Bose-Einstein condensation in complex networks

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Abstract

The evolution of many complex systems, including the world wide web, business and citation networks is encoded in the dynamic web describing the interactions between the system’s constituents. Despite their irreversible and non-equilibrium nature these networks follow Bose statistics and can undergo Bose-Einstein condensation. Addressing the dynamical properties of these non-equilibrium systems within the framework of equilibrium quantum gases predicts that the 'first-mover-advantage', 'fit-get-rich' and 'winner-takes-all' phenomena observed in competitive systems are thermodynamically distinct phases of the underlying evolving networks.

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Competition for links is a common feature of complex systems: on the world wide web sites compete for URLs to enhance their visibility [1], in business world companies compete for links to consumers [2] and in the scientific community scientists and publications compete for citations, a measure of their impact on the field [3]. A common feature of these systems is that the nodes self-organize into a complex network, whose topology and evolution closely reflects the dynamics and outcome of this competition [1,3–6]. The reversibility of the microscopic processes that govern the evolution of these systems, such as growth through the addition of new nodes, or the impossibility (citation network) and the high cost (business networks) to modify an established links give a distinctly far-from equilibrium character to the evolution of these complex systems. Here we show that despite their non-equilibrium and irreversible nature, evolving networks can be mapped into an equilibrium Bose gas [7], nodes corresponding to energy levels and links representing particles. This mapping predicts that the common epithets used to characterize competitive systems, such as 'winner-takes-all', 'fit-get-rich', or 'first-mover-advantage' emerge naturally as thermodynamically and topologically distinct phases of the underlying complex evolving network. In particular, we predict that such networks can undergo Bose-Einstein condensation, in which a single node captures a macroscopic fraction of links.

**Fitness model**—Consider a network that grows through the addition of new nodes such as the creation of new webpages, the emergence of new companies or the publication of new papers. At each time step we add a new node, connecting it with $m$ links to the nodes already present in the system. The rate at which nodes acquire links can vary widely as supported by measurements on the www [5], and by empirical evidence in citation [3] and economic networks. To incorporate the different ability of the nodes to compete for links we assign a fitness parameter to each node, $\eta$, chosen from a distribution $\rho(\eta)$, accounting for the differences in content of webpages, the quality of products and marketing of companies, or the importance of the findings reported in a publication. The probability $\Pi_i$ that a new node connects one of its $m$ links to a node $i$ already present in the network depends on the number of links, $k_i$, and on the fitness $\eta_i$ of node $i$, such that
\[ \Pi_i = \frac{\eta_i k_i}{\sum_{\ell} \eta_{\ell} k_{\ell}} \]  

Equation (1) incorporates in the simplest possible way the fact that new nodes link preferentially to nodes with higher \( k \) (i.e. connecting to more visible websites; favoring more established companies; or citing more cited papers), and with larger fitness (i.e. websites with better content; companies with better products and sales practice; or papers with novel results). Thus fitness (\( \eta_i \)) and the number of links (\( k_i \)) jointly determine the attractiveness and evolution of a node.

**Mapping to a Bose gas**—Using a simple parameterization, we assign an energy \( \epsilon_i \) to each node, determined by its fitness \( \eta_i \) through the relation

\[ \epsilon_i = -\frac{1}{\beta} \log \eta_i, \]

where \( \beta \) is a parameter playing the role of inverse temperature, \( \beta = 1/T \), whose relevance to real networks will be discussed later. A link between two nodes \( i \) and \( j \) with energy \( \epsilon_i \) and \( \epsilon_j \) (e.g. fitnesses \( \eta_i \) and \( \eta_j \)) corresponds to two non-interacting particles on the energy levels \( \epsilon_i \) and \( \epsilon_j \) (Fig. 1). Adding a new node to the network corresponds to adding a new energy level \( \epsilon_i \) and 2\( m \) particles to the system. Of these 2\( m \) particles \( m \) are deposited on the level \( \epsilon_i \) (corresponding to the \( m \) outgoing link node \( i \) has), while the other \( m \) particles are distributed between the other energy levels (representing the links pointing to \( m \) nodes present in the system), the probability that a particle lands on level \( i \) being given by (1). Deposited particles are inert, i.e. they are not allowed to jump to other energy levels.

Each node (energy level) added to the system at time \( t_i \) with energy \( \epsilon_i \) is characterized by the occupation number \( k_i(t, \epsilon_i, t_i) \), denoting the number of links (particles) the node (energy level) has at time \( t \). The rate at which level \( \epsilon_i \) acquires new particles is

\[ \frac{\partial k_i(\epsilon_i, t, t_i)}{\partial t} = m \frac{e^{-\beta \epsilon_i} k_i(\epsilon_i, t, t_i)}{Z_t}, \]

where \( Z_t \) is the partition function, defined as

\[ Z_t = \sum_{j=1}^{t} e^{-\beta \epsilon_j} k_j(\epsilon_j, t, t_j). \]
We assume that each node increases its connectivity following a power-law (we demonstrate the self-consistency of this assumption later)

\[ k_i(\epsilon_i, t, t_i) = m \left( \frac{t}{t_i} \right)^{f(\epsilon_i)}, \tag{5} \]

where \( f(\epsilon) \) is the energy dependent dynamic exponent. Since \( \eta \) is chosen randomly from the distribution \( \rho(\eta) \), the energy levels are chosen from the distribution \( g(\epsilon) = \beta \rho(e^{-\beta \epsilon})e^{-\beta \epsilon}. \)

We can now determine \( Z_t \) by averaging over \( g(\epsilon) \), i.e.

\[
< Z_t > = \int d\epsilon g(\epsilon) \int_{t_0}^t dt_0 e^{-\beta \epsilon} k(\epsilon, t, t_0)
= m z^{-1} t \left( 1 + O(t^{-\alpha}) \right), \tag{6}
\]

where

\[ \frac{1}{z} = \int d\epsilon g(\epsilon) \frac{e^{-\beta \epsilon}}{1 - f(\epsilon)} \tag{7} \]

is the inverse fugacity. Since \( z \) is positive for any \( \beta \neq 0 \) we introduce the chemical potential, \( \mu \), as \( z = e^{\beta \mu} \), which allow us to write (8) and (7) as

\[ e^{-\beta \mu} = \lim_{t \to \infty} \frac{< Z_t >}{t}. \tag{8} \]

Using (8) we can solve the continuum equation (3) finding in a self-consistent way solutions of form (5), where the dynamic exponent is

\[ f(\epsilon) = e^{-\beta(\epsilon - \mu)}. \tag{9} \]

Combining (7) and (8), we find that the chemical potential is the solution of the equation

\[ I(\beta, \mu) = \int d\epsilon g(\epsilon) \frac{1}{e^{\beta(\epsilon - \mu)} - 1} = 1. \tag{10} \]

The system defined above has a number of properties that make it an unlikely candidate for an equilibrium Bose gas [7]. First, the inertness of the particles is a non-equilibrium feature, in contrast with the ability of particles in a quantum gas to jump between energy levels, leading to a temperature driven equilibration. Second, both the number of eligible energy
levels (nodes) and particles populating them (links) increase linearly in time, in contrast with the fixed system size employed in quantum systems. Despite these apparent conflicts, Eq. (10) indicates that in the thermodynamic limit \( t \to \infty \) the fitness model maps into a Bose gas. Indeed, since in an ideal gas of volume \( v = 1 \) we have

\[
\int d\epsilon g(\epsilon) n(\epsilon) = 1, \tag{11}
\]

where \( n(\epsilon) \) is the occupation number of a level with energy \( \epsilon \) Equation (10) indicates that for the inert gas inspired by the fitness model the occupation number follows the familiar Bose statistics

\[
n(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} - 1}, \tag{12}
\]

i.e. the evolving network maps into a Bose gas. Thus the irreversibility and the inertness of the network are resolved by the stationarity of the asymptotic distribution, allowing the occupation number to follow Bose statistics in the thermodynamic limit \( t \to \infty \).

*Bose-Einstein condensation*—The solutions (5), (8) and (9) exist only when there is a \( \mu \) that satisfies Eq. (10). However, \( I(\beta, \mu) \) defined in (10) takes its maximum at \( \mu = 0 \), thus when \( I(\beta, 0) < 1 \) for a given \( \beta \) and \( g(\epsilon) \), Eq. (10) has no solution. The absence of a solution is a well-known signature of Bose-Einstein condensation [7], indicating that a finite \( n_0(\beta) \) fraction of particles condensate on the lowest energy level. Indeed, due to mass conservation at time \( t \) we have \( t \) energy levels populated by \( 2mt \) particles, i.e.

\[
2mt = \sum_{t_0=1}^{t} k(\epsilon_{t_0}, t, t_0) = mt + mtI(\beta, \mu). \tag{13}
\]

When \( I(\beta, 0) < 1 \), Eq. (13) has to be replaced with

\[
2mt = mt + mtI(\beta, \mu) + n_0(\beta), \tag{14}
\]

where \( n_0(\beta) \) is given by [7]

\[
\frac{n_0(\beta)}{mt} = 1 - I(\beta, 0). \tag{15}
\]
The occupancy of the lowest energy level corresponds to the number of links the node with the largest fitness has. Thus the emergence of a nonzero $n_0(\beta)$, a signature of Bose-Einstein condensation in quantum gases, represents a 'winner-takes-all' phenomena for networks, the fittest node acquiring a finite fraction of the links, independent of the size of the network.

The mapping to a Bose gas and the possibility of Bose-Einstein condensation in random networks predicts the existence of three distinct phases characterizing the dynamical properties of evolving networks: (a) a scale-free phase, (b) a fit-get-rich phase and (c) a Bose-Einstein condensate. Next we discuss each of these possible phase separately.

(a) Scale-free phase– When all nodes have the same fitness, i.e. $\rho(\eta) = \delta(\eta - 1)$, $(g(\epsilon) = \delta(\epsilon))$, the model reduces to the scale-free model \[8\], introduced to account for the power-law connectivity distribution observed in diverse systems, such as the www \[4, 6\], actor network \[9, 8\], Internet \[10\] or citation networks \[3\]. The model describes a 'first-mover-wins' behavior, in which the oldest nodes acquire most links. Indeed, \(f(\epsilon) = 1/2\), i.e. according to \(5\) all nodes increase their connectivity as $t^{1/2}$, the older nodes with smaller $t_i$ having larger $k_i$. However, the oldest and 'richest' node is not an absolute winner, since its share of links, $k_{\text{max}}(t)/(mt)$, decays to zero as $t^{-1/2}$ in the thermodynamic limit. Thus a continuous hierarchy of large nodes coexist, such that the connectivity distribution $P(k)$, giving the probability to have a node with $k$ links, follows a power law $P(k) \sim k^{-3}$ \[8, 11\]. Rewiring, aging, and other local processes can modify the scaling exponents or introduce exponential cutoffs in $P(k)$ \[11\] while leaving the thermodynamic character of the phase unchanged.

(b) Fit-get-rich (FGR) phase– This phase emerges in systems for which nodes have different fitnesses and Eq. \(10\) has a solution (i.e. $I(\beta, \mu) = 1$). Equation \(3\) indicates that each node increases its connectivity in time, but the dynamic exponent depends on the fitness, being larger for nodes with higher fitness. This allows fitter nodes to join the system at a later time and surpass the less fit but older nodes by acquiring links at higher rate \[3\]. Consequently, this phase predicts a 'fit-get-rich' phenomena, in which, with time, the fitter prevails. But, while there is a clear winner, similar to the scale-free phase the fittest node’s
share of all links decreases to zero in the thermodynamic limit. Indeed, since \( f(\epsilon) < 1 \), the relative connectivity of the fittest node decrease as \( k(\epsilon_{\text{min}},t)/(mt) \sim t^{f(\epsilon_{\text{min}})-1} \). This competition again leads to the emergence of a hierarchy of a few large 'hubs' accompanied by many less connected nodes, \( P(k) \) following a power-law \( P(k) \sim k^{-\gamma} \), where \( \gamma \) can be calculated analytically if \( \rho(\eta) \) is known.

(c) Bose-Einstein (BE) condensate – Bose-Einstein condensation appears when (10) has no solution, at which point (3), (4), (10) break down. In the competition for links the node with the largest fitness emerges as a clear winner, a finite fraction of particles \( (n_0(\beta)) \) landing on this energy level. Thus BE condensation predicts a real ‘winner-takes-all’ phenomena, in which the fittest node is not only the largest, but despite the continuous emergence of new nodes that compete for links, it always acquires a finite fraction of links (Eq. (15)).

To demonstrate the existence of a phase transition from the FGR phase to a BE condensate in a network, we consider a formally simple case, assuming that the energy (fitness) distribution follows

\[
g(\epsilon) = C \epsilon^\theta,\tag{16}
\]

where \( \theta \) is a free parameter and the energies are chosen from \( \epsilon \in (0, \epsilon_{\text{max}}) \), normalization giving \( C = (\theta+1)/\epsilon_{\text{max}}^\theta \). For this class of distributions the condition for a Bose condensation is

\[
\frac{\theta+1}{(\beta \epsilon_{\text{max}})^\theta+1} \int_{\beta \epsilon_{\text{min}}(t)}^{\beta \epsilon_{\text{max}}} dx \frac{x^\theta}{e^x - 1} < 1,\tag{17}
\]

where \( \epsilon_{\text{min}}(t) \) corresponds to the lowest energy (fittest) node present in the system at time \( t \). Extending the limits of integration to zero and to infinity, we find the lower bound for the critical temperature \( T_{BE} = 1/\beta_{BE} \)

\[
T_{BE} > \epsilon_{\text{max}}(\zeta(\theta+1)\Gamma(\theta+2))^{1/(\theta+1)}.\tag{18}
\]

To demonstrate the emergence of Bose-Einstein condensation near the predicted \( T_{BE} \), we simulated numerically the discrete network model described above, using the energy distribution (16). The chemical potential, \( \mu \), measured numerically indicates a sharp transition
from a positive to a negative value (Fig. 2a), corresponding to the predicted phase transition between the BE and the FGR phases. The difference between the network dynamics in the two phases is illustrated in Fig. 2b, where we plot the relative occupation number of the most connected node for different temperatures. We find that the ratio $k_{\text{max}}(t)/mt$ is independent on time in the BE phase, indicating that the largest node maintains a finite fraction of the total number of links even as the network continues to expand, a signature of BE condensation. In contrast, for $T > T_{BE}$, the most connected node gradually loses its share of links, $k_{\text{max}}(t)/mt$ decreasing continuously with time. The numerically determined phase diagram (Fig. 2b) confirms that the analytical prediction (18) offers a lower bound for $T_{BE}$. To predict the precise value of $T_{BE}$ one needs to incorporate the interplay between the convergence of $\epsilon_{\text{min}}(t)$ to zero and the thermodynamics of BE condensation.

Since real networks have a fixed ($T$ independent) $\rho(\eta)$ fitness distribution, whether they are in the BE or FGR phase is independent of $T$. Indeed, for $\rho(\eta) (1 - \eta)^{\lambda}$ the network undergoes a BE condensation for $\lambda > \lambda_{BE} = 1$, thus $T$, introduced during the calculations, plays the role of a dummy variable that at the end vanishes from all topologically relevant quantities. The existence of $T_{BE}$ in the numerically studied model (Fig. 2) is rooted in our technically simpler choice of defining $g(\epsilon)$ to be independent of $T$, providing a richer phase space to demonstrate the existence of a phase transition in a discrete network model. However, as the inset in Fig. 2b shows, by changing $\theta$ the phase transition emerges for fixed $T$ as well.

Do the www, business or citation networks represent a Bose-Einstein condensate, or are described by the FGR phase without a dominating winner? It is well known that in certain markets some companies did gain and maintain an unusually high market share. A much publicized case is the continued dominance of Microsoft in the rapidly expanding operating systems market, indicating that business networks do develop ‘winner-takes-all’ phenomena, similar to a Bose-Einstein condensate. The situation of the www is more complex: while indeed some webpages did capture an unusually high number of links, it is less clear weather they maintain a finite fraction of all links as the www grows (as expected if the www is
a BE condensate), or will lose market share, leading to the coexistence of a continuous hierarchy of large websites, a signature of the FGR phase. While the power-law connectivity distribution holding over six order of magnitude support the FGR scale-free phase [3,4], the most recent study involving more that 200 million nodes did identify a run-away node of approximately $10^7$ links [4], an apparently clear and dominating winner. Since $\rho(\eta)$ could be explicitly determined from network data, the phase to which various networks belong to could be decided in the future. Uncovering the prevailing behavior in these and other complex evolving networks (e.g. social networks [5], ecological and transportation webs [6,7], etc.) is a formidable task that will require careful quantitative studies on real networks. The identification of the possible phases offered in this paper might provide the appropriate analytical framework for such studies.
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FIG. 1. Schematic illustration of the mapping between the network model and the Bose gas. 

(a) On the left we have a network of five nodes (continuous circles and lines), each characterized by a fitness, $\eta_i$, chosen randomly from a distribution $\rho(\eta)$. Equation (2) assigns an energy $\epsilon_i$ to each $\eta_i$, generating a system of random energy levels (right). A link from node $i$ to node $j$ corresponds to a particle at level $\epsilon_i$ and one at $\epsilon_j$. The network evolves by adding a new node (dashed circle, $\eta_6$) at each timestep which connects to $m = 2$ other nodes (dashed lines), chosen randomly following (2). In the gas this results in the addition of a new energy level ($\epsilon_6$, dashed) populated by $m = 2$ particles, and the deposition of $m = 2$ other particles at energy levels to which $\eta_6$ is connected to ($\epsilon_2$ and $\epsilon_5$). The number of energy levels and particles increase linearly with time, as $t$ and $2mt$, respectively. 

(b) In the FGR phase we have a continuous connectivity distribution, the several highly connected nodes linking the numerous small nodes together. In the energy diagram this corresponds to a decreasing occupation level with increasing energy. 

(c) In the Bose-Einstein condensate the fittest node attracts a finite fraction of all links, corresponding to a highly populated energy level, and only sparsely populated higher energies. In (b) and (c) the energy diagram shows only incoming links, ignoring the default $m = 2$ particles on each energy level corresponding to the outgoing links to simplify the picture.
FIG. 2. Numerical evidence for Bose-Einstein condensation in a network model. (a) Choosing the energies from the distribution (16) with $\epsilon_{\text{max}} = 1$, we calculated the chemical potential $\mu$ numerically as the network evolved in time, using (18). Due to the wide range over which $\mu$ varies, we plot $|\mu|$ on a logarithmic scale. The temperature at which $\mu$ changes sign corresponds to the sharp drop in $|\mu|$ on the figure, and identifies the critical temperature $T_{BE}$ for Bose-Einstein condensation. The data are shown for $\theta = 1$ and for different times (i.e. system sizes) $t = 10^3$ (continuous), $10^4$ (dashed), $10^5$ (long-dashed), being averaged over 500, 100 and 30 runs respectively. The inset shows the chemical potential for different values of the exponent $\theta$ in (16), i.e. $\theta = 0.5, 1.0, 2.0$, indicating the $\theta$ dependence of $T_{BE}$. (b) Fraction of the total number of links connected to the most connected ('winner') node, $k_{\text{max}}/(mt)$, plotted as a function of $T$, shown for $m = 2$ and $\theta = 1$. The three curves, corresponding to different system sizes recorded at $t = 10^3, 10^4, 10^5$ indicate the difference between the two phases: in the BE phase (left) the fittest node maintains a finite fraction of links even as the system expands, while in the FGR phase (right) the fraction of links connected to the most connected node decreases with time. The inset shows the $(\theta, T_{BE})$ phase diagram, the continuous line corresponding to the lower bound predicted by Eq. (18), while the symbols represent the numerically measured $T_{BE}$ as indicated by the position of the peaks in Inset(a).