Lines on Fermat surfaces

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Abstract. We prove that the Néron-Severi groups of several complex Fermat surfaces are integrally generated by lines. Specifically, we obtain these new results for degrees 5, 7, 11, 13. The proof uses reduction modulo a supersingular prime. The technique is developed in detail. It can be applied to other surfaces and varieties as well.

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1 Introduction

Fermat surfaces have been a classical object of study in geometry and arithmetic. Here we consider the smooth projective surface of degree \( m \in \mathbb{N} \)

\[
S : \{ x_0^m + x_1^m + x_2^m + x_3^m = 0 \} \subset \mathbb{P}^3.
\]

This paper is concerned with the Néron-Severi group \( \text{NS}(S) \) of \( S \) over the complex numbers, consisting of divisors up to algebraic equivalence.

In general, it is hard to compute the Néron-Severi group. The cohomology of Fermat surfaces, however, admits a decomposition into eigenspaces with character for the induced action of an abelian subgroup of the automorphism group. Combinatorial data thus give the Picard number \( \rho \), the rank of NS. In fact, for any degree \( m \), a rational basis of \( \text{NS}(S) \) was determined in [1].

This rational basis involves some particularly prominent divisors on \( S \), namely the \( 3m^2 \) obvious lines. The lines generate \( \text{NS}(S) \) rationally if and only if \( m \leq 4 \) or \( (m, 6) = 1 \). In Prop. 3 we will provide an explicit rational basis of lines.

A natural question now is in which of the above cases the lines generate the full Néron-Severi group. Integral generation is known to hold true for \( m \leq 4 \), as we will review in section 2. Our main result is the following:

**Theorem 1**

Let \( m < 17 \). Then the Néron-Severi group of the complex Fermat surface \( S \) of degree \( m \) is integrally generated by lines if and only if \( m \leq 4 \) or \( (m, 6) = 1 \).

We shall use supersingular reduction to prove the statement for the Fermat surfaces of degree 5, 7, 11, and 13. The technique will be introduced in sect. 4. Separately for each degree, the proof of Thm. 1 will be given in sections 7–10. In section 11 we comment on other possible techniques and applications.
2 Rational generation of NS

The cohomology of Fermat varieties admits a decomposition into eigenspaces with respect to an abelian subgroup of the automorphism group. According to work by Katz and Ogus, it splits into one-dimensional eigenspaces with character for the induced action of $\mu^3_m$. The latter corresponds to coordinate multiplication by $m$-th roots of unity (see [11]). It is well known which eigenspaces are algebraic, and in the surface case, even which correspond to lines.

**Theorem 2 (Shioda [14])**

The $\mathbb{Q}$-vector space $NS(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by divisor classes of lines if and only if $m \leq 4$ or $m$ is coprime to 6.

The proof of the theorem will be recalled at the end of this section, since it involves concepts that we will need to find an explicit rational basis of $NS(S)$.

This paper investigates the problem whether, for the appropriate degrees, lines generate $NS(S)$ fully or only up to finite index. We now review the current knowledge about this problem.

For $m \leq 3$, the generation problem has a positive answer. These Fermat surfaces are rational. For $m = 1, 2$, the statement is almost trivial, corresponding to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Any smooth projective cubic complex surface contains 27 lines. Their configuration has been classically studied in great detail. In fact, any smooth cubic surface is isomorphic to the projective plane $\mathbb{P}^2$ blown up in six distinct points.

For $m = 4$, the answer was conjectured to be positive, but unknown until Mizukami in 1975 proved the affirmative [9]. We will review the history of the original proof and provide an alternative proof using our technique of supersingular reduction in section 6. Our Thm. 1 provides the first answer to the question for Fermat surfaces of general type.

For later use, we shall now sketch the proof of Thm. 2 in [14]. First we fix notation for the 3 $m^2$ lines on $S$. Throughout the paper, we denote by $\mu_n$ the group of $n$-th roots of unity over a given field. Let $\omega \in \mu_{2m}$ such that $\omega^m = -1$. Then for any $\eta, \zeta \in \mu_m$ we have the lines

\[
\ell_1(\eta, \zeta) = \{[\lambda, \omega \eta \lambda, \mu, \omega \zeta \mu]; \ [\lambda, \mu] \in \mathbb{P}^1\},
\]

\[
\ell_2(\eta, \zeta) = \{[\lambda, \mu, \eta \omega \lambda, \omega \zeta \mu]; \ [\lambda, \mu] \in \mathbb{P}^1\},
\]

\[
\ell_3(\eta, \zeta) = \{[\lambda, \mu, \omega \zeta \mu, \omega \eta \lambda]; \ [\lambda, \mu] \in \mathbb{P}^1\}.
\]

The triple product of $m$-th roots of unity $\mu^3_m$ acts on $S$ by coordinate multiplication:

\[g = (\zeta_1, \zeta_2, \zeta_3) \in \mu^3_m: \ [x_0, x_1, x_2, x_3] \mapsto [x_0, \zeta_1 x_1, \zeta_2 x_2, \zeta_3 x_3]. \quad (1)\]

We shall consider the eigenspaces of $H^2(S)$ for the induced action of $\mu^3_m$ with character $\alpha$ in the character group

\[ \mathfrak{A}_m := \{\alpha = (a_0, a_1, a_2, a_3) \in (\mathbb{Z}/m\mathbb{Z})^4 \mid a_i \neq 0 \pmod{m}, \sum_{i=0}^3 a_i \equiv 0 \pmod{m}\}. \]

For $\alpha \in \mathfrak{A}_m$, the corresponding eigenspace $V(\alpha) \subset H^2(S)$ with character $\alpha$ is defined by the condition

\[ g^*|V(\alpha) = \alpha(g) = \zeta_1^{a_1} \zeta_2^{a_2} \zeta_3^{a_3} \quad \forall \ g = (\zeta_1, \zeta_2, \zeta_3) \in \mu^3_m. \]
By results of Katz [7, §6] and Ogus [11, §3] (which hold more generally true for Fermat varieties of any dimension), each $V(\alpha)$ is one-dimensional, and
\[
H^2(S) = V_0 \oplus \bigoplus_{\alpha \in \mathfrak{A}_m} V(\alpha).
\]
Here $V_0$ corresponds to the trivial character and is spanned by the hyperplane section.

It is easy to decide whether $V(\alpha)$ is algebraic: Let $(\mathbb{Z}/m\mathbb{Z})^*$ operate on $\mathfrak{A}_m$ coordinatewise by multiplication. Let $\mathfrak{B}_m \subset \mathfrak{A}_m$ consist of all those $\alpha \in \mathfrak{A}_m$ such that the $(\mathbb{Z}/m\mathbb{Z})^*$-orbit of $\alpha$ consists exclusively of elements $(b_0, \ldots, b_3)$ written in canonical representatives $0 < b_i < m$ such that
\[
\sum_{i=0}^3 b_i = 2m.
\]
Then $V(\alpha)$ is algebraic if and only if $\alpha \in \mathfrak{B}_m$. Even the span of the lines is known: In $\text{NS}(S) \otimes \mathbb{C}$, this consists of $V_0$ and $V(\alpha)$ for all $\alpha \in \mathfrak{B}_m$ which are decomposable, i.e. there is some index $j > 0$ such that $a_0 + a_j \equiv 0 \pmod{m}$. Let $\mathfrak{D}_m \subseteq \mathfrak{B}_m$ denote the subset of decomposable elements. Then one easily computes
\[
\# \mathfrak{D}_m = 3(m-1)(m-2) + \begin{cases} 0, & \text{if } m \text{ is odd,} \\ 1, & \text{if } m \text{ is even.} \end{cases} \tag{2}
\]
We now recall why the lines generate the eigenspaces $V(\alpha)$ for the decomposable elements $\alpha \in \mathfrak{D}_m$. This will be achieved by establishing a $\mathbb{C}$-linear combination of lines which is a non-zero eigendivisor for the character $\alpha \in \mathfrak{D}_m$.

More specifically, let $\mathfrak{D}_m^j$ denote the subset of decomposable elements in $\mathfrak{D}_m$ such that $a_0 + a_j \equiv 0 \pmod{m}$. Note that $\mathfrak{D}_m^j \cap \mathfrak{D}_m^k \neq \emptyset$ – a fact that will be crucial to our later analysis of an explicit basis of lines. Depending on $j$, we give an eigendivisor with character for each $\alpha \in \mathfrak{D}_m^j$:
\[
\begin{align*}
\alpha \in \mathfrak{D}_m^1 : & \quad w_1(\alpha) = \sum_{\zeta, \eta} \zeta a_1 \eta a_2 \ell_1(\zeta, \eta), \\
\alpha \in \mathfrak{D}_m^2 : & \quad w_2(\alpha) = \sum_{\zeta, \eta} \zeta a_2 \eta a_3 \ell_2(\zeta, \eta), \\
\alpha \in \mathfrak{D}_m^3 : & \quad w_3(\alpha) = \sum_{\zeta, \eta} \zeta a_3 \eta a_3 \ell_3(\zeta, \eta).
\end{align*}
\]
By construction, almost all of these eigendivisors are orthogonal (independent of the index):
\[
w(\alpha).H = 0, \quad w(\alpha),w(\beta) = 0 \quad \text{if } \alpha \neq -\beta. \tag{3}
\]
To see that the lines form a basis of $\text{NS}(S) \otimes \mathbb{C}$, it remains to be shown that $w(\alpha) \neq 0$. This follows from the intersection number
\[
w_j(\alpha),w_j(-\alpha) = -m^3. \tag{4}
\]
which is easily computed thanks to the following intersection behaviour:
\[
\ell_i(\zeta, \eta) \ell_j(\zeta', \eta') \neq 0 \Leftrightarrow \begin{cases} 
\zeta = \zeta' \text{ or } \eta = \eta', & i = j, \\
\zeta \eta' = \zeta' \eta, & (i, j) = (1, 2), \\
\zeta' = \omega^2 \eta \zeta \eta', & (i, j) = (1, 3), \\
\zeta \eta = \zeta' \eta', & (i, j) = (2, 3).
\end{cases} \tag{5}
\]
3 Rational basis of lines

Hence the lines span all the eigenspaces for decomposable elements. Clearly, also $H$ and thus $V_0$ can be expressed by lines (cf. (6), (7)). Denote the span of the lines by $L$. Then

$$\text{rank}(L) = 1 + \# \mathcal{D}_m.$$ 

On the other hand, it can be seen that $\mathcal{D}_m = \mathfrak{B}_m \iff m \leq 4$ or $(m, 6) = 1$ (cf. [14, Thm. 6]). This completes the proof that lines generate $\text{NS}(S)$ rationally (or after tensoring with $\mathbb{C}$ as done here) exactly in the cases of Thm. 2.

3 Rational basis of lines

In this section, we will work out an explicit rational basis of the lattice $L$ generated by the lines in $\text{NS}(S)$. For this, we fix another notation for the lines. Since we are concerned with odd $m$, we can set $\omega = -1$. Then we fix a primitive $m$-th root of unity $\gamma$. We introduce the short-hand notation $\ell_j(\gamma^k, \gamma^l) = \ell_j(k, l)$

When the index $j$ is clear or if the claim is independent of the index, we will sometimes omit the index completely for brevity.

**Proposition 3 (Rational basis for $m$ coprime to 6)**

If $(m, 6) = 1$, then the following lines form a rational basis of $\text{NS}(S)$:

$$\mathcal{B} = \{\ell_j(k, l); \ j = 1, 2, 3, \ 0 \leq k < m - 1, \ 0 < l < m - 1\} \cup \{\ell_1(m - 1, 1)\}$$

**Proof:** We shall use relations between lines and the hyperplane class $H$. Clearly

$$H = \sum_{\zeta} \ell(\zeta, \eta) = \sum_{\eta} \ell(\zeta, \eta)$$

for any fixed $\eta$ resp. $\zeta$ and independent of the index. Taking the sum of the lines $\ell_1(\cdot, 1)$, we see that $H$ is in the span of $\mathcal{B}$. In consequence, all $\ell_i(m - 1, l)$ for $1 < l < m - 1$ can be expressed by $\mathcal{B}$ as well. It remains to write the lines $\ell(\cdot, 0), \ell(\cdot, m - 1)$ in terms of the previous lines.

A second group of relations is derived for all those $\alpha \in \mathcal{D}_m^i \cap \mathcal{D}_m^j$ for some $i \neq j$. Since $V(\alpha)$ is always one-dimensional, we have

$$V(\alpha) = \mathbb{C} w_i(\alpha) = \mathbb{C} w_j(\alpha),$$

so the two eigendivisors are multiples of each other. Recall that each eigendivisor intersects its complex conjugate $w(-\alpha)$ with intersection multiplicity $-m^3$.

**Claim:** Let $i \neq j$ and $\alpha \in \mathcal{D}_m^i \cap \mathcal{D}_m^j$. Then

$$w_i(\alpha) = -w_j(\alpha).$$

Recall the orthogonality for eigendivisors with character from (3). To see the claim, it thus suffices to compute the intersection number

$$w_i(\alpha) . w_j(-\alpha) = m^3.$$ 

This is easily verified thanks to the intersection behaviour of the lines in (5).
The coefficients of the lines in the relations (3) are still fairly complicated. To obtain a more convenient shape of the coefficients of the lines, we shall now simplify the above relations by multiplying with fixed powers of a varying root \( \varepsilon \in \mu_m \).

Here we use that fixing \((i, j)\) with \(i \neq j\) gives a unique map

\[
\alpha_{i,j} : \mathbb{Z}/m\mathbb{Z} - \{0\} \to \mathcal{D}^m_\ell \cap \mathcal{D}^m_j,
\]

by setting \(a_0 = 0\). Then \(a_i = a_j = -a\) and \(a_k = o\) with \(\{i, j, k\} = \{1, 2, 3\}\). For any \(\varepsilon \in \mu_m\) and \((i, j)\) with \(i \neq j\), we then consider the relations of divisors obtained from (3)

\[
\sum_{o \in \mathbb{Z}/m\mathbb{Z} - \{0\}} \varepsilon^o w_i(\alpha_{i,j}(o)) = -\sum_{o \in \mathbb{Z}/m\mathbb{Z} - \{0\}} \varepsilon^o w_j(\alpha_{i,j}(o)).
\]

Both sums simplify greatly. For instance,

\[
\sum_{o \in \mathbb{Z}/m\mathbb{Z} - \{0\}} \varepsilon^o \ell_1(\alpha_{1,2}(o)) = \sum_{\zeta, \eta} \sum_o \left(\frac{\varepsilon \eta}{\zeta}\right)^o \ell_1(\zeta, \eta)
\]

\[
= (m - 1) \sum_{\zeta = \varepsilon \eta} \ell_1(\zeta, \eta) - \sum_{\zeta \neq \varepsilon \eta} \ell_1(\zeta, \eta)
\]

\[
= m \left( \sum_{\zeta = \varepsilon \eta} \ell_1(\zeta, \eta) - H \right).
\]

Analogous sums for the other indices result in the following 3 \(m\) relations (depending on the choice of \(\varepsilon \in \mu_m\)):

\[
\sum_{\zeta = \varepsilon \eta} \ell_1(\zeta, \eta) = -\sum_{\zeta = \varepsilon \eta} \ell_2(\zeta, \eta) \quad (9)
\]

\[
\sum_{\zeta \eta = \varepsilon} \ell_1(\zeta, \eta) = -\sum_{\zeta = \varepsilon \eta} \ell_3(\zeta, \eta) \quad (10)
\]

\[
\sum_{\zeta \eta = \varepsilon} \ell_2(\zeta, \eta) = -\sum_{\zeta \eta = \varepsilon} \ell_3(\zeta, \eta) \quad (11)
\]

Prop. 3 states that the lines \(\ell_j(\cdot, 0), \ell_j(\cdot, m - 1)\) can be expressed purely in terms of the remaining lines. To prove this, we work with the \(6m \times 6m\) matrix \(M\) whose entries are the coefficients of the "missing" lines in the relations (3) and (9)–(11).

The matrix \(M\) is defined as follows:

- columns: \(\ell_1(0, 0), \ldots, \ell_1(m - 1, 0), \ell_1(0, m - 1), \ldots, \ell_1(m - 1, m - 1), \ell_2(0, 0), \ldots, \ell_3(m - 1, m - 1)\)
- rows: \(\eta = \gamma^l, l = 0, \ldots, m - 1\) and \(j = 1, 2, 3\)
- \(\varepsilon = \gamma^i, i = 0, \ldots, m - 1\)

By the relations, all entries of \(M\) are either 0 or 1. It will be convenient to write \(M\) as a block matrix whose entries are 36 matrices of type \(m \times m\). In fact, the blocks arising from relation (3) are just the identity Matrix \(I\). For the other relations, we need two permutation matrices of order \(m\):

\[
D = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
& & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad B = D'.
\]
Then $M$ is given as follows:

$$M = \begin{pmatrix}
I & I & 0 & 0 & 0 \\
0 & 0 & I & I & 0 \\
0 & 0 & 0 & 0 & I \\
I & B & I & B & 0 \\
I & D & 0 & 0 & I \\
0 & 0 & I & D & I
\end{pmatrix}$$

Then we want to solve the system of linear equations

$$M \text{ (fixed lines)} = \text{(expressions in remaining lines given by relations)}.$$ 

The above matrix $M$ is not invertible, but we are only looking for a solution in $\text{NS}(S)$. Hence we can still modify $M$ using relation (7). In other words, we can add constant rows (i.e. with constant entries) to the building blocks of $M$ at our leisure in order to derive an invertible matrix. Of course, this changes the expression on the right-hand side by adding a multiple of $H$, but we will not need to consider this expression at all.

Elementary linear algebra simplifies the problem of invertibility greatly:

$$\begin{pmatrix}
I & I & 0 & 0 & 0 \\
0 & 0 & I & I & 0 \\
0 & 0 & 0 & 0 & I \\
I & B & I & B & 0 \\
I & D & 0 & 0 & I \\
0 & 0 & I & D & I
\end{pmatrix} \rightarrow
to$$

$$\begin{pmatrix}
I & I & 0 & 0 & 0 \\
0 & 0 & I & I & 0 \\
0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & I
\end{pmatrix} \rightarrow$$

$$\begin{pmatrix}
B - I & 0 & 0 \\
0 & B - I & 0 \\
0 & 0 & B - I \\
0 & 0 & 0 & B - I
\end{pmatrix} \rightarrow$$

Hence to solve the system of equations, it suffices to modify the following matrices by adding constant rows such that they become invertible:

$$B - I, \quad D - I, \quad B + D.$$ 

**Lemma 4**

(i) The matrices $B - I, D - I$ have rank $m - 1$. No non-zero constant vector is contained in the span of the rows.

(ii) The matrix $B + D$ is invertible over $\mathbb{Q}$.

**Proof:** The rank of $B - I, D - I$ can be computed through the characteristic polynomial:

$$\chi_{D - I}(\lambda) = \det(\lambda I - (D - I)) = \det((\lambda + 1)I - D) = \chi_D(\lambda + 1).$$

Here $D$ describes a cyclic permutation of order exactly $m$, so

$$\chi_D(\lambda) = \prod_{\varepsilon \in \mathbb{Z}_m} (\lambda - \varepsilon).$$

The claim follows. The same argument applies to $B - I$.

Assume that a constant vector $c = (c, \ldots, c)$ is written as a linear combination of the rows of $D - I$:

$$u \cdot (D - I) = c, \quad u = (u_1, \ldots, u_m).$$
The second column gives $u_2 = u_1 - c$ etc. Ultimately we deduce $u_m = u_1 - (m-1)c$. Thus $u_1 = 0$ and therefore $u = 0$. □

As for $B + D$, the equation $u \cdot (B + D) = 0$ subsequently gives $u_3 = -u_1, u_5 = -u_3 = u_1, \ldots, u_{m-1} = u_1$. Hence the last column returns $0 = u_1 + u_{m-1} = 2u_1$. Thus $u_1 = 0$ and therefore $u = 0$. □

As a consequence of the lemma, we can add any non-zero constant vector to any row of $B - I$ and $D - I$ to obtain an invertible matrix. The accordingly modified matrix $M$ becomes invertible over $\mathbb{Q}$. Thus we can express all lines missing from $B$ rationally through the lines in $B$. Since lines generate $\text{NS}(S)$ rationally by Thm. 2, this completes the proof of Proposition 3.

In fact, the above proof does not require that $(m, 6) = 1$, but only that $m$ is odd. Essentially the oddness is required because otherwise we cannot choose $\omega = -1$. For general $\omega$, the relations for $\alpha \in \mathcal{D}_m \cap \mathcal{D}_m^1$ change to

$$w_1(\alpha) = -\omega^{2\alpha} w_3(\alpha).$$

Summing up as for odd $m$, we obtain

$$\sum_{\zeta \eta = \varepsilon} \ell_1(\zeta, \eta) = - \sum_{\omega^2 \zeta \varepsilon = \eta} \ell_3(\zeta, \eta).$$

Hence the matrix $M$ takes a different shape.

**Lemma 5**

Let $m$ be odd. Then the lines in $B$ generate the span of all lines $L$ rationally.

Actually we can get fairly close to integral generation. For instance, by the proof of Lemma 4, the matrix $B + D$ is invertible mod $p \neq 2$. In fact, subtracting 1 from the top row, we obtain a matrix that is invertible modulo any prime, hence over $\mathbb{Z}$.

The matrices $B - I, D - I$, however, can only become invertible over $\mathbb{Z}[\frac{1}{m}]$, even after adding constant vectors to their rows. This is because the proof of Lemma 4 relies on the implication $mc = 0 \Rightarrow c = 0$. If $p | m$, any constant vector thus lies in the span of rows mod $p$.

Note that integral generation might still hold, since the expression on the right-hand side might be divisible in $\text{NS}(S)$. In the cases of this paper with $(m, 6) = 1$, the integral generation will be checked as part of the proof of Thm. 2.

For some small odd degrees $m$, we calculated the determinant of the intersection form of the lines in $B$. In each case, the determinant turned out to be a power of $m$:

$$\det(\ell, \ell')_{\ell, \ell' \in B} = m^{e(m)}.$$  

[This determinant will always divide $m^r$ for $r = 3 \# \mathcal{D}_m + 1$ because the latter is the determinant of the intersection form of the $\mathbb{Z}[\zeta_m]$-basis $\{H, w_\alpha; \alpha \in \mathcal{D}_m\}$ of $\text{NS}(S) \otimes \mathbb{Z}[\zeta_m]$ by [3] and [4].] The exponents $e(m)$ show an interesting pattern which we document in the next table. For prime $m$, the conjectural exponent from [15] is attained by the given rational basis; for composite $m$, however, different exponents show up:

| $m$         | $e(m)$          |
|-------------|-----------------|
| $3 \leq m \leq 25$ squarefree, odd | $3(m-3)^2$         |
| $9, 25$     | $6(m-3)^2$      |
4 Supersingular reduction technique

Consider the reductions of the complex Fermat surface $S$ mod $p$. Denote the resulting surface by $S_p$. Then $S_p$ is smooth for any $p \nmid m$. For all these $p$, reduction induces a specialisation embedding

$$NS(S) \hookrightarrow NS(S_p).$$

We call a surface $X$ supersingular if its Picard number is maximal: $\rho(X) = b_2(X)$. For Fermat surfaces, we have the following result of Katsura and Shioda:

**Theorem 6 (Katsura-Shioda [6])**

The reduction $S_p$ is supersingular if and only if there is some $\nu \in \mathbb{N}$ such that

$$p^\nu \equiv -1 \mod m.$$

One advantage of working with supersingular surfaces is that we have good knowledge about the discriminant of their Néron-Severi groups. The following result is a generalisation of Artin’s classification of supersingular K3 surfaces [2].

**Theorem 7 (Ekedahl [4], Schütt–Schweizer [13])**

Let $X$ be a smooth projective surface over a finite field $k$ of characteristic $p$. Assume that $X$ is supersingular. Let $N_0 = NS(X)/NS(X)_{tor}$. Then

$$|\text{disc}(N_0)| = p^{2\sigma} \quad (\sigma \in \mathbb{N}_0).$$

The proof in [13] uses exactly the same techniques as Artin’s original paper, mainly the Artin-Tate conjecture. The proof in [4] is based on cohomological results by Illusie and even allows to bound the (Artin) invariant $\sigma$.

We now explain the method by which we will prove Thm. 1. For this we recall the second betti number of $S$:

$$b = b_2(S) = (m - 1)(m^2 - 3m + 3) + 1.$$

We shall also use the Lefschetz number $\lambda(S) = b_2(S) - \rho(S)$.

**Supersingular reduction technique**

Fix the degree $m$. Let $p$ be a prime of supersingular reduction for $S$.

1. Compute a basis of $NS(S)_{\mathbb{Q}}$ consisting of lines $\ell_j$.

2. Let $N = \langle \ell_j; j = 1, \ldots, \rho \rangle \subseteq NS(S)$. Compute $\text{disc}(N)$ in terms of the Gram matrix of the intersection numbers of the lines. Then $\text{disc}(N) = v^2 \text{disc}(NS(S))$.

3. Complement the reductions of the lines $\ell_j(j = 1, \ldots, \rho)$ by $\lambda(S)$ divisor classes $d_k$ on the supersingular reduction $S_p$ for a basis of $NS(S_p)_{\mathbb{Q}}$.

4. Let $N_p = \langle \ell_j, d_k; j = 1, \ldots, \rho; k = 1, \ldots, b - \rho \rangle \subseteq NS(S_p)$. Compute $\text{disc}(N_p)$.

If $(m, 6) = 1$, then we will work with the rational basis $\mathcal{B}$ from Prop. 8 in step 1. At the end of the previous section, we computed the discriminants of the lattice $N$ generated by these lines for several $m$. Recall that this discriminant was always a power of $m$ (and in general it is a divisor of some power of $m$).

**Criterion 8**

If $N$ and $N_p$ have coprime discriminants, then $N = NS(S)$ (i.e. $\nu = 1$).
Proof: Otherwise $\nu > 1$. In consequence, the lattice generated by the image of $\text{NS}(S)$ in $\text{NS}(S_p)$ under the embedding (12), and the divisor classes $d_k$ would have discriminant $\frac{1}{\nu} \text{disc}(N_p)$. By assumption, this number is not in $\mathbb{Z}$, contradiction. \qed

In sections 6–10, we will apply the supersingular reduction technique to the Fermat surfaces of degree 4, 5, 7, 11 and 13. For a generalisation of Crit. 8, the reader is referred to section 11. There we also compare the supersingular reduction technique to a method by van Luijk to compute the Picard number of projective surfaces.

5 Additional lines mod $p$

For later use, we investigate additional lines on Fermat surfaces of several degrees and characteristics. Throughout we let $q = p^r$ and $m = q + 1$ and perform our calculations over $\mathbb{F}_q$. In this situation, Tate and Thompson realised that the unitary group over $\mathbb{F}_q^2$ acts irreducibly on the primitive part of $H^2(S_p)$ (cf. [16]). This provided the first proof for the if-part in Theorem 6. In consequence, the images of any line on $S_p$ under the action of the unitary group generate $\text{NS}(S_p)$ rationally together with the hyperplane section.

In the sequel, we shall exhibit very specific lines for different choices of $m$. In each case, we shall only give one line. For simplicity, we will content ourselves to applying the action of $\mu_3^m$ to this line.

5.1 General $m$

Let $\alpha \in \mathbb{F}_q^*$ with $\alpha^2 \neq -1$. Then consider the solutions $\beta \in \mathbb{F}_q^*$ of

$$\beta^2 = 1 + \alpha^2. \tag{13}$$

Since $m - 2 = q - 1$, we have $\alpha^{m-2} = 1$. Thus one can easily check that

$$\beta^{2(m-2)} = 1.$$

It follows that there are several $\alpha \in \mathbb{F}_q^*$ such that there is a solution $\beta$ of (13) with

$$\beta^{m-2} = -1.$$

For each such pair $(\alpha, \beta)$, we have the following line on $S_p$:

$$\ell_p = \{[\lambda, \alpha \lambda + \beta \mu, \beta \lambda + \alpha \mu, \mu]; \ [\lambda, \mu] \in \mathbb{P}^1\}.$$

For many $m = q + 1$, we can find simpler lines on $S_p$. We consider two cases:

5.2 $m \equiv 2 \mod 3$

If $m \equiv 2 \mod 3$, i.e. $q \equiv 1 \mod 3$, then let $\alpha \in \mathbb{F}_q$ be a primitive third root of unity: $\alpha^2 + \alpha + 1 = 0$. Then $S_p$ contains the following line:

$$\ell_p = \{[\lambda, \alpha(\lambda + \alpha \mu), \alpha(\lambda - \mu), \mu]; \ [\lambda, \mu] \in \mathbb{P}^1\}.$$
5.3 $p = 3$

Let $p = 3$. For any $q = p^r$ and $m = q + 1$, $S_p$ contains the following line:

$$\ell_p = \{[\lambda, (\lambda + \mu), (\lambda - \mu), \mu]; \ [\lambda, \mu] \in \mathbb{P}^1\}.$$  

In the sequel, we shall always fix one line $\ell_p$ as above. Then we let the subgroup $\mu_m^3$ of $\text{Aut}(S)$ act on $\ell_p$. Here we normalise the action as

$$g = (\zeta, \eta, \xi) \in \mu_m^3 : \ [x_0, x_1, x_2, x_3] \mapsto [\zeta x_0, \eta x_1, \xi x_2, x_3].$$  

As before, we denote the resulting $m^3$ lines by $\ell_p(\zeta, \eta, \xi)$ or $\ell_p(j, k, l)$.

To define the latter lines, we shall always consider the reduction of the primitive root of unity $\gamma \in \mu_m$ that was used to enumerate the lines $\ell_j(k, l)$ on $S$ in characteristic zero.

Remark 9

In the supersingular case, $V(\alpha) \subset H^2(S_p)$ is algebraic for any character $\alpha \in \mathfrak{A}_m$. Given a line $\ell_p$ as above, we can mimic the construction from section 2 to produce an eigendivisor with character $\alpha = (a_0, a_1, a_2, a_3)$:

$$w_p(\alpha) = \sum_{\zeta, \eta, \xi} \zeta^{a_0} \eta^{a_1} \xi^{a_2} \ell_p(\zeta, \eta, \xi).$$

However, it is non-trivial to decide whether $w_p(\alpha)$ is non-zero in $\text{NS}(S_p)$ (cf. Rem. 14).

6 The Fermat quartic revisited

In this section, we let $m = 4$. Thus $S$ is a singular K3 surface (in the sense that $\rho(S) = 20$, the maximum possible over $\mathbb{C}$). It was shown by Pjateckii-Šapiro and Šafarevič [11] that $\text{NS}(S)$ has discriminant $d = -16$ or $-64$. The latter is the case if the Néron-Severi group is generated by lines. Depending on a claim by Demjanenko, Pjateckii-Šapiro and Šafarevič deduced $d = -64$. However, Demjanenko’s argument contained a mistake. A correction was given by Cassels in 1978 [3].

In the meantime, Mizukami had investigated the following family of K3 surfaces:

$$X_\lambda : \ \{x^4 + y^4 + z^4 + w^4 = 2 \lambda (x^2 y^2 + z^2 w^2)\} \subset \mathbb{P}^3.$$  

The following result was part of his Master’s thesis in 1975 [9]:

Proposition 10 (Mizukami)

Let $X_\lambda$ as above. Then $\rho(X_\lambda) \geq 19$, and

$$\text{Disc}(\text{NS}(X_\lambda)) = \begin{cases} -64, & \text{if } \lambda = 0, \\ 128, & \text{if } \rho(X_\lambda) = 19. \end{cases}$$

For the Fermat quartic, this result implied $d = -64$. Thus it follows that lines generate $\text{NS}(S)$ integrally (Prop. 11). An alternative proof can be based on another result about certain Kummer surfaces by Inose [5].

Here we present an alternative argument using the supersingular reduction technique from sect. 4 at the prime $p = 3$. (It can be seen from Thm. 6 that $p$ is a supersingular prime if and only if $p \equiv 3 \mod 4$.)
1. A rational basis $\mathcal{B}'$ of $\NS(S)$ can be obtained from $\mathcal{B}$ by switching $l \mapsto l - 1$ and adding $\ell_2(0, m - 2)$:

$$\mathcal{B}' = \{ \ell_j(k, l); \ell_j(k, l + 1) \in \mathcal{B} \} \cup \{ \ell_2(0, m - 2) \}.$$ 

2. Let $N = \langle \ell; \ell \in \mathcal{B}' \rangle$. Then $\text{discr}(N) = -64$.

3. On the supersingular reduction $S_3$, we have the additional line

$$\ell_3 = \{ [\lambda, (\lambda + \mu), (\lambda - \mu), \mu]; [\lambda, \mu] \in \mathbb{P}^1 \}.$$ 

from section [5.2] Recall $\gamma$, the fixed square root of $-1$. Let

$$\ell'_3 = \{ [\lambda, \gamma(\lambda + \mu), (\lambda - \mu), \mu]; [\lambda, \mu] \in \mathbb{P}^1 \}.$$ 

Then we compute that the $\ell \in \mathcal{B}'$ together with $\ell_3, \ell'_3$ constitute a rational basis $\mathcal{B}_3$ of $\NS(S_3)$:

4. Let $N_3 = \langle \ell; \ell \in \mathcal{B}_3 \rangle$. Then $\text{discr}(N_3) = -9$.

By Crit. [8] we deduce that $N = \NS(S)$. In other words we have reproven the following

**Proposition 11 (Mizukami, Inose)**

*The complex Fermat quartic surface has Néron-Severi group generated by lines. Its discriminant is $-64$.*

The next result was first pointed out to the second author by Mizukami in the 1970's (unpublished report). Mizukami’s proof was based on the computation of the intersection matrix for a suitable collection of lines on $S_3$.

**Lemma 12 (Mizukami)**

*The reduction $S_3$ of the Fermat quartic mod 3 has Néron-Severi group generated by lines over $\mathbb{F}_9$.***

*Proof:* Since $S_3$ is a supersingular K3 surface, the exponent $\sigma$ from Thm. [6] is the Artin invariant of $S_3$. By Artin’s stratification [2], $\sigma \in \{1, \ldots, 10\}$. Since the sublattice $N_3$ of $\NS(S_3)$ has discriminant $-9$, we deduce $N_3 = \NS(S_3)$. 

## 7 Fermat quintic

In this section we shall prove Thm. [11] for the complex Fermat quintic surface $S$. Note that $\rho(S) = 37, b_2(S) = 53$. It follows from Thm. [8] that $p = 2$ is a supersingular prime. We now apply the supersingular reduction technique from sect. [4].

1. Take the rational basis $\mathcal{B}$ of $\NS(S)$ from Prop. [3].
2. Then $N = \langle \ell; \ell \in \mathcal{B} \rangle$ has discriminant $5^{12}$.

On the supersingular reduction $S_2$ mod 2, section [5.2] gives 125 additional lines $\ell_2(j, k, l)$ (plus their conjugates with respect to $\alpha \mapsto \alpha^2$).

We express these lines through one parameter $\nu = 1, \ldots, 125$ as $\ell_p(\nu)$ where

$$\nu = \nu(j, k, l) = 25j + 5k + l + 1.$$ 

3. Let $\mathcal{N} = \{ 32, 33, 34, 35, 36, 37, 38, 39, 44, 80, 81, 82, 83, 84, 93, 95 \}$ and $\mathcal{B}_2 = \{ \ell_p(\nu) ; \nu \in \mathcal{N} \}$. Then $\mathcal{B} \cup \mathcal{B}_2$ constitutes a rational basis of $\NS(S_2)$. 


4. Let $N_2 = < \ell; \ell \in \mathcal{B} \cup \mathcal{B}_2 >$. Then $\text{discr}(N_2) = 2^{16}$.

By Crit. 8, we deduce that $N = \text{NS}(S)$ with discriminant $5^{12}$. In other words we have proven Thm. 1 for the Fermat quintic surface.

By [4, p. 12], $\text{NS}(S_p)$ has discriminant $p^{16}$ for all primes $p \equiv 2, 3 \mod 5$. Hence we deduce

**Lemma 13**

The reduction $S_2$ of the Fermat quintic mod 2 has Néron-Severi group generated by lines over $\mathbb{F}_{16}$.

## 8 Fermat septic

The Fermat septic surface $S$ has $\rho(S) = 91$, $b_2(S) = 187$. In characteristic zero, we have

1. rational basis $\mathcal{B}$ of $\text{NS}(S)$ from Prop. 3

2. lattice $N = < \ell; \ell \in \mathcal{B} >$ of discriminant $7^{48}$.

Since section 5 only applies to $m = q + 1$, the Fermat septic $S$ does not admit any supersingular reduction with apparent additional lines. Instead we consider a suitable covering Fermat surface and push down the additional lines on a supersingular reduction.

Here we can work with the Fermat surface $\hat{S}$ of degree 14 and consider the reduction $\hat{S}_p$ mod $p = 13$. In order to define a line mod $p$, we fix a primitive root $\gamma \in \mu_7$ as a zero of $\gamma^2 + 5 \gamma + 1$. Let $\ell_p$ denote the line from 5.1 for $\alpha = 2, \beta = 3 (\gamma - 4)$. Denote the push-down to $S$ by $D_p$. The action of $\mu_3^2$ as in section 5 gives divisors $D_p(j, k, l)$. We compute the following rational basis of $\text{NS}(S_p)$:

$$\mathcal{B}_p = \{D_p(j, k, l); (j, k, l) \in I\}$$

where

$$\begin{align*}
I &= I_1 \cup I_2 \\
I_1 &= \{(j, k, l); 0 \leq j, k < m - 1, 0 < l < m - 1\} \\
I_2 &= \{(j, 0, 0); 0 \leq j < m - 1\} \cup \{(m - 1, m - 2, m - 2)\}.
\end{align*}$$

The discriminant of the intersection form of the divisors in $\mathcal{B}_p$ is $2^{38} 7^2 13^{48}$.

In order to combine the above divisors with the original lines from characteristic zero, we number them as follows:

$$\begin{align*}
I_1 \ni (j, k, l) &\mapsto \nu(j, k, l) = 1 + j + (m - 1)k + (m - 1)^2(l - 1), \\
I_2 \ni (j, k, l) &\mapsto \nu(j, k, l) = b_2(S) - (m - 1) + j.
\end{align*}$$

With this notation, we can refer to $D_p(\nu)$ for $1 \leq \nu \leq b_2(S)$. We then find a mixed basis using certain multiples of all $\nu$ in the range $1, \ldots, \lambda(S)$ modulo $b_2(S)$:

3. Let $\mathcal{N} = \{[78 \nu \mod b_2(S)]; 1 \leq \nu \leq \lambda(S)\}$ and $\mathcal{B}_p' = \{D_p(\nu); \nu \in \mathcal{N}\}$.

Then $\mathcal{B} \cup \mathcal{B}_p'$ constitutes a rational basis of $\text{NS}(S_p)$.

4. Let $N_p = < C; C \in \mathcal{B} \cup \mathcal{B}_p' >$. Then $\text{discr}(N_p) = 13^{40}$. 

By Crit. 8 we deduce that $N = \text{NS}(S)$ with discriminant $7^{48}$. Thus we have proven Thm. 1 for the Fermat septic surface.

By [4, p. 12], twice the geometric genus $p_g(S)$ gives a lower bound for the Artin invariant $\sigma_0(S)$ for all $p \equiv -1 \mod m$ ($m$ being the degree of the Fermat surface $S$). For $m = 7$ and $p = 13$, the latter condition is fulfilled, and $p_g(S) = 20$. Hence we deduce $N_p = \text{NS}(S_p)$ and $\text{disc}(\text{NS}(S_p)) = 13^{40}$. In particular, $\text{NS}(S_p)$ can be generated by divisors defined over $\mathbb{F}_{p^2}$.

**Remark 14**

The choice $\alpha = 1$ and $\beta = \sqrt{2}$ would yield another set of $m^3$ divisors on $S$. It is easily verified that the divisors from $B_p$, even combined with the original lines from $B$, only generate a sublattice of rank 133 inside $\text{NS}(S_p)$. This shows that non-trivial linear combinations as in Rem. 9 might return zero for particular choices of $\alpha, \beta$.

### 9 Fermat surface of degree 11

The Fermat surface $S$ of degree $m = 11$ has $\rho(S) = 271, b_2(S) = 911$. In characteristic zero, we have

1. rational basis $B$ of $\text{NS}(S)$ from Prop. 3
2. lattice $N = \langle \ell; \ell \in B \rangle$ of discriminant $11^{192}$.

Consider the supersingular reduction $S_p \mod p = 2$. In order to exhibit additional divisors on $S_p$, we consider the Fermat surface $\hat{S}$ of degree 33. The covering map $\hat{S} \to S$ has degree 27. By section 5 the reduction $\hat{S}_p$ admits many additional lines. These will be pushed down to $S_p$.

A primitive root $\gamma \in \mu_m$ is given as non-trivial zero of $\gamma^m - 1$. I.e. $\gamma \in \mathbb{F}_{p^{10}}$. Let $\ell_p$ denote the line from 5.1 for $\alpha = \gamma^8 + \gamma^7 + \gamma^6 + \gamma^5 + \gamma^4 + \gamma^3, \beta = \alpha + 1$.

Denote the push-down to $S$ by $D_p$. The action of $\mu_m^3$ as in section 3 gives divisors $D_p(j, k, l)$. We compute the same rational basis $B_p = B_p(m)$ of $\text{NS}(S_p)$ as in section 8. The lattice generated by the divisors in $B_p$ has discriminant

$$2^{1200} 3^2 11^2 23^{64} 43^{24} 67^8 131^{16} 197^4 307^8 331^8 463^{12} 593^8 3541^8.$$

With $m$ and $p$ replaced, we employ the same numbering of $D_p(\nu)$ for $1 \leq \nu \leq b_2(S)$. As before we determine a mixed basis by using appropriate multiples of all $\nu$ in the range $1, \ldots, \lambda(S)$ modulo $b_2(S)$:

3. Let $N = \{ [311 \nu \mod b_2(S)]; \ 1 \leq \nu \leq \lambda(S) \}$ and $B_p = \{ D_p(\nu) ; \nu \in N \}$. Then $B \cup B_p'$ constitutes a rational basis of $\text{NS}(S_p)$.
4. Let $N_p = \langle C; C \in B \cup B_p' \rangle$. Then $N_p$ has discriminant

$$2^{1204} 23^{48} 43^{16} 131^{16} 3063908564512744812.$$

By Crit. 9 we deduce that $N = \text{NS}(S)$ with discriminant $11^{192}$. This completes the proof of Thm. 1 for the Fermat surface of degree 11.


10 Fermat surface of degree 13

The Fermat surface $S$ of degree $m = 13$ has $\rho(S) = 397, b_2(S) = 1597$. In characteristic zero, we have

1. rational basis $\mathcal{B}$ of $\text{NS}(S)$ from Prop. 3

2. lattice $N = \langle \ell; \ell \in \mathcal{B} \rangle$ of discriminant $13^{300}$.

Consider the supersingular reduction $S_p \mod p = 5$. In order to derive additional divisors on $S_p$, we consider the Fermat surface $\tilde{S}$ of degree 26 which is a degree 8-covering of $S$. The reduction $\tilde{S}_p$ admits many additional lines by section 5.

Here, we fix a primitive root $\gamma \in \mu_m$ as a zero of $\gamma^4 + 2\gamma^3 + \gamma^2 + 2\gamma + 1$. Let $\ell_p$ denote the line from 5.1 for $\alpha = 2\gamma^3 + \gamma^2 + 2\gamma + 1$, $\beta = 2\gamma^3 + 3\gamma^2 + 1$.

Denote the push-down to $S$ by $D_p$. The action of $\mu_m^3$ as in section 5 gives divisors $D_p(j, k, l)$. We compute the same rational basis $\mathcal{B}_p = \mathcal{B}_p(m)$ of $\text{NS}(S_p)$ as in section 8 and 9. The discriminant of the intersection form of the divisors in $\mathcal{B}_p$ is

$2^{26} 3^{192} 5^{912} 13^2 53^{24} 79^{24} 103^{32} 181^8 233^8 313^8 677^{16} 883^4 2003^8 2729^8 3847^8$.

Employ the same numbering of $D_p(\nu)$ for $1 \leq \nu \leq b_2(S)$. Again we find a mixed basis using appropriate multiples of all $\nu$ in the range $1, \ldots, \lambda(S)$ modulo $b_2(S)$:

3. Let $\mathcal{N} = \{[5 \nu \mod b_2(S)]; 1 \leq \nu \leq \lambda(S)\}$ and $\mathcal{B}_p' = \{D_p(\nu); \nu \in \mathcal{N}\}$. Then $\mathcal{B} \cup \mathcal{B}_p'$ constitutes a rational basis of $\text{NS}(S_p)$.

4. Let $N_p = \langle C; C \in \mathcal{B} \cup \mathcal{B}_p' \rangle$. Then $N_p$ has discriminant

$3^{144} 5^{914} 7^2 17^2 53^{16} 103^{42} 677^{16} 5531^2 2500543^2 1044508180948322868415061^2$.

By Crit. 8, we deduce that $N = \text{NS}(S)$ with discriminant $13^{300}$. This completes the proof of Thm. 1 for the Fermat surface of degree 13.

11 Generalisations

This sections presents a generalisation of Crit. 8. In terms of the supersingular reduction technique from sect. 4 our formulation replaces $\text{NS}(S)$ by a lattice denoted by $M$.

Criterion 15

Given a commutative diagram of integral lattices

$$
\begin{align*}
N & \hookrightarrow M \\
\cap & \quad \cap \\
N' & \hookrightarrow M'
\end{align*}
$$

where the vertical inclusions have finite index, assume that the embedding $N \hookrightarrow M$ is primitive. Then $N = N'$, if no prime divides both $\text{discr}(N)$ and $\text{discr}(M)$ with multiplicity at least two, i.e.

$$
v_p(\text{discr}(N), \text{discr}(M)) \leq 1 \text{ for any prime } p.
$$
Proof: If \( N \neq N' \), then \( N \) has index \( \nu^2 > 1 \) in \( N' \). Let \( p|\nu \). By primitivity, the integral lattice \( N' + M \) has discriminant dividing \( \frac{1}{p^2} \cdot \text{disc}(M) \). By assumption \( p^2 \nmid \text{disc}(M) \). Hence the discriminant of \( N' + M \) is not an integer. This gives a contradiction, since \( N' + M \subseteq M' \) is an integral lattice.

In the context of the supersingular reduction technique, \( N \) is chosen as the sub-lattice of \( \text{NS}(S) \) generated by lines in characteristic zero. Then we consider the embedding of \( \text{NS}(S) \) into \( \text{NS}(S_p) \) at a supersingular prime \( p \). Complementing \( N \) by exactly \( \lfloor \text{rk}(\text{NS}(S_p)) - \text{rk}(N) \rfloor \) elements to give a finite index sublattice \( M \) of \( \text{NS}(S_p) \) guarantees the primitivity of the embedding \( N \hookrightarrow M \).

We note that Criterion 15 does not necessarily improve Crit. 8 when we want to apply the supersingular reduction technique to Fermat surfaces of degree \( m \) coprime to 6: By Thm. 7, the discriminant of the Néron-Severi group of a supersingular reduction is a square up to sign; on the other hand, conjecturally \( \text{disc}(\text{NS}(S)) = m^{2r} \). In this case, both criteria would be equivalent.

**Example 16**

We want to test Crit. 15 for the complex Fermat quartic surface and its supersingular reduction mod 3. We employ the notation from sect. 6 and consider the following two sublattices of \( \text{NS}(S_3) \)

\[
L_1 = \langle N, \ell_3 \rangle \quad L_2 = \langle N, \ell'_3 \rangle.
\]

Here \( \text{disc}(L_1) = 24 = \text{disc}(L_2) = 24 \), so Crit. 15 does not apply to \( L_1 \) or \( L_2 \).

In spirit the supersingular reduction technique is related to a method to compute the Picard number of a projective surface which was pioneered by van Luijk in [8]. Van Luijk proved that certain K3 surfaces have Picard number one by reducing modulo two different primes. From the Lefschetz fixed point formula, he would derive that the reductions had Picard number (at most) two. Then he would find divisors peculiar to the respective characteristic and compare the resulting discriminants of the Néron-Severi lattices. Once they did not match up to a square factor, it would follow that the original surface had Picard number one.

In contrast, we only consider one reduction of the initial Fermat surface. We choose a supersingular prime, so that we have good control of the discriminant of the Néron-Severi lattice of the reduction. Then we also determine some divisors peculiar to the chosen characteristic and compare discriminants of certain sublattices of the Néron-Severi lattices. For all Fermat surfaces in consideration, this method suffices to prove that the sublattice generated by lines is already the full Néron-Severi lattice (Thm. 1).

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