On the $p$-biharmonic submanifolds and stress $p$-bienergy tensors

Khadidja Mouffoki and Ahmed Mohammed Cherif

Abstract. In this paper, we consider $p$-biharmonic submanifolds of a space form. We give the necessary and sufficient conditions for a submanifold to be $p$-biharmonic in a space form. We present some new properties for the stress $p$-bienergy tensor.

1 Introduction

Consider a smooth map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds, and let $p \geq 2$, for any compact domain $D$ of $M$ the $p$-energy functional of $\varphi$ is defined by

$$E_p(\varphi; D) = \frac{1}{p} \int_D |d\varphi|^p v_g,$$

(1)

where $|d\varphi|$ is the Hilbert-Schmidt norm of the differential $d\varphi$, and $v^g$ is the volume element on $(M, g)$. A map is called $p$-harmonic if it is a critical point of the $p$-energy functional over any compact subset $D$ of $M$. Let $\{\varphi_t\}_{t \in (-\epsilon, \epsilon)}$ be a smooth variation of $\varphi$ supported in $D$. Then

$$\frac{d}{dt} E_p(\varphi_t; D) \bigg|_{t=0} = - \int_D h(\tau_p(\varphi), v) v_g,$$

(2)

where $v = \frac{\partial \varphi_t}{\partial t} \bigg|_{t=0}$ denotes the variation vector field of $\varphi$,

$$\tau_p(\varphi) = \text{div}^M(|d\varphi|^{p-2}d\varphi).$$

(3)

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Let $\tau(\varphi)$ be the tension field of $\varphi$ defined by

$$
\tau(\varphi) = \text{trace}_g \nabla d\varphi = \sum_{i=1}^{m} \left\{ \nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i) \right\}.
$$

(4)

(see [2]), where $\{e_1, \ldots, e_m\}$ is an orthonormal frame on $(M, g)$, $m = \dim M$, $\nabla^M$ is the Levi-Civita connection of $(M, g)$, and $\nabla^\varphi$ denotes the pull-back connection on $\varphi^{-1}TN$. If $|d\varphi|_x \neq 0$ for all $x \in M$, the map $\varphi$ is $p$-harmonic if and only if (see [1], [3], [7])

$$
|d\varphi|^p - \tau(\varphi) + (p - 2)|d\varphi|^{p-2}d\varphi(\text{grad}^M|d\varphi|) = 0.
$$

(5)

Example 1.1. Let $n \geq 2$. The inversion map

$$
\varphi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\},
\quad x \mapsto \frac{x}{|x|^l}
$$

is $p$-harmonic if and only if $l = \frac{n+p-2}{p-1}$.

A natural generalization of $p$-harmonic maps is given by integrating the square of the norm of $\tau_p(\varphi)$. More precisely, the $p$-bienergy functional of $\varphi$ is defined by

$$
E_{2,p}(\varphi; D) = \frac{1}{2} \int_D |\tau_p(\varphi)|^2 \nu_g.
$$

(6)

We say that $\varphi$ is a $p$-biharmonic map if it is a critical point of the $p$-bienergy functional, that is to say, if it satisfies the Euler-Lagrange equation of the functional (6), that is (see [11])

$$
\tau_{2,p}(\varphi) = -|d\varphi|^p - \text{trace}_g R^\varphi(\tau_p(\varphi), d\varphi)d\varphi - \text{trace}_g \nabla^\varphi |d\varphi|^{p-2}\nabla^\varphi \tau_p(\varphi) - (p - 2) \text{trace}_g \nabla(\nabla^\varphi \tau_p(\varphi), d\varphi)|d\varphi|^{p-4}d\varphi = 0.
$$

(7)

Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame on $(M, g)$, we have

$$
\text{trace}_g R^\varphi(\tau_p(\varphi), d\varphi)d\varphi = \sum_{i=1}^{m} R^\varphi(\tau_p(\varphi), d\varphi(e_i))d\varphi(e_i),
$$

$$
\text{trace}_g \nabla^\varphi |d\varphi|^{p-2}\nabla^\varphi \tau_p(\varphi) = \sum_{i=1}^{m} \left( \nabla^\varphi_{e_i} |d\varphi|^{p-2}\nabla^\varphi_{e_i} \tau_p(\varphi) - |d\varphi|^{p-2}\nabla^\varphi_{\nabla^M_{e_i} e_i} \tau_p(\varphi) \right),
$$

$$
\langle \nabla^\varphi \tau_p(\varphi), d\varphi \rangle = \sum_{i=1}^{m} h \left( \nabla^\varphi_{e_i} \tau_p(\varphi), d\varphi(e_i) \right),
$$

$$
\text{trace}_g \nabla(\nabla^\varphi \tau_p(\varphi), d\varphi)|d\varphi|^{p-4}d\varphi = \sum_{i=1}^{m} \left( \nabla^\varphi_{e_i} \langle \nabla^\varphi \tau_p(\varphi), d\varphi \rangle |d\varphi|^{p-4}d\varphi(e_i) 
- \langle \nabla^\varphi \tau_p(\varphi), d\varphi \rangle |d\varphi|^{p-4}d\varphi(\nabla^M_{e_i} e_i) \right).
$$
The $p$-energy functional (resp. $p$-bienergy functional) includes as a special case ($p = 2$) the energy functional (resp. bi-energy functional), whose critical points are the usual harmonic maps (resp. bi-harmonic maps), for more details on the concept of harmonic and bi-harmonic maps see [6], [9].

$p$-harmonic maps are always $p$-biharmonic maps by definition. In particular, if $(M, g)$ is a compact orientable Riemannian manifold without boundary, and $(N, h)$ is a Riemannian manifold with non-positive sectional curvature. Then, every $p$-biharmonic map from $(M, g)$ to $(N, h)$ is $p$-harmonic.

**Example 1.2 ([11]).** Let $M$ the manifold $\mathbb{R}^2 \setminus \{(0, 0)\} \times \mathbb{R}$ equipped with the Riemannian metric $g = (x_1^2 + x_2^2)^{\frac{1}{p}}(dx_1^2 + dx_2^2 + dx_3^2)$, and let $N$ the manifold $\mathbb{R}^2$ equipped with the Riemannian metric $h = dy_1^2 + dy_2^2$. The map

$$\varphi : (M, g) \to (N, h) \quad \text{defined by} \quad \varphi(x_1, x_2, x_3) = \left(\sqrt{x_1^2 + x_2^2}, x_3\right)$$

is proper $p$-biharmonic.

A submanifold in a Riemannian manifold is called a $p$-biharmonic submanifold if the isometric immersion defining the submanifold is a $p$-biharmonic map. We will call proper $p$-biharmonic submanifolds a $p$-biharmonic submanifolds which is non $p$-harmonic.

In this paper, we will focus our attention on $p$-biharmonic submanifolds of space form, we give the necessary and sufficient conditions for submanifolds to be $p$-biharmonic. Then, we apply this general result to many particular cases. We also consider the stress $p$-bienergy tensor associated to the $p$-bienergy functional, and we give the relation between the divergence of the stress $p$-bienergy tensor and the $p$-bitension field (7). Finally, we classify maps between Riemannian manifolds with vanishing stress $p$-bienergy tensor.

### 2 Main Results

Let $M$ be a submanifold of space form $N(c)$ of dimension $m$, $\mathbf{i} : M \hookrightarrow N(c)$ be the canonical inclusion, and let $\{e_1, \ldots, e_m\}$ be an orthonormal frame with respect to induced Riemannian metric $g$ on $M$ by the inner product $\langle , \rangle$ on $N(c)$. We denote by $\nabla^N$ (resp. $\nabla^M$) the Levi-Civita connection of $N^n(c)$ (resp. of $(M, g)$), by grad$^M$ the gradient operator in $(M, g)$, by $B$ the second fundamental form of the submanifold $(M, g)$, by $A$ the shape operator, by $H$ the mean curvature vector field of $(M, g)$, and by $\nabla^\perp$ the normal connection of $(M, g)$ (see for example [2]). Under the notation above we have the following results.

**Theorem 2.1.** The canonical inclusion $\mathbf{i}$ is $p$-biharmonic if and only if

$$\begin{cases}
-\Delta^\perp H + \text{trace}_g B(\cdot, A_H(\cdot)) - m(c - (p - 2)|H|^2) H = 0; \\
2 \text{trace}_g A_{\nabla^\perp H}(\cdot) + (p - 2 + \frac{m}{2}) \text{grad}^M |H|^2 = 0,
\end{cases}$$

where $\Delta^\perp$ is the Laplacian in the normal bundle of $(M, g)$. 


Proof. First, the $p$-tension field of $i$ is given by

$$\tau_p(i) = |di|^{p-2}\tau(i) + (p-2)|di|^{p-3}di(\nabla^M M|di|),$$

since $\tau(i) = mH$ (see [1], [2]), and $|di|^2 = m$, we get $\tau_p(i) = m^p H$. Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1, \ldots, m$, then calculating at $x$

$$\text{trace}_g R^N(\tau_p(i), di) di = \sum_{i=1}^m R^N(\tau_p(i), di(e_i)) di(e_i) = m^p \sum_{i=1}^m R^N(H, e_i)e_i.\quad (9)$$

By the following equation $R^N(X, Y)Z = c(\langle Y, Z \rangle X - \langle X, Z \rangle Y)$, with $\langle H, e_i \rangle = 0$, for all $X, Y, Z \in \Gamma(TN(c))$ and $i = 1, \ldots, m$, the last equation becomes

$$\text{trace}_g R^N(\tau_p(i), di) di = m^{p+2} cH.\quad (9)$$

We compute the term $\text{trace}_g (\nabla^i)^2 \tau_p(i)$ at $x$

$$\sum_{i=1}^m \nabla^i_{e_i} \nabla^i_{e_i} H = \sum_{i=1}^m \nabla^i_{e_i} \left( - A_H(e_i) + (\nabla^i_{e_i} H)^\perp \right)$$

$$= - \sum_{i=1}^m \nabla^M_{e_i} A_H(e_i) - \sum_{i=1}^m B(e_i, A_H(e_i))$$

$$- \sum_{i=1}^m A(\nabla^i_{e_i} H)^\perp(e_i) + \sum_{i=1}^m (\nabla^i_{e_i} (\nabla^i_{e_i} H)^\perp)^\perp,\quad (10)$$

since $\langle A_H(X), Y \rangle = \langle B(X, Y), H \rangle$ for all $X, Y \in \Gamma(TM)$, we get

$$\sum_{i=1}^m \nabla^M_{e_i} A_H(e_i) = \sum_{i,j=1}^m \langle \nabla^M_{e_i} A_H(e_i), e_j \rangle e_j$$

$$= \sum_{i,j=1}^m e_i(\langle A_H(e_i), e_j \rangle) e_j$$

$$= \sum_{i,j=1}^m e_i(\langle B(e_i, e_j), H \rangle) e_j$$

$$= \sum_{i,j=1}^m e_i(\langle \nabla^N_{e_j} e_i, H \rangle) e_j,$$
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Since \( \nabla^N_X \nabla^N_Y Z = R^N(X, Y)Z + \nabla^N_Y \nabla^N_X Z + \nabla^N_{[X,Y]} Z \), for all \( X, Y, Z \in \Gamma (TN(c)) \), we conclude

\[
\sum_{i=1}^{m} \nabla^M_{e_i} A_H(e_i) = \sum_{i,j=1}^{m} \langle \nabla^N_{e_i} \nabla^N_{e_j} e_i, H \rangle e_j + \sum_{i,j=1}^{m} \langle \nabla^N_{e_i} e_i, \nabla^N_{e_j} H \rangle e_j
\]

\[
= \sum_{i,j=1}^{m} \langle R^N(e_i, e_j) e_i, H \rangle e_j + \sum_{i,j=1}^{m} \langle \nabla^N_{e_j} \nabla^N_{e_i} e_i, H \rangle e_j
\]

\[
+ \sum_{i,j=1}^{m} \langle B(e_i, e_j), (\nabla^N_{e_i} H)^\perp \rangle e_j,
\]

since \( R^N(X, Y)Z = c \langle (Y, Z)X - \langle X, Z \rangle Y \rangle \), for all \( X, Y, Z \in \Gamma (TN(c)) \), we have

\[
\sum_{i=1}^{m} \nabla^M_{e_i} A_H(e_i) = \sum_{i,j=1}^{m} e_j(\langle \nabla^N_{e_i} e_i, H \rangle) e_j - \sum_{i,j=1}^{m} \langle \nabla^N_{e_i} e_i, \nabla^N_{e_j} H \rangle e_j
\]

\[
+ \sum_{i,j=1}^{m} \langle A(\nabla^N_{e_i} H)^\perp(e_i), e_j \rangle e_j
\]

\[
= m \sum_{j=1}^{m} e_j(\langle H, H \rangle) e_j - m \sum_{j=1}^{m} \langle H, \nabla^N_{e_j} H \rangle e_j
\]

\[
+ \sum_{i=1}^{m} A(\nabla^N_{e_i} H)^\perp(e_i)
\]

\[
= \frac{m}{2} \sum_{j=1}^{m} e_j(\langle H, H \rangle) e_j + \sum_{i=1}^{m} A(\nabla^N_{e_i} H)^\perp(e_i).
\]

(11)

From equations (10) and (11), we obtain

\[
\text{trace}_g(\nabla^1)^2 \tau_p(i) = -\frac{m + 2}{2} \text{grad}^M |H|^2 - 2 m^2 \text{trace}_g A(\nabla^N H)^\perp(\cdot)
\]

\[
- m^2 \text{trace}_g B(\cdot, A_H(\cdot)) + m^2 \Delta^N H.
\]

(12)

Now, we compute the term \( \text{trace}_g \nabla \langle \nabla^1 \tau_p(i), d\alpha \rangle d\alpha \) at \( x \)

\[
\sum_{i,j=1}^{m} \nabla^1_{e_i} \langle \nabla^1_{e_j} \tau_p(i), d\alpha(e_j) \rangle d\alpha(e_i) = m^2 \sum_{i,j=1}^{m} \nabla^1_{e_i} \langle \nabla^1_{e_j} H, e_j \rangle e_i,
\]

by the compatibility of pull-back connection \( \nabla^1 \) with the Riemannian metric of \( N(c) \), and
the definition of the mean curvature vector field $H$ of $(M, g)$, we have
\[
\sum_{j=1}^{m} \langle \nabla_{e_j}^i H, e_j \rangle = \sum_{j=1}^{m} \{ e_j \langle H, e_j \rangle - \langle H, \nabla_{e_j}^i e_j \rangle \} = - \sum_{j=1}^{m} \langle H, B(e_j, e_j) \rangle
\]
\[
= -m |H|^2,
\]
by the last two equations, we have the following
\[
\text{trace}_g \nabla \langle \nabla^i \tau_p (i), d \rangle d i = -m^\frac{p-2}{2} \text{grad}^M |H|^2 - m^\frac{p-4}{2} |H|^2 H. \tag{13}
\]

The Theorem 2.1 followed by (7), (9), (12), and (13).

If $p = 2$ and $N = S^n$, we arrive at the following Corollary.

**Corollary 2.2.** Let $M$ be a submanifold of sphere $S^n$ of dimension $m$, then the canonical inclusion $i : M \hookrightarrow S^n$ is biharmonic if and only if
\[
\begin{cases}
\frac{m}{2} \text{grad}^M |H|^2 + 2 \text{trace}_g A(\nabla^i H)(\cdot) = 0, \\
-mH + \text{trace}_g B(\cdot, A_H(\cdot)) - \Delta^\perp H = 0.
\end{cases}
\]

This result was deduced by B-Y. Chen and C. Oniciuc [4], [12].

**Theorem 2.3.** If $M$ is a hypersurface with nowhere zero mean curvature of $N^{m+1}(c)$, then $M$ is $p$-biharmonic if only if
\[
\begin{cases}
-\Delta^\perp H + (|A|^2 + m(p-2)|H|^2 - mc) H = 0; \\
2A(\text{grad}^M |H|) + (2(p-2) + m)|H| \text{grad}^M |H| = 0.
\end{cases} \tag{14}
\]

**Proof.** Consider $\{e_1, \ldots, e_m\}$ to be a local orthonormal frame field on $(M, g)$, and let $\eta$ the unit normal vector field at $(M, g)$ in $N^{m+1}(c)$. We have
\[
H = \langle H, \eta \rangle \eta
= \frac{1}{m} \sum_{i=1}^{m} \langle B(e_i, e_i), \eta \rangle \eta
= \frac{1}{m} \sum_{i=1}^{m} g(A(e_i), e_i) \eta
= \frac{1}{m} \text{trace}_g A \eta.
\]
Let $i = 1, \ldots, m$, we compute

$$A_H(e_i) = \sum_{j=1}^{m} g(A_H(e_i), e_j)e_j$$

$$= -\sum_{j=1}^{m} \langle \nabla_{e_i}^N H, e_j \rangle e_j$$

$$= -\sum_{j=1}^{m} e_i \langle H, e_j \rangle e_j + \sum_{j=1}^{m} \langle H, B(e_i, e_j) \rangle e_j$$

$$= \langle H, \eta \rangle \sum_{j=1}^{m} \langle \eta, B(e_i, e_j) \rangle e_j,$$

by the last equation and the formula $\langle \eta, B(e_i, e_j) \rangle = g(Ae_i, e_j)$, we obtain the following equation $A_H(e_i) = \langle H, \eta \rangle A(e_i)$. So that

$$\sum_{i=1}^{m} B(e_i, A_H(e_i)) = \sum_{i=1}^{m} B(e_i, \langle H, \eta \rangle A(e_i))$$

$$= \langle H, \eta \rangle \sum_{i=1}^{m} B(e_i, A(e_i))$$

$$= \langle H, \eta \rangle \sum_{i=1}^{m} g(A(e_i), A(e_i))\eta$$

$$= |A|^2H. \quad (15)$$

In the same way, with $\eta = H/|H|$, we find that

$$\sum_{i=1}^{m} A_{\nabla^H_{\perp}}(e_i) = \sum_{i,j=1}^{m} \langle A_{\nabla^H_{\perp}}(e_i), e_j \rangle e_j$$

$$= -\sum_{i,j=1}^{m} \langle \nabla_{e_i}^N \nabla_{e_i}^{\perp} H, e_j \rangle e_j$$

$$= -\sum_{i,j=1}^{m} \langle e_i \langle H, \eta \rangle \nabla_{e_i} \eta, e_j \rangle e_j$$

$$= A(\text{grad}^M |H|). \quad (16)$$

The Theorem 2.3 followed by equations (15), (16), and Theorem 2.1.

**Corollary 2.4.** (i) A submanifold $M$ with parallel mean curvature vector field in $N^n(c)$ is $p$-biharmonic if and only if

$$\text{trace}_g B(\cdot, A_H(\cdot)) = m \left( c - (p - 2)|H|^2 \right) H, \quad (17)$$
(ii) A hypersurface $M$ of constant non-zero mean curvature in $N^{m+1}(c)$ is proper $p$-biharmonic if and only if

$$|A|^2 = mc - m(p - 2)|H|^2. \quad (18)$$

**Example 2.5.** We consider the hypersurface

$$S^m(a) = \{(x^1, \ldots, x^m, x^{m+1}, b) \in \mathbb{R}^{m+2} : \sum_{i=1}^{m+1} (x^i)^2 = a^2\} \subset S^{m+1},$$

where $a^2 + b^2 = 1$. We have

$$\eta = \frac{1}{r}(x^1, \ldots, x^{m+1}, -\frac{a^2}{b}),$$

with $r^2 = \frac{a^2}{b^2}$ ($r > 0$), is a unit section in the normal bundle of $S^m(a)$ in $S^{m+1}$. Let $X \in \Gamma(TS^m(a))$, we compute

$$\nabla^m_X \eta = \frac{1}{r} \nabla X \eta = \frac{1}{r} (x^1, \ldots, x^{m+1}, -\frac{a^2}{b}) = \frac{1}{r} X.$$

Thus, $\nabla^m \eta = 0$ and $A = -\frac{1}{r} Id$. This implies that $H = -\frac{1}{r} \eta$, and so $S^m(a)$ has constant mean curvature $|H| = \frac{1}{r}$ in $S^{m+1}$. Since $|A|^2 = \frac{m}{r^2}$, according to Corollary 2.4, we conclude that $S^m(a)$ is proper $p$-biharmonic in $S^{m+1}$ if and only if $p = 1/b^2$.

## 3 Stress $p$-bienergy tensors

Let $\varphi : (M, g) \rightarrow (N, h)$ be a smooth map between two Riemannian manifolds and $p \geq 2$. Consider a smooth one-parameter variation of the metric $g$, i.e. a smooth family of metrics $(g_t)$ ($-\epsilon < t < \epsilon$) such that $g_0 = g$, write $\delta = \frac{\partial}{\partial t} \big|_{t=0}$, then $\delta g \in \Gamma(\otimes^2 T^*M)$ is a symmetric 2-covariant tensor field on $M$ (see [2]). Take local coordinates $(x^i)$ on $M$, and write the metric on $M$ in the usual way as $g_t = g_{ij}(t, x) dx^i dx^j$, we now compute

$$\frac{d}{dt} E_{2,p}(\varphi; D) \big|_{t=0} = \frac{1}{2} \int_D \delta(|\tau_p(\varphi)|^2) v_g + \frac{1}{2} \int_D |\tau_p(\varphi)|^2 \delta(v_g). \quad (19)$$

The calculation of the first term breaks down in three lemmas.

**Lemma 3.1.** The vector field $\xi = (\text{div}^M \delta g)^t - \frac{1}{2} \text{grad}^M(\text{trace} \delta g)$ satisfies

$$\delta(|\tau_p(\varphi)|^2) = -(p - 2)|d\varphi|^{p-4} \langle \varphi^* h, \delta g \rangle h(\tau(\varphi), \tau_p(\varphi))$$

$$-2|d\varphi|^{p-2} \langle h(\nabla d\varphi, \tau_p(\varphi)), \delta g \rangle - 2|d\varphi|^{p-2} h(\delta g, |\nabla \tau_p(\varphi)|)$$

$$- (p - 2) |d\varphi|^{p-5} \langle \delta g, h(\text{grad}^M |d\varphi|, \tau_p(\varphi)) \rangle$$

$$- 2(p - 2)|d\varphi|^{p-3} \langle d\varphi \otimes h(\varphi, \tau_p(\varphi)), \delta g \rangle$$

$$- (p - 2)|d\varphi|^{p-4} h(\delta g, \text{grad}^M \langle \varphi^* h, \delta g \rangle, \tau_p(\varphi)),$$

where $\varphi^* h$ is the pull-back of the metric $h$, and $\langle , \rangle$ is the induced Riemannian metric on $\otimes^2 T^*M$. 

Proof. In local coordinates \((x^i)\) on \(M\) and \((y^\alpha)\) on \(N\), we have

\[
\delta(|\tau_p(\varphi)|^2) = \delta(\tau_p(\varphi)^\alpha \tau_p(\varphi)^\beta h_{\alpha\beta}) = 2\delta(\tau_p(\varphi)^\alpha \tau_p(\varphi)^\beta h_{\alpha\beta}).
\]  \((20)\)

By the definition of \(\tau_p(\varphi)\) we get

\[
\delta(\tau_p(\varphi)^\alpha) = \delta(|d\varphi|^{p-2}\tau(\varphi)^\alpha + \theta^\alpha)
= \delta(|d\varphi|^{p-2}\tau(\varphi)^\alpha + |d\varphi|^{p-2}\delta(\tau(\varphi)^\alpha) + \delta(\theta^\alpha)).
\]  \((21)\)

where \(\tau(\varphi)^\alpha = g^{ij}(\varphi^\alpha_{ij} + N \Gamma^\alpha_{\mu\sigma} \varphi^\mu_{ij} \varphi^\sigma_k - M \Gamma^k_{ij} \varphi^\alpha_k)\) is the component of the tension field \(\tau(\varphi)\), and \(\theta^\alpha = (p - 2)|d\varphi|^{p-3}g^{ij}|d\varphi|_{ij} \varphi^\alpha_j\).

The first term in the right-hand side of \((21)\) is given by

\[
\delta(|d\varphi|^{p-2}\tau(\varphi)^\alpha) = (p - 2)|d\varphi|^{p-4}\delta\left(\frac{|d\varphi|^2}{2}\right)\tau(\varphi)^\alpha
= -\frac{p - 2}{2}|d\varphi|^{p-4}\langle \varphi^* h, \delta g \rangle \tau(\varphi)^\alpha.
\]  \((22)\)

The second term on the right-hand side of \((21)\) is (see \([10]\))

\[
|d\varphi|^{p-2}\delta(\tau(\varphi)^\alpha) = -|d\varphi|^{p-2}g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)_{ij}^\alpha - |d\varphi|^{p-2}\xi^k \varphi^\alpha_k,
\]  \((23)\)

Now, we compute the third term on the right-hand side of \((21)\)

\[
\delta(\theta^\alpha) = (p - 2)(p - 3)|d\varphi|^{p-5}\delta\left(\frac{|d\varphi|^2}{2}\right)g^{ij}|d\varphi|_{ij} \varphi^\alpha_j
+ (p - 2)|d\varphi|^{p-3}\delta(\tau(\varphi)^\alpha)|d\varphi|_{ij} \varphi^\alpha_j
+ (p - 2)|d\varphi|^{p-3}g^{ij}\delta(|d\varphi|_{ij}) \varphi^\alpha_j.
\]  \((24)\)

By using \(\delta\left(\frac{|d\varphi|^2}{2}\right) = -\frac{1}{2}\langle \varphi^* h, \delta g \rangle\) with \(\delta(|d\varphi|_{ij}) = (\delta(|d\varphi|))_{ij}\), the equation \((24)\) becomes

\[
\delta(\theta^\alpha) = -\frac{(p - 2)(p - 3)}{2}|d\varphi|^{p-5}\langle \varphi^* h, \delta g \rangle g^{ij}|d\varphi|_{ij} \varphi^\alpha_j
+ (p - 2)|d\varphi|^{p-3}\langle \varphi^* h, \delta g \rangle |d\varphi|_{ij} \varphi^\alpha_j
- \frac{p - 2}{2}|d\varphi|^{p-4}g^{ij}\langle \varphi^* h, \delta g \rangle \varphi^\alpha_j
+ \frac{p - 2}{2}|d\varphi|^{p-5}g^{ij}|d\varphi|_{ij} \langle \varphi^* h, \delta g \rangle \varphi^\alpha_j.
\]  \((25)\)

Note that

\[
2\delta(|d\varphi|^{p-2})\tau(\varphi)^\alpha \tau_p(\varphi)^\beta h_{\alpha\beta} = -(p - 2)|d\varphi|^{p-4}\langle \varphi^* h, \delta g \rangle \tau(\varphi)^\alpha \tau_p(\varphi)^\beta h_{\alpha\beta}
= -(p - 2)|d\varphi|^{p-4}\langle \varphi^* h, \delta g \rangle h(\tau(\varphi), \tau_p(\varphi)),
\]  \((26)\)
Lemma 3.3. We set 

\[ 2|d\varphi|^{p-2}\delta(\tau(\varphi)\alpha)\tau_p(\varphi)\beta h_{\alpha\beta} = -2|d\varphi|^{p-2}g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)\alpha\beta \tau_p(\varphi)h_{\alpha\beta} \]

\[ = -2|d\varphi|^{p-2}\delta(h(\nabla d\varphi, \tau_p(\varphi)), \delta g) \]

and the following

\[ 2\delta(\theta^a)\tau_p(\varphi)\beta h_{\alpha\beta} = -(p-2)(p-3)|d\varphi|^{p-5}\langle \varphi^* h, \delta g \rangle g^{ij}|d\varphi|^{\alpha\beta} \tau_p(\varphi)h_{\alpha\beta} \]

\[ + 2(p-2)|d\varphi|^{p-3}\delta(g^{ij})|d\varphi|^{\alpha\beta} \tau_p(\varphi)h_{\alpha\beta} \]

\[ - (p-2)|d\varphi|^{p-4}g^{ij}\varphi^* h, \delta g)\varphi^\beta \tau_p(\varphi)h_{\alpha\beta} \]

\[ + (p-2)|d\varphi|^{p-5}g^{ij}|d\varphi|^{\alpha\beta} \varphi^\beta \tau_p(\varphi)h_{\alpha\beta} \]

\[ = -(p-2)(p-3)|d\varphi|^{p-5}\langle \varphi^* h, \delta g \rangle h(d\varphi(\text{grad}^M |d\varphi|), \tau_p(\varphi)) \]

\[ - 2(p-2)|d\varphi|^{p-3}|d\varphi| \odot h(d\varphi, \tau_p(\varphi)), \delta g) \]

\[ = -(p-2)|d\varphi|^{p-4}h(d\varphi(\text{grad}^M \langle \varphi^* h, \delta g \rangle), \tau_p(\varphi)) \]

\[ + (p-2)|d\varphi|^{p-5}(\varphi^* h, \delta g)h(d\varphi(\text{grad}^M |d\varphi|), \tau_p(\varphi)). \]

(27)

Substituting (21), (26), (27) and (28) in (20), the Lemma 3.1 follows. □

Lemma 3.2 ([5]). Let \( D \) be a compact domain of \( M \). Then

\[ \int_D |d\varphi|^{p-2}h(d\varphi(\xi), \tau_p(\varphi))v_g = \int_D \bigl\langle -\text{sym} (\nabla |d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi))) \bigr\rangle \]

\[ + \frac{1}{2} \text{div}^M (|d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi))^2) g, \delta g \bigr\rangle v_g. \]

Lemma 3.3. We set \( \omega = |d\varphi|^{p-4}h(d\varphi, \tau_p(\varphi)) \). Then

\[ - \int_D |d\varphi|^{p-4}h(d\varphi(\text{grad}^M \langle \varphi^* h, \delta g \rangle), \tau_p(\varphi))v_g = \int_D \langle \varphi^* h, \delta g \rangle \text{div} \omega v_g. \]

Proof. Note that

\[ \text{div}(\langle \varphi^* h, \delta g \rangle \omega) = \langle \varphi^* h, \delta g \rangle \text{div} \omega + \omega(\text{grad}^M \langle \varphi^* h, \delta g \rangle), \]

and consider the divergence Theorem, Lemma 3.3 follows. □

Theorem 3.4. Let \( \varphi : (M, g) \rightarrow (N, h) \) be a smooth map such that \( |d\varphi| x \neq 0 \) for all \( x \in M \), and let \( \{g_t\} \) a one parameter variation of \( g \). Then

\[ \frac{d}{dt} E_{2,p}(\varphi, D) \bigg|_{t=0} = \frac{1}{2} \int_D \langle S_{2,p}(\varphi), \delta g \rangle v_g, \]

where \( S_{2,p}(\varphi) \in \Gamma(\otimes^2 T^* M) \) is given by

\[ S_{2,p}(\varphi)(X, Y) = -\frac{1}{2} |\tau_p(\varphi)|^2 g(X, Y) - |d\varphi|^{p-2} \langle d\varphi, \nabla^p \tau_p(\varphi) \rangle g(X, Y) \]

\[ + |d\varphi|^{p-2}h(d\varphi(X), \nabla^p \tau_p(\varphi)) + |d\varphi|^{p-2}h(d\varphi(Y), \nabla^p \tau_p(\varphi)) \]

\[ + (p-2)|d\varphi|^{p-4} \langle d\varphi, \nabla^p \tau_p(\varphi) \rangle h(d\varphi(X), d\varphi(Y)). \]
$S_{2,p}(\varphi)$ is called the stress $p$-bienergy tensor of $\varphi$.

**Proof.** By using $\delta(v_g) = \frac{1}{2} \langle g, \delta g \rangle v_g$ (see [2]), Lemmas 3.1, 3.2, and 3.3, the equation (19) becomes

$$S_{2,f}(\varphi) = -(p-2)|d\varphi|^{p-4}h(\tau(\varphi), \tau_p(\varphi))\varphi^* h$$

$$-2|d\varphi|^{p-2}h(\nabla d\varphi, \tau_p(\varphi)) + 2 \text{sym} (\nabla|d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi)))$$

$$- \text{div}^M ([|d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi))^2])g$$

$$-(p-2)(p-4)|d\varphi|^{p-5}h(d\varphi(\text{grad}^M|d\varphi|), \tau_p(\varphi))\varphi^* h$$

$$-2(p-2)|d\varphi|^{p-3}|d\varphi| \circ h(d\varphi, \tau_p(\varphi))$$

$$+(p-2) \text{div}^M [|d\varphi|^{p-4}h(d\varphi, \tau_p(\varphi))]| \varphi^* h + \frac{1}{2} |\tau_p(\varphi)|^2 g.$$

(29)

Note that, for all $X, Y \in \Gamma(TM)$, we have

$$2 \text{sym} (\nabla|d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi)))(X, Y) = 2|d\varphi|^{p-2}h(\nabla d\varphi(X, Y), \tau_p(\varphi))$$

$$+|d\varphi|^{p-2}h(d\varphi(X), \nabla^\varphi \tau_p(\varphi))$$

$$+|d\varphi|^{p-2}h(d\varphi(Y), \nabla^\varphi \tau_p(\varphi))$$

$$+X(|d\varphi|^{p-2}h(d\varphi(Y), \tau_p(\varphi))$$

$$+Y(|d\varphi|^{p-2}h(d\varphi(X), \tau_p(\varphi))).$$

(30)

and the following formula

$$-2|d\varphi| \circ h(d\varphi, \tau_p(\varphi))(X, Y) = -X(|d\varphi|)h(d\varphi(Y), \tau_p(\varphi))$$

$$-Y(|d\varphi|)h(d\varphi(X), \tau_p(\varphi)).$$

(31)

Calculating in a normal frame at $x$, we have

$$\text{div}^M ([|d\varphi|^{p-4}h(d\varphi, \tau_p(\varphi))^2]) = \sum_{i=1}^m e_i(g(|d\varphi|^{p-2}h(d\varphi, \tau_p(\varphi))^2, e_i))$$

$$= \sum_{i=1}^m e_i(|d\varphi|^{p-2}h(d\varphi(e_i), \tau_p(\varphi)))$$

$$= \sum_{i=1}^m e_i(|d\varphi|^{p-2}h(d\varphi(e_i), \tau_p(\varphi))$$

$$+ \sum_{i=1}^m |d\varphi|^{p-2}h(\nabla^\varphi_{e_i} d\varphi(e_i), \tau_p(\varphi))$$

$$+ \sum_{i=1}^m |d\varphi|^{p-2}h(d\varphi(e_i), \nabla^\varphi_{e_i} \tau_p(\varphi))$$

$$= (p-2)|d\varphi|^{p-3}h(d\varphi(\text{grad}^M|d\varphi|), \tau_p(\varphi))$$

$$+|d\varphi|^{p-2}h(\tau(\varphi), \tau_p(\varphi))$$

$$+|d\varphi|^{p-2}(d\varphi, \nabla^\varphi \tau_p(\varphi)).$$

(32)
From the definition of $\tau_p(\varphi)$, and equation (32), we get

$$
\text{div}^M \left( |d\varphi|^{p-2} h(d\varphi, \tau_p(\varphi)) \right) = |\tau_p(\varphi)|^2 + |d\varphi|^{p-2} \langle d\varphi, \nabla^\varphi \tau_p(\varphi) \rangle.
$$

(33)

With the same method of (32), we find that

$$
\text{div}^M \left( |d\varphi|^{p-4} h(d\varphi, \tau_p(\varphi)) \right) = (p-4)|d\varphi|^{p-5} h(\tau(\varphi), \tau_p(\varphi)) + |d\varphi|^{p-4} \langle d\varphi, \nabla^\varphi \tau_p(\varphi) \rangle + |d\varphi|^{p-4} \langle d\varphi, \nabla^\varphi \tau_p(\varphi) \rangle.
$$

(34)

Substituting (30), (31), (33) and (34) in (29), the Theorem 3.4 follows.

By using the definition of divergence for symmetric $(0,2)$-tensors (see [2], [5]) we have the following result.

**Theorem 3.5.** Let $\varphi : (M, g) \to (N, h)$ be a smooth map such that $|d\varphi|_x \neq 0$ for all $x \in M$. Then

$$
\text{div}^M S_{2,p}(\varphi)(X) = -h(\tau_{2,p}(\varphi), d\varphi(X)), \quad \forall X \in \Gamma(TM).
$$

**Remark 3.6.** When $p = 2$, we have $S_{2,p}(\varphi) = S_2(\varphi)$, where $S_2(\varphi)$ is stress bienergy tensor in [10].

**Corollary 3.7.** Let $\varphi : (M, g) \to (N, h)$ be a smooth map. (1) Then $S_{2,m}(\varphi) = 0$ implies that $\varphi$ is $m$-harmonic, where $m = \dim M$. (2) If $M$ is compact without boundary, and $p \neq m$. Then $S_{2,p}(\varphi) = 0$ implies $\varphi$ is $p$-harmonic.

**Proof.** Let $\{e_i\}$ be an orthonormal frame on $(M, g)$. (1) We have

$$
0 = \sum_{i=1}^m S_{2,p}(\varphi)(e_i, e_i) = -\frac{m}{2} |\tau_p(\varphi)|^2 + (p-m)|d\varphi|^{p-2} \langle d\varphi, \nabla^\varphi \tau_p(\varphi) \rangle.
$$

For $p = m$, the last equation becomes $-\frac{m}{2} |\tau_m(\varphi)|^2 = 0$. So $\varphi$ is $m$-harmonic map. (2) We set $\theta(X) = h(|d\varphi|^{p-2} d\varphi(X), \tau_p(\varphi))$, for all $X \in \Gamma(TM)$. The trace of $S_{2,p}(\varphi)$ gives the equality

$$
0 = \sum_{i=1}^m S_{2,p}(\varphi)(e_i, e_i) = \left( \frac{m}{2} - p \right) |\tau_p(\varphi)|^2 + (p-m) \text{div}^M \theta.
$$

By using the Green Theorem, we get

$$
\left( \frac{m}{2} - p \right) \int_M |\tau_p(\varphi)|^2 v^g = 0.
$$

Since $p \neq \frac{m}{2}$, we obtain $|\tau_p(\varphi)|^2 = 0$, that is $\varphi$ is $p$-harmonic map.
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