ON THE CONTINUITY OF THE INTEGRATED DENSITY OF STATES IN THE DISORDER

MIRA SHAMIS

ABSTRACT. Recently, Hislop and Marx studied the dependence of the integrated density of states on the underlying probability distribution for a class of discrete random Schrödinger operators, and established a quantitative form of continuity in weak* topology. We develop an alternative approach to the problem, based on Ky Fan inequalities, and establish a sharp version of the estimate of Hislop and Marx. We also consider a corresponding problem for continual random Schrödinger operators on $\mathbb{R}^d$.

1. INTRODUCTION

Recently, Hislop and Marx [3] studied the dependence of the integrated density of states (IDS) of random Schrödinger operators on the distribution of the potential.

Let $\{V(n)\}_{n \in \mathbb{Z}^d}$ be independent identically distributed random variables (i.i.d.r.v.) with the common probability distribution $\mu$. Let $H$ be the random Schrödinger operator acting on $\ell^2(\mathbb{Z}^d)$ by

$$H = -\Delta + V,$$

where $\Delta$ is the discrete Laplacian operator.

The IDS corresponding to the operator $H$ is the function

$$N(E) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \# \{\text{eigenvalues of } H_\Lambda \text{ in } (-\infty, E]\},$$

where $H_\Lambda$ is the restriction of $H$ to a finite box $\Lambda \subset \mathbb{Z}^d$, i.e. $H_\Lambda = P_\Lambda H P_\Lambda^*$, where $P_\Lambda : \ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda)$ is the coordinate projection. The measure with cumulative distribution function $N$ is denoted by $\rho$.

To discuss the dependence of $\rho$ on the distribution of the potential $\mu$, we introduce two metrics on the space of Borel probability measures on $\mathbb{R}$.

The Kantorovich-Rubinstein (Wasserstein) metric is defined via

$$d_{KR}(\mu, \bar{\mu}) = \sup \left\{ \left| \int f \, d\mu - \int f \, d\bar{\mu} \right| : f : \mathbb{R} \to \mathbb{R} \text{ is } 1\text{-Lipschitz} \right\}.$$
By the Kantorovich-Rubinstein duality theorem
\begin{equation}
    d_{KR}(\mu, \tilde{\mu}) = \inf \{ \mathbb{E}|X - \tilde{X}| \},
\end{equation}
where the infimum is taken over $\mathbb{R}^2$-valued random variables $(X, \tilde{X})$, such that $X \sim \mu, \tilde{X} \sim \tilde{\mu}$. Following [3], we also consider the bounded Lipschitz metric, defined by
\begin{equation}
    d_{BL}(\mu, \tilde{\mu}) = \sup \left\{ \left| \int f \, d\mu - \int f \, d\tilde{\mu} \right| : f : \mathbb{R} \to \mathbb{R} \text{ is 1-Lip, } \|f\| \leq 1 \right\}.
\end{equation}
Our definition differs from [3] by a multiplicative constant. Observe that
\begin{equation}
    d_{BL}(\mu, \tilde{\mu}) \leq d_{KR}(\mu, \tilde{\mu}),
\end{equation}
and if $\text{supp } \mu, \text{supp } \tilde{\mu} \subset [-A, A]$, then
\begin{equation}
    d_{KR}(\mu, \tilde{\mu}) \leq \max(A, 1) d_{BL}(\mu, \tilde{\mu}).
\end{equation}
In this notation Theorem 1.1 of [3] (formulated here in slightly less general setting than in the cited work) asserts the following.

**Theorem** (Hislop–Marx). Suppose $H, \tilde{H}$ are random Schrödinger operators of the form (1.1) with potentials $\{V(n)\}, \{\tilde{V}(n)\}$ sampled from a probability distributions $\mu, \tilde{\mu}$, respectively. Denote by $N, \tilde{N}$ the IDS corresponding to $H, \tilde{H}$, and let $\rho, \tilde{\rho}$ be the measures with cumulative distribution functions $N, \tilde{N}$, respectively. If $\text{supp } \mu, \text{supp } \tilde{\mu} \subset [-A, A]$, then
\begin{equation}
    d_{BL}(\rho, \tilde{\rho}) \leq C_A \frac{1}{d_{BL}(\mu, \tilde{\mu})^{1/(1+2d)}},
\end{equation}
\begin{equation}
    \sup |N(E) - \tilde{N}(E)| \leq \frac{C_A}{\log_{+} \frac{1}{d_{BL}(\mu, \tilde{\mu})}},
\end{equation}
where $C_A$ depends only on $A$.

Hislop and Marx [3] presented several applications, particularly, to the continuity of the Lyapunov exponent of a one-dimensional operator as a function of the underlying distribution of the potential. The proof of the theorem in [3] is based on the approximation of the function $f$ (in (1.5)) by polynomials.

We suggest a different approach to estimates of the form (1.6) using the Ky Fan inequalities. Our first result is the following theorem.

**Theorem 1.** Suppose $H, \tilde{H}$ are random Schrödinger operators of the form (1.1), where $\{V(n)\}, \{\tilde{V}(n)\}$ are i.i.d.r.v. distributed accordingly to $\mu, \tilde{\mu}$ respectively. Let $N, \tilde{N}$ be the IDS corresponding $H, \tilde{H}$, and let $\rho, \tilde{\rho}$ be the measures with cumulative distribution functions $N, \tilde{N}$ respectively. Then,
\begin{equation}
    d_{KR}(\rho, \tilde{\rho}) \leq d_{KR}(\mu, \tilde{\mu}),
\end{equation}
\begin{equation}
    \sup |N(E) - \tilde{N}(E)| \leq \frac{C}{\log_{+} \frac{1}{d_{KR}(\mu, \tilde{\mu})}},
\end{equation}
where $C > 0$ is a numerical constant.

**Remark 1.1.** The power 1 as well as the prefactor 1 in (1.8) are optimal in general.

**Remark 1.2.** This result can be extended to other models in which the potential is of the form

\[(1.10) \quad \sum v_j P_j,\]

where $v_j$ are i.i.d.r.v. with common Borel distribution supported on a finite interval and $P_j$ are finite rank projections (see [3]).

**Remark 1.3.** Theorem 1 can be extended to different underlying lattices, since the proof does not rely on the structure of $\mathbb{Z}^d$.

In the forthcoming paper [4], Hislop and Marx prove a version of their results for continual Anderson model, which is not of the form (1.10). A modification of our argument can be applied to the continual setting as well. We illustrate it by the following theorem.

Let $H$ be a random Schrödinger operator acting on $L^2(\mathbb{R}^d)$, defined by

\[(1.11) \quad H = -\Delta + V,\]

where the potential $V$ is of the form

\[(1.12) \quad V(x) = \sum_{j \in \mathbb{Z}^d} v_j u(x - j), \quad x \in \mathbb{R}^d,\]

where $v_j$ are i.i.d.r.v. distributed accordingly to $\mu$, and $u$ is real-valued continuous compactly supported function: $u \in C_c(\mathbb{R})$. Denote by $\Lambda$ the cube of side length $L$ around the origin

$$\Lambda = \left[ -\frac{L}{2}, \frac{L}{2} \right]^d.$$

Let $H_\Lambda$ be the restriction of $H$ to $L^2(\Lambda)$ with Dirichlet boundary conditions. Define the IDS corresponding to $H$ similarly to (1.2)

\[(1.13) \quad N(E) = \lim_{L \to \infty} \frac{1}{L^d} \# \{\text{eigenvalues of } H_\Lambda \text{ in } (-\infty, E] \},\]

and let $\rho$ be the measure with cumulative distribution function $N$.

**Theorem 2.** Suppose $H, \tilde{H}$ are random Schrödinger operators of the form (1.11), and suppose that $d = 1, 2, 3$, supp $\mu, \text{supp } \tilde{\mu} \subset \mathbb{R}_+$, and $u \geq 0$. Let $N, \tilde{N}$ be the IDS corresponding $H, \tilde{H}$, and let $\rho, \tilde{\rho}$ be the measures with cumulative distribution functions $N, \tilde{N}$ respectively. Then,

\[(1.14) \quad \left| \int f \left( \frac{1}{1+E} \right) d\rho(E) - \int f \left( \frac{1}{1+E} \right) d\tilde{\rho}(E) \right| \leq C d_{KR}(\mu, \tilde{\mu}),\]

for any 1-Lipschitz function $f$, for which $\int f \left( \frac{1}{1+E} \right) d\rho(E)$ converges.
If \( \text{supp } \mu \subset [0, A] \) then for any \( E_0 \in \mathbb{R} \)

\[
\sup_{E \leq E_0} |N(E) - \tilde{N}(E)| \leq \frac{C(d, E_0, A)}{\log^{\kappa_d}(d) + d_{KR}(\mu, \lambda)},
\]

where \( C(d, E_0, A) > 0 \), and \( \kappa_1 = 1, \kappa_2 = 1/4, \kappa_3 = 1/8 \).

**Remark 1.4.** The restriction on \( V \) in the second part of Theorem 2 comes from the work of Bourgain and Klein [1] which we use to deduce (1.15) from (1.14).

**Remark 1.5.** A similar statement to (1.14) can be proved for \( d \geq 4 \) and for sign-indefinite \( V \). We stick to the case \( u \geq 0, d \leq 3 \) for the simplicity of presentation.

## 2. Preliminaries

### 2.1. Discrete case.

The main ingredient of the proof of Theorem 1 is the Ky Fan inequality [5]:

Assume that \( A, B, \) and \( \tilde{A} = A + B \) are linear self-adjoint operators that act on \( n \)-dimensional Euclidean space. Let \( \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1, e_n \leq e_{n-1} \leq \cdots \leq e_1, \tilde{\lambda}_n \leq \tilde{\lambda}_{n-1} \leq \cdots \leq \tilde{\lambda}_1 \) be the eigenvalues of \( A, B, \) and \( \tilde{A} \) respectively. Then, for any continuous convex function \( \phi : \mathbb{R} \to \mathbb{R} \)

\[
\sum_{j=1}^{n} \phi(\tilde{\lambda}_j - \lambda_j) \leq \sum_{j=1}^{n} \phi(e_j).
\]

In particular,

\[
\sum_{j=1}^{n} |\tilde{\lambda}_j - \lambda_j| \leq \sum_{j=1}^{n} |e_j|.
\]

To deduce (1.9) from (1.8) (similarly to [3]) we shall use the following result due to Craig and Simon [2]. Denote by

\[
\omega(\delta) = \sup \{|\rho(E) - \rho(E')| : E' < E \leq E + \delta\},
\]

the modulus of continuity of \( \rho \). Then ([2]) the measure \( \rho \) with the cumulative distribution function \( N \) (the IDS) of any ergodic Schrödinger operator on \( \ell^2(\mathbb{Z}^d) \) is log-Hölder continuous, namely, for any \( \delta \in (0, \frac{1}{2}] \)

\[
\omega(\delta) \leq \frac{C}{\log \frac{1}{\delta}},
\]

where \( C > 0 \) is a universal constant.
2.2. **Continual case.** First, recall that for $1 \leq p < \infty$ the Schatten class $S_p$ is the class of all compact operators in a given Hilbert space such that

$$
\|A\|_p = \left(\sum_{n=1}^{\infty} s_n(A)^p\right)^{1/p} < \infty,
$$

where $\{s_n(A)\}$ is the sequence of all singular values of the operator $A$ enumerated with multiplicities taken into account. The class $S_{\infty}$ consists of all compact operators.

The main ingredient in the proof of Theorem 2 is the following version of the Ky Fan inequality (see Markus [6]).

If $A \in S_1$, $B \in S_{\infty}$ that are self-adjoint, and $\tilde{A} = A + B$, $\lambda_1 \geq \lambda_2 \geq \cdots$, $e_1 \geq e_2 \geq \cdots$, $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots$, are the eigenvalues of $A, B$, and $\tilde{A}$ respectively, then, for any continuous convex function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$

$$
\sum_{j=1}^{\infty} \phi(\tilde{\lambda}_j - \lambda_j) \leq \sum_{j=1}^{\infty} \phi(e_j).
$$

(2.5)

In particular,

$$
\sum_{j=1}^{\infty} |\tilde{\lambda}_j - \lambda_j| \leq \sum_{j=1}^{\infty} |e_j| = \|B\|_1.
$$

(2.6)

To deduce (1.15) from (1.14) we will need the following result due to Bourgain and Klein [1].

**Theorem (BK).** Assume that $H$ as in (1.11)-(1.12) on $L^2(\mathbb{R}^d)$, $d = 1, 2, 3$, with $\text{supp } \mu \subset [-A, A]$. Let $N$ be the corresponding IDS. Then, given $E_0 \in \mathbb{R}$, for all $E \leq E_0$ and $\delta \leq 1/2$

$$
|N(E) - N(E + \delta)| \leq \frac{C(d, E_0, A)}{\log^{\kappa_1} \frac{1}{\delta}},
$$

(2.7)

where $C(d, E_0, A) > 0$, and $\kappa_1 = 1, \kappa_2 = 1/4, \kappa_3 = 1/8$.

3. **Proof of Theorem 1 and Theorem 2**

3.1. **Proof of Theorem 1.** Denote by $\Lambda \subset \mathbb{Z}^d$ a finite box and let (in the notation of Ky Fan’s inequality)

$$
A = H_\Lambda = (-\Delta + V)_\Lambda, \quad \tilde{A} = \tilde{H}_\Lambda = (-\Delta + \tilde{V})_\Lambda,
$$

be the restrictions of the operators $H$ and $\tilde{H}$ to the box $\Lambda$. Then,
\[ |\text{tr} f(A) - \text{tr} f(\tilde{A})| = \left| \sum_{j=1}^{|A|} f(\lambda_j) - \sum_{j=1}^{|\Lambda|} f(\tilde{\lambda_j}) \right| \]

(3.1)

\[ \leq \sum_{j=1}^{|A|} |f(\lambda_j) - f(\tilde{\lambda_j})| \leq \sum_{j=1}^{|\Lambda|} |\lambda_j - \tilde{\lambda_j}| \]

\[ \leq \sum_{j=1}^{|\Lambda|} |e_j| = \sum_{x \in \Lambda} |V(x) - \tilde{V}(x)|, \]

where the second inequality holds since \( f \) is 1-Lipschitz and the last inequality follows from (2.1).

By (1.4) there is a realization of \( V \) and \( \tilde{V} \) on a common probability space such that

\[ \mathbb{E}|V(x) - \tilde{V}(x)| \leq d_{KR}(\mu, \tilde{\mu}). \]

Thus, using (3.1) for any 1-Lipschitz function \( f \), we obtain

(3.2) \[ |\mathbb{E} \text{tr} f(A) - \mathbb{E} \text{tr} f(\tilde{A})| \leq \mathbb{E} \sum_{x \in \Lambda} |V(x) - \tilde{V}(x)| \leq |\Lambda| d_{KR}(\mu, \tilde{\mu}). \]

Since

\[ \int f \, d\rho = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathbb{E} \text{tr} f(A), \]

we obtain by passing to the limit \( \Lambda \nearrow \mathbb{Z}^d \)

(3.3) \[ d_{KR}(\rho, \tilde{\rho}) \leq d_{KR}(\mu, \tilde{\mu}), \]

thus concluding the proof of (1.8).

To deduce (1.9), we choose

(3.4) \[ f(x) = \begin{cases} 
\delta, & x \leq E \\
0, & x \geq E + \delta,
\end{cases} \]

for \( \delta > 0 \). Then, by definition of the IDS, we get for any \( E \in \mathbb{R} \)

(3.5) \[ \delta N(E) \leq \int f(E) d\rho(E) \leq \delta N(E + \delta), \]

(3.6) \[ \delta \tilde{N}(E) \leq \int f(E) d\tilde{\rho}(E) \leq \delta \tilde{N}(E + \delta). \]

Since

\[ \int f(E) d\tilde{\rho}(E) = \int f(E) d\rho(E) + \int f(E) d(\tilde{\rho} - \rho)(E), \]

combining (3.3), (3.5), and (3.6), we obtain

\[ \delta \tilde{N}(E) \leq \delta N(E + \delta) + d_{KR}(\mu, \tilde{\mu}), \]
namely

\[(3.7) \quad \tilde{N}(E) \leq N(E + \delta) + \frac{d_{KR}(\mu, \tilde{\mu})}{\delta}. \]

In the same way we get

\[(3.8) \quad \tilde{N}(E) \geq N(E - \delta) - \frac{d_{KR}(\mu, \tilde{\mu})}{\delta}. \]

Let \( \omega \) be the modulus of continuity of \( N \). Combining (2.4), (3.7), and (3.8), we obtain

\[(3.9) \quad \sup_{E} |N(E) - \tilde{N}(E)| \leq \inf_{\delta} \left( \omega(\delta) + \frac{d_{KR}(\mu, \tilde{\mu})}{\delta} \right) \leq \frac{C}{\log(C d_{KR}(\mu, \tilde{\mu}))}, \]

where \( C > 0 \) is a constant and we choose \( \delta = \frac{d_{KR}(\mu, \tilde{\mu})}{\omega(d_{KR}(\mu, \tilde{\mu}))} \). This finishes the proof of (1.9).

\[\square\]

**Remark 3.1.** If the operator \( H \) is such that the modulus of continuity \( \omega \) satisfies

\[ \omega(\delta) \leq C \delta^a, \]

for some \( C, a > 0 \), then (3.9) implies that

\[ \sup_{E} |N(E) - \tilde{N}(E)| \leq \inf_{\delta} \left( C \delta^a + \frac{d_{KR}(\mu, \tilde{\mu})}{\delta} \right) \leq \tilde{C} d_{KR}(\mu, \tilde{\mu})^{1/(1+a)}. \]

### 3.2. Proof of Theorem 2

Let

\[ H_\Lambda = (-\Delta + V)_\Lambda, \quad \tilde{H}_\Lambda = (-\Delta + \tilde{V})_\Lambda, \]

be the restrictions of the operators \( H \) and \( \tilde{H} \) to a finite box \( \Lambda \in \mathbb{R}^d \), \( d = 1, 2, 3 \), with Dirichlet boundary conditions. Let (in the notation of Ky Fan’s inequality)

\[ A = (H_\Lambda + 1)^{-1}, \quad \tilde{A} = (\tilde{H}_\Lambda + 1)^{-1}. \]

Using that \( V, \tilde{V} \geq 0 \), \( u \in C_c(\mathbb{R}) \), and \( d \leq 3 \), it is easy to see that

\[ u^{1/4} A u^{1/4}, \quad u^{1/4} \tilde{A} u^{1/4} \in S_2, \]

with 2-Schatten norms bounded by a \( C \), depending only on \( u \), uniformly in \( |\Lambda| \) and in the realization of \( V \) and \( \tilde{V} \). Thus,

\[(3.10) \quad \|A u^{1/2}\|_2, \quad \|u^{1/2} \tilde{A}\|_2 \leq C, \quad \text{and consequently} \quad \|A u \tilde{A}\|_1 \leq C. \]

Thus, by the second resolvent identity

\[ \mathbb{E}\|A - \tilde{A}\|_1 = \mathbb{E}\|A(\tilde{V} - V)\tilde{A}\|_1 = \mathbb{E}\|A \sum_{j \in 2\Lambda \cap \mathbb{Z}^d} (\tilde{v}_j - v_j)u(\cdot - j) \tilde{A}\|_1 \]

\[ \leq \sum_{j \in 2\Lambda \cap \mathbb{Z}^d} \mathbb{E}|\tilde{v}_j - v_j| \|A u(\cdot - j) \tilde{A}\|_1 \leq C \sum_{j \in 2\Lambda \cap \mathbb{Z}^d} \mathbb{E}|\tilde{v}_j - v_j| \leq \tilde{C} |\Lambda| d_{KR}(\mu, \tilde{\mu}). \]
The eigenvalues of $A$ are exactly $\frac{1}{1+\lambda_j}$, where $\lambda_j$ are the eigenvalues of $H_\Lambda$, thus using (2.6), we obtain for any 1-Lipschitz function $f$ with $f(0) = 0$

\begin{equation}
\left| \sum_{j=1}^{\infty} f\left(\frac{1}{1+\lambda_j}\right) - \sum_{j=1}^{\infty} f\left(\frac{1}{1+\lambda_j}\right) \right| \leq \sum_{j=1}^{\infty} \left| f\left(\frac{1}{1+\lambda_j}\right) - f\left(\frac{1}{1+\lambda_j}\right) \right| \\
\leq \sum_{j=1}^{\infty} |\lambda_j - \tilde{\lambda}_j| \leq C |\Lambda| d_{KR}(\mu, \tilde{\mu}).
\end{equation}

Using the definition (1.13) of $\rho$ and passing to the limit $\Lambda \nearrow \mathbb{R}^d$, we conclude that

\begin{equation}
d_{KR}(\rho, \tilde{\rho}) \leq C d_{KR}(\mu, \tilde{\mu}),
\end{equation}

where the constant $C > 0$ depends only on $u$. Thus we complete the proof of (1.14).

To deduce (1.15), we define

\begin{equation}
f(x) = \begin{cases} \frac{\delta}{2(1+E)^2}, & x \geq \frac{1}{1+E} \\ 0, & x \leq \frac{1}{1+E+\delta}, \end{cases}
\end{equation}

for $\delta > 0$. Then, by the definition of the IDS (1.13) we get for a fixed $E_0 \in \mathbb{R}$ in the same way as in the proof of Theorem 1

\begin{equation}
\tilde{N}(E_0) \leq N(E_0 + \delta) + \frac{2d_{KR}(\mu, \tilde{\mu})(1+E_0)^2}{\delta},
\end{equation}

\begin{equation}
\tilde{N}(E_0) \geq N(E_0 - \delta) - \frac{2d_{KR}(\mu, \tilde{\mu})(1+E_0)^2}{\delta}.
\end{equation}

Let $\omega$ be the modulus of continuity of $N$. Then, by (2.7) for any $E \leq E_0$ and $\delta \leq 1/2$

$$|N(E) - N(E + \delta)| \leq \frac{C(d, E_0, A)}{\log^{\frac{d}{2}} \delta}.$$ 

Thus, choosing $\delta = \frac{C(d, E_0, A) d_{KR}(\mu, \tilde{\mu})}{\omega(C(d, E_0, A) d_{KR}(\mu, \tilde{\mu}))}$, we obtain

$$\sup_{E \leq E_0} |\tilde{N}(E) - N(E)| \leq \frac{\tilde{C}(d, E_0, A)}{\log^{\frac{d}{2}} d_{KR}(\mu, \tilde{\mu})},$$

therefore completing the proof. \[\square\]
Acknowledgements: I am grateful to Peter Hislop and Christoph Marx for helpful correspondence. I would also like to thank Sasha Sodin for pleasant and useful discussions pertaining to the Ky Fan inequalities.

REFERENCES

[1] J. Bourgain, A. Klein, Bounds on the density of states for Schrödinger operators, Inventiones mathematicae 194, 1–72 (2013)

[2] W. Craig, B. Simon, Log Hölder continuity of the integrated density of states for stochastic Jacobi matrices, Comm. Math. Phys. 90(2) 207–218 (1983).

[3] P. D. Hislop, C. A. Marx, Dependence of the density of states on the probability distribution for discrete random Schrödinger operators, to appear in International Mathematics Research Notices (2018)

[4] P. D. Hislop, C. A. Marx, Dependence of the density of states on the probability distribution - part II: Schrödinger operators on $\mathbb{R}^d$ and non-compactly supported probability measures, in preparation (2018)

[5] V. B. Lidskii, Inequalities for eigenvalues and singular values, appendix to: F. R. Gantmaher. Theory of matrices. Second supplemented edition. Izdat. "Nauka", Moscow 1966 576 pp.

[6] A. S. Markus, Eigenvalues and singular values of the sum and product of linear operators. (Russian) Uspehi. Mat. Nauk 19, no. 4 (118), 93–123 (1964)

School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, England

E-mail address: m.shamis@qmul.ac.uk