New code for equilibriums and quasiequilibrium initial data of compact objects. II. Convergence tests and comparisons of binary black hole initial data

Kōji Uryū,1 Antonios Tsokaras,2 and Philippe Grandclément3

1Department of Physics, University of the Ryukyus, Senbaru, Nishihara, Okinawa 903-0213, Japan
2Department of I.C.S.E., University of Aegean, Karlovassi 83200, Samos, Greece
3Laboratoire Univers et Théories, UMR 8102 du CNRS, Observatoire de Paris, Université Paris Diderot, F-92190 Meudon, France

(Dated: May 3, 2014)

Cocal is a code for computing equilibriums or quasiequilibrium initial data of single or binary compact objects based on finite difference methods. We present the results of supplementary convergence tests of Cocal code using time symmetric binary black hole data (Brill-Lindquist solution). Then, we compare the initial data of binary black holes on the conformally flat spatial slice obtained from cocal and KADATH, where KADATH is a library for solving a wide class of problems in theoretical physics including relativistic compact objects with spectral methods. Data calculated from the two codes converge nicely towards each other, for close as well as largely separated circular orbits of binary black holes. Finally, as an example, a sequence of equal mass binary black hole initial data with corotating spins is calculated and compared with data in the literature.

I. INTRODUCTION

Various methods have been developed in the past couple of decades for computing numerical solutions of compact objects in equilibrium or quasiequilibrium. Those include methods for computing relativistic rotating stars in equilibrium or binary black hole initial data (see e.g. 1,2). For example, numerical solutions of binary black holes (BBH) in quasicircular orbits have been widely used for initial data of merger simulations 3,4, and sequences of such data with fixed irreducible mass of each black hole (BH) have been also applied to approximate the inspiral evolution of BBH 5.

The authors have been developing, independently, numerical codes for computing such compact objects, Cocal 3 (Paper I hereafter) and KADATH 5 (Paper II). Cocal is a code for computing various kinds of astrophysical compact objects - isolated or binary systems of neutron stars and black holes which may be associated with strong magnetic fields. KADATH is a library for solving a wide class of problems in theoretical physics including those of general relativity, and is capable of computing such compact objects.

In the first part of this paper, we present the results of supplementary convergence tests of Cocal to those presented in Paper I. With straightforward changes in the radial coordinate grid spacings and in the finite difference formula for the integration over the zenith angle, errors in the gravitational fields especially near the compact objects decrease substantially, which is necessary to improve the accuracy of widely separated BBH solutions. In the second part of the paper, we carefully compare the spatially conformally flat BBH initial data in circular orbit calculated from cocal and KADATH code. Comparison of the solutions is the most effective test to confirm the reliability of the codes in which a system of complicated equations is solved. Such comparison had been done for the codes for relativistic rotating stars in 6.

To our knowledge, this is the first attempt to compare the BBH data calculated from totally different methods as the spectral method and the finite difference method. Finally, we present a sequence of spatially conformally flat BBH initial data in circular orbits for the case with equal mass and corotating spin, and compare the result with those presented in 5. Throughout the paper we use geometric units with $G = c = 1$.

II. CONVERGENCE TESTS FOR COCAL CODE

In this section, we present convergence tests of Cocal code supplementary to those presented in Paper I. The setup for the test problem is the same as in Paper I: the Brill-Lindquist solution for the time symmetric BBH data is generated numerically, and it is compared with the analytic value. We briefly review the setup of the problem and discuss the modification of the finite difference scheme from the previous paper.

A. A test problem for binary black holes

We assume the spacetime $\mathcal{M}$ is foliated by a family of spacelike hypersurfaces $(\Sigma_t)_{t \in \mathbb{R}}$, $\mathcal{M} = \mathbb{R} \times \Sigma$ parametrized by $t \in \mathbb{R}$. We assume the line element in the neighborhood of $\Sigma_t$ to be

$$ds^2 = -\alpha^2 dt^2 + \psi^4 f_{ij} dx^i dx^j, \quad (1)$$

where $f_{ij}$ is the flat spatial metric, so that the data on $\Sigma_t$ becomes time symmetric - the extrinsic curvature $K_{ab}$ on $\Sigma_t$ vanishes.
Decomposing Einstein’s equation \( G_{\alpha\beta} = 0 \) with respect to the foliation using hypersurface normal \( n^\alpha \) to \( \Sigma_t \), and the projection tensor \( \gamma^{ab} = g^{\alpha\beta} + n^\alpha n^\beta \) to it, we write the Hamiltonian constraint \( G_{\alpha\beta} n^\alpha n^\beta = 0 \), and a combination of the spatial trace of Einstein’s equation and the constraint \( G_{\alpha\beta} (\gamma^{\alpha\beta} + \frac{1}{2} n^\alpha n^\beta) = 0 \), as

\[
\nabla^2 \psi = 0 \quad \text{and} \quad \nabla^2 (\alpha \psi) = 0 . \tag{2}
\]

These equations have solutions, which correspond to the first and second BH, respectively, and \( M_r \) and second BH; where subscripts 1 and 2 corresponds to those of the first and second BH are excised to avoid singularities. Therefore, instead of solving two Laplace equations as Eq. (2), we write an equation for \( \psi \) with a source on the whole domain of \( \Sigma_t \):

\[
\nabla^2 \psi = 0 \quad \text{and} \quad \nabla^2 \alpha = -\frac{2}{\psi} f^{ij} \partial_i \psi \partial_j \alpha . \tag{4}
\]

In an actual computation, spherical regions near the center of BH are excised to avoid singularities. Therefore, boundary conditions for these elliptic equations are imposed at the radius \( r = r_a \) of the excised sphere \( S_a \), and at the radius \( r = r_b \) of the boundary of computational domain \( S_b \). We also set the mass parameters \( M_1 \) and \( M_2 \) as 0.8 \( r_a \) of each BH to avoid the lapse to be negative at \( S_a \). In the following tests, we impose Dirichlet boundary conditions on \( S_a \) and \( S_b \) whose values are taken from the analytic solution [3].

B. Coordinates, grid setup, and finite difference scheme of COCAL code

As explained in Paper I, three spherical coordinate patches are introduced for solving binary compact objects with COCAL. Two of them are the compact objects coordinate patches (COCP-I and II) and one is the asymptotic region coordinate patch (ARCP). In each spherical patch, coordinates cover the region \( (r, \theta, \phi) \in [r_a, r_b] \times [0, \pi] \times [0, 2\pi] \). The two COCPs are centered at the center of compact objects and extend up to about \( r_b \sim O(10^3 M) \), while ARCP is centered at the center of mass of the binary, and extends from \( r_a \sim O(10^4 M) \) to \( r_b \sim O(10^6 M) \), where \( M \) is the total mass of the system. Definitions of the parameters for the grid setups are listed in Table I

In solving a system of elliptic equations such as Eq. [4], we rewrite them in integral form using Green’s function that satisfies given boundary conditions, and apply a finite difference scheme to discretize those integral equations on the spherical coordinates of each domain. We use the midpoint rule for numerical quadrature formula, and hence compute the source terms at the midpoints of the grids. Coordinate grids \( (r_i, \theta_j, \phi_k) \) with \( i = 0, \ldots, N_r \), \( j = 0, \ldots, N_\theta \), and \( k = 0, \ldots, N_\phi \), are freely specifiable except for the endpoint of each coordinate grid that corresponds to the boundary of the computational region, \( (r_0, \theta_0, \phi_0) = (r_a, 0, 0) \) and \( (r_{N_r}, \theta_{N_\theta}, \phi_{N_\phi}) = (r_b, \pi, 2\pi) \). The grid setup for COCP and ARCP is the same as Paper I except for the radial grid of COCP which will be explained later. For angular coordinate grids \( (\theta_j, \phi_k) \), we choose equally spaced grids.

In Paper I, we have used for the finite difference formulas, (1) 2nd order midpoint rule for the quadrature formula, (2) 2nd order finite difference formula for the \( \theta \) and \( \phi \) derivatives evaluated at the mid points \( (r_i + \frac{1}{2}, \theta_j + \frac{1}{2}, \phi_k + \frac{1}{2}) = ((r_i + r_{i+1})/2, (\theta_j + \theta_{j+1})/2, (\phi_k + \phi_{k+1})/2) \), (3) 3rd order finite difference formula for the \( r \) derivative evaluated at the mid points, and (4) 4th order finite difference formula for the derivatives evaluated at the grid points, \( (r_i, \theta_j, \phi_k) \). In the present computations, we use the same finite difference formulas mentioned above except for the numerical quadrature formula in \( \theta \) integrations.

Differences from the previous Paper I are the spacings of radial grids \( \Delta r_i := r_i - r_{i-1} \) in COCP, and the quadrature formula used for the integration in zenith angle \( \theta \).

1. Radial grid spacings for COCP

When we compute a sequence of BBH data from larger to smaller separations in the COCAL code, we change the BH excision radius \( r_a \) from smaller to larger values and fix the separation \( d_s \) (instead of fixing \( r_a \) and varying \( d_s \)). In this way, the number of grid points are kept to the same, and the structures of coordinate grids are almost the same for all solutions of the sequence. As

---

**Table I. Summary of grid parameters for COCP.**

| Parameter | Description |
|-----------|-------------|
| \( r_a \) | Radial coordinate where the radial grids start. |
| \( r_b \) | Radial coordinate where the radial grids end. |
| \( r_c \) | Radial coordinate between \( r_a \) and \( r_b \) where the radial grid spacing changes. |
| \( r_e \) | Radius of the excised sphere. |
| \( N_r \) | Number of intervals \( \Delta r_i \) in \( r \in [r_a, r_b] \). |
| \( N_\theta \) | Number of intervals \( \Delta \theta_i \) in \( \theta \in [0, \pi] \). |
| \( N_\phi \) | Number of intervals \( \Delta \phi_i \) in \( \phi \in [0, 2\pi] \). |
| \( \Delta r \) | Coordinate distance between the center of \( S_a \) \((r = 0)\) and the center of mass. |
| \( d_s \) | Coordinate distance between the center of \( S_a \) \((r = 0)\) and the center of \( S_b \). |
| \( L \) | Order of included multipoles. |
For each region, \( \Delta r \lambda \) which correspond to regions I, II, III, and IV, respectively, where

\[ N = \frac{r_a}{\Delta r} \]


\[ \text{FIG. 1. The radial grid points } r_i \text{ of COCP are plotted against the grid number } i = 0, \ldots, N_r \text{ for the case with H3 grid in Table III.} \]

a result the discretization error behaves systematically from one solution to the other, and hence the quantities such as mass or angular momentum vary smoothly along a sequence of solutions.

It is important to notice that, by changing the BH radius, we change the mass of the solution and hence change the length scale of the system. Therefore, to maintain the accuracy of the gravitational fields near the BH, the intervals near the hole should be proportional to the mass of the BH, or in our case, the BH excision radius \( r_a \). Therefore, we modify the construction of the grid spacing in the radial direction \( r \) of COCP for computing a sequence from smaller to larger BH as follows. Without loss of generality, we set the radius of BH excision sphere \( S_a \) as \( r_a < 1 \). We divide the radial coordinate to four regions, I: \( r \in [r_a, 1] \), II: \( r \in [1, r_c] \), III: \( r \in [r_c, 3r_c] \), and IV: \( r \in [3r_c, r_b] \). We set the first interval by

\[ \Delta r_1 = \frac{r_a}{\lambda N_r^f}, \]  

where \( N_r^f \) is the number of intervals in the region I: \( [r_a, 1] \), and \( \lambda \) is a constant factor which is chosen to be \( \lambda = 0.75 \). For each region, \( \Delta r_i := r_i - r_{i-1} \), are defined by

\[ \Delta r_{i+1} = h_i \Delta r_i, \text{ for } i = 1, \ldots, N_r^f - 1 \]  
\[ \Delta r_i = \Delta r_1, \text{ for } i = N_r^f, \ldots, N_r^m \]  
\[ \Delta r_{i+1} = h_3 \Delta r_i, \text{ for } i = N_r^m, \ldots, N_r^m + N_r^f - 1 \]  
\[ \Delta r_{i+1} = h_4 \Delta r_i, \text{ for } i = N_r^m + N_r^f, \ldots, N_r - 1 \]

which correspond to regions I, II, III, and IV, respectively. The ratios \( h_i (> 1) \) (\( i = 1, 3, 4 \)) are respectively determined from relations

\[ 1 - r_a = \Delta r_1 \frac{h_1^{N_r^f} - 1}{h_1 - 1}, \]  
\[ 2 r_c = \Delta r_1 \frac{h_3 (h_1^{N_r^f} - 1)}{h_3 - 1}, \]  
\[ r_b - 3 r_c = \Delta r_1 \frac{h_4 (h_1^{N_r^f} - N_r^m - N_r^f - 1)}{h_4 - 1}. \]

Values of the parameters for the coordinate grids of COCAL used in computing the results presented in this paper are listed in Table III. In Fig. 1, an example of the radial grid points is plotted for the case with H3 grid setup in Table III. Because of the construction, the grid structure in the region larger than \( r \geq 1 \) is the same for all solutions with different BH radius \( r_a \) once a grid setup (resolution) as in Table III is selected.

2. 4th order midpoint rule for the quadrature formula of \( \theta \) integration

As discussed in Paper I, our Poisson solver is a system of integral equations, and it is numerically integrated with a quadrature formula of midpoint rule. Therefore, the sources of the integrals are always evaluated at the midpoints of \((r_i, \theta_j, \phi_k)\) grids. As summarized above, the 2nd order midpoint rule was used for a quadrature formula in Paper I. With the above mentioned choice for finite difference formulas, the 2nd order convergence of the error has been achieved. As shown in the top panel of Fig. 2 (as well as figures 3, 4, and 6 in Paper I), however, the fractional error of the potentials normally increases near the excision surfaces of the BH, \( S_a \) \( (r = r_a) \), although it converges in 2nd order.

One might expect that the increase of the error near \( S_a \) is due only to a lack of resolution in radial grid points. It turns out, however, that the finite difference errors in the potentials near the boundaries of computational domains are dominated by the discretization error in the \( \theta \) coordinate. In particular, the \( \theta \) integration of the source involving the Legendre function turns out to be the source of error. Therefore, we replace the quadrature formula of \( \theta \) integration to 4th order accurate midpoint rule whose weights are

\[ \int_{\theta_j}^{\theta_{j+4}} f(\theta) d\theta \simeq \Delta \theta \left( \frac{13}{12} f_{j+\frac{1}{2}} + \frac{11}{12} f_{j+\frac{3}{2}} + \frac{11}{12} f_{j+\frac{5}{2}} + \frac{13}{12} f_{j+\frac{7}{2}} \right) \]
For the excision radius \( r_a \) for a variable parameter assigned to each solution. In the test problems in Sec. II C, they are chosen to be \( r_a = 0.2 \) and 0.4 for close BBH, and \( r_a = 0.05 \) and 0.1 for separated BBH for COCP-1 and 2, respectively.

### C. Convergence tests

We perform convergence tests to examine that the above two modifications improve the accuracy of the COCAL code. In Table II grid setups for the computations are listed. The grids H1–H4 correspond to different levels of resolutions. At each level, the resolution is double the previous one.

In Figs. 2–4 the fractional errors in the lapse that are averaged over the angular coordinate grids \((\theta_i, \phi_j)\) at fixed radial coordinate \( r \) are plotted against \( r \) of each coordinate patch,

\[
\left\langle \frac{\delta \alpha}{\alpha} \right\rangle := \frac{1}{\#(G_i)} \sum_{p \in G_i} \left| \frac{\alpha - \alpha_{\text{exact}}}{\alpha_{\text{exact}}} \right| ,
\]

where writing a grid point \((r_i, \theta_j, \phi_k)\) by \( p \), we define a set \( G_i \) by \( G_i := \{p \mid p \in V \setminus S_c^{\text{in}} \text{ and } r_i = \text{const}\} \), where \( S_c^{\text{in}} \) is an interior domain of \( S_c \), and \( \#(G_i) \) is the number of points included in \( G_i \).

In the top panel of Fig. 2 the same finite difference scheme as presented in Paper I is used for computing closer BBH solution with the BH excision radius \( r_a = 0.2 \) and 0.4, and with the separation \( d_s = 2.5 \). The midpoint rule in the \( \theta \) integration is 2nd order accurate in this panel. In the bottom panel, the 4th order midpoint rule Eq. (13) for the \( \theta \) integration is used for the same model and the same grid spacings. Clearly, the fractional error substantially decreases by this change for the H2 to H4 levels. We notice that the error near the BH converges in 4th order, that is, the error decreases about 1/16 at each level of resolution. The errors near the BH as well as in the asymptotic region are dominated by those from the \( \theta \) integrations of the surface integral terms.

In Fig. 3 we calculate more separated BBH solutions, decreasing the radius of BH excision surface to 1/4 (therefore effectively separating BBH 4 times apart) as \( r_a = 0.05 \) and 0.1 with the same separation \( d_s = 2.5 \), and using the 4th order integration in \( \theta \) as in the bottom panel of Fig. 2. Although the errors near the BH excision surfaces are of the same order of magnitude as those of the corresponding resolutions plotted in Fig. 4 bottom.

| Type | Patch | \( r_a \) | \( r_b \) | \( r_c \) | \( r_e \) | \( N_r^i \) | \( N_{\phi}^i \) | \( N_{\theta} \) | \( N_{\phi} \) | \( L \) |
|------|-------|------|------|------|------|-------|-------|-------|-------|------|
| H1   | COCP-1 var. | 10^2 | 1.25 | 1.125 | 32 | 40 | 96 | 24 | 24 | 12 |
|      | COCP-2 var. | 10^2 | 1.25 | 1.125 | 32 | 40 | 96 | 24 | 24 | 12 |
|      | ARCP     | 5.0  | 10^6 | 6.25  | —  | 8  | 10 | 96 | 24 | 24 | 12 |
| H2   | COCP-1 var. | 10^2 | 1.25 | 1.125 | 64 | 80 | 192 | 48 | 48 | 12 |
|      | COCP-2 var. | 10^2 | 1.25 | 1.125 | 64 | 80 | 192 | 48 | 48 | 12 |
|      | ARCP     | 5.0  | 10^6 | 6.25  | —  | 16 | 20 | 192 | 48 | 48 | 12 |
| H3   | COCP-1 var. | 10^2 | 1.25 | 1.125 | 128 | 160 | 384 | 96 | 96 | 12 |
|      | COCP-2 var. | 10^2 | 1.25 | 1.125 | 128 | 160 | 384 | 96 | 96 | 12 |
|      | ARCP     | 5.0  | 10^6 | 6.25  | —  | 32 | 40 | 384 | 96 | 96 | 12 |
| H4   | COCP-1 var. | 10^2 | 1.25 | 1.125 | 256 | 320 | 768 | 192 | 192 | 12 |
|      | COCP-2 var. | 10^2 | 1.25 | 1.125 | 256 | 320 | 768 | 192 | 192 | 12 |
|      | ARCP     | 5.0  | 10^6 | 6.25  | —  | 64 | 80 | 768 | 192 | 192 | 12 |

TABLE II. Grid parameters of COCAL used for computation of BBH data. The separation of two BHs is fixed as \( d_s = 2.5 \). For the excision radius \( r_a \) for COCP-1 and 2, “var.” stands for a variable parameter assigned to each solution. In the test problems in Sec. II C, they are chosen to be \( r_a = 0.2 \) and 0.4 for close BBH, and \( r_a = 0.05 \) and 0.1 for separated BBH for COCP-1 and 2, respectively.

\footnote{We have also tested different combinations of grid numbers \((N_r, N_{\phi}, N_{\theta})\) for the first level of resolution and performed convergence tests. We found the combination of type H was better than others. For example, the accuracy was not improved by increasing the grid points in the \( \phi \) direction.}
panel, the errors once increase as the radial coordinate $r$ increases. It turns out that the BBH initial data discussed later can not be calculated accurately with this grid setup for largely separated orbits.

In Fig. 3, convergence tests for the close ($r_a = 0.2$ and 0.4) and the separated ($r_a = 0.05$ and 0.1) BBH are calculated with scaled radial spacing near the BH introduced in Sec. II B 1, as well as the 4th order midpoint rule in $\theta$ integration. The size of the errors around the BH excision radius for the largely separated BBH data (bottom panel) is now comparable to those for the close BBH data (top panel) for each level of resolution. This improvement turns out to be important for accurately computing the separated BBH data in COCAL code.

III. COMPARISON OF BBH INITIAL DATA

A. The KADATH library

In this section, we compare the circular solutions of BBH initial data on a conformally flat spacelike hypersurface calculated from COCAL and the KADATH library. KADATH [7] is a library designed to solve a wide class of problems in theoretical physics including those of general relativity such as the above compact objects. It is based on spectral methods (see for instance [10] and references therein) where the various fields are approximated by finite sums of known functions typically trigonometrical functions and orthogonal polynomials. One of the main advantages of spectral methods is their fast convergence to the true solution (typically exponentially), when one increases the order of the expansion. For instance, in this paper, a relative accuracy of about $10^{-4}$ is achieved with 15 coefficients in each dimension.

Spectral methods enable one to translate a set of partial differential equations into an algebraic system on the coefficients of the expansions. This system is then solved by a standard Newton-Raphson iteration. The computation of the Jacobian as well as its inversion are parallelized.

The code used in this paper is essentially the same as the one used in Sec. 7.3 of [11]. In order to check the overall accuracy of the computations, one monitors the convergence of some global quantities (like the orbital frequency), as a function of $N$, the number of points in each dimensions. Let us mention that, in the case of a large separation, the code was slightly modified to maintain accuracy, probably due to a stretch of the bispherical coordinates when the distance between the holes gets much bigger than the size of the holes themselves. In particular, a spherical shell was added between the bispherical coordinates and the outer compactified domain and the

FIG. 3. Same as Fig. 2 but for the BH radius $r_a = 0.05$ and 0.1 for COCP-1 and 2, respectively. The results are calculated using the 4th order midpoint rule for $\theta$ integration as in the bottom panel of Fig. 2, otherwise the same finite differencing scheme (in particular, the same radial grid spacing $\Delta r_i$) as in Paper 1.

FIG. 4. Same as Fig. 2. The results are calculated using 4th order integration in $\theta$ coordinate Eq. (13), and the scaled radial spacing discussed in Sec. II B 1. Top panel: data with the BH radius $r_a = 0.2$ and 0.4 for COCP-1 and 2, respectively. Bottom panel: data with the BH radius $r_a = 0.05$ and 0.1 for COCP-1 and 2, respectively.
determination of the orbital velocity had to be changed (see Sec. [III.C]).

B. Conformally flat BBH initial data

The circular solution of BBH initial data is calculated by solving the Hamiltonian and momentum constraints, and the spatial trace of the Einstein’s equation on a conformally flat spacelike hypersurface $\Sigma_t$. The spacetime metric on $\Sigma_t$ is written in 3+1 form as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

where the spatial three metric $\gamma_{ij}$ on the slice $\Sigma_t$ is assumed to be $\gamma_{ij} = \psi^4 f_{ij}$. Here, field variables $\psi$, $\alpha$, and $\beta^i$ are the conformal factor, lapse, and shift vector, respectively, and $f_{ij}$ is a flat three dimensional metric. We also assume maximal slicing to $\Sigma_t$, so that the trace of the extrinsic curvature $K_{ij} := -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - \partial_j \gamma_{ti})$ vanishes. Writing its tracefree part $A_{ij}$, the conformally rescaled quantity $\tilde{A}_{ij}$ becomes

$$\tilde{A}_{ij} = \frac{1}{2\alpha} \left( \partial_i \tilde{\beta}_j + \partial_j \tilde{\beta}_i - \frac{2}{3} f_{ij} \partial_k \tilde{\beta}^k \right),$$

(16)

where the derivative $\partial_i$ is associated with the flat metric $f_{ij}$, and conformally rescaled quantities with tilde are defined by $\tilde{A}^i_j = A^i_j$ and $\tilde{\beta}^i = \beta^i$, whose indexes are lowered (raised) by $f_{ij}$ ($f^{ij}$). The system to be solved, which are Hamiltonian and momentum constraints and the spatial trace of the Einstein’s equation, becomes

$$\Delta \psi = -\frac{\psi^6}{8} \tilde{A}_{ij} \tilde{A}^{ij},$$

$$\Delta \beta_i = -2\alpha \tilde{A}^j_i \partial_j \ln \frac{\psi^6}{\alpha} - \frac{1}{3} \partial_i \partial_j \tilde{\beta}^j,$$

$$\Delta(\alpha) = \frac{7}{8} \alpha \psi^5 \tilde{A}_{ij} \tilde{A}^{ij},$$

(17)

(18)

(19)

where $\Delta := \partial_i \partial^i$ is a flat Laplacian [2,11,12].

For the boundary conditions at the BH excision boundary $S_a$, we choose approximate irrotational apparent horizon boundary conditions,

$$\frac{\partial \psi}{\partial r} \bigg|_{r=r_a} = -\psi^3 \frac{1}{4} K_{ij} s^i s^j,$$

$$\beta^i \bigg|_{r=r_a} = \frac{n_0}{\psi^2 s^i} + \Omega y^i_{em},$$

$$\alpha \bigg|_{r=r_a} = n_0,$$

(20)

(21)

(22)

where $n_0$ is an arbitrary positive constant for which we choose $n_0 = 0.1$, $s^i$ is the unit normal to the sphere $S_a$, and $\Omega$ represents a parameter for orbital angular velocity. The vector $y^i_{em} := (0, d, 0)$ is the translational vector with respect to the center of mass. With these conditions, the sphere $S_a$ becomes an apparent horizon (AH) in quasiequilibrium [3,7,13].

| Code | Res. | $\Omega M_{\text{int}}$ | $M_{\text{ADM}}/M_{\text{int}}$ | $J/M^2_{\text{int}}$ | $M_{\text{int}}/r_a$ | $d_s/r_a$ |
|------|------|----------------------|-------------------------------|------------------|------------------|-------------|
| KADATH | 11   | 0.127171            | 0.982866                      | 0.757608          | 3.93366          | 12          |
| KADATH | 13   | 0.127299            | 0.983041                      | 0.758390          | 3.93405          |             |
| KADATH | 15   | 0.127346            | 0.983072                      | 0.758698          | 3.93417          |             |
| COCAL | H2   | 0.127243            | 0.982848                      | 0.754592          | 3.93191          |             |
| COCAL | H3   | 0.127383            | 0.983125                      | 0.758389          | 3.93392          |             |

TABLE III. Comparison of BBH initial data from COCAL and KADATH for the cases $d_s/r_a = 12$ and $d_s/r_a = 30$. Fractional differences in $\%$ between the highest resolution cases of KADATH and COCAL are shown in the lines indicated by “error”. A column “Res.” stands for the resolution.

At the asymptotics, the boundary conditions are

$$\psi \bigg|_{r=\infty} = 1.0,$$

$$\beta^i \bigg|_{r=\infty} = 0.0,$$

$$\alpha \bigg|_{r=\infty} = 1.0.$$  

(23)

(24)

(25)

When using KADATH , the whole spacelike slice $\Sigma_a$ is compactified, and hence the above conditions are imposed at the spatial infinity, while in COCAL , the computational domain is truncated at the radius $r_b \sim O(10^6 M)$ and it is at $r_b$ that the above conditions are imposed.

C. Comparison of the circular solutions of BBH initial data for KADATH and COCAL

Following [14], we obtain the angular velocity $\Omega$ of a circular orbit of BBH initial data from an assumption that an equality of ADM mass and Komar mass, $M_{\text{ADM}} = M_K$, is satisfied for the circular orbit, where $M_{\text{ADM}}$ and $M_K$ are defined by

$$M_{\text{ADM}} = \frac{1}{16\pi} \int_{S_{\infty}} \left( f^{ac} f^{bd} - f^{ab} f^{cd} \right) \partial_b \gamma_{cd} dS_a$$

$$= -\frac{1}{2\pi} \int_{S_{\infty}} D^a \psi dS_a,$$

$$M_K = -\frac{1}{4\pi} \int_{S_{\infty}} \nabla^a \beta \cdot dS_{\alpha\beta}$$

$$= \frac{1}{4\pi} \int_{S_{\infty}} D^a \alpha dS_a.$$  

(26)

(27)

In COCAL , the surface integrals are calculated at a certain finite radius, typically around $r \sim (10^3 M)$. 


FIG. 5. The panels show the scaled values of \( M_{\text{ADM}}/M_{\text{irr}} \) as a function of \( M_{\text{irr}}/r_a \) and \( J_{\text{ADM}}/M_{\text{irr}}^2 \) as a function of \( \Omega M_{\text{irr}} \). The circles denote the results from \textsc{KADATH} and the triangles the ones from \textsc{Cocal}. The runs using \textsc{KADATH} are labeled by the number of points in each dimension. The left top and bottom panels correspond to a separation of \( d_s/r_a = 12 \) and the right ones to \( d_s/r_a = 30 \) (in this case the lower resolution results from \textsc{KADATH} are outside the range of the plot).

In \textsc{KADATH}, these integrals are usually evaluated at spatial infinity as its definition, \( \int S_r := \lim_{r \to \infty} \int_{S_r} \) with \( S_r \) the sphere of radius \( r \). However, in the case of a large separation, it turned out that this was not giving good results. The precision of the results, measured by convergence of the value of \( \Omega \), is much better when one demands that, at spatial infinity,

\[ 1 - a\psi^2 \to O(r^{-2}). \tag{28} \]

The difference between the Komar and ADM mass is then of the order of \( 10^{-4} \) for the higher resolution, thus giving a measure of the overall error of the code.

For a converged circular solution, we also calculate ADM angular momentum:

\[ J = -\frac{1}{8\pi} \int_{S_\infty} K^{ab} \phi_b dS_a. \tag{29} \]

The circles denote the results from \textsc{KADATH} and the triangles the ones from \textsc{Cocal}. The runs using \textsc{KADATH} are labeled by the number of points in each dimension. The left top and bottom panels correspond to a separation of \( d_s/r_a = 12 \) and the right ones to \( d_s/r_a = 30 \) (in this case the lower resolution results from \textsc{KADATH} are outside the range of the plot).

In \textsc{KADATH}, these integrals are usually evaluated at spatial infinity as its definition, \( \int S_r := \lim_{r \to \infty} \int_{S_r} \) with \( S_r \) the sphere of radius \( r \). However, in the case of a large separation, it turned out that this was not giving good results. The precision of the results, measured by convergence of the value of \( \Omega \), is much better when one demands that, at spatial infinity,

\[ 1 - a\psi^2 \to O(r^{-2}). \tag{28} \]

The difference between the Komar and ADM mass is then of the order of \( 10^{-4} \) for the higher resolution, thus giving a measure of the overall error of the code.

For a converged circular solution, we also calculate ADM angular momentum:

\[ J = -\frac{1}{8\pi} \int_{S_\infty} K^{ab} \phi_b dS_a. \tag{29} \]

The above quantities are normalized by the irreducible mass of AH, \( M_{\text{irr}} \), which is defined from the surface area of AH, namely the area integral over the BH excision surface \( S_a \),

\[ A_{AH} = \int_{S_a} \psi^2 r_a^2 d\Omega. \tag{30} \]

We write \( M_{\text{irr}} := \sqrt{A_{AH}/16\pi} \) and \( M_{\text{irr}} := \sqrt{A_{AH}/16\pi} \) for each BH, and write \( M_{\text{irr}} = M_{\text{irr}} + M_{\text{irr}} \) for a total mass.

In Table III, global quantities normalized by \( M_{\text{irr}} \) are presented for the irrotational BBH data computed from \textsc{KADATH} and \textsc{Cocal} at the separations \( d_s/r_a = 12 \) and \( d_s/r_a = 30 \). Three different resolutions for \textsc{KADATH} are given, mainly 11, 13 and 15 points in each dimension, and two (lower and higher) resolutions, H2 and H3, are used for the computations of \textsc{Cocal}. The relative differences between the highest resolution results from both codes are also indicated. The convergence of these quantities is plotted in Fig. 6 for both separations, \( d_s/r_a = 12 \) (left panels) and \( d_s/r_a = 30 \) (right panels). All plots indicate a nice convergence of the global quantities, when the resolutions are increased.

In Fig. 6 plotted are fractional errors of metric potentials between those calculated by \textsc{KADATH} and \textsc{Cocal}. Errors in the conformal factor \( \psi \), lapse \( \alpha \), and \( y \)-component of the shift \( \beta_y \) are plotted along the \( x \)-axis that intersects with the centers of two BHs. The model is the same as that in Table III, with the separation \( d_s/r_a = 12 \). The runs using \textsc{KADATH} are labeled by the number of points in each dimension. The left top and bottom panels correspond to a separation of \( d_s/r_a = 12 \) and the right ones to \( d_s/r_a = 30 \) (in this case the lower resolution results from \textsc{KADATH} are outside the range of the plot).

In \textsc{KADATH}, these integrals are usually evaluated at spatial infinity as its definition, \( \int S_r := \lim_{r \to \infty} \int_{S_r} \) with \( S_r \) the sphere of radius \( r \). However, in the case of a large separation, it turned out that this was not giving good results. The precision of the results, measured by convergence of the value of \( \Omega \), is much better when one demands that, at spatial infinity,

\[ 1 - a\psi^2 \to O(r^{-2}). \tag{28} \]

The difference between the Komar and ADM mass is then of the order of \( 10^{-4} \) for the higher resolution, thus giving a measure of the overall error of the code.

For a converged circular solution, we also calculate ADM angular momentum:

\[ J = -\frac{1}{8\pi} \int_{S_\infty} K^{ab} \phi_b dS_a. \tag{29} \]

The above quantities are normalized by the irreducible mass of AH, \( M_{\text{irr}} \), which is defined from the surface area of AH, namely the area integral over the BH excision surface \( S_a \),

\[ A_{AH} = \int_{S_a} \psi^2 r_a^2 d\Omega. \tag{30} \]

We write \( M_{\text{irr}} := \sqrt{A_{AH}/16\pi} \) and \( M_{\text{irr}} := \sqrt{A_{AH}/16\pi} \) for each BH, and write \( M_{\text{irr}} = M_{\text{irr}} + M_{\text{irr}} \) for a total mass.

In Table III, global quantities normalized by \( M_{\text{irr}} \) are presented for the irrotational BBH data computed from \textsc{KADATH} and \textsc{Cocal} at the separations \( d_s/r_a = 12 \) and \( d_s/r_a = 30 \). Three different resolutions for \textsc{KADATH} are given, mainly 11, 13 and 15 points in each dimension, and two (lower and higher) resolutions, H2 and H3, are used for the computations of \textsc{Cocal}. The relative differences between the highest resolution results from both codes are also indicated. The convergence of these quantities is plotted in Fig. 6 for both separations, \( d_s/r_a = 12 \) (left panels) and \( d_s/r_a = 30 \) (right panels). All plots indicate a nice convergence of the global quantities, when the resolutions are increased.

In Fig. 6 plotted are fractional errors of metric potentials between those calculated by \textsc{KADATH} and \textsc{Cocal}. Errors in the conformal factor \( \psi \), lapse \( \alpha \), and \( y \)-component of the shift \( \beta_y \) are plotted along the \( x \)-axis that intersects with the centers of two BHs. The model is the same as that in Table III, with the separation \( d_s/r_a = 12 \). The runs using \textsc{KADATH} are labeled by the number of points in each dimension. The left top and bottom panels correspond to a separation of \( d_s/r_a = 12 \) and the right ones to \( d_s/r_a = 30 \) (in this case the lower resolution results from \textsc{KADATH} are outside the range of the plot).

In \textsc{KADATH}, these integrals are usually evaluated at spatial infinity as its definition, \( \int S_r := \lim_{r \to \infty} \int_{S_r} \) with \( S_r \) the sphere of radius \( r \). However, in the case of a large separation, it turned out that this was not giving good results. The precision of the results, measured by convergence of the value of \( \Omega \), is much better when one demands that, at spatial infinity,

\[ 1 - a\psi^2 \to O(r^{-2}). \tag{28} \]

The difference between the Komar and ADM mass is then of the order of \( 10^{-4} \) for the higher resolution, thus giving a measure of the overall error of the code.

For a converged circular solution, we also calculate ADM angular momentum:

\[ J = -\frac{1}{8\pi} \int_{S_\infty} K^{ab} \phi_b dS_a. \tag{29} \]
D. Solution sequence for corotating BBH data

Finally, we present a sequence of solutions for the conformally flat BBH initial data computed from COCAL code focusing on the BBH data. Especially the conformally flat initial data of BBH in circular orbit calculated from COCAL code are compared with those from KADATH code. We demonstrated that the results from both codes converge toward each other for large and small separations (see also [16]).

As fully discussed in [5], a corotating sequence presented in Sec. II D is not considered as a model for inspiraling BBH, because BBH tides do not effectively work to spin up the BH to synchronize the BH spin with orbital motion within the time of inspirals. In [5], the authors describe a more realistic sequence of BBH inspiral where the spin angular momentum of the AH is conserved. We will present elsewhere the performance of the COCAL code for computing those sequences to model BBH inspirals, which is a necessary step to compute more complex binary systems including black hole-neutron star binaries.

ACKNOWLEDGMENTS

This work was supported by JSPS Grant-in-Aid for Scientific Research(C) 23540314 and 22540287, and MEXT Grant-in-Aid for Scientific Research on Innovative Area 20105004. We thank Eric Gourgoulhon for discussion. KU thanks Charalampos Markakis, and Noriyuki Sugiyama for discussion.
TABLE IV. Solution sequence of equal mass BBH with corotating spins. In the computation, parameters H3 listed in Table 3 are used with a fixed separation $d_s = 2.5$, and with varying the excision radius $r_a$ of BH.

| $r_a$  | $d_s/r_a$ | $d_s/M_{\text{Hir}}$ | $\Omega M_{\text{Hir}}$ | $M_{\text{ADM}}/M_{\text{Hir}}$ | $E_b/M_{\text{Hir}}$ | $J/M_{\text{Hir}}^2$ | $M_{\text{Hir}}$ |
|-------|-----------|----------------------|--------------------------|-------------------------------|----------------------|---------------------|--------------|
| 0.03  | 83.333    | 24.327               | 0.0077690                | 0.99501                       | -0.0049873          | 1.3211              | 0.10277      |
| 0.04  | 62.500    | 18.126               | 0.011925                 | 0.99360                       | -0.0064032          | 1.1824              | 0.13793      |
| 0.05  | 50.000    | 14.403               | 0.016575                 | 0.99226                       | -0.0077416          | 1.0910              | 0.17357      |
| 0.06  | 41.667    | 11.921               | 0.021647                 | 0.99100                       | -0.0089952          | 1.0268              | 0.20972      |
| 0.07  | 35.714    | 10.147               | 0.027806                 | 0.98984                       | -0.010159           | 0.97988             | 0.24638      |
| 0.08  | 31.250    | 8.8151               | 0.032848                 | 0.98777                       | -0.011228           | 0.94470             | 0.28360      |
| 0.09  | 27.778    | 7.7785               | 0.038898                 | 0.98780                       | -0.012201           | 0.91792             | 0.32140      |
| 0.10  | 25.000    | 6.9483               | 0.045207                 | 0.98693                       | -0.013074           | 0.89736             | 0.35980      |
| 0.11  | 22.727    | 6.2682               | 0.051746                 | 0.98615                       | -0.013847           | 0.88156             | 0.39884      |
| 0.12  | 20.833    | 5.7007               | 0.058495                 | 0.98548                       | -0.014518           | 0.86946             | 0.43854      |
| 0.13  | 19.231    | 5.2197               | 0.065430                 | 0.98491                       | -0.015087           | 0.86032             | 0.47896      |
| 0.14  | 17.857    | 4.8067               | 0.072533                 | 0.98445                       | -0.015555           | 0.85355             | 0.52011      |
| 0.15  | 16.667    | 4.4481               | 0.079787                 | 0.98408                       | -0.015922           | 0.84873             | 0.56204      |
| 0.16  | 15.625    | 4.1337               | 0.087174                 | 0.98381                       | -0.016191           | 0.84550             | 0.60479      |
| 0.17  | 14.706    | 3.8557               | 0.094679                 | 0.98364                       | -0.016362           | 0.84360             | 0.64839      |
| 0.18  | 13.889    | 3.6081               | 0.10229                  | 0.98356                       | -0.016439           | 0.84279             | 0.69289      |
| 0.19  | 13.158    | 3.3860               | 0.10998                  | 0.98358                       | -0.016424           | 0.84290             | 0.73834      |
| 0.20  | 12.500    | 3.1856               | 0.11775                  | 0.98368                       | -0.016320           | 0.84377             | 0.78477      |

[1] N. Stergioulas, "Rotating Stars in Relativity", Living Rev. Relativity 6, (2003), 3. URL (cited on July 1st 2011): http://www.livingreviews.org/lrr-2003-3

[2] G. B. Cook, Living Rev. Rel. 3, 5 (2000): http://www.livingreviews.org/lrr-2000-5

[3] H. P. Pfeiffer, arXiv:1203.5166 [gr-qc].

[4] G. B. Cook, Phys. Rev. D 50, 5025 (1994); S. Brandt and B. Brügmann, Phys. Rev. Lett. 78, 3606 (1997); H. P. Pfeiffer, S. A. Teukolsky and G. B. Cook, Phys. Rev. D 62, 104018 (2000); P. Marronetti and R. A. Matzner, Phys. Rev. Lett. 85, 5500 (2000); G. B. Cook, Phys. Rev. D 65, 084003 (2002); H. P. Pfeiffer, G. B. Cook and S. A. Teukolsky, Phys. Rev. D 66, 024047 (2002); G. Lovelace, R. Owen, H. P. Pfeiffer and T. Chu, Phys. Rev. D 78, 084017 (2008); M. Ansorg, Phys. Rev. D 72, 024018 (2005); M. Ansorg, B. Brügmann and W. Tichy, Phys. Rev. D 70, 064011 (2004); W. Tichy, B. Brügmann, M. Campanelli and P. Diener, Phys. Rev. D 67, 064008 (2003); B. J. Kelly, W. Tichy, M. Campanelli and B. F. Whiting, Phys. Rev. D 76, 024008 (2007); B. C. Mundim, B. J. Kelly, Y. Zlochower, H. Nakano and M. Campanelli, Class. Quant. Grav. 28, 134003 (2011); L. T. Buchman, H. P. Pfeiffer, M. A. Scheel and B. Szilagyi, Phys. Rev. D 86, 084033 (2012).