A NOTE ON COMMUTATION RELATIONS AND FINITE DIMENSIONAL APPROXIMATIONS

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ABSTRACT. In this article we show that the main C*-algebras describing the canonical commutation relations of quantum physics, i.e., the Weyl and resolvent algebras, are in the class of Følner C*-algebras, a class of C*-algebras admitting a kind of finite approximations of Følner type. In particular, we show that the tracial states of the resolvent algebra are uniform locally finite dimensional.

1. INTRODUCTION

One of the most fundamental relations in quantum mechanics is the canonical commutation relation (CCR) between position and momentum, which when written in the most simple case (and taking \(\hbar = 1\)) is
\[
QP - PQ = i\mathbb{I}.
\]
Heisenberg attributed this “ingenious” relation to Born in a famous letter to Pauli in 1925. It soon became clear that neither matrices nor bounded operators on a Hilbert space can represent the relation (1.1) (cf. \([37, 36]\)). This fact opens several possibilities to analyze mathematically these relations. A first natural development is to study the commutation relations (or more generally commutators) of unbounded operators affiliated to finite von Neumann algebras, i.e., von Neumann algebras where the identity \(\mathbb{I}\) is a finite projection in the sense of Murray and von Neumann. In the case of the CCRs this restricts necessarily to type \(\text{II}_1\) von Neumann algebras which are infinite dimensional but still allow some kind of finiteness through the existence of a normalized trace. We refer to \([24, 25]\) for further motivations, results and an interesting historical account. Alternatively, the commutation relations can be reformulated in terms of more tractable bounded operators using bounded functions of the canonical variables (or fields) and encode the commutation relations in terms of them. There are two main options in the mathematical physics literature: the well known Weyl (or CCR) algebra \([32]\) generated by complex exponentials of the fields and the more recent resolvent algebra introduced by Buchholz and Grundling in \([18]\). The connection between the generators of the Weyl and resolvent algebras and the fields satisfying the commutation relations is obtained in terms of the notion of regular states. For example, in the case of the Weyl algebra \(\mathcal{W}(X, \sigma)\) associated to a symplectic vector space \((X, \sigma)\) a state \(\omega\) is called regular if the corresponding GNS-representation \(\pi = \pi_\omega\)
satisfies that the one-parameter groups

\[ \mathbb{R} \ni t \rightarrow \pi(W(tf)), \text{ where } f \in X, \]

are strongly continuous. Observe that \( W(f) \) denotes the Weyl elements generating the C*-algebra. Quantum fields appear as unbounded self-adjoint generators of the one-parameter groups associated to the Weyl elements in a regular representation. For the definition and further results concerning regular states in the context of the resolvent algebra see Section 4 in [18].

In a completely different context the theory of C*-algebras has been developed in the last decades focusing on different approximation patterns in terms of finite structures and leaving the representation theory in the background. For instance, central notions in operator algebras like nuclearity, exactness or quasidiagonality (see, e.g., [11, 8]) are nowadays defined using nets of contractive completely positive (c.c.p.) maps from the C*-algebra \( A \) into matrices highlighting different approximation patterns of the algebra:

\[ \varphi_n : A \to M_{k(n)}(\mathbb{C}). \]

Quasidiagonality, for example, requires these maps to be asymptotically multiplicative and asymptotically isometric in the C*-norm. The class of Følner C*-algebras then weakens the convergence in the multiplicativity condition to a convergence in mean (see Definition 2.1 for details), hence broadening the class of allowed approximations. These type of approximations show that the linear, the involutive and the order structure of the C*-algebra are respected by the approximating matrix algebras, while the product and the norm of \( A \) are captured only asymptotically. This class of C*-algebras has several equivalent characterizations which can be useful in the description of quantum theory. Indeed, let \( A \subset B(\mathcal{H}) \) be a unital C*-algebra modeling the quantum observables in a complex separable Hilbert space \( \mathcal{H} \) not containing the compact operators. If \( A \) is a Følner C*-algebra, then there is a net \( \{\rho_n\}_n \) positive trace-class operators with \( \text{Tr}(\rho_n) = 1 \) and such that

\[ \lim_n \|\rho_n A - A \rho_n\|_1 = 0 \text{ for every } A \in A, \]

where \( \| \cdot \|_1 \) denotes the trace-class norm in \( B(\mathcal{H}) \). Moreover, it is well known that the net of states \( \rho_n \) may be taken of finite range. We refer to [6, 8, 2] for additional remarks and motivation concerning this class of algebras. Finally, Følner C*-algebras and, in particular quasidiagonal C*-algebras, admit a good spectral approximation behavior, e.g., weak convergence of spectral measures associated to selfadjoint elements (see, e.g., [4, 7, 3, 30, 13]).

The aim of this article is to show that the main C*-algebras modeling the canonical commutation relations (the Weyl and the resolvent algebras associated with a real symplectic vector space \((X, \sigma)\)) are Følner C*-algebras and, hence, admit finite dimensional approximations in mean. Note that since the commutation relations (either in terms of Weyl elements or resolvents (see Eqs. (3.1) and (4.1) below) involve products, one obtains a finite-dimensional approximation of the C*-algebra in terms of matrices and where the commutation relations appear only asymptotically. Recall that the Weyl and resolvent C*-algebras are both non-separable (except in trivial cases), and hence we refer to [6, 1, 2] for the analysis of non-separable of Følner C*-algebras. The CAR-algebra, which models the canonical anti-commutation relations, is also in this class (see, e.g., [31]), as it is even quasidiagonal (see Definition 2.1 and comments below). Finally, we will show that the traces of the resolvent algebra admit a strong approximation in terms of tracial states on matrix algebras.

The article is structured as follows. In the next section we extend the notion of Følner C*-algebra to the nonunital case and prove some useful results, such as stability under inductive limits. We also mention some related notions needed later like, e.g., algebraic amenability or the notion of an amenable trace. In Section 3 we show that the Weyl algebra is a Følner C*-algebra. In the last section we study the resolvent algebra and show that it is a Følner C*-algebra.
where any trace is uniform locally finite dimensional, regardless of the dimension of the starting symplectic space.

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## 2. Følner C*-algebras

The structure of the resolvent algebra and, in particular, the importance of compact operators in the case of finite dimensional symplectic vector spaces suggests to introduce the definition of Følner C*-algebra in the non-unital case. Concretely, if the underlying symplectic vector space \((X, \sigma)\) is finite dimensional then a characteristic and useful feature of the associated resolvent algebra is its unique minimal ideal is isomorphic to the compact operators (see \[14\] Theorem 4.5 and Section 4).

In the following we generalize the definition of Følner C*-algebra (cf. Definition 4.1 in [3]) to the non-unital case. Recall that a linear map between C*-algebras \(\varphi : A \to B\) is called **completely positive (c.p.)** if it preserves positivity for all matrix amplifications. If \(\varphi\) is contractive (i.e., if \(\|\varphi\| \leq 1\)) we abbreviate it by c.c.p. If the C*-algebras are unital and \(\varphi\) sends the unit of \(A\) to the unit of \(B\) we say \(\varphi\) is u.c.c.p. (see [33]). Recall that u.c.p. maps are neither multiplicative nor do they preserve orthogonality in general. Define the multiplicative domain of the u.c.p. map \(\varphi : A \to B\) as

\[
A_\varphi := \{ A \in A \mid \varphi(A^*A) = \varphi(A)^*\varphi(A) \quad \text{and} \quad \varphi(AA^*) = \varphi(A)\varphi(A)^* \},
\]

which turns out to be the largest algebra on which \(\varphi\) restricts to a *-homomorphism.

**Definition 2.1.** A C*-algebra \(A\) is a **Følner C*-algebra** if there exists a net of contractive completely positive (c.c.p.) maps \(\varphi_n : A \to M_{k(n)}(\mathbb{C})\) being both

i) asymptotically multiplicative, i.e.,

\[
\lim_n \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|_{2,\text{tr}} = 0 \quad \text{for every} \quad A, B \in A,
\]

where \(\|F\|_{2,\text{tr}} := \sqrt{\text{tr}(F^*F)}\), \(F \in M_n(\mathbb{C})\) and \(\text{tr}(\cdot)\) denotes the unique tracial state on a matrix algebra \(M_n(\mathbb{C})\);

ii) asymptotically isometric, i.e.,

\[
\|A\| = \lim_n \|\varphi_n(A)\|, \quad \text{for every} \quad A \in A.
\]

This definition is analogous to the general definition of quasi-diagonal C*-algebra (see, e.g., Definition 7.1.1 in [11]). In fact, a quasi-diagonal C*-algebra satisfies the previous two conditions except that asymptotic multiplicativity is defined in terms of the C*-norm \(\|\cdot\|\). Since the \(\|\cdot\|_{2,\text{tr}}\)-norm is weaker than the C*-norm it is clear that quasi-diagonal algebras are a subclass of Følner C*-algebras. It is also an immediate consequence of Definition 2.1 that the class of Følner C*-algebras is stable when passing to subalgebras. The same definition above may also be used to define Følner pre-C*-algebras.

**Examples 2.2.**

(i) The set of compact operators \(K(\mathcal{H})\) on an infinite dimensional Hilbert space is a non-unital quasi-diagonal C*-algebra, hence a Følner C*-algebra. The c.c.p. maps \(\varphi_n : K(\mathcal{H}) \to M_{k(n)}(\mathbb{C})\) are given by compressions \(\varphi_n(K) := P_nKP_n\), where \(\{P_n\}_{n \in \mathbb{N}}\) is any sequence of non-zero finite rank projections strongly converging to \(1\) and \(k(n) = \dim(P_n)\). Note that the quasi-diagonalising condition follows from the fact that \(\{P_n\}_{n \in \mathbb{N}}\) is an approximate unit for \(K(\mathcal{H})\).

(ii) The Toeplitz algebra \(T\) generated by the unilateral shift on \(l^2(\mathbb{N})\) is an example of a Følner C*-algebra which is not quasi-diagonal. Indeed, note that it is not quasi-diagonal, as it has a proper isometry (see [11] Proposition 7.1.15). Meanwhile, the maps
\[ \varphi_n(A) := P_n A P_n, \] where \( P_n \in B(\ell^2(\mathbb{N})) \) projects onto the first \( n \) vectors of the canonical orthonormal basis of \( \ell^2(\mathbb{N}) \), witness the Følner approximation of \( T \).

The following lemma is the main technical result, and assures that if the \( \text{C}^* \)-algebras are unital one can replace the approximating c.c.p. maps by unit completely positive maps (see Proposition 2.4).

**Lemma 2.3.** Let \( A \) be a unital Følner \( \text{C}^* \)-algebra and denote by \( \varphi_n: A \to M_{k(n)}(\mathbb{C}) \) a net of c.c.p. maps which are asymptotically multiplicative in the \( \| \cdot \|_{2, \text{tr}} \)-norm and asymptotically isometric in the operator norm. Define \( e_n := \varphi_n(1) \in M_{k(n)}(\mathbb{C}) \) and \( P_n := E_n([1 - \varepsilon_n, 1]) \), where \( E_n(\cdot) \) denotes the resolution of the identity associated to the self-adjoint matrix \( e_n \) and \( \varepsilon_n > 0 \) tends to 0 when \( n \) grows. Then

\[ \lim_{n} \|e_n - P_n\|_{2, \text{tr}} = 0 . \]

**Proof.** Let \( \varphi_n: A \to M_{k(n)}(\mathbb{C}) \) be a net of c.c.p. maps as in Definition 2.1. Consider the self-adjoint matrix \( e_n := \varphi_n(1) \in M_{k(n)}(\mathbb{C}) \) and denote its spectrum by \( \{\lambda^{(n)}_1, \ldots, \lambda^{(n)}_{k(n)}\} \subset [0, 1] \). The fact that \( \varphi_n \) is asymptotically isometric implies that \( \|e_n\| \to 1 \), while from asymptotic multiplicativity one obtains

\[ (2.4) \quad \|e_n - e_n^2\|_{2, \text{tr}}^2 = \frac{1}{k(n)} \sum_{s=1}^{k(n)} \left( \lambda^{(n)}_s - \left( \lambda^{(n)}_s \right)^2 \right)^2 \to 0 . \]

Denoting by \( E_n(\cdot) \) the spectral resolution of the identity associated to \( e_n \), we may split the spectrum of \( e_n \) into three regions (close to 0, intermediate and close to 1):

\[ \Lambda_0(n) := \text{supp}(E_n) \cap [0, \varepsilon_n] , \quad \Lambda_{\text{mid}}(n) := \text{supp}(E_n) \cap (\varepsilon_n, 1 - \varepsilon_n) \quad \text{and} \quad \Lambda_1(n) := \text{supp}(E_n) \cap [1 - \varepsilon_n, 1] . \]

Note that \( k(n) = |\Lambda_0(n)| + |\Lambda_{\text{mid}}(n)| + |\Lambda_1(n)| \) and since \( \|e_n\| \to 1 \) we have that \( \Lambda_1(n) \neq \emptyset \) for large \( n \). By definition of the \( \| \cdot \|_{2, \text{tr}} \)-norm we have

\[ (2.5) \quad \|e_n - P_n\|_{2, \text{tr}} = \frac{1}{k(n)} \left( \sum_{\lambda \in \Lambda_0(n)} \lambda^2 + \sum_{\lambda \in \Lambda_1(n)} (1 - \lambda)^2 + \sum_{\lambda \in \Lambda_{\text{mid}}(n)} \lambda^2 \right) \]

\[ \leq \frac{|\Lambda_0(n)|}{k(n)} \frac{\varepsilon_n}{n} + \frac{|\Lambda_1(n)|}{k(n)} \frac{\varepsilon_n}{n} + \frac{|\Lambda_{\text{mid}}(n)|}{k(n)} (1 - \varepsilon_n)^2 \]

\[ \leq \frac{2\varepsilon_n}{n} + \frac{|\Lambda_{\text{mid}}(n)|}{k(n)} (1 - \varepsilon_n)^2 . \]

Observe that, from Eq. (2.4), we may, by passing to a subsequence if necessary, assume that

\[ \frac{1}{k(n)} \sum_{s=1}^{k(n)} \left( \lambda^{(n)}_s - \left( \lambda^{(n)}_s \right)^2 \right)^2 \leq \varepsilon_n \cdot (\varepsilon_n - \varepsilon_n^2) \]

and, therefore, we have

\[ \varepsilon_n \cdot (\varepsilon_n - \varepsilon_n^2) \geq \frac{1}{k(n)} \sum_{\lambda \in \Lambda_{\text{mid}}(n)} (\lambda - \lambda)^2 \geq \frac{|\Lambda_{\text{mid}}(n)|}{k(n)} (\varepsilon_n - \varepsilon_n^2)^2 \]

or, equivalently,

\[ \frac{|\Lambda_{\text{mid}}(n)|}{k(n)} \leq \varepsilon_n . \]

Using this estimate in the r.h.s. of Eq. (2.5) above we obtain finally the claim, i.e., \( \lim_n \|e_n - P_n\|_{2, \text{tr}} = 0 \). \( \square \)
**Proposition 2.4.** Let $\mathcal{A}$ be a unital Følner C*-algebra. Then there is a net of u.c.p. maps $\psi_n : \mathcal{A} \to M_{k(n)}(\mathbb{C})$ which is both asymptotically multiplicative $\| \cdot \|_{2,\text{tr}}$-norm and asymptotically isometric in C*-norm.

**Proof.** Let $\varphi_n : \mathcal{A} \to M_{r(n)}(\mathbb{C})$ be a net of c.c.p. maps witnessing the Følner condition of Definition 2.1. As in the preceding lemma we define $e_n := \varphi_n(1) \in M_{r(n)}(\mathbb{C})$ and denote its spectral resolution by $E_n(\cdot)$. Putting $P_n := E_n\left(\left[1 - \frac{1}{n}, 1\right]\right)$ note that $P_n e_n$ is an invertible element in $P_n M_{r(n)}(\mathbb{C}) P_n$ and, by functional calculus, we can introduce the matrices $f_n := (P_n e_n)^{-\frac{1}{2}}$. Finally defining $k(n) := \text{ran} P_n$ we have that the c.c.p. maps given by $\psi_n : \mathcal{A} \to M_{k(n)}(\mathbb{C})$, with $\psi_n(A) := f_n \varphi_n(A) f_n$, satisfy the claim. In fact, note that the maps are unital since for any $n$ we have $\psi(1) = f_n e_n f_n = f_n^2 e_n = P_n$, which is the unit of $P_n M_{r(n)} P_n$. Asymptotic multiplicativity in $\| \cdot \|_{2,\text{tr}}$-norm follows from Lemma 2.3 and the fact that, by functional calculus, we have $\| f_n - P_n \| \to 0$. Finally, note that, in the unital case, the condition of asymptotic isometry can be obtained just by taking direct sums of the u.c.p. maps $\psi_n$ (see [11] Proposition 4.2 for details).

**Remark 2.5.** The previous result shows that for unital C*-algebras c.c.p. maps may be replaced by unital completely positive (u.c.p.) maps, i.e., completely positive maps sending 1 to the matrix unit. Note that the proof requires a different approach as the asymptotic condition in the quasidiagonal case of Proposition 2.3 shows that in general $\Lambda_{\text{mid}}(n)$ may be non-empty.

As many C*-algebras used in the physical literature appear as inductive limits of simpler C*-algebras we show next that the class of Følner C*-algebras is stable under inductive limits as long as the connecting maps are injective.

**Proposition 2.6.** The inductive limit of Følner C*-algebras with injective connecting maps is also a Følner C*-algebra.

**Proof.** Consider the inductive limit $\mathcal{A} = \lim_{\longrightarrow} A_n$ of Følner C*-algebras $A_n$ with injective connecting maps. Given $\varepsilon > 0$ and a finite subset $\mathfrak{F} \subset A_\infty$ we may, without loss of generality, assume that $\mathfrak{F} \subset A_{n_0}$ for a sufficiently large $n_0$. Let $\varphi_{n_0} : A_{n_0} \to M_{k_{n_0}}(\mathbb{C})$ be a c.c.p. map witnessing the Følner condition for $A_{n_0}$, and note that, by Arveson’s extension theorem (cf. [11] Theorem 1.6.1)), there is an extension to a c.c.p. map $\overline{\varphi_{n_0}} : \mathcal{A} \to M_{k_{n_0}}(\mathbb{C})$. It is then clear that the map $\overline{\varphi_{n_0}}$ witnesses the Følner property of $\mathcal{A}$ with respect to $\varepsilon, \mathfrak{F}$, which proves the claim.

The class of Følner C*-algebras has a rich variety of equivalent characterizations. We begin mentioning the relation with an algebraic version of amenability which will be needed in the analysis of the Weyl algebra in the next section. We refer to [1] Section 3 and [5] 20 for additional results and motivation. We will be interested in the case of *-subalgebras of C*-algebras, but the definition and the results in the references mentioned above are true for arbitrary algebras over arbitrary fields.

**Definition 2.7.** Let $\mathfrak{A} \subset \mathcal{A}$ be a *-subalgebra of a C*-algebra $\mathcal{A}$. We say $\mathfrak{A}$ is algebraically amenable if there is a net $\{V_n\}_n$ of nonzero finite dimensional subspaces of $\mathfrak{A}$ satisfying

$$\lim_{n} \frac{\dim(AV_n + V_n)}{\dim(V_n)} = 1 \quad \text{for every} \quad \mathcal{A} \in \mathfrak{A}.$$
Theorem 2.8. Let $\mathfrak{A} \subset \mathcal{A}$ be a dense *-subalgebra of a unital separable C*-algebra $\mathcal{A}$. If $\mathfrak{A}$ is algebraically amenable, then $\mathcal{A}$ is a Følner C*-algebra.

We conclude this brief survey mentioning amenable traces, which are also in close relationship with the class of Følner C*-algebras. Recall that a tracial state on a C*-algebra $\mathcal{A}$ is a positive and normalized functional $\tau: \mathcal{A} \to \mathbb{C}$ that satisfies the usual tracial property $\tau(AB) = \tau(BA)$ for any $A, B \in \mathcal{A}$. In the next definition we specify the subclass of amenable traces (see, e.g., [11 Chapter 6]).

Definition 2.9. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a unital C*-algebra. An amenable trace $\tau$ on $\mathcal{A}$ is a tracial state on $\mathcal{A}$ that extends to a state $\psi$ on $\mathcal{B}(\mathcal{H})$ that has $\mathcal{A}$ in its centralizer, i.e.,

$$\tau = \psi|_{\mathcal{A}} \quad \text{and} \quad \psi(XA) = \psi(AX) \quad \text{for every} \quad A \in \mathcal{A} \quad \text{and} \quad X \in \mathcal{B}(\mathcal{H}).$$

Remark 2.10. (i) Kirchberg uses in [26 Proposition 3.2] the name liftable trace instead of amenable trace. Moreover, the state $\psi$ in the preceding definition is called a hypertrace in the literature, and the class of Følner C*-algebras is also referred to as weakly hypertracial (see [6] and references therein). The hypertrace $\psi$ can be interpreted as an operator-algebraic generalization of the invariant mean of an amenable group.

(ii) It is well known (see, e.g., Proposition 6.2.2 in [11]) that the definition of amenable trace does not depend on the choice of the embedding $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$.

(iii) Nuclearity and Følner type conditions for C*-algebras are, in general, independent notions. The relation between the class of nuclear and Følner C*-algebras is given by the following statement (see [3 Corollary 4.8]). Let $\mathcal{A}$ be a unital nuclear C*-algebra. Then $\mathcal{A}$ is a Følner C*-algebra iff $\mathcal{A}$ admits a tracial state. Recall also that any trace on a nuclear C*-algebra is amenable (cf. [11 Proposition 6.3.4]). We will relate these results when analyzing the resolvent algebras in the final section.

(iv) However, even though nuclearity and Følner type conditions are, in general, independent, they do agree in the class of C*-algebras associated from discrete groups. Indeed, Lance proved in [28 Theorem 4.2] that $C^*_r(G)$ is nuclear iff $G$ is amenable, which happens iff $C^*_r(G)$ is Følner in the sense of Definition 2.1.

The following result gives a first relation between unital Følner C*-algebras and amenable traces (see [26 and [12 Theorem 3.1.6]). Observe that we consider unital C*-algebras since the Weyl and resolvent algebras are unital. Recall from the proof of Proposition 2.4 that if the algebra is unital then any net of u.c.p. maps which is asymptotically multiplicative in mean can be turned into a net of asymptotically u.c.p. maps which are, in addition, asymptotically isometric in the C*-norm. The following results shows that amenable traces may be approximated in the weak*-topology by matrix traces.

Theorem 2.11. Let $\mathcal{A}$ be a unital C*-algebra and let $\tau$ be a tracial state on $\mathcal{A}$. Then $\tau$ is an amenable trace if and only if there is a net of u.c.p. maps $\varphi_n: \mathcal{A} \to M_{k(n)}(\mathbb{C})$ which are asymptotically multiplicative in the $\|\cdot\|_{2,\text{tr}}$-norm (cf. Eq. (2.2)) and where $(\text{tr} \circ \varphi_n)(A) \to \tau(A)$ for all $A \in \mathcal{A}$.

We conclude this section mentioning some additional characterizations of the class of Følner C*-algebras in terms of the different notions mentioned before (see [2 Theorem 3.8] and [6 Theorem 1.1]).

Theorem 2.12. Let $\mathcal{A}$ be a unital C*-algebra. Then the following conditions are equivalent.

(i) $\mathcal{A}$ is a Følner C*-algebra.

(ii) Every faithful representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies that $\pi(\mathcal{A})$ has an amenable trace.

(iii) Every faithful and essential representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ (i.e., $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}) = \{0\}$) satisfies that there is a net of positive trace-class operators $\{\rho_n\}_n \subset \mathcal{B}(\mathcal{H})$ with
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\[ \operatorname{Tr}(\rho_n) = 1 \text{ and such that} \]
\[ \lim_n \| \rho_n \pi(A) - \pi(A) \rho_n \|_1 = 0 \text{ for every } A \in A, \]

where \( \| \cdot \|_1 \) denotes the trace-class norm in \( \mathcal{B}(\mathcal{H}) \).

**Remark 2.13.** We mention some additional useful facts around the class of Følner C*-algebras in relation to representations:

(i) In part (iii) of Theorem 2.12 above one can choose the \( \rho_n \) to have finite dimensional range, and also use equivalently the Hilbert-Schmidt norm instead of the trace-class norm.

(ii) If a nonzero quotient of a unital C*-algebra \( A \) is Følner, then \( A \) itself is a Følner C*-algebra. In particular, if \( A \) admits a finite-dimensional representation then it is immediately Følner. This property shows, for instance, that the universal C*-algebra generated by two projections is a Følner C*-algebra, as it admits one dimensional representations. This algebra is also 2-subhomogeneous and hence also type I (see [8, IV.1.4.2]).

(iii) Vaillant introduced in [35, Definition 2.1] the Følner-Voiculescu condition on a separable C*-algebra \( A \), and related it to growth notions (see also the previous [27]). It is clear that if \( A \) has the Følner-Voiculescu condition, in the sense of [35], then \( A \) itself is algebraically amenable, and hence Følner in the sense of Definition 2.1 by Theorem 2.8 above. However, it is unclear whether these conditions are equivalent. Indeed, note that the Følner-Voiculescu condition [35] requires the subspaces \( X,Y \) to be contained in \( A \), whereas for the Følner condition the approximations \( \varphi_n \) might not be inner, i.e., coming from subspaces of \( A \). This is a subtle difference, very similar to the difference between quasidiagonality and inner quasidiagonality (see [9]).

3. **The Weyl Algebra**

Given a non-degenerate symplectic vector space \((X, \sigma)\), the Weyl algebra is defined abstractly as the C*-algebra generated by the unitary elements (or Weyl elements) \( \{ W(f) \mid f \in X \} \) satisfying the relations \( W(f)^* = W(-f) \) and encoding the canonical commutation relations as

\[ W(f)W(g) = e^{-\frac{i}{2} \sigma(f,g)}W(f+g) \quad f,g \in X. \]

We denote by \( \mathcal{W}_0(X, \sigma) \) the *-algebra generated by the Weyl elements and by \( \mathcal{W}(X, \sigma) \) the corresponding C*-closure (the Weyl-algebra). Note that the set of Weyl elements forms an (uncountable) basis for \( \mathcal{W}_0(X, \sigma) \). The Weyl algebra is a unital nonseparable simple C*-algebra, and we refer to [32, 34, 10, 29] for proofs, additional results and some applications.

The proof of the fact that the Weyl algebra is a Følner C*-algebra exploits the nice multiplicative structure of the Weyl elements that generate this algebra.

**Theorem 3.1.** Let \((X, \sigma)\) be a non-degenerate symplectic vector space. Then the *-algebra \( \mathcal{W}_0(X, \sigma) \) is algebraically amenable. In particular, the Weyl algebra \( \mathcal{W}(X, \sigma) \) is a Følner C*-algebra.

**Proof.** To show that the *-algebra \( \mathcal{W}_0(X, \sigma) \) is algebraically amenable we reformulate Definition 2.7 as a local condition. In fact, it is enough to show that for any \( \varepsilon > 0 \) and any finite \( \mathcal{F} \subset \mathcal{W}_0(X, \sigma) \) there is a nonzero finite-dimensional subspace \( V \) of \( \mathcal{W}_0(X, \sigma) \) such that for all \( A \in \mathcal{F} \) one has

\[ \frac{\dim(AV + V)}{\dim(V)} \leq 1 + \varepsilon. \]

Since any element in \( \mathcal{F} \) is a (finite) linear combination of Weyl elements it suffices to show condition (3.2) for \( \mathcal{F} \) being any finite collection of Weyl elements. Consider \( \varepsilon > 0 \) and \( \mathcal{F} := \)
\{W(g_1), \ldots, W(g_n)\}$ for some $g_1, \ldots, g_n \in X$. Define the vector space as follows: choose $N > 1/\varepsilon$ and

$$V := \text{span}\left\{W(k_1g_1 + \cdots + k_ng_n) \mid k_1, \ldots, k_n \in \{1, \ldots, N\}\right\}.$$  

Note that since the Weyl elements are linearly independent we have $\dim(V) = N^n$. Moreover, for any $g_i, i = 1, \ldots, n$, we have

$$\dim(W(g_i)V + V) = \dim(V) + N^n - 1$$

since only linear combinations of Weyl elements with $k_i = N$ will contribute to the dimension of $W(g_i)V + V$ not already counted in $V$. Therefore, we have

$$\frac{\dim(W(g_i)V + V)}{\dim(V)} \leq \left(1 + \frac{1}{N}\right) < 1 + \varepsilon.$$  

Finally, since $W_0(X, \sigma)$ is dense in $W(X, \sigma)$ it follows from Theorem 2.8 that the Weyl algebra is a Følner C$^*$-algebra. \hfill \Box

**Remark 3.2.** It is well-known that the Weyl algebra has a unique tracial state. By the results of Section 2 it is clear that it is an amenable trace and that it can be approximated by a net of states $\{\rho_n\}_n$ in any essential representation. Note nevertheless that, from a physical point of view, this state has not been considered in the physical literature, since it is non-regular and, hence, one lacks the connection with the usual language of quantum fields. We refer to [22, 23] for other results concerning the importance of nonregular states in the presence of quantum constraints.

4. **The resolvent algebra and finite dimensional approximations**

An alternative C$^*$-algebra also encoding the canonical commutation relations, this time motivated by resolvents of quantum fields, is the so-called resolvent algebra. It is well-known that the Weyl algebra has a unique tracial state. By the results of Remark 3.2. this algebra is a Følner C$^*$-algebra.

The resolvent algebra can be defined as an abstract C$^*$-algebra generated by elements $\{R(\lambda, f) \mid \lambda \in \mathbb{R}^* \text{ and } f \in X\}$, where for simplicity we denote $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ (see Section 3 in [18]). The generators of this algebra satisfy the following relations, which encode in terms of resolvents the usual properties of fields: normalization, linearity, self-adjointness and resolvent identity. For $f, g \in X$ and $\lambda, \nu \in \mathbb{R}^*$ the generators satisfy the following six relations:

$$R(\lambda, 0) = -\frac{i}{\lambda} \mathbb{1}, \quad R(\lambda, f)^* = \mathbb{R}(-\lambda, f), \quad \nu R(\nu \lambda, \nu f) = R(\lambda, f),$$  

$$R(\lambda, f) - R(\nu, f) = i(\nu - \lambda)R(\lambda, f)R(\nu, f),$$  

$$R(\lambda, f)R(\nu, g) = R(\lambda + \nu, f + g)R(\lambda, f) + R(\nu, g) + i\sigma(f, g)R(\lambda, f)^2R(\nu, g).$$

In addition, and motivated by the commutation relations of the fields expressed in terms of their resolvents, one requires

$$[R(\lambda, f), R(\nu, g)] = i\sigma(f, g)R(\lambda, f)R(\nu, g)^2R(\lambda, f).$$

Note that these relations imply that the generators $R(\lambda, f)$ are normal operators. We denote the resolvent algebra by

$$\mathcal{R}(X, \sigma) = C^*\left(R(\lambda, f) \mid \lambda \in \mathbb{R}^*, \ f \in X\right).$$

There is a remarkable difference between the Weyl algebra, which is simple and has a unique (non-regular) tracial state, and the resolvent algebra, which has a rich variety of ideals. This
Remark 4.2. (i) If one is only interested in the C*-algebra there are different ways to accommodate the description of relevant quantum dynamics as pointed out on [18, p. 2767]. Moreover, the ideals and the structure of the algebra depend on the dimension of the underlying symplectic space. In fact, if \( \dim X < \infty \) then \( \mathcal{R}(X, \sigma) \) is a Type I C*-algebra, in particular it is nuclear, which means it has a nice representation theory (see [8, Section IV.1]). If \( \dim X = \infty \) then the resolvent algebra is merely nuclear.

From Theorem 4.6 in [14] one can construct traces exploring the maximal ideals of \( \mathcal{R}(X, \sigma) \). Let \( \mathcal{Z} \) be a symplectic subspace of \( (X, \sigma) \), i.e., \( \mathcal{Z} \subset \mathcal{Z}^\perp \), and \( \chi \) a pure state on the Abelian C*-algebra \( \mathcal{A}(Z) \) := \( C^*(\{\mu g : \mu \in \mathbb{R}^*, g \in Z\}) \). For each pair \( (Z, \chi) \) one can associate a proper maximal ideal \( \mathcal{I} = \mathcal{I}(Z, \chi) \) generated by the sets
\[
\{ R(f, \lambda) - \chi(R(f, \lambda))1 \mid \lambda \in \mathbb{R}^*, f \in Z \} \cup \{ R(g, \mu) \mid \mu \in \mathbb{R}^*, f \in X \setminus Z \}.
\]
In this case one has \( \mathcal{R}(X, \sigma) = \mathbb{C}1 + \mathcal{I} \), i.e., \( \mathcal{I} \) has codimension 1. Tracial states can then be defined by choosing \( \tau(D) = 0 \) if \( D \in \mathcal{I} \). Note that, since the resolvent algebra is nuclear (cf. [14, Theorem 3.8 (i)]) these traces are amenable. We will show next that these traces allow a much stronger approximation by means of tracial states of matrix algebras. We will base our reasoning on well-known results from Chapters 3 and 4 in [12]. Recall that a tracial state \( \tau \) on a unital C*-algebra \( A \) is uniformly locally finite dimensional if there exist a net of u.c.p. maps \( \varphi_n : A \to M_{k(n)}(\mathbb{C}) \) satisfying the following two conditions:
\[
(4.2) \quad \| \tau - \text{tr} \circ \varphi_n \|_{A^*} \to 0,
\]
where \( \| \cdot \|_{A^*} \) is the (uniform) norm on the dual of \( A \); and
\[
(4.3) \quad d(A, A_{\varphi_n}) \to 0 \quad \text{for all} \quad A \in A,
\]
where \( A_{\varphi_n} \) is the multiplicative domain of \( \varphi_n \) (cf. Eq. (2.1)), i.e., for any \( A \) there are \( A_n \in A_{\varphi_n} \) with \( \|A - A_n\| \to 0 \).

Finally, we will exploit the fact that, in general, the resolvent algebra \( \mathcal{R}(X, \sigma) \) is the inductive limit of the resolvent algebras associated with all the finite dimensional subspaces of \( X \).

**Theorem 4.1.** Given a non-degenerate symplectic vector space \( (X, \sigma) \), any trace in the resolvent algebra \( \mathcal{R}(X, \sigma) \) is uniform locally finite dimensional, i.e., there exist u.c.p. maps \( \varphi_n : A \to M_{k(n)}(\mathbb{C}) \) satisfying conditions (4.2) and (4.3) above. In particular, the resolvent algebra is a Følner C*-algebra and for any tracial state \( \tau \) the von Neumann algebra \( \pi_\tau(\mathcal{R}(X, \sigma))'' \) is hyperfinite, where \( \pi_\tau \) denotes the GNS representation corresponding to \( \tau \).

**Proof.** Consider first the case where \( \dim X < \infty \). Then by Theorem 3.8 in [14] \( \mathcal{R}(X, \sigma) \) is type I and from Corollary 4.4.4 in [12] we conclude that any trace is uniform locally finite dimensional. This implies that any trace is amenable and hence \( \mathcal{R}(X, \sigma) \) is a Følner C*-algebra.

Second, if \( \dim X = \infty \) then from Proposition 4.9 (ii) in [18] \( \mathcal{R}(X, \sigma) \) is the inductive limit of the net of all \( R(S, \sigma) \) where \( S \subset X \) ranges over all finite-dimensional nondegenerate subspaces of \( X \). Moreover, by Proposition 4.9 (i) of [18] the connecting maps are injective and so \( \mathcal{R}(X, \sigma) \) is a Følner C*-algebra as well.

Applying again Corollary 4.4.4 in [12] to the inductive limit algebra we have that any trace in the resolvent algebra is uniform locally finite dimensional. Finally, since any trace is, in particular, a uniform amenable trace it follows from Theorem 3.2.2 in [12] that \( \pi_\tau(\mathcal{R}(X, \sigma))'' \) is a hyperfinite von Neumann algebra.

**Remark 4.2.** (i) If one is only interested in the C*-algebra there are different ways to show that the resolvent algebra is a Følner C*-algebra. If fact, since \( \mathcal{R}(X, \sigma) \) is a unital nuclear C*-algebra with a tracial state it is in the Følner class (see Remark 2.13(iii)). Alternatively, from Proposition 4.7 in [14] the ideal \( \mathcal{I}_c \) generated by commutators of generators of the algebra is proper and its quotient is an Abelian C*-algebra. Therefore, by Remark 2.13(ii), the resolvent algebra must also be Følner.
Finally, it is an interesting question if the Weyl or resolvent algebras are quasidiagonal. Note that local finite dimensional approximation of the trace \( \tau \) implies that the trace is, in particular, quasidiagonal since one can approximate in norm any pair of elements \( A_n, B_n \in \mathcal{A} \) by elements in the multiplicative domain. In fact, from Eq. (4.3) there exist \( A_n, B_n \in \mathcal{A} \) such that \( \| A - A_n \| \to 0, \| B - B_n \| \to 0 \) and so
\[
\varphi_n(AB) \approx \varphi_n(A_nB_n) = \varphi_n(A_n)\varphi_n(B_n) \approx \varphi_n(A)\varphi_n(B)
\]
(see [12] Section 3.4) for more details). From Theorem 2.11 it follows that the traces are amenable as well. However, the traces we consider for the resolvent algebra are manifestly not faithful, as they vanish in the ideals \( \mathcal{I}_c \). Therefore, should one wish to prove that \( \mathcal{R}(X, \sigma) \) is always quasi-diagonal it is not enough to only consider these traces.

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