A note on the fractal dimensions of invariant measures associated with $C^{1+\alpha}$-diffeomorphisms, expanding and expansive homeomorphisms

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Abstract

We show in this work that the upper and the lower generalized fractal dimensions $D_{\pm}^\mu(q)$, for each $q \in \mathbb{R}$, of an ergodic measure associated with an invertible bi-Lipschitz transformation over a Polish metric space are equal, respectively, to its packing and Hausdorff dimensions. This is particularly true for hyperbolic ergodic measures associated with $C^{1+\alpha}$-diffeomorphisms of smooth compact Riemannian manifolds, from which follows an extension of Young’s Theorem [31]. Analogous results are obtained for expanding systems. Furthermore, for expansive homeomorphisms (like $C^1$-Axiom A systems), we show that the set of invariant measures with zero correlation dimension, under a hyperbolic metric, is generic (taking into account the weak topology). We also show that for each $s \geq 0$, $D_{+}^\mu(s)$ is bounded above, up to a constant, by the topological entropy, also under a hyperbolic metric.

Key words and phrases. Expansive homeomorphisms, expanding maps, Hausdorff dimension, packing dimension, invariant measures, generalized fractal dimensions.

1 Introduction

The dimension theory of invariant measures plays a very important role in the theory of dynamical systems.

There are several different notions of dimension for more general sets, some easier to compute and others more convenient in applications. One of them is, and could be said to be the most popular of all, the Hausdorff dimension, introduced in 1919 by Hausdorff, which gives a notion of size useful for distinguishing between sets of zero Lebesgue measure.

Unfortunately, the Hausdorff dimension of relatively simple sets can be very hard to calculate; besides, the notion of Hausdorff dimension is not completely adapted to the dynamics per se (for instance, if $Z$ is a periodic orbit, then its Hausdorff dimension is zero, regardless to whether the
Thus, in order to obtain relevant information about the dynamics, one should consider not only the geometry of the measurable set $Z \subset X$ (where $X$ is some Borel measurable space), but also the distribution of points on $Z$ under $f$ (which is assumed to be a measurable transformation). That is, one should be interested in how often a given point $x \in Z$ visits a fixed subset $Y \subset Z$ under $f$. If $\mu$ is an ergodic measure for which $\mu(Y) > 0$, then for a typical point $x \in Z$, the average number of visits is equal to $\mu(Y)$. Thus, the orbit distribution is completely determined by the measure $\mu$. On the other hand, the measure $\mu$ is completely specified by the distribution of a typical orbit.

This fact is widely used in the numerical study of dynamical systems where the distributions are, in general, non-uniform and have a clearly visible fine-scaled interwoven structure of hot and cold spots, that is, regions where the frequency of visitations is either much greater than average or much less than average respectively.

In this direction, the so-called correlation dimension of a probability measure was introduced by Grassberger, Procaccia and Hentschel [19] in an attempt to produce a characteristic of a dynamical system that captures information about the global behavior of typical (with respect to an invariant measure) trajectories by observing only one them.

This dimension plays an important role in the numerical investigation of chaotic behavior in different models, including strange attractors. The formal definition is as follows (see [16, 17, 18]): let $(X, r)$ be a complete and separable (Polish) metric space, and let $f : X \to X$ be a continuous map. Given $x \in X$, $\varepsilon > 0$ and $n \in \mathbb{N}$, one defines the correlation sum of order $q \in \mathbb{N} \setminus \{1\}$ (specified by the points $\{f^i(x)\}, i = 1, \ldots, n$) by

$$C_q(x, n, \varepsilon) = \frac{1}{n^q} \text{card } \{(i_1 \cdots i_q) \in \{0, 1, \cdots, n\}^q | r(f^{i_j}(x), f^{i_l}(x)) \leq \varepsilon \text{ for any } 0 \leq j, l \leq q\},$$

where $\text{card } A$ is the cardinality of the set $A$. Given $x \in X$, one defines (when the limit $n \to \infty$ exists) the quantities

$$\underline{\alpha}_q(x) = \frac{1}{q - 1} \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\log C_q(x, n, \varepsilon)}{\log(\varepsilon)}, \quad \overline{\alpha}_q(x) = \frac{1}{q - 1} \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\log C_q(x, n, \varepsilon)}{\log(\varepsilon)},$$

the so-called lower and upper correlation dimensions of order $q$ at the point $x$ or the lower and the upper $q$-correlation dimensions at $x$. If the limit $\varepsilon \to 0$ exists, we denote it by $\alpha_q$, the so-called $q$-correlation dimension at $x$. In this case, if $n$ is large and $\varepsilon$ is small, one has the asymptotic relation

$$C_q(x, n, \varepsilon) \sim \varepsilon^{\alpha_q}.$$  

$C_q(x, n, \varepsilon)$ gives an account of how the orbit of $x$, truncated at time $n$, “folds” into an $\varepsilon$-neighborhood of itself; the larger $C_q(x, n, \varepsilon)$, the “tighter” this truncated orbit is. $\underline{\alpha}_q(x)$ and $\overline{\alpha}_q(x)$ are, respectively, the lower and upper growing rates of $C_q(x, n, \varepsilon)$ as $n \to \infty$ and $\varepsilon \to 0$ (in this order).

**Definition 1.1** (Energy function). Let $X$ be a general metric space and let $\mu$ be a Borel probability measure on $X$. For $q \in \mathbb{R} \setminus \{1\}$ and $\varepsilon \in (0, 1)$, one defines the so-called energy
function of \( \mu \) by the law

\[
I_\mu(q, \varepsilon) = \int_{\text{supp}(\mu)} \mu(B(x, \varepsilon))^{q-1} d\mu(x),
\]

where \( \text{supp}(\mu) \) is the topological support of \( \mu \).

The next result shows that the two previous definitions are intimately related.

**Theorem 1.1** (Pesin [17, 18]). Let \( X \) be a Polish metric space, assume that \( \mu \) is ergodic and let \( q \in \mathbb{N} \setminus \{1\} \). Then, there exists a set \( Z \subset X \) of full \( \mu \)-measure such that, for each \( R, \eta > 0 \) and each \( x \in Z \), there exists an \( N = N(x, \eta, R) \in \mathbb{N} \) such that

\[
|C_q(x, n, \varepsilon) - I_\mu(q, \varepsilon)| \leq \eta
\]

holds for each \( n \geq N \) and each \( 0 < \varepsilon \leq R \). In other words, \( C_q(x, n, \varepsilon) \) tends to \( I_\mu(q, \varepsilon) \) when \( n \to \infty \) for \( \mu \)-almost every \( x \in X \), uniformly over \( \varepsilon \in (0, R] \).

Taking into account Theorem 1.1, it is natural to introduce the following dimensions.

**Definition 1.2** (Generalized fractal dimensions). Let \( X \) be a general metric space, let \( \mu \) be a Borel probability measure on \( X \), and let \( q \in \mathbb{R} \setminus \{1\} \). The so-called upper and lower \( q \)-generalized fractal dimensions of \( \mu \) are defined, respectively, as

\[
D^+_{\mu}(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log I_\mu(q, \varepsilon)}{(q-1) \log \varepsilon}
\]

and

\[
D^-_{\mu}(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log I_\mu(q, \varepsilon)}{(q-1) \log \varepsilon}.
\]

If the limit \( \varepsilon \to 0 \) exists, we denote it by \( D_{\mu}(q) \), the so-called \( q \)-generalized fractal dimension (also known as \( q \)-Hentzel-Procaccia dimension). For \( q = 1 \), one defines the so-called upper and lower entropy dimensions (see [1] for a discussion about the connection between entropy dimensions and Rényi information dimensions), respectively, as

\[
D^+_{\mu}(1) = \limsup_{\varepsilon \downarrow 0} \frac{\int_{\text{supp}(\mu)} \log \mu(B(x, \varepsilon)) d\mu(x)}{\log \varepsilon},
\]

and

\[
D^-_{\mu}(1) = \liminf_{\varepsilon \downarrow 0} \frac{\int_{\text{supp}(\mu)} \log \mu(B(x, \varepsilon)) d\mu(x)}{\log \varepsilon}.
\]

**Definition 1.3** (lower and upper packing and Hausdorff dimensions of a measure [13]). Let \( \mu \) be a positive Borel measure on \( (X, \mathcal{B}) \). The lower and upper packing and Hausdorff dimensions of \( \mu \) are defined, respectively, as

\[
\dim^-_{K}(\mu) = \inf\{\dim_{K}(E) \mid \mu(E) > 0, \ E \in \mathcal{B}\},
\]

\[
\dim^+_{K}(\mu) = \inf\{\dim_{K}(E) \mid \mu(X \setminus E) = 0, \ E \in \mathcal{B}\},
\]

where \( K \) stands for \( H \) (Hausdorff) or \( P \) (packing); here, \( \dim_{H(P)}(E) \) represents the Hausdorff (packing) dimension of the Borel set \( E \) (see [13] for details).

**Definition 1.4** (lower and upper local dimensions of a measure). Let \( \mu \) be a positive finite Borel measure on \( X \). One defines the upper and lower local dimensions of \( \mu \) at \( x \in X \) as

\[
\overline{d}_{\mu}(x) = \limsup_{\varepsilon \to 0} \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon}
\]

and

\[
\underline{d}_{\mu}(x) = \liminf_{\varepsilon \to 0} \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon},
\]

if, for every \( \varepsilon > 0 \), \( \mu(B(x; \varepsilon)) > 0 \); if not, \( \overline{d}_{\mu}(x) := +\infty \).
Some useful relations involving the generalized, Hausdorff and packing dimensions of a probability measure are given by the following inequalities, which combine Propositions 4.1 and 4.2 in [1] with Proposition 1.1 in [5] (although Propositions 4.1 and 4.2 in [1] were originally proved for probability measures defined on \( \mathbb{R} \), one can extend them to probability measures defined on a general metric space \( X \); see also [21]).

**Proposition 1.1** ([1, 21]). Let \( \mu \) be a Borel probability measure over \( X \), let \( q > 1 \) and let \( 0 < s < 1 \). Then,

\[
D_{\mu}^- (q) \leq \mu \text{-ess inf } d_{\mu}(x) = \dim_H^-(\mu) \leq \mu \text{-ess sup } d_{\mu}(x) = \dim_H^+(\mu) \leq D_{\mu}^-(s),
\]

and

\[
D_{\mu}^+ (q) \leq \mu \text{-ess inf } d_{\mu}(x) = \dim_P^- (\mu) \leq \mu \text{-ess sup } d_{\mu}(x) = \dim_P^+(\mu) \leq D_{\mu}^+(s).
\]

Moreover, \( D_{\mu}^\pm (q) \leq D_{\mu}^\pm (1) \leq D_{\mu}^\pm (s) \).

Our main result states that if \( X \) is a Polish metric space, \( f : X \to X \) is an invertible transformation such that both \( f \) and \( f^{-1} \) are Lipschitz, and \( \mu \in \mathcal{M}_{e}(f) \) (where \( \mathcal{M}_{e}(f) \) is the set of the \( f \)-ergodic measures, which we suppose that is nonempty), then the inequalities in (3) and (4) become identities.

**Theorem 1.2.** Let \( X \) be a Polish space and let \( f : X \to X \) be an invertible transformation such that both \( f \) and \( f^{-1} \) are Lipschitz. Then, for each \( \mu \in \mathcal{M}_{e}(f) \) and each \( q \in \mathbb{R} \), one has

\[
D_{\mu}^- (q) = \dim_H^\pm (\mu) \quad \text{and} \quad D_{\mu}^+ (q) = \dim_P^\pm (\mu).
\]

There are numerous important examples in the literature of dynamical systems that satisfy the hypotheses of Theorem 1.2, such as:

1. \( X \) is a smooth \( n \)-dimensional compact manifold and \( f : X \to X \) is a \( C^1 \)-diffeomorphism.
2. \( X = \prod_{i=\infty}^{\infty} M \) is the bilateral product of a countable number of copies of the Polish metric space \( (M, \rho) \), endowed with the metric

\[
d(x, y) = \sum_{|n| \geq 0} \frac{1}{2^{|n|}} \frac{\rho(x_n, y_n)}{1 + \rho(x_n, y_n)},
\]

and \( f : X \to X \) is the full-shift operator.
3. \( X \) is a compact metric space and \( f : X \to X \) is an expansive homeomorphism (see Definition 1.7 and Theorem 1.5 for details).

We also explore throughout the text and provide, for Lipschitz and expanding transformations, some results that relate the local dimensions of invariant measures with their respective metric entropy (see Lemma 4.1 and Section 4 for details).
1.1 $C^{1+\alpha}$-diffeomorphisms of smooth compact Riemannian manifolds

In case $\mu \in \mathcal{M}_e(f)$ is such that $d_\mu(x) = d_\mu(y)$ for $\mu$-a.e. $x \in X$, it follows from Proposition [11] and Theorem [12] that the upper and lower fractal dimensions of $\mu$ coincide with its Hausdorff and packing dimensions, respectively. This is particularly true if $f : M \to M$ is a $C^{1+\alpha}$-diffeomorphism of a compact Riemannian manifold $M$ and $\mu$ is a hyperbolic ergodic measure (Barreira-Pesin-Schmeling’s Theorem [2]), or if $f : M \to M$ is a $C^{1+\alpha}$-diffeomorphism of a smooth compact surface $M$ and $\mu$ is a hyperbolic ergodic measure with Lyapunov exponents $\lambda_1(\mu) \geq \lambda_2(\mu)$ (Young’s Theorem [31]).

Given a point $x \in M$ in a $p$-dimensional compact Riemannian manifold and $v \in T_xM$, one defines the Lyapunov exponent of $v$ at $x$ by

$$\lambda(x,v) = \limsup_{n \to \infty} \frac{\log \|df^n x\v|}{n} > 0.$$ 

If $x$ is fixed, then the function $\lambda(x,\cdot)$ can assume only finite many distinct values $\lambda^1(x) > \ldots > \lambda^q(x)$, where $q = q(x)$, $q \in\{1,\ldots,p\}$. The functions $q(x), \lambda^i(x)$ are measurable and $f$-invariant. If $\mu \in \mathcal{M}_e(f)$, then the above functions are $\mu$-a.e. constant. One denotes the corresponding values by $\lambda^i_\mu$. A measure $\mu \in \mathcal{M}_e(f)$ is said to be hyperbolic if there exists a $k \in \{1,\ldots,q-1\}$ such that $\lambda^1_\mu > \ldots > \lambda^k_\mu > 0 > \lambda^{k+1}_\mu > \ldots > \lambda^q_\mu$.

**Corollary 1.1.** Let $f : M \to M$ be a $C^{1+\alpha}$-diffeomorphism of a smooth compact Riemannian manifold $M$ and let $\mu$ be a hyperbolic ergodic measure. Then, for $\mu$-a.e. $x \in X$ and for each $q \in \mathbb{R}$, one has

$$D^+_\mu(q) = D^-_\mu(q) = d^{(u)} + d^{(s)}, \quad (5)$$

where $d^{(u)}$ and $d^{(s)}$ stand, respectively, for the (exact) pointwise dimension of the unstable (stable) component of $\mu$ (see [18]) for details.

**Corollary 1.2.** Let $f : M \to M$ be a $C^{1+\alpha}$-diffeomorphism of a smooth compact surface $M$ and let $\mu$ be a hyperbolic ergodic measure with Lyapunov exponents $\lambda^1_\mu > 0 > \lambda^2_\mu$. Then, for each $q \in \mathbb{R}$,

$$D^+_\mu(q) = D^-_\mu(q) = h_\mu(f) \left[ \frac{1}{\lambda^1_\mu} - \frac{1}{\lambda^2_\mu} \right]. \quad (6)$$

**Remark 1.1.** We note that, as in Theorem 4.4 in [31], one has for each $q \in \mathbb{R}$,

$$D^+_\mu(q) = C(\mu) = C(\mu) = C(L)(\mu) = C(L)(\mu) = R(\mu) = R(\mu) = h_\mu(T) \left[ \frac{1}{\lambda^1_\mu} - \frac{1}{\lambda^2_\mu} \right],$$

where $C(\mu), C(L)(\mu), C(L)(\mu), R(\mu)$ are the (lower and upper) capacity and the (lower and upper) modified capacity dimensions of $\mu$, respectively, and $R(\mu), R(\mu)$ are the (lower and upper) Renyi dimensions of $\mu$. So, for such measures, the main notions of dimension presented in the literature coincide. Let us call such number $\dim(\mu)$.

We also have something to say about Axiom A systems. For that, some preparation is required.
Definition 1.5. A closed subset $\Lambda \subset M$ is said to be hyperbolic if $f(\Lambda) = \Lambda$ and if each tangent fiber $T_xM$, $x \in \Lambda$, can be written as a direct sum $T_xM = E^u_x \oplus E^s_x$ of subspaces so that:

(a) $df_x(E^s_x) = E^s_{f(x)}$, $df_x(E^u_x) = E^u_{f(x)}$;
(b) there exist constants $c > 0$ and $\lambda \in (0,1)$ so that

$$
\|df^n_x(v)\| \leq c\lambda^n\|v\| \quad \text{if } v \in E^s_x, \ n \geq 0,
$$

and

$$
\|df^{-n}_x(v)\| \leq c\lambda^n\|v\| \quad \text{if } v \in E^u_x, \ n \geq 0;
$$

(c) $E^s_x, E^u_x$ vary continuously with $x$.

A point $x \in M$ is said to be non-wandering if, for each neighborhood $U$ of $x$,

$$
U \cap \bigcup_{n>0} f^n U \neq \emptyset.
$$

The set $\Omega = \Omega(f)$ of all non-wandering points is closed and $f$-invariant. A point $x$ is periodic if $f^n x = x$ for some $n > 0$; clearly, each periodic point belongs to $\Omega$.

Definition 1.6. $f$ is called an Axiom A diffeomorphism if $\Omega$ is a hyperbolic set and the set of periodic points is dense in $\Omega$.

Remark 1.2. Smale’s Spectral Decomposition Theorem (see [26]) states that if $f$ satisfies the Axiom A conditions, then one can write $\Omega = \Omega_1 \cup \cdots \Omega_M$, where the $\Omega_i$ are disjoint closed $f$-invariant sets and $f \mid \Omega_i$ is topologically transitive. In what follows, let $T : X \to X$, where $X := \Omega_l$ and $T := f \mid \Omega_l$. The system $(T, X)$ is called an Axiom A system.

Let $T : X \to X$ be a $C^{1+\alpha}$-Axiom A system over a compact smooth manifold $M$. Let $\mathcal{M}(T)$ be the space of all $T$-invariant probability measures, endowed with the weak topology (that is the coarsest topology for which the net $\{\mu_\alpha\}$ converges to $\mu$ if, and only if, for each bounded and continuous function $\varphi$, $\int \varphi d\mu_\alpha \to \int \varphi d\mu$). Such space is metrizable (take, for instance, the Lévy-Prohorov metric; see [3]). Note that every $\mu \in \mathcal{M}(T)$ is hyperbolic (see [18]).

Theorems 4 and 6 in [22] state that $\{\mu \in \mathcal{M}_c(T) \mid h_\mu(T) = 0\}$ is a residual subset of $\mathcal{M}(T)$. The next result is a direct consequence of this fact, Corollary [12] and Remark [11].

Theorem 1.3. Let $T : X \to X$ be a $C^{1+\alpha}$-Axiom A system over a compact smooth manifold $M$. Then, the set $D_0 := \{\mu \in \mathcal{M}_c(T) \mid \dim(\mu) = 0\}$ is residual in $\mathcal{M}(T)$.

Theorems [11] and [13] may be combined in order to produce the following result. Let $q \in \mathbb{N} \setminus \{1\}$; if $\mu \in D_0$, then there exists a Borel set $Z \subset X$, $\mu(Z) = 1$, such that for each $x \in Z$, one has $\mathcal{C}_q(x) = D^{\gamma_q}_\mu(q) = 0$.

This means that for each $x \in Z$ and each $\alpha > 0$, there exists a $\delta = \delta(x, \alpha) > 0$ such that if $0 < \varepsilon < \delta$, then there exists an $N = N(x, \alpha, \beta) \in \mathbb{N}$ such that, for each $n > N$, one has $C_q(x, n, \varepsilon) \geq \varepsilon^{(q-1)\alpha}$. Thus, one has $\gamma = \text{card} \{\{i_1 \cdots i_q\} \in \{0,1,\cdots,n\}^q \mid d(T^{i_l}(x), T^{i_j}(x)) \leq \varepsilon \}$ for each $j, l = 0, \ldots, q \geq \varepsilon^{(q-1)\alpha} n^q$, which means that $\gamma$ is of order $n^q$ for $n$ large enough. The conclusion is that the orbit of a typical point (with respect to $\mu$) is very “tight” (it is some sense, similar to a periodic orbit).
1.2 Full-shift operator over a perfect and separable metric space

Let \( X = \prod_{i=-\infty}^{+\infty} M \) be the bilateral product of a countable number of copies of the perfect and separable metric space \((M, \rho)\), endowed with the metric

\[
d(x, y) = \sum_{|n| \geq 0} \frac{1}{2^{|n|}} \frac{\rho(x_n, y_n)}{1 + \rho(x_n, y_n)},
\]

and let \( f : X \to X \) be the respective full-shift operator. The next result shows that dimensional properties of a typical invariant measure of such system starkly differs from the typical behavior depicted in Theorem 1.3.

**Theorem 1.4.** Let \((X, f, \mathcal{B})\) be the full-shift dynamical system over \( X = \prod_{i=-\infty}^{+\infty} M \), where the alphabet \( M \) is a perfect and separable metric space. Then, the set \( \mathcal{P}H := \{ \mu \in \mathcal{M}_e(f) \mid 0 = D^{-}_\mu(q) < D^{+}_\mu(q) = +\infty, \text{ for each } q \in \mathbb{R} \} \) is generic in \( \mathcal{M}(f) \).

Theorem 1.4 follows from Theorem 1.2 and Theorem 1.1(III-IV) in [4]. One should compare it to Theorems 1.2 and 1.3 in [4]; here, \( M \) is only separable; there, \( M \) is compact. The metric here makes \( f \) a bi-Lipschitz transformation; there, the metric is sub-exponential, and \( f \) is not a bi-Lipschitz transformation.

Let \( q \in \mathbb{N} \setminus \{1\} \); if \( \mu \in \mathcal{P}H \), then there exists a Borel set \( Z \subset X \), \( \mu(Z) = 1 \), such that for each \( x \in Z \), one has \( \alpha_q(x) = D^{-}_\mu(q) = 0 \) and \( \overline{\alpha_q}(x) = D^{+}_\mu(q) = \infty \).

This means that if \( x \in Z \), since \( \alpha_q(x) = 0 \), it follows that given \( 0 < \alpha \ll 1 \) and \( R > 0 \), there exist a radial sequence \( (\varepsilon_k) \), with \( \varepsilon_k \in (0, R) \), and an \( N_k = N_k(x, \alpha, R) \in \mathbb{N} \) such that, for each \( n > N_k \), one has \( C_q(x, n, \varepsilon_k) \geq \varepsilon^{(q-1)\alpha}_k \). Thus, there exists a scale (defined by \( (\varepsilon_k) \)) such that \( F_k = \text{card} \{(i_1 \cdot \cdot \cdot i_q) \in \{0,1,\ldots,n\}^q \mid r(f^{i_j}(x), f^{i_l}(x)) \leq \varepsilon_k \text{ for each } 0 \leq j, l \leq q \} \geq \varepsilon^{(q-1)\alpha}_k n^q \); in this scale, the quantity \( F_k \) is of order \( n^q \) for each \( n \) and each \( k \) large enough. This means that, at least in this scale, the orbit of a typical point (with respect to \( \mu \)) is similar to a periodic orbit.

Nonetheless, since \( \overline{\alpha_q}(x) = +\infty \), it follows that given \( \beta \gg 1 \) and \( S > 0 \), there exist a radial sequence \( (s_k) \), with \( s_k \in (0, S) \), and an \( N_\ell \in \mathbb{N} \) such that, for each \( n > N_\ell \), one has \( C_q(x, n, s_\ell) \leq s_\ell^{(q-1)\beta} \). Thus, there exists a scale such that \( P_\ell = \text{card} \{(i_1 \cdot \cdot \cdot i_q) \in \{0,1,\ldots,n\}^q \mid r(f^{i_j}(x), f^{i_l}(x)) \leq s_\ell \text{ for each } 0 \leq j, l \leq q \} \leq s_\ell^{(q-1)\beta} n^q \); in this scale, \( P_\ell \) is of lesser order than \( n^q \), which means that (at least in this scale) the orbit of a typical point spreads fast.

In summary, the orbit of a point \( x \in Z \) has a very complex structure, being “tight” for some spatial scale, and spreading fast throughout the space for another scale.

1.3 Expansive homeomorphisms

We are also interested in dimensional properties of invariant (not necessarily ergodic) measures for expansive homeomorphisms.

**Definition 1.7.** Let \( X \) be a metrizable space, and let \( f : X \to X \) be a homeomorphism. \( f \) is said to be expansive if there exists a \( \delta > 0 \) such that, for each pair of different points \( x, y \in X \), there exist an \( n \in \mathbb{Z} \) such that \( d(f^n(x), f^n(y)) > \delta \), where \( d \) is any metric which induces the topology of \( X \).
Note that expansivity is a topological notion, i.e., it does not depend on the choice of a particular (compatible) metric under consideration, although the expansivity constant $\delta$ may depend on $d$.

Examples of expansive homeomorphisms are: Axiom A systems (see [8]), homeomorphisms that admit a Lyapunov function (see [12]), examples 1 and 2 in [29], the shift system with finite alphabet, pseudo-Anosov homeomorphisms, quasi-Anosov diffeomorphisms, etc.

The following result shows that if $X$ is a compact metrizable space, then a homeomorphism $f : X \to X$ is expansive if $X$ admits a hyperbolic metric (the converse of this statement is also true; see Theorem 5.3 in [10]).

**Theorem 1.5** (Theorem 5.1 in [10]). If $f : X \to X$ is an expansive homeomorphism over the compact metrizable space $X$, then there exist a metric $d$ on $X$, compatible with its topology, and numbers $k > 1$, $\varepsilon > 0$ such that, for each $x, y \in X$,

$$\max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} \geq \min\{k d(x, y), \varepsilon\}. \quad (7)$$

Moreover, both $f$ and $f^{-1}$ are Lipschitz for $d$. The metric $d$ is called a hyperbolic metric for $X$.

Aside from the results stated in Theorem 1.2, we also have some estimates for the generalized fractal dimensions of invariant measures of expansive homeomorphisms (with respect to the hyperbolic metric given by Theorem 1.5) in terms of the metric and the topological entropies.

**Proposition 1.2.** Let $f : X \to X$ be an expansive homeomorphism over a compact metric space $X$, and let $d$ be the respective hyperbolic metric. Then, for each invariant measure $\mu \in \mathcal{M}(f)$ and each $q \in [0, 1)$, one has $D^+_\mu(q) \leq \frac{2h(f)}{\log k}$, where $k$ is defined in the statement of Theorem 1.5.

**Remark 1.3.** 1. If $\mu \in \mathcal{M}_e(f)$, then Proposition 1.2 is a direct consequence of Theorem 1.2 and Theorem 5.4 in [10], given that $\dim^+\mu$ is bounded above by the upper capacity $\overline{C}(X) := \limsup_{\varepsilon \to 0} \frac{N(X, \varepsilon)}{-\log \varepsilon}$, where $N(X, \varepsilon)$ is the minimum number of balls of radius $\varepsilon > 0$ needed to cover $X$.

2. One should compare Proposition 1.2 with Theorem 5.4 in [10].

**Proposition 1.3.** Let $f : X \to X$ be an expansive homeomorphism over a compact metric space $X$, and let $d$ be the respective hyperbolic metric. Then, for each invariant measure $\mu \in \mathcal{M}(f)$ and each $q \geq 1$, one has $D^+_\mu(q) \leq h_\mu(f) \log k$, where $k$ is defined in the statement of Theorem 1.5.

**Remark 1.4.** One may wonder if there is a contradiction between Proposition 1.3 and Theorem 1.4 (since, by Corollary 4.2, $\{\mu \in \mathcal{M}(T) \mid h_\mu(T) = 0\}$ is a residual subset of $\mathcal{M}(T)$). Although the full-shift over a finite alphabet is an example of expansive homeomorphism, this is not the case when $M$ is a perfect compact metric space: such result can be seen as a consequence of Mañé’s Theorem (see Corollary 5.4 in [10]).

**Corollary 1.3.** Let $X$ be a compact metric space, let $f : X \to X$ be an expansive homeomorphism and let $q \geq 1$. If there exists $\mu \in \mathcal{M}(f)$ such that $D^+_\mu(q) > 0$ (with respect to a hyperbolic metric), then $h(f) \geq h_\mu(f) > 0$.

Since each $C^1$-Axiom A system over a compact smooth manifold $M$ is an expansive homeomorphism (see [8]), the next result is a consequence of this fact, Theorems 4 and 6 in [22], Theorems 1.2, 1.3 and Proposition 1.3 (it is easy to check that the usual metric in $M$ satisfies (7)).
Theorem 1.6. Let \( T : X \to X \) be a \( C^1 \)-Axiom A over a compact smooth manifold \( M \) and let \( q \in \mathbb{R} \). Then, the set \( \{ \mu \in \mathcal{M}_e(T) \mid D_\mu^+(q) = 0 \} \) is residual in \( \mathcal{M}(T) \).

Remark 1.5. Theorem 1.6 extends Theorem 1.3 to \( C^1 \)-Axiom A systems.

Corollary 1.4. Let \( T : X \to X \) be a \( C^1 \)-Axiom A over a compact smooth manifold \( M \). Then, the set \( \{ \mu \in \mathcal{M}_e(T) \mid \dim(\mu) = 0 \} \) is residual in \( \mathcal{M}(T) \) (see Remark 1.1 for the definition of \( \dim(\mu) \)).

Sigmund has proved in [24] that if \( T : X \to X \) is a \( C^1 \)-Axiom A over a compact smooth manifold \( M \), then the set \( \{ \mu \in \mathcal{M}_e(T) \mid h_\mu(T) > 0 \} \) is dense in \( \mathcal{M}(T) \). It is natural to ask if there exists an \( \alpha > 0 \) such that \( \{ \mu \in \mathcal{M}_e(T) \mid h_\mu(T) \geq \alpha \} \) is also dense.

Corollary 1.5. Let \( T : X \to X \) be a \( C^1 \)-Axiom A over a compact smooth manifold \( M \), and let \( \alpha > 0 \). Then, the set \( \{ \mu \in \mathcal{M}_e(T) \mid h_\mu(T) \geq \alpha \} \) is nowhere dense in \( \mathcal{M}(T) \).

1.4 Expanding maps

Let \((X, f)\) be a topological dynamical system (that is, \(X\) is a compact metric space and \(f\) is continuous). One says that \(f\) is expanding if it is a local homeomorphism and if there exist constants \(1 < \lambda \leq \Lambda \) (the so-called expanding constants) and \(\varepsilon_0 > 0\) such that, for each \(x \in X\) and each \(0 < \varepsilon < \varepsilon_0\),

\[
B(f(x), \lambda \varepsilon) \subset f(B(x, \varepsilon)) \subset B(f(x), \Lambda \varepsilon).
\]

It follows from this definition that for each \(0 < \varepsilon < \varepsilon_0\), one has:

i) if \(x, z \in X\) are such that \(d(f(x), f(z)) < \lambda \varepsilon\), then \(d(x, z) < \varepsilon\);

ii) if \(x, z \in X\) are such that \(d(x, z) < \varepsilon\), then \(d(f(x), f(z)) < \Lambda \varepsilon\);

iii) \(f\) is locally bi-Lipschitz.

Without loss of generality, one may assume that for any \(x \in X\), the map \(f\) restricted to the ball \(B(x, \varepsilon_0)\) is a homeomorphism.

Examples of expanding maps include conformal, weakly-conformal, quasi-conformal maps, the non-conformal expanding maps of the torus, given by \(T(x, y) = (lx \mod 1, my \mod 1)\), \(l > m \geq 2\), \(l, m \in \mathbb{N}\) (see [30], see also [11] and [6] for more examples).

The next result is a consequence of Lemma 4.1 and a modified version of Theorem 1.2 (see Remark 2.2).

Theorem 1.7. Let \(f : X \to X\) be an expanding mapping with expanding constants \(1 < \lambda \leq \Lambda\), and let \(\mu \in \mathcal{M}_e(f)\). Then, for each \(q \in \mathbb{R}\), one has

\[
\frac{h_f(\mu)}{\log \lambda} \leq \dim^\pm_H(\mu) = D^-_\mu(q) \leq D^+_\mu(q) = \dim^\pm_P(\mu) \leq \frac{h_f(\mu)}{\log \lambda}.
\]

There are some situations in the literature of expanding maps where the local dimensions of their respective ergodic measures are known. This is particularly true for the ergodic measures of positive metric entropy supported on a conformal repeller \(J\).
**Definition 1.8.** Let \( f : M \to M \) be a \( C^{1+\alpha} \) map of a smooth Riemannian manifold \( M \) and let \( J \subset M \) be a compact invariant subset (i.e., \( f^{-1}(J) = J \)). One says that \( f \) is expanding on \( J \) and that \( J \) is a repeller for \( f \) if

i) there exists an open neighborhood \( V \) of \( J \) (called a basin) such that
\[
J = \{ x \in V \mid f^n(x) \in V \text{ for each } n \geq 0 \};
\]

ii) there exist \( C > 0 \) and \( \lambda > 1 \) such that, for each \( x \in J \), each \( v \in T_xM \) and each \( n \in \mathbb{N} \),
\[
\| df^n_x v \| \geq C \lambda^n \| v \| \text{ (with respect to a Riemannian metric on } M).\]

A smooth map \( f : M \to M \) is called conformal if \( df_x = a(x) \text{Isom}_x \) for each \( x \in M \), where \( \text{Isom}_x \) is an isometry of \( T_xM \) and \( a(x) \) is a Hölder continuous function on \( M \). A conformal map \( f \) is expanding if, for each \( x \in M \), \(|a(x)| > 1\). A repeller \( J \) for a conformal expanding map is called a conformal repeller.

The next result, which follows from Theorem 21.3 in [18] and Remark 2.2, should be compared to Corollary 1.2.

**Corollary 1.6.** Let \( \mu \) be a Borel ergodic measure of positive entropy supported on a conformal repeller \( J \). Then, one has
\[
\dim(\mu) = \frac{h_f(\mu)}{\lambda_\mu},
\]
where \( \dim(\mu) \) is defined in Remark 1.4 and \( \lambda_\mu = \lim_{n \to \infty} \frac{\log \| df^n_x \|}{n} > 0 \) (such limit exists \( \mu \)-a.e. as a constant).

### 1.5 Organization

The paper is organized as follows. In Section 2, we present some auxiliary results used in the proof of Theorem 1.2, which we also present there. Section 3 is devoted to the proofs of Propositions 1.2, 1.3 and Corollary 1.5. In Section 4, we present some relations between the metric entropy and the local dimensions of an invariant measure and show that, for some dynamical systems, the metric entropy of an invariant measure is typically zero, settling a conjecture posed by Sigmund in [25] for Lipschitz transformations which satisfy the specification property.

### 2 Generalized fractal dimensions of ergodic measures of bi-Lipshitz transformations

This section is devoted to the proof of Theorem 1.2 for which some preparation is required.

For each \( t > 0 \), let \( \varepsilon = 1/t \). Since, for each \( x \in X \),
\[
\overline{d}_\mu(x) = \lim_{s \to \infty} \sup \left( \inf_{t \geq s} \frac{\log \mu(B(x, \varepsilon))}{\log t} \right),
\]
one sets, for each $s \in \mathbb{N}$,
\[
\bar{\beta}_\mu(x, s) = \sup_{t>s} \frac{\log \mu(B(x, \varepsilon))}{-\log t} \quad \text{and} \quad \underline{\beta}_\mu(x, s) = \inf_{t>s} \frac{\log \mu(B(x, \varepsilon))}{-\log t};
\]
note that $\bar{\beta}_\mu(x, s)$ is non-decreasing, whereas $\underline{\beta}_\mu(x, s)$ is non-increasing in $s$.

**Lemma 2.1.** Let $(X, f)$ be as in the statement of Theorem 1.2 and let $\mu \in \mathcal{M}(f)$. Then, for each $x \in X$, $\bar{\beta}_\mu(x) = d_\mu(f(x))$, $\underline{\beta}_\mu(x) = d_\mu(f(x))$. Furthermore, for each $\alpha_1, \alpha_2 > 0$, the sets
\[
Z_1 = \{x \in X \mid \bar{\beta}_\mu(x, s) \text{ converges uniformly to } d_\mu(x) < \alpha_1\},
\]
\[
Z_2 = \{x \in X \mid \underline{\beta}_\mu(x, s) \text{ converges uniformly to } d_\mu(x) > \alpha_2\}
\]
are $f$-invariant.

**Proof.** This is basically Proposition 2.1 in [5]. We present the proof in details.

It follows from Birkhoff’s Ergodic Theorem that, for each $z \in X$ and each $\varepsilon > 0$, the limit
\[
\bar{\varphi}_{B(z, \varepsilon)}(y) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{B(z, \varepsilon)}(f^i(y))
\]
exists for $\mu$-a.e. $y \in X$, and
\[
\int \bar{\varphi}_{B(z, \varepsilon)}(y) d\mu(y) = \int \chi_{B(z, \varepsilon)}(y) d\mu(y) = \mu(B(z, \varepsilon)).
\]

Fix $x \in \text{supp}(\mu)$. It is straightforward to show that, for each $y \in X$ and each $i \in \mathbb{N} \cup \{0\}$, one has $\chi_{B(x, \varepsilon/\Lambda)}(f^i(y)) \leq \chi_{B(f^i(x), \varepsilon)}(f^i(y))$. Letting $z = x$ and $z = f(x)$ in [8], respectively, one gets $\bar{\varphi}_{B(x, \varepsilon/\Lambda)}(y) \leq \bar{\varphi}_{B(f(x), \varepsilon)}(y)$ for $\mu$-a.e. $y \in X$, from which follows that $\mu(B(x, \varepsilon/\Lambda)) \leq \mu(B(f(x), \varepsilon))$.

**Case 1:** $x \in \text{supp}(\mu)$. Note that, for each $\eta > 0$, $\mu(B(x, \eta)) > 0$. Let $\varepsilon = 1/t$, $t = l/\Lambda$ and $s \geq 1 + 1/\Lambda$; then,
\[
\sup_{t \geq s} \frac{\log \mu(B(f(x), 1/t))}{-\log t} \leq \sup_{t \geq \Lambda s} \frac{\log l - \log \mu(B(x, 1/l))}{-\log l} \leq A_\Lambda(s) \sup_{t \geq \Lambda s} \frac{\log \mu(B(x, 1/l))}{-\log l},
\]
where $A_\Lambda(s) := \frac{\log s + \log \Lambda}{\log s}$ (since $s \geq 1 + 1/\Lambda$, one has $l \geq \Lambda + 1$).

Using the same idea, one can prove that $\mu(B(x, \varepsilon/\Lambda')) \leq \mu(B(f^{-1}(x), \varepsilon))$; letting $z = f(x)$, one gets $\mu(B(f(x), \varepsilon/\Lambda')) \leq \mu(B(x, \varepsilon))$. Thus, the previous discussion leads to
\[
\bar{\beta}_\mu(f(x), s) \leq A_\Lambda(s) \bar{\beta}_\mu(x, \Lambda s) \quad \text{and} \quad \underline{\beta}_\mu(x, s) \leq A_{\Lambda'}(s) \underline{\beta}_\mu(f(x), \Lambda' s);
\]
one can combine these inequalities and obtain, for each $x \in X$ and each $s \geq \max\{1 + 1/\Lambda, 1 + 1/\Lambda'\}$,
\[
\bar{\beta}_\mu(f(x), s) \leq A_\Lambda(s) \bar{\beta}_\mu(x, \Lambda s) \leq A_\Lambda(s) A_{\Lambda'}(s) \bar{\beta}_\mu(f(x), \Lambda' s).
\]

Now, taking the limit $s \to \infty$ in the inequalities above and observing that $A_\Lambda(s)$ and $A_{\Lambda'}(s)$ are decreasing functions such that $\lim_{s \to \infty} A_{\Lambda'(\Lambda)}(s) = 1$, one gets $\bar{\beta}_\mu(f(x)) = \bar{\beta}_\mu(x)$. 

Case 2: $x \not\in \text{supp}(\mu)$. It follows from the $f$-invariance of $\text{supp}(\mu)$ that $f(x) \not\in \text{supp}(\mu)$; thus, $\overline{d}_\mu(f(x)) = +\infty = \overline{d}_\mu(x)$.

The proof that, for each $x \in X$ $\overline{d}_\mu(f(x)) = \overline{d}_\mu(x)$, is analogous; therefore, we omit it.

It remains to prove that $Z_1$ and $Z_2$ are $f$-invariant; since the arguments in both proofs are similar, we just prove the statement for $Z_1$.

Let $\varepsilon \in (0, 1)$. Since $\overline{\beta}_\mu(x, n)$ converges uniformly to $\overline{d}_\mu(x)$ on $Z_1$, there exists an $n_0 \in \mathbb{N}$ such that for each $x \in Z_1$ and each $n \geq n_0$, $|\overline{\beta}_\mu(x) - \overline{\beta}_\mu(x, \Lambda n)| < \varepsilon/4$. Let also $n_1 \in \mathbb{N}$ be such that, for each $n \geq n_1$, $A_\Lambda(n) - 1 < \frac{\varepsilon}{\alpha^4}$. Then, it follows from (11) that, for each $x \in Z_1$ and each $n \geq \max\{n_0, n_1\}$,

$$\left|\overline{\beta}_\mu(f(x), n) - \overline{d}_\mu(f(x))\right| \leq A_\Lambda(n)\overline{\beta}_\mu(s, \Lambda n) - \overline{d}_\mu(x) = \overline{d}_\mu(x)(A_\Lambda(n) - 1) + A_\Lambda(n)(\overline{\beta}_\mu(x, \Lambda n) - \overline{d}_\mu(x))$$

$$< \alpha \frac{\varepsilon}{2\alpha} + \varepsilon \left(1 + \frac{\varepsilon}{2\alpha}\right) < \varepsilon. \tag{10}$$

This show that $f(Z_1) \subset Z_1$. Since one can also prove (using the same reasoning as above for $f^{-1}$) that $f^{-1}(Z_1) \subset Z_1$, and thus $Z_1 \subset f(Z_1)$, it follows that $Z_1$ is $f$-invariant.

Remark 2.1. The results stated in Lemma 2.1 can be generalized to the following setting: $(X, f)$ is a topological dynamical system, $f$ is surjective, locally invertible and locally bi-Lipschitz. Namely, although the Lipschitz constant for $f$ is locally defined, one can bound it from above by some global constant $\Lambda$, due to the fact that $X$ is a compact space. Thus, one can replace $A_\Lambda(x)(n)$ by $A_\Lambda(n)$ in (11), where $\Lambda(x)$ is the local Lipschitz constant at $x \in X$. Note also that one just needs to check, in this setting, that $f(Z_i) \subset Z_i$.

Lemma 2.2. Let $\mu \in \mathcal{M}_e(f)$.

1. if $\dim_H^\mu(\mu) < \infty$, then for each $\eta > 0$,

$$\mu(\{x \in X \mid \overline{\beta}_\mu(x, s_0) \text{ converges uniformly to } \overline{d}_\mu(x), \overline{d}_\mu(x) < \dim_H^\mu(\mu) + \eta\}) = 1;$$

2. if $\dim_H^\mu(\mu) > 0$, then for each $0 < \eta < \dim_H^\mu(\mu)$,

$$\mu(\{x \in X \mid \overline{\beta}_\mu(x, s_0) \text{ converges uniformly to } \overline{d}_\mu(x), \overline{d}_\mu(x) > \dim_H^\mu(\mu) - \eta\}) = 1.$$

Proof. The proof of this statement is presented in the proof of Lemma 2.2 in [5]. We discuss it here for reader’s sake. Since the proofs of both equalities follow the same ideas, we just present the proof of the first one.

Let $\eta > 0$, set $\alpha := \dim_H^\mu(\mu) + \eta$, and set

$$Z^\mu_{\text{unif}}(\alpha) := \{x \in X \mid \overline{\beta}_\mu(x, s_0) \text{ converges uniformly to } \overline{d}_\mu(x), \overline{d}_\mu(x) < \alpha\}.$$

It follows from Proposition 1.1 that there exists a measurable subset $Z \subset X$, with $\mu(Z) = 1$, such that for each $x \in Z$, $\overline{d}_\mu(x) < \alpha$. Moreover, for each $x \in X$, $\lim_{s \to \infty} \overline{\beta}_\mu(x, s) = \overline{d}_\mu(x)$; it follows from Egoroff’s theorem that given $\gamma > 0$, there exists a measurable $U \subset Z$, with $\mu(U) > 1 - \gamma$, such that $\overline{\beta}_\mu(x, s)$ converges uniformly to $\overline{d}_\mu(x) < \alpha$ on $U$. Given that $U \subset Z^\mu_{\text{unif}}(\alpha)$ and $\mu \in \mathcal{M}_e(f)$, one has $\mu(Z^\mu_{\text{unif}}(\alpha)) = 1$, by Lemma 2.1. \qed
Proof (Theorem 1.2). It follows from Proposition 1.1 that one just needs to show that, for each $q > 1$ and each $s < 1$, $\dim_p^-(\mu) \leq D^-_p(q)$ and $D^+_p(s) \leq \dim^+_p(\mu)$. Since the arguments used in the proof of the first and the last inequalities are similar, we just present the proof of the second one.

If $\dim_p^+(\mu) = \infty$, there is nothing to prove, so assume that $\alpha := \dim_p^+(\mu) < \infty$. Fix $s < 1$, and let $\eta > 0$. It follows from Lemma 2.2 that there exist a $\mu$-measurable set, $Z^\mu_{unif}(\alpha) \subset X$, with $\mu(Z^\mu_{unif}(\alpha)) = 1$, and a number $0 < \varepsilon_0 < 1$ such that, for each $0 < \varepsilon < \varepsilon_0$ and each $x \in Z^\mu_{unif}(\alpha)$,

$$\frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon} < \alpha + \eta,$$

from which follows that

$$I_\mu(s, \varepsilon) < \varepsilon^{(s-1)(\alpha+\eta)};$$

here, we have used the fact that $I_\mu(s, \varepsilon) = \int_{Z^\mu_{unif}(\alpha)} \mu(B(x, \varepsilon))^{s-1} d\mu(x)$. Thus, one has, for each $0 < \varepsilon < \varepsilon_0$,

$$\frac{\log I_\mu(s, \varepsilon)}{(s-1) \log \varepsilon} < \alpha + \eta,$$

and therefore, $D^+_\mu(s) \leq \alpha + \eta$. The result is now a consequence of the fact that $\eta > 0$ is arbitrary.

Remark 2.2. The conclusions of Theorem 1.2 also follow if $(X, f)$ is a topological dynamical system such that $f$ is surjective, locally invertible and locally bi-Lipschitz. Namely, it follows from Remark 2.1 that the conclusions of Lemma 2.2 are also valid, and so the results stated in Theorem 1.2.

3 Proofs of Propositions 1.2, 1.3, and Corollary 1.5

Let $(X, d)$ be a compact metric space, and let $f: X \to X$ be a homeomorphism. For each $n \in \mathbb{N}$, one defines a new metric $d_n$ on $X$ by the law

$$d_n(x, y) = \max\{d(f^k(x), f^k(y)) : k = 0, \cdots, n-1\}.$$

Note that, for each $\varepsilon > 0$, the open ball of radius $\varepsilon$ centered at $x \in X$ with respect to $d_n$ coincides with the Bowen dynamical ball of size $n$ and radius $\varepsilon > 0$, centered at $x$:

$$B(x, n, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\}.$$

Proposition 3.1. The metrics $d_n$ and $d$ induce the same topology on $X$.

Proof. This is a direct consequence of the fact that $f$ is a homeomorphism. □

Thus, for each $x \in X$, each $n \in \mathbb{N}$ and each $\varepsilon > 0$, $\overline{B}(x, n, \varepsilon) = \{y \in X \mid d_n(x, y) \leq \varepsilon\}$ is a closet set and $B(x, n, \varepsilon) = \{y \in X \mid d_n(x, y) < \varepsilon\}$ is an open set (both with respect to the topology induced by $d$).

Let $n \in \mathbb{N}$ and $\varepsilon > 0$. A subset $F$ of $X$ is said to be an $(n, \varepsilon)$-generating set if, for each $x \in X$, there exists $y \in F$ such that $d_n(x, y) < \varepsilon$. 

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Let \( R(n, \varepsilon) \) be the smallest cardinality of an \((n, \varepsilon)\)-generating set for \( X \) with respect to \( f \). Then, the following limit exists, and it is called the topological entropy of \( f \) (see [28]):

\[
h(f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log R(n, \varepsilon) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log R(n, \varepsilon).
\]

**Proof (Proposition 1.2).** It follows from Theorem 1.5 that there exist \( k > 1 \) and \( \varepsilon > 0 \) such that, for each \( x \in X \), each \( n \in \mathbb{N} \) and each \( 0 < r < \varepsilon/k \), one has \( B(x, n, r) \subset B(x, k^{-n}r) \). Thus, \( \mu(B(x, n, r)) \geq \mu(B(x, k^{-n}r))^{q-1} \), and (taking \( r \) sufficiently small so that \( k^{-n}r < 1 \))

\[
\frac{\log \int_{\supp(\mu)} \mu(B(x, k^{-n}r))^{q-1} d\mu(x)}{(q-1) \log k^{-n}r} \leq \frac{\log \int_{\supp(\mu)} \mu(B(x, n, r))^{q-1} d\mu(x)}{(q-1) \log k^{-n}r}.
\]

Let \( E = \{x_i\} \) be an arbitrary \((2n+1, r/2)\)-generating set (which, in particular, implies that \( X \subset \bigcup_{x_i \in E} B(x_i, n, r/2) \)). Since \( X \) is compact, one may take \( E \) as a finite subset of \( X \). Let also \( \tilde{F} = \{x_j\} \) be a subset of \( E \) such that \( \{B(x_j, n, r/2)\}_{i}^{\tilde{F}} \) is a covering of \( \supp(\mu) \). Then, one has

\[
\int_{\supp(\mu)} \mu(B(x, n, r))^{q-1} d\mu(x) \leq \sum_{x_j \in \tilde{F}} \int_{B(x_j, n, r/2)} \mu(B(x, n, r))^{q-1} d\mu(x)
\]

\[
\leq \sum_{x_j \in \tilde{F}} \mu(B(x_j, n, r/2))^{q} \leq \sum_{x_i \in E} \mu(B(x_i, n, r/2))^{q},
\]

where we have used the fact that, for each \( x \in B(x_j, n, r/2) \), \( B(x_j, n, r/2) \subset B(x, n, r) \).

Now, by \(13\) and \(14\), one has

\[
\frac{\log \int_{\supp(\mu)} \mu(B(x, k^{-n}r))^{q-1} d\mu(x)}{(q-1) \log k^{-n}r} \leq \frac{\log \int_{\supp(\mu)} \mu(B(x, n, r))^{q-1} d\mu(x)}{(q-1) \log k^{-n}r} \leq \frac{\log \sum_{x_i \in E} \mu(B(x_i, n, r/2))^{q}}{(q-1) \log k^{-n}r}.
\]

Thus, for \( q = 0 \),

\[
\frac{\log \int_{\supp(\mu)} \mu(B(x, k^{-n}r))^{-1} d\mu(x)}{- \log k^{-n}r} \leq \frac{\log R(n, r/2)}{- \log k^{-n}r} = \frac{\log R(n, r/2)}{n(\log k - \frac{\log r}{n}).
\]

Given that \( D^+(\mu) \) is a decreasing function of \( q \) (see [7] and [11]), one has \( D^+(\mu) \leq D^+(\mu) \). Furthermore, since the function \( \varphi : (0, \infty) \to \mathbb{R} \), \( \varphi(\varepsilon) = \int_{\supp(\mu)} \mu(B(x, \varepsilon))^{-1} d\mu(x) \) is decreasing, it follows from Lemma 6.2 in [9] that

\[
D^+(\mu) = \limsup_{n \to \infty} \frac{\log \int_{\supp(\mu)} \mu(B(x, k^{-n}r))^{-1} d\mu(x)}{- \log k^{-n}r}.
\]

Thus, it follows from \(13\) and \(16\) that

\[
D^+(\mu) \leq D^+(\mu) = \limsup_{n \to \infty} \frac{\log \int_{\supp(\mu)} \mu(B(x, k^{-n}r))^{-1} d\mu(x)}{- \log k^{-n}r} \leq \limsup_{n \to \infty} \frac{\log R(n, r/2)}{2n} \frac{2}{\log k}.
\]

Therefore, taking \( r \to 0 \), the result follows from \(12\). \(\square\)
Corollary 3.1. Let \( (X, d, f) \) be as in the statement of Proposition 1.2. If \( h(f) = 0 \), then for each \( \mu \in \mathcal{M}(f) \) and each \( s \geq 0 \), one has \( D^{\pm}_\mu(s) = 0 \).

Remark 3.1. It follows from Corollary 5.5 in [10] that if a compact metric space admits an expansive homeomorphism whose topological entropy is zero, then its topological dimension is zero. See Section 3 in [10] for examples of systems with zero topological entropy.

**Proof (Proposition 1.3).** It suffices, from Proposition 1.1, to prove the result for \( q = 1 \). It follows from Theorem 1.5 that there exist a hyperbolic metric \( d \) which induces an equivalent topology on \( X \), and numbers \( k > 1 \), \( \varepsilon > 0 \) such that \( f \) is expansive under this metric and, for each \( 0 < r < \varepsilon/k \) and each \( x \in X \), \( B(x, n, r) \subset B(x, k^{-n}r) \). Thus,

\[
\frac{\int \log \mu(B(x, k^{-n}r))d\mu(x)}{\log k^{-n}r} \leq \frac{\int \log \mu(B(x, n, r))d\mu(x)}{\log k^{-n}r}.
\]

(17)

**Claim.**

\[
\limsup_{n \to \infty} \frac{\int \log \mu(B(x, n, r))d\mu(x)}{\log k^{-n}r} \leq h_\mu(f) \log k.
\]

Following the proof of Brin-Katok’s Theorem, fix \( r > 0 \) and consider a finite measurable partition \( \xi \) such that \( \text{diam} \xi = \max_{C \in \xi} \text{diam}(C) < r \). Let \( \xi(x) \) be the element of \( \xi \) such that \( x \in \xi(x) \), and let \( C^\xi_n(x) \) be the element of the partition \( \xi_n = \bigvee_{i=-n}^n f^{-i}\xi \) such that \( x \in C^\xi_n(x) \). Given that \( \xi(x) \subset B(x, r) \), one has

\[
C^\xi_n(x) = \bigcap_{i=-n}^n f^{-i}(\xi(f^i x)) \subset \bigcap_{i=-n}^n f^{-i}(B(f^i x, r)) = B(x, n, r),
\]

from which follows that

\[
\frac{\int \log \mu(B(x, n, r))d\mu(x)}{-n} \leq \frac{\int \log \mu(C^\xi_n(x))d\mu(x)}{-n} = \frac{H(\xi_n)}{n},
\]

where \( H(\xi_n) = -\sum_{C^\xi_n(x) \in \xi_n} \mu(C^\xi_n(x)) \log \mu(C^\xi_n(x)) = \int -\log \mu(C^\xi_n(x))d\mu(x) \). Thus,

\[
\limsup_{n \to \infty} \frac{\int \log \mu(B(x, n, r))d\mu(x)}{-n} \leq \limsup_{n \to \infty} \frac{H(\xi_n)}{n} = H(f, \xi) \leq h_\mu(f),
\]

proving the claim.

Now, since for each \( r > 0 \), each \( k > 1 \) and each \( n \in \mathbb{N} \), \( \int \log \mu(B(x, k^{-n}r))d\mu(x) \) is finite (by (17) and Lemma 2.12 in [27]), it follows from an adaptation of Lemma A.6 in [14] that

\[
\limsup_{r \to 0} \frac{\int \log \mu(B(x, r))d\mu(x)}{\log r} = \limsup_{n \to \infty} \frac{\int \log \mu(B(x, k^{-n}r))d\mu(x)}{\log k^{-n}r}.
\]

(18)

One concludes the proof of the proposition combining relations (17) and (18) with Claim. \( \square \)

**Proof (Corollary 1.5).** Suppose, by absurd, that \( H_\alpha := \{ \mu \in \mathcal{M}_e(T) \mid h_\mu(T) \geq \alpha \} \) is dense in some closed ball \( A \subset \mathcal{M}_e(T) \). Then, it follows from Proposition 2.2 in [5] and Lemma 1.1 that \( \{ \mu \in A \mid \text{dim}_p(\mu) \geq \frac{\alpha}{\log A} \} \) is a generic subset of \( A \) (where \( A \) is endowed with the induced metric); here, \( A \) is the Lipschitz constant of \( T \). Now, one has from Proposition 1.1 that, for each \( s \in (0, 1) \), \( \{ \mu \in A \mid D^+_\mu(s) \geq \frac{h_\mu(T)}{\log A} \} \) is a residual subset of \( A \). This contradiction with Theorem 1.6 finishes the proof. \( \square \)
4 Relation between metric entropy and local dimensions of an invariant measure

We prove here estimates on the local dimensions of ergodic measures, associated with some particular dynamical systems, in terms of their metric entropies.

Lemma 4.1. Let $(X, f, \mu)$ be a dynamical system such that $X$ is a Polish metric space and $\mu \in \mathcal{M}(f)$.

i) If $f$ is a continuous function for which there exist constants $\Lambda > 1$ and $\delta > 0$ such that, for each $x, y \in X$ so that $d(x, y) < \delta$, $d(f(x), f(y)) \leq \Lambda d(x, y)$, then for each $x \in X$,

$$d_\mu(x) \geq \frac{h_\mu(f, x)}{\log \Lambda}. \quad (19)$$

Moreover, if $\mu \in \mathcal{M}_e(f)$, it follows that

$$\dim_H^{-}(\mu) \geq \frac{h_\mu(f)}{\log \Lambda}. \quad (20)$$

ii) If $f$ is a continuous function for which if there exist constants $\lambda > 1$ and $\delta > 0$ such that, for each $x, y \in X$ so that $d(x, y) < \delta$, $\lambda d(x, y) \leq d(f(x), f(y))$, then for each $x \in X$,

$$\overline{d}_\mu(x) \leq \frac{\overline{h}_\mu(f, x)}{\log \lambda}. \quad (21)$$

Moreover, if $X$ is compact and $\mu \in \mathcal{M}_e(f)$, it follows that

$$\dim_P^{+}(\mu) \leq \frac{h_\mu(f)}{\log \lambda}. \quad (22)$$

Here, $\overline{h}_\mu(f, x) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{- \log \mu(B(x, n, \varepsilon))}{n}$ is the upper (lower) local entropy of $(f, \mu)$ at $x \in X$.

Proof. i) Claim 1. One has, for each $x \in X$, each $n \in \mathbb{N}$ and each $0 < \varepsilon \leq \min\{1/2, \delta/2\}$, $B(x, \varepsilon \Lambda^{-n}) \subset B(x, n, \varepsilon)$, where $B(x, n, \varepsilon) := \{ y \in X \mid d(f^i(x), f^i(y)) < \varepsilon, \forall i = 0, \ldots, n \}$ is the Bowen ball of size $n$ and radius $\varepsilon$, centered at $x$. Namely, fix $x \in X$, $n \in \mathbb{N}$ and $0 < \varepsilon \leq \min\{1/2, \delta/2\}$, and let $y \in B(x, \varepsilon \Lambda^{-n})$; then, since $\varepsilon \Lambda^{-n} < \delta$, one has, for each $i = 0, \ldots, n$, $d(f^i(x), f^i(y)) < \varepsilon$, proving the claim.

Now, it follows from Claim 1 that, for each $x \in X$ and each $0 < \varepsilon \leq \min\{1/2, \delta/2\}$,

$$d_\mu(x) = \liminf_{n \to \infty} \frac{\log \mu(B(x, \varepsilon \Lambda^{-n}))}{\log \varepsilon \Lambda^{-n}} \geq \liminf_{n \to \infty} \frac{\log \mu(B(x, n, \varepsilon))}{-n} \frac{- \log \varepsilon}{n} + \log \Lambda$$

$$= \liminf_{n \to \infty} \frac{\log \mu(B(x, n, \varepsilon))}{-n} \frac{1}{\log \Lambda}.$$

Thus, taking $\varepsilon \to 0$ in both sides of the inequalities above, the result follows.
Now, if $\mu \in \mathcal{M}_e(f)$, it follows from Lemma 2.8 in [20] that $h_\mu(f, x) = \mu - \text{ess inf } h_\mu(T, y)$ is valid for $\mu$-a.e. $x$, and then, by Theorem 2.9 in [20], that $h_\mu(f, x) \geq h_\mu(T)$ is also valid for $\mu$-a.e. $x$. Relation (21) is now a consequence of relation (19) and Definition 1.3.

\textbf{Claim 2.} One has, for each $x \in X$, each $n \in \mathbb{N}$ and each $0 < \varepsilon \leq \delta$, $B(x, n, \varepsilon) \subset B(x, \varepsilon \lambda^{-n})$. Namely, fix $x \in X$, $n \geq 1$ and $0 < \varepsilon \leq \delta$, and let $y \in B(x, n, \varepsilon)$ so that, for each $j = 0, \ldots, n$, $d(f^j(x), f^j(y)) < \varepsilon \leq \delta$; it follows from the hypothesis that $\lambda^n d(x, y) \leq d(f^n(x), f^n(y)) < \varepsilon$, and therefore that $d(x, y) < \varepsilon \lambda^{-n}$.

Now, it follows from Claim 2 that, for each $x \in X$ and each $0 < \varepsilon \leq \delta$,

$$\limsup_{n \to \infty} \frac{\log \mu(B(x, n, \varepsilon))}{-n} \frac{1}{\log \lambda} \geq \limsup_{n \to \infty} \frac{\log \mu(B(x, n, \varepsilon))}{-n - \log \varepsilon + \log \lambda} = d_\mu(x).$$

Thus, taking $\varepsilon \to 0$ in both side of the inequalities above, the result follows.

Now, if $\mu \in \mathcal{M}_e(f)$, it follows from Brin-Katok’s Theorem that, for $\mu$-a.e. $x \in X$, $h_\mu(f, x) = h_\mu(f, x) = h_\mu(f, x) = h_\mu(f, x)$. Relation (22) is now a consequence of relation (21) and Definition 1.3. \qed

\textbf{Remark 4.1.} It follows from Theorem 2.10 in [20] that if $X$ is a complete (non-compact) Riemannian manifold and $\mu \in \mathcal{M}_e(f)$, then (22) is also valid.

Since every map considered in this work satisfies the conditions of Lemma 4.1(i), one may combine this fact with Theorem 1.2 in order to obtain the following result.

\textbf{Corollary 4.1.} Let $X$ be a Polish space and let $f : X \to X$ be an invertible transformation such that both $f$ and $f^{-1}$ are Lipschitz, with Lipschitz constant $\Lambda$. Then, for each $\mu \in \mathcal{M}_e(f)$ and each $q \in \mathbb{R}$, one has

$$\frac{h_\mu(f)}{\log \Lambda} \leq \dim_{H}^+(\mu) = D^+_{\mu}(q) \leq D^+_{\mu}(q) = \dim_{P}^+(\mu).$$

Furthermore, one can combine Lemma 4.1 with some results presented in [5] in order to show that, in some situations, the set of invariant measures whose metric entropy is zero is residual.

In what follows, denote by $\mathcal{M}^{co}(f)$ the set of $f$-invariant periodic measures, that is, the set of measures of the form $\mu_x(\cdot) := \frac{1}{k_x} \sum_{i=0}^{k_x-1} \delta_{f^i(x)}(\cdot)$, where $x \in X$ is an $f$-periodic point of period $k_x$, and $\delta_x(A) = 1$ if $x \in A$ and zero otherwise.

\textbf{Theorem 4.1.} Let $X$ be a Polish space and let $f : X \to X$ be an invertible transformation such that both $f$ and $f^{-1}$ are Lipschitz. Suppose that $\mathcal{M}^{co}(f) = \mathcal{M}(f)$. Then,

$$\{\mu \in \mathcal{M}(f) | h_\mu(f) = 0\}$$

is a residual subset of $\mathcal{M}(f)$.

\textbf{Proof.} Firstly, we note that $\mathcal{M}_e(f)$ is a generic subset of $\mathcal{M}(f)$. Namely, the measures in $\mathcal{M}^{co}(f)$ are obviously ergodic. Hence, $\mathcal{M}_e(f) = \mathcal{M}(f)$. Since one has from Theorem 2.1 in [15] that $\mathcal{M}(f)$ is a $G_\delta$ subset of $\mathcal{M}(f)$, the result follows.
Thus, one gets from Propositions 2.2 and 2.5 in [5] that \( \{ \mu \in \mathcal{M}_c(f) \mid \dim_H^+(\mu) = 0 \} \) is a generic subset of \( \mathcal{M}(f) \) (although Proposition 2.2 in [5] was proven for the full-shift system presented in Subsection 1.2, the result can be extended to the dynamical system \((X, f)\) considered here). The result is now a consequence of Lemma 4.1(i).

**Corollary 4.2.** Let \((X, f, \mathcal{B})\) be the full-shift dynamical system over \(X = \prod_{i=-\infty}^{+\infty} M\), where the alphabet \(M\) is a Polish space. Then,

\[
\{ \mu \in \mathcal{M}(f) \mid h_\mu(f) = 0 \}
\]

is a residual subset of \( \mathcal{M}(f) \).

Corollary 4.2 generalizes Theorem 1 in [23] (originally proved for \(M = \mathbb{R}\)) for any Polish space.

We also have a version of Theorem 4.1 for topological dynamical systems.

**Theorem 4.2.** Let \((X, f)\) be a topological dynamical system such that \(f\) is Lipschitz, and suppose that \(\mathcal{M}^{\text{co}}(f) = \mathcal{M}(f)\). Then,

\[
\{ \mu \in \mathcal{M}(f) \mid h_\mu(f) = 0 \}
\]

is a residual subset of \( \mathcal{M}(f) \).

**Proof.** Theorem 1.2 in [4] states that, for each \(q \in (0, 1)\), \( \{ \mu \in \mathcal{M}(f) \mid D^-\mu(q) = 0 \} \) is a residual subset of \( \mathcal{M}(f) \). The result is now a consequence of Proposition 1.1 and Lemma 4.1(i).

Theorem 4.2 partially settles a conjecture posed by Sigmund in [25], which states that if a topological dynamical system \((X, f)\) satisfies the specification property (and consequently, \(\mathcal{M}^{\text{co}}(f) = \mathcal{M}(f)\); see [25]), then \( \{ \mu \in \mathcal{M}(f) \mid h_\mu(f) = 0 \} \) is a residual subset of \( \mathcal{M}(f) \).

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