Cohomological descent theory for a morphism of stacks and for equivariant derived categories

A. D. Elagin

Abstract. In the paper, we find necessary and sufficient conditions under which, if $X \to S$ is a morphism of algebraic varieties (or, in a more general case, of stacks), the derived category of $S$ can be recovered by using the tools of descent theory from the derived category of $X$. We show that for an action of a linearly reductive algebraic group $G$ on a scheme $X$ this result implies the equivalence of the derived category of $G$-equivariant sheaves on $X$ and the category of objects in the derived category of sheaves on $X$ with a given action of $G$ on each object.

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§ 1. Introduction

As is well known, sheaves on a variety can be defined locally. Namely, let $S = \bigcup U_i$ be a cover of a variety $S$ by open subsets. Defining a sheaf on $S$ is equivalent to defining a family of sheaves $F_i$ on $U_i$ together with gluing isomorphisms $\varphi_{ij} : F_i|_{U_i \cap U_j} \to F_j|_{U_i \cap U_j}$ satisfying the cocycle condition $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on the intersections $U_i \cap U_j \cap U_k$. A sheaf can also be defined by a cover of a more general class. Let $p : X \to S$ be a covering map (for example, a covering of topological spaces or a flat finite morphism of schemes) and let $p_i$ and $p_{ij}$ be the projections of the fibred products $X \times_S X$ and $X \times_S X \times_S X$ onto the factors. Then defining a sheaf on the base $S$ is equivalent to defining a sheaf $F$ on $X$ together with a gluing isomorphism $\theta : p_1^*F \to p_2^*F$ of the sheaves on $X \times_S X$ which satisfies the cocycle condition claiming that the isomorphisms $p_{13}^* \theta$ and $p_{23}^* \theta \circ p_{12}^* \theta$ of the sheaves on $X \times_S X \times_S X$ coincide.

We note that the first of the above statements (concerning an open cover) is in essence a special case of the other for $X = \bigsqcup U_i$.

A natural question arises: Are similar statements valid if we replace the sheaves by objects of the derived category of coherent sheaves on an algebraic variety?

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Suppose that an algebraic group \( G \) acts on an algebraic variety \( X \). By an \textit{equivariant sheaf on} \( X \) we mean a sheaf \( F \) equipped with an action of the group (for a finite group the action is given by isomorphisms \( \theta_g: F \to g^*F \) for \( g \in G \), and these maps are compatible, which means that \( g^*\theta_h \circ \theta_g = \theta_{hg} \) for any pair \( g, h \in G \)). For a given action (of a group \( G \) on a variety \( X \)) the quasi-coherent equivariant sheaves form an Abelian category \( \text{qcoh}^G(X) \). If \( F^\bullet \) is an object of the corresponding derived category \( \mathcal{D}^G(X) = \mathcal{D}(\text{qcoh}^G(X)) \), then on the complex \( F^\bullet \) obtained from \( F^\bullet \) by forgetting the group action there is an action of \( G \) (defined for finite groups in a similar way, as compatible isomorphisms \( \theta_g: F^\bullet \to g^*F^\bullet \) in the category \( \mathcal{D}(X) \)). Is the converse true, that is, is it true that introducing an action of an algebraic group \( G \) on a complex \( F^\bullet \in \mathcal{D}(X) \) defines an object in the category \( \mathcal{D}^G(X) \)?

In the present paper we answer these two questions. We note that, in essence, the second question is a special case of the first one; here the role of the covering space is played by the variety \( X \) and the role of the base by the stack \( X/\!/G \) (the quotient stack of \( X \) by the action of \( G \)). This makes it necessary to work in the category of stacks rather than in that of schemes.

The correct statement of the task is the equivalence problem for the corresponding categories, namely, the derived category of sheaves on the base and some descent category associated with the derived category of sheaves on the covering variety. A classical way to define a descent category was described above; any object of this category is an object \( F \) of \( \mathcal{D}(X) \) together with gluing data that are given as an isomorphism of \( p_1^*F \to p_2^*F \) on \( X \times_S X \) which satisfies the cocycle condition. The answer is given by the following theorem (Theorem 7.3).

\textbf{Theorem.} For any flat morphism of stacks \( p: X \to S \) the unbounded derived category \( \mathcal{D}(S) \) is equivalent to the descent category \( \mathcal{D}(X)/p \) associated with \( p \) if and only if the sheaf \( \mathcal{O}_S \) is singled out as a direct summand under the natural map \( \mathcal{O}_S \to Rp_*\mathcal{O}_X \).

When comparing the equivariant and ordinary derived categories, we obtain the following corollary (Theorem 9.6).

\textbf{Corollary.} For any action of a linearly reductive group \( G \) (that is, a group all of whose linear representations are semisimple) on a scheme \( X \) the derived category of the equivariant sheaves, \( \mathcal{D}^G(X) \), is equivalent to the descent category \( \mathcal{D}(X)^G \) formed by the objects of \( \mathcal{D}(X) \) with the given action of \( G \).

In the proof we use another way of defining the descent data; it originates from comonad theory. A \textit{comonad descent category} is a category of comodules over a comonad (on the category \( \mathcal{D}(X) \)) associated with a pair of adjoint functors \( p^* \) and \( p_* \) between the unbounded derived categories \( \mathcal{D}(X) \) and \( \mathcal{D}(S) \). As is proved in the paper, these two descent categories are equivalent for a flat morphism \( p \) (Proposition 4.2). This makes it possible to use the more convenient language of comonad theory, which helps to prove Theorem 7.3; the proof involves the classical Beck theorem.

The structure of the paper is as follows. The sections 2–6 are mainly of an informative nature. The information concerning cosimplicial categories and related
descent categories is accumulated in §2. The theory concerning comonads is presented in §3 in accordance with the monograph by Barr and Wells [1]: definitions, the comparison theorem, and a criterion for the comparison functor to be an equivalence. Here the specification of this criterion in the case of triangulated categories is new. The equivalence of two ways to define the descent data, namely, the classical one and that using comodules over a comonad, is shown in §4. The definitions and statements concerning subcategories of the descent categories needed when working with bounded derived categories and categories of perfect complexes are given in §5. In §6 we present information on the derived categories of sheaves on stacks.

Section 7 is the central part of the paper; here, using the results of §3, we obtain a criterion for the derived category of the base, as a descent category, to be recoverable from the derived category of the covering space. In §8 we study the SCDT property; it is shown that the finite flat morphisms and also smooth projective morphisms in characteristic zero have this property. In §9, the results concerning the descent for stacks are applied to a particular case of comparing the derived category of coherent sheaves and the derived category of equivariant coherent sheaves.

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§2. Cosimplicial constructions

Let $\Delta_0$ be the category whose objects are the sets of the form $[1, \ldots, n]$, $n \in \mathbb{N}$, and the empty set and the morphisms are the non-decreasing maps between sets of this form. Let $\Delta \subset \Delta_0$ be the full subcategory consisting of the nonempty sets.

By definition, a cosimplical object of some category $\mathcal{C}$ (for example, a cosimplicial set, a cosimplicial scheme, and so on) is a functor from $\Delta$ to $\mathcal{C}$. If for $\mathcal{C}$ we take the 2-category of categories $\text{Cats}$, we obtain the definition of cosimplicial category.

Definition 2.1. By a cosimplicial category we mean a covariant 2-functor from $\Delta$ to the 2-category $\text{Cats}$ of categories and functors and by an augmented cosimplicial category we mean a covariant 2-functor $\Delta_0 \to \text{Cats}$. In other words, a cosimplicial category $\mathcal{C}_\bullet$ (an augmented cosimplicial category $\mathcal{C}_\bullet$, respectively) consists of the following data:

1) a family of categories $\mathcal{C}_k$, $k = 0, 1, 2, \ldots$ ($k = -1, 0, 1, 2, \ldots$, respectively) corresponding to the objects of $\Delta$ (of $\Delta_0$, respectively), where $\mathcal{C}_k$ corresponds to the set $[1, \ldots, k + 1]$;

2) a family of functors $P_k^*: \mathcal{C}_m \to \mathcal{C}_n$ corresponding to the morphisms in $\Delta$ (in $\Delta_0$, respectively), that is, to the monotone maps $f: [1, \ldots, m + 1] \to [1, \ldots, n + 1]$;

3) a family of isomorphisms of the functors $\epsilon_{f,g}: P_k^* P_l^* \to P_{k+l}^*$ corresponding to the pairs of maps $f, g$ for which the composition $f \circ g$ makes sense.
The isomorphisms 3) must satisfy the cocycle condition: the diagram

\[
P^*_f P^*_g P^*_h \xrightarrow{\epsilon_{f,g}} P^*_f P^*_g P^*_h \\
\downarrow \epsilon_{g,h} \downarrow \epsilon_{f,g,h} \\
P^*_f P^*_g h \xrightarrow{\epsilon_{f,g,h}} P^*_f g h
\]

must be commutative for any triple of maps \(f, g, h\) for which the composition \(f \circ g \circ h\) makes sense.

The simplicial and augmented simplicial categories are defined as contravariant 2-functors \(\Delta \rightarrow \text{Cats}\) and \(\Delta_0 \rightarrow \text{Cats}\).

Let \(\mathcal{C}_\bullet = [\mathcal{C}_0, \mathcal{C}_1, \ldots, P^*_f]\) be an augmented cosimplicial category. The category \(\mathcal{C}_-\) and the functors \(P^*_f\) defined on \(\mathcal{C}_-\) are referred to as augmentation. The cosimplicial category \([\mathcal{C}_0, \mathcal{C}_1, \ldots, P^*_f]\) obtained by deleting the augmentation from \(\mathcal{C}_\bullet\) is denoted by \(\text{Sk}_{\geq 0}(\mathcal{C}_\bullet)\).

Using the terminology of 19.1 in [2], we may say that a simplicial category is a pre-stack on the category \(\Delta\).

Example 2.2. Let \(X \rightarrow S\) be a morphism of schemes. Then the schemes \(S, X, X \times_S X, X \times_S X \times_S X, \ldots\) and the morphisms between these schemes

\[
p_f: \underbrace{X \times_S X \times \cdots \times X}_n \rightarrow \underbrace{X \times_S X \times \cdots \times X}_m
\]

that are defined by the rule

\[
p_f(x_1, \ldots, x_n) = (x_f(1), \ldots, x_f(m))
\]

for \(f \in \text{Hom}_{\Delta_0}([1, \ldots, m], [1, \ldots, n])\) form an augmented simplicial scheme. The categories of sheaves on these schemes and the inverse image functors ('pull-backs') between them are an important example of an augmented cosimplicial category.

It is useful to imagine the categories \(\mathcal{C}_-, \mathcal{C}_0, \mathcal{C}_1, \ldots\) as categories of sheaves related to \(S, X, X \times_S X, X \times_S X \times_S X, \ldots\). For the functors \(P^*_f\) we use below the natural notation recalling the inverse image functors between categories of sheaves. For example, the functor \(P^*_f: \mathcal{C}_1 \rightarrow \mathcal{C}_2\) for the map \(f: [1, 2] \rightarrow [1, 2, 3]\) such that \(f(1) = 1\) and \(f(2) = 3\) is denoted by \(P^*_{13}\). It is convenient to think about this functor as the inverse image under the projection \(p_{13}: X \times_S X \times_S X \rightarrow X \times_S X\). We denote by \(P^*\) the functor \(P^*_f: \mathcal{C}_- \rightarrow \mathcal{C}_0\) corresponding to the only map \(\emptyset \rightarrow [1]\). This functor plays the role of the inverse image functor under the morphism \(p: X \rightarrow S\). We denote by \(D^*\) the functor \(P^*_f: \mathcal{C}_1 \rightarrow \mathcal{C}_0\) for a unique map \(f: [1, 2] \rightarrow [1]\). It is useful to view this functor as the inverse image under the diagonal embedding \(d: X \rightarrow X \times_S X\).

For any cosimplicial category \(\mathcal{C}_\bullet = [\mathcal{C}_0, \mathcal{C}_1, \ldots, P^*_f]\) one can define the following descent category denoted by \(\text{Kern}(\mathcal{C}_\bullet)\).

**Definition 2.3** (classical descent category; see 19.3 in [2]). The objects of \(\text{Kern}(\mathcal{C}_\bullet)\) are the pairs \((F, \theta)\), where \(F \in \text{Ob} \mathcal{C}_0\) and \(\theta\) is an isomorphism \(P^*_1 F \rightarrow P^*_2 F\).
subjected to the cocycle condition which claims that the diagram

\[
\begin{array}{c}
P^*_{12} P^*_1 F \\
\sim \downarrow P^*_{13} \theta \downarrow \sim \downarrow P^*_{13} \theta \downarrow \sim \downarrow P^*_{23} \theta \downarrow \sim \downarrow P^*_{23} \theta \downarrow \\
P^*_{12} P^*_2 F \\
\end{array}
\]

is commutative. In this diagram, the arrows with the symbols ~ show functor isomorphisms from the definition of cosimplicial category. The morphisms in Kern(\(\mathcal{C}_\bullet\)) from \((F_1, \theta_1)\) into \((F_2, \theta_2)\) are the morphisms \(f \in \text{Hom}_{\mathcal{C}_0}(F_1, F_2)\) such that \(P^*_2 f \circ \theta_1 = \theta_2 \circ P^*_1 f\).

Remark 2.4. Identifying canonically isomorphic objects, one customarily represents the cocycle condition in the form

\[P^*_{23} \circ P^*_{12} = P^*_{13} \theta.\]

The category Kern(\(\mathcal{C}_\bullet\)) admits another (equivalent) description.

**Definition 2.5.** An object of the category Kern(\(\mathcal{C}_\bullet\)) is a family of objects \(F_i \in \mathcal{C}_i, i = 0, 1, 2, \ldots,\) and a family of isomorphisms \(\varphi_f : P^*_f F_m \xrightarrow{\sim} F_n\) for any morphism \(f : [1, \ldots, m+1] \to [1, \ldots, n+1]\) in \(\Delta\) that satisfy the equations \(\varphi_{gf} = \varphi_g \circ P^*_{\varphi_f}\) for any pair \((f, g)\) for which the composition is well defined. A morphism in Kern(\(\mathcal{C}_\bullet\)) from \((F_\bullet, \varphi_\bullet)\) to \((F'_\bullet, \varphi'_\bullet)\) is a family of morphisms \(\rho_i : F_i \to F'_i\) compatible with \(\varphi_\bullet\) and \(\varphi'_\bullet\).

**Proposition 2.6.** Definitions 2.3 and 2.5 are equivalent.

**Proof.** For the convenience of the reader, we present a sketch of the proof of this proposition.

Let an object of the category in Definition 2.5 consist of families \(F_\bullet = (F_i)\) and \(\varphi_\bullet = (\varphi_f)\). To this object we assign an object of the category in Definition 2.3 as follows: we set \(F = F_0\) and set \(\theta : P^*_1 F \to P^*_2 F\) to be equal to the composition

\[
\varphi_{i_2}^{-1} \varphi_{i_1} : P^*_i F_0 \xrightarrow{\varphi_{i_1}} F_1 \xrightarrow{\varphi_{i_2}^{-1}} P^*_2 F_0.
\]

Here \(i_1\) and \(i_2\) stand for two morphisms \([1] \to [1, 2]\) in \(\Delta\) given by \(i_1(1) = 1\) and \(i_2(1) = 2\). The compatibility condition for \(\varphi_f\) implies the cocycle condition for \(\theta\). Conversely, let \((F, \theta)\) be an object of the category in Definition 2.3. To this object we assign an object of the category in Definition 2.3. Let \(f_i : [1] \to [1, \ldots, i+1]\) be the function taking 1 to 1. We set \(F_i = P^*_f F\) and define an isomorphism \(\varphi_f\) for the morphism \(f : [1, \ldots, m+1] \to [1, \ldots, n+1]\) as follows. Let \(f(1) = r\) and let \(g : [1, 2] \to [1, \ldots, n+1]\) be the map such that \(g(1) = 1\) and \(g(2) = r\). For \(\varphi_f\) we take the composition of the isomorphisms

\[
P^*_f F_m = P^*_f P^*_m F \xrightarrow{\epsilon_{f,f_m}} P^*_{f,f_m} F \xrightarrow{\epsilon_{g_1}} P^*_g F = P^*_f F.
\]
It can readily be seen that the cocycle condition for \( \theta \) implies the compatibility condition for \( \varphi_f \).

The category \( \text{Kern}(\mathcal{C}_\bullet) \) constructed from a cosimplicial category \( \mathcal{C}_\bullet \) admits the natural forgetful functors \( \text{Kern}(\mathcal{C}_\bullet) \to \mathcal{C}_k \). It turns out that these functors complete \( \mathcal{C}_\bullet \) to an augmented cosimplicial category. We write \( \mathcal{C}_{-1} = \text{Kern}(\mathcal{C}_\bullet) \) and define the functor \( P^*_f : \mathcal{C}_{-1} \to \mathcal{C}_n \) for any morphism \( f : \emptyset \to [1, \ldots, n+1] \) as the forgetting \( (F_\bullet, \varphi_\bullet) \mapsto F_n \) on the objects and \( \rho_\bullet \mapsto \rho_n \) on the morphisms (here we use Definition 2.5).

**Proposition 2.7.** 1) The categories \( \mathcal{C}_{-1} = \text{Kern}(\mathcal{C}_\bullet), \mathcal{C}_0, \mathcal{C}_1, \ldots \) and the functors \( P^*_f \) between them form an augmented cosimplicial category \( \mathcal{C}_\bullet \).

2) The category \( \mathcal{C}_\bullet \) has the following universal property: for any completion of \( \mathcal{C}_\bullet \) to an augmented cosimplicial category \( \mathcal{C}'_\bullet = [\mathcal{C}'_{-1}, \mathcal{C}_0, \mathcal{C}_1, \ldots, P^*_f] \) there is a functor \( \Phi : \mathcal{C}'_{-1} \to \text{Kern}(\mathcal{C}_\bullet) = \mathcal{C}_1 \) (unique up to isomorphism) which can be extended by a family of identity functors to a functor \( \mathcal{C}'_\bullet \to \mathcal{C}_\bullet \) between augmented cosimplicial categories.

**Remark 2.8.** The part 2) can be viewed as an analogue of the comparison theorem 3.6 for the category of comodules over a comonad (see below).

**Proof.** 1) For a pair of morphisms

\[
f : \emptyset \to [1, \ldots, m+1], \quad g : [1, \ldots, m+1] \to [1, \ldots, n+1]
\]

we define a functor isomorphism \( \epsilon_{g,f} : P^*_g\big|_{P^*_f} \cong \rho_{g\circ f} \) on an object \((F_\bullet, \varphi_\bullet) \in \text{Kern}(\mathcal{C}_\bullet)\) by the rule

\[
P^*_g\big|_{P^*_f}((F_\bullet, \varphi_\bullet)) = P^*_g F_m \xrightarrow{\varphi_{g\circ f}} F_n = P^*_g((F_\bullet, \varphi_\bullet)).
\]

The compatibility condition for \( \varphi \) implies the cocycle condition for the ‘new’ \( \epsilon \).

2) Let \( H \) be an object of the category \( \mathcal{C}'_{-1} \). We define a functor \( \Phi \) on an object \( H \) as the pair \((F_\bullet, \varphi_\bullet)\), where

\[
F_k = P^*_{f_k} H \in \mathcal{C}_k = \mathcal{C}_k
\]

for the (unique) map \( f_k : \emptyset \to [1, \ldots, k+1] \) and \( \varphi_f \) is defined for \( f : [1, \ldots, m+1] \to [1, \ldots, n+1] \) by

\[
P^*_{f_m} F_m = P^*_{f_m} P^*_{f_m} H = P^*_{f_m} P^*_{f_m} H \xrightarrow{\epsilon_{f_m}} P^*_{f_m} H = P^*_{f_m} H = F_n.
\]

On a morphism \( u : H_1 \to H_2 \) we set \( \Phi \) to be equal to \( \rho_\bullet \), where \( \rho_k : P^*_{f_k} H_1 \to P^*_{f_k} H_2 \) is equal to \( P^*_{f_k} u \). Obviously, the functor \( \Phi \) has the desired properties.

We are interested in the augmented cosimplicial categories

\[[\mathcal{C}_{-1}, \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots, P^*_\bullet]\]

satisfying two additional conditions.

**Condition 1.** All functors \( P^*_f : \mathcal{C}_m \to \mathcal{C}_n \) have right adjoint functors \( P_{f_*} : \mathcal{C}_n \to \mathcal{C}_m \).
It can be proved (see 3.6 in [3]) that the functors $P_{\bullet*}$ form an augmented simplicial category

$$[C_{-1}, C_0, C_1, \ldots, P_{\bullet*}].$$

It is useful to think of these functors as direct image functors (‘push-forward functors’) between categories of sheaves.

Let us consider a commutative square in the category $\Delta_0$ and the corresponding square of categories and functors,

$$\begin{array}{ccc}
[1, \ldots, m + n - r + 1] & \xrightarrow{f'} & [1, \ldots, n + 1] \\
\uparrow g' & & \uparrow g \\
[1, \ldots, m + 1] & \xleftarrow{f} & [1, \ldots, r + 1]
\end{array}$$

$$\begin{array}{ccc}
C_{m+n-r} & \xleftarrow{P_{f'}} & C_n \\
P_{g'} & & P_g \\
C_m & \xleftarrow{P_{f'}} & C_r
\end{array}$$

If the maps $f$ and $f'$ (or $g$ and $g'$) are injective and if $[1, \ldots, m + n - r + 1] = \text{Im } f' \cup \text{Im } g'$, then we refer to these squares as exact Cartesian ones. To any square we assign two natural base-change morphisms,

$$P_{g'} P_{f*} \to P_{f'*} P_{g*}, \quad P_{g*} P_{f*} \to P_{g'*} P_{f'}. \quad (2.1)$$

For example, the morphism $P_{g'} P_{f*} \to P_{f'*} P_{g*}$ can be defined as the composition

$$P_{g'} P_{f*} \xrightarrow{\eta} P_{g'} P_{f*} P_{g*} \xrightarrow{\sim} P_{g'} P_{g*} P_{f'*} P_{g*} \xrightarrow{\varepsilon} P_{f'*} P_{g*},$$

or as the composition

$$P_{g'} P_{f*} \xrightarrow{\eta} P_{f'*} P_{f*} P_{g*} \xrightarrow{\sim} P_{f'*} P_{g*} P_{f*f} \xrightarrow{\varepsilon} P_{f'*} P_{g*},$$

where $\eta$ and $\varepsilon$ stand for the canonical adjunction morphisms. A simple verification shows that these two ways give equal results.

The other condition is an axiomatization of the theorem on the flat base change.

**Condition 2.** The base-change morphisms (2.1) are isomorphisms for any exact Cartesian square.

**Proposition 2.9.** Let $\mathcal{C}_\bullet$ be a cosimplicial category and let $\tilde{\mathcal{C}}_\bullet$ be the augmented cosimplicial category obtained from $\mathcal{C}_\bullet$ by adjoining the category $\text{Kern}(\mathcal{C}_\bullet)$. If $\mathcal{C}_\bullet$ satisfies the conditions 1 and 2, then so does $\tilde{\mathcal{C}}_\bullet$.

**Proof.** Condition 1 consists in the existence of right adjoint functors $P_{f*}$ in the category $\tilde{\mathcal{C}}_\bullet$ for all morphisms $f$ in $\Delta_0$. For morphisms $f$ in $\Delta$ the adjoint functor exists by assumption, and one must consider morphisms of the form $f : \varnothing \to [1, \ldots, n]$. Expanding $f$ into a composition, we see that it is sufficient to verify the existence of the adjoint functor to the forgetful functor $P^* : \text{Kern}(\mathcal{C}_\bullet) \to \mathcal{C}_0$.

Let us define a functor $P_* : \mathcal{C}_0 \to \text{Kern}(\mathcal{C}_\bullet)$ as follows. Let $F \in \mathcal{C}_0$. We set

$$P_* F = (P_{2*} P_{1*} F, \theta_F).$$
Here $\theta_F : P_1^* P_2^* P_1^* F \rightarrow P_2^* P_2^* P_1^* F$ is defined as the composition

$$P_1^* P_2^* P_1^* F \xrightarrow{\sim} P_{23}^* P_{12}^* P_1^* F \xrightarrow{\sim} P_{23}^* P_{13}^* P_1^* F \xrightarrow{\sim} P_2^* P_2^* P_1^* F,$$

where the outermost isomorphisms are the base-change maps and the isomorphism in the middle occurs in the definition of cosimplicial category. On the morphisms we set $P_* f = P_2^* P_1^* f$. The proof of the fact that $\theta_F$ satisfies the cocycle condition is left to the reader.

To show that the functors $P^*$ and $P_*$ are adjoint to each other, we construct morphisms of the functors

$$\eta : \text{Id}_{\text{Kern}(\mathcal{C}_*)} \rightarrow P_* P^*, \quad \varepsilon : P^* P_* \rightarrow \text{Id}_{\mathcal{C}_0}.$$

We define $\eta$ on the objects $(F, \theta) \in \text{Kern}(\mathcal{C}_*)$ as the composition

$$\eta(F, \theta) : F \xrightarrow{\eta_F} P_2^* P_2^* F \xrightarrow{P_2^* \theta^{-1}} P_2^* P_1^* F.$$

It can readily be shown that the morphism $\eta(F, \theta)$ is compatible with $\theta$ and $\theta_F$ and is indeed a morphism in $\text{Kern}(\mathcal{C}_*)$.

We define $\varepsilon$ on the objects $F \in \mathcal{C}_0$ as follows:

$$\varepsilon_F : P_2^* P_1^* F \xrightarrow{\varepsilon} P_2^* D_* D_* P_1^* F \xrightarrow{\sim} \text{Id}_* \text{Id}^* F = F.$$

It follows from the above definitions of morphisms $\eta$ and $\varepsilon$ and from the properties of cosimplicial categories that the compositions of functors

$$P_* \xrightarrow{\eta P_*} P_* P_* P_* \xrightarrow{P_* \varepsilon} P_* \quad \text{and} \quad P^* \xrightarrow{P^* \eta} P^* P_* P^* \xrightarrow{\varepsilon P^*} P^*$$

are identity functors. This implies that the functors $P^*$ and $P_*$ are adjoint.

Condition 2 is that the base-change formula must hold for the exact Cartesian squares in $\Delta_0$. This formula holds by assumption for the squares lying in $\Delta$, and therefore it is sufficient to consider the squares of the form

$$[1, \ldots, m + n] \xleftarrow{f'} [1, \ldots, n] \xrightarrow{g} [1, \ldots, m] \xleftarrow{f} \emptyset$$

Expanding $f$ and $g$ in compositions, we reduce the proof to the case of the square

$$[1, 2] \xleftarrow{i_2} [1] \xrightarrow{i_1} [1] \xleftarrow{f} \emptyset \xrightarrow{g} [1]$$

that is, to the proof of the fact that the natural morphisms

$$P^* P_* \rightarrow P_2^* P_1^*, \quad P^* P_* \rightarrow P_1^* P_2^*$$

in $\mathcal{C}_0$ are isomorphisms. In this case, the argument is simple and uses the definitions.

This completes the proof of Proposition 2.9.
§ 3. Comonads and comodules

Let us recall some facts of the comonad theory. For details, see Ch. 3 in Barr-Wells’ book [1] and Ch. 6 in Maclane’s book [4].

We denote by $\mathcal{C}$ an arbitrary category.

**Definition 3.1.** A *comonad* $T = (T, \varepsilon, \delta)$ (which is also referred to as a *standard construction*) on the category $\mathcal{C}$ consists of a functor $T: \mathcal{C} \to \mathcal{C}$ and natural transformations of the functors $\varepsilon: T \to \text{Id}_{\mathcal{C}}$ and $\delta: T \to T^2 = TT$ such that the following diagrams are commutative:

$$
\begin{array}{ccc}
T & \xrightarrow{\delta} & T^2 \\
\downarrow{\delta} & & \downarrow{\delta} \\
T^2 & \xrightarrow{\varepsilon T} & T \\
\end{array}
\quad
\begin{array}{ccc}
T & \xrightarrow{\delta} & T^2 \\
\downarrow{\delta} & & \downarrow{T \delta} \\
T^2 & \xrightarrow{\delta T} & T^3 \\
\end{array}
$$

**Example 3.2.** Consider a pair of adjoint functors $P^*: \mathcal{B} \to \mathcal{C}$ (left) and $P_*: \mathcal{C} \to \mathcal{B}$ (right). Let $\eta: \text{Id}_\mathcal{B} \to P_*P^*$ and $\varepsilon: P^*P_* \to \text{Id}_\mathcal{C}$ be the canonical adjunction morphisms. We define a triple $(T, \varepsilon, \delta)$ by setting $T = P^*P_*$ with $\varepsilon: P^*P_* \to \text{Id}_\mathcal{C}$ and $\delta = P^*\eta P_*: P^*P_* \to P^*P_*P^*P_*$. Then $T = (T, \varepsilon, \delta)$ forms a comonad on the category $\mathcal{C}$.

In fact, an arbitrary comonad can be obtained from a pair of adjoint functors in the above way. This follows from the construction presented below, which is due to Eilenberg and Moore.

**Definition 3.3.** Let $T = (T, \varepsilon, \delta)$ be a comonad on the category $\mathcal{C}$. By a *comodule* over $T$ (or a *$T$-coalgebra*) we mean a pair $(F, h)$, where $F \in \text{Ob} \mathcal{C}$ and $h: F \to TF$ is a morphism satisfying the following two conditions: the composition

$$
F \xrightarrow{h} TF \xrightarrow{\varepsilon F} F
$$

is equal to the identity morphism, and the diagram

$$
\begin{array}{ccc}
F & \xrightarrow{h} & TF \\
\downarrow{h} & & \downarrow{T h} \\
TF & \xrightarrow{\delta F} & T^2 F \\
\end{array}
$$

is commutative. By a *morphism* between comodules $(F_1, h_1)$ and $(F_2, h_2)$ we mean a morphism $f: F_1 \to F_2$ in the category $\mathcal{C}$ for which the diagram

$$
\begin{array}{ccc}
F_1 & \xrightarrow{f} & F_2 \\
\downarrow{h_1} & & \downarrow{h_2} \\
TF_1 & \xrightarrow{T f} & TF_2 \\
\end{array}
$$

is commutative.
The comodules over a given comonad $T$ on $C$ form a category, which is denoted by $\mathcal{C}_T$. Let us define a functor $Q_*: C \to \mathcal{C}_T$ by setting

$$Q_*F = (TF, \delta F), \quad Q_*f = Tf,$$

and define $Q^*: \mathcal{C}_T \to C$ as the forgetful functor, $(F, h) \mapsto F$. Then the functors $(Q^*, Q_*)$ form an adjoint pair, and this pair generates the comonad $T$ by the construction of Example 3.2.

The category $\mathcal{C}_T$ inherits many properties of $C$. One can readily see that the following statement holds.

**Proposition 3.4.** Let $T = (T, \varepsilon, \delta)$ be a comonad on a category $C$. If $C$ is an additive category and the functor $T$ is additive, then the category $\mathcal{C}_T$ is also additive. If $C$ is Abelian and $T$ is left exact, then $\mathcal{C}_T$ is also Abelian.

On the contrary, it is unclear whether or not the category $\mathcal{C}_T$ constructed from a comonad $T = (T, \varepsilon, \delta)$ is triangulated if the functor $T$ is exact on a triangulated category $C$. It is natural to try to define a triangulated structure on $\mathcal{C}_T$ as follows.

**Definition 3.5.** We define a shift functor on $\mathcal{C}_T$ by the equations

$$(F, h)[1] = (F[1], h[1]), \quad f[1] = f[1].$$

The triangles $(F', h') \to (F, h) \to (F'', h'') \to (F', h')[1]$ for which $F' \to F \to F'' \to F'[1]$ is a distinguished triangle in $C$ are said to be distinguished in $\mathcal{C}_T$.

Unfortunately, the operation of taking a cone in $C$ is not functorial, and therefore one cannot prove without additional assumptions that any morphism in $\mathcal{C}_T$ can be completed to a distinguished triangle. However, it happens sometimes that the above definition does indeed introduce a triangulated structure on $\mathcal{C}_T$; if this is the case, then we simply say that the category $\mathcal{C}_T$ is triangulated (meaning that the triangulated structure is that defined above). Below (Proposition 3.13) we shall see that $\mathcal{C}_T$ is triangulated under certain conditions; in fact, it will be proved that $\mathcal{C}_T$ is equivalent (as an abstract category) to some triangulated category.

The following statement shows that the Eilenberg-Moore construction gives a terminal object among all adjoint pairs defining the same comonad.

**Proposition 3.6** (comparison theorem; see 3.2.3 in [1] and 6.3 in [4]). Let $T = (T, \varepsilon, \delta)$ be a comonad on a category $C$, which is defined by a pair of adjoint functors $P^*: \mathcal{B} \to C$ and $P_*: C \to \mathcal{B}$. Then there is a functor $\Phi: \mathcal{B} \to \mathcal{C}_T$, which is unique up to isomorphism (and is referred to as a comparison functor), for which the diagram of categories

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{P_*} & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{C}_T & \xrightarrow{Q_*} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow{Q^*} & \mathcal{C}_T \\
\end{array}
$$

is commutative, that is, the outer and inner triangles commute,

$$\Phi P_* \cong Q_*, \quad Q^* \Phi \cong P^*.$$
Proof. Let us define $\Phi: \mathcal{B} \to \mathcal{C}$ as a functor assigning to any object $H \in \text{Ob} \mathcal{B}$ a pair $(P^*H, h)$ in which $h: P^*H \to P^*P_*P^*H$ is $P^*\eta$ for the canonical morphism $\eta: H \to P_*P^*H$ and assigning the morphism $P^*f$ to any morphism $f$. A simple verification shows that $\Phi$ is the desired functor and any functor for which the diagram is commutative is isomorphic to $\Phi$.

Criteria for a comparison functor to be fully faithful or an equivalence are of importance for our purposes. Before formulating these criteria, we recall the notions of equalizer and of contractible equalizer.

**Definition 3.7.** By an equalizer of a pair of morphisms $d_1: F_1 \to F_2$, $d_2: F_1 \to F_2$ we mean a morphism $d: F \to F_1$ such that

1) $d_1d = d_2d$,

2) for any morphism $d': F' \to F_1$ such that $d_1d' = d_2d'$ there is a unique morphism $f: F' \to F$ for which $df = d'$.

**Definition 3.8.** By a contractible equalizer of a pair of morphisms $d_1, d_2: F_1 \to F_2$ we mean a morphism $d: F \to F_1$ equipped with morphisms $s$ and $t$,

$$
\begin{array}{ccc}
F & \xleftarrow{d} & F_1 \\
\downarrow{s} & & \downarrow{d_1} \\
F_2 & \xrightarrow{d_2} & F_2
\end{array}
$$

for which

\begin{align*}
d_1d &= d_2d, \\
sd &= \text{Id}, \\
td_1 &= \text{Id}, \\
td_2 &= ds.
\end{align*}

We note that any contractible equalizer is an equalizer and any equalizer is a monomorphism. We also note that contractible equalizers are preserved by any functor.

A morphism $f: H' \to H$ in a category $\mathcal{B}$ is said to be a split embedding or, briefly, splits if $f$ admits a left inverse morphism $f'$, $f'f = \text{Id}_{H'}$. If the category $\mathcal{B}$ is additive, it is also said that $f$ is an embedding of a direct summand.

We recall that a functor $\Phi$ is said to be conservative if any morphism $f$ for which $\Phi(f)$ is an isomorphism is also an isomorphism.

**Theorem 3.9** (Beck, see 3.14 in [1] and 6.7 in [4]). 1) The functor $\Phi$ is fully faithful if and only if for any $H \in \text{Ob} \mathcal{B}$ the natural morphism $\eta_H: H \to P_*P^*H$ is an equalizer (of some pair).

2) The functor $\Phi$ is an equivalence if and only if $P^*$ is conservative and for any pair $d_1, d_2: H_1 \to H_2$ of morphisms in $\mathcal{B}$ for which the pair $P^*d_1, P^*d_2: P^*H_1 \to P^*H_2$ admits a contractible equalizer $f: F \to P^*H_1$ there is an equalizer $d: H \to H_1$ of $(d_1, d_2)$ for which $P^*d \cong f$.

**Corollary 3.10.** 1) If the categories $\mathcal{B}$ and $\mathcal{C}$ are Abelian, then the comparison functor $\Phi$ turns out to be fully faithful if and only if the morphism $\eta_H: H \to P_*P^*H$ is injective for any object $H$ in $\mathcal{B}$. Suppose in addition that the functor $P^*$ is exact. Then the full faithfulness of the functor $\Phi$ is equivalent to the condition that $\Phi$ is an equivalence and also to the condition that $P^*H \neq 0$ for all $H \neq 0 \in \mathcal{B}$.

2) If the categories $\mathcal{B}$ and $\mathcal{C}$ are triangulated, then the comparison functor $\Phi$ is fully faithful if and only if the morphism $\eta_H: H \to P_*P^*H$ is a split embedding for any object $H$ in $\mathcal{B}$.
We note that adjoint functors between additive categories are automatically additive.

Proof of Corollary 3.10. We note that, in an additive category, the equalizer of a pair of morphisms \( (f_1, f_2) \) is precisely the kernel of the difference \( f_1 - f_2 \). Therefore, in an additive category, the conditions ‘to be an equalizer’ and ‘to be a kernel’ are equivalent.

1) In an Abelian category, the condition ‘to be a kernel’ is ‘to be injective’. Let us prove the part 2) of the statement. If a morphism \( H \to P_*P^*H \) is injective, then, clearly, \( P^*H \neq 0 \) for \( H \neq 0 \). Conversely, if \( P^*H \neq 0 \) for any nonzero \( H \) and \( \ker(H \to P_*P^*H) \neq 0 \), then \( \ker(P^*H \to P^*P_*P^*H) \neq 0 \). However, the morphism \( P^*H \to P^*P_*P^*H \) is a split embedding. Thus, we arrive at a contradiction. In order to prove that \( \Phi \) is an equivalence under the above assumptions, let us verify the conditions in Theorem 3.9. Indeed, equalizers always exist in any Abelian category, and they are preserved by all exact functors. The functor \( P^* \) is conservative because it is exact and \( P^*H \neq 0 \) for \( H \neq 0 \).

2) In any triangulated category, the kernels are precisely the split embeddings, which implies the desired statement.

We recall that a category \( \mathcal{B} \) is said to be Karoubian complete or a Karoubi category if any projector in \( \mathcal{B} \) splits, that is, if for any object \( H \) in \( \mathcal{B} \) and any morphism \( f: H \to H \) such that \( f^2 = f \) there are an object \( H' \) (referred to as the image of \( f \)) and morphisms \( \sigma: H' \to H \) and \( \rho: H \to H' \) such that \( \rho \sigma = \Id_{H'} \) and \( \sigma \rho = f \).

Corollary 3.11. Let \( (P^*, P_*) \) be a pair of adjoint functors between categories \( \mathcal{B} \) and \( \mathcal{C} \). Suppose that \( \mathcal{B} \) is a Karoubi category. If the natural morphism of functors \( \eta: \Id_{\mathcal{B}} \to P_*P^* \) splits, then the comparison functor is an equivalence.

Proof. Let a morphism of functors \( \eta: \Id_{\mathcal{B}} \to P_*P^* \) split. Then for any object \( H \) in \( \mathcal{B} \) there is a projector \( \pi: P_*P^*H \to P_*P^*H \) whose image is \( H \), and this projector depends on \( H \) in a natural way (that is, \( \pi \) is a morphism of functors \( P_*P^* \to P_*P^* \)). Let us verify two conditions in Beck’s theorem. First, \( P^* \) is conservative. Indeed, if \( f \) is a morphism in \( \mathcal{B} \) such that the morphism \( P^*f \) is an isomorphism, then \( P_*P^*f \) is also an isomorphism, and thus \( f \) is an isomorphism as well. Second, suppose that \( P^* \) applied to some pair of morphisms \( (f_1, f_2) \) in \( \mathcal{B} \) gives a pair with a contractible equalizer. Then \( P_*P^* \) applied to this pair also gives a pair with a contractible equalizer. Passing to the images of the projector \( \pi \) by using the next lemma, we see that the pair \( (f_1, f_2) \) also has a contractible equalizer, and this equalizer is preserved by the functor \( P^* \) by the remark after Definition 3.8.

Lemma 3.12. Let \( f_1, f_2: H_1 \to H_2 \) be two morphisms in a Karoubi category \( \mathcal{B} \) that is compatible with the projectors \( \pi_1: H_1 \to H_1 \) and \( \pi_2: H_2 \to H_2 \). In this case, the existence of a contractible equalizer for the pair \( (f_1, f_2) \) implies the existence of a contractible equalizer for a pair of the corresponding morphisms on the images of the projectors \( \pi_1 \) and \( \pi_2 \).

Proof. By assumption, there are objects \( H'_1 \) and \( H'_2 \) in \( \mathcal{B} \) (the images of \( \pi_1 \) and \( \pi_2 \)) and morphisms \( \sigma_i: H'_i \to H_i \) and \( \rho_i: H_i \to H'_i \) (the embeddings of the images and the projections onto these images) for which \( \rho_i \sigma_i = \Id_{H'_i} \), \( \sigma_i \rho_i = \pi_i \), \( i = 1, 2 \). The
morphisms $f_1$ and $f_2$ induce morphisms $f'_1 = \rho_2 f_1 \sigma_1$ and $f'_2 = \rho_2 f_2 \sigma_1$ from $H'_1$ to $H'_2$. Let

\[
\begin{array}{c}
H \xrightarrow{f} H_1 \xrightarrow{f_1} H_2 \\
\downarrow s \quad \downarrow f_2 \quad \downarrow t
\end{array}
\]

be a diagram associated with some contractible equalizer for $(f_1, f_2)$. Consider the morphism $\pi_1 f : H \to H_1$. We see that $f_1(\pi_1 f) = \pi_2 f_1 f = \pi_2 f_2 f = f_2(\pi_1 f)$. Thus, by the definition of equalizer, there is a morphism $\pi : H \to H$ such that $f \pi = \pi_1 f$. This morphism $\pi$ is a projector. Indeed, $f \pi^2 = \pi_1 f \pi = \pi_2 f = \pi_1 f = f \pi$ and, since $f$ is a monomorphism, it follows that $\pi^2 = \pi$. By assumption, there are an object $H'$ (an image of $\pi$) and morphisms $\sigma : H' \to H$ and $\rho : H \to H'$ such that $\rho \sigma = \Id_{H'}$ and $\sigma \rho = \pi$. We define $f' : H' \to H'_1$ by the rule $\rho_1 f \sigma$.

Let $s : H_1 \to H$ and $t : H_2 \to H_1$ be morphisms from the definition of contractible equalizer. We write $s' = \rho s \sigma_1$ and $t' = \rho_1 t \sigma_2$. Simple manipulations show that $f'$, $s'$, and $t'$ satisfy the definition of contractible equalizer for the pair $(f'_1, f'_2)$.

Let us present an example of conditions sufficient for the comodules over a comonad on a triangulated category to form a triangulated category.

**Proposition 3.13.** Let $(P^*, P_*)$ be a pair of adjoint exact functors between triangulated categories $\mathcal{B}$ and $\mathcal{C}$. Let $\mathcal{T}$ be the comonad on $\mathcal{C}$ associated with this pair. Suppose that $\mathcal{B}$ is a Karoubi category and that the natural functor morphism $\eta : \Id_{\mathcal{B}} \to P_* P^*$ splits. Then the category $\mathcal{C}_\mathcal{T}$ is triangulated in the sense of Definition 3.5.

**Proof.** By Corollary 3.11, the comparison functor $\Phi : \mathcal{B} \to \mathcal{C}_\mathcal{T}$ is an equivalence. By the comparison theorem, we have $P^* \cong Q^* \Phi$, where $Q^* : \mathcal{C}_\mathcal{T} \to \mathcal{C}$ stands for the forgetful functor. We must prove that the triangle $F' \to F \to F'' \to F'[1]$ is distinguished in $\mathcal{B}$ if and only if $P^*$ takes this triangle to a distinguished triangle in $\mathcal{C}$. The ‘only if’ part holds because the functor $P^*$ is exact. To prove the ‘if’ part, suppose that $P^*(F' \to F \to F'' \to F'[1])$ is a distinguished triangle. Then $P_* P^*(F' \to F \to F'' \to F'[1])$ is also a distinguished triangle, and so is $F' \to F \to F'' \to F'[1]$ as well, as a direct summand of a distinguished triangle (see Proposition 1.2.3 in [5]).

§ 4. Two ways to introduce descent data

Let $\mathcal{C}_\bullet$ be an augmented cosimplicial category satisfying the condition 1 (see § 2). Two categories of descent data are defined for this category. The first of them is the category $\text{Kern}(\text{Sk}_0(\mathcal{C}_\bullet))$ introduced in § 2. Its definition uses neither the augmentation nor adjoint functors to $P^*_\bullet$. On the contrary, the other category is defined only by using the categories $\mathcal{C}_{-1}$ and $\mathcal{C}_0$ and functors between them. This is the category of comodules over the comonad $\mathcal{T}$ on $\mathcal{C}_0$ associated with the adjoint pair $(P^*, P_*)$. Let us recall the definition (see Definition 3.3).

**Definition 4.1** (comonad descent category). The objects of $\mathcal{C}_{\bullet \mathcal{T}}$ are pairs $(F, h)$, where $F \in \text{Ob} \mathcal{C}_0$ and $h : F \to P^* P_* F$ is a morphism for which the composition
$F \xrightarrow{h} P^*P_*F \xrightarrow{\varepsilon^F} F$ is the identity morphism and the diagram

\[
\begin{array}{c}
F \xrightarrow{h} P^*P_*F \\
\downarrow h \quad \downarrow P^*P_*h \\
P^*P_*F \xrightarrow{P^*\eta P_*F} P^*P_*P_*P_*F
\end{array}
\]

is commutative. The morphisms from $(F_1, h_1)$ to $(F_2, h_2)$ in the category $\mathcal{C}_{\bullet T}$ are the morphisms $f: F_1 \to F_2$ in $\mathcal{C}_0$ such that $h_2 \circ f = P^*P_*f \circ h_1$.

**Proposition 4.2.** Let Conditions 1 and 2 hold. Then the categories $\mathcal{C}_{\bullet T}$ and $\text{Kern}(\text{Sk}_{\geq 0}(\mathcal{C}_\bullet))$ are equivalent.

**Proof.** The objects of the category $\text{Kern}(\text{Sk}_{\geq 0}(\mathcal{C}_\bullet))$ are pairs $(F, \theta)$, where $F \in \text{Ob} \mathcal{C}_0$ and $\theta: P^*_1F \to P^*_2F$ is a morphism (satisfying some conditions). The objects of $\mathcal{C}_{\bullet T}$ are also pairs $(F, h)$, where $F \in \text{Ob} \mathcal{C}_0$ and $h: F \to P^*P_*F$ is a morphism (satisfying some other conditions). By the adjunction condition, for $F \in \mathcal{C}_0$ we have

$$\text{Hom}(P^*_1F, P^*_2F) = \text{Hom}(F, P^*_1P^*_2F) = \text{Hom}(F, P^*P_*F).$$

Obviously, a map $F_1 \to F_2$ is compatible with $\theta: P^*_1F_i \to P^*_2F_i$ if and only if it is compatible with $h: F_i \to P^*P_*F_i$. The only remaining task is to prove that the conditions on $h$ in the definition of comodule over a comonad, that is, the conditions

(C1) the composition $F \xrightarrow{h} P^*P_*F \xrightarrow{\varepsilon^F} F$ is the identity morphism,

(C2) the diagram

\[
\begin{array}{c}
F \xrightarrow{h} P^*P_*F \\
\downarrow h \quad \downarrow P^*P_*h \\
P^*P_*F \xrightarrow{P^*\eta P_*F} P^*P_*P_*P_*F
\end{array}
\]

is commutative,

are equivalent to the following conditions:

(C1′) $\theta$ is an isomorphism,

(C2′) (the cocycle condition on $\theta$) the morphisms $P^*_1\theta$ and $P^*_2\theta \circ P^*_1\theta$ from $P^*_1P^*_1F$ to $P^*_2P^*_2F$ coincide.

Let us show first that (C2) is equivalent to (C2′). We note that

$$\text{Hom}(P^*_1P^*_1F, P^*_2P^*_2F) = \text{Hom}(F, P^*_1P^*_1P^*_2P^*_2F) = \text{Hom}(F, P^*_1P^*_1P^*_2P^*_2F)$$

$$= \text{Hom}(F, P^*_1P^*_2P^*_1P^*_2F) = \text{Hom}(F, P^*P_*P^*P_*F).$$

Under this identification, the morphism

$$P^*_1\theta \in \text{Hom}(P^*_1P^*_1F, P^*_2P^*_2F)$$

corresponds to the morphism

$$P^*\eta P_*F \circ h \in \text{Hom}(F, P^*P_*P^*P_*F),$$
and the morphism
\[ P_{23}^* \theta \circ P_{12}^* \theta \in \text{Hom}(P_{12}^* P_1^* F, P_{23}^* P_2^* F) \]
corresponds to the morphism
\[ P^* P_* h \circ h \in \text{Hom}(F, P^* P_* P^* P_* F). \]

Let us now prove that (C1) is a consequence of (C1′) + (C2′). We note first that the composition \( f : F \xrightarrow{h} P^* P_* F \rightarrow F \) is equal to the inverse image (the pull-back) \( D^* \theta : F = D^* P_1^* F \rightarrow D^* P_2^* F = F \). Since \( \theta \) is an isomorphism, it follows that \( f \) is also an isomorphism. Further, it follows from the cocycle condition for \( \theta \) that \( f^2 = f \). Hence, \( f = \text{Id} \). It remains to show that (C1′) follows from (C1) + (C2).

We recall that the morphism \( \theta \) corresponding to \( h \) by adjunction is the following composition:
\[ \theta : P_1^* F \xrightarrow{P^* h} P_1^* P_* P_* F \sim P_2^* P_* P_* F \xrightarrow{P^* \varepsilon} P_2^* F. \]

It can readily be seen that the map
\[ \theta' : P_2^* F \xrightarrow{P^* h} P_2^* P_* P_* F \sim P_1^* P_* P_* F \xrightarrow{P^* \varepsilon} P_1^* F \]
is inverse to \( \theta \). Hence, \( \theta \) is an isomorphism.

Proposition 4.2 shows that the category \( C_{•T} \) does not depend on the augmentation; it depends only on the cosimplicial part \([C_0, C_1, \ldots, P_*]\). In fact, the comonad \( T \) by itself does not depend on the augmentation either.

**Corollary 4.3.** To any cosimplicial category \( C_{•} = [C_0, C_1, \ldots, P_*] \) for which Conditions 1 and 2 hold, one can assign a comonad \( T \) on the category \( C_0 \). This comonad coincides with that in Definition 4.1 for any extension of \( C_{•} \) to an augmented cosimplicial category satisfying Conditions 1 and 2.

**Proof.** Let \( \widetilde{C}_{•} \) be an augmented cosimplicial category for which Conditions 1 and 2 hold and which coincides with \( C_{•} \) when deleting the augmentation. (For example, for \( \widetilde{C}_{•} \) one can take the category constructed in Proposition 2.9.) Let us introduce a comonad by applying Definition 4.1 to \( \widetilde{C}_{•} \). We see that the functor \( T = P^* P_* = P_2^* P_1^* \) does not depend on the augmentation. It can readily be seen that the natural transformations of the functors \( T = P^* P_* \rightarrow \text{Id} \) and \( T = P^* P_* \rightarrow P^* P_* P^* P_* = TT \) are of the form
\[ P_2^* P_1^* \xrightarrow{\eta} P_2^* D_* D^* P_1^* \sim \text{Id} \circ \text{Id} = \text{Id}, \]
\[ P_2^* P_1^* \xrightarrow{\eta} P_2^* P_{13}^* P_{13}^* P_1^* \sim P_2^* P_{23}^* P_{12}^* P_1^* \sim P_2^* P_1^* P_2^* P_1^*, \]
respectively, and do not depend on the augmentation either.
§ 5. Restriction to subcategories

In this section we present facts related to subcategories in descent categories. Suppose that there is a cosimplicial subcategory

\[ \mathcal{C}' = [\mathcal{C}'_0, \mathcal{C}'_1, \ldots, P'_*] \quad \text{in} \quad [\mathcal{C}_0, \mathcal{C}_1, \ldots, P_*]. \]

This means that a subcategory \( \mathcal{C}'_i \) of any category \( \mathcal{C}_i \) is given and these subcategories are compatible with the functors \( P'_* \); that is, \( P'_f \mathcal{C}'_m \subset \mathcal{C}'_n \) (and need not be compatible with the adjoint functors \( P'_* \) if they exist). In this case, we may consider the classical descent category \( \text{Kern}(\mathcal{C}'_0) \). This is a subcategory of \( \text{Kern}(\mathcal{C}_0) \).

Remark 5.1. We note that, in fact, the category \( \text{Kern}(\mathcal{C}'_0) \) does not depend on \( \mathcal{C}'_1, \mathcal{C}'_2, \ldots \). This category is defined uniquely provided that the subcategory \( \mathcal{C}'_0 \subset \mathcal{C}_0 \) is given, namely, one can take \( \mathcal{C}'_k = \mathcal{C}_k \) for \( k > 0 \).

Suppose that \( T \) is a comonad on a category \( \mathcal{C} \) and \( \mathcal{C}' \subset \mathcal{C} \) is a subcategory. Let us introduce a category \( \mathcal{C}'_T \) as follows.

Definition 5.2. The objects of \( \mathcal{C}'_T \) are pairs \((F, h)\) in \( \text{Ob} \mathcal{C}_T \) such that \( F \in \text{Ob} \mathcal{C}' \). The morphisms in \( \mathcal{C}'_T \) are the morphisms in \( \mathcal{C}_T \) which belong to \( \mathcal{C}' \), that is,

\[ \text{Hom}_{\mathcal{C}_T}((F_1, h_1), (F_2, h_2)) = \text{Hom}_{\mathcal{C}_T}((F_1, h_1), (F_2, h_2)) \cap \text{Hom}_{\mathcal{C}}(F_1, F_2). \]

Obviously, \( \mathcal{C}'_T \) is a subcategory of \( \mathcal{C}_T \), and the subcategory \( \mathcal{C}'_T \) is full if \( \mathcal{C}' \) is a full subcategory of \( \mathcal{C} \).

This definition is caused by the following reason: for many important examples, the functor \( T : \mathcal{C} \to \mathcal{C} \) is sufficiently ‘large’ and does not preserve ‘small’ subcategories of \( \mathcal{C} \). Therefore, Definition 3.3 gives no possibility to consider a category of ‘small’ objects equipped with some descent data. A typical example here is given by the situation in which \( p : X \to S \) is a nonproper morphism of schemes, \( \mathcal{C} = \text{qcoh}(X), \mathcal{C}' = \text{coh}(X) \) and \( T = p^*p_* \).

The following proposition holds, which is a natural analogue of Proposition 3.4.

Proposition 5.3. Let \( T = (T, \varepsilon, \delta) \) be a comonad on a category \( \mathcal{C} \).

If \( \mathcal{C}' \subset \mathcal{C} \), \( \mathcal{C}' \) and \( \mathcal{C} \) are additive categories, and \( T \) is an additive functor, then the category \( \mathcal{C}'_T \) is also additive.

If \( \mathcal{C}' \subset \mathcal{C} \), \( \mathcal{C}' \) and \( \mathcal{C} \) are Abelian categories, and \( T \) is a left exact additive functor, then the category \( \mathcal{C}'_T \) is Abelian as well.

If \( \mathcal{C}' \subset \mathcal{C} \), \( \mathcal{C}' \) and \( \mathcal{C} \) are triangulated categories, the functor \( T \) is exact, and \( \mathcal{C}'_T \) is a triangulated category in the sense of Definition 3.5, then \( \mathcal{C}'_T \subset \mathcal{C}_T \) is a triangulated subcategory.

Let \( \mathcal{C}_* \) be an augmented cosimplicial category for which Conditions 1 and 2 in § 2 hold and let \( \mathcal{C}'_0 \subset \mathcal{C}_* \) be a cosimplicial subcategory (possibly without augmentation). In this situation, two descent categories associated with the subcategory \( \mathcal{C}'_0 \) are defined, namely, the category \( \text{Kern}(\mathcal{C}'_0) \) and the category \( \mathcal{C}'_0 \subset \mathcal{C}_T = \mathcal{C}'_0 \subset \mathcal{C}_T \) defined above. The following simple corollary to Proposition 4.2 holds.

Corollary 5.4. Under the above assumptions, the categories \( \text{Kern}(\mathcal{C}'_0) \) and \( \mathcal{C}'_0 \subset \mathcal{C}_T \) are equivalent.
Let \((P^*, P_*)\) be a pair of adjoint functors on the categories \(B\) and \(C\) and let \(T\) be the comonad on \(C\) defined by these functors. If \(P^*\) takes some subcategory \(B' \subset B\) to \(C' \subset C\), then one can consider the restriction of the comparison functor,
\[
\Phi|_{B'} : B' \to C'_T.
\]
In particular, for \(B'\) one can take the preimage of \((P^*)^{-1}(C')\); this is a subcategory of \(B\) whose objects/morphisms are precisely the objects/morphisms of \(B\) that are taken by the functor \(P^*\) to objects/morphisms belonging to \(C'\).

**Lemma 5.5.** If the comparison functor \(\Phi : B \to C_T\) is an equivalence and if \(C' \subset C\) is a subcategory, then the restriction of \(\Phi\) to \(B' = (P^*)^{-1}(C')\) is an equivalence \(B' \to C'_T\).

The proof of the lemma is clear.

§ 6. Coherent sheaves on schemes and stacks and their derived categories

Usually, one works with coherent and quasi-coherent sheaves on schemes under the assumptions that the scheme is quasi-compact and quasi-separated. We recall (see 1.1 and 1.2 in [6]) that a scheme is said to be **quasi-compact** if it can be covered by finitely many open affine subschemes and **quasi-separated** if the intersection of any two affine subsets of the scheme can be covered by finitely many affine subsets. For example, every Noetherian scheme (in particular, every quasi-projective scheme) is quasi-compact and quasi-separated. All schemes used below are assumed to be quasi-compact and quasi-separated.

In this section we present a survey concerning coherent and quasi-coherent sheaves on stacks and derived categories of sheaves on stacks. The reader who is interested in a more complete exposition of the material is referred to the papers of Laumon and Moret-Bailly [7], Laszlo and Olsson [8] and Arinkin and Bezrukavnikov [9]. All stacks under consideration are assumed to be algebraic stacks of finite type over a field. In particular, every stack \(X\) is Noetherian, quasi-compact and quasi-separated and can be covered by a scheme of finite type over a field. When speaking of sheaves on \(X\), we mean sheaves of \(\mathcal{O}_X\)-modules in the smooth topology (see 6.1 in [7]).

Thus, we use the following assumption.

**Assumption (†).** Every stack under consideration is assumed to be either an algebraic stack of finite type over an arbitrary field \(k\) or a quasi-compact quasi-separated scheme.

Let \(X\) be a stack. The category \(\mathcal{O}_X-\text{Mod}\) of sheaves of \(\mathcal{O}_X\)-modules on \(X\) has enough injective objects. We denote by \(D(\mathcal{O}_X-\text{Mod})\) the unbounded derived category of the Abelian category \(\mathcal{O}_X-\text{Mod}\); it has arbitrary direct sums. Let us use the notation \(\text{qcoh}(X)\) and \(\text{coh}(X)\) for the categories of quasi-coherent and coherent sheaves on \(X\), respectively. Let \(D_{\text{qcoh}}(X)\) be the full subcategory of \(D(\mathcal{O}_X-\text{Mod})\) formed by complexes with quasi-coherent cohomology sheaves; this category is also closed with respect to the direct sums. Let \(D_{\text{coh}}^b(X) \subset D_{\text{qcoh}}(X)\) be the full subcategory consisting of the complexes whose cohomology sheaves are coherent and almost
all of them vanish. Other versions of derived categories are defined analogously. By Assertion 2.5 in [9], one can consider the smooth-étale topology on $X$ instead of the smooth one, and also use the Zariski topology in the case of schemes, and this does not modify the (quasi)coherent categories $\mathcal{O}_{\text{qcoh}}(X)$, $\mathcal{O}_{\text{coh}}(X)$, and so on.

Suppose that stacks $X$ and $Y$ satisfy Assumption (†) and $f: X \to Y$ is a morphism. Then a morphism of the ringed smooth sites $(X_{\text{lis}}, \mathcal{O}_X) \to (Y_{\text{lis}}, \mathcal{O}_Y)$ is well defined, together with the functors

$$f^*: \mathcal{O}_Y - \text{Mod} \to \mathcal{O}_X - \text{Mod}, \quad f_*: \mathcal{O}_X - \text{Mod} \to \mathcal{O}_Y - \text{Mod},$$

where $f^*$ is left adjoint to $f_*$. The functor $f^*$ takes the quasi-coherent sheaves to quasi-coherent ones (see 6.8 in [7]) and coherent sheaves to coherent ones. Further, if $f$ is quasi-compact and quasi-separated, then $f_*$ takes a quasi-coherent sheaf to a quasi-coherent sheaf again; however, this fails in general for coherent sheaves.

Following Spaltenstein [10], one can define derived functors of direct and inverse image on unbounded derived categories. By Proposition 2.1.4 in [8], every complex $F$ in $\mathcal{D}_{\text{qcoh}}(X)$ has a $K$-injective resolution. Applying the functor $f_*$ to this resolution termwise, we obtain an object of the category $\mathcal{D}(\mathcal{O}_Y - \text{Mod})$. This defines a functor $Rf_*: \mathcal{D}_{\text{qcoh}}(X) \to \mathcal{D}(\mathcal{O}_Y - \text{Mod})$. If the morphism $f$ is quasi-compact and quasi-separated and if $F \in \mathcal{D}_{\text{qcoh}}^+(X)$, then it follows from 6.8 in [7] that $Rf_* F \in \mathcal{D}_{\text{qcoh}}^+(X)$. If, in addition, the morphism $f$ is representable, then the functor $Rf_*$ has finite cohomological dimension. Thus, it follows from Lemma 2.1.10 in [8] that the functor $Rf_*$ takes $\mathcal{D}_{\text{qcoh}}^+(X)$ to $\mathcal{D}_{\text{qcoh}}(Y)$.

The derived inverse image functor

$$L f^*: \mathcal{D}(\mathcal{O}_Y - \text{Mod}) \to \mathcal{D}(\mathcal{O}_X - \text{Mod})$$

on unbounded derived categories is defined by using $K$-flat resolutions. It takes $\mathcal{D}_{\text{qcoh}}(Y)$ to $\mathcal{D}_{\text{qcoh}}(X)$ and $\mathcal{D}_{\text{coh}}^-(Y)$ to $\mathcal{D}_{\text{coh}}^-(X)$ (see 6.8 and 8.7 in [7]). The derived direct and inverse image functors on unbounded derived categories on stacks satisfy the expected properties, namely, these functors are adjoint, $R(fg)_* \cong Rf_* Rg_*$ (and a similar formula holds for the inverse image), and the projection formula and the flat base-change formula also hold. For representable morphisms these functors commute with arbitrary direct sums.

If a stack $X$ is quasi-compact and semi-separated (that is, the diagonal morphism $X \to X \times X$ is affine), then the category $\mathcal{D}_{\text{qcoh}}^+(X)$ is equivalent to $\mathcal{D}^+(\text{qcoh}(X))$, that is, to the left bounded derived category of an Abelian category $\text{qcoh}(X)$; similarly, $\mathcal{D}_{\text{coh}}^b(X)$ is equivalent to $\mathcal{D}^b(\text{coh}(X))$ (see Claim 2.7 and Corollary 2.11 in [9]). If $X$ is a scheme here, then there is also an equivalence of unbounded derived categories, namely, $\mathcal{D}_{\text{qcoh}}(X) \cong \mathcal{D}(\text{qcoh}(X))$ (see Corollary 5.5 in [11]). In what follows, we mainly deal with the ‘big’ category $\mathcal{D}_{\text{qcoh}}(X)$ and, in the Noetherian case, with the ‘small’ category $\mathcal{D}_{\text{coh}}^b(X)$. We write

$$\mathcal{D}(X) = \mathcal{D}_{\text{qcoh}}(X).$$

By a perfect complex on a stack or a scheme $X$ we mean a complex locally quasi-isomorphic to a bounded complex of vector bundles, that is, an
$F \in \mathcal{D}(\mathcal{O}_X-\text{Mod})$ such that the inverse image $f^*F$ is quasi-isomorphic to a bounded complex of locally free finite-rank sheaves for some cover $f: U \to X$ by a scheme. The full subcategory of $\mathcal{D}(\mathcal{O}_X-\text{Mod})$ formed by the perfect complexes is denoted by $\mathcal{D}^{\text{perf}}(X)$. Obviously, $\mathcal{D}^{\text{perf}}(X) \subset \mathcal{D}_{\text{qcoh}}(X)$, and $\mathcal{D}^{\text{perf}}(X) \subset \mathcal{D}_{\text{coh}}^b(X)$ for any Noetherian stack. We note that the inverse image functor $Lf^*: \mathcal{D}(\mathcal{O}_Y-\text{Mod}) \to \mathcal{D}(\mathcal{O}_X-\text{Mod})$ takes perfect complexes to perfect complexes.

If $X$ is a scheme, then, by Theorem 3.1.1 in [12], the category of perfect complexes on $X$ is precisely the category of compact objects in $\mathcal{D}_{\text{qcoh}}(X)$, which generates the category $\mathcal{D}_{\text{qcoh}}(X)$. The category of perfect complexes coincides with the bounded derived category of coherent sheaves for any smooth scheme over a field.

We recall that a morphism of stacks is said to be \textit{faithfully flat} if it is flat and surjective. One can give an equivalent definition, namely, a flat morphism $f: X \to S$ is faithfully flat if and only if $H = 0$ for any sheaf $H$ on $S$ such that $f^*H = 0$. It is also true that $f$ is a faithfully flat morphism if and only if $f$ is flat and the inverse image functor $f^*$ is conservative.

§ 7. Derived descent theory for stacks

In this section we apply the results of comonad theory to study the cohomological descent for derived categories of sheaves on schemes and stacks. We work in the category of stacks to simultaneously cover the cases of descent for schemes and for equivariant derived categories.

Let us recall our assumption (†): all stacks are assumed to be algebraic stacks of finite type over a field or quasi-compact quasi-separated schemes. Every time we speak of coherent sheaves, it is assumed in addition that the stack is Noetherian.

Let $X$ and $S$ be stacks and let $p: X \to S$ be a flat representable morphism. Consider the augmented simplicial stack

$$(X \to S)_{\bullet} = [S, X, X \times_S X, X \times_S X \times_S X, \ldots, p_{\bullet}].$$

The fibred products $X \times_S X, X \times_S X \times_S X, \ldots$ satisfy the assumption (†). The Abelian categories of quasi-coherent sheaves on $S, X, X \times_S X, \ldots$ and the inverse image functors between them form an augmented cosimplicial category

$$[\text{qcoh}(S), \text{qcoh}(X), \text{qcoh}(X \times_S X), \text{qcoh}(X \times_S X \times_S X), \ldots, p^*_{\bullet}].$$

(7.1)

This category satisfies Conditions 1 and 2 in § 2, namely, the functors $p^*_{\bullet}$ admit right adjoint functors $p^*_{\bullet*}$ and the flat base-change formula holds. We also consider the augmented cosimplicial subcategory of (7.1) formed by the categories of coherent sheaves,

$$[\text{coh}(S), \text{coh}(X), \text{coh}(X \times_S X), \text{coh}(X \times_S X \times_S X), \ldots, p^*_{\bullet}].$$

(7.2)

We note that this category does not satisfy Condition 1 of § 2 because the direct-image functors need not preserve coherent sheaves. We denote by

$$\text{qcoh}(X)/p = \text{Kern}([\text{qcoh}(X), \text{qcoh}(X \times_S X), \ldots]),$$

$$\text{coh}(X)/p = \text{Kern}([\text{coh}(X), \text{coh}(X \times_S X), \ldots])$$

the descent categories associated with (7.1) and (7.2), respectively (see Definition 2.3).
Theorem 7.1. Let $X$ and $S$ be stacks satisfying the assumption (†) and let $p: X \to S$ be a flat representable morphism. In this case, the category $\text{qcoh}(X)/p$ is equivalent to the category $\text{qcoh}(S)$ of quasi-coherent sheaves on $S$ if and only if the morphism $p$ is faithfully flat. If the stacks $S$ and $X$ are Noetherian, then these conditions hold if and only if the categories $\text{coh}(X)/p$ and $\text{coh}(S)$ are equivalent.

Remark 7.2. This result is well known in the case of schemes (see, for example, Proposition 2.22 in [13]). We present a proof only to show how this result follows from Corollary 3.10.

Proof. We denote by $\text{qcoh}(X)_{T_p}$ the descent category associated with the comonad $T_p = (p^*p_*, \varepsilon, \delta)$ on $\text{qcoh}(X)$ (see Definition 4.1). The category (7.1) satisfies Conditions 1 and 2 of §2. Hence, by Proposition 4.2, the categories $\text{qcoh}(X)/p$ and $\text{qcoh}(X)_{T_p}$ are equivalent. Let us apply Corollary 3.10,1). The functor $p^*$ is exact. Hence, the comparison functor $\Phi: \text{qcoh}(S) \to \text{qcoh}(X)_{T_p}$ is an equivalence if and only if $H = 0$ for any $H \in \text{qcoh}(S)$ such that $p^*H = 0$. The last condition is equivalent to the strict flatness of $p$.

We claim now, assuming that $S$ and $X$ are Noetherian, that the comparison functor $\Phi$ is an equivalence for the categories $\text{coh}$ if and only if this functor is an equivalence for the categories $\text{qcoh}$. Let us denote by $\text{coh}(X)_{T_p}$ the subcategory of $\text{qcoh}(X)_{T_p}$ corresponding to the subcategory $\text{coh}(X) \subset \text{qcoh}(X)$ (see Definition 5.2). By Corollary 5.4, the categories $\text{coh}(X)/p$ and $\text{coh}(X)_{T_p}$ are equivalent.

Suppose that a functor $\Phi: \text{qcoh}(S) \to \text{qcoh}(X)_{T_p}$ is an equivalence. Let us prove that the restriction of $\Phi$ is an equivalence between fully faithful subcategories $\text{coh}(S) \subset \text{qcoh}(S)$ and $\text{coh}(X)_{T_p} \subset \text{qcoh}(X)_{T_p}$. Lemma 5.5 claims that the restriction of $\Phi$ gives an equivalence between the categories $(p^*)^{-1}(\text{coh}(X))$ and $\text{coh}(X)_{T_p}$. Hence, one must prove that for $H \in \text{qcoh}(S)$ the sheaf $p^*H$ is coherent if and only if $H$ is coherent. This is Lemma 7.5.

Let us prove the converse statement. Suppose that the comparison functor

$$\Phi_{|\text{coh}(S)}: \text{coh}(S) \to \text{coh}(X)_{T_p}$$

is an equivalence. We claim that the functor $\Phi: \text{qcoh}(S) \to \text{qcoh}(X)_{T_p}$ is also an equivalence and note that the morphism $H \xrightarrow{\eta H} p_*p^*H$ is injective for any sheaf $H \in \text{coh}(S)$. Indeed, let us consider the kernel $K = \ker(H \to p_*p^*H)$. The morphism $p^*\eta H: p^*H \to p_*p_*p^*H$ is a split embedding whose inverse is the canonical adjunction morphism $\varepsilon p^*H: p_*p^*H \to p^*H$. We have $p^*K = 0$. Thus, $\Phi(K) = 0$. However, the sheaf $K$ is coherent on $S$. Hence, $K = 0$. By the projection formula, $p_*p^*H \cong H \otimes p_*\mathcal{O}_X$. We see that the morphism $\mathcal{O}_S \to p_*\mathcal{O}_X$ remains injective after tensor multiplication by an arbitrary coherent sheaf $H \in \text{coh}(S)$. This means that $\text{Tor}_1(p_*\mathcal{O}_X/\mathcal{O}_S, H) = 0$ for any $H \in \text{coh}(S)$ and implies that $p_*\mathcal{O}_X/\mathcal{O}_S$ is a flat $\mathcal{O}_S$-module, and therefore the morphism $H \to p_*p^*H$ is injective for any $H \in \text{qcoh}(S)$. Applying now Corollary 3.10,1), we see that $\Phi$ is an equivalence $\text{qcoh}(S) \to \text{qcoh}(X)_{T_p}$.

An analogue of Theorem 7.1 for derived categories is more interesting. Consider the augmented cosimplicial category

$$[\mathcal{D}(S), \mathcal{D}(X), \mathcal{D}(X \times S X), \mathcal{D}(X \times S X \times S X), \ldots, Lp^*_k]$$

(7.3)
formed by the unbounded derived categories of quasi-coherent sheaves on the stacks $S$, $X$, $X \times_S X$, \ldots. We denote by

$$D(X)/p = \text{Kern}([D(X), D(X \times_S X), D(X \times_S X \times_S X), \ldots, Lp_\bullet])$$

the classical descent category (see Definition 2.3). The cosimplicial category (7.3) satisfies Conditions 1 and 2 in §2, namely, the functors $Lp_\bullet$ have right adjoint functors $Rp_\bullet$ and the base-change formula holds. Let us also consider the classical descent categories $D^b_{coh}(X)/p$ and $D^{perf}(X)/p$ (see Definition 2.3 and Remark 5.1) associated with (7.3) and with the subcategories $D^b_{coh}(X)$ and $D^{perf}(X)$ in $D(X)$.

**Theorem 7.3.** Let stacks $X$ and $S$ satisfy the assumption (†) and let $p: X \to S$ be a flat representable morphism. Then the comparison functor $D(S) \to D(X)/p$ is an equivalence if and only if the natural morphism $\mathcal{O}_S \to Rp_\bullet \mathcal{O}_X$ in the category $D(S)$ is an embedding of a direct summand. If this condition holds, then the comparison functor $D^{perf}(S) \to D^{perf}(X)/p$ and the comparison functor $D^b_{coh}(S) \to D^b_{coh}(X)/p$ (if $S$ and $X$ are Noetherian) are equivalences.

**Remark 7.4.** We note that the condition that the morphism $\mathcal{O}_S \to Rp_\bullet \mathcal{O}_X$ splits is not necessary for $D^b_{coh}(S) \to D^b_{coh}(X)/p$ to be the equivalence of the categories (see Example 8.9 below).

**Proof.** We denote by $D(X)_{T_p}$ the category of comodules over the comonad $T_p = (p^*Rp_\bullet, \varepsilon, \delta)$ on the category $D(X)$ (see Definition 4.1). By Proposition 4.2, the descent categories $D(X)/p$ and $D(X)_{T_p}$ are equivalent. Let us use results of comonad theory.

If the comparison functor $\Phi: D(S) \to D(X)_{T_p}$ is an equivalence, then it is fully faithful and, by Corollary 3.10, 2), the map $\mathcal{O}_S \to Rp_\bullet p^* \mathcal{O}_S = Rp_\bullet \mathcal{O}_X$ is a split embedding. The converse statement follows from Corollary 3.11. Indeed, $D(S)$ is a Karoubi category. By the projection formula, for $H \in D(S)$ the natural morphism $H \to Rp_\bullet p^* H = H \otimes^L Rp_\bullet \mathcal{O}_X$ is of the form

$$H \otimes^L (\mathcal{O}_S \to Rp_\bullet \mathcal{O}_X).$$

Therefore, the morphism of functors $\text{Id} \to Rp_\bullet p^*$ is split provided that the morphism $\mathcal{O}_S \to Rp_\bullet \mathcal{O}_X$ is.

Suppose now that

$$\Phi: D(S) \to D(X)_{T_p}$$

is an equivalence. We denote by $D^b_{coh}(X)_{T_p}$ and $D^{perf}(X)_{T_p}$ the comonad descent categories associated with the subcategories $D^b_{coh}(X) \subset D(X)$ and $D^{perf}(X) \subset D(X)$. As above, it follows from Corollary 5.4 that

$$D^b_{coh}(X)/p \cong D^b_{coh}(X)_{T_p}, \quad D^{perf}(X)/p \cong D^{perf}(X)_{T_p}.$$ 

We must prove that the restriction of $\Phi$ gives an equivalence between the fully faithful subcategories $D^b_{coh}(S) \subset D(S)$ and $D^b_{coh}(X)_{T_p} \subset D(X)_{T_p}$. By Lemma 5.5, the restriction of $\Phi$ gives an equivalence between $(p^*)^{-1}(D^b_{coh}(X))$ and $D^b_{coh}(X)_{T_p}$. Hence, we must prove that for $H \in D(S)$ the complex $p^*H$ belongs to $D^b_{coh}(X)$ if and only if $H$ belongs to $D^b_{coh}(S)$. This follows from Lemma 7.5 and from the fact that the functor $p^*$ is exact.
Similar considerations for the categories of perfect complexes need the relation
\((p^*)^{-1}(\mathcal{D}^{\text{perf}}(X)) = \mathcal{D}^{\text{perf}}(S)\). Obviously, the inverse image of a perfect complex is a perfect complex. Let us prove that the converse also holds. It is sufficient here to consider the case of schemes.

Let us choose a cover \(f: U \to S\) of the stack \(S\) by a scheme. The morphism \(p\) is representable. Hence, the map \(f': U' = U \times_S X \to X\) is also a cover by a scheme. By definition, an object \(H \in \mathcal{D}(S)\) is a perfect complex on \(S\) if and only if \(f^*H\) is a perfect complex on \(U\), and a similar claim holds for \(f': U' \to X\). The perfect complexes on the scheme \(U\) are precisely the compact objects in the category \(\mathcal{D}(U)\) (see 3.1.1 in \([12]\)), and the same is true for \(U'\).

Suppose that \(H \in \mathcal{D}(U)\) and that \(p^*H\) is compact in \(\mathcal{D}(U')\). Let \((F_\alpha)\) be an arbitrary family of objects in \(\mathcal{D}(U)\). Consider the commutative diagram

\[
\begin{array}{ccc}
\bigoplus \text{Hom}(H, F_\alpha) & \longrightarrow & \bigoplus \text{Hom}(H, Rp^*_p p'^* F_\alpha) \\
\downarrow & & \downarrow \\
\text{Hom}(H, \bigoplus F_\alpha) & \longrightarrow & \text{Hom}(H, Rp^*_p p'^* \left( \bigoplus F_\alpha \right))
\end{array}
\] (7.4)

We note that the sheaf \(\mathcal{O}_U\) is a direct summand in \(Rp^*_p \mathcal{O}_{U'}\) (see Proposition 8.3). Thus, the functor \(\text{Id}_{\mathcal{D}(U)}\) is a direct summand in \(Rp^*_p p'^*\). Hence, the left column of (7.4) is a direct summand of the right column. Let us prove that the morphism in the right column is an isomorphism. Indeed, we have

\[
\bigoplus \text{Hom}(H, Rp^*_p p'^* F_\alpha) = \bigoplus \text{Hom}(p'^*H, p'^* F_\alpha) = \text{Hom}(p'^*H, \bigoplus p'^* F_\alpha) = \text{Hom}(p'^*H, p'^* \left( \bigoplus F_\alpha \right)) = \text{Hom}(H, Rp^*_p p'^* \left( \bigoplus F_\alpha \right)).
\]

This implies that the left column of (7.4) is also an isomorphism, and \(H\) is a compact object in \(\mathcal{D}(U)\).

**Lemma 7.5.** Let \(X\), \(S\), and \(p\) be as in Theorem 7.3. Suppose that the functor \(\Phi: \text{qcoh}(S) \to \text{qcoh}(X)_T\) is an equivalence. If \(H\) is a quasi-coherent sheaf on \(S\) and if the sheaf \(p^*H\) is coherent, then \(H\) is also coherent.

**Proof.** By Proposition 15.4 in \([7]\), the sheaf \(H\) is a union of its coherent subsheaves. Suppose that \(H\) is not coherent. Then one can choose a strictly ascending sequence \(H_1 \to H_2 \to H_3 \to \cdots\) of coherent subsheaves of \(H\). Since \(\Phi\) is an equivalence, it follows that the sequence \(p^*H_i\) of subsheaves in \(p^*H\) is also strictly ascending. However, the stack \(X\) is Noetherian and the sheaf \(p^*H\) is coherent. A contradiction.

§ 8. Morphisms with the SCDT property

Let \(p: X \to S\) be a flat morphism. By Theorem 7.3, the condition that the morphism \(\mathcal{O}_S \to Rp^*_p \mathcal{O}_X\) splits is a criterion for a derived category of \(S\) to be recoverable from the derived category of \(X\). In this connection, the following SCDT property is of interest.
Definition 8.1. We say that a morphism of schemes or stacks $p: X \to S$ has the SCDT property if $p$ is flat and the natural map $\mathcal{O}_S \to Rp_*\mathcal{O}_X$ is an embedding of a direct summand in the derived category of sheaves of $\mathcal{O}_S$-modules (SCDT stands for ‘strictly cohomological descent type’).

In this section we present some facts concerning the morphisms with the SCDT property (or simply ‘SCDT morphisms’) and give sufficient conditions for the validity of this property. We recall that a functor $\Psi$ is said to be faithful if for any pair of objects $A$, $B$ the map

$$\text{Hom}(A, B) \to \text{Hom}(\Psi(A), \Psi(B))$$

induced by the functor is injective.

Lemma 8.2. Let $X$ and $S$ be stacks satisfying the assumption $(†)$ and let $p: X \to S$ be a flat morphism. Then $p$ is an SCDT morphism if and only if the functor $p^*: \mathcal{D}(S) \to \mathcal{D}(X)$ is faithful.

Proof. Suppose that $p$ is an SCDT morphism. By the projection formula, the functor $\text{Id}: \mathcal{D}(S) \to \mathcal{D}(S)$ is a direct summand of the functor $Rp_*p^*$. Hence, $Rp_*p^*$ does not vanish on morphisms, and thus $p^*$ does not vanish on morphisms either.

Suppose that $p^*$ is faithful. Consider a distinguished triangle

$$\mathcal{O}_S \to Rp_*\mathcal{O}_X \to K \xrightarrow{f} \mathcal{O}_S[1].$$

We claim that $f = 0$. Applying $p^*$ to this triangle, we obtain $p^*\mathcal{O}_S \to p^*Rp_*p^*\mathcal{O}_S \to p^*K \xrightarrow{p^*f} p^*\mathcal{O}_S[1]$. The adjunction properties of $p^*$ and $Rp_*$ imply that the latter triangle splits. Thus, $p^*f = 0$. Since $p^*$ is faithful, it follows that $f = 0$.

Proposition 8.3. Let stacks $S$, $S'$, $X$, and $Y$ satisfy the assumption $(†)$.

1) If morphisms $q: Y \to X$ and $p: X \to S$ are SCDT morphisms, then so is $p \circ q$. If $p$ is flat and $p \circ q$ is an SCDT morphism, then so is $p$. If $q$ is an SCDT morphism and $p \circ q$ is flat, then $p$ is also flat.

2) The SCDT property is preserved under any base change, namely, if $p: X \to S$ is an SCDT morphism and $s: S' \to S$ is a base change, then $p': X' = X \times_S S' \to S'$ also is an SCDT morphism.

3) If the base-change morphism has in addition the SCDT property, then $p$ and $p'$ have or do not have this property simultaneously.

4) Let $k \subset K$ be a field extension. Then the morphism $p: X \to S$ of stacks over $k$ has the SCDT property if and only if the morphism $p': X \times_k K \to S \times_k K$ has.

Proof. The first statement in 1) is elementary. Let us prove the second statement in 1). We know that the composition of the natural morphisms

$$\mathcal{O}_S \xrightarrow{\eta_p} Rp_*\mathcal{O}_X \xrightarrow{Rp_*\eta_q} Rp_*Rq_*\mathcal{O}_Y$$

splits. Let $\sigma: Rp_*Rq_*\mathcal{O}_Y \to \mathcal{O}_S$ be the corresponding splitting morphism. Then $\sigma \circ Rq_*\eta_q$ is a splitting morphism for $\eta_p$. We now prove the third statement in 1) by
showing that the functor $p^*: \text{qc}(S) \to \text{qc}(X)$ is exact. Let $H_1 \to H_2 \to H_3$ be an exact sequence of quasi-coherent sheaves on $S$ and let $F$ be the middle term cohomology of the sequence $p^*H_1 \to p^*H_2 \to p^*H_3$. Since the functors $q^*$ and $(p \circ q)^*$ are exact, it follows that $q^*F = 0$. Since $q$ has the SCDT property, we also have $F = 0$.

The statement 2) follows from the flat base-change formula (see 2.4 in [14]), namely, $Rp^!\mathcal{O}_{X'} = s^!Rp_*\mathcal{O}_X$. The statement 3) follows from 1) and 2), and 4) follows from 3).

**Proposition 8.4.** A finite flat morphism $p: X \to S$ of quasi-compact quasi-separated schemes over a field of characteristic zero has the SCDT property.

**Proof.** Since the morphism $p$ is affine, we have $Rp_*\mathcal{O}_X = p_*\mathcal{O}_X$ (there are no non-trivial higher direct images). Let us denote by $C$ the quotient $p_*\mathcal{O}_X / \mathcal{O}_S$. We must prove that the extension $0 \to \mathcal{O}_S \to p_*\mathcal{O}_X \to C \to 0$ is trivial. Tensoring this extension with $p_*\mathcal{O}_X$ gives a trivial extension, because the morphism $p_*\mathcal{O}_X \to p_*\mathcal{O}_X \otimes p_*\mathcal{O}_X$ is a split embedding (the splitting is defined by the multiplication). The morphism $p$ is flat and finite. Thus, the sheaf $E = p_*\mathcal{O}_X$ is a vector bundle on $S$. The tensor product on $E$ induces a map

$$\sigma: \text{Ext}^1(C, \mathcal{O}_S) \to \text{Ext}^1(C \otimes E, \mathcal{O}_S \otimes E) = \text{Ext}^1(C, \mathcal{E}\text{nd}(E)).$$

We claim that $\sigma$ is a monomorphism. Indeed, the left inverse to $\sigma$ is given by taking the trace $\mathcal{E}\text{nd}(E) \to \mathcal{O}_S$,

$$\text{Ext}^1(C, \mathcal{E}\text{nd}(E)) \xrightarrow{\text{Tr}} \text{Ext}^1(C, \mathcal{O}_S).$$

A morphism $p: X \to S$ is said to have a multi-section if there is a subscheme $Y \subset X$ such that the restriction $p|_Y$ is a finite morphism $Y \to S$. If a subscheme $Y$ can be chosen to be flat over $S$, we say that $p$ has a flat multi-section.

**Corollary 8.5.** Suppose that a flat morphism $p: X \to S$ of quasi-compact quasi-separated schemes has a flat multi-section. Then $p$ has the SCDT property.

The proof follows from Proposition 8.4 and Proposition 8.3, 1).

**Proposition 8.6.** Let $p: X \to S$ be a smooth projective morphism of smooth schemes over a field of characteristic zero. Then $p$ has the SCDT property.

**Proof.** By a result of Deligne (see Theorem 6.1 in [15]), the complex $Rp_*\mathcal{O}_X$ is a direct sum of its cohomology. Hence, $p_*\mathcal{O}_X$ (the direct image in the category of sheaves) is a direct summand of $Rp_*\mathcal{O}_X$. Here $p_*\mathcal{O}_X$ is a vector bundle by Theorem 5.5 in [15]. The argument in the proof of Proposition 8.4 shows that $\mathcal{O}_S$ is a direct summand in $p_*\mathcal{O}_X$, and thus in $Rp_*\mathcal{O}_X$ as well.

Let us now present an example of a flat affine (and even a locally trivial) morphism of smooth varieties over a field which fails to have the SCDT property and for which the derived descent category is not equivalent to the derived category of the base.
Example 8.7. Let $V$ be a vector space over a field $k$. Let $X$ be the group $GL(V)$ and let $P \subset X$ be a parabolic subgroup. Consider the homogeneous space $S = X/P$. This is a smooth projective variety. We denote its dimension by $d$ and the factorization $X \to S$ by $p$. Let us choose linear bundles $L_1 = \mathcal{O}_S$ and $L_2 = \omega_S$ in such a way that

$$\text{Hom}_{\mathcal{D}_{\text{coh}}(S)}(L_1, L_2[d]) = \text{Ext}^d(L_1, L_2) \neq 0.$$  

Applying the comparison functor, we obtain

$$\text{Hom}_{\mathcal{D}_{\text{coh}}(X)/p}(\Phi(L_1), \Phi(L_2[d])) \subset \text{Hom}_{\mathcal{D}_{\text{coh}}(X)}(p^*L_1, p^*L_2[d])$$

because $X$ is affine and $\text{Pic} X = 0$. It is clear that the comparison functor $\Phi: \mathcal{D}_{\text{coh}}(S) \to \mathcal{D}_{\text{coh}}(X)/p$ is not fully faithful.

In the following example we demonstrate that objects of the derived category of coherent sheaves cannot be defined locally in the Zariski topology.

Example 8.8. Let $S$ be a scheme, let $S = \bigcup U_i$ be a cover of $S$ by affine schemes, let $X = \bigsqcup U_i$, let $p: X \to S$ be the natural map, and let

$$\Phi: \mathcal{D}_{\text{coh}}(S) \to \mathcal{D}_{\text{coh}}(X)/p$$

be the comparison functor. In this case, for any coherent sheaf $F$ on $S$ and any $k > 0$ we obtain

$$\text{Hom}_{\mathcal{D}(X)/p}(\Phi(\mathcal{O}_S), \Phi(F[k])) \subset \text{Hom}_{\mathcal{D}(X)}(\mathcal{O}_X, p^*F[k]) = H^k(X, \oplus F|_{U_i}) = 0.$$  

On the other hand, if the scheme $S$ is not affine, then for some coherent sheaf $F$ and for some $k > 0$ we have

$$\text{Hom}_{\mathcal{D}(S)}(\mathcal{O}_S, F[k]) = H^k(S, F) \neq 0.$$  

In this case, the functor $\Phi$ is not an equivalence, and the morphism $p$ does not have the SCDT property.

We claim now that the comparison functor can be an equivalence for a bounded derived category for a morphism $p: X \to S$ which fails to have the SCDT property.

Example 8.9. Let $S = \mathbb{A}^1$ be an affine line over an algebraically closed field $k$ and let $P_1, P_2 \in \mathbb{A}^1$ be two different points. Let $X = (\mathbb{A}^1 \setminus P_1) \sqcup (\mathbb{A}^1 \setminus P_2)$ be a disjoint union of two punctured lines and let $p: X \to S$ be the natural map. We claim that the comparison functor

$$\Phi: \mathcal{D}_{\text{coh}}(S) \to \mathcal{D}_{\text{coh}}(X)/p$$

is an equivalence, whereas the morphism $\mathcal{O}_S \to Rp_*\mathcal{O}_X$ is not split and the comparison functor

$$\mathcal{D}(S) \to \mathcal{D}(X)/p$$

is not an equivalence.
Proof. The cohomological dimension of the category \( \text{coh}(\mathbb{A}^1) \) is equal to one, and therefore every object \( \mathscr{D}^b_{\text{coh}}(\mathbb{A}^1) \) is quasi-isomorphic to a direct sum of its cohomology sheaves. Further, every coherent sheaf on \( \mathbb{A}^1 \) is a direct sum of indecomposable sheaves of the form \( \mathcal{O}_{\mathbb{A}^1} \) or \( \mathcal{O}_P \), \( P \in \mathbb{A}^1 \), where \( \mathcal{O}_P \) stands for the structure sheaf of the \( r \)th neighbourhood of a point \( P \). We introduce the following notation:

\[
U_1 = \mathbb{A}^1 \setminus P_1, \quad U_2 = \mathbb{A}^1 \setminus P_2, \quad U_{12} = U_1 \cap U_2 = \mathbb{A}^1 \setminus \{P_1, P_2\}.
\]

To prove that \( \Phi \) is fully faithful, let us show that

\[
\text{Hom}_{\mathscr{D}^b_{\text{coh}}(\mathbb{A}^1)}(H_1, H_2[k]) = \text{Hom}_{\mathscr{D}^b_{\text{coh}}(\mathbb{A}^1)/p}(\Phi(H_1), \Phi(H_2[k]))
\]

for any indecomposable sheaves \( H_1 \) and \( H_2 \) and for all \( k \). This holds for \( H_1 = \mathcal{O}_{\mathbb{A}^1} \) and for \( k = 0 \), because the comparison functor is fully faithful for the Abelian categories (see Theorem 7.1). For \( H_1 = \mathcal{O}_{\mathbb{A}^1} \) and for \( k \neq 0 \) both the left- and right-hand sides of (8.1) vanish. Let \( H_1 = \mathcal{O}_P \) and \( P \neq P_1, P_2 \). Then

\[
\begin{align*}
\text{Hom}_{\mathscr{D}^b_{\text{coh}}(\mathbb{A}^1)}(\mathcal{O}_P, H_2[k]) &= \text{Hom}_{\mathscr{D}^b_{\text{coh}}(\mathbb{A}^1)/p}(\mathcal{O}_P, H_2[k]|_{U_{12}}) \\
&= \text{Hom}_{\mathscr{D}^b_{\text{coh}}(\mathbb{A}^1/\mathbb{A}^1)}(\mathcal{O}_P, H_2[k]|_{U_{12}}) = \text{Hom}_{\mathscr{D}^b_{\text{coh}}(\mathbb{A}^1)/p}(\Phi(\mathcal{O}_P), \Phi(H_2[k])),
\end{align*}
\]

where \( X' = U_{12} \sqcup U_{12} \), the morphism \( p' : X' \to U_{12} \) is the natural map, and \( \Phi' : \mathscr{D}^b_{\text{coh}}(U_{12}) \to \mathscr{D}^b_{\text{coh}}(\mathbb{A}^1)/p' \) is the comparison functor. The middle equality in (8.2) holds because \( p' \) has the SCDT property. Finally, if \( H_1 = \mathcal{O}_P \) and \( P = P_1 \) or \( P = P_2 \), then (8.1) holds for trivial reasons.

We claim now that \( \Phi \) is essentially surjective. Indeed, any object of the category

\[
\mathscr{D}^b_{\text{coh}}(\mathbb{A}^1)/p = \mathscr{D}^b_{\text{coh}}(U_1 \sqcup U_2)/p
\]

is an object of the derived category \( \mathscr{D}^b_{\text{coh}}(\mathbb{A}^1) \) together with the gluing data. Every object \( F \) in \( \mathscr{D}^b_{\text{coh}}(\mathbb{A}^1) \) is of the form \( F = F_1 \oplus F_2 \), where the summands

\[
F_1 = \bigoplus_i \mathcal{O}_{U_1}[k_i] \bigoplus \bigoplus_{i : Q_i \neq P_1} \mathcal{O}_{Q_i}[l_i],
\]

\[
F_2 = \bigoplus_i \mathcal{O}_{U_2}[k'_i] \bigoplus \bigoplus_{i : Q'_i \neq P_2} \mathcal{O}_{Q'_i}[l'_i]
\]

are shifts of sheaves with support in \( U_1 \subset X \) and \( U_2 \subset X \), respectively, and all sums are finite. The gluing data are an isomorphism

\[
\theta : F_1|_{U_{12}} \simto F_2|_{U_{12}}.
\]

It follows from the existence of an isomorphism of this kind that the sums

\[
\left( \bigoplus_i \mathcal{O}_{U_{12}}[k_i] \right) \bigoplus \left( \bigoplus_{i : Q_i \neq P_1, P_2} \mathcal{O}_{Q_i}[l_i] \right),
\]

\[
\left( \bigoplus_i \mathcal{O}_{U_{12}}[k'_i] \right) \bigoplus \left( \bigoplus_{i : Q'_i \neq P_1, P_2} \mathcal{O}_{Q'_i}[l'_i] \right)
\]
We recall that by a \( F \) and \( G \) which the following condition holds on the corresponding factors.

In this section, the symbol \( X \) denotes a scheme of finite type over \( k \) (where \( k \) is an arbitrary field) and by an algebraic group \( G \) we mean a group scheme of finite type over \( k \).

Let an algebraic group \( G \) act on a scheme \( X \). We denote by \( a: G \times X \to X \) the morphism of the action and by \( \mu: G \times G \to G \) the structure morphism of the group. We also denote by \( p_i \) and \( p_{jk} \) the projections of \( G \times X\) and \( G \times G \times X \) onto the corresponding factors.

**Definition 9.1.** We recall that by a \( G \)-equivariant sheaf on \( X \) one means a sheaf \( F \) on \( X \) together with an isomorphism \( \theta: p_2^*F \to a^*F \) of sheaves on \( G \times X \) for which the following condition holds on \( G \times G \times X \):

\[
(1 \times a)^*\theta \circ p_{23}^*\theta = (\mu \times 1)^*\theta.
\]

The morphisms of equivariant sheaves from \((F_1, \theta_1)\) to \((F_2, \theta_2)\) are the morphisms \( f: F_1 \to F_2 \) compatible with \( \theta \), that is, such that \( \theta_2 \circ p_2^*f = a^*f \circ \theta_1 \).

In the important case of finite groups, the definition acquires the following form.

**Definition 9.2.** By a \( G \)-equivariant sheaf on \( X \) we mean a sheaf \( F \) on \( X \) together with fixed isomorphisms \( \theta_g: F \to g^*F \) for any \( g \in G \) such that \( \theta_{gh} = h^*(\theta_g) \circ \theta_h \) for every pair \( g, h \in G \). By a morphism of \( G \)-equivariant sheaves from \((F_1, \theta_{1,g})\) to \((F_2, \theta_{2,g})\) we mean a morphism of sheaves \( f: F_1 \to F_2 \) such that \( \theta_{2,g} \circ f = g^*f \circ \theta_{1,g} \) for every \( g \in G \).

We denote the Abelian category of \( G \)-equivariant quasi-coherent (coherent) sheaves on \( X \) by \( \text{qcoh}^G(X) \) (by \( \text{coh}^G(X) \), respectively). Let us consider the unbounded derived category of all \( G \)-equivariant \( \mathcal{O}_X \)-modules and denote by \( \mathcal{D}^G(X) \) its full subcategory consisting of complexes with \( G \)-equivariant quasi-coherent cohomology sheaves. We denote by \( \mathcal{D}_{\text{ coh}}^G(X) \) the full subcategory of \( \mathcal{D}^G(X) \) formed by the complexes whose cohomology sheaves are coherent and almost all of them vanish. Further, by a \( G \)-equivariant perfect complex on \( X \) we mean an object of the category \( \mathcal{D}^G(X) \) which becomes a perfect complex on \( X \) when forgetting the equivariant structure. We denote by \( \mathcal{D}^{\text{ perf},G}(X) \) the category of perfect \( G \)-equivariant complexes on \( X \).

One can look at the definition of equivariant sheaf in a somewhat different way. Consider the simplicial scheme

\[
(X/G)_\bullet = [X_0, X_1, X_2, \ldots, p_\bullet] = [X, G \times X, G \times G \times X, \ldots, p_\bullet],
\]

where \( p_\bullet \) is a sequence of projection morphisms.
where the morphisms $p_{\bullet}$ are defined as follows. For a monotone map $f: [1, \ldots, m] \to [1, \ldots, n]$ we introduce a morphism of schemes

$$
p_{f}: \underbrace{G \times \cdots \times G \times X}_{n \text{ times}} \to \underbrace{G \times \cdots \times G \times X}_{m \text{ times}}
$$

by the rule

$$(g_{n}, \ldots, g_{2}, x_{1}) \mapsto (g_{f(m)} \cdots g_{f(m-1)+1}, \ldots, g_{f(2)} \cdots g_{f(1)+1}; g_{f(1)} \cdots g_{2} x_{1}).$$

For small values of $n = m \pm 1$ and for strictly monotone maps $f$ the morphisms $p_{\bullet}$ are of the form

$$
\begin{array}{ccc}
G \times G \times X & \xrightarrow{\mu \times 1} & G \times X \\
& \xleftarrow{e \times 1} & X \\
& \xleftarrow{p_{23}} & \\
\end{array}
\begin{array}{ccc}
\xrightarrow{1 \times a} & \quad & \xrightarrow{p_{1} \times e \times p_{2}} \\
\xleftarrow{p_{\bullet}} & \quad & \xleftarrow{e \times 1} \\
\xleftarrow{a} & \quad & \xleftarrow{p_{\bullet}} \\
\end{array}
$$

Consider the descent category

$$\text{Kern}([\text{qcoh}(X), \text{qcoh}(G \times X), \text{qcoh}(G \times G \times X), \ldots, p_{\bullet}^{*}])
$$

associated with the cosimplicial category formed by categories of sheaves on schemes $X_{i} = G^{i} \times X$ and the inverse image functors between them. Comparing Definitions 2.3 and 9.1, we see that the category (9.2) is equivalent to the category of $G$-equivariant quasi-coherent sheaves on $X$ (see also 6.1.2b in [16]).

In § 7, to any morphism of stacks $p: X \to S$ we have assigned an augmented simplicial stack formed by fibred products

$$(X \to S)_{\bullet} = [S, X, X \times_{S} X, X \times_{S} X \times_{S} X, \ldots, p_{\bullet}].$$

It turns out that the cosimplicial scheme (9.1) has the same form if for $p$ one takes the covering by a scheme $X$ of the quotient stack $X//G$ (the quotient of the scheme $X$ by the action of the group $G$). For the definitions and the simplest properties of the stack $X//G$, see 1.3.2 and 4.14.1.1 in [7]. In this section it is assumed that $X//G$ is an algebraic stack of finite type over $k$. The morphism $p$ is faithfully flat, and therefore, by Theorem 7.1, the quasi-coherent sheaves on the stack $X//G$ can be defined locally by using the cover $p$ (see also 6.23 in [7]). We obtain

$$\text{qcoh}(X//G) = \text{Kern}([\text{qcoh}(X), \text{qcoh}(X \times_{X//G} X), \text{qcoh}(X \times_{X//G} X \times_{X//G} X), \ldots, p_{\bullet}^{*}])
$$

$$= \text{Kern}([\text{qcoh}(X), \text{qcoh}(G \times X), \text{qcoh}(G \times G \times X), \ldots, p_{\bullet}^{*}]) = \text{qcoh}^{G}(X).$$

In other words, the sheaves on $X//G$ are $G$-equivariant sheaves on $X$. This enables us to use the language of stacks to study equivariant sheaves. We note that the categories $\mathcal{D}^{G}(X)$, $\mathcal{O}_{\text{coh}}^{G}(X)$, and $\mathcal{D}^{\text{perf},G}(X)$ coincide with the corresponding derived
categories of sheaves on the stack $X/G$. We also note that, if the scheme $X$ (and hence the stack $X/G$; see Lemma 2.1 in [9]) is semi-separated, then the category $\mathcal{D}_{\text{coh}}^{b,G}(X)$ is equivalent to the bounded derived category $\mathcal{D}^b(\text{coh}^G(X))$ of the equivariant coherent sheaves on $X$.

There is an interesting question: When can the objects of the derived category of equivariant sheaves be defined by using an action of the group on objects of the ordinary derived category of sheaves? In other words: When is the category $\mathcal{D}^G(X)$ equivalent to the descent category $\text{Kern}$ associated with the cosimplicial category $[\mathcal{D}(X), \mathcal{D}(G \times X), \mathcal{D}(G \times G \times X), \ldots, Lp^*_\bullet]$? (9.3)

As was noted above, the category (9.3) is of the form $[\mathcal{D}(X), \mathcal{D}(X \times S X), \mathcal{D}(X \times S X \times S X), \ldots, Lp^*_\bullet]$ for the morphism of stacks $X \to S = X//G$, which enables us to use the results of §7. We answer this question below.

We are to consider infinite-dimensional representations of algebraic groups. Let us recall the corresponding notion.

**Definition 9.3.** Let $G$ be an algebraic group over a field $k$. A *rational linear representation* of $G$ over $k$ can be defined in each of the following ways (see Ch. 1, §1 in [17]):

1) as an inductive limit of finite-dimensional linear representations of $G$ over $k$;
2) as an object of the descent category (see Definitions 2.3 and 4.1) for the morphism of stacks $p: \text{Spec} \, k \to (\text{Spec} \, k)//G$;
3) as a $G$-equivariant sheaf on $\text{Spec} \, k$;
4) as a comodule over the coalgebra $k[G]$ (this way fits for affine group schemes only).

**Definition 9.4.** An algebraic group $G$ over $k$ is said to be *linearly reductive* (see Definition 1.4 in [17]) if the category of finite-dimensional representations (or, equivalently, of all rational representations) of $G$ over $k$ is semisimple. We denote the category of all rational (finite-dimensional rational) linear representations of $G$ by $\text{Rep}(G)$ (by $\text{rep}(G)$, respectively).

**Proposition 9.5.** Let $G$ be a group scheme of finite type over $k$. Then the following conditions are equivalent:

1) the group $G$ is linearly reductive;
2) the natural homomorphism $k \to k[G]$ is an embedding of a direct summand in the category $\text{Rep}(G)$;
3) the comparison functor $\Phi$ between the derived categories $\mathcal{D}^b(\text{rep}(G))$ and $\mathcal{D}^b(k-\text{mod})/p$ is an equivalence.

If, in addition, the group scheme $G$ is affine and smooth over $k$, then the conditions 1)–3) are equivalent to the following condition:

4) the connected identity component $G_0 \subset G$ is a diagonalizable torus $(\mathbb{G}_m)^r$ and the order of the finite group $G/G_0$ is coprime to $l$ for $\text{char} \, k = l > 0$; the group scheme $G$ is reductive for $\text{char} \, k = 0$. 

Proof. Condition 1) implies condition 2) by definition.

Let us prove that 1) implies 3). This follows from Theorem 7.3 applied to the morphism of stacks $p: X \to S$, where $X = \text{Spec}k$, $S = (\text{Spec}k)\sslash G$ and $p$ stands for the canonical morphism. Indeed, the sheaf $p_*\mathcal{O}_X = H^0(Rp_*\mathcal{O}_X)$ is a direct summand in the complex $Rp_*\mathcal{O}_X$ because the category $\text{qcoh}(S)$ is semisimple. The homomorphism $\mathcal{O}_S \to p_*\mathcal{O}_X$ is the natural homomorphism of the representations $k \to k[ G]$, and it splits by condition 2).

Let us prove that 2) implies 1). Let $f: U \to V$ be an embedding of representations of the group $G$. We claim that $f$ has a left inverse. Consider the commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\downarrow{\eta_U} & & \downarrow{\eta_V} \\
p_*p^*U & \xrightarrow{p_*p^*f} & p_*p^*V
\end{array}
$$

of quasi-coherent sheaves on $S$ (that is, representations of $G$). By the projection formula, the morphism $\eta_U$ is of the form $U \otimes (\mathcal{O}_S \to p_*\mathcal{O}_X)$, and it splits by assumption. The morphism $p_*f$ is an embedding of vector spaces, and it splits. Thus, the morphism $p_*p^*f$ splits as well. Hence, the composition $\eta_V \circ f$ splits. Therefore, $f$ also splits.

Let us prove that 3) implies 1). In essence, one must prove that

$$\text{Ext}^1_{\text{rep}(G)}(V, V') = 0$$

for any $V, V' \in \text{rep}(G)$. By the assumption concerning the comparison functor, we have

$$\text{Ext}^1_{\text{rep}(G)}(V, V') = \text{Hom}_{D^b(\text{rep}(G))}(V, V'[1]) = \text{Hom}_{D^b(k-\text{mod})/p}(\Phi(V), \Phi(V'[1])) \subset \text{Hom}_{D^b(k-\text{mod})}(V, V'[1]) = 0.$$

The equivalence of conditions 1) and 4) is a result of Nagata (see [18]).

Let us introduce the following notation:

$$D(X)^G = \text{Kern}([D(X), D(G \times X), \ldots, Lp^*_i]),$$

$$D^\text{perf}(X)^G = \text{Kern}([D^\text{perf}(X), D^\text{perf}(G \times X), \ldots, Lp^*_i]),$$

$$D^b_{\text{coh}}(X)^G = \text{Kern}([D^b_{\text{coh}}(X), D^b_{\text{coh}}(G \times X), \ldots, Lp^*_i]).$$

**Theorem 9.6.** Let $G$ be a group scheme of finite type over a field $k$ and let $G$ act on a scheme $X$ of finite type over $k$. Suppose that $G$ is linearly reductive. Then the following equivalences hold:

$$D^G(X) \cong D(X)^G, \quad D_{\text{perf},G}(X) \cong D_{\text{perf}}(X)^G, \quad D_{\text{coh},G}(X) \cong D_{\text{coh}}(X)^G.$$

**Proof.** All statements follow from Theorem 7.3 applied to the morphism of stacks $X \xrightarrow{p} X\sslash G$. To prove that the morphism $p$ has the SCDT property, we consider
the fibred square of stacks

\[
\begin{array}{ccc}
X & \longrightarrow & pt \\
\downarrow^p & & \downarrow^t \\
X//G & \longrightarrow & pt//G
\end{array}
\]

The morphism \( \mathcal{O}_{pt//G} \to Rt_* t^* \mathcal{O}_{pt//G} \) is the canonical homomorphism of representations \( k \to k[G] \) of the group \( G \). It splits because \( G \) is linearly reductive. Thus, \( t \) has the SCDT property. By Proposition 8.3, 2), the morphism \( p \) also has this property.

**Corollary 9.7.** The equivalencies of categories in the previous theorem hold provided that \( G \) is a reductive group over an algebraically closed field of characteristic zero or \( G \) is a finite group and the characteristic of the field \( k \) does not divide the order of \( G \).

**Proof.** Under these conditions, the group \( G \) is linearly reductive (see Proposition 9.5).

**Remark 9.8.** If a group \( G \) is finite, then, by definition, the objects of the descent category \( \mathcal{D}(X)^G \) are the objects \( F \in \mathcal{D}(X) \) equipped for any \( g \in G \) with isomorphisms \( \theta_g : F \overset{\sim}{\to} g^* F \) such that \( g^* \theta_h \circ \theta_g = \theta_{hg} \). In other words, an object of \( \mathcal{D}(X)^G \) is an object of \( \mathcal{D}(X) \) together with an action of the group \( G \) on this object.

The statement of Theorem 9.6 can fail to hold for a group \( G \) which is not linearly reductive.

**Example 9.9.** Let \( V \) be a vector space, let \( X = GL(V) \), and let \( G = P \) be a parabolic subgroup in \( GL(V) \) acting on \( X \) by shifts. The action of \( G \) on \( X \) is free, and we have an equivalence

\[
\mathcal{D}_{coh}^b(G)(X) \cong \mathcal{D}^b(coh^G(X)) \cong \mathcal{D}^b(coh(S)) \cong \mathcal{D}_{coh}^b(S)
\]

to the derived category of the quotient (the homogeneous space \( S = GL(V)/P \)). Thus, we arrive at the situation of Example 8.7, and the comparison functor is not an equivalence.

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A. D. Elagin

Institute for Information Transmission Problems, the Russian Academy of Sciences, Moscow; Laboratory of Algebraic Geometry and its Applications, National Research University “Higher School of Economics,” Moscow

E-mail: alexelagin@rambler.ru

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