On Semipositivity of Sheaves of Differential Operators
and the degree of a unipolar \( \mathbb{Q} \)-Fano variety

Z. Ran
Math Dept. UCR, Riverside CA 92521 USA
ziv@math.ucr.edu

Abstract

We consider normal projective \( n \)-dimensional varieties \( X \) whose anticanonical divisor class \(-K\) is ample and where every Weil divisor is a rational multiple of \( K \). The index \( i \) is the largest integer such that \( K/i \) exists as a Weil divisor. We show (i) if \( X \) has log-terminal singularities and locally, 1-forms on \( X_{\text{smooth}} \) extend holomorphically to a resolution, then \((-K)^n \leq (\max(in, n+1))^n\); (ii) if the tangent sheaf of \( X \) is semistable, then \((-K)^n \leq (2n)^n\). The proof is based on some elementary but possibly surprising slope estimates on sheaves of differential operators on plurianticanonical sheaves. Unlike previous arguments in the smooth case (Nadel, Campana, Kollar-Miyaoka-Mori), rational curves and rational connectedness are not used. Actually, the proof yields the stronger result that bounds as above hold on the ‘local degree’ of \((X, -K)\), and as such the bound in case (ii) is sharp.

Contents:

Introduction

1. Preliminaries
   1.1 Differential operators.
   1.2 The sheaves \( D^{i,1} \).
   1.3 Splitting properties.
   1.4 Slopes: curve case.
   1.5 Slopes: higher-dimensional case.
   1.6 Slopes in an extension.

2. Slopes of differential-operator sheaves.
   2.1 First-order estimates for rank 1.
   2.2 First-order estimates for higher rank
   2.3 Higher-order estimates.

3. Positivity of \( T_X \).

4. Conclusion
Our purpose is to give some boundedness results for \(\mathbb{Q}\)-Fano varieties of Picard number 1. We begin with some basic definitions. Recall that a \(\mathbb{Q}\)-Fano variety is by definition a normal projective variety \(X\) such that the anticanonical divisor class \(-K = -K_X\) is \(\mathbb{Q}\)-Cartier and ample. For such \(X\) we define the (Weil, resp. Cartier) index \(i = i(X)\) (resp. \(i_C = i_C(X)\)) to be the largest (resp. smallest) integer such that \(K_X/i\) (resp. \(i_C K_X\)) exists as a Weil (resp. Cartier) divisor (see [R] for a discussion of Weil divisors and reflexive sheaves, and also Lemma 2 below), the divisor class group \(\mathcal{N}(X)\) to be the group of Weil divisors modulo rational equivalence, and the Picard number \(\rho = \rho(X)\) to be the rank of \(\mathcal{N}(X)\). When \(\rho = 1\), \(X\) is said to be unipolar; note that in this case the singularities of \(X\) are automatically \(\mathbb{Q}\)-factorial. We will say that \(X\) is \(t\)-factorial, for a natural number \(t\), if for every Weil divisor \(L\), \(tL\) is Cartier; the smallest such \(t\) (that is, the minimum annihilator of \(\mathcal{N}(X) / \text{Pic}(X)\) may be called the denominator of \(X\). Now if \((X, L)\) is any \(\mathbb{Q}\)-polarized variety and \(p \in X\) a smooth point, the 'local degree'

\[
\delta(L, p)
\]

is defined to be the supremum of real numbers \(\delta\) such that for all rational numbers \(d < \delta\) and \(k \gg 0\), the natural map

\[
H^0(kL) \to kL \otimes \mathcal{O}_X / m_p^{kd},
\]

is not injective, i.e. such that

\[
\{ D \in |kL| : \text{ord}_p(D) \geq kd \} \neq \emptyset
\]

and

\[
\delta(L) = \inf(\delta(L, p) : p \in X\text{smooth}).
\]

By Riemann-Roch, clearly

\[
L^n \leq \delta(L)^n,
\]

an inequality which is sometimes strict (see Example 3.4 below).

To state our result we need one more definition. A normal, Cohen-Macaulay variety \(X\) is said to be 1-canonical if for any resolution \(\epsilon : X' \to X\), the differential

\[
\epsilon^*(\Omega_X) \xrightarrow{d\epsilon} \Omega_Y
\]

factors through a map

\[
\epsilon^*(\Omega^*_{X'}) \to \Omega_Y.
\]

In other words, 1-forms on the smooth part of \(X\) lift to holomorphic forms on \(Y\). Thus, 1-canonical is precisely the analogue for 1-forms of the condition defining canonical singularities. Note that the 1-canonical condition is automatically satisfied if \(\Omega^*_{X'} = \Omega_X / (\text{torsion})\). This condition is not too restrictive in view of the following Lemma, whose proof is given at the end of the Introduction:

**Lemma 0.** If \(X\) is locally a finite quotient of a complete intersection \(Z\) with \(Z\) nonsingular in codimension 2, then \(X\) has 1-canonical singularities.

Our main result is the following (we work over \(\mathbb{C}\):
Theorem 1. Let $X$ be an $n$-dimensional unipolar $\mathbb{Q}$-Fano variety of index $i$. Then

(i) if $X$ has log-terminal 1-canonical singularities, we have

$$(0.1) \quad \delta(-K_X) \leq \max(in, n+1);$$

(ii) if the tangent sheaf $T_X = (\Omega^1_X)^*$ is semistable (with respect to $-K_X$) (singularities only assumed normal), we have the sharp estimate

$$(0.2) \quad \delta(-K_X) \leq 2n.$$

Corollary 2. For $X$ log-terminal 1-canonical $t$-factorial as above, we have

$$( -K_X)^n \leq t^n(n+1)^2;$$

if $X$ is locally factorial (e.g. smooth) of index 1, then

$$( -K_X)^n \leq (n+1)^n;$$

if $X$ is smooth and not $\mathbb{P}^n$ or a quadric, then

$$( -K_X)^n \leq ((n-1)(n+1))^n;$$

if $X$ is terminal and $n = 3$, then

$$( -K_X)^3 \leq 4^6i^3_C.$$
The existence of a universal bound on the anticanonical degrees of all smooth unipolar Fano manifolds of given dimension, and hence their boundedness as a family, was known previously by works of Campana, Kollár-Miyaoka-Mori and Nadel, see [K] for an exposition and references. In particular Kollár-Miyaoka-Mori give the bound \((-K_X)^n \leq (n(n+1))^n\) for \(X\) smooth. The singular case is rather different, and I am grateful to J. McKernan for a crash course on this. Already in dimension 2 the set of log-terminal unipolar \(\mathbb{Q}\)-Fanos, as well as their denominators \(t\) and degrees \(K^2\), are unbounded, the simplest example, suggested by McKernan, being the cone over a rational normal curve; see [KeMcK] for many more examples. It has been conjectured by Alexeev, and proven by him in dim. 2 [A], that bounding the log-discrepancy by \(\epsilon > 0\) yields a bounded family, and Kawamata [Ka] and Borisov [Bor] have proven that in dim. 3, assuming \(X\) terminal (resp. bounding the Cartier index) does the same. Batyrev [B3] has conjectured that the \(\mathbb{Q}\)-Fano \(n\)-folds of bounded Cartier index form a bounded family.

The bound \((-K_X)^n \leq (n+1)^n\), which would evidently be sharp (e.g. \(X = \mathbb{P}^n\)) has apparently been conjectured for \(X\) smooth; it is false in the non-unipolar case by Batyrev’s example [b], namely \(X = \mathbb{P}^n_{p_{n-1}}(\mathcal{O} \oplus \mathcal{O}(n-1))\).

Corollary 5. Let \(X\) be a smooth Fano \(n\)-fold with \(\rho = 1\) which is deformable to one admitting a Kähler-Einstein metric. Then \((-K_X)^n \leq (2n)^n\).

**proof.** It is well known that existence of a Kähler-Einstein metric implies \(T_X\) stable, so we can use Theorem 1 (ii).

Apparently, it is generally conjectured that all (smooth) Fano manifolds with \(\rho = 1\) should admit Kähler-Einstein metrics (a fortiori be deformable to ones that do). On the other hand, it is easy to make examples of singular Fano \(n\)-folds \(X\) with \((-K_X)^n > (2n)^n\), and we conclude that these all have unstable tangent sheaves.

Our proof resembles others in focusing on the existence of a section \(s \in H^0(-kK)\) having a zero of high order (roughly \(k\delta(-K_X)\)) at a general point \(p \in X\), but is more ‘elementary’ in that rational curves and bend-and-break are not used. Rather, the basic idea is to consider a sheaf \(D^j(-kK, \mathcal{O})\) of differential operators on a plurianticanonical bundle, \(k \gg 0\), showing that this restricts to a semipositive bundle on a sufficiently ample and general curve-section \(C\) of \(X\) (through \(p\)), provided, roughly, that \(j/k \geq i(n+1)\) (log-terminal 1-canonical case) or \(j/k \geq 2n\) (semistable case). Evaluation on \(s\) reduces the vanishing order by at most \(j\), and this yields a contradiction if \(\delta(-K)\) is too large.

The paper is organized as follows. Sect. 1 gives various preliminary results, mostly well known, on differential operators and slopes in general. Sect. 2 develops slope estimates for sheaves of differential operators. In sect. 3 we prove generic positivity of the tangent sheaf \(T_X\); the method is cohomological, and it is here that assumptions on the singularities of \(X\) come into play. The argument is concluded in Sect. 4. For a rough idea of the proof, the reader may wish to start with Sect. 4, referring back as necessary.

**Acknowledgement.** I would like to thank Professor Dr. E. Viehweg for pointing out Lemma 1.1, Professors P. Burchard, J. Kollár and R.K. Lazarsfeld for helpful communications, Professor J. McKernan for several helpful comments, in particular for pointing out an error in an earlier statement of Lemma 2 as well as suggesting the correct statement, and most especially Professor H. Clemens for patiently and
generously sifting through many error-filled versions of this paper. In addition, I would like to emphasize my debt to several important papers by Y. Miyaoka.

**proof of Lemma 0.** First, if $Z$ is a local complete intersection with singular locus $S$ of codimension $\geq 3$, we have an exact conormal sequence

$$0 \rightarrow r\mathcal{O}_Z \rightarrow (n+r)\mathcal{O}_Z \rightarrow \Omega_Z \rightarrow 0.$$ 

Now it is standard from this sequence that $\Omega_Z$ is reflexive: indeed, denoting by $j : Z - S \rightarrow Z$ the inclusion, we have $\mathcal{O}_Z = j_* j^* \mathcal{O}_Z$ by normality and $R^1 j_* j^* \mathcal{O}_Z = \mathcal{H}_Z^2(\mathcal{O}_Z) = 0$ by depth considerations, hence $\Omega_Z \rightarrow j_* j^* \Omega_Z$ is an isomorphism.

Next, writing locally $X = Z/G$, note that we have, for any $G$-modules $A, B$, split epimorphisms

$$A \rightarrow A^G, B \rightarrow B^G, \text{Hom}(A,B)^G \rightarrow \text{Hom}(A^G,B^G),$$

whence a split epimorphism

$$\text{Hom}(\Omega_Z, \mathcal{O}_Z)^G \rightarrow \text{Hom}(\Omega^G_Z, \mathcal{O}_Z^G).$$

Applying this twice and using the previous result, we get a split epimorphism (where $^+$ temporarily denotes $\text{Hom}(\cdot, \mathcal{O}_Z^G = \mathcal{O}_X)$)

$$\Omega_Z^G = (\Omega_Z^G)^G \rightarrow (\Omega_Z^G)^{++},$$

which easily implies that $\Omega_Z^G$ is a reflexive $\mathcal{O}_X$-module. In view of the natural map $\Omega_X \rightarrow \Omega_Z^G$, which is generically an isomorphism, it follows that

$$\Omega_Z^G = \Omega_X^{++}.$$

Now consider a resolution $\epsilon : X' \rightarrow X$, and let $Z'$ be a resolution of $X' \times_X Z$, so we have a diagram

$$\begin{array}{ccc}
Z' & \rightarrow & Z \\
\downarrow & & \downarrow \\
X' & \rightarrow & X
\end{array}$$

We have a pullback map $\Omega_Z^G \subset \Omega_Z \rightarrow \Omega_Z'$. On the other hand as $X'$ is smooth, we can map $\Omega_Z' \rightarrow \Omega_X'$ by the trace, whence a map $\Omega_X^{++} \rightarrow \Omega_X'$. \qed

**proof of Lemma 3.** (i) This argument is due in the smooth case to Kobayashi-Ochiai, see [K]. We will prove more generally that the largest rational $r \in \mathbb{Q}$ such that $-K_X = rD$ with $D$ Cartier satisfies

$$r \leq n + 1.$$ 

Write $-K_X = iL$ with $tL$ Cartier. It suffices to prove that

$$i/t \leq n + 1.$$ 

Let $\epsilon : Y \rightarrow X$ be a resolution and write

$$K_Y = \epsilon^* K_X + F.$$

Where
with $E$ an exceptional $\mathbb{Q}$-divisor (not necessarily effective), and $M = \epsilon^* L$ (pullback of a $\mathbb{Q}$–Cartier divisor) which is nef and big and $tM$ is integral. On the one hand, for all integers $0 \leq j < n, a > 0$ we have by Kawamata-Viehweg

$$H^j(-atM) = 0.$$ 

On the other hand, we have by Serre duality

$$h^n(-atM) = h^0(K_Y + atM).$$

Now take $u$ sufficiently large and divisible and write

$$h^0(u(K_Y + atM)) = h^0(u(at - i)M + uE).$$

As $bM$ is effective and nontrivial for $b \gg 0$ and as there is clearly no nonconstant rational function on $Y$ with poles only on $\text{supp} E$, the latter is 0 whenever $at < i$, hence in this case

$$h^n(-atM) = 0.$$

Thus the $n$th degree polynomial $\chi(-atM)$ vanishes for all integers $a \in (0, i/t)$, hence $i/t \leq n + 1$, which proves our assertion.

(ii) See [K].

(iii) See [Kea], 6.7.2 (see also 6.11.5 for a more complicated bound for log-terminal threefolds). □

1. Preliminaries.

1.1 Differential operators.

For a normal variety $X$, a torsion-free $\mathcal{O}_X$-module $M$ and a locally free module $N$ we denote by $D^i(M, N)$ or $D^i_{\mathcal{O}_X}(M, N)$ the sheaf of (holomorphic) $i$-th order differential operators on $M$ with values in $N$, i.e. $\text{Hom}(P^i(M), N)$ where $P^i$ denotes the $i$-th principal parts (or jet) sheaf. For $i = 1$ we have an exact sequence

$$\Omega_X \otimes M \to P^1(M) \to M \to 0$$

which is left-injective and locally split over the open set $\text{reg}(M)$ where $M$ is locally free, and induces

$$0 \to \text{Hom}(M, N) \to D^1(M, N) \to \text{Hom}(M, N \otimes T_X)$$

(where $T_X = (\Omega_X)^*$), which is right-surjective over $\text{reg}(M)$, and in particular induces an exact sequence (called the Atiyah sequence)

$$0 \to \text{Hom}(M, N) \to D^1(M, N) \to G \to 0$$

where $G$ is isomorphic over $\text{reg}(M)$ (a fortiori in codimension 1) to $T_X \otimes M^* \otimes N$ Note that $D^i(M, N)$ forms an $\mathcal{O}_X$-bimodule, and in fact there is a natural map

$$N \otimes D^i(\mathcal{O}, \mathcal{O}) \otimes M^* \to D^i(M, N)$$

which is an isomorphism over the open set where $M$ is locally free and $X$ is smooth. As is well known for $X$ smooth, the action of $T_X$ by Lie derivative on $K_X$ gives rise to an identification

$$D^i(\mathcal{O}, \mathcal{O}) = D^i(K_X, K_X)^{\text{op}}, \quad \text{op} = \text{opposite bimodule},$$

hence for $M, N$ locally free $D^i(M, N) = D^i(N^* \otimes K_X, M^* \otimes K_X)^{\text{op}}$. 


1.2 The sheaves $D^{i,1}$.

As a convenient intermediate object for passing from $D^i(M, N)$ to $D^{i+1}(M, N)$ we will consider the sheaf

$$D^{i,1}(M, N) = D^1(P^i(M), N).$$

Note the following sequences which are defined for all $M, N$ and exact over $\text{reg}(M)$:

(1.1) $$0 \to D^{i,1}(M, N) \to D^{i+1,1}(M, N) \to D^1(S^{i+1}(\Omega_X) \otimes M, N) \to 0$$

(1.2) $$0 \to D^i(M, N) \to D^{1,i}(M, N) \to T_X \otimes D^i(M, N) \to 0$$

Combining the evident pairing $D^{i,1}(M, N) \times P^i(M) \to N$ with the canonical $i$-th order differential operator $M \to P^i(M)$, we get a pairing $D^{i,1}(M, N) \times M \to N$ which is easily seen to be a differential operator of order $i + 1$ on $M$, hence yields a natural map

$$D^{i,1}(M, N) \to D^{i+1}(M, N).$$

By an induction using (1.1) we see easily that this map is surjective locally over the open set where $M, N$ are locally free and $X$ is smooth: indeed over this open set we have an exact diagram

$$
\begin{array}{cccccc}
0 & \to & D^{i,1}(M, N) & \to & D^{i+1,1}(M, N) & \to & D^1(S^{i+1}(\Omega_X) \otimes M, N) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & D^{i+1}(M, N) & \to & D^{i+2}(M, N) & \to & M^* \otimes N \otimes S^{i+2}T_X & \to & 0
\end{array}
$$

where the left (resp. right) vertical arrow is surjective by induction (resp. the Atiyah sequence).

Using the natural map $H^0(M) \to H^0(P^i(M))$, the sequence (1.2) gives rise to a pairing

(1.3) $$* : D^{i,1}(M, N) \otimes_C H^0(M) \to T \otimes N$$

which is clearly left $O_X$-linear.

Now as functor in $M$, note that $P^i(M)$ is exact over the smooth part of $X$, because the $i$-th neighborhood of the diagonal in $X \times X$ is flat over the smooth part of $X$. Consequently $D^i(M, N)$ is right-left exact over the smooth part, i.e. an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

induces

(1.4) $$0 \to A'' \to D^i(M, N) \to A' \to 0$$

exact with $A''$ and $A'$ isomorphic in codimension 1 to $D^i(M'', N)$ and $D^i(M', N)$, respectively.

Note in particular that all of the above exactness statements hold in a neighborhood of a generic curve-section.
1.3 Splitting properties.

As is well known for $M, N$ locally free and $X$ smooth the extension class of the Atiyah extension

$$0 \to M^* \otimes N \to D^1(M, N) \to T_X \otimes M^* \otimes N \to 0$$

as left modules is induced from the Atiyah Chern class of $M$, and in particular is nontrivial if $M$ has some nontrivial (ordinary) Chern class and $N \neq 0$. As Viehweg kindly pointed out to me, this remark may be considerably strengthened if $c_1(M)^n \neq 0$. First a definition. An exact sequence of locally free sheaves on $X$:

$$0 \to E \to F \to G \to 0$$

is said to be strongly nonsplit if the associated extension element in $H^1(G^* \otimes E)$ does not lie in the image of $H^1(A) \to H^1(G^* \otimes E)$ for any lower-rank subsheaf $A \subset G^* \otimes E$. If $E$ is invertible, this means precisely that the extension $F$ does not come from a locally split extension of any lower-rank subsheaf of $G$. Notice that if $F$ is of rank 2, i.e. $E, G$ are both invertible, strongly nonsplit is equivalent to nonsplit.

On the other hand an extension of torsion-free sheaves (1.5) is said to be quasisplit if it lies in the image of the natural map

$$\text{Ext}^1(G', E) \to \text{Ext}^1(G, E)$$

for some torsion-free lower-rank quotient $G'$ of $G$; $G'$ or the corresponding (non-trivial, saturated) subsheaf of $G$ is called a quasisplitting. Thus, a non-quasisplit extension is one that does not come from any extension, locally split or not, of a lower-rank subsheaf of $G$. Note that a strongly nonsplit extension (even of vector bundles) may well be quasisplit, as there will in general be nonlocally-free subsheaves $G'$ and they will admit nonlocally split extensions inducing locally split extensions of $G$.

The following result, pointed out by Viehweg, will not be needed in the sequel, but is good for motivation:

Lemma 1.1. (Viehweg): Let $X^n$ be smooth compact, $L$ a line bundle on $X$ with $c_1(L)^n \neq 0, A \subset \Omega^1_X$ a subsheaf of rank $< n$. Then the extension class $c_1(L) \in H^1(\Omega^1_X)$ is outside the image of $H^1(A)$. Hence the extensions defining $P^1(L)$ and $D^1(L, O)$ are strongly nonsplit.

Proof. Suppose $c_1(L)$ comes from an element $\alpha \in H^1(A)$. Then we may represent $c_1(L)$ by a suitable Čech cocycle $z$ with values in $A$. As $A$ has rank $< n$, the cup-power $z^n$, which represents $c_1(L)^n \in H^n(\Lambda^n \Omega_X^1)$ must vanish ‘point-by-point’(even as a cocycle), against our hypothesis $c_1(L)^n \neq 0$. □

1.4 Slopes: curve case.

We begin by reviewing some definitions and facts about bundles on curves and their slopes (see [S],[SB], [MehR] for details). For a vector bundle $E$ on a smooth curve, we denote by $\mu(E)$ its slope, i.e.

$$\mu(E) = \text{deg}(E)/\#(E)$$
by $\mu'(E)$ the 'shifted' slope

$$\mu'(E) = \deg(E)/(rk(E) + 1)$$

and by $\mu_{\max}(E)$ and $\mu_{\min}(E)$ (resp $\mu'_{\max}(E)$ and $\mu'_{\min}(E)$) the largest (resp. smallest) slopes (resp. shifted slopes) of a subbundle (resp. quotient bundle) of $E$. As is well known, the former coincide respectively with the slopes of the first and last associated graded of the Harder-Narasimhan (HN) filtration of $E$. See op. cit. for various basic properties of these invariants. One property we need which is not mentioned there is behaviour with respect to duality, viz.

$$\mu_{\min}(E) = \mu_{\max}(E^*) .$$

This can be checked easily.

1.5 Slopes: higher-dimensional case.

Now given a (normal projective) variety $X$ we shall henceforth denote by $C$ a sufficiently general sufficiently ample curve-section of $X$ (say with respect to a given polarization $H$), and define slopes of a torsion-free sheaf $E$ on $X$ by restricting on $C$. The results of [MeR], which show that an $H$–semistable sheaf of $X$ restricts to a semistable one on $C$, imply that these slopes coincide with those based on $H$–semistability on $X$; in particular they are independent of the choice of $C$ and $\mu_{\min}(E)$ coincides with the slope of the last associated graded of the $HN$ filtration of both $E$ and $E|_C$, which are compatible (i.e. the former restricts to the latter). Also various linear algebra type properties of slopes carry over from $C$ to $X$, and two torsion-free sheaves which are isomorphic in codimension one have the same slopes. $E$ is said to be generically (semi)positive if $\mu_{\min}(E) > (\geq)0$. Naturally the slopes of an arbitrary coherent sheaf are defined to be those of its largest torsion-free quotient.

1.6 Slopes in an extension.

Now we want to give a simple remark concerning the behavior of slopes under extension. To this end we introduce the following invariant

$$a = a(X, H) = \min\{A.H^{n-1} : A \subset X \text{ nontrivial effective Weil divisor}\} .$$

Note that for $X$ unipolar $\mathbb{Q}$-Fano and notations as above) we have

$$a \geq -K_X.H^{n-1}/i \geq -K_X.H^{n-1}/(n+1)$$

as $i \leq n+1$ (Lemma 2). It will be convenient to abuse notation a bit and assume $C \sim H^{n-1}$.

Lemma 1.2. Let $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$ be an extension of torsion-free sheaves on $X$ with $F$ of rank $r+1$ and $E$ reflexive of rank 1. Then for any torsion-free quotient $F'$ of $F$ that is not a quasisplitting, we have

$$(1.5) \quad \mu(F') \geq \min\left(\frac{a + E.H^{n-1}}{s} + \frac{s}{\mu_{\min}(G)}, s = 0, \ldots, r\right) .$$
Proof. Let \( E' \subset F' \) be the saturation of the image of \( E \). Then \( G' = F'/E' \) is torsion-free of rank \( s \) say, and we have an exact diagram

\[
\begin{array}{cccccc}
0 & \to & E & \to & F & \to & G & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & E' & \to & F' & \to & G' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & 0 & & 0
\end{array}
\]

If \( E' = 0 \), then \( G \cong G' \), and the lower bound on \( \mu(F') \) clearly holds. Otherwise, \( E \to E' \) is injective. Suppose this map is surjective in codimension 1. Then so is the composite map to the double dual \( E \to E' \to E'' \). But \( E'' \) is reflexive, and an injection of rank-1 reflexive sheaves which is surjective in codimension 1 is an isomorphism (cf. [R]). It follows that \( E = E' \) which contradicts our hypothesis that \( F' \) is not a quasisplitting. Thus \( E \to E' \) must vanish on some nontrivial effective divisor and again our estimate follows easily.

2. Slopes of differential-operator sheaves.

The purpose of this section is to give some slope estimates for sheaves of differential operators, culminating in Proposition 2.6 below, which is a (generic) semipositivity result for operators on suitable plurianticanonical bundles.

2.1 First-order estimates for rank 1.

The basic idea here is to apply Lemma 1.2 to sheaves of the form \( F = D^1(L, O) \), where \( L \) is an ample line bundle on our unipolar \( \mathbb{Q} \)-Fano \( X \), and \( F \) is considered as an Atiyah extension. The problem is to deal with possible quasisplittings.

Lemma 2.1. For any rank-1 torsion-free sheaf \( L \) on \( X \) with \( L.C \neq 0 \) we have

\[
\mu_{\min}(D^1(L, O)) \geq -L.C + b
\]

where

\[
b = \min(a, \mu'_{\min}(T)).
\]

Proof. To begin with, there exists \( s > 0 \) so that \( L^{[s]} := (L^{\otimes s})^{**} \) is invertible \( D^1(L, O) \) is isomorphic in codimension 1 to (hence has the same slopes as) \( D^1_{T}(L^{[s]}, O) \otimes L^{[-s+1]} \). Thus there is no loss of generality in assuming \( L \) invertible.

Now suppose given a torsion-free quotient \( F' \) of \( D^1(L, O) \). If \( F' \) is not a quasisplitting, i.e. \( L^* \) does not map isomorphically to a saturated subsheaf of \( F' \), then Lemma 2.1 applies and note that the RHS of (2.1) in this case is just

\[
-L.C + \min(a, \mu_{\min}(T)) \geq -L.C + b,
\]

hence

\[
\mu(F') \geq -L.C + b.
\]

Otherwise, \( F' \) is a quasisplitting, so we get a saturated subsheaf \( T_0 \subset T \) so that the Atiyah extension splits over \( T_0 \otimes L^* \). If \( T \neq T_0 \), then obviously \( \mu(F') \geq -L.C + b \).
If \( T_0 = T \), the Atiyah extension for \( L \) splits at least over the smooth part of \( X \) and in particular over \( C \). This clearly implies that the Atiyah extension for \( L \otimes \mathcal{O}_C \) splits, which implies that \( L \otimes \mathcal{O}_C \) is flat, hence \( L.C = 0 \) which is impossible. \( \square \)

2.2 First-order estimates for higher rank.

We shall next extend Lemma 2.1 to higher-order operators.

**Lemma 2.2.** If \( E \) is a torsion-free generically positive sheaf on \( X \), then

\[
\mu_{\text{min}}(D^1(E, \mathcal{O}_X)) \geq \mu_{\text{min}}(E^*) + b
\]

(Here and elsewhere \( D^i \) are considered as left \( \mathcal{O}_X \)-modules unless otherwise specified).

**Proof.** If \( E \) has rank 1, the assertion follows directly from Lemma 2.2. In the general case rank \( E = \rho \) we may use the HN filtration \( E \) and the map

\[
D^1(E_{i-1}, \mathcal{O}) \to D^1(E_i, \mathcal{O})/D^1(E_i/E_{i-1}, \mathcal{O}),
\]

(cf. (1.3)), which is an isomorphism in codimension 1, to reduce to the case \( E \) semistable. Then use in a similar manner a Seshadri stable filtration to reduce to the case \( E \) stable (i.e. \( E|_C \) stable, cf. [S]). At this point let us fix \( C \) and consider a suitable unbranched degree- \( d \) cyclic cover

\[
\pi : U' \to U
\]
of a tubular neighborhood of \( C \) in \( X \) where \( \pi^* \det(E) \) admits a \( \rho \)-th root \( L \), so we may write

\[
\pi^* E = F \otimes L
\]

where \( c_1(F) = 0 \). Let \( C' = \pi^{-1}(C) \). Note that \( L.C' = dc_1(E).C = d\mu_{\text{min}}(E) \). We compute:

\[
\mu_{\text{min}}(D^1_U(L, \mathcal{O}_{C'})) = \mu_{\text{min}}(L^\rho - 1 \otimes D^1_U(L^\rho, \mathcal{O}_{C'})) =
\]

\[
= (\rho - 1)L.C' + \mu_{\text{min}}(\pi^* D^1_X(\det E, \mathcal{O}_C)) = (\rho - 1)L.C' + d\mu_{\text{min}}(D^1(\det E, \mathcal{O}_X)) \geq
\]

\[
\geq (\rho - 1)L.C' + d(\det E^* . C + b) = -L.C' + db = d(\mu_{\text{min}}(E^*) + b).
\]

(The inequality follows from the rank-1 case) Now note that \( D^1_U(\pi^* E, \mathcal{O}_{C'}) \) depends only on the restriction of \( \pi^* E \) on the first-order neighborhood \( C_1 \) of \( C' \) in \( U' \), and that \( C_1 \) coincides with the first-order neighborhood of the zero-section of the normal bundle \( N \) to \( C' \) in \( U' \), hence admits a projection map \( p : C_1 \to C' \) and a scaling action, i.e. a family of endomorphisms \( \{ \phi_t, t \in \mathbb{C} \} \) given by multiplication by \( t \) on a fibre of \( N \). This gives rise to a family of sheaves

\[
\{ F_t = \phi_t^*(F) : t \in \mathbb{C} \}
\]

with \( F_t \simeq F \) for all \( t \neq 0 \) and \( F_0 = p^*(F|_{C'}) \). Now by an easy general remark about bundles on curves, the pullback of a stable bundle by an unramified cyclic cover is stable. Consequently, as \( F|_{C'} \) is stable, so is \( \pi^* F|_{C'} \); hence also \( F|_{C_1} \), so that \( F|_{C_1} \)...
has the form \( F \otimes \mathcal{O}_{C'} \) for some locally constant sheaf \( F \) on \( C' \). As \( \mu_{\min} \) decreases under specialization, we get the estimate

\[
\mu_{\min}(D^1(E, \mathcal{O}_C)) = \frac{1}{d} \mu_{\min}(D^1_U(\pi^*E, \mathcal{O}_{C'})) = \frac{1}{d} \mu_{\min}(D^1_U(F \otimes L, \mathcal{O}_{C'})) \\
\geq \frac{1}{d} \mu_{\min}(D^1_U(F_0 \otimes L, \mathcal{O}_{C'})) = \frac{1}{d} \mu_{\min}(\mathcal{F}^\vee \otimes_C D^1_U(L, \mathcal{O}_{C'})) \\
= \frac{1}{d} \mu_{\min}(D^1_U(L, \mathcal{O}_{C'})) \geq \frac{1}{d} \mu_{\min}(\mathcal{F}^\vee \otimes_C D^1_U(L, \mathcal{O}_{C'})) \geq b + \mu_{\min}(E^*). 
\]

\[\square\]

### 2.3 Higher-order estimates.

Next we extend this slope estimate to higher order, via the sheaves \( D^{m, 1} \). First observe the standard formula

\[
c_1(D^m(E, \mathcal{O}_X)) = \binom{r + m}{r + 1} \rho c_1(T) + \binom{r + m + 1}{r} c_1(E^*). 
\]

Then a straightforward induction based on Lemma 2.2 plus the surjection \( D^{1, m}(E, \mathcal{O}_X) \rightarrow D^{m+1}(E, \mathcal{O}_X) \) yield:

**Lemma 2.3.** *In the above situation we have*

\[
\mu_{\min}(D^{m+1}(E, \mathcal{O}_X)) \geq \mu_{\min}(D^{m, 1}(E, \mathcal{O})) \geq \min(0, \mu_{\min}(E^*) + (m + 1)b).
\]

*Proof.* To begin with, the first inequality is immediate from the map \( D^{m, 1}(E, \mathcal{O}) \rightarrow D^{m+1}(E, \mathcal{O}) \) which is surjective in codimension 1. Now by induction on \( m \), if \( Q \) is any quotient of the HN filtration \( F \) of \( D^m(E, \mathcal{O}) \), then

\[
\mu_{\min}(E^*) + mb \leq \mu(Q).
\]

Now either \( \mu(Q) < 0 \), in which case, \( Q^* \) being semistable, \( Q^* \) is generically positive and so by Lemma 2.3

\[
\mu_{\min}(D^1(Q^*, \mathcal{O})) \geq \mu_{\min}(E^*) + (m + 1)b;
\]

or else \( \mu(Q) \geq 0 \), in which case clearly \( \mu_{\min}(D^1(Q^*, \mathcal{O})) \geq 0 \). Using (1.3) as above \( F \) induces a filtration on \( D^{m, 1}(E, \mathcal{O}) \) whose quotients are isomorphic in codimension 1 to the \( D^1(Q^*, \mathcal{O}) \), hence satisfy the above inequality, so the Lemma holds. \( \square \)

Specializing to the case of a line bundle, we conclude

**Lemma 2.5.** *Let \( L \) be a line bundle on \( X \) and \( m \) any integer satisfying*

\[
m \geq \frac{L.C}{b} - 1
\]

*Then, in the above situation, \( D^{m, 1}(L, \mathcal{O}_X) \) and \( D^{m+1}(L, \mathcal{O}_X) \) are generically (left) semipositive*

Specializing further to the case \( L = k \det(T) \), we get the following useful estimate...
Proposition 2.6. Let $X$ be a complex unipolar $\mathbb{Q}$-Fano variety. Then $D^{\alpha k,1}(-kK,\mathcal{O}_X)$ and $D^{\alpha k+1}(-kK,\mathcal{O}_X)$ are generically semipositive provided $\alpha k$ is an integer and

$$\alpha \geq \max(-K.C/\mu'_\min(T), -K.C/a).$$

proof. Observe that the RHS of the above inequality on $\alpha$ is just $-K.C/b$, hence by hypothesis there exists an integer $m$ with $-K.C/bk-1 \leq m < k(r+1).$ so that the hypotheses of Lemma 2.5 are satisfied, hence $D^{m,1}(-kK,\mathcal{O}_X)$ is generically semipositive, which easily implies that so is $D^{\alpha k,1}(-kK,\mathcal{O}_X)$ since $\alpha k \geq m$, hence so is $D^{\alpha k+1}(-kK,\mathcal{O}_X)$. \qed

To take advantage of this result, it is convenient to introduce the following defi-

cition. For a $\mathbb{Q}$-divisor $L$, we define the 'differential degree'

$$\gamma(L) \in \mathbb{R} \cup \{\infty\}$$

as the inf of all $m \in \mathbb{Q}$ such that for all $\alpha > m$ and all $k$ sufficiently large and divisible, $D^{\alpha k}(kL,\mathcal{O}_X)$ is generically semipositive. Thus the conclusion of Proposition 2.6 can be rephrased as the estimate

$$\gamma(-K) \leq \max(-K.C/\mu'_\min(T), -K.C/a).$$

3. Positivity of $T_X$.

By definition, a Fano variety $X$ has a tangent sheaf $T_X$ which is positive 'on average'. The purpose of this section is to show that, with suitable extra hypotheses, $T_X$ is actually positive on a generic curve-section. We will prove the following result, which is apparently well known in the smooth case (and which also is the only place where log-terminal 1-canonical singularities are used).

Proposition 3. If $X$ is $\mathbb{Q}$-Fano unipolar with log-terminal 1-canonical singulari-

ties then $T_X$ is generically positive.

proof. Let

$$T_X \to Q \to 0$$

be a quotient of rank $r > 0$ and $c_1 \leq 0$, corresponding to a reflexive saturated subsheaf $Q^* \subset \Omega_X^{**}$ and to a section

$$u \in H^0(Q^{**} \otimes \Omega_X^{**}).$$

Note $r < n$ and set

$$M = c_1(Q^*) = (\bigwedge^r Q^*)^{**}.$$ 

This is a divisorial sheaf which is either numerically trivial or ample (i.e. for some $s > 0$, $M^{[s]}$ is Cartier and either numerically trivial or ample; note that $M^{[*]} = M^{[-s]}$ is then Cartier as well ). We consider resolutions

$$\xi : Y \to X.$$
Lemma 3.2. Given a torsion-free sheaf $S$ on $Y$, there is a blowup $\eta : Z \to Y$ such that $\eta^*S/(\text{torsion})$ is locally free.

**proof.** Let $\alpha : F \to S$ be a surjection with $F$ locally free (in fact of the form $\oplus H^*$ with $H$ sufficiently ample), and consider the 'canonical resolution' of $\alpha$, i.e. let $Y'$ be the subvariety of the Grassmann bundle

$$\pi : G(r, F) \to Y, \ r = \text{rank}(F) - \text{rank}(S)$$

defined as the zero-locus (with the reduced structure) of the natural map $\text{Sub} \to \pi^*S$, where $\text{Sub}$ denotes the tautological subbundle, with natural map $\beta : Y' \to Y$. Clearly $\beta$ is birational. Let $Z$ be a desingularization of $Y'$ with natural map $\eta : Z \to Y$, and $Q$ the pullback of the tautological quotient bundle on $G(r, F)$ to $Z$. Then there is an induced surjection $Q \to \eta^*S$, and the induced map $Q \to \eta^*S/(\text{torsion})$ is surjective and generically injective, hence an isomorphism. \[ \square \]

It follows in our situation that we may assume

$$-N := \epsilon^*(M^*)/\text{torsion}$$

is invertible and that $\epsilon^*(Q^{**})/(\text{torsion})$ is locally free. Note that

$$c_1(\epsilon^*(Q^{**})/(\text{torsion})) = -N.$$ 

Now the multiplication map

$$(\epsilon^*(M^*))^{\otimes s} = \epsilon^*((M^*)^{\otimes s}) \to \epsilon^*((M)^{[-s]})$$

gives rise to a sheaf inclusion $-sN \subset -sN_1$, which is an equality locally off the exceptional locus. Consequently we may write in case $M$ is numerically trivial

$$N = R + S$$

with $R$ numerically trivial and $S = \sum e_i E_i, e_i \in \mathbb{Q}^{\geq 0}$. If $M$ is ample we write

$$N = A + B + F$$

with $A \mathbb{Q}$-ample, $B$ effective integral and $F$ $\mathbb{Q}$-effective with normal crossing support and $[F] = 0$(i.e. $B + F = S$ as above).

Now by our hypothesis of 1-canonical singularities the differential of $\epsilon$ factors through a map

$$de : \epsilon^*(\Omega_X^{**}) \to \Omega_Y$$

and $df(f^*u)$ yields a section $v \in H^0(\Omega_Y \otimes f^*(Q^{**}))$; by rank considerations the component of $\text{Sym}^{2r}v$ in $H^0(\Omega_Y \otimes \bigwedge^r f^*(Q^{**}))$ is nonzero, whence a nonzero section $O \to \Omega_Y(-N)$.

Now our assertion follows from
Lemma 3.3. In the above situation, we have $H^0(\Omega_Y^r(-N)) = 0$.

proof. First, if $M = O_X$ then $N = O_Y$. By Hodge symmetry it suffices to prove $H^r(O_Y) = 0$.

Now by [KaMaMa], Thm. 1-2-5 (log-terminal Kodaira vanishing), we have $H^i(O_X) = 0$ for $i > 0$, and by op. cit. Thm 1-3-6 (rationality of log-terminal singularities) we have $R^i\epsilon_*(O_Y) = 0$ for $i > 0$. Hence by Leray $H^r(O_Y) = 0$, as required.

Next, if $M$ is numerically trivial, then as we have just proven $H^1(O_Y) = 0$, we have $R = 0$, hence $N$ is $\mathbb{Q}$-effective exceptional. On the other hand, if $N$ is effective in a neighborhood of any fiber of $\epsilon$, and it follows that $N$ is trivial, so we are done as above.

Now assume $M$ is ample. Mimicking the usual proof of Kawamata-Viehweg as in [KaMaMa], Thm. 1-1-1, consider a suitable finite Galois cover

$$\tau : Z \to Y$$

with $Z$ smooth so that $\tau^*A$ is integral and

$$\Omega_Y^r(-A - B - F) \to \tau_*(\Omega_Y^r(-A - B))$$

is a direct summand inclusion, and note the injection

$$\tau^*(\Omega_Y^n) \to \Omega_Z^n.$$

By Nakano,

$$H^0(\Omega_Z^n(-\tau^*A)) = 0,$$

hence $H^0(\Omega_Z^n(-\tau^*(A + B)) = 0$, hence $H^0(\tau^*(\Omega_Y^n(-A - B)) = 0$, hence clearly

$$H^0(\Omega_Y^r(-A - B - F)) = 0. \Box$$

4. Conclusion.

We continue with the notation of Proposition 2.6, and seek firstly to estimate the RHS of (2.5).

Proposition 4.1. In the above situation we have (i) if $X$ is log-terminal 1-canonical,

$$(3.1) \quad \mu_{\text{min}}'(T) \geq \frac{-K.H^{n-1}}{\max(in, n + 1)};$$

(ii) if $T$ is semistable,

$$(3.2) \quad \mu_{\text{min}}'(T) \geq \frac{-K.H^{n-1}}{2n}.$$

proof. (i) This follows directly from Proposition 3.1 and the definition of index.

(ii) Semistability means that any torsion-free quotient $Q$ of $T$ of rank $r > 0$ has

$$\mu(Q) \geq \frac{-K.H^{n-1}}{n},$$

hence

$$\mu'(Q) = \frac{r}{r+1} \mu(Q) \geq \frac{1}{2} \mu(Q) \geq \frac{-K.H^{n-1}}{2n}.$$

$\Box$

In light of Propositions 4.1 and 2.6, Theorem 1 follows immediately from the following easy remark.
Lemma 4.2. For any $\mathbb{Q}$-ample divisor $L$, we have

$$\delta(L) \leq \gamma(L).$$

proof. If not, pick a rational number

$$\alpha \in (\gamma(L), \delta(L)).$$

Then we have that for $k$ sufficiently large and divisible,

$$D^{\alpha k}(kL, \mathcal{O}_C)$$ is semipositive.

Now the evaluation map gives rise to a (left $\mathcal{O}_C$-linear) map

$$D^{\alpha k}(kL, \mathcal{O}_C) \to H^0(kL)^* \otimes \mathcal{O}_C.$$

By definition of $\delta(L)$, we can find, for a suitably small $\varepsilon > 0$, a nontrivial section

$$s \in H^0(kL) \quad \text{with} \quad \text{ord}_p(s) \geq (\alpha + \varepsilon)k, \ p \in X \text{ general}.$$

Projecting $H^0(kL)^*$ onto $(\mathbb{C}s)^* = \mathbb{C}$, we get a map

$$\varphi: D^{\alpha k}(kL, \mathcal{O}_C) \to \mathcal{O}_C.$$

Choosing $C, s$ sufficiently general mutually, clearly we can assume $\varphi$ is nontrivial. Moreover $\varphi$ evidently factors through $\mathcal{O}_C(-\varepsilon kp)$, which is a negative sheaf, contradicting semipositivity of $D^{\alpha k}(kL, \mathcal{O}_C)$. □

Example 4.3 Let $X$ be a smooth hypersurface of degree $d \leq n+1$ in $\mathbb{P}^{n+1}$, $n \geq 3$. Then $X$ is Fano of index $i = n+2-d$ and the tangent bundle $T_X$ is stable by [PeW]. By Propositions 2.6 and 4.1, $D^{\alpha k}(-kK_X, \mathcal{O}_X)$ is semipositive for $\alpha \geq 2n$.

On the other hand, given a general point $p \in X$, let $u \in H^0(\mathcal{O}_X(1))$ be the section defining the tangent hyperplane at $p$, and $s = u^{ik} \in H^0(-kK_X)$, which has order $2ik$ at $p$, showing in particular that $\delta(-K_X) \geq 2i$, so the estimate (0.2) is sharp for $d = 2$. Further, choosing things generally enough so that $C$ passes through $p$ but is not contained in the tangent hyperplane at $p$, we get as above a nonzero map

$$\varphi: D^{\alpha k}(-kK_X, \mathcal{O}_C) \to \mathcal{O}_C.$$

For any $\alpha < 2i$, this map clearly factors through $\mathcal{O}_C(-p)$, and consequently $D^{\alpha k}(-kK_X, \mathcal{O}_X)$ is not generically semipositive. In particular, for $d = 2$ the estimate of Propositions 2.6 and 4.1 is sharp as well, i.e. $D^{\alpha k}(-kK_X, \mathcal{O}_X)$ is generically semipositive iff $\alpha \geq 2n$, and thus $\delta(-K_X) = \gamma(-K_X)$.

Note that this example shows that it can happen that $L^n < \delta(L)^n$. This means that the natural map $H^0(-kK_X) \to -kK_X \otimes \mathcal{O}_X/m_p^N$ is not necessarily of maximal rank for all $k, N \gg 0$ and $p \in X$ general.
References

[A] Alexeev, V.: 'Boundedness of $K^2$ for log surfaces' Internat. J. Math 5, (1994), 779-810.

[B] Batyrev, V.V.: 'Boundedness of the degree of multidimensional Fano varieties', Vestnik MGU (1982), 22-27.

[B3] Batyrev, V.V.: 'The cone of effective divisors of threefolds' Contemp. math. 131 (1989), part 3, 337-352.

[Bor] Borisov, A.: 'Boundedness theorem for Fano log-threefolds' J. Algebraic Geometry 5, (1996), 119-133.

[Ka] Kawamata, Y.: 'Boundedness of $Q$-Fano threefolds' Contemp. Math 131 (1989), part 3, 439-445. Amer. math. soc.

[KaMaMa] Kawamata, Y., Matsuda, K., Matsuki, K.: 'Introduction to the minimal model program' Adv. studies in Pure Math. 10 (1987), 283-360.

[KeMcK] Keel, S., McKernan, J.: (to appear)

[Ko] Kollár, J.: 'Rational curves on algebraic varieties' Springer 1996.

[Koef] Kollár, J.: 'Effective base point freeness'. Math. Ann. 296 595-605 (1994).

[Kea] Kollár, J., et al.: 'Flips and abundance for algebraic threefolds' Astérisque 211 (1992).

[KoMiMo] Kollár, J., Miyaoka, Y., Mori, S.: 'Rationally connected varieties'. J. Alg. Geom. 1 (1992), 429-448; 'Rational connectedness and boundedness of Fano manifolds' J. Diff. Geom. 36 (1992), 765-769.

[MeR] Mehta, V., Ramanathan, A.: 'Semi-stable sheaves on projective varieties and their restriction to curves'. Math. Ann. 258 (1982), 213-224.

[Mi1] Miyaoka, Y.: 'Deformations of a morphism along a foliation and applications'. Proc. Symp. Pure Math 46 (1987), 245–268.

[Mi2] __________: 'The Chern classes and Kodaira dimension of a minimal variety'. Adv. Studies in Pure Math. 10 (1981), 449-476.

[Mu] Mukai, S.: 'Biregular classification of Fano threefolds and Fano manifolds of coin-
dex 3', Proc. Nat. Acad. Sci USA 86 (1989), 3000-3002.

[PeW] Peternell, T., Wisniewski, J.: 'On stability of tangent bundles of Fano manifolds with $b_2 = 1$' (preprint).

[ReMi] Reid, M.: 'Canonical 3-folds'. In: A. Beauville, ed.: 'Journées d’Angers', North-Holland 1980.

[ReMii] Reid, M.: 'Bogomolov’s theorem $c_1^2 \leq 4c_2$', in: Algebraic Geometry Kyoto 1977, 623-642, Tokyo, Kinokuniya.

[SP] Shepherd-Barron, N.I.: 'Miyaoka’s theorems...'. Astérisque 211 (1992), 103-114.
[S] Siu, Y.T.: 'Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics', DMV 8, Birkhauser 1987.