An inverse problem for a three-dimensional heat equation in bounded regions with several convex cavities

Mishio Kawashita

Abstract

In this paper, an inverse initial-boundary value problem for the heat equation in three dimensions is studied. Assume that a three-dimensional heat conductive body contains several cavities of strictly convex. In the outside boundary of this body, a single pair of the temperature and heat flux is given as an observation datum for the inverse problem. It is found the minimum length of broken paths connecting arbitrary fixed point in the outside, a point on the boundary of the cavities and a point on the outside boundary in this order, if the minimum path is not line segment.

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1 Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^3$ with $C^2$ boundary. Let $D$ be an open subset of $\Omega$ with $C^2$ boundary and satisfy that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. We denote by $\nu_\xi$ and $\nu_y$ the unit outward normal vectors at $\xi \in \partial D$ and $y \in \partial \Omega$ on $\partial D$ and $\partial \Omega$ respectively.

Let $T > 0$ be a fixed constant and $\rho$ be a continuous function on $\partial D$. Consider the following initial boundary value problem of the usual heat equation:

$$
\begin{align*}
(\partial_t - \Delta)u(t, x) &= 0 \quad \text{in } (0, T) \times \Omega \setminus \overline{D}, \\
\partial_\nu u(t, x) &= f(t, x) \quad \text{on } (0, T) \times \partial \Omega, \\
(\partial_\nu + \rho(x))u(t, x) &= 0 \quad \text{on } (0, T) \times \partial D, \\
u(0, x) &= 0 \quad \text{on } \Omega \setminus \overline{D},
\end{align*}
$$

where $\partial_\nu = \sum_{j=1}^{3}(\nu_x)_j \partial_{x_j}$ for $x \in \partial D \cup \partial \Omega$.

Mathematical studies on inverse problems arising thermal imaging are formulated as the boundary inverse problems for the usual heat equation. In this inverse problem, pairs of the measurement data $(u, f)$ on the outside boundary, i.e. the temperature $u$ and the heat flux $f$ on $(0, T) \times \partial \Omega$, are given as observation data. The problem is to understand what information on $\partial D$ can be extracted by using these data on the outside boundary.

Elayyan and Isakov [4] investigated the uniqueness problem corresponding to this type of inverse problem, which determine $D$ and $\rho$ uniquely by using infinitely many observation data. In this paper,
the case that infinitely many observation data are used to obtain inside information is called by “infinite measurement”. The completely opposite case to infinite measurement is “one measurement”. This is the case that only one pair of the observation data \((u, f)\) is allowed to use as the observation data for inverse problem. In one measurement case, as in Bryan and Caudill [1], the uniqueness results fail if the initial condition does not vanish. Hence, to handle one measurement case, as in (1.1), we need to assume that \(u(0, x) = 0\) on \(\Omega\).

Other important problems are for stability and reconstruction. The stability problem is to show continuous properties between the observation data and the unknown objects \((D, \rho)\). For stability problems, see Vessella [15], and references therein.

In this paper, the problems concerning reconstruction procedure, that is to find information on \(D\) or \(\rho\) from the observation data, are treated. For this problem, several methods are proposed by many authors. For a one space dimensional case, Daido, Kang, and Nakamura [2] gives an approach for this type of inverse problem by using an analogue of the probe method introduced by Ikehata [7]. This procedure is numerically simulated by Daido, Lei, Liu and Nakamura [3].

Various approaches for reconstruction procedures are proposed, however, there are relationships among them although the formulations are not similar to each other. These relations are found by Honda, Nakamura, Potthast and Sini [6]. In the author’s best knowledge, only the enclosure method is different from many of other approaches. Thus, it is worth investigating inverse problems by the enclosure method.

The enclosure method is originally developed by Ikehata [8] and [9] for static problems formulated by elliptic boundary value problems. About boundary inverse problems for the heat equations, infinite measurement cases are treated by [10]. In [11], the case of inclusions (i.e. the case that \(D\) is filled by other medium) is considered. In the case that the inside boundary of the inclusion may depend on time variable \(t\) and is strictly convex for all \(t\), Gaitan, Isozaki, Poisson, Siltanen and Tamminen [5] also investigated by using a similar approach to the enclosure method.

Usually, to give reconstruction procedure, functions called “indicator” defined by using the observation data are introduced. From asymptotic behaviors of indicator functions, one can obtain informations for the inside. In [10] and [11], \(h_D(\omega) = \sup_{x \in \partial D} x \cdot \omega\) (\(\omega \in S^2\) for the three dimensional case), \(d_D(p) = \inf_{x \in D} |x - p|\) (\(p \in \mathbb{R}^3 \setminus \overline{\Omega}\)), and \(R_D(q) = \sup_{x \in D} |x - q|\) (\(q \in \mathbb{R}^3\)) are extracted. Hence \(D\) is enclosed by the sets such \(\cap_{\omega \in S^2} \{x \in \mathbb{R}^3 | x \cdot \omega < h_D(\omega)\}\) as \(\cap_{p \in \mathbb{R}^3} \{x \in \mathbb{R}^3 | |x - p| > d_D(p)\}\) and \(\cap_{q \in \mathbb{R}^3} \{x \in \mathbb{R}^3 | x - q < R_D(q)\}\), which are the origin of the word “the enclosure method” as introduced in [8] and [9].

As stated in [13], infinite observation cases are different from a one measurement case. This comes from how to choose the indicator functions \(I_\tau\), which contain a (real or complex) large parameter \(\tau\) from the observation data \((u, f)\) on \((0, T) \times \partial \Omega\). We take functions \(v_\tau(t, x)\) with large parameter \(\tau\) satisfying \((\partial_t + \nabla) v = 0\) in \((0, T) \times \partial \Omega\). From these functions, \(I_\tau\) is defined by

\[
I_\tau = \int_{\partial \Omega} \int_0^T (\partial_{\nu_\tau} v_\tau(t, y) u(t, y) - v_\tau(t, y) f(t, y)) dt dS_y. \tag{1.2}
\]

For infinite measurement cases, the boundary data \(f\) in (1.1) can be changed as it suits for \(\partial_\nu v_\tau\) on \((0, T) \times \partial \Omega\). Thus, the observation data \((u, f)\) are designed to obtain information of \(\partial D\). Hence, as above, various amounts related to \(\partial D\) are obtained. Note that most of the works stated above are for infinite measurement cases.

For one measurement case, only one pair of the observation data \((u, f)\) is given. This means that we can not design the indicator functions like as infinite measurement cases. Only we can do is to choose \(v_\tau(t, x)\) for given \(f\). One possibility of a choice of \(v_\tau(t, x)\) is to take \(v_\tau(t, x) = e^{-\tau^2 t} q(x; \tau)\), where \(q(x, \tau)\) is the solution of

\[
\begin{align*}
\Delta q - \tau^2 q &= 0 \quad \text{in } \Omega, \\
\partial_\nu q(y; \tau) &= \int_0^T e^{-\tau^2 t} f(t, y) dt \quad \text{on } \partial \Omega.
\end{align*}
\]

In [11], by using \(I_\tau\) with this choice of \(v_\tau(t, x)\), \(\text{dist}(D, \partial \Omega) = \inf \{|x - y| | x \in D, y \in \partial \Omega\}\) is extracted.
Another idea choosing \(v_\tau(t,x)\) is to put \(v_\tau(t,x) = e^{-\tau^2}q(x;\tau)\) with functions \(q(x;\tau)\) independent of \(f\) and satisfying \((\Delta - \tau^2)q(x;\tau) = 0\) in \(\Omega\). For a fixed \(p \in \mathbb{R}^3 \setminus \Omega\) arbitrary taken, we put \(q(x;\tau) = e^{-\tau|p-p|/(2\pi|x-p|)}\). This is a good example of \(q(x;\tau)\). In [13], it is shown that asymptotic behaviors of the indicator function \(I_\tau\) defined in (1.2) by using this function \(q(x;\tau)\) give

\[
I(p,D) = \inf_{(\xi,y) \in \partial D \times \partial \Omega} l_p(\xi, y),
\]

where

\[
l_p(\xi, y) = |p - \xi| + |\xi - y|, \quad (\xi, y) \in \mathbb{R}^3 \times \mathbb{R}^3.
\]

In [13], however, we need to assume strict convexity of \(\partial D\). In this paper, we treat the case that \(D\) consists of several strictly convex cavities.

To describe main theorems, we introduce notations. Take an arbitrary point \(x \in \partial D\). Throughout this paper, we put the following assumptions on \(\partial D\):

1. \(\partial D\) consists of several disjoint convex domains, namely \(D = \bigcup_{j=1}^N D_j\), where \(D_j\) \((j = 1, 2, \ldots, N)\) are disjoint domains of strictly convex with the boundaries \(\partial D_j\) of class \(C^2\).

2. \(\mathcal{M}_g(p) \cup \mathcal{M}_g^+(p) = \emptyset\).

Now we state what is obtained from the indicator function \(I(\lambda, p)\) for \(\lambda \geq 0\). We set \(g(y;\lambda)\)

\[
g(y; \lambda) = \int_0^T e^{-\lambda^2 t} f(t,y) dt \quad (y \in \partial \Omega, \lambda \in \mathbb{C}_\delta). \tag{1.4}
\]
Theorem 1.1 Assume that ∂Ω and ∂D are class of $C^2$ satisfying (I.1) and (I.2). Assume also that $f \in L^2((0, T) \times \partial \Omega)$, and there exists a constant $\beta_0 \in \mathbb{R}$ such that the function $g(y; \lambda)$ defined by (1.4) belongs to $C(\partial \Omega)$ for all $\lambda \in \mathbb{C}_{\delta_0}$ with large $|\lambda|$ and satisfies

$$0 < \inf_{y \in \partial \Omega} \lim_{|\lambda| \to \infty} \inf \text{Re} |\lambda^{\beta_0} g(y; \lambda)| \leq \limsup_{|\lambda| \to \infty} |\lambda|^{\beta_0} \|g(\cdot; \lambda)\|_{C(\partial \Omega)} < \infty$$

\[ \text{(1.5)} \]

(\text{uniformly in } \lambda \in \mathbb{C}_{\delta_0}).

Then there exists a sufficiently small $\delta_1 > 0$ such that

$$\lim_{|\lambda| \to \infty} \frac{1}{\lambda} \log |I(\lambda, p)| = -l(p, D) \quad \text{(uniformly in } \lambda \in \Lambda_{\delta_1}).$$

(1.6)

Remark 1.2 (1) There exist many $f \in L^2((0, T) \times \partial \Omega)$ satisfying (1.5). Take $f \in C^1([0, T]; C(\partial \Omega))$ with $\inf_{y \in \partial \Omega} f(0, y) > 0$. As is in Remark 1 of [13], integration by parts implies that

$$\|\lambda^2 g(\cdot; \lambda) - f(0, \cdot)\|_{C(\partial \Omega)} \leq \frac{\max_{0 \leq t \leq T} \|\partial_t f(t, \cdot)\|_{C(\partial \Omega)}}{\mu(1 - \delta_0^2)} \quad (\lambda \in \mathbb{C}_{\delta_0}).$$

Note that $\delta_0$ is taken as $0 < \delta_0 < 1$. Thus, this $f$ satisfies (1.5) with $\beta_0 = 2$.

(2) As is in (4) of Proposition 2 in p.1090 of [13], if $(\xi_0, y_0) \in M_1(p) \cup M_2^+(p) \cup M_2^-(p)$, the points $p$, $\xi_0$ and $y_0$ consist of a line, and the point $\xi_0$ is on the line segment connecting $p$ and $y_0$. Hence, for $(\xi_0, y_0) \in M_2^+(p)$, there exists a point $(\xi_1, y_1) \in M_2^+(p)$ respectively. If $\partial D$ itself is strictly convex, this point $(\xi_1, y_1)$ is uniquely determined, however, for non-strictly convex $\partial D$, it is possible to be several points satisfying $(\xi_1, y_1) \in M_2^+(p)$. In any case, $M_2^+(p) = \emptyset$ if and only if $M_2^+(p) = \emptyset$. Thus, $M_2(p) \cup M_2^+(p) \cup M_2^- = \emptyset$ if (I.2) is assumed.

(3) A sufficient condition that (I.2) holds is given in Proposition 4 of [13]. Note that strict convexity of $\partial D$ does not used to show Proposition 4 of [13].

(4) In Theorem 1.1, it does not assume that $l_p(\xi, y)$ is non-degenerate at $M_1(p)$ (see (I.3) below for the precise description). In this sense, Theorem 1.1 is better than the main result in [13], since in [13], non-degenerate assumptions are also assumed even if the case that $\partial D$ consists of one strictly convex surface.

Formula (1.6) holds only for $\lambda \in \Lambda_{\delta_1}$. This can be improved for $\lambda \in \mathbb{C}_{\delta_0}$ if we add the following assumption:

(I.3) Every point $(\xi_0, y_0) \in \partial D \times \partial \Omega$ attaining $l(p, D)$ is non-degenerate critical point of $l_p(\xi, y)$.

Note that as introduced in (4) of Remark 1.2, (I.3) and strict convexity of $\partial D$ are always assumed in [13].

Theorem 1.3 Assume that $\partial \Omega$ and $\partial D$ satisfy (I.1), (I.2) and (I.3). Assume also that $f \in L^2((0, T) \times \partial \Omega)$, and there exists a constant $\beta_0 \in \mathbb{R}$ such that the function $g(y; \lambda)$ defined by (1.4) belongs to $C(\partial \Omega)$ for all $\lambda \in \mathbb{C}_{\delta_0}$ with large $|\lambda|$ and satisfies (1.5) for some $\beta_0 \in \mathbb{R}$. Further, assume that $\lambda^{\beta_0} g(y; \lambda)$ is uniformly continuous in $y \in \partial \Omega$ with respect to $\lambda \in \mathbb{C}_{\delta_0}$. Then,

$$\lim_{|\lambda| \to \infty} \frac{1}{\lambda} \log |I(\lambda, p)| = -l(p, D) \quad \text{(uniformly in } \lambda \in \mathbb{C}_{\delta_0}).$$

(1.6)

Remark 1.4 (1) Assumption (I.3) in Theorem 1.3 is used to obtain an asymptotic behavior of $I(\lambda, p)$ as $|\lambda| \to \infty$ uniformly in $\lambda \in \mathbb{C}_{\delta_0}$. In this sense, for non-degenerate case, we can say that the asymptotic behavior is better.
\(2\) If \(\partial D\) and \(\partial \Omega\) are \(C^{2,\alpha_0}\) for some \(0 < \alpha_0 < 1\), and \(g(\cdot; \lambda) \in C^{0,\alpha_0}(\partial \Omega)\), \(I(p, \lambda)\) has the following asymptotics:

\[
I(\lambda, p) = \frac{1}{\lambda} e^{-\lambda(p;D)} \left\{ A(\lambda, p)g + \|g(\cdot; \lambda)\|_{C^{0,\alpha_0}(\partial \Omega)} O(\lambda^{-\alpha_0}/2) \right\} + O(\lambda^{-\frac{3}{2}} e^{-\lambda^2 T})
\]

as \(\lambda \to \infty\) uniformly with \(\lambda \in \mathbb{C}_{\delta_0}\) for each \(\delta_0 > 0\), where

\[
A(\lambda, p)g = \sum_{(\xi_0, y_0) \in \mathcal{M}_1(p)} C(\xi_0, y_0) H^+(\xi_0, y_0, p) g(y_0, \lambda).
\]

In the above, \(C(\xi_0, y_0)\) for each \((\xi_0, y_0) \in \mathcal{M}_1(p)\) is a positive constant independent of \(g\) and

\[
H^+(\xi, y, p) = \frac{1}{|\xi - p||\xi - y|} \nu_\xi \cdot \left\{ \frac{p - \xi}{|p - \xi|} + \frac{y - \xi}{|y - \xi|} \right\}, (\xi, y) \in \partial \Omega \times \partial \Omega.
\]

This is the same asymptotic formula as in [13] for the case of one strictly convex cavity. Note that \((\xi, y) \in \mathcal{M}_1(p)\) means that \(\nu_\xi \cdot (p - \xi) > 0\) and \(\nu_\xi \cdot (y - \xi) > 0\), which yields

\[
H^+(\xi, y, p) > 0 \quad ((\xi, y) \in \mathcal{M}_1(p)).
\]

Thus, from (1.5), \(\text{Re} A(\lambda, p) g > 0\) holds.

Basic approaches for showing Theorem 1.1 and 1.3 are similar to our previous work [13] for the strictly convex case. As is in Section 3 of [13], a decomposition of \(I(\lambda, p)\) and the representation formula of the main term \(I_0(\lambda, p)\) of \(I(\lambda, p)\) are deduced by using usual potential theory. In Section 2, a brief review of this decomposition and the formula of \(I_0(\lambda, p)\) are given (cf. Proposition 2.1). Note that the formula is of the form of Laplace integrals on \(\partial \Omega \times \partial \Omega\) with exponential term \(e^{-\lambda_0(\xi, y)}\).

The amplitude functions of the Laplace integrals contain the inverse of the form \(K(\lambda)(I - K(\lambda))^{-1}\) deduced from an integral operator \(K(\lambda)\) on \(\partial D\) with the integral kernel \(K(\xi, \zeta; \lambda)\) estimated by

\[
|K(\xi, \zeta; \lambda)| \leq C e^{-\text{Re}_\lambda|\xi - \zeta|}\quad (\xi, \zeta \in \partial D, \lambda \in \mathbb{C}_{\delta_0}).
\]

To obtain asymptotic behavior of \(I_0(\lambda, p)\), it is crucial to get an estimate for the integral kernel \(K^\infty(\xi, \zeta; \lambda)\) of \(K(\lambda)(I - K(\lambda))^{-1}\) with the same exponential term \(e^{-\text{Re}_\lambda|\xi - \zeta|}\) as for \(K(\xi, \zeta; \lambda)\). For the case \(N = 1\), i.e. \(\partial D\) is strictly convex, such type of estimates is given in [12], and applied for an approach to an inverse problem via the enclosure method, which is the main subject of the previous work [13].

For arbitrary \(\partial D\), it seems to be hard to obtain good estimates described above for \(K^\infty(\xi, \zeta; \lambda)\). For the case that \(\partial D\) consists of several components \(\partial D_j\) \((j = 1, 2, \ldots, N)\), however, contributions to the estimates of the integral kernel \(K^\infty(\xi, \zeta; \lambda)\) from the different components, i.e. the case \(\xi \in \partial D_j\) and \(\zeta \in \partial D_k\) with \(j \neq k\) are weaker than the same components, i.e. the case \(\xi, \zeta \in \partial D_j\). In this paper, we call the parts coming from the different components and the parts coming from the same components off-diagonal parts and diagonal parts respectively. Since the dominant part is given by the diagonal parts, in Section 3, we introduce the estimates of the integral kernels for the diagonal parts and the amplitude functions of \(I_0(\lambda, p)\). To control off-diagonal parts, we need to give additional argument, which is handled in Section 5.

In Section 4, proofs of the main theorems are given. The main contributions for these Laplace integrals come from the points in \(\mathcal{M}(p)\). To pick up the main terms, we need to study on structures of \(\mathcal{M}(p)\). Here, we use assumption (I.2), i.e. \(\mathcal{M}_2(p) = \mathcal{M}_0(p) = \emptyset\). By using local coordinate systems near \(\mathcal{M}(p)\), eventually, the problems are reduced to investigating the asymptotic behaviors of Laplace integrals. Since the appeared integrals seems to be different from usual and typical ones, we give a brief outline to handle these integrals in Section 6 for the paper to be self-contained.

To obtain the estimates of the diagonal parts, the kernel estimates for the case of one strictly convex cavity is essentially used. These estimates are given in [12] and [13] by assuming that \(\partial D\) is \(C^{2,\alpha_0}\) for some \(0 < \alpha_0 < 1\). Note that this regularities assumption can be reduced to \(C^2\) regularity,
however, additional arguments are needed. This is performed in [14] by using strict convexity. Here, we can give a different approach showing equi-continuous properties for a class of local coordinate systems of $\partial D$, which is handled in Appendix.

In the last of Introduction, we explain why assumption (I.2) is needed. As is in (2) of Remark 1.2, for the points $(\xi_0, y_0) \in M_2(p) \cup M_3^+(p) \cup M_3(p)$, attaining the minimum length $l(p, D)$, the point $\xi_0$ places on the line segment $p D$. Hence, contributions from the off-diagonal parts are the same levels as that from the diagonal parts. Thus, in this case, the off-diagonal parts can not be negligible. This is essentially different from the case $(\xi_0, y_0) \in M_1(p)$. Hence, the approach picking up the diagonal parts works only the case $(\xi_0, y_0) \in M_1(p)$, which is why assumption (I.2) is needed.

2 Decomposition of the indicator functions

In the beginning, we give a remark on the class of the solutions of (1.1) to make sure the meaning of integrals in $I(\lambda, p)$. We denote by $L^2(0, T; H)$ the space of $H$-valued $L^2$ functions in $t \in [0, T]$. For a Hilbert space $V$ with $V \subset L^2(\Omega \setminus \bar{D}) \subset V'$, we introduce the space $W(0, T; V, V') = \{ u \mid u \in L^2(0, T; V) \}$, where $V'$ is the dual space of the Hilbert space $V$, and $u'$ means the (weak) derivative in $t \in [0, T]$. Throughout this paper, we always assume that the heat flux $f(t, y)$ belongs to the space $L^2((0, T) \times \partial \Omega)$. Note that for any $f \in L^2((0, T) \times \partial \Omega)$, the weak solution $u \in W(0, T; H^1(\Omega \setminus \bar{D}), (H^1(\Omega \setminus \bar{D}))')$ of (1.1) uniquely exists. For the weak solutions see Section 1.5 of [13] and the references in it. Hence, the indicator function $I(\lambda, p)$ defined by (1.3) is well-defined.

To show Theorem 1.1 and 1.3, we need to pick up the main term $I_0(\lambda, p)$ of the original indicator function $I(\lambda, p)$, Define

$$w(x; \lambda) = \int_0^T e^{-\lambda^2 t} u(t, x) dt, \ x \in \Omega \setminus \bar{D},$$

which satisfies

$$\begin{cases} (\Delta - \lambda^2)w = u(T, x)e^{-\lambda^2 T} & \text{in } \Omega \setminus \bar{D}, \\ (\partial_{\nu} + \rho)w = 0 & \text{on } \partial D. \end{cases}$$

Let $w_0(x; \lambda)$ be the solution of

$$\begin{cases} (\Delta - \lambda^2)w_0 = 0 & \text{in } \Omega \setminus \bar{D}, \\ (\partial_{\nu} + \rho)w_0 = 0 & \text{on } \partial D, \\ \partial_{\nu}w_0 = g & \text{on } \partial \Omega, \end{cases}$$

where $g(x; \lambda)$ is defined by (1.4). As in Section 2 of [10] or Appendix C of [13], $w(x; \lambda) = w_0(x; \lambda) + O(e^{-\lambda^2 T})$ in $H^1(\Omega \setminus \bar{D})$ as weak sense, integration by parts implies

$$I(\lambda, p) = I_0(\lambda, p) + O(\lambda^{-\frac{1}{2}} e^{-\lambda^2 T}) \quad \text{(as } |\lambda| \to \infty \text{ uniformly in } \lambda \in \mathbb{C}_{\delta_0}),$$

where

$$I_0(\lambda, p) = \int_{\partial \Omega} (\partial_{\nu} E_{\lambda}(y, p) w_0(y; \lambda) - E_{\lambda}(y, p) \partial_{\nu} w_0(y; \lambda)) dS_y.$$ 

As is in [13], we use the layer potentials to construct $w_0(x; \lambda)$. From the layer potentials and the density functions, we can get the integral representation of $I_0(\lambda, p)$. The procedure is the same as in Section 3.1 of [13]. We give a brief review for it.

Given $g \in C(\partial \Omega)$ and $h \in C(\partial D)$ define

$$V_\Omega(\lambda)g(x) = \int_{\partial \Omega} E_{\lambda}(x, y) g(y) dS_y, \ x \in \mathbb{R}^3 \setminus \partial \Omega$$

and

$$V_D(\lambda)h(x) = \int_{\partial D} E_{\lambda}(x, \zeta) h(\zeta) dS_\zeta, \ x \in \mathbb{R}^3 \setminus \partial D.$$
We construct \( w_0 \) in the form
\[
  w_0(x; \lambda) = V_\Omega(\lambda)\varphi(x; \lambda) + V_D(\lambda)\psi(x; \lambda),
\]
where \( \varphi(\cdot; \lambda) \in C(\partial \Omega) \) and \( \psi(\cdot; \lambda) \in C(\partial D) \) are called the density functions satisfying the following equations in \( C(\partial \Omega) \times C(\partial D) \):
\[
  \begin{align*}
    \varphi(y; \lambda) - Y_{11}(\lambda)\varphi(y; \lambda) - Y_{12}(\lambda)\psi(y; \lambda) &= g(y; \lambda) \quad \text{on } \partial \Omega, \\
    \psi(\xi; \lambda) - Y_{21}(\lambda)\varphi(\xi; \lambda) - Y_{22}(\lambda)\psi(\xi; \lambda) &= 0 \quad \text{on } \partial D.
  \end{align*}
\]
(2.2)

In (2.2), \( Y_{ij}(\lambda) \) \((i, j = 1, 2)\) are defined by
\[
  \begin{align*}
    Y_{11}(\lambda)\varphi(y; \lambda) &= -\int_{\partial \Omega} \partial_{y_j} E_\lambda(y, z)\varphi(z; \lambda)dS_z \quad (y \in \partial \Omega), \\
    Y_{12}(\lambda)\psi(y; \lambda) &= -\int_{\partial \Omega} \partial_{y_j} E_\lambda(y, \zeta)\psi(\zeta; \lambda)dS_\zeta \quad (y \in \partial \Omega), \\
    Y_{21}(\lambda)\varphi(\xi; \lambda) &= \int_{\partial \Omega} (\partial_{\xi_i} E_\lambda(\xi, z) + \rho(\xi)E_\lambda(\xi, z))\varphi(z; \lambda)dS_z \quad (\xi \in \partial D), \\
    Y_{22}(\lambda)\psi(\xi; \lambda) &= \int_{\partial D} (\partial_{\xi_i} E_\lambda(\xi, \zeta) + \rho(\xi)E_\lambda(\xi, \zeta))\psi(\zeta; \lambda)dS_\zeta \quad (\xi \in \partial D).
  \end{align*}
\]
and
\[
  \begin{align*}
    Y_{22}(\lambda)\psi(\xi; \lambda) &= \int_{\partial D} (\partial_{\xi_i} E_\lambda(\xi, \zeta) + \rho(\xi)E_\lambda(\xi, \zeta))\psi(\zeta; \lambda)dS_\zeta \quad (\xi \in \partial D).
  \end{align*}
\]

Note that for \( \lambda \in \mathbb{C}, \text{Re } \lambda > 0 \), \( Y_{11}(\lambda) \in B(C(\partial \Omega)), Y_{22}(\lambda) \in B(C(\partial D)), Y_{12}(\lambda) \in B(C(\partial \Omega), C(\partial D)) \) and \( Y_{21}(\lambda) \in B(C(\partial D), C(\partial \Omega)) \), where \( B(E, F) \) is the set of bounded linear operators from \( E \) to \( F \), and \( B(E) = B(E, E) \). The operator norms of \( Y_{ij}(\lambda) \) are estimated by
\[
  \|Y_{11}(\lambda)\|_{B(C(\partial \Omega))} + \|Y_{22}(\lambda)\|_{B(C(\partial D))} + \|Y_{12}(\lambda)\|_{B(C(\partial \Omega), C(\partial D))} + \|Y_{21}(\lambda)\|_{B(C(\partial D), C(\partial \Omega))} \leq C(\text{Re } \lambda)^{-1} \quad (\lambda \in \mathbb{C}, \text{Re } \lambda > 0).
\]
(2.3)

Hence, for \( \lambda \in \mathbb{C}_0 \) with sufficiently large \( \text{Re } \lambda \), equation (2.2) can be solved by using the Neumann series. Since the inverse \((I - Y_{22}(\lambda))^{-1}\) is also constructed, from the second equation of (2.2), it follows that \( \psi(\xi; \lambda) = (I - Y_{22}(\lambda))^{-1}Y_{21}(\lambda)\varphi(\xi; \lambda) \), which yields
\[
  \varphi(y; \lambda) = \{I - Y_{11}(\lambda) - Y_{12}(\lambda)(I - Y_{22}(\lambda))^{-1}Y_{21}(\lambda)\}^{-1}g(y; \lambda).
\]
From this and (2.3), we obtain
\[
  \varphi(y; \lambda) = g(y; \lambda) + O(\lambda^{-1})\|g(\cdot; \lambda)\|_{C(\partial \Omega)} \quad \text{uniformly in } y \in \partial \Omega, \lambda \in \mathbb{C}_0 \text{ as } |\lambda| \to \infty,
\]
(2.4)

which is used in Section 5.

From (2.2), (2.1) and the equality
\[
  I_0(\lambda, p) = \int_{\partial D} \left( \frac{\partial E_\lambda}{\partial \nu}(\xi, p) + \rho E_\lambda(\xi, p) \right) w_0(\xi; \lambda)dS_\xi
\]
given by integration by parts, we can write \( I_0(\lambda, p) \) using only \( \varphi \). This is given in Section 3.1 in [13] for strictly convex case. Note that this argument works even if \( \partial D \) is not strictly convex. Thus, we can obtain the same formula of \( I_0(\lambda, p) \) as given in Proposition 1 of [13].

To obtain the formula of \( I_0(\lambda, p) \), the transposed operator \( {^t}Y_{22}(\lambda) \) of \( Y_{22}(\lambda) \) is frequently used. Note that the operator \( {^t}Y_{22}(\lambda) \) is given by
\[
  {^t}Y_{22}(\lambda)h(\zeta) = \frac{1}{2\pi} \int_{\partial D} e^{-\lambda|\zeta - \xi|}H(\xi, \zeta; \lambda)h(\xi)dS_\xi, \quad h \in C(\partial D), \zeta \in \partial D
\]
(2.5)
with the kernel $H(\xi, \zeta; \lambda) = \lambda H_0(\xi, \zeta) + H_1(\xi, \zeta)$, where

$$H_0(\xi, \zeta) = \frac{\nu_{\xi} \cdot (\zeta - \xi)}{|\zeta - \xi|^2} \quad \text{and} \quad H_1(\xi, \zeta) = \frac{1}{|\zeta - \xi|} \left( \frac{\nu_{\zeta} \cdot (\zeta - \xi)}{|\zeta - \xi|^2} + \rho(\xi) \right).$$

For $H_0(\xi, \zeta)$ and $H_1(\xi, \zeta)$, we define the operators $M^{(0)}(\lambda)$ and $\tilde{M}(\lambda)$ by

$$M^{(0)}(\lambda) h(\xi) = \frac{\lambda}{2\pi} \int_{\partial D} e^{-\lambda |\xi - \zeta|} H_0(\xi, \zeta) h(\zeta) dS_{\zeta} \quad (2.6)$$

and

$$\tilde{M}(\lambda) h(\xi) = \frac{1}{2\pi} \int_{\partial D} e^{-\lambda |\xi - \zeta|} H_1(\xi, \zeta) h(\zeta) dS_{\zeta}, \quad (2.7)$$

respectively. Note that $t^T Y_{22}(\lambda)$ is decomposed into $t^T Y_{22}(\lambda) = M^{(0)}(\lambda) + \tilde{M}(\lambda)$.

Using $t^T Y_{22}(\lambda)$, we can represent $I_0(\lambda, p)$ as follows:

$$I_0(\lambda, p) = \frac{1}{(2\pi)^2} \int_{\partial D} dS_y \varphi(y; \lambda) \int_{\partial D} e^{-\lambda \varphi(\xi, \eta)} \left\{ \frac{H(\xi, \eta; \lambda)}{\xi - \eta} - \frac{H(\xi, \eta; \lambda)}{|\xi - \eta|} + 2H(\xi, \eta; \lambda) F(\xi, \eta; \lambda) \right\} dS_{\xi},$$

where

$$F(\xi, \eta; \lambda) = e^{\lambda |\xi - \eta|} \left( (I - t^T Y_{22}(\lambda))^{-1} e^{-\lambda |\cdot - \eta|} \right)(\xi).$$

This is just (35) in p.1088 of [13].

Next we decompose $F(\xi, \eta; \lambda)$ to pick up the main term of $I_0(\lambda, p)$. We put $M(\lambda) = t^T Y_{22}(\lambda)(I - t^T Y_{22}(\lambda))^{-1}$ and $M^{(1)}(\lambda) = M(\lambda) + t^T Y_{22}(\lambda) M(\lambda)$. From $M(\lambda) = t^T Y_{22}(\lambda) + (t^T Y_{22}(\lambda))^2 (I - t^T Y_{22}(\lambda))^{-1}$, it follows that

$$M(\lambda) = M^{(0)}(\lambda) + M^{(1)}(\lambda), \quad M^{(1)}(\lambda) = \tilde{M}(\lambda) + t^T Y_{22}(\lambda) M(\lambda). \quad (2.8)$$

Using these $M^{(j)}(\lambda)$, we set

$$F^{(j)}(\xi, \eta; \lambda) = e^{\lambda |\xi - \eta|} \left( M^{(j)}(\lambda) \left( e^{-\lambda |\cdot - \eta|} \right)(\xi) \right) \quad j = 0, 1. \quad (2.9)$$

Since $(I - t^T Y_{22}(\lambda))^{-1} = I + M^{(0)}(\lambda) + M^{(1)}(\lambda)$, $F(\xi, \eta; \lambda)$ can be decomposed into

$$F(\xi, \eta; \lambda) = \frac{1}{|\eta - \xi|} + F^{(0)}(\xi, \eta; \lambda) + F^{(1)}(\xi, \eta; \lambda).$$

Using these notations and the function $H^+(\xi, \eta, \lambda)$ introduced in (1.7), we can give an integral representation of $I_0(\lambda, p)$.

**Proposition 2.1** The decomposition

$$I_0(\lambda, p) = \lambda I_{00}(\lambda, p) + I_{01}(\lambda, p),$$

is valid, where

$$G_0(\xi, \eta; \lambda) = H^+(\xi, \eta, \lambda) + 2H_0(\xi, \eta) F^{(0)}(\xi, \eta; \lambda) + F^{(1)}(\xi, \eta; \lambda),$$

$$G_1(\xi, \eta; \lambda) = H_1(\xi, \eta; \lambda) + \frac{H_1(\xi, \eta; \lambda)}{|\eta - \xi|} + 2H_1(\xi, \eta) F^{(0)}(\xi, \eta; \lambda) + F^{(1)}(\xi, \eta; \lambda)$$

and

$$I_{0j}(\lambda, p) = \left( \frac{1}{2\pi} \right)^2 \int_{\partial D} dS_y \varphi(y; \lambda) \int_{\partial D} e^{-\lambda \varphi(\xi, \eta)} G_j(\xi, \eta, \lambda) dS_{\xi}, \quad j = 0, 1.$$
3 Estimates of integral kernels

To show Theorem 1.1 and 1.3, we need to give estimates of $I_{0,j}(\lambda, p)$, which is reduced to getting estimates of $F^{(k)}(\xi, p; \lambda)$ ($k = 0, 1$) defined by (2.9). In this section, necessary estimates of functions $F^{(k)}(\xi, p; \lambda)$ ($k = 0, 1$) are given.

Since $D = \bigcup_{j=1}^{N} D_j$ and each $D_j$ is disjoint, by the map

$$C(\partial D) \ni f \mapsto (f|_{\partial D_1}, f|_{\partial D_2}, \ldots, f|_{\partial D_N}) \in C(\partial D_1) \times \cdots \times C(\partial D_N),$$

we can identify the space $C(\partial D)$ to $C(\partial D_1) \times \cdots \times C(\partial D_N)$. In what follows, we put $f_j = f|_{\partial D_j}$ ($j = 1, 2, \ldots, N$) for $f \in C(\partial D)$. From (2.5), the integral kernel of $I_{22}(\xi, \zeta; \lambda)$ is given by

$$t_{I_{22}}(\xi, \zeta; \lambda) = \frac{1}{2\pi} e^{-\lambda|\xi - \zeta|} H(\xi, \zeta; \lambda).$$

Hence, $t_{I_{22}}(\xi, \zeta; \lambda)$ is a measurable function on $\partial D \times \partial D$ with parameter $\lambda \in \mathbb{C}_{b_0}$, and continuous for $\xi \neq \eta$. From the well known estimate

$$|\nu_\xi \cdot (\xi - \zeta)| \leq C(|\xi - \zeta|^2), \quad (\xi, \zeta \in \partial D = \bigcup_{j=1}^{N} \partial D_j) \tag{3.1}$$

for $C^2$ surface, it follows that there exists a constant $C > 0$ such that

$$|t_{I_{22}}(\xi, \zeta; \lambda)| \leq C \left( \text{Re} \lambda + \frac{1}{|\xi - \zeta|} \right) e^{-C|\xi - \zeta|} \quad (\xi, \zeta \in \partial D, \xi \neq \zeta, \lambda \in \mathbb{C}_{b_0}). \tag{3.2}$$

For this integral kernel $t_{I_{22}}(\xi, \zeta; \lambda)$, we put

$$t_{Y_{22}^{ij}}(\xi, \zeta; \lambda) = t_{I_{22}}(\xi, \zeta; \lambda) \quad (\xi \in \partial D_i, \zeta \in \partial D_j, \lambda \in \mathbb{C}_{b_0}),$$

and define

$$t_{Y_{22}^{ij}}(\lambda) f_j(\xi) = \int_{\partial D_j} t_{Y_{22}^{ij}}(\xi, \zeta; \lambda) f_j(\zeta) dS_\zeta.$$

Note that for $f \in C(\partial D)$, $t_{Y_{22}^{ij}}(\lambda) f(\xi)$ for $\xi \in \partial D_i$ can be written by

$$t_{Y_{22}^{ij}}(\lambda) f(\xi) = \sum_{j=1}^{N} t_{Y_{22}^{ij}}(\lambda) f(\xi) \quad (\xi \in \partial D_i, i = 1, 2, \ldots, N, f \in C(\partial D)).$$

In what follows, for simplicity, we write $\mu = \text{Re} \lambda$. For $\mu > 0$, $t_{Y_{22}^{ij}}(\lambda)$ is a bounded linear operator on $C(\partial D)$, namely, each $t_{Y_{22}^{ij}}(\lambda)$ is a bounded linear operator from $C(\partial D_j)$ to $C(\partial D_i)$. From (2.3), it follows that

$$\|t_{Y_{22}^{ij}}(\lambda)\|_{B(C(\partial D_j), C(\partial D_i))} \leq C \mu^{-1} \quad (i, j = 1, 2, \ldots, N)$$

for some constant $C > 0$.

For each integral operator $t_{Y_{22}^{ij}}(\lambda) \in B(C(\partial D_j))$ on $\partial D_j$ with the integral kernels $t_{Y_{22}^{ij}}(\xi, \zeta; \lambda)$, $t_{Y_{22}^{ij}}(\lambda)(I - t_{Y_{22}^{ij}}(\lambda))^{-1} \in B(C(\partial D_j))$ exists for $\lambda \in \mathbb{C}_{b_0}$, $\mu \geq \mu_0$ by choosing $\mu_0 > 0$ larger if necessary. In what follows, we put $M_{D_j}(\lambda) = t_{Y_{22}^{ij}}(\lambda)(I - t_{Y_{22}^{ij}}(\lambda))^{-1}$. Note that $M_{D_j}(\lambda)$ corresponds to the operator $M(\lambda)$ for the case that $\partial D$ consists of $\partial D_j$ only. According to one cavity case as in [13], we define $M_{D_j}^{(0)}(\lambda)$, $\tilde{M}_{D_j}(\lambda)$ and $M_{D_j}^{(1)}(\lambda)$ by

$$M_{D_j}^{(0)}(\lambda) h_j(\xi) = \frac{\lambda}{2\pi} \int_{\partial D_j} e^{-\lambda|\xi - \zeta|} H_0(\xi, \zeta) h_j(\zeta) dS_\zeta \quad (\xi \in \partial D_j, h_j \in C(\partial D_j)),$$

$$\tilde{M}_{D_j}(\lambda) h_j(\xi) = \frac{1}{2\pi} \int_{\partial D} e^{-\lambda|\xi - \zeta|} H_1(\xi, \zeta) h_j(\xi) dS_\xi \quad (\xi \in \partial D_j, h_j \in C(\partial D_j))$$

and

$$M_{D_j}^{(1)}(\lambda) = \tilde{M}_{D_j}(\lambda) + t_{Y_{22}^{ij}}(\lambda) M_{D_j}(\lambda). \tag{3.3}$$
Note that the operators $M_D^{(0)}(\lambda)$, $M_D^{(1)}(\lambda)$ and $M^{(1)}_D(\lambda)$ correspond to $M^{(0)}(\lambda)$, $M(\lambda)$ and $M^{(1)}(\lambda)$ for one cavity case, respectively, and the relation $M_D(\lambda) = M_D^{(0)}(\lambda) + M_D^{(1)}(\lambda)$ holds.

Since each $\partial D_j$ is strictly convex, as in Theorem 6.1 of [12], the integral kernel $M_D^{(1)}(\xi, \eta; \lambda)$ of $M^{(1)}_D(\lambda)$ has the following estimates:

**Proposition 3.1** Assume that $\partial D_j$ is bounded, $C^2$ and strictly convex. Then, there exist positive constants $C$ and $\mu_0 \geq 1$ such that for all $\lambda \in \mathbb{C}_{\theta_0}$ with $\mu = \text{Re} \lambda \geq \mu_0$ the operator $M_D^{(1)}(\lambda)$ has an integral kernel $M_D^{(1)}(\xi, \eta; \lambda)$ which is measurable for $(\xi, \eta) \in \partial D \times \partial D$, continuous for $\xi \neq \eta$ and has the estimate

$$|M_D^{(1)}(\xi, \eta; \lambda)| \leq Ce^{-\mu|\xi - \eta|} \left(1 + \frac{1}{|\xi - \eta|} + \min \left\{\mu(\mu|\xi - \eta|^3)^{1/2}, \frac{1}{|\xi - \eta|^3}\right\}\right).$$

**Remark 3.2** In [12], the above estimate in Proposition 3.1 is obtained for strictly convex $\partial D_j$ with $C^{2, \alpha_0} (0 < \alpha_0 < 1)$ regularities. As is in [14], this regularity assumption can be relaxed to $C^2$. A proof of this relaxation is given in [14], which uses strict convexity of $\partial D_j$. A different proof is given in Appendix for the paper to be self-contained.

Note that since $\min \{\sqrt{a}, a^{-1}\} \leq 1$ for all $a > 0$, from Proposition 3.1, we get

$$|M_D^{(1)}(\xi, \eta; \lambda)| \leq C \left(\mu + \frac{1}{|\xi - \eta|}\right)e^{-\mu|\xi - \eta|}. \quad (3.4)$$

From (3.4), (3.1) and the form of $M_D^{(0)}(\xi, \eta; \lambda)$, we obtain

$$|M_D(\xi, \eta; \lambda)| \leq C \left(\mu + \frac{1}{|\xi - \eta|}\right)e^{-\mu|\xi - \eta|} \quad (3.5)$$

$$(\xi, \eta) \in \partial D_j, \lambda \in \mathbb{C}_{\theta_0}, \mu \geq \mu_0, j = 1, 2, \ldots, N,$$

where $M_D(\xi, \eta; \lambda)$ is the integral kernel of $M_D(\lambda)$.

Next, we introduce estimates of the integral kernel of $M^{(k)}(\lambda) (k = 0, 1)$. We denote by $M^{(1)}(\xi, \eta; \lambda)$, $M^{(0),ij}(\xi, \eta; \lambda)$ and $M^{(1),ij}(\xi, \eta; \lambda)$ the $(i, j)$-components of the integral kernel of $M(\lambda)$, $M^{(0)}(\lambda)$ and $M^{(1)}(\lambda)$, respectively. In what follows we put

$$d_1 = \frac{1}{2} \min \{\text{dist}(\partial D_i, \partial D_j) > 0, \quad (3.6)$$

where $\text{dist}(\partial D_i, \partial D_j) = \inf \{|\xi - \eta| \mid \xi \in \partial D_i, \eta \in \partial D_j\}$. Note that from (3.2), for $\xi \in \partial D_i$, $\eta \in \partial D_j$, $i \neq j$

$$|\mathcal{Y}_{22}^{ij}(\xi, \eta; \lambda)| \leq C \left(\mu + \frac{1}{|\xi - \eta|}\right)e^{-\mu|\xi - \eta|} \leq C \left(\mu + \frac{1}{2d_1}\right)e^{-\mu(2d_1)}e^{-(1-\delta)\mu|\xi - \eta|}

\leq C \left[\frac{1}{d_1}e^{-\mu \delta_1} + \frac{1}{2d_1}\right]e^{-(1-\delta)\mu|\xi - \eta|} \quad (3.7)$$

$$(\xi \in \partial D_i, \eta \in \partial D_j, i \neq j, 0 < \delta \leq 1).$$

From (2.6), the similar argument to getting (3.7) implies that there exists a constant $C > 0$ such that for all $i, j = 1, 2, \ldots, N$, $i \neq j$ and $0 < \delta \leq 1$,

$$|M^{(0),ij}(\xi, \eta; \lambda)| \leq C \delta e^{-2\delta_1 \mu}e^{-(1-\delta)\mu|\xi - \eta|} \quad (\xi \in \partial D_i, \eta \in \partial D_j, \lambda \in \mathbb{C}_{\theta_0}, \mu \geq 1).$$

For the diagonal parts, (3.1) implies that there exists a constant $C > 0$ such that

$$|M^{(1),ij}(\xi, \eta; \lambda)| \leq C \mu e^{-\mu|\xi - \eta|} \quad (\xi, \eta) \in \partial D_i, i \neq j, \lambda \in \mathbb{C}_{\theta_0}, \mu \geq 1) \quad (3.9)$$

for all $i = 1, 2, \ldots, N$.

The problem is to give estimates for $M^{(1),ij}(\xi, \eta; \lambda)$. 

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Proposition 3.3 There exist constants $C > 0$, $\mu_1 > 0$ and $0 < \delta_1 \leq 1$ such that

1. the integral kernel $M^{(1),ij}(\xi, \zeta; \lambda)$ is estimated by
   \[
   |M^{(1),ij}(\xi, \zeta; \lambda)| \leq C\delta^{-4}e^{-\delta d_{ij}\mu}e^{-(1-\delta_1)\mu|\xi-\zeta|} \quad (\xi \in \partial D_i, \zeta \in \partial D_j, \lambda \in \mathcal{C}_{\delta_0}, \mu \geq \mu_1\delta^{-3})
   \]
   for all $i, j = 1, 2, \ldots, N$, $i \neq j$ and $0 < \delta \leq \delta_1$;

2. the integral kernel $M^{(1),jj}(\xi, \zeta; \lambda)$ is estimated by
   \[
   |M^{(1),jj}(\xi, \zeta; \lambda)| \leq C\delta^{-4}e^{-\delta d_{ij}\mu}e^{-\mu|\xi-\zeta|} \quad (\xi, \zeta \in \partial D_j, \lambda \in \mathcal{C}_{\delta_0}, \mu \geq \mu_1\delta^{-3})
   \]
   for all $j = 1, 2, \ldots, N$ and $0 < \delta \leq \delta_1$.

Remark 3.4 (2) of Proposition 3.3 and (3.4) imply that

\[
|M^{(1),jj}(\xi, \zeta; \lambda)| \leq C\left(\mu + \frac{1}{|\xi-\zeta|}\right)e^{-\mu|\xi-\zeta|} \quad (\xi, \zeta \in \partial D_j, \lambda \in \mathcal{C}_{\delta_0}, \mu \geq \mu_1, 0 < \delta \leq \delta_1).
\]

Choosing $\delta = \delta_1$ in the above, and replacing $\mu$ with $\mu_1\delta_1^{-3}$ denoted by $\mu_1$ again, we obtain

\[
|M^{(1),jj}(\xi, \zeta; \lambda)| \leq C\left(\mu + \frac{1}{|\xi-\zeta|}\right)e^{-\mu|\xi-\zeta|} \quad (\xi, \zeta \in \partial D_j, \lambda \in \mathcal{C}_{\delta_0}, \mu \geq \mu_1).
\]

Proposition 3.3 can be obtained by decomposing off-diagonal parts of the integral kernels. These procedures and a proof of Proposition 3.3 are given in Section 5. Here, we proceed to introduce the estimates of $F^{(k)}(\xi, p; \lambda)$ $(k = 0, 1)$ given by Proposition 3.3.

For $\varepsilon > 0$, we define

\[
\mathcal{G}_\varepsilon(p) = \{\xi \in \partial D \mid \text{dist}(\xi, G(p)) \geq \varepsilon\}, \quad \mathcal{G}_\varepsilon^+(p) = \mathcal{G}_\varepsilon(p) \cap \mathcal{G}_\varepsilon^+(p).
\]

We also put $\mathcal{G}_\varepsilon^{+,0}(p) = \{\xi \in \mathcal{G}_\varepsilon^+ \mid tp + (1-t)\xi \notin \partial D(0 < t \leq 1)\}$ and $\mathcal{G}_\varepsilon^{+,0}(p) = \mathcal{G}_\varepsilon(p) \cap \mathcal{G}_\varepsilon^{+,0}(p)$. The definitions (2.9) of $F^{(k)}(\xi, p; \lambda)$ $(k = 0, 1)$ in Section 2 imply that

\[
F^{(k)}(\xi, p; \lambda) = \sum_{j=1}^N F^{(k),ij}(\xi, p; \lambda) \quad (\xi \in \partial D_i, i = 1, 2, \ldots, N, k = 0, 1), \quad (3.10)
\]

where

\[
F^{(k),ij}(\xi, p; \lambda) = e^{\lambda|\xi-p|} \int_{\partial D_j} M^{(k),ij}(\xi, \zeta; \lambda) \frac{e^{-\lambda|\zeta-p|}}{|\xi-p|} dS_\zeta \quad (\xi \in \partial D_i). \quad (3.11)
\]

To obtain the estimates of $I_{0,j}(\lambda, p)$, the following estimates of $F^{(k),ij}(\xi, p; \lambda)$ are necessary:

Proposition 3.5 There exists a positive constant $\mu_1$ such that the following assertions hold:

1. There exist positive constants $C$ and $d_+$ such that if $i, j = 1, 2, \ldots, N$, $i \neq j$, $\lambda \in \mathcal{C}_{\delta_0}$, $\mu = \text{Re} \lambda \geq \mu_1\delta^{-3}$ and $0 < \delta \leq 1$, then
   \[
   |F^{(k),ij}(\xi, p; \lambda)| \leq C\delta^{-4}e^{\delta d_+ \mu} \quad (\xi \in \partial D_i, i \neq j, k = 0, 1).
   \]

2. There exists a positive constant $C$ such that if $\lambda \in \mathcal{C}_{\delta_0}$ and $\mu = \text{Re} \lambda \geq \mu_1$, then
   \[
   |F^{(k),jj}(\xi, p; \lambda)| \leq C \mu \quad (\xi \in \partial D_j, k = 0, 1, j = 1, 2, \ldots, N).
   \]

3. Given $\varepsilon > 0$ and an open set $U \subset \partial D$ satisfying $\overline{U} \subset \mathcal{G}_\varepsilon^{+,0}(p) \cap \partial D_i$ for some $i = 1, 2, \ldots, N$, there exist positive constants $C$, $d_2$, $0 < \delta_2 \leq 1$ such that if $j = 1, 2, \ldots, N$, $j \neq i$, $\lambda \in \mathcal{C}_{\delta_0}$, $\mu = \text{Re} \lambda \geq \mu_1\delta^{-3}$ and $0 < \delta \leq \delta_2$, then
   \[
   |F^{(k),ij}(\xi, p; \lambda)| \leq C\delta^{-4}e^{-2\delta d_2 \mu} \quad (\xi \in \overline{U}, i \neq j, k = 0, 1).
   \]

4. Given $\varepsilon > 0$ there exists a positive constant $\mu_0$ such that if $\lambda \in \mathcal{C}_{\delta_0}$ and $\text{Re} \lambda \geq \mu_0$, then
   \[
   |F^{(k),jj}(\xi, p; \lambda)| \leq C\varepsilon^{-1} \quad (\xi \in \mathcal{G}_\varepsilon^+(p) \cap \partial D_j, j = 1, 2, \ldots, N, k = 0, 1).
   \]
To show Proposition 3.5 and various estimates of the integral kernels, we need to use local coordinate systems of \( \partial D \). For \( a \in \mathbb{R}^3 \) and \( r > 0 \), we put \( B(a, r) = \{ x \in \mathbb{R}^3 | |x - a| < r \} \). We denote by \( B^2(\mathbb{R}^2) \) the set of \( C^2 \) functions \( f \) in \( \mathbb{R}^2 \) such that the norm \( \|f\|_{B^2(\mathbb{R}^2)} = \max_{|\alpha| \leq 2} \sup_{x \in \mathbb{R}^2} |\partial^\alpha_x f(x)| \) is finite. Since \( \partial D \) is compact, we can take the following coordinate systems:

**Lemma 3.6** There exists \( 0 < r_0 \) such that, for all \( \xi \in \partial D, \partial D \cap B(\xi, 2r_0) \) can be represented as a graph of a function on the tangent plane of \( \partial D \) at \( \xi \), that is, there exist an open neighborhood \( U_\xi \) of \((0, 0)\) in \( \mathbb{R}^2 \) and a function \( g = g_\xi \in B^2(\mathbb{R}^2) \) with \( g(0, 0) = 0 \) and \( \nabla g(0, 0) = 0 \) such that the map

\[
U_\xi \ni \sigma = (\sigma_1, \sigma_2) \mapsto \xi + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma_1, \sigma_2) \nu_\xi \in \partial D \cap B(\xi, 2r_0)
\]
gives a system of local coordinates around \( \xi \), where \( \{e_1, e_2\} \) is an orthogonal basis for \( T_\xi(\partial D) \). Moreover the norm \( \|g\|_{B^2(\mathbb{R}^2)} \) has an upper bound independent of \( \xi \in \partial D \).

In what follows, we call this system of coordinates the standard system of local coordinates around \( \xi \).

As is given in Lemma 3.1 of [12] or Lemma 5.3 of [13], the following estimates, which are frequently used, are shown by the standard system of local coordinates:

**Lemma 3.7** Let \( r_0 \) be the same constant as that of Lemma 3.6. There exists a positive constant \( C \) depending only on \( \partial D \) such that

(i) for all \( \xi \in \partial D, 0 < \rho_0 \leq r_0, \mu > 0, 0 \leq k < 2 \)

\[
\int_{B(\xi, \rho_0) \cap \partial D} e^{-\mu|\xi - \zeta|} |\xi - \zeta|^k \, dS_\zeta \leq \frac{C}{2 - k} \min \{ \mu^{-2+k}, (\rho_0)^{2-k} \};
\]

(ii) for all \( \xi \in \partial D, \mu > 0, 0 \leq k < 2 \)

\[
\int_{\partial D} e^{-\mu|\xi - \zeta|} |\xi - \zeta|^k \, dS_\zeta \leq \frac{C}{2 - k} \mu^{-2-k} \left( 1 + \frac{\mu^{2-k} e^{-\mu r_0}}{\rho_0^k} \right).
\]

Although \( C^{2,\kappa_0} \) regularities for \( \partial D \) is assumed in [12] and [13], the proofs given in [12] and [13] work even if \( \partial D \) is \( C^2 \). Hence, Lemma 3.7 holds for \( C^2 \) boundary case.

Take \( \mu = 1 \) and \( k = 1 \) in (ii) of Lemma 3.7, it follows that

\[
e^{-\inf_{\xi, \zeta \in \partial D} |\xi - \zeta|} \int_{\partial D_\rho} \frac{dS_\zeta}{|\xi - \zeta|} \leq \int_{\partial D} e^{-|\xi - \zeta|} \, dS_\zeta \leq C,
\]

which yields

\[
\int_{\partial D_\rho} \frac{dS_\zeta}{|\xi - \zeta|} \leq C \quad (\xi \in \partial D_\rho, p = 1, 2, \ldots, N). \tag{3.12}
\]

Now we are in the position to give a proof of Proposition 3.5 assuming Proposition 3.3 holds.

**Proof of Proposition 3.5:** From (3.8), (3.9), the estimate for \( M^{(1),ij}(\xi, \zeta; \lambda) \) given in (1) of Proposition 3.3 and the forms of \( F^{(k),ij}(\xi, p; \lambda) \) given in (3.11), it follows that for any \( i, j = 1, 2, \ldots, N, i \neq j \) and \( 0 < \delta \leq 1 \),

\[
|F^{(k),ij}(\xi, p; \lambda)| \leq e^{\mu|\xi - p|} \int_{\partial D_j} C\delta^{-4} e^{-\delta d_+} e^{-|\delta| \mu|\xi - \zeta| - \mu|\xi - p|} |\xi - \zeta| \, dS_\zeta \leq \frac{C\delta^{-4}}{\text{dist}(p, \partial D)} \int_{\partial D_j} e^{\mu|\xi - p| - |\delta| |\xi - \zeta| - |\xi - p|} |\xi - \zeta| \, dS_\zeta. \tag{3.13}
\]

Put \( d_+ = \max\{|\xi - \zeta| | \xi, \zeta \in \partial D \} > 0 \). Noting

\[
e^{\mu(|\xi - p| - |\delta| |\xi - \zeta| - |\xi - p|)} \leq e^{\mu(|\xi - p| - |\xi - \zeta| - |\xi - p|)} e^{\mu|\xi - \zeta|} \leq e^{\mu d_+} \quad (\xi, \zeta \in \partial D),
\]

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we obtain (1) of Proposition 3.5.

When \( i = j \), the second estimate in Remark 3.4 and estimate (3.12) imply that

\[
|F^{(k),ij}(\xi, p; \lambda)| \leq C e^{j(\xi-p)} \int_{\partial D_j} C' (\mu + \frac{1}{|\xi-p|}) e^{-|\xi-p| |\xi-p|} dS_\zeta
\]

\[
\leq \frac{C'}{\text{dist}(p, \partial D_j)} \int_{\partial D_j} e^{j(\xi-p|\xi-p|)} \left( \mu + \frac{1}{|\xi-p|} \right) dS_\zeta \leq C' \mu, \quad (3.14)
\]

which follows (2) of Proposition 3.5.

Next is for the proof of (3) in Proposition 3.5. We need the following lemma:

**Lemma 3.8** Given \( \varepsilon > 0 \) and an open set \( U \subset \partial D \) satisfying \( \overline{U} \subset G_{\varepsilon}^+(p) \cap \partial D_i \) for some \( i = 1, 2, \ldots, N \), there exist \( 0 < \delta_2 \leq 1 \) and \( d_2 > 0 \) such that

\[
|\xi - p| + (1-\delta)|\xi - \zeta| \geq |\xi - p| + 2d_2 \quad (\xi \in \overline{U}, \zeta \in \partial D_j, 0 < \delta \leq \delta_2)
\]

for any \( j = 1, 2, \ldots, N \) with \( j \neq i \).

**Proof:** From the definition of \( G_{\varepsilon}^+(p) \), it follows that for any \( \xi \in \overline{G}_{\varepsilon}^+(p) \cap \partial D_i \) and \( \zeta \in \partial D_j \) with \( j \neq i \), \( |p - \zeta| + |\xi - \zeta| > |p - \xi| \), which yields

\[
|\xi - p| + |\zeta - \xi| > |p - \xi| \quad (\xi \in \overline{U}, \zeta \in \partial D_j, j \neq i).
\]

Since \( U \times \cup_{j \neq i} \partial D_j \) is a bounded closed set, from the above estimate, there exists a constant \( d_2 > 0 \) such that

\[
|p - \xi| + |\zeta - \xi| \geq |p - \xi| + 3d_2 \quad (\xi \in \overline{U}, \zeta \in \partial D_j, i \neq j).
\]

We put \( \varphi(\xi, \zeta, \delta) = |p - \xi| + (1-\delta)|\xi - \zeta| - |p - \xi| \) and \( d'_i = \max \{ |\xi - \zeta| | \xi \in \partial D_i, \zeta \in \partial D_j, i, j = 1, 2, \ldots, N, i \neq j \} \). Note that the above estimate implies that \( \varphi(\xi, \zeta, 0) = |p - \xi| + |\zeta - \xi| - |p - \xi| \geq 3d_2 \) for \( (\xi \in \overline{U}, \zeta \in \partial D_j) \). We define \( \delta_2 = \min\{1, d_2/d'_i\} \) so that \( 0 < \delta_2 \leq 1 \).

Noting

\[
|\varphi(\xi, \zeta, \delta) - \varphi(\xi, \zeta, 0)| = \delta |\xi - \zeta| \leq d'_i \delta \quad \text{for } 0 < \delta \leq \delta_2, \text{ (\xi, \zeta) } \in \overline{U} \times \cup_{j \neq i} \partial D_j,
\]

we obtain

\[
\varphi(\xi, \zeta, \delta) \geq \varphi(\xi, \zeta, 0) - |\varphi(\xi, \zeta, \delta) - \varphi(\xi, \zeta, 0)| \geq 3d_2 - d'_i \delta \geq 2d_2,
\]

which completes the proof of Lemma 3.8.

Take any \( \varepsilon > 0 \) and an open set \( U \subset \partial D \) satisfying \( \overline{U} \subset G_{\varepsilon}^+(p) \cap \partial D_i \). Lemma 3.8 yields

\[
e^{\mu(|p-p|-(1-\delta)|\xi-\zeta|)/|\xi-\zeta|}) \leq e^{-2\mu d_2} \quad (\xi \in \overline{U}, \zeta \in \cup_{j \neq i} \partial D_j, 0 < \delta \leq \delta_2).
\]

This estimate and (3.13) imply that

\[
|F^{(k),ij}(\xi, p; \lambda)| \leq \frac{C \delta^{-4}}{\text{dist}(p, \partial D)} \text{Vol}(\partial D_j) e^{-2\mu d_2} \quad (0 < \delta \leq \delta_2),
\]

which shows (3) of Proposition 3.5.

Last, we show (4) of Proposition 3.5. Since \( \partial D_j \) is strictly convex, as in (i) of Lemma 5.2 in [13], p.1095, for any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon > 0 \) such that

\[
|\xi - \zeta| + |\zeta - p| \geq |\xi - p| + C_\varepsilon |\xi - \zeta| \quad (\zeta \in \partial D_j, \xi \in G_{\varepsilon}^+(p) \cap \partial D_j).
\]

Hence, from (ii) of Lemma 3.7, it follows that for \( \xi \in G_{\varepsilon}^+(p) \cap \partial D_j \),

\[
\int_{\partial D} e^{\mu(|p-p|-(1-\delta)|\xi-\zeta|)/|\xi-\zeta|}) \left( \mu + \frac{1}{|\xi-\zeta|} \right) dS_\zeta \leq \int_{\partial D} e^{-C_\varepsilon \mu |\xi-\zeta|} \left( \mu + \frac{1}{|\xi-\zeta|} \right) dS_\zeta
\]

\[
\leq C (\mu(C_\varepsilon \mu)^{-2} + (C_\varepsilon \mu)^{-1}) = CC_\varepsilon^{-2}(1 + C_\varepsilon)\mu^{-1}.
\]

The above estimate and (3.14) give (4) of Proposition 3.5.
4 Proofs of the main theorems

For \( \xi(0) \in \partial D \) and \( y(0) \in \partial \Omega \) and \( \epsilon > 0 \), we put \( U_\epsilon(\xi(0)) = \{ \xi \in \partial D \mid |\xi - \xi(0)| < \epsilon \} \) and \( V_\epsilon(y(0)) = \{ y \in \partial \Omega \mid |y - y(0)| < \epsilon \} \). We need the following properties of points in \( \mathcal{M}(p) \):

Lemma 4.1 Assume that \( C^2 \) surface \( \partial D \) satisfies (1.1) and (1.2). Then, the following properties hold:

(1) For each \( m = 1, 2, \ldots, N \), \( G^{+,0}(p) \cap \partial D_m \) is an open set in \( \partial D \).

(2) For any point \( (\xi(0), y(0)) \in \mathcal{M}(p) = \mathcal{M}_1(p) \), there exist constants \( \epsilon > 0 \) and \( \epsilon' > 0 \), and a number \( m \in \{1, 2, \ldots, N\} \) satisfying \( U_{2\epsilon}(\xi(0)) \subset \partial D_m \cap G^{+,0}(p) \).

Proof: Note that \( \mathcal{G}^+(p) \) is an open set and \( \mathcal{G}^+(p) \cup \mathcal{G}(p) \) is a closed set since the function \( x \mapsto (p-\xi) \cdot \nu_\xi \) is continuous. For (1), it suffices to show that \( \mathcal{G}^+(p) \) is open. Take any \( \xi_0 \in \mathcal{G}^+(p) \). Then \( \xi_0 \in \partial D_j \) for some \( j \in \{1, 2, \ldots, N\} \). We can assume that \( j = 1 \) without loss of generality. We denote by \( l[p, \xi_0] \) the line segment \( p \xi_0 \). Since \( l[p, \xi_0] \) does not intersect \( \cup_{j=2}^N \partial D_j \), there exists a constant \( \delta > 0 \) such that \( U_{\delta} \cap (\cup_{j=2}^N \partial D_j) = \emptyset \), where \( U_{\delta} = \{ z \in \mathbb{R}^3 \mid \text{dist}(z, l[p, \xi_0]) < \delta \} \). We can take this \( \delta > 0 \) small enough that \( U_{\delta} \cap \partial D_1 \subset \mathcal{G}^+(p) \) since \( \xi_0 \in \mathcal{G}^+(p) \). Note that \( U_{\delta} \cap \partial D_1 \) is an open set in \( \partial D_1 \). To obtain (1), it suffices to show \( U_{\delta} \cap \partial D_1 \subset \mathcal{G}^+(p) \).

Take any \( \xi \in U_{\delta} \cap \partial D_1 \). Since \( \partial D_1 \) is convex set, \( l[p, \xi] \subset U_{\delta} \), which yields \( l[p, \xi] \cap (\cup_{j=2}^N \partial D_j) = \emptyset \). From \( U_{\delta} \cap \partial D_1 \subset \mathcal{G}^+(p) \), \( \nu_\xi \cdot (p - \xi) > 0 \), which means \( l[p, \xi] \cap D_1 = \emptyset \) since \( D_1 \) is convex and \( \nu_\xi \) is the outer normal of \( \partial D_1 \) at \( \xi \). Thus, \( l[p, \xi] \cap \partial D = \{ \xi \} \), i.e. \( \xi \in \mathcal{G}^+(p) \) is shown, which implies (1) of Lemma 4.1.

Next, we show (2). Take any point \( (\xi(0), y(0)) \in \mathcal{M}(p) \). Since \( (\xi(0), y(0)) \in \mathcal{M}_1(p) \), \( \xi(0) \in \partial D_m \) holds for some \( m \in \{1, 2, \ldots, N\} \). Since \( \partial D_1 \cap \partial D_m = \emptyset \) if \( i \neq m \), this \( m \) is unique. For this \( m \), we show \( \xi(0) \in \mathcal{G}^{+,0}(p) \). If we obtain this, from (1) of Lemma 4.1, there exists \( \epsilon > 0 \) such that \( U_{2\epsilon}(\xi(0)) \subset \partial D_m \cap \mathcal{G}^{+,0}(p) \). Since \( \mathcal{G}^{+,0}(p) = \mathcal{G}_c^{+}(p) \cap \mathcal{G}^{+,0}(p) \) by the definition of \( \mathcal{G}_c^{+,0}(p) \), we obtain \( U_{2\epsilon}(\xi(0)) \subset \partial D_m \cap \mathcal{G}_c^{+,0}(p) \) if we choose \( \epsilon' > 0 \) sufficiently small enough.

To obtain \( \xi(0) \in \mathcal{G}^{+,0}(p) \), it suffices to show that the line segment \( p \xi(0) \) crosses \( \partial D \) at only \( \xi(0) \). Assume that \( p \xi(0) \) crosses \( \partial D \) at \( \zeta = tp + (1-t)\xi(0) \) in \( \partial D \) for some \( 0 < t \leq 1 \). If \( y(0) \) does not contain the line \( \xi(0), p \) it follows that \( |\zeta - \xi(0)| + |\xi(0) - y(0)| > |\zeta - y(0)| \). If not, since \( (\xi(0), y(0)) \in \mathcal{M}(p) \), \( \nu_{\xi(0)} \cdot (p - \xi(0)) > 0 \) and \( \nu_{\xi(0)} \cdot (y(0) - \xi(0)) > 0 \), \( y(0) \) is on the line segment \( l[\xi(0), p] \), which yields \( |\zeta - \xi(0)| + |\xi(0) - y(0)| > |\zeta - y(0)| \). In any case, we obtain

\[
I(p, D) = I_p(\xi(0), y(0)) = |p - \xi(0)| + |\xi(0) - y(0)| = |p - \zeta| + |\zeta - \xi(0)| + |\xi(0) - y(0)|
\]

\[
> |p - \zeta| + |\zeta - y(0)| = I_p(\zeta, y(0)) = \lambda(I, D),
\]

which is a contradiction. This completes the proof of Lemma 4.1.

Proof of Theorem 1.1: From the proof of (2) of Lemma 4.1, we can take \( \epsilon > 0 \) and \( \epsilon' > 0 \) in (2) of Lemma 4.1 arbitrary small. From (1.8), we can also assume that

\[
\inf_{(\xi, y) \in U_{2\epsilon}(\xi(0)) \times V_{2\epsilon}(y(0))} H^+(\xi, y, p) > 0.
\]

(4.1)

Take cut-off functions \( \Psi_j \in C_0^2(U_{\epsilon_j}(\xi(j)) \times V_{\epsilon_j}(y(j))) \) with \( \Psi_j(\xi, y) = 1 \) in \( U_{\epsilon_j/2}(\xi(j)) \times V_{\epsilon_j/2}(y(j)) \) and \( \Psi_j(\xi, y) = 0 \) in \( (U_{\epsilon_j/3}(\xi(j)) \times V_{\epsilon_j/3}(y(j)))^c \). and put

\[
I_{0k_1}\lambda(p) = \int_{V_{\epsilon_j}(y(j))} dS_y \lambda(y; \lambda) \int_{U_{\epsilon_j}(\xi(j))} e^{\lambda p(y, \xi)} \Psi_j(\xi, y) G_k(\xi, y, p; \lambda) dS_\xi,
\]

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and define $I_0^{(j)}(\lambda, p)$ by $I_0^{(j)}(\lambda, p) = I_{0j}(\lambda, p) + \lambda^{-1}I_{0j}(\lambda, p)$. Note that there exists a positive constant $c_0$ such that

$$I_p(\xi, y) \geq l(p, D) + c_0 \text{ for } (\xi, y) \in (\partial D \times \partial \Omega) \setminus \left( \cup_{j=1}^{N_1} U_{\varepsilon_j/3}(\xi^{(j)}) \times V_{\varepsilon_j/3}(y^{(j)}) \right)$$

since $I_p(\xi, y) > l(p, D)$ for all $(\xi, y) \in (\partial D \times \partial \Omega) \setminus \left( \cup_{j=1}^{N_1} U_{\varepsilon_j/3}(\xi^{(j)}) \times V_{\varepsilon_j/3}(y^{(j)}) \right)$ being a compact set. The above estimate, (1) and (2) of Proposition 3.5, and Proposition 2.1 imply that there exist constants $C > 0$, $\mu_1 \geq 1$, $0 < \delta_1 \leq 1$ such that

$$\left| \lambda^{\delta_1}I_0(\lambda, p) - \frac{1}{(2\pi)^2} \sum_{j=1}^{N_1} I_0^{(j)}(\lambda, p) \right| \leq Ce^{-\mu_1(p, D)}e^{-c_0\mu}(\mu + \delta^{-4}e^{\delta_1\mu})\|g\|_{C(\partial\Omega)} \quad (4.2)$$

for any $0 < \delta \leq \delta_1$, $\lambda \in C_{\delta_0}$, $\mu \geq \delta^{-3}\mu_1$.

Take local coordinates $\xi = s^{(j)}(\sigma)$ and $y = \tilde{s}^{(j)}(\tilde{\sigma})$ in $U_{2\varepsilon_j}(\xi^{(j)})$ and $V_{2\varepsilon_j}(y^{(j)})$ with $\xi^{(j)} = s^{(j)}(0)$ and $y^{(j)} = \tilde{s}^{(j)}(0)$ respectively. We put $\tilde{\Psi}_j(\sigma, \tilde{\sigma}) = \Psi_j(s^{(j)}(\sigma), \tilde{s}^{(j)}(\tilde{\sigma}))$ and $\tilde{l}_p^{(j)}(\sigma, \tilde{\sigma}) = \tilde{l}_p(s^{(j)}(\sigma), \tilde{s}^{(j)}(\tilde{\sigma}))$, and write $J_j(\sigma, \tilde{\sigma})$ as the local coordinate expressions of the surface elements. Using these coordinates and notations, we obtain

$$I_0^{(j)}(\lambda, p) = \int_{\mathbb{R}^4} e^{-\lambda\tilde{l}^{(j)}_p(\sigma, \tilde{\sigma})} \tilde{\Psi}_j(\sigma, \tilde{\sigma})\alpha^{(j)}(\sigma, \tilde{\sigma}; \lambda) d\sigma d\tilde{\sigma},$$

where $\alpha^{(j)}(\sigma, \tilde{\sigma}; \lambda)$ is defined by

$$\alpha^{(j)}(\sigma, \tilde{\sigma}; \lambda) = J_j(\sigma, \tilde{\sigma})\lambda^{\beta_0} A^{(j)}(\tilde{\sigma}; \lambda)
\phi_{\beta_0}(\tilde{\sigma}^{(j)}(\tilde{\sigma}); \lambda)(A_0^{(j)}(\tilde{\sigma}^{(j)}(\tilde{\sigma}); \lambda)
|\sigma^{(j)}(\tilde{\sigma})|^{\lambda}(\sigma^{(j)}(\tilde{\sigma}), \tilde{\sigma}; \lambda)\lambda^{\beta_0}(\sigma^{(j)}(\tilde{\sigma}), \tilde{\sigma}; \lambda)\lambda^{-1}(\sigma^{(j)}(\tilde{\sigma}), \tilde{\sigma}; \lambda)\lambda^{-1}(\sigma^{(j)}(\tilde{\sigma}), \tilde{\sigma}; \lambda)\lambda^{-1}(\sigma^{(j)}(\tilde{\sigma}), \tilde{\sigma}; \lambda).$$

Since $\overline{U_{2\varepsilon_j}(\xi^{(j)})} \subset D_{m_j} \cap \mathcal{G}_{\varepsilon_j}(\rho)$ for each $j = 1, 2, \ldots, N_1$, (3.10), Proposition 2.1 and (3) and (4) of Proposition 3.5 imply that there exist constants $C > 0$, $d_2 > 0$ and $0 < \delta_2 \leq 1$ such that

$$|G_0(\xi, y, p; \lambda) - H^+(\xi, y, p; \lambda)| \leq C(\mu^{-1} + \delta_2^{-1}e^{-\mu_2d_2}), \quad |G_1(\xi, y, p; \lambda)| \leq C \quad (4.3)$$

$$((\xi, y) \in \overline{U_{\varepsilon_j}(\xi^{(j)})} \times \partial\Omega, \lambda \in C_{\delta_0} \text{ with } \text{Re} \lambda \geq \mu_1\delta_2^{-3})$$

From (2.4) and (1.5), it follows that

$$\lambda^{\beta_0} \phi(y; \lambda) = \lambda^{\beta_0} g(y; \lambda) + O(\lambda^{-1}) \quad \text{(uniformly in } y \in \partial\Omega, \lambda \in C_{\delta_0} \text{ as } |\lambda| \to \infty),$$

and there exist constants $C > 0$ and $\mu_2 > 0$ such that

$$\text{Re} [\lambda^{\beta_0} g(y; \lambda)] \geq C \quad (y \in \partial\Omega, \lambda \in C_{\delta_0}, \mu \geq \mu_2).$$

Combining these estimates with (4.1), we obtain the following decomposition of $\alpha^{(j)}$:

$$\alpha^{(j)}(\sigma, \tilde{\sigma}; \lambda) = \alpha_1^{(j)}(\sigma, \tilde{\sigma}; \lambda) + \lambda^{-1}\tilde{\alpha}_1^{(j)}(\sigma, \tilde{\sigma}; \lambda),$$

$$\text{Re} [\alpha_1^{(j)}(\sigma, \tilde{\sigma}; \lambda)] \geq C, \quad |\alpha_1^{(j)}(\sigma, \tilde{\sigma}; \lambda)| + |\tilde{\alpha}_1^{(j)}(\sigma, \tilde{\sigma}; \lambda)| \leq C' \quad ((\sigma, \tilde{\sigma}) \in \text{supp} \tilde{\Psi}_j)$$

for some constants $C > 0$ and $C' > 0$. Note that there exists a constant $C > 0$ such that

$$l(p, D) \leq \tilde{l}_p^{(j)}(\sigma, \tilde{\sigma}) \leq l(p, D) + C|\sigma|^2 + |\tilde{\sigma}|^2 \quad ((\sigma, \tilde{\sigma}) \in \text{supp} \tilde{\Psi}_j)$$

since $\tilde{l}_p^{(j)}(\sigma, \tilde{\sigma})$ is $C^2$ and $\nabla_{(\sigma, \tilde{\sigma})}\tilde{l}_p^{(j)}(0, 0) = 0$.

From these properties of $\alpha^{(j)}$ and $\tilde{l}_p^{(j)}(\sigma, \tilde{\sigma})$, it easily follows that there exists a constant $C > 0$ such that

$$|e^{\lambda(l(p, D)}I_0^{(j)}(\lambda, p)| \leq C \quad \text{(uniformly in } \lambda \in \Lambda_{\delta_0} \text{ as } |\lambda| \to \infty).$$
For lower bounds, the arguments for the Laplace integrals of some type given in Section 7 of [14] implies that there exist constants \( \delta_1 > 0 \) and \( C > 0 \) such that

\[
\text{Re}[e^{\lambda(p, D)} I^{(j)}_0(\lambda, p)] \geq C \lambda^{-1} \quad \text{uniformly in } \lambda \in \Lambda_{\delta_1} \text{ as } |\lambda| \to \infty.
\]

Note that the Laplace integrals appeared in [14] are of the cases that the principal part of the amplitude functions, corresponding to the part \( a^{(j)}_1(\cdot) \) of \( a^{(j)} \) for our case, does not contain the parameter \( \lambda \). Thus, the types of the integrals are slightly different from each other. From this reason and for the paper to be self-contained, a proof for the above estimate is given in Section 6 (cf. Proposition 6.1). Combining (4.2) with the above estimates, we obtain Theorem 1.1.

**Proof of Theorem 1.3:** In this case, since (I.1), (I.2) and (I.3) are assumed, \( M^+_{\sigma}(p) \cup M^-_{\sigma}(p) \cup M_{\sigma}(p) = \emptyset \), and each point in \( M(p) \) is non-degenerate critical point of \( l_p(\xi, y) \). These imply that \( M(p) \) is discrete set, which is expressed by \( M(p) = M_1(p) = \{ (\xi^{(j)}, y^{(j)}) \mid j = 1, 2, \ldots, N_1 \} \). From (2) of Lemma 4.1, for any \( j = 1, 2, \ldots, N_1 \), there exist constants \( \varepsilon_j > 0 \) and \( \varepsilon'_j > 0 \) such that \( \big( U_{2\varepsilon'}(\xi^{(j)}) \times V_{2\varepsilon'}(y^{(j)}) \big) \cap M(p) = \{ (\xi^{(j)}, y^{(j)}) \} \) and \( \overline{U_{2\varepsilon'}(\xi^{(j)})} \subset \partial D_{\varepsilon'} \cap G^{-0}_{\sigma}(p) \). In this case, we can also obtain (4.2) and (4.3).

Taking local coordinates \( \xi = s^{(j)}(\sigma) \) and \( y = \tilde{s}^{(j)}(\tilde{\sigma}) \) in \( U_{\varepsilon'}(\xi^{(j)}) \) and \( V_{\varepsilon'}(y^{(j)}) \) with \( \xi^{(j)} = s^{(j)}(0) \) and \( y^{(j)} = \tilde{s}^{(j)}(0) \) respectively, in this case, we decompose \( \tilde{I}^{(j)}_0(\lambda, p) \) into \( \tilde{I}^{(j)}_0(\lambda, p) = \tilde{I}_{01j}(\lambda, p) + \lambda^{-1} \tilde{I}_{01j}(\lambda, p) \), where for each \( j = 1, 2, \ldots, N_1 \),

\[
\tilde{I}_{0kj}(\lambda, p, \sigma) = \int_{\varepsilon'} e^{-L_{\sigma}(\sigma, \tilde{\sigma})} \phi_j(\sigma, \tilde{\sigma}) \beta^{(j)}(\sigma; \lambda) d\tilde{\sigma} \quad (k = 0, 1),
\]

\[
\beta^{(j)}_0(\sigma; \lambda) = \lambda^{\delta_0} g(s^{(j)}(\tilde{\sigma}; \lambda)) H^{+}(s^{(j)}(\tilde{\sigma}; \lambda), \tilde{s}^{(j)}(\tilde{\sigma}; \lambda), p) J_j(\sigma, \tilde{\sigma}), \text{ and } \beta^{(j)}_1(\sigma; \lambda) = \beta^{(j)}_0(\sigma; \lambda) \text{ is given by}
\]

\[
\beta^{(j)}_1(\sigma; \lambda) = \lambda^{\delta_0 + 1} \varphi(s^{(j)}(\tilde{\sigma}); \lambda) - g(s^{(j)}(\tilde{\sigma}; \lambda)) H^{+}(s^{(j)}(\tilde{\sigma}); \lambda, \tilde{s}^{(j)}(\tilde{\sigma}; \lambda), p) J_j(\sigma, \tilde{\sigma})
\]

\[
+ \lambda^{\delta_0} \varphi(s^{(j)}(\tilde{\sigma}); \lambda) \lambda(G_0(s^{(j)}(\sigma), \tilde{s}^{(j)}(\tilde{\sigma}; \lambda), p, \lambda) - H^{+}(s^{(j)}(\sigma), \tilde{s}^{(j)}(\tilde{\sigma}; \lambda), p)) J_j(\sigma, \tilde{\sigma})
\]

Each \( (\xi^{(j)}, y^{(j)}) \) is non-degenerate, \( \text{Hess}(\tilde{t}_p(\xi^{(j)}))(0, 0) > 0 \) holds. Since \( \lambda^{\delta_0} g(s^{(j)}(\sigma); \lambda) \) is uniformly continuous in \( \sigma \in U_{2\varepsilon'}(\xi^{(j)}) \) with respect to \( \lambda \in \mathbb{C}_{\delta_0} \), \( \lim_{\sigma \to 0} \beta^{(j)}_0(\sigma; \lambda) = \beta^{(j)}_0(0; \lambda) \) uniformly in \( \lambda \in \mathbb{C}_{\delta_0} \). From (1.5), \( \beta^{(j)}_0(0; \lambda) \) is bounded for \( \lambda \in \mathbb{C}_{\delta_0} \). Further, (4.3) and (4.4) yield that \( \beta^{(j)}_1(\sigma; \lambda) \) is uniformly bounded for \( \sigma \in U_{2\varepsilon'}(\xi^{(j)}) \) and \( \lambda \in \mathbb{C}_{\delta_0} \), Hence, Laplace method (cf. Proposition 6.2) implies

\[
\tilde{I}_{01j}(\lambda, p) = e^{-\lambda(\xi, p)} \| g(\cdot, \lambda) \|_{C^2(\partial \Omega)} O(\lambda^{\delta_0 - 2})
\]

and

\[
\tilde{I}_{00j}(\xi, y) = \frac{J_j(0, 0) e^{-\lambda(\xi, p)}}{\sqrt{\det(\text{Hess}(\tilde{t}_p(\xi^{(j)}))(0, 0))}} \left( \frac{2\pi}{\lambda} \right)^2 \left( \lambda^{\delta_0} g(y^{(j)}; \lambda) H^{+}(\xi^{(j)}, y^{(j)}, p) + o(1) \right)
\]

as \( |\lambda| \to \infty \) uniformly for \( \lambda \in \mathbb{C}_{\delta_0} \). From (1.8), it follows that \( H^{+}(\xi^{(j)}, y^{(j)}, p) > 0 \) holds since \( (\xi^{(j)}, y^{(j)}) \in M_1(p) \). This completes the proof of Theorem 1.3.

Note that if \( \partial D \) and \( \partial \Omega \) are \( C^{2, \alpha_0} \) for some \( 0 < \alpha_0 < 1 \), and \( g(\cdot, \lambda) \in C^{0, \alpha_0}(\partial \Omega) \), it holds that \( \beta^{(j)}_k(\cdot; \lambda) \in C^{0, \alpha_0} \) near \( \sigma = 0 \). Hence, from Remark 6.3 in Section 6, we obtain (2) of Remark 1.4.

### 5 The influence from the off-diagonal parts

In this section, a proof of Proposition 3.3 is given. As in Proposition 3.3 and estimate (3.5), the integral kernels of the operators \( M^{(j)}_{D_j}(\lambda) \) and \( M_{D_j}(\lambda) = iY^{22}_{j}(\lambda)(I - iY^{22}_{j}(\lambda))^{-1} \), which are for the
There exist constants $C > 0$ and $d_1 > 0$ such that
\[
\int_{\partial D_p} \left( \mu + \frac{1}{|\xi - \zeta|} \right) e^{-\mu |\xi - \zeta|} dS_\zeta \leq C \delta^{-1} e^{-d_1 \mu} \quad (\xi \in \partial D_i, i, p = 1, 2, \ldots, N, i \neq p, 0 < \delta \leq 1, \mu > 0),
\]
\[
\int_{\partial D_i} \left( \mu + \frac{1}{|\xi - \zeta|} \right) e^{-\mu |\xi - \zeta|} dS_\zeta \leq C \delta^{-1 - q} \mu^{-q} \quad (\xi \in \partial D_i, i = 1, 2, \ldots, N, q = 0, 1, 0 < \delta \leq 1, \mu > 0).
\]

Proof: Recalling (3.6), we obtain $|\xi - \zeta| \geq 2d_1$ ($\xi \in \partial D_i, \zeta \in \partial D_p$) for $i \neq p$. This implies that there exists a constant $C > 0$ such that
\[
\int_{\partial D_p} \left( \mu + \frac{1}{|\xi - \zeta|} \right) e^{-\mu |\xi - \zeta|} dS_\zeta \leq \left( \mu + \frac{1}{2d_1} \right) e^{-2\mu \delta d_1} \text{Vol}(\partial D_p)
\]
\[
\leq C \delta^{-1} e^{-\mu \delta d_1} \quad (0 < \delta \leq 1, \xi \in \partial D_i).
\]

For the case $i = p$, from $\mu |\xi - \zeta| e^{-\mu |\xi - \zeta|} \leq 1$, it follows that
\[
\int_{\partial D_p} \left( \mu + \frac{1}{|\xi - \zeta|} \right) e^{-\mu |\xi - \zeta|} dS_\zeta \leq 2\delta^{-1} \int_{\partial D_p} \frac{dS_\zeta}{|\xi - \zeta|} \quad (0 < \delta \leq 1).
\]

The above estimate and (3.12) imply the estimate for the case $q = 0$. For the case $q = 1$, from (ii) of Lemma 3.7, for $0 < \delta \leq 1$, it follow that
\[
\int_{\partial D_p} \left( \mu + \frac{1}{|\xi - \zeta|} \right) e^{-\mu |\xi - \zeta|} dS_\zeta \leq C \left( \mu (\delta \mu)^{-2} + (\delta \mu)^{-1} \right) \leq C \delta^{-2} \mu^{-1},
\]
which completes the proof of Lemma 5.1.

We put $Y_D(\lambda) = \text{diag}(^t Y_{22}^1(\lambda), ^t Y_{22}^2(\lambda), \ldots, ^t Y_{22}^N(\lambda))$, where $\text{diag}(a_1, a_2, \ldots, a_N)$ is the diagonal matrix with $(p, q)$-component $a_p \delta_{pq}$, and $\delta_{pq}$ is Kronecker’s delta. Note that $Y_D(\lambda)(I - Y_D(\lambda))^{-1}$ is given by
\[
Y_D(\lambda)(I - Y_D(\lambda))^{-1} = \text{diag}(M_D(\lambda), M_D(\lambda), \ldots, M_D(\lambda)).
\]

To handle off-diagonal parts, we introduce $W(\lambda) = (^t Y_{22}(\lambda) - Y_D(\lambda))(I - Y_D(\lambda))^{-1}$ and $\tilde{W}(\lambda) = (^t Y_{22}(\lambda) - Y_D(\lambda))Y_D(\lambda)(I - Y_D(\lambda))^{-1}$. Noting
\[
I - ^t Y_{22}(\lambda) = I - Y_D(\lambda) - (^t Y_{22}(\lambda) - Y_D(\lambda))
\]
\[
= (I - (^t Y_{22}(\lambda) - Y_D(\lambda))(I - Y_D(\lambda))^{-1})(I - Y_D(\lambda)),
\]
we obtain
\[
(I - ^t Y_{22}(\lambda))(I - Y_D(\lambda))^{-1} = I - W(\lambda). \quad (5.1)
\]

We define the operators $W^{ij}(\lambda)$ and $\tilde{W}^{ij}(\lambda)$ by
\[
W(\lambda)f(\xi) = \sum_{j=1}^{N} W^{ij}(\lambda)f_j(\xi), \quad \tilde{W}(\lambda)f(\xi) = \sum_{j=1}^{N} \tilde{W}^{ij}(\lambda)f_j(\xi) \quad (\xi \in \partial D_i)
\]
for $f \in C(\partial D)$ and $i = 1, 2, \ldots, N$. Since each $(i, j)$-component of
\[
(^t Y_{22}(\lambda) - Y_D(\lambda))Y_D(\lambda)(I - Y_D(\lambda))^{-1}
\]
with \( i \neq j \) and \( i, j = 1, 2, \ldots, N \) is given by \( Y_{22}^{ij}(\lambda)M_{D_j}(\lambda) \), and

\[
W(\lambda) = (tY_{22}(\lambda) - Y_D(\lambda))(I - Y_D(\lambda))^{-1}
= (tY_{22}(\lambda) - Y_D(\lambda)) + (tY_{22}(\lambda) - Y_D(\lambda))(I - Y_D(\lambda))^{-1},
\]

we obtain the following relations:

\[
W^{ii}(\lambda) = \tilde{W}^{ii}(\lambda) = 0 \quad (i = 1, 2, \ldots, N),
W^{ij}(\lambda) = tY_{22}^{ij}(\lambda) + \tilde{W}^{ij}(\lambda), \quad \tilde{W}^{ij}(\lambda) = tY_{22}^{ij}(\lambda)M_{D_j}(\lambda) \quad (i, j = 1, 2, \ldots, N, i \neq j).
\]

From the definition of \( W(\lambda) \), \( (I - W(\lambda))^{-1} \) exists for \( \lambda \in C_{\delta_0}, \mu \geq \mu_0 \) by choosing \( \mu_0 > 0 \) larger if necessary. In what follows, we put \( W^\infty(\lambda) = W(\lambda)(I - W(\lambda))^{-1} \), which can also be written as

\[
W^\infty(\lambda)f(\xi) = \sum_{j=1}^{N} W^\infty_{ij}(\lambda)f_j(\xi) \quad (\xi \in \partial D_1, f \in C(\partial D))
\]

by using the operators \( W^\infty_{ij}(\lambda) \in B(C(\partial D_j), C(\partial D_i)) \). We denote by \( W^{ij}(\xi, \zeta; \lambda) \) and \( W^\infty_{ij}(\xi, \zeta; \lambda) \) the integral kernel of \( W^{ij}(\lambda) \) and \( W^\infty_{ij}(\lambda) \) respectively.

We need the following estimates of \( W^{ij}(\xi, \zeta; \lambda) \):

**Proposition 5.2** There exist constants \( d_1 > 0 \) and \( C_1 > 0 \) such that for all \( i, j = 1, 2, \ldots, N \) with \( i \neq j \) and \( 0 < \delta \leq 1 \), the integral kernel \( W^{ij}(\xi, \zeta; \lambda) \) is estimated by

\[
|W^{ij}(\xi, \zeta; \lambda)| \leq C_1 \delta^{-2} e^{-\delta d_1} e^{-\delta(1-\delta)\mu|\xi - \zeta|} \quad (\xi \in \partial D_1, \zeta \in \partial D_2, \lambda \in C_{\delta_0}, \mu \geq \mu_0).
\]

Note that from the definition of \( W^{ij}(\lambda) \), \( W^{ij}(\xi, \zeta; \lambda) = 0 \) \( (\xi, \zeta \in \partial D_j, j = 1, 2, \ldots, N) \).

**Proof of Proposition 5.2**: Assume that \( i \neq j \). Since \( \tilde{W}^{ij}(\lambda) = tY_{22}^{ij}(\lambda)M_{D_j}(\lambda) \), the integral kernel \( \tilde{W}^{ij}(\xi, \zeta; \lambda) \) of \( \tilde{W}^{ij}(\lambda) \) has the following integral representation:

\[
\tilde{W}^{ij}(\xi, \zeta; \lambda) = \int_{\partial D_j} tY_{22}^{ij}(\xi, \eta; \lambda)M_{D_j}(\eta, \zeta; \lambda)dS_\eta \quad (\xi \in \partial D_1, \zeta \in \partial D_2). \tag{5.2}
\]

The above representation, estimates (3.2) and (3.5), and \( |\xi - \eta| \geq 2d_1 \) \( (\xi \in \partial D_1, \eta \in \partial D_2) \) imply that

\[
|\tilde{W}^{ij}(\xi, \zeta; \lambda)| \leq C \int_{\partial D_j} \left( \mu + \frac{1}{|\xi \eta \zeta|} \right) e^{-\mu |\xi - \eta|} \left( \mu + \frac{1}{|\eta - \zeta|} \right) e^{-\mu|\eta - \zeta|} dS_\eta
\]

\[
\leq C \mu \int_{\partial D_j} \left( \mu + \frac{1}{|\eta - \zeta|} \right) e^{-\mu(\delta d_1)\mu|\eta - \zeta|} dS_\eta
\]

\[
(\xi \in \partial D_1, \zeta \in \partial D_2, \lambda \in C_{\delta_0}, \mu \geq \mu_0).
\]

For \( \xi \in \partial D_1, \eta, \zeta \in \partial D_2 \), it follows that

\[
|\xi - \eta| + |\eta - \zeta| \geq \delta|\xi - \eta| + \delta|\eta - \zeta| + (1 - \delta)|\xi - \zeta|
\geq 2d_1 \delta + \delta|\eta - \zeta| + (1 - \delta)|\xi - \zeta|.
\]

From this estimate and Lemma 5.1, we obtain

\[
\int_{\partial D_j} \left( \mu + \frac{1}{|\eta - \zeta|} \right) e^{-\mu(\delta d_1)\mu|\eta - \zeta|} dS_\eta
\leq e^{-2d_1 \delta \mu} e^{-\mu(1-\delta)\mu|\xi - \zeta|} \int_{\partial D_j} \left( \mu + \frac{1}{|\eta - \zeta|} \right) e^{-\mu \delta \mu|\eta - \zeta|} dS_\eta
\leq C \delta^{-2} e^{-2d_1 \delta \mu} e^{-\mu(1-\delta)\mu|\xi - \zeta|} \quad (0 < \delta \leq 1, \xi \in \partial D_1, \zeta \in \partial D_2).
\]
Thus, we can find a constant $C > 0$ satisfying
\[
| \overline{W}^{ij}(\xi, \zeta; \lambda) | \leq C \delta^{-2} e^{-2d_1 \delta \mu} e^{-(1-\delta) \mu |\xi - \zeta|} \quad (\xi \in \partial D_i, \zeta \in \partial D_j, i \neq j, 0 < \delta \leq 1). \tag{5.3}
\]
Estimate (5.3) and (3.2), and $W^{ij}(\xi, \zeta; \lambda) = \overline{Y}^{ij}_{22}(\xi, \zeta; \lambda) + \overline{W}^{ij}(\xi, \zeta; \lambda)$ for $i \neq j$ imply Proposition 5.2.

From Proposition 5.2, we can give estimates of $W^{\infty,ij}(\xi, \zeta; \lambda)$.

**Proposition 5.3** There exists a constant $\mu_1 > 0$ such that for all $i, j = 1, 2, \ldots, N$ and $0 < \delta \leq 1$, the integral kernel $W^{\infty,ij}(\xi, \zeta; \lambda)$ is estimated by
\[
| W^{\infty,ij}(\xi, \zeta; \lambda) | \leq 2C_1 \delta^{-2} e^{-d_1 \delta \mu} e^{-(1-\delta) \mu |\xi - \zeta|} \quad (\xi \in \partial D_i, \zeta \in \partial D_j, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_1 \delta^{-3}),
\]
where $C_1 > 0$ and $d_1 > 0$ are the constants given in Proposition 5.2. Further, there also exist constants $0 < \delta_1 \leq 1$ and $d_3 > 0$ such that for any $j = 1, 2, \ldots, N$ and $0 < \delta \leq \delta_1$, $W^{\infty,ij}(\xi, \zeta; \lambda)$ is estimated by
\[
| W^{\infty,ij}(\xi, \zeta; \lambda) | \leq 2C_1 \delta^{-2} e^{-d_1 \delta \mu} e^{-2d_3 \delta \mu} e^{-\mu |\xi - \zeta|} \quad (\xi, \zeta \in \partial D_j, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_1 \delta^{-3}).
\]

Proof: We start to get estimates of the repeated kernels of the integral operator $W(\lambda)$. We put $W^{(n)}(\lambda) = (W(\lambda))^n (n = 1, 2, \ldots)$, and denote by $W^{(n)}_{ij}(\lambda)$ the $(i, j)$-components of $W^{(n)}(\lambda)$, and by $W^{(n)}_{ij}(\xi, \zeta; \lambda) (\xi \in \partial D_i, \zeta \in \partial D_j)$ the integral kernel of $W^{(n)}_{ij}(\lambda)$. Then it follows that
\[
W^{(n)}_{ij}(\lambda) f_j(\xi) = \int_{\partial D_j} W^{(n)}_{ij}(\xi, \zeta; \lambda) f_j(\zeta) dS_\zeta \quad (\xi \in \partial D_i).
\]
By induction, we show
\[
| W^{(n)}_{ij}(\xi, \zeta; \lambda) | \leq C_2^{n-1} (C_1 \delta^{-2} e^{-\mu d_1}) n e^{-(1-\delta) \mu |\xi - \zeta|} \quad (\xi \in \partial D_i, \zeta \in \partial D_j, 0 < \delta \leq 1, i, j = 1, 2, \ldots, N, n = 1, 2, \ldots), \tag{5.4}
\]
where $C_2 = \text{Vol}(\partial D) = \sum_{p=1}^{N} \text{Vol}(\partial D_p) > 0$. From $W^{(1)}_{ij}(\xi, \zeta; \lambda) = W^{(1)}_{ij}(\xi, \zeta; \lambda)$, Proposition 5.2 shows that the case $n = 1$ is true. Assume that the case less than or equal to $n$ are true.

Note that the kernel $W^{(n+1)}_{ij}(\xi, \zeta; \lambda)$ is given by
\[
W^{(n+1)}_{ij}(\xi, \zeta; \lambda) = \sum_{p=1}^{N} \int_{\partial D_p} W^{(n)}_{ip}(\xi, \eta; \lambda) W^{(n)}_{pj}(\eta, \zeta; \lambda) dS_\eta. \tag{5.5}
\]
Hence Proposition 5.2 and the assumption of induction imply that
\[
| W^{(n+1)}_{ij}(\xi, \zeta; \lambda) | \leq C_2^{n-1} (C_1 \delta^{-2} e^{-\mu d_1}) n+1 \sum_{p=1}^{N} \int_{\partial D_p} e^{-(1-\delta) \mu (|\xi - \eta| + |\eta - \zeta|)} dS_\eta.
\]
From $e^{-(1-\delta) \mu (|\xi - \eta| + |\eta - \zeta|)} \leq e^{-(1-\delta) \mu |\xi - \zeta|}$, it follows that
\[
\int_{\partial D_p} e^{-(1-\delta) \mu (|\xi - \eta| + |\eta - \zeta|)} dS_\eta \leq e^{-(1-\delta) \mu |\xi - \zeta|} \text{Vol}(\partial D_p).
\]
This implies
\[
| W^{(n+1)}_{ij}(\xi, \zeta; \lambda) | \leq C_2^{n} (C_1 \delta^{-2} e^{-\mu d_1}) n+1 e^{-(1-\delta) \mu |\xi - \zeta|},
\]
which means that the case $n + 1$ is also true. Thus, we obtain (5.4).

For handling the diagonal parts $W^{(n)}_{jj}(\xi, \zeta, \lambda)$, we need the following lemma:
Lemma 5.4 There exist $0 < \delta_1 \leq 1$ and $d_3 > 0$ such that

$$(1 - \delta)(|\xi - \eta| + |\eta - \zeta|) \geq |\xi - \zeta| + 2d_3 \quad (\xi, \zeta \in \partial D_j, \eta \in \partial D_p, j \neq p, 0 < \delta \leq \delta_1).$$

Proof: We put $\varphi(\xi, \zeta, \eta, \delta) = (1 - \delta)(|\xi - \eta| + |\eta - \zeta|) - |\xi - \zeta|$. The function $\varphi(\xi, \zeta, \eta, 0)$ is continuous on the compact set $\bigcup_{j=1}^N \partial D_j \times \partial D_j \times (\partial D \setminus \partial D_j)$, and $\varphi(\xi, \zeta, \eta, 0) > 0$ if $((\xi, \zeta, \eta) \in \bigcup_{j=1}^N \partial D_j \times \partial D_j \times (\partial D \setminus \partial D_j))$, which yields

$$d_3 = 3^{-1} \inf \{ \varphi(\xi, \zeta, \eta, 0) \mid (\xi, \zeta, \eta) \in \bigcup_{j=1}^N \partial D_j \times \partial D_j \times (\partial D \setminus \partial D_j) \} > 0.$$  

We put $d'_+ = \max\{|\xi - \zeta| \mid \xi \in \partial D_i, \zeta \in \partial D_j, i, j = 1, 2, \ldots, N, i \neq j \} > 0$. Note that

$$|\varphi(\xi, \zeta, \eta, \delta) - \varphi(\xi, \zeta, \eta, 0)| = \delta(\xi - \eta) + \delta(\eta - \zeta) \leq 2d'_+\delta$$

$$(\xi, \zeta, \eta) \in \bigcup_{j=1}^N \partial D_j \times \partial D_j \times (\partial D \setminus \partial D_j),$$

we put $\delta_1 = \min\{1, (2d'_+)^{-1}d_3\}$. Then $0 < \delta_1 \leq 1$, and for $0 < \delta \leq \delta_1$ and $(\xi, \zeta, \eta) \in \bigcup_{j=1}^N \partial D_j \times \partial D_j \times (\partial D \setminus \partial D_j)$,

$$\varphi(\xi, \zeta, \eta, \delta) \geq \varphi(\xi, \zeta, \eta, 0) - |\varphi(\xi, \zeta, \eta, \delta) - \varphi(\xi, \zeta, \eta, 0)| \geq 3d_2 - 2d'_+\delta \geq 2d_3,$$

which completes the proof of Lemma 5.4.

Now estimates of $W^{(n)}_{jj}(\xi, \zeta; \lambda)$ are given as follows: Noting $W^{(n)}_{jj}(\xi, \zeta; \lambda) = 0$ ($j = 1, 2, \ldots, N$) and (5.5), for $n \geq 2$ we have

$$W_{jj}^{(n)}(\xi, \zeta; \lambda) = \sum_{p \neq j} \int_{\partial D_p} W_{jp}(\xi, \eta; \lambda) W_{pj}^{(n-1)}(\eta, \zeta; \lambda) dS_\eta.$$ 

The above equality, Proposition 5.2 and (5.4) imply that for any $\xi, \zeta \in \partial D_j$,

$$|W_{jj}^{(n)}(\xi, \zeta; \lambda)| \leq C_2^{n-2}(C_1 \delta^{-2}e^{-\mu d_1})^n \sum_{p \neq j} \int_{\partial D_p} e^{-(1-\delta)|\xi - \eta| + |\eta - \zeta|} dS_\eta.$$ 

Since Lemma 5.4 yields that there exist $0 < \delta_1 \leq 1$ and $d_2 > 0$ such that for any $0 < \delta \leq \delta_1$ and $j \neq p$,

$$(1 - \delta)(|\xi - \eta| + |\eta - \zeta|) \geq |\xi - \zeta| + 2d_3 \quad (\xi, \zeta \in \partial D_j, \eta \in \partial D_p, 0 < \delta \leq \delta_1),$$

which yields

$$\int_{\partial D_p} e^{-(1-\delta)|\xi - \eta| + |\eta - \zeta|} dS_\eta \leq \text{Vol}(\partial D_p) e^{-\mu \xi - \zeta} e^{-2\mu d_3} \quad (\xi, \zeta \in \partial D_j, p \neq j).$$

From these estimates, we obtain

$$|W_{jj}^{(n)}(\xi, \zeta; \lambda)| \leq C_2^{n-1}(C_1 \delta^{-2}e^{-\mu d_1})^n e^{-\mu |\xi - \zeta|} e^{-2\mu d_3} \quad (\xi, \zeta \in \partial D_j, 0 < \delta \leq \delta_1, j = 1, 2, \ldots, N).$$

(5.6)

Now we put $\mu_1 = \max\{\mu_0, 2C_1C_2/d_1\} \geq \mu_0$. For $\mu \geq \delta^{-3}\mu_1$, it follows that

$$C_1C_2 \delta^{-2}e^{-\mu d_1} \leq C_1C_2 \delta^{-2}(\mu d_1)^{-1}(\mu d_1)e^{-\mu d_1} \leq C_1C_2d_1^{-1}\delta^{-3}\mu^{-1} \leq 1/2.$$ 

This estimate, (5.4) and (5.6) imply

$$|W_{ij}^{(n)}(\xi, \zeta; \lambda)| \leq C_1 \delta^{-2}e^{-\mu d_1} \left(\frac{1}{2}\right)^{n-1} e^{-(1-\delta)|\xi - \zeta|} \quad (\xi \in \partial D_i, \zeta \in \partial D_j, 0 < \delta \leq 1).$$
\[ |W_{ij}^{(n)}(\xi, \zeta; \lambda)| \leq C_1 \delta^{-2} e^{-\mu_1 d_1} \left(\frac{1}{2}\right)^{n-1} e^{-\mu |\xi - \zeta| e^{-2\mu_2}} \quad (\xi, \zeta \in \partial D_2, 0 < \delta \leq \delta_1). \]

Noting that \( W_{ij}^{(n)}(\xi, \zeta; \lambda) = \sum_{n=1}^{\infty} W_{ij}^{(n)}(\xi, \zeta; \lambda) \) \( W_{ij}^{(\infty)}(\lambda) = \sum_{n=1}^{\infty} W_{ij}^{(n)}(\lambda) \), we obtain
\[
|W_{ij}^{(\infty)}(\xi, \zeta; \lambda)| \leq 2C_1 \delta^{-2} e^{-\mu_1 d_1} e^{-(1-\delta)\mu |\xi - \zeta|} \quad (\xi, \zeta \in \partial D_1, 0 < \delta \leq \delta_1),
\]
\[
|W_{ij}^{(\infty)}(\xi, \zeta; \lambda)| \leq 2C_1 \delta^{-2} e^{-\mu_1 d_1} e^{-(1-\delta)\mu |\xi - \zeta|} e^{-2\mu_2} \quad (\xi, \zeta \in \partial D_2, 0 < \delta \leq \delta_1),
\]
which completes the proof of Proposition 5.3.

Now, we proceed to get estimates for the integral kernel of \( M(\lambda) = \left( Y_{22}(\lambda) - Y_{22}(\lambda) \right)^{-1} \). From (5.1), it follows that
\[
\left( I - Y_{22}(\lambda) \right)^{-1} = \left( I - Y_D(\lambda) \right)^{-1} \left( I - W(\lambda) \right)^{-1}
\]
\[
= I + Y_D(\lambda)(I - Y_D(\lambda))^{-1} + W(\lambda)(I - W(\lambda))^{-1}
\]
\[
+ Y_D(\lambda)(I - Y_D(\lambda))^{-1} W(\lambda)(I - W(\lambda))^{-1},
\]
which yields
\[
M(\lambda) = Y_D(\lambda)(I - Y_D(\lambda))^{-1} + W(\lambda)(I - W(\lambda))^{-1}
\]
\[
+ Y_D(\lambda)(I - Y_D(\lambda))^{-1} W(\lambda)(I - W(\lambda))^{-1}
\]
since \( M(\lambda) = \left( Y_{22}(\lambda) - Y_{22}(\lambda) \right)^{-1} = \left( I - Y_{22}(\lambda) \right)^{-1} - I \). We denote by \( M^{ij}(\xi, \zeta; \lambda) \) the \((i, j)\)-components of the integral kernel of \( M(\lambda) \). The above expression implies that
\[
M^{ij}(\xi, \zeta; \lambda) = \delta_{ij} M_{D_j}(\xi, \zeta; \lambda) + W_{ij}^{(\infty)}(\xi, \zeta; \lambda)
\]
\[
+ \int_{\partial D_1} M_{D_i}(\xi, \eta; \lambda) W_{ij}^{(\infty)}(\eta, \zeta; \lambda) dS_\eta \quad (\xi \in \partial D_1, \zeta \in \partial D_j). \quad (5.7)
\]
From (2.8) and (2.7), for \( \xi \in \partial D_1, \zeta \in \partial D_j \), the \((i, j)\)-components of the integral kernel \( M^{(1), ij} = M^{(1)}(\xi, \zeta; \lambda) \) of \( M^{(1)}(\lambda) \) is given by
\[
M^{(1), ij}(\xi, \zeta; \lambda) = \frac{1}{2\pi} e^{-\lambda |\xi - \zeta|} H_1(\xi, \zeta) + \sum_{p=1}^{N} \int_{\partial D_p} Y_{22}^{ij}(\xi, \eta; \lambda) M_{D_j}(\eta, \zeta; \lambda) dS_\eta. \quad (5.8)
\]
Since \( M_{D_j}^{(1)} \) is defined by (3.3), \( M_{D_j}^{(1)}(\xi, \zeta; \lambda) \) are written by
\[
M_{D_j}^{(1)}(\xi, \zeta; \lambda) = \frac{1}{2\pi} e^{-\lambda |\xi - \zeta|} H_1(\xi, \zeta) + \int_{\partial D_j} Y_{22}^{ij}(\xi, \eta; \lambda) M_{D_j}(\eta, \zeta; \lambda) dS_\eta. \quad (5.9)
\]

**Lemma 5.5** There exist constants \( C > 0 \) and \( \mu_1 > 0 \) such that for all \( 0 < \delta \leq 1 \) and \( i, j = 1, 2, \ldots, N \) with \( i \neq j \), the integral kernel \( M^{ij}(\xi, \zeta; \lambda) \) given by (5.7) is estimated by
\[
|M^{ij}(\xi, \zeta; \lambda)| \leq C \delta^{-3} e^{-\delta d_1} e^{-(1-\delta)\mu |\xi - \zeta|} \quad (\xi \in \partial D_1, \zeta \in \partial D_j, \lambda \in \mathbb{C}_{b_0}, \mu \geq \mu_1 \delta^{-3}).
\]

There also exist constants \( C > 0, \mu_1 > 0 \) and \( 0 < \delta_1 \leq 1 \) such that for all \( j = 1, 2, \ldots, N \) and \( 0 < \delta \leq \delta_1 \), the integral kernel \( M^{(1), ij}(\xi, \zeta; \lambda) \) is estimated by
\[
|M^{(1), ij}(\xi, \zeta; \lambda) - M_{D_j}(\xi, \zeta; \lambda)| \leq C \delta^{-2} e^{-\delta d_1} e^{-\delta d_\mu} e^{-\delta \mu |\xi - \zeta|} \quad (\xi, \zeta \in \partial D_2, \lambda \in \mathbb{C}_{b_0}, \mu \geq \mu_1 \delta^{-3}).
\]

**Remark 5.6** From Lemma 5.5 and (3.5), we obtain
\[
|M^{ij}(\xi, \zeta; \lambda)| \leq C \left( \mu + \delta^{-2} e^{-\delta d_1} + \frac{1}{|\xi - \zeta|} \right) e^{-\mu |\xi - \zeta|} \quad (\xi, \zeta \in \partial D_j, \lambda \in \mathbb{C}_{b_0}, \mu \geq \mu_1 \delta^{-3}, 0 < \delta \leq \delta_1),
\]
where \( 0 < \delta_1 \leq 1 \) and \( \mu_1 > 0 \) are given in Lemma 5.5.
Proof: We show the first estimate in Lemma 5.5. Assume that $i \neq j$. It suffices to show

$$\int_{\partial D_i} |M_{D_i}(\xi, \eta; \lambda)W^{\infty,ij}(\eta, \zeta; \lambda)|dS_{\eta} \leq C\delta^{-3}e^{-\delta d_1\mu}e^{-(1-\delta)|\mu|\zeta - \xi|}$$  \hspace{1cm} (5.10)

$$(\xi \in \partial D_i, \zeta \in \partial D_j, i \neq j, \lambda \in C_{\delta_0}, \mu \geq \mu_1\delta^{-3})$$

since other terms in the representation (5.7) of the integral kernel $M^{ij}(\xi, \zeta; \lambda)$ are given by Proposition 5.3.

Keeping the case $i \neq j$ in mind, and using (3.5) and Proposition 5.3, for $0 < \delta \leq 1$, we obtain

$$\int_{\partial D_i} |M_{D_i}(\xi, \eta; \lambda)W^{\infty,ij}(\eta, \zeta; \lambda)|dS_{\eta} \leq 2CC_1\delta^{-2}e^{-\delta d_1\mu}e^{-(1-\delta)|\mu|\zeta - \xi|} \int_{\partial D_i} \frac{1}{|\xi - \eta|}\leq \frac{1}{\xi - \eta|}e^{-(1-\delta)|\mu|\zeta - \xi|}dS_{\eta}.$$

Since

$$e^{-\mu|\xi - \eta|}e^{-(1-\delta)|\mu|\zeta - \xi|} \leq e^{-(1-\delta)|\mu|\zeta - \xi|}e^{-\mu|\xi - \eta|} \quad (\xi \in \partial D_i, \zeta, \eta \in \partial D_j),$$

$0 < \delta \leq 1$ and Lemma 5.1 imply

$$\int_{\partial D_i} |M_{D_i}(\xi, \eta; \lambda)W^{\infty,ij}(\eta, \zeta; \lambda)|dS_{\eta} \leq 2CC_1\delta^{-2}e^{-\delta d_1\mu}e^{-(1-\delta)|\mu|\zeta - \xi|} \int_{\partial D_i} \frac{1}{|\xi - \eta|}\leq \frac{1}{\xi - \eta|}e^{-(1-\delta)|\mu|\zeta - \xi|}dS_{\eta} \leq C\delta^{-3}e^{-\delta d_1\mu}e^{-(1-\delta)|\mu|\zeta - \xi|},$$

which shows (5.10).

For the case $i = j$, Proposition 5.3 and (5.7), it suffices to show

$$\int_{\partial D_j} |M_{D_j}(\xi, \eta; \lambda)W^{\infty,jj}(\eta, \zeta; \lambda)|dS_{\eta} \leq C\delta^{-2}e^{-\delta d_1\mu}e^{-d_3\mu}e^{-\mu|\xi - \zeta|}$$  \hspace{1cm} (5.11)

$$(\xi, \zeta \in \partial D_j, \lambda \in C_{\delta_0}, \mu \geq \mu_1\delta^{-3}, 0 < \delta \leq \delta_1),$$

where $0 < \delta_1 \leq 1$ is the constant given in Proposition 5.3.

From the estimate of $W^{\infty,jj}(\xi, \zeta; \lambda)$ in Proposition 5.3 and (3.5), it follows that for $0 < \delta_1 \leq 1$,

$$\int_{\partial D_j} |M_{D_j}(\xi, \eta; \lambda)W^{\infty,jj}(\eta, \zeta; \lambda)|dS_{\eta} \leq 2CC_1\delta^{-2}e^{-\delta d_1\mu}e^{-d_3\mu} \int_{\partial D_j} \frac{1}{|\xi - \eta|}\leq \frac{1}{\xi - \eta|}e^{-(1-\delta)|\mu|\zeta - \xi|}dS_{\eta}.$$

Since (3.12) implies

$$e^{-d_3\mu} \int_{\partial D_j} \frac{1}{|\xi - \eta|}\leq \frac{1}{\xi - \eta|}dS_{\eta} \leq Ce^{-d_3\mu} \leq C \quad (\mu \geq 1),$$

we obtain (5.11), which completes the proof of Lemma 5.5.

Now we are in the position to show Proposition 3.3.

Proof of Proposition 3.3: We put

$$A^{||P}(\xi, \zeta; \lambda) = \int_{\partial D_p} Y^{IP}_{22}(\xi, \eta; \lambda)M^{Pj}(\eta, \zeta; \lambda)dS_{\eta} \quad (\xi \in \partial D_i, \zeta \in \partial D_j),$$

where...

...Finally, we have the following equality...
which are in the integral representation (5.8) of the integral kernel $M^{(1)}_{ij}(\xi, \zeta; \lambda)$ of $M^{(1)}(\lambda)$. Here we consider the following four cases (i)-(iv), though they do not correspond to the partition of the possible cases.

(i) The case $j \neq p$: From the first estimate in Lemma 5.5 and (3.2), there exists a constant $C > 0$ such that for any $0 < \delta \leq 1$ and $\lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_1\delta^{-3}$,

$$|A^{ijp}(\xi, \zeta; \lambda)| \leq C\delta^{-3}e^{-\delta_d \mu} \int_{\partial D_p} \left( \mu + \frac{1}{|\xi - \eta|} \right) e^{-\mu|\xi - \eta|} e^{-(1-\delta)\mu|\eta - \zeta|} dS_{\eta}$$

$$\leq C\delta^{-3}e^{-\delta_d \mu} e^{-(1-\delta)\mu|\xi - \zeta|} \int_{\partial D_p} \left( \mu + \frac{1}{|\xi - \eta|} \right) e^{-\mu|\xi - \eta|} dS_{\eta}.$$

Hence, Lemma 5.1 implies

$$|A^{ijp}(\xi, \zeta; \lambda)| \leq C\delta^{-4}e^{-\delta_d \mu} e^{-(1-\delta)\mu|\xi - \zeta|}$$

$(\xi \in \partial D_1, \xi \in \partial D_2, i, j, p = 1, 2, \ldots, N, j \neq p, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_1\delta^{-3}, 0 < \delta \leq 1).$ (5.12)

(ii) The case $j \neq p$ and $i = j$: For these $i$ and $j$, as in the case (i), it follows that

$$|A^{ijp}(\xi, \zeta; \lambda)| \leq C\delta^{-3}e^{-\delta_d \mu} \int_{\partial D_p} \left( \mu + \frac{1}{|\xi - \eta|} \right) e^{-(1-\delta)\mu(|\eta - \zeta| + |\xi - \eta|)} e^{-\delta_d \mu|\eta - \zeta|} dS_{\eta}$$

$(\xi, \zeta \in \partial D_j, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_1\delta^{-3}, 0 < \delta \leq 1).$

Hence, Lemma 5.4 and Lemma 5.1 yield

$$|A^{ijp}(\xi, \zeta; \lambda)| \leq C\delta^{-4}e^{-\delta_d \mu} e^{-\mu|\xi - \zeta|} e^{-2d_3 \mu} \int_{\partial D_p} \left( \mu + \frac{1}{|\xi - \eta|} \right) e^{-\delta_d \mu|\eta - \zeta|} dS_{\eta}$$

$(\xi, \zeta \in \partial D_j, j, p = 1, 2, \ldots, N, j \neq p, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_1\delta^{-3}, 0 < \delta \leq \delta_1).$ (5.13)

(iii) The case $j = p$: If this is the case, the second estimate in Lemma 5.5 and (3.2) implies that there exist constants $C > 0, \mu_1 > 0$ and $0 < \delta_1 \leq 1$ such that for any $0 < \delta \leq \delta_1$ and $\lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_1\delta^{-3}$,

$$|A^{ijj}(\xi, \zeta; \lambda)| = \left| \int_{\partial D_j} t Y^{ijj}_{22}(\xi, \eta; \lambda) M_{D_j}(\eta, \xi; \zeta; \lambda) dS_{\eta} \right|$$

$$\leq C \int_{\partial D_j} \left( \mu + \frac{1}{|\xi - \eta|} \right) e^{-\mu|\xi - \eta|} e^{-2d_3 \mu} e^{-\mu|\xi - \zeta|} dS_{\eta}$$

$$\leq C\delta^{-2}e^{-\delta_d \mu} e^{-\mu|\xi - \zeta|} e^{-2d_3 \mu} \int_{\partial D_j} \left( \mu + \frac{1}{|\xi - \eta|} \right) dS_{\eta}.$$.

From (3.12) and $\mu e^{-\mu d_3} \leq d_3^{-1}$, there exist constants $C > 0$ and $0 < \delta_1 \leq 1$ such that

$$|A^{ijj}(\xi, \zeta; \lambda)| = \left| \int_{\partial D_j} t Y^{ijj}_{22}(\xi, \eta; \lambda) M_{D_j}(\eta, \xi; \zeta; \lambda) dS_{\eta} \right| \leq C\delta^{-2} e^{-\delta_d \mu} e^{-\mu|\xi - \zeta|}$$

$(\xi \in \partial D_1, \zeta \in \partial D_j, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_1\delta^{-3}, 0 < \delta \leq \delta_1).$ (5.14)

(iv) The case $j = p$ and $i \neq j$: If this is the case, (5.2) and (5.14) yield

$$|A^{ijj}(\xi, \zeta; \lambda)| \leq C\delta^{-2} e^{-\delta_d \mu} e^{-\mu|\xi - \zeta|}.$$
Hence, estimate (5.3) for $\tilde{W}^{ij}(\xi, \zeta; \lambda)$ ($i \neq j$) implies that

$$|A^{ij}(\xi, \zeta; \lambda)| \leq C\delta^{-2}e^{-\delta d_1}e^{-(1-\delta)|\xi - \zeta|}$$

(5.15)

$$(\xi \in \partial D_i, \zeta \in \partial D_j, i \neq j, \lambda \in C_{\delta_0}, \mu \geq \mu_1\delta^{-3}, 0 < \delta \leq \delta_1).$$

Now we are in the position to give the estimate of $M_{ij}^{(1)}(\xi, \zeta; \lambda)$ for the case $i \neq j$. For (5.8), it follows that

$$M_{ij}^{(1)}(\xi, \zeta; \lambda) = \frac{1}{2\pi}e^{-\lambda|\xi - \zeta|}H_1(\xi, \zeta) + \sum_{p=1}^{N} A_{ijp}(\xi, \zeta; \lambda).$$

From (5.12), (5.15) and the argument for getting (3.7) implies

$$|M_{ij}^{(1)}(\xi, \zeta; \lambda)| \leq C\delta^{-4}e^{-\delta d_1}e^{-(1-\delta)|\xi - \zeta|}$$

(5.13)

$$(\xi \in \partial D_i, \zeta \in \partial D_j, i \neq j, \lambda \in C_{\delta_0}, \mu \geq \mu_1\delta^{-3}, 0 < \delta \leq \delta_1).$$

Next is the case $i = j$. The representations (5.8) and (5.9) yield

$$M_{ij}^{(1)}(\xi, \zeta; \lambda) - M_{ij}^{(1)}(\xi, \zeta; \lambda) = \sum_{p \neq j} A_{ijp}(\xi, \zeta; \lambda) + A_{ij}^{ij}(\xi, \zeta; \lambda) - \int_{\partial D_j} Y_{ij}^{ij}(\xi, \eta; \lambda)M_{ij}(\eta, \zeta; \lambda)dS_\eta.$$  

This equality, (5.13) and (5.14) imply that

$$|M_{ij}^{(1)}(\xi, \zeta; \lambda) - M_{ij}^{(1)}(\xi, \zeta; \lambda)| \leq C\delta^{-4}e^{-\delta d_1}e^{-(1-\delta)|\xi - \zeta|}$$

(5.14)

$$(\xi, \zeta \in \partial D_j, \lambda \in C_{\delta_0}, \mu \geq \mu_1\delta^{-3}, 0 < \delta \leq \delta_1),$$

which completes the proof of Proposition 3.3.

6 Estimate of some Laplace integrals

Let $U \subset \mathbb{R}^n$ be a bounded open set, and $S(\sigma)$ is a $C^2$ function in $U$, and $h(\sigma; \lambda)$ be a continuous function in $\sigma \in U$ with a parameter $\lambda \in \mathbb{C}_{\delta_0}$ for some $\delta_0 > 0$. For $S$ and $h$, assume that

(S.1) $\tau_{-\infty} = \inf_{\sigma \in U} S(\sigma)$ exists and $\tau_{-\infty} = S(0),$

(S.2) there exists a constant $C_0 > 0$ such that $\tau_{-\infty} \leq S(\sigma) \leq \tau_{-\infty} + C_0|\sigma|^2$ $(\sigma \in U),$

(H.1) $h$ is of the form: $h(\sigma; \lambda) = h_1(\sigma; \lambda) + \lambda^{-1}\tilde{h}_1(\sigma; \lambda)$ $(\sigma \in U, \lambda \in \mathbb{C}_{\delta_0}),$

(H.2) there exist constants $C_1 > 0, C'_1 > 0$ and $\mu_0 > 0$ such that

$$\Re[h_1(\sigma; \lambda)] \geq C_1, \quad |h_1(\sigma; \lambda)| + |\tilde{h}_1(\sigma; \lambda)| \leq C'_1 \quad (\sigma \in U, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0).$$

For the functions $S$ and $h$, and a cutoff function $\varphi \in C^3_0(U)$ with $0 \leq \varphi \leq 1$ and $\varphi(0) = 1$, we introduce a Laplace integral $I(\lambda)$ of the form:

$$I(\lambda) = \int_U e^{-\lambda S(\sigma)}\varphi(\sigma)h(\sigma; \lambda)d\sigma.  \tag{6.1}$$

**Proposition 6.1** For integral (6.1), assume that $S$ and $h$ satisfy (S.1), (S.2), (H.1) and (H.2) in the above. Then there exist constants $0 < \delta_1 < \delta_0, \mu_1 > 0$ and $C > 0$ such that

$$\Re[e^{\lambda\tau_{-\infty}}I(\lambda)] \geq C\mu^{-n/2} \quad (\lambda \in \Lambda_{\delta_1}, \mu \geq \mu_1).$$
Proof: We put \( \tau_\infty = \sup_{\sigma \in U} S(\sigma), \ E_\tau = \{ \sigma \in U \mid S(\sigma) \leq \tau \} \) for \( \tau \in \mathbb{R} \), and

\[
\beta_\lambda(\tau) = \int_{E_\tau} \varphi(\sigma) h(\sigma; \lambda) d\sigma \quad (\tau \in \mathbb{R}).
\]

Note that \( \beta_\lambda(\tau) \) is a function of bounded variation, \( \beta_\lambda(\tau) = 0 \) for \( \tau < \tau_\infty \) and \( \beta_\lambda(\tau) = \beta_\lambda(\tau_\infty) \) for \( \tau \geq \tau_\infty \). Note also that \( \beta_\lambda \) is a right continuous function in \( \tau \in \mathbb{R} \) since for any \( \tau_0 \in \mathbb{R} \), \( \lim_{\tau \to \tau_0+0} \chi_{E_\tau}(\sigma) = \chi_{E_{\tau_0}}(\sigma) \), where \( \chi_{E_\tau}(\sigma) \) is the characteristic function of the set \( E_\tau \). From Stieltjes integral with respect to \( \beta_\lambda \), for any \( \tau_\infty < \tau_\infty \), it follows that

\[
I(\lambda) = \int_{-\infty}^{\tau_\infty} e^{-\lambda \tau} d\beta_\lambda(\tau) = e^{-\lambda \tau_\infty} \beta_\lambda(\tau_\infty) + \lambda \int_{-\infty}^{\tau_\infty} e^{-\tau \lambda} \beta_\lambda(\tau) d\tau. \tag{6.2}
\]

We put

\[
\beta_{\lambda,0}(\tau) = \int_{E_\tau} \varphi(\sigma) h_1(\sigma; \lambda) d\sigma \quad (\tau \in \mathbb{R}).
\]

From (H.2), it follows that

\[
|\beta_{\lambda,0}(\tau) - \beta_{\lambda,0}(\tau_\infty)| \leq C_1' \int_{E_{\tau}\setminus E_{\tau_\infty}} \varphi(\sigma) d\sigma \leq C_1' \|\varphi\|_{L^1(U)} \quad (\tau \in \mathbb{R}, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0). \tag{6.3}
\]

Note also that (H.1) and (H.2) yield that

\[
|\beta_\lambda(\tau) - \beta_{\lambda,0}(\tau_\infty)| \leq C_1' \int_{E_{\tau}\setminus E_{\tau_\infty}} \varphi(\sigma) d\sigma \leq C_1' \|\varphi\|_{L^1(U)} \quad (\tau \in \mathbb{R}, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0), \tag{6.4}
\]

where \( C_2 = C_1' / C_1 > 0 \). A similar argument for getting (6.4) implies

\[
|\beta_\lambda(\tau)| \leq C_1' \|\varphi\|_{L^1(U)}, \quad |\text{Im} \beta_{\lambda,0}(\tau)| \leq C_2 \text{Re} \beta_{\lambda,0}(\tau) \quad (\tau \in \mathbb{R}, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0), \tag{6.5}
\]

which yields

\[
\text{Re} \left( e^{\lambda \tau_\infty} \lambda \int_{-\infty}^{\tau_\infty} e^{-\tau \lambda} \beta_\lambda(\tau) d\tau \right) \geq J_5(\lambda) - C_1' \|\varphi\|_{L^1(U)} \frac{|\lambda|}{\mu} e^{-\mu \delta} \tag{6.6}
\]

for any \( 0 \leq \delta \leq \tau_0 \), where \( \tau_0 = \tau_\infty - \tau_\infty \) and

\[
J_5(\lambda) = \text{Re} \left( e^{\lambda \tau_\infty} \lambda \int_{-\infty}^{\tau_\infty+\delta} e^{-\tau \lambda} \beta_{\lambda,0}(\tau) d\tau \right).
\]

Further, (H.2) implies an estimate of \( \text{Re} \beta_{\lambda,0} \) from below:

\[
\text{Re} \beta_{\lambda,0}(\tau) \geq C_1 \gamma(\tau) \quad (\tau \in \mathbb{R}, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0), \tag{6.7}
\]

where

\[
\gamma(\tau) = \int_{E_\tau} \varphi(\sigma) d\sigma \quad (\tau \in \mathbb{R}).
\]

We can divide the following three cases: Case 1: \( \tau_\infty = \tau_\infty \), Case 2: \( \tau_\infty < \tau_\infty \) and \( \gamma(\tau_\infty) > 0 \), Case 3: \( \tau_\infty < \tau_\infty \) and \( \gamma(\tau_\infty) = 0 \).

Case 1: In this case, \( E_{\tau_\infty} = U \). This and assumption (H.2) imply

\[
\text{Re} \beta_{\lambda,0}(\tau_\infty) \geq C_1 \int_U \varphi(\sigma) d\sigma = C_1 \|\varphi\|_{L^1(U)} \quad (\lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0).
\]
This estimate, (6.2) and (6.4) yield

\[ \text{Re}[e^{\lambda r(-\infty)}I(\lambda)] = \text{Re} \beta_\lambda(\tau_{-\infty}) \geq (1 - C_2|\lambda|^{-1}) \text{Re} \beta_{\lambda_0}(\tau_{-\infty}) \geq C_1(1 - C_2|\lambda|^{-1})\|\varphi\|_{L^1(U)}, \]

which implies \( \text{Re}[e^{\lambda r(-\infty)}I(\lambda)] \geq C \) for some constant \( C > 0 \) for large \( |\lambda| \) in \( \lambda \in \mathbb{C}_{\delta_0} \) uniformly.

Case 2: In this case, we put \( C_3 = C_1\gamma(\tau_{-\infty}) > 0 \). Since \( E_\tau \supset E_{\tau_{-\infty}} \) for \( \tau \geq \tau_{-\infty} \) and \( \lim_{\tau \to \tau_{-\infty}+0} \chi_{E_\tau \setminus E_{\tau_{-\infty}}} = 0 \), (6.3) implies that there exists a constant \( \delta > 0 \) such that

\[ |\beta_{\lambda_0}(\tau) - \beta_{\lambda_0}(\tau_{-\infty})| \leq \frac{C_3}{2(1 + \delta_0)} (\tau_{-\infty} \leq \tau \leq \tau_{-\infty} + \delta). \]

From the above estimate, (6.4) and (6.7), it follows that

\[ J_\delta(\lambda) \geq \text{Re} \left( e^{\lambda r(-\infty)} \int_{\tau_{-\infty}}^{\tau_{-\infty}+\delta} e^{-\tau r} \beta_{\lambda_0}(\tau_{-\infty}) d\tau \right) - |\lambda| \int_{\tau_{-\infty}}^{\tau_{-\infty}+\delta} e^{-\tau r} m(|\beta_\lambda(\tau) - \beta_{\lambda_0}(\tau)| + |\beta_{\lambda_0}(\tau) - \beta_{\lambda_0}(\tau_{-\infty})|) d\tau \]

\[ \geq \text{Re} \left[ \beta_{\lambda_0}(\tau_{-\infty})(1 - e^{-\lambda \delta}) \right] - \frac{C_1 C_2}{\mu} \|\varphi\|_{L^1(U)} - \frac{C_1|\lambda|}{2(1 + \delta_0)\mu} \geq \frac{C_3}{2} - C(\mu^{-1} + e^{-\lambda \delta}). \]

Combining the above estimate and (6.6) with (6.2), we obtain \( \text{Re}[e^{\lambda r(-\infty)}I(\lambda)] \geq C \) for some constant \( C > 0 \) for large \( |\lambda| \) in \( \lambda \in \mathbb{C}_{\delta_0} \) uniformly.

Case 3: In this case, take \( r_1 > 0 \) with \( B(0, r_1) \subset U \) and \( \varphi(\sigma) \geq \varphi(0)/2 (|\sigma| \leq r_1) \), where \( B(0, r_1) \) is the open ball with the center 0 and the radius \( r_1 \). Note that (S.2) implies that \( B(0, \sqrt{(r - \tau_{-\infty})/C_0}) \subset E_\tau \) for \( \tau_{-\infty} \leq \tau \leq \tau_{-\infty} + C_0 r_1^2 \), which yields

\[ \gamma(\tau) \geq \frac{\varphi(0)}{2C_0^{n/2}} \text{Vol}(B(0, 1))(\tau - \tau_{-\infty})^{n/2} \quad (\tau_{-\infty} \leq \tau \leq \tau_{-\infty} + C_0 r_1^2). \]

Hence, taking \( C_4 = 2^{-1}\varphi(0)\text{Vol}(B(0, 1)) \min\{C_0^{-n/2}, \frac{r_1}{(\tau_{-\infty} - \tau_{-\infty})^{n/2}}\} \), we obtain

\[ \gamma(\tau) \geq C_4(\tau - \tau_{-\infty})^{n/2} \quad (\tau_{-\infty} \leq \tau \leq \tau_{\infty}). \quad (6.8) \]

From (6.5), for any \( 0 < \delta \leq \tau_0 \), it follows that

\[ J_\delta(\lambda) \geq \text{Re} \left( \int_0^\delta e^{-\tau r} \beta_{\lambda_0}(\tau + \tau_{-\infty}) d\tau \right) - C_2 \int_0^\delta |\text{Im}(\lambda e^{-\tau r})| \text{Re} \beta_{\lambda_0}(\tau + \tau_{-\infty}) d\tau. \]

From this estimate, (6.6) and (6.2), for any \( 0 < \delta \leq \tau_0 \) and \( \lambda \in \Lambda_\delta \), we obtain

\[ \text{Re}[e^{\lambda r(-\infty)}I(\lambda)] \geq \mu \int_0^\delta e^{-\tau r} \Phi(\tau, \lambda) \text{Re} \beta_{\lambda_0}(\tau + \tau_{-\infty}) d\tau - C_1 \|\varphi\|_{L^1(U)}(e^{-\tau_0 \mu} + \frac{|\lambda|}{\mu} e^{-\delta \mu}), \]

where

\[ \Phi(\tau, \lambda) = \cos(\text{Im} \lambda r) - C_2|\sin(\text{Im} \lambda r)| + \frac{\text{Im} \lambda}{\mu} \sin(\text{Im} \lambda r) - \frac{C_2|\text{Im} \lambda|}{\mu} \cos(\text{Im} \lambda r). \]

We take constants \( 0 < c_0 < 1 \) and \( 0 < \theta_0 < \pi/2 \) satisfying \( \cos x - C_2|\sin x| \geq 2c_0 \) for \( |x| \leq \theta_0 \), and choose \( \delta = \min\{\theta_0/|\text{Im} \lambda|, \tau_0\} \) and \( \mu_1 = e^{\delta_1(C_2 + 1)/c_0} \). Since

\[ \Phi(\tau, \lambda) \geq \cos(\text{Im} \lambda r) - C_2|\sin(\text{Im} \lambda r)| - \frac{\delta_1(C_2 + 1)}{\log \mu} \quad (\lambda \in \Lambda_\delta), \]

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it follows that $\Phi(\tau, \lambda) \geq c_0$ for $\lambda \in \Lambda_{\delta_1}$, $\mu \geq \mu_1$ and $0 \leq \tau \leq \delta$. From this fact, (6.7) and (6.8), and $|\lambda|/\mu \leq 1 + \delta_1 (\log \mu)^{-1} \leq 2$ for $\lambda \in \Lambda_{\delta_1}$ with $\mu \geq \mu_1$ and $0 < \delta_1 \leq 1$, it follows that

$$\operatorname{Re} [e^{-\nu \lambda} I(\lambda)] \geq c_0 C_1 C_4 \mu^{-n/2} \int_0^{\delta_1} e^{-\nu \tau n/2} d\tau - 3 C_1 \| \varphi \|_{L^1(U)} e^{-\delta \mu}.$$  

If $|\operatorname{Im} \lambda| \leq \theta_0/\tau_0$, $\delta \mu = \tau_0 \mu \geq \tau_0 \epsilon$, and if $|\operatorname{Im} \lambda| \geq \theta_0/\tau_0$, $\delta \mu = \theta_0 \mu/|\operatorname{Im} \lambda| \geq \theta_0 \delta_1^{-1} \log \mu \geq \theta_0$ ($0 < \delta_1 \leq 1$ and $\lambda \in \Lambda_{\delta_1}$). Hence, in any case, we obtain

$$\operatorname{Re} [e^{-\nu \lambda} I(\lambda)] \geq c_0 C_1 C_4 \mu^{-n/2} \int_0^{\min\{\tau_0, \delta_0\}} e^{-\nu \tau n/2} d\tau - 3 C_1 \| \varphi \|_{L^1(U)} (\mu^{-\theta_0 \delta_1} + e^{-\tau_0 \mu})$$

$(\lambda \in \Lambda_{\delta_1}, \mu \geq \mu_1)$.

This implies $\operatorname{Re} [e^{\nu \lambda} I(\lambda)] \geq C \mu^{-n/2}$ for some constant $C > 0$ for large $|\lambda|$ in $\lambda \in \Lambda_{\delta_1}$ uniformly if we take $\delta_1$ sufficiently small. This completes the proof of Proposition 6.1.

Next, we treat the non-degenerate case, i.e.

(S.3) $\operatorname{grad} S(0) = 0$, $\operatorname{Hess} S(0) > 0$ and $S(\sigma) > \tau_{-\infty}$ $(0 \neq \sigma \in U)$

is assumed. For the amplitude function $h(\sigma; \lambda)$, we also assume

(H.3) there exists a constant $\mu_0 > 0$ such that $\lim_{\sigma \to 0} h(\sigma; \lambda) = h(0; \lambda)$ uniformly in $\lambda \in \mathbb{C}_{\delta_0}$ with $\mu \geq \mu_0$,

(H.4) $h(\sigma; \lambda)$ is bounded for $\sigma \in U$ and $\lambda \in \mathbb{C}_{\delta_0}$.

**Proposition 6.2** Assume that $S(\sigma)$ satisfies (S.1) and (S.3). If $h(\sigma; \lambda)$ $(\sigma \in U$, $\lambda \in \mathbb{C}_{\delta_0})$ is continuous in $\sigma \in U$, then there exists a constant $C > 0$ such that

$$|I(\lambda)| \leq C e^{-\tau_{-\infty} \mu^{-n/2}} \| \varphi(\cdot) h(\cdot; \lambda) \|_{C(U)} \quad (\lambda \in \mathbb{C}_{\delta_0}),$$

where $I(\lambda)$ is given by (6.1). Further, assume also that $h$ satisfies (H.3) and (H.4). Then the following asymptotic formula holds:

$$I(\lambda) = \frac{e^{-\tau_{-\infty}}}{\sqrt{\operatorname{Hess} S(0)}} \lambda^{-n/2} \left( h(0; \lambda) + o(1) \right) \quad (\mu \to \infty \text{ uniformly in } \lambda \in \mathbb{C}_{\delta_0}).$$

Proof: Take a cutoff function $\psi \in C_0^\infty(U)$ satisfies $\psi = 1$ near $\operatorname{supp} \varphi$ and $0 \leq \psi \leq 1$, and decompose the integral $I(\lambda)$ in (6.1) as follows:

$$I(\lambda) = \varphi(0) h(0; \lambda) \int_U e^{-\lambda S(\sigma)} \psi(\sigma) d\sigma + \int_U e^{-\lambda S(\sigma)} \psi(\sigma) \tilde{h}(\sigma; \lambda) d\sigma,$$

where $\tilde{h}(\sigma; \lambda) = \varphi(\sigma) h(\sigma; \lambda) - \varphi(0) h(0; \lambda)$. We write the first and second terms of the right side of (6.9) as $I_1(\lambda)$ and $I_2(\lambda)$ respectively. From the usual Laplace method, $I_1(\lambda)$ is expanded as

$$I_1(\lambda) = h(0; \lambda) \frac{e^{-\tau_{-\infty}}}{\sqrt{\operatorname{Hess} S(0)}} \lambda^{-n/2} \left( 1 + O(\lambda^{-1}) \right) \quad (\lambda \in \mathbb{C}_{\delta_0}, \operatorname{Re} \lambda \to \infty).$$

From (S.3), it follows that there exists a constant $C'_0 > 0$ such that $S(\sigma) \geq \tau_{-\infty} + C'_0 |\sigma|^2$ $(\sigma \in U)$, which yields

$$|I_2(\lambda)| \leq \mu^{-n/2} e^{-\nu_{-\infty}} \int_{\mathbb{R}^n} e^{-C'_0 |\sigma|^2} |\psi(\mu^{-1/2} \sigma)| \tilde{h}(\mu^{-1/2} \sigma; \lambda) d\sigma.$$
Put $M = \sup_{\sigma \in U, \lambda \in \mathbb{C}_0} |h(\sigma; \lambda)| < \infty$ for (H.4). There exists a constant $C > 0$ such that

$$|\tilde{h}(\sigma; \lambda)| \leq |\varphi(\sigma)||h(\sigma; \lambda) - h(0; \lambda)| + |h(0; \lambda)||\varphi(\sigma) - \varphi(0)|$$

$$\leq C\{ |h(\sigma; \lambda) - h(0; \lambda)| + M|\sigma| \} \quad (\sigma \in U, \lambda \in \mathbb{C}_0). \quad (6.12)$$

For any $\eta_0 > 0$, it follows that

$$|\psi(\mu^{-1/2})\tilde{h}(\mu^{-1/2}; \lambda) - h(0; \lambda)| \leq \sup_{|\sigma| \leq \eta_0, \mu^{-1/2}} |h(\sigma; \lambda) - h(0; \lambda)| \quad (|\sigma| \leq \eta_0, \lambda \in \mathbb{C}_0)$$

and

$$|\psi(\mu^{-1/2})\tilde{h}(\mu^{-1/2}; \lambda) - h(0; \lambda)| \leq 2M \quad (|\sigma| \geq \eta_0, \lambda \in \mathbb{C}_0).$$

These estimates and (6.11) imply that there exists a constant $C > 0$ independent of $\eta_0 > 0$ such that

$$|I_2(\lambda)| \leq C\mu^{-n/2}e^{-\mu\tau - \infty}\left(\mu^{-1/2}M + \sup_{|\sigma| \leq \eta_0} |h(\sigma; \lambda) - h(0; \lambda)|\right)$$

$$+ M \int_{|\sigma| \geq \eta_0} e^{-C\delta|\sigma|^2} d\sigma \quad (\lambda \in \mathbb{C}_0).$$

Hence, taking $\eta_0 = \mu^{1/4}$ in the above estimate, and noting (H.3), (6.10) and (6.9), we obtain the asymptotic behavior of $I(\lambda)$ in Proposition 6.2.

Similarly to (6.11), we have

$$|I(\lambda)| \leq \mu^{-n/2}e^{-\mu\tau - \infty}\int_{\mathbb{R}^n} e^{-C\delta|\sigma|^2}|\varphi(\mu^{-1/2})\tilde{h}(\mu^{-1/2}; \lambda)|d\sigma$$

$$\leq \mu^{-n/2}e^{-\mu\tau - \infty}\|\varphi(\cdot; \lambda)|_{C(U)} \int_{\mathbb{R}^n} e^{-C\delta|\sigma|^2} d\sigma,$$

which shows the estimate of $I(\lambda)$ in Proposition 6.2. \hfill \qed

**Remark 6.3** Instead of (H.3) and (H.4), assume that $h(\cdot; \lambda)$ is Hölder continuous in $\sigma \in U$ of order $0 < \alpha_0 < 1$. In this case, from (6.12), it follows that

$$|\psi(\mu^{-1/2})\tilde{h}(\mu^{-1/2}; \lambda)| \leq C\mu^{-\alpha_0/2}\|h(\cdot; \lambda)|_{C^{0,\alpha_0}(U)} \quad (\lambda \in \mathbb{C}_0).$$

Hence, there exist a constant $C > 0$ and a neighborhood $V$ of $0$ with $\nabla \subset U$ such that $|I_2(\lambda)| \leq C\mu^{-n/2-\alpha_0/2}e^{-\mu\tau - \infty}\|h(\cdot; \lambda)|_{C^{0,\alpha_0}(V)} \quad (\lambda \in \mathbb{C}_0)$. This estimate, (6.10) and (6.9) imply

$$I(\lambda) = \frac{e^{-\lambda\tau - \infty}}{\sqrt{\text{Hess}S(0)}} \lambda^{-n/2}\left(h(0; \lambda) + O(\lambda^{-\alpha_0/2})\|h(\cdot; \lambda)|_{C^{0,\alpha_0}(V)}\right)$$

(as $\mu \to \infty$ uniformly in $\lambda \in \mathbb{C}_0$).

**A The case of one strictly convex cavity with $C^2$ boundary**

We discuss reducing regularities of $\partial D$ to obtain the estimates of $M^{(1)}_{ij}(\xi, \zeta; \lambda)$ in Proposition 3.1. Since this estimate is for the case of one strictly convex boundary, from now on, we assume that $\partial D$ is a strictly convex $C^2$ surface. As described in Remark 3.2, this estimate is given for $C^{2,\alpha_0}$ boundary with some $\alpha_0 \in (0, 1)$. In [12], for any $\xi \in \partial D$, standard local coordinates

$$U_\xi \ni \sigma = (\sigma_1, \sigma_2) \mapsto \xi + \sigma_1e_1 + \sigma_2e_2 - g_\xi(\sigma_1, \sigma_2)u_\xi \in \partial D \cap B(\xi, r_0) \quad (A.1)$$
are used to show the estimate of the integral kernels. In this case, $g_\xi$ can be extended as $g_\xi \in B^{2,\alpha_0}(\mathbb{R}^2)$ (i.e. $g_\xi \in B^2(\mathbb{R}^2)$ and each derivative $\frac{\partial}{\partial \xi} g_\xi$ for $|\alpha| = 2$ is uniform Hölder continuous in $\mathbb{R}^2$). Since $g_\xi$ is uniformly bounded in $B^{2,\alpha_0}(\mathbb{R}^2)$ with respect to $\xi \in \partial D$, there exists a constant $C > 0$ such that $|\frac{\partial^\alpha g_\xi(\sigma)}{\partial \xi^\alpha} - \frac{\partial^\alpha g_\xi(\sigma)}{\partial \xi^\alpha}| \leq C|\sigma - \sigma'|^{\alpha_0}$ for any $\sigma, \sigma' \in \mathbb{R}^2$, $|\alpha| = 2$ and $\xi \in \partial D$. Thus, we can use perturbation arguments. When $\partial D$ is $C^2$, more delicate arguments than that in [12] are necessary since we only have $g_\xi \in B^2(\mathbb{R}^2)$.

For $C^2$ class boundary, we need to show the following properties:

**Lemma A.1** All derivatives $\frac{\partial^\alpha g_\xi}{\partial \xi^\alpha} \in B(\mathbb{R}^2)$ for $|\alpha| \leq 2$ of the functions $g_\xi \in B^2(\mathbb{R}^2)$ for $\xi \in \partial D$ given in Lemma 3.6 are equi-continuous, that is, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $|\frac{\partial^\alpha g_\xi(\sigma)}{\partial \xi^\alpha} - \frac{\partial^\alpha g_\xi(\sigma)}{\partial \xi^\alpha}| < \varepsilon$ holds for $|\sigma - \sigma'| < \delta_\varepsilon$ and $\xi \in \partial D$.

A proof of Lemma A.1 is given later. We proceed to show how to treat the $C^2$ boundary case.

Take any $\xi \in \partial D$ and a standard local coordinate (A.1) around $\xi$. Note that we can choose $r_0 > 0$ in Lemma 3.6 sufficiently small enough. In what follows, we change $r_0 > 0$ to be small several finite times. Since $\partial D$ is strictly convex and compact, and (3.1) holds for any $C^2$ surface, there exist constants $M_1 > M_0 > 0$ independent of $r_0$ such that $M_1|\xi - \xi| \geq \sigma \cdot (\xi - \eta) \geq M_0|\xi - \xi|^2$ ($\xi \in \partial D$ and $\xi \in \partial D \cap B(\xi, 2r_0)$). Choose $r_0 > 0$ satisfying $M_1r_0 < 1/2$. For $\sigma \in U_\xi$, we put $\zeta = \xi + \sigma_1 e_1 + \sigma_2 e_2 - g_\xi(\sigma)\nu_\xi \in \partial D \cap B(\xi(2r_0))$. From

$$M_0|\sigma|^2 \leq g_\xi(\sigma) \leq M_1|\zeta - \xi|^2 \leq \frac{|\zeta - \xi|}{2} \leq \frac{|\sigma| + |g_\xi(\sigma)|}{2} \quad (\sigma \in U_\xi),$$

$M_0|\sigma|^2 \leq g_\xi(\sigma) \leq |\sigma|$ holds. Since $|\zeta - \xi|^2 = |\sigma|^2 + |g_\xi(\sigma)|^2 \leq 2|\sigma|^2$, we obtain

$$M_0|\sigma|^2 \leq g_\xi(\sigma) \leq 2M_1|\sigma|^2 \quad (\sigma \in U_\xi, \xi \in \partial D). \quad (A.2)$$

We put $r_1 = 2r_0/\sqrt{1 + 16M_1^2r_0^2} < 2r_0$. For $\sigma \in U_\xi$, it follows that $(2r_0)^2 > |\zeta - \xi|^2 \geq |\sigma|^2$ and $|\zeta - \xi|^2 = |\sigma|^2 + |g_\xi(\sigma)|^2 \leq |\sigma|(1 + 4M_1^2|\sigma|^2)$, which imply

$$|\sigma| \leq |\zeta - \xi| < \sqrt{1 + 16M_1^2r_0^2}|\sigma| \quad (\xi = \xi + \sigma_1 e_1 + \sigma_2 e_2 - g_\xi(\sigma)\nu_\xi \in \partial D \cap B(\xi, 2r_0)). \quad (A.3)$$

Take any $\eta \in \partial D \cap B(x, 2r_0)$ with $\xi \neq \eta$ and fixed. Choose $\{e_1, e_2\}$ in the standard system of local coordinates (A.1) around $\xi$ in such a way that $\eta - \xi$ is perpendicular to $e_2$ and $(\eta - \xi) \cdot e_1 > 0$. Thus, one can write

$$\eta = \xi + \sigma_1^0 e_1 - g_\xi(\sigma_1^0, 0)\nu_\xi$$

with $(\sigma_1^0)^2 + g_\xi(\sigma_1^0, 0)^2 < (2r_0)^2$ and $\sigma_1^0 > 0$.

**Proposition A.2** Assume that $\partial D$ is of class $C^2$ and strictly convex.

(i) It follows that

$$|\xi - \zeta| + |\zeta - \eta| \geq |\xi - \eta| + \frac{\sigma_2^2}{2|\zeta - \xi|} \quad (\zeta \in \partial D \cap B(\xi, 2r_0)).$$

(ii) If $r_0$ is chosen small enough, it follows that

$$|\xi - \zeta| + |\zeta - \eta| \geq |\xi - \eta| + \frac{c_0}{|\zeta - \xi|}((\sigma_1^0)^2\sigma_2^2 + \sigma_2^3)$$

for all $\sigma = (\sigma_1, \sigma_2)$ and $\sigma^0 = (\sigma_1^0, 0)$ with $\sigma_1 < 2\sigma_1^0 / 3$, $|\sigma| < r_1$ and $|\sigma^0| < r_1$, where $r_1 = 2r_0/\sqrt{1 + 16M_1^2r_0^2}$ and $c_0$ is a positive constant depending only on $\partial D$. 29
Proof: For $\zeta = \xi + \sigma_1 \epsilon_1 + \sigma_2 \epsilon_2 - g(\sigma) \nu_\epsilon \in \partial D \cap B(\xi, 2r_0)$, we put $\rho_0 = |\eta - \xi|$, $\rho = |\zeta - \xi|$, and denote by $\theta$ the angle made by the line segments $\zeta \xi$ and $\xi \eta$. The cosine theorem implies $|\zeta - \eta| = \sqrt{\rho_0^2 - 2\rho_0 \rho \cos \theta + \rho^2} \geq \rho_0 - \rho \cos \theta$, which yields
\[
|\xi - \zeta| + |\zeta - \eta| \geq \rho_0 + \rho (1 - \cos \theta) = \rho_0 + \frac{\rho \sin^2 \theta}{1 + \cos \theta} \geq \rho_0 + \frac{\rho}{2} \sin^2 \theta. \quad (A.4)
\]
Since $\rho_0 \rho \cos \theta = \sigma_1^0 \sigma_1 + g_\xi(\sigma_0, 0) g_\xi(\sigma)$, it follows that
\[
|\sin \theta|^2 = \frac{\rho_0^2 \sigma_2^2 + (g_\xi(\sigma) \sigma_1^0 - \sigma_1 g_\xi(\sigma_0, 0))^2}{\rho_0^2 \rho^2} \geq \frac{\sigma_2^2}{\rho^2}, \quad (A.5)
\]
which implies (i) of Proposition A.2.

We put $r = |\sigma|$ and $\omega_j = \sigma_j / r$ $(j = 1, 2)$. Take any $0 < \epsilon < 1/2$ fixed later. For $\omega_1 \leq 1 - \epsilon$, $\omega_2^2 \geq 1 - (1 - \epsilon)^2 > \epsilon$ holds, which yields $\sigma_2^2 = r^2 \omega_2^2 \geq \epsilon |\sigma|^2$. Thus, we get
\[
|\zeta - \zeta| + |\zeta - \eta| \geq \rho_0 + \frac{\sigma_2^2}{2 \rho} \geq \rho_0 + \frac{\epsilon}{2 \rho} |\sigma| \geq \rho_0 + \frac{\epsilon}{2 \rho} \left( \frac{(\sigma_1^0)^2}{2 r_0^2 \sigma_1^0} + \sigma_2^2 \right)
\]
\[
(\zeta \in \partial D \cap B(\xi, 2r_0), \omega_1 \leq 1 - \epsilon).
\]
Hence, to obtain (ii) of Proposition A.2, from (A.4) and (A.5), and
\[
\frac{(\sigma_1^0)^2}{\rho_0^2} \geq \frac{1}{1 + 16 M_1^{-2} r_0^2}
\]
given by (A.3), it suffices to show that
\[
\frac{g_\xi(\sigma_1^0, 0)}{\sigma_1^0} - \frac{g_\xi(\sigma)}{\sigma_1} \geq \frac{M_0}{12} \frac{1}{\sigma_1} \quad (|\sigma_1| < r_1, |\sigma| < r_1, \sigma_1 < \frac{2}{3} \sigma_1, \omega_1 \geq 1 - \epsilon) \quad (A.6)
\]
if we choose $0 < \epsilon < 1$ sufficiently small.

Since $\partial D$ is $C^2$ and $\partial D$ is strictly convex, $g_\xi$ is expressed by
\[
g_\xi(\sigma) = \sum_{i,j=1}^2 a_{ij}^\xi(\sigma) \sigma_i \sigma_j \quad \text{and} \quad a_{ij}^\xi(\sigma) = \int_0^1 (1 - \theta) \partial_{\sigma_i} \partial_{\sigma_j} g_\xi(\sigma \theta) d\theta \quad (i, j = 1, 2).
\]
Note that each $a_{ij}^\xi \in (C(U_\xi))$ is uniformly bounded for $|\sigma| \leq r_1$. Hence, there exists a constant $M_2 > 0$ such that $|a_{ij}^\xi(\sigma)| \leq M_2$ for $|\sigma| \leq r_1$. Note that this constant $M_2 > 0$ does not depend on $\xi \in \partial D$ and $r_1 > 0$.

From (A.2), $a_{11}^\xi(\sigma_1^0, 0) \geq M_0 ((\sigma_1^0, 0) \in U_\xi)$. For this $M_0 > 0$, $|a_{11}^\xi(\sigma) - a_{11}^\xi(0)| < M_0 / 8 (|\sigma| \leq r_1)$ if we take $r_1 > 0$ sufficiently small. Note that this $r_1 > 0$ (and $r_0 > 0$ also) can be chosen as a constant independent of $\xi \in \partial D$ since Lemma A.1 implies that $g_\xi$ is equi-continuous with respect to $\xi \in \partial D$.

Hence, it follows that
\[
|a_{11}^\xi(\sigma_1^0, 0) - a_{11}^\xi(\sigma)| \leq |a_{11}^\xi(\sigma_1^0, 0) - a_{11}^\xi(0, 0)| + |a_{11}^\xi(0, 0) - a_{11}^\xi(\sigma)| \leq M_0 / 4,
\]
which yields
\[
a_{11}^\xi(\sigma_1^0, 0) - \frac{2}{3} a_{11}^\xi(\sigma) \geq \frac{1}{3} a_{11}^\xi(\sigma_1^0, 0) - \frac{2}{3} a_{11}^\xi(\sigma_1^0, 0) - a_{11}^\xi(\sigma) | \geq \frac{1}{6} M_0,
\]
for $|\sigma_1^0| < r_1$ and $|\sigma| < r_1$, and
\[
a_{11}^\xi(\sigma) \geq a_{11}^\xi(\sigma_1^0, 0) - a_{11}^\xi(\sigma_1^0, 0) - a_{11}^\xi(\sigma) | \geq M_0 - M_0 / 4 > 0
\]
for $|\sigma| < r_1$.  

When \( \omega_1 \geq 1 - \varepsilon \) and \( 0 < \varepsilon \leq 1/2 \), \( |\omega_2| \leq \sqrt{2\varepsilon} \) holds, which yields \( |\omega_2|/\omega_1 \leq \sqrt{2\varepsilon}/(1-\varepsilon) \leq 2\sqrt{2\varepsilon} \leq 2 \). Hence, for any \( |\sigma_0^0| \) is small enough, \( |\sigma| < r_1 \) and \( |\sigma| \) is small enough with \( 0 < r_1 = \sigma_1 \leq 2\sigma_0^0/3 \), it follows that
\[
\frac{g_\varepsilon(\sigma_0^0, 0)}{\sigma_1^0} - \frac{g_\varepsilon(\sigma)}{\sigma_1} \geq a_{11}^\varepsilon(\sigma_0^0, 0)\sigma_1^0 - a_{11}^\varepsilon(\sigma)\sigma_1 - 2M_2\sigma_1\frac{|\omega_2|}{\omega_1} - M_2 \frac{\omega_2^2}{\omega_1^2} \sigma_1 \\
\geq a_{11}^\varepsilon(\sigma_0^0, 0)\sigma_1^0 - \frac{2}{3} a_{11}^\varepsilon(\sigma)\sigma_1^0 - 2M_2^2 2\sqrt{2\varepsilon} - M_2 \frac{2}{3} \sigma_1^0 4\sqrt{2\varepsilon} \\
\geq \left( \frac{1}{6} M_0 - \frac{16}{3} M_2 \sqrt{2\varepsilon} \right) \sigma_1^0 \geq \frac{M_0}{6} \left( 1 - \frac{32M_2 \sqrt{2\varepsilon}}{M_0} \right) \sigma_1^0.
\]
which implies (A.6) if we choose \( \varepsilon = \min\{1/2, M_0^2/(2(64M_2^2))^3\} \). This completes the proof of Proposition A.2.

Last, we show Lemma A.1 used to show Proposition A.2.

Proof of Lemma A.1: Since \( \partial D \) is \( C^2 \) class, for any \( \xi \in \partial D \), there exist a constant \( r_\xi > 0 \), an open neighborhood \( U_\xi \) of the origin \( 0 \) in \( \mathbb{R}^2 \) and a function \( g_\xi \in B^2(\mathbb{R}^2) \) with \( g_\xi(0) = 0 \) and \( \nabla g_\xi(0) = 0 \) such that
\[
U_\xi \ni \sigma = (\sigma_1, \sigma_2) \mapsto s_\xi(\sigma) = \xi + \sigma_1 e_1(\xi) + \sigma_2 e_2(\xi) - g_\xi(\sigma)\nu_\xi \in \partial D \cap B(\xi, r_\xi),
\]
where \( \{e_1(\xi), e_2(\xi)\} \) is an orthogonal basis for \( T_\xi(\partial D) \). Take any \( \varepsilon_1 \) with \( 0 < \varepsilon_1 \leq 1/4 \) fixed later. We can also assume that \( |\nu_\xi \cdot \nu_\eta| \geq 1 - \varepsilon_1 \) holds for any \( \xi \in \partial D \) and \( \zeta, \eta \in \partial D \cap B(\xi, r_\xi) \) since for \( C^2 \) class surfaces, it is well known that there exists a constant \( C > 0 \) such that \( |\nu_\xi - \nu_\eta| \leq C|\xi - \eta| \) (\( \xi, \zeta \in \partial D \)). In what follows, we write \( e_3(\xi) = -\nu_\xi \).

Since \( \nu_\xi(\sigma) \cdot \nu_\xi = 1/\sqrt{1 + |\nabla_\sigma g_\xi(\sigma)|^2} \), \( \nu_\xi(\sigma) \cdot \nu_\xi \geq 1 - \varepsilon_1 \) implies \( |\nabla_\sigma g_\xi(\sigma)|^2 \leq 1/(1 - \varepsilon_1)^2 \leq 4 \varepsilon_1/(1 - \varepsilon_1)^2 \), which yields \( |\nu_\xi(\sigma)| \leq 2\varepsilon_1/(1 - \varepsilon_1)^2 \), which implies \( |\nu_\xi(\sigma)| \leq 2\varepsilon_1 \) for \( k = 1, 2 \) and \( 0 < \varepsilon_1 \leq 1/4 \).

From compactness of \( \partial D \), we can choose finitely many points \( \xi(j) \) (\( j = 1, 2, \ldots, N \)) satisfying \( \partial D \subset \cup_{j=1}^N B(\xi(j), r_{\xi(j)}/4) \). Put \( r_0 = \min_{j=1, 2, \ldots, N} r_{\xi(j)}/8 > 0 \). Note that
\[
\partial D = \cup_{j=1}^N \{ \zeta \in \partial D | B(\xi(j), 2r_0) \subset B(\xi(j), r(j)/2) \}. \tag{A.7}
\]
Indeed, for any \( \zeta \in \partial D \), there exists some \( \xi(j) \in \partial D \) satisfying \( \zeta \in B(\xi(j), r_{\xi(j)}/4) \). For this \( \xi(j) \) and \( z \in B(\zeta, 2r_0) \), \( |z - \xi(j)| \leq |z - \zeta| + |\xi(j) - \zeta| < 2r_0 + r_{\xi(j)}/4 \leq r_{\xi(j)}/2 \), which yields \( B(\zeta, 2r_0) \subset B(\xi(j), r_{\xi(j)}/2) \).

We take any \( j \in \{ 1, 2, \ldots, N \} \) and \( \zeta \in \partial D \) satisfying \( B(\zeta, 2r_0) \subset B(\xi(j), r_{\xi(j)}/2) \). We define \( V_\zeta \subset \mathbb{R}^2 \) by \( V_\zeta = \{ \tau = (\tau_1, \tau_2) \in \mathbb{R}^2 | \tau_1 = \tau_1 e_1(\xi(j)) + \tau_2 e_2(\xi(j)) + \tau_3 e_3(\xi(j)) \in \partial D \cap B(\xi(j), r(j)/2) \} \) for some \( \tau \in \mathbb{R} \). Note that for any \( \tau \in V_\zeta \), there exists a unique \( \tau_3 \in \mathbb{R} \) satisfying \( \eta = \zeta + \tau_1 e_1(\xi(j)) + \tau_2 e_2(\xi(j)) + \tau_3 e_3(\xi(j)) \in \partial D \cap B(\xi(j), r(j)/2) \). Hence \( \tau_3 \) is a function in \( \tau \), which is written by \( \tau_3 = \tilde{h}_\zeta(\tau) \). This fact is shown as follows: Assume that there exists different \( \tilde{\tau}_3 \in \mathbb{R} \) from \( \tau_3 \) satisfying \( \tilde{\eta} = \zeta + \tau_1 e_1(\xi(j)) + \tau_2 e_2(\xi(j)) + \tilde{\tau}_3 e_3(\xi(j)) \in \partial D \cap B(\xi(j), r(j)/2) \). From \( \partial D \cap B(\xi(j), r(j)/2) \subset \partial D \cap B(\xi(j), r_{\xi(j)}/2) \), \( \eta \) and \( \tilde{\eta} \) are written as \( \eta = \xi(j) + \sigma_1 e_1(\xi(j)) + \sigma_2 e_2(\xi(j)) + \eta e_3(\xi(j)) \) and \( \tilde{\eta} = \xi(j) + \tilde{\sigma}_1 e_1(\xi(j)) + \tilde{\sigma}_2 e_2(\xi(j)) + \tilde{\eta} e_3(\xi(j)) \). By taking some \( \sigma \) and \( \tilde{\sigma} \) in \( U_{\xi(j)} \), respectively. Put \( \eta_t = ((1 - t)\sigma + t\tilde{\sigma}) e_1(\xi(j)) + ((1 - t)\sigma + t\tilde{\sigma}) e_2(\xi(j)) + \eta e_3(\xi(j)) \in \partial D \cap B(\xi(j), r_{\xi(j)}/2) \) (\( 0 \leq t \leq 1 \)). From mean value theorem, it follows that \( g_{\xi(j)}(\tilde{\sigma}) - g_{\xi(j)}(\sigma) = (\tilde{\sigma} - \sigma) \cdot \partial_\sigma g_{\xi(j)}(\sigma) \) where \( \sigma^0 = (1 - t_0)\sigma + t_0\tilde{\sigma} \) for some \( 0 < t_0 < 1 \). Hence, we obtain
\[
\hat{\eta} - \eta = (\hat{\sigma} - \sigma_1) e_1(\xi(j)) + (\hat{\sigma}_2 - \sigma_2) e_2(\xi(j)) + (g_{\xi(j)}(\hat{\sigma}) - g_{\xi(j)}(\sigma)) e_3(\xi(j))
= \sum_{k=1}^2 (\tilde{\sigma}_k - \sigma_k) e_k(\xi(j)) + \partial_{\tilde{\sigma}_k} g_{\xi(j)}(\sigma^0) e_3(\xi(j)) \in T_{\eta^0}(\partial D).
\]
Thus \( (\hat{\eta} - \eta) \cdot \nu_{\eta^0} = 0 \), which yields \( (\tilde{\tau}_3 - \tau_3)\nu_\xi \cdot \nu_{\eta^0} = 0 \). This gives a contradiction since \( \nu_\xi \cdot \nu_{\eta^0} \geq 1 - \varepsilon_1 > 0 \) holds. Hence, \( \tau_3 \) is uniquely determined.
From the above argument, the map \( V_\tau \in \tau \mapsto \zeta + \tau e_1(\zeta) + \tau e_2(\zeta) + h_\zeta(\tau)e_3(\zeta) \in \partial D \cap B(\zeta, 2r_0) \) is bijective, and the function \( h_\zeta \) is related to \( g_{\xi^{(j)}} \) by the equality \( \xi^{(j)} = \sigma_1 e_1(\xi^{(j)}) + \sigma_2 e_2(\xi^{(j)}) + g_{\xi^{(j)}}(\sigma)e_3(\xi^{(j)}) \). This is equivalent to the following equalities:

\[
\begin{align*}
\sigma_1 &= e_1(\xi^{(j)}) - \zeta + \sigma_1 e_1(\xi^{(j)}) + \sigma_2 e_2(\xi^{(j)}) + g_{\xi^{(j)}}(\sigma)e_3(\xi^{(j)}) \\
\sigma_2 &= e_2(\xi^{(j)}) - \zeta + \sigma_1 e_1(\xi^{(j)}) + \sigma_2 e_2(\xi^{(j)}) + g_{\xi^{(j)}}(\sigma)e_3(\xi^{(j)}) \\
\end{align*}
\]

We put \( \tau = \Phi_{\xi^{(j)}, \zeta}(\sigma) \), which has the inverse \( \sigma = \Psi_{\xi^{(j)}, \zeta}(\tau) \) for \( \tau \in V_\zeta \). Since \( \{ e_1(\zeta), e_2(\zeta), e_3(\zeta) \} \) and \( \{ e_1(\xi^{(j)}), e_2(\xi^{(j)}), e_3(\xi^{(j)}) \} \) are orthogonal basis, it follows that

\[
\begin{align*}
\det(\frac{\partial \tau}{\partial \sigma}) &\geq \det \left( \begin{array}{cc}
e_1(\zeta)e_1(\xi^{(j)}) & e_1(\zeta)e_2(\xi^{(j)}) \\
e_2(\zeta)e_1(\xi^{(j)}) & e_2(\zeta)e_2(\xi^{(j)}) \end{array} \right) - 2(|\partial_{\sigma_1} g_{\xi^{(j)}}(\sigma)| + |\partial_{\sigma_2} g_{\xi^{(j)}}(\sigma)|) \\
&\quad (\sigma \in U_{\xi^{(j)}}, s_{\xi^{(j)}}(\sigma) \in \partial D \cap B(\zeta, 2r_0)) \quad \zeta \in \partial D \cap B(\zeta, 2r_0)
\end{align*}
\]

and

\[
\begin{align*}
1 &\leq \left| \det \left( \begin{array}{cc}
e_1(\zeta)e_1(\xi^{(j)}) & e_1(\zeta)e_2(\xi^{(j)}) \\
e_2(\zeta)e_1(\xi^{(j)}) & e_2(\zeta)e_2(\xi^{(j)}) \end{array} \right) \left| e_3(\zeta)e_3(\xi^{(j)}) \right| + 2\sqrt{2} \sum_{k=1}^{2} |e_k(\zeta)e_3(\xi^{(j)})|^2 \\
&\leq \left| \det \left( \begin{array}{cc}
e_1(\zeta)e_1(\xi^{(j)}) & e_1(\zeta)e_2(\xi^{(j)}) \\
e_2(\zeta)e_1(\xi^{(j)}) & e_2(\zeta)e_2(\xi^{(j)}) \end{array} \right) \left| e_3(\zeta)e_3(\xi^{(j)}) \right| + 2\sqrt{2}(1 - |e_3(\zeta)e_3(\xi^{(j)})|^2). \\
\end{align*}
\]

From these estimates and \( e_3(\zeta)e_3(\xi) = \nu_\xi \nu_\xi \geq 1 - \nu_1 \), we obtain

\[
\begin{align*}
\left| \det(\frac{\partial \tau}{\partial \sigma}) \right| &\geq 1 - 4\sqrt{2} \epsilon_1 - 2(|\partial_{\sigma_1} g_{\xi^{(j)}}(\sigma)| + |\partial_{\sigma_2} g_{\xi^{(j)}}(\sigma)|) \geq 1 - 4\sqrt{2} \epsilon_1 - 8\sqrt{2} \epsilon_1 \\
&= 1 - 4\sqrt{2} \epsilon_1 \quad (\sigma \in U_{\xi^{(j)}}, s_{\xi^{(j)}}(\sigma) \in \partial D \cap B(\zeta, 2r_0)) \quad \zeta \in \partial D \cap B(\zeta, 2r_0).
\end{align*}
\]

From now on, take \( \epsilon_1 = 1/1024 \) to be \( \left| \det(\frac{\partial \tau}{\partial \sigma}) \right| \geq 1/2 \) for \( \sigma \in U_{\xi^{(j)}}, s_{\xi^{(j)}}(\sigma) \in \partial D \cap B(\zeta, 2r_0) \) and \( \zeta \in \partial D \cap B(\zeta, 2r_0) \). Thus, the implicit function theorem implies that \( \Psi_{\xi^{(j)}, \zeta} \in C^2(V_\zeta) \) and

\[
\frac{\partial \Psi_{\xi^{(j)}, \zeta}}{\partial \tau}(\tau) = \left( \frac{\partial \xi^{(j)}}{\partial \sigma}(\tau) \right)^{-1} \quad (\tau \in V_\zeta).
\]

From this facts and \( h_\zeta(\tau) = g_{\xi^{(j)}}(\Psi_{\xi^{(j)}, \zeta}(\tau)) \), we can see that for any \( |a| \leq 2 \), the function \( \partial^a h_\zeta(\tau) \) is equi-continuous with respect to \( \zeta \) and \( j = 1, 2, \ldots, N \). Thus, we obtain Lemma A.1 if we note (A.7).

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KAWASHITA, MISHIO

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY,

HIGASHI-HIROSHIMA 739-8526 JAPAN