Multifractal properties of growing networks

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\textbf{Abstract.} – We introduce a new family of models for growing networks. In these networks new edges are attached preferentially to vertices with higher number of connections, and new vertices are created by already existing ones, inheriting part of their parent’s connections. We show that combination of these two features produces multifractal degree distributions, where degree is the number of connections of a vertex. An exact multifractal distribution is found for a nontrivial model of this class. The distribution tends to a power-law one, $\Pi(q) \sim q^{-\gamma}$, $\gamma = \sqrt{2}$ in the infinite network limit. Nevertheless, for finite networks’s sizes, because of multifractality, attempts to interpret the distribution as a scale-free would result in an ambiguous value of the exponent $\gamma$.

Networks of various kinds, such as the World Wide Web, citation network of scientific papers, social networks, neural networks, etc. (see \textsuperscript{[1–8]}) are very popular objects of studies nowadays. There crucial role is played by the degree distribution function (DDF) $\Pi(q)$, where the degree $q$ is the number of connections of a vertex (sometimes it is called connectivity). Data, obtained from the observations of many existing networks were interpreted as if they are scale-free, that is degrees of their vertices were distributed according to $\Pi(q) \sim q^{-\gamma}$ \textsuperscript{[5]}. Such networks have some specific properties, as compared, e.g., to classic random graphs \textsuperscript{[9]}, where degree distribution follows a Poisson’s law. In particular, they are extremely resilient to random breakdowns \textsuperscript{[10–12]}.

To describe a scale-free growing network, the mechanism of preferential linking \textsuperscript{[5]} was proposed. This principle is similar to the one introduced in the well known Simon model \textsuperscript{[13]}, used to explain power-law distributions in various social and economic systems. In fact, all these models belong to the class of stochastic multiplicative processes \textsuperscript{[14]}. Here we show that a natural generalization of studied models of evolving networks leads to multifractal degree distribution with much reacher properties than a power-law one.

In general, as network is growing, two parallel processes take place: (i) New edges are formed between vertices. They attach preferentially to vertices of a growing network with a

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high number of connections. (ii) New vertices appear. Several types of such a model were proposed, that differ in details of how new vertices and edges are entered into the network. They produced scale-free networks with the $\gamma$ exponent of the DDF, either in the range $(2, \infty)$, which means, that the average degree of a vertex remains finite, or $\gamma < 2$, when number of edges is growing faster then of vertices.

The questions are: whether all the scenarios of the evolution of networks, produced by preferential linking, lead to scale-free DDF? What other types of DDF do arise? These questions were considered in several recently published papers. However, in most of the models, where the idea of preferential linking was used, it was assumed, that new vertices appear with the same properties, independent of the state of network at this moment, i.e., all vertices are born equal. (Nevertheless, see the counter-example of a “breathing network” in Ref. [20]). One could say, that new vertices are created by some external source, whose properties are independent of the network’s current state. Here we put forward a different concept: new vertices are born with random properties, which reflects the state of the network at the moment of birth. In this respect, one could say, that they are created by the network itself.

Our model may be formulated as follows: (i) New edges are introduced between existing vertices (see Fig.1a). This happens with probability $m$ per unit time. We assume the linear preferential linking: the probability, a new edge to point to a vertex $i$ is $p_i = q_i/Q$. Here $Q(t) = \sum_i q_i$ is total degree (number of edges) of the network. We shall look only at the in-degrees (numbers of incoming edges) of vertices, putting aside the question of what vertices these edges come from. For example, one may assume that an edge comes from any vertex with equal probability. (ii) New vertices appear with probability $n$ per unit time (Fig.1b) (in the following we assume $n = 1$, which can always be done by the proper choice of the time scale). We suppose that every new vertex has a parent — some vertex, randomly chosen among $N(t)$ existing ones. Degree of a daughter vertex is assumed to be a random number, whose distribution depends upon parent’s degree. Namely, we assume, that every edge, pointing to the parent with some probability $c$ is inherited by the daughter. We emphasize, that here inheritance means copying of edges, — parent loses nothing, simply some new edges are created, pointing at its heir. If parent’s degree is $q$, the probability for its heir to have the degree $q_1 \leq q$ is $\left(\frac{q}{q_1}\right)c^{q_1} (1-c)^{q-q_1}$.

It is the way of introducing new vertices into the network, that makes our model differ significantly from previously studied ones. Usually it is assumed, that new vertices have some preset properties, which are independent of the current state of the network. For example, in most cases new vertices are entered with some given initial degree $q_0$ (see Fig.1a). In our model new vertices are being born by network’s components themselves, and with properties (initial degrees), reflecting the current state of the network.

DDF may be introduced as: $\Pi(t,q) = \left\langle \left[1/N(t)\right] \sum_{j=1}^{N(t)} \delta_K [q_j (t) - q] \right\rangle$, where $\delta_K (q - q')$ is the notation for Kronecker’s $\delta$-symbol. Averaging is over the statistical ensemble of networks, whose evolution is governed by the above described rules. Here $N(t)$ is an integer, growing in time in a random fashion, — introduction of new vertices (as well as new links) may be considered as a random process (e.g., Poissonian one). Its average is $\bar{N} = \int dt n(t) = t$. For $t \gg 1$, this function may be shown to obey the following master equation:

$$\frac{\partial}{\partial t} \Pi(t,q) + \frac{m}{\bar{q}(t)} [q \Pi(t,q) - (q - 1) \Pi(t, q - 1)] = \sum_{q' = q}^{\infty} \left(\frac{q'}{q}\right)c^q (1-c)^{q'-q} \Pi(t,q') . \quad (1)$$

Here $\bar{q}(t) = Q(t)/t$ is the average degree of network’s vertices. The above equation may easily be derived, but, as one can see, it is a direct generalization of the master equation, introduced...
Coefficient $c$ reflects the “succession right” of a given network. In principle, it can also be a random number, characterized by some distribution density $h(c)$. The general evolution equation acquires much simpler form, if to pass to a new representation of the DDF, which is a bit modified Z-transform: $\Pi(t, q) \rightarrow \Phi(t, y) = \sum_{q=0}^{\infty} \Pi(t, q) (1 - y)^q = \langle 1/N(t) \rangle \sum_{j=1}^{N(t)} (1 - y)^{\eta_j}$. Then the evolution equation becomes:

$$
\frac{\partial}{\partial t} \Phi(t, y) - (1 - c) y (1 - y) \frac{\partial \Phi(t, y)}{\partial y} - \int_0^1 dc \: h(c) \Phi(t, cy) = 0.
$$

It is to be supplied with the initial condition: $\Phi(t_0, y) = \Phi_0(y)$, $t_0 \gg 1$ (recall, that the equation is valid for $t \gg 1$). After rescaling of the size variable $t \rightarrow t/t_0$ the initial condition becomes $\Phi(1, y) = \Phi_0(y)$.

General solution of Eq. (1) may be found in the continuous approximation, when it is assumed, that $\Pi(t, q)$ is a slowly varying function of $q$. Then this equation, taking into account the above introduced randomness of $c$, takes the form:

$$
\frac{\partial}{\partial t} t\Pi(t, q) + (1 - c) \frac{\partial}{\partial q} q\Pi(t, q) - \int_0^1 dc \: h(c) \Pi\left(t, \frac{q}{c}\right) = 0.
$$

It can easily be solved after transition to Mellin’s representation with respect to $q$: $\Xi(t, \xi) = \int_0^\infty dq \: \Pi(t, q) q^{\xi - 1}$. Note, that $\Xi(t, n + 1) = M_n(t)$ are the moments of the distribution. The solution is:

$$
\Xi(t, \xi) = \Xi_0(\xi) t^{\xi(\xi - 1)} \: \tau(\xi) = (1 - c) \xi - 1 + \chi(\xi + 1), \; \chi(\xi) = \int_0^q dc \: h(c) \: c^{\xi - 1}.
$$

One can see, that moments of the distribution scale with network as $M_n \sim t^\tau(n)$, where $\tau(n)$ is a nonlinear function of $n$. These distributions are usually referred to as multifractals [21 23], as distinct from fractal ones, where $\tau(n)$ depends on $n$ linearly, $\tau(n) = (n - 1) D$, $D$ is called fractal’s dimensionality. Such a distributions were found in many objects, ranging from localized excitations in disordered solids to the distribution of matter in the Universe. Multifractals may be thought of as a statistical mixture of fractals with different dimensionalties. Indeed, if to introduce the $f(\alpha)$-spectrum of dimensionalities $\alpha$, then a mixture of $\alpha$-dimensional fractals with statistical weights $g(\alpha) t^{-f(\alpha)}$, ascribed to $\alpha$-dimensional fractal, yields the distribution moments $M_n = \int d\alpha \: g(\alpha) t^{\alpha(n - 1) - f(\alpha)}$. At large $t$ they scale with network’s size as $M_n \sim t^{\tau(n)}$. The prefactor $g(\alpha)$ gives only the coefficients of proportionality, and is not relevant. Functions $f(\alpha)$ and $\tau(n)$ are connected by the Legendre transform: $\tau(n) + f(\alpha) = \alpha n, \; n = df/da, \; a = d\tau/da$.

As an example, let us consider the case of the homogeneous distribution of the inheritance coefficient $c$, $h(c) = \theta(c) \theta(1 - c)$. The motivation for such a choice is that, as we shall see below, in this case an exact solution is possible without continuous approximation. We have $\chi(\xi) = 1/\xi$, and $\tau(\xi) = \xi/2 - 1 + 1/(\xi + 1)$. This gives multifractal $f(\alpha)$-spectrum...
to be: $f(\alpha) = (1 - \sqrt{1/2 - \alpha})^2$, where $-\infty < \alpha < 1/2$. Note, that it includes negative dimensionalities (see [24]).

If we set the initial condition for Eq. (3) as $\Pi(1, q) = \Pi_0(q) = \delta(q - q_0)$, then the solution is a Green’s function $\Pi(t; q, q_0)$, whose expression is:

$$
\Pi(t; q, q_0) = \frac{1}{q_0 t^{3/2}} \int_{-\infty+\delta}^{+i\infty+\delta} \frac{d\xi}{2\pi i} \left( \frac{q_0}{q} \right)^\xi t^{\xi/2+1/\xi}
$$

$$
= \frac{\theta(q_0 \sqrt{t} - q)}{q_0 t^{3/2}} \sqrt{\frac{\ln t}{\ln(q_0 \sqrt{t}/q)}} I_1(2 \sqrt{\ln t \ln(q_0 \sqrt{t}/q)}) + \frac{1}{t} \delta(q - q_0 \sqrt{t}),
$$

(5)

where $I_1$ is the modified Bessel’s function. If $t$ is large enough, the above formula everywhere, except the tail region $q \approx q_0 \sqrt{t}$, may be replaced by:

$$
\Pi(t; q, q_0) \approx \frac{2^{-1/4}}{q_0 t^{3/2} \sqrt{\pi \ln t}} \exp \left( 2 \sqrt{\ln t \ln(q_0 \sqrt{t}/q)} \right).
$$

(6)

At $t \to \infty$ this expression is asymptotically equal to:

$$
\Pi(t; q, q_0) \approx \frac{2^{-1/4}}{q_0 \sqrt{\pi}} \left( \frac{q_0}{q} \right)^{\sqrt{\gamma}} t^{-3/2+\sqrt{\gamma}} \ln t^{1/2}.
$$

(7)

However, this latter result is valid only if $\ln(q/q_0) \ll \sqrt{\ln t}$.

At $t \to \infty$ this region is small as compared with the one of the validity of Eq. (3), which is $\ln q \lesssim (1/2) \ln t$. This means that, in spite of that the distribution asymptotically assumes a scale-free form, $\sim q^{-\gamma}$ with $\gamma = \sqrt{2}$, at finite network’s sizes this is true only within rather restricted region of degrees, small compared with the upper cut-off $q_0 \sqrt{t}$. However, it is extremely important, that one has to compare not the sizes of the regions themselves, but their logarithms. This ratio is of the order of $1/\sqrt{\ln t}$. Even for the largest known network, the World Wide Web, whose size is $\sim 10^9$, this value is of the order of 0.2. These features make the analysis of the distribution in terms of scale-free functions ambiguous, the value of the exponent $\gamma$ becomes dependent on the choice of the degree region, from which it is extracted. The (negative) slope of a log-log plot of the distribution steadily grows with $\ln q$ from $\sqrt{2}$ at $\ln q \lesssim \sqrt{\ln t}$, until the exponential cut-off is reached at $\ln q \sim \ln t$.

There is another peculiar feature of the distribution (3), obtained in the continuum approximation. Namely, for any $q_c > 0$ at large enough $t$ the fraction of vertices with $q > q_c$ scales with $t$ as $t^{-3/2+\sqrt{2}}$. This means, that the distribution concentrates within the region of small $q$, where continuum approach is invalid. Fortunately, for the homogeneous distribution of inheritance coefficient $c$, the exact solution is possible. Indeed, for the homogeneous $h(c)$, after application of the operator $\partial y(y^r)$, Eq. (3), may be reduced to a linear partial differential equation. This equation, after the Mellin’s transformation with respect to time, $\psi(\eta, y) = \int_1^\infty dt t^{\eta-1} \Phi(t, y)$, takes the form,

$$
y^2(1 - y) \partial^2 y \psi + y (2\eta - 3y) \partial_y \psi + 2\eta \psi = -2\partial_y(\psi \Phi_0),
$$

(8)

where $\Phi_0(y)$ is the initial distribution in the Z-representation. For the Green’s function it is: $\Phi_0(y) = (1 - y)^{10}$. Let us denote Mellin’s time-transform of the Green’s function as $\Psi(\eta; q, q_0)$. Eq. (8) may be reduced to an inhomogeneous hypergeometric one after the substitution $\psi(y) = y^\zeta x(y)$, where $\zeta$ is one of the roots of the characteristic equation: $\zeta^2 -$
(1 - 2\eta) \zeta + 2\eta = 0. Here we present the result in terms of \Psi (\eta; q, k), which is the q-th term of Taylor’s series of \psi (y) around the point y = 1. After lengthy calculations, we obtain

\[
\Psi (\eta; q, k) = \begin{cases} 
-\phi_1 (\zeta_1, k) \frac{\phi_2 (\zeta_1, q) - \phi_2 (\zeta_2, q) }{\zeta_1 - \zeta_2}, & k > q > 0; \\
-\phi_2 (\zeta_1, k) - \phi_2 (\zeta_2, k), & q \geq k > 0;
\end{cases}
\]

\[
\Psi (\eta; 0, k) = -\phi_1 (\zeta_1, k); \quad \Psi (\eta; q, 0) = -\frac{\delta (q)}{\eta},
\]

where \(\zeta_1\) is the root of the characteristic equation, which is positive for \(\Re \eta < 0\), \(\zeta_2\) is the other root, and functions \(\phi_{1,2}\) may be expressed as:

\[
\phi_1 (\zeta, q_0) = \frac{4q_0 \Gamma (\zeta) \Gamma (2 + \zeta) / (1 - \zeta) \Gamma (2 + \zeta - 2 / (1 + \zeta))}{1 + \zeta + 2 / (1 + \zeta) \Gamma (2 + \zeta - 2 / (1 + \zeta))} \\
\int_0^1 \frac{dz}{z} z^{1 - 2 / (1 + \zeta)} (1 - z)^{q_0 - 1} 2 F_1 \left( 1 - \frac{2}{1 + \zeta}; 1 - \frac{2}{1 + \zeta}; 2 + \zeta - \frac{2}{1 + \zeta}; z \right),
\]

\[
\phi_2 (\zeta, q) = \frac{\Gamma \left( 1 + \zeta + \frac{2}{1 + \zeta} \right) \Gamma \left( \frac{2}{1 + \zeta} - \zeta \right) \sin \pi \zeta}{\Gamma \left( \frac{2}{1 + \zeta} - 1 \right) \Gamma \left( \frac{2}{1 + \zeta} + 1 \right)}, \pi \\
\int_0^\infty \frac{dy}{y^\zeta} (1 + y)^{q - 1} \frac{2 F_1 \left( \zeta, q + \zeta - \frac{2}{1 + \zeta}; y \right)}{2 F_1 \left( \zeta, q + \zeta - \frac{2}{1 + \zeta}; 1 \right)}.
\]

Here \(2 F_1\) is the hypergeometric function.

Two formulas may be derived from Eqs.(10,11), which indicate: i) that the results of the continuous approximation are valid for large \(q\), except for some minor corrections, and ii) that concentration of the distribution at low degree values is to be interpreted as in the large size limit, almost all vertices have zero degree. In the large \(q\), large \(t\) limit, the following expression may be written for the Green’s function:

\[
\Pi (t; q, q_0) \approx \omega' g (q_0) \frac{\ln (aq)}{(t \ln t)^{3/2}} \exp \left[ 2 \sqrt{\ln t \ln \left( \sqrt{t}/q \right)} \right], \quad g (q_0) = \left. \frac{\partial \phi_1 (\zeta, q_0)}{\partial \zeta} \right|_{\zeta = \sqrt{t} - 1},
\]

where: \(\omega' = 0.0823\ldots\) and \(a = 0.840\ldots\). It differs from Eq.3 in some details, but the main conclusions remain the same. The formula 4 is reproduced, apart from the appearance of the additional multiple \(\ln q/\ln t\) and from different numerical factors. Also, the dependence on the initial degree value \(q_0\) is different (one may reproduce completely continuous approximation result 3 from the exact solution, if to assume the limit \(q \to \infty, q_0 \to \infty, q/q_0 \) and \(t\) fixed).

Anyway, apart from some power of logarithm, DDF values for any \(q > 0\) decay as \(t^{-3/2+\sqrt{2}}\) with network’s size increase. For \(q = 0\), we have:

\[
\Pi (t; 0, q_0) \approx 1 - g (q_0) \frac{t^{-3/2+\sqrt{2}}}{2^{3/4} \sqrt{\pi} \ln^{3/2} t}.
\]

One can see from Eq. (13) that at long times the fraction of zero-degree vertices tends to 1. These vertices do not have incoming edges but only outgoing ones (recall, that degree, in
the present paper, is defined as in-degree, the number of incoming edges). They are passive constituents of the network, i.e., their degree remains unchanged all the time. Although the fraction of active vertices (with non-zero degree) tends to zero as the network grows, their total number increases with time as $t^{\sqrt{2} - 1/2} / \ln^{3/2} t$. One can introduce the DDF of active vertices, $\Pi_1(t; q, q_0)$. It follows from Eqs. (12) and (13), that in the large size limit, it tends to the distribution, $\Pi_1(q, t)$, which does not depend on the initial conditions:

$$\Pi_1(t, q) = (1 - \delta_{q,0}) \frac{\Pi(t; q, q_0)}{1 - \Pi(t; 0, q_0)} \to \omega t^{-\sqrt{2} \ln (aq)} \exp \left[ 2 \sqrt{\ln \ln \left( \sqrt{t}/q \right)} \right]. \quad (14)$$

$\omega = 0.174 \ldots$ One can obtain from Eq. (14) the following asymptotic expression at $t \to \infty$:

$$\Pi_1(q) = \frac{\omega}{q^{\sqrt{2}}} \ln (aq). \quad (15)$$

Apart from the logarithmic factor, this is a power-law (scale-free) DDF with exponent $\gamma < 2$. However, the range of validity is small, $\ln q \ll \sqrt{\ln t}$, and the moments of the distribution are defined by its behavior outside “scale-free” region. More general expression (14) is valid if $q < q_0 \sqrt{t}$, at $q \approx q_0 \sqrt{t}$ there exist an exponential cut-off, which replaces the $\delta$-functional term at the point of abrupt cut-off, found in continuous approximation (Eq. (13)).

In conclusion: natural generalization of network’s evolutionary dynamics with preferential linking was suggested. We have found that evolving networks with preferential linking, in which new vertices are born by previously existing ones, and inherited edges from their parent, arrive at the multifractal degree distribution. For a specific model, exact large-size solution was found. Degree distribution tends to nearly power-law type in the infinite network limit, $\Pi(q) \sim q^{-\gamma}$, $1 < \gamma < 2$ (specifically, $\gamma = \sqrt{2}$ for the exactly solvable model). Nevertheless, for finite network the region $q < \exp \left( \sqrt{\ln t} \right)$ where this behavior may be observed, is rather small as compared to one of relevance, $q < q_0 \sqrt{t}$. With probability close to 1 a vertex, randomly chosen among ones with nonzero degree (active vertex), has its degree within the power-law dependence region. For a specific model, exact large-size solution was found. Degree distribution tends to nearly power-law type in the infinite network limit, $\Pi(q) \sim q^{-\gamma}$, $1 < \gamma < 2$ (specifically, $\gamma = \sqrt{2}$ for the exactly solvable model). Nevertheless, for finite network the region $q < \exp \left( \sqrt{\ln t} \right)$ where this behavior may be observed, is rather small as compared to one of relevance, $q < q_0 \sqrt{t}$. With probability close to 1 a vertex, randomly chosen among ones with nonzero degree (active vertex), has its degree within the power-law dependence region. However, distribution’s moments will be defined by the broad large-degrees tail, containing a small fraction of vertices. Results, obtained here for the model with a particular (homogeneous) distribution of inheritance coefficient, may easily be generalized to a more generic case of any distribution.

In most of empirical scale-free distributions of degrees the exponent $\gamma$ was found to be $\gamma > 2$. Also, in most models, $\gamma > 2$, that means that the average degree remains finite as network grows. Note that the degree distributions with $\gamma < 2$ were obtained analytically and by simulation for networks with accelerating growth [17], where $\bar{q} \to \infty$ as $t \to \infty$. In the present case, the number of edges per active site also grows with time in a power law manner, i.e., as $t^{3/2-\sqrt{2}} \ln^{3/2} t$. However, the case of multifractal distribution is essentially different from a scale-free one. Here the distribution can not be described, using one, or finite set of scaling exponents. In particular, the attempt to analyze the distribution in terms of a scale-free one would lead to the value of the exponent $\gamma$, dependent on the method of analysis.

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Fig. 1 — Elementary processes, defining the evolution of a network. (a) — creation of new link, pointing at a node. Probability of this process per unit time is \( m_q(t)/Q(t) \). (b) — creation of new node by a randomly chosen member of the network. Its probability is \( W(q, q') = n(q') c q (1 - c)^{q - q'} \). (c) — creation of new node by external source. This process has the probability \( n \).

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