Heralded dissipative preparation of nonclassical states in a Kerr oscillator

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(Dated: June 13, 2019)

We present a heralded state preparation scheme for driven nonlinear open quantum systems. The protocol is based on a continuous photon counting measurement of the system’s decay channel. When no photons are detected for a period of time, the system has relaxed to a measurement-induced pseudo-steady state. We illustrate the protocol by the creation of states with a negative Wigner function in a Kerr oscillator, a system whose unconditional steady state is strictly positive.

Introduction.— Nonlinearity is a crucial prerequisite for quantum algorithms to outperform their classical counterparts in quantum information processing because it gives rise to states or operations that cannot be efficiently described in a classical framework [1]. An important property to evaluate the usefulness of a quantum state in this context is the occurrence of negative values in its phase-space quasiprobability distribution [2–4].

However, such nonclassical states are challenging to prepare and stabilize because of unavoidable decoherence due to interaction with an unmonitored environment. For example, the perhaps simplest nonlinear quantum system, a driven and damped quantum oscillator with a Kerr nonlinearity, has a steady-state Wigner function that is strictly positive [5–7].

In this Letter, we circumvent this restriction and quantify the potential of such a system to stabilize nonclassical states with negative Wigner density. To this end, we take into account the information leaking out of the system, i.e., we consider setups where a detector continuously monitors the emitted photons.

In contrast to most heralded state preparation protocols relying on a photon detection event that heralds the collapse to a target state [8–15], we explore the opposite approach and use the photon-counting measurement to identify a time evolution which continuously relaxes the system into the target state, similar to [16]. Because the system will stay in this state conditioned on no further photon detection events, we will refer to it as a pseudo-steady state, to distinguish it from dissipative steady-state stabilization [17–20]. We apply this protocol also to a parametrically driven Kerr oscillator and show that it is feasible to stabilize Schrödinger kitten states without feedback [21].

On one hand, our results shed light on the actual dynamics of an open quantum system when the information leaking out to the environment is not discarded. On the other hand, they can be seen as a practical protocol for heralded state preparation in open quantum systems that is feasible with current technology.

System.— We consider an open quantum system exchanging photons with a finite-temperature environment.

Figure 1. (a) A driven nonlinear open quantum system (gray box) is monitored by a photon-counting measurement of detection efficiency $\eta$. The detection signal provides a herald for the creation of a pseudo-steady state in the system. (b) In a homodyne detection setup, a local oscillator (LO) signal is added before the detection, which allows one to modify the pseudo-steady state.

Its quantum master equation is ($\hbar = 1$)

$$\frac{d}{dt} \hat{\rho} = \mathcal{L}_0 \hat{\rho} + \kappa(n_{\text{th}} + 1) \mathcal{D}[a] \hat{\rho} + \kappa n_{\text{th}} \mathcal{D}[a^\dagger] \hat{\rho}, \quad (1)$$

where $\hat{a}$ is the photon annihilation operator, $\kappa$ denotes the decay rate, $n_{\text{th}}$ is the thermal photon number, and $\mathcal{D}[\hat{O}] \hat{\rho} = \hat{O} \hat{\rho} \hat{O}^\dagger - \{\hat{O}^\dagger \hat{O}, \hat{\rho}\}/2$ is a Lindblad dissipator. In general, $\mathcal{L}_0$ can be any completely positive and trace-preserving linear superoperator such that Eq. (1) has a steady-state solution $\hat{\rho}_{ss}$. For now, we choose $\mathcal{L}_0 \hat{\rho} = -i[\hat{H}_0, \hat{\rho}]$, where

$$\hat{H}_0 = -\Delta \hat{a}^\dagger \hat{a} + K \hat{a}^\dagger \hat{a} \hat{a}^2 + (\alpha_1 \hat{a}^\dagger + \alpha_2 \hat{a} \hat{a}^\dagger + \text{H.c.}) \quad (2)$$

describes an anharmonic oscillator with a Kerr nonlinearity of strength $K$ that is subjected to semiclassical and parametric drives of strength $\alpha_1$ and $\alpha_2$, respectively. We work in a frame rotating at the semiclassical drive frequency $\omega_{\text{drive}}$, and $\Delta = \omega_{\text{drive}} - \omega_0$ is the detuning with respect to the natural frequency $\omega_0$. The photon emission of the system is constantly monitored by a photon detector, as shown in Fig. 1(a). To illustrate the basic principle of the protocol, we focus on the case of a zero-temperature environment, $n_{\text{th}} = 0$, and unit detection efficiency of the photon-counting measurement, $\eta = 1$. The generalization to a more general $\mathcal{L}_0$, finite temperature, and non-unit detection efficiency is given in [22].

To model the photon-counting measurement, Eq. (1) is rewritten to a stochastic Schrödinger equation [23],

$$d |\psi\rangle = \mathcal{H} |\psi\rangle \, dt + \left( \frac{\hat{a} |\psi\rangle}{\sqrt{\langle\psi| \hat{a}^\dagger \hat{a} |\psi\rangle}} - |\psi\rangle \right) dN. \quad (3)$$
The term in brackets describes sudden quantum jumps of the state vector \( |\psi\rangle \) due to photon detection events. The Poissonian stochastic increment \( dN \) is unity if the photon detector clicks and zero otherwise. It has an ensemble-averaged expectation value \( E(dN) = 2 \langle |\psi\rangle \hat{M} |\psi\rangle dt \), where we have introduced the abbreviation \( \hat{M} = \kappa \hat{a}^\dagger \hat{a} / 2 \). The continuous time evolution of \( |\psi\rangle \) in the absence of photon detection events is captured by the nonlinear operator

\[
\mathcal{H} |\psi\rangle = [-i(\hat{H}_0 - i\hat{M}) + \langle |\psi\rangle \hat{M} |\psi\rangle] |\psi\rangle.
\]

The non-Hermitian correction \(-i\hat{M}\) to the Hamiltonian \(\hat{H}_0\) introduces relaxation and a decay of the norm of \( |\psi\rangle \). To preserve the norm, we include the nonlinear term \(\langle |\psi\rangle \hat{M} |\psi\rangle\) in \(\mathcal{H}\). By construction, an ensemble average over many solutions of Eq. (3) for different realizations of the stochastic jump process, so-called quantum trajectories, recovers the solution of Eq. (1) [23].

Protocol. The stochastic Schrödinger equation (3) describes a continuous time evolution of the state \( |\psi\rangle \) that is interrupted by discontinuous quantum jump events. This will lead to an interplay of two timescales: After initial transient dynamics, the quantum trajectories fluctuate on average around the steady state \( \rho_{ss} \) of Eq. (1), as shown in Fig. 2(a). Quantum jumps happen at a rate \( \Gamma_{\text{jump}} = 2\text{Tr}(\hat{M} \rho_{ss}) \). Between two adjacent quantum jumps, the state \( |\psi\rangle \) evolves continuously according to the operator \(\mathcal{H}\), which has a steady-state solution fulfilling \(\mathcal{H} |\psi\rangle_{ps} = 0\) and an associated relaxation rate \(\Gamma_{\text{rel}}\). In the following, we will call \( |\psi\rangle_{ps} \) the pseudo-steady state of the stochastic Schrödinger equation (3) because it is a steady state conditioned on the absence of photon detection events. In the regime \(\Gamma_{\text{rel}} \gtrsim \Gamma_{\text{jump}}\), the waiting time between two adjacent quantum jumps can be much larger than the relaxation time and \( |\psi\rangle \) relaxes exponentially to \( |\psi\rangle_{ps} \) as shown in Fig. 2(b). Hence, a photon detection event followed by no further click of the detector for several relaxation times \(\Gamma_{\text{rel}}^{-1} \) heralds the preparation of the state \( |\psi\rangle_{ps} \) and the waiting time determines the state preparation fidelity.

Pseudo-steady state and relaxation rate. We now derive explicit expressions for the pseudo-steady state and the relaxation rate. We assume that \(\hat{H}_0 - i\hat{M}\) has a non-degenerate spectrum \(\{h_\nu\}\), a more general calculation can be found in [22]. Under this assumption, each normalized eigenstate \( |\psi_\nu\rangle \) of \(\hat{H}_0 - i\hat{M}\) solves \(\mathcal{H} |\psi_\nu\rangle = 0\). However, since \(\mathcal{H}\) is a nonlinear operator, some states \( |\psi_\mu\rangle \) may be unstable to perturbations. A linear stability analysis reveals that in general, the pseudo-steady state \( |\psi\rangle_{ps} \) is the eigenstate \( |\psi_\mu\rangle \) whose eigenvalue \(h_\mu\) has the largest imaginary part, as shown in Fig. 2(c). The decay rate of an unstable eigenstate \( |\psi_{\nu \neq \mu}\rangle \) towards \( |\psi_\mu\rangle \) is determined by the imaginary part of the corresponding spectral gap, \(\Gamma_{rel\nu \rightarrow \mu} = \text{Im}(h_\mu - h_\nu)\). Thus, the long-term decay rate of a state \( |\psi\rangle \) to \( |\psi\rangle_{ps} \) is determined by the smallest imaginary gap between the stable and unstable eigenstates, \(\Gamma_{rel} = \min_{\nu \neq \mu}(\Gamma_{rel\nu \rightarrow \mu})\), as shown in Fig. 2(c).

Nonclassical states in a Kerr oscillator. The state \(\hat{\rho}\) of a quantum system can be represented by the Wigner function \(W_\rho(\alpha) = \text{Tr}[\hat{\rho} \hat{D}(\alpha) \hat{D}(\alpha)^\dagger] / \pi\), where \(D(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^\dagger \hat{a}}\) is the displacement operator and \(\hat{P} = e^{\alpha \hat{a}^\dagger - \alpha^\dagger \hat{a}}\) is the parity operator [24]. The Wigner function is a quasi-probability distribution in phase space and negative values of \(W_\rho(\alpha)\) indicate a nonclassical state \(\hat{\rho}\) [25]. We now show that the pseudo-steady state \( |\psi\rangle_{ps} \) of a Kerr oscillator can have a negative Wigner function \(W_{\rho_{ps}}(\alpha)\), whereas the steady-state Wigner function \(W_{\rho_{ss}}(\alpha)\) has been proven to be strictly positive [5–7]. As negativity measure, we use the modulus of the minimum of the Wigner function, \(N(\hat{\rho}) = |\text{min}_\alpha[W_\rho(\alpha)]|\), which is non-zero if \(W_\rho(\alpha)\) takes negative values and zero otherwise.

Semiclassical drive. We consider a semiclassical drive, \(\alpha_1 \geq 0\), and set \(\alpha_2 = 0\), such that the steady-state solution is characterized by the detuning \(\Delta / \kappa\), the rescaled drive power \(|\alpha_1|^2 K / \kappa^3\), and the ratio \(K / \kappa\) [26]. For fixed values of the first two quantities and \(K \gg \kappa\), the pseudo-steady state \( |\psi\rangle_{ps} \) is positive, as shown in Fig. 2(c).
Fig. 3(a). This is due to the fact that the steady state of a Kerr oscillator is strictly positive. If the relaxation rate dominates, \( \Gamma_{\text{rel}} \gg \Gamma_{\text{jump}} \), the system is almost always in the pseudo-steady state and, therefore, \( |\psi\rangle_{\text{ps}} \) must be identical to \( \tilde{\rho}_{\text{ss}} \) to ensure that an ensemble average over many trajectories reproduces the steady state. However, if relaxation rate and jump rate are comparable, \( \Gamma_{\text{rel}} \gtrapprox \Gamma_{\text{jump}} \), the pseudo-steady state differs from \( \tilde{\rho}_{\text{ss}} \) and can be nonclassical. Quantum jumps let \( |\psi\rangle \) explore many different states that compensate the nonclassicality of \( |\psi\rangle_{\text{ps}} \) and average out to a positive steady state. Finally, for \( K \ll \kappa \) the quantum trajectory is dominated by stochastic quantum jump events. Then, \( |\psi\rangle \) can no longer relax to \( |\psi\rangle_{\text{ps}} \) because the intervals between two quantum jumps are much shorter than the relaxation time, \( \Gamma_{\text{jump}} \gg \Gamma_{\text{rel}} \). Considering this, we define the maximum observable negativity \( N_{\text{max}} \) as the maximum of \( N(|\psi\rangle_{\text{ps}}) \) in the regime \( \Gamma_{\text{rel}} \gtrapprox \Gamma_{\text{jump}} \). The left panel of Fig. 3(c) displays \( N_{\text{max}} \) as a function of the dimensionless detuning and the rescaled drive power. Usually, the negativity \( N(|\psi\rangle_{\text{ps}}) \) decreases monotonically as a function of \( K/\kappa \), such that the maximum observable negativity \( N_{\text{max}} \) is achieved for \( \Gamma_{\text{rel}} = \Gamma_{\text{jump}} \). However, in the regime where two stable semiclassical solutions exist, enclosed by the gray lines in Fig. 3(c), the largest negativity is observed for \( \Gamma_{\text{rel}} > \Gamma_{\text{jump}} \).

Different unravelings.—The unraveling of the quantum master equation (1) is not unique [23]. Thus, the operator \( \hat{H} \) is not unique and many different pseudo-stationary states \( |\psi\rangle_{\text{ps}} \) can be stabilized to a given steady-state solution \( \tilde{\rho}_{\text{ss}} \). To illustrate this point, we consider the homodyne detection setup shown in Fig. 1(b). A beamsplitter is placed between the system and the photon detector, such that the signal \( \sqrt{\kappa} \xi \) of a local oscillator is added to the system’s output and the jump probability is modified, \( E(d\hat{N}^\prime) = \kappa (\hat{a}^\dagger + \xi^\dagger)(\hat{a} + \xi)|\psi\rangle \langle \psi| \). This corresponds to a photon-counting measurement in a displaced frame \( |\chi\rangle = \sqrt{\kappa} \xi |\psi\rangle \). With a modified Hamiltonian \( \hat{H}_0^{\prime} = \hat{D}(\xi)|\psi\rangle \hat{D}(\xi^\dagger) = \hat{D}(\xi) \hat{H}_0^{\prime} \hat{D}(\xi^\dagger) - i\kappa (\xi^\ast \hat{a} - \xi \hat{a}^\dagger)/2 \). The local oscillator signal \( \sqrt{\kappa} \xi \) now allows us to modify the ratio \( \Gamma_{\text{rel}}(\xi)/\Gamma_{\text{jump}}(\xi) \) and the pseudo-steady state \( |\psi(\xi)\rangle_{\text{ps}} \), as shown in Fig. 3(b). In contrast to the standard homodyne detection limit \( |\xi| \ll (\hat{a}) \), where the local oscillator signal dominates and the quantum trajectory is a continuous Wiener process [23], we consider the opposite limit \( |\xi| \gtrapprox (\hat{a}) \), such that the detection of photons is still a Poissonian quantum jump process. Moreover, a state \( |\psi(\xi)\rangle_{\text{ps}} \) can only be prepared if \( \Gamma_{\text{rel}} \leq \Gamma_{\text{jump}} \), which restricts \( \xi \) to the area inside the black curve in Fig. 3(b). Nevertheless, an optimization of the local oscillator signal \( \xi \) under these constraints significantly increases the maximum observable negativity \( N_{\text{max}} \) over the case of \( \xi = 0 \), as shown in the right panel of Fig. 3(c).

Parametric drive.—Our protocol can be used to stabilize a Schrödinger kitten state in a Kerr oscillator without feedback [21]: We consider a resonant parametric drive, i.e., \( \Delta = 0 \), \( \alpha_1 = 0 \), and \( \alpha_2 \geq 0 \), such that the non-Hermitian Hamiltonian \( \hat{H}_0 - i\hat{M} \) commutes with the parity operator \( \hat{P} \) and the spectrum consists of two subspaces of eigenstates having different parity, \( \{ h_{\mu}^\pm \} \). The operator \( \hat{H} \) does not mix these subspaces, therefore, both the even and the odd-parity eigenstate \( |\psi_{\mu}^\pm\rangle \) with largest imaginary part of the eigenvalue \( h_{\mu}^\pm \) are stable, as shown in Fig. 4(a), and their relaxation rates are determined by the imaginary parts of the spectral gaps to the unstable eigenstates of the corresponding parity.

While we redefined here the relaxation rate \( \Gamma_{\text{rel}} \) to take into account parity conservation, the relevant quantity to be compared to \( \Gamma_{\text{jump}} \) in the heralding protocol is still the first spectral gap, \( \Gamma_{\text{asy}} = \text{Im}(h_{\mu}^+ - h_{\mu}^-) \): Photon detection events change the parity of \( |\psi\rangle \) and approximately map the stable states \( |\psi_{\mu}^\pm\rangle \) to one another, such that the quantum trajectories jump between the two states, as shown in Fig. 4(b). The rate \( \Gamma_{\text{asy}} \) measures the asymmetry in the jump rates of \( |\psi_{\mu}^\pm\rangle \), which reflects their different photon-number expectation values. If \( \Gamma_{\text{asy}} \geq \Gamma_{\text{jump}} \).
holds, the states can be discriminated in the photon detection signal and the longer-lived state $|\psi^+_{\text{ps}}\rangle$ can be heraldedly prepared, $|\psi\rangle_{\text{ps}} = |\psi^+_{\text{ps}}\rangle$. The relaxation rate $\Gamma_{\text{rel}}$ towards $|\psi\rangle_{\text{ps}}$ is given by the second spectral gap, as shown in Fig. 4(c), and since $\Gamma_{\text{rel}} > \Gamma_{\text{asy}}$ holds, the relaxation to the target state within the heralding interval is guaranteed. Similar to the case of a semiclassical drive, Figs. 4(d) and (e) show that $|\psi\rangle_{\text{ps}}$ can have a negative Wigner function if $\Gamma_{\text{asy}} \approx \Gamma_{\text{jump}}$ and $K \gtrsim \kappa$ hold, but $N(|\psi\rangle_{\text{ps}})$ is zero in the limit $\Gamma_{\text{asy}} \gg \Gamma_{\text{jump}}$ because $|\psi\rangle_{\text{ps}}$ converges to the positive steady state $\rho_{\text{ss}}$.

Importantly, in the limit $K \gg \kappa$ the states $|\psi^\pm_{\text{ps}}\rangle$ converge to the even and odd Schrödinger cat states $|C_k\rangle = (|\alpha\rangle \pm |-\alpha\rangle)/\sqrt{2(1 \pm e^{-2|\alpha|^2})}$ [25], where $\alpha = i\sqrt{\alpha_2/K}$. In this regime, the steady-state solution $\rho_{\text{ss}}$ is a statistical mixture of the two indistinguishable cat states $|C_k\rangle$. The small correction $-iM \propto \kappa$ due to the photon detection breaks this symmetry and allows us to stabilize a Schrödinger kitten state $|C_k\rangle = |\psi\rangle_{\text{ps}}$ without feedback.

**Experimental implementation.**— Our results show that quantum oscillators with Kerr nonlinearities of the order of the decay rate $\kappa$ are sufficient to observe negative pseudo-steady state Wigner functions. Such nonlinear resonators can be realized in a variety of platforms, e.g., superconducting circuits [27, 28] and trapped ions [29, 30]. Potentially even hybrid optomechanical systems could reach the required nonlinearities [27, 31–33]. To ensure $\Gamma_{\text{jump}} \lesssim \Gamma_{\text{rel}}$, the steady-state photon number needs to be small, $\langle a^\dagger a \rangle_{\text{ss}} \lesssim 1$.

For finite temperature $n_{th} > 0$ or imperfect detection $0 \leq \eta < 1$, the unobserved dissipative processes mix different eigenstates of $\hat{H}_0 - iM$ such that negativities in the Wigner function get averaged out. As discussed in [22], the observation of a negative Wigner function requires a low thermal photon number $n_{th} \lesssim 0.1$. This is well satisfied in the optical frequency range, but requires cryogenic environments or pre-cooling for microwave frequencies. However, imperfect efficiency of the single photon detection process is not a major challenge, as analyzed in [22]. Relatively low efficiencies of $\eta \gtrsim 0.25$ and $\eta \gtrsim 0.5$ are already sufficient to observe negativities in a Kerr oscillator with single-photon and two-photon coherent drives, respectively. Thus, current photon detection efficiencies in the optical and infrared range of above 88% are promising to resolve nonclassical states [34, 35]. The single-photon detection efficiency in the microwave regime is still lower [36], but recently more than 70% detection efficiency have been reached [37, 38].

**Conclusion.**— We have shown that continuous photon detection can stabilize nonclassical pseudo-steady states in a driven and damped Kerr nonlinear oscillator, whose steady-state Wigner function is known to be strictly positive. The required nonlinearities and photon detection efficiencies are feasible with current technology. Furthermore, we have applied this protocol to a Kerr parametric oscillator to prepare Schrödinger kitten states. Making use of the jump-rate asymmetry between the states of different parity, we demonstrated that such nonclassical states can be stabilized only by observation and without feedback. Finally, seen from a different angle, the proposed scheme is a heralding protocol to stabilize quantum states in open systems.

**Acknowledgments.**— We thank P. Treutlein and K. Hammerer for fruitful discussions. This work was financially supported by the Swiss National Science Foundation (SNSF) and the NCCR Quantum Science and Technology. The numerical simulations have been implemented using the qutip package [39]. Parts of the calculations were preformed at the sciCORE scientific com-

![Figure 4](link-to-image)
puting core facility at University of Basel [40].

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**Pseudo-steady state of a stochastic quantum master equation**

**Stochastic quantum master equation.**– We consider the general case of the quantum master equation (1) of the main text,

$$\frac{d}{dt} \hat{\rho} = \mathcal{L}_0 \hat{\rho} + \kappa(n_{th} + 1) \mathcal{D}[\hat{a}] \hat{\rho} + \kappa n_{th} \mathcal{D}[\hat{a}^\dagger] \hat{\rho} \quad (5)$$

where $\mathcal{L}_0$ is a completely positive and trace-preserving linear superoperator. We assume that the output mode $\hat{a}$ is displaced by a local oscillator signal of strength $\sqrt{\kappa(n_{th} + 1)} \eta \xi$ before photon detection, as sketched in Fig. 1(b) of the main text. Note that the case $\xi = 0$ reproduces the conventional photon detection scenario. The stochastic quantum master equation is given by [23]

$$\frac{d\hat{\rho}}{dt} = \mathcal{L} \hat{\rho} dt + \left[ \frac{(\hat{a} + \xi)\hat{\rho}(\hat{a}^\dagger + \xi^*)}{\text{Tr}[(\hat{a}^\dagger + \xi^*)(\hat{a} + \xi)]} - \hat{\rho} \right] dN + \mathcal{N} \hat{\rho} \quad (6)$$

where we introduced the abbreviations

$$\mathcal{L} \hat{\rho} = \mathcal{L}_0 \hat{\rho} - i[\kappa(n_{th} + 1) \eta \frac{i}{2} (\xi \hat{a}^\dagger - \xi^* \hat{a}), \hat{\rho}] + \kappa(n_{th} + 1)(1 - \eta) \mathcal{D}[\hat{a}] \hat{\rho} + \kappa n_{th} \mathcal{D}[\hat{a}^\dagger] \hat{\rho} \quad (7)$$

$$\mathcal{N} \hat{\rho} = -\frac{\kappa}{2} (n_{th} + 1) \eta \{(\hat{a}^\dagger + \xi^*)(\hat{a} + \xi), \hat{\rho} \} \quad (8)$$

The Poissonian increment $dN = dN^2$ has the ensemble-averaged expectation value $E(dN) = -\text{Tr}(\mathcal{N} \hat{\rho}) dt$. The superoperator $\mathcal{N} \hat{\rho}$ describes the modification of the dynamics if no photons are detected and causes a decay of the norm of $\hat{\rho}$. To compensate this, we include the nonlinear term $-\text{Tr}(\mathcal{N} \hat{\rho}) \hat{\rho}$ into $\mathcal{L} \hat{\rho}$.

In the following we require that the quantum master equation (5) has a steady-state solution $\hat{\rho}_{ss}$ and that the superoperator $\mathcal{L} + \mathcal{N}$ has a set of left and right eigenvectors

$$(\mathcal{L} + \mathcal{N}) \hat{\rho}_\mu = \lambda_\mu \hat{\rho}_\mu \quad (10)$$

$$(\mathcal{L} + \mathcal{N})^\dagger \hat{\rho}_\mu = \lambda_\mu^* \hat{\rho}_\mu \quad (11)$$

that can suitably be normalized to form a complete orthonormal basis with respect to the Hilbert-Schmidt scalar product, $(\hat{\rho}_\mu, \hat{\rho}_\nu) = \text{Tr}(\hat{\rho}_\mu^\dagger \hat{\rho}_\nu) = \delta_{\nu,\mu}$. This assumption is valid for all systems that do not have exceptional points [41].

**Pseudo-steady state.**– A pseudo-steady state of Eq. (6) is a density matrix $\hat{\rho}$ that is Hermitian, positive semidefinite, normalized to unit trace, and that satisfies $\mathcal{L} \hat{\rho} = 0$. We decompose $\hat{\rho}$ with respect to the basis of eigenstates of $\mathcal{L} + \mathcal{N}$, $\hat{\rho} = \sum_\mu c_\mu \hat{\rho}_\mu$, and obtain the following conditions for the expansion coefficients:

$$\forall \mu : \quad c_\mu \left[ \lambda_\mu - \sum_\beta c_\beta \lambda_\beta \text{Tr}(\hat{\rho}_\beta) \right] = 0 \quad (12)$$

For a non-degenerate eigenvalue $\lambda_\nu$, all but the coefficient $c_\nu$ of the corresponding eigenstate $\hat{\rho}_\nu$ must be zero. Thus, each eigenstate $\hat{\rho}_\nu$ to a non-degenerate eigenvalue $\lambda_\nu$ is a valid solution provided that it is Hermitian, positive semidefinite, and has a non-zero trace such that it can be normalized by $c_\nu = 1/\text{Tr}(\hat{\rho}_\nu)$.

In the following we consider the case where $\mathcal{L}_0$ describes Hamiltonian dynamics, $\mathcal{L}_0 \hat{\rho} = -i[\hat{H}_0, \hat{\rho}]$. Furthermore, if the relations $n_{th} = 0$ and $\eta = 1$ hold, an initial pure state $\hat{\rho} = |\psi \rangle \langle \psi|$ will always stay pure and we can rewrite Eq. (6) as a stochastic Schrödinger equation for the state vector $|\psi \rangle$.
we make the ansatz
\[ \text{where } \sum \mathcal{N} \text{ space is a pseudo-steady state where }\]
\[ E_{\mu} - F \text{ or a non-degenerate eigenvalue } \]
\[ \text{Hamiltonian } \partial_t \]
\[ \text{Similar to the general calculation given above, we re-}
\[ \text{associate eigenvalue } \xi \text{ with associated eigenvalue } \]
\[ \text{order in } \varepsilon \text{. We now expand } d \chi = \mathcal{H} |\chi\rangle \text{ dt in terms of}
\[ \text{and decompose } |\sigma\rangle = \sum \mathcal{C}_\mu |\psi_{\mu}\rangle \text{ with respect to the}
\[ \text{basis of eigenstates } |\psi_{\mu}\rangle \text{ of } \mathcal{H} - i \mathcal{M} \text{, which yields}
\[ \sum \mathcal{C}_\mu \mathcal{P}_\perp |\psi_{\mu}\rangle = -i \sum \mathcal{C}_\mu (h_{\mu} - h) \mathcal{P}_\perp |\psi_{\mu}\rangle, \]
where } \mathcal{P}_\perp \text{ is the projector on the subspace perpendicular to } |\psi\rangle \text{. The state } |\psi\rangle \text{ is stable if all expansion coefficients } \mathcal{C}_\mu \text{ associated to perturbations orthogonal to } |\psi\rangle \text{ decay to zero. For a non-degenerate spectrum } \{h_{\mu}\}, \text{ } |\psi\rangle = |\psi_{\alpha}\rangle \text{ is an eigenstate of } \mathcal{H} - i \mathcal{M} \text{ to eigenvalue } h_{\alpha} \text{ and we can rewrite Eq. (21) to}
\[ \forall \mu \neq \alpha : \frac{d\mathcal{C}_\mu}{dt} = -i(h_{\mu} - h_{\alpha})\mathcal{C}_\mu.\]
Hence, the state } |\psi_{\alpha}\rangle \text{ is stable if } \text{Im}(h_{\mu} - h_{\alpha}) \leq 0 \text{ holds for all } \mu \neq \alpha, \text{ i.e., if } h_{\alpha} \text{ is the eigenvalue of the spectrum with the largest imaginary part.}
\[ \text{The decay rate of any state } |\psi_{\mu}\rangle \text{ towards } |\psi_{\alpha}\rangle \text{ is given by } \Gamma_{rel, \mu \rightarrow \alpha} = -\text{Im}(h_{\mu} - h_{\alpha}) = \langle \psi_{\mu} | \mathcal{M} | \psi_{\mu}\rangle - \langle \psi_{\alpha} | \mathcal{M} | \psi_{\alpha}\rangle, \text{ which is the imaginary part of the spectral gap between the two eigenstates. The overall relaxation rate } \Gamma_{rel} \text{ observed in the decay of any state } |\psi\rangle \text{ towards the pseudo-steady state } |\psi_{\alpha}\rangle \text{ is dominated by the smallest decay rate } \Gamma_{rel, \alpha \rightarrow \mu}, \text{ which corresponds to the spectral gap between the stable pseudo-steady state } |\psi_{\alpha}\rangle \text{ and the closest unstable state.}

Unmonitored dissipative processes
In the limit } \mathcal{L}_0 \rho \rightarrow -i[\mathcal{H}, \rho], \text{ } n_{th} \rightarrow 0, \text{ and } \eta \rightarrow 1, \text{ the stochastic Schrödinger equation (3) considered in the main text and the stochastic quantum master equation (6) can be mapped to one another. The right eigenstates } |\psi_{\mathcal{N}}\rangle \text{ of } \mathcal{H} - i \mathcal{M}, \text{ cf. Eq. (18), can be used to construct the right eigenstates } \rho_{\mu} = \rho_{\mu,j} |\psi_{\mu}\rangle |\psi_{\mu}\rangle^\dagger \text{ of } \mathcal{L} \mathcal{N}, \text{ cf. Eq. (10), and the corresponding eigenvalues fulfill } \lambda_{\mu} = \lambda_{i,j} = -i(h_i - h_j^\dagger). \text{ For finite temperature } n_{th} > 0, \text{ imperfect detection efficiency } 0 \leq \eta < 1, \text{ or additional dissipation channels in } \mathcal{L}_0, \text{ this relation breaks down because the associated Lindblad dissipators mix different basis states } \rho_{i,j}. \text{ Note that non-Hermitian states } \rho_{i,j} \text{ are never mixed with Hermitian states } \rho_{i,j} \text{ because } \mathcal{L} \rho \text{ must preserve the Hermiticity of } \rho. \text{ Physically, these processes correspond to unmonitored dissipative interactions such that the system state can no longer be described by a pure state } |\psi\rangle. \text{ As a consequence, different states } \rho_{i,j}, \text{ each of them possibly having a negative Wigner function, are mixed and their negativity is ultimately averaged out to a non-negative pseudo-steady-state Wigner function.}

In Fig. 5, the change of the negativity } N(\rho_{mm}) \text{ due to finite temperature or imperfect detection is shown. Note that the latter can either be caused by loss of photons}
Figure 5. (a) Impact of finite temperature or imperfect detection on the pseudo-steady state of a Kerr oscillator subject to a semiclassical drive. The main plot shows the negativity $-N(\hat{\rho}_{ps})$ as a function of the thermal photon number $n_{th}$ and the detection efficiency $\eta$. The smaller plots show the Wigner function $W(\alpha)$ of selected states. The origin has been shifted to the steady-state expectation value $\langle \hat{a} \rangle_{ss}$. Top row: Wigner function of steady-state $\hat{\rho}_{ss}$ and pseudo-steady state $\hat{\rho}_{ps}$ for $n_{th} = 0$ and $\eta = 1$. Bottom row: Wigner function of pseudo-steady state $\hat{\rho}_{ps}$ for $n_{th} = 0.5$ and $\eta = 1$ (left) and $n_{th} = 0$ and $\eta = 0.25$ (right). Parameters are $\Delta/\kappa = 1.5$, $|\alpha_1|^2 K/\kappa^3 = 1.5$, $\alpha_2/\kappa = 0$, $K/\kappa = 2.2$, $\xi/\sqrt{\kappa} = 0.9 \times \exp(1.8i)$. (b) Same plots for a Kerr oscillator subject to a parametric drive. Top row: Wigner function of steady-state $\hat{\rho}_{ss}$ and pseudo-steady state $\hat{\rho}_{ps}$ for $n_{th} = 0$ and $\eta = 1$. Bottom row: Wigner function of pseudo-steady state for $n_{th} = 0.1$ and $\eta = 1$ (left) and $n_{th} = 0$ and $\eta = 0.5$ (right). Parameters are $\Delta/\kappa = 0$, $\alpha_1/\kappa = 0$, $\alpha_2/\kappa = 5.3$, $K/\kappa = 10$, $\xi = 0$. on the way to the detector or by a non-unit detection efficiency at the detector itself. Thermal effects average out the negativity at a thermal photon number of about $n_{th} \approx 0.1$, such that cryogenic environments or active cooling of the system prior to the protocol are necessary. However, the negativity is found to be quite robust against imperfect detection. Even for a relatively low detection efficiency of $\eta \approx 0.25$ for a semiclassical drive and $\eta \approx 0.5$ for a parametric drive, negativities in the Wigner function are still present.