Coresets Meet EDCS: Algorithms for Matching and Vertex Cover on Massive Graphs

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Abstract

Maximum matching and minimum vertex cover are among the most fundamental graph optimization problems. Recently, randomized composable coresets were introduced as an effective technique for solving these problems in various models of computation on massive graphs. In this technique, one partitions the edges of an input graph randomly into multiple pieces, compresses each piece into a smaller subgraph, namely a coreset, and solves the problem on the union of these coresets to find the final solution. By designing small size randomized composable coresets, one can obtain efficient algorithms, in a black-box way, in multiple computational models including streaming, distributed communication, and the massively parallel computation (MPC) model.

We develop randomized composable coresets of size $\tilde{O}(n)$ that for any constant $\varepsilon > 0$, give a $(3/2 + \varepsilon)$-approximation to matching and a $(3 + \varepsilon)$-approximation to vertex cover. Our coresets improve upon the previously best approximation ratio of $O(1)$ for matching and $O(\log n)$ for vertex cover. Most notably, our result for matching goes beyond a 2-approximation, which is a natural barrier for maximum matching in many models of computation. Our coresets lead to improved algorithms for the simultaneous communication model with randomly partitioned input, the streaming model when the input arrives in a random order, and the MPC model with $\tilde{O}(n^{1/2})$ memory per machine and only two MPC rounds.

Furthermore, inspired by the recent work of Czumaj et al. (arXiv 2017), we study algorithms for matching and vertex cover in the MPC model with only $\tilde{O}(n)$ memory per machine. Building on our coreset constructions, we develop parallel algorithms that give an $O(1)$-approximation to both matching and vertex cover in only $O(\log \log n)$ MPC rounds and $\tilde{O}(n)$ memory per machine. We further improve the approximation ratio of our matching algorithm to $(1 + \varepsilon)$ for any constant $\varepsilon > 0$. Our results settle multiple open questions posed by Czumaj et al.

A key technical ingredient of our paper is a novel application of edge degree constrained subgraphs (EDCS) that were previously introduced in the context of maintaining matchings in dynamic graphs. At the heart of our proofs are new structural properties of EDCS that identify these subgraphs as sparse certificates for large matchings and small vertex covers which are quite robust to sampling and composition.

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1 Introduction

As massive graphs become more prevalent, there is a rapidly growing need for scalable algorithms that solve classical graph problems on large datasets. The salient challenge is that the entire input graph is orders of magnitude larger than the amount of storage on a single processor. For massive inputs, several different computational models have been introduced, each focusing on certain additional resources needed to solve large-scale problems. One example is the streaming model, in which algorithms are allowed to make a single (or a few) passes over the input graph and the target resource is the space being used. Another example is the distributed communication model in which the input is partitioned across multiple parties that can communicate with each other and the resource of interest is the amount of communication between the parties. Yet another example is the massively parallel computation (MPC) model that is a common abstraction of MapReduce-style computation (see Section 2.1 for a definition). The target resources here are the number of rounds of computation and the local storage on each machine.

The three models above, along with their seemingly different target resources, turn out to be closely related. As a result, researchers have focused on designing general algorithmic techniques that can be applicable across wide range of settings. Popular examples of such techniques are linear sketches (see, e.g., [5,6,13,26,28,29,53,54,62]) and composable coresets (see, e.g. [14,15,17,49,64,65]; see also [61], Section 2.2, for natural composable coresets for connectivity and cut sparsifiers). The main idea behind both these approaches is to partition the data into smaller parts, compute a small-size representative summary of each part separately, combine the summaries, and then recover the final solution from their combination. These techniques have been successfully applied to design efficient algorithms for a wide range of problems across these models. Nevertheless, for the two prominent problems of maximum matching and minimum vertex cover, strong impossibility results are known for these techniques [13]; even a weak approximation ratio of $n^{o(1)}$ requires summaries of size $n^{2-o(1)}$, essentially no better than outputting the original subgraph itself as the summary.

Very recently, Assadi and Khanna [11] turned to the notion of a randomized composable coreset in order to bypass these strong impossibility results. In this technique, originally introduced by [64] (see also [33]) in the context of submodular maximization, one partitions the input graph randomly and computes a suitable subgraph of each piece, i.e., a coreset, as its representative summary. These subgraphs are said to be composable in that their union yields a subgraph in which the optimal solution to the problem at hand is a good approximation to the original optimal solution. The authors in [11] demonstrated the effectiveness of this technique by designing randomized composable coresets of size $\tilde{O}(n)$ that give an $O(1)$ and $O(\log n)$ approximation to matching and vertex cover, respectively. These results directly translate to streaming, distributed, and MPC algorithms (under random partitioning of the input in the first two cases). However, while the results in [11] showed a strong gap between the power of randomized composable coresets compared to previous approaches on adversarial partitions, the implication of these results were rather weak and could not compete with the state-of-the-arts algorithms that were designed specifically for each model.

In this paper, we continue the study of randomized composable coresets for matching and vertex cover and present coresets with significantly improved approximation ratios for both problems. Our results imply a unified approach for solving these two problems across different settings and improve the state-of-the-art in the aforementioned computational models in some or all parameters involved. The generality of randomized composable coresets can, in principle, make the problem of designing them harder or even impossible compared to solving the problem on each specific computational model. It is therefore perhaps surprising that using this unified approach, we can design essentially a single algorithm that can improve the state-of-the-art algorithms in all these models simultaneously.
1.1 Randomized Composable Coresets

Let $E$ be an edge-set of a graph $G(V, E)$; we say that a collection of edges $\{E^{(1)}, \ldots, E^{(k)}\}$ is a random $k$-partition of $E$ if the sets are constructed by assigning each edge $e \in E$ to some $E^{(i)}$ chosen uniformly at random. A random $k$-partition of $E$ naturally results in partitioning the graph $G$ into $k$ subgraphs $G^{(1)}, \ldots, G^{(k)}$ whereby $G^{(i)} := G(V, E^{(i)})$ for all $i \in [k]$ (we use random partitions for both the edge-set and the input graph interchangeably).

**Definition 1** (Randomized Composable Coreset [11, 64]). Consider an algorithm ALG that given a graph $G(V, E)$ outputs a subgraph $\text{ALG}(G) \subseteq G$ with at most $s$ edges. Let $G^{(1)}, \ldots, G^{(k)}$ be a random $k$-partition of a graph $G$. We say that ALG outputs an $\alpha$-approximation randomized composable coreset of size $k$ for a graph optimization problem $P$ iff $P(\text{ALG}(G^{(1)}) \cup \ldots \cup \text{ALG}(G^{(k)}))$ is an $\alpha$-approximation to $P(G)$ with high probability (over the randomness of the random $k$-partition).

For brevity, we use randomized coresets to refer to randomized composable coresets. Following [11], we augment the definition of randomized coresets by allowing the coresets to also contain a fixed solution (which is counted in the size of the coreset) to be directly added to the final solution of the composed coresets (this is only needed for our vertex cover coreset; see [11] on necessity of this definition for this problem). We recite some of the well-known applications of randomized composable coresets (see Appendix A for definition of these models in details and proofs, and [49, 64] for further applications).

**Proposition 1.1.** Suppose ALG outputs an $\alpha$-approximation randomized coreset of size $s$ for a problem $P$. Let $G(V, E)$ be a graph with $m = |E|$ edges. Then, ALG implies:

1. A parallel algorithm in the MPC model that with high probability outputs an $\alpha$-approximation to $P$ in two rounds with $O(\sqrt{m/s})$ machines, each with $O(\sqrt{ms} + n)$ memory.
2. A streaming algorithm that on random arrival streams outputs an $\alpha$-approximation to $P$ with high probability using $O(\sqrt{ms})$ space.
3. A simultaneous communication protocol that on randomly partitioned inputs computes an $\alpha$-approximation to $P$ with high probability using $O(s)$ communication per machine/player.

1.2 Our Results

We start by studying the previous randomized coreset of [11] for matching, which was simply to pick a maximum matching of each machine’s subgraph as its coreset. This is arguably the most natural approach to the problem and results in truly sparse subgraphs (maximum degree one). As a warm-up to our main results, we present a simpler and improved analysis (compared to that in [11]), which shows that this coreset achieves a 3-approximation (vs. 9-approximation proven in [11]). Our analysis of this randomized coreset relies on a new concentration result for size of maximum matchings in sampled subgraphs that could be of independent interest (it is in fact used in other parts of this paper as well). We also show that there exist graphs on which the approximation ratio of this coreset is at least 2. This suggests that to achieve a better than 2 approximation, fundamentally different ideas are needed which brings us to our first main result.

Our first main result gives new randomized composable coresets for matching and vertex cover.

**Result 1.** There exist randomized composable coresets of size $O(n)$ that for any constant $\varepsilon > 0$, give a $(3/2 + \varepsilon)$-approximation for maximum matching and $(3 + \varepsilon)$-approximation for minimum vertex cover with high probability.
Result 1 improves upon the randomized coresets of [11] that obtained $O(1)$ and $O(\log n)$ approximation for matching and vertex cover, respectively. Additionally, size of our coresets are optimal up to polylog($n$) factor by a lower bound of [11]. Result 1 yields several algorithms for matching and vertex cover across different computational models. Let us exhibit the most interesting ones.

**The MPC Model.** Maximum matching and minimum vertex cover are among the most studied graph optimization problems in the MPC and similar MapReduce-style computation models [4, 5, 11, 19, 32, 58]. As an application of Result 1 (by Proposition 1.1), we obtain efficient MPC algorithms for matching and vertex cover in only two rounds of computation.

**Corollary 1.** There exists MPC algorithms that with high probability achieve (almost) $(3/2)$-approximation to matching and (almost) $3$-approximation to vertex cover in two MPC rounds and $\tilde{O}(n\sqrt{n})$ memory per machine (in general $\tilde{O}(\sqrt{mn})$ for graphs with $m$ edges)\footnote{The approximation factor for vertex cover degrades to 4 if one requires local computation on each machine to be polynomial time; see Remarks A.1 and 5.2.}

The number of rounds of our algorithms in Corollary 1 is optimal among all MPC algorithms that use $n^{2-o(1)}$ memory per machine, as it follows from the results in [13] that any single round MPC algorithm that uses $n^{2-o(1)}$ memory per machine cannot achieve a better than $n^{o(1)}$ approximation to either problem. Furthermore, if the input is distributed randomly in the first place, our algorithms can be implemented in only one MPC round (see [64] for details on when this assumption applies).

Our algorithms outperform the previous algorithms of [11] for matching and vertex cover in terms of approximation ratio ($3/2$ vs. $O(1)$ and $3$ vs. $O(\log n)$), while memory and round complexity are the same. Our matching algorithm outperforms the 2-approximate maximum matching algorithm of Lattanzi et al. [58] in terms of both the approximation ratio ($3/2$ vs. 2) and round complexity (2 vs. 6) within the same memory. Our result for the matching problem is particularly interesting as all other MPC algorithms [4, 5, 19] that can achieve a better than two approximation (which is a natural barrier for matching algorithms across different models) require a large (unspecified) constant number of rounds. The improvement on the number of rounds is significant in this context; the round complexity of MPC algorithms determines the dominant cost of the computation (see, e.g. [18, 58]), and hence minimizing the number of rounds is the primary goal in this model.

**Streaming.** Obtaining a 2-approximation streaming algorithm for matching (and vertex cover) is trivial within $O(n)$ space as one can simply maintain a maximal matching in the stream. Beating the factor of 2 in the approximation ratio of this naive algorithm for matching however has remained one of the central open questions in the graph streaming literature since the introduction of the field in [41]. Currently no $o(n^2)$-space algorithm is known for this task on adversarially ordered streams and the best lower bound result by Kapralov [51] proves that $(\frac{e}{\epsilon-1})$-approximation requires $n^{1+\Omega(1/\log \log n)}$ space by single-pass streaming algorithms. To make progress on this fascinating open problem, Konrad et al. [57] suggested the study of matching in random arrival streams. They presented an algorithm with approximation ratio strictly better than 2, namely $2-\delta$ for $\delta \approx 0.002$, in $O(n)$ space over random streams. Our Result 1 combined with Proposition 1.1 improves the approximation ratio of this algorithm significantly albeit at a cost of larger space requirement.

**Corollary 2.** There exists a single-pass streaming algorithm on random arrival streams that uses $\tilde{O}(n\sqrt{n})$ space and with high probability (over the randomness of the stream) achieves an (almost) $(3/2)$-approximation to the maximum matching problem.

This is the first streaming algorithm for matching that beats the ratio of $(\frac{e}{\epsilon-1})$ which is known to be “hard” on adversarial streams. In particular, while the lower bound of [51] does not preclude the
existence of streaming algorithms with $\tilde{O}(n^{1/2})$ space that achieve better than $\left(\frac{\varepsilon}{\varepsilon-1}\right)$-approximation on adversarial streams, the proof in [51] (see also [43]) suggests that achieving such algorithm is ultimately connected to further understanding of Ruzsa-Szemerédi graphs, a notoriously hard problem in additive combinatorics (see, e.g., the survey by Gowers [45]). We refer the interested reader to [8,42] for details on Ruzsa-Szemerédi graphs and to [12,43,51] for their connection to the streaming matching problem.

**Simultaneous Communication Model.** Maximum matching (and to a lesser degree vertex cover) has been studied previously in the simultaneous communication model owing to many applications of this model in achieving round-optimal distributed algorithms [11] to proving lower bounds in dynamic graph streams [7,12,13,56] or applications to mechanism design [9,35,36]. As another application of Result 1, we obtain the following corollary in this model.

**Corollary 3.** There exists simultaneous communication protocols on randomly partitioned inputs that achieve (almost) $(3/2)$-approximation to matching and (almost) $3$-approximation to vertex cover with high probability (over the randomness of the input partitioning) with only $\tilde{O}(n)$ communication per machine/player.

These results improve upon the $O(1)$ and $O(\log n)$ approximation simultaneous protocols of [11] (on randomly partitioned inputs) for matching and vertex cover that were also designed by using randomized coresets. Our protocols achieve optimal communication complexity (up to polylog$(n)$ factors) [11]. Interestingly, when the input is adversarially partitioned, the best approximation ratio achievable by any simultaneous protocol for either matching or vertex cover with $\tilde{O}(n)$ communication is only $\Theta(n^{1/3})$ [13] (see also [11]). It is thus remarkable that a simple data oblivious partitioning scheme, namely, the random partitioning, can make these problems so much more tractable.

Our second main result concerns the MPC model specifically. We build on our coresets in Result 1 to design a memory efficient MPC algorithm for matching and vertex cover in a small number of rounds.

**Result 2.** There exists an MPC algorithm that with high probability gives an $O(1)$ approximation to both maximum matching and minimum vertex cover in $O(\log \log n)$ MPC rounds using only $\tilde{O}(n)$ memory per machine.

The approximation ratio of the matching algorithm in Result 2 can be reduced to (almost) 2 by standard techniques (see Corollary 8). Additionally, using the reduction of [60], this approximation ratio can be further improved to $(1+\varepsilon)$ for any constant $\varepsilon > 0$ with asymptotically the same memory and number of rounds (see Corollary 9).

Prior to [32], all MPC algorithms for matching and vertex cover [4,5,58] required $\Omega\left(\frac{\log n}{\log \log n}\right)$ rounds to achieve $O(1)$ approximation when the memory per machine was restricted to $\tilde{O}(n)$ (which is arguably the most natural choice of parameter, similar-in-spirit to the semi-streaming restriction [41,61]). Classical PRAM algorithms [50,59] already achieved an $O(1)$-approximation for these problems in $O(\log n)$ rounds, implying that the previous MPC algorithms could not benefit from the additional power of MPC model (more storage and local computational power) compared to classical parallel settings such as PRAM unless the memory per machine become as large as $n^{1+\Omega(1)}$.

In a recent breakthrough, Czumaj et al. [32] presented an (almost) 2-approximation algorithm for maximum matching that requires $O(n)$ (even $n/(\log n)^{O(\log \log n)}$) memory per machine and only $O\left((\log \log n)^2\right)$ MPC rounds. Result 2 improves upon this result on several fronts: (i) we improve the round complexity of the matching algorithm to $O(\log \log n)$, resolving a conjecture of [32] in the
affirmative, (ii) we obtain an $O(1)$ approximation to vertex cover, answering another open question of [32], and (iii) we achieve all these using a considerably simpler algorithm and analysis than [32].

1.3 Our Techniques

We obtain our coresets in Result 1 using a novel application of edge degree constrained subgraphs (EDCS) that were previously introduced by Bernstein and Stein [20] for maintaining large matchings in dynamic graphs. While previous work on EDCS [20, 21] focused on how large a matching an EDCS contains and how it can be maintained efficiently in a dynamic graph, in this paper we study several new structural properties of the EDCS itself. Our results identify these subgraphs as sparse certificates for large matchings and small vertex covers which are quite robust to sampling and composition, an ideal combination for a randomized coreset.

To prove Result 2, we borrow one simple high-level technique from [32], namely the vertex sampling approach. In this technique, instead of having each machine work on a subgraph obtained by randomly sampling edges in the original graph (as is the case in randomized coresets), each machine samples some fraction of the vertices, and then works with the induced subgraph defined by those vertices. We show that with proper modifications, EDCS used in our coresets in Result 1 are robust enough even under this vertex sampling approach. We use this property to design a recursive procedure in which we repeatedly compute an EDCS of the underlying graph in a distributed fashion, redistribute it again via the vertex sampling approach, and recursively solve the problem on this EDCS to compute an $O(1)$-approximation to matching and vertex cover. We therefore limit the memory on each machine to only $O(n)$ at the cost of increasing the number of rounds from $O(1)$ to $O(\log \log n)$. Additional ideas are needed to ensure that the approximation ratio of the algorithm does not increase beyond a fixed constant as a result of repeatedly computing an EDCS of the current graph in $O(\log \log n)$ iterations.

Remark 1.2. While our Result 1 was obtained completely independently of the results by Czumaj et al. [32] (indeed it deals with a different setting), our Result 2 was influenced by the results in [32] in the following way. First, our result was influenced by the choice of the model to consider, i.e., MPC with only $O(n)$ memory per machine. More importantly, we also borrowed one simple high-level technique from [32], namely the vertex sampling approach mentioned above. Other than this starting point, our approach proceeds along entirely different lines from [32], in terms of both the local algorithm computed on each subgraph and in the analysis. The main approach in [32] is round compression, which corresponds to compressing multiple rounds of a particular distributed algorithm into smaller number of MPC rounds by maintaining a consistent state across the local algorithms computed on each subgraph (using a highly non-trivial local algorithm and analysis). Our results on the other hand do not correspond to a round compression approach at all and we do not require any consistency in the local algorithm on each machine. Instead, we rely on structural properties of the EDCS that we prove in this paper, independent of the algorithms that compute these subgraphs.

1.4 Further Related Work

Maximum matching and minimum vertex cover are among the most studied problems in the context of massive graphs including in MPC model and MapReduce-style computation [4, 5, 11, 19, 32, 58], streaming algorithms [3–6, 12, 13, 29–31, 37–41, 43, 46, 51, 52, 56, 57, 60, 61, 63, 72], simultaneous communication model and similar distributed models [9, 11–13, 36, 46, 48], dynamic graphs [16, 20–25, 67, 70, 74], and sub-linear time algorithms [47, 68, 69, 71, 76]. Beside the results mentioned already, most relevant to our work are the polylog($n$)-space polylog($n$)-approximation algorithm of [52] for estimating the size of a maximum matching in random stream, and the (3/2)-approximation communication protocol of [43] when the input is (adversarially) partitioned between two parties and the
communication is from one party to the other one (as opposed to simultaneous which we studied). However, the techniques in these results and ours are completely disjoint.

Coresets, composable coresets, and randomized composable coresets are respectively introduced in [2, 49], and [64]. Composable coresets have been studied previously in nearest neighbor search [1], diversity maximization [49, 77], clustering [15, 17], and submodular maximization [14, 33, 34, 49, 64]. Moreover, while not particularly termed a composable coreset, the “merge and reduce” technique in graph streaming literature (see [61], Section 2.2) is identical to composable coresets.

2 Preliminaries

Notation. For a graph $G(V, E)$, we use $\text{MM}(G)$ to denote the maximum matching size in $G$ and $\text{VC}(G)$ to denote the minimum vertex cover size. For any subset of vertices $V' \subseteq V$ and any subset of edges $E' \subseteq E$, we use $V'(E')$ to denote the set of vertices in $V'$ that are incident on edges of $E'$ and $E'(V')$ to denote the set of edges in $E'$ that are incident on vertices of $V'$. For any vertex $v \in V$, we use $d_G(v)$ to denote the degree of $v$ in the graph $G$.

We use capital letters to denote random variables. Let $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ be a sequence of random variables on a common probability space such that $\mathbb{E}[X_i | Y_1, \ldots, Y_{i-1}] = X_{i-1}$ for all $i$. The sequence $\{X_i\}$ is referred to as a martingale with respect to $\{Y_i\}$. A summary of concentration bounds we use in this paper appears in Appendix B.1.

Sampled Subgraphs. Throughout the paper, we work with two different notions of sampling a graph $G(V, E)$. For a parameter $p \in (0, 1)$,

- A graph $G_p^E(V, E_p)$ is an edge sampled subgraph of $G$ iff the vertex set of $G_p^E$ and $G$ are the same and every edge in $E$ is picked independently and with probability $p$ in $E_p$.
- A graph $G_p^V(V_p, E_p)$ is a vertex sampled (induced) subgraph of $G$ iff every vertex in $V$ is sampled in $V_p$ independently and with probability $p$ and $G_p^V$ is the induced subgraph of $G$ on $V_p$.

2.1 The Massively Parallel Computation (MPC) Model

We adopt the most stringent model of modern parallel computation among [10, 18, 44, 55], the so-called Massively Parallel Computation (MPC) model of [18]. Let $G(V, E)$ with $n := |V|$ and $m := |E|$ be the input graph. In this model, there are $p$ machines, each with a memory of size $s$ and one typically requires that both $p, s = m^{1-\Omega(1)}$ i.e., polynomially smaller than the input size [10, 55]. Computation proceeds in synchronous rounds: in each round, each machine performs some local computation and at the end of the round machines exchange messages to guide the computation for the next round. All messages sent and received by each machine in each round have to fit into the local memory of the machine. This in particular means that the length of the messages on each machine is bounded by $s$ in each round. At the end, the machines collectively output the solution.

2.2 Basic Graph Theory Facts

Fact 2.1. For any graph $G(V, E)$, $\text{MM}(G) \leq \text{VC}(G) \leq 2 \cdot \text{MM}(G)$.

The following propositions are well-known.

Proposition 2.2. Suppose $M$ and $V'$ are respectively, a matching and a vertex cover of a graph $G$ such that $\alpha \cdot |M| \geq |V'|$; then, both $M$ and $V'$ are $\alpha$-approximation to their respective problems.

Proof. $\text{VC}(G) \stackrel{\text{Fact 2.1}}{\geq} \text{MM}(G) \geq |M| \geq \frac{1}{\alpha} \cdot |V'| \geq \frac{1}{\alpha} \cdot \text{VC}(G) \stackrel{\text{Fact 2.1}}{\geq} \frac{1}{\alpha} \cdot \text{MM}(G)$. □

Proposition 2.3. Suppose $G(V, E)$ is a graph with maximum degree $\Delta$ and $V_{\text{high}}$ is the set of all vertices with degree at least $\gamma \cdot \Delta$ in $G$ for $\gamma \in (0, 1)$. Then, $\text{MM}(G) \geq \frac{\gamma \Delta}{\Delta + 1} \cdot |V_{\text{high}}|$. 

6
implies that an EDCS of a graph
2.3 Edge Degree Constrained Subgraph (EDCS)
We introduce edge degree constrained subgraphs (EDCS) in this section and present several of their properties which are proven in previous work. We emphasize that all other properties of EDCS proven in the subsequent sections are new to this paper.

An EDCS is defined formally as follows.

Definition 2 ([20]). For any graph \(G(V,E)\) and integers \(\beta \geq \beta^- \geq 0\), an edge degree constraint subgraph (EDCS) \((G,\beta,\beta^-)\) is a subgraph \(H := (V,E_H)\) of \(G\) with the following two properties:

(P1) For any edge \((u,v) \in E_H\) we have \(d_H(u) + d_H(v) \leq \beta\).

(P2) For any edge \((u,v) \in E \setminus E_H\) we have \(d_H(u) + d_H(v) \geq \beta^-\).

We sometimes abuse the notation and use \(H\) and \(E_H\) interchangeably.

In the remainder of the paper, we use the terms “Property (P1)” and “Property (P2)” of EDCS to refer to the first and second items in Definition 2 above.

One can prove the existence of an EDCS\((G,\beta,\beta^-)\) for any graph \(G\) and parameters \(\beta^- < \beta\) using the results in [21] (Theorem 3.2) which in fact shows how to maintain an EDCS efficiently in the dynamic graph setting. As we are only interested in existence of EDCS in this paper, we provide a simpler and self-contained proof of this fact in Appendix B.2, which also implies a simple polynomial time algorithm for computing any EDCS of a given graph \(G\).

Lemma 2.4. Any graph \(G\) contains an EDCS\((G,\beta,\beta^-)\) for any parameters \(\beta > \beta^-\).

It was shown in [20] (bipartite graphs) and [21] (general graphs) that for appropriate parameters and EDCS always contains an (almost) 3/2-approximate maximum matching of \(G\). Formally:

Lemma 2.5 ([20,21]). Let \(G(V,E)\) be any graph and \(\varepsilon < 1/2\) be a parameter. For parameters \(\lambda \geq \frac{\Delta}{10\Delta}, \beta \geq 32\lambda^{-3}\), and \(\beta^- \geq (1 - \lambda) \cdot \beta\), in any subgraph \(H := \text{EDCS}(G,\beta,\beta^-)\), \(\text{MM}(G) \leq \left(\frac{3}{2} + \varepsilon\right) \cdot \text{MM}(H)\).

Lemma 2.5 implies that an EDCS of a graph \(G(V,E)\) preserves the maximum matching of \(G\) approximately. We also show a similar result for vertex cover. The basic idea is that in addition to computing a vertex cover for the subgraph \(H\) (to cover all the edges in \(H\)), we also add to the vertex cover all vertices that have degree at least \(\beta^-/2\) in \(H\), which by Property (P2) of an EDCS covers all edges in \(G \setminus H\).

Lemma 2.6. Let \(G(V,E)\) be any graph, \(\varepsilon < 1/2\) be a parameter, and \(H := \text{EDCS}(G,\beta,\beta^-)\) for parameters \(\beta \geq \frac{1}{\varepsilon} \cdot \beta\) and \(\beta^- \geq \beta \cdot (1 - \varepsilon/4)\). Suppose \(V_{\text{high}}\) is the set of vertices \(v \in V\) with \(d_H(v) \geq \beta^-/2\) and \(V_{\text{vc}}\) is a minimum vertex cover of \(H\); then \(V_{\text{high}} \cup V_{\text{vc}}\) is a vertex cover of \(G\) with size at most \((3 + \varepsilon) \cdot \text{VC}(H)\) (note that \(\text{VC}(H) \leq \text{VC}(G)\)).

Proof. We first argue that \(V_{\text{high}} \cup V_{\text{vc}}\) is indeed a feasible vertex cover of \(G\). To see this, notice that any edge \(e \in H\) is covered by \(V_{\text{vc}}\), and moreover by Property (P2) of EDCS, any edge \(e \in E \setminus H\) has at least one endpoint with degree at least \(\beta^-/2\) in \(H\) and hence is covered by \(V_{\text{high}}\). In the following, we bound the size of \(V_{\text{high}}\) by \((2 + \varepsilon) \cdot \text{VC}(H)\), which finalizes the proof as clearly \(|V_{\text{vc}}| = \text{VC}(H)\).

By Property (P1) of EDCS, the maximum degree of each vertex in \(H\) is \(\beta\). Moreover, for any vertex \(v \in V_{\text{high}}\), we have \(d_H(v) \geq \beta^-/2 = (1 - \varepsilon/4) \cdot \beta/2\). Hence, we can apply Proposition 2.3 on
the graph $H$ with parameters $D = \beta$ and $\gamma = \frac{1}{2} \cdot (1 - \varepsilon/4)$, and obtain,

$$\text{VC}(H) \geq \text{MM}(H) \geq \frac{1}{(2 + \varepsilon)} \cdot |V_{\text{high}}|.$$  

finalizing the proof. ■

3 Warmup: A 3-Approximation Coreset for Matching

A natural randomized coreset for the matching problem was previously proposed by [11]: simply compute a maximum matching of each graph $G^{(i)}$. We refer to this randomized coreset as the MaxMatching coreset. It was shown in [11] that MaxMatching is an $O(1)$-approximation randomized coreset for the matching problem (the hidden constant in the $O$-notation was bounded by 9 in [11]). As a warm up, we propose a better analysis of this randomized coreset in this section.

**Theorem 4.** Let $G(V, E)$ be a graph with $\text{MM}(G) = \omega(k \log n)$ and $G^{(1)}, \ldots, G^{(k)}$ be a random $k$-partition of $G$. Any maximum matching of the graph $G^{(i)}$ is a $(3 + o(1))$-approximation randomized composable coreset of size $O(n)$ for the maximum matching problem.

**Assumption on MM($G$).** In this section, we follow [11] in assuming that $\text{MM}(G) = \omega(k \log n)$ since otherwise we can immediately obtain a (non-randomized) composable coreset with approximation ratio one (an exact maximum matching) and size $\tilde{O}(k^2)$ for the matching problem using the results in [29]. We emphasize that this assumption is only needed for the results in this section.

A crucial building block in our proof of Theorem 4 is a new concentration result for the size of maximum matching in edge sampled subgraphs that we prove in the next section. This result is quite general and can be of independent interest.

3.1 Concentration of Maximum Matching Size under Edge Sampling

Let $G(V, E)$ be any arbitrary graph and $p \in (0, 1)$ be a parameter (possibly depending on size of the graph $G$). Define $G^E_p(V, E_p)$ as a subgraph of $G$ obtained by sampling each edge in $E$ independently and with probability $p$, i.e., an edge sampled subgraph of $G$. We show that $\text{MM}(G_p)$ is concentrated around its expected value.

**Lemma 3.1.** Let $G(V, E)$ be any arbitrary graph, $p \in (0, 1)$ be a parameter, and $\mathbb{E}[\text{MM}(G^E_p)] \leq \mu$. For any $\lambda > 0$,

$$\Pr \left( \left| \text{MM}(G^E_p) - \mathbb{E}[\text{MM}(G^E_p)] \right| \geq \lambda \right) \leq 2 \cdot \exp \left( -\frac{\lambda^2 \cdot p}{2 \cdot \mu} \right).$$

**Proof.** For simplicity, define $G_p := G^E_p$. Let $C$ be any minimum vertex cover in the graph $G$. We use vertex exposure martingales over vertices in $C$ to prove this result. Fix an arbitrary ordering of vertices in $C$ and for any $v \in C$, define $C^{<v}$ as the set of vertices in $C$ that appear before $v$ in this ordering. For each $v \in C$, we define a random variable $Y_v \in \{0, 1\}^{V \setminus C^{<v}}$ as a vector of indicators whether a possible edge (i.e., an edge already in $G$) between the vertices $v$ and $u \in V \setminus C^{<v}$ appears in $G_p$ or not. Since $C$ is a vertex cover of $G$, every edge in $G$ is incident on some vertex of $C$. As a result, the graph $G_p$ is uniquely determined by the vectors $Y_1, \ldots, Y_{|C|}$. Define a sequence of random variables $\{X_i\}_{i=1}^{|C|}$, whereby $X_i = \mathbb{E}[\text{MM}(G_p) | Y_1, \ldots, Y_i]$. The following claim is standard.

**Claim 3.2.** The sequence $\{X_i\}_{i=1}^{|C|}$ is a martingale with respect to the sequence $\{Y_i\}_{i=1}^{|C|}$.

**Proof.** For any $i \leq |C|$,

$$\mathbb{E}[X_i | Y_1, \ldots, Y_{i-1}] = \mathbb{E}_{(y_1, \ldots, y_{i-1})}[\mathbb{E}[\text{MM}(G_p) | Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}, Y_i] | Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}]$$

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\[ \begin{align*}
&= \mathbb{E}_{(y_1, \ldots, y_{i-1})} \left[ \text{MM}(G_p) \mid Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1} \right] \\
&= \mathbb{E} \left[ \text{MM}(G_p) \mid Y_1, \ldots, Y_{i-1} \right] = X_{i-1}. \quad \blacksquare
\end{align*} \]

Notice that \( X_0 := \mathbb{E} [\text{MM}(G_p)] \) and \( X_{|C|} = \text{MM}(G_p) \) as fixing \( Y_1, \ldots, Y_{|C|} \) uniquely determines the graph \( G_p \). Hence, we can use Azuma’s inequality to show that value of \( X_{|C|} \) is close to \( X_0 \) with high probability. To do this, we need a bound on \( |C| \), as well as each term \( |X_i - X_{i-1}| \). Bounding each \( |X_i - X_{i-1}| \) term is quite easy; the set of edges incident on the vertex \( i \) can only change the maximum matching in \( G_p \) by 1 (as \( i \) can only be matched once), and hence \( |X_i - X_{i-1}| \leq 1 \). In the following, we also bound the value of \( |C| \).

**Claim 3.3.** \( |C| \leq 2 \cdot \mu/p \).

**Proof.** Since size of a minimum vertex cover of a graph \( G \) is at most twice the size of its maximum matching (Fact 2.1), we have that \( |C| \leq 2 \cdot \text{MM}(G) \). It is also straightforward to verify that \( p \cdot \text{MM}(G) \leq \mathbb{E} \left[ \text{MM}(G_p) \right] \leq \mu \), since \( p \) fraction of the edges of any maximum matching of \( G \) appear in \( G_p \) in expectation; hence \( |C| \leq 2 \cdot \mu/p \). \( \blacksquare \)

We are now ready to finalize the proof. By setting \( c_i = 1 \) for all \( i \leq |C| \), we can use Azuma’s inequality (Proposition B.1) with parameters \( \lambda \) and \( c_i \) for the martingales \( \{X_i\}_i \), and obtain that,

\[
\Pr \left( |\text{MM}(G_p) - \mathbb{E}[\text{MM}(G_p)]| \geq \lambda \right) = \Pr \left( |X_{|C|} - X_0| \geq \lambda \right) \leq 2 \cdot \exp \left( -\frac{\lambda^2}{\sum_{i \in C} c_i^2} \right) \quad \text{Proposition B.1}
\]

\[
= 2 \cdot \exp \left( -\frac{\lambda^2}{|C|} \right) \quad \text{Claim 3.3} \leq 2 \cdot \exp \left( -\frac{\lambda^2 \cdot p}{2 \cdot \mu} \right)
\]

finalizing the proof. \( \blacksquare \)

### 3.2 Proof of Theorem 4

Let \( G(V, E) \) be any arbitrary graph and \( G^{(1)}, \ldots, G^{(k)} \) be a random \( k \)-partition of \( G \). Recall that \( \text{MaxMatching} \) coreset simply computes a maximum matching \( M_i \) on each graph \( G^{(i)} \) for \( i \in [k] \); hence, we only need to show that the graph \( H(V, M_1 \cup \ldots \cup M_k) \) has a large matching compared to the graph \( G \).

Let \( M^* \) be any fixed maximum matching in \( G \), and let \( \mu := |M^*| = \text{MM}(G) \). Our approach is to show that either each graph \( G^{(i)} \) has a large matching already, i.e., \( |M_i| \geq \mu/3 \), or many edges of \( M^* \) are picked in \( M_i \) as well. In the latter case, the union of edges in \( M_i \) for \( i \in [k] \) has a large intersection with \( M^* \) and hence contains a large matching.

Define \( G^-(V, E^-) \) whereby \( E^- := E \setminus M^* \). Let \( G^-_i := G^- \cap G^{(i)} \) be the intersection of the graph \( G^{(i)} \) and \( G^- \). Finally, define \( \mu^-_i \) as the maximum matching size in \( G^-_i \). Using our concentration result from the previous section, we can show that,

**Claim 3.4.** Let \( \varepsilon \in (0, 1) \) be a parameter. Suppose \( \mu \geq 4 \cdot \varepsilon^{-2} \cdot k \log n \); then, there exists an integer \( \mu^- \in [n] \) such that with probability \( 1 - o(1) \) (over the random k-partition), \( \mu^-_i = \mu^- \pm \varepsilon \cdot \mu \) simultaneously for all \( i \in [k] \).

**Proof.** Let \( p = 1/k \); the graph \( G^-_i \) is a subgraph of \( G^- \) obtained by picking each edge in \( G^- \) independently and with probability \( p \). Let \( \mu^- := \mathbb{E} \left[ \text{MM}(G^-_i) \right] \leq \mu \) (notice that the marginal
distribution of $G_i^-$ graphs for all $i \in [k]$ are identical). By setting $\lambda = \varepsilon \cdot \mu$ in Lemma 3.1, we have that,
\[
\Pr \left( |\mu_i^- - \mu^-| \geq \lambda \right) \leq 2 \cdot \exp \left( -\frac{\varepsilon^2 \cdot \mu^2 \cdot p}{2 \cdot \mu} \right) \leq 2 \cdot \exp (-2 \log n) \leq \frac{1}{n^2}
\]
where the second inequality is by the assumption on the value of $\mu$. Taking a union bound over all $k \leq n$ subgraphs $G_i^-$ for $i \in [k]$ finalizes the proof.

In the following, we condition on the event in Claim 3.4. We now have,

**Lemma 3.5.** Let $\mu$, $\mu^-$ and $\varepsilon$ be as in Claim 3.4. If $\mu^- \leq \mu/3$, then $\left| \bigcup_{i=1}^k M_i \cap M^* \right| \geq \mu/3 - 3\varepsilon \cdot \mu$ w.p. $1 - o(1)$.

*Proof.* Fix an index $i \in [k]$ and notice that conditioning on the event in Claim 3.4, only fixes the set of edges in $G_i^-$. Let $M_i^-$ be any maximum matching in $G_i^-$. By definition, $\mu_i^- = |M_i^-|$. By conditioning on the event in Claim 3.4, we have $\mu_i^- \leq \mu^- + \varepsilon \cdot \mu$. It is straightforward to verify that there are at least $|M^*| - 2 \cdot |M_i^-| = \mu - \mu_i^- \geq \mu/3 - 3\varepsilon \cdot \mu$ edges $e$ in $M^*$ such that neither endpoints of $e$ are matched by $M_i^-$. We refer to these edges as free edges and use $M^*_f \subseteq M^*$ to denote them.

Note that even after conditioning on $G_i^-$, the edges in $M^*$, and consequently $M^*_f$, appear in the graph $G^{(i)}$ independently and with probability $1/k$. As such, using a Chernoff bound (by assumption on the value of $\mu$), w.p. $1 - 1/n^2$, $|M^*_f|/k - \varepsilon \cdot \mu/k$ edges of $M^*_f$ appear in $G^{(i)}$. Since these edges can be directly added to the matching $M_i^-$ (as neither endpoints of them are matched in $M_i^-$), this implies that there exists a matching of size $\mu_i^- + \frac{1}{k} \cdot (\mu/3 - 3\varepsilon \cdot \mu)$ in $G^{(i)}$ w.p. $1 - 1/n^2$.

Now let $M_i$ be the maximum matching computed by MaxMatching; the above argument implies that $|M_i| \geq \mu_i^- + \frac{1}{k} \cdot (\mu/3 - 3\varepsilon \cdot \mu)$. On the other hand, notice that $|M_i \cap G_i^-| \leq \mu_i^-$ as $M_i \cap G_i^-$ forms a matching in the graph $G_i^-$ and $\mu_i^-$ denotes the maximum matching size in this graph. This means that $|M_i \cap M^*| = |M_i \setminus G_i^-| \geq \frac{1}{k} \cdot (\mu/3 - 3\varepsilon \cdot \mu)$. To finalize the proof, notice that by a union bound over all $k$ matchings $M_i$, we have that with probability $1 - 1/n$,
\[
\left| \bigcup_{i=1}^k M_i \cap M^* \right| = \sum_{i=1}^k |M_i \cap M^*| \geq k \cdot \frac{1}{k} \cdot (\mu/3 - 3\varepsilon \cdot \mu) = \mu/3 - 3\varepsilon \cdot \mu.
\]

We can now easily prove Theorem 4.

*Proof of Theorem 4.* By our assumption that $\mu = \text{MM}(G) = \omega(k \log n)$, we can take $\varepsilon$ in Claim 3.4 and Lemma 3.5 to be some arbitrary small constant, say $\varepsilon = o(1)$. Define $\mu^-$ as in Lemma 3.5. If $\mu^- > \mu/3$, we are already done as by Claim 3.4, for any $i \in [k]$, $|M_i| \geq \mu^- - o(\mu) \geq \mu/3 - o(\mu)$ and hence the union of matchings $M_1, \ldots, M_k$ surely has a $(3 + o(1))$ approximate matching. On the other hand, if $\mu^- \leq \mu/3$, we can apply Lemma 3.5, and argue that $\mu/3 - o(\mu)$ edges of the matching $M^*$ appear in the union of matchings $M_1, \ldots, M_k$, which finalizes the proof.

We also show that there exists a graph for which the approximation ratio of MaxMatching is arbitrarily close to 2. This implies that one cannot improve the analysis of MaxMatching much further and in particular beat the approximation ratio of 2 using this coreset.

**Lemma 3.6.** There exists a graph $G(V, E)$ such that for any random $k$-partition of $G$ ($k \leq n^{1-\delta}$ for any constant $\delta > 0$), the MaxMatching coreset can only find a matching of size at most $\left( \frac{1}{2} + \frac{1}{\varepsilon} \right) \cdot \text{MM}(G)$ with high probability.

We defer the proof of this lemma to Appendix C.
4 New Properties of Edge Degree Constrained Subgraphs

We study further properties of EDCS in this section. Although EDCS was used prior to our work, all the properties proven in this section are entirely new to this paper and look at the EDCS from a different vantage point.

Previous work in [20, 21] studied the EDCS from the perspective of how large of matching it contains and how it can be maintained efficiently in a dynamically changing graph. In this paper, we prove several new interesting structural properties of the EDCS itself. In particular, while it is easy to see that in terms of edge-sets there can be many different EDCS of some fixed graph $G(V, E)$ (consider $G$ being a complete graph), we show that the degree distributions of every EDCS (for the same parameters $\beta$ and $\beta^-$) are almost the same. In other words, the degree of any vertex $v$ is almost the same in every EDCS of $G(V, E)$. This is in sharp contrast with similar objects such as maximum matchings or $b$-matchings, which can vary a lot within the same graph.

This semi-uniqueness renders the EDCS extremely robust under sampling and composition as we prove next in this section. These new structural results on EDCS are the main properties that allows their use in our coresets and parallel algorithms in the rest of the paper. In fact, our parallel algorithms in Section 6 are entirely based on these results and do not rely at all on the fact that an EDCS contains a large matching (i.e., do not depend on Lemma 2.5 at all).

4.1 Degree Distribution Lemma

**Lemma 4.1** (Degree Distribution Lemma). Fix a graph $G(V, E)$ and parameters $\beta, \beta^- = (1 - \lambda) \cdot \beta$ (for $\lambda < 1/100$). For any two subgraphs $A$ and $B$ that are EDCS($G, \beta, \beta^-$), and any vertex $v \in V$, 

$$|d_A(v) - d_B(v)| = O(\log n) \cdot \lambda^{1/2} \cdot \beta.$$ 

In the rest of this section, we fix the parameters $\beta, \beta^-$ and the two EDCS $A$ and $B$ in Lemma 4.1. The general strategy of the proof is as follows. We start with a set $S_1$ of vertices that have the most difference in degree between $A$ and $B$. For simplicity, assume vertex degrees are all larger by an additive factor of $\Delta$ in $A$ compared to $B$. We look at all neighbors of $S_1$ in $A \setminus B$, denoted by $T_1$, and then all neighbors of $T_1$ in $B \setminus A$ plus the original set $S_1$ which we denote by $S_2$. We then use the two properties of EDCS in Definition 2 to prove that the vertices in $S_2$ still have a larger degree in $A$ compared to $B$ by an additive factor which is only slightly smaller than $\Delta$, while the size of $S_2$ is a constant factor larger than $S_1$. The main observation behind this claim is that since vertices in $S$ have a “large” degree in $A$, their neighbors in $A \setminus B$ in $T_1$ should have a “small” degree to satisfy Property (P1) of EDCS $A$. Similarly, since vertices in $S_1$ have a “small” degree in $B$, and since the edges in $A \setminus B$ are missing from the EDCS $B$, by Property (P2) of EDCS $B$ the vertices in $T_1$ should have a “large” degree in $B$. Applying this idea one more time to vertices in $T_1$ in order to obtain the set $S_2$, we can see that vertices in $S_2$ have a “large” degree in $A$ and a “small” degree in $B$, and that the decrease in the original gap $\Delta$ between the degree of vertices in $A$ vs $B$ is only $O(\beta - \beta^-) = O(\lambda \cdot \beta)$.

We use this argument repeatedly to construct the next set $S_3$ and so on. With each iteration the new set $S_i$ is larger than the previous one $S_{i-1}$ by a constant multiplicative factor, while the $A$-$B$ degree gap only decreases by a small additive factor from $S_{i-1}$ to $S_i$. Now on one hand we can keep iterating this process as long as the $A$-$B$ degree gap remains $\geq \Delta/2$. On the other hand, a multiplicative increase in set size can only happen for $O(\log(n))$ steps before we run out of vertices. Thus, the gap must decrease from $\Delta$ to $\Delta/2$ after only $O(\log(n))$ steps of a small additive decrease, which gives us an upper bound on the original gap $\Delta$.

We now formalize this argument, starting with a technical lemma which allows us to obtain each set $S_i$ from the set $S_{i-1}$ in the above argument.
Lemma 4.2. Fix an integer $D > 2\lambda^{1/2} \cdot \beta$ and suppose $S \subseteq V$ is such that for all $v \in S$, we have $d_A(v) - d_B(v) \geq D$. Then, there exists a set of vertices $S' \supseteq S$ such that $|S'| \geq (1 + 2\lambda^{1/2}) \cdot |S|$ and for all $v \in S'$, $d_A(v) - d_B(v) \geq D - 2\lambda \cdot \beta$.

Proof. We define the following two sets $T$ and $T'$:

- $T$ is the set of all neighbors of vertices in $S$ using only the edges in $A \setminus B$. In other words, $T := \{v \in V \mid \exists u \in S \land (u, v) \in A \setminus B\}$.
- $T'$ is the set of all neighbors of vertices in $T$ using only the edges in $B \setminus A$. In other words, $T' := \{w \in V \mid \exists v \in T \land (v, w) \in B \setminus A\}$.

We start by proving the following property on the degree of vertices in the sets $T$ and $T'$.

Claim 4.3. We have,

- for all $v \in T$, $d_B(v) - d_A(v) \geq D - \lambda \cdot \beta$.
- for all $w \in T'$, $d_A(w) - d_B(w) \geq D - 2\lambda \cdot \beta$.

Proof. For the first part, since $v \in T$, it means that there exists an edge $(u, v) \in A \setminus B$ such that $u \in S$. Since $(u, v)$ belongs to $A$, by Property (P1) of an EDCS we have $d_A(v) \leq \beta - d_A(u) \leq \beta - d_B(u) - D$. On the other hand, since $(u, v)$ does not belong to $B$, by Property (P2) of an EDCS we have $d_B(v) \geq \beta - \lambda \cdot \beta - d_B(u)$, completing the proof for vertices in $T$.

For the second part, since $w \in T'$, it means that there exists an edge $(v, w) \in B \setminus A$ such that $v \in T$. Since $(v, w)$ does not belong to $A$, by Property (P2) of an EDCS we have $d_A(w) \geq \beta - \lambda \cdot \beta$. Moreover, since $(u, v)$ belongs to $B$, by Property (P1) of an EDCS, we have, $d_B(w) \leq \beta - d_B(v)$. This means that $d_A(w) - d_B(w) \geq d_B(v) - d_A(v) - \lambda \cdot \beta$ which is at least $D - 2\lambda \cdot \beta$ by the first part.

Notice that since $D > 2\lambda \cdot \beta$, by Claim 4.3, for any vertex $v \in T$, we have $d_B(v) > d_A(v)$ and hence $S \cap T = \emptyset$ (similarly, $T \cap T' = \emptyset$, but $S$ and $T'$ may intersect). We define the set $S'$ in the lemma statement to be $S' := S \cup T'$. The bound on the degree of vertices in $S'$ follows immediately from Claim 4.3 (recall that vertices in $S$ already satisfy the degree requirement for the set $S'$). In the following, we show that $|T' \setminus S| \geq 2\lambda^{1/2} \cdot |S|$, which finalizes the proof.

Recall that $E_{A \setminus B}(S)$ and $E_{A \setminus B}(S, T)$ denote the set of edges in subgraph $A \setminus B$ incident on vertices $S$, and between vertices $S$ and $T$, respectively. We have,

$$|E_{B \setminus A}(T, T' \setminus S)| = |E_{B \setminus A}(T)| - |E_{B \setminus A}(T, S)|$$

(as all the edges in $B \setminus A$ that are incident on $T$ are going to $T'$)

$$\geq |E_{A \setminus B}(T)| - |E_{B \setminus A}(T, S)|$$

(as by Claim 4.3, the degree of vertices in $T$ is larger in $B \setminus A$ compared to $A \setminus B$)

$$\geq |E_{A \setminus B}(S)| - |E_{B \setminus A}(S)|$$

(as all edges in $A \setminus B$ incident on $S$ are also incident on $T$)

$$\geq |S| \cdot D$$

(by the assumption on the degree of vertices in $S$ in subgraphs $A$ and $B$)

Finally, since $B$ is an EDCS, the maximum degree of any vertex in $T' \setminus S$ is at most $\beta$ and hence there should be at least $|S| \cdot \frac{D}{\beta} \geq 2\lambda^{1/2} \cdot |S|$ vertices in $T' \setminus S$ (as $D > 2\lambda^{1/2} \cdot \beta$).

Proof of Lemma 4.1. Suppose towards a contradiction that there exists a vertex $v \in V$ s.t. $D := d_A(v) - d_B(v) \geq 3\ln(n) \cdot \lambda^{1/2} \cdot \beta$ (the other case is symmetric). Let $D_0 = D$ and $S_0 = \{v\}$ and for $i = 1$ to $t := \lambda^{-1/2} \cdot (\ln(n) + 1)$: define the set $S_i$ and integer $D_i$ by applying Lemma 4.2 to $S_{i-1}$
and $D_{i-1}$ (i.e., $S_i = S'$ and $D_i = D_{i-1} - 2\lambda \cdot \beta$). By the lower bound on the value of $D$, for any $i \in [t]$, we have that $D_i \geq D - i \cdot 2\lambda \cdot \beta > 2\lambda^{1/2} \cdot \beta$, and hence we can indeed apply Lemma 4.2. As a result, we have,

$$|S_t| \geq \left(1 + 2\lambda^{1/2}\right) |S_{t-1}| \geq \left(1 + 2\lambda^{1/2}\right)^t |S_0| \geq \exp\left(\lambda^{1/2} \cdot t\right) > \exp(\ln(n)) = n.$$  

which is a contradiction as there are only $n$ vertices in the graph $G$. Consequently, we obtain that for any vertex $v$, $|d_A(v) - d_B(v)| = O(\log n \cdot \lambda^{1/2} \cdot \beta)$, finalizing the proof.

### 4.2 EDCS in Sampled Subgraphs

In this section, we prove two lemmas regarding the structure of different EDCS across sampled subgraphs. The first lemma concerns edge sampled subgraphs. We show that the degree distributions of any two EDCS for two different edge sampled subgraphs of $G$ is almost the same no matter how the two EDCS are selected or even if the choice of the two subgraphs are not independent.

**Lemma 4.4 (EDCS in Edge Sampled Subgraphs).** Fix any graph $G(V, E)$ and $p \in (0, 1)$. Let $G_1$ and $G_2$ be two edge sampled subgraphs of $G$ with probability $p$ (chosen not necessarily independently). Let $H_1$ and $H_2$ be arbitrary EDCSs of $G_1$ and $G_2$ with parameters $(\beta, (1 - \lambda) \cdot \beta)$. Suppose $\beta \geq 750 \cdot \lambda^{-2} \cdot \ln(n)$, then, with probability $1 - 4/n^9$, simultaneously for all $v \in V$:

$$|d_{H_1}(v) - d_{H_2}(v)| \leq O(\log n \cdot \lambda^{1/2} \cdot \beta).$$

We also prove a qualitatively similar lemma for vertex sampled subgraphs. The main difference here is that there will be a huge gap between the degree of a vertex between the two EDCS if the vertex is sampled in one subgraph but not the other one. However, we show that the degree of vertices that are sampled in both subgraphs are almost the same across the two different (and arbitrarily chosen) EDCS for the subgraphs.

**Lemma 4.5 (EDCS in Vertex Sampled Subgraphs).** Fix any graph $G(V, E)$ and $p \in (0, 1)$. Let $G_1$ and $G_2$ be two vertex sampled subgraphs of $G$ with probability $p$ (chosen not necessarily independently). Let $H_1$ and $H_2$ be arbitrary EDCSs of $G_1$ and $G_2$ with parameters $(\beta, (1 - \lambda) \lambda/\beta)$. If $\beta \geq 750 \cdot \lambda^{-2} \cdot \ln(n)$, then, with probability $1 - 4/n^9$, simultaneously for all $v \in V_1 \cap V_2$:

$$|d_{H_1}(v) - d_{H_2}(v)| \leq O(\log n \cdot \lambda^{1/2} \cdot \beta).$$

The proofs of both these lemmas proceed along the following lines. We start with any EDCS $H$ of the original graph $G$ with parameters (almost) $(\beta/p, \beta^-/p)$. We then consider the set of edges from $H$ in each of the sampled subgraphs $G_1$ and $G_2$, i.e., the two subgraphs $H'_1 := G_1 \cap H$ and $H'_2 := G_2 \cap H$. We use the randomness in the process of sampling subgraphs $G_1$ and $G_2$ to prove that, with high probability, $H'_1$ and $H'_2$ are both an EDCS of $G_1$ and $G_2$, respectively, with parameters $(\beta, \beta^-)$. Moreover, since they are sampled from the same underlying EDCS $H$, $H'_1$ and $H'_2$ end up having similar degree distributions (denoted by $H'_1 \sim H'_2$). Finally, we use our degree distribution lemma (Lemma 4.1) to argue that for any arbitrary EDCS $H_1$ of $G_1$ and $H_2$ of $G_2$, degree distribution of $H_1$ (resp. $H_2$) is close to $H'_1$ (resp. $H'_2$), namely $H_1 \sim H'_1$ and $H_2 \sim H'_2$. We thus have $H_1 \sim H'_1 \sim H'_2 \sim H_2$, completing the proof.

There are some technical differences in implementing the above intuition between the edge sampled and vertex sampled subgraphs and hence we provide a separate proof for each case.

#### 4.2.1 EDCS in Edge Sampled Subgraphs: Proof of Lemma 4.4

*Proof of Lemma 4.4.* We first prove that edge sampling an EDCS results in another EDCS for the sampled subgraph.
Claim 4.6. Let $H$ be an EDCS($G, \beta_H, \beta_H^-$) for parameters $\beta_H := (1 - \frac{\lambda}{2}) \cdot \frac{\beta}{5}$ and $\beta_H^- := \beta_H - 1$. Suppose $G_p := G_p^E(V, E_p)$ is an edge sampled subgraph of $G$ and $H_p := H \cap G_p$, then, with probability $1 - 2/n^9$:

1. For any vertex $v \in V$, $|d_{H_p}(v) - p \cdot d_H(v)| \leq \frac{\lambda}{5} \cdot \beta$.
2. $H_p$ is an EDCS of $G_p$ with parameters $(\beta, (1 - \lambda) \cdot \beta)$.

Proof. For any vertex $v \in V$, $\mathbb{E}[d_{H_p}(v)] = p \cdot d_H(v)$ and $d_H(v) \leq \beta$ by Property (P1) of EDCS $H$. Moreover, since each neighbor of $v$ in $H$ is sampled in $H_p$ independently, by Chernoff bound (Proposition B.2), we have,

$$\Pr\left(|d_{H_p}(v) - p \cdot d_H(v)| \geq \frac{\lambda}{5} \cdot \beta\right) \leq 2 \cdot \exp\left(-\frac{\lambda^2 \cdot \beta}{75}\right) \leq 2 \cdot \exp(-10 \ln n) = \frac{2}{n^{10}},$$

where the second inequality is by the lower bound on $\beta$ in Lemma 4.5 statement. In the following, we condition on the event that:

$$\forall v \in V \ |d_{H_p}(v) - p \cdot d_H(v)| \leq \frac{\lambda}{5} \cdot \beta.$$ (1)

This event happens with probability at least $1 - 2/n^9$ by above equation and a union bound on $|V| = n$ vertices. This finalizes the proof of the first part of the claim. We are now ready to prove that $H_p$ is indeed an EDCS($G_p, \beta, (1 - \lambda) \cdot \beta$) conditioned on this event.

Consider any edge $(u, v) \in H_p$. Since $H_p \subseteq H, (u, v) \in H$ as well. Hence, we have,

$$d_{H_p}(u) + d_{H_p}(v) \leq \mathbb{E}[d_H(u) + d_H(v)] + \frac{2 \lambda}{5} \cdot \beta \leq p \cdot \beta + 2 \lambda \cdot \beta = (1 - \frac{\lambda}{2}) \cdot \beta + \frac{2 \lambda}{5} \cdot \beta < \beta,$$

where the second inequality is by Property (P1) of EDCS $H$ and the equality is by the choice of $\beta_H$. As a result, $H_p$ satisfies Property (P1) of EDCS for parameter $\beta$.

Now consider an edge $(u, v) \in G_p \setminus H_p$. Since $H_p = G_p \cap H, (u, v) \notin H$ as well. Hence,

$$d_{H_p}(u) + d_{H_p}(v) \geq \mathbb{E}[d_H(u) + d_H(v)] - \frac{2 \lambda}{5} \cdot \beta \geq p \cdot \beta - \frac{2 \lambda}{5} \cdot \beta = (1 - \frac{\lambda}{2}) \cdot \beta - \frac{2 \lambda}{5} \cdot \beta > (1 - \lambda) \cdot \beta,$$

where the second inequality is by Property (P2) of EDCS $H$ and the equality is by the choice of $\beta_H^-$. As such, $H_p$ satisfies Property (P2) of EDCS for parameter $(1 - \lambda) \cdot \beta$ and hence $H_p$ is indeed an EDCS($G_p, \beta, (1 - \lambda) \cdot \beta$). \qed

We continue with the proof of Lemma 4.4. Let $H$ be an EDCS($G, \beta_H, \beta_H^-$) for the parameters $\beta_H, \beta_H^-$ in Claim 4.6. The existence of $H$ follows from Lemma 2.4 as $\beta_H^- < \beta_H$. Define $\hat{H}_1 := H \cap G_1$ and $\hat{H}_2 := H \cap G_2$. By Claim 4.6, $\hat{H}_1$ (resp. $\hat{H}_2$) is an EDCS of $G_1$ (resp. $G_2$) with parameters $(\beta, (1 - \lambda) \beta)$ with probability $1 - 4/n^9$. In the following, we condition on this event.

By Lemma 4.1 (Degree Distribution Lemma), since both $H_1$ (resp. $H_2$) and $\hat{H}_1$ (resp. $\hat{H}_2$) are EDCS for $G_1$ (resp. $G_2$), the degree of vertices in both of them should be “close” to each other. Moreover, since by Claim 4.7 the degree of each vertex in $\hat{H}_1$ and $\hat{H}_2$ is close to $p$ times its degree in $H$, we can argue that the vertex degrees in $H_1$ and $H_2$ are close. Formally, for any $v \in V$, we have,

$$|d_{H_1}(v) - d_{H_2}(v)| \leq |d_{H_1}(v) - d_{\hat{H}_1}(v)| + |d_{\hat{H}_1}(v) - d_{\hat{H}_2}(v)| + |d_{\hat{H}_2}(v) - d_{H_2}(v)| \leq O(\log n) \cdot \lambda^{1/2} \cdot \beta + O(\log n) \cdot \lambda^{1/2} \cdot \beta + O(1) \cdot \lambda \cdot \beta,$$

finalizing the proof. \qed
4.2.2 EDCS in Vertex Sampled Subgraphs: Proof of Lemma 4.5

Proof of Lemma 4.5. We first prove that vertex sampling an EDCS results in another EDCS for the sampled subgraph.

Claim 4.7. Let $H$ be an EDCS($G, \beta_H, \beta_H^-$) for parameters $\beta_H := (1 - \frac{1}{p}) \cdot \beta$ and $\beta_H^- := \beta_H - 1$. Suppose $G_p := G^V_p(V_p, E_p)$ is a vertex sampled subgraph of $G$ and $H_p := H \cap G_p$; then, with probability $1 - 2/n^9$:

1. For any vertex $v \in V_p$, $|d_{H_p}(v) - p \cdot d_H(v)| \leq \frac{\lambda}{5} \cdot \beta$.
2. $H_p$ is an EDCS of $G_p$ with parameters $(\beta, (1 - \lambda) \cdot \beta)$.

Proof. For any vertex $v \in V_p$, $\mathbb{E}[d_{H_p}(v)] = p \cdot d_H(v)$ by the independent sampling of vertices and $d_H(v) \leq \beta_H$ by Property (P1) of EDCS $H$. Moreover, since each neighbor of $v$ in $H$ is sampled in $H_p$ independently, by Chernoff bound (Proposition B.2), we have,

$$\Pr \left( |d_{H_p}(v) - \mathbb{E}[d_{H_p}(v)]| \geq \frac{\lambda}{5} \cdot \beta \right) \leq 2 \cdot \exp \left( - \frac{\lambda^2 \cdot \beta}{75} \right) \leq 2 \cdot \exp (-10 \ln n) = \frac{2}{n^{10}},$$

where the second inequality is by the lower bound on $\beta$ in Lemma 4.5 statement. In the following, we condition on the event that:

$$\forall v \in V_p \quad |d_{H_p}(v) - p \cdot d_H(v)| \leq \frac{\lambda}{5} \cdot \beta. \quad (2)$$

which happens with probability at least $1 - 2/n^9$ by above equation and a union bound on $|V_p| \leq n$ vertices. This finalizes the proof of the first part of the claim. We are now ready to prove that $H_p$ is indeed an EDCS($G_p, \beta, (1 - \lambda) \cdot \beta$) conditioned on this event.

Consider any edge $(u, v) \in H_p$. Since $H_p \subseteq H$, $(u, v) \in H$ as well. Hence, we have,

$$d_{H_p}(u) + d_{H_p}(v) \leq \mathbb{E}[d_H(u)] + \mathbb{E}[d_H(v)] + \frac{2 \lambda}{5} \cdot \beta \leq p \cdot \beta_H + \frac{2 \lambda}{5} \cdot \beta = (1 - \frac{\lambda}{2}) \cdot \beta + \frac{2 \lambda}{5} \cdot \beta < \beta,$$

where the second inequality is by Property (P1) of EDCS $H$ and the equality is by the choice of $\beta_H$. As a result, $H_p$ satisfies Property (P1) of EDCS for parameter $\beta$.

Now consider an edge $(u, v) \in G_p \setminus H_p$. Since $H_p = G_p \cap H$, $(u, v) \notin H$ as well. Hence,

$$d_{H_p}(u) + d_{H_p}(v) \geq \mathbb{E}[d_H(u)] + \mathbb{E}[d_H(v)] - \frac{2 \lambda}{5} \cdot \beta \geq p \cdot \beta_H^- - \frac{2 \lambda}{5} \cdot \beta = (1 - \frac{\lambda}{2}) \cdot \beta - \frac{2 \lambda}{5} \cdot \beta > (1 - \lambda) \cdot \beta,$$

where the second inequality is by Property (P2) of EDCS $H$ and the equality is by the choice of $\beta_H^-$. As such, $H_p$ satisfies Property (P2) of EDCS for parameter $(1 - \lambda) \cdot \beta$ and hence $H_p$ is indeed an EDCS($G_p, \beta, (1 - \lambda) \cdot \beta$).

We continue with the proof of Lemma 4.5. Let $H$ be an EDCS($G, \beta_H, \beta_H^-$) for the parameters $\beta_H, \beta_H^-$ in Claim 4.7. The existence of $H$ follows from Lemma 2.4 as $\beta_H^- < \beta_H$. Define $H_1 := H \cap G_1$ and $H_2 := H \cap G_2$. By Claim 4.7, $H_1$ (resp. $H_2$) is an EDCS of $G_1$ (resp. $G_2$) with parameters $(\beta, (1 - \lambda) \beta)$ with probability $1 - 4/n^9$. In the following, we condition on this event.

By Lemma 4.1 (Degree Distribution Lemma), since both $H_1$ (resp. $H_2$) and $\tilde{H}_1$ (resp. $\tilde{H}_2$) are EDCS for $G_1$ (resp. $G_2$), the degree of vertices in both of them should be “close” to each other. Moreover, since by Claim 4.7 the degree of each vertex in $\tilde{H}_1$ and $\tilde{H}_2$ is close to $p$ times its degree
in $H$, we can argue that the degree of shared vertices in $H_1$ and $H_2$ are close. Formally, let $v$ be a vertex in both $G_1$ and $G_2$; we have,

$$|d_{H_1}(v) - d_{H_2}(v)| \leq |d_{H_1}(v) - d_{\tilde{H}_1}(v)| + |d_{\tilde{H}_1}(v) - d_{\tilde{H}_2}(v)| + |d_{\tilde{H}_2}(v) - d_{H_2}(v)|$$

$$\leq O(\log n) \cdot \lambda^{1/2} \cdot \beta + |d_{\tilde{H}_1}(v) - p \cdot d_H(v)| + |d_{\tilde{H}_2}(v) - p \cdot d_H(v)|$$

by Lemma 4.1

finalizing the proof.  

5 Randomized Coresets for Matching and Vertex Cover

We introduce our randomized coresets for matching and vertex cover in this section. Both of these results are achieved by computing an EDCS of the input graph (for appropriate choice of parameters) and then applying Lemmas 2.5 and 2.6.

5.1 Computing an EDCS from Random $k$-Partitions

Let $G(V, E)$ be any arbitrary graph and $G^{(1)}, \ldots, G^{(k)}$ be a random $k$-partition of $G$. We show that if we compute an arbitrary EDCS of each graph $G^{(i)}$ (with no coordination across different graphs) and combine them together, we obtain an EDCS for the original graph $G$.

1. Let $G^{(1)}, \ldots, G^{(k)}$ be a random $k$-partition of the graph $G$.
2. For any $i \in [k]$, compute $C^{(i)} := \text{EDCS}(G, \beta, (1 - \lambda) \cdot \beta)$ for parameters

   $$\lambda = \Theta \left( \frac{\varepsilon}{\log n} \right)^2 \quad \text{and} \quad \beta := \Theta(\lambda^{-3} \cdot \log n).$$

3. Let $C := \bigcup_{i=1}^{k} C^{(i)}$.

**Lemma 5.1.** With probability $1 - 4/n^7$, the subgraph $C$ is an EDCS($G, \beta_C, \beta_C^{-}$) for parameters:

$$\lambda_C := O(\log n) \cdot \lambda^{1/2} \quad \text{,} \quad \beta_C := (1 + \lambda_C) \cdot k \cdot \beta \quad \text{and} \quad \beta_C^{-} := (1 - 2\lambda_C) \cdot k \cdot \beta.$$

**Proof.** Recall that each graph $G^{(i)}$ is an edge sampled subgraph of $G$ with sampling probability $p = \frac{1}{k}$. By Lemma 4.4 for graphs $G^{(i)}$ and $G^{(j)}$ (for $i \neq j \in [k]$) and their EDCSs $C^{(i)}$ and $C^{(j)}$, with probability $1 - 4/n^9$, for all vertices $v \in V$:

$$|d_{C^{(i)}}(v) - d_{C^{(j)}}(v)| \leq O(\log n) \cdot \lambda^{1/2} \cdot \beta = \lambda_C \cdot \beta. \tag{3}$$

By taking a union bound on all $\dbinom{k}{2} \leq n^2$ pairs of subgraphs $G^{(i)}$ and $G^{(j)}$ for $i \neq j \in [k]$, the above property holds for all $i, j \in [k]$, with probability at least $1 - 4/n^7$. In the following, we condition on this event.

We now prove that $C$ is indeed an EDCS($G, \beta_C, \beta_C^{-}$). First, consider an edge $(u, v) \in C$ and let $j \in [k]$ be such that $(u, v) \in C^{(j)}$ as well. We have,

$$d_C(u) + d_C(v) = \sum_{i=1}^{k} d_{C^{(i)}}(u) + \sum_{i=1}^{k} d_{C^{(i)}}(v) \leq k \cdot (d_{C^{(j)}}(u) + d_{C^{(j)}}(v)) + k \cdot \lambda_C \cdot \beta$$

$$\leq k \cdot \beta + k \cdot \lambda_C \beta = \beta_C. \quad \text{(by Property (P1) of EDCS $C^{(j)}$ with parameter $\beta$)}$$
Hence, $C$ satisfies Property (P1) of EDCS for parameter $\beta_C$.

Now consider an edge $(u, v) \in G \setminus C$ and let $j \in [k]$ be such that $(u, v) \in G^{(j)} \setminus C^{(j)}$ (recall that each edge in $G$ is sent to exactly one graph $G^{(j)}$ in the random $k$-partition). We have,

$$d_C(u) + d_C(v) = \sum_{i=1}^{k} d_{C^{(i)}}(u) + \sum_{i=1}^{k} d_{C^{(i)}}(v) \geq k \cdot (d_{C^{(j)}}(u) + d_{C^{(j)}}(v)) - k \cdot \lambda_C \cdot \beta \geq k \cdot (1 - \lambda) \cdot \beta - k \cdot \lambda_C \cdot \beta \geq (1 - 2\lambda_C) \cdot k \cdot \beta = \beta_C^- .$$

(by Property (P2) of EDCS $C^{(j)}$ with parameter $(1 - \lambda) \cdot \beta$)

Hence, $C$ also satisfies the second property of EDCS for parameter $\beta_C^-$, finalizing the proof.

5.2 EDCS as a Coreset for Matching and Vertex Cover

We are now ready to present our randomized coresets for matching and vertex cover using the EDCS as the coreset, formalizing Result 1.

**Theorem 5.** Let $G(V, E)$ be a graph and $G^{(1)}, \ldots, G^{(k)}$ be a random $k$-partition of $G$. For any $\varepsilon \in (0, 1)$, any EDCS$(G^{(i)}, \beta, (1 - \lambda) \cdot \beta)$ for $\lambda := \Theta\left(\left(\frac{\varepsilon}{\log n}\right)^2\right)$ and $\beta := \Theta(\varepsilon^{-6} \cdot \log^7 n)$ is a $(3/2 + \varepsilon)$-approximation randomized composable coreset of size $O(n \cdot \beta)$ for the maximum matching problem.

**Proof.** By Lemma 5.1, the union of the coresets, i.e., the $k$ EDCSes, is itself an EDCS$(G, \beta_C, \beta_C^-)$, such that $\beta_C^- = (1 - \Theta(\varepsilon)) \cdot \beta_C$. Hence, by Lemma 2.5, the maximum matching in this EDCS is of size $(2/3 - \varepsilon) \cdot \MM(G)$. The bound on the size of the coreset follows from Property (P1) of EDCS as maximum degree in the EDCS computed by each machine is at most $\beta$ and hence size of each coreset is $O(n \cdot \beta) = O_{\varepsilon}(n)$.

Recall that in our randomized coreset for vertex cover, we allow each coreset to contain, in addition to a subgraph of the input piece, a set of vertices to be inserted to the final vertex cover.

**Theorem 6.** Let $G(V, E)$ be a graph and $G^{(1)}, \ldots, G^{(k)}$ be a random $k$-partition of $G$. For any $\varepsilon \in (0, 1)$, any EDCS$(G^{(i)}, \beta, (1 - \lambda) \cdot \beta)$ for $\lambda := \Theta\left(\left(\frac{\varepsilon}{\log n}\right)^2\right)$ and $\beta := \Theta(\varepsilon^{-6} \cdot \log^7 n)$ plus the set of vertices with degree larger than $(1 - \Theta(\varepsilon)) \cdot \beta/2$ in the EDCS (to be added directly to the final vertex cover) is a $(3 + \varepsilon)$-approximation randomized composable coreset of size $O(n \cdot \beta)$ for the minimum vertex cover problem.

**Proof.** By Lemma 5.1, the union of the coresets, i.e., the $k$ EDCSes, is itself an EDCS$(G, \beta_C, \beta_C^-) C$, such that $\beta_C^- = (1 - \Theta(\varepsilon)) \cdot \beta_C$. Suppose first that instead of each coreset fixing the set of vertices to be added to the final vertex cover, we simply add all vertices with degree more than $\beta_C^-/2$ to the vertex cover and then compute a minimum vertex cover of $C$. In this case, by Lemma 2.6, the returned solution is a $(3 + \varepsilon)$-approximation to the minimum vertex cover of $G$.

To complete the argument, recall that the degree of any vertex $v \in V$ is essentially the same across all machines (up to an additive term of $\varepsilon \cdot \beta$) by Lemma 4.4, and hence the set of vertices with degree more than $\beta_C^-/2$ would be a subset of the set of fixed vertices across all machines. Moreover, any vertex added by any machine to the final vertex cover has degree at least $(1 - \Theta(\varepsilon)) \cdot \beta_C^-/2$ and hence we can apply Lemma 2.6, with a slightly smaller parameter $\varepsilon$ to argue that the returned solution is still a $(3 + \varepsilon)$-approximation.

**Remark 5.2.** In the proof of Theorem 6, we neglected the time necessary to compute a vertex cover in the union of the coresets (as is consistent with the definition of randomized coresets). In case we require this algorithm to run in polynomial time, we need to approximate the final vertex cover in
the union of the coresets as opposed to recover it exactly. Recall that by Lemma 2.6, the set of high
degree vertices added to the final vertex cover are of size (almost) $2 \cdot \text{VC}(G)$, and the vertex cover in
the union of the coresets has size at most $\text{VC}(G)$. Hence, by approximating the latter one to within
a factor of 2, we obtain an (almost) 4 approximation to the vertex cover of $G$.

6 MPC Algorithms for Matching and Vertex Cover

In this section, we show that a careful adaptation of our coresets construction together with the
structural results proven for EDCS in Section 4 can be used to obtain MPC algorithms with memory
requirement of $O(n)$ while increasing the number of required rounds to only $O(\log \log n)$.

**Theorem 7.** There exists an MPC algorithm that given a graph $G(V, E)$ with high probability
computes an $O(1)$ approximation to both maximum matching and minimum vertex cover of $G$ in
$O(\log \log n)$ MPC rounds using only $O(n)$ memory per machine.

The overall idea of the algorithm in this result is as follows. Instead of the edge sampled
subgraphs used by our randomized coresets, we start by picking $k = O(n)$ vertex sampled subgraphs
of $G$ with sampling probability roughly $1/\sqrt{n}$ and send each to a separate machine. Each machine
then locally computes an EDCS of its input (with parameters $\beta = \text{polylog}(n)$ and $\beta^{-1} \approx \beta$) with no
coordination across the machines. Unlike the MPC algorithm obtained by our randomized coreset
approach (Corollary 1), where the memory per machine was as large as $\Theta(n\sqrt{n})$, here we cannot
collect all these smaller EDCSes on a single machine of memory $\tilde{O}(n)$. Instead, we repartition them
across the machines again (and discard remaining edges) and repeat the previous process on this
new graph. The main observation is that after each step, the maximum degree of the remaining
graph (i.e., the union of all EDCSes) would drop quadratically (e.g., from $\Omega(n)$ to $\tilde{O}(\sqrt{n})$ in the
first step). As such, in each subsequent step, we can pick a smaller number of vertex sampled
subgraphs, each with a higher sampling probability than previous step, and still each graph fits into
the memory of a single machine. Repeating this process for $O(\log \log n)$ steps reduces the maximum
degree of the remaining graph to polylog($n$). At this point, we can store the final EDCS on a single
machine and solve the problem locally.

Unfortunately this approach on its own would only yield a $(3/2)^{O(\log \log n)} = \text{polylog}(n)$ approximation
to matching, since by Lemma 2.5 each recursion onto an EDCS of the graph could introduce
a $3/2$-approximation. A similar problem exists for vertex cover. In the proof of Lemma 2.6, computing
a vertex cover of $G$ from its EDCS $H$ involves two steps: we add to the vertex cover all
vertices with high degree in $H$ to cover the edges in $G \setminus H$, and then we separately compute a vertex
cover for the edges in $H$. Since $H$ cannot fit into a single machine, the second computation is done
recursively: in each round, we find an EDCS of the current graph (which is partitioned amongst
many machines), add to the vertex cover all high degree vertices in this EDCS, and then recurse onto
the sparser EDCS. A straightforward analysis would only lead to an $O(\log \log n)$ approximation.

We improve the approximation factor for both vertex cover and matching by showing that they
can serve as witnesses to each other. Every time we add high-degree vertices to the vertex cover,
we will also find a large matching incident to these vertices: we show that this can be done in $O(1)$
parallel rounds. We then argue that their sizes are always within a constant factor of each other,
so both are a constant approximation for the respective problem (by Proposition 2.2).

The rest of this section is organized as follows. We first present our subroutine for computing
the EDCS of an input graph in parallel using vertex sampled subgraphs. Next, we present a simple
randomized algorithm for finding a large matching incident on high degree vertices of an input graph.
Finally, we combine these two subroutines to provide our main parallel algorithm for approximating
matching and vertex cover. We finish this section by specifying the MPC implementation of our
parallel algorithm and finalize the proof of Theorem 7.
### 6.1 A Parallel Algorithm for EDCS

We now present our parallel algorithm for computing an EDCS via vertex sampling. Recall that in this algorithm, the edges of the input graph as well as the output EDCS will be partitioned across multiple machines. Each machine has $O(n)$ memory, and we assume that there are at least $\text{polylog}(n)$ machines. In the following, we use a slightly involved method of sampling the vertices using limited independence. This is due to technical reasons in the MPC implementation of this algorithm which are described in Section 6.4.1 (see also Remark 6.1 below).

**ParallelEDCS**($G, k$). A parallel algorithm for computing an EDCS of a given graph on $k$ machines.

1. Create $k$ vertex sampled subgraphs $G^{(1)}, \ldots, G^{(k)}$ on $k$ different machines as follows:
   
   (a) Let $\kappa := (20 \log n)$ and $p := \left(\frac{200 \log n}{\sqrt{k}}\right)$.
       Each vertex $v$ in $G$ independently picks a $\kappa$-wise independent hash function $h_v : [k] \to [1/p]$.
   
   (b) The graph $G^{(i)}$ is the induced subgraph of $G$ on vertices $v \in V$ with $h_v(i) = 0$.

2. Define parameters $\lambda := (2 \cdot \log n)^{-3}$ and $\beta := 750 \cdot \lambda^{-2} \cdot \ln (n)$.

3. For $i = 1$ to $k$ in parallel: Compute $C^{(i)} = \text{EDCS}(G^{(i)}, \beta, (1 - \lambda) \cdot \beta)$ locally on machine $i$.

4. Define the multi-graph $C(V, E_C)$ with $E_C := \bigcup_{i=1}^{k} C^{(i)}$ (allowing for multiplicities). Notice that this multi-graph is edge partitioned across the machines.

For any vertex $v \in V$, define $I(v) \subseteq k$ as the set of indices of the machines that sampled vertex $v$ in their subgraphs. Notice that indices in $I(v)$ are $\kappa$-wise independent random variables. Additionally, it is easy to see that each graph $G^{(i)}$ is a vertex sampled subgraph of $G$ with sampling probability $p$.

**Remark 6.1.** We point out that as opposed to the previous vertex sampling approach of Czumaj et al. [32] that resulted in a partitioning of vertices of $G$ across different machines, our way of sampling subgraphs in ParallelEDCS results in each vertex appearing in $\Theta(\sqrt{\Delta})$ different machines. This is necessary for our algorithm as we need to ensure that every edge of the input graph is sampled in this process. However, this property introduces new challenges in the MPC implementation of our algorithm as a naive implementation of this idea requires communicating $O(\sqrt{\Delta})$ messages per each edge of the graph which cannot be done within the memory restrictions of the MPC model. This is the main reason that we sample these subgraphs in ParallelEDCS with limited independence as opposed to truly independently to reduce the communication necessary per each edge to only $O(\log n)$.

We first prove several simple properties of ParallelEDCS.

**Proposition 6.2.** For $k \geq (400 \cdot \log^{12}(n))$, with probability $1 - 2/n^8$,

1. For any vertex $v \in V$, $|I(v)| = p \cdot k \pm \lambda \cdot p \cdot k$.

2. For any edge $e \in E$, there exists at least one index $i \in [k]$ such that $e$ belongs to $G^{(i)}$.

**Proof.** Fix any vertex $v \in V$. Clearly, $\mathbb{E}|I(v)| = p \cdot k$. Moreover, $|I(v)|$ is sum of zero-one $\kappa$-wise independent random variables and hence by Chernoff bound with bounded independence (Proposition B.3)

$$\Pr \left( |I(v)| - \mathbb{E}|I(v)| \geq \lambda \cdot \mathbb{E}|I(v)| \right) \leq 2 \cdot \exp \left( -\frac{\kappa}{2} \right) \leq 2 \cdot \exp (-10 \log n) \leq 1/n^{10}.$$
Note that $\mathbb{E}[I(v)] = 200 \cdot \sqrt{k}\log n \geq 4000 \cdot \log^7 n$, $\lambda = (2 \cdot \log n)^{-3}$ and hence $\lambda^2 \cdot \mathbb{E}[I(v)] / 2 \geq (200/64) \cdot \log n > 20 \log n = \kappa$, and hence we can indeed apply Proposition B.3 here. By a union bound on all $n$ vertices, the first part holds w.p. $\geq 1 - 1/n^9$.

We now prove the second part. Fix an edge $e \in E$ and define the indicator random variables $X_1, \ldots, X_k$ where $X_i = 1$ iff $e$ is contained in the graph $G^{(i)}$. Define $X := \sum_{i=1}^k X_i$ to denote the number of graphs the edge $e$ belongs to. Clearly, $\mathbb{E}[X_i] = p^2$ for all $i \in [k]$ and hence $\mathbb{E}[X] = p^2 \cdot k$. Moreover, the random variables $X_i$’s are $\kappa$-wise independent for $\kappa = 20 \log n$. Hence, by Chernoff bound with bounded independence (Proposition B.3), the probability that $e$ belongs to no graph $G^{(i)}$, i.e., $X = 0$ is at most,

$$\Pr(X = 0) \leq \Pr\left(|X - \mathbb{E}[X]| \geq \mathbb{E}[X]\right) \leq 2 \cdot \exp\left(-\frac{\kappa}{2}\right) \leq 2 \cdot \exp(-10 \log n) \leq 1/n^{10}.$$ 

Again, note that $\mathbb{E}[X] = p^2 \cdot k = 40000 \log^2 n > 400 \log n = 2\kappa$ and hence we could apply Proposition B.3. By a union bound on all $O(n^2)$ edges, the second part also holds w.p. at least $1 - 1/n^8$. Another union bound on this event and the event in the first part finalizes the proof.

We now prove that the graph $C$ defined in the last line of ParallelEDCS is also an EDCS of $G$ with appropriate parameters. The proof is quite similar to that of Lemma 5.1 with some additional care to handle the difference between vertex sampled subgraphs and edge sampled ones.

**Lemma 6.3.** For $k \geq (400 \cdot \log^{12}(n))$, with probability $1 - 5/n^7$, $C$ is an EDCS($G, \beta_C, \beta'_C$) for parameters:

$$\lambda_C := \lambda^{1/2} \cdot \Theta(\log n) = o(1), \quad \beta_C := p \cdot k \cdot (1 + \lambda_C) \cdot \beta, \quad \text{and} \quad \beta'_C := p \cdot k \cdot (1 - \lambda_C) \cdot \beta.$$ 

**Proof.** Recall that each graph $G^{(i)}$ is a vertex sampled subgraph of $G$ with sampling probability $p$. Hence, by Lemma 4.5 and a union bound, with probability $1 - 4/n^7$, for any two subgraphs $C^{(i)}$ and $C^{(j)}$ for $i, j \in [k]$, and any vertex $v \in V^{(i)} \cap V^{(j)}$, we have,

$$|d_{C^{(i)}}(v) - d_{C^{(j)}}(v)| \leq O(\log n) \cdot \lambda^{1/2} \cdot \beta. \quad (4)$$

In the following, we condition on the events in Eq (4) and Proposition 6.2 which happen together with probability at least $1 - 5/n^7$.

We now prove that $C$ is indeed an EDCS($G, \beta_C, \beta'_C$) for the given parameters. First, consider an edge $(u, v) \in C$ and let $j \in [k]$ be such that $(u, v) \in C^{(j)}$ as well. We have,

$$d_C(u) + d_C(v) = \sum_{i \in I(u)} d_{C^{(i)}}(u) + \sum_{i \in I(v)} d_{C^{(i)}}(v) \leq Eq (4) \cdot |I(u)| \cdot d_{C^{(j)}}(u) + |I(v)| \cdot d_{C^{(j)}}(v) + (|I(u)| + |I(v)|) \cdot O(\log n) \cdot \lambda^{1/2} \cdot \beta \leq p \cdot k \cdot \beta + O(\lambda) \cdot p \cdot k \cdot \beta + p \cdot k \cdot O(\log n) \cdot \lambda^{1/2} \cdot \beta \quad \text{(by Property (P1) of EDCS $C^{(j)}$ with parameter $\beta$)}$$

$$\leq p \cdot k \cdot (1 + \lambda_C) \cdot \beta = \beta_C.$$ 

Hence, $C$ satisfies Property (P1) of EDCS for parameter $\beta_C$.

Now consider an edge $(u, v) \in G \setminus C$ and let $j \in [k]$ be such that $(u, v) \in G^{(j)} \setminus C^{(j)}$ (the existence of $j$ follows from conditioning on the event in Proposition 6.2). We have,

$$d_C(u) + d_C(v) = \sum_{i \in I(u)} d_{C^{(i)}}(u) + \sum_{i \in I(v)} d_{C^{(i)}}(v)$$

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\[
\begin{align*}
\sum_{v \in V} |I(v)| \cdot d_{E(C)}(v) - (|I(u)| + |I(v)|) \cdot O(\log n) \cdot \lambda_{1/2} \cdot \beta \\
\geq p \cdot k \cdot (1 - \lambda) \cdot \beta - O(\lambda) \cdot p \cdot k \cdot \beta - p \cdot k \cdot O(\log n) \cdot \lambda_{1/2} \cdot \beta
\end{align*}
\]
(by Property (P2) of EDCS \(C^{(j)}\) with parameter \(\beta\))

\[\geq p \cdot k \cdot (1 - \lambda_C) \cdot \beta = \beta_{C}.\]

Hence, \(C\) also satisfies Property (P2) of EDCS for parameter \(\beta_{C}\), finalizing the proof. \(\blacksquare\)

### 6.2 Random Match Algorithm

In our main algorithm, we need a subroutine for finding a large matching incident on the set of “high” degree vertices of a given graph \(G\) which its edges are initially partitioned across many machines. In this section, we provide such an algorithm based on a simple randomized procedure that is easily implementable in constant number of MPC rounds.

**RandomMatch** \((G, S, \Delta)\). A parallel algorithm for finding a matching \(M\) incident on given vertices \(S\) in a graph \(G\) with maximum degree \(\Delta\).

1. If \(|S| \geq \Delta \cdot \log^2(n)\), sample each edge \(e\) incident on \(S\) w.p. \(p := 1/\Delta\) independently to form a set \(E_{\text{smpl}}\).
2. Otherwise, let \(E_{\text{smpl}}\) be the set of all edges incident on \(S\).
3. Send \(E_{\text{smpl}}\) to a single machine and find a maximum matching \(M\) in \(E_{\text{smpl}}\) locally on this machine.

We prove that if the set \(S\) consists of high degree vertices of \(G\), then \(\text{RandomMatch}(G, S, \Delta)\) finds a large matching in \(S\) with high probability. Formally,

**Lemma 6.4.** Suppose \(G(V, E)\) is a graph with maximum degree \(\Delta\) and \(S \subseteq V\) is such that for all \(v \in S\), \(d_G(v) \geq \Delta/3\). With probability \(1 - 1/n^{10}\), for the matching \(M := \text{RandomMatch}(G, S, \Delta)\) we have \(|M| = \Theta(|S|)\).

**Proof.** The proof when \(|S| < \Delta \cdot \log^2(n)\) immediately follows from Proposition 2.3 as there is a matching of size \(\Omega(|S|)\) in \(E_{\text{smpl}}\) in this case. We now prove the complementary case. In the following, we first show that the expected size of the maximum matching in the set \(E_{\text{smpl}} = \Omega(|S|)\) and then use our result on the concentration of maximum matching size in edge sampled subgraphs (Lemma 3.1) to finalize the proof.

Let \(G_S(V, E_S)\) be the subgraph of \(G\) containing only the edges incident on vertices \(S\). Moreover, define \(H(V, E_{\text{smpl}})\) as the edge sampled subgraph of \(G_S\) chosen in \(\text{RandomMatch}\). For any edge \(e \in E_S\), we define an indicator random variable \(X_e \in \{0, 1\}\) which is 1 iff: (i) edge \(e = (u, v)\) is sampled in \(E_{\text{smpl}}\) and (ii) no other edge incident on \(u\) or \(v\) is sampled in \(E_{\text{smpl}}\). It is easy to see that \(\text{MM}(H) \geq \sum_{e \in E} X_e\) as all edges \(e\) with \(X_e = 1\) form a matching together. We have,

\[
\mathbb{E}[\text{MM}(H)] \geq \sum_{e \in E} \mathbb{E}[X_e] \geq |E| \cdot p \cdot (1-p)^{2\Delta} \geq |S| \cdot \Delta/3 \cdot p \cdot \Theta(1) \geq \Theta(|S|).
\]

In the second inequality above, we use the fact that each edge is sampled independently in \(E_{\text{smpl}}\) and that the maximum degree of any vertex is at most \(\Delta\). In the third inequality we used the fact that each vertex in \(S\) has degree at least \(\Delta/3\) and that \(p = 1/\Delta\) and hence \((1-p)^{2\Delta}\) is larger than some absolute constant independent of \(\Delta\).
Notice that the random variables \( X_e \) have a non-trivial dependency with each other (for example, they are neither negatively nor positively correlated). As a result, we cannot simply use Chernoff bound to prove a concentration result for their sum. Instead, we use our Lemma 3.1 to prove a concentration on the size of the maximum matching in \( H \) directly. We have,

\[
\Pr \left( |\text{MM}(H) - \mathbb{E}[\text{MM}(H)]| \geq \frac{\mathbb{E}[\text{MM}(H)]}{2} \right) \leq \exp \left( -\frac{\mathbb{E}[\text{MM}(H)]^2 \cdot p}{8 \cdot |S|} \right) \leq \exp \left( -\frac{\Theta(|S|)}{\Delta} \right) \leq \frac{1}{n^{10}}.
\]

In the first inequality, we applied Lemma 3.1 with parameter \( \mu = |S| \) (as clearly \( \text{MM}(H) \leq |S| \) always). In the second inequality, we used the fact that \( \mathbb{E}[\text{MM}(H)] = \Theta(|S|) \), and in the last inequality we used the fact that \( |S| \geq \Delta \cdot \log^2(n) \) and assumed \( n \) is sufficiently large.

### 6.3 A Parallel Algorithm for Matching and Vertex Cover

We now present our main parallel algorithm.

```
ParallelAlgorithm(G, \Delta). A parallel algorithm for computing a vertex cover \( V_{\text{alg}} \) and a matching \( M_{\text{alg}} \) of a given graph \( G \) with maximum degree at most \( \Delta \).

1. If \( \Delta \leq (400 \cdot \log^{12} n) \) send \( G \) to a single machine and run the following algorithm locally.
   Compute a maximal matching \( M_{\text{alg}} \) in \( G \) and let \( V_{\text{alg}} \) be the set of vertices matched by \( M_{\text{alg}} \). Return \( V_{\text{alg}} \) and \( M_{\text{alg}} \).
2. If \( \Delta > (400 \cdot \log^{12} n) \), we run the following algorithm.
3. Compute an EDCS \( C := \text{ParallelEDCS}(G, \Delta) \) in parallel. Let \( \beta_C, \beta_C^{-} \) be the parameters of this EDCS (as specified in Claim 6.5 below).
4. Define \( V_{\text{high}} := \{ v \in V \mid d_C(v) \geq \beta_C^{-}/2 \} \) be the set of “high” degree vertices in \( C \).
5. Compute a matching \( M_{\text{high}} := \text{RandomMatch}(C, V_{\text{high}}, \beta_C) \).
6. Define \( V^{-} := V \setminus \left( V_{\text{high}} \cup V(M_{\text{high}}) \right) \) as the set of vertices that are neither high degree in \( C \) nor matched by \( M_{\text{high}} \). Let \( C^{-} \) be the induced subgraph of \( C \) on vertices \( V^{-} \).
7. Recursively compute \( (V_{\text{rec}}, M_{\text{rec}}) := \text{ParallelAlgorithm}(C^{-}, \beta_C) \).
8. Return \( V_{\text{alg}} := V_{\text{high}} \cup V(M_{\text{high}}) \cup V_{\text{rec}} \) and \( M_{\text{alg}} := M_{\text{high}} \cup M_{\text{rec}} \).
```

We start by proving some simple properties of \( \text{ParallelAlgorithm} \). The following claim is a direct corollary of Lemma 6.3 by setting \( k = \Delta \).

**Claim 6.5.** The subgraph \( C := \text{ParallelEDCS}(G, \Delta) \) computed in \( \text{ParallelAlgorithm}(G, \Delta) \) is an EDCS(\( G, \beta_C, \beta_C^{-} \)) for parameters:

\[
\lambda_C := o(1) \quad \beta_C := \sqrt{\Delta} \cdot O(\log^5 n) \quad \beta_C^{-} := (1 - \lambda_C) \cdot \beta_C
\]

with probability at least \( 1 - 1/n^5 \).

Similarly, the following claim follows easily from Lemma 6.4.

**Claim 6.6.** Conditioned on \( C = \text{EDCS}(G, \beta_C, \beta_C^{-}) \), matching \( M_{\text{high}} = \text{RandomMatch}(C, V_{\text{high}}, \beta_C^{-}/2) \) is of size \( |M_{\text{high}}| = \Omega(|V_{\text{high}}|) \) with probability \( 1 - 1/n^{10} \).
Proof. With this conditioning, the maximum degree of $G$ is at most $\beta_C$, while the degree of vertices $V_{\text{high}}$ is at least $\frac{\beta_C}{2} \geq \beta_C/3$. Hence, we can apply Lemma 6.4 and prove the statement.

Let $T$ be the number of recursive calls made by $\text{ParallelAlgorithm}(G, \Delta)$. We refer to any $t \in [T]$ as a step of $\text{ParallelAlgorithm}$. We bound the total number of steps in $\text{ParallelAlgorithm}$ as follows.

Claim 6.7. The total number of steps made by $\text{ParallelAlgorithm}(G, \Delta)$ is $T = O(\log \log \Delta)$.

Proof. Define a function $F(\Delta)$ denoting the number of recursive calls made by $\text{ParallelAlgorithm}$ with maximum degree $\Delta$. As $\text{ParallelAlgorithm}(G, \Delta)$ runs $\text{ParallelAlgorithm}(C^-, \beta_C)$ for $\beta_C < \Delta^{2/3}$, we have, $F(\Delta) \leq F(\Delta^{2/3}) + 1$ for $\Delta > (400 \cdot \log^{12} n)$ and $F(\Delta) = 1$ otherwise. It is now easy to see that $F(\Delta) = O(\log \log \Delta)$, finalizing the proof.

In each step, $\text{ParallelAlgorithm}$ runs the subroutines $\text{ParallelEDCS}$ and $\text{RandomMatch}$ once. We say that a run of $\text{ParallelEDCS}$ (resp. $\text{RandomMatch}$) is valid in this step iff the high probability event in Claim 6.5 (resp. Claim 6.6) happens. Roughly speaking, this means that $\text{ParallelEDCS}$ and $\text{RandomMatch}$ are valid when they return the “correct” output. Additionally, we say that a step of $\text{ParallelAlgorithm}$ is valid if both of the subroutines in this step are valid. We define $\mathcal{E}_{\text{valid}}$ as the event that all $T$ steps of $\text{ParallelAlgorithm}(G, \Delta)$ are valid. By Claims 6.5 and 6.6 each step of $\text{ParallelAlgorithm}$ is valid with probability at least $1 - \frac{1}{2^n}$. As there are in total $T = O(\log \log n)$ steps by Claim 6.7, $\mathcal{E}_{\text{valid}}$ happens with probability at least $1 - \frac{1}{n^4}$.

We are now ready to prove the correctness of $\text{ParallelAlgorithm}$. Let $G = (V, E)$.

Lemma 6.8. For any graph $G(V, E)$, $\text{ParallelAlgorithm}(G, n)$, with high probability, outputs a matching $M_{\text{ALG}}$ which is an $O(1)$-approximation to the maximum matching of $G$ and a vertex cover $V_{\text{ALG}}$ which is an $O(1)$-approximation to the minimum vertex cover of $G$.

Proof. It is clear that the second parameter in $\text{ParallelAlgorithm}(G, n)$ is an upper bound on the maximum degree of $G$ and hence $G$ satisfies the requirement of $\text{ParallelAlgorithm}$. In the following, we condition on the event $\mathcal{E}_{\text{valid}}$ which happens with high probability by the above discussion.

We first argue that $V_{\text{ALG}}$ and $M_{\text{ALG}}$ are respectively a feasible vertex cover and a feasible matching of $G$. The case for $M_{\text{ALG}}$ is straightforward; the set of vertices matched by $M_{\text{high}}$ is disjoint from the vertices in $M_{\text{rec}}$ as all vertices matched by $M_{\text{high}}$ are removed in $C^-$, and hence (by induction) $M_{\text{ALG}} = M_{\text{high}} \cup M_{\text{rec}}$ is a valid matching in $G$. Now consider the set of vertices $V_{\text{ALG}}$. By conditioning on the event $\mathcal{E}_{\text{valid}}$, $C$ is indeed an EDCS$(G, \beta_C, \beta_C^-)$. Hence, by Property (P2) of EDCS $C$, any edge $e \in G \setminus C$ has at least one neighbor in $V_{\text{high}}$ and is thus covered by $V_{\text{high}}$. Additionally, as we pick $V(M_{\text{high}})$ in the vertex cover, any edge incident on these vertices are also covered. This implies that $V_{\text{high}} \cup V(M_{\text{high}})$ plus any vertex cover of the remaining graph $C^{-}$ is a feasible vertex cover of $G$. As $V_{\text{rec}}$ is a feasible vertex cover of $C^-$ by induction, we obtain that $V_{\text{ALG}}$ is also a feasible vertex cover of $G$ (the analysis for the base case in step 1 where a maximal matching is compute offline is trivial).

We now show that sizes of $M_{\text{ALG}}$ and $V_{\text{ALG}}$ are within a constant factor of each other. By Proposition 2.2 this implies that both are an $O(1)$-approximation to their respective problem. By conditioning on $\mathcal{E}_{\text{valid}}$, we know from Claim 6.6 that $V_{\text{high}}$ and $M_{\text{high}}$ are within constant factor of each other. Moreover, $|V(M_{\text{high}})| \leq 2|V_{\text{high}}|$ clearly and hence the vertex cover $V_{\text{high}} \cup V(M_{\text{high}})$ and the matching $M_{\text{high}}$ chosen at each step are within a constant factor of each other. By induction, this implies that sizes of $V_{\text{ALG}}$ and $M_{\text{ALG}}$ are within a constant factor of each other as well (the base case is again trivial), hence finalizing the proof.
6.4 MPC Implementation of the Parallel Algorithm

In this section, we specify the details in implementing ParallelAlgorithm in the MPC model on machines of memory $O(n)$. Throughout this section, we assume that the event $\mathcal{E}_{\text{valid}}$ defined in the previous sections holds and hence we are implicitly conditioning on this (high probability) event. The first step is to specify the details of storing the graph $G(V, E)$ that allows for the functionalities required by our algorithm.

To each vertex $v \in V$, we dedicate a single machine, namely machine $V[v]$, of memory $O(n)$ that stores all edges incident on $v$. Throughout the algorithm, we never use these machines for any purpose other than storing the current graph on which ParallelAlgorithm is supposed to run. We say that a vertex (resp. an edge) is active if it belongs to the current graph. We maintain the invariant that for any $v$, machine $V[v]$ knows whether $v$ is active and if so, what are its active edges (we show how to maintain this invariant in the following). With this in mind, we are now ready to present our main lemma of this section.

**Lemma 6.9.** For a given graph $G(V, E)$, one can implement the following algorithms in the MPC model with at most $O(n)$ machines with memory $O(n)$ with probability $1 - 1/n^{13}$:

1. Each call to ParallelEDCS in ParallelAlgorithm($G, n$) in $O(1)$ MPC rounds.
2. Each call to RandomMatch in ParallelAlgorithm($G, n$) in $O(1)$ MPC rounds.
3. ParallelAlgorithm($G, n$) in $O(\log \log n)$ MPC rounds.

In the following, we prove each part of this lemma separately. We then finish this section by showing how to further optimize the number of machines (in addition to their memory) in our MPC implementation.

6.4.1 MPC Implementation of ParallelEDCS

In ParallelAlgorithm, each call to ParallelEDCS is on some graph, say $H$, with maximum degree $\Delta$ and $k = \Delta \leq n$ machines. By construction, in ParallelEDCS, each machine $i \in [k]$ is responsible for sampling, storing, and finding an EDCS in a vertex sampled subgraph $H^{(i)}$ of $H$ w.p. $p = \frac{20 \log n}{\sqrt{\Delta}}$. In the following, we show how to perform each of these tasks.

We create $k$ machines, namely machines $G[1], \ldots, G[k]$, dedicated to the vertex sampled subgraphs $H^{(1)}, \ldots, H^{(k)}$. Each machine $V[v]$ uses a $\kappa$-wise independent hash function $h_v$ (for $\kappa = 20 \log n$) as described in ParallelEDCS to determine the set $I(v)$, i.e., the set of vertex sampled subgraphs this vertex $v$ belongs to. Next, machine $V[v]$ informs any machine corresponding to the neighbors of $v$ of the indices $I(v)$ to which $v$ belongs to. To do so, it only needs to communicate a message of size $\text{polylog}(n)$ over any of its edges as $h_v$ only requires $\hat{O}(\kappa) = \text{polylog}(n)$ bits for representation [66]. This way, machine $V[v]$ only sends and receives $\hat{O}(n)$ size messages in total which would fit its local memory. After this, for any edge $(u, v)$ both machines $V[u]$ and $V[v]$ know the set $I(u) \cap I(v)$, i.e., the set of graphs to which the edge $(u, v)$ belongs to. This edge can then be sent to every machine $G[i]$ for $i \in I(u) \cap I(v)$ by either of machines $V[u]$ or $V[v]$.

We now argue that the memory $\hat{O}(n)$ on each machine $G[i]$ is also sufficient for storing the graph $H^{(i)}$. To do this, it suffices to bound the total number of edges in $H^{(i)}$.

**Claim 6.10.** With probability $1 - 1/n^{13}$, the total number of edges in $H^{(i)}$ is $\hat{O}(n)$.

**Proof.** Let $v$ be a vertex sampled in $H^{(i)}$. By the independent sampling of vertices in a vertex sampled subgraph, we have that $\mathbb{E}[d_{H^{(i)}}(v)] = p \cdot d_v \leq p \cdot \Delta = \hat{O}(\sqrt{\Delta})$. By Chernoff bound, with probability $1 - 1/n^{20}$, degree of $v$ is $\hat{O}(\sqrt{\Delta})$. We can then take a union bound on all vertices in $H^{(i)}$ and have that with probability $1 - 1/n^{19}$, the maximum degree of $H^{(i)}$ is $\hat{O}(\sqrt{\Delta})$. At the same time, the expected number of vertices sampled in $H^{(i)}$ is at most $p \cdot n = \Theta(n/\sqrt{\Delta})$. Another
application of Chernoff bound ensures that the total number of vertices sampled in $H^{(i)}$ is $\tilde{O}(n/\sqrt{\Delta})$ with probability $1 - 1/n^{19}$. As a result, the total number of edges in $H^{(i)}$ is $\tilde{O}(n/\sqrt{\Delta}) \cdot \tilde{O}(\sqrt{\Delta}) = \tilde{O}(n)$ with probability at least $1 - 1/n^{18}$ as desired.

Finally, as for any $i \in [k]$, the graph $H^{(i)}$ resides on a single machine $G[i]$ we can simply compute any EDCS of $H^{(i)}$ locally on this machine in $\tilde{O}(n)$ memory (using for example the algorithm in Lemma 2.4). After this, each machine $G[i]$ informs any machine $V[v]$ where $v$ belongs to the graph $H^{(i)}$, which edges of $v$ participates in the computed EDCS $C^{(i)}$ (and hence final EDCS $C$). As there are at most $\tilde{O}(n)$ edges in the EDCS $C^{(i)}$, $G[i]$ needs to communicate $\tilde{O}(n)$ size message in total. Moreover, each machine $v$ is only participating in $\tilde{O}(\sqrt{n})$ sampled subgraphs and each subgraph may want to send at most $O(\log n)$ messages to this vertex; hence in total $V[v]$ is going to receive a message of length $\tilde{O}(\sqrt{n})$ which fits its local memory. After this step, every machine $V[v]$ knows the edges of $v$ in EDCS $C$. This finalizes the proof of the first part of Lemma 6.9 as all the procedures described above only requires $O(1)$ MPC rounds.

6.4.2 MPC Implementation of RandomMatch

In ParallelAlgorithm, each call to RandomMatch is on a graph, say $C$, with maximum degree $\Delta = \beta C = \tilde{O}(\sqrt{n})$ and set $V_{high}$ of vertices with degree at least $\Delta/3$ in $C$. We first argue that $E_{smpl}$ fits the memory of a single machine with $\tilde{O}(n)$ size. If $|S| \geq \Delta \log^2 n$, then each edge in $C$ is sampled with probability $1/\Delta$ and since $\Delta$ is the maximum degree of $C$, the total number of edges in $E_{smpl}$ would be $\tilde{O}(n)$ with high probability by Chernoff bound. If $|S| < \Delta \cdot \log^2 n$, this means that the total number of edges incident on $S$, i.e., the set $E_{smpl}$ in this case, is at most $\tilde{O}(\Delta^2) = \tilde{O}(n)$ as $\Delta = \tilde{O}(\sqrt{n})$ whenever RandomMatch is called by ParallelAlgorithm. Notice that every machine $V[v]$ for which $v \in V_{high}$, i.e., the set $S$, can identify itself based on its degree in $C$ and hence either performs the sampling locally or send all its edges in case $|S| < \Delta \log^2 n$ (determining which of the two cases we are in can be trivially done in $O(1)$ MPC rounds). After $E_{smpl}$ resides on a single machine, we can locally compute a maximum matching of $G$ using any offline algorithm. We can also inform the machine corresponding to any vertex $v$, whether $v$ is matched in $M_{high}$ and if so to which vertex. All of this can be implemented in $O(1)$ MPC rounds easily, hence proving the second part of Lemma 6.9.

6.4.3 MPC Implementation of ParallelAlgorithm

We can now combine the results in the previous two section to show how to implement ParallelAlgorithm in the MPC model as well. Consider one of the steps of ParallelAlgorithm. We saw that ParallelEDCS and RandomMatch can both be implemented in $O(1)$ MPC models. Moreover, after running ParallelEDCS and RandomMatch the machines $V[v]$ for all $v \in V$ know whether $v$ still belongs to the graph $C^-$ for the next step and which edges are still active. We can hence recursively solve the problem on the graph $C^-$ in the subsequent steps and each step requires $O(1)$ MPC rounds, resulting in an algorithm with $O(\log \log n)$ MPC rounds in total.

6.4.4 Optimizing the Number of Machines

We conclude this section by making a remark about optimizing the number of machines (in addition to their memory) in our results as well.

As it is, the total number of machines needed to implement ParallelAlgorithm in the MPC model is $\tilde{O}(n)$. This means that the total memory across all machines is $\tilde{O}(n^2)$, which is proportional to the input size (up to polylog($n$) factors) whenever the input graph is completely dense, i.e., has $O(n^2)$ edges. However for sparser graphs with $n^{1+\Omega(1)}$, this can be larger than the input size by a factor of $n^{1-\Omega(1)}$. This is consistent with some definitions of MapReduce-style computation such
as [55, 58] but not with the strictest definitions in [10, 18], which require that the total memory of
the system for a graph with $m$ edges to be only $\tilde{O}(m)$, i.e., proportional to the input size.

Nevertheless, a straightforward modification of our algorithm can reduce the number of ma-
chines down to $\tilde{O}(m/n)$ which ensures that the total memory used by our algorithm is $\tilde{O}(m)$ which
adheres to the strictest restrictions of the MPC model. The only change we need to do is to
work with the average degree of the graph in ParallelAlgorithm as opposed to its maximum degree.
Concretely, in each call to ParallelAlgorithm$(G, \Delta)$, instead of computing ParallelEDCS$(G, \Delta)$, we
compute ParallelEDCS$(G, \tilde{\Delta})$, where $\tilde{\Delta} := m/n$ denotes the average degree of the graph $G$. It is
easy to see that in this case, we still only need $\tilde{O}(n)$ memory per machine (essentially the same
argument in Claim 6.10 proves this), but now the total number of machines needed is only $\tilde{O}(m/n)$
and hence we only need $\tilde{O}(m)$ memory in total. It is easy to verify that the arguments in the proof
of correctness of ParallelEDCS can be immediately applied to this version; we omit the details.

6.5 Wrap-up and Further Improvements

As was shown by Lemma 6.9, ParallelAlgorithm can be implemented in the MPC model with machines
of memory $\tilde{O}(n)$ and $O(\log \log n)$ MPC rounds. Combining this with Lemma 6.8 on the correctness
of ParallelAlgorithm algorithm, we immediately obtain Theorem 7, i.e., an MPC algorithm with $O(1)$
approximation to both matching and vertex cover.

In the remainder of this section, we show that using standard techniques, one can improve the
approximation ratio of our matching algorithm significantly. In particular,

Corollary 8. There exists an MPC algorithm that given a graph $G$ and $\varepsilon \in (0, 1)$, with high
probability computes a $(2 + \varepsilon)$-approximation to maximum matching of $G$ in $O(\log (1/\varepsilon)) \cdot (\log \log n)$
MPC rounds using only $\tilde{O}(n)$ memory per machine.

Corollary 9. There exists an MPC algorithm that given a graph $G$ and $\varepsilon \in (0, 1)$, with high
probability computes a $(1 + \varepsilon)$-approximation to the maximum matching of $G$ in $(1/\varepsilon)^{O(1/\varepsilon) \cdot (\log \log n)}$
MPC rounds using only $\tilde{O}(n)$ memory per machine.

We prove each of the above corollaries in the next two sections.

6.5.1 Proof of Corollary 8

The idea is to simply run our MPC algorithm in Theorem 7, to compute a matching $M_{\text{ALG}}$, remove
all vertices matched by $M_{\text{ALG}}$ from the graph $G$, and repeat. Clearly, the set of all matchings
computed like this is itself a matching of $G$. In the following, we show that only after $O(\log 1/\varepsilon)$
repetition of this procedure, one obtains a $(2 + \varepsilon)$-approximation to the maximum matching of $G$.

Let $\alpha = O(1)$ be the approximation ratio of the algorithm in Theorem 7. Suppose we repeat
the above process for $T := (\alpha \cdot \log (1/\varepsilon))$ steps. For any $t \in [T]$, let $M_t$ be the matching computed
so far, i.e., the union of the all the matchings in the first $t$ applications of our $\alpha$-approximation
algorithm. Also let $G_{t+1} := G \setminus V(M_t)$, i.e., the graph remained after removing vertices matched
by $M_t$. Note that $M_{t+1}$ is an $\alpha$-approximation to the maximum matching of $G_{t+1}$. Moreover,
$\text{MM}(G_{t+1}) \geq \text{MM}(G) - 2|M_t|$ as each edge in $M_t$ can only match (and hence remove) two vertices
of any maximum matching of $G$. This implies that $|M_{t+1}| \geq |M_t| + \frac{1}{\alpha} \cdot (\text{MM}(G) - 2|M_t|)$ for all
$t \in [T]$. We now have,

$$\text{MM}(G) - 2|M_T| \leq \left(1 - \frac{2}{\alpha}\right) \cdot \text{MM}(G) - 2 \cdot \left(1 - \frac{2}{\alpha}\right) \cdot |M_{T-1}|$$

$$= \left(1 - \frac{2}{\alpha}\right) \cdot (\text{MM}(G) - 2 \cdot |M_{T-1}|)$$

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\[
\leq \left(1 - \frac{2}{\alpha}\right)^2 (\text{MM}(G) - 2 \cdot |M_{T-2}|) \\
\text{(by applying the second equation to } M_{T-1}) \\
\leq \left(1 - \frac{2}{\alpha}\right)^T \cdot \text{MM}(G) \\
\text{(by recursively applying the previous equation)} \\
\leq \exp\left(-\frac{2}{\alpha} \cdot \alpha \cdot \log \left(\frac{1}{\epsilon}\right)\right) \cdot \text{MM}(G) \leq \epsilon \cdot \text{MM}(G).
\]

Hence, after \( T = O(\log 1/\epsilon) \) steps, the matching computed by the above algorithm is of size \((2 + \epsilon) \cdot \text{MM}(G)\). It is immediate to verify that the new algorithm can be implemented in the MPC model with machines of memory \( \tilde{O}(n) \) and \( O(\log 1/\epsilon) \cdot (\log \log n) \) MPC rounds. Note that as the probability of error in the algorithm in Theorem 7 is at most \( 1/n^4 \), by a union bound, the new algorithm also outputs the correct answer with probability at least \( 1 - 1/n^3 \).

### 6.5.2 Proof of Corollary 9

Corollary 9 can be proven using Theorem 7 plus a simple adaption of the multi-pass streaming algorithm of McGregor [60] for maximum matching to the MPC model. The high level approach in [60] is to reduce the problem of finding a \((1 + \epsilon)\)-approximate maximum matching in \( G \) to many instances of finding a maximal matching in multiple adaptively chosen subgraphs of \( G \). It was then shown that there exists a single pass streaming algorithm that can both determine the appropriate subgraph of \( G \) needed in each step of this reduction and compute a maximal matching of this subgraph. Hence, after repeating this streaming algorithm in multiple passes over the stream, one can fully implement the reduction and obtain a \((1 + \epsilon)\)-approximation to the maximum matching.

We show that essentially the same approach can also be used in the MPC model. The main difference is to switch from computing a maximal matching to finding an \( O(1) \)-approximate maximum matching using our Theorem 7. In the following, we briefly describe the approach in [60] and point out the modifications needed to make it work in Corollary 9. The purpose of this section is only to convince the reader that the reduction [60] can be seamlessly implemented in the MPC model and hence we do not delve into the full details of the algorithm and analysis and instead refer the reader to [60] for more details and formal proofs.

**The Streaming Algorithm of [60].** The idea behind the algorithm is to start with some maximal matching \( M \) (which is easy to compute in one pass over the stream) and then *augment* this matching further over multiple *phases* by finding a large set of vertex disjoint augmenting paths of length \( O(1/\epsilon) \) in a *layered* graph created from \( G \) and the current matching \( M \). We first introduce the concept of the layered graph that is used to reduce the task of finding augmenting paths to multiple instances of approximate matching. Given a graph \( G \), a matching \( M \), and an odd integer \( k \) (which is the target length of the augmenting paths to be found), we create a graph \( \mathcal{L}(G, M, k) \) using the following randomized procedure:

1. The vertices in \( \mathcal{L} \) are the same as \( G \) and are partitioned into \( k+1 \) subsets \( L_1, \ldots, L_{k+1} \) called *layers*. The layer of each vertex \( v \) is determined as follows:
   
   (a) For any vertex \( u \) left unmatched by \( M \), \( u \) is assigned to either \( L_1 \) or \( L_{k+1} \) chosen uniformly at random.

   (b) For any matched edge \((u, v) \in M\), the vertex \( u \) is assigned to a uniformly at random chosen even layer \( L_2, L_4, \ldots, L_{k-1} \) and \( v \) is assigned to the subsequent layer.

2. For any edge \((u, v) \in E \setminus M\), if \( u \) belongs to an odd layer and \( v \) belongs to the next (even)
layer then \((u, v)\) is added to \(\mathcal{L}\) as well.

3. For any edge \((u, v)\) \(\in M\), \((u, v)\) is added to \(\mathcal{L}\) as well.

One could easily verify that the edges in \(\mathcal{L}\) are only between two consecutive layers and for any edge in \(\mathcal{L}\) there is a unique edge in \(G\). The main property of the above construction is that a collection of vertex disjoint paths between vertices in \(L_1\) and \(L_{k+1}\) corresponds to a set of vertex disjoint augmenting paths of length \(k\) for \(M\) in \(G\). It is also relatively easy to prove that as long as \(|M| < (1-\varepsilon) \cdot \text{MM}(G)\), then, there exists some \(k = O(1/\varepsilon)\), for which the corresponding graph \(\mathcal{L}(G, M, k)\) has at least \(\text{MM}(G) \cdot \varepsilon^{O(1/\varepsilon)}\) vertex disjoint paths between \(L_1\) and \(L_{k+1}\) with high probability (see Theorem 1 in [60]).

We now describe how to find a large fraction of these augmenting paths in \(\mathcal{L}\) (in each phase) by finding different maximal matchings between consecutive layers of \(\mathcal{L}\). We first compute an approximate matching \(M_{1,2}\) between \(L_1\) and \(L_2\). Let \(L_3' \subseteq L_3\) be the set of vertices that are neighbor to matched vertices of \(M_{1,2}\) (in \(L_2\)). We then, compute a matching \(M_{3,4}\) between \(L_3'\) and \(L_4\) and we continue like this. One can see this approach as growing vertex disjoint (augmenting) paths from layer \(L_1\) to (eventually) layer \(L_{k+1}\). If at some point, size of the matching between \(L_i'\) and \(L_{i+1}\) (for some odd \(i\)) goes below a certain threshold, we remove all vertices in \(L_i'\) from the graph (as they are mostly “dead ends”) and backtrack (similar to a DFS procedure). By choosing a relatively large threshold, one can ensure that the number of backtracks is bounded by some function of \((1/\varepsilon)\) only and hence is not too large. At the same time, we like the threshold to be small enough also so that not many actual vertex disjoint paths are marked (incorrectly) as dead ends. Once we complete some paths from \(L_1\) to \(L_{k+1}\) we remove these paths (to be augmented later) and recurse on the remaining graph until no vertices are left. We again emphasize that at each step of this procedure, we simply find a maximal matching between some set of nodes in \(\mathcal{L}\) and the above algorithm determines which sets of vertices to choose for finding the next approximate matching. We refer the reader to Section 2.3 (in particular Fig 2) of [60] for a formal definition and a pseudo-code of this algorithm.

We now argue that essentially the same algorithm can also be implemented in the MPC model using our Theorem 7 (instead of picking maximal matchings).

**From Maximal to \(O(1)\)-Approximate Matching.** We point out that the algorithm of [60] uses a maximal matching in its construction as it is easy to compute in one pass over a stream and results in a 2-approximate matching. However, we emphasize that the analysis in [60] in no way uses the “maximality” property of this matching and only relies on its approximation ratio. We do not know how to compute a maximal matching in the MPC model efficiently, however, we can use Theorem 7 to compute an \(O(1)\)-approximate matching. By a simple adjustment of the thresholds used in the above algorithm, we can use any \(O(1)\)-approximation algorithm to maximum matching in place of a maximal matching algorithm, while blowing up the number of calls to the matching subroutine between two layers by a constant factor in total.

**From Streaming to MPC Model.** It is easy to see that the construction of the layered graph as well as the choices to which subgraph to compute the next matching on in the above algorithm can be easily performed in constant number of rounds in the MPC model. As argued above, in place of the maximal matching algorithm in reduction of [60], we use our algorithm in Theorem 7 which requires \(O(\log \log n)\) MPC rounds (compared to one pass in [60]). As a result, number of rounds in our algorithm is \(O_{\varepsilon}(\log \log n)\) larger than the passes in the streaming algorithm of [60]. Overall, the new algorithm requires \((1/\varepsilon)^{O(1/\varepsilon)} \cdot (\log \log n)\) MPC rounds and \(\tilde{O}(n)\) memory per machine.
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A Some Applications of Randomized Composable Coresets

In the following, we provide more details on the applications of randomized coresets to different computational models and in particular prove Proposition 1.1.

MPC Algorithms. The MPC model is defined in Section 2.1. We can use any $\alpha$-approximation randomized coreset $\text{ALG}$ of size $s$ for a problem $P$ to obtain a parallel algorithm in only two MPC rounds. Suppose $G(V, E)$ is the input graph and let $k := \sqrt{m/s}$. The algorithm is as follows:

1. In the first round, create a random $k$-partition $G^{(1)}, \ldots, G^{(k)}$ and allocate each graph $G^{(i)}$ to the machine $i \in [k]$.
2. Each machine $i \in [k]$ creates a randomized composable coreset $C_i = \text{ALG}(G^{(i)})$.
3. In the second round, collect the union of all coresets to create $H := H(V, C_1, \ldots, C_k)$ on one machine and solve $P$ in $H$ using any offline algorithm.

It is straightforward to verify that this MPC algorithm requires $O(k) = O(\sqrt{ms})$ machines each with $O(\sqrt{ms} + n)$ memory. Moreover, by Definition 1, the output of this algorithm is an $\alpha$-approximation to $P(G)$ with high probability.

Streaming Algorithms. In the streaming model, the edges of the input graph are presented to the algorithm one by one in a sequence and the algorithm is allowed to make a single pass (or a few passes) over this sequence.

Similar to the case for MPC algorithms, any $\alpha$-approximation randomized coreset $\text{ALG}$ of size $s$ for a problem $P$ also imply a single-pass streaming algorithm for $P$ in random arrival streams. Suppose $G(V, E)$ is the input graph which its edges are arriving in a random order in a stream of length $m$. The algorithms is as follows:

1. Randomly partition the edges in the stream into $k := \sqrt{m/s}$ consecutive pieces such that the distribution of the graphs $G^{(1)}, \ldots, G^{(k)}$ created by edges in each piece is a random $k$-partition of $G$ (using the randomness in the arrival of the stream).
2. Let $H$ be the empty graph on vertices $V$. For $i = 1$ to $k$:
   (a) Read the graph $G^{(i)}$ completely and store it in the memory temporarily.
   (b) Compute a randomized composable coreset $C_i = \text{ALG}(G^{(i)})$.
   (c) Update $H$ by adding all edges in $C_i$ to it and discard all edges in $G^{(i)}$ to reuse the memory in the next iteration.
3. At the end of stream, solve $P$ in $H$ using any offline algorithm.

It is again easy to see that the total memory required by this algorithm is $O(m/k) = O(\sqrt{ms})$ (to store each graph $G^{(i)}$ in the memory temporarily) plus $O(k \cdot s) = O(\sqrt{ms})$ (to store all coresets during the stream). The output of this algorithm is then an $\alpha$-approximation to $P(G)$ with high probability by Definition 1.

Simultaneous Communication Model. In this model, the input graph is edge partitioned across $k$ machines/players and the goal is to solve the problem on the union of these graphs. In order to this, the players simultaneously each send a single message to an additional party called the coordinator who then outputs the solution.

Any $\alpha$-approximation randomized coreset $\text{ALG}$ of size $s$ for a problem $P$ immediately implies a simultaneous protocol for $P$ on randomly partitioned inputs. Suppose $G(V, E)$ is the input graph
which its edges are partitioned across \( k \) parties \textit{randomly}. Each party only needs to compute a coreset of its input graph and communicate it with the coordinator. The communication by each party then would be \( O(s) \) and the coordinator can recover an \( \alpha \)-approximation to \( P \) with high probability by Definition 1.

Remark A.1. In the above algorithms, we neglected the computation time needed for solving \( P \) on the union of the coresets. This is consistent with the definitions of the models considered here as they all allow unbounded computation time to the algorithm. However, if one insists on having efficient time algorithms (e.g., polynomial time) then the approximation ratio of the resulting algorithm would be \( \rho \cdot \alpha \) where \( \rho \) is the approximation guarantee of any offline algorithm we use for solving \( P \) in the end (in many scenarios however this naive blow-up in the approximation ratio can be avoided by additional care, although not in a black-box way anymore).

B Missing Details from Section 2

B.1 Useful Concentration of Measure Inequalities

Azuma’s inequality proves a concentration bound for martingales.

Proposition B.1 (Azuma’s inequality). Let \( \{X_i\}_{i=0}^n \) be a martingale (with respect to some random variables \( \{Y_i\}_{i=0}^n \)). Suppose there exists a sequence of integers \( \{c_i\}_{i=1}^n \) such that \( |X_i - X_{i-1}| \leq c_i \); then,

\[
\Pr\left( |X_n - X_0| \geq \lambda \right) \leq 2 \cdot \exp\left(-\frac{\lambda^2}{\sum_{i=1}^n c_i^2}\right).
\]

We also use the following standard variant of the Chernoff bound as well as its generalization for variables with bounded independence.

Proposition B.2 (Chernoff bound). Let \( X_1, \ldots, X_n \) be independent random variables taking value in \([0, 1]\) and \( X := \sum_{i=1}^n X_i \). Then, for any \( \delta \in (0, 1) \)

\[
\Pr\left( |X - \mathbb{E}[X]| \geq \delta \cdot \mathbb{E}[X]\right) \leq 2 \cdot \exp\left(-\frac{\delta^2 \cdot \mathbb{E}[X]}{3}\right).
\]

Proposition B.3 (Chernoff bound with bounded independence [73]). Let \( X_1, \ldots, X_n \) be \( \kappa \)-wise independent variables taking value in \([0, 1]\) and \( X := \sum_{i=1}^n X_i \). For any \( \delta \in (0, 1) \), if \( \kappa \leq \delta^2 \cdot \mathbb{E}[X]/2 \), then,

\[
\Pr\left( |X - \mathbb{E}[X]| \geq \delta \cdot \mathbb{E}[X]\right) \leq 2 \cdot \exp\left(-\frac{\kappa}{2}\right).
\]

B.2 Proof of Lemma 2.4

Lemma. Any graph \( G \) contains an EDCS\((G, \beta, \beta^-)\) for any parameters \( \beta > \beta^- \).

Proof. Consider the following simple procedure for creating an EDCS \( H \) of a given graph \( G \):

```
While \( H \) is not an EDCS\((G, \beta, \beta^-)\) of \( G \):
  (a) Find an edge \( e \) which violates one of the properties of EDCS.
  (b) Fix the edge \( e \), i.e., remove it from \( H \) if it was violating Property (P1) and add it to \( H \) if it was violating Property (P2).
```

The output of the above procedure is clearly an EDCS of graph \( G \). However, a-priori it is not clear that this procedure ever terminates as fixing one edge \( e \) can result in many edges violating the
EDCS properties, potentially undoing the previous changes. In the following, we argue that the maximum matching in the graph always exists.

We define the following potential function $\Phi$:

$$\Phi := (\beta - 1/2) \cdot \sum_{u \in V} d_H(u) - \sum_{(u,v) \in H} d_H(u) + d_H(v).$$

We argue that in any step of the procedure above, the value of $\Phi$ increases by at least 1. Since the maximum value of $\Phi$ is at most $O(n \cdot \beta^2)$, this immediately implies that this procedure terminates in $O(n \cdot \beta^2)$ steps.

Define $\Phi_1 := (\beta - 1/2) \cdot \sum_{u \in V} d_H(u)$ and $\Phi_2 := -\sum_{(u,v) \in H} d_H(u) + d_H(v)$ (note the minus sign) and hence $\Phi = \Phi_1 + \Phi_2$. Let $(u, v)$ be the edge we choose to fix at this step, $H$ be the subgraph before fixing the edge $(u, v)$, and $H'$ be the resulting subgraph.

Suppose first that the edge $(u, v)$ was violating Property (P1) of EDCS. As the only change is in the degrees of vertices $u$ and $v$, $\Phi_1$ decreases by $(2\beta - 1)$. On the other hand, $d_H(u) + d_H(v) \geq \beta + 1$ originally (as $(u, v)$ was violating Property (P1) of EDCS) and hence after removing $(u, v)$ $\Phi_2$ increases by $\beta - 1$. Additionally, for each neighbor $w$ of $u$ and $v$ in $H'$, after removing the edge $(u, v)$, $d_{H'}(w)$ decreases by one. As there are at least $d_H(u) + d_H(v) = d_H(u) + d_H(v) - 2 \geq \beta - 1$ choices for $w$, this means that in total, $\Phi_2$ increases by at least $(\beta + 1) + (\beta - 1) = 2\beta$. As a result, in this case $\Phi$ increases by at least 1 after fixing the edge $(u, v)$.

Now suppose that the edge $(u, v)$ was violating Property (P2) of EDCS instead. In this case, degree of vertices $u$ and $v$ both increase by one, hence $\Phi_1$ increases by $2\beta - 1$. Additionally, not that since edge $(u, v)$ was violating Property (P2) we have $d_H(u) + d_H(v) \leq \beta - 1$, so the addition of edge $(u, v)$ decreases $\Phi_2$ by at most $d_{H'}(u) + d_{H'}(v) = d_H(u) + d_H(v) - 2 \leq \beta - 1$. Moreover, for each neighbor $w$ of $u$ and $v$, after adding the edge $(u, v)$, $d_{H'}(w)$ increases by one and since there are at most $d_H(u) + d_H(v) \leq \beta - 1$ choices for $w$, $\Phi_2$ decreases in total by at most $(\beta - 1) + (\beta - 1) = 2\beta$. Since $\beta \leq \beta - 1$, we have that $\Phi$ increases by at least $(2\beta - 1) - (2\beta) \geq 1$ after fixing the edge $(u, v)$, finalizing the proof.

We remark that this proof also implies a natural polynomial time algorithm for computing any EDCS of a given graph $G$.

## C Lower Bound on Approximation Ratio of MaxMatching Coreset

We prove Lemma 3.6 in this section.

**Lemma.** There exists a graph $G(V, E)$ such that for any random $k$-partition of $G$ ($k \leq n^{1-\delta}$ for any constant $\delta > 0$), the MaxMatching coreset can only find a matching of size at most $(\frac{1}{2} + \frac{1}{k}) \cdot \text{MM}(G)$ with high probability.

**Proof.** The vertex set of the graph $G$ consists of four sets of vertices $L_1, L_2, R_1, R_2$ with $|L_1| = \frac{n}{2} + \frac{n}{k}$ and $|L_2| = |R_1| = |R_2| = \frac{n}{2}$. $G$ is a bipartite graph with $L_1 \cup L_2$ on one side of the bipartition and $R_1 \cup R_2$ on the other side. There is a complete bipartite graph between $L_1$ and $R_2$, a perfect matching between $L_2$ and $R_2$ and a matching of size $|R_1|$ between $L_1$ and $R_1$.

It is easy to verify that there exists a matching of size $|R_1| + |R_2| = n$ in $G$ and hence $\text{MM}(G) \geq n$.

Suppose we create a random $k$-partition $G^{(1)}, \ldots, G^{(k)}$ of $G$ and each machine $i \in [k]$ computes an arbitrary maximum matching $M_i$ of its input graph (i.e., compute the MaxMatching coreset). In the following, we argue that the maximum matching in the graph $H(V, M_1 \cup \ldots, M_k)$ is of size $(\frac{1}{2} + \frac{1}{k}) \cdot n$ with high probability, which concludes the proof.

To prove the lemma, we need the following simple claim about the maximum matching in the edge sampled subgraphs of $G$.

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Claim C.1. Suppose $G_p^E(V,E_p)$ is an edge sampled subgraph of $G$ with probability $p = 1/k$; then, w.p. $1 - 1/n^5$, there exists a matching $M_p$ in $G$ such that:

1. $M_p$ is a maximum matching in $G_p$, i.e., $|M_p| = 	ext{MM}(G)$.
2. No edges between $L_2$ and $R_2$ belong to $M_p$.

Proof. A simple application of Chernoff bound ensures that the total number of edges between $L_1$ and $R_1$ in $G_p$ is at most $2p \cdot n = n/k$ with probability at least $1 - 1/n^{10}$. In the following, we condition on this event. Define $M_{1,1}$ as the matching consisting of the edges between $L_1$ and $R_1$ in $G_p$ and let $L_1^- := L_1 \setminus L_1(M_{1,1})$ be the set of vertices in $L_1$ that are not incident on $M_{1,1}$.

Consider the graph between $L_1^-$ and $R_2$. Note that since $|M_{1,1}| \leq n/k$, we have $|L_1^-| \geq |R_2|$. By the independence in the sampling of edges and the fact that in $G$, $L_1^-$ and $R_1$ forms a bipartite clique, the set of edges between $L_1^-$ and $R_2$ in $G_p$ form a random bipartite graph with probability of having each edge equal to $p = 1/k = \omega(\log n)$. Using standard facts about random graphs (see, e.g., [27], Chapter 7), this implies that there exists a matching of size $|R_2|$ between $L_1^-$ and $R_2$ in $G_p$ with probability $1 - 1/n^{10}$. Let $M_p$ be the union of this matching and $M_{1,1}$.

It is clear that $M_p$ does not have any edges between $L_2$ and $R_2$. To see that $M_p$ is indeed a maximum matching of $G_p$, notice that all vertices in $R_1 \cup R_2$ in $G_p$ that have non-zero degree are matched by $M_p$ and hence there cannot be any larger matching in $G$.

We are now ready to finalize the proof of Lemma 3.6. Recall that each graph $G^{(i)}$ is an edge sampled subgraph of $G$ with probability $1/k$. We can apply Claim C.1 to each graph $G^{(i)}$ and by a union bound, w.p. $1 - 1/n^4$, there exists a suitable maximum matching $M_p^{(i)}$ in each graph $G^{(i)}$. Since we are choosing an arbitrary maximum matching of $G^{(i)}$ as its coreset, we can assume that $M_p^{(i)}$ would be chosen from each graph $G^{(i)}$, i.e., $M_i = M_p^{(i)}$ for all $i \in [k]$. This implies that no edge incident to vertices in $L_2$ are chosen among all coresets $M_1 \cup \ldots \cup M_k$. As a result, the maximum matching in the graph $H(V, M_1 \cup \ldots \cup M_k)$ can have size at most $|L_1| = \left(\frac{1}{2} + \frac{1}{k}\right) \cdot n$, finalizing the proof as $\text{MM}(G) = n$.