Further results on $A$-numerical radius inequalities

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Abstract
Let $A$ be a bounded linear positive operator on a complex Hilbert space $\mathcal{H}$. Furthermore, let $B_A \subseteq \mathcal{H}$ denote the set of all bounded linear operators on $\mathcal{H}$ whose $A$-adjoint exists, and $\mathcal{A}$ signify a diagonal operator matrix with diagonal entries are $A$. Very recently, several $\mathcal{A}$-numerical radius inequalities of $2 \times 2$ operator matrices were established. In this paper, we prove a few new $\mathcal{A}$-numerical radius inequalities for $2 \times 2$ and $n \times n$ operator matrices. We also provide a new proof of an existing result by relaxing a sufficient condition “$A$ is strictly positive”. Our proofs show the importance of the theory of the Moore–Penrose inverse of a bounded linear operator in this field of study.

Keywords $\mathcal{A}$-numerical radius · Moore–Penrose inverse · Positive operator · Semi-inner product · Inequality · Operator matrix

Mathematics Subject Classification 47A12 · 47A30 · 47A63 · 47A05

1 Introduction
Throughout, $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. By $B(\mathcal{H})$, we mean the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$. Let $\| \cdot \|$ be the norm induced from $\langle \cdot, \cdot \rangle$. For $A \in B(\mathcal{H})$, $R(A)$ stands for the range space of $A$ and $\overline{R(A)}$ for the norm closure of $R(A)$ in $\mathcal{H}$. And $A^*$ represents the adjoint operator of $A$. An operator $A \in B(\mathcal{H})$ is called selfadjoint if $A = A^*$. A selfadjoint operator $A \in B(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and is called strictly positive if $\langle Ax, x \rangle > 0$ for all non-zero $x \in \mathcal{H}$. If $A$ is a positive (strictly positive)
operator, then we use the notation \( A \geq 0 \) \((A > 0)\). Let \( \mathbb{A} \) be an \( n \times n \) diagonal operator matrix whose diagonal entries are positive operator \( A \) for \( n = 1, 2, \ldots \). Then, \( \mathbb{A} \in \mathcal{B}(\bigoplus_{i=1}^{n} \mathcal{H}) \) and \( \mathbb{A} \geq 0 \). If \( A \geq 0 \), then it induces a positive semidefinite sesquilinear form, \( \langle \cdot, \cdot \rangle_{A} : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) defined by \( \langle x, y \rangle_{A} = \langle A x, y \rangle_{A} \), \( x, y \in \mathcal{H} \). Let \( \| \cdot \|_{A} \) denote the seminorm on \( \mathcal{H} \) induced by \( \langle \cdot, \cdot \rangle_{A} \), i.e., \( \| x \|_{A} = \sqrt{\langle x, x \rangle_{A}} \) for all \( x \in \mathcal{H} \). Then, \( \| x \|_{A} \) is a norm if and only if \( A > 0 \). Also, \( (\mathcal{H}, \| \cdot \|_{A}) \) is complete if and only if \( \mathcal{R}(A) \) is closed in \( \mathcal{H} \). Henceforth, we use the symbol \( A \) and \( \mathbb{A} \) for positive operators on \( \mathcal{H} \) and \( \bigoplus_{i=1}^{n} \mathcal{H} \), respectively. We retain the notations \( O \) and \( I \) for the null operator and the identity operator on \( \mathcal{H} \), respectively. Given \( T \in \mathbb{B}\mathcal{L}(\mathcal{H}) \), the \( A \)-operator seminorm \( \| T \|_{A} \) is defined as follows:

\[
\| T \|_{A} = \sup_{x \in \mathcal{R}(A), x \neq 0} \frac{\| Tx \|_{A}}{\| x \|_{A}} = \inf \left\{ c > 0 : \| Tx \|_{A} \leq c \| x \|_{A}, 0 \neq x \in \overline{\mathcal{R}(A)} \right\} < \infty.
\]

We set \( \mathcal{B}^{A} \triangleq \mathcal{H} = \{ T \in \mathbb{B}\mathcal{L}(\mathcal{H}) : \| T \|_{A} < \infty \} \). Then \( \mathcal{B}^{A} \triangleq \mathcal{H} \) is not a subalgebra of \( \mathcal{B}\mathcal{L}(\mathcal{H}) \). It is pertinent to point out that \( \| T \|_{A} = 0 \) if and only if \( A T A = O \). For \( T \in \mathcal{B}^{A} \triangleq \mathcal{H} \), we have

\[
\| T \|_{A} = \sup \{ \| \langle Tx, y \rangle_{A} \| : x, y \in \overline{\mathcal{R}(A)}, \| x \|_{A} = \| y \|_{A} = 1 \}.
\]

If \( AT \geq 0 \), then the operator \( T \) is called \( A \)-positive. Note that if \( T \) is \( A \)-positive, then

\[
\| T \|_{A} = \sup \{ \langle Tx, x \rangle_{A} : x \in \mathcal{H}, \| x \|_{A} = 1 \}.
\]

Before we proceed further, it is necessary to introduce the concept of \( A \)-adjoint operator. We say an operator \( X \in \mathbb{B}\mathcal{L}(\mathcal{H}) \) to be \( A \)-adjoint operator of \( T \in \mathbb{B}\mathcal{L}(\mathcal{H}) \) if \( \langleTx, y\rangle_{A} = \langle Xy, x\rangle_{A} \) for every \( x, y \in \mathcal{H} \), i.e., \( AX = T^*A \). By Douglas Theorem [5], the existence of an \( A \)-adjoint operator is not guaranteed. An operator \( T \in \mathbb{B}\mathcal{L}(\mathcal{H}) \) may admit none, one or many \( A \)-adjoints. A rather well-known result states that \( A \)-adjoint of an operator \( T \in \mathbb{B}\mathcal{L}(\mathcal{H}) \) exists if and only if \( \mathcal{R}(T^*A) \subseteq \mathcal{R}(A) \). Let us now denote \( \mathcal{B}_{A} \triangleq \mathcal{H} = \{ T \in \mathbb{B}\mathcal{L}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A) \} \). Note that \( \mathcal{B}_{A} \triangleq \mathcal{H} \) is a subalgebra of \( \mathbb{B}\mathcal{L}(\mathcal{H}) \) which is neither closed nor dense in \( \mathbb{B}\mathcal{L}(\mathcal{H}) \). Moreover, we have the following inclusion relations:

\[
\mathcal{B}_{A} \triangleq \mathcal{H} \subseteq \mathcal{B}^{A} \triangleq \mathcal{H} \subseteq \mathcal{B}\mathcal{L}(\mathcal{H}) \Rightarrow \mathcal{B}_{A} \triangleq \mathcal{H} \triangleq \mathcal{B}\mathcal{L}(\mathcal{H}) \Rightarrow \mathcal{B}_{A} \triangleq \mathcal{H} \triangleq \mathcal{B}\mathcal{L}(\mathcal{H}) \Rightarrow \mathcal{B}_{A} \triangleq \mathcal{H} \triangleq \mathcal{B}\mathcal{L}(\mathcal{H})
\]

And the equality holds if \( A \) is injective and has a closed range.

For \( T \in \mathbb{B}\mathcal{L}(\mathcal{H}) \), \( w_{A}(T) \), the \( A \)-numerical radius of \( T \) was proposed by Saddi [13]. And is defined as follows:

\[
w_{A}(T) = \sup \{ \| \langle Tx, x \rangle_{A} \| : x \in \mathcal{H}, \| x \|_{A} = 1 \}.
\]

When \( T = (T_{ij}) \) is an \( n \times n \) operator matrix with \( T_{ij} \in \mathbb{B}\mathcal{L}(\mathcal{H}) \), then (1) can be written as

\[
w_{A}(T) = \sup \left\{ \| \langle Tx, x \rangle_{A} \| : x \in \bigoplus_{n=1}^{n} \mathcal{H}, \| x \|_{A} = 1 \right\}.
\]
Very recently, Zamani [16] obtained the following $A$-numerical radius inequality for $T \in B_A(\mathcal{H})$:

$$\frac{1}{2} \|T\|_A \leq w_A(T) \leq \|T\|_A. \tag{2}$$

The first inequality in (2) becomes an equality if $T^2 = O$ and the second inequality becomes an equality if $T$ is $A$-selfadjoint. The $A$-Crawford number of $T \in B_A(\mathcal{H})$ is defined as

$$c_A(T) = \inf\{|\langle Tx, x \rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}.$$

This terminology was introduced by Zamani [16]. Furthermore, if $T$ is $A$-selfadjoint, then $w_A(T) = \|T\|_A$. Moslehian et al. [8] continued the study of $A$-numerical radius and obtained some new $A$-numerical radius inequalities. In this year, Bhunia et al. [3] presented several $A$-numerical radius inequalities for a strictly positive operator $A$. We refer the interested reader to [12, 14–16] and the references cited therein for further generalizations and refinements of $A$-numerical radius inequalities.

The objective of this paper is to present a few new $A$-numerical radius inequalities for $2 \times 2$ and $n \times n$ operator matrices. Besides these, we aim to establish some existing $A$-numerical radius inequalities by relaxing sufficient condition $A > 0$. To this end, the paper is sectioned as follows. In Sect. 2, we define additional mathematical constructs including the definition of the Moore–Penrose inverse of an operator, $A$-adjoint, $A$-selfadjoint and $A$-unitary operator, that are required to state and prove the results in the subsequent sections. Section 3 contains several new $A$-numerical radius inequalities. More interestingly, it also provides some recent existing results in the literature on $A$-numerical radius inequalities by dropping a sufficient condition.

\section{2 Preliminaries}

This section gathers a few more definitions and results that are useful in proving our main results. It starts with the definition of the Moore–Penrose inverse of a bounded operator $A$ in $\mathcal{H}$. The \textit{Moore–Penrose inverse} of $A \in B(\mathcal{H})$ [9] is the operator $X : \mathcal{R}(A) \oplus \mathcal{N}(A)^\perp \rightarrow \mathcal{H}$ which satisfies the following four equations:

$$(1) AXA = A, \quad (2) XAX = X, \quad (3) XA = P_N(A)^\perp, \quad (4) AX = P_{\mathcal{R}(A)}|_{\mathcal{R}(A) \oplus \mathcal{N}(A)^\perp}.$$

Here, $\mathcal{N}(A)$ and $P_L$ denote the null space of $A$ and the orthogonal projection onto $L$, respectively. The Moore–Penrose inverse is unique, and is denoted by $A^\dagger$. In general, $A^\dagger \not\in B(\mathcal{H})$. It is bounded if and only if $\mathcal{R}(A)$ is closed. If $A \in B(\mathcal{H})$ is invertible, then $A^\dagger = A^{-1}$. If $T \in B_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished $A$-adjoint operator of $T$, which is denoted by $T^{\#A}$ (see [2, 7]). Note that $T^{\#A} = A^\dagger T^*A$. If $T \in B_A(\mathcal{H})$, then $AT^{\#A} = T^*A$, $\mathcal{R}(T^{\#A}) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(T^{\#A}) = \mathcal{N}(T^*A)$ (see [5]). One can observe that
\[ T^{\#_{A}} = A^{\dagger} T^{*} A = A^{\dagger} A = P_{\mathcal{R}(A)} \quad (\because \mathcal{N}(A)^{\perp} = \mathcal{R}(A^{*})). \] (3)

Besides, we derive below two new properties of \( A \)-adjoint of an operator \( T \in \mathcal{B}_{A}(\mathcal{H}) \), which are crucial in providing some new proofs of the existing results and in proving new results on \( A \)-numerical radius inequalities.

\[ T^{\#_{A}} P_{\mathcal{R}(A)} = A^{\dagger} T^{*} A A^{\dagger} = A^{\dagger} T^{*} A = T^{\#_{A}}, \] (4)

and

\[ P_{\mathcal{R}(A)} T^{\#_{A}} = A^{\dagger} A A^{\dagger} T^{*} A = A^{\dagger} T^{*} A = T^{\#_{A}}. \] (5)

An operator \( T \in \mathcal{B}_{\mathcal{H}} \) is said to be \( A \)-selfadjoint if \( AT = T^{*} A \). Observe that if \( T \) is \( A \)-selfadjoint, then \( T \in \mathcal{B}_{A}(\mathcal{H}) \). However, in general, \( T \neq T^{\#_{A}} \). But, \( T = T^{\#_{A}} \) if and only if \( T \) is \( A \)-selfadjoint and \( \mathcal{R}(T) \subseteq \mathcal{R}(A) \). If \( T \in \mathcal{B}_{A}(\mathcal{H}) \), then \( T^{\#_{A}} \in \mathcal{B}_{A}(\mathcal{H}) \), \( (T^{\#_{A}})^{\#_{A}} = P_{\mathcal{R}(A)} T P_{\mathcal{R}(A)} \), and \( ((T^{\#_{A}})^{\#_{A}})^{\#_{A}} = T^{\#_{A}} \). Also, \( T^{\#_{A}} T \) and \( TT^{\#_{A}} \) are \( A \)-positive operators, and

\[ \| T^{\#_{A}} T \|_{A} = \| TT^{\#_{A}} \|_{A} = \| T \|_{A}^{2} = \| T^{\#_{A}} \|_{A}^{2}. \] (6)

For any \( T_{1}, T_{2} \in \mathcal{B}_{A}(\mathcal{H}) \), we have

\[ \| T_{1}^{\#_{A}} T_{2} \|_{A} = \sup \{ \| \langle T_{1}^{\#_{A}} T_{2} x, y \rangle \| : x, y \in \mathcal{H}, \| x \|_{A} = \| y \|_{A} = 1 \} \]

\[ = \sup \{ \| \langle T_{2} x, T_{1} y \rangle \| : x, y \in \mathcal{H}, \| x \|_{A} = \| y \|_{A} = 1 \} \]

\[ = \sup \{ \| \langle x, T_{2}^{\#_{A}} T_{1} y \rangle \| : x, y \in \mathcal{H}, \| x \|_{A} = \| y \|_{A} = 1 \} \]

\[ = \sup \{ \| \langle T_{2}^{\#_{A}} T_{1} y, x \rangle \| : x, y \in \mathcal{H}, \| x \|_{A} = \| y \|_{A} = 1 \} \]

\[ = \| T_{2}^{\#_{A}} T_{1} \|_{A}. \] (7)

However, the above proof is a very simple one and directly follows from the definition of \( A \)-norm. An operator \( U \in \mathcal{B}_{A}(\mathcal{H}) \) is said to be \( A \)-unitary if \( \| U x \|_{A} = \| U^{\#_{A}} x \|_{A} = \| x \|_{A} \) for all \( x \in \mathcal{H} \). If \( T \in \mathcal{B}_{A}(\mathcal{H}) \) and \( U \) is \( A \)-unitary, then \( w_{A}(U^{\#_{A}} T U) = w_{A}(T) \). For \( T, S \in \mathcal{B}_{A}(\mathcal{H}) \), we have \( (TS)^{\#_{A}} = S^{\#_{A}} T^{\#_{A}} \), \( (T + S)^{\#_{A}} = T^{\#_{A}} + S^{\#_{A}} \), \( \| TS \|_{A} \leq \| T \|_{A} \| S \|_{A} \) and \( \| T x \|_{A} \leq \| T \|_{A} \| x \|_{A} \) for all \( x \in \mathcal{H} \). The real and imaginary part of an operator \( T \in \mathcal{B}_{A}(\mathcal{H}) \) as \( \text{Re}_{A}(T) = \frac{T + T^{\#_{A}}}{2} \) and \( \text{Im}_{A}(T) = \frac{T - T^{\#_{A}}}{2i} \). An interested reader may refer [1, 2] for further properties of operators on semi-Hilbertian space. From the definition of \( A \)-numerical radius of an operator, it follows that

\[ w_{A}(T) = w_{A}(T^{\#_{A}}) \quad \text{for any} \ T \in \mathcal{B}_{A}(\mathcal{H}). \] (8)

Some interesting results are collected hereunder for further use.

The next result is a combination of [3, Lemma 2.4(i)] and [11, Lemma 2.2].

**Lemma 2.1** Let \( T_{1}, T_{2}, T_{3}, T_{4} \in \mathcal{B}_{A}(\mathcal{H}) \). Then
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(i) \( \max\{w_A(T_1), w_A(T_4)\} = w_A\left(\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right) \leq w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \).

(ii) \( w_A\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \).

The other parts of [3, Lemma 2.4] assume the condition \( A \) is strictly positive. Rout et al. [11] proved the same result for positive \( A \), and the same is stated below.

**Lemma 2.2 [11, Lemma 2.4]**

Let \( T_1, T_2 \in B_A(\mathcal{H}) \). Then

(i) \( w_A\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) = w_A\left(\begin{bmatrix} O & T_2 \\ T_1 & O \end{bmatrix}\right) \).

(ii) \( w_A\left(\begin{bmatrix} O & e^{i\theta}T_2 \\ T_1 & O \end{bmatrix}\right) = w_A\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) \) for any \( \theta \in \mathbb{R} \).

(iii) \( w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}\right) = \max\{w_A(T_1 + T_2), w_A(T_1 - T_2)\} \). In particular,

\( w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}\right) = w_A(T_2) \).

The next result establishes upper and lower bounds for the \( A \)-numerical radius of a particular type of \( 2 \times 2 \) operator matrix that is a generalization of (2).

**Lemma 2.3 [11, Theorem 2.6]**

Let \( T_1, T_2 \in B_A(\mathcal{H}) \). Then

\[
\max\{w_A(T_1), w_A(T_2)\} \leq w_A\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \leq w_A(T_1) + w_A(T_2). \tag{9}
\]

**Lemma 2.4 [11, Lemma 2.9]**

Let \( T_1, T_2 \in B_A(\mathcal{H}) \). Then,

\[
w_A\left(\begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix}\right) = \max\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.
\]

### 3 Main results

This section begins with the power inequality for semi-Hilbert space that has been proved by Moslehian et al. [8], which states that for \( T \in B(\mathcal{H}) \), \( w_A(T^n) \leq w_A^n(T) \) for \( n \in \mathbb{N} \). Using this, we prove the following theorem.
Theorem 3.1 Let $T_1, T_2, T_3, T_4 \in B_A(\mathcal{H})$ and $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$. Then

$$\max\{w_A^{1/2}(T_2T_3), w_A^{1/2}(T_3T_2)\} \leq w_A\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right).$$

Proof Here,

$$w_A\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \text{ by Lemma 2.1.}$$

Now,

$$\max\{w_A(T_2T_3), w_A(T_3T_2)\} = w_A\left(\begin{bmatrix} T_2T_3 & O \\ O & T_3T_2 \end{bmatrix}\right) = w_A\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) = w_A\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^2\right) \leq w_A^2\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right).$$

This implies

$$\max\{w_A^{1/2}(T_2T_3), w_A^{1/2}(T_3T_2)\} \leq w_A\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right).$$

Remark 3.2 Using Lemma 2.1 and a known inequality $w_A(T) \leq \frac{1}{2}(||T||_A + ||T^2||_A^{1/2})$, Theorem 3.1 implies that

$$\max\{w_A^{1/2}(T_2T_3), w_A^{1/2}(T_3T_2)\} \leq w_A\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq \frac{1}{2}(||T||_A + ||T^2||_A^{1/2}).$$

Example Let $T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, T_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then,

$$w_A(T_2T_3) = w_A(T_3T_2) = w_A\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 1.$$ 

By Theorem 3.1, we thus have

$$w_A\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \geq 1.$$ 

We generalize some of the results of [6] now. Using Lemma 2.2, one can now state [3, Theorem 3.1] without assuming the condition $A > 0$. Its proof takes the
same steps as in the [3, proof of Theorem 3.1] after the use of Lemma 2.2, and hence the proof is omitted.

**Lemma 3.3**  Let $T, S, X, Y \in \mathcal{B}_A(\mathcal{H})$. Then

$$w_A(TXS^\#_A \pm SYT^\#_A) \leq 2\|T\|_A\|S\|_AW_A\begin{bmatrix} O & X \\ Y & O \end{bmatrix}.$$ 

*In particular, putting $Y = X$*

$$w_A(TXS^\#_A \pm SXT^\#_A) \leq 2\|T\|_A\|S\|_AW_A(X).$$

Considering $X = Y = Q$ and $T = I$ in Lemma 3.3, we get the following corollary.

**Corollary 3.4**  Let $Q, S \in \mathcal{B}_A(\mathcal{H})$. Then,

$$w_A(QS^\#_A \pm SQ) \leq 2\|S\|_AW_A(Q).$$

It is well known that $P_{\mathcal{R}(A)}T \neq TP_{\mathcal{R}(A)}$ for $T \in \mathcal{B}_A(\mathcal{H})$ (even if $A$ and $T$ are finite matrices). And the equality holds if $N(A)^{±}$ is invariant under $T$. The following result shows that $w_A(P_{\mathcal{R}(A)}T)$ and $w_A(TP_{\mathcal{R}(A)})$ are same for all $T \in \mathcal{B}_A(\mathcal{H})$ even though $N(A)^{±}$ is not invariant under $T$.

**Theorem 3.5**  $w_A(P_{\mathcal{R}(A)}T) = w_A(TP_{\mathcal{R}(A)}) = w_A(T)$ for any $T \in \mathcal{B}_A(\mathcal{H})$.

$$w_A(P_{\mathcal{R}(A)}T) = w_A((P_{\mathcal{R}(A)}T)^\#_A) \begin{array}{l} \because w_A(T) = w_A(T^\#_A) \\ = w_A(T^{\#_A}P_{\mathcal{R}(A)}) \begin{array}{l} \because (TS)^{\#_A} = S^{\#_A}T^\#_A & (P_{\mathcal{R}(A)})^{\#_A} = P_{\mathcal{R}(A)} \end{array} \\ = w_A(T^{\#_A}) \begin{array}{l} \text{by (4)} \end{array} \end{array} = w_A(T).$$

**Proof**

Again,

$$w_A(TP_{\mathcal{R}(A)}) = w_A((TP_{\mathcal{R}(A)})^{\#_A}) \begin{array}{l} \because w_A(T) = w_A(T^\#_A) \\ = w_A(P_{\mathcal{R}(A)}T^{\#_A}) \begin{array}{l} \because (TS)^{\#_A} = S^{\#_A}T^\#_A & (P_{\mathcal{R}(A)})^{\#_A} = P_{\mathcal{R}(A)} \end{array} \\ = w_A(T^{\#_A}) \begin{array}{l} \text{by (5)} \end{array} \end{array} = w_A(T).$$

We, therefore, have

$$w_A(P_{\mathcal{R} \leftarrow A \rightarrow} T) = w_A(TP_{\mathcal{R} \leftarrow A \rightarrow}) = w_A(T).$$

□

The next result provides an estimate for lower bound of $A$-numerical radius of a $2 \times 2$ operator matrix.
Theorem 3.6 Let $T_1, T_2, T_3, T_4 \in B_A(\mathcal{H})$. Then $w_A \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \geq \frac{1}{2} \max \{ \alpha, \beta \}$, where

$\alpha = \max \{ w_A(T_1 + T_2 + T_3 + T_4), w_A(T_1 + T_4 - T_2 - T_3) \}$ and

$\beta = \max \{ w_A(T_1 + T_4 + i(T_2 - T_3)), w_A(T_1 + T_4 - i(T_2 - T_3)) \}$.

**Proof** Let $T = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix}$ and $Q = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$. To show that $Q$ is $\mathcal{A}$-unitary, we need to prove that $\|x\|_\mathcal{A} = \|Qx\|_\mathcal{A} = \|Q^\#x\|_\mathcal{A}$. So,

$$Q^\# = \begin{bmatrix} O & P_{\mathcal{R}(A)}^A \\ I^\# & O \end{bmatrix} \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix} = \begin{bmatrix} O & P_{\mathcal{R}(A)}^A \\ P_{\mathcal{R}(A)}^A & O \end{bmatrix} \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix} \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix}$$

This in turn implies $QQ^\# = \begin{bmatrix} O & P_{\mathcal{R}(A)}^A \\ O & P_{\mathcal{R}(A)}^A \end{bmatrix} = Q^\#Q$. Now, for $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$, we have

$$\|Qx\|^2_\mathcal{A} = \langle Qx, Qx \rangle_\mathcal{A} = \langle Q^\#Qx, x \rangle_\mathcal{A} = \left\langle \begin{bmatrix} P_{\mathcal{R}(A)}^A & O \\ O & P_{\mathcal{R}(A)}^A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle_\mathcal{A} = \left\langle \begin{bmatrix} AP_{\mathcal{R}(A)}^A & O \\ O & AP_{\mathcal{R}(A)}^A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} AA^\dagger & O \\ O & AA^\dagger A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = \|x\|^2_\mathcal{A}.$$

So, $\|Qx\|_\mathcal{A} = \|x\|_\mathcal{A}$. Similarly, it can be proved that $\|Q^\#x\|_\mathcal{A} = \|x\|_\mathcal{A}$. Thus, $Q$ is an $\mathcal{A}$-unitary operator. By Corollary 3.4, we obtain

$$w_A(TQ^\# \pm QT) \leq 2w_A(T). \quad (12)$$

So,
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By Lemma 2.4, we, therefore, achieve the following:

\[
2w_{\mathcal{A}}(T) \geq w_{\mathcal{A}} \left( \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} \begin{bmatrix} O & P_{R(A)} \\ P_{R(A)} & O \end{bmatrix} + \begin{bmatrix} O & I \\ I & O \end{bmatrix} \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} \right)
\]

\[
= w_{\mathcal{A}} \left( \begin{bmatrix} T^\#_{3}P_{R(A)} & T^\#_{4}P_{R(A)} \\ T^\#_{2}P_{R(A)} & T^\#_{1}P_{R(A)} \end{bmatrix} + \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} \right)
\]

\[
= w_{\mathcal{A}} \left( \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} + \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} \right)
\]

\[
= w_{\mathcal{A}} \left( \begin{bmatrix} T^\#_{3} + T^\#_{4} & T^\#_{3} + T^\#_{4} \\ T^\#_{2} + T^\#_{1} & T^\#_{2} + T^\#_{1} \end{bmatrix} \right)
\]

\[
= w_{\mathcal{A}} \left( \begin{bmatrix} T_{2} + T_{3} & T_{4} + T_{1} \\ T_{3} + T_{1} & T_{2} + T_{3} \end{bmatrix} \right)
\]

\[
= w_{\mathcal{A}} \left( \begin{bmatrix} T_{2} + T_{3} & T_{4} + T_{1} \\ T_{3} + T_{1} & T_{2} + T_{3} \end{bmatrix} \right).
\]

Hence, we have

\[
2w_{\mathcal{A}} \left( \begin{bmatrix} T_{1} & T_{2} \\ T_{3} & T_{4} \end{bmatrix} \right) = 2w_{\mathcal{A}} \left( \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} \right) \geq w_{\mathcal{A}} \left( \begin{bmatrix} T_{2} + T_{3} & T_{4} + T_{1} \\ T_{3} + T_{1} & T_{2} + T_{3} \end{bmatrix} \right).
\] (13)

By (13) and Lemma 2.2, we obtain

\[
w_{\mathcal{A}} \left( \begin{bmatrix} T_{1} & T_{2} \\ T_{3} & T_{4} \end{bmatrix} \right) \geq \frac{1}{2} \max \{ w_{\mathcal{A}}(T_{1} + T_{2} + T_{3} + T_{4}), w_{\mathcal{A}}(T_{2} + T_{3} - T_{4} - T_{1}) \}.
\] (14)

Again, applying Corollary 3.4 and taking \( T = \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ \end{bmatrix} \) and \( Q = \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \), it can be shown that \( Q \) is \( \mathcal{A} \)-unitary and \( w_{\mathcal{A}}(TQ^\# + QT) \leq 2w_{\mathcal{A}}(T) \). Therefore,

\[
2w_{\mathcal{A}}(T) \geq w_{\mathcal{A}} \left( \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} \begin{bmatrix} O & \frac{-P_{R(A)}}{O} \\ \frac{-P_{R(A)}}{O} & O \end{bmatrix} \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} \right)
\]

\[
= w_{\mathcal{A}} \left( \begin{bmatrix} T^\#_{3}P_{R(A)} - T^\#_{4}P_{R(A)} \\ T^\#_{2}P_{R(A)} - T^\#_{1}P_{R(A)} \end{bmatrix} - \begin{bmatrix} T^\#_{3} & T^\#_{4} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} \right)
\]

\[
= w_{\mathcal{A}} \left( \begin{bmatrix} -T^\#_{2} + T^\#_{3} & -T^\#_{4} + T^\#_{1} \\ T^\#_{2} & T^\#_{1} \end{bmatrix} \right)
\]

\[
= w_{\mathcal{A}} \left( \begin{bmatrix} -T_{2} + T_{3} & T_{4} + T_{1} \\ -T_{4} + T_{1} & -T_{2} + T_{3} \end{bmatrix} \right).
\]

By Lemma 2.4, we, therefore, achieve the following:
From (14) and (15), we get the desired result. \(\square\)

**Example** Let \(T_1 = T_2 = T_3 = T_4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\) and \(A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\). Then \(w_A\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 1\).

By Theorem 3.6, we have \(\alpha = 4\) and \(\beta = 2\) and

\[
w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq 2.
\]

**Remark 3.7** Using [11, Theorem 2.1] in the above example, it is easy to find that

\[
w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \leq 2.
\]

So, we have

\[
w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) = 2.
\]

**Remark 3.8** Using [11, Theorem 2.15] in the above example, it is easy to find

\[
w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq 1.
\]

So, Theorem 3.6 is a refinement of Theorem 2.15 of [11].

As a special case of the above result, we obtain a corollary in Krein space setting. Let \(J\) be a non-scalar Hermitian involution operator in \(\mathcal{B}(\mathbb{H})\), where \(\mathbb{H}\) is a separable Hilbert space. Then, space \((\mathbb{H}, \langle \cdot, \cdot \rangle_J)\) is called a Krein space [4]. Note that the \(J\)-adjoint operator of \(T \in \mathcal{B}(\mathbb{H})\) is the unique operator in \(\mathcal{B}(\mathbb{H})\) such that \(\langle Tx, y \rangle_J = \langle x, T^J y \rangle_J\), for all \(x, y \in \mathbb{H}\). Therefore, we have \(T^J = JT^* J\) and \((T^J)^J = T\). One can observe that Lemmas 2.1 and 2.2 hold in the setting of Krein spaces for \(T_1, T_2 \in \mathcal{B}(\mathbb{H})\). We provide below a lemma for the \(J\)-adjoint operator of an \(n \times n\) operator matrix.

**Lemma 3.9** Let \(T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}\), where \(T_{ij} \in \mathcal{B}(\mathbb{H})\). Then

\[
\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n}^J \\ T_{21} & T_{22} & \cdots & T_{2n}^J \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn}^J \end{bmatrix} = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \cdots & \cdots & \cdots \\ T_{n1} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} T_{11}^J & \cdots & T_{1n}^J \\ \cdots & \cdots & \cdots \\ T_{n1}^J & \cdots & T_{nn}^J \end{bmatrix},
\]

\(\mathcal{B}(\mathbb{H})\)
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\[
\begin{bmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{21} & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{bmatrix}^{\#_j}
\]

$$
\begin{bmatrix}
J & 0 & \cdots & 0 \\
0 & J & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J
\end{bmatrix}
\begin{bmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{21} & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{bmatrix}^{\#_j}
\begin{bmatrix}
J & 0 & \cdots & 0 \\
0 & J & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J
\end{bmatrix}
$$

**Proof**

We state below another lemma for Krein space operators whose proof follows from [3, Theorem 3.1].

**Lemma 3.10** Let $T, S, X, Y \in \mathcal{B}(\mathbb{H})$. Then,

$$w_j(TXS^{\#_j} \pm SYT^{\#_j}) \leq 2\|T\|_J \|S\|_J w_j\left(\begin{bmatrix} O & X \\ Y & O \end{bmatrix}\right).$$

In particular, if $X = Y$, then

$$w_j(TXS^{\#_j} \pm SYT^{\#_j}) \leq 2\|T\|_J \|S\|_J w_j(X),$$

and if $X = Y = Q$, $T = I$, then

$$w_j(QS^{\#_j} \pm SQ) \leq 2\|S\|_J w_j(Q).$$

Based on these results, we have the following corollary of Theorem 3.6.

**Corollary 3.11** Let $T_1, T_2, T_3, T_4 \in \mathcal{B}(\mathbb{H})$. Then $w_j\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq \frac{1}{2} \max\{\alpha, \beta\}$, where

$$\alpha = \max\{w_j(T_1 + T_2 + T_3 + T_4), w_j(T_1 + T_4 - T_2 - T_3)\}$$

and

$$\beta = \max\{w_j(T_1 + T_4 + i(T_2 - T_3)), w_j(T_1 + T_4 - i(T_2 - T_4))\}.$$
We provide below a lower bound for $\mathcal{A}$-numerical radius inequality of an operator matrix.

**Theorem 3.12** Let $T_1, T_2 \in \mathcal{B}_\mathcal{A}(\mathcal{H})$. Then,

$$w_\mathcal{A}\left(\begin{bmatrix} T_1 & T_2 \\ O & O \end{bmatrix}\right) \geq \frac{1}{2} \max\{w_\mathcal{A}(T_1 + iT_2), w_\mathcal{A}(T_1 - iT_2)\}.$$  

**Proof** Suppose that $T = \begin{bmatrix} T_1 & O \\ T_2 & O \end{bmatrix}$ and $Q = \begin{bmatrix} O & -I \\ I & O \end{bmatrix}$. It then follows that $Q$ is $\mathcal{A}$-unitary. So, $\|Q\|_\mathcal{A} = 1$. Using Corollary 3.4, we get

$$2w_\mathcal{A}(T) \geq w_\mathcal{A}(TQ^\# - QT).$$

Now,

$$w_\mathcal{A}(T) \geq \frac{1}{2} w_\mathcal{A}(TQ^\# - QT)$$

$$= \frac{1}{2} w_\mathcal{A}(\begin{bmatrix} T_1 & O \\ T_2 & O \end{bmatrix} \begin{bmatrix} O & P_{\mathcal{R}(\mathcal{A})} \\ -P_{\mathcal{R}(\mathcal{A})} & O \end{bmatrix} - \begin{bmatrix} O & -I \\ I & O \end{bmatrix} \begin{bmatrix} T_1 & O \\ T_2 & O \end{bmatrix})$$

$$= \frac{1}{2} w_\mathcal{A}(\begin{bmatrix} O & T\#_1 P_{\mathcal{R}(\mathcal{A})} \\ T\#_2 P_{\mathcal{R}(\mathcal{A})} & O \end{bmatrix} - \begin{bmatrix} O & -T\#_1 O \\ T\#_2 O & O \end{bmatrix})$$

$$= \frac{1}{2} w_\mathcal{A}(\begin{bmatrix} T\#_1 & T\#_2 \\ -T\#_1 & T\#_2 \end{bmatrix}) \text{by (4)}$$

$$= \frac{1}{2} w_\mathcal{A}(\begin{bmatrix} T_2 - T_1 \\ T_1 - T_2 \end{bmatrix}^\#)$$

$$= \frac{1}{2} w_\mathcal{A}(\begin{bmatrix} T_2 - T_1 \\ T_1 - T_2 \end{bmatrix}).$$

By Lemma 2.4, we thus have

$$w_\mathcal{A}\left(\begin{bmatrix} T_1 & T_2 \\ O & O \end{bmatrix}\right) = w_\mathcal{A}\left(\begin{bmatrix} T\#_1 & O \\ T\#_2 & O \end{bmatrix}\right) \geq \frac{1}{2} \max\{w_\mathcal{A}(T_1 + iT_2), w_\mathcal{A}(T_1 - iT_2)\}. \square$$

**Remark 3.13** One can observe that Theorem 3.12 can also be proved by taking $T_3 = T_4 = O$ in Theorem 3.6, and is shown below.

By Theorem 3.6, we have
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Theorem 3.17 Let \( T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \), where \( T_{ij} \in \mathcal{B}_A(H) \) for \( 1 \leq i, j \leq n \). Then

\[
\begin{bmatrix} T_{11} & O & \cdots & O \\ O & T_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & T_{nn} \end{bmatrix} 
\]

Proof Let \( z = e^{2\pi i/n} \) and \( U = \begin{bmatrix} I & O & \cdots & O \\ O & zI & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & z^{n-1}I \end{bmatrix} \). It is easy to see that \( \bar{z} = z^{-1} = z^{n-1} \) and \( |z| = 1 \). To show that \( U \) is \( A \)-unitary, we need to prove that \( \|x\|_A = \|Ux\|_A = \|U^{#A}x\|_A \), for \( x = (x_1, x_2, \cdots, x_n) \in \bigoplus_{i=1}^n H \). Here,

\[
U^{#A} = \begin{bmatrix} I & O & \cdots & O \\ O & zI & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & z^{n-1}I \end{bmatrix}^{#A}
\]

\[
= \begin{bmatrix} \overline{I^{#A}} & O & \cdots & O \\ O & \overline{zI^{#A}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \overline{z^{n-1}I^{#A}} \end{bmatrix}
\]

\[
= \begin{bmatrix} \overline{P_{R(A)}} & O & \cdots & O \\ O & \overline{zP_{R(A)}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \overline{z^{n-1}P_{R(A)}} \end{bmatrix}.
\]

This in turn implies \( UU^{#A} = \begin{bmatrix} \overline{P_{R(A)}} & O & \cdots & O \\ O & \overline{P_{R(A)}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \overline{P_{R(A)}} \end{bmatrix} = U^{#A} U \).

Now, for \( x = (x_1, x_2, \cdots, x_n) \in \bigoplus_{i=1}^n H \), we have

\[
\|Ux\|_A^2 = \langle Ux, Ux \rangle_A = \langle U^{#A}Ux, x \rangle_A = \|x\|_A^2.
\]

So, \( \|Ux\|_A = \|x\|_A \). Similarly, \( \|U^{#A}x\|_A = \|x\|_A \). Thus, \( U \) is an \( A \)-unitary operator. Further, a simple calculation shows that
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Therefore,

\[
\begin{bmatrix}
T_{11}^# & O & \cdots & O \\
O & T_{22}^# & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & T_{nn}^#
\end{bmatrix} = \frac{1}{n} \sum_{k=0}^{n-1} U_{\#}^k T_{\#} U^k.
\]

This implies that

\[
\begin{bmatrix}
T_{11}^# & O & \cdots & O \\
O & T_{22}^# & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & T_{nn}^#
\end{bmatrix} = \frac{1}{n} \sum_{k=0}^{n-1} w_{\#}^k (U_{\#}^k T_{\#} U^k)
\]

\[
= \frac{1}{n} \sum_{k=0}^{n-1} w_{\#} (T_{\#}) \\
= w_{\#} (T).
\]

The last result provides a relation between $\#_\#$-numerical radius of two diagonal operator matrices, where $\text{diag}(T_1, \ldots, T_n)$ means an $n \times n$ diagonal operator matrix with entries $T_1, \ldots, T_n$.

**Theorem 3.18** Let $T_i \in B_\#(\mathcal{H})$ for $1 \leq i \leq n$. Then

\[
w_{\#} \left( \text{diag} \left( \sum_{i=1}^{n} T_i, \ldots, \sum_{i=1}^{n} T_i \right) \right) \leq n w_{\#} (\text{diag}(T_1, \ldots, T_n)).
\]

**Proof** Here,
Concluding remarks

In this paper, we have further studied $A$-numerical radius inequalities for operators on semi-Hilbertian space and numerical radius inequalities for operators in Krein space. The important findings are summarized as follows:

- Several upper and lower bounds for a $2 \times 2$ operator matrices are demonstrated.
- Some existing $A$-numerical radius inequalities are provided by relaxing sufficient condition $A > 0$. This shows the importance of the theory of the Moore–Penrose inverse in this field of research.
- Some numerical radius inequalities for operator matrices in Krein space setting are shown as corollaries to our main results.

This paper ends with the note that further work on $A$-numerical radius for $n \times n$ operator matrices and numerical radius inequalities for operators in Krein space can be studied.

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