Exact solitary waves of the Korteveg – de Vries – Burgers equation

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Abstract

New approach is presented to search exact solutions of nonlinear differential equations. This method is used to look for exact solutions of the Korteveg – de Vries – Burgers equation. New exact solitary waves of the Korteveg – de Vries – Burgers equation are found.

Keywords: exact solution, traveling wave, nonlinear differential equation, Korteveg – de Vries – Burgers equation, simplest equation method.

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1 Introduction

At the present great interest has been expressed by searching exact solutions of nonlinear differential equations. This is because many mathematical models are described by nonlinear differential equations.

The inverse scattering transform [1] and the direct method by R. Hirota [2] are known as impressive methods to look for solutions of exactly solvable differential equations.

The singular manifold method [3–5], the transformation method [6,7] the tanh-function method [8–11], the elliptic function method [12,13], and the Weierstrass function method [14,15] are useful in many applications to search exact solutions of nonlinear partially solvable differential equations.
In this letter we present new approach to search exact solutions of nonlinear differential equations. The first advantage of our approach that this one generalizes a number of known methods for searching exact solutions of nonlinear ordinary differential equations. The second advantage of the method is simplicity of realization.

This letter has two parts. One of them is devoted to discussion of our method. We apply our approach to look for new exact solutions of the Korteveg – de Vries – Burgers equation in the second part.

2 Method applied

First of all let us discuss our approach in detail. Consider nonlinear ordinary differential equation in the polynomial form

\[ M[y] = 0 \quad (2.1) \]

Assume we look for exact solutions of equation (2.1). Take nonlinear ordinary differential equation of lesser order than equation (2.1)

\[ E[Y] = 0 \quad (2.2) \]

We call any nonlinear ordinary differential equation (2.2) of lesser order than (2.1) with known general solution as the simplest equation.

First example of the simplest equation is the Riccati equation

\[ E[Y] = Y_z + Y^2 - aY - b = 0 \quad (2.3) \]

We emphasize that the general solutions of equation (2.3) have first order singularity. We are going to use this property of equations (2.3) later.

Suppose we have the relation between solutions of equations (2.1) and (2.2)

\[ y = F(Y) \quad (2.4) \]

In addition, suppose that substitution (2.4) into equation (2.1) leads to the relation in the form

\[ M[F(Y)] = \hat{A}E[Y] \quad (2.5) \]

where \( \hat{A} \) is a operator which we have to find.
From relation (2.5) we can see that for any solution of the simplest equation (2.2) there exist a special solution of equation (2.1) by the formula (2.4).

To look for exact solutions of nonlinear ordinary differential equation (2.1) in this letter we firstly introduce and use formula (2.4) in the form

\[
y(z) = A_0 + A_1 Y + A_2 Y^2 + \ldots + A_n Y^n + \\
+ B_1 \left( \frac{Y_z}{Y} \right) + B_2 \left( \frac{Y_z}{Y} \right)^2 + \ldots + B_n \left( \frac{Y_z}{Y} \right)^n
\]  

(2.6)

where \( n \) is the singularity order of general solution of equation (2.1) and \( Y(z) \) is the general solution of equation (2.3), coefficients \( A_k \) \((k = 0, \ldots, n)\) and \( B_k \) \((k = 1, \ldots, n)\) are found after substitution of expression (2.6) into equation (2.1). To find coefficients \( A_k \) \((k = 0, \ldots, n)\) and \( B_k \) \((k = 1, \ldots, n)\) we take the following simple theorem into consideration.

**Theorem 2.1.** Let \( Y(z) \) be solution of equation (2.3) than equations

\[
Y_{zz} = 2 Y^3 - 3a Y^2 + (a^2 - 2b)Y + ab
\]  

(2.7)

\[
Y_{zzz} = -6 Y^4 + 12 Y^3 a + (8b - 7a^2) Y^2 + \\
(-8a b + a^3) Y - 2b^2 + a^2b
\]  

(2.8)

have special solutions that are expressed via the general solution of equation (2.3).

**Proof.** Theorem 2.1 is proved by differentiation of (2.3) with respect to \( z \) and substitution \( Y_z \) from equation (2.3) into expressions obtained.

\( \square \)

Substituting expression (2.6) into equation (2.1) and taking theorem 2.1 into account we have the relation

\[
M[y] = \sum_{k=0}^{2n} P_k(a, b, A_0, \ldots, A_n, B_1, \ldots, B_n) Y^{k-n}
\]  

(2.9)

In the case of nontrivial solutions of the equations for coefficients \( A_k \) \((k = 0, \ldots, n)\) and \( B_k \) \((k = 1, \ldots, n)\)

\[
P_k(a, b, A_0, \ldots, A_n, B_1, \ldots, B_n) = 0, \quad k = (0, \ldots, 2n)
\]  

(2.10)
we have special solution of equation (2.1) by the formula (2.6) where $Y(z)$ is
general solution of equation (2.3).

In this manner the method applied allows us to look for exact solutions
of the origin equation (2.1). Note that our approach is the generalization
of a number methods to look for exact solutions of nonlinear differential
equations. For example using $B_k = 0 \ (k = 1, \ldots, n)$ in (2.6) and $a = 0$ in
(2.3) we have the tanh - function method as special case of our approach.

3 New exact solitary waves of
the Korteveg – de Vries – Burgers equation

Let us apply our approach to look for exact solutions of the the Korteveg –
de Vries – Burgers equation. This equation takes the form

$$u_t + uu_x + \beta u_{xxx} = \alpha u_{xx} \quad (3.1)$$

Nonlinear evolution equation (3.1) describes nonlinear waves taking disper-
sion and dissipation into account. At $\nu = 0$ and $\beta \neq 0$ we have the famous
Korteveg – de Vries equation from equation (3.1). The Causchy problem for
this equation can be solved by the inverse scattering transform [1]. In the
case $\beta = 0$ and $\nu \neq 0$ we have the Burgers equation from equation (3.1)
that can be linearized by the Cole – Hopf transformation into the heat equa-
tion [16,17]. At $\beta \neq 0$ and $\nu \neq 0$ equation (3.1) is not integrable one because
this one does not pass the Painleve test. However we are going to look for
exact solutions of this equation in this letter.

There is the special solution in the form of the solitary wave of equation
(3.1) at $\beta \neq 0$ and $\nu \neq 0$ that was firstly found in [4]. Later this solution
was rediscovered time and again. However this exact solution has only one
arbitrary constant and this one is of limited usefulness.

Let us show that using our approach we can obtain more general solitary
waves.

Taking travelling wave into consideration

$$u(x,t) = y(z), \quad z = x - C_0 t \quad (3.2)$$

we have from equation (3.1)

$$C_1 - C_0 y - \alpha y_x + \beta y_{xx} + \frac{1}{2} y^2 = 0 \quad (3.3)$$
We can see that general solution of equation (3.3) has the singularity order equal two and we have to take \( n = 2 \) by formula (2.6). Therefore we will look for exact solution of equation (3.3) in the form

\[ y(z) = A_0 + A_1 Y + A_2 Y^2 + \frac{B_1 Y_2}{Y} + \frac{B_2 Y^2}{Y^2} \]  

(3.4)

Taking equation (2.3) into account we have from equation (3.3)

\[ y(z) = A_0 + 2 B_1 a + 4 B_2 a^2 - 2 B_2 b + (A_1 - B_1 - 4 B_2 a) Y + \]

\[ + (A_2 + B_2) Y^2 + \frac{B_1 b + 4 B_2 a b}{Y} + \frac{B_2 b^2}{Y^2} \]  

(3.5)

Substituting (3.4) into equation (3.3) we have

\[ B_2^{(1)} = -12 \beta, \quad B_2^{(2)} = 0, \]  

(3.6)

Consider first case: \( B_2 = B_2^{(1)} = -12 \beta \). We have

\[ B_1 = -\frac{12}{5} \alpha + 24 \beta a \]  

(3.7)

\[ A_0 = -4 \beta a^2 + C_0 + \frac{1}{25} \frac{\alpha^2}{\beta} + \frac{12}{5} \alpha a - 16 \beta b \]  

(3.8)

\[ A_1 = \frac{4}{5} \frac{\alpha a^2}{b} - 24 \beta a - \frac{8}{5} \alpha - \frac{1}{125} \frac{\alpha^3}{b \beta^2} \]  

(3.9)

\[ A_2^{(1)} = 12 \beta, \quad A_2^{(2)} = 0 \]  

(3.10)

Assuming at the beginning \( A_2 = A_2^{(1)} = 12 \beta \) we find additionally

\[ b = -\frac{1}{100} \frac{100 \beta^2 a^2 - a^2}{\beta^2} \]  

(3.11)
As the result of calculations we obtain special solutions of the Korteweg - de Vries - Burgers equation in the form

\[ y(z) = C_0 - 12 \beta a^2 - \frac{3}{125} (z - C_2) \left( \alpha + 10 \beta a \right)^2 - \left( \alpha + 10 \beta a \right)^2 Y^{-1} + \]

\[ + \frac{3 \alpha^2}{25 \beta} - \frac{12 \alpha a}{5} - \frac{3}{2500 \beta^3} \left( \alpha - 10 \beta a \right)^2 \left( \alpha + 10 \beta a \right)^2 Y^{-2} \]  

where \( Y(z) \) is determined by the formula

\[ Y(z) = a + \frac{\alpha}{10 \beta} \tanh \left( \frac{z - C_2}{10 \beta^2} \right) \]  

and \( Y(z) \) satisfies the equation

\[ Y_z + Y^2 - 2aY + \frac{1}{100} \frac{100 \beta^2 a^2 - \alpha^2}{\beta^2} = 0 \]  

Exact solitary waves \((3.14)\) are new solutions of equation \((3.3)\). They have two arbitrary constants \( a \) and \( C_2 \) and refer to the solitary waves in the form of kinks.

Assuming \( B_2 = B_2^{(1)} \) and \( A_2 = 0 \) we have relations

\[ A_1 = -\frac{24 \alpha}{5}, \quad b = \frac{\alpha^2}{400 \beta^2}, \quad a = 0 \]  

\[ C_1 = \frac{1}{2} C_0^2 - \frac{18 \alpha^4}{625 \beta^2} \]  

\[ y(z) = C_0 + \frac{3 \alpha^2}{50 \beta} - \frac{12 \alpha Y - 12 \beta Y^2}{500 \beta^2 Y} - \frac{3 \alpha^3}{40000 \beta^3 Y^2} \]
Where \( Y(z) \) is determined by the function

\[
Y(z) = \frac{\alpha}{20 \beta} \tanh \left( \frac{\alpha (z - C_2)}{20 \beta} \right)
\]  
(3.19)

Substituting solution (3.19) into (3.18) we have the new solution of the Korteveg - de Vries - Burgers equation again but this solution is the singular one.

\[
y(z) = C_0 + \frac{3}{50} \alpha^2 \frac{\alpha}{\beta} - \frac{3}{25} \alpha^2 \tanh \left( \frac{\alpha (z - C_2)}{20 \beta} \right) - \\
- \frac{3}{25} \alpha^2 \left( \tanh \left( \frac{\alpha (z - C_2)}{20 \beta} \right) \right)^{-1} - \frac{3}{100} \alpha^2 \left( \tanh \left( \frac{\alpha (z - C_2)}{20 \beta} \right) \right)^2 - \\
- \frac{3}{100} \alpha^2 \left( \tanh \left( \frac{\alpha (z - C_2)}{20 \beta} \right) \right)^{-2}
\]  
(3.20)

Consider case: \( B_2 = 0 \). Now we have to obtain the known solution. After calculations we find

\[
B_1 = 0, \quad A_2 = -12 \beta, \quad A_1 = 24 \alpha \beta - \frac{12}{5} \alpha
\]  
(3.21)

\[
A_0 = C_0 + \frac{12}{5} \alpha a - 4 \beta a^2 + \frac{1}{25} \alpha^2 + 8 \beta b
\]  
(3.22)

\[
b = -\frac{1}{100} \frac{100 \beta^2 a^2 - \alpha^2}{\beta^2}
\]  
(3.23)

\[
C_1 = \frac{1}{2} C_0^2 - \frac{18 \alpha^4}{625 \beta^2}
\]  
(3.24)

Exact solution of the Korteveg - de Vries - Burgers equation is found by the formula

\[
y(z) = C_0 + \frac{12}{5} \alpha a - 12 \beta a^2 + \frac{3}{25} \alpha^2 \beta + \left( 24 \beta a - \frac{12}{5} \alpha \right) Y - 12 \beta Y^2
\]
Where $Y(z)$ is determined by the function

$$Y(z) = a \frac{1}{10} \frac{\alpha}{\beta} \tanh \left( \frac{1}{10} \frac{\alpha}{\beta} (z - C_2) \right)$$ (3.26)

It is hoped that we have two arbitrary constants again but this is not the case because substituting solutions (3.26) into (3.25) we obtain known solution with one arbitrary constant in the form

$$y(z) = C_0 + \frac{3}{25} \frac{\alpha^2}{\beta} - \frac{3}{25} \frac{\alpha^2}{\beta} \left( \tanh \left( \frac{\alpha}{10\beta} (z - C_2) \right) \right)^2 + \frac{6}{25} \frac{\alpha^2}{\beta} \tanh \left( \frac{\alpha}{10\beta} (z - C_2) \right)$$ (3.27)

The last solution is solitary wave in the form of kink with one arbitrary constant at given $\beta$ and $\nu$. This solution was found in work [4] and rediscovered more than once later.

4 Conclusion

Let us emphasize in brief the results of this work. In this paper we presented new approach to look for exact solutions of nonlinear ordinary differential equations that we called the simplest equation method. The idea of our approach is to use simplest nonlinear equation with known general solution in order to express special solution of nonlinear differential equation of higher order. This method was applied to look for exact solitary waves of the Korteveg - de Vries - Burgers equation. Application of our method allowed us to find new exact solitary waves with two arbitrary constants of the Korteveg - de Vries - Burgers equation.

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