Existence results for a superlinear singular equation of Caffarelli-Kohn-Nirenberg type

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Abstract

In this paper, using Mountain Pass Lemma and Linking Argument, we prove the existence of nontrivial weak solutions for the Dirichlet problem for the superlinear equation of Caffarelli-Kohn-Nirenberg type in the case where the parameter $\lambda \in (0, \lambda_2)$, $\lambda_2$ is the second positive eigenvalue of the quasilinear elliptic equation of Caffarelli-Kohn-Nirenberg type.

Key Words: singular equation, Caffarelli-Kohn-Nirenberg inequality, Mountain Pass Lemma, Linking Argument
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1 Introduction.

In this paper, we shall investigate the existence of weak solutions for the following Dirichlet problem for the superlinear singular equation of Caffarelli-Kohn-Nirenberg type:

$$
\begin{cases}
-\text{div} (|x|^{-ap}|Du|^{p-2}Du) = \lambda |x|^{-(a+1)p+c}|u|^{p-2}u + |x|^{-bq}f(u), & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
$$

(1.1)

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with $C^1$ boundary and $0 \in \Omega$, $1 < p < n$, $0 \leq a < \frac{n-p}{p}$, $a \leq b \leq a+1$, $q < p^*(a,b) = \frac{np}{n-dp}$, $d = 1 + a - b \in [0, 1]$, $c > 0$.

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For $a = 0$, $c = p$, many results on linking-type critical points have been obtained (eg. [1, 2, 6] for $p = 2$, [12] for $p \neq 2$ and [14] for the case with indefinite weights).

The starting point of the variational approach to these problems with $a \geq 0$ is the following weighted Sobolev-Hardy inequality due to Caffarelli, Kohn and Nirenberg [4], which is called the Caffarelli-Kohn-Nirenberg inequality. Let $1 < p < n$. For all $u \in C^\infty_0(\mathbb{R}^n)$, there is a constant $C_{a,b} > 0$ such that

$$
\left( \int_{\mathbb{R}^n} |x|^{-a+b} |u|^q \, dx \right)^{p/q} \leq C_{a,b} \int_{\mathbb{R}^n} |x|^{-ap} |Du|^p \, dx,
$$

(1.2)

where

$$
-\infty < a < \frac{n-p}{p}, \quad a \leq b \leq a+1, \quad q = p^*(a,b) = \frac{np}{n-dp}, \quad d = 1 + a - b. \tag{1.3}
$$

Let $\Omega \subset \mathbb{R}^n$ is an open bounded domain with $C^1$ boundary and $0 \in \Omega$, $D^{1,p}_a(\Omega)$ be the completion of $C^\infty_0(\mathbb{R}^n)$, with respect to the norm $\| \cdot \|$ defined by

$$
\|u\| = \left( \int_{\Omega} |x|^{-a} |Du|^p \, dx \right)^{1/p}.
$$

From the boundedness of $\Omega$ and the standard approximation argument, it is easy to see that (1.2) holds for any $u \in D^{1,p}_a(\Omega)$ in the sense:

$$
\left( \int_{\Omega} |x|^{-a} |u|^r \, dx \right)^{p/r} \leq C \int_{\Omega} |x|^{-ap} |Du|^p \, dx,
$$

(1.4)

for $1 \leq r \leq \frac{np}{n-p}$, $\alpha \leq (1+a)r + n(1 - \frac{r}{p})$, that is, the embedding $D^{1,p}_a(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is continuous, where $L^r(\Omega, |x|^{-\alpha})$ is the weighted $L^r$ space with norm:

$$
\|u\|_{r,\alpha} := \|u\|_{L^r(\Omega, |x|^{-\alpha})} = \left( \int_{\Omega} |x|^{-\alpha} |u|^r \, dx \right)^{1/r}.
$$

In fact, we have the following compact embedding result which is an extension of the classical Rellich-Kondrachov compactness theorem (cf. [7] for $p = 2$ and [16] for the general case). For the convenience of readers, we include the proof here.

**Theorem 1.1 (Compact embedding theorem)** Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded domain with $C^1$ boundary and $0 \in \Omega$, $1 < p < n$, $-\infty < a < \frac{n-p}{p}$. The embedding $D^{1,p}_a(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is compact if $1 \leq r < \frac{np}{n-p}$, $\alpha < (1+a)r + n(1 - \frac{r}{p})$.  


Proof. The continuity of the embedding is a direct consequence of the Caffarelli-Kohn-Nirenberg inequality (1.2) or (1.4). To prove the compactness, let \( \{u_m\} \) be a bounded sequence in \( D_{1,p}^a(\Omega) \). For any \( \rho > 0 \) with \( B_\rho(0) \subset \Omega \) is a ball centered at the origin with radius \( \rho \), there holds \( \{u_m\} \subset W^{1,p}(\Omega \setminus B_\rho(0)) \). Then the classical Rellich-Kondrachov compactness theorem guarantees the existence of a convergent subsequence of \( \{u_m\} \) in \( L^r(\Omega \setminus B_\rho(0)) \). By taking a diagonal sequence, we can assume without loss of generality that \( \{u_m\} \) converges in \( L^r(\Omega \setminus B_\rho(0)) \) for any \( \rho > 0 \).

On the other hand, for any \( 1 \leq r < \frac{np}{n-p}\), there exists a \( b \in (a,a+1] \) such that \( r < q = p^*(a,b) = \frac{np}{n-p} \), \( d = 1 + a - b \in [0, 1) \). From the Caffarelli-Kohn-Nirenberg inequality (1.2) or (1.4), \( \{u_m\} \) is also bounded in \( L^q(\Omega, |x|^{-bq}) \). By the Hölder inequality, for any \( \delta > 0 \), there holds

\[
\int_{|x|<\delta} |x|^{-a}|u_m - u_j|^r \, dx \leq \left( \int_{|x|<\delta} |x|^{-(a-br)\frac{q}{q-r}} \, dx \right)^{1-q} \times \left( \int_{\Omega} |x|^{-br}|u_m - u_j|^r \, dx \right)^{r/q} \\
\leq C \left( \int_{0}^{\delta} r^{n-1-(a-br)\frac{q}{q-r}} \, dr \right)^{1-\frac{q}{q}} \\
= C \delta^{n-(a-br)\frac{a}{q-r}},
\]

where \( C > 0 \) is a constant independent of \( m \). Since \( \alpha < (1+a)r + n(1-\frac{r}{p}) \), there holds \( n - (\alpha - br)\frac{q}{q-r} > 0 \). Therefore, for a given \( \varepsilon > 0 \), we first fix \( \delta > 0 \) such that

\[
\int_{|x|<\delta} |x|^{-a}|u_m - u_j|^r \, dx \leq \frac{\varepsilon}{2}, \quad \forall \ m, j \in \mathbb{N}.
\]

Then we choose \( N \in \mathbb{N} \) such that

\[
\int_{\Omega \setminus B_\delta(0)} |x|^{-a}|u_m - u_j|^r \, dx \leq C_\alpha \int_{\Omega \setminus B_\delta(0)} |u_m - u_j|^r \, dx \leq \frac{\varepsilon}{2}, \quad \forall \ m, j \geq N,
\]

where \( C_\alpha = \delta^{-\alpha} \) if \( \alpha \geq 0 \) and \( C_\alpha = (\text{diam} (\Omega))^{-\alpha} \) if \( \alpha < 0 \). Thus

\[
\int_{\Omega} |x|^{-a}|u_m - u_j|^r \, dx \leq \varepsilon, \quad \forall \ m, j \geq N,
\]

that is, \( \{u_m\} \) is a Cauchy sequence in \( L^q(\Omega, |x|^{-bq}) \). \( \square \)

Our results will mainly rely on the results of the eigenvalue problem correspondent to problem (1.1) in [15]. Let us first recall the main results of [15]. Consider the nonlinear eigenvalue problem:

\[
\begin{align*}
-\text{div} \left( |x|^{-ap} Du \right) |Du|^{p-2} Du &= \lambda |x|^{-(a+1)p+c} |u|^{p-2} u, \quad \text{in} \ \Omega \\
u &= 0, \quad \text{on} \ \partial \Omega,
\end{align*}
\]

(1.6)
where \( \Omega \subset \mathbb{R}^n \) is an open bounded domain with \( C^1 \) boundary and \( 0 \in \Omega \), \( 1 < p < n \), \( 0 \leq a < \frac{n-p}{p} \), \( c > 0 \).

Let us introduce the following functionals in \( \mathcal{D}_a^{1,p}(\Omega) \):

\[
\Phi(u) := \int_{\Omega} |x|^{-ap} |Du|^p \, dx, \quad \text{and} \quad J(u) := \int_{\Omega} |x|^{-(a+1)p+c} |u|^p \, dx.
\]

For \( c > 0 \), \( J \) is well-defined. Furthermore, \( \Phi, J \in C^1(\mathcal{D}_a^{1,p}(\Omega), \mathbb{R}) \), and a real value \( \lambda \) is an eigenvalue of problem (1.6) if and only if there exists \( u \in \mathcal{D}_a^{1,p}(\Omega) \setminus \{0\} \) such that \( \Phi'(u) = \lambda J'(u) \). At this point, let us introduce set

\[
\mathcal{M} := \{ u \in \mathcal{D}_a^{1,p}(\Omega) : J(u) = 1 \}.
\]

Then \( \mathcal{M} \neq \emptyset \) and \( \mathcal{M} \) is a \( C^1 \) manifold in \( \mathcal{D}_a^{1,p}(\Omega) \). It follows from the standard Lagrange multiples arguments that eigenvalues of (1.6) correspond to critical values of \( \Phi\mid_{\mathcal{M}} \). From Theorem 1.1, \( \Phi \) satisfies the (PS) condition on \( \mathcal{M} \). Thus a sequence of critical values of \( \Phi\mid_{\mathcal{M}} \) comes from the Ljusternik-Schnirelman critical point theory on \( C^1 \) manifolds. Let \( \gamma(A) \) denote the Krasnoselski’s genus on \( \mathcal{D}_a^{1,p}(\Omega) \) and for any \( k \in \mathbb{N} \), set

\[
\Gamma_k := \{ A \subset \mathcal{M} : A \text{ is compact, symmetric and } \gamma(A) \geq k \}.
\]

Then values

\[
\lambda_k := \inf_{A \in \Gamma_k} \max_{u \in A} \Phi(u) \tag{1.7}
\]

are critical values and hence are eigenvalues of problem (1.6). Moreover, \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty \).

One can also define another sequence of critical values minimaxing \( \Phi \) along a smaller family of symmetric subsets of \( \mathcal{M} \). Let us denote by \( S^k \) the unit sphere of \( \mathbb{R}^{k+1} \) and

\[
\mathcal{O}(S^k, \mathcal{M}) := \{ h \in C(S^k, \mathcal{M}) : h \text{ is odd} \}.
\]

Then for any \( k \in \mathbb{N} \), the value

\[
\mu_k := \inf_{h \in \mathcal{O}(S^k, \mathcal{M})} \max_{t \in S^{k-1}} \Phi(h(t)) \tag{1.8}
\]

is an eigenvalue of (1.6). Moreover \( \lambda_k \leq \mu_k \). This new sequence of eigenvalues was first introduced by [11] and later used in [10, 9] for \( a = 0, c = p \).

In [15], we proved that the first positive eigenvalue \( \lambda_1 = \mu_1 \) is simple, isolated and it is the unique eigenvalue with positive eigenfunction, and \( \lambda_2 := \inf\{ \lambda \in \mathbb{R} : \lambda \text{ is eigenvalue and } \lambda > \lambda_1 \} = \lambda_2 = \mu_2 \).

In this paper, based on the Mountain Pass Lemma and the Linking Argument, we shall prove the existence of nontrivial weak solutions problem (1.1) in the case where the parameter \( \lambda \in (0, \lambda_2) \).
2 Linking results

Let \( e_k \in \mathcal{M} \) be the eigenfunction associated to \( \lambda_k \), then \( \| e_k \|_{\mathcal{D}_a^{1,p}(\Omega)}^p = \lambda_k \). Denote \( G = \{ u \in \mathcal{M} : \Phi(u) < \lambda_2 \} \). Obviously, \( G \) is an open set containing \( e_1 \) and \( e_2 \). Moreover \( -G = G \). First we shall prove the following Lemma.

Lemma 2.1 \( e_1 \) and \( -e_1 \) do not belong to the same connected component of \( G \).

Proof. Otherwise, there exists a continuous curve \( \sigma \) connecting \( e_1 \) and \( -e_1 \) in \( G \). Let \( A = \sigma \cup \{-\sigma\} \), then from the definition of \( \mathcal{M} \), \( 0 \not\in A \), hence \( \gamma(A) > 1 \). By connectedness of \( A \), so \( A \in \Gamma_2 \). Hence, as \( A \) is a compact set in \( G \), and from the definition of \( \mathcal{M} \), we will have \( \max\{\Phi(u); u \in A\} < \lambda_2 \), and this contradicts the definition of \( \lambda_2 \). Q.E.D.

Let \( G_1 \) be the connected component of \( G \) containing \( e_1 \), then \( -G_1 \) is the connected component of \( G \) containing \( -e_1 \). Let

\[
K_1 = \{ tu : u \in G_1, t > 0 \}, \quad K = K_1 \cup \{-K_1\}.
\]

Then, we have

\[
\int_\Omega |x|^{-ap} |Du|^p dx < \lambda_2 \int_\Omega |x|^{-(a+1)p+c} |u|^p dx, \quad \forall u \in K, \tag{2.1}
\]

and

\[
\int_\Omega |x|^{-ap} |Du|^p dx = \lambda_2 \int_\Omega |x|^{-(a+1)p+c} |u|^p dx, \quad \forall u \in \partial K, \tag{2.2}
\]

where \( \partial K \) is the boundary of \( K \) in \( X = \mathcal{D}_a^{1,p}(\Omega) \). Let \( (\partial K)_\rho = \{ u \in \partial K : \| u \| = \rho \} \).

Set

\[
\mathcal{E}_1 = \text{span}\{e_1\}, \quad \mathcal{E}_2 = \text{span}\{e_1, e_2\},
\]

\[
\mathcal{Z} = \{ u \in X : \int_\Omega |Du|^p = \lambda_2 \int_\Omega V(x)|u|^p \}.
\]

(2.2) implies \( \partial K \subset \mathcal{Z} \).

Similar to Proposition 2.1-2.2 in [12] and Lemma 2.1-2.2 in [14], we obtain the following two linking results.

Theorem 2.2 Assume that \( v \in \mathcal{E}_1, \ v \neq 0, \ Q = [-v, \ v] \) is the line segment connecting \( -v \) and \( v \), \( \partial Q = \{-v, \ v\} \). Then \( \partial Q \subset Q \) and \( \mathcal{Z} \) link in \( X \), that is,

(i) \( \partial Q \cap \mathcal{Z} = \emptyset \) and
(ii) For any continuous map \( \psi : Q \to X \) with \( \psi|_{\partial Q} = \text{id} \), there holds \( \psi(Q) \cap Z \neq \emptyset \).

**Proof.** It is obvious that \( \partial Q \cap Z = \emptyset \). Now let \( \psi : Q = [-v, v] \to X \) be continuous and \( \psi|_{\partial Q} = \text{id} \). From the definition of \( K \) and Lemma 2.1, \( K \) has two connected components \( K_1 \) and \( -K_1 \). Assume \( v \in K_1 \), \( -v \in -K_1 \), then \( \psi(Q) \) is a continuous curve connecting \( v \) and \( -v \), therefore there holds \( \psi(Q) \cap \partial K \neq \emptyset \) and thus \( \psi(Q) \cap Z \neq \emptyset \). \( \square \)

**Theorem 2.3** Assume \( 0 < \rho < r < \infty \), let \( \tilde{e}_1 = e_1/\lambda_1^{1/p} \), \( \tilde{e}_2 = e_2/\lambda_2^{1/p} \), and

\[
Q = Q_r = \{ u = t_1 \tilde{e}_1 + t_2 \tilde{e}_2 : \|u\| \leq r, t_2 \geq 0 \},
\]

\[
\partial Q = \partial Q_r = \{ u = t_1 \tilde{e}_1 : |t_1| \leq r \} \cup \{ u \in Q_r : \|u\| = r \},
\]

\[
Z_\rho = \{ u \in Z : \|u\| = \rho \}.
\]

Then \( \partial Q_r \subset Q_r \) and \( Z_\rho \) link in \( X \).

**Proof.** \( \partial Q_r \cap Z_\rho = \emptyset \) is obvious. Let \( \psi : Q_r \to X \) be continuous and \( \psi|_{\partial Q_r} = \text{id} \). Denote \( d_1 = \text{dist} (\tilde{e}_1, \partial K) \) and define mapping \( P : X \to \mathcal{E}_2 \) as follows:

\[
P(u) = \begin{cases} 
( \min \{ \text{dist} (u, \partial K), rd_1 \} ) \tilde{e}_1 + (\|u\| - \rho) \tilde{e}_2, & \text{if } u \not\in -K_1; \\
- ( \min \{ \text{dist} (u, \partial K), rd_1 \} ) \tilde{e}_1 + (\|u\| - \rho) \tilde{e}_2, & \text{if } u \in -K_1.
\end{cases}
\]

It is easy to see that \( P \) is continuous, and \( P \) maps \( v = r \tilde{e}_1 \) to \( v_1 = P(0) = P\tilde{e}_1 = rd_1 \tilde{e}_1 + (r - \rho) \tilde{e}_2 \), the origin \( 0 \) to \( v_1 = P(0) = -\rho \tilde{e}_2 \), the line segment \([v, 0]\) onto the line segment \([v_1, 0]\) homeomorphically; \(-v = -r \tilde{e}_1 \) to \( v_2 = P(-v) = -rd_1 \tilde{e}_1 + (r - \rho) \tilde{e}_2 \), the line segment \([0, -v]\) onto a line segment \([0_1, v_2]\) homeomorphically; and the half circle \( \{ u \in \partial Q : \|u\| = r \} \) which is from \( v \) to \(-v \) in \( \partial Q \) onto the line segment \([v_1, v_2]\), where \( P(r \tilde{e}_2) = (r - \rho) \tilde{e}_2 \).

Let \( f = P \circ \psi : Q \to \mathcal{E}_2 \). From the discussion above, there holds \( 0 \not\in f(\partial Q) \), and when \( u \) turns a circuit along \( \partial Q \) anticlockwise, \( f(u) \) also moves a circuit around the original 0 in \( \mathcal{E}_2 \) anticlockwise. Hence by a degree argument, there holds \( \deg (f, Q, 0) = 1 \). So there exists some \( u \in Q \) such that \( f(u) = 0 \), i.e., \( P(\psi(u)) = 0 \), which implies that \( \psi(u) \in \partial K \); and \( \|\psi(u)\| = \rho \). Thus \( \psi(u) \in (\partial K)_\rho \) and \( \psi(Q) \cap (\partial K)_\rho \neq \emptyset \). Since \( (\partial K)_\rho \subset Z_\rho \), hence \( \psi(Q) \cap Z \neq \emptyset \). \( \square \)
3 Existence results for problem (1.1)

In this section, we will give some conditions on $f(u)$ to guarantee the functional associated to problem (1.1) satisfies the Palais-Smale condition ((PS) condition) for $\lambda \in (0, \lambda_2)$, the geometric assumptions of Mountain Pass Lemma (cf. Theorem 6.1 in Chapter 2 of [13]) in the case of $0 < \lambda < \lambda_1$, and those of the linking theorem (cf. Theorem 8.4 in Chapter 2 of [13]) in the case of $\lambda_1 \leq \lambda < \lambda_2$.

Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies:

(f1) (Subcritical growth) $|f(s)| \leq c_1 |s|^{q-1} + c_2$, $\forall s \in \mathbb{R}$, where $1 < q < p^*(a, b) = \frac{np}{N - dp}$;

(f2) $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $uf(u) \geq 0$, $u \in \mathbb{R}$;

(f3) (Asymptotic property at infinity) $\exists \theta \in (p, p^*(a, b))$ and $M > 0$ such that $0 < \theta F(u) \leq uf(u)$ for $|u| \geq M$, where $F(s) = \int_0^s f(t)dt$;

(f4) (Asymptotic property at $u = 0$) $\lim_{s \to 0} f(s)/|s|^{p-1} = 0$.

Theorem (1.1) and (f1) imply that functional $I : X \to \mathbb{R}$:

$$I(u) = \frac{1}{p} \int_{\Omega} |x|^{-ap}|Du|^p dx - \frac{\lambda}{p} \int_{\Omega} |x|^{-(a+1)p+c}|u|^p dx - \int_{\Omega} |x|^{-bq}F(u)dx$$

is well-defined and $I \in C^1(X; \mathbb{R})$, and the weak solutions of problem (1.1) is equivalent to the critical points of $I$. (f2) implies that 0 is a trivial solution to problem (1.1).

**Lemma 3.1** If $f$ satisfies assumptions (f1)-(f3), then $I$ satisfies the (PS) condition for $\lambda \in (0, \lambda_1)$.

**Proof. 1.** The boundedness of (PS) sequence of $I$.

Suppose $\{u_m\}$ is a (PS) sequence of $I$, that is, there exists $C > 0$ such that $|I(u_m)| \leq C$ and $I'(u_m) \to 0$ in $X'$, the dual space of $X$, as $m \to \infty$. The properties of the first eigenvalue $\lambda_1$ imply that for any $u \in X$, there holds

$$\lambda_1 \int_{\Omega} |x|^{-(a+1)p+c}|u|^p dx \leq \int_{\Omega} |x|^{-ap}|Du|^p dx.$$
Let $c := \sup \limits_\mathcal{M} I(u_m)$. Then by the above inequality and $(f_3)$, as $m \to \infty$, there holds
\[
c - \frac{1}{\theta} o(1) \|u_m\| = \left(\frac{1}{p} - \frac{1}{\theta}\right) \int_\Omega |x|^{-ap}|Du_m|^p \, dx \\
- \lambda \left(\frac{1}{p} - \frac{1}{\theta}\right) \int_\Omega |x|^{-(a+1)p+c}|u_m|^p \, dx + \int_\Omega |x|^{-bq} \left(\frac{1}{\theta} f(u_m) u_m - F(u_m)\right) \, dx \\
\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)(1 - \frac{\lambda}{\lambda_1}) \int_\Omega |x|^{-ap}|Du_m|^p \, dx \\
+ \int_{\Omega(u_m \geq M)} |x|^{-bq} \left(\frac{1}{\theta} f(u_m) u_m - F(u_m)\right) \, dx \\
+ \int_{\Omega(u_m < M)} |x|^{-bq} \left(\frac{1}{\theta} f(u_m) u_m - F(u_m)\right) \, dx \\
\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)(1 - \frac{\lambda}{\lambda_1}) \|u_m\|^p - C_1,
\]
where $C_1 \geq 0$ is a constant independent of $u_m$. The above estimate implies the boundedness of $\{u_m\}$ for $0 < \lambda < \lambda_1$.

2. By $(f_1)$, $f$ satisfies the subcritical growth condition, by a standard argument, one can obtain that there exists a convergent subsequence of $\{u_m\}$ from the boundedness of $\{u_m\}$ in $X$.

Theorem 3.2 If $f$ satisfies assumptions $(f_1)$-$(f_4)$, then problem (1.1) has a non-trivial weak solution $u \in W^{1,p}_0(\Omega)$ provided that $0 < \lambda < \lambda_1$.

Proof. We will verify the geometric assumptions of the Mountain Pass Lemma (cf. [13] Chapter 2, Theorem 6.1):

1. $I(0) = 0$ is obvious;
2. $\exists \rho > 0, \alpha > 0 : \|u\| = \rho \implies I(u) \geq \alpha$;

In fact, $\forall u \in X$, there holds
\[
I(u) \geq \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_1}\right) \int_\Omega |x|^{-ap}|Du|^p \, dx - \int_\Omega |x|^{-bq} F(u) \, dx. \tag{3.1}
\]

From $(f_4)$, $\forall \epsilon > 0, \exists \rho_0 = \rho_0(\epsilon)$ such that: if $0 < \rho = \|u\| < \rho_0$, then $|f(u)| < \epsilon |u|^{p-1}$, thus
\[
\int_\Omega |x|^{-bq} F(u) \, dx \leq \int_\Omega |x|^{-bq} \int_0^{u(x)} f(t) \, dt \, dx \leq \frac{\epsilon}{p} \int_\Omega |x|^{-bq} |u|^p \, dx \leq \frac{c_0 \epsilon}{p} \|u\|.
\]
Choose $c_0\epsilon_0 = (1 - \frac{\lambda}{\lambda_1})/2 > 0$, $\rho = \frac{p_0(\epsilon_0)}{2}$, from (3.1), one has

$$I(u) \geq \frac{1}{p} (1 - \frac{\lambda}{\lambda_1} - c_0\epsilon_0) \int_\Omega |x|^{-ap} |Du|^p \, dx \geq \frac{\lambda_1 - \lambda}{2\lambda_1 p} \cdot \rho =: \alpha > 0.$$  

(3) $\exists u_1 \in X: \|u_1\| \geq \rho$ and $I(u_1) < 0$.

In fact, from (f2) and (f3), one can deduce that there exist constants $c_3, c_4 > 0$ such that

$$F(s) \geq c_3|s|^\theta - c_4, \; \forall s \in \mathbb{R}.$$  

(3.3)

Since $\theta > p$, a simple calculation shows that as $t \to \infty$, there holds

$$I(te_1) \leq \frac{p}{p} \int_\Omega |x|^{-ap} |D_1|^p \, dx - \frac{M p}{p} \int_\Omega |x|^{-(a+1)p+e} |e_1|^p \, dx$$

$$- c_3 t^\theta \int_\Omega |x|^{-bq} |e_1|^{\theta} \, dx + c_4 \int_\Omega |x|^{-bq} \, dx$$

$$\to -\infty,$$

which implies that $I(te_1) < 0$ for $t > 0$ large enough.

Thus Lemma 3.1 and the Mountain Pass Lemma imply that value

$$\beta = \inf_{p \in P} \sup_{u \in p} E(u) \geq \alpha > 0$$

is critical, where $P = \{p \in C^0([0,1]; X) : p(0) = 0, p(1) = u_1\}$. That is, there is a $u \in X$, such that

$$E'(u) = 0, \; E(u) = \beta > 0.$$  

$E(u) = \beta > 0$ implies $u \neq 0$.  

\[ \square \]

**Lemma 3.3** Assume that $\lambda_1 \leq \lambda < \lambda_2$ and $f$ satisfies assumptions $(f_1)-(f_3)$. Then $I$ satisfies the (C) condition introduced by Cerami in [5], that is, any sequence $\{u_m\} \subset X$ such that $I(u_m) \to c$ and $(1 + \|u_m\|)\|I'(u_m)\|_X \to 0$ possesses a convergent subsequence.

**Proof.** 1. The boundedness of (C) sequence in $X$.

Let $\{u_m\} \subset X$ be such that $I(u_m) \to c$ and $(1 + \|u_m\|)\|I'(u_m)\|_X \to 0$. Then from (f2), (f3) and (3.3), as $m \to \infty$, there holds

$$pc + o(1) = pI(u_m) - < I'(u_m), u_m >$$

$$= \int_\Omega |x|^{-bq} (u_m f(u_m) - p F(u_m)) \, dx$$

$$= \int_\Omega |x|^{-bq} (u_m f(u_m) - \theta F(u_m)) \, dx + (\theta - p) \int_\Omega \theta |x|^{-bq} F(u_m) \, dx$$

$$\geq -C_1 + (\theta - p) C_3 |u_m|_{L^p(\Omega, |x|^{-bq})}^\theta - C_4 \int_\Omega |x|^{-bq} \, dx.$$  

(3.5)
Thus $\theta > p$ implies the boundedness of $\{u_m\}$ in $L^p(\Omega, |x|^{-bq})$.

On the other hand, a simple calculation shows that

$$\theta c + o(1) = \theta I(u_m) - < I'(u_m), u_m >$$

$$= \left(\frac{\theta}{p} - 1\right) \| Du_m \|_{L^p(\Omega, |x|^{-ap})}^p - \lambda \left(\frac{\theta}{p} - 1\right) \int_\Omega |x|^{-\left(a+1\right)p+c} |u_m|^p \, dx$$

$$+ \int_\Omega (|x|^{-bq} (u_m f(u_m) - \theta F(u_m))) \, dx$$

$$\geq \left(\frac{\theta}{p} - 1\right) \int_\Omega |x|^{-ap} |Du_m|^p \, dx - C$$

$$+ \int_{\Omega(u_m < M)} |x|^{-bq} (u_m f(u_m) - \theta F(u_m)) \, dx$$

$$+ \int_{\Omega(u_m \geq M)} |x|^{-bq} (u_m f(u_m) - \theta F(u_m)) \, dx$$

$$\geq \left(\frac{\theta}{p} - 1\right) \| Du_m \|_{L^p(\Omega, |x|^{-ap})}^p - C,$$

where $C > 0$ is a universal constant independent of $u_m$, which may be different from line to line. Thus $\theta > p$ and (3.6) imply the boundedness of $\{u_m\}$ in $X$.

2. By $(f_1)$, $f$ satisfies the subcritical growth condition, by a standard argument, one can obtain that there exists a convergent subsequence of $\{u_m\}$ from the boundedness of $\{u_m\}$ in $X$.

\begin{proof}
It was shown in [8] that $(C)_c$ condition actually suffices to get a deformation theorem (Theorem 1.3 in [3]), and it also remarked in [8] that the proofs of the standard Mountain Pass Lemma and saddle-point theorem go through without change once the deformation theorem (Theorem 1.3 in [3]) is obtained with $(C)_c$ condition. Here we verify the assumptions of standard Linking Argument Theorem(cf. [13] Chapter 2, Theorem 8.4) hold with $(C)_c$ condition replacing $(PS)_c$ condition.

Since $\partial Q_r \subset Q_r$ and $Z_\rho$ link in $X$, it suffice to show that:

1. $\alpha_0 = \sup_{u \in \partial Q_r} I(u) \leq 0$, when $r > 0$ is large enough;

2. $\alpha = \inf_{u \in Z_\rho} I(u) > 0$, when $\rho > 0$ is small enough.

\end{proof}

\textbf{Theorem 3.4} Suppose that $f$ satisfies assumptions $(f_1)$-$(f_4)$. Then problem (1.1) has a nontrivial weak solution $u \in X$ provided that $\lambda_1 \leq \lambda < \lambda_2$.

\textbf{Proof.} It was shown in [3] that $(C)_c$ condition actually suffices to get a deformation theorem (Theorem 1.3 in [3]), and it also remarked in [8] that the proofs of the standard Mountain Pass Lemma and saddle-point theorem go through without change once the deformation theorem (Theorem 1.3 in [3]) is obtained with $(C)_c$ condition. Here we verify the assumptions of standard Linking Argument Theorem(cf. [13] Chapter 2, Theorem 8.4) hold with $(C)_c$ condition replacing $(PS)_c$ condition.

Since $\partial Q_r \subset Q_r$ and $Z_\rho$ link in $X$, it suffice to show that:

1. $\alpha_0 = \sup_{u \in \partial Q_r} I(u) \leq 0$, when $r > 0$ is large enough;

2. $\alpha = \inf_{u \in Z_\rho} I(u) > 0$, when $\rho > 0$ is small enough.
In fact, let \( u = te_1 \in E_1 \), from assumption \((f_2)\), \( F(x, s) \geq 0 \) for all \( s \in \mathbb{R} \) and almost every \( x \in \Omega \), thus there holds

\[
I(u) = I(te_1) \leq \frac{|t|^p}{p} \int_{\Omega} |x|^{-ap} |Du_1|^p \, dx - \frac{|t|^p \lambda}{p} \int_{\Omega} |x|^{-(a+1)p+c} |e_1|^p \, dx
= \frac{|t|^p}{p} (1 - \frac{\lambda}{\lambda_1}) \|e_1\| \leq 0.
\]  

(3.7)

Noticing that

\[
|u_m|_{L^\theta(\Omega, |x|^{-b\theta})} = \left( \int_{\Omega} |x|^{-b\theta} |u|^\theta \right)^{1/\theta},
\]

is a norm on \( E_2 \), and the norms of finite dimensional space are equivalent, thus there exists a constant \( c_5 > 0 \) such that

\[
\int_{\Omega} |x|^{-b\theta} |u|^\theta \, dx \geq c_5 \|u\|^\theta,
\]

From (3.3), there holds

\[
I(u) \leq \frac{1}{p} \|u\|^p - c_3 c_5 \|u\|^\theta + c_4 |\Omega|.
\]  

(3.8)

Since \( \theta > p \), there holds

\[
I(u) \to -\infty, \text{ as } \|u\| \to \infty, \ u \in E_2.
\]

This implies (1).

From \((f_4)\) and \((f_1)\), there holds

\[
\int_{\Omega} |x|^{-bq} F(u) \, dx = o(\|u\|^p) \text{ as } u \to 0 \text{ in } X,
\]

then for any \( u \in Z \), there holds

\[
I(u) = \frac{1}{p} (1 - \frac{\lambda}{\lambda_2}) \int_{\Omega} |x|^{-ap} |Du|^p \, dx + o(\|u\|^p).
\]  

(3.9)

Since \( \lambda < \lambda_2 \), (3.9) implies (2).

Thus the Linking Argument Theorem (cf. [13] Chapter 2, Theorem 8.4) implies that value

\[
\beta = \inf_{h \in \Gamma} \sup_{u \in Q} E(h(u)) \geq \alpha > 0
\]

is critical, where \( \Gamma = \{ h \in C^0(\mathcal{X}; \mathcal{X}); \ h|_{\partial Q} = \text{id} \} \). That is, there is a \( u \in X \), such that

\[
E'(u) = 0, \ E(u) = \beta > 0.
\]

\( E(u) = \beta > 0 \) implies \( u \neq 0 \). \( \square \)
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