Unique continuation for the gradient of eigenfunctions and Wegner estimates for random divergence-type operators

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Abstract

We prove a scale-free quantitative unique continuation estimate for the gradient of eigenfunctions of divergence-type operators, i.e. operators of the form $-\text{div} A \nabla$, where the matrix function $A$ is uniformly elliptic. The proof uses a unique continuation principle for elliptic second order operators and a lower bound on the $L^2$-norm of the gradient of eigenfunctions corresponding to strictly positive eigenvalues.

As an application, we prove an eigenvalue lifting estimate that allows us to prove a Wegner estimate for random divergence-type operators. Here our approach allows us to get rid of a restrictive covering condition that was essential in previous proofs of Wegner estimates for such models.

1. Introduction

The analysis of divergence-type operators is motivated, among others, by the study of propagation of electromagnetic and classical waves in media, including random ones.

Since such operators are elliptic second order operators they obey unique continuation estimates. In fact, it was recently shown that they even satisfy so-called scale-free unique continuation estimates, that have first been established for Schrödinger operators, cf. [17] and references therein. These estimates compare the $L^2$-norm of an eigenfunction $\psi$ (or a function in the range of an appropriate spectral projector of the operator under consideration) on the full domain with its $L^2$-norm on a collection of small balls that are evenly distributed throughout the domain. For elliptic second order operators, analogous, but somewhat weaker, bounds were proven in [11] [20]. The methods used there rely on those developed for classical (i.e. local) unique continuation estimates for elliptic second order differential operators, see e.g. [13] [16] and the literature cited there.

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An important application of scale-free unique continuation estimates is the theory of random operators, where unique continuation principles are used to prove e.g. Wegner and initial length scale estimates, see \cite{3, 21, 12, 17, 26, 22}. However, in all these references and in most of the existing literature, the random part of the operator is assumed to be the zeroth order term. In other words, the randomness is introduced by adding a random potential.

In this paper we consider more challenging operators, where the leading order term is random. This situation was studied in \cite{5} and \cite{23} as a model for propagation of waves in random media, see also \cite{6}. These paper provide a Wegner estimate, assuming however, that the random perturbations satisfy a covering condition.

Our proof demonstrates how to remove this covering condition assumed in \cite{5, 23} using a scale-free unique continuation estimate for the gradient of eigenfunctions. In contrast to usual scale-free unique continuation estimates, we compare the $L^2$-norm of the gradient of an eigenfunction on the union of balls described above with the $L^2$-norm of the eigenfunction on the full domain. To the best of our knowledge, previously only qualitative unique continuation for the gradient has been studied, see \cite{18}. We use ideas of the latter paper and combine it with a unique continuation estimate of \cite{26} to obtain the desired unique continuation estimate for the gradient.

The energy zero is not a fluctuation boundary of random divergence-type operators. This is illustrated by the fact that if one restricts the operator to a cube and imposes Neumann boundary conditions, zero is an eigenvalue regardless of the random configuration. Therefore we will not only exclude high energies from our consideration, but also energies close to zero. Consequently, our unique continuation estimate for the gradient is only valid for eigenfunctions corresponding to strictly positive eigenvalues.

The structure of the paper is as follows: In the next section 2 we introduce the notation and formulate the main results concerning unique continuation for the gradient, the proof of which is postponed to section 3. Thereafter, in section 4 we consider applications of our unique continuation estimate for the gradient to random divergence-type operators. section 5 is dedicated to stronger bounds that hold true for small energies. Some of these are based on a remark made in \cite{26} which allows us to partly remove some assumptions of our main result. The proof of the latter remark is postponed to the appendix A. Finally, in section 6 we provide scaled variants of our results.

Let us emphasize that, having in mind future applications in the theory of Anderson localization for divergence-type operators, we formulate a number of similar results displaying the explicit dependence of the constants on the model parameters. This is necessary because proofs of localization depend on a delicate interplay of a number or parameters.

2. Notation and the main result

Let $d \in \mathbb{N}$ and let $\Lambda_L = (-\frac{L}{2}, \frac{L}{2})^d$ denote the cube with side length $L \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$, i.e. $\Lambda_\infty = \mathbb{R}^d$. Let $B(x, r)$ denote the ball with center $x \in \mathbb{R}^d$ and radius $r \geq 0$ and let $A : \Lambda_L \to \mathbb{R}^{d \times d}$ be a matrix function such that $A(x)$ is
symmetric for all \( x \in \Lambda_L \) and there are constants \( \theta_{\text{Ellip},-}, \theta_{\text{Ellip},+} > 0 \) such that
\[
\theta_{\text{Ellip},-} |\xi|^2 \leq \xi \cdot A(x) \xi \leq \theta_{\text{Ellip},+} |\xi|^2
\] (Ellip)
holds for all \( x \in \Lambda_L \) and all \( \xi \in \mathbb{R}^d \). We use the abbreviation \( \theta_{\text{Ellip}} := \max\{\theta_{\text{Ellip},-}, \theta_{\text{Ellip},+}\} \). For \( \mathcal{D}(\mathfrak{h}^L) = H^1_0(\Lambda_L) \subseteq H^1(\Lambda_L) \), consider the form \( \mathfrak{h}^L : \mathcal{D}(\mathfrak{h}) \times \mathcal{D}(\mathfrak{h}) \to \mathbb{C} \) defined by
\[
\mathfrak{h}^L(u,v) := \int_{\Lambda_L} \nabla u \cdot A \nabla v,
\] (1)
where \( \nabla \) denotes the weak gradient. The form \( \mathfrak{h}^L \) is densely defined, closed, symmetric and sectorial so that there exists a unique self-adjoint operator \( H^L(A) \) associated with the form \( \mathfrak{h}^L \). Note that in general the operator domain \( \mathcal{D}(H^L(A)) \) does not contain smooth functions. This is the reason why we rely on the form approach.

However, if the matrix function \( A \) is Lipschitz continuous, i.e. if there is a constant \( \theta_{\text{Lip}} > 0 \) such that for all \( x,y \in \Lambda_L \)
\[
\|A(x) - A(y)\|_\infty \leq \theta_{\text{Lip}} |x-y|,
\] (Lip)
we have \( C^\infty_c(\Lambda_L) \subseteq \mathcal{D}(H^L(A)) \) and on \( C^\infty_c(\Lambda_L) \) the operator \( H^L(A) \) coincides with the operator
\[
\mathcal{H}_L : C^\infty_c(\Lambda_L) \to L^2(\Lambda_L), u \mapsto -\text{div} A \nabla u.
\]
The latter illustrates that the operator \( H^L(A) \) defined above is a realization of the divergence-type operator \( -\text{div} A \nabla \) on the cube \( \Lambda_L \) and due to the choice \( \mathcal{D}(\mathfrak{h}^L) = H^1_0(\Lambda_L) \) the operator \( H^L(A) \) has Dirichlet boundary conditions. While the main body of the paper is devoted to Dirichlet boundary conditions, we treat at several instances Neumann boundary conditions (at least for energies close to zero), see subsection 5.1.2 below. Note that in this case the form domain is given by \( H^1(\Lambda_L) \).

The notion of a scale-free unique continuation principle relies on the following

**Definition 2.1.** Let \( G > 0 \) and \( \delta \in (0,G/2) \). A sequence \( Z = (z_j)_{j \in (G\mathbb{Z})^d} \subseteq \mathbb{R}^d \) is said to be \((G,\delta)\)-equidistributed, if \( B(z_j,\delta) \subseteq \Lambda_G(j) \) for all \( j \in \mathbb{Z}^d \). For \( L \in \mathbb{N} \) we set
\[
S_{Z,\delta}(\infty) := \bigcup_{j \in (G\mathbb{Z})^d} B(z_j,\delta) \text{ and } S_{Z,\delta}(GL) := S_{Z,\delta}(\infty) \cap \Lambda_{GL}.
\]
Initially, we only consider the case of \((1,\delta)\)-equidistributed sequences. The general case will follow from this by a scaling argument, see section 6. In order to formulate our main result, we need to introduce a technical assumption from \cite{26}: Assume that \( L \in \mathbb{N} \) and \( A = (a_{j,k})_{j,k=1,\ldots,d} \), then
\[
\forall k,j \in \{1,\ldots,d\}, j \neq k \forall x \in \overline{\Lambda_L} \cap (\Lambda_L + L e_k) : a_{j,k}(x) = a_{k,j}(x) = 0. \quad \text{(Dir)}
\]
This is in particular satisfied, if all off-diagonal coefficients of \( A \) vanish on \( \partial \Lambda_L \).

With this notation our first main result reads as follows:
Theorem 2.2. Let $L \in \mathbb{N}_\infty$. Assume that $A$ satisfies $\{\text{Ellip}\}$, $\{\text{Lip}\}$ and $\{\text{Dir}\}$, let $0 < E_- < E_+ < \infty$ and let $\delta_0$ be sufficiently small (depending only on $d, \vartheta_{\text{Ellip}}$ and $\vartheta_{\text{Lip}}$). Then for all $\delta \in (0, \delta_0)$, there exists a constant $C_{\text{JUCP}}^\nabla > 0$ depending only on $d, \vartheta_{\text{Ellip}}$ and $\vartheta_{\text{Lip}}$. Then for all $\delta \in (0, \delta_0)$, there exists a constant $C_{\text{JUCP}}^\nabla > 0$ depending only on $d, \vartheta_{\text{Ellip}}$, $\vartheta_{\text{Lip}}$, $E_-$, $E_+$ and $\delta$ such that for all $E \in (E_-, E_+)$, all $\psi \in \mathcal{D}(H^1(A))$ satisfying $H^1(A)\psi = E\psi$ and all $(1, \delta)$-equidistributed sequences $Z$ we have
\[
\|\nabla \psi\|_{L^2(S_Z(A))}^2 \geq C_{\text{JUCP}}^\nabla \|\psi\|_{L^2(A_L)}^2.
\] (2)

The constant $C_{\text{JUCP}}^\nabla$ will be given in (9).

Remark 2.3. one can choose $\delta_0 = 2(330de^2\vartheta_{\text{Ellip}}^{11/2}(\vartheta_{\text{Ellip}} + 1)^5/3(\vartheta_{\text{Lip}} + 1))^{-1}$ in the previous theorem, cf. [26]. Moreover, an inequality like (2) holds true for $\delta \geq \delta_0$ as well but with a different constant, cf. Remark 2.4 in [26] and Remark 3.4 below.

Remark 2.4. The theorem fails for $E_- = 0$. More precisely it is possible to construct a sequence of normalized eigenfunctions $\psi_L$ corresponding to eigenvalues converging to 0 as $L$ increases such that
\[
\lim_{L \to \infty} \|\nabla \psi_L\|_{L^2(A_L)} = 0.
\]
Thus (2) must fail.

The stated theorem contains two assumptions which one can hope to eliminate eventually: Assumption $\{\text{Dir}\}$ is needed for a certain extension argument used in [26]. It is quite possible that this step could be replaced by a generalization of an extension (possibly by a smoothing procedure) which does not require the assumption $\{\text{Dir}\}$. Beside condition $\{\text{Dir}\}$, the Lipschitz continuity of $A$ needed in Theorem 2.2 is a drawback for the application we have in mind. However, it is possible to use an approximation argument to allow for discontinuous coefficient matrices $A$, at least for eigenfunctions corresponding to eigenvalues close to zero. Moreover, the assumption $\{\text{Dir}\}$ is no longer needed in this case, see section 5.

3. Proof of Theorem 2.2

First we prove a lemma that establishes a relation between an eigenfunction and its gradient. The proof is inspired by the arguments in [18], where the author proves qualitative unique continuation for the gradient of eigenfunctions of second order elliptic operators, however under a strict condition on the sign of the zeroth order term. In our context, this condition is partly replaced in the assumptions for the energy interval.

Lemma 3.1. Let $0 < E_- < \infty$, $r > 0$ and assume that $A$ fulfills $\{\text{Ellip}\}$. Then there is a constant $C^\nabla(r) > 0$ depending only on $\vartheta_{\text{Ellip}}$, $E_-$ and $r$ such that for all points $x_0 \in \Lambda_L$ for which $B(x_0, 2r) \subseteq \Lambda_L$, all $E > E_-$ and all solutions $\psi \in \mathcal{D}(H^1(A))$ of $H^1(A)\psi = E\psi$ we have
\[
\|\nabla \psi\|_{L^2(B(x_0, 2r))}^2 \geq C^\nabla(r) \|\psi\|_{L^2(B(x_0, r))}^2.
\] (3)

The constant $C^\nabla(r)$ will be given in (5).
Proof. Choose a smooth cutoff-function $\varphi : \Lambda_L \to [0,1]$, satisfying $\varphi \equiv 1$ on $B(x_0,r)$, $\varphi \equiv 0$ on $\Lambda_L \setminus B(x_0,2r)$ and $\|\nabla \varphi\|_{\infty} \leq 2/r$. By the definition of the operator $H^L(A)$, we have $H^L(A)\psi = E\psi$ if and only if $\psi \in D(h^L)$ and $h^L(v,\psi) = (v,E\psi)_{L^2(\Lambda_L)}$ holds for all $v \in D(h^L)$. Choosing $v = \psi\varphi^2 \in D(h^L)$ we get

$$
(\psi\varphi^2,E\psi)_2 \leq |h^L(\psi\varphi^2,\psi)| \leq \int_{\Lambda_L} \varphi^2 |\nabla \psi \cdot A\nabla \psi| + 2\varphi|\psi||\nabla \varphi \cdot A\nabla \psi|.
$$

(4)

Using the ellipticity of $A$, Cauchy-Schwartz and Young’s inequality, the right hand side is bounded from above by

$$
\gamma(\psi\varphi^2,\psi)_2 + \int_{\Lambda_L} \left( \vartheta_{Ellip,+}\varphi^2 + \frac{\vartheta_{Ellip,+}^2}{\gamma}|\nabla \varphi|^2 \right) |\nabla \psi|^2
$$

for $\gamma = E_-/2$. Hence (4) and the estimate for the gradient of $\varphi$ imply

$$
(E - \frac{E_-}{2}) \int_{\Lambda_L} \varphi^2 |\psi|^2 \leq \left( \vartheta_{Ellip,+} + \frac{2\vartheta_{Ellip,+}^2}{E_-} \cdot \frac{4}{r^2} \right) \int_{B(x_0,2r)} |\nabla \psi|^2.
$$

Since $E - E_-/2 \geq E_-/2$ and $\varphi \geq 1_{B(x_0,r)}$ we obtain

$$
\|\psi\|_{L^2(B(x_0,r))}^2 \leq \frac{2}{E_-} \left( \vartheta_{Ellip,+} + \frac{8\vartheta_{Ellip,+}^2}{r^2E_-} \right) \|\nabla \psi\|_{L^2(B(x_0,2r))}^2.
$$

Thus with

$$
C^\nabla(r) := \frac{r^2E_-}{2\vartheta_{Ellip,+} + (8\vartheta_{Ellip,+}^2 + r^2E_-)} \leq \frac{E_-}{2\vartheta_{Ellip,+}}
$$

(5)

the inequality (3) holds true. \qed

Remark 3.2. (i) The lemma gives in a sense a reverse Cacciopoli-type inequality valid under certain conditions,

(ii) for $r \leq 1$ the constant given in (5) can be estimated from below by

$$
\frac{E_-}{2\vartheta_{Ellip,+} + (8\vartheta_{Ellip,+}^2 + r^2E_-)} \leq C^\nabla(r),
$$

(iii) $C^\nabla(r)$ does not depend on $x_0$ and only the ellipticity of $A$ was used, not the Lipschitz continuity,

(iv) the proof applies to all operators that correspond to forms $h^L$ as above with form domain $D(h^L)$ satisfying $\psi\varphi^2 \in D(h^L)$ for all cut-off functions $\varphi$ and all $\psi \in D(h^L)$.

The proof of Theorem 2.2 relies on the main result of [26]. For the sake of completeness, we here recap a version of it suited to our purpose.
Theorem 3.3. Assume that $A$ fulfills \(\text{Ellip}\), \(\text{Lip}\) and \(\text{Dir}\). Then there exists a constant $N > 0$ depending only on $d$, $\vartheta_{\text{Ellip}}$ and $\vartheta_{\text{Lip}}$ such that for all $L \in \mathbb{N}_\infty$, all measurable and bounded $V : \Lambda_L \to \mathbb{R}$, all $\delta \in (0, \delta_0/2]$, all $(1, \delta)$-equidistributed sequences $Z$ and all $\psi \in D(H^L(A))$ satisfying $|H^L(A)\psi| \leq |V\psi|$ almost everywhere on $\Lambda_L$, we have
\[
\|\psi\|^2_{L^2(S_{Z,\delta}(L))} \geq C_{\text{sfUCP}} \|\psi\|^2_{L^2(\Lambda_L)}.
\] Here
\[
C_{\text{sfUCP}} = C_{\text{sfUCP}}(\delta) = \delta^{N(1+\|V\|^2/3)}.
\]
Combining this theorem with Lemma 3.1 yields the Proof of Theorem 2.2.

Proof of Theorem 2.2. First, note that the sequence $Z$ is also $(1, \delta/2)$-equi-distributed. We apply Lemma 3.1 followed by Theorem 3.3 with $V \equiv E_+$ on $\Lambda_L$ to obtain
\[
\|\nabla\psi\|^2_{L^2(S_{Z,\delta}(L))} \geq C^\nabla(\delta) \|\psi\|^2_{L^2(S_{Z,\delta/2}(L))} \geq C_{\text{sfUCP}} \|\psi\|^2_{L^2(\Lambda_L)},
\]
where the constant equals
\[
C_{\text{sfUCP}} = C_{\text{sfUCP}}(\delta) = \frac{\delta^2 E_+^2}{2\vartheta_{\text{Ellip},+} \left( 8\vartheta_{\text{Ellip},+} + \delta^2 E_- \right)} \left( \frac{\delta}{2} \right)^{N(1+|E_+^{2/3})}. \tag{9}
\]

Remark 3.4. (a) We can allow arbitrary $\delta \in (0, 1/2]$ if we modify the definition of the constant in (7) to
\[
C_{\text{sfUCP}} = C_{\text{sfUCP}}(\delta) = (\min\{\delta, \delta_0\})^{N(1+\|V\|^2/3)}
\]
and similarly substitute $\min\{\delta, \delta_0\}$ for $\delta$ in the constant $C_{\text{sfUCP}}^\nabla$ appearing in Theorem 2.2 and (9).

(b) Since $\delta \leq 1$, we see that
\[
C_1 \left( \frac{\delta}{2} \right)^{2+N(1+|E_+^{2/3})} \leq C_{\text{sfUCP}}^\nabla(\delta) \leq C_2 \left( \frac{\delta}{2} \right)^{N(1+|E_+^{2/3})} \tag{10}
\]
for constants $C_1, C_2$ depending only on $E_-$ and $\vartheta_{\text{Ellip},+}$.

4. Applications

Here we apply our main results in the theory of random operators. As noted in the introduction, we want to understand the case where the second order term is random. Therefore, we need to understand how a modification of the coefficient matrix affects the eigenvalues. In particular we investigate the movement of eigenvalues if we perturb the matrix function $A$ by some non-negative function $W$ times the identity matrix. The novelty is that we are able to treat the case that $W$ has, in some sense, small support. In a second step, we use this result to prove a Wegner estimate for random divergence-type operators.

Throughout this section we assume that $L \in \mathbb{N}$, i.e. the geometric domain is a finite cube. Note that in this situation the divergence-type operators under consideration have compact resolvent and therefore purely discrete spectrum.
4.1. Eigenvalue lifting

Given $\delta \in (0, 1/2)$ and a $(1, \delta)$-equidistributed sequence $Z$, we want to investigate how the eigenvalues of the operator $H^L_t := H^L(A + tW \text{Id})$ vary as $t$ increases. Here $A$ is a matrix function as defined above that at least satisfies $[\text{Ellip}]$ and $W$ satisfies $W \geq 1_{S_{Z, \delta}(L)}$. Now we fix $T > 0$ and recall that for $t \in [0, T]$ the operator $H^L_t$ is associated to the form

$$\frac{d}{dz} h^L_t : \mathcal{D}(h^L_t) \times \mathcal{D}(h^L_t) \to \mathbb{C}, (u, v) \mapsto \int_{\Lambda_L} \nabla u \cdot (A + tW \text{Id}) \nabla v$$  \hspace{1cm} (11)

with $\mathcal{D}(h^L_t) = H^0_t(\Lambda_L)$. It is not hard to see that there is a domain $D \subseteq \mathbb{C}$ such that $[0, T] \subseteq D$ and such that $h^L_t$ is densely defined and sectorial for all $t \in D$. Thus the family of forms $(h^L_t)_{t \in D}$ turns out to be a holomorphic family of type (a) in the sense of Kato [9]. Moreover, the quadratic form is increasing in $t$, i.e.

$$h^L_t(u, u) \leq h^L_s(u, u)$$  \hspace{1cm} (12)

for $t \leq s$ and we easily calculate

$$\left( \frac{d}{dz} h^L_t \right)(u, u) := \lim_{w \to z} \frac{h^L_w(u, u) - h^L_z(u, u)}{w - z} = \int_{\Lambda_L} W |\nabla u|^2.$$  \hspace{1cm} (13)

Thus $(H^L_t)_{t \in D}$ is a holomorphic family of type (B) in the sense of Kato and $H^L_t$ is self-adjoint, lower-semibounded and has compact resolvent for $t \in [0, T]$. We denote by $(E^L_n(t))_{n \in \mathbb{N}}$ the eigenvalues of $H^L_t$, enumerated non-decreasingly and counting multiplicities.

In order to exploit our unique continuation estimate for the gradient, we will first assume that $W$ and $A$ are Lipschitz continuous and prove the next

**Theorem 4.1.** Let $T > 0$, $\delta \in (0, 1/2]$, $K_1, K_2 > 0$, $0 < E_- < E_+ < \infty$ and assume that $A$ satisfies $[\text{Ellip}]$, $[\text{Lap}]$ and $[\text{Dir}]$. Then there is a constant $C_{\text{evl}} > 0$ such that for all $(1, \delta)$-equidistributed sequences $Z$, all Lipschitz continuous $W \in L^\infty(\Lambda_L)$ satisfying $\text{Lip}(W) \leq K_1$, $\|W\|_\infty \leq K_2$ and $W \geq 1_{S_{Z, \delta}(L)}$, and all $n \in \mathbb{N}$ such that $E_- < E^L_n(0) \leq E^L_n(T) < E_+$ we have

$$E^L_n(t) \geq E^L_n(0) + t C_{\text{evl}}.$$  \hspace{1cm} (14)

The constant $C_{\text{evl}}$ will be given in [18].

**Proof.** There are two sequences $(\lambda_{\ell})_{\ell \in \mathbb{N}}$ and $(\varphi_{\ell})_{\ell \in \mathbb{N}}$ of analytic functions on $D \cap \mathbb{R}$ such that $H^L_\ell \varphi_{\ell}(t) = \ell(t) \varphi_{\ell}(t)$ for all $t \in [0, T]$, see VII. Remark 4.22 and VII. Theorem 3.9 in [9]. The $(\lambda_{\ell}(t))_{\ell}$ are all the repeated eigenvalues of $H^L_t$ with corresponding normalized eigenfunctions $(\varphi_{\ell}(t))_{\ell}$. The functions $E^L_n$ are formed by connecting several $\lambda_{\ell}$ in a continuous manner, i.e. $E^L_n$ may jump from one $\lambda_{\ell}$ to another at every crossing point between the different $\lambda_{\ell}$. Since the $\lambda_{\ell}$ are analytic and the $H^L_t$ are lower-semibounded the jumps occur at at most finitely many points.

More precisely, there exists $M \in \mathbb{N}$, $\ell_1, \ldots, \ell_M \in \mathbb{N}$ and $0 = t_1 < \cdots < t_{M+1} = T$ such that

$$E^L_n(t) = \lambda_{\ell_j}(t) \text{ for all } t \in [t_j, t_{j+1}],$$  \hspace{1cm} (15)
where the overlap is continuous. Especially the function \( E^L_n \) agrees piecewise with some \( \lambda_t \) and is therefore piecewise analytic. The argument in the proof of VII. Theorem 4.21 in [9] shows that

\[
\left( \frac{d}{dt} E^L_n \right) (t) = \left( \frac{d}{dt} b^L_t \right) (\varphi_t(t), \varphi_t(t))
\]

(16)

if \( E^L_n (\cdot) = \lambda_t (\cdot) \) in a neighborhood of \( t \). Only at the points \( t \in \{ t_1, \ldots, t_{N+1} \} \), at which a crossing from one \( \lambda_t \) to another occurs, it is possible that \( E^L_n (\cdot) \) is not differentiable. It follows that (16) holds true for all but finitely many \( t \in [0, T] \).

By (13), the right hand side of (16) satisfies

\[
\left( \frac{d}{dt} b^L_t \right) (\varphi_t(t), \varphi_t(t)) = \int_{A_L} W |\nabla \varphi_t(t)|^2 \geq \| \nabla \varphi_t(t) \|^2_{L^2(S_Z, L)}.
\]

(17)

Now we want to apply the unique continuation estimate for the gradient on the right hand side of (17). Therefore we need to verify the assumptions of Theorem 2.2. For fixed \( t \), the matrix function \( x \mapsto A(x) + tW(x) \text{Id} \) is Lipschitz continuous with Lipschitz-constant \( \hat{\vartheta}_{\text{Lip}}(t) = \vartheta_{\text{Lip}} + tK_1 \) and elliptic with \( \hat{\vartheta}_{\text{Ellip,}+}(t) = \vartheta_{\text{Ellip,}+} \) and \( \hat{\vartheta}_{\text{Ellip,}+}(t) = \vartheta_{\text{Ellip,}+} + tK_2 \). Moreover, the assumption on \( E^L_n (\cdot) \) shows that \( \varphi_t(t) \) is a normalized eigenfunction with eigenvalue between \( E_- \) and \( E_+ \).

For all \( t \in [0, T] \) we apply Theorem 2.2 which provides us with a constant \( \bar{C}^\nabla_{\text{sfUCP}} = \bar{C}^\nabla_{\text{sfUCP}} (\delta, d, \hat{\vartheta}_{\text{Ellip,} \pm}(t), \hat{\vartheta}_{\text{Lip}}(t)) \) such that

\[
\| \nabla \varphi_t(t) \|^2_{L^2(S_Z, L)} \geq \bar{C}^\nabla_{\text{sfUCP}} \| \varphi_t(t) \|^2_{L^2(A_L)} = \bar{C}^\nabla_{\text{sfUCP}}.
\]

Thus

\[
\| \nabla \varphi_t(t) \|^2_{L^2(S_Z, L)} \geq C_{\text{evl}} := \inf_{t \in [0, T]} \bar{C}^\nabla_{\text{sfUCP}}
\]

where \( C_{\text{evl}} \) does not depend on \( t \), indeed

\[
C_{\text{evl}} = \frac{\delta^2 E^2_2}{2 \hat{\vartheta}_{\text{Ellip,}+}(T) \left( \hat{\vartheta}_{\text{Ellip,}+}(T) + \delta^2 E_- \right) } \left( \frac{\delta}{2} \right)^{N(1+E^2_2/3)}
\]

(18)

where \( N = N(\hat{\vartheta}_{\text{Lip}}(T), \hat{\vartheta}_{\text{Ellip,} \pm}(T)) \). Consequently,

\[
\left( \frac{d}{dt} E^L_n \right) (t) \geq C_{\text{evl}}
\]

(19)

for all \( t \in (0, T) \setminus \{ t_2, \ldots, t_M \} \). For fixed \( t \in [0, T] \), we let \( \tilde{M} \in \{ 1, \ldots, M \} \) be the smallest number such that \( t_n > t \) for all \( n > \tilde{M} \) and we finally obtain

\[
E^L_n (t) = E^L_n (0) + \sum_{j=1}^{\tilde{M}-1} \int_{t_j}^{t_{j+1}} \frac{d}{ds} E^L_n (s) \, ds + \int_{t_{\tilde{M}}}^{t} \frac{d}{ds} E^L_n (s) \, ds
\]

\[
\geq E^L_n (0) + t C_{\text{evl}}.
\]

\( \square \)
Remark 4.2. Let us stress the difference of our situation to the one we encounter for Schrödinger operators with alloy-type potentials: To that end, let us define $H^L_t = -\Delta + tW$ with associated form $b^L_t$ on $H^0_0(\Lambda_L)$. Using in this case scale-free unique continuation estimates for eigenfunctions of $H^L_t$, that were proven in e.g. [21, 17], we obtain

$$\left( \frac{d}{dt} b^L_t \right) (\tilde{\varphi}_t(t), \tilde{\varphi}_t(t)) = \int_{\Lambda_L} W|\tilde{\varphi}_t(t)|^2 \geq ||\tilde{\varphi}_t(t)||^2_{L^2(S_{Z,\delta}(L))} \geq C_\delta \text{UCP}.$$  

In particular, we see that in the latter case the gradient does not appear, in contrast to the present situation in [17].

Remark 4.3. If we assume that $W \geq 1$ on the whole cube $\Lambda_L$, Theorem 4.1 is already implicit in [23]. Unique continuation for the gradient is not needed, and consequently we do not need the Lipschitz continuity of the coefficients either, since in this particular case (14) follows from the following elementary argument: As shown in the proof above

$$\left( \frac{d}{dt} E^L_n \right)(t) \geq ||\nabla \psi_n(t)||^2_{L^2(\Lambda_L)},$$  

where $H^L_t \psi_n(t) = E^L_n(t) \psi_n(t)$. The upper bound on the coefficient matrix and the lower bound on the eigenvalue give:

$$(E^L_n(t) \psi_n(t), \psi_n(t))_{L^2(\Lambda_L)} = (H^L_t \psi_n(t), \psi_n(t))_{L^2(\Lambda_L)}$$

$$= ((A + tW)\nabla \psi_n(t), \nabla \psi_n(t))_{L^2(\Lambda_L)}$$

$$\leq (\vartheta_{\text{Ellip}^+} + T ||W||_{\infty}) ||\nabla \psi_n(t)||^2_{L^2(\Lambda_L)},$$

hence

$$\left( \frac{d}{dt} E^L_n \right)(t) \geq \frac{E_-}{\vartheta_{\text{Ellip}^+} + T ||W||_{\infty}}.$$  

In case that only the matrix $A$ satisfies (Lip), we may use a monotonicity argument to prove eigenvalue lifting for perturbations $W$ which are not Lipschitz continuous. This is the tenor of our next

Corollary 4.4. Let $T > 0, \delta \in (0, 1/2), 0 < E_- < E_+ < \infty$ and $K > 0$. Assume that $A$ satisfies (Ellip), (Lip) and (Dir). Then there is a constant $\hat{C}_{\text{eval}} > 0$ such that for all $(1, \delta)$-equidistributed sequences $Z$, all $W \in L^\infty(\Lambda_L)$ satisfying $K \geq W \geq 1_{S_{Z,\delta}(L)}$ and all $n \in \mathbb{N}$ such that $E_- \leq E^L_n(0) \leq E^L_n(T) < E_+$ we have

$$E^L_n(t) \geq E^L_n(0) + t \hat{C}_{\text{eval}}.$$  

Remark 4.5. The disadvantage of Corollary 4.4 compared to Theorem 4.1 is the fact that we can not quantify the dependence of the constant $\hat{C}_{\text{eval}}$ on $\delta$. The reason for that is an approximation argument used in the proof: The Lipschitz-constant of the approximation depends on $\delta$.

Proof. Fix $\delta = \delta/2$. Then there exists a Lipschitz continuous function $\tilde{W}$ with Lipschitz-constant $\text{Lip}(\tilde{W}) = 1/\delta$ satisfying $W \geq \tilde{W} \geq 1_{S_{Z,\delta}(L)}$. We define the
operator $\tilde{H}_L^t := H^L(A + t\tilde{W}\text{Id})$ and denote its eigenvalues by $\tilde{E}_n^L(t)$. Since $\tilde{W}$ is Lipschitz continuous and we have

$$E_- < E_n^L(0) = \tilde{E}_n^L(0) \leq \tilde{E}_n^L(T) \leq E_n^L(T) < E_+,$$

Theorem 4.1 shows that

$$\tilde{E}_n^L(t) \geq \tilde{E}_n^L(0) + C_{\text{evl}}.$$

The minimax-principle finally implies

$$E_n^L(t) \geq \tilde{E}_n^L(t) \geq \tilde{E}_n^L(0) + C_{\text{evl}} = E_n^L(0) + C_{\text{evl}}.$$

The constant is given by

$$C_{\text{evl}} = \frac{\hat{\delta}^2 E_-^2}{2(\vartheta_{\text{Ellip},+} + TK)(8\vartheta_{\text{Ellip},+} + 8TK + \hat{\delta}^2 E_-)} \left(\frac{\delta}{2}\right)^{N(1+\epsilon^{2/3})},$$

where $N$ is a constant that depends on $\delta$, $T$, $\vartheta_{\text{Ellip}}$, and $\vartheta_{\text{Lip}}$.

4.2. Wegner estimate

In this section we will prove a Wegner estimate for random divergence-type operators. We consider a generalized alloy-type random perturbation as introduced in [21, 12]. Let us introduce the model at hand:

**Model (A) (Generalized alloy-type).** Let $0 < \delta_- < \delta_+ < \infty$, $\delta_- \in (0, 1/2)$, $0 < C_- < C_+ < \infty$ and let $\omega = (\omega_j)_{j \in \mathbb{Z}^d}$ be a sequence of independent random variables with probability distributions $(\mu_j)_{j \in \mathbb{Z}^d}$ satisfying $\text{supp} \mu_j \subseteq [0, m]$ for some $m > 0$. Denote by $s : [0, \infty) \to [0, 1]$

$$\sup_{j \in \mathbb{Z}^d} \{\mu_j ([E - \epsilon/2, E + \epsilon/2]) : E \in \mathbb{R}\} \leq s(\epsilon),$$

the global modulus of continuity of the family $(\mu_j)_{j \in \mathbb{Z}^d}$. Moreover, let $Z = (z_j)_{j \in \mathbb{Z}^d}$ be a $(1, \delta_-)$-equidistributed sequence and let $(u_j)_{j \in \mathbb{Z}^d}$ be a sequence of functions on $\Lambda_L$ such that

$$C_- \mathbb{1}_{B(z_j, \delta_-)} \leq u_j \leq C_+ \mathbb{1}_{B(z_j, \delta_+)}.$$ (22)

We fix a matrix-function $A$ that satisfies $\text{Ellip}$, $\text{Lip}$ and $\text{Dir}$ for all $\Lambda_L$, $L \in \mathbb{N}$. Then the generalized alloy-type random perturbation is defined as

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^d} \omega_j u_j(x)$$ (23)

and we consider the random matrix $A_\omega := A + V_\omega\text{Id}$. 
It is easy to see that the assumptions imply \(|V_\omega|_\infty \leq m(2 + \delta_+)^dC_+\) and since \(V_\omega\) is non-negative, the random matrix function \(A_\omega\) is uniformly elliptic with respect to \(\omega\) and \(x\). In particular, the ellipticity constants of \(A_\omega\) are given by
\[
\delta_{\text{Ellip},-} = \delta_{\text{Ellip},-} \quad \text{and} \quad \delta_{\text{Ellip},+} = \delta_{\text{Ellip},+} + m(2 + \delta_+)^dC_+.
\]

We consider the random operator \(H^L_\omega := H^L(A_\omega)\) and denote its eigenvalues by \(E_n^L(\omega)\), enumerated non-decreasingly and counting multiplicities. Using the results on eigenvalue lifting, we may prove a Wegner estimate for the random divergence-type operator \((H^L_\omega)\).

**Remark 4.6.** In general, the random operator \((H_\omega)\) is not ergodic. However, if we assume that the random variables \((\omega_j)\) are i.i.d., the single-site perturbations satisfy \(u_j = u(\cdot - j)\) for some bounded and compactly supported function \(u \geq 1\) and the matrix \(A\) is \(Z^d\)-periodic, then the random operator is ergodic and has therefore almost sure spectrum \(\Sigma\). Note that for \(Z^d\)-periodic \(A\) the assumption \([\text{Dir}]\) is satisfied for all cubes \(\Lambda_L\) if it is satisfied for \(\Lambda_1\).

**Theorem 4.7** (Wegner estimate). Consider a model of type \([A]\) and let \(0 < E_- < E_+ < \infty\). Then there exists a constant \(C_W > 0\) such that for every \(L \in \mathbb{N}\), \(E > 0\) and \(\varepsilon > 0\) satisfying \([E - 3\varepsilon, E + 3\varepsilon] \subseteq [E_-, E_+]\) we have
\[
\mathbb{E}\left[\text{Tr} \chi_{[E-\varepsilon,E+\varepsilon]}(H^L_\omega)\right] \leq C_W s(\varepsilon)|\Lambda|^2.
\] (24)

The proof closely follows [7] and uses the following partial integration formula for singular distributions proved in this paper.

**Lemma 4.8** (Lemma 6 in [7]). Let \(\mu\) be a probability measure with support contained in \((a, b), a < b\), and global modulus of continuity \(s: [0, \infty) \to [0, 1]\). Let \(\Phi\) be a continuously differentiable, non-decreasing function on \([a, b]\). Then
\[
\int_{\mathbb{R}} \Phi(\lambda + \varepsilon) - \Phi(\lambda) \, d\mu(\lambda) \leq s(\varepsilon) (\Phi(b + \varepsilon) - \Phi(a))
\] (25)
holds for all \(\varepsilon > 0\).

**Proof of Theorem 4.7.** Let \(Q := \Lambda_{L+2\delta_+} \cap Z^d\) be the cube that contains all indices \(j\) such that \(\omega_j\) can affect the random perturbation \(V_\omega\) inside \(\Lambda_L\). In other words, \(H^L_\omega\) depends only on the randomness in \(Q\).

Let \(r = (r_j)_{j=1,\ldots,\#Q}\) be an enumeration of the lattice points \(Q\). We define the following vectors in \(\{0, 1\}^{\#Q}\): Let \(e := (1, \ldots, 1)\), let \(e_\ell\) be the vector where only the \(\ell\)-th entry is 1 and let
\[
e^{(r)}_\ell := \sum_{j=1}^\ell e_{r_j} \quad \text{for} \quad \ell \in \{1, \ldots, \#Q\}.
\]

We set
\[
V^Q_{\omega + t e^{(r)}_\ell} := V^Q_\omega + t \sum_{j=1}^\ell e_{r_j} u_j, \quad \text{where} \quad V^Q_\omega := \sum_{j \in Q} \omega_j u_j.
\]
Choose a monotone increasing function $\rho_\varepsilon \in C^\infty(\mathbb{R}, [-1, 0])$ satisfying $\rho \equiv -1$ on $(-\infty, -\varepsilon]$, $\rho \equiv 0$ on $[\varepsilon, \infty)$ and $\|\rho_\varepsilon\|_\infty \leq \varepsilon$. Herewith
\[
1_{[E-\varepsilon, E+\varepsilon]} \leq \rho_\varepsilon(\cdot - E - 2\varepsilon + 4\varepsilon) - \rho_\varepsilon(\cdot - E - 2\varepsilon) \tag{26}
\]
and the spectral theorem implies
\[
\mathbb{E} \left( \text{Tr} \left[ \chi_{[E-\varepsilon, E+\varepsilon]}(H_n^L) \right] \right) \\
\leq \mathbb{E} \left( \text{Tr} \left[ \rho_\varepsilon(H_n^L - E + 4\varepsilon - 2\varepsilon) - \rho_\varepsilon(H_n^L - E - 2\varepsilon) \right] \right) \\
= \mathbb{E} \left( \sum_{n \in \mathbb{N}} \left[ \rho_\varepsilon(E_n^L(\omega) - E - 2\varepsilon + 4\varepsilon) - \rho_\varepsilon(E_n^L(\omega) - E - 2\varepsilon) \right] \right) \tag{27}
\]
Since the support of the right hand side of (26) is contained in $[E_-, E_+]$, only the eigenvalues lying inside $[E_-, E_+]$ give a non-zero contribution in (27).

Choose $T = \varepsilon + m + 1$. Then, by Corollary 4.4, there exists a constant $\tilde{C}_{\text{evl}} > 0$ such that
\[
E_n^L(\omega + \varepsilon \cdot e) \geq E_n^L(\omega) + \varepsilon \tilde{C}_{\text{evl}} \tag{28}
\]
for all $n$ where $E_- < E_n^L(\omega) \leq E_n^L(\omega + \varepsilon \cdot e) < E_+$. Since the latter is satisfied for all summands that give a non-zero contribution on the right hand side of (27), we further estimate it using (28). Thus setting $\varepsilon' := 4\varepsilon/\tilde{C}_{\text{evl}}$ we obtain
\[
\mathbb{E}(\text{Tr} \left[ \chi_{[E-\varepsilon, E+\varepsilon]}(H_n^L) \right]) \\
\leq \mathbb{E} \left( \sum_{n \in \mathbb{N}} \left[ \rho_\varepsilon(E_n^L(\omega + \varepsilon' \cdot e) - E - 2\varepsilon) - \rho_\varepsilon(E_n^L(\omega) - E - 2\varepsilon) \right] \right) \\
= \mathbb{E} \left( \text{Tr} \left[ \rho_\varepsilon(H_n^L - E - 2\varepsilon) - \rho_\varepsilon(H_n^L - E + 2\varepsilon) \right] \right) \\
= \sum_{\ell=1}^{\#Q} \mathbb{E} \left( \text{Tr} \left[ \rho_\varepsilon(H_{\omega^\perp + e'_{\ell} e_\ell}^L(\omega) - E - 2\varepsilon) - \rho_\varepsilon(H_{\omega^\perp + e'_{\ell} e_\ell}^L(\omega) - E - 2\varepsilon) \right] \right). \tag{29}
\]

We will handle each summand separately. Therefore we fix a $\ell \in \{1, \ldots, \#Q\}$ and define
\[
\omega^\perp := (\omega_k^\perp)_{k \in Q} \text{ where } \omega_k^\perp = \begin{cases} 0 & \text{if } k = r_\ell \\ \omega_k & \text{otherwise} \end{cases}
\]
and
\[
\Phi_\ell(t) := \text{Tr} \left[ \rho_\varepsilon(\omega^\perp + e'_{\ell} e_\ell(\rho) - E - 2\varepsilon) \right] < 0.
\]

For fixed $\ell$, $\omega^\perp$ and $\varepsilon'$ we consider the operator
\[
\hat{H}_t^L = H^L \left( A + V_{\omega^\perp + e'_{\ell} e_\ell(\rho)} + t e_\ell \text{Id} \right)
\]
As seen in [4,1], there is a domain $D \subseteq \mathbb{C}$ satisfying $[0, m] \subseteq D$ and such that $(\hat{H}_t^L)_{t \in D}$ is a holomorphic family of type (B) in the sense of Kato and $\hat{H}_t^L$ is self-adjoint, lower-semibounded and has compact resolvent for real $t$. In fact, it is possible to choose $D$ such that $[0, m] \subset (a, b) \subseteq D$ holds true with
\[
a = -\gamma, \ b = m + \gamma \text{ and } \gamma = \frac{\psi_{\text{Ellip}, -}}{4m(2 + \delta_+)\delta_+}.\]
We will show that the function $\Phi_\ell$ is continuously differentiable, bounded and non-decreasing on $(a, b)$ and apply Lemma 4.8. Here the last two assertions follow from the definition of $\rho_\varepsilon$ and the minimax-principle and the differentiability follows from the identity

$$\Phi_\ell(t) = \sum_{n \in \mathbb{N}} \rho_\varepsilon(\bar{E}_n^L(t) - E - 2\varepsilon) = \sum_{m \in \mathbb{N}} \rho_\varepsilon(\hat{\lambda}_m(t) - E - 2\varepsilon),$$

where $\bar{E}_n^L(t)$ denotes the eigenvalues of $\bar{H}_n^L$, enumerated non-decreasingly and counting multiplicities, and $\hat{\lambda}_m(t)$ denotes the eigenvalues of $\bar{H}_n^L$ given by VII. Theorem 3.9 in [9], that depend analytically on $t$. Now each summand in (29) satisfies

$$E \left( \operatorname{Tr} \left[ \rho_\varepsilon(H^L_{\omega + \varepsilon', \varepsilon'_j} - E - 2\varepsilon) - \rho_\varepsilon(H^L_{\omega + \varepsilon', \varepsilon'_j} - E - 2\varepsilon) \right] \right) = E^{Q \setminus \{r_j\}} \left( \int \Phi_\ell(\omega_{r_\varepsilon} + \varepsilon') - \Phi_\ell(\omega_{r_\varepsilon}) \, d\mu_{r_\varepsilon}(\omega_{r_\varepsilon}) \right),$$

where $E^{Q \setminus \{r_j\}}$ denotes the expectation with respect to all random variables $\omega_j$ with $j \in Q \setminus \{r_j\}$. We apply Lemma 4.8 with $a$ and $b$ as above to estimate the inner integral which shows

$$\int \Phi_\ell(\omega_{r_\varepsilon} + \varepsilon') - \Phi_\ell(\omega_{r_\varepsilon}) \, d\mu_{r_\varepsilon}(\omega_{r_\varepsilon}) \leq s(\varepsilon')(\Phi_\ell(b + \varepsilon') - \Phi_\ell(a)) \leq s(\varepsilon'(-\Phi_\ell(a))),$$

and since $|\rho_\varepsilon| \leq 1$, we further estimate

$$-\Phi_\ell(a) = \sum_{n \in \mathbb{N}} (-\rho_\varepsilon(E_n^L(\omega_\perp + \varepsilon' \cdot e_\ell + a \cdot e_{r_\varepsilon}) - E - 2\varepsilon) \leq \# \{E_n^L(\omega_\perp + \varepsilon' \cdot e_\ell + a \cdot e_{r_\varepsilon}) \in \operatorname{supp} \rho_\varepsilon(\cdot - E - 2\varepsilon) \} \leq \# \{E_n^L(\omega_\perp + \varepsilon' \cdot e_\ell + a \cdot e_{r_\varepsilon}) \leq E + 3\varepsilon \} \leq \# \{E_n^L(\omega_\perp + \varepsilon' \cdot e_\ell + a \cdot e_{r_\varepsilon}) \leq E_+ \}.$$ 

Applying Weyl asymptotics, cf. Corollary 4.1.26 in [24], there is a constant $C_{E_+} > 0$, depending only on the dimension $d$ and the ellipticity constant $\vartheta_{\text{Ellip}, -1}$, such that

$$-\Phi_\ell(a) \leq \# \{E_n^L(\omega_\perp + \varepsilon' \cdot e_\ell + a \cdot e_{r_\varepsilon}) \leq E_+ \} \leq C_{E_+} L^d.$$ 

Therefore

$$\int \Phi_\ell(\omega_{r_\varepsilon} + \varepsilon') - \Phi_\ell(\omega_{r_\varepsilon}) \, d\mu_{r_\varepsilon}(\omega_{r_\varepsilon}) \leq C_{E_+} s(\varepsilon') L^d. \quad (30)$$

Bringing everything together and using $\sharp Q \leq (2 + \delta_+)^d L^d$, we have thus shown

$$E \left( \operatorname{Tr} \left[ \chi_{[E-\varepsilon, E+\varepsilon]}(H^L_{\omega}) \right] \right) \leq C_{E_+} s(\varepsilon') L^d \sharp Q \leq C_{E_+} s(\varepsilon')(2 + \delta_+)^d L^{2d}$$

which proves (24) with

$$C_W = C_{E_+}(2 + \delta_+)^d \left[ \frac{4}{C_{evl}} \right]. \quad (31)$$
Remark 4.9. If we could prove (30) with a right hand side independent from $L$, we would obtain a Wegner estimate that is linear in the volume of the cube. This would yield some information on the regularity of the integrated density of states.

If we assume that the single site perturbations $u_j$ additionally satisfy
\[
|u_j(x) - u_j(y)| \leq \ell |x - y| \quad \text{for all } x, y \in \Lambda_L, j \in \mathbb{Z}^d
\]  
and some $\ell > 0$, then the constant $C_W$ can be given more precisely.

Corollary 4.10 (Wegner estimate with Lipschitz continuous perturbations). Consider a model of type (A) assume that (32) holds true and let $0 < E_- < E_+ < \infty$. Then, there exists a constant $C'_W > 0$ such that for all $L \in \mathbb{N}$, $E > 0$ and $\varepsilon > 0$ with $[E - 3\varepsilon, E + 3\varepsilon] \subseteq [E_-, E_+]$,
\[
\mathbb{E} \left( \text{Tr} \chi_{[E-\varepsilon,E+\varepsilon]}(H^L_L) \right) \leq C'_W \varepsilon |\Lambda_L|^2.
\]  
(33)

Proof. It is not hard to see that the matrix function $A_\omega$ satisfies (Lip). Following verbatim the proof of Theorem 4.7 but using Theorem 4.1 instead of Corollary 4.4 we obtain (33) with
\[
C'_W = C_{E_+} (2 + \delta_+)^d \left[ \frac{4}{C_{\text{evl}}} \right].
\]

Since $C_{\text{evl}}$ is given in (18), the dependence of $C'_W$ on $\delta_-$ is explicitly known.

Remark 4.11. It would also be possible to give a proof of Theorem 4.10 that is very similar to the proof given in [10], see also [11, 23, 27]. These proofs are rigorous variants of Wegner’s original idea in [28].

5. Stronger results for small energies

5.1. Unique continuation estimates for the gradient

As noted in remark 3.2 above, Lemma 3.1 applies to discontinuous matrix functions as well. In order to make use of this, we need to replace Theorem 3.3 by some appropriate version that holds true for such coefficient functions. Such versions are at disposal if we consider only energies near the minimum of the spectrum, in fact both in the case of Dirichlet and of Neumann boundary conditions on $\partial \Lambda$.

5.1.1. Dirichlet b.c.

In [26], the authors prove an uncertainty relation that implies an unique continuation estimate with a constant independent of the Lipschitz-constant $\vartheta_{\text{Lip}}$, provided the considered energies are close to zero.

In fact, the independence of the Lipschitz-constant is crucial, since it (as already noted in [26]) allows us to combine the uncertainty relation with an approximation argument to allow matrices that do not satisfy (Lip). Since the proof of the approximation argument was not spelled out in [26], we will provide it in the appendix A.

As a corollary of the latter we obtain
Corollary 5.1. Let $L \in \mathbb{N}$, let $A$ be a matrix function satisfying \([\text{Ellip}]\) and let $\delta \in (0, \delta_0/2)$. Then there is a constant $\kappa'$ (given in (54)) such that for all $(1, \delta)$-equidistributed sequences $Z = (z_j)_{j \in \mathbb{Z}^d}$, all $\lambda < \kappa'$, and all $\psi \in \text{Ran} \chi_{(-\infty, \lambda)}(H^L(A))$ we have

$$\|\psi\|_{L^2(S_{Z, \delta}(L))}^2 \geq \kappa' \|\psi\|_{L^2(A_L)}^2. \quad (34)$$

We may now replace the use of Theorem 3.3 in the proof of Theorem 2.2 with the last mentioned Corollary 5.1 to obtain a version of our main result for small energies without assuming Lipschitz continuity of the matrix function $A$.

Theorem 5.2. Let $L \in \mathbb{N}$, assume that $A$ satisfies \([\text{Ellip}]\) and let $\delta \in (0, \delta_0)$. Then there exists $\kappa > 0$, depending on $\delta, \vartheta_{\text{Ellip}}, - \frac{\vartheta_{\text{Ellip}}}{2} + \frac{\delta}{2}$ and the dimensions $d$, such that for all $0 < E_- < E_+ \leq \kappa$, all $\psi \in \mathcal{D}(H^L(A))$ satisfying $H^L(A)\psi = E\psi$ for some $E \in (E_-, E_+)$ and all $(1, \delta)$-equidistributed sequences $Z$

$$\|\nabla \psi\|_{L^2(S_{Z, \delta}(L))}^2 \geq \tilde{C}_{\text{sfUCP}} \|\psi\|_{L^2(A_L)}^2. \quad (35)$$

holds true. Here the constant $\tilde{C}_{\text{sfUCP}}$ is given in (37) and the constant $\kappa$ in (36).

Proof. Let $\kappa'$ be as in (54) and choose

$$\kappa = \kappa(\delta) := \kappa'(\delta/2) \quad (36)$$

An application of Lemma 3.3 provides us with

$$\|\nabla \psi\|_{L^2(S_{Z, \delta}(L))}^2 \geq C^\nabla(\delta) \|\psi\|_{L^2(S_{Z, \delta/2}(L))}^2$$

and since $E_+ \leq \kappa(\delta) = \kappa'(\delta/2)$ and $Z$ is also $(1, \delta/2)$-equidistributed, Corollary 5.1 shows

$$\|\psi\|_{L^2(S_{Z, \delta/2}(L))}^2 \geq \kappa'(\delta/2) \|\psi\|_{L^2(A_L)}^2.$$ 

Combining these inequalities, we obtain (35) and the constant is given by

$$\tilde{C}_{\text{sfUCP}} = \frac{1}{2} \frac{\delta^2 E_+^2}{2\vartheta_{\text{Ellip}, +} (8\vartheta_{\text{Ellip}, +} + \delta^2 E_-)} \left( \frac{\delta}{2} \right)^{M(1+\vartheta_{\text{Ellip}, -}^{-2/3})}, \quad (37)$$

where $M$ depends only on the dimension $d$. 

For energy intervals $(E_-, E_+)$ higher up in the spectrum it is unclear whether one can expect an estimate like (35) to hold without assuming Lipschitz continuous coefficients. It is well known that there are operators with Hölder continuous coefficients which do not obey the (local) unique continuation principle, see [19, 15, 14].
5.1.2. Neumann b.c.

We may compare Corollary 5.1 with a recent result due to Stollmann and Stolz [25] for divergence-type operators with Neumann boundary conditions in dimensions \( d \geq 3 \). In order to formulate it, let \( H^{L}_N(A) \) be the unique operator associated with the form \( h^L \) given in [11] but with the domain \( \mathcal{D}(h^L) := H^1(\Lambda^L) \).

Then the following theorem is a special case of [25, Theorem 1.1].

**Theorem 5.3 (25).** Let \( L \in \mathbb{N}_\infty \). Assume that \( d \geq 3 \) and that the matrix function \( A \) satisfies \( \text{Ellip} \). There are constants \( C, a, b, c > 0 \) depending only on the dimension \( d \), such that for all \( \delta \in (0, 1/2) \), all \((1, \delta)\)-equidistributed sequences \( Z \) and all \( \psi \in \chi_I(H^{L}_N(A)) \), where \( I = [0, C \varphi \text{Ellip} \cdot \delta^{d-2}] \), we have

\[
\|\psi\|_{L^2(S_{\delta, Z}(L))}^2 \geq C_{NsfUCP}^N \|\psi\|_{L^2(\Lambda^L)}^2.
\]

The constant \( C_{NsfUCP}^N \) is explicitly given by

\[
C_{NsfUCP}^N = C_{NsfUCP}(\delta) = c \varphi_{\text{Ellip}, -} \delta^d \left[ \frac{b}{\min\{\sqrt{d}, L/2\})^2} + \left| \log \left( a \delta^{d-2} \right) \right| \right]^{-2}.
\]

Note that Theorem 5.3, as Corollary 5.1 above, does not need the Lipschitz condition \( \text{Lip} \) nor the assumption \( \text{Dir} \).

**Remark 5.4.** Actually the result of [25] only uses the lower ellipticity constant \( \varphi_{\text{Ellip}, -} \) of \( A \). Since \( A \) is assumed to satisfy \( \text{Ellip} \) in our application, we do not want to elaborate on this fact. It should also be mentioned that the main result of [25] is applicable in more general situations.

With Theorem 5.3 at hand, it is possible to prove a unique continuation estimate for the gradient of eigenfunctions of divergence-type operators with Neumann boundary conditions.

**Theorem 5.5.** Let \( L \in \mathbb{N}_\infty \) and assume \( d \geq 3 \) and that \( A \) satisfies \( \text{Ellip} \). Let \( \delta \in (0, 1/2) \) and let \( Z = (z_j)_{j \in \mathbb{Z}^d} \) be a \((1, \delta)\)-equidistributed sequence. Then there exists \( \kappa^N > 0 \), depending on \( \delta, \varphi_{\text{Ellip}, -} \) and the dimension \( d \), such that for all \( 0 < E_- < E_+ \leq \kappa^N \) the following holds: There is a constant \( C_{SFUCP}^{N, \nabla} > 0 \) such that for all \( \psi \in \mathcal{D}(H^{L}_N(A)) \) satisfying \( H^{L}_N(A)\psi = E\psi \) for some \( E \in (E_-, E_+) \)

\[
\|\nabla \psi\|_{L^2(S_{\delta, Z}(L))}^2 \geq C_{SFUCP}^{N, \nabla} \|\psi\|_{L^2(\Lambda^L)}^2.
\]

holds true. Here the constants are given by

\[
C_{SFUCP}^{N, \nabla} = C^N(\delta)C_{SFUCP}^N(\delta/2) \quad \text{and} \quad \kappa^N = C^N(\delta) = \varphi_{\text{Ellip}, -} \left( \frac{\delta}{2} \right)^{d-2},
\]

where \( C \) depends only on the dimension \( d \).

**Proof.** The proof is an easy adaption of the proof of Theorem 5.2. \( \square \)
In contrast to Theorem 5.2, we do not need to assume that \( L \) is finite. This assumption was needed in the last mentioned theorem since it relies on an approximation argument that uses that the limit operator has purely discrete spectrum.

Remark 5.6. In the case of Neumann b.c. it is trivial to see that unique continuation for the gradient fails at 0: In fact, for all \( L \in \mathbb{N} \) the constant function \( \psi \equiv 1 \) is an eigenfunction of \( H^L_N(A) \) to the eigenvalue \( 0 \in \sigma(H^L_N(A)) \). Since \( \nabla \psi \equiv 0 \) on the whole cube \( \Lambda_L \), unique continuation for the gradient cannot hold for this eigenfunction.

5.2. Eigenvalue lifting and Wegner estimates at low energies

The unique continuation estimates for the gradient of eigenfunctions corresponding to eigenvalues close to zero proven above allow us to prove some of our results in section 4 for more general models.

To begin with, replacing Theorem 2.2 in the proof of Theorem 4.1 by Theorem 5.2, allows us to prove the following variant of our eigenvalue lifting estimate for energies close to zero. Here we do not need the assumption \((\text{Lip})\) for the matrix function \( A \).

Theorem 5.7. Let \( T > 0 \), \( \delta \in (0,1/2) \), assume that \( A \) satisfies \((\text{Ellip})\), let \( \kappa \) be the constant from Theorem 5.2 and let \( 0 < E_- < E_+ < \kappa \). Then there is a constant \( \hat{C}_{\text{evl}} > 0 \), such that for all \( (1,\delta) \)-equidistributed sequences \( Z \), all \( W \in L^\infty(\Lambda_L) \) satisfying \( W \geq 1_{S_{Z,\delta}(L)} \) and all \( n \in \mathbb{N} \) such that \( E_- < E_n^L(0) \leq E_n^L(T) < E_+ \) we have

\[
E_n^L(t) \geq E_n^L(0) + t \hat{C}_{\text{evl}}. \tag{42}
\]

The constant is given by

\[
\hat{C}_{\text{evl}} = \frac{\delta^2 E_-^2}{4\vartheta_{\text{Ellip}_+} + (8\vartheta_{\text{Ellip}_+} + \delta^2 E_-)} \left( \frac{\delta}{2} \right)^{M(1+\vartheta_{\text{Ellip}_-}^{-2/3})}.
\]

where \( M \) depends only on the dimension \( d \).

It is not surprising that an application of our Theorem 5.5 (that is based on \cite{[25]}), provides us with the same result for Neumann boundary conditions and energies close to zero. We again consider the forms defined in \cite{[11]} but with domain given by \( \mathcal{D}(b_T^L) = H^1(\Lambda_L) \) and the operator \( H^L_{N,t} := H^L_N(A + tW \text{Id}) \). Then the same arguments as above apply and we obtain the following

Theorem 5.8. Let \( d \geq 3 \), \( T > 0 \), \( \delta \in (0,1/2) \), \( Z \) be a \((1,\delta)\)-equidistributed sequence, assume that \( A \) satisfies \((\text{Ellip})\) and let \( \kappa^N \) be as in \cite{[41]}. Moreover, let \( W \in L^\infty(\Lambda_L) \) satisfy \( W \geq 1_{S_Z,\delta}(L) \) and let \( 0 < E_- < E_+ \leq \kappa^N \). Then for all \( n \in \mathbb{N} \) such that \( E_- < E_n^L(0) \leq E_n^L(T) < E_+ \) the eigenvalues of the divergence-type operator \( H^L_{N,t} \) with Neumann boundary conditions obey

\[
E_n^L(t) \geq E_n^L(0) + t C_{\text{evl}}^N, \tag{43}
\]
where
\[ C_{\text{evl}}^N = C_{NJCP}^N(\delta) = cC_{\text{Ellip}}^N(\delta)\delta^d \left( \frac{b}{\min\{\sqrt{n}, L/2\}} \right)^2 + \left| \log \left( a\delta^d \right) \right|^2 \]

with constants \( a, b, c > 0 \) depending only on the dimension \( d \).

The new eigenvalue-lifting results provide us, as already noted in subsection 4.2, with more general Wegner estimates for small energies. In our model (A) we needed the matrix function \( A \) to be Lipschitz continuous. However, since Theorem 5.2 does not require the Lipschitz continuity of \( A \) it is not necessary to require this property for the Wegner estimate for small energies. Hence, we consider in the following

Model (B). A model of type (A) where we do not assume that the matrix function \( A \) satisfies (Lip) nor (Dir).

For the operator \( H^L_\omega := H^L(A_\omega) \) we obtain the following

**Theorem 5.9.** Consider a model of type (B). Let \( \kappa \) be as in Theorem 5.2. Then for all \( 0 < E_- < E_+ < \kappa \) there exists a constant \( C_W > 0 \) such that for all \( L \in \mathbb{N}, E \in \mathbb{R} \) and \( \varepsilon > 0 \) with \( [E - 3\varepsilon, E + 3\varepsilon] \subseteq [E_-, E_+] \) we have
\[
\mathbb{E} \left[ \text{Tr} \chi_{[E-\varepsilon,E+\varepsilon]}(H^L_\omega) \right] \leq C_W \varepsilon |\Lambda_L|^2.
\]

For small energies the same proof applies for Neumann boundary conditions. Therefore let \( H^L_{N,\omega} := H^L_N(A_\omega) \). The result reads as follows:

**Theorem 5.10.** Consider a model of type (B). Let \( \kappa^N \) be as in Corollary 5.5. Then for all \( 0 < E_- < E_+ \leq \kappa^N \) there exists a constant \( C_W > 0 \) such that for all \( L \in \mathbb{N}_\infty, E \in \mathbb{R} \) and \( \varepsilon > 0 \) with \( [E - 3\varepsilon, E + 3\varepsilon] \subseteq [E_-, E_+] \) we have
\[
\mathbb{E} \left[ \text{Tr} \chi_{[E-\varepsilon,E+\varepsilon]}(H^L_{N,\omega}) \right] \leq C_W \varepsilon |\Lambda_L|^2.
\]

6. Scaling

Further applications we have in mind require scaled variants of our main results. In particular, we are interested in scaled variants of Theorem 2.2 and variants of the eigenvalue lifting spelled out in Theorem 4.1 and Corollary 4.4.

Therefore, let \( G > 0 \) and consider the cube \( \Lambda_{GL} = G\Lambda_L \). We define the scaling \( S: \mathbb{R}^d \to \mathbb{R}^d \) given by \( S(x) := Gx \) and set \( h_G = h \circ S \) for all functions \( h \) defined on \( \Lambda_{GL} \). Using the definition of the divergence-type operators via forms, it is easy to see that every eigenfunction \( \psi \in L^2(G\Lambda_L) \) to some eigenvalue \( E \in \sigma(H^L_{G\Lambda_L}(A)) \) satisfies \( H^L_{G\Lambda_L}(A)\psi_G = G^2 E\psi_G \), where \( H^L_{G\Lambda_L}(A) \) is the operator associated to the form
\[
h_G^L: H^1_0(\Lambda_L) \times H^1_0(\Lambda_L) \to \mathbb{C}, (u, v) \mapsto \int_{\Lambda_L} \nabla u \cdot A_G \nabla v.
\]

Note that \( A_G \) satisfies (Ellip) with the ellipticity constants \( \vartheta^G_{\text{Ellip}, \pm} = \vartheta_{\text{Ellip}, \pm} \), (Lip) with Lipschitz-constant \( \vartheta^G_{\text{Lip}} = G\vartheta_{\text{Lip}} \) and if \( A \) satisfies (Dir) on the cube
ΛGL then \( A_G \) satisfies (Dir) on the cube \( \Lambda_L \). For some \( \delta \in (0, G/2) \), we let 
\[
Z = (z_j)_{j \in \mathbb{Z}^d} \ 	ext{be a (} G, \delta \text{-equidistributed sequence and calculate using the chain rule}
\]
\[
\int_{S_{Z,\delta}(GL)} |\nabla u|^2 = G^d \int_{G^{-1}S_{Z,\delta}(GL)} |(\nabla u)_G|^2 = G^{d-2} \int_{G^{-1}S_{Z,\delta}(GL)} |\nabla_{\delta/u}|^2
\]
for any function \( u \in H^1(\Lambda_{GL}) \). Since \( G^{-1}S_{Z,\delta}(GL) = S_{Z,G,\delta/G}(L) \) for some 
\((1, \delta/G)\)-equidistributed sequence \( Z_G \), we are in the position to apply our main results. This proves our next two Corollaries.

**Remark 6.1.** The results in this section are only stated for Dirichlet boundary conditions. However, as seen in the previous sections, it is possible to treat Neumann boundary conditions with similar arguments.

**Corollary 6.2.** Let \( L \in \mathbb{N}_\infty \). Assume that \( A \) satisfies (Ellip), (Lip) and (Dir) on the cube \( \Lambda_{GL} \). Let \( 0 < E_- < E_+ < \infty \) and let \( \delta_0 \) be sufficiently small, depending on \( d, \vartheta_{\text{Ellip}}, \vartheta_{\text{Lip}}, G \). Then for all \( \delta \in (0, \delta_0) \) there exists a constant \( C_{sfUCP,G}^\dagger > 0 \), depending on \( d, \vartheta_{\text{Ellip}}, \vartheta_{\text{Lip}}, E_- < E_+ \) and \( \delta/G \), such that for all \( E_- < E < E_+ \), all \( \psi \in \mathcal{D}(H^{\dagger}(A)) \) satisfying \( H^{\dagger}(A)\psi = E \psi \), all \( (G, \delta) \)-equidistributed sequences \( Z \) we have
\[
\|\nabla \psi\|^2_{L^2(S_{Z,\delta}(GL))} \geq C_{sfUCP,G}^\dagger \|\psi\|^2_{L^2(GA_L)}.
\]
The constant is given by
\[
C_{sfUCP,G}^\dagger = C_{sfUCP,G}^\dagger(\delta) = \frac{\delta^2 E_-}{2\vartheta_{\text{Ellip}} + 8(\vartheta_{\text{Ellip}} + \delta^2 E_-)} \left( \frac{\delta}{2G} \right)^{N(1+\delta^{2/3}E_+^{1/3})},
\]
where \( N = N(\delta, \vartheta_{\text{Ellip}}, \vartheta_{\text{Lip}}) \) is the constant from Theorem 3.3 with \( \vartheta_{\text{Lip}} \) replaced by \( \vartheta_{\text{Ellip}} \).

**Remark 6.3.** Note that the scaling procedure described above also provides an appropriate choice for \( \delta_0 \), namely \( \delta_0 = 2G(330d^{1/2} \vartheta_{\text{Ellip}} + 1)^{1/3}(\vartheta_{\text{Ellip}} + 1)^{-1} \).

**Corollary 6.4.** Let \( L \in \mathbb{N} \) and \( G > 0 \). Assume that \( A \) satisfies (Ellip) on the cube \( \Lambda_{GL} \) and let \( \delta \in (0, \delta_0) \). Then there are a constants \( \kappa_G > 0 \) and \( C_{sfUCP,G}^\dagger > 0 \), depending on \( \delta, \vartheta_{\text{Ellip}}, G \) and the dimension \( d \), such that for all \( 0 < E_- < E_+ < \kappa_G \), all \( \psi \in \mathcal{D}(H^{\dagger}(A)) \) satisfying \( H^{\dagger}(A)\psi = E \psi \) for some \( E \in (E_-, E_+) \) and all \( (G, \delta) \)-equidistributed sequences \( Z \)
\[
\|\nabla \psi\|^2_{L^2(S_{Z,\delta}(GL))} \geq C_{sfUCP,G}^\dagger \|\psi\|^2_{L^2(GA_L)}
\]
holds true. The constants are given by
\[
\kappa_G = \frac{1}{2G^2} \left( \frac{\delta}{2G} \right)^{M(1+\delta^{2/3}E_+^{1/3})}
\]
and
\[
C_{sfUCP,G}^\dagger = \frac{\delta^2 E_-}{2\vartheta_{\text{Ellip}} + 8(\vartheta_{\text{Ellip}} + \delta^2 E_-)} \left( \frac{\delta}{2G} \right)^{M(1+\delta^{2/3}E_+^{1/3})},
\]
where \( M \) is a constant that depends only on the dimension.
Obviously, there are scaled variants of the results on eigenvalue lifting, Theorem 4.1 and Corollary 4.4. These can be formulated in the following way. Here we denote by $E_n^{GL}(t)$ the eigenvalues of the operator $H^L_t := H^{GL}(A + t W \text{Id})$ enumerated non-decreasingly and counting multiplicities.

**Corollary 6.5.** Let $T, G > 0, \delta \in (0, 1/2)$ and let $0 < E_- < E_+ < \infty$. Assume that $A$ satisfies $(\text{Ellip}), (\text{Lip})$ and $(\text{Dir})$ on the cube $\Lambda_{GL}$. Then there exists a constant $C_{evl,G} > 0$ such that for all $(G, \delta)$-equidistributed sequence $Z$, all Lipschitz continuous $W \in L^\infty(\Lambda_{GL})$ satisfying $W \geq 1_{S_{Z,\delta}(GL)}$ and all $n \in \mathbb{N}$ such that $E_- < E_n^{GL}(0) \leq E_n^{GL}(T) < E_+$ we have

$$E_n^{GL}(t) \geq E_n^{GL}(0) + t C_{evl}^G.$$  

(49)

Denoting the Lipschitz constant of $W$ by Lip($W$), the constant is given by

$$C_{evl,G} = \frac{\delta^2 E_-^2}{2 \vartheta'_{\text{Ellip},+}(8 \vartheta'_{\text{Ellip},+} + \delta^2 E_-)} \left( \frac{\delta}{2G} \right)^{N(1+G^{4/3}E_-^{2/3})}.$$  

(50)

Here $\vartheta'_{\text{Ellip},-} = \vartheta_{\text{Ellip},-} - \vartheta'_{\text{Ellip},+} = \vartheta_{\text{Ellip},+} + T \|W\|_\infty$, $\vartheta'_{\text{Lip}} = \vartheta_{\text{Lip}} + T\text{Lip}(W)$ and $N = N(\delta, \vartheta_{\text{Ellip}}, G\vartheta'_{\text{Lip}})$.

**Corollary 6.6.** Let $T, G > 0, \delta \in (0, 1/2)$ and let $0 < E_- < E_+ < \infty$. Assume that $A$ satisfies $(\text{Ellip}), (\text{Lip})$ and $(\text{Dir})$ on the cube $\Lambda_{GL}$. Then there exists a constant $\tilde{C}_{evl,G}$, such that for all $(G, \delta)$-equidistributed sequences $Z$, all $W \in L^\infty(\Lambda_{GL})$ satisfying $W \geq 1_{S_{Z,\delta}(GL)}$ and all $n \in \mathbb{N}$ such that $E_- < E_n^{GL}(0) \leq E_n^{GL}(T) < E_+$ we have

$$E_n^{GL}(t) \geq E_n^{GL}(0) + t \tilde{C}_{evl,G}.$$  

(51)

**Remark 6.7.** As already noted in remark 4.5, we do not have control of the dependence of $\tilde{C}_{evl,G}$ on $\delta$.

In addition, we may formulate the scaled variant of Theorem 5.7.

**Corollary 6.8.** Let $T, G > 0, \delta \in (0, \delta_0)$ and assume that $A$ satisfies $(\text{Ellip})$ on the cube $\Lambda_{GL}$, let $\kappa_G$ be the constant from Corollary 6.4 and let $0 < E_- < E_+ \leq \kappa_G$. Then there exists a constant $\tilde{C}_{evl,G}$, such that for all $(G, \delta)$-equidistributed sequences $Z$ and all $W \in L^\infty(\Lambda_{GL})$ satisfying $W \geq 1_{S_{Z,\delta}(GL)}$ we have

$$E_n^{GL}(t) \geq E_n^{GL}(0) + t \tilde{C}_{evl,G}.$$  

(52)

for all $n \in \mathbb{N}$ such that $E_- < E_n^{GL}(0) \leq E_n^{GL}(T) < E_+$. The constant is given by

$$\tilde{C}_{evl,G} = \frac{\delta^2 E_-^2}{2 \vartheta'_{\text{Ellip},+}(8 \vartheta'_{\text{Ellip},+} + \delta^2 E_-)} \left( \frac{\delta}{2G} \right)^{M(1+\vartheta_{\text{Ellip},-}^{2/3})}.$$  

(53)
A. Proof of a remark made in [26]

Let us first cite a special case of Theorem 3.8 in [26] that eliminates the dependence on the constant $\vartheta_{\text{Lip}}$ and the condition [Dir].

**Theorem A.1** ([26]). Assume that the matrix function $A$ satisfies [Lip] and [Ellip]. Then for all $\delta \in (0, \delta_0/2)$ and all $(1, \delta)$-equidistributed sequences $Z$ the uncertainty relation

$$\chi_I(H^L(A))\mathbb{1}_{S_{Z,\delta}(L)}\chi_I(H^L(A)) \geq \kappa' \chi_I(H^L(A))$$

(53)

holds for all measurable $I \subseteq (-\infty, \kappa']$. Here

$$\kappa' = \kappa'(\delta) = \frac{1}{2} \delta^{M(1+\vartheta_{\text{Ellip},-}^{-2/3})}$$

(54)

with some constant $M$ depending only on the dimension.

**Remark A.2.** The uncertainty relation (53) implies that

$$\|\psi\|^2_{L^2(S_{Z,\delta}(L))} \geq \kappa' \|\psi\|^2_{L^2(\Lambda_L)}$$

holds for an eigenfunction $\psi$ of $H^L(A)$ corresponding to an eigenvalue $E \leq \kappa'$.

Now let $A: \Lambda_L \to \mathbb{R}^{d \times d}$ satisfy [Ellip]. In order to approximate the operator $H^L := H^L(A)$ by a sequence of operators $(H^L_\ell)$ satisfying the assumption of Theorem A.1, we need to approximate $A$ by a sequence of Lipschitz continuous and uniformly elliptic matrix functions ($A_\ell$).

**Lemma A.3.** Let $\emptyset \neq U \subseteq \mathbb{R}^d$ be bounded open set. Let $A = (a_{j,k})_{j,k = 1}^d: U \to \text{Sym}(d, \mathbb{R})$ be a matrix-function that satisfies [Ellip] and let $0 < \varepsilon < \vartheta_{\text{Ellip},-}$. Then there exists a sequence of symmetric, uniformly elliptic, and Lipschitz continuous matrices $(A_\ell)$ with ellipticity constants $\vartheta_{\text{Ellip},-}(A_\ell) = \vartheta_{\text{Ellip},-} - \varepsilon$ and $\vartheta_{\text{Ellip},+}(A_\ell) = \vartheta_{\text{Ellip},+}$, converging to $A$ pointwise almost everywhere.

**Remark A.4.** Note that by the polarization identity the ellipticity of $A$ implies that $a_{j,k} \in L^\infty(U) \subseteq L^1(U)$ for all $j, k \in \{1, \ldots, d\}$.

**Proof.** We consider $A$ as a matrix-function $\mathbb{R}^d \to \text{Sym}(d, \mathbb{R})$ by setting $A \equiv 0$ on $\mathbb{R}^d \setminus U$. Let $B := (\vartheta_{\text{Ellip},-} - \varepsilon) \text{Id}$ and $\hat{A} = A - B$. Then $\hat{A}$ is elliptic with ellipticity constants $\vartheta_{\text{Ellip},-}(\hat{A}) = \varepsilon$ and $\vartheta_{\text{Ellip},+}(\hat{A}) = \vartheta_{\text{Ellip},+} - \varepsilon$. Let $g \in C_0^\infty(\mathbb{R})$ satisfy $\text{supp} \ g = [-1, 1]$, $g \geq 0$ and $\int_{\mathbb{R}} g = 1$. Moreover, suppose that $\|g\|_\infty \leq M$ for some $M > 0$. We define $\varphi: \mathbb{R}^d \to \mathbb{R}$ by $\varphi(x) := g(|x|)$ and $\varphi_\ell: \mathbb{R}^d \to \mathbb{R}$ by $\varphi_\ell(x) := \ell^d \varphi(\ell x)$ for $\ell \in \mathbb{N}$. Then $\|\nabla \varphi_\ell\|_\infty \leq \ell^{d+1}M$ and by Young’s inequality for convolutions it is easy to see that $\|a_{j,k} * \varphi_\ell\|_\infty \leq \ell^{d+1}M \max_{j,k} \|a_{j,k}\|_1$. Furthermore, each $a_{j,k} * \varphi_\ell$ is Lipschitz continuous with Lipschitz-constant at most $\ell^{d+1}M \max_{j,k} \|a_{j,k}\|_1$.

We define

$$\hat{A}_\ell = \hat{A} * \varphi_\ell := (\tilde{a}_{j,k} * \varphi_\ell)_{j,k = 1, \ldots, d}.$$
Then $\tilde{A}_\ell$ converges pointwise almost everywhere to $\tilde{A}$ and $\tilde{A}_\ell$ is Lipschitz continuous for all $\ell \in \mathbb{N}$. Since for all $\xi \in \mathbb{R}^d$

$$\xi \cdot \tilde{A}_\ell \xi = \sum_{j,k=1}^{d} (\tilde{a}_{j,k} \ast \varphi_\ell) \xi_j \xi_k = \left( \sum_{j,k=1}^{d} \tilde{a}_{j,k} \xi_j \xi_k \right) \ast \varphi_\ell,$$

it easily follows $0 \leq \xi \cdot \tilde{A}_\xi \leq \vartheta_{\text{Ellip,}+}(\tilde{A})$. Thus the approximation $A_\ell := B + \tilde{A}_\ell$ is Lipschitz continuous and uniformly elliptic with $\vartheta_{\text{Ellip,}-(A_\ell)} = \vartheta_{\text{Ellip,}+} - \varepsilon$ and $\vartheta_{\text{Ellip,}+(A_\ell)} = \vartheta_{\text{Ellip,}+}$.

Corresponding to a sequence of matrices as in Lemma [A.3] we define forms

$$h^L_\ell(u,v) = \int_{\Lambda_L} \nabla u \cdot A_\ell \nabla v$$

on $D(h^L_\ell) = H^1_0(\Lambda_L)$. Each form $h^L_\ell$ generates a unique self-adjoint operator $H^L_\ell = H^L(A_\ell)$ and since all the $A_\ell$ are Lipschitz continuous, Proposition [A.1] applies to $H^L_\ell$ for all $\ell \in \mathbb{N}$. We aim to prove that the sequence of operators $(H^L_\ell)_\ell$ converges to $H^L$ in some appropriate sense, such that we can conclude an inequality like (53) for $H^L$.

For the sake of completeness we recap the the notion of $\Gamma$-convergence.

**Definition A.5.** Let $F$ be a topological vector space, $f \in F$ and denote by $\mathcal{U}(f)$ the system of all open neighbourhoods of $f$ in $F$. Then a sequence $(x_n)_n \in \mathbb{N}$ of functions $x_n : F \to \mathbb{R}$ is said to $\Gamma$-converge to a function $x : F \to \mathbb{R}$ iff for all $f \in F$

$$x(f) = \sup_{U \in \mathcal{U}(f)} \liminf_{n \to \infty} \inf_{g \in U} x_n(g) = \sup_{U \in \mathcal{U}(f)} \limsup_{n \to \infty} \inf_{g \in U} x_n(g).$$

The next proposition shows how the $\Gamma$-convergence of the forms $(h^L_\ell)_\ell$ implies that the sequence $(H^L_\ell)_\ell$ converges to $H$ in the strong resolvent sense.

**Proposition A.6.** With the notation from above, the sequence of quadratic forms $(h^L_\ell)_\ell$ $\Gamma$-converges to $h^L$ in the weak topology of $H^1_0(\Lambda_L)$. This implies that the corresponding operators $H^L_\ell$ converge to $H^L$ in the strong resolvent sense.

**Proof.** The pointwise convergence

$$\lim_{\ell \to \infty} \xi \cdot A_\ell(x) \xi = \xi \cdot A(x) \xi.$$

for every $\xi \in \mathbb{R}^d$ and almost every $x \in \Lambda_L$ and the uniform ellipticity of the matrices $A_\ell$ imply by Theorem 5.14 in [4] that $h^L_\ell$ $\Gamma$-converges to $h^L$ in the weak topology of $H^1_0(\Lambda_L)$. Moreover Theorem 13.12 (cf. also Example 13.13) in [4] implies that in this case $H^L_\ell$ converges to $H^L$ in the strong resolvent sense.

**Remark A.7.** If it is possible to approximate $A$ in a, in some sense, monotone way, one could replace the use of $\Gamma$-convergence by simpler or more elementary arguments, cf. [29, 8].
Combining these results, we may prove the next theorem, which was the goal of this appendix:

**Theorem A.8.** Let $L \in \mathbb{N}$, let $A$ be a matrix function satisfying (Ellip) and let $\delta \in (0, \delta_0)$. With $\kappa'$ as in Theorem A.1 for all $(1, \delta)$-equidistributed sequences $Z = (z_j)_{j \in \mathbb{Z}}$ and all intervals $I := (-\infty, \lambda) \subseteq (-\infty, \kappa']$ we have

$$\chi_I(H^L) 1_{S_{Z,\delta}(L)} \chi_I(H^L) \geq \kappa' \chi_I(H^L) \quad \text{(55)}$$

**Proof.** Since $H^L$ has purely discrete spectrum, there exists $\tilde{\lambda} \in [\lambda, \kappa'] \setminus \sigma(H^L)$. Set $J = (-\infty, \tilde{\lambda})$. Since $A_\ell$ is Lipschitz continuous, Theorem A.1 applies to $H^L_\ell$ and we obtain

$$\chi_J(H^L_\ell) 1_{S_{Z,\delta}(L)} \chi_J(H^L_\ell) \geq \kappa' \chi_J(H^L_\ell) \quad \text{(56)}$$

for all $\ell \in \mathbb{N}$. By Theorem VIII.24 in [20], the convergence $H^L_\ell \to H^L$ in the strong resolvent sense and $\|\chi_J(H^L_\ell)\| \leq 1$ implies that

$$\lim_{\ell \to \infty} \chi_J(H^L_\ell) \psi = \chi_J(H^L) \psi \quad \text{for all } \psi \in L^2(\Lambda_L),$$

since $\chi_{\{\tilde{\lambda}\}}(H^L) = 0$. Note that the strong convergence of $\chi_I(H^L_\ell)$ implies, that

$$\chi_J(H^L_\ell) 1_{S_{Z,\delta}(L)} \chi_J(H^L_\ell) \to \chi_I(H^L) 1_{S_{Z,\delta}(L)} \chi_J(H^L)$$

strongly as $\ell \to \infty$. The right hand side of (56) converges strongly to $\kappa' \chi_J(H^L)$ whilst the left hand side converges strongly to $\chi_J(H^L) 1_{S_{Z,\delta}(L)} \chi_J(H^L)$. Multiplying both sides by $\chi_I(H^L)$, we see that (55) holds true. \qed

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