CONCRETE EXAMPLES OF $H(b)$ SPACES

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Abstract. In this paper we give an explicit description of de Branges-Rovnyak spaces $H(b)$ when $b$ is of the form $q^r$, where $q$ is a rational outer function in the closed unit ball of $H^\infty$ and $r$ is a positive number.

1. Introduction

The purpose of this paper is to explicitly describe the elements of the de Branges-Rovnyak space $H(b)$ for certain $b \in b(H^\infty)$. Here $H^\infty$ denotes the space of bounded analytic functions on the open unit disk $\mathbb{D}$ normed by $\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|$, and $b(H^\infty) := \{g \in H^\infty : \|g\|_\infty \leq 1\}$ is the closed unit ball in $H^\infty$ and, for $b \in b(H^\infty)$, the de Branges-Rovnyak space $H(b)$ is the reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ whose kernel is

$$k^b_\lambda(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$ 

Besides possessing a fascinating internal structure [9], $H(b)$ spaces play an important role in several aspects of function theory and operator theory, most importantly, in the model theory for many types of contraction operators [3] [4].

Despite the important role $H(b)$ spaces play in operator theory, the exact contents of $H(b)$ often remain mysterious. What functions belong to $H(b)$? Certainly the kernel functions $k^b_\lambda, \lambda \in \mathbb{D}$, do (and have dense linear span). What else?

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In this paper, we give a precise description of the elements of \( \mathcal{H}(b) \) for certain relatively simple \( b \), namely positive powers of rational outer functions. Our description needs the following setup. If \( b \in \mathfrak{b}(H^\infty) \) is a non-extreme point of \( \mathfrak{b}(H^\infty) \), equivalently, \( \log(1 - |b|) \in L^1(\mathbb{T}, m) \) (where \( \mathbb{T} := \{ \zeta \in \mathbb{C} : |\zeta| = 1 \} \) and \( m \) Lebesgue measure on \( \mathbb{T} \) normalized so that \( m(\mathbb{T}) = 1 \)), then there exists a unique outer function \( a \in \mathfrak{b}(H^\infty) \), called the Pythagorean mate for \( b \), such that \( a(0) > 0 \) and \( |a|^2 + |b|^2 = 1 \) almost everywhere on \( \mathbb{T} \). The pair \((a, b)\) is said to be a Pythagorean pair.

Our first observation says that in certain situations \( \mathcal{H}(b^r) \) does not depend on \( r > 0 \).

**Theorem 1.1.**

1. Suppose \( b \in \mathfrak{b}(H^\infty) \) is outer. The following are equivalent:
   
   (a) For any \( r > 0 \) we have \( \mathcal{H}(b^r) = \mathcal{H}(b) \) as sets.
   
   (b) \( \mathcal{H}(b^2) = \mathcal{H}(b) \) as sets.
   
   (c) \( b \mathcal{H}(b) \subset \mathcal{H}(b) \).

2. If \( b \) is non-extreme, i.e., \( \log(1 - |b|) \in L^1(\mathbb{T}) \), with Pythagorean mate \( a \), then conditions (a), (b), and (c) are equivalent to the condition
   
   \[ \inf\{|a(z)| + |b(z)| : z \in \mathbb{D}\} > 0. \]

3. If \( b \) extreme, i.e., \( \log(1 - |b|) \not\in L^1(\mathbb{T}) \), then conditions (a), (b), and (c) are equivalent to the condition
   
   \[ b \text{ is invertible in } H^\infty. \]

**Remark 1.4.**

1. Since \( b \) is outer, it has no zeros on \( \mathbb{D} \) and so we can define \( b^r \) by taking any logarithm of \( b \). Note that \( b^r \in \mathfrak{b}(H^\infty) \).

2. Statement (a) of Theorem 1.1 says that \( \mathcal{H}(b^r) = \mathcal{H}(b) \) as sets. Though the norms on \( \mathcal{H}(b^r) \) and \( \mathcal{H}(b) \) are different, one sees from the closed graph theorem that they are equivalent.

3. Statement (c) of the theorem says that \( b \) is a multiplier of \( \mathcal{H}(b) \). We refer the reader to Sarason’s book [9] for further information and references about multipliers of \( \mathcal{H}(b) \).

4. By Carleson’s corona theorem [6], the condition (1.2) is equivalent to existence of \( \phi, \psi \in H^\infty \) so that \( a\phi + b\psi = 1 \) on \( \mathbb{D} \). Such a pair \((a, b)\) satisfying this condition is called a corona pair.
When \( b \) is a rational outer function, or any positive power of a rational function (which is necessarily non-extreme (see Lemma 3.1)), we obtain the following complete description of \( \mathcal{H}(b) \) involving the derivatives of the reproducing kernels. Indeed, when \( b = q^r \), where \( q \) is outer and rational and \( r > 0 \), we set

\[
v_{r,\lambda}^\ell(z) := \frac{d\ell}{d\lambda} k_{q^r}(\lambda) = \frac{d\ell}{d\lambda} \left( \frac{1 - \overline{q^r(\lambda)} q^r(z)}{1 - \lambda z} \right),
\]

for any \( z \in \mathbb{D} \), \( \lambda \in \mathbb{D}^\ast \), and \( \ell \geq 0 \). We let \( H^2 \) denote the classical Hardy space \([6]\). By means of the F\'ejer-Riesz theorem (see Section 6), one can prove that if \( q \) is a rational function then so is its Pythagorean mate \( a \). In this case, also notice that for \( \zeta \in \mathbb{T} \) we have \( |q(\zeta)| = 1 \) if and only if \( a(\zeta) = 0 \).

**Theorem 1.5.** Suppose \( q \in \mathfrak{b}(H^\infty) \) is a rational outer function and \( r \) is a positive real number. Then

1. \( \mathcal{H}(q^r) = \mathcal{H}(q) \) as sets.

2. If \( a \) is the Pythagorean mate for \( q \) and \( a \) has distinct zeros \( \zeta_1, \ldots, \zeta_n \) on \( \mathbb{T} \) with corresponding multiplicities \( m_1, \ldots, m_n \), then

   a) the functions \( v_{r,\zeta_j}^\ell := v_{r,\lambda}^\ell \) are well-defined and belong to \( \mathcal{H}(q^r) \) for \( 1 \leq j \leq n \) and \( 0 \leq \ell \leq m_j - 1 \). Moreover, they are orthogonal to

\[
aH^2 = \left( \prod_{j=1}^{n} (z - \zeta_j)^{m_j} \right) H^2.
\]

b) \( \mathcal{H}(q^r) \) is equal to

\[
\left( \prod_{j=1}^{n} (z - \zeta_j)^{m_j} \right) H^2 \oplus \bigvee \{ v_{r,j}^\ell : 0 \leq \ell \leq m_j - 1, 1 \leq j \leq n \},
\]

where the orthogonal decomposition is in terms of the inner product in \( \mathcal{H}(q^r) \).

Writing \( v_j^\ell = v_{1,j}^\ell \), the theorem above implies that \( \mathcal{H}(q^r) \) is equal to

\[
\left( \prod_{j=1}^{n} (z - \zeta_j)^{m_j} \right) H^2 + \bigvee \{ v_j^\ell(z) : 0 \leq \ell \leq m_j - 1, 1 \leq j \leq n \},
\]

where the sum is no longer necessarily orthogonal.
It was shown in [2], and rediscovered in [1], that
\[
\mathcal{H}(q) = \left( \prod_{j=1}^{n} (z - \zeta_j)^{m_j} \right) H^2 + \mathcal{P}_{N-1},
\]
(1.6)
where \( N = \sum_{j=1}^{n} m_j \), \( \mathcal{P}_{N-1} \) is the \( N \)-dimensional vector space of polynomials of degree at most \( N - 1 \), and the sum is an algebraic direct sum (not necessarily an orthogonal one). The novelty of our result is that we can precisely identify the orthogonal complement of \( aH^2 = \left( \prod_{j=1}^{n} (z - \zeta_j)^{m_j} \right) H^2 \) in \( \mathcal{H}(q) \) without using (1.6).

In a recent preprint, Lanucha and Nowak [7] examined when an \( \mathcal{H}(b) \) space is isomorphic to a Dirichlet type space. Their discussion naturally leads to the situation when \( a \) is a polynomial with simple zeros on \( \mathbb{T} \) and a similar description of \( \mathcal{H}(b) \) for such \( a \).

A key ingredient used to show statement (1) of Theorem 1.5, and an added bonus to our result, is that if \( a_r \) is the Pythagorean mate for \( q^r \) then the co-analytic Toeplitz operators \( T_\pi \) and \( T_{\overline{\pi}} \) on \( H^2 \) have the same range, namely \( \mathcal{H}(q) \).

2. Preliminaries

There are several equivalent definitions of the de Branges-Rovnyak space \( \mathcal{H}(b) \). We can, for instance, define it in the standard way [3] as the reproducing kernel Hilbert space associated with the (positive definite) reproducing kernel
\[
k^b_\lambda(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \lambda z}, \quad \lambda, z \in \mathbb{D}.
\]
By definition, \( f(\lambda) = \langle f, k^b_\lambda \rangle_b \) for all \( f \in \mathcal{H}(b) \) and \( \lambda \in \mathbb{D} \), where \( \langle \cdot, \cdot \rangle_b \) represents the scalar product in \( \mathcal{H}(b) \).

The space \( \mathcal{H}(b) \) can also be defined as the range space \( (I - T_bT_b^*)^{1/2} H^2 \) equipped with the norm which makes \( (I - T_bT_b^*)^{1/2} \) a partial isometry. Here \( T_\varphi \) is the Toeplitz operator on \( H^2 \) with symbol \( \varphi \in L^\infty(\mathbb{T}) \) defined by
\[
T_\varphi f = P_+(\varphi f), \quad f \in H^2,
\]
where \( P_+ \) is the orthogonal projection of \( L^2(\mathbb{T}) \) onto \( H^2 \). The book [9] is the classic reference for \( \mathcal{H}(b) \) spaces.
When \( \|b\|_\infty < 1 \), \( \mathcal{H}(b) \) turns out to be a renormed version of \( H^2 \) while if \( b \) is an inner function, then \( \mathcal{H}(b) \) turns out to be one of the classical and well-studied model spaces \( H^2 \ominus bH^2 \).

When \( b \) is non-extreme and \( a \) is its Pythagorean mate, two important (not necessarily closed) vector spaces of functions in \( \mathcal{H}(b) \) are
\[
\mathcal{M}(a) := T_aH^2 \quad \text{and} \quad \mathcal{M}(\overline{a}) := T_{\overline{a}}H^2.
\]

It follows from the Douglas factorization theorem and the operator inequalities
\[
T_aT_{\overline{a}} \leq T_{\overline{a}}T_a \quad \text{and} \quad T_{\overline{a}}T_a = I - T_bT_b \leq I - T_bT_{\overline{b}}
\]
that \( \mathcal{M}(a) \subset \mathcal{M}(\overline{a}) \subset \mathcal{H}(b) \) (see [9, p. 24]).

For technical reasons, we will make use of the space \( \mathcal{H}(\overline{b}) \) which, for any \( b \in \mathfrak{b}(H^\infty) \), is defined similarly as with \( \mathcal{H}(b) \) but as the range space \((I - T_bT_\overline{b})^{1/2}H^2 \). The operator inequalities from (2.1) show that \( \mathcal{H}(\overline{b}) \) is contractively contained in \( \mathcal{H}(b) \).

### 3. Corona pairs

This following lemma is well-known but we record it here along with a proof for the sake of completeness and for the discussion of the examples in Section [3].

**Lemma 3.1.** Suppose \( q \in \mathfrak{b}(H^\infty) \) is rational and not inner. Then \( q \) is non-extreme and, if \( a \) is the Pythagorean mate for \( q \), then \( a \) is also rational.

**Proof.** Since \( q \) is rational then \( q = p_1/p_2 \) where \( p_1 \) and \( p_2 \) are analytic polynomials and \( p_2 \) has no zeros on \( \mathbb{D}^- \). We can, of course, choose \( p_2 \) such that \( p_2(0) > 0 \). Since \( q \in \mathfrak{b}(H^\infty) \), we see that \( 1 - |q(e^{i\theta})|^2 \geq 0 \) for all \( \theta \) and so \( |p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 \) is a non-negative trigonometric polynomial. Furthermore, \( |p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 \) is not the zero function since we are assuming that \( q \) is not an inner function. By the Fényer-Riesz theorem, \( |p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 = |p(e^{i\theta})|^2 \), where \( p \) is an analytic polynomial which is zero free in \( \mathbb{D} \) and \( p(0) > 0 \).

Let \( a = p/p_2 \). Note that \( a \) is rational and zero free in \( \mathbb{D} \), hence outer. Moreover, \( a(0) > 0 \).

Furthermore, on \( \mathbb{T} \) we have
\[
|a|^2 = \left| \frac{p}{p_2} \right|^2 = \frac{|p_2|^2 - |p_1|^2}{|p_2|^2} = 1 - \frac{|p_1|^2}{|p_2|^2} = 1 - |q|^2.
\]
This means that \((a,q)\) is a Pythagorean pair which, in particular, implies that \(q\) is non-extreme. \(\square\)

**Lemma 3.2.** Suppose \(b \in b(H^\infty)\) is outer and \(r\) is a positive real number. Then \(b\) and \(b^r\) are simultaneously non-extreme. Moreover, if \(a_r\) is the Pythagorean mate for \(b^r\), the pairs \((a,b)\) and \((a_r,b^r)\) are simultaneously corona.

**Proof.** Since

\[
\frac{1 - x^r}{1 - x} \approx 1, \quad x \in [0, 1),
\]

we see that \(1 - |b|^r \approx 1 - |b|\) when \(b \in b(H^\infty)\), from which we deduce the first part of the Lemma.

Now observe that

\[
\frac{|a|^2}{|a_r|^2} = \frac{1 - |b|^2}{1 - |b^r|^2} \approx 1,
\]

and since \(a\) and \(a_r\) are outer, Smirnov’s theorem (which says that if the boundary function for the quotient of two outer functions is bounded on \(\mathbb{T}\), then \(f \in H^\infty\)), shows that \(a/a_r\) is invertible in \(H^\infty\). Thus both expressions

\[
\inf_{z \in \mathbb{D}}(|a(z)| + |b(z)|) \quad \text{and} \quad \inf_{z \in \mathbb{D}}(|a_r(z)| + |b^r(z)|)
\]

are strictly positive (or not) simultaneously. Indeed, if there is a sequence \(\{z_n\}_{n \geq 1} \) in \(\mathbb{D}\) such that one expression goes to 0 then, since both \(a(z_n)\) and \(b(z_n)\) go to zero, the other expression will go to zero as well. \(\square\)

A special situation where \(b\) forms a corona pair with its Pythagorean mate is when \(b\) is rational.

**Lemma 3.4.** Suppose \(q \in b(H^\infty)\) is rational and not inner. If \(a\) is the Pythagorean mate for \(q\), then \((a,q)\) is a corona pair.

**Proof.** According to the proof of Lemma 3.1, we know that \(a\) is rational, \(a = p/p_2\), where \(p\) and \(p_2\) are polynomials, \(p_2\) has no zeros in \(\mathbb{D}^-\) and \(p\) is zero free in \(\mathbb{D}\). In particular, \(a\) is analytic in an open neighborhood of \(\mathbb{D}^-\) and thus has a finite number of zeros on \(\mathbb{T}\), say \(\{\xi_1, \ldots, \xi_n\}\). Note that, due to the identity \(|a|^2 + |q|^2 = 1\) on \(\mathbb{T}\), the zeros of \(a\) (on \(\mathbb{T}\)) must lie where \(q\) is unimodular on \(\mathbb{T}\).
Let $D_j$ be disjoint open disks with center at the zeros $\zeta_j$ of $a$ and let

$$F = \mathbb{D}^- \setminus \bigcup_{j=1}^{n} D_j.$$ 

By making the disks smaller, one can, by using the continuity of $|q|$ on $\mathbb{D}^-$, arrange things so that $|q| \geq \frac{1}{2}$ on each $D_j \cap \mathbb{D}^-$. Notice that $F$ is closed and omits all of the zeros of $a$ in $\mathbb{D}^-$ and so

$$\inf_{z \in F} |a(z)| = \delta > 0.$$ 

Thus

$$\inf_{z \in \mathbb{D}^c} (|a(z)| + |q(z)|) \geq \min(\frac{1}{2}, \delta) > 0$$

concluding the proof. $\square$

The first statement of Theorem 1.1 depends on the following two results. The first is from Sarason’s book [9, p. 62].

**Proposition 3.5.** For $b \in b(H^\infty)$ and non-extreme, the following are equivalent:

1. $(a,b)$ is a corona pair;
2. $H(b) = \mathcal{M}(\overline{a}).$

The second is the following.

**Proposition 3.6.** If $a, a_1 \in H^\infty$ are two outer functions such that $a/a_1$ and $a_1/a$ belong to $L^\infty$, then $\mathcal{M}(a) = \mathcal{M}(a_1)$.

**Proof.** Again, by Smirnov’s theorem, we know that $a/a_1$ and $a_1/a$ belong to $H^\infty$. Thus $T_{a/a_1}$ and hence $T_{a_1/a}$, are invertible operators on $H^2$. From here we get

$$\mathcal{M}(\overline{a}) = T_{\overline{a}}H^2 = T_{\overline{a}/a_1}H^2 = T_{\overline{a_1}}H^2 = \mathcal{M}(\overline{a_1}).$$

$\square$

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** The implication $(a) \implies (b)$ is trivial.

To show $(b) \implies (c)$ note from [2, I-10] we have

$$\mathcal{H}(b^2) = \mathcal{H}(b) + b\mathcal{H}(b).$$
But since we are assuming that $\mathcal{H}(b^2) = \mathcal{H}(b)$ is follows that $b\mathcal{H}(b) \subset \mathcal{H}(b)$.

For the implication $(c) \implies (1.2)$, we use the fact that $b$ is non-extreme and [5] VIII-1, VIII-7] to see that $b$ being a multiplier of $\mathcal{H}(b)$ is equivalent to $(a, b)$ being a corona pair.

To show that $(1.2) \implies (a)$, we proceed as follows. By Lemma 3.2 we know that since $(a, b)$ is a corona pair, then so is $(a_r, b^r)$. Thus from Proposition 3.6 we see that $\mathcal{H}(b) = \mathcal{M}(\overline{a})$ and $\mathcal{H}(b^r) = \mathcal{M}(\overline{a_r})$. As in the proof of Lemma 3.2 a/a and $a_r/a$ belong to $H^\infty$ so, by Proposition 3.6 we get $\mathcal{M}(\overline{a}) = \mathcal{M}(\overline{a_r})$. Putting this all together we get the desired set equality $\mathcal{H}(b^r) = \mathcal{H}(b)$.

For the implication $(c) \implies (1.3)$, we use the fact that $b$ is extreme and [5] VIII-1, VIII-5] to see that $b$ being a multiplier of $\mathcal{H}(b)$ is equivalent to $b$ being an invertible element of $H^\infty$.

It remains to show $(1.3) \implies (a)$. Assuming that $b$ is invertible in $H^\infty$, we use, once again, [5] VIII-1] to see that $\mathcal{H}(b) = \mathcal{H}(\overline{b})$. But, since $b$ is invertible in $H^\infty$, then so is $b^r$ and we thus also have $\mathcal{H}(b^r) = \mathcal{H}(\overline{b^r})$.

Remember that

$$\frac{1 - |b|^2}{1 - |b^r|^2} \gtrsim 1$$

and thus there are two constants $c_1, c_2 > 0$ such that

$$c_1(I - T_b^*T_b) \leq I - T_{\overline{b}^*}T_{\overline{b}} \leq c_2(I - T_{\overline{b_r}}T_{\overline{b}}).$$

The Douglas factorization theorem implies that $\mathcal{H}(\overline{b}) = \mathcal{H}(\overline{b^r})$ which concludes the proof.

\textbf{Remark 4.1.} In Theorem 1.1 we see from the above proofs that one can add the condition $\mathcal{H}(b) = \mathcal{H}(\overline{b})$ to the list of equivalent conditions.

5. The contents of $\mathcal{H}(b)$

We now can give the proof of Theorem 1.5. Indeed, statement (1) of the theorem follows from Lemma 3.4 and Theorem 1.1. Let us consider statement (2).

In [5] it was shown, for an outer function $b$, that if $\zeta \in \mathbb{T}$ and

$$\int_{\mathbb{T}} \frac{\log |b(w)|}{|w - \zeta|^{2n+2}} dm(w) < \infty,$$

(5.1)
then every function in $\mathcal{H}(b)$, as well as its derivatives up to order $n$, has a finite non-tangential limit at $\zeta$.

Recalling the notation $v^\ell_{r,\lambda}$ for the $\ell$-th derivative in the variable $\lambda$ of the reproducing kernel in $\mathcal{H}(q^r)$, the results of [5] also show that $v^\ell_{r,\zeta} \in \mathcal{H}(q^r)$, $0 \leq \ell \leq n$, and

$$f^{(\ell)}(\zeta) = \langle f, v^\ell_{r,\zeta} \rangle_{q^r}, \quad f \in \mathcal{H}(q^r), \ 0 \leq \ell \leq n.$$

Let us check condition (5.1) for our situation. Since $q$ is rational its Pythagorean mate $a$ is also rational and can be written as

$$a(z) = s(z) \prod_{j=1}^{n} (z - \zeta_j)^{m_j},$$

where $s$ is a rational function whose poles and zeros lie on the complement of $D^-$. Pick $w = e^{it}$ near one of the zeros $\zeta_j = e^{i\theta_j}$ of $a$. Then

$$|\log |q^r(e^{it})|| \asymp |\log |q(e^{it})|^2| = |\log(1 - |a(e^{it})|^2)|
\asymp |a(e^{it})|^2 \asymp |e^{it} - e^{i\theta_j}|^{2m_j}$$

This means that for $t$ near $\theta_j$ we have

$$\frac{|\log |q^r(e^{it})||}{|e^{it} - e^{i\theta_j}|^{2(m_j-1)+2}} \asymp 1$$

and so, by (5.1), every function in $\mathcal{H}(b)$ as well as its derivatives up to order $m_j - 1$ admits non-tangential limits at $\zeta_j$, and $v^\ell_{r,\zeta_j} \in \mathcal{H}(q^r)$ for all $0 \leq \ell \leq m_j - 1$.

The following interesting observation will be very useful in the proof of our main theorem.

**Lemma 5.3.** Suppose $a(z) = \prod_{j=1}^{n} (z - \zeta_j)^{m_j}$, where $\zeta_j \in \mathbb{T}$ and $m_j$ is the corresponding multiplicity. If the non-tangential limits of an $f = T_\pi g \in \mathcal{M}(\overline{a})$, along with the non-tangential limits of its derivatives up to order $m_j - 1$, vanish at every point $\zeta_j$, $j = 1, \ldots, n$, then

$$\widehat{g}(0) = \widehat{g}(1) = \cdots = \widehat{g}(N - 1) = 0,$$

where $N = \sum_{j=1}^{n} m_j$.

**Proof of Lemma.** We prove (5.4) as follows. Consider the kernels

$$k_{\lambda,\ell}(z) = c_\ell \frac{z^\ell}{(1 - \lambda z)^{\ell+1}},$$
where $c_\ell$ is adjusted so that these are the reproducing kernels for $\ell$-th derivatives at point $\lambda \in \mathbb{D}$ in the Hardy space $H^2$, that is to say,

$$f^{(\ell)}(\lambda) = \langle f, k_{\lambda,\ell} \rangle_{H^2} = \int_{\mathbb{T}} f(\zeta) \overline{k_{\lambda,\ell}(\zeta)} dm(\zeta), \quad f \in H^2.$$ 

Observe, for $1 \leq j \leq n$ and $0 \leq \ell \leq m_j - 1$, that

$$a(z)k_{i\zeta_j,\ell}(z) = c_\ell z^{\ell}(z - \zeta_j)^{m_j} \prod_{k \neq j} (z - \zeta_k)^{m_k}$$

$$= c_\ell z^{\ell}(z - \zeta_j)^{m_j-(\ell+1)} \left( \frac{z - \zeta_j}{1 - t\zeta_j z} \right) \prod_{k \neq j} (z - \zeta_k)^{m_k}.$$ 

Writing

$$\frac{z - \zeta_j}{1 - t\zeta_j z} = -\zeta_j \left( 1 - \frac{\zeta_j z}{1 - t\zeta_j z} \right),$$

we see that $a(z)k_{i\zeta_j,\ell}(z)$ is uniformly bounded in $z \in \mathbb{D}$ and $t \in [0,1)$, and moreover

$$\frac{z - \zeta_j}{1 - t\zeta_j z} \to -\zeta_j, \quad t \to 1,$$

for every $z$. Thus, by the dominated convergence theorem,

$$a k_{i,\ell} \to c z^{\ell}(z - \zeta_j)^{m_j-(\ell+1)} \prod_{k \neq j} (z - \zeta_k)^{m_k}$$

in the norm of $H^2$, where $c$ is some non zero constant depending on $\ell$ and $j$.

Choose any function $f = T_\pi g \in \mathcal{M}(\pi)$ with $(T_\pi g)^{(\ell)}(\zeta_j) = 0$ for all $1 \leq j \leq n$, $0 \leq \ell \leq m_j - 1$. Recall that $\mathcal{M}(\pi) \subset \mathcal{H}(b)$ and so $f$, as well as all its derivatives up to order $m_j - 1$, admits non-tangential limits at $\zeta_j$ for all $1 \leq j \leq n$. Then

$$0 = (T_\pi g)^{(\ell)}(\zeta_j) = \lim_{t \to 1^-} (T_\pi g)^{(\ell)}(t\zeta_j) = \lim_{t \to 1^-} \langle T_\pi g, k_{t\zeta_j,\ell} \rangle_{H^2}$$

$$= \lim_{t \to 1^-} \langle g, a k_{t\zeta_j,\ell} \rangle_{H^2}$$

$$= \overline{\langle g, c z^{\ell}(z - \zeta_j)^{m_j-(\ell+1)} \prod_{k \neq j} (z - \zeta_k)^{m_k} \rangle}_{H^2}.$$ 

In order to prove the lemma, it suffices to show that the set

$$\left\{ \varphi_{j,\ell}(z) := z^{\ell}(z - \zeta_j)^{m_j-(\ell+1)} \prod_{k \neq j} (z - \zeta_k)^{m_k} \right\},$$

where $j = 1, \ldots, n$ and $\ell = 0, \ldots, m_j - 1$, is a basis for the space of polynomials of degree at most $N - 1$. Clearly each $\varphi_{j,\ell}$ is a polynomial
of degree $N - 1$ and there are $N - 1$ functions $\varphi_{j,\ell}$. It remains to show that the elements of this family are linearly independent. Obviously, for fixed $1 \leq r \leq n$ and $0 \leq k \leq m_r - 1$, we have

$$\varphi^{(k)}_{j,\ell}(\zeta_r) = 0, \quad j \neq r, \quad 0 \leq k \leq m_r - 1,$$

and

(5.5) $$\varphi^{(k)}_{r,\ell}(\zeta_r) = 0, \quad 0 \leq k \leq m_r - (\ell + 2).$$

In particular, if $\sum_{j,\ell} \alpha_{j,\ell} \varphi_{j,\ell} \varphi_{j,\ell} = 0$, then, for fixed $r$ and $0 \leq k \leq m_r - 1$, $\sum_{j,\ell} \alpha_{j,\ell} \varphi^{(k)}_{j,\ell}(\zeta_r) = 0$ which reduces to $\sum_{\ell} \alpha_{r,\ell} \varphi^{(k)}_{r,\ell}(\zeta_r) = 0$. Writing $\varphi_{j,\ell}(z) = (z - \zeta_j)^{m_j - (\ell + 1)}p_{j,\ell}(z)$, where $p_{j,\ell}$ does not vanish at $\zeta_j$, Leibniz’s formula gives

$$\varphi_{j,\ell}(z)^{m_j - (\ell + 1)}(z) = \sum_{k=0}^{m_j - (\ell + 1)} \binom{m_j - (\ell + 1)}{k} \times \frac{(m_j - (\ell + 1))!}{(m_j - (\ell + 1) - k)!} (m_j - (\ell + 1) - k)! p_{j,\ell}(m_j - (\ell + 1) - k)(z).$$

Evaluating this expression at $\zeta_j$ makes all terms of the sum vanish except for $k = m_j - (\ell + 1)$, and thus

$$\varphi_{j,\ell}(z)^{m_j - (\ell + 1)}(\zeta_j) = (m_j - (\ell + 1))! p_{j,\ell}(\zeta_j) \neq 0.$$

This together with (5.5) generates a triangular system of linear equations with non-zero diagonal entries. Thus $\alpha_{r,\ell} = 0, \quad 0 \leq \ell \leq m_r - 1$. □

We are now in a position to prove Theorem 1.5.

Our arguments so far yield

(5.6) $$\mathcal{H}(q^r) = \mathcal{M}(\overline{a_r})$$

and

(5.7) $$\mathcal{M}(\overline{a_r}) = \mathcal{M}(\overline{a}) \supset \mathcal{M}(a) + \sqrt{\{ v_{r,\zeta_j}^\ell : 1 \leq j \leq n, 0 \leq \ell \leq m_j - 1 \}}.$$

First we show that the sum is orthogonal in the $\mathcal{H}(q^r)$ inner product:

$$v_{r,\zeta_j}^\ell \perp \mathcal{M}(a), \quad 1 \leq j \leq n, 0 \leq \ell \leq m_j - 1.$$

Indeed, for each $f \in \mathcal{H}(q^r)$ the radial limits $f^{(\ell)}(t\zeta_j)$ exist as $t \to 1^-$. Since

$$f^{(\ell)}(t\zeta_j) = \langle f, v_{r,\zeta_j}^\ell \rangle_{q^r},$$

we can apply the principle of uniform boundedness to see that $\|v_{r,\zeta_j}^\ell\|_{q^r}$ is uniformly bounded as $t \to 1^-$. Since $v_{r,t\zeta_j}^\ell$ converges pointwise to $v_{r,\zeta_j}^\ell$ as $t \to 1^-$ we see that $v_{r,t\zeta_j}^\ell$ converges weakly to $v_{r,\zeta_j}^\ell$. Thus, since $v_{r,t\zeta_j}^\ell$
reproduces the $\ell$-th derivative of $\mathcal{H}(q^r)$-functions at point $t_\zeta_j$, for any $g \in H^2$, we have
\[
\langle ag, v^\ell_{r,\zeta_j} \rangle_{q^r} = \lim_{t \to 1^-} \langle ag, v^\ell_{r, t_\zeta_j} \rangle_{q^r} = \lim_{t \to 1} (ag)^{(\ell)}(t_\zeta_j) = \lim_{t \to 1} \sum_{p=0}^\ell \binom{\ell}{p} a^{(p)}(t_\zeta_j) g^{(\ell-p)}(t_\zeta_j).
\]
Using the estimate
\[
|a^{(p)}(t_\zeta_j)| \lesssim (1 - t)^{m_j - p}
\]
along with the following standard $H^2$ estimate on the growth of the derivative of an $H^2$ function
\[
|g^{(\ell-p)}(t_\zeta_j)| \lesssim \frac{1}{(1 - t)^{(\ell-p)+1/2}},
\]
we see that
\[
|a^{(p)}(t_\zeta_j)g^{(\ell-p)}(t_\zeta_j)| \lesssim (1 - t)^{m_j - p - ((\ell-p)+1/2)}.
\]
But since $0 \leq \ell \leq m_j - 1$ we see that
\[
m_j - p - ((\ell-p)+1/2) = m_j - \ell - \frac{1}{2} \geq \frac{1}{2}
\]
and so
\[
\lim_{t \to 1^-} |a^{(p)}(t_\zeta_j)g^{(\ell-p)}(t_\zeta_j)| = 0.
\]
Thus $\langle ag, v^\ell_{r,\zeta_j} \rangle_{q^r} = 0$ and $v^\ell_{r,\zeta_j} \perp \mathcal{M}(a)$ in $\mathcal{H}(q^r)$, for all $0 \leq \ell \leq m_j - 1$.

This upgrades (5.6) and (5.7) to
\[
(5.8) \quad \mathcal{H}(q^r) = \mathcal{M}(\overline{\pi}) \supset \mathcal{M}(a) \bigoplus \{v^\ell_{r,\zeta_j} : 1 \leq j \leq n, 0 \leq \ell \leq m_j - 1\},
\]
and orthogonality is with respect to the norm in $\mathcal{H}(q^r)$.

To show equality in (5.8), our second step is to show that if $f \in \mathcal{M}(\overline{\pi})$ and $f \perp v^\ell_{r,\zeta_j}$ for all $1 \leq j \leq n, 0 \leq \ell \leq m_j - 1$, then $f \in \mathcal{M}(a)$. Since $\mathcal{M}(\overline{\pi}) = T_{\overline{\pi}}H^2$ this is equivalent to prove that if $g \in H^2$ and
\[
0 = (T_{\overline{\pi}g})^{(\ell)}(\zeta_j) = \lim_{t \to 1^-} (T_{\overline{\pi}g})^{(\ell)}(t_\zeta_j)
\]
for all $1 \leq j \leq n, 0 \leq \ell \leq m_j - 1$ then $T_{\overline{\pi}g} \in \mathcal{M}(a)$. To simplify matters a bit, let us recall the formula for $a$ from (5.2). Since $s$ is a rational function with zeros and poles outside $\mathbb{D}^-$ then certainly the Toeplitz operators $T_{1/s}$ and $T_{1/\overline{\pi}s}$ are invertible, and so $\mathcal{M}(\overline{\pi}) = \mathcal{M}(a/s)$. We can therefore make the simplifying assumption that
\[
a(z) = \prod_{j=1}^n (z - \zeta_j)^{m_j}.
\]
We will show that
\[(5.9) \quad (T_\pi g)^{(n)}(\zeta_j) = 0, \, 1 \leq j \leq n, \, 0 \leq \ell \leq m - 1 \implies T_\pi g \in aH^2.\]
With \(N = \sum_{j=1}^n m_j\), one can verify the identity
\[
\overline{a(\zeta)} = \zeta^N a(\zeta) \prod_{j=1}^n (-\overline{\zeta_j})^{m_j}, \quad \zeta \in \mathbb{T}.
\]
Thus
\[
T_\pi g = \prod_{j=1}^n (-\overline{\zeta_j})^{m_j} P_+ (a\overline{\zeta}^N g).
\]
By Lemma 5.3 we have \(\hat{g}(0) = \hat{g}(1) = \cdots = \hat{g}(N - 1) = 0\), which shows that \(\overline{\zeta}^N g \in H^2\) and so
\[
T_\pi g = \left( \prod_{j=1}^n (-\overline{\zeta_j})^{m_j} \right) P_+ (a\overline{\zeta}^N g) \in aH^2.
\]
This completes the proof. \(\square\)

### 6. Examples

**Example 6.1.** Consider the function
\[
q(z) = \frac{1}{2}(1 + z)
\]
and notice that \(q\) is outer and \(\|q\|_\infty = 1\). One can easily guess the Pythagorean mate for \(q\) to be \(a(z) = \frac{1}{2}(1 - z)\). The function \(a(z)\) has one zero of order 1 at \(z = 1\) and a computation reveals that
\[
v_{1,1}^0(z) = \frac{1 - \overline{q(1)}q(z)}{1 - z} = \frac{1}{2}.
\]
In this case
\[
\mathcal{H}(q) = (z - 1)H^2 \oplus \mathbb{C}.
\]
Moreover, for any \(r > 0\) we get \(\mathcal{H}(q^r) = \mathcal{H}(q)\) and
\[
\mathcal{H}(q^r) = (z - 1)H^2 \oplus \mathbb{C} = (z - 1)H^2 \oplus \mathbb{C} \frac{1 - \left(\frac{1 + z}{2}\right)^r}{1 - z}.
\]
For more general \(q\) we need to review the proof of the Féjer-Riesz theorem which says that if
\[
w(e^{i\theta}) = \sum_{j=-n}^n c_je^{ij\theta}
\]
is a non-zero trigonometric polynomial which assumes non-negative values for all $\theta$, then there is an analytic polynomial

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

so that $w(e^{i\theta}) = |p(e^{i\theta})|^2$. Since the proof gives us the algorithm for computing $p$, we give a quick sketch. Indeed, as a function of the complex variable $z$, we see that if

$$w(z) = \sum_{j=-n}^{n} c_j z^j$$

then $w(1/z) = w(z)$, $z \in \mathbb{T}$. Assuming that $c_{-n} \neq 0$ we see that $s(z) = z^n w(z)$, $z \in \mathbb{C}$, is a polynomial of degree $2n$ and the roots of $s$ occur in the pairs $\alpha, 1/\alpha$ of equal multiplicity. It follows that

$$w(z) = c \prod_{j=1}^{n} (z - \alpha_j) \left(\frac{1}{z} - \frac{1}{\alpha_j}\right)$$

for some positive constant $c$ and where $\alpha_1, \ldots, \alpha_n$ satisfy $|\alpha_j| \geq 1$ for $1 \leq j \leq n$. The desired polynomial $p$ is

$$p(z) = \sqrt{c} \prod_{j=1}^{n} (z - \alpha_j).$$

Note that $p$ is zero free in $\mathbb{D}$ and we can multiply $p$ by a unimodular constant so that $p(0) > 0$.

Recall from the proof of Lemma 3.1 that if $q = p_1/p_2$ is rational then the Pythagorean mate $a$ for $q$ is given by $a = p/p_2$, where $p$ is the analytic polynomial (guaranteed by the Féjer-Riesz theorem) which satisfies $|p(e^{i\theta})|^2 = w(e^{i\theta}) = |p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 \geq 0$, and $p$ is chosen to that $a(0) > 0$.

**Example 6.2.** Consider the function

$$q(z) = \frac{1}{2} (1 - z)(1 + z)$$

and note that $q \in \mathcal{B}(H^\infty)$ and is outer. A computation shows that

$$1 - |q(e^{it})|^2 = \frac{1}{4} e^{-2it} + \frac{1}{4} e^{2it} + \frac{1}{2}.$$  

Define

$$w(z) = \frac{z^{-2}}{4} + \frac{z^2}{4} + \frac{1}{2}$$
and
\[ s(z) = z^2w(z) = \frac{z^4}{4} + \frac{z^2}{2} + \frac{1}{4} = \frac{1}{4}(z - i)^2(z + i)^2. \]

Notice how the zeros occur in pairs \( i = \frac{1}{i} \) and \(-i = \frac{1}{-i} \) as guaranteed by the above proof of the Féjer-Riesz theorem. Thus the Pythagorean mate \( a \) for \( q \) is of the form
\[ a(z) = c(z - i)(z + i) \]
for some \( c \) adjusted so that \( a(0) > 0 \) and \( 1 - |q(e^{i\theta})|^2 = |a(e^{i\theta})|^2 \). One can check by direct calculation that \( c = 1/2 \) works and so \( a(z) = \frac{1}{2}(z - i)(z + i) \). Of course the exact value of \( c \) is not important for our calculations since we only need to identify the zeros of \( a \) along with their multiplicities.

The zeros of \( a \) are at \( z = i \) and \( z = -i \) and each has order one. Thus
\[ \mathcal{H}(q) = (z - i)(z + i)H^2 \oplus \bigvee \{v_{1,i}^0, v_{1,-i}^0\}, \]
where the kernels can be computed directly as
\[ v_{1,i}^0(z) = \frac{1}{2i}(z + i), \quad v_{1,-i}^0(z) = \frac{1}{2i}(z - i). \]

Again, as in the previous example, \( \mathcal{H}(q^*) = \mathcal{H}(q) \) and so
\[ \mathcal{H}(q^*) = (z - i)(z + i)H^2 + \bigvee \{z + i, z - i\}. \]

**Example 6.3.** Consider the function
\[ q(z) = \frac{1}{4}(z + 1)^2 \]
and note that \( q \) is outer and belongs to \( \mathfrak{b}(H^\infty) \). Following our Fejer-Riesz computations as in the previous example, note that
\[ 1 - |q(e^{i\theta})|^2 = -\frac{e^{-it}}{4} - \frac{e^{it}}{4} - \frac{1}{16}e^{-2it} - \frac{1}{16}e^{2it} + \frac{5}{8}. \]

Define
\[ w(z) = -\frac{z^2}{16} - \frac{1}{16z^2} - \frac{z}{4} - \frac{1}{4z} + \frac{5}{8} \]
and
\[ s(z) = z^2w(z) \]
\[ = -\frac{z^4}{16} - \frac{z^3}{4} + \frac{5z^2}{8} - \frac{z}{4} - \frac{1}{16} \]
\[ = -\frac{1}{16}(-1 + z)^2(1 + 6z + z^2). \]

The zeros of \( s \) are at
\[ z = -1, z = -1, z = -3 - 2\sqrt{2} \approx -5.82843, z = -3 + 2\sqrt{2} \approx -0.171573. \]
Notice how these roots occur in the pairs $\alpha, 1/\alpha$). The function $a$ is then $a(z) = c(z - 1)(z + 3 + 2\sqrt{2})$ for some appropriate constant $c$. There is one zero of $a$ at $z = 1$ with multiplicity one and so

$$H(q) = (z - 1)H^2 \oplus \mathbb{C}v^0_{1,1}(z).$$

The kernel can be computed to be

$$v^0_{1,1}(z) = \frac{z + 3}{4}.$$

As in our previous examples, note that

$$H(q^*) = (z - 1)H^2 + \mathbb{C}(z + 3).$$

Observe that the $q$ from this example is the square of the $q$ from Example 6.1 and thus the corresponding spaces should be the same. Indeed, a little algebra will show that

$$(z - 1)H^2 + \mathbb{C} = (z - 1)H^2 + \mathbb{C}(z + 3).$$

**Example 6.4.** Reversing the roles of $a$ and $q$ in the preceding example:

$$a(z) = \frac{1}{4}(z + 1)^2, \quad q(z) = c(z - 1)(z + 3 + 2\sqrt{2}),$$

with suitable $c$ so that $\|q\|_\infty = 1$ (the maximum modulus on $\mathbb{D}^-$ being attained at $-1$, one has $c = (4(1 + \sqrt{2}))^{-1}$, and $q(-1) = -1$ corresponding to the normalization $a(0) > 0$), we obtain a function $a$ with double zero, and so

$$H(q) = (z + 1)^2H^2 \oplus \sqrt{\{v^0_{1,-1}, v^1_{1,-1}\}}$$

where

$$v^0_{1,-1}(z) = \frac{1 - q(-1)q(z)}{1 - (-1)z} = \frac{1 + q(z)}{1 + z}.$$

Using the facts that $q(-1) = -1$, $q'(z) = c(2 + 2\sqrt{2})$, and $q'(-1) = -1/2$, we obtain

$$v^1_{1,-1}(z) = \frac{1}{2}q(z)(1 + z) + z(1 + q(z)) = \frac{1}{2}q(z)(1 + 3z) + \frac{2z}{(1 + z)^2}.$$

**Question 6.5.** So far we have computed the exact contents of $H(b)$ when $b$ outer, rational, and non-extreme. Can one compute the contents of $H(b)$ when $b$ is outer and extreme. For example if $b$ is the outer function corresponding to the outer function which satisfies $|b(e^{i\theta})| = 1$ for $0 \leq \theta \leq \pi$ and $|b(e^{i\theta})| = \frac{1}{2}$ for $\pi < \theta < 2\pi$, can one describe the functions in $H(b)$?
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