Dynamical analysis in scalar field cosmology

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A general method to extract exact cosmological solutions for scalar field dark energy in the presence of perfect fluids is presented. We use as a selection rule the existence of invariant transformations for the Wheeler De Witt (WdW) equation. We show that the existence of point transformation in which the WdW equation is invariant is equivalent to the existence of conservation laws for the field equations. Mathematically, the existence of extra integrals of motion indicates the existence of analytical solutions. We extend previous work by providing exact solutions for the Hubble parameter and the effective dark energy equation of state parameter for cosmologies containing a combination of perfect fluid and a scalar field whose self-interaction potential is a power of hyperbolic functions. Finally, we perform a dynamical analysis by studying the fixed points of the field equations using dimensionless variables. Amongst the variety of dynamical cases, we find that if the current cosmological model is Liouville integrable (admits conservation laws) then there is a unique stable point which describes the de-Sitter phase of the universe.

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1. INTRODUCTION

The discovery of the accelerated expansion of the universe (see [1] and references therein) has opened a new window in cosmological studies. Indeed, the underlying physical process responsible for this phenomenon is considered as one of the fundamental problems in cosmology. Within the framework of general relativity, scalar fields provide possible dark energy models which can describe, but not so far explain, the acceleration. This approach attempts to alleviate the coincidence and the cosmological constant problems of the standard cosmological model with a simple cosmological constant, \( \Lambda \). Scalar field models require a choice of self-interaction potential \( V(\phi) \) for the scalar field \( \phi \). Various candidates have been proposed in the literature, such as an inverse power law, exponential, hyperbolic and the list goes on (for review see [2] and references therein). An example, Rubano and Barrow [3] (see also [4]) found that if the scalar field behaves as a perfect fluid then one can show that the corresponding potential obeys \( V(\phi) \propto \sinh^p(q\phi) \), where the constants \( q, p \) are given in terms of observable cosmological parameters, namely the dark-energy equation of state (EoS) parameter and \( \Omega_{m0} \). It is therefore important to develop a way of selecting potentials which are realistic and whose dynamics are exactly soluble in the FLRW spacetime.

This has inspired many authors to propose the Noether (point and dynamical) symmetry approach as a selection rule for dark-energy models, including those of modified gravity [5–16]. The underlying mathematical idea is that the geometry of the field equations can be used as a selection criterion in order to discriminate the dark-energy models [13]. One may check that the dynamical Noether symmetries are associated with the Killing tensors of the Lagrangian and the Noether point symmetries of the minisuperspace are related to the homothetic algebra of the minisuperspace (see [19] and references therein). Therefore, using the Noether symmetry approach one can extract conserved quantities (first integrals) which can be used to simplify a given system of differential equations and thus to determine the integrability of the system and finally solve it.
In this work by using a more general geometric criterion, employing the Lie symmetries of the Wheeler-DeWitt (WdW) equation, we attempt to extend the work of Rubano and Barrow\(^3\) to a general family of hyperbolic scalar-field potentials \(V(\phi)\). In section 2 we give the basic theory of the scalar-field cosmology in a FLRW spacetime. The basic definitions and results from the Lie and Noether point symmetries of partial differential equations and the application in the WdW equation are presented in section 3. In section 4 we consider our cosmological model, which includes a hyperbolic family of scalar field potentials with a perfect fluid with constant equation of state parameter \(w_m\). We study the existence of Lie point symmetries of the WdW equation, where we find that for our model the WdW equation admits Lie point symmetries if the free parameters of the potential and the parameter \(w_m\) are related. In section 5 we apply the Lie point symmetries of the WdW equation in order to construct invariant solutions of the WdW equation and exact solutions of the field equations. Also, in section 6 we perform a dynamical analysis by studying the fixed points of the field equations in the dimensionless variables for the general model and we show that when the cosmological model is Liouville integrable (the model admits conservation laws), there is a unique stable point which describes the de Sitter universe. Finally, in section 7 we draw some conclusions.

2. SCALAR-FIELD COSMOLOGY

We start with the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with line element \((c \equiv 1)\)

\[ds^2 = -dt^2 + a^2(t) \left( \frac{1}{1 + \frac{k}{a^2}} \right) (dx^2 + dy^2 + dz^2).\]  

(1)

The total action of the field equations is written as

\[S = S_{EH} + S_\phi + S_m,\]

(2)

where \(S_{EH} = \int dx^4 \sqrt{-g}R\) is the Einstein-Hilbert action, \(R\) is the Ricci scalar of the underlying space, \(S_\phi\) is the action of the scalar field

\[S_\phi = \int dx^4 \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + V(\phi) \right],\]

(3)

and \(S_m = \int dx^4 \sqrt{-g} L_m\) is the matter term. We assume that \(\phi\) inherits the symmetries of the metric \(\mathbf{11}\). The FLRW spacetime \(\mathbf{41}\) admits a six dimensional Killing algebra \(\mathbf{18}\). Hence, the scalar field depends only on the cosmic time \(t\) and consequently \(\phi_{,\mu} = \phi_0 \delta_0^\mu\) where \(\phi = \frac{d\phi}{dt}\).

From the action \(\mathbf{2}\), we have the Einstein field equations \(\mathbf{2}\)

\[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu},\]

(4)

where \(\kappa = 8\pi G\), \(R_{\mu\nu}\) is the Ricci tensor and \(\tilde{T}_{\mu\nu}\) is the total energy momentum tensor given by \(\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} + T_{\mu\nu}(\phi)\). \(T_{\mu\nu}\) is the energy-momentum tensor of baryonic matter and radiation and \(T_{\mu\nu}(\phi)\) is the energy-momentum tensor associated with the scalar field \(\phi\). Modeling the expanding universe as a fluid (which includes radiation, matter and DE) with 4-velocity \(u_{\mu}\), proper isotropic density \(\rho_m\) and proper isotropic pressure \(P_m\) gives \(T_{\mu\nu} = -\rho g_{\mu\nu} + (\rho + P) u_{\mu} u_{\nu}\), where \(\rho = \rho_m + \rho_\phi\) and \(P = P_m + P_\phi\). The variable \(\rho_\phi\) denotes the energy density of the scalar field and \(P_\phi\) is the corresponding isotropic pressure. Moreover the parameters \((\rho_\phi, P_\phi)\) of the scalar field are given by

\[\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad P_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi).\]

(5)

For the FLRW spacetime, for comoving observers \((u^\mu = \delta_0^\mu)\), the Einstein field equations \(\mathbf{1}\) are

\[H^2 = \frac{\kappa}{3} (\rho_m + \rho_\phi) - \frac{K}{a^2},\]

(6)

and

\[3H^2 + 2\dot{H} = -\kappa (P_m + P_\phi) - \frac{K}{a^2},\]

(7)

where \(H(t) \equiv \dot{a}/a\) is the Hubble function.
Furthermore, assuming that the scalar field and matter do not interact, we have the two conservation laws

$$\dot{\rho}_m + 3H(\rho_m + P_m) = 0$$

(8)

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0$$

(9)

while the corresponding equation of state (EoS) parameters are given by $w_m = P_m/\rho_m$ and $w_\phi = P_\phi/\rho_\phi$. In what follows we assume a constant $w_m$, so that $\dot{\rho}_m = \rho_{m0}a^{-3(1+w_m)}$ $(w_m = 0$ for cold matter and $w_m = 1/3$ for radiation), where $\rho_{m0}$ is the matter density at the present time. Generically, some high-energy field theories suggest that the dark energy EoS parameter may be a function of cosmic time (see, for instance, [21]).

Replacing (5) in (9) we have the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0$$

(10)

where $V_\phi = \frac{dV}{d\phi}$. Furthermore the corresponding dark energy EoS parameter is

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\langle \dot{\phi}^2/2 \rangle - V(\phi)}{\langle \dot{\phi}^2/2 \rangle + V(\phi)}$$

(11)

which means that when $w_\phi < -\frac{1}{3}$ then $\dot{\phi}^2 < V(\phi)$. On the other hand, if the kinetic term of the scalar field is negligible with respect to the potential energy, i.e. $\dot{\phi}^2 \ll V(\phi)$, then the equation of state parameter is $w_\phi \approx -1$.

From the above analysis it becomes clear that the unknown quantities of the problem are $a(t)$, $\phi(t)$ and $V(\phi)$ whereas we have only two independent differential equations available namely Eqs. (7) and (10). Therefore, in order to solve the system of differential equations we need to assume an additional assumption which will determine the functional form of the scalar field potential $V(\phi)$. In the literature, due to the unknown nature of DE, there are many forms of this potential which describe differently the physical features of the scalar field (for instance see [2–4, 22–29]).

As far as the exact solution of the field equations (6), (7) and (10) is concerned there are few solutions with spatial curvature [30, 31] and even fewer solutions are known for a perfect fluid and a scalar field [16, 32–35]. A special solution for a spatially flat FLRW spacetime ($K = 0$) which contains a perfect fluid with a constant equation of state parameter $P_m = (\gamma - 1) \rho_m$ and a scalar field with a constant equation of state parameter $w_\phi = \gamma_0 - 1 = P_\phi/\rho_\phi$, has been found in [2]. Specifically in [3] it has been shown that under these assumptions one solves the field equations and finds the potential $V(\phi)$:

$$V(\phi) = 3H_0^2 (1 - \Omega_{m0}) \left(1 - \frac{\gamma_0}{2}\right) \left(\frac{1 - \Omega_{m0}}{\Omega_{m0}}\right)^{\frac{\gamma_0}{2}} \sinh \left(\sqrt{3} \frac{\gamma - \gamma_0}{\sqrt{\gamma}} (\phi - \phi_0)\right) - \frac{2\gamma_0}{\gamma}. \quad (12)$$

Evidently, this solution is a special solution, in the sense that it exists for specific initial conditions, for example with $w_\phi (z)$ constant.

In the following we consider a spatially-flat FLRW spacetime with a perfect fluid $P_m = (\gamma - 1) \rho_m$ and a scalar field and assume that the potential has the generic form

$$V(\phi) = V_0 [\alpha \cosh (p\phi) + \beta \sinh (p\phi)]^q,$$  

(13)

where $V_0, \alpha, \beta, p, q$ are constants. This potential [13] is a generalization of (12) and our aim is to provide the corresponding conservation laws which lead to exact solutions of the field equations for a particular relation between the constants $p, q$ of the potential (13) and the barotropic parameter $\gamma$ of the perfect fluid.

3. PRELIMINARIES

In this section, we show that if the WdW equation admits Lie symmetries which form an Abelian Lie algebra, then the WdW equation admits oscillatory terms in the solution specified by the dimension of the minisuperspace and the Hamiltonian system defined by the field equations is Liouville integrable; that is, the field equations can be solved by quadratures.
3.1. Lie symmetries and invariant functions

Consider the infinitesimal point transformations
\[ \begin{align*}
\bar{x}^i &= x^i + \varepsilon \xi^i(x^k, u^B), \\
\bar{u}^A &= u^A + \varepsilon \eta^A(x^k, u^B),
\end{align*} \tag{14} \]
with generator
\[ \mathbf{X} = \xi^i(x^k, u^B) \partial_i + \eta^A(x^k, u^B) \partial_A. \tag{16} \]

A differential equation \( H = H(x^i, u^A, u^A_{ij}) \) is invariant under the action of the point transformation \(14,15\) if there exists a function \( \lambda \) such that the following condition holds \[36\]
\[ \mathbf{X}^2[H] = \lambda H, \quad \text{mod} H = 0, \tag{17} \]
where
\[ \mathbf{X}^2 = \mathbf{X} + \eta^i \partial_{u^i} + \eta^A_{ij} \partial_{u^A_{ij}}, \tag{18} \]
is the second prolongation vector of \( \mathbf{X} \). The fields \( \eta^i \) and \( \eta^A_{ij} \) are defined as follows
\[ \eta^i = \eta^i_i + u^B \eta^A_{ij} - \xi^i_j u^A - u^i u^B \xi^j_B, \tag{19} \]
and
\[ \eta^A_{ij} = \eta^A_{ij} + 2 \eta^A_{i(B} u^B_{j)} - \xi^B_{i(B} u^B_{j)} - \xi^A_{ij} u^C - 2 \eta^A_{C(A} u^B_{ij)} u^B_k
\[ - \xi^B_{i(B} u^B_{j)} u^A_k + \eta^B_{ij} u^B_{ik} - 2 \xi^B_{(i} u^A_{j)} + \xi^A_{ik} u^B_{ij} + \xi^A_{iB} u^B_{ik} + 2 \eta^B_{ij} u^A_{ik}. \tag{20} \]

If condition (17) is satisfied, the vector field \( \mathbf{X} \) is called a Lie symmetry vector for the differential equation \( H = H(x^i, u^A, u^A_{ij}) \). The importance of Lie symmetries is that each symmetry can be used to reduce the number of the dependent variables. When a reduction is possible, one determines invariant solutions or transforms them into other solutions \[36\]. From condition (17) we define the Lagrange system
\[ \frac{dx^i}{\xi^i} = \frac{du^A}{\eta^A} = \frac{du^A}{\eta^A_{ij}} = \frac{du^A_{ij}}{\eta^A_{ij}}, \]
whose solution provides the characteristic functions
\[ W^{[0]}(x^k, u^A), W^{[1]}(x^k, u^A, u^A_{ij}), W^{[2]}(x^k, u^A, u^A_{ij}, u^A_{ij}), \]
The solution \( W^{[k]} \) is called the \( k \)th-order invariant of the Lie symmetry vector \(16\). By writing the differential equation in terms of the invariants \( W^{[k]} \), we can reduce the order of the differential equation.

3.2. Noether Symmetries

For differential equations which arise from a variational principle there exists a special class of Lie symmetries which are called Noether symmetries. Noether symmetries are the generators of one-parameter point transformations which transform the Lagrangian so that the Euler-Lagrange equations are invariant. The Noether point symmetries provide first integrals/conservation laws \[36\].

The condition for a Noether symmetry is that there exists a vector field \( A^i = A^i(x^i, u) \) such that the following condition is satisfied:
\[ \mathbf{X}^{[1]} L + LD_i \xi^i = D_i A^i. \tag{21} \]
The corresponding Noether conservation flow \( I^i \) is defined by the expression
\[ I^i = \xi^k \left( u^B_k \frac{\partial L}{\partial u^B_i} - L \right) - \eta^A \frac{\partial L}{\partial u^A_i} + A^i. \tag{22} \]
and it is conserved, that is, it satisfies the relation $D_i I^i = 0$.

As we discussed above, the method of using the Noether symmetries of the cosmological field equations has been applied by many authors in scalar-field cosmology, in $f(R)$ gravity and other modified gravity theories. Recently [20, 37], it has been proposed that the cosmological model will be determined by the existence of Lie symmetries of the WdW equation of quantum cosmology.

This selection rule is more general than that imposed by the Noether symmetries of the field equations. This is because it has been shown in [37] that the WdW equation is possible to admit Lie symmetries while the Lagrangian of the field equations does not admit Noether point symmetries. In the following we discuss the application of Lie symmetries in the WdW equation. Specifically, we discuss the reduction process and we show how to construct Noetherian conservation laws for a conformally related Lagrangian of the field equations from the Lie point symmetries of the WdW equation.

3.3. Minisuperspace and invariant solutions of the WdW equation

The Lagrangian of the field equations in minimally coupled scalar field cosmology in a spatially flat FLRW spacetime with a perfect fluid with a constant equation of state parameter $P_m = (\gamma - 1) \rho_m$ is

$$L (a, \dot{a}, \phi, \dot{\phi}) = -3a\dot{a}^2 + \frac{1}{2} a^3 \dot{\phi}^2 - a^3 V (\phi) - \rho_m a^{-3(\gamma - 1)}. \quad (23)$$

The field equations are the Euler-Lagrange equations of (23) with respect to the variables $(a, \phi)$ and are equations (7) and (10). As the Lagrangian is independent of time, we also have the Hamiltonian constraint (6), which, in terms of the momenta $p_a = \frac{\partial L}{\partial \dot{a}}, p_\phi = \frac{\partial L}{\partial \dot{\phi}}$, becomes

$$-\frac{1}{12a} p_a^2 + \frac{1}{2a^3} p_\phi^2 + a^3 V \left( \dot{\phi} \right) + \rho_m a^{-3(\gamma - 1)} = 0. \quad (24)$$

Finally, the field equations are equivalent to the following Hamiltonian system:

$$\dot{a} = -\frac{1}{6a} p_a, \quad \dot{\phi} = \frac{1}{a^2} p_\phi, \quad \dot{p}_a = -\frac{1}{12a} p_a^2 + \frac{3}{2a^4} p_\phi^2 - 3a^2 V \left( \phi \right) + (3\gamma - 3) \rho_m a^{-3\gamma + 2} = 0.$$

The WdW equation is the Klein Gordon equation which is defined by the conformal Laplacian operator. The general conformal Klein Gordon equation is:

$$\Delta \Psi + \frac{n - 2}{4(n - 1)} R (x^k) \Psi + V_{eff} (x^k) \Psi = 0, \quad (25)$$

where $\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} \frac{\partial}{\partial x^i} \right)$ is the Laplacian operator, $g_{ij}$ is the metric of the minisuperspace and $n = \text{dim } g_{ij}$. From Lagrangian (23), we see that the minisuperspace is

$$ds^2 = -6a da^2 + a^3 d\phi^2 \quad (26)$$

with dimension $n = 2$, and $V_{eff} (a, \phi) = 2a^3 \left[ V (\phi) + \rho_m a^{-3\gamma} \right]$ denotes the effective potential. Therefore, eq. (25) becomes

$$\Delta \Psi + 2a^3 \left[ V (\phi) + \rho_m a^{-3\gamma} \right] \Psi = 0, \quad (27)$$

where the new Laplacian operator $\Delta$ is defined by

$$\Delta \equiv -\frac{1}{6a} \left( \frac{\partial^2}{\partial a^2} + \frac{\partial}{\partial a} \right) + \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2}. \quad (28)$$

In [20], it was proved that the Lie symmetries of equation (25) are related to the conformal algebra of the minisuperspace $g_{ij}$. More specifically, it has been shown that:
1. The general form of the Lie point symmetry vector is

\[ X = \xi^i (x^k) \partial_i + \left[ \frac{(2 - n)\psi}{2} \Psi + a_0 \Psi \right] \partial_\psi, \tag{29} \]

where \( \xi^i (x^k) \) is a conformal Killing vector of the minisuperspace, with conformal factor \( \psi (x^k) \).

2. The Lie point symmetry condition which constrains the potential is \( \mathcal{L}_\xi V_{\text{eff}} + 2 \psi V = 0 \).

Assume that equation (25) admits as Lie point symmetry the vector (29). Then under the coordinate transformation \( x^i \to y^i \) so that \( \xi^i (x^k) \partial_i \to \partial_j \), the Lie symmetry vector (30) becomes

\[ X = \partial_j + \left( \frac{2 - n}{2} \psi (y^k) \Psi + a_0 \Psi \right) \partial_\psi. \tag{30} \]

There exist two equivalent methods to reduce the WdW equation by means of the symmetry vector (30).

a) In the first method we calculate the zero-order invariants from the Lagrange system (\( b \neq J \)),

\[ \frac{dy^b}{0} = \frac{dy^J}{1} = \frac{d\Psi}{(\frac{2 - n}{2} \psi + a_0) \Psi}, \tag{31} \]

which turn out to be

\[ y^b, u (y^b, y^J) = \Phi (y^b) \exp \left[ \int \left( \frac{2 - n}{2} \psi + a_0 \right) dy^J \right]. \tag{32} \]

b) The second method is to write the Lie point symmetry as a Lie Bäcklund symmetry. The Lie symmetry (30) is equivalent to the contact symmetry (see (36)).

\[ \tilde{X} = \left( \Psi_J - \left( \frac{2 - n}{2} \psi + a_0 \right) \Psi \right) \partial_\psi, \tag{33} \]

from which we obtain the differential equation

\[ \Psi_J - \left( \frac{2 - n}{2} \psi + a_0 \right) \Psi = a_1 \Psi. \tag{34} \]

We set \( a_0 + a_1 = Q_0 \) and find that the solution of the reduced equation is

\[ \Psi (y^b, y^J) = \Phi (y^b) \exp \left[ \int \left( \frac{2 - n}{2} \psi + Q_0 \right) dy^J \right]. \tag{35} \]

from which it follows again that the coordinate \( y^J \) is factored out from the solution of the wavefunction \( \Psi (y^b, y^J) \).

In the WKB approximation, \( \Psi (x^k) \sim e^{iS(x^k)} \) the WdW equation reduces to a (null) Hamilton-Jacobi equation. The latter can be seen as the Hamilton-Jacobi equation of a Hamiltonian system moving in the same geometry under the conformal Laplace operator of the WdW equation and with the same potential. Specifically, the WdW equation (27) in scalar field cosmology provides the null Hamilton-Jacobi equation:

\[ - \frac{1}{12a} \left( \frac{\partial S}{\partial a} \right)^2 + \frac{1}{2a^3} \left( \frac{\partial S}{\partial \phi} \right)^2 + a^4 V (\phi) + \rho_0 a^{-3(\gamma - 1)} = 0. \tag{36} \]

Furthermore, in (21) it was also shown that the symmetries of the WdW equation can be used in order to find Noether point symmetries for classical particles. However, the null Hamilton-Jacobi equation is separable if the \( n \)-dimensional Hamiltonian system admits \( n \) conservation laws \( \Phi_I (\text{symmetries}) \) i.e. \( n \) corresponding Noether symmetries which are independent and in involution, i.e. \( \{ \Phi_I, \Phi_J \} = 0 \) where \( \{ , \} \) denotes Poisson bracket. If this is the case, then the Hamiltonian system is Liouville integrable (38). That means that it is possible for the WdW equation to admit an invariant solution and at the same time the classical Hamiltonian system to be not integrable. Therefore, in order for the WdW equation to admit an invariant solution and the Hamiltonian system to be Liouville integrable, the \( n \)-dimensional WdW equation must admit at least \( n - 1 \) independent Lie point symmetries, \( X_I \), which form an Abelian Lie algebra. If this is the case, the zero-order invariants of these \( n - 1 \) Lie symmetries will give the solution of the WdW equation in the form

\[ \Psi (\bar{x}^n, \bar{x}^J) = \Phi (\bar{x}^n) \exp \left[ \sum_{j=1}^{n-1} \int \left( \frac{2 - n}{2} \psi_J - Q_J \right) d\bar{x}^J \right], \tag{37} \]
where $Q_j$ are constants, $J = 1, 2, \ldots, n-1$, and the function $\Phi(\bar{x}^a)$ satisfies a linear second-order ODE. That is, when the field equations are Liouville integrable by Noether point symmetries then there exists a coordinate system where the WdW equation admits $n$ oscillatory terms in the solution and \textit{vice versa}. It is important to note that this result is more general and includes the one given in \cite{10} when $\psi(\bar{x}^a) = 0$; that is, if one considers the Killing algebra of the minisuperspace only. We conclude that for the reduction/solution of the WdW we may consider directly the Lie point symmetries of the WdW equation which are given in terms of the CKVs of the space instead of restricting ourselves to the Noether point symmetries only, as has been done in \cite{10}.

In the case of scalar-field cosmology \cite{23}, where the dimension of the minisuperspace is $n = 2$, if the WdW equation \cite{23} admits a Lie point symmetry then the field equations are Liouville integrable. Below, we study the Lie point symmetries and the WdW equation for the potentials of the form \cite{13} which generalize the work done in \cite{3, 4}.

4. LIE POINT SYMMETRIES OF THE WHEELER-DEWITT EQUATION

We are considering a scalar field cosmological model which contains a quintessence scalar field with the potential of Eq.\cite{13} and a perfect fluid with equation of state parameter $w_m = (\gamma - 1)$. Under these assumptions the Lagrangian of the field equations \cite{23} becomes

$$L \left( a, \dot{a}, \phi, \dot{\phi} \right) = -3a a^2 + \frac{1}{2} a^3 \dot{\phi}^2 - V_0 a^3 [\alpha \cosh (p \phi) + \beta \sinh (p \phi)]^q - \rho_m a^{-3(\gamma-1)}. \quad (38)$$

From previous work on Noether point symmetries in scalar-field cosmology \cite{16} and on dynamical symmetries, \cite{16} we know that this Lagrangian admits conservation laws when

a. The potential reduces to the exponential potential i.e. $\beta = \pm \alpha$ and

b. We have the so called \textit{Unified Dark Matter} (UDM) potential (see Paliathanasis et al. \cite{16} and references therein), i.e. $p = \sqrt{\frac{\alpha}{\beta}}$ and $q = 2$ when the extra fluid is dust, namely $(w, \gamma) = (0, 1)$.

In the following we consider $\alpha \neq \beta$ which implies that the current analysis generalizes the previous works of \cite{3, 10}. The Hamiltonian \cite{24} of the field equations for the Lagrangian \cite{35} in terms of the momenta $p_a$ and $p_\phi$ becomes

$$- \frac{1}{12 a} p_a^2 + \frac{1}{2 a^3} p_\phi^2 + \left( V_0 a^3 [\alpha \cosh (p \phi) + \beta \sinh (p \phi)]^q + \rho_m a^{-3(\gamma-1)} \right) = 0, \quad (39)$$

and the WdW equation \cite{27} is

$$- \frac{1}{12 a} \Psi_{,a a} + \frac{1}{2 a^3} \Psi_{,\phi \phi} - \frac{1}{12 a^2} \Psi_{,a} + \left( V_0 a^3 [\alpha \cosh (p \phi) + \beta \sinh (p \phi)]^q + \rho_m a^{-3(\gamma-1)} \right) \Psi = 0. \quad (40)$$

Applying the results of \cite{20}, we find that the second-order partial differential equation \cite{40} admits the generic Lie point symmetry vector

$$X = \alpha X_1 + \beta X_2 + \alpha_0 \Psi \partial_\psi \quad (41)$$

where $\alpha_0$ is a constant and

$$X_1 = a^{\frac{3q}{2}} \left[ \frac{\sqrt{6}}{6} a \sinh \left( \frac{\sqrt{6}}{4} \mu \phi \right) \partial_a + \cosh \left( \frac{\sqrt{6}}{4} \mu \phi \right) \partial_\phi \right] \quad (42)$$

$$X_2 = a^{\frac{3q}{2}} \left[ \frac{\sqrt{6}}{6} a \cosh \left( \frac{\sqrt{6}}{4} \mu \phi \right) \partial_a + \sinh \left( \frac{\sqrt{6}}{4} \mu \phi \right) \partial_\phi \right] \quad (43)$$

where the constants $p, q, \gamma$ are related as follows:

$$p = \frac{\sqrt{6}}{4} \mu, \quad q = - \frac{4}{\mu} - 2, \quad \gamma = \mu + 2. \quad (44)$$

That is, the effective potential of the field equations is

$$V_{eff} = \left( V_0 a^3 \left[ \alpha \cosh \left( \frac{\sqrt{6}}{4} \mu \phi \right) + \beta \sinh \left( \frac{\sqrt{6}}{4} \mu \phi \right) \right]^{-\frac{q}{p} - 2} + \rho_m a^{-3(\mu+1)} \right). \quad (45)$$
Therefore, for $\mu = -1$, we have that $q = 2$, $\gamma = 1$ i.e. we have the UDM potential with dust (for the exact solution and the observation constraints of that model see [16]).

If the perfect fluid is a barotropic fluid, that is the barotropic index $\gamma \in [1, 2]$ then, from (44), $\mu \in [-1, 0]$ since $\mu \neq 0$. However, if we require the perfect fluid to have a negative equation of state parameter, like a cosmological constant, then $\gamma \in [0, 2)$ which means that $\mu \in [-2, 0)$. Furthermore, when $\rho_{m0} = 0$, i.e. there is no extra fluid, we have that $\mu \in \mathbb{R}^*$. In the following, we apply the Lie symmetry vector (41) in order to construct the invariant solution of the WdW equation (40) and to solve the null Hamilton-Jacobi equation of the Hamiltonian (39) in order to reduce the order of the field equations. In the following section we study the case $\alpha\beta = 0$ and in appendix B we present the general solution for $\alpha\beta \neq 0$.

5. EXACT SOLUTIONS FOR THE $\cosh / \sinh$ POTENTIAL

In this section we determine the exact solution of the field equations and of the WdW equation for the quintessence scalar field. We consider the case $\alpha = 1$, $\beta = 0$ (the case $\alpha = 0$, $\beta = 1$ is equivalent to that case). In appendix B we consider the general case $\alpha\beta \neq 0$.

When $\alpha = 1$, $\beta = 0$ the scalar field potential is $V(\phi) = V_0 \left[ \cosh \left( \frac{\sqrt{3}}{4\mu} \phi \right) \right]^{-\frac{8}{\mu} - 2}$. We apply the coordinate transformation\(^1\):

$$
a = (x^2 - y^2)^{-\frac{1}{\mu}}, \quad \phi = \frac{2\sqrt{3}}{3\mu} \arctan h \left( \frac{y}{\sqrt{x^2 - y^2}} \right),
$$

the effective potential (46) becomes

$$
V_{\text{eff}} = V_0 \left( x^2 - y^2 \right)^{\frac{4}{\mu} - 2}
$$

hence the WdW equation is

$$
(x^2 - y^2)^{\frac{1}{\mu} + 1} \left[ \Psi_{,yy} - \Psi_{,xx} + \left( 2V'_0 x^{-\frac{2}{\mu} - 2} + 2\rho_{m0} \right) \Psi \right] = 0
$$

where $V'_0 = \frac{3}{4\mu^2} V_0$, $\rho_{m0} = \frac{3}{4\mu^2} \rho_{m0}$. In these coordinates the Lie point symmetry vector (41) is $X = \partial_y + a_0 \Psi \partial_x$. Therefore, the solution of equation (48) admits an oscillatory term, i.e. $\Psi(x, y) = e^{a_0 y} \Phi(x)$ where

$$
\Phi_{,xx} - \left( 2V'_0 x^{-\frac{2}{\mu} - 2} + 2\rho_{m0} + a_0^2 \right) \Phi = 0.
$$

Furthermore, in the WKB approximation, $\Psi \propto e^{iS}$ and equation (48) becomes

$$
(x^2 - y^2)^{\frac{1}{\mu} + 1} \left[ \left( \frac{\partial S}{\partial y} \right)^2 - \left( \frac{\partial S}{\partial x} \right)^2 + 2V'_0 x^{-\frac{2}{\mu} - 2} + 2\rho_{m0} \right] = 0,
$$

which is the null Hamilton-Jacobi equation which describes the field equations. The solution of (50) is

$$
S(x, y) = c_1 y \pm \int \sqrt{c_1^2 + 2\rho_{m0} + 2V'_0 x^{-\frac{2}{\mu} - 2}}.
$$

Therefore, the field equation is reduced to the following two-dimensional system,

$$
(x^2 - y^2)^{-\frac{1}{\mu} + 1} \dot{x} = \mp \sqrt{c_1^2 + 2\rho_{m0} + 2V'_0 x^{-\frac{2}{\mu} - 2}}, \quad (x^2 - y^2)^{-(\frac{1}{\mu} + 1)} \dot{y} = c_1.
$$

In order to simplify the system (52) further we apply the transformation $d\tau = (x^2 - y^2)^{\frac{1}{\mu} + 1} dt = a^{-3(\mu+1)} dt$ and the dynamical system becomes

$$
x' = \mp \sqrt{c_1^2 + 2\rho_{m0} + 2V'_0 x^{-\frac{2}{\mu} - 2}}, \quad y' = c_1
$$

\(^1\) We assume $\mu < 0$. However when $\rho_{m0} = 0$ it is possible to have $\mu > 0$. In that case all calculations remain valid provided we replace $\mu = -\nu$ in (55) and in the subsequent coordinate transformations.
where we have set $w = 0$. We perform a numerical integration of the non-linear system (52) and in fig. 1 we give the evolution of the equation of state parameter $w$ for various values of the constant $c_1$ in the case $\mu = -1$. For $\mu = -1$ the extra perfect fluid is dust, i.e. $(\omega_m, \gamma) = (0, 1)$. From fig. 1 we observe that the scalar field mimics the cosmological constant for small values of the constant $c_1$, however for large values of $c_1$ the scalar field has an EoS parameter $w \phi > -1$. Furthermore, from the evolution of $w_{\text{tot}}(a)$ we see that there is a matter-dominated epoch. However, as the parameter $c_1$ increases, this epoch has shorter duration. In what follows we study the case $c_1 = 0$ and express analytically the scalar field and the Hubble function in terms of the scale factor.

5.1. Subcase $c_1 = 0$.

When $c_1 = 0$, from (52) we have $y(t) = y_0$, hence from (50) it follows that $x^2 = y_0^2 + a^{-3\mu}$. Furthermore, from the transformation (10) and from (52) we find that

$$\frac{1}{2} y^2 = a^{-6} \left[ \rho_{m0} y_0^2 + y_0^2 V_0 (y_0^2 + a^{-3\mu})^{-\frac{2}{\mu} - 1} \right],$$

$$V(\phi) = V_0 a^{-3(\mu+2)} (y_0^2 + a^{-3\mu})^{-\frac{2}{\mu} - 1},$$

that is,

$$w_{\phi} = \frac{y_0^2}{y_0^2 + a^{-3\mu}} \left[ \frac{\Omega_{\Lambda 0}}{\Omega_{\Lambda 0} + (y_0^2 + a^{-3\mu})^{-\frac{2}{\mu} - 1}} - a^{-3\mu} (y_0^2 + a^{-3\mu})^{-\frac{2}{\mu} - 1} \right] - a^{-3\mu} (y_0^2 + a^{-3\mu})^{-\frac{2}{\mu} - 1}$$

where we have set

$$\Omega_{\Lambda 0} = \frac{V_0}{3 H_0^2}, \quad \Omega_{m0} = \frac{\rho_{m0}}{3 H_0^2}.$$  \hspace{1cm} (57)

---

2 It is easy to see that when $\mu = -1$ then $d\tau = dt$, which is the UDM solution for $\omega_2 = 0$ of (16).
Here, we would like to note that we call $\Omega_{A0}$ the density parameter of the cosmological constant-like term and $\Omega_{m0}$ the parameter of the perfect fluid.

Therefore, for the scalar field density $\rho_\phi(a)$ we have:

$$\rho_\phi(a) = 3\Omega_{A0}H_0^2a^{-6}\left[y_0^2\left(\frac{\Omega_{m0}}{\Omega_{A0}} + (y_0^2 + a^{-3\mu})^{-\frac{2}{3} - 1}\right) + a^{-3\mu}(y_0^2 + a^{-3\mu})^{-\frac{2}{3} - 1}\right].$$

Then (58) implies that

$$E^2(a) = \frac{H^2(a)}{H_0^2} = \Omega_{m0}a^{-3(\mu+2)} + \Omega_{A0}a^{-6}\left(y_0^2\left[\frac{\Omega_{m0}}{\Omega_{A0}} + (y_0^2 + a^{-3\mu})^{-\frac{2}{3} - 1}\right] + a^{-3\mu}(y_0^2 + a^{-3\mu})^{-\frac{2}{3} - 1}\right).$$

We note that if $y_0^2 + a^{-3\mu} \approx a^{-3\mu}$ then

$$E^2(a) = \Omega_{m0}a^{-3(\mu+2)} + \Omega_{A0}\left(1 + y_0^2\left[\frac{\Omega_{m0}}{\Omega_{A0}}a^{-6} + a^{3\mu}\right]\right).$$

If $y_0 = 0$, then (59) becomes $E^2(a) = \Omega_{m0}a^{-3(\mu+2)} + \Omega_{A0}$ which is obvious because when $y_0 = 0$, we have $\phi = 0$ and $V(\phi) = V_0$, which means that the scalar field acts as a cosmological constant. Furthermore, from (59) and $H(a = 1) = H_0$, we have the constraint

$$(1 + y_0^2)\left[\Omega_{m0} + \Omega_{A0}(1 + y_0^2)^{-\frac{2}{3} - 1}\right] - 1 = 0.$$  

In the following section we consider special values of the barotropic constant $\gamma = \mu + 2$.

5.1.1. Dust fluid versus the effective dark energy EoS

When the perfect fluid is dust then $\mu = -1$, $\gamma = 1$ ($w_m = 0$ for other cases see appendix A) Eq. (59) takes the following form

$$E^2(a) = \Omega_{m0}a^{-3} + \Omega_{A0}\left[1 + 2y_0^2a^{-3} + y_0^2\left(\frac{\Omega_{m0}}{\Omega_{A0}} + y_0^2\right)a^{-6}\right] = \Omega_{m0}a^{-3} + \Delta H(a).$$

It should be mentioned that the last term $\Delta H(a)$ of the normalized Hubble function (62) introduces a cosmological constant-like fluid, dust and stiff matter. Furthermore, from (62), we have the following algebraic equation

$$\Omega_{A0}y_0^4 + (2\Omega_{A0} + \Omega_{m0})y_0^2 + (\Omega_{m0} + \Omega_{A0} - 1) = 0$$

hence

$$D = (2\Omega_{A0} + \Omega_{m0})^2 + 4\Omega_{A0}(1 - \Omega_{m0} - \Omega_{A0})$$

where $D \geq 0$ when $(1 - \Omega_{m0} - \Omega_{A0}) \geq 0$. Recall that $\Omega_{m0} \in [0, 1], \Omega_{A0} \in (0, 1]$, and because $y_0^2 > 0$ we have the solution

$$y_0^2 = \frac{\sqrt{(\Omega_{m0})^2 + 4\Omega_{A0} - (2\Omega_{A0} + \Omega_{m0})}}{2\Omega_{A0}}.$$

Let us now compute the effective dark energy EoS $w_{\phi, eff}$ for the scalar field model introduced above. Generally, it is well known that one can express the effective dark energy EoS parameter in terms of the normalized Hubble parameter $30$

$$w_{\phi, eff}(a) = \frac{-1 - \frac{2}{3} \frac{d\ln E}{d\ln a}}{1 - \Omega_m(a)},$$

\[3\] In general hold $\Omega_{m0} + \Omega_{A0} \neq 1$; however the equality holds only when the scalar field act as a cosmological constant, i.e. $w_{\phi} = -1$. 

where \( \Omega_m(a) = \frac{\Omega_m a^{-3}}{E^2(a)} \). Inserting the second equality of Eq. (62) into Eq. (66), the effective dark energy EoS parameter takes the following form (see [40]):

\[
w_{\phi,\text{eff}}(a) = -1 - \frac{1}{3} \frac{d \ln \Delta H}{d \ln a}.
\]  

(67)

which implies that any modifications to the effective EoS parameter are included in the second term of Eq. (67). Inserting Eq. (62) into Eq. (67) it is straightforward to obtain a simple analytical expression for the effective dark energy EoS parameter:

\[
w_{\phi,\text{eff}}(a) = -1 + \frac{2y_0^2a^{-3} + 2y_0^2(\frac{\Omega_{m0}}{\Omega_m}) + y_0^2a^{-6}}{1 + 2y_0^2a^{-3} + 6y_0^2(\frac{\Omega_{m0}}{\Omega_m}) + y_0^2a^{-6}}.
\]  

(68)

5.1.2. The total case

Our dynamical system is integrable in the case of a single perfect fluid. Here, we introduce two perfect fluids (for example dust and radiation). Consider that we have dust \( (w_m = 0) \) and another perfect fluid with equation of state parameter \( P_f = w_f \rho_f \) (for radiation \( w_f = 1/3 \)), and \( \left( \frac{\Omega_m}{\Omega_{m0}} \right) \ll 1 \). The latter implies that the equation of state parameter which is associated with the two perfect fluids is

\[
w_m = \frac{P_m + P_f}{\rho_m + \rho_f} = \frac{w_f \rho_f}{\rho_m + \rho_f} = \frac{w_f}{1 + \left( \frac{\Omega_{m0}}{\Omega_m} \right)} \approx w_f \left( \frac{\Omega_f}{\Omega_m} \right), \quad w_m^2 \approx 0
\]  

(69)

that is, \( \gamma = 1 + w_m \) and \( \mu = -1 + w_m \). When \( w_f > 0 \) we have that \( w_m > 0 \) and when \( w_f < 0 \) holds we have \( w_m < 0 \). We replace (69) in (69) and perform a Taylor expansion near \( \bar{w}_m = 0 \) (\( \gamma = 1 \) or \( \mu = -1 \)). We find

\[
E^2(a) = E_{\gamma=1}^2(a) - 3\Omega_{m0}a^{-3} \ln(a) \bar{w}_m + \Omega_{m0}F(a) \bar{w}_m
\]  

(70)

where the normalized Hubble parameter \( E_{\gamma=1}^2(a) \) is given by Eq. (62) and

\[
F(a) = 2 \ln \left( \gamma_0^2 + a^3 \right) \left( 1 + y_0^2a^{-3} \right)^2 - 6 \left( y_0^2 + a^3 \right)a^{-3} \ln a
\]

where \( F(a \rightarrow 1) = 2 \ln \left( \gamma_0^2 + 1 \right) \left( 1 + y_0^2 \right)^2 \) and when \( y_0 = 0, F(a) = 0 \). In fig. (2) we give the numerical solutions of the total EoS parameter \( w_{\phi,\text{tot}}(a) \) for \( \mu = 1 \pm 0.05 \) and \( c_1 = 0 \).

In appendix A we give the exact solutions for the Hubble function, \( H(a) \), for the cases where \( \gamma = \frac{4}{3} \) (radiation fluid) and \( \gamma = \frac{1}{3} \) (curvature-like fluid).

6. DYNAMICAL ANALYSIS

In order to complete our analysis of the model with Lagrangian (38), we perform a dynamical analysis of the field equations by studying the fixed points of the field equations. We introduce the new dimensionless variables \( \frac{\phi}{\sqrt{6}H} = x, \sqrt{\frac{V}{3}} = y, \Omega_m = \frac{\rho_m}{3H^2}, \lambda = -\frac{V_{\phi}}{V} \)

(71)

and the lapse time \( N = \ln a \). In the new variables the field equations reduce to the following first-order ODEs

\[
\frac{dx}{dN} = -3x_1 + \frac{\sqrt{6}}{2} \lambda y^2 + \frac{3}{2} x_1 \left[ (1 - w_m) x^2 + (1 + w_m) (1 - y^2) \right]
\]  

(72)

\[
\frac{dy}{dN} = -\frac{\sqrt{6}}{2} \lambda x_1 y + \frac{3}{2} y \left[ (1 - w_m) x^2 + (1 + w_m) (1 - y^2) \right]
\]  

(73)

\[
\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2 (\Gamma - 1) x
\]  

(74)
where $\Gamma = \frac{V_{,\phi\phi} V}{V_{,\phi}^2}$ and the Friedmann equation (69) gives the constraint $\Omega_m = 1 - \Omega_\phi$, where $\Omega_\phi = x^2 + y^2$.

In this case the second Friedmann equation (71) becomes

$$\frac{2}{3} \frac{\dot{H}}{H^2} = -1 - w_m - (1 - w_m) x^2 - (1 + w_m) (1 - y^2)$$

which gives that the total EoS parameter $w_{\text{tot}}$ as a function of $w_m$, $x$ and $y$:

$$w_{\text{tot}} = w_m + (1 - w_m) x^2 - (1 + w_m) y^2. \quad (76)$$

Furthermore, the EoS parameter $w_\phi$ for the scalar field is $w_\phi = -\frac{x^2 - y^2}{x^2 + y^2}$. Note that at any point $(x_0, y_0, \lambda)$, from (76) the solution of the scalar factor is a power law as long as $w_{\text{tot}} = \text{const.}$; that is, $a(t) \propto t^{\frac{1}{3(1+w_{\text{tot}})}}$ for $w_{\text{tot}} \neq -1$ and $a(t) = a_0 e^{H_0 t}$ for $w_{\text{tot}} = -1$.

In the following we consider in (69) $\beta = 0$, then the potential of the scalar field is $V(\phi) = V_0 \cosh^2(\mu \phi)$ [42–44]. For this potential we write $\Gamma(\phi)$ as a function of $\lambda$, i.e. $\Gamma(\lambda) = 1 + \frac{\mu^2}{q} - \frac{1}{q}$, and equation (74) becomes

$$\frac{d\lambda}{dN} = -\sqrt{6q} (qp - \lambda) (qp + \lambda) x. \quad (77)$$

Equations (72), (73) and (77) describe an autonomous dynamical system in the $E^3$ space. Furthermore from the constraints $0 \leq \Omega_\phi \leq 1$, $y \geq 0$, the variables $(x, y)$ are bounded in the ranges $x \in [-1, 1]$, $y \in [0, 1]$ from which follows that that the points $(x, y)$ belong to a half disk; however for the parameter $\lambda$ there is no constraint that implies that $\lambda \in \mathbb{R}$ [41, 45]. In the following we consider $w_{\text{m}} \in (-1, 1)$. The fixed points of the dynamical system (72), (73) and (77) and the corresponding cosmological parameters are given in Table I

The eigenvalues of the linearized dynamical system near the fixed points are given in Table II.

Point $O$ exists for all values of the parameter $\lambda$ and corresponds to the matter epoch ($\Omega_m = 1$); the total EoS parameter is $w_{\text{tot}} = w_m$. Since there exists at least one positive eigenvalue, $m_2 > 0$, the point $O$ is always unstable. At this point the universe accelerates if and only if $w_m < -\frac{1}{3}$. At the points $A(\pm)$ and $B(\pm)$ the universe is dominated

---

4 Where in table $\Delta = \sqrt{(1 - w_m) \left( 24 (1 + w_m)^2 - (7 + 9w_m) (qp)^2 \right)}$
by the kinetic energy of the scalar field \((\Omega_m = 1, V(\phi) = 0)\) which means that the scalar field acts as a stiff fluid, i.e. \(\rho_\phi \propto a^{-6}\) which provides a decelerating universe. These points exist when \(\lambda = \pm qp\), for arbitrary \(q, p\). For these points there exist positive eigenvalues of the linearized system, \(m_1 > 0\), for \(w_m \in (-1, 1)\), hence these critical points are always unstable.

Point \(C\) is the de Sitter solution, \((\Omega_m = 0, w_{tot} = -1)\) where the scalar field acts as a cosmological constant. This point exists for all values of the constants \(q, p\) and could be the future attractor of the universe. From the eigenvalues of Table II for that point we have that it is stable when \(q < 0\). The points \(D_{(\pm)}\) corresponds to a scalar field dominated universe \((\Omega_\phi = 0)\) and exist only when \(|qp| < \sqrt{2}\). The total EoS parameter is that of the scalar field \(w_\phi = -1 + \frac{(qp)^2}{3}\) which gives an accelerated universe when \(|qp| < \sqrt{2}\). The points \(D_{(\pm)}\) are stable for \(q < 0\) and \(|qp| < \sqrt{2}\). Hence, we see that \(D_{(\pm)}\) are stable points and describe an accelerated universe when \(qp < -\frac{1}{3}\) and \(|qp| < \sqrt{2}\). Furthermore, in the limit \(q \to 0^+\), these points correspond to the de Sitter universe: when \(q \to 0^-\) then \(V(\phi) \to V_0\).

Finally, the points \(E_{(\pm, \pm)}\) are the so called ‘scaling’ solutions where \(\Omega_m = 1 - \frac{1}{4}(1 + w_m)\) and the scalar field mimics the matter component of the universe, i.e. \(w_\phi = w_m\). The points \(E_{(\pm, +)}\) exist when \(qp > \sqrt{3}(1 + w_m)\) and they are stable when \(q, p < 0\) whereas the points \(E_{(\pm, -)}\) exist when \(qp < -\sqrt{3}(1 + w_m)\) and are stable when \(q < 0, p > 0\). The total EoS parameter is \(w_{tot} = w_m\) so they lead to an accelerated universe when \(w_m < -\frac{1}{3}\).

In fig. 3 we give the two-dimensional phase portrait in the \(x - y\) plane and the three-dimensional phase portrait of the model with values \((p, q, w_m) = (1, -1.5, 0)\) and \((p, q, w_m) = (1, -3, 0)\). We observe that for \((pq)^2 > 3\) the two stable points are the points \(D_{(\pm)}\) whereas for \((pq)^2 > 3\) the stable points are \(E_{(\pm, -)}\).

It is important to study the case when the constants \(q = q(\mu), p = p(\mu), w_m = w_m(\mu)\) are related to the constant \(\mu\) so as to render the field equations integrable. This is the case we studied in the previous section. Since we considered \(w_m \in (-1, 1)\) we have that \(\mu \in (-2, 0)\).

Hence, for the integrable case we have that the points \(O, A_{(\pm)}, B_{(\pm)}, D_{(\pm)}\) exist and they are always unstable. The point \(O\) has \(w_{tot} < -\frac{1}{3}\) when \(\mu \in (-2, -\frac{4}{3})\) and the points \(D_{(\pm)}\) describe an accelerated universe so long as \(\mu \in (-2, -\frac{2 + 2\sqrt{3}}{3})\). The point \(C\) exists and it is the unique stable point. Finally, the tracker solutions, i.e. points \(E_{(\pm, \pm)}\), do not exist for \(\mu \in (-2, 0)\). The existence and the stability of the fixed points for general values \(q, p\) and for the integrable case \(q = q(\mu), p = p(\mu), w_m = w_m(\mu)\) are given in Table III.

In fig. 4 we give the two-dimensional phase portrait in the \(x - y\) plane and the three-dimensional phase portrait of the model with values \((p, q, w_m) = \left(-\frac{\sqrt{3}}{3}, 2, 0\right)\), which corresponds to the integrable case for \(\mu = -1\). The point

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**TABLE I: Fixed points and cosmological parameters**

| Point  | \((x, y, \lambda)\) | \(\Omega_m\) | \(w_{tot}\) | \(w_\phi\) | Acceleration |
|--------|---------------------|--------------|-------------|-------------|-------------|
| \(O\)  | \((0, 0, \lambda)\) | 1            | \(w_m\)    | \(\frac{\lambda}{3}\) | \(w_m < -\frac{1}{3}\) |
| \(A_{(\pm)}\) | \((1, 0, \pm qp)\) | 0            | 1           | 1           | No          |
| \(B_{(\pm)}\) | \((-1, 0, \pm qp)\) | 0            | 1           | 1           | No          |
| \(C\)  | \((0, 1, 0)\)       | 0            | -1          | -1          | Yes         |
| \(D_{(\pm)}\) | \((\pm \sqrt{\frac{3}{2}} qp, \sqrt{1 - (\frac{qp}{3})^2}, \pm qp)\) | 0            | -1 + \(\frac{(qp)^2}{3}\) | \(\frac{1}{3}\) |
| \(E_{(\pm, +)}\) | \((\pm \sqrt{\frac{3}{2}(1 + w_m)} \frac{1}{2qp}, \sqrt{1 - \frac{w_m}{2qp}} \frac{1}{2qp}, \pm qp)\) | 1 - \(3(1 + w_m) \frac{1}{(qp)^2}\) | \(w_m\) | \(w_m\) | \(w_m < -\frac{1}{3}\) |
| \(E_{(\pm, -)}\) | \((\pm \sqrt{\frac{3}{2}(1 + w_m)} \frac{1}{2qp}, -\sqrt{\frac{w_m}{2qp}} \frac{1}{2qp}, \pm qp)\) | 1 - \(3(1 + w_m) \frac{1}{(qp)^2}\) | \(w_m\) | \(w_m\) | \(w_m < -\frac{1}{3}\) |

**TABLE II: Eigenvalues of fixed points**

| Point  | \(m_1\) | \(m_2\) | \(m_3\) |
|--------|---------|---------|---------|
| \(O\)  | 0       | \(\frac{1}{2}(1 + w_m)\) | \(-\frac{1}{2}(1 - w_m)\) |
| \(A_{(\pm)}\) | 3(1 - w_m) | \(\pm 2p\sqrt{6}\) | \(3 \pm \frac{\sqrt{2}}{3} qp\) |
| \(B_{(\pm)}\) | 3(1 - w_m) | \(\pm 2p\sqrt{6}\) | \(3 \pm \frac{\sqrt{2}}{3} qp\) |
| \(C\)  | -3(1 + w_m) | \(\frac{3}{2} \left(1 - \sqrt{1 - 4qp^2}\right)\) | \(\frac{3}{2} \left(1 + \sqrt{1 - 4qp^2}\right)\) |
| \(D_{(\pm)}\) | -3(1 + w_m) + \((qp)^2\) | \(-3 + \frac{(qp)^2}{2}\) | \(2qp^2\) |
| \(E_{(\pm, \pm)}\) | \(\frac{\delta}{\gamma}(1 + w_m)\) | \(-\frac{\delta}{\gamma}(1 - w_m) + \frac{\delta}{\gamma}\) | \(-\frac{\delta}{\gamma}(1 - w_m) - \frac{\delta}{\gamma}\) |
FIG. 3: Phase portrait in the $x−y$ plane and in the $E^3$ space for the potential $V(\phi) = V_0 \cosh^q (p\phi)$. Left-hand figures are for the variables $(p, q, w_m) = (1, -1.5, 0)$ and the right-hand figures are for the variables $(p, q, w_m) = (1, -3, 0)$. For $(pq)^2 > 3$, the stable points are the points $E_{\pm}^i$ (scaling solutions) while for $(pq)^2 < 3$ the two stable points are the points $D_{\pm}$. The solid lines are for the initial condition $\lambda = pq$, the dashed lines for $\lambda = -pq$, and the dotted lines for $\lambda = 0$.

TABLE III: Fixed points and their stability for the general potential and for the integrable subcases

| Point | Existence | Stability | Stability for $\mu \in (-2, 0)$ | Acceleration |
|-------|-----------|-----------|-------------------------------|--------------|
| $O$   | $p, q \in \mathbb{R}^*$ | Unstable  | Unstable                      | $\mu \in (-2, -\frac{4}{3})$ |
| $A_{\pm}$ | $p, q \in \mathbb{R}^*$ | Unstable  | Unstable                      | No           |
| $B_{\pm}$ | $p, q \in \mathbb{R}^*$ | Unstable  | Unstable                      | No           |
| $C$   | $p, q \in \mathbb{R}^*$ | Stable for $q \in \mathbb{R}^{++}$ | Stable | Yes                         |
| $D_{\pm}$ | $\vert qp \vert < \sqrt{6}$ | Stable for $q \in \mathbb{R}^{--}$, $\vert qp \vert < \sqrt{3(1 + w_m)}$ | Unstable | $\mu \in (-2, -2 + \frac{2\sqrt{3}}{3})$ |
| $E_{\pm, +}$ | $qp > \sqrt{3(1 + w_m)}$ | Stable for $q \in \mathbb{R}^{--}$, $p < 0$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $E_{\pm, -}$ | $qp < -\sqrt{3(1 + w_m)}$ | Stable for $q \in \mathbb{R}^{--}$, $p > 0$ | $\frac{3}{2}$ | $\frac{3}{2}$ |

$C$ is the unique stable point. We observe that the points $D_{\pm}$ act as attractors in the plane $(x − y)$ for $\lambda = \pm \frac{\sqrt{6}}{3}$ and the solutions reach the boundary where $\Omega_m = 0$, and move to the de Sitter points ($w_\phi = -1$). It is important to note that the existence of conservation laws in the field equations which follow from the Lie symmetries of the WdW equation, i.e. the dynamical system is Liouville integrable, gives us constraints on the free parameters of the model so that there exists a unique stable point which describes the de Sitter universe.

7. CONCLUSIONS

We have applied knowledge of Lie and Noether symmetries with an attempt to extend the original works of Rubano and Barrow [3] and Paliathanasis et al. [16] for a general family of potentials, $V(\phi)$. We have shown that there exists a unique connection between the Lie point symmetries of the WdW equation and the conservation laws of the field equations. We considered a general form of $V(\phi)$ which contains hyperbolic functions for a scalar field with a perfect fluid and we have investigated the existence of Lie point symmetries of the WdW equation. This approach is more
FIG. 4: Phase portrait in the \(x−y\) plane for \(\mu = −1\) (Dust) and in the \(E^3\) space for the potential \(V(\phi) = V_0 \cosh^q (p\phi)\) with variables \((p, q, w_m) = \left(-\sqrt{6}/4, 2, 0\right)\), which corresponds to the integrable case for \(\mu = −1\). The point \(C\) is the unique stable point. The points \(D_{±}\) act as attractors in the plane \((x−y)\) for \(λ = ±\sqrt{6}/2\). From the right-hand plot we observe that the solutions reach a boundary where \(Ω_{m0} = 0, \, w_φ > −1\) and from there they move to the de Sitter point \(C\). The solid lines are for initial condition \(λ = pq\), the dashed lines for \(λ = −pq\), and the dotted lines for \(λ = 0\).

general than the application of Noether’s theorem. We confirm the result of ref. [3] that, amongst the variety of \(V(\phi)\) potentials, the hyperbolic types play a key role in scalar field cosmology since they admit conservation laws. Moreover, based on the Lie symmetries of the WdW equation, we have obtained the exact solutions of the field equations. Finally, we have performed a dynamical analysis by studying the fixed points of the field equations in the dimensionless variables. We found various dynamical cases among which, if the current cosmological model is Liouville integrable (i.e. admits conservation laws), there is a unique stable point which describes the de-Sitter universe as a late-time attractor for the dynamics.

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Appendix A: Solutions with radiation and curvature

Here we provide some more details concerning the solutions of section 5.1. Specifically, for a radiation perfect fluid, \(γ = \frac{4}{3}\) (hence \(μ = −\frac{2}{3}\) \((w_m = 1/3)\) and \(Ω_{r0} \equiv Ω_{m0}\)). Then Eq. (59) gives

\[
E(a) = \left(\frac{H(a)}{H_0}\right)^2 = Ω_{r0} a^{-2} + Ω_{Λ0} \left[1 + 3y_0^2 a^{-2} + 3y_0^4 a^{-4} + y_0^2 \left(\frac{Ω_{r0}}{Ω_{Λ0}} + y_0^2\right) a^{-6}\right].
\]  

(A1)

On the other hand, when \(γ = \frac{2}{3}\) (or \(μ = −\frac{4}{3}\)), the perfect fluid has an equation of state \(w_m = −\frac{1}{3}\) (which can be seen also as the curvature term in a non-flat FLRW spacetime); then \(Ω_{K0} \equiv Ω_{m0}\) and:

\[
E(a) = \left(\frac{H(a)}{H_0}\right)^2 = Ω_{K0} a^{-2} + Ω_{Λ0} \left[\sqrt{y_0^2 + a^4 a^{-2}} + y_0^2 \left(\frac{Ω_{K0}}{Ω_{Λ0}} + \sqrt{y_0^2 + a^4}\right) a^{-6}\right].
\]  

(A2)

This is a solution of the scalar field cosmology in a curved FLRW spacetime.
Appendix B: Exact solution for a general potential

In the general case where \( \alpha \neq 0 \) and \( \beta = 1 \) we apply the following coordinate transformations,

\[
a = \left[ \left( \frac{x - \frac{y}{\alpha}}{x} \right)^2 - y^2 \right]^{\frac{1}{2\alpha}} , \quad \phi = \frac{2\sqrt{6}}{3\mu} \arctan h \left( \frac{y}{x - \frac{y}{\alpha}} \right),
\]

(B1)

and the effective potential (45) becomes

\[
V_{eff} = \left[ \left( \frac{x - \frac{y}{\alpha}}{x} \right)^2 - y^2 \right]^{\frac{1}{2\alpha}} \left( V_0 \alpha - \frac{4}{\alpha} - \frac{4}{\alpha} - \frac{\rho_{m0}}{\alpha} \right).
\]

(B2)

In the new coordinate system the WdW equation becomes

\[
\left[ \left( \frac{x - \frac{y}{\alpha}}{x} \right)^2 - y^2 \right]^{\frac{1+\alpha}{2\alpha}} \left[ - \left( 1 - \frac{1}{\alpha^2} \right) \Psi_{,xx} + \frac{2}{\alpha} \Psi_{,xy} + \Psi_{,yy} + \left( \frac{2V_0' \alpha - \frac{4}{\alpha} - \frac{4}{\alpha} + 2\rho_{m0}' \alpha} {2\alpha^2} \right) \Psi \right] = 0,
\]

(B3)

where \( V_0' = \frac{2}{3} \mu^2 \alpha - \frac{4}{\alpha} V_0', \rho_{m0}' = \frac{2}{3} \mu^2 \rho_{m0} \) and the Lie symmetry vector is \( \partial y \) \( X = \partial y + a_0 \Psi \partial \psi \). Therefore, the invariant solution of the WdW equation (B3) is \( \Psi (x, y) = e^{a_0 \Phi (x)} \) where \( \Phi (x) \) satisfies the following second-order ODE

\[
\left[ - \left( 1 - \frac{1}{\alpha^2} \right) \Phi_{,xx} + \frac{2a_0}{\alpha} \Phi_{,x} + \left( \frac{2V_0' \alpha - \frac{4}{\alpha} - \frac{4}{\alpha} + 2\rho_{m0} + a_0^2}{\alpha^2} \right) \Phi \right] = 0.
\]

(B4)

When \( \alpha = 1 \), which is the case of the exponential scalar field, we have the invariant solution

\[
\Phi (x) = \exp \left( \frac{-(2\rho_{m0} + a_0^2)}{2a_0} x + \frac{V_0'}{a_0 \left( \frac{4}{\alpha} + 1 \right)} x^{-\frac{4}{\alpha} - 1} \right).
\]

(B5)

Furthermore, in the WKB approximation the WdW equation (B3) becomes the null Hamilton-Jacobi equation

\[
\left[ \left( \frac{x - \frac{y}{\alpha}}{x} \right)^2 - y^2 \right]^{\frac{1+\alpha}{2\alpha}} \left[ - \left( 1 - \frac{1}{\alpha^2} \right) \left( \partial S \right)_{,xx} + \frac{2}{\alpha} \left( \partial S \right)_{,xy} + \left( \partial S \right)_{,yy} + \left( \partial S \right)_{,yy} + \left( \frac{2V_0' \alpha - \frac{4}{\alpha} - \frac{4}{\alpha} + 2\rho_{m0}'} {2\alpha^2} \right) \right] = 0
\]

with solution

\[
\tilde{S} (x) = S_1 (x) + c_1 y
\]

(B6)

where

\[
S_1 (x) = \mp \frac{\alpha}{1 - \alpha^2} \int \left( c_1 \sqrt{2V_0' (\alpha^2 - 1) x^{-\frac{4}{\alpha} - 2} + \alpha^2 c_1^2 + 2 (\alpha^2 - 1) \rho_{m0}' c_1^2} \right) , \quad \alpha \neq 1
\]

(B7)

and

\[
\frac{dS_1 (x)}{dx} = - \frac{V_0'}{c_1} x^{-\frac{4}{\alpha} - 2} - \frac{1}{2} c_1 \frac{\rho_{m0}'}{c_1} , \quad \alpha = 1.
\]

(B8)

From the solution of the Hamilton-Jacobi function (B6) we can reduce the equivalent Hamiltonian system of the field equation to the following system of first order equations

\[
\dot{x} = \left[ - \left( 1 - \frac{1}{\alpha^2} \right) p_x + \frac{1}{\alpha} p_y \right] \left[ \left( \frac{x - \frac{y}{\alpha}}{x} \right)^2 - y^2 \right]^{\frac{1+\alpha}{2\alpha}},
\]

(B9)

\[
\dot{y} = \frac{1}{\alpha} p_x + p_y \left[ \left( \frac{x - \frac{y}{\alpha}}{x} \right)^2 - y^2 \right]^{\frac{1+\alpha}{2\alpha}},
\]

(B10)
where \( p_x = \frac{\partial S}{\partial x} \) and \( p_y = \frac{\partial S}{\partial y} \). We note that for \( \alpha = 1 \) the reduced Hamiltonian system is
\[
\dot{x} = p_y \left( x^2 - 2xy \right)^{\frac{\mu + 1}{\mu}}, \\
\dot{y} = (p_x + p_y) \left( x^2 - 2xy \right)^{\frac{\mu + 1}{\mu}}.
\]
In both cases, in order to make the reduced system simpler, we apply the transformation
\[
d\tau = a^{-3(\mu+1)} dt
\]
and the system (B9)-(B10) becomes
\[
x' = -\left(1 - \frac{1}{\alpha^2}\right) p_x + \frac{1}{\alpha} p_y, \quad y' = \frac{1}{\alpha} p_x + p_y.
\]
For the exponential potential \((\alpha = 1)\), the corresponding system (B11)-(B12) is.
\[
x' = p_y, \quad y' = p_x + p_y.
\]
For the system (B15) \((\alpha = 1)\) we have the solution
\[
x(\tau) = c_1 \tau + c_0, \\
y(\tau) = \frac{V_0' c_1}{c_1^2 \left( \frac{\mu}{\mu + 1} \right)} (c_1 \tau + c_0)^{-\frac{1}{\mu} - 1} - \frac{\rho_{m0}}{c_1} \tau + \frac{c_1}{2} \tau + y_0,
\]
where the scale factor is \( a(\tau) = \left( x^2 (\tau) - 2x(\tau) y(\tau) \right)^{\frac{1}{3\mu}} \). This is the solution of the exponential scalar field with matter in the Einstein frame. Recall that the matter has an equation of state parameter of the form \( w_m = \mu + 1 \).

1. **Special solution with \( \rho_{m0} = 0, c_1 = 0 \)**

When \( \rho_{m0} = 0 \) and \( c_1 = 0 \), the dynamical system (B14) becomes
\[
x' = x_0 x^{-\frac{1}{\mu} - 1}, \quad y' = -\frac{\alpha x_0}{\alpha^2 - 1} x^{-\frac{1}{\mu} - 1},
\]
where \( x_0 = \frac{\xi}{\delta} \sqrt{2V_0 (\alpha^2 - 1)} \) and \( \varepsilon = \pm 1 \). Hence, we have that
\[
x(\tau) = \left[ \left( \frac{1}{\mu} + 1 \right) x_0 \tau + x_1 \right]^{\frac{1}{\mu + 1}}, \quad y(\tau) = \frac{\alpha}{\alpha^2 - 1} x(\tau) - y_0.
\]

In the case of \( \mu = -1 \) (with \( (w_m, \gamma) = (0, 1) \)) the system (B18) becomes
\[
x' = x_0 x, \quad y' = -\frac{\alpha x_0}{\alpha^2 - 1} x,
\]
a solution of which is
\[
x(\tau) = x_1 e^{x_0 \tau}, \quad y(\tau) = -\frac{\alpha}{\alpha^2 - 1} x(\tau) - y_0
\]
(recall that for \( \mu = -1 \), we have \( dt = d\tau \)). Moreover, the solution for the scale factor is
\[
a(t) = \left[ \frac{\alpha^2 x_1^2}{\alpha^2 - 1} e^{2x_0 t} - \frac{\alpha^2 - 1}{\alpha^2} y_0^2 \right]^{\frac{1}{3}}.
\]
Furthermore, from the singularity condition \( a(0) = 0 \), we have that \( y_0^2 = \frac{\alpha^2 x_1^2}{\alpha^2 - 1} \) which implies
\[
a(t) = a_1 \left( e^{2x_0 t} - 1 \right)^{\frac{1}{3}},
\]
(B23)
where \( \alpha_1 = \left( \frac{\alpha_2}{\alpha_2 + 1} \right)^{\frac{1}{3}} \). From Eq. \( \text{[123]} \), we obtain \( H(t) = \frac{2x_0}{3} e^{2x_0 t - 1} \) and \( t(a) = \frac{1}{2x_0} \ln \left[ 1 + \left( \frac{a}{a_1} \right)^3 \right] \). Therefore, we can express the Hubble function in terms of the scale factor, i.e.

\[
E^2(a) = \frac{H^2(a)}{H_0^2} = \Omega_{\Lambda 0} + \Omega_{m0} a^{-3} + \Omega_{sf0} a^{-6}
\]

(B24)

where

\[
\Omega_{\Lambda 0} = \frac{4}{9} x_0^2, \quad \Omega_{m0} = \frac{8}{9} x_0^2 a_1^3, \quad \Omega_{sf0} = \frac{4}{9} x_0^2 a_1^{-6}
\]

which means that the scalar field introduces an effective dark matter component, namely \( \tilde{\Omega}_m(a) = \tilde{\Omega}_m a^{-3}/E^2(a) \) in the cosmic expansion.

[1] M. Tegmark et al., Astrophys. J. 606, 702 (2004); D. N. Spergel et al., Astrophys. J. Suppl. 170, 377 (2007); T. M. Davis et al., Astrophys. J. 666, 716 (2007); M. Kowalski et al., Astrophys. J. 686, 749(2008); G. Hinshaw et al., Astrophys. J. Suppl. 180, 225 (2009); J. A. S. Lima and J. S. Alcaniz, Mon. Not. Roy. Astron. Soc. 317, 893 (2000); J. F. Jesus and J. V. Cunha, Astrophys. J. Lett. 690, L85 (2009); S. Basilakos and M. Plionis, Astrophys. J. Lett. 714, 185 (2010); E. Komatsu E. et al., 2011, Astrophys. J. Sup., 192, 18 (2011); G. Hinshaw et al., Astrophys. J. Sup. 208, 19 (2013); O. Farooq, D. Mania and B. Ratra, Astrophys. J., 764, 138 (2013); P. A. R. Ade et al., (Planck Collaboration), Astronomy and Astrophysics 571, A16 (2014)

[2] E. J. Copeland, M. Sami and S. Tsujikawa, Intern. Journal of Modern Physics D, 15, 1753,(2006); L. Amendola and S. Tsujikawa, Dark Energy Theory and Observations, Cambridge University Press, Cambridge UK, (2010).

[3] C. Rubano and J. D. Barrow, Phys. Rev. D. 64, 127301 (2001)

[4] L. A. Urena-Lopez, T. Matos, Phys. Rev. D 62, 081302 (2000); V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D 9 373 (2000); J.A.E. Carrillo, J.M. Silva and J.A.S. Lima, Astr. Relativ. Astroph.: New Phenomena and New States of Matter in the Universe, Proceedings of the Third Workshop (IWARA07) (arXiv:0806.3299)

[5] S. Capozziello, E. Pidipatumba, C. Rubano and P. Scudellaro, Phys. Rev. D. 80 104030 (2009); M. Szydłowski et al., Gen. Rel. Grav., 38, 795, (2006); Yi Zhang, Yun-gui Gong and Zong-Hong Zhu, Phys. Lett. B., 688 13 (2010); M. Sharif and I. Shafique, Phys. Rev. D 90 084033 (2014); D. Momeni, R. Myrzakulov and E. Güdekli (arXiv:1502.00977); L. Collodel and G.M. Kremer, (arXiv:1411.3580)

[6] H. Dong, J. Wang and X. Meng, Eur. Phys. J. C 73 2543 (2013)

[7] F. Darabi, K. Atazadeh and A. Rezaei-Aghdam, Eur. Phys. J. C 74 2967 (2014)

[8] H. Wei, X.J. Guo and L.F. Wang, Phys. Lett. B. 707 298 (2012)

[9] Y. Kucukakca, U. Camici and I. Semiz, Gen. Relativ. Gravit. 44 1893 (2012); Y. Kucukakca, Eur. Phys. J. C 73 2327 (2013); Y. Kucukakca, Eur. Phys. J. C 74 3086 (2014)

[10] S. Capozziello, M. De Laurentis and S.D. Odintsov, Eur. Phys. J. C 72 2068 (2012)

[11] Vakili B., Phys. Lett. B. 669 (2008) 206; B. Vakili, F. Khazaie, Class. Quant. Grav. 29 035015 (2012); B. Vakili, Phys. Lett. B 738 488 (2014)

[12] N. Dimakis, T. Christodoulakis and P.A. Terzis, J. Geom. Phys. 77 97 (2014)

[13] P.A Terzis, N. Dimakis and T. Christodoulakis, Phys. Rev. D 90 123543 (2014)

[14] A. Paliathanasis, M. Tsamparlis and S. Basilakos, Phys. Rev. D. 84 123541 (2011)

[15] A. Paliathanasis, M. Tsamparlis, S. Basilakos and S. Capozziello, Phys. Rev. D. 89 063532 (2014); A. Paliathanasis, S. Basilakos, E.N. Saridakis, S. Capozziello, K. Atazadeh, F. Darabi and M. Tsamparlis, Phys. Rev. D. 89 104042 (2014); A. Paliathanasis and M. Tsamparlis, Phys. Rev. D. 90 (2014) 043529; S. Basilakos, S. Capozziello, M. De Laurentis, A. Paliathanasis and M. Tsamparlis, Phys. Rev. D. 88 103526 (2013)

[16] A. Paliathanasis, M. Tsamparlis and S. Basilakos, Phys. Rev. D. 90 103524 (2014); S. Basilakos, M. Tsamparlis and A. Paliathanasis, Phys. Rev. D. 83 103512 (2011)

[17] T.M. Kalotas and B.G. Wybourne, J. Phys. A: Math. Gen. 15 2077 (1982)

[18] R. Maartens and S.D. Maharaj, Class. WdWm Grav. 3 1005 (1986)

[19] M. Tsamparlis and A. Paliathanasis, Gen. Relativ. Gravit. 43 1861 (2011)

[20] A. Paliathanasis and M. Tsamparlis, Int. J. Geom. Methods Mod. Phys. 11 (2014) 1450037

[21] J.R. Ellis, N.E. Mavromatos and D.V. Nanopoulos, Phys. Lett. B. 619 17 (2005)

[22] B. Ratra and P.J.E. Peebles, Phys. Rev. D. 37 3406 (1988)

[23] J.L. Sievers et al, Astrophys. J. 591 599 (2003)

[24] D. Bertacca, S. Matarrese and M. Pietroni, Mod. Phys. Lett. A. 22 2893 (2007)

[25] P. Brax and J. Martin, Phys. Lett. B. 468 40 (1999)

[26] V. Gorini, A. Kamenshchik, U. Moschella and V. Pasquier, Phys. Rev. D 69 123512 (2004)

[27] J.A. Frieman, C.T. Hill, A. Stebbins and I. Waga, Phys. Rev. Lett. 75 2077 (1995)
[28] V. Sahni and L.M. Wang, Phys. Rev. D 62 103517 (2000)
[29] J.D. Barrow and P. Parsons, Phys. Rev. D 52 5576 (1995)
[30] J.J. Halliwell, Phys. Lett. B 185 341 (1987)
[31] R. Easther, Class. Quantum Grav. 10 2203 (1993)
[32] J.D. Barrow and P. Saich, Class. Quantum Grav. 10 279 (1993)
[33] L.P. Chimento and A.E. Cossarini, Class. Quantum Grav. 11 1177 (1994)
[34] L.P. Chimento and A. Jakubi, Int. J. Mod. Phys. D 5 71 (1996)
[35] E. Piedipalumbo, P. Scudellaro, G. Esposito and C. Rubano, Gen. Relativ. Gravit. 44 2611 (2012)
[36] G.W. Bluman and S. Kumei, Symmetries of Differential Equations, (Springer-Verlag, New York, (1989))
[37] A. Paliathanasis, Symmetries of differential equations and applications in relativistic physics, PhD Thesis, University of Athens, Athens, Greece (2014) (arXiv:1501.05129)
[38] V.I. Arnol’d, Mathematical Methods of Classical Mechanics, Graduate Texts in Mathematics, Vol. 60 (2nd ed.), Springer (1989)
[39] T. D. Saini, S. Raychaudhury, V. Sahni, and A. A. Starobinsky, Phys. Rev. Lett. 85, 1162 (2000); D. Huterer and M. S. Turner, Phys. Rev. D 64 123527 (2001).
[40] E. V. Linder and A. Jenkins, Mon. Not. Roy. Astron. Soc. 346 573 (2003); E. V. Linder, Phys. Rev. D 72, 043529 (2005)
[41] E.J. Copeland, A.R. Liddle and D. Wands, Phys. Rev. D 57 4686 (1998)
[42] E.J. Copeland, S. Mizuno and M. Shaeri, Phys. Rev. D 79 103515 (2009)
[43] S.A. Pavluchenko, Phys. Rev. D 67, 103518 (2003)
[44] C.R. Fadragas, G. Leon and E.N. Saridakis, Class. Quantum Grav. 31 075018 (2014)
[45] N. Tamanini, Phys. Rev. D 89 083521 (2014)