On Hausdorff dimension of polynomial not totally disconnected Julia sets

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Abstract

We prove that for every polynomial of one complex variable of degree at least 2 and Julia set not being totally disconnected nor a circle, nor interval, Hausdorff dimension of this Julia set is larger than 1. Till now this was known only in the connected Julia set case.

We give also an example of a polynomial with non-connected but not totally disconnected Julia set and such that all its components comprising more than single points are analytic arcs, thus resolving a question by Christopher Bishop, who asked whether every such component must have Hausdorff dimension larger than 1.

1. Introduction

In the sequel, we refer to connected components of subsets of the complex plane $\mathbb{C}$ that are not singletons (sets comprising of single points) as non-trivial components.

Christopher Bishop in [2], commenting on his result on the existence of an entire transcendental function with one-dimensional Julia set, asked the following question:

The connected components of the Julia set constructed in this paper are all either points or continua of Hausdorff dimension one. (…) However, the situation for polynomials is open. If a polynomial Julia set is connected, then it is either a generalized circle/segment or has Hausdorff dimension strictly greater than 1 (this follows from work of Zdunik [51] and Przytycki [36]). Is this also true of the non-trivial connected components when the Julia set is disconnected? In other words, if $J(p)$ is disconnected, its every connected component is either a point or a set of Hausdorff dimension strictly greater than 1?

We recall that for every polynomial $p$ of degree at least 2, the Julia set $J(p)$ is the boundary of the basin of attraction to $\infty$ under the action by $p$.

In this article, we answer the above questions and related ones. We start with key.

Theorem 1. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. If the Julia set $J(f)$ is disconnected, then every non-trivial connected component $J'$ of $J(f)$ is eventually periodic, that is, $f^\ell(J')$ is periodic with period $k$ for some integers $\ell, k$. Furthermore, either $f^\ell(J')$ is an analytically embedded interval (that is, analytic arc), and $f^k$ on it is analytically conjugate to a $\pm$ Chebyshev polynomial, or $J'$ has Hausdorff dimension, and even hyperbolic dimension, greater than 1.

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This theorem is proved in Section 2. An example is presented in Figure 1.

In Section 3, we complete the picture, including possible analytic components and proving the following theorem, which is the main result of this note.

**THEOREM 2.** Let \( f : \mathbb{C} \to \mathbb{C} \) be a polynomial of degree \( d \geq 2 \), such that \( J(f) \) is not totally disconnected. Assume also that \( f \) is not complex affine conjugate to a map \( z \mapsto z^d \) or a \( \pm \) Chebyshev polynomial. Then \( \text{HD}(J(f)) > 1 \). Even more, the hyperbolic dimension, \( \text{HD}_{\text{hyp}}(J(f)) \), is greater than 1.

Recall that hyperbolic dimension of the Julia set \( J(f) \), analogously of any other \( f \)-invariant subset of \( J(f) \), \( \text{HD}_{\text{hyp}}(J(f)) \) is defined as supremum of Hausdorff dimensions of isolated hyperbolic forward invariant subsets of \( J(f) \). A compact set \( L \subset \mathbb{C} \) is hyperbolic (or expanding) forward invariant for \( f \) if \( f(L) \subset L \) and there exists \( n \in \mathbb{N} \) such that \( |(f^n)'(z)| > 1 \) for all \( z \in L \). The set \( L \) is said to be isolated (or repelling) if there exists its neighbourhood \( U \) in \( J(f) \) such that if a trajectory \( (f^j(z))_{j \geq 0} \) is in \( U \), then it is in \( L \).

Thus, the inequality

\[
\text{HD}(J(f)) \geq \text{HD}_{\text{hyp}}(J(f))
\]

holds trivially. See [25, Chapter 12] for the discussion on equivalent definitions of the hyperbolic dimension, in particular the one we shall use here, introduced in Proposition 4.

So, in view of Theorem 2, if \( J(f) \) is disconnected but it has a non-trivial connected component, then \( \text{HD}_{\text{hyp}}(J(f)) > 1 \), even if the dimension of every non-trivial component is equal to 1. In the latter case, the dimension larger than 1 is achieved due to a Cantor set of other components. To prove it, we follow a strategy by Irene Inoquio-Renteria and Juan Rivera-Letelier in [13], where they proved that geometric pressure \( P(t) \) is larger than \(-t\) times Lyapunov exponent of any probability invariant measure on the Julia set \( J(f) \) of a rational function on the Riemann sphere \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) (for the definition, see Section 2), provided the...
exponent is positive. See Proposition 4 for the definition of the geometric pressure and [25, Remark 12.5].

Theorem 2 strengthens the result of [28], where the analogous theorem was proved for polynomials with connected Julia set.

Finally, we show in Section 4 that the situation we deal with in the proof of Theorem 2 really may happen: in Proposition 10, we provide an example of a family of polynomials of degree 3 with non-trivial components of the Julia set $J(f)$, each of which is an analytically embedded interval. This resolves Christopher Bishop’s question quoted above in the negative.

**Remark.** One can also consider radial (or conical) Julia set $J_r(f) \subset J(f)$ which consists of all those points $z \in J(f)$ for which there exists $\delta > 0$ such that for infinitely many $n \in \mathbb{N}$ the map $f^n$ admits a holomorphic inverse branch $f^{-n} : B = B(f^n(z), \delta) \to \mathbb{C}$ sending $f^n(z)$ to $z$, for the Euclidean balls $B$ in the case $f$ is a polynomial and balls in the spherical metric for $f$ a rational function. Then $\text{HD}_{\text{hyp}}(J(f)) = \text{HD}(J_r(f))$, see, for example, [8, Section 3] for this, in the setting including also rational and meromorphic functions, and see references therein; other versions of conical Julia sets for rational functions were introduced in [18]. Therefore in Theorem 2 we can write

$$\text{HD}(J_r(f)) > 1. \quad (1)$$

2. **Basic notions, repelling boundary domains and proof of Theorem 1**

For a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$, one considers the set of points escaping to infinity

$$A_{\infty}(f) := \{ z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty \}.$$  

Here $f^n = f \circ f \ldots \circ f$ $n$-times the composition of $f$. It is clear that this is exactly the set of points with the forward orbits $(f^n(z))_{n=0,1,\ldots}$ unbounded. The set $A_{\infty}(f)$ is called the **basin of attraction to infinity**. It is open and connected (since otherwise its bounded component would contain a pole for an iterate of $f$ by Maximum Principle). Define the filled-in Julia set $K(f) := \mathbb{C} \setminus A_{\infty}(f)$ and Julia set $J(f) := \partial K(f) = \partial A_{\infty}(f)$ where $\partial$ means the boundary, as in the Introduction. Clearly both $K(f)$ and $J(f)$ are completely invariant, namely $f^{-1}(K(f)) = K(f)$ and $f^{-1}(J(f)) = J(f)$.

A point $z \in \mathbb{C}$ is said to be **critical** (or $f$-critical) if the derivative $f'(z)$ is equal to 0. The basin $A_{\infty}(f) \cup \infty$ is simply-connected, that is, $J(f)$ is connected, if and only if the forward trajectories of all critical points are bounded. At the other extreme, if all critical points escape to the infinity, then $J(f)$ is totally disconnected. However, the opposite implication is false! For more details see, for example, [6, Section III.4.], and [5] in particular Theorem 3.8 and an example following it.

Another definition of Julia set is that it is the complement of Fatou set $F(f)$ on which the sequence $f^n$ is normal, that is for every its subsequence and a compact set $K \subset F(f)$ a subsequence of it is uniformly convergent on $K$. This definition is valid also for a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ acting on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. The Julia set $J(f)$ is compact, completely $f$-invariant, namely $f^{-1}(J(f)) = J(f)$, and a closure of repelling periodic orbits contained in it. For entire transcendental functions $f$, the definition is the same.

Intuitively, the action of $f$ on $J(f)$ is ‘chaotic’ and at points in $J_r(f)$ yields a similarity of infinitesimal parts of $J(f)$ with its big pieces, thus exhibiting a fractal local geometry/nature of $J(f)$.

We go back to a polynomial $f$. Choose $R$ so large that $K(f)$ is a subset of $\mathbb{D}(0,R)$, the Euclidean disc with radius $R$ and origin at $0$, and moreover

$$f^{-1}(\mathbb{D}(0,R)) \subset \mathbb{D}(0,R). \quad (2)$$
Then, by its complete invariance, \( K(f) \subset \bigcap_{n=0}^{\infty} f^{-n}(\mathbb{D}(0, R)) \). Directly by the definition of \( K(f) \), the opposite inclusion holds, hence the equality holds.

Let \( C \) be a connected component of the filled-in Julia set \( K(f) \). Denote by \( U_n(C) \) the unique connected component of \( f^{-n}(\mathbb{D}(0, R)) \) containing \( C \). The boundary \( \partial U_n(C) \) is disjoint from \( K(f) \) since \( \partial \mathbb{D}(0, R) \) is, and \( f(K(f)) = K(f) \). Note also that by (2)

\[
U_{n+1}(C) \subset U_n(C). \tag{3}
\]

Finally note that all \( U_n \) are simply-connected (topological discs) by Maximum Principle.

Define \( C' := \bigcap_{n=0}^{\infty} U_n(C) \). By \( U_n(C) \supset C \), we get \( C' \supset C \). On the other hand, note that \( C' \) is connected as the intersection of the decreasing sequence of compact connected sets \( U_n(C) \). It is also clear from the definition of \( C' \) that it consists of non-escaping points, hence it is contained in \( K(f) \). Therefore \( C' \subset C \). We conclude with

\[
C = \bigcap_{n=0}^{\infty} U_n(C).
\]

One fact pivotal for our paper is

**Theorem A** [14, 26]. For a polynomial \( f \) of degree at least 2 if a component of filled-in Julia set \( K(f) \) is not a point, then its forward orbit contains a periodic component containing a critical point.

We refer to periodic components that contain a critical point as critical periodic components.

In the proof of Theorem 2, we shall use in fact a weaker version of Theorem A, that if the Julia set of a polynomial is not totally disconnected, then there exists a non-trivial critical periodic component of \( K(f) \).

Theorem A was proved for degree 3 polynomials by Branner and Hubbard [4] and independently by Yoccoz, but for higher degree polynomials stayed not proven for a long time.

Consider any non-trivial component of \( K(f) \). We can replace it by a periodic component \( C \) which is its image under an iterate of \( f \), that is, \( f^k(C) = C \) for some \( k \geq 1 \). Since replacing \( f \) by \( f^k \) does not change the Julia set, we may assume and we do assume from now on that \( f(C) = C \).

**Definition** (polynomial-like maps). A polynomial-like map is a triple \((U, U', \Phi)\), where \( U \) and \( U' \) are open subsets of \( \mathbb{C} \) homeomorphic to a disc, with \( U' \) relatively compact in \( U \) and \( \Phi : U' \to U \) holomorphic, proper, of degree \( \deg \Phi \geq 2 \).

Proper means that pre-image of every compact set is compact.

Degree is the number of preimages \( \#(\Phi^{-1}(z)) \) for an arbitrary \( z \in U \) counted with multiplicities. It does not depend on \( z \).

The notion polynomial-like was introduced by Adrien Douady and John Hubbard in [9], Section I.1. See also [6, Section VI.1] or, for example, [3, Section 7.1], where a smoothness or analyticity of \( \partial U \) were additionally assumed. This allowed to extend \( \Phi \) continuously to \( \partial U' \) and define proper by \( \Phi(\partial U') = \partial U \). One can always achieve this analyticity by a slight decreasing of the Douady and Hubbard \( U \) and \( U' \), see [3, Remark 7.2]. In our case, we adjust \( R \), see below.

Note that degree above is well defined for \( U' \) and \( U \) being open connected domains not necessarily simply connected (topological discs). This will be used in the proof of Lemma 3.

For a polynomial-like \( \Phi \), the filled-in Julia set and the Julia set can be defined in the same way as for a polynomial, namely

\[
K(\Phi) = \{ z \in U' : \Phi^n(z) \in U' \text{ for all } n \in \mathbb{N} \} \quad \text{and} \quad J(\Phi) = \partial K(\Phi). \tag{4}
\]
Figure 2. A capture of a critical point.

Back to our component $C$, for an arbitrary $n \in \mathbb{N}$, $F := f|_{U_{n+1}(C)}$ maps $U_{n+1}(C)$ onto $U_n(C)$ (the same $C$). So since $F$ is proper and due to (3), the triple $(U_n(C), U_{n+1}(C), F)$ is a polynomial-like map. The boundaries of the domain and range are analytic if $\partial \mathbb{D}(0, R)$ is taken disjoint from the forward orbits of critical points. We shall use the latter property to guarantee that for every $n \in \mathbb{N}$, each two distinct components of $f^{-n}(\mathbb{D}(0, R))$ have disjoint closures.

**Lemma 3.** If $C$ is a fixed by $f$ non-trivial critical component of $K(f)$, then for each $n$ large enough the map $F$ defined above is polynomial-like, with $C$ being its filled-in Julia set.

Although $F$ is polynomial-like for each $n$, there is no reason to expect that $K(F) = C$; only $K(F) \supset C$ must hold, since forward trajectories of points in $C$ do not leave $U_{n+1}(C)$ by the forward invariance of $C$. So Lemma 3 is not void.

**Proof.** Let us write here $U_n$ for $U_n(C)$ for all $n$. We claim that for $n$ large enough, $U_{n+1}$ is the only connected component of $f^{-1}(U_n)$ contained in $U_n$. No other component intersects $U_n$.

Indeed, if a component $V$ of $f^{-1}(U_n)$ intersects $U_n$, it must have its closure entirely in $U_n$. Otherwise $\overline{V}$ would intersect $\partial U_n$. Taking the images under $f^{n+1}$, we would conclude that $\mathbb{D}(0, R) \cap f(\partial \mathbb{D}(0, R)) \neq \emptyset$ contradicting (2).

Suppose now there is a component $V$ of $f^{-1}(U_n)$ with closure entirely in $U_n$, different from $U_{n+1}$. Then there is a critical point of $f$ in $W' := U_n \setminus (\overline{V} \cup \overline{U_{n+1}})$. Otherwise $f : W' \to U_{n-1} \setminus U_n = W$ would be a covering map from $W'$ to the annulus $W$. This contradicts the equality $\chi(W') = \deg(f|_{W'}) \chi(W)$ for Euler’s characteristics since $\chi(W')$ is negative and $\chi(W) = 0$. See Figure 2.

Since the number of critical points of $f$ is finite, they cannot happen in $W'$ for $n$ large enough. The claim follows.

We conclude that $F^{-k}(U_n) = U_{n+k}$ for all $k > 1$, so $\bigcap_{k=0}^{\infty} F^{-k}(U_n(C)) = \bigcap_{k=0}^{\infty} U_{n+k}(C) = C$.

Note finally that degree of $F$ is larger than 1, since $C$ hence $U_{n+1}$ is assumed to contain a critical point.

**Remarks.** (1) Note that the assumption that $C$ is critical need not be specified. It follows automatically from the assumption that $C$ is non-trivial. Indeed, if there were no critical point in $U_{n+1}$, then the moduli of all annuli $U_n \setminus U_{n+1}$ for $n$ large enough would be positive equal to each other so $C$ would be a point, contradicting the non-triviality of $C$.\[\square\]
(2) Note that for $F$ as in Lemma 3, $\deg(F) < \deg(f)$. Indeed, $C = K(F)$ is by definition completely invariant for $F$, namely $F^{-1}(C) = C$. At the same time, there exists a component $C'$ of $f^{-1}(C)$, in $K(f)$ by its complete invariance for $f$, different from $C$, since otherwise $C$ would be completely invariant also for $f$ so $K(f) = K(F)$. Then $f(U_{n+1}(C')) = U_n(C)$ for all $n \in \mathbb{N}$, hence $f$ maps $C'$ onto $C$ thus adding to $\deg(F)$ an additional contribution.\footnote{A more general approach to the proof, more standard, is via the topological exactness of $f$ on $J(f)$, compare a footnote to Comment 1 in Section 5, yielding the density of $\bigcup_{n=0}^{\infty} f^{-n}(\{z\})$ in $J(f)$ for each $z \in J(f)$. This density will be used also in Proof of Proposition 10.}

Below we recall the definition of RB-domain, which was introduced in [17], with the name introduced in [24] together with crucial applications of this notion.

**Definition (RB-domain).** Let $\Omega$ be a simply connected open domain in the Riemann sphere $\hat{\mathbb{C}}$ with $\#(\hat{\mathbb{C}} \setminus \Omega) > 2$. Assume there exists a holomorphic map defined on a neighbourhood $U$ of $\partial \Omega$ such that

$$f(U \cap \Omega) \subset \Omega, \quad f(\partial \Omega) = \partial \Omega \quad \text{and} \quad \bigcap_{k=0}^{\infty} f^{-k}(U \cap \Omega) = \partial \Omega. \quad (5)$$

Then $\Omega$ is called a repelling boundary domain (RB-domain).

**Observation 1.** The domain $\Omega := \hat{\mathbb{C}} \setminus C$ together with $f$ restricted to $U := U_{n+1}(C)$ with $n$ sufficiently large, as in Lemma 3, is a repelling boundary domain (RB-domain). Indeed, by this Lemma, $f|_{U_{n+1}^{-1}} (C) \subset C$ which yields $f(U \cap \Omega) \subset \Omega$.

Now we rely on:

**Theorem B** (see [19, Theorem A']). If $\Omega$ is an RB-domain, then either $\text{HD}_{\text{hyp}}(\partial \Omega) > 1$ or $\partial \Omega$ is an analytic Jordan curve or an analytically embedded interval, where $f$ is topologically conjugate to $z^d$ or to a $\pm$ Chebyshev polynomial, respectively.

Theorem B was stated as a conjecture in [24]. A comparison of harmonic measure $\omega$ on $\partial \Omega$ to $H^1$ which is the Hausdorff measure in dimension 1, is the key. For $\omega$ absolutely continuous it was proved in [29] that $\partial \Omega$ was analytic. It was also claimed in [29] that for $\omega$ singular $\text{HD}(\partial \Omega) > 1$ could be proved by adapting the methods of [28]. A detailed proof of this appeared in [19]. Analogously to [28], the proof not only showed that $\text{HD}(\partial \Omega) > 1$, but, actually, the hyperbolic dimension of $\partial \Omega$ for $f$ restricted to it, was larger than 1. For a survey, see [20].

Combining the above Observation and Theorem B concludes the proof of Theorem 1, except it does not exclude $C$ such that $\partial C = \partial \Omega$ is an analytic Jordan curve.

So, now we consider the latter case. Denote $\partial C$ by $\gamma$. Denote by $\Omega'$ the bounded (internal) component of $\mathbb{C} \setminus \gamma$, and consider now $\Omega'$ instead of the external one $\hat{\mathbb{C}} \setminus C$, which was considered above. The domain $\Omega'$ is forward invariant for $f$ since $f(C) = C$ (as we replaced $f$ by its adequate iterate) and due to Maximum Principle.

We already know that $\Omega$ is an RB-domain. Let $U$ be a neighbourhood coming from the definition of RB-domain (see (5)). We know (see Observation 1) that any set $U_n$, with $n$ sufficiently large can be taken as $U$ in this definition.

Since $\gamma = \partial C$ is an analytic curve, there exists a neighbourhood $W$ of $\gamma$, for which the Schwarz Reflection Principle applies.
Define \( \hat{U}_n := U_n \cap \Omega \) and choose \( n \), additionally, so large that \( \hat{U}_n \subset W \). Take \( \hat{U} := \hat{U}_n \) for this \( n \). We have by (5):

\[
f(U \cap \Omega) \subset \Omega, \quad f(\partial \Omega) = \partial \Omega \quad \text{and} \quad \bigcap_{k=0}^{\infty} f^{-k}(U \cap \overline{\Omega}) = \partial \Omega = \gamma.
\]

Denoting by \( U' \) the reflection of \( \hat{U} \) with respect to \( \gamma \), we see that the same holds true with \( U \) replaced by the set \( U'' \), being the union of \( U' \), \( \hat{U} \) and the arc \( \gamma \) along which their closures intersect, and \( \Omega \) replaced by \( \hat{\Omega} \), that is,

\[
f(U'' \cap \Omega') \subset \Omega', \quad f(\partial \Omega') = \partial \Omega' \quad \text{and} \quad \bigcap_{k=0}^{\infty} f^{-k}(U'' \cap \overline{\Omega'}) = \partial \Omega' = \gamma.
\]

In other words, \( \Omega' \) is also a repelling boundary domain. Since \( f(\hat{U}) \) contains the closure of \( \hat{U} \) in \( \Omega \), by symmetry \( f(U') \) contains the closure of \( U' \) in \( \Omega' \). In other words, the compact set \( \Omega' \setminus U'' \) is mapped by holomorphic \( f \) into its interior. It follows that there is an attracting fixed point (a sink) in \( \Omega' \), see, for example, [6, Wolff-Denjoy Theorem], and \( \Omega' \) is the immediate basin of attraction to this sink. One can also refer to the observation that (due to RB) \( f \) is a contraction on \( \hat{\Omega} \setminus U'' \) in the hyperbolic metric on \( \hat{\Omega} \).

Using [5, Lemma 9.1], we now conclude that \( \gamma = \partial C \) is a circle and \( f(z) = z^d \) in some complex affine coordinates. (See also [19, Theorem A] or [20, Theorem 8.1]). But this contradicts the assumption that \( J(f) \) is disconnected.\(^5\)

**Remark.** A key feature on the domain \( \Omega' \) is that it is \( f \)-invariant, unlike \( \Omega \) containing components of \( f^{-1}(C) \). So having the non-empty interior of \( C \) we can apply [5, Lemma 9.1], which does not work for \( \Omega \). The RB-property is weaker than being an immediate basin of attraction to a sink.

The common feature is the repulsion of the boundary to the side of a domain \( \Omega \), which for a pullback \( g = R^{-1} \circ f \circ R \) of \( f \) for a Riemann mapping \( R : \mathbb{D}(0,1) \to \Omega \) allows to extend \( g \) holomorphically beyond \( \partial \mathbb{D}(0,1) \), expanding, see [17, Section 'Resolving Singularities'], thus allowing easily to invoke Gibbs measures, see, for example, [20].

### 3. Proof of Theorem 2

We shall use a version of Bowen’s formula, which can be found in [18], see also [25, Section 12.5], for a strengthened version.

**Proposition 4.** Let \( f \) be a rational map of degree \( d \geq 2 \). There exists an exceptional set \( E \subset \hat{\mathbb{C}} \) of Hausdorff dimension 0, such that for every \( t \geq 0 \) (even every real \( t \)) and for every \( z \notin E \), the limit

\[
P(t, f; z) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{v \in f^{-n}(z)} \frac{1}{(|f^n)'(v)|^t}
\]

exists, and is independent of \( z \in \hat{\mathbb{C}} \setminus E \). Denote the common value as \( P(t, f) \). It is called the geometric pressure. The function \( t \mapsto P(t, f) \) is continuous and non-increasing. Moreover,

1 "Immediate" stands for a connected component of the basin, which contains the sink.

2 One can also argue that, as \( \Omega' \) is connected bounded and invariant, it is a component of the Fatou set for \( f \). As being an RB-domain it cannot be neither parabolic, nor Siegel disc, nor Herman ring, so it must be an immediate basin of attraction to a sink. See, for example, [6, Theorem IV.2.1].

3 We are grateful to Fei Yang for bringing our attention to this absence of analytic Jordan curves.
$P(0, f)$ is positive (equal to $\log d > 0$, the topological entropy). The following formula holds:

$$\text{HD}_{\text{hyp}}(J(f)) = \inf\{t > 0 : P(t, f) \leq 0\}.$$ 

In words, $\text{HD}_{\text{hyp}}(J(f))$ is the first zero of $P(t, f)$. There is an abundance of points not belonging to $E$ (that is, non-exceptional). Those are, for example, all points which are not post-critical (not in the forward $f$-trajectory of an $f$-critical point) and which belong to basins of attraction to periodic orbits.

In view of Theorem 1, to prove Theorem 2 it remains only to prove the following.

**Proposition 5.** Let $f : C \to C$ be a polynomial of degree $d \geq 2$ with disconnected Julia set. Suppose there exists a periodic connected component $C$ of $K(f)$ being an analytic arc. Then $\text{HD}_{\text{hyp}}(J(f)) > 1$.

**Proof of Proposition 5.** We shall prove that $P(1, f) > 0$, which implies, by Proposition 4, that $\text{HD}_{\text{hyp}}(J(f)) > 1$. Consider $F = f : U_{m+1}(C) \to U_m(C)$, with some fixed $m$ large enough, so that $F$ is a polynomial-like map and $C$ its Julia set, here equal to the arc $K(F)$, see Lemma 3 (there $m$ was denoted $n$).

For $x \in U_m \setminus C$, denote

$$L_n(F, x) := \log \sum_{F^n(y)=x} \exp(-\log |(F^n)'(y)|) = \log \sum_{F^n(y)=x} |(F^n)'(y)|^{-1}. \quad (6)$$

To continue the proof of Proposition 5, we need two lemmas (Lemmas 6 and 7), which we formulate and prove below.

**Lemma 6.** There exists $C_0 > 0$ such that for all $x \in U_m \setminus C$ and $n > 0$

$$L_n(F, x) \geq -C_0. \quad (7)$$

**Proof.** Let $\Omega = \hat{C} \setminus C$. We know from the proof of Theorem 1 that $\Omega$ is an RB-domain. Recall that $C = \partial\Omega$ is an analytically embedded interval, as assumed in Proposition 5. We recall the final step of the proof of the analyticity of $C$ provided in \cite[Proposition 6]{29}, which will allow us to deduce also the inequality (7).

So assume that $C$ is an analytically embedded interval, with endpoints, say $-1, 1$. Now we use Zhukovsky’s function, namely the ramified (branched) covering map $\Pi : \hat{C} \to \hat{C}$, ramified over $-1$ and $1$: $\Pi(z) = \frac{1}{2}(z + \frac{1}{2})$ and the pre-image $\gamma = \Pi^{-1}(C)$, see Figure 3. Then $\gamma$ is an analytic Jordan curve, as proved in \cite[Proposition 6]{29}.

The curve $\gamma$ divides the sphere into two disc $D_i$, $i = 1, 2$ and $\Pi(D_1) = \Pi(D_2) = \hat{C} \setminus C$. The map $F : U_{m+1} \setminus C \to U_m \setminus C$ can be lifted to holomorphic maps $G_i$ defined on the respected components $W_i := D_i \cap \Pi^{-1}(U_{m+1} \setminus C)$, so that $G_i(W_i) \subset D_i$. Next consider just one of these components, say $W_1$.

By Carathéodory’s Theorem (local version), see, for example, \cite[Chapter II.3, Theorem 4]{11}, and Schwarz reflection principle, we can extend $G_1$ holomorphically from $D_1$ to a map $\hat{G}$ acting on a neighbourhood of $\gamma$. Then the map $\hat{G}$ is expanding on $\gamma$ because of uniform convergence of $\hat{G}^{-n}$ on a neighbourhood of $\gamma$ to $\gamma$ which is a nowhere dense set. This implies that the limit functions are constant, hence uniform convergence of derivatives $(\hat{G}^{-n})'$ to 0,
by normality, see, for example, [17, Section 7] or proof of [5, Theorem 6.5]. Due to the just mentioned expanding property, the domain $D_1$ bounded by $\gamma$ is an RB-domain for the action of $\hat{G}$. (It is seen also directly, by taking as a neighbourhood of $\gamma = \partial D$ in the definition of RB, the set $\hat{U} = \Pi^{-1}(U_m)$).

**Remark.** Here is a different argument for the existence of the extension of $G$ from $W_1$ beyond $\gamma$ not using Schwarz reflection principle: Consider the lift of $F$ on say $U_{m+1}$ to $G$ on the set $W := \Pi^{-1}(U_{m+1})$, which is a neighbourhood of $\gamma$ (the both sides and $\gamma$ itself), for the branched covering $\Pi$, directly by the formula

$$G := \Pi^{-1} \circ F \circ \Pi. \tag{8}$$

Because of presence of $\Pi^{-1}$, it is not a priori clear that $G$ in this formula is well defined. Here is an explanation. First define $G : W_1 \to D_1$ as $G_1$ above. Fix an arbitrary $z \in W_1$ and denote $w := G(z)$. Next extend $G$ using (8) along curves in $W$ starting from $z$ mapped to $w$, using (8), the curves omitting

$$A := \Pi^{-1}(\text{Crit}(F) \cup \{-1, 1\}),$$

where Crit$(F)$ is the set of $F$-critical points. Note that $A$ is precisely the set mapped by $F \circ \Pi$ to 1 or $-1$, the critical values of $\Pi$, namely where $\Pi^{-1}$ has singularities. This is so because $F$ is topologically conjugate to $\pm$ Chebyshev polynomial. So all the singularities of $G$ defined by (8) are holomorphically removable.

Note finally that $W$ is a topological annulus, with the fundamental group generated by the homotopy class of any curve $\gamma'$ in $W_1$ starting and ending at $z$ running once along $W_1$. The growth of the prolongation $G$ along $\gamma'$ is 0, because it starts and ends at $w$. It is so because $\gamma' \subset W_1$, where $G$ has been well defined.

So $G$ being an analytic continuation along curves is a well-defined (single-valued) holomorphic function on $W$. In fact, this $G$ coincides with the extension $\hat{G}$ found before, since these maps are holomorphic and coincide in $D_1$.

Let us return to the proof of Lemma 6. Let $x \in U_m \setminus C$. Our aim is to find a lower bound for $L_n(F, x)$. Denote by $w$ the unique pre-image of $x$ under $\Pi$ in $D_1$. The map $\Pi$ gives a bijection between the sets $\{v \in G^{-n}(w)\}$ and $\{y \in F^{-n}(x)\}$. Putting $y = \Pi(v)$, we obtain:

$$(F^n)'(y) = (G^n)'(v) \cdot \frac{\Pi'(w)}{\Pi'(v)}$$
and, consequently,
\[ \sum_{y \in F^{-n}(x)} \frac{1}{|f^n(y)|} = \frac{1}{|\Pi'(w)|} \cdot \sum_{v \in G^{-n}(w)} \frac{1}{|(G^n)_v'(v)|} \cdot |\Pi'(v)|. \] (9)

Let us estimate the expression above, but without \( \Pi' \), that is, let us estimate \( L_n(G, w) \)
(defined for \( G \) as we did for \( F \) in (6)). First assume \( w \in \gamma \). Denote by \( \text{Le} \) the Euclidean length of curves contained in our analytic curve \( \gamma \). There exists a constant \( c > 0 \) such that for each \( n \) and every \( w \in \gamma \), choosing an arbitrary \( w' \in \gamma \setminus \{w\} \), denoting by \( \gamma(w) \) the open arc \( \gamma \setminus \{w'\} \), we have
\[ \sum_{G^n(v) = w} |(G^n)'(v)|^{-1} \geq c \cdot \sum_{G^n(v) = w} \text{Le}(G^{-n}(\gamma(w))) / \text{Le}(\gamma(w)) \geq (10) \]
\[ c \cdot \text{Le}(\gamma)/\text{Le}(\gamma) = c > 0. \]

This is due to bounded distortion for iterates (or by Koebe distortion lemma) by a constant \( c \), see [25, Section 6.2]. The subscript \( v \) at \( G^{-n} \) means the component containing \( v \). Due to bounded distortion \( w \in \gamma \) can be replaced by an arbitrary point \( w \) in a neighbourhood of \( \gamma \) and the same estimate as in (10) is achieved for such points.

We could define so-called transfer operator (Perron-Frobenius-Ruelle) for potential \( \psi := -\log |G'| \) acting on a continuous function \( \varphi \)
\[ \mathcal{L}_\psi(\varphi)(w) := \sum_{G^n(v) = w} (\exp \psi(v)))\varphi(v). \]

Then the expression estimated in (10) could be written as \( \mathcal{L}_\psi^0(\varphi) \). Compare (6).

To continue (9) we need to estimate from below the value \( \mathcal{L}_\psi^0(\varphi)(w) \) for potential \( -\log |G'| \), where \( \varphi = |\Pi'| \). The function \( \varphi \) has value zero at two points, but \( \mathcal{L}_\psi^0(\varphi) \) is positive, thus bounded from below by some constant \( c_1 > 0 \). So,
\[ \mathcal{L}_\psi^n(\varphi)(w) = \mathcal{L}_\psi^{n-2}(\mathcal{L}_\psi^2(\varphi))(w) \geq \mathcal{L}_\psi^{n-2}(c_1 \cdot \varphi)(w) = c_1 \cdot \mathcal{L}_\psi^{n-2}(\varphi)(w), \]
and the last term is bounded below by \( c > 0 \) as in (10).

Since \( \frac{1}{\Pi'(w)} \) is bounded away from 0 in \( \Pi^{-1}(U_m) \), since \( \Pi' \) is upper bounded, we are done. \( \square \)

For all \( n = 0, 1, \ldots \) write \( F_n := f|_{U_{n+1}(C)} \) the polynomial-like mappings considered in Section 2. Consider now an arbitrary \( m \in \mathbb{N} \) such that \( C \) is the filled-in Julia set for \( F_m \), existing by Lemma 3.

**Lemma 7.** There exists an integer \( N_1 > 0 \) and a connected component \( V \) of \( f^{-N_1}(U_m) \), such that \( V \subset U_m \setminus U_{m+1} \).

**Proof.** Notice that there exists \( k : 0 \leq k \leq m \) for which there exists a component \( U' \) of \( f^{-1}(U_{m-k}) \) in \( U_{m-k} \), different from \( U_{m+1-k} \). Otherwise \( C = K(f) \) which is not the case as \( C \) is a connected component and \( K(f) \) is disconnected. Compare Proof of Lemma 3.

Consider \( V' \), an arbitrary component of \( (f^k|_{U_m})^{-1}(U') \). It is a subset of \( U_m \) and a component of \( f^{-(m+1)}(\mathbb{D}(0, R)) \). It is disjoint from \( U_{m+1} \), hence bounded away from \( C \).

Note that \( f^{m+1} \) maps \( V' \) onto the disc \( \mathbb{D}(0, R) \) and \( U_m \subset \mathbb{D}(0, R) \). Select \( V \), an arbitrary component of \( f^{-(m+1)}(U_m) \) in \( V' \). It satisfies the assertions of our Lemma with \( N_1 := m + 1 \). \( \square \)

Note that \( V \) intersects \( J(f) \) because \( U_0 \) does and \( J(f) \) is completely invariant for \( f \). More precisely \( V \) contains branched holomorphic images of \( C \), namely components of \( f^{-N_1}(J(f)) \). These are small ‘basilicas’ surrounding the big critical fixed one in Figure 1.
Having Lemmas 6 and 7 at our disposal, we continue the proof of Proposition 5. In order to simplify the notation, we pass again to the iterate of $f$, replacing now $f$ by $f^{N_1}$ and $F$ by $F^{N_1}$. Recall again that this modification does not change the Julia set. To simplify further the notation, denote $U_0 := U_m(C)$, $U_1 := U_{m+N_1}(C)$. So, from now on, after this modification, the component $V$ becomes just a component of $f^{-1}(U_0)$ in $U_0$ different from $U_1 = U_1(C)$. We denote $f|_V$ by $F_V$. This pairs up with $F$ on $U_1$ denoted also as $F_{U_1}$.

Now we build an infinite collection of multivalued maps $\phi_\ell$, $\ell \in \mathbb{N}$, as follows: Let us note that $F_V : V \to U_0$ is a holomorphic proper map onto $U_0$. Since the degree of this map may be larger than 1, the inverse may be not well defined. However, we shall use the notation inverses of $F$, where we can restrict to $x \in U$.

Consider a family of multivalued maps $\phi_\ell : U_0 \to U_0$, $\ell = 0, 1, \ldots$, defined as multivalued (branched) inverses

$$\phi_\ell := F^{-\ell} \circ h,$$

where $F^{-\ell} : U_0 \to U_0$ are multivalued holomorphic maps given by multivalued (branched) inverses of $F^\ell$. Since $F^{-\ell}$ is pre-composed by $h$, it is meaningful even when restricted to $V$.

So, each multivalued map $\phi_\ell$ assigns to a point $x \in U_0$ some collection of its pre-images under $f^n$, with $n = n_\ell = \ell + 1$, all of them being in $U_0$, even in $U_\ell$.

For $x \in U_0$ and $\ell \in \mathbb{N} \cup \{0\}$, we write $\sum |\phi_j'(x)|$ to denote the summation of derivatives which runs over all branches of the multivalued map $\phi_\ell$. If a critical point $c$ and its $f$-image are met, then we can put in this sum $\infty$, as inverse of forward derivative 0. In fact, it does not matter since we can restrict to $x$ not post-critical.

For every $\ell \in \mathbb{N} \cup \{0\}$ and $x \in U_0$, denote

$$L_\ell^*(f, x) = \log \sum |\phi_{\ell}(x)|,$$

where the summation runs over all branches of the multivalued map $\phi_{\ell}$. Denote also by $\underline{\ell}$ a sequence $\underline{\ell} = (\ell_1, \ldots, \ell_k)$, and put

$$n_{\underline{\ell}} = n_{\ell_1} + \cdots + n_{\ell_k}.$$

Finally, for a sequence $\underline{\ell} = (\ell_1, \ldots, \ell_k)$, denote

$$L_{\underline{\ell}}(f, x) := L_{\ell_k}(f, y_{k-1}) + \cdots + L_{\ell_2}(f, y_1) + L_{\ell_1}(f, x),$$

where $y_1 = \phi_{\ell_1}(x), \ldots, y_{k-1} = \phi_{\ell_k}(y_{k-2})$ (remember that the maps $\phi_j$ are multivalued, so we consider sums over their values). The star $*$ means we consider only pre-images along first $h$ and next $F^{-\ell_1}$. In other words, more formally,

$$L_{\underline{\ell}}(f, x) = \log \sum |\phi_{\underline{\ell}}(x)|,$$

where the summation runs over all branches of the multivalued function

$$\phi_{\underline{\ell}} = \phi_{\ell_k} \circ \phi_{\ell_{k-1}} \circ \cdots \circ \phi_{\ell_1}.$$

It is important to note that all the multivalued functions $\phi_{\ell}$ are mutually distinct, compare free Iterated Function System in [13]. Indeed, let $\underline{\ell} \neq \underline{\ell}'$, but $n_{\ell} = n_{\ell'}$. Let $i$ be the first integer such that $\ell_i \neq \ell'_i$. Then, supposing that $n_{\ell_i} > n_{\ell'_i}$, we compose in $\ell_i'$ after $\ell'_i$ with $h$ with range in $V$, whereas in $\ell_i$ still within $\ell_i$ with $F^{-1}$ having range $U_1$. Further compositions by branches of $f^{-1}$ preserve distinction.

Denote by $\Sigma^*$ the set of all finite sequences $\underline{\ell} = (\ell_1, \ldots, \ell_k)$, $k \geq 1$. For an arbitrary $x \in U_0$, denote

$$\Lambda_N(x) = \sum_{\underline{\ell} \in \Sigma^* : n_{\ell} = N} \exp L_{\underline{\ell}}(f, x).$$
The star * again means we consider only pre-images along blocks of first $h$ next $F^{-1}$’s.

**Proposition 8.** For every $N \geq 1$ and non-exceptional, in particular not post-critical, $x \in U_0$,

$$P(1, f; x) \geq \liminf_{N \to \infty} \frac{1}{N} \log \Lambda_N(x).$$

**Proof.** In view of Proposition 4, it is sufficient to prove

$$\sum_{y \in f^{-N}(x)} \frac{1}{|(f^N)'(y)|} \geq \Lambda_N(x).$$

This is, however, obvious, because on the left-hand side all $y \in f^{-N}(x)$ appear, whereas on the right-hand side only selected ones. \hfill  □

**Proposition 9.** Let $\Lambda_N := \inf_{x \in U_0} \Lambda_N(x)$. Then

$$\liminf_{N \to \infty} \frac{1}{N} \log \Lambda_N > 0.$$

**Proof.** Put $a := \inf_{x \in U_0}(\sum |h'(x)|)$, and $b := \exp(-C_0)$, where $C_0$ comes from Lemma 6.

Consider all sequences $\ell = (\ell_1, \ldots, \ell_k)$ such that $n_\ell = N$, for each $k \leq N$. The number of such sequences can be calculated in the following way:

For each $k \leq N$, there are $(N-1)_{k-1}$ ways of choosing $k - 1$ positions $m_1, \ldots, m_{k-1}$ from the sequence $\{1, \ldots, N-1\}$.\footnote{The $\ell$ has length $N$, but its first place is occupied by the beginning of the first block. So the only choices for the beginnings of remaining $k - 1$ blocks are the remaining $N - 1$ places.} Having these positions chosen, we assign to them the values $\ell_1 = m_1 - 1$, $\ell_2 = (m_2 - m_1) - 1$, $\ell_k = (N - m_{k-1}) - 1$ and the (multivalued) map $\phi_\ell$ with $\ell = (\ell_1, \ldots, \ell_k)$ and $n_\ell = (\ell_1 + 1) + (\ell_2 + 1) + \cdots + (\ell_k + 1) = N$.

Then we have the estimate

$$\exp L_\ell(f, x) \geq a^k b^k$$

and, consequently,

$$\sum_{\ell \in \Sigma^*: n_\ell = N} \exp L_\ell(f, x) \geq \sum_{k=1}^{N} \binom{N-1}{k-1} (ab)^k = ab(1 + ab)^{N-1}. \hfill \Box$$

**Remark.** Let us note that a calculation in a similar spirit appeared in [13] and in [15]. The authors used there a method of generating functions (some power series with coefficients related to a value similar to $\Lambda_N$). Our case is simpler, in a sense that the set of admissible values $n_\ell$ forms an arithmetical progression. So, the straightforward calculation provided above is sufficient.\footnote{However, in Section 5, Comment 1, we use the general, power series, version.}

Obviously, combining Propositions 8 and 9, together with Proposition 4, we conclude the proof of Proposition 5.

Indeed, Propositions 8 and 9 imply immediately that for every non-exceptional point $x \in U_0$, we have that

$$P(1, f; x) > 0,$$
which implies (see Proposition 4) that \( P(1, f) > 0. \) Moreover, Proposition 4 asserts that
\[
\text{HD}_{\text{hyp}}(J(f)) = \inf\{t > 0 : P(t, f) \leq 0\}.
\]
Since the function \( t \mapsto P(t, f) \) is continuous and non-increasing for \( t \geq 0 \), we conclude immediately that
\[
\text{HD}_{\text{hyp}}(J(f)) > 1.
\]

4. Example

To complete the answer to the question of Christopher Bishop, it remains to ask whether there exists a polynomial with disconnected Julia set, and a connected component of \( J(f) \) being an analytically embedded interval.

In the following proposition, we provide a family of maps with this property.

**Proposition 10.** Consider a family of cubic polynomials
\[
f_{\varepsilon, \beta}(z) := \varepsilon z^3 + z^2 - \beta, \quad (13)
\]
with \( \varepsilon \) and \( \beta \) real, except \( \varepsilon = 0 \) for which the polynomials are quadratic. Then there exists an analytic curve \( \Gamma \) of parameters \( (\varepsilon, \beta) \), which is the graph of an analytic function \( \varepsilon \mapsto \beta(\varepsilon) \) for \( \varepsilon \approx 0 \), passing through the point \((0,2)\), and such that for every \( (\varepsilon, \beta) \in \Gamma \) with \( \varepsilon > 0 \) the Julia set of \( f_{\varepsilon, \beta} \) is disconnected, and contains infinitely many components being analytic arcs.

**Proof.** We start with the quadratic Chebyshev polynomial
\[
f_{0,2}(z) = z^2 - 2.
\]
Its Julia set is just the interval \( I := [-2, 2] \).

Consider now the polynomials \( f_{\varepsilon, \beta} \) with \( \varepsilon \) real and close to 0, and \( \beta \) real and close to 2. The parameters \( (\varepsilon, \beta) = (0,2) \) will be called initial parameters.

The map \( f_{\varepsilon, \beta} \) has a real repelling fixed point, denoted by \( p_{\varepsilon, \beta} \), close to \( p_{0,2} = 2 \), and 0 is a (not moving) critical point of \( f_{\varepsilon, \beta} \).

**Lemma 11.** There exists an analytic curve \( \Gamma \) of parameters \( (\varepsilon, \beta) \), passing through the initial parameters \((0,2)\) for which
\[
f_{\varepsilon, \beta}^2(0) = p_{\varepsilon, \beta}.
\]
Moreover, \( \partial \Gamma / \partial \beta \neq 0 \) at the initial parameters and in consequence \( \Gamma \) is the graph of an analytic function \( \varepsilon \mapsto \beta(\varepsilon) \).

**Proof.** This is a straightforward calculation. Denoting \( \gamma(\varepsilon, \beta) = f_{\varepsilon, \beta}^2(0) \), we have \( \gamma(\varepsilon, \beta) = -\varepsilon \beta^3 + \beta^2 - \beta \), so
\[
\text{grad}(\gamma)(0, 2) = (-8, 3).
\]
Denoting by \( p(\varepsilon, \beta) \) the fixed point of \( f_{\varepsilon, \beta} \) close to 2, we calculate (differentiating the implicit function)
\[
\text{grad}(p)(0, 2) = (-8/3, 1/3).
\]
Thus,
\[
\text{grad}(\gamma - p)(0, 2) \neq (0, 0)
\]
and therefore, there exists a smooth curve of parameters \( (\varepsilon, \beta) \), passing through the initial parameters \((0,2)\) for which
\[
f_{\varepsilon, \beta}^2(0) = p_{\varepsilon, \beta}.
\]
In particular, \((\partial(\gamma - p)/\partial \beta)(0, 2) = 3 - 1/3 \neq 0\) hence by implicit function theorem \(\Gamma\) is indeed the graph of an analytic function \(\beta(\varepsilon)\).

Note that \(f_{\varepsilon, \beta}(0) = -\beta\). The interval \(I_{\varepsilon, \beta} = [-\beta, p_{\varepsilon, \beta}]\) is thus invariant under \(f_{\varepsilon, \beta}\) and the map \(f_{\varepsilon, \beta} : I_{\varepsilon, \beta} \to I_{\varepsilon, \beta}\) is two-to-one, with critical point at 0, hence topologically conjugate to the quadratic Chebyshev polynomial \(z^2 - 2\) on \(f\) as above.

Suppose from now on that \(\varepsilon > 0, \varepsilon \approx 0\) and \(\beta = \beta(\varepsilon)\). Let us note that the Julia set \(J(f_{\varepsilon, \beta})\) is not connected, as the trajectory of the second critical point \(c = -2/\varepsilon\) tends to infinity. The latter holds since \(f_{\varepsilon, \beta}(\frac{-2}{\varepsilon}) > p_{\varepsilon, \beta}\) is repelled to infinity under further action of \(f_{\varepsilon, \beta}\).

Denote by \(C\) the connected component of \(J(f_{\varepsilon, \beta})\) containing \(I_{\varepsilon, \beta}\), so in particular it is fixed under \(f_{\varepsilon, \beta}\) and is non-trivial.

By Lemma 3, there exist connected neighbourhoods of \(C\),

\[
C \subset U_1 \subset \overline{U}_1 \subset U_0,
\]

such that the map \(F := (f_{\varepsilon, \beta})_{|U_1} : U_1 \to U_0\) is a polynomial-like map, and \(C\) is its filled-in Julia set. Since the Julia set of \(f_{\varepsilon, \beta}\) is disconnected, then by Remark 2, after the proof of Lemma 3, the degree of \(F\) is not maximal, so it is equal to 2. By [9, Theorem 1 (The Straightening Theorem)], see also more recent [3, Theorem 7.4], \(F\) is quasi-conformally conjugate to a true quadratic polynomial. Therefore, pre-images of every point in \(C\) are dense in \(C\). But the pre-images under \(F\) of points from \(I_{\varepsilon, \beta}\) remain in \(I_{\varepsilon, \beta}\), which implies

\[
C = I_{\varepsilon, \beta}.
\]

So, each map \(f_{\varepsilon, \beta}\) with \((\varepsilon, \beta) \in \Gamma\) is a polynomial of degree 3 with disconnected Julia set, for which the Julia set has an invariant component being a true interval on which the degree of the map is equal to 2. So, for each such map the filled-in Julia set, here equal to the Julia set, has a collection of countably many non-trivial components, each of them being an analytic arc; this collection is formed by the invariant analytic arc and all its pre-images under the iterates of \(f_{\varepsilon, \beta}\).

Note also that these are the only non-trivial components of the filled-in Julia set \(K(f_{\varepsilon, \beta})\). Indeed, by Theorem A all non-trivial components are eventually periodic and by Remark 1 after Lemma 3 every non-trivial periodic component of the filled-in Julia set has to contain a critical point in its orbit. In our situation, there are two critical points; one of them is escaping, and the other one is already contained in the invariant interval \(I_{\varepsilon, \beta}\).

5. Final remarks and questions

Comment 1. For \(f\) as in Proposition 5 and \(\partial C\) the boundary of a connected component \(C\) of the filled-in Julia set \(K(f)\), one can ask under what assumptions it holds

\[
\text{HD}(\partial C) < \text{HD}(J(f)).
\]  \hspace{1cm} (14)

It does not hold in general. It may happen that the component \(C\) has empty interior (so is the Julia set of the map \(F\) introduced in Lemma 3), but it satisfies \(\text{HD}(J(F)) = \text{HD}(J(f)) = 2\) and even \(\text{HD}_{\text{hyp}}(J(F)) = 2\). Indeed, just in the example \(f_{\varepsilon, \beta}(z) = \varepsilon z^3 + z^2 - \beta\) with \(\varepsilon > 0\) (but \(\beta \neq \beta(\varepsilon)\)), we find \(\varepsilon \approx 0\) and complex \(\beta \approx 2\) such that \(\text{HD}_{\text{hyp}}(J(F)) = 2\), by finding\(^1\) parameters

\(^1\)See [9] or [3, Theorem 7.8]. The family \(F\) of quadratic-like maps \(f_{\varepsilon, \beta}\) for an arbitrary small \(\varepsilon\) for \(|\beta| < 3\) and \(|z| < 3\) is Mandelbrot-like, as a small perturbation of the family \(z^2 - \beta\). So there is 1-to-1 (homeomorphic) correspondence by hybrid (in particular quasiconformal) equivalences of its elements with connected filled-in Julia sets, to polynomials \(z^2 + c\) with \(c\) in the Mandelbrot set. This is a parameter version of Douady–Hubbard’s Straightening Theorem.
so that the quadratic polynomial hybrid equivalent to \( F \) satisfies this, [27]. Note that a quasi-conformal conjugacy cannot drop neither Hausdorff nor hyperbolic dimension down from 2, see [1].

Proving Proposition 5 we proved and used the fact that \( f \) is ‘almost hyperbolic’, here: topologically (analytically) conjugate to a ± Chebyshev polynomial on a neighbourhood of \( C \). An even easier case is where \( f \) is hyperbolic (expanding) on \( C \). Then Lemma 6, leading to (14), holds easily. In fact, (14) holds under more general assumptions, namely if \( F \) is non-uniformly hyperbolic here in the version that it satisfies Topological Collet-Eckmann, TCE, see [23]. One out of several definitions equivalent to TCE, called Backward Exponential Shrinking, says:

There exist \( \lambda > 1 \) and \( r > 0 \) such that for every \( x \in \partial C = J(F) \), every \( n > 0 \) and every connected component \( W_n \) of \( F^{-n}(\mathbb{D}(x,r)) \) for the Euclidean disc \( \mathbb{D}(x,r) \) with radius \( r \) and origin at \( x \), it holds \( \text{diam}(W_n) \leq \lambda^{-n} \).

**Sketch of proof of (14) for \( C \) and \( F \) satisfying TCE.** Assume for simplification that \( \partial C \) contains just one \( f \)-critical point. We shall apply an analogue of Lemma 6, where \( F \) is replaced by a canonical induced map \( G \) (see [22], Subsection 3.2), which is a map of return (not first return !) to a nice set \( W \) for iteration of \( F \), \( G(x) := F^{m(x)}(x) \). We refer here to a theory developed by the first author and Rivera-Letelier in [21, 22]. In particular, see Section 3 of the latter paper. \( G^{-1} \) forms an Iterated Function System with infinitely many branches from \( W \) into itself, with bounded distortion. Consider the pressure

\[
P(t,p,G) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{v \in G^{-n}(w)} \exp S_{n,G}(\psi_G)(v) \tag{15}
\]

for \( \psi_G := -t \log(|G'| - pm) \) where \( S_{n,G}(\psi_G) := \sum_{j=0}^{n-1} \psi_G \circ G^j \), not depending on \( w \) due to bounded distortion. Compare Proposition 4 and (6). Now we apply [22, Subsection 7.2] which says that \( P(t,P(t,F),G) = 0 \) for \( t = t_0 \) being zero of \( P(t,F) \). Next denote \( \psi := -t_0 \log |F'| \).

Let \( h \) be an additional branch of \( f^{-m_0} \) from \( W \) into itself for some \( m_0 \in \mathbb{N} \), compare Lemma 7. Now, for each \( \ell \in \mathbb{N} \) consider the multivalued function \( \phi_\ell := G^{-\ell} \circ h \). We obtain the key estimate similar to (11) in proof of Proposition 5

\[
\log \sum_{y \in \phi_\ell(x)} \exp S_{m_\ell}(\psi)(y) > -C_0 \tag{16}
\]

for \( m_\ell := m(y) + m(G(y)) + \cdots + m(G^{\ell-1}(y)) + m_0 \). Here \( S_n \) means \( S_{n,F} := \sum_{j=0}^{n-1} \psi \circ F^j \). The estimate (16) follows from [16, Section 6] and [22, Subsection 4.3. Existence]; \( G \) is hyperbolic, but with infinitely many branches. For each \( \ell \), define the series \( \Phi_\ell(s) := \sum_{N=1}^{\infty} A_\ell,N s^N \) for complex \( s \), with coefficients

\[
A_\ell,N = \sum_{y \in \phi_\ell(x), m(y) = N} \exp S_N \psi(y). \tag{17}
\]

Note that (16) yields

\[
\Phi_\ell(1) = \sum_{N} A_\ell,N > \exp -C_0.
\]

\( ^{†} \)The nice set \( W \) is a small neighbourhood of the set \( \text{Crit}(F,J) \) of all \( F \)-critical points in \( J(F) \). If \( \# \text{Crit}(F,J) > 1 \), it has more than one component. So this needs some care. In place of the IFS, one applies a Graph Directed Markov System and defines \( P \) appropriately, see [16]. The branch of \( h \) above means a branch for each component of \( W \), where it is comfortable to choose \( m_0 \) common. Such a choice is possible due to the topological exactness of \( f \) on \( J(f) \), which means that for every non-empty set \( A \subset J(f) \) open in \( J(f) \) there exists \( n \in \mathbb{N} \) such that \( f^n(A) = J(f) \). Compare proof of [21, Lemma 4.1].
So, for every $L$, there exists $N$, such that for every $\ell < L$,
\[
\sum_{n \leq N} A_{n,\ell} \geq \frac{1}{2} \exp -C_0.
\]

So, for $\Phi(s) := \sum \Phi_\ell(s)$, we get
\[
\Phi(1) \geq \sum_{n \leq N, \ell < L} A_{n,\ell} \geq \frac{1}{2} L \exp -C_0 > 2
\]
for $L$ large enough. Hence also $\Phi(s) > 2$ for some $s < 1$. 

So, $\Phi(s) + \Phi(s)^2 + \cdots = \infty$, hence the radius of convergence of $\Phi$ is at most $s$. So $P(t_0, f) > P(t_0, F) = 0$. Hence the (unique) zero of the function $P(t, f)$, equal to $\text{HD}_{\text{hyp}}(J(f))$ ($= \text{HD}(J(f))$ due to TCE), is strictly larger than $t_0 = \text{HD}_{\text{hyp}}(J(F)) = \text{HD}(J(F))$. \hfill \Box

Note that in the situation of Proposition 5, given $\ell$, $\Phi_\ell = A_{m_\ell, s}$, where the time of each backward branch for its iterate of $f$ is fixed, equal to $m_\ell$, whereas here it is not.

**Question.** How can TCE be weakened so that (14) still holds?

**Comment 2.** Theorems 1 and 2 hold for a polynomial-like map $f$, so in the situation more general than for a polynomial $f$. The assumption of Theorem 2 for a general case of polynomial-like maps $f$ is slightly different: the excluded exceptional cases are the ones listed in Theorem B (with $\partial \Omega$ replaced by the Julia set $J(f)$). Proofs are the same.

**Comment 3.** A positive answer to the following conjecture would generalize Theorem 2 to rational maps:

**Conjecture 12.** Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree at least 2. Assume that the Julia set $J(f)$ is connected. Then $f$ is either a finite Blaschke product in some holomorphic coordinates or a quotient of a Blaschke product by a rational function of degree 2, or $\text{HD}(J(f)) > 1$.

If $f$ has an attracting periodic orbit, then the positive answer is given by Theorem B applied to $\Omega$ being the immediate basin of attraction to a point of the orbit for an iterate of $f$. Indeed $\Omega$ is simply connected by the connectivity of $J(f)$. More precisely, either $\text{HD}_{\text{hyp}}(J(f)) > 1$ or due to [19, Theorem A] it is like in the assertion of the conjecture.

We believe that the same holds if $f$ has a parabolic periodic orbit.

Another possibility is that the Fatou set consists only of Siegel discs and their pre-images for iterates of $f$.

Before discussing this, consider $f = P_\alpha$, a quadratic polynomial $e^{2\pi i \alpha} z + z^2$ such that $\alpha$ is real irrational of sufficiently high type, namely all the coefficients in its continued fraction expansion are larger than certain constant. Then, if $\alpha \in \mathcal{B} \setminus \mathcal{H}$, that is, satisfying Brjuno condition, (equivalent to the linearizability at 0, yielding a Siegel disc $S$), but not satisfying Herman condition (in particular not Diophantine), the Siegel disc is hairy outside. Namely there is a Cantor bouquet of lines in $J(P_\alpha)$ having together Hausdorff dimension 2. See [7] and references therein.

If $\alpha$ is Diophantine, of constant type, namely all the coefficients in its continued fraction expansion are smaller than a constant, then $\text{HD}(\partial S) > 1$, see [12].

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1. We do not know whether we can refer to Theorems 1 and 2 for polynomials using straightening, since quasi-conformal homeomorphisms may in general change Hausdorff dimension larger than 1 to equal to 1.
In those situations, however, \( P_\alpha \) are polynomials so they have basins of attraction to infinity in their Fatou sets and Theorem 2 proves the Conjecture. But maybe for a rational map \( f \) the above theory can also be applied, so in the ‘Siegel’ cases above, Conjecture 12 holds true.

In the remaining case of Herman rings, the situation is expected to be similar to Siegel’s.

Note that a positive answer to Conjecture 12 follows from the Comment 2, for \( f \) being renormalizable. Renormalizable means here that there exist an integer \( r \geq 2 \), a compact connected set \( C \) having more than one point, intersecting \( J(f) \), and its neighbourhoods \( U \) and \( U' \), such that \( F = f^r|_{U'} \colon U' \to U \) is polynomial-like with the filled-in Julia set equal to \( C \). Unlike before, we do not assume here that \( \partial C \) is a connected component of \( J(f) \). See examples of such renormalizations for rational Newton maps in [10].

References

1. K. Astala, ‘Area distortion of quasiconformal mappings’, Acta Math. 173 (1994) 37–60.
2. Ch. Bishop, ‘A transcendental Julia set of dimension 1’, Invent. Math. 212 (2018) 407–460.
3. B. Branner and N. Fagella, Quasiconformal surgery in holomorphic dynamics (Cambridge University Press, Cambridge, 2014).
4. B. Branner and J. H. Hubbard, ‘The iteration of cubic polynomials, Part II: patterns and parapatterns’, Acta Math. 169 (1992) 229–325.
5. H. Brolin, ‘Invariant sets under iteration of rational functions’, Ark. Mat. 6.6 (1965) 103–144.
6. L. Carleson and T. W. Gamelin, Complex dynamics (Springer, Berlin, 1993).
7. D. Cheraghi, A. DeZotti and Fei Yang, ‘Dimension paradox of irrationally indifferent attractors’, Preprint, 2003, arXiv:2003.12340v1.
8. T. Das, L. Fishman, D. Simmons and M. Urbański, ‘Badly approximable vectors and fractals defined by conformal dynamical systems’, Math. Res. Lett. 25 (2018) 437–467.
9. A. Douady and J. Hubbard, ‘On the dynamics of polynomial-like mappings’, Ann. Sci. Éc. Norm. Supér. (4) 18 (1985) 287–343.
10. K. Drach and D. Schleicher, ‘Rigidity of Newton dynamics’, Preprint, 2018, arXiv:1812.11919v2.
11. G. M. Goluzin, Geometric theory of functions of a complex variable, Russian Edition Izd. Nauka, Translation of Mathematical Monographs 26 (American Mathematical Society, Providence, RI, 1966).
12. J. Graczyk and P. Jones, ‘Dimension of the boundary of quasicomformal Siegel discs’, Invent. Math. 148 (2002) 465–493.
13. I. Inoquio-Renteria and J. Rivera-Letelier, ‘A characterization of hyperbolic potentials of rational maps’, Bull. Braz. Math. Soc. (N.S.) 43 (2012) 99–127.
14. O. Kozlovski and S. van Strien, ‘Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials’, Proc. Lond. Math. Soc. (3) 99 (2009) 275–296.
15. N. Makarov and S. Smirnov, ‘On thermodynamics of rational maps. II. Non-recurrent maps’, J. Lond. Math. Soc. (2) 67 (2003) 417–432.
16. R. D. Mauldin and M. Urbański, Graph directed Markov systems, Cambridge Tracts in Mathematics 148 (Cambridge University Press, Cambridge, 2003).
17. F. Przytycki, ‘Riemann map and holomorphic dynamics’, Invent. Math. 85 (1986) 439–455.
18. F. Przytycki, ‘Conical limit set and Poincaré exponent for iterations of rational functions’, Trans. Amer. Math. Soc. 351 (1999) 2081–2099.
19. F. Przytycki, ‘On the hyperbolic Hausdorff dimension of the boundary of a basin of attraction for a holomorphic map and of quasirepellers’, Bull. Pol. Acad. Sci. Math. 54 (2006) 41–52.
20. F. Przytycki, ‘Thermodynamic formalism methods in one-dimensional real and complex dynamics’, Proceedings of the International Congress of Mathematicians 2018, Volume 2, Rio de Janeiro (ed. B. Sirakov, P. N. de Souza and M. Viana; World Scientific Publishing, Singapore, 2018) 2081–2106.
21. F. Przytycki and J. Rivera-Letelier, ‘Statistical properties of Topological Collet–Eckmann maps’, Ann. Sci. Éc. Norm. Supér. (4) 40 (2007) 135–178.
22. F. Przytycki and J. Rivera-Letelier, ‘Nice inducing schemes and the thermodynamics of rational maps’, Comm. Math. Phys. 301 (2011) 661–707.
23. F. Przytycki, J. Rivera-Letelier and S. Smirnov, ‘Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps’, Invent. Math. 151 (2003) 29–63.
24. F. Przytycki, M. Urbański and A. Zdunik, ‘Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps. I’, Ann. of Math. (2) 130 (1989) 1–40.
25. F. Przytycki and M. Urbański, Conformal fractals: Ergodic theory methods, London Mathematical Society Lecture Note Series 371 (Cambridge University Press, Cambridge, 2010).
26. W. Qiu and Y. Yin, ‘Proof of the Branner-Hubbard conjecture on Cantor Julia sets’, Sci. China. Ser. A. Math. Phys. Astron. 52 (2009) 45–65.
27. M. Shishikura, ‘The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets’, Ann. of Math. (2) 147 (1998) 225–267.
28. A. Zdunik, ‘Parabolic orbifolds and the dimension of the maximal measure for rational maps’, Invent. Math. 99 (1990) 627–649.
29. A. Zdunik, ‘Harmonic measure versus Hausdorff measures on repellers for holomorphic maps’, Trans. Amer. Math. Soc. 326 (1991) 633–652.

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