Universal point sets for planar three-trees*

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Abstract

For every \( n \in \mathbb{N} \), we present a set \( S_n \) of \( O(n^{3/2} \log n) \) points in the plane such that every planar 3-tree with \( n \) vertices has a straight-line embedding in the plane in which the vertices are mapped to a subset of \( S_n \). This is the first subquadratic upper bound on the size of universal point sets for planar 3-trees, as well as for the class of 2-trees and serial parallel graphs.

1 Introduction

Every planar graph has a straight-line embedding in the plane \([21]\) where the vertices are mapped to distinct points and the edges to pairwise noncrossing straight line segments between the corresponding vertices. A set \( S \subset \mathbb{R}^2 \) of points in the plane is called \( n \)-universal if every \( n \)-vertex planar graph has a straight-line embedding in \( \mathbb{R}^2 \) such that the vertices are mapped into a subset of \( S \). Similarly, \( S \subset \mathbb{R}^2 \) is \( n \)-universal for a family \( G \) of planar graphs if every \( n \)-vertex planar graph in \( G \) has a straight-line embedding in \( \mathbb{R}^2 \) such that the vertices are mapped into a subset of \( S \). It is a longstanding open problem to determine the minimum size \( f(n) \) of an \( n \)-universal point set for all \( n \in \mathbb{N} \). Our main result is that there is an \( n \)-universal point set of size \( O(n^{3/2} \log n) \) for the class of planar graphs of treewidth at most three.

Theorem 1 For every \( n \in \mathbb{N} \), there is an \( n \)-universal point set of size \( O(n^{3/2} \log n) \) for planar 3-trees.

A graph is called a \( k \)-tree, for some \( k \in \mathbb{N} \), if it can be constructed by the following iterative process: start with a \( k \)-vertex clique and successively add new vertices such that each new vertex has exactly \( k \) neighbors that form a clique in the current graph. For example, 1-trees are the same as trees; 2-trees are maximal series-parallel graphs, and include also all outerplanar graphs. In general, \( k \)-trees are the maximal graphs with treewidth \( k \). A planar 3-tree is a 3-tree that is planar. Theorem 1 is the first subquadratic upper bound on the size of \( n \)-universal point sets for planar 3-trees, for 2-trees, and for series-parallel graphs.

Related previous work. In a pivotal paper, de Fraysseix, Pach and Pollack \([13]\) showed that an \( n \)-universal set must have at least \( n + (1 - o(1)) \sqrt{n} \) points. Chrobak and Karloff \([11]\) improved the lower bound to \( 1.098n \) and later Kuowski \([26]\) to \((1.235 - o(1))n\). This is the currently known best lower bound for \( n \)-universal sets in general. De Fraysseix et al. \([13]\) and Schnyder \([29]\) independently showed that there are \( n \)-universal sets of size \( O(n^2) \). In fact, an \((n - 1) \times (n - 1)\) section of the integer lattice is \( n \)-universal \([10, 29]\) for every \( n \geq 3 \). Alternatively, an \( \frac{4}{3} n \times \frac{2}{3} n \) section of the integer lattice is also \( n \)-universal \([7]\). The quadratic upper bound is the best possible if the point set is restricted to sections of the integer lattice: Frati and Patrignani \([24]\) showed (based on earlier work by Dolev et al. \([15]\)) that if a rectangular section of the integer lattice is \( n \)-universal, then it must contain at least \( n^2 / 9 + \Omega(n) \) points.

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Grid drawings have been studied intensively due to their versatile applications. It is known that sections of the integer lattice with $o(n^2)$ points are $n$-universal for certain classes of graphs. For example, Di Battista and Frati [14] proved that an $O(n^{1.48})$ size integer grid is $n$-universal for outerplanar graphs. Frati [23] showed that 2-trees on $n$ vertices require a grid of size at least $\Omega(n^{2\sqrt{\log n}})$. Biedl [3] observed that the grid embedding of all $n$-vertex 2-trees requires an $\Omega(n) \times \Omega(n)$ section of the integer lattice if the combinatorial embedding (i.e., all vertex-edge and edge-face incidences) is given. On the other hand, Zhou et al. [30] showed recently that every $n$-vertex series-parallel graph, and thus, every 2-tree, has a straight-line embedding in a $\frac{2}{3}n \times \frac{2}{3}n$ section of the integer lattice and a section of the integer lattice of area $0.3941n^2$. Researchers have studied classes of planar graphs that admit $n$-universal point sets of size $o(n^2)$. A classical result in this direction, due to Gritzmann et al. [25] (see also [6]), is that every set of $n$ points in general position is $n$-universal for outerplanar graphs. Angelini et al. [1] generalized this result and showed that there exists an $n$-universal point set of size $O(n(\log n/\log \log n)^2)$ for so-called simply nested planar graphs. A planar graph is simply nested if it can be reduced to an outerplanar graph by successively deleting chordless cycles from the boundary of the outer face. Recently, Bannister et al. [2] found $n$-universal point sets of size $O(n \log n)$ for simply nested planar graphs, and $O(n \log n)$ for planar graphs of bounded pathwidth.

Organization. We briefly review some structural properties of planar 3-trees (Section 2), then construct a point set $S_n \subset \mathbb{R}^2$ for every $n \in \mathbb{N}$ (Section 3), and show that it is $n$-universal for planar 3-trees (Section 4).

2 Basic Properties of Planar Three-Trees

A graph $G$ is a planar 3-tree if it can be constructed by the following iterative procedure. Initially, let $G = K_3$, the complete graph with three vertices. Successively augment $G$ by adding one new vertex $u$ and three new edges that join $u$ to three vertices of a triangle such that no two vertices are connected to all the vertices of the same triangle. A planar 3-tree can be embedded in the plane simultaneously with the iterative process: the initial triangle forms the outer-face and each new vertex $u$ is inserted in the interior of the face corresponding to the triangle it is attached to.

The iterative augmentation process that produces a 3-tree $G$ can be represented by a rooted tree $T = T(G)$ as follows (this is called a face-representative tree in [22]). Refer to Fig. 1. The nodes of $T$ correspond to the triangles of $G$. For convenience we denote a vertex of $T$ by its corresponding triangle in $G$. The root of $T$ corresponds to the initial triangle of $G$. When $G$ is augmented by a new vertex $u$ connected to the vertices of the triangle $\Delta = v_1v_2v_3$, we attach three new leaves to $\Delta$ corresponding to the triangles $v_1v_2u$, $v_1uv_3$ and $uv_2v_3$. For a node $\Delta$ of $T$, let $T_\Delta$ denote the subtree of $T$ rooted at $\Delta$. Let $V_\Delta$ denote the set of vertices of $G$ embedded in the interior of $\Delta$. 

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In Section 4 we embed the vertices of a planar 3-tree on a point set by traversing the tree $T$ from the root. The initial triangle $abc$ will be the outer face in the embedding such that the edge $ab$ is a horizontal line segment, and the vertex $c$ is the top vertex (i.e., it has maximal $y$-coordinate). We then successively insert the remaining $n - 3$ vertices of $G$, each of which subdivides a triangular face into three triangles. We label the vertices of each triangle of $G$ arbitrarily as left, right and top, respectively. These labels are assigned (without knowing the specifics of our embedding algorithm) as follows. Label the three vertices of the initial triangle in $G$ arbitrarily as left, right and top, respectively. When $G$ is augmented by a new vertex $u$ and edges $uv_1, uv_2,$ and $uv_3,$ where $v_1$ is the left, $v_2$ is the right, and $v_3$ is the top vertex of an existing triangle $v_1v_2v_3,$ then let $v_1,$ $v_2,$ and $v_3$ keeps their labels left, right, top, respectively) in the new triangles $v_1v_2u,$ $v_2v_3u$ and $v_1v_3u;$ while vertex $u$ becomes the top vertex of $v_1v_2u,$ the left vertex of $v_2v_3u,$ and the right vertex of $v_1v_3u.$ The triangles $v_1v_2u, v_1v_3u$ and $uv_2v_3,$ respectively, will be called the bottom, left and right triangles within $v_1v_2v_3$. In the tree $T = T(G),$ the three children of a node corresponding to a vertex can be labeled as bottom, left, and right child, analogously.

**Weighted Nodes in $T(G)$.** Let $G$ be a planar 3-tree with $n$ vertices. Our embedding algorithm (in Section 4) is guided by the tree $T = T(G)$, which represents an incremental process that constructs $G$ from a single triangle. Recall that $T_\Delta$ denotes the subtree of $T$ rooted at a node $\Delta$; and $V_\Delta$ denotes the set of vertices of $G$ that correspond to nodes in $T_\Delta$. Let the weight of a node $\Delta$ of $T$ be $weight(\Delta) = |V_\Delta|$. The tree $T$ is a partition tree: for every node $\Delta$, $weight(\Delta)$ equals one plus the total weight of the children of $\Delta$.

Let $\alpha \in (0, 1]$ be a constant. A node $\Delta$ is heavy (resp., light) in $T$ if its weight is at least (resp., less than) $n^\alpha$. We designate some of the nodes in $T$ as hubs recursively in a top-down traversal of the tree $T$: Let the root of $T$ be a hub. Let a node $\Delta \in V_T$ be a hub if $n^\alpha \leq weight(\Delta) \leq weight(\Delta') - n^\alpha$, for every hub $\Delta'$ that is an ancestor of $\Delta$. We note a few immediate consequences of the definition.

**Lemma 2** If $\Delta_1, \Delta_2 \in V_T$ are heavy siblings, then they are both hubs.

**Proof.** Let $\Delta$ denote the common parent of $\Delta_1$ and $\Delta_2$. Then $weight(\Delta) > weight(\Delta_1) + weight(\Delta_2)$. Since both $\Delta_1$ and $\Delta_2$ are heavy, we have $weight(\Delta) > 2n^\alpha$. If $\Delta' = \Delta$ or $\Delta'$ is an ancestor of $\Delta$, we have $n^\alpha \leq weight(\Delta_i) \leq weight(\Delta') - n^\alpha$ for $i = 1, 2$. \qed

**Lemma 3** The tree $T(G)$ has at most $2n^{1-\alpha}$ hubs.

**Proof.** Let $T'$ be the subtree of $T$ induced by all heavy nodes. By definition, every hub of $T$ is in $T'$. Denote by $T''$ the tree obtained from $T'$ by adding a sibling leaf to every hub that is a single child in $T'$. Note that
every hub of $T$ is in $T''$, and its parent has at least two children in $T''$. The tree $T''$ has at most $n^{1-\alpha}$ leaves, since every leaf of $T''$ accounts for at least $n^\alpha$ vertices of $G$. Therefore, $T''$ has at most $n^{1-\alpha} - 1$ vertices with two or three children. Together with the root, there are at most $2n^{1-\alpha}$ hubs in $T$. □

3 Construction of a Point Set

Let $\alpha \in (0, 1]$ be a constant. In this section, we construct a point set $S_n$ of size $\Theta(n^{2-\alpha} \log n)$ for every $n \in \mathbb{N}$. In Section 4, we show that for $\alpha = 1/2$, the point set $S_n$ of size $\Theta(n^{3/2} \log n)$ is $n$-universal for planar 3-trees. Assume for the remainder of this section that $\alpha = 2q$ for some positive integer $q \in \mathbb{N}$; otherwise let $S_n = S_{n'}$ for $n' = 2^{\lceil \log_2 n \rceil}$.

The point set $S_n$ is constructed in two steps: we first choose a “sparse” set $B_n$ of points from a $14n \times 14n$ section of the integer lattice, and then “stretch” the points by the transformation $(x, y) \rightarrow (x, (28n)^y)$, as described below.

Sparse grid. Let $A_n = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i, j \leq 14n\}$ be an $14n \times 14n$ section of the integer lattice. Let $B_n \subset A_n$ be the set of points in $A_n$ with at least one of the following three properties (refer to Fig. 2):

- $(i, j)$ such that $n^\alpha | ij$;
- $(i + k, j + k)$ such that $n^\alpha | i, n^\alpha | j$, and $k \in \{1, 2, \ldots, n^\alpha\}$ (forward diagonals);
- $(i + k, j - k)$ such that $n^\alpha | i, n^\alpha | j$, and $k \in \{1, 2, \ldots, n^\alpha\}$ (backward diagonals).

Note that for every $0 \leq i \leq 14n$, if $n^\alpha | i$, then all points $(i, j) \in A_n$ are in $B_n$. We say that these points form a full row. Similarly, for every $0 \leq j \leq 14n$, if $n^\alpha | j$, then all points $(i, j) \in A_n$ are in $B_n$, forming a full column. The points $(i, j) \in A_n$, with $n^\alpha | i$ and $n^\alpha | j$ lie at the intersection points of full rows and full columns.

![Figure 2: (a) A pattern of points $(i, j) \in \mathbb{Z}^2$ with $8|ij$. (b) A pattern of forward and backward diagonals.](image)

Stretched grid. We deform the plane by the following transformation.

$$\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \rightarrow (x, (28n)^y).$$

For an integer point $(i, j) \in \mathbb{Z}^2$, we use the shorthand notation $\tau(i, j) = \tau((i, j))$. If $A \subset \mathbb{R}^2$ is a rectangular section of the integer lattice (a grid), then we call the point set $\tau(A) = \{\tau(p) : p \in A\}$ a stretched grid. Note that $\tau$ translates every point vertically, and it translates points of the same $y$-coordinate by the same vector.
Universal point set for 3-trees. We are now in a position to define \( S_n \). Let \( S_n = \tau(B_n) \).

Similarly to [8], our illustrations show the “unstretched” point set \( B_n = \tau^{-1}(S_n) \) instead of \( S_n \). The transformation \( \tau^{-1} \) maps line segments between points in \( S_n \) to Jordan arcs between grid points in \( B_n \). In our figures, line segments are drawn as Jordan arcs that correctly represent the above-below relationship between segments and points (Fig. 3).

### 3.1 Properties of Sparse Grids

We first show that \( B_n \) contains \( O(n^{2-\alpha} \log n) \) points.

**Lemma 4** For every \( \alpha \in (0, 1] \), the sparse grid \( B_n \) contains \( O(n^{2-\alpha} \log n) \) points.

**Proof.** It is enough to consider the case that \( n^\alpha = 2^q \) for some positive integer \( q \in \mathbb{N} \). We count first the points \( (i, j) \in A_n \) such that \( 2^q | ij \). The grid \( A_n \) has \( 14n + 1 \) rows and \( 14n + 1 \) columns. If \( (i, j) \in A \), then \( 2^q | ij \), hence \( 2^k | i \) and \( 2^q-k | j \) for some \( k = 0, 1, \ldots, q \). There are exactly \( 14n/2^k + 1 \) values \( j \), \( 0 \leq j \leq 14n \), with \( 2^q-k | j \), and so the number of pairs \( (i, j) \in A_n \) with \( 2^k | i \) and \( 2^q-k | j \) is \( (14n/2^k + 1)(14n/2^{q-k} + 1) \). Therefore, the total number of points \( (i, j) \in A_n \) is bounded above by

\[
\sum_{k=0}^{q} \left( \frac{14n}{2^k} + 1 \right) \left( \frac{14n}{2^{q-k}} + 1 \right) \leq q \cdot \frac{(14n)^2}{2^q} + 2 \cdot 2 \cdot 14n + 1 = O(n^{2-\alpha} \log n).
\]

Consider now the points of the forward and backward diagonals. Every \( n^\alpha \)-th row and every \( n^\alpha \)-th column is full, and so \( B_n \) contains \( (14n^{1-\alpha} + 1) \cdot (14n^{1-\alpha} + 1) = O(n^{2-\alpha}) \) points lying at a full column and a full row. Each such point column generates at most \( n^\alpha - 1 \) points in a forward diagonal and \( n^\alpha - 1 \) points in a backward diagonal. The total number of these points is \( O(n^{2-\alpha}) \). \( \square \)

The convex hull of \( A_n \), denoted \( \text{conv}(A_n) \), is a closed square of side length \( 14n \). For an axis-aligned (open) rectangle \( R = (x_1, x_2) \times (y_1, y_2) \), we introduce the following parameters:

- the **width** of \( R \) is \( w(R) = x_2 - x_1 \);
- the **height** of \( R \) is \( h(R) = y_2 - y_1 \);
- the **area** of \( R \) is \( \text{area}(R) = (x_2 - x_1)(y_2 - y_1) \).

For example, if \( R_0 = \text{int}(\text{conv}(A_n)) \), then \( w(R_0) = 14n, h(R_0) = 14n, \) and \( \text{area}(R_0) = (14n)^2 \).

Note also that the sparse grid contains at least one point in every sufficiently large axis-aligned rectangle.

**Lemma 5** Let \( R \subseteq \text{conv}(A_n) \) be an open axis-aligned rectangle such that \( w(R) > 1, h(R) > 1 \) and \( \text{area}(R) \geq 4 \cdot n^\alpha \). Then \( B_n \cap R \neq \emptyset \).

**Proof.** Let \( k \in \mathbb{N} \) be the largest integer such that \( 2^k < w(R) \); and \( \ell \in \mathbb{N} \) be the largest integer such that \( 2^\ell < h(R) \). Then \( 2^k \geq \frac{w(R)}{2^\ell} \), \( 2^\ell \geq \frac{h(R)}{2^k} \), and so \( 2^{k+\ell} \geq \frac{\text{area}(R)}{4} \geq n^\alpha = 2^q \). That is, we have \( k + \ell \geq q \). Now \( R \) intersects a vertical line \( \ell_x : x = i \) such that \( 2^k | i \); and it also intersect a horizontal line \( \ell_y : y = j \) such that \( 2^\ell | j \). The point \( (i, j) \in A_n \) is in \( R \), and \( 2^{k+\ell} | ij \), as required. \( \square \)

**Lemma 6** Let \( R \subseteq \text{conv}(A_n) \) be an open axis-aligned rectangle such that \( \{w(R) > 1 \text{ and } h(R) > n^\alpha\} \) or \( \{w(R) > n^\alpha \text{ and } h(R) > 1\} \). Then there is a point \( (i, j) \in B_n \cap R \) on a forward and a backward diagonal.

**Proof.** On any vertical line \( \ell : x = i \) (resp., horizontal line \( \ell : y = i \)), \( 0 \leq i \leq 14n \), the distance between two consecutive points on forward diagonals is \( n^\alpha \). If \( h(R) \geq n^\alpha \), then \( R \) contains a point of a forward diagonal on any line \( \ell : x = i \) that intersects \( R \). Similarly, if \( w(R) \geq n^\alpha \), then \( R \) contains a point of a forward diagonal on any line \( \ell : y = i \) that intersects \( R \). \( \square \)
In our embedding algorithms (in Section 4), we translate some points $p \in B_n$ vertically or horizontally by $n^\alpha$. We note here that the translated image of $p \in B_n$ is either in $B_n$ or outside of the bounding box of $B_n$.

**Lemma 7** Let $p \in B_n$, and translate $p$ by a horizontal or vertical vector of length $n^\alpha$ to another point $p'$. If $p' \in \text{conv}(B_n)$, then $p' \in B_n$.

**Proof.** Assume first that $p = (i, j) \in A_n$ such that $n^\alpha|i,j$. It is clear that $n^\alpha|i \pm n^\alpha|j$ and $n^\alpha|i(j \pm n^\alpha)$. Assume now that $p = (i + k, j + k) \in A_n$ such that $n^\alpha|i$ and $n^\alpha|j$. Then $n^\alpha|i \pm n^\alpha$ and $n^\alpha|j \pm n^\alpha$. In both cases, if $p'$ is still within $\text{conv}(B_n)$, then it is also in $B_n$. □

### 3.2 Properties of Stretched Grids

The purpose of transformation $\tau$ is to establish the following property for the stretched grid $\tau(A_n)$.

**Lemma 8** Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in A_n$ such that $(a_2, b_2)$ lies in the interior of the axis-aligned rectangle spanned by $(a_1, b_1)$ and $(a_3, b_3)$ (formally, $a_1 < a_2 < a_3$ and either $b_1 < b_2 < b_3$ or $b_3 < b_2 < b_1$). Then $\tau(a_2, b_2)$ lies below the line segment between $\tau(a_1, b_1)$ and $\tau(a_3, b_3)$. (See Fig. 3)

**Proof.** We may assume that $b_1 < b_2 < b_3$, since the other case can be treated analogously. Denote by $L$ the line through $\tau(a_1, b_1)$ and $\tau(a_3, b_3)$. Consider the following function in two variables:

$$D(x, y) = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_3 & x \\ c_1 & c_3 & y \end{vmatrix} = a_3 y - c_3 x - a_1 y + c_1 x + a_1 c_3 - c_1 a_3.$$  

The function $D(x, y)$ is negative for all the points below $L$ and positive for all the points above $L$. For $\tau(a_2, b_2)$, we have $D(a_2, c_2) = c_3(a_1 - a_2) + a_2 c_1 + c_2(a_3 - a_1) - c_1 a_3 < c_3(a_1 - a_2) + a_2 c_1 + c_2 a_3 < -c_3 + 14nc_1 + 14nc_2 < -(28n)^b_3 + 14n(28n)^b_3 - 1 + 14n(28n)^b_3 - 1 = 0$. Hence $\tau(a_2, b_2)$ is below the line $L$, as required. □

![Figure 3: A grid and three points $p_1 = (a_1, b_1), p_2 = (a_2, b_2)$ and $p_3 = (a_3, b_3)$ with $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$. The Jordan arcs between the points represent straight-line segments between the stretched points $\tau(a_1, b_1), \tau(a_2, b_2)$ and $\tau(a_3, b_3)$. The rectangle $\square(\Delta)$ defined for the triangle for $\Delta = \Delta(p_1, p_2, p_3)$ is shaded.](image)

For each triangle $\Delta$ determined by $(a_1, b_1), (a_2, b_2)$, and $(a_3, b_3)$, we define an open axis-aligned rectangle $\square(\Delta)$. Assume, by permuting the indices if necessary, that $\max(b_1, b_2) < b_3$, and let

$$\square(\Delta) = (\min(a_1, a_2), \max(a_1, a_2)) \times (\max(b_1, b_2), b_3).$$
Lemma 9 Let \((a_1, b_1), (a_2, b_2), (a_3, b_3) \in A_n\) such that \(\max(b_1, b_2) < b_3\). Then for all \(p \in A_n \cap \Box(\Delta)\), the point \(\tau(p)\) lies in the interior of the triangle determined by \(\tau(p_1), \tau(p_2), \text{and} \tau(p_3)\).

Proof. By Lemma 8, all points \(\tau(a_2, b_2)\) lie above the line spanned by \(\tau(a_1, b_1)\) and \(\tau(a_2, b_2)\). Consider now the position of \(a_3\) relative to \(a_1\) and \(a_2\). Assume, without loss of generality, that \(a_1 < a_2\) if \(a_1 = a_2\), then \(\Box(\Delta) = \emptyset\), and the conclusion trivially holds.

If \(a_3 = a_1\), then \(\tau(a_1, b_1)\) and \(\tau(a_3, b_3)\) span a vertical line, and the point \(\tau(a_2, b_2)\) and all points \(\tau(p_1)\), \(p \in A_n \cap \Box(\Delta)\), lie to the right of this line. If \(a_3 < a_1\) (resp., \(a_3 > a_1\)), then \(\tau(a_2, b_2)\) and all points \(\tau(p_1)\), \(p \in A_n \cap \Box(\Delta)\), lie above (resp., below) the line spanned by \(\tau(a_1, b_1)\) and \(\tau(a_3, b_3)\) by Lemma 8.

Similarly, if \(a_3 = a_2\), then \(\tau(a_2, b_2)\) and \(\tau(a_3, b_3)\) span a vertical line, and the point \(\tau(a_1, b_1)\) and all points \(\tau(p_1)\), \(p \in A_n \cap \Box(\Delta)\), lie on the left of this line. If \(a_3 < a_2\) (resp., \(a_3 > a_2\)), then \(\tau(a_1, b_1)\) and all points \(\tau(p_1)\), \(p \in A_n \cap \Box(\Delta)\), lie below (resp., above) the line spanned by \(\tau(a_2, b_2)\) and \(\tau(a_3, b_3)\) by Lemma 8. In all cases, all points \(\tau(p_1)\), \(p \in A_n \cap \Box(\Delta)\), lie in the interior of \(\Delta\). $\Box$

Grid-embedding in a stretched grid. The grid-embedding algorithm by de Fraysseix et al. [13] embeds every \(n\)-vertex planar graph on an \((2n - 4) \times (n - 2)\) section of the integer lattice. Their algorithm also works on the stretched grid in place of the integer grid.

Specifically, we use their result in the following form. Suppose that \(G_m\) is a planar graph with \(m \in \mathbb{N}\) vertices and endowed with a given combinatorial embedding in which \(u, v\) and \(z\) are the vertices of the outer face. Let \(X, Y \subseteq \mathbb{N}\) be two sets of cardinality \(|X| \geq 2m\) and \(|Y| \geq m\). Then \(G\) has a straight-line embedding such that the vertices are mapped to the stretched cross product \(\tau(X \times Y)\) of size at least \(2m^2\); the two endpoints of edge \(uv\) are mapped to \(\tau(\min X, \min Y)\) and \(\tau(\max X, \min Y)\), respectively; and \(z\) is mapped to an arbitrary point in the top row \(\tau(X \times \max Y)\). Furthermore, By Lemma 8, we can shift \(u\) or \(v\) vertically down to another point of the stretched grid (while keeping all other vertices fixed) without introducing any edge crossings. Similarly, we can shift \(z\) horizontally to any other point of the stretched grid without introducing any edge crossings.

Lemma 10 Let \(G_m\) be a planar 3-tree with \(m \in \mathbb{N}\) vertices and a combinatorial embedding in which \(u, v\) and \(z\) are the vertices of the outer face. Let \(p_1, p_2, p_3 \in B_n\), and let \(R \subseteq \Box(\Delta(p_1, p_2, p_3))\) be a rectangle such that \(\text{area}(R) > 8m^2n^a, w(R) > 2m,\) and \(h(R) > m\). Then \(G_m\) admits a planar straight-line embedding such that \(u, v,\) and \(z\) are mapped to \(p_1, p_2,\) and \(p_3,\) respectively, and the interior vertices of \(G_m\) are mapped to points \(\tau(p), p \in B_n \cap R\).

Proof. It is enough to show that \(B_n \cap R\) contains a cross product \(X \times Y\) such that \(|X| \geq 2m\) and \(|Y| \geq m\). Recall that \(n^a = 2^l\) for some \(q \in \mathbb{N}\). By decreasing the width and height of \(R\), we obtain a rectangle \(R' \subseteq R\) with width \(w(R') > 2^k(2m)\), height \(h(R') > 2^q-km\), and \(\text{area}(R') > 2^q(2m^2)\), for some integer \(k, 0 \leq k \leq q\). The points \((i, j) \in R'\), with \(i \equiv 0 \mod 2^k\) and \(j \equiv 0 \mod 2^q-k\), are in \(B_n \cap \text{int}(R)\) and form a required cross product \(X \times Y\) such that \(|X| \geq 2m\) and \(|Y| \geq m\).

We can now embed \(u, v,\) and \(z\) at \(p_1, p_2,\) and \(p_3,\) respectively. The algorithm by de Fraysseix et al. [13] embeds the interior vertices of \(G_m\) to points \(\tau(p), p \in X \times Y \subseteq B_n \cap R,\) as required. $\Box$

4 Embedding Algorithm

In this section, we show that for \(\alpha = 1/2\), every \(n\)-vertex planar 3-tree admits a straight-line embedding such that the vertices are mapped to the set \(S_n\) of size \(\Theta(n^{3/2} \log n)\).
Overview. Let $G$ be a planar 3-tree with $n$ vertices. We describe our embedding algorithm in term of the “unstretched” grid $B_n$. The function $\tau$ maps this embedding into a straight-line embedding into $S_n$. Our embedding algorithm is guided by the tree $T = T(G)$, which represents an incremental process that constructs $G$ from a single triangle. Recall that $T_\Delta$ denotes the subtree of $T$ rooted at a node $\Delta$; and $V_\Delta$ denotes the set of vertices of $G$ that correspond to nodes in $T_\Delta$.

Our algorithm processes the nodes $\Delta \in V(T)$ in a breath-first traversal of $T$. When a triangle $\Delta$ is already embedded in the point set $S_n$, then the rectangle $\square(\Delta)$ is well defined, the vertices $V_\Delta$ are mapped to points in $B_n \cap \square(\Delta)$. In order to maintain additional properties (invariant $I_2$ below), we also maintain an open rectangle $R(\Delta) \subseteq \square(\Delta)$, and require that the vertices $V_\Delta$ be mapped into $B_n \cap R(\Delta)$. Intuitively, $R(\Delta)$ is the region "allocated" for the vertices in $V_\Delta$.

When the breath-first traversal of $T$ reaches a node $\Delta \in V(T)$ such that $\text{area}(R(\Delta)) > 8n^\alpha \text{weight}^2(\Delta)$, $w(R(\Delta)) > 2\text{weight}(\Delta)$, and $h(R(\Delta)) > \text{weight}(\Delta)$, then we complete the embedding of the vertices $V_\Delta$ by Lemma 10. We call the set of nodes of $T$ where these conditions are first satisfied the fringe of $T$. We show below that Lemma 10 becomes applicable by the time weight($\Delta$) drops below $n^{1-\alpha}$. For nodes $\Delta \in V(T)$ below the fringe, there is no need to assign rectangles $R(\Delta)$.

When we process a node $\Delta \in V(T)$ that is a parent of a hub, we shift some of the previously embedded vertices in horizontal or vertical direction (as described below), and shift the corners of rectangles $R(\Delta')$ corresponding to previously processed nodes $\Delta' \in V(T)$, as well. Each shift operation changes the $x$- or $y$-coordinate of a point (with respect to the “unstretched” point set $B_n$) by 0, $n^\alpha$, or $2n^\alpha$. The number of hubs is at most $2n^{1-\alpha}$ by Lemma 3, so the $x$- and $y$-coordinate of each vertex may be shifted by at most $2n^{1-\alpha} \cdot 2n^\alpha = 4n$. To allow sufficient space for these operations, we initially start with a $10n \times 10n$ section of the sparse grid $B_n$, and the shift operations may expand the bounding box to up to $14n \times 14n$.

Invariants. For all nodes $\Delta \in V(T)$ on or above the fringe of $T$, we maintain the following invariants.

$I_1$ $R(\Delta) \subseteq \square(\Delta)$.

$I_2$ If weight$(\Delta) \geq n^\alpha$, then the lower-left corner of $R(\Delta)$ is in a forward diagonal, and the lower-right corner of $R(\Delta)$ is in a backward diagonal of $B_n$.

$I_3$ At least one of the following two conditions is satisfied:

\[ \text{area}(R(\Delta)) \geq 100n \text{weight}(\Delta); \]
\[ \text{area}(R(\Delta)) > 8n^\alpha \text{weight}^2(\Delta), w(R(\Delta)) > 2\text{weight}(\Delta), \text{ and } h(R(\Delta)) > \text{weight}(\Delta). \]

Condition 2 implies that $\Delta$ is on the fringe, and the vertices $V_\Delta$ can be embedded by Lemma 10.
Initialization. Denote by $abc$ the initial triangle of $G$, with $a$ labeled left, $b$ labeled right and $c$ labeled top. Then we have $T = T_{abc}$. Let $R(abc)$ be the interior of bounding box of a $10n \times 10n$ section of $B_n$. Embed $a$ and $b$ to the lower-left and lower-right corners of $R(abc)$, respectively. Embed $c$ in the upper-right corner of $R(abc)$ (see Fig. 4). It is clear that invariants $I_1$–$I_3$ are satisfied for $abc$.

One recursive step. Assume that the vertices of triangle $\Delta \in V(T)$ have already been embedded and we are given a rectangle $R(\Delta)$ satisfying invariants $I_1$–$I_3$. If $\Delta$ is on the fringe of $T$, then the embedding of the vertices $V_\Delta$ is completed by Lemma 10 and the subtree $T_\Delta$ is removed from further consideration.

In the remainder of this section, we assume that node $\Delta$ is strictly above the fringe, where invariant $I_3$ is satisfied with (1) rather than (2). Note that $w(R(\Delta)) \leq 14n$ and $h(R(\Delta)) \leq 14n$ since the bounding box of $B_n$ is a $14n \times 14n$ square. These upper bounds, combined with (1), yield the following lower bounds for heavy nodes:

$$w(R(\Delta)) > 7\text{weight}(\Delta) \quad \text{and} \quad h(R(\Delta)) > 7\text{weight}(\Delta).$$

(3)

Since $\Delta$ does not satisfy (2) despite (3), we have $\text{area}(R(\Delta)) \leq 8n^2\text{weight}^2(\Delta)$. This, combined with (1), yields

$$\text{weight}(\Delta) \geq 12.5n^{1-\alpha}.$$  

(4)

Ideal location for a vertex. Denote the bottom, left and right child of $\Delta$, respectively, by $\Delta_1$, $\Delta_2$ and $\Delta_3$. Suppose that $R(\Delta) = (a, b) \times (c, d)$. We wish to place the vertex $v$ corresponding to $\Delta$ at some point $p \in B_n \cap R(\Delta)$. The point $p \in B_n \cap R(\Delta)$ subdivides $R(\Delta)$ into a bottom, left, and right rectangle, $\Delta_1 \cap R(\Delta)$, $\Delta_2 \cap R(\Delta)$, and $\Delta_3 \cap R(\Delta)$ corresponding to $\Delta_1$, $\Delta_2$, and $\Delta_3$. We choose an “ideal” location for $v$ (which may not be a point in $B_n$) that would ensure that the area of $R(\Delta)$ is distributed among the rectangles $R(\Delta_i)$, $i = 1, 2, 3$, proportionally to weight($\Delta_i$), maintaining (1). Refer to Fig. 5a. Recall that

$$\text{weight}(\Delta) = \text{weight}(\Delta_1) + \text{weight}(\Delta_2) + \text{weight}(\Delta_3) + 1.$$ 

Partition the area of $R(\Delta)$ into a top and a bottom part by a horizontal line $\ell_y$ in ratio

$$\left(\text{weight}(\Delta_2) + \text{weight}(\Delta_3) + \frac{2}{3}\right) : \left(\text{weight}(\Delta_1) + \frac{1}{3}\right).$$

Partition the top area of $R(\Delta)$ into a left and a right part by a vertical line $\ell_x$ in ratio

$$\left(\text{weight}(\Delta_2) + \frac{1}{3}\right) : \left(\text{weight}(\Delta_3) + \frac{1}{3}\right).$$

The ideal location for vertex $v$ is the intersection point $\ell_x \cap \ell_y$. Note that the ideal location is in the interior of $R(\Delta)$ even if weight($\Delta_1$), weight($\Delta_3$), or weight($\Delta_3$) is 0. If we place vertex $v$ at $\ell_x \cap \ell_y$, then $R(\Delta)$ is partitioned among the rectangles $R(\Delta_i)$, $i = 1, 2, 3$, such that

$$\text{area}(R(\Delta_i)) \geq \text{area}(R(\Delta))\frac{\text{weight}(\Delta_i)}{\text{weight}(\Delta)}.$$  

(5)

It is clear that if $\Delta$ satisfies (2) and its children satisfy (5) for $i = 1, 2, 3$, then the children also satisfy (2).

Unfortunately, the ideal location is not necessarily in $B_n$ (and often not in $A_n$). Inevitably, some of the constraints have to be relaxed. We distinguish several cases: When one of the children of $\Delta$ is a hub, we snap vertex $v$ to a point in $B_n$ near the ideal location, and shift some of the previously embedded vertices (as explained below) to restore the lower bounds in (5). When none of the children of $\Delta$ is a hub, then two or three of its children are light by Lemma 2, we maintain invariant $I_3$ by establishing either (1) or (2) for every child of $\Delta$. 

Case 1: one of the children of \( \Delta \) is a hub. We wish to embed the vertex \( v \) corresponding to \( \Delta \) so that invariant \( I_2 \) is maintained for all children of \( \Delta \) (at least one of them is a hub). We will shift the previously embedded vertices and rectangles by 0, \( n^\alpha \) or \( 2n^\alpha \), and embed the vertex \( v \) corresponding to \( \Delta \) into a point “near” its ideal location.

\[
\ell_x \\
\ell_y
\]

\[
\text{weight}(\Delta_1) + \frac{1}{3} \quad \text{weight}(\Delta_2) + \frac{1}{3} \\
\text{weight}(\Delta_3) + \frac{1}{3}
\]

\( R(\Delta) \)

(a)

Let \( \ell_x \) and \( \ell_y \) be the vertical and horizontal lines that define the ideal location for \( v \) (Fig. 5b). By definition, both intersect the interior of rectangle \( R(\Delta) \). For all previously embedded vertices and all corners of previously defined rectangles \( R(\cdot) \), if they lie on or to the right of line \( \ell_x \), then increase their \( x \)-coordinates by \( 2n^\alpha \); if they lie on or above the line \( \ell_y \), then increase their \( y \)-coordinates by \( n^\alpha \). Note that the width (resp., height) of all previously defined rectangles increases by 0 or \( 2n^\alpha \) (resp., 0 or \( n^\alpha \)). In particular, \( R(\Delta) \) is expanded to a rectangle \( R'(\Delta) \) whose width and height are \( \text{w}(R(\Delta)) + 2n^\alpha \) and \( \text{h}(R(\Delta)) + n^\alpha \), respectively. Let \( \ell_y' \) be a horizontal line at distance \( n^\alpha \) above \( \ell_y \), and let \( \ell_x' \) and \( \ell_x'' \) be two vertical lines at distance \( n^\alpha \) and \( 2n^\alpha \) to the right of \( \ell_x \) (Fig. 5c).

We now embed vertex \( v \) at a point \( B_n \cap R'(\Delta) \), and define rectangles \( R(\Delta_i) \) for \( i = 1, 2, 3 \). We shall choose the rectangles \( R(\Delta_i), i = 1, 2, 3 \), such that their widths and heights are at least as large as if \( v \) were placed at the ideal location in \( R(\Delta) \). In addition, we also ensure that the lower left (resp., lower right) corner of \( R(\Delta_i), i = 1, 2, 3 \), lies on a forward diagonal (resp., backward diagonal), thereby establishing invariant \( I_2 \).
for all children of $\Delta$.  

We define $R(\Delta_{i})$, $i = 1, 2, 3$, as follows (refer to Fig. 6a). The bottom child of $\Delta$ is $\Delta_{1}$. Let the bottom side of $R(\Delta_{1})$ be the bottom side of $R'(\Delta)$, and let its top side be the unique segment between $\ell_y$ and $\ell_y'$ such that the height of $R(\Delta_{1})$ is a multiple of $n^\alpha$. Since the lower left (right) corner of both $R(\Delta)$ and $R'(\Delta)$ are on a forward (backward) diagonal by invariant $I_2$, this is also true for $R(\Delta_{1})$. Since the height of $R(\Delta_{1})$ is a multiple of $n^\alpha$, the upper left (right) corner of $R(\Delta_{1})$ is also on a forward (backward) diagonal.

Let the lower left corner of $R(\Delta_{2})$ be the upper left corner of $R(\Delta_{1})$, and choose $x$-coordinate of the lower right corner of $R(\Delta_{2})$ between lines $\ell_x$ and $\ell_x'$ such that it is on a backward diagonal. Similarly, let the lower right corner of $R(\Delta_{3})$ be the upper right corner of $R(\Delta_{1})$, and choose the $x$-coordinate of its lower left corner between lines $\ell_x'$ and $\ell_x''$ such that it is on a forward diagonal. Let top side of both $R(\Delta_{2})$ and $R(\Delta_{3})$ be part of the top side of $R'(\Delta)$. Finally, embed vertex $v$ at the lower right corner of $R(\Delta_{2})$.

This ensures $R(\Delta) \subseteq \square(\Delta_{i}) \cap R'(\Delta)$ for $i = 1, 2, 3$, maintaining invariant $I_1$. Note that the width and height of the rectangle $R(\Delta_{i})$ are at least as large as if $v$ were placed at the ideal location within in $R(\Delta)$, establishing (3) hence (1) for $i = 1, 2, 3$. Invariants $I_1$–$I_3$ are maintained for $\Delta_1$, $\Delta_2$, and $\Delta_3$.

**Preliminaries for Cases 2 and 3.** In the remaining cases, the children of $\Delta$ are not hubs. By Lemma 2, $\Delta$ has at most one heavy child. For the one possible heavy child $\Delta_{i}$, $i \in \{1, 2, 3\}$, we shall choose a rectangle $R(\Delta_{i})$ satisfying (1); and we establish (2) for two light children of $\Delta$.

**Case 2: the children of $\Delta$ are not hubs, and the left and right children are light.** In this case, we use the following strategy (refer to Fig. 6a). We choose pairwise disjoint “preliminary” rectangles $R_0(\Delta_{1})$, $R_0(\Delta_{2})$, and $R_0(\Delta_{3})$ in $R(\Delta)$ such that $R_0(\Delta_{1})$ satisfies (1); and $R_0(\Delta_{2})$ and $R_0(\Delta_{3})$ satisfy (2). We also choose a rectangular region $Q \subset R(\Delta)$ such that placing $v$ at any point in $Q$ yields $R_0(\Delta_{i}) \subset \square(\Delta_{i})$, for $i = 1, 2, 3$. Finally, we show that $Q$ contains a point from a full column. We place $v$ at an arbitrary point in $Q \cap B_n$, and put $R(\Delta_{i}) = \square(\Delta_{i}) \cap R(\Delta)$.

Let $\delta_1 = \text{weight}(\Delta_{2}) + \text{weight}(\Delta_{3}) + 2/3$. Recall that the horizontal line $\ell_y$ partitions the area of $R(\Delta)$ into a top and a bottom part in ratio $\delta_1 : (\text{weight}(\Delta_{1}) + 1/3)$. Decompose the top part of $R(\Delta)$ into 6 congruent rectangles by a horizontal line and two vertical lines. Now, let $R_0(\Delta_{2})$ and $R_0(\Delta_{3})$ be the upper left and upper right congruent rectangles, respectively, and let $Q$ be the lower middle rectangle (see Fig. 6a).

Figure 6: (a) When the left and right children are light (Case 2), vertex $v$ is placed on a full column. (b) When the left or right child is heavy (Case 3), vertex $v$ is placed on a forward or backward diagonal.

The height $h(Q) = h(R_0(\Delta_{2})) = h(R_0(\Delta_{3}))$ of the congruent rectangles is bounded by

$$h(Q) = \frac{h(R(\Delta))}{2} \cdot \frac{\delta_1}{\text{weight}(\Delta)} \geq \frac{\text{weight}(\Delta)}{2} \cdot \frac{\delta_1}{\text{weight}(\Delta)} = \frac{7\delta_1}{2} \geq \frac{7(2/3)}{2} = \frac{7}{3}.$$
Their width is \( w(Q) = w(R_0(\Delta_2)) = w(R_0(\Delta_3)) = \frac{1}{3} w(R(\Delta)) > 29n^{1-\alpha} \), using (3) and (4). Hence \( w(Q) \geq n^\alpha \) when \( \alpha \leq 1/2 \), and so \( Q \) contains a point on a full column of \( B_n \).

We place \( v \) at an arbitrary point in \( Q \cap B_n \), and put \( R(\Delta_i) = \Box(\Delta_i) \cap R(\Delta) \) for \( i = 1, 2, 3 \). The lower left (resp., lower right) corner of \( R(\Delta_i) \) is the same as that of \( R(\Delta) \), which establishes invariant \( I_2 \) for \( \Delta_1 \). Note that \( \Delta_1 \) satisfies (1), since \( \text{area}(R(\Delta(1))) \geq \text{area}(R_0(\Delta_1))) \). We show that \( \Delta_2 \) and \( \Delta_3 \) satisfy (2). For \( i = 2, 3 \), we have \( w(R(\Delta_i)) \geq w(R_0(\Delta_i)) \geq 29n^{1-\alpha} \geq n^\alpha \geq \text{weight}(\Delta_i) \); \( h(R(\Delta_i)) \geq h(R_0(\Delta_i)) \geq \frac{7}{3} \delta_1 \geq \text{weight}(\Delta_i) \); and \( \text{area}(R(\Delta_i)) \) is bounded by

\[
\text{area}(R(\Delta_i)) = w(R(\Delta_i)) h(R(\Delta_i)) \geq \frac{w(R(\Delta))}{3} \cdot \frac{h(R(\Delta)) \delta_1}{\text{weight}(\Delta)} + 1/3 \geq \frac{\text{area}(R(\Delta)) \delta_1}{3 \text{weight}(\Delta)} \\
\geq \frac{100n \text{weight}(\Delta) \delta_1}{3 \text{weight}(\Delta)} > 33n \delta_1 > 33n \text{weight}(\Delta_i) \geq 33n^{1-\alpha} \text{weight}^2(\Delta_i),
\]

which is more than \( n^\alpha \text{weight}^2(\Delta_i) \) when \( \alpha \leq 1/2 \).

**Case 3: the children of \( \Delta \) are not hubs, and the left or right child of \( \Delta \) is heavy**  Assume that \( \Delta_2 \) is heavy (the case that \( \Delta_3 \) is heavy is treated analogously). Let \( \delta_2 = \text{weight}(\Delta_1) + \text{weight}(\Delta_3) + 1 \). We distinguish between two possibilities.

**Case 3A:** \( 4\delta_2 n^\alpha \leq w(R(\Delta)) \) and \( 4\delta_2 n^\alpha \leq h(R(\Delta)) \)  Refer to Fig. 6b. Place vertex \( v \) corresponding to \( \Delta \) at \((c+2\delta_2, b-2\delta_2)\) on a forward diagonal, and assign the rectangles \( R(\Delta_1) = (a, b) \times (c, c+2\delta_2) \), \( R(\Delta_2) = (a+2\delta_2, b-2\delta_2) \times (c+2\delta_2, d) \); and \( R(\Delta_3) = (d-2\delta_2, d) \times (c+2\delta_2, d) \). Note that the lower-left (resp., lower-right) corner of \( R(\Delta_2) \) is on a forward (resp., backward) diagonal of \( B_n \), establishing invariant \( I_3 \).

By construction, rectangles \( R(\Delta_i), i = 1, 2, 3 \), satisfy invariants \( I_1 \) and \( I_2 \). We establish (1) for \( \Delta_2 \):

\[
\text{area}(R(\Delta_2)) = [w(R(\Delta)) - 4\delta_2] \cdot [h(R(\Delta)) - 2\delta_2] \geq \text{area}(R(\Delta)) - [2w(\Delta) + 4h(\Delta)] \delta_2 \\
\geq 100n \text{weight}(\Delta) - 6 \cdot 14 \delta_2 \geq 100n (\text{weight}(\Delta) - \delta_2) = 100n \text{weight}(\Delta_2).
\]

For \( \Delta_1 \) and \( \Delta_3 \), we establish (2). We have \( w(R(\Delta_1)) = w(R(\Delta)) \geq 4\delta_2 n^\alpha \geq 4 \text{weight}(\Delta_1) n^\alpha \) and \( h(R(\Delta_3)) = h(R(\Delta)) - 2\delta_2 \geq (4\delta_2 - 2\delta_2 / n) n^\alpha \geq (4 \text{weight}(\Delta_3) - 2) n^\alpha \). On the other hand, \( h(R(\Delta_1)) = w(R(\Delta_3)) = 2\delta_2 \geq 2 \text{weight}(\Delta_1) + 2, 2 \text{weight}(\Delta_3) + 2 \) by construction.

**Case 3B:** \( 4\delta_2 n^\alpha > w(R(\Delta)) \) or \( 4\delta_2 n^\alpha > h(R(\Delta)) \)  In this case, we follow a strategy similar to Case 2 (refer to Fig. 7b). We choose pairwise disjoint “preliminary” rectangles \( R_0(\Delta_1), R_0(\Delta_2), \) and \( R_0(\Delta_3) \) in \( R(\Delta) \) such that \( R_0(\Delta_2) \) satisfies (1); and \( R_0(\Delta_1) \) and \( R_0(\Delta_3) \) satisfy (2). We also choose a rectangular region \( Q \subset R(\Delta) \) such that placing vertex \( v \) at any point in \( Q \) yields \( R_0(\Delta_i) \subset \Box(\Delta_i) \), for \( i = 1, 2, 3 \). Finally, we show that \( Q \) contains a point on backward diagonal by Lemma 5. We place \( v \) at such a point \( p \in Q \). Choose \( R(\Delta_3) \) such that \( R_0(\Delta_3) \subset R(\Delta_3) \), its lower right corner is \( p \) on a backward diagonal, and its lower left corner is a symmetric point \( p' \) on a forward diagonal.

Partition the area of \( R(\Delta) \) into a top and a bottom part by a horizontal line \( \ell_0 \) in ratio (Fig. 7a)

\[
\left( \frac{\text{weight}(\Delta_2)}{2} \right) : \frac{2}{3} \delta_2.
\]

Partition the top area of \( R(\Delta) \) into a left, middle, and right part by two vertical lines, \( \ell_1 \) and \( \ell_2 \), in ratio

\[
\frac{2}{3} \delta_2 : \frac{2}{3} \text{weight}(\Delta_2).
\]
Universal point sets for three-trees

Let $R_0(\Delta_1)$ be the lower half of the part of $R(\Delta)$ below $\ell_0$, as indicated in Fig. 7b. Let $R_0(\Delta_2)$ be the upper middle part of $R(\Delta)$. Let $R_0(\Delta_3)$ be the right half of the upper right part of $R(\Delta)$. Note that $\text{area}(R_0(\Delta_2)) = \text{area}(R(\Delta)) \cdot \text{weight}(\Delta_2) / \text{weight}(\Delta)$. We choose $Q$ to be the rectangle above $R_0(\Delta_1)$, below $\ell_0$, to the right of $R_0(\Delta_2)$, and to the left of $R_0(\Delta_3)$. Observe that placing vertex $v$ at any point in $Q$ yields $R_0(\Delta_i) \subset \square(\Delta_i)$ for $i = 1, 2, 3$.

We show that $Q$ contains a point on backward diagonal of $B_n$. The width of $Q$ is bounded by

$$w(Q) = \frac{1}{2} w(R(\Delta)) \cdot \frac{\delta_2/3}{\text{weight}(\Delta_3) + 2\delta_2/3} \geq \frac{w(R(\Delta)) \delta_2}{6 \text{weight}(\Delta)}. \quad (6)$$

Similarly, the height of $Q$ is bounded by

$$h(Q) = \frac{1}{2} h(R(\Delta)) \cdot \frac{\delta_2/3}{\text{weight}(\Delta_3)} = \frac{h(R(\Delta)) \delta_2}{6 \text{weight}(\Delta)}. \quad (7)$$

Using the trivial bound $\delta_2 \geq 1$ and (5), we derive $w(Q) > 7/6 > 1$ and $h(Q) > 7/6 > 1$. However, we have $4\delta_2 n^\alpha > w(R(\Delta))$ or $4\delta_2 n^\alpha > h(R(\Delta))$ in Case 3B. Combining $4\delta_2 n^\alpha > h(R(\Delta))$ with (6), we obtain

$$w(Q) > \frac{w(R(\Delta)) h(R(\Delta))}{24n^\alpha \text{weight}(\Delta)} = \frac{\text{area}(R(\Delta))}{24n^\alpha \text{weight}(\Delta)} \geq \frac{100n \text{weight}(\Delta)}{24n^\alpha \text{weight}(\Delta)} = \frac{25n^{1-\alpha}}{6}.$$  

Similarly, the combination of $4\delta_2 n^\alpha > w(R(\Delta))$ and (7) gives

$$h(Q) > \frac{h(R(\Delta)) w(R(\Delta))}{24n^\alpha \text{weight}(\Delta)} = \frac{\text{area}(R(\Delta))}{24n^\alpha \text{weight}(\Delta)} \geq \frac{100n \text{weight}(\Delta)}{24n^\alpha \text{weight}(\Delta)} = \frac{25n^{1-\alpha}}{6}.$$  

Consequently, there is a point $p \in B_n \cap Q$ on a backward diagonal by Lemma 6 when $\alpha \leq 1/2$. Since $R(\Delta)$ satisfies invariant $I_2$, there is a point $p' \in B_n$ on a forward diagonal with the same $y$-coordinate as $p$, to the left of $R_0(\Delta_2)$. We place vertex $v$ at $p$; define $R(\Delta_1) = \square(\Delta_1) \cap R(\Delta)$, $R(\Delta_3) = \square(\Delta_3) \cap R(\Delta)$. Let $R(\Delta_2)$ be a rectangle in $\square(\Delta_2)$ such that its lower left and lower right corners are $p'$ and $p$, respectively, and its top side is contained in the top side of $R(\Delta)$.

Note that $R_0(\Delta_i) \subset R(\Delta_i)$, for $i = 1, 2, 3$. This establishes (1) for $\Delta_2$. Since $\delta_2 = \text{weight}(\Delta_1) + \text{weight}(\Delta_3) + 1$, this also implies the bound $\text{area}(R(\Delta_i)) \geq \frac{25}{3} n \text{weight}(\Delta_i)$ for $i = 1, 3$, which immediately implies (2) for $\Delta_1$ and $\Delta_3$.

Figure 7: (a) The lines $\ell_0$, $\ell_1$, and $\ell_2$ partition the area of $R(\Delta)$ proportionally to $\delta_2/3$, $\text{weight}(\Delta_2)$, $\delta_2/3$, and $\delta_2/3$. (b) We define “preliminary” rectangles $R_0(\Delta_i)$, $i = 1, 2, 3$, and a rectangle $Q$. (c) Vertex $v$ is placed at a point in $B_n \cap Q$ lying on a backward diagonal.
This concludes the description of the embedding algorithm. Since all invariants are maintained, our algorithm embeds $G$ into $B_n$. The function $\tau$ maps this embedding to a straight-line embedding in $S_n$, completing the proof of Theorem 1.

5 Conclusion

We have presented a set $S_n$ of $O(n^{3/2} \log n)$ points in the plane such that every $n$-vertex planar 3-tree has a straight-line embedding where the vertices are mapped into $S_n$. We do not know what is the minimum size of an $n$-universal point set for planar 3-trees. The point set $S_n$, $n \in \mathbb{N}$, certainly admits some other $n$-vertex planar graphs, as well. It remains to be seen whether $S_n$ is $n$-universal for all $n$-vertex planar graphs.

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