Global algebraic linear differential operators

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Abstract

In this note, we want to investigate the question, given a projective algebraic scheme $X/k$ and coherent sheaves $F_1$ and $F_2$ on $X$, when do global differential operators of order $N > 0$, $D : F_1 \to F_2$ exist. After remembering the necessary foundational material, we prove one result in this direction, namely that on $\mathbb{P}^n_k$, $n > 1$, for each locally free sheaf $\mathcal{E}$ and each $n \in \mathbb{Z}$, there exist global differential operators $\mathcal{E} \to \mathcal{E}(n)$ of order $N$, if $N \geq N(n)$ is sufficiently large. Also, we calculate the order of growth of differential operators of order $\leq N$, as $N$ tends to infinity. As a final result, we prove, that, with the standard definition, for ”most” projective algebraic varieties, algebraic elliptic operators on a locally free sheaf do not exist.

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1 Notation and Conventions and Basic Definitions

Convention 1 By $\mathbb{N}$ we denote the natural numbers, by $\mathbb{N}_0$ the set of non-negative integers.

We use multi index notation: if $x_1, \ldots, x_n$ is a set of variables, we denote $\underline{x}^m : x_1^{m_1} \cdot x_2^{m_2} \ldots x_n^{m_n}$ where $\underline{m} := (m_1, m_2, \ldots, m_n)$ is a multi-index of length $n$. By $|\underline{m}|$ we denote the number $m_1 + \ldots + m_n$. The partial derivatives of a function $f(x_1, \ldots, x_m)$ in the variables $x_i$ we denote by $\partial^{\underline{m}} f(x_1, \ldots, x_m)$. 
Notation 1 Let $X \to S$ be a morphism of schemes. By $\Omega^{(1)}(X/S)$ we denote the usual sheaf of Kähler differentials. The ideal sheaf of the relative diagonal $\Delta_{X/S} \hookrightarrow X \times_S X$ we denote by $\mathcal{I}_{X/S}$. Thus $\Omega^{(1)}(X/S) \cong \mathcal{I}_{X/S}/\mathcal{I}_{X/S}^2$.

We collect some basic facts about jet bundles and differential operators. Proofs and more detailed information can be found in [1].

**Facts 1.1** If $\pi : X \to S$ is a morphism of finite type of noetherian schemes and $\mathcal{E}$ is a quasi coherent sheaf on $X$, for each $N \in \mathbb{N}_0 \cup \{\infty\}$, we denote by $\mathcal{J}^N(\mathcal{E}/S)$, the $N^{th}$ jet bundle of $\mathcal{E}$, relative to $S$, which is a quasi coherent sheaf on $X$. If $\mathcal{E}$ is coherent, so is $\mathcal{J}^N(\mathcal{E}/S)$ for $N \in \mathbb{N}_0$. Locally, if $\text{Spec } B \to \text{Spec } A$ is the restriction of $\pi$ to Zariski-open subsets and $\mathcal{E}$ restricted to the Spec $B$ corresponds to the $B$-module $M$, $\mathcal{J}^N(\mathcal{E}/S)$ is given by

$$\mathcal{J}^N(M/A) = B \otimes_A M/I_{B/A} \cdot (B \otimes_A M),$$

where $I_{B/A}$ is the ideal in $B \otimes_A B$ which is the kernel of the multiplication map. If $\mathcal{E}' \subset \mathcal{E}$ is a quasi coherent subsheaf, we denote by $\mathcal{J}^N(\mathcal{E}'/S)'$ the image of the homomorphism $\mathcal{J}^N(i/S)$, where $i : \mathcal{E}' \hookrightarrow \mathcal{E}$ is the inclusion.

We denote the universal derivation $\mathcal{E} \to \mathcal{J}^N(\mathcal{E}/S)$ by $d^N_{\mathcal{E}/S}$.

If $\mathcal{E} = \mathcal{O}_X$, the jet sheaf $\mathcal{J}^N(\mathcal{O}_X/S)$ is an $\mathcal{O}_X$-algebra and we denote by $J^N(X/S) := \text{Spec } X \mathcal{J}^N(i(\mathcal{O}_X/S))$, the associated affine bundle over $X$ with projection $p_X = p_{1,X} : J^N(X/S) \to X$. There is a second projection $p_{2,X} : J^N(X/S) \to X$ which corresponds to the universal derivation $d^N_{\mathcal{O}_X/S}$.

**Proposition 1.2** 1 Let $X \to S$ be a morphism of finite type between noetherian schemes. For each $N \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{J}^N(-/S)$ be the functor from quasi coherent $\mathcal{O}_X$-modules to quasi coherent $\mathcal{J}^N(X/S)$-modules sending $\mathcal{F}$ to $\mathcal{J}^N(\mathcal{F}/S)$. Then, this functor is right exact and there is a canonical natural isomorphism $\mathcal{J}^N(-/S) \xrightarrow{\cong} p^*_2(-)$.

2 If $\pi : X \to S$ is flat, then $\mathcal{J}^N(-/S)$ is an exact functor.

3 If $\pi : X \to S$ is a smooth morphism of schemes, then for each $N \in \mathbb{N}$, the functor $\mathcal{J}^N(-/S)$, sending quasi coherent $\mathcal{O}_X$-modules to quasi coherent $\mathcal{J}^N(X/S)$-modules, is exact and equal to $(p^*_2)^N$.

**Proof:** (see [1][section 3.5, Proposition 3.33, p.19].)

**Definition 1.3** Let $X \to S$ be an arbitrary morphism of schemes, or more generally of algebraic spaces, and $\mathcal{F}_i$, $i = 1, 2$ be quasi coherent sheaves on $X$. Then, a differential operator of order $\leq N$ is an $\mathcal{O}_S$-linear map $D : \mathcal{F}_1 \to \mathcal{F}_2$ that can be factored as $\mathcal{F}_1 \xrightarrow{\mathcal{O}_S^{d_{\mathcal{F}_1}}} \mathcal{J}^N(\mathcal{F}_1/S)$ and an $\mathcal{O}_X$-linear map $\tilde{D} : \mathcal{J}^N(\mathcal{F}_1/S) \to \mathcal{F}_2$.

A differential operator of order $N$ is a differential operator that is of order $\leq N$ but not of order $\leq N - 1$.

Thus, in this situation, there is a 1-1 correspondence between differential operators $\mathcal{F}_1 \to \mathcal{F}_2$ relative to $S$ and $\mathcal{O}_X$-linear maps $\mathcal{J}^N(\mathcal{F}_1/S) \to \mathcal{F}_2$. 


Facts 1.4 Let \( q : X \to S \) be a morphism of finite type of noetherian schemes and \( \mathcal{F}_1, \mathcal{F}_2 \) be quasi coherent sheaves on \( X \). By \( DO^N_{X/S}(\mathcal{F}_1, \mathcal{F}_2) \) we denote the \( \mathcal{O}_S \)-module of global differential operators \( D : \mathcal{F}_1 \to \mathcal{F}_2 \) relative to \( S \), which is by definition the quasi coherent \( \mathcal{O}_X \)-module

\[
DO^N_{X/S}(\mathcal{F}_1, \mathcal{F}_2) := \text{Hom}_{\mathcal{O}_X}(\mathcal{J}^N(\mathcal{F}_1/S), \mathcal{F}_2).
\]

If \( \mathcal{F}_1, \mathcal{F}_2 \) are coherent, so is \( DO^N(\mathcal{F}_1, \mathcal{F}_2) \).

If \( S = \text{Spec } k \) is a point, we use the simplified notation \( DO^N(\mathcal{F}_1, \mathcal{F}_2) \).

Facts 1.5 Let \( q : X \to S \) be a morphism of finite type between noetherian schemes and \( \mathcal{E} \) be a coherent sheaf on \( X \). Then we denote for each \( N \in \mathbb{N} \) the standard jet bundle exact sequence by

\[
\mathcal{j}^N(\mathcal{E}/S) : 0 \to \mathcal{T}^N_{X/S}\mathcal{E}/\mathcal{T}^{N+1}_{X/S}\mathcal{E} \to \mathcal{J}^N(\mathcal{E}/S) \to \mathcal{J}^{N-1}(\mathcal{E}/S) \to 0.
\]

If \( \mathcal{E} \) is locally free and the morphism \( q : X \to S \) is smooth, the natural homomorphism

\[
\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \otimes^N \cong \mathcal{E} \otimes_{\mathcal{O}_X} (\mathcal{I}_{X/S}/\mathcal{I}_{X/S}^2)^{N+1}
\]

\[
\to \mathcal{T}^N_{X/S}/\mathcal{T}^{N+1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{T}^N_{X/S} \cdot \mathcal{E}/\mathcal{T}^{N+1}_{X/S} \cdot \mathcal{E}
\]

is an isomorphism, which follows from the fact, that in this case \( \Delta_{X/S} : X \to X \times_S X \) is a regular embedding.

Notation 2 If \( q : X \to S \) is a smooth morphism of finite type of noetherian schemes and \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are locally free sheaves on \( X \), for each \( N \in \mathbb{N} \) we denote the standard exact sequence of differential operators by

\[
\text{do}^N_{X/S}(\mathcal{E}_1, \mathcal{E}_2) : 0 \to \text{DO}^N_{X/S}(\mathcal{E}_1, \mathcal{E}_2) \to \text{DO}^N_{X/S}(\mathcal{E}_1, \mathcal{E}_2)
\]

\[
\to \text{Hom}_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{E}_2) \otimes_{\mathcal{O}_X} \mathcal{T}^1(X/S)^{N+1} \to 0,
\]

which is \( \text{Hom}_{\mathcal{O}_X}(\mathcal{j}^N(\mathcal{E}_1/S), \mathcal{E}_2) \).

2 Introduction

In this note, we want to investigate the question, given a complete algebraic projective scheme \( X/k \) and a coherent sheaves \( \mathcal{E}_1, \mathcal{E}_2 \) on \( X \), under which conditions does there exist a differential operator of order \( N \), \( D : \mathcal{E}_1 \to \mathcal{E}_2 \).

We in particular treat the case where \( X = \mathbb{P}^n_k, n > 1 \) is projective \( n \)-space over \( k \). One result (see Proposition 3.1), is that for each \( n \in \mathbb{Z} \) and each locally free sheaf \( \mathcal{E} \) on \( \mathbb{P}^n_k \) there exists for \( N >> 0 \) a global linear partial differential operator \( D : \mathcal{E} \to \mathcal{E}(n) \) of order \( N \). This is in contrast to the case of \( \mathcal{O}_X \)-linear homomorphisms \( \phi : \mathcal{E} \to \mathcal{E}(n) \) where for \( n << 0 \) thanks to Serre duality no global homomorphisms exist. We have in addition that for fixed \( N > 0 \) and given nontorsion coherent \( \mathcal{O}_X \)-modules \( \mathcal{F}_1, \mathcal{F}_2 \) there exists a global section \( D \in \Gamma(X, DO^N(\mathcal{F}_1, \mathcal{F}_2(n))) \) for \( n >> 0 \) which does not lie in \( \Gamma(X, DO^{N-1}(\mathcal{F}_1, \mathcal{F}_2(n))) \) (see Lemma 3.1). This is a consequence of Serre vanishing and the fact, that the inclusion of coherent sheaves
We also calculate for fixed locally free $\mathcal{E}_1, \mathcal{E}_2$ the number of global sections $\Gamma(\mathbb{P}^n_k, DO^N(\mathcal{E}_1, \mathcal{E}_2))$ as $N$ tends to infinity.

Next, we study the behavior of differential operators $D : \mathcal{E} \longrightarrow \mathcal{E}$ on arbitrary smooth projective $X/k$, with respect to the Harder-Narhasimhan filtration (HN-filtration for short) of $\mathcal{E}$. Our main result is \textbf{Proposition 3.9}, which says, that if $X$ is not uniruled, then $D$ respects the Harder-Narhasimhan-filtration of $\mathcal{E}$. If moreover, $HN^*(\Omega^1(X/k))$ is the HN-filtration of the cotangent sheaf and the minimal slope $\mu_{\text{min}}(\Omega^1(X/k)) > 0$, then $D$ respects the HN-filtration and the differential operator $gr^1HN(\mathcal{E}) \longrightarrow gr^1HN(\mathcal{E})$ is $\mathcal{O}_X$-linear (see \textbf{Proposition 3.9}).

Finally we study the question of global elliptic differential operators on locally free sheaves. We show that they exist on abelian varieties but on smooth projective varieties with $\mu_{\text{min}}(\Omega^1(X/k)) > 0$ they do not exist at all. In $\textbf{[5]}$, an algebraic index theorem has been proved, and the result proved here puts an end to speculations that there could exist an algebraic index theorem equivalent to the Hirzebruch-Riemann-Roch theorem.

### 3 Global differential operators on coherent sheaves

If $X/k$ is a complete algebraic scheme, or more generally a complete algebraic space over a field $k$ and $\mathcal{F}$ is a coherent sheaf on $X$, the question arises, are there global differential operators $\mathcal{F} \longrightarrow \mathcal{F}$? To get a feeling for this subtle question, we prove a few lemmas and give some examples.

\textbf{Lemma 3.1} Let $X/k$ be a projective scheme, $\mathcal{F}_1, \mathcal{F}_2$ be coherent nontorsion sheaves on $X$ and $\mathcal{O}_X(1)$ be an ample invertible sheaf on $X$. Then, for each $N \in \mathbb{N}$ there is $M = M(\mathcal{F}_1, \mathcal{F}_2, N)$ such that for all $m \geq M$ there is a differential operator of order $N$, $D : \mathcal{F}_1 \longrightarrow \mathcal{F}_2(m)$. In particular for $m >> 0$ there always exist non-$\mathcal{O}_X$-linear operators.

\textbf{Proof}: By $\textbf{[1]}$[section 3.7, Proposition 3.44, pp. 26-29], for any $N \in \mathbb{N}$, the sheaf

$$DO^N(\mathcal{F}_1, \mathcal{F}_2) := \mathcal{H}om_X(\mathcal{J}^N(\mathcal{F}_1/k), \mathcal{F}_2)$$

is stricly larger then the sheaf $DO^{N-1}_X(\mathcal{F}_1, \mathcal{F}_2)$. There is $M = M(\mathcal{F}_1, \mathcal{F}_2)$ such that

$$DO^N(\mathcal{F}_1, \mathcal{F}_2)(m) = DO^N(\mathcal{F}_1, \mathcal{F}_2(m)) \quad \text{and} \quad DO^{N-1}_X(\mathcal{F}_1, \mathcal{F}_2)(m) = DO^{N-1}(\mathcal{F}_1, \mathcal{F}_2(m))$$

and the quotient sheaf

$$Q(m, N) = DO^N(\mathcal{F}_1, \mathcal{F}_2(m))/(DO^{N-1}(\mathcal{F}_1, \mathcal{F}_2(m)))$$

are nonzero, globally generated for $m \geq M$ and have no higher cohomology. The sheaf $Q(m, N)$ is for $\mathcal{F}_i, i = 1, 2$ locally free and $X/k$ smooth the sheaf

$$Q(m, N) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}(m)) \otimes S^N\mathcal{T}^1(X/k).$$
Then we get an exact sequence
\[ 0 \rightarrow H^0(X, DO^{N-1}(\mathcal{F}_1, \mathcal{F}_2(m))) \rightarrow H^0(X, DO^N(\mathcal{F}_1, \mathcal{F}_2(m))) \rightarrow H^0(X, Q(m, N)) \rightarrow 0. \]

All cohomology groups are nonzero (because the sheaves are nonzero and globally generated) and thus there exists a global differential operator \( D_N \in H^0(X, DO^N(\mathcal{F}, \mathcal{F}(m))) \) not contained in \( H^0(X, DO^{N-1}(\mathcal{F}, \mathcal{F}(m))) \). Putting \( N = 0 \) and observing that \( DO^0(\mathcal{F}, \mathcal{F}(m)) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}(m)) \) we get the last claim. ■

**Lemma 3.2** For each projective morphism \( f : X \rightarrow S \) there exists a locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \) and a global differential operator \( \mathcal{E} \rightarrow \mathcal{E} \) of arbitrary high order relative to \( S \).

**Proof:** Let \( \mathcal{O}_X(1) \) be an \( f \)-ample invertible sheaf. By the previous proposition, there exists differential operators \( D_{12} : \mathcal{O}_X(m) \rightarrow \mathcal{O}_X(n + m) \) for \( n \gg 0 \) of arbitrary high order. Put \( \mathcal{E} := \mathcal{O}_X(m) \oplus \mathcal{O}_X(n + m) \) and define \( D \) to be given by the matrix of differential operators
\[
D = \begin{pmatrix}
\text{Id}_{\mathcal{O}_X(m)} & D_{12} \\
0 & \text{Id}_{\mathcal{O}_X(m+n)}
\end{pmatrix}
\]

Then \( D \) is a differential operator on \( \mathcal{E} \). The Harder-Narasimhan filtration on \( \mathcal{E} \) is the filtration \( \mathcal{H}N^i\mathcal{E} \) with \( \mathcal{H}N^1(\mathcal{E}) = \mathcal{O}_X(n + m) \) and \( gr^2\mathcal{H}N^i(\mathcal{E}) = \mathcal{O}_X(m) \) and the induced differential operators on the graded pieces are \( \mathcal{O}_X \)-linear. By taking direct sums of \( \mathcal{O}_X(n_i) \), \( n_i \) appropriately choosen, we can manage to get the rank of \( \mathcal{E} \) arbitrarily high. ■

We have the following

**Proposition 3.3** Let \( q : X \rightarrow S \) be a proper morphism of noetherian schemes and \( D : \mathcal{E}_1 \rightarrow \mathcal{E}_2 \) be a differential operator of coherent sheaves relative to \( S \) of some order \( \leq N \). Then, there are \( \mathcal{O}_S \)-linear maps
\[ R^iD : R^iq_*\mathcal{E}_1 \rightarrow R^iq_*\mathcal{E}_2. \]

**Proof:** There is the standard factorization of \( D \) as
\[
\epsilon_1 \xrightarrow{d_{\epsilon_1/S}^N} \mathcal{J}^N(\epsilon_1/S) \xrightarrow{\overline{D}} \mathcal{E}_2.
\]

The first map \( d_{\epsilon_1/S}^N \) is \( \mathcal{O}_X \)-linear if the \( \mathcal{O}_X \)-bimodule \( \mathcal{J}^N(X/S) \) is considered as a coherent \( \mathcal{O}_X \)-module via the second \( \mathcal{O}_X \)-module structure. For a detailed discussion of this, see [1][section 3.5, pp. 16-17, Lemma 3.27]. Hence, we get an \( \mathcal{O}_S \)-linear map
\[
R^iq_*\mathcal{E}_1 \xrightarrow{R^iq_*d_{\epsilon_1/S}^N} R^iq_*(\mathcal{J}^N(\epsilon_1/S)^{(2)}).
\]

Regarding \( \mathcal{J}^N(\epsilon_1/S) \) with its first \( \mathcal{O}_X \)-module-structure, we get an \( \mathcal{O}_S \)-linear map
\[ R^iq_*\overline{D} : R^iq_*(\mathcal{J}^N(\epsilon_1/S)^{(1)}) \rightarrow R^iq_*\mathcal{E}_2. \]
Given a differential operator $D$ we may assume that $D$ be an exact sequence, where

$$j^N(E_1/S) = p_1^*(O_S \otimes O_S E_1) = p_2^*(O_X \otimes O_S E_1),$$

where $p_1, p_2 : J^N(X/S) \to X$ are the two projection morphisms, which are morphisms of $S$-schemes and where $J^N(X/S) = \text{Spec}_X J^N(X/S)$. Since $q \circ p_1 = q \circ p_2$ and $p_1$ and $p_2$ are affine, we have

$$R^i q_* j^N(E_1/S) = R^i (q \circ p_2)_* O_X \otimes O_S E_1 = R^i (q \circ p_1)_* O_X \otimes O_S E_1 = R^i q_* j^N(E_1/S).$$

Thus, we can compose $R^i q_*, \tilde{D}$ with the map $R^i q_* \tilde{D}$ to get the required $O_S$-linear map $R^i q_* D : R^i q_* E_1 \to R^i q_* E_2$.

### 3.1 Extensions of differential operators

The basic question is the following: If

$$0 \to E_1 \to E \to E_2 \to 0$$

is an exact sequence of coherent sheaves on an $S$-scheme $X/S$ and $D_1 : E_1 \to E_1$ and $D_2 : E_2 \to E_2$ are differential operators on $E_1, E_2$, respectively, relative to $S$, can one find a differential operator $D : E \to E$ on $E$ fitting the exact sequence. We want to show that this is not always the case.

**Proposition 3.4** Let $X/S$ be an integral scheme of finite type and

$$0 \to E_1 \to E \to T \to 0$$

be an exact sequence, where $E_1, E$ are locally free and $T$ is a torsion sheaf. Given a differential operator $D_1 : E_1 \to E_1$, there is at most one differential operator $D : E \to E$, extending $D_1$.

**Proof:** Suppose, $D_1$ is of order $N$ and there are two differential operators $D, D'$ of order $\leq N'$ with $N' \geq N$ on $E$ extending $D_1$. Without loss of generality, we may assume that $N = N'$. Considering the difference $D - D'$, we may assume that $D_1 = 0$. We look for an $O_X$-linear homomorphism

$$\tilde{D} : J^N(E/k) \to E.$$

By assumption, $\tilde{D}$ restricted to $J^N(E_1/k)' \subset J^N(E/k)$ is the zero homomorphism. But, by the right exactness of the jet module functor, the quotient module is the jet-module $J^N(E/E_1/S)$. By [1][section 3.5, Lemma 3.30, p.18], this is a torsion sheaf, since $E/E_1 \cong T$ is so. Thus $\tilde{D}$ factors through $J^N(E/k)/J^N(E_1/k)' \to E$. But the first sheaf is a torsion $O_X$-module so $\tilde{D}$ must be the zero homomorphism.

**Example 3.5** Let $(X, O_X(H))$ be a polarized projective scheme with $O_X(H)$ very ample and $H = \text{div}(s), s \in \Gamma(X, O_X(H))$ be a smooth section. Let
$n \in \mathbb{N}$ be chosen such that there exists a global differential operator $D_{12} : \mathcal{O}_X \rightarrow \mathcal{O}_X(nH)$. Consider the extension of locally free sheaves

$$0 \rightarrow \mathcal{O}_X \oplus \mathcal{O}_X(nH) \rightarrow \mathcal{O}_X(H) \oplus \mathcal{O}_X((n+1)H) \rightarrow \mathcal{O}_H(H) \oplus \mathcal{O}_H((n+1)H) \rightarrow 0.$$ 

The global $\mathcal{O}_X$-(\mathcal{O}_H-linear) endomorphisms of the last sheaf contain the endomorphisms in diagonal form and thus

$$h^0(H, \text{Hom}_H(\mathcal{O}_H(H) \oplus \mathcal{O}_H((n+1)H), \mathcal{O}_H(H) \oplus \mathcal{O}_H((n+1)H))) \geq 2.$$ 

By the previous proposition, the differential operator $D : \mathcal{O}_X \oplus \mathcal{O}_X(nH) \rightarrow \mathcal{O}_X \oplus \mathcal{O}_X(nH)$ constructed in \textbf{Lemma 3.2} has at most one extension to an operator $\mathcal{E}$ on $\mathcal{O}_X(H) \oplus \mathcal{O}_X((n+1)H)$ so not every pair $(D, \phi)$ where $\phi$ is an $\mathcal{O}_H$-linear map on the right hand side extends. □

### 3.2 Differential operators and semistable sheaves

Let $X/k$ be a projective algebraic scheme of dimension $n$ and $\mathcal{O}_X(H) = \mathcal{O}_X(1)$ be an ample invertible sheaf on $X$. Let $\mathcal{E}$ be a $\mu$-semi-stable sheaf on $X$ with respect to the polarization $\mathcal{O}_X(1)$. We want to investigate the question under which circumstances there exists a global differential operator $\mathcal{E} \rightarrow \mathcal{E}$. We first give a criterion, when the answer is always negative.

**Proposition 3.6** Let $X/k, \mathcal{E}, \mathcal{O}_X(H)$ be as above and suppose, that for some $m_0 \in \mathbb{N}$, there are effective Cartier divisors $D_i, i = 1, \ldots, N$ with $D_i \cdot H^{n-1} > 0$ and a homomorphism with torsion kernel

$$\bigoplus_{i=1}^{M[m]} \mathcal{O}_X(D_i) \rightarrow \Omega^{(1)}(X/k) \otimes^m \forall m \geq m_0$$

Then, for every torsion free coherent semistable sheaf $\mathcal{E}$, every global differential operator $D : \mathcal{E} \rightarrow \mathcal{E}$ has order $< m_0$. In particular if $m_0 = 1$, then every differential operator on $\mathcal{E}$ is $\mathcal{O}_X$-linear. If $\mathcal{E}$ is arbitrary torsion free, then each $D$ respects the $HN$-filtration and the differential operators on the graded pieces are $\mathcal{O}_X$-linear.

**Proof:** Let $\mathcal{E}$ be a torsion free semistable sheaf on $X$ and $D : \mathcal{E} \rightarrow \mathcal{E}$ be a differential operator corresponding to an $\mathcal{O}_X$-linear map: $\tilde{D} : J^N(\mathcal{E}/k) \rightarrow \mathcal{E}$ with $N$ minimal with $N > m_0$. From the jet bundle exact sequence $j_N^X(\mathcal{E}_1)$, we get a homomorphism of coherent sheaves

$$\bigoplus_{i=1}^{N} \mathcal{O}_X(D_i) \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \Omega^{(1)}(X/k) \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{E}.$$

If $f$ is not the zero homomorphism, there must be a direct summand $\mathcal{E}(D_i)$ such that the above homomorphism restricted to $\mathcal{E}(D_i)$ gives a nonzero homomorphism $\mathcal{E}(D_i) \rightarrow \mathcal{E}$. Then $\mathcal{E}(D_i)$ is semistable and

$$\mu(\mathcal{E}(D_i)) = \mu(\mathcal{E}) + D_i \cdot H^{n-1} > \mu(\mathcal{E}).$$
By [chapter 1, Proposition 1.2.7, p.11], this homomorphism must be zero. Thus, \( D \) restricted to \( \Omega^{(1)}(X/k)^{⊗^N} \otimes \mathcal{E} \) is zero. By the same standard exact sequences for the jet modules \( j^{N}_{X/S}(\mathcal{E}) \), \( \overline{D} \) factors over \( \mathcal{J}^{N-1}(\mathcal{E}/k) \). This is in contradiction with the assumed minimality of \( N \). Thus we must have had \( N \leq m_0 \).

To the last point, if \( m_0 = 1 \) and \( \mathcal{E} \) is arbitrary torsion free, let \( D : \mathcal{E} \rightarrow \mathcal{E} \) and the Harder Narhasimhan filtration \( HN^\bullet(\mathcal{E}) \) be given. Let \( j \) be minimal such that \( D(HN^1(\mathcal{E})) \subset HN^j(\mathcal{E}) \) so we get \( \overline{D} : gr^jHN(\mathcal{E}) \rightarrow gr^jHN(\mathcal{E}) \), since by [section 3.6 Lemma 3.43, p.24], differential operators restricted to subsheaves and quotient sheaves are again differential operators). Arguing as above, we see that \( j \) must be equal to 1 and by the first part of the proposition, \( \overline{D} \) must be \( \mathcal{O}_X \)-linear. So we know that \( D(HN^1(\mathcal{E})) \subset HN^1(\mathcal{E}) \).

We get a differential operator \( D' : \mathcal{E}/HN^1(\mathcal{E}) \rightarrow \mathcal{E}/HN^1(\mathcal{E}) \) and we can continue by induction on the length of the \( HN \)-filtration, the start of the induction given by the first part of the proposition.

**Proposition 3.7** Let \( X/k \) be a smooth projective algebraic scheme, \( \mathcal{O}_X(H) \) be a very ample invertible sheaf on \( X \) and \( \mathcal{E} \) be a coherent sheaf on \( X \). Suppose, \( \Omega^{(1)}(X/k) \) is globally generated, e.g., \( X \) is an abelian variety. Then each differential operator \( D : \mathcal{E} \rightarrow \mathcal{E} \) respects the Harder-Narasimhan-Filtration \( HS^i(\mathcal{E}) \).

**Proof:** Let \( D : \mathcal{E} \rightarrow \mathcal{E} \) and the Harder-Narhasimhan filtration \( HN^\bullet(\mathcal{E}) \) be given. Let \( j \) be minimal such that \( D(HN^1(\mathcal{E})) \subset HN^j(\mathcal{E}) \). Then, we get a nonzero differential operator \( \overline{D} : HN^1(\mathcal{E}) \rightarrow gr^jHN(\mathcal{E}) \). If \( \Omega^{(1)}(X/k) \) is globally generated, so is each symmetric power. Let \( N \in \mathbb{N} \) be minimal such that \( \overline{D} \) factors over \( \mathcal{J}^N(HN^1(\mathcal{E})/k) \). We get homomorphisms of \( \mathcal{O}_X \)-modules

\[
(\bigoplus \mathcal{O}_X) \otimes HN^1 \mathcal{E} \rightarrow \Omega^{(1)}(X/k)^{⊗^N} \otimes \mathcal{O}_X \mathcal{E} \rightarrow gr^jHN(\mathcal{E}),
\]

Restricting to each single direct summand \( \mathcal{O}_X \otimes HN^1(\mathcal{E}) \cong HN^1(\mathcal{E}) \rightarrow gr^jHN(\mathcal{E}) \) the standard argument shows, that this map must be zero unless \( j = 1 \). Since this homomorphism is then zero for each direct summand, so is the homomorphism

\[
\Omega^{(1)}(X/k)^{⊗^N} \otimes \mathcal{O}_X HN^1(\mathcal{E}) \rightarrow gr^jHN(\mathcal{E})
\]

But then \( \overline{D} \) factors through \( \mathcal{J}^{N-1}(HN^1(\mathcal{E})/k) \), a contradiction. Thus, \( j = 1 \), we take the quotient differential operator \( D_1 : \mathcal{E}/HN^1(\mathcal{E}) \rightarrow \mathcal{E}/HN^1(\mathcal{E}) \) and go on by induction.

**Remark 3.8** The same argument shows that it suffices to assume that \( \Omega^{(1)}(X/k) \) is almost globally generated, i.e. that there is a homomorphism \( \bigoplus \mathcal{O}_X \rightarrow \Omega^{(1)}(X/k) \), with torsion cokernel.

**Proposition 3.9** Let \( (X, \mathcal{O}_X(1)) \) be a smooth polarized projective variety over a field \( k \) of characteristic zero and suppose that \( \mu_{min}(\Omega^{(1)}(X/k)) \geq 0 \). If \( \mathcal{E} \) is an arbitrary torsion free sheaf on \( X \), then each differential operator \( D : \mathcal{E} \rightarrow \mathcal{E} \) respects the \( HN \)-filtration, and if \( \mu_{min}(\Omega^{(1)}(X/k)) > 0 \), the differential operator \( \overline{D}_1 : gr^iHN(\mathcal{E}) \rightarrow gr^iHN(\mathcal{E}) \) is \( \mathcal{O}_X \)-linear.
Proof: The argument is similar to the proof of the previous proposition. Observe first, that if \( \mu_{\text{min}}(\Omega^{(1)}(X/k)) > 0 \) then also \( \mu_{\text{min}}(\Omega^{(1)}(X/k)^{\otimes m}) > 0 \). Here we need that the characteristic of \( k \) is zero, because we need that the symmetric tensor product of semi-stable sheaves is again semistable. Now, let \( E \) be torsion free and \( D : E \to E \) be given. Let again \( j \) be minimal such that \( D(HN^1(E)) \subset HN^j(E) \) so we get \( \overline{D} : HN^1(E) \to gr^jHN(E) \). Let \( N \in \mathbb{N} \) be minimal such that \( \overline{D} \) factors over \( \mathcal{J}^N(HN^1(E)/k) \). We then get a nonzero homomorphism

\[
\Omega^{(1)}(X/k)^{\otimes N} \otimes_{\mathcal{O}_X} HN^1(E) \to gr^jHN(E).
\]

We show by induction that this homomorphism is zero if \( j > 1 \) and \( N > 1 \). Let \( HN^*(\Omega^{(1)}(X/k)^{\otimes N}) \) be the Harder-Narasimhan-filtration. Let \( \Omega_{1,N} = HN^1 \) be the maximal destabilizing subsheaf. We get by restriction a homomorphism

\[
\Omega_{1,N} \otimes HN^1(E) \to gr^jHN^*(E).
\]

\( \Omega_{1,N} \otimes HN^1(E) \) is semistable and

\[
\mu(\Omega_{1,N} \otimes HN^1(E)) = \mu(\Omega_{1,N}) + \mu(HN^1(E))
\]

\[
> \mu_{\text{min}}(\Omega^{(1)}(X/k)^{\otimes N}) + \mu(HN^1(E)) > \mu_j(E),
\]

so by \[\mathbb{B}\][chapter 1, Proposition 1.2.7, p.11], this homomorphism is zero. At this point, we also need that the characteristic is zero since we need the semi-stability of the tensor product. Suppose that for some \( n \in \mathbb{N} \) the restricted homomorphism

\[
HN^n(\Omega^{(1)}(X/k)^{\otimes N}) \otimes HN^1(E) \to gr^jHN(E)
\]

is nonzero. We then get a homomorphism

\[
\Omega_{n+1,N} \otimes HN^1(E) := gr^{n+1}HN^*(\Omega^{(1)}(X/k)^{\otimes N}) \otimes HN^1(E) \to gr^jHN(E)
\]

and the same as the previous argument shows, since \( \mu(\Omega_{n+1,N}) > \mu_{\text{min}} > 0 \) that this map must be zero if \( j > 1 \). Thus, the entire map \( \Omega^{(1)}(X/k)^{\otimes N} \otimes HN^1(E) \to gr^jHN(E) \) is zero. Since \( N \) was chosen minimal, we get by the \( N^{\text{th}} \) exact sequence of jet bundles a contradiction. Thus \( j = 1 \). We get a quotient differential operator \( D_1 : E/HN^1(E) \to E/HN^1(E) \) and we argue by induction on the length of the HN-filtration.

To the last point, if \( E \) is semistable and a differential operator \( D : E \to E \) is given, choose again \( N \) minimal such that \( D \) factors over \( \mathcal{J}^N(E/k) \) and show, as in the first part of this proof, that the homomorphism

\[
\Omega^{(1)}(X/k)^{\otimes N} \otimes E \to E
\]

is the zero map if \( N > 1 \).

\[\blacksquare\]

Remark 3.10 By \[\mathbb{A}\][Lecture III, 2.14 Theorem, p. 67], the condition that \( \mu_{\text{min}}(\Omega^{(1)}(X/k)) \geq 0 \) is equivalent to the uniruledness of \( X \).
3.3 Vector bundles with global differential operators on projective n-space

**Proposition 3.11** Let $X = \mathbb{P}^n_k$ be projective n-space with $n \geq 2$. Then for each pair of locally free sheaves $\mathcal{F}_i, i = 1, 2$, there exists an $N = N(\mathcal{F}_i)$ such that for each $N \geq N(\mathcal{F}_i)$ there exist operators $D : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ of order $\geq N$.

**Proof:** The tangent sheaf $\mathcal{T}^1(\mathbb{P}^n_k/k)$ is ample. For each $N \in \mathbb{N}$, we consider the exact sequence

$$j^N(\mathcal{F}_1) : 0 \rightarrow \Omega^{(1)}(X/k)^{\otimes N} \otimes_{\mathcal{O}_X} \mathcal{F}_1 \rightarrow \mathcal{J}^N(\mathcal{F}_1/k) \rightarrow \mathcal{J}^{N-1}(\mathcal{F}_1/k) \rightarrow 0.$$

Taking duals and tensoring with $\mathcal{F}_2$ we get exact sequences

$$do^N(\mathcal{F}_1, \mathcal{F}_2) : 0 \rightarrow DO^{N-1}(\mathcal{F}_1, \mathcal{F}_2) \rightarrow DO^N(\mathcal{F}_1, \mathcal{F}_2) \rightarrow \mathcal{T}^1(\mathbb{P}^n_k/k)^{\otimes N} \otimes Hom_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2) \rightarrow 0.$$

We have $DO^0(\mathcal{F}_1, \mathcal{F}_2) \cong Hom_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)$. The claim is that for $N >> 0$

$$H^0(\mathbb{P}^n_k, DO^N(\mathcal{F}_1, \mathcal{F}_2)) \supseteq H^0(\mathbb{P}^n_k, DO^{N-1}(\mathcal{F}_1, \mathcal{F}_2)) \supseteq H^0(\mathbb{P}^n_k, Hom_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)).$$

By [2][II, chapter 6.1.B, Theorem 6.1.10, p.11] for each coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n_k$ and for all $m \geq m(\mathcal{E})$ we have $H^1(\mathbb{P}^n_k, S^m \mathcal{T}^1(\mathbb{P}^n_k) \otimes \mathcal{E}) = 0 \forall i > 0$ and the sheaf $S^m \mathcal{T}^1(\mathbb{P}^n_k) \otimes \mathcal{E}$ is globally generated. Putting $\mathcal{E} = Hom_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)$ and $i = 1$ we get

$$H^1(\mathbb{P}^n_k, S^m \mathcal{T}^1(\mathbb{P}^n_k) \otimes Hom_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)) = 0 \forall m \geq m(\mathcal{E}).$$

Writing out the long exact cohomology sequences for the exact sequence $do^N(\mathcal{F}_1, \mathcal{F}_2), N \geq m(\mathcal{E})$, we get

$$0 \rightarrow H^0(\mathbb{P}^n_k, DO^{N-1}(\mathcal{F}_1, \mathcal{F}_2)) \rightarrow H^0(\mathbb{P}^n_k, DO^N(\mathcal{F}_1, \mathcal{F}_2)) \rightarrow H^0(\mathbb{P}^n_k, S^N \mathcal{T}^1(\mathbb{P}^n_k/k) \otimes Hom_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)) \rightarrow \cdots \rightarrow \cdots \rightarrow H^1(\mathbb{P}^n_k, DO^{N-1}(\mathcal{F}_1, \mathcal{F}_2)) \rightarrow H^1(\mathbb{P}^n_k, DO^N(\mathcal{F}_1, \mathcal{F}_2)) \rightarrow H^1(\mathbb{P}^n_k, S^N \mathcal{T}^1(\mathbb{P}^n_k/k) \otimes Hom_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)) = 0.$$
Remark 3.12 By global generation of the sheaf \( DO^N(F_1, F_2) \) after a sufficiently high ample twist, on gets immediately differential operators \( E \rightarrow E(n) \) for \( n >> 0 \). This result implies, that for each \( n > 0 \) there exist differential operators of arbitrary high degree \( E \rightarrow E(-n) \).

Using the ampleness of \( T_1^{1}(\mathbb{P}^n/k) \) we now want to estimate the growth of \( DO^N(F_1, F_2) \) as \( N \) tends to infinity.

Let \( E_1 \) and \( E_2 \) be locally free sheaves on \( \mathbb{P}^n \), \( n \geq 2 \) be given. By the ampleness of \( T_1^{1}(\mathbb{P}^n/k) \), we know that for some \( N \in \mathbb{N}_0 \) and all \( d \geq N \), we have that \( H^0(\mathbb{P}^n, S^d T_1^{1}(\mathbb{P}^n/k) \otimes \text{Hom}_\mathcal{O}(E_1, E_2)) \) is globally generated and all higher cohomology groups vanish. As we have shown, there is \( M \in \mathbb{N} \), that only depends on \( E_1, E_2 \) such that the sequence

\[
0 \rightarrow H^0(\mathbb{P}^n, DO^{d-1}(E_1, E_2)) \rightarrow H^0(\mathbb{P}^n, DO^d(E_1, E_2)) \rightarrow H^0(\mathbb{P}^n, S^d T_1^{1} \otimes \text{Hom}_\mathcal{O}(E_1, E_2)) \rightarrow 0
\]

is exact for all \( d \geq M \). For \( d >> M \) we can thus write for \( h^0(\mathbb{P}^n, DO^d(E_1, E_2)) \)

\[
h^0(\mathbb{P}^n, DO^d(E_1, E_2)) = h^0(\mathbb{P}^n, DO^M(F_1, F_2)) + \sum_{M+1 \leq k \leq d} h^0(\mathbb{P}^n, S^k T_1^{1}(\mathbb{P}^n/k) \otimes \text{Hom}_\mathcal{O}(E_1, E_2)) = h^0(\mathbb{P}^n, DO^M(F_1, F_2)) + \sum_{M+1 \leq k \leq d} \chi(\mathbb{P}^n, S^k T_1^{1}(\mathbb{P}^n/k) \otimes \text{Hom}_\mathcal{O}(E_1, E_2)).
\]

If we tensor the symmetric power of the Euler sequence \((\mathbb{P}^n = \mathbb{P}(V))\)

\[
0 \rightarrow S^{k-1}V \otimes \mathcal{O}_\mathbb{P}(-1) \rightarrow S^k V \otimes \mathcal{O}_\mathbb{P} \rightarrow S^k T_1(\mathbb{P}^n/k)(-k) \rightarrow 0, \quad \text{or}
\]

\[
0 \rightarrow S^{k-1}V \otimes \mathcal{O}(k-1) \rightarrow S^k V \otimes \mathcal{O}(k) \rightarrow S^k T_1(\mathbb{P}^n/k) \rightarrow 0
\]

(see \[3\] [chapter 1.4, p. 19]) with \( \text{Hom}_\mathcal{O}(E_1, E_2) \) and use the additivity of the Euler characteristic, we get

\[
\chi(\mathbb{P}^n, S^k T_1^{1}(\mathbb{P}^n/k) \otimes \mathcal{O}_\mathbb{P} \text{Hom}_\mathcal{O}(E_1, E_2)) = \chi(\mathbb{P}^n, S^k V \otimes \mathcal{O}(k) \otimes \text{Hom}_\mathcal{O}(E_1, E_2)) - \chi(\mathbb{P}^n, S^{k-1}V \otimes \mathcal{O}(k-1) \otimes \text{Hom}_\mathcal{O}(E_1, E_2)) = \binom{n+k}{k} \cdot \chi(\mathbb{P}^n, \text{Hom}_\mathcal{O}(E_1, E_2(k))) - \binom{n+k-1}{k-1} \cdot \chi(\mathbb{P}^n, \text{Hom}_\mathcal{O}(E_1, E_2(k-1))).
\]

Summing over \( M+1 \leq k \leq d \) we get for \( d >> M \)

\[
h^0(\mathbb{P}^n, DO^d(E_1, E_2)) = h^0(\mathbb{P}^n, DO^M(E_1, E_2)) + \sum_{M+1 \leq k \leq d} \binom{n+k}{k} \cdot \chi(\mathbb{P}^n, \text{Hom}_\mathcal{O}(E_1, E_2(k))) - \binom{n+k-1}{k-1} \cdot \chi(\mathbb{P}^n, \text{Hom}_\mathcal{O}(E_1, E_2(k-1))) = h^0(\mathbb{P}^n, DO^M(E_1, E_2)) + \binom{n+d}{d} \cdot \chi(\mathbb{P}^n, \text{Hom}_\mathcal{O}(E_1, E_2(d))) - \binom{n+M}{M} \cdot \chi(\mathbb{P}^n, \text{Hom}_\mathcal{O}(E_1, E_2(M))).
\]

We thus have proved the following
Proposition 3.13 Let $\mathcal{E}_1, \mathcal{E}_2$ be locally free sheaves on $\mathbb{P}^n_k, n \geq 2$. Let $F(N) := h^0(\mathbb{P}^n_k, DO^N(\mathcal{E}_1, \mathcal{E}_2))$. There is a polynomial of degree $2n$, $P(N) = P(\mathcal{E}_1, \mathcal{E}_2)(N)$ such that for $N >> 0$

$$F(N) = P(N).$$

The polynomial $P(N)$ is explicitly given by

$$P(N) := \binom{n + N}{N} \cdot \chi(\mathbb{P}^n_k, \text{Hom}_O(\mathcal{E}_1, \mathcal{E}_2)(N)) + h^0(\mathbb{P}^n_k, DO^M(\mathcal{E}_1, \mathcal{E}_2)) - n \cdot \chi(\mathbb{P}^n_k, \text{Hom}_O(\mathcal{E}_1, \mathcal{E}_2)(M)),$$

where $M = M(\mathcal{E}_1, \mathcal{E}_2)$ is a fixed natural number.

Proof:

For the special case where $\mathcal{E}_1 = \mathcal{O}(m_1)$ and $\mathcal{E}_2 = \mathcal{O}(m_2)$, the formula then reads,

$$h^0(\mathbb{P}^n_k, DO^d(\mathcal{O}(m_1), \mathcal{O}(m_2))) = h^0(\mathbb{P}^n_k, DO^M(\mathcal{O}(m_1), \mathcal{O}(m_2))) + \binom{n + d}{d} \cdot \chi(\mathbb{P}^n_k, \mathcal{O}(m_2 - m_1 + d)) - \binom{n + M}{M} \cdot \chi(\mathbb{P}^n_k, \mathcal{O}(m_2 - m_1 + M)).$$

3.4 Elliptic operators in algebraic geometry

Definition 3.14 (Definition of the symbol of a differential operator)

Let $X/k$ be a smooth variety and $D : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ be a differential operator of order $N \in \mathbb{N}$. Consider the standard exact sequence

$$\text{do}^N(\mathcal{E}_1, \mathcal{E}_2) : 0 \rightarrow DO^{N-1}_{(X/k)}(\mathcal{E}_1, \mathcal{E}_2) \rightarrow DO^N_{(X/k)}(\mathcal{E}_1, \mathcal{E}_2) \rightarrow S^N T^1(X/k).$$

The element $\sigma^N(D) \in \text{Hom}_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{E}_2) \otimes_{\mathcal{O}_X} S^N T^1(X/k)$ is called the symbol of the differential operator $D$.

Recall, that if $\mathcal{F}$ is a coherent sheaf on $X$ and $F := \text{Spec}_X \text{Sym}^\bullet(\mathcal{F})$ with projection $\pi : F \rightarrow X$ is the associated affine bundle, for each $N \in \mathbb{N}$, there is a canonical homomorphism $c_{N, \mathcal{F}} : \pi^* S^N \mathcal{F} \rightarrow \mathcal{O}_F$. Zariski locally, if $X = \text{Spec} A$ and $\mathcal{F}$ corresponds to the $A$-module $M$, we have

$$\Gamma(\text{Spec} A, \mathcal{O}_F) = \text{Sym}^\bullet M \quad \text{and} \quad \pi^* S^N \mathcal{F} \big|_{\text{Spec} A} = S^N M \otimes_A \text{Sym}^\bullet M$$

and the required map is simply the tensor multiplication map $S^N M \otimes_A \text{Sym}^\bullet M \rightarrow \text{Sym}^\bullet M$.

In our case, $\mathcal{F} = T^1(X/k)$ and $T^1(X/k) := \text{Spec}_X \text{Sym}^\bullet T^1_{X/k}$ is the classical cotangent bundle of $X$ with projection $\pi_X : T^1(X/k) \rightarrow X$. Put $T'(X/S) := T^1(X/S) \setminus s_0(X)$ where $s_0 : X \hookrightarrow T^1(X/k)$ is the zero section with same projection $\pi_X : T'(X/k) \rightarrow X$.

We have the following classical


Definition 3.15 (Definition of an elliptic operator in algebraic geometry)

With notation as just introduced, let $X/k$ be a smooth scheme of finite type over the base field $k$, $\mathcal{E}_1, \mathcal{E}_2$ be locally free sheaves on $X$ and $D : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ be a differential operator of order $N$. The operator $D$ is called elliptic, if

$$c_{N, T^1} \circ \pi_X^*(\sigma^N(D)) : \pi_X^*\mathcal{E}_1 \rightarrow \pi_X^*\mathcal{E}_2$$

is an isomorphism of locally free sheaves on $T'(X/S)$.

This is just an adoption of the classical definition of an elliptic operator in differential geometry to the setting of locally free sheaves. Observe that $\mathcal{E}_1$ and $\mathcal{E}_2$ must then have the same rank.

Proposition 3.16 Let $X/k$ be a smooth complete variety such that $\mu_{\min}(\Omega^1(X/k)) > 0$ with respect to some polarization $H$. Then, there is no locally free sheaf $\mathcal{E}$ plus an elliptic differential operator $D : \mathcal{E} \rightarrow \mathcal{E}$ on $X$.

Proof: The proof is relatively straightforward. Let $\mathcal{E}_1$ be the maximal destabilizing subsheaf. By Proposition 3.9, $D$ respects the Harder Narasimhan-filtration and $D|_{\mathcal{E}_1}$ is $\mathcal{O}_X$-linear. The sheaves $\mathcal{E}_1$ and $\mathcal{E}/\mathcal{E}_1$ are torsion free and are locally free outside a codimension 2 subset of $X$. So there exists a Zariski-open Spec $A \subset X$ such that $\mathcal{E}$ and $\mathcal{E}_1$ and $\mathcal{E}/\mathcal{E}_1$ are free on $X$. Furthermore, by shrinking Spec $A$, we may assume that $T^1(X/k)$ is free on Spec $A$. Choose a trivialization

$$\mathcal{E}|_{\text{Spec } A} \cong A^{\oplus n} \cong A^{\oplus k} \oplus A^{\oplus l}$$

with

$$\mathcal{E}_1|_{\text{Spec } A} \cong A^{\oplus k}$$

and $\mathcal{E}/\mathcal{E}_1|_{\text{Spec } A} \cong A^{\oplus l}$.

The symbol $\sigma(D)$ is given by an $n \times n$-matrix of $N$th-order differential operators on Spec $A$. But the $(k \times k)$-submatrix that corresponds to $\sigma(D)|_{\mathcal{E}_1}$ is the zero matrix since $D$ restricted to $\mathcal{E}_1$ is $\mathcal{O}_X$-linear. By definition of ellipticity, the symbol $\sigma(D)$ can never be a fibrewise isomorphism outside the zero section.

Remark 3.17 This just says, that for "most" varieties, e.g., if $\Omega^1(X/k)$ is ample, elliptic operators on locally free sheaves do not exist. So there is no hope to prove the existence of an algebraic Atiyah-Singer-Index theorem that is equivalent to the Hirzebruch-Riemann-Roch theorem.

In order to give an example of an elliptic operator on a smooth complete variety, we prove

Proposition 3.18 Let $A/\mathbb{C}$ be an abelian variety, or, more generally a complex torus. Then, there exist elliptic operators $\mathcal{O}_A \rightarrow \mathcal{O}_A$ of arbitrary high order.

Proof: Let $A = \mathbb{C}^n/\Lambda$ with $\Lambda$ generated by the vectors $\lambda_j + i\mu_j, j = 1, \ldots, 2n$. Then, if $D$ is a differential operator on $\mathbb{C}^n$ with constant coefficients, then $D$ is obviously translation invariant, since $d^1 z_i = d^1(z_i + c)$ for all $c \in \mathbb{C}$ and thus $D$ descends to a differential operator $D_A : \mathcal{O}_A \rightarrow \mathcal{O}_A$.

Now, on $\mathbb{C}^n$ on can construct for arbitrary $N >> 0$ elliptic operators with constant coefficients.
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