Elliptic Fibrations and Elliptic Models

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Abstract. We study the Seiberg-Witten curves for $\mathcal{N} = 2$ SUSY gauge theories arising from type IIA string configurations with two orientifold sixplanes. Such theories lift to elliptic models in M-theory. We express the M-theory background for these models as a nontrivial elliptic fibration over $\mathbb{C}$. We discuss singularities of this surface, and write the Seiberg-Witten curve for several theories as a subvariety of this surface.

1. Introduction

In the Seiberg-Witten approach to $\mathcal{N} = 2$ supersymmetric gauge theory \[1\], one identifies a family of algebraic curves associated to each choice of a gauge group and matter content. One approach to this is via M-theory \[2\]. One identifies a configuration of branes in type IIA string theory such that the induced theory on the world volume of the D4 branes is the desired gauge theory. The D4 branes and NS5 branes in the IIA theory then lift to an M-theory five brane, whose world volume is $\mathbb{R}^4 \times \Sigma$, where $\Sigma$ is the desired algebraic curve. This curve is embedded in an algebraic surface $Q$, where $\mathbb{R}^7 \times Q$ is the eleven-dimensional M-theory background. (For brevity, we will refer to $Q$ as the M-theory background.)

In some cases, the type IIA configuration contains D6 branes and orientifold six planes as well as the NS5 and D4 branes. These affect the geometry of $Q$. In cases with D6 branes and no orientifold planes, $Q$ is a multi-Taub-NUT space \[2\]. In cases with one (negatively-charged) orientifold plane, $Q$ is the Atiyah-Hitchin monopole moduli space \[3, 4, 5\]. In cases with two orientifold planes, $Q$ is an elliptic surface, and the M-theory model is an elliptic model. This is the case that we study in this paper.

For elliptic models without orientifold planes, the M-theory background $Q$ is of the form $\mathbb{R}^2 \times T^2$, which can be given the complex structure $\mathbb{C} \times E$, where $E$ is an elliptic curve. More generally, the background can be an affine $\mathbb{C}$ bundle over $E$. The Seiberg-Witten curve is then written as a cover over $E$. For elliptic models with orientifold planes, in cases where the orientifold plane charge is cancelled locally by D6 branes \[4, 7\], the M-theory background can be viewed as the quotient of an

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affine $\mathbb{C}$ bundle over $E$ by a $\mathbb{Z}_2$ action. It is not clear, however, how to extend this to more general orientifold backgrounds.

In this paper, we adopt a different approach, viewing the M-theory background $Q$ as an elliptic fibration over $\mathbb{C}$, and the Seiberg-Witten curve as a cover of $\mathbb{C}$. This approach is partially motivated by the results of ref. [8], in which the Seiberg-Witten curves were written in terms of theta functions with a varying modular parameter. In section 2, we give the explicit form of the elliptic surface $Q$ for the background corresponding to two negatively-charged orientifold six-planes with coincident D6 branes. We briefly discuss the singularities of this surface in the context of M-theory and F-theory. In section 3, we derive the explicit form of the Seiberg-Witten curves for three $\mathcal{N}=2$ gauge theories with this background: $\text{Sp}(2k) + 1$ antisymmetric + 4 fundamental hypermultiplets, $\text{Sp}(2k) \times \text{Sp}(2k) + 1$ bifundamental + 4 fundamental hypermultiplets, and $\text{SU}(N) + 2$ antisymmetric + 4 fundamental hypermultiplets. We show that our results are in agreement with the curves for these theories derived using different approaches [6, 7, 9].

We expect that the description of the M-theory background $Q$ as an elliptic fibration will generalize to the situation where the D6 branes are displaced from the orientifold sixplanes, by a deformation of the elliptic fibration, as occurs in F-theory [10]. Knowing the precise form of the M-theory background for these more general brane configurations is crucial to determining the Seiberg-Witten curve for theories in which the fundamental hypermultiplets have nonzero masses, particularly the terms of the curve equation subleading in the QCD scale $\Lambda$, which are currently unknown [9].

2. The surface

We consider the M-theory background corresponding to type IIA models with two orientifold six planes. An orientifold six plane can have charge +4 or -4 relative to the D6-brane charge; we consider models with two negatively-charged O6\(^-\) planes, and add four pairs of D6 branes to the theory to cancel the charge. Each orientifold six plane and D6 brane is extended in the 0123789 directions. We combine the $x_4$ and $x_5$ coordinates into the complex coordinate $v = x_4 + ix_5$, and place one of the orientifold planes at $(v, x_6) = (0, 0)$ and the other at $(v, x_6) = (m, \pi L)$, where $m$ is the global mass [6]. Each orientifold plane is the fixed point set of a reflection, and the two reflections generate a translation symmetry $(v, x_6) \rightarrow (v + 2m, x_6 + 2\pi L)$ of the IIA model.

The corresponding M-theory background, which is invariant under $x_{10} \rightarrow x_{10} + 2\pi R$, is thus doubly periodic, and gives rise to elliptic models. The metric in the $x_6$ and $x_{10}$ directions is not generally a simple product $S^1 \times S^1$, but is such that travelling around the $x_6$ direction results in a shift in the $x_{10}$ direction by $\theta R$. The orientifold reflections lift to

\[
\begin{align*}
(v, x_6, x_{10}) &\rightarrow (-v, -x_6, -x_{10}) \\
(v, x_6, x_{10}) &\rightarrow (2m - v, 2\pi L - x_6, \theta R - x_{10}),
\end{align*}
\]

which have four fixed points

\[
\begin{align*}
(v, x_6, x_{10}) &= (0, 0, 0), \quad (0, 0, \pi R), \quad (m, -\pi L, -\frac{1}{2}\theta R), \quad (m, -\pi L, (\pi - \frac{1}{2}\theta) R).
\end{align*}
\]

The fundamental parallelogram, with the four fixed points, is shown in fig. 1.
Now we consider the case where the global mass $m$ vanishes. Defining
\begin{equation}
\nu = \frac{x_6 + i x_{10}}{2\pi i R},
\end{equation}
the background is therefore invariant under $\nu \to \nu + 1$ and $\nu \to \nu + \tau$, where $\tau = -\frac{\theta}{2\pi} + i \frac{L}{2\pi}$. This gives $\mathbb{R}^2 \times T^2$ the complex structure of $\mathbb{C} \times E$, where $\mathbb{C}$ is the $\nu$-plane and $E$ is the quotient of the $\nu$-plane by the lattice $\mathbb{Z} \oplus \mathbb{Z} \tau$. We can now embed the elliptic curve $E$ as a cubic in $\mathbb{C}P^2$ with local equation
\begin{equation}
y^2 = (x^2 - 4)(x - \lambda).
\end{equation}

If in addition the D6 branes are coincident with the O6$^{-}$ planes, the M-theory background $Q$ is precisely the $\mathbb{Z}_2$ quotient of the product $\mathbb{C} \times E$, where the $\mathbb{Z}_2$-involution sends $(v, x, y) \to (-v, x, -y)$ in the cubic equation (2.4). We now express this quotient as an elliptic fibration. Define invariant variables $u = v^2$ and $\eta = yv$. Then $Q$ is characterized locally in $\mathbb{C}^3$ by the equation
\begin{equation}
\eta^2 = u(x^2 - 4)(x - \lambda),
\end{equation}
an elliptic fibration over the complex $u$-plane. For nonzero values of $u$, the fiber is isomorphic (via rescaling) to the original cubic curve $E$, so the parameter $\tau$ is constant for $u \neq 0$. At $u = 0$, the fiber is singular and consists of the line $u = \eta = 0$, with three singular points at $x = 2, -2, \lambda$, plus a line at infinity. The local equation for each singular point is $\eta^2 \sim u(x - e_i)$, where $e_i = 2, -2, \lambda$, so these are $A_1$ type singularities. If we blow down the line at infinity, we get one more $A_1$ singularity on this fiber at $x = \infty$.

The elliptic fibration (2.3) is somewhat analogous to the one that arises in the F-theory background corresponding to an O7 plane and coincident D7 branes in type IIB string theory [10]. However, there is an important difference. The F-theory fibration contains a $D_4$ type singularity, whereas the M-theory fibration above contains four distinct $A_1$ type singularities on the singular fiber at $u = 0$. This difference reflects the fact that, in F-theory, the value of $\tau$ at each fiber is the dilaton-axion modulus, but coordinates along the fiber have no physical meaning whereas, in M-theory, the coordinates along the fiber correspond to the $x_6$ and $x_{10}$ coordinates (as discussed in sec. 4.3 of ref. [11]). If the $D_4$ singularity of the F-theory model is resolved, the fiber is a chain of five genus zero curves arranged so that their dual graph is the affine $\tilde{D}_4$ Dynkin diagram. If the four curves on the ends are contracted, the result is a genus zero curve with four $A_1$ singularities,
which is the fiber of the M-theory model. This is very similar to the phenomenon discussed in refs. [10, 11].

We can perform a check on the fibration (2.5) by considering the limit \( \tau \to i\infty \), causing \( \lambda \to \infty \). This limit corresponds to sending one of the orientifold planes, and its two accompanying pairs of D6 branes, to infinity. The resulting M-theory background, which corresponds to one orientifold plane and two pairs of coincident D6 branes, was shown in ref. [4] to be the “\( D_2 \)” surface \( a^2 + b^2 z = 4z \) (which has a pair of \( A_1 \) singularities). By rescaling the surface (2.5) to

\[
\eta^2 = u(x^2 - 4)(-\frac{x}{\lambda} + 1)
\]

and taking the limit \( \lambda \to \infty \), we obtain \( \eta^2 = u(x^2 - 4) \), which is identical to the surface above, with \( a = \eta \), \( b = x \), and \( z = -u \).

3. The curves

We will consider three \( \mathcal{N} = 2 \) supersymmetric gauge theories arising from type IIA brane configurations with two O6\(^-\) planes, with global mass \( m = 0 \) and coincident D6 branes. In each case, the IIA configuration contains parallel NS5 branes, extended in the 012345 directions and separated in the \( x_6 \) direction, and D4 branes extended in the 01236 directions, ending on the NS5 branes in the \( x_6 \) direction. Brane configurations for the three theories under consideration are shown in fig. 2.

![Fig. 2: Brane configurations for the three theories. Circles denote the positions of the O6\(^-\) planes.](image)

For each theory, the IIA brane configuration lifts to an M5-brane, whose embedding in \( \mathbb{R}^7 \times Q \) is given by \( \mathbb{R}^4 \times \Sigma \), where \( \Sigma \) is the Seiberg-Witten curve. In the cases with two NS5 branes, we will express \( \Sigma \) as a double cover of the \( u \)-plane. The NS5 branes correspond to the sheets of the cover, and each D4 brane becomes a “tube” connecting the two sheets of the cover. More precisely, each D4 brane (and its orientifold mirror) corresponds to a branch cut in the \( u \) plane.
This agrees with what we know from nonelliptic models, where a pair of NS5 branes connected by a pair of D4-branes (at \( v = \pm a \)) is represented by

\[ t^2 + (u - a^2)t + 1 = 0, \]

where \( t = \exp(-x_6 + i x_{10}/R) \). The branch points occur at \( u = a^2 \pm 2, t = \pm 1 \), i.e., \((x_6, x_{10}) = (0, 0)\) and \((0, \pi R)\). The branch cut in the \( u \) plane, parallel to the real axis from \( a^2 - 2 \) to \( a^2 + 2 \), corresponds to \(|t| = 1\), i.e., the \( x_{10} \) circle at the fixed value \( x_6 = 0 \) where the NS5 branes join. The “position” of the D4-brane in the IIA picture, viz. \( u = a^2 \), corresponds to \( t = \pm i \), i.e. points on the \( x_{10} \) circle midway between the pre-images of the branch points.

For each of the gauge theories in this section, we will give an equation \( F(u, x, \eta) = 0 \) for \( \Sigma \) as a curve in the surface \( Q \) (3.3).

### 3.1. \( \text{Sp}(2k) + 1 \) antisymmetric + 4 fundamental hypermultiplets. The IIA configuration giving rise to the \( \mathcal{N} = 2 \) \( \text{Sp}(2k) \) gauge theory with hypermultiplets in the \( \Box + 4 \Box \) representations \( \Box \) contains only one NS5 brane, intersecting the O6\(^-\) plane at \( u = \eta = 0, x = 2 \). Each physical D4 brane, located at \( u = a_i^2 \) (corresponding to a mirror pair at \( v = \pm a_\mu \)), wraps around the \( x_6 \) direction and comes back to the same point on the NS5 brane. In the M-theory lift, the brane wraps the \( x_{10} \) direction as well, so it is represented by the entire fiber torus at \( u = a_i^2 \). The NS5 brane is represented by \( x = 2 \). Thus the equation for the Seiberg-Witten curve is given by

\[ (x - 2) \prod_{i=1}^{k} (u - a_i^2) = 0 \]

within the surface \( Q \) given by equation (2.3). It is a reducible curve made up of the \( x = 2 \) line along with the fiber torus at each \( u = a_i^2 \). This agrees with the results obtained in refs. [3.1] and [3.2].

### 3.2. \( \text{Sp}(2k) \times \text{Sp}(2k) + 1 \) bifundamental + 4 fundamental hypermultiplets. The IIA configuration corresponding to the \( \mathcal{N} = 2 \) \( \text{Sp}(2k) \times \text{Sp}(2k) \) gauge theory with hypermultiplets in the \( (\Box, \Box) + 2(\Box, 1) + 2(1, \Box) \) representations consists of two NS5 branes between the orientifold planes, and a total of \( 2k \) physical D4 branes stretched between them \( \Box \). Half of the D4 branes go around the \( x_6 \) circle in one direction and the other half go in the other direction. The two NS5 branes are symmetric with respect to the orientifold \( \mathbb{Z}_2 \) action. Furthermore, we can choose them to be symmetric under the reflection \((u, x, \eta) \rightarrow (u, x, -\eta)\). The Seiberg-Witten curve can then be written as a double cover of the \( u \)-plane by giving an equation

\[ x = f(u), \]

with \( f(u) \) to be determined. To simplify calculations, we shift \( x \) by a fractional linear transformation so that the fibration locally has the form

\[ \eta^2 = u(x^2 - 4)(x^2 - \mu^2). \]

The two sheets of the cover correspond to \((u, x, \eta)\) and \((u, x, -\eta)\) satisfying (3.2) and (3.3). The branch points of the cover therefore occur at points \( u \) where \( f(u) = 2, -2, \mu, \) or \(-\mu\).

To determine the form of \( f(u) \), consider the picture on the universal cover, \( \mathbb{C}_\nu \times \mathbb{C}_\nu \) of the surface \( Q \), where \( \nu = (x_{10} - i x_6)/2\pi R \). Choose \( x \) so that \( x = 2, -2, \)

\[ ... \]
μ, and −μ correspond to ν = 0, 1/2, and 1/2 + τ/2 respectively (see fig. 3). For each value of ν, the two NS5 brane positions are at ν and −ν.

Consider D4 branes at positions u = a_i^2 going around the x_6 circle in one direction. These pull the NS5 branes together to meet at x_6 = 0. The branch cut centered on a_i^2 in the u plane (i.e. going around the “tube”) is the image under the covering map of the x_10 circle at x_6 = 0. The branch points are the images of x_10 = 0 and πR, corresponding to x = 2 and −2 respectively. The D4-brane position u = a_i^2 is the image of points on the x_10 circle midway between the branch point pre-images, viz. x_10 = 1/2πR (or 3/2πR), corresponding to x = 0. Therefore f(u) = 0 at u = a_i^2.

The D4 branes at positions u = b_i^2 going around the x_6 circle in the other direction pull the NS5 branes together to meet at x_6 = −πL. The branch cut centered on b_i^2 in the u plane is the image of the x_10 circle at x_6 = −πL, with the branch point pre-images at x_{10} = −1/2θR and (π − 1/2θ)R corresponding to x = μ and −μ respectively. The D4-brane position u = b_i^2 is the image of x_{10} = (1/2π − 1/2θ)R (or (3/2π − 1/2θ)R), corresponding to x = ∞. Therefore f(u) = ∞ at u = b_i^2.

With the poles and zeros of f(u) determined, we may write the Seiberg-Witten curve as the double cover

\[ x = x_0 \prod_{i=1}^{k} \frac{(u - a_i^2)}{(u - b_i^2)} \]

for some x_0. Written as a polynomial in u, this becomes

\[ u^k + \left( A_1 + \frac{B_1}{x - x_0} \right) u^{k-1} + \ldots + \left( A_k + \frac{B_k}{x - x_0} \right) = 0 \]

with A_i and B_i constants. The points (x_0, η_0) and (x_0, −η_0), where η_0^2 = u(x_0^2 − 4)(x_0^2 − μ^2) correspond to the asymptotic positions of the NS5 branes as u → ∞. When rewritten in terms of ν, this curve agrees exactly with the results in ref. 7.

3.3. SU(N) + 2 antisymmetric + 4 fundamental hypermultiplets. The \( \mathcal{N} = 2 \) SU(N) gauge theory with hypermultiplets in representations 2 + 4 corresponds to a IIA brane configuration with two NS5 branes, each of them intersecting one of the O6− planes 7. There are a total of 2N D4 branes stretched between the two NS5 branes: N going around the x_6 circle one way, at positions v = a_i,
and $N$ going around the $x_6$ circle the other way, at positions $v = -a_i$. In invariant coordinates, there are $N$ mirror pairs of branes at positions $u = a_i^2$. As in sec. 3.2, the curve we seek is a degree two cover of the $u$-plane, with a branch cut for each pair of D4 branes.

We can describe a double cover of the $u$ plane with an equation

$$\frac{\eta}{x + 2} - f(u) = 0,$$

where $f(u)$ is to be determined, and $Q$ is described by the elliptic fibration $\eta^2 = u(x^2 - 4)(x - \lambda)$. For each value of $u$, the left hand side of (3.6) is a rational function with poles at $x = -2$ and $x = \infty$, and two zeroes, which correspond to the sheets of the cover.

Again, consider the universal cover $\mathbb{C}_v \times \mathbb{C}_\nu$ of $Q$, choosing $x$ such that $x = 2$, $-2$, $\infty$ correspond to $\nu = 0, \frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ respectively (see fig. 4). Then the two sheets of the cover will be at points $\nu_1$ and $\nu_2$ in each fiber, where $\nu_1 + \nu_2 + \frac{1}{4} + \frac{1}{4} \in \mathbb{Z} + \mathbb{Z}\tau$ (since these are the zeroes and poles of a rational function on $E$). The sheets coincide when $2\nu + \frac{1}{2} \in \mathbb{Z} + \mathbb{Z}\tau$, or $\nu = \frac{1}{4}\tau, \frac{1}{2}\tau + \frac{1}{4}, \frac{3}{4}\tau$, and $\frac{1}{2}\tau + \frac{3}{4}$; these correspond to the branch points of the cover.

We wish to choose $f(u)$ such that the locations of the D4 branes, $u = a_i^2$, are the images of points $\nu$ midway (along the $x_{10}$ circles) between the branch point pre-images, namely, $\nu = \frac{1}{4}\tau + \frac{1}{4}$ and $\frac{1}{4}\tau + \frac{3}{4}$ (corresponding to $x_6 = -\frac{1}{2}\pi L$), and $\frac{1}{4}\tau + \frac{1}{4}$ and $\frac{3}{4}\tau + \frac{3}{4}$ (corresponding to $x_6 = -\frac{3}{2}\pi L$). These points satisfy $2\nu + \frac{1}{2} + \frac{1}{4} \in \mathbb{Z} + \mathbb{Z}\tau$; in terms of the cubic curve, each of them is a point of tangency between the curve and a line through the point corresponding to $\frac{1}{2} + \frac{1}{2}$. It is easy to check that these points correspond to points with coordinates $(x, \eta)$ in the quotient surface $Q$ with $x = 2 \pm 2\sqrt{2} - \lambda$, and therefore satisfying

$$\frac{\eta}{x + 2} = \pm \sqrt{u\sqrt{2} - \lambda}.$$

The choice of sign in (3.7) corresponds to the choice $x_6 = -\frac{1}{2}\pi L$ or $-\frac{3}{2}\pi L$ at which the NS5-branes meet. This in turn corresponds to choosing the direction in which the D4 branes wrap the $x_6$ circle at $v = a_i$. (If the D4 branes wrap one direction at $v = a_i$, then due to the orientifold, they wrap the other direction at $v = -a_i$, so a different choice changes the sign of $v = \sqrt{u}$.)
Thus, comparing (3.6) and (3.7), we require \( f(u) \) to satisfy

\[
\frac{f(u)^2}{u} = 2 - \lambda \quad \text{when} \quad u = a_i^2.
\]

This can be attained by choosing

\[
f(u) = \frac{\sqrt{2 - \lambda} F_1(u)}{F_2(u)}
\]

where \( F_1(u) \) and \( F_2(u) \) satisfy either

\[
F_1(u)^2 - u F_2(u)^2 = \prod_{i=1}^{N} (u - a_i^2),
\]

or, if \( F_1(u) \) has a factor of \( u \),

\[
\frac{F_1(u)^2}{u} - F_2(u)^2 = \prod_{i=1}^{N} (u - a_i^2).
\]

Let us compare this to the results in sec. 5.1 of ref. [9]. In the notation of that paper,

\[
H_0(v) = \prod_{i=1}^{N} (v - a_i) = \sum_{j=0}^{N} u_j v^{N-j}, \quad H_1(v) = H_0(-v) = (-1)^N \prod_{i=1}^{N} (v + a_i)
\]

\[
H_0(v) = H_{\text{even}}(v) + H_{\text{odd}}(v), \quad H_1(v) = (-1)^N (H_{\text{even}}(v) - H_{\text{odd}}(v)),
\]

so that \( H_{\text{even}}(v) \) is the even degree part of \( H_0(v) \) if \( N \) is even, and the odd degree part if \( N \) is odd.

If \( N \) is even, then \( H_{\text{even}}(v) \) and \( H_{\text{odd}}(v)/v \) have only even powers of \( v \), so can be written as polynomials \( G_{\text{even}}(u) \) and \( G_{\text{odd}}(u) \), where \( u = v^2 \). Then

\[
G_{\text{even}}(u)^2 - u G_{\text{odd}}(u)^2 = H_{\text{even}}(v) - H_{\text{odd}}(v) = H_0(v) H_1(v) = \prod_{i=1}^{N} (u - a_i^2).
\]

Since \( F_1(u) = G_{\text{even}}(u) \) and \( F_2(u) = G_{\text{odd}}(u) \) obey condition (3.10), we may choose \( f(u) = \sqrt{2 - \lambda} G_{\text{even}}(u)/G_{\text{odd}}(u) \).

Similarly, if \( N \) is odd, then \( v H_{\text{even}}(v) \) and \( H_{\text{odd}}(v) \) have only even powers of \( v \), so we write them also as \( G_{\text{even}}(u) \) and \( G_{\text{odd}}(u) \). Then \( G_{\text{even}}(u) \) has a factor of \( u \), and we have

\[
\frac{G_{\text{even}}(u)^2}{u} - G_{\text{odd}}(u)^2 = H_{\text{even}}(v) - H_{\text{odd}}(v) = -H_0(v) H_1(v) = \prod_{i=1}^{N} (u - a_i^2).
\]

Since \( F_1(u) = G_{\text{even}}(u) \) and \( F_2(u) = G_{\text{odd}}(u) \) obey condition (3.11), we may again choose \( f(u) = \sqrt{2 - \lambda} G_{\text{even}}(u)/G_{\text{odd}}(u) \).

The Seiberg-Witten curve for this theory is therefore

\[
\frac{\eta}{x + 2} = \sqrt{2 - \lambda} \frac{G_{\text{even}}(u)}{G_{\text{odd}}(u)}.
\]
Observe that, since $G_{\text{even}}$ has higher degree than $G_{\text{odd}}$, the right hand side goes to $\infty$ as $u \to \infty$, thus the asymptotic positions of the NS5-branes are $x = -2$ and $x = \infty$, corresponding to $(x_6, x_{10}) = (0, \pi R)$ and $(-\pi L, (\pi - \frac{1}{4} \theta) R)$.

In terms of the double cover $C_v \times E$, the curve (3.15) becomes

$$y \frac{x+2}{x+2} = \sqrt{2} \prod_{i=1}^{N} (v - a_i) + \prod_{i=1}^{N} (v + a_i),$$

This agrees with the result (5.1.7) of ref. [9] and therefore with ref. [7], upon rescaling $u_i$, which amounts to a redefinition of the $a_i$.

Finally, we observe that, when $v = 0$, the right hand side of eq. (3.16) goes to infinity (and thus $x = -2$ or $\infty$) for even $N$, whereas it goes to zero (and thus $x = 2$ or $\lambda$) for odd $N$. That is, when $N$ is even, the $(x_6, x_{10})$ position of the NS5 branes (in the $v \to \infty$ limit) coincides with the $(x_6, x_{10})$ position of those fixed points of the orientifold (2.3) through which the curve (3.16) passes, whereas when $N$ is odd, the $(x_6, x_{10})$ position of the NS5 branes (in the $v \to \infty$ limit) coincides with the $(x_6, x_{10})$ position of those fixed points through which the curve does not pass. This is precisely in accordance with the discussion in section 4.4 of ref. [6].

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