Fixed-Parameter Tractable Algorithms for Corridor Guarding Problems

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Abstract. Given an orthogonal connected arrangement of line-segments, Minimum Corridor Guarding (MCG) problem asks for an optimal tree/closed walk such that, if a guard moves through the tree/closed walk, all the line-segments are visited by the guard. This problem is referred to as Corridor-MST/Corridor-TSP (CMST/CTSP) for the cases when the guarding walk is a tree/closed walk, respectively. The corresponding decision version of MCG is shown to be NP-Complete[1]. The parameterized version of CMST/CTSP referred to as $k$-CMST/$k$-CTSP, asks for an optimal tree/closed walk on at most $k$ vertices, that visits all the line-segments. Here, vertices correspond to the endpoints and intersection points of the input line-segments. We show that $k$-CMST/$k$-CTSP is fixed-parameter tractable (FPT) with respect to the parameter $k$. Next, we propose a variant of CTSP referred to as Minimum Link CTSP (MLC), in which the link-distance of the closed walk has to be minimized. Here, link-distance refers to the number of links or connected line-segments with same orientation in the walk. We prove that the decision version of MLC is NP-Complete, and show that the parameterized version, namely $b$-MLC, is FPT with respect to the parameter $b$, where $b$ corresponds to the link-distance. We also consider another related problem, the Minimum Corridor Connection (MCC). Given a rectilinear polygon partitioned into rectilinear components or rooms, MCC asks for a minimum length tree along the edges of the partitions, such that every room is incident to at least one vertex of the tree. The decision version of MCC is shown to be NP-Complete[2]. We prove the fixed parameter tractability of the parameterized version of MCC, namely $k$-MCC with respect to the parameter $k$, where $k$ is the number of rooms.

Keywords: Computational geometry, Parameterized complexity, Corridor guarding

1 Introduction

Geometric covering is one of the fundamental problems in computational geometry, in which one has to cover geometric objects (e.g., points, lines, disks, squares or rectangles) with other geometric objects, satisfying some optimization requirements. Most geometric covering problems are NP-hard even in rectilinear (lines/line-segments parallel to $x$ or $y$ axis) domains [3]. In this paper, we focus on one of such problems referred to as corridor guarding problems. Given a connected orthogonal arrangement of line-segments, Minimum Corridor Guarding (MCG) problem asks for an optimal tree/closed walk, such that if a guard moves through the tree/closed walk, all the line-segments are visited by the guard. The length of the tree/closed walk is the total length of edges used by tree/closed walk. If the guarding walk is a tree/closed walk, then the problem is referred to as Corridor-MST/Corridor-TSP (Figure 1). The decision version of Corridor-MST/Corridor-TSP asks whether there exists a tree/closed walk with edge length at most $W$ that visits all the line-segments in $L$, where $W$ is an integer given as part of the input.

Motivated by the applications in VLSI, we pose another related problem referred to as minimum link corridor-TSP (MLC) as follows: Given an orthogonal connected arrangement $L$ of line-segments, find a minimum link-distance closed walk visiting all the line-segments (Figure 2). Decision version of MLC asks whether there exists a closed-walk,
Fig. 1: (a) represents input instance of CMST and CTSP. Bold lines in (b) and (c) represent the tree and closed walk respectively with link-distance at most $b$ that visits all the line-segments in $L$, where $b$ is an integer given as part of the input.

Fig. 2: Input and Output Instances of MLC. (a) represents the input arrangement of line-segments. Bold lines in (b) and (c) represent two closed walks in (a) with link-distance four ($ac, ch, hf$, and $fa$ are the links) and six ($ac, ce_1, d_1, dg, gf$ and $fa$ are the links) respectively. Note that $cee_1$ in (c) is counted as a single link.

Another problem that we consider is the Minimum Corridor Connection (MCC) problem. Given a rectilinear polygon partitioned into rectilinear components/rooms, MCC asks for a minimum length tree such that every room in the polygon is incident to one of the vertices of the tree (Figure 3). The decision version of MCC checks whether there exists a tree with edge length at most $W$, such that at least one of the vertices of every room is incident to the tree, where $W$ is an integer given as part of the input.

Fig. 3: Input and Output instances of MCC. (a) corresponds to the input rectilinear polygon partitioned into rooms. In (b) the bold lines represent the tree visiting all rooms. In (c) the bold lines represent a minimal tree visiting all rooms

Applications [2, 4, 5, 6]: Applications of geometric covering problems are encountered in telecommunications, especially in VLSI design. One of the design issues in layout design of VLSI is to minimize the length of the wire used, which reduces the cost of physical wiring required and also reduces the electrical hazards of having long wires in the interconnection. Another design issue is to reduce the number of links (bends) in a path connecting two points in the board. Signals are transmitted in two layers of wires, with one set
running horizontally and the other vertically. Switching from one direction to another, introduces some resistance along the path. Hence, it is desirable to limit the number of such switching activity. Reducing the number of links has applications in robotics, wireless communications, space travel, and others where turning is a relatively expensive operation.

**Complexity Status:** The decision version of MCG is proved to be NP-Complete [1]. Dumitrescu et al. [7] presented the only polynomial-time approximation algorithm for MCG with a ratio of $O(\log^3 n)$, where $n$ is the number of line-segments in the input. The decision version of MCC is NP-Complete [2]. Bodlaender et al. [2], proved that MCC is strongly NP-hard and provided a subexponential-time exact algorithm. For some special cases depending on the type of rooms, Bodlaender et al. [2] also proposed a polynomial-time constant factor approximation algorithm. To the best of our knowledge, the parameterized complexity of MCG, MLC and MCC are not studied till date.

**Parameterized Complexity** [8,9,10,11]: Parameterized complexity offers another framework for solving NP-hard problems by measuring their running time in terms of one or more parameters, in addition to the input size. A problem with input instance of size $n$, and with a non-negative integer parameter $k$, is fixed-parameter tractable (FPT), if it can be solved by an algorithm that runs in $O(f(k).n^c)$-time, where $f$ is a computable function depending only on $k$, and $c$ is a constant independent of $k$ [10]. A parameterized reduction from a parameterized language $L$ to another parameterized language $L'$ is a function such that, given an instance $(x,k)$, an instance $(x',k')$ is computed in time $f(k).n.O(1)$ such that $(x,k) \in L \iff (x',k') \in L'$. We recommend the interested reader to [10] for a more comprehensive overview of the topic. We refer to the parameterized version of the problems as $k$-CMST/$k$-CTSP, $b$-MLC and $k$-MCC for Corridor-MST/Corridor-TSP, Minimum link CTSP and Minimum Corridor Connection respectively.

| $k$-CMST/$k$-CTSP ($k$-Corridor-MST/$k$-Corridor-TSP) |
|--------------------------------------------------------|
| **Input:** A connected arrangement of line-segments (corridors) $L = \{L_1, L_2, \ldots, L_n\}$, and an integer $k$ |
| **Parameter:** $k$ |
| **Output:** A minimum length tree/closed walk on at most $k$ vertices, along the edges of the corridor, such that all the line-segments are visited. |

| $b$-MLC ($b$-Minimum link Corridor-TSP) |
|----------------------------------------|
| **Input:** A connected arrangement of line-segments (corridors) $L = \{L_1, L_2, \ldots, L_n\}$ with bounded number of intersections $m$ for every line-segment in $L$ & an integer $b$ |
| **Parameter:** $b$ |
| **Output:** A minimum length closed walk on at most $b$ link distance along the edges of the corridor, such that all the line-segments are visited. |

| $k$-MCC ($k$-Minimum Corridor Connection) |
|-------------------------------------------|
| **Input:** A rectilinear polygon $P$ partitioned into $\{P_1, P_2, \ldots, P_k\}$ rectilinear components or rooms. |
| **Parameter:** $k$, The number of partitions or rooms. |
| **Output:** A minimum length tree along the edges of the partitions such that all $k$ rooms are visited. |

In the above definitions, a line-segment $l$ is said to be visited by a tree/walk, if any of the vertices in the tree/walk is incident to one of the endpoints or intersection points created by $l$ with other line-segments. A room is said to be visited by a tree when it is incident to one of the vertices of the tree.

**Our Results:** We investigate the fixed-parameter tractability of Corridor guarding and corridor connection problems (defined above). We show that the decision version of MLC is NP-Complete and devise an FPT algorithm for an input instance with bounded number of intersections, $m$ in each line-segment. The FPT results of MCG, MLC and MCC have been summarized in the following table:
| Problem    | Complexity Status | FPT results                                      |
|------------|-------------------|-------------------------------------------------|
| $k$-CMST   | NP-Complete [1]    | $O^*(2k^k)$ (Section 3.1), $O^*(k(4^k))$ (Section 3.2) |
| $k$-CTSP   | NP-Complete [1]    | $O^*(4k^k)$ (Section 3.1), $O^*((k/2)4^k)$ (Section 3.2) |
| $l$-MLC    | NP-Complete(Section 3.3) | $O^*(m(4(m + 1))^k)$ (Section 3.4) |
| $k$-MCC    | NP-Complete[2]     | $O^*(2^k\log k)$ (Section 3.5)                 |

**Organization of the paper:** In Section 2, we discuss the preliminaries and existing FPT results referred in our proposed algorithms. In Section 3, we present the parameterized reductions and algorithms for the problems under consideration. Conclusion and future work are discussed in Section 4.

## 2 Preliminaries

In this section we define terms and notations that are used in this paper.

**Graphs** [12]: An undirected graph $G$ is defined as a tuple $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set of vertices and $E = \{e_1, e_2, \ldots, e_m\}$ is a finite set of edges where each undirected edge $e_i$ is denoted by $\{u, v\}$ for any $u, v \in V$. A Directed graph (or a digraph) is defined as $G' = (V', E')$ where $V' = \{v_1, v_2, \ldots, v_n\}$, a set of edges $E' = \{e_1, e_2, \ldots, e_m\}$, where each edge is denoted by ordered pair of vertices $(v_i, v_j)$. Let $G'$ be a graph with a vertex set $V'$ and an edge set $E'$. Then, we say that $H$ is a subgraph of $G$ if $V' \subseteq V$ and $E' \subseteq E$. A subgraph of $G$ whose vertex set is $V'$ and whose edge set is the set of those edges of $G$ that have both ends in $V'$ is called the subgraph of $G$ induced by $V'$ (Induced subgraph). A walk in a graph $G$ is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. A vertex may appear more than once in a walk, but edges never repeat. If a walk begins and ends at the same vertex, it is referred to as a Closed Walk. A walk that is not closed is called an open walk. An open walk in which no vertex appears more than once is called a path. A graph $G$ is said to be connected when there is a path between every pair of vertices in $V$. A closed path is also called as a cycle. A tree is a connected graph without any cycles. A rooted tree has a distinguished vertex called root. A directed rooted tree having all its edges pointing away from the root is called a directed out-tree. We refer the reader to [12,13] for the standard definitions and notations related to graphs.

**Connected vertex cover:** A vertex cover $S \subseteq V$ of an undirected graph $G$ is a subset of its vertices such that for every edge $\{u, v\}$ of the graph, either $u$ or $v$ is in $S$. If the vertex cover induces a connected subgraph, then it is referred to as connected vertex cover (CVC). Given a graph $G = (V, E)$ and an integer $k \geq 0$, decision version of CVC checks whether, there exists a vertex cover $C$ for $G$ containing at most $k$ vertices, such that the $C$ induces a connected subgraph and this is proved to be NP-Complete [14].

Two weighted variants of Connected vertex cover namely Tree Cover and Tour Cover [15,16,17] are derived by introducing a weight function $w: E \rightarrow \mathbb{R}^+$, and requires that the cover must induce a connected subgraph satisfying some weight constraints. Both the problems were introduced by Arkin et.al and were proved to be NP-Complete [16] and has a constant 3-factor approximation algorithm. Given a graph $G = (V, E, w)$ where $w: E \rightarrow \mathbb{R}^+$, an integer $l \geq 0$, and a real number $W > 0$, $l$-Tree Cover finds a Tree $T = (V', E')$ of $G$ with $V' \subseteq V$ and $E' \subseteq E$, $V' \leq l$ and $\sum_{e \in E'} w(e) \leq W$ such that $V'$ is a vertex cover for $G$, where $l$ is the parameter. If the problem asks for a closed walk instead of tree, then it is referred to as $l$-Tour Cover. Both $l$-Tree Cover and $l$-Tour Cover were later shown to be FPT by Guo.

**Lemma 1 ([18], Corollary 3).** $l$-Tree Cover and $l$-Tour Cover can be solved in $O((2l)!)$ and $O((4l)!)$-time, respectively.

**Group Steiner Tree (GST):** GST is a generalization of Steiner tree (ST) problem for graphs. ST problem can be defined in a variety of settings [19]. The general version of ST...
is defined as: Given an undirected graph $G = (V, E, w)$ with non-negative edge-weights and a subset of vertices $Y \subseteq V$, usually referred to as terminals, the ST problem in a graph finds a tree of minimum weight that contains all terminals (but may include additional vertices). Dreyfus and Wagner [20] designed a dynamic programming algorithm for ST problem which runs in FPT time with the number of terminals as parameter. Given a connected undirected edge-weighted graph $G = (V, E, w)$ and $k$ disjoint sets of terminals \( \{S_1, S_2, \ldots, S_k\} \), GST finds a minimal tree $T$ such that at least one terminal from each set $S_i$ is in $T$. The decision version of GST, DEC-DST decides whether there exists a tree $T$ of total edge-weight at most $m$ such that at least one terminal from each set $S_i$ is in $T$. DEC-DST is shown to be NP-Complete as it is a generalization of ST problem for graphs [21]. An FPT algorithm for parameterized version of unweighted GST is proposed recently, with the number of terminals as parameter [22].

**Directed Steiner tree (DST):** For DST, the input is a directed graph $D = (V, A)$, a distinguished vertex $r$ called as root, a set of terminals $Y$, and an additional integer $k$. This problem asks for a tree with at most $k$ vertices such that, it has a directed path from $r$ to all the vertices in $Y$. If the output tree needs to be an out-tree, then the problem is referred to as Directed Steiner out-tree. The weighted version of the problem has an additional integer input $m$, and the total edge weight of the output tree has to be at most $m$.

Given a connected undirected graph $G = (V, E, w)$ where $w : E \rightarrow \mathbb{R}^+\$, vertex-disjoint subsets \( \{S_1, S_2, \ldots, S_k\} \) where each $S_i \subseteq V \ \forall \ 1 \leq i \leq k$, and an integer $m$, $k$-edgewt-GST finds a tree in $G$ with total edge-weight at most $m$, that includes at least one vertex from each $S_i \ \forall \ 1 \leq i \leq k$. Parameterized version of Unweighted GST is shown to be in FPT by reducing it to parameterized version of unweighted DST for which, an FPT algorithm using Inclusion-Exclusion principle was proposed [22].

**Lemma 2** ([22], Lemma 1). The unweighted GST problem reduces to unweighted DST problem.

In this paper, we reduce the weighted version of GST, $k$-edgewt-GST to weighted version DST which we refer to as $k$-edgewt-DST. Given a Directed graph $G' = (V', E', w')$ where $w' : E' \rightarrow \mathbb{R}^+$, a distinguished vertex $r \in V'$, a set of terminals $S \subseteq V$ where $|S| = k$ and an integer $m$, $k$-edgewt-DST finds an out-tree with edge-weight at most $m$ in $G'$ that is rooted at $r$ and that contains all the vertices of $S$. In our work, the algorithm we use for $k$-edgewt-DST is essentially the same as that of optimal steiner tree by Fomin et al. [23].

**Lemma 3** ([23], Corollary 1). There is a $O^*(2^{O(k \log k)})$-time algorithm for $k$-edgewt-DST.

**Computational geometry** [21-25] deals with designing algorithms and data structures for problems defined in terms of basic geometrical objects such as points, line-segments and polygons. A line is an unbounded infinite set of points with zero-width that is convex and contains the shortest path between any of the two points in it. Line-segment is a closed subset of a line contained between two points, which are called endpoints. A polygon is a region of plane bounded by a finite collection of line-segments, forming a simple closed curve. The edges of a polygon are the maximal line-segments on the boundary of the polygon. The vertices of a polygon are the intersection points of its edges. In this paper, the lengths mentioned are Euclidean distance. The number of straight line-segments in a path/tour is often referred to as the link-distance or the link length of the path/tour. A covering tour having a minimum link-distance is called a minimum link tour.

**Arrangements:** Let $S$ be a finite set of line-segments in the Euclidean plane. An arrangement of $S$ is defined as the subdivision of the plane induced by these line-segments. It consists of vertices, edges, and faces. To avoid confusion with vertices and edges of a graph, we use the term segment-vertices and segment-edges respectively for vertices and edges of an arrangement. Segment vertices contain all the intersection points and may also contain some of the endpoints. We define segment-degree as the number of segment-edges connected to a segment-vertex.

**Sweep line algorithms** use an imaginary sweep line or sweep surface to solve various problems in Euclidean space. The idea behind algorithms of this type is to imagine that a line
is swept or moved across the plane, stopping at some events defined. A Recilinear Tour on a set of objects (points, lines, segments and polygons) is a closed walk that visits a set of input objects and is made up of rectilinear lines. A lot of work has been done in the area of point covering by rectilinear tour with minimum links [27,28,29,30]. Point covering rectilinear tour of $b$ links or $b$ link point-tour is defined as: Given a set of $n$ points in a plane, is there a rectilinear tour of $b$ links which covers all the points?

**Lemma 4 ([3], Theorem 1).** The decision version of $b$-link point tour is NP-complete.

### 3 Our Results

In this section, we present FPT algorithms for $k$-CMST, $k$-CTSP, $b$-MLC and $k$-MCC. $k$-CMST and $k$-CTSP are proved to be FPT by transforming the input instance to a graph instance, and then using the FPT results of Tree Cover and Tour Cover problem of graphs ([Lemma 1]). Another FPT algorithm is also proposed for $k$-CMST and $k$-CTSP by considering the geometric instance as such. A branching algorithm is proposed for $b$-MLC assuming that each line-segment in the input instance has a bounded number of intersections $m$. $k$-MCC is proved to be FPT by a series of parameterized reductions and then using the FPT result for $k$-edgewt-DST ([Lemma 3]).

#### 3.1 An FPT algorithm for $k$-CMST/$k$-CTSP

Here, we show that the problem of solving $k$-CMST or $k$-CTSP on an instance $(L,k)$, is equivalent to the problem of solving $l$-Tree Cover or $l$-Tour Cover on an instance $(G_{ls},l)$ constructed from the arrangement of line-segments. Before transforming the instance into a graph instance, we preprocess the input by removing those segment-edges, whose removal does not affect the existence or absence of a solution. For the preprocessing of $L$, we have to identify isolated segment-edges and segment-bounding rectangles in $L$ which we define as follows:

**Definition 1.** (Isolated segment-edge): An endpoint of a line-segment which does not intersect with any other line-segment is an isolated endpoint. An edge between an isolated endpoint and an intersection point created by the arrangement, is referred to as an isolated segment-edge (In Figure 4(a), ab is an isolated segment edge).

**Definition 2.** (Segment-bounding rectangle): When two or more horizontal(vertical) line-segments in $L$ is intersected by three or more vertical(horizontal) line-segments in $L$, then the rectangular component formed by the set of topmost and bottommost horizontal line-segments ($h_T$ and $h_B$), and leftmost and rightmost vertical line-segments ($v_L$ and $v_R$) is called a Segment-bounding rectangle (In Figure 4, the bold lines labeled with $h_T$, $h_B$, $v_L$ and $v_R$ shows a segment bounding rectangle).

For identifying isolated segment edges and segment-bounding rectangles, sweep line algorithms can be used.

**Preprocessing:** After identifying the isolated segment-edges and segment-bounding rectangles, we perform the following preprocessing steps on $(L,k)$:

1. Remove isolated segment-edges from the arrangement, if any.
2. Remove those line segments which have both their end-points in the boundary of a segment-bounding rectangle, if any.

Removal of isolated segment-edges do not affect the parameter value $k$ in $k$-CMST/$k$-CTSP. Removal of line-segments inside the segment-bounding rectangle reduces the parameter by the number of line-segments removed i.e. if the number of line-segments removed from $(L,k)$ is $k_1$, the reduced instance becomes $(L',k-k_1)$. Note that removal of one line-segment may affect existence of another segment bounding rectangle. So, the output is dependant on the orientation of sweep lines in sweep line algorithm. To find a minimal tree/closed walk sweep lines are traversed in two directions left to right (vertical sweep line) and top to bottom (horizontal sweep line).
Fig. 4: Figure (a) shows the $k$-CMST input instance and Figure (b) shows the corresponding graph instance after preprocessing. The edge weights in the transformed graph is the Euclidean length of the line-segments.

Let the preprocessed arrangement of line-segments be $L' = \{L_1, L_2, \ldots, L_n'\}$. We transform $L$ into a Line-Segment graph $G_{ls} = \{V, E, w\}$ where $w : E \to \mathbb{R}^+$ by the following steps:

1. Vertex set $V = V_1 \cup V_2$ where $V_1 = \{ ep | ep \text{ corresponds to an endpoint of a line-segment} \}$, and $V_2 = \{ ep_{ij} | ep_{ij} \text{ corresponds to the intersection point made by the line-segments } l_i, l_j \text{ or the endpoint shared by } l_i \text{ and } l_j \}.$

2. The edge set $E$ is constructed as follows: $E = \{ (x_i, y_i) | x_i \text{ and } y_i \text{ are the endpoints of } l_i \text{ or the intersection points } l_i \text{ creates with other line-segments where } l_i \in L' \}.$

For example, if a line-segment $l_j$ with endpoints $ep_1$ and $ep_2$ intersects at $c_1, c_2, \ldots, c_m$ with a series of $m$ line-segments $l_i, l_{i+1}, \ldots, l_m$, then create edges $\{ep_1, c_1\}, \{c_1, c_2\}, \ldots, \{c_m, ep_2\}$. The edges are assigned with weights equal to the corresponding line-segment’s Euclidean length.

Now we show that, solving $k$-CMST/$k$-CTSP on the input instance $(L, k)$ is equivalent to solving the $l$-Tree Cover/$l$-Tour Cover on the instance $(G_{ls}, l)$ where $l = k - k_1$ (The parameter $k$ is updated after preprocessing).

**Lemma 5.** $k$-CMST/$k$-CTSP on an arrangement $L$ has an YES-instance iff $k$-CMST/$k$-CTSP on the preprocessed instance $L'$ has an YES-instance.

**Proof.** Step 1 in the preprocessing removes those segment-edges whose corresponding isolated endpoint is not part of the tree/closed walk. Any minimal tree/tour cover which visits the removed line-segment definitely visits the intersection point and not the isolated endpoint. This is to avoid the length of the isolated segment-edge in tree/closed walk. In step 2, any tree/closed walk that visits the line-segments $h_T, h_B, v_L$ and $v_R$, visits all the line-segments contained in the component formed by them. Hence, the solution for $L'$ is a solution for the original input $L$ of the $k$-CMST/$k$-CTSP problem. Hence, $k$-CMST/$k$-CTSP on an arrangement $L$ has an YES-instance iff $k$-CMST/$k$-CTSP on the preprocessed instance $L'$ has an YES-instance.

**Lemma 6.** $k$-CMST/$k$-CTSP on the preprocessed instance $(L', l)$ has an YES-instance iff $l$-Tree Cover/$l$-Tour Cover on its corresponding $G_{ls}$ has an YES-instance.

**Proof.** It is clear by construction and preprocessing that, each of the line-segments in $L'$ is mapped to one of the edges in $G_{ls}$. So, if there exists a tree/tour visiting all line-segments in $L'$ with at most $l$ vertices, then the corresponding vertices in $G_{ls}$ induces a tree/tour which covers all the edges. Also, if any of the line-segments in $L'$ is not visited by $k$-CMST/$k$-CTSP, then the corresponding edge in $G_{ls}$ is not covered by $l$-Tree Cover/$l$-Tour cover. Hence, there exists a CMST/CTSP on $l$ vertices in $L'$ if and only if there exists a Tree/Tour Cover in $G_{ls}$ of size $l$.

Since the upper bound of $l$ is $O(k)$, $k$-CMST and $k$-CTSP is shown to be FPT as follows:
Theorem 1. $k$-CMST and $k$-CTSP on an arrangement $L$ is FPT with a run-time of $O^*(2k^k)$ and $O^*(4k^k)$ respectively.

Proof. The proof follows from Lemma 15 and 6. Since the sweep line algorithms for preprocessing, and construction of graph instance can be done in polynomial time, $k$-CMST and $k$-CTSP can be solved in $O^*(2k^k)$ and $O^*(4k^k)$ respectively. 

An Improved FPT algorithm for $k$-CMST/$k$-CTSP

In this section, we propose an alternate algorithm for $k$-CMST/$k$-CTSP which considers the geometric instance, rather than converting it to a graph instance. The algorithm uses the following lemma:

Lemma 7. Let $L$ be the connected arrangement of line-segments. If there is line-segment $l$ in $L$ intersected by more than $k$ line-segments, then the instance $(L,k)$ is a NO instance for $k$-CMST. If $l$ is intersected by more than $k/2$ line-segments, then the instance is a NO instance for $k$-CTSP.

Proof. Let $l_i$ be the line-segment intersected by $m$ line-segments, where $m > k$. The proof is obvious since, to visit the $m$ line-segments which intersect $l_i$, an optimal solution of $k$-CMST and $k$-CTSP selects the segment-vertices in the following ways:
1. All the $m$ intersection points in $l_i$
2. any one of the endpoint of $m$ line-segments or
3. the intersection points it makes with line-segments other than $l_i$.
In all the above cases, the number of vertices in the tree/closed walk is greater than $k$. So any input containing $L_i$ will be a NO instance. 

Corollary 1. The maximum intersections possible for a line-segment $l$ in a YES instance of $k$-CMST is $k$, and $k/2$ for $k$-CTSP.

The input for branching algorithm Rec-CMST/CTSP for $k$-CMST/CTSP has an additional argument $v$, which is a segment-vertex in the arrangement. Initially, this vertex is the start vertex for the tree/closed walk. If we start with a vertex which is not part of the tree/closed walk, the branching algorithm may return a NO, even when the input is a YES instance. Hence, the algorithm is invoked for a maximum of $k$ times for $k$-CMST and $k/2$ times for $k$-CTSP (Corollary 1).

Algorithm Rec-CMST/CTSP($L,v,k$)

**Input:** <$L,v,k$>, where $L$ is an orthogonal arrangement of line-segments, $v$ is a segment-vertex with segment-degree $\geq 2$ and $k$ is a non-negative integer

**Output:** $S$, a minimal tree/closed walk on at most $k$ vertices

1: If $L$ has a line-segment which is intersected by more than $k$ or $(k/2)$ line-segments, then return NO for $k$-CMST($k$-CTSP).
2: If all line-segments in $L$ are visited and $S$ is a tree/closed walk with $k\geq 0$,
   then return Yes.
3: If $k<0$ then return No.

**Branching step:**
4: If $L$ contains a line-segment which is not visited
   Branch into at most four cases corresponding to segment edges $(v,u_1),(v,u_2),(v,u_3)$ and $(v,u_4)$ connected to $v$.
   Mark as visited the line-segments which are visited by the selected segment edge.
   update $k = k-1$
   $S_i = S_i \cup (v,u_i)$
   For some $i \in \{1,2,3,4\}$, if Rec-CMST/CTSP($L,u_i,k$) returns Yes,
   then return $S_i$
   $S = S_i$ with min(Euclidean length of segment-edges constituting tree/closed walk)
   return $S$
   else if all instances return No,
   then return No.
**Theorem 2.** There is an $O^*(kA^k)$-time algorithm for $k$-CMST and $O^*((k/2)A^k)$-time algorithm for $k$-CTSP. Consequently, these problems are FPT.

**Proof.** In the proposed algorithm, branching starts with one of the $k$ intersection points in any one of the line-segments for $k$-CMST and $k/2$ for $k$-CTSP respectively (Note that if we start with an isolated end-point, the branching algorithm may return a NO, even when the input is a YES instance. So we restrict the degree of segment vertex to be $\geq 2$). One of the $k$ or $k/2$ points in any line-segment has to be the part of the tree/closed walk since each and every line-segment has to be visited. The maximum depth of the tree with four way branching is $k$, and maximum number of invocations is $k$ or $(k/2)$ (Corollary 1). So, the algorithm runs in time $O^*(kA^k)$ for $k$-CMST and $O^*((k/2)A^k)$ for $k$-CTSP.

**Hardness proof of $b$-MLC**

In this section, we reduce $b$-link point tour to $b$-MLC. Since $b$-link point tour is proven to be NP-Complete (Lemma 4), $b$-MLC is also NP-complete.

**Theorem 3.** $b$-MLC is NP-complete.

**Proof.** To show that the decision version of the problem is in NP, we show that a certificate consisting of sequence of line-segments can be verified in polynomial time, and the verifying algorithms checks if the line-segments in the sequence forms a closed walk, visits all the line-segments, and has at most $b$ link-distance.

To prove that $b$-MLC is NP-hard, we show that $b$-link point tour $\leq_P b$-MLC. Given an instance of covering $n$ by a tour of $b$ links, we construct a $b$-MLC instance as follows: Enclose the points in a rectangular bounding box. Build an orthogonal line arrangement of the points, such that the bounding box includes all the intersection points of the lines in its interior. But it may have the original points in the boundary. The endpoints in the line-segments of $b$-MLC is either one of the original $n$ points, or the intersection points made by the lines with the bounding box. Every point in the input of point covering corresponds to four line-segments in $b$-MLC. So, the instance of $b$-MLC has $4n$ connected line-segments(Figure 5).

![Fig. 5: Example of reduction from point covering by a $b$-link tour to $b$-MLC.](image)

Now, we want to prove that if there is a $b$-link tour connecting the $n$ points, then there is a closed walk visiting all $4n$ line-segments with at most $b$ link-distance. It is obvious from the construction, that each of the line-segments share one of its endpoints with at least one of the $n$ points.

Also, if the input instance is a NO-instance, the reduced instance is also a NO instance. Suppose the points cannot be connected with an $b$-link tour, then any closed walk connecting the line-segments has link-distance more than $b$, since any closed walk of line-segments using the intersection points has more number of links than the tour traversed using $n$ points. The construction of the new instance can be done in polynomial time. Therefore, $b$-link point cover can be reduced to $b$-MLC in polynomial time. Thus, $b$-MLC is NP-complete.

\qed
3.2 An FPT algorithm for $b$-MLC with bounded intersections

In this section, we propose an FPT algorithm for $b$-MLC. We assume that the maximum number of intersections possible in a line-segment is $m$. We first preprocess the input connected arrangement of line-segments $L$ using the steps mentioned in Section 3.1. Note that, here the parameter $b$ is not affected in the preprocessing. The branching algorithm takes an additional argument $v$ which is a segment-vertex in the arrangement. Initially, this vertex is the start vertex of the closed walk. If we start with a segment-vertex which is not part of the closed walk, the algorithm returns a NO, even when the input is a YES instance. Hence, the algorithm is executed a maximum of $m$ times.

**Algorithm Rec-MLC($L,v,b$)**

**Input:** $<L,v,b>$ where $L$ is an orthogonal arrangement of line-segments, $v$ is a segment-vertex with segment-degree $\geq 2$ and $b$ is a non-negative integer

**Output:** $S$, a minimum length closed walk on at most $b$ links.

1. If all line-segments in $L$ are visited and $S$ is a closed walk with $b>=0$, then return Yes.
2. If $b<0$ then return No.

**Branching step:**

3. If $L$ contains a line-segment which is not visited

   Branch into at most $4(m+1)$ links connected to $v$ i.e select $(v, u_i)$ in each branch for $1 \leq i \leq 4(m+1)$

   Mark as visited the line-segments which are visited by the selected link.

   update $b = b - 1$

   $S_i = S_i \cup (v, u_i)$

   For some $i \in \{1, 2, \ldots, 4(m+1)\}$, if Rec-MLC($L, u_i, k$) returns Yes,

   then return $S_i$

   $S=S_i$ with min(Euclidean length of links constituting the closed walk)

   return $S$

   else if all instances return No,

   then return No.

**Theorem 4.** $b$-MLC problem on a connected orthogonal arrangement of line-segments $L$, can be solved in $O^*((m)((4m+4)^b))$-time, where $b$ denotes the link-distance of the closed walk. Consequently it is FPT.

**Proof.** In the branching algorithm described above, the maximum depth of the tree with four branches is $4((m+1)$ (segment degree for a segment-vertex is at most four, and maximum link-distance possible in one of the four orientations is $(m+1)$). The number of invocations is at most $m$, the algorithm runs in time $O^*((m)(4(m+1)^b))$.

3.3 An FPT algorithm for $k$-MCC

In this section, we use two parameterized reductions to prove that $k$-MCC is FPT. Firstly, we transform the instance of $k$-MCC, an orthogonal polygon $P$ partitioned into rectilinear partitions $P = \{P_1, P_2, \ldots, P_k\}$ to a Disj-Polygon-decomp graph $G_{pd} = \{V_{pd}, E_{pd}, w\}$ where $w: E_{pd} \rightarrow \mathbb{R}^+$ by the following steps(Figure 3):

1. The vertex set $V_{pd}$ is constructed as follows: $V_{pd}=V_{s_1} \cup V_{s_2} \cup \ldots, V_{s_k}$ where $V_{s_i}=\{v \mid v$ corresponds to an endpoint of a line-segment $\in P_i$, for $1 \leq i \leq k. \}$

2. The edge set $E_{pd}$ is constructed as follows: $E_{pd}=E_{in} \cup E_{between}$ where $E_{in}=\{(x_i, y_i) \mid x_i$ and $y_i$ are endpoints of line-segments $\in P_i,\}$

   $E_{between}=\{(x_i, y_j) \mid x_i$ and $y_j$ corresponds to an endpoint shared by $P_i$ and $P_j,\}$

3. The weight of edge $e$ corresponding to a line-segment $l_i$ is assigned as follows:

   $w(e) = \begin{cases} 
   \text{Euclidean length of } l_i, & \text{if } e \in E_{in} \\
   0, & \text{if } e \in E_{between} 
   \end{cases}$
Lemma 8. \( (P,k) \) is an Yes instance of \( k\)\(-\text{MCC} \) iff \( (G_{pd}, k) \) is an yes instance of \( k\)-edgewt-GST.

Proof. If there is a corridor connection of length at most \( m \) visiting all \( k \) rooms, the corresponding vertices in \( G_{pd} \) induces a GST of edge-weight at most at most \( m \). If any of the \( k \) rooms is not visited, then none of the terminals in the corresponding group is included in GST. In the other direction, if there is a GST with total edge length at most \( m \), then the corresponding line-segments in MCC form a corridor connection of total length at most \( m \). So, there exists an instance of \( k\)-MCC with length at most \( m \) visiting all \( k \) rooms if and only if there exists an instance in \( k\)-edgewt-GST with edge length \( m \) covering all \( k \) groups of terminals.

Lemma 9. \( k\)-edgewt-GST has a parameter preserving reduction to \( k\)-edgewt-DST.

Proof. The reduction for the weighted version of GST to DST is analogous to the reduction from unweighted-GST to unweighted-DST (Lemma 2). Details have been omitted due to space constraints and are included in the appendix.

Theorem 5. The problem \( k\)-MCC can be solved in \( O^*(2^{O(k \log k)}) \) time and in poly-space. Hence, it is FPT.

Proof. Proof follows from Lemma 3, Lemma 8 and Lemma 9. Thus, \( k\)-MCC is fixed-parameter tractable.

4 Conclusion

In this paper, we investigated the parameterized complexity of Corridor guarding and connection problems, which comes under the class of covering problems in computational geometry. We developed parameterized algorithms for both of the corridor guarding problems (CMST and CTSP) which runs in \( O^*((2k)^k) \) and \( O^*((4k)^k) \)-time respectively. An improved algorithm which considers the geometric instance is also proposed for the same. We proposed a variant of corridor-TSP in which the number of links in the closed walk has to be minimized. We showed that the decision version of the problem is NP-Complete and the parameterized version, \( b\)-MLC is FPT with a run-time of \( O^*(m(4(m + 1))^b) \), where \( b \) is the number of links and \( m \) is the bound on number of intersections in one line-segment. We also showed that \( k\)-MCC is FPT and takes \( O^*(2^{k \log k}) \)-time. Future work includes the notion of incorporating visibility of line-segments or rooms, in addition to the notion of visiting line-segments or rooms. Another direction of work related to MLC problem is finding a tree with minimum number of links or link-diameter.

References

1. Xu, N.: Complexity of minimum corridor guarding problems. Information Processing Letters 112(17-18) (2012) 691–696
2. Bodlaender, H.L., Feremans, C., Grigoriev, A., Penninx, E., Sitters, R., Wolle, T.: On the minimum corridor connection problem and other generalized geometric problems. Computational Geometry 42(9) (2009) 939–951
3. Wang, J., Yao, J., Feng, Q., Chen, J.: Improved fpt algorithms for rectilinear k-links spanning path. In: International Conference on Theory and Applications of Models of Computation, Springer (2012) 560–571
4. Lee, D., Yang, C.D., Wong, C.: On bends and distances of paths among obstacles in two-layer interconnection model. IEEE Transactions on Computers 43(6) (1994) 711–724
5. Feremans, C., Labbé, M., Laporte, G.: Generalized network design problems. European Journal of Operational Research 148(1) (2003) 1–13
6. Wagner, D.P.: Path planning algorithms under the link-distance metric. (2006)
7. Dumitrescu, A., Tóth, C.D.: Watchman tours for polygons with holes. Computational Geometry 45(7) (2012) 326–333
8. Flum, J., Grohe, M.: Parameterized complexity theory. Springer Science & Business Media (2006)
9. Downey, R.G., Fellows, M.R.: Fundamentals of parameterized complexity. Volume 4. Springer (2013)
10. Niedermeier, R.: Invitation to fixed-parameter algorithms. (2006)
11. Cygan, M., Fomin, F.V., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., Saurabh, S.: Parameterized algorithms. Volume 3. Springer (2015)
12. West, D.B., et al.: Introduction to graph theory. Volume 2. Prentice hall Upper Saddle River (2001)
13. Deo, N.: Graph theory with applications to engineering and computer science. Courier Dover Publications (2017)
14. Garey, M.R., Johnson, D.S.: Computers and intractability. Volume 29. WH freeman New York (2002)
15. Alber, J., Gramm, J., Niedermeier, R.: Faster exact algorithms for hard problems: a parameterized point of view. Discrete Mathematics 229(1-3) (2001) 3–27
16. Arkin, E.M., Halldórsson, M.M., Hassin, R.: Approximating the tree and tour covers of a graph. Information Processing Letters 47(6) (1993) 275–282
17. Könemann, J., Konjevod, G., Parekh, O., Sinha, A.: Improved approximations for tour and tree covers. Algorithmica 38(3) (2004) 441–449
18. Guo, J., Niedermeier, R., Wernicke, S.: Parameterized complexity of generalized vertex cover problems. In: Workshop on Algorithms and Data Structures, Springer (2005) 36–48
19. Hauptmann, M., Karpinski, M.: A compendium on steiner tree problems (cit. on p. 25). (2014)
20. Dreyfus, S.E., Wagner, R.A.: The steiner problem in graphs. Networks 1(3) (1971) 195–207
21. Garey, M.R., Graham, R.L., Johnson, D.S.: Some np-complete geometric problems. In: Proceedings of the eighth annual ACM symposium on Theory of computing, ACM (1976) 10–22
22. Misra, N., Philip, G., Raman, V., Saurabh, S., Sikdar, S.: Fpt algorithms for connected feedback vertex set. Journal of Combinatorial Optimization 24(2) (2012) 131–146
23. Fomin, F.V., Grandoni, F., Kratsch, D., Lokshtanov, D., Saurabh, S.: Computing optimal steiner trees in polynomial space. Algorithmica 65(3) (2013) 584–604
24. o’Rourke, J.: Computational geometry in C. Cambridge university press (1998)
25. De Berg, M., Van Kreveld, M., Overmars, M., Schwarzkopf, O.: Computational geometry. In: Computational geometry. Springer (1997) 1–17
26. Preparata, F.P., Shamos, M.I.: Computational geometry: an introduction. Springer Science & Business Media (2012)
27. Heednacram, A.: The NP-hardness of covering points with lines, paths and tours and their tractability with FPT-algorithms. Griffith University (2010)
28. Estivill-Castro, V., Heednacram, A., Suraweera, F.: The rectilinear k-bends tsp. In: International Computing and Combinatorics Conference, Springer (2010) 264–277
29. Wang, J., Tan, P., Yao, J., Feng, Q., Chen, J.: On the minimum link-length rectilinear spanning path problem: complexity and algorithms. IEEE Transactions on Computers 63(12) (2014) 3092–3100
30. Jiang, M.: On covering points with minimum turns. In: Frontiers in Algorithmics and Algorithmic Aspects in Information and Management. Springer (2012) 58–69
31. Mölle, D., Richter, S., Rossmanith, P.: Enumerate and expand: Improved algorithms for connected vertex cover and tree cover. Theory of Computing Systems 43(2) (2008) 234–253
5 Appendix

5.1 FPT Results for \( k \)-Tree Cover and \( k \)-Tour Cover[18]

We revisit the FPT algorithm for \( k \)-CVC proposed by Guo et al., where the parameter is the size of the desired vertex cover.

This algorithm proceeds by enumerating all minimal vertex covers with at most \( k \) vertices. If any one of the \( k \) sized vertex covers is connected, the algorithm terminates by providing the same as minimal connected vertex cover. If none of the vertex covers is connected, the dynamic programming algorithm for Steiner tree (Dreyfus-Wagner [20]) is used to compute a minimal Steiner tree with the vertices in enumerated vertex cover as the terminals. If one minimal vertex cover has a minimal Steiner tree with at most \( k - 1 \) edges, then return the \( k \)-size vertex set of the tree as output; otherwise, the input has no connected vertex cover of size at most \( k \).

Since the Dreyfus-Wagner algorithm for Steiner minimum tree for a set of at most \( k \) terminals runs in \( O(3^k n + 2^k n^2 + n^2 \log n + nm) \)-time where \( n \) is the number of vertices and \( m \) is the number of edges, the algorithm for connected vertex cover takes \( 2^k \) times, since we consider all possible subsets of size \( k \).

Theorem 6 ([18], Theorem 2). \( k \)-CVC is solved in \( O(6^k n + 4^k n^2 + 2^k n^2 \log n + 2^k nm) \)-time.

This algorithm is modified to solve both \( k \)-Tree Cover and \( k \)-Tour Cover problems.

Corollary 2 ([18], Corollary 3). \( k \)-Tree Cover and \( k \)-Tour Cover can be solved in \( O((2k)^k \tilde{A} \tilde{u} km) \) and \( O((4k)^k \tilde{A} \tilde{u} km) \)-time, respectively.

5.2 FPT Results for \( k \)-edgewt-DST

In this section, we show that parameterized version of Weighted Directed Steiner out-tree, \( k \)-edgewt-DST is fixed parameter tractable. For more than 30 years, the fastest parameterized algorithm for the Steiner Tree (ST) problem was the Dreyfus-Wagner dynamic programming algorithm [20]. It uses the Optimal Decomposition Property of Steiner tree, which can be stated as follows: Let \( S \) be any optimal Steiner tree for a graph \( G = (V, E) \) connecting the set of terminals \( Y \), where \( Y \subseteq V \). Let \( q \) be any node of \( Y \). If \( Y \) contains at least 3 terminals, then there exists a node \( p \in V \) and a subset \( D \) of \( Y \) such that, \( S \) consists of three disjoint Steiner subtrees \( S_1, S_2, \) and \( S_3 \), in which \( S_1 \) connects \( \{ p, q \} \), \( S_2 \) connects \( \{ p \} \cup D \) and \( S_3 \) connects \( \{ p \} \cup \{ Y \setminus D \setminus \{ q \} \} \) (Figure 7).

ST for a given graph \( G = X \cup \{ p \} \) can be computed as follows:

\[
ST(X \cup \{ p \}) = \min_{X' \subseteq Y, X'' \subseteq Y} \{ P_{qp} \cup ST(X' \cup \{ p \}) \cup ST(X'' \cup \{ p \}) \}
\]

In equation (1), \( X' = \{ D \}, X'' = \{ Y \setminus D \setminus \{ q \} \} \) and \( P_{qp} \) is the length of the path between \( p \) and \( q \).

Fig.7: Optimal decomposition property of Steiner tree. Here, \( T = \{ q, r, s \} \) and let \( D = \{ r, s \} \). The tree is decomposed into \( ST(\{ p, q \}) \), \( ST(\{ p \} \cup X') \) and \( ST(\{ p \} \cup X'') \) where \( X' = \{ r, s \} \) and \( X'' = \phi \).
In this dynamic programming algorithm, the computation is made in a bottom-up fashion, saving all the partial solutions which are needed for later computations. This approach takes exponential space complexity. To alleviate this problem, equation (1) is applied in top down manner which results in an increased running time. We can avoid this by considering only some selected partitions called roughly balanced partitions.

**Roughly balanced Partitions:** An \( \alpha \)-separator of a graph \( G \) is a set of nodes \( S \), where \( \alpha \) lies between 0 and 1, if the vertex set \( V \setminus S \) can be partitioned into two vertex sets \( V_X \) and \( V_{X'} \) of size at most \( \alpha n \) each, such that no vertex of \( V_{X'} \) is adjacent to any vertex of \( V_X \). A Steiner separator partitions the tree into two partitions \( X' \) and \( X'' \), each one containing at most \( \alpha k \) terminals. Fomin et.al [23] uses the fact that every Steiner tree on a terminal set \( Y \), \( |Y| = k \geq 3 \), has a \( 2/3 \) Steiner separator of cardinality one. Using the Steiner separator, partitions are created which are referred as roughly balanced partitions. Let \( \text{Rough}(s,Y) \) be the set of partitions created by the steiner separator \( s \) for ST with the set of terminals \( Y \).

\[
ST(X \cup \{p\}) = \min_{\forall X' \in \text{Rough}(s,Y)} \{ P_{qp} \cup ST(X' \cup \{p\}) \cup ST(X'' \cup \{p\}) \} \quad (2)
\]

Roughly balanced partitions are useful only when top down approach is used.

**Theorem 7** ([23], Theorem 1). \( k \)-edgewt-DST is solved in \( O(\frac{27}{4})^k n^{O(\log k)} \)-time and polynomial space.

Let \( A \) be the algorithm for \( k \)-edgewt-DST. Due to the \( n^{O(\log k)} \) factor, \( A \) comes under the class of quasi-fpt algorithms. Here, it is sufficient to run \( A \) only when \( k \geq \log n \) (in all other cases Dreyfus-wagner can be run in poly-space). Since the value of \( n \leq 2^k \) in \( A \), the running time can be expressed as \( O^*\left((\frac{27}{4})^k n^{O(\log k)} = O^*\left(2^{O(\log k)}\right) \right) \). So, there is a \( O^*(2^{O(\log k)}) \) polynomial space FPT algorithm for optimal Steiner tree.

**Corollary 3** ([23], Corollary 1). There is a \( O^*(2^{O(\log k)}) \)-time poly-space algorithm for \( k \)-edgewt-DST.

**6 Reduction from \( k \)-edgewt-GST to \( k \)-edgewt-DST**

**Theorem 8.** \( k \)-edgewt-GST has a parameter preserving reduction to \( k \)-edgewt-DST.

**Proof.** Given an instance \( G = (V, E, w), S_1, \ldots, S_k, m > 0 \) of \( k \)-edgewt-GST, an instance of \( k \)-edgewt-DST \( D = (X, A, w') \), \( r, S, m + k + 1 > 0 \) is constructed as follows (Figure 8): Let \( S = \{s_1, s_2, \ldots, s_k\} \) and root \( r \) be the \( (k+1) \) new vertices, where \( r \notin V \) and \( s_i \notin V \) for \( 1 \leq i \leq k \). So, the set of vertices in \( D \) becomes \( X = \{V \cup S\} \). For each edge \( \{u, v\} \) in \( E \), edges \( (u, v) \) and \( (v, u) \) are added in \( A \) with \( w'(uv) = w(uv) = w(uv) \). For every vertex \( v \in S_i \), an arc \( vs_i \) with weight 1 is added to \( A \) and for every \( u \in V \), an arc \( ru \) with weight 1 is added to \( A \).

The size of \( D \) is polynomial in the size of \( G \) (only an addition of \( k+1 \) vertices and \( 2(E+V) \) edges), and hence we can construct \( D \) in time polynomial in the size of \( G \). The parameter \( k \) is preserved in this reduction.

Now we must show that \( G \) has a GST of total edge-weight at most \( m \) only if \( D \) has an out-tree rooted at \( r \) with total edge-weight \( m \). Suppose \( G \) contains a tree \( T \) with edge-weight at most \( m \) that includes at least one vertex from each \( S_i \). Then this tree with the same weight \( m \) is also contained in \( D \) which can be accessed from \( r \) using one of the \( (r, u) \) arc for some \( u \in V \). Thus we have a directed out-tree with edge-weight at most \( m + k + 1 \) containing \( r \) and all vertices in \( S \).

Conversely, let \( T \) be a directed Steiner out-tree with total edge-weight at most \( m + k + 1 \). \( T \) has paths from \( r \) to each of the \( S_i \)‘s. Here, there exists exactly one path from root to each of the terminal set \( S_i \), \( 1 \leq i \leq k \). If two paths are selected from the same group of terminals, the resultant path has a total edge-weight of \( (2k + m + 1) \) which contradicts our assumption of total edge-weight being at most \( m + k + 1 \). If any one of the
Fig. 8: Transformation of instance from weighted GST in $G$ to weighted DST in $D$. Additional $k+1$ vertices $\{s_1, s_2, \ldots, s_k, r\}$ are included in DST instance. For each edge $(u, v)$ in $G$, edges $(u, v)$ and $(v, u)$ with the same edge weights is added in $D$. An arc of length 1 is added from $r$ to all vertices in $S_i$. An arc of length 1 is added from vertices of $S_i$ to corresponding $s_i$, $\forall 1 \leq i \leq k$.

If group $S_i$ is omitted, then $T$ must omit $s_i$. Since $s_i$ is connected to all of the terminals in group $S_i$, it is a contradiction. Thus $G$ contains a tree with edge-weight at most $m$ that includes at least one vertex from each $S_i$, if and only if there exists an out-tree in $D$ with edge-weight at most $m + k + 1$ containing all vertices of $S$, and rooted at $r$. 

\[ \square \]