CHIP-FIRING ON TREES OF LOOPS

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ABSTRACT. Cools, Draisma, Payne, and Robeva proved that generic metric graphs that are “paths of loops” are Brill-Noether general. We show that Brill-Noether generality does not hold for “trees of loops”: the only trees of loops that are Brill-Noether general are paths of loops. We study various notions of generality and examine which of these graphs satisfy them.

1. INTRODUCTION

Let \( \Gamma \) be a compact tropical curve, or a metric graph. The polyhedral subset \( W^r_d(\Gamma) \) of \( \text{Pic}^d(\Gamma) \), consisting of linear equivalence classes of divisors of degree \( d \) and rank at least \( r \), has expected dimension

\[
\rho(g,r,d) = g - (r + 1)(g - d + r).
\]

This expectation is formalized by the notion of Brill-Noether generality, in which a graph \( \Gamma \) is Brill-Noether general if \( W^r_d(\Gamma) \) is empty whenever \( \rho < 0 \) and \( \dim(W^r_d(\Gamma)) = \min\{\rho, g\} \) whenever \( \rho \geq 0 \). Very few examples of Brill-Noether general graphs are known. The most studied is the set of metric graphs that are combinatorially a path, or chain, of \( g \) loops, with generic edge lengths. In [5], these are shown to be Brill-Noether general. A natural combinatorial generalization is to consider metric graphs \( \Gamma \) that are trees of \( g \) loops, with generic edge lengths. In this paper we examine these graphs and show that the expectation \( \rho \) for the dimension of \( W^r_d(\Gamma) \) is never accurate unless \( \Gamma \) is a path of loops.

Theorem A. A tree of loops \( \Gamma \) is Brill-Noether general if and only if \( \Gamma \) is a path of loops.

This negative result inspires the definition of other notions of Brill-Noether generality. One natural approach is to weaken Brill-Noether generality to the boolean condition that \( \rho \) is nonnegative if and only if \( W^r_d(\Gamma) \) is nonempty. We refer to this boolean condition as weak Brill-Noether generality. However, examining this condition for general trees of loops gives another negative result.

Theorem B. Let \( \Gamma \) be a tree of loops of genus \( g \) such that the longest path of loops consists of at most \( g - 2 \) loops. Then \( \Gamma \) is not weakly Brill-Noether general.

Another approach, introduced in [7], is to examine the Brill-Noether rank \( w^r_d(\Gamma) \) as opposed to the dimension of the space \( W^r_d(\Gamma) \). The Brill-Noether rank also has expected value \( \rho \). Similarly, the condition of rank Brill-Noether generality is identical to the condition of Brill-Noether generality, but measuring \( w^r_d(\Gamma) \) (instead of \( \dim(W^r_d(\Gamma)) \)) against \( \rho \). The authors in [7] show that for \( \Gamma \) a loop of loops of genus 4, \( w^3_1(\Gamma) = 0 = \rho(4,1,3) \). As a result, rank Brill-Noether generality may hold in some cases where geometric Brill-Noether generality does not. This paper concludes by introducing a possible technique for examining the rank Brill-Noether generality of trees of loops.

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2. Background

We will be working within the realm of divisors on metric graphs. A metric graph $\Gamma$ is a compact connected metric space such that for all points $p \in \Gamma$, $p$ has a neighborhood isometric to a star-shaped set. A divisor $D$ on a metric graph $\Gamma$ is an element of the free abelian group generated by $\Gamma$; we write $\text{Div}(\Gamma)$ for the group of these divisors under addition. For a divisor $D = a_1 v_1 + \cdots + a_n v_n$ with $a_i \in \mathbb{Z}$ for all $i$, the degree of $D$ is the sum of the coefficients $a_1 + \cdots + a_n$, and a divisor is called effective if all coefficients $a_i$ are nonnegative. We denote by $D(v)$ the coefficient of $v$ in the divisor $D$, so that

$$D = \sum_{v \in \Gamma} D(v) \cdot v.$$ 

For $f$ a continuous piecewise linear function on $\Gamma$ with integer slopes, we may consider the divisor of $f$ given by

$$\text{div}(f) = \sum_{v \in \Gamma} \text{ord}_v(f) \cdot v$$

where for all $v \in \Gamma$, $\text{ord}_v(f)$ is the sum of the incoming slopes of $f$ at $v$. Divisors of piecewise linear functions are called principal, and $\text{Prin}(\Gamma)$ denotes the additive group of principal divisors.

Two divisors $D$ and $D'$ on a metric graph $\Gamma$ are said to be equivalent, chip-firing equivalent, or linearly equivalent, written $D \sim D'$, if $D - D' \in \text{Prin}(\Gamma)$. Informally, $D$ and $D'$ are equivalent if and only if one can move from $D$ to $D'$ via moves in the chip-firing game on metric graphs. In this game, a divisor $D$ is thought of as a configuration of finitely many chips placed on a metric graph, with $D(v)$ the number of chips at any point $v$. Negative coefficients are taken to be piles of “anti-chips” instead, where an anti-chip and a chip cancel whenever they collide. One can then “fire” a point on the graph with a certain speed. When a point $p$ is fired, a chip is sent along every edge adjacent to $p$ at the same speed, so that each chip lands the same distance away from the point $p$. On a metric graph, any closed subset can in fact be fired, so that chips are sent along outgoing edges. The intuition of firing a closed subset exactly aligns with the concept of adding a principal divisor. Chip-firing on metric graphs is also discussed in [3], [4], [5], and [9]. In [10], Osserman examines chip-firing on metric graphs with an approach akin to the one presented here.

For a point $v \in \Gamma$ and a divisor $D$, one can consider the unique $v$-reduced divisor $D_0$ equivalent to $D$, which satisfies the following two conditions.

1. $D_0$ is effective away from $v$.
2. Let $A \subseteq \Gamma \setminus \{v\}$ be any closed connected set. Then there exists $p \in \partial A$ with $\text{outdeg}_A(p) < D(p)$. Here $\text{outdeg}_A(p)$ is the degree of $p$ in the graph $\Gamma \setminus A \cup \{p\}$. Intuitively, this condition means that no more chips may be fired towards $v$ while preserving the effectiveness of $D_0$ away from $v$.

The $v$-reduced divisor $D_0$ can be obtained from $D$ via Dhar’s burning algorithm, which is explained in [8]. For each divisor $D$ on a metric graph $\Gamma$, and for each $v \in \Gamma$, there is a unique $v$-reduced divisor equivalent to $D$. Among divisors equivalent to $D$ and effective away from $v$, the unique $v$-reduced divisor has maximal coefficient at $v$.

The Picard group $\text{Pic}^0(\Gamma)$ of $\Gamma$ is the quotient group $\text{Div}^0(\Gamma)/\sim$, which can be viewed as the $g$-dimensional torus $H_1(\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{Z}) \cong \text{Jac}(\Gamma)$ via the Abel-Jacobi map. Similarly, for any degree $d$ we define $\text{Pic}^d(\Gamma) = \text{Div}^d(\Gamma)/\sim$, which is a $\text{Pic}^0(\Gamma)$-torsor for every $d$. Moreover, $\Gamma$ and thus $H_1(\Gamma, \mathbb{R})$ is endowed with a natural “cycle intersection” bilinear form (see [9], [2], [11]).

The question at hand is then which equivalence classes of divisors contain an effective divisor, and the robustness of this containment. This concept is made rigorous by the definition of a divisor’s rank.
**Definition 2.1.** The rank \( r(D) \) of a divisor \( D \) on a metric graph \( \Gamma \) is the largest nonnegative integer \( r \) such that for every effective divisor \( E \) of degree \( r \) on \( \Gamma \), the divisor \( D - E \) is equivalent to an effective divisor. If \( D \) is not equivalent to an effective divisor, \( r(D) \) is defined to be \(-1\).

One particular divisor, known as the **canonical divisor** \( K \) on a metric graph \( \Gamma \) is defined as
\[
K = \sum_v (\deg v - 2)v,
\]
ranging over all vertices \( v \in \Gamma \). Then \( \deg K = 2g - 2 \), which can be checked by examining the Euler characteristic of \( \Gamma \). This divisor is a key part of the tropical Riemann-Roch Theorem, which holds for metric graphs and is a useful result in the study of divisors on metric graphs.

**Theorem 2.2** (Tropical Riemann-Roch Theorem). ([9], [6])

Let \( D \) be a divisor on a metric graph \( \Gamma \) of genus \( g \). Then
\[
r(D) - r(K - D) = \deg(D) + 1 - g.
\]

Our main application of the tropical Riemann-Roch Theorem will be the case when \( \Gamma \) is a single loop. In that case the divisor \( K \) has no chips whatsoever. For any divisor \( D \) with positive degree, \( K - D \) then has negative degree. Thus the rank \( r(K - D) = -1 \), so the Riemann-Roch theorem tells us that
\[
r(D) + 1 = \deg(D) + 1 - 1 = \deg(D).
\]
So for the graph \( \Gamma \) consisting of a single loop, for any divisor \( D \) with \( \deg(D) \geq 1 \), the rank \( r(D) \) is given by \( r(D) = \deg(D) - 1 \). We will frequently use this fact.

For each degree \( d \) and rank \( r \), we would like to examine \( W^r_d(\Gamma) \subseteq \text{Pic}^d(\Gamma) \), which denotes the set of all divisors of \( \Gamma \) that have degree exactly \( d \) and rank at least \( r \). The set \( W^r_d(\Gamma) \) is a polyhedral subset of \( \text{Pic}(\Gamma) \), but it is not necessarily pure dimensional (see, for example, [7]). It is then natural to ask what its dimension is, where the dimension of \( W^r_d(\Gamma) \) is defined as the largest dimension of any cell in \( W^r_d(\Gamma) \). If \( \Gamma \) has genus \( g \), the algebraic-geometric analogue suggests that the dimension of \( W^r_d(\Gamma) \) is the Brill-Noether estimate \( \rho(g, r, d) = g - (r + 1)(g - d + r) \). We spend the remainder of this paper examining the accuracy of this estimate for metric graphs that are combinatorially trees of loops.

3. **Geometric Brill-Noether Generality**

**Definition 3.1.** A metric graph \( \Gamma \) is **geometric Brill-Noether general** if:

1. \( W^r_d(\Gamma) \) is empty whenever \( \rho(g, r, d) \) is negative.
2. \( W^r_d(\Gamma) \) has dimension \( \min\{\rho, g\} \) whenever \( \rho(g, r, d) \) is nonnegative.

The authors in [5] have shown that a general path of loops is geometric Brill-Noether general. This leads to the question of whether or not the same holds for trees of loops, which we address using inequalities. These inequalities, presented in [3.4] might be of independent interest. However, we will first present a tree of loops of small genus that is not geometric Brill-Noether general.

**Lemma 3.2.** Let \( \Gamma \) be a tree of loops of genus 4 that is not a path of loops. Then \( \Gamma \) is not geometric Brill-Noether general; in particular, \( \dim(W^3_d(\Gamma)) > \rho(4, 1, 3) \).

**Proof.** Since \( \Gamma \) is a tree of loops of genus 4 that is not a path of loops, \( \Gamma \) must be given by the following picture with some edge lengths.
For any angle $0 \leq \theta < 2\pi$, let $D_\theta$ be the divisor on $\Gamma$ of the form

In other words, $\Gamma$ has one chip at each of $A$ and $B$ and a third chip at an angle $\theta$ on the central circle. Each $D_\theta$ has degree 3. Let $E$ be any divisor of degree 1; then $E$ consists of one chip at some point $p \in \Gamma$. Crucially, the chip-firing game may be played independently on any loop, by firing all points on one of the other loops whenever a chip is in danger of going from one to the next. Then if $p$ is a point in the central loop, $D_\theta - E = D_\theta - (p)$ has degree 2 when restricted to the central loop, which by Riemann-Roch is linearly equivalent to an effective divisor. If $D_\theta$ is $A$-, $B$-, or $C$-reduced, it has at least two chips at $A$, $B$, or $C$, respectively. Thus if $p$ is in the right, left, or bottom loop, we may take the $A$-, $B$- or $C$-reduced divisor equivalent to $D_\theta$, respectively. The degree of this divisor with one chip removed from $p$, and restricted to the right, left, or bottom loop, is 1. By Riemann-Roch this loop is linearly equivalent to an effective divisor, so $D_\theta - E$ must be linearly equivalent to an effective divisor.

In particular, for all $\theta$, $D_\theta$ has rank at least 1. Thus we have a one-dimensional subset of $W^1_3(\Gamma)$, with parameter $\theta$, so $\dim(W^1_3(\Gamma)) \geq 1$. However, the genus of $\Gamma$ is 4, and $\rho(4,1,3) = 0 < 1$. □

**Definition 3.3.** Suppose $\Gamma_1$ and $\Gamma_2$ are metric graphs. For any $p \in \Gamma_1$ and $q \in \Gamma_2$, the *wedge sum* $\Gamma_1 \wedge_{p,q} \Gamma_2$ is the metric graph obtained by gluing $\Gamma_1$ and $\Gamma_2$ via the identification $p \sim q$.

As mentioned in the proof of Lemma 3.2, chip-firing may be performed independently on either side of a wedge point. It is therefore possible to examine the relationship between different values of $W^r_d$ for a wedge sum and its summands. For $\Gamma$ a metric graph and $\Gamma_1 \subset \Gamma$ a metric subgraph, for any divisor $D \in \text{Div}(\Gamma)$, we will denote by $D|_{\Gamma_1}$ the *restriction* of $D$ to $\Gamma_1$, namely the divisor $D_1$ on $\Gamma_1$ with $D_1(p) = D(p)$ for all $p \in \Gamma_1$.  

\[ \Gamma_1 \quad q \quad \Gamma_2 \]
Theorem 3.4. Let $\Gamma$ be any metric graph, $\Gamma_2$ a loop, and $\Gamma = \Gamma_1 \land \Gamma_2$ an arbitrary wedge sum with wedge point $q$. If $W^r_d(\Gamma_1)$ is nonempty, then

$$\dim(W^r_{d+1}(\Gamma)) \geq \dim(W^r_d(\Gamma_1)) + 1$$

Proof. Let $D \in W^r_d(\Gamma_1)$ and $p \in \Gamma_2$ be arbitrary; we may assume, without loss of generality, that $D$ is $q$-reduced. We claim $r(D + (p)) \geq r$ as a divisor on $\Gamma$. To see this, let $E \in \text{Div}(\Gamma)$ be any effective divisor of degree $r$, which is $q$-reduced without loss of generality, and denote $A := D + (p) - E$. Let $A_1$ and $A_2$ be the unique divisors on $\Gamma$ such that:

- $A = A_1 + A_2$,
- $\text{supp}(A_i) \subset \Gamma_i$ for $i = 1, 2$,
- $\deg(A_1) = d - r$ and $\deg(A_2) = 1$

Since chip firing moves can be conducted independently on $\Gamma_1$ and $\Gamma_2$, to show that $A$ is equivalent to an effective divisor, we need only show that $A_1$ and $A_2$ are equivalent to effective divisors on $\Gamma_1$ and $\Gamma_2$ respectively. By construction, $A_1 = D - E_1$ for some effective divisor $E_1$ of degree $r$ with $\text{supp}(E_1) \subset \Gamma_1$. Because $D$ has rank $r$ in $\Gamma_1$, it follows $A_1$ is equivalent to an effective divisor on $\Gamma_1$. The Riemann-Roch theorem implies $A_2$ is equivalent to an effective divisor on $\Gamma_2$, since $\deg(A_2) = 1 = \text{genus}(\Gamma_2)$. Thus, we conclude $D + (p)$ has rank at least $r$ on $\Gamma$.

We may identify $W^r_d(\Gamma_1)$ with a polyhedral subset of $W^r_d(\Gamma)$ and $W^0_1(\Gamma_2)$ with a polyhedral subset of $W^0_1(\Gamma)$ via the natural inclusion. Then, working inside $\text{Pic}^{d+1}(\Gamma)$, the above argument furnishes an injective map from the Minkowski sum $W^r_d(\Gamma_1) + W^0_1(\Gamma_2)$ into $W^r_{d+1}(\Gamma)$. It follows that

$$\dim(W^r_{d+1}(\Gamma)) \geq \dim(W^r_d(\Gamma_1) + W^0_1(\Gamma_2)) = \dim(W^r_d(\Gamma_1)) + 1$$

where the second equality follows from the fact that images under the Abel-Jacobi map of the edges of $\Gamma_1$ are orthogonal to the image of $\Gamma_2$ with respect to the “cycle intersection” bilinear form, since the intersection of $\Gamma_2$ and any subset of $\Gamma_1$ is at most a point.

This theorem and technique is all that is necessary to prove Theorem A, which we restate and prove.

Theorem 3.5. A tree of loops $\Gamma$ is geometric Brill-Noether general if and only if $\Gamma$ is a path of loops.

Proof. If $\Gamma$ is a path of loops, then $\Gamma$ is geometric Brill-Noether general from the main result of [5]. Now suppose that $\Gamma$ is a tree of loops, but not a path of loops; we will show $\Gamma$ cannot be geometric Brill-Noether general. Since $\Gamma$ is not a path, it must contain a subgraph of genus 4 that is a tree of loops but not a path of loops, denoted $\Gamma_0 \subset \Gamma$. Trees are connected, so we can construct $\Gamma_i$ by adding one loop at a time to $\Gamma_0$. Let $\Gamma_i$ be the $i$th step in this process, once $i$ cycles have been added to $\Gamma_0$. Then $\Gamma_i$ has genus $i + 4$ for all $i$, and if $\Gamma$ has genus $g$, then $\Gamma = \Gamma_{g-4}$. We will prove by induction that $\Gamma_i$ is not geometric Brill-Noether general for all $i$, and thus that $\Gamma$ is not geometric Brill-Noether general.

As a base case, $\Gamma_0$ is not geometric Brill-Noether general, since by Lemma [3,2] $\dim(W^1_3(\Gamma_0)) \geq 1 > \rho(4, 1, 3) = 0$. Now assume that $\Gamma_i$ is not geometric Brill-Noether general, i.e. that there exist $r, d$ such that $\dim(W^r_d(\Gamma_i)) > \rho(i + 4, r, d)$. By Theorem 3.4 we see

$$\dim(W^r_{d+1}(\Gamma_{i+1})) \geq \dim(W^r_d(\Gamma_i)) + 1 \geq \rho(i + 4, r, d) + 1$$

$$= \rho(i + 5, r, d + 1) + 1$$

Thus, $\Gamma_{i+1}$ is not geometric Brill-Noether general, so by induction, $\Gamma$ is not geometric Brill-Noether general. \qed

4. Weakly Geometric Brill-Noether Generality

Our second notion of Brill-Noether generality is a weakening of geometric Brill-Noether generality. One might wonder if $\rho$ is at minimum an indicator on trees of loops $\Gamma$ of whether the set $W_d^r(\Gamma)$ is empty.

Definition 4.1. A metric graph $\Gamma$ is weakly geometric Brill-Noether general if $W_d^r(\Gamma)$ is nonempty whenever $\rho \geq 0$ and empty otherwise.

Since this condition is strictly weaker than that of geometric Brill-Noether generality, it is conceivable that it all trees of loops would be weakly geometric Brill-Noether general. Nevertheless, most trees of loops still do not satisfy this condition.

Proposition 4.2. Let $\Gamma$ be a tree of loops of genus $g$ such that the longest path of loops consists of at most $g - 2$ loops. Then $\Gamma$ is not weakly geometric Brill-Noether general.

Proof. We will consider two cases. Let $l$ be the number of loops in the longest path of loops in $\Gamma$. First, consider the case where $l$ is even. In this case, let $q$ be the intersection point of the middle two loops of a path of length $l$, as pictured.

Let $D = \left( \frac{l}{2} + 1 \right) (q)$. By the tropical Riemann-Roch theorem, the rank of $D$ when restricted to either of the loops containing $q$ is $\frac{l}{2}$. In particular, for $s$ the connection point as in the diagram, $D - \frac{l}{2}(s)$ is equivalent to an effective divisor. In other words, $D$ is equivalent to some divisor $D'$ with $\frac{l}{2}$ chips placed on $s$. The rank of $D'$ when restricted to one of the loops containing $s$ is then $\frac{l}{2} - 1$, so $D'$ has rank $\frac{l}{2} - 1$ when restricted to one of these loops. Thus $D' - \left( \frac{l}{2} - 1 \right) (t)$ is equivalent to an effective divisor, since chip-firing can be performed on each loop in isolation, so $D'$ is equivalent to some effective divisor $D''$ with $\frac{l}{2} - 1$ chips placed on $t$.

We will show by induction on $k$ that for $L$ any loop at distance $k \leq \frac{l}{2}$ from one of the two loops containing $q$, $D$ is equivalent to some effective divisor with degree at least $\frac{l}{2} - k + 1$ when restricted to the loop $L$. The base case is precisely the argument that $\frac{l}{2}$ chips may be placed on the point $s$ above. For the inductive step, let $L$ be a loop at distance $k$ from one of the loops containing $q$, and let $L'$ be the loop at distance $k - 1$ that is adjacent to $L$. Let $v$ be the intersection point between $L'$ and $L$. By the inductive hypothesis, $D$ is equivalent to an effective divisor $D^{(k-1)}$ with degree at least $\frac{l}{2} - k + 2$ when restricted to $L'$. Since $v$ is in both loops $L$ and $L'$, it suffices to show that $D^{(k-1)}$...
is equivalent to some divisor with \( \frac{l}{2} - k + 1 \) chips placed on \( v \), or that \( D^{(k-1)} - (\frac{l}{2} - k + 1) \) (\( v \)) is an effective divisor. Since \( k \leq \frac{l}{2} \), \( \deg(D^{(k-1)}) \geq 2 \), so the rank of \( D^{(k-1)} \) is

\[
\rho(D^{(k-1)}) = \deg(D^{(k-1)}) - 1 = \frac{l}{2} - k + 1.
\]

But then \( D^{(k-1)} - (\frac{l}{2} - k + 1) \) (\( v \)) is equivalent to an effective divisor, just as desired. This completes the inductive argument.

Any loop in \( \Gamma \) is connected to a loop containing \( q \) via a path of loops of length at most \( \frac{l}{2} - 1 \), since otherwise we would have a longer maximal path. Let \( v \) be an arbitrary point in \( \Gamma \). Let \( L \) be the closest loop to \( q \) containing \( v \) and let \( k \leq \frac{l}{2} - 1 \) be the distance between \( L \) and a loop containing \( q \). Then by our inductive argument, \( D \) is equivalent to an effective divisor \( D^{(L)} \) with at least \( \frac{l}{2} - k + 1 \) chips placed on \( L \). The bound \( k \leq \frac{l}{2} - 1 \) implies that \( D^{(L)} \) has at least 2 chips placed on \( L \) itself. But then by the tropical Riemann-Roch Theorem, \( D^{(L)} \) has rank 1 when restricted to \( L \), so \( D^{(L)} - (v) \) is linearly equivalent to an effective divisor. Since \( D^{(L)} \) and \( D \) are linearly equivalent, \( D - (v) \) must also be linearly equivalent to an effective divisor. Thus \( D \) has rank 1, so

\[
W_{\frac{l}{2}+1}(\Gamma) \neq \emptyset.
\]

However, since \( \Gamma \) has genus \( g \) where \( l \leq g - 2 \),

\[
\rho(g, 1, \frac{l}{2} + 1) = g - 2 \cdot \left( g - \frac{l}{2} - 1 + 1 \right)
= -g + l
\leq -2 < 0.
\]

Thus \( \Gamma \) is not Brill-Noether general.

Now we consider the case where \( l \) is odd; it is very similar. In this case, let \( q \) be any point on the middle loop of the longest path. Let \( D = \frac{l+3}{2}(q) \). By an inductive argument identical to that of the even case, if \( L \) is any loop at distance \( k \leq \frac{l-1}{2} \) from the center loop, then \( D \) is equivalent to some effective divisor with degree at least \( \frac{l+3}{2} - k \) when restricted to \( L \). All loops are at distance at most \( \frac{l-1}{2} \) from the center loop, so for any loop \( L \), \( D \) is equivalent to some effective divisor \( D^{(L)} \) with degree at least \( \frac{l+3}{2} - \frac{l-1}{2} = 2 \) when restricted to \( L \). Then for any point \( v \in L \), \( D^{(L)} - (v) \) is linearly equivalent to some effective divisor just as above, so \( D - (v) \) must also be linearly equivalent to an effective divisor. Thus \( D \) has rank 1, so

\[
W_{\frac{l+1}{2}}(\Gamma) \neq \emptyset.
\]

However, \( \Gamma \) has genus \( g \), with \( l \leq g - 2 \), so

\[
\rho(g, 1, \frac{l+3}{2}) = g - 2 \cdot \left( g - \frac{l+3}{2} + 1 \right)
= g - 2g + l + 3 - 2
= -g + l + 1
\leq -1 < 0.
\]

Thus \( \Gamma \) is not Brill-Noether general. \( \square \)

However, note that this argument proves that for negative values of \( \rho \), the set \( W'_f(\Gamma) \) may still be nonempty. This is in fact the only way in which weakly geometric Brill-Noether generality may
be violated. In [11], van der Pol proved that for trees of loops $\Gamma$, referred to in his paper as cactus graphs, whenever the value of $\rho$ is positive, the set $W_d^r(\Gamma)$ is nonempty.

5. Rank Brill-Noether Generality

Following the work done in [7], we examine a more combinatorial definition of rank.

**Definition 5.1.** Suppose $\Gamma$ is a metric graph and $r, d$ are natural numbers such that $W_d^r(\Gamma)$ is nonempty. The Brill-Noether rank $w^r_d(\Gamma)$ is the largest integer $k$ such that for every effective divisor $E$ of degree $r + k$, there exists $D$ of rank at least $r$ and degree $d$ such that $D - E$ is effective.

**Definition 5.2.** A metric graph $\Gamma$ of genus $g$ is rank Brill-Noether general if:

1. $w^r_d(\Gamma) = -1$ whenever $\rho(g, r, d) < 0$.
2. $w^r_d(\Gamma) = \rho(g, r, d)$ whenever $0 \leq \rho(g, r, d) \leq g$.

The negative result obtained in examining weak Brill-Noether generality carries over, so certainly rank Brill-Noether generality is out of reach for most trees of loops.

**Theorem 5.3.** Let $\Gamma$ be a tree of loops of genus $g$ such that the longest path of loops consists of at most $g - 2$ loops. Then $\Gamma$ is not rank Brill-Noether general.

**Proof.** This follows immediately from Proposition 4.2. If for some $r, d$, the set $W_d^r(\Gamma) \neq \emptyset$, then $w^r_d(\Gamma) \geq 0$. If $\Gamma$ has genus $g$ where the longest path of loops consists of at most $g - 2$ loops, then $\Gamma$ is not weakly geometric Brill-Noether general and in particular there exist $r, d$ such that $W_d^r(\Gamma) \neq \emptyset$ despite $\rho(g, r, d)$ being negative. Thus $w^r_d(\Gamma) \geq 0$ despite $\rho(g, r, d)$ being negative, so $\Gamma$ is not rank Brill-Noether general.

We do not know the relationship between $\rho(g, r, d)$ and $w^r_d$ for a general tree of loops $\Gamma$ when $\rho > 0$. However, there is a constraint relating $w^r_d$ of a metric graph and $w^r_{d+1}$ of the graph formed from gluing a loop onto that metric graph. This constraint may prove useful in further explorations.

**Theorem 5.4.** Let $\Gamma_1$ be any metric graph, $\Gamma_2$ a loop, and $\Gamma = \Gamma_1 \land \Gamma_2$ an arbitrary wedge sum (with wedge point $q$). If $W_d^r(\Gamma_1)$ is nonempty, then

$$w^r_d(\Gamma_1) \leq w^r_{d+1}(\Gamma) \leq w^r_{d-1}(\Gamma_1).$$

**Proof.** Set $k = w^r_d(\Gamma_1); k \geq 0$, since $W_d^r(\Gamma_1)$ is nonempty.

First, we will prove the lower bound. Let $E \in \text{Div}^{r+k}(\Gamma)$ be an arbitrary effective divisor of degree $r + k$; we will construct a divisor $D \in W_d^r(\Gamma_1)$ with $D - E$ effective. We may first $q$-reduce $E$ to obtain $E' \sim E$, where $E'$ must still be effective and must contain either one chip or zero chips placed on $\Gamma_2 \setminus \{q\}$. Note that if there exists a divisor $D'$ with $D' - E'$ effective, we can let $D = (D' - E') + E = D' + (E - E')$, which is then a divisor, still in $W_d^r(\Gamma_1)$, with $D - E = D' - E'$ effective. Thus it suffices to find a satisfactory divisor for the $q$-reduced case.

If $E'$ contains exactly one chip on a point $p \in \Gamma_2 \setminus \{q\}$, let $D_2$ be the divisor $(p)$ on $\Gamma_2$. $E'|_{\Gamma_1}$ is an effective divisor of degree $r + k - 1$, so there is a divisor $D_1 \in W_d^r(\Gamma_1)$ with $D_1 - E'|_{\Gamma_1}$ effective. Then $D' = D_1|_{\Gamma} + D_2|_{\Gamma}$ is an element of $W_d^r(\Gamma)$ by the argument in Theorem 3.4, with $D' - E'$ effective; thus there is a divisor $D \in W_d^r(\Gamma_1)$ with $D - E$ effective.

Now assume that $E'$ contains no chips on $\Gamma_2 \setminus \{q\}$. Then $E'|_{\Gamma_1}$ has degree $r + k$, so we can pick a divisor $D_1 \in W_d^r(\Gamma_1)$ with $D_1 - E'|_{\Gamma_1}$ effective. Let $p$ be any point in $\Gamma_2$; then the divisor $D' = D_1|_{\Gamma} + (p)$ is an element of $W_d^r(\Gamma)$ with $D' - E'$ effective. Since $E \sim E'$, there exists a divisor $D \in W_d^r(\Gamma)$ with $D - E$ effective as well, so $w^r_{d+1}(\Gamma) \geq k = w^r_d(\Gamma_1)$. 

We will now prove the upper bound. Let \( l = w_{d+1}^{r} (\Gamma) \). We will show that \( l \) is a lower bound for \( w_{d+1}^{r-1} (\Gamma_1) \), or equivalently, that for any effective divisor \( E_1 \) on \( \Gamma_1 \) of degree \( r - 1 + l \), there exists a divisor \( D_1 \in W_{d+1}^{r-1} (\Gamma_1) \) with \( D_1 - E_1 \geq 0 \).

Fix any effective divisor \( E_1 \) on \( \Gamma_1 \) of degree \( r - 1 + l \). Just as above, we may restrict to addressing the case when \( E_1 \) is \( q \)-reduced, so assume that this is the case. Let \( E = (E_1 + (q)) |^\Gamma \), which is then an effective divisor of degree \( r + l \) on \( \Gamma \). Note that \( E \) remains \( q \)-reduced. Since \( w_{d+1}^{r-1} (\Gamma_1) = l \) and \( E \) is \( q \)-reduced, there exists a \( q \)-reduced divisor \( D \) of degree \( d + 1 \) on \( \Gamma \) with \( D - E \geq 0 \).

Since \( D \) is a \( q \)-reduced divisor on \( \Gamma \), \( D \) either has one chip on the loop \( \Gamma \setminus \Gamma_1 = \Gamma_2 \setminus \{ q \} \), or all chips are on \( \Gamma_1 \). If \( D \) has one chip on \( \Gamma_2 \setminus \{ q \} \), let \( p \) be the point at which this chip is located. If not, let \( p = q \).

Then \( D - (p) \) is a divisor with no chips on \( \Gamma \setminus \Gamma_1 \). Since \( D \) has rank at least \( r \), as an element of \( W_{d+1}^{r} (\Gamma) \), \( D - (p) \) has rank at least \( r - 1 \), since any additional \( r - 1 \) chips may be removed from \( D \) after the removal of \( (p) \), and the result will still be equivalent to an effective divisor. Thus \( D \in W_{d+1}^{r-1} (\Gamma) \); since \( D - (p) \) has support on \( \Gamma_1 \), \( D_1 = D - (p) \in W_{d}^{r-1} (\Gamma_1) \), as well.

It suffices to show that \( D_1 = D - (p) \) satisfies \( D_1 - E_1 \geq 0 \); then for every choice of \( E_1 \) we will have found a divisor \( D_1 \in W_{d+1}^{r-1} (\Gamma_1) \) with \( D_1 - E_1 \geq 0 \). If \( p = q \), then
\[
D_1 - E_1 = D - (p) - (E - (q)), \quad \text{since} \ E = E_1 + (q) \\
= D - E - (q) + (q) \\
= D - E \geq 0, \quad \text{just as desired.}
\]

Now if \( p \neq q \), we know that \( D|_{\Gamma_1} = D_1 = D - (p) \). Since \( E = E_1 + (q) \) lies entirely on \( \Gamma_1 \), \( D|_{\Gamma_1} - E \geq 0 \), since \( D - E \geq 0 \). But then
\[
D|_{\Gamma_1} - E \geq 0 \\
\Rightarrow D_1 - E \geq 0 \\
\Rightarrow D_1 - E_1 - (q) \geq 0 \\
\Rightarrow D_1 - E_1 \geq 0, \quad \text{just as desired.}
\]

Thus we have proven that \( D_1 - E - 1 \geq 0 \), so \( w_{d+1}^{r-1} (\Gamma_1) \geq w_{d+1}^{r} (\Gamma) \), which is exactly the desired result. \( \square \)

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