Apéry-Type Series and Colored Multiple Zeta Values

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\textbf{Abstract.} In this paper, we study new classes of Apéry-type series involving the central binomial coefficients and the multiple $t$-harmonic sums by combining the methods of iterated integrals and Fourier–Legendre series expansions, where the multiple $t$-harmonic sums are a variation of multiple harmonic sums in which all the summation indices are restricted to odd numbers only. Our approach also enables us to generalize some old classes of Apéry-type series involving harmonic sums to those with products of multiple harmonic sums and multiple $t$-harmonic sums. We show that these series can be expressed as either the real or the imaginary part of a $Q$-linear combination of colored multiple zeta values of level 4. Hopefully, these relations will shed some new lights on their properties which may lead to novel approaches to irrationality questions on their properties and may lead to new approaches to irrationality questions on the Riemann zeta values, or more generally, the multiple zeta values.

\textbf{Keywords:} Apéry-type series, central binomial coefficients, (colored) multiple zeta values, multiple $t$-values, multiple ($t$-)harmonic sums, Legendre polynomials.

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1 Introduction

The study of infinite series involving central binomial coefficients was first brought to he attention to the math world by Apéry with his celebrated proof of irrationality of $\zeta(2)$ and $\zeta(3)$. These series are called Apéry-type series or Apéry-like sums. Although many new identities have been found they lead to no more irrationality proofs for other odd Riemann zeta values.

However, in recent years a large number of research work have been focus on the study of multiple zeta values (MZVs) and their generalizations. Many surprising connections to other objects in mathematics and physics have been discovered. In this paper, we will present many new families of relations between Apéry-type series and the multiple colored zeta values, among which the Riemann zeta values are the special cases. Hopefully, these will shed some new lights to the properties of these series and may lead to new approaches to irrationality questions.

We begin with some basic notations. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A finite sequence $k := (k_1, \ldots, k_r) \in \mathbb{N}^r$ is called a \textit{composition}. We put

$$|k| := k_1 + \cdots + k_r, \quad \text{dep}(k) := r,$$

and call them the weight and the depth of $k$, respectively. If $k_1 > 1$, $k$ is called \textit{admissible}.

For a composition $k = (k_1, \ldots, k_r)$ and a positive integer $n$, the \textit{multiple harmonic sum} and \textit{multiple harmonic star sum} are defined by

$$\zeta_n(k) := \sum_{n \geq n_1 > \cdots > n_r \geq 0} \frac{1}{n_1 \cdots n_r k_1 \cdots k_r} \quad \text{and} \quad \zeta^*_n(k) := \sum_{n \geq n_1 \geq \cdots \geq n_r > 0} \frac{1}{n_1 \cdots n_r k_1 \cdots k_r}, \quad (1.1)$$

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respectively. If \( n < k \) then \( \zeta_n(k) := 0 \) and \( \zeta_n(0) = \zeta_n^*(0) := 1 \). When taking the limit \( n \to \infty \) in (1.1) we get the multiple zeta values (MZVs) and the multiple zeta star values (MZSVs), respectively:

\[
\zeta(k) := \lim_{n \to \infty} \zeta_n(k), \quad \zeta^*(k) := \lim_{n \to \infty} \zeta_n^*(k),
\]

defined for admissible compositions \( k \) to ensure convergence of the series. The systematic study of MZVs began in the early 1990s with the works of Hoffman [11] and Zagier [26]. Due to their surprising and sometimes mysterious appearance in the study of many branches of mathematics and theoretical physics, these special values have attracted a lot of attention and interest in the past three decades (for example, see the book by the second author [28]).

In general, let \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \) and \( z = (z_1, \ldots, z_r) \) where \( z_1, \ldots, z_r \) are \( N \)th roots of unity. We can define the colored MZVs (CMZVs) of level \( N \) as

\[
\text{Li}_k(z) := \sum_{n_1 > \cdots > n_r > 0} \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}},
\]

(1.2)

which converges if \( (k_1, z_1) \neq (1, 1) \) (see [16] and [28, Ch. 15]), in which case we call \( (k; z) \) admissible. The level two colored MZVs are often called Euler sums or alternating MZVs. In this case, namely, when \( (z_1, \ldots, z_r) \in \{\pm 1\}^r \) and \( (k_1, z_1) \neq (1, 1) \), we set \( \zeta(k; z) = \text{Li}_k(z) \). Further, we put a bar on top of \( k_j \) if \( z_j = -1 \). For example,

\[
\zeta(2, 3, 1, 4) = \zeta(2, 3, 1, 4; -1, 1, -1, 1).
\]

More generally, for any composition \( (k_1, \ldots, k_r) \in \mathbb{N}^r \), the classical multiple polylogarithm function with \( r \)-variable is defined by

\[
\text{Li}_{k_1, \ldots, k_r}(x_1, \ldots, x_r) := \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{x_1^{n_1} \cdots x_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}},
\]

(1.3)

which converges if \( |x_1 \cdots x_j| < 1 \) for all \( j = 1, \ldots, r \). They can be analytically continued to a multi-valued meromorphic function on \( \mathbb{C}^r \) (see [27]). In particular, if \( x_1 = x, x_2 = \cdots = x_r = 1 \), then \( \text{Li}_{k_1, \ldots, k_r}(x, 1, \ldots, 1) \) is the classical single-variable multiple polylogarithm function.

Similar to the multiple harmonic sums and the multiple harmonic star sums, for a composition \( k = (k_1, \ldots, k_r) \) and a positive integer \( n \), we define the multiple \( t \)-harmonic sum and multiple \( t \)-harmonic star sum respectively by

\[
t_n(k) := \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{1}{(2n_1 - 1)^{k_1} \cdots (2n_r - 1)^{k_r}} \quad \text{and} \quad t_n^*(k) := \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{1}{(2n_1 - 1)^{k_1} \cdots (2n_r - 1)^{k_r}}.
\]

We set \( t_n(\emptyset) = t_n^*(\emptyset) := 1 \) and \( t_n(k) := 0 \) if \( n < k \). When taking the limit \( n \to \infty \) we get the so-called the multiple \( t \)-values (MtVs) and the multiple \( t \)-star values (MtSVs), respectively. These values has been studied first by Hoffman who introduced them as the odd variants of MZVs and MZSVs in [13], respectively. In the above definition of MtVs, we put a bar on the top of \( k_j \) for \( j = 1, 2, \ldots, r \) if there is a sign \((-1)^n_j \) appearing in the numerator on the right. Or we can also use similar notation \( t(s; \varepsilon) \) where \( \varepsilon \) is a composition of \( \pm 1 \)'s. Those involving one or more of the \( k_j \) barred are called the alternating multiple \( t \)-values. For example,

\[
t(k_1, k_2, k_3, k_4) = \sum_{n_1 > n_2 > n_3 > n_4 \geq 1} \frac{(-1)^{n_1 + n_2}}{(2n_1 - 1)^{k_1}(2n_2 - 1)^{k_2}(2n_3 - 1)^{k_3}(2n_4 - 1)^{k_4}}.
\]

Obviously, the alternating MtVs can be expressed in terms of CMZVs of level 4.

We point out that we have defined more generally the multiple mixed values for an admissible composition \( k = (k_1, \ldots, k_r) \) and \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \{\pm 1\}^r \) (see [21]) by

\[
M(k; \varepsilon) := \sum_{n_1 > n_2 > \cdots > n_r > 0} \prod_{j=1}^r \frac{1 + \varepsilon_j(-1)^{n_j}}{n_j^{k_j}}.
\]

(1.4)
These can be regarded as variants of the multiple zeta value of level two. Also, as special cases of multiple mixed values, the first author [19] defined the values

\[ R(k_1, k_2, \ldots, k_r) := 2^{k_1 + \cdots + k_r} \sum_{n_1 > \cdots > n_r > 0} \frac{1}{(2n_1 - 1)^{k_1} (2n_2)^{k_2} \cdots (2n_r)^{k_r}}, \]  

for positive integers \( k_1, k_2, \ldots, k_r \) with \( k_1 \geq 2 \), which are called multiple \( R \)-values. If \( r = 1 \) and \( k_1 = k \geq 2 \), we have

\[ R(k) = 2^k \sum_{n=1}^{\infty} \frac{1}{(2n-1)^k} = (2^k - 1) \zeta(k) \quad (k \geq 2). \]

In fact, the first author showed that (see [19, Thm. 4.1 and Cor. 4.2])

\[ R(m + 1, 1_{n-1}) \in \mathbb{Q}[\log 2, \zeta(2), \zeta(3), \zeta(4), \ldots], \]

where \( 1_d \) is the sequence of 1’s with \( d \) repetitions.

The primary goal of this paper is to study the explicit relations of Apéry-type series and colored multiple zeta values by using the method of iterated integrals and Fourier–Legendre series expansions. This is partially motivated by the recent work of Campbell and his collaborators, see [3, 4, 5]. For example, in Thm. 6.3 we will express the Apéry-type series involving central binomial coefficients explicitly in terms of alternating multiple \( t \)-values by computing the Fourier–Legendre series expansions of \( \log(x)/\sqrt{x} \). As two general results, we show in Cor. 9.8 that for any positive integers \( p, m \) and \( k \in \mathbb{N}^r \), the series involving multiple \( t \)-harmonic sums

\[ \sum_{n=1}^{\infty} \frac{1}{4^n \binom{2n}{n}} \frac{t_n(k)}{(2n+1)^m} \quad (\text{resp. } \sum_{n=1}^{\infty} \frac{4^n}{n^m} t_n(k)), \quad m \in \mathbb{N}, \]

can be expressed as the imaginary (resp. real) part of a \( \mathbb{Q} \)-linear combination of CMZVs of level 4 and weight \( |k| + m \) (resp. \( |k| + m + 1 \)). Furthermore, in Thm. 6.3 we prove that the Apéry-type series

\[ \sum_{n=1}^{\infty} \frac{1}{4^n \binom{2n}{n}} \frac{t_n(1_k)}{n^{m+1}}, \quad m, k \in \mathbb{N}_0, \]

can be expressed in terms of rational linear combinations of products of \( \log 2 \) and the Riemann zeta values. Some related results may be found in [1, 3, 9, 15, 20] and references therein.

The remainder of this paper is organized as follows. After stating three preliminary lemmas in section 2 we shall consider the Fourier–Legendre series expansion of \( \log^m(x) \) (resp. \( \log^m(x)/\sqrt{x} \)) and obtain an expansion of their higher derivatives in terms the multiple harmonic sums in section 3 (resp. section 4). The result of section 3 will then be applied in section 5 to show that some Apéry-type series can be expressed in terms of CMZVs (see Thm. 5.1) and Prop. 5.2). Here the Fourier–Legendre series expansion of the complete elliptic integral of the first kind \( K(x) \), defined by \( \frac{1}{\sqrt{x}} \), plays a crucial role. Next, in a similar vein, the result of section 4 will be applied in sections 6 and 7 to show that some MtV and MtSV variants of Apéry-type series can be expressed in terms of CMZVs (see Thm. 6.3) and Thm. 7.3). In sections 8 and 9 using the theory of iterated integrals we prove a few results for more general forms of variation of Apéry-type series involving products of multiple \( t \)-harmonic sums and multiple harmonic sums, and relate them to CMZVs of level 2 and level 4. In particular, we consider series in which the central binomial coefficients appear both as numerator and denominator. We conclude this paper with some further questions for future research along this direction in the last section. Our numerical computation throughout the paper was done using MAPLE.
2 Some preliminary lemmas

In this section, we collect three lemmas which will be used throughout the rest of the paper.

Lemma 2.1. (H) Faà di Bruno’s formula may be stated in terms of Bell polynomials as follows:
\[
\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} \frac{f^{(k)}(g(x))B_{n,k}(g^{(1)}(x), g^{(2)}(x), \ldots, g^{(n-k+1)}(x)), \ n \in \mathbb{N}}{k!} \tag{2.1}
\]
where $B_{n,k}$ is the exponential partial Bell polynomials defined by
\[
\frac{1}{k!} \left( \sum_{j=1}^{\infty} x_j j^j \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, \ldots, x_{n-k+1}) \frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots. \tag{2.2}
\]
Moreover, we have the following recurrence relation
\[
B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{i=1}^{n-k+1} \left( \frac{n-1}{i-1} \right) x_i B_{n-i,k}(x_1, \ldots, x_{n-k-i+2}), \tag{2.3}
\]
with $B_{0,0}(x_1) = 1, \ B_{n,0}(x_1, \ldots, x_{n+1}) = 0 \ (n \geq 1), \ B_{0,k}() = 0 \ (k \geq 1)$.

Lemma 2.2. ([L] Thm. 4.1) Define two sequences $A_m(n)$ and $B_m(n)$ by
\[
A_m(n) = (m-1)! \sum_{i=0}^{m-1} \frac{A_i(n)}{i!} \sum_{k=1}^{n} x_k^{m-i}, \quad A_0(n) = 1, \quad (x_k \in \mathbb{C}, k = 1, 2, \ldots, n),
\]
\[
B_m(n) = \sum_{k_1=1}^{n} x_{k_1} \sum_{k_2=1}^{k_1} x_{k_2} \cdots \sum_{k_m=1}^{k_{m-1}} x_{k_m}, \quad B_0(n) = 1, \quad (x_k \in \mathbb{C}, k = 1, 2, \ldots, n).
\]
Then
\[
A_m(n) = m! B_m(n).
\]

Lemma 2.3. ([L] Thm. 4.2) Define two sequences $\tilde{A}_m(n)$ and $\tilde{B}_m(n)$ by
\[
\tilde{A}_m(n) = (m-1)!(-1)^{m-1} \sum_{i=0}^{m-1} \frac{\tilde{A}_i(n)}{i!} \sum_{k=1}^{n} x_k^{m-i}, \quad \tilde{A}_0(n) = 1,
\]
\[
\tilde{B}_m(n) = \sum_{k_1=1}^{n} x_{k_1} \sum_{k_2=1}^{k_1} x_{k_2} \cdots \sum_{k_m=1}^{k_{m-1}} x_{k_m}, \quad \tilde{B}_0(n) = 1.
\]
Then
\[
\tilde{A}_m(n) = m! \tilde{B}_m(n).
\]

3 Expansion of $\log^m(x)$ and their higher derivatives

In this section, we will apply Legendre polynomials to obtain some expansions of $\log^m(x)$ and its higher derivatives. Recall that the Rodrigues formula for the Legendre polynomials has the form
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \tag{3.1}
\]
Then we can find easily that
\[
P_n(2x - 1) = \frac{1}{n!} \frac{d^n}{dx^n} [x^n (x - 1)^n]. \tag{3.2}
\]
Theorem 3.1. For any positive integers \(m, n\) and any real number \(x > 0\),
\[
\frac{\log^m(x)}{m!(-1)^m} = 1 + \sum_{k=1}^{m} \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)\zeta_{n-1}(1k-1)\zeta_{n+1}(1m-k)}{n(n+1)} P_n(2x-1), \tag{3.3}
\]
\[
\frac{d^n}{dx^n} \left(\log^m(x)\right) = \frac{(-1)^n(n-1)!}{x^n} \sum_{k=1}^{m} (-1)^k k! \left(\begin{array}{c}m \\ k \end{array}\right) \zeta_{n-1}(1k-1) \log^{m-k}(x). \tag{3.4}
\]

Proof. We first prove (3.4). Assume \(n \geq 1\). Using (2.2) we set
\[
B_{n, k} := B_{n, k} \left(x^{-1}, -1!x^{-2}, \ldots, (-1)^{n-k}(n-k)!x^{-(n-k+1)}\right).
\]
By recurrence formula (2.3) and by induction on \(k\), we can prove easily that
\[
B_{n, k} = \frac{(-1)^{n-k}(n-1)!}{x^n} \zeta_{n-1}(1k-1). \tag{3.5}
\]
We then apply Lemma 2.1 with \(f(t) = t^n\) and \(g(x) = \log(x)\) to find that
\[
f^{(k)}(g(x)) = \begin{cases} \frac{k!(m)}{k} \log^{m-k}(x) & (k \leq m), \\ 0 & (k > m). \end{cases}
\]
Further,
\[
\frac{d^n}{dx^n} \left(\log^m(x)\right) = \sum_{k=0}^{n} f^{(k)}(g(x)) B_{n, k}. \tag{3.6}
\]
Hence, adding up these three contributions yields (3.4).

We now prove (3.3). By Fourier–Legendre series expansions, we have
\[
\log^m(x) = \sum_{n=0}^{\infty} \left\{ (2n+1) \int_{0}^{1} P_n(2x-1) \log^m(x)dx \right\} P_n(2x-1). \tag{3.7}
\]
Using integration by parts and (3.4), we see that
\[
\int_{0}^{1} P_n(2x-1) \log^m(x)dx = \frac{1}{m} \int_{0}^{1} \left\{ \frac{d^n}{dx^n} \log^m(x)dx \right\} x^n(1-x)^n dx
\]
\[
= \frac{(-1)^n}{n} \sum_{k=1}^{m} (-1)^k k! \left(\begin{array}{c}m \\ k \end{array}\right) \zeta_{n-1}(1k-1) \int_{0}^{1} (1-x)^n \log^{m-k}(x)dx
\]
\[
= \frac{(-1)^m+1}{n(n+1)} \sum_{k=1}^{m} \zeta_{n-1}(1k-1) \zeta_{n+1}(1m-k), \tag{3.8}
\]
where we have used the well-known identity ([18] Eq. (2.5))
\[
\int_{0}^{1} x^{n-1} \log^m(1-x)dx = (-1)^m m! \frac{\zeta_n(1m)}{n}. \tag{3.9}
\]
Thus, (3.3) follows from (3.7) and (3.8) immediately. We have now completed the proof of Thm. 3.1. \(\square\)

4 Expansion of \(\log^m(x)/\sqrt{x}\) and their higher derivatives

In this section, we will apply Legendre polynomials to obtain an expansion of \(\log^m(x)/\sqrt{x}\). In order to do this, we will first find an expression of the higher \(m\) of \(\log^m(x)/\sqrt{x}\) using multiple \(t\)-harmonic
Therefore, differentiating this equality \( m = \psi \) Observe that for any positive integers \( \alpha, \beta \) and the digamma function defined by

\[
\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}
\]

where \( \Gamma(x) \) is the usual gamma function.

From the definition, it is obvious that

\[
\frac{\partial B(\alpha, \beta)}{\partial \alpha} = B(\alpha, \beta)[\psi(\alpha) - \psi(\alpha + \beta)].
\]

Therefore, differentiating this equality \( m - 1 \) times by the Leibniz rule, we can deduce that

\[
\frac{\partial^m B(\alpha, \beta)}{\partial \alpha^m} = \sum_{i=0}^{m-1} \binom{m-1}{i} \frac{\partial^i B(\alpha, \beta)}{\partial \alpha^i} \left[ \psi^{(m-i-1)}(\alpha) - \psi^{(m-i-1)}(\alpha + \beta) \right].
\] (4.1)

Here, \( \psi^{(m)}(x) \) stands for the polygamma function of order \( m \) defined as the \( (m + 1) \)-st derivative of the logarithm of the gamma function:

\[
\psi^{(m)}(x) := \frac{d^m}{dx^m} \psi(x) = \frac{d^{m+1}}{dx^{m+1}} \log \Gamma(x).
\]

Observe that \( \psi^{(m)}(x) \) satisfy the following relations

\[
\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+z} \right), \quad z \notin \mathbb{N}_0 := \{0, -1, -2, \ldots\},
\]

\[
\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}, \quad n \in \mathbb{N},
\]

\[
\psi(x + n) = \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+n-1} + \psi(x), \quad n \in \mathbb{N}.
\]

Here, \( \gamma \) denotes the Euler-Mascheroni constant, defined by

\[
\gamma := \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = -\psi(1) \approx 0.57721566490153286.
\]

Note that if \( \alpha = 1/2 \), then

\[
\psi^{(m-i-1)}(1/2 + n) - \psi^{(m-i-1)}(1/2) = (-1)^{m-i-1}(m-i-1)!2^{m-i}t_n(m-i). \quad (4.2)
\]

Hence, setting \( \alpha = 1/2 \) and \( \beta = n + 1 \) in (4.1) yields

\[
\left. \left( \frac{-1}{2m!} \right)^m \frac{\partial^m B(\alpha, \beta)}{\partial \alpha^m} \right|_{\alpha=1/2, \beta=n+1} = \frac{1}{m} \sum_{i=0}^{m-1} \frac{(-1)^i}{2^i i!} \frac{\partial^i B(\alpha, \beta)}{\partial \alpha^i} \bigg|_{\alpha=1/2, \beta=n+1} t_{n+1}(m-i). \quad (4.3)
\]

In particular, if \( m = 0 \) then

\[
B\left( \frac{1}{2}, n+1 \right) = \frac{2 \cdot 4^n}{(2n + 1) \binom{2n}{n}}. \quad (4.4)
\]

Proposition 4.1. For any positive integers \( n \) and \( k \),

\[
a_n(k) := \frac{\zeta_{n-1}(1k-1)}{n} + \sum_{i+j=n, \ i,j \geq 1} \frac{\zeta_{i-1}(1k-1)}{i} \left( \frac{2j}{i} \right) = \frac{1}{4n} \binom{2n}{n} t_n(1k) 2^k. \quad (4.5)
\]
Proof. By the well-known binomial expansion,
\[
\frac{1}{\sqrt{1-x}} - 1 = \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} x^n, \quad x \in [-1, 1).
\] (4.6)

Further, using the shuffle product we see that
\[
\log^k(1-x) = (-1)^k \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(1-k-1)}{n} x^n, \quad x \in [-1, 1).
\] (4.7)

Hence, by the Cauchy product, we obtain
\[
\frac{(-1)^k \log^k(1-x)}{k!} \sqrt{1-x} = \sum_{n=1}^{\infty} \left\{ \frac{\zeta_{n-1}(1-k-1)}{n} + \sum_{i+j=n, \ i,j \geq 1} \frac{\zeta_{i-1}(1-k-1) \binom{2j}{4j}}{i} \right\} x^n
= \frac{1}{k!} \lim_{\alpha \to 1/2} \frac{\partial^k}{\partial \alpha^k} \frac{1}{(1-x)^\alpha} = \sum_{n=1}^{\infty} a_n(k) x^n.
\] (4.8)

Denote by \((\alpha)_n\) the Pochhammer symbol (or the shifted factorial) given by
\[
(\alpha)_0 := 1, \quad (\alpha)_n := \alpha(\alpha+1) \cdots (\alpha+n-1), \ \forall n \geq 1.
\]

Then, combining (4.8) with the binomial series expansion \((1-x)^{-\alpha} = \sum_{n=0}^{\infty} (\alpha)_n x^n/n!, x \in (-1, 1),\) we get
\[
a_n(k) = \frac{1}{k!} \lim_{\alpha \to 1/2} \frac{\partial^k}{\partial \alpha^k} (\alpha)_n.
\] (4.9)

In particular, when \(k = 0\) we obtain
\[
a_n(0) = \frac{1}{4^n} \binom{2n}{n}.
\] (4.10)

By a simple calculation, the \(\frac{\partial^k(\alpha)_n}{\partial x^k}\) satisfy a recurrence relation in the form of
\[
\frac{\partial^k(\alpha)_n}{\partial x^k} = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{\partial^i(\alpha)_n}{\partial x^i} \left[ \psi(k-i-1)(\alpha + n) - \psi(k-i-1)(\alpha) \right], \quad k \in \mathbb{N}.
\] (4.11)

Hence, taking \(m = k\) in (4.11) to evaluate (4.11) and using (4.9), we find the recurrence relation
\[
a_n(k) = \frac{(-1)^{k-1}}{k} \sum_{i=0}^{k-1} (-1)^{i} 2^{k-i} a_n(i) t_n(k-i).
\] (4.12)

In Lemma 2.3 letting \(x_k = \frac{1}{2k-1}\) yields
\[
t_n(1_m) = \frac{(-1)^{m-1}}{m} \sum_{i=0}^{m-1} (-1)^{i} t_n(1_i) t_n(m-i).
\] (4.13)
Proof. By a direct calculation, we have

\[ a_n(m) = \frac{2^m}{4^n} \binom{2n}{n} t_n(1_m). \]  

(4.14)

Finally, setting \( m = k \) in (4.14) yields the desired formula (4.5).

We now consider the expansions of the higher derivatives of \( \log^m(x)/\sqrt{x} \).

**Theorem 4.2.** For any \( m, n \in \mathbb{N}_0 \) and any real number \( x > 0 \),

\[ \frac{d^n}{dx^n} \left( \frac{\log^m(x)}{\sqrt{x}} \right) = (-1)^n \frac{(2n)!}{n!4^n} \sum_{k=0}^{m} (-1)^k k! \binom{m}{k} 2^k t_n(1_k) \frac{\log^{m-k}(x)}{x^{n+1/2}}. \]  

(4.15)

**Proof.** By a direct calculation, we have

\[ \frac{d^n}{dx^n} \left( \frac{1}{\sqrt{x}} \right) = (-1)^n \frac{(2n)!}{n!4^n} x^{-1/2-n}. \]  

(4.16)

Then, using (3.4) and (4.16), we can deduce by the Leibniz rule that

\[ \frac{d^n}{dx^n} \left( \frac{\log^m(x)}{\sqrt{x}} \right) = \sum_{i+j=n, i,j \geq 0} \frac{n!}{i!j!} \frac{d^i}{dx^i} \left( \log^m(x) \right) \frac{d^j}{dx^j} \left( \frac{1}{\sqrt{x}} \right) \]

\[ = (-1)^n \frac{(2n)!}{n!4^n} x^{-n+1/2} \]

\[ + (-1)^n n! \sum_{k=1}^{m} (-1)^k k! \binom{m}{k} \left\{ \zeta_{n-1}(1_{k-1}) \right. \left. + \sum_{i+j=n, i,j \geq 1} \zeta_{i-1}(1_{j-1}) \frac{(2j)}{4j} \right\} \frac{\log^{m-k}(x)}{x^{n+1/2}}. \]  

(4.17)

Thus, combining (4.17) with (4.5), we can complete the proof of the theorem immediately.

**Theorem 4.3.** For any \( m \in \mathbb{N}_0 \) and any real number \( x > 0 \),

\[ \frac{\log^m(x)}{\sqrt{x}} = (-1)^m m! 2^{m+1} \sum_{k=0}^{m} \sum_{n=0}^{\infty} (-1)^n t_n(1_k) t_{n+1}(1_{m-k}) P_n(2x-1). \]  

(4.18)

**Proof.** By Fourier–Legendre series expansions, we have

\[ \frac{\log^m(x)}{\sqrt{x}} = \sum_{n=0}^{\infty} \left( 2n + 1 \right) \int_0^1 P_n(2x-1) \frac{\log^m(x)}{\sqrt{x}} dx \right) P_n(2x-1). \]  

(4.19)

Using integration by parts and (4.15), we deduce that

\[ \int_0^1 P_n(2x-1) \frac{\log^m(x)}{\sqrt{x}} dx = \frac{1}{n!} \int_0^1 \left( \frac{d^n}{dx^n} \frac{\log^m(x)}{\sqrt{x}} \right) x^n(1-x)^n dx \]

\[ = (-1)^n \frac{1}{4^n} \binom{2n}{n} \sum_{k=0}^{m} (-1)^k k! \binom{m}{k} 2^k t_n(1_k) \int_0^1 x^{-1/2}(1-x)^n \log^{m-k}(x) dx. \]  

(4.20)

Observe that

\[ \int_0^1 x^{-1/2}(1-x)^n \log^m(x) dx = \frac{\partial^m B(\alpha, \beta)}{\partial \alpha^m} \bigg|_{\alpha=1/2, \beta=n+1}. \]  

(4.21)
On the other hand, from Lemma 2.4, changing \( n \) to \( n + 1 \) and letting \( x_k = \frac{1}{2k - 1} \), we obtain
\[
t_{n+1}^*(1_m) = \frac{1}{m} \sum_{i=0}^{m-1} t_{n+1}^*(1) t_{n+1}(m - i). \tag{4.22}
\]
Hence, comparing (4.3) with (4.22) gives
\[
\frac{\partial^m B(\alpha, \beta)}{\partial \alpha^m} \bigg|_{\alpha=1/2, \beta=n+1} = (-1)^m m! 2^{m+1} \frac{4^n}{(2n+1)(2n)} t_{n+1}^*(1_m). \tag{4.23}
\]
Thus, combining (4.20), (4.21) and (4.23), we get
\[
\int_0^1 P_n(2x - 1) \frac{\log^m(x)}{\sqrt{x}} \, dx = \frac{(-1)^{n+m} m! 2^{m+1}}{2n+1} \sum_{k=0}^{m} t_n(1_k) t_{n+1}^*(1_{m-k}). \tag{4.24}
\]
Finally, combining (4.24) with (4.19) we arrive at (4.18). This finishes the proof of the theorem. \( \square \)

\textbf{Remark 4.4.} In fact, we can find the following more general results
\[
\frac{d^m}{dx^n} \left( \frac{\log^m(x)}{x^\alpha} \right) = (-1)^m m! \sum_{k=0}^{m} (-1)^k k! \binom{m}{k} \zeta_n(1_k; \alpha) \frac{\log^{m-k}(x)}{x^{n+\alpha}} \tag{4.25}
\]
for \( \alpha \neq 0, -1, -2, -3, \ldots \), and
\[
\frac{\log^m(x)}{x^\alpha} = (-1)^m m! \sum_{k=0}^{m} \sum_{n=0}^{\infty} (-1)^n \frac{(2n + 1)(\alpha)_n}{(1 - \alpha)_{n+1}} \zeta_n(1; \alpha) \zeta_{n+1}^*(1_{m-k}; 1 - \alpha) P_n(2x - 1) \tag{4.26}
\]
for \( \alpha \notin \mathbb{Z} \), by a similar argument as in the proofs of (4.18) and (4.19). Here, for a composition \( k = (k_1, \ldots, k_r) \) and \( \alpha \neq 0, -1, -2, -3, \ldots \),
\[
\zeta_n(k; \alpha) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r > 0} \frac{1}{(n_1 + \alpha - 1)^{k_1} \cdots (n_r + \alpha - 1)^{k_r}},
\]
\[
\zeta_n^*(k; \alpha) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r > 0} \frac{1}{(n_1 + \alpha - 1)^{k_1} \cdots (n_r + \alpha - 1)^{k_r}}.
\]
If \( n < k \) then \( \zeta_n(k; \alpha) := 0 \) and \( \zeta_n(0; \alpha) = \zeta_n^*(0; \alpha) := 1. \)

\section{Apéry-type series involving central binomial coefficients}

Recall that the complete elliptic integral of the first kind is defined by
\[
K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x \sin^2 t}} = \frac{\pi}{2} F_1 \left( \frac{1}{2}, \frac{1}{2}; 1 \bigg| x \right). \tag{5.1}
\]
By \( [4, (8)] \) we see that
\[
K(x) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \binom{2n}{n} \right] x^n = 2 \sum_{n=0}^{\infty} \frac{P_n(2x - 1)}{2n+1}. \tag{5.2}
\]
In this section, using the above Fourier–Legendre expansion of \( K(x) \) we will derive the evaluation of some Apéry-type series involving squares of central binomial coefficients in terms of colored MZVs.
Theorem 5.1. For any $m \in \mathbb{N}_0$ we have
\[
\sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right) \right]^2 \frac{1}{(n+1)(2n+1)} = \frac{4}{\pi} \left\{ 1 + \sum_{k=1}^{m} \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(1) \zeta_{n+1}^*(1-m-k)}{n(n+1)(2n+1)} (-1)^n \right\}.
\]
\[
\sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right) \right]^2 \frac{\zeta_{n+1}^*(1-m)}{n+1} = \frac{4}{\pi} \left\{ 1 + \sum_{k=1}^{m} \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(1) \zeta_{n+1}^*(1-m-k)}{n(n+1)(2n+1)} \right\}.
\]

Proof. Multiplying (5.2) by $\log^m(x)$ and integrating over the interval $(0, 1)$, we can prove (5.3) by a straight-forward calculation with the aid of (3.3).

Similarly, by [18, Eq. (2.5)] we have
\[
\int_0^1 K(x) \log^m(1-x) \, dx = (-1)^m m! \pi^2 \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right) \right]^2 \frac{\zeta_{n+1}^*(1-m)}{n+1}.
\]

Now, multiplying (5.2) by $\log^m(1-x)$ and integrating over the interval $(0, 1)$, we see that
\[
\int_0^1 K(x) \log^m(1-x) \, dx = 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} \int_0^1 P_n(2x-1) \log^m(1-x) \, dx.
\]

Since the Legendre polynomial satisfies $P_n(-x) = (-1)^n P_n(x)$ by (5.1), applying (3.3) we obtain
\[
\int_0^1 P_n(2x-1) \log^m(1-x) \, dx = (-1)^n m! \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(1) \zeta_{n+1}^*(1-m-k)}{n(n+1)(2n+1)} (-1)^n \in \text{CMZV} \leq 4.
\]

Thus, (5.4) follows from (5.5) to (5.7) immediately. This concludes the proof of the theorem. \qed

In fact, we have the following results.

Proposition 5.2. For any integers $m \geq k \geq 0$ we have
\[
\sum_{n=1}^{\infty} \frac{\zeta_{n-1}(1) \zeta_{n+1}^*(1-m-k)}{n(n+1)(2n+1)} \in \text{CMZV} \leq 2 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(1) \zeta_{n+1}^*(1-m-k)}{n(n+1)(2n+1)} (-1)^n \in \text{CMZV} \leq 4,
\]

where $\text{CMZV} \leq N$ is the $\mathbb{Q}$-span of CMZVs of level $\leq N$. In particular, $\text{CMZV} \leq 1 = \mathbb{Q} + \text{MZV}$ where MZV is the $\mathbb{Q}$-span of all MZVs.

Proof. According to the definition of multiple harmonic (star) sums, we know that
\[
\zeta_{n-1}(1) \zeta_{n+1}^*(1-m-k) \in \mathbb{Q} \left[ \frac{1}{n}, \frac{1}{n+1}, \zeta_n(\cdot \cdot \cdot) \right].
\]

Hence, to prove the proposition we need to evaluate the following sums
\[
\sum_{n=1}^{\infty} \frac{\zeta_n(k_1, \ldots, k_r)}{n^p(n+1)^q(2n+1)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\zeta_n(k_1, \ldots, k_r)}{n^p(n+1)^q(2n+1)} (-1)^n.
\]

By induction of $p + q$ we get the following partial fraction expansion,
\[
\frac{1}{n^p(n+1)^q(2n+1)} = \sum_{i=2}^{p} \frac{a_i}{n^i} + \sum_{j=2}^{q} \frac{b_j}{(n+1)^j} + \frac{c}{n(2n+1)} + \frac{d}{(n+1)(2n+1)},
\]

where $a_i, b_j, c$ and $d$ are rational numbers. Obviously,
\[
\sum_{n=1}^{\infty} \frac{\zeta_n(k_1, \ldots, k_r)}{n^i} \in \text{MZV} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\zeta_n(k_1, \ldots, k_r)}{(n+1)^j} \in \text{MZV}.
\]

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Moreover, noting the fact that
\[ \zeta_n(s_1, s_2, \ldots, s_m) = 2^{s_1 + s_2 + \cdots + s_m - m} \sum_{2n \geq n_1 > n_2 > \cdots > n_m \geq 1} \frac{(1 + (-1)^{n_1})(1 + (-1)^{n_2}) \cdots (1 + (-1)^{n_m})}{n_1^{s_1} n_2^{s_2} \cdots n_m^{s_m}} \]

Hence, we have
\[ \sum_{n=1}^{\infty} \frac{\zeta_n(k_1, \ldots, k_r)}{(2n+1)} = 2^{k_1 + k_2 + \cdots + k_r - r+1} \sum_{\sigma_j \in \{\pm 1\}} \sum_{j=1}^{\infty} \frac{\zeta_n(s_1, \ldots, s_m; \sigma_1, \ldots, \sigma_m)}{2n(2n+1)} \]
\[ = 2^{k_1 + k_2 + \cdots + k_r - r} \sum_{\sigma_j \in \{\pm 1\}} \sum_{j=1}^{\infty} \frac{\zeta_n(s_1, \ldots, s_m; \sigma_1, \ldots, \sigma_m)}{n(n+1)} (1 + (-1)^n). \quad (5.12) \]

Then, applying [22, Thms. 3.1] (by taking \( n = 1 \) there), we obtain
\[ \sum_{n=1}^{\infty} \frac{\zeta_n(k_1, \ldots, k_r)}{n(2n+1)} \in \text{CMZV}^\leq. \quad (5.13) \]

Similarly, we have
\[ \sum_{n=1}^{\infty} \frac{\zeta_n(k_1, \ldots, k_r)}{(n+1)(2n+1)} \in \text{CMZV}^\leq. \quad (5.14) \]

Hence,
\[ \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(k_1-1)}{n(n+1)^2} \zeta^*_n(1, \ldots, 1) \cdot \zeta_{n+1}(1, \ldots, 1) \in \text{CMZV}^\leq. \quad (5.15) \]

We can prove the second formula in the proposition in a similar fashion. \( \square \)

**Corollary 5.3.** For any \( m \in \mathbb{N}_0 \) we have
\[ \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \frac{1}{(n+1)^m+1} \in \frac{1}{\pi} \text{CMZV}^\leq, \quad \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \frac{2n}{n+1} \zeta^*_n(1, \ldots, 1) \zeta^*_{n+1}(1, \ldots, 1) \in \frac{1}{\pi} \text{CMZV}^\leq. \]

**Example 5.4.** By straightforward calculations and Au's package [1], we have the following cases (which can also be found in [4, 5])
\[ \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \frac{1}{n+1} = \frac{4}{\pi}, \]
\[ \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \frac{1}{(n+1)^2} = \frac{16}{\pi} - 4, \]
\[ \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \frac{1}{(n+1)^3} = \frac{16}{\pi} (3 - 2G - \pi + \pi \log 2), \]
\[ \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \frac{\zeta^*_n(1, \ldots, 1)}{n+1} = \frac{4}{\pi} (1 - \log 2), \]
where \( i = \sqrt{-1} \) and \( G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \) is Catalan’s constant. Note that \( \text{Li}_3\left(\frac{1+i}{2}\right) \in \text{CMZV}^4 \) by the next theorem which is of independent interest.

**Theorem 5.5.** For all \( k \in \mathbb{N}^r \) we have

\[
\text{Li}_k\left(\frac{1+i}{2}\right), \text{Li}_k\left(\frac{1-i}{2}\right) \in \text{CMZV}^4_{[k]}.
\]

**Proof.** We know that

\[
\text{Li}_k\left(\frac{1+i}{2}\right) = \int_0^{1+i} \frac{dt}{1-t} \left( \frac{dt}{t} \right)^{k-1} \cdots \left( \frac{dt}{t} \right)^{k-1}
\]

by using Chen’s iterated integral expression (see [28, pp. 13-14] where the convention of the 1-form order is opposite to the one used in this paper). By the change of variables \( t \to \frac{1+i}{2}(1-t) \) we see that \( d \log t \to d \log(1-t) = -\frac{dt}{1-t} \) and

\[
-\frac{dt}{1-t} = d \log(1-t) \to d \log\left(\frac{1-i}{2} + \frac{1+i}{2}t\right) = d \log(-i + t) = -\frac{dt}{i-t}.
\]

Thus

\[
\text{Li}_k\left(\frac{1+i}{2}\right) = (-1)^r \int_0^1 \left( \frac{dt}{1-t} \right)^{k_1-1} \left( \frac{dt}{i-t} \right) \cdots \left( \frac{dt}{1-t} \right)^{k_r-1} \frac{dt}{i-t} \in \text{CMZV}^4_{[k]}.
\]

Taking complex conjugation we see that \( \text{Li}_k\left(\frac{1-i}{2}\right) \in \text{CMZV}^4_{[k]} \).

\[\Box\]

### 6 Some special MtV variant of Apéry-type series

In this section, by applying Thm. 1.3 and using the Fourier–Legendre expansion of \( K(x) \) we will derive explicit evaluations of some MtV variant of Apéry-type series involving squares of central binomial coefficients.

In order to prove the next theorem, we will need the following result.

**Proposition 6.1.** Let \( \eta = \pm 1 \). For all \( m, l \in \mathbb{Z} \) with \( l \geq 2 \) and \( m \geq 0 \), we have

\[
\sum_{k=0}^{m} \sum_{n \geq 0} \eta^n t_n \frac{\zeta_n(1, m-k) \zeta_{n+1}(1, k)}{(n+1)^l} = \sum_{d=1}^{m+1} \sum_{|s|=m+2, s_1 \geq 2} 2^{d-1} \zeta(s_1 + l - 2, s_2, \ldots, s_d; \eta, 1_{d-1}),
\]

\[
\sum_{k=0}^{m} \sum_{n \geq 0} \eta^n t_n \frac{\zeta_n(1, m-k) \zeta_{n+1}(1, k)}{(2n+1)^l} = \sum_{d=1}^{m+1} \sum_{|s|=m+2, s_1 \geq 2} 2^{d-1} t(s_1 + l - 2, s_2, \ldots, s_d; \eta, 1_{d-1}),
\]

where \( s = (s_1, \ldots, s_d) \).
Proof. Fix any positive integer \( n \). Then both \( \zeta_n \) and \( t_n \) satisfy stuffle relations. We now prove the claims by induction on \( m \). If \( m = 1 \) then both sides are equal to \( ff(l + 1; \eta) + 2f(l, 1; \eta, 1) \) for \( f = \zeta \) or \( f = t \). In general, we see that

\[
\sum_{k=0}^{m} \sum_{n \geq 0} \eta^{n} \frac{\zeta_{n}(1_{m-k}) \zeta_{n+1}^*(1_{k})}{(n+1)!} = \sum_{k=1}^{m} \sum_{n \geq 0} \eta^{n} \frac{\zeta_{n}(1_{m-k}) \zeta_{n+1}^*(1_{k-1})}{(n+1)!+1} + \sum_{k=0}^{m} \sum_{n \geq 0} \eta^{n} \frac{\zeta_{n}(1_{m-k}) \zeta_{n}^*(1_{k})}{(n+1)!}
\]

by induction. By the following lemma about words and noticing that

\[
\sum_{d=1}^{m} \sum_{|s|=m+1, s \geq 2} 2^{d-1} \zeta(s_{1} + l - 1, s_{2}, \ldots, s_{d}; \eta, 1_{d-1}) + \sum_{k=0}^{m} \sum_{n \geq 0} \eta^{n} \frac{\zeta_{n}(1_{m-k}) \zeta_{n}^*(1_{k})}{(n+1)!}
\]

Similar identity holds for \( f = t \), too. The proposition follows quickly.

Lemma 6.2. Let \( (N, \ast) \) be Hoffman’s \( \mathbb{Q} \)-algebra of words generated by \( z_{j}(j \in \mathbb{N}) \) satisfying the stuffle relations (see (6.1)). For any \( s = (s_{1}, \ldots, s_{d}) \in \mathbb{N}^{d} \) we put \( z_{s} = z_{s_{1}}z_{s_{2}} \cdots z_{s_{d}} \). Then for any non-positive integer \( m \) we have

\[
\left( \sum_{k=0}^{m} \sum_{q=0}^{k} \sum_{|s|=k-1, r \in \mathbb{N}^{q}} 2^{m-k} \ast z_{r} \right) = \sum_{k=0}^{m} \sum_{|s|=m} \sum_{|r|=k} 2^{d} \ast z_{s} = \sum_{s \ast m} 2^{d} \ast z_{s} = \sum_{d=0}^{m} \sum_{s \ast m} 2^{d} \ast z_{s}.
\]

Proof. We proceed by induction on \( m \). When \( m = 0 \) both sides of the equation in the lemma are empty word. Assume \( m \geq 1 \). For any \( r \in \mathbb{N}^{q} \), by the recursive definition of the stuffle product, we have (after setting \( a = r_{1} \) and \( r = (r_{2}, \ldots, r_{q}) \))

\[
\sum_{k=0}^{m} \sum_{|s|=k} 2^{m-k} \ast z_{r} = \sum_{k=0}^{m-1} \sum_{|s|=k} z_{1}(z_{1}^{m-1-k} \ast z_{r}) + \sum_{a=1}^{m} \sum_{k=a}^{m} \sum_{|s|=a} z_{a}(z_{1}^{m-k} \ast z_{r}) + \sum_{a=1}^{m-1} \sum_{k=a}^{m-1} \sum_{|s|=a} z_{1+a}(z_{1}^{m-1-k} \ast z_{r}).
\]

Breaking the middle term into two cases (namely, \( a = r_{1} = 1 \) and \( a = r_{1} \geq 2 \)) and shifting the indices \( k \) and \( a \) suitably, we easily get

\[
\sum_{k=0}^{m} \sum_{|s|=k} 2^{m-k} \ast z_{r} = 2 \sum_{k=0}^{m-1} \sum_{|s|=k} z_{1}(z_{1}^{m-1-k} \ast z_{r}) + 2 \sum_{a=2}^{m} \sum_{k=0}^{m-a} z_{a}(z_{1}^{m-a-k} \ast z_{r})
\]
\[= 2 \sum_{a=1}^{m} \sum_{d=0}^{m-a} \sum_{|s|=m-a} 2^{\text{deg}(s)} z_a z_s\]

by inductive assumption. The lemma follows immediately. \(\square\)

The following theorem and ([5, Thm. 3]) can be regarded as explicit versions of [5, Thm. 3].

**Theorem 6.3.** For any \(m \in \mathbb{N}_0\) we have

\[
\sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right) \right]^{2} \frac{1}{(2n+1)^{m+1}} = \frac{4}{\pi} \sum_{k=0}^{m} \sum_{n=0}^{\infty} (-1)^n \frac{t_n(1k)t^*_n+1(1m-k)}{(2n+1)^2} \tag{6.2}
\]

\[= -\frac{2}{\pi} \sum_{d=1}^{m+1} \sum_{|s|=m+2, s \in \mathbb{N}^d, s \geq 2} 2^d t(s; -1, 1_d-1) \tag{6.3}
\]

where \(\text{AMtV}_w\) (resp. \(\text{CMZV}^N_w\)) denotes the \(\mathbb{Q}\)-span of all the alternating multiple \(t\)-values (resp. CMVZs of level \(N\) and) of weight \(w\).

**Proof.** We consider the integral

\[
\int_{0}^{1} K(x) \frac{\log^m(x)}{\sqrt{x}} dx = 2^{m+1} \int_{0}^{1} K(x^2) \log^m(x) dx = 2^m \pi \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right) \right]^{2} \int_{0}^{1} x^{2n} \log^m(x) dx
\]

\[= 2^m m! (-1)^m \pi \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right) \right]^{2} \frac{1}{(2n+1)^{m+1}}
\]

\[= 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} \int_{0}^{1} P_n(2x-1) \frac{\log^m(x)}{\sqrt{x}} dx
\]

\[= 2^{m+2} m! (-1)^m \sum_{k=0}^{m} \sum_{n=0}^{\infty} (-1)^n \frac{t_n(1k)t^*_n+1(1m-k)}{(2n+1)^2}, \tag{6.4}
\]

where we have used (1.24) and (6.2). Hence, we can prove (6.2) by a straight-forward calculation. Finally, (6.3) follows immediately from Prop. (6.1) \(\square\)

**Example 6.4.** Observe that for any \(s \in \mathbb{N}^d\), we have the explicit expression in \(\text{CMZV}^4\)

\[2^d t(s; -1, 1_d-1) = i \sum_{\eta_1=\pm 1} \cdots \sum_{\eta_d=\pm 1} \eta_1 \cdots \eta_d \text{Li}_d(i\eta_1, \eta_2, \ldots, \eta_d).\]

For example,

\[
\sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right) \right]^{2} \frac{1}{2n+1} = -\frac{4t(\bar{2})}{\pi} = \frac{4G}{\pi} \approx 1.166243616, \tag{6.5}
\]

\[
\sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right) \right]^{2} \frac{1}{(2n+1)^2} = -\frac{4}{\pi} \left( t(\bar{3}) + 2t(\bar{2}, 1) \right)
\]

\[= \frac{3\pi^2}{8} + \frac{\log^2(2)}{2} - \frac{16}{\pi} \text{Im} \text{Li}_3 \left( \frac{1+i}{2} \right) \approx 1.037947765, \tag{6.6}
\]

\[
\sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right) \right]^{2} \frac{1}{(2n+1)^3} = -\frac{4}{\pi} \left( t(\bar{4}) + 2t(\bar{3}, 1) + 2t(\bar{2}, 2) + 4t(\bar{2}, 1, 1) \right)
\]

\[= \frac{3\pi^2}{4} \log 2 + \frac{1}{3} \log^3(2) + \frac{64}{\pi} \text{Im} \text{Li}_4 \left( \frac{1+i}{2} \right) - \frac{48}{\pi} G \approx 1.010879510,
\]
where $G$ is Catalan’s constant as before and $G(4) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^4}$.

**Theorem 6.5.** For any $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n(1_k)}{n^{m+1}} = \frac{1}{2^k} R(k + 1, 1, m) \in \mathbb{Q}[\log 2, \zeta(2), \zeta(3), \zeta(4), \ldots]. \quad (6.7)$$

**Proof.** From (4.8) and (4.14), we deduce

$$\frac{1}{2^k \sqrt{1 - x}} \int_0^x \left( \frac{dt}{1 - t} \right)^k = \frac{(-1)^k \log^k (1 - x)}{2^k k! \sqrt{1 - x}} = \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1_k)x^n. \quad (6.8)$$

If $k = 0$ we need to modify this as follows:

$$\frac{1}{\sqrt{1 - x}} - 1 = \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} x^n. \quad (6.9)$$

Multiplying (6.8) by $\frac{\log^m(x)}{x}$ and integrating over $(0, 1)$ yields

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n(1_k)}{n^{m+1}} = \frac{(-1)^{m+k} \log^m(1-x)}{2k^m k!} \int_0^1 \frac{\log^m(x) \log^{m-1}(1-t)}{x \sqrt{1-x}} (1-t)^{-1/2} dt. \quad (6.10)$$

In [19], the first author proved that

$$R(m + 1, 1, \ldots, 1) = \sum_{n=1}^{\infty} \frac{1}{m!(n-1)!} \int_0^1 \frac{\log^m(t) \log^{n-1}(1-t)}{1-t} (1-t)^{-1/2} dt. \quad (6.11)$$

Hence, applying the change of variables $x \to 1 - x$ in the integral on the right-hand side of (6.10) and using (6.11), we obtain the desired formula

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n(1_k)}{n^{m+1}} = \frac{1}{2^k} R(k + 1, 1, m).$$

We can now finish the proof of Thm. 6.5 by applying (1.6).

**Example 6.6.** Taking $m \leq 2$, we get

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n(1)}{n} = \frac{2^{k+1} - 1}{2^k} \zeta(k + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n(1)}{n^2} = \frac{7}{2} \zeta(3) - 3 \zeta(2) \log 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n(1, 1)}{n^2} = \frac{45}{16} \zeta(4) - \frac{7}{2} \zeta(3) \log 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n(1)}{n^3} = \frac{15}{4} \zeta(4) + 3 \log^2(2) \zeta(2) - 7 \zeta(3) \log 2.$$

In fact, from [17] Thm. 3.4 we know that the “$\in$” part (but not the “$=$” part as $R(1, \ldots)$ is undefined) of the Thm. 6.5 holds for $k = 0$, too.
7 Some special product variant of Apéry-type series

In this section, we use the idea similar to the one in section 5 to derive explicit evaluations of some product variants of Apéry-type series involving the central binomial coefficients. By product variant we mean products of multiple harmonic sums and multiple $t$-harmonic sums (or their star version) appear in the terms of such series.

**Theorem 7.1.** For any $m, k \in \mathbb{N}$ we have

$$\sum_{n=1}^{\infty} \frac{4^n \zeta_{n-1}(1-m-1)t_n^*(1_k)}{(2^n n)^2} = \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n(1_k)\zeta_{n}^*(1_m)}{n} = \binom{m+k}{k} \frac{2^{m+k+1}-1}{2^k} \zeta(m+k+1), \tag{7.1}$$

where if $m = 0$, the second equal sign also holds.

**Proof.** Applying (4.23) and noting that $t = \zeta$, we have

$$\int_0^1 x^{n-1} \log^k (1-x) \frac{dx}{\sqrt{1-x}} = (-1)^k k! t^k\zeta_n^*(1_k). \tag{7.2}$$

Applying (3.9), (4.7), (6.8), (6.11) and (7.2), we may compute the integral

$$\int_0^1 \frac{\log^{m+k}(1-x)}{x} \frac{dx}{\sqrt{1-x}} = (-1)^k m! R(m+k+1)$$

$$= (-1)^k m! 2^k \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1_k) \int_0^1 x^{n-1} \log^m (1-x) \frac{dx}{\sqrt{1-x}}$$

$$= (-1)^{k+m} m! 2^k \sum_{n=1}^{\infty} \frac{t_n(1_k)\zeta_n^*(1_m)}{n4^n} \binom{2n}{n}$$

$$= (-1)^k m! \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(1-m-1)}{n} \int_0^1 x^{n-1} \log^k (1-x) \frac{dx}{\sqrt{1-x}}$$

$$= (-1)^{k+m} m! 2^k \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(1-m-1)t_n^*(1_k)}{n^2(2^n n)} \frac{dx}{4^n}. \tag{7.3}$$

Finally, we may use $R(k) = \frac{2^k - 1}{\zeta(k)} \ (k > 1)$ to get the desired formula. \qed

**Example 7.2.** Taking $m \leq 2$ and $k \leq 2$, we get

$$\sum_{n=1}^{\infty} \frac{4^n}{(2^n n)^2} t_n^*(1) = \sum_{n=1}^{\infty} \frac{t_n(1)\zeta_n^*(1)}{n4^n} \binom{2n}{n} = 7\zeta(3),$$

$$\sum_{n=1}^{\infty} \frac{4^n}{(2^n n)^2} t_n^*(1, 1) = \sum_{n=1}^{\infty} \frac{t_n(1, 1)\zeta_n^*(1)}{n4^n} \binom{2n}{n} = \frac{45}{4} \zeta(4),$$

$$\sum_{n=1}^{\infty} \frac{4^n}{(2^n n)^2} \zeta_{n-1}(1) t_n^*(1) = \sum_{n=1}^{\infty} \frac{t_n(1)\zeta_n^*(1, 1)}{n4^n} \binom{2n}{n} = \frac{45}{2} \zeta(4).$$

Furthermore, from (7.1), changing $(m, k)$ to $(k, m)$, we obtain the duality relation

$$2^k \sum_{n=1}^{\infty} \frac{4^n}{(2^n n)^2} \zeta_{n-1}(1-m-1)t_n^*(1_k) = 2m \sum_{n=1}^{\infty} \frac{4^n}{(2^n n)^2} \zeta_{n-1}(1-k-1)t_n^*(1_m)$$

$$= 2^k \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1_k)\zeta_n^*(1_m) = 2^m \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1_m)\zeta_n^*(1_k). \tag{7.4}$$
Theorem 7.3. For any \( m \in \mathbb{N}_0 \) we have
\[
\sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_{n+1}^*(1_m) \frac{1}{2n+1} \frac{t_n(1_k) t_{n+1}(1_{m-k})}{(2n+1)^2} = \frac{4}{\pi} \sum_{k=0}^{m} \sum_{n=0}^{\infty} t_n(1_k) t_{n+1}(1_{m-k}) (2n+1)^2 \tag{7.5}
\]
\[
= \frac{4}{\pi} \sum_{d=1}^{m+1} \sum_{s=(s_1, \ldots, s_d) \in [n]^d} 2^d-1 \tau(s) \in \frac{1}{\pi} \text{MSV}_{m+2} \subset \frac{1}{\pi} \text{CMZV}_{m+2}. \tag{7.6}
\]

Proof. Applying \( x \to 1 - x \) in \((5.2)\) and using \( P_n(-x) = (-1)^n P_n(x) \) we get
\[
K(1-x) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left( \frac{1}{4^n} \binom{2n}{n} \right)^2 (1-x)^n = 2 \sum_{n=0}^{\infty} (-1)^n \frac{P_n(2x-1)}{2n+1}. \tag{7.7}
\]

We consider the integral
\[
\int_0^1 K(1-x) \frac{\log^m(x)}{\sqrt{x}} dx = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \int_0^1 (1-x)^n \frac{\log^m(x)}{\sqrt{x}} dx
\]
\[
= \frac{\pi}{2} \sum_{n=0}^{\infty} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \int_0^1 x^n \frac{\log^m(1-x)}{\sqrt{1-x}} dx
\]
\[
= (-1)^m m! 2^m \pi \sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_{n+1}(1_m)}{2n+1}
\]
\[
= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 P_n(2x-1) \frac{\log^m(x)}{\sqrt{x}} dx
\]
\[
= 2^{m+2} m! (-1)^m \sum_{n=0}^{\infty} \sum_{k=0}^{m} \frac{t_n(1_k) t_{n+1}(1_{m-k})}{(2n+1)^2}, \tag{7.8}
\]
where we have used formulas \((4.24)\) and \((7.24)\). Finally, the equation \((7.0)\) in the theorem follows from the Prop. \((6.1)\) by taking \( l = 2 \). This completes the proof of the theorem. \( \square \)

Example 7.4. Taking \( m \leq 3 \), we can verify easily that
\[
\sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{1}{2n+1} = \frac{4}{\pi} \zeta(2) = \frac{3}{\pi} \zeta(2) = \frac{\pi}{2},
\]
\[
\sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_{n+1}(1)}{2n+1} = \frac{4}{\pi} (2t(2,1) + t(3)) = \pi \log 2,
\]
\[
\sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_{n+1}(1,1)}{2n+1} = \frac{4}{\pi} (4t(2,1,1) + 2t(3,1) + 2t(2,2) + t(4)) = \frac{\pi^3}{24} + \pi \log^2(2),
\]
\[
\sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_{n+1}(1,1,1)}{2n+1} = \frac{\pi}{4} \zeta(3) + \frac{2\pi}{3} \log^3(2) + \frac{\pi}{12} \log^2(2).
\]

8 General MSV-MtV product variant of Apéry-type series, I

The theory of iterated integrals was developed first by K.T. Chen in the 1960’s \([7, 8]\). It has played important roles in the study of algebraic topology and algebraic geometry in the past half century. For real values \( a, b \) its simplest form is
\[
\int_a^b f_p(t)dt f_{p-1}(t)dt \cdots f_1(t)dt := \int_{a<t_p<\cdots<t_t<b} f_p(t_p) f_{p-1}(t_{p-1}) \cdots f_1(t_1)dt_1 dt_2 \cdots dt_p.
\]
In this section, we use the iterated integrals to establish a recurrence relation of Apéry series and then derive the evaluations of some MZSV-MtV product variants of Apéry-type series.

Recall that the Hoffman dual of a composition \( k = (k_1, \ldots, k_r) \) is \( k' = (k'_1, \ldots, k'_{r'}) \) determined by
\[
|k| := k_1 + \cdots + k_r = k'_1 + \cdots + k'_{r'} \quad \text{and} \quad
\{1, 2, \ldots, |k| - 1\} = \left\{ \sum_{i=1}^{j} k_i \right\}_{j=1}^{r-1} \prod_{j=1}^{r-1} \left\{ \sum_{i=1}^{j} k'_i \right\}_{j=1}^{r'-1}.
\]
Equivalently, \( k' \) can be obtained from \( k \) by swapping the commas "," and the plus signs "+" in the expression
\[
k = (1 + \cdots + 1, \ldots, 1 + \cdots + 1)_{\overset{k_1 \text{ times}}{k_r \text{ times}}}.
\]
For example, we have \((1, 1, 2, 1)' = (3, 2)\) and \((1, 2, 1, 1)' = (2, 3)\). More generally, we have
\[
k' = (1, \ldots, 1 + 1, \ldots, 1 + 1, \ldots, 1, 1).\]

To save space, for any \( i, j \in \mathbb{N} \) we put
\[
\overrightarrow{k}_{i,j} := \begin{cases} (k_i, \ldots, k_j), & \text{if } i \leq j \leq r; \\ \emptyset, & \text{if } i > j; \end{cases} \quad \text{and} \quad \overleftarrow{k}_{i,j} := \begin{cases} (k_j, \ldots, k_i), & \text{if } i \leq j \leq r; \\ \emptyset, & \text{if } i > j. \end{cases}
\]
Set \( \overrightarrow{k}_i = \overrightarrow{k}_{1,i} = (k_1, \ldots, k_i) \) and \( \overleftarrow{k}_i = \overleftarrow{k}_{i,r} = (k_r, \ldots, k_i) \) for all \( 1 \leq i \leq p \).

**Lemma 8.1.** For any composition \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \), we have
\[
\int_0^1 \frac{\text{Li}_k(t)}{t^{1-t}} dt = 2^{|k|+1} t((1, k_r, k_{r-1}, \ldots, k_1)').
\]

**Proof.** According to the definition of classical multiple polylogarithm function, we have the iterated integral expression
\[
\text{Li}_{k_1, \ldots, k_r}(x) = \int_0^x \left( \frac{dt}{1-t} \right)^{k_1-1} \cdots \left( \frac{dt}{1-t} \right)^{k_r-1}.
\]

Hence
\[
\int_0^1 \frac{\text{Li}_k(t)}{t^{1-t}} dt = \int_0^1 \left( \frac{dt}{1-t} \right)^{k_1-1} \cdots \left( \frac{dt}{1-t} \right)^{k_r-1} \frac{dt}{t^{1-t}}
\]
\[
= 2^{|k|+1} t((1, k_r, k_{r-1}, \ldots, k_1)').
\]
This completes the proof of the lemma.

**Lemma 8.2.** ([22] Thm. 2.1) Let \( r, n \in \mathbb{N} \), \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \). Then
\[
\int_0^1 x^{n-1} \text{Li}_k(x) dx = \frac{(-1)^{|k|}}{n^{k_1}} \zeta_n(\overrightarrow{k}_{2,r}, 1) + \sum_{j=0}^{k_1-2} (-1)^j \frac{\zeta(k_1-j, \overrightarrow{k}_{2,r})}{j^{k_1+1}}
\]
\[
+ \sum_{l=1}^{r-1} (-1)^{|k|+l-2} \sum_{j=0}^{k_{l+2}-1} (-1)^j \frac{\zeta_{n,k_{l+2}}(\overrightarrow{k}_{2,l+2}, j+1)}{j^{k_{l+2}}} \zeta(k_{l+1} - j, \overrightarrow{k}_{l+2,r}).
\]
Theorem 8.3. Let $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$. If $p \in \mathbb{N}$ then we have

$$
\frac{(-1)^p}{2^p p!} \int_0^1 \frac{\text{Li}_k(x) \log^p(1-x)}{x^{1-x}} \, dx = \sum_{n=1}^{\infty} \frac{4^n}{n(2n)} \frac{t_n(1_p) \zeta_n-1(k_2, \ldots, k_r)}{n^{k_1+1}} \tag{8.6}
$$

$$
= (-1)^{|k|-r} \frac{\sum_{n=1}^{\infty} \frac{1}{4^n} \left( \frac{(2n)}{n} \right) t_n(1_p) \zeta_n^+(\vec{k}_{2,r}, 1)}{n^{k_1}} + \sum_{j=0}^{k_1-2} (-1)^j \zeta(k_1 - j, \vec{k}_{2,r}) \sum_{n=1}^{\infty} \frac{1}{4^n} \left( \frac{(2n)}{n} \right) t_n(1_p) \zeta_n^+(\vec{k}_{2,r}, j+1) \sum_{i=1}^{r-1} (-1)^{|k_i|-i} \frac{1}{4^n} \left( \frac{(2n)}{n} \right) t_n(1_p) \zeta_n^+(\vec{k}_{2,r}, i+1) \tag{8.7}
$$

If $p = 0$, then we have

$$
\int_0^1 \frac{\text{Li}_k(x)}{x^{1-x}} \, dx = \sum_{n=1}^{\infty} \frac{4^n}{n(2n)} \frac{\zeta_n-1(k_2, \ldots, k_r)}{n^{k_1+1}} = 2^{|k|+1} t((1, k_1, k_2, \ldots, k_1)^v) \tag{8.8}
$$

Proof. From (8.2), we obtain

$$
\frac{\sum_{n=1}^{\infty} \frac{4^n}{n(2n)} \frac{t_n(1_p)}{n^{k_1+1}}} = \frac{(-1)^p}{2^p p!} \int_0^1 \frac{\log^p(1-x)}{x^{1-x}} \, dx. \tag{8.9}
$$

According to the definition of classical multiple polylogarithm function, we have

$$
\text{Li}_{k_1, \ldots, k_r}(x) = \sum_{n=1}^{\infty} \frac{\zeta_n-1(k_2, \ldots, k_r)}{n^{k_1}} x^n. \tag{8.10}
$$

Multiplying (8.3) by $\frac{\zeta_n-1(k_2, \ldots, k_r)}{n^{k_1}}$, summing up, and applying (8.8) and (8.5) we get

$$
\sum_{n=1}^{\infty} \frac{4^n}{n^{k_1+1}(2n)} t_n(1_p) \zeta_n-1(k_2, \ldots, k_r) = \frac{(-1)^p}{2^p p!} \int_0^1 \frac{\text{Li}_{k_1, \ldots, k_r}(t) \log^p(1-t)}{t^{1-t}} \, dt \int_0^1 x^{n-1} \text{Li}_{k_1, \ldots, k_r}(x) \, dx
$$

by (4.8). Hence (8.7) follows immediately from Lemma 8.2. Next, according to the iterated integral shuffle relation, we know that

$$
\text{Li}_{k_1, \ldots, k_r}(t) \log^p(1-t) = (-1)^p \text{Li}_{k_1, \ldots, k_r}(t) \text{Li}_1(t)^p
$$

can be expressed in terms of a rational linear combination of $\text{Li}_m(t)$ with weight $|m| = |k| + p$ and depth $r + p$. For example, we have

$$
\text{Li}_{2,2}(t) \log(1-t) = -2 \text{Li}_{2,2,1}(t) - 2 \text{Li}_{2,1,2}(t) - \text{Li}_{1,2,2}(t).
$$

Thus we see that the left-hand side of (8.6) lies in $\text{MtV}$ by Lemma 8.1.

Similarly, applying (4.6) and the fact

$$
\int_0^1 \frac{x^{n-1}}{\sqrt{1-x}} \, dx = B\left(\frac{1}{2}, n\right) = \frac{4^n}{n(2n)} \tag{b.4.31}
$$

we can also deduce (8.8) by an argument similar to that used in the proof of (8.7).
Remark 8.4. It should be pointed out that Chen [9, Eq. (5.2)] already obtained the explicit formula (8.6).

Corollary 8.5. Let \( p \in \mathbb{N}_0, m \in \mathbb{N} \) and \( k \in \mathbb{N}' \). Then we have

\[
\sum_{n=1}^{\infty} \frac{\binom{4^n}{2n}}{n^{m+1}} t_n(1_p) \zeta_n(k) \in \text{CMZV}^2_{|k|+m+p+1}.
\]

Proof. This follows from Thm. 8.3 since all MtVs lie in CMZV^2 and

\[
\zeta_n(k) = \frac{1}{n^2} \zeta_{n-1}(k_2, \ldots, k_r) + \zeta_{n-1}(k).
\]

Corollary 8.6. For integers \( m, r \in \mathbb{N} \) and \( p \in \mathbb{N}_0 \), we have

\[
\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1_p) \zeta_n^*(1_r) \frac{1}{n^m} \in \text{CMZV}^2_{p+r+m}.
\] (8.11)

Proof. Letting \( k = (m, 1_{r-1}) \) in Thm. 8.3 yields

\[
\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \zeta_{n-1}(1_r) t_n^*(1_p) \frac{1}{n^{m+1}} = \sum_{j=1}^{m-1} (-1)^{j-1} \zeta(m + 1 - j, 1_{r-1}) \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1_p) \frac{1}{n^j} + (-1)^{m-1} \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1_p) \zeta_n^*(1_r) \frac{1}{n^m} + \delta_{0,p} \zeta(m + 1, 1_{r-1}),
\]

where \( \delta_{0,0} = 1 \) and \( \delta_{0,p} = 0 \) for \( p > 0 \). In particular, if letting \( m = 1 \) and \( p \in \mathbb{N} \) in the above formula, we obtain Thm. 8.3 by replacing \( r \) by \( m \).

Applying (6.7) and noting that the (6.7) also holds for \( p = 0 \) (see [17, Thm. 3.4]), we obtain the desired description.

Example 8.7. Setting \( m = 1, p = 0 \) yields

\[
\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \zeta_n^*(1_r) \frac{1}{n^2} = \zeta(r + 1, 1) - 2^{r+2}(r + 1, 1) + 2\zeta(r + 1) \log 2.
\]

Setting \( r = m = 2, p = 1 \) we get

\[
\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}} \zeta_{n-1}(1_r) t_n^*(1_p) \frac{1}{n^3} = 16(3t(4, 1) + t(3, 2)) = 45\zeta(4) \log 2 - 31\zeta(5),
\]

\[
\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1_p) \zeta_n^*(1_r, 1, 1) \frac{1}{n^2} = \frac{3}{2} \zeta(2) \zeta(3) + 31\zeta(5) - 45\zeta(4) \log 2.
\]

Corollary 8.8. For integers \( m, r \in \mathbb{N} \) and \( p \in \mathbb{N}_0 \), we have

\[
\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1_p) \zeta_n^*(2_{r-1}, 1) \frac{1}{n^m} \in \text{CMZV}^2_{p+m+2r-1},
\] (8.12)

where \( 2_p \) means the string of \( p \) repetitions of \( 2 \)'s.

Proof. Similarly to the proof of Cor. 8.6 setting \( k = (m, 2_{r-1}) \) in Thm. 8.3 we can quickly arrive at (8.12). □
Example 8.9. Taking $p \leq 1$ and $r = 2$, we have
\[
\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\zeta_n^*(2,1)}{n^2} = 75\frac{\zeta(5)}{8} - 4\zeta(4) \log 2 - 3\zeta(2)\zeta(3),
\]
\[
\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} t_n(1) \frac{\zeta_n^*(2,1)}{n^2} = \frac{1055}{32} \zeta(6) - \frac{69}{4} \zeta^2(3) + 32\zeta(5,1) - 62\zeta(5) \log 2
\]
\[+ 28\zeta(2)\zeta(3) \log 2 - 16\zeta(2)\zeta(3,1).
\]

9 General MSV-MtV product variant of Apéry-type series, II

In this section, we will use a different approach to derive more general formulas of some MZSV-MtV product variant of Apéry-type series treated in the last section. The key idea is to use the shuffle regularization of CMZVs developed in Racinet’s thesis (see [16]). For an exposition in English, the reader can consult the book by the second author [28].

Lemma 9.1. (cf. [23 Thm. 2.1]) For any composition $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$, $n \in \mathbb{N}$, and $|x| \leq 1$, we have
\[
\int_0^x \frac{t^n}{1-t} \left(\frac{dt}{t}\right)^{k_1-1} \cdots \frac{dt}{1-t} \left(\frac{dt}{t}\right)^{k_r-1} = (-1)^r \zeta_n^*(k;x) - \sum_{j=1}^r (-1)^j \zeta_n^*(\vec{k}_{j-1}) \text{Li}_{k_j}(x),
\]
where
\[
\zeta_n^*(k;x) := \sum_{n \geq n_1 \geq \cdots \geq n_r \geq 1} \frac{x^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}.
\]

Lemma 9.2. Let $p, m \in \mathbb{N}_0$, $r \in \mathbb{N}$ and $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$. If $|x| < 1$ then
\[
(-1)^r \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\zeta_n^*(k;x)t_n(1_p)}{n^{m+1}} - \sum_{j=1}^r (-1)^j \text{Li}_{k_j}(x) \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\zeta_n^*(\vec{k}_{j-1})t_n(1_p)}{n^{m+1}}
\]
\[= 2^r \int_0^1 \frac{2tdt}{1-t^2} \left(\frac{2tdt}{1-t^2}\right)^{k_r-1} \cdots \left(\frac{2tdt}{1-t^2}\right)^{k_1-1} \frac{dt}{t} \left(\frac{2tdt}{1-t^2}\right)^m \frac{dt}{t} \chi_p \left(\frac{dt}{t}\right)^p ,
\]
where $\chi_0 = \frac{2dt}{1+t}$ and $\chi_p = \frac{2dt}{1-t^2}$ for $p \geq 1$.

Proof. If $p > 0$ then from (8) we have
\[
2^p \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n(1_p)}{n^{m+1}} x^n = \int_0^x \left(\frac{dt}{1-t}\right)^p \frac{dt}{t} \left(\frac{dt}{t}\right)^m.
\]

Multiplying (8.1) by $\frac{2^p}{4^n} \binom{2n}{n} \frac{t_n(1_p)}{n^{m+1}}$ we see that
\[
2^p (-1)^r \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\zeta_n^*(m;x)t_n(1_p)}{n^{m+1}} - 2^p \sum_{j=1}^r (-1)^j \text{Li}_{k_j}(x) \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\zeta_n^*(\vec{k}_{j-1})t_n(1_p)}{n^{m+1}}
\]
\[= \int_0^x \left(\frac{dt}{1-t}\right)^p \frac{dt}{t} \left(\frac{dt}{t}\right)^m \frac{dt}{1-t} \left(\frac{dt}{t}\right)^{k_1-1} \cdots \frac{dt}{1-t} \left(\frac{dt}{t}\right)^{k_r-1}
\]
\[\int_0^{t^{1-2^p}} \frac{1}{1-t} \left(\frac{2tdt}{1-t^2}\right)^{k_r-1} \cdots \left(\frac{2tdt}{1-t^2}\right)^{k_1-1} \frac{dt}{t} \left(\frac{2tdt}{1-t^2}\right)^m \frac{2dt}{t} \left(\frac{dt}{t}\right)^p .
\]
If $p = 0$ then we need to use (6.9) and see that the

\[
2^p(-1)^r \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\zeta''(m; x)}{n^{m+1}} - 2^p \sum_{j=1}^{r} (-1)^j \text{Li}_{k_j}(x) \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\zeta''(k_{j-1})}{n^{m+1}}
\]

\[
= \int_0^x \left( \frac{1}{\sqrt{1-t}} - 1 \right) \frac{dt}{t} \left( \frac{dt}{t} \right)^m \frac{dt}{1-t} \left( \frac{dt}{1-t} \right)^{k_{r-1}} \cdots \frac{dt}{1-t} \left( \frac{dt}{1-t} \right)^{k_1-1} t^{r+p} \int_1^{\tau(\varepsilon)} \left( \frac{2dt}{1-t^2} \right)^{k_{r-1}} \cdots \left( \frac{2dt}{1-t^2} \right)^{k_1-1} \frac{dt}{1-t^2}.
\]

This completes the proof of the lemma. \qed

Put $\mathbf{a} = \frac{dt}{\xi - \ell}$ and for $N \in \mathbb{N}$ denote by $\Gamma_N$ the group of $N$th roots of unity. Set $x_\varepsilon = \frac{dt}{\xi - \ell}$ for any $\varepsilon \in \Gamma_N$.

**Lemma 9.3.** Let $r \in \mathbb{N}$, $\ell \in \mathbb{N}_0$ and $\alpha_1, \ldots, \alpha_r \in \{ \mathbf{x}_\varepsilon : \varepsilon \in \Gamma_N \} \cup \{ \mathbf{a} \}$ with $\alpha_1 \neq \mathbf{a}$ and $\alpha_r \neq \mathbf{x}_1$, namely, $\boldsymbol{\alpha} = \alpha_1 \ldots \alpha_r$ is admissible. Suppose $\varepsilon \in (0, 1)$ and the function $\tau(\varepsilon) = O(\varepsilon)$ satisfies $\log \tau(\varepsilon) = A + \lambda \log \varepsilon + O(\varepsilon)$ for some positive $\lambda \in \mathbb{Q}$, $A = 0$ or $A \in \text{CMZV}_1^N$. Then we have

\[
\int_0^1 \mathbf{a}^\ell \mathbf{\alpha} = P_1(\log \varepsilon) + O(\varepsilon \log^{j+r-1} \varepsilon)
\]

for some polynomial $P_1(T) \in \text{CMZV}_1^N$. Moreover, the coefficient of $T^{\ell-j}$ has weight $j + r$ for all $0 \leq j \leq \ell$.

**Proof.** We proceed by induction on the total weight $w = r + \ell$. Let $\shuffle$ denote the shuffle product of words used when multiplying iterated integrals first studied by K.T. Chen [8].

The case $w = 1$ is obvious. In general, if $\ell = 0$ then this is trivial since $\boldsymbol{\alpha}$ is admissible and the value

\[
\int_0^1 \mathbf{a}^\ell \mathbf{\alpha} \text{ is finite. If } \ell \geq 1 \text{ then we see that}
\]

\[
\int_0^1 \mathbf{a}^\ell \mathbf{\alpha} - \int_0^1 \mathbf{a}^\ell \int_0^1 \mathbf{\alpha} = \int_0^1 (\mathbf{a}^\ell \mathbf{\alpha} - \mathbf{a}^\ell \shuffle \mathbf{\alpha}) = Q_{\ell-1}[\log \varepsilon]
\]

where by induction $Q_{\ell-1}[T] \in \text{CMZV}_1^N[T]$ with its coefficient of $T^{\ell-j}$ having weight $j + r$ for all $1 \leq j \leq \ell$. But $\log \tau(\varepsilon) = A + \lambda \log \varepsilon + O(\varepsilon)$ and therefore

\[
\int_0^1 \mathbf{a}^\ell = \left( \frac{-1}{\ell!} \right) \log^\ell \tau(\varepsilon) = \left( \frac{-\lambda}{\ell!} \right) (A + \lambda \log \varepsilon)^\ell + O(\varepsilon \log^{j-1} \varepsilon).
\]

On the other hand,

\[
\int_0^1 \mathbf{\alpha} - \int_0^1 \mathbf{\alpha} = \int_0^1 \mathbf{\alpha} + \sum_{j=1}^{r-1} \int_0^1 \mathbf{\alpha}_1 \ldots \mathbf{\alpha}_j \int_0^1 \mathbf{\alpha}_{j+1} \ldots \mathbf{\alpha}_r.
\]

For all $j \leq r$, since $\alpha_i \neq \mathbf{a}$ we may assume $\alpha_1 \ldots \alpha_j = x_\varepsilon^s_1 \mathbf{a}^{s_1-1} \ldots x_\varepsilon^s_d \mathbf{a}^{s_d-1}$ for some $(s_1, \ldots, s_d) \in \mathbb{N}^d$. Then $\int_0^1 \mathbf{\alpha}_1 \ldots \mathbf{\alpha}_j = L_{s_1, \ldots, s_d}(\tau(\varepsilon)/\xi_1, \ldots) = O(\varepsilon)$. Furthermore, by induction we see that

\[
\int_0^1 \mathbf{\alpha}_j \ldots \mathbf{\alpha}_r = O(\log^{r-1} \varepsilon). \text{ Hence}
\]

\[
\int_0^1 \mathbf{\alpha} - \int_0^1 \mathbf{\alpha} = O(\varepsilon \log^{j-1} \varepsilon).
\]
Combining (9.4) and (9.5) we have
\[
\int_{\tau(\epsilon)}^1 a^t \int_{\tau(\epsilon)}^1 x = \int_{\tau(\epsilon)}^1 a^t \int_0^1 x + O(\varepsilon \log^{t+r-1} \varepsilon) = \frac{(-\lambda)^t}{t!} (A + \lambda \log \varepsilon)^t \int_0^1 x + O(\varepsilon \log^{t+r-1} \varepsilon),
\]
where \( \int_0^1 x \in \text{CMZV}_r^N \). Together with (9.3) this yields that
\[
\int_{\tau(\epsilon)}^1 a^t x = \frac{(-\lambda)^t}{t!} (A + \lambda \log \varepsilon)^t \int_0^1 x + Q_{t-1}(\log \varepsilon) + O(\varepsilon \log^{t+r-1} \varepsilon)
\]
\[
= P_t(\log \varepsilon) + O(\varepsilon \log^{t+r-1} \varepsilon).
\]
This completes the proof of the lemma.

\[\square\]

**Lemma 9.4.** Suppose \( m, p, r \in \mathbb{N} \) and \( k \in \mathbb{N}^r \). If a function \( f(n, p) \) satisfies that there is a constant \( C_p \) such that \( |f(n, p)| < C_p \) for all \( n \) then we have
\[
\lim_{x \to 1^-} \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{f(n, p)(\zeta_n^*(k) - \zeta_n^*(k; x))}{n^m} = 0,
\]
\[
\lim_{x \to 1^-} \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{f(n, p)(t_n^*(k) - t_n^*(k; x))}{(2n+1)^m} = 0,
\]
\[
\lim_{x \to 1^-} \sum_{n=1}^{\infty} \frac{4^n f(n, p)(t_n^*(k) - t_n^*(k; x))}{(2n)^n^{m+1}} = 0,
\]
where
\[
t_n^*(k; x) := \sum_{n \geq n_1 \geq \cdots \geq n_r \geq 1} x^{2n_{r-1}}.
\]

**Proof.** Let \( x \in (1/2, 1] \). Then
\[
|\zeta_n^*(k) - \zeta_n^*(k; x)| \leq \sum_{n \geq n_1 \geq n_2 \geq \cdots \geq n_r \geq 0} (1-x^n \prod_{j=1}^{r} \frac{1}{nj}) \leq (1-x^n) \zeta_n^*(1_r) \ll (1-x^n) \log^{r}(n)
\]
and
\[
|t_n^*(k) - t_n^*(k; x)| = \sum_{n \geq n_1 \geq n_2 \geq \cdots \geq n_r \geq 0} (1-x^{2n_{r-1}}) \prod_{j=1}^{r} \frac{1}{2n_j - 1}
\]
\[
= \sum_{n \geq n_1 \geq n_2 \geq \cdots \geq n_r \geq 0} (1-x^{n_r}) \prod_{j=1}^{r} \frac{1-(-1)^{n_j}}{n_j}
\]
\[
\leq 2^r \sum_{n \geq n_1 \geq n_2 \geq \cdots \geq n_r \geq 0} (1-x^{2n}) \prod_{j=1}^{r} \frac{1}{n_j} \ll (1-x^{2n}) \log^{r}(n).
\]

Observing that \( \frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{2n}} \) by Stirling’s formula we see that for \( m \in \mathbb{N} \) all the series
\[
\sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{t_n^*(k) - t_n^*(k; x)}{(2n+1)^m}, \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\zeta_n^*(k) - \zeta_n^*(k; x)}{n^m}, \sum_{n=1}^{\infty} \frac{4^n (t_n^*(k) - t_n^*(k; x))}{(2n)^n^{m+1}}
\]
converge absolutely and uniformly for \( x \in (1/2, 1] \), which yields the lemma immediately. \[\square\]
Theorem 9.5. For any $p \in \mathbb{N}_0, r, m \in \mathbb{N}$ and $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$, we have

\[
\sum_{n=1}^{\infty} \frac{1}{n!} \binom{2n}{n} \frac{\zeta^n_{r}(k)t_n(1_p)}{n^m} \in \text{CMZV}_{|k|+m+p}^2.
\] (9.6)

Proof. Taking $N = 2$ and $\tau(\varepsilon) = \sqrt{\varepsilon}$ in Lemma 9.3 we see that the regularized value of the iterated integral in (9.2)

\[
\int_{\sqrt{\varepsilon}}^{1} \left( \frac{2\varepsilon t}{1-t^2} \right)^{k_r-1} \frac{dt}{t} \cdots \frac{dt}{t} \left( \frac{2\varepsilon t}{1-t^2} \right)^{k_1-1} \frac{dt}{t} \left( \frac{2\varepsilon t}{1-t^2} \right)^{m-1} \chi_p \left( \frac{dt}{t} \right)^p
\]
is a multiple mixed value which lies in $\text{CMZV}_{|k|+m+p}^2$. Further, for all $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$

\[
\zeta^*_m(m; x) = \sum_{\circ = "-" \or +"} \zeta^*(m_1 \circ \cdots \circ m_d; x),
\]
where, by [28] Lemma 13.3.29, Remark 13.3.23 each $\zeta^*(s; x)$ term is regarded as a shuffle regularization of $\zeta(s)$ (setting $s = (s_1, \ldots, s_t) \in \mathbb{N}$)

\[
\zeta^*(s; x) = \mathrm{Li}_s(1 - \varepsilon) = \int_{0}^{1-\varepsilon} x_1 a^{s_1-1} \cdots x_1 a^{s_t-1} = F(\log \varepsilon) + O(\varepsilon \log^l \varepsilon)
\]
for some polynomial $F[T] \in \text{CMZV}_{1}^1[T]$ whose constant term has weight $|s| = |m|$. The theorem follows immediately from Lemma 9.2 by induction on $r$ after taking constant terms of the regularization of (9.2) (i.e., taking $x = 1 - \varepsilon \to 1$), using Lemma 9.3. \hfill \Box

Theorem 9.6. Suppose $m, p \in \mathbb{N}$ and $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$ with $r \in \mathbb{N}_0$, where $k = \emptyset$ and $|k| = 0$ if $r = 0$. Set $t^*_n(\emptyset) = 1$. Then we have

\[
\sum_{n=1}^{\infty} \frac{(2n)!}{n!} \binom{2n}{n} t_n(2p-1) t^*_n(k) \in i\text{CMZV}_{2p+m+|k|-2}^4,
\] (9.7)

\[
\sum_{n=1}^{\infty} \frac{4^n}{{n}^m+1} \zeta_{n-1}(2p-1) t_n^*(k) \in \text{CMZV}_{2p+m+|k|-1}^4.
\] (9.8)

Proof. By the proof of [2] p. 262, Prop. 15], for $p \in \mathbb{N}$ we have

\[
\frac{(\arcsin x)^{2p-1}}{(2p-1)!} = \sum_{n=p-1}^{\infty} \frac{(2n)!}{n!} t_n(2p-1) \frac{x^{2n+1}}{2n+1} \]

(9.9)

and

\[
\frac{(\arcsin x)^{2p}}{(2p)!} = \sum_{n=p-1}^{\infty} \frac{4^{n+1-p} \zeta_{n}(2p-1)}{(2n+1)(2n)!} \frac{x^{2n+2}}{2n+2} = \sum_{n=p}^{\infty} \frac{4^{n-p} \zeta_{n-1}(2p-1)}{n^2(2n)!} x^{2n}.
\]

Moreover, we have the following iterated integral

\[
(\arcsin x)^p = \left( \int_{0}^{x} \frac{dt}{\sqrt{1-t^2}} \right)^p = p! \int_{0}^{x} \left( \frac{dt}{\sqrt{1-t^2}} \right)^p.
\]

Let

\[
F_{p,m}(x) := \int_{0}^{x} \left( \frac{dt}{\sqrt{1-t^2}} \right)^p \left( \frac{dt}{t} \right)^{m-1}.
\]

(9.11)
Then (9.9) and (9.10) yield
\[ F_{2p-1,m}(x) = \sum_{n=p-1}^{\infty} \frac{(2n)!}{4^n} t_n(2p-1) x^{2n+1} \quad \text{and} \quad F_{2p,m}(x) = \sum_{n=p}^{\infty} \frac{4^n\zeta_{n-1}(2p-1)}{(2n+1)^{n+1}} x^{2n}. \] (9.12)

By Thm. 3.6], for any \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r, n \in \mathbb{N} \) and \( |x| < 1 \), we have
\[
\int_0^x \frac{t^{2n-1} dt}{(1-t^2)^{k_1+1}} = (2n-1) \sum_{j=1}^{r} (-1)^{j-1} t_n^{(j-1)}(\kappa_{j-1}) t_{n}(\kappa_j; x) + (-1)^r t_n^{(r)}(k; x).
\] (9.13)

Multiplying (9.13) by \( \frac{(2n)!}{4^n} t_n(2p-1) \) and \( \frac{4^n\zeta_{n-1}(2p-1)}{2m-1\binom{2n}{n} n^{m+1}} \), respectively, then summing up, we see that
\[
\sum_{j=1}^{r} (-1)^{j-1} t_n^{(j-1)}(\kappa_{j-1}) x^{2n+1} \quad \text{and} \quad \sum_{j=1}^{r} (-1)^r t_n^{(r)}(k; x) x^{2n+1}
\] (9.14)

and by (9.11)
\[
\sum_{j=1}^{r} (-1)^{j-1} t_n^{(j-1)}(\kappa_{j-1}) x^{2n+1} \quad \text{and} \quad \sum_{j=1}^{r} (-1)^r t_n^{(r)}(k; x) x^{2n+1}
\] (9.15)

Applying \( t \to \frac{1 - t^2}{1 + t^2} \) to (9.14) and (9.15), setting \( y = x_{-i} + x_i - x_{-1} - x_1 \) and \( z = a - x_{-i} - x_i \), we get
\[
\begin{align*}
a &= \frac{dt}{t} \to -\left( \frac{2dt}{1 + t^2} + \frac{2dt}{1 - t^2} \right) = y, \\
\omega_2 &= \frac{dt}{1 - t^2} \to -\frac{dt}{t} = -a, \\
\omega_3 &= \frac{dt}{1 + t^2} = i(x_{-i} - x_i), \\
\frac{dt}{\sqrt{1 - t^2}} &= \frac{dt}{t} + \frac{tdt}{1 - t^2} \to y + z = -a - x_{-1} - x_1.
\end{align*}
\] (9.16)

Therefore, setting
\[
\tau(\varepsilon) := \sqrt{\frac{\varepsilon}{2 - \varepsilon}}
\]
we have
\[
\sum_{j=1}^{r} (-1)^{j-1} t_{n}(\kappa; 1 - \varepsilon) \sum_{n=1}^{\infty} \frac{(2n)!}{4^n} t_n(2p-1) t_n^{(j-1)}(\kappa_{j-1}) + \sum_{n=1}^{\infty} \frac{(2n)!}{4^n} t_n(2p-1) t_n^{(r)}(k; 1 - \varepsilon)
\]
\[= \sum_{n=1}^{\infty} \frac{4^n\zeta_{n-1}(2p-1)}{(2n+1)^{n+1}} x^{2n}.
\] (9.18)
which lies in $i\text{CMZV}^4[T]$ after regularization by Lemma 9.3, with $N = 4$, and

$$
\sum_{j=1}^{r}(-1)^{j-1}t^*(k_j; 1 - \varepsilon) \sum_{n=1}^{\infty} \frac{4^{n-p}n^{2n}t_n(2p-1)}{(2n+1)^n n^{m+1}} = (-1)^{|k|+m-1} \int_{i(x-x_i)} y^{k-1}z \cdots y^{k-1}z y^{k-1}(y+z)y^{m-1}(i(x-x_i))^{2p-1}
$$

(9.19)

which lies in $\text{CMZV}^4[T]$ after regularization by Lemma 9.3, with $N = 4$. Here we need the fact that $\log 2 = -\text{Li}_1(-1) \in \text{CMZV}^4_1$. Hence, by induction on $r$ using the constant terms of the regularization of (9.19) and (9.20), and Lemma 9.4, we finish the proof of the theorem.

**Remark 9.7.** One may want to compare statement (9.8) with [1, Thm. 4.1] in which the product is lowered to 2 in that case. Furthermore, Au showed that the level of the CMZVs can be lowered to 2 in that case.

**Corollary 9.8.** For any $p, m, r \in \mathbb{N}$ and $k \in \mathbb{N}^r$ or $k = \emptyset$, we have

$$
\sum_{n=1}^{\infty} \frac{1}{4^n} \frac{(2n)!}{(2n+1)!} t_n(k) \in i\text{CMZV}^4_{|k|+m},
$$

(9.21)

$$
\sum_{n=1}^{\infty} \frac{4^n}{n^{m+1}} t_n(k) \in \text{CMZV}^4_{|k|+m+1},
$$

(9.22)

where $t_n(\emptyset) = 1$ and $|\emptyset| = 0$.

**Proof.** Setting $p = 1$ in Thm. 9.6, we see that the corollary follows from the fact that

$$
t_n(k) = \sum_{o = \text{"-" or "+"}} (-1)^{t_1 \circ \cdots \circ t_r} t_n(k_1 \circ \cdots \circ k_r)
$$

(9.23)

which, in turn, follows easily from

$$
t_n(k) = \sum_{o = \text{"-" or "+"}} t_n(k_1 \circ \cdots \circ k_r)
$$

(9.24)

by the Principle of Inclusion and Exclusion.

**Example 9.9.** Explicitly, set $k = (1)$ in (9.21) we have

$$
\sum_{n=1}^{\infty} \frac{1}{4^n} \frac{(2n)!}{(2n+1)!} t_n(1) = i(-1)^m \int_{0}^{1} y^{m-1}(x_{-i} - x_i) a + \sum_{j=1}^{m-1} y^{m-j}a y^{j-1}(x_{-i} - x_i) - (x_1 + x_{-1})y^{m-1}(x_{-i} - x_i).
$$

Further setting $m = 1, 2$ we get

$$
\sum_{n=1}^{\infty} \frac{1}{4^n} \frac{(2n)!}{(2n+1)!} t_n(1) = 4(G + \text{Im Li}_{1,1}(-i, i)) \approx 1.088793045,
$$

$$
\sum_{n=1}^{\infty} \frac{1}{4^n} \frac{(2n)!}{(2n+1)!} t_n(1) = i \int_{0}^{1} (y(x_{-i} - x_i) a + y a(x_{-i} - x_i) - (x_1 + x_{-1})y(x_{-i} - x_i))
$$

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Second, letting \( t \) in (9.27) we get
\[
\sum_{n=1}^{\infty} \frac{1}{2n+1} \frac{t_n(2)}{4n} = -it(2) \int_0^1 (x_i-x_i) - i \int_0^1 y(a+x_1+x_1)(x_i-x_i) \approx 0.6459640977,
\]
\[
\sum_{n=1}^{\infty} \frac{1}{2n} \frac{t_n(2)}{2n+1} = it(2) \int_0^1 y(x_i-x_i) + i \int_0^1 y(a+x_1+x_1)y(x_i-x_i) \approx 0.0937132114.
\]

Setting \( k = (2), p = 1 \) and \( m = 1 \) in (9.8) we get
\[
\sum_{n=1}^{\infty} \frac{4^p n t_n(2)}{n^2} = -4 \left( t(2) \int_0^1 (x_i-x_i)^2 + \int_0^1 ya(x_i-x_i)^2 \right) \approx 5.4641926215,
\]
\[
\sum_{n=1}^{\infty} \frac{4^p n \zeta_{n-1}(2)t_n(2)}{n^2} = 2^4 \left( t(2) \int_0^1 (x_i-x_i)^4 + \int_0^1 ya(x_i-x_i)^4 \right) \approx 4.822651414.
\]

**Theorem 9.10.** For any \( p, k, m \in \mathbb{N} \) we have
\[
\sum_{n=p-1}^{\infty} \left[ \frac{1}{4n} \frac{t_n(2p-1)}{(2n+1)m} \right]^2 \frac{t_n(2p-1)}{(2n+1)m} \approx \frac{i}{\pi} \text{CMZV}^4_{2p+m-1}, \tag{9.25}
\]
\[
\sum_{n=p}^{\infty} \left[ \frac{4^n}{2n} \right]^2 \frac{\zeta_{n-1}(2p-1)t_n^*(1_k)}{n^2} \approx \text{CMZV}^4_{2p+k+m}. \tag{9.26}
\]

**Proof.** The proof is similar to that of Thm. 9.6 First, it is well-known that
\[
\frac{1}{2n} \frac{t_n(2p-1)}{(2n+1)m} \approx \frac{2}{\pi} \int_0^{\pi/2} \sin^{2n} \theta \, d\theta = \frac{2}{\pi} \int_0^1 \frac{t^{2n}}{\sqrt{1-t^2}} \, dt.
\]
Multiplying this by \( \frac{t_n(2p-1)}{2n} \frac{t_n(2p-1)}{(2n+1)m} \) and summing up, we see that
\[
\sum_{n=1}^{\infty} \left[ \frac{1}{4n} \frac{t_n(2p-1)}{(2n+1)m} \right]^2 \frac{t_n(2p-1)}{(2n+1)m} = \frac{2}{\pi} \int_0^1 \frac{t^{2n}}{\sqrt{1-t^2}} \, dt. \tag{9.27}
\]
Second, letting \( t \to t^2 \) in the integral on the right-hand side of (8.3), we get
\[
2^k \frac{4^n}{n^2} t_n^*(1_k) = 2 \int_0^1 \left( \frac{2t^2 dt}{1-t^2} \right)^k \frac{t^{2n-1} dt}{\sqrt{1-t^2}}. \tag{9.28}
\]
Multiplying (9.28) by \( \frac{t_n(2p-1)}{2n} \frac{t_n(2p-1)}{(2n+1)m+1} \) and summing up, we have
\[
\sum_{n=1}^{\infty} \frac{\zeta_{n-1}(2p-1)t_n^*(1_k)}{n^2} \approx 2^{2p+m-k} \int_0^1 \left( \frac{2t^2 dt}{1-t^2} \right)^k \frac{F_{2p+m}(t) dt}{t^{1-t^2}}. \tag{9.29}
\]
Applying \( t \to 1 - t^2 \), using (9.16)-(9.18) and

\[
\frac{dt}{\sqrt{1 - t^2}} \to -\frac{2dt}{1 - t^2} = x_{-1} - x_1
\]

we get

\[
\text{RHS of (9.27)} = \frac{2i(-1)^{p+m}}{\pi} \int_0^1 (x_{-1} - x_1) y^{m-1}(x_{-1} - x_i)^{2p-1},
\]

\[
\text{RHS of (9.29)} = (-1)^{p+m+k} 2^{2p+m} \int_0^1 (x_{-1} - x_1) \left(y^{m-1}(x_{-1} - x_i)^{2p} \uparrow (x_0 - x_i - x_{-1})^k\right),
\]

both of which lie in CMZV\(^4\). This concludes the proof of the theorem.

\[\square\]

**Example 9.11.** Setting \( p = 1, k = 1, m = 1 \), we get

\[
\sum_{n=1}^\infty \left[\frac{4^n}{(2n)}\right]^2 \frac{t_n(1)}{n^3} = -8 \sum_{n=1}^\infty \frac{\int_0^1 (x_{-1} - x_1)\left((x_{-1} - x_i)^2 \uparrow (x_0 - x_i - x_{-1})\right)}{n^3} = 7.7112698415,
\]

which is consistent with (5.5). Now taking \( p = 2 \) or \( 3, k = 1 \) and \( m = 1 \), we have

\[
\sum_{n=1}^\infty \left[\frac{4^n}{(2n)}\right]^2 \frac{\int_0^1 (x_{-1} - x_1)\left((x_{-1} - x_i)^2 \uparrow (x_0 - x_i - x_{-1})\right)}{n^3} \approx 4.8416943704,
\]

\[
\sum_{n=1}^\infty \left[\frac{4^n}{(2n)}\right]^2 \frac{\int_0^1 (x_{-1} - x_1)\left((x_{-1} - x_i)^2 \uparrow (x_0 - x_i - x_{-1})\right)}{n^3} \approx 1.3105783945,
\]

\[
\sum_{n=1}^\infty \left[\frac{1}{4^n}\right]^2 \frac{\int_0^1 (x_{-1} - x_1)\left((x_{-1} - x_i)^2 \uparrow (x_0 - x_i - x_{-1})\right)}{2n+1} \approx 0.179386942,
\]

\[
\sum_{n=1}^\infty \left[\frac{1}{4^n}\right]^2 \frac{\int_0^1 (x_{-1} - x_1)\left((x_{-1} - x_i)^2 \uparrow (x_0 - x_i - x_{-1})\right)}{2n+1} \approx 0.0139754925.
\]

If we let \( p = 1 \) or \( 2, k = 1 \) and \( m = 2 \), then we have

\[
\sum_{n=1}^\infty \left[\frac{4^n}{(2n)}\right]^2 \frac{\int_0^1 (x_{-1} - x_1)\left(y(x_{-1} - x_i)^2 \uparrow (x_0 - x_i - x_{-1})\right)}{n^4} \approx 5.0319188594,
\]

\[
\sum_{n=1}^\infty \left[\frac{4^n}{(2n)}\right]^2 \frac{\int_0^1 (x_{-1} - x_1)\left(y(x_{-1} - x_i)^4 \uparrow (x_0 - x_i - x_{-1})\right)}{n^4} \approx 1.1896632248,
\]

\[
\sum_{n=1}^\infty \left[\frac{1}{4^n}\right]^2 \frac{\int_0^1 (x_{-1} - x_1)\left(y(x_{-1} - x_i)^3 \uparrow (x_0 - x_i - x_{-1})\right)}{2n+1} \approx 1.037947765,
\]

\[
\sum_{n=1}^\infty \left[\frac{1}{4^n}\right]^2 \frac{\int_0^1 (x_{-1} - x_1)\left(y(x_{-1} - x_i)^3 \uparrow (x_0 - x_i - x_{-1})\right)}{2n+1} \approx 0.0393547288.
\]
We see that (9.31) is consistent with (6.6).

In fact, we can slightly generalize (9.25) to involve a factor of multiple harmonic star sum, which requires the following lemma.

**Lemma 9.12.** For all $m, n \in \mathbb{N}$, we have

$$\sum_{i=1}^{m} \zeta_n^{*}(1_{m-i}) \zeta_n(i) = m \zeta_n^{*}(1_m).$$

**Proof.** This is an immediate consequence of Lemma 2.2 if we take $x_k = 1/k$ for all $k$. Alternatively, we can prove it directly by induction. The lemma is clearly true if $m = 1$. Suppose $m \geq 2$. Fix $n$ and let $\star$ denote the stuffle product of the word algebra representing the multiple harmonic star sums by induction. The lemma follows immediately by a simple index shifting. ζ

$$\zeta_n^{*}(1_m)$$

and let $\zeta_n^{*}(1_m)$ by $b^m$. Then we have

$$\sum_{i=1}^{m} b^{m-i} \ast a^{i-1} b = a^{m-1} b + \sum_{i=1}^{m-1} (a^{i-1} b^{m-i+1} + b(a^{i-1} b \ast b^{m-i}) - a^i b^{m-i})$$

$$= b^m + \sum_{i=2}^{m} a^{i-1} b^{m-i} + (m-1)b^m - \sum_{i=1}^{m-1} a^i b^{m-i}$$

by induction. The lemma follows immediately by a simple index shifting. □

**Proposition 9.13.** Let $b_0 = 1$ and for all $j \geq 1$ define recursively

$$b_j = -\frac{1}{j} \sum_{i=1}^{j} a^i b_{j-i} \zeta(i).$$

Then $b_j$ is in the weight $j$ piece of $\mathbb{Q}[\log 2, \zeta(2), \zeta(3), \zeta(4), \ldots]$ (assuming $\log 2$ has weight one) for all $j \geq 1$. Moreover, we have

$$\frac{\partial^k B(a, b)}{\partial b^k} \bigg|_{a=n+1/2, b=1/2} = \frac{(-1)^k k! \pi (2n)}{4^n} \sum_{j=0}^{k} b_j \zeta_n^{*}(1_{k-j}).$$

(9.32)

**Proof.** By induction it is easy to see that $b_j \in \mathbb{Q}[\log 2, \zeta(2), \zeta(3), \zeta(4), \ldots]$ and has weight $j$.

We now prove (9.32) by induction on $k$. The case $k = 0$ is clear. Now suppose (9.32) is true for all partial derivatives of order up to $k - 1$. Observe that

$$\frac{\partial^k B(a, b)}{\partial b^k} = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{\partial^i B(a, b)}{\partial b^i} \left[ \psi^{(k-i-1)}(b) - \psi^{(k-i-1)}(a+b) \right]$$

(9.33)

and

$$\psi^{(k)}(1/2) - \psi^{(k)}(n+1) = (-1)^{k+1} k! \sum_{m=0}^{\infty} \left\{ \frac{1}{(1/2 + m)^{k+1}} - \frac{1}{(n+1+m)^{k+1}} \right\}$$

$$= (-1)^{k+1} k! \left\{ 2^{k+1} t(k+1) - \zeta(k+1) + \zeta_n(k+1) \right\}$$

$$= (-1)^{k+1} k! \left\{ -2^{k+1} \zeta(k+1) + \zeta_n(k+1) \right\}.$$

Hence, setting $a = n + 1/2, b = 1/2$ we get

$$\frac{\partial^k B(a, b)}{\partial b^k} \bigg|_{a=n+1/2, b=1/2} = (-1)^k (k-1)! \sum_{i=0}^{k-1} \frac{(-1)^i}{i!} \frac{\partial^i B(a, b)}{\partial b^i} \bigg|_{a=n+1/2, b=1/2} \left\{ \zeta_n(k-i) - 2^{k-i} \zeta(k-i) \right\}.$$
This yields that
\[
\frac{(-1)^k 4^n}{\pi (2n)_k (k-1)!} \frac{\partial^k B(a, b)}{\partial b^k} \bigg|_{a=\frac{n+1}{2}, \quad b=\frac{1}{2}} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} b_j \zeta_n^*(1_{i-j}) \left\{ \zeta_n(k-i) - 2^{k-i} \zeta(k-i) \right\} \\
= \sum_{j=0}^{k-1} \sum_{i=0}^{j} b_j \zeta_n^*(1_{k-i-j}) \zeta_n(i) - \sum_{i=0}^{k-1} \sum_{j=0}^{i} b_{i-j} \zeta_n^*(1_{j}) 2^{k-i} \zeta(k-i) \\
= \sum_{j=0}^{k-1} \sum_{i=1}^{j} b_j \zeta_n^*(1_{k-i-j}) \zeta_n(i) - \sum_{j=1}^{k} \sum_{i=1}^{j} b_{j-i} \zeta_n^*(1_{k-j}) 2^i \zeta(i) \\
= \sum_{j=0}^{k-1} b_j (k-j) \zeta_n^*(1_{k-j}) + \sum_{j=1}^{k} j b_j \zeta_n^*(1_{k-j})
\]
by Lemma 9.12 and inductive assumption, respectively. The proposition follows readily.

**Theorem 9.14.** For any \( p, m \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \),
\[
\sum_{n=p-1}^{\infty} \left[ \frac{(2n)_n}{4^n} \right] \frac{t_n(2p-1) \zeta_n^*(1_{k})}{(2n+1)^m} \in \frac{i}{\pi} \text{CMZV}_{2p+k+m-1}^4.
\]

**Proof.** Note that for any integers \( n, k \geq 0 \),
\[
\int_0^{1} \left( \frac{tdt}{1-t^2} \right)^k \frac{2n}{\sqrt{1-t^2}} dt = \frac{(-1)^k}{2k+1} \int_0^{1} 2n \log^k(1-t^2) \sqrt{1-t^2} dt \\
= \frac{(-1)^k}{2k+1} \int_0^{1} x^{n-1/2} \log^k(1-x) \sqrt{1-x} dx \\
= \frac{(-1)^k}{2k+1} \frac{\partial^k B(a, b)}{\partial b^k} \bigg|_{a=\frac{n+1}{2}, \quad b=\frac{1}{2}} = \frac{\pi}{2k+1} \left[ \frac{(2n)_n}{4^n} \right] \sum_{j=0}^{k} b_j \zeta_n(1_{k-j})
\]
by Prop. 9.13. Multiplying this by \( \frac{(2n)_n t_n(2p-1)}{4^n(2n+1)^m} \) and summing up, we see that
\[
\sum_{j=0}^{k} b_j \sum_{n=p-1}^{\infty} \left[ \frac{(2n)_n}{4^n} \right] \frac{t_n(2p-1) \zeta_n^*(1_{k-j})}{(2n+1)^m} \\
= \frac{2^{k+1}}{\pi} \int_0^{1} \left( \frac{tdt}{1-t^2} \right)^k \sum_{n=p-1}^{\infty} \left[ \frac{(2n)_n}{4^n} \right] \frac{t_n(2p-1)}{4^n(2n+1)^m} \sqrt{1-t^2} dt \\
= \frac{2^{k+1}}{\pi} \int_0^{1} \left( \frac{tdt}{1-t^2} \right)^k F_{2p-1, m}(t) \frac{dt}{t \sqrt{1-t^2}} dt \in \frac{i}{\pi} \text{CMZV}_{2p+k+m-1}^4.
\]
The theorem follows easily from an induction on \( k \).

**Theorem 9.15.** For any \( p, m \in \mathbb{N}_0 \) we have
\[
t(m+2, 2p) = \int_0^{1} \left( \frac{dt}{\sqrt{1-t^2}} \right)^{2p+1} \left( \frac{dt}{t} \right)^m \frac{dt}{\sqrt{1-t^2}}.
\]
Proof. Replacing $n$ by $n+1$ in (9.29), multiplying by $\frac{\binom{2n}{n}}{4^n(2n+1)^{m+1}}$ and summing up, we obtain

$$
\int_0^1 \left(\frac{2x}{1-x^2}\right)^k \frac{F_{2p+1,m+1}(t)dt}{\sqrt{1-t^2}} = 2^k \sum_{n=1}^\infty \frac{t_{n+1}(1_k)\zeta_n(2p)}{(2n+1)^{m+2}}.
$$

The theorem follows immediately if we set $k = 0$. \hfill \Box

The following result can be found at the end of the proof of [6, Thm. 2.8]. We can now prove it as an corollary of Thm. 9.15.

**Corollary 9.16.** For any $p \in \mathbb{N}_0$ we have

$$
t(3, 2p) = \frac{1}{(2p+1)!} \int_0^1 \frac{(\arcsin t)^{2p+1} \arccos t dt}{t}.
$$

**Proof.** Note that

$$
\int_0^t \left(\frac{dx}{\sqrt{1-x^2}}\right)^{2p+1} = \frac{(\arcsin t)^{2p+1}}{(2p+1)!}.
$$

Taking $m = 1$ in the Thm. 9.15 we get

$$
t(3, 2p) = \frac{1}{(2p+1)!} \int_0^1 \int_0^t \frac{(\arcsin x)^{2p+1} dx}{x} \frac{dt}{\sqrt{1-t^2}}.
$$

The corollary follows easily from integration by parts once and the fact that $\arccos t = \pi/2 - \arcsin t$. \hfill \Box

**Theorem 9.17.** For integers $p, k, m \in \mathbb{N}_0$ with $m \geq 3$, we have

$$
\sum_{n \geq 0} \left[\frac{4^n}{\binom{2n}{n}}\right] t_{n+1}^m(1_k)\zeta_n(2p) \in \text{CMZV}^{4}_{m+2p+k}.
$$

**Proof.** For all $k \geq 1$ we have

$$
t_{n+1}^m(1_k) = t_{n+1}^m(1_{k-1}) - \frac{t_{n+1}^m(1_{k-1})}{2n+1}.
$$

Thus Thm. 9.17 follows immediately from the next theorem. \hfill \Box

**Theorem 9.18.** For integers $s \geq 0$ and $p \geq 1$, we have

$$
\sum_{n \geq 0} \left[\frac{4^n}{\binom{2n}{n}}\right] t_{n+1}^s(1_k)\zeta_n(2p-1) \in \text{CMZV}^{4}_{s+2p+k+1}.
$$

**Proof.** Differentiating (9.19) and then dividing by $x$ we get

$$
\sum_{n=p-1}^\infty \frac{4^{n+1-p}\zeta_n(2p-1)}{(2n+1)\binom{2n}{n}} x^{2n} = \frac{1}{x^\sqrt{1-x^2}} \frac{(\arcsin x)^{2p-1}}{(2p-1)!} = \frac{1}{x^\sqrt{1-x^2}} \int_0^x \left(\frac{dt}{\sqrt{1-t^2}}\right)^{2p-1}.
$$

Repeatedly integrating and then dividing by $x$ exactly $s$ times leads to

$$
\sum_{n \geq 0} \frac{4^{n+1-p}\zeta_n(2p-1)}{(2n+1)^{s+2}\binom{2n}{n}} x^{2n} = \frac{1}{x^\sqrt{1-x^2}} \int_0^x \left(\frac{dt}{\sqrt{1-t^2}}\right)^{2p-1} \frac{dt}{t^{s+1}} \left(\frac{dt}{t}\right)^{s}.
$$

(9.37)

where we have used the convention that $\zeta_n(2p-1) = 0$ if $n < p - 1$. Since

$$
\binom{2n+2}{n+1}^{-1} \frac{1}{n+1} = \frac{1}{2} \binom{2n}{n}^{-1} \frac{1}{2n+1},
$$

$$
\binom{2n+1}{n}^{-1} \frac{1}{n} = \frac{1}{2n+1}.
$$

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we see that
\[
\sum_{n \geq 0} \left[ \frac{4^n}{(2n)!} \right]^2 \zeta_n(2p, 1) \zeta_n(2p-1) = \sum_{n \geq 0} \frac{4^n}{(2n)!} \zeta_n(2p-1) \cdot \frac{1}{2(1+(2n+2))} \frac{4^n+1}{(2n+1)^{s+1}} \frac{4^n}{(2n+1+(2n+2)^{s+1})} \frac{t^{*+1}}{n_{n+1}} (1_k)
\]
\[
= \sum_{n \geq 0} \frac{4^n}{(2n)!} \zeta_n(2p, 1) \zeta_n(2p) \int_0^1 \left( \int_0^x \left( \frac{2t}{1-t^2} \right)^k \right) \frac{x^{2n+1}}{\sqrt{1-x^2}} \frac{dx}{x} \quad (by \ (11.28))
\]
\[
= 4^{p-1} \int_0^1 \left( \int_0^x \left( \frac{2t}{1-t^2} \right)^k \right) \frac{x^{2p-1}}{\sqrt{1-x^2}} \frac{dx}{x} \quad (by \ (11.37))
\]
\[
= 4^{p-1} \int_0^1 \frac{\left( \frac{2t}{1-t^2} \right)^k}{\sqrt{1-t^2}} \int_{\sqrt{1-t^2}}^{\frac{t}{\sqrt{1-t^2}}} \left( \frac{dt}{t} \right)^{s} \frac{dt}{t}
\]

By the change of variables \( t \to \frac{1-t}{1+t^2} \) using (9.10)-(9.18) and (9.34) we see easily that this iterated integral lies in \( \text{CMZV}^{4}_{+2p+k+1} \). This completes the proof of the theorem.

\[ \square \]

10 Concluding remarks and questions

In this paper, we have proved there are a few variant families of Apéry type series such that each one can be expressed in terms of the real and imaginary part of CMZVs of the same weight and level (divided by \( \pi \) sometimes). In general, however, as manifested by Example 5.4 not every Apéry type series has this property. Hence, we would like to conclude our paper with the following questions.

**Question 10.1.** Let \( a_n = \frac{1}{4^n} \binom{2n}{n} \). Is it true that for all \( m \in \mathbb{N} \), \( p \in \mathbb{N}_{\geq 2} \), \( q \in \mathbb{N}_{\geq 3} \), and all compositions of positive integers \( k \) and \( l \) (including the cases \( k = \emptyset \) or \( l = \emptyset \)),

(i) \( \sum_{n=1}^{\infty} a_n \zeta_n(k) t_n(l) \in \text{CMZV}_{|k|+|l|+m}^{4} \),

(ii) \( \sum_{n=0}^{\infty} a_n^2 \zeta_n(k) t_n(l) \in \frac{1}{\pi} \text{CMZV}_{|k|+|l|+m+1}^{4} \),

(iii) \( \sum_{n=1}^{\infty} a_n^{-1} \zeta_n(k) t_n(l) \in \text{CMZV}_{|k|+|l|+p}^{4} \),

(iv) \( \sum_{n=1}^{\infty} a_n^{-2} \zeta_n(k) t_n(l) \in \text{CMZV}_{|k|+|l|+q}^{4} \),

(v) \( \sum_{n=0}^{\infty} a_n \zeta_n(k) t_n(l) \in i \text{CMZV}_{|k|+|l|+m}^{4} \),

(vi) \( \sum_{n=0}^{\infty} a_n^2 \zeta_n(k) t_n(l) \in i \frac{1}{\pi} \text{CMZV}_{|k|+|l|+m+1}^{4} \),

(vii) \( \sum_{n=0}^{\infty} a_n^{-1} \zeta_n(k) t_n(l) \in i \text{CMZV}_{|k|+|l|+p}^{4} \),

(viii) \( \sum_{n=0}^{\infty} a_n^{-2} \zeta_n(k) t_n(l) \in i \text{CMZV}_{|k|+|l|+q}^{4} \).

Can any of these be improved to \( \text{CMZV}^{2} \)?

In our recent papers [24, 25] we use a completely different approach to answer (i)-(vii) affirmatively. Note that Thm. 9.16 (resp. Thm. 9.11) provides the affirmative answer for some special cases of (iii) and (v) (resp. (iv) and (vi)). Moreover, Thm. 9.17 confirm (viii) in some special cases. Furthermore, when \( l = \emptyset \) (i) and (ii) follow from [11 Thm. 4.1] and [11 Thm. 4.14], respectively.

To conclude, we remark that many results in this paper can be regarded as special cases of the above eight families of Apéry type series even though they use the star version, because we have the well-known relations (9.23) and (9.24) between the star and non-star versions. These two relations hold for the multiple harmonic sums \( \zeta_n \), too.

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