ON THE NUMERICAL SOLUTION OF SOME NONLINEAR
STOCHASTIC DIFFERENTIAL EQUATIONS USING THE
SEMI-DISCRETE METHOD

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Abstract. In this paper we are interested in the numerical solution of stochastic differential equations with non-negative solutions. Our goal is to construct explicit numerical schemes that preserve positivity, even for super-linear stochastic differential equations. It is well known that the usual Euler scheme diverges on super-linear problems and the Tamed-Euler method does not preserve positivity. In that direction, we use the semi-discrete method that the first author has proposed in two previous papers. We propose a new numerical scheme for a class of stochastic differential equations which are super-linear with non-negative solutions. In this class of stochastic differential equations belongs the Heston 3/2-model that appears in financial mathematics.

1. Introduction

Throughout, let $T > 0$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a complete probability space, meaning that the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfies the usual conditions, i.e. is right continuous and $\mathcal{F}_0$ includes all $\mathbb{P}$–null sets. Let $W_{t,\omega}: [0, T] \times \Omega \to \mathbb{R}$ be a one dimensional Wiener process adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. Let $D \subset \mathbb{R}_+$ be an open set. Consider the following stochastic differential equation (SDE),

\begin{equation}
    x_t = x_0 + \int_0^t a(s, x_s)ds + \int_0^t b(s, x_s)dW_s, \quad t \in [0, T],
\end{equation}

where the coefficients $a, b: D \times \mathbb{R} \mapsto \mathbb{R}$ are measurable functions such that (1.1) has a unique strong solution and $x_0 > 0$, a.s. SDE (1.1) has non-autonomous coefficients, i.e. $a(t, x), b(t, x)$ depend explicitly on $t$.

To be more precise, we assume the existence of a predictable stochastic process $x: [0, T] \times \Omega \mapsto D$ such that \cite[Definition 2.1]{26}],

\{a(t, x_t)\} \in \mathcal{L}^1([0, T]; \mathbb{R}), \quad \{b(t, x_t)\} \in \mathcal{L}^2([0, T]; \mathbb{R})

and

\[ P \left[ x_t = x_0 + \int_0^t a(s, x_s)ds + \int_0^t b(s, x_s)dW_s \right] = 1, \quad \text{for every } t \in [0, T]. \]

The drift coefficient $a$ is the infinitesimal mean of the process $x_t$ and the diffusion coefficient $b$ is the infinitesimal variance of the process $x_t$. SDEs of the form (1.1) have rarely explicit solutions, thus numerical approximations are necessary for simulations of the paths $x_t(\omega)$, or for approximation of functionals of the form $\mathbb{E}F(x)$, where $F: \mathcal{C}([0, T], \mathbb{R}) \mapsto \mathbb{R}$ can be for example in the area of finance, the discounted payoff of European type derivative.

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We are interested in strong approximations (mean-square) of \[(1.1)\], in the case of super or sub linear drift and diffusion coefficients. This kind of numerical schemes have applications in many areas, such as simulating scenarios, filtering, visualizing stochastic dynamics (see for instance [21, Section 4] and references therein), have theoretical interest (they provide fundamental insight for weak-sense schemes) and generally do not involve simulations over long-time periods or of a significant number of trajectories.

We present some models that are not linear both in the drift and diffusion coefficient:

- The following linear drift model had been initially proposed for the dynamics of the inflation rate in ([7, Relation 50]) and has taken its name, CIR, by the initials of the authors in the aforementioned paper. It is used in the field of finance as a description of the stochastic volatility procedure in the Heston model [14], but also belongs to the fundamental family of SDEs that approximate Markov jump processes [9]. The CIR model is described by the following SDE,

\[
x_t = x_0 + \int_0^t \kappa (\lambda - x_s) ds + \int_0^t \sigma \sqrt{x_s} dW_s, \quad t \in [0,T],
\]

where \(x_0\) is independent of all \(\{W_t\}_{t \geq 0}\), \(x_0 > 0\) a.s. and the parameters \(\kappa, \lambda, \sigma\) are positive. Parameter \(\lambda\) is the level of the interest rate \(x_t\) where the drift is zero, meaning that when \(x_t\) is below \(\lambda\) the drift is positive, whereas in the other case is negative. As \(\lambda\) grows, the range of the positive drift becomes wider. Parameter \(\kappa\) defines the slope of the drift. The condition \(\kappa > 0\) is necessary for the stationarity of the process \(x_t\). When \(\kappa\) is negative, the main term of the slope, \(-\kappa\), is positive and given the diffusion \(\sigma \sqrt{x_t}\), the process \(x_t\) blows up. The condition \(\sigma^2 < 2\kappa\lambda\) implied by the Feller test [10, Case (ii), p.173] is necessary and sufficient for the process not to reach the boundary zero in finite time.

- The \(3/2\)-model [15] or the inverse square root process [1], that is used for modeling stochastic volatility,

\[
x_t = x_0 + \int_0^t (\alpha x_s - \beta x_s^2) ds + \int_0^t \sigma x_s^{3/2} dW_s, \quad t \in [0,T],
\]

where \(x_0\) is independent of \(\{W_t\}_{0 \leq t \leq T}\), \(x_0 > 0\) a.s. and \(\sigma \in \mathbb{R}\). The conditions \(\alpha > 0\) and \(\beta > 0\) are necessary and sufficient for the stationarity of the process \(x_t\) and such that zero and infinity is not attainable in finite time [1, Appendix A].

- The constant elasticity of variance model [6], which is used for pricing assets,

\[
x_t = x_0 + \int_0^t \mu x_s ds + \int_0^t \sigma x_s^\gamma dW_s, \quad t \in [0,T],
\]

where \(x_0\) is independent of \(\{W_t\}_{0 \leq t \leq T}\), \(x_0 > 0\), a.s., \(\mu \in \mathbb{R}, \sigma > 0\) and \(0 < \gamma \leq 1\). SDE (2.4) has a unique strong solution if and only if \(\gamma \in [1/2, 1]\) and takes values in \([0, \infty)\). The case \(\gamma = 1/2\) corresponds to CIR model (1.2), whereas \(\gamma = 1\) corresponds to a Brownian motion, i.e. the famous Black-Scholes model (11).

- Superlinear models, i.e. models of the form (1.1) where one of the coefficients \(a(\cdot), b(\cdot)\) is superlinear, i.e. when we have that

\[
a(x) \geq \frac{|x|^\beta}{C}, \quad b(x) \leq C|x|^\alpha, \quad \text{for every } |x| \geq C,
\]
or

\[(1.6)\quad b(x) \geq \frac{|x|^\beta}{C}, a(x) \leq C|x|^\alpha, \text{ for every } |x| \geq C,\]

where \(\beta > 1, \beta > \alpha \geq 0, C > 0\).

For some of the aforementioned problems there are methods of simulation ([5], [29]). However, if a full sample path of the SDE has to be simulated or the SDEs under study are a part of a bigger system of SDEs, then numerical schemes are in general more effective.

Problems like (1.2) and (1.3) are meant for non-negative values, since they represent rates or pricing values. Thus “good” numerical schemes preserve positivity([2], [22]). The explicit Euler scheme has not that property, since its increments are conditional Gaussian. For example, the transition probability of the Euler scheme in case of (1.2) reads as

\[p(y|x) = \frac{1}{\sqrt{2\pi \sigma^2 x^\Delta}} \exp\left\{- \frac{(y - (x + \kappa(\lambda - x)\Delta))^2}{2\sigma^2 x^\Delta}\right\}, \quad y \in \mathbb{R}, x > 0,\]

thus, even in the first step there is an event of negative values with positive probability.

We refer to [24], between other papers, that considers Euler type schemes, modifications of them to overcome the above drawback, and the importance of positivity. Thus, for the same problem, the truncated Euler scheme [8] has been proposed, as well as a modification of it, [16], where in a step the numerical scheme can leave \((0, \infty)\) but is forced to come back in the next steps.

One more drawback, that appears in case of superlinear problems (1.5) or (1.6), like (1.3), is that the moments of the scheme may explode [20, Theorem 1]. A method that overcomes this drawback is the Tamed-Euler method, [19, Relation 8] and reads:

\[(1.7)\quad Y_{n+1}^N(\omega) := Y_n^N(\omega) + \frac{T/N \cdot a(Y_n^N(\omega))}{1 + T/N \cdot |a(Y_n^N(\omega))|} + b(Y_n^N(\omega)) \left(W_{n+1/N}^\omega - W_n^\omega\right),\]

for every \(n \in \{0, 1, \ldots, N - 1\}, N \in \mathbb{N}\) and all \(\omega \in \Omega\). (1.7) is explicit, does not explode and converges strongly to the exact solution \(x_t\) of SDE (1.1), i.e.,

\[(1.8)\quad \lim_{N \to \infty} \left(\sup_{0 \leq t \leq T} E \left| x_t - Y_t^N \right|^q \right) = 0,\]

for some \(q > 0\), where \(Y_t^N := (n + 1 - \frac{tn}{N})Y_n^N + (\frac{tn}{N} - n)Y_{n+1}^N\) are continuous versions of (1.7) through linear interpolation. It still does not preserve positivity.

For the aforementioned reasons there is an interest in the construction of numerical schemes to simulate the corresponding SDEs, that have the desired properties. An attempt to this direction has been made by the first author in [12], [13] suggesting the semi-discrete method (where, briefly saying, we discretize a part of the SDE). Using this method in [12] the author produced a new numerical scheme (but not unique in this situation) for the first aforementioned problem and proves the strong convergence of the scheme in mean square sense. Later on, in [13], the author generalizes the idea of the semi-discrete method and uses this generalization to approximate a class of super linear problems, suggesting a new numerical scheme that preserves positivity in that case, proving again the strong convergence in the mean square sense.

A basic feature of the semi-discrete method is that it is explicit, compared to other interesting, but implicit methods ([28], [27]), and converges strongly in the mean square sense to
the exact solution of the original SDE. Moreover, the semi-discrete method preserves positivity [12, Section 3] and it does not explode in some super-linear problems [13, Section 3]. The purpose of this paper is to generalize further the method to include nonautonomous coefficients, \( a(t, x), b(t, x) \) in (1.1) and cover cases like that of the Heston 3/2-model.

**Assumption A** Let \( f(s, r, x, y), g(s, r, x, y) : D^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \) be such that \( f(s, s, x, x) = a(s, x), g(s, s, x, x) = b(s, x) \), where \( f, g \) satisfy the following conditions

\[
|f(s_1, r_1, x_1, y_1) - f(s_2, r_2, x_2, y_2)| \leq C_R \left( |s_1 - s_2| + |r_1 - r_2| + |x_1 - x_2| + |y_1 - y_2| \right)
\]

and

\[
|g(s_1, r_1, x_1, y_1) - g(s_2, r_2, x_2, y_2)| \leq C_R \left( |s_1 - s_2| + |r_1 - r_2| + |x_1 - x_2| + |y_1 - y_2| + \sqrt{|x_1 - x_2|} \right),
\]

for any \( R > 0 \) such that \( |x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R \), where the constant \( C_R \) depends on \( R \) and \( x \vee y \) denotes the maximum of \( x, y \).

Let the equidistant partition \( 0 = t_0 < t_1 < \ldots < t_N = T \) and \( \Delta = T/N \). We propose the following semi-discrete numerical scheme

\[
y_t = y_n + \int_{t_n}^{t} f(t, s, y_t, s)ds + \int_{t_n}^{t} g(t, s, y_t, s)dW_s, \quad t \in [t_n, t_{n+1}],
\]

where we assume that for every \( n \leq N - 1 \), (1.9) has a unique strong solution and \( y_n = y_{t_n}, y_0 = x_0, \) a.s. In order to compare with the exact solution \( x_t \), which is a continuous time process, we consider the following interpolation process of the semi-discrete approximation, in a compact form,

\[
y_t = y_n + \int_{0}^{t} f(\hat{s}, s, y_{\hat{s}}, \hat{s})ds + \int_{0}^{t} g(\hat{s}, s, \hat{y}_{\hat{s}}, \hat{s})dW_s,
\]

where \( \hat{s} = t_n \), when \( s \in [t_n, t_{n+1}] \). The first and third variable in \( f, g \) denote the discretized part of the original SDE. We observe from (1.10) that in order to solve for \( y_t \), we have to solve an SDE and not an algebraic equation, thus in this context, we cannot reproduce implicit schemes, but we can reproduce the Euler scheme if we choose \( f(s, r, x, y) = a(r, y) \) and \( g(s, r, x, y) = b(r, y) \).

The numerical scheme (1.10) converges to the true solution \( x_t \) of SDE (1.1) and this is stated in the following, which is our main result.

**Theorem 1.1.** Suppose Assumption A holds and (1.9) has a unique strong solution for every \( n \leq N - 1 \), where \( x_0 \in L^p(\Omega, \mathbb{R}), x_0 > 0 \) a.s. Let also

\[
E(\sup_{0 \leq t \leq T} |x_t|^p) \vee E(\sup_{0 \leq t \leq T} |y_t|^p) < A,
\]

for some \( p > 2 \) and \( A > 0 \). Then the semi-discrete numerical scheme (1.10) converges to the true solution of (1.1) in the mean square sense, that is

\[
\lim_{\Delta \to 0} E(\sup_{0 \leq t \leq T} |y_t - x_t|^2) = 0.
\]

In [13] the case with no square root term is treated, thus Theorem 1.1 is a generalization of [13, Theorem 1]. Section 2 provides all the necessary and finally the proof of Theorem 1.1 Section 3 gives applications to super linear drift and diffusion problems with non negative

\[1\] By the fact that we want the problem (1.1) to be well posed and by the conditions on \( f \) and \( g \) we get that \( f, g \) are bounded on bounded intervals.
solution, one of which includes the Heston 3/2-model. The Semi-Discrete scheme is strongly convergent in the mean square sense and preserves positivity of the solution.

2. Proof of Theorem 1.1

We denote the indicator function of a set $A$ by $\mathbb{1}_A$. The constant $C_R$ may vary from line to line and it may depend apart from $R$ on other quantities, like time $T$ for example, which are all constant, in the sense that we don’t let them grow to infinity.

2.1. Error bound for the explicit semi-discrete scheme.

Lemma 2.1. Let the assumption of Theorem 1.1 hold. Let $R > 0$, and set the stopping time $\theta_R = \inf\{t \in [0, T] : |y_t| > R\text{ or } |y_t| > R\}$. Then the following estimate holds

\begin{equation}
\mathbb{E}|y_{s^{\wedge}\theta_R} - y_{s^{\wedge}\theta_R}|^2 \leq C_R \Delta,
\end{equation}

where $C_R$ does not depend on $\Delta$, implying $\sup_{t \in [t_{n_s}, t_{n_s+1}]} \mathbb{E}|y_{s^{\wedge}\theta_R} - y_{s^{\wedge}\theta_R}|^2 = O(\Delta)$, as $\Delta \downarrow 0$.

Proof. Let $n_s$ integer such that $s \in [t_{n_s}, t_{n_s+1}]$. It holds that

\[ |y_{s^{\wedge}\theta_R} - y_{s^{\wedge}\theta_R}|^2 = \left| \int_{t_{n_s}^{\wedge}\theta_R}^{s^{\wedge}\theta_R} f(\hat{u}, u, y_{\hat{u}}, y_u)du + \int_{t_{n_s}^{\wedge}\theta_R}^{s^{\wedge}\theta_R} g(\hat{u}, u, y_{\hat{u}}, y_u)dW_u \right|^2 \]

\[
\leq 2 \left( \int_{t_{n_s}^{\wedge}\theta_R}^{s^{\wedge}\theta_R} f(\hat{u}, u, y_{\hat{u}}, y_u)du \right)^2 + 2 \left( \int_{t_{n_s}^{\wedge}\theta_R}^{s^{\wedge}\theta_R} g(\hat{u}, u, y_{\hat{u}}, y_u)dW_u \right)^2 \]

\[
\leq 2\Delta \int_{t_{n_s}^{\wedge}\theta_R}^{s^{\wedge}\theta_R} f^2(\hat{u}, u, y_{\hat{u}}, y_u)du + 2 \left( \int_{t_{n_s}^{\wedge}\theta_R}^{s^{\wedge}\theta_R} g(\hat{u}, u, y_{\hat{u}}, y_u)dW_u \right)^2 \]

\[
\leq C_R \Delta^2 + 2 \left( \int_{t_{n_s}^{\wedge}\theta_R}^{s^{\wedge}\theta_R} g(\hat{u}, u, y_{\hat{u}}, y_u)dW_u \right)^2,
\]

where we have used Cauchy-Schwarz inequality and Assumption A for the function $f$. Taking expectations in the above inequality gives

\[ \mathbb{E}|y_{s^{\wedge}\theta_R} - y_{s^{\wedge}\theta_R}|^2 \leq C_R \Delta^2 + 8\mathbb{E} \int_{t_{n_s}^{\wedge}\theta_R}^{t_{n_s+1}^{\wedge}\theta_R} g^2(\hat{u}, u, y_{\hat{u}}, y_u)du \]

\[ \leq C_R \Delta^2 + C_R \Delta,
\]

where in the first step we have used Doob’s martingale inequality \[23\] Theorem 3.8 on the diffusion term, in the second step Assumption A for the function $g$. Thus,

\[ \lim_{\Delta \downarrow 0} \sup_{t \in [t_{n_s}, t_{n_s+1}]} \frac{\mathbb{E}|y_{s^{\wedge}\theta_R} - y_{s^{\wedge}\theta_R}|^2}{\Delta} \leq C_R,
\]

which justifies the $O(\Delta)$ notation, (see for example \[31\]).
2.2. Convergence of the semi-discrete scheme in $L^1$.

**Proposition 2.2.** Let the assumptions of Theorem 1.1 hold. Let $R > 0$, and set the stopping time $\theta_R = \inf\{t \in [0, T]: |y_t| > R \text{ or } |x_t| > R\}$. Then we have

$$\sup_{0 \leq t \leq T} E|y_{t \wedge \theta_R} - x_{t \wedge \theta_R}| \leq \left( C_R + \frac{C_R}{m_e} \right) \sqrt{\Delta} + \left( \frac{C_R}{m_e} + C_R \right) \Delta + \frac{C_R}{m_e} \Delta^2 + \frac{C_R}{m} + e_{m-1} \right) e^{a_{R,m} T},$$

for any $m > 1$, where

$$e_m = e^{-m(m+1)/2}, \quad a_{R,m} := C_R + \frac{C_R}{m},$$

and $C_R$ does not depend on $\Delta$. It holds that $\lim_{m \to \infty} e_m = 0$.

**Proof.** Let the non increasing sequence $\{e_m\}_{m \in \mathbb{N}}$ with $e_m = e^{-m(m+1)/2}$ and $e_0 = 1$. We introduce the following sequence of smooth approximations of $|x|$, (method of Yamada and Watanabe, [32])

$$\phi_m(x) = \int_0^{[x]} dy \int_0^y \psi_m(u) du,$$

where the existence of the continuous function $\psi_m(u)$ with $0 \leq \psi_m(u) \leq 2/(mu)$ and support in $(e_m, e_m-1)$ is justified by $\int_{e_m}^{e_m-1} (du/u) = m$. The following relations hold for $\phi_m \in C^2(\mathbb{R}, \mathbb{R})$ with $\phi_m(0) = 0$,

$$|x| - e_{m-1} \leq \phi_m(x) \leq |x|, \quad |\phi'_m(x)| \leq 1, \quad x \in \mathbb{R},$$

$$|\phi''_m(x)| \leq \frac{2}{m|x|}, \quad \text{when } e_m < |x| < e_{m-1} \text{ and } |\phi''_m(x)| = 0 \text{ otherwise.}$$

We have that

$$(2.3) \quad E|y_{t \wedge \theta_R} - x_{t \wedge \theta_R}| \leq e_{m-1} + E\phi_m(y_{t \wedge \theta_R} - x_{t \wedge \theta_R}).$$
Applying Ito’s formula to the sequence \( \{ \phi_m \}_{m \in \mathbb{N}} \), we get

\[
\phi_m(y_{t \wedge \theta_R} - x_{t \wedge \theta_R}) = \int_0^{t \wedge \theta_R} \phi'_m(y_s - x_s)(f(\hat{s}, s, y_s, s) - f(s, s, x_s, x_s))ds + M_t
\]

\[
+ \frac{1}{2} \int_0^{t \wedge \theta_R} \phi''_m(y_s - x_s)(g(\hat{s}, s, y_s, s) - g(s, s, x_s, x_s))^2ds
\]

\[
\leq \int_0^{t \wedge \theta_R} C_R(|y_s - x_s| + |y_s - x_s| + |\hat{s} - s|) ds + M_t
\]

\[
+ \frac{1}{2} \int_0^{t \wedge \theta_R} \frac{2}{m|y_s - x_s|} C_R(|y_s - x_s|^2 + |y_s - x_s|^2 + |y_s - x_s| + |\hat{s} - s|^2) ds
\]

\[
\leq C_R \int_0^{t \wedge \theta_R} |y_s - y_s| ds + C_R \int_0^{t \wedge \theta_R} |y_s - x_s| ds + C_R \int_0^{t \wedge \theta_R} |\hat{s} - s| ds + M_t
\]

\[
+ \frac{C_R}{m} \int_0^{t \wedge \theta_R} \frac{2|y_s - y_s|^2 + 3|y_s - x_s|^2 + |y_s - x_s| + |\hat{s} - s|^2}{|y_s - x_s|} ds
\]

\[
\leq \left( C_R + \frac{C_R}{m \epsilon_m} \right) \int_0^{t \wedge \theta_R} |y_s - y_s| ds + \frac{C_R}{m \epsilon_m} \int_0^{t \wedge \theta_R} |y_s - y_s|^2 ds + \left( C_R + \frac{C_R}{m} \right) \int_0^{t \wedge \theta_R} |y_s - x_s| ds
\]

\[
+ \frac{C_R}{m} \int_0^{t \wedge \theta_R} \frac{2|y_s - y_s|^2 + 3|y_s - x_s|^2 + |y_s - x_s| + |\hat{s} - s|^2}{|y_s - x_s|} ds
\]

\[
\leq \left( C_R + \frac{C_R}{m \epsilon_m} \right) \int_0^{t \wedge \theta_R} |y_s - y_s| ds + \frac{C_R}{m \epsilon_m} \int_0^{t \wedge \theta_R} |y_s - y_s|^2 ds + \left( C_R + \frac{C_R}{m} \right) \int_0^{t \wedge \theta_R} |y_s - x_s| ds
\]

\[
+ \frac{C_R}{m} \Delta^2 + C_R \Delta + M_t,
\]

where in the second step we have used Assumption A for the functions \( f, g \) and

\[
M_t := \int_0^{t \wedge \theta_R} \phi'_m(y_u - x_u)(g(\hat{u}, u, y_u, y_u) - g(u, u, x_u, x_u))dW_u.
\]

Taking expectations in the above inequality yields

\[
\mathbb{E}[\phi_m(y_{t \wedge \theta_R} - x_{t \wedge \theta_R})] \leq \left( C_R + \frac{C_R}{m \epsilon_m} \right) \int_0^{t \wedge \theta_R} \mathbb{E}|y_s - y_s| ds + \left( C_R + \frac{C_R}{m} \right) \int_0^{t \wedge \theta_R} \mathbb{E}|y_s - x_s| ds
\]

\[
+ \frac{C_R}{m \epsilon_m} \int_0^{t \wedge \theta_R} \mathbb{E}|y_s - y_s|^2 ds + \frac{C_R}{m} \Delta^2 + C_R \Delta + \mathbb{E}M_t
\]

\[
\leq \left( C_R + \frac{C_R}{m \epsilon_m} \right) \sqrt{\Delta} + \left( \frac{C_R}{m \epsilon_m} + C_R \right) \Delta + \frac{C_R}{m} \Delta^2 + \frac{C_R}{m} \int_0^{t \wedge \theta_R} \mathbb{E}|y_s - x_s| ds,
\]
where we have used Lemma 2.1 and the fact that $\mathbb{E}M_t = 0$\(^2\) Thus \((2.3)\) becomes
\[
\mathbb{E}\left|y_{t\wedge \theta_R} - x_{t\wedge \theta_R}\right| \leq \left( C_R + \frac{C_R}{m e_m} \right) \Delta + \frac{C_R}{m e_m} \Delta ^2 + \frac{C_R}{m} + \epsilon_{m-1}
\]
\[
+ \left( C_R + \frac{C_R}{m} \right) \int_0^{t\wedge \theta_R} \mathbb{E}\left|y_s - x_s\right|ds
\]
\[
\leq \left[ \left( C_R + \frac{C_R}{m e_m} \right) \Delta + \frac{C_R}{m e_m} \Delta ^2 + \frac{C_R}{m} + \epsilon_{m-1} \right] e^{a_{R,m} t},
\]
where in the last step we have used the Gronwall inequality \([11]\) Relation 7] and $a_{R,m} = C_R + \frac{C_R}{m}$. Taking the supremum over all $0 \leq t \leq T$ gives \((2.2)\).

2.3. Convergence of the semi-discrete scheme in $L^2$. Set the stopping time $\theta_R = \inf \{ t \in [0,T] : |y_t| > R \text{ or } |x_t| \geq R \}$, for some $R > 0$ big enough. We have that
\[
\mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^2 = \mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^2 \mathbb{1}_{(\theta_R > t)} + \mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^2 \mathbb{1}_{(\theta_R \leq t)}
\]
\[
\leq \mathbb{E} \sup_{0 \leq t \leq T} |y_{t\wedge \theta_R} - x_{t\wedge \theta_R}|^2 + \frac{2\delta}{p} \mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^p + \frac{(p-2)}{p \delta ^2/(p-2)} \mathbb{P} (\theta_R \leq T)
\]
\[
\leq \mathbb{E} \sup_{0 \leq t \leq T} |y_{t\wedge \theta_R} - x_{t\wedge \theta_R}|^2 + \frac{2p \delta}{p} \mathbb{E} \sup_{0 \leq t \leq T} (|y_t|^p + |x_t|^p) + \frac{(p-2)}{p \delta ^2/(p-2)} \mathbb{P} (\theta_R \leq T)
\]
\[
(2.4)
\]
\[
\leq \mathbb{E} \sup_{0 \leq t \leq T} |y_{t\wedge \theta_R} - x_{t\wedge \theta_R}|^2 + \frac{2p+1 \delta A}{p} + \frac{(p-2)}{p \delta ^2/(p-2)} \mathbb{P} (\theta_R \leq T),
\]
where in the second step we have applied Young inequality,
\[
ab \leq \frac{\delta}{r} a^r + \frac{1}{q \delta ^{q/r}} b^q,
\]
for $a = \sup_{0 \leq t \leq T} |y_t - x_t|^2$, $b = \mathbb{1}_{(\theta_R \leq t)}$, $r = p/2$, $q = p/(p-2)$ and $\delta > 0$, in the third step we have used the elementary inequality $(\sum_{i=1}^n a_i)^p \leq n^{p-1} \sum_{i=1}^n a_i^p$, with $n = 2$, and $A$ comes from the moment bound assumption. It holds that
\[
\mathbb{P} (\theta_R \leq T) \leq \mathbb{E} \left( \mathbb{1}_{(\theta_R \leq T)} \frac{|y_{\theta_R}|^p}{R^p} \right) + \mathbb{E} \left( \mathbb{1}_{(\theta_R \leq T)} \frac{|x_{\theta_R}|^p}{R^p} \right) \leq \frac{1}{R^p} \left( \mathbb{E} \sup_{0 \leq t \leq T} |x_t|^p + \mathbb{E} \sup_{0 \leq t \leq T} |y_t|^p \right) \leq \frac{2A}{R^p},
\]
thus \((2.4)\) becomes
\[
\mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^2 \leq \mathbb{E} \sup_{0 \leq t \leq T} |y_{t\wedge \theta_R} - x_{t\wedge \theta_R}|^2 + \frac{2p+1 \delta A}{p} + \frac{2(p-2)A}{p \delta ^2/(p-2) R^p},
\]

\(^2\)The function $h(u) = \phi_m'(y_u - x_u)(g(\hat{u}, u, y_u, y_u) - g(u, u, x_u, x_u))$ belongs to the space $\mathcal{M}^2([0, t \wedge \theta_R]; \mathbb{R})$ of real valued measurable $\mathcal{F}_t$-adapted processes such that $\mathbb{E} \int_0^{t \wedge \theta_R} |h(u)|^2 du < \infty$ thus \([20]\) Theorem 1.5.8] implies $\mathbb{E}M_t = 0$.  

We estimate the difference $|y_{t∧θ_R} - x_{t∧θ_R}|^2$. It holds that
\[
|y_{t∧θ_R} - x_{t∧θ_R}|^2 = \left| \int_0^{t∧θ_R} (f(\hat{s}, s, \hat{y}, y) - f(s, s, x, x)) \, ds + \int_0^{t∧θ_R} (g(\hat{s}, s, \hat{y}, y) - g(s, s, x, x)) \, dW_s \right|^2 \\
\leq 2T \int_0^{t∧θ_R} C_R \left( |y_s - x_s|^2 + |y_s - x_s|^2 + |\hat{s} - s|^2 \right) \, ds + 2|M_t|^2 \\
\leq C_R \int_0^{t∧θ_R} |y_s - \hat{y}|^2 \, ds + C_R \int_0^{t∧θ_R} |y_s - x_s|^2 \, ds + 2|M_t|^2 \\
\leq C_R \int_0^{t∧θ_R} |y_s - \hat{y}|^2 \, ds + C_R \int_0^{t∧θ_R} |y_s - x_s|^2 \, ds + C_R \sum_{k=0}^{[t/\Delta-1]} \int_{t_k}^{t_{k+1 ∧ t∧θ_R}} |t_k - s|^2 \, ds + 2|M_t|^2,
\]
where in the second step we have used Cauchy-Schwarz inequality and Assumption A for $f$ and
\[
M_t := \int_0^{t∧θ_R} (g(\hat{s}, s, \hat{y}, y) - g(s, s, x, x)) \, dW_s.
\]
Taking the supremum over all $t \in [0, T]$ and then expectations we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} |y_{t∧θ_R} - x_{t∧θ_R}|^2 \leq C_R \mathbb{E} \left( \int_0^{T∧θ_R} |y_s - \hat{y}|^2 \, ds \right) + 2\mathbb{E} \sup_{0 \leq t \leq T} |M_t|^2 \\
+ C_R \int_0^{T∧θ_R} \mathbb{E} |y_s - \hat{y}|^2 \, ds + C_R \Delta^2
\]
(2.6) \leq C_R \int_0^{T∧θ_R} \mathbb{E} |y_s - y|^2 \, ds + 8\mathbb{E} |M_T|^2 + C_R \int_0^{T} \mathbb{E} \sup_{0 \leq t \leq s} |y_{t∧θ_R} - x_{t∧θ_R}|^2 \, ds + C_R \Delta^2,
where in the last step we have used Holder’s inequality and Doob’s martingale inequality with $p = 2$, since $M_t$ is an $\mathbb{R}$-valued martingale that belongs to $L^2$. It holds that
\[
\mathbb{E} |M_T|^2 := \mathbb{E} \left( \int_0^{T∧θ_R} (g(\hat{s}, s, \hat{y}, y) - g(s, s, x, x)) \, dW_s \right)^2 \\
= \mathbb{E} \left( \int_0^{T∧θ_R} (g(\hat{s}, s, \hat{y}, y) - g(s, s, x, x))^2 \, ds \right) \\
\leq C_R \mathbb{E} \left( \int_0^{T∧θ_R} (|y_s - x_s|^2 + |y_s - x_s|^2 + |\hat{y}_s - x_s| + |\hat{s} - s|^2) \, ds \right) \\
\leq C_R \int_0^{T∧θ_R} \mathbb{E} |y_s - y|^2 \, ds + C_R \int_0^{T} \mathbb{E} \sup_{0 \leq t \leq s} |y_{t∧θ_R} - x_{t∧θ_R}|^2 \, ds + C_R \Delta^2,
\]
where we have used Assumption A for $g$. Relation (2.6) becomes
\[
\mathbb{E} \sup_{0 \leq t \leq T} |y_{t∧θ_R} - x_{t∧θ_R}|^2 \leq C_R \int_0^{T∧θ_R} \mathbb{E} |y_s - y|^2 \, ds + C_R \int_0^{T} \mathbb{E} \sup_{0 \leq t \leq s} |y_{t∧θ_R} - x_{t∧θ_R}|^2 \, ds \\
+ C_R \int_0^{T∧θ_R} (\mathbb{E} |y_s - y| + \mathbb{E} |y_s - x_s|) \, ds + C_R \Delta^2 \\
\leq C_R \sqrt{\Delta} + C_R \Delta + C_R \Delta^2 + C_R \int_0^{T} \mathbb{E} \sup_{0 \leq t \leq s} |y_{t∧θ_R} - x_{t∧θ_R}|^2 \, ds + C_R \int_0^{T∧θ_R} \mathbb{E} |y_s - x_s| \, ds,
\]
where we have used Lemma 2.1 and Jensen’s inequality for the concave function \( \phi(x) = \sqrt{x} \). The integrand of the last term is bounded, from Proposition 2.2, by

\[
K_{R,\Delta,m}(s) := \left[ \left( C_R + \frac{C_R}{m e_m} \right) \sqrt{\Delta} + \frac{C_R}{m e_m} \Delta + \left( \frac{C_R}{m e_m} + C_R \right) \Delta + \frac{C_R}{m e_m} \Delta^2 + \frac{C_R}{m} + e_{m-1} \right] e^{a_{R,m}s},
\]

where \( s \in [0, T \wedge \theta_R] \). Application of the Gronwall inequality implies

\[
\mathbb{E} \sup_{0 \leq t \leq T} |y_{t \wedge \theta_R} - x_{t \wedge \theta_R}|^2 \leq \left( C_R \sqrt{\Delta} + C_R \Delta + C_R K_{R,\Delta,m}(T) \right) e^{C_R} \leq C_{R,\Delta,m}.
\]

Note that, given \( R > 0 \), the quantity \( C_{R,\Delta,m} \) can be arbitrarily small by choosing big enough \( m \) and small enough \( \Delta \). Relation (2.5) becomes,

\[
\mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^2 \leq C_{R,\Delta,m} + \frac{2^{p+1} \delta A}{p} + \frac{2(p-2)A}{p^{p^2/(p-2)} R^p} := I_1 + I_2 + I_3.
\]

Given any \( \epsilon > 0 \), we may first choose \( \delta \) such that \( I_2 < \epsilon/3 \), then choose \( R \) such that \( I_3 < \epsilon/3 \), then \( m > 1 \) and finally \( \Delta \) such that \( I_1 < \epsilon/3 \) concluding \( \mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^2 < \epsilon \) as required to verify (1.11).

### 3. Superlinear examples

#### 3.1. Example I.

We study the numerical approximation of the following SDE,

\[
x_t = x_0 + \int_0^t (k_1(s)x_s - k_2(s)x_s^2)ds + \int_0^t k_3(s)x_s^{3/2}\phi(x_s)dW_s, \quad t \in [0, T],
\]

where \( \phi(\cdot) \) is a locally Lipschitz and bounded function with locally Lipschitz constant \( C_\phi \), bounding constant \( K_\phi \), \( x_0 \) is independent of all \( \{W_t\}_{0 \leq t \leq T}, x_0 \in \mathcal{L}^p(\Omega, \mathbb{R}) \) for some \( 2 < p \) and \( x_0 > 0, \text{a.s.}, \mathbb{E}(x_0)^{-2} < A, k_1(\cdot), k_2(\cdot), k_3(\cdot) \) are positive and bounded functions with \( k_{2,\text{min}} > \frac{7}{2}(K_\phi k_{3,\text{max}})^2 \). Model (3.1) has super linear drift and diffusion coefficients.

We propose the following semi-discrete numerical scheme

\[
y_t = y_n + \int_{t_n}^t (k_1(s) - k_2(s)y_s)y_sds + \int_{t_n}^t k_3(s)\sqrt{y_s}\phi(y_s)y_s dW_s, \quad t \in [t_n, t_{n+1}],
\]

where \( y_n = y_n(t_n) \), for \( n \leq T/\Delta \) and \( y_0 = x_0, \text{a.s.} \), or in a more compact form,

\[
y_t = y_0 + \int_0^t (k_1(s) - k_2(s)y_s)y_sds + \int_0^t k_3(s)\sqrt{y_s}\phi(y_s)y_s dW_s,
\]

where \( \hat{s} = t_n, \) when \( s \in [t_n, t_{n+1}) \). The linear SDE (3.3) has a solution which, by use of Ito’s formula, has the explicit form

\[
y_t = x_0 \exp \left\{ \int_0^t \left[ k_1(s) - k_2(s)y_s - k_3(s)\frac{y_s^2}{2} \right] ds + \int_0^t k_3(s)\sqrt{y_s}\phi(y_s)dW_s \right\},
\]

where \( y_t = y_t(t_0, x_0) \).

**Proposition 3.1.** The semi-discrete numerical scheme (3.3) converges to the true solution of (3.1) in the mean square sense, that is

\[
\lim_{\Delta \to 0} \mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^2 = 0.
\]
3.1.1. Proof of Proposition 3.1. In order to prove Proposition 3.1 we need to verify the assumptions of Theorem 1.1. Let

\[ a(s, x) = k_1(s)x - k_2(s)x^2, \quad f(s, r, x, y) = (k_1(s) - k_2(s)x)y, \]

\[ b(s, x) = k_3(s)x^{3/2}\phi(x), \quad g(s, r, x, y) = k_3(s)\sqrt{x}\phi(x)y. \]

**Assumption A for \( f \).** Let \( R > 0 \) such that \(|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \vee |s| \vee |r| \leq R \). We have that

\[
|f(s, r, x_1, y_1) - f(s, r, x_2, y_2)| = |(k_1(s) - k_2(s)x_1)y_1 - (k_1(s) - k_2(s)x_2)y_2|
\]

\[
\leq |k_1(s)||y_1 - y_2| + |k_2(s)||x_2||y_1 - y_2| + |y_1||x_1 - x_2|
\]

\[
\leq (|k_{1,\max}| + |k_{2,\max}|R)|y_1 - y_2| + |k_{2,\max}|R|x_1 - x_2|
\]

\[
\leq C_R \left(|x_1 - x_2| + |y_1 - y_2|\right),
\]

thus, Assumption A holds for \( f \) with \( C_R := (|k_{1,\max}| + |k_{2,\max}|)R \).

**Assumption A for \( g \).** Let \( R > 0 \) such that \(|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \vee |s| \vee |r| \leq R \). We have that

\[
|g(s, r, x_1, y_1) - g(s, r, x_2, y_2)| = |k_3(s)\sqrt{x_1}\phi(x_1)y_1 - k_3(s)\sqrt{x_2}\phi(x_2)y_2|
\]

\[
\leq |k_3(s)| \left(\sqrt{|x_1|}\phi(x_1)||y_1 - y_2| + |y_2||\sqrt{x_1}\phi(x_1) - \sqrt{x_1}\phi(x_2) + \sqrt{x_1}\phi(x_2) - \sqrt{x_1}\phi(x_2)|\right)
\]

\[
\leq |k_{3,\max}| \left(R\sqrt{|y_1 - y_2|} + R\sqrt{|x_1|}\phi(x_1) - \phi(x_2) + K_\phi|\sqrt{x_1} - \sqrt{x_2}|\right)
\]

\[
\leq |k_{3,\max}| \left(K_\phi\sqrt{R}|y_1 - y_2| + R^{3/2}C_R^\phi|x_1 - x_2| + K_\phi\sqrt{|x_1 - x_2|}\right)
\]

\[
\leq C_R \left(|x_1 - x_2| + |y_1 - y_2| + \sqrt{|x_1 - x_2|}\right),
\]

where we have used the fact that the function \( \sqrt{x} \) is \( 1/2 \)-Holder continuous and \( C_R := |k_{3,\max}| \left(C_R^\phi R^{3/2} \vee K_\phi\sqrt{R} \vee K_\phi\right) \). Thus, Assumption A holds for \( g \).

**Moment bound for original SDE.**

**Lemma 3.2.** In the previous setting it holds that \( x_t > 0 \) a.s.
Proof. Set the stopping time \( \theta_R = \inf\{t \in [0,T] : x_t^{-1} > R \} \), for some \( R > 0 \), with the convention that \( \inf \emptyset = \infty \). Application of Ito’s formula on \( x_t^{-2} \) implies,

\[
(x_{t \wedge \theta_R})^{-2} = (x_0)^{-2} + \int_0^{t \wedge \theta_R} (-2)x_s^{-3}(k_1(s)x_s - k_2(s)x_s^2)ds + \int_0^{t \wedge \theta_R} (2k_3(s)x_s^{-3}k_2(s)x_s^{-1} + 3k_3^2(s)k_\phi^2x_s^{-1})ds \\
+ \int_0^{t \wedge \theta_R} \frac{(-2)(-3)}{2}(x_s)^{-4}k_3^2(s)x_s^3\phi^2(x_s)ds \\
= (x_0)^{-2} + \int_0^{t \wedge \theta_R} \left(-2k_1(s)x_s^{-2} + 2k_2(s)x_s^{-1} + 3k_3^2(s)k_\phi^2x_s^{-1}\right)ds \\
+ \int_0^{t \wedge \theta_R} (-2k_3(s)x_s^{-3}k_\phi^2\phi(x_s))I_{(0,t \wedge \theta_R)}(s)dW_s \\
\leq (x_0)^{-2} + \int_0^{t \wedge \theta_R} \left(-2k_1(s)x_s^{-2} + 2k_2(s) + 3k_3^2(s)k_\phi^2\right)\left(x_s^{-1}I_{[0,1]}(x_s) + x_s^{-1}I_{(1,\infty]}(x_s)\right)ds \\
+ M_t,
\]

where

\[ M_t := \int_0^{t} (-2k_3(s)x_s^{-3}k_\phi^2\phi(x_s))I_{(0,t \wedge \theta_R)}(s)dW_s. \]

Taking expectations in the above inequality and using the fact that \( E M_t = 0 \) we get that

\[
E(x_{t \wedge \theta_R}^{-2}) \leq E(x_0)^{-2} + 2k_{2,\max}T + 3(k_{3,\max}k_\phi)^2T + (2k_{2,\max} + 3(k_{3,\max}k_\phi)^2) \int_0^{t} E(x_s^{-2})ds \\
\leq \left( E(x_0)^{-2} + 2k_{2,\max}T + 3(k_{3,\max}k_\phi)^2T \right)e^{(2k_{2,\max} + 3(k_{3,\max}k_\phi)^2)T} < C,
\]

where we have used Gronwall inequality with \( C \) independent of \( R \). We have that

\[
(x_{t \wedge \theta_R})^{-2} = (x_{\theta_R})^{-2}I_{(\theta_R \leq t)} + (x_t)^{-2}I_{(t < \theta_R)} = R^2I_{(\theta_R \leq t)} + (x_t)^{-2}I_{(t < \theta_R)}.
\]

By relation \( 3.6 \) we have that,

\[
E \left( \frac{1}{x_{t \wedge \theta_R}} \right) = R^2P(\theta_R \leq t) + E \left( \frac{1}{x_{t \wedge \theta_R}}I_{(t < \theta_R)} \right) < C,
\]

thus

\[
P(x_t \leq 0) = P \left( \bigcap_{R=1}^{\infty} \left\{ x_t < \frac{1}{R} \right\} \right) = \lim_{R \to \infty} P \left( \left\{ x_t < \frac{1}{R} \right\} \right) \leq \lim_{R \to \infty} P(\theta_R \leq t) = 0.
\]

We conclude that \( x_t > 0 \) a.s.

\[ \square \]

Lemma 3.3. In the previous setting it holds that

\[ E( \sup_{0 \leq t \leq T} (x_t)^p) < A_1, \]

for some \( A_1 > 0 \) and any \( 2 < p \leq k_{2,\min}/(K_\phi k_{3,\max})^2 \).

\[ \text{The function } h(u) = (-2)k_3(u)x_u^{-3/2}\phi(x_u)I_{(0,t \wedge \theta_R)}(u) \text{ belongs to the space } M^2([0,t];\mathbb{R}) \text{ thus } \text{[26], Theorem 1.5.8} \text{ implies } E M_t = 0. \]
Proof. In the case of \( x \)'s outside a finite ball of radius \( R \), with \( R > 1 \), and \( s \in [0, T] \) we have that
\[
J(s, x) := \frac{xa(s, x) + (p - 1)b^2(s, x)/2}{1 + x^2} = \frac{x(k_1(s)x - k_2(s)x^2) + (p - 1)k_3^2(s)/2[x^{3/2}\phi(x)]^2}{1 + x^2}
\]
\[
= \frac{k_1(s)x^2 - k_2(s)x^3 + 0.5(p - 1)k_3^2(s)x^3\phi^2(x)}{1 + x^2}
\]
\[
\leq k_{1,\text{max}}x^2 + \left(0.5(p - 1)(k_{3,\text{max}}K_\phi)^2 - k_{2,\text{min}}\right)x^3
\]
where the last inequality is valid for all \( p \) such that \( p \leq 1 + 2k_{2,\text{min}}/(K_\phi k_{3,\text{max}})^2 \). Thus \( J(s, x) \) is bounded for all \( (s, x) \in [0, T] \times \mathbb{R} \), since when \( |x| \leq R \) we have that \( J(s, x) \) is finite and say \( J(s, x) \leq C \). Since \( C \) is positive, application of \([26, \text{Theorem 2.4.1}]\) implies
\[
E(x_t)^p \leq 2^{(p-2)/2}(1 + E(x_0)^p)e^{Cp t},
\]
for any \( 2 < p \leq 1 + 2k_{2,\text{min}}/(K_\phi k_{3,\text{max}})^2 \) and all \( t \in [0, T] \). Using Ito’s formula on \( (x_t)^p \), with \( p \leq k_{2,\text{min}}/(K_\phi k_{3,\text{max}})^2 \) (in order to use Doob’s martingale inequality later) we have that
\[
(x_t)^p = (x_0)^p + \int_0^t p(x_s)^{p-1}(k_1(s)x_s - k_2(s)x_s^2)ds
\]
\[
+ \int_0^t \frac{p(p - 1)}{2}(x_s)^{p-2}[k_3(s)x_s^{3/2}\phi(x_s)]^2ds + \int_0^t pk_3(s)(x_s)^{p-1}x_s^{3/2}\phi(x_s)dW_s
\]
\[
\leq (x_0)^p + p \int_0^t \left[k_1(s)(x_s)^p + \left(\frac{p - 1}{2}k_{3,\text{max}}K_\phi^2 - k_2\right)(x_s)^{p+1}\right]ds + \int_0^t pk_3(s)\phi(x_s)(x_s)^{p+1/2}dW_s
\]
\[
\leq (x_0)^p + p \int_0^t k_1(s)(x_s)^pds + M_t,
\]
where \( M_t = \int_0^t pk_3(s)\phi(x_s)(x_s)^{p+1/2}dW_s \). Taking the supremum and then expectations in the above inequality we get
\[
E(\sup_{0 \leq t \leq T} (x_t)^p) \leq E(x_0)^p + pk_{1,\text{max}}E(\sup_{0 \leq t \leq T} \int_0^t (x_s)^pds) + E\sup_{0 \leq t \leq T} M_t
\]
\[
\leq E(x_0)^p + pk_{1,\text{max}} \int_0^t E(\sup_{0 \leq s \leq t} (x_s)^p)ds + \sqrt{E\sup_{0 \leq t \leq T} M_t^2}
\]
\[
\leq \left(E(x_0)^p + \sqrt{4EM_T^2}\right)e^{pk_1T} := A_1,
\]
where in the last step we have used Doob’s martingale inequality to the diffusion term \( M_t^2 \) and Gronwall inequality. \qed

Moment bound for semi-discrete approximation.

Lemma 3.4. In the previous setting it holds that
\[
E(\sup_{0 \leq t \leq T} (y_t)^p) < A_2,
\]
\footnote{The function \( h(u) = pk_3(u)\phi(x_u)(x_u)^{p+1/2} \) belongs to the family \( \mathcal{M}^2([0, T]; \mathbb{R}) \) thus \([26, \text{Theorem 1.5.8}]\) implies \( EM_T^2 = E(\int_0^T h(u)dW_u)^2 = E\int_0^T h^2(u)du \), i.e. \( M_t \in L^2(\Omega; \mathbb{R}) \).}
for some $A_2 > 0$ and for any $2 < p \leq 1/4 + \frac{k_{2, \min}}{2 (k_{3, \max} K\rho)T}$.

**Proof.** Set the stopping time $\theta_R = \inf\{t \in [0, T] : y_t > R\}$, for some $R > 0$, with the convention that $\inf \emptyset = \infty$. Application of Ito’s formula on $(y_{t \wedge \theta_R})^q$, with $q = 4p$ implies,

$$
(y_{t \wedge \theta_R})^q = (y_0)^q + \int_0^{t \wedge \theta_R} q(y_s)^q (k_1(s) - k_2(s)y_s) y_s ds
$$

$$
+ \int_0^{t \wedge \theta_R} \frac{q(q-1)}{2} (y_s)^{q-2} k_3(s) \sqrt{y_s} \phi(y_s)^2 ds + \int_0^{t \wedge \theta_R} q k_3(s) (y_s)^{q-1} \sqrt{y_s} \phi(y_s) y_s dW_s
$$

$$
= (x_0)^q + \int_0^{t \wedge \theta_R} q (k_1(s) - k_2(s)y_s) + \frac{q(q-1)}{2} k_3^2(s) y_s \phi^2(y_s) (y_s)^q ds
$$

$$
+ \int_0^{t \wedge \theta_R} q k_3(s) \sqrt{y_s} \phi(y_s) (y_s)^q dW_s
$$

$$
\leq (x_0)^q + q \int_0^t \left[ k_1(s) + \left( \frac{q-1}{2} k_3^2(s) - k_{2, \min} \right) y_s \right] (y_s)^q \mathbb{I}_{[0, t \wedge \theta_R]}(s) ds + M_t
$$

where the last inequality is valid for $q \leq 1 + 2k_{2, \min}/(k_{3, \max} K\phi)^2$ and

$$
M_t := \int_0^{t \wedge \theta_R} q k_3(s) \sqrt{y_s} \phi(y_s) (y_s)^q dW_s.
$$

Taking expectations and using that $\mathbb{E} M_t = 0$ we get

$$
\mathbb{E}(y_{t \wedge \theta_R})^q \leq \mathbb{E}(x_0)^q + 2p k_{1, \max} \int_0^t \mathbb{E}(y_{s \wedge \theta_R})^q ds,
$$

Application of the Gronwall inequality implies

$$
\mathbb{E}(y_{t \wedge \theta_R})^q \leq \mathbb{E}(x_0)^q e^{q k_{1, \max} T}.
$$

We have that

$$
(y_{t \wedge \theta_R})^q = (y_{\theta_R})^q \mathbb{I}_{[\theta_R \leq t]} + (y_t)^q \mathbb{I}_{[t < \theta_R]} = R^q \mathbb{I}_{[\theta_R \leq t]} + (y_t)^q \mathbb{I}_{[t < \theta_R]},
$$

thus taking expectations in the above inequality and using the estimated upper bound for $\mathbb{E}(y_{t \wedge \theta_R})^q$ we arrive at

$$
\mathbb{E}(y_t)^q \mathbb{I}_{[t < \theta_R]} \leq \mathbb{E}(x_0)^q e^{q k_{1, \max} T}
$$

and taking limits in both sides as $R \to \infty$ we get that

$$
\lim_{R \to \infty} \mathbb{E}(y_t)^q \mathbb{I}_{[t < \theta_R]} \leq \mathbb{E}(x_0)^q e^{q k_{1, \max} T}.
$$

Fix $t$. The sequence $(y_{t \wedge \theta_R})^q \mathbb{I}_{[t < \theta_R]}$ is nondecreasing in $R$ since $\theta_R$ is increasing in $R$ and $t \wedge \theta_R \to t$ as $R \to \infty$ and $(y_t)^q \mathbb{I}_{[t < \theta_R]} \to (y_t)^q$ as $R \to \infty$, thus the monotone convergence theorem implies

$$
(3.7) \quad \mathbb{E}(y_t)^q \leq \mathbb{E}(x_0)^q e^{q k_{1, \max} T},
$$

for any $q \leq 1 + \frac{2k_{2, \min}}{2(k_{3, \max} K\rho)}$. Following the same lines as in Lemma 3.3, i.e. using again Ito’s formula on $(y_t)^p$, taking the supremum and then using Doob’s martingale inequality
Remark 3.5.  
(i) Proposition 3.1 implies that our explicit numerical scheme converges in the mean square sense. Thus, by (3.4) we get that our numerical scheme preserves positivity, which is a desirable modelling property [2], [22]. Example (3.1) covers the \(3/2\)-model (3.3), in the case where \(\phi(\cdot), k_1(\cdot), k_2(\cdot), k_3(\cdot)\) are constant, and super-linear problems both in drift and diffusion.

(ii) Moreover, note that in the analysis that we followed, we did not discretize the coefficients \(k_i\). In general, by Theorem 1.1 we are free to discretize any of the \(k_i(\cdot), i = 1, 2, 3\), functions at any degree. Thus, we can fully discretize every \(k_i(\cdot), i = 1, 2, 3\), meaning that (3.2) will become

\[
y_t = y_n + \int_{t_n}^{t} (k_1(t) - k_2(t)y_n) y_s ds + \int_{t_n}^{t} k_3(t) \sqrt{y_n} \phi(y_n) y_s dW_s, \quad t \in [t_n, t_{n+1}],
\]

or semi-discretize every \(k_i(\cdot), i = 1, 2, 3\),

\[
y_t = y_n + \int_{t_n}^{t} (\hat{k}_1(s, t) - \hat{k}_2(s, t)y_n) y_s ds + \int_{t_n}^{t} \hat{k}_3(s, t) \sqrt{y_n} \phi(y_n) y_s dW_s, \quad t \in [t_n, t_{n+1}],
\]

where \(\hat{k}_i(t, t) = k_i(t), i = 1, 2, 3\). The only difference in that situation is that we require, \(\hat{k}_i(\cdot, \cdot), i = 1, 2, 3\) to be locally Lipschitz in both variables.

(iii) One more point of discussion is the dependence on \(\omega\) that we can assume on the coefficients \(k_i\)’s. Specifically, we consider the more general SDE

\[
x_t = x_0 + \int_0^{t} a_\omega(s, x_s) ds + \int_0^{t} b_\omega(s, x_s) dW_s, \quad t \in [0, T].
\]

Then, assuming that it admits a unique strong solution, our method seems to work. In the example discussed here, an extra condition on the \(k_i\)’s would be of the form

\[|k_i(t, \omega)| \leq C, t \in [0, T], \omega \in \Omega, i = 1, 2, 3.\]

(iv) We illustrate our method in the case \(\phi(x) = \sin(x)\). Then the diffusion term \(b(x)\) takes positive and negative values and thus method [30] does not work since it requires \(b(x) > 0\) in order to use the Lamperti-type transformation, as well as Milstein method [17] since for the same reason their Assumption 2.7 is violated. The only method that we know and can be used for this situation is the Tamed-Euler method ([19], [18]) but the drawback is that it does not preserve positivity.

Below, we compare our scheme, in the case where \(k_1(\cdot), k_2(\cdot), k_3(\cdot)\) are constant, with Tamed-Euler method in [18] and see that for “good” data the two methods are close. Choosing different data, we see that Tamed-Euler takes negative values.
3.2. Example II. Consider the following stochastic differential equation (SDE),

\begin{equation}
    x_t = x_0 + \int_0^t (k_1(s)x_s - k_2(s)x_s^{2r-1})ds + \int_0^t k_3(s)x_s^{r}dW_s, \quad t \in [0,T],
\end{equation}

Figure 1. Difference between the semi-discrete scheme and Tamed-Euler scheme for $x_0 = 1$, $\Delta = 10^{-3}$, $k_1 = 1$, $k_2 = 4$, $k_3 = 1$, $T = 1$.

Figure 2. Tamed-Euler method does not preserve positivity, $\Delta = 10^{-3}$, $k_1 = 10^{-4}$, $k_2 = 1100$, $k_3 = 10^{-4}$, $x_0 = 0.99$, $T = 1$. 

\[\begin{array}{c}
\end{array}\]
where \( x_0 \) is independent of all \( \{ W_t \}_{0 \leq t \leq T} \), \( x_0 \in L^p(\Omega, \mathbb{R}) \) for some \( 2 < p \) and \( x_0 > 0 \), a.s., \( k_1(\cdot), k_2(\cdot), k_3(\cdot) \) are positive and bounded functions with \( 2k_{2,\min} > k_{3,\max}^2 \), and \( 1 < r < 3/2 \). The above conditions on the parameters imply

\[
E\left( \sup_{0 \leq t \leq T} |x_t|^p \right) < A_1,
\]

for some \( A_1 > 0 \) and any \( p \) such that \( 2 < p \leq 1 + 2k_{2,\min}/k_{3,\max}^2 \). Model (3.11) has super linear drift and diffusion coefficients.

We study the numerical approximation of (3.11). We propose the following semi-discrete numerical scheme for the transformed process \( z_t = x_t^{2r-2}, \) of (3.11),

\[
y_t = y_n + \int_{t_n}^t (K_1(s) - K_2(s)y_n)y_sds + \int_{t_n}^t K_3(s)\sqrt{y_n}y_sdW_s, \quad t \in [t_n, t_{n+1}],
\]

where \( y_n = y_n(t_n) \), for \( n \leq T/\Delta \) and \( y_0 = x_0 \), a.s., where

\[
K_1(s) = (2r-2)k_1(s), \quad K_2(s) = (2r-2)k_2(s) - \frac{(2r-2)(2r-3)}{2} k_3^2(s), \quad K_3(s) = (2r-2)k_3(s),
\]

or in a more compact form,

\[
y_t = y_0 + \int_0^t (K_1(s) - K_2(s)y_s)y_sds + \int_0^t K_3(s)\sqrt{y_s}y_sdW_s,
\]

where \( \hat{s} = t_n \), when \( s \in [t_n, t_{n+1}] \). The linear SDE (3.14) has a solution which, by use of Ito’s formula, has the explicit form

\[
y_t = x_0 \exp \left\{ \int_0^t \left( K_1(s) - K_2(s)y_s - K_3^2(s)\frac{y_s}{2} \right) ds + \int_0^t K_3(s)\sqrt{y_s}dW_s \right\},
\]

where \( y_t = y_t(t_0, x_0) \).

The transformation of (3.11). Application of Ito’s formula to the function \( z(t, x) = x^{2r-2} \), implies

\[
z_t = z_0 + \int_0^t \left[ (2r-2)x_s^{2r-3}(k_1(s)x_s - k_2(s)x_s^{2r-1}) + \frac{(2r-2)(2r-3)}{2} x_s^{2r-4}k_3^2(s)x_s^{2r} \right] ds
\]

\[
+ \int_0^t (2r-2)k_3(s)x_s^{2r-3}x_\hat{s}dW_s
\]

\[
= z_0 + \int_0^t \left[ k_1(s)(2r-2)x_s^{2r-2} - (2r-2)k_2(s)x_s^{4r-4} + \frac{(2r-2)(2r-3)}{2} k_3^2(s)x_s^{4r-4} \right] ds
\]

\[
+ \int_0^t (2r-2)k_3(s)x_s^{3r-3}dW_s
\]

\[
= z_0 + \int_0^t (K_1(s)z_s - K_2(s)z_s^2)ds + \int_0^t K_3(s)z_s^{3/2}dW_s,
\]

where \( K_1(\cdot), K_2(\cdot), K_3(\cdot) \) are given by (3.13).

In order to use Proposition 3.1, we have to verify that

\[
K_1(s) > 0, \quad K_2(s) > 0, \quad K_3(s) > 0, \quad 2K_{2,\min} > 7K_{3,\max}^2.
\]
Since $1 < r < 3/2$ we immediately have $K_1(s) > 0$ and $K_3(s) > 0$. Moreover
\[ K_2(s) = (2r - 2)k_2(s) - \frac{(2r - 2)(2r - 3)}{2}k_3^2(s) > \frac{(2r - 2)K_{3,\text{max}}^2(4 - 2r)}{2}, \]
and
\[ 2K_{2,\text{min}} > 7K_{3,\text{max}}^2. \]

**Assumption B** Suppose that the following moment bounds holds for some $p > 2$,
\[ \mathbb{E}\sup_{0 \leq t \leq T} \left( |z_t|^{4p} + |z_t|^{q-1} + |y_t|^{4p} + |y_t|^{q-1} \right). \]

**Proposition 3.6.** The following convergence to the true solution of (3.11) in the mean square sense holds,
\[ \lim_{\Delta \to 0} \mathbb{E}\sup_{0 \leq t \leq T} |y_t^{1/2} - x_t|^2 = 0, \]
under Assumption B.

### 3.2.1. Proof of Proposition 3.6

In order to prove Proposition 3.6 we first transform the original SDE (3.11) to a SDE (3.1), later on verify the assumptions of Example I to use Proposition 3.5, and in the end make the necessary arrangements for the approximation of the original SDE.

**Convergence result.** We use the following inequality implied by the mean value theorem
\[ |y_t^{1/2} - x_t| = |y_t^{1/2} - z_t^{1/2}| \leq \frac{1}{2r - 2} \left( |y_t|^{1/2} - 1 + |z_t|^{1/2} - 1 \right) |z_t - y_t|. \]
thus we get that
\[ |y_t^{1/2} - x_t|^2 \leq \frac{2}{(2r - 2)^2} \left( |y_t|^{1/2} + |z_t|^{1/2} \right)^2 |z_t - y_t|^2. \]
Set the stopping time $\theta_R = \inf\{t \in [0, T]: |y_t| > R \text{ or } |x_t| > R\}$, for some $R > 0$ big enough. Taking the supremum and then expectations in the above inequality yields,
\[ \mathbb{E}\sup_{0 \leq t \leq T} |y_t^{1/2} - x_t|^2 \leq c_r \mathbb{E}\sup_{0 \leq t \leq T} \left( |y_{t\wedge \theta_R}|^{1/2} + |z_{t\wedge \theta_R}|^{1/2} \right)^2 |y_{t\wedge \theta_R} - y_{t\wedge \theta_R}|^2 + \mathbb{E}\sup_{0 \leq t \leq T} \left( |y_t|^{1/2} + |z_t|^{1/2} \right)^2 |z_t - y_t|^2 \mathbb{P}(\theta_R \leq t), \]
\[ \leq c_{r,R} \mathbb{E}\sup_{0 \leq t \leq T} |z_{t\wedge \theta_R} - y_{t\wedge \theta_R}|^2 + c_r \frac{2\delta}{p}\mathbb{E}\sup_{0 \leq t \leq T} \left( |y_t|^{1/2} + |z_t|^{1/2} \right)^p |z_t - y_t|^p \]
\[ + c_r \frac{(p - 2)}{p\delta^2(p - 2)} \mathbb{P}(\theta_R \leq T), \]
where in the second step we have applied Young inequality,
\[ ab \leq \frac{\delta}{w}a^w + \frac{1}{wq^{q/w}}b^q, \]
for $a = \sup_{0 \leq t \leq T} \left( |y_t|^{1/2} + |z_t|^{1/2} \right) |z_t - y_t|^2, b = \mathbb{I}(\theta_R \leq t), w = p/2, q = p/(p - 2), \delta > 0$, and
\[ c_r = \frac{2}{(2r - 2)^2}, \quad c_{r,R} = 2c_r R^{1/2}. \]
It holds that
\[
\Pr(\theta \leq T) \leq \mathbb{E}\left( \mathbb{I}_{(\theta \leq T)} \frac{|y_{\theta R}|^p}{R^p} \right) + \mathbb{E}\left( \mathbb{I}_{(\theta \leq T)} \frac{|x_{\theta R}|^p}{R^p} \right) \leq \frac{1}{R^p} \left( \mathbb{E} \sup_{0 \leq t \leq T} |y_t|^p + \mathbb{E} \sup_{0 \leq t \leq T} |x_t|^p \right) \leq \frac{2A}{R^p},
\]
where \(A\) is the maximum of the bounding moment constants of \(y\) and \(x\). Moreover, we have that,
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left( |y_t|^{\frac{3-2r}{1-r}} + |z_t|^{\frac{3-2r}{1-r}} \right)^p |z_t - y_t|^{2p} \leq 2^{p-1}2^{2p-1} \mathbb{E} \sup_{0 \leq t \leq T} \left( |y_t|^{\frac{3-2p}{r-1}} + |z_t|^{\frac{3-2p}{r-1}} \right) \left( |z_t|^{2p} + |y_t|^{2p} \right) \leq 2^{3p-2} \mathbb{E} \sup_{0 \leq t \leq T} \left( |y_t|^{\frac{3-2r}{r-1}2p} + |y_t|^{\frac{3-2r}{r-1}2p} + |z_t|^{\frac{3-2r}{1-r}2p} + |z_t|^{\frac{3-2r}{1-r}2p} \right) \leq 2^{3p-2} \mathbb{E} \sup_{0 \leq t \leq T} \left( |z_t|^{4p} + |y_t|^{4p} + |y_t|^{\frac{p}{r-1}} + |z_t|^{\frac{p}{r-1}} \right) \leq 2^{3p} A
\]
where we have used again Young inequality. Collecting all the estimates together,
\[
\mathbb{E} \sup_{0 \leq t \leq T} |y_t|^{\frac{1}{r-1}} - x_t|^2 \leq c_{r,R} \mathbb{E} \sup_{0 \leq t \leq T} |z_{t \wedge \theta_R} - y_{t \wedge \theta_R}|^2 + c_{r} \frac{2^{3p+1}A}{p} \delta + c_{r} \frac{2(p-2)A}{p} \frac{1}{\delta^{2/(p-2)}R^p}
\]
\[
:= I_1 + I_2 + I_3.
\]
Given any \(\varepsilon > 0\), we may first choose \(\delta\) such that \(I_2 < \varepsilon/3\), then choose \(R\) such that \(I_3 < \varepsilon/3\), and finally \(\Delta\) such that \(I_1 < \varepsilon/3\), which is justified by Proposition 3.1 to get that
\[
\mathbb{E} \sup_{0 \leq t \leq T} |y_t|^{\frac{1}{r-1}} - x_t|^2 < \varepsilon, \quad \text{as required to verify (3.17)}.
\]
**Lemma 3.7.** In the previous setting it holds that \(x_t > 0\) a.s.

**Proof.** Set the stopping time \(\theta_R = \inf \{ t \in [0,T] : x_t^{-1} > R \}\), for some \(R > 0\), with the convention that \(\inf \emptyset = \infty\). Application of Itô’s formula on \(x_{t \wedge \theta_R}^{-2}\) implies,
\[
(x_{t \wedge \theta_R})^{-2} = (x_0)^{-2} + \int_0^{t \wedge \theta_R} \left( -2 \right) x_s^{-3} (k_1(s)x_s - k_2(s)x_s^{2r-1}) ds + \int_0^{t \wedge \theta_R} \left( -2 \right) k_3(s)(x_s)^{-3} x_s r dW_s
\]
\[
= (x_0)^{-2} + \int_0^{t \wedge \theta_R} \left( -2 \right) k_1(s)x_s^{-2} + 2k_2(s)x_s^{2r-4} + 3k_3^2(s)x_s^{2r-4}) ds
\]
\[
+ \int_0^{t \wedge \theta_R} \left( -2 \right) k_3(s)x_s^{-3} \mathbb{I}_{(0,t \wedge \theta_R)}(s) dW_s + M_t
\]
\[
\leq (x_0)^{-2} + 2k_2 \sup_{0 \leq t \leq T} + 3k_3^2 \sup_{0 \leq t \leq T} + \int_0^{t} \left( 2k_2(s) + 3k_3^2(s) \right) x_s^{-2} \mathbb{I}_{(0,t \wedge \theta_R)}(s) ds + M_t,
\]
where
\[ M_t := \int_0^t (-2)k_3(s)x_s^{-3}\mathbb{I}_{(0,t \wedge \theta_R)}(s)dW_s. \]

Taking expectations in the above inequality and using the fact that \( \mathbb{E}M_t = 0 \)^5 we get that
\[
\mathbb{E}(x_{t \wedge \theta_R}^{-2}) \leq \mathbb{E}(x_0^{-2}) + 2k_{2, \max}T + 3k_{3, \max}^2T + (2k_{2, \max} + 3k_{3, \max}^2) \int_0^t \mathbb{E}(x_{s \wedge \theta_R})^{-2}ds
\]
\[
\leq \mathbb{E}(x_0^{-2}) + 2k_{2, \max}T + 3k_{3, \max}^2T e^{(2k_2 + 3k_3^2)T} < C,
\]

where we have used Gronwall inequality with \( k = \theta_R \) and \( x \in (3.19) \).

**Remark 3.8.** Proposition 3.6 implies that our explicit numerical scheme converges in the mean square sense. Moreover, we get that our numerical scheme preserves positivity. Example 3.11 covers super-linear problems both in drift and diffusion.

### 3.3. Example III.

Consider the following stochastic differential equation (SDE),
\[
(3.19) \quad x_t = x_0 + \int_0^t (k_1(s)x_s - k_2(s)x_s^q)ds + \int_0^t k_3(s)x_s^r\varphi(x_s)dW_s, \quad t \in [0,T],
\]
where \( \varphi(\cdot) \) is a locally Lipschitz and bounded function with locally Lipschitz constant \( C_R^\varphi \), bounding constant \( K_\varphi \), \( x_0 \) is independent of all \{\( W_t \)\}_{t \leq T}, \( x_0 \in \mathcal{L}^p(\Omega, \mathbb{R}) \) for every \( 2 < p, \mathbb{E}[\ln |x_0|] < \infty \) and \( x_0 > 0 \) a.s., \( k_1(\cdot), k_2(\cdot), k_3(\cdot) \) are positive and bounded functions and \( q \) is odd with \( q > 2r - 1 \) where \( 3/2 < r < 2 \). The above conditions on the parameters imply
\[
\mathbb{E}(\sup_{0 \leq t \leq T} |x_t|^p) < A_1,
\]
for some \( A_1 > 0 \) and any \( p > 2 \). Model (3.19) has super linear drift and diffusion coefficients.

We study the numerical approximation of (3.19). We propose the following semi-discrete numerical scheme for (3.19)
\[
(3.20) \quad y_t = y_n + \int_{t_n}^t (k_1(s) - k_2(s)y_n^{q-1})y_sds + \int_{t_n}^t k_3(s)y_s^{r-1}\varphi(y_s)y_sdw_s, \quad t \in [t_n, t_{n+1}],
\]
where \( y_n = y_n(t_n) \), for \( n \leq T/\Delta \) and \( y_0 = x_0 \), a.s., or in a more compact form,
\[
(3.21) \quad y_t = y_0 + \int_0^t (k_1(s) - k_2(s)y_s^{q-1})y_sds + \int_0^t k_3(s)y_s^{r-1}\varphi(y_s)y_sdw_s,
\]

^5The function \( h(u) = (-2)k_3(u)x_u^{-3}\mathbb{I}_{(0,t \wedge \theta_R)}(u) \) belongs to the space \( \mathcal{M}^2([0,t]; \mathbb{R}) \) thus 26 Theorem 1.5.8 implies \( \mathbb{E}M_t = 0 \).
where \( s = t_n \), when \( s \in [t_n, t_{n+1}) \). The linear SDE (3.21) has a solution which, by use of Ito’s formula, has the explicit form \( y = t \exp \left\{ \int_0^t \left( k_1(s) - k_2(s) y_s^{q-1} - \frac{k_3(s)}{2} y_s^{2r-2} \phi(y_s) \right) ds + \int_0^t k_3(s) y_s^{r-1} \phi(y_s) dW_s \right\} \),

\[
\lim_{\Delta \to 0} \mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^2 = 0.
\]

**Proposition 3.9.** The following convergence to the true solution of (3.10) in the mean square sense holds,

\[
(3.23)
\]

**Proof of Proposition 3.9.** In order to prove Proposition 3.9 we just need to verify the assumptions of Theorem 1.1. Let

\[
\begin{align*}
a(s, x) &= k_1(s)x - k_2(s)x^q, \\
b(s, x) &= k_3(s)x^r \phi(x), \\
f(s, r, x, y) &= (k_1(s) - k_2(s)x^{q-1})y \\
g(s, r, x, y) &= k_3(s)x^{r-1} \phi(x) y.
\end{align*}
\]

**Assumption A for \( f \).** The conditions on the parameters imply that \( q > 2 \). Let \( R > 0 \) such that \( |x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R \). We have that

\[
|f(s, r, x_1, y_1) - f(s, r, x_2, y_2)| = |(k_1(s) - k_2(s)x_1^{q-1})y_1 - (k_1(s) - k_2(s)x_2^{q-1})y_2|
\leq |k_1(s)||y_1 - y_2| + |k_2|_{\text{max}} (|x_2|^{q-1}|y_1 - y_2| + |y_1||x_1^{q-1} - x_2^{q-1}|)
\leq (|k_1|_{\text{max}} + |k_2|_{\text{max}}|R^{q-1}|)|y_1 - y_2| + |k_2|_{\text{max}}|R|x_1^{q-1} - x_2^{q-1}|
\leq C_R (|x_1 - x_2| + |y_1 - y_2|),
\]

where we have applied the mean value theorem for the function \( x^{q-1} \), thus Assumption A holds for \( f \) with \( C_R := (|k_1|_{\text{max}} + |k_2|_{\text{max}}|R^{q-1}) \vee (2|k_2|(|q - 1)|R^{q-1}) \).

**Assumption A for \( g \).** Since \( 1/2 < r - 1 < 1 \) we have that \( g_1(x) = x^{r-1} \) is locally \( 1/2 \)-Hölder continuous in \( x \), i.e.

\[
|g_1(x_1) - g_1(x_2)| \leq \sqrt{|x_1 - x_2|}.
\]

Let \( R > 0 \) such that \( |x_1| \vee |x_2| \vee |y_1| \vee |y_2| \vee |s| \vee |r| \leq R \). We have that

\[
|g(s, r, x_1, y_1) - g(s, r, x_2, y_2)| = |k_3(s)x_1^{r-1}\phi(x_1)y_1 - k_3(s)x_2^{r-1}\phi(x_2)y_2|
\leq |k_3|_{\text{max}} (|x_1|^{r-1}\phi(x_1)|y_1 - y_2| + |y_2||x_1^{r-1}\phi(x_1) - x_2^{r-1}\phi(x_2) + x_1^{r-1}\phi(x_2) - x_2^{r-1}\phi(x_2)|)
\leq |k_3|_{\text{max}} (K_\phi R^{r-1}|y_1 - y_2| + Rx_1^{r-1}\phi(x_1) - \phi(x_2)| + K_\phi|x_1^{r-1} - x_2^{r-1}|
\leq |k_3|_{\text{max}} (K_\phi R^{r-1}|y_1 - y_2| + R\sqrt{C_R|x_1 - x_2|} + K_\phi\sqrt{|x_1 - x_2|})
\leq C_R (|x_1 - x_2| + |y_1 - y_2| + \sqrt{|x_1 - x_2|}),
\]

where we have used (3.24) and \( C_R := |k_3|_{\text{max}} \left( C_R R^r \vee K_\phi R^{r-1} \vee K_\phi \right) \). Thus, Assumption A holds for \( g \).
Proof. In the case of $x$’s outside a finite ball of radius $R$, with $R > 1$, and when $s \in [0, T]$ we have that

$$J(s, x) := \frac{xa(s, x) + (p - 1)b^2(s, x)/2}{1 + x^2} = \frac{x(k_1(s)x - k_2(s)x^q) + (p - 1)k_3^2(s)/2[x^r \phi(x)]^2}{1 + x^2}$$

$$\leq k_{1, \text{max}},$$

where the the last inequality is valid for all $p > 2$ and we have used $q + 1 > 2r$ and that $q$ is odd. Thus $J(s, x)$ is bounded for all $(s, x) \in [0, T] \times \mathbb{R}$, since when $|x| \leq R$ we have that $J(s, x)$ is finite and say $J(s, x) \leq C$. Application of [26, Theorem 2.4.1] implies

$$\mathbb{E}|x_t|^p \leq 2^{(p-2)/2}(1 + \mathbb{E}|x_0|^p)e^{CpT},$$

for any $2 < p$ and all $t \in [0, T]$. Using Ito’s formula on $|x_t|^p$, we have that

$$|x_t|^p = |x_0|^p + \int_0^t p|x_s|^{p-2}x_s(k_1(s)x_s - k_2(s)x_s^2)ds$$

$$+ \int_0^t \left( |x_s|^{p-2} + (p - 2)|x_s|^{p-4}x_s^2 \right) [k_3(s)x_s^r \phi(x_s)]^2 ds + \int_0^t pk_3(s)|x_s|^{p-2}x_s^r \phi(x_s)dW_s$$

$$\leq |x_0|^p + p \int_0^t \left[ k_1(s) - k_2(s)(x_s)^q + \frac{p - 1}{2}k_3^2(s)K^2 \phi(x_s)^{2r-2} \right] |x_s|^p ds$$

$$+ \int_0^t pk_3(s)\phi(x_s)|x_s|^p(x_s)^{r-1}dW_s$$

$$\leq |x_0|^p + C \int_0^t |x_s|^p ds + M_t,$$

where we have used that $0 < 2r-2 < q-1$, that $q$ is odd and $M_t = \int_0^t pk_3(s)\phi(x_s)|x_s|^p(x_s)^{r-1}dW_s$. Taking the supremum and then expectations in the above inequality we get

$$\mathbb{E}\left( \sup_{0 \leq t \leq T} |x_t|^p \right) \leq \mathbb{E}|x_0|^p + C\mathbb{E}\left( \sup_{0 \leq t \leq T} \int_0^t |x_s|^p ds \right) + \mathbb{E}\sup_{0 \leq t \leq T} M_t$$

$$\leq \mathbb{E}|x_0|^p + C\int_0^t \mathbb{E}\left( \sup_{0 \leq s \leq t} |x_t|^p \right) ds + \sqrt{\mathbb{E}\sup_{0 \leq t \leq T} M_t^2}$$

$$\leq \left( \mathbb{E}|x_0|^p + \sqrt{4 \mathbb{E}M_T^2} \right) e^{CT} := A_1,$$

where in the last step we have used Doob’s martingale inequality to the diffusion term $M_T^2$ and Gronwall inequality.

\[6\text{The function } h(u) = pk_3(u)\phi(x_u)|x_u|^p\sqrt{\phi(x_u)} \text{ belongs to the family } \mathcal{M}^2([0, T]; \mathbb{R}) \text{ thus [26] Theorem 1.5.8 implies } \mathbb{E}M_T^2 = \mathbb{E}\left( \int_0^T h(u)dW_u \right)^2 = \mathbb{E} \int_0^T h^2(u)du, \text{ i.e. } M_T \in \mathcal{L}^2(\Omega; \mathbb{R}).\]
Lemma 3.11. In the previous setting it holds that $x_t > 0$ a.s.

Proof. Set the stopping time $\theta_R = \inf \{t \in [0, T] : x_t^{-1} > R \}$, for some $R > 0$, with the convention that $\inf \emptyset = \infty$. Application of Ito’s formula on $\ln x_{t \wedge \theta_R}$ implies,

\[
\ln x_{t \wedge \theta_R} = \ln x_0 + \int_0^{t \wedge \theta_R} \frac{1}{x_s}(k_1(s)x_s - k_2(s)x_s^2)ds
\]

\[
+ \int_0^{t \wedge \theta_R} \left(-\frac{1}{x_s^2}\right) k_2^2(s)x_s^{2q}\phi^2(x_s)ds + \int_0^{t \wedge \theta_R} \frac{1}{x_s}k_3(s)x_s^r\phi(x_s)dW_s
\]

\[
= \ln x_0 + \int_0^{t \wedge \theta_R} \left(k_1(s) - k_2(s)x_s^{q-1} - k_3(s)\phi^2(x_s)\right)ds
\]

\[
+ \int_0^t k_3(s)x_s^{-1}\phi(x_s)I_{\{0, t \wedge \theta_R\}}(s)dW_s,
\]

where we have used that $q$ is odd. Taking absolute values in the above inequality and then expectations and using Jensen inequality and then Ito’s isometry on the diffusion term we get

\[
E|\ln x_{t \wedge \theta_R}| \leq E|\ln x_0| + |k_{1, \max}|T + |k_{2, \max}|T E\sup_{0 \leq t \leq T} |x_t|^{q-1} + |k_{3, \max}|^2K_0^2T + E|M_t|
\]

\[
\leq E|\ln x_0| + (|k_{1, \max}| + |k_{2, \max}|A_1 + |k_{3, \max}|^2K_0^2)T + \sqrt{4EM_t^2} < C,
\]

where $A_1$ is as in Lemma 3.10 and $M_t := \int_0^t k_3(s)x_s^{-1}\phi(x_s)I_{\{0, t \wedge \theta_R\}}(s)dW_s$. Now we proceed as in Lemmata 3.2 and 3.7 to get first that $\lim_{R \to \infty} P(\theta_R \leq t) = 0$ and then conclude that $P(x_t \leq 0)$, i.e. $x_t > 0$ a.s. \qed

Moment bound for semi-discrete approximation.

Lemma 3.12. In the previous setting it holds that

\[
E(\sup_{0 \leq t \leq T} (y_t)^p) < A_2,
\]

for some $A_2 > 0$ and for every $p > 2$.

Proof. Set the stopping time $\theta_R = \inf \{t \in [0, T] : y_t > R \}$, for some $R > 0$, with the convention that $\inf \emptyset = \infty$. Application of Ito’s formula on $(y_{t \wedge \theta_R})^p$, implies,

\[
(y_{t \wedge \theta_R})^p = (y_0)^p + \int_0^{t \wedge \theta_R} p(y_s)^{p-1}(k_1(s) - k_2(s)y_s^{q-1})y_sds
\]

\[
+ \int_0^{t \wedge \theta_R} \frac{p(p-1)}{2} (y_s)^{p-2} k_3(s)y_s^{q-1}\phi(y_s)y_s^2 ds + \int_0^{t \wedge \theta_R} pk_3(s)(y_s)^{p-1}y_s^{q-1}\phi(y_s)y_s^2dW_s
\]

\[
= (x_0)^p + \int_0^{t \wedge \theta_R} \left(p(k_1(s) - k_2(s)y_s^{q-1}) + \frac{p(p-1)k_3^2(s)}{2}y_s^{2q-2}\phi^2(y_s)\right)(y_s)^p ds
\]

\[
+ \int_0^{t \wedge \theta_R} pk_3(s)y_s^{q-1}\phi(y_s)(y_s)^p dW_s
\]

\[
\leq (x_0)^p + p \int_0^t \left[-k_2(s)(y_s)^{q-1} + \frac{p-1}{2}k_{3, \max}^2K_0^2y_s^{2q-2} + k_{1, \max} \right] (y_s)^p I_{\{0, t \wedge \theta_R\}}(s)ds + M_t
\]

\[
\leq (x_0)^p + C \int_0^t (y_s)^p I_{\{0, t \wedge \theta_R\}}(s)ds + M_t,
\]
where we have used that $q - 1 > 2r - 2 > 1$, the last inequality is valid for $p > 2$, the constant $C$ is independent of $R$ and $M_t := \int_0^t \phi(s) (y_t(s))^{r-1} \phi(y_t) dW_s$. Taking expectations and using that $EM_t = 0$ we get

$$E(y_{t \land \theta_R})^p \leq E(x_0)^p + C \int_0^t E(y_{s \land \theta_R})^p ds \leq E(x_0)^p e^{CT},$$

where in the second step we have applied Gronwall inequality. We have that

$$E \quad using that

$$E \quad thus taking expectations in the above inequality and using the estimated upper bound for $E(y_{t \land \theta_R})^p$ we arrive at

$$E(y_t)^p \leq E(x_0)^p e^{CT}$$

and taking limits in both sides as $R \to \infty$ we get that

$$\lim_{R \to \infty} E(y_t)^p_{t \land \theta_R} \leq E(x_0)^p e^{CT}$$

Fix $t$. The sequence $(y_t)^p_{t \land \theta_R}$ is nondecreasing in $R$ since $\theta_R$ is increasing in $R$ and $t \land \theta_R \to t$ as $R \to \infty$ and $(y_t)^p_{t \land \theta_R} \to (y_t)^p$ as $R \to \infty$, thus the monotone convergence theorem implies

$$E(y_t)^p \leq E(x_0)^p e^{CT},$$

for any $2 < p$. Following the same lines as in Lemma 3.10 i.e. using again Ito’s formula on $(y_t)^p$, taking the supremum and then using Doob’s martingale inequality on the diffusion term we obtain the desired result. \hfill \Box

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