How many invariant polynomials are needed to decide local unitary equivalence of qubit states?

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Given $L$-qubit states with the fixed spectra of reduced one-qubit density matrices, we find a formula for the minimal number of invariant polynomials needed for solving local unitary (LU) equivalence problem, that is, problem of deciding if two states can be connected by local unitary operations. Interestingly, this number is not the same for every collection of the spectra. Some spectra require less polynomials to solve LU equivalence problem than others. The result is obtained using geometric methods, i.e. by calculating the dimensions of reduced spaces, stemming from the symplectic reduction procedure.
I. INTRODUCTION

We consider a quantum system consisting of \( L \) isolated qubits with the Hilbert space \( \mathcal{H} = (\mathbb{C}^2)^\otimes L \). We assume that states are normalized to one and neglect a global phase. In this way the space of pure states is isomorphic to the complex projective space \( \mathbb{P}(\mathcal{H}) \). In the following, by \([\phi] \in \mathbb{P}(\mathcal{H})\) we will denote a state corresponding to a vector \( \phi \in \mathcal{H} \). Each qubit is located in a different laboratory and the available operations are restricted to the local unitaries described by the group \( K = SU(2)^\times L \). Two states are called locally unitary equivalent (LU equivalent) if and only if they can be connected by the action of \( K \), that is, belong to the same \( K \)-orbit. The problem of local unitary equivalence of states can be in principle solved by finding the set of \( K \)-invariant polynomials, i.e. polynomials that are constant on \( K \)-orbits (see\(^{10,11}\) for another approach). When the number of qubits is large this, however, becomes hard as the number of polynomials grows exponentially with the number of constituents of the system. The problem of LU equivalence for bipartite and three-qubit pure states was recently studied form the symplecto-geometric perspective\(^{16,17}\) (see also\(^{13}\)). In particular, the connection with the symplectic reduction was established. The current paper can be seen as a generalization of these ideas to an arbitrary number of qubits.

Among the \( K \)-invariant polynomials there are \( L \) polynomials \( \{ \text{tr} (\rho_l^2(\phi)) \}_{l=1}^L \), where \( \rho_l(\phi) \) are the reduced one-qubit density matrices. Consequently, for two LU equivalent states \( [\phi_{1,2}] \in \mathbb{P}(\mathcal{H}) \) the spectra of the corresponding reduced one-qubit density matrices are the same. If we denote by \( \Psi \) the map which assigns to \( [\phi] \in \mathbb{P}(\mathcal{H}) \) the shifted spectra of its reduced one-qubit density matrices, i.e. \( \Psi([\phi]) = \{\text{diag}(-\lambda_1,\lambda_1),\ldots,\text{diag}(-\lambda_L,\lambda_L)\} \), where \( \lambda_i = \frac{1}{2} - p_i \) and \( \{p_i,1-p_i\} \) is the increasingly ordered spectrum of \( \rho_i([\phi]) \), then the states satisfying the above necessary condition form a fiber of \( \Psi \). Fibers of \( \Psi \) are connected collections of \( K \)-orbit.\(^8\) Moreover, the image, \( \Psi(\mathbb{P}(\mathcal{H})) \), is a convex polytope.\(^7\) The polynomials \( \{ \text{tr} (\rho_l^2([\phi])) \}_{l=1}^L \) restricted to a fiber of \( \Psi \) are constant functions. Therefore, typically, for two states \([\phi_1]\) and \([\phi_2]\) with \( \alpha := \Psi([\phi_1]) = \Psi([\phi_2]) \), where \( \alpha \) denote the collection of spectra of one-qubit density matrices, some additional \( K \)-invariant polynomials are needed to decide the LU equivalence. The number of these polynomials is given by the dimension of the reduced space \( M_\alpha := \Psi^{-1}(\alpha) / K \) (see\(^{16,17}\)). Interestingly, \( \dim M_\alpha \) may not be the same for every \( \alpha \in \Psi(\mathbb{P}(\mathcal{H})) \), that is, some collections of spectra of reduced one-qubit density matrices require less additional polynomials to solve LU equivalence problem than others. In particular, if the fiber \( \Psi^{-1}(\alpha) \) contains exactly one \( K \)-orbit,
i.e. \( \dim M_\alpha = 0 \), no additional information is needed and any two states \([\phi_{1,2}] \in \Psi^{-1} (\alpha)\) are LU equivalent.

In this paper we find the formula for the dimension of the reduced space \( M_\alpha = \Psi^{-1}(\alpha)/K \), for any \( \alpha \in \Psi(\mathcal{P}(\mathcal{H})) \) and for an arbitrary number \( L \) of qubits. Our result is obtained in two steps. First, we consider the points \( \alpha_{\text{gen}} \in \Psi(\mathcal{P}(\mathcal{H})) \) which belong to the interior of the polytope \( \Psi(\mathcal{P}(\mathcal{H})) \). In this case the map \( \Psi \) is regular and the calculation is rather straightforward. The dimension of \( M_{\alpha_{\text{gen}}} \) does not depend on \( \alpha_{\text{gen}} \). Moreover, \( \dim M_{\alpha_{\text{gen}}} + L \) is equal to the cardinality of the spanning set of \( K \)-invariant polynomials. For points \( \alpha_b \in \Psi(\mathcal{P}(\mathcal{H})) \) which belong to the boundary of \( \Psi(\mathcal{P}(\mathcal{H})) \) the problem requires more advanced methods and turns out to be more interesting. In particular, for a large part of the boundary of \( \Psi(\mathcal{P}(\mathcal{H})) \) we have \( \dim M_{\alpha_b} = 0 \). We also observe that for \( \alpha_b \in \Psi(\mathcal{P}(\mathcal{H})) \) corresponding to the \( \{ \rho_l([\phi]) \}_{L=1}^L \) such that \( k \) matrices are maximally mixed \( \dim M_{\alpha_b} = \dim M_{\alpha_{\text{gen}}} - 2k \).

II. HOW MANY INVARIANT POLYNOMIALS ARE NEEDED TO DECIDE LU EQUIVALENCE OF 4-QUBIT STATES?

In this section we briefly discuss the considered problem and present the main results of the paper on the 4 qubits example.

Recently, the problem of finding \( \dim M_\alpha \) was considered for three qubits\(^{[17]} \). In particular it was shown that for points in the interior of the polytope \( \Psi(\mathcal{P}(\mathcal{H})) \), \( \dim M_\alpha = 2 \), whereas for points on the boundary \( \dim M_\alpha = 0 \). The uniform behaviour of \( \dim M_\alpha \) on the boundary of \( \Psi(\mathcal{P}(\mathcal{H})) \) in case of three qubits is, as already indicated in ref.\(^{[17]} \), a low dimensional phenomenon. As we explain in section III for an arbitrary number of \( L \) qubits the boundary consists of three parts characterized by a different behaviour of \( \dim M_\alpha \). The first part is the polytope \( \Psi(\mathcal{P}(\tilde{\mathcal{H}})) \) that corresponds to a system with one qubit less, that is, \( \tilde{\mathcal{H}} = (\mathbb{C}^2)^{\otimes (L-1)} \). The second corresponds to changing one of the non-trivial inequalities \( (6) \) into an equality. The third represents situations when \( k \) one-qubit density matrices are maximally mixed. The clear distinction between these parts of the boundary can be seen already in the four qubits case.

The four-qubit polytope \( \Psi(\mathcal{P}(\mathcal{H})) \) is a 4-dimensional convex polytope spanned by 12 vertices (see appendix for the proof and the list of vertices). The dimension of the reduced space in the interior of \( \Psi(\mathcal{P}(\mathcal{H})) \) is \( \dim M_{\alpha_{\text{gen}}} = 14 \) (see formula \( (8) \)). In figure 1 the above mentioned three different parts of the boundary are shown. In particular, in figure 1(a) we see that for 3-dimensional
face of $\Psi(\mathbb{P}(\mathcal{H}))$ corresponding to three qubits, $\dim M_\alpha = 2$ in the interior and $\dim M_\alpha = 0$ on the boundary which agrees with results of ref.\cite{17}. On the other hand, inside the 3-dimensional face shown in figure [1](b) corresponding to one of $\{\rho_i\}_{i=1}^4$ being maximally mixed we have $\dim M_\alpha = 12$. The boundary of this face contains: 2-dimensional faces corresponding to two of $\{\rho_i\}_{i=1}^4$ being maximally mixed - $\dim M_\alpha = 10$, 1-dimensional faces - three of $\{\rho_i\}_{i=1}^4$ are maximally mixed and $\dim M_\alpha = 8$, and finally, the vertex denoted by $v_{\text{GHZ}}$ when all one-particle reduced density matrices are maximally mixed - $\dim M_\alpha = 6$. Therefore, as mentioned in the introduction, $\dim M_{\alpha_b} = \dim M_{\alpha_{\text{gen}}} - 2k$. Finally, in figure [1](c) we see the 3-dimensional face of $\Psi(\mathbb{P}(\mathcal{H}))$ with $\dim M_\alpha = 0$.

![Fig. 1](image_url)

**FIG. 1.** Three parts of the boundary of $\Psi(\mathbb{P}(\mathcal{H}))$. The numbers denote $\dim M_\alpha$. If the number is missing, $\dim M_\alpha = 0$. The vertices are defined in appendix.

In the next sections we show how to calculate $\dim M_\alpha$ for any point of $\Psi(\mathbb{P}(\mathcal{H}))$ for an arbitrary number of qubits.

### III. LU EQUIVALENCE OF QUBITS AND THE REDUCED SPACES $M_\alpha$

We start with a rigorous statement of the problem and the solution of its easy part. For a detailed description of symplecto-geometric methods in quantum information theory see for example\cite{2,15,18}.

Let $\mathcal{H} = (\mathbb{C}^2)^\otimes L$ be the $L$-qubit Hilbert space and denote by $\mathbb{P}(\mathcal{H})$ the corresponding complex projective space. It is known that $\mathbb{P}(\mathcal{H})$ is a symplectic manifold with the Fubini-Study symplectic form $\omega_{\text{FS}}$. The action of $K = SU(2)^\times L$ on $\mathbb{P}(\mathcal{H})$ is symplectic, i.e. it preserves $\omega_{\text{FS}}$. Consequently, there is the *momentum map* for this action. In the considered setting this map is given by

$$\mu : \mathbb{P}(\mathcal{H}) \to \mathfrak{k}$$
\[
\mu([\phi]) = \left\{ \rho_1([\phi]) - \frac{1}{2} I, \ldots, \rho_L([\phi]) - \frac{1}{2} I \right\},
\]

where \( \mathfrak{k} \) is the Lie algebra of \( K \), \( \{\rho_l([\phi])\}_{l=1}^L \) are the reduced one-qubit density matrices, \( I \) is a 2 \( \times \) 2 identity matrix and \( i^2 = -1 \). The map \( \mu \) is equivariant, i.e. for any \( g \in K \)

\[
\mu([g\phi]) = \left\{ g \left( \rho_1([\phi]) - \frac{1}{2} I \right) g^*, \ldots, g \left( \rho_L([\phi]) - \frac{1}{2} I \right) g^* \right\} = g \mu([\phi]) g^* = \text{Ad}_g \mu([\phi]),
\]

where \( g^* \) is the Hermitian conjugate of \( g \) and \( \text{Ad}_g X := g X g^* \), for any \( X \in \mathfrak{k} \). The equivariance of \( \mu \) implies that \( K \)-orbits in \( \mathbb{P}(\mathcal{H}) \) are mapped onto adjoint orbits in \( \mathfrak{k} \). Consequently, the necessary condition for two states \([\phi_1, \phi_2]\) \( \in \mathcal{H} \) to be on the same \( K \)-orbit in \( \mathbb{P}(\mathcal{H}) \) is \( \mu(K.[\phi_1]) = \mu(K.[\phi_2]) \).

On the other hand, adjoint orbits in \( \mathfrak{k} \) are determined by the spectra of matrices \( \{\rho_l([\phi])\}_{l=1}^L \). The characteristic polynomial \( w_l(\nu) \) for \( \rho_l([\phi]) \) reads

\[
w_l(\nu) = \nu^2 - \text{tr} \left( \rho_l([\phi]) - \frac{1}{2} I \right) \nu + \left( \text{tr} \left( \rho_l([\phi]) - \frac{1}{2} I \right) \right)^2 - \text{tr} \left( \rho_l([\phi]) - \frac{1}{2} I \right)^2.
\]

Using \( \text{tr} (\rho_l([\phi])) = 1 \), equation (3) reduces to

\[
w_l(\nu) = \nu^2 - \text{tr} \left( \rho_l^2([\phi]) \right) - \frac{1}{2}.
\]

One can thus say that the necessary condition for two states \([\phi_1, \phi_2]\) \( \in \mathbb{P}(\mathcal{H}) \) to be on the same \( K \)-orbit in \( \mathbb{P}(\mathcal{H}) \) is that

\[
\forall l \ \text{tr} \left( \rho_l^2([\phi_1]) \right) = \text{tr} \left( \rho_l^2([\phi_2]) \right).
\]

Let \( \mathfrak{t}_+ \) be the positive Weyl chamber in \( \mathfrak{k} \), i.e.

\[
\mathfrak{t}_+ = \left\{ \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} -\lambda_L & 0 \\ 0 & \lambda_L \end{pmatrix} : \lambda_i \in \mathbb{R}_+ \right\}.
\]

We define the map \( \Psi : \mathbb{P}(\mathcal{H}) \to \mathfrak{t}_+ \) to be

\[
\Psi([\phi]) := \mu(K.[\phi]) \cap \mathfrak{t}_+ = \left\{ \tilde{\rho}_1([\phi]) - \frac{1}{2} I, \ldots, \tilde{\rho}_L([\phi]) - \frac{1}{2} I \right\}.
\]
where each \( \tilde{\rho}_l(\phi) \) is a diagonal \( 2 \times 2 \) matrix whose diagonal elements are given by the increasingly ordered spectrum \( \sigma(\rho_l(\phi)) = \{ p_l, 1 - p_l \} \) of \( \rho_l(\phi) \), that is, \( \frac{1}{2} \geq p_l \geq 0 \). The shifted spectrum, i.e. the spectrum of \( \rho_l(\phi) - \frac{1}{2} I \) is given by \( \{- \lambda_l, \lambda_l\} \), where

\[
0 \leq \lambda_l = \frac{1}{2} - p_l \leq \frac{1}{2}.
\]

Under these assumptions the image \( \Psi(\mathbb{P}(\mathcal{H})) \) is known to be a convex polytope, defined by the following set of inequalities:

\[
\forall_l \ 0 \leq \lambda_l \leq \frac{1}{2}, \quad \text{and} \quad \left( \frac{1}{2} - \lambda_l \right) \leq \sum_{j \neq l} \left( \frac{1}{2} - \lambda_j \right).
\]

Note that the polytope \( \Psi(\mathbb{P}(\mathcal{H})) \) gives 1-1 parametrization of \( \mu(\mathbb{P}(\mathcal{H})) \), that is, of all available adjoint \( K \)-orbits. Moreover, for each point \( \alpha \in \Psi(\mathbb{P}(\mathcal{H})) \) the set \( \Psi^{-1}(\alpha) \) is a connected \( K \)-invariant stratified symplectic space. We will next briefly describe the structure of this space. Note that isotropy groups of points belonging to a fixed \( K \)-orbit are conjugated. First, one can decompose \( \mathbb{P}(\mathcal{H}) \) according to the conjugacy classes of isotropy groups, i.e. into sets of points, whose isotropies are conjugated. This decomposition divides \( \mathbb{P}(\mathcal{H}) \) into sets of orbits characterized by the same isotropy type. These sets are in general non-connected, thus performing the decomposition further into connected components, one obtains the stratification

\[
\mathbb{P}(\mathcal{H}) = \bigcup S_\nu.
\]

There exists an unique highest dimensional stratum \( S^{\text{max}} \), which is open and dense. Moreover, it can be shown that for each \( \alpha \in \Psi(\mathbb{P}(\mathcal{H})) \) the set \( \mu^{-1}(\alpha) \cap S_\nu \) is smooth and that the decomposition

\[
\mu^{-1}(\alpha) = \bigcup_\nu \mu^{-1}(\alpha) \cap S_\nu,
\]

(7)

gives the stratification of \( \mu^{-1}(\alpha) \). Let \( K_\alpha \) be the isotropy subgroup of \( \alpha \) with respect to the adjoint action, i.e. \( K_\alpha = \{ g \in K : \text{Ad}_g \alpha = \alpha \} \). Because \( \mu^{-1}(\alpha) \) is invariant to the action of \( K_\alpha \), one has that the reduced space \( M_\alpha = \Psi^{-1}(\alpha)/K \cong \mu^{-1}(\alpha)/K_\alpha \). The stratification given by (7) induces the stratification of \( M_\alpha \) in a natural way. What is more, every \( M_\alpha \) also has an unique highest dimensional stratum \( M^{\text{max}}_\alpha \). If \( \mu^{-1}(\alpha) \cap S^{\text{max}} \neq \emptyset \) then

\[
M^{\text{max}}_\alpha = (\mu^{-1}(\alpha) \cap S^{\text{max}})/K.
\]

Therefore it is clear that the dimension of the highest dimensional stratum in \( M_\alpha \), or more precisely \( \dim M_\alpha + L \) is the number of \( K \)-invariant polynomials needed to decide LU equivalence of states.
satisfying $\Psi([\phi]) = \alpha$. The problem of finding $\dim M_\alpha$ is essentially different for points $\alpha$ in the interior and on the boundary of $\Psi(\mathbb{P}(\mathcal{H}))$. Using (6) the boundary of $\Psi(\mathbb{P}(\mathcal{H}))$ can be divided into three parts.

**Case 1:** $k$ of $\lambda_l$s are equal to $\frac{1}{2}$.

**Case 2:** At least one of the inequalities $(\frac{1}{2} - \lambda_l) \leq \sum_{j \neq l} (\frac{1}{2} - \lambda_j)$ is an equality.

**Case 3:** $k$ of $\lambda_l$s are equal to 0.

By the permutation symmetry of inequalities (6), it is enough to consider one example for each of these cases. In the remaining two paragraphs of this section we find $\dim M_\alpha$ for the interior and the boundary points satisfying the condition of the first case.

**A. Interior of $\Psi(\mathbb{P}(\mathcal{H}))$**

For the points $\alpha$ in the interior of the polytope $\Psi(\mathbb{P}(\mathcal{H}))$ the calculation of $\dim M_\alpha$ turns out to be rather straightforward.

To begin, let us denote by $\mathbb{P}(\mathcal{H})^{max}$ the union of $K$-orbits of maximal dimension in $\mathbb{P}(\mathcal{H})$. The principal isotropy theorem implies that this set is connected, open and dense. In order to calculate the maximal dimension of $K$-orbits in $\mathbb{P}(\mathcal{H})$ we use the following fact (see ref. 17).

**Proposition 1.** Assume that $\dim \Psi(\mathbb{P}(\mathcal{H})) = \dim t_+$. Then a generic $K$-orbit has dimension of the group $K$.

For $L$-qubits, $L \geq 3$, one easily checks that the polytope $\Psi(\mathbb{P}(\mathcal{H}))$ given by inequalities (6), is $L$-dimensional. On the other hand, for the Weyl chamber defined by (5), we have $\dim t_+ = L$. Therefore, by proposition 1 the $K$-orbits belonging to $\mathbb{P}(\mathcal{H})^{max}$ have dimension of the group $K = SU(2)^L$, i.e $3L$. Moreover, $\Psi(\mathbb{P}(\mathcal{H})^{max})$ contains the interior of the polytope $\Psi(\mathbb{P}(\mathcal{H}))$ (see ref. 12) and for $\alpha$ in the interior of the polytope $\Psi(\mathbb{P}(\mathcal{H}))$ the set $\Psi^{-1}(\alpha) \cap \mathbb{P}(\mathcal{H})^{max}$ is the highest dimensional stratum of $\Psi^{-1}(\alpha)$ (see ref. 12). Hence

$$\dim M_\alpha = \dim (\Psi^{-1}(\alpha)/K) = (\dim \mathbb{P}(\mathcal{H}) - \dim \Psi(\mathcal{H})) - \dim K =$$

$$= ((2^{L+1} - 2) - L) - 3L = 2^{L+1} - 4L - 2. \quad (8)$$

Note that already in the case of three qubits we have $\dim M_\alpha = 2$, that is, one needs $3 + 2 = 5$, $K$-invariant polynomials to decide LU equivalence of states whose spectra of reduced density
matrices are the same and belong to the interior of $\Psi(\mathbb{P}(\mathcal{H}))$. The exponential growth of $\dim M_\alpha$, which for large $L$ is of order $2^L$, can be seen as a usual statement that the number of $K$-invariant polynomials needed to distinguish between generic $K$-orbits grows exponentially with the number of particles.

B. Calculation of $\dim M_\alpha$ for case 1

The Case 1 is once again straightforward. To see this note that when the first $k$ of $\lambda_l$’s are equal to $\frac{1}{2}$ inequalities (6) reduce to the inequalities for the polytope of $L - k$ qubits. Moreover, a generic state belonging to $\Psi^{-1}(\alpha)$ is of the form $\phi_1 \otimes \phi_2$ where $\phi_1$ is a $k$-qubit separable state and $\phi_2$ is an arbitrary state of $L - k$ qubits. Therefore, using (8), we get

$$\dim M_\alpha = \left( (2^{L-k+1} - 2) - (L - k) \right) - 3(L - k) = 2^{L-k+1} - 4(L - k) - 2.$$ 

In the next two sections we find $\dim M_\alpha$ for cases 2 and 3.

IV. CALCULATION OF $\dim M_\alpha$ FOR CASE 2

In this section we show that $\dim M_\alpha = 0$ for $\alpha$’s satisfying assumptions of case 2.

Assume that one of the inequalities \( \left( \frac{1}{2} - \lambda_l \right) \leq \sum_{j \neq l} \left( \frac{1}{2} - \lambda_j \right) \) is an equality, e.g.

\[-\lambda_1 + \sum_{i=2}^{L} \lambda_i = \frac{1}{2} L - 1. \tag{9}\]

Using the fact that states mapped by $\Psi$ onto $\alpha$ are $K$-orbits through states mapped by $\mu$ onto $\alpha$, i.e. $\Psi^{-1}(\alpha)/K = \mu^{-1}(\alpha)/K_\alpha$, one can, if more convenient, use map $\mu$ instead of $\Psi$ to calculate $\dim M_\alpha$. It turns out that this is the case for the considered $\alpha$’s, as we have the following:

**Proposition 2.** Let $\phi \in \mathcal{H}$ be such that $\mu([\phi])$ satisfies (9). Let $\xi = [\xi_1, \ldots, \xi_L]$ be a vector perpendicular to the plane given by (9). Then $\phi$ is an eigenvector of $X = X_1 \otimes I \otimes \ldots \otimes I + \ldots + I \otimes I \otimes \ldots \otimes X_L$, where $X_l = \text{diag}\{\xi_l, -\xi_l\}$.

In order to prove this, we will use the fact characterizing the image of the differential of the momentum map $\mathbb{P}^2$. 

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Proposition 3. The image of \( d\mu|_{\mathfrak{g}} : T_{[\mathfrak{g}]}M \to i\mathfrak{k} \) is equal to the annihilator of \( \mathfrak{g}_{[\mathfrak{g}]} \), the Lie algebra of the isotropy subgroup \( K_{[\mathfrak{g}]} \subset K \).

Proof. (Proposition[2]) The Lie algebra of \( K \) is a real vector space equipped with the inner product given by \( \langle A|B \rangle = -\frac{1}{2} \text{tr}(AB) \). The matrices

\[
\mathcal{X}_k = iI \otimes I \otimes \ldots \otimes \sigma_x \otimes I \otimes \ldots \otimes I,
\]

\[
\mathcal{Y}_k = iI \otimes I \otimes \ldots \otimes \sigma_y \otimes I \otimes \ldots \otimes I,
\]

\[
\mathcal{Z}_k = iI \otimes I \otimes \ldots \otimes \sigma_z \otimes I \otimes \ldots \otimes I,
\]

where \( \sigma_{x,y,z} \) are Pauli matrices, form an orthogonal basis of \( \mathfrak{k} \). The matrix representing the collection \( \{ \rho_1([\phi]) - \frac{1}{2}I, \ldots, \rho_L([\phi]) - \frac{1}{2}I \} \) is of the form:

\[
(\rho_1([\phi]) - \frac{1}{2}I) \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes (\rho_L([\phi]) - \frac{1}{2}I).
\]

Note that \( d\mu|_{\mathfrak{g}} \) is the map that transports vectors tangent to \( \mathbb{P}(\mathcal{H}) \) at the point \( [\phi] \) to the tangent space to \( \mathfrak{k} \), \( T_{\mu([\phi])}\mathfrak{k} \cong \mathfrak{k} \). Assume that \( \alpha := \mu([\phi]) \) belongs to \( \Psi(\mathbb{P}(\mathcal{H})) \) and that spectra of the corresponding matrices \( \{ \rho_l([\phi]) \}_{l=1}^L \) are nondegenerate.

In this case the Lie algebra \( \mathfrak{t}_\alpha \) of the isotropy subgroup \( K_\alpha \subset K \) is given by the diagonal matrices in \( \mathfrak{k} \), the set of which we denote by \( \mathfrak{t} \). On the other hand by the equivariance of the momentum map \([1] \) we have \( \mathfrak{t}_{[\mathfrak{g}]} \subset \mathfrak{t}_\alpha = \mathfrak{t} \). In other words, states \([\phi] \) mapped by \( \mu \) to \( \alpha \in \Psi(\mathbb{P}(\mathcal{H})) \) with non-degenerate spectra of \( \{ \rho_l([\phi]) \}_{l=1}^L \) can have isotropy given by at most \( \mathfrak{t} \). As the off-diagonal matrices in the image of \( d\mu|_{\mathfrak{g}} \) are orthogonal to \( \mathfrak{t} \), by proposition[3] in order to find \( \mathfrak{t}_{[\mathfrak{g}]} \) we need to find matrices in \( \mathfrak{t} \) orthogonal to (annihilated by) diagonal matrices from \( d\mu|_{[\mathfrak{g}]} \). Note that since \( \Psi(\mathbb{P}(\mathcal{H})) \subset \mathfrak{t} \) and \( \dim \Psi(\mathbb{P}(\mathcal{H})) = \text{dim} \mathfrak{t} \), for \( \alpha \) inside \( \Psi(\mathbb{P}(\mathcal{H})) \) the diagonal matrices in \( T_{\mu([\phi])}\mathfrak{t} \), for \( [\phi] \in \mathbb{P}(\mathcal{H})^{max} \), span the space \( \mathfrak{t} \) and hence the isotropy \( \mathfrak{t}_{[\mathfrak{g}]} = 0 \). On the other hand, for \( \alpha \) satisfying \([2] \) the space \( T_{\mu([\phi])}\mathfrak{k} \cap \mathfrak{t} \neq \mathfrak{t} \). In order to find matrices in \( \mathfrak{t} \) orthogonal to \( T_{\mu([\phi])}\mathfrak{k} \cap \mathfrak{t} \) note that any element of \( \mathfrak{t} \) can be written as \( \sum_{k=1}^L \sum_{l=1}^L \alpha_k \mathcal{Z}_k \mathcal{Z}_l \). The inner product of two matrices of this type reads

\[
\langle \sum_{k=1}^L \sum_{l=1}^L b_l \mathcal{Z}_l, \sum_{k=1}^L \sum_{l=1}^L a_k \mathcal{Z}_k \rangle = -\frac{1}{2} \sum_{k=1}^L \sum_{l=1}^L a_k b_l \text{tr} (\mathcal{Z}_k \mathcal{Z}_l) = \bar{a} \cdot \bar{b}
\]

i.e. is equal to the standard inner product of vectors \( \bar{a} = [a_1, \ldots, a_L] \) and \( \bar{b} = [b_1, \ldots, b_L] \) in \( \mathbb{R}^L \). The vectors \( \bar{a} \) corresponding to the diagonal matrices from the image of \( d\mu|_{\mathfrak{g}} \) are tangent to
\( \Psi(\mathcal{P}(\mathcal{H})) \) at the point \( \mu([\phi]) \) satisfying (9). Therefore, if \( \xi = [\xi_1, \ldots, \xi_L] \) is a vector perpendicular to the plane given by (9) then the corresponding operator \( X = X_1 \otimes I \otimes \ldots \otimes I + \ldots + I \otimes I \otimes \ldots \otimes X_L, \) where \( X_i = \text{diag}\{\xi_i, -\xi_i\} \) is the element of the Lie algebra of the isotropy group \( \mathfrak{e}[\phi]. \) Consequently \( X\phi = \lambda\phi, \) for some \( \lambda. \)

The vector \( v = [-1, 1, \ldots, 1] \) is perpendicular to the plane given by (9). The corresponding operator \( X \) reads

\[
X = X_1 \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes X_L, \tag{10}
\]

where \( X_1 = \text{diag}\{-1, 1\}, \) \( X_2 = \ldots = X_L = \text{diag}\{1, -1\}. \) By proposition 2 we need to consider eigenspaces of \( X. \) We have the following:

**Proposition 4.** The matrix \( X, \) defined by (10), is a diagonal \( 2^L \times 2^L \) matrix. The eigenvalues of \( X \) are the integers chosen from \(-L \) to \( L \) with the step 2, that is \( \sigma(X) = \{-L, -L + 2, \ldots, L - 2, L\}. \) The multiplicity of eigenspace \( \mathcal{H}_{-L+2k} \) is \( \dim\mathcal{H}_{-L+2k} = \binom{L}{k}. \)

**Proof.** The matrices \( X_i \) are diagonal and their spectra are \( \sigma(X_i) = \{-1, 1\}. \) Consequently, the matrix \( X \) is also diagonal and its eigenvalues are sums of eigenvalues of \( X_i \)'s. One can easily verify that the eigenvalues of \( X \) belong to the set \( \sigma(X) = \{-L, -L + 2, \ldots, L - 2, L\}. \) To see this note that the eigenvalue \(-L + 2k\) arises as a sum of eigenvalues of \( \{X_i\}_{i=1}^L, \) where \( k \) out of \( L \) eigenvalues of \( X_i \)'s are positive (+1) and \( L - k \) negative (−1). Therefore the multiplicity of \( \mathcal{H}_{-L+2k} \) is \( \dim\mathcal{H}_{-L+2k} = \binom{L}{k}. \)

As a direct consequence of proposition 4 we need to consider \( L + 1 \) eigenspaces of \( X. \) In the following we describe the structure of these spaces and show that only one of them, that is, \( \mathcal{H}_{-L+2} \) contains states \([\phi]\) for which \( \mu([\phi]) \) consists of diagonal matrices whose diagonal elements satisfy (9). The result is obtained in two steps. First in proposition 5 we determine \( \mathcal{H}_{-L+2k} \) and show that condition (9) is not satisfied for \( k \in \{0, \ldots, L\} \setminus \{1\}. \) Then in proposition 6 we prove that for \( \mathcal{H}_{-L+2} \) condition (9) is satisfied.

Denote by \( D_k^L \) the subspace of \( (\mathbb{C}^2)^\otimes L \) spanned by separable states of \( L \) qubits such that \( k \) out of \( L \) qubits are in the ground state \([0]\) and the remaining \( L - k \) qubits are in the excited state \([1]\), for example, \( D_2^3 = \text{Span}_\mathbb{C}\{[001], [010], [100]\}. \) Assume that \( D_k^L = \{0\} \) if \( k > L. \)

**Proposition 5.** States which belong to eigenspace \( \mathcal{H}_{-L+2k} \) are of the form \( \phi = p_0[0] \otimes \psi_1 + p_1[1] \otimes \psi_2, \) where \( \psi_1 \in D_k^{L-1} \) and \( \psi_2 \in D_k^{L-1}. \) The reduced one-qubit density matrices for any
\( \phi \in \mathcal{H}_{-L+2k} \) are diagonal. For any \( \phi \in \mathcal{H}_{-L+2k} \) condition (9) is equivalent to \( (L-k-1)||\phi||^2 = L - 2 \).

Proof. We first determine vectors spanning eigenspace \( \mathcal{H}_{-L+2k} \) of \( X \). The eigenvalue \(-L + 2k\) arises as the sum of eigenvalues of \( \{X_l\}_{l=1}^L \) with \( k \) out of \( L \) eigenvalues equal to \(+1\) and \( L - k \) equal to \(-1\). Note, however, that matrix \( X_1 = \text{diag}\{-1, 1\} \), whereas \( X_2 = \ldots = X_L = \text{diag}\{1, -1\} \). Therefore, the eigenvalue \(-L + 2k\) corresponds to separable states with either the first qubit in the ground state \(|0\rangle\) and the remaining \( L - 1 \) qubits in a state from \( D_{k-1}^{L-1} \) or with the first qubit in the excited state \(|1\rangle\) and the remaining \( L - 1 \) qubits in a state from \( D_k^{L-1} \). Thus the generic state belonging to \( \mathcal{H}_{-L+2k} \) can be written as

\[
\phi = \sum_{l=1}^{(L-1)/2} a_l |0\rangle \otimes e_l + \sum_{l=1}^{(L-1)/2} b_l |1\rangle \otimes f_l,
\]

where \( \{e_i\} \) and \( \{f_i\} \) are the separable states spanning \( D_k^{L-1} \) and \( D_{k-1}^{L-1} \) respectively. It is straightforward to see that the reduced one-qubit density matrices of \( \phi \) are diagonal. The first one is of the form:

\[
\rho_1([\phi]) - \frac{1}{2} I = \begin{pmatrix}
0 & \sum_{l=1}^{(L-1)/2} |a_l|^2 - \frac{1}{2} \\
0 & \sum_{l=1}^{(L-1)/2} |b_l|^2 - \frac{1}{2}
\end{pmatrix},
\]

and hence

\[
\lambda_1 = \sum_{l=1}^{(L-1)/2} |b_l|^2 - \frac{1}{2}.
\]

We now show that (9) is equivalent to \((L-k-1)||\phi||^2 = L - 2\). As matrices \( \rho_l([\phi]) \) are diagonal, the \((\rho_l([\phi]))_{11}\) entry of each \( \rho_l \), which is equal to \( \lambda_l + \frac{1}{2} \), is the sum of \( |b_j|^2 \) and \( |a_j|^2 \) coefficients corresponding to vectors with the \( l \)-th qubit in the excited state \(|1\rangle\). Consequently, in the sum \( \sum_{l=2}^{L} (\rho_l([\phi]))_{11} \), each \( |b_j|^2 \) coefficient occurs \( L - k \) times and each \( |a_j|^2 \) coefficient \( L - k - 1 \) times.

Therefore,

\[
\sum_{l=2}^{L} \lambda_l = (L-k-1) \sum_{i=1}^{(L-1)/k} |a_i|^2 + (L-k) \sum_{i=1}^{(L-1)/k-1} |b_i|^2 - \frac{1}{2}(L-1) = \]

\[
= (L-k)||\phi||^2 - \sum_{i=1}^{(L-1)/k} |a_i|^2 - \frac{1}{2}(L-1).
\]
Using (11)

\[-\lambda_1 + \sum_{i=2}^{L} \lambda_i = (L - k - 1)||\phi||^2 - \frac{1}{2}L + 1.\]

Hence equation (9) reads

\[(L - k - 1)||\phi||^2 = L - 2.\]

Using fact 5 one easily finds that for normalized state, i.e. when ||\phi|| = 1 condition (9) can be satisfied only when \(k = 1\). The following proposition ensures that indeed this is the case.

**Proposition 6.** The reduced one-qubit density matrices of states \(\phi \in \mathcal{H}_{-L+2}\) are diagonal and satisfy condition (9).

**Proof.** The eigenspace \(\mathcal{H}_{-L+2}\) is \(L\)-dimensional. Any vector \(\phi \in \mathcal{H}_{-L+2}\) can be written as

\[
\phi = c_1|1\rangle \otimes |1\ldots1\rangle + c_2|0\rangle \otimes |01\ldots1\rangle + c_3|0\rangle \otimes |101\ldots1\rangle + \ldots + c_L|0\rangle \otimes |1\ldots110\rangle,
\]

that is, \(\phi\) is a linear combination of separable state where all qubits are in the excited state and the states for which the first and one additional qubits are in the ground state (while other are in the excited state). Assume that \(\phi\) is normalized, i.e. \(\sum_{k=1}^{L} |c_i|^2 = 1\). It is straightforward to calculate

\[
\rho_i([\phi]) - \frac{1}{2}I = \begin{pmatrix}
|c_i|^2 - \frac{1}{2} & 0 \\
0 & -\frac{1}{2} + \sum_{k \neq i}^{L} |c_k|^2
\end{pmatrix}, \quad i \in \{2, \ldots, L\},
\]

\[
\rho_1([\phi]) - \frac{1}{2}I = \begin{pmatrix}
-\frac{1}{2} + \sum_{k=2}^{L} |c_k|^2 & 0 \\
0 & |c_1|^2 - \frac{1}{2}
\end{pmatrix}.
\]

Note that we can assume that \(|c_1|^2 \geq \sum_{k=2}^{L} |c_k|^2\). This means that for all \(i \in \{1, \ldots, L\}\) we have \((\rho_i([\phi]) - \frac{1}{2}I)_{11} \leq 0\) as required. It is also easy to see that condition (9) is equivalent to

\[
\sum_{k=2}^{L} |c_k|^2 = \sum_{k=2}^{L} |c_k|^2,
\]

and is satisfied. \(\square\)
By proposition 6, states mapped by $\Psi$ onto $\alpha$’s satisfying (9) belong to $K$-orbits through $\phi \in \mathcal{H}_{-L+2}$. On the other hand states $\phi \in \mathcal{H}_{-L+2}$ are $K_C$-equivalent to $L$-qubit $W$ state, where

$$K_C = SL(2, \mathbb{C})^x \times L$$

is complexification of $K = SU(2)^x L$ and

$$[W] = |01\ldots1\rangle + |10\ldots1\rangle + \ldots + |1\ldots0\rangle.$$ 

This can be easily seen by changing $|0\rangle \leftrightarrow |1\rangle$ on the first qubit of (13). It was shown in ref. 17 that the variety $K_C \cdot [W]$ is spherical, i.e. reduced spaces stemming from the restriction $\Psi|_{K_C \cdot [W]}$ are zero-dimensional. Therefore:

**Theorem 7.** Let $\alpha \in \Psi(\mathbb{P}(\mathcal{H}))$ be such that at least one of the inequalities $(\frac{1}{2} - \lambda_i) \leq \sum_{j \neq i} (\frac{1}{2} - \lambda_j)$ is equality. Then $\dim M_\alpha = 0$.

**V. CALCULATION OF $\dim M_\alpha$ FOR CASE 3**

As we showed in section III for points $\alpha$ in the interior of $\Psi(\mathbb{P}(\mathcal{H}))$ the dimension of the reduced space is

$$\dim M_{\alpha_{gen}} = 2^{L+1} - 4L - 2.$$ 

In the following we show that for $\alpha = (\alpha_1, \ldots, \alpha_L) \in \Psi(\mathbb{P}(\mathcal{H}))$ with matrices $\alpha_1 = \ldots = \alpha_k = 0$

$$\dim M_\alpha = (2^{L+1} - 4L - 2) - 2k = \dim M_{\alpha_{gen}} - 2k.$$ (14)

This means that the dimension $\dim M_\alpha$ drops by 2 every time one of the reduced density matrices becomes maximally mixed. The argument for this is based on the existence of *stable states* which we discuss first.

**A. Stable states**

In the following we assume that $\hat{K}$ is any compact semisimple group acting in the symplectic way on the complex projective space $\mathbb{P}(\mathcal{H})$, where $\mathcal{H}$ can be, for example, the Hilbert space of $L$ qubits. We will denote by $\tilde{\mu}$ the corresponding momentum map and assume that $\tilde{\mu}^{-1}(0) \neq \emptyset$. Let $\hat{G} = \hat{K}^C$ be the complexification of $\hat{K}$. Following ref. 15 we denote

$$X(\tilde{\mu}) = \{[\phi] \in \mathbb{P}(\mathcal{H}) : \overline{\hat{G} \cdot [\phi] \cap \tilde{\mu}^{-1}(0)} \neq \emptyset\}.$$
It is known that the set $X(\tilde{\mu})$ is an open dense subset of $\mathbb{P}(\mathcal{H})$. Moreover the set $G.\tilde{\mu}^{-1}(0) \subset X(\tilde{\mu})$ is also an open dense subset of $\mathbb{P}(\mathcal{H})$ (see ref.\cite{14}). We will use the following terminology, typical for geometric invariant theory:

1. $[\phi]$ is unstable iff $[\phi] \notin X(\tilde{\mu})$,
2. $[\phi]$ is semistable iff $[\phi] \in X(\tilde{\mu})$.

Among semistable states we distinguish the class of stable states. By definition, a semistable state $[\phi]$ is stable if and only if $\tilde{\mu}([\phi]) = 0$ and $\dim \tilde{K}.[\phi] = \dim \tilde{K}$. Note that since for $[\phi] \in \tilde{\mu}^{-1}(0)$ one has $\dim \tilde{K}^C. [\phi] = 2 \dim \tilde{K}. [\phi]$ (see ref.\cite{8}), the condition $\dim \tilde{K}.[\phi] = \dim \tilde{K}$ can be phrased as $\dim \tilde{G}. [\phi] = \dim \tilde{G}$. Remarkably, the existence of a stable state implies that almost all semistable states are stable, in particular almost all states in $\tilde{\mu}^{-1}(0)$ are stable \cite{5}. Note that since $\tilde{G}.\tilde{\mu}^{-1}(0)$ is open and dense in $\mathbb{P}(\mathcal{H})$ and a generic $\tilde{G}$-orbit in $\tilde{G}.\tilde{\mu}^{-1}(0)$ has dimension $\dim \tilde{G}$ we get

$$\dim \mathbb{P}(\mathcal{H}) = \dim \tilde{G}.\tilde{\mu}^{-1}(0) = \dim \tilde{G} + \dim \left( \tilde{G}.\tilde{\mu}^{-1}(0) / \tilde{G} \right).$$ \hspace{1cm} (15)

One of the central results in the geometric invariant theory reads \cite{8}

$$\left( \tilde{G}.\tilde{\mu}^{-1}(0) \right) / \tilde{G} = \tilde{\mu}^{-1}(0) / \tilde{K}. \hspace{1cm} (16)$$

Hence, under the assumption of stable states existence and using (15) and (16) we get

$$\tilde{\mu}^{-1}(0) / \tilde{K} = \dim \mathbb{P}(\mathcal{H}) - 2 \dim \tilde{K}. \hspace{1cm} (17)$$

As we will see formula (17) plays a major role in showing (14).

B. The strategy for showing (14)

Let $K = K_1 \times K_2$, where $K_1 = SU(2)^k$ and $K_2 = SU(2)^{(L-k)}$. We first consider the natural action of $K_1$ on the first $k$ qubits in $\mathbb{P}(\mathcal{H})$, where $\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$. The momentum map $\mu_1$ for this action gives the first $k$ reduced density matrices. Therefore, $\mu_1^{-1}(0)$ consists of all states with the first $k$ reduced density matrices maximally mixed, but no assumption is made on the spectra of the remaining $(L-k)$ matrices. In the following we assume that there exists a stable state for
$K_1$-action on $\mathbb{P}(\mathcal{H})$ (see lemma 8 for proof). Under this assumption and using formula (17) the dimension of $\dim \mu_1^{-1}(0)/K_1$ is

$$\dim \mu_1^{-1}(0)/K_1 = \dim \mathbb{P}(\mathcal{H}) - 2\dim K_1 = 2^{L+1} - 6k - 2.$$  

Recall that $\mu_1^{-1}(0)/K_1$ is a stratified symplectic space and we consider the highest dimensional stratum which is a symplectic manifold. Removing $K_1$ freedom does not affect $K_2$ action, i.e. the actions of $K_1$ and $K_2$ commute. Therefore, we can consider action of $K_2$ on the highest dimensional stratum of $\mu_1^{-1}(0)/K_1$. The momentum map $\mu_2$ for $K_2$ action on $\mu_1^{-1}(0)/K_1$ gives the remaining $L - k$ reduced density matrices. Moreover, using inequalities (6) with $\lambda_1 = \ldots = \lambda_k = 0$ it is straightforward to see that the image of the corresponding map $\Psi_2$ is $L - k$ dimensional polytope. Using fact 1 and formula (8), for a point inside of this polytope, e.g. when $\lambda_{k+1}, \ldots, \lambda_L \neq 0$, the dimension of $\Psi_2$-fiber is

$$\left(\left(\dim \mu_1^{-1}(0)/K_1\right) - (L - k)\right) - \dim K_2 = \left(2^{L+1} - 6k - 2\right) - (L - k) - 3(L - k)$$

$$= 2^{L+1} - 4L - 2k - 2.$$  

But the $\Psi_2$-fiber is exactly the reduced space we look for, i.e. the one which corresponds to $\lambda_1 = \ldots = \lambda_k = 0$ and $\lambda_{k+1}, \ldots, \lambda_L \neq 0$. Therefore, as promised

$$\dim M_\alpha = \dim M_{\alpha_{gen}} - 2k = 2^{L+1} - 4L - 2k - 2.$$  

In order to complete the above reasoning we now show that an appropriate stable state indeed exists.

**Lemma 8.** Let $K_1 = SU(2)^{\times k}$, $k \leq L$. If $L \geq 5$ then the $L$-qubit state

$$[\phi] = (|0\ldots0\rangle + |1\ldots1\rangle) + (|110\ldots0\rangle + |001\ldots1\rangle) + (|1010\ldots0\rangle + |0101\ldots1\rangle) +$$

$$+ \ldots + (|10\ldots01\rangle + |01\ldots10\rangle),$$

is $K_1$-stable. For $L = 4$

$$[\phi] = \alpha \left( |0000\rangle + |1111\rangle \right) + (|1100\rangle + |0011\rangle) + (|1010\rangle + |0101\rangle) +$$

$$+ (|1001\rangle + |0110\rangle), \quad \alpha \in \mathbb{R} \setminus \{1, -3\},$$

is $K_1$-stable.
Proof. We need to show that the first $k$ reduced density matrices of $[\phi]$ are maximally mixed and that $\dim K_1[\phi] = \dim K_1$. Note first that if the state $[\phi]$ is stable with respect to $K = SU(2)^\times L$ action it is also stable with respect to $K_1 \subset K$ action. Therefore, we will show that $[\phi]$ is $K$-stable.

The state $[\phi]$ consists of $L \geq 5$ pairs of separable states. In each pair the second vector is the first vector with the swap $|0\rangle \leftrightarrow |1\rangle$ performed on every qubit. The first pair is GHZ state. In the remaining pairs the first vector is such that the first and one additional qubits are in the excited state $|1\rangle$ while the remaining $L - 2$ qubits are in the ground state $|0\rangle$. The construction ensures that $\mu([\phi]) = 0$. What is left is to calculate the dimension $\dim G.[\phi]$, where $G = K_1 = SL(2, \mathbb{C})^\times L$.

This is equivalent to calculating the dimension of the tangent space $T_{[\phi]}G.[\phi]$ which is generated by the action of Lie algebra $\mathfrak{g} = sl(2, \mathbb{C})^\times L$ on $[\phi]$. More precisely let

$$E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

be the basis of $sl(2, \mathbb{C})$. Let

$$E_{12}^{(l)} = \mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes E_{12} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I},$$

$$E_{21}^{(l)} = \mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes E_{12} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I},$$

$$H^{(l)} = \mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes H \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}.$$

Our goal is to show that vectors $\{E_{12}^{(l)} \phi, E_{21}^{(l)} \phi, H^{(l)} \phi\}_{l=1}^L$ are linearly independent and orthogonal to $\phi$. Denote by $|00\ldots0\rangle \otimes |1\rangle_1$ a separable state whose $l$-th qubit is in the excited state $|1\rangle$ and remaining qubits are in the ground state $|0\rangle$ (e.g. a 5 - qubit state $|00\ldots0\rangle \otimes |1\rangle_2 = |01000\rangle$ and a 6 - qubit state $|00\ldots0\rangle \otimes |1\rangle_1 \otimes |1\rangle_4 \otimes |1\rangle_6 = |100101\rangle$). It is straightforward to verify that

$$E_{21}^{(1)} \phi = |00\ldots0\rangle \otimes |1\rangle_1 + \sum_{\nu=2}^L |11\ldots1\rangle \otimes |0\rangle_\nu,$$

$$E_{21}^{(l)} \phi = |00\ldots0\rangle \otimes |1\rangle_1 + |11\ldots1\rangle \otimes |0\rangle_1 + \sum_{\nu \neq l} |00\ldots0\rangle \otimes |1\rangle_1 \otimes |1\rangle_1 \otimes |1\rangle_\nu, \ l \geq 2.$$

Moreover, the action of $E_{12}$ gives vectors that can be obtained from the above set of vectors by preforming the swap operation on every qubit. The vectors obtained by the action of $E_{21}$ are linearly independent, because each vector from this group contains a unique separable state of the form $|00\ldots0\rangle \otimes |1\rangle_1$ (an analogous argument can be applied to the set of vectors obtained by the action of $E_{12}$). We next examine the linear independence of vectors from groups $E_{12}^{(l)}$ and $E_{21}^{(l)}$. 

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First, note that $E^{(l)}_{12} \phi$, $l \geq 2$ are linearly independent from all vectors $E^{(l)}_{21} \phi$, because they consist of separable states of the form $|11\ldots1\rangle \otimes |0\rangle_1 \otimes |0\rangle_l \otimes |0\rangle_{l'}$ that do not appear in the vectors $E^{(l)}_{21} \phi$ (it is not true in the 4-qubit case, but we will return to this problem later). The last thing to show is the linear independence of the vector $E^{(1)}_{12} \phi = |11\ldots1\rangle \otimes |0\rangle_1 + \sum_{l'=2}^{L} (00\ldots0) \otimes |1\rangle_{l'}$ from the vectors $E^{(l)}_{21} \phi$, $l \geq 2$. Note that this vector is orthogonal to $E^{(1)}_{21} \phi$. Assume that $E^{(1)}_{12} \phi$ can be expressed as the linear combination of the remaining vectors

$$E^{(1)}_{12} \phi = \sum_{l=2}^{L} \lambda_l E^{(l)}_{21} \phi. \quad (18)$$

We will show that this leads to a contradiction. To this end, let us calculate the sum

$$\sum_{l=2}^{L} E^{(l)}_{21} \phi = \sum_{l=2}^{L} |00\ldots0\rangle \otimes |1\rangle_l + (L-1)|11\ldots1\rangle \otimes |0\rangle_1 + \sum_{l} \sum_{l' \neq l} |00\ldots0\rangle \otimes |1\rangle_1 \otimes |1\rangle_l \otimes |1\rangle_{l'} =$$

$$= E^{(1)}_{12} \phi + (L-2)|11\ldots1\rangle \otimes |0\rangle_1 + \sum_{l} \sum_{l' \neq l} |00\ldots0\rangle \otimes |1\rangle_1 \otimes |1\rangle_l \otimes |1\rangle_{l'}.$$

Using (18) we get

$$\sum_{l=2}^{L} (1-\lambda_l) E^{(l)}_{21} \phi = (L-2)|11\ldots1\rangle \otimes |0\rangle_1 + \sum_{l} \sum_{l' \neq l} |00\ldots0\rangle \otimes |1\rangle_1 \otimes |1\rangle_l \otimes |1\rangle_{l'} \quad (19)$$

On the other hand,

$$\sum_{l=2}^{L} (1-\lambda_l) E^{(l)}_{21} \phi = \sum_{l=2}^{L} (1-\lambda_l) |00\ldots0\rangle \otimes |1\rangle_l + \sum_{l=2}^{L} (1-\lambda_l) |11\ldots1\rangle \otimes |0\rangle_1 +$$

$$+ \sum_{l} (1-\lambda_l) \sum_{l' \neq l} |00\ldots0\rangle \otimes |1\rangle_1 \otimes |1\rangle_l \otimes |1\rangle_{l'}. \quad (20)$$

Thus, comparing the coefficients by $|11\ldots1\rangle \otimes |0\rangle_1$ and $|00\ldots0\rangle \otimes |1\rangle_l$ in equations (19) and (20), we get

$$\sum_{l=2}^{L} (1-\lambda_l) = L-2, \quad \forall l \quad 1 - \lambda_l = 0 \quad (21)$$

This is a contradiction, because the first equation (21) is equivalent to $\sum_{l=2}^{L} \lambda_l = 1$, which cannot be satisfied by $\lambda_l = 1$ for all $l$ (which is implied by the second equation (21)). Clearly for any $l \in \{1,\ldots,L\}$ we also have $\langle E^{(l)}_{12} | \phi \rangle = 0 = \langle E^{(l)}_{21} | \phi \rangle$. We are left with vectors $H^{(l)} \phi$. It
is straightforward to see that \( \langle H^{(l)} \phi \vert \phi \rangle = \langle H^{(l)} \phi \vert E^{(l)}_{21} \phi \rangle = \langle H^{(l)} \phi \vert E^{(l)}_{12} \phi \rangle = 0 \). The matrix of coefficients for \( \{ H^{(l)} \phi \} \) is given by

\[
C' = \begin{pmatrix}
1 & 1 & . & . & . & 1 \\
1 & 1 & -1 & . & . & -1 \\
1 & -1 & 1 & -1 & . & -1 \\
1 & -1 & -1 & 1 & -1 & . \\
& & & & & \\
1 & -1 & -1 & . & . & 1
\end{pmatrix}
\]

in the basis \( \{ (|0000\rangle - |1111\rangle), (-|1100\rangle + |0011\rangle), (-|1010\rangle + |0101\rangle), \ldots, (-|10\rangle + |01\rangle) \} \). By direct calculation one checks that \( \det C' \neq 0 \). Therefore the dimension of \( G.\phi \) is equal to the dimension of \( G \), which is \( 6L \) and \( \phi \) is \( K \)-stable. In the case of 4 qubits we need to consider a slightly different state

\[
[\phi] = \alpha (|0000\rangle + |1111\rangle) + (|1100\rangle + |0011\rangle) + (|1010\rangle + |0101\rangle) + \\
+ (|1001\rangle + |0110\rangle), \alpha \in \mathbb{R} \setminus \{1, -3\}.
\]

It can be shown by similar calculation that this state is \( K \)-stable.

VI. SUMMARY

Given spectra of one-qubit reduced density matrices, we found the formula for the minimal number of polynomials needed to decide LU equivalence of \( L \)-qubit pure states. As we showed this number is the same for spectra belonging to the interior of the polytope \( \Psi(\mathbb{P}(\mathcal{H})) \). This is not the case on the boundary where the behaviour of \( \dim M_{\alpha} \) is not uniform. In particular, for a large part of the boundary of \( \Psi(\mathbb{P}(\mathcal{H})) \) we have \( \dim M_{\alpha_b} = 0 \). We also observed that for \( \alpha_b \in \Psi(\mathbb{P}(\mathcal{H})) \) corresponding to the \( \{ \rho_i((\phi)) \}_{i=1}^L \) such that \( k \) matrices are maximally mixed \( \dim M_{\alpha_b} = \dim M_{\alpha_{gen}} - 2k \).

The methods used in this paper can be in principle applied to \( L \)-particle systems with an arbitrary finite-dimensional one-particle Hilbert spaces. The argument for points in the interior of \( \Psi(\mathbb{P}(\mathcal{H})) \) can be used \textit{mutatis mutandis} in this case. We note, however, that inequalities describing
the polytope $\Psi(\mathbb{P}(\mathcal{H}))$ are much more complicated when $\mathcal{H} \neq (\mathbb{C}^2)^{\otimes L}$ (see ref. [9]) and therefore
the problem for boundary points is of higher computational complexity. Nevertheless, one should
expect that, similarly to the qubit case, there is a large part of the boundary characterized by
$\dim M_\alpha = 0$.

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2010.
Appendix: Vertices of the polytope $Ψ(\mathbb{P}(\mathcal{H}))$ for $L$ qubits

In order to find vertices of the polytope $Ψ(\mathbb{P}(\mathcal{H}))$, it is more convenient to view the inequalities describing the polytope in terms of the minimal eigenvalues of the reduced one-qubit density matrices (remembering that the shifted spectra are given by $λ_i = \frac{1}{2} - p_i$). As proven in\(^\text{6}\) these inequalities read:

\begin{align*}
p_i &\leq \sum_{j \neq i} p_j, \; i = 1, \ldots, L, \quad (A.1a) \\
p_i &\geq 0, \; i = 1, \ldots, L, \quad (A.1b) \\
p_i &\leq \frac{1}{2}, \; i = 1, \ldots, L, \quad (A.1c)
\end{align*}

where each $p_i$ denotes the minimum eigenvalue of the reduced one-qubit density matrix describing $i$-th qubit, $ρ_i$. The general algorithm of finding the vertices of the polytope given by a set of inequalities is to choose the set of $L$ of the inequalities, write them as a set of equations and check whether there exists a unique solution\(^\text{3}\). If obtained solution satisfies the remaining $2L$ inequalities then it defines a vertex of $Ψ(\mathbb{P}(\mathcal{H}))$. The following fact describes the structure of vertices of $Ψ(\mathbb{P}(\mathcal{H}))$.
Proposition 9. Vertices of the polytope $\Psi(\mathbb{P}(\mathcal{H}))$ in the case of $L$ qubits are given by equations

$$p_i = 0 \text{ or } p_i = \frac{1}{2}, \ i = 1, \ldots, L \tag{A.2}$$

such that the number of indecies $i$ for which $p_i = \frac{1}{2}$ belongs to $\{0, 2, 3, \ldots, L\}$, that is, $|\{i : p_i = \frac{1}{2}\}| \in \{0, 2, 3, \ldots, L\}$.

**Proof.** First note that each vertex of this form can be obtained by picking $|\{i : p_i = \frac{1}{2}\}|$ equations from (A.1c) and $L - |\{i : p_i = \frac{1}{2}\}|$ linearly independent equations from (A.1b). One easily checks that in each case $2L$ remaining inequalities are trivially satisfied. Thus, points described by conditions (A.2) are indeed vertices of $\Psi(\mathbb{P}(\mathcal{H}))$. A single exceptional case occurs when $|\{i : p_i = \frac{1}{2}\}| = 1$, e.g. $p_1 = \frac{1}{2}$. Then by (A.1a) we get $p_1 = \frac{1}{2} \leq 0$, which is a contradiction. We will now show that these are all solutions. We achieve our goal by considering all remaining possibilities of choosing $L$ out of $3L$ inequalities. Before we proceed let us introduce some useful notation. Any choice of $L$ out of $3L$ inequalities (and turning them into equalities) is uniquely given when the following three auxiliary sets are specified:

$$I_1 = \{i : i^{th} \text{ inequality of the form (A.1a) was chosen}\},$$
$$I_2 = \{i : i^{th} \text{ inequality of the form (A.1b) was chosen}\},$$
$$I_3 = \{i : i^{th} \text{ inequality of the form (A.1c) was chosen}\}.$$

We have $|I_1| + |I_2| + |I_3| = L$, where $|I|$ denotes the number of elements of the finite set $I$. We have already covered cases when $I_1 = \emptyset$. What remains to be checked are three possibilities: $I_2 = \emptyset$ or $I_3 = \emptyset$ and the case when each $I_i$ is non-empty. Firstly, let us consider the case when $I_3 = \emptyset$. If additionally $I_2 = \emptyset$, one can check by direct calculations that the set of $L$ equations from $I_1$ gives $p_i = 0$ for all $i$. Next, if the set $I_2$ is non-empty, i.e. we choose some number of $p_i$’s equal to zero and combine these conditions with the equations from $I_1$, our problem either reduces to a problem analogous to the previous case with $I_2 = \emptyset$, or there exists such $i \in I_1$ that reads $p_i = \sum_{j \neq i} p_j$ with $p_i = 0$. Because all $p_i$’s are positive or equal to zero, we obtain in both cases that $p_j = 0$ for all $j$. One of the last things to check is the case when the only empty set is $I_2$, i.e. $I_2 = \emptyset$, $I_3 \neq \emptyset$ and $I_1 \neq \emptyset$. If, in addition $|I_3| = k > 1$, there exists such $i \in I_1$ that reads $p_i = \sum_{j \neq i} p_j = \sum_{j \notin I_3} p_j + \frac{k}{2}$. This implies that $p_i > \frac{1}{2}$, which is a contradiction. Now, if $|I_3| = 1$ from the same equation we get that for all $j \notin I_3$, $p_j = 0$ and $p_i = \frac{1}{2}$. This is a contradiction, because we assumed that $|I_3| = 1$. This argument remains also true in the case when all sets are non-empty. \qed
By the above fact, the number of the vertices of the polytope $\Psi(\mathbb{P}(\mathcal{H}))$ for $L$ qubits is the number of ways to place $k$ out of $L$ $p_i$’s equal to $\frac{1}{2}$ and the remaining $p_i$’s equal to zero on $L$ places:

$$V = \binom{L}{0} + \binom{L}{2} + \ldots + \binom{L}{L} = \sum_{k=0}^{N} \binom{L}{k} - \binom{L}{1} = 2^L - L.$$  

Moreover, to find the $L - 1$ dimensional faces of the polytope, one has to change one of the inequalities (A.1a), (A.1b), (A.1c) into equality and find vertices that satisfy this condition (the minimal number of vertices sufficient to span such face is $L$). In this way, for $L \geq 4$ qubits, one obtains that there are $3L$ such faces.

1. The 4 qubits example

The four-qubit polytope is a 4-dimensional convex polytope spanned by 12 vertices (in terms of $\lambda$’s):

| $\{i: p_i = \frac{1}{2}\}$ | Vertices in terms of $\lambda_i = \frac{1}{2} - p_i$ |
|-----------------------------|-------------------------------------------------|
| 0                           | $v_{\text{SEP}} = \{\text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(-\frac{1}{2}, \frac{1}{2})\}$ |
|                            | $v_{B1} = \{\text{diag}(0,0), \text{diag}(0,0), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(-\frac{1}{2}, \frac{1}{2})\}$ |
|                            | $v_{B2} = \{\text{diag}(0,0), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(0,0), \text{diag}(-\frac{1}{2}, \frac{1}{2})\}$ |
|                            | $v_{B3} = \{\text{diag}(0,0), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(0,0)\}$ |
|                            | $v_{B4} = \{\text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(0,0), \text{diag}(0,0), \text{diag}(-\frac{1}{2}, \frac{1}{2})\}$ |
|                            | $v_{B5} = \{\text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(0,0), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(0,0)\}$ |
|                            | $v_{B6} = \{\text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(0,0), \text{diag}(0,0)\}$ |
| 2                           | $v_4 = \{\text{diag}(0,0), \text{diag}(0,0), \text{diag}(0,0), \text{diag}(-\frac{1}{2}, \frac{1}{2})\}$ |
|                            | $v_3 = \{\text{diag}(0,0), \text{diag}(0,0), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(0,0)\}$ |
|                            | $v_2 = \{\text{diag}(0,0), \text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(0,0), \text{diag}(0,0)\}$ |
|                            | $v_1 = \{\text{diag}(-\frac{1}{2}, \frac{1}{2}), \text{diag}(0,0), \text{diag}(0,0), \text{diag}(0,0)\}$ |
| 3                           | $v_{\text{GHZ}} = \{\text{diag}(0,0), \text{diag}(0,0), \text{diag}(0,0), \text{diag}(0,0)\}$ |
| 4                           | |

**TABLE I.** The vertices of the four-qubit polytope $\Psi(\mathbb{P}(\mathcal{H}))$.

The 3-dimensional faces can be divided into three groups obtained by: (1) changing one inequality from (A.1a) into equality: the face is spanned by $v_{\text{SEP}}$ and three $v_{B_j}$ vertices, (2) choosing
one $\lambda_i = 0$: the face is spanned by three $v_{B_j}$ vertices, three $v_j$ vertices and $v_{\text{GHZ}}$, (3) choosing one $\lambda_i = \frac{1}{2}$: the face is spanned by $v_{\text{SEP}}$, three $v_{B_j}$ vertices and one $v_j$ vertex.