4] Bohm, D., 1951. Quantum Theory. Prentice-Hall, New York.
[5] Cohen, L., 1988. Rules of probability in quantum mechanics. Found. Phys. 18(10), 983-998.
[6] Cohen, L., 2017. Are there quantum operators and wave functions in standard probability theory?, in: Wong, M. W., Zhu, H. (Eds.), Pseudo-Differential Operators: Groups, Geometry and Applications, pp. 133-147. Birkhäuser Mathematics.
[7] Feller, W., 1971. An Introduction to Probability Theory and Its Applications, Vol. 2. John Wiley and Sons, New York.
[8] Heyde, C. and Kou, S., 2004. On the controversy over tailweight of distributions. Oper. Res. Letters 32, 399-408.
[9] Khinchin, A., 1937. On a property of characteristic functions. Bull. Univ. Moscow, Vol. 1.
[10] Lukacs, E., 1970. Characteristic Functions, 2nd ed. Griffin & Co., London.
[11] Marks II, R. J., 2009. Handbook of Fourier Analysis & Its Applications. Oxford Univ. Press.
[12] Merzbacher, E., 1998. Quantum Mechanics. John Wiley & Sons, Inc.
[13] Morse, P. and Feshbach, H., 1953. Methods of Theoretical Physics, Part I. McGraw-Hill Book Company, New York.
[14] Sala Mayato, R., Loughlin, P. and Cohen, L., 2018. M-indeterminate distributions in quantum mechanics and the non-overlapping wave function paradox. Physics Letters A 382, 2914–2921.
[15] Sala Mayato, R., Loughlin, P. and Cohen, L., 2022. Generating M-indeterminate probability densities by way of quantum mechanics. J. Theor. Probab., 35, 1537-1555.
[16] Semon, M.D., Taylor, J.R., 1987. Expectation values in the Aharonov-Bohm effect. Il Nuovo Cimento B 97(1), 25-40.
[17] Semon, M.D. and Taylor, J.R., 1987. Expectation values in the Aharonov-Bohm effect. - II. Il Nuovo Cimento B 100(3), 389-401.

[18] Slepian, D., 1976. On bandwidth. Proc. IEEE, 64(3), 292-300.

[19] Slepian, D., 1983. Some comments on Fourier analysis, uncertainty and modeling. SIAM Review 25(3), 379-393.

[20] Stoyanov, J., 2004. Stieltjes classes for moment-indeterminate probability distributions. J. Applied Probability 41A, 281-294.

[21] Stoyanov, J. and Tolmatz, L., 2005. Method for constructing Stieltjes classes for m-indeterminate probability distributions. Appl. Math. and Comp. 165, 669-685.

[22] Stoyanov, J., 2013. Counterexamples in Probability, 3rd ed. Dover.

[23] Tu, L. W., 2011. An Introduction to Manifolds, Second Edition. Springer, NY.

[24] Wilcox, R.M., 1967. Exponential operators and parameter differentiation in quantum physics. J. Math. Phys. 8, 962-981.