The Quantum and Statistical Theory of Early Universe and Its Possible Applications to Cosmology

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Abstract

In this paper a new approach to investigation of Quantum and Statistical Mechanics of the Early Universe (Planck scale) - density matrix deformation - is proposed. The deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition. In the first part of this chapter Quantum Mechanics of the Early Universe is treated as a Quantum Mechanics with Fundamental Length considering the fact that different approaches to quantum gravitation exhibited in the Early Universe were inevitably leading to the notion of fundamental length on the order of Planck's. Besides, this is possible due to the involvement in this theory of the Generalized Uncertainty Relations. And Quantum Mechanics with Fundamental Length is obtained as a deformation of Quantum Mechanics. The distinguishing feature of the proposed approach as compared to the previous ones is the fact that here the density matrix is subjected to deformation, rather than commutators or (that is the same) Heisenberg’s Algebra. In this chapter the density matrix obtained by deformation of the quantum-mechanical one is referred to as a density pro-matrix. Within our
approach two main features of Quantum Mechanics are conserved: the probabilistic interpretation of the theory and the well-known measuring procedure associated with this interpretation. The proposed approach allows for description of dynamics, in particular, the explicit form of deformed Liouville equation and the deformed Shrodinger’s picture. Some implications of the obtained results are discussed including the singularity problem, hypothesis of cosmic censorship, possible improvement of the definition for statistical entropy. It is shown that owing to the obtained results one is enabled to deduce in a simple and natural way the Bekenstein-Hawking formula for black hole entropy in a semiclassical approximation. In the second part of the chapter it is demonstrated that Statistical Mechanics of the Early Universe is a deformation of the conventional Statistical Mechanics. The statistical-mechanics deformation is constructed by analogy to the earlier quantum mechanical results. As previously, the primary object is a density matrix, but now the statistical one. The obtained deformed object is referred to as a statistical density pro-matrix. This object is explicitly described, and it is demonstrated that there is a complete analogy in the construction and properties of quantum-mechanics and statistical density matrices at Plank scale (i.e. density pro-matrices). It is shown that an ordinary statistical density matrix occurs in the low-temperature limit at temperatures much lower than the Plank’s. The associated deformation of a canonical Gibbs distribution is given explicitly. Also consideration is being given to rigorous substantiation of the Generalized Uncertainty Relations as applied in thermodynamics. And in the third part of the chapter the results obtained are applied to solution of the Information Paradox (Hawking) Problem. It is demonstrated that involvement of black holes in the suggested approach in the end twice causes nonunitary transitions resulting in the unitarity. In parallel this problem is considered in other terms: entropy density, Heisenberg algebra deformation terms, respective deformations of Statistical Mechanics, - all showing the identity of the basic results. From this an explicit solution for Hawking’s information paradox has been derived. Besides, it is shown that owing to the proposed approach a new small parameter is derived in physics, the principal features of which are its dimensionless character and its association with all the fundamental constants. In the last part of this chapter it is shown that on the basis of the above parameter the Universe may be considered as nonuniform lattice in the finite-dimensional hypercube. Besides, possible applications of the
results are proposed.

1 Introduction

In the last few years the Early Universe has aroused considerable interest of the researchers. This may be caused by several facts. First, a Big Bang theory is presently well grounded and has established experimental status. Second, acknowledged success of the inflation model and its interface with high-energy physics. Third, various approaches to topical problems of the fundamental physics, specifically to the problem of divergence in a quantum theory or singularity in the General Relativity Theory, in some or other way lead to the problem of quantum-gravitational effects and their adequate description. And all the above aspects are related to the Early Universe. Because of this, investigation of the Early Universe is of particular importance. The Early Universe is understood as a Universe at the first Plancks moments following the Big Bang when energies and scales were on the order of Plancks.

In this chapter a new approach to investigation of Quantum and Statistical Mechanics of the Early Universe - density matrix deformation - is proposed. The deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition. The most clear example is QM being a deformation of Classical Mechanics. The parameter of deformation in this case is the Planck’s constant $h$. When $h \to 0$ QM goes to Classical Mechanics.

In the first part of this chapter Quantum Mechanics of the Early Universe is treated as a Quantum Mechanics with Fundamental Length. This becomes possible since different approaches to quantum gravitation exhibited in the Early Universe unavoidably involve the notion of fundamental length on the order of Plancks (see [1] and the references). Also this is possible due to the involvement in this theory of the Generalized Uncertainty Relations. And Quantum Mechanics with Fundamental Length is obtained as a deformation of Quantum Mechanics. The distinguishing feature of the proposed approach as compared to the previous ones is the fact that here the density matrix is subjected to deformation, rather than commutators or (that is the same) Heisenberg’s Algebra. In this chapter the density matrix obtained by deformation of the quantum-mechanical one is referred to as a density pro-matrix. Within our approach two main features
of Quantum Mechanics are conserved: the probabilistic interpretation of the theory and the well-known measuring procedure associated with this interpretation. The proposed approach allows for description of dynamics, in particular, the explicit form of deformed Liouville equation and the deformed Shrodinger’s picture. Some implications of the obtained results are discussed including the singularity problem, hypothesis of cosmic censorship, possible improvement of the definition for statistical entropy. It is shown that owing to the obtained results one is enabled to deduce in a simple and natural way the Bekenstein-Hawking formula for black hole entropy in a semiclassical approximation. In the second part of the chapter it is demonstrated that Statistical Mechanics of the Early Universe is a deformation of the conventional Statistical Mechanics. The statistical-mechanics deformation is constructed by analogy to the earlier quantum mechanical results. As previously, the primary object is a density matrix, but now the statistical one. The obtained deformed object is referred to as a statistical density pro-matrix. This object is explicitly described, and it is demonstrated that there is a complete analogy in the construction and properties of quantum-mechanics and statistical density matrices at Plank scale (i.e. density pro-matrices). It is shown that an ordinary statistical density matrix occurs in the low-temperature limit at temperatures much lower than the Plank’s. The associated deformation of a canonical Gibbs distribution is given explicitly. Also consideration is being given to rigorous substantiation of the Generalized Uncertainty Relations as applied in thermodynamics. And in the third part of the chapter the results obtained are applied to solution of the Information Paradox (Hawking) Problem. It is demonstrated that involvement of black holes in the suggested approach in the end twice causes nonunitary transitions resulting in the unitarity. In parallel this problem is considered in other terms: entropy density, Heisenberg algebra deformation terms, respective deformations of Statistical Mechanics, - all showing the identity of the basic results. From this an explicit solution for Hawking’s Information paradox has been derived. Besides, it is shown that owing to the proposed approach a new small parameter is derived in physics, the principal features of which are its dimensionless character and its association with all the fundamental constants. In the last part of the chapter it is shown that on the basis of the above parameter the Universe may be considered as nonuniform lattice in the finite-dimensional hypercube. Besides, possible applications of the results are proposed.
This chapter is devoted by the author to two anniversaries, namely: the 70-th anniversary of Academician Ludvig Dmitrievich Faddeev, Russian Academy Sciences, whose work [2] and report at the round-table Conference of the XI International Congress on Mathematical Physics in Paris, July 1994, became a guiding star for the author in his research; and the 60-th anniversary of my first science manager Professor Vassilii Ivanovich Strazhev, presently Rector of the Belarusian State University.

2 Fundamental Length and Density Matrix

Using different approaches (String Theory [3], Gravitation [4], etc.), the authors of numerous papers issued over the last 14-15 years have pointed out that Heisenberg’s Uncertainty Relations should be modified. Specifically, a high energy correction has to appear

\[ \Delta x \geq \frac{\hbar}{\Delta p} + \alpha' L_p^2 \frac{\Delta p}{\hbar}. \]  

(1)

Here \( L_p \) is the Planck’s length: \( L_p = \sqrt{\frac{G \hbar}{c^3}} \approx 1.6 \times 10^{-35} \text{ m} \) and \( \alpha' > 0 \) is a constant. In [4] it was shown that this constant may be chosen equal to 1. However, here we will use \( \alpha' \) as an arbitrary constant without giving it any definite value. Equation (1) is identified as the Generalized Uncertainty Relations in Quantum Mechanics.

The inequality (1) is quadratic in \( \Delta p \):

\[ \alpha' L_p^2 (\Delta p)^2 - \hbar \Delta x \Delta p + \hbar^2 \leq \Delta p \leq 0, \]  

(2)

from whence the fundamental length is

\[ \Delta x_{\text{min}} = 2 \sqrt{\alpha' L_p^2}. \]  

(3)

Since in what follows we proceed only from the existence of fundamental length, it should be noted that this fact was established apart from GUR as well. For instance, from an ideal experiment associated with Gravitational Field and Quantum Mechanics a lower bound on minimal length was obtained in [9], [10] and improved in [11] without using GUR to an estimate of the form \( \sim L_p \). As reviewed previously in [1], the fundamental length appears quite naturally at Planck scale, being related to the
quantum-gravitational effects. Let us consider equation (3) in some detail.
Squaring both its sides, we obtain

$$\overline{(\Delta X^2) \geq 4\alpha' L_p^2} ,$$

(4)

Or in terms of density matrix

$$Sp[(\rho \hat{X}^2) - Sp^2(\rho \hat{X})] \geq 4\alpha' L_p^2 = l_{\text{min}}^2 > 0 ,$$

(5)

where \(\hat{X}\) is the coordinate operator. Expression (5) gives the measuring
rule used in QM. As distinct from QM, however, in the are considered here
the right-hand side of (5) can not be brought arbitrary close to zero as it is
limited by \(l_{\text{min}}^2 > 0\), where because of GUR \(l_{\text{min}} \sim L_p\).

Apparently, this may be due to the fact that QMFL is unitary non-
equivalent to QM. Actually, in QM the left-hand side of (5) can be chosen
arbitrary close to zero, whereas in QMFL this is impossible. But if
two theories are unitary equivalent then the form of their spurs should
be retained. Besides, a more important aspect is contributing to unitary
non-equivalence of these two theories: QMFL contains three fundamenta-
lar constants (independent parameters) \(G, c\) and \(\hbar\), whereas QM contains
only one: \(\hbar\). Within an inflationary model (see [12]), QM is the low-
energy limit of QMFL (QMFL turns to QM) for the expansion of the
Universe. This is identical for all cases of transition from Planck's energies
to the normal ones [1]. In special case of using GUR, the second term
in the right-hand side of (1) vanishes and GUR turn to UR [8]. A nat-
ural way for studying QMFL is to consider this theory as a deformation
of QM, turning to QM at the low energy limit (during expansion of the
Universe after the Big Bang). We will consider precisely this option.
In this paper, unlike the works of other authors (e.g. see [5]) the density matrix is
deformed rather than commutators, whereas the fundamental fundamental
quantum-mechanical measuring rule (5) is left without changes. Here the
following question may be formulated: how should be deformed density
matrix conserving quantum-mechanical measuring rules in order to obtain
self-consistent measuring procedure in QMFL? To answer this question,
we use the R-procedure. First consider R-procedure both at the Planck's
and low-energy scales. At the Planck's scale \(a \approx i l_{\text{min}}\) or \(a \sim i L_p\), where \(i\)
is a small quantity. Further \(a\) tends to infinity and we obtain for density
matrix [13–17]:

$$Sp[\rho a^2] - Sp[\rho a]Sp[\rho a] \simeq l_{\text{min}}^2 \text{ or } Sp[\rho] - Sp^2[\rho] \simeq l_{\text{min}}^2/a^2 .$$
Therefore:

1. When \( a < \infty \), \( Sp[\rho] = Sp[\rho(a)] \) and \( Sp[\rho] - Sp^2[\rho] > 0 \). Then \( Sp[\rho] < 1 \) that corresponds to the QMFL case.

2. When \( a = \infty \), \( Sp[\rho] \) does not depend on \( a \) and \( Sp[\rho] - Sp^2[\rho] \to 0 \). Then \( Sp[\rho] = 1 \) that corresponds to the QM case.

Interesting, how should be interpreted 1 and 2? Does the above analysis agree with the main result from [36]? Note the agreement is well. Indeed, any time when the state vector reduction (R-procedure) place in QM always an eigenstate (value) is chosen exactly. In other words, the probability is equal to 1. As it was pointed out in statement 1, the situation changes when we consider QMFL: it is impossible to measure coordinates exactly, they never will be absolutely reliable. In all cases we obtain a probability less than 1 \( (Sp[\rho] = p < 1) \). In other words, any R-procedure in QMFL leads to an eigenvalue, but only with a probability less than 1. This probability is as near to 1 as far the difference between measuring scale \( a \) and \( l_{\text{min}} \) is growing, or in other words, the second term in (1) becomes insignificant and we turn to QM. Here there is no contradiction with [36]. In QMFL there are no exact coordinate eigenstates (values) as well as there are no pure states. In this paper we consider not the operator properties in QMFL as it was done in [36] but density-matrix properties.

The properties of density matrix in QMFL and QM have to be different. The only reasoning in this case may be as follows: QMFL must differ from QM, but in such a way that in the low-energy limit a density matrix in QMFL be coincident with the density matrix in QM. That is to say, QMFL is a deformation of QM and the parameter of deformation depends on the measuring scale. This means that in QMFL \( \rho = \rho(x) \), where \( x \) is the scale, and for \( x \to \infty \) \( \rho(x) \to \hat{\rho} \), where \( \hat{\rho} \) is the density matrix in QM.

Since on the Planck’s scale \( Sp[\rho] < 1 \), then for such scales \( \rho = \rho(x) \), where \( x \) is the scale, is not a density matrix as it is generally defined in QM. On Planck’s scale \( \rho(x) \) is referred to as ”density pro-matrix”. As follows from the above, the density matrix \( \hat{\rho} \) appears in the limit [13]-[17]:

\[
\lim_{x \to \infty} \rho(x) \to \hat{\rho},
\]  

(6)

\footnote{"... there cannot be any physical state which is a position eigenstate since an eigenstate would of course have zero uncertainty in position"}
when QMFL turns to QM.

Thus, on Planck’s scale the density matrix is inadequate to obtain all information about the mean values of operators. A "deformed" density matrix (or pro-matrix) $\rho(x)$ with $Sp[\rho] < 1$ has to be introduced because a missing part of information $1 - Sp[\rho]$ is encoded in the quantity $l_{min}^2/a^2$, whose specific weight decreases as the scale $a$ expressed in units of $l_{min}$ is going up.

3 QMFL as a deformation of QM

3.1 Main Definitions

Here we describe QMFL as a deformation of QM using the above-developed formalism of density pro-matrix. Within it density pro-matrix should be understood as a deformed density matrix in QMFL. As fundamental parameter of deformation we use the quantity $\alpha = l_{min}^2/x^2$, where $x$ is the scale. The following deformation is not claimed as the only one satisfying all the above properties. Of course, some other deformations are also possible. At the same time, it seems most natural in a sense that it allows for minimum modifications of the conventional density matrix in QM.

Definition 1.\[13\]-[17]

Any system in QMFL is described by a density pro-matrix of the form $\rho(\alpha) = \sum_i \omega_i(\alpha)|i><i|$, where

1. $0 < \alpha \leq 1/4$.
2. Vectors $|i>$ form a full orthonormal system;
3. Coefficients $\omega_i(\alpha) \geq 0$ and for all $i$ the limit $\lim_{\alpha \to 0} \omega_i(\alpha) = \omega_i$ exists;
4. $Sp[\rho(\alpha)] = \sum_i \omega_i(\alpha) < 1$, $\sum_i \omega_i = 1$.
5. For every operator $B$ and any $\alpha$ there is a mean operator $B$ depending on $\alpha$:

$$< B >_\alpha = \sum_i \omega_i(\alpha) < i|B|i >.$$
Finally, in order that our definition 1 be in agreement with the result of section 2, the following condition must be fulfilled:

\[ Sp[\rho(\alpha)] - Sp^2[\rho(\alpha)] \approx \alpha. \]  

(7)

Hence we can find the value for \( Sp[\rho(\alpha)] \) satisfying the condition of definition 1:

\[ Sp[\rho(\alpha)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}. \]  

(8)

According to statement 5 \( \langle 1 \rangle_\alpha = Sp[\rho(\alpha)] \). Therefore, for any scalar quantity \( f \) we have \( \langle f \rangle_\alpha = fSp[\rho(\alpha)] \). In particular, the mean value \( \langle [x_\mu, p_\nu] \rangle_\alpha \) is equal to

\[ \langle [x_\mu, p_\nu] \rangle_\alpha = i\hbar\delta_{\mu, \nu}Sp[\rho(\alpha)]. \]  

(9)

We denote the limit \( \lim_{\alpha \to 0} \rho(\alpha) = \rho \) as the density matrix. Evidently, in the limit \( \alpha \to 0 \) we return to QM.

As follows from definition 1, \( \langle |j> < j| \rangle_\alpha = \omega_j(\alpha) \), from whence the completeness condition by \( \alpha \) is

\[ \langle \sum_i |i> < i| \rangle_\alpha = \langle 1 \rangle_\alpha = Sp[\rho(\alpha)] \].

The norm of any vector \( |\psi> \) assigned to \( \alpha \) can be defined as

\[ \langle \psi|\psi \rangle_\alpha = \langle \psi|\sum_i |i> < i|\rangle_\alpha|\psi \rangle = \langle \psi|(1)_\alpha|\psi \rangle = \langle \psi|\psi \rangle Sp[\rho(\alpha)], \]

where \( \langle \psi|\psi \rangle \) is the norm in QM, i.e. for \( \alpha \to 0 \). Similarly, the described theory may be interpreted using a probabilistic approach, however requiring replacement of \( \rho \) by \( \rho(\alpha) \) in all formulae.

### 3.2 Some obvious implications

It should be noted:

1. The above limit covers both Quantum and Classical Mechanics. Indeed, since \( \alpha \sim L_p^2/x^2 = G\hbar/c^3x^2 \), we obtain:

   a. \( (h \neq 0, x \to \infty) \Rightarrow (\alpha \to 0) \) for QM;

   b. \( (h \to 0, x \to \infty) \Rightarrow (\alpha \to 0) \) for Classical Mechanics;
II. As a matter of fact, the deformation parameter $\alpha$ should assume the value $0 < \alpha \leq 1$. However, as seen from $[\ddag]$, $Sp[\rho(\alpha)]$ is well defined only for $0 < \alpha \leq 1/4$. That is if $x = il_{\text{min}}$ and $i \geq 2$, then there is not any problem. At the point where $x = l_{\text{min}}$ there is a singularity related to complex values assumed by $Sp[\rho(\alpha)]$, i.e. to the impossibility of obtaining a diagonalized density pro-matrix at this point over the field of real numbers. For this reason definition 1 has no sense at the point $x = l_{\text{min}}$. We will return to this question when considering singularity and hypothesis of cosmic censorship in the following section.

III. We consider possible solutions for (7). For instance, one of the solutions of (7), at least to the first order in $\alpha$, is

$$\rho^*(\alpha) = \sum_i \alpha_i \exp(-\alpha) |i><i|,$$

where all $\alpha_i > 0$ are independent of $\alpha$ and their sum is equal to 1. In this way $Sp[\rho^*(\alpha)] = \exp(-\alpha)$. Indeed, we can easily verify that

$$Sp[\rho^*(\alpha)] - Sp^2[\rho^*(\alpha)] = \alpha + O(\alpha^2).$$

The exponential ansatz for $\rho^*(\alpha)$ given here will be included in subsequent sections. Note that in the momentum representation $\alpha = p^2/p_{\text{max}}^2 \sim p^2/p^2_{\text{pl}}$, where $p_{\text{pl}}$ is the Planck's momentum. When present in matrix elements, $\exp(-\alpha)$ can damp the contribution of great momenta in a perturbation theory.

IV. It is clear that within the proposed description the states with a unit probability, i.e. pure states, can appear only in the limit $\alpha \to 0$, when all $\omega_i(\alpha)$ except one are equal to zero or when they tend to zero at this limit. In our treatment pure states are states, which can be represented in the form $|\psi><\psi|$, where $<\psi|\psi> = 1$.

V. We suppose that all definitions concerning a density matrix can be carried over to the above-mentioned deformation of Quantum Mechanics (QMFL) changing the density matrix $\rho$ by the density pro-matrix $\rho(\alpha)$ with subsequent passing to the low-energy limit $\alpha \to 0$. Specifically, for statistical entropy we have

$$S_\alpha = -Sp[\rho(\alpha \ln(\rho(\alpha))].$$

10
The quantity of \( S_\alpha \) seems never to be equal to zero as \( \ln(\rho(\alpha)) \neq 0 \) and hence \( S_\alpha \) may be equal to zero at the limit \( \alpha \to 0 \) only.

The following statements are essential for our study:

I. If we carry out a measurement at a pre-determined scale, it is impossible to regard the density pro-matrix as a density matrix with an accuracy better than the limit \( \sim 10^{-66+2n} \), where \( 10^{-n} \) is the measuring scale. In the majority of known cases this is sufficient to consider the density pro-matrix as a density matrix. But at Planck’s scale, where quantum gravitational effects and Planck’s energy levels cannot be neglected, the difference between \( \rho(\alpha) \) and \( \rho \) should be taken into consideration.

II. Proceeding from the above, on Planck’s scale the notion of Wave Function of the Universe (as introduced in \[18\]) has no sense, and quantum gravitation effects in this case should be described with the help of density pro-matrix \( \rho(\alpha) \) only.

III. Since density pro-matrix \( \rho(\alpha) \) depends on the measuring scale, evolution of the Universe within the inflationary model paradigm \[12\] is not a unitary process, or otherwise the probabilities \( p_i = \omega_i(\alpha) \) would be preserved.

4 Applications of the Quantum-Mechanical Density Pro-Matrix

In this section some apparent applications of the primary definitions and methods derived in the previous section are given \[15\]-\[17\].

4.1 Dynamic aspects of QMFL. Deformed Liouville equation

Let’s suppose that in QMFL a density pro-matrix has the form \( \rho[\alpha(t), t] \), in other words, it depends on two parameters: time \( t \) and parameter of deformation \( \alpha \), which also depends on time \( (\alpha = \alpha(t)) \). Then, we have

\[
\rho[\alpha(t), t] = \sum \omega_i[\alpha(t)]|i(t)> <i(t)|. 
\] (12)
Differentiating the last expression with respect to time, we obtain
\begin{equation}
\frac{d \rho}{dt} = \sum_i \frac{d \omega_i(\alpha(t))}{dt} |i(t)><i(t)| - i[H, \rho(\alpha)] = d[\text{ln} \omega(\alpha)]\rho(\alpha) - i[H, \rho(\alpha)].
\end{equation}
(13)

Where \( \text{ln}[\omega(\alpha)] \) is a row-matrix and \( \rho(\alpha) \) is a column-matrix. In such a way we have obtained a prototype of Liouville’s equation.

Let’s consider some cases of particular importance.

I. First we consider the process of time evolution at low energies, i.e. when \( \alpha \approx 0, \alpha(t) \approx 0 \) and \( t \to \infty \). Then it is clear that \( \omega_i(\alpha) \approx \omega_i \approx \text{constant} \). The first term in (13) vanishes and we obtain Liouville equation.

II. Also we obtain the Liouville’s equation when using inflationary approach and going to large-scales. Within the inflationary approach the scale \( a \approx e^{Ht} \), where \( H \) is the Hubble’s constant and \( t \) is time. Therefore \( \alpha \sim e^{-2Ht} \) and when \( t \) is quite big \( \alpha \to 0 \). In other words, \( \omega_i(\alpha) \to \omega_i \), the first term in (13) vanishes and again we obtain the Liouville’s equation.

III. At very early stage of the inflationary process or even before it took place \( \omega_i(\alpha) \) was not a constant and hence, the first term in (13) should be taking into account. This way we obtain a deviation from the Liouville’s equation.

IV. Finally, let us consider the case when \( \alpha(0) \approx 0, \alpha(t) > 0 \) where \( t \to \infty \). In this case we are going from low-energy to high-energy scale one and \( \alpha(t) \) grows when \( t \to \infty \). The first term in (13) is significant and we obtain an addition to the Liouville’s equation of the form
\[ d[\text{ln} \omega(\alpha)]\rho(\alpha). \]
This could be the case when matter goes into a black hole and is moving in direction of the singularity (to the Planck’s scale).

### 4.2 Singularity, entropy and information loss in black holes

Note that remark II in section 3.2 about complex meaning assumed by the density pro-matrix at the point with fundamental length is directly related
to the singularity problem and cosmic censorship in the General Theory of Relativity [19]. For instance, considering a Schwarzchild’s black hole ([20]) with metrics:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2d\Omega^2,$$

we obtain a well-known a singularity at the point $r = 0$. In our approach this corresponds to the point with fundamental length ($r = l_{\text{min}}$). At this point we are not able to measure anything, since at this point $\alpha = 1$ and $Sp[\rho(\alpha)]$ becomes complex. Thus, we carry out a measurement, starting from the point $r = 2l_{\text{min}}$ that corresponds to $\alpha = 1/4$. Consequently, the initial singularity $r = l_{\text{min}}$, which cannot be measured, is hidden of observation. This could confirm the hypothesis of cosmic censorship in this particular case. By this hypothesis “a naked singularity cannot be observed”. Thus, QMFL in our approach ”feels” the singularity compared with QM, that does not [16 [17]. Statistical entropy, associated with the density pro-matrix and introduced in the remark V section 3 is written

$$S_\alpha = -Sp[\rho(\alpha) \ln(\rho(\alpha))],$$

and may be interpreted as a density of entropy on the unit minimal area $l_{\text{min}}^2$ depending on the scale $x$. It could be quite big close to the singularity, i.e. for $\alpha \rightarrow 1/4$. This does not contradict the second law of Thermodynamics since maximal entropy of a specific object in the Universe is proportional to the square of its surface $A$, measured in units of minimal square $l_{\text{min}}^2$ or Planck’s square $L_p^2$, as shown in some papers (see, for instance [21]). Therefore, in the expanded Universe since surface $A$ grows, entropy does not decrease.

The obtained results enable one to consider anew the information loss problem associated with black holes [22 [23], at least, for the case of ”mini” black holes [16 [17]. Indeed, when we consider these black holes, Planck’s scale is a factor. It was shown that entropy of matter absorbed by a black hole at this scale is not equal to zero, supporting the data of R.Myers [24]. According to his results, the pure state cannot form a black hole. Then, it is necessary to reformulate the problem per se, since so far in all publications on information paradox zero entropy at the initial state has been compared to nonzero entropy at the final state. According to our analysis at the Planck’s scale there is not initial zero entropy and ”mini” black holes with masses of the order $M_{pl}$ should not radiate at all. Similar results were
deduced by A.D.Helfer [37] using another approach: "The possibility
that non-radiating "mini" black holes should be taken seriously; such holes
could be part of the dark matter in the Universe". Note that in [37] the
main argument in favor of the existence of non-radiating "mini" black holes
developed with consideration of quantum gravity effects. In our analysis
these effects are considered implicitly since, as stated above, any approach
in quantum gravity leads to the fundamental-length concept [1]. Besides,
it should be noted that in some recent papers for all types of black holes
QMFL with GUR is considered from the start [26, 38]. By this approach
stable remnants with masses of the order of Planck’s ones $M_{\text{pl}}$ emerge
during the process of black hole evaporation. From here it follows that black
holes should not evaporate fully. We arrive to the conclusion that results
given in [20, 27] are correct only in the semi-classical approximation and
they should not be applicable to the quantum back hole analysis.

At least at a qualitative level the above results can clear up the answer to
the question, how information may be lost at black holes formed due to
the star collapse. Our point of view is close to that of R.Penrose’s one [28]
who considers that information in black holes is lost when matter meets
a singularity. In our approach information loss takes place in the same
form. Indeed, near the horizon of events an approximately pure state with
the initial entropy practically equal to zero $S^{\text{in}} = -S[p\ln(p)]$, that corre-
sponds to $\alpha \to 0$, when approaching a singularity (of reaching the Planck’s
scale) gives yet non zero entropy $S_\alpha = -S[p(\alpha)\ln(p(\alpha))] > 0$ for $\alpha > 0$.
Therefore, entropy increases and information is lost in this black hole. We
can (at the moment, at a qualitative level) evaluate the entropy of black
holes. Actually, starting from a density matrix for the pure state at the
"entry" to a black hole $\rho_{\text{in}} = \rho_{\text{pure}}$ with zero entropy $S^{\text{in}} = 0$, we obtain
with a straightforward "naive" calculation (that is (7) is considered an ex-
act relation). Then, for the singularity in the black hole the corresponding
entropy of the density pro-matrix $S[p(\rho_{\text{out}})] = 1/2$ at $\alpha = 1/4$ is

$$S^{\text{out}} = S_{1/4} = -1/2 \ln 1/2 \approx 0.34657.$$  

Taking into account that total entropy of a black hole is proportional to
the quantum area of surface $A$, measured in Planck’s units of area $L_p^2$ [29],
we obtain the following value for quantum entropy of a black hole:

$$S_{\text{BH}} = 0.34657 \frac{A}{L_p^2} \quad (15)$$
This formula differs from the well-known one given by Bekenstein-Hawking for black hole entropy \( S_{BH} = \frac{1}{4} A \) \[30\]. This result was obtained in the semi-classical approximation. At the present moment quantum corrections to this formula are an object of investigation \[31\]. As it was mentioned above we carry out straightforward calculation. Otherwise, using the ansatz of statement remark III in section 3 and assuming that spur of density pro-matrix is equal to \( Sp[\rho^*(\alpha)] = exp(-\alpha) \), we obtain for \( \alpha = 1/4 \) that entropy is equal to
\[
S^{*\text{out}} = S^{1/4}_1 = -Sp[exp(-1/4) \ln exp(-1/4)] \approx 0.1947,
\]
and consequently we arrive to the following value of entropy
\[
S_{BH} = 0.1947 \frac{A}{L_p^2},
\] \[16\]
that is the closest the result obtained in \[31\]. Our approach leading to formula \[16\] is from the very beginning based on quantum nature of black holes. Note here that in the approaches used up to now to modify Liouville’s equation due to information paradox \[32\] the added member appearing in the right side of \[13\] takes the form
\[
-\frac{1}{2} \sum_{\xi, \gamma \neq 0} (Q^\xi Q^\gamma \rho + \rho Q^\gamma Q^\xi - 2Q^\xi \rho Q^\gamma),
\]
where \( Q^\xi \) is a full orthogonal set of Hermitian matrices with \( Q^0 = 1 \). In this case either locality or conservation of energy-impulse tensor is broken down. By the approach offered in this paper the member added in the deformed Liouville’s equation, in our opinion, has a more natural and beautiful form:
\[
d[\ln(\omega(\alpha))]\rho(\alpha).
\]
In the limit \( \alpha \to 0 \) all properties of QM are conserved, the added member vanishes and we obtain Liouville’s equation.
The information paradox problem at black holes is considered in greater detail in section 7, where the above methods provide a new approach to this problem.

### 4.3 Bekenstein-Hawking formula

The problem is whether can the well-known semiclassical Bekenstein-Hawking formula for Black Hole entropy \[29\], \[38\] can be obtained within the proposed approach? We show how to do it \[17\]. To obtain black hole quantum
entropy, we use the formula $S_\alpha = -S p[\rho(\alpha) \ln(\rho(\alpha))] = - < \ln(\rho(\alpha)) >_\alpha$ when $\alpha$ takes its maximal meaning ($\alpha = 1/4$). In this case (15) and (16) can be written as

$$S_{BH} = - < \ln(\rho(1/4)) >_{1/4} \frac{A}{L_p^2},$$

(17)

for different $\rho(\alpha)$ in (15) and (16) but for the same value of $\alpha$ ($\alpha = 1/4$). Semiclassical approximation works only at large-scales, therefore measuring procedure is also defined at large scales. In other words, all mean values must be taken when $\alpha = 0$. However, for the operators whose mean values are calculated the dependence on $\alpha$ should be taken into account since according to the well-known Hawking’s paper [21], operator of superscattering $\$ \$ translates $\$ \$ : $\rho_{in} \mapsto \rho_{out}$, where in the case considered $\rho_{in} = \rho_{pure}$ and $\rho_{out} = \rho_{pure}^\star(\alpha) = \exp(-\alpha)\rho_{pure} = \exp(-1/4)\rho_{pure}$ conforming to the exponential ansatz of statement III, section 3. Therefore we have

$$S_\alpha^{\text{semiclass}} = - < \ln(\rho(\alpha)) >$$

and formula for semiclassical entropy of a black hole takes the form

$$S_{BH}^{\text{semiclass}} = - < \ln(\rho(1/4)) > \frac{A}{L_p^2} = - < \ln[\exp(-1/4)\rho_{pure}] > \frac{A}{L_p^2} = \frac{1}{4} \frac{A}{L_p^2}$$

(18)

that coincides with the well-known Bekenstein-Hawking formula. It should be noted that $\alpha = 1/4$ in our approach appears in section 3 quite naturally as a maximal meaning for which $Sp\rho(\alpha)$ still stays real, according to (7) and (8). Apparently, if considering corrections of order higher than 1 on $\alpha$, then members from $O(\alpha^2)$ in the formula for $\rho_{out}$ in (10) can give quantum corrections [31] for $S_{BH}^{\text{semiclass}}$ (18) in our approach.

### 4.4 Some comments on Shrödinger’s picture

As it was indicated above in the statement 1 section 3.2, we are able to obtain from QMFL two limits: Quantum and Classical Mechanics. The deformation described here should be understood as “minimal” in the sense that we have deformed only the probability $\omega_i \rightarrow \omega_i(\alpha)$, whereas the state vectors have been not deformed. In a most complete treatment we have to consider vectors $|i(\alpha) > < i(\alpha)|$ instead $|i > < i|$, and in this
case the full picture will be very complicated. It is easy to understand how Shrodinger’s picture is transformed in QMFL [17]. The prototype of Quantum Mechanical normed wave function \( \psi(q) \) with \( \int |\psi(q)|^2 dq = 1 \) in QMFL is \( \theta(\alpha)\psi(q) \). The deformation parameter \( \alpha \) assumes the value \( 0 < \alpha \leq 1/4 \). Its properties are \( |\theta(\alpha)|^2 < 1, \lim_{\alpha \to 0} |\theta(\alpha)|^2 = 1 \) and the relation \( |\theta(\alpha)|^2 - |\theta(\alpha)|^4 \approx \alpha \) takes place. In such a way the full probability always is less than 1: \( p(\alpha) = |\theta(\alpha)|^2 = \int |\theta(\alpha)|^2 |\psi(q)|^2 dq < 1 \) tending to 1 when \( \alpha \to 0 \). In the most general case of arbitrarily normed state in QMFL \( \psi = \psi(\alpha, q) = \sum_n a_n \theta_n(\alpha)\psi_n(q) \) with \( \sum_n |a_n|^2 = 1 \) the full probability is \( p(\alpha) = \sum_n |a_n|^2 |\theta_n(\alpha)|^2 < 1 \) and \( \lim_{\alpha \to 0} p(\alpha) = 1 \).

It is natural that in QMFL Shrodinger’s equation is also deformed. It is replaced by equation

\[
\frac{\partial \psi(\alpha, q)}{\partial t} = \frac{\partial [\theta(\alpha)\psi(q)]}{\partial t} = \frac{\partial \theta(\alpha)}{\partial t} \psi(q) + \theta(\alpha) \frac{\partial \psi(q)}{\partial t} ,
\]

where the second term in the right side generates the Shrodinger’s equation since

\[
\theta(\alpha) \frac{\partial \psi(q)}{\partial t} = \frac{-i\theta(\alpha)}{\hbar} H \psi(q).
\]

Here \( H \) is the Hamiltonian and the first member is added, similarly to the member appearing in the deformed Loiuville’s equation and vanishing when \( \theta(\alpha(t)) \approx \text{const} \). In particular, this takes place in the low energy limit in QM, when \( \alpha \to 0 \). Note that the above theory is not a time-reversal as QM, since the combination \( \theta(\alpha)\psi(q) \) breaks down this property in the deformed Shrodinger’s equation. Time-reversal is conserved only in the low energy limit, when quantum mechanical Shrodinger’s equation is valid.

## 5 Density Matrix Deformation in Statistical Mechanics of Early Universe

### 5.1 Main definition and properties

First we revert to the Generalized Uncertainty Relations ”coordinate -momentum” (section 2, formula (11)):

\[
\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' L_p^2 \frac{\Delta p}{\hbar}.
\]
Using relations (21) it is easy to obtain a similar relation for the “energy-time” pair. Indeed (21) gives

\[ \frac{\Delta x}{c} \geq \frac{\hbar}{\Delta p c} + \alpha'L_p^2 \frac{\Delta p}{\hbar}, \]  

(22)

then

\[ \Delta t \geq \frac{\hbar}{\Delta E} + \alpha'L_p^2 \frac{\Delta p}{c^2 \hbar} = \frac{\hbar}{\Delta E} + \alpha't_p^2 \frac{\Delta E}{\hbar}. \]  

(23)

where the smallness of \( L_p \) is taken into account so that the difference between \( \Delta E \) and \( \Delta(\text{pc}) \) can be neglected and \( t_p \) is the Planck time \( t_p = L_p/c = \sqrt{G\hbar/c^3} \approx 0.54 \times 10^{-43} \text{sec} \). From whence it follows that we have a maximum energy of the order of Planck’s:

\[ E_{\text{max}} \sim E_p \]

Proceeding to the Statistical Mechanics, we further assume that an internal energy of any ensemble \( U \) could not be in excess of \( E_{\text{max}} \) and hence temperature \( T \) could not be in excess of \( T_{\text{max}} = E_{\text{max}}/k_B \sim T_p \). Let us consider density matrix in Statistical Mechanics (see [39], Section 2, Paragraph 3):

\[ \rho_{\text{stat}} = \sum_n \omega_n |\varphi_n\rangle <\varphi_n|, \]  

(24)

where the probabilities are given by

\[ \omega_n = \frac{1}{Q} \exp(-\beta E_n) \]

and

\[ Q = \sum_n \exp(-\beta E_n). \]

Then for a canonical Gibbs ensemble the value

\[ \overline{\Delta(1/T)^2} = Sp[\rho_{\text{stat}}(\frac{1}{T})^2] - Sp^2[\rho_{\text{stat}}(\frac{1}{T})], \]  

(25)

is always equal to zero, and this follows from the fact that \( Sp[\rho_{\text{stat}}] = 1 \). However, for very high temperatures \( T \gg 0 \) we have \( \Delta(1/T)^2 \approx 1/T^2 \geq \)
Thus, for $T \gg 0$ a statistical density matrix $\rho_{\text{stat}}$ should be deformed so that in the general case 

$$Sp[\rho_{\text{stat}}(1/T)]^2 - Sp^2[\rho_{\text{stat}}(1/T)] \approx \frac{1}{T_{\text{max}}^2},$$

or

$$Sp[\rho_{\text{stat}}] - Sp^2[\rho_{\text{stat}}] \approx \frac{T^2}{T_{\text{max}}^2}.$$ 

(26)

(27)

In this way $\rho_{\text{stat}}$ at very high $T \gg 0$ becomes dependent on the parameter $\tau = T^2/T_{\text{max}}^2$, i.e. in the most general case

$$\rho_{\text{stat}} = \rho_{\text{stat}}(\tau)$$

and

$$Sp[\rho_{\text{stat}}(\tau)] < 1$$

and for $\tau \ll 1$ we have $\rho_{\text{stat}}(\tau) \approx \rho_{\text{stat}}$ (formula (24)).

This situation is identical to the case associated with the deformation parameter $\alpha = \sqrt[2]{\rho_{\text{min}}/x^2}$ of QMFL given in section 3. That is the condition $Sp[\rho_{\text{stat}}(\tau)] < 1$ has an apparent physical meaning when:

I. At temperatures close to $T_{\text{max}}$ some portion of information about the ensemble is inaccessible in accordance with the probability that is less than unity, i.e. incomplete probability.

II. And vice versa, the longer is the distance from $T_{\text{max}}$ (i.e. when approximating the usual temperatures), the greater is the bulk of information and the closer is the complete probability to unity.

Therefore similar to the introduction of the deformed quantum-mechanics density matrix in section 3 we give the following

**Definition 2. (Deformation of Statistical Mechanics)**

Deformation of Gibbs distribution valid for temperatures on the order of the Planck’s $T_p$ is described by deformation of a statistical density matrix (statistical density pro-matrix) of the form

$$\rho_{\text{stat}}(\tau) = \sum_n \omega_n(\tau)|\varphi_n><\varphi_n|$$

having the deformation parameter $\tau = T^2/T_{\text{max}}^2$, where
I. $0 < \tau \leq 1/4$.

II. The vectors $|\varphi_n>$ form a full orthonormal system;

III. $\omega_n(\tau) \geq 0$ and for all $n$ at $\tau \ll 1$ we obtain $\omega_n(\tau) \approx \omega_n = \frac{1}{Q} \exp(-\beta E_n)$

In particular,

$$\lim_{T_{\text{max}} \to \infty (\tau \to 0)} \omega_n(\tau) = \omega_n$$

IV. $Sp[\rho_{\text{stat}}] = \sum_n \omega_n(\tau) < 1$, $\sum_n \omega_n = 1$;

V. For every operator $B$ and any $\tau$ there is a mean operator $B$ depending on $\tau$

$$<B>_{\tau} = \sum_n \omega_n(\tau) <n|B|n>$$

Finally, in order that our Definition 2 agree with the formula (27), the following condition must be fulfilled:

$$Sp[\rho_{\text{stat}}(\tau)] - Sp^2[\rho_{\text{stat}}(\tau)] \approx \tau. \quad (28)$$

Hence we can find the value for $Sp[\rho_{\text{stat}}(\tau)]$ satisfying the condition of Definition 2 (similar to Definition 1):

$$Sp[\rho_{\text{stat}}(\tau)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \tau}. \quad (29)$$

It should be noted:

I. The condition $\tau \ll 1$ means that $T \ll T_{\text{max}}$ either $T_{\text{max}} = \infty$ or both in accordance with a normal Statistical Mechanics and canonical Gibbs distribution (23).

II. Similar to QMFL in Definition 1, where the deformation parameter $\alpha$ should assume the value $0 < \alpha \leq 1/4$. As seen from (29), here $Sp[\rho_{\text{stat}}(\tau)]$ is well defined only for $0 < \tau \leq 1/4$. This means that the feature occurring in QMFL at the point of the fundamental length $x = l_{\text{min}}$ in the case under consideration is associated with the fact that highest measurable temperature of the ensemble is always $T \leq \frac{1}{2}T_{\text{max}}$.

III. The constructed deformation contains all four fundamental constants: $G, \hbar, c, k_B$ as $T_{\text{max}} = \varsigma T_p$, where $\varsigma$ is the denumerable function of $\alpha'$ (21) and $T_p$, in its turn, contains all the above-mentioned constants.
IV. Again similar to QMFL, as a possible solution for \( \rho_{\text{stat}}(\tau) \) we have an exponential ansatz

\[
\rho_{\text{stat}}(\tau) = \sum_n \omega_n(\tau) |n\rangle < n\rangle = \sum_n \exp(-\tau) \omega_n |n\rangle < n\rangle
\]

\[
Sp[\rho_{\text{stat}}(\tau)] - Sp^2[\rho_{\text{stat}}(\tau)] = \tau + O(\tau^2).
\]  \( \text{(30)} \)

In such a way with the use of an exponential ansatz (30) the deformation of a canonical Gibbs distribution at Planck scale (up to factor 1/Q) takes an elegant and completed form:

\[
\omega_n(\tau) = \exp(-\tau) \omega_n = \exp\left(-\frac{T^2}{T_{\text{max}}^2} - \beta E_n\right)
\]  \( \text{(31)} \)

where \( T_{\text{max}} = \zeta T_p \)

5.2 Some implications

Using in this section only the exponential ansatz of (30), in the coordinate representation we have the following:

\[
\rho(x, x', \tau) = \sum_i \frac{1}{Q} e^{-\beta E_i - \tau} \varphi_i(x) \varphi_i^*(x')
\]  \( \text{(32)} \)

However, as \( H | \varphi_i >= E_i | \varphi_i > \), then

\[
\rho(\beta, \tau) = \frac{1}{Q} \sum_i e^{-\beta H - \tau} | \varphi_i > < \varphi_i | = \frac{e^{-\beta H - \tau}}{Q},
\]  \( \text{(33)} \)

where \( Q = \sum_i e^{-\beta E_i} = Sp e^{-\beta H} \). Consequently,

\[
\rho(\beta, \tau) = \frac{e^{-\beta H - \tau}}{Sp e^{-\beta H}}
\]  \( \text{(34)} \)

In this way the deformed average energy of a system is obtained as

\[
U_\tau = Sp \rho(\tau) H = \frac{H e^{-\beta H - \tau}}{Sp e^{-\beta H}}
\]  \( \text{(35)} \)
The calculation of deformed entropy is also a simple task. Indeed, in the general case of the canonical Gibbs distribution the probabilities are equal to

$$P_n = \frac{1}{Q} e^{-\beta E_n}.$$  \hspace{1cm} (36)

Nevertheless, in case under consideration they are replenished by $e^{\exp(-\tau)}$ factor and hence are equal to

$$P_n^{\tau} = \frac{1}{Q} e^{-(\tau + \beta E_n)}.$$  \hspace{1cm} (37)

Thus, a new formula for entropy in this case is as follows:

$$S_\tau = -k_B e^{-\tau} \sum_n P_n (\ln P_n - \tau)$$  \hspace{1cm} (38)

It is obvious that $\lim_{\tau \to 0} S_\tau = S$, where $S$ - entropy of the canonical ensemble, that is a complete analog of its counterpart in quantum mechanics at the Planck scale $\lim_{\alpha \to 0} S_\alpha = S$, where $S$ - statistical entropy in quantum mechanics, and deformation parameter $\tau$ is changed by $\alpha$ of section 3.

Given the average energy deformation in a system $U_\tau$ and knowing the entropy deformation, one is enabled to calculate the deformed free energy $F_\tau$ as well:

$$F_\tau = U_\tau - TS_\tau$$  \hspace{1cm} (39)

Consider the counterpart of Liouville equation \[39\] for the unnormed $\rho(\beta, \tau)$:

$$-\frac{\partial \rho(\beta, \tau)}{\partial \beta} = -\frac{\partial}{\partial \beta} e^{-\tau - \beta H},$$  \hspace{1cm} (40)

where

$$\tau = \frac{T^2}{T_{\text{max}}^2} = \frac{\beta^2_{\text{max}}}{\beta^2},$$

where $\beta_{\text{max}} = 1/k_B T_{\text{max}} \sim 1/k_B T_P \equiv \beta_P$, $\tau = \tau(\beta)$. Taking this into consideration and expanding the right-hand side of equation \[40\], we get deformation of Liouville equation further referred to as $\tau$-deformation:

$$-\frac{\partial \rho(\beta, \tau)}{\partial \beta} = -e^{-\tau} \frac{\partial}{\partial \beta} + e^{-\tau} H \rho(\beta) = e^{-\tau} [H \rho(\beta) - \frac{\partial}{\partial \beta}]$$  \hspace{1cm} (41)
where \( \rho(\beta) = \rho(\beta, \tau = 0) \).

The first term in brackets (41) generates Liouville equation. Actually, taking the limit of the left and right sides (41) for \( \tau \rightarrow 0 \), we derive the normal Liouville equation for \( \rho(\beta) \) in statistical mechanics [39]:

\[
- \frac{\partial \rho(\beta)}{\partial \beta} = H \rho(\beta)
\]  

(42)

By this means we obtain a complete analog of the quantum-mechanical results for the associated deformation of Liouville equation derived in section 4.1 and [15]-[17]. Namely:

(1) Early Universe (scales approximating those of the Plancks, original singularity, \( \tau > 0 \)). The density pro-matrix \( \rho(\beta, \tau) \) is introduced and a \( \tau \)-deformed Liouville equation (41), respectively;

(2) after the inflation extension (well-known scales, \( \tau \approx 0 \)) the normal density matrix \( \rho(\beta) \) appears in the limit \( \lim_{\tau \rightarrow 0} \rho(\beta, \tau) = \rho(\beta) \). \( \tau \)-deformation of Liouville equation (41) is changed by a well-known Liouville equation (42);

(3) and finally the case of the matter absorbed by a black hole and its tendency to the singularity. Close to the black hole singularity both quantum and statistical mechanics are subjected to deformation as they do in case of the original singularity [15]-[17]. Introduction of temperature on the order of the Plancks [43],[44] and hence the deformation parameter \( \tau > 0 \) may be taken as an indirect evidence for the fact. Because of this, the case is associated with the reverse transition from the well-known density matrix in statistical mechanics \( \rho(\beta) \) to its \( \tau \)-deformation \( \rho(\beta, \tau) \) and from Liouville equation (42) to its \( \tau \)-deformation (41).

6 Generalized Uncertainty Relation in Thermodynamics

Now we consider the thermodynamic uncertainty relations between the inverse temperature and interior energy of a macroscopic ensemble

\[
\Delta \frac{1}{T} \geq \frac{k}{\Delta U},
\]  

(43)

where \( k \) is the Boltzmann constant.

N.Bohr [45] and W.Heisenberg [46] first pointed out that such kind of
uncertainty principle should take place in thermodynamics. The thermodynamic uncertainty relations (43) were proved by many authors and in various ways [47]. Therefore their validity does not raise any doubts. Nevertheless, relation (43) was established using a standard model for the infinite-capacity heat bath encompassing the ensemble. But it is obvious from the above inequalities that at very high energies the capacity of the heat bath can no longer be assumed infinite at the Planck scale. Indeed, the total energy of the pair heat bath - ensemble may be arbitrary large but finite, merely as the Universe is born at a finite energy. Thus the quantity that can be interpreted as a temperature of the ensemble must have the upper limit and so does its main quadratic deviation. In other words the quantity Δ(1/T) must be bounded from below. But in this case an additional term should be introduced into (43) [48, 49, 41]

\[ \Delta \frac{1}{T} \geq \frac{k}{\Delta U} + \eta \Delta U, \] (44)

where \( \eta \) is a coefficient. Dimension and symmetry reasons give

\[ \eta \sim \frac{k}{E_p^2} \text{ or } \eta = \alpha' \frac{k}{E_p^2} \]

As in the previous cases inequality (44) leads to the fundamental (inverse) temperature.

\[ T_{\text{max}} = \frac{\hbar}{2\sqrt{\alpha' t_p k}} = \frac{\hbar}{\Delta t_{\text{min}} k}, \quad \beta_{\text{min}} = \frac{1}{kT_{\text{max}}} = \frac{\Delta t_{\text{min}}}{\hbar} \] (45)

It should be noted that the same conclusion about the existence of maximal temperature in Nature can be made also considering black hole evaporation [50].

Thus, we obtain the system of generalized uncertainty relations in the symmetric form

\[
\begin{align*}
\Delta x & \geq \frac{\hbar}{\Delta p} + \alpha' \left( \frac{\Delta p}{P_{pl}} \right) \frac{\hbar}{P_{pl}} + \ldots \\
\Delta t & \geq \frac{\hbar}{\Delta E} + \alpha' \left( \frac{\Delta E}{E_p} \right) \frac{\hbar}{E_p} + \ldots \\
\Delta \frac{1}{T} & \geq \frac{k}{\Delta U} + \alpha' \left( \frac{\Delta U}{E_p} \right) \frac{k}{E_p} + \ldots
\end{align*}
\] (46)
or in the equivalent form

\[
\begin{align*}
\Delta x & \geq \frac{\hbar}{\Delta p} + \alpha' L_p^2 \frac{\Delta p}{\hbar} + ...
\end{align*}
\]

\[
\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar} + ...
\]

\[
\Delta \frac{1}{T} \geq \frac{k}{\Delta U} + \alpha' \frac{1}{T_p^2} \frac{\Delta U}{k} + ...
\]

(47)

where dots mean the existence of higher order corrections as in \[34\]. Here \( T_p \) is the Planck temperature: \( T_p = \frac{E_p}{k} \).

In conclusion of this section we would like to note that the restriction on the heat bath made above makes the equilibrium partition function non-Gibbsian \[51\].

Note that the last-mentioned inequality is symmetrical to the second one with respect to substitution \[52\]

\[ t \mapsto \frac{1}{T}, \hbar \mapsto k, \Delta E \mapsto \Delta U. \]

However this observation can by no means be regarded as a rigorous proof of the generalized uncertainty relation in thermodynamics.

There is reason to believe that rigorous justification for the latter (thermodynamic) inequalities in systems \[16\] and \[17\] may be made by means of a certain deformation of Gibbs distribution. One of such deformations that, by the authors opinion, is liable to give the indicated result has been considered in the previous section of this chapter and in some other papers \[40, 41\].

7 Non-Unitary and Unitary Transitions in Generalized Quantum Mechanics and Information Problem Solving

In this section the earlier obtained results are used for the unitarity study in Generalized Quantum Mechanics and Information Paradox Problem \[22, 20, 23\]. It is demonstrated that the existence of black holes in the suggested approach in the end twice causes nonunitary transitions resulting in the unitarity. In parallel this problem is considered in other terms:
entropy density, Heisenberg algebra deformation terms, respective deformations of Statistical Mechanics, - all showing the identity of the basic results. From this an explicit solution for Information Paradox Problem has been derived. This section is based on the results presented in [53, 54, 55]

7.1 Some comments and unitarity in QMFL

As has been indicated in section 4.4, time reversal is retained in the large-scale limit only. The same is true for the superposition principle in Quantum Mechanics. Indeed, it may be retained in a very narrow interval of cases for the functions \(\psi_1(\alpha, q) = \theta(\alpha)\psi_1(q)\) \(\psi_2(\alpha, q) = \theta(\alpha)\psi_2(q)\) with the same value \(\theta(\alpha)\). However, as for all \(\theta(\alpha)\), their limit is \(\lim_{\alpha \to 0} |\theta(\alpha)|^2 = 1\) or equivalently \(\lim_{\alpha \to 0} |\theta(\alpha)| = 1\), in going to the low-energy limit each wave function \(\psi(q)\) is simply multiplied by the phase factor \(\theta(0)\). As a result we have Hilbert Space wave functions in QM. Comparison of both pictures (Neumann’s and Shrödinger’s) is indicative of the fact that unitarity means the retention of the probabilities \(\omega_i(\alpha)\) or retention of the squared modulus (and hence the modulus) for the function \(\theta(\alpha): |\theta(\alpha)|^2, (|\theta(\alpha)|). That is

\[
\frac{d\omega_i[\alpha(t)]}{dt} = 0
\]

or

\[
\frac{d|\theta[\alpha(t)]|}{dt} = 0.
\]

In this way a set of unitary transformations of QMFL includes a group \(U\) of the unitary transformations for the wave functions \(\psi(q)\) in QM. It is seen that on going from Planck’s scale to the conventional one, i.e. on transition from the Early Universe to the current one, the scale has been rapidly changing in the process of inflation expansion and the above conditions failed to be fulfilled:

\[
\frac{d\omega_i[\alpha(t)]}{dt} \neq 0, \frac{d|\theta[\alpha(t)]|}{dt} \neq 0.
\]  \(48\)

In terms of the density pro-matrices of sections 2,3 this is a limiting transition from the density pro-matrix in QMFL \(\rho(\alpha), \alpha > 0\), that is a prototype of the pure state at \(\alpha \to 0\) to the density matrix \(\rho(0) = \rho\) representing a
pure state in QM. Mathematically this means that a nontotal probability (below 1) is changed by the total one (equal to 1). For the wave functions in Schrödinger picture this limiting transition from QMFL to QM is as follows:

\[ \lim_{\alpha \to 0} \theta(\alpha)\psi(q) = \psi(q) \]

up to the phase factor.

It is apparent that the above transition from QMFL to QM is not a unitary process, as indicated in [13]-[17] and section 3.2. However, the unitarity may be recovered when we consider in a sense a reverse process: absorption of the matter by a black hole and its transition to singularity conforming to the reverse and nonunitary transition from QM to QMFL. Thus, nonunitary transitions occur in this picture twice:

\[ I. (QMFL, OS, \alpha \approx 1/4) \xrightarrow{Big \ Bang} (QM, \alpha \approx 0) \]

\[ II. (QM, \alpha \approx 0) \xrightarrow{absorbing BH} (QMFL, SBH, \alpha \approx 1/4). \]

Here the following abbreviations are used: OS for the Origin Singularity; BH for a Black Hole; SBH for the Singularity in Black Hole.

As a result of these two nonunitary transitions, the total unitarity may be recovered:

\[ III. (QMFL, OS, \alpha \approx 1/4) \rightarrow (QMFL, SBH, \alpha \approx 1/4). \]

In such a manner the total information quantity in the Universe remains unchanged, i.e. no information loss occurs.

In terms of the deformed Liouville equation [15]-[17] and section 4.1 we arrive to the expression with the same right-hand parts for \( t_{\text{initial}} \sim t_{\text{Planck}} \).
and $t_{\text{final}}$ (for $\alpha \approx 1/4$).

$$\frac{d\rho[\alpha(t),t]}{dt} = \sum_i \frac{d\omega_i[\alpha(t)]}{dt}|i(t)><i(t)| - i[H,\rho(\alpha)] = d[\ln\omega(\alpha)]\rho(\alpha) - i[H,\rho(\alpha)].$$  \hspace{1cm} (49)

It should be noted that for the closed Universe one can consider Final Singularity (FS) rather than the Singularity of Black Hole (SBH), and then the right-hand parts of diagrams II and III will be changed:

$$IIa.(QM,\alpha \approx 0) \xrightarrow{\text{Big Crunch}} (QMFL,FS,\alpha \approx 1/4),$$

$$IIIa.(QMFL,OS,\alpha \approx 1/4) \rightarrow (QMFL,FS,\alpha \approx 1/4)$$

At the same time, in this case the general unitarity and information are still retained, i.e. we again have the unitary product of two nonunitary arrows:

$$IV.(QMFL,OS,\alpha \approx 1/4) \xrightarrow{\text{Big Bang}} (QM,\alpha \approx 0) \xrightarrow{\text{Big Crunch}} (QMFL,FS,\alpha \approx 1/4).$$

Finally, arrow III may appear directly, i.e. without the appearance of arrows I II, when in the Early Universe mini BH are arising:

$$IIIb.(QMFL,OS,\alpha \approx 1/4) \rightarrow (QMFL,\text{mini BH},SBH,\alpha \approx 1/4).$$

Note that here, unlike the previous cases, a unitary transition occurs immediately, without any additional nonunitary ones, and with retention of the total information.

Another approach to the information paradox problem associated with the
above-mentioned methods (density matrix deformation) is the introduction and investigation of a new value, namely entropy density per minimum unit area. This approach is described in the following subsection.

### 7.2 Entropy density matrix and information loss problem

In [13]-[17] the authors were too careful, when introducing for density matrix $\rho(\alpha)$ the value $S_\alpha$ generalizing the ordinary statistical entropy:

$$S_\alpha = -Sp[\rho(\alpha) \ln(\rho(\alpha))] = - < \ln(\rho(\alpha)) >_\alpha.$$  

In [16],[17] it was noted that $S_\alpha$ means the entropy density on a minimum unit area depending on the scale. In fact a more general concept accepts the form of the entropy density matrix [53]:

$$S_{\alpha_1}^{\alpha_2} = -Sp[\rho(\alpha_1) \ln(\rho(\alpha_2))] = - < \ln(\rho(\alpha_2)) >_{\alpha_1},$$  

where $0 < \alpha_1, \alpha_2 \leq 1/4$.

$S_{\alpha_2}^{\alpha_1}$ has a clear physical meaning: the entropy density is computed on the scale associated with the deformation parameter $\alpha_2$ by the observer who is at a scale corresponding to the deformation parameter $\alpha_1$. Note that with this approach the diagonal element $S_{\alpha} = S_{\alpha_1}^{\alpha}$ of the described matrix $S_{\alpha_2}^{\alpha_1}$ is the density of entropy, measured by the observer who is at the same scale as the measured object associated with the deformation parameter $\alpha$. In [22] and section 4.3 such a construction was used implicitly in derivation of the semiclassical Bekenstein-Hawking formula for the Black Hole entropy:

a) for the initial (approximately pure) state

$$S_{in} = S_{0}^{0} = 0,$$

b) using the exponential ansatz [10], we obtain:

$$S_{out} = S_{1/4}^{0} = - < ln[exp(-1/4)]\rho_{pure} >= - < \ln(\rho(1/4)) > = \frac{1}{4}.$$  

So increase in the entropy density for an external observer at the large-scale limit is 1/4. Note that increase of the entropy density
loss) for the observer crossing the horizon of the black hole’s events and moving with the information flow to singularity will be smaller:

\[ S_{\text{out}} = S_{\alpha}^{1/4} = -Sp(\exp(-1/4)\ln[\exp(-1/4)]\rho_{\text{pure}}) = - < \ln(\rho(1/4)) >^{1/4} \approx 0.1947. \]

It is clear that this fact may be interpreted as follows: for the observer moving together with information its loss can occur only at the transition to smaller scales, i.e. to greater deformation parameter \( \alpha \).

Now we consider the general Information Problem. Note that with the well-known Quantum Mechanics (QM) the entropy density matrix \( S_{\alpha_1}^{\alpha_2} \) is reduced only to one element \( S_0^0 \). So we can not test anything. Moreover, in previous works relating the quantum mechanics of black holes and information paradox \cite{22,20,23} the initial and final states when a particle hits the hole are treated proceeding from different theories (QM and QMFL respectively), as was indicated in diagram II:

(Large-scale limit, QM, density matrix) \( \rightarrow \) (Black Hole, singularity, QMFL, density pro-matrix).

Of course in this case any conservation of information is impossible as these theories are based on different concepts of entropy. Simply saying, it is incorrect to compare the entropy interpretations of two different theories (QM and QMFL) where this notion is originally differently understood. So the chain above must be symmetrized by accompaniment of the arrow on the left, so in an ordinary situation we have a chain (diagram III):

(Early Universe, origin singularity, QMFL, density pro-matrix) \( \rightarrow \) (Large-scale limit, QM, density matrix) \( \rightarrow \) (Black Hole, singularity, QMFL, density pro-matrix).

So it’s more correct to compare entropy close to the origin and final (Black hole) singularities. In other words, it is necessary to take into account not only the state, where information disappears, but also that whence it appears. The question arises, whether in this case the information is lost for every separate observer. For the event under consideration this question sounds as follows: are the entropy densities \( S(\text{in}) \) and \( S(\text{out}) \) equal for every separate observer? It will be shown that in all conceivable cases they are equal.
1) For the observer in the large-scale limit (producing measurements in the semiclassical approximation) $\alpha_1 = 0$

\[ S(in) = S^0_1 \text{ (Origin singularity)} \]

\[ S(out) = S^0_1 \text{ (Singularity in Black Hole)} \]

So $S(in) = S(out) = S^0_1$. Consequently, the initial and final densities of entropy are equal and there is no information loss.

2) For the observer moving together with the information flow in the general situation we have the chain:

\[ S(in) \rightarrow S(large-scale) \rightarrow S(out), \]

where $S(large-scale) = S^0_0 = S$. Here $S$ is an ordinary entropy of Quantum Mechanics (QM), but $S(in) = S(out) = S^1_1$ - value considered in QMFL. So in this case the initial and final densities of entropy are equal without any loss of information.

3) This is a special case of 2), when we do not leave out of the Early Universe considering the processes with the participation of black mini-holes only. In this case the originally specified chain becomes shorter by one section (diagram IIIb):

(Early Universe, origin singularity, QMFL, density pro-matrix)$\rightarrow$(Black Mini-Hole, singularity, QMFL, density pro-matrix),

and member $S(large-scale) = S^0_0 = S$ disappears at the corresponding chain of the entropy density associated with the large-scale:

\[ S(in) \rightarrow S(out), \]

It is, however, obvious that in case $S(in) = S(out) = S^1_1$ the density of entropy is preserved. Actually this event was mentioned in [17], where from the basic principles it has been found that black mini-holes do not radiate, just in agreement with the results of other authors [25], [37], [36]. As a result, it’s possible to write briefly
\[ S(in) = S(out) = S_{\alpha}^{0}, \]

where \( \alpha \) - any value in the interval \( 0 < \alpha \leq 1/4 \).

Actually our inferences are similar to those of section 4.1 in terms of the Liouville’s equation deformation:

\[
\frac{d\rho}{dt} = \sum_i \frac{d\omega_i[\alpha(t)]}{dt} [i(t) > < i(t)] - i[H, \rho(\alpha)] = d[ln\omega(\alpha)]\rho(\alpha) - i[H, \rho(\alpha)].
\]

The main result of this section is a necessity to account for the member \( d[ln\omega(\alpha)]\rho(\alpha) \), deforming the right-side expression of \( \alpha \approx 1/4 \).

### 7.3 Unitarity, non-unitarity and Heisenbergs algebra deformation

The above-mentioned unitary and non-unitary transitions may be described in terms of Heisenbergs algebra deformation (deformation of commutators) as well. We use the principal results and designations from [7]. In the process the following assumptions are resultant: 1) The three-dimensional rotation group is not deformed; angular momentum \( J \) satisfies the undeformed \( SU(2) \) commutation relations, whereas the coordinate and momenta satisfy the undeformed commutation relations \([J_i, x_j] = i\epsilon_{ijk}x_k, [J_i, p_j] = i\epsilon_{ijk}p_k\).

2) The momenta commute between themselves: \([p_i, p_j] = 0\), so the translation group is also not deformed. 3) Commutators \([x, x]\) and \([x, p]\) depend on the deformation parameter \( \kappa \) with the dimension of mass. In the limit \( \kappa \to \infty \) with \( \kappa \) much larger than any energy the canonical commutation relations are recovered.

For a specific realization of points 1) to 3) the generating GUR are of the form [7]: (\( \kappa \)-deformed Heisenberg algebra)

\[
[x_i, x_j] = -\frac{\hbar^2}{\kappa^2} i\epsilon_{ijk}J_k \quad (51)
\]

\[
[x_i, p_j] = i\hbar\delta_{ij}(1 + \frac{E^2}{\kappa^2})^{1/2}. \quad (52)
\]

Here \( E^2 = p^2 + m^2 \). Note that in this formalism the transition from GUR to UR, or equally from QMFL to QM with \( \kappa \to \infty \) or from Planck scale to the conventional one, is nonunitary exactly following the transition from
density pro-matrix to the density matrix in previous sections:

$$\rho(\alpha \neq 0) \xrightarrow{\alpha \to 0} \rho.$$  

Then the first arrow I in the formalism of this section may be as follows:

$$I'.(GUR, OS, \kappa \sim M_p) \xrightarrow{\text{Big Bang}} (UR, \kappa = \infty)$$

or what is the same

$$I''.(QMFL, OS, \kappa \sim M_p) \xrightarrow{\text{Big Bang}} (QM, \kappa = \infty),$$

where $M_p$ is the Planck mass.

In some works of the last two years Quantum Mechanics for a Black Hole has been already considered as a Quantum Mechanics with GUR [25, 37]. As a consequence, by this approach the Black Hole is not completely evaporated but rather some stable remnants always remain in the process of its evaporation with a mass $\sim M_p$. In terms of [17] this means nothing else but a reverse transition: $(\kappa = \infty) \rightarrow (\kappa \sim M_p)$. And for an outside observer this transition is of the form:

$$II'.(UR, \kappa = \infty) \xrightarrow{\text{absorbing BH}} (GUR, SBH, \kappa \sim M_p),$$

that is

$$II''.(QM, \kappa = \infty) \xrightarrow{\text{absorbing BH}} (QMFL, SBH, \kappa \sim M_p).$$

So similar to the previous section, two nonunitary inverse transitions a) $I'$, (II'') and b) $II'$, (II'') are liable to generate a unitary transition:

$$III'.(GUR, OS, \kappa \sim M_p) \xrightarrow{\text{Big Bang}} (UR, \kappa = \infty) \xrightarrow{\text{absorbing BH}} (GUR, SBH, \kappa \sim M_p),$$

or to summerize

$$III''.(GUR, OS, \kappa \sim M_p) \rightarrow (GUR, SBH, \kappa \sim M_p)$$
In conclusion of this section it should be noted that an approach to the Quantum Mechanics at Planck Scale using the Heisenberg algebra deformation (similar to the approach based on the density matrix deformation from the section 3) gives a deeper insight into the possibility of retaining the unitarity and the total quantity of information in the Universe, making possible the solution of Hawkings Information Paradox Problem [22], [20], [23].

7.4 Statistical mechanics deformation and transitions

Naturally, deformation of Quantum Mechanics in the Early Universe is associated with the Statistical Mechanics deformation as indicated in [40], [41]. In case under consideration this simply implies a transition from the Generalized Uncertainty Relations (GUR) of Quantum Mechanics to GUR in Thermodynamics [41], [48], [49]. The latter are distinguished from the normal uncertainty relations by:

$$\Delta \frac{1}{T} \geq \frac{k}{\Delta U},$$  \hspace{1cm} (53)

i.e. by inclusion of the high-temperature term into the right-hand side (section 6 of this chapter)

$$\Delta \frac{1}{T} \geq \frac{k}{\Delta U} + \alpha \frac{1}{T_p^2} \frac{\Delta U}{k} + \ldots$$  \hspace{1cm} (54)

Thus, denoting the Generalized Uncertainty Relations in Thermodynamics as GURT and using abbreviation URT for the conventional ones, we obtain a new form of diagram I from section III ($I'$ of section IV respectively):

$$I^T.(GURT,OS) \xrightarrow{Big} \xrightarrow{Bang} (URT).$$

In [40], [41] and section 5 of this chapter the Statistical Mechanics deformation associated with GURT is implicitly assumed by the introduction of the respective deformation for the statistical density matrix $\rho_{stat}(\tau)$ where $0 < \tau \leq 1/4$. Obviously, close to the Origin Singularity $\tau \approx 1/4$. Because of this, arrow $I^T$ may be represented in a more general form as

$$I^{Stat}.(GURT,OS,\rho_{stat}(\tau),\tau \approx 1/4) \xrightarrow{Big} \xrightarrow{Bang} (URT,\rho_{stat},\tau \approx 0).$$
The reverse transition is also possible. In [25], [37] it has been shown that Statistical Mechanics of Black Hole should be consistent with the deformation of a well-known Statistical Mechanics. The demonstration of an *upper* bound for temperature in Nature, given by Planck temperature and related to Black Hole evaporation, was provided in [50]. It is clear that emergence of such a high temperatures is due to GURT. And we have the following diagram that is an analog of diagrams II and II’ for Statistical Mechanics:

\[ II^{Stat}.(URT, \rho_{stat}, \tau \approx 0) \overset{absorbing}{\longrightarrow} BH (GURT, SBH, \rho_{stat}(\tau), \tau \approx 1/4). \]

By this means, combining I^{Stat} and II^{Stat}, we obtain III^{Stat} representing a complete statistical-mechanics analog for quantum-mechanics diagrams III and III’:

\[ III^{Stat}.(GURT, OS, \tau \approx 1/4) \overset{Big Bang, absorbing}{\longrightarrow} BH (GURT, SBH, \tau \approx 1/4). \]

And in this case two nonunitary transitions I^{Stat} and II^{Stat} in the end lead to a unitary transition III^{Stat}.

8 The Universe as a Nonuniform Lattice in Finite-Volume Hypercube

In this section a new small parameter associated with the density matrix deformation (density protomatrix) studied in previous sections is introduced into the Generalized Quantum Mechanics (GQM), i.e., quantum mechanics involving description of the Early Universe. It is noted that this parameter has its counterpart in the Generalized Statistical Mechanics. Both parameters offer a number of merits: they are dimensionless, varying over the interval from 0 to 1/4 and assuming in this interval a discrete series of values. Besides, their definitions contain all the fundamental constants. These parameters are very small for the conventional scales and temperatures, e.g., the value of the first parameter is on the order of \( 10^{-66+2n} \), where \( 10^{-n} \) is the measuring scale and the Planck scale \( \sim 10^{-33} cm \) is assumed. The second one is also too small for the conventional temperatures, that is those much below the Plancks. It is demonstrated that
relative to the first of these parameters the Universe may be considered as
a nonuniform lattice in the four-dimensional hypercube with dimension-
less finite-length (1/4) edges. And the time variable is also described by
one of the above-mentioned dimensions due to the second parameter and
Generalized Uncertainty Relation in thermodynamics. In this context the
lattice is understood as a deformation rather than approximation [56].

8.1 Definition of lattice

It should be noted that according to subsection 3.2 a minimum measur-
able length is equal to \( l_{\text{min}}^* = 2l_{\text{min}} \) being a nonreal number at point
\( l_{\text{min}}, Sp[\rho(\alpha)] \). Because of this, a space part of the Universe is a lattice
with a spacing of \( a_{\text{min}} = 2l_{\text{min}} \sim 2l_p \). In consequence the first issue con-
cerns the lattice spacing of any lattice-type model(for example [57, 58]):
a selected lattice spacing \( a_{\text{lat}} \) should not be less than \( a_{\text{min}} \), i.e. always
\( a_{\text{lat}} \geq a_{\text{min}} > 0 \). Besides, a continuum limit in any lattice-type model is
meaning \( a_{\text{lat}} \to a_{\text{min}} > 0 \) rather than \( a_{\text{lat}} \to 0 \).

Proceeding from \( \alpha \), for each space dimension we have a discrete series
of rational values for the inverse squares of even numbers nonuniformly
distributed along the real number line \( \alpha = 1/4, 1/16, 1/36, 1/64, ... \). A
question arises,is this series somewhere terminated or, on the contrary, is
it infinite? The answer depends on the answers to two other questions:
(1) Is there theoretically a maximum measurability limit for the scales
\( l_{\text{max}} \) and
(2) Is our Universe closed in the sense that its extension may be some-
time replaced by compression, when a maximum extension precisely gives
a maximum scale \( l_{\text{max}} \)?

Should an answer to one of these questions be positive, we should have
\( 0 < l_{\text{min}}^2/l_{\text{max}}^2 \leq \alpha \leq 1/4 \) rather than condition 1 of Definition 1, sub-
section 3.1 of this chapter

Note that in the majority of cases all three space dimensions are equal,
at least at large scales, and hence their associated values of \( \alpha \) parameter
should be identical. This means that for most cases, at any rate in the
large-scale (low-energy) limit, a single deformation parameter \( \alpha \) is suffi-
cient to accept one and the same value for all three dimensions to a high
degree of accuracy. In the general case, however, this is not true, at least
for very high energies (on the order of the Plancks), i.e. at Planck scales,
due to noncommutativity of the spatial coordinates [5, 7]:
\[ [x_i, x_j] \neq 0. \]

In consequence in the general case we have a point with coordinates \( \tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \) in the normal (three-dimensional) cube \( I_{1/4}^3 \) of side \( I_{1/4} = (0; 1/4] \).

It should be noted that this universal cube may be extended to the four-dimensional hypercube by inclusion of the additional parameter \( \tau, \tau \in I_{1/4} \) that is generated by internal energy of the statistical ensemble and its temperature for the events when this notion is the case. It will be recalled that \( \tau \) parameter occurs from a maximum temperature that is in its turn generated by the Generalized Uncertainty Relations of energy-time pair in GUR (see Definition 2 in subsection 5.1 and [40, 41]).

So \( \tau \) is a counterpart (twin) of \( \alpha \), yet for the Statistical Mechanics. At the same time, originally for \( \tau \) nothing implies the discrete properties of parameter \( \alpha \) indicated above:

for \( \tau \) there is a discrete series (lattice) of the rational values of inverse squares for even numbers not uniformly distributed along the real number line: \( \tau = 1/4, 1/16, 1/36, 1/64, \ldots \)

Provided such a series exists actually, the finiteness and infinity question for this series amounts to two other questions:

1. Is there theoretically any minimum measurability limit for the average temperature of the Universe \( T_{\text{min}} \neq 0 \) and
2. Is our Universe closed in a sense that its extension may be sometime replaced by compression? Then maximum extension just gives a minimum temperature \( T_{\text{min}} \neq 0 \).

The question concerning the discretization of parameter \( \tau \) is far from being idle. The point is that originally by its nature this parameter seems to be continuous as it is associated with temperature. Nevertheless, in the following section we show that actually \( \tau \) is dual in nature: it is directly related to time that is in turn quantized, in the end giving a series \( \tau = 1/4, 1/16, 1/36, 1/64, \ldots \).

### 8.2 Dual nature of parameter \( \tau \) and its temporal aspect

In this way when at point \( \tilde{\alpha} \) of the normal (three-dimensional) cube \( I_{1/4}^3 \) of side \( I_{1/4} = (0; 1/4] \) an additional temperature variable \( \tau \) is added, a
nonuniform lattice of the point results, where we denote \( \tilde{\alpha}_\tau = (\tilde{\alpha}, \tau) = (\alpha_1, \alpha_2, \alpha_3, \tau) \) at the four-dimensional hypercube \( I_{1/4}^4 \), every coordinate of which assumes one and the same discrete series of values: \( 1/4, 1/16, 1/36, 1/64, \ldots \), \( 1/4n^2, \ldots \). (Further it is demonstrated that \( \tau \) is also taking on a discrete series of values.) The question arises, whether time falls within this picture. The answer is positive. Indeed, parameter \( \tau \) is dual (thermal and temporal) in nature owing to introduction of the Generalized Uncertainty Relations in Thermodynamics (GURT) ([48], [49], [41] and section 6):

\[
\Delta \frac{1}{T} \geq \frac{k}{\Delta U} + \alpha' \frac{1}{T_p^2} \frac{\Delta U}{k} + \ldots,
\]

where \( k \) - Boltzmann constant, \( T \) - temperature of the ensemble, \( U \) - its internal energy. A direct implication of the latter inequality is occurrence of a maximum temperature \( T_{\text{max}} \) that is inversely proportional to minimal time \( t_{\text{min}} \sim t_p^2 \):

\[
T_{\text{max}} = \frac{\hbar}{2\sqrt{\alpha' t_p k}} = \frac{\hbar}{\Delta t_{\text{min}} k}
\]

However, \( t_{\text{min}} \) follows from the Generalized Uncertainty Relations in Quantum Mechanics for energy-time\(^\text{ii}\) pair ([40], [41] and section 5):

\[
\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar}.
\]

Thus, \( T_{\text{max}} \) is the value relating GUR and GURT together (see sections 5,6 and [48], [49], [41])

\[
\begin{align*}
\Delta x & \geq \frac{\hbar}{\Delta p} + \alpha' t_p^2 \frac{\Delta p}{\hbar} + \ldots \\
\Delta t & \geq \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar} + \ldots \\
\Delta \frac{1}{T} & \geq \frac{k}{\Delta U} + \alpha' \frac{1}{T_p^2} \frac{\Delta U}{k} + \ldots,
\end{align*}
\]

, since the thermodynamic value \( T_{\text{max}} \) (GURT) is associated with the quantum-mechanical one \( E_{\text{max}} \) (GUR) by the formula from section 5:

\[
T_{\text{max}} = \frac{E_{\text{max}}}{k}
\]

38
The notion of value \( t_{\text{min}} \sim 1/T_{\text{max}} \) is physically crystal clear, it means a minimum time for which any variations in the energy spectrum of every physical system may be recorded. Actually, this value is equal to \( t_{\text{min}}^* = 2t_{\text{min}} \sim t_p \) as at the initial points \( l_{\text{min}} \) and \( T_{\text{max}} \) the spurs of the quantum-mechanical and statistical density pro-matrices \( \rho(\alpha) \) and \( \rho_{\text{stat}}(\tau) \) are complex, determined only beginning from \( 2l_{\text{min}} T_{\text{max}} = \frac{1}{2} T_{\text{max}} \) [17], [41]

that is associated with the same time point \( t_{\text{min}}^* = 2t_{\text{min}} \). For QMFL this has been noted in the previous section.

In such a manner a discrete series \( l_{\text{min}}^*, 2l_{\text{min}}^*, \ldots \) generates in QMFL the discrete time series \( t_{\text{min}}^*, 2t_{\text{min}}^*, \ldots \) that is in turn associated (due to GURT) with a discrete temperature series \( T_{\text{max}}^*, \frac{1}{2} T_{\text{max}}^*, \ldots \). From this it is inferred that a temperature scale \( \tau \) may be interpreted as a temporal one \( \tau = t_{\text{min}}^2/t^2 \). In both cases the generated series has one and the same discrete set of values for parameter \( \tau : \tau = 1/4, 1/16, 1/36, 1/64, \ldots \). Thus, owing to time quantization in QMFL, one is enabled to realize quantization of temperature in the generalized Statistical Mechanics with the use of GURT.

Using \( \text{Lat}_{\tilde{\alpha}} \), we denote the lattice in cube \( I_{1/4}^3 \) formed by points \( \tilde{\alpha} \), and through \( \text{Lat}_{\tilde{\alpha}}^\tau \) we denote the lattice in hypercube \( I_{1/4}^4 \) that is formed by points \( \tilde{\alpha}_{\tau} = (\tilde{\alpha}, \tau) \).

### 8.3 Quantum theory for the lattice in hypercube

Any quantum theory may be defined for the indicated lattice in hypercube. To this end, we recall the principal result of subsection 4.4 as Definition 1’ in this section with \( \alpha \) changed by \( \tilde{\alpha} 

**Definition 1’ Quantum Mechanics with Fundamental Length (Shrödinger’s picture)**

Here, the prototype of Quantum Mechanical normed wave function (or the pure state prototype) \( \psi(q) \) with \( \int |\psi(q)|^2 dq = 1 \) in QMFL is \( \psi(\tilde{\alpha}, q) = \theta(\tilde{\alpha}) \psi(q) \). The parameter of deformation \( \tilde{\alpha} \in I_{1/4}^3 \). Its properties are \( |\theta(\tilde{\alpha})|^2 < 1, \lim_{|\tilde{\alpha}|\to 0} |\theta(\tilde{\alpha})|^2 = 1 \) and the relation \( |\theta(\alpha_i)|^2 - |\theta(\alpha_i)|^4 \approx \alpha_i \) takes place. In such a way the total probability always is less than 1: \( p(\tilde{\alpha}) = |\theta(\tilde{\alpha})|^2 = \int |\theta(\tilde{\alpha})|^2 |\psi(q)|^2 dq < 1 \) tending to 1, when \( \|\tilde{\alpha}\| \to 0 \). In the most general case of the arbitrarily normed state in QMFL(mixed state prototype) \( \psi = \psi(\tilde{\alpha}, q) = \sum_n a_n \theta_n(\tilde{\alpha}) \psi_n(q) \) with \( \sum_n |a_n|^2 = 1 \) the total
probability is 
\[ p(\tilde{\alpha}) = \sum_n |a_n|^2 |\theta_n(\tilde{\alpha})|^2 < 1 \quad \text{and} \quad \lim_{\|\tilde{\alpha}\|\to 0} p(\tilde{\alpha}) = 1. \]

It is natural that Shrödinger equation is also deformed in QMFL. It is replaced by the equation
\[ \frac{\partial \psi(\tilde{\alpha}, q)}{\partial t} = \frac{\partial}{\partial t} \left[ \theta(\tilde{\alpha}) \psi(q) \right] = \theta(\tilde{\alpha}) \frac{\partial \psi(q)}{\partial t} + \frac{\theta(\tilde{\alpha})}{\hbar} H \psi(q), \]
where the second term in the right-hand side generates the Shrödinger equation as
\[ \theta(\tilde{\alpha}) \frac{\partial \psi(q)}{\partial t} = -i \theta(\tilde{\alpha}) \hbar H \psi(q). \]

Here \( H \) is the Hamiltonian and the first member is added similarly to the member that appears in the deformed Liouville equation, vanishing when \( \theta[\tilde{\alpha}(t)] \approx \text{const.} \). In particular, this takes place in the low energy limit in QM, when \( \|\tilde{\alpha}\| \to 0 \). It should be noted that the above theory is not a time reversal of QM because the combination \( \theta(\tilde{\alpha}) \psi(q) \) breaks down this property in the deformed Shrödinger equation. Time-reversal is conserved only in the low energy limit, when a quantum mechanical Shrödinger equation is valid.

According to Definition 1' everywhere \( q \) is the coordinate of a point at the three-dimensional space. As indicated in [13]–[17] and section 3.2, for a density pro-matrix there exists an exponential ansatz satisfying the formula (7) of Definition 1, section 3.1:
\[ \rho^*(\alpha) = \sum_i \omega_i \exp(-\alpha_i |i\rangle < i|), \]
where all \( \omega_i > 0 \) are independent of \( \alpha \) and their sum is equal to 1. In this way \( Sp[\rho^*(\alpha)] = \exp(-\alpha) \). Then in the momentum representation \( \alpha = p^2/p_{\text{max}}^2, p_{\text{max}} \sim p_{\text{pl}} \), where \( p_{\text{pl}} \) is the Planck momentum. When present in matrix elements, \( \exp(-\alpha) \) damps the contribution of great momenta in a perturbation theory.

It is clear that for each of the coordinates \( q_i \) the exponential ansatz makes a contribution to the deformed wave function \( \psi(\tilde{\alpha}, q) \) the modulus of which equals \( \exp(-\alpha_i/2) \) and, obviously, the same contribution to the conjugate function \( \psi^*(\tilde{\alpha}, q) \). Because of this, for exponential ansatz one may write
\[ \psi(\tilde{\alpha}, q) = \theta(\tilde{\alpha}) \psi(q), \]
where \( |\theta(\tilde{\alpha})| = \exp(- \sum_i \alpha_i/2) \). As noted above, the last exponent of the momentum representation reads \( \exp(- \sum_i p_i^2/2p_{\text{max}}^2) \) and in this way it removes UV (ultra-violet) divergences in the theory.
It follows that $\tilde{\alpha}$ is a new small parameter. Among its obvious advantages one could name:

1) its dimensionless nature,
2) its variability over the finite interval $0 < \alpha_i \leq 1/4$. Besides, for the well-known physics it is actually very small: $\alpha \sim 10^{-66+2n}$, where $10^{-n}$ is the measuring scale. Here the Planck scale $\sim 10^{-33} cm$ is assumed;
3) and finally the calculation of this parameter involves all three fundamental constants, since by Definition 1 of subsection 3.1

\[ \alpha_i = \frac{l_{\min}^2}{x_i^2}, \]

where $x_i$ is the measuring scale on $i$-coordinate and $l_{\min}^2 \sim l_{pl}^2 = G\hbar/c^3$.

Therefore, series expansion in $\alpha_i$ may be of great importance. Since all the field components and hence the Lagrangian will be dependent on $\tilde{\alpha}$, i.e. $\psi = \psi(\tilde{\alpha})$, $L = L(\tilde{\alpha})$, quantum theory may be considered as a theory of lattice $Lat_{\tilde{\alpha}}$ and hence of lattice $Lat_{\tau_{\tilde{\alpha}}}$.

### 8.4 Introduction of quantum field theory and initial analysis

With the use of this approach for the customary energies a Quantum Field Theory (QFT) is introduced with a high degree of accuracy. In our context customary means the energies much lower than the Planck ones.

It is important that as the spacing of lattice $Lat_{\tilde{\alpha}}^\tau$ is decreasing in inverse proportion to the square of the respective node, for a fairly large node number $N > N_0$ the lattice edge beginning at this node $\ell_{N,N+1}$ will be of length $\ell_{N,N+1} \sim 1/4N^3$, and by this means edge lengths of the lattice are rapidly decreasing with the spacing number. Note that in the large-scale limit this (within any preset accuracy) leads to parameter $\alpha = 0$, pure states and in the end to QFT. In this way a theory for the above-described lattice presents a deformation of the originally continuous variant of this theory as within the developed approach continuity is accurate to $\approx 10^{-66+2n}$, where $10^{-n}$ is the measuring scale and the Planck scale $\sim 10^{-33} cm$ is assumed. Whereas the lattice per se $Lat_{\tilde{\alpha}}^\tau$ may be interpreted as a deformation of the space continuum with the deformation parameter equal to the varying edge length $\ell_{\alpha_1, \alpha_2}$, where $\alpha_1, \alpha_2$ are two adjacent points of the lattice $Lat_{\tau_{\tilde{\alpha}}}^\tau$. Proceeding from this, all well-known theories including $\phi^4$, QED, QCD and so on may be studied based on the above-described lattice.

Here it is expedient to make the following remarks:

1) going on from the well-known energies of these theories to
higher energies (UV behavior) means a change from description of the theory's behavior for the lattice portion with high edge numbers to the portion with low numbers of the edges;

(2) finding of quantum correction factors for the primary deformation parameter \( \bar{\alpha} \) is a power series expansion in each \( \alpha_i \). In particular, in the simplest case (Definition 1' of subsection 8.3) means expansion of the left side in relation

\[
|\theta(\alpha_i)|^2 - |\theta(\alpha_i)|^4 \approx \alpha_i:
\]

and calculation of the associated coefficients \( a_0, a_1, ... \). This approach to calculation of the quantum correction factors may be used in the formalism for density pro-matrix (Definition 1 of subsection 3.1). In this case, the primary relation (7) of Definition 1, section 3.1 may be written in the form of a series

\[
Sp[\rho(\alpha)] - Sp^2[\rho(\alpha)] = \alpha + a_0\alpha^2 + a_1\alpha^3 + ... \tag{60}
\]

As a result, a measurement procedure using the exponential ansatz may be understood as the calculation of factors \( a_0, a_1, ... \) or the definition of additional members in the exponent destroying \( a_0, a_1, ... \) \[55\]. It is easy to check that the exponential ansatz gives \( a_0 = -3/2 \), being coincident with the logarithmic correction factor for the Black Hole entropy \[30\].

Most often a quantum theory is considered at zero temperature \( T = 0 \), in this context amounting to nesting of the three-dimensional lattice \( \text{Lat}_{\bar{\alpha}} \) into the four-dimensional one: \( \text{Lat}_{\bar{\alpha}}^4 \subset \text{Lat}_{\bar{\alpha}}^3 \) and nesting of the cube \( I_{1/4}^3 \) into the hypercube \( I_{1/4}^4 \) as a bound given by equation \( \tau = 0 \). However, in the most general case the points with nonzero values of \( \tau \) may be important as there is a possibility for nonzero temperature \( T \neq 0 \) (quantum field theory at finite temperature) that is related to the value of \( \tau \) parameter, though very small but still nonzero: \( \tau \neq 0 \). To illustrate: in QCD for the normal lattice \[59\] a critical temperature \( T_c \) exists so that the following is fulfilled:

at

\[ T < T_c \]

the confinement phase occurs,

and for

\[ T > T_c \]

42
the deconfinement is the case.
A critical temperature $T_c$ is associated with the critical parameter $\tau_c = \frac{T_c^2}{T_{\text{max}}^2}$ and the selected bound of hypercube $I_{1/4}$ set by equation $\tau = \tau_c > 0$.

9 Conclusion

In conclusion the scope of problems associated with the above-mentioned methods is briefly outlined.

I. Involvement of Heisenbergs Algebra Deformation
One of the major problems associated with the proposed approach to investigation of Quantum Mechanics of the Early Universe is an understanding of its relation to the Heisenberg's algebra deformation (e.g. see [7]). It should be noted that from the authors point of view the latter has two serious disadvantages:
1) the deformation parameter is a dimensional variable $\kappa$ with a dimension of mass;
2) in the limiting transition to QM this parameter goes to infinity and fluctuations of other values are hardly sensitive to it.
At the same time, the merit of this approach is its ability with particular assumptions to reproduce the Generalized Uncertainty Relations. The proposed approach is free from such limitations as 1) and 2), since the deformation parameter is represented by the dimensionless quantity $\alpha$ and the variation interval $\alpha$ is finite $0 < \alpha \leq 1/4$. However, it provides no direct reproduction of the Generalized Uncertainty Relations. This approach is applicable in the general cases of Quantum Mechanics with Fundamental Length irrespective of the fact whether it is derived from the Generalized Uncertainty Relations or in some other way. Because of this, involvement of the both approaches in deformation of Quantum Mechanics is of particular importance.

II. The Approach as Applied to a Quantum Theory of Black Holes
2.1 Bekenstein-Hawking formula strong derivation
This paper presents certain results pertinent to the application of the above methods in a Quantum Theory of Black Holes (subsections 4.2, 4.3). Further investigations are still required in this respect, specifically for the
complete derivation of a semiclassical Bekenstein-Hawking formula for the Black Hole entropy, since in subsection 4.3 the treatment has been based on the demonstrated result: a respective number of the degrees of freedom is equal to $A$, where $A$ is the surface area of a black hole measured in Plancks units of area $L_p^2$ (e.g. [21], [28]). Also it is essential to derive this result from the basic principles given in this paper. Problems 2.1 and 2.2 are related.

2.2 Calculation of quantum corrections to the Bekenstein-Hawking formula.
In subsection 8.4 for the introduced logarithmic correction it has been noted (see for example [30]) that it is coincident with coefficient $a_0$ in formula (60):

$$Sp[\rho(\alpha)] - Sp^2[\rho(\alpha)] = \alpha + a_0 \alpha^2 + a_1 \alpha^3 + ...$$

when using the exponential ansatz. It is clear that such a coincidence is not accidental and further investigations are required to elucidate this problem.

2.3 Quantum mechanics and thermodynamics of black holes with GUR
Of interest is to consider the results of [25], [37] as related to the quantum-mechanical studies and thermodynamics of black holes with GUR assumed valid rather than the Heisenberg Uncertainty Relations. This is directly connected to the above-mentioned problem of the associations between the density matrix deformation considered in this work and Heisenbergs algebra deformation.

2.4 Singularities and cosmic censorship hypothesis
In subsection 4.2 a slight recourse has been made to the case when Schwarzshild radius is $r = 0$ that is associated with going to value $\alpha = 1$ and finally to a complex value of the density pro-matrix trace $Sp[\rho(\alpha)]$. It should be noted that the problem of singularities is much more complex [60] and is presently treated both physically and mathematically. It seems interesting to establish the involvement of the results obtained by the author in solving of this problem.

III. Divergence in Quantum Field Theory
It is obvious that once the fundamental length is included into a Quantum
Theory, ultra-violet (UV) divergences should be excluded due to the presence of a maximum momentum determining the cut-off \[1\]. In case under study this is indicated by the presence of an exponential ansatz (subsection 3.2). Note, however, that for any particular theory it is essential to derive the results from the basic principles with high accuracy and in good agreement with the already available ones and with the experimental data of QFT for the UV region without renormalization \[61, 62\].

IV. The Approach as Applied to Inflation Cosmology

As we concern ourselves with the Early Universe (Planck's energies), the proposed methods may be applied in studies of inflation cosmology \[12, 34, 63\], especially as Wheeler-DeWitt Wave Function of the Universe $\Psi$ \[18\] is reliably applicable in a semiclassical approximation only \[64\]. The problem is formulated as follows: on what conditions and in what way the density pro-matrix $\rho(\alpha)$ or its respective modification may be a substitute for $\Psi$ in inflation models?

V. High-Energy Deformation of Gravitation

Since this work actually presents a study of physics at Planck scales, it is expedient to consider quantum-gravitational effects which should be incorporated for specific energies. As a development of the proposed approach this means the construction of an adequate deformation of the General Relativity including parameter $\alpha$, i.e. deformation of Einstein’s Equations and the associated Lagrangian involving parameter $\alpha$. Then the question arises: and what about the space-time quantization? The author holds the viewpoint that as the first approximation of a quantized space-time one can use a portion of the Nonuniform Lattice $Lat^\alpha_\tau$ described in section 8 that is associated with small-number nodes or with high-valued parameters $\tilde{\alpha}$ and $\tau$, just which are used to define the physics at Planck scale where the quantum-gravitational effects are considerable. In this approximation for the prototype of a point in the General Relativity may be taken an elementary cell, i.e. as an element of the above-mentioned lattice with small-number neighboring nodes. Then the associated deformation of Einstein’s should be considered exactly in this cell.

Note that this section involves all the problems considered in I-IV.

In summary it might be well to make three general remarks.

1) It should be noted that in some well-known papers on GUR and Quantum Gravity (e.g. see \[1, 4, 6, 7\]) there is no mention of any measuring
procedure. However, it is clear that this question is crucial and it cannot be ignored or passed over in silence. We would like to remark that the measuring rule used in [35], (formula (5)) is identical to the ours. In this paper the proposed measuring rule [3] is a good initial approximation to the exact measuring procedure of QMFL. Corrections to this procedure could be defined by an adequate and fully established description of the space-time foam (see [33], [65]) at Planck’s scale.

2) One of the principal issues of the present work is the development of a unified approach to study all the available quantum theories without exception owing to the proposed small dimensionless deformation parameter $\tilde{\alpha}_r \in \text{Lat}_{r\tilde{\alpha}}$ that is in turn dependent on all the fundamental constants $G, c, \hbar$ and $k$.

Thus, there is reason to believe that lattices $\text{Lat}_{\tilde{\alpha}}$ and $\text{Lat}_{r\tilde{\alpha}}^x$ may be a universal means to study different quantum theories. This poses a number of intriguing problems:

1) description of a set of lattice symmetries $\text{Lat}_{\tilde{\alpha}}$ and $\text{Lat}_{r\tilde{\alpha}}^x$;

2) for each of the well-known physical theories ($\phi^4$, QED, QCD and so on) definition of the selected (special) points (phase transitions, different symmetry violations and so on) associated with the above-mentioned lattices.

3) As it was noted in [2], advancement of a new physical theory implies the introduction of a new parameter and deformation of the precedent theory by this parameter. In essence, all these deformation parameters are fundamental constants: $G, c$ and $\hbar$ (more exactly in [2] $1/c$ is used instead of $c$). As follows from the above results, in the problem from [2] one may redetermine, whether a theory we are seeking is the theory with the fundamental length involving these three parameters by definition: $L_p = \sqrt{\frac{G\hbar}{c^3}}$.

Notice also that the deformation introduced in this paper is stable in the sense indicated in [2].

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