FIELDS OF DEFINITION FOR REPRESENTATIONS OF ASSOCIATIVE ALGEBRAS

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Abstract We examine situations, where representations of a finite-dimensional $F$-algebra $A$ defined over a separable extension field $K/F$, have a unique minimal field of definition. Here the base field $F$ is assumed to be a field of dimension $\leq 1$. In particular, $F$ could be a finite field or $k(t)$ or $k((t))$, where $k$ is algebraically closed. We show that a unique minimal field of definition exists if (a) $K/F$ is an algebraic extension or (b) $A$ is of finite representation type. Moreover, in these situations the minimal field of definition is a finite extension of $F$. This is not the case if $A$ is of infinite representation type or $F$ fails to be of dimension $\leq 1$. As a consequence, we compute the essential dimension of the functor of representations of a finite group, generalizing a theorem of Karpenko, Pevtsova and the second author.

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1. Introduction

1.1. Notational conventions

Throughout this paper $F$ will denote a base field and $A$ a finite-dimensional associative algebra over $F$. If $K/F$ is a field extension (not necessarily algebraic), we will denote the tensor product $K \otimes_F A$ by $A_K$. Let $M$ be an $A_K$-module. Unless otherwise specified, we will always assume that $M$ is finitely generated (or equivalently, finite-dimensional as a $K$-vector space). If $L/K$ is a field extension, we will write $M_L$ for $L \otimes_K M$.

An intermediate field $F \subset K_0 \subset K$ is called a field of definition for $M$ if there exists a $K_0$-module $M_0$ such that $M \cong (M_0)_K$. In this case we will also say that $M$ descends to $K_0$.

1.2. Minimal fields of definition

A field of definition $K_0$ of $M$ is said to be minimal if whenever $M$ descends to a field $L$ with $F \subset L \subset K$, we have $K_0 \subset L$. 

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Minimal fields of definition do not always exist. For example, let $F = \mathbb{Q}$ and $A$ be the quaternion algebra

$$A = \mathbb{Q}\{i, j, k\}/(i^2 = j^2 = k^2 = ijk = -1).$$

Then $A_K$ has a two-dimensional module $M$ given by

\[
i \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad j \mapsto \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

over any field $K$ of characteristic 0 having two elements $a$ and $b$ such that $a^2 + b^2 = -1$. Examples of such fields include $\mathbb{C}$, $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-5})$. If we take $K$ to be 'the generic field' of this type, i.e., the field of fractions of $\mathbb{Q}[a, b]/(a^2 + b^2 + 1)$, then $M$ has no minimal field of definition; see Proposition 6.3(b).

### 1.3. Fields of dimension $\leq 1$

Such examples arise because of the existence of non-commutative finite-dimensional division algebras over $F$. So, it makes sense to develop a theory over those fields $F$ over which these division algebras do not exist. More precisely, we require that

$$\text{Br}(E) = 0 \quad \text{for every algebraic field extension } E/F,$$

(1.1)

where $\text{Br}(E)$ denotes the Brauer group of $E$. This class of fields was studied in detail by Serre in connection with his celebrated Conjecture I; see [14, §II.3]. Serre referred to fields satisfying (1.1) as ‘fields of dimension $\leq 1$’. If $F$ is perfect, this condition is equivalent to the cohomological dimension of the absolute Galois group $\text{Gal}(F)$ being $\leq 1$; see [14, Proposition II.3.1.6]. In particular, this condition is satisfied by all finite fields, all algebraically closed fields and all field extensions of transcendence degree 1 over an algebraically closed field. For proofs of these assertions and further examples, see [14, §II.3.3].

Our first main result is as follows.

**Theorem 1.2.** Let $F$ be a field satisfying (1.1) $A$ be a finite-dimensional $F$-algebra, $K/F$ be a separable algebraic field extension and $M$ be an $A_K$-module. Then $M$ has a minimal field of definition $F \subset K_0 \subset K$ such that $[K_0 : F] < \infty$.

To illustrate Theorem 1.2, let us consider a simple case, where $\text{char}(F) = 0$, $A := FG$ is the group algebra of a finite group $G$, and $M$ is an absolutely irreducible $KG$-module. Denote the character of $G$ associated to $M$ by $\chi : G \to K$. We claim that in this case the minimal field of definition is $F(\chi)$, the field generated over $F$ by the character values $\chi(g)$, as $g$ ranges over $G$. Indeed, it is clear that $F(\chi)$ has to be contained in any field of definition $F \subset K_0 \subset K$ of $M$. Thus to prove the above assertion, we only need to show that $M$ descends to $F(\chi)$. The minimal degree of a finite field extension $L/F(\chi)$, such that $M$ is defined over $L$ (i.e., there exists an $LG$-module with character $\chi$), is the Schur index $s_M$; cf. [4, Definition 41.4]. Thus it suffices to show that $s_M = 1$. By [4, Theorem (70.15)], $s_M$ is the index of the endomorphism algebra $\text{End}_A(M)$ of $M$, which is a central simple algebra over $F(\chi)$. Since $F$ satisfies condition (1.1) and $[F(\chi) : F] < \infty$, the index of every central simple algebra over $F(\chi)$ is 1. In particular, $s_M = 1$, and $M$ descends to $F(\chi)$, as claimed.
1.4. Algebras of finite representation type

A finite-dimensional \( F \)-algebra \( A \) is said to be of finite representation type if there are only finitely many indecomposable finitely generated \( A \)-modules (up to isomorphism).

Our next result shows that for algebras of finite representation type Theorem 1.2 remains valid even if the field extension \( K/F \) is not assumed to be algebraic.

**Theorem 1.3.** Let \( F \) be a field satisfying (1.1) \( A \) be a finite-dimensional \( F \)-algebra of finite representation type, \( K/F \) be a field extension, and \( M \) be an \( A_K \)-module. Assume further that \( F \) is perfectly closed in \( K \). Then \( M \) has a minimal field of definition \( F \subset K_0 \subset K \) such that \([K_0 : F] < \infty \).

1.5. Essential dimension

Given the \( A_K \)-module \( M \), the essential dimension \( \text{ed}(M) \) of \( M \) over \( F \) is defined as the minimal value of the transcendence degree \( \text{trdeg}(K_0/F) \), where the minimum is taken over all fields of definition \( F \subset K_0 \subset K \). The integer \( \text{ed}(M) \) may be viewed as a measure of the complexity of \( M \). Note that \( \text{ed}(M) \) is well defined, irrespective of whether \( M \) has a minimal field of definition or not. We also remark that this number implicitly depends on the base field \( F \), which is assumed to be fixed throughout. As a consequence of Theorem 1.3, we will deduce the following.

**Theorem 1.4.** Let \( F \) be a field satisfying (1.1), \( A \) be finite-dimensional \( F \)-algebra of finite representation type, \( K/F \) be a field extension, and \( M \) be an \( A_K \)-module. Then \( \text{ed}(M) = 0 \).

Both Theorems 1.3 and 1.4 fail if we do not require \( F \) to satisfy (1.1); see §6.

1.6. The essential dimension of the functor of \( A \)-modules

We will also be interested in the essential dimension \( \text{ed}(	ext{Mod}_A) \) of the functor \( \text{Mod}_A \) from the category of field extensions of \( F \) to the category of sets, which associates to a field \( K \), the set of isomorphism classes of \( A_K \)-modules. By definition,

\[
\text{ed}(	ext{Mod}_A) := \sup \text{ed}(M),
\]

where the supremum is taken over all field extensions \( K/F \) and all finitely generated \( A_K \)-modules \( M \). The value of \( \text{ed}(	ext{Mod}_A) \) may be viewed as a measure the complexity of the representation theory of \( A \). For generalities on the notion of essential dimension we refer the reader to [2,10–13]. Essential dimensions of representations of finite groups and finite-dimensional algebras are studied in [8] and [3, §3].

Note that while \( \text{ed}(M) < \infty \), for any given \( A_K \)-module \( M \) (see Lemma 2.1), \( \text{ed}(	ext{Mod}_A) \) may be infinite. In particular, in the case, where \( A = FG \) is the group algebra of a finite group \( G \) over a field \( F \), it is shown in [8, Theorem 14.1] that \( \text{ed}(	ext{Mod}_A) = \infty \), provided that \( F \) is a field of characteristic \( p > 0 \) and \( G \) has a subgroup isomorphic to \((\mathbb{Z}/p\mathbb{Z})^2\). Our final main result is the following amplification of [8, Theorem 14.1].

**Theorem 1.5.** Let \( G \) be a finite group and \( F \) be a field of characteristic \( p \). Then the following conditions are equivalent:
(1) the $p$-Sylow subgroup of $G$ is cyclic;
(2) $\text{ed}(\text{Mod}_{FG}) = 0$;
(3) $\text{ed}(\text{Mod}_{FG}) < \infty$.

Note that by a theorem of Higman [7], the condition that the $p$-Sylow subgroup of $G$ is cyclic is equivalent to the group algebra $FG$ being of finite representation type.

2. Preliminaries on fields of definition

**Lemma 2.1.** Let $A$ be a finite-dimensional $F$-algebra, $K/F$ be a field extension and $M$ be an $A_K$-module. Then $M$ descends to an intermediate subfield $F \subset E \subset K$, where $E/F$ is finitely generated.

**Proof.** Suppose $a_1, \ldots, a_r$ generate $A$ as an $F$-algebra. Choose an $F$-vector space basis for $M$. Then the $A$-module structure of $M$ is completely determined by the matrices representing multiplication by $a_1, \ldots, a_r$ in this basis. Each of these matrices has $n^2$ entries in $K$, where $n = \dim_F(M)$. Let $E \subset K$ be the field extension of $F$ obtained by adjoining these $rn^2$ entries to $F$. Then $M$ descends to $E$. □

Next we recall the classical theorem of Noether and Deuring. For a proof, see [4, (29.7)] or [1, Lemma 5.1].

**Theorem 2.2 (Noether–Deuring theorem).** Let $K/E$ be a field extension, $A$ be a finite-dimensional $E$-algebra, and $M$, $M'$ be $A$-modules. If $M_K = K \otimes_E M$ and $M'_K = K \otimes_E M'$ are isomorphic as $A_K$-modules, then $M$ and $M'$ are isomorphic as $A$-modules.

**Lemma 2.3.** Let $F$ be a field, $A$ be a finite-dimensional $F$-algebra, $F \subset E \subset K$ be field extensions, $N$ be an $A_E$-module, and $F \subset E_0 \subset E$ be an intermediate field. Then:

(a) $N_K$ descends to $E_0$ if and only if $N$ descends to $E_0$;
(b) if $F \subset E_{\min} \subset K$ is a minimal field of definition for $N_K$, then $E_{\min}$ is a minimal field of definition for $N$.

**Proof.** (a) If $N$ descends to $E_0$, then clearly so does $N_K$. Conversely, suppose $N_K$ descends to $E_0$. That is, there exists an $E_0$-module $M$ such that $K \otimes_{E_0} M \simeq N_K$ as an $A_K$-module. The $A_E$-modules $M_E = E \otimes_{E_0} M$ and $N$ become isomorphic to $M_K = N_K$ over $K$. By Theorem 2.2, $M_E \simeq N$ as $A_E$-modules. Thus $N$ descends to $E_0$, as desired.

(b) Clearly $E$ is a field of definition for $N_K$. Hence, by definition of $E_{\min}$, $E_{\min} \subset E$. On the other hand, by part (a), $E_{\min}$ is a field of definition for $N$, and part (b) follows. □

We finally come to the main result of this section.

**Proposition 2.4.** Suppose $F$ is a field satisfying (1.1), $A$ is a finite-dimensional $F$-algebra, $K/F$ is a field extension, $M$ is an indecomposable $A_K$-module, and $F \subset K_0 \subset K$ is an intermediate field, such that $[K_0 : F] < \infty$.

If $M^n$ is defined over $K_0$ for some positive integer $n$, then so is $M$. 
Proof. Set $\text{End}_{A_K}^{ss}(M)$ to be the quotient of $\text{End}_{A_K}(M)$ by its Jacobson radical. By our assumption $M^n \simeq K \otimes_{K_0} N$ for some $A_{K_0}$-module $N$. By Fitting’s lemma,

$$\text{End}_{A_K}^{ss}(M^n) \simeq M_n(D),$$

where $D$ is a finite-dimensional division algebra over some field extension $K'$ of $K$, where $[K' : K] < \infty$. On the other hand,

$$M_n(D) \simeq \text{End}_{A_K}^{ss}(M^n) \simeq \text{End}_{A_K}(K \otimes_{K_0} N) \simeq K \otimes_{K_0} \text{End}_{A_{K_0}}^{ss}(N). \tag{2.5}$$

We conclude that $\text{End}_{A_{K_0}}^{ss}(N)$ is a simple algebra over $K_0$, i.e.,

$$\text{End}_{A_{K_0}}^{ss}(N) \simeq M_m(D_0) \tag{2.6}$$

over $K_0$, for some integer $m \geq 0$ and some finite-dimensional central division algebra $D_0$ over a field extension $K'_0$ of $K_0$ such that $[K'_0 : K_0] < \infty$. On the other hand, $M_n(D)$ is a simple algebra over $K_0$, i.e.,

$$M_n(D) \simeq K \otimes_{K_0} \text{End}_{A_{K_0}}^{ss}(N) \simeq K \otimes_{K_0} M_m(K'_0). \tag{2.5}$$

Since $M_n(D)$ is a simple algebra, we conclude that $K \otimes_{K_0} K'_0$ is a field. Moreover, the index of $M_m(K \otimes_{K_0} K'_0)$ is 1; hence, $D = K'$ is commutative, $K \otimes_{K_0} K'_0 = K'$, and $m = n$.

Now (2.6) tells us that $N \simeq M_0^n$ as a $A_{K_0}$-module, for some indecomposable $A_{K_0}$-module $M_0$. Since $K \otimes_{K_0} N \simeq M^n$, by the Krull–Schmidt theorem $K \otimes_{K_0} M_0 \simeq M$. Thus $M$ descends to $K_0$, as claimed. □

3. Proof of Theorem 1.2

We begin with a simple criterion for the existence of a minimal field of definition.

Lemma 3.1. Let $A$ be a finite-dimensional $F$-algebra, and $K/F$ be a field extension, and $M$ be an $A_K$-module, satisfying conditions (a) and (b) below. Then $M$ has a minimal field of definition.

(a) Suppose $M$ descends to an intermediate field $F \subset L \subset K$, i.e., $M \simeq K \otimes_L N$ for some $A_L$-module $N$. Then $N$ further descends to a subfield $F \subset E \subset L$, where $[E : F] < \infty$.

(b) Suppose $M$ descends to an intermediate field $F \subset E \subset K$ such that $[E : F] < \infty$. That is, $M \simeq K \otimes_E N$ for some $A_E$-module $N$. Then $N$ has a minimal field of definition $E_{\text{min}} \subset E$.

Proof. Condition (a) implies that $M$ is defined over some $F \subset E \subset K$ with $[E : F] < \infty$. Let the $A_E$-module $N$ and the field $E_{\text{min}} \subset E$ be as in (b).
We claim that $E_{\text{min}}$ is independent of the choice of $E$. That is, suppose $F \subset E' \subset K$ is another field of definition of $M$ with $[E' : F] < \infty$, $M := K \otimes_{E'} N'$ for some $A_{E'}$-module $N'$. Let $E''_{\text{min}} \subset E'$ be the minimal field of definition of $N'$, so that $N' := E' \otimes_{E''_{\text{min}}} N''_{\text{min}}$. Then our claim asserts that $E_{\text{min}} = E'_{\text{min}}$. If we can prove this claim, then clearly $E_{\text{min}}$ is the minimal field of definition for $M$. Our proof of the claim will proceed in two steps.

First assume $E \subset E'$. By Lemma 2.3(b), $E''_{\text{min}}$ is a minimal field of definition for $N$. By uniqueness of the minimal field of definition for $N$, $E_{\text{min}} = E'_{\text{min}}$. Now suppose $F \subset E \subset K$ and $F \subset E' \subset K$ are fields of definition for $M$ such that $[E : F] < \infty$ and $[E' : F] < \infty$. Let $E''$ be the composite of $E$ and $E'$ in $K$ and $E''_{\text{min}}$ be the minimal field of definition of $N_{E''} \cong N'_{E''}$. (Note that $N_{E''}$ and $N'_{E''}$ become isomorphic over $K$; hence, by Theorem 2.2, they are isomorphic over $E''$.) Then, $[E' : F] < \infty$, and $E, E' \subset E''$. As we just showed, $E_{\text{min}} = E''_{\text{min}}$ and $E'_{\text{min}} = E''_{\text{min}}$. Thus $E_{\text{min}} = E'_{\text{min}}$, as desired.

We now proceed with the proof of Theorem 1.2.

**Reduction 3.2.** For the purpose of proving Theorem 1.2, we may assume without loss of generality that:

(i) $K$ is a finite extension of $F$;

(ii) $K$ is a Galois extension of $F$.

**Proof.** (i) follows from Lemma 3.1. Indeed, we are assuming that Theorem 1.2 holds whenever $K$ is a finite extension of $F$. That is, condition (b) of Lemma 3.1 holds. On the other hand, condition (a) of Lemma 3.1 follows from Lemma 2.1.

(ii) By part (i), we may assume that $K/F$ is finite. Let $L$ be the normal closure of $K$ over $F$. Then $L/F$ is finite Galois. Lemma 2.3(b) now tells us that if $M_L := L \otimes_K M$ has a minimal field of definition then so does $M$.

We are now ready to finish the proof of Theorem 1.2. In view of Reduction 3.2, it remains to establish the following.

**Lemma 3.3.** Let $F$ be a field satisfying (1.1), $A$ be a finite-dimensional $F$-algebra, $K/F$ be a finite Galois extension, and $M$ be an $A_K$-module. The Galois group $G := \text{Gal}(K/F)$ acts on the set of isomorphism classes of $A_K$-modules via

$$g : N \rightarrow ^g N := K \otimes_g N.$$ 

Let $G_M$ be the stabilizer of $M$ under this action. Then the fixed field $K^{G_M}$ of $G_M$ is the minimal field of definition for $M$.

**Proof.** Suppose $M$ is defined over $K_0$, where $F \subset K_0 \subset K$. Then clearly $^g M \simeq M$ for every $g \in \text{Gal}(K/K_0)$. Hence, $\text{Gal}(K/K_0) \subset G_M$ and consequently, $K^{G_M} \subset K_0$. This shows that $K^{G_M}$ is contained in every field of definition of $M$.

It remains to show that $M$ descends to $K_0 := K^{G_M}$. Write $M = M_{d_1} \oplus \cdots \oplus M_{d_r}$, where $M_1, \ldots, M_r$ are distinct indecomposables. The condition that $^g M \simeq M$ for any $g \in G_M$ is equivalent to the following: if $M_j \simeq ^g M_i$ for some $g \in \text{Gal}(K/K_0)$, then $d_i = d_j$. 
Grouping $G_M$-conjugate indecomposables together, we see that $M \cong S_1 \oplus \cdots \oplus S_m$, where each $S_1, \ldots, S_m$ is the $G_M$-orbit sum of one of the indecomposable modules $M_i$. (Here the orbit sums $S_1, \ldots, S_m$ may not be distinct.) It thus suffices to show that each orbit sum is defined over $K_0$.

Consider a typical $G_M$-orbit sum $S := M_1 \oplus \cdots \oplus M_s$, where we renumber the indecomposable factors of $M$ so that $M_1, \ldots, M_s$ are the $G_M$-translates of $M_1$. Let $H$ be the stabilizer of $M_1$ in $G_M$. That is,

$$ H := \{ h \in G_M \mid hM_1 \cong M_1 \}. $$

Let $K_1 := K^H$. Then

$$ K \otimes_{K_1} (M_1)_{|K_1} = \bigoplus_{h \in H} hM_1 = M_1^{[H]}. $$

In particular, this tells us that $M_1^{[H]}$ descends to $K_1$. By Proposition 2.4, so does $M_1$. In other words, $M_1 \cong K \otimes_{K_1} N_1$ for some $K_1$-module $N_1$. We claim that

$$ K \otimes_{K_0} (N_1)_{|K_0} \cong S. \quad (3.4) $$

If we can prove this claim, then $S$ descends to $K_0$ and we are done.

To prove the claim, note that on the one hand,

$$ K \otimes_{K_0} (M_1)_{|K_0} = \prod_{g \in G_M} gM_1 = S^{[H]}. \quad (3.5) $$

On the other hand, since $M_1 \cong K \otimes_{K_1} N_1$, we have

$$ (M_1)_{|K_0} \cong ((M_1)_{|K_1})_{|K_0} \cong (N_1^{[H]})_{|K_0}, $$

and thus

$$ K \otimes_{K_0} (M_1)_{|K_0} = (K \otimes_{K_0} ((N_1)_{|K_0})^{[H]}) \cong (K \otimes_{K_0} (N_1)_{|K_0})^{[H]}. \quad (3.6) $$

Comparing (3.5) and (3.6), we obtain

$$ (K \otimes_{K_0} (N_1)_{|K_0})^{[H]} \cong S^{[H]} \quad (3.7) $$

The desired isomorphism (3.4) follows from (3.7) by the Krull-Schmidt theorem. This completes the proof of Lemma 3.3 and thus of Theorem 1.2.

\[ \square \]

4. Algebras of finite representation type

A finite-dimensional $F$-algebra $A$ is said to be of finite representation type if there are only finitely many indecomposable finitely generated $A$-modules (up to isomorphism).

**Theorem 4.1.** Let $F$ be a field satisfying (1.1), $A$ be finite-dimensional $F$-algebra of finite representation type, and $K/F$ be a field extension (not necessarily algebraic) such that $F$ is perfectly closed in $K$. (That is, for every subextension $F \subset E \subset K$ with
[\[E:F\]<\infty, E is separable over F.] Suppose \(M\) is an indecomposable \(A_K\)-module. Then:

(a) \(M\) descends to an intermediate subfield \(F \subset E \subset K\) such that \([E:F]<\infty;\)

(b) \(M\) is a direct summand of \(K \otimes_F N\) for some indecomposable \(A_F\)-module \(N\).

**Proof.** (a) Consider the \(A\)-module \(M_{1,F}\). Generally speaking this module is not finitely generated over \(A\). Nevertheless, since \(A\) has finite representation type, thanks to a theorem of Tachikawa [15, Corollary 9.5], \(M_{1,F}\) can be written as a direct sum of finitely generated indecomposable \(A\)-modules. Denote one of these modules by \(N\). This, \(M_{1,F} \cong N \oplus N'\), (4.2)

for some \(A\)-module \(N'\) (not necessarily finitely generated).

Let us now take a closer look at \(N\). By Fitting’s lemma, \(E := \text{End}_{A}^{ss}(N)\) is a finite-dimensional division algebra over \(F\). Since \(F\) is a field satisfying (1.1), \(E\) is a field extension of \(F\). Now set \(F' := E \cap K\) and \(m = [F' : F]\). Since \(F\) is perfectly closed in \(K\), \(F'\) is finite and separable over \(F\). Thus \(\text{End}_{A}^{ss}(F' \otimes_F N) \cong F' \otimes_F \text{End}_{A}^{ss}(N) \cong E \times \cdots \times E\).

This tells us that over \(F'\), \(N\) decomposes into a direct sum of \(m\) indecomposables,

\[F' \otimes_F N = N_1 \oplus \cdots \oplus N_m,\] (4.3)

By the definition of \(F'\), \(K \otimes_{F'} E\) is a field. Hence, each indecomposable \(A_{F'}\)-module \(N_i\) remains indecomposable over \(K\).

Tensoring both sides of (4.2) with \(K\), we obtain an isomorphism of \(A_K\)-modules

\[K \otimes M_{1,F} \cong (K \otimes_F N) \oplus (K \otimes_F N')\]

\[= \left( \bigoplus_{i=1}^{m} K \otimes_{F'} N_i \right) \oplus (K \otimes_F N')\]

\[= (K \otimes_F N_1) \oplus N',\]

where \(N' := (\bigoplus_{i=2}^{m} K \otimes_{F'} N_i) \oplus (K \otimes_F N')\). Note that

\[K \otimes M_{1,F'} \cong \bigoplus B M,\]

where \(B\) is a basis of \(K\) as an \(F'\)-vector space. As we mentioned above, \(K \otimes_{F'} N_1\) is an indecomposable \(A_K\)-module. Since \(K \otimes_{F'} N_1\) is finitely generated and is contained in \(\bigoplus B M\), it lies in the direct sum of finitely many copies of \(M\), say, in \(M^r := M \oplus \cdots \oplus M\) \((r\) copies). Thus we have maps

\[K \otimes_F N_1 \hookrightarrow M^r \hookrightarrow \bigoplus B M \twoheadrightarrow K \otimes_F N_1\]

whose composite is the identity, and so \(K \otimes_F N_1\) is isomorphic to a direct summand of \(M^r\). By the Krull–Schmidt theorem, \(K \otimes_{F'} N_1 \cong M\). In particular, \(M\) descends to \(F'\), as claimed.
(b) By (4.3), $N$ is an indecomposable $A$-module, and $N_1$ is a direct summand of $F' \otimes_F N$. Hence, $M \cong K \otimes_F N_1$ is a direct summand of $K \otimes_F N$, as desired. \qed

**Corollary 4.4.** Let $F$ be a field satisfying (1.1), $A$ be finite-dimensional $F$-algebra of finite representation type, and $K/F$ be a field extension such that $F$ is perfectly closed in $K$. Then $A_K$ is also of finite representation type.

**Proof.** By our assumption $A$ has finitely many indecomposable modules $N^{(1)}, \ldots, N^{(d)}$. By Theorem 4.1(b) every indecomposable $A_K$-module is isomorphic to a direct summand of $K \otimes_F N^{(i)}$ for some $i$. By the Krull–Schmidt theorem, each $K \otimes_F N^{(i)}$ has finitely many direct summands (up to isomorphism), and the corollary follows. \qed

5. **Proof of Theorems 1.3 and 1.4**

We will deduce Theorem 1.3 from Lemma 3.1. $M$ satisfies condition (b) of Lemma 3.1 by Theorem 1.2. It thus remains to show that $M$ satisfies condition (a) of Lemma 3.1. For notational simplicity, we may assume that $K = L$ and $M = N$. That is, we want to show that $M$ descends to some intermediate field $F \subset E \subset K$ with $[E : F] < \infty$. Note that in the case, where $M$ is indecomposable, this is precisely the content of Theorem 4.1(a).

In general, write $M = M_1 \oplus \cdots \oplus M_r$ as a direct product of (not necessarily distinct) indecomposables. By Theorem 4.1(a), each $M_i$ descends to an intermediate field $F \subset K_i \subset K$ such that $[K_i : F] < \infty$. Let $E$ be the compositum of $K_1, \ldots, K_r$ inside $K$. Then $[E : F] < \infty$, and $M$ descends to $E$. This completes the proof of Theorem 1.3. \qed

We now proceed with the proof of Theorem 1.4. Denote the perfect closure of $F$ in $K$ by $F^{pf}$. By Theorem 1.3, $M$ descends to an intermediate field $F^{pf} \subset K_0 \subset K$ such that $[K_0 : F^{pf}] < \infty$. Hence, $K_0$ is algebraic over $F$, and consequently, $\text{ed}(M) \leq \text{trdeg}_F(K_0) = 0$, as desired. \qed

6. **An example**

In this section we will show by example that both Theorems 1.3 and 1.4 fail if we do not require $F$ to be a field satisfying (1.1). Let $F = \mathbb{Q}$ and $A$ be the quaternion algebra

$$A = \mathbb{Q}\{x, y\}/(x^2 = y^2 = -1, \ xy = -yx).$$

and $K/F$ be any field having two elements $a$ and $b$ satisfying $a^2 + b^2 = -1$. Then $A$ has a two-dimensional $A_K$-module $M$ given by

$$x \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad y \mapsto \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}. \quad (6.1)$$

Note that the multiplicative subgroup of $A$ generated by $x$ and $y$ is isomorphic to the quaternion group $Q_8$. Thus $A$ is naturally a quotient of the group algebra $\mathbb{Q}Q_8$ of $Q_8$ over $\mathbb{Q}$. Since $\mathbb{Q}Q_8$ is of finite representation type, one readily concludes that so is $A$.

**Lemma 6.2.** The following conditions on an intermediate field $\mathbb{Q} \subset E \subset K$ are equivalent:
(a) \( \varphi \) descends to \( E \);
(b) \( A \) splits over \( E \);
(c) there exist elements \( a_0, b_0 \in E \) such that \( a_0^2 + b_0^2 = -1 \).

**Proof.** (a) \( \implies \) (b). Suppose \( M \) descends to an \( AE \)-module \( N \). Since \( AE := E \otimes \mathbb{Q} A \) is a central simple four-dimensional algebra over \( E \), the homomorphism of algebras given by

\[
AE \rightarrow \text{End}_E(N) \simeq M_2(E)
\]

is an isomorphism. In other words, \( E \) splits \( A \).

(b) \( \implies \) (a). Conversely, suppose \( E \) splits \( A \). Then the representation of \( A \rightarrow \text{End}_K(M) \) factors as follows:

\[
A \rightarrow E \otimes \mathbb{Q} A \simeq M_2(E) \rightarrow M_2(K).
\]

This shows that \( \varphi \) descends to \( E \).

The equivalence of (b) and (c) a special case of Hilbert’s criterion for the splitting of a quaternion algebra; see the equivalence of conditions (1) and (7) in [9, Theorem III.2.7] as well as Remark (B) on [9, p. 59]. \( \square \)

**Proposition 6.3.** Let \( a \) and \( b \) be independent variables over \( F = \mathbb{Q} \), \( E \) be the field of fractions of \( \mathbb{Q}[a, b]/(a^2 + b^2 + 1) \), and \( M \) be the two-dimensional \( AE \)-module given by (6.1). Then:

(a) \( \text{ed}(M) = 1 \);
(b) \( M \) does not have a minimal field of definition.

**Proof.** (a) The assertion of part (a), follows from [8, Example 6.1]. For the sake of completeness, we will give an independent proof.

Suppose \( M \) descends to an intermediate subfield \( \mathbb{Q} \subset E_0 \subset E \). Since \( \text{trdeg}_\mathbb{Q}(E_0) = 0 \) or 1. Our goal is to show that \( \text{trdeg}_\mathbb{Q}(E_0) \neq 0 \). Assume the contrary, i.e., \( E_0 \) is algebraic over \( \mathbb{Q} \).

Note that \( E \) is the function field of the conic curve \( a^2 + b^2 + c^2 = 0 \) in \( \mathbb{P}^2 \). Since this curve is absolutely irreducible, \( \mathbb{Q} \) is algebraically closed in \( E \). Since \( E_0 \) is algebraic over \( \mathbb{Q} \), we conclude that \( E_0 = \mathbb{Q} \). On the other hand, \( M \) does not descend to \( \mathbb{Q} \) by Lemma 6.2, a contradiction.

(b) Suppose \( M \) descends to \( E_1 \subset E \). Our goal is to show that \( M \) descends to a proper subfield \( E_3 \subset E_1 \). By Lemma 6.2(c) there exist \( a_1 \) and \( b_1 \) in \( E_1 \) such that \( a_1^2 + b_1^2 = -1 \). If \( \mathbb{Q}(a_1, b_1) \) is properly contained in \( E_1 \), then we are done. Thus we may assume without loss of generality that \( E_1 = \mathbb{Q}(a_1, b_1) \). Set \( E_3 := \mathbb{Q}(a_3, b_3) \), where \( a_3 := a_1^3 - 3a_1b_1^2 \) and \( b_3 = 3a_1^2b_1 - b_1^3 \). We claim that (i) \( A \) splits over \( E_3 \), and (ii) \( E_3 \subsetneq E_1 \).
In order to establish (i) and (ii), let us consider the following diagram

\[
\begin{array}{ccc}
E_1(i) & \rightarrow & E_3(i) \\
\downarrow & & \downarrow \\
E_3 & \rightarrow & E_1
\end{array}
\]

of field extensions. Here as usual, \(i\) denotes a primitive 4th root of 1. It is easy to see that \(E_1(i) = \mathbb{Q}(i)(a_1, b_1) = \mathbb{Q}(i)(z)\) is a purely transcendental extension of \(\mathbb{Q}(i)\), where \(z = a_1 + b_1i\) and \((1/z) = -a_1 + b_1i\). Similarly \(E_3(i) = \mathbb{Q}(i)(z^3)\), where \(z^3 = a_3 + b_3i\) and \((1/z^3) = -a_3 + b_3i\). In particular, this shows \(a_3^2 + b_3^2 = -1\), thus proving (i). Moreover, since \(z\) is transcendental over \(\mathbb{Q}(i)\), we have

\[
[E_1(i) : E_3(i)] = [\mathbb{Q}(i)(z) : \mathbb{Q}(i)(z^3)] = 3
\]

and thus

\[
[E_1 : E_3] = \frac{[E_3(i) : E_3] \cdot [E_1(i) : E_3(i)]}{[E_1(i) : E_1]} = \frac{2 \cdot 3}{2} = 3.
\]

This proves (ii). \(\Box\)

**Remark 6.4.** Write \(z^n = a_n + b_ni\) for suitable \(a_n, b_n \in E_1\) and set \(E_n = \mathbb{Q}(a_n, b_n)\). We showed above that \([E_1 : E_3] = 3\) and thus \(E_3 \subsetneq E_1\). The same argument yields \([E_1 : E_n] = n\) for any positive integer \(n\).

7. Proof of Theorem 1.5

We shall actually prove a stronger, more natural theorem, about blocks of finite group algebras. Theorem 1.5 will follow from the fact that \(p\)-Sylow subgroups of a finite group \(G\) are cyclic if and only if every block over a field \(F\) of characteristic \(p\) has cyclic defect; see [6] or [5, Theorem 62.21].

**Theorem 7.1.** Let \(B\) be a block of a finite group algebra \(FG\), where \(F\) is a field of characteristic \(p\). Then the following are equivalent:

1. \(B\) has cyclic defect;
2. \(\text{ed} (\text{Mod}_B) = 0\);
3. \(\text{ed} (\text{Mod}_B) < \infty\).

The implication (1) \(\implies\) (2) is a direct consequence of Theorem 1.4. The implication (2) \(\implies\) (3) is obvious.

The remainder of this section will be devoted to proving that (3) \(\implies\) (1). We shall show that if \(B\) has non-cyclic defect, then \(\text{ed} (\text{Mod}_B) = \infty\). Let \(K\) be an extension field of
$F$, let $e$ be the block idempotent of $B$, let $D$ be a defect group of $B$, and let $N = \Phi(D)$, the Frattini subgroup of $D$. If $D$ is not cyclic, $D/N$ is elementary abelian of rank $r \geq 2$, with basis the images of elements $g_1, \ldots, g_r \in D$. Since $D$ is a defect group of $B$, any $KD$-module $M$ is a summand of $\text{Res}_{G,D}(e.\text{Ind}_{D,G}(M))$.

Now let $n > 0$, and let $K = F(t_{1,1}, \ldots, t_{n,r})$ be a function field in $nr$ indeterminates, and let $M_i$ ($1 \leq i \leq n$) be the two-dimensional $KD$-module

$$g_j \mapsto \begin{pmatrix} 1 & t_{i,j} \\ 0 & 1 \end{pmatrix}.$$ 

Then $J^2(KD)$ is in the kernel of $M_i$, so $M_i$ is really a module for $KD/J^2(KD)$, which has a basis $1, (g_1 - 1), \ldots, (g_r - 1)$. The last $r$ elements of this list form a basis for $J(KD)/J^2(KD)$, and we form a vector space $V$ with basis $(g_1 - 1), \ldots, (g_r - 1)$. The kernel of $M_i$ as a module for $KD/J^2(KD)$ is the codimension one subspace $H_i$ of

$$J(KD)/J^2(KD) \cong V$$

given by

$$H_i := \left\{ \lambda_j(g_j - 1) \mid \sum_j t_{i,j}\lambda_j = 0 \right\}. \quad (7.2)$$

By the Mackey decomposition theorem, the module $M'_i = \text{Res}_{G,D}(e.\text{Ind}_{D,G}(M_i))$ is a direct sum of at least one copy of $M_i$, some conjugates of $M_i$ by elements of $N_G(D)$, and some modules of the form $\text{Ind}_{D_n^g.D,D} \text{Res}_{D,D_{g^e}D} \varrho M$. It follows that the Jordan canonical form of elements of $V$ on $M'_i$ is constant, except on a set $S_i$, which is a finite union of hyperplanes $N_G(D)$-conjugates of $H_i$ and linear subspaces of smaller dimension.

Now let $M := \bigoplus_i M_i$. Our goal is to show that

$$\text{ed}(e.\text{Ind}_{D,G}(M)) \geq n(r - 1).$$

This will imply that $\text{ed}(\text{Mod}_B) \geq n(r - 1)$ for every $n > 0$ and thus $\text{ed}(\text{Mod}_B) = \infty$, as desired.

Note that $e.\text{Ind}_{D,G}(M)$ is a module whose restriction to $D$ is $\bigoplus_i M'_i$. If $e.\text{Ind}_{D,G}(M)$ descends to an intermediate subfield $F \subset K_0 \subset K$, then so does the set $\bigcup_i S_i \subset V$ and its natural image in $\mathbb{P}(V) = \mathbb{P}^{r-1}$, which we will denote by $S$. To complete the proof of Theorem 7.1, it remains to show that if $S$ descends to $K_0$, then

$$\text{trdeg}_F(K_0) \geq n(r - 1). \quad (7.3)$$

**Lemma 7.4.** Let $S \subset \mathbb{P}^{r-1}$ be a projective variety defined over a field $K$. Assume that a hyperplane $H$ given by $a_1x_1 + a_2x_2 + \cdots + a_rx_r = 0$ is an irreducible component of $S$ for some $a_1, \ldots, a_r \in K$ (not all zero). Suppose $S$ descends to a subfield $K_0 \subset K$. Then each ratio $a_j/a_l$ is algebraic over $K_0$, as long as $a_l \neq 0$.

To deduce the inequality (7.3) from Lemma 7.4, recall that in our case $S$ is the union of the hyperplanes $H_1, \ldots, H_n$, a finite number of other hyperplanes (translates of $H_1, \ldots, H_n$ by elements of $N_G(D)$) and lower-dimensional linear subspaces of $\mathbb{P}(V) = \mathbb{P}^{r-1}$. In the basis $(g_1 - 1), \ldots, (g_r - 1)$ of $V$, $H_i$ is given by
Fields of definition for representations of associative algebras

\[ t_{i,1}x_1 + t_{i,2}x_2 + \cdots + t_{i,r}x_r = 0; \]  
see (7.2). Thus by Lemma 7.4 the elements \( t_{i,j}/t_{i,1} \) are algebraic over \( K_0 \) for every \( i = 1, \ldots, n \) and every \( j = 2, \ldots, r \). In other words, if \( K_1 \) is the algebraic closure of \( K_0 \) in \( K \), then each \( t_{i,j}/t_{i,1} \in K_1 \), and thus \( \text{trdeg}_F(K_0) = \text{trdeg}_F(K_1) \geq n(r - 1) \), as desired.

**Proof of Lemma 7.4.** We may assume without loss of generality that \( K_0 \) is algebraically closed. To reduce to this case, we replace \( K_0 \) by its algebraic closure \( \overline{K_0} \) and \( K \) by a compositum of \( K \) and \( \overline{K_0} \). If we know that each \( a_{i,j} \) is algebraic over \( \overline{K_0} \) (or equivalently, is contained in \( \overline{K_0} \)), then \( a_{i,j} \) is algebraic over \( K_0 \).

Now assume that \( K_0 \) is algebraically closed. Since \( S \) is defined over \( K_0 \), every irreducible component of \( S \) is defined over \( K_0 \). In particular, \( H \) is defined over \( K_0 \). That is, the point \( (a_1 : \cdots : a_r) \) of the dual projective space \( \mathbb{P}^{r-1} \) is defined over \( K_0 \). Equivalently, \( a_i/a_j \in K_0 \) whenever \( a_l \neq 0 \). This completes the proof of the claim and thus of Lemma 7.4 and Theorem 7.1. \( \square \)

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