VALUE REGIONS OF UNIVALENT SELF-MAPS WITH TWO
BOUNDARY FIXED POINTS

PAVEL GUMENYUK, DMITRI PROKHOROV

ABSTRACT. In this paper we find the exact value region \( \mathcal{V}(z_0, T) \) of the point evaluation functional \( f \mapsto f(z_0) \) over the class of all holomorphic injective self-maps \( f : \mathbb{D} \to \mathbb{D} \) of the unit disk \( \mathbb{D} \) having a boundary regular fixed point at \( \sigma = -1 \) with \( f'(-1) = e^T \) and the Denjoy–Wolff point at \( \tau = 1 \).

1. Introduction

Since the seminal paper [11] by Cowen and Pommerenke, the study of holomorphic functions with finite angular derivative at prescribed boundary points has been an active field of research in Complex Analysis, see, e.g., [2, 3, 10, 15, 17, 31, 36], just to mention some works in the topic.

Given a holomorphic function \( f \) in the unit disk \( \mathbb{D} := \{ z : |z| < 1 \} \) and a point \( \sigma \in \partial \mathbb{D} \) such that there exists finite angular limit \( f(\sigma) := \angle \lim_{z \to \sigma} f(z) \), the angular derivative at \( \sigma \) is \( f'(\sigma) := \angle \lim_{z \to \sigma} (f(z) - f(\sigma)) / (z - \sigma) \).

On the one hand, for univalent (i.e., holomorphic and injective) functions \( f \), existence of the angular derivative \( f'(\sigma) \) different from 0 and \( \infty \) is closely related to the geometry of \( f(\mathbb{D}) \) near \( f(\sigma) \); moreover, if there exists \( f'(\sigma) \neq 0, \infty \), then the behaviour of \( f \) at the boundary point \( \sigma \) resembles conformality, see, e.g., [30, §§4.3, 11.4].

On the other hand, for the dynamics of a holomorphic (but not necessarily univalent) self-map \( f : \mathbb{D} \to \mathbb{D} \), a crucial role is played by the points \( \sigma \in \partial \mathbb{D} \) for which \( f(\sigma) = \sigma \) (or, more generally, \( f(\sigma) \in \partial \mathbb{D} \)) and the angular derivative \( f'(\sigma) \) is finite, see, e.g., [5, 7, 8, 9, 14, 16, 29]. Such points \( \sigma \) are called boundary regular fixed points, see Section 2 for precise definitions and some basic theory. In particular, a classical result due to Wolff and Denjoy asserts that if \( f \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) has no fixed points in \( \mathbb{D} \), then it possesses the so-called (boundary) Denjoy–Wolff point, i.e., a unique boundary regular fixed point \( \tau \) such that \( f'(\tau) \leq 1 \).

In this paper we study univalent self-maps \( f : \mathbb{D} \to \mathbb{D} \) with a given boundary regular fixed point \( \sigma \in \partial \mathbb{D} \) and the Denjoy–Wolff point \( \tau \in \partial \mathbb{D} \setminus \{ \sigma \} \). Using automorphisms of \( \mathbb{D} \), we may suppose that \( \tau = 1 \) and \( \sigma = -1 \). Our main result is the sharp value region of \( f \mapsto f(z_0) \) for all such self-maps of \( \mathbb{D} \) with \( f'(-1) \) fixed. To give a detailed statement, fix \( z_0 \in \mathbb{D}, T > 0 \) and let \( \zeta_0 = x_1^0 + ix_2^0 := \ell(z_0) \), where

\[ \ell : \mathbb{D} \to \mathbb{S}; \quad z \mapsto \log\left( \frac{(1 + z)}{(1 - z)} \right) \]

\[ \]
is a conformal map of \( \mathbb{D} \) onto the strip \( S := \{ \zeta : -\pi/2 < \text{Im} \zeta < \pi/2 \} \). Define:

\[
a_\pm(T) := e^{-T/2} \sin x_2^0 \pm (1 - e^{-T/2}), \quad R(a, T) := \log \frac{1 - a}{1 - a_+(T)} \log \frac{1 + a}{1 + a_-(T)},
\]

\[
V(\zeta_0, T) := \left\{ x_1 + ix_2 \in S : a_-(T) \leq \sin x_2 \leq a_+(T), \ |x_1 - x_0^0 - \frac{T}{2}| \leq \sqrt{R(\sin x_2, T)} \right\}.
\]

**Theorem 1.** Let \( f \in \text{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{ \text{id}_\mathbb{D} \} \) and \( T > 0 \). Suppose that:

(i) \( f \) is univalent in \( \mathbb{D} \);

(ii) the Denjoy–Wolff point of \( f \) is \( \tau = 1 \);

(iii) \( \sigma = -1 \) is a boundary regular fixed point of \( f \) and \( f'(-1) = e^T \).

Then

\[
f(z_0) \in V(z_0, T) := \ell^{-1}(V(\ell(z_0), T)) \setminus \{z_0\} \quad \text{for any } z_0 \in \mathbb{D}.
\]

This result is sharp, i.e., for any \( w_0 \in V(z_0, T) \) there exists \( f \in \text{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{ \text{id}_\mathbb{D} \} \) satisfying (i)–(iii) and such that \( f(z_0) = w_0 \).

We can also characterize functions \( f \) delivering boundary points of \( V(z_0, T) \). In many extremal problems for univalent functions \( f : \mathbb{D} \to \mathbb{C} \) normalized by \( f(0) = f'(1) - 1 = 0 \), the Koebe function \( f_0(z) := z/(1 - z)^2 \) mapping \( \mathbb{D} \) onto \( \mathbb{C} \setminus (-\infty, \frac{1}{4}] \), and its rotations \( f_0(\theta z) = e^{i\theta f_0(e^{-i\theta} z)), \theta \in \mathbb{R}, \) are known to be extremal. For bounded univalent functions \( f : \mathbb{D} \to \mathbb{D} \) normalized by \( f(0) = 0, f'(0) > 0 \), the role of the Koebe function is played by the Pick functions \( p_\alpha(z) := f_0^{-1}(\alpha f_0(z)), \alpha \in (0, 1), \) mapping \( \mathbb{D} \) onto \( \mathbb{D} \setminus [-1, -r] \), \( r = r(\alpha) \in (0, 1) \). In our case, it would be natural to expect that some functions of the form \( f = h_1 \circ p_\alpha \circ h_2 \), where \( h_1, h_2 \in \text{Aut}(\mathbb{D}) \), are extremal.

**Theorem 2.** For any \( w_0 \in \partial V(z_0, T) \setminus \{z_0\} \), there exists a unique \( f = f_{w_0} \) satisfying conditions (i)–(iii) in Theorem 1 and such that \( f_{w_0}(z_0) = w_0 \). If \( w_0 = \ell^{-1}(\zeta + T) \), then \( f_{w_0} \) is a hyperbolic automorphism of \( \mathbb{D} \), namely \( f_{w_0}(z) = \ell^{-1}(\ell(z) + T) \). Otherwise, \( f_{w_0} \) is a conformal mapping of \( \mathbb{D} \) onto \( \mathbb{D} \) minus a slit along an analytic Jordan arc \( \gamma \) orthogonal to \( \partial \mathbb{D} \), with \( f'_{w_0}(1) = 1 \). Moreover, \( f_{w_0} = h_1 \circ p_\alpha \circ h_2 \) for some \( h_1, h_2 \in \text{Aut}(\mathbb{D}) \) and \( \alpha \in (0, 1) \) if and only if \( w_0 = \ell^{-1}(x_1^0 + \frac{T}{2} + i \arcsin a_\pm(T)) \).

**Remark 1.1.** Note that \( z_0 \) is a boundary point of the value region \( V(z_0, T) \), but does not belong to \( V(z_0, T) \). The proof of the above theorem, given in Section 4 shows that \( z_0 \) would be included, and this would be the only modification of the value region, if we replaced the equality \( f'(-1) = e^T \) in condition (iii) of Theorem 1 by the inequality \( f'(-1) \leq e^T \) and removed the requirement \( f \neq \text{id}_\mathbb{D} \) assuming as a convention that \( \text{id}_\mathbb{D} \) satisfies (ii). Note also that under the conditions of Theorem 1 modified in this way, \( f(z_0) = z_0 \) if and only if \( f = \text{id}_\mathbb{D} \), see Remark 2.3.

If \( f \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) has boundary regular fixed points at \( \pm 1 \), then replacing \( f \) by \( h \circ f \), where \( h \) is a suitable hyperbolic automorphism with the same boundary fixed points, we may suppose that \( \tau = 1 \) is the Denjoy–Wolff point. In this way, as a corollary of Theorems 1 and 2 we easily deduce a sharp estimate for \( f'(-1)f'(1) \), which was obtained earlier with the help of the extremal length method in [15, Section 4].
Corollary 1. Let \( z_0 \in \mathbb{D} \) and let \( f \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) be a univalent function with boundary regular fixed points at 1 and −1. Then

\[
\sqrt{f'(1)f'(1)} \geq L \left( \sin \text{Im} \ell(z_0), \sin \text{Im} \ell(f(z_0)) \right), \quad L(a, b) := \max \left\{ \frac{1+a}{1+b}, \frac{1-a}{1-b} \right\}.
\]

Inequality \( 1.2 \) is sharp. The equality can occur only for hyperbolic automorphisms and functions of the form \( f = h_1 \circ p_\alpha \circ h_2 \), \( h_1, h_2 \in \text{Aut}(\mathbb{D}), \alpha \in (0, 1). \)

Recently, the sharp value regions of \( f \mapsto f(z_0) \) have been determined for other classes of univalent self-maps \([22, 33, 35]\). The main instrument is the classical parametric representation of univalent functions, going back to the seminal work by Loewner \([27]\). In this paper, we use a new variant of Loewner’s parametric method, which is specific for functions satisfying conditions of Theorem 1. This variant of parametric representation was discovered quite recently, see \([19, 20]\). We discuss it in Section 3.

It is also worth mentioning that in \([17]\), using another specific variant of the parametric representation, Goryainov obtained the sharp value region of \( f \mapsto f'(0) \) in the class of all univalent \( f \in \text{Hol}(\mathbb{D}, \mathbb{D}), f(0) = 0 \), having a boundary regular fixed point at \( \sigma = 1 \) with a given value of \( f'(1) \).

To complete the Introduction, we recall another related result announced by Goryainov \([18]\). Dropping the univalence requirement, one can study holomorphic self-maps \( f : \mathbb{D} \to \mathbb{D} \) satisfying conditions (ii) and (iii) in Theorem 1 by using relationships between boundary regular fixed points and the Alexandrov–Clark measures. In particular, according to \([18]\), the value region \( \mathcal{D}(0, T) \) of \( f \mapsto f(0) \) over all such self-maps \( f \) is the closed disk whose diameter is the segment \([0, \ell^{-1}(T)]\), with the boundary point \( z_0 = 0 \) excluded. Analyzing the functions delivering the boundary points of \( \mathcal{D}(0, T) \), one can conclude that \( \partial \mathcal{D}(0, T) \cap \partial \mathcal{V}(0, T) = \{0, \ell^{-1}(T)\} \).

2. Holomorphic self-maps of the unit disk

In this section we cite some basic theory of holomorphic self-maps of \( \mathbb{D} \). More details can be found, e.g., in the monograph \([1]\).

Let \( f \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) and \( \sigma \in \partial \mathbb{D} \). According to the classical Julia–Wolff–Carathéodory Theorem, see, e.g., \([1]\) Theorem 1.2.5, Proposition 1.2.6, Theorem 1.2.7, if

\[
\alpha_f(\sigma) := \liminf_{\mathbb{D} \ni z \to \sigma} \frac{1 - |f(z)|}{1 - |z|} < +\infty,
\]

then

\[
\exists \lim_{z \to \sigma} f(z) =: f(\sigma) \in \partial \mathbb{D}, \quad \exists \lim_{z \to \sigma} \frac{f(z) - f(\sigma)}{z - \sigma} =: f'(\sigma) = \alpha_f(\sigma) \frac{f(\sigma)}{\sigma}, \quad \text{and}
\]

\[
\frac{|f(z) - f(\sigma)|^2}{1 - |f(z)|^2} \leq |f'(\sigma)| \frac{|z - \sigma|^2}{1 - |z|^2} \quad \text{for all} \ z \in \mathbb{D},
\]

with the equality sign if and only if \( f \in \text{Aut}(\mathbb{D}) \). Note that in its turn, existence of the limits in \( 2.2 \) satisfying \( f(\sigma) \in \partial \mathbb{D} \) and \( f'(\sigma) \neq \infty \) immediately implies \( 2.1 \).
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The value region $V(z_0, T)$ and the disks $D_1$, $D_2$ for $z_0 := i/2$, $T \in \{\log 2, \log 4, \log 6\}$ and for $z_0 := 0$, $T := \log 6$. The right picture also shows the disk $D(0, T)$. Notation $\Gamma^\pm$ is explained in Section 4.}
\end{figure}

Definition 2.1. Points $\sigma \in \partial \mathbb{D}$ satisfying (2.2) are referred to as regular contact points of $f$. If in addition to (2.2), $f(\sigma) = \sigma$, then $\sigma$ is said to be a regular fixed point of $f$. The number $f'(\sigma)$ is called the angular derivative of $f$ at $\sigma$.

Among all fixed points (boundary and internal) of a self-map $f \neq \text{id}_\mathbb{D}$, there is one point of special importance for dynamics. On the one hand, if $f(\tau) = \tau$ for some $\tau \in \mathbb{D}$, then by the Schwarz Lemma, $\tau$ is the only fixed point of $f$ in $\mathbb{D}$. If in addition, $f$ is not an elliptic automorphism, then $|f'(\tau)| < 1$ and hence the sequence of iterates $(f^n)$, $f^1 := f$, $f^{n+1} := f \circ f^n$, converges (to the constant function equal) to $\tau$ locally uniformly in $\mathbb{D}$. On the other hand, if $f$ has no fixed points in $\mathbb{D}$, then by the Denjoy–Wolff Theorem, see, e.g. [1, Theorem 1.2.14, Corollary 1.2.16, Theorem 1.3.9], $f$ has a unique boundary regular fixed point $\tau \in \partial \mathbb{D}$ such that $f'(\tau) \leq 1$ and moreover, $f^n \to \tau$ locally uniformly in $\mathbb{D}$ as $n \to +\infty$.

Definition 2.2. The point $\tau$ above is referred to as the Denjoy–Wolff point of $f$.

Remark 2.3. Since the strict inequality holds in (2.3) unless $f \in \text{Aut}(\mathbb{D})$, a self-map $f$ can have a fixed point in $\mathbb{D}$ and a boundary regular fixed point $\sigma$ with $f'(\sigma) \leq 1$ only if $f = \text{id}_\mathbb{D}$.

Remark 2.4. Let $f_n(z) := -\sigma H^{-1}(\alpha/H(z/\sigma) + \beta/(H(z/\sigma) + n))$, where $n \in \mathbb{N}$, $\alpha, \beta > 0$, and $H(z) := (1 + z)/(1 - z)$. Note that $f_n(\mathbb{D}) \subset \mathbb{D}$ for all $n \in \mathbb{N}$ and that $f_n(z) \to$
\[ f(z) := (z + c)/(1 + cz), \quad c := \sigma(1 - \alpha)/(1 + \alpha), \] locally uniformly in \( \mathbb{D} \) as \( n \to +\infty \). Moreover, \( f_n(\sigma) = f(\sigma) = \sigma \) and \( f_n'(\sigma) = \alpha + \beta \) for all \( n \in \mathbb{N} \), but \( f'(\sigma) = \alpha \). This example shows that the map \( f \mapsto f'(\sigma) \) is not continuous. However, it turns out to be semicontinuous in the following sense. Suppose that \( f_n(z) \to f(z) \) as \( n \to +\infty \) and that \( \sigma \in \partial \mathbb{D} \) is a boundary regular fixed point of \( f_n \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) for all \( n \in \mathbb{N} \) with \( \alpha := \liminf_{n \to +\infty} f_n'(\sigma) < +\infty \). Then passing in Julia's inequality (2.3) applied for functions \( f_n \) to the limit, we conclude that \( f \) satisfies (2.3) with \( |f'(\sigma)| \) replaced by \( \alpha \). It follows that \( \alpha f(\sigma) \leq \alpha < +\infty \). Therefore, either \( f \equiv \sigma \) or \( f \not\equiv \sigma \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) and \( \sigma \) is a regular boundary fixed point of \( f \) with \( f'(\sigma) \leq \alpha \). As a consequence, the set of all \( f \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) sharing two different boundary regular fixed points \( \sigma_1 \) and \( \sigma_2 \) and satisfying \( f'(\sigma_j) \leq \alpha_j < +\infty, \ j = 1, 2, \) is compact.

According to inequality (2.3), the value region \( \mathcal{V}(z_0, T) \) in Theorem 3 lies in the intersection of two closed disks \( D_1, D_2 \subset \overline{\mathbb{D}} \) whose boundaries pass through \( z_0 \) and \( \tau = 1 \) and through \( \ell^{-1}(\ell(z_0) + T) \) and \( \sigma = -1 \), respectively. Comparison of \( \mathcal{V}(z_0, T) \) with \( D_1 \cap D_2 \) is shown in Figure 1. On the right picture, for which \( z_0 = 0 \), we also place the value range \( \mathcal{D}(0, T) \) of \( f \mapsto f(0) \) over all holomorphic but not necessary injective maps \( f : \mathbb{D} \to \mathbb{D} \) satisfying conditions (ii) and (iii) in Theorem 3. See [18].

3. Parametric representation

Denote the class of all \( f \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) satisfying conditions (i)–(iii) in Theorem 3 by \( \mathcal{U}(T) \). The following theorem, proved in [20], gives a parametric representation for \( \mathcal{U}(T) \) in terms of a Loewner–Kufarev-type ODE.

**Theorem 3 ([20, Corollary 1.2]).** The class \( \mathcal{U}(T) \) coincides with the set of all functions representable in the form \( f(z) = w_z(T) \) for all \( z \in \mathbb{D} \), where \( w_z(t) \) is the unique solution to the initial value problem

\[
(3.1) \quad \frac{dw_z}{dt} = \frac{1}{4}(1 - w_z)^2(1 + w_z)q(w_z, t), \quad t \in [0, T], \quad w_z(0) = z,
\]

with some function \( q : \mathbb{D} \times [0, T] \to \mathbb{C} \) satisfying the following conditions:

(i) for every \( z \in \mathbb{D} \), \( q(z, \cdot) \) is measurable on \( [0, T] \);

(ii) for a.e. \( t \in [0, T] \), \( q(\cdot, t) \) has the following integral representation

\[
(3.2) \quad q(z, t) = \int_{\partial \mathbb{D} \setminus \{1\}} \frac{1 - \kappa}{1 + \kappa z} \, d\nu_t(\kappa),
\]

where \( \nu_t \) is a probability measure on \( \partial \mathbb{D} \setminus \{1\} \).

**Remark 3.1.** A related parametric representation for a class of univalent self-maps of a strip was considered in [13].

**Remark 3.2.** In many cases, it is more convenient to deal with the the union \( \mathcal{U}(T) := \bigcup_{0 < T' < T} \mathcal{U}(T') \), where we define \( \mathcal{U}(0) := \{ \text{id}_\mathbb{D} \} \). Indeed, it is evident from the argument of Remark 2.4 that in contrast to \( \mathcal{U}(T) \), the class \( \mathcal{U}(T) \) is compact. Moreover, it is easy
to see that Theorem 3 gives representation of $\Omega'(T)$ if all probability measures $\nu_t$ in (3.2) are replaced with all positive Borel measures $\nu_t$ satisfying
\begin{equation}
\nu_t(\partial \mathbb{D} \setminus \{1\}) \in [0, 1].
\end{equation}
Note that the possibility of $\nu_t = 0$ is not excluded.

Remark 3.3. Obviously, the right-hand side of (3.1) can be written as $G(w_z, t)$, where $G(z, t) := \frac{1}{4}(1 - z)^2(1 + z)q(z, t)$ with $q$ satisfying conditions (i) and (ii) in Theorem 3. By [19, Theorem 1], $G(\cdot, t)$ is an infinitesimal generator in $\mathbb{D}$ for each $t \in [0, T]$. For simplicity, extend $G$ to all $t \geq 0$ by setting $G(z, t) \equiv 0$ for any $t > T$. Then according to the general theory of Loewner–Kufarev-type equations, see [4, Sections 3–5], for any $s \geq 0$ and any $z \in \mathbb{D}$, the initial value problem $dw/dt = G(w, t)$, $t \geq s$, $w(s) = z$, has a unique solution $w = w_{z, s}(t)$ defined for all $t \geq s$ and the functions $\varphi_{z, t}(z) := w_{z, s}(t)$, $z \in \mathbb{D}$, $t \geq s \geq 0$, form an evolution family, see [4, Definition 3.1].

**Proposition 1.** Let $\vartheta : [0, T] \to (-\pi, \pi) \setminus \{0\}$, $T > 0$, be a $C^1$-smooth function. Suppose that in the conditions of Theorem 3, $d\nu_t(e^{i\vartheta}) = \delta(\theta - \vartheta(t))d\theta$ for all $t \in [0, T]$, where $\delta$ stands for the Dirac delta function. Then $f$ maps $\mathbb{D}$ onto $\mathbb{D} \setminus \gamma$, where $\gamma$ is a slit in $\mathbb{D}$, i.e., $\gamma$ is the image of a homeomorphism $\gamma : [0, 1] \mapsto \overline{\mathbb{D}}$ with $\gamma([0, 1]) \subset \mathbb{D}$ and $\gamma(1) \in \partial \mathbb{D}$.

Moreover:
\begin{itemize}
  \item[(i)] if $\vartheta$ is a real-analytic function on $[0, T]$, then $\gamma$ is a real-analytic Jordan arc orthogonal to $\mathbb{D}$;
  \item[(ii)] $\gamma$ is a circular arc or a straight line segment orthogonal to $\partial \mathbb{D}$ if and only if
\end{itemize}
\begin{equation}
\lambda(t) := i\frac{1 + e^{i\vartheta(t)}}{1 - e^{i\vartheta(t)}} = C_1 e^{-t/2} \left( C_2 e^{t/2} + \sqrt{C_2^2 (e^t - 1) + 1} \right)^3
\end{equation}
for all $t \in [0, T]$ and some constants $C_1, C_2 \in \mathbb{R}$, $C_1 \neq 0$.

**Proof.** In the conditions of the proposition, (3.1) takes the following form:
\begin{equation}
\frac{dw_z}{dt} = \frac{1}{4}(1 - w_z)^2(1 + w_z)\frac{1 - e^{i\vartheta(t)}}{1 + e^{i\vartheta(t)}w_z}, \quad t \in [0, T], \quad w_z(0) = z.
\end{equation}
The change of variables $\omega_z := H(w_z)$, where $H(w) := i(1 + w)/(1 - w)$ maps $\mathbb{D}$ conformally onto $\mathbb{H} := \{\omega : \text{Im}\omega > 0\}$, transforms the above problem to
\begin{equation}
\frac{d\omega_z}{dt} = \frac{\omega_z}{1 - \lambda(t)\omega_z}, \quad t \in [0, T], \quad \omega_z(0) = H(z),
\end{equation}
where $\lambda(t) := H(e^{i\vartheta(t)})$ for all $t \in [0, T]$. Making further change of variables
\begin{align*}
\bar{\omega}_z(t) := \omega_z(t) + \int_0^t \frac{ds}{\lambda(s)}, \quad \xi(t) := \frac{1}{\lambda(t)} + \int_0^t \frac{ds}{\lambda(s)}, \quad \tau = v(t) := \frac{1}{2} \int_0^t \frac{ds}{\lambda(s)^2},
\end{align*}
we obtain the chordal Loewner equation
\begin{equation}
\frac{d\bar{\omega}_z}{d\tau} = \frac{2}{\xi - \bar{\omega}_z}, \quad \tau \in [0, v(T)], \quad \bar{\omega}_z(0) = H(z).
The geometry of solutions to (3.1) is well-studied, see, e.g., [23, 28, 31, 21, 37]; see also [23]. In particular, since the function \( s \mapsto \xi(v^{-1}(s)) \) is \( C^1 \)-smooth, it follows that \( z \mapsto \hat{\omega}_s(T) \) maps \( \mathbb{D} \) onto \( \mathbb{H} \) minus a slit along some Jordan arc \( \gamma_0 \). Taking into account that \( w_s(T) = H^{-1}(\hat{\omega}_s(T) - C) \), where \( C := \int_0^T \lambda(t)^{-1} \, dt \), this proves the first part of the proposition.

If \( \vartheta \) is real-analytic, then \( s \mapsto \xi(v^{-1}(s)) \) is real-analytic on \([0, T]\) as well, and hence by [26, Theorem 1.4], \( \gamma_0 \) is a real-analytic Jordan arc. Moreover, the argument of [26, Section 6.1] shows that in such a case, \( \gamma_0 \) is orthogonal to \( \mathbb{R} \). This proves (i).

It remains to prove (ii). Suppose that \( \gamma \) is a circular arc or a straight line segment orthogonal to \( \partial \mathbb{D} \). Then we can find a linear-fractional transformation \( H_\gamma \) of \( \mathbb{D} \) onto \( \mathbb{H} \) such that \( H_\gamma(\gamma) = \{0, i\} \). Let \((\varphi_{\gamma,t})\) be the evolution family associated with equation (3.5), see Remark 3.3. Note that \( \varphi_{\gamma,t}(\mathbb{D}) \supset \varphi_{\gamma,t}(\varphi_{\gamma,0}(\mathbb{D})) = \varphi_{\gamma,t}(\mathbb{D}) = f(\mathbb{D}) \) for any \( t \in [0, T] \).

It follows that the intersection of a sufficiently small neighbourhood of \( H_\gamma^{-1}(\infty) \) with \( \partial \varphi_{\gamma,t}(\mathbb{D}) \) is an open arc of \( \partial \mathbb{D} \) containing \( H_\gamma^{-1}(\infty) \). Therefore, for each \( t \in [0, T] \), there exists a unique \( h_t \in \text{Aut}(\mathbb{D}) \) such that \( g_t := H_\gamma \circ \varphi_{\gamma,t} \circ h_t \circ H_{\gamma}^{-1} \in \text{Hol}(\mathbb{H}, \mathbb{H}) \) satisfies the Laurent expansion \( g_t(z) = z - c(t)/z + \ldots \) at \( \infty \) with some \( c(t) \in \mathbb{R} \).

Denote \( H_t := H_\gamma \circ h_t^{-1} \) for all \( t \in [0, T] \). By construction, \( \mathbb{H} \setminus \{0, i\} = g_0(\mathbb{H}) \subset g_t(\mathbb{H}) \subset g_T(\mathbb{H}) = \mathbb{H} \) for all \( t \in [0, T] \). Thanks to continuity of \( \vartheta \), the function \( t \mapsto c(t) \) is \( C^1 \)-smooth. Therefore, according to the classical result [24] by Kufarev et al, see also [12], for any \( z \in \mathbb{D} \), \( \hat{\omega}_t(z) := g_t^{-1} \circ g_0(H_0(z)), t \in [0, T], \) is the unique solution to the initial value problem \( d\hat{\omega}_t/dt = -c'(t)/\hat{\omega}_t, \hat{\omega}_t(0) = H_0(z) \in \mathbb{H} \).

By construction, \( \hat{\omega}_t(t) = H_t(w_t) \) for all \( t \in [0, T] \) and all \( z \in \mathbb{D} \). Comparing the differential equations for \( \hat{\omega}_t \) and \( w_t \), one can conclude that for all \( t \in [0, T] \),

\[
H_t(w) := \frac{\lambda(t)H(w) - 1}{a(t)\left(\lambda(t)H(w) - 1\right) + b(t)}
\]

with real coefficients \( a(t) \) and \( b(t) \) satisfying

\[
da/dt = a^2/b^2, \quad db/dt = -3a + b + 3a^2/b, \quad t \in [0, T],
\]

and such that \( \lambda(t)/\lambda(t) = 1 - 3a(t)/b(t) \) and \( b(t)\lambda(t) > 0 \) for all \( t \in [0, T] \). System (3.9) can be solved by introducing a new unknown function \( k(t) := a(t)/b(t) \). In this way, one can easily check that \( \lambda \) must be of the form (3.4).

Conversely, if \( \lambda \) is given by (3.4), then system (3.9) has a real-valued solution satisfying \( \lambda'(t)/\lambda(t) = 1 - 3a(t)/b(t) \) and \( b(t)\lambda(t) > 0 \) for all \( t \in [0, T] \). It follows that for any \( z \in \mathbb{D} \), the function \( \hat{\omega}_t(z) := H_t(w_t) \), where \( H_t \) is given by (3.8), is a solution to \( d\hat{\omega}_t/dt = -1/(b(t)^2\hat{\omega}_t), t \in [0, T], \hat{\omega}_t(0) = H_0(z) \in \mathbb{H} \). Solving the latter initial value problem for \( \hat{\omega}_t \), we conclude that the image of the map \( \mathbb{D} \ni z \mapsto \hat{\omega}_z(T) \) is the domain \( \mathbb{H} \setminus [0, i\sqrt{Q_T}], \) where \( Q_T := 2\int_0^T b(t)^{-2} \, dt \). Thus, \( \gamma = H_T^{-1}([0, i\sqrt{Q_T}]) \) is a circular arc or a straight line segment orthogonal to \( \partial \mathbb{D} \). The proof is now complete. \( \square \)

4. Proof of the main results

In this section we prove Theorems 1 and 2. Fix \( T > 0 \). We start by considering the problem to determine the compact value region \( \{f(z_0) : f \in \mathfrak{A}(T)\} \). Thanks to
Pontryagin’s maximum principle, by the driving functions we can rewrite (4.1) in the following form

\[
\frac{d\zeta}{dt} = \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1-i\lambda e^{\zeta}}, \quad t \in [0,T]; \quad \zeta|_{t=0} = \zeta_0 := \ell(z_0),
\]

where \( \mu_t \)'s are positive Borel measures on \( \mathbb{R} \) with \( \mu_t(\mathbb{R}) \leq 1 \). By using the prime in the notation \( \Omega'_T \) we emphasize that this reachable set corresponds to the class \( T' \).

Denote \( x_1 := \text{Re} \zeta \) and \( x_2 := \text{Im} \zeta \). Note that \( x_2 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). For any fixed \( \zeta = x_1 + ix_2 \in \mathbb{S} \), the range of the right-hand side in (4.1), regarded as a function of the measure \( \mu_t \), is the disk

\[
\left\{ \omega \in \mathbb{C} : \left| \omega - \frac{e^{-ix_2}}{2\cos x_2} \right| \leq \frac{1}{2\cos x_2} \right\}.
\]

Therefore, replacing the measure-valued control \( t \mapsto \mu_t \) with the complex-valued control

\[
u(t) := 2e^{ix_2} \cos x_2 \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1-i\lambda e^{x_1+ix_2}},
\]

we can rewrite (4.1) in the following form

\[
\begin{align*}
\frac{dx_1}{dt} &= \text{Re} \frac{u(t)e^{-ix_2}}{2\cos x_2} = \frac{1}{2} \text{Re} u(t) + \frac{\text{tg} x_2}{2} \text{Im} u(t), \quad x_1(0) = x_1^0 := \text{Re} \zeta_0, \\
\frac{dx_2}{dt} &= \text{Im} \frac{u(t)e^{-ix_2}}{2\cos x_2} = \frac{1}{2} \text{Im} u(t) - \frac{\text{tg} x_2}{2} \text{Re} u(t), \quad x_2(0) = x_2^0 := \text{Re} \zeta_0,
\end{align*}
\]

where \( u : [0,T] \to U := \{ u : |u - 1| \leq 1 \} \) is an arbitrary measurable function.

Introduce the Hamilton function

\[
H(x_1, x_2, \Psi_1, \Psi_2, u) := \Psi_1 \text{Re} \frac{ue^{-ix_2}}{2\cos x_2} + \Psi_2 \text{Im} \frac{ue^{-ix_2}}{2\cos x_2} = \text{Re} \frac{ue^{-ix_2}(\Psi_1 - i\Psi_2)}{2\cos x_2},
\]

where \( \Psi_1, \Psi_2 \) satisfy the adjoint system of ODEs

\[
\begin{align*}
\frac{d\Psi_1}{dt} &= -\frac{\partial H}{\partial x_1} = 0, \quad \frac{d\Psi_2}{dt} = -\frac{\partial H}{\partial x_2} = -\frac{\text{Im} u(t)(\Psi_1 - i\Psi_2)}{2\cos^2 x_2}.
\end{align*}
\]

Boundary points of the reachable set \( \Omega'_T \), forming a dense subset of \( \partial \Omega'_T \), are generated by the driving functions \( u^* \) satisfying the necessary optimal condition in the form of Pontryagin’s maximum principle,

\[
\max_{u \in U} H(x_1(t), x_2(t), \Psi_1(t), \Psi_2(t), u) = H(x_1(t), x_2(t), \Psi_1(t), \Psi_2(t), u^*(t))
\]

for all \( t \in [0,T] \), see, e.g., [32]. Trajectories \( (x_1(t), x_2(t)) \) in (4.5) are optimal in the reachable set problem, and \( (\Psi_1(t), \Psi_2(t)) \) satisfy the adjoint system (4.4) with the optimal
trajectories. In particular, \((\Psi_1(t), \Psi_2(t))\) does not vanish, and hence the maximum in (4.5) is attained at the unique point \(u^* = 1 + e^{ix_2 + \varphi}\), where \(\varphi := \arg(\Psi_1 + i\Psi_2)\). Therefore, from (4.2) – (4.4) for the optimal trajectories we obtain

\[
\frac{dx_1}{dt} = \frac{\cos \varphi + \cos x_2}{2 \cos x_2}, \quad x_1(0) = x_1^0,
\]

\[
\frac{dx_2}{dt} = \frac{\sin \varphi - \sin x_2}{2 \cos x_2}, \quad x_2(0) = x_2^0,
\]

\[
\frac{d\Psi_1}{dt} = 0,
\]

\[
\frac{d\Psi_2}{dt} = \frac{\sin \varphi - \sin x_2}{2 \cos^2 x_2} |\Psi_1 - i\Psi_2|.
\]

System (4.6) – (4.9) is invariant w.r.t. multiplication of \((\Psi_1, \Psi_2)\) by a positive constant. Therefore, we may assume that either \(\Psi_1 \equiv 0\), or \(\Psi_1 \equiv 1\), or \(\Psi_1 \equiv -1\).

If \(\Psi_1 \equiv 0\), then \(\varphi = \pm \pi/2\) and we easily get that for all \(t \geq 0\),

\[
x_1(t) = x_1(0) + t/2, \quad \sin x_2(t) = a_\pm(t) := e^{-t/2} \sin x_2(0) \pm (1 - e^{-t/2}).
\]

Now let \(\Psi_1 \equiv 1\). Then \(\varphi \in (-\pi/2, \pi/2)\) and equation (4.9) takes the following form

\[
\frac{d\varphi}{dt} = \frac{\sin \varphi - \sin x_2}{2 \cos^2 x_2} \cos \varphi = \frac{\cos \varphi}{\cos x_2} \frac{dx_2}{dt}.
\]

System (4.6), (4.11) admits the first integral

\[
I(x_2, \varphi) := \frac{1 - \sin \varphi}{1 + \sin \varphi} \frac{1 + \sin x_2}{1 - \sin x_2} > 0,
\]

and as a result it can be integrated in quadratures. Namely, if \(C := I(x_2(0), \varphi(0)) \neq 1\), we obtain the following identities

\[
B_1(t) - CB_2(t) = (C - 1)t/2,
\]

\[
x_1(t) - x_1(0) = \frac{B_1(t) - \sqrt{CB_2(t)}}{\sqrt{C} - 1},
\]

where \(B_1(t) := \log \frac{1 - \sin x_2(t)}{1 - \sin x_2(0)}\), \(B_2(t) := \log \frac{1 + \sin x_2(t)}{1 + \sin x_2(0)}\).

Excluding \(C\) from (4.12), (4.13) and setting \(t := T\) gives

\[
x_1(T) = x_1(0) + \frac{1}{2} \left( T + \sqrt{(T + 2B_1(T))(T + 2B_2(T))} \right)
\]

\[
= x_1(0) + \frac{T}{2} + \sqrt{R(\sin x_2(T), T)},
\]

where we took into account that according to (4.12),

\[
\frac{d}{dt} \left( t + 2B_1(t) \right) = \frac{2C}{1 + \sin x_2(t) + C(1 - \sin x_2(t))} > 0
\]
and therefore, \( T + 2B_1(T) > 0 \).

For \( C = 1 \), we have \( \varphi(t) = x_2(t) \) and hence \( d\varphi/dt = dx_2/dt = 0, dx_1/dt = 1 \). Therefore, if \( C = 1 \), then (4.12) and (4.14) hold as well.

Since \( C > 0 \), from (4.12) we obtain that \( x_2(T) \in J(T) := (\arcsin a_-(x), \arcsin a_+(x)) \). On the other hand, for any \( x \in J(T) \) there exists a unique \( C = C(x) > 0 \) that verifies (4.12) with \( T \) and \( x \) substituted for \( t \) and \( x_2(t) \), respectively. Solving \( I(x_2(0), \varphi(0)) = C(x) \) provides us with the initial condition in equation (4.11) for which \( x_2(T) = x \).

Investigating the case \( \Psi_1 \equiv -1 \) in a similar way, we conclude that \( \partial \Omega_T^+ \) is the union of the two Jordan arcs

\[
\Gamma^\pm(T) := \left\{ x_1 + ix_2 \in \mathbb{S} : a_-(T) \leq x_2 \leq a_+(T), \ x_1 = x_1^0 + \frac{T}{2} \pm \sqrt{R(x_2, T)} \right\},
\]

which do not intersect except for the common end-points \( \omega^\pm := x_1^0 + T/2 + i \arcsin a_+(T) \), delivered by solutions (4.10). Taking into account that by the very definition, \( \mathcal{U}'(T') \subset \mathcal{U}'(T) \) for any \( T' \in [0, T] \), it follows that \( \Omega_T^+ = V(\zeta_0, T) \).

The next step in the proof is to pass from the class \( \mathcal{U}'(T) \) to the class \( \mathcal{U}(T) \). In the problem of finding the value region of the functional \( f \mapsto f(z_0) \), this is equivalent to replacing the range \( U \) of the admissible controls \( u \) in (4.2) – (4.3) by \( U \setminus \{0\} \). Denote by \( \Omega_T^u \) the corresponding reachable set. By re-scaling the time, the problem to find \( \Omega_T^u \), \( T' \in (0, T) \), can be restated as the reachable set problem at the same time \( T \) and for the same controllable system, but with the value range of admissible controls restricted to \( \alpha(U \setminus \{0\}) \), \( \alpha := T'/T \). Note also that \( \Gamma^+(T) \cup \Gamma^-(T) \setminus \{\zeta_0\} \subset \Omega_T^u \) for any \( T > 0 \). Since \( \alpha(U \setminus \{0\}) \subset U \setminus \{0\} \) for any \( \alpha \in (0, 1) \), it follows that

\[
\Gamma^+(T') \cup \Gamma^-(T') \setminus \{\zeta_0\} \subset \Omega_{T'}^u \subset \Omega_T^u \quad \text{for any } T \in (0, T].
\]

Thus \( \Omega_T = V(\zeta_0, T) \setminus \{\zeta_0\} \), which completes the proof of Theorem 1.

To prove Theorem 2, we have to identify the functions delivering the boundary points of \( \mathcal{V}(z_0, T) \). They correspond to the controls \( u^* \) satisfying Pontryagin’s maximum principle (4.5). It is easy to see from the above argument that every point \( \omega \in \partial \Omega_T^u \setminus \{0\} \) corresponds to a unique control, which is \( C^1 \)-smooth and takes values on \( \partial U \setminus \{0\} \). It follows that the corresponding measures \( \mu_t \) in (4.1) and the measures \( \nu_t \) in the Loewner-type representation (3.1), (3.2) are also unique. They are probability measures concentrated at one point that moves smoothly with \( t \). Namely, \( d\mu_t(\lambda) = \delta(\lambda - \lambda^*(t)) \, d\lambda \), where

\[
\lambda^*(t) := \frac{1 - 2 \cos x_2(t)/(e^{-iz_2(t)} + e^{iz_2(t)})}{i(e^{-i\varphi(t)} + e^{i\varphi(t)})} = e^{-x_1(t)} \frac{\sin \varphi(t) - x_2(t)}{\cos \varphi(t) + x_2(t)}.
\]

The point \( \omega = \omega_0 := \zeta_0 + T \in \Gamma^+ \) corresponds to \( C = 1 \), in which case \( \varphi(t) = x_2(t) \) for all \( t \in [0, T] \) and hence \( \lambda^*(t) \equiv 0 \). Therefore, from (4.1) we see that the unique \( f \in \mathcal{U}(T) \) delivering the boundary point \( \ell^{-1}(\omega_0) \) of \( \mathcal{V}(z_0, T) \) is the hyperbolic automorphism

\[
f(z) = \frac{z + c(T)}{1 + c(T)z}, \quad c(T) := \frac{e^T - 1}{e^T + 1}, \quad \text{for all } z \in \mathbb{D}.
\]
For the common end-points $\omega^\pm$ of $\Gamma^+$ and $\Gamma^-$, which correspond to $\varphi = \pm \pi/2$, formula (4.15) simplifies to $\lambda^*(t) = \pm e^{-x_1(t)}$. In view of (4.10), the latter expression coincides with $\lambda(t)$ given by (3.4) if we set $C_1 := \pm e^{-x_1^0}$ and $C_2 := 0$. Taking into account the correspondence between $\mu$, $\nu$ and $\nu_t$ and applying Proposition 1 we conclude that the unique functions $f \in \mathcal{U}(T)$ delivering the points $\ell^{-1}(\omega^\pm)$ map $\mathbb{D}$ onto $\mathbb{D}$ minus a slit along a circular arc or a segment of a straight line orthogonal to $\partial \mathbb{D}$.

It remains to compare $\lambda^*(t)$ given by (4.15) with $\lambda(t)$ given by (3.4) for the case $\omega \in \partial \Omega_T \setminus \{\zeta_0, \omega_0, \omega^+, \omega^--\}$. Suppose $\omega \in \Gamma^+ \setminus \{\omega_0, \omega^+, \omega^--\}$. Using equations (4.6), (4.7), (4.11) and taking into account the first integral $I(x_2, \varphi) = C$, we find that

$$\left(1 + 2 \frac{d}{dt} \log \lambda^*(t)\right)^2 = \left(\frac{\cos \varphi(t)}{\cos x_2(t)}\right)^2 = \frac{C(1 - a^2)}{(1 + C)(a + (1 - C)a^2)^2}, \quad a := \sin x_2(t),$$

while $(1 + 2(d/dt) \log \lambda(t))^2 = 9C_2^2 e^t/(1 + C_2^2(e^t - 1))$. However, according to (4.12), $e^t$ cannot be expressed as a rational function of $\sin x_2(t)$. This shows that $\lambda^*$ is not of the form (3.4) and hence, by Proposition 1, the unique function $f \in \mathcal{U}(T)$ that delivers the boundary point $\ell^{-1}(\omega)$ maps $\mathbb{D}$ onto $\mathbb{D}$ minus a slit along a real-analytic arc $\gamma$ orthogonal to $\partial \mathbb{D}$ but different from a circular arc or a segment of a straight line. A similar argument applied to the case $\omega \in \Gamma^- \setminus \{\zeta_0, \omega^+, \omega^--\}$ completes the proof of Theorem 2. \qed

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P. Gumenyuk: Department of Mathematics and Natural Sciences, University of Stavanger, N-4036 Stavanger, Norway  
*E-mail address*: pavel.gumenyuk@uis.no

D. Prokhorov: Department of Mathematics and Mechanics, Saratov State University, Saratov 410012, Russia  
*E-mail address*: ProkhorovDV@info.sgu.ru