Shrinking the Sample Covariance Matrix using Convex Penalties on the Matrix-Log Transformation

David E. Tyler* and Mengxi Yi*
Department of Statistics & Biostatistics
Rutgers, The State University of New Jersey, Piscataway, NJ 08854, U.S.A.
dtyler@stat.rutgers.edu; mengxiyi@stat.rutgers.edu

Abstract

For $q$-dimensional data, penalized versions of the sample covariance matrix are important when the sample size is small or modest relative to $q$. Since the negative log-likelihood under multivariate normal sampling is convex in $\Sigma^{-1}$, the inverse of its covariance matrix, it is common to add to it a penalty which is also convex in $\Sigma^{-1}$. More recently, Deng and Tsui (2013) and Yu et al. (2017) have proposed penalties which are functions of the eigenvalues of $\Sigma$, and are convex in $\log \Sigma$, but not in $\Sigma^{-1}$. The resulting penalized optimization problem is not convex in either $\log \Sigma$ or $\Sigma^{-1}$. In this paper, we note that this optimization problem is geodesically convex in $\Sigma$, which allows us to establish the existence and uniqueness of the corresponding penalized covariance matrices. More generally, we show the equivalence of convexity in $\log \Sigma$ and geodesic convexity for penalties on $\Sigma$ which are strictly functions of their eigenvalues. In addition, when using such penalties, we show that the resulting optimization problem reduces to a $q$-dimensional convex optimization problem on the eigenvalues of $\Sigma$, which can then be readily solved via Newton-Raphson. Finally, we argue that it is better to apply these penalties to the shape matrix $\Sigma/(\det \Sigma)^{1/q}$ rather than to $\Sigma$ itself. A simulation study and an example illustrate the advantages of applying the penalty to the shape matrix.

Keywords: geodesic convexity; M-estimation; Newton-Raphson algorithm; penalized covariance matrices.

1 Introduction and Motivation

For a $q$ dimensional sample $x_1, \ldots, x_n$, the sample covariance matrix $S_n = n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T$ is not well-conditioned and can be highly variable when $q$ is of the same order as $n$. In such cases, one may wish to consider a regularized or a penalized version of the sample covariance matrix. Since the loss function obtained from the negative log likelihood under multivariate normal sampling

$$l(\Sigma; S_n) = \text{tr}(\Sigma^{-1}S_n) + \log \det \Sigma,$$

is convex in $\Sigma^{-1}$, it is natural to consider additive penalties which are also convex in $\Sigma^{-1}$ such as the graphical lasso penalty $\sum_{i \neq j} |(\Sigma^{-1})_{ij}|$ (Yuan and Lin 2007; Friedman et al. 2008). Minimizing the penalized loss function

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$$L(\Sigma; S_n, \eta) = l(\Sigma; S_n) + \eta \Pi(\Sigma),$$
over the set of symmetric positive definite matrices \( \Sigma > 0 \), with \( \Pi(\Sigma) \) being a non-negative penalty function and \( \eta \geq 0 \) being a tuning parameter, is then a convex optimization problem.

More recently, Deng and Tsui (2013) consider the penalty \( \Pi_R(\Sigma) \equiv \| \log \Sigma \|_F \), where the norm refers to the Frobenius norm. This penalty is strictly convex in \( \log \Sigma \) but not in \( \Sigma^{-1} \). By letting \( A = \log \Sigma \), they observe that, when using this penalty, (2) can be express in terms of a penalized loss function over the set of symmetric matrices of order \( q \), namely

\[
\mathcal{L}(A; S_n, \eta) = \text{tr}(e^{-A} S_n) + \text{tr} A + \eta \| A \|_F^2,
\]

with the penalty \( \| A \|_F^2 = \text{tr}(A^2) \) being strictly convex in \( A \). As noted in section 2, the function \( \text{tr}(e^{-A} S_n) \) is not in general a convex function of \( A \), and consequently minimizing (3) over \( A \) does not correspond to a convex optimization problem. Hence, there is no assurance as to the existence and uniqueness of a minimum to (3).

One of our objectives in this paper is to argue that rather than using the concept of convexity in \( \log \Sigma \) in problem (3), a more appropriate setting is based on the notion of geodesic convexity, or \( g \)-convexity for short. The function \( \| \log \Sigma \|_F \) has been well studied within Riemannian geometry and corresponds to the Riemannian or geodesic distance between \( \Sigma \) and the identity matrix (Moakher, 2005; Bhatia, 2009), and is known to be strictly \( g \)-convex in \( \Sigma \). For \( S_n \neq 0 \), the loss function (1) is also strictly \( g \)-convex, and consequently the penalized loss function (2) is strictly \( g \)-convex, when choosing \( \Pi = \Pi_R \). Moreover, (2), with \( \Pi = \Pi_R \), can be shown to be \( g \)-coercive, which implies it has a unique critical point, with this unique critical point corresponding to its global minimum; see Lemmas 2.2 and 2.3.

The concept of \( g \)-convexity can be mathematically challenging, and in practice it can be difficult to prove that a given function is \( g \)-convex. A further contribution of this paper is to show that for an orthogonally invariant penalties, i.e., penalties which are strictly functions of the eigenvalues of \( \Sigma \), as is the case for \( \Pi_R \), (strict) \( g \)-convexity in \( \Sigma \) and (strict) convexity in \( \log \Sigma \) are equivalent. Furthermore, it is shown that \( g \)-convexity for such function reduces to the simpler task of establishing (strict) convexity when viewed as function on the logs of the eigenvalues; see Theorem 3.1. For example, if we express \( \Pi_R(\Sigma) = \sum_{j=1}^q a_j^2 \), where \( a_j = \log \lambda_j \) with \( \lambda_1 \geq \ldots \geq \lambda_q > 0 \) being the eigenvalues of \( \Sigma \), then it is strictly convex as a function of \( a \in \mathbb{R}^q \), and hence \( \Pi_R(\Sigma) \) is strictly convex in \( \log \Sigma \) and strictly \( g \)-convex in \( \Sigma \).

Deng and Tsui (2013) also propose an iterative quadratic programming algorithm over the class of symmetric matrices \( A \) of order \( q \) for finding the minimum of (3). We show in Theorem 3.1 though, that the solution to this problem has the same eigenvectors as \( S_n \). This leads to a simpler algorithm based on finding the minimum of a strictly convex univariate function for each eigenvalue, namely \( g(a; d) \equiv d e^{-a} + a - \eta a^2 \), with \( d \) corresponding to an eigenvalue of \( S_n \) and \( a \) being the corresponding log eigenvalue of \( \Sigma \). The solution to this univariate convex optimization problem can be readily obtained via a Newton-Raphson algorithm.

As recently noted by Yu et al. (2017), the penalty \( \| A \|_F^2 \) shrinks the sample covariance matrix towards the identity matrix. They proposed using the alternative penalty \( \| A - \hat{m} I_q \|_F^2 \), with \( \hat{m} \) being an estimate of the mean of the log of the eigenvalues of \( \Sigma \), i.e. of \( m(A) = \text{tr} A/q \). Since \( \hat{m} \) is first determined from the data, this does not correspond to a pure penalty function for \( \Sigma \). Rather than using a preliminary estimate of \( m \), we propose replacing \( \hat{m} = m(A) \). This approach yields an estimate of \( m(A) \) consistent with the penalized estimate of \( \Sigma \), i.e. \( m(A) = \text{tr} A/q \). The resulting penalized objective function (2), when using the penalty \( \| A - m(A) I_q \|_F^2 = \sum_{j=1}^q (a_j - m_j)^2 \) is shown, within section 4, to also be strictly \( g \)-convex and \( g \)-coercive. Consequently, the global minimum of (2) corresponds to the unique critical point. The solution to this optimization problem reduces to finding the minimum of a strictly convex function in \( \mathbb{R}^q \).

Summarizing, this article is organized as follows. In section 2, the concept of geodesic convexity is briefly reviewed, and results on the existence and uniqueness of penalized sample covariance matrices based
on g-convex penalty functions in general are presented. Results on the relationship between convexity in \( \log \Sigma \) and g-convexity in \( \Sigma \) are given in section [3]. In section [4] convexity results for (2) are given when applying the penalty to the shape matrix \( \Sigma/(\det \Sigma)^{1/2} \) rather than to \( \Sigma \) itself, with \( \| A - m(A)I_q \|_F^2 \) being a special case of such a shape penalty. Algorithms for computing the penalized sample covariance matrices, based on orthogonally invariant g-convex penalties are given in section [5]. We emphasize that this paper treats g-convex penalties in general, with applications to \( \Pi \) treated as a special case. The results of a simulation study discussed in section [6], together with an example given in section [7], demonstrate the advantages of g-convex when \( S_n \neq 0 \), which follows as a special case of Theorem 1 in Zhang et al. (2013), is relatively recent.

The set of symmetric positive definite matrices of order \( q \) can be viewed as a Riemannian manifold with the geodesic path from \( \Sigma_0 > 0 \) to \( \Sigma_1 > 0 \) being given by \( \Sigma_t = \Sigma_0^{1/2}(\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2})^{1/2}\Sigma_0^{1/2} \) for \( 0 \leq t \leq 1 \), see Bhatia (2009) or Wiesel and Zhang (2015) for more details. An alternative representation for this path is given by \( \Sigma_t = Be^{t\Delta}B^T \), where \( \Sigma_0 = BB^T \) and \( \Sigma_1 = Be^\Delta B^T \) with \( \Delta \) being a diagonal matrix of order \( q \). A function \( f(\Sigma) \) is said to be g-convex if and only if \( f(\Sigma_t) \leq (1-t)f(\Sigma_0) + tf(\Sigma_1) \) for \( 0 < t < 1 \), and it is strictly g-convex if strict inequality holds for \( \Sigma_0 \neq \Sigma_1 \). Analogous to convexity in \( \log \Sigma \), for which convexity in \( \log \Sigma^{-1} \), g-convexity in \( \Sigma \) implies g-convexity in \( \Sigma^{-1} \).

As with convexity, any local minimum of a g-convex function is a global minimum, and when differentiable any critical point is a global minimum, with the set of all minima being g-convex. In addition, if a minimum exists, then the minimum is unique when the function is strictly g-convex. Finally, the sum of two g-convex functions is g-convex, and the sum is strictly g-convex if either of the two g-convex summands is strictly g-convex. Consequently, the following lemma holds.

**Lemma 2.1.** If \( \Pi(\Sigma) \) is g-convex and \( S_n \neq 0 \), then \( L(\Sigma; S_n, \eta) \) is strictly g-convex on \( \Sigma > 0 \), and the set of all local minima \( \mathcal{A}_n \) is either empty or contains a single element. That is, if there exists a minimizer \( \hat{\Sigma}_n > 0 \) to \( L(\Sigma; S_n, \eta) \), then it is unique.

The existence of a minima for a g-convex function requires some additional conditions, with a necessary and sufficient condition being that it be geodesic coercive (Dümbgen and Tyler, 2016). A g-convex function \( F(\Sigma) \) is said to be g-coercive if and only if \( F(\Sigma) \to \infty \) as \( \|\log \Sigma\|_F \to \infty \). For \( S_n > 0 \), \( l(\Sigma; S_n) \) is g-coercive and so, since \( \Pi(\Sigma) \) is bounded below, \( L(\Sigma; S_n, \eta) \) is g-coercive and hence has a unique minimizer. Moreover, since a g-convex function is continuous on \( \Sigma > 0 \), it follows that the solution is a continuous function of \( \eta \geq 0 \). This is summarized in the following lemma.

**Lemma 2.2.** Under the conditions of Lemma 2.1, if \( S_n > 0 \), then there exists a unique critical point \( \hat{\Sigma}_n > 0 \) to \( L(\Sigma; S_n, \eta) \), with \( \hat{\Sigma}_n \) being its unique minimizer. Furthermore, \( \hat{\Sigma}_n \) is a continuous function of \( \eta \geq 0 \).

For singular \( S_n \), some conditions on the penalty function are needed since it is possible for \( tr(\Sigma^{-1}S_n) \) to be bounded as \( \log \det \Sigma \to -\infty \), and hence \( l(\Sigma; S_n) \) is not g-coercive in this case. A sufficient condition for \( L(\Sigma; S_n, \eta) \) to be g-coercive when \( S_n \) singular is that \( \Pi(\Sigma) \) be g-coercive and \( \eta > 0 \). This condition, however, is too strong, and does not hold for the scale invariant or shape penalties discussed in section [4].
Some weaker conditions are given in the following lemma, with these conditions holding when $\Pi(\Sigma)$ is g-coercive. Note that under each of the three conditions below, $\Pi(\Sigma) \to \infty$.

**Lemma 2.3.** Under the conditions of Lemma 2.1 if

1. $\Pi(\Sigma) \to \infty$ whenever $|\log \det \Sigma|$ is bounded above and $\|\log \Sigma\|_F \to \infty$,
2. $(\log \det \Sigma)/\Pi(\Sigma) \to 0$ whenever $\log \det \Sigma \to -\infty$ but with $\lambda_j$ bounded away from 0, and
3. $\{\log(\lambda_i/\lambda_q)\}/\Pi(\Sigma)$ is bounded above whenever $\lambda_1 \to 0$ but with $\lambda_1/\lambda_q$ bounded away from 1.

then the conclusions stated in Lemma 2.2 hold when $S_n \neq 0$ is singular and $\eta > 0$.

### 3 Geodesic Convexity and Convexity in Log

In the following, we show that for orthogonally invariant functions, g-convexity in $\Sigma$ is equivalent to convexity in $\log \Sigma$. We say that a function $F$ is (strictly) convex in $\log \Sigma$ whenever $F(\log \Sigma)$ is (strictly) convex in $\Sigma$.

**Theorem 3.1.** For an orthogonally invariant function $F(\Sigma)$, the following three conditions are equivalent:

1. $F(\Sigma)$ is (strictly) g-convex.
2. $F(\Sigma)$ is (strictly) convex in $\log \Sigma$.
3. The corresponding function $f$, as defined in lemma 3.1, is (strictly) convex.

A clarifying point regarding Theorem 3.1 may be helpful. It should be noted, for example, that the corresponding function on $\mathbb{R}^q$ for the log concave function $F(\Sigma) = \log \lambda_q$ is $f(a_1, \ldots, a_q) = \min\{a_1, \ldots, a_q\}$ which is symmetric but concave. Rather, its corresponding function is $f(a_1, \ldots, a_q) = \min\{a_1, \ldots, a_q\}$ which is symmetric but concave.

Outside of orthogonal invariant functions, g-convexity and convexity in log do not necessarily coincide. For example, as previously noted, $l(\Sigma; S_n)$ is strictly g-convex, but not necessarily convex in log. In particular, although $\log \det \Sigma = \text{tr}(A)$ is linear and hence convex in $A$, whether or not the term $\text{tr}(\Sigma^{-1} S_n) = \text{tr}(e^{-A} S_n)$ is convex in $A$ depends on the value of $S_n$. For example, when $S_n = I$, the convexity of $\text{tr}(e^{-A})$ follows from Theorem 3.1 since $\sum_{j=1}^q e^{-a_j}$ is convex. As far as we are aware, general conditions on $S_n$ needed for $\text{tr}(e^{-A} S_n)$ to be convex have not been formally addressed in the literature. An example of $S_n$ for which $\text{tr}(e^{-A} S_n)$ is not convex is given in the appendix. On the other hand, an example of a function which is convex in log but not g-convex is also presented in the appendix.

We now apply these results to the penalty studied by Deng and Tsui (2013), i.e. $\Pi_R(\Sigma) = \|\log \Sigma\|_F^2$. This penalty is orthogonally invariant and can be expressed as $\Pi_R(\Sigma) = \sum_{j=1}^q a_j^2$, which is symmetric and strictly convex as a function of $a \in \mathbb{R}^q$. Hence, by Theorem 3.1 $\Pi_R$ is strictly g-convex, and so Lemma 2.2 holds. Furthermore, Lemma 2.3 also holds since $\log \det \Sigma/\Pi_R(\Sigma) = \{\sum_{j=1}^q a_j\}/\{\sum_{j=1}^q a_j^2\} \to 0$ as $\sum_{j=1}^q a_j \to -\infty$.

The geodesic convexity of $\|\log \Sigma\|_F^2$ has been previously established using more involved proofs, see Bhatia (2009) for comparison. The importance of Theorem 3.1 is that for penalty functions which are strictly functions of the eigenvalues of $\Sigma$, it completely characterizes g-convexity, as well as provides a simple condition for verifying g-convexity. For example, it readily follows that the Kullback-Leibler divergence from the identity matrix, i.e. $\text{tr}(\Sigma^{-1}) + \log \det(\Sigma)$, which is convex in $\Sigma^{-1}$ is also g-convex. The condition number

4
where $m$ and $\lambda_1/\lambda_q$ and the penalty $\text{tr}(\Sigma) + \text{tr}(\Sigma^{-1})$, among others considered by Wiesel (2012) and D"umbgen and Tyler (2016), are also seen to be $g$-convex.

4 Penalizing the shape matrix

Any penalty on $\Sigma > 0$ can also be applied to its shape matrix $V(\Sigma) = \Sigma / \det(\Sigma)^{1/q}$. Here $\det V(\Sigma) = 1$, with the orbits of $V(\Sigma)$ form equivalence classes over $\Sigma > 0$ Paindaveine (2008). This then generates the new penalty $\Pi_s(\Sigma) \equiv \Pi(V(\Sigma))$. If the original penalty is minimized e.g. at $\Sigma = I$, then the new penalty is minimized at any $\Sigma \propto I$. Applying the penalty studied by Deng and Tsui (2013) to the shape matrix yields

$$
\Pi_{R,s}(\Sigma) \equiv \Pi_R\{V(\Sigma)\} = \| \log \Sigma - q^{-1}\{\log \det \Sigma\}I\|_F^2 = \|A - mI\|_F^2,
$$

where $m = q^{-1}\{\log \det \Sigma\} = \text{tr} A/q$. Since $\Pi_{R,s}$ is orthogonally invariant, with $\Pi_{R,s}(\Sigma) = \sum_{j=1}^{q} (\sigma_j - \bar{a})^2$ being convex, it follows from Theorem 3.1 that $\Pi_{R,s}$ is convex in log as well as $g$-convex, although the convexity is not strict in this case. Thus, for non-singular $S_n$, Lemma 2.1 on existence and uniqueness applies when using $\Pi_{R,s}$ as the penalty term. Also, as shown in the appendix, the additional conditions given in Lemma 2.3 needed to assure existence and uniqueness when $S_n$ is singular also holds when using this penalty.

More generally, applying any $g$-convex penalty or penalty which is convex in $\log \Sigma$ to the shape matrix of $\Sigma$, yields respectively a new $g$-convex penalty or penalty convex in $\log \Sigma$. The following theorem applies to any such penalties and does not presume $\Pi$ is orthogonally invariant.

Theorem 4.1.

(i) If $\Pi(\Sigma)$ is $g$-convex, then $\Pi_s(\Sigma)$ is also $g$-convex.

(ii) If $\Pi(\Sigma)$ is convex in $\log \Sigma$, then $\Pi_s(\Sigma)$ is also convex in $\log \Sigma$.

Thus, for $g$-convex $\Pi(\Sigma)$, Lemma 2.1 on existence and uniqueness for the case when $S_n$ is non-singular still applies when the penalty term $\Pi$ is replaced by $\Pi_s$. As another example, if we apply the Kullback-Leibler divergence from the identity to the shape matrix of $\Sigma$, one obtains the penalty $\Pi_s(\Sigma) = \text{tr}\{V(\Sigma)^{-1}\} = q\bar{\lambda}_g / \bar{\lambda}_h$, where $\bar{\lambda}_g$ and $\bar{\lambda}_h$ are respectively the geometric mean and the harmonic mean of the eigenvalues of $\Sigma$. This ratio represents a measure of eccentricity for $\Sigma$, and is minimized at any $\Sigma \propto I$. By the previous theorem, this new penalty is $g$-convex, and hence Lemma 2.1 applies. It can be verified that Lemma 2.3 also applies for this case.

5 Optimizing the penalized loss function

As noted in the introduction, Deng and Tsui (2013) propose a quadratic iterative programming algorithm over $A = \log \Sigma$. The algorithm is derived by a repeated application of the Volterra integral equation for $e^{tA}$ to obtain a second order expansion. Although they state in their introduction that some other previously proposed “methods have retained the use of the eigenvectors of $S_n$ in estimating $\Sigma$ or $\Sigma^{-1}$,” it is not clear if they recognize that the minimum $\hat{A}_n$ to \{3\}, and hence $\hat{\Sigma}_n = e^{\hat{A}_n}$, also retain the same eigenvectors as $S_n$.

As shown in the following theorem, this is true for any orthogonally invariant penalty.

Theorem 5.1. Suppose $\Pi(\Sigma)$ is orthogonally invariant. Using the spectral value decomposition, express $S_n = P_n D_n P_n^T$ with $P_n$ being an orthogonal matrix of order $q$, and where $D_n = \text{diag}\{d_1, \ldots, d_q\}$. Then

$$
L(\Sigma; S_n, \eta) \geq L(P_n\Lambda P_n^T, S_n, \eta),
$$

where $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_q\}$.
This lemma then implies that for orthogonally invariant penalties, the penalized covariance matrix has the form \( \hat{\Sigma}_n = P_n D_n P_n^T \) for some diagonal matrix \( D_n \). In particular, \( D_n = \text{diag}\{e^{\hat{\alpha}_1}, \ldots, e^{\hat{\alpha}_q}\} \) with \( \hat{\alpha}_n \) being the minimizer over \( a \in \mathbb{R}^q \) of

\[
L_q(a; d; \eta) = \sum_{j=1}^q d_j e^{-\alpha_j} + a_j + \eta \pi(a_1, \ldots, a_q). \tag{4}
\]

Here \( \pi \) is the function on \( \mathbb{R}^q \) corresponding to the function \( \Pi \) as defined in Lemma 3.1 and \( d_1 \geq \cdots \geq d_q \) are the eigenvalues of \( S_n \). The function \( L_q(a; d; \eta) \) is strictly convex whenever \( \pi(a) \) is convex in \( a \in \mathbb{R}^q \), which by Theorem 3.1 holds whenever \( \Pi(\Sigma) \) is g-convex, or equivalently convex in \( \log \Sigma \). Thus, for orthogonally invariant g-convex penalties, the minimization problem (2) reduces to the simpler and numerically well studied problem of minimizing a strictly convex function over \( \mathbb{R}^q \).

The penalties proposed by Deng and Tsui (2013) and Yu et al. (2017) are both of the form \( \Pi(\Sigma) = \|A - cI_q\|_F^2 \) with \( c \) not dependent on \( \Sigma \). For these cases, we have \( \pi(a) = \sum_{j=1}^q (a_j - c)^2 \) and so \( L_q(a; d; \eta) = \sum_{j=1}^q d_j e^{-a_j} + a_j + \eta (a_j - c)^2 \). Rather than using their proposed quadratic iterative programing algorithm over the set of symmetric matrices of order \( q \) for this problem, one only needs to solve \( q \) univariate strictly convex optimization problems, namely \( \min \{d_j e^{-\alpha_j} + a_j + \eta (a_j - c)^2 \} \) for \( j = 1, \ldots, q \). The Newton-Raphson algorithm for this problem is simply

\[
a_j \leftarrow a_j + \frac{d_j e^{-\alpha_j} - 2\eta(a_j - c) - 1}{d_j e^{-\alpha_j} + 2\eta} \tag{5}
\]

It can be readily shown that the solution to these \( q \) optimization problems produces \( \hat{\alpha}_{n,1} \geq \cdots \geq \hat{\alpha}_{n,q} \), with the inequalities being strict whenever the corresponding inequalities for the corresponding sample eigenvalues are strict.

For the shape version of this penalty, i.e. \( \Pi_{R,s} \), we have \( \pi_{R,s}(a) = \sum_{j=1}^q (a_j - \bar{a})^2 \) and hence \( L_q(a; d; \eta) = \sum_{j=1}^q d_j e^{-a_j} + a_j + \eta (a_j - \bar{a})^2 \). For this case, the Newton-Raphson algorithm is given by

\[
a_j \leftarrow a_j + \frac{g_j + \beta \sum_{k=1}^q \delta_k}{\delta_j}, \tag{6}
\]

where \( g_j = d_j e^{-\alpha_j} - 1 - 2\eta(a_j - \bar{a}), \delta_j = d_j e^{-\alpha_j} + 2\eta \) and \( \beta = 2\eta q^{-1}/\{1 - 2\eta q^{-1} \sum_{i=1}^q \delta_j^{-1}\} \).

6 Simulation study

In this section, we conduct a simulation study to compare the performance of the following five covariance estimators:

- \( S \): the sample covariance matrix,
- \( \text{LogF} \): the penalized covariance matrix proposed by Deng and Tsui (2013) with penalty \( \|A\|_F^2 \), where \( A = \log \Sigma \),
- \( \text{sLogF} \): our proposed shape penalized covariance matrix based on \( \|A - \{\text{tr}(A)/q\} I_q\|_F^2 \),
- \( \text{mLogF} \): the adjusted penalized covariance matrix proposed by Yu et al. (2017) based on \( \|A - \hat{m} I_q\|_F^2 \), with \( \hat{m} \) being an estimate of \( m(\Sigma) = \text{tr}(A)/q \), and
- \( \text{dLogF} \): an adjusted penalized covariance matrix based on \( \|A - \log(\bar{d}) I_q\|_F^2 \), where \( \bar{d} = \text{tr}(S_n)/q \), i.e. the average of the sample eigenvalues.
Comparisons of LogF and mLogF to other penalized covariance estimators are given in Deng and Tsui (2013) and Yu et al. (2017).

As the tuning constant $\eta \to \infty$, the estimator LogF goes to the identity matrix and so one would anticipate its performance would be poor whenever $\hat{\lambda} = \text{tr}\Sigma/q$ is far from one. This would be particularly problematic when heavy tuning is needed, as would be the case whenever the roots of $\Sigma$ are not well separated or in general when $S_n$ is singular. As noted by Yu et al. (2017), this weakness can be alleviated by using the estimator mLogF. Alternatively, the estimators sLogF or dLogF can be considered. As shown in the appendix, the estimator sLogF goes to $dI_q$ as $\eta \to \infty$. On the other hand, an adjusted estimator, i.e. one using a penalty of the form $\|A - cI_q\|_F^2$, goes to $e^\eta I_q$ as $\eta \to \infty$. Consequently, the estimators mLogF and dLogF go to $e^\eta I_q$ and $dI_q$ respectively as $\eta \to \infty$.

The performance of the estimator mLogF depends on the definition of $\hat{m}$. Yu et al. (2017) observe that the simple choice $\hat{m}_0 = m(S_n) = \text{tr}(S_n)/q$ is known to underestimate $m(A)$. They propose using a bias corrected estimator of the form $\hat{m}_1 = m(S_n) + b_{n,q}$ when $q < n$, and a Bayesian estimator for $\hat{m}_3$ when $q \geq n$; see Yu et al. (2017) for details. We use their proposed choices of $\hat{m}$ in our simulation study. When using $\hat{m}_1$ the estimator mLogF shrinks the eigenvalues of $S_n$ towards $e^{\hat{m}} \propto (\det S_n)^{1/q} = d_q$, the geometric mean of the eigenvalues of $S_n$. We surmise it would be better to shrink them towards the arithmetic mean since $d$ is the minimum variance unbiased estimator of $\lambda$ when random sampling from a spherical multivariate normal distribution with $\Sigma = \lambda I$. In particular, we anticipate our proposed estimators sLogF and dLogF, which both shrink the eigenvalues of $S_n$ towards $d$, will have a better performance in settings where heavy tuning is needed. The results of our simulation study, reported in Table 1, support this heuristic argument.

For the simulations, we consider $q = 60$ dimensional data arising as a random sample from a multivariate normal distribution with mean $\mu = 0$ and covariance matrix $\Sigma = \{\sigma_{ij}\}$. The four different covariance models used in the simulations are listed below, along with the corresponding mean $\bar{\lambda}$ and standard deviation $s_\lambda$ of their eigenvalues.

Model 1: An MA(2) model where $\sigma_{ii} = 10, \sigma_{i,i-1} = \sigma_{i-1,i} = 0.1, \sigma_{i,i-2} = \sigma_{i-2,i} = 0.05$, and $\sigma_{ij} = 0$ otherwise. Here $\bar{\lambda} = 10.0$ and $s_\lambda = 1.16$.

Model 2: An AR(1) model where $\sigma_{ij} = 0.5^{|i-j|} \rho$ and $\rho = 0.3$. Here $\bar{\lambda} = 0.5$ and $s_\lambda = 0.22$.

Model 3: $\Sigma^{-1} = \{\sigma^{ij}\}$ where $\sigma^{ii} = 1$ and $\sigma^{ij} = 0.6$ for $i \neq j$. Here $\bar{\lambda} = 2.46$ and $s_\lambda = 0.32$.

Model 4: $\Sigma = 5 I_q$. Here $\bar{\lambda} = 5.0$, and $s_\lambda = 0.0$.

To evaluate the performance of the different estimators under the various covariance models, four measures of the discrepancy between the estimated covariance matrix and the true $\Sigma$ are computed.

\[ \text{Fnorm: } ||\hat{\Sigma} - \Sigma||_F = \sqrt{\sum_{i,j}(\hat{\sigma}_{ij} - \sigma_{ij})^2}. \]
\[ L_1: ||\hat{\Sigma} - \Sigma||_1 = \max_j \sum_i |\hat{\sigma}_{ij} - \sigma_{ij}|. \]
\[ \text{op-norm: } ||\hat{\Sigma} - \Sigma||_{op} = \max_j |\hat{\sigma}_j|, \text{ where } \hat{\sigma}_j \text{ s are the singular values of } \hat{\Sigma} - \Sigma. \]
\[ \Delta_1: |\hat{\lambda}_1 - \lambda_1|, \text{ the absolute difference between the largest eigenvalues of } \hat{\Sigma} \text{ and } \Sigma, \]

We follow the simulation protocol used by both Deng and Tsui (2013) and Yu et al. (2017). For each covariance model, $2n$ data points are generated, with the first $n$ observations serving as a training set and last $n$ observations serving as a validation set. The tuning parameter for any particular method is selected to be the value of $\eta$ which minimizes the non-penalized loss $l(\hat{\Sigma}_{\eta,1}, \Sigma_{n,1})$ defined by (1), where $\hat{\Sigma}_{\eta,1}$ is the penalized covariance estimate based on the training set, and $\Sigma_{n,1}$ is the sample covariance matrix of the validation set. Since the true mean $\mu = 0$ and interest lies in the performance of the estimators of $\Sigma$, the non-centered sample covariance matrices $S_n = n^{-1} \sum_{i=1}^n x_i x_i^T$ are used in the simulations. We consider three values
for the sample size \( n = 30, 60, 120 \), with the dimension being \( q = 60 \) in each case. The simulations are repeated 100 times and the means and standard deviations (in parenthesis) over the 100 trials for each of the discrepancy measures are reported in Table 1. The means and standard deviations over the 100 trials of the value of the selected tuning parameter \( \eta \) are also reported.

The five estimators in Table 1 are listed by the order of the overall performance over the various models. The estimator which performed the best for a given sample size and given discrepancy measure is noted with an asterisk (*). For every discrepancy measure, our proposed \( s\text{LogF} \) outperforms all the other estimators under models 1 and 4 at every sample size. Under models 2 and 3, with one exception, either \( s\text{LogF} \) or \( d\text{LogF} \) is the best performing estimator, depending on the particular discrepancy measure used. The notable exception is under model 3, for which \( \text{LogF} \) performs best under the operator norm. Overall, the performance of our two proposed estimators \( s\text{LogF} \) and \( d\text{LogF} \) are similar for all four models. The performance of \( m\text{Log} \) is also similar to these two estimators when \( n = 120 \), but performs considerably worse when \( n = 60 \) or \( n = 30 \). As previously surmised, \( s\text{LogF} \) and \( d\text{LogF} \) have particularly better performance than the other estimators whenever their tuning parameters tend to be large. Finally, as suspected, the sample covariance matrix uniformly performs the worse.

7 An example: Sonar data

As an example, we consider the sonar data set obtained from University of California Irvine Machine Learning Repository, which was developed and first analyzed by Gorman and Sejnowski (1988). This data set consists of 208 multivariate observations of dimension \( q = 60 \). For each observation, the 60 variables correspond to the average energy over a particular frequency band obtained by bouncing sonar signals off of an object under various conditions, with 111 observations labeled M (metal cylinder) and the other 97 observations labeled R (rock).

Our goal here is to study the relative performance of covariance estimators when used within linear discriminant analysis (LDA) to classify an observation as either M or R. As in the simulation study, the data set is randomly partitioned into a training set of size 78 for estimating the covariance matrix, a validation set of size 78 for selecting the tuning parameter and a test set of size 52 for computing the misclassification error. The covariance estimators being compared are those considered in the simulation study in section 6. Here, though, the estimators are based on the pooled sample covariance matrix of the two groups M and R.

The above procedure is independently repeated for 100 times. A boxplot of the misclassification errors over these 100 trials are displayed in Figure 1 and the mean and standard deviation of the misclassification errors are showed in Table 2. Finally, Table 3 displays the frequency over the 100 trials that a given estimator (row) has a lower classification rate than another estimator (column). For example, \( s\text{LogF} \) has less misclassification errors than \( m\text{LogF} \) in 46 of the 100 runs, and more misclassification errors in 22 of the runs, with the two estimators having the same misclassification rate in the other 32 runs. Among the estimators of the covariance matrix considered here, our proposed \( s\text{LogF} \) estimator performs best.
Tab. 1: Simulation results for the performance of the five estimators of covariance under Models 1-4 based on four performance measures. Averages and standard deviation are calculated from 100 runs.

| n   | Method | Model 1 | Model 2 |
|-----|--------|---------|---------|
|     |        | L₁      | op-norm |        |        | L₁      | op-norm |
|     |        | Δ₁      | η       |        |        | Δ₁      | η       |
| n = 120 | sLogF | 1.81* 1.09* 0.46* 0.23* 69.83 | 1.54 0.84* 0.46 0.28 2.21 |
|     | dLogF  | 1.87 1.13 0.47 0.23 66.83 | 1.46* 0.99 0.42* 0.15* 1.34 |
|     | mLLogF | 1.98 1.16 0.50 0.24 66.38 | 1.49 0.95 0.42 0.21 1.56 |
|     | LogF   | 41.58 36.42 9.05 7.42 0.13 | 2.29 2.07 0.70 0.45 0.36 |
|     | S      | 55.36 57.46 17.90 17.60 NA | 73.63 12.30 10.01 8.61 NA |
| n = 60 | sLogF | 2.08* 1.15* 0.52* 0.28* 78.29 | 1.64 0.72* 0.46 0.34 5.46 |
|     | dLogF  | 2.22 1.36 0.56 0.28 73.48 | 1.57* 0.94 0.43* 0.19* 2.58 |
|     | mLLogF | 50.83 20.80 9.45 3.18 1.10 | 2.76 2.53 0.88 0.61 0.50 |
|     | LogF   | 78.19 85.01 27.86 27.56 NA | 73.70 13.57 10.13 8.11 NA |
|     | S      | 59.99 21.15 10.28 3.37 NA | 47.52 32.02 9.54 3.65 NA |

| n = 30 | sLogF | 2.67 2.51 2.20 0.16* 17.39 | 0.79* 0.45* 0.17* 0.12* 78.54 |
|     | dLogF  | 2.54 2.41 2.31 0.28 9.80 | 0.65 0.40 0.15 0.14 117.74 |
|     | mLLogF | 6.40 15.75 9.35 6.07 4.13 | 3.20 2.40 1.94 1.10 |
|     | LogF   | 78.19 85.01 27.86 27.56 NA | 73.70 13.57 10.13 8.11 NA |
|     | S      | 59.99 21.15 10.28 3.37 NA | 47.52 32.02 9.54 3.65 NA |

| n =120 | sLogF | 2.45* 2.43 2.31 0.08* 16.05 | 0.61* 0.37* 0.14* 0.12* 124.21 |
|     | dLogF  | 2.45* 2.40* 2.33 0.15 13.48 | 0.65 0.40 0.15 0.14 117.74 |
|     | mLLogF | 2.61 2.51 2.21 0.08 13.91 | 0.72 0.41 0.17 0.16 118.41 |
|     | LogF   | 9.03 5.12 1.82* 0.16 0.83 | 26.31 27.04 8.26 8.26 0.01 |
|     | S      | 59.99 21.15 10.28 3.37 NA | 47.52 32.02 9.54 3.65 NA |

| n = 60 | sLogF | 2.67 2.51 2.20 0.16* 17.39 | 0.79* 0.45* 0.17* 0.12* 78.54 |
|     | dLogF  | 2.54 2.41 2.31 0.28 9.80 | 0.65 0.40 0.15 0.14 117.74 |
|     | mLLogF | 6.40 15.75 9.35 6.07 4.13 | 3.20 2.40 1.94 1.10 |
|     | LogF   | 78.19 85.01 27.86 27.56 NA | 73.70 13.57 10.13 8.11 NA |
|     | S      | 59.99 21.15 10.28 3.37 NA | 47.52 32.02 9.54 3.65 NA |

| n = 30 | sLogF | 2.67 2.51 2.20 0.16* 17.39 | 0.79* 0.45* 0.17* 0.12* 78.54 |
|     | dLogF  | 2.54 2.41 2.31 0.28 9.80 | 0.65 0.40 0.15 0.14 117.74 |
|     | mLLogF | 6.40 15.75 9.35 6.07 4.13 | 3.20 2.40 1.94 1.10 |
|     | LogF   | 78.19 85.01 27.86 27.56 NA | 73.70 13.57 10.13 8.11 NA |
|     | S      | 59.99 21.15 10.28 3.37 NA | 47.52 32.02 9.54 3.65 NA |
Fig. 1: Boxplot of misclassification errors over 100 runs.

Tab. 2: Means and standard deviations of the misclassification error over 100 runs.

|       | sLogF | dLogF | mLogF | LogF  | S   |
|-------|-------|-------|-------|-------|-----|
| Mean  | 0.239 | 0.247 | 0.259 | 0.267 | 0.349 |
| S.D.  | 0.048 | 0.051 | 0.062 | 0.059 | 0.071 |

Tab. 3: Frequency of less misclassifications using row estimator versus column estimator out of 100 runs.

|       | sLogF | dLogF | mLogF | LogF  | S  |
|-------|-------|-------|-------|-------|----|
| sLogF | 0     | 38    | 46    | 60    | 89 |
| dLogF | 28    | 0     | 31    | 56    | 93 |
| mLogF | 22    | 18    | 0     | 54    | 84 |
| LogF  | 26    | 31    | 35    | 0     | 83 |
| S     | 5     | 6     | 12    | 14    | 0  |
8 Appendix: Proofs and some technical details

Counterexamples to the equivalence of g-convexity and convexity in log.

Lemma 1.14 in Wiesel and Zhang (2015) states that \( a^T \Sigma^{-1} x \) is a strictly g-convex function of \( \Sigma \), which implies that \( \text{tr}(\Sigma^{-1} S_n) \) is g-convex for \( S_n \neq 0 \). It is difficult to show analytically whether or not \( \text{tr}(\Sigma^{-1} S_n) \) is convex in \( \log \Sigma \) for a given \( S_n \), and almost all randomly generated counterexamples tend to imply that it true. After extensive trials, though, the following counterexample was found which shows that \( \{\Sigma^{-1}\}_{11} \) is not a convex function of \( \log \Sigma \), and consequently \( \text{tr}(\Sigma^{-1} S_n) \) cannot be convex in \( \log \Sigma \) in general. For \( q = 2 \), let \( A = \log \Sigma \) and choose

\[
A_0 = \begin{bmatrix} 0 & -1 \\ -1 & 300 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 0 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}.
\]

This gives \( \{e^{-0.5 A_1 + 0.5 A_2}\}_{11} = 1.001690296 > 1.001688939 = 0.5 \{e^{-A_0}\}_{11} + 0.5 \{e^{-A_1}\}_{11} \), and so \( \{e^{-A}\}_{11} = \{\Sigma^{-1}\}_{11} \) is not convex in \( A \).

On the other hand, a function may be convex in \( \log \Sigma \) but not g-convex in \( \Sigma \). For example, the matrix \( L_1 \) norm on the elements of \( \log \Sigma \), i.e. \( H(\Sigma) = \max_{1 \leq k \leq q} \sum_{j=1}^{n} |\{\log \Sigma\}_{j,k}| \), is convex in \( \log \Sigma \). The following counterexample, though, shows that it is not g-convex. For \( q = 3 \), choose

\[
\Sigma_0 = \begin{bmatrix} 1.00 & 0.30 & 0.09 \\ 0.30 & 1.00 & 0.30 \\ 0.09 & 0.30 & 1.00 \end{bmatrix} \quad \text{and} \quad \Sigma_1 = \begin{bmatrix} 1.00 & 0.90 & 0.81 \\ 0.90 & 1.00 & 0.90 \\ 0.81 & 0.90 & 1.00 \end{bmatrix}.
\]

This gives \( H(\Sigma_0) = 2.289438 > 2.284073 = 0.5 H(\Sigma_0) + 0.5 H(\Sigma_1) \), and so \( H(\Sigma) \) is not g-convex.

Proof of Lemma 2.3

The lemma follows if \( L(\Sigma; S_n, \eta) \) is g-coercive. Consider any sequence in \( \Sigma \) such that \( \| \log \Sigma \|_F \to \infty \). Divide the proof into the following three cases: a) \( \log \det \Sigma \to \infty \), b) \( \log \det \Sigma \) is bounded above, and c) \( \log \det \Sigma \to -\infty \). For case (a), the result holds since both \( \text{tr}(\Sigma^{-1} S_n) \geq 0 \) and \( \Pi(\Sigma) \geq 0 \). For case (b), the result follows from condition (i) since \( \text{tr}(\Sigma^{-1} S_n) \geq 0 \) and \( \Pi(\Sigma) \to \infty \).

When case (c) holds, consider the two sub-cases: c1) \( \lambda_1 \) is bounded away from zero, and c2) \( \lambda_1 \to 0 \). If (c1) holds, condition (ii) implies \( (\log \det \Sigma) / (\Pi(\Sigma) \to 0 \) and so \( \Pi(\Sigma) \to \infty \). Hence, for \( \eta > 0 \),

\[
L(\Sigma; S_n, \eta) = \text{tr}(\Sigma^{-1} S_n) + \Pi(\Sigma) (\log \det \Sigma) / (\Pi(\Sigma) + \eta) \to \infty.
\]

If (c2) holds, since \( \text{tr}(\Sigma^{-1} S_n) \geq \text{tr}(S_n) / \lambda_1 \) and \( \log \det \Sigma \geq \log \lambda_1 - (q - 1) \log \lambda_q \), it follows that

\[
L(\Sigma; S_n, \eta) \geq \text{tr}(S_n) / \lambda_1 + q \log \lambda_1 + (q - 1) \log (\lambda_q / \lambda_1) + \eta \Pi(\Sigma),
\]

with \( \text{tr}(S_n) / \lambda_1 + q \log \lambda_1 \to \infty \). So, if \( \lambda_1 / \lambda_q \to 1 \), then \( L(\Sigma; S_n, \eta) \to \infty \). Whereas, if \( \lambda_1 / \lambda_q \) is bounded away from one, then by condition (iii), \( (q - 1) \log (\lambda_q / \lambda_1) + \eta \Pi(\Sigma) = \Pi(\Sigma) (\{q - 1\} / (\log (\lambda_q / \lambda_1) / (\Pi(\Sigma) + \eta)) \) is bounded below and so \( L(\Sigma; S_n, \eta) \to \infty \).

Proof of Theorem 3.7

First, we show \( (i) \Rightarrow (iii) \). Suppose that \( F(\Sigma) \) is (strictly) g-convex, then by Lemma 3.6 of Dümbgen and Tyler (2016), \( F(BD(e^x)B^T) \) is (strictly) convex in \( x \in \mathbb{R}^q \setminus \{0\} \) for any non-singular \( B \) of order \( q \). Here, for
$y \in \mathbb{R}^q$, $D(y)$ represents the diagonal matrix with the elements of $y$ corresponding to its diagonal elements. Thus, by Lemma 3.1 $f(x) = F(D(e^x))$ is (strictly) convex.

Next, we show $(iii) \Rightarrow (i)$. Here, the concept of majorization plays an important role. For a vector $v \in \mathbb{R}^q$, denote its ordered values by $v(1) \geq \cdots \geq v(q)$. A vector $y \in \mathbb{R}^q$ is then said to majorize a vector $x \in \mathbb{R}^q$, denoted $x \prec y$ if and only if $\sum_{j=1}^k x(j) \leq \sum_{j=1}^k y(j)$, with equality when $k = q$. As stated in Theorem 1.3 of [Ando] (1957), $x \prec y$ if an only if $x$ is a convex combination of coordinate permutations of $y$, i.e.

$$x \prec y \iff x = \sum_{j=1}^q w_j P_j y,$$  \hspace{1cm} (7)

where, for $j = 1, \ldots, q$, $P_j$ is a permutation matrix of order $q$, hence orthogonal, and $w_j \geq 0$ with $\sum_{j=1}^q w_j = 1$. As a side note, the Birkhoff-von Neumann Theorem notes that $Q$ is a doubly stochastic matrix of order $q$ if and only if it has the representation $Q = \sum_{j=1}^q w_j P_j$. For $\Sigma > 0$, let $\lambda(\Sigma) = (\lambda_1(\Sigma), \ldots, \lambda_q(\Sigma))$ denote the vector of the ordered eigenvalues of $\Sigma$. An important result given by Lemma 2.17 in [Sra and Hosseini] (2015) states

$$\log(\lambda(\Sigma_t)) \prec (1 - t) \log(\lambda(\Sigma_0)) + t \log(\lambda(\Sigma_1)),$$  \hspace{1cm} (8)

where $\Sigma_t$ is the geodesic curve from $\Sigma_0$ and $\Sigma_1$. So, by (7), we can express

$$\log \lambda(\Sigma_t) = Q\{(1 - t) \log \lambda(\Sigma_0) + t \log \lambda(\Sigma_1)\},$$  \hspace{1cm} (9)

with $Q = \sum_{j=1}^q w_j P_j$ being defined as in (7). Thus,

$$F(\Sigma_t) = f(\log \lambda(\Sigma_t)) = f(Q\{(1 - t) \log \lambda(\Sigma_0) + t \log \lambda(\Sigma_1)\})$$

$$\leq (1 - t)f(Q \log \lambda(\Sigma_0)) + tf(Q \log \lambda(\Sigma_1))$$

$$\leq (1 - t) \sum_{j=1}^q w_j f(P_j \log \lambda(\Sigma_0)) + t \sum_{j=1}^q w_j f(P_j \log \lambda(\Sigma_1))$$

$$= (1 - t) \sum_{j=1}^q w_j f(\log \lambda(\Sigma_0)) + t \sum_{j=1}^q w_j f(\log \lambda(\Sigma_1))$$

$$= (1 - t)F(\Sigma_0) + tF(\Sigma_1).$$

The two inequalities above follow from condition $(iii)$, i.e. $f$ is convex. Suppose now that $f$ is strictly convex, then the first inequality is strict unless $\lambda(\Sigma_0) = \lambda(\Sigma_1)$, and the second inequality is strict unless $Q = I$. Thus, both equality holds if and only if $\lambda(\Sigma_t) = \lambda(\Sigma_0)$ for $0 \leq t \leq 1$. However, since $\text{tr}\Sigma$ is strictly g-convex, see e.g. Lemma 1.15 in [Wiesel and Zhang] (2015), it follows that $\text{tr}(\Sigma_t) < (1 - t)\text{tr}(\Sigma_0) + t \text{tr}(\Sigma_1) = \text{tr}(\Sigma_0)$ for $0 < t < 1$, unless $\Sigma_0 = \Sigma_1$. Thus, $F(\Sigma)$ is strictly g-convex.

Finally, we note the statement $(ii) \Leftrightarrow (iii)$ follows from the main theorem in [Davis] (1957), at least in the convex case. The strictly convex case can be shown to hold by applying arguments analogous to those used in the $(i) \Leftrightarrow (iii)$ case.

**Proof that $\Pi_{R,s}$ satisfies the conditions of Lemma 2.3**

Again let $a_j = \log \lambda_j$, and so $\pi = q^{-1} \log \det \Sigma$, $\sum_{j=1}^q a_j^2 = \|\log \Sigma\|^2_F$ and $ss(a) = \sum_{j=1}^q (a_j - \bar{a})^2 = \Pi(\Sigma)$. Condition $(i)$ states that if $|\bar{a}|$ is bounded above and $\sum_{j=1}^q a_j^2 \to \infty$, then $ss(a) \to \infty$, which
holds since \(ss(a) = \sum_{j=1}^{q} a_j^2 - q\sigma^2\). Condition (ii) states that if \(\bar{\sigma} \to -\infty\) and \(a_1\) is bounded below, then \(\bar{\sigma}/ss_a \to 0\). To show this, express \(\bar{\sigma}/ss(a) = \left\{\bar{\sigma}ss(b)\right\}^{-1}\), where \(b_j = a_j/\bar{\sigma}\). Since \(a_1\) is bounded below, \(b_1 \to 0\) and \(\sum_{j=1}^{q} b_j \to q\). Hence, \(ss(b)\) must be bounded away from zero, which implies \(\bar{\sigma}/ss(a) \to 0\). Condition (iii) states that if \(a_1 \to -\infty\) and \(a_1 - a_q \to \epsilon > 0\), then \((a_1 - a_q)/ss(a)\) is bounded above. This follows since \(ss(a) \geq (a_1 - a_q)^2\) and so \((a_1 - a_q)/ss(a) \leq 1/(a_1 - a_q) \leq 1/\epsilon\).

**Proof of Theorem 4.1**

i) Let \(\Sigma_0 \#_t \Sigma_1 := \Sigma_0^{1/2} (\Sigma_0^{-1/2} \Sigma_1 \Sigma_0^{-1/2})^{1/2} \Sigma_1^{1/2}\), \(t \in [0, 1]\), and so \(\Sigma_t = \Sigma_0 \#_t \Sigma_1\) (Sra and Hosseini, 2015). It readily follows that \(V(\Sigma_t) = V(\Sigma_0) \#_t V(\Sigma_1)\), and so

\[
\Pi_s(\Sigma_t) = \Pi\{V(\Sigma_t)\} = \Pi\{V(\Sigma_0) \#_t V(\Sigma_1)\} \\
\leq (1 - t) \Pi\{V(\Sigma_0)\} + t \Pi\{V(\Sigma_1)\} = (1 - t) \Pi_s(\Sigma_0) + t \Pi_s(\Sigma_1).
\]

ii) Let \(A = \log \Sigma\), and define \(G(A) = \Pi(e^A)\) and \(G_s(A) = \Pi_s(e^A)\). The goal is to show that if \(G(A)\) is convex in \(A\), then \(G_s(A)\) is also convex in \(A\). Since \(G_s(A) = G(\hat{A})\), where \(\hat{A} \equiv A - (\text{tr}(A)/q)I\), and so

\[
G_s((1-t)A_0 + tA_1) = G((1-t)\hat{A}_0 + t\hat{A}_1) \\
\leq (1-t)G(A_0) + tG(A_1) = (1-t)G_s(A_0) + tG_s(A_1)
\]

**Proof of Theorem 5.1**

The proof relies on the following well known extremal property of eigenvalues of symmetric matrices. Let \(B\) be a symmetric matrix of order \(q\), and let \(C = [c_1 \cdots c_m]\) be of order \(q \times m, m \leq q\), with orthonormal columns. Then \(\text{tr}\{C^TBC\} = \sum_{j=1}^{m} c_j^T B c_j\) is bounded above and below by the sum of the largest \(m\) and the sum of the smallest \(m\) eigenvalues of \(B\) respectively.

Expressing \(\Sigma = \Pi A \Pi^T\) in terms of its spectral value decomposition, let \(H = [h_1 \cdots h_q] = P^Tn\), which is itself an orthogonal matrix. Define \(\kappa_1 = 1/\lambda_1\) and \(\kappa_j = 1/\lambda_j - 1/\lambda_{j-1}\) for \(j \neq 1\). Inverting this relationship gives \(\lambda_j^{-1} = \sum_{k=1}^{j} \kappa_k\). Since \(\kappa_j \geq 0\), the above noted extremal property of eigenvalues of a symmetric matrix implies

\[
\text{tr}\{\Sigma^{-1} S_n\} = \text{tr}\{\Lambda^{-1} H^T D_n H\} = \sum_{j=1}^{q} \lambda_j^{-1} h_j^T D_n h_j = \sum_{k=1}^{q} \kappa_k \left\{\sum_{j=k}^{q} h_j^T D_n h_j\right\} \\
\geq \sum_{j=1}^{q} \kappa_j \left\{\sum_{k=j}^{q} d_k\right\} = \sum_{j=1}^{q} d_j/\lambda_j = \text{tr}\{\Lambda^{-1} D_n\}
\]

with equality when \(Q = P\). The lemma follows since \(\det \Sigma = \det \Lambda\) and \(\Pi(\Sigma) = \Pi(\Lambda)\).

**Limiting behavior of the sLogF estimator as \(\eta \to \infty\).**

As \(\eta \to \infty\), the penalty term \(\Pi_{R,s}(\hat{\Sigma}_n)\) must go to 0, which implies \(\hat{\Sigma}_n\) is proportional to \(I_q\) in the limit. The eigenvalues of \(\hat{\Sigma}_n\) correspond to the unique critical point of \(\sum_{j=1}^{q} (d_j e^{-a_j} + a_j + \eta (a_j - \bar{\pi})^2)\), where again \(a_j = \log \lambda_j\), which in turn corresponds to the unique solution to the set of equations \(d_j e^{-a_j} = 1 + 2\eta (a_j - \bar{\pi})\) for \(j = 1, \ldots, q\). By taking the sum, we obtain \(q = \sum_{j=1}^{q} d_j e^{-a_j} = \sum_{j=1}^{q} d_j/\lambda_j\) for any \(q \geq 0\). Hence, since the eigenvalues of \(\hat{\Sigma}_n\) approach each other as \(\eta \to \infty\), it follows that \(\bar{\lambda}_j \to d\) or \(\bar{\Sigma}_n \to dI_q\).
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