Self-Similar Structures and Fractal Transforms in Approximation Theory

V.I. Yukalov\textsuperscript{1} and E.P. Yukalova\textsuperscript{2}

\textsuperscript{1} Bogolubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, Dubna 141980, Russia
\textsuperscript{2} Laboratory of Informational Technologies
Joint Institute for Nuclear Research, Dubna 141980, Russia

Abstract

An overview is given of the methods for treating complicated problems without small parameters, when the standard perturbation theory based on the existence of small parameters becomes useless. Such complicated problems are typical of quantum physics, many-body physics, physics of complex systems, and various aspects of applied physics and applied mathematics. A general approach for dealing with such problems has been developed, called \textit{Self-Similar Approximation Theory}. A concise survey of the main ideas of this approach is presented, with the emphasis on the basic notion of group self-similarity. The techniques are illustrated by examples from quantum field theory.
1 Preliminaries

It would not be an exaggeration to say that practically all interesting realistic problems cannot be solved exactly, so that one almost always has to resort to some methods of approximate solution. In obtaining a solution, it is highly desirable, before plunging to numerical calculations, to get an approximate analytical solution to the problem, which could help to understand the basic properties and specific features of the considered case. It is just the possibility of deriving analytical presentations for approximate solutions that is our main concern in this paper. After deriving and studying such analytical presentations, nothing prohibits one to pass to a numerical procedure. Moreover, a prefactory analytical consideration can help in devising an approximate numerical algorithm, and the knowledge of the basic peculiarities of the problem under consideration can save plenty of computer time.

A general approach for treating complicated real world problems has been developed, called Self-Similar Approximation Theory. In this paper, we aim at delineating the principal ideas of the approach and in presenting its several new developments. To clearly distinguish the pivotal aspects of our theory from the characteristic points of other known techniques, we feel it necessary to say several words on the latter. There are, roughly speaking, three common ways of obtaining approximate solutions: single step estimates, asymptotic perturbation theory, and methods of successive iteration.

A. Single Step Estimates

This kind of estimates is often related to minimizing or maximizing the corresponding part of an inequality. Probably, the most known and widely used such a tool is based on the Gibbs-Bogolubov inequalities, which are formulated as follows. Let $A$ and $B$ be Hermitian operators on a Hilbert space, for which the form

$$F[A] \equiv -T \ln \text{Tr} e^{-\beta A}$$

exists, where $\beta T \equiv 1$, with $T$ being real. Define an average of $A$ with respect to $B$ as

$$<A>_B \equiv \frac{\text{Tr} e^{-\beta B} A}{\text{Tr} e^{-\beta B}} ,$$

and similarly, an average of $B$ with respect to $A$. Then the Gibbs-Bogolubov inequalities are

$$<A - B>_A \leq F[A] - F[B] \leq <A - B>_B .$$

(1)

This is the generalized presentation of the inequalities which are better known for the case when, instead of arbitrary Hermitian operators, one deals with Hamiltonians $H$ and $H_0$. In that case, the right-hand side of Eq. (1) becomes the inequality for the free energies,

$$F[H] \leq F[H_0] + <H - H_0>_0 ,$$

where $<\ldots>_0$ implies the averaging with respect to $H_0$. This inequality was derived by Gibbs [1] for classical statistics. Bogolubov [2] generalized it for quantum statistics and added the left-hand side of Eq. (1). When $T \to 0$, the free energy reduces to the ground-state energy. Then, as a particular case, one has the Peierls inequality. Introducing an effective Hamiltonian

$$H_{eff} \equiv H_0 + <H - H_0>_0 ,$$
one may also write
\[ F[H] \leq F[H_{eff}] . \]

In the standard way, one chooses the approximating Hamiltonian \( H_0 = H_0(\omega_q) \) depending on a set of parameters or functions, say on a trial spectrum \( \omega_q \), so that \( H_0 \) could model the considered system and would allow one to calculate the free energy \( F[H_{eff}] \) for the effective Hamiltonian \( H_{eff} = H_{eff}(\omega_q) \). Then, one minimizes \( F[H_{eff}] \) with respect to the trial functions \( \omega_q \) given by the equation
\[
\frac{\delta}{\delta \omega_q} F[H_{eff}(\omega_q)] = 0 .
\]

When \( <H - H_0>_0 > 0 \) is positively defined, an approximate minimization can be done with the help of equation
\[
<H - H_0(\omega_q)>_0 = 0 .
\]

The Gibbs-Bogolubov inequality is constantly used in various problems of statistical mechanics and condensed matter theory. Among thousands of examples, let us mention the self-consistent photon approximation [3]. In quantum mechanics, the minimization of an effective ground-state energy with respect to trial parameters incorporated in a trial wave function is usually named the Ritz variational method. Such methods are termed single step since they give just a single approximation, without hinting on how to obtain subsequent corrections.

B. Asymptotic Perturbation Theory

Contrary to the single step estimates, perturbation theory is a systematic procedure defining a sequence of approximations. It may, of course, happen for some complicated problems that one technically is able to calculate only a few terms of the perturbation sequence, but perturbation theory is systematic in the sense of prescribing a general way for calculating perturbative terms of arbitrary order. There is, especially in physical literature, quite a mess in the usage of the terms ”perturbation theory” as opposed to ”nonperturbative methods”. Therefore, to avoid in what follows linguistic confusion, it is necessary to concretize several principal points and definitions.

The standard perturbation theory presupposes the existence of small parameters permitting one to present solutions in the form of asymptotic series [4]. Because of the latter, the standard perturbation theory may be named asymptotic perturbation theory. Usually, when talking about perturbation theory, one keeps in mind exactly the standard asymptotic perturbation theory.

One can distinguish three main types of asymptotic series occurring in perturbation theory. Suppose the problem is in calculating a real function \( f(x) \) of a variable \( x \in \mathbb{X} \subset \mathbb{R} \). The case of one function of one variable is taken just for the simplicity of notations and is not principal. In general, the function \( f(x) \) can depend on any number of other variables, but we separate and explicitly write down only one variable that is assumed to play the role of a small parameter. Perturbation theory with respect to a small parameter \( |x| \ll 1 \) yields a sequence of approximants \( \varphi_k(x) \), where \( k = 0, 1, 2, \ldots \), approximating the sought function \( f(x) \) in the vicinity of \( x = 0 \),
\[
f(x) \simeq \varphi_k(x) \quad (|x| \ll 1) .
\]
The approximants $\varphi_k(x)$ have the structure of asymptotic series of one of the following types.

(i) Expansion over small parameters:

$$\varphi_k(x) = \sum_{n=0}^{k} a_n x^n . \quad (5)$$

Generally, this can be an expansion over one or several parameters.

(ii) Expansion over asymptotic sequences:

$$\varphi_k(x) = \sum_{n=0}^{k} a_n \varepsilon_n(x) . \quad (6)$$

Here, the sequence $\{\varepsilon_n(x)\}$ is asymptotic in the sense of Poincaré, so that

$$\left| \frac{\varepsilon_{n+1}(x)}{\varepsilon_n(x)} \right| \rightarrow 0 \quad (x \rightarrow 0) . \quad (7)$$

In particular, $\varepsilon_n(x)$ can be $\varepsilon^n(x)$, with $\varepsilon(x)$ being a given function of $x$.

(iii) Generalized asymptotic expansion:

$$\varphi_k(x) = \sum_{n=0}^{k} a_n(x) \varepsilon_n(x) . \quad (8)$$

In the latter, the coefficients $a_n(x)$ retain dependence on $x$ in order to satisfy some additional conditions, but so that the sequence $\{a_n(x)\varepsilon_n(x)\}$ be asymptotic.

The general feature of all the cases above is that the difference

$$\Delta \varphi_k(x) \equiv \varphi_k(x) - \varphi_{k-1}(x) \quad (9)$$

forms an asymptotic sequence $\{\Delta \varphi_k(x)\}$, such that

$$\left| \frac{\Delta \varphi_{k+1}(x)}{\Delta \varphi_k(x)} \right| \rightarrow 0 \quad (x \rightarrow 0) .$$

The construction of generalized asymptotic expansions can often be rather elaborate, aiming at improving the accuracy of calculations. For example, in the Lindstedt-Poincaré method [5], one expands over a small parameter the sought solution and the frequency choosing the expansion coefficients so that to kill secular terms. In the Krylov-Bogolubov averaging technique [6], the generalized asymptotic expansion is constructed above an initial approximation including nonlinearity in order to model well the main properties of a nonlinear system. In the theory of anharmonic crystals [7,8], the expansion in powers of a small anharmonic parameter starts with a self-consistent phonon approximation partly taking account of anharmonicity. Nevertheless, no matter how elaborate is an initial approximation and how complicated is the structure of the resulting generalized expansion, all abovementioned cases presuppose the existence of small parameters and are typical examples of asymptotic perturbation theory.
C. Methods of Successive Iteration

For numerical iterative algorithms, the existence of small parameters is not required. When the considered equation can be presented in the form

$$Af(x) = h(x),$$

in which $A$ is an operator and $h(x)$, a given function, then an iterative solution of this equation reads

$$\varphi_{k+1}(x) = \varphi_k(x) + B_k [A \varphi_k(x) - h(x)],$$

(10)

where $B_k$ are operators chosen so that to make the sequence $\{\varphi_k(x)\}$ convergent [9]. The iterative algorithm (10) leads to approximants $\varphi_k(x)$ that, in general, are not asymptotic series, even if $x \to 0$. This is because an expansion of $\varphi_k(x)$ at $x = 0$, provided it exists, has the structure of a series

$$\varphi_k(x) = \sum_{n=0}^{k} a_n^k x^n,$$

with coefficients $a_n^k$ labelled by two indices. Then the difference (9) is

$$\Delta \varphi_k(x) = \sum_{n=0}^{k-1} (a_n^k - a_{n+1}^{k-1}) x^n + a_n^k x^k,$$

which is not of order $x^k$ but contains lower powers of $x$, since $a_n^k \neq a_{n+1}^{k-1}$. Hence, $\{\Delta \varphi_k(x)\}$ is not an asymptotic sequence.

Though the method of successive iteration looks more general than perturbation theory, it has a weak point of being, in the majority of cases, purely numerical, without providing analytical formulas. And also, for very complicated problems a numerical procedure can be too much time consuming or even unsolvable.

2 Optimized Perturbation Theory

A qualitatively different systematic approach was advanced [10] for treating the problems without small parameters and permitting one to obtain, at least in first orders, approximate analytical solutions. This approach was motivated by and applied to many-particle systems with strong interactions [11–21]. Technically, the approach is based on the methods of perturbation and iteration theories combined with optimal control theory.

The pivotal idea of the approach [10] is the introduction of control functions whose role is to govern the convergence of approximation sequences. Generally, an approximation theory consists of three parts: an initial approximation, a calculational algorithm, and a sequence analysis. Control functions can be introduced in any of these parts. The main is that the resulting approximation sequence $\{F_k(x, u_k)\}$ be convergent due to control functions $u_k = u_k(x)$. Optimized approximants are

$$f_k(x) \equiv F_k(x, u_k(x)) \quad (k = 0, 1, 2, \ldots),$$

(11)

forming a convergent sequence $\{f_k(x)\}$. In obtaining the approximations $\{F_k(x, u_k)\}$, it is not required to have any small parameters, but, if the techniques of perturbation theory
are employed, an expansion is made with respect to a formal parameter $\varepsilon$ that at the end is set to unity

$$F_k(x, u_k) = \sum_{n=0}^{k} a_{nk}(x)\varepsilon^n \quad (\varepsilon \to 1).$$

This expansion is not asymptotic, even when $x \to 0$, since the coefficients $a_{nk}$ are labelled by two indices. In this way, the expressions $F_k(x, u_k)$, although can be obtained by means of formal techniques of perturbation theory, have the structure typical of the terms of an iterative procedure. Thus, the approach is not an asymptotic perturbation theory. Moreover, the approximations $F_k(x, u_k)$ may also be obtained by means of an iterative algorithm. Therefore, the optimized perturbation theory is principally different from the standard asymptotic perturbation theory, since the former does not require any small parameters and the structure of its approximate terms is not that of asymptotic series.

**Definition.** *Optimized Perturbation Theory* is a systematic method defining a sequence of successive approximants, whose convergence is governed by control functions.

How could we formulate general rules for finding such control functions? Since the role of the latter is to govern convergence, let us write down the Cauchy criterion of uniform convergence for the sequence $\{F_k(x, u_k)\}$, with $x \in X$. The sequence is uniformly convergent on $X$ if and only if for each given $\varepsilon$ there exists $N_\varepsilon$ such that

$$|F_{k+p}(x, u_{k+p}) - F_k(x, u_k)| < \varepsilon$$

for all $k \geq N_\varepsilon$ and $p \geq 1$. We also wish that convergence be as fast as possible. A general way of defining control functions, providing for the considered system the required property, is formulated by the *optimal control theory* [22]. According to this theory, control functions are given by minimizing a cost functional. In our case, the cost functional guaranteeing the fastest convergence can be constructed as the *fastest-convergence cost functional*

$$F_u = \frac{1}{2} \sum_{n=0}^{\infty} [F_{n+p}(x, u_{n+p}) - F_n(x, u_n)]^2,$$

in line with the Cauchy criterion (12). The fastest convergence is achieved by means of control functions, minimizing the cost functional (13), that is, from the condition

$$\frac{\delta F_u}{\delta u_k} = 0, \quad \frac{\delta^2 F_u}{\delta u_k^2} > 0.$$  

(14)

The variational derivatives are

$$\frac{\delta F_u}{\delta u_k} = (2F_k - F_{k+p} - F_{k-p})F'_k,$$

$$\frac{\delta^2 F_u}{\delta u_k^2} = 2(F'_k)^2 + (2F_k - F_{k+p} - F_{k-p})F''_k,$$

(15)

where $F_k = F_k(x, u_k), 1 \leq p \leq k,$ and

$$F'_k = \frac{\partial F_k}{\partial u_k}, \quad F''_k = \frac{\partial^2 F_k}{\partial u_k^2}.$$
The first of Eqs. (14) possesses two solutions yielding two possible optimization conditions.

(i) **Differential optimization condition**

\[
\frac{\partial}{\partial u_k} F_k(x, u_k) = 0 \tag{16}
\]

gives an extremum of the cost functional, but it is not clear what this extremum is, since the sign of the second derivative

\[
\frac{\delta^2 F_u}{\delta u_k^2} = (2F_k - F_{k+p} - F_{k-p})F_k''
\]

is not defined. To concretize the situation, we need to invoke some additional constraints. Consider a particular case of \( p = 1 \) and assume the validity of the *fast convergence condition*

\[
F_k(x, u_k) \approx F_{k+1}(x, u_{k+1}) \tag{17}
\]

Then the differential condition (16) is an approximate condition for the minimum of the cost functional, since

\[
\frac{\delta^2 F_u}{\delta u_k^2} \approx 0, \quad \frac{\delta^3 F_u}{\delta u_k^3} \approx 0, \quad \frac{\delta^4 F_u}{\delta u_k^4} \approx 6(F_k'')^2 > 0.
\]

(ii) **Difference optimization condition**

\[
F_{k+p} - 2F_k + F_{k-p} = 0 \tag{18}
\]

clearly correspond to the minimum of the cost functional, as far as

\[
\frac{\delta^2 F_u}{\delta u_k^2} = 2(F_k')^2 > 0.
\]

However, this does not uniquely define control functions, since Eq. (18) contains three of them, \( u_{k+p}, u_k, \) and \( u_{k-p}. \) To resolve the problem, consider again the case of \( p = 1 \) and assume that the value \( F_k(x, u_k) \) weakly depends on the change of \( u_k \) by \( u_{k+1}, \) which can be formulated as the *weak sensitivity condition*

\[
F_k(x, u_k) \approx F_k(x, u_{k+1}) \tag{19}
\]

Then the difference condition (18) reduces to the equation

\[
F_k(x, u_k) - F_{k-1}(x, u_k) = 0, \tag{20}
\]

making it possible to define \( u_k = u_k(x). \)

In this way, neither Eq. (16) nor Eq. (18) can serve as exact equations for minimizing the cost functional (13) and for uniquely defining control functions. But the latter are unambiguously defined and the cost functional (13) is approximately minimized by either optimization condition (16) or condition (20). These conditions are equivalent to each other. Both of them are approximate. Both are variational, following from the variation of a cost functional. Both invoke the notion of weak sensitivity with respect to the variation.
of control functions. Both are designed for supplying fastest convergence of the sequence of optimized approximants by minimizing the fastest-convergence cost functional. Being completely equivalent by their meaning, the optimization conditions (16) and (20) differ only by their form, being expressed either through a derivative or through a finite difference.

Solving one of the optimization equations, either (16) or (20), one gets control functions \( u_k = u_k(x) \) governing convergence of the optimized approximants (11). It may happen for some \( k \) that an optimization equation does not have an exact solution. Then, since the optimization conditions themselves are approximate, it is admissible to look for an approximate solution for a control function. For example, if the latter becomes complex, one may take only its real part. Or, when neither the derivative (16) nor difference (20) are exactly zero, one may look for control functions minimizing one of the following forms:

\[
\min_{u_k} \left| \frac{\partial}{\partial u_k} F_k(x, u_k) \right|, \quad \min_{u_k} \left| F_k(x, u_k) - F_{k-1}(x, u_k) \right|
\]

As is mentioned at the beginning of this section, control functions can be incorporated at any step of the theory. A straightforward way is to introduce them in the initial approximation. For instance, if one considers a problem described by a Hamiltonian \( H \), one may take for the initial approximation a Hamiltonian \( H_0(u) \) containing a set of trial parameters \( u \). Then the Hamiltonian of the problem can be presented as

\[
H = H_0(u) + \varepsilon [H - H_0(u)]
\]

with a formal parameter \( \varepsilon \to 1 \). Accomplishing perturbative calculations with respect to the formal parameters \( \varepsilon \), one sets it to unity and obtains perturbative terms \( F_k(x, u) \). After this, one finds control functions from an optimization condition and substitute them into \( F_k(x, u_k) \). Absolutely the same procedure can be realized if the considered problem is characterized not by a Hamiltonian but by a Lagrangian or an action. It is also possible to incorporate trial parameters into an initial approximation for a wave function or for Green functions and to derive the subsequent approximations by means of an iterative procedure.

As is evident, there can be a variety of particular technical ways of introducing initial approximations and control functions. But all such variants are based on the same fundamental idea of control functions governing convergence for the sequence of optimized approximants [10]. Several years after Ref. [10], there appeared a series of papers [23–28] advertizing the same idea of introducing control functions for rendering perturbation theory convergent. Nowadays the optimized perturbation theory is widely used for various problems, being employed under different guises and called by different names, such as modified perturbation theory, renormalized perturbation theory, variational perturbation theory, controlled perturbation theory, self-consistent perturbation theory, oscillator-representation method, delta expansion, optimized expansion, nonperturbative expansion, and so on [23–32]. Many problems of quantum mechanics, statistical mechanics, condensed matter physics, and quantum field theory are successively treated by this optimized approach. Not to list numerous applications, let us cite a couple of recent reviews [33,34].

Despite a number of very successful applications, optimized perturbation theory does not provide answers to the following important questions:
(i) How to improve the accuracy with a given number of approximate terms?
(ii) How to realize a step-by-step control of the stability of the method, if no exact solutions for the problem are known?
(iii) How to choose the best initial approximation, if several of them are admissible?

To answer these questions it is necessary to look at the problem from the point of view of a more general approach, which is described in the next section.

3 Self-Similar Approximation Theory

An approach, more general than optimized perturbation theory, has been developed [35–57], being based on the techniques of the theory of dynamical systems and optimal control theory. The underlying idea of the approach is to consider the passage from one successive approximation to another as the motion on the manifold of approximations. Then, the approximation order $k = 0, 1, 2, \ldots$ should play the role of discrete time, and each approximant should represent a point of a trajectory in the phase space of approximants.

To correctly describe evolution, one has to construct a dynamical system. For this purpose, we again need to introduce control functions. Employing some variant of the method of successive approximations, we get the terms $F_k(x, u_k)$. The role of control functions $u_k = u_k(x)$ is to make convergent the sequence $\{f_k(x)\}$ of the optimized approximants

$$f_k(x) \equiv F_k(x, u_k(x)). \quad (21)$$

Convergence implies the existence of the limit

$$\lim_{k \to \infty} f_k(x) = f^*(x). \quad (22)$$

Let all approximating functions $f_k(x)$, with $k = 0, 1, 2, \ldots$ and $x \in X$, together with the limit (22) pertain to a complete space $\mathcal{A}$, which we call approximation space and which plays the role of a phase space for the evolution of $f_k(x)$ with respect to the discrete time $k$.

Introduce a function $x_k(\varphi)$ by the reonomic constraint

$$F_0(x, u_k(x)) = \varphi, \quad x = x_k(\varphi). \quad (23)$$

Changing the variables, we define

$$y_k(\varphi) \equiv f_k(x_k(\varphi)). \quad (24)$$

The transformation $y_k : \mathcal{A} \to \mathcal{A}$ is an endomorphism of the phase space $\mathcal{A}$, with a unitary element given by the equation

$$y_0(\varphi) = \varphi. \quad (25)$$

By definition (24), each $f_k(x)$ corresponds to $y_k(\varphi)$ and, conversely, to each $y_k(\varphi)$ we set in correspondence

$$f_k(x) = y_k(F_0(x, u_k(x))). \quad (26)$$
By this construction, the sequences \( \{y_k(\varphi)\} \) and \( \{f_k(x)\} \) are bijective. The existence of the limit (22) implies the existence of the limit
\[
\lim_{k \to \infty} y_k(\varphi) = y^*(\varphi) ,
\]
so that
\[
f^*(x) = y^*(F_0(x, u^*(x))) ,
\]
where \( u^*(x) = \lim_{k \to \infty} u_k(x) \). For the map \( \{y_k(\varphi)\} \), the limit \( y^*(\varphi) \) is a fixed point, when
\[
y_k(y^*(\varphi)) = y^*(\varphi) .
\]

The dependence of the fixed point \( y^*(\varphi) \) on the starting point \( \varphi \) is due to the reonomic constraint (23).

A particular form of the endomorphism (24) depends on the choice of an initial approximation \( F_0(x, u) \) and of control functions \( u_k(x) \). These are to be chosen so that to guarantee the fastest convergence of the sequence \( \{y_k(\varphi)\} \). Uniform convergence on \( A \) implies the validity of the Cauchy criterion
\[
|y_{k+p}(\varphi) - y_k(\varphi)| < \varepsilon
\]
for all \( k \geq N_\varepsilon \) and \( p \geq 1 \). This suggests that the evolution in the map \( \{y_k(\varphi)\} \) is to be such that to minimize the fastest-convergence cost functional
\[
F_y = \frac{1}{2} \sum_{n=0}^{\infty} [y_{n+p}(\varphi) - y_n(\varphi)]^2 .
\]
The minimization is to be done with respect to any \( y_n(\varphi) \), including \( y_0(\varphi) = \varphi \). As is evident, the absolute minimum of the functional (31) is realized if and only if the initial point \( \varphi \) coincides with the fixed point \( y^*(\varphi) \), when
\[
F_y = 0 , \quad \varphi = y^*(\varphi) .
\]

**Proposition.** For the fastest-convergence cost functional to be minimal it is necessary that the self-similarity relation
\[
y_{k+p}(\varphi) = y_k(y_p(\varphi))
\]
be valid.

**Proof.** The absolute minimum of the cost functional (31) is \( F_y = 0 \). This is realized if and only if \( \varphi = y^*(\varphi) \), when equation (33) becomes an identity.

Clearly, the self-similarity relation (33) is a necessary but not sufficient condition for minimizing the cost functional (31). This relation describes the property of self-similarity in the general sense, which includes as a particular case the scaling self-similarity \( y_k(\lambda \varphi) = \lambda^{\alpha_k} y_k(\varphi) \) taking place for the power-law functions \( y_k(\varphi) = \varphi^{\alpha_k} \), with the powers such that \( \alpha_{k+p} = \alpha_k \cdot \alpha_p \). The scaling self-similarity is what one usually keeps in mind when mentioning this property. However, this type of self-similarity is just a particular trivial case of the relation (33). The latter for the endomorphism \( y_k \), together with the unitary element (25), defines the semigroup properties
\[
y_{k+p} = y_k \cdot y_p , \quad y_0 = 1 ,
\]
because of which the relation (33) can be called the \textit{group self-similarity}. In the theory of dynamical systems \cite{58,59}, the family of endomorphisms \( \{ y_k \mid k \in \mathbb{Z}_+ \} \), where \( \mathbb{Z}_+ \equiv \{0, 1, 2, \ldots \} \), is termed a cascade, which is a dynamical system in discrete time. Since in our case this family of endomorphisms is formed of the sequence \( \{ y_k(\varphi) \} \) of the approximants \( y_k(\varphi) \), the corresponding cascade can be named the \textit{approximation cascade}.

To deal with discrete time is less convenient than with continuous time, when the latter is given on \( \mathbb{R}_+ \equiv [0, \infty) \). Therefore, it is useful to embed the cascade into a flow. The \textit{embedding}

\[
\{ y_k \mid k \in \mathbb{Z}_+ \} \subset \{ y(t, \ldots) \mid t \in \mathbb{R}_+ \}
\]  

implies that the flow possesses the same semigroup property

\[
y(t + t', \varphi) = y(t, y(t', \varphi))
\]  

and the trajectory \( \{ y(t, \varphi) \} \) of the flow passes through all points of the cascade,

\[
y(t, \varphi) = y_k(\varphi) \quad (t = k \in \mathbb{Z}_+).
\]  

The flow embedding the approximation cascade is termed the \textit{approximation flow}.

The advantage of dealing with a flow is that differentiating the self-similarity relation (36) one comes to a differential Lie equation

\[
\frac{\partial}{\partial t} y(t, \varphi) = v(y(t, \varphi)),
\]  

where \( v(y) \) is a velocity field. Integrating the evolution equation (38) between \( y_{k-1} = y_{k-1}(\varphi) \) and \( y_k^* = y_k^*(\varphi) \), where \( y_k^* \) is an approximate fixed point, we get the \textit{evolution integral}

\[
\int_{y_{k-1}}^{y_k^*} \frac{dy}{v(y)} = \tau_k,
\]  

with \( \tau_k \) being the \textit{effective approximation time} required for reaching the quasifixed point \( y_k^* \). For short, we may call \( \tau_k \) the \textit{control time}. Due to the relations (23) to (28), the integral (39) may be presented as

\[
\int_{f_{k-1}}^{f_k} \frac{d\varphi}{v_k(\varphi)} = \tau_k,
\]  

where \( f_k = f_k(x), \) \( f_k^* = f_k^*(x) \), with

\[
f_k^*(x) \equiv y_k^*(F_0(x, u_k(x))),
\]  

and \( v_k(\varphi) \) is the cascade velocity given by a discretization of the flow velocity. The Euler discretization of the flow velocity is defined as

\[
v_k(\varphi) \equiv V_k(x_k(\varphi))
\]  

where

\[
V_k(x) = F_k(x, u_k) - F_{k-1}(x, u_k) + (u_k - u_{k-1}) \frac{\partial}{\partial u_k} F_k(x, u_k),
\]  

with \( u_k = u_k(x) \) and \( k = 1, 2, \ldots \).
Our aim is to find a fixed point $y^*(\varphi)$ of the map $\{y_k(\varphi)\}$, which, by construction, corresponds to the sought function $f(x)$. The fixed point for a flow is given by the zero velocity $v(y^*) = 0$. For the approximation cascade, we have

$$V_k(x) = 0 .$$  \hspace{1cm} (44)

This condition is to be treated as an equation for control functions $u_k(x)$. However, the cascade velocity (43) contains two control functions, $u_k(x)$ and $u_{k-1}(x)$, hence one equation (44) cannot define them both. Thus, we either have to invoke some additional constraints or can use an approximate equation for finding control functions, for instance, by minimizing the absolute value of the cascade velocity

$$|V_k(x)| \leq |F_k(x, u_k) - F_{k-1}(x, u_k)| + \left| (u_k - u_{k-1}) \frac{\partial}{\partial u_k} F_k(x, u_k) \right|. \hspace{1cm} (45)$$

There are three ways of formulating fixed-point conditions.

**Fixed-point condition 1.** Find $u_1(x)$ from an additional condition, say, from the differential optimization condition (16) or from the difference optimization condition (20). Then, all other $u_k(x)$, with $k \geq 2$, are given by Eq. (44), that is, by the equation

$$F_k(x, u_k) - F_{k-1}(x, u_k) + (u_k - u_{k-1}) \frac{\partial}{\partial u_k} F_k(x, u_k) = 0 . \hspace{1cm} (46)$$

The disadvantage of this way is that control functions for $k = 1$ and $k \geq 2$ are defined by different conditions. Also, if for some $k$ equation (46) has no solutions, then an ambiguity arises requiring some additional assumptions.

**Fixed-point condition 2.** Minimizing the first term in the right-hand side of Eq. (45), one has

$$F_k(x, u_k) - F_{k-1}(x, u_k) = 0 , \hspace{1cm} (47)$$

which coincides with the difference condition (20). Then the cascade velocity (43) is

$$V_k(x) = (u_k - u_{k-1}) \frac{\partial}{\partial u_k} F_k(x, u_k) . \hspace{1cm} (48)$$

Since Eq. (47) gives $u_k(x)$ starting from $k = 1$, the function $u_0(x)$ is left undefined. Hence, the velocity (48) is valid for $k \geq 2$. If Eq. (47) has no solution for some $k$, then one could find $u_k(x)$ by minimizing $|F_k - F_{k-1}|$. But in that case, the cascade velocity is not given by Eq. (48).

**Fixed-point condition 3.** Minimize the second term in the right-hand side of Eq. (45), which yields

$$(u_k - u_{k-1}) \frac{\partial}{\partial u_k} F_k(x, u_k) = 0 . \hspace{1cm} (49)$$

This is to be understood as the equation

$$\frac{\partial}{\partial u_k} F_k(x, u_k) = 0 , \hspace{1cm} u_k \neq u_{k-1} , \hspace{1cm} (50)$$

provided a solution for $u_k(x)$ exists, or as the equality

$$u_k = u_{k-1} , \hspace{1cm} \frac{\partial}{\partial u_k} F_k(x, u_k) \neq 0 , \hspace{1cm} (51)$$
when the differential condition (50) does not possess a solution for \( u_k(x) \). In all the cases under condition (49), the cascade velocity becomes

\[
V_k(x) = F_k(x, u_k) - F_{k-1}(x, u_k),
\]

where \( u_k = u_k(x) \) and \( k \geq 1 \).

Comparing the possible fixed-point conditions (46), (47), and (49), we see that the latter is more general and provides a unique unambiguous way for defining control functions \( u_k(x) \) for all \( k \geq 1 \). In all three cases, the function \( u_0(x) \) is, generally, not defined, so is the term \( f_0(x) = F_0(x, u_0(x)) \). Therefore, the evolution integral (40) has to be considered starting with \( k = 2 \). The effective time \( \tau_k \) in Eq. (40) can be treated as another control function. By its definition, \( \tau_k \) is the minimal time required for reaching a quasifixed point \( f_k^* \) from an approximant \( f_{k-1} \). In general, the minimal time should correspond to one step, which means that \( k\tau_k \) should be close to one. Therefore, the effective approximation time can be evaluated as

\[
\tau_k = \frac{1}{k}.
\]

In this way, an approximate fixed point \( f_k^*(x) \), representing a \( k \)-th order self-similar approximant for the sought function \( f(x) \), is completely defined by the evolution integral (40).

An important advantage of the self-similar approximation theory is the possibility of controlling the stability of the procedure, which can be done by invoking the ideas of dynamical theory. For this purpose, for a map \( \{y_k(\varphi)\} \), one may define the local multipliers

\[
\mu_k(\varphi) \equiv \frac{\partial}{\partial \varphi} y_k(\varphi).
\]

The multiplier at a quasifixed point \( y_k^*(\varphi) \) is given by

\[
\mu_k^*(\varphi) \equiv \mu_k(y_k^*(\varphi)).
\]

The images of these multipliers on the manifold \( \mathbb{X} \) are obtained by means of the change of variables (23). The image of Eq. (54) reads

\[
m_k(x) \equiv \mu_k(F_0(x, u_k(x))) \equiv \mu_k^*(x),
\]

and that of Eq. (55) is

\[
m_k^*(x) \equiv \mu_k^*(f_k^*(x)) \equiv m_k^*(x).
\]

A quasifixed point is stable provided that

\[
|\mu_k^*(\varphi)| < 1, \quad |m_k^*(x)| < 1.
\]

It is useful to consider uniform stability characterized by the maximal local multipliers

\[
\mu_k^* \equiv \sup_{\varphi} |\mu_k^*(\varphi)|, \quad m_k^* \equiv \sup_x |m_k^*(x)|.
\]

Then a \( k \)-th order approximant is uniformly stable if

\[
\mu_k^* < 1, \quad m_k^* < 1.
\]
Generally, because of the definition of the multipliers (55) to (57) through the same local multiplier (54) given on the phase space \( \mathcal{A} \), the maximal multipliers (59) coincide, 
\[
\mu_k^\ast = m_k^\ast
\]
so that it is sufficient to consider one of them.

The stability analysis also makes it possible to answer the question "which of several admissible initial approximations should one prefer in calculating higher-order approximants?" The answer is straightforward: One has to prefer that initial approximation which guarantees the best stability of the procedure [57].

In conclusion to this section, let us emphasize a principal aspect distinguishing our approach from different variants of the standard renormalization-group techniques [60–62]. In the latter, one tries to establish either an exact or an approximate relation between a function \( f(x) \) and its value \( f(\lambda x) \) for the scaled physical variable. Such a relation describes the motion with respect to the scaling parameter. Contrary to this, in self-similar approximation theory, we do not scale physical variables. But the group self-similarity describes an evolution in the phase space of approximants, with the approximation order playing the role of time.

\section{Method of Fractal Transforms}

One of the basic ideas in self-similar approximation theory is the introduction of control functions which govern the evolution of an approximation dynamical system to be close to a fixed point. As was mentioned earlier, control functions can be introduced at any part of calculational procedure. For instance, this can be done at the step of choosing an initial approximation, which results in the sequence of optimized approximants. But this can also be accomplished in the last part of calculations, after deriving a sequence of perturbative terms.

As is discussed in Section 1, employing the standard perturbation theory, one obtains approximations having the structure of asymptotic series. Such series are usually divergent and have no sense for finite values of expansion parameters. There exist the so-called resummation methods ascribing finite values to divergent series [63]. The most often used among such techniques are the Borel summation [63] and the construction of Padé approximants [64], including the two-point [65] and multivalued [66,67] Padé approximants. These techniques have many known limitations. Thus, to get a good accuracy, they require to invoke a number of perturbative terms which often are not available. And also, such techniques are, actually, numerical.

In order to incorporate control functions into a given asymptotic series, one needs to resort to a transformation including some trial parameters. The transformation involved must decode the self-similarity property hidden in the given perturbative sequence. For power series, it looks natural to employ the power-law transformations [68–75]. Since power laws are typical of fractals [76,77] the power-law transformation can also be called the fractal transformation [75].

For a function \( f(x) \), the fractal transform is

\[
F(x, s) \equiv x^s f(x) ,
\]

with a real \( s \). The inverse transform is

\[
f(x) \equiv x^{-s} F(x, s).
\]
The fractal transform satisfies the scaling relation
\[
\frac{F(\lambda x, s)}{f(\lambda x)} = \lambda^s \frac{F(x, s)}{f(x)}
\]
reminding us those typical of fractals.

Assume that for a finite function \( f(x) \), the standard perturbation theory in powers of \( x \) results in a set of approximations
\[
\varphi_k(x) = \sum_{n=0}^{k} \varphi_n x^{\gamma_n}, \quad (63)
\]
where \( \varphi_0(x) = \varphi_0 x^{\gamma_0} \neq 0 \) and \( x \to 0 \). The powers are arranged in the ascending order
\[
\gamma_n < \gamma_{n+1} \quad (n = 0, 1, 2, \ldots, k - 1) ; \quad (64)
\]
the signs of \( \gamma_n \) being arbitrary. The structure of the series (63) is that of an expansion (6) over the asymptotic sequence \( \varepsilon_n(x) = x^{\gamma_n} \).

Since it is always more convenient to work with dimensionless quantities, we assume that the variable \( x \) is dimensionless. Define the dimensionless scale-invariant function
\[
g_k(x) \equiv \varphi_k(x) / \varphi_0(x). \quad (65)
\]
Employing the notation
\[
a_n \equiv \frac{\varphi_n}{\varphi_0}, \quad \alpha_n \equiv \gamma_n - \gamma_0 ,
\]
we have
\[
g_k(x) = \sum_{n=0}^{k} a_n x^{\alpha_n}, \quad (66)
\]
where
\[
g_0(x) = a_0 = 1 \quad \alpha_0 = 0 , \quad (67)
\]
and the powers are such that
\[
0 < \alpha_n < \alpha_{n+1} \quad (n = 1, 2, \ldots, k - 1) . \quad (68)
\]

Introduce the \textit{fractal transform}
\[
F_k(x, s) \equiv x^s g_k(x) \quad (69)
\]
and the inverse one
\[
g_k(x) = x^{-s} F_k(x, s) . \quad (70)
\]
For the series (66), this gives
\[
F_k(x, s) = \sum_{n=0}^{k} a_n x^{s+\alpha_n}. \quad (71)
\]
Then we apply the same procedure of self-similar approximation theory, described in the previous section, to the sequence \( \{F_k(x, s)\} \). The difference is that here the trial parameter
s is treated as a quasi-invariant, that is, it is kept fixed under the evolution of the map \( \{ F_k(x, s) \} \) and it is transformed in a control function \( s_k(x) \) aposteriori, the latter being defined from convergence and boundary conditions.

The reonomic constraint (23) now becomes

\[ F_0(x, s) = \varphi , \quad x = x(\varphi, s) . \]  

(72)

With the form (71), from where \( F_0(x, s) = x^s \), this gives \( x(\varphi, s) = \varphi^{1/s} \). The endomorphism (24) now is

\[ y_k(\varphi, s) \equiv F_k(x(\varphi, s), s) , \]  

(73)

which results in

\[ y_k(\varphi, s) = \sum_{n=0}^{k} a_n \varphi^{1+\alpha_n/s} . \]  

(74)

For \( s \) being a quasi-invariant, the cascade velocity (42) is

\[ u_k(\varphi, s) = y_k(\varphi, s) - y_{k-1}(\varphi, s) . \]  

(75)

From Eqs. (74) and (75), one gets

\[ u_k(\varphi, s) = a_k \varphi^{1+\alpha_k/s} . \]

This is to be substituted in the evolution integral

\[ \int_{y_{k-1}^*}^{y_k^*} \frac{dy}{u_k(y, s)} = \tau_k , \]  

(76)

which is similar to the integral (39), and where \( y_k^* = y_k^*(\varphi, s) \) is a quasifixed point, with \( y_0^*(\varphi, s) \equiv \varphi \) and \( k \geq 1 \). Note a slight difference between the integrals (39) and (76). In the present case, the evolution integral (76) is obtained by integrating the Lie equation (38) between two quasifixed points, \( y_{k-1}^* \) and \( y_k^* \). After changing variables according to the constraint (72), the integral (76) reduces to

\[ \int_{F_{k-1}}^{F_k} \frac{d\varphi}{u_k(\varphi, s)} = \tau_k , \]  

(77)

where \( F_k^* = F_k^*(x, s) \), \( k \geq 1 \) and

\[ F_k^*(x, s) \equiv y_k^*(F_0(x, s), s) . \]  

(78)

For \( k = 0 \), since \( y_0^*(\varphi, s) = \varphi \), one has \( F_0^*(x, s) = x^s \).

Calculating the evolution integral (77), with the cascade velocity (75), results in the iterative equation

\[ (F_k^*)^{\delta_k} = \left( F_{k-1}^* \right)^{\delta_k} + A_k , \]  

(79)

in which

\[ A_k \equiv a_k \delta_k \tau_k , \quad \delta_k \equiv -\frac{\alpha_k}{s} . \]

If we recall that, to return to the original physical quantities, we have to accomplish the inverse fractal transform (70), then we define

\[ g_k(x, s) \equiv x^{-s}F_k^*(x, s) , \]  

(80)
with \( g_0(x, s) = 1 \). For the transform (80), the iterative equation (79) reads
\[
g_k^{\delta_k} = g_{k-1}^{\delta_k} + A_k x^{\alpha_k}.
\] (81)

At this stage, we have to convert the trial parameter \( s \) into a control function \( s_k = s_k(x) \). Taking this into account, we come to a self-similar approximant
\[
g^*_k(x) \equiv g_k(x, s_k).
\] (82)
Equation (81) for the approximant (82) can be written as
\[
g^*_k(x) = \left\{ \left[ g_{k-1}^*(x) \right]^{\delta_k} + A_k x^{\alpha_k} \right\}^{1/\delta_k},
\] (83)
where \( \delta_k \equiv -\alpha_k/s_k \) and
\[
g_0^*(x) = 1.
\] (84)
The solution to Eq. (83), with the initial condition (84), gives \( g_k^*(x) \), which defines the self-similar approximant
\[
f^*_k(x) = \varphi_0(x) g_k^*(x)
\] (85)
for the sought function \( f(x) \). The quantities \( A_k \) and \( \delta_k \) are expressed through control functions \( \tau_k \) and \( s_k \), which actually means that \( A_k \) and \( \delta_k \) can be considered themselves as control functions. The latter are to be defined from additional conditions, such as convergence and boundary conditions.

5 Self-Similar Root Approximants

When the behaviour of the sought function is known in the asymptotic vicinity of two boundaries of the domain \( \mathbb{X} \), the control parameters \( A_k \) and \( \delta_k \) can be found from the related boundary conditions [71,73,74]. Here, we generalize this procedure to the case when the boundary asymptotic expansions contain arbitrary powers of \( x \), including noninteger powers.

Suppose, for concreteness, that the variable \( x \) is given on the real semiaxes \( \mathbb{X} = \mathbb{R}_+ \equiv [0, \infty) \). If this is not so, then it is always possible to resort to a change of variables reducing the domain of \( x \) to \( \mathbb{R}_+ \). As earlier, we keep in mind that the variable \( x \) as well as the sought function are normalized to dimensionless units, so that the considered function is presented in a scale-invariant form \( g(x) \). Assume that the asymptotic behaviour of \( g(x) \) in the vicinity of the left boundary,
\[
g(x) \simeq g_k(x) \quad (x \to 0),
\] (86)
is given by the asymptotic series
\[
g_k(x) = \sum_{n=0}^{k} a_n x^{\alpha_n},
\] (87)
where
\[
g_0(x) = a_0 = 1, \quad \alpha_0 = 0.
\] (88)
and the powers are arbitrary real numbers arranged in the ascending order
\[ 0 < \alpha_n < \alpha_{n+1} \quad (n = 1, 2, \ldots, k - 1) . \] (89)

And let us assume that the asymptotic behaviour of \( g(x) \) at the right boundary is also known,
\[ g(x) \simeq G_k(x) \quad (x \to \infty), \] (90)
being presented by the asymptotic series
\[ G_k(x) = \sum_{n=1}^{k} b_n x^{\beta_n}, \] (91)
in which
\[ b_1 \neq 0, \quad \beta_1 \neq 0, \] (92)
and the powers are arranged in the descending order
\[ \beta_n > \beta_{n+1} \quad (n = 1, 2, \ldots, k - 1). \] (93)

Note that \( \beta_n \) can be of any sign.

Iterating \( k \) times Eq. (83), and using the notation
\[ n_p \equiv \frac{\delta_{p+1}}{\delta_p} \quad (p = 1, 2, \ldots, k - 1), \quad n_k \equiv \frac{1}{\delta_k}, \] (94)
we obtain the self-similar root approximant
\[ g^*_k(x) = (\cdots ((1 + A_1 x^{\alpha_1})^{n_1} + A_2 x^{\alpha_2})^{n_2} + \ldots + A_k x^{\alpha_k})^{n_k}, \] (95)
which can also be called a nested root or superroot. The control parameters \( A_p \) and \( n_p \), with \( p = 1, 2, \ldots, k \), are to be defined from the asymptotic coincidence of expressions (95) and (91) at the right boundary, that is from the asymptotic boundary condition
\[ g^*_k(x) \simeq G_k(x) \quad (x \to \infty). \] (96)

In this way, the crossover formula (95) extrapolates the sought function from the left boundary \( x \to 0 \) to the whole interval \([0, \infty)\).

As is easy to observe, reexpanding Eq. (95) in small \( x \to 0 \) does not reproduce the structure of \( g_k(x) \) in the series (87). For this to hold, one, first, should require that \( \alpha_p = p\alpha \), and even then the expansion coefficients would not coincide with \( a_p \) for \( p > 1 \). However, there is no need to demand that the asymptotic behaviour of \( g^*(x) \) be identical with \( g_k(x) \). Vice versa, such a restriction would essentially spoil the accuracy of the approximant for large \( x \). It is important to bring to mind that the main aim of the self-similar approximation theory is to construct accurate expression uniformly approximating the sought function in the whole interval of the variable \( x \in [0, \infty) \). The asymptotic expansion (87) at \( x \to 0 \) are used only as constructing blocks. In general, it would be possible to modify the form of the self-similar approximant so that it could exactly reproduce \( g_k(x) \) at \( x \to 0 \). But this, from the point of view of an accurate uniform extrapolation, is neither necessary nor correct.
At the same time, by the asymptotic condition (96), the control parameters $A_p$ and $n_p$ are defined so that the asymptotic expansion of $g_k^*(x)$ at $x \to \infty$ exactly coincide with the series (91). Such a construction, as is evident, can be reverted in the following sense. One could derive self-similar root approximants starting from the right asymptotic expansion (91) and fitting the corresponding control parameters so that the expansion of the derived root approximants at $x \to 0$ be coinciding with the series (87). Nevertheless, the construction of crossover formulas from the left to right seems more preferable because of the following. The region of validity for the expansion (87) is $|x| \ll 1$ and that for the expansion (91) is $|x| \gg 1$, that is, the region of validity of the right expansion is essentially larger than that of the left expansion. This conclusion is confirmed by a number of particular cases demonstrating that the radius of convergence of the series (87) is usually zero, while that of the series (91) is finite.

In order to define the control parameters $A_p$ and $n_p$ from the asymptotic condition (96), one has to know how to present an asymptotic form of the root approximant (95) at $x \to \infty$. To the first glance, the procedure of obtaining an asymptotic, as $x \to \infty$, expression from the superroot (95) looks ambiguous, since the powers $n_p$ are yet not known, hence it is not clear how to classify larger and smaller terms. In particular cases, really, there can be several ways, depending on the relation between the values of $\alpha_p$ and $n_p$, of classifying asymptotic terms of the superroot (95) at $x \to \infty$. This ambiguity can be overcome by requiring the uniqueness and generality of the procedure.

**Definition.** The self-similar root approximant (95) is called to be uniquely defined by the asymptotic condition (96) if and only if all control parameters can be uniquely determined from a general rule, whose form is invariant with respect to the values of $\alpha_p$ and which is valid for arbitrary $p = 1, 2, \ldots$.

**Theorem.** The self-similar root approximant (95) is uniquely defined by the asymptotic condition (96) if and only if the powers $n_p$ are given by the equations

$$\begin{align*}
\alpha_p n_p - \alpha_{p+1} &= \text{const}, \\
\alpha_p n_p &= \alpha_{p+1} - \beta_{k-p} + \beta_{k-p+1}, \\
\alpha_k n_k &= \beta_1 \quad (p = 1, 2, \ldots, k - 1).
\end{align*}$$

(97)

**Proof.** When $x \to \infty$, it is convenient to introduce the small parameter $\varepsilon \equiv x^{-1}$. In terms of the latter, the superroot (95) can be identically presented as

$$g_k^*(x) = A_k^{n_k} x^{\alpha_k n_k} \left(1 + B_k \varepsilon^{\alpha_k - \alpha_{k-1} n_{k-1}} \left(1 + B_{k-1} \varepsilon^{\alpha_{k-1} - \alpha_{k-2} n_{k-2}} (1 + \right.ight.$$

$$
+ B_2 \varepsilon^{\alpha_2 - \alpha_1 n_1} (1 + B_1 \varepsilon^{\alpha_1})^{n_2} \ldots)^{n_{k-1}})^{n_k},
$$

(98)

where

$$B_1 \equiv \frac{1}{A_1}, \quad B_{p+1} \equiv \frac{A_p}{A_{p+1}} \quad (p = 1, 2, \ldots, k - 1).$$

To prove sufficiency, let us assume that Eqs. (97) hold true. Then,

$$\alpha_{p+1} - \alpha_p n_p = \beta_{k-p} - \beta_{k-p+1}.$$
Due to the descending order (93) of $\beta_n$, one has
\[ \alpha_{p+1} - \alpha_p n_p = \text{const} > 0. \]
This unambiguously defines the classification of powers of $\varepsilon$ in the form (98). Expanding Eq. (98) in powers of $\varepsilon$, we observe that the first $k$ terms of the expansion coincide with all $k$ terms of the series (91), that is the asymptotic condition (96) is satisfied. To prove necessity, we assume that the superroot (95) is uniquely defined by the asymptotic condition (96). This implies, according to the above definition, that there exists a unique general expansion of the form (98) in powers of $\varepsilon$, which is invariant with respect to $\alpha_p$ and $p$. As is evident from expression (98), such a unique general expansion is possible if and only if $\alpha_{p+1} - \alpha_p n_p = \text{const} > 0$. Then Eq. (98) can be unambiguously expanded in powers of $\varepsilon$. This expansion is to be compared with the series (91) that can be written as
\[ G_k(x) = b_1 x^{\beta_1} \left( 1 + \frac{b_2}{b_1} \varepsilon^{\beta_1 - \beta_2} \left( 1 + \frac{b_2}{b_2} \varepsilon^{\beta_2 - \beta_3} \left( 1 + \ldots + \frac{b_k}{b_{k-1}} \varepsilon^{\beta_{k-1} - \beta_k} \right) \right) \right) \ldots . \]
Comparing the first $k$ terms of the expansion of Eq. (98) with the series (91), we obtain Eqs. (97).

This theorem makes it possible to apply the same general rules for constructing various crossover formulas. Let us stress that the theorem is a new result that has not yet been published.

Till now, we have considered the situation when the order $k$ of the left expansion (87) coincides with that of the right expansion (91). How could we proceed if these orders were different?

If the number of available terms from the left is less than that from the right, this is not as important, provided we know the law prescribing the values of $\alpha_n$, which is usually known or can be easily guessed. This is because we, actually, do not need to have all coefficients $a_n$, which are incorporated in the control parameters $A_n$, and these are determined through the coefficients $b_n$ and powers $\beta_n$ of the right expansion (91). Therefore, when only $\alpha_n$ are available, nevertheless, we may add to the left expansion the required number of terms up to the order of the right expansion.

When the number of terms in the right expansion is less than that of the left expansion, the situation again is not dangerous. Say, the left expansion $g_k(x)$ is of order $k$, while the right expansion $G_m(x)$ is of order $m < k$. In that case, we iterate Eq. (83) till $g_{k-m+1}(x)$ and at the next step, we set $g_{k-m}(x)$. Iterating in this way Eq. (83) $m$ times, we come to the self-similar root approximant
\[ g_{km}^*(x) = \left( \ldots \left( \left[ (g_{k-m}(x))^{1/n_{k-m+1}} + A_{k-m+1} x^{\alpha_{k-m+1}} \right]^{n_{k-m+1}} + A_{k-m+2} x^{\alpha_{k-m+2}} \right]^{n_{k-m+2}} + \ldots + A_k x^{\alpha_k} \right)^{n_k}, \]
where the notation (94) is used. The superroot (99) for the case $m = k$ returns to the form (95), since $g_0(x) = 1$. The control parameters $A_p$ and $n_p$, with $p = k - m + 1, k - m + 2, \ldots, k$, are defined by the asymptotic boundary condition
\[ g_{km}^*(x) \approx G_m(x) \quad (x \to \infty). \]
Consequently, the self-similar root approximants can always be constructed, even when the number of terms in the left and right asymptotic expansions are not equal to each other.
6 Self-Similar Exponential Approximants

A different strategy is to be pursued when only a single-side asymptotic expansion, say at \( x \to 0 \), is available. Then there are no boundary conditions determining control parameters. The latter are to be specified in a different way appealing to convergence properties. A particular choice of control parameters results in a nice structure of nested exponentials [70,72,75]. Here we present a more refined derivation of the exponential approximants and suggest some novel ways of constructing the cost functionals defining the effective approximation time, that is, the control time.

Looking at the fractal transform (71), it is easy to notice that the convergence properties of the sequence \( \{F_k\} \) improve if \( |x|^s \to 0 \). The latter can be realized when \( x \to 0 \), i.e. in the same situation as for the asymptotic expansion (66). But there are in addition two other possibilities for \( |x|^s \) to tend to zero, when

\[
s \to \begin{cases} +\infty , & |x| < 1 \\ -\infty , & |x| > 1 \end{cases} .
\]

(101)

By taking these limits, we may extend the region of applicability of the function presented by the asymptotic expansion (66), valid only at \( x \to 0 \), to the regions \( [0,1) \) and \( (1,\infty) \).

To derive a self-similar approximant for the case (101), by employing the method of fractal transforms, we need to obtain a kind of an iterative equation, similar to Eq. (83). For this purpose, let us introduce a set of functions

\[
\psi_k(z_n) \equiv 1 + z_n \quad (n = 1, 2, \ldots, k) ,
\]

\[
z_n = z_n(x) , \quad \psi_k(z_{k+1}) \equiv 1 ,
\]

(102)

being iteratively connected with each other by means of the relation

\[
z_n(x) = \frac{a_n}{a_{n-1}} x^{\alpha_n - \alpha_{n-1}} \psi_k(z_{n+1}) .
\]

(103)

Then the series (66) can be identically presented as

\[
g_k(x) = \psi_k(z_1) .
\]

(104)

The self-similar renormalization of \( \psi_k(z_n) \), accomplished by means of the method of fractal transforms, is

\[
\psi_k^*(z_n, s) = \left( 1 - \frac{1}{s} \tau_n z_n \right)^{-s} .
\]

(105)

Realizing the \( k \)-step renormalization for the iterative relations (102) and (103), we find

\[
g_k(x, s) \equiv \psi_k^*(z^*_n, s) ,
\]

(106)

where \( z^*_n = z^*_n(x, s) \), with \( n = 1, 2, \ldots, k \), and

\[
z^*_n(x, s) = \frac{a_n}{a_{n-1}} x^{\alpha_n - \alpha_{n-1}} \psi_k^*(z^*_{n+1}, s) .
\]

(107)

According to the last identity in Eqs. (102),

\[
\psi_k^*(z^*_{k+1}, s) = 1 .
\]
Irrespectively to what limit, either \( s \to +\infty \) or \( s \to -\infty \), is taken in the form (105), one gets the same result
\[
\lim_{s \to \pm \infty} \left( 1 - \frac{1}{s} \tau z \right)^{-s} = \exp(\tau z) .
\]
Therefore, in what follows, we may write \( |s| \to \infty \), keeping in mind any of the limits \( s \to \pm \infty \). Let us define
\[
g^*_k(x) = \lim_{|s| \to \infty} g_k(x, s) .
\]
And introduce the notation
\[
c_n \equiv \frac{a_n}{a_{n-1}} \tau_n , \quad \nu_n \equiv \alpha_n - \alpha_{n-1} .
\]
Taking the limit (108) in the iterative relations (106) and (107), we come to the \textit{self-similar exponential approximant}
\[
g^*_k(x) = \exp \left( c_1 x^{\nu_1} \exp \left( c_2 x^{\nu_2} \ldots \exp \left( c_k x^{\nu_k} \right) \ldots \right) \right) ,
\]
for short, called \textit{superexponential}.

Expression (110) contains the coefficients \( c_n \) that, as is seen from the notation (109), are proportional to the control time \( \tau_n \), which is not yet defined. The simplest way would be to set \( \tau_n = 1/n \), as in Eq. (53). It could also be possible to find \( \tau_n \) from a fixed-point condition. However, the most general and refined way is to determine the control time by minimizing a cost functional [75]. In optimal control theory, one constructs cost functionals by formulating the desired properties of the system. For our case, the procedure can be as follows. If it is recalled that the control time \( \tau_n \) describes the minimal time necessary for reaching a fixed point at the \( n \)-th step of the calculational procedure, then \( n \tau_n \) approximately corresponds to the total time \( \tau^* \) required for reaching the fixed point. When \( n \tau_n \sim \tau^* \), this implies that \( \tau_n \sim \tau^*/n \). The time of reaching a fixed point depends on how far this point is. The shorter is the distance from the point, the faster is the way to it. The distance passed at the \( n \)-th step can be evaluated as \( v^*_n \tau_n \), with \( v^*_n \) being a characteristic velocity at this step. In this manner, we need to find a minimal time \( \tau_n \) that is close to \( \tau^*/n \) and which corresponds to the fastest passage of the distance \( v^*_n \tau_n \). These requirements suggest to construct the \textit{fastest passage cost functional}
\[
F_\tau = \frac{1}{2} \sum_n \left[ \left( \tau_n - \frac{\tau^*}{n} \right)^2 \right] ,
\]
in which the parameter \( \zeta \geq 0 \) is included for generality. The value of \( \zeta \) can be chosen if some additional information on the system is available. In the absence of such an additional information, we set \( \zeta = 1 \).

Defining the characteristic velocity \( v^*_n \), it is natural to associate it with a cascade velocity \( v_n(x) \) taken at the most dangerous value of \( x \), where convergence is the worst and, respectively, the deviation \( v^*_n \tau_n \) should be the largest. Thinking back to the fractal transform (71), we know that the sequence \( \{ F_k \} \) converges, under condition (101), if either \( |x| < 1 \) or \( |x| > 1 \). This means that the dangerous point is \( |x| = 1 \). Therefore, we define
\[
v^*_n = v_n(x) \quad (|x| = 1) .
\]
With the cascade velocity
\[ v_n(x) = g_n(x) - g_{n-1}(x) = a_n x^{\alpha_n}, \quad (113) \]
we have
\[ (v_n^*)^2 = a_n^2. \quad (114) \]
So that the fastest passage cost functional (111) becomes
\[ F_\tau = \frac{1}{2} \sum_n \left[ \left( \tau_n - \frac{\tau^*}{n} \right)^2 + \zeta a_n^2 \tau_n^2 \right]. \quad (115) \]

The control time \( \tau_n \) is given by the minimization of the cost functional (115), i.e. from the conditions
\[ \frac{\delta F_\tau}{\delta \tau_n} = 0, \quad \frac{\delta^2 F_\tau}{\delta \tau_n^2} > 0. \quad (116) \]
The extremum condition leads to
\[ \tau_n = \frac{\tau^*}{n(1 + \zeta a_n^2)}. \quad (117) \]
The found extremum is a minimum, since
\[ \frac{\delta^2 F_\tau}{\delta \tau_n^2} = 1 + \zeta a_n^2 > 0. \]

What has been yet left undefined is the effective total time \( \tau^* \), which can be derived from the following reasoning. If the sought fixed point is reached in one step, this implies that \( \tau_1 = 1 \). Applying this condition to formula (117) yields
\[ \tau^* = 1 + \zeta a_1^2 \quad (\tau_1 = 1). \quad (118) \]

Then the control time is
\[ \tau_n = \frac{1 + \zeta a_1^2}{n(1 + \zeta a_n^2)}. \quad (119) \]
The parameters \( c_n \), defined in Eq. (109), are proportional to the control time \( \tau_n \), because of which they can be called the control parameters or, simply, controllers. With the control time (119), the controllers are
\[ c_n = \frac{a_n(1 + \zeta a_1^2)}{n a_n(1 + \zeta a_n^2)}. \quad (120) \]
In this way, the superexponential (110) is completely defined.

If the function \( g(x) \) was introduced as a scale-invariant form of the sought function \( f(x) \), then the self-similar exponential approximant for the latter is \( f_k^*(x) = \varphi_0(x) g_k^*(x) \). Recall that the function \( g(x) \) has been assumed to be finite on the manifold \( \mathbb{X} \). In the case of a function \( g(x) \) divergent at some point \( x_0 \in \mathbb{X} \), one should consider its inverse \( g^{-1}(x) \), provided this is a finite everywhere on \( \mathbb{X} \). In the example of deriving the control time (119) from the cost functional (115), it is supposed that the function \( f(x) \) is sign definite so that the function \( g(x) \) is nonnegative. When it is known that \( f(x) \) changes its sign, this information has to be encompassed in the procedure. This can be done, for example, by factoring \( f(x) = \varphi(x) g(x) \), with \( g(x) \) being positive. Another possibility could be to incorporate information on the points of the sign change into the constructed cost functional. For describing oscillating functions, it could be conceivable to deal with complex control times.
7 Examples

Self-similar approximation theory has been applied to various physics problems. These applications can be found in the cited references [35–57,68–75]. Among recent works, we may mention the usage of this approach to barrier crossing processes [78,79], critical phenomena [80], and to the rupture of mechanical systems [81]. In the present section, we give several examples which illustrate some new possibilities of the approach.

A. Amplitude of Elastic Scattering

This example is interesting by demonstrating the use of superroots when the number of terms in the left asymptotic expansion is much larger than that in the right expansion.

Consider the scattering of two particles of masses \( m_1 \) and \( m_2 \), with momenta \( p_1 \) and \( p_2 \) before collision and \( p_1' \) and \( p_2' \) after it. The four-momenta are normalized on the mass shell so that \( p_i^2 = m_i^2 \). The scattering amplitudes are usually presented as functions of the Mandelstam variables [82] which are

\[
 s \equiv (p_1 + p_2)^2 = (p_1' + p_2')^2 , \quad t \equiv (p_1 - p_1')^2 = (p_2 - p_2')^2 , \\
 u \equiv (p_1 - p_2')^2 = (p_2 - p_1')^2 , \quad s + t + u = 2(m_1^2 + m_2^2) .
\]

The amplitude of elastic scattering can be expressed, by means of perturbation theory, as an asymptotic expansion in powers of the coupling parameter \( g \),

\[
 T(g,s,t) \simeq g + \sum_{n=2}^{\infty} T_n(s,t)g^n \quad (g \to 0) , \quad (121)
\]

where \( T_n(s,t) \sim \exp(ns) \) as \( s \to \infty \). It is known that for any \( g \) there exists the Froissart upper bound given by the inequality \(|T(g,s,t)| \leq |A(s,t)|\), where a particular form of \( A(s,t) \) depends on whether the considered theory is local [82] or nonlocal [83]. Since the Froissart upper bound is valid for any \( g \), including \( g \to \infty \), let us assume that

\[
 T(g,s,t) \simeq A(s,t) \quad (g \to \infty) . \quad (122)
\]

Our aim is to construct a crossover formula between the left and right expansions (121) and (122), respectively.

Following the general scheme, we, first, have to introduce the scale-invariant function

\[
 f(g) \equiv \frac{1}{g} T(g,s,t) , \quad (123)
\]

in which, for brevity, we do not write explicitly other variables, except \( g \). Denoting for the function (123) the weak-coupling,

\[
 f(g) \simeq \varphi_k(g) \quad (g \to 0) , \quad (124)
\]

and the strong-coupling,

\[
 f(g) \simeq F_1(g) \quad (g \to \infty) , \quad (125)
\]

asymptotic expansions, from Eqs. (121) and (122), we have

\[
 \varphi_k(g) = 1 + a_1 g + a_2 g^2 + \ldots + a_k g^k , \quad F_1(g) = b g^{-1} , \quad (126)
\]
where \( a_n = T_{n+1}(s, t) \) and \( b = A(s, t) \).

The crossover formula for the scattering amplitude

\[
T_k^*(g) = g\varphi^*_{k1}(g)
\]

is obtained by constructing the self-similar root approximant (99) for

\[
\varphi^*_{k1}(g) = \left( (\varphi_{k-1}(g))^{1/n_k} + A_k g^k \right)^{n_k},
\]

where the control parameters \( n_k \) and \( A_k \) are defined by the asymptotic boundary condition (100), which gives

\[
n_k = -\frac{1}{k}, \quad A_k = \frac{1}{b^k}.
\]

Thus, a self-similar root approximant for the scattering amplitude is

\[
T_k^*(g) = \frac{b\varphi_{k-1}(g)g}{[b^k + \varphi^*_k(g)g^k]^{1/k}}.
\]

Note that the expansion \( \varphi_{k-1}(g) \) here can also be converted to \( \varphi^*_k-1(g) \) given by the superexponential (110).

**B. Summation of Numerical Series**

Poorly convergent or divergent numerical series can be summed by means of the superexponentials in the following way. Let us consider a series \( S_\infty = \sum_{n=0}^\infty a_n \), whose particular sums are

\[
S_k = \sum_{n=0}^k a_n .
\]

Introduce the function

\[
S_k(x) \equiv \sum_{n=0}^k a_n x^n ,
\]

for which \( S_k(1) = S_k \). Construct the self-similar exponential approximant \( S_k^*(x) \), according to formula (110). Setting \( x = 1 \) in \( S_k^*(x) \), we get \( S_k^* = S_k^*(1) \), which is the sought self-similar approximant

\[
S_k^* = a_0 \exp \left( c_1 \exp \left( c_2 \ldots \exp(c_k) \right) \ldots \right) ,
\]

with the controllers \( c_n \) given by Eq. (120), where we set \( \zeta = 1 \). If the exact value of \( S_\infty \) is known, one can compare the accuracy of the particular sums (131), characterized by the percentage error

\[
\varepsilon_k \equiv \left( \frac{S_k}{S_\infty} - 1 \right) \cdot 100\% ,
\]

with the accuracy of the self-similar approximants (133), described by the error

\[
\varepsilon_k^* \equiv \left( \frac{S_k^*}{S_\infty} - 1 \right) \cdot 100\% .
\]

As an illustration, let us consider the sum (131), with the coefficients

\[
a_n = \frac{(-1)^n}{2n + 1} .
\]
The sequence \( \{S_k\} \) converges to \( S_\infty = \pi/4 \). This convergence is rather slow, for example, the percentage errors (134) for the first five terms are

\[-15\% , \ 10\% , \ -7.8\% , \ 6.3\% , \ -5.3\% ,\]

while the superexponential (133) gives the errors (135) for the first five approximants as

\[-8.8\% , \ -3.6\% , \ -1.9\% , \ -1.6\% , \ -1.6\% ,\]

demonstrating a much faster convergence.

\textbf{C. Multiloop Feynman Integrals}

Employing the Feynman diagram techniques in quantum field theory or quantum statistical mechanics, one confronts with the so-called multiloop integrals. These can be calculated by means of perturbation theory [84,85] resulting in asymptotic series. The latter can be summed with the help of the superexponentials.

Let us start the illustration with a simple one-loop integral

\[ I(a,D) \equiv \frac{1}{(2\pi)^D} \int \frac{d^D p}{(1 + p^2)^a} , \]  

(136)

where \( a \) is a positive parameter, \( D \) is space dimensionality. The exact value of the integral (136) is known to be

\[ I(a,D) = \frac{\Gamma(a-D/2)}{(4\pi)^{D/2} \Gamma(a)} , \]  

(137)

where \( \Gamma(\cdot) \) is a gamma-function. A perturbative procedure for Eq. (136) can be defined [84,85] by introducing

\[ I(a,D,\varepsilon) \equiv \frac{1}{(2\pi)^D} \int \frac{d^D p}{(1 + \varepsilon p^2)^a/\varepsilon} \]  

(138)

and expanding the integrand in powers of \( \varepsilon \), which, after the integration term by term, results in a series

\[ I(a,D,\varepsilon) = \sum_n \varphi_n \varepsilon^n . \]  

(139)

Since, as follows from Eqs. (136) and (138), \( I(a,D,1) = I(a,D) \), the answer is obtained by setting \( \varepsilon = 1 \).

Accomplishing a partial self-similar exponentization of the series (139), one gets

\[ I^*(a,D,\varepsilon) = \frac{1}{(4\pi a)^{D/2}} \exp \left\{ \frac{D(D+2)}{8a} \varepsilon g(\varepsilon) \right\} , \]  

(140)

with

\[ g(\varepsilon) \equiv \sum_n a_n \varepsilon^n , \]  

(141)

the coefficients \( a_n \) being

\[ a_0 = 1 , \quad a_1 = \frac{D + 1}{6a} , \quad a_2 = \frac{D(D + 2)}{24a^2} , \]

\[ a_3 = \frac{(D + 1)(3D^2 + 6D - 4)}{240a^3} , \quad a_4 = \frac{D(D + 2)(D^2 + 2D - 2)}{240a^4} , \quad \ldots \]
The partial sums of Eq. (141), after setting $\varepsilon = 1$, become
\begin{equation}
g_k = \sum_{n=0}^{k} a_n .
\end{equation}
\tag{142}

The corresponding superexponentials are
\begin{equation}
g_k^* = \exp \left( c_1 \exp \left( c_2 \ldots \exp(c_k) \right) \right) .
\end{equation}
\tag{143}

Finally, for the integral (136), we find the self-similar approximants
\begin{equation}
I_k^*(a, D) = \frac{1}{(4\pi a)^{D/2}} \exp \left\{ \frac{D(D+2)}{8a} g_{k-1}^* \right\} ,
\end{equation}
\tag{144}
with $g_0^* = 1$.

Consider the case of $a = 1$ and $D = 1$, when $I(1, 1) = 1/2$. The perturbation series (139) take the form
\begin{equation}
I(1, 1, \varepsilon) \simeq \frac{1}{\sqrt{4\pi}} \left( 1 + \frac{3}{8} \varepsilon + \frac{25}{128} \varepsilon^2 + \frac{105}{1024} \varepsilon^3 + \frac{1659}{32768} \varepsilon^4 \right) .
\end{equation}
\tag{145}

The coefficients $a_n$ in the sum (142) are
\begin{align*}
a_0 &= 1 , \quad a_1 = \frac{1}{3} , \quad a_2 = \frac{1}{8} , \quad a_3 = \frac{1}{24} , \quad a_4 = \frac{1}{80} .
\end{align*}

For the self-similar approximant (144), we have
\begin{equation}
I_k^*(1, 1) = \frac{1}{\sqrt{4\pi}} \exp \left( \frac{3}{8} g_{k-1}^* \right) .
\end{equation}
\tag{146}

The errors of the perturbative expression (145) at $\varepsilon = 1$ are
\begin{align*}
-44\% , \quad -22\% , \quad -11\% , \quad -5.6\% , \quad -2.8\% ,
\end{align*}
which is to be compared with the errors of the self-similar approximants (146),
\begin{align*}
-7\% , \quad -4.8\% , \quad -0.77\% , \quad -0.14\% , \quad -0.08\% ,
\end{align*}
which are an order smaller.

For the case $a = 2$, $D = 3$, one has $I(2, 3) = 1/8\pi$. The perturbative expression (139) reads
\begin{equation}
I(2, 3, \varepsilon) \simeq \frac{1}{(8\pi)^{3/2}} \left( 1 + \frac{15}{16} \varepsilon + \frac{385}{512} \varepsilon^2 + \frac{4725}{8192} \varepsilon^3 + \frac{228459}{524288} \varepsilon^4 \right) .
\end{equation}
\tag{147}

The coefficients $a_n$ from Eq. (142) are
\begin{align*}
a_0 &= 1 , \quad a_1 = \frac{1}{3} , \quad a_2 = \frac{5}{32} , \quad a_3 = \frac{41}{480} , \quad a_4 = \frac{13}{256} .
\end{align*}

The self-similar approximant (144) becomes
\begin{equation}
I_k^*(2, 3) = \frac{1}{(8\pi)^{3/2}} \exp \left( \frac{15}{16} g_{k-1}^* \right) .
\end{equation}
\tag{148}
The direct expansion (147) yields the errors

\[-80\% , \ -61\% , \ -46\% , \ -35\% , \ -26\% ,\]

while those of the self-similar approximants (148) are lower:

\[-30\% , \ -26\% , \ -16\% , \ -13\% , \ -12\% .\]

Now, let us turn to a \(D\)-dimensional three-loop Feynman integral

\[
J(D) \equiv \frac{1}{(2\pi)^3} \int \frac{d^Dp_1 \ d^Dp_2 \ d^Dp_3}{(1+p_1^2)(1+p_2^2)(1+p_3^2)[1+(p_1+p_2+p_3)^2]}.
\]  

(149)

Following the same procedure as in the calculation of the previous Feynman integrals, one defines \(J(D,\varepsilon)\) and then set \(\varepsilon = 1\). For concreteness, let us take \(D = 2\). Then the self-similar approximants for integral (149) are defined as

\[
J^*_k(2) = \frac{1}{256\pi^3} \exp \left( \frac{9}{4} g^*_k \right),
\]

(150)

with \(g^*_k\) having the form (143), which is obtained from \(g_k\) of Eq. (142), where the coefficient \(a_n\) are

\[
a_0 = 1 , \quad a_1 = \frac{7}{24} , \quad a_2 = \frac{13}{144} , \quad a_3 = \frac{59}{768} , \quad a_4 = \frac{373}{3840} ,
\]

\[
a_5 = \frac{2324}{18432} , \quad a_6 = \frac{15243}{86016} , \quad a_7 = \frac{150379}{393216}.
\]

The accuracy of the approximants (150) again is much better than that of simple perturbative expressions. The first seven approximants demonstrate a fast increase of accuracy. The related errors, calculated by comparing the self-similar form (150) with the numerical value \(J(2) = 0.00424027\), are

\[-46\% , \ -40\% , \ -29\% , \ -24\% , \ -21\% , \ -20\% , \ -19\% .\]

This demonstrates a monotonic convergence, while the standard perturbation theory in powers of \(\varepsilon\) wildly diverges.

Thus, the self-similar approximants provide rather good approximations even for very bad, fastly divergent series derived by means of standard perturbation theory.

In conclusion, we may mention that the self-similar approximation theory has been successfully applied not only to a number of physical problems [35–57, 75, 78–81] but also to other complex systems, such as financial markets [86–90]. Time series, related to financial, economic, biological, and social systems, are known to possess special fractal properties [91–95]. This is why these series can be naturally described by self-similar approximants, especially by those that explicitly display their self-similar structure as in self-similar roots and self-similar exponentials.
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