GIRY ALGEBRAS VIA THE SUPPORT OF A PROBABILITY MEASURE

KIRK STURTZ

Abstract. We show that the subcategory of standard measurable spaces consisting of the single object of the set of natural numbers with the hom set of monotonic functions (=countably affine maps between the natural numbers) is codense in the category of standard measurable space. This leads directly to the result that the category of Giry-algebras for standard measurable spaces is isomorphic to the category of super convex spaces because, as shown, the set of probability measures on the natural numbers is dense in the category of super convex spaces.

Keywords. Standard measurable spaces, super convex spaces, Giry algebras, categorically compact objects, compact dense subcategory, Giry monad, support of a probability measure.

Contents

1. The basic schema for determining an equivalent representation of $\mathcal{T}$-algebras 1
2. Applying the scheme to the Giry monad for standard measurable spaces 3
3. Super convex spaces
   3.1. Types of spaces, basic properties, and examples 8
   3.2. Limits and Colimits in $\text{SCvx}$ 12
   3.3. The compact objects $\mathbb{N}$ and $\Delta_{\mathbb{N}}$ 15
4. The dense functor $\iota: \Omega \rightarrow \text{SCvx}$ 20
5. The dense functor $\kappa: \Omega \rightarrow \text{Std}^{G}$ 22
6. Standard measurable spaces 24
7. The codense functor $\Sigma': \Omega \rightarrow \text{Std}_{G}$ 26
8. Barycenter maps are determined by the support of a probability measure 28
9. The adjunction $\mathcal{P} \dashv \Sigma$ and isomorphism $\text{Std}^{G} \cong \text{SCvx}$ 31
10. Remarks 33
References 35

1. The Basic Schema for Determining an Equivalent Representation of $\mathcal{T}$-Algebras

To understand a monad $\mathcal{T}$ on any category $\mathcal{C}$ it is necessary to understand its $\mathcal{T}$-algebras. This is usually not such a simple task because the category of $\mathcal{T}$-algebras rarely occur as in the representation given by Eilenberg-Moore. Their definition of the category of algebras, denoted $\mathcal{C}^{\mathcal{T}}$, was descriptive and not constructive. Hence the question of the existence of $\mathcal{T}$-algebras is of considerable relevance and importance.
One scheme to obtain a useful representation of $T$-algebras is to find a dense subcategory of $T$-algebras and recognize that the objects and arrows in that subcategory can also be viewed as having a different structure. Thus that subcategory, call it $\Omega$, is a subcategory of another category $\mathcal{R}$, and when it is a dense subcategory of $\mathcal{R}$ and consist of categorically compact objects of $\mathcal{R}$, then $\Omega$ is called a compact dense subcategory of $\mathcal{R}$.

Recall, that an object $A$ in any category $\mathcal{R}$ is called categorically compact if we have

$$\mathcal{R}(A, \text{colim} D) = \text{colim}_i \mathcal{R}(A, D_i)$$

for every filtered diagram $D : I \to \mathcal{R}$ such that the colimit exists. The benefit of having a compact full subcategory of $\mathcal{R}$ is due to

**Lemma 1.1.** Let $\mathcal{R}$ and $\mathcal{C}$ be two big categories having filtered colimits. Let $\Omega \subset \mathcal{R}$ be a small full subcategory of $\mathcal{R}$ consisting of categorically compact objects of $\mathcal{R}$ such that every object of $\mathcal{R}$ is a filtered colimit of objects of $\Omega$. Then every functor $F' : \Omega \to \mathcal{C}$ has a unique extension $F : \mathcal{R} \to \mathcal{C}$ commuting with filtered colimits.

A proof of this result can be found in reference [4].

If the inclusion functor $\Omega \hookrightarrow \mathcal{R}$ is a dense functor, then it is necessarily a full functor, and so provided that $\Omega$ consist of compact objects of $\mathcal{R}$, then given any functor $j : \Omega \to \mathcal{C}$ there exists an extension of the functor $j$ to all of $\mathcal{R}$. The objective of the scheme is choose the space $\Omega$ such that we also can choose $j$ as a codense functor on the full subcategory of $\mathcal{C}$ consisting of all those objects which have a $T$-algebra.

![Diagram 1](image)

**Diagram 1.** Finding a useful representation of the category $\mathcal{C}^T$ by finding a compact dense subcategory $\Omega$ of $\mathcal{R}$ and a codense functor $j$ on the full subcategory of $\mathcal{C}$ consisting of those objects of $\mathcal{C}$ which have a $T$-algebra. Thus $j$ factors through $\mathcal{R}$ via the extension of $j$. The category $\Omega$ itself can be surmised by the condition that the functor $\kappa$ is a dense functor.

Intuitively, the idea is that $\mathcal{R}$ is a “reflection” of $\mathcal{C}^T$ about the codense functor $j$, with both $\iota$ and $\kappa$ dense functors, although the reflection is only up to an equivalence. This scheme requires that $\Omega$ must contain at least one $T$-algebra $h : TC \to C$ in the sense that the objects $TC$ and $C$ are both objects in $\Omega$, although viewed with a different structure, and the map $h$ likewise is an arrow in $\Omega$ viewed with different properties.

Under such a circumstance, an adjunction between two functors $L : \mathcal{C} \to \mathcal{R}$ and $R : \mathcal{R} \to \mathcal{C}$ can often be established by using the codense functor to obtain a canonical representation of the object $R(A) \in \mathcal{C}$ as the limit of the functor $D : RA \downarrow j \to \Omega \Rightarrow \mathcal{C}$ while the dense
Giry algebras via the support of a probability measure

The functor is used to obtain a canonical representation of the object \( LX \in \mathcal{R} \) as the colimit of the functor \( \mathcal{E} : \iota \downarrow LX \xrightarrow{\pi} \Omega \xrightarrow{\iota} \mathcal{R} \). The condition of having an adjunction \( L \dashv R \) given by the bijective correspondence of hom sets

\[
\mathcal{C}(X, RA) \cong \mathcal{R}(LX, A),
\]

which is natural in both arguments, \( X \) and \( A \), reduces to showing that

\[
\lim_i \mathcal{C}(X, D_i) \cong \lim_j \mathcal{R}(E_j, A),
\]

where we have used the fact that \( \mathcal{R}(\text{colim} \mathcal{E}, A) = \lim_j \mathcal{R}(\mathcal{E}_j, A) \). The component objects and arrows in computing these limits are precisely the objects and arrows in \( \Omega \) viewed with the appropriate structure, via the functor \( j \) or the functor \( \iota \), making them lie in both \( \mathcal{C} \) and \( \mathcal{R} \), respectively. The functor \( R \) of interest is generally the right Kan extension of \( \iota \) along the functor \( j \), while \( L \) is the left Kan extension of \( j \) along the functor \( \iota \).

2. Applying the scheme to the Giry monad for standard measurable spaces

This section provides an overview of how the preceding scheme is applied to the Giry monad, with references to the necessary theorems which are required to prove a given claim.

The Giry monad \((G, \eta, \mu)\) for standard measurable spaces provides a nice illustration of this scheme described in §1 because it illustrates the fact that, in practice, the \( G \)-algebras do not arise via the Eilenberg-Moore representation, but rather as an isomorphic category which is more familiar to us. Recall that the category of standard measurable space, \( \text{Std} \), consist of measurable spaces \( X \) which have a \( \sigma \)-algebra \( \Sigma_X \) generated by a countably generated field \( \mathcal{F} \), so that \( \Sigma_X = \sigma(\mathcal{F}) \). A measurable space is said to be separated if for any two elements in \( X \) there exists a measurable set in the \( \sigma \)-algebra of \( X \) which contains precisely one of those elements. Categorically this is equivalent to saying that the object \((2, \emptyset)\) is a coseparator, and we denote the full subcategory of \( \text{Std} \) consisting of the separated measurable spaces by \( \text{Std}_2 \). Note that that \( \text{Std}^G = \text{Std}^G_2 \) because any measurable space \( X \) which is not separated cannot have any \( G \)-algebras since there exists elements \( x_1, x_2 \in X \) such that \( \delta_{x_1} = \delta_{x_2} \), and hence the required condition for a \( G \)-algebra \( h : G\mathcal{X} \to X \) given by \( h \circ \eta_X = 1_X \) can’t be satisfied. The last property about standard spaces we require is that if \( X \) is a standard measurable space then \( G\mathcal{X} \) is a Polish space, and hence a standard space, and consequently \( G : \text{Std} \to \text{Std} \)[7, Lemma 8.2.1]

To describe how this scheme is applied to the Giry monad, our starting point requires we find a dense subcategory of \( \text{Std}^G \) which specifies the small category \( \Omega \). Because the key defining property for a standard measurable space \( X \) is countability, arising from the condition of having a countably generated finite field which generates the \( \sigma \)-algebra on \( X \), we take the set of natural numbers, \( \mathbb{N} \), with some appropriate structure to be an object in \( \Omega \).

The space \((\mathbb{N}, \emptyset\mathbb{N})\) is trivially a standard measurable space, and hence, as noted above, this implies \( G\mathbb{N} \) is a Polish Space which is a standard measurable space. Moreover, both
(N, \mathcal{P}N) and \mathcal{G}N are separated measurable spaces. The function
\[
\mathcal{G}N \xrightarrow{\epsilon_N} (N, \mathcal{P}N)
\]
is, by Lemma 3.25, a measurable function, and it satisfies the unit law \(\epsilon_N \circ \eta_N = 1_N\) where \(\eta_N(j) = \delta_j\) is the unit of the Giry monad, and the associative law \(\epsilon_N \circ \mathcal{G}(\epsilon_N) = \epsilon_N \circ \mu_N\) follows from analysis of the given expressions \(\mu_N(Q)\) and \(\mathcal{G}\epsilon_N(Q)\) on the measurable subset \(\{k\} \in \mathcal{P}N\). We have
\[
\mu_N(Q)(k) = \int_{p \in \mathcal{G}N} ev_k(p) \, dQ = \int_{p \in \mathcal{G}N} p_k \, dQ
\]
and hence
\[
\epsilon_N(\mu_N(Q)) = \epsilon_N\left(\sum_{i=0}^{\infty} \left(\int_{p \in \mathcal{G}N} p_i \, dQ\right) \delta_i\right).
\]
(4)

On the other hand we have
\[
\left(\mathcal{G}\epsilon_N(Q)\right)(k) = Q(\epsilon_N^{-1}(k)) = Q\left(\left\{ \sum_{j=k}^{\infty} p_j \delta_j \middle| \sum_{j=k}^{\infty} p_j = 1 \text{ and } p_k > 0 \right\}\right)
\]
and hence
\[
\epsilon_N(\mathcal{G}\epsilon_N(Q)) = \min_i \{i \mid Q\left(\left\{ \sum_{j=i}^{\infty} p_j \delta_j \middle| \sum_{j=i}^{\infty} p_j = 1 \text{ and } p_i > 0 \right\}\right) > 0\}.
\]
(5)

A moment's thought shows that the two expressions given in equations (4) and (5) are just two representations of the same quantity. Consequently the function \(\epsilon_N\) is a \(\mathcal{G}\)-algebra, and it is called the barycenter map of \(N\).

The set of all probability measures on the measurable space \((N, \mathcal{P}N)\) can be characterized as
\[
\mathcal{G}N = \left\{ \sum_{i \in \mathbb{N}} p_i \delta_i \mid \text{for all sequences } p : \mathbb{N} \to [0,1] \text{ such that } \lim_{N \to \infty} \left\{ \sum_{i=0}^{N} p_i \right\} = 1 \right\},
\]
where each \(\delta_i\) is the Dirac measure on \(i \in \mathbb{N}\), and \(p(i) = p_i\). Thus a probability measure on \(N\) is equivalent to specifying a sequence \(p : \mathbb{N} \to [0,1]\) satisfying the condition \(\lim_{N \to \infty} \left\{ \sum_{i=0}^{N} p_i \right\} = 1\), and hence we write \(p \in \mathcal{G}(\mathcal{P}(\mathbb{N}))\), or more briefly, \(p \in \mathcal{G}N\).

Given any set \(A\), a sequence \(a : \mathbb{N} \to A\), and any \(p \in \mathcal{G}N\) we refer to the formal expression \(\sum_{i \in \mathbb{N}} p_i a_i\) as a countably affine sum of elements of \(A\), and for brevity we use the notation \(\sum_{i \in \mathbb{N}} p_i a_i\) to refer to a countably affine sum dropping the explicit reference to the condition that the limit of partial sums \(\sum_{i=0}^{N} p_i\) converges to one. An alternative notation to the countable affine sum notation \(\sum_{i \in \mathbb{N}} p_i a_i\) is the notation \(\int_{A} a \, dp\). While our derivations use only the summation notation, we present the axioms of a super convex space using both notations because the integral notation removes the mystery behind super convex spaces.

We say a set \(A\) has the structure of a super convex space \(A\) if and only if the following three axioms are satisfied:
Axiom 1. For every sequence $a : \mathbb{N} \to A$ and every $p \in \mathcal{G}(\mathbb{N})$ it follows that
\[
\int_{\mathbb{N}} a \, dp := \sum_{i \in \mathbb{N}} p_i a_i \in A.
\]

Axiom 2. For every sequence $a : \mathbb{N} \to A$ and every integer $j \in \mathbb{N}$ the property
\[
\int_{\mathbb{N}} a \, d\delta_j = a_j
\]
holds. Stated alternatively, $\sum_{i \in \mathbb{N}} p_i a_i = a_j$ whenever $p = \delta_j$.

Axiom 3. If $p \in \mathcal{G}(\mathbb{N})$ and $Q : \mathbb{N} \to \mathcal{G}(\mathbb{N})$ is a sequence of probability measures on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, then
\[
\int_{j \in \mathbb{N}} \left( \int_{\mathbb{N}} a \, dQ^j \right) \, dp = \int_{\mathbb{N}} a \left( \int_{j \in \mathbb{N}} Q^j \, dp \right).
\]

Stated alternatively, $\sum_{j \in \mathbb{N}} p_j (\sum_{i \in \mathbb{N}} q_i^j a_i) = \sum_{i \in \mathbb{N}} (\sum_{j \in \mathbb{N}} p_j q_i^j) a_i$.

In the third axiom the probability measure $p \in \mathcal{G}(\mathbb{N})$ is operating like an element $p \in \mathcal{G}(\mathcal{G}(\mathbb{N}))$ to obtain a "probability mixture", which amounts to using the multiplicative natural transformation $\mu$ of the Giry monad at component $\mathbb{N}$, $\mu_\mathbb{N} : \mathcal{G}^2(\mathbb{N}) \to \mathcal{G}(\mathbb{N})$. In other words, at the measurable set \{i\} the probability mixture has weight
\[
(\mu_\mathbb{N}(p))(i) = \int_{\mathbb{N}} ev_i(Q) \, dp = \sum_{j \in \mathbb{N}} Q^j_i p_j = \sum_{j \in \mathbb{N}} p_j q_i^j.
\]

A morphism of super convex spaces, called a countably affine map, is a set function $m : A \to B$ such that
\[
m(\int_{\mathbb{N}} a \, dp) = \int_{\mathbb{N}} m(a) \, dp,
\]
or using the summation notation, $m(\sum_{i \in \mathbb{N}} p_i a_i) = \sum_{i \in \mathbb{N}} p_i m(a_i)$. Composition of countably affine maps is the set-theoretical composition. Super convex spaces with morphisms the countably affine maps form a category denoted $\text{SCvx}$.

The underlying set of $\mathcal{G}\mathbb{N}$ with its natural super convex space structure is the prototype super convex space which is used to define an arbitrary super convex space. Moreover, the set $\mathbb{N}$ itself has a super convex space structure defined, for all $p \in \mathcal{G}\mathbb{N}$, by
\[
\sum_{i \in \mathbb{N}} p_i \underline{i} = \min_{i \in \mathbb{N}} \{i | \text{such that } p_i > 0\},
\]
and it is this structure which makes the barycenter map $\epsilon_\mathbb{N}$ a countably affine map because
\[
\epsilon_\mathbb{N}(\sum_{i \in \mathbb{N}} p_i \delta_i) = \sum_{i \in \mathbb{N}} p_i \underline{i} = \min_{i \in \mathbb{N}} \{i | p_i > 0\},
\]
where the underline notation "$\underline{i}$" is used to distinguish the coefficients "$p_i$" from the elements of $\mathbb{N}$ because the coefficients can assume the values 0 and 1.

Due to the super convex space structure associated with the two measurable space, $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and $\mathcal{G}\mathbb{N}$, along with the property that the barycenter map $\epsilon_\mathbb{N}$ is a countably affine map we therefore take the category $\Omega$ to be the full subcategory of super convex spaces, $\text{SCvx}$, consisting of the two objects $\mathbb{N}$ and $\Delta_\mathbb{N}$, where $\Delta_\mathbb{N}$ is $\mathcal{G}\mathbb{N}$ viewed as a super convex space, rather than a measurable space. In Theorem 3.23 we show both the objects in $\Omega$ are categorically compact objects in $\text{SCvx}$. The proof that $i$ is a dense functor is given by Lemma 4.2 and Corollary 4.3. While we have chosen to view $\Omega$ as a subcategory of
It would perhaps be more descriptive to view it as a subcategory of the category of measurable super convex spaces just as one uses the category of topological groups or topological rings.

The scheme illustrated in Diagram 1, applied to the problem of finding a useful representation of $\text{Std}^G$, is given by

$$\Omega \xrightarrow{\kappa} \text{Std}^G \xrightarrow{\Upsilon} \text{Std}_2 \xrightarrow{\Sigma'} \text{SCvx}$$

$$\Sigma' = \Upsilon \circ \kappa \text{ and } \Upsilon \dashv \Sigma$$

$$\Sigma' = \Upsilon \circ \iota \text{ and } \iota \dashv \Sigma$$

**Diagram 2.** Two dense factorizations of the codense functor $\Sigma'$. The dense functor $\iota$ is the inclusion functor, while the dense functor $\kappa$ sends $N \mapsto (N, \epsilon_N)$ and $\Delta_N \mapsto (G^N, \mu_N)$. On arrows, $\kappa$ sends every countably affine map $f \mapsto \Sigma'f$ which is a $G$-algebra map. The extended functor $\Sigma$ assigns to each super convex space $A$ the largest $\sigma$-algebra such that all of the countably affine maps $m : G(N) \to A$ are measurable.

The functor $\Sigma' : \Omega \to \text{Std}_2$ assigns to each object in $\Omega$ the largest $\sigma$-algebra such that every countably affine map from the measurable space $G\mathbb{N}$ into that object is a measurable function. Since the identity map $G\mathbb{N} \to \Delta_N$ is a countably affine map $\Sigma'\Delta_N = G\mathbb{N}$. Since $\epsilon_N$ is a countably affine map and $\epsilon^{-1}_N(\{k\}) = \{ \sum_{i \in \mathbb{N}} p_i \delta_i \in G\mathbb{N} | p_k > 0 \text{ and } p_i = 0 \text{ for all } i < k - 1 \}$ it follows by Lemma 3.25 that $\Sigma'(\mathbb{N}) = (N, \mathcal{P}N)$.

The extended functor $\Sigma$ is in fact the left Kan extension of $\Sigma'$ along $\iota$; however, to prove this requires showing that the objects in $\Omega$ are categorically compact objects because one must show that a super convex space $A$ endowed with the largest $\sigma$-algebra such that all the countably affine maps from $G\mathbb{N}$ into $A$ are measurable yields a standard measurable space. Trying to show that result without knowledge of the property of categorical compactness can be perplexing. However once we have shown that both $\mathbb{N}$ and $\Delta_N$ are categorically compact objects in $\text{SCvx}$ then Lemma 1.1 achieves precisely what we need without explicit knowledge of the endowed $\sigma$-algebra on a space $A$.

For any measurable space $X$ the set $GX$ has a super convex space structure defined on it by the fact that any countably affine sum of elements of $GX$ is itself an element of $GX$ under the point wise evaluation definition, $\left(\sum_{i \in \mathbb{N}} p_i P_i\right) U = \sum_{i \in \mathbb{N}} p_i P_i(U)$ for each $P_i \in GX$ and all measurable sets $U$ in $X$. Consequently, the functor $G$ can be viewed as a functor $P : \text{Std}_2 \to \text{SCvx}$. Using the functor $P$, the super convex space $\Delta_N$ can be written as

$$\Delta_N = P(\Sigma'N) = P(\Sigma\mathbb{N}),$$

where the notation “$\Delta_N$” originates from the idea that it is the countably infinite-dimensional simplex. After §5, which shows the functor $\kappa : \Omega \to \text{Std}^G$ is a dense functor, I use
the notation \( \mathcal{P}(\Sigma N) \) because this subtle change in notation relates to the bijection of sets \( \text{Std}_2(\Sigma N, \Sigma A) \cong \text{SCvx}(\Delta_N, A) \), which is elementary to prove (Lemma 3.1), becomes \( \text{Std}_2(\Sigma N, \Sigma A) \cong \text{SCvx}(\mathcal{P}(\Sigma N), A) \) which suggest the more general natural bijection \( \text{Std}_2(X, \Sigma A) \cong \text{SCvx}(\mathcal{P}X, A) \).

That bijection can be viewed in terms of the compact dense subcategory \( \Omega \), from the perspective of Isbells’ duality, in the diagram

\[
\begin{align*}
\text{Set}^{\Omega^\text{op}} & \xrightarrow{\text{Spec}} \text{Set}^{\Omega^\text{op}} \\
\text{Std}_2 & \xrightarrow{\mathcal{P}} \text{SCvx} \\
\Sigma & \xrightarrow{\iota} \Omega
\end{align*}
\]

\( \mathcal{O} \rightarrow \text{Spec} \)

\( \mathcal{O}(\mathcal{F})[\omega] = \text{Set}^{\Omega^\text{op}}(\mathcal{F}, \Omega(\cdot, \omega)) \quad \text{for all } \omega \in \Omega \)

\( \text{Spec}(\mathcal{G})[\omega] = \text{Set}^{\Omega}(\Omega(\omega, \cdot), \mathcal{G}) \quad \text{for all } \omega \in \Omega \)

**Diagram 3.** Isbell duality between the functor categories, \( \text{Set}^{\Omega^\text{op}} \) (geometric spaces) and \( \text{Set}^{\Omega} \) (algebraic spaces) where a canonical adjunction exists, and its restriction to subcategories embedded into those functor categories.

where, due to the fact \( \Sigma' \) is codense, as shown in Corollary 7.5, the truncated Yoneda functor \( \mathbf{y} : \text{Std}_2 \rightarrow (\text{Set}^{\Omega})^{\text{op}} \) is full and faithful, and because \( \iota \) is a dense functor, shown in Corollary 4.3, the truncated (dual) Yoneda functor \( \mathbf{y}^{\text{op}} : \text{SCvx} \rightarrow \text{Set}^{\Omega^\text{op}} \) is full and faithful. The term “truncated Yoneda” map is due to Isbell\[9, 10\] and is descriptive. It is only by representing large categories \( C \) via the truncated Yoneda functor, using a small category \( \Omega \), that we are able to apply Isbells duality theorem.

Once we have the adjoint factorization of the Giry monad given by \( \mathcal{P} \rightarrow \mathcal{G} = \Sigma \circ \mathcal{P} \), we have the comparison functor \( \mathcal{K} : \text{SCvx} \rightarrow \text{Std}^\mathcal{G} \) mapping an object \( A \mapsto (\Sigma A, \Sigma \varepsilon_A) \).

But there is also have a functor \( \text{Std}^\mathcal{G} \xrightarrow{W} \text{SCvx} \) mapping the object \( (X, h) \mapsto X_h \) where \( X_h \) is the super convex space consisting of the underlying set of \( X \) with the super convex space structure defined by \( \sum_{i \in \mathbb{N}} p_i x_i = h(\sum_{i \in \mathbb{N}} p_i \delta_{x_i}) \). If \( f : (X, h) \rightarrow (Y, k) \) is a morphism of two \( \mathcal{G} \)-algebras then, under the super convex space structure specified on the two spaces, \( X_h \) and \( Y_k \), it follows by verification, given in Lemma 5.1, that \( f \) is a countably affine map. The comparison functor \( \mathcal{K} \) and the functor \( W \) specify the isomorphism \( \text{Std}^\mathcal{G} \cong \text{SCvx} \).

**Advice on reading:** To prove the fundamental result that barycenter maps exist it is only necessary to know that the subcategory consisting of the single object \( (\mathbb{N}, \mathcal{P}\mathbb{N}) \) and the morphisms all monotonic functions \( \phi : \mathbb{N} \rightarrow \mathbb{N} \) are right-adequate (codense) in \( \text{Std}_2 \). The material in section 6 provides a concise overview of \( \text{Std} \), while the other supporting material provides for a complete description of the category \( \text{SCvx} \) and its relationship to \( \text{Std} \). (The property that \( \mathbb{N} \) and \( \mathcal{G}(\mathbb{N}) \) are categorically compact objects in \( \text{SCvx} \) is not necessary to prove the main result that \( \text{Std}^\mathcal{G} \cong \text{SCvx} \).)
Notation: In this article, we use the following terminology and naming convention which is used throughout the entire article. The term “space” will always refer to a super convex space. The symbol $A$ is always a super convex space, while the symbol “$X$” refers to a standard measurable space, and the term “measurable space” always refers to a standard measurable space unless noted otherwise. The symbol “$\mathbb{N}$” refers to the set of all natural numbers $\{0, 1, 2, \ldots\}$, and unless the context dictates otherwise, the symbol “$\mathbb{N}$” refers to the set $\mathbb{N}$ with its super convex space structure. By writing “$\mathcal{G}\mathbb{N}$” we are of course implying $\mathbb{N}$ is a measurable space (with the power set $\sigma$-algebra); otherwise the term would not make sense since the Giry monad is applied only to measurable spaces.

When we are using the elements $0, 1, \ldots$ to represent elements of a space, we may emphasize the elements by $0, 1, \ldots$ to distinguish the elements of the space from the indexing set. The symbol $1$ without subscripts denotes the terminal object which for our categories of interest is always a singleton set $\{\star\}$. For the notation “$1$” with subscripts, such as $1_\mathbb{N}$, denote the identity map on the object denoted in the subscript. The notation $\text{Std}(X, \Sigma A)$ and $\text{SCvx}(\mathcal{P} X, A)$ refers to the hom set of measurable functions and the hom set of countably affine maps. The symbol “$\bigcup_{i \in \mathbb{N}} A_i$” refers to a disjoint union of sets. For explanations on proofs we use the abbreviation “cam” for the phrase “countably affine map”.

3. SUPER CONVEX SPACES

3.1. Types of spaces, basic properties, and examples. We can classify a super convex space into one of three types.

(1) A space that can be embedded into a vector space is of geometric type.

(2) Let $\Delta^\dagger_\mathbb{N} = \{p \in (0, 1)^\mathbb{N} \mid \sum_{i \in \mathbb{N}} p_i = 1\}$. A space $A$, consisting of at least two points, is of discrete type if, for each fixed sequence $a : \mathbb{N} \to A$, the function

\[
\Delta^\dagger_\mathbb{N} \to A
\]

\[p \mapsto \sum_{i \in \mathbb{N}} p_i a_i\]

is a constant function.

(3) A space which is neither of discrete type or geometric type is of mixed type.

The super convex spaces that we use are given in the following examples. Note that by restricting the countable affine sums to finite affine sums every super convex spaces determines a convex space. While the definition given above is defined in terms of a sequence $a \in A^\mathbb{N}$ of points of a space $A$, we can simply choose an arbitrary point $a \in A$ and have the sequence constant after the $n^{th}$ term with value $a$. We then associate a coefficient (weight) $p_{n+1} = 0$ with all those terms. While every super convex space restricts to a convex space, as Example 3.9 shows, not every convex space is a super convex space without some modification of that convex space.

The super convex space $\mathbb{N}$ is a space of discrete type. The super convex spaces $\emptyset$ and $\mathbf{1}$ are geometric types which embed into $\mathbb{R}$, while $\Delta^\dagger_\mathbb{N}$ is also of geometric type since it embeds into the $\mathbb{R}$-vector space $\mathbb{R}^\mathbb{N}$ via the projection map $\pi : \Delta^\dagger_\mathbb{N} \to \mathbb{R}^\mathbb{N}$ which maps $p = \{p_0, p_1, p_2, \ldots\}$ to the sequence $\{p_0, p_1, p_2, \ldots\}$.

Lemma 3.1. For $A$ any super convex space every countably affine map $m \in \text{SCvx}(\Delta^\dagger_\mathbb{N}, A)$ is uniquely specified by a sequence in $A$. Associating the powerset $\sigma$-algebra with $\mathbb{N}$,
and letting $\Sigma A$ denote $A$ viewed as a measurable space - for any $\sigma$-algebra - the sequence $a : (N, P N) \to \Sigma A$ is a measurable function so we have a bijective correspondence $SCvX(\Delta_N, A) \cong \text{Std}_2((N, P N), \Sigma A)$.

Proof. Every element $p \in \Delta_N$ has a unique representation as a countable affine sum $p = \sum_{i \in N} p_i \delta_i$, and hence a countably affine map $m : \Delta_N \to A$ is uniquely determined by where it maps each Dirac measure $\delta_i$. Thus $i \mapsto m(\delta_i)$ specifies a sequence in $A$.

For $N$ having the powerset $\sigma$-algebra, every map from $N$ to any other measurable space is measurable, so a sequence can be viewed as a measurable function. Thus the bijective correspondence. $\square$

Given any sequence of elements $a : N \to A$ we denote the countably affine map $\Delta_N \to A$ specified by $\delta_i \to a_i$ with the notation $\langle a \rangle$ or just $\langle a_i \rangle$ with the brackets implying we are viewing it as a countably affine map from $\Delta_N$. A sequence of elements $Q : N \to \Delta_N$ specifies a countably affine map $\langle Q \rangle : \Delta_N \to \Delta_N$ which can be interpreted as a countably affine transformation of the spanning set of elements, mapping $\delta_i \mapsto Q_i$ where $Q_i = \sum_{j \in \mathbb{N}} q^j_i \delta_j$, and the transformation amounts to

$$
\Delta_N \xrightarrow{\langle a \rangle} A \quad \text{and} \quad b_i = \sum_{j \in \mathbb{N}} q^j_i a_j.
$$

When the sequence $a : N \to A$ is itself a countably affine map then the $SCvX$-diagram

$$
\begin{array}{ccc}
\Delta_N & \xrightarrow{\langle a \rangle} & A \\
\epsilon_N & \downarrow & \downarrow a \\
N & \xrightarrow{a} & A \\
\end{array}
$$

commutes.

Countably affine endomaps on the super convex space $N$ have a simple characterization.

**Lemma 3.2.** A function $f : N \to N$ is a countably affine map if and only if $f$ is monotone, $i < j$ implies $f(i) \leq f(j)$.

**Proof.** Necessary condition Suppose that $f : N \to N$ is a countably affine map. Let $i < j$. By the super convex space structure on $N$ it follows, for all $\alpha \in (0, 1)$, that $\alpha i + (1 - \alpha) j = i$. If $f$ is not monotone then there exist a pair of elements $i, j \in N$ such that $i < j$ with $f(j) < f(i)$. This implies, for all $\alpha \in (0, 1)$, that $f(j) = \alpha f(i) + (1 - \alpha) f(j) < f(\alpha i + (1 - \alpha) j) = f(i)$, which contradicts our hypothesis that $f$ is a countably affine map.

Sufficient condition Suppose $f$ is a monotone function, and that we are given an arbitrary countably affine sum $\sum_{i=0}^\infty p_i \delta_i = n$ in $N$, so that for all $i = 0, 1, \ldots, n - 1$ we have $p_i = 0$. Since the condition defining the super convex structure is conditioned on the property “for $p_i \neq 0$”, the countably affine sum is not changed by removing any number of terms $i$ in the countable sum whose coefficient $p_i = 0$. Hence for all $j$ such that $n < j$ it follows that $f(n) \leq f(j)$ so...
that
\[ f(\sum_{i=0}^{\infty} p_i i) = f(n) = \sum_{i=n}^{\infty} p_i f(i) \]
where the last equality follows from the definition of the super convex space structure on \( \mathbb{N} \).

**Corollary 3.3.** Any \( m \in \text{SCvx}(\Delta_{\mathbb{N}}, \mathbb{N}) \) can be written uniquely as \( m = \phi \circ \epsilon_{\mathbb{N}} \) where \( \phi \in \text{SCvx}(\mathbb{N}, \mathbb{N}) \).

**Proof.** By Lemma 3.1 every element \( m \in \text{SCvx}(\Delta_{\mathbb{N}}, \mathbb{N}) \) can be represented by a map \( (u) : \Delta_{\mathbb{N}} \to \mathbb{N} \) which is determined by the sequence \( u : \mathbb{N} \to \mathbb{N} \), which maps \( i \mapsto u_i \). The map \( \epsilon_{\mathbb{N}}(\delta_i) = i \) acts as an idempotent operator on the right since \( u(\epsilon_{\mathbb{N}}(\delta_i)) = u_i \). Therefore \( m = u \circ \epsilon_{\mathbb{N}} \). \( \square \)

In particular, this implies that given any countably affine map \( (Q) : \Delta_{\mathbb{N}} \to \Delta_{\mathbb{N}} \) that there exists a countably affine map \( \phi : \mathbb{N} \to \mathbb{N} \) such that the diagram

\[
\begin{array}{ccc}
\Delta_{\mathbb{N}} & \xrightarrow{\epsilon} & \mathbb{N} \\
\downarrow{(Q)} & & \downarrow{\phi} \\
\Delta_{\mathbb{N}} & \xrightarrow{\epsilon} & \mathbb{N}
\end{array}
\]

commute.

**Corollary 3.4.** For every countably affine map \( \phi : \mathbb{N} \to \mathbb{N} \) the property \( \phi \circ \epsilon_{\mathbb{N}} = \epsilon_{\mathbb{N}} \circ \mathcal{P}(\Sigma \phi) \) holds.

**Proof.** A direct verification gives
\[
(\phi \circ \epsilon_{\mathbb{N}})(\sum_{i\in\mathbb{N}} p_i \delta_i) = \phi\left(\sum_{i\in\mathbb{N}} p_i \delta_i\right) = \phi\left(\min_i\{i \mid p_i > 0\}\right)
\]
Suppose \( \min_i\{i \mid p_i > 0\} = k \) which implies \( p_i = 0 \) for all \( i = 0, 1, \ldots, k - 1 \) and \( p_k > 0 \). Hence \( (\phi \circ \epsilon_{\mathbb{N}})(\sum_{i\in\mathbb{N}} p_i \delta_i) = \phi(k) \).

On the other hand we have
\[
(\epsilon_{\mathbb{N}} \circ \mathcal{P}(\Sigma \phi))(\sum_{i\in\mathbb{N}} p_i \delta_i) = \epsilon_{\mathbb{N}}\left(\sum_{i\in\mathbb{N}} p_i \delta_{\phi(i)}\right) = \sum_{i\in\mathbb{N}} p_i \phi(i) = \min_i\{\phi(i) \mid p_i > 0\}
\]
using the \( \text{SCvx} \) structure on \( \mathbb{N} \)

Since \( p_i = 0 \) for all \( i = 0, 1, 2, \ldots, k - 1 \) and \( p_k > 0 \) the minimum is given by \( \phi(k) \), which proves the lemma. \( \square \)

**Lemma 3.5.** If \( D \) a discrete type space and \( G \) a geometric type space the set \( \text{SCvx}(D, G) \) consist of constant maps.

**Proof.** Since \( D \) is a discrete space, it follows that for all \( d_1, d_2 \in D \) and all \( r \in (0, 1) \), that the property \( rd_1 + (1-r)d_2 \in \{d_1, d_2\} \) holds, which implies that the condition \( m(rd_1 + (1-r)d_2) = rm(d_1) + (1-r)m(d_2) \) can only be satisfied if \( m \) is a constant function. \( \square \)
Lemma 3.5 shows that no useful information can be obtained about a geometric space $G$ by looking at maps from a discrete type space $D$ into $G$. On the other hand, the maps $\text{SCvx}(G, D)$ can provide quite useful information.

**Example 3.6.** The set $2$ with the super convex structure defined by

$$(1 - r)0 + r1 = \begin{cases} 0 & \text{if and only if } r \in [0, 1) \\ 1 & \text{otherwise} \end{cases}$$

is of discrete type. The inclusion map $2 \hookrightarrow \mathbb{N}$ is a countably affine map.

There is also a super convex space $2$ with the structure defined by

$$(1 - r)0 + r1 = \begin{cases} 1 & \text{if and only if } r \in (0, 1] \\ 0 & \text{otherwise} \end{cases}$$

and there is a $\text{SCvx}$-isomorphism $\text{sw} : 2 \rightarrow 2$ defined by $\text{sw}(0) = 1$ and $\text{sw}(1) = 0$.

Whenever we use the characteristic function, because its defined as $\chi_U(x) = 1$ if and only if $x \in U$ rather than the alternative, we are implicitly using $\chi_U : X \rightarrow 2$, and similarly with respect to the evaluation map $G(\chi_U) = \text{ev}_U : G(X) \rightarrow G(2)$ where $G(2) \cong [0, 1]$ via the $\text{Std}$-isomorphism and $\text{SCvx}$-isomorphism mapping $(1 - r)\delta_0 + r\delta_1 \mapsto r$.

As a consequence, when we employ characteristic functions and want to view $\chi_U$ as a measurable map into $\mathbb{N}$ it is necessary to use the swap isomorphism to obtain

$$X \xrightarrow{\chi_U} 2 \xrightarrow{\text{sw}_2} 2 \hookrightarrow \mathbb{N},$$

and similarly,

$$G(X) \xrightarrow{\text{ev}_U} G(2) \xrightarrow{G(\text{sw}_2)} G(2) \xrightarrow{\epsilon_2} 2 \hookrightarrow \mathbb{N},$$

where $\epsilon_2 : G(2) \rightarrow 2$ is defined by

$$\epsilon_2((1 - r)\delta_0 + r\delta_1) = \begin{cases} 0 & \text{for all } r \in [0, 1) \\ 1 & \text{if and only if } r = 1 \end{cases}$$

or, as is usually done, by first using the isomorphism $G(2) \cong [0, 1]$ mapping $(1 - r)\delta_0 + r\delta_1 \mapsto r$ and then defining $\epsilon_2(r) = 0$ for all $r \in [0, 1)$ and $\epsilon_2(r) = 1$ if and only if $r = 1$.

For every finite number $n \in \mathbb{N}$, let $n = \{0, 1, \ldots, n - 1\}$, and endow it with a super convex space structure by defining $\sum_{i \in \mathbb{N}} p_i u_i = \min_i \{u_i | u_i \in n \text{ s.t. } p_i > 0\}$. The inclusion map $n \hookrightarrow \mathbb{N}$ is a countably affine map. Just as for the space $2$, one can choose the max function instead to obtain the super convex space $n$, and there is a swap map $\text{sw}_n : n \rightarrow n$ defined by $i \mapsto (n - 1) - i$ which is an isomorphism. It is only for the case $\mathbb{N}$ that we must choose the minimum function since we need to employ the well-ordered property.

**Example 3.7.** The $n-1$ dimensional affine simplex $\Delta_n = \{ \sum_{i=0}^{n-1} p_i \delta_i | \sum_{i=0}^{n-1} p_i = 1 \text{ and } p_i \in [0, 1] \}$ is the space of all probability measures on the set of $n$ points. It is a space of geometric type since it can be embedded into the vector space $\mathbb{R}^{n-1}$. The 1-dimensional simplex $\Delta_2 \cong [0, 1]$ via the $\text{SCvx}$-isomorphism $(1 - r)\delta_0 + r\delta_1 \mapsto r$. 
Example 3.8. Given any finite \( n \in \mathbb{N} \), let \( n \) be the super convex space with the structure defined by the min function as defined in Example 3.6 and let \( \Delta_n \) denote the \( n-1 \) dimensional simplex given by the preceding example. The function \( \epsilon_n \) defined by

\[
\Delta_n \stackrel{\epsilon_n}{\longrightarrow} \sum_{i=0}^{n-1} p_i \delta_i \mapsto \min_i \{ i \mid p_i > 0 \}
\]

is a countable affine map. Moreover, \( \epsilon_n \) is the only countably affine map in \( \text{SCvx}(\Delta_n, n) \) which satisfies the property that \( \epsilon_n(\delta_i) = i \). Using the argument given in Lemma 3.3, every countably affine maps \( m \in \text{SCvx}(\Delta_n, n) \) can be represented as a composite \( \phi \circ \epsilon_n \) where \( \phi : \mathbb{N} \to \mathbb{N} \) is a monotonic function.

For every finite \( n \), the affine map \( \epsilon_n \) is in fact a Giry-algebra when we endow \( n \) with the powerset \( \sigma \)-algebra, and endow \( \Delta_n \) with the same \( \sigma \)-algebra as \( \mathcal{G}n \), which is equivalent to saying that \( \Delta_n \) is endowed with the smallest \( \sigma \)-algebra such that all of the the point evaluation maps \( p \mapsto p_i \) are measurable. The proof that these affine maps are Giry-algebras follows the same argument as that given for \( \epsilon_n \) given in §2.

Example 3.9. A space of mixed type is given by \( \mathbb{R}_\infty \) which is the real line \( \mathbb{R} \) with one point adjoined, denoted \( \infty \), which satisfies the property that any countably affine sum \( \sum_{i \in \mathbb{N}} p_i r_i = \infty \) if either (1) \( r_j = \infty \) and \( p_j > 0 \) for any index \( j \), or (2) the sequence of partial sums does not converge. It is for the latter reason that \( \mathbb{R} \) is not a super convex space since we could take \( p_i = \frac{1}{2^i} \) and \( r_i = 2^{i+1} \) and the limit of the sequence does not exist in \( \mathbb{R} \). Thus while \( \mathbb{R} \) is a convex space it is not a super convex space.

The only nonconstant countably affine map \( j : \mathbb{R}_\infty \to 2 \) is given by \( j(u) = 0 \) for all \( u \in \mathbb{R} \) and \( j(\infty) = 1 \).

Example 3.10. The three point space \( \{0, u, 1\} \) with the structure defined by \( r \cdot 0 + (1-r) \cdot u = u \) for all \( r \in (0, 1) \), \( r \cdot 1 + (1-r) \cdot u = u \) for all \( r \in (0, 1) \), and \( r \cdot 0 + (1-r) \cdot 1 = u \) for all \( r \in (0, 1) \). This three point space is a space of mixed type due to the last condition. We will subsequently see that this space arises as a quotient space of \( \Delta_2 \) which is \( \text{SCvx} \)-isomorphic to \([0, 1] \).

3.2. Limits and Colimits in \( \text{SCvx} \).

Lemma 3.11. Products exists in \( \text{SCvx} \).

Proof. Given any family of super convex spaces \( \{A_i\}_{i \in I} \) the cartesian product of the spaces is defined as

\[
\prod_{i \in I} A_i = \{ f : I \to \bigcup_{i \in I} A_i \mid f(i) \in A_i \},
\]

and we endow it with a super convex space structure defined componentwise,

\[
(\sum_{j \in \mathbb{N}} p_j f_j)(i) = \sum_{j \in \mathbb{N}} p_j f_j(i) \in A_i.
\]

The terminology “countable” in the expression “countable affine map” is rather meaningless for spaces with a finite number of points because any countable affine sum of \( n \) elements can be reduced to a finite affine sum of at most \( n \) elements. Despite that fact, I still occasionally use the terminology to emphasize we are working in the category \( \text{SCvx} \) rather than \( \text{Cvx} \).
The two axioms of a super convex space structure are satisfied because each component space \( A_i \) satisfies them.

For each index \( j \in I \) we define \( \pi_j : \prod_{i \in I} \pi_i A_i \to A_j \) to be the evaluation map sending \( f \mapsto f(j) \). The pair \( (\prod_{i \in I} A_i, \{\pi_j\}_{j \in I}) \) specify the product of the family \( \{A_i\}_{i \in I} \). The verification that \( (\prod_{i \in I} A_i, \{\pi_j\}_{j \in I}) \) satisfies the universal property of being a product is straightforward using the coordinates \( i \in I \).

\[ \square \]

Lemma 3.12. The category \( \text{SCvx} \) is a monoidal closed category under the tensor monoidal structure \( A \otimes B \) which is characterized by the equations \( (\sum_{i \in \mathbb{N}} p_i a_i) \otimes b = \sum_{i \in \mathbb{N}} p_i (a_i \otimes b) \) and \( a \otimes (\sum_{i \in \mathbb{N}} p_i b_i) = \sum_{i \in \mathbb{N}} p_i (a \otimes b_i) \) which holds for all \( a, a \in A \), all \( b, b \in B \) and all \( p \in \Delta_n \).

Proof. The proof of this fact is identical to the proof found in Mengs' thesis [8, Proposition 5, pp1.6-1.7], with the only exception being that there she showed the category of convex spaces was monoidal closed under the tensor product. Hence, one needs to replace the finite affine sums with countable affine sums for the category of super convex spaces. \( \square \)

Lemma 3.13. All coproducts exist in \( \text{SCvx} \).

Proof. Let \( J \) be a set (=discrete category), and \( \mathcal{D} : J \to \text{SCvx} \) a diagram in \( \text{SCvx} \). Let \( \bigcup_{j \in J} \mathcal{D}_i \) denote the disjoint union of the spaces \( \mathcal{D}_i \) (viewed as sets). Let \( n = \{0, 1, \ldots, n-1\} \) and \( \Delta_n = \{p : n \to [0, 1] | \sum_{i \in n} p_i = 1\} \).

We define the free coproduct of the set of all the spaces \( \mathcal{D}_j \) as the set of all formal countable affine sums

\[
\text{Free}(\{\mathcal{D}_j\}_{j \in J}) = \left\{ \sum_{i \in n} p_i d_{s(i)} \mid n \longrightarrow \bigcup_{j \in J} \mathcal{D}_i, p \in \Delta_n, \text{ for all } n \in \mathbb{N} \right\}
\]

We then define a relation \( \mathcal{R} \) on this set as consisting of all elements of the form

\[
(\sum_{i \in n} p_i d_{s(i)}, \sum_{j \in J} \left( \sum_{i \in s^{-1}(\mathcal{D}_j)} p_i \right) \left( \sum_{i \in s^{-1}(\mathcal{D}_j)} \frac{p_i}{\sum_{i \in s^{-1}(\mathcal{D}_j)} p_i} d_{s(i)} \right)) \in \mathcal{R}
\]

where, provided that \( \sum_{i \in s^{-1}(\mathcal{D}_j)} p_i > 0 \), the countable affine sum

\[
\sum_{i \in s^{-1}(\mathcal{D}_j)} \frac{p_i}{\sum_{i \in s^{-1}(\mathcal{D}_j)} p_i} d_{s(i)}
\]

is computed in \( \mathcal{D}_j \). Note that the second coordinate of the term in \( \mathcal{R} \) of expression (8) is a countable sum because \( s \) is a countable sequence in \( \bigcup_{j \in J} \mathcal{D}_i \).

Now let \( S \) denote the smallest equivalence relation on \( \text{Free}(\{\mathcal{D}_j\}_{j \in J}) \) containing all the elements of the relation \( \mathcal{R} \), as given in expression (8). The coproduct of the family of spaces \( \{\mathcal{D}_j\}_{j \in I} \) is defined as the free coproduct modulo the equivalence relation,

\[
\bigsqcup_{j \in J} \mathcal{D}_j = \text{Free}(\{\mathcal{D}_j\}_{j \in J})/S.
\]
The equivalence relation $\mathcal{S}$ is a normalization procedure so that we identify, for example, the two countably affine sums
\[
\frac{1}{4} d_{1,1} + \frac{1}{2} d_{1,2} + \frac{1}{4} d_2 + 0 d_3 + 0 d_4 + \ldots \equiv \frac{3}{4} \left( \frac{1}{3} d_{1,1} + \frac{2}{3} d_{1,2} \right) + \frac{1}{4} d_2 + 0 d_3 + \ldots
\]
whenever $d_{1,1}, d_{1,2} \in D_1$. This equivalencing (normalization procedure) is necessary to satisfy the third axiom defining a super convex space.

A sequence in $\bigsqcup_{j \in J} D_j$ is a countable family of countably affine sums, $\{\sum_{k \in \mathbb{N}} q_k^j d_{s(k)}\}_{i \in \mathbb{N}}$, and given an element $p \in \Delta_n$ the super convex space structure on $\bigsqcup_{j \in J} D_j$ is defined by
\[
\sum_{i \in \mathbb{N}} p_i (\sum_{k \in \mathbb{N}} q_k^j d_{s(k)}) = \sum_{i \in \mathbb{N}} \sum_{k \in \mathbb{N}} p_i q_k^j d_{s(k)}
\]
which is just a countable affine sum of elements of the disjoint union of all the spaces $D_j$, and hence an element in $\bigsqcup_{j \in J} D_j$. The axioms defining a super convex space are now easily, albeit tediously, verified.

The insertion maps $\iota_k : D_k \twoheadrightarrow \bigsqcup_{j \in J} D_j$ are specified by the map $d \mapsto 1d$. More formally, the insertion map sends the element $d \in D_k$ to the countable affine sum specified by $d : 1 \rightarrow \bigsqcup_{j \in J} D_j$ mapping $0 \mapsto d$ and the (only) element $1 \in \Delta_1$.

To verify that $(\bigsqcup_{j \in J} D_j, \{\iota_j\}_{j \in J})$ is a coproduct, let $\{m_j : D_j \rightarrow A\}_{j \in J}$ be a family of countably affine maps. The map $m : \bigsqcup_{j \in J} D_j \rightarrow A$ specified by $m\left(\sum_{i \in \mathbb{N}} p_i d_{s(i)}\right) = \sum_{i \in \mathbb{N}} p_i m_{s(i)}(d_{s(i)})$ satisfies $m_j = m \circ \iota_j$ for all indices $j \in J$. If $\theta : \bigsqcup_{j \in J} D_j \rightarrow A$ is a countably affine map also satisfying $m_j = \theta \circ \iota_j$ then for every index $j$ we have
\[
\theta(1d) = (\theta \circ \iota)(d) = m_j(d) = (m \circ \iota_k)d = m(1d).
\]
Since $\theta$ and $m$ are countably affine maps this completely defines the mappings because given any countable sequence $s : n \rightarrow \bigsqcup_{j \in J} D_j$ and $p \in \Delta_n$ it follows that
\[
\theta(\sum_{i \in \mathbb{N}} p_i d_{s(i)}) = \sum_{i \in \mathbb{N}} p_i \theta(1d_{s(i)}) = \sum_{i \in \mathbb{N}} p_i m(1d_{s(i)}) = m(\sum_{i \in \mathbb{N}} p_i d_{s(i)}).
\]
Thus $\theta = m$ and it follows that $(\bigsqcup_{j \in J} D_j, \{\iota_j\}_{j \in J})$ is a coproduct.

We say a countable affine sum $\sum_{i \in \mathbb{N}} p_i a_i$ of elements in a coproduct $\bigsqcup_{i \in \mathbb{N}} A_i$ are in “standard normal form” whenever there is at most one term $p_i a_i$ in the summation with an element of one of the component spaces $A_i$ of the coproduct. As noted in the proof, this can always be done by taking multiple such terms which all lie in one component space and compute a normalized countably affine sum in that component.

**Corollary 3.14.** For every finite $n \in \mathbb{N}$ the coproduct $\bigsqcup_{i \in \mathbb{N}} 1_i \cong \mathcal{P}(\Sigma n)$, and $\bigsqcup_{i \in \mathbb{N}} 1_i \cong \mathcal{P}(\Sigma N)$.

**Proof.** This follows from the definition of a coproduct. The required isomorphism is given by $1_i \leftrightarrow \delta_i$.

**Lemma 3.15.** Coequalizers exist in $\text{SCvx}$.

**Proof.** Let $f, g : A \rightarrow B$ be two countably affine maps. The coequalizer of $f$ and $g$ is constructed by choosing the smallest congruence relation $\mathcal{R}$ on $B$ such that $(f(a), g(a)) \in \mathcal{R}$ for all $a \in A$. This relation is indeed the smallest congruence relation with this property, since if $(f(a), g(a)) \in \mathcal{R}$ then $(f(b), g(b)) \in \mathcal{R}$ whenever $b$ is related to $a$ by any relation $\mathcal{R}'$.

We now define a function $\delta : A \rightarrow B$ such that $\delta(a) = b$ for some $b \in B$ which is a coequalizer of $f$ and $g$. Since $(f(a), g(a)) \in \mathcal{R}$, there exists a chain of relations $\mathcal{R}'_1, \ldots, \mathcal{R}'_n$ such that $a \mathcal{R}'_1 \ldots \mathcal{R}'_n b$. Define $\delta(a) = b$ for some $b \in B$. This function is indeed a coequalizer of $f$ and $g$ since if $f(a) = g(a)$ then $b \mathcal{R} b$ for all $b \in B$. Thus $\mathcal{R}$ is the smallest congruence relation with this property.

We now show that $\mathcal{R}$ is indeed the smallest congruence relation with this property. Let $\mathcal{R}'$ be any relation on $B$ such that $(f(a), g(a)) \in \mathcal{R}'$ for all $a \in A$. Then for any $a \mathcal{R}' b$, there exists a chain of relations $\mathcal{R}'_1, \ldots, \mathcal{R}'_n$ such that $a \mathcal{R}'_1 \ldots \mathcal{R}'_n b$. Define $\delta(a) = b$. This function is indeed a coequalizer of $f$ and $g$ since if $f(a) = g(a)$ then $b \mathcal{R} b$ for all $b \in B$. Thus $\mathcal{R}$ is indeed the smallest congruence relation with this property.

Therefore, coequalizers exist in $\text{SCvx}$. \qed
for all \( a \in A \). A congruence relation \( \mathcal{R} \) on a super convex space \( B \) is an equivalence relation on \( B \) such that, for all sequences \( b : \mathbb{N} \to B \), and for all \( p \in \Delta_\mathbb{N} \),
\[
[\sum_{i \in \mathbb{N}} p_i b_i]_{\mathcal{R}} = \sum_{i \in \mathbb{N}} p_i [b_i]_{\mathcal{R}}.
\]

The quotient space \( B/\mathcal{R} \) consisting of equivalence classes of elements of \( B \) is a super convex space with the super convex space structure specified by equation (9), with the two axioms being satisfied for \( B/\mathcal{R} \) following from the fact that the two axioms are satisfied for \( B \). The projection map \( \pi : B \to B/\mathcal{R} \) sending an element to its equivalence class is the coequalizer of the pair \( \{f, g\} \) in \( \text{SCvx} \). \( \pi \) is a countably affine map by equation (9). \( \square \)

For geometric type super convex spaces, the coequalizer of a parallel pair collapses subspaces to a point.

**Example 3.16.** Given the inclusion map \( i : [0,1] \hookrightarrow \mathbb{R}_\infty \) and \( f : [0,1] \to \mathbb{R}_\infty \) given by \( x \mapsto \frac{1}{2} x + \frac{1}{2} \) the coequalizer is \( j : \mathbb{R}_\infty \to 2 \) specified in Example (3.3) because the equivalence class of any \( u \in \mathbb{R} \) is all of \( \mathbb{R} \), and hence the coequalizer is given by \( j \). For discrete type spaces the effect is less dramatic. For example, choose two monotonic functions \( f, g : \mathbb{N} \to \mathbb{N} \) which disagree on a finite number of terms. (See Lemma 3.2.)

**Example 3.17.** Take the two points \( x = \frac{1}{2} \) and \( y = \frac{2}{3} \) specifying a parallel pair \( 1 \rightrightarrows [0,1] \).

The coequalizer of \( \{x, y\} \) is the three point space \( \{0, u, 1\} \), with \( u \) being the equivalence class of the interval \((0,1)\), and with the structure defined by \( r_0 + (1-r)u = u \) for all \( r \in (0,1) \), \( r_1 + (1-r)u = u \) for all \( r \in (0,1) \), and \( r_0 + (1-r)1 = 1 \) for all \( r \in (0,1) \).

**Example 3.18.** Because \( 2 \) is a discrete type space and \( \Delta_2 \) is a geometric type space it follows by Example (3.3) that \( \text{SCvx}(\Delta_2, \Delta_2) \) has only constant functions. However for the quotient space of \( \Delta_2 \) given in Example (3.17) the set \( \text{SCvx}(2, \{0, u, 1\}) \) has two nonconstant maps given by (1) \( 1 \mapsto u \) and \( 0 \mapsto 0 \), and (2) \( 1 \mapsto u \) and \( 0 \mapsto 1 \).

While we make no use of equalizers in this article, equalizers exists in \( \text{SCvx} \) and are constructed in the usual fashion. For \( f, g : A \to B \) two countably affine maps the equalizer of the pair \( \{f, g\} \) is given by the subobject \( \text{Eq}(f, g) = \{ a \in A \mid f(a) = g(a) \} \) of \( A \) which is a super convex space because any countably affine sum of elements of \( \text{Eq}(f, g) \) is itself an element of \( \text{Eq}(f, g) \). Thus, all told, \( \text{SCvx} \) has all limits and all colimits.

### 3.3. The compact objects \( \mathbb{N} \) and \( \Delta_\mathbb{N} \)

**Lemma 3.19.** The hom set functor \( \text{SCvx}(\Delta_\mathbb{N}, \bullet) : \text{SCvx} \to \text{Set} \) preserves coproducts.

**Proof.** Let \( J \) be a set, viewed as a discrete category, and \( D : J \to \text{SCvx} \) a diagram. Let the colimit of the diagram be given by \( (\coprod_{j \in J} D_j, \lambda) \) denote the coproduct where the maps \( \lambda_i : D_i \to \coprod_{j \in J} D_j \) are the insertion maps. We show that the colimit of the composite functor \( J \to \text{SCvx} \xrightarrow{D} \text{SCvx} \to \text{Set} \), denoted by \( \text{colim}_J \text{SCvx}(\Delta_\mathbb{N}, D_j) \), is the colimit of \( \text{SCvx}(\Delta_\mathbb{N}, \bullet) \). When spelled out in detail coincides directly with \( (\text{SCvx}(\Delta_\mathbb{N}, \coprod_{j \in J} D_j), \text{SCvx}(\Delta_\mathbb{N}, \lambda)) \) where \( \text{SCvx}(\Delta_\mathbb{N}, \lambda) : \text{SCvx}(\Delta_\mathbb{N}, D_j) \to \text{SCvx}(\Delta_\mathbb{N}, \coprod_{j \in J} D_j) \) are the universal arrows to the vertex \( \text{SCvx}(\Delta_\mathbb{N}, \coprod_{j \in J} D_j) \).

The set \( \text{SCvx}(\Delta_\mathbb{N}, \coprod_{j \in J} D_j) \) consist of countably affine maps \( \psi : \Delta_\mathbb{N} \to \coprod_{j \in J} D_j \), and by Lemma 3.1 each such map \( \psi \) is specified by where it maps the Dirac measures. Hence
ψ(∑_{i ∈ N} p_i δ_i) = ∑_{i ∈ N} p_i ψ(δ_i), where ψ(δ_i) ∈ ∪_{j ∈ J} D_j, and hence can be written as ψ(δ_i) = ∑_{k ∈ N} q_k^i d_{s_i(k)} where s_i : n → ∪_{j ∈ J} D_j, with n ∈ N. Hence we have

\[ \psi(\sum_{i ∈ N} p_i δ_i) = \sum_{k ∈ N} \sum_{i ∈ N} p_i q_k^i d_{s_i(k)}. \]

This expression is generally not in standard normal form because there exist multiple elements \( d_{s_i(k)}, d_{s_j(k)} \in D_j \), and by rewriting it as a countable sum over the index set \( J \),

\[ \psi(\sum_{i ∈ N} p_i δ_i) = \sum_{j ∈ J} (\sum_{i ∈ N, k ∈ s_i^{-1}(j)} p_i q_k^i d_{s_i(k)}), \]

and we can put this expression into standard normal form to obtain

\[ (10) \quad \psi(\sum_{i ∈ N} p_i δ_i) = \sum_{k ∈ N} p_k d_{s(k)} \]

with \( s : m → ∪_{j ∈ J} D_j \) an injective function, and \( m ∈ N \). Hence every \( ψ ∈ \text{SCvx}(Δ_N, \bigcup_{j ∈ J} D_j) \) is equivalent to specifying a countable affine sum of elements of the coproduct \( \bigcup_{j ∈ J} D_j \) by way of an injective map \( s : m → ∪_{j ∈ J} D_j \) where \( m ∈ N \).

On the other hand, the coproduct \( \bigcup_{j ∈ J} \text{SCvx}(Δ_N, D_j) \) consists of all countable affine sums \( ∑_{k ∈ N} q_k^i ψ_{s(k)} \) where \( s : n → ∪_{j ∈ J} \text{SCvx}(Δ_N, D_j) \), \( n ∈ N \), and \( ψ_{s(k)} ∈ \text{SCvx}(Δ_N, D_{s(k)}) \). But because the functions \( ψ_{s(k)} \) are defined pointwise, a countable affine sum of such functions is just another function into the disjoint union. In other words, we have \( (∑_{k ∈ N} q_k^i ψ_{s(k)})(∑_{i ∈ N} p_i δ_i) = ∑_{k ∈ N} q_k(∑_{i ∈ N} p_i ψ_{s(k)}(δ_i)) \), where each \( ψ_{s(k)}(δ_i) = d_{s(k)}^i \in D_{s(k)} \). Making the substitution yields

\[ (∑_{k ∈ N} q_k^i ψ_{s(k)})(∑_{i ∈ N} p_i δ_i) = ∑_{k ∈ N} q_k (∑_{i ∈ N} p_i d_{s(k)}^i), \]

where the term \( d_{s(k)}^i \) is computed using the super convex space structure in the space \( D_{s(k)} \).

Thus the countable affine sum \( ∑_{k ∈ N} q_k^i ψ_{s(k)} \) is equivalent to a function \( Ψ : Δ_N → ∪_{j ∈ J} D_j \) specified by \( Ψ(δ_i) = ∑_{k ∈ N} q_k^i d_{s(k)}^i, \) so that

\[ (11) \quad Ψ(∑_{i ∈ N} p_i δ_i) = ∑_{i ∈ N} p_i Ψ(δ_i) = ∑_{i ∈ N} p_i (∑_{k ∈ N} q_k^i d_{s(k)}^i) = ∑_{k ∈ N} q_k (∑_{i ∈ N} p_i d_{s(k)}^i) = (∑_{k ∈ N} q_k^i ψ_{s(k)})(∑_{i ∈ N} p_i δ_i) \]

Comparing the two function spaces, characterized by the equations (10) and (11), we conclude that both spaces consist of all countable affine sums of elements of the disjoint union \( ∪D_j \), and hence \( \bigcup_{k ∈ N} \text{SCvx}(Δ_N, D_k) = \text{SCvx}(Δ_N, \bigcup_{k ∈ N} D_k) \).

The insertion map \( \text{SCvx}(Δ_N, D_i) → \bigcup_{k ∈ N} \text{SCvx}(Δ_N, D_k) \) is specified by \( ψ → 1ψ \) whereas the insertion map \( \text{SCvx}(Δ_N, D_i) → \text{SCvx}(Δ_N, \bigcup_{k ∈ N} D_k) \) is specified by \( ψ → ψ \) where, for all \( i ∈ N, \), \( \tilde{ψ}(δ_i) = ψ(δ_i) \). Since the last equation holds for all \( i ∈ N \) it follows the insertions maps yield the identical function which we view as \( ψ : Δ_N → ∪_{j ∈ J} D_j. \)

**Lemma 3.20.** The functor \( \text{SCvx}(N, \bullet) : \text{SCvx} → \text{Set} \) preserves coproducts.

**Proof.** Let \( D : J → \text{SCvx} \) where \( J \) is a discrete category and let \( D_i \) denote the object in \( \text{SCvx} \) which node \( i \) it gets mapped to. Like the preceding proof, writing out the sets \( \text{SCvx}(N, \bigcup_{j ∈ J} D_j) \) and \( \bigcup_{j ∈ J} \text{SCvx}(N, D_j) \) shows they are the same set of functions.
The set \( \mathbf{SCvx}(\mathbb{N}, \bigsqcup_{j \in J} D_j) \) consist of all countably affine maps \( \psi : \mathbb{N} \to \bigsqcup_{j \in J} D_j \), and so

\[
\psi(i) = \sum_{k \in \mathbb{N}} p_k d_{s_i(k)} \quad \text{by def. of the coproduct} \quad \bigsqcup_{j \in J} D_j \quad \text{where} \quad d_{s_i(k)} \in D_{s_i(k)}.
\]

On the other hand, by definition of the coproduct, \( \bigsqcup_{j \in J} \mathbf{SCvx}(\mathbb{N}, D_j) \) consists of all countably affine sums \( \sum_{k \in \mathbb{N}} p_k \psi_{s_i(k)} \) where \( s : \mathbb{N} \to \bigsqcup_{j \in J} \mathbf{SCvx}(\mathbb{N}, D_j) \), \( n \in \mathbb{N} \), and \( \psi_{s_i(k)} : \mathbb{N} \to D_{s_i(k)} \), and hence at each \( i \in \mathbb{N} \) we have by the pointwise definition of functions

\[
(\sum_{k \in \mathbb{N}} p_k \psi_{s_i(k)})(i) = \sum_{k \in \mathbb{N}} p_k \psi_{s_i(k)}(i) = \sum_{k \in \mathbb{N}} p_k d_{s_i(k)} \quad \text{for all} \quad i \in \mathbb{N}.
\]

Comparing the characterization of the elements of \( \mathbf{SCvx}(\mathbb{N}, \bigsqcup_{i \in I} D_i) \) given in equation (12) with the characterization of the elements of \( \bigsqcup_{i \in I} \mathbf{SCvx}(\mathbb{N}, D_i) \) given in equation (13), it follows that both elements \( \psi \) and \( \sum_{k \in \mathbb{N}} p_k \psi_{s_i(k)} \) are, for all \( i \in \mathbb{N} \), a countable affine sum of elements of the disjoint union \( \bigsqcup_{j \in J} D_j \). Since the two sets, \( \mathbf{SCvx}(\mathbb{N}, \bigsqcup_{i \in I} D_i) \) and \( \bigsqcup_{i \in I} \mathbf{SCvx}(\mathbb{N}, D_i) \), consist of all countable affine sums the two sets are equal.

For the insertion maps, we have \( \mathbf{SCvx}(\mathbb{N}, D_j) \to \mathbf{SCvx}(\mathbb{N}, \bigsqcup_{i \in I} D_i) \) maps \( \psi \to 1\psi \), while the insertion map \( \mathbf{SCvx}(\mathbb{N}, \bigsqcup_{i \in I} D_i) \to \bigsqcup_{i \in I} \mathbf{SCvx}(\mathbb{N}, D_i) \) sends \( \psi \to 1\psi \), where \( 1\psi = \psi(i) \) for all \( i \in \mathbb{N} \). Thus the two insertion maps coincide as well.

**Lemma 3.21.** The hom set functor \( \mathbf{SCvx}(\Delta_N, \bullet) : \mathbf{SCvx} \to \text{Set} \) preserves coequalizers.

**Proof.** Suppose \( f, g : A \to B \) in \( \mathbf{SCvx} \) with the coequalizer given by \( q : B \to B/\mathcal{R} \) where \( \mathcal{R} \) is the smallest congruence relation on \( B \) such that \( (f(a), g(a)) \in \mathcal{R} \) for all \( a \in A \). Thus \( \mathcal{R} \) is a subobject of \( B \times B \) given by the inclusion map \( \mathcal{R} \hookrightarrow B \times B \). Since every element in \( A \) is the image of some countably affine map \( \psi \in \mathbf{SCvx}(\Delta_N, A) \), which includes all the constant maps \( \langle a \rangle : \Delta_N \to A \) which sends every \( p \mapsto a \), we can restate the characterization of \( \mathcal{R} \) as the smallest congruence relation on \( B \) such that

\[
\forall p \in \Delta_N, \forall \psi \in \mathbf{SCvx}(\Delta_N, A) \quad ((f \circ \psi)(p), (g \circ \psi)(p)) \in \mathcal{R}.
\]

In **Set** the coequalizer of the parallel pair \( \{ \mathbf{SCvx}(\Delta_N, f), \mathbf{SCvx}(\Delta_N, g) \} \) is given by taking the smallest equivalence relation \( \mathcal{S} \) on the set \( \mathbf{SCvx}(\Delta_N, B) \) such that the property

\[
\forall \psi \in \mathbf{SCvx}(\Delta_N, A) \quad (\mathbf{SCvx}(\Delta_N, f) \psi, \mathbf{SCvx}(\Delta_N, g) \psi) = (f \circ \psi, g \circ \psi) \in \mathcal{S}
\]

holds, with the coequalizer given by the projection map \( \pi : \mathbf{SCvx}(\Delta_N, B) \to \mathbf{SCvx}(\Delta_N, B)/\mathcal{S} \).

Hence the distinction between the generating set for the congruence relation \( \mathcal{R} \) and the generating set for the equivalence relation \( \mathcal{S} \) resides in the evaluation map \( ev : \mathbf{SCvx}(\Delta_N, B) \otimes \Delta_N \to B \). Let \( \mathcal{R} \) denote the generating set for the relation \( \mathcal{R} \) while \( \mathcal{S} \) denotes the generating set for the relation \( \mathcal{S} \). If \( (f \circ \psi, g \circ \psi) \in \mathcal{S} \) then for every \( p \in \Delta_N \) we have \( (f(\psi(p)), g(\psi(p))) \in \mathcal{S} \). Conversely, if \( (b_1, b_2) \in \mathcal{S} \) then there exists \( \psi \in \mathbf{SCvx}(\Delta_N, A) \) and there exists a \( p \in \Delta_N \) such that \( b_1 = f(\psi(p)) \) and \( b_2 = g(\psi(p)) \), which implies \( (f \circ \psi, g \circ \psi) \in \mathcal{S} \). Hence it follows that the two sets \( \mathbf{SCvx}(\Delta_N, B/\mathcal{R}) \) and \( \mathbf{SCvx}(\Delta_N, B)/\mathcal{S} \) are identical. More explicitly, the equivalence relation \( \mathcal{S} \) is a subobject of the cartesian product space \( \mathbf{SCvx}(\Delta_N, B) \times \mathbf{SCvx}(\Delta_N, B) \), which has the structure of a super convex space since \( \mathbf{SCvx} \) is monoidal closed, and because \( \mathbf{SCvx}(\Delta_N, \bullet) \) preserves all limits, we have

\[
\mathcal{S} \hookrightarrow \mathbf{SCvx}(\Delta_N, B) \times \mathbf{SCvx}(\Delta_N, B) \overset{iso}{\to} \mathbf{SCvx}(\Delta_N, B \times B).
\]
Using the monoidal closed structure the above map is equivalent to the countably affine evaluation map $S \otimes \Delta_n \to B \times B$ sending $((\phi_1, \phi_2) \otimes p) \mapsto (\phi_1(p), \phi_2(p))$, and hence $\sum_{i \in \mathbb{N}} q_i((\phi_{1,i}, \phi_{2,i}) \otimes p_i) \mapsto (\sum_{i \in \mathbb{N}} q_i(\phi_{1,i}(p_i)), \sum_{i \in \mathbb{N}} q_i(\phi_{2,i}(p_i)))$. The relationship between the two generating sets $S$ and $R$ holds, with the coequalizer given by the projection map $\pi : SCvx(\mathbb{N}, B) \to SCvx(\mathbb{N}, B)/S$.

Hence the distinction between the generating set for the congruence relation $\mathcal{R}$ and the generating set for the equivalence relation $\mathcal{S}$ resides in the evaluation map $ev : SCvx(\mathbb{N}, B) \otimes \mathbb{N} \to B$. Let $R$ denote the generating set for the relation $\mathcal{R}$ while $S$ denotes the generating set for the relation $\mathcal{S}$. If $(f \circ \psi, g \circ \psi) \in \mathcal{S}$ then for every $i \in \mathbb{N}$ we have $(f(\psi(i)), g(\psi(i))) \in S$. Conversely, if $(b_1, b_2) \in S$ then there exists $\psi \in SCvx(\mathbb{N}, A)$ and there exists an $i \in \mathbb{N}$ such that $b_1 = f(\psi(i))$ and $b_2 = g(\psi(i))$, which implies $(f \circ \psi, g \circ \psi) \in \mathcal{S}$. Hence it follows that the two sets $SCvx(\mathbb{N}, B/R)$ and $SCvx(\mathbb{N}, B)/S$ are identical. More explicitly, the equivalence relation $\mathcal{S}$ is a subobject of the cartesian product space $SCvx(\mathbb{N}, B) \times SCvx(\mathbb{N}, B)$, which has the structure of a super convex space since $SCvx$ is monoidal closed, and because $SCvx(\mathbb{N}, \bullet)$ preserves all limits, we have

$$\mathcal{S} \Rightarrow SCvx(\mathbb{N}, B) \times SCvx(\mathbb{N}, B) \xrightarrow{iso} SCvx(\mathbb{N}, B \times B).$$

Using the monoidal closed structure the above map is equivalent to the countably affine evaluation map $S \otimes \mathbb{N} \to B \times B$ sending $((\phi_1, \phi_2) \otimes j) \mapsto (\phi_1(j), \phi_2(j))$, and hence $\sum_{i \in \mathbb{N}} q_i((\phi_{1,i}, \phi_{2,i}) \otimes j_i) \mapsto (\sum_{i \in \mathbb{N}} q_i(\phi_{1,i}(j_i)), \sum_{i \in \mathbb{N}} q_i(\phi_{2,i}(j_i)))$. The relationship between the two generating sets $S$ and $R$ shows that $\mathcal{R} \Rightarrow B \times B$ is isomorphic to $S \otimes \mathbb{N} \Rightarrow B \times B$.

**Theorem 3.23.** The objects $\Delta_n$ and $\mathbb{N}$ are categorically compact objects in $SCvx$.

**Proof.** By Lemma 3.19 and Lemma 3.21 the functor $SCvx(\Delta_n, \bullet)$ preserves both coproducts and coequalizers, and hence $SCvx(\Delta_n, \bullet)$ preserves all colimits in $SCvx$. 

\[ \square \]
Similarly, by Lemma 3.20 and Lemma 3.22 the functor $\text{SCvx}(\mathbb{N}, \bullet)$ preserves both coproducts and coequalizers, and hence $\text{SCvx}(\mathbb{N}, \bullet)$ preserves all colimits in $\text{SCvx}$.

By the definition of categorically compact objects, given in equation (11), it follows that both $\text{SCvx}(\Delta_\mathbb{N}, \bullet)$ and $\text{SCvx}(\mathbb{N}, \bullet)$ are categorically compact objects in $\text{SCvx}$.

**Corollary 3.24.** There is a unique extension of the functor $\Sigma' : \Omega \to \text{Std}_2$ along the inclusion functor $\iota : \Omega \to \text{SCvx}$ to $\Sigma : \text{SCvx} \to \text{Std}_2$.

**Proof.** By Theorem 3.23 the category $\Omega$, consisting of the two objects $\Delta_\mathbb{N}$ and $\mathbb{N}$, is a compact subcategory of $\text{SCvx}$. By Lemma 3.15 $\text{SCvx}$ has coequalizers, and by Lemma 3.13 $\text{SCvx}$ has all coproducts. Hence $\text{SCvx}$ has all colimits, thus all filtered colimits. $\text{Std}_2$ also has filtered colimits. The coproduct of any two standard measurable spaces $X$ and $Y$ is generated by the disjoint union of the two $\sigma$-algebras $\Sigma_X$ and $\Sigma_Y$, and hence is has a countably generated field $\mathcal{F}$ which gives $\Sigma_{X+Y} = \sigma(\mathcal{F})$. Given any two parallel arrows $f, g : X \to Y$ a standard measurable space $Z$, which is a quotient space of $Y$, can be defined using the finite fields $F_i$, which are used to asymptotically generate the field $\mathcal{F}$ for $\Sigma_Y = \sigma(\mathcal{F})$, by requiring $\{f(x), g(x)\} \in U \in F_i$ for all $x \in X$. Whenever $f(x) \in U$ and $g(x) \in V$ with $U \neq V$ then the two measurable sets in that field $F_i$ are equivalent.

Thus we can apply Lemma 3.11 to obtain the functor $\Sigma$ which is the unique extension of $\Sigma'$ along the inclusion functor $\iota$. □

The extended functor $\Sigma$ is characterized just as $\Sigma'$ was characterized - a super convex space $A$ is assigned the largest $\sigma$-algebra such that every countably affine map $m : \mathcal{G}(\mathbb{N}) \to A$ is a measurable function. The basic measurable sets of $\mathcal{G}(\mathbb{N})$ can be characterized as the sets

\[
\{ \sum_{i \in \mathbb{N}} p_i \delta_i | p \in \mathcal{G}(\mathbb{N}) \text{ with } p_i \in U_i \text{ for all } i \in \mathbb{N} \text{ where each } U_i \in \mathcal{B}([0, 1]) \},
\]

where $\mathcal{B}([0, 1])$ is the standard $\sigma$-algebra on the $[0, 1]$ generated by the intervals. For $a : \mathbb{N} \to A$ any sequence in $A$ it follows $(a) : \mathcal{G}(\mathbb{N}) \to A$ is a countably affine map, and hence the sets

\[
\{ \sum_{i \in \mathbb{N}} p_i a_i | p \in \mathcal{G}(\mathbb{N}) \text{ with } p_i \in U_i \text{ for all } i \in \mathbb{N} \text{ where each } U_i \in \mathcal{B}([0, 1]) \}
\]

are measurable sets in $A$.

The extension property $\Sigma' = \Sigma \circ \iota$ says that $\Sigma' \mathbb{N} = \Sigma \mathbb{N}$ and $\Sigma' \Delta_\mathbb{N} = \Sigma \Delta_\mathbb{N}$, and we subsequently use either $\Sigma \mathbb{N}$ or $\Sigma' \mathbb{N}$, and similarly for $\Delta_\mathbb{N}$, depending upon our viewpoint.

The reader should keep this in mind, and not hesitate to employ the alternative viewpoint.

**Lemma 3.25.** The countably affine map $\epsilon_\mathbb{N} : \Delta_\mathbb{N} \to \mathbb{N}$ is, when viewed in $\text{Std}$, a measurable map $\epsilon_\mathbb{N} : \mathcal{G}\mathbb{N} \to (\mathbb{N}, \mathcal{P}\mathbb{N})$.

**Proof.** If $W \in \mathcal{P}\mathbb{N}$, by definition of the $\sigma$-algebra on $\mathcal{G}\mathbb{N}$, each function $ev_W : \mathcal{G}\mathbb{N} \to [0, 1]$ given by $ev_W(\sum_{i=0}^{\infty} p_i \delta_i) = \sum_{i=0}^{\infty} p_i$ is measurable. Taking $W = \downarrow n = \{0, 1, \ldots, n-1\}$, it follows that $ev_W^{-1}(0) = \{ \sum_{i=0}^{\infty} p_i \delta_i \in \mathcal{G}\mathbb{N} | p_i = 0 \text{ for all } i < n \}$ is a measurable set in $\mathcal{G}\mathbb{N}$. Since $\epsilon_\mathbb{N}^{-1}(n) = ev_W^{-1}(0)$ we conclude the function $\epsilon_\mathbb{N}$ is measurable. □

**Lemma 3.26.** For every space $A$ the measurable space $\Sigma A$ is a separated measurable space, and for every $a \in A$ the set $\{a\}$ is a measurable set.
Conversely, if all the evaluation maps are measurable then the subsets measurable in \( \sigma \)-algebra \( \Sigma \) are measurable. Therefore on objects, \( \Sigma \)-algebra \( \Sigma \) with the largest \( \sigma \)-algebra such that all of the countably affine maps \( m : \mathcal{G}N \to A \) are measurable. If \( m = a \) with \( a = \sum_{i \in \mathbb{N}} p_i a_i \) for some \( p \in \mathcal{G}N \). Because each \( \{ p_i \} \) is a measurable set in \([0,1]\) it follows that \( m^{-1}(a) \) is, by the characterization of measurable sets in \( \mathcal{G}N \) given in equation (14), a measurable set. Thus \( \{ a \} \) is a measurable set in \( \Sigma A \).

Lemma 3.27. The composite functor \( \Sigma \circ \mathcal{P} = \mathcal{G} \).

Proof. Given any measurable space \( X \) we have \( \mathcal{P}(X) \) is the super convex space whose underlying set coincides with the underlying set of \( \mathcal{G}(X) \). The functor \( \Sigma \) assigns to the underlying set \( \mathcal{P}(X) \) the largest \( \sigma \)-algebra such that all of the countably affine maps \( m : \mathcal{G}N \to \mathcal{P}(X) \) are measurable.

The set \( \text{SCvx}(\Delta_\mathbb{N}, \mathcal{P}X) \) is determined by all sequences \( Q : \mathbb{N} \to \mathcal{P}(X) \), giving all the possible countably affine maps \( \langle Q \rangle \in \text{SCvx}(\Delta_\mathbb{N}, \mathcal{P}(X)) \) where each \( Q_i \in \mathcal{P}(X) \). Let \( \Sigma_{\mathcal{P}(X)} \) denote the \( \sigma \)-algebra on \( \mathcal{P}(X) \) such that all the countably affine maps \( \langle Q \rangle \) are measurable. Given any measurable set \( U \) in \( X \) let \( ev_U : \mathcal{P}(X) \to [0,1] \) denote the evaluation map sending \( Q_i \mapsto Q_i(U) \). Since the composite map \( ev_U \circ \langle Q \rangle : \Delta_\mathbb{N} \to [0,1] \) given by \( p \mapsto \sum_{i \in \mathbb{N}} p_i Q_i(U) \) is measurable and \( \mathcal{P}(X) \) has the largest \( \sigma \)-algebra on \( X \) such that the maps \( \langle Q \rangle \) are measurable it follows that \( ev_U \) is a measurable function. Hence the \( \sigma \)-algebra \( \Sigma_{\mathcal{P}(X)} \) on \( \mathcal{P}(X) \) contains the \( \sigma \)-algebra on \( \mathcal{P}(X) \) as specified by the Giry monad - where the \( \sigma \)-algebra on \( \mathcal{P}(X) \) is specified as the smallest \( \sigma \)-algebra such that all of the evaluation maps \( ev_U \) are measurable. Conversely, if all the evaluation maps are measurable then the subsets \( ev_U^{-1}((r,1]) \) are measurable in \( \mathcal{P}(X) \), and \( \langle Q \rangle^{-1}(ev_U^{-1}((r,1]) = \{ p \in \Delta_\mathbb{N} \mid \sum_{i \in \mathbb{N}} p_i Q_i(U) \in (r,1] \} \) which is a measurable set in \( \mathcal{G}(\mathbb{N}) \), and hence \( \langle Q \rangle \) is measurable since \( \Sigma_{\mathcal{P}(X)} \) endows the set \( \mathcal{P}(X) \) with the largest \( \sigma \)-algebra. Therefore on objects, \( \Sigma \circ \mathcal{P} = \mathcal{G} \).

Now suppose that \( f : X \to Y \). Then by the preceding argument we only need to show that the underlying set functions are identical. But this is obvious because \( \mathcal{P}(f) \) is defined as the set function \( \mathcal{G}(f) \) viewed as a countably affine map, \( \mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y) \). Because \( \mathcal{G}(f) \) is a measurable function it follows that \( \mathcal{P}(f) \) is a measurable function under the \( \sigma \)-algebra on \( \mathcal{P}(X) \) and \( \mathcal{P}(Y) \) specified by the functor \( \Sigma \).

4. The dense functor \( \iota : \Omega \to \text{SCvx} \)

Lemma 4.1. In \( \text{SCvx} \) the countably affine map \( \epsilon_\mathbb{N} \) is an epimorphism.

Proof. Suppose that \( f, g : \mathbb{N} \to A \) are any two parallel arrows in \( \text{SCvx} \), and that \( f \circ \epsilon_\mathbb{N} = g \circ \epsilon_\mathbb{N} \). The countably affine maps are completely specified by where they send the elements of \( \mathbb{N} \),
and hence \((f \circ \epsilon_n)(\delta_i) = (g \circ \epsilon_n)(\delta_i)\) for all \(i \in \mathbb{N}\), which says \(f(i) = g(i)\) for all \(i \in \mathbb{N}\). Hence \(f = g\).

**Lemma 4.2.** The full subcategory of \(\mathbf{SCvx}\) consisting of the single object \(\Delta_\mathbb{N}\) is a dense subcategory of \(\mathbf{SCvx}\).

**Proof.** We show the functor \(\mathbf{SCvx} \to \mathbf{Set}^{\Delta_\mathbb{N}^{op}}\) given by \(A \mapsto \mathbf{SCvx}(\bullet, A)\) is full and faithful. 

*Faithful* Let \(\overline{\pi} : \Delta_\mathbb{N} \to A\) given by \(\delta_i = a\) for all \(i\) denote the constant maps with value \(a\). Suppose that \(f, g : A \to B\) with \(f(a) \neq g(a)\). Then the induced natural transformations 

\[ f_* = \bigtriangleup N \Delta_\mathbb{N} \partial_\mathbb{N} \partial_\mathbb{N} \tag{72}\]

\[ g_* : \mathbf{SCvx}(\bullet, A) \to \mathbf{SCvx}(\bullet, B) \]

specified by composition on the left with \(f\) and \(g\), respectively, yield two distinct natural transformations since \(f_*(\overline{\pi}) = f(a) \neq g(a) = g_*(\overline{\pi})\).

*Fullness* Suppose that \(J \in \text{Nat}(\mathbf{SCvx}(\bullet, A), \mathbf{SCvx}(\bullet, B))\). Then at the component \(\Delta_N\), for every \(a \in A\), we have \(J(\overline{\pi}) \in \mathbf{SCvx}(\Delta_N, B)\) which is a constant map into \(B\), hence specifies a unique point in \(B\). Thus the function \(J_{\Delta_N}\) determines a set mapping \(A \to B\) specified by \(a \mapsto J_{\Delta_N}(\overline{\pi})\).

The fact that \(J_{\Delta_N}\) specifies a countably affine map follows from naturality. We have the commutative \(\mathbf{SCvx}\)-diagram

\[
\begin{array}{ccc}
\Delta_\mathbb{N} & \xrightarrow{\overline{p}} & \Delta_\mathbb{N} \\
\sum_{i \in \mathbb{N}} p_i a_i & \\ A & \xrightarrow{\langle a_i \rangle} & \langle a_i \rangle
\end{array}
\]

where \(\overline{p}\) is a constant map into the element \(p \in \Delta_\mathbb{N}\), and the composite map \(\langle a_i \rangle \circ \overline{p}\) is the constant map \(\sum_{i \in \mathbb{N}} p_i a_i : \Delta_\mathbb{N} \to 1 \to A\) with value \(\sum_{i \in \mathbb{N}} p_i a_i \in A\). By naturality we have the commutative \(\mathbf{Set}\)-diagram

\[
\begin{array}{ccc}
\Delta_\mathbb{N} & \xrightarrow{\overline{p}} & \Delta_\mathbb{N} \\
\mathbf{SCvx}(\Delta_\mathbb{N}, A) & \xrightarrow{J_{\Delta_N}} & \mathbf{SCvx}(\Delta_\mathbb{N}, B) \\
\mathbf{SCvx}(\overline{p}, A) & \xrightarrow{J_{\Delta_N}} & \mathbf{SCvx}(\overline{p}, B)
\end{array}
\]

Since \(J_{\Delta_N}(\langle a_i \rangle) \in \mathbf{SCvx}(\Delta_\mathbb{N}, B)\) it is specified by a family of points, \(J_{\Delta_N}(\langle a_i \rangle) = (b_i)\). The equality in the lower right hand corner thus shows that the map defined by \(J_{\Delta_N}\) on the constant functions \(\overline{\pi}\) specifies a countably affine \(A \to B\), and hence \(\overline{y}\) is full, i.e., \(J_{\Delta_N} = y(m) = \mathbf{SCvx}(\bullet, m)\) for some \(m \in \mathbf{SCvx}(A, B)\). \(\square\)

**Corollary 4.3.** The inclusion functor \(i : \Omega \to \mathbf{SCvx}\) is dense. Equivalently, the restricted dual Yoneda mapping \(\mathcal{Y}_{\Omega^{op}} : \mathbf{SCvx} \to \mathbf{Set}^{\Omega^{op}}\) is full and faithful.
Proof. Suppose \( J \in \text{Nat}(\text{SCvx}(\iota, A), \text{SCvx}(\iota, B)) \). By naturality we have the commutative \textbf{Set}-diagram

\[
\begin{array}{ccc}
\text{SCvx}(\Delta_N, A) & \xrightarrow{J_{\Delta_N}} & \text{SCvx}(\Delta_N, B) \\
\text{SCvx}(\iota, A) & \xrightarrow{f \circ \iota} & \text{SCvx}(\iota, B) \\
\text{SCvx}(N, A) & \xrightarrow{J_N} & \text{SCvx}(N, B) \\
\end{array}
\]

and by Lemma 4.2 it follows that \( J_{\Delta_N} = \text{SCvx}(\Delta_N, m) \) for some \( m \in \text{SCvx}(A, B) \). Hence the upper right hand corner in the diagram is given by \( m \circ f \circ \iota = J_N(f) \). Since, by Lemma 4.1, \( \iota \) is an epimorphism it follows that \( J_N(f) = m \circ f \), and hence \( J_N = \text{SCvx}(N, m) \).

\[\square\]

5. The dense functor \( \kappa : \Omega \to \text{Std}^G \)

The basic idea that there are two distinct factorizations of the codense functor \( \Sigma' \), as illustrated in Diagram 6, was the motivation for our scheme for searching for a category which is isomorphic (or more generally, equivalent) to the category \( \text{Std}^G \). This material is not necessary to arrive at the conclusion that \( \text{SCvx} \) is isomorphic to \( \text{Std}^G \), however it does explicitly show how every Giry algebra specifies and super convex space and that morphisms of \( G \)-algebras are countably affine maps.

**Lemma 5.1.** Let \( \mathcal{F} : \text{Std} \to \text{Std}^G \) denote the functor in the Eilenberg-Moore adjoint factorization of \( \mathcal{G} \). The functor \( \kappa : \Omega \to \text{Std}^G \), defined on objects by \( \kappa(N) = (\Sigma' N, \epsilon_N) \) and \( \kappa(\Delta_N) = (\mathcal{G} N, \mu_N) \), and on arrows by \( m \mapsto \Sigma m \) for every \( f \in \text{Arr}(\Omega) \) is a dense functor.

**Proof.** We first make the observation that every \( \mathcal{G} \)-algebra \( h : \mathcal{G}(X) \to X \) specifies a super convex space structure on \( X \), denoted \( X_h \), and defined by \( \sum_{i \in \mathbb{N}} p_i x_i := h(\sum_{i \in \mathbb{N}} p_i \delta_{x_i}) \). The super convex space structure of \( X_h \) makes the measurable map \( h \) a countably affine map because for all countably affine sums \( \sum_{i \in \mathbb{N}} p_i \delta_{P_i} \in \mathcal{G}^2(X) \), we have

\[
(h \mu_X)(\sum_{i \in \mathbb{N}} p_i \delta_{P_i}) = (h \circ \mathcal{G}(h))(\sum_{i \in \mathbb{N}} p_i \delta_{P_i}) \quad \text{because } h \text{ is a } \mathcal{G} \text{-algebra}
\]

\[
(15) \quad h(\sum_{i \in \mathbb{N}} p_i P_i) = h(\sum_{i \in \mathbb{N}} p_i \delta_{h(P_i)}) \quad \text{property of } \mu_X \text{ and } \mathcal{G}(f)
\]

\[
= \sum_{i \in \mathbb{N}} p_i h(P_i) \quad \text{def. of } h \text{ on count. aff. sums in } X_h
\]

Thus, every \( \mathcal{G} \)-algebra has the property of being a countably affine map with respect to the induced super convex space structure on \( X \).
Moreover, given any map of \( \mathcal{G} \)-algebras, \( f : (X, h) \to (Y, k) \) it follows that \( f \) is a countably affine map because

\[
\begin{align*}
    f(\sum_{i \in \mathbb{N}} p_i x_i) &= f(h(\sum_{i \in \mathbb{N}} p_i \delta_i)) & \text{by definition of } X_h \\
    &= \delta(\sum_{i \in \mathbb{N}} p_i \delta_i) & \text{using } f \circ h = k \circ \mathcal{G}(f) \\
    &= \sum_{i \in \mathbb{N}} p_i \delta_i f(x_i) & \text{by definition of pushforward} \\
    &= \sum_{i \in \mathbb{N}} p_i f(x_i) & \text{because } k \text{ is countably affine}
\end{align*}
\]

Hence, it follows that any map of \( \mathcal{G} \)-algebras \( f : (\mathcal{G} \mathbb{N}, \mu_{\mathbb{N}}) \to (X, h) \) is uniquely specified by choosing a countably infinite number of points of \( X \), so that \( f(\sum_{i \in \mathbb{N}} p_i \delta_i) = \sum_{i \in \mathbb{N}} p_i f(\delta_i) = \sum_{i \in \mathbb{N}} p_i x_i \).

To prove the functor \( \kappa \) is dense we proceed to show that the functor \( \textbf{Std}^{\mathcal{G}} \to \textbf{Set}^{\text{op}} \) given by \( (X, h) \mapsto \textbf{Std}^{\mathcal{G}}(\kappa(\bullet), (X, h)) \) is full and faithful. The reasoning precisely follows that of the preceding proof, using the induced super convex space structure \( X_h \). Hence, to a large extent, we mimic the preceding proof.

**Faithful** Let \( \overline{\tau} : \mathcal{G} \mathbb{N} \to X \) given by \( \delta_i = x \) for all \( i \) denote the constant maps with value \( x \). Suppose that \( f, g : X \to Y \) with \( f(x) \neq g(x) \). Then the induced natural transformations \( f_*, g_* : \textbf{Std}^{\mathcal{G}}(\bullet, (X, h)) \to \textbf{Std}^{\mathcal{G}}(\bullet, (Y, k)) \) specified by composition on the left with \( f \) and \( g \), respectively, yield two distinct natural transformations since \( f_*(\overline{\tau}) = f(x) \neq g(x) = g_*(\overline{\tau}) \).

(Note that because \( 1 = (\mathcal{G}(\mathbf{1}) \xrightarrow{id} \mathbf{1}) \) is a separator in \( \textbf{Std}^{\mathcal{G}} \) the points of a \( \mathcal{G} \)-algebra are specified by a pair \( (P, x) : \mathbf{1} \to (X, h) \), with \( P \in \mathcal{G}X \) and \( x \in X \) where \( h(P) = x \). Thus two points \( (P_1, x_1) \) and \( (P_2, x_2) \) are distinct if either \( P_1 \neq P_2 \) or \( x_1 \neq x_2 \). Hence, since the given natural transformations \( f_* \) and \( g_* \) differ in the second component, the natural transformations are distinct.)

**Fullness** Suppose that \( J \in \text{Nat}(\textbf{Std}^{\mathcal{G}}(\kappa(\bullet), (X, h)), \textbf{Std}^{\mathcal{G}}(\kappa(\bullet), (Y, k))) \). At component \( \Delta_{\mathbb{N}} \in \Omega \) we have \( \kappa(\Delta_{\mathbb{N}}) = (\mathcal{G} \mathbb{N}, \mu_{\mathbb{N}}) \), and for every \( x \in X \) we have \( J(\overline{x}) \in \textbf{Std}^{\mathcal{G}}((\mathcal{G} \mathbb{N}, \mu_{\mathbb{N}}), (Y, k)) \) which is a constant map into \( Y \), hence specifies a unique point in \( Y \). Thus the function \( J_{\mathbb{N}} \) determines a set mapping \( X \to Y \) specified by \( x \mapsto J_{\mathbb{N}}(\overline{x}) \).

The fact that \( J_{\mathbb{N}} \) specifies a countably affine map follows from naturality. We have the commutative \( \textbf{Std}^{\mathcal{G}} \)-diagram

\[
\begin{array}{c}
\text{(} \mathcal{G} \mathbb{N}, \mu_{\mathbb{N}}) \\
\sum_{i \in \mathbb{N}} p_i x_i
\end{array} \xrightarrow{\overline{p}} \begin{array}{c}
\text{(} \mathcal{G} \mathbb{N}, \mu_{\mathbb{N}}) \\
\sum_{i \in \mathbb{N}} p_i x_i
\end{array}
\]

where \( \overline{p} \) is a constant map into the element \( p \in \mathcal{G} \mathbb{N} \), and the composite map \( \langle x_i \rangle \circ \overline{p} \) is the constant map \( \sum_{i \in \mathbb{N}} p_i x_i : \mathcal{G} \mathbb{N} \to \mathbf{1} \to X \) with value \( \sum_{i \in \mathbb{N}} p_i x_i \in X \). By naturality we have the
Since $J_N(\langle x_i \rangle) \in \text{Std}^G((G^N, \mu \Sigma^N), (Y,k))$ it is specified by a family of points, $J_N(\langle x_i \rangle) = \langle y_i \rangle$. The equality in the lower right hand corner thus shows that the map defined by $J_N$ on the constant functions $x$ specifies a countably affine map $m : X \to Y$, and hence $y$ is full, i.e., $J_N = y(m) = \text{Std}^G(\bullet, m)$ for some $m \in \text{Std}^G((X,h), (Y,k))$. \hfill $\square$

6. STANDARD MEASURABLE SPACES

The two defining characteristics of a standard measurable space $X$ are

(1) Its $\sigma$-algebra $\Sigma_X$ is asymptotically generated by a countably generated field $\mathcal{F}$, so that $\Sigma_X = \sigma(\mathcal{F})$. Let \{\(F_i\)\}_{i=1}^{\infty}$ be the sequence of finite fields which asymptotically generate the field $\mathcal{F} = \lim_{n \to \infty} \bigcup_{i=1}^{n} F_i$.

Each such field $F_n$ itself is generated by a partition of $X$, and we take $F_n$ as the field generated by the partition $\text{Atoms}_n : X \to \mathfrak{n}$ which specifies the atoms of the finite field $F_n$. Thus every element $U \in F_n$ is a union of atoms which are obtained from the partition map $\text{Atoms}_n$. Thus $F_1$ has $\text{Atoms}_1 : X \to \mathfrak{1}$ given by the trivial partition of making no distinction among the elements of $X$, and hence $F_1 = \{X, \emptyset\}$. The finite field $F_2$ is determined by a partition $\text{Atoms}_2 : X \to \mathfrak{2}$, hence has two atoms, say $U_1 \subset X$ and $U_2 \subset X$, and therefore the field $F_2 = \{X, U_1, U_2, \emptyset\}$. The finite field $F_3$ is determined by a partition $\text{Atoms}_3 : X \to \mathfrak{3}$, hence has three atoms, say $U_1 \subset X$, $U_{2,1} \subset X$, and $U_{2,2} \subset X$, where $U_{2,1} \cup U_{2,2} = U_2$, and therefore the field $F_3$ has $2^3$ elements. At each iteration, one of the partitions consisting of more than one element is refined by splitting it into two separate nonempty subsets.

To view the partitioning as a function to $\mathfrak{n}$ simply take the composite map
Diagram 4. Every standard measurable space $X$ is generated by a countable family of maps $A_n$ which partition $X$ into atoms.

For brevity, denote the composite map as $A_n$.

Since $X$ is a standard space the finite fields $F_n$ which generate the field $\mathcal{F}$ on $X$ satisfy the property that every element $X_{n,i} = A_n^{-1}(i)$ which is an atom of the field $F_n$ either remains an atom in $F_{n+1}$ or gets split into two separate atoms of the finite field $F_{n+1}$. Thus $X_{n,i}$ is an atom of $F_{n+1}$ or $X_{n,i}$ gets split into two, say $X_{n,i_1}$ and $X_{n,i_2}$. Using the isomorphism

$$\{0, 1, 2, \ldots, i - 1, i_1, i_2, i + 1, i + 2, \ldots, n - 1\} \cong \{0, 1, 2, \ldots, n\}$$

we have a monotonic decreasing function $\phi : \mathbb{N} \to \mathbb{N}$ specified by

$$\phi : \mathbb{N} \to \mathbb{N} : k \mapsto \begin{cases} k & \text{for all } k \leq i \\ i & \text{for } k = i + 1 \\ k - 1 & \text{for all } k > i + 1 \end{cases}$$

such that the $\text{Std}$-diagram

$$\begin{array}{ccc}
(X, F_{n+1}) & \xrightarrow{1_X} & (X, F_n) \\
\mathcal{A}_{n+1} & \downarrow & \mathcal{A}_n \\
(\mathbb{N}, \mathcal{P}\mathbb{N}) & \xrightarrow{\Sigma \phi} & (\mathbb{N}, \mathcal{P}\mathbb{N})
\end{array}$$

Diagram 5. The refinement process used in generating the finite fields associated with every standard measurable space satisfy the commutativity condition depicted in the diagram.

(2) If $\{G_n\}_{n=1}^\infty$ is a sequence of atoms with $G_n \in F_n$ such that $G_{n+1} \subset G_n$ for $n = 1, 2, \ldots$, then $\bigcap_{n=1}^\infty G_n \neq \emptyset$. Since $G_n \in F_n$ is an atom it is given by $G_n = A_n^{-1}(i)$ for some index $i \in \mathbf{n}$.

A basis for a field is an asymptotically generating sequence of finite fields with the property that a decreasing sequence of atoms cannot converge to the empty set. Property (2) of a sequence of fields is called the finite intersection property.
A measurable space \((X, \Sigma_X)\) is called standard if \(\Sigma_X = \sigma(\mathcal{F})\) for some field \(\mathcal{F}\) which possesses a basis.

Since the generating fields are given to us it may be the case that some \(F_n\) are omitted. The fact some \(F_n\) may be omitted is immaterial; what matters is the refinement of the partition at each step which makes the partitioning of \(X\) finer as the indexing set increases. Regardless of the \(\mathcal{F}\) characterizing the finite field \(\mathcal{F}\) which the sets \(F_i\) generate, the number of atoms is a monotonically increasing sequence in \(\mathbb{N}\) as a function of the indexing set. If the finite fields are given to us with say \(F_n\) having \(k\) atoms and \(F_{n+1}\) having \(k + m\) atoms then in Diagram 5 the monotonic function \(\phi\) will be the composite of \(m\) monotonic decreasing functions, which is a monotonic decreasing function. Hence we will assume without loss of generality that the sequence of finite fields \(\{F_n\}_{n=1}^\infty\) is such that \(F_{n+1}\) has one more atom than the finite field \(F_n\).

## 7. The codense functor \(\Sigma' : \Omega \to \text{Std}_2\)

Let \([\mathbb{N}]\) denote the full subcategory of \(\text{SCvx}\) consisting of the single object \(\mathbb{N}\), and let \(\iota_{\mathbb{N}} : [\mathbb{N}] \to \Omega\) denote the inclusion functor.

**Lemma 7.1.** Suppose that \(U : X \to \Sigma n\) is a measurable function with image \(n\). Then the composite map \(X \xrightarrow{U} \Sigma n \to \Sigma \mathbb{N}\) is a measurable function, and

\[
\text{if } J \in \text{Nat}(\text{Std}_2(X, \Sigma'(\iota_{\mathbb{N}})), \text{Std}_2(1, \Sigma'(\iota_{\mathbb{N}}))) \text{ then } J(U^n) \in n.
\]

**Proof.** Proof by contradiction. Suppose, to obtain a contradiction, that \(J_{\mathbb{N}}(U) = m\), with \(m \geq n\) which does not lie in the image of \(n\). Choose a monotonic function \(\phi : \mathbb{N} \to \mathbb{N}\) which sends all elements \(k \in \mathbb{N}\) such that \(k \geq m\) to \(m\), and all elements \(k \in \mathbb{N}\) such that \(k < m\) get mapped to 0. Thus \((\phi \circ U)(x) = \chi_X(x)\mathbb{1} + \chi_{\phi}(x)m = 10 + 0m = 0\), which is the constant function with value 0. The naturality condition \(\phi(J(U)) = J_{\mathbb{N}}(\phi \circ U)\), obtained by the monotonic mapping sending everything to zero, requires \(J_{\mathbb{N}}(\mathbb{1}) = 0\) where \(\mathbb{1}\) is the constant function. The hypothesis \(J_{\mathbb{N}}(U) = m\) implies \(\phi(J_{\mathbb{N}}(U)) = \phi(m) = m\). We thus have

\[
m = \phi(m) = \phi(J(U)) = J_{\mathbb{N}}(\phi \circ U) = J_{\mathbb{N}}(\mathbb{1}) = 0.
\]

Hence we must conclude that \(J(U) \in n\). \(\square\)

**Lemma 7.2.** The composite functor \([\mathbb{N}] \xrightarrow{\iota_{\mathbb{N}}} \Omega \xrightarrow{\Sigma'} \text{Std}_2\) is a codense functor.

**Proof.** The functor \(\Sigma' \circ \iota_{\mathbb{N}}\) is codense if and only if the functor \(y : \text{Std}_2^{\text{op}} \to \text{Set}^{[\mathbb{N}]},\) specified on objects by \(y(X) = \text{Std}_2(X, \Sigma'(\iota_{\mathbb{N}}))\) is full and faithful.

The functor \(y\) is faithful follows because if \(x_1, x_2 \in X\) then there exists a measurable set \(U\) in \(X\) such that \(x_1 \in U\) and \(x_2 \notin U\), and hence the characteristic function \(\chi_U : X \to 2 \xrightarrow{\text{incl}} 2 \xrightarrow{\iota_{\mathbb{N}}} \mathbb{N}\) suffices to separate the two points. More explicitly, the measurable function \(U : X \to \mathbb{N}\) defined, for each \(x \in X\) by \(U(x) = \chi_U(x)\mathbb{1} + \chi_{U^c}(x)\mathbb{1}\) suffices.

Since \(\text{Std}_2\) has the object \(1\) as a separator, to prove that \(y\) is a full functor it suffices to consider natural transformations \(J \in \text{Nat}(\text{Std}_2(X, \Sigma'(\iota_{\mathbb{N}})), \text{Std}_2(1, \Sigma'(\iota_{\mathbb{N}})))\). We proceed to show that \(J_{\mathbb{N}} = ev_x\) for a unique point \(x \in X\).
For $n \geq 1$ let $A_n : X \to N$ be the measurable function which partitions $X$, giving the atoms of the finite field $F_n$ which are used to generate the $\sigma$-algebra on $X$. By Lemma 7.1 it follows that $J_n(A_n) = k$ for some $k \in n$, and this implies that $J_n(A_n) = ev_x(A_n) = A_n(x)$ for every point $x \in A_n^{-1}(k)$. Let $W_n = A_n^{-1}(k)$.

Next we compute $J_n(A_{n+1})$, where $A_{n+1}$ is a refinement of the partition $A_n$. By Lemma 7.1 it follows that $J_n(A_{n+1}) \in n + 1$. By the relationship $\phi \circ A_{n+1} = A_n$ given in equation (5), where $\phi$ is a monotonic decreasing function, it follows that

$$
\phi(J_n(A_{n+1})) = J_n(\phi \circ A_{n+1}) = J_n(A_n) = k.
$$

If the atom $A_n^{-1}(k)$ was partitioned into two separate atoms then $\phi^{-1}(k) = \{k, k + 1\}$, and hence $J_n(A_{n+1}) = k$ or $J_n(A_{n+1}) = k + 1$. Let $k_*$ denote either the index $k$ or $k + 1$, depending upon which of the equations is true. This implies that $J_n(A_{n+1}) = ev_x(A_{n+1}) = A_{n+1}(x)$ for every point $x \in A_n^{-1}(k_*)$. Let $W_{n+1} = A_{n+1}^{-1}(k_*) \subseteq A_n^{-1}(k) = W_n$.

Now suppose the atom $A_n^{-1}(k)$ was not partitioned and $J_n(A_{n+1}) = m$, where by Lemma 7.1 it follows that $m \in n + 1$. Equation (16) always holds, regardless of where the given partition refinement of an atom occurs, and hence it follows that

$$
m = J_n(A_{n+1}) = \begin{cases} 
k & \text{if the partition refinement occurs at an atom with index } k \\
k + 1 & \text{if the partition refinement occurs at an atom with index } < k \end{cases}.
$$

Let $W_{n+1} = A_{n+1}^{-1}(k_*)$, where $k_*$ is either $k$ or $k + 1$ depending upon where the partition refinement occurred. Regardless of where the partition refinement occurred we always have the property that $W_{n+1} \subseteq W_n$.

We can continue this process by partitioning the map $A_{n+1}$ just as we partitioned the map $A_n$. In this manner we obtain a monotone decreasing sequence of sets $W_n \supset W_{n+1} \supset W_{n+2} \supset \ldots$ such that $J_n(A_n) = ev_x(A_n)$ for every $x \in W_n$. Since $X$ is a standard measurable space and the atoms satisfy the finite intersection property it follows that $Z = \bigcap_{i=1}^{\infty} W_i \neq \emptyset$. Since $X$ is a separated measurable space the set $Z$ must be a singleton set $\{x\}$, and it follows that $J_n = ev_x$. \hfill \Box

We say that a measurable function $U : X \to GN$ is deterministic if and only if for every $x \in X$ it follows that $U(x) := \sum_{i \in N} U_i(x) \delta_i$, with $\sum_{i \in N} U_i(x) = 1$, satisfies $U(x) = \delta_{\kappa(x)}$ for some index $\kappa(x) \in \mathbb{N}$. The function $\kappa : X \to \mathbb{N}$ generally varies with the point $x \in X$.

**Lemma 7.3.** Let $X$ be an object in $Std_2$, and let $U : X \to GN$ be a measurable function $U(x) = \sum_{i=0}^{\infty} U_i(x) \delta_i$ where, for every $x \in X$, $\sum_{i=0}^{\infty} U_i(x) = 1$. Then for every natural transformation $J \in Nat(Std_2(X, \Sigma'), Std_2(1, \Sigma'))$ and every permutation $\phi : \mathbb{N} \to N$ it follows that if $J_{GN}(U) = \sum_{i=0}^{\infty} p_i \delta_i$ then $J_{GN}(\epsilon \circ \phi \circ U) = \min_i \{ \phi(i) | p_i > 0 \}$.

**Proof.** By naturality $J_{GN}(G(\phi) \circ U) = G(\phi) J_{GN}(U)$. Since $J_{GN}(U) \in GN$ it can be written as a countably affine sum of the Dirac measures, $J_{GN}(U) = \sum_{i=0}^{\infty} p_i \delta_i$. Consequently we have

$$
J_n(\epsilon_n \circ \phi \circ U) = \epsilon_n(G(\phi)(J_{GN}(U)))
= \epsilon_n\left(\sum_{i=0}^{\infty} p_i \delta_{\phi(i)}\right)
= \sum_{i=0}^{\infty} p_i \phi(i)
= \min_i \{ \phi(i) | p_i > 0 \}
$$
Lemma 7.4. The functor $\Sigma^\prime: \Omega \to \text{Std}_2$ is a codense functor.

Proof. Let $y: \text{Std}_2^{op} \to \text{Set}^\Omega$ be given by $X \mapsto \text{Std}_2(X, \Sigma^\prime)$. We have the $\text{Cat}$-diagram

\[
\begin{array}{ccc}
\text{Std}_2^{op} & \xrightarrow{y} & \text{Set}^\Omega \\
\text{Set}^{\text{Std}_2} & \xrightarrow{\text{Set}^{\text{Std}_2} \circ y} & \text{Set}^{[\Omega]} \\
\end{array}
\]

where we know, from Lemma 7.2, that for every measurable space $X$ that the composite $\text{Set}^{\text{Std}_2} \circ y$ satisfies the property that every natural transformation

\[J \in \text{Nat}(\text{Std}_2(X, \Sigma^\prime(\iota_{\Omega})), \text{Std}_2(1, \Sigma^\prime(\iota_{\Omega})))\]

is, at component $\Omega$, given by $J_{\Omega} = ev_\epsilon$ for some $x \in X$.

Let $X \xrightarrow{U} \mathcal{G}N$ be any measurable function, hence for each $x \in X$ we have the countably affine sum $U(x) = \sum_{i=0}^\infty U_i(x)\delta_i$. By naturality the $\text{Set}$ diagram

\[
\begin{array}{ccc}
\mathcal{G}(N)^X & \xrightarrow{J_{\mathcal{G}(N)}} & \mathcal{G}N \\
\epsilon_X \downarrow & & \downarrow \epsilon_N \\
N^X & \xrightarrow{J_N} & N \\
\end{array},
\]

commutes. Suppose that $J_N(\epsilon_N \circ U) = ev_\epsilon(\epsilon_N \circ U) = n$, and that $J_{\mathcal{G}N}(U) = \sum_{i=0}^\infty p_i\delta_i$. Since $\epsilon_N(\sum_{i=0}^\infty p_i\delta_i) = \sum_{i=0}^\infty p_i\epsilon_\delta$ the equation in the bottom right hand corner of the above diagram forces the coefficients $p_i = 0$ for $i = 0, 1, \ldots, n - 1$. We claim that $J_{\mathcal{G}N}(U) = \delta_n \in \epsilon_N^{-1}(n)$.

To obtain a contradiction suppose that $J_{\mathcal{G}(N)}(U)$ is nondeterministic, and hence there exists an $m > n$ such that $p_m > 0$. Let $\phi: \Omega \to \Omega$ be the simple permutation interchanging the two elements $n$ and $m$, which yields the countably affine map $\mathcal{G}(\phi): \mathcal{G}N \to \mathcal{G}N$. By Lemma 7.3 it follows that $J_N(\epsilon_N \circ \mathcal{G}(\phi) \circ U) = m$, whereas, under the hypothesis that $J_{\mathcal{G}N}(U) = \sum_{i=n}^\infty p_i\delta_i$ is nondeterministic,

\[
\epsilon_N(J_{\mathcal{G}N}(\mathcal{G}(\phi) \circ U)) = \epsilon_N(\mathcal{G}(\phi)(J_{\mathcal{G}N}(\mathcal{U}))) = \epsilon_N(\sum_{i=n}^\infty p_i\delta_{\phi(i)}) = \sum_{i=n}^\infty p_i\phi(i) = n.
\]

Thus, to avoid a contradiction, we must conclude that $J_{\mathcal{G}N}(U) = \delta_n = U(x) = ev_\epsilon(U)$. \qed

The preceding lemma can be recast as

Corollary 7.5. The truncated Yoneda functor $\mathcal{Y}': \text{Std}_2^{op} \to \text{Set}^\Omega$ is a full and faithful functor. Equivalently, the functor $\mathcal{Y}: \text{Std}_2 \to (\text{Set}^\Omega)^{op}$ is full and faithful.

Proof. By the preceding lemma the given functor is full and faithful. \qed
8. Barycenter maps are determined by the support of a probability measure

**Lemma 8.1.** For every super convex space $A$ there exists a measurable map, called the barycenter map,

$$
\epsilon_A : \mathcal{G}(\Sigma A) \to \Sigma A
$$

such that

1. $\epsilon_A \circ \eta_{\Sigma A} = 1_{\Sigma A}$, and
2. $\epsilon_A$ satisfies $\epsilon_A \circ \mathcal{G}(\epsilon_A) = \epsilon_A \circ \epsilon_{\mathcal{G}(\Sigma A)}$ where $\epsilon_{\mathcal{G}(\Sigma A)} := \mu_{\Sigma A}$.

In other words, $\epsilon_A$ is a $\mathcal{G}$-algebra.

**Proof.** Given $A$ apply the functor $\Sigma$ to obtain the separated standard measurable space $\Sigma A$, and using Lemma 7.2 represent $\Sigma A$ as the canonical colimit of the functor

$$
\mathcal{D} = (\Sigma A \downarrow \iota_N) \xrightarrow{\pi} \Omega \xrightarrow{\iota_N} \Sigma' \xrightarrow{\text{Std}_2} \Omega \Sigma_N \xrightarrow{\text{Std}_2},
$$

so we have $\lim \mathcal{D} = (\Sigma A, \text{Std}(\Sigma A, \Sigma N))$ where the universal projection arrows are given by the set of all measurable functions $f : \Sigma A \to \Sigma N$. We can construct a cone over $\mathcal{D}$ with vertex $\mathcal{G}(\Sigma A)$ and natural transformation $\omega$ from the constant functor assigning the object $\mathcal{G}(\Sigma A)$ to every component of $\mathcal{D}$ by specifying for each measurable function $f : \Sigma A \to \Sigma N$ that $\omega_f : \mathcal{G}(\Sigma A) \to \Sigma N$ be the measurable function $\epsilon_N \circ \mathcal{G} f$. The Std-diagram\(^2\) is

\[
\begin{array}{ccc}
\mathcal{G}(\Sigma A) & \xrightarrow{\epsilon_N} & \Sigma N \\
\downarrow \mathcal{G} g & & \downarrow \iota_N \\
\mathcal{G}(\Sigma N) & \xrightarrow{\Sigma \phi} & \Sigma N
\end{array}
\]

$(\mathcal{G}(\Sigma A), \omega)$ specifies a cone over $\mathcal{D}$ because if $g = \Sigma \phi \circ f$, where $\phi : N \to N$, then $\mathcal{G} g = \mathcal{G}(\Sigma \phi \circ f)$, and hence

$$
\begin{align*}
\Sigma \phi \circ \omega_f &= \Sigma \phi \circ \epsilon_N \circ \mathcal{G} f \\
&= (\epsilon_N \circ \mathcal{G}(\Sigma \phi)) \circ \mathcal{G} f & \text{by def. of } \omega_f \\
&= \epsilon_N \circ \mathcal{G}(\Sigma \phi \circ f) & \text{by naturality } \epsilon_N \circ \mathcal{G}(\Sigma \phi) = \Sigma \phi \circ \epsilon_N \\
&= \epsilon_N \circ \mathcal{G} g & \text{factoring out common } \mathcal{G} \\
&= \omega_g & \text{by hypothesis } g = \Sigma \phi \circ f \\
&= \omega_g & \text{by def. of } \omega_g
\end{align*}
$$

Because $\Sigma A$ is the limit of the diagram $\mathcal{D}$ it follows by universality that there exists a unique measurable map $\epsilon_A : \mathcal{G}(\Sigma A) \to \Sigma A$ such that $\omega_f = \epsilon_A \circ f$ for all measurable maps $f : \Sigma A \to \Sigma N$.

---

\(^2\)All the objects in the diagram are separated measurable spaces so whether we say “Std\(_2\)-diagram or Std-diagram is a matter of choice.
Property (1): To show that $\epsilon_A(\delta_a) = a$ for all $a \in A$ note that at each component $f$ we have

$$\left(\epsilon_{\Sigma} \circ Gf\right)(\delta_a) = \epsilon_{\Sigma}(\delta_{f(a)}) \quad \text{because } Gf(\delta_a) = \delta_a f^{-1} = \delta_{f(a)}$$

and we also have

$$\left(\epsilon_{\Sigma} \circ Gf\right)(\delta_a) = f(\epsilon_a(\delta_a)) \quad \text{because } \epsilon_{\Sigma} \circ Gf = f \circ \epsilon_A.$$ 

By Lemma 3.26 the space $\Sigma A$ is a separated measurable space so given any two distinct points $a, b \in A$ there exists a measurable set $U$ in $\Sigma A$ such that $a \in U$ and $b \notin U$, and hence the function $\chi_U$ coseparates the points. Since the equation $\epsilon_{\Sigma} \circ Gf = f \circ \epsilon_A$ must hold at every $f : \Sigma A \to \Sigma N$, including $f = \chi_U$, it follows that $\epsilon_A(\delta_a) = a$. The property $\eta_{\Sigma A}(a) = \delta_a$ is equivalent to the statement $\epsilon_A \circ \eta_{\Sigma A} = 1_{\Sigma A}$.

Property (2):

Note that the $\text{Std}$-diagram

$$
\begin{array}{c}
\text{G}^2(\Sigma A) \\
\downarrow G(\epsilon_A) \quad \downarrow \quad \downarrow \quad \downarrow \epsilon_G(\Sigma N) \\
\text{G}(\Sigma A) \\
\downarrow \quad \quad \downarrow f \\
\Sigma A \quad \rightarrow \quad \Sigma N
\end{array}
$$

is serially commutative (both squares in the top diagram are commutative), and which commutes for every measurable function $f : \Sigma A \to \Sigma N$. Because $\epsilon_N$ is $G$-algebra it follows that $\epsilon_N \circ G(N) = \epsilon_N \circ \epsilon_G(N)$ (where $\epsilon_G(N) = \mu_N$). Hence, for all measurable $f : \Sigma A \to \Sigma N$ it follows that $f \circ \epsilon_A \circ G(\epsilon_A) = f \circ \epsilon_A \circ \epsilon_G(\Sigma A)$. Then because $N$ is codense in $\text{Std}_2$ the set of all arrows to $N$ is jointly monic, and hence $\epsilon_A$ coequalizes the pair of arrows $G(\epsilon_A)$ and $\epsilon_G(\Sigma A)$. (More specifically, $\Sigma : \text{SCvx} \to \text{Std}_2$ is codense.)

Corollary 8.2. $\epsilon_A$ is a countably affine map.

Proof. By the preceding lemma $\epsilon_A$ is a $G$-algebra. Since every $G$-algebra yields an induced super convex space structure on the underlying (measurable) set, via

$$\sum_{i \in N} p_i a_i = \epsilon_A(\sum_{i \in N} p_i \delta_{a_i}) \quad \text{where } \sum_{i \in N} p_i = 1 \text{ and } p_i \in [0, 1]$$

we only need to verify that this super convex space structure coincides with the super convex space structure of $A$. This follows using the identity $\epsilon_A \circ \eta_{\Sigma A} = 1_{\Sigma A}$ because we have $\epsilon_A(\eta_{\Sigma A} (\sum_{i \in N} p_i a_i)) = \epsilon_A(\delta_{\sum_{i \in N} p_i a_i}) = \sum_{i \in N} a_i p_i$, and we also have, using definition (19) that $\sum_{i \in N} a_i p_i = \epsilon_A(\sum_{i \in N} p_i \delta_{a_i}) = \epsilon_A(\sum_{i \in N} p_i \eta_{\Sigma A}(a_i))$. 

\end{proof}
Because $\epsilon_A$ is a $\mathcal{G}$-algebra it follows that, by the analysis given in Lemma 5.1 equation (15), that $\epsilon_A$ is in fact countably affine.

Note that the property of $\epsilon_A$ being countably affine implies that the $\text{Std}$-diagram

$$
\begin{array}{ccc}
\Sigma A & \xrightarrow{\epsilon_A} & \Sigma A \\
\downarrow{\mathrlap{(\epsilon_A(P_i))}} & & \\
\Sigma N & \xrightarrow{(\epsilon_A(P_i))} & \Sigma A
\end{array}
$$

commutes because

$$(\epsilon_A \circ (P))(\sum_{i \in \mathbb{N}} q_i \delta_i) = \epsilon_A (\sum_{i \in \mathbb{N}} q_i P_i) \quad \text{whereas} \quad (\epsilon_A(P_i))(\sum_{j \in \mathbb{N}} q_j \delta_j) = \sum_{j \in \mathbb{N}} q_j \epsilon_A(P_i).$$

Of course, by forgetting measurability, we can also view the preceding diagram as a $\text{SCvx}$-diagram.

9. THE ADJUNCTION $\mathcal{P} \leftrightarrow \Sigma$ AND ISOMORPHISM $\text{Std}^\mathcal{G} \cong \text{SCvx}$

**Lemma 9.1.** The family of maps $\epsilon_A : \mathcal{P}(\Sigma A) \to A$, one for each super convex space $A$ form the components of a natural transformation $\epsilon : \mathcal{P} \circ \Sigma \Rightarrow 1_{\text{SCvx}}$.

**Proof.** Suppose $m : A \to B$ is a countably affine map. Using Lemma 8.1 construct the countably affine measurable maps $\epsilon_A$ and $\epsilon_B$. Let $\mathcal{D}$ be the diagram

$$
\mathcal{D} = (\Sigma A \downarrow \mu) \xrightarrow{\pi} \mathbb{N} \xrightarrow{\iota} \Omega \xrightarrow{\Sigma'} \text{Std}_2,
$$

and let $\mathcal{E}$ be the diagram

$$
\mathcal{E} = (\Sigma B \downarrow \mu) \xrightarrow{\pi} \mathbb{N} \xrightarrow{\iota} \Omega \xrightarrow{\Sigma'} \text{Std}_2.
$$

For every measurable function $f : \Sigma B \to \Sigma N$ the composite function $f \circ \Sigma m : \Sigma A \to \Sigma N$ is a projection arrow in the limit of $\mathcal{D}$, and hence because $(\mathcal{G}(\Sigma A), \{\epsilon_N \circ \mathcal{G}(f') \mid \text{all } f' : \Sigma A \to \Sigma N\})$ is a cone over $\mathcal{D}$ the unique arrow $\epsilon_A : \mathcal{G}(\Sigma A) \to \Sigma A$ satisfies $(f \circ \Sigma m) \epsilon_A = \epsilon_N \circ \mathcal{P}(f \circ \Sigma m)$. Hence the outer square of the $\text{Std}_2$ diagram

$$
\begin{array}{ccc}
\mathcal{G}(\Sigma A) & \xrightarrow{\mathcal{G}(\Sigma m)} & \mathcal{G}(\Sigma B) \\
\downarrow{\mathrlap{\epsilon_A}} & & \downarrow{\mathrlap{\mathcal{G}(f)}} \\
\Sigma A & \xrightarrow{\Sigma m} & \Sigma B \\
\downarrow{\mathrlap{f}} & & \downarrow{\mathrlap{\epsilon_N}} \\
\Sigma N & & \Sigma N
\end{array}
$$

commutes, and the right square commutes because $\epsilon_B : \mathcal{G}(\Sigma B) \to \Sigma B$ is the unique arrow from the vertex of the cone $(\mathcal{G}(\Sigma B), \{\epsilon_N \circ \mathcal{G}(f) \mid f : \Sigma B \to \Sigma N\})$ over $\mathcal{E}$ to the limit $\mathcal{E} = (\Sigma B, \{f : \Sigma B \to \Sigma N\})$.

Thus we have the two equations, $f \circ \Sigma m \circ \epsilon_A = \epsilon_N \circ \mathcal{G}(f) \circ \mathcal{G}(\Sigma m)$ and $f \circ \epsilon_B = \epsilon_N \circ \mathcal{G}(f)$. Using the second equation and substituting into the first equation, replacing the expression $\epsilon_N \circ \mathcal{G}(f)$, we obtain $\epsilon_B \circ \mathcal{G}(\Sigma m) = \Sigma m \circ \epsilon_A$ which proves naturality. \qed
Theorem 9.2. The pair of functors $\mathcal{P} : \text{Std}_2 \to \text{SCvx}$ and $\Sigma : \text{SCvx} \to \text{Std}_2$ specify an adjunction $\langle \mathcal{P}, \Sigma, \eta, \epsilon \rangle$ with $\mathcal{P} \dashv \Sigma$.

Proof. The unit of the adjunction is $\eta_X(x) = \delta_x$ while the counit of the adjunction is the natural transformation $\epsilon$ specified in Lemma 9.1.

We verify the two triangular identities. For $X$ any measurable space we have the commutative $\text{SCvx}$-diagram,

\[
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{\mathcal{P}\eta_X} & \mathcal{P}(\Sigma\mathcal{P}(X)) \\
\downarrow \cong \mathcal{P}(X) & & \downarrow \cong \mathcal{P}(\Sigma\mathcal{P}(X)) \\
\mathcal{P}(X) & \xrightarrow{\delta_{\mathcal{P}(X)}} & \mathcal{P}(X)
\end{array}
\]

and for $A$ any super convex space, we have the commutative $\text{Std}_2$-diagram,

\[
\begin{array}{ccc}
\Sigma(\mathcal{P}\Sigma A) & \xrightarrow{\Sigma\epsilon_A} & \Sigma A \\
\downarrow \eta_{\Sigma A} & & \downarrow \eta_{\Sigma A} \\
\Sigma A & \xrightarrow{id_{\Sigma A}} & \Sigma A \\
\end{array}
\]

where in both triangular identities we have used the property that, for all spaces $A$ and all elements $a \in A$, $\epsilon_A(\delta_a) = a$ as shown in Lemma 8.1.

Thus $\langle \mathcal{P}, \Sigma, \eta, \epsilon \rangle$ specifies an adjunction with $\mathcal{P} \dashv \Sigma$. \qed

Corollary 9.3. The adjunction $\langle \mathcal{P}, \Sigma, \eta, \epsilon \rangle$ is an adjoint factorization of $G$.

Proof. This follows from Theorem 9.2 and Lemma 3.27. \qed

Because each countably affine map $m : A \to \mathbb{R}_\infty$ yields a measurable function, $\Sigma m$, we have

Corollary 9.4. Viewing a probability measure as an functional, $\hat{\mathcal{P}} : \text{Std}_2(\Sigma A, \mathbb{R}_\infty) \to \mathbb{R}_\infty$, specified by $f \mapsto \int_A f \, d\mathcal{P}$, we have the result that the restriction of $\mathcal{P} \in G(A, \mathbb{R}_\infty)$ to the countably affine (measurable) functions $\text{SCvx}(A, \mathbb{R}_\infty)$ is an evaluation map, $\mathcal{P} = ev_a$, for a unique point $a \in A$. In other words, every $m \in \text{SCvx}(A, \mathbb{R}_\infty)$ is a measurable function $\Sigma m \in \text{Std}_2(\Sigma A, \mathbb{R}_\infty)$, and for every such $m$ it follows that $\hat{\mathcal{P}}(m) = m(a)$ for a unique point $a \in A$.

Proof. This result is a translation of the naturality of $\epsilon$. For every $m \in \text{SCvx}(A, \mathbb{R}_\infty)$ the square

\[
\begin{array}{ccc}
\mathcal{P}(\Sigma A) & \xrightarrow{\mathcal{P}(\Sigma m)} & \mathcal{P}(\Sigma \mathbb{R}_\infty) \\
\downarrow \epsilon_A & & \downarrow \epsilon_{\mathbb{R}_\infty} = \mathbb{E} \\
A & \xrightarrow{m} & \mathbb{R}_\infty \\
\end{array}
\]

commutes. Using the fact that $\epsilon_{\mathbb{R}_\infty}$ is the expectation operator, $\mathbb{E}$, we have

\[
ev_{\epsilon_A(P)}(m) = m(\epsilon_A(P)) = \epsilon_{\mathbb{R}_\infty}(\mathbb{P}m^{-1}) = \mathbb{E}(\mathbb{P}m^{-1}) = \int_{\mathbb{R}_\infty} id_{\mathbb{R}_\infty} d\mathbb{P}(m^{-1}) = \int_A m \, d\mathcal{P}.
\]
For a given \( P \in \mathcal{G}(\Sigma A) \), the uniqueness property is simply the statement that \( \epsilon_A(P) = a \). Thus every probability measure \( P \in \mathcal{G}(\Sigma A) \) “appears like” a Dirac delta measure when integrating a countably affine map into \( \mathbb{R}_{\infty} \).

**Theorem 9.5.** The category \( SCvx \) is isomorphic to \( Std^G \).

*Proof.* The comparison functor \( K : SCvx \to Std^G \) is specified on objects by \( A \mapsto (\Sigma A, \Sigma \epsilon_A) \). (In our constructions, we have shown the existence of the barycenter maps as countably affine measurable maps. The use of the notation \( \Sigma \epsilon_A \) is used to distinguish between the measurable function \( \Sigma \epsilon_A \) viewed in \( Std^2 \) versus viewing it just as the countably affine map \( \epsilon_A \) in \( SCvx \).) The functor \( Std^G \xrightarrow{W} SCvx \) is defined on objects by \( W((X,h)) = X_h \) where \( X_h \) is the super convex space consisting of the underlying set of \( X \) with the super convex space structure defined by \( \sum_{i \in \mathbb{N}} p_i \delta_{x_i} = h(\sum_{i \in \mathbb{N}} p_i \delta_{x_i}) \) is the inverse to the comparison functor \( K \). If \( f : (X,h) \to (Y,k) \) is a morphism of two \( G \)-algebras then, under the super convex space structure specified on the two spaces, \( X_h \) and \( Y_k \), it follows by the proof given in Lemma 5.1 that \( f \) is a countably affine map. Hence, on arrows, the functor \( W \) is defined by \( W(f) = f \), and it is thus obvious that \( W \) is functorial. The comparison functor \( K \) and the functor \( W \) specify the isomorphism \( Std^G \cong SCvx \).

\( \square \)

**10. Remarks**

We comment on two different aspects of this work: (1) The advantage of representing \( Std^G \) as \( SCvx \), and (2) other research directly related to this article.

(1) In attempting to understand the \( T \)-algebras of any monad \( T \) on a category \( C \) it is, generally speaking, more efficient to search for codense functors from a small category \( \Omega \) into \( C \) rather than studying functors into \( C \) whose codensity monad is the monad \( T \). In the latter approach, one can, at best, obtain qualitative information about the monad, and hence about the \( T \)-algebras. The strategy presented in §1 gives a general approach which can be applied to any monad to find a dense functor and a codense functor pair which permits us to represent the two categories as subobjects of the two functor categories, \( Set^\Omega_{\text{op}} \) and \( Set^\Omega \), respectively, precisely like we did in Diagram 3.

By using codensity monads, useful qualitative information about the Giry monad was obtained by Ruben Van Belle\(^{[1]}\) and presented in the article *Probability monads as codensity monads*. Belles’ research showed that we could limit our focus of attention on *countability*. That article shows the Giry monad for \( \text{Meas} \) can be viewed as arising from the codensity monad of a functor \( G : Set_c \to \text{Meas} \), where \( Set_c \) is the category of countable sets. It is equivalent to say that \( G \) is the Giry monad restricted to countable measurable spaces with the powerset \( \sigma \)-algebra. He employs the countable-dimensional simplexes in defining the functor, and it is clear that the finite-dimensional simplexes can be viewed as subspaces of \( \Delta_N \). Consequently he could have chosen his functor \( G : \Omega \to \text{Meas} \) where \( \Omega \) is the category with the one object \( \Delta_N \), and with arrows as he defined in the article, \( \Delta_f : \Delta_N \to \Delta_N \), which is the pushforward map induced by a function \( f : \mathbb{N} \to \mathbb{N} \). While the pushforward maps are countably affine maps, most countably affine maps \( \Delta_N \to \Delta_N \) are not pushforward maps. But even if one defines the category \( \Omega \) as consisting of all countably affine maps \( \Delta_N \to \Delta_N \),
one still obtains the right Kan extension of $G$ along itself as the Giry monad rather than a codense functor which has the right Kan extension along itself as the identity functor. That point is the point of distinction between using functors into $\mathbf{Meas}$ whose codensity monad is $G$, or using codense functors into $\mathbf{Meas}$ or some subcategory thereof to study the monad.

The article “The Monad of Probability Measures over Compact Ordered Spaces” by Klaus Keimel\[5\] is the most closely related work to this article because he specifically addresses ordered spaces. It is clear that the countable (discrete) super convex spaces $n$ and $\mathbb{N}$ that we use, which are in essence the building blocks of more complex spaces, are linearly ordered. However the algebras employed in that article are topological barycenter maps.

Barycenter maps (algebras) makes modeling nondeterminism possible by making the connection between analysis on continuous spaces, $G(\Sigma A)$, and analysis on the “underlyings” spaces $A$, which is of a combinatorial/discrete nature (countable affine sums). As such, the algebras we need are generally measurable maps which may or may not be continuous with respect to some topology that may (or may not) exist.

The general case of finding the $G$-algebras for $\mathbf{Meas}$ by trying to find a codense subcategory (or codense functor to $\mathbf{Meas}$) is not possible because one can choose measurable cardinals that can’t be “measured” by any object.

(2) The advantage of working with $\mathbf{SCvx}$ rather than $\mathbf{Std}^G$ are numerous. Basic facts, like that $\mathbf{SCvx}$ has no coseparator show that $\mathbf{Std}^G$ has no coseparator. For example, take the measurable space $\mathbb{R}_\infty$, where the $\sigma$-algebra is generated by the measurable sets of $\mathbb{R}$ and the singleton set $\{\infty\}$.

In Example (3.3) we noted the only nonconstant countably affine map $j : \mathbb{R}_\infty \to \mathbb{2}$ is given by $j(u) = 0$ for all $u \in \mathbb{R}$ and $j(\infty) = 1$. Consequently, given two distinct points of the $G$-algebra $E : G(\mathbb{R}_\infty) \to \mathbb{R}_\infty$ given by $(P, u) : 1 \to E$ and $(P, v) : 1 \to E$ where both $u, v \in \mathbb{R}$ and $u \neq v$, the two points cannot be coseparated by $(2, \epsilon_2)$, or any other object in $\mathbf{SCvx}$.\[3\]

The category $\mathbf{SCvx}$ has two monoidal structures associated with it, both a cartesian monoidal structure and a tensor monoidal structure, and both appear useful. While the Giry monad is not a strong functor with respect to the tensor monoidal structure,\[13\] emphasizing only the cartesian monoidal structure is a mistake as can be made perfectly clear by the fact we can employ probability amplitudes using super convex spaces, and we should be able to model various nondeterministic processes such as quantum computation using the $G$-algebras.

To use probability amplitudes in defining “countably affine sums” it is only necessary to view

$$
G\mathbb{N} = \{\text{all sequences } p : \mathbb{N} \to D_2 \text{ such that } \lim_{N \to \infty} \{\sum_{i=1}^{N} p_i p_i^*\} = 1\}
$$

where $D_2 = \{re^{\theta} \in \mathbb{C} \mid r \in [0, 1], \text{ and } \theta \in [0, 2\pi)\}$ and $p_i^*$ is the complex conjugate of $p_i$. Using probability amplitudes, the point wise definition of the super convex space structure on $\mathcal{P}X$ is given by “$(\sum_{i \in \mathbb{N}} p_i P_i)(U) = \sum_{i \in \mathbb{N}} p_i p_i^* P_i(U)$ for all measurable sets $U$ in $X$”, in other words, whenever we evaluate a probability measure we use the $\ell_2$-norm in evaluating the countable affine sum, which is precisely what is done in quantum mechanics. That same
principal can be applied to any countably affine sum. For example, the super convex space structure of \( \mathbb{N} \) reads as \( \sum_{i \in \mathbb{N}} p_i \cdot i = \min \{ i \mid p_i \cdot i > 0 \} \). Nowhere in any of our theorems or lemmas do we use any special properties arising from the unit interval. We only use the property of “countably affine sums” which can be defined using either the \( \ell_2 \)-norm or the \( \ell_1 \)-norm. (I am speaking loosely here; by the “\( \ell_1 \)-norm” I am referring to the conditions \( \sum_{i \in \mathbb{N}} p_i = 1 \) and \( p_i \geq 0 \). It is the second condition which allows us to think of the first condition as \( \sum_{i \in \mathbb{N}} |p_i| = 1 \).)

The big advantage of using probability amplitudes only arises with the use of a dynamic model and measurement model where cancellations can occur which never arise when we restrict ourselves to using the \( \ell_1 \)-norm in defining a super convex space. The axioms of a super convex space make no preference on whether we choose the \( \ell_1 \)-norm or the \( \ell_2 \)-norm in defining countable affine sums. The importance of the tensor monoidal structure should now be clear to anyone who is familiar with either quantum mechanics or quantum computation[12]. Under the tensor monoidal structure the no copying rule is just the statement that the function \( a \mapsto a \otimes a \) is not permitted because it is not a countably affine map.

**References**

[1] Ruben Van Belle, *Probability monads as codensity monads*, https://arxiv.org/abs/2111.01250, 2021.
[2] Phillipe Biane, *The Riemann Zeta function and probability theory*, https://www.dam.brown.edu/people/menon/publications/notes/biane.pdf
[3] Reinhard Börger and Ralf Kemper, *There are no cogenerators for totally convex spaces*, Cahiers Differential Geometry, XXXV-4, 1994, http://www.numdam.org/article/CTGDC_1994__35_4_335_0.pdf
[4] Categories.pdf at https://stacks.math.columbia.edu/download/categories.pdf
[5] Klaus Keimel, *The Monad of Probability Measures over Compact Ordered Spaces*. Topology and its Applications, 2008.
[6] M. Giry, A categorical approach to probability theory, in Categorical Aspects of Topology and Analysis, Vol. 915, pp 68-85, Springer-Verlag, 1982.
[7] Robert M. Gray, Probability, Random Processes, and Ergodic Properties, Springer-Verlag, 2008.
[8] Xiao-qing Meng, Categories of convex sets and metric spaces, with applications to stochastic programming and related areas, Ph.D. dissertation, NYU Buffalo, 1987. https://ncatlab.org/nlab/show/metric+space
[9] John R. Isbell, *General Functorial Semantics, I*, American Journal of Mathematics, Vol. 94, No2, pp535-596.
[10] John R. Isbell, *Adequate Subcategories*, Illinois J. Math, 4 (1960) pp541-552.
[11] Saunders Mac Lane, Categories for the working mathematician, Springer Verlag, 1971.
[12] Eleanor Rieffel and Wolfgang Polak, Quantum Computation: A Gentle Introduction. The MIT press, Cambridge Massachusetts, 2011.
[13] Tetsuya Sato, *The Giry monad is not strong for the canonical symmetric monoidal closed structure on Meas*, Journal of Pure and Applied Algebra, Vol. 222, Issue 10, Oct. 2018, pp2888-2896.

**Nong Han, Thailand**

*Email address: kirksturtz@yandex.com*