Non-exactness of the SWKB Quantization Condition for Several Classes of Solvable Systems and the Higher Order Corrections

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(Dated: August 31, 2021)

The SWKB quantization condition is an exact quantization condition for the conventional shape-invariant potentials. This exactness is partly understood in the context of the quantum Hamilton–Jacobi (QHJ) formalism. On the other hand, this condition equation does not hold for other known solvable systems. We have already shown numerically for the so-called Krein–Adler systems that the condition is not exactly but approximately satisfied [Mod. Phys. Lett. A 36, 2150025 (2021)]. In this paper, we study the non-exactness of the SWKB condition for the Krein–Adler system and further general classes of the conditionally exactly solvable systems in detail. We carry out the calculations in the QHJ approach to show properties of integrands in the quantization conditions in the complex plane, which gives us a qualitative understanding of the non-exactness. Moreover, we propose a novel method to analyze the deviation of the SWKB condition equation.

I. INTRODUCTION

The supersymmetric quantum mechanics (SUSY QM) [1,2] is a powerful tool for analyzing spectra of many potential problems in quantum mechanics with one space coordinate. The study is usually performed for the cases of some special classes of potential, i.e., the solvable potentials, which are often models for the realistic bound-state problems. The well-known example is the systems possessing the shape invariance (SI) [3]. It is a sufficient (but not a necessary) condition for the exact solvability of the Schrödinger equation. There are numerous studies concerning the shape-invariant (SI) potentials; the authors compared some solvable potentials with the related SI ones [5,6] to inquire the origin of the exactness. Also, several novel types of SI potentials [6,12] have been proposed so far.

In 1980’s, Comtet and co-authors proposed a WKB-like integral form of the quantization con-
dition in the context of SUSY QM [13]:

\[ I_{SWKB} = n \pi \hbar, \quad I_{SWKB} := \int_{a_{cl}}^{a_{fl}} \sqrt{E_n - W(x)^2} \, dx, \quad n \in \mathbb{Z}_{\geq 0}, \tag{1} \]

which is called the supersymmetric WKB (SWKB) quantization condition. This condition seems to have a deep physical insight, for it successfully reproduces the exact bound-state spectra for all conventional SI potentials, as was demonstrated by Dutt et al. [14]. Many people at the period thought that the SWKB condition became exact if and only if the system possesses SI.

After that, several efforts have been made for this criterion. Khare et al. derived the non-exactness of the SWKB condition for the Ginocchio potential [15] and a potential iso-spectral to the one-dimensional harmonic oscillator (1-d H.O.) [16], both of which are exactly solvable but are not SI [17]. DeLaney et al. showed that the Abraham–Moses systems [16], which are another class of solvable potentials without SI, does not satisfy the SWKB condition [18]. In our previous letter [19], we have provided yet another example; the Krein–Adler systems [20, 21] which are a class of solvable potentials but not SI, do not satisfy the SWKB condition. Obviously, the exactly solvable systems without SI almost always break the exactness of SWKB condition. Then, what about cases of the novel class of SI systems? The Multi-indexed systems [7–11] are the case, where such systems do not satisfy the SWKB condition [19, 22]. This clearly indicates that the SI of a potential does not always give a sufficient condition for the exactness of the SWKB condition. Now, we conclude that SI does not account for the exactness of the SWKB condition.

In 1996, Bhalla et al. gave a qualitative explanation for the (non-)exactness of the SWKB condition by means of the quantum Hamilton–Jacobi (QHJ) formalism [23, 24]. The QHJ formalism was first developed by Leacock and Padgett, and is another approach toward the bound-state problems [25, 26]. By employing this, one can obtain the exact bound-state spectra without any information of the eigenstate wave functions. The energy eigenvalues are given by the result of the quantization condition for a quantum analogue of the action variable, which is similar to the relation between the frequency and the action variable in classical mechanics [27]. The quantization condition for the quantum action variable is

\[ J_{QHJ} = nh, \quad J_{QHJ} := \frac{1}{2\pi} \oint_{C} p(x; \mathcal{E}) \, dx, \quad n \in \mathbb{Z}_{\geq 0}, \tag{2} \]

where \( p(x; \mathcal{E}) \) is called the quantum momentum function (QMF). Bhalla et al. pointed out that the quantization condition for the quantum action variable is equivalent to the SWKB quantization condition for the conventional SI systems. They concluded that the SWKB condition becomes an exact quantization condition when the integrand (in which the real variable \( x \) is extended
to a complex variable $x$) has the common singularity nature with the QMF. If it is correct, we have an intuition for the non-exactness of the SWKB condition; it depends on the existence of extra singularities of the integrand. For the case of the conventional SI potentials, the complex integrations for the QMFs can be deformed into simple WKB-like integrations along the real axis. For the other potentials, the deformation is not so straightforward, thus the SWKB condition takes some advantages.

Incompatibility of these singularities thus concern the higher order corrections of the SWKB condition, i.e., the residual $\Delta := I_{SWKB} - \pi J_{QHJ}$. In Ref. [24], $\Delta$ is estimated with the residues of the poles of the SWKB integrand that essentially is different from those of the QMF. For this point of view, they got a quite natural explanation about why the SWKB condition of a potential can not reproduce the exact bound-state spectra. However, yet another difficulty exists in due course; one cannot always calculate the poles analytically in some potentials and the integration is not straightforward in these cases. Further, an essential question arises: why the singular natures coincide for the conventional SI potentials?

In order to achieve a deeper understanding of the SWKB condition, we study $\Delta$ somewhat different point of view. For a given potential, suppose we have a quantized integral and the SWKB integral $I_{SWKB}$ is a leading order of an expansion with a parameter which is not yet fixed (the “unknown parameter”). All the higher order corrections are gathered in $\Delta$ and then, an “exact” SWKB condition is formally expressed as follows: $I_{SWKB} + \Delta = n\pi \hbar$. Note that for the conventional SI systems, $\Delta = 0$. The evaluation of $\Delta$ is important because it brings us a good intuition for the meaning of the condition, and also one could obtain an “exact” SWKB formula for a given potential, which is certainly a final goal of the study on the SWKB condition. Moreover, as DeLaney et al. already mentioned, the SWKB condition is a better approximation than the WKB formalism for the estimation of the energy eigenvalues. This indicates that the SWKB condition gives a better formula for the energy. The analysis on $\Delta$ will enable us to obtain an even more accurate formula for the energy. For the studies of the SWKB condition and the method of evaluating $\Delta$, several approaches has already formulated so far [22, 28, 30].

As was mentioned above, the investigation of $\Delta$ is a significant issue and has several future applications. We mention another one here. Although the conjecture that the SWKB condition is an exact condition if and only if we deal with the conventional SI potentials is solved negatively, there is still a chance the condition is somehow involved with the solvability of the Schrödinger equation. By getting to know how $\Delta$ behaves around $\Delta = 0$, the role that the SWKB condition plays in the solvability of the Schrödinger equation could be unveiled.
In this paper, we study the non-exactness of the SWKB condition and the residual $\Delta$ for several classes of solvable systems, and propose a novel approach to analyze and evaluate $\Delta$. In Sec. II we introduce two quantization conditions: the one in the QHJ formalism and the SWKB quantization condition. We also write down the three classes of solvable systems that will appear in this paper. Sec. III is devoted to show examples where the SWKB condition does not hold and explain how one can understand the breaking of the condition equation in comparison to the QHJ approach. Sec. IV deals with the detailed analysis on the non-exactness of the SWKB condition. By employing the conditionally exactly solvable systems, we show where the non-exactness comes from and formulate a perturbative approach toward the SWKB integral and the residual $\Delta$. We also discuss the properties of $\Delta$ here. We make concluding remarks and point out a few future directions of this sequence of studies in Sec. V.

II. PRELIMINARIES

In 1996, Bhalla et al. gave a qualitative explanation for the (non-)exactness of the SWKB condition by means of the quantum Hamilton–Jacobi (QHJ) formalism [23, 24]. It is instructive for better understanding of implication the SWKB. Here we give brief reviews for both formulations.

A. SWKB condition

The potential $V(x)$ is formally given by the ground-state wave function $\psi_0(x)$, corresponding to the vanishing ground-state energy, $\mathcal{E}_0 = 0$:

$$V(x) = h^2 \left[ (\partial_x \ln |\phi_0(x)|)^2 + \partial_x^2 \ln |\phi_0(x)| \right]. \quad (3)$$

We set $2m = 1$ but retain $\hbar$ for discussing explicit $\hbar$-dependency. In the standard WKB, the quantization condition is given in terms of the potential $V(x)$. On the other hand, for the SWKB quantization condition it only concerns the first term of (3), where $-\hbar \partial_x \ln |\phi_0(x)|$ is referred to as the superpotential, denoted by $W(x)$.

The SWKB condition (whose l.h.s is sometimes referred as the SWKB integral) [13] is

$$I_{\text{SWKB}} = n \pi h, \quad I_{\text{SWKB}} := \int_{a_L}^{a_R} \sqrt{\mathcal{E}_n - W(x)^2} \, dx, \quad n \in \mathbb{Z}_{\geq 0}, \quad (4)$$

where $a_L, a_R$ ($a_L < a_R$) are the turning points which are obtained as the roots of $W(x)^2 = \mathcal{E}_n$. For the ground state the condition is always exact and also, it is well-known that the condition is conserved for a wide class of potentials called the conventional SI potentials. The most essential
aspect with the condition is that $\hbar$ can always be factored out from the SWKB integral \cite{19}, which was often missed in the literatures.

In some cases, the equation $W(x)^2 = \mathcal{E}_n$ possesses more than two roots: \{$(a_{L,i}, a_{R,i})$; $i = 1, \ldots, N$\}. We employ a prescription of estimating $I_{\text{SWKB}}$ by summing up the SWKB integrals of all $i$ \cite{19}:

$$I_{\text{SWKB}} = \sum_i \int_{a_{L,i}}^{a_{R,i}} \sqrt{\mathcal{E}_n - W(x)^2} \, dx . \tag{5}$$

In this paper, we only consider the cases where the equation has two solutions, \textit{i.e.}, $i = 1$, for the rigorous study on the exactness of the SWKB condition.

\textbf{B. Exact quantization condition for quantum action variable in QHJ formalism}

The QHJ formalism was first developed by Leacock and Padgett \cite{25, 26}. In the QHJ formalism, the quantum momentum function (QMF):

$$p(x; \mathcal{E}) = \frac{\hbar}{i} \frac{\partial_x \psi(x)}{\psi(x)} , \tag{6}$$

plays the central role. The QMF satisfies the QHJ equation:

$$p(x; \mathcal{E})^2 + \frac{\hbar}{i} \partial_x p(x; \mathcal{E}) = \mathcal{E} - V(x) , \tag{7}$$

where $\mathcal{E}$ and $V(x)$ are the energy and the potential of the system. $\psi(x)$ is a wave function and relates to the QMF via $\psi(x) = \exp[iS(x)/\hbar]$ and $p(x, \mathcal{E}) = \partial_x S(x)$. The quantum action variable $J_{\text{QHJ}}$ is defined as an analogy with the classical one and the quantization condition for the quantum action variable is written as

$$J_{\text{QHJ}} = nh , \quad J_{\text{QHJ}} := \frac{1}{2\pi} \oint_C p(x; \mathcal{E}) \, dx , \quad n \in \mathbb{Z}_{\geq 0} , \tag{8}$$

where $C$ is a counterclockwise contour in the complex $x$-plane enclosing the classical turning points $x_L, x_R$: $V(x_L) = V(x_R) = \mathcal{E}$. This condition equation is guaranteed by the fact that $n$-th eigenfunction has $n$ nodes, which produce the poles of $p(x; \mathcal{E})$ having residue $\hbar/i$, on the real $x$ axis between the two classical turning points.

As Bhalla \textit{et al.} pointed out, when the SWKB integrand is extended into the complex plane $x \in \mathbb{C}$, the singularity structure of the QMF $p(x, \mathcal{E})$ and that of the SWKB integrand in the conventional SI systems perfectly coincide, \textit{i.e.}, $I_{\text{SWKB}} = \pi J_{\text{QHJ}} = n\pi \hbar$. Contrary, for the case where the quantization of $I_{\text{SWKB}}$ is not exact, it is easy to guess the singularity structures are
different from those of the corresponding QMF. In this paper, we examine the singular nature of the SWKB integral for the Krein–Adler systems and also the conditionally exactly solvable systems, and see how their exactness are broken.

C. Several exactly solvable systems

In this subsection, we give a brief review on the conventional SI systems, the conditionally exactly solvable (CES) systems and the several Krein–Adler systems.

1. Conventional shape-invariant systems

We start with the conventional SI potentials. The term “conventional” reflects the fact that these potentials are already known in the 1950’s. A dozen quantum mechanical systems belong to this class. In this paper, we consider systems that are obtained by a systematic deformation from the following three conventional SI systems. The Hamiltonians for the three conventional SI systems are defined as

\[
\mathcal{H}^{(\ast)} = -\hbar^2 \frac{\partial^2}{\partial x^2} + V^{(\ast)}(x),
\]

\[
V^{(\ast)}(x) = \begin{cases} 
\omega^2 x^2 - \hbar \omega, & x \in (-\infty, \infty) \quad \ast = H \\
\omega^2 x^2 + \frac{\hbar^2 g(g-1)}{x^2} - \hbar \omega(2g + 1), & x \in (0, \infty) \quad \ast = L, \\
\frac{\hbar^2 g(g-1)}{\sin^2 x} + \frac{\hbar^2 h(h-1)}{\cos^2 x} - \hbar^2 (g + h)^2, & x \in \left(0, \frac{\pi}{2}\right) \quad \ast = J
\end{cases}
\]

and the Schrödinger equations are

\[
\mathcal{H}^{(\ast)} \phi_n^{(\ast)}(x) = \mathcal{E}_n^{(\ast)} \phi_n^{(\ast)}(x),
\]

with

\[
\mathcal{E}_n^{(\ast)} = \begin{cases} 
2n \hbar \omega & \ast = H \\
4n \hbar \omega & \ast = L, \\
4\hbar^2 n(n + g + h) & \ast = J
\end{cases}
\]

\[
\phi_n^{(\ast)}(x) = \begin{cases} 
e^{-\frac{x^2}{2}} H_n(\xi) & \ast = H \\
e^{-\frac{z^2}{2}} L_n^{\left(g - \frac{1}{2}\right)}(z) & \ast = L, \\
(1 - y)^{\frac{1}{2}} (1 + y)^{\frac{1}{2}} P_n^{\left(g - \frac{1}{2}, h - \frac{1}{2}\right)}(y) & \ast = J
\end{cases}
\]
where $H_n, L_n^{(\alpha)}, P_n^{(\alpha, \beta)}$ are Hermite, Laguerre, Jacobi polynomials respectively, and $\xi \equiv \sqrt{\omega/\hbar} x$, $z \equiv \xi^2$ and $y \equiv \cos 2x$. These three systems are of importance in that they have the classical orthogonal polynomials as the main parts of their eigenfunctions, and the ground-state wave functions coincide with the square root of the weight functions for the polynomials. Note that the SI transformations are $g \rightarrow g + 1$ and $h \rightarrow h + 1$.

When the SWKB condition is exact, one can directly deduce the energy spectra from the condition. The analytical computations of the energy spectra for this class of solvable potentials are first appeared in the literature in 1997 [31]. Here we demonstrate the procedure for the case of $\ast = H$. All the others can be done in the same manner. The superpotential is

$$W(x) = -\hbar \partial_x \ln \left| \phi^{(H)}_0(x) \right| = -\omega x$$

in this case, and the SWKB integral is calculated as

$$I_{SWKB} = \int_{-\sqrt{\mathcal{E}/\omega}}^{\sqrt{\mathcal{E}/\omega}} \sqrt{\mathcal{E} - \omega^2 x^2} \, dx = \frac{2\mathcal{E}}{\omega} \int_0^1 \sqrt{1 - t^2} \, dt = \frac{\pi \mathcal{E}}{2\omega},$$

(13)

where $t \equiv \omega x/\sqrt{\mathcal{E}}$. From the SWKB quantization it reduces

$$I_{SWKB} = \frac{\pi \mathcal{E}}{2\omega} = n\pi \hbar \quad i.e. \quad \mathcal{E}_n = 2n\hbar \omega,$$

(14)

which indeed agrees with Eq. (11).

2. Conditionally exactly solvable systems

A system is conditionally exactly solvable, when the eigenvalues and eigenfunctions are obtained explicitly for some specific choices of potential parameters [32, 33]. In this paper, we follow the construction method of the CES systems based on SUSY QM, which was developed by Junker and Roy [34]. In their new class of exactly solvable systems, two continuous parameters exist; one concerns with the modification of the level structure and the other describes iso-spectral deformation of a potential.

The Hamiltonians for the CES systems are

$$\mathcal{H}^{(C, \ast)} := -\hbar^2 \partial_x^2 + \hbar^2 \left[ \left( \partial_x \ln \left| \phi^{(\ast)}_0(x) \right| \right)^2 + \partial_x^2 \ln \left| \frac{\phi^{(\ast)}_0(x)}{u^{(\ast)}(x)} \right| \right], \quad \ast = H, L, J,$$

(15)

where $u^{(\ast)}(x)$ satisfies

$$\hbar^2 \partial_x^2 u^{(\ast)}(x) - 2\hbar^2 \left( \partial_x \ln \left| \phi^{(\ast)}_0(x) \right| \right) \partial_x u^{(\ast)}(x) - \tilde{b} u^{(\ast)}(x) = 0.$$  

(16)
The Schrödinger equation is

$$\mathcal{H}^{(C,\ast)} \phi^{(C,\ast)}_n(x) = \mathcal{E}^{(C,\ast)}_n \phi^{(C,\ast)}_n(x)$$  \hspace{1cm} (17)$$

with

$$\mathcal{E}^{(C,\ast)}_0 = 0, \quad \mathcal{E}^{(C,\ast)}_n = \mathcal{E}^{(\ast)}_n + \tilde{b} \quad \text{for} \ n \geq 1,$$

$$\phi^{(C,\ast)}_0(x) = \frac{\phi^{(\ast)}_0(x)}{u^{(\ast)}(x)} \quad \phi^{(C,\ast)}_n(x) = \hbar \left( -\partial_x - \partial_x \ln \left| \phi^{(C,\ast)}_n(x) \right| \right) \phi^{(\ast,\pm)}_{n-1}(x) \quad \text{for} \ n \geq 1.  \hspace{1cm} (19)$$

Here, $$\phi^{(\ast,\pm)}_n(x)$$ denotes the $$n$$-th eigenstate of the SUSY partner of $$\mathcal{H}^{(\ast)}$$. Note that from the positivity of $$\mathcal{H}^{(C,\ast)}$$, $$\mathcal{E}^{(C,\ast)}_1 > \mathcal{E}^{(C,\ast)}_0 = 0$$, which gives a condition on the parameter $$\tilde{b}$$.

For the case of $$\ast = H$$,

$$\tilde{b} \equiv b \hbar \omega > -2 \hbar \omega,$$  \hspace{1cm} (20)$$

and the general solution for Eq. (16) is

$$u^{(H)}(x) = \alpha_1 F_1 \left( -\frac{b}{4}; \frac{1}{2} \xi^2 \right) + \beta \xi \frac{1}{2} F_1 \left( \frac{1}{2} + \frac{b}{4}; \frac{3}{2} \xi^2 \right).$$  \hspace{1cm} (21)$$

The parameters $$\alpha$$ and $$\beta$$ must satisfy the following condition so that the resulting system does not have singularities in the domain:

$$\left| \frac{\beta}{\alpha} \right| < \frac{2 \Gamma \left( \frac{1}{2} + 1 \right)}{\Gamma \left( \frac{b}{4} + \frac{1}{2} \right)}.$$  \hspace{1cm} (22)$$

For $$\ast = L$$,

$$u^{(L)}(x) = \alpha_1 F_1 \left( -\frac{b}{4}; \frac{1}{2} - g; -z \right) + \beta z \frac{1}{2} F_1 \left( \frac{1}{2} + g - \frac{b}{4}; \frac{3}{2} + g; -z \right),$$  \hspace{1cm} (23)$$

and the parameters must satisfy

$$\tilde{b} \equiv b \hbar \omega > -4 \hbar \omega, \quad \alpha > 0, \quad \frac{\beta}{\alpha} > -\frac{\Gamma \left( \frac{1}{2} - g \right)}{\Gamma \left( \frac{1}{2} - g + \frac{b}{4} \right)} \frac{\Gamma \left( \frac{1}{2} + 1 \right)}{\Gamma \left( \frac{3}{2} + g \right)}.$$  \hspace{1cm} (24)$$

For $$\ast = J$$,

$$u^{(J)}(x) = \alpha_2 F_1 \left( -\frac{g}{2} - \frac{h}{2} - \frac{\sqrt{(g+h)^2 - b}}{2}; -\frac{g}{2} - \frac{h}{2} + \frac{\sqrt{(g+h)^2 - b}}{2}, \frac{1}{2} - h; \frac{1}{2} + y \right)$$

$$+ \beta y^{\frac{b}{2} + 1} F_1 \left( \frac{1}{2} - \frac{g}{2} + \frac{h}{2} - \frac{\sqrt{(g+h)^2 - b}}{2}; \frac{1}{2} - \frac{g}{2} + \frac{h}{2} + \frac{\sqrt{(g+h)^2 - b}}{2}, \frac{3}{2} + h; \frac{1}{2} + y \right),$$  \hspace{1cm} (25)$$
and the parameters satisfy
\[
\tilde{b} \equiv bh^2 > -4h^2(g + h + 1),
\]
\[
\alpha - \beta > 0, \quad \frac{\beta}{\alpha} > -\frac{\Gamma\left(\frac{1}{2} - h\right)}{\Gamma\left(\frac{3}{2} + h\right)} \cdot \frac{\Gamma\left(1 + \frac{g}{2} + \frac{b}{2} + \frac{\sqrt{(g+h)^2-b}}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{g}{2} - \frac{b}{2} - \frac{\sqrt{(g+h)^2-b}}{2}\right)}.
\]

(26)

Hereafter, we fix \(\alpha = 1\) without loss of generality. Thus, as was mentioned above, the CES systems have two model parameters; \(b\) is responsible for the energy shift while \(\beta\) is a parameter describing an iso-spectral deformation of the system. The case where \(b = \beta = 0\) is identical to the conventional SI system.

Junker and Roy pointed out in Ref. [34] that the CES systems of \((H) b = 4N, \beta = 0, (L) b = 8N, \beta = 0\) with \(N \in \mathbb{Z}_{>0}\) are obtained by the Krein–Adler transformation of the corresponding conventional SI potentials. Note that for the case of \(\ast = J\) there has no such correspondence.

The Krein–Adler transformation concerns the deletion of the eigenstates of the original exactly solvable system whose indices are designated by \(\mathcal{D}\). Generally, one can take \(\mathcal{D} = \{d_1, d_1 + 1 < d_2, d_2 + 1 < \cdots < d_M, d_M + 1\}\) and \(\{d_i \in \mathbb{Z}_{>0}; i = 1, \ldots, M\}\), but we consider the deletion of eigenstates indicated by \(2N\) consecutive integers, i.e., \(\mathcal{D} = \{d, d + 1, \ldots, d + 2N - 1\}\) in this paper. Moreover, we restrict ourselves mainly to \(N = 1\) for simplicity. The Hamiltonian for the Krein–Adler systems are

\[
\mathcal{H}_\mathcal{D}^{(K, \ast)} \equiv \mathcal{H}^{(\ast)} - 2h^2 \partial_x^2 \ln \left| \mathcal{W}\left[ \phi_\ast^{(d)}, \phi_\ast^{(d+1)} \right](x) \right|, \quad \ast = H, L, J,
\]

(27)
in which \(\mathcal{W}[f_1, \ldots, f_n](x) \equiv \det \left( \partial_{x}^{-1} f_k(x) \right)\) is the Wronskian. The Schrödinger equation is

\[
\mathcal{H}_\mathcal{D}^{(K, \ast)} \phi_{\mathcal{D}; n}^{(K, \ast)}(x) = \mathcal{E}_{\mathcal{D}; n}^{(K, \ast)} \phi_{\mathcal{D}; n}^{(K, \ast)}(x)
\]

(28)

with

\[
\mathcal{E}_{\mathcal{D}; n}^{(K, \ast)} = \mathcal{E}_{\tilde{n}}^{(\ast)} \quad \text{and} \quad \phi_{\mathcal{D}; n}^{(K, \ast)}(x) = \frac{\mathcal{W}\left[ \phi_d^{(\ast)}, \phi_{d+1}^{(\ast)}, \phi_\ast^{(d)} \right](x)}{\mathcal{W}\left[ \phi_d^{(\ast)}, \phi_{d+1}^{(\ast)} \right](x)}.
\]

(29)

Here, \(\tilde{n}\) is defined as

\[
\tilde{n} := \begin{cases} 
  n & (0 \leq n \leq d - 1) \\
  n + 2 & (n \geq d)
\end{cases}
\]

(30)

with \(n\) being the number of nodes.
III. NON-EXACTNESS OF SWKB CONDITION: EXAMPLES

A. SWKB condition for Krein–Adler systems

It was already confirmed that all the conventional SI systems exactly satisfy the SWKB condition. As we mentioned in Sec. I, the common singularity structures of the integrands of the SWKB and the QMF supports the exactness of the SWKB condition. This is the qualitative understanding of the exactness.

The Krein–Adler systems are notable examples where the SWKB condition is broken. The SWKB integral (4) for the Krein–Adler systems are

\[ I_{SWKB} = \int_{a_L}^{a_R} \sqrt{\mathcal{E}_{D,n}^{(K,*)} - \left( \hbar \partial_x \ln \left| \phi_{D,0}^{(K,*)}(x) \right| \right)^2} \, dx. \] (31)

For the case of \( * = H \), Eq. (31) reduces to

\[ I_{SWKB} = \hbar \int_{a'_L}^{a'_R} \sqrt{2\tilde{n} - \left( \partial_z \ln \left| \phi_{D,0}^{(K,H)}(x) \right| \right)^2} \, d\xi \equiv \hbar I^{(K,H)}, \] (32)

while for the cases of \( * = L, J \), Eq. (31) becomes

\[ I_{SWKB} = \hbar \int_{a'_L}^{a'_R} \sqrt{\tilde{n} - (\partial_z \ln \left| \phi_{D,0}^{(K,L)}(x) \right|)^2} \, dz \equiv \hbar I^{(K,L)}, \] (33)

\[ I_{SWKB} = \hbar \int_{a'_L}^{a'_R} \sqrt{\tilde{n}(\tilde{n} + g + \hbar) - (1 - y^2)(\partial_y \ln \left| \phi_{D,0}^{(K,J)}(x) \right|)^2} \, \frac{dy}{\sqrt{1 - y^2}} \equiv \hbar I^{(K,J)}. \] (34)

For each case above, \( a'_L, a'_R \) denote the two solutions of the equation obtained by setting the inside of the square root equals zero. The SWKB condition is then

\[ I^{(K,*)} = n\pi, \quad n \in \mathbb{Z}_{\geq 0}. \] (35)

Since the SWKB conditions (32), (34) are totally independent of \( \hbar \) (and \( \omega \)) [19], for the rest of this section we fix \( \hbar = \omega = 1 \) without loss of generality.

Since no analytical method is known so far, we numerically solve the SWKB integrals \( I^{(K,*)} \). We show in Fig. [1] the numerical evaluations of \( I^{(K,*)} \) for the case of \( d = 1 \). We also display in Fig. [1] the accuracy of the SWKB conditions calculated by

\[ \text{Err}(n) := \frac{I^{(K,*)} - n\pi}{I^{(K,*)}}. \] (36)

As is easy to find from Fig. [1], though the SWKB condition is not exact except for the ground state, the error \( \text{Err}(n) \) remains small, at most \( \text{Err}(n) \sim 10^{-1} \). \( \text{Err}(n) \) has its maximal value at \( n = 1 \), and as \( n \) gets larger, \( \text{Err}(n) \) monotonically decay; \( \text{Err}(n) \) goes to zero as \( n \to \infty \). We thus conclude
that the SWKB condition is not exact but still reproduces approximate bound-state spectra for these cases. The similar can be said for other choices of parameters. The maximal value of $\text{Err}(n)$ is seen around $n = d$. More comprehensive results are given in [19].

B. Non-exactness of SWKB condition from point of the QHJ formalism

The conventional SI systems, where the SWKB condition is exact, and the QMF $p(x; \mathcal{E})$ have the equal singular nature. For the broken cases, a notable difference appears on the complex $x$-plane.

The original SWKB integral is for the real variable. Now we need to extend the variable to complex $x \in \mathbb{C}$ for the comparison. We consider the corresponding contour integral

$$J_{\text{SWKB}} := \frac{1}{2\pi} \oint_{C'} \sqrt{\mathcal{E}_n - W(x)^2} \, dx,$$

where the contour $C'$ encloses the turning points $a_{\text{L,R}}$ counterclockwise. In Fig. 2 we show the singularity structures of the SWKB integrand and the QMF for the first excited state of a Krein–Adler system $\mathcal{H}^{(K,H)}_{\{1, 2\}}$. Apparently they do not coincide with each other and the quantization of the SWKB integral is not exact, i.e., $\text{Err}(n) \neq 0$.

The next question is where the essential difference of the singular nature of the Krein–Adler systems from that of the conventional SI potentials of which they perfectly coincide with the $J_{\text{QHJ}}$, comes from. The quantization condition on $J_{\text{QHJ}}$ is supposed to be exact for any solvable systems, because the analytical contour integrations are definitely executable. Originally, the calculation of $J_{\text{QHJ}}$ was performed in a direct manner, by means of the node theorem and the Cauchy’s argument principle. We, on the other hand, employ a different method to calculate $J_{\text{QHJ}}$, intending to make it easier to compare to the SWKB integral $J_{\text{SWKB}}$. Here we show an example with the Krein–Adler system. The QMF of the Krein–Adler system (27) is

$$p(x, \mathcal{E}) = \frac{1}{i} \left( -x - \frac{\partial_x W[H_d, H_{d+1}]}{W[H_d, H_{d+1}]}(x) + \frac{\partial_x W[H_d, H_{d+1}, \hat{n}]}{W[H_d, H_{d+1}, \hat{n}]}(x) \right)$$

$$= \frac{1}{i} \left( -x - \frac{\partial_x W[H_d, H_{d+1}]}{W[H_d, H_{d+1}]}(x) + \frac{\partial_x W[H_d, H_{d+1}, 1]}{W[H_d, H_{d+1}, 1]}(x) 
- \frac{\partial_x W[H_d, H_{d+1}, 1]}{W[H_d, H_{d+1}, 1]}(x) + \frac{\partial_x W[H_d, H_{d+1}, \hat{n}]}{W[H_d, H_{d+1}, \hat{n}]}(x) \right).$$

As we mentioned, $C$ in Eq. (8) is the counterclockwise contour enclosing the two classical turning points $x_{\text{L,R}}$. For the Krein–Adler system, the QMF has an isolated pole at $x \to \infty$, $4d - 2$ fixed poles other than that and $\hat{n}$ moving poles, including $n$ moving poles on the real axis. The contour
FIG. 1. The values of the SWKB integral $I^{(K, \ast)}$ (blue closed dots) and the accuracy of the SWKB condition $\text{Err}(n)$ (red squares) for the Krein–Adler systems are plotted. We choose $d = 1$ for each case, and we set (b) $g = 3$ and (c) $g = 3, h = 4$. The condition is exact when the blue dots are on the blue dashed line $I^{(K, \ast)}/\pi = n$. The maximal errors are (a) $1.1 \times 10^{-1}$, (b) $1.2 \times 10^{-1}$, and (c) $9.8 \times 10^{-2}$, respectively, which are found at $n = 1$ for every case.
The poles (x-marks) and branch cuts (wavy lines) on a complex $x$-plane are displayed for (a) the SWKB integrand $\sqrt{\mathcal{E}}_1 - W(x)^2$, and (b) QMF $p(x; \mathcal{E}_1)$ for a Krein–Adler system $\mathcal{H}_{\{1,2\}}^{(K,H)}$. We also plot the zeros with closed dots on both figures. The poles are at (a) $x = \pm i/\sqrt{2}$, (b) $x = 0, \pm i/\sqrt{2}, \pm i\sqrt{3}/2$, and the nodes exist at (a) $x = \pm \sqrt{3/2}$, (b) $x = \pm 1/\sqrt{2}$, $\left(\pm \sqrt{33 - 12\sqrt{6}} \pm \sqrt{33 + 12\sqrt{6}}\right)/2\sqrt{2}$.

$\Gamma_R$ is of the radius $R$, enclosing all $4d - 2$ fixed poles counterclockwise, where each of the poles is enclosed by a counterclockwise contour $\gamma_i$. $\bar{n} - n$ moving poles, which are off the real axis, are enclosed by a counterclockwise contour $\tilde{\gamma}_j$ one by one. See Fig. 3a. Hence, the following equation holds:

$$J_{\Gamma_R} = J_{QH} + \sum_{i=1}^{4d-2} J_{\gamma_i} + \sum_{j=1}^{\bar{n}-n} J_{\tilde{\gamma}_j},$$  \hspace{1cm} (38)

where $J_\bullet$ is defined as

$$J_\bullet := \frac{1}{2\pi} \oint_\bullet p(x; \mathcal{E}) \, dx. \hspace{1cm} (39)$$

Here, considering $W[H_d, H_{d+1}](x)$, $W[H_d, H_{d+1}, 1](x)$ and $W[H_d, H_{d+1}, H_{\bar{n}}](x)$ are polynomials of $2d$, $2d - 2$ and $2d - 2 + \bar{n}$ degrees respectively, the second and the third terms of the r.h.s. of Eq. (38) are

$$\sum_{i=1}^{4d-2} J_{\gamma_i} = (2d - 2) - 2d = -2, \hspace{1cm} (40)$$

$$\sum_{j=1}^{\bar{n}-n} J_{\tilde{\gamma}_j} = -(2d - 2) + (2d - 2 + \bar{n} - n) = \bar{n} - n. \hspace{1cm} (41)$$
In order to evaluate $J_{\Gamma_R}$, we change variable as $x \to w \equiv x^{-1}$, and

$$J_{\Gamma_R} = \frac{1}{2\pi} \oint_{\Gamma_R} p(x; \mathcal{E}) \, dx = \frac{1}{2\pi} \oint_{\gamma_0} \bar{p}(w; \mathcal{E}) \, \frac{dw}{w^2}, \tag{42}$$

with $\bar{p}(w; \mathcal{E}) \equiv p(w^{-1}; \mathcal{E})$ and $\gamma_0$ enclosing counterclockwise the only pole in the $w$-plane, i.e., the one at $w = 0$ (see Fig. 3b). Note that $\bar{p}(w; \mathcal{E})$ satisfies the QHJ equation:

$$\bar{p}(w; \mathcal{E})^2 + iw^2 \partial_w \bar{p}(w; \mathcal{E}) = \mathcal{E} - \bar{V}^{(K,H)}(w). \tag{43}$$

$\bar{V}^{(K,H)}(w)$ means the potential for the Krein–Adler system with $* = H$ in terms of the variable $w$.

We employ the Laurent expansion of $\bar{p}(w; \mathcal{E})$ about $w = 0$:

$$\bar{p}(w; \mathcal{E}) \cong \sum_{n=0}^{\infty} a_n w^n + \sum_{q=1}^{k} \frac{b_q}{w^q} \tag{44}$$

to calculate $J_{\Gamma_R}$. Substituting this expansion into Eq. (43) and comparing the l.h.s. and the r.h.s., one obtains

$$a_0^2 + 2a_1 b_1 - i b_1 = \mathcal{E} - 3, \tag{45}$$
$$b_1^2 = -1, \tag{46}$$
$$2a_0 b_1 = 0, \tag{47}$$

and $b_q = 0$ for $q \geq 2$. The asymptotic behavior of $\bar{p}$ leads $b_1 = i$, and $a_0 = 0$. Given $\mathcal{E} = 2\tilde{n}$,

$$a_1 = \frac{-i\mathcal{E}}{2} + 2i = (2 - \tilde{n})i, \tag{48}$$
and one gets

\[ J_{\Gamma_R} = \tilde{n} - 2. \]  

(49)

Therefore, from Eq. (38), the quantum action variable is quantized, \( J_{QH} = n \). We note that this calculation is consistent with the original one.

On the other hand, the contour integral for \( J_{SWKB} \) is not so straightforward. The different number of the poles is responsible for the breaking of the quantization condition. Further, existence of the branch cuts (which is sometimes referred as “other branch cuts”) cause additional difficulty, and is also responsible for the breaking. As a result, we only say that we have many explicit reasons for the broken quantization condition of SWKB but no way of concretely calculating it.

Instead, a perturbative treatment works well for the discrepancy \( \Delta := I_{SWKB} - \pi J_{QH} \). It enables us to discuss how close the systems of Eqs. (52)–(64) with the corresponding SI ones. In the latter half of the next section, we describe the method in detail.

C. Discussions on non-exactness of SWKB condition

Before closing this section, we would like to mention a little about the numerical results of the Krein–Adler systems. The maximal errors of the SWKB condition for the Krein–Adler systems occur around the deleted levels \( D \). (See Fig. 1 for the case of (II) with \( d = 1 \), and Figs. 4–6 in Ref. [19] for other cases.) As was already discussed in our previous paper [19], this result clearly suggests that modulation of such level structures strongly correlates with the discrepancy of the SWKB condition. Therefore, it is quite natural to guess that the modifications of a conventional SI potential move the level structure of the system, and so do the values of the SWKB integrals. For every case in Fig. 1, the non-exactness of the SWKB condition is as a result of the deletion of eigenstates from the original conventional SI systems. The influence of deformations of the original systems on the SWKB integral is seen through how much the condition breaks, \( i.e., \) the error \( \text{Err}(n) \). The behaviors of the red lines in Figs. 1a–1c are almost identical, which gives us an intuition concerning the relations between the level structure of a system and the SWKB integral.

Moreover, though the SWKB condition is broken, still the value of the SWKB integral tends to be in the vicinity of \( n\pi \). This is also the case with the Krein–Adler systems with other choices of parameters and the Multi-indexed systems [19]. Similarly, DeLaney et al. claimed that the SWKB condition gives better approximation than the WKB condition [18]. This feature of the SWKB condition implies that a modulation of the level structure causes a small perturbation on
the SWKB integral.

IV. NUMERICAL ANALYSIS ON BREAKING OF SWKB CONDITION

The contour integral for the SWKB condition is generally impossible and then, we are to employ a kind of perturbative treatment with an unknown parameter. The parameter describing the discrepancy between Krein–Adler systems and the corresponding SI ones is not known so far. Here, there is a hint for tackle the problem. As we already pointed out, the Krein–Adler systems with $d = 1$ are regarded as the CES systems with the specific choice of parameters. Therefore, we can employ those parameters connecting between the Krein–Adler systems and the corresponding SI ones, and they are concerned with the parameter of the perturbation.

A. SWKB condition for CES systems

Just as almost the same manner with the Krein–Adler systems, the SWKB integrals for the CES systems are

$$I_{SWKB} = \int_{a_L}^{a_R} \sqrt{\mathcal{E}_n^{(C,\ast)} - \left( \hbar \partial_x \ln \left| \phi_0^{(C,\ast)}(x) \right| \right)^2} \, dx \ .$$

(50)

For the case of $\ast = H$, Eq. (50) reduces to

$$I_{SWKB} = \hbar \int_{a'_L}^{a'_R} \sqrt{2n + b - \tilde{W}^{(H)}(\xi)^2} \, d\xi \equiv \hbar I^{(C,H)}(\xi), \quad \tilde{W}^{(H)}(\xi) \equiv \partial_\xi \ln \left| \phi_0^{(C,H)}(x) \right| ,$$

and for the cases of $\ast = L, J,$

$$I_{SWKB} = \hbar \int_{a'_L}^{a'_R} \sqrt{n + \frac{b}{4} - \tilde{W}^{(L)}(z)^2} \, \frac{dz}{\sqrt{z}} \equiv \hbar I^{(C,L)}(z), \quad \tilde{W}^{(L)}(z) \equiv \sqrt{z} \partial_x \ln \left| \phi_0^{(C,L)}(x) \right| ,$$

$$I_{SWKB} = \hbar \int_{a'_L}^{a'_R} \sqrt{4n(n + g + h) + b - \tilde{W}^{(J)}(y)^2} \, \frac{dy}{2\sqrt{1 - y^2}} \equiv \hbar I^{(C,J)}(y), \quad \tilde{W}^{(J)}(y) \equiv \sqrt{1 - y^2} \partial_y \ln \left| \phi_0^{(C,J)}(x) \right| ,$$

(52)

(53)

where $a'_L, a'_R, (a'_L < a'_R)$ are the two solutions for the equation obtained by setting the inside of the square root equals zero. The SWKB condition is

$$I^{(C,\ast)} = n\pi , \quad n \in \mathbb{Z}_{\geq 0} .$$

(54)

Again, the SWKB conditions (51)-(53) are totally independent of $\hbar$ (and $\omega$), which means that this condition equation is not to be discussed in the context of the semi-classical regime of the quantum system. Hereafter, we fix $\hbar = \omega = 1$ without loss of generality.
FIG. 4. The values of the SWKB integral $I^{(C,H)}$ for $n = 1, 2, 3$. Range of each plot is determined so that the systems will not have more than two turning points. The parametric conditions (20) and (22) yield $b > -2$ and $|\beta| < 2/\sqrt{\pi}$ respectively.

Here, the SWKB integrals $I^{(C,\ast)}$ are only calculated numerically. Fig. 4a shows the $b$-dependency of the SWKB integral $I^{(C,H)}$ for $\beta = 0$, and Fig. 4b is the $\beta$-dependency of $I^{(C,H)}$ for $b = 0$. The SWKB integrals $I^{(C,H)}$ grow with the parameter $b$ around $b = 0$, while $I^{(C,H)}$ exhibit plateau behavior (but the condition is never exactly satisfied) around $\beta = 0$. Different behaviors are seen as the parameters approach their boundary (20), (22). These statements hold for general $b \neq 0$ and $\beta = 0$ case. Similar results can be obtained for the cases of $\ast = L, J$. The numerical calculations Fig. 4 support our conjecture that the level structure approximately guarantees the exactness of the SWKB condition.

In order to calculate exact bound-state spectra through the SWKB condition, one needs to evaluate $\Delta$. Since the quantization condition for the quantum action variable gives exact results, one may think that all one has to do is to evaluate $J_{QHJ}$. However, in general, one cannot calculate the analytical relation between $J_{QHJ}$ and the energy because of the complicated singularity structures of the QMF. In the latter half of the next section, we propose a noble method for evaluating $\Delta$ by means of series expansion of the SWKB integrand.

B. Non-exactness of SWKB condition and the quantization condition in QHJ formalism

As is seen in Fig. 4, the SWKB condition is not exact. We plot the singularity structures of both the SWKB integrand and the QMF again. For the exactness of the quantization condition of $J_{QHJ}$ for the CES systems, more careful analysis is mandatory. We present the pole structures of the QMF in Fig. 5. Note that these figures are made from our numerical results with various
where we set $\beta = 0$ and $n = 1$. They reveal notable features of the QMF of the CES systems. Except for $b = 0$ (conventional SI) and $b = 4N$ (Krein–Adler), the QMF has infinite number of poles in the complex plane. At $b = 0$, there is just one pole at the origin $x = 0$ (Fig. 5b). For $b \neq 0$, infinite number of poles appear in the complex plane (Fig. 5c) and also $4N + 1$ poles on the imaginary axis for $4(N - 1) < b \leq 4N$. The poles on the imaginary axis approach the origin $x = 0$ as $b$ grows, while the other poles remain almost the same locations (Fig. 5d). When $b$ reaches $4N$, all the poles except the ones on the imaginary axis disappear (Fig. 5e). Again, as $b$ grows further, infinite poles appear in the complex plane (Fig. 5f). A notable feature is that these poles except for the origin $x = 0$ (and the one at $x \to \infty$) are pairwise with the residues $\hbar$ and $-\hbar$, respectively. Therefore, for the contour integral of $J_{QHJ}$, these contributions exactly vanish and only the residue at $x \to \infty$ contributes to the integral:

$$J_T = J_{QHJ} + \sum_{i=1}^{\infty} J_{\gamma_i} + \sum_{j=1}^{\infty} J_{\tilde{\gamma}_j} = J_{QHJ},$$

which we have numerically verified. For the definitions of the contours, see Fig. 6. Note that this is just the quantization of the quantum action variable, not the quantization of the energy and then, there is no direct method for calculating the energy $E$ from the quantization condition.

For the SWKB integration, the situation is worse. Infinite number of the poles appeared in the complex plane are not pairwise and then, no cancelation of residue of the poles occurs (see Fig. 7). Also, there appear other branch cuts, which has nonzero contribution on the contour integral $J_{SWKB}$. The other branch cuts spread all over the complex plane (which we do not plot in Fig. 7 for making easier to see). These are the origin of the non-exactness of the SWKB condition and also the essential difficulty for the explicit calculation of $J_{SWKB}$. Then, we have to rely on a perturbative treatment; here we introduce the method for the residual $\Delta$ in Sec. IV D.

As a result, both $J_{QHJ}$ and $J_{SWKB}$ cannot analytically reproduce the energy spectra. The SWKB formalism has some advantages because the original $I_{SWKB}$ can be integrated along the real axis, without summing up all the residues of the poles in the complex plane. Therefore, if we successfully prove that a system satisfies the SWKB condition within some uncertainty, we are able to compute all the energy spectra approximately.

C. Non-exactness of SWKB condition and the series expansion of SWKB integrand

Before moving on the detailed analysis on the residual $\Delta$, we first investigate how the SWKB integrals $I^{(C,*)}$ change as the parameters $b, \beta$ grow by a series expansion of the SWKB integrand.
FIG. 5. The singularity structures of the QMFs for the first excited states of the CES (H) systems with various $b$. Poles are plotted by x-marks. The location of each pole is calculated numerically. Note that (b) and (f) are identical to that of the 1-d H.O. and the Krein–Adler (H) system with $d = 1$ respectively.
Note that for $b = \beta = 0$, the condition becomes exact, since the systems are equivalent to the original conventional SI ones.

Our basic idea for the formulation is to consider small perturbations from the exact case: $b = \beta = 0$. We employ Taylor expansion for the SWKB integrand around the point where the SWKB condition is exact. For the case of $* = H$,

$$I^{(C,H)} = \int_{a_L}^{a_R} \sqrt{(x-a_L)(a_R-x)} \sqrt{1 + \frac{2n + b - \tilde{W}^{(H)}(x)^2 - (x-a_L)(a_R-x)}{(x-a_L)(a_R-x)}} \, dx$$

$$\approx \frac{(a_R-a_L)^2}{8} \pi + \sum_{k=1}^{\infty} \frac{(-1)^k (2k)!}{(1 - 2k)(k!)^2 4^k} \int_{a_L}^{a_R} \frac{2n + b - \tilde{W}^{(H)}(x)^2 - (x-a_L)(a_R-x)}{[(x-a_L)(a_R-x)]^{k-\frac{1}{2}}} \, dx. \quad (56)$$

After the series expansion, we use the fact to obtain Eq. (56) that the integrand converges uniformly where one can swap the orders of the integration and the limit to infinity. Note that the second term in Eq. (56) is not zero, except for $b = \beta = 0$, where the second term in Eq. (56) vanishes and
FIG. 7. The singularity structures of the SWKB integrands for the first excited states of the CES (H) systems with various $b$. The poles are plotted by x-marks and the branch cut on the real axis is shown by wavy lines. Other branch cuts are removed from these cartoons. The location of each pole is calculated numerically. Note that (b) and (f) are identical to that of the 1-d H.O. and the Krein–Adler (H) system with $d = 1$ respectively.
\[
(a_R - a_L)^2 \frac{\pi}{8} = n \pi.
\]

We expand the square root in the second line about the parameter
\[
\frac{2n + b - \tilde{W}^{(1)}(x)^2 - (x - a_L)(a_R - x)}{(x - a_L)(a_R - x)},
\]
and the radius of convergence for the expansion is thus
\[
\left| \frac{2n + b - \tilde{W}^{(1)}(x)^2 - (x - a_L)(a_R - x)}{(x - a_L)(a_R - x)} \right| = 1.
\]

Similarly, one can consider the expansion formulae for the cases of \( * = L, J \):
\[
I^{(C,L)} \approx \frac{\pi}{4} \left( \sqrt{a_R} - \sqrt{a_L} \right) \\
+ \sum_{k=1}^{\infty} \frac{(-1)^k(2k)!}{(1-2k)(k!)^2} \int_{a_L}^{a_R} \left\{ \left[ n + \frac{b}{4} - \tilde{W}^{(L)}(z)^2 \right] z - (z - a_L)(a_R - z) \right\}^k \frac{dz}{z},
\]
\[
I^{(C,J)} \approx \frac{2n + g + h}{2} \left\{ \frac{\pi}{2} \left( 2 - \sqrt{(1-a_L')(1-a_R') - \sqrt{(1+a_L')(1+a_R')} \right) \\
+ \sum_{k=1}^{\infty} \frac{(-1)^k(2k)!}{(1-2k)(k!)^2} \int_{a_L}^{a_R} \left[ \frac{4n(n+g+h)+b-\tilde{W}^{(J)}(y)^2}{(2n+g+h)^2} \left( 1 - y^2 \right) - (y - a_L')(a_R' - y) \right]^k \frac{dy}{1 - y^2} \right\}.
\]

For the series (59) and (60) to be convergent,
\[
\left| \frac{n + \frac{b}{4} - \tilde{W}^{(L)}(z)^2}{(z - a_L')(a_R' - z)} \right| \leq 1,
\]
\[
\left| \frac{4n(n+g+h)+b-\tilde{W}^{(J)}(y)^2}{(2n+g+h)^2} \left( 1 - y^2 \right) - (y - a_L')(a_R' - y) \right| \leq 1,
\]
respectively.

The radius of convergence for the series (56), (59) and (60) are given by Eqs. (58), (61) and (62). A choice of parameters \((b, \beta)\) within these radius of convergence correspond to a CES system which is connected to the original conventional SI potential in terms of Eqs. (56), (59) and (60). When a choice of parameters \((b, \beta)\) is outside the radius, such system simply does not relate to the original conventional SI potential in terms of those series. We plot the domains where the series converges for \( n = 1, 2, 3 \) on \((b, \beta)\)-plane in Fig. 8. One can obtain the domains by solving Eq. (58) numerically. Let us call the each domain \( D_n \) respectively. We have checked numerically that as \( n \) grows, the radius of convergence are enlarged; \( D_1 \subset D_2 \subset D_3 \subset \cdots \). A quantitative argument
FIG. 8. The domains $D_n$ for $n = 1, 2, 3$. Here, one can see that $D_1 \subset D_2 \subset D_3$. The parameters $b, \beta$ must satisfy Eqs. (20) and (22) by construction.

of the inclusion relation of domains $D_n$ supports this result. Thus we conclude that there always exist sets of model parameters where the expansion (56) is possible for any $n$, and it is enough to consider the domain $D_1$ so that the expansion formula (56) holds for any $n$.

D. Numerical analysis on $\Delta$

We evaluate the residual

$$\Delta := n\pi h - I_{SWKB}, \quad \Delta^{(C,*)} := n\pi - I^{(C,*)}$$

as a function of $b$ and $\beta$. Since our initial aim is to investigate the higher order correction of the SWKB condition, it would be relevant when we express the residual $\Delta$ in a series. By employing our formulation, which is basically $I_{SWKB} \cong \sum_{k=0}^{\infty} I^{(k)}$, $\Delta$ can be evaluated in the form of a series:

$$\Delta \cong n\pi h - \sum_{k=0}^{\infty} I^{(k)}.$$  

Especially for $* = H$, using Eq. (56),

$$\Delta^{(C,H)} \cong \left[n - \frac{(a_R - a_L)^2}{8}\right] \pi - \sum_{k=1}^{\infty} \frac{(-1)^k (2k)!}{(1-2k)(k)!^2} \int_{a_L}^{a_R} \frac{2n + b - \tilde{W}^{(H)}(x)^2 - (x - a_L)(a_R - x)}{[(x - a_L)(a_R - x)]^{k-\frac{1}{2}}} \, dx.$$  

All terms in Eq. (64) vanish when $b = \beta = 0$, while no term is equals to zero for other cases. We display the numerical calculation of $\Delta^{(C,H)}$ in Fig. 9. First, we fix $\beta = 0$ and see $\Delta^{(C,H)}$ as a function of $b$ (Fig. 9a). The value of $\Delta^{(C,H)}$ declines as $b$ grows. $\Delta^{(C,H)}$ is negative when $b > 0,$
FIG. 9. Plots of $\Delta^{(C,H)}$ for $n = 1$ (numerical results). The range of each plot is determined so that the systems will not have more than two turning points. The parametric conditions (20) and (22) yield $b > -2$ and $|\beta| < 2/\sqrt{\pi}$ respectively. The light gray lines at (a) $b \approx -1.64, 1.44$, (b) $|\beta| \approx 1.01$ shows the radius of convergence [58].

while $\Delta^{(C,H)}$ is positive for $b < 0$. Note that for the case of $b = 0$, i.e., 1-d H.O., $\Delta^{(C,H)}$ equals zero. Next, we fix $b = 0$ and see $\Delta^{(C,H)}$ as a function of $\beta$ (Fig. 9b). $\Delta^{(C,H)}$ shows plateau behavior around $\beta = 0$, and it grows around $b \approx 2/\sqrt{\pi}$.

In Fig. 10, we plot $\Delta^{(C,H)}$ with different orders of the power series approximation. For glancing behavior of $\Delta^{(C,H)}$, the first few orders of the expansion formula is sufficient. Similar analysis can be done for the cases of $* = L, J$.

V. CONCLUSION

In this paper, we have studied the non-exactness of the SWKB condition for the Krein–Adler and the CES systems. First we calculated the SWKB integral numerically and showed that the condition equation is not an exact one for those systems. As pointed out in our previous letter, the deviations of the SWKB condition relate to the modifications of the whole distribution of the energy eigenvalues, which confirmed by the analysis on the $b$- and $\beta$-dependency of the value of SWKB integral fro the case of the CES systems. Then we showed that the non-exactness comes from the properties of the singularity structure of the SWKB integrand. As was already discussed by Bhalla et al., there are significant disagreements between the singularity structures of the QMF and the SWKB integrand for cases where the SWKB condition does not reproduce the exact bound-state spectra. For the case of the CES systems, we found the singularity structures possess the following properties. The QMF has $n$ poles on the real axis and infinite number of poles on the complex plane.
FIG. 10. Plots of $\Delta^{(C,H)}$ with different orders of the power series approximation for $n = 1$. We plot the 0th order of the r.h.s. of Eq. (56) (gray), up to first order (blue), second order (green) and third order (red), while the black curve shows the numerical result $\Delta^{(C,H)}$. The range of each plot is determined so that the systems will not have more than two turning points. The parametric conditions (20) and (22) yield $b > -2$ and $|\beta| < 2/\sqrt{\pi}$ respectively. The light gray lines at (a) $b \approx -1.64, 1.44$, (b) $|\beta| \approx 1.01$ shows the radius of convergence.

other than that, whose effect on the contour integral vanishes pairwisely. This feature proves the quantization condition in the QHJ formalism to be exact. On the other hand, the SWKB integrand does not have the pairwise-cancellation property, and it just has infinite number of general poles on the complex plane. Those poles are not to be treated analytically. More than that, branch cuts spread all over the complex plane, which is also impossible to evaluate the effect on the contour integral. One can see the non-exactness of the SWKB condition for the CES systems from the above properties.

Also, we have shed light on the residual $\Delta$ and introduced a novel way of evaluating it for a case of the CES system. We employed a perturbative approach, where we chose the non-perturbed system as a conventional SI potential and expanded the SWKB integral in powers of a parameter. Our formulation realizes the change of the value of SWKB integral, i.e., the level structure of a system, according to the change of model parameters describing the modifications of the level structure. One can understand the behavior of the SWKB integral or the residual by a few simple integrations along with the real line. Our approach unintentionally classifies the CES systems into two according to whether a CES system is inside or outside the radius of convergence for the expansion, i.e., whether or not a CES system connected to a conventional SI potential in terms of the series. The physical interpretation of the radius of convergence is still an open question.

As was mentioned in Sec. [1] an “exact” SWKB formula, whose leading term corresponds to the
current SWKB formalism surely exists. We do not know what the exact formula may look like so far. However, by virtue of the expansion formula, hopefully the “unknown parameter” will be identified, as a result of which “exact” SWKB is to be formulated. We also mentioned in Sec. I that the relation between the SWKB condition and the solvability of the Schrödinger equation has been discussed. We believe that our analysis on the residual $\Delta$ may give a clue to understand the inherent meaning of the SWKB condition in connection with the solvability of the Schrödinger equation.

ACKNOWLEDGMENT

The authors would like to thank Ryu Sasaki for his careful and useful advice and comments. We also appreciate Naruhiko Aizawa and Atsushi Nakamura for valuable discussions. Discussions during the YITP workshop YITP-W-20-03 on “Strings and Fields 2020” , and YITP workshop YITP-W-21-04 on “Strings and Fields 2021” have been useful to complete this work. N.S. was supported in part by JSPS KAKENHI Grant Number JP B20K03278(1).

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