We present a bosonic model, in which two bosons may form a bound pair with d-wave symmetry via the four-site ring exchange interaction. A d-wave pairing superfluid as well as a d-wave density wave (DDW) state, are proposed to be achievable in this system. By the mean field approach, we find that at low densities, the d-wave pairs may condensate, leading to a d-wave bosonic paired superfluid. At half filling, a d-wave Mott insulator could be realized in a superlattice structure. At some particular filling factors, there exists a novel phase: d-wave density wave state, which preserves the d-wave symmetry within plaquette while spontaneously breaks the translational symmetry. The DDW state and its corresponding quantum phase transition in a two-leg ladder are studied by the time-evolving block decimation (TEBD) method. We show that this exotic bosonic system can be realized in the BEC zone of cold Fermi gases loaded in a two-dimensional (2D) spin-dependent optical lattice.

PACS numbers: 05.30.Jp, 03.75.Nt, 74.20.Mn, 73.43.Nq

Recently, ultracold atoms in optical lattice have provided a perfect platform for simulating quantum many-body model in condensed matter physics. Because of the flexible tunability of parameters such as the hopping amplitudes, interaction or even the dimensionality of the system, ultracold atomic systems allow us to directly study some fundamental Hamiltonian systems and their associated phase transitions, such as boson Hubbard model and the superfluid to Mott insulator phase transition, or the recent realization of the repulsive or attractive fermionic Hubbard model. In addition, some exotic phases emerged from the low-dimensional strongly correlated systems, such as resonating valence bond (RVB) states, d-wave superfluidity, deconfined Coulomb phase as well as the topological insulator with fractional statistic and topological order, can also be investigated in the cold atom systems. Furthermore, the uniqueness of cold atomic system also provides new playgrounds for physicists, such as the strongly correlated model for high spin systems, higher orbital systems or the optical superlattice. A two-dimensional (2D) optical superlattice may be constructed by imposing two optical lattices with different periods to form an array of plaquettes. The hopping amplitude and interaction for atoms between these plaquettes are much smaller than that within the plaquette. One of the exotic phase emerges in the superlattice is the d-wave Mott insulator and d-wave superfluid. The d-wave Mott insulator is the insulator state with local d-wave symmetry, i.e., if we rotate the site within a plaquette by \( \pi/2 \), the wave function reverses its sign. When we introduce holes into the d-wave Mott insulator, two holes tend to bind together within the plaquette to form a Cooper pair with local d-wave symmetry, and the propagation of the d-wave pairs between different plaques leads to the d-wave superfluid.

The mechanism of pairing with d-wave symmetry has played an important role in the high-Tc superconductor. Though without rigorous proof, numerous evidences strongly support the existence of d-wave superconductor (or superfluidity in cold atomic system) near the half-filling. The background of Neel state with antiferromagnetic correlations plays a key role in this mechanism of d-wave symmetry. In cold fermionic atom system, however, the binding energy of a d-wave pair is much smaller than the hopping amplitude, which makes it difficult to directly simulate the d-wave mechanism of the high-Tc superconductor in cold atomic systems. One solution is to trap the d-wave pair within a plaquette of the optical superlattice. In this paper, we propose a novel mechanism for realizing the d-wave pairing. Different from that in high-Tc superconductor induced by antiferromagnetic correlation, the d-wave pairing here is induced by the four-site ring exchange interaction.

Before discussing the physical realization of the effective Hamiltonian, we first present the Hamiltonian, which is a hard-core bosonic model with a strong nearest neighbor (NN) repulsive interaction and a four-site ring interaction:

\[
H = \sum_{\langle ij \rangle} \left[ a_i^\dagger a_j + V n_i n_j \right] - \mu \sum_i n_i + K \sum_{\langle ijkl \rangle} a_i^\dagger a_j a_k^\dagger a_l + h.c
\]

where \( \langle ij \rangle \) denotes a pair of nearest-neighbor sites and \( \langle ijkl \rangle \) are sites on the corners of a plaquette. In this paper, we focus on the parameters region \( V \gg K \gg |t| > 0 \). The four-site ring exchange interaction with positive coefficient (\( K > 0 \)) may be very important in determining...
the properties of this system (at least at low densities) and leads to the exotic boson pairs with the d-wave symmetry. Though in most systems, the effect of this four-site ring exchange interaction is much smaller compared to the two-site hopping amplitudes because it usually comes from the fourth order perturbation, below we would show that we can realize the Hamiltonian as well as the corresponding parameter region \( V \gg K \gg |t| > 0 \) in the BEC zone of cold Fermi gases loaded in a two-dimensional (2D) spin-dependent optical. Similar model without the NN repulsive interaction has been proposed to study the exotic phases in cold atom system such as the deconfined phase or Bose-metal phase. However, as we will show below, the strong NN repulsive interaction in Eq. makes the novel Bose-metal phase unstable.

First, we would discuss the experimental realization of our Hamiltonian as well as the corresponding parameter regions. Most of above discussion are based on the parameter region: \( V \gg K \gg |t| > 0 \) in Hamiltonian. However, in most systems, no matter in solid physics or cold atom physics, the effect of this four-site ring exchange interaction is much smaller comparing to the two-site hopping amplitudes because it usually comes from the fourth order perturbation. Below we would show that not only the Hamiltonian but also the corresponding parameter region \( V \gg K \gg |t| > 0 \) could be realized in the BEC zone of cold Fermi gases loaded in a two-dimensional (2D) spin-dependent optical. A fermionic Hubbard model with spin-dependent hopping has been proposed, by tuning the lasers between hyperfine structure levels of \(^{40}K\) atoms. The tunneling matrix elements for the two spin components are spin-dependent and can be tuned with different anisotropy. In our case, we tune the hopping so that spin up \( |\uparrow\rangle \) atoms prefer to hop along the x axis and spin down \( |\downarrow\rangle \) atoms prefer to hop along the y axis, which means there are two kind of typical hopping amplitude \( t_\pi \) and \( t_\sigma \). \( t_\sigma \) represents the hopping amplitude for the \( |\uparrow\rangle (|\downarrow\rangle \) fermions along x (y) axis, while \( t_\pi \) denotes the hopping amplitude for the \( |\uparrow\rangle (|\downarrow\rangle \) fermions along y (x) axis (as shown in Fig. a)). The ratio \( \delta = t_\sigma/t_\pi \) can be tuned experimentally and we choose the high anisotropic condition: \( t_\pi \ll t_\sigma \). In addition, we can use Feshbach resonances to manipulate the interactions and adjust it from repulsive to attractive. The unconventional pairing with attractive interaction in the anisotropic spin-dependent optical lattice has been analyzed recently.

We load the fermions into the 2D spin-dependent optical lattice defined above. Then we use Feshbach resonances to make the two fermions occupying the same sites binding together to form a bosonic molecule. Obviously this molecule is hard-core in nature. We assume that the binding energy is large enough that all fermions are tightly bound into bosonic molecule and the system enter a BEC zone. Next we will analyze the dynamics and interactions of these new bosons to show how can we construct the Hamiltonian as well as the corresponding parameter region in this spin-dependent optical lattice.

As shown in Fig. both the two-site hopping and interaction involve the second order perturbation via a virtual process. Taking the hopping term for example (Fig.1 (b)), from the standard second order perturbation theory we can obtain the effective hopping amplitude of the boson: \( t = -t_\sigma t_\pi/U \), \( t_\sigma \) and \( t_\pi \) has been defined above. U is the binding energy for a bosonic molecule formed by two fermions. Similarly, the virtual process in Fig.1 (c) plays a role similar to the term \( V S^z_i S^z_p (V = t_\sigma^2/U > 0) \) in the spin model, which means the NN repulsive interaction \( V_{nn\pi} \) in the boson language (those terms proportional to \( n_i \) are absorbed into the chemical potential). Because of the strong anisotropic hopping of the fermions in the spin-dependent optical lattice: \( t_\pi \ll t_\sigma \), we have \( V \gg t \).

The virtual process shown in Fig.1(d) is the leading term from the fourth order perturbation. Apparently it involves four sites within a plaquette and thus results in a ring exchange interaction in Eq. From the standard perturbation theory, we can get \( K = t_\pi^2/U^3 > 0 \), which is much larger than the contribution of all the other four-site virtual processes. Due to the strongly anisotropic hopping of the fermions in the spin-dependent lattice, it is possible to adjust the parameters \( (\delta = t_\pi/t_\sigma \rightarrow 0) \) in Hamiltonian to satisfy \( K \gg t \), which means that the pair binding energy is much larger than the single-particle hopping energy.

To define the symmetry of a pair, we introduce a local operator \( D_p \), which rotates the four sites within the plaquette p cyclically by an angle \( \pi/2 \). The sign of K is important for the symmetry of this bosonic pair. When \( K < 0 \), it is s-wave \( (D_p = 1) \); while \( K > 0 \) is d-wave \( (D_p = -1) \). This can be seen with just one plaquette. Loading two bosons into one plaquette: due to the strong NN repulsive interaction, the only two possible configura-
tions in the plaquette is two bosons occupying the diagonal sites. There are two eigenstates of the ring-exchange term, denoted as $|s\rangle$ and $|d\rangle$ (Fig.1(e)), with eigenvalues $K$ and $-K$, respectively. We neglect the single particle hopping term because $K \gg t$. Notice that when $K > 0$, the ground state of this plaquette is a d-wave state. At low densities, since the pair binding energy is much larger than the single particle hopping energy, the bosons prefer to move as pairs rather than hopping independently. At low density, these bosonic pairs with d-wave symmetry will condensate to form a d-wave superfluid.

A natural question arose here is whether the ring exchange interaction would make the system to form bound state with more than two bosons? Without the NN repulsive interaction, this is true: at low density the ring exchange, just like an attractive potential, causes not only two but many bosons to clump together, which leads to phase separation between an isolated boson metal cluster and vacuum. In our case, however, the strong repulsive interaction will make the isolated clusters consisted of more than two bosons unstable and break up into many bosonic pairs.

At a low density, the interaction between the pairs is not important, thus, we can analyze the problem using the mean field theory. We introduce a d-wave bosonic pair order parameter to decouple the four-site ring exchange interaction in our original Hamiltonian Eq. (1): $\langle a_{i}^{\dagger}a_{j}^{\dagger}\rangle = -\langle a_{i}a_{j}\rangle = \Delta$, where the minus sign is due to the d-wave character. As we concentrate on the low-density limit, the hard-core constraint is expected to be irrelevant. We decouple the NN interaction by the Hartree-Fock approximation: $Vn_{i}n_{j} = V\langle n_{i}\rangle n_{j} = V\langle n_{i}\rangle(\langle n_{j}\rangle - \langle n_{i}\rangle)$. The Hamiltonian in Eq. (1) can be rewritten as:

$$H = \sum_{k} \xi_{k}a_{k}^{\dagger}a_{k} + \Delta_{k}a_{k}a_{-k} - Vn^{2} + h.c + 2K\Delta^{2} \quad (2)$$

with

$$\Delta_{k} = 2K\Delta \sin k_{x} \sin k_{y}$$

$$\xi_{k} = 2t(\cos k_{x} + \cos k_{y}) - \mu + 2Vn,$$

where $n$ is the average value of the total particle number. The mean field Hamiltonian (2) is diagonalized by using the Bogoliubov transformation for bosons and we obtain the energy spectrum: $E_{k} = \sqrt{(\xi_{k}/2)^{2} - (\Delta_{k})^{2}}$. We focus on the zero temperature case and the ground state energy is given as: $E_{g} = \sum_{k} E_{k} + 2\Delta^{2}K + \mu/2 - Vn - Vn^{2}$. The sum is over all the $k$ in the first Brillouin zone. The self-consistent equations are:

$$n = -\frac{1}{2} + \frac{1}{4L} \sum_{k} \frac{\xi_{k}}{E_{k}},$$

$$1 = \frac{K}{L} \sum_{k} \frac{\sin^{2}k_{x}\sin^{2}k_{y}}{E_{k}}. \quad (3)$$

FIG. 3: The optical superlattice for (a) square lattice and (b) ladder with periodic boundary condition

Since we are dealing with a bosonic system, it is possible that another BEC state with single particle condensation will compete with our d-wave pairing state. In this case, Eq. (3) should be replaced by $n = n_{c} - \frac{\partial E_{w}}{\partial n}$, where $n_{c}$ is the density of bosons with single particle condensation. We find that at least in our parameter regime $t \ll K \ll V$, there is no positive self-consistent solution for $n_{c}$. The absence of single particle condensation has been observed previously in a bosonic system with correlated hopping. It is shown that the bosons prefer to pairing with each other due to the strong effective attractive interaction. To clarify this point, we also calculate the energy of the single particle BEC state without d-wave pairing via the standard Bogoliubov approximation: $a_{i} = \sqrt{n_{c}} + \delta a_{i}$ to decouple the Hamiltonian (1). The result is shown in Fig.2 (we set $K=1$ and $t=0.1$). Notice that at least at low densities with $K \gg t$, the ground state energy of d-wave pairing state is always lower than that of the single particle BEC state.

Next we would turn to another limit, when the filling factor is 1/2. In this case, the strong NN repulsive interaction induces a conventional $(\pi, \pi)$ density wave phase, rather than the boson metal or d-wave bosonic pairs. However, if we load the half filling bosons into a 2D...
optical superlattice, it is possible to recover the d-wave symmetry within plaquette and lead to a d-wave Mott insulator. Next we will clarify this point by the exact diagonalization (ED) of the small size systems. We classify all the plaquettes in the superlattice as two classes: P (grey plaquette in Fig.3) and P’ (white plaquette) and the Hamiltonian in this case is given by:

\[ H_s = \sum_{\square \in P} H_0 + \lambda \sum_{\square \in P'} H_0, \]

where \( H_0 \) is the Hamiltonian defined by Eq.(1) in one plaquette and \( 0 < \lambda < 1 \). Notice that when \( \lambda \ll 1 \), the situation is similar to that in a single plaquette, the system forms a d-wave Mott insulator with \( \langle D_p \rangle \approx -1 \). When \( \lambda \approx 1 \), the ground state should be a \((\pi, \pi)\) DDW with \( \langle D_p \rangle = 0 \) due to the strong repulsive NN interaction. \( \langle D_p \rangle \) is the expectation value of the rotating operator defined above to measure the d-wave symmetry within one plaquette. We calculate \( \langle D_p \rangle \) in a \( 4 \times 4 \) superlattice (Fig.4(a)) and a \( 2 \times 8 \) ladder (Fig.4(b)) with periodic boundary conditions, to show how it changes when we increase \( \lambda \) from 0 to 1.

At some particular filling factors \( f=1/3 \) for the two-leg ladder system and \( 1/4 \) for the 2D system, a novel phase emerges. The ring exchange interaction makes two bosons prefer to form a d-wave pair, while the strong NN interaction prevents two pairs from being too close. The conspiracy of them makes these pairs localized and separated as far as possible to avoid the strong NN interaction. The crystallization of these d-wave pairs leads to a novel phase: d-wave density wave state, which preserves the d-wave symmetry within plaquette and spontaneously breaks the translational symmetry, as shown in Fig.4(b) (two-leg ladder) and Fig.4(c) (2D). Notice that unlike the half filling case, the translational symmetry breaking is spontaneously in DDW state thus we don’t need any superlattice structure.

Before discussing the TEBD result, we first briefly discuss the global phase diagram of our ladder system in the limit of \( t \rightarrow 0 \) in Hamiltonian. Below we would study the DDW state and the properties of the quantum phase transition in the two-leg ladder system by the TEBD method. We focus on the case the filling factor \( f \approx 1/3 \). The open boundary condition is used to artificially shift the ground state degeneracy of DDW state due to the spontaneous translational symmetry breaking, therefore the filling factor is not exactly 1/3 in the ladder with finite length. For example in a \( 2 \times L_x \) ladder, the number of boson \( N_0 = (2L_x + 2)/3 \). In the thermodynamic limit, \( L_x \rightarrow \infty \), the filling factor is exactly 1/3. Because the DBL state and the phase separation as well as the quantum phase transition between them have been explicitly discussed in Ref.\( 21 \), thus we would not discuss them here and mainly focus on the DDW state and the properties of the corresponding quantum phase transition.

Because of the different structure of the phase separation and the DDW state, the phase transition between them can been seen most directly from the particle number distribution in the real place, as shown in Fig.8 where we set \( t = 0.1, K = 1.0, L_x = 26 \). In the phase
order derivative of the ground state energy \( (\partial E / \partial V) \), while a sharp peak appears in the second order derivative \( (\partial^2 E / \partial V^2) \), which indicates that a second order phase transition occurs at the point \( V_c = 0.43 \sim 0.44 \).

The pairing between two bosons have recently attracted considerable attentions, while most of previous pairing mechanisms are based on the direct attractive interspecies interactions tuned by Feshbach Resonance, or on a three-body onsite hardcore constraint. All of these mechanisms are due to the uniqueness of the cold atomic system and have no counterpart in traditional condensed matter physics. In this paper, we proposed a novel pairing mechanism for bosons via strong four-site ring exchange interaction, which is also thanks to the unique feature of the ultracold atoms in optical lattice. Recently a proposal has been provided to experimentally detect these bosonic pairs, which would be helpful to detect the bosonic pair in our case.

In summary, we propose a strongly correlated bosonic Hamiltonian with four-site ring exchange interaction. We focus on the parameter region \( V \gg K \gg |t| > 0 \) and investigate the exotic phases with d-wave symmetry emerging at different filling factors. A physical realization of the Hamiltonian as well as the parameter region has also been discussed.

The authors are grateful to Congjun Wu for helpful discussion. ZC is supported by AROW911NF0810291. The work is also supported in part by NSF-China, MOST-China and US-DOE-FG02-04ER46124 (XCX).

---

1 M. Greiner, O. Mandel, T. Esslinger, T. W. Hansch and I. Bloch. Nature. 415, 39 (2002).

2 R. Jördens, N. Strohmaier, K. Günter, H. Moritz, T. [...]

---

FIG. 6: The particle distribution in the real space in a 2 \times 26 ladder, we set \( t=0.1, K=1.0 \) and (a)\( V=0.01 \) (b)\( V=0.41 \), (c)\( V=0.44 \), (d)\( V=0.46 \), (e)\( V=3.0 \)

FIG. 7: (a) The dependence of \( \Delta N \) on \( V \) in a two-leg ladder, we set \( t=0.1, K=1.0 \); (b) The average energy per boson \( E \) vs. \( V \); (c) \( \partial^2 E / \partial V^2 \) vs. \( V \); (d) \( \partial E / \partial V \) vs. \( V \).
Esslinger, Nature. 445, 204 (2008).

3 S. Trebst, U. Schollwöck, M. Troyer, and P. Zoller, Phys. Rev. Lett. 96, 250402 (2006).

4 M.R. Peterson, C.W. Zhang, S.T. Tewari, and S. Das Sarma, Phys. Rev. Lett. 101, 150406 (2008).

5 B. Paredes and I. Bloch, Phys. Rev. A 77, 023603 (2008).

6 W. Hofstetter, J.I. Cirac, P. Zoller, E. Demler, and M. D. Lukin, Phys. Rev. Lett. 89, 220407 (2002).

7 C. Honerkamp and W. Hofstetter, Phys. Rev. Lett. 92, 170403 (2004).

8 H.P. Büchler, M. Hermel, S.D. Huber, Matthew.P.A. Fisher, and P. Zoller, Phys. Rev. Lett. 95, 040402 (2005).

9 S. Tewari, V.W. Scarola, T. Senthil and S. Das Sarma, Phys. Rev. Lett. 97, 200401 (2006).

10 Y.J. Han, R. Raussendorf and L.M. Duan, Phys. Rev. Lett. 98, 150404 (2007).

11 C.J. Wu, Phys. Rev. Lett. 100, 200406 (2008); Phys. Rev. Lett. 101, 186807 (2008); C.J. Wu and S. Das Sarma, Phys. Rev. B 77, 235107 (2008); W.C. Lee and C.J. Wu, arXiv: 0905.1146; H. H. Hung, W.C. Lee and C.J. Wu, arXiv: 0910.0507;

12 E. H. Zhao and W.V. Liu, Phys. Rev. Lett. 100, 160403 (2008).

13 L. Wang, X. Dai, S. Chen, and X.C. Xie, Phys. Rev. A 78, 023603 (2008).

14 B. Paredes and I. Bloch, Phys. Rev. A 71, 063608 (2005).

15 L. Santos, M. A. Baranov, J. I. Cirac, H.-U. Everts, H. Fehrmann, and M. Lewenstein, Phys. Rev. Lett. 95, 060403 (2005); Phys. Rev. A 72, 053612 (2005).

16 T. Goodman and L.M. Duan, Phys. Rev. A 74, 052711 (2006).

17 H. Yao, W.F. Tsai and S.A. Kivelson, Phys. Rev. B 76, 161104(R) (2007).

18 A.M. Rey, R. Sensarma, S. Foelling, M. Greiner, E. Demler, and M.D. Lukin, [arXiv:0806.0166]

19 D.J. Scalapino, S.A. Trugman, Philos. Mag. B 74, 607 (1996); K. Le Hur and T. M. Rice, Annals of Physics 324, 1452 (2009)

20 A. Parameswanti, L. Balslev, and M.P.A. Fisher, Phys. Rev. B 66, 054526 (2002); O.I. Motrunich and M.P.A. Fisher, Phys. Rev. B 75, 235116 (2007);

21 D.N. Sheng, O.I. Motrunich, S. Trebst, E. Gull, and M.P.A. Fisher, Phys. Rev. B 78, 054520 (2008)

22 A. E. Feiguin and M. P. A. Fisher, Phys. Rev. Lett. 103, 025303 (2009).

23 O. Mandel et al., Nature (London) 425, 937 (2003); Phys. Rev. Lett. 91, 010407 (2003).

24 W. Vincent Liu, F. Wilczek and P. Zoller, Phys. Rev. A 70, 033603 (2004).

25 V.G. Rousseau, R.T. Scalettar, and G.G. Batrouni, Phys. Rev. B 72 054524 (2005); V.G. Rousseau, G.G. Batrouni and R.T. Scalettar, Phys. Rev. Lett. 93 110404 (2004).

26 R. Bendjama, B. Kumar, and F. Mila, Phys. Rev. Lett. 95, 110406 (2005).

27 G. Vidal, Phys. Rev. Lett. 91, 147902 (2003); Phys. Rev. Lett. 93, 040502 (2004).

28 M.W. J. Romans, R. A. Duine, Subir Sachdev, and H.T.C. Stoof, Phys. Rev. Lett. 93, 020405 (2004).

29 M. J. Bhaseen, A. O. Silver, M. Hohenadler, B. D. Simons, Phys. Rev. Lett. 103, 265302 (2009).

30 S. Diehl, M. Baranov, A. J. Daley, P. Zoller, arXiv:0910.1859 (2009).

31 C. Menotti and S. Stringari, arXiv: 0912.4452 (2009)