I. INTRODUCTION

Nonlinear structures, solitons, and vortices have been actively studied in atomic Bose-Einstein condensate (BEC) and boson-fermion mixtures situated in magnetic and optical traps [1]-[5]. Solitons exist in quest of nonlinearity of interactionally conditioned terms. These nonlinear terms compensate the dispersion emerging particularly as a consequence of a free motion of quantum particles. Solitons in the BEC are described via the Gross-Pitaevskii (GP) equation [6] which has form of one-particle nonlinear Schrödinger equation (NLSE). The NLSE plays an important role when describing the dynamics of various physical systems; name just a few degenerated chargeless bosons and fermions as well as superconductors. A more detailed account of the interaction in comparison with the GP equation leads to the existence of a new type of solitons. We use a set of QHD equations in the third order by the interaction radius (TOIR), which corresponds to the GP equation in a first order by the interaction radius. The solution for the soliton in a form of expression for the particle concentration is obtained analytically. The conditions of existence of the soliton are studied. It is shown what solution exists if the interaction between the particles is repulsive. Particle concentration of order of \(10^{12}-10^{14} \text{ cm}^{-3}\) has been achieved experimentally for the BEC, the solution exists if the scattering length is of the order of 1 \(\mu\text{m}\), which can be reached using the Feshbach resonance. It is one of the limit case of existence of new solution. The corresponding scattering length decrease with the increasing of concentration of particles. The investigation of effects in the TOIR approximation gives a more detail information on interaction potentials between the atoms and can be used for a more detail investigation into the potential structure.

The possibility to derive the GP equation from a microscopic many-particle Schrödinger equation (MPSE) is substantiated in [12]. The method of direct derivation of the GP equation from the MPSE was suggested in Ref. [7]. It was made by means of QHD method [13], [14]. It is well-known that the GP equation can be present in the form of hydrodynamic equations [6]

\[
\frac{\partial n(r,t)}{\partial t} + \frac{\partial}{\partial r} [n(r,t) v^\alpha(r,t)] = 0
\]

and

\[
\frac{\partial n(r,t)}{\partial t} + \frac{\partial}{\partial r} [n(r,t) v^\alpha(r,t)] + \frac{1}{2} \frac{\hbar^2}{m} \frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} - \frac{1}{4m} \frac{\partial^2}{\partial r^2} n(r,t) + \frac{1}{2} \frac{\hbar^2}{m} \frac{\partial^2}{\partial r^2} n(r,t) + \frac{1}{2} \frac{\hbar^2}{m} \frac{\partial^2}{\partial r^2} n(r,t) = -n(r,t) \frac{\partial}{\partial r} \nabla^2 V_{\text{ext}}(r,t),
\]

where

\[
g = \int \frac{d^3r}{m} U(r),
\]

and \(n(r,t)\) is the concentration of particles and \(v^\alpha(r,t)\) is the velocity field. The quantity \(\nabla\) is the Laplace operator.

For dilute gases the quantity \(g\) can be express via scattering length by formula

\[
g = \frac{4\pi \hbar^2 a}{m},
\]

where \(a\) is the scattering length.

In this paper we use set of equations derived with the QHD method. There are different methods for obtaining
equations describing the BEC evolution. For example, in Ref. [15], equation for BEC evolution was derived in the framework of nonequilibrium Thermo Field Dynamics. In the QHD method the system of equation is appeared directly from many-particle Schrödinger equation. The first step of derivation is the definition of concentration of particles in three dimensional physical space. Differentiation of concentration with respect to time and applying of the Schrödinger equation leads to continuity equation, a current of density is arisen there. Next step of derivation is differentiation of the current density. In this way we obtain a momentum balance equation, in another terms the Euler equation. A force field exists in the obtained Euler equation. For neutral particles with the short-range interaction the force field could be present in the form of a expansion in a series. In this case, the GP equation emerges when we take into account the first member of decomposition of the force field by the interaction radius. The next nonzero term appears in the third order by the interaction radius (TOIR).

Different types of solitons occur in the BEC and boson-fermion mixtures. If the interaction between the bosons is repulsive $a > 0$, dark solitons that are the regions with a lowered concentration of the particles can propagate in the BEC [16], [17], [18]. Bright solitons, i.e., solitons of compression, can exist in the system of Bose particles coupled by attractive forces in quasi-one-dimensional (1D) traps [19], [20]. Gap solitons manifest themselves in periodic structures, particularly, the existence of gap bright solitons are found experimentally in the system of bosons with $a > 0$ [21]. Solitons of compression occur in boson-fermion mixtures if repulsive forces $a_{bb} > 0$ act between the bosons, while the interaction between bosons and fermions is attractive with force $a_{bf} < 0$ [22]. In the work [11] authors obtain a change of the form of the well-known bright soliton due to TOIR terms. The bright soliton solution arise from GP equation. More detailed account of interaction with accuracy to TOIR leads to change of form of bright soliton.

In this paper, we report about the existence of a new soliton solution in a one dimensional (1D) BEC. This solution appears when we account the interaction accurate to the TOIR. To obtain this soliton we solve the set of QHD equations by the perturbative method suggested Washimi et al. [23], which is widely used in the plasma physics, see for example [24], [25]. The obtained solution is the soliton of compression; it exists under the condition $a > 0$, i.e., in the case of repulsion between the particles. The existence of such a solution can be conditioned by higher spatial concentration derivatives in the term for interaction in the TOIR. We consider a two cases it are 1D configuration and quasi-1D trap. Let us notice the limiting cases of existence of the solution. One of the limiting cases in the region of parameters when the scattering length (SL) $a$ is of the order $10^{-8} cm$, and the corresponding equilibrium concentration is $10^{12} cm^{-3}$. That could be actual in connection with the development of cooling methods for dense gases [26], [27]. Another limiting case is the region of parameters with the SL of the order $10^{-3} cm$. This case corresponds to concentrations about of $10^{12} - 10^{14} cm^{-3}$, which are usually dealt with in experiments with BEC. Thus, in order to form the conditions for soliton occurrence, the Feshbach resonance (FR) phenomenon should be used [28], [29]. They attain the wide-limit SL change in FR experiments, particularly the values $10^{3} - 10^{4}a_{0}$ (a-Bohr radius) can be reached for magnetically trapped $^{85}Rb$ [30].

We use in this paper short-range interaction potential quantum hydrodynamic equation derived for the system of ultracold neutral particles. In connection with this, our attention should be paid to the fact that an increase in the SL can be caused both by a decrease and an increase in the depth or width of the interaction potential. Assuming that an increase in the SL is caused by a decrease in the interaction potential depth, the conditions of existence of equations could be considered as fulfilled.

In a general case, the fact that under the FR condition larger values of SL $a$ are attained, can point to the fact that a more successive account for the interaction should be necessary.

The processes and effects in the TOIR, along with the effects in the spinor BEC [31], magnetically [32], [33], [34] and electrical [35], [36] polarized BEC, can play an important role when investigating BEC and interatomic interaction.

Our paper is organized as follows. In Sect. 2 we present basic equation and describe using model. In Sect. 3 we consider a solitons in 1D BEC and describe a method of getting of solution. We show that with solution is a new solution and receive a condition of existence of this solution. In Sect. 4 we obtain system of QHD equations for the quasi-one-dimensional case. In Sect. 5 we investigate soliton solution obtained in sect. 3 for quasi-one-dimensional case. In Sect. 6 brief summary of obtained results is presented.

II. MODEL

To investigate solitons in BEC, we use the set of QHD equations up to the TOIR approximation [7]. The calculation of the first member in a quantum stress tensor that corresponds to the GP equation is fulfilled in [7] under the condition that the particles do not interact. A more complete investigation into the conditions of derivation of the GP equation from the MPSE shows that the GP equation appears in the first order by the interaction radius (POIR), if the particles are in an arbitrary state that can be simulated by a single-particle wave function. Such a state can particularly appears as a result of strong interaction between the particles that takes place in quantum fluids.

The QHD equations set for the atoms with a two-particle interaction with the potential $U(r)$ and located in external
field $V_{\text{ext}}(r, t)$ in the TOIR approximation has the form \cite{7}

$$\partial_t n(r, t) + \partial^n (n(r, t)v^n(r, t)) = 0$$

(3)

and

$$mn(r, t)\partial_t v^n(r, t) + \frac{1}{2}mn(r, t)\partial^n v^2(r, t)$$

$$-\frac{\hbar^2}{4m}\partial^n n(r, t) + \frac{\hbar^2}{4m}\partial^n v^2(r, t)$$

$$-\Upsilon n(r, t)\partial_n n(r, t) - \frac{1}{16}\Upsilon_2\partial_n n^2(r, t)$$

$$= -n(r, t)\partial^n V_{\text{ext}}(r, t),$$

(4)

where

$$\Upsilon = \frac{4\pi}{3} \int dr(r) \frac{\partial U(r)}{\partial r}$$

(5)

and

$$\Upsilon_2 = \frac{4\pi}{15} \int dr(r) \frac{\partial U(r)}{\partial r}.$$ (6)

We also have $\Upsilon = -g$. Equations (3) and (4) determine the dynamic of concentration of particles $n(r, t)$ and velocity field $v^n(r, t)$. From equation (4) we see that dynamics of BEC depends on different moments of interaction potential $\Upsilon, \Upsilon_2$. The system of equations (3) and (4) is differ from (1) and (2) by existence of one new term. It is a last term in left hand side of equation (4). This term appears at interaction account up to TOIR approximation.

In diluted alkali gases, the interaction between particles can be considered as scattering. In this case, the FOIR interaction constant can be expressed in terms of SL $\Upsilon = -4\pi\hbar^2a/m$ \cite{6}. The second interaction constant $\Upsilon_2$ emerges in the TOIR. In a general case, parameter $\Upsilon_2$ is independent of $\Upsilon$. In \cite{7}, the approximate expression $\Upsilon_2$ via $\Upsilon$ is considered. We use this expression in our work when we investigate the existence region of the soliton solution.

Considering the dispersion equation for elementary excitations in BEC accurate to TOIR

$$\omega^2(k) = \left(\frac{\hbar^2}{4m^2} + \frac{n_0\Upsilon_2}{8m}\right)k^4 - \frac{\Upsilon n_0}{m}k^2,$$

which obtained in \cite{7} we can see that coefficient at $k^4$ could be negative. It is realized at condition $\Upsilon_2 < -2\hbar^2/mn_0$. Consequently, we can expect that value $\Upsilon_2 = -2\hbar^2/mn_0$ could play important role at investigation of nonlinear processes.

III. BRIGHT-LIKE SOLITON IN 1D BEC

In this section, we consider the solitons in the 1D BEC. For this purpose, we use the perturbative method \cite{23,24}. Here we present some detail of calculations and describe the perturbative method.

We investigate the case when the stretched variables include the expansion parameter in follows combination:

$$\xi = \varepsilon^{1/2}(z - ut), \quad \tau = \varepsilon^{3/2}ut$$

(7)

where $u$ is the phase velocity of the wave, $\varepsilon$ is a small nondimension parameter.

An operational relations are arisen from (7)

$$\partial_\xi = \varepsilon^{1/2}\partial_\tau, \quad \partial_t = u(\varepsilon^{3/2}\partial_\tau - \varepsilon^{1/2}\partial_\xi)$$

(8)

The decomposition of the concentration and velocity field involves a small parameter $\varepsilon$ in the following form:

$$n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + ...$$

(9)

$$v = \varepsilon v_1 + \varepsilon^2 v_2 + ...$$

(10)

Presented in (9) equilibrium concentration $n_0$ is a constant. We put expansions (8)-(10) in equations (3) and (4). Then, the system of equation is divided into systems of equations in different orders on $\varepsilon$.

Equations emerging in the first order by $\varepsilon$ from the system of equations (28) have form

$$-u\partial_\xi n_1 + n_0\partial_\xi v_1 = 0,$$

$$-mun_0\partial_\xi v_1 = \Upsilon n_0\partial_\xi n_1$$

(11)

and lead to the following expression for the phase velocity $u$:

$$u^2 = -\frac{\Upsilon n_0}{m}.$$ (12)

Square of phase velocity $u^2$ must be positive. Consequently $\Upsilon$ is negative, i.e.

$$\Upsilon < 0.$$ (13)

It corresponds to the repulsive SRI.

Also, from (11) we obtain relation between $n_1$ and $v_1$ and their derivatives

$$\partial_\xi n_1 = \frac{n_0}{u}\partial_\xi v_1.$$ (14)

Integrating this equation and using a boundary conditions

$$n_1, \quad v_1 \rightarrow 0 \text{ at } x \rightarrow \pm \infty$$ (14)

we have

$$n_1 = \frac{n_0}{u}v_1.$$ (15)

In the second order by $\varepsilon$, from equations (3) and (4), we derive

$$-u\partial_\xi n_2 + u\partial_\xi n_1 + \partial_\xi (n_0v_2 + n_1v_1) = 0$$ (16)
and

\[-mu(n_0 \partial_\xi v_2 + n_1 \partial_\xi v_1) + mn_0 \partial_\tau v_1 + mn_0 v_1 \partial_\xi v_1\]

\[-\frac{\hbar^2}{4m} \partial_\xi^2 n_1 = \Upsilon n_0 \partial_\xi n_2 + \Upsilon n_1 \partial_\xi n_1 + \frac{1}{8} \Upsilon_2 n_0 \partial_\xi^3 n_1.\]  

(17)

In (16) we can express \(n_2\) via \(v_2\) and \(n_1, v_1\) and put it in equation (17). Using (12), we exclude \(v_2\) from the obtained equation (17). Thus, we obtain an equation which contain \(n_1\) and \(v_1\), only. Using (15), expressing \(v_1\) via \(n_1\) we get a Korteweg-de Vries equation for \(n_1\)

\[\partial_\tau n_1 + p_{1D} n_1 \partial_\xi n_1 + q_{1D} \partial_\xi^2 n_1 = 0.\]  

(18)

In this equation the coefficients \(p_{1D}\) and \(q_{1D}\) arise in the form

\[p_{1D} = \frac{3}{2n_0},\]  

(19)

and

\[q_{1D} = \frac{\hbar^2}{2m} + \frac{1}{8} n_0 \Upsilon_2 \frac{1}{2n_0 \Upsilon}.\]  

(20)

From equation (18) we can find the solution in the form of a solitary wave using transformation \(\eta = \xi - V \tau\) and taking into account boundary condition \(n_1 = 0\) and \(\partial_\eta^2 n_1 = 0\) at \(\eta \rightarrow \pm \infty\), we get

\[n_1 = \frac{1}{p_{1D}} \frac{3V}{\cosh^2 \left( \frac{V}{q_{1D}} \eta \right)},\]  

(21)

where \(V\) is the velocity of soliton propagation to the right. From expression \(p_{1D} = 3/2n_0\) and solution (21) we can find that a perturbation of concentration is positive. Consequently, obtained solution is the bright like soliton (BLS). A width of the soliton is given with formula \(d = 2\sqrt{q_{1D}/V}\). BLS exists in the case \(q_{1D}\) is positive. From condition \(q_{1D} > 0\) (13) we have

\[\frac{\hbar^2}{2m} + \frac{1}{8} n_0 \Upsilon_2 < 0.\]  

(22)

Relation (22) is fulfilled only in the case when \(\Upsilon_2\) is negative. In the absence of the second interaction constant \(\Upsilon_2\) (i.e. in the Gross-Pitaevskii approximation) the relation (22) does not fulfill and, consequently, BLS does not exist. From (22) we receive that the second interaction constant \(\Upsilon_2\) must be negative and its module must be more than \(4\hbar^2/mn_0\)

\[|\Upsilon_2| > \frac{4\hbar^2}{mn_0}.\]  

(23)

Using representation \(\Upsilon_2\) via the s-wave SL \(a\) (7) we get

\[\Upsilon_2 = \theta a^2 \Upsilon = -4\pi \theta \hbar^2 a^3/m,\]  

(24)

where \(\theta\) is a constant, which is determined by an explicit form of the interaction potential, \(\theta > 0, \theta \sim 1\) (7).

From (22) and (24) we obtain

\[\pi a^3 n_0 > 0.\]  

(25)

It is the condition of BLS existence.

Due to used method perturbation \(n_1\) must be smaller than equilibrium concentration \(n_0\): \(n_0 \gg n_1\). Here we consider the rate \(n_1/n_0\) at the centre of soliton at \(\cosh(\sqrt{V}/q_{1D}/2n) = 1:\)

\[\frac{n_1(\text{centre})}{n_0} = 3V \frac{p_{1D}}{p_{1D}} = 2V.\]  

Correspondingly, dimensionless velocity \(V\) must be much smaller than one.

At equality in formula (25) the BLS has infinite width, with the increasing of interaction solitons width becomes finite. Formula (25) shows the bound condition for existence of soliton.

Below we consider the same problem for cigar-shaped trap.

IV. THE QUANTUM HYDRODYNAMICS EQUATION IN THE CIGAR-SHAPED TRAPS

Let us to consider the variation of the form of equations of QHD in the case of cigar-shaped magnetic traps:

\[V_{\text{ext}} = \frac{m \omega_0^2}{2} (\rho^2 + \lambda^2 z^2),\]

where \(\omega_0\) and \(\lambda \omega_0\) are angular frequencies in radial and axial directions and \(\lambda\) is the anisotropy parameter. In a quasi-1D geometry, the anisotropy parameter in the axially-free motion approximation becomes zero \(\lambda = 0\). Thus, the solution for the radial wave-function appears in the form

\[|\Phi(\rho)|^2 = n(\rho) = \frac{m \omega_0}{\pi \hbar} \exp \left( -\frac{m \omega_0 \rho^2}{\hbar} \right).\]  

(26)

In the TOIR during the quasi-1D motion of bosons in magnetic traps, the GP equation preserves the form but the interaction \(\Upsilon_2\) changes. A complete 3D particle concentration \(n_w(\rho, z, t)\) can be presented as product of one-dimensional time dependent concentration \(n(z, t)\) and static radial two-dimensional concentration \(n(\rho)\)

\[n_w(\rho, z, t) = n(z, t) n(\rho),\]  

(27)

where the value \(n(\rho)\) is presented by the formula (26).

Applying the procedure described in (37), from the set of equations (4) and using the corresponding NLSE, we can acquire the system of QHD equations for cigar-shaped trap. Here, we describe basic steps of this procedure. Starting from equation (4) we get a equation of evolution of a following function, which sometimes called wave function in medium or order parameter,

\[\Phi(\mathbf{r}, t) = \sqrt{n(\mathbf{r}, t)} \exp(m \theta(\mathbf{r}, t)/\hbar),\]
where \( \theta \) is the potential of velocity field, i.e. \( \mathbf{v} = \nabla \theta \). Equation for \( \Phi(\mathbf{r}, t) \) is the NLSE corresponding to system of equations (4). Approximately we can present \( \Phi(\mathbf{r}, t) \) in the form \( \Phi(\mathbf{r}, t) = \Phi(\rho, z, t) = \Phi(z, t)\Phi(\rho) \), where \( \Phi(\rho) \) is the wave function of the ground state of harmonic oscillator and the square of module of \( \Phi(\rho) \) presented by formula (25). Since, we get a NLSE for \( \Phi(z, t) \). This equation describes the evolution of BEC in quasi-one dimensional trap. From obtained NLSE we derive the system of QHD equations for quasi-1D trap. In the results we have
\[
\partial_t n(z, t) + \partial_z (n(z, t)v(z, t)) = 0
\]
and
\[
m n(z, t)\partial_t v(z, t) + \frac{1}{2} mn(z, t)\partial_z v^2(z, t) - \frac{\hbar^2}{4m} \partial^2_z n(z, t) + \frac{\hbar^2}{4m} \left( \partial_z n(z, t) \right)^2 - \frac{\eta n(z, t)\partial_z n(z, t) + \alpha_2 n(z, t)\partial_z^2 n(z, t)}{n(z, t)} - \frac{7}{2} \alpha_2 (\partial_z n(z, t))\partial_z n^2(z, t) - \alpha_2 \frac{(\partial_z n(z, t))^2}{n(z, t)} = 0.
\]
(28)
The following parameters appear in equation (28):
\[
\alpha_1 = - \frac{1}{2} \frac{m\omega_0}{\pi\hbar} + \frac{5}{2} \frac{\eta m\omega_0}{\pi\hbar},
\]
and
\[
\alpha_2 = - \frac{3}{16} \frac{m\omega_0}{\pi\hbar}.
\]

The form of nonlinear terms that describe the interaction in equation (28) differs from corresponding terms in (4). This leads to varying the form of solutions and conditions of their existence in a quasi-1D geometry compared with a 1D case.

V. THE SMALL AMPLITUDE SOLITONS IN QUASI-1D BEC

In this section, we consider the solitons in the BEC for the case of small nonlinearity taking into account the TOIR. For this purpose, we use the perturbative method (23), (24).

We investigate the case when the stretched variables include the expansion parameter in follows combination:
\[
\xi = \varepsilon^{1/2}(z - ut), \quad \tau = \varepsilon^{3/2}ut
\]
(29)
where \( u \) is the phase velocity of the wave.
The decomposition of the concentration and velocity field involves a small parameter \( \varepsilon \) in the following form:
\[
n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + ...
\]
(30)

\[v = \varepsilon v_1 + \varepsilon^2 v_2 + ...
\]
(31)
Presented in (30) equilibrium concentration \( n_0 \) is a constant. Equations emerging in the first order by \( \varepsilon \) from the set of equations (28) lead to the following expression for the phase velocity \( u \):
\[
u^2 = \frac{n_0^2\alpha_1}{m}
\]
\[= \frac{n_0}{m} \left( - \frac{1}{2} \frac{m\omega_0}{\pi\hbar} + \frac{5}{2} \frac{\eta m\omega_0}{\pi\hbar} \right)^2
\]
(32)
It is evident from (32) that the wave can exist under the condition \( \alpha_1 > 0 \). The obtained condition means that in the repulsive forces should act in the case under consideration taking into account the difference in the contribution of terms in TOIR prevails over the TOIR terms.

Relationship (32) is the analog of the dispersion dependence. The use of scaling (29) leads to simplifying the dispersion relationship compared with the case when we consider small perturbations proportional to \( \varepsilon \exp(-\omega t + ikz) \). In the latter case, we obtained the dispersion relation \( \omega(k) \) form the set of equations (28) in the form:
\[
\omega^2 = \left( \frac{\hbar^2}{4m^2} + \frac{3n_0}{16n\omega_0} \frac{m\omega_0}{\pi\hbar} \right)k^4
\]
\[+ n_0k^2 \left( - \frac{1}{2} \frac{m\omega_0}{\pi\hbar} + \frac{5}{2} \frac{\eta m\omega_0}{\pi\hbar} \right)^2.
\]
(33)
Thus, relationship (32) corresponds to the phonon part of the dispersion dependence (33).

From the second-order set of equations (28) by \( \varepsilon \) we find that the concentration \( n_1 \) satisfies the Korteweg-de Vries equation:
\[
\partial_{\xi} n_1 + pn_1\partial_{\xi} n_1 + q\partial^2_{\xi} n_1 = 0,
\]
(34)
where
\[
p = \frac{3}{2n_0},
\]
(35)
\[
q = \frac{4mn_0\alpha_2 - h^2}{8mn_0\alpha_1}.
\]
(36)

When we derive this equation, we used boundary conditions \( n_1 = 0 \) and \( v_1 = 0 \) at \( \xi \to \pm \infty \). Using transformation \( \eta = \xi - V\tau, \) and taking into account boundary condition \( n_1 = 0 \) and \( \partial_{\eta} n_1 = 0 \) at \( \eta \to \pm \infty \), we can obtain the solution in the form of a solitary wave from equation (34)
\[
n_1 = \frac{3V}{p} \frac{1}{\cosh^2 \left( \frac{1}{2} \sqrt{\frac{q}{p}} \eta \right)},
\]
(37)
where \( V \) is the velocity of soliton propagation to the right. Sign of perturbation is determined by the sign of \( p \). From formulas (35), (36) we can see the quantity \( p \) is positive.
Consequently, obtained solution is the soliton of compression or bright like soliton solution, by analogy with well-known bright soliton in BEC [11], [19], [20]. As it will be shown below, this solution exists only when taking into account the TOIR.

Let us pass on to a detail consideration of the conditions of existence of the solution (37). The solution (37) of the equation (34) exists as the conditions $q > 0$ and $\alpha_1 > 0$ are fulfilled. We start with consideration the condition $q > 0$.

As $\alpha_1 > 0$, then to fulfill the condition $q > 0$ we need $-\hbar^2 + 4mn_0\alpha_2 > 0$.

In the case when $\alpha_2$ is vanish (i.e. in FOIR approximation) the solution (37) is not exist. It means, that solution arises in TOIR approximation which developed in [7]. One of the condition of existence of the solution (37) is:

$$\alpha_2 > \frac{\hbar^2}{4mn_0}. \quad (38)$$

Consequently, for the second interaction constant $\Upsilon_2$ we obtain:

$$\Upsilon_2 < -\frac{4\pi\hbar^3}{3m^2n_0\omega_0}. \quad (39)$$

Using relation (24) we can make estimation for corresponding SL. It is useful to present the value of possible SL in the terms of space parameter of the trap $a_\perp = \sqrt{\hbar/m\omega_0}$. From conditions (39), (24), the conditions for SL $a$ appear:

$$a > \sqrt{\frac{a_\perp^2}{30n_0}}. \quad (40)$$

In addition, from $\alpha_1 > 0$ and (24), we obtain

$$a < \frac{a_\perp}{\sqrt{50}}. \quad (41)$$

Using equation (24), the particle concentration $n$ can be presented in the form

$$n = n_0 + 2Vn_0\varepsilon \cdot \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{V}{q'^2}}\right), \quad (42)$$

where

$$q' = \frac{3\theta mn_0\omega_0 a^3h - h^2}{16mn_0\omega_0 a(h - 5a^2\theta m\omega_0)} \quad (43)$$

The soliton width $d$ arises in the form $d = 2\sqrt{q'/V}$.

The numerical analysis of formula (43) is presented in Fig. 1. It is evident from Fig. 1 that there is a narrow interval of the SL values, for which the solution (42), (43) exists. The dependence of the soliton width on the SL is resonant-shaped. The resonant value of the SL $a_r$ depends on the particle concentration $n_0$ and trap parameter. The values of $a_r$ become lower at increasing of equilibrium concentration $n_0$. The SL $a_r$ reaches 0.1 nm at the concentration of the order $10^{18}$ cm$^{-3}$. As the concentration decreases to values $10^{12}-10^{14}$ cm$^{-3}$, which are usually used in BEC experiments, the SL increases to the values of the order of 1 µm. Such values of the SL can be attained when using the FR.

![FIG. 1. The dependence of the soliton width $d$ on the scattering length $a$ at fixed parameter of the trap $a_\perp = \sqrt{\hbar/m\omega_0} = 10^{-5}$ cm and equilibrium concentration $n_0 = 10^6$ cm$^{-1}$ and at $V = 1$, $\theta = 1$. On Fig. 1a we can see what width of soliton is the positive in small range of the values of the SL. On Fig. 1b the area of the resonance is presented more detailed than on Fig. 1a.](image)

![FIG. 2. (Color online) The dependence of soliton width $d$ on radial parameter of the trap $a_\perp$ and the nonperturbative concentration of the particles $n_0$ at the fixed scattering length $a = 10^{-6}$ cm, and at $V = 1$, $\theta = 1$. The soliton width $d$ is positive in two area. But in area with smaller value of the particles concentration $n_0$ the square of phase velocity is negative. Thus, the solution exist in area of bigger concentrations.](image)
VI. CONCLUSION

In this article, we showed that at a more exact accounting of the interaction, specifically, taking into account the TOIR, a new type of solitons emerges in the BEC. We also studied the conditions of existence of such a solution. For this problem solving we used the set of QHD equations where the interactions included up to TOIR approximation. The TOIR approximation is an example of the nonlocal interaction. The GP approximation gives us the force density in the right hand side of the Euler equation $F = -g\nabla n^2/2 = \Upsilon \nabla n^2/2$. It is corresponds to the first order on interaction radius. The interaction including up to TOIR approximation gives us the second term in the force field $F = \Upsilon \nabla n^2/2 + \Upsilon_2 \nabla^2 n^2/16$. The new term contain the third spatial derivative of the concentration square and the new interaction constant. For obtained results analysis and estimation we used approximate estimation of $\Upsilon_2$ and its approximate connection with the $\Upsilon$ or SL $a$.

We found that BLS (soliton of compression) exists in 1D case (one dimensional propagation in three dimensional medium) in the case of strong enough repulsive interaction. BLS appearance strongly connects with the second interaction constant $\Upsilon_2$. If we consider the interaction in the first order by the interaction radius there is no BLS. We also studied the BLS behavior in the case of quasi-1D trap and describe contribution of external fields on BLS amplitude and width.

In a general case, the second interaction constant $\Upsilon_2$, which appears in the TOIR, is independent of $\Upsilon$ and, consequently, of SL $a$. Thus, the relationships obtained in this article \cite{34, 35, 36} can be used for an independent experimental determination of $\Upsilon_2$. In this case, parameter $\hbar^2/\mu v_0$ can be used for the qualitative evaluation of the second interaction constant $\Upsilon_2$.

Thus, in this paper we showed that new physical effects appear at account of interaction up to TOIR approximation. The processes and effects in the TOIR approximation, along with the effects in the spinor and polarized BEC, can play an important role at investigation of BEC and interatomic interaction.

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