Abstract

A recent form of the Slavnov-Taylor identity for the ghost-gluon vertex of QCD is compared with perturbative results. It is found that this identity, derived assuming ghost-ghost scattering can be neglected, is not consistent with perturbation theory. A new identity is derived at the one-loop perturbative level.
1 Introduction

Recently, it has been realised that ghosts may play a key role in the confining behaviour of quantum chromodynamics (QCD) in covariant gauges. Schwinger-Dyson studies of Landau gauge QCD have revealed that the previously neglected ghost contributions have an important effect on the infrared behaviour of the propagators [1, 2]. To arrive at this conclusion, the authors were necessarily forced to make certain truncating assumptions, namely neglecting ghost-ghost scattering-like contributions in the case of von Smekal et al. [1] and simply using bare vertices in the case of Atkinson and Bloch [2].

The problem faced in studying the Schwinger-Dyson equations is that the full (non-perturbative) three and four-point functions that occur in the equations for the propagators are unknown. One way to overcome this is to use the appropriate Slavnov-Taylor identities to find the so-called longitudinal part of the vertex, from which it is hoped that sensible physics can be obtained. One example of this is the Ball-Chiu vertex in QED [3].

In the case of the ghost-gluon vertex, the complete Slavnov-Taylor identity is unknown and identification of longitudinal and transverse parts cannot be made. In [1], under their truncation scheme, an identity was derived and the vertex function constructed from the solution.

In [4], Davydychev et al. have calculated the ghost two and three-point functions perturbatively at one loop order (using dimensional regularisation) in arbitrary covariant gauge and dimension. They have gone on, in a more recent paper [5], to compute the same quantities at two-loop order just in the zero-momentum limit. Since the weak-coupling solution to the Schwinger-Dyson equations is perturbation theory, these results allow a comparison of the perturbative and non-perturbative results.

In this paper, the perturbative results for the ghost-gluon vertex are substituted into the identity put forward by von Smekal et al. [1] in order to check its validity, the justification being that algebraically the full Slavnov-Taylor identity must be identical to the perturbative expression at a given order in the coupling. It is found that the identity is not satisfied. The perturbative result at one-loop is then used to derive a form for an identity, which is a candidate for the Slavnov-Taylor identity.

2In simple cases, the longitudinal part can be either the part of the vertex that does not vanish under contraction with the vector boson momentum or the solution obtained from the Slavnov-Taylor identity alone.
2 Notation and Conventions

Briefly, the Feynman rules used are written as follows (in Minkowski space). The full gluon propagator, with its dressing parameterised by the function $J^{-1}$ is

$$D_{ab}^{\mu\nu}(p) = \delta^{ab} \frac{1}{p^2} \left( t_{\mu\nu}(p) J(-p^2)^{-1} + \xi \frac{p_{\mu} p_{\nu}}{p^2} \right),$$ (2.1)

where $t_{\mu\nu}(p) = g_{\mu\nu} - p_{\mu} p_{\nu}/p^2$ is the transverse projector and $\xi = 0$ gives the Landau gauge. The ghost propagator with its associated dressing function $G$, is written as

$$D_{G}^{ab}(p) = \delta^{ab} \frac{1}{p^2} G(-p^2)$$ (2.2)

The ghost-gluon vertex is defined with all momenta incoming and can be written as (see Fig. 1)

$$\tilde{\Gamma}^{abc}_{\mu}(p, q; r) \equiv -ig f^{abc} \tilde{\Gamma}_{\mu}(p, q; r) = -ig f^{abc} p^{\nu} \tilde{\Gamma}_{\nu\mu}(p, q; r)$$ (2.3)

At tree-level, $J = G = 1$ and $\tilde{\Gamma}_{\mu\nu}(p, q; r) = g_{\mu\nu}$. The colour factors are related to the Casimir invariant $C_A$ by $f^{abc} f^{dbc} = \delta^{ad} C_A$. One-loop expressions for these quantities have been calculated [4] in terms of two non-trivial functions $\kappa$ and $\phi$, defined in the following way. The basic one-loop integral in $n = 4 - 2\varepsilon$ dimensions can be written as

$$I(\nu_1, \nu_2, \nu_3) \equiv \int \frac{d^n\omega}{[(q - \omega)^2]^{\nu_1} [(p + \omega)^2]^{\nu_2} [\omega^2]^{\nu_3}},$$ (2.4)

The unashamed use of their notation is by virtue of the fact that the present paper is based almost entirely on their work.
so that

\[ I(1, 1, 1) \equiv i\pi^2 \eta \varphi, \quad I(1, 1, 0) \equiv i\pi^2 \eta \kappa(r^2) \quad (2.5) \]

(similarly for \( I(0, 1, 1) \) and \( I(1, 0, 1) \)). \( \eta \) is a combination of \( \Gamma \) functions

\[ \eta = \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)} \Gamma(1 + \varepsilon). \quad (2.6) \]

The function \( \kappa \) corresponds to the two-point scalar integral and is

\[ \kappa(p) \equiv \kappa_p = \frac{1}{\varepsilon(1 - 2\varepsilon)}(-p^2)^{-\varepsilon} \quad (2.7) \]

The two-point function \( G \) to one-loop is then

\[ G(-p^2) \equiv G_p = 1 + \frac{g^2 \eta}{(4\pi)^{n/2} C_A/4} [(n - 1) - \xi(n - 3)] \kappa(p). \quad (2.8) \]

The function \( \varphi \equiv \varphi(p^2, q^2, r^2) \) is totally symmetric and in four dimensions can be written as (see for example [4],[6] and references therein)

\[ \varphi = \frac{1}{2p^2\sqrt{q^2-x}} \left\{ \ln(x) \ln \left( \frac{1+y+\sqrt{y^2-x}}{1+y-\sqrt{y^2-x}} \right) + 2\text{Li}_2 \left( \frac{x+y-\sqrt{y^2-x}}{1+x+2y} \right) - 2\text{Li}_2 \left( \frac{x+y+\sqrt{y^2-x}}{1+x+2y} \right) \right\} \quad (2.9) \]

where \( x = q^2/p^2, y = p\cdot q/p^2 \). The function \( \varphi \) therefore encapsulates all the dilogarithmic content inherent in any one-loop vertex expression. The result for the one-loop vertex function in the Feynman gauge is (see [4])

\[ \Gamma_\mu(p, q; r) = p_\mu + \frac{g^2 \eta}{(4\pi)^{n/2} \Delta^2} \times \]

\[ \left\{ p_\mu \varphi \left[ p^4 r^2 - \frac{1}{2} p^2 (p\cdot r)^2 + \frac{5}{4} p^2 (p\cdot r)^2 + \frac{3}{4} p^2 r^4 - \Delta^2 \left( p^2 + \frac{1}{2} (p\cdot r) + \frac{1}{2} r^2 \right) \right] \right. \]

\[ + r_\mu \varphi \left[ -\frac{1}{2} p^4 (p\cdot r) + \frac{1}{4} p^2 (p\cdot r)^2 - \frac{3}{2} p^2 (p\cdot r)^2 - \frac{3}{4} p^2 (p\cdot r)^2 - \frac{1}{2} \Delta^2 p^2 \right] + p_\mu \kappa_\mu \left[ \frac{1}{2} p^4 + \frac{3}{4} p^2 (p\cdot r)^2 \right] \]

\[ + p_\mu \kappa_\nu \left[ \Delta^2 + \frac{1}{2} p^2 (p\cdot r) - \frac{1}{4} p^2 r^2 + \frac{3}{2} (p\cdot r)^2 + \frac{3}{4} (p\cdot r) r^2 \right] \]

\[ + r_\mu \kappa_\nu \left[ \frac{1}{2} \Delta^2 - \frac{1}{2} p^4 - \frac{5}{4} p^2 (p\cdot r) - \frac{3}{4} p^2 r^2 \right] + p_\mu \kappa_\tau \left[ -\Delta^2 + p^2 r^2 - \frac{3}{2} (p\cdot r)^2 - \frac{3}{4} (p\cdot r) r^2 \right] \]

\[ + r_\mu \kappa_\tau \left[ -\frac{1}{2} \Delta^2 + \frac{1}{2} p^2 (p\cdot r) + \frac{3}{4} p^2 r^2 \right] \} \quad (2.10) \]

where \( \Delta^2 = p^2 r^2 - (p\cdot r)^2 \).
3 Truncating the Slavnov-Taylor Identity

In this section, a short review of the work of von Smekal et al.\[1\] in deriving a form for the Slavnov-Taylor identity is presented. In their scheme, a truncation was made, neglecting all four-point interactions including the connected ghost-ghost scattering. By utilising the BRS invariance of the pure Yang-Mills theory, the following identity was found, relating the difference of two three-point reducible correlation functions (a combination of propagators and contracted ghost-gluon vertices) to the four-point function (comprised solely of ghost fields)

\[
\frac{1}{\xi} \langle C^c(z) \overline{C}^b(y) \partial A^a(x) \rangle - \frac{1}{\xi} \langle C^c(z) \overline{C}^a(x) \partial A^b(y) \rangle = -\frac{g}{2} f^{cde} \langle C^d(z) C^e(z) \overline{C}^a(x) \overline{C}^b(y) \rangle
\]  

(3.1)

By then replacing the four-point function with only the disconnected ghost propagation terms, they obtain

\[
G^{-1}_r r^\mu \overline{\Gamma}_\mu(p, q; r) + G^{-1}_q q^\mu \overline{\Gamma}_\mu(p, r; q) + p^2 G^{-1}_p = 0.
\]

(3.2)

This equation should be true in all gauges. One point to notice immediately is that as either \( q \to 0 \) or \( r \to 0 \), one simply obtains

\[
p^\mu \overline{\Gamma}_\mu(p, 0; -p) = p^2
\]

(3.3)

from which one can infer that the vertex in this limit remains bare to all orders. This is known to be true non-perturbatively [7]. In the wider context it is possible that this behaviour is responsible for the similarities of the results for the infrared properties of the gluon propagator obtained when using a vertex derived from the above Eq. (3.2) and when using simply a bare vertex [4].

Returning to Eq. (3.2), one finds that this is not true perturbatively. Given that the one-loop expression for the vertex is known, it is a straightforward matter to check the identity. In fact, one does not need the full vertex function, but only the part dependent on the integral \( \varphi \) in the Feynman gauge to see that the identity is not valid except under the truncation, since \( \varphi \) does not enter the two-point function at this order. Doing this, one obtains

\[
\left[ G^{-1}_r r^\mu \overline{\Gamma}_\mu(p, q; r) + G^{-1}_q q^\mu \overline{\Gamma}_\mu(p, r; q) + p^2 G^{-1}_p \right]_{\varphi \to \infty} = \frac{g^2 \eta}{(4\pi)^{n/2}} \frac{CA}{4} p^4 \varphi
\]

(3.4)
In a general covariant gauge, the result is

\[
\left[ G_{r}^{-1} r^\mu \bar{\Gamma}_\mu(p, q; r) + G_{q}^{-1} q^\mu \bar{\Gamma}_\mu(p, r; q) + p^2 \ G_{p}^{-1} \right] \varphi_{-dep} = \frac{g^2 \eta}{(4\pi)^{n/2}} \frac{C_A}{8} \varphi \left\{ (1 + \xi) p^4 - (1 - \xi) (2 - n) p^2 (q \cdot r) \right\} .
\]

Thus one can see that the truncation used is not consistent with perturbation theory.

### 4 Deriving the Slavnov-Taylor Identity from Perturbation Theory

In the absence of a full functional derivation of the Slavnov-Taylor identity, we turn to one of the crucial properties of such identities, namely that they are true to all orders in perturbation theory. Eq. (3.1) gives us our first clue as to the nature of the identity. The left-hand side is some combination of contracted vertices and two-point functions. The right-hand side is not so clear. In their more familiar guise, Slavnov-Taylor identities relate three-point functions to two-point functions, but here we are faced with a four-point correlation function which may or may not be some combination of two and three-point functions. However, it will be shown that there is a simple form for the identity based solely on the contracted vertex and the two-point dressing function \( G \).

As a first step, consider the \( \varphi \) dependence of the quantity \( r^\mu \bar{\Gamma}_\mu(p, q; r) \) in the Feynman gauge at one-loop.

\[
\left[ r^\mu \bar{\Gamma}_\mu(p, q; r) \right] \varphi_{-dep} = \frac{g^2 \eta}{(4\pi)^{n/2}} \frac{C_A}{\Delta^2} \varphi \left\{ \frac{1}{2} (p \cdot r)^4 + \frac{1}{2} (p \cdot r)^3 (p^2 + r^2) - \frac{1}{4} (p \cdot r)^2 p^2 r^2 - \frac{1}{2} (p \cdot r) p^2 r^2 (p^2 + r^2) - \frac{1}{4} p^4 r^4 \right\} .
\]

The function \( \varphi \) contains all the dilogarithmic content of the vertex at this order. This function does not appear in the ghost two-point function to this order. Consequently, it is desirable that it be eliminated explicitly. By inspection, one sees that this part of the contraction is symmetric under interchange of \( p \) and \( r \). Thus it seems as though a good starting point for our identity is

\[
r^\mu \bar{\Gamma}_\mu(p, q; r) - p^\mu \bar{\Gamma}_\mu(r, q; p) = \ ?
\]
The right hand side of Eq. (4.2) is readily deduced using FORM [8] for the full vertex function at one-loop [4] and one finds that

\[ r^\mu \tilde{\Gamma}_\mu(p, q; r) - p^\mu \tilde{\Gamma}_\mu(r, q; p) = g^2 \eta \left( \frac{CA}{8} \right) \left[ (n-1) - \xi(n-3) \right] \times \left\{ \kappa_p p^2 - \kappa_r r^2 + \kappa_q \left( r^2 - p^2 \right) \right\} \]

\[ = \frac{1}{2} p^2 \left[ G_p - G_q \right]_{\text{one-loop}} - \frac{1}{2} r^2 \left[ G_r - G_q \right]_{\text{one-loop}} \] (4.3)

This equation is true in all covariant gauges and dimensions. That this is the only way of eliminating the \( \varphi \) dependence of the vertex and gives precisely the right structure for the right-hand side to be expressed in terms of only the ghost dressing function \( G \) leads us to the conclusion that this is the one-loop form of the Slavnov-Taylor identity. The two quantities on the right-hand side each admit four possible non-perturbative forms, indistinguishable at this order

\[ [G_p - G_q]_{\text{one-loop}} \rightarrow \begin{cases} G_p - G_q \\ 1/G_q - 1/G_p \\ 1 - G_q/G_p \\ G_q/G_p - 1 \end{cases} \] (4.4)

One can clearly see that Eq. (4.3) is unlike the more familiar Slavnov-Taylor identities, since it does not lead to a unique expression for the so-called longitudinal part of the vertex. This is disappointing since it is usual for the starting point of a vertex ansatz in Schwinger-Dyson studies to be based around this.

5 Conclusions

A one-loop identity for the ghost-gluon vertex of QCD, valid in all gauges and dimensions has been derived from perturbation theory. It is postulated that substituting one of the forms of Eq. (4.4) into the right hand side of Eq. (4.3) yields the one-loop form of the Slavnov-Taylor identity. It is shown to differ from a previous non-perturbative identity, obtained using the truncating assumption that connected ghost-ghost scattering could be neglected. The identity does not lend itself to a unique definition of the longitudinal part of the vertex. Nevertheless, the hope is that a practical solution of this identity exists, which will be useful in Schwinger-Dyson studies of the ghost and gluon propagators, and so determine their infrared behaviour crucial to confinement.
Acknowledgements. The author would particularly like to thank M.R. Pennington for many useful discussions and proof-readings. He is grateful to the U.K. Particle Physics and Astronomy Research Council (PPARC) for a research studentship.

References

[1] L.v. Smekal, A. Hauck, and R. Alkofer, Phys. Rev. Lett. 79 (1997) 3591; Ann. Phys. 267 (1998) 1; Erratum-ibid. 269 (1998) 182.

[2] D. Atkinson, and J.C.R. Bloch, Phys. Rev. D58 (1998) 094036; Mod.Phys.Lett. A13 (1998) 1055-1062.

[3] J.S. Ball, and T-W. Chiu, Phys. Rev. D22 (1980) 2542.

[4] A.I. Davydychev, P. Osland, and O.V. Tarasov, Phys. Rev. D54 (1996) 4087.

[5] A.I. Davydychev, P. Osland, and O.V. Tarasov, Phys. Rev. D58 (1998) 036007.

[6] J.M. Campbell, E.W.N. Glover, and D.J. Miller, Nucl. Phys. B498 (1997) 397.

[7] W. Marciano, and H. Pagels, Phys. Rep. 36C (1978) 137.

[8] J.A.M. Vermaseren, *Symbolic Manipulation with FORM* (Computer Algebra Nederland, Amsterdam, 1991).