Mod $p$ points on Shimura varieties of parahoric level

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Structure of the talk

Introduction to the Langlands-Rapoport conjecture and a quick survey of previous work

Statement of the main results

Idea of the proof
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The Langlands-Rapoport conjecture I

Langlands and Rapoport conjectured the existence of integral models of Shimura varieties with good properties. For example, the modular curve $Y_0(N)$ has a smooth integral model over $\mathbb{Z}(p)$ with $p \nmid N$, using the moduli interpretation in terms of families of elliptic curves. The modular curve $Y_0(Np)$ also has an integral model over $\mathbb{Z}(p)$ with $p \nmid N$, but it is no longer smooth.

Understanding these integral models has interesting applications, e.g., construction of Galois representations (Deligne, Langlands), Ribet's proof of the $\epsilon$-conjecture.
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Understanding these integral models has interesting applications, e.g. construction of Galois representations (Deligne, Langlands), Ribet’s proof of the $\epsilon$-conjecture.
Let \((G, X)\) be a Shimura datum, i.e., \(G\) is a reductive group over \(\mathbb{Q}\) and \(X\) is a Hermitian symmetric domain with an action of \(G(\mathbb{R})\). Let \(p\) be a prime number, \(K_p \subset G(\mathbb{A}_f)\) be a compact open subgroup and \(K_p \subset G(\mathbb{Q}_p)\) a parahoric subgroup and let \(K = K_p K_p \subset G(\mathbb{A}_f)\).

Let \(\text{Sh} K(G, X)\) be the associated Shimura variety, which is an algebraic variety over a number field \(E\), the reflex field. If \(v|p\) is a place of \(E\), then the conjecture predicts that there should be a 'good' integral model \(\text{S} K(G, X)\) over \(\mathcal{O}_E(v)\).

For example, \(G = \text{GL}_2\), \(X = \mathbb{H}^+\) and \(K_p = \text{GL}_2(\mathbb{Z}_p)\) or \(K_p = \Gamma_0(p)\), then \(E = \mathbb{Q}\) and the integral models from the previous slide are 'good'.

The Langlands-Rapoport conjecture II
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For example \(G = \text{GL}_2\), \(X = \mathbb{H}^\pm\) and \(K_p = \text{GL}_2(\mathbb{Z}_p)\) or \(K_p = \Gamma_0(p)\), then \(E = \mathbb{Q}\) and the integral models from the previous slide are ‘good’.
The conjecture then predicts that there is a partition into 'isogeny classes' $S_{K}(G, X)(F_p) \cong \bigoplus_{\phi} S_{\phi}$,\(^{(1)}\) compatible with the action of prime to $p$ Hecke operators.

Moreover, the $S_{\phi} \subset S_{K}(G, X)(F_p)$ have the following description ('Rapoport-Zink uniformisation') $S_{\phi} \cong I_{\phi}(Q) \setminus X_{p}(\phi) \times X_{p}(\phi) / K_{p}$,\(^{(2)}\)

Here $X_{p}(\phi)$ is an affine Deligne-Lusztig variety of level $K_{p}$. 
The conjecture then predicts that there is a partition into ‘isogeny classes’

\[ S_K(G, X)(\overline{\mathbb{F}}_p) \simeq \bigsqcup_{\phi} S_{\phi}, \]  

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Previous Work

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**Theorem (Kisin, 2008 and 2013)**

Let $(G, X)$ be a Shimura datum of abelian type, let $p > 2$ and suppose that $G_{\mathbf{Q}_p}$ is unramified and that $K_p$ is hyperspecial. Then the Langlands-Rapoport conjecture holds for $(G, X, p)$.

**Theorem (Zhou, 2017)**

Let $(G, X)$ be a Shimura datum of Hodge type, let $p > 2$ and suppose that $G_{\mathbf{Q}_p}$ is residually split, then isogeny classes have Rapoport-Zink uniformisation for arbitrary parahorics $K_p$. 
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Main Results I

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**Theorem 1 (-)**

*Suppose that \(G\) has no factors of type \(A\) and that \(\text{Sh}_K(G, X)\) is proper. Then the Langlands-Rapoport conjecture holds for the Kisin-Pappas integral models of \(\text{Sh}_K(G, X)\).*
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**Remarks**

The assumption that \(G_{\mathbb{Q}_p}\) is unramified can be removed for most \((G, X)\).
Idea of the proof I

Since we know the results at hyperspecial level, it suffices to understand the fibers of the forgetful map. When $\mathbb{G} = \text{GL}_2$, then the forgetful map has the following description:

$$\mathcal{Y}_0(Np) \{ (E, \alpha_N, H \subset E[p]) \}$$

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Here $E$ is an elliptic curve, $\alpha_N$ is a $\Gamma_0(N)$ level structure and $H \subset E[p]$ is a subgroup of order $p$. An elliptic curve over $\mathbb{F}_p$ has either one or two choices for $H$, depending on whether it is supersingular or ordinary. We observe that the fiber only depends on the $p$-divisible group $E[p]$. 
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Idea of the proof II

For moduli spaces of abelian varieties with extra structures, these fibers are more complicated and usually not finite, for example the fibers can be projective lines. However, it is still true that the fibers only depend on the $p$-divisible group with extra structures. This means that we can use Dieudonné theory to understand the fibers.

For Hodge type Shimura varieties, the integral models do not have a moduli interpretation, which makes it difficult to make the above strategy work. We can still associate a $p$-divisible group with extra structures $X$ to an $\mathbb{F}_p$-point, but it is no longer clear that the fiber only depends on this $X$. 
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Idea of the proof III

Let $K_p$ be a hyperspecial parahoric and $K'_p \subset K_p$ another parahoric. Let $S_{K, F_p}(G, X)$ be the special fiber of the Kisin-Pappas integral model, then it has a morphism to the 'moduli space of $p$-divisible groups with extra structures'. This map fits into a commutative diagram together with its variant for $K'_p S_{K, F_p}(G, X)$.

(4)

Here $S_{G, \mu, K_p}$ is the pre-stack of $G$-shtukas of type $\mu$ and parahoric $K_p$. These were introduced by Xiao-Zhu and generalised by Shen-Yu-Zhang. The LR conjecture holds for the Shimura variety in the top left corner if and only if the diagram is Cartesian.
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The LR conjecture holds for the Shimura variety in the top left corner if and only if the diagram is Cartesian.
Idea of the proof IV

So let $Y$ be the fiber product of the diagram and consider the morphism $i: S_K', F_p(G, X) \to Y$, we will show it is an isomorphism in three steps:

We show that $i$ is a closed immersion.

We show that $Y$ is equidimensional.

We prove that $Y$ has the same number of irreducible components as $S_K', F_p(G, X)$.

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This last result is new even for $S_{K', \overline{\mathbb{F}}_p}(G, X)$!
The modular curve $\Gamma_0(N)$ comes equipped with the Ekedahl-Oort stratification; the stratum that an $\overline{\mathbb{F}}_p$ point $(E, \alpha_n)$ is in is determined by the $p$-torsion $E[p]$. More generally, this defines a stratification on the moduli space of abelian varieties. Ekedahl and van der Geer showed that Ekedahl-Oort strata are irreducible precisely when they are not contained in the supersingular locus. Let $(G, X)$ be as above, and let $K_p$ be a hyperspecial subgroup. Theorem 2 (-) Suppose that $G$ has no factors of type $A$, that $\text{Sh}_{K_p}(G, X)$ is proper and that $G_{ad}$ is $\mathbb{Q}$-simple. Then Ekedahl-Oort strata that are not contained in the basic locus are 'irreducible'.
The modular curve $\Gamma_0(N)$ comes equipped with the Ekedahl-Oort stratification; the stratum that an $\overline{\mathbb{F}}_p$ point $(E, \alpha_n)$ is in is determined by the $p$-torsion $E[p]$. More generally, this defines a stratification on the moduli space of abelian varieties.
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