Acoustic metafluids

Andrew N. Norris

Mechanical and Aerospace Engineering, Rutgers University, Piscataway NJ 08854

(Dated: November 18, 2008)

Acoustic metafluids are defined as the class of fluids that allow one domain of fluid to acoustically mimic another, as exemplified by acoustic cloaks. It is shown that the most general class of acoustic metafluids are materials with anisotropic inertia and the elastic properties of what are known as pentamode materials. The derivation uses the notion of finite deformation to define the transformation of one region to another. The main result is found by considering energy density in the original and transformed regions. Properties of acoustic metafluids are discussed, and general conditions are found which ensure that the mapped fluid has isotropic inertia, which potentially opens up the possibility of achieving broadband cloaking.

PACS numbers: 43.20.Fn, 43.20Tb, 43.40Sk, 43.35.Bf

I. INTRODUCTION

Ideal acoustic stealth is provided by the acoustic cloak, a shell of material that surrounds the object to be rendered acoustically “invisible”. Stealth can also be achieved by “hiding under the carpet”\textsuperscript{1} as shown in Fig. 1. A simpler situation but one that displays the essence of the acoustic stealth problem is depicted in Fig. 2. The common issue is how to make one region of fluid acoustically mimic another region of fluid. The fluids are different as are the domains they occupy; in fact the mimicking region is typically smaller in size, it can be viewed as a compacted version of the original.

The subject of this paper is not acoustic cloaks, or carpets, or ways to hide things, but rather the type of material necessary to achieve stealth. We define these materials as \textit{acoustic metafluids}, which as we will see can be considered fluids with microstructure and properties outside those found in nature. The objective is to derive the general class of acoustic metafluids, and in the process show that there is a closed set which can be mapped from one to another. Acoustic metafluids are defined as the class of fluids that (a) acoustically mimic another region as in the examples of Figs. 1 and 2, and (b) can themselves be mimicked by another acoustic metafluid in the same sense. The requirement (b) is important, implying that there is a closed set of acoustic metafluids. The set includes as a special case the “normal” acoustic fluid of uniform density and bulk modulus. Acoustic metafluids can therefore be used to create stealth devices in a normal fluid. But, in addition, acoustic metafluids can provide stealth in any type of acoustic metafluid. The reciprocal nature of these fluids make them a natural generalization of normal acoustic fluids.

The acoustic cloaks that have been investigated to date fall into two categories in terms of the type of acoustic metafluid proposed as cloaking material. Most studies, e.g.\textsuperscript{2–7}, consider the cloak to comprise fluid with the normal stress-strain relation but anisotropic inertia, what we call \textit{inertial cloaking}. Particular realization of inertial cloaks are in principle feasible using layers of isotropic “normal” fluid\textsuperscript{12–15}, the layers are introduced in order to achieve an homogenized medium that approximates a fluid with anisotropic inertia. An alternative and more general approach\textsuperscript{8,11} is to consider anisotropic inertia combined with anisotropic elasticity. The latter is intro-

\textsuperscript{a)Electronic address: norris@rutgers.edu}
The paper is organized as follows. The concept of acoustic metafluids is introduced in §II through two “acoustic mirage” examples. The methods used to find the acoustic metafluid in these examples are simple but not easily generalized. An alternative and far more powerful approach is discussed in §III: the transformation method. This is based on using the change of variables between the coordinates of the two regions combined with differential relations to identify the metafluid properties of the transformed domain. Leonhardt and Philbin provide an instructive review of the transformation method in the context of optics. The transformation method does not however define the range of material properties capable of being transformed. This is the central objective of the paper and it is resolved in §IV by considering the energy density in the original and transformed domains. Physical properties of acoustic metafluids are discussed in §V, including the unusual property that the top surface is not horizontal when at rest under gravity. The subset of acoustic metafluids that have isotropic inertia is considered in §VI, and a concluding summary is presented in §VII.

II. ACOUSTIC MIRAGES AND SIMPLE METAFLUIDS

The defining property of an acoustic metafluid is its ability to mimic another acoustic fluid that occupies a different domain. The simplest type of acoustic illusion is what may be called an acoustic mirage where an observer hears, for example, a reflection from a distant wall, but in reality the echo originates from a closer boundary. Two examples of acoustic mirages are discussed next.

A. 1D mirage

Consider perhaps the simplest configuration imaginable, a one dimensional semi-infinite medium. The upper picture in Fig. 2 shows the left end of an acoustic half space \( x \geq 0 \) with uniform density \( \rho_0 \) and bulk modulus \( K_0 \). The wave speed is \( c_0 = \sqrt{K_0/\rho_0} \). Now replace the region \( 0 \leq x < b \) by a shorter section \( 0 < b - a \leq x < b \) filled with an acoustic metafluid. The acoustic mirage effect requires that an observer in \( x > b \) hears a response as if the half space is as shown in the top of Fig. 2. This occurs if the metafluid region is such that: (i) no reflection occurs at the interface \( x = b \), i.e. the acoustic impedance in the modified region is the same as before; and (ii) the round trip travel time of a wave incident from the right is unchanged. The impedance condition and the travel time requirement ensure the amplitude and phase of any signal is exactly the same as in the original half-space, and hence the mirage is accomplished.

Let the acoustic metafluid have material properties \( \rho \) and \( \kappa \), with speed \( c = \sqrt{\kappa/\rho} \). Conditions (i) and (ii) are satisfied if

\[
\rho c = \rho_0 c_0, \quad \frac{a}{c} = \frac{b}{c_0}.
\]

respectively, implying the density and bulk modulus in the shorter region are

\[
\rho = \frac{b}{a} \rho_0, \quad \kappa = \frac{a}{b} K_0.
\]

In this example the acoustic metafluid is another acoustic fluid, although with different density and bulk modulus. Note that the total mass of the metafluid region is unchanged from the original: \( \rho_0 b = \rho a \).

B. 2D mirage

Consider the same problem in 2D under oblique wave incidence, Fig. 3. A wave incident at angle \( \theta_0 \) from the normal has travel time \( \sqrt{b/(c_0 \cos \theta_0)} \) in the original layer. If the shortened region has wave speed \( c \), then the modified travel time is \( a/(c \cos \theta) \) where \( \theta \) is defined by the Snell-Descartes law, \( \frac{1}{2} \sin \theta = \frac{1}{2} \sin \theta_0 \). At the same time, the reflectivity of the modified layer is \( R = (Z - Z_0)/(Z + Z_0) \) where \( Z = \rho c/\cos \theta \). The impedance and travel time conditions are now

\[
\frac{\rho c}{\cos \theta} = \frac{\rho_0 c_0}{\cos \theta_0}, \quad \frac{a}{c \cos \theta} = \frac{b}{c \cos \theta_0}.
\]

Solving for the modified parameters implies \( \kappa \) is again given by eq. (2) but the density is now

\[
\rho = \frac{b}{a} \frac{\rho_0}{2 \cos^2 \theta_0} \left[ 1 + \frac{a^2}{b^2} \sin^2 2\theta_0 \right].
\]

The mirage works only for a single direction of incidence, \( \theta_0 \), and is therefore unsatisfactory. The underlying problem here is that three conditions need to be met: Snell-Descartes’ law, matched impedances and equal travel times, with only two parameters, \( \rho \) and \( \kappa \). Some additional degree of freedom is required.
2. Solution of the 2D mirage problem

The travel time is \( t = \frac{a}{w \cdot e_1} \), and the impedance is now \( Z = \rho_1 v / \cos \theta \), where, referring to Fig. 3, \( n_1 = \cos \theta \) and \( n_2 = \sin \theta \). Hence, the conditions for zero reflectivity and equal travel times are

\[
\frac{\rho_1 v}{\cos \theta} = \frac{\rho_0 c_0}{\cos \theta_0}, \quad \frac{a v}{c_1 \cos \theta} = \frac{b}{c_0 \cos \theta_0}.
\]

Dividing the latter two relations implies \( k \) is given by (2), Snell-Descartes' law, \( v^{-1} \sin \theta = c^{-1}_0 \sin \theta_0 \), combined with eq. (9) yields

\[
\frac{c_1^2}{v^2} = \frac{\rho_1}{\rho_0} + (1 - \frac{\rho_0^2}{\rho_0^2}) \sin^2 \theta_0, \quad (10)
\]

while Snell-Descartes' law together with eq. (8) implies

\[
\frac{c_0^2}{v^2} = \frac{\rho_1}{\rho_2} + (1 - \frac{\rho_1}{\rho_2}) \sin^2 \theta_0. \quad (11)
\]

Comparison of (10) and (11) implies two identities for \( \rho_1 \) and \( \rho_2 \). In summary, the three parameters of the modified region are

\[
\kappa = \frac{a}{b} \kappa_0, \quad \rho_1 = \frac{b}{a} \rho_0, \quad \rho_2 = \frac{a}{b} \rho_0. \quad (12)
\]

These give the desired result: no reflection, and the same travel time for any angle of incidence. The metafluid layer faithfully mimics the wave properties of the original layer as observed from exterior vantage points.

3. Anisotropic stiffness

An alternative solution to the quardardian raised by eq. (4) is to keep the density isotropic but to relax the standard isotropic constitutive relation between stress \( \sigma \) and strain \( \varepsilon = [\nabla u + (\nabla u)^t]/2 \) to allow for material anisotropy. Thus, the standard relation \( \sigma = -p I \) with \( p = \kappa \text{tr} \varepsilon \) is replaced by the stress-strain relation for pentamode materials

\[
\sigma = \kappa (Q : \varepsilon) Q, \quad \text{div} \, Q = 0. \quad (13)
\]

The physical meaning of the symmetric second order tensor \( Q \) is discussed later within the context of a more general constitutive theory. The requirement \( \text{div} \, Q = 0 \) was first noted by Norris and is discussed in §V. Standard acoustics corresponds to \( Q = I \).

Rewriting (13) as \( \sigma = -pQ \) and using the divergence free property of \( Q \), the equation of motion and the constitutive relation in the metafluid are now

\[
\rho \dot{v} = -Q \nabla p, \quad \dot{p} = -\kappa Q : \nabla v. \quad (14)
\]

Eliminating \( v \) yields the scalar wave equation

\[
\ddot{p} - \kappa Q : \nabla (\rho^{-1} Q \nabla p) = 0. \quad (15)
\]

General properties of this equation have been discussed by Norris and will be examined later in §V. For the
purpose of the problem in Fig. 3 $Q$ is assumed constant of the form

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad (16)$$

then it follows that the phase speed and group velocity in direction $n$ are

$$v^2 = C_1 n_1^2 + C_2 n_2^2, \quad w = v^{-1} (C_1 n_1 e_1 + C_2 n_2 e_2), \quad (17)$$

where $C_j = Q_j \sqrt{\kappa_j / \rho_j}, \; j = 1, 2$. Proceeding as before, using Snell-Descartes’ law and the conditions of equal travel time and matched impedance, yields

$$Q_2^2 = \frac{a}{b} \kappa_0, \quad Q_2^2 = \frac{b}{a} \rho_0, \quad Q_1 Q_2 = b/a. \quad (18)$$

Since the important physical quantity is the product of $\kappa$ with $Q \otimes Q$, any one of the three parameters $\kappa, Q_1$ and $Q_2$ may be independently selected. A natural choice is to impose $\text{div} Q = 0$ at the interface, which means that the “flux” vector $Qn$ is continuous, where $n$ is the interface normal. In this case $n = e_1$ so that $Q_1 = 1$ and

$$\rho = \frac{b}{a} \rho_0, \quad \kappa = \frac{a}{b} \kappa_0, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & \frac{b}{a} \end{bmatrix}. \quad (18)$$

Comparing the alternative metafluids defined by eqs. (12) and (18) note that in each case the density and the stiffness associated with the normal $e_1$ direction both equal their 1D values given by eq. (2). The first metafluid of eq. (12) has a smaller inertia in the transverse direction $e_2$ ($\rho_2 < \rho_1$). The second metafluid defined by eq. (18) has increased stiffness in the transverse direction ($Q_2 > Q_1$). The net effect in each case is an increased phase speed in the transverse direction as compared with the normal direction ($c_2 > c_1, C_2 > C_1$).

The 1D and 2D mirage examples illustrate the general idea of acoustic metafluids as fluids that replicate the wave properties of a transformed region. However, the methods used to find the metafluids are not easily generalized to arbitrary regions. How does one find the metafluid that can, for instance, mimic a full spherical region by a smaller shell? This is the cloaking problem. The key to the generalized procedure are the related notions of transformation and finite deformation, which are introduced next.

III. THE TRANSFORMATION METHOD

A. Preliminaries

Let $\Omega$ and $\omega$ denote the original and the deformed domains (the regions $0 \leq X < b$ and $b - a \leq x < b$ in the examples of §II). The coordinates in each configuration are $X$ and $x$, respectively; the divergence operators are $\nabla$ and $\nabla$, and the gradient (nabla) operators are $\nabla$ and $\nabla$. Upper and lower case indices indicate components, $X_i, x_i$, and the component form of $\text{div} A$ is $\partial A_i / \partial x_i = A_{i,i}$ or $\partial A_{i,j} / \partial x_i = A_{i,j,i}$ when $A$ is a vector and a second order tensor-like quantity, respectively, and repeated indices imply summation (case sensitive!). Similarly $\text{Div} B = B_{i,j,i}$.

The finite deformation or transformation is defined by the mapping $\Omega \rightarrow \omega$ according to $x = \chi(X)$. In the terminology of finite elasticity $X$ describes a particle position in the Lagrangian or undeformed configuration, and $x$ is particle location in the Eulerian or deformed physical state. The transformation $\chi$ is assumed to be one-to-one and invertible.$^{17}$ The deformation gradient is defined $F = \nabla x$ with inverse $F^{-1} = \nabla X$, or in component form $F_{it} = \partial x_i / \partial X_t$, $F_{it}^{-1} = \partial X_i / \partial x_t$. The Jacobian of the finite deformation is $\Lambda = \det F = |F|$, or in terms of volume elements in the two configurations, $\Lambda = dV / dV$. The polar decomposition implies $F = VR$, where the rotation $R$ is proper orthogonal ($RR^T = R^T R = I$, $\det R = 1$) and the stretch tensor $V \in \text{Sym}^+$ is the positive definite solution of $V^2 = FF^T$. Note for later use the kinematic identities$^{18}$

$$\text{div} (\Lambda^{-1} F) = 0, \quad \text{Div} (\Lambda F^{-1}) = 0, \quad (19)$$

and the expression for the Laplacian in $X$ in terms of derivatives in $x$, i.e., the chain rule$^8$

$$\nabla^2 f = \Lambda \text{div} (\Lambda^{-1} \nabla^2 f). \quad (20)$$

B. The transformation method

The undeformed domain $\Omega$ is of arbitrary shape and comprises a homogeneous acoustic fluid with density $\rho_0$ and bulk modulus $\kappa_0$. The goal is to mimic the scalar wave equation in $\Omega$,

$$\ddot{p} - (\kappa_0 / \rho_0) \nabla^2 p = 0, \quad X \in \Omega, \quad (21)$$

by the wave equation of a metafluid occupying the deformed region $\omega$. The basic result$^6,8,19$ is that eq. (21) is exactly replicated in $\omega$ by the equation

$$\ddot{p} - \kappa \text{div} (\rho^{-1} \nabla p) = 0, \quad x \in \omega, \quad (22)$$

where the bulk modulus and inertia tensor are

$$\kappa = \kappa_0 \Lambda, \quad \rho = \rho_0 \Lambda V^{-2}. \quad (23)$$

The equivalence of eq. (21) with eqs. (22) and (23) is evident from the differential equality (20). The idea is to use the change of variables from $X$ to $x$ to identify the metafluid properties. Equation (23) describes a metafluid with anisotropic inertia and isotropic elasticity. It can be used for modeling acoustic cloaks but has the unavoidable feature that the total effective mass of the cloak is infinite. This problem, discussed by Norris$^8$, arises from the singular nature of the finite deformation in a cloak which makes $\Lambda V^{-2}$ non-integrable. This type of fluid, which could be called an inertial fluid, appears to be the main candidate considered for acoustic cloaking to date. The major exception is Milton et al.$^{11}$ who considered fluids with properties of pentamode materials, although as we will discuss in Section IV, their findings are of limited use for acoustic cloaking.
1. Pentamode materials

Norris\(^8\) showed that eq. (20) is a special case of a more general identity:

\[
\nabla^p = \Lambda \mathbf{Q} : \nabla (\Lambda^{-1} \mathbf{Q}^{-1} \mathbf{V}^2 \nabla \rho),
\]

(24)

where \(\mathbf{Q}\) is any symmetric, invertible and divergence free (\(\text{div} \mathbf{Q} = 0\)) second order tensor. The increased degrees of freedom afforded by the arbitrary nature of \(\mathbf{Q}\) means that (21) is equivalent to the generalized scalar wave equation in \(\omega\),

\[
\ddot{p} - \kappa \mathbf{Q} : \nabla (\rho^{-1} \mathbf{Q} \nabla p) = 0, \quad \mathbf{x} \in \omega,
\]

(25)

where the modulus \(\kappa\) and the inertia follow from a comparison of eqs. (21), (24) and (25) as

\[
\kappa = \kappa_0 \Lambda, \quad \rho = \rho_0 \Lambda \mathbf{Q} \mathbf{V}^{-2} \mathbf{Q}.
\]

(26)

As will become apparent later, these metafluid parameters describe a pentamode material with anisotropic inertia. For the moment we return to the acoustic mirages in light of the general transformation method.

C. Mirages revisited

The 2D mirage problem corresponds to the following finite deformation \(x_1 = b - a + ab^{-1} X_1, x_2 = X_2\) for \(0 \leq X_1 < b, -\infty < X_2 < \infty\). The deformation gradient is

\[
\mathbf{F} = \begin{bmatrix}
\frac{a}{b} & 0 \\
0 & 1
\end{bmatrix},
\]

(27)

implying \(\mathbf{R} = \mathbf{I}, \mathbf{V} = \mathbf{F}\) and \(\Lambda = a/b\). Equation (23) therefore implies

\[
\kappa = \frac{a}{b} \kappa_0, \quad \rho = \begin{bmatrix}
\frac{b}{a} \rho_0 & 0 \\
0 & \frac{a}{b} \rho_0
\end{bmatrix}.
\]

(28)

These agree with the parameters found in §II, eq. (12).

Using the more general formulation of (25) and (26) with the arbitrary tensor chosen as \(\mathbf{Q} = \Lambda^{-1} \mathbf{V}\) yields the metafluid described by eq. (18). It is interesting to note that although \(\rho\) of eq. (26) is, in general, anisotropic, it can be made isotropic in this instance by any \(\mathbf{Q}\) proportional to \(\mathbf{V}\). Keeping in mind the requirement seen above that \(\chi = 0\), we consider as a second example of (25) the case \(\mathbf{Q} = \Lambda^{-1} \mathbf{V}\). The mirage can then be achieved with material properties

\[
\rho = \frac{b}{a} \rho_0, \quad \kappa = \frac{a}{b} \kappa_0, \quad \mathbf{Q} = \begin{bmatrix}
1 & 0 \\
0 & \frac{a}{b}
\end{bmatrix}.
\]

(29)

This again corresponds to a pentamode material with isotropic inertia, equal to that of (18). These two examples illustrate the power associated with the arbitrary nature of the divergence free tensor \(\mathbf{Q}\). There appears to be a multiplicative degree of freedom associated with \(\mathbf{Q}\) that is absent using anisotropic inertia. As will be evident later, this degree of freedom is related to a gauge transformation.

IV. THE MOST GENERAL TYPE OF ACOUSTIC METAFLUID

A. Summary of main result

In order to make it easier for the reader to assimilate, the paper’s main result is first presented in the form of a theorem. In the following \(\mathbf{Q}_0\) and \(\mathbf{Q}\) are arbitrary symmetric, invertible and divergence free second order tensors.

Theorem 1 The kinetic and strain energy densities in \(\Omega\), of the form

\[
T_0 = \mathbf{U}^T \rho_0 \mathbf{U}, \quad W_0 = \kappa_0 (\mathbf{Q}_0 : \nabla \mathbf{U})^2,
\]

(30)

respectively, are equivalent to the current energy densities in \(\omega\):

\[
T = \mathbf{U}^T \rho \mathbf{U}, \quad W = \kappa (\mathbf{Q} : \nabla \mathbf{u})^2,
\]

(31)

where

\[
\kappa = \Lambda \kappa_0, \quad \rho = \Lambda \mathbf{Q} \mathbf{F}^{-1} \mathbf{Q}_0^{-1} \rho_0 \mathbf{Q}_0^{-1} \mathbf{F}^{-1} \mathbf{Q}.
\]

(32)

Discussion of the implications are given following the proof.

B. Gauge transformation

The energy functions per unit volume in the undeformed configuration, \(T_0\) and \(W_0\), depend upon the infinitesimal displacement \(\mathbf{U}\) in that configuration. The kinetic energy is defined by the density \(\rho_0\), while the strain energy is \(W_0 = C_{ijkl} U_{ij} U_{kl}\), where \(C_{ijkl}\) are elements of the stiffness. The density and stiffness possess the symmetries \(\rho_{ij} = \rho_{ji}, C_{ijkl} = C_{klij}, C_{ijkl} = C_{klij}\). The total energy is \(E_0 = T_0 + W_0\) and the total energy \(E = T + W\) per unit deformed volume is, using \(E_0 \text{d}V = E \text{d}v\), simply \(E = \Lambda^{-1} E_0\).

Our objective is to find a general class of material parameters \(\{\rho_0, \mathbf{C}\}\) that maintain the structure of the energy under a general transformation \(\chi\). Structure here means that the energy remains quadratic in velocity and strain. In order to achieve the most general form for the transformed energy introduce a gauge transformation for the displacement. Let

\[
\mathbf{U} = \mathbf{G} u,
\]

(33)

or \(U_i = u_i G_{i\ell}\) in components, where \(\mathbf{G}\) is independent of time but can be spatially varying. Thus, using the chain rule \(U_{ij} = U_{i\ell} F_{\ell j}\), yields \(E = T + W\), where the kinetic and strain energy densities are

\[
T = \mathbf{U}^T \rho \mathbf{U},
\]

(34a)

\[
W = \Lambda^{-1} C_{ijkl} F_{ij} F_{kl} (G_{j\ell} u_j)_i (G_{i\ell} u_i)_k,
\]

(34b)

and the transformed inertia is

\[
\rho = \Lambda^{-1} \mathbf{G} \rho_0 \mathbf{G}^T.
\]

(35)
The kinetic energy has the required structure, quadratic in the velocity; the strain energy however is not in the desired form. The objective is to obtain a strain energy of standard form

\[ W = C_{ijkl}u_{j,i}u_{k,l}, \tag{36} \]

where \( C \) has the usual symmetries: \( C_{ijkl} = C_{klij}, C_{ijkl} = C_{jikl} \).

The question of how to convert \( W \) of (34b) into the form of (36) for generally anisotropic elasticity stiffness \( C_0 \) will be discussed in a separate paper. The objective here is to find the largest class of metafluids that includes all those previously found.

C. Pentamode to pentamode

In order to proceed assume that the initial stiffness tensor is of pentamode form\(^\text{10}\)

\[ C_0 = \kappa_0 Q_0 \otimes Q_0, \quad \text{Div} \, Q_0 = 0, \tag{37} \]

that is, \( C_{ijkl} = \kappa_0 Q_{0ijkl} \) where \( Q_0 \) is a positive definite symmetric second order tensor. The tensor \( Q_0 \) is necessarily divergence free\(^6\).

The strain energy density in the physical space after the general deformation and gauge transformation is now

\[ W = \kappa_0 \Lambda^{-1} [Q_{0ijkl}(u_{j,j}g_{j,j}),j]^2. \tag{38} \]

Consider

\[ Q_{0ijkl}(u_{j,j}g_{j,j}),l = Q_{0ijkl}g_{j,j}u_{j,l} + Q_{0ijkl}g_{j,l}u_{j,j}. \tag{39} \]

In order to achieve the quadratic structure of (36) the final term in (39) must vanish. Since \( u \) is considered arbitrary this in turn implies that \( Q_{0ijkl}g_{j,j} \) must vanish for all \( j \). With no loss in generality let

\[ G^t = Q_0^{-1}P, \tag{40} \]

or \( G_{j,j} = Q_0^{-1}_{0jM}P_{M,j} \) in components. Then using the identity for the derivative of a second order tensor, \((A^{-1})_{\alpha} = -A^{-1}_{\alpha\beta}A_{\beta}^{-1}\) where \( \alpha \) can be any parameter, gives

\[
\begin{align*}
0 &= Q_{0ijkl}g_{j,j},l \\
&= -Q_{0ijkl}Q_{0jkl}^{-1}Q_{0klm}^{-1}P_{M,j} + P_{lj},l \\
&= -Q_{0ijkl}Q_{0klm}^{-1}P_{M,j} + P_{lj},l \\
&= P_{lj},l, \tag{41}
\end{align*}
\]

where the important property (37)\(_2\) has been used. Hence, \( \text{Div} \, P = 0 \). Then using (39) yields

\[
\begin{align*}
Q_{0ijkl}(u_{j,j}g_{j,j}),l &= P_{lj}u_{j,l} \\
&= P_{lj}F_{ij}u_{j,i} \\
&= \Lambda Q_{ij}u_{j,i}, \tag{42}
\end{align*}
\]

where the tensor \( Q \) is defined by

\[
P = \Lambda F^{-1}Q \quad \Leftrightarrow \quad Q = \Lambda^{-1}FP. \tag{43}
\]

The condition \( \text{Div} \, P = 0 \) becomes, using (19)\(_2\),

\[ P_{lj},l = (\Lambda F_{ij}^{-1}Q_{ij})_j = \Lambda F_{ij}^{-1}Q_{ij},l = \Lambda Q_{ij},l, \tag{44} \]

implying

\[ \text{div} \, Q = 0. \tag{45} \]

It has therefore been demonstrated that if the gauge transformation is defined by

\[ G^t = \Lambda Q_0^{-1}F^{-1}Q, \tag{46} \]

where \( Q \) is divergence free in physical coordinates, then the strain energy (38) is \( W = \kappa_0 \Lambda (Q_{ij}u_{j,i})^2 \). This is of the desired form, eq. (36), with \( C_{ijkl} = \kappa Q_{ij}Q_{kl} \), hence completing the proof of Theorem 1.

D. Discussion

I. Equivalence of physical quantities

Theorem 1 states that the pentamode material defined by stiffness \( \kappa_0 \) with stress-like tensor \( Q_0 \) and anisotropic inertia \( \rho_0 \) is converted into another pentamode material with anisotropic inertia. The properties of the new metafluid are defined by the original metafluid and the deformation-gauge pair \( \{F, G\} \) where \( F \) is arbitrary and possibly inhomogeneous, and \( G \) is given by eq. (46) with \( Q \) symmetric, positive definite and divergence free but otherwise completely arbitrary. The special case of a fluid with isotropic stiffness but anisotropic inertia, eq. (23), is recovered from Theorem 1 when the starting medium is a standard acoustic fluid and \( Q \) is taken to be \( I \).

It is instructive to examine how physical quantities transform: we consider displacement, momentum and pseudopressure. Eliminating \( G \), it is possible to express the new displacement vector in terms of the original,

\[ u = \Lambda^{-1}Q_0^{-1}FQ_0U. \tag{47} \]

Physically, this means that as the metafluid in \( \omega \) acoustically replicates that in \( \Omega \), particle motion in the latter is converted into the mimicked motion defined by (47).

Define the momentum vectors in the two configurations,

\[ m_0 = \rho_0 \dot{U}, \quad m = \rho \dot{u}. \tag{48} \]

Equations (35) and (46) imply that they are related by

\[ m = QF^{-1}Q_0^{-1}m_0. \tag{49} \]

The transformation of momentum is similar to that for displacement, eq. (47), but with the inverse tensor, i.e. \( u = G^{-1}U \) while \( m = \Lambda^{-1}Gm_0 \).

Stress in the two configurations is defined by Hooke’s law in each:

\[ \sigma_0 = C_0 : \nabla U, \quad \sigma = C : \nabla u, \tag{50} \]

where \( C_0 \) and \( C \) are fourth order elasticity tensors,

\[ C_0 = \kappa_0 Q_0 \otimes Q_0, \quad C = \kappa Q \otimes Q, \tag{51} \]
that is, \( C_{ijkl} = \kappa Q_{ij} Q_{kl} \), etc. Using (37), (42) and (51), yields

\[
\sigma_0 = -p Q_0, \quad \sigma = -p Q,
\]

where \( p \) is the same in each configuration,

\[
p = -\kappa Q : \nabla \mathbf{u} = -\kappa_0 Q_0 : \nabla \mathbf{U}.
\]

The quantity \( p \) is similar to pressure, and can be exactly identified as such when \( Q \) is diagonal, but it is not pressure in the usual meaning. For this reason it is called the pseudopressure. It is interesting to compare the equal values of \( p \) in \( \Omega \) and \( \omega \) with the more complicated relations (47) and (49) for the displacement and momentum.

2. Equations of motion

The equations of motion can be derived as the Euler-Lagrange equation for the Lagrangian \( T - W \). A succinct form is as follows, in terms of the the momentum density \( \mathbf{m} \) and the stress tensor \( \sigma \):

\[
\mathbf{m} = \nabla \sigma.
\]

The constitutive relation may be expressed as an equation for the pseudopressure \( p^8 \),

\[
\bar{p} = -\kappa Q : \nabla \mathbf{u},
\]

while eqs. (53), (48) and (54) imply that the acceleration is

\[
\ddot{\mathbf{u}} = -\rho^{-1} Q \nabla \bar{p}.
\]

Eliminating \( p \) between the last two equations implies that the displacement satisfies

\[
\kappa Q Q : \nabla \ddot{\mathbf{u}} - \rho \ddot{\mathbf{u}} = 0.
\]

This is, as expected, the elastodynamic equation for a pentamode material with anisotropic inertia. Alternatively, eliminating \( \ddot{\mathbf{u}} \) between eqs. (55) and (56) yields a scalar wave equation for the pseudopressure,

\[
\bar{p} - \kappa Q : \nabla (\rho^{-1} Q \nabla \bar{p}) = 0.
\]

This clearly reduces to the standard acoustic wave equation when \( Q = I \) and \( \rho \) is isotropic.

3. Relation to the findings of Milton et al.\(^{11} \)

The present findings appear to contradict those of Milton et al.\(^{11} \) who found the negative result that it is not in general possible to find a metafluid that replicates a standard acoustic medium after arbitrary finite deformation. However, their result is based on the assumption that \( G \equiv F \) (their eq. (2.2)). Equation (46) implies that \( Q \) must then be

\[
Q = \Lambda^{-1} F Q_0 F^t.
\]

Using (19)\(_1\) and (37)\(_2\) yields \( Q_{ij,l} = \Lambda^{-1} Q_{0l,l} F_{i,l,l} \). Hence, this particular \( Q \) can only satisfy the requirement (45) that \( \text{div} \ Q = 0 \) if

\[
Q_{0l,l} \frac{\partial^2 x_l}{\partial X_i \partial X_j} = 0.
\]

Milton et al.\(^{11} \) considered \( Q_0 \) isotropic (diagonal), in which case (60) means that the only permissible finite deformations are harmonic, i.e. those for which \( \nabla^2 \phi = 0 \). In short, the negative findings of Milton et al.\(^{11} \) are a consequence of constraining the gauge to \( G \equiv F \), which in turn severely restricts the realizability of metafluids except under very limited types of transformation deformation. The main difference in the present analysis is the inclusion of the general gauge transformation which enables us to find metafluids under arbitrary deformation.

V. PROPERTIES OF ACOUSTIC METAFLUIDS

The primary result of the paper, summarized in Theorem 1, states that the class of acoustic metafluids is defined by the most general type of pentamode material with elastic stiffness \( \kappa Q \otimes Q \) where \( \text{div} \ Q = 0 \), and anisotropic inertia \( \rho \). We now examine some of the unusual physical properties, dynamic and static, to be expected in acoustic metafluids. Some of the dynamic properties have been discussed by Norris\(^8 \), but apart from Milton and Cherkaev\(^10 \) no discussion of static effects has been given.

A. Dynamic properties: plane waves

Consider plane wave solutions for displacement of the form \( \mathbf{u}(x,t) = \mathbf{q} e^{i \kappa |\mathbf{n} \cdot \mathbf{x} - v t|} \), for \( |\mathbf{n}| = 1 \) and constant \( \mathbf{q} \), \( k \) and \( v \), and uniform metafluid properties. Non trivial solutions satisfying the equations of motion eq. (57) require

\[
(\kappa (\mathbf{Q} \mathbf{n}) \otimes (\mathbf{Q} \mathbf{n}) - \rho v^2) \mathbf{q} = 0.
\]

The acoustical or Christoffel\(^20 \) tensor \( \kappa (\mathbf{Q} \mathbf{n}) \otimes (\mathbf{Q} \mathbf{n}) \) is rank one and it follows that of the three possible solutions for \( v^2 \), only one is not zero, the quasi-longitudinal solution

\[
v^2 = \kappa \mathbf{n} \cdot \mathbf{Q} \rho^{-1} \mathbf{Q} \mathbf{n}, \quad \mathbf{q} = \rho^{-1} \mathbf{Q} \mathbf{n}.
\]

The slowness surface is an ellipsoid. The energy flux velocity\(^20 \) (or wave velocity or group velocity) is

\[
\mathbf{w} = v^{-1} \kappa \mathbf{Q} \rho^{-1} \mathbf{Q} \mathbf{n}.
\]

\( \mathbf{w} \) is in the direction \( \mathbf{Q} \mathbf{n} \), and satisfies \( \mathbf{w} \cdot \mathbf{n} = v \), a well known relation for generally anisotropic solids with isotropic density.
B. Static properties

1. Five easy modes

The static properties of acoustic metafluids are just as interesting, if not more so. Hooke’s law (52) is

$$\sigma = C \varepsilon,$$  \hspace{1cm} (64)

where $\varepsilon = \varepsilon^t$ is strain and the stiffness $C$ is defined by eq. (50). The strain energy is $W = \kappa(Q : \varepsilon)^2$. Note that $C$ is not invertible in the usual sense of fourth order elasticity tensors. If considered as a $6 \times 6$ matrix mapping strain to stress then the stiffness is rank one: it has only one non-zero eigenvalue. This means that there are five independent strains each of which will produce zero stress and zero strain energy, hence the name pen-tamode\textsuperscript{10}. The five “easy” strains are easily identified in terms of the principal directions and eigenvalues of $Q$. Let

$$Q = \lambda_1 q_1 q_1 + \lambda_2 q_2 q_2 + \lambda_3 q_3 q_3,$$ \hspace{1cm} (65)

where $\{q_1, q_2, q_3\}$ is an orthonormal triad. Three of the easy strains are pure shear: $q_i q_j + q_j q_i, \ i \neq j$ and the other two are $\lambda_3 q_3 q_2 - \lambda_2 q_2 q_3$ and $\lambda_1 q_2 q_1 - \lambda_1 q_1 q_2$. Any other zero-energy strain is a linear combination of these. Note that there is no relation analogous to (64) for strain in terms of stress because only the single “component” $Q : \varepsilon$ is relevant, i.e., energetic.

It is possible to write $\sigma$ in the form (52) where $p = -\kappa Q : \varepsilon$. Under static load in the absence of body force choose $\kappa$ such that $p = \text{constant}$, or equivalently, eq. (37). The relevant strain component is then $Q : \varepsilon = -\kappa^{-1}p$ and the surface tractions supporting the body in equilibrium are $t = \sigma \cdot n = -p Q \cdot n$. Figure 4 illustrates the tractions required to maintain a block of metafluid in static equilibrium. Note that the traction vectors act obliquely to the surface, implying that shear forces are necessary. Furthermore, the tractions are not of uniform magnitude. These properties are to be compared with a normal acoustic fluid which can be maintained in static equilibrium by constant hydrostatic pressure.

2. div $Q = 0$

The gedanken experiment of Fig. 4 also implies that $Q$ has to be divergence free. Thus, imagine a smoothly varying but inhomogeneous metafluid $C = \kappa Q \otimes Q$ in static equilibrium under the traction $t = -p Q \cdot n$ for constant $p$. The divergence theorem then implies $\text{div} \ Q = 0$ everywhere in the interior. This argument is a bit simplistic but it provides the basis for a more rigorous proof. Thus, stress in the metafluid must be of the form $f Q$ where $Q$ is a scalar multiple of $Q$. Local equilibrium requires $\text{div} f Q = 0$, or $\nabla \ln f = -Q^{-1} \text{div} Q$. This can be integrated to find $f$ to within a constant. Now define $Q = f Q$, and note that the tractions must be of the form $t = -p Q \cdot n$ for constant $p$. The normalized $Q$ is divergence free.

3. Non-horizontal free surface

Consider the same metafluid in equilibrium under a body force, e.g. gravity. Assuming the inertia is isotropic (cf. the comments about inertia at zero frequency in §VI),

$$\text{div} \ \sigma + \rho g = 0.$$  \hspace{1cm} (66)

Use eq. (52) with $\text{div} Q = 0$ and the invertibility of $Q$, implies

$$\nabla p = \rho Q^{-1} g.$$  \hspace{1cm} (67)

For constant $Q^{-1} \rho g$ this can be integrated to give an explicit form for the pseudopressure,

$$p = (x - x_0) \cdot Q^{-1} \rho g,$$  \hspace{1cm} (68)

where $x_0$ is any point lying on the surface of zero pressure. Unlike normal fluids, the surface where $p = 0$ does not have to be horizontal, see Fig. (5). The pseudopressure increases in the direction of $g$, as in normal fluids. However, it is possible that $p$ varies in the plane $x \cdot g = 0$. For instance, the traction along the lower surface in Fig. (5) decreases in magnitude from left to right.

VI. METAFLUIDS WITH ISOTROPIC DENSITY

A. Necessary constraints on the finite deformation

The most practical case of interest is of course where the initial properties are those of a standard acoustic fluid with isotropic density and isotropic stress. The circumstances under which the mapped inertia $\rho$ is also isotropic are now investigated. Acoustic metafluids with isotropic inertia are an important subset since it can be argued
that achieving anisotropic inertia could be more difficult than the anisotropic elasticity. Indeed, the very concept of anisotropic inertia is meaningless at zero frequency, unlike anisotropic stiffness.

Assuming $\rho_0 = \rho_0 I$ and $Q_0 = I$ then the current density becomes, using eq. (32b) and the fact that $Q$ is symmetric,

$$\rho = \rho_0 \Lambda Q V^{-2} Q. \quad (69)$$

If $\rho = \rho I$ then eq. (69) implies $Q$ must be of the form

$$Q = \rho^{1/2} (\rho_0 \Lambda)^{-1/2} V. \quad (70)$$

Thus, $Q$ is proportional to the stretch tensor $V$ and the coefficient of proportionality defines the current density. It is not in general possible to choose $Q$ in the form

$$Q = \alpha V, \quad (71)$$

where $\alpha \neq 0$. Certainly, $Q$ of (71) is symmetric and invertible but not necessarily divergence free. The latter condition requires $\text{div}(\alpha V) = 0$ which in turn may be expressed $\nabla \ln \alpha = -V^{-1} \text{div} V$. The necessary and sufficient condition that $V^{-1} \text{div} V$ is the gradient of a scalar function, and hence $\alpha$ can be found which makes $Q$ of (71) possible, is that $V$ satisfy

$$\text{curl} \ V^{-1} \text{div} V = 0. \quad (72)$$

This condition is not very useful. It does, however, indicate that the possibility of achieving isotropic $\rho$ depends on the underlying finite deformation; there is a subset of general deformations that can yield isotropic inertia. The deformation gradient $F$ has nine independent elements, $V$ has six, and $R$ has three. The condition (72) is therefore a differential constraint on six parameters. We now demonstrate an alternative statement of the condition $\text{div}(\alpha V) = 0$ in terms of the rotation $R$. This will turn out to be more useful, leading to general forms of potential deformation gradients.

Substituting $F = \alpha^{-1} Q R$ into the identity (19) and using (45) implies that

$$Q_{ij} (\Lambda^{-1} \alpha^{-1} R_{jK})_{,i} = 0. \quad (73)$$

Using $Q = \alpha F R^i$ and the identity $F^i \nabla = \nabla$ along with $\alpha \neq 0$, yields

$$R_{jM} (\Lambda^{-1} \alpha^{-1} R_{jK})_{,M} = 0. \quad (74)$$

Then using the identity $(R_{jK} R_{jM})_{,M} = 0$, eq. (74) yields

$$\beta_{,K} = R_{jK} R_{jM,M} \leftrightarrow \nabla \beta = R^i \text{Div} R^j, \quad (75)$$

where $\beta = -\ln(\Lambda \alpha)$.

The necessary and sufficient condition that (75) can be integrated to find $\beta$ is $\nabla \wedge \nabla \beta = 0$, or using (75)

$$\text{Curl} R^i \text{Div} R^j = 0. \quad (76)$$

The integrability condition (76) is in general not satisfied by $R$, except in trivial cases. Norris\textsuperscript{8} noted that isotropic
density can be obtained if \( R = I \). This corresponds to \( \beta = \text{constant} \), and it can be realized more generally if \( R \) is constant. Hence,

**Lemma 1** If the rotation \( R \) is constant then a normal acoustic fluid can be mapped to a unique metafluid with isotropic inertia:

\[
E_0 = \kappa_0 (\text{Div } U)^2 + \rho_0 \hat{U} \cdot \hat{U} \quad \text{in } \Omega, \tag{77}
\]

is equivalent to the current energy density

\[
E = \lambda (V : \nabla u)^2 + \rho \hat{u} \cdot \hat{u} \quad \text{in } \omega, \tag{78}
\]

where

\[
\lambda = \Lambda^{-1} \kappa_0, \quad \rho = \Lambda^{-1} \rho_0. \tag{79}
\]

The total mass of the deformed region \( \omega \) is the same as the total mass contained in \( \Omega \).

The parameter \( \lambda \) is used to distinguish it from \( \kappa = \Lambda \kappa_0 \), because in this case \( Q = \Lambda^{-1} V \). Also, the displacement fields are related simply by \( \mathbf{u} = RU \).

As an example of a deformation satisfying Lemma 1: \( \mathbf{x} = f(\mathbf{X} \cdot \mathbf{A} \mathbf{X}) \mathbf{A} \mathbf{X} \) for any constant positive definite symmetric \( \mathbf{A} \). This type of finite deformation includes the important cases of radially symmetric cloaks. Thus, Norris\(^8\) showed that radially symmetric cloaks can be achieved using pentamode materials with isotropic inertia.

**B. General condition on the rotation**

The results so far indicate that isotropic inertia is achievable for transformation deformations with constant rotation. We would however like to understand the broader implications of eq. (76). The rotation can be expressed in Euler form

\[
R = \exp(\theta \text{axt} \mathbf{a}), \tag{80}
\]

where \( \theta \) is the angle of rotation, the unit vector \( \mathbf{a} \) is the rotation axis, and the axial tensor \( \text{axt}(\mathbf{a}) \) is a skew-symmetric tensor defined by \( \text{axt}(\mathbf{a}) \mathbf{b} = \mathbf{a} \times \mathbf{b} \). The vector \( \theta \mathbf{a} \) encapsulates the three independent parameters in \( R \). The integrability condition (76) is now replaced with a more explicit one in terms of \( \theta(\mathbf{X}) \) and \( \mathbf{a}(\mathbf{X}) \). It is shown in the Appendix that

\[
\mathbf{R}^t \text{Div } \mathbf{R}^t = \mathbf{a} \times \nabla \theta + \mathbf{Z}, \tag{81}
\]

where the vector \( \mathbf{Z} \) follows from eq. (A6). In particular, \( \mathbf{Z} \) vanishes if the axis of rotation \( \mathbf{a} \) is constant. In general, for arbitrary spatial dependence, (81) implies that the integrability condition (76) is equivalent to the following constraint on the rotation parameters:

\[
\mathbf{a} \nabla^2 \theta - (\mathbf{a} \nabla) \nabla \theta + (\nabla \theta \cdot \nabla) \mathbf{a} - (\nabla \cdot \mathbf{a}) \nabla \theta + \text{Curl } \mathbf{Z} = 0. \tag{82}
\]

In summary,

**Lemma 2** If the rotation \( \mathbf{R} \) satisfies (76), or equivalently, if \( \theta \) and \( \mathbf{a} \) satisfy the condition (82), then a normal acoustic fluid can be mapped to a unique metafluid with isotropic inertia according to eqs. (77) and (78) with

\[
\lambda = \Lambda^{-1} e^{-2\beta} \kappa_0, \quad \rho = \Lambda^{-1} e^{-2\beta} \rho_0, \tag{83}
\]

where the function \( \beta \) is defined by eq. (75).

**C. Simplification in 2D**

The integrability condition (76) simplifies for the important general configuration of two-dimensional spatial dependence. In this case \( \mathbf{a} \) is constant, \( \theta = \hat{\theta}(X_2) \) where \( X_2 \cdot \mathbf{a} = 0 \). Equation (82) then reduces to

\[
\nabla^2 \hat{\theta} = 0, \tag{84}
\]

**1. Example**

Consider finite deformations with inhomogeneous rotation

\[
\theta = \theta_0 + \gamma X_1, \quad \mathbf{a} = \mathbf{e}_3, \tag{85}
\]

for constants \( \theta_0 \) and \( \gamma \). This satisfies (84) and therefore eq. (75) can be integrated. The metafluid in \( \omega \) has isotropic density and pentamode stiffness given by Lemma 2, where \( \beta = \gamma(X_2 - X_{02}) \). The constants \( \theta_0 \) and \( X_{02} \) may be set to zero, with no loss in generality.

As an example of a deformation that has rotation of the form (85), consider deformation of the region \( \Omega = \left[ -\frac{\pi}{2\gamma}, \frac{\pi}{2\gamma} \right] \times [0, L] \times \mathbb{R} \) according to

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} =
\begin{bmatrix}
  A_{11} & A_{12} & 0 \\
  A_{12} & A_{22} & 0 \\
  0 & 0 & \alpha
\end{bmatrix}
\begin{pmatrix}
  \gamma^{-1} \sin \gamma X_1 e^{-\gamma X_2} \\
  \gamma^{-1} \sin \gamma X_1 e^{-\gamma X_2} \\
  X_3
\end{pmatrix}, \tag{86}
\]

where \( \alpha > 0 \) and the \( 2 \times 2 \) symmetric matrix \( \mathbf{A} \) with elements \( A_{ij} \) is positive definite. The deformation gradient is \( \mathbf{F} = \mathbf{V}(X_2) \mathbf{R}(X_1) \) with \( \mathbf{V} = \mathbf{A} e^{-\gamma X_2} + \alpha \mathbf{e}_3 \mathbf{e}_3 \), and

\[
\mathbf{R} =
\begin{bmatrix}
  \cos \gamma X_1 - \sin \gamma X_1 & 0 \\
  \sin \gamma X_1 & \cos \gamma X_1 \\
  0 & 0 & 1
\end{bmatrix}. \tag{87}
\]

The mapped metafluid is defined by the energy density in \( \omega \)

\[
E = \frac{1}{\alpha \text{det } \mathbf{A}} \left[ \kappa_0 \left( \mathbf{A} : \nabla \mathbf{u} \right)^2 e^{-2\gamma X_2} + (\alpha \mathbf{u}_{3,3})^2 \right] + \rho_0 \hat{u} \cdot \hat{u}.
\]

In particular, it has isotropic inertia.

This example is not directly applicable to modeling a complete acoustic cloak. However, it opens up the possibility of patching together metafluids with different local properties, each with isotropic inertia so that the entire cloak has isotropic mass density.
VII. SUMMARY AND CONCLUSION

Whether it is the simple 1D acoustic mirage of Fig. 2 or a three-dimensional acoustic cloak, we have seen how acoustic stealth can be achieved using the concept of domain transformation. The fluid in the transformed region exactly replicates the acoustical properties of the original domain. The most general class of material that describes both the mimic and the mimicked fluids is defined as an acoustic metafluid. A general procedure for mapping/transforming one acoustic metafluid to another has been described in this paper.

The results, particularly Theorem 1 in §IV, show that acoustic metafluids are characterized by as few as two parameters \((\rho, \sqrt{\kappa})\) and as many as twelve \((\rho, \sqrt{\kappa}Q)\). This broad class of materials can be described as pentamode materials with anisotropic inertia. It includes the restricted set of fluids with anisotropic inertia and isotropic stress \((Q = I)\).

The arbitrary nature of the divergence free tensor \(Q\) adds an enormous amount of latitude to the stealth problem. It may be selected in some circumstances to guarantee isotropic inertia in cloaking materials, examples of which are given elsewhere. In this paper we have derived and described the most general conditions required for \(\rho\) to be isotropic. The conditions have been phrased in terms of the rotation part of the deformation, \(R\). If this is a constant then the cloaking metafluid is defined by Lemma 1. Otherwise the condition is eq. (76) with the metafluid given by Lemma 2. The importance of being able to use metafluids with isotropic inertia should not be underestimated. Apart from the fact that it resolves questions of infinite total effective mass and isotropic inertia removes frequency bandwidth issues that would be an intrinsic drawback in materials based on anisotropic inertia.

This paper also describes, for the first time, some of the unusual physical features of acoustic metafluids. Strange effects are to be expected in static equilibrium, as illustrated in Figs. 4 and 5. These properties can be best appreciated in terms of the rotation part of the deformation, \(R\). It is also possible that the large contrasts required in specific cloak design may be underestimated. Apart from the fact that it resolves questions of infinite total effective mass\(^8\), isotropic inertia removes frequency bandwidth issues that would be an intrinsic drawback in materials based on anisotropic inertia.

In addition to the degrees of freedom associated with the tensor \(Q\), the properties of metafluids depend upon the finite deformation through \(V\). Even in the simple example of the 1D mirage, one could arrive at the lower picture in Fig. 2 through different finite deformations. This raises the question of how to best choose the nonunique deformation gradient \(F\). The present results indicate some strategies for choosing \(F\) to ensure the cloak inertia has isotropic mass, and the cloaking properties are in effect determined by the elastic pentamodal material.

Li and Pendry\(^1\) consider other optimal choices for the finite deformation. Combined with the enormous freedom afforded by the arbitrary nature of \(Q\), there are clearly many optimization strategies to be considered.

Acknowledgments

Thanks to the Laboratoire de Mécanique Physique at the Université Bordeaux I for hosting the author, and in particular Dr. A. Shuvalov. And to the reviewers.

APPENDIX A: DERIVATION OF EQ. (81)

Equation (80) can be written in Euler-Rodrigues form\(^21\)

\[
R = I + \sin \theta S + (1 - \cos \theta)S^2, \quad (A1)
\]

where

\[
S \equiv \text{axt} (a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \quad (A2)
\]

Equation (A1) can be used, for instance, to find \(\theta = \cos^{-1}[(1/2)(1 - \text{tr} R)]\) and hence \(a\) from

\[
S = (R - R^T)/(2\sin \theta). \quad \text{For the sake of notational brevity, in the remainder of the Appendix \(c = \cos \theta\) and \(s = \sin \theta\).}
\]

Explicit differentiation of (A1) yields

\[
R_{i,j} = (cS_{i,j} - sP_{i,j})\theta_{j,i} + sS_{i,j} - (1 - c)P_{i,j}, \quad (A3)
\]

where \(P = -S^2 = I - aa\). Noting that \(S_{i,j} = -(\nabla \wedge a)_i\) and \(P_{i,j} = -a_i\nabla \cdot a - a \cdot \nabla a_i\), implies

\[
\text{Div } R' = c a \wedge \nabla \theta - s \nabla \cdot a - s(I - aa) \cdot \nabla \theta + (1 - c)(a \nabla \cdot a) + (a \cdot \nabla) a. \quad (A4)
\]

Multiplying by \(R'\) using (A1), gives after some elimination and simplification,

\[
R' \text{ Div } R' = a \wedge \nabla \theta + Z, \quad (A5)
\]

where

\[
Z = -s[cI + (1 - c)aa] \cdot \nabla \wedge a + (1 - c) a(\nabla \cdot a) - (1 - c)[I - (1 - c)aa] (a \cdot \nabla) a - s(1 - c)a \wedge (a \cdot \nabla) a. \quad (A6)
\]

1. J. Li and J. B. Pendry. Hiding under the carpet: a new strategy for cloaking. http://arxiv.org/abs/0806.4396, Jun 2008.
2. S. A. Cummer and D. Schurig. One path to acoustic cloakung. New J. Phys., 9(3):45+, March 2007.
3. H. Chen and C. T. Chan. Acoustic cloaking in three dimensions using acoustic metamaterials. Appl. Phys. Lett., 91(18):183518+, 2007.
4. L-W Cai and J. Sánchez-Dehesa. Analysis of Cummer-Schurig acoustic cloakung. New J. Phys., 9(12):450+, December 2007.
5 S. A. Cummer, B. I. Popa, D. Schurig, D. R. Smith, J. Pendry, M. Rahm, and A. Starr. Scattering theory derivation of a 3D acoustic cloaking shell. Phys. Rev. Lett., 100(2):024301+, 2008.

6 A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann. Comment on “Scattering theory derivation of a 3D acoustic cloaking shell”. http://arxiv.org/abs/0801.3279v1, Jan 2008.

7 A. N. Norris. Acoustic cloaking in 2D and 3D using finite mass. http://arxiv.org/abs/0802.0701, Feb 2008.

8 A. N. Norris. Acoustic cloaking theory. Proc. R. Soc. A, 464:2411–2434, 2008.

9 The name pentamode is based on the defining property that the material supports five easy modes of infinitesimal strain. See §V for details.

10 G. W. Milton and A. V. Cherkaev. Which elasticity tensors are realizable? J. Eng. Mater. Tech., 117(4):483–493, 1995.

11 G. W. Milton, M. Briane, and J. R. Willis. On cloaking for elasticity and physical equations with a transformation invariant form. New J. Phys., 8:248–267, 2006.

12 Y. Cheng, F. Yang, J. Y. Xu, and X. J. Liu. A multilayer structured acoustic cloak with homogeneous isotropic materials. Appl. Phys. Lett., 92:151913+, April 2008.

13 D. Torrent and J. Sánchez-Dehesa. Anisotropic mass density by two-dimensional acoustic metamaterials. New J. Phys., 10(2):023004+, 2008.

14 D. Torrent and J. Sánchez-Dehesa. Acoustic cloaking in two dimensions: a feasible approach. New J. Phys., 10(6):063015+, June 2008.

15 H-Y Chen, T. Yang, X-D Luo, and H-R Ma. The impedance-matched reduced acoustic cloaking with realizable mass and its layered design. http://arxiv.org/abs/0805.1114, May 2008.

16 U. Leonhardt and T. G. Philbin. Transformation optics and the geometry of light. http://arxiv.org/abs/0805.4778, June 2008.

17 The bijective property of the mapping does not extend to acoustic cloaks, where there is a single point in Ω mapped to a surface in ω, see Norris8 for details.

18 R. W. Ogden. Non-Linear Elastic Deformations. Dover Publications, 1997.

19 A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann. Full-wave invisibility of active devices at all frequencies. Comm. Math. Phys., 275(3):749–789, November 2007.

20 M. J. P. Musgrave. Crystal Acoustics. Acoustical Society of America, New York, 2003.

21 A. N. Norris. Euler-Rodrigues and Cayley formulas for rotation of elasticity tensors. Math. Mech. Solids, 3:243–260, 2008.