Novel symmetries in $\mathcal{N} = 2$ supersymmetric quantum mechanical models

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Abstract

We demonstrate the existence of a novel set of discrete symmetries in the context of $\mathcal{N} = 2$ supersymmetric (SUSY) quantum mechanical model with a potential function $f(x)$ that is a generalization of the potential of the 1D SUSY harmonic oscillator. We perform the same exercise for the motion of a charged particle in the $X − Y$ plane under the influence of a magnetic field in the $Z$-direction. We derive the underlying algebra of the existing continuous symmetry transformations (and corresponding conserved charges) and establish its relevance to the algebraic structures of the de Rham cohomological operators of differential geometry. We show that the discrete symmetry transformations of our present general theories correspond to the Hodge duality operation. Ultimately, we conjecture that any arbitrary $\mathcal{N} = 2$ SUSY quantum mechanical system can be shown to be a tractable model for the Hodge theory.

Keywords: $\mathcal{N} = 2$ supersymmetric quantum mechanics, continuous and discrete symmetries, de Rham cohomological operators, Hodge theory

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1. Introduction

The supersymmetric (SUSY) quantum mechanical models represent mathematically one of the most beautiful and elegant examples in the realm of theoretical physics which have found applications in diverse domains of physical phenomena (see, e.g. [1,2]). At the level of quantum mechanics, supersymmetry connects two Hamiltonians and corresponding states and, at the classical level, this symmetry transforms the commuting dynamical variables into the anticommuting ones and vice-versa. The central theme of our present investigation is to explore new set of continuous and discrete symmetries of totally different kinds of SUSY models and demonstrate that these symmetries provide physical realizations of abstract properties associated with the cohomological operators of differential geometry [3-5]. As a result, we conjecture that $\mathcal{N} = 2$ SUSY models, obeying $sl(1/1)$ superalgebra, belong to a special class which can be shown to be physical models for the Hodge theory.

From the physical point of view, the above kind of studies are very important. For instance, in our earlier series of research works [6-10], we have established that the Abelian 1-form, 2-form and 3-form gauge theories (in 2D, 4D and 6D dimensions of spacetime) are models for the Hodge theory within the framework of Becchi–Rouet-Stora-Tyutin (BRST) formalism. Furthermore, the (non-)Abelian 1-form gauge theory in 2D [6], interacting 2D Abelian 1-form theory with Dirac fields [7], 4D free Abelian 2-form gauge theory [8], 6D free Abelian 3-form gauge theory, etc., are endowed with continuous and discrete set of symmetry transformations that provide physical realizations for the de Rham cohomological operators, Hodge duality operation, degree of a form, etc., within the purview of BRST formalism. The culmination of all the above studies is the proof that the 2D (non-)Abelian gauge theories (without any interaction with matter fields) belong to a new class of topological field theories [11] and 4D free Abelian 2-form as well as 6D free Abelian 3-form gauge theories turn out to be the models for quasitopological field theory [12,10].

All the above cited theoretical models, however, belong to a special class of field theories (i.e. gauge theories) that are endowed with first-class constraints in the terminology of Dirac’s prescription for the classification scheme [13]. In a very recent paper [14], we have taken a one (0 + 1)-dimensional (1D) $\mathcal{N} = 2$ SUSY model of harmonic oscillator and shown that it provides a physical model for the Hodge theory. In our present endeavor, we consider two different kinds of physically interesting $\mathcal{N} = 2$ SUSY models and
demonstrate that they respect all the pertinent symmetries that provide a basis for these physical systems to be the models for Hodge theory because all the cohomological operators (and connected Hodge duality operation) find their physical realizations in terms of continuous and discrete symmetry transformations of the theory. Furthermore, the conserved charges, obeying the \( sl(1/1) \) superalgebra, are found to be the analogues of de Rham cohomological operators of differential geometry from various points of view. As a consequence, we conjecture that the specific class of models, corresponding to \( \mathcal{N} = 2 \) SUSY theories, represent physical models for the Hodge theory.

In our present endeavor, we discuss explicitly the continuous fermionic symmetry transformations (corresponding to \( \mathcal{N} = 2 \) supersymmetry) and derive the corresponding supercharges by exploiting the Noether's theorem. We also derive the conserved charge corresponding to a bosonic symmetry that is an anticommutator of the above two SUSY transformations. As expected, we observe that this bosonic symmetry transformation turns out to be equivalent to a time translation. This observation is sacrosanct for any well-defined SUSY theory where it is a crucial requirement that two successive SUSY transformations must produce the spacetime translations in a given dimension of spacetime. In our present couple of systems (corresponding to \( \mathcal{N} = 2 \) SUSY models), the generator of the bosonic symmetry transformations turns out to be connected with the Hamiltonian of the theory because, as is well-known, the latter is the generator of the time translation.

Our present investigation is essential on the following counts. First, we have shown, in our earlier work [14], that the 1D SUSY harmonic oscillator provides a prototype example of a Hodge theory. Thus, it is very tempting to study other \( \mathcal{N} = 2 \) SUSY models and check whether they also respect similar kinds of symmetries as does the SUSY oscillator. Second, the motion of an electrically charged particle under influence of an EM field is a physically very important topic. Thus, its \( \mathcal{N} = 2 \) SUSY quantum mechanical version is interesting in its own right. To say something new about this model is always challenging. We show, in our present endeavor, that this system, too, is a tractable model for the Hodge theory. Finally, we go a step further and conjecture that any arbitrary \( \mathcal{N} = 2 \) SUSY quantum mechanical model would be endowed with symmetries that would turn out to be the realizations of cohomological operators. As a consequence, these SUSY systems represent a special class of models that provide a realization of Hodge theory.

Besides the above motivations, our present study of simple SUSY quantum mechanical systems would provide insights into the understanding of
$\mathcal{N} = 2$ SUSY gauge theories of phenomenological importance where the cohomological structure might appear. As a consequence, one would be able to apply the celebrated Hodge decomposition theorem in defining the physical state of the theory (which would be chosen to be the harmonic state). The latter would be, naturally, annihilated by the operator form of $Q, \bar{Q}$ and $W$ [cf.(16)]. This would put constraints on the theory which will be useful in the counting of degrees of freedom of the theory. This information would enable us to study the topological nature of the SUSY gauge theory. In fact, the fermionic charges $Q$ and $\bar{Q}$ would play important roles in expressing the Lagrangian density as well as energy-momentum tensor of such theories. As a consequence, one would be able to state that the energy excitation of the physical state would be zero if the physical state is chosen to be the harmonic state in the Hodge decomposition theorem. Such kind of study has been performed in the context of usual (non-)Abelian 2D gauge theories [11].

The material of our present investigation is organized as follows. To set up the notations and conventions, we start off with a brief synopsis of $\mathcal{N} = 2$ supersymmetric harmonic oscillator and discuss its various continuous as well as discrete symmetry transformations in Sec. 2. Our Sec. 3 is devoted to the discussion of continuous symmetries and the derivation of corresponding Noether conserved charges for two different $\mathcal{N} = 2$ supersymmetrical models. Our Sec. 4 deals with the discrete symmetries of the above two supersymmetric quantum mechanical systems. We deduce the algebraic structures of the symmetry operators (and corresponding conserved charges) and establish their connection with the algebra of cohomological operators in Sec. 5. Finally, we make some concluding remarks in Sec. 6.

In our Appendix A, we discuss simpler ways of deriving the $sl(1/1)$ closed superalgebra amongst the conserved charges of $\mathcal{N} = 2$ SUSY quantum mechanical models that are topics of discussion in our present endeavor.

Conventions and Notations: Throughout the whole body of our text, the fermionic ($s_1^2 = 0, s_2^2 = 0$) symmetries [that are the analogue of the nilpotent (co-)exterior derivatives] have been denoted by $s_1$ and $s_2$ and their anticommutator (which is an analogue of the Laplacian operator) is represented by $s_\omega = \{s_1, s_2\}$ for all the models of $\mathcal{N} = 2$ SUSY quantum mechanics. The corresponding conserved charges have been expressed by $Q, \bar{Q}, W$. This has been done purposely, so that, some common features of the above SUSY models could be expressed in a concise fashion (see, e.g. Sec. 5 below).
2. Preliminaries: SUSY oscillator

We begin with the Lagrangian for a one (0 + 1)-dimensional (1D) supersymmetric harmonic oscillator which is described by the ordinary bosonic position variable \( x \) and a pair of Grassmannian variables \( (\psi, \bar{\psi}) \) (with \( \psi^2 = \bar{\psi}^2 = 0, \psi \bar{\psi} + \bar{\psi} \psi = 0 \)) at the classical level. For the sake of simplicity, we take the mass \( m \) of the oscillator to be one (i.e. \( m = 1 \)) in the following Lagrangian (with natural oscillator frequency \( \omega \)) (see, e.g. [14] for details)

\[
L_0 = \frac{\dot{x}^2(t)}{2} - \frac{1}{2} \omega^2 x^2(t) + i \bar{\psi}(t) \dot{\psi}(t) - \omega \bar{\psi}(t) \psi(t),
\]

(1)

where \( \dot{x} = dx/dt \) and \( \dot{\psi} = d\psi/dt \) are the generalized “velocities” in terms of the variation of the instantaneous bosonic and fermionic variables \( x \) and \( \psi \) with respect to the evolution parameter \( t \).

The above starting Lagrangian is endowed with the following on-shell nilpotent \( (s_1^2 = s_2^2 = 0) \) infinitesimal symmetry transformations [14]

\[
s_1 x = \frac{-i \psi}{\sqrt(2 \omega)}, \quad s_1 \psi = 0, \quad s_1 \bar{\psi} = \frac{1}{\sqrt(2 \omega)} (\dot{x} + i \omega x),
\]

\[
s_2 x = \frac{i \bar{\psi}}{\sqrt(2 \omega)}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = \frac{1}{\sqrt(2 \omega)} (-\dot{x} + i \omega x), \quad (2)
\]

because the Lagrangian transforms to a total time derivative under \( s_1 \) and \( s_2 \). As a consequence, the action integral \( (S = \int dt \ L_0) \) remains invariant under the above continuous and infinitesimal SUSY transformations.

There is yet another continuous symmetry in the theory that is obtained by taking the anticommutator of the above SUSY transformations \( s_1 \) and \( s_2 \) modulo a factor of \( i \). The infinitesimal version of this bosonic symmetry \( s_\omega = \{s_1, s_2\} \), for the relevant dynamical variables of the theory, are

\[
s_\omega x = \frac{1}{\omega} \dot{x}, \quad s_\omega \psi = \frac{1}{2 \omega} (\dot{\psi} - i \omega \psi), \quad s_\omega \bar{\psi} = \frac{1}{2 \omega} (\dot{\bar{\psi}} + i \omega \bar{\psi}). \quad (3)
\]

It can be checked that the Lagrangian in (1) transforms to a total derivative under the above infinitesimal transformations, too, thereby rendering the action integral invariant [14]. Thus, ultimately, we have three continuous symmetries in the theory, out of which, two are fermionic and one is bosonic.

Now we dwell a bit on the existence of a discrete set of symmetries in the theory. These transformations are responsible for the beautiful connection
between the two SUSY continuous symmetries $s_1$ and $s_2$ that have been discussed above. These explicit and useful discrete transformations are

$$x \rightarrow -x, \quad t \rightarrow +t, \quad \omega \rightarrow -\omega, \quad \psi \rightarrow \pm i \bar{\psi}, \quad \bar{\psi} \rightarrow \mp i \psi,$$  

(4)

under which the Lagrangian (1) transforms to itself (i.e. $L_0 \rightarrow L_0$). Thus, finally, we conclude that there are, in totality, five symmetries in the theory. Three of them are continuous in nature and two are discrete.

We note that the SUSY symmetry transformation $s_1$ corresponds to the exterior derivative $d$ (with $d^2 = 0$) of differential geometry. On the other hand, the nilpotent ($s_2^2 = 0$) SUSY symmetry transformation $s_2$ stands for the co-exterior derivative $\delta$ (with $\delta^2 = 0$). This is due to the fact that we have the following operator relationships (see, e.g. [14])

$$s_2 \Phi = \pm * s_1 * \Phi, \quad s_1^2 \Phi = 0, \quad s_2^2 \Phi = 0, \quad \Phi = x, \psi, \bar{\psi},$$  

(5)

which mimic the relationship $\delta = \pm * d*, d^2 = \delta^2 = 0$ of differential geometry. It should be noted that the $*$, in the above equation (5), corresponds to the discrete set of symmetries quoted in (4). Thus, the discrete symmetry transformations (4) stand for the Hodge duality $\star$ operation of differential geometry which connects the (co-)exterior derivatives by: $\delta = \pm * d*$.

Pertinent to the above discussions, we note that the outcome of two successive discrete transformations on the generic variable $\Phi(t)$ is

$$\star [\star \Phi] = + \Phi, \quad \Phi = x, \psi, \bar{\psi}.$$  

(6)

Following the strictures, laid down by the duality invariant theories [15], there would be only a positive sign in the relationship (5) due to the positive sign present in the above generic equation (6). Thus, the correct version of (5), consistent with a correct duality-invariant theory, is [15]

$$s_2 = + * s_1 *, \quad s_1^2 = 0, \quad s_2^2 = 0,$$  

(7)

As a consequence, only one of the two transformations, listed in (4), would be physically useful. This can be succinctly expressed as

$$x \rightarrow -x, \quad t \rightarrow +t, \quad \omega \rightarrow -\omega, \quad \psi \rightarrow +i \bar{\psi}, \quad \bar{\psi} \rightarrow -i \psi.$$  

(8)

To sum up, we have precisely a single discrete symmetry in the theory as given in the above equation. Thus, we conclude that, for the one dimensional theory under consideration, the analogue of the exact relationship between the
(co-)exterior derivative is captured by the relationship \( s_2 = + \ast s_1 \ast \). Dimensionality of our problem also allows the validity of an inverse relationship (i.e. \( s_1 = - \ast s_2 \ast \)) between SUSY transformations \( s_1 \) and \( s_2 \).

We have discussed a bosonic symmetry transformation \( s_\omega = \{s_1, s_2\} \) in the theory that corresponds to the Laplacian operator \( \Delta = (d + \delta)^2 = \{d, \delta\} \). The operator form of the algebra of the transformations \( s_1, s_2, s_\omega \) match precisely with the algebra of the de Rham cohomological operators of differential geometry because we have the following exact relationships, namely;

\[
\begin{align*}
\{s_1, s_2\} &= 0, \\
\{s_\omega, s_1\} &= 0, \\
\{s_\omega, s_2\} &= 0, \\
d^2 &= 0, \\
\delta^2 &= 0, \\
\Delta &= \{d, \delta\}, \\
[\Delta, d] &= 0, \\
[\Delta, \delta] &= 0.
\end{align*}
\]

Finally, we have shown, in our earlier work [14], that conserved charges of the 1D SUSY oscillator have one-to-one correspondence with the cohomological operators of differential geometry [3-5]. We shall follow the logistics of our present discussion and establish the above kind of correspondence in the cases of potential functions which are (i) the generalizations of a harmonic oscillator potential, and (ii) motion of a charged particle in a plane under the influence of a magnetic field which is perpendicular to the plane.

3. Continuous symmetries: conserved charges

In this section, we take two different kinds of example of \( \mathcal{N} = 2 \) supersymmetric quantum mechanical models and discuss their continuous symmetry transformations and derive the corresponding conserved charges by exploiting the fundamental techniques of Noether’s theorem. We also establish that these conserved Noether charges are the generators of the above continuous and infinitesimal symmetry transformations.

3.1. A model with the generalized SUSY potential

We begin with the 1D general Lagrangian \( L_g \), which is a generalization of the starting Lagrangian \( L_0 \) [cf. (1)] with an arbitrary potential \( f(x) \), as

\[
L_g = \frac{\dot{x}^2(t)}{2} - \frac{1}{2} \omega^2 (f(x))^2 + i \bar{\psi}(t) \dot{\psi}(t) - \omega f'(x) \bar{\psi}(t) \psi(t),
\]

where \( \omega \) is a parameter in the theory and \( f'(x) = df/dx \) is the first order derivative on the potential function. It is evident that, in the limit \( f(x) = x \), we retrieve our original Lagrangian \( L_0 \) for the harmonic oscillator. We would
like to lay stress on the fact that potential function $f(x)$ is any arbitrary (but physically well-defined) potential function and other symbols (in $L_g$) denote their standard meanings as we have elaborated in our previous section.

The following nilpotent ($s_1^2 = 0, s_2^2 = 0$) SUSY transformations

$$s_1 x = \frac{-i \psi}{\sqrt{2 \omega}}, \quad s_1 \psi = 0, \quad s_1 \bar{\psi} = \frac{1}{\sqrt{2 \omega}} [\dot{x} + i \omega f(x)],$$

$$s_2 x = \frac{i \bar{\psi}}{\sqrt{2 \omega}}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = \frac{1}{\sqrt{2 \omega}} [-\dot{x} + i \omega f(x)],$$

are the symmetry transformations for the Lagrangian $L_g$ because

$$s_1 L_g = - \frac{d}{dt} \left[ \frac{\omega f(x)}{\sqrt{2 \omega}} \psi \right], \quad s_2 L_g = + \frac{d}{dt} \left[ \frac{i \dot{x} \bar{\psi}}{\sqrt{2 \omega}} \right]. \quad (11)$$

As a consequence, the action integral ($S = \int dt L_g$) remains invariant under the above fermionic transformations. The nilpotency properties of $s_1$ and $s_2$ is valid only on the on-shell where the following equations of motion

$$\ddot{x} + \omega^2 f f' \bar{\psi} \psi = 0, \quad \dot{\psi} + i \omega f' \psi = 0, \quad \dot{\bar{\psi}} - i \omega f' \bar{\psi} = 0,$$

$$\ddot{\psi} + i \omega f'' \dot{x} \psi + \omega^2 (f')^2 \psi = 0, \quad \ddot{\bar{\psi}} - i \omega f'' \dot{x} \bar{\psi} + \omega^2 (f')^2 \bar{\psi} = 0. \quad (13)$$

are satisfied. The last two equations, in the above, have been derived from the basic equations of motion $\dot{\psi} + i \omega f' \psi = 0, \dot{\bar{\psi}} - i \omega f' \bar{\psi} = 0$. Furthermore, it is interesting to check that, under the symmetry transformations $s_1$ and $s_2$, the above equations of motion go to one-another.

There exists a bosonic symmetry $s_\omega = \{s_1, s_2\}$ in the theory modulo a factor of $i$, under which, the physical variables transform as

$$s_\omega x = \frac{1}{\omega} \dot{x}, \quad s_\omega \psi = \frac{1}{2 \omega} (\dot{\psi} - i \omega f' \psi), \quad s_\omega \bar{\psi} = \frac{1}{2 \omega} (\ddot{\bar{\psi}} + i \omega f' \bar{\psi}). \quad (14)$$

The key point to be noted here is the fact that, if we use the equations of motion, the r.h.s of the above transformations can be written as the time derivative on the individual variables. This verifies the existence of supersymmetry in the theory. It is a decisive feature of any arbitrary supersymmetric theory that two consecutive supersymmetric transformations always generate the spacetime translation. This implies that, for a 1D system, two
supersymmetric transformations should lead to the time translation (which is satisfied in our case). Under \( s_\omega \) [cf. (14)], the Lagrangian changes as

\[
s_\omega L_g = \frac{d}{dt} \left[ \frac{1}{2\omega} (\dot{x}^2 - \omega^2 f^2 + i \bar{\psi} \dot{\psi} - \omega f' \bar{\psi} \dot{\psi}) \right]. \tag{15}
\]

As a consequence, the action integral of our present theory remains invariant under the infinitesimal transformations \( s_\omega = \{ s_1, s_2 \} \).

According to Noether’s theorem, the above continuous symmetry transformations would lead to the derivation of conserved charges which would turn out to be the generators of the transformations \( s_1, s_2, s_w \). To derive these charges \( (Q, \bar{Q}, W) \), corresponding to the above continuous symmetry transformations \( (s_1, s_2, s_w) \), we exploit the standard techniques and obtain the following explicit expressions in terms of the variables of the theory:

\[
Q = \frac{1}{\sqrt{2\omega}} \left[ (-i \dot{x}) + (\omega f(x)) \right] \psi \equiv \frac{1}{\sqrt{2\omega}} \left[ (-i p) + (\omega f(x)) \right] \psi,
\]

\[
\bar{Q} = \frac{\bar{\psi}}{\sqrt{2\omega}} \left[ (+i \dot{x}) + (\omega f(x)) \right] \equiv \frac{\bar{\psi}}{\sqrt{2\omega}} \left[ (+i p) + (\omega f(x)) \right],
\]

\[
W = \frac{1}{\omega} H_g \equiv \frac{1}{\omega} \left[ \frac{p^2}{2} + \frac{\omega^2 f^2}{2} + \omega f' \bar{\psi} \dot{\psi} \right], \tag{16}
\]

where \( p = \partial L_g / \partial \dot{x} = \dot{x} \) is the canonically conjugate momentum w.r.t. the position variable \( x \) and \( H_g \) is the Hamiltonian for the system under consideration. Furthermore, the equation of motion \( \dot{\psi} + i\omega f' \psi = 0 \) has been used to express \( \dot{\psi} \) in terms of \( \psi \) in the derivation of the Noether charge \( W \).

The conservation laws for these charges can be proven by directly exploiting the equations of motion (13) and substituting them into the expressions for \( \dot{Q}, \dot{\bar{Q}}, \dot{W} \). The other way to prove the conservation laws is by computing the commutator of the above charges with the Hamiltonian \( H_g \) by exploiting the canonical brackets that emerge from the Lagrangian (10) of our system.

### 3.2. Motion of a charged particle under influence of a magnetic field

We consider here the well-known example of the motion of a charged particle in the \( X-Y \) plane where the magnetic field \( (B_z) \) is in the \( Z \)-direction. For the sake of simplicity, we take here the natural units \( \hbar = c = 1 \) as well as the mass \( (m) \) and charge \( (e) \) to be unity (i.e. \( m = e = 1 \)). The Hamiltonian \( H_{em} \) of such a charged particle, under the above magnetic field, is [2]

\[
H_{em} = \frac{1}{2} (p_x + A_x)^2 + \frac{1}{2} (p_y + A_y)^2 - B_z \bar{\psi} \dot{\psi}, \tag{17}
\]
where $A_x(x,y), A_y(x,y)$ are the components of the vector potential in the $X–Y$ plane, $p_x = \dot{x}, p_y = \dot{y}$ are the $x$ and $y$ components of the 2D momenta and $B_z = \partial_x A_y - \partial_y A_x$ is the $z$-component of the magnetic field. The Lagrangian for the above system (due to Legendre transformation) is

$$L_{em} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - (\dot{x} A_x + \dot{y} A_y) + i \bar{\psi} \psi + B_z \bar{\psi} \psi,$$

(18)

where $\dot{x} = dx/dt, \dot{y} = dy/dt, \dot{\psi} = d\psi/dt$ are the generalized “velocities” for a pair of bosonic coordinates $(x(t), y(t))$ and the fermionic variable $\psi(t)$ (in terms of their variations w.r.t. the evolution parameter $t$). The bosonic variables are commuting in nature whereas the pair of fermionic variables $(\psi(t), \bar{\psi}(t))$ are anticommuting (i.e. $\psi^2 = \bar{\psi}^2 = 0, \psi \bar{\psi} + \bar{\psi} \psi = 0$).

The following continuous and infinitesimal nilpotent ($s_1^2 = 0, s_2^2 = 0$) fermionic transformations ($s_1, s_2$):

$$s_1 x = \frac{\psi}{\sqrt{2}}, \quad s_1 y = -i \frac{\psi}{\sqrt{2}}, \quad s_1 \psi = 0, \quad s_1 \bar{\psi} = \frac{i}{\sqrt{2}} (\dot{x} - i \dot{y}),$$

$$s_1 A_x = \frac{1}{\sqrt{2}} (\partial_x A_x - i \partial_y A_x) \psi, \quad s_1 A_y = \frac{1}{\sqrt{2}} (\partial_x A_y - i \partial_y A_y) \psi,$$

$$s_2 x = \frac{\bar{\psi}}{\sqrt{2}}, \quad s_2 y = i \frac{\bar{\psi}}{\sqrt{2}}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = \frac{i}{\sqrt{2}} (\dot{x} + i \dot{y}),$$

$$s_2 A_x = \frac{\bar{\psi}}{\sqrt{2}} (\partial_x A_x + i \partial_y A_x), \quad s_2 A_y = \frac{\bar{\psi}}{\sqrt{2}} (\partial_x A_y + i \partial_y A_y),$$

(19)

are the symmetry transformation for the action integral ($S = \int dt L_{em}$) because the Lagrangian (18) transforms to the total derivatives as

$$s_1 L_{em} = - \frac{d}{dt} \left[ \frac{(A_x - i A_y) \psi}{\sqrt{2}} \right],$$

$$s_2 L_{em} = + \frac{d}{dt} \left[ \frac{\bar{\psi}}{\sqrt{2}} \left\{ \dot{x} + i \dot{y} - (A_x + i A_y) \right\} \right].$$

(20)

Thus, according to Noether’s theorem, we shall have conserved charges which would turn out to be the generators for the above continuous symmetries. These SUSY fermionic ($Q^2 = \bar{Q}^2 = 0$) charges, corresponding to the nilpotent SUSY transformations $s_1$ and $s_2$, are as follows

$$Q = \frac{1}{\sqrt{2}} \left[ (p_x + A_x) - i (p_y + A_y) \right] \psi,$$

$$\bar{Q} = \frac{\bar{\psi}}{\sqrt{2}} \left[ (p_x + A_x) + i (p_y + A_y) \right],$$

(21)
which are derived using the standard techniques of Noether’s theorem.

The anticommutator \( \{ s_1, s_2 \} = s_\omega \) leads to the derivation of a bosonic symmetry in the theory. The continuous and infinitesimal version of these transformations (modulo a factor of \( i \)) on the generic variable \( \Phi \) is

\[
s_\omega \Phi = \dot{\Phi}, \quad \Phi = x(t), \ y(t), \ \psi(t), \ \bar{\psi}(t), \ A_x(x, y), \ A_y(x, y). \quad (22)
\]

In the derivation of the above bosonic symmetry transformations, for obvious reasons, we have used the following straightforward inputs, namely;

\[
\partial_x \psi(t) = 0, \quad \partial_y \psi(t) = 0, \quad \partial_x \bar{\psi}(t) = 0, \quad \partial_y \bar{\psi}(t) = 0,
\]

\[
\frac{d}{dt} A_x(x, y) = \dot{x} \partial_x A_x + \dot{y} \partial_y A_x, \quad \frac{d}{dt} A_y(x, y) = \dot{x} \partial_x A_y + \dot{y} \partial_y A_y. \quad (23)
\]

The Lagrangian of our present theory transforms to a total derivative rendering the action integral invariant. Mathematically, this statement is

\[
s_\omega L_{em} = \frac{d}{dt} [L_{em}] \Rightarrow s_\omega S = \int dt \ (s_\omega L_{em}) = 0. \quad (24)
\]

Applying the standard techniques of Noether’s theorem, we obtain the expression of the conserved charge as follows

\[
W = H_{em} \equiv \left[ \frac{(p_x + A_x)^2}{2} + \frac{(p_y + A_y)^2}{2} - B_z \bar{\psi} \psi \right]. \quad (25)
\]

Thus, we note that the conserved charge, corresponding to the bosonic symmetry transformations, is nothing but the Hamiltonian of the theory itself. The conservation laws \( \dot{Q} = \dot{\bar{Q}} = \dot{W} = 0 \) can be proven by using the equations of motion that emerge from \( L_{em} \). There is another way to prove the conservation laws, too. One can compute the commutator of the Hamiltonian with the charges \( Q, \bar{Q}, W \) by exploiting the canonical (anti)commutators that are deduced from the Lagrangian \( L_{em} \) [cf. (38) and Appendix, below].

We wrap up this section with a couple of general remarks. First, the conserved charges in (16), (21) and (25) are the generators of the infinitesimal transformations \( s_1, s_2, s_\omega \) because we have the following relationships

\[
s_r \Phi = \mp i \ [\Phi, Q_r](\pm), \quad s_r = s_1, s_2, s_\omega, \quad Q_r = Q, \bar{Q}, W, \quad (26)
\]

where the \((+)-(+)\) signs, as the subscripts on the square bracket, stand for the bracket to be (anti)commutator for the generic variable \( \Phi = x, \psi, \bar{\psi} \) and
\[ \Phi = x, y, \psi, \bar{\psi}, A_x, A_y \] being (fermionic) bosonic in nature for both the SUSY examples of our present endeavor. The \((+)-\) signs, in front of the square bracket, are chosen judiciously. A detailed discussion about the choice of the latter can be found in our earlier work (see, e.g. [16]). Second, it is clear that the fermionic variables \(\psi\) and \(\bar{\psi}\) remain invariant under the nilpotent transformations \(s_1\) and \(s_2\), respectively. As pointed out in equation (4), there is a duality symmetry (i.e. discrete symmetry) in the theory. One can argue that the nilpotent symmetry transformations \(s_1\) and \(s_2\) are dual to each-other because they leave \(\psi\) and \(\bar{\psi}\) invariant which are connected to each-other by the duality transformations in (4) and its generalized form (see below).

4. Discrete symmetries: duality transformations

In this section, we discuss the presence of a set of discrete symmetry transformations for both the \(N = 2\) supersymmetric quantum mechanical models under consideration and establish their relevance to the Hodge duality \((\ast)\) operation of differential geometry.

4.1. A model with the generalized SUSY potential

It is very interesting to note that there is a set of discrete symmetries in the theory because the Lagrangian \(L_g\) [cf. (10)] remains invariant under these specific transformations. Let us focus on the discrete transformations

\[
\begin{align*}
x &\to -x, \quad \omega \to -\omega, \quad \psi \to \pm i \bar{\psi}, \quad \bar{\psi} \to \mp i \psi, \\
t &\to t, \quad f(x) \to -f(x), \quad f'(x) \to f'(x),
\end{align*}
\]

(27)

where we have denoted the actual transformations: \(x \to x' = -x, f(x) \to f(-x) = -f(x)\), etc., in an abbreviated form. Furthermore, as is evident, there are two symmetry transformations that are hidden in the above transformations. The discrete transformations (27) are actually the generalization of transformations in (4) because, in the limit \(f(x) = x\), we retrieve (4) from (27). It is clear that, physically, we are talking about the parity transformation operator \((\hat{P})\), under which, the potential function \(f(x)\) has to be odd because we are theoretically compelled to choose \(f'(-x) = f'(x)\) to incorporate the useful discrete symmetry transformations in our present theory.

In the case where the first-order derivative on the potential function is chosen to be odd under the parity operator (i.e. \(\hat{P}f'(x) \equiv f'(-x) = -f'(x)\)),
we are theoretically forced to rely on the following transformations

\[ x \rightarrow -x, \quad \omega \rightarrow -\omega, \quad \psi \rightarrow \pm i \bar{\psi}, \quad \bar{\psi} \rightarrow \pm i \psi, \]
\[ t \rightarrow -t, \quad f(x) \rightarrow f(x), \quad f'(x) \rightarrow -f'(x), \]

(28)

under which, the Lagrangian \( L_g \) remains invariant. A close look at the above transformations implies, physically, that there is time-reversal (\( \hat{T} \)) as well as parity (\( \hat{P} \)) invariance in the theory where the potential function is an even function under parity (i.e., \( \hat{P}f(x) = +f(x) \)). As a consequence, the first-order derivative on the potential function, automatically, becomes an odd function of parity (i.e. \( \hat{P}f'(x) = -f'(x) \)). Furthermore, we note that the fermionic variables transform, under the time-reversal operator, as

\[ \hat{T}: t \rightarrow -t, \quad \hat{T} \psi(t) = \pm i \bar{\psi}(t), \quad \hat{T} \bar{\psi}(t) = \pm i \psi(t). \]

(29)

We observe, in passing, that the generalized velocity \( \dot{x} \) is an invariant quantity under the combined operations of parity and time-reversal as is evident from \( \hat{T}\hat{P} (\dot{x}) = +(\dot{x}) \). Physically, this observation shows that the kinetic term for the bosonic variable of our present supersymmetric system is \( PT \)-invariant. However, the kinetic term (i.e. \( i \bar{\psi} \dot{\psi} \)), for the fermionic part of our present model, for obvious reasons, is time-reversal invariant (i.e. \( \hat{T} (i \bar{\psi} \dot{\psi}) = i \bar{\psi} \dot{\psi} \)).

The Lagrangian \( L_g \), with the general potential function \( f(x) \), has also only time-reversal symmetry. It is elementary to check that the following transformations (corresponding to the time-reversal operator), namely;

\[ x \rightarrow +x, \quad \omega \rightarrow +\omega, \quad \psi \rightarrow \pm i \bar{\psi}, \quad \bar{\psi} \rightarrow \pm i \psi, \]
\[ t \rightarrow -t, \quad f(x) \rightarrow f(x), \quad f'(x) \rightarrow f'(x), \]

(30)

leave the Lagrangian \( L_g \) invariant. It is to be pointed out that, in the above, there is no reflection symmetry in the theory because \( x \rightarrow x \). As a consequence, the potential function as well as its first-order derivative remain unaffected due to the presence of time-reversal symmetry alone [cf. (18)].

We dwell a bit now on the importance of the discrete symmetry transformations we have discussed so far. The discrete symmetry transformations (27) [that are generalization of (4)] correspond to the Hodge duality * operation of differential geometry. This can be proven by checking that relations (5) are satisfied by the interplay of continuous and discrete symmetry transformations (11) and (27) when they blend together in a meaningful manner.
Furthermore, we find that relation (6) is also valid in the case of general potential function \( f(x) \) for our first \( \mathcal{N} = 2 \) SUSY example. As a consequence, we observe that relations (7) and (9) are also true. Thus, it is crystal clear that the relationship (7) is the analogue of the relationship that exists between the (co-)exterior derivatives \([\delta d]\) of differential geometry.

Now we comment on the existence of discrete symmetry transformations (28) and (30), under which, the Lagrangian \( L_g \) remains invariant, too. It turns out that neither set of these symmetries leads to the exact derivation of relationship like (7) and its counterpart \( s_1 = -s_2^* \). As a consequence, these symmetries are not interesting from the point of view of the duality-invariant physical theories [15]. To elaborate on this statement, first of all, we note that the transformations (30) outrightly do not yield \( s_2 = \pm s_1^* \). Furthermore, two successive operations of (28) produces: \( * (\star x) = +x, \quad * (\star \Phi) = -\Phi \) where \( \Phi = \psi, \bar{\psi} \). As a consequence, we have the relationships \( s_2 x = +s_1^* x \) and \( s_2 \Phi = -s_1^* \Phi \) for \( \Phi = \psi, \bar{\psi} \). These are very nicely satisfied by (28). However, the reverse relationships \( s_1 x = -s_2^* x \) and \( s_1 \Phi = +s_2^* \Phi \) are not satisfied by the discrete symmetry transformations (28). Thus, we ignore (28) as the physical realization of the Hodge duality \((\star)\) operation.

We emphasize that, ultimately, the following unique transformations

\[
\begin{align*}
x & \to -x, \quad \omega \to -\omega, \quad \psi \to +i \bar{\psi}, \quad \bar{\psi} \to -i \psi, \\
t & \to t, \quad f(x) \to -f(x), \quad f'(x) \to f'(x),
\end{align*}
\]

(31)
correspond to the Hodge duality \((\star)\) operation of differential geometry. At this stage, there are a couple of remarks. First, it is physically very important that the potential function turns out to be odd under parity (see, e.g. [17,2] for details). Second, it can be checked that the reverse relationship \((s_1 = -s_2^*)\) of (7) also exists in the theory because of its dimensionality. Henceforth, we shall concentrate on the above unique transformations as the analogue of the Hodge duality \((\star)\) operation (as far as the physical discussions of our present theory, with a general super potential function \( f(x) \), is concerned).

We close this section with the remark that the super charges \( Q \) and \( \bar{Q} \) transform under the the duality transformations (31) as follows

\[
* (Q) = \bar{Q}, \quad * (\bar{Q}) = -Q, \quad * (\star Q) = -Q, \quad * (\star \bar{Q}) = -\bar{Q}.
\]

(32)

We point out that \( Q \) and \( \bar{Q} \) transform in exactly same manner as the electromagnetic duality transformations for the source-free Maxwell’s equations where \( \mathbf{E} \to \mathbf{B}, \mathbf{B} \to -\mathbf{E} \). Thus, there is a perfect duality symmetry in our
Furthermore, we note that, in contrast to the transformations (6), we find that the double $\ast$ operations on the fermionic charges results in a negative sign. Another interesting observation is:

\[ *W = -W, \quad (\ast \ast W) = +W. \]

As a consequence, we find that the $sl(1/1)$ algebraic structure:

\[ Q^2 = 0, \quad \bar{Q}^2 = 0, \quad \{Q, \bar{Q}\} = (H_g/\omega) \]

remains invariant under the $\ast$ duality transformations (applied any arbitrary number of times on this algebra). Finally, we note that the Hamiltonian of the theory remains duality invariant (because $\ast H_g = H_g$) as is the case with the Lagrangian ($\ast L_g = L_g$) of our present theory.

### 4.2. Motion of a charged particle under influence of a magnetic field

Unlike the previous subsection 4.1, we shall focus here only on those discrete symmetry transformations which are useful to us as far as the derivation of connection between SUSY transformations $s_1$ and $s_2$ is concerned. In fact, it can be checked that the Lagrangian $L_{em}$ remains invariant under the following useful discrete symmetry transformations

\[
x \to \mp x, \quad \psi \to \mp \bar{\psi}, \quad A_x \to \pm A_x, \quad t \to -t, \\
y \to \pm y, \quad \bar{\psi} \to \pm \psi, \quad A_y \to \mp A_y, \quad B_z \to B_z. \tag{33}
\]

A few remarks are in order at this stage. First, it should be noted that, in reality, there are two discrete transformations in (33) that leave the Lagrangian $L_{em}$ invariant. Second, there is always a time reversal ($t \to -t$) symmetry in the theory irrespective of how the coordinates $(x, y)$, in the plane, transform. Third, as a consequence of transformations ($x \to \mp x, y \to \pm y$), the space derivatives transform as $\partial_x \to \mp \partial_x, \partial_y \to \pm \partial_y$. Fourth, the kinetic terms ($\dot{x}^2/2$) and ($\dot{y}^2/2$) remain invariant under (33). Finally, we would like to remark that the vector potentials change explicitly, under the parity-type transformations for the space variables ($x \to \mp x, y \to \pm y$), as

\[
A_x(x, y) \to A_x(\mp x, \pm y) = \pm A_x(x, y), \\
A_y(x, y) \to A_y(\mp x, \pm y) = \mp A_y(x, y), \tag{34}
\]

which leave the magnetic field $B_z = \partial_x A_y - \partial_y A_x$ invariant (because we have to take into account the corresponding transformations for the derivatives $\partial_x \to \mp \partial_x, \partial_y \to \pm \partial_y$ under the above transformations $x \to \mp x, y \to \pm y$).

We would like to re-state the following. As discussed in subsection 4.1, we can also pay attention to various possibilities of the existence of discrete symmetries in our present model of $\mathcal{N} = 2$ supersymmetric quantum mechanical
system. However, we have concentrated only on the symmetry transformations (33) that are very useful to us as far as the derivation of the analogue of the relationship $\delta = \pm * d *$ (in the language of symmetry transformations) is concerned. In fact, as it turns out, we shall see that it is the interplay of the continuous nilpotent ($s_1^2 = 0, s_2^2 = 0$) transformations $s_1, s_2$ and discrete symmetry transformations (33) that provide the analogue of the relationship between the (co-)exterior derivatives ($\delta, d$) of differential geometry.

To appreciate the importance of the discrete symmetry transformations (33), first of all, we observe that two successive operations of these discrete symmetry transformations on any generic variable $\Phi$ yields either plus or minus sign. This can be mathematically stated as $* [ * \Phi ] = \pm \Phi$ where the $*$ operation is nothing but the discrete symmetry transformations (33) and the generic variable $\Phi = x, y, \psi, \bar{\psi}, A_x, A_y$. To be more specific, it can be seen that the following is true (for $\Phi_1$ and $\Phi_2$ components of $\Phi$), namely;

\[
\begin{align*}
* [ * ] \Phi_1 &= + \Phi_1, & \Phi_1 &= x, y, A_x, A_y, \\
* [ * ] \Phi_2 &= - \Phi_2, & \Phi_2 &= \psi, \bar{\psi}.
\end{align*}
\]

(35)

The connection between the (co-)exterior derivatives ($\delta, d$) (i.e. $\delta = \pm * d *$) can be realized between the nilpotent fermionic symmetry transformations $s_1$ and $s_2$ (i.e. $s_2 = \pm * s_1 *$). To pin-point this relationship in a more specific fashion (see, e.g. [15]), we have the following explicit relationships, namely;

\[
\begin{align*}
s_2 \Phi_1 &= + * s_1 * \Phi_1 \Rightarrow s_2 = + * s_1 * , \\
s_2 \Phi_2 &= - * s_1 * \Phi_2 \Rightarrow s_2 = - * s_1 * ,
\end{align*}
\]

(36)

where, as is evident, $\Phi_1 = x, y, A_x, A_y$ and $\Phi_2 = \psi, \bar{\psi}$. We note that the reverse relationships $s_1 \Phi_1 = - * s_2 * \Phi_1$ and $s_1 \Phi_2 = + * s_2 * \Phi_2$ are also true.

As a final remark, we mention here that the Lagrangian and Hamiltonian of our present model are duality invariant (i.e. $* L_{em} = L_{em}, * H_{em} = H_{em}$) and the fermionic conserved charges transform under (33) as

\[
* Q = \mp \bar{Q}, \quad * \bar{Q} = \pm Q, \quad [ * Q ] = - Q, \quad [ * \bar{Q} ] = - \bar{Q}.
\]

(37)

Thus, once again, we observe the duality transformations (i.e. $E \rightarrow \pm B, B \rightarrow \mp E$) of source-free Maxwell equations being replicated here in the duality transformations of $Q$ and $\bar{Q}$. Furthermore, we note that the $sl(1/1)$ closed superalgebra here does not remain invariant under the duality $*$ operation because the Hamiltonian turns out to be duality invariant (i.e. $* H_{em} = \bar{H}_{em}$).
This discrepancy (from the model of our previous subsection 4.1) appears because of the fact that parameter $\omega$ is absent in our present model. This is the reason that, in our case, we have $* W = +W$ (as $W = H_{em}$). On the contrary, in our previous subsection, we had $* W = -W$ because $* (H/\omega) = -(H_g/\omega)$ (due to $* H_g = H_g$ and $* \omega = -\omega$).

5. Algebraic structures: cohomological aspects

In the present section, we shall discuss the algebraic structures of the conserved charges $(Q, \bar{Q}, W)$ for both the $\mathcal{N} = 2$ SUSY quantum mechanical models together that have been considered in our present endeavor.

We have already noted that the fermionic SUSY transformations $s_1$ and $s_2$ are nilpotent of order two (i.e. $s_1^2 = s_2^2 = 0$) on the on-shell where the Euler-Lagrange equations of motion (13) are satisfied for our first model of SUSY example. In the case of the motion of a charged particle, we observe that the nilpotency of $s_1$ and $s_2$ ensue from the fermionic (i.e. $\psi^2 = \bar{\psi}^2 = 0$) nature of variables $\psi$ and $\bar{\psi}$. Furthermore, it can be explicitly checked that the bosonic symmetry transformation $s_\omega$, that is equal to the anticommutator (i.e. $s_\omega = \{s_1, s_2\}$) of the fermionic transformations, commutes with both the SUSY transformations $s_1$ and $s_2$. As a consequence, the operator form of the transformations $s_\omega$ is the Casimir operator for the whole algebra. Thus, we conclude that the operators $s_1, s_2, s_\omega$ satisfy exactly the same algebra as is the case of SUSY harmonic oscillator [cf. (9)].

It turns out that the conserved charges of (16), (21) and (25) obey exactly the same algebra as the operator form of the transformations $s_1, s_2, s_\omega$. Mathematically, this super algebra $sl(1/1)$ can be succinctly written as

$$Q^2 = 0 \quad \bar{Q}^2 = 0, \quad W = \{Q, \bar{Q}\}, \quad [W, Q] = 0, \quad [W, \bar{Q}] = 0. \quad (38)$$

In view of the fact that $W = (H_g/\omega)$ and $W = H_{em}$ for both the SUSY models, respectively, it is obvious that the last two entries in the above equation are nothing but the conservation laws (i.e. $\dot{Q} = \dot{\bar{Q}} = 0$) for $Q$ and $\bar{Q}$. Furthermore, it is crystal clear that the conserved bosonic charge $W$ is the Casimir operator for the whole algebra. A close look at (38) shows that its algebraic structure is exactly same as the algebraic structure of the de Rham cohomological operators of differential geometry [(cf. (9))].

Due to the above observations, it is very tempting to identify the set of conserved charges $(Q, \bar{Q}, W)$ with the set of cohomological operators $(d, \delta, \Delta)$
of differential geometry. However, the identification is not yet complete because the cohomological operators satisfy specific properties when they operate on the differential form of a definite degree. For instance, it is a well-known fact that the (co-)exterior derivatives (lower)raise the degree of a form by one when they operate on it. On the contrary, the Laplacian operator does not change the degree of the form on which it acts. We have to capture these properties in the language of conserved charges (i.e. $Q, \bar{Q}, W$) for the completion and correctness of an exact identification.

To achieve the above goal, we have taken the help of bosonic as well as fermionic number operators (and their eigen-values) in the context of SUSY quantum mechanical harmonic oscillator where $f(x) = x$ [14]. For an arbitrary potential function $f(x)$, the above arguments fail because the bosonic creation and annihilation operators become non-trivial and their commutation relation produce a first-order derivative $f'(x)$ on the potential function $f(x)$. In exactly similar fashion, the second example of our $\mathcal{N} = 2$ SUSY model (connected with the motion of a charged particle) also does not obey the above logic. However, it is illuminating to note that the following algebra, amongst the set of operators $(Q, \bar{Q}, W)$, is true, namely;

$$[Q \bar{Q}, Q] = +W Q, \quad [Q \bar{Q}, \bar{Q}] = -W \bar{Q},$$
$$[\bar{Q} Q, Q] = -W Q, \quad [\bar{Q} Q, \bar{Q}] = +W \bar{Q},$$

(39)

where, as is evident from (38), the charge $W = \{Q, \bar{Q}\}$ is the Casimir operator (i.e. $[W, Q] = [W, \bar{Q}] = 0$). We assume that the inverse of the Casimir operator ($W^{-1}$) is well-defined and the latter logically commutes with both the nilpotent super charges (i.e. $[W^{-1}, Q] = [W^{-1}, \bar{Q}] = 0$).

As a consequence of the above arguments, the algebra (39) can be re-expressed, in a theoretically useful and handy manner, as follows

$$\left[\frac{Q \bar{Q}}{W}, Q\right] = +Q, \quad \left[\frac{Q \bar{Q}}{W}, \bar{Q}\right] = -\bar{Q},$$
$$\left[\frac{\bar{Q} Q}{W}, Q\right] = -Q, \quad \left[\frac{\bar{Q} Q}{W}, \bar{Q}\right] = +\bar{Q},$$

(40)

In this situation, one can define a state $|\chi >_p$, in the quantum Hilbert space of states (QHSS), which satisfies $(QQ/W)|\chi >_p = p |\chi >_p$ where $p$ is the eigen-value of operator $(QQ/W)$. Using the top two relations of (40), it can be checked that the states $Q |\chi >_p, \bar{Q} |\chi >_p, W |\chi >_p$ satisfy

$$\left(\frac{Q \bar{Q}}{W}\right) Q |\chi >_p = (p + 1) Q |\chi >_p,$$
\[
\left( \frac{Q}{W} \right) \bar{Q} \chi_{>p} = (p - 1) \bar{Q} \chi_{>p}, \\
\left( \frac{Q}{W} \right) W \chi_{>p} = (p) W \chi_{>p}. \tag{41}
\]

As a consequence, we note that the states \( Q \chi_{>p}, \bar{Q} \chi_{>p}, W \chi_{>p} \) have the eigen-values \((p + 1), (p - 1), (p)\), respectively. This establishes the fact that if the degree of a form is identified with the eigen-value of a specific state in the QHSS for the operator \((QQ/W)\), the result of the operation of conserved charges \((Q, \bar{Q}, W)\) on this particular state is exactly same as the consequences that follow after the operation of the cohomological operators \((d, \delta, \Delta)\) on the specific degree of a form (which is equal to the above eigen-value). Thus, ultimately, we have the following one-to-one mapping

\[
(Q, \bar{Q}, W) \leftrightarrow (d, \delta, \Delta), \tag{42}
\]

between the conserved charges corresponding to the physical symmetries of the theory and the cohomological operators of differential geometry.

Now we exploit the lower two relations of (40) and define an arbitrary state \( |\xi >_q \) to possess the eigen-value \( q \) w.r.t. the operator \((\bar{Q}Q/W)\) [i.e. \((\bar{Q}Q/W) |\xi >_q = q |\xi >_q \)]. In view of this definition, the following theoretically interesting relationships automatically ensue

\[
\left( \frac{Q}{W} \right) Q |\xi >_q = (q - 1) Q |\xi >_q, \\
\left( \frac{\bar{Q}}{W} \right) \bar{Q} |\xi >_q = (q + 1) \bar{Q} |\xi >_q, \\
\left( \frac{Q}{W} \right) W |\xi >_q = (q) W |\xi >_q. \tag{43}
\]

The above relationships establish that the states \( Q |\xi >_q, \bar{Q} |\xi >_q, W |\xi >_q \) have the eigen-values \((q - 1), (q + 1), (q)\), respectively. Thus, we conclude that if the degree of a form is identified with the eigen-value \( q \) of a state in the QHSS corresponding to the operator \((\bar{Q}Q/W)\), there is one-to-one relationship between the conserved charges \((\bar{Q}, Q, W)\) corresponding to the continuous symmetries of the theory and the cohomological operators:

\[
(\bar{Q}, Q, W) \leftrightarrow (d, \delta, \Delta), \tag{44}
\]

as far as the analogy between the eigen-values and the degree \( q \) of a given form is concerned. Thus, we have proven that our present couple of \( \mathcal{N} = 2 \) SUSY
models are very interesting physical models for the Hodge theory where all the de Rham cohomological operators, Hodge duality operation, degree of a form, etc., find their physical realizations in the language of discrete and continuous symmetry transformations (and corresponding generators).

6. Conclusions

In our present investigation, we have shown that a triplet of well-known SUSY quantum mechanical systems are tractable models for the Hodge theory. We have touched very briefly upon the proof that the 1D SUSY harmonic oscillator is a model for the Hodge theory. An extensive discussion on this observation can be found in [14]. We have provided definite proofs, however, for the other two \( \mathcal{N} = 2 \) SUSY systems of our present investigation and demonstrated that these systems are also models for the Hodge theory. We conjecture, in our present endeavor, that any arbitrary \( \mathcal{N} = 2 \) SUSY quantum mechanical model could be shown to respect continuous and discrete symmetries that are physical realizations of the de Rham cohomological operators and Hodge duality operation of differential geometry, respectively. As a consequence, the above set of \( \mathcal{N} = 2 \) SUSY models are very special.

All the above SUSY models are endowed with two SUSY transformations \((s_1, s_2)\) and a bosonic symmetry transformation \(s_\omega\). In our present investigation, we have defined the latter symmetry as an anticommutator of the above two SUSY transformations modulo a factor of \( i \) because \( s_\omega \) corresponds to the Laplacian operator which is, as is well-known, a hermitian operator with a positive real eigen-value [3-5]. In fact, it is because of the above choice that the conserved charge \( W \) (which is the generator of the bosonic symmetry transformation \( s_\omega \)) turns out to be hermitian (i.e. \( W = (H_g/\omega) \) and \( W = H_{em} \)) for both the \( \mathcal{N} = 2 \) SUSY models under consideration. Furthermore, the above observation (at the symmetry level) is also reflected in the \( sl(1/1) \) algebra satisfied by the conserved charges \((Q, \bar{Q}, W)\) which are the generators of the continuous symmetry transformations \((s_1, s_2, s_\omega)\).

We observe that, for the 1D system of \( \mathcal{N} = 2 \) SUSY model, we obtain only one discrete symmetry transformation that is consistent with the strictures laid down by the duality-invariant physical theories [15]. As a consequence, we have only one relationship between the SUSY transformations \( s_1 \) and \( s_2 \) (i.e. \( s_2 = + * s_1 * \)) as an analogue of the well-known connection between the (co-)exterior derivatives: \( \delta = \pm * d * \). On the contrary, for the 2D case of the motion of a charged particle (corresponding to our second example of
$\mathcal{N} = 2$ SUSY model), we have a set of two discrete symmetries [cf. (33)]. As a result, we have two relationships $s_2 = \pm * s_1 *$ that are precise analogues of the relationships between the (co-)exterior derivatives: $\delta = \pm * d *$. In addition to the above observations, we note that there is always a time-reversal ($t \to -t$) discrete symmetry in the case of 2D $\mathcal{N} = 2$ SUSY theory [cf. (33)] which is not present in the case of 1D $\mathcal{N} = 2$ SUSY model of our present investigation [which is clear from equation (31)].

It is an open question as to why there is only one physically consistent discrete symmetry for the 1D $\mathcal{N} = 2$ SUSY model whereas there are two physically consistent discrete symmetry transformations for the 2D model of $\mathcal{N} = 2$ SUSY system. In our earlier works on Abelian $p$-form ($p = 1, 2, 3...$) gauge theories (within the framework of BRST formalism) [6-10], we have established that such theories are examples of Hodge theories when the space-time dimension $D$ is equal to $2p$ (i.e. $D = 2p$). In these theories, we have shown the existence of two physically important discrete symmetry transformations. We do not know, at the moment, whether there is any type of connection between the SUSY theories and gauge theories (as far as theoretical aspects of models for the Hodge theory are concerned). These are some of the issues that we plan to address in our future investigations.

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Appendix A. Simpler ways of deriving $sl(1/1)$

Here we discuss the simpler ways of deriving the closed super algebra $sl(1/1)$ amongst the conserved charges $(Q, \bar{Q}, W)$ by exploiting the canonical definition of the generator of a continuous symmetry transformation [cf. (26)]. For the first example of $\mathcal{N} = 2$ SUSY model with the generalized potential function $f(x)$, we can exploit the nilpotent ($s_1^2 = s_2^2 = 0$) symmetry transformations (11) to compute the l.h.s. of the following equation

\begin{align*}
s_1 \bar{Q} = i \{ \bar{Q}, Q \} \equiv i W, \\
s_2 Q = i \{ Q, \bar{Q} \} \equiv i W.
\end{align*} \quad (A.1)
With the inputs from (16) for the expressions of charges, we can demonstrate that the l.h.s. matches with the r.h.s. with \( iW = i \left( \frac{H_g}{\omega} \right) \). Thus, we derive the relationship \( \{ Q, \bar{Q} \} = \left( \frac{H_g}{\omega} \right) \equiv W \). The whole beauty of this simple derivation is the mere use of (11) and (16) in the derivation of one of the most important ingredients of the \( sl(1/1) \) superalgebra.

To prove the nilpotency (i.e. \( Q^2 = \bar{Q}^2 = 0 \)) of the super charges \( Q \) and \( \bar{Q} \), we exploit the following appropriate relationships

\[
\begin{align*}
\text{s}_1 Q &= i \{ Q, Q \} \equiv 0, \\
\text{s}_2 \bar{Q} &= i \{ \bar{Q}, \bar{Q} \} \equiv 0,
\end{align*}
\]

where, once again, the SUSY transformations \( s_1 \) and \( s_2 \) from (11) and expressions for the charges \( Q \) and \( \bar{Q} \) from (16) have been used in the evaluation of the l.h.s. of the above relationships. The above equations (45) and (46) show the validity and deduction of \( sl(1/1) \) closed super algebra (38) amongst the conserved nilpotent charges \( Q, \bar{Q} \) and the bosonic charge \( W = \left( \frac{H_g}{\omega} \right) \).

We wrap up this Appendix with the remarks that the analogues of computations (45) and (46) can be performed for the second \( N = 2 \) SUSY example of the motion of a charged particle under influence of a magnetic field where the nilpotent transformations (19) and expressions for the charges in (21) and (25) can be exploited for the evaluation of variations \( s_1 \bar{Q} \) and \( s_2 Q \) which lead to the derivation of \( \{ Q, \bar{Q} \} = W \) where \( W \) turns out to be equal to the Hamiltonian \( H_{em} \) (i.e. \( W = H_{em} \)). Similarly, the nilpotency of the charges \( Q \) and \( \bar{Q} \) [cf. (21)] can be proven by exploiting the nilpotent transformations (19). In other words, we evaluate \( s_1 Q \) and \( s_2 \bar{Q} \) which turn out to yield \( Q^2 = \bar{Q}^2 = 0 \). We wish to lay emphasis on the fact that it is the definition of the generator of a continuous symmetry transformation [cf. (26)] that plays a key role in the derivation of the superalgebra \( sl(1/1) \).

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