QUANTUM COHOMOLOGY OF FLAG VARIETIES

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INTRODUCTION

The quantum cohomology ring of a Kähler manifold $X$ is a deformation of the usual cohomology ring which appears naturally in theoretical physics in the study of the supersymmetric nonlinear sigma models with target $X$. In [W], Witten introduces the quantum multiplication of cohomology classes on $X$ as a certain deformation of the usual cup-product, obtained by adding to it the so-called instanton corrections (see also [V]). These can be interpreted as intersection numbers on a sequence of moduli spaces of (holomorphic) maps $\mathbb{P}^1 \to X$. To make this interpretation rigorous according to mathematical standards, one encounters severe problems, mainly because these moduli spaces are not compact and they may have the wrong dimension. Recently, substantial efforts have been made to put the theory on firm mathematical footing, and a proof for the existence of the quantum cohomology ring, using methods of symplectic topology, has been given by Ruan and Tian [RT], for a large class of manifolds (semi-positive symplectic manifolds).

Nevertheless, computing the quantum cohomology ring is typically a difficult task (the method of proving existence relies on changing the complex structure of $X$ to a generic almost complex structure, hence it is not suited for computations). Several examples have been worked out (Batyrev ([Bat]) for toric varieties - see also [MP] - , Bertram ([Be2]) and Siebert -Tian ([ST]) for Grassmannians), but the problem is far from solved. In [GK], based on the conjectures following from conformal field theory, Givental and Kim proposed a presentation of the quantum cohomology ring in the case of flag varieties.

In the present paper we describe a method for computing these rings, building on the ideas in [Be1], [Be2] and [BDW]. We give a rigorous construction of the (genus 0) Gromov-Witten invariants for the flag varieties (i.e. the intersection numbers we mentioned above) and using it we complete our computations, recovering the statement in [GK]. Fulton’s main result in [F1] will be essential for the proof. It is also worth noting that the method provides a new (algebraic-geometric!) proof for the existence of quantum cohomology, as in [Be2]. This will be done elsewhere. (The proof in [RT] has been recently redone in an algebraic setting for the case of homogeneous spaces by J. Li and G. Tian - see [LT].)}
Finally, we should remark that the method described here, coupled with the results in [F2], should apply also to more general homogeneous spaces (at least for the classical groups) giving a presentation for the quantum cohomology ring in these cases as well.

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**Notations and Statements of Results**

We start by recalling some well-known facts about flag varieties and their cohomology.

Let $V \cong \mathbb{C}^n$ be a complex vector space and define $F(n) = F(V^*)$ to be the variety of complete flags: $U_i : \{0\} = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{n-1} \subset U_n = V^*$.

On $F(V^*)$ there is an universal flag of subbundles

$$E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = V^* \otimes \mathcal{O}_{F(V^*)}$$

and an universal sequence of quotient bundles

$$V^* \otimes \mathcal{O}_{F(V^*)} = L_n \rightarrow L_{n-1} \rightarrow \cdots \rightarrow L_1 \rightarrow 0,$$

where $L_i = V^* \otimes \mathcal{O}_{F(V^*)}/E_{i-1}$ for $i = 1, \ldots, n-1$.

Fix a complete flag of subspaces $V_i^* : \{0\} = V_0^* \subset V_1^* \subset \cdots \subset V_{n-1}^* \subset V_n^* = V^*$.

We have then induced maps $f_{pq} : V_p^* \otimes \mathcal{O}_{F(V^*)} \rightarrow L_q$. Let $S_n$ be the symmetric group on $n$ letters. For $w \in S_n$, let $r_w(q, p) = \text{card}\{i \mid i \leq q, w(i) \leq p\}$. Set

$$\Omega_w = \Omega_w(V_i^*) = \{U_i \in F(V^*) \mid \text{rank}\_\text{\_}\_\text{U}_i f_{pq} \leq r_w(q, p), 1 \leq p, q \leq n-1\}$$

and

$$X_w = X_w(V_i^*) = \{U_i \in F(V^*) \mid \dim(U_q \bigcap V_p^*) \geq r_w(q, p), 1 \leq p, q \leq n-1\}.$$

$\Omega_w$ is a subvariety of $F(V^*)$ of codimension $\ell(w) = \text{the length of the permutation } w \in S_n$. If we let $w_0 \in S_n$ be the permutation of longest length (given by $w_0(i) = n - i + 1, i = 1, \ldots, n$), then $\Omega_w = X_{w_0}$ for all $w \in S_n$.

**Facts**: 1 \{\Omega_w \}_{w \in S_n} and \{\Omega_w \}_{w \in S_n} form dual additive bases for $CH^*(F(V^*); \mathbb{Z})$.

2 Let $x_i = c_1(\text{ker}(L_i \rightarrow L_{i-1})) = c_1(E_{n-i+1}/E_{n-i})$, $I = 1, \ldots, n$. Then \{\begin{array}{c} x_1, x_2, \ldots, x_n \end{array} | i_j \leq n-j\} form an additive basis as well, and

$$H^*(F(V^*); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, \ldots, x_n]/(R_1(n), R_2(n), \ldots, R_n(n)),$$

where $R_i(n)$ is the $i^{th}$ elementary symmetric function in $x_1, x_2, \ldots, x_n$.

To express $\Omega_w$ in $H^*(F(V^*); \mathbb{Z})$ we need the notion of Schubert polynomials (see [LS1], [LS2]). Define operators $\partial_i, i = 1, \ldots, n-1$ on $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$\partial_i P = \frac{P(x_1, \ldots, x_n) - P(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)}{x_i - x_{i+1}}.$$
For any \( w \in S_n \), write \( w = w_0 \cdot s_{i_1} \cdot \ldots \cdot s_{i_k} \), with \( k = \frac{n(n-1)}{2} - \ell(w) \), where \( s_i = (i, i+1) \) is the transposition interchanging \( i \) and \( i+1 \). The polynomial \( \sigma_w(x) \in \mathbb{Z}[x_1, \ldots, x_n] \) defined by

\[
\sigma_w(x) = \partial_{i_k} \circ \cdots \circ \partial_{i_1}(x_1^{n-1}x_2^{n-2} \cdots x_{n-1})
\]

is the Schubert polynomial associated to \( w \). With this definition, we have a Giambelli type formula:

\[
\Omega_w = \sigma_w(x_1, \ldots, x_n)
\]

in \( H^*(F(V^*); \mathbb{Z}) \).

\( F(n) \) embeds in a projective space by a Plücker embedding as a linear section of a product of \( n-1 \) Grassmannians, and if \( H \) is the hyperplane section class we have \( \mathcal{O}_{F(n)}(H) = \mathcal{O}_{F(n)}(1, 1, \ldots, 1) = \mathcal{O}_{F(n)}(\Omega_{s_1} + \cdots + \Omega_{s_{n-1}}) \). Also, the canonical class is \( K_{F(n)} = -2H \). (see [M])

A map \( f : \mathbb{P}^1 \to F(V^*) \) of multidegree \( \overline{d} = (d_1, d_2, \ldots, d_{n-1}) \) with respect to \( H \) is given by specifying a flag of subbundles \( S_1 \subset S_2 \subset \cdots \subset S_{n-1} \subset V^* \otimes \mathcal{O}_{\mathbb{P}^1} \) with \( \text{rank}(S_i) = i \), \( \deg(S_i) = -d_i \). Fixing the multidegree of the map amounts to fixing the homology class \( f_*[\mathbb{P}^1] = \sum d_i Y_i \), where \( Y_i \in H_2(F(V^*); \mathbb{Z}) \) is the Poincaré dual of \( \Omega_{s_i} \). Since \( F(V^*) \) is a homogeneous space, the moduli space of such maps \( \text{Hom}_{\mathbb{P}^1}(\mathbb{P}^1, F(V^*)) \) is a smooth quasi-projective variety of dimension

\[ h^0(\mathbb{P}^1, f^*F(V)) = \dim F(V^*) + f_*[\mathbb{P}^1] \cdot (-K_{F(V^*)}) = \frac{n(n-1)}{2} + 2 \sum_{i=1}^{n-1} d_i. \]

Our main tool is a certain compactification of \( \text{Hom}_{\mathbb{P}^1}(\mathbb{P}^1, F(V^*)) \), generalizing Grothendieck’s Quot scheme, which we will introduce now.

Let \( X \) be a smooth projective variety over an algebraically closed field \( k \) and \( E \) a vector bundle on \( X \). For any scheme \( S \) over \( k \), let \( \pi : X \times S \to X \) be the projection. Consider the functor \( F(X, E) : \{ \text{Schemes over } k \} \to \{ \text{Sets} \} \) given by

\[
F(X, E)(S) = \left\{ \begin{array}{l}
\text{equivalence classes of flagged quotient sheaves } \\
\pi^*E \to Q_{n-1} \to \cdots \to Q_1, \text{which are flat over } S
\end{array} \right\},
\]

and for a morphism \( S \to T \),

\[
F(X, E)(\varphi) = \text{pull-back by } \varphi.
\]

Here \( \pi^*E \to Q_{n-1} \to \cdots \to Q_1 \) is equivalent to \( \pi^*E \to Q'_{n-1} \to \cdots \to Q'_1 \) if there are isomorphisms \( \theta_i : Q_i \to Q'_i \) such that all the squares commute.

Let \( \overline{P} = (P_1(m), \ldots, P_{n-1}(m)) \) be numerical polynomials and define the subfunctor \( \mathcal{F}_{\tau}(X, E) \) by requiring that \( \chi(X_s, Q_s) = P_i(m) \) for all \( s \in S \). Extending the construction of the Quot scheme we have the following

**Theorem 1.** For fixed \( \overline{P}(m) \), \( \mathcal{F}_{\tau}(X, E) \) is represented by a projective scheme.

We will denote this scheme by \( \mathcal{H}Q_{\overline{P}}(X, E) \) and refer to it as the hyper-quot scheme associated to \( X, E \) and \( \overline{P} \). In general this scheme may be very complicated, but in the case of interest for our purposes it is well-behaved. More precisely, let \( P_i(m) = (m + 1)i + d_i \) which is the Hilbert polynomial of a vector bundle of rank \( i \) and degree \( d_i \) on \( \mathbb{P}^1 \). Then \( \overline{P} \) is determined by \( \overline{d} = (d_1, \ldots, d_{n-1}) \) only. Denote the hyper-quot scheme associated to \( \mathbb{P}^1, V^* \otimes \mathcal{O}_{\mathbb{P}^1} \) and \( \overline{d} \) by \( \mathcal{H}Q_{\overline{d}}(\mathbb{P}^1, F(V^*)) \).
Theorem 2. $\mathcal{H}_\Omega(\mathbb{P}^1, F(V^*))$ is a smooth projective variety of dimension $\frac{n(n-1)}{2} + 2 \sum_{i=1}^{n-1} d_i$, containing $\text{Hom}_\Omega(\mathbb{P}^1, F(V^*))$ as an open subscheme.

As a fine moduli space, $\mathcal{H}_\Omega(\mathbb{P}^1, F(V^*))$ comes equipped with an universal sequence of sheaves

$$0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_{n-1} \hookrightarrow S_n = V^* \otimes \mathcal{O} \rightarrow \mathcal{T}_{n-1} \rightarrow \cdots \rightarrow \mathcal{T}_2 \rightarrow \mathcal{T}_1 \rightarrow 0$$
on $\mathbb{P}^1 \times \mathcal{H}_\Omega(\mathbb{P}^1, F(V^*))$, where $S_i = \text{ker}(V^* \otimes \mathcal{O} \rightarrow \mathcal{T}_{n-i})$. For $i = 1, \ldots, n-1$, $S_i$ is a vector bundle of rank $i$ and relative degree $-d_i$ (this follows from flatness), but the inclusions are injective as maps of sheaves only!

In fact, $\text{Hom}_\Omega(\mathbb{P}^1, F(V^*))$ is the largest subscheme $U$ of $\mathcal{H}_\Omega(\mathbb{P}^1, F(V^*))$ with the property that on $\mathbb{P}^1 \times U$ all the inclusions are injections of vector bundles.

The main technical result needed for our computations describes the locus where these maps degenerate. Let $\bar{c} = (e_1, \ldots, e_{n-1})$ with $e_i \leq \max(i, d_i)$ and $\bar{c}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$.

Theorem 3. There are rational maps $j_{\bar{c}} : \mathbb{P}^1 \times \mathcal{H}_\Omega(\mathbb{P}^1, F(V^*)) \rightarrow \mathcal{H}_\Omega(\mathbb{P}^1, F(V^*))$ with the following properties:

(i) $j_{\bar{c}}$ is defined on $\mathbb{P}^1 \times \text{Hom}_\Omega(\mathbb{P}^1, F(V^*))$ and its restriction to it is an embedding.

(ii) If $s \in \mathcal{H}_\Omega(\mathbb{P}^1, F(V^*))$ is in the image of $j_{\bar{c}}$, then rank$(S_i \hookrightarrow S_{i+1}) \leq i - e_i$ at $(t, s)$, for some $t \in \mathbb{P}^1$.

(iii) The images of the maps $j_{\bar{c}}$ cover $\mathcal{H}_\Omega(\mathbb{P}^1, F(V^*)) \setminus \text{Hom}_\Omega(\mathbb{P}^1, F(V^*))$.

From the previous theorem, one can see that the boundary of the hyper-quot scheme compactifying the moduli space of maps $\mathbb{P}^1 \rightarrow F(V^*)$ consists of $n-1$ divisors $\mathbb{D}_1, \ldots, \mathbb{D}_{n-1}$, which are birational to $\mathbb{P}^1 \times \text{Hom}_\Omega(\mathbb{P}^1, F(V^*)), \ldots, \mathbb{P}^1 \times \mathcal{H}_\Omega(\mathbb{P}^1, F(V^*))$.

Following [Be2], we will define now the quantum multiplication for cohomology classes on $F(V^*)$. There is an evaluation morphism

$$ev : \mathbb{P}^1 \times \text{Hom}_\Omega(\mathbb{P}^1, F(V^*)) \rightarrow F(V^*),$$

given by $ev(t, f) = f(t)$.

For $t \in \mathbb{P}^1$, $w \in S_n$, define a subscheme of $\text{Hom}_\Omega(\mathbb{P}^1, F(V^*))$ by

$$\Omega_w(t) = ev^{-1}(\Omega_w) \bigcap \{\{t\} \times \text{Hom}_\Omega(\mathbb{P}^1, F(V^*))\}.$$

Set theoretically,

$$\Omega_w(t) = \{ f \in \text{Hom}_\Omega(\mathbb{P}^1, F(V^*)) \mid f(t) \in \Omega_w \}.$$

Also, we define $\Omega_w(t)$ to be the following subscheme of $\mathcal{H}_\Omega(\mathbb{P}^1, F(V^*))$: consider the dual sequence

$$V \otimes \mathcal{O} \rightarrow S_{n-1}^* \rightarrow \cdots \rightarrow S_1^*$$
on $\mathbb{P}^1 \times \mathcal{H}_\Omega(\mathbb{P}^1, F(V^*))$ and the fixed flag $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$.

Let

$D_{w, q}^\nu = \text{locus where rank}(V_p \otimes \mathcal{O} \rightarrow S_q^*) \leq r_w(q, p)$
and
\[ \Delta^{p,q}(t) = D^{p,q}_w \cap \{ \{t\} \times \mathcal{H}\mathcal{Q}_\pi(\mathbb{P}^1, F(V^*)) \} . \]

Then
\[ \mathfrak{P}_w(t) := \bigcap_{p,q=1}^{n-1} D^{p,q}_w(t) . \]

The following lemma and its corollaries will allow us to define intersection numbers on \( \text{Hom}_\pi(\mathbb{P}^1, F(V^*)) \). The proof of (ii) of the lemma depends heavily on the analysis in Theorem 3.

(Moving) Lemma. (i) For any \( w_1, \ldots, w_N \in S_n \); \( t_1, \ldots, t_n \in \mathbb{P}^1 \) and general translates of \( \Omega_{w_i} \subset F(V^*) \), the intersection \( \bigcap_{i=1}^N \Omega_{w_i}(t_i) \) has pure codimension \( \sum_{i=1}^N \ell(w_i) \) in \( \text{Hom}_\pi(\mathbb{P}^1, F(V^*)) \) (or is empty).

(ii) If \( t_1, \ldots, t_N \) are distinct, then for general translates of the \( \Omega_{w_i} \), the intersection \( \bigcap_{i=1}^N \mathfrak{P}_w(t_i) \) has pure codimension \( \sum_{i=1}^N \ell(w_i) \) in \( \mathcal{H}\mathcal{Q}_\pi(\mathbb{P}^1, F(V^*)) \) and is the (Zariski) closure of \( \bigcap_{i=1}^N \Omega_{w_i}(t_i) \) (or is empty).

Corollary 1. The class of \( \mathfrak{P}_w(t) \) in \( CH^{\ell(w)}(\mathcal{H}\mathcal{Q}_\pi(\mathbb{P}^1, F(V^*))) \) is independent of \( t \in \mathbb{P}^1 \) and the flag \( V^*_r \subset V^*_r \).

Let \( D = \frac{n(n-1)}{2} + 2 \sum_{i=1}^{n-1} d_i = \dim(\mathcal{H}\mathcal{Q}_\pi(\mathbb{P}^1, F(V^*))) \).

Corollary 2. If \( \sum_{i=1}^N \ell(w_i) = D \), and \( t_1, \ldots, t_N \) are distinct, then the number of points in \( \bigcap_{i=1}^N \Omega_{w_i}(t_i) \) can be computed as the degree of \( (\bigcap_{i=1}^N \mathfrak{P}_w(t_i)) \) in \( CH^D(\mathcal{H}\mathcal{Q}_\pi(\mathbb{P}^1, F(V^*))) \) (hence it is independent of \( t_i \) and the translates of \( \Omega_{w_i} \)).

The corollaries imply that we have a well defined intersection number
\[ \langle \Omega_{w_1}, \ldots, \Omega_{w_N} \rangle := \begin{cases} \text{number of points in } \bigcap_{i=1}^N \Omega_{w_i}(t_i), & \text{if } \sum_{i=1}^N \ell(w_i) = D, \\ 0, & \text{otherwise}. \end{cases} \]

This is the Gromov-Witten invariant associated to the classes \( \Omega_{w_1}, \ldots, \Omega_{w_N} \).

Definition: The quantum multiplication map is the linear map
\[ m_q : \text{Sym}(H^*(F(V^*); \mathbb{C})[q_1, \ldots, q_{n-1}]) \to H^*(F(V^*); \mathbb{C})[q_1, \ldots, q_{n-1}] \]
given by
\[ m_q(\prod_{i=1}^N \Omega_{w_i}q^\ell_{\pi_i}) = q^{\sum_{w \in S_n} \ell(w)} \sum_{\pi \in \Pi^N} \bar{q}^{\ell(\pi)}(\sum_{w \in S_n} \langle \Omega_{w_1}, \ldots, \Omega_{w_N} \rangle \mathcal{P} X_w). \]

Here \( \bar{q}^{\ell(\pi)} \) denotes as usual the monomial \( \prod_{i=1}^{n-1} q_i^{m_i} \).

[RT] proves that the pairing induced by \( m_q \) on \( \text{Sym}(H^*(F(V^*); \mathbb{C})[q_1, \ldots, q_{n-1}] \) determines a ring structure on \( H^*(F(V^*); \mathbb{C})[q_1, \ldots, q_{n-1}] \) (associativity of quantum multiplication; see also [LT]). The pair \( (H^*(F(V^*); \mathbb{C})[q_1, \ldots, q_{n-1}], m_q) \) is the quantum cohomology ring of \( F(V^*) \). It is easy to prove (see e.g. [Be2]), using the Moving Lemma, that \( m_q \) is the identity map on \( \text{Sym}(H^*(F(V^*); \mathbb{C})[q_1, \ldots, q_{n-1}] \). The restriction
\[ m_q : \mathbb{C}[x_1, x_2, \ldots, x_n, q_1, q_2, \ldots, q_{n-1}] \to H^*(F(V^*); \mathbb{C})[q_1, \ldots, q_{n-1}] \]
is surjective (see [ST] for an easy argument by induction on degree) and by [RT] it is a ring homomorphism. Let \( I \) be the kernel of this map. Then the quantum cohomology ring of \( F(n) \) is

\[
(H^*(F(V^*); \mathbb{C})[q_1, \ldots, q_{n-1}], m_q) \cong \mathbb{C}[x_1, x_2, \ldots, x_n, q_1, q_2, \ldots, q_{n-1}]/I.
\]

We compute here the generators of \( I \).

Recall that in \( H^*(F(V^*); \mathbb{C}) \) we have \( R_k(n) = 0 \), where \( R_k(n) \) is the \( k \)th symmetric function in \( x_1, \ldots, x_n \), \( k = 1, \ldots, n \). In the quantum ring however, \( m_q(R_k(n)) \) is a polynomial \( R'_k(n)(x_1, \ldots, x_n, q_1, \ldots, q_{n-1}) \) which doesn’t vanish anymore (unless \( k = 1 \)).

**Definition.** The quantum deformation of \( R_k(n) \) is \( R^q_k(n) := R_k(n) - R'_k(n) \).

**Theorem 4.** (i) The quantum deformations of the relations \( R_k(n) \) can be computed recursively with the formula

\[
R^q_k(n) = R^q_k(n-1) + x_n \cdot R^q_{k-1}(n-1) + q_{n-1} \cdot R^q_{k-2}(n-2).
\]

(Here \( R^q_k(n-1) \) is set to be 0 and \( R^q_0(n-2) \) is set to be 1).

(ii) The ideal \( I \) is generated by \( R^q_1(n), \ldots, R^q_n(n) \).

**Remarks:** 1. From Theorem 4 one can get a presentation for the quantum cohomology ring of \( F(n) \) for all \( n > 1 \), once the ring is known for \( F(1) \). But \( F(1) \cong \mathbb{P}^1 \) and for projective spaces quantum cohomology is well-known. In this case the ring is isomorphic to \( \mathbb{C}[x_1, x_2, q_1]/(x_1 + x_2, x_1 x_2 + q_1) \).

2. In [GK], Givental and Kim gave the following (conjectural at that time) compact description of the generators for the ideal \( I \):

\[
R^q_k(n) \text{ is the coefficient of } \lambda^{n-k} \text{ in the expansion of the determinant:}
\]

\[
\begin{vmatrix}
  x_1 + \lambda & q_1 & 0 & 0 & \ldots & 0 & 0 \\
  -1 & x_2 + \lambda & q_2 & 0 & \ldots & 0 & 0 \\
  0 & -1 & x_3 + \lambda & q_3 & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \ldots & x_{n-1} + \lambda & q_{n-1} \\
  0 & 0 & 0 & 0 & \ldots & -1 & x_n
\end{vmatrix}.
\]

If one expands this determinant along the last column, the formula of Theorem 4 (i) is obtained.

**Sketch of Proofs**

For Theorem 1 and Theorem 2 the proofs, although lengthy, are straightforward following the same line as for the corresponding results in the case of the Quot functor.

For the proof of Theorem 3 we will change notation slightly and let \( S^i_{\mathbb{P}^1} \mathcal{P} \) be the \( i \)th universal bundle on \( \mathbb{P}^1 \times \mathcal{Q}^i_{\mathbb{P}^1}(\mathbb{P}^1, F(V^*)) \).

The maps \( j_{i-1, \mathcal{P}} \) can be constructed by downward recursion. Therefore, we will outline here the construction for \( j_{i-1, \mathcal{P}}, i = 1, \ldots, n - 1 \). Let

\[
\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{Q}^i_{\mathbb{P}^1}(\mathbb{P}^1, F(V^*)) \xrightarrow{\pi_1} \mathbb{P}^1 \times \mathcal{Q}^i_{\mathbb{P}^1}(\mathbb{P}^1, F(V^*))
\]
be the projection. If we denote by $\Delta$ the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, then
\[
\Delta \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*)) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*))
\]
is a divisor. Pull-back by $\pi_1$ of the universal sequence on $\mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*))$ gives the following sequence on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*))$:
\[
0 \to \pi_1^* S_1 \to \pi_1^* S_2 \to \cdots \to \pi_1^* S_n \to V^* \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \times \mathcal{H}_d(-\mathcal{F})
\]
Let $\mathcal{S}_i = (\pi_1^* S_i^\mathcal{F}) \otimes \mathcal{O}(-\Delta \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*))$. Then $\mathcal{S}_i$ is a vector bundle of rank 1 and relative degree $-d_1$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*))$, which is a subsheaf of $\pi_1^* S_i^\mathcal{F}$. Since $\mathcal{H}_d(\mathbb{P}^1, F(V^*))$ is a fine moduli space, we get a morphism
\[
j_d: \mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*)) \to \mathcal{H}_d(\mathbb{P}^1, F(V^*))
\]
such that
\[
(id, j_d)^* \mathcal{S}_i = \begin{cases} 
\mathcal{S}_i, & \text{for } i = 1 \\
\mathcal{S}_i^\mathcal{F}, & \text{for } i \neq 1.
\end{cases}
\]
It is easy to see that if $\text{rank}(\mathcal{S}_i^\mathcal{F} \to \mathcal{S}_j^\mathcal{F}) = 0$ at $(t, x) \in \mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F})$, then $x$ is in the image of $j_d$, and that the restriction of $j_d$ to $\mathbb{P}^1 \times \text{Hom}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*))$ is an embedding.

For $2 \leq i \leq n - 1$, let
\[
\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*)) \xrightarrow{j_i} \mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*))
\]
be the projection and consider as before the sequence
\[
0 \to \pi_i^* S_{i-1} \to \cdots \to \pi_i^* S_{i-1} \to \cdots \to \pi_i^* S_n \to V^* \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \times \mathcal{H}_d(-\mathcal{F})
\]
Let $\mathcal{L}_i^\mathcal{F}$ be the quotient $\pi_i^* S_i^\mathcal{F} / \pi_i^* S_{i-1}^\mathcal{F}$. The map $\mathcal{L}_i^\mathcal{F} (-\Delta \times \mathcal{H}_d(-\mathcal{F})) \to \mathcal{L}_i^\mathcal{F}$ induces
\[
\text{Ext}^1 (\mathcal{L}_i^\mathcal{F}, \pi_i^* S_{i-1}^\mathcal{F}) \to \text{Ext}^1 (\mathcal{L}_i^\mathcal{F}, (-\Delta \times \mathcal{H}_d(-\mathcal{F}) \pi_i^* S_{i-1}^\mathcal{F})
\]
Thus we have a diagram of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F}) (\mathbb{P}^1, F(V^*))$:
\[
\begin{array}{cccccc}
0 & \to & \pi_i^* S_{i-1} & \to & \pi_i^* S_i & \to & \mathcal{L}_i^\mathcal{F} & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \pi_i^* S_{i-1} & \to & S_i & \to & \mathcal{L}_i^\mathcal{F} & \to & 0
\end{array}
\]
The restriction of $S_i^\mathcal{F}$ to the open set $\pi_i^{-1}(\mathcal{U}) = \mathbb{P}^1 \times \mathcal{U}$, where
\[
\mathcal{U} = \{ y \in \mathbb{P}^1 \times \mathcal{H}_d(-\mathcal{F}) | \text{rank}_y (S_i^\mathcal{F} \to S_{i-1}^\mathcal{F}) = i - 1 \},
\]
is a vector bundle of rank $i$ and relative degree $-d_i$. Also, the restriction of
$$S_i^\rightarrow_{\mathcal{P}} \to \pi_1^* S_i^\rightarrow_{\mathcal{P}},$$
to $\mathbb{P}^1 \times \mathcal{U}$ is injective as a map of sheaves. Note that $\mathbb{P}^1 \times \text{Hom}_{\mathcal{P}}(\mathbb{P}^1, F(V^*)) \subset \mathcal{U}$.

It follows that there is a rational map
$$j_{\mathcal{P}}: \mathbb{P}^1 \times \mathcal{H}_{\mathcal{Q}_{\mathcal{P}}}(\mathbb{P}^1, F(V^*)) \to \mathcal{H}_{\mathcal{Q}_{\mathcal{P}}}(\mathbb{P}^1, F(V^*))$$
which is defined on $\mathcal{U}$ and has the following properties:

1. On $\mathbb{P}^1 \times \mathcal{U}$,
$$\left(id, j_{\mathcal{P}}\right)^* S_k = \begin{cases} S_i^\rightarrow_{\mathcal{P}}, & \text{for } k = i \\ \pi_1^* S_i^\rightarrow_{\mathcal{P}}, & \text{for } k \neq i. \end{cases}$$

2. If $S_i^\rightarrow_{\mathcal{P}} \to S_{i+1}^\rightarrow_{\mathcal{P}}$ degenerates at $(t, x) \in \mathbb{P}^1 \times \mathcal{H}_{\mathcal{Q}_{\mathcal{P}}}$ and $S_i^\rightarrow_{\mathcal{P}} \to S_{i-1}^\rightarrow_{\mathcal{P}}$ has maximal rank at $(t, x)$, then $x$ is in the image of $j_{\mathcal{P}}$.

3. The restriction of $j_{\mathcal{P}}$ to $\mathbb{P}^1 \times \text{Hom}_{\mathcal{P}}(\mathbb{P}^1, F(V^*))$ is an embedding.

It is clear that
$$\mathcal{H}_{\mathcal{Q}_{\mathcal{P}}}(\mathbb{P}^1, F(V^*)) \setminus \text{Hom}_{\mathcal{P}}(\mathbb{P}^1, F(V^*)) = \bigcup_{i=1}^{n-1} \text{Im}(j_{\mathcal{P}}).$$

**Remark 3.** We can describe now the preimages of the degeneracy loci $\overline{\Omega}_w(t)$ by the maps $j_{\mathcal{P}}$. Note first that the map $S_i^\rightarrow_{\mathcal{P}} \to \pi_1^* S_i^\rightarrow_{\mathcal{P}}$ is an isomorphism outside $\Delta \times \mathcal{H}_{\mathcal{Q}_{\mathcal{P}}}$ and that when restricted to $\Delta \times \mathcal{H}_{\mathcal{Q}_{\mathcal{P}}}$ it factors through $\pi_1^* S_i^\rightarrow_{\mathcal{P}}$. Consequently, $j_{\mathcal{P}}^{-1}(\overline{\Omega}_w(t))$ splits as the union of two subschemes of $\mathbb{P}^1 \times \mathcal{H}_{\mathcal{Q}_{\mathcal{P}}}$:
$$j_{\mathcal{P}}^{-1}(\overline{\Omega}_w(t)) = \mathbb{P}^1 \times \overline{\Omega}_w(t) \bigcup \overline{\Omega}_w(t),$$
where $\overline{\Omega}_w(t)$ is the locus $\bigcap_{q \neq i} \text{D}_{w,q}(t)$ in $\{t\} \times \mathcal{H}_{\mathcal{Q}_{\mathcal{P}}}$.

Using the previous remark, the same proof as in [Be2], Lemma 2.2 and 2.2 A, gives the Moving Lemma in our case too.

We will give now in more detail the proof of Theorem 4.

The following result, due to Fulton (Theorem 8.2. in [F1]), will be crucial. It generalizes the Giambelli formula mentioned in the introduction. We give here only the version needed for our purposes and refer to the original paper for the full statement.

We need to establish some notations first. Let $X$ be a scheme of finite type over a field $k$, $V$ an $n$-dimensional vector space, $V_\bullet \subset V$ a fixed complete flag of subspaces and $w \in S_{n+1}$ a permutation. Consider a flag of vector bundles on $X$:
$$B_{n-1} \to B_{n-2} \to \cdots \to B_1,$$
with $\text{rank}(B_i) = i$. To each map $h : V \otimes \mathcal{O}_X \to B_{n-1}$ one can associate degeneracy loci $\Omega_w(h)$ defined by the conditions
$$\text{rank}\{V_p \otimes \mathcal{O}_X \to B_q\} \leq r_w(q, p)$$
for $1 \leq p \leq n$ and $1 \leq q \leq n - 1$. Let $x_1 = c_1(B_1)$, $x_i = c_1(\text{ker}\{B_i \to B_{i-1}\})$ for $i = 2, \ldots, n-1$, $x_n = 0$. 
Theorem (Fulton). If $X$ is a purely $d$-dimensional scheme, there is a class $\hat{\Omega}_w(h)$ in $CH_{d-\ell(w)}(\Omega_{\ell(w)}(h))$, satisfying the following.

(i) The image of $\Omega_w(h)$ in $CH_{d-\ell(w)}(X)$ is $\sigma_w(x) \cap [X]$.

(ii) Each irreducible component of $\Omega_w(h)$ has codimension at most $\ell(w)$ in $X$. If $\text{codim}(\Omega_w(h), X) = \ell(w)$, then $\hat{\Omega}_w(h)$ is a positive cycle whose support is $\Omega_w(h)$.

(iii) If $\text{codim}(\Omega_w(h), X) = \ell(w)$ and $X$ is Cohen-Macaulay, then $\hat{\Omega}_w(h)$ is Cohen-Macaulay and $\hat{\Omega}_w(h) = [\Omega_w(h)]$.

Remark 4. It will be important later that the statement is formulated for $w \in S_{n+1}$ rather than $S_n$. Of course, the theorem applies to the degeneracy loci we employed so far, since both the loci and the Schubert polynomials are invariant under the natural embedding $S_n \subset S_m$, for $m > n$.

By the previous theorem, if the maps $S_i^* \to S_{i-1}^*$ were surjective for $i = 2, \ldots, n-1$, one could express the degeneracy locus $\overline{\Omega}_w(t)$ in $CH^*(\mathbb{P}^1 \times \mathcal{H}(\mathbb{P}^1, F(V^*))$ as the Schubert polynomial $\sigma_w$ evaluated in the Chern classes $c_1(S_i^*) = c_1(\ker \{S^*_i \to S^*_i\}), c_1(\ker \{S^*_{n-1-1} \to S^*_{n-2-1}\})$. In our case, this is true only if we restrict to the open set $U$ of $\mathbb{P}^1 \times \mathcal{H}(\mathbb{P}^1, F(V^*))$ where $S_i$ is a subbundle of $S_{i+1}$ for all $i = 1, \ldots, n-2$.

Hence, if we let

$$x_i(t) = c_1(S_i^*) \cap \{ \{ t \} \times \mathcal{H}(\mathbb{P}^1, F(V^*)) \} - c_1(S_{i-1}^*) \cap \{ \{ t \} \times \mathcal{H}(\mathbb{P}^1, F(V^*)) \}$$

$$= \overline{\Omega}_w(t) - \overline{\Omega}_{w_{i-1}}(t),$$

we have in $CH_*(\mathcal{H}(\mathbb{P}^1, F(V^*)))$ the following identity:

$$\overline{\Omega}_w(t) - \sigma_w(x_1(t), \ldots, x_{n-1}(t)) = j_*(G_w(t));$$

where $G_w(t) \in CH_*(\bigcup_{i=1}^{n-2} D_i)$ and $j : \bigcup_{i=1}^{n-2} D_i \to \mathcal{H}(\mathbb{P}^1, F(V^*))$ is the inclusion.

Let $w_1, \ldots, w_N \in S_n$ such that $\ell(w) + \sum_{i=1}^{N} \ell(w_i) = D$ and $t, t_1, \ldots, t_N \in \mathbb{P}^1$ distinct points. Then we get

$$\overline{\Omega}_w(t) \cdot \prod_{i=1}^{N} \overline{\Omega}_{w_i}(t) = \sigma_w(x_1(t), \ldots, x_{n-1}(t)) \cdot \prod_{i=1}^{N} \overline{\Omega}_{w_i}(t_i) = j_*(G_w(t)) \cdot \prod_{i=1}^{N} \overline{\Omega}_{w_i}(t_i)$$

in $CH^*(\mathcal{H}(\mathbb{P}^1, F(V^*)))$. From the Moving Lemma and its corollaries, we see that $j_*(G_w(t)) \cdot \prod_{i=1}^{N} \overline{\Omega}_{w_i}(t_i)$ is (the negative of) the (signed) number of points in $\sigma_w(x_1(t), \ldots, x_{n-1}(t)) \cdot \prod_{i=1}^{N} \overline{\Omega}_{w_i}(t)$ supported on $\bigcup_{i=1}^{n-2} D_i$.

Note that for distinct points $u_1, \ldots, u_{n-1}, t_1, \ldots, t_N \in \mathbb{P}^1$ the intersection

$$\sigma_w(\overline{\Omega}_{1}(u_1), \ldots, \overline{\Omega}_{n-2}(u_{n-2})) \cap \bigcap_{i=1}^{N} \overline{\Omega}_{w_i}(t_i)$$

avoids the boundary. Allowing the points $u_1, \ldots, u_{n-1}$ to come together “moves” part of the intersection in the boundary.
For $m \leq n$, $k \leq m - 1$, let $\alpha_k(m) \in S_n$ be the permutation
\[
\begin{pmatrix}
1 & 2 & \ldots & m - k - 1 & m - k & \ldots & m - 1 & m & \ldots & n - 1 & n \\
1 & 2 & \ldots & m - k - 1 & m - k & \ldots & m - 1 & m & \ldots & n - 1 & n
\end{pmatrix}.
\]
Its Schubert polynomial is the $k - 1$st elementary symmetric function in variables
\[x_1, \ldots, x_m.\]
\[\sigma_{\alpha_k(m)}(x) = R_{k-1}(m-1).\]
For the proof of the formula in Theorem 4 (i) we will need to compute the number given by the LHS of (2) for $w = \alpha_k(m).
\]
Note first that $\sigma_{\alpha_k(m)}(x) = R_{k-1}(m-1)$ is the sum of all degree $k - 1$ “square-free” monomials in $x_1, \ldots, x_m$. Recall that $x_p(t) = \overline{\Omega}_{s_p}(t) - \overline{\Omega}_{s_{p-1}}(t)$.
\]
We are therefore led to compute the part supported on $\bigcup_{i=1}^{n-2} D_i$ for intersections of the type
\[
\overline{\Omega}_{s_1}(t) \cap \cdots \cap \overline{\Omega}_{s_{k-1}}(t) \cap \bigcap_{i=1}^{N} \overline{\Omega}_{w_i}(t_i),
\]
with $i_j \in \{1, \ldots, m - 1\}$. The $i_j$’s are not necessarily distinct, but each index in \(\{1, \ldots, m - 2\}\) may occur at most twice, while \(m - 1\) may occur at most once.

By Remark 3,
\[
\overline{\Omega}_{s_i}(t) = \begin{cases}
\mathbb{P}^1 \times \overline{\Omega}_{s_p}(t) \cup \{t\} \times \mathcal{H} \mathcal{Q}_{\mathcal{D} - \mathcal{P}_s}, & \text{for } i = p \\
\mathbb{P}^1 \times \overline{\Omega}_{s_p}(t) \cup \{t\} \times \overline{\Omega}_{s_p}(t), & \text{for } i \neq p
\end{cases}
\]
in $\mathbb{P}^1 \times \mathcal{H} \mathcal{Q}_{\mathcal{D} - \mathcal{P}_s}$, for $i = 1, \ldots, n - 2$.
Since the points $t, t_1, \ldots, t_N \in \mathbb{P}^1$ are distinct, the only time we will get nonempty intersections is when at least one index occurs twice. Say that $\overline{\Omega}_{s_p}(t)$ occurs twice, for some $1 \leq p \leq m - 2$ (this is equivalent to saying that the product $x_p x_{p+1}$ is in the corresponding square-free monomial $M$). Then the intersection will be
\[
\overline{\Omega}_{s_1}(t) \cap \cdots \cap \overline{\Omega}_{s_{k-1}}(t) \cap \bigcap_{i=1}^{N} \overline{\Omega}_{w_i}(t_i) \subset \{t\} \times \mathcal{H} \mathcal{Q}_{\mathcal{D} - \mathcal{P}_s},
\]
where the collection of indices $l_1, \ldots, l_{k-3}$ is obtained from $i_1, \ldots, i_{k-1}$ by removing $p$ (twice). This corresponds to the monomial $M/x_p x_{p+1}$. If $l_1, \ldots, l_{k-3}$ are not distinct we can repeat the procedure, working now with $\mathcal{H} \mathcal{Q}_{\mathcal{D} - \mathcal{P}_s}$. This says the following: in the quantum ring of $F(n)$, for all $m \leq n$, we have
\[
\begin{equation}
\tag{3}
m_q(\Omega_{\alpha_k(m)}, \bullet) = m_q(\sigma_{\alpha_k(m)}(x), \bullet) + m_q(\sigma'_{\alpha_k(m)}(x, q), \bullet),
\end{equation}
\]
where $\sigma'_{\alpha_k(m)}(x, q)$ is the sum of all square-free monomials of weighted degree $k - 1$ in $x_1, \ldots, x_m, q_1, \ldots, q_{m-2}$ (here $\deg(x_i) = 1$ and $\deg(q_i) = 2$), with the additional condition that if $q_i$ occurs in such a monomial, then none of $x_i, x_{i+1}, q_{i+1}$ can occur.

It is clear that $m_q(\sigma_{\alpha_k(m)}(x) + \sigma'_{\alpha_k(m)}(x, q))$ satisfies the recursion relation in Theorem 4(i), i.e.
\[
\begin{equation}
\tag{4}
m_q(\sigma_{\alpha_k(m)}(x) + \sigma'_{\alpha_k(m)}(x, q)) = m_q(\sigma_{\alpha_k(m-1)}(x) + \sigma'_{\alpha_k(m-1)}(x, q)) + m_q(x_{m-1}(\sigma_{\alpha_{k-1}(m-1)}(x) + \sigma'_{\alpha_{k-1}(m-1)}(x, q))) + m_q(q_{m-2}(\sigma_{\alpha_{k-2}(m-2)}(x) + \sigma'_{\alpha_{k-2}(m-2)}(x, q))).
\end{equation}
\]
Remark 5. Recently, W. Fulton informed us that he computed the coefficients of the characteristic polynomial obtained by expanding the determinant in [GK], i.e. the generators of the ideal \( I \). His formula, which is easy to prove by induction, is (with our notations):

\[
R_k(n) = \sigma_{\alpha_{k+1}(n+1)}(x) + \sigma'_{\alpha_{k+1}(n+1)}(x, q).
\]

This suggested us to make the explicit calculation in (3), rather than showing the recursion (4), which is the only thing we need to complete our proof.

Monk’s formula (see [M]) gives an expression for the intersection of Schubert subvarieties on \( F(n) \):

\[
\Omega_{s_p} \cdot \Omega_w = \sum_{t_{ij}} \Omega_{w_{t_{ij}}},
\]

where the sum is over all transpositions \( t_{ij} \) of integers \( i \leq p < j \) such that \( \ell(w \cdot t_{ij}) = \ell(w) + 1 \). As noted in [LS2], this is not an identity among the corresponding Schubert polynomials, unless one embeds \( S_n \) in \( S_{n+1} \) (as usual, by setting \( w(n+1) = n+1 \)). Using Monk’s formula to multiply \( \sigma_{\alpha_{n-1}}(x) = x_1 + x_2 + \cdots + x_{n-1} \) and \( \sigma_{\alpha_{k-1}(n)}(x) \) one sees easily that we get

\[
(x_1 + x_2 + \cdots + x_{n-1}) \cdot \sigma_{\alpha_{k-1}(n)}(x) = \sigma_{\alpha_k(n)}(x) + \sigma_{\beta_k}(x),
\]

with \( \beta_k = \alpha_{k-1}(n) \cdot t_{n-1, n+1} \in S_{n+1} \).

We can also define the degeneracy locus \( \Omega_{\beta_k}(t) \in \mathcal{H}_{\mathbb{Q}[\mathbb{P}^1, F(V^*)]} \) in a similar manner by

\[
\Omega_{\beta_k}(t) = \bigcap_{p=1}^{n-1} \nabla_{t \cdot q}^{\beta_k}(t).
\]

It is easy to see that \( \Omega_{\beta_k}(t) \) is given by the conditions \( \operatorname{rank}(V^n \otimes \mathcal{O} \rightarrow S_i^*) \leq i - 1 \), for \( n - k + 1 \leq i \leq n - 1 \). In particular, it is supported on \( \mathbb{D}_{n-1} \).

It is immediate from the definition that \( x_1 + \cdots + x_n = 0 \) in the quantum ring. Using (5), we obtain for \( k \geq 2 \) that

\[
\begin{align*}
\Omega_{\beta_k}(t) &= \Omega_{\beta_k}(t) - \langle \Omega_{\beta_k}(t), \sigma_{\beta_k}(x_1, x_2, \ldots, x_{n-1}) \rangle_{\beta_k} X_n.
\end{align*}
\]

Now, by our definition and Fulton’s Theorem 8.2 again,

\[
\begin{align*}
-\langle \Omega_{\beta_k}(t), \sigma_{\beta_k}(x_1, x_2, \ldots, x_{n-1}) \rangle_{\beta_k}^2 &= -\Omega_{\beta_k}(t) \cdot \sigma_{\beta_k}(x_1(t), \ldots, x_{n-1}(t)),
\end{align*}
\]

for distinct points \( t, u \in \mathbb{P}^1 \). (The same proof as in the Moving Lemma shows that the codimension of \( \Omega_{\beta_k}(t) \) in \( \mathcal{H}_{\mathbb{Q}[\mathbb{P}^1, \mathcal{F}(V^*)]} \) is equal to \( \ell(\beta_k) \).

The description of \( \Omega_{\beta_k}(t) \) in Remark 3 gives, for any \( w \in S_n \),

\[
\begin{align*}
\Omega_{\beta_k}(t) &= \Omega_{\beta_k}(t) - \Omega_{\beta_k}(t) \cdot j_s(G_{\beta_k}(t)) - \Omega_{\beta_k}(t) \cdot j_s(G_{\beta_k}(t)),
\end{align*}
\]

for distinct points \( t, u \in \mathbb{P}^1 \).
where the first product is computed in $CH^*(\mathbb{H}_Q(\mathbb{P}^1, F(V^*)))$ and the others are in $CH^*(\mathbb{H}_Q(\mathbb{P}^1, F(V^*)))$.

Indeed, $\overline{\Omega}_w(t) = \int_{D^p_q} D^{p,q}_{\beta_\ell}$ and $r_{\alpha_k-2(n-1)}(q,p) = r_{\beta_\ell}(q,p)$ for all $1 \leq p \leq n$, $1 \leq q \leq n-2$, while $r_{\alpha_k-2(n-1)}(1,n,p) = p$ for all $1 \leq p \leq n$.

Note that the Schubert polynomial of $\alpha_k-1(n-1)$ is the $(k-2)^{nd}$ elementary symmetric function in $x_1, \ldots, x_{n-2}$ (and 1 if $k = 2$).

Using (5) and Remark 3 again to compute the intersections supported on the boundary, we get

\begin{equation}
\overline{\Omega}_w(u) \cdot j^*(G_{\beta_\ell}(t)) = -\overline{\Omega}_w(u) \cdot j^*(G_{\alpha_k(n)}(t)) + \overline{\Omega}_w(u) \cdot (x_1(t) + \cdots + x_{n-1}(t)) \cdot j^*(G_{\alpha_k-1(n)}(t))
\end{equation}

This is easy once we observe that $x_1(t) + \cdots + x_{n-1}(t) = \overline{\Omega}_{s_{n-1}}(t)$ and

$$j_{\pi_{n-1}^{-1}}(\overline{\Omega}_{s_{n-1}}(t)) = \mathbb{P}^1 \times \overline{\Omega}_{s_{n-1}}(t) \cup \{t\} \times \overline{\Omega}_{s_{n-1}}(t).$$

Combining (3), (4), (6), (7), (8) and (9), the formula in Theorem 4 (i) follows by induction on $n$.

The proof of (ii) is standard (see [Bat], [Be2], [MP], [ST]).

Note added: After this work was completed, B. Kim informed us that he obtained independently Theorems 1 and 2.

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