Albert Einstein and the fifth dimension.
A new interpretation of the papers published in 1927

Giulio Peruzzi* and Alessio Rocci†

1,2Department of Physics and Astronomy “G. Galilei”, via Marzolo 8, I-35131 Padova (Italy)

November 8, 2018

Abstract

In 1927 Einstein sent two brief communications to the Prussian Academy of Sciences on Kaluza’s five-dimensional theory. In his Einstein biography, Abraham Pais asserted that he could not understand the reasons that pushed Einstein to communicate his work. Indeed, Einstein’s paper seems to be very close to Klein’s approach, published in 1926. The question seems to be yet unanswered, also in recent works. The idea of adding a new space-like dimension fascinated Einstein for a long period with oscillating enthusiasm. We review his 1927 brief communications and we propose a reinterpretation of his papers. We argue that Einstein’s work contains some fundamental innovations and that a deeper analysis shows the differences between Einstein’s and Kaluza’s approaches. Firstly, Einstein realized that Kaluza’s cylinder condition admitted a covariant formulation. Secondly he explicitly tried to use Weyl’s scale invariance in order to construct a theory covering the macroscopic as well the microscopic phenomena. In constructing a five-dimensional action, Einstein used in a modern way, the idea of “gauge invariance”. We argue that he did not undergo the mistake of setting as constant the fifth component of the five-dimensional metric. We urge that Einstein knew Klein’s work and that he did not mention Klein’s and Fock’s contributions in his first communication because he was trying to construct a new approach to the five-dimensional Universe. Furthermore, we reiterate that he cited them in the second part, when he realized that he failed his attempt. We found a new implementation of Einstein’s idea in the Seventies.

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*giulio.peruzzi@unipd.it
†a_rocci@hotmail.com
1 Prologue: inconsistent!

In his first paper on the five-dimensional Universe [Klein 1926a] [Klein 1984], in order to unify gravitational, electromagentic and quantum phenomena, Oskar Klein introduced a five-dimensional space-time. In Klein’s model, Einstein’s General Relativity (GR) and Maxwell’s electromagnetism (EM) emerge as part of a five-dimensional GR-like theory. Indeed, Klein showed that, using suitable Ansatz discussed later, the five-dimensional Einstein equations, obtained using a five-dimensional curvature scalar $\tilde{R}$ as Lagrangian density, are equivalent to the four-dimensional Einstein equations coupled with Maxwell’s equations. The equivalence is based on the two following hypotheses.

First, Klein imposed the so-called cylinder condition, namely $\partial_5 \gamma_{\mu \bar{\nu}} = 0$, where $\gamma_{\mu \bar{\nu}}$ is the five-dimensional metric. As a consequence, using modern language, there is a residual four-dimensional general coordinate invariance and an invariance associated with the transformations of the fifth coordinate. Indeed, the cylinder condition means that the five-dimensional space-time admits a Killing vector, i.e. a preferred direction in five dimensions.

Second, Klein noted that scalar quantities are particular invariants of the transformation laws, thus he imposed the constancy of the 55-component of the metric, i.e. $\gamma_{55} = \text{constant}$, as a further condition. Then, he inserted both hypotheses into the five-dimensional action, the integral over the whole space-time of the Lagrangian density. Finally, Klein used the resulting action for obtaining his five-dimensional field equations, which are equivalent to the system formed by Einstein’s and Maxwell’s four-dimensional equations.

The second hypothesis, i.e. the constancy of the scalar function $\gamma_{55}$, is sometimes claimed as responsible for an inconsistency of Klein’s model ([Overduin 1997]; p. 315), ([O’Raifeartaigh 2000]; p. 10). This remark was pointed out for the first time by Pascual Jordan [Jordan 1947] and Yves Thiry [Thiry 1948] in 1947 and in 1948 respectively. The inconsistency emerges if we reverse the order stated above by calculating firstly the five-dimensional field equations, treating $\gamma_{55}$ as a function of the space-time coordinates, and then imposing $\gamma_{55} = \text{constant}$. Indeed, after having defined the electromagnetic potentials as follows, namely

$$\kappa A_{\mu} = \frac{\gamma_{5\mu}}{\gamma_{55}}$$

where $\kappa = \frac{8\pi G}{c^4}$, the 55-component of the five-dimensional Einstein equations reads

$$\Box \gamma_{55} = \frac{\kappa^2}{4} (\sqrt{\gamma_{55}})^3 F_{\alpha \beta} F^{\alpha \beta},$$

where the four-dimensional operator $\Box$, when acting on the scalar function $\gamma_{55}(x)$ is defined by $\Box \gamma_{55} = g^{\mu \nu} \nabla_\mu \partial_\nu \gamma_{55}$ for a curved four-dimensional space-time, and $\nabla_\mu$, $g^{\mu \nu}$ and $F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ represent the covariant derivative, the four-dimensional metric and the Maxwell’s antisymmetric tensor respectively. Hence, from the five-dimensional point of view, equation (2) implies that the choice $\gamma_{55} = \text{constant}$ would be compatible only with the too restrictive condition $F_{\alpha \beta} F^{\alpha \beta} = 0$, which means that the moduli of the electric and the magnetic field should be proportional to each other.

Soon after Klein’s work, in February 1927, Einstein published his first paper on the five-dimensional Universe. Abraham Pais pointed out how Einstein’s attitude toward the five-dimensional approach changed between the beginning and the end of August 1926. Then, referring to Einstein’s 1927 papers, he emphasized: ‘I should explain why these papers are a mystery to me. [...] What does puzzle me is a note added to the second paper [...] I fail to understand why he published his two notes in the first place.’ (Pais 2005; p. 333). Einstein’s papers are two brief communications [Einstein 1927a] and [Einstein 1927b]. He used the same hypotheses and obtained the same equations. Furthermore, he declared also, in an added note to the second communication, that he did not obtained new results. This is the note to which Pais is referring. It is worth noting that Klein also introduced a five-dimensional wave function, in order to describe the undulatory nature of the emerging quantum phenomena. Unlike Klein, Einstein considered only the gravitational and the electromagnetic forces. Hence, many unanswered questions emerge. Why did Einstein publish these two communications? The question seems to be yet unanswered, also in recent works [Halpern 2007]. Was Einstein aware of the inconsistency of the model? Did Einstein know Klein’s work at that time?
Even if he claimed that his results were essentially the same as obtained by Klein, is there any difference between Einstein’s and Klein’s approach? In October, 1927, the fifth Solvay conference took place. This conference is known to historians because the famous Einstein-Bohr debate started. Einstein’s communications, where any reference to the Schrödinger apparatus is absent, were published in March. Did Einstein only want to unify electromagnetic and gravitational forces, or did he want to incorporate microscopic phenomena as well?

The aim of our paper is to address these questions. In the following section we start with a closer inspection of Einstein’s communications. It shall reveal that Einstein’s approach was different with respect to Klein’s. It will also innovatively engage with Einstein’s communications. Indeed, we argue that he wanted to build up what we call, in modern language, a Gauge Theory. The final note of Klein’s work suggested that he started from Theodore Kaluza’s original theory [Kaluza 1921]. Indeed, Einstein discussed many features of the original Kaluza’s model with the author. Hence, in the third section we shall reconsider Einstein-Kaluza correspondence at the time when Kaluza asked Einstein to communicate his paper to the Prussian Academy. We conclude that Einstein knew, in a sense to be specified, equation (2). We shall try to address another question: why Einstein decided to set \( \gamma_{55} = \text{constant} \) in the action before calculating the field equation. We shall discuss again the inconsistency of the model stated above.

Finally, in the fourth section, we shall address the question about the description of microscopic phenomena. Using the conclusions of our preceding section, we argue that Einstein tried to introduce Hermann Weyl’s ideas in order to avoid the introduction of Schrödinger’s wave function. Finally, the section “Conclusions” is a brief summary of our paper, and in the Epilogue we briefly reconsider how the discussion on the inconsistency of the model emerged again in the Seventies. We argue that Einstein declared that he did not conceived new results, because he did not have, at that time, the mathematical apparatus he needed to proceed.

2 Einstein’s communications

2.1 The first communication

In the introduction of his first communication, Einstein considered the two main attempts to unify gravitational and electromagnetic forces. Indeed, he referred to Weyl’s and Arthur Eddington’s attempt on one side, and he quoted Kaluza’s paper [Kaluza 1921] on the other side. The aim of the first two authors, in Einstein’s words, was “to bring together Gravitation and Electricity into a unifying framework [...] through a generalization of the Riemannian geometry” ([Einstein 1927a]; p. 23). Kaluza’s attempt, instead, “maintained Riemannian metric, but introduced a five-dimensional space-time, which could be reduced to some extent to a four-dimensional space-time through the »cylinder condition«.” ([Einstein 1927a]; p. 23). The objective of Einstein’s first communication was “to draw attention to a disregarded point of view, which is essential for the Kaluza’s theory.” [our emphasis] ([Einstein 1927a]; p. 23).

The extremely important ingredient of Kaluza’s theory is the cylinder condition he cited above. Einstein reformulated the cylinder condition in terms of Killing vectors. After having introduced the five-dimensional space-time, Einstein associated the cylindrical shape of the space-time manifold with the existence of an infinitesimal unit-vector \( \xi^\mu \) that preserves the metric. Indeed, Einstein considered two points separated by an infinitesimal displacement \( dx^\mu \): when the vector is parallel transported from one point to another, it must satisfy a particular equation, in order to be the infinitesimal generator of an isometry of space-time. This equation is today known as the Killing equation, because it was first introduced by Wilhelm K. J. Killing ([Killing 1892]; p. 167). Einstein wrote explicitly the same equation ([Einstein 1927a]; p. 23, eq.2), namely:

\[
\xi^\beta \partial_\beta \gamma_{\mu\nu} + \gamma_{\beta\nu} \partial_\mu \xi^\beta + \gamma_{\beta\mu} \partial_\nu \xi^\beta = 0,
\]

which means that the five-dimensional manifold admits a preferred direction or, said another way, it has a cylindrical shape. Using the freedom to choose the coordinate-system, Einstein pointed out that the cylinder condition assumes the form stated by Kaluza, i.e. \( \partial_5 \gamma_{\mu\nu} = 0 \), if the invariant direction points to the fifth coordinate.\(^7\) It is worth noting that Einstein underlined, in a footnote, the role of equation (3), which states explicitly that Kaluza’s cylinder condition can be recast in a manifestly covariant form ([Einstein 1927a]; p. 23). Hence, we can infer that Einstein considered it essential for the theory to express the cylinder condition in a manifestly covariant form. The cylinder condition was presented for the first time in a physics paper by Einstein with a modern language.

After this close inspection of the beginning of Einstein’s paper, two questions arise: why did Einstein consider it fundamental to understand if the cylinder condition can be recast in a manifestly covariant form? How long did Einstein struggle to find it? In order to answer these questions, we shall reconsider, in section 3 the Kaluza-Einstein

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\(^7\)Einstein explicitly chose \( \xi^5 \) as the only non-vanishing component. Hence, the Killing vector assumes the following form: \( \xi^\beta = (0,0,0,0,1) \) and, inserting it into eq. (3), Einstein obtained Kaluza’s condition.
correspondence at the time when both authors started to discuss it. We shall follow also the development of the discussion and new questions that will arise. We argue that Einstein struggled with the cylinder condition from the very beginning and that one of the motivations of publishing his communications was because he found its manifestly covariant form. As we shall see in the next section, this question also played an important role in the history of Kaluza’s paper.

After having considered the cylinder condition, Einstein was concerned with the role of $\gamma_{55}$ itself. As we said in section [1] Klein decided to set $\gamma_{55} = \text{constant}$ after having presented the invariance group of transformations implied by the cylinder condition. Unlike Klein, Einstein gave this condition first a geometrical meaning. Indeed, even if the non-zero component of the Killing vector is constant, its modulus, on a curved manifold, is, in general, not constant. Hence, Einstein noted that setting the modulus of the Killing vector to be constant over the whole five-dimensional space-time implied the constancy of $\gamma_{55}$. Indeed, squared modulus of the Killing vector $\xi^2$ can be rewritten as follows: $\xi^2 = \gamma_{\mu\nu}\xi^\mu\xi^\nu = \gamma_{55}\xi^5\xi^5 = \gamma_{55}$, then $\xi^2 = \text{constant}$ implied $\gamma_{55} = \text{constant}$. Einstein called this further condition strengthen cylinder condition. As Einstein promised, he tried to analyse Kaluza’s theory from a different perspective with respect to the other authors. Indeed, in his approach geometry came first and the coordinate transformation induced by the structure of the five-dimensional space-time are a consequence. Having introduced the idea of a constant $\gamma_{55}$, Einstein observed that at the beginning either $\gamma_{55} = 1$ or $\gamma_{55} = -1$ are equally acceptable. He considered the first option, promising to discuss the other option on another occasion. As we shall see, he will come back to this point in the second communication. Indeed, at this stage, there is no reason for choosing a space-like or a time-like extra-dimension. Before proceeding we note that at that time Einstein did not yet understand the meaning of considering a space-time which admits a Killing vector with constant modulus. The strengthen cylinder condition is equivalent to the request that the Killing trajectories would be geodesic lines. As far as we know, Heinrich Mandel pointed out this fact two years later and used for the first time the expression “Killing equations” [Mandel 1929], while Einstein introduced this idea after the end of the Second World War.

Einstein’s aim was to analyse Kaluza’s theory, discussing the effect of every single hypothesis introduced by Kaluza himself. After having analyzed the geometric meaning of the strengthen cylinder condition, he inserted the two hypotheses into the five-dimensional line element and showed how gravitational as well as electromagnetic potentials emerge from the five-dimensional metric tensor, pointing out how Kaluza’s theory should unify both forces in a natural way. Subsequently, Einstein considered only the cylinder condition in the ‘adapted coordinate system’ [Einstein 1927a; p. 24], where the invariant direction has been set along to the fifth coordinate. Like Klein, using modern language, Einstein underlined that the cylinder condition implied that Kaluza’s theory invariance group can be written as a product of the four-dimensional diffeomorphism group and a one-dimensional group. In order to investigate the role of this residual symmetry, Einstein relaxed explicitly the strengthen cylinder condition and wrote the two coordinate transformations. Einstein called the one-dimensional group that lives invariant the four-dimensional metric ‘the $x^5$-transformations’ [Einstein 1927a; p. 24-25].

Then, he pointed out the invariance of the four-dimensional metric tensor under the action of the $x^5$-transformations, when the strengthen cylinder condition is introduced, and wrote the transformation law for the electromagnetic potentials. As we shall argue in the following section, Einstein received Klein’s paper, where the author had introduced the same coordinates transformations, before he would send his communications to the Prussian Academy. What emerges from our analysis is the different emphasis that Einstein gave them and the distinct sources they had. Klein condensed in just a few lines his arguments, while Einstein devoted three pages.

Having introduced the geometrical framework, what he called ‘an hyper-cylinder $x^5 = \text{constant}$’ [Einstein 1927a; p. 25], Einstein made explicit his strategy for constructing the physical five-dimensional theory. Einstein’s purpose was to construct a Lagrangian density in order to obtain the usual gravitational and electromagnetic field equations. Kaluza and Klein proposed the five-dimensional curvature scalar using the argument of the analogy with GR and then they showed how it reduces to the sum of the Hilbert-Einstein and Maxwell Lagrangian densities. Einstein instead proposed a specific argument. His approach was the same procedure we apply today to construct what we call a gauge theory. He argued that the Lagrangian density should be constructed using combinations that remain invariant with respect to a certain group of transformations. In this context, as we shall argue in the following pages, from Einstein’s point of view the group represented by the $x^5$-transformations played the role of a gauge group, while Klein did not make any similar statement. Hence, new questions arise. Was this the first time Einstein used this approach? Why did Einstein introduce this gauge principle? We shall return to these questions in section [3]. Now we continue to follow Einstein’s arguments. In the few lines concluding the first communication, Einstein did not tackle the whole problem. Indeed, he presented only an argument concerning electromagnetic phenomena. In order to understand it, we must go deeper into mathematical details. Using Einstein’s notation [Einstein 1927a; p. 25], the coordinates

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8It is worth remembering that Klein’s work is quoted in the post-scriptum at the end of the second communication. But as we said, Einstein knew Klein’s work, hence we shall continue to compare Einstein’s work also with Klein’s approach.
transformation read:

\[ x^\mu = \frac{x^\mu}{\gamma_{\mu\nu}} \]
\[ x^5 = \frac{x^5 + \psi(x^0, x^1, x^2, x^3)}{\gamma_{5\nu}} \]

where a bar over a quantity, in this section, will indicate the same object but in the transformed coordinate system and \( \psi \) is an arbitrary function of the four-dimensional coordinates. Equations (4) represent the four-dimensional diffeomorphisms and eq. (5) represents Einstein’s \( x^5 \)-transformations. The components of the five-dimensional metric tensor transform as follow:

\[ \gamma_{\mu\nu} = \gamma_{\mu\nu} + \frac{\partial \psi}{\partial x^\mu} \gamma_{5\nu} + \frac{\partial \psi}{\partial x^\mu} \frac{\partial \psi}{\partial x^5} \gamma_{5\nu} \]  
\[ \gamma_{5\nu} = \frac{\gamma_{5\nu}}{\gamma_{55}} \]
\[ \gamma_{5\nu} = \frac{\gamma_{5\nu} - \gamma_{5\mu} \gamma_{\nu\mu}}{\gamma_{55}} \]  

It is worth noting that equations (6), (7) and (8) did not appear explicitly in Klein’s paper and that in writing them Einstein did not use the strengthen cylinder conditions. Like Klein, Einstein defined \( A_\mu \) as in eq. (11), which we rewrite here for convenience \( \kappa A_\mu = \frac{\gamma_{5\mu}}{\gamma_{55}} \), but, in the concluding part of the first communication, he set explicitly \( \gamma_{55} = 1 \) and implicitly \( \kappa = 1 \). Hence, using both cylinder conditions, the electromagnetic potentials and the four-dimensional metric tensor read:

\[ A_\mu = \gamma_{5\mu} \]
\[ g_{\mu\nu} = \gamma_{\mu\nu} - \gamma_{5\mu} \gamma_{5\nu} \]

Einstein pointed out that the Lagrangian density must be constructed using the metric tensor, the electromagnetic potentials and their derivatives, because these quantities are all invariant with respect to the \( x^5 \)-transformations. This is exactly the approach of the modern gauge theory. Then, Einstein concluded his first communication by applying his gauge principle only to the electromagnetic phenomena. He underlined the importance of equation (7). Indeed, it implied that the only invariant combination of the electromagnetic potentials, under \( x^5 \)-transformations, is the usual electromagnetic tensor, namely \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), the mathematical object which represented physical variable from Einstein’s point of view. This result remains valid also without imposing the strengthen cylinder condition, as he will emphasize in the second communication. This fact, which is connected with the Ansatz that the space-time manifold has a symmetry group \( G \), is well known nowadays. These features have been implemented in modern Kaluza-Klein theories, where the massless states include Yang-Mills gauge fields with gauge group \( G \). Klein also emphasized the analogy between the transformation law induced by the \( x^5 \)-transformations and the usual gauge freedom of the electromagnetic potentials in Maxwell’s theory. But in his second communication, Einstein extends his gauge approach, in order to incorporate microscopic phenomena.

Are we allowed to define Einstein’s procedure as a gauge approach, in a modern sense? It is worth remembering that nowadays the isometry group is regarded like an external symmetry group, while a gauge group is usually called an internal symmetry group. As already said, this fact is more than an analogy and this is ‘the whole beauty of Kaluza-Klein theories’ ([Duff 1986]; p. 15), but did Einstein consider the \( x^5 \)-transformation as a gauge transformation? Like John Norton emphasized, after Einstein elaborated the so called “hole argument” and after his reply to Erich Kretschmann objection, he was aware of the “passive” as well of the “active” reading of the general covariance. Passive general covariance means that “if we have some system of fields, we can change our space-time coordinate system as we please and the new descriptions of the fields in the new coordinate system will still solve the theory’s equations.” ([Norton 2003]; p. 113). Active general covariance “licenses the generation of many solutions of the equations of the theory in the same coordinate system” ([Norton 2003]; p. 113) and the new fields are mathematically but not physically distinct fields, like the “hole argument” proves ([Norton 2003]; p. 114). Hence, in this sense, general covariance can be interpreted as a gauge freedom. In this paper, Einstein proposed the same interpretation for his \( x^5 \)-transformation, when he connected it with the transformation laws of the electromagnetic potentials. From our point of view, when Einstein emphasized the necessity of constructing invariant objects, he was proposing an “active” reading for the \( x^5 \)-transformations and the resulting one-dimensional group. As we shall see in the next section, he will extend his proposal also to the conformal group.

\[ ^{9} \text{The physical reality of the electromagnetic potentials would be recognized only after the Aharonov-Bohm effect} \quad \text{([Ehrenberg et al. 1949])} \quad \text{([Aharonov et al. 1959])} \]
Before proceeding with the second communication, Einstein pointed out that the field would change without imposing the strengthen cylinder condition. Indeed, a non-constant \( \gamma_{55} \) would imply in addition the presence of a scalar and of a symmetric tensor field\(^{10}\).

### 2.2 The second communication

The second communication opens with the arguments that lead to the Lagrangian density. In order to discuss every single step, Einstein started by relaxing the strengthen cylinder condition. He wrote the only invariant combinations emerging from equations (6), (7) and (8), namely (Einstein 1927b; p. 26):

\[
\frac{\gamma_{\mu\nu}}{\gamma_{55}} - \frac{\gamma_{\mu\nu}}{\gamma_{55}} \frac{\gamma_{\mu\nu}}{\gamma_{55}} ; \qquad \frac{\partial}{\partial x^\mu} \left( \frac{\gamma_{\mu\nu}}{\gamma_{55}} \right) - \frac{\partial}{\partial x^\mu} \left( \frac{\gamma_{\mu\nu}}{\gamma_{55}} \right) ; \quad \gamma_{55}
\]

(11)

where \( \gamma_{55} \) is a function of the four-dimensional coordinates. Hence, Einstein underlined again that only combinations (11) must be used, in order to construct the Lagrangin density.

As we shall see in section 3, Einstein was aware of the effects introduced by the scalar field \( \gamma_{55} \), which is an uncomfortable presence. Einstein introduced an argument in order to eliminate it from the scene. He used the following physical principle for justifying his choice. ‘Let us suppose that only the ratios of the components of the metric tensor \( \gamma_{\mu\nu} \) have objective meaning’ [emphasis added] (Einstein 1927b; p. 27). Einstein tried to specify his idea by presenting it from a more geometrical point of view. In GR all the components of the metric tensor are needed in order to describe the four-dimensional reality. Einstein supposed that this fact should not be true for the five-dimensional manifold: the objective meaning of the ratio is equivalent to ask that it is sufficient to determine the null-geodesics of the five-dimensional metric. Why then did Einstein choose this specific physical principle? In Klein’s paper, after having discussed the possibility of setting \( \gamma_{55} = \) constant, the author made a similar statement: ‘As it is obviously conjectured that only the ratios of \( \gamma_{\mu\nu} \) have physical meaning, this assumption is always possible as a convention.’ (Klein 1984; p. 13). Did Einstein simply replicate Klein’s argument? Why did both make this assumption? In section 4.4, we argue that Einstein was the father of this statement. It is worth noting now that Klein underlined that assigning physical reality only to the ratios is equivalent to assume the constancy of \( \gamma_{55} \). He did not use this assumption again in his paper. Unlike Klein, Einstein used this principle in order to assert that only the first two quantities in eq. (11) must be enter the construction of the five-dimensional Lagrangian density. Hence, Einstein concluded, the condition \( \gamma_{55} = \) constant, i.e. the strengthen cylinder condition, is not in contrast with any fundamental theoretical principle, and he finally set \( \gamma_{55} = 1 \). Assuming that only the five-dimensional null-geodesics have physical meaning is equivalent to assuming that a conformal transformation of the five-dimensional metric tensor does not have any influence on the four-dimensional Universe. Indeed, even if \( \gamma'_{\mu\nu} = \Omega(x)\gamma_{\mu\nu} \), where \( \Omega(x) \) is a scalar function, represents a different five-dimensional metric, the first two quantities of equation (11), that correspond to the four-dimensional metric and the electromagnetic potentials, remain the same. Hence, Einstein extended his gauge principle to the five-dimensional group, by demanding the invariance of the four-dimensional physics. Then, Einstein used this gauge freedom to set \( \gamma_{55} = 1 \). Indeed, it is sufficient to choose \( \Omega = (\gamma_{55})^{-1} \). From our point of view, Einstein’s choice can be interpreted as a sort of gauge fixing. As we shall see in section 4, Einstein declared his appreciation for this idea during his correspondence with Hermann Weyl. As a matter of fact, this was one of the ingredients of Weyl’s theory. In section 4 we shall reconsider the Einstein-Weyl correspondence and its developments in order to understand why Einstein needed this principle.

In the second paragraph of (Einstein 1927b), like Klein, Einstein showed that projecting the five-dimensional geodesics onto the hyperplane \( x^5 = \) constant yields the four-dimensional geodesic for charged particles in an electromagnetic and gravitational field. Einstein obtained his result identifying a particular scalar quantity with the ratio between the particle electric charge and its mass. Then, he observed also that this quantity is invariant under the action of the \( x^5 \)-transformations. In the third part of the communication, Einstein constructed his Lagrangian density, which was different from Klein’s, but it is equivalent up to a total derivative to the Ricci’s scalar, and that generalized Kaluza’s approach. Klein used the five-dimensional curvature scalar. Einstein introduced the following Lagrangian density, namely:

\[
\mathcal{L} = \sqrt{-\gamma} \left[ \gamma^\alpha^\beta \left[ \Gamma^\mu_{\alpha\beta} \gamma^\beta_{\alpha\bar{\beta}} - \gamma^\beta_{\alpha\bar{\beta}} \Gamma^\beta_{\mu\bar{\beta}} \right] \right],
\]

(12)

where the five-dimensional Christoffel symbols are defined, as usual in GR, by using the metric tensor \( \gamma_{\mu\nu} \) and its first derivatives and he specified that eq. (12) ‘is expressed using \( g_{\mu\nu} \) and \( A_\mu \)’ (Einstein 1927b; p. 28). This statement follows after that Einstein fixed \( \gamma_{55} = 1 \), hence he referred to equations (9) and (10).

\(^{10}\)Kaluza made a similar observation.
Then, Einstein showed that equation (12) reduces to the sum of the usual Einstein-Hilbert and Maxwell Lagrangian densities, namely
\[
\tilde{\mathcal{L}} = \sqrt{-g} \left[ g^{\mu\nu} \left( \Gamma^\gamma_{\mu\nu} \Gamma_{\gamma\alpha\beta} - \Gamma^\gamma_{\alpha\beta} \Gamma_{\gamma\mu\nu} \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right].
\] (13)

By identifying \( g_{\mu\nu} \) and \( A_\mu \) with the first two expressions of eq. (11), the Lagrangian density (13) is invariant under conformal transformations of the five-dimensional metric\(^{[11]}\) as Einstein wanted. But the original Lagrangian density in not invariant under conformal transformations, if impose only the cylinder condition. Indeed, equations (12) and (13) can coincide only if additional conditions are imposed.\(^{[12]}\)

In the last lines of the second communication, Einstein finally discussed the possibility of setting \( \gamma_{55} = -1 \). If this second choice was used, a wrong sign between the gravitational and the electromagnetic Lagrangian densities would appear in eq. (13). Hence, Einstein concluded that this fact forced the extra-dimension to be space-like, instead of time-like. For the first time in the history of the Unified Field Theories, this fact has been explicitly stated. Indeed, we are aware of the fact that a similar discussion would take place in the same year between Klein and Louis de Broglie, who tried to introduce a time-like extra-dimension, because he did not discuss the Lagrangian approach for the gravitational and the electromagnetic fields (\cite{Peruzzi2018}; p. 199-200). In the following sections of our paper, we shall address all the questions arisen. We start considering Kaluza-Einstein correspondence and then we shall discuss Einstein’s ideas on Weyl’s theory.

### 3 Einstein-Kaluza correspondence revisited

The modern multidimensional theories are often called \textit{Kaluza-Klein theories}. Indeed, before Klein, Theodore Kaluza introduced also a five-dimensional space-time \cite{Kaluza1921,Kaluza1984}, in order to unify GR and EM\(^{[1]}\). Kaluza’s theory has been largely analysed from the historical point of view, see e.g. \cite{Vizgin2010}, but a brief review is needed, in order to set the stage. Kaluza sent Einstein a draft of his paper in 1919, but Einstein communicated Kaluza’s work only 2 years later. Why did Einstein wait for so long time? The aim of this section is to answer this question, which is connected with the introduction of the cylinder condition. Kaluza introduced the fifth dimension interpreting the electromagnetic tensor \( F_{\alpha\beta} \) as a sort of truncated Christoffel symbol. Hence, using five-dimensional Christoffel symbols \( \Gamma_{\mu\nu}^{\gamma} \), Kaluza was able to introduce both gravitational and electromagnetic potentials. In order to take into account the non-observability of the fifth dimension, Kaluza introduced the cylinder condition, stating that: ‘one has to take into account the fact that we are only aware of the space-time variation of state-quantities, by making the derivatives with respect to the new parameter vanish or by considering them to be small as they are of higher order’ (\cite{Kaluza1984}; p. 8). Kaluza did not, therefore, write explicitly the cylinder condition in its non-covariant form, namely \( \partial_\mu \gamma_{5\mu} = 0 \). It is worth noting that a few lines after its introduction, Kaluza stated that he extended Einstein’s hypothesis in five dimensions. Hence, it can be inferred that Kaluza’s theory should be invariant under the group of the full five-dimensional diffeomorphism. Maybe because he had introduced the cylinder condition so vaguely, he never discussed the implications on the symmetry group of his theory.

Unlike Klein, Kaluza treated \( \gamma_{55} \) as variable, even if he considered the linearised five-dimensional Einstein equations only, i.e. in the weak-field limit. Kaluza called this hypothesis \textit{approximation I}. In Kaluza’s paper there is no explicit discussion of the full field equations. The author considered the fifth dimension as a physical ingredient of his theory, because he discussed the consequences implied by the introduction of a non-constant \( \gamma_{55} \) for our four-dimensional world. Kaluza was concerned with the particle’s geodesic motion in five dimensions, which should be connected, in Kaluza’s theory, with the motion of charged particles in four dimensions in the presence of gravitational and electromagnetic interactions. It is worth noting that in the concluding paragraph of his paper, Kaluza maintained a neutral position with respect to the physical meaning of the new “world parameter”, i.e. the fifth dimension. As Kaluza said, he encountered ‘physical as well epistemological difficulties’ (\cite{Kaluza1984}; p. 8) in giving a physical meaning to the new formalism. Kaluza’s difficulties are described in the following.

Kaluza investigated the five-dimensional geodesic motion in the small-velocities limit and called it \textit{approximation II}. Thanks to this approximation, an extra term coming from the non-constancy of \( \gamma_{55} \) disappeared from the geodesic equation. Hence, the five-dimensional geodesics corresponded to the Lorentz equation for charged particles in the presence of gravitational and electromagnetic interaction. The small-velocities approximation implied also that the

\(^{[11]}\) The inverse components of \( g_{\mu\nu} \) read \( g_{\mu\nu} = \gamma_{55} g^{\mu\nu} \), which are also conformally invariant quantities as well as the determinant of the four-dimensional metric

\(^{[12]}\) The strengthen cylinder condition is one possibility, another one is to consider a five-dimensional conformally flat space \cite{Grumiller2006}.

\(^{[13]}\) Gunnar Nordstr"{o}m also considered a five-dimensional space-time before Kaluza \cite{Nordstrom1914}, but the Norwegian mathematician described the gravitational interaction using a scalar field instead of a tensor field.
charged matter should have ‘a very tiny specific charge \(\rho_0/\mu_0\)’ \(\text{[Kaluza 1984; p. 5]}\), where \(\rho_0\) and \(\mu_0\) are the electric charge and the particle’s mass and their ratio is what Kaluza called the specific charge. Exploring the possibility of applying his model to microscopic phenomena, Kaluza emphasized that Einstein has noted an inconsistency of his model in this context \(\text{[Kaluza 1984; p. 7, cf. footnote 6]}\). Indeed, approximation II cannot be satisfied by electrons.

Kaluza’s comment is important for two reasons. First, it emphasizes that the fifth dimension was connected with measurable quantities, even if it should not be observable by itself. Kaluza was afraid that the new quantum theory, concerning the microscopic phenomena, should be a menace for the validity of his model. Second, it points out how Einstein was concerned with the physical meaning of the fifth dimension and with the physical consequences of Kaluza’s approach.

Was Einstein aware of the consequences of setting \(\gamma_{55} = \text{constant}\) in 1921, for instance when he communicated Kaluza’s paper to the Preussische Akademie? What was exactly Einstein’s role in pointing out the incompatibility of approximation II with electrons’ motion? In order to answer these questions, we reconsidered Einstein-Kaluza correspondence between 1919, when Einstein received Kaluza’s manuscript for the first time, and 1921, when Einstein convinced himself to communicate Kaluza’s results.

### 3.1 Kaluza-Einstein correspondence

Einstein first reaction to Kaluza’s introduction of the extra-dimension was enthusiastic in April 1919. Indeed, Einstein wrote: ‘your idea [Kaluza’s] has great appeal for me. It seems to me to have decidedly more promise from the physical point of view than the mathematically probing exploration by Weyl’ \(\text{[Einstein Papers vol. 9bis; p. 21]}\). In making his assertion, Einstein contrasted Kaluza’s theory with Hermann Weyl’s approach. Einstein defined Weyl’s theory like a ‘mathematically probing exploration’ for various reasons. For our purpose it is worth mentioning that Einstein pointed out that Weyl’s theory was in contrast with the empirical facts. Furthermore, Weyl considered the potentials \(A_{\mu}\) as a fundamental object, instead of as electric and magnetic fields, but in Einstein’s opinion they were physically meaningless. On the contrary, Kaluza’s idea is based on the electromagnetic tensor.

In his first letter and in the following correspondence, it emerges how Einstein analysed the weaknesses of Kaluza’s model. From our point of view, the most important comment is the following: ‘It now all depends on whether your idea will withstand on physical scrutiny.’ \(\text{[Einstein Papers vol. 9bis; p. 21]}\). This comment emphasizes that Einstein was concerned with the measurable consequences of the introduction of the fifth dimension, as we will see in the following.

In his second answer to Kaluza, Einstein underlined again the importance of empirical proofs in considering the projection of five-dimensional geodesics onto four-dimensional slices \(x^5 = \text{constant}\). ‘I myself definitely would not publish the idea – if it had occurred to me – before having done this test, which seems to be a secure and relatively simply criterion.’ \(\text{[Einstein Papers vol. 9bis; p. 26]}\). In our opinion, this is the reason why Kaluza performed his approximations: he wanted to compare his mathematical approach with the empirical reality. In his third letter, Einstein convinced himself, with the help of Kaluza, that ‘from the point of view of realistic experiments, your [Kaluza’s] theory has nothing to fear.’ \(\text{[Einstein Papers vol. 9bis; p. 32]}\).

Third Einstein’s letter shall play an important role in the following section of our paper. Indeed, Einstein pointed out two important arguments against Kaluza’s approach, that he would continue to grapple with in his future developments of Kaluza’s theory. First he noted the incompatibility between the request of the general covariance in five-dimensions and the cylinder condition. As we said, also in the published version of Kaluza’s paper, the author made no comments about this fact. Second, Einstein judged as very unsatisfactory the cylinder condition: ‘One requires: 1) General covariance in \(R_5\). 2) In combination with this the non covariant condition \(\partial_5 \gamma_{\mu\nu} = 0\). Obviously, this is very unsatisfactory’ \(\text{[Einstein Papers vol. 9bis; p. 32]}\). This comment answers one of our questions: Einstein was struggled from the very beginning by the fact that the cylinder condition was not formulated in a covariant form.

In his fourth answer, Einstein pointed out finally the inconsistency of Kaluza’s model, when applied to microscopic phenomena, i.e. the motion of an electron. ‘[…] upon more careful reflections about the consequences of your interpretation, I did hit upon another difficulty, which I have not been able to resolve until now.’ \(\text{[Einstein Papers vol. 9bis; p. 36]}\). In order to understand the limits of applicability of Kaluza’s model, Einstein considered the field equations with a non-constant \(\gamma_{55}\). Indeed, Einstein claimed that he had calculated the full field equations in the first approximation, but he reported the \(55\)–component only. He used this equation for estimating the order of magnitude of \(\gamma_{55}\), and used it to estimate the magnitude of the fifth component of the particle’s velocity in the case of an electron. Indeed, the two quantities appear in the fifth component of the geodesic equation. The huge order of magnitude he

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14 Using the modern values of electron’s charge and mass, the order of the ratio is \(10^{11}\). We are using the Lorentz-Heaviside units.

15 We will investigate Weyl’s role in section 4.

16 English translation has been modified.
obtained was incompatible with the small-velocities approximation proposed by Kaluza for eliminating the physical effects of the fifth dimension.

Einstein’s calculation assumes a new and important role for our purpose. Let us take a closer look to Einstein’s letter. He considered a weak field limit, what he called ‘a first approximation’. Hence, in order to calculate the field equations, Einstein set, for the determinant of the metric, \( \gamma = 1 \) and used the following Lagrangian density

\[ \mathcal{L} = \gamma^\rho_\sigma \gamma^\beta_\mu \tilde{F}^\rho_\sigma \tilde{F}^\beta_\mu. \]

The Lagrangian density \((14)\) is equivalent, up to a total derivative and in the weak-field limit, to the five-dimensional curvature scalar \( \hat{R} \). Following Kaluza’s Ansatz and using \((14)\), the 55–component of Einstein equations reads:

\[ \frac{1}{2} \Box \gamma_{55} = \frac{\kappa^2}{4} F_{\alpha \beta} \hat{F}^{\alpha \beta}. \]  

\((15)\)  

\([\text{Einstein Papers vol. 9bis}]; \text{p. 36}\), where, unlike section \(2\) Einstein defined Kaluza’s electromagnetic tensor \( F_{\alpha \beta} \) using Kaluza’s electromagnetic potentials \( A_{\mu} = \gamma_{5 \mu} \), which are related with the usual electromagnetic potential \( A_{\mu} \) as follows: \( A_{\mu} = \frac{\hat{A}_{\mu}}{\gamma_{55}} \) (cf. with definition \([1]\)). By starting with r.h.s of equation \((2)\), inserting \( A_{\mu} = \frac{\hat{A}_{\mu}}{\gamma_{55}} \) in the definition of \( F_{\alpha \beta} \) and neglecting the non-linear terms, equation \(2\) reduces to \((15)\). From eq. \((15)\) Einstein would have been able to infer that \( \gamma_{55} = \text{constant} \) implied \( \tilde{F}_{\alpha \beta} \hat{F}^{\alpha \beta} = 0 \). Indeed, in 1919, he did not suggest this short cut to eliminate the term proportional to \( \partial_{\mu} \gamma_{55} \) from the geodesic equation, which describes the effect of \( \gamma_{55} \) on four-dimensional particle’s dynamic. As we said above, Kaluza eliminated this extra term using a different trick, at the price to accept the inapplicability of his model to the motion of electrons.

Before proceeding, it is worth remembering that in the communications published by Einstein in 1927 he decided to set \( \gamma_{55} = \text{constant} \). Hence, a new question arises: why did Einstein change his mind? As we said in section \(2\) Einstein choice was connected with Weyl’s ideas, hence, to be addressed in section \(4\).

In his subsequent answer to Kaluza (\[\text{Einstein Papers vol. 9bis}; \text{pp. 41-42}\], dated May 1919, Einstein emphasized again the importance of his objections, underlining that it would prevent him communicating Kaluza’s paper to the Preussische Akademie. Notwithstanding this fact, Einstein considered Kaluza’s model of mathematical interesting. He suggested that Kaluza submit the paper to a journal of mathematics, i.e. the (\textit{Matematische Zeitschrift}). From the same letter we know that Einstein sent Kaluza his paper, \[\text{Einstein 1919}\] (see \[\text{Einstein Papers vol. 7bis}\]; p. 80–88 for an english translation.), where he reflected on one of the problems with which Einstein was concerned in that period: the applicability of GR to microscopic phenomena and the problem of the cosmological constant.

From our point of view, all these considerations show that the need to find a covariant formulation for the cylinder condition and the physical meaning of the fifth dimension played a fundamental role, for Einstein, in considering Kaluza’s theory as physically acceptable. Kaluza’s draft of the paper remained unpublished for the following two years. In October 1921 Einstein decided to give a second chance to Kaluza’s theory. Why did Einstein write again to Kaluza? Once again, the answer is connected with Weyl’s idea, but we address this question now. In his letter Einstein declared that after having reconsidered Kaluza’s theory he decided that it had ‘more in its favour than the one of H. Weyl’ (\[\text{Einstein Papers vol. 12bis}; \text{p. 178}\]). In that period, Einstein was working with Jakob Grommer on a possible extension of GR. While considering multidimensional spaces, in August 1921 Grommer communicated to Einstein that he tried to consider the special case of five dimensions (\[\text{Einstein Papers vol. 12bis}; \text{p. 146}\). As we know from a footnote of the following Grommer’s letter, in October Einstein and Grommer continued to discuss the five-dimensional case\([17]\) while Einstein was sending his letter to Kaluza (\[\text{Einstein Papers vol. 12bis}; \text{p. 333, footnote 3}\]). Now let us follow again Kaluza-Einstein correspondence. From the letter sent by Kaluza in response to Einstein’s offer for a second chance, we know that during these years Kaluza tried to fix the problems that Einstein pointed out (\[\text{Einstein Papers vol. 12bis}; \text{p. 178}\). Even if he did not fix the problem of the magnitude of \( \gamma_{55} \), Kaluza declared ‘it does not appear to be quite insurmountably imposing as then’ (\[\text{Einstein Papers vol. 12bis}; \text{p. 178}\). In a subsequent letter, Kaluza admitted he was not able to eliminate the discrepancy and proposed to Einstein the solution which appeared in the published paper. Kaluza also pointed out different proposals which occurred in his mind. It is worth mentioning that he also tried to introduce a \textit{time-like} extra-dimension (\[\text{Einstein Papers vol. 12bis}; \text{p. 191}\), but he did not refer to the Lagrangian density, as Einstein would do in 1927.

Finally, Einstein communicated Kaluza’s paper and it was published. But the paper he published with Grommer in 1923 contains another important contribution to the question of the cylinder condition.

\[17\] Einstein and Grommer work would be published only at the beginning of 1923.
3.2 The importance of being “covariant”

One year after the publication of Kaluza’s paper, in January 1922, Einstein and Grommer sent a communication to *Scripta Universitatis atque Bibliothecae Hierosolymitanarum. Mathematica et Physica*, where they pointed out, already in the title, the non-existence of every-conventional centrally symmetric field according to the theory of Kaluza. What Einstein and Grommer meant was underlined by the authors in the final assertion of the paper: ‘Thus it is proven that Kaluza’s theory possesses no centrally symmetric solutions dependent on the $\gamma_{\mu\nu}$’s alone that could be interpreted as a (singularity free) electron.’ (Einstein Papers vol. 13bis; p. 33). Already in 1909 Einstein had considered the possibility of reducing particles to solutions of field equations in electromagnetic theory (Einstein Papers vol. 13bis, 32, footnote 9). Furthermore, as already said in section 3.1, Kaluza’s theory was not able to describe electrons. Einstein never stopped to consider the possibility of applying GR to the microscopic phenomena. Indeed he published two papers trying to address this question in 1919 and 1921, i.e. Einstein 1919 and Einstein 1921. Einstein would go on to try to merge Kaluza’s and Weyl’s theory.

The paper of Einstein and Grommer is important also for another reason. Reviewing Kaluza’s theory, the authors discussed the advantages of Kaluza’s theory, briefly contrasting it with Weyl’s approach, and pointed out its weakness. The advantages are summed up by the fact that Kaluza’s theory offered a more natural way of unifying GR with Maxwell’s theory from Einstein’s point of view, because the two theories emerge from a unique five-dimensional Lagrangian density. The weaknesses are based on his critics about the non-covariant form of the cylinder condition, emerged in his correspondence with Kaluza, as the authors describe in the following. ‘In the general theory of relativity [...] the $dx^2 = \gamma_{\mu\nu}dx^\mu dx^\nu$ means a directly measurable magnitude for a local inertial system using measured rods and clocks, whereas the $d\bar{r}^2$ of the five-dimensional manifold in Kaluza’s extension initially stands for a pure abstraction that seems not to deserve direct metrical significance.’ [emphasis added] (Einstein Papers vol. 13bis, p. 31). Einstein and Grommer referred to the fact that the physical effects of the fifth dimension must be eliminated in order to obtain the usual four-dimensional particle’s dynamic. The emphasis shows how Einstein started to lose his faith on the physical reality of the five-dimensional space-time and we can infer that he was wondering also what should have physical meaning in five dimensions. As we pointed out in section 2 he would use a physical principle belonging to Weyl’s approach. In addition, as they had underlined few lines before, the $\gamma_{55}$ function still awaited interpretation. It is worth mentioning that in Einstein and Grommer’s analysis $\gamma_{55}$ played the role of a variable. The authors continued: ‘Therefore, from the physical point of view, the requirement of general covariance of all equations in the five-dimensional continuum appears completely unfounded.’ (Einstein Papers vol. 13bis, p. 31). As we underlined, Kaluza did not discuss the consequence that had the cylinder condition on the group of the coordinates transformations. Einstein and Grommer concluded: ‘Moreover, it is a questionable asymmetry that the requirement of the cylinder property distinguish one dimension above the others and yet with reference to the structure of the equations all five dimensions should be equivalent.’ (Einstein Papers vol. 13bis, p. 31). This last comment reflects once again the fact that the cylinder condition was not formulated in a covariant form.

In the following years Einstein tried to develop different approaches in order to unify the gravitational and the electromagnetic interactions. His new starting point was to consider a generalized affine connection which would contain the Christoffel symbols as a particular case, an idea initiated by Weyl and Eddington, as it is widely explained in Goenner 2004. These attempts failed for various reasons. Like Vizgin writes: ‘he was clearly very disappointed in the efforts so far made to create unified geometrized field theories.’ (Vizgin 2010; p. 219) Hence, ‘having lost faith in the affine and affine-metric theories related to the theories of Weyl and Eddington, Einstein returned to Kaluza’s five-dimensional scheme.’ (Vizgin 2010; p. 220). In addition, Vizgin observes that ‘it was an entirely natural step for Einstein to turn to the five-dimensional approach. Of course, this may have been due to the revival of five-dimensional field theory in connection with quantum mechanics.’ (Vizgin 2010; p. 221). Goenner reported that ‘Einstein became interested in Kaluza’s theory again due to O. Klein’s paper’, i.e. Klein 1926a. (Goenner 2004; p. 64) and that then Einstein wrote to his friend and colleague Paul Ehrenfest. A closer inspection on Einstein correspondence conserved at the Albert Einstein Archives shows that it was Ehrenfest himself who drew Einstein’s attention on Klein’s paper. Indeed, we know that Ehrenfest asked Grommer to send Klein’s paper to Einstein and that he received it in August 23, 1926, when he wrote an acknowledgement letter to Ehrenfest Einstein to Ehrenfest, 23 August 1926 (AEA, 10 141). Indeed, Klein visited Ehrenfest in Leiden soon after the cited letter, in order to give a talk on his ideas and Ehrenfest worried about the absence of Einstein, as he wrote in August 26 Ehrenfest to Einstein, 26 August 1926 (AEA, 10 143). Two days later, among various critics, Einstein also declared that he distrusted the n-dimensional generalizations of wave mechanics Einstein to Ehrenfest, 28 August 1926 (AEA, 10 145). During this period, Einstein corresponded also with Klein who outlined the future developments he tried to achieve with his five-dimensional approach, i.e. to connect the periodicity of the fifth coordinate with quantum phenomena Klein to Einstein, 29 August 1926 (AEA, 14 279).

It is worth noting that Klein’s letter was sent in August 29, while Klein sent his note to Nature five days later, September 3. Vizgin writes ‘From the middle of 1926, there had appeared in the *Zeitschrift für Physik* alone not less
than ten papers on the application of the five-dimensional approach to quantum mechanics. It appears that Einstein did not pay attention to these papers, since otherwise he would not have failed to mention the new aspect of the five-dimensional approach discovered by O. Klein and Fock’ (Vizgin 2010, p. 221). Vizgin refers to the note added in proof [Einstein 1927b], that we commented in section 2. The contents of the letters tell us that Einstein was aware both of the five-dimensional generalization of wave mechanics and the work of Klein, but that he did not believe that that research topic was the right way to construct a unifying theory.

After having received Klein’s letter and having read Klein’s paper on five-dimensional world, Einstein realized that not much had changed about the state of the cylinder condition since he stopped working on it. Corresponding with Ehrenfest, Einstein emphasized that ‘Klein’s paper is good, but that Kaluza’s theory is too unnatural’ (Pais 2005, p. 350). A closer inspection of the postcard Einstein to Ehrenfest, 3 September 1926 (AEA, 10 147) showed that Einstein pointed out once again his disappointment regarding the contrast between the request of general covariance and the cylinder condition, defining it as unlikely. This comment is dated September 3. Hence, we can infer that at that time he started again to investigate how to write the cylinder condition in a covariant form. Many events took place between the beginning of September and January 20, 1927, i.e. when the first communication we commented in section 2 was presented. In October the fifth Solvay Conference took place, where the Einstein-Bohr debate started and where the statistical interpretation of the wave function was discussed. Furthermore, Einstein was working with Grommer on another topic. Einstein realized that the cylinder condition can be formulated in a covariant form. This fact and the idea of merging Kaluza’s approach with that of Weyl, an aspect that we shall consider in the next section, convinced Einstein to write the following statement to Lorentz on February 16th, four days before the Prussian Academy would receive the second communication [Einstein 1927b]: ‘It appears that the union of gravitation and Maxwell’s theory is achieved in a completely satisfactory way by the five-dimensional theory (Kaluza-Klein-Fock).’ (Goenner 2004, 65).

Before ending this section, we shall briefly address the following question: why did Einstein continue to struggle with the problem of the non-covariant form of the cylinder condition? In the process of creating GR, Einstein achieved the covariant form of the gravitational field equations with hundreds of stop and go. He had also formulated an argument ad hoc, the already mentioned “hole argument” before 1915, when he was convinced that such form cannot exist. As Norton emphasized, from Einstein’s point of view the principle of general covariance was ‘a principle with significant physical content, and [...] that content is the character of a generalized relativity principle.’ (Norton 1992, p. 283). Indeed, in October 1916, Einstein wrote that the principle of equivalence is always satisfied if equations are covariant (Einstein Papers vol. 6bis; p. 239). Once achieved, the principle of general covariance was an essential character of physical laws. Indeed, in his founding paper of GR, Einstein remarked that ‘the laws of Nature have to be expressed by generally covariant equations’ (Einstein Papers vol. 6bis; p. 153). Furthermore, Norton asserted: ‘Einstein also predicated the covariance property not directly to the model set but to the equations that define the model set [...] if the equations defining a model set are covariant under a group $G$ then the model set must also be covariant under that group and vice versa.’ (Norton 1992, p. 292). Hence, we argued that general covariance was an essential character both for all of the equations defining GR and for searching for new theories. And we finally understand why the original formulation of the cylinder condition bothered Einstein until he solved this puzzle in 1927.

4 Einstein, Weyl and the scale invariance

Having investigated the role played by Kaluza’s theory on Einstein’s view, many questions are yet unanswered. Did Einstein want to unify electromagnetic and gravitational forces only or did he want to incorporate the microscopic phenomena also? Was 1927 the first time he used a “gauge” approach? Why did Einstein choose the scale invariance as a physical principle for generalizing GR? In order to answer these questions we shall reconsider Einstein-Weyl correspondence between 1918 and 1921.

4.1 Weyl’s theory

A brief review of Weyl’s theory in a historical context can be found in O’Raifeartaigh 2000. Here we summarize the most important aspects. Weyl’s starting point was purely mathematical. He introduced an affine connection, which generalized the Levi-Civita connection. The geometrical idea is related to the concept of parallel transportation. In GR, the parallel transportation of a vector from a point to another could result in a rotation of the vector, but its modulus remains unchanged. Using Weyl’s connection, the magnitude of the vector’s modulus change. The advantage of this new connection is that the new connection’s symbols contain both the gravitational potential and the electromagnetic potentials. Weyl’s purpose was to construct what he called ‘a true infinitesimal geometry’ (O’Raifeartaigh 1997, p. 25). As a consequence of these assumptions, using modern language, the four-dimensional space-time manifold
of Weyl’s theory is equipped with a conformal structure, i.e. with a set of conformally equivalent Lorentz metrics and not with a definite metric as in GR. With the help of this new internal symmetry group, Weyl realized his idea of a truly infinitesimal geometry, giving the vectors the possibility of changing their magnitude. The idea of gauge-invariance emerged in those years, but the meaning attributed by the author was different from our modern concept (see [O’Raifeartaigh 1997] for further details). Before giving up his ideas, Weyl proposed many Lagrangian densities in order to achieve a unified theory of gravity and electromagnetism. All of them were conformally invariant objects: in this sense Weyl used an approach analogous to the modern idea of gauge-invariance. Before proceeding, it is worth noting that Weyl introduced this new symmetry, because he was convinced that ‘it is not very probable that the Einstein gravitational field equations are strictly correct, particularly, since the gravitational constant contained in them is quite out of place with respect to the other natural constants, so that the gravitational radius of the mass and charge of an electron, for example, is of a completely different order of magnitude (about $10^{10}$ resp. $10^{40}$ times smaller) than the radius of the electron itself’ ([O’Raifeartaigh 1997], p. 34).

The essential weakness of Weyl’s theory was pointed out by Einstein very early. The main consequence of the introduction of the conformal symmetry is that the behaviour of clocks would depend on their history, a fact that does not have any empirical justification. From an epistemological point of view, what disturbed Einstein was that the line element $ds$ lost the physical meaning which he had in GR: ‘I reported to you more exactly the objection bothering me with regard to your new theory. (Objective meaning for $ds$, not just for the ratios of different $ds$’s originating from one point.)’ ([Einstein Papers vol. 8bis]; p. 532). Indeed, as Weyl’s himself underlined, in his conformal invariant theory ‘only the ratios of the components of the metric tensor [...] have a direct physical meaning’ ([O’Raifeartaigh 1997], p. 26).

4.2 ‘A new idea occurred to me...’

One year after the correspondence discussed above, which took place at the end of Summer 1918, Einstein received the first draft of Kaluza’s paper (cfr. section 3). Hence the two stories start to intersect. After having delayed the publication of Kaluza’s paper because of the critics we pointed out in section 3, Einstein explored different directions. But around the end of 1920 his friend Michele Besso reawakened Einstein’s interest in Weyl’s theory. Indeed, in December 1920 Besso wrote Einstein wondering under which transformations Weyl’s theory is invariant ([Einstein Papers vol. 10bis]; p. 540). From this moment, Weyl’s conformal transformations started to make their way in Einstein’s mind a little at a time, as we shall see in the following. At the beginning of 1921 Einstein visited Prague and Vienna, where he was invited to give lectures on the theory of relativity. During his Vienna’s lecture, Einstein should have talked about conformal transformations, because after returning to Berlin he corresponded with Wilhelm Wirtinger on this topic. In February, the Austrian mathematician wrote: ‘I pursued your remark in Vienna further, whether it would not be possible to form tensors that depended solely on the ratios of the $g_{\mu\nu}$’s, and arrived at some quite satisfactory and interesting results.’ ([Einstein Papers vol. 12bis]; p. 44). In his first letter, Wirtinger analysed the effect on particle’s paths obtained by varying an action principle using a new line element, ‘which also depends only on the ratios of the $g_{\mu\nu}$’s.’ ([Einstein Papers vol. 12bis]; p. 45). From Einstein’s answer we know his purpose: ‘I am convinced that you have thus done relativity theory an inestimable service. For it is now a simple matter to construct a theory of relativity that assigns meaning to the ratios of the $g_{\mu\nu}$’s, or the equation $g_{\mu\nu}dx^\mu dx^\nu = 0$, without—as with Weyl,—in my conviction, the physically meaningless quantities $A_\mu$ (electromagnetic potentials) explicitly appearing in the equations. There only remains the problem of whether Nature really has made use of this possibility available to her to constrain herself accordingly. As soon as I have formed a judgement about this, I shall give myself the pleasure of informing you of the details.’ ([Einstein Papers vol. 12bis]; p. 52) Hence, Einstein tried to introduce a sort of scale invariance in GR, in order to generalize it and unify gravitational and electromagnetic phenomena. Einstein was interested in finding all conformally invariant tensors. In his answer, Wirtinger suggested Einstein to consider the forthcoming paper of Roland Weitzenböck on this topic and concluded: ‘I am already very much looking forward to your so kindly promised news about the “$A_\mu$”-free physics.’ ([Einstein Papers vol. 12bis]; p. 65).

In the same period Einstein wrote both to Hendrick A. Lorentz and to Ehrenfest with a great enthusiasm about his new attempt. February 22, to Lorentz: ‘I now have another hope of throwing a light on the realm of the molecular with the aid of relativity theory. For there is a possibility that the following two postulates can be united with each other. 1) The natural laws depend only on the ratios of the $g_{\mu\nu}$’s [...] [Weyl’s postulate]. 2) The electromagnetic potentials do not explicitly enter into the laws, just the field strengths. I am very curious to see if these hypotheses will prove correct.’ [emphasis added] ([Einstein Papers vol. 12bis]; p. 51) On March 1, to Ehrenfest: ‘A good idea occurred to me about relativity. One can, like Weyl, assign physical meaning just to the relative value of the $g_{\mu\nu}$’s (i.e., to the light cone $ds^2 = 0$), without therefore having to resort to the characteristic $A$-metric with the non-integrable changes in the...’

\footnote{We changed the notation in order to make it uniform throughout our paper.}
distances or measuring rods. The mathematical apparatus is, relatively speaking, quite simple. Whether the business is physically valid will be determinable in a relatively short time. I’ll send you the preliminary paper then.’ [emphasis added] (Einstein Papers vol. 12bis; p. 60). Even if Einstein was not explicit, we can infer that he started to think to merge Kaluza’s and Weyl’s theory in some way. Einstein’s appreciation for the conformal invariance emerged in the paper he published in March. As emphasized in the above quotations, Einstein liked the conformal invariance because it could be a viable physical principle in order to reconcile GR with the microscopic phenomena, but he tried to introduce it in GR avoiding the problem of the history-dependent clocks and the use of the electromagnetic potentials as independent variables. He emphasized this fact even with other colleagues, e.g. Arnold Sommerfeld on March 9: ‘I introduce it in GR avoiding the problem of the history-dependent clocks and the use of the electromagnetic potentials it could be a viable physical principle in order to reconcile GR with the microscopic phenomena, but he tried to paper he published in March. As emphasized in the above quotations, Einstein liked the conformal invariance because it is physically valid will be determinable in a relatively short time. I’ll send you the preliminary paper then.’ [emphasis added] (Einstein Papers vol. 12bis; p. 119).

### 4.3 Reconciling GR with microscopic phenomena

Einstein had tried by the beginning of 1919 to use GR for constructing a theory describing electrons (Einstein Papers vol. 7; p. 131). The importance of establishing the character of geometry at the subatomic scale emerged often in this period. At the end of 1920 the editor of Nature informed Einstein that in January a special issue of the journal would be dedicated to relativity and asked him for a contribution. Einstein wrote: ‘Do gravitational fields play a part in the constitution of matter, and is the continuum within the atomic nucleus to be regarded as appreciably non-Euclidean?’ (Einstein Papers vol. 7bis; original manuscript p. 377, translation published in Nature p. 409).

In the following Prague’s lecture, entitled Geometry and Experience, he mentioned again the question posed by Weyl on the applicability of GR to microscopic phenomena domain. In the lecture he analysed the connection between geometry and experience starting by the following question: ‘At this point an enigma presents itself which in all ages has agitated inquiring minds. How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things?’ (Einstein Papers vol. 7bis; p. 209).

Einstein argued that the character of the geometry of the world should be established only by using phenomenological evidences: ‘The existence of sharp spectral lines is a convincing experimental proof of the above-mentioned principle of practical geometry. This, in the last analysis, is the reason which enables us to speak meaningfully of a Riemannian metric of the four-dimensional space-time continuum.’ (Einstein Papers vol. 7bis; p. 214). Hence he inferred the following. ‘According to the view advocated here, the question whether this continuum [space-time] has a Euclidean, Riemannian, or any other structure is a question of physics proper which must be answered by experience, and not a question of a convention to be chosen on grounds of mere expediency.’ (Einstein Papers vol. 7bis; p. 214). He came to Weyl’s critic. ‘Riemann’s geometry will hold if the laws of disposition of practically-rigid bodies approach those of Euclidean geometry the more closely the smaller the dimensions of the region of space-time under consideration. It is true that this proposed physical interpretation of geometry breaks down when applied immediately to spaces of sub-molecular order of magnitude. But nevertheless, even in questions as to the constitution of elementary particles, it retains part of its significance. For even when it is a question of describing the electrical elementary particles constituting matter, the attempt may still be made to ascribe physical meaning to those field concepts which have been physically defined for the purpose of describing the geometrical behaviour of bodies which are large as compared with the molecule. Success alone can decide as to the justification of such an attempt, which postulates physical reality for the fundamental principles of Riemann’s geometry outside of the domain of their physical definitions. It might possibly turn out that this extrapolation has no better warrant than the extrapolation of the concept of temperature to parts of a body of molecular order of magnitude.’ (Einstein Papers vol. 7bis; p. 214) After this brief note, Einstein considered the extension of the concept of geometry to the scale of Universe.

As a result of his correspondence with Wirtinger, Einstein sent his contribution to the Prussian Academy: *On natural addition to the foundation of general relativity*. In the paper, received on March 3, 1921, Einstein suggested considering a theory where only the ratios of the metric have physical meaning and emphasized that ‘[it] seems to me to be a lucky and natural one, even though one cannot a priori know whether or not it can lead to a useful physical theory.’ (Einstein Papers vol. 7bis; p. 225). From the correspondence we described above, we know that Einstein was concerned with the description of microscopic phenomena and that he considered Weyl’s idea as a tool for...
reconciling them with GR. Furthermore, in order to avoid the fact that Weyl’s theory implied history depending rods, in this paper Einstein proposed to drop from the assumptions of the theory ‘the existence of transferable measuring rods (and clocks, resp.).’ ([Einstein Papers vol. 7bis]; p. 225). We infer that he did not consider them as useful tools for describing, for example, electrons dynamics. Indeed, he proposed to describe it using a new conformal invariant action^{19}, constructed by referring explicitly to the useful correspondence with Wirtinger. He concluded his paper with the following emphasis. ‘Our only intention was to point out a logical possibility that is worthy of publication; it may be useful for physics or not. Only further investigations can show whether one or the other is the case […]’ ([Einstein Papers vol. 7bis]; p. 228).

4.4 Merging Weyl’s and Kaluza’s theory

In May, 1921, Einstein gave four lectures at Princeton, where he only mentioned Weyl’s and Kaluza’s attempts: ‘A theory in which the gravitational field and the electromagnetic field do not enter as logically distinct structures would be much preferable. H. Weyl, and recently Th. Kaluza, have put forward ingenious ideas along this direction; but concerning them, I am convinced that they do not bring us nearer to the true solution of the fundamental problem. I shall not go into this further […]’ ([Einstein Papers vol. 7bis]; p. 358) About Weyl’s principle, as we mentioned at the end of §22 Einstein communicated his conviction also to Lorentz. About Kaluza’s attempt, as we pointed out in section 3, he was aware of its incompatibility with electrons dynamics.

Even if Einstein maintained a sceptical approach, he continued to consider every possible development of these theories. In January, as we discussed in section 3, Einstein and Grommer sent their analysis on the existence of centrally symmetric solutions in Kaluza’s theory. In 1922 he wrote Weyl and wondered if Weyl had read Kaluza’s approach ([Einstein Papers vol. 13bis]; p. 181). Einstein convinced himself step by step of the necessity of introducing further abstractions. In July 1923, he delivered a lecture to the Assembly of Nordic Naturalists in Gothenburg and declared: ‘The theory of gravitation—that is, viewed from the standpoint of mathematical formalism, Riemannian geometry—shall be generalized in such a way as to include the laws of the electromagnetic field. Unfortunately, in this endeavour, unlike the case of the derivation of the theory of gravitation (equivalence of inertial and gravitational mass), we cannot base our efforts on empirical facts. Instead, we have to base them on the criterion of mathematical simplicity, which is not free from arbitrariness.’ [emphasis added] ([Einstein Papers vol. 13bis]; p. 80).

Worried about the fact that GR was not able to explain microscopic phenomena, he continued to develop different attempts using new ideas, for instance, the introduction of over-determined equations in December, 1923. In his searching for connections between the theory of electron and GR he persevered to follow the unification way, always starting from the inclusion of the electromagnetic theory and GR in a wider framework. As pointed out by Daniela Wünsch, ‘between 1919 and 1921 [...] he became convinced that this method of unification represented a significantly new path in physics’ ([Wünsch 2005]; p. 277). She emphasized that in proposing an extension of the Eddington–Weyl theory, ‘Einstein borrowed one element of Kaluza’s theory: the Hamilton function contains a single tensor to describe the unified field.’ ([Wünsch 2005]; p. 286). Hence, Einstein tried constantly to merge in different ways the principles he appreciated of both theories. In 1926 Klein published his work and as we underlined in section 3, Ehrenfest attracted Einstein’s attention to it. Finally, Einstein tried to merge Weyl’s and Klein’s ideas. Conformally equivalent metrics are not physically equivalent, but Einstein considered Weyl’s transformations for the five-dimensional metric that leave invariant the four-dimensional electromagnetic and gravitational potentials. Einstein used this “mathematical simplicity principle” in order to select the $\gamma_{55} = \text{constant}$ scenario, among all of the possible ones. Despite the non-equivalence of different five-dimensional metrics, the conformal transformations did not affect the four-dimensional physics. From this point of view, the $\gamma_{55} = \text{constant}$ scenario appears as a gauge choice. As we pointed out in section 3, Klein also made a statement about the physical meaning of the $\gamma_{\mu\nu}$’s ratios. Indeed, Klein read about Kaluza’s attempt in Einstein’s Princeton lectures, where, as quoted above, there is no description of the two approaches. In Heilbron’s interview, session III, the interviewer asked Klein to confirm that he did not know at that time Kaluza’s approach. Klein answered: ‘Yes, yes. I had seen a notice but I never knew what it meant – in Einstein’s Princeton lectures which had appeared about ’22, or ’21, or something like that. He said that there were such attempts by Weyl and Kaluza to make a unified theory. \textit{Weyl’s I knew,} but he never mentioned Kaluza’s. I don’t think there was a quotation. I wondered later on that I never saw Kaluza’s work, because I read rather much at Ann Arbor and browsed in the library. But I would imagine that those volumes of the Berlin Academy which were rather near to the war were missing, that they hadn’t gotten them yet. I never heard anything more than this notice in Einstein’s lectures about Kaluza before the spring ’26 when I showed Pauli my manuscript on the five dimension theory.’ [emphasis added]. Hence, Klein knew Weyl’s approach and this fact explains Klein’s assertion on the reality of $ds^2 = 0$ equations. But while Einstein used it for justifying explicitly his choice for the Lagrangian density, Klein considered it only like a “convention” and simply

\[ds^2 = 0\]
assumed the five-dimensional curvature scalar as a viable Lagrangian density: ‘The problem now is to set up field equations for the quantities $\gamma_{\mu\bar{\nu}}$ [...] Here we do not want to get into this difficult problem any further but only wish to show that the ordinary field equations are simply unified from the point of view of the five-dimensional geometry.’ (Klein 1984; p. 13). Then Klein wrote the five-dimensional curvature scalar and specified that in its expression all the quantities were assumed to be independent to the fifth dimension and that $\gamma_{55} = \text{constant}$. But after this statement he varied the five-dimensional action with respect to $\gamma_{\mu\bar{\nu}}$ and their first derivatives, and obtained a set of field equations which are equivalent to the coupled Einstein’s and Maxwell’s equations. As we pointed out in our Prologue, this choice generates the inconsistency of Klein’s model. Einstein did not vary explicitly the five-dimensional action: he observed that under his conditions the five-dimensional Lagrangian density is equivalent to the sum of the Einstein-Hilbert and the Maxwell’s Lagrangian densities. Hence, the physical equivalence emerged from the Lagrangian density, and he always considered only the $g_{\mu\nu}$ and the $A_{\mu}$ as Lagrangian variables. This means that Einstein’s procedure is equivalent to the modern procedure of constructing a reduced action starting from a constrained action. Even if the action to be constrained is not known, the constrained action can be viewed as an effective action, describing the physics only up to a specific energy scale. From this point of view, Einstein’s approach was different from Klein’s and it was not inconsistent.

As far as we know, the role of the cylinder conditions as a constraint condition for the action was pointed out in the Mid Seventies by Elhanan Leibovitz and Nathan Rosen [Leibovitz et al. 1973]. They proved that the Einstein-Maxwell field equations can be consistently derived by the five-dimensional curvature scalar, by imposing the cylinder and the strengthen cylinder conditions as constraints of the action. They also proved that this action can be further generalized. Furthermore, Einstein’s idea of merging Weyl’s and Kaluza’s approach was reconsidered by Yu S. Vladimirov at the beginning of the Eighties. He considered a more general model where $\gamma_{55}$ is not constant, but he introduced the five-dimensional conformal transformation and the $4+1$-splitting ([Vladimirov 1982; p. 1173]). He also discussed more widely which of the different conformal equivalent five-dimensional metrics should represent our four-dimensional world and the 1927 Einstein’s papers where Einstein was the first to point out the importance to choose a space-like extra-dimension.

5 Epilogue: really inconsistent?

We started the Prologue by considering Klein’s approach to five-dimensional relativity. We emphasized that Klein’s approach was inconsistent from a fully five-dimensional point of view. The communications published by Einstein in 1927 are very similar to Klein’s approach, in the sense that he did not obtained new results, as he himself declared. But Einstein’s path into the five-dimensional world started many years before Klein, because of his role in Kaluza’s early formulation of the theory. We raised many questions at the beginning of the paper and during our analysis. Hence, by summing up the conclusions we reached, we shall also answer briefly to the questions we posed.

The main question we addressed was why did Einstein publish these two communications? Einstein was convinced that a unified theory should fulfil some precise requirements. One of them was that all the equations defining the model must have a covariant formulation. He struggled for years, from 1919 until 1927, with the non-covariant form of the cylinder condition. In the communications we analyzed, he finally exhibited the covariant formulation. This was at least one good reason for communicating them. But he also discussed with Weyl the consequences of the introduction of a conformal structure. And in his attempts to construct a field theory able to shed light on the new microscopic phenomena, Einstein considered the conformal transformations as a viable theoretical principle. In many attempts he tried to melt together different theoretical principles, struggling with the fact that he had no empirical facts to be guided by. In these communications, Einstein tried to merge Weyl’s principle with Kaluza’s ideas explicitly. Hence, this is the second reason for publishing his papers.

Was Einstein aware of the inconsistency of Klein’s model? We found evidence, as far as we know for the first time, that Einstein could have inferred it, because he calculated the full field equations at least in the linear approximation, but he did not make any statement about it. Did Einstein know Klein’s work at that time? He knew it, but their path and their aims were completely different. Even if he claimed that his results were essentially the same as Klein’s, is there any difference between Einstein’s and Klein’s approach? The are some subtle differences between the two approaches. Klein spent very little time introducing the five-dimensional manifold, while Einstein analysed the cylinder condition, the strengthened cylinder condition and the classical character of the fifth dimension in more detail. Furthermore, Klein declared explicitly that he obtained his four-dimensional field equations by setting the

20Klein will note this fact only in his subsequent work.
21He also stated that the two steps can be reversed, obtaining the same results
22In his correspondence with Kaluza he investigated the consequence of considering $\gamma_{55}$ as scalar function, from the point of view of the four-dimensional particle dynamics.
23Klein was more interested in the quantum interpretation of the fifth dimension.
constancy of $\gamma_{55}$ after varying the five-dimensional equation. This is the sequence responsible of the inconsisteny of the model. Einstein inserted explicitly the cylinder and the strengthened cylinder condition into the action, obtaining a function of the four-dimensional electromagnetic and gravitational potentials. Hence, using modern language, he treated the two conditions as a sort of constraint and considered a reduced action for constructing a sort of effective field theory model. This procedure is perfectly consistent.

Did Einstein want to unify electromagnetic and gravitational forces only or did he want to incorporate the microscopic phenomena also? Einstein viewed the unification path as a first step towards a full understanding of the microscopic phenomena. Vizgin observed: ‘In his paper of 1927, Einstein ignored all aspects of the fifth dimension associated with quantum mechanics, probably because he hoped to obtain particles as singular solutions of unified field equations and the quantum features of their behaviour as properties of these solutions.’ (Vizgin 2010; p. 232).

By analysing Einstein’s correspondence we found that he convinced himself that the conformal invariance could have been a viable physical principle for including microscopic phenomena in the framework of GR.

Why Einstein decided to set $\gamma_{55} = \text{constant}$ in the action, i.e. before calculating the field equation? The modern procedure for managing constrained system in Lagrangian formalism was not yet developed at that time. We know that Einstein followed the development of tensor calculus in the work of Tullio Levi-Civita, especially in the area of GR. In [Levi-Civita 1917] Levi-Civita considered the static solutions of the Einstein equations in GR and the equations of motions of a massive point. Levi-Civita imposed the static character by reducing the line-element $ds^2$ and inserted the “constraints” into the action or into the full field equations. In addition, when considering the variation of the action for a point particle, Levi-Civita explicitly emphasized the fact that time must be varied, even if the metric does not depend on the time variable, because of its static character ([Levi-Civita 1917]; p. 465). Hence, we can infer that Einstein was aware of the correct procedure to follow in his analogous five-dimensional framework. This fact enforced our conviction that Einstein did not set consciously $\gamma_{55} = \text{constant}$ after calculating the equations of motion.

Why did Einstein consider it fundamental to understand if the cylinder condition could be recast in a manifestly covariant form? He grasped the concept of general covariance after a long path and finally he was convinced that this was one of the concepts to be maintained in generalizing GR (as in every physical theory). Hence, the cylinder conditions also had to be expressed in this form.

How long did Einstein struggle to find the covariant form of the cylinder condition? Einstein started to criticised the cylinder condition in his correspondence with Kaluza in 1919 and this was one of the reasons that forced him to delay the publication of Kaluza’s theory. He criticized it again, after he was convinced to communicate Kaluza’s paper to the Prussian Academy, in his published work and in his correspondence until the end of 1926. Soon after, he wrote the cylinder condition using the Killing vectors language.

Einstein used a sort of modern gauge approach in order to write the action. Was this the first time he used it and why did he introduce it? As far as we know, this was the first time he used it in order to motivate his choice of the Lagrangian density, a point that Klein explicitly left out.

Why did Einstein decide to introduce the idea that only the ratios of the five-dimensional metric tensor components have physical meaning? Einstein considered the conformal transformations as a viable tool for describing microscopic phenomena in the context of GR. He always criticized Weyl’s theory because of the history-dependent rods and clocks, but he always declared to like the concept of conformal invariance. Hence he tried to separate the two concept in order to use the latter for generalizing GR.

A similar statement on the ratios of the metric tensor components also appeared in Klein’s first paper. Did Einstein simply replicate Klein’s argument? Why did both make this statement? Klein considered the possibility of giving physical meaning to the ratios only as a possible convention. Einstein saw it as an important physical principle, which played a fundamental role in the construction of the Lagrangian density of his unified field theory, like the role played by gauge-group in modern gauge-theory. Hence, we emphasize that Einstein did not replicate Klein’s argument. Notwithstanding this, he did not see any possible future development of his approach at that time and decided to add a note in the proof, where he cited Klein’s work and where he underlined that no new results were presented in his communications.

By analysing the Kaluza-Einstein correspondence we discussed also Einstein’s role in pointing out the incompatibility of the original Kaluza’s theory with the motion of the electrons. Furthermore we pointed out how his friend Besso and his colleague Ehrenfest attracted Einstein’s attention again on the five-dimensional universe.

In the end, some questions arise. In correspondence with Kaluza, Einstein considered the physical consequences of a “real” fifth dimension. Then, by giving physical meaning to the ratios of the five-dimensional metric only, he pointed to the opposite direction, i.e. he assumed that the fifth dimension has no physical significance. What changed Einstein’s attitude toward the fifth dimension from 1919 to 1927? How did it change after 1927? Did Einstein realize the connection between geodesic lines and the Killing trajectories we mentioned in the prologue? Did he reconsider the framework he settled in 1927? We will not address these questions here, but we postpone the answers to future work.
6 Acknowledgment

We express our gratitude to Morgan E. Aronson for invaluable comments and suggestions. This work has been supported in part by the DOR 2016 funds of the University of Padua and by the Smithsonian Libraries Dibner Library Resident Scholar Program, Washington DC.

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