IMPROVED BOUND FOR THE BILINEAR BOCHNER-RIESZ OPERATOR

EUNHEE JEONG, SANGHYUK LEE, AND ANA VARGAS

ABSTRACT. We study $L^p \times L^q \rightarrow L^r$ bounds for the bilinear Bochner-Riesz operator $B^\alpha$, $\alpha > 0$ in $\mathbb{R}^d$, $d \geq 2$, which is defined by

$$B^\alpha(f, g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{2\pi i x \cdot (\xi + \eta)}(1 - |\xi|^2 - |\eta|^2)^\alpha \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta.$$  

We make use of a decomposition which relates the estimates for $B^\alpha$ to those of the square function estimates for the classical Bochner-Riesz operators. In consequence, we significantly improve the previously known bounds.

1. Introduction

Let $d \geq 2$. The Bochner-Riesz operator in $\mathbb{R}^d$ of order $\alpha \geq 0$ is the multiplier operator defined by

$$R^\alpha_t(f)(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} (1 - |\xi|^2/t^2)^\alpha \hat{f}(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad t > 0,$$

where $x \cdot y$ is the usual inner product in $\mathbb{R}^d$, $r_+ = r$ if $r > 0$ and $r_+ = 0$ if $r \leq 0$. Here $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space in $\mathbb{R}^d$ and $\hat{f}$ is the Fourier transform of $f$. Related to summability of Fourier series and integral in $L^p$, boundedness of the Bochner-Riesz operators in $L^p$ spaces has been of interest and it is known as one of most fundamental problems in harmonic analysis which is also connected to the outstanding open problems such as restriction problem for the sphere and the Kakeya conjecture ([38]). For $1 \leq p \leq \infty$ and $p \neq 2$, it is conjectured that $R^\alpha_t$ is bounded on $L^p(\mathbb{R}^d)$ if and only if

$$\alpha > \max \left\{ d \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2}, 0 \right\}. \quad (1.1)$$

When $\alpha = 0$, $R^0_t$ is the disc multiplier (and ball multiplier) operator and Fefferman [21] verified that it is unbounded on $L^p(\mathbb{R}^d)$ except $p = 2$. For $d = 2$ the conjecture was shown to be true by Carleson and Sjölin [11], but in higher dimensions $d \geq 3$ the conjecture is verified on a restricted range and remains open. To be more specific, the sharp $L^p$-boundedness of $R^\alpha_t$ for $p$ satisfying $\max\{p, p'\} \geq 2(d+1)/(d-1)$ follows from the argument due to Stein [20] and the sharp $L^2$ restriction estimate for the sphere which is also known as Stein-Tomas theorem. Subsequently, progress has been made by Bourgain [7], and Tao-Vargas in [40] when $d = 3$. One of the authors

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We are concerned with the estimate, for \( f, g \) \( f, g \) \( f, g \) for \( B \) \( B \) \( B \) Riesz operator multiplier operator which is called bilinear Bochner-Riesz operator.

\[
\|B\|_{\mathcal{L}^r} \leq C\|f\|_{\mathcal{L}^p} \|g\|_{\mathcal{L}^q}
\]

for \( f, g \in \mathcal{S}(\mathbb{R}^d) \). For simplicity we set \( m^\alpha(\xi, \eta) = (1 - |\xi|^2 - |\eta|^2)^\alpha \) in what follows. We are concerned with the estimate, for \( f, g \in \mathcal{S}(\mathbb{R}^d) \),

\[
\|B^\alpha(f, g)\|_{\mathcal{L}^r(\mathbb{R}^d)} \leq C\|f\|_{\mathcal{L}^p(\mathbb{R}^d)} \|g\|_{\mathcal{L}^q(\mathbb{R}^d)}.
\]

Since \( B^\alpha \) is commutative under simultaneous translation, (1.3) holds only if \( 1 \leq p, q \leq \infty \) and \( 0 < r \leq \infty \) satisfies \( 1/p + 1/q = 1/r \). In view of this, the case in which Hölder relation \( 1/p + 1/q = 1/r \) holds may be regarded as a critical case. This case is also important since (1.3) becomes scaling invariant. Thus, by the standard density argument one can deduce from (1.3) the convergence

\[
\lim_{\lambda \to \infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i x \cdot (\xi + \eta)} \left( 1 - \frac{|\xi|^2 + |\eta|^2}{\lambda^2} \right)^\alpha \left( \hat{f}(\xi) \hat{g}(\eta) \right) d\xi d\eta = f(x)g(x)
\]
in $L^r$ whenever $f \in L^p$ and $g \in L^q$, $p, q \neq \infty$. Studies on boundedness of $B^\alpha$ under H"older relation were carried out recently by several authors [24, 19, 4, 5]. When $d = 1$, the problem was almost completely solved when the involved $L^p, L^q, L^r$ are Banach spaces (see [5, Theorem 4.1] and [24, 4]), that is to say, all of $p, q, r$ are in $[1, \infty]$. For higher dimensions $d \geq 2$, Diestel and Grafakos [19] proved that for $\alpha = 0$ (1.3) cannot hold if exactly one of $p, q, r$ is $r/(r - 1)$ is less than $2$, by modifying Fefferman’s counterexample to the (linear) disk multiplier conjecture [21].

Boundedness of $B^\alpha$ for general $\alpha > 0$ was studied by Bernicot, Grafakos, Song, and Yan in [5]. They obtained some positive and negative results for the boundedness for $B^\alpha$ for any $p$ and $q$ between $1$ and $\infty$. However, to state their results in full detail is a bit complicated. So, focusing on Banach cases, we summarize some of them in the following, which are the most recent result regarding boundedness of $B^\alpha$ as far as we are aware.

**Proposition 1.1.** [5, Proposition 4.10, 4.11] Let $d \geq 2$ and $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1/r$. Then (1.3) holds if exponents $p, q, r$ and $\alpha$ satisfy one of the following conditions:

- $2 \leq p, q < \infty$, $1 \leq r \leq 2$ and $\alpha > (d - 1)(1 - \frac{1}{q});$
- $2 \leq p, q, r < \infty$ and $\alpha > d\frac{q - 1}{2} + d(\frac{1}{2} - \frac{1}{r});$
- $2 \leq q < \infty$, $1 \leq p, r < 2$ and $\alpha > d(1/2 - 1/q) - 1/r;$
- $2 \leq p < \infty$, $1 \leq q, r < 2$ and $\alpha > d(1/2 - 1/p) - 1/r.$

In particular, $B^\alpha$ is bounded from $L^2 \times L^2$ to $L^1$ if and only if $\alpha > 0$. In [5] $L^2 \times L^2 \to L^1$ boundedness was shown for general bilinear multiplier operator $T_m$ of which the multiplier $m$ is bi-radial and compactly supported and satisfying some regularity condition. The authors took advantage of bi-radial structure of $m$, which makes it possible to reduce a $2d$-dimensional symbol to 2-dimensional one. By verifying a minimal regularity condition for $m^\alpha$ they showed $B^\alpha$ is bounded from $L^2 \times L^2 \to L^1$ for all $\alpha > 0$. For the other exponents $p, q, r$ they used the standard argument which has been used to prove $L^p$-boundedness for the classical Bochner-Riesz operator. To be precise, regarding $R^n_\alpha$ as a multiplier operator acting on $\mathbb{R}^{2d}$, they decomposed the multiplier dyadically away from the set $\{(\xi, \eta) : m^\alpha(\xi, \eta) = 0\} = \{(\xi, \eta) : |\xi|^2 + |\eta|^2 = 1\}$ and used estimates for the kernels of bilinear multiplier operators which result from dyadic decomposition. From this, they showed that (1.3) holds on a certain range of $\alpha$ when $(p, q, r) = (1, \infty, 1), (\infty, 1, 1), (2, \infty, 2), (\infty, 2, 2)$, and $(\infty, \infty, \infty)$. Then, complex interpolation was used to obtain results for general exponents.

However, as is well known in studies of multiplier operators of Bochner-Riesz type, the kernel estimate alone is not enough to show sharp results except for some specific exponents. Regarding such problem the heart of matter lies in quantitative understanding of oscillatory cancellation. In contrast with the classical Bochner-Riesz operator of which boundedness is almost characterized by the frequency near the singularity on the sphere, for the bilinear Bochner-Riesz operator we need to understand interaction between the two frequency variables $\xi, \eta$ as well as behavior related to the singularity of the multiplier of $B^\alpha$. From (1.2) it is natural to expect that the worst scenario may arise from the contribution near the intersection of the sets $|\xi|^2 + |\eta|^2 = 1$ and $\xi = -\eta$, where the oscillation effect disappears. Our main novelty is in exploiting this observation. First, following the usual way we
decompose $m^\alpha$ away from the singularity and then make further decomposition so that the interaction between two $\xi$ and $\eta$ can be minimized. Then, to handle the resulting operators we use square function estimates for the Bochner-Riesz operator about which we give more details below.

There have been various works which are related to so called bilinear approach to various linear problems, such as bilinear restriction estimates (see, for example [42, 44, 40, 28, 29, 38]). Since $B^\alpha$ has bilinear structure, it seems natural to expect that such bilinear methodology can be useful to obtain improved bounds but this doesn’t seem to work well for $B^\alpha$, especially, because of the interaction between two frequencies near the set $\xi = -\eta$. This is the reason why we rely on the square function estimate instead of following the typical bilinear approach.

We now consider the square function $S^\alpha$ for the Bochner-Riesz means, which is defined by

$$S^\alpha f(x) = \left( \int_0^\infty \left| \frac{\partial}{\partial t} R^\alpha_t f(x) \right|^2 t \, dt \right)^{1/2}.$$  

This was introduced by Stein [36] in order to study pointwise convergence of the Bochner-Riesz means and finding the optimal $\alpha$ for which the estimate

$$\|S^\alpha f\|_p \leq C\|f\|_p$$

holds has been investigated and it is related to various problems. See Cabery-Gasper-Trebels [10] and Lee-Rogers-Seeger [31]. The estimate (1.4) is well understood for $1 < p \leq 2$. For $p > 2$, however, it was conjectured that $S^\alpha$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $\alpha > \max\{d(1/2 - 1/p), 1/2\}$. When $d = 2$ the conjecture was proved by Carbery [9], and in higher dimensions partial results are known (see [13, 31, 30]) and the best known results can be found in [31, 30].

Let $0 < \delta \ll 1$, $\phi$ be a smooth function supported in $[-1, 1]$, and define a square function with localized frequency which is given by

$$S^\phi_{\delta} f(x) = \left( \int_{1/2}^2 \left| \phi \left( \frac{\sqrt{D^2 - t}}{\delta} \right) f(x) \right|^2 dt \right)^{1/2}.$$  

The conjectured $L^p$ $(2 < p \leq \infty)$ estimate for $S^\alpha$ is essentially equivalent to the following: For $p \geq \frac{2d}{d-1}$ and $\epsilon > 0$, there exists $C = C(\epsilon)$ such that

$$\|S^\phi_{\delta} f\|_p \leq C\delta^{\frac{2d}{d-1} + \frac{1}{p} - \epsilon} \|f\|_p.$$  

Implication from (1.6) to (1.4) is easy to see from dyadic decomposition and using easy $L^2$ estimate $\|S^\phi_{\delta} f\|_2 \leq C\delta^2 \|f\|_2$ and interpolation. We don’t draw direct connection from (1.3) to (1.4). Instead we show that the estimate (1.3) can be deduced from $L^p$ bound for $S^\phi_{\delta}$.

To present our results, we introduce some notations: For $\nu \in [0, 1/2]$, we set

$$\Delta_1(\nu) = \{(u, v) \in [0, 1/2]^2 : u, v \leq \nu\}, \quad \Delta_2(\nu) = \{(u, v) \in [0, 1/2]^2 : u, v \geq \nu\},$$

$$\Delta_3(\nu) = \{(u, v) \in [0, 1/2]^2 : u < \nu < v \text{ or } v < \nu < u\}.$$  

The regions $\Delta_j(\nu)$, $1 \leq j \leq 3$, are pairwise disjoint and $\bigcup_{j=1}^3 \Delta_j(\nu) = [0, 1/2]^2$. 

...
For $u \in [0,1]$ set
$$\beta_*(u) = \frac{d-1}{2} - ud.$$  
Let us define a real valued function $\alpha_\nu : [0,1/2]^2 \to \mathbb{R}$ by
$$\alpha_\nu(u,v) = \begin{cases} 
\beta_*(u) + \beta_*(v) = (d-1) - d(u+v), & (u,v) \in \Delta_1(\nu), \\
\frac{2-2u-2v}{1-2v} \beta_*(\nu), & (u,v) \in \Delta_2(\nu), \\
\max\{\beta_*(u), \beta_*(v)\} + \beta_*(\nu) \min\{\frac{1-2u}{1-2v}, \frac{1-2v}{1-2v}\}, & (u,v) \in \Delta_3(\nu). 
\end{cases}$$

The following is our first result.

**Theorem 1.2.** Let $d \geq 2$, $p_\circ \geq 2d/(d-1)$ and let $2 \leq p, q \leq \infty$ and $r$ with $1/r = 1/p + 1/q$. Suppose that for $p \geq p_\circ$ the estimate (1.6) holds with $C$ independent of $\phi$ whenever $\phi \in C_N([-1,1])$ for some positive integer $N$. Here $C_N([-1,1])$ is defined by (2.1). Then for any $\alpha > \alpha_{p_\circ}(1/p,1/q)$ (1.3) holds.

For $d \geq 2$ we set $p_0(d)$ and $p_s$ to be
$$p_0(d) = 2 + \frac{12}{4d-6-k}, \quad d \equiv k \pmod{3}, \quad k = 0,1,2,$$

(1.7) $$p_s = p_s(d) = \min \left\{ p_0(d), \frac{2(d+2)}{d} \right\}.$$  
We will prove (Lemma 2.6) that (1.6) holds for $p \geq p_s$. Hence, this and Theorem 1.2 yields the following.

**Corollary 1.3.** Let $d \geq 2$, and let $2 \leq p, q \leq \infty$ and $r$ be given by $1/r = 1/p + 1/q$. Then (1.3) holds provided that $\alpha > \alpha_{p_\circ}(1/p,1/q)$

Remarkably, when $d = 2$ Corollary 1.3 gives sharp estimates for some $p, q$ other than $p = q = 2$. Indeed, note that $p_s(2) = 4$ and, thus, for $2 \leq p, q \leq 4$ we have $\alpha_{4/3}(1/4,1/4) = 0$. By Corollary 1.3 it follows that (1.3) holds for $\alpha > 0$ if $(p,q) \in [2,4]^2$. This result is clearly sharp in view of Diestel-Grafakos’s result [19].
Corollary 1.3 provides improved estimates over those in Proposition 1.1 except the case \( p = 2 \) and \( q = 2 \). This can be clearly seen by considering the boundedness of \( B^\alpha \) from \( L^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \) to \( L^{p/2}(\mathbb{R}^d) \). See Figure 2. However, we do not know whether the exponents in Corollary 1.3 are sharp for most of the cases and we are only able to provide improved lower bounds for \( \alpha \) which is slightly better than the one known before. (See the section 4.3.)

The main new idea of this work is a decomposition lemma (Lemma 3.1) which enables us to split frequency interaction between two variables \( \xi \) and \( \eta \). The decomposition lemma basically reduces the problem to dealing with the operator \( B_{\phi_1, \phi_2}^{\delta, \rho} \) which is a sum of products of two linear operators with localized frequency. See (3.3) for the precise definition of \( B_{\phi_1, \phi_2}^{\delta, \rho} \). This lemma makes the problem much simpler. For example, various previous result can be easily obtained by making use of the lemma. Moreover, \( S_{\rho, \delta}^\phi \) (see (2.2)) appeared in \( B_{\phi_1, \phi_2}^{\delta, \rho} \) are closely related to the (linear) Bochner-Riesz operator \( R_i^\alpha \) and its bounds are now better understood. Since \( B_{\phi_1, \phi_2}^{\delta, \rho} \) has product structure, by Cauchy-Schwarz inequality we can simply bounds this with a product of discretized square function \( S^\phi_\delta \) defined by (2.5), of which sharp bounds can be deduced from the well-known estimates for the square function \( S^\phi_\delta \).

The rest of this paper is organized as follows. In Section 2 we consider two different types of square functions \( S^\phi_\delta \) and \( D^\phi_\delta \) and make observation that their \( L^p \)-boundedness properties are more or less equivalent. In Section 3 we introduce a decomposition lemma which convert our problem to estimates for bilinear operators \( B_{\phi_1, \phi_2}^{\delta, \rho} \). In Section 4 we prove Theorem 1.2 and discuss the boundedness (1.3) for \( B^\alpha \) under sub-critical relation \( 1/p + 1/q > 1/r \). Finally, in Section 4 we find a new lower bound for \( \alpha \).
Throughout the paper, the positive constant $C$ may vary line to line. For $A, B > 0$, by $A \lesssim B$, we mean $A \leq CB$ for some constant $C$ independent of $A, B$. We write $A \sim B$ to denote $A \lesssim B$ and $A \gtrsim B$. Also, $\hat{f}$ and $f'$ denote the Fourier and inverse Fourier transforms of $f$, respectively: $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi ix \cdot \xi} f(x) dx$, $f'(x) = \int_{\mathbb{R}^d} e^{2\pi ix \cdot \xi} f(\xi) d\xi$. We also use $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$ for the Fourier and the inverse Fourier transforms of $f$, respectively. For a bilinear operator $T$ we denote by $\|T\|_{L^p \times L^q \to L^r}$ the operator norm of $T$ from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$.

2. Preliminaries

In this section we obtain several preliminary results which we need in the course of proof.

Let $I \subset \mathbb{R}$ be an interval, and $N$ be nonnegative integer. We define $C_N(I)$ to be a class of smooth functions $\phi$ on $\mathbb{R}$ satisfying
\begin{equation}
\text{supp } \phi \subset I \text{ and } \sup_{t \in I} |\phi^{(k)}(t)| \leq 1, \quad k = 0, \ldots, N.
\end{equation}
For a smooth $\phi$ and $0 < \delta \ll 1$, we define linear operators $S_{\rho, \delta}^{\phi}$ by
\begin{equation}
S_{\rho, \delta}^{\phi} f(\xi) = \phi \left( \frac{|\xi|^2 - \rho}{\delta} \right) \hat{f}(\xi), \quad f \in \mathcal{S}(\mathbb{R}^d).
\end{equation}

2.1. Kernel estimate. For $\omega \in \mathbb{R}^d$ with $|\omega| = 1$ and $0 < l \leq 1$, let $\chi_{l, i}^{\omega} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a homogeneous function of degree 0 such that $\chi_{l, i}^{\omega}$ is supported in $\Gamma_{l, i}^{\omega} := \{ \xi : |\xi|/|\xi - \omega| \leq 2l \}$ and
\begin{equation}
|\partial_\xi^\alpha \chi_{l, i}^{\omega}(\xi)| \leq C_{\alpha} l^{-|\alpha|}|\xi|^{-|\alpha|}
\end{equation}
for all multi-indices $\alpha$. We also set
\begin{equation}
K_{\nu, \delta}^{\omega, l}(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \phi \left( \frac{|\xi|^2 - \rho}{\delta} \right) \chi_{l, i}^{\omega}(\xi) d\xi.
\end{equation}

Lemma 2.1. Let $d \geq 2$, $0 < \delta \ll 1$, $2\delta \leq \rho \leq 1$. Suppose that $l \sim (\delta/\rho)^{1/2}$. Then there is a constant $C$, independent of $\delta, \rho, \omega$, such that
\begin{equation}
|K_{\nu, \delta}^{\omega, l}(x)| \leq C \rho^{-1/2} \delta^{(d+1)/2} (1 + \delta^{1/2}|x - (\omega \cdot x)\omega| + \delta \rho^{-1/2}|\omega \cdot x|)^{-N}
\end{equation}
whenever $\phi \in C_N([-1, 1])$.

Proof. By scaling $\xi \to \sqrt{\rho} \xi$ and $x \to \rho^{-1/2} x$, it is sufficient to show that
\begin{equation}
|K_{\nu, l}^{\omega, l}(x)| \leq C l^{d+1}(1 + l^1|x - (\omega \cdot x)\omega| + l^2|\omega \cdot x|)^{-N}
\end{equation}
with $C$ independent of $\phi$. And this can be obtained by routine integration by parts. \(\square\)

Making use of a homogeneous partition of unity which is given by $\{\chi_{l, i}^{\omega}\}$ with $l \sim (\delta/\rho)^{1/2}$ and $\{\omega\}$ which is a $\sim (\delta/\rho)^{1/2}$ separated subset of $S^{d-1}$ and Lemma 2.1, one can easily obtain the following.
Lemma 2.2. For $0 < \delta \ll 1$, $\rho \geq 0$, and $\phi \in C_N([-1, 1])$, let us set

$$K_{\rho, \delta} = F^{-1}\left(\phi\left(\frac{|\xi|^2 - \rho}{\delta}\right)\right).$$

Then, there exists a constant $C$, independent of $\phi$ (also $\rho$ and $\delta$), such that

$$|K_{\rho, \delta}(x)| \leq C\begin{cases} \rho^{\frac{d-2}{2}}(1 + \rho^{-\frac{1}{2}}\delta|\xi|)^{-N}, & \text{for } \rho \geq C\delta \\ \delta^2(1 + \delta^2|\xi|)^{-N}, & \text{for } \rho \leq C\delta \end{cases}$$

for some large $C > 1$. In particular, $|K_{\rho, \delta}(x)| \leq C\delta(1 + \delta|\xi|)^{-N}$ for all $\rho \in [0, 1]$.

2.2. Discretized square function. For a compactly supported smooth function $\phi$ and $0 < \delta \ll 1$, we define a discrete square functions $D_\phi^0$ by

$$D_\phi^0 f(x) = \left(\sum_{\rho \in \delta Z \cap [1/2, 1]} |S_{\rho, \delta}^\phi f(x)|^2\right)^{1/2},$$

and let $D_\phi^1$ be defined by (1.5). In what follows we show that, for $p \geq 2$, $L^p$-boundedness properties of these two square functions are essentially equivalent.

Lemma 2.3. Let $1 \leq p \leq \infty$, $N$ be a positive integer, and $0 < \delta \leq \delta_0 \leq 1/8$. Suppose that

$$\left\| \left(\int_0^1 |S_{t, \delta}^\phi f(x)|^2 dt\right)^{1/2}\right\|_p \leq A\|f\|_p$$

holds with $A$ independent of $\phi$ whenever $\phi \in C_N([-1, 1])$. Then $\|D_\phi^0 f\|_p \leq 2\delta^{-\frac{1}{2}}A\|f\|_p$ holds whenever $\phi \in C_{N+1}([-1, 1])$.

Proof. We fix $\phi \in C_{N+1}([-1, 1])$. From the fundamental theorem of calculus we have

$$\phi\left(\frac{|\xi|^2 - \rho}{\delta}\right) = \phi\left(\frac{|\xi|^2 - \rho - t}{\delta}\right) + \delta^{-1} \int_0^{\delta} \phi'\left(\frac{|\xi|^2 - \rho - t}{\delta}\right) dt.$$

Thus $S_{\rho, \delta}^\phi f = S_{\rho+t, \delta}^\phi f + \delta^{-1} \int_0^{\delta} S_{\rho+t, \delta}^\phi f dt$. Using this and taking additional integration in $t$ over $[0, \delta]$ give

$$S_{\rho, \delta}^\phi f(x) = \delta^{-1} \int_0^{\delta} S_{\rho+t, \delta}^\phi f(x) dt + \delta^{-2} \int_0^{\delta} \int_0^{\delta} S_{\rho+\tau, \delta}^\phi f(x) d\tau dt.$$

By Cauchy-Schwarz and triangle inequalities, we have

$$D_\phi^1 f(x) \leq \delta^{-1/2}\left(I_1 + \delta^{-1}I_2\right),$$

where

$$I_1 = \left(\sum_{\rho \in \delta Z \cap [1/2, 1]} \int_0^{\delta} |S_{\rho+t, \delta}^\phi f(x)|^2 dt\right)^{1/2}, \quad I_2 = \left(\sum_{\rho \in \delta Z \cap [1/2, 1]} \int_0^{\delta} \int_0^{\delta} |S_{\rho+\tau, \delta}^\phi f(x)|^2 d\tau dt\right)^{1/2}.$$

Then, it is clear that

$$I_1 \leq \left(\sum_{\rho \in \delta Z \cap [1/2, 1]} \int_0^{\rho+\delta} |S_{t, \delta}^\phi f(x)|^2 dt\right)^{1/2} \leq \left(\int_0^{2} \int_{1/2} |S_{t, \delta}^\phi f(x)|^2 dt\right)^{1/2}.$$
Since $\delta < 1/2$, we have

$$I_2 \leq \left( \sum_{\rho \in \delta Z} \int_0^\delta \int_0^\delta |S_{\rho+t,\delta}^\phi f(x)|^2 \, dr \, dt \right)^{1/2} \leq \delta \left( \int_{1/2}^2 |S_{t,\delta}^\phi f(x)|^2 \, dt \right)^{1/2}.$$ 

Thus, combining this with (2.7) we have $D_\delta^\phi f(x) \leq \delta^{-1/2}(\mathcal{S}_\delta^\phi f(x) + \mathcal{S}_\delta^\phi \varphi(x))$. Since $\phi, \phi' \in C_N([-1,1]),$

$$\|D_\delta^\phi f\|_p \leq \delta^{-\frac{1}{2}}(\|\mathcal{S}_\delta^\phi f\|_p + \|\mathcal{S}_\delta^\phi \varphi\|_p) \leq 2A\delta^{-\frac{1}{2}}\|f\|_p.$$ 

This completes the proof. \hfill \Box

The implication in Lemma 2.3 is reversible for a certain range of $p$. We record the following lemma even though we do not use it in this paper.

**Lemma 2.4.** Let $2 \leq p < \infty$, $N$ be a positive integer, and $0 < \delta \leq \delta_0 \leq 1/8$. Suppose that $\|D_\delta^\phi f\|_p \leq A\|f\|_p$ holds with $A$ independent of $\phi$ whenever $\phi \in C_N([-1,1])$. Then, there is a constant $C$, independent of $\delta$ and $\phi$, such that $\|\mathcal{S}_\delta^\phi f\|_p \leq C\delta^{1/2}\|f\|_p$ holds for all $\phi \in C_N([-1/2,1/2]).$

Decomposing $\phi$ into functions supported in smaller intervals we may replace the interval $[-1/2,1/2]$ with $[-1,1]$.

**Proof.** Let $\phi \in C_N([-1/2,1/2]).$ To begin with, observe that

$$S_{\rho,\delta}^\phi f(x) = S_{\lambda,\lambda\delta}^\phi(f(\lambda^{-1}x))$$

Thus decomposing the interval $[1/2,1]$ into finite subintervals and using the above rescaling identity it is sufficient to show that

$$\left\| \left( \int_{1/2}^1 |S_{t,\delta}^\phi f(x)|^2 \, dt \right)^{1/2} \right\|_p \leq A\delta^{1/2} \|f\|_p.$$ 

Since $\delta < 1/8$ we note that

$$\int_{1/2}^1 |S_{t,\delta}^\phi f(x)|^2 \, dt \leq \int_{-\delta/2}^{\delta/2} \sum_{\rho \in \delta Z} |S_{\rho+t,\delta}^\phi f(x)|^2 \, dt.$$ 

For $|t| \leq \delta/2$, set $\psi_t(s) = \phi(s - \frac{t}{\delta})$. Then we see $\psi_t \in C_N([-1,1])$ and $S_{\rho+t,\delta}^\phi f(x) = S_{\rho+\psi_t,\delta}^\phi f(x)$. Hence

$$\int_{1/2}^1 |S_{t,\delta}^\phi f(x)|^2 \, dt \leq \int_{-\delta/2}^{\delta/2} \sum_{\rho \in \delta Z} |S_{\rho,\delta}^\phi f(x)|^2 \, dt.$$ 

Since $\rho \geq 2$ and $\psi_t \in C_N([-1,1])$, by Minkowski’s inequality and the assumption, we have

$$\left\| \left( \int_{1/2}^1 |S_{t,\delta}^\phi f(x)|^2 \, dt \right)^{1/2} \right\|_p \leq \left( \int_{-\delta/2}^{\delta/2} \|D_{\rho,\delta}^\phi f\|_p^2 \, dt \right)^{1/2} \leq A \left( \int_{-\delta/2}^{\delta/2} \|f\|_p^2 \, dt \right)^{1/2}.$$

This gives the desired bound. \hfill \Box
2.3. Estimates for $\mathcal{G}_2^N$. Let $I = [-1, 1]$ and set $\mathcal{E}(N)$ to be a class of smooth functions $\eta \in C^\infty(I^d \times I)$ satisfying $\|\eta\|_{C^N(I^d \times I)} \leq 1$ and $1/2 \leq \eta \leq 1$. We denote by $\mathcal{E}(\epsilon_0, N)$ the class of smooth functions defined on $I^{d-1} \times I$ which satisfy

$$\|\psi - \psi_0 - t\|_{C^N(I^{d-1} \times I)} \leq \epsilon_0,$$

where $\psi_0(\zeta) = |\zeta|^2/2$ for $\zeta \in I^{d-1}$. We now recall the following from [30].

Proposition 2.5. [30, Proposition 3.2] Let $\phi$ be a smooth function supported in $[-1, 1]$. If $p > \min \{p_0(d), \frac{2(d+2)}{d} \}$ and $\epsilon_0$ is sufficiently small, then for $\epsilon > 0$ there is a positive integer $M = M(\epsilon)$ such that

$$(2.9) \quad \left\| \left( \int_{-1}^1 \phi\left( \frac{\eta(D,t)(D_D - \psi(D',t))}{\delta} \right)f \right)^2 dt \right\|_p \leq B\delta^{-\frac{d-2}{2} + \frac{1}{2}}\|f\|_p$$

holds uniformly for $\psi \in \mathcal{E}(\epsilon_0, M)$ and $\eta \in \mathcal{E}(M)$ whenever $\text{supp} \hat{f} \subset [-1/2, 1/2]^d$. Here, we denote by $m(D)f$ the multiplier operator given by $\mathcal{F}(m(D)f)(\xi) = m(\xi)\hat{f}(\xi)$ and also write $D = (D',D_d)$ where $D',D_d$ correspond to the frequency variables $\xi', \xi_d$, respectively, where $\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

It is not difficult to see that the constant $B$ in (2.9) only depends on the $C^N$-norm of $\phi$ for some $N$ large enough, hence one can find $N = N(\epsilon)$, $\epsilon > 0$, such that (2.9) holds uniformly for $\psi \in \mathcal{E}(\epsilon_0, N)$, $\eta \in \mathcal{E}(N)$, and $\phi \in C_N([-1,1])$. In fact, $C^N$-is involved with kernel estimate which is needed for localization argument and $N$ can be taken to be as large as $\sim d$. As mentioned in Remark 3.3 in [30], Proposition 2.5 implies the following.

Lemma 2.6. For any $\epsilon > 0$, there is an $N$ such that (1.6) holds uniformly for all $\phi \in C_N([-1,1])$, if $p > p_\epsilon(d) = \min \{p_0(d), \frac{2(d+2)}{d} \}$.

In fact, let $\epsilon_0 > 0$ be sufficiently small. By finite decompositions, rotation, scaling, and change of variable, it suffices to prove that

$$\left\| \left( \int_{I_{\epsilon_0}} \phi\left( \frac{\eta^2 - |\tau|^2}{\delta} \right)f \right)^2 dt \right\|_p \leq C\delta^{\frac{2d}{2} - \frac{1}{2}}\|f\|_p, \quad \forall \text{supp} \hat{f} \subset B(-e_d, c\epsilon_0^2),$$

for $I_{\epsilon_0} = (1 - \epsilon_0^2, 1 + \epsilon_0^2)$ and $e_d = (0, \cdots, 0, 1)$. Note that $t^2 - |\xi|^2 = -(\tau + \sqrt{t^2 - |\xi|^2})(\tau - \sqrt{t^2 - |\xi|^2})$ for $\xi = (\zeta, \tau) \in B(-e_d, c\epsilon_0^2)$. Here $(\zeta, \tau) = \mathbb{R}^{d-1} \times \mathbb{R}$. Then the simple change of variables in Remark 3.3 in [30] transforms $\phi\left( \frac{t^2 - |\xi|^2}{\delta} \right)$ to $\phi\left( \frac{2n(\zeta, \tau)}{e_0^2\delta} \right)$ for some $\psi \in C(C\epsilon_0^2, N)$ and $\eta \in \mathcal{E}(N)$. Applying Proposition 2.5 we obtain Lemma 2.6.

Proposition 2.7 below follows from Lemma 2.6 and Lemma 2.3.

Proposition 2.7. Let $0 < \delta_0 \ll 1$. Then, for $p > p_\epsilon(d)$ and any $\epsilon > 0$ there is $N = N(\epsilon)$ so that

$$\|\mathcal{O}_\phi^N f\|_p \leq C\delta^{\frac{1}{2} + \frac{1}{d}-\epsilon}\|f\|_p$$

holds uniformly for $\phi \in C_N([-1,1])$ and $0 < \delta \leq \delta_0$. 
2.4. \(L^p - L^q\) estimates for \(\mathcal{D}_\delta^\phi\). Note that the multiplier of \(S^\phi_{\rho,\delta}\) in \(\mathcal{D}_\delta^\phi\) is supported in a \(C\delta\)-neighborhood of \(\sqrt{\rho}\)-sphere in \(\mathbb{R}^d\). Thus, by using Stein-Tomas theorem and well-known space localization argument we can obtain \(L^p - L^q\) estimates for \(\mathcal{D}_\delta^\phi\).

**Proposition 2.8.** Let \(q \geq \frac{2(d+1)}{d-1}\) and \(2 \leq p \leq q\). Then for any \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that, for any \(\phi \in \mathcal{C}\mathcal{N}([-1,1])\) and \(0 < \delta < 1\),

\[
\|\mathcal{D}_\delta^\phi f\|_q \lesssim \delta^{1-d/2 + \frac{d}{q} - \epsilon}\|f\|_p.
\]

Here the implicit constant is independent of \(\delta\) and \(\phi\).

Interpolation between these estimates and those in Proposition 2.7 give additional estimates. The loss \(\delta^{-\epsilon}\) can be removed by using better localization argument. See, for example, [31]. But we don’t attempt to this here.

Before proving Proposition 2.8, we recall Stein-Tomas theorem ([37]): For any \(q \geq \frac{2(d+1)}{d-1}\), there is \(C = C(p,d) > 0\) such that

\[
\|\hat{f}d\sigma\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^{d-1})},
\]

where \(\mathbb{S}^{d-1}\) is the unit sphere in \(\mathbb{R}^d\) and \(d\sigma\) is the induced Lebesgue measure on \(\mathbb{S}^{d-1}\). Using the polar coordinate, Stein-Tomas theorem, and mean-value theorem it is easy to see that, for \(q \geq \frac{2(d+1)}{d-1}\),

\[
(2.10) \quad \|S^\phi_{\rho,\delta} f\|_q \lesssim \delta^{1/2}\|f\|_2 \quad \text{for } 1/2 \leq \rho \leq 2.
\]

**Proof of Proposition 2.8.** We denote by \(K_{\rho}\) the kernel of \(S^\phi_{\rho,\delta}\) in short. By Lemma 2.2, we see that for \(1/2 \leq \rho \leq 1\), \(|K_{\rho}(x)| \leq C_{N}\delta(1 + \delta|x|)^{-N}\). Recall that \(C_N\) is independent of \(\delta\) and the choice of \(\phi \in \mathcal{C}\mathcal{N}([-1,1])\). This means that the kernel \(K_{\rho}\) is essentially supported in a ball of radius \(\sim \delta^{-1}\). This enable us to use spatial localization argument, which deduces \(L^p\) estimates for \(\mathcal{D}_\delta^\phi\) from \(L^2\to L^p\) bound.

Let \(\epsilon' > 0\). We first restrict \(f\) into balls of radius \(\delta^{-1-\epsilon'}\): set \(f_l = f\chi_{B(l,\delta^{-1-\epsilon'})}\), \(l \in \delta^{-1}\mathbb{Z}^d\). For \(x \in B(l,\delta^{-1-\epsilon'})\), we see that

\[
|S^\phi_{\rho,\delta}(f - f_l)(x)| \leq C_N\delta \int_{|y| \geq 2\delta^{-1-\epsilon'}} (1 + \delta|y|)^{-d-1}|f(x - y)|dy \leq E\|f\|_2(x),
\]

where \(E(x) = C_N\delta^K(1 + \delta|x|)^{-d-1}\) and \(K = N - d - 1\). Since \(q > 2\), we have

\[
||\left( \sum_{\rho \in \delta \mathbb{Z}^d \cap [1/2,1]} |S^\phi_{\rho,\delta} f|^2 \right)^{1/2}\|_q^q \lesssim \sum_{l \in \delta^{-1}2^d} \int_{B(l,\delta^{-1-\epsilon'})} \left( \sum_{\rho} |S^\phi_{\rho,\delta} f_l|^2 \right)^{q/2} \left( \sum_{\rho} |S^\phi_{\rho,\delta} (f - f_l)|^2 \right)^{q/2} \, dx
\]

\[
\lesssim \delta^{-C\epsilon'} \sum_{l} \left( \sum_{\rho \in \delta \mathbb{Z}^d \cap [1/2,1]} |S^\phi_{\rho,\delta} f_l|^2 \right)^{q/2} + \delta^\epsilon' \int_{\mathbb{R}^d} \left( \sum_{\rho \in \delta \mathbb{Z}^d \cap [1/2,1]} (E \|f\|_2)^2 \right)^{q/2} \, dx.
\]
Here the implicit constant depends only on $d$. Notice that $S^d_{\rho,\delta} f_i = S^d_{\rho,\delta} P_{\rho} f_i$, where $P_{\rho} h$ is defined by

$$
(2.12) \quad \hat{P}_{\rho} h(\xi) = \chi_{\rho}(\xi) \hat{h}(\xi)
$$

and $\chi_{\rho}$ is a characteristic function of $\Delta_{\rho} := \{ \xi \in \mathbb{R}^d : |\xi|^2 \in [\rho - \delta, \rho + \delta] \}$. Since $\Delta_{\rho}$ are overlapping at most twice, $\sum_{\rho \in \delta \mathbb{Z}^2 \cap [1/2,1]} \| P_{\rho} f_i \|_2^2 \leq 2 \| f_i \|_2^2$. By using this, the first term of (2.11) is bounded by $(C\delta^{-(\frac{d}{2} - s) - \epsilon - (\frac{d}{2} - t) + C)q \| f \|_p^q$ because of (2.10), $l^p \subset l^q$, and $p \geq 2$. Since $0 < \delta < 1$ and $p \leq q$, the second term of (2.11) is bounded by

$$
\delta^{-C'} \left( \sum_{\rho \in \delta \mathbb{Z}^2 \cap [1/2,1]} \| E * |f| \|_q^2 \right)^{q/2} \leq C K_1 \delta^{-C'} \left( \sum_{\rho \in \delta \mathbb{Z}^2 \cap [1/2,1]} \| E * |f| \|_q^2 \right)^{q/2} \leq \| f \|_p^q.
$$

if $K$ is sufficiently large (i.e., $N$ is large enough). Thus, taking $\epsilon' = \epsilon/C$ for some large $C$, we get the desired inequality. \hfill \Box

3. Reduction; decomposition lemma

In this section, we will break the operator $B^\alpha$ so that our problem is reduced to obtaining bounds for a simpler bilinear operator which is given by products of $S^d_{\rho,\delta}$ with different $\rho$ which is defined by (2.2). This reduction enables us to draw connection to the square function estimate. To do this, we first consider an auxiliary bilinear operators $B_\delta$, $0 < \delta \ll 1$ which is given by dyadically decomposing the multiplier of $B^\alpha$ away from its singularity $\{ (\xi, \eta) : |\xi|^2 + |\eta|^2 = 1 \}$.

Let us denote by $D$ the set of positive dyadic numbers, that is to say $D = \{ 2^k : k \in \mathbb{Z} \}$. Fix $\alpha > 0$ and let $\psi$ be a function $\in C_c^\infty(1/2, 2)$ satisfying $\sum_{\delta \in D} \delta^\alpha \psi(t/\delta) = \delta^\alpha$, $t > 0$. Then we may write

$$
(1 - t)^\alpha = \sum_{\delta \in D, \delta \leq 2^{-1}} \delta^\alpha \psi \left( \frac{1 - t}{\delta} \right) + \psi_0(t), \quad t \in [0, 1),
$$

where $\psi_0$ is a smooth function supported in $[0, 3/4]$. Using this we decompose $B^\alpha$ so that

$$
(3.1) \quad B^\alpha = \sum_{\delta \in D} \delta^\alpha B_\delta + \tilde{B}_0,
$$

where

$$
(3.2) \quad B_\delta(f, g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i x \cdot (\xi + \eta)} \psi \left( \frac{1 - |\xi|^2 - |\eta|^2}{\delta} \right) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta
$$

and $\tilde{B}_0$ is similarly defined by $\psi_0$. Since $\psi_0 \in C_c([0, 3/4])$, it is easy to see that

$$
\| \tilde{B}_0(f, g) \|_r \leq C \| f \|_p \| g \|_q
$$

whenever $1/r \leq 1/p + 1/q$. Thus, in order to show (1.3) for $\alpha > \kappa$ it is sufficient to show that, for any $\epsilon > 0$, there exits $C_\epsilon$ such that

$$
\| \tilde{B}_\delta(f, g) \|_r \leq C_\epsilon \delta^{-\kappa - \epsilon} \| f \|_p \| g \|_q.
$$
For $\phi_1, \phi_2$ be smooth functions supported in $[-1, 1]$, and $q \in [1/2, 2]$, we define the bilinear operators $B_{\delta, \epsilon}^{\phi_1, \phi_2}$ by setting

$$B_{\delta, \epsilon}^{\phi_1, \phi_2}(f, g)(x) := \sum_{\rho \in \delta Z \cap [0, 1]} S_{\rho, \delta}^{\phi_1} f(x) S_{\epsilon - \rho, \delta}^{\phi_2} g(x)$$

Thanks to the above argument and Lemma 3.1 below, instead of $B_{\delta}$ it suffices to obtain bounds for $B_{\delta, \epsilon}^{\phi_1, \phi_2}$ of which product structure makes the problem easier.

**Lemma 3.1.** Let $\kappa \geq -1$, $0 < \delta_0 < 1$ and $1 \leq p, q, r \leq \infty$ satisfy $1/p + 1/q \geq 1/r$. Suppose that, for any $0 < \delta \ll \delta_0$ and $q \in [1/2, 2]$.

$$\|B_{\delta, \epsilon}^{\phi_1, \phi_2}\|_{L^p \times L^q \rightarrow L^r} \leq A \delta^{-\kappa}$$

holds uniformly with $A > 0$ independent of $\delta, q$, and $\phi_1, \phi_2$, whenever $\phi_1, \phi_2 \in C_N([-1, 1])$ for some $N$. Then, for any $\epsilon > 0$ there exists a constant $A_\epsilon$, independent of $\delta$, such that

$$\|\tilde{B}_{\delta}\|_{L^p \times L^q \rightarrow L^r} \leq A_\epsilon \delta^{-\kappa - \epsilon(1 + \delta)}.$$

It is not difficult to see that (3.4) does not hold for $\kappa < -1$. In fact, let $f, g$ be smooth functions such that $\text{supp} \tilde{f}, \text{supp} \tilde{g} \subset B(0, 4)$ and $\tilde{f} = \tilde{g} = 1$ on $B(0, 3)$ and $\phi = \phi_1 = \phi_2$ be nontrivial nonnegative functions with $\text{supp} \phi \subset [-1, 1]$. Then, it is easy to see that $|B_{\delta, \epsilon}^{\phi_1, \phi_2}(f, g)(x)| \gtrsim \delta$ if $|x| \leq c$ with sufficiently small $c > 0$. Thus $\|B_{\delta, \epsilon}^{\phi_1, \phi_2}(f, g)\|_r \gtrsim \delta$ while $\|f\|_p, \|g\|_q \lesssim 1$. This implies $\|B_{\delta, \epsilon}^{\phi_1, \phi_2}\|_{L^p \times L^q \rightarrow L^r} \gtrsim \delta$.

**Remark 3.2.** Using Lemma 3.1 and the trivial $L^2$-estimate for $S_{\rho, \delta}^{\phi}$, we can easily recover the boundedness of $B^\alpha$ from $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ for $\alpha > 0$ (Proposition 1.1). Applying Schwarz’s inequality and Plancherel’s theorem, we have

$$\|B_{\delta, \epsilon}^{\phi_1, \phi_2}(f, g)\|_1 \leq \left( \sum_{\rho \in \delta Z \cap [0, 1]} \|S_{\rho, \delta}^{\phi_1} f\|_2^2 \right)^{1/2} \left( \sum_{\rho \in \delta Z \cap [0, 1]} \|S_{\epsilon - \rho, \delta}^{\phi_2} g\|_2^2 \right)^{1/2}.$$

Since $\sum_{\rho \in \delta Z \cap [0, 1]} \|S_{\rho, \delta} f\|_2^2 \lesssim \|f\|_2^2$, $\sum_{\rho \in \delta Z \cap [0, 1]} \|S_{\epsilon - \rho, \delta} g\|_2^2 \lesssim \|g\|_2^2$, it follows from the above that $\|B_{\delta, \epsilon}^{\phi_1, \phi_2}\|_{L^2 \times L^2 \rightarrow L^1} \leq 1$. By Lemma 3.1 $\|\tilde{B}_{\delta}\|_{L^2 \times L^2 \rightarrow L^1} \leq A_\epsilon \delta^{-\kappa - \epsilon}$ and, hence, from (3.1) we see that $B^\alpha$ is bounded from $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ for all $\alpha > 0$.

**Proof of Lemma 3.1.** Let $\varphi \in C_c^\infty([-1, 1])$ satisfy

$$\sum_{k \in \mathbb{Z}} \varphi(t + k) = 1, \quad t \in \mathbb{R}.$$

Using this, we will decompose the multiplier of $\tilde{B}_{\delta}$ into sum of multipliers which are given by (tensor) product of two multipliers supported in thin annuli. More precisely, we fix small $\epsilon > 0$ and $0 < \delta \leq \delta_0$, and set $\delta = \delta^{1+\epsilon} < \delta$. Then

$$\psi\left( \frac{1 - |\xi|^2 - |\eta|^2}{\delta} \right) = \sum_{\rho \in \delta Z \cap [0, 1]} \sum_{\rho \in \delta Z} \varphi\left( \frac{\rho - |\xi|^2}{\delta} \right) \varphi\left( \frac{\rho - |\eta|^2}{\delta} \right) \psi\left( \frac{1 - |\xi|^2 - |\eta|^2}{\delta} \right).$$

Note that $\varphi\left( \frac{\rho - |\xi|^2}{\delta} \right) \psi\left( \frac{1 - |\xi|^2 - |\eta|^2}{\delta} \right) \neq 0$ implies $1 - 3\delta \leq |\eta|^2 + \rho \leq 1 + \delta$, since $\text{supp} \varphi \subset [-1, 1]$ and $\text{supp} \psi \subset [1/2, 2]$. The summands vanish if we take the sum
over $g \in \delta \mathbb{Z} \cap (\mathbb{R} \setminus [1 - 4\delta, 1 + 2\delta])$. Thus we can write

\begin{equation}
\bar{B}_\delta(f, g)(x) = \sum_{\rho \in \delta \mathbb{Z} \cap [1 - 4\delta, 1 + 2\delta]} \sum_{\rho \in \delta \mathbb{Z} \cap [0, 1]} \iint_{\mathbb{R}^d \times \mathbb{R}_d} e^{2\pi i x \cdot (\xi + \eta)} \times \psi\left(1 - \frac{1}{\delta} \frac{\left|\xi^2 - \eta^2\right|}{\left|\rho^2 - \eta^2\right|}\right) \varphi\left(\frac{\rho - \left|\xi^2 - \eta^2\right|}{\delta}\right) \varphi\left(\frac{\rho - \eta^2}{\delta}\right) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta.
\end{equation}

Let $N$ be a constant to be chosen later. By Taylor’s theorem we may write

\begin{equation}
E^{2\pi i \left(\frac{\xi^2 - \eta^2}{\delta}\right)} = \sum_{0 \leq \beta + \gamma \leq N} C_{\beta, \gamma} \tau^{\beta + \gamma} \left(\frac{\rho - \left|\xi^2 - \eta^2\right|}{\delta}\right) \left(\frac{\rho - \eta^2}{\delta}\right) E\left(2\pi i \left(\frac{\rho - \left|\xi^2 - \eta^2\right|}{\delta}\right)\right),
\end{equation}

for any $\rho \in \delta \mathbb{Z} \cap [0, 1]$ and the remainder $E$ satisfies, for $0 \leq k \leq N$,

\begin{equation}
|E^{(k)}(t)| \leq C_k |t|^{N-k}.
\end{equation}

Using inversion, for any $\rho$ we have

\begin{equation}
\psi\left(1 - \frac{1}{\delta} \frac{\left|\xi^2 - \eta^2\right|}{\left|\rho^2 - \eta^2\right|}\right) = \int_{\mathbb{R}} \hat{\psi}(t) e^{2\pi i \frac{\xi^2 - \eta^2}{\delta}} e^{2\pi i (\rho - \left|\xi^2 - \eta^2\right|) \tau} d\tau.
\end{equation}

For $0 \leq \beta \leq N$ we set $\phi_\beta(t) = t^\beta \varphi(t) \in C_0^\infty(-1, 1)$ and also set

\begin{equation}
\hat{\psi}_\beta = \hat{\psi}(t) e^{2\pi i \frac{1}{\delta} \tau}.
\end{equation}

Then, putting the above in the right hand side of (3.8), we have

\begin{equation}
\psi\left(1 - \frac{1}{\delta} \frac{\left|\xi^2 - \eta^2\right|}{\left|\rho^2 - \eta^2\right|}\right) = \sum_{0 \leq \beta + \gamma \leq N} C_{\beta, \gamma} \left( \int \hat{\psi}(t) \tau^{\beta + \gamma} d\tau \right) \times \phi_\beta \left(\frac{\rho - \left|\xi^2 - \eta^2\right|}{\delta}\right) \varphi\left(\frac{\rho - \eta^2}{\delta}\right) + \int \hat{\psi}(t) E\left(2\pi i \left(\frac{\rho - \left|\xi^2 - \eta^2\right|}{\delta}\right)\right) d\tau
\end{equation}

For $0 \leq \beta, \gamma \leq N$, we set

\begin{equation}
I_{\beta, \gamma}^\rho(f, g) = \left(\int \hat{\psi}(t) \tau^{\beta + \gamma} d\tau \right) \times \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0, 1]} S_{\rho, \delta}^\phi \right) f(x) S_{\rho, \delta}^\eta g(x).
\end{equation}

Inserting (3.9) in (3.6), we express $\bar{B}_\delta$ as a sum of bilinear operators which are given by products of $S_{\rho, \delta}^{\phi, \psi}$.

\begin{equation}
\bar{B}_\delta(f, g) = \sum_{\rho \in \delta \mathbb{Z} \cap [1 - 4\delta, 1 + 2\delta]} \left[ \sum_{0 \leq \beta + \gamma \leq N} C_{\beta, \gamma} \delta^{(\beta + \gamma)} I_{\beta, \gamma}^\rho(f, g) \right],
\end{equation}

where

\begin{equation}
I_{E}^{\delta}(f, g) = \sum_{\rho} \int \hat{\psi}(t) \int e^{2\pi i x \cdot (\xi + \eta)} E_{\delta, \delta}(\xi, \eta, \rho, \varphi, \tau) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta d\tau,
\end{equation}

and

\begin{equation}
E_{\delta, \delta}(\xi, \eta, \rho, \varphi, \tau) = E\left(2\pi i \left(\frac{\left|\xi^2 - \eta^2\right|}{\delta}\right)\right) \varphi\left(\frac{\rho - \left|\xi^2 - \eta^2\right|}{\delta}\right) \varphi\left(\frac{\rho - \eta^2}{\delta}\right).
\end{equation}
Set $M = \max \{\|\phi\|_{C^N([-1,1])} : 0 \leq \beta \leq N\}$. Then $M^{-1}\phi \in C_N([-1,1])$ for all $0 \leq \beta \leq N$. Thus, from the assumption (3.4) we have that, for each $I^g_{\beta,\gamma}$, there exists a constant $A$ such that

\[(3.11) \quad \|I^g_{\beta,\gamma}(f, g)\|_r \leq A M^\delta N\|f\|_p\|g\|_q,\]

since $\psi$ is a Schwartz function.

Now, in order to complete the proof, it is sufficient to show

\[(3.12) \quad \|I^g_E(f, g)\|_r \leq A^\delta N\|f\|_p\|g\|_q.\]

For the purpose, we use Lemma 3.3 below, which is a simple consequence of the bilinear interpolation.

**Lemma 3.3.** Let $0 < \delta < 1$ and $\tau \in \mathbb{R}$. Fix a large integer $N > 2d$. Suppose that $m_{\delta, \tau} \in C^\infty_0(\mathbb{R}^d \times \mathbb{R}^d)$ is a smooth function supported in the cube $[-2, 2]^d$ in $\mathbb{R}^d$, and suppose that $m_{\delta, \tau}$ satisfies

\[|\partial_{\xi}^\alpha \partial_{\eta}^\beta m_{\delta, \tau}(\xi, \eta)| \leq C_{\alpha, \beta}(1 + |\tau|)^N \delta^{-|\alpha| - |\beta|}\]

for all multi-indices $\alpha, \beta$ with $|\alpha| + |\beta| \leq N$. Let $T_{\delta, \tau}$ be defined by

\[T_{\delta, \tau}(f, g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{2\pi i x \cdot (\xi + \eta)} m_{\delta, \tau}(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta, \quad f, g \in \mathcal{S}(\mathbb{R}^d)\]

Then, for $p, q, r \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have

\[\|T_{\delta, \tau}(f, g)\|_r \leq C(1 + |\tau|)^N \delta^{-d(2 - \frac{1}{p} - \frac{1}{q})}\|f\|_p\|g\|_q.\]

**Proof.** By definition, we can write

\[T_{\delta, \tau}(f, g)(x) = \int \int \widehat{m_{\delta, \tau}}(y - x, z - x) f(y) g(z) dy dz.\]

Applying usual integration by parts, we have $|\widehat{m_{\delta, \tau}}(y, z)| \leq C_K(1 + |\tau|)^N (1 + \delta|y|)^{-N_1} (1 + \delta|z|)^{-N_2}$ for all $N_1 + N_2 \leq N$. Since $N$ is an integer bigger than $2d$, in particular, we have

\[|\widehat{m_{\delta, \tau}}(y, z)| \leq C(1 + |\tau|)^N (1 + \delta|y|)^{-d - \frac{1}{2}} (1 + \delta|z|)^{-d - \frac{1}{2}}.\]

Thus, for any $p, q \geq 1$,

\[\|T_{\delta, \tau}(f, g)\|_\infty \leq C(1 + |\tau|)^N \delta^{-d(2 - \frac{1}{p} - \frac{1}{q})}\|f\|_p\|g\|_q.\]

On the other hand, by Fubini’s theorem we have

\[\|T_{\delta, \tau}(f, g)\|_1 \leq C(1 + |\tau|)^N \delta^{-d(2 - \frac{1}{p} - \frac{1}{q})}\|f\|_p\|g\|_q\]

for any $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} \geq 1$. The bilinear interpolation between these two estimates gives all the desired estimates. \qed

In order to apply Lemma 3.3 to $I^g_E$, we define a function $m$ on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ by

\[m(\xi, \eta, \tau) = \delta^{-CN} E^\delta_{\beta}(\xi, \eta, \rho, q, \tau)\]
for some \( \rho \in \mathbb{Z} \cap [0, 1] \), then \( m(\cdot, \cdot, \tau) \) satisfies all properties of the function \( m_{3, \tau} \) in Lemma 3.3, because of (3.7). More precisely, using (3.7) we see that

\[
|m(\xi, \eta, \tau)| \leq C_0 |\tau|^N \left( \frac{\rho - |\xi|^2}{\delta} + \left| \frac{\rho - \rho - |\eta|^2}{\delta} \right| \right)^N \left| \varphi \left( \frac{\rho - |\xi|^2}{\delta} \right) \varphi \left( \frac{\rho - \rho - |\eta|^2}{\delta} \right) \right|
\]

\[
\leq C_0 (1 + |\tau|)^N 2^N \| \varphi \|_\infty^2,
\]

and similarly, by direct differentiation and using (3.7) we also have, for \( \beta, \gamma \) with \( |\beta| + |\gamma| \leq N \),

\[
|\partial_\beta^\gamma m(\xi, \eta, \tau)| \leq C (1 + |\tau|)^N (\delta)^{-\beta - \gamma}.
\]

Here \( C \) depends only on \( C_k \) in (3.7) and \( M \). We note that \( I_E(f, g) \) is expressed by

\[
I_E(f, g)(x) = \delta^{\alpha N} \sum_{\rho \in \mathbb{Z} \cap [0, 1]} e^{2\pi i \frac{\rho x}{\delta}} \int T_{3, \tau} \psi(\tau) T_{3, \tau}(f, g)(x) d\tau,
\]

where \( T_{3, \tau} \) is defined as in Lemma 3.3. Thus, by Lemma 3.3 and Minkowski’s inequality we obtain

\[
\|I_E(f, g)\|_r \leq C \delta^{\alpha N} (\delta)^{-2d} \| f \|_p \| g \|_q \int |\psi(\tau)| (1 + |\tau|)^N d\tau
\]

\[
\lesssim \delta^{\alpha N} (\delta)^{-2d - 1} \| f \|_p \| g \|_q,
\]

provided that \( \frac{1}{p} + \frac{1}{q} \geq \frac{1}{r} \). Thus, combining this estimate, (3.10) and (3.11), we obtain

\[
\| \mathcal{B}_\delta(f, g) \|_r \lesssim \sum_{\rho \in \mathbb{Z} \cap [1 - 4\delta, 1 + 2\delta]} \left( \sum_{0 \leq \beta + \gamma \leq N} |C_{\beta, \gamma}| \delta^{\beta + \gamma} (\delta)^{-\kappa} + \delta^{(N-2d-1)-2d-1} \right) \| f \|_p \| g \|_q.
\]

Therefore, choosing sufficiently large \( N \), we have

\[
\| \mathcal{B}_\delta(f, g) \|_r \lesssim \delta^{-\kappa - \epsilon(1 + \kappa)} \| f \|_p \| g \|_q.
\]

Here the implicit constant is independent of \( \delta \).

\[\square\]

### 4. Boundedness of bilinear Bochner-Riesz operators

In this section we prove Theorem 1.2 and also obtain results for the sub-critical case \( \frac{1}{p} + \frac{1}{q} > \frac{1}{r} \) mostly relying on Stein-Tomas’s theorem. In addition, we find a necessary condition for \( \mathcal{B}^* \) by using duality and asymptotic behavior of localized kernel.

#### 4.1. Proof of Theorem 1.2

To verify Theorem 1.2, by (3.1) and Lemma 3.1 it is enough to show Proposition 4.1 below by using the argument in Section 3.

**Proposition 4.1.** Let \( d \geq 2 \), \( 0 < \delta \leq \delta_0 \ll 1 \), \( 2 \leq p, q \leq \infty \) and \( 1/r = 1/p + 1/q \). Suppose that for \( p \geq p_0 \) the estimate (1.6) holds with \( C \) independent of \( \delta \) and \( \phi \) whenever \( \phi \in C_N([-1, 1]) \) for some \( N \). Then, for any \( \epsilon > 0 \), there exist \( N = N(\epsilon) \) and \( C_\epsilon \) such that for \( \rho \in [1/2, 2] \)

\[
(4.1) \quad \| \mathcal{B}_\rho \|_{L^r(\mathbb{R}^d)} \leq C_\epsilon \delta^{-\kappa - \epsilon} \| f \|_{L^p(\mathbb{R}^d)} \| g \|_{L^q(\mathbb{R}^d)}
\]

holds uniformly provided that \( \phi_1 \) and \( \phi_2 \) in \( C_N([-1, 1]) \).
Proof. In view of the interpolation it suffices to prove (4.1) for critical pairs of exponents $(1/p, 1/q)$ which are in $\Delta_1 = \Delta_1 \left( \frac{1}{p_0} \right)$, $\{(1/2, 1/p_0)\}$, $\{(1/p_0, 1/2)\}$, $\{(1/2, 0)\}$, $\{(0, 1/2)\}$, and $\{(1/2, 1/2)\}$.

We first consider the case $(1/p, 1/q) \in \Delta_1$. We fix $(1/p, 1/q) \in \Delta_1$, i.e., $p, q \geq p_0$. Then it is sufficient to show that for any $\epsilon > 0$ there is an $N$ such that

$$\|B_{\phi_1, \phi_2}^{\rho, \delta}(f, g)\|_p \lesssim \delta^{-\beta}(1-\beta) - \epsilon \|f\|_p \|g\|_q,$$

where $B_{\phi_1, \phi_2}^{\rho, \delta}$ is associated with $\phi_j \in C_N([-1, 1])$ and the implicit constant is independent of the choice of $\phi_j$’s and $\delta, \rho$. Recall that

$$B_{\phi_1, \phi_2}^{\rho, \delta}(f, g) = \sum_{\rho \in \delta \mathbb{Z} \cap [0,1]} (S_{\phi_1, \rho, \delta}^\rho f(S_{\phi_2, \rho, \delta}^\rho g)).$$

By Schwarz’s inequality, for any $x \in \mathbb{R}^d$

$$\|B_{\phi_1, \phi_2}^{\rho, \delta}(f, g)(x)\| \leq \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0,1]} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \right)^{1/2} \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0,1]} |S_{\phi_2, \rho, \delta}^\rho g(x)|^2 \right)^{1/2}. \tag{4.2}$$

In this case we only deal with the triple pair of exponents $(p, q, r)$ satisfying Hölder’s relation. Hence, by Hölder’s inequality, it suffices to show that

$$\left\| \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0,1]} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \right)^{1/2} \right\|_p \leq C \delta^{-\beta(1/p) - \epsilon} \|f\|_p \tag{4.3}$$

for $p \geq p_0$. Indeed, since each $g$ is small perturbation of 1 and $q \geq p_0$, the same argument which shows (4.3) implies the uniform bounds for $L^q$-estimate for

$$\left( \sum_{\rho \in \delta \mathbb{Z} \cap [0,1]} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \right)^{1/2} \|g\|_q.$$

We now prove (4.3). It is a consequence of the estimates for square functions $\mathcal{D}_\phi$ in subsection 2.3. In fact, set $C_1 = \delta_0^{-1}$ and decompose the interval $[C_1 \delta, 1]$ dyadically, i.e., we set

$$\bigcup_{k=0}^{k_\delta} I_k := \bigcup_{k=0}^{k_\delta} [2^{-k-1}, 2^{-k}] \cap [C_1 \delta, 1] = [C_1 \delta, 1],$$

where $k_\delta + 1$ is the smallest integer satisfying $[2^{-k-1}, 2^{-k}] \cap [C_1 \delta, 1] = \emptyset$. By triangle inequality, we have

$$\left( \sum_{\rho \in \delta \mathbb{Z} \cap [0,1]} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \right)^{1/2} \leq \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0, C_1 \delta]} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \right)^{1/2} + \sum_{k=0}^{k_\delta} \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_k} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \right)^{1/2}. \tag{4.4}$$

When $k = 0$, Lemma 2.3 implies

$$\|\langle \sum_{\rho \in \delta \mathbb{Z} \cap I_0} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \rangle \|_p \leq C \delta^{-\beta(1/p) - \epsilon} \|f\|_p$$

holds uniformly for $\phi \in C_N([-1, 1])$. By scaling $\xi \to 2^{-k} \xi$, it is easy to see that $S_{\phi_1, \rho, \delta}^\rho f(x) = S_{2^{k} \phi, 2^k \rho, \delta}^{2^k \rho}(f(2^{-k} \cdot))(2^{-k} x)$. Thus we have that

$$\|\langle \sum_{\rho \in \delta \mathbb{Z} \cap I_k} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \rangle \|_p = 2^{\frac{k_\delta}{2} - \frac{k}{2}} \|\langle \sum_{\rho \in \delta \mathbb{Z} \cap I_k} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \rangle \|_p.$$

By scaling, we get

$$\|\langle \sum_{\rho \in \delta \mathbb{Z} \cap I_0} |S_{\phi_1, \rho, \delta}^\rho f(x)|^2 \rangle \|_p \leq C \delta^{-\beta(1/p) - \epsilon} \|f\|_p.$$
Now, since $2^k \delta \leq \delta_0$, using Proposition 2.7 and recalling, we have for $k \geq 1$
\begin{equation}
\| \left( \sum_{\rho \in \delta \mathbb{Z}^n \setminus \Delta_k} |S_{\rho, \delta} f(x)|^2 \right)^{1/2} \|_p \leq C\epsilon (2^k \delta)^{-\beta_*(1/p)-\epsilon} \|f\|_p.
\end{equation}
Since $\beta_*(1/p) > 0$, summing over $k$ we see that
\begin{equation}
\sum_{k=0}^{\infty} \left( \sum_{\rho \in \delta \mathbb{Z}^n \setminus \Delta_k} |S_{\rho, \delta} f(x)|^2 \right)^{1/2} \|_p \leq C\epsilon \delta^{-\beta_*(1/p)-\epsilon} \|f\|_p.
\end{equation}
For the first term in (4.4), we recall from Lemma 2.2 that $\|K_{\rho, \delta}\|_1 = O(1)$ for $\rho \leq \delta$.
Thus, by Young’s convolution inequality we have that $\|S_{\rho, \delta} f\|_p \leq \|f\|_p$ for $\rho \leq \delta$.
There are only $O(1)$ many $\rho \in \delta \mathbb{Z} \cap [0, C_1 \delta]$. Thus it follows that
\begin{equation}
\left( \sum_{\rho \in \delta \mathbb{Z} \cap [0, C_1 \delta]} |S_{\rho, \delta} f(x)|^2 \right)^{1/2} \|_p \leq \|f\|_p.
\end{equation}
Combining this with the above, we obtain (4.3).

We now consider the remaining cases $(p, q) = (2, 2), (2, \infty), (\infty, 2), (2, p_0), (p_0, 2)$. The case $(p, q) = (2, 2)$ is already handled in Remark 3.2. It is sufficient to show (4.1) for $(\infty, 2), (p_0, 2)$ since the other cases symmetrically follow by the same argument. The proof of these two cases are rather straight forward. From (4.2), Hölder’s inequality, (4.3), and Plancherel’s theorem we have, for $p \geq p_0$ and $\epsilon > 0$,
\begin{equation}
\|B^\alpha_{k\Phi^1, \phi^2}(f, g)\|_r \leq \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0, 1]} |S_{\rho, \delta} f(x)|^2 \right)^{1/2} \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0, 1]} |S_{\rho, \delta} g(x)|^2 \right)^{1/2} \leq C \delta^{-\beta_*(1/p)-\epsilon} \|f\|_p \|g\|_2,
\end{equation}
where $1/r = 1/p + 1/2$. This completes the proof.

4.2. Sub-critical case: $\frac{1}{p} + \frac{1}{q} > \frac{1}{r}$. In this subsection, we consider $L^p \times L^q \rightarrow L^r$ boundedness for the case $1/p + 1/q \geq 1/r$. For the rest of this section we set
\begin{equation}
r_1 = \frac{2(d + 1)}{d - 1}, \quad r_2 = \frac{2d}{d - 2}.
\end{equation}
The following is the main result of this section.

Theorem 4.2. Let $d \geq 2, 2 \leq p, q \leq \infty$ and $r \geq \frac{d + 1}{d - 1}$. If $1/p + 1/q \geq 1/r$, then
\begin{equation}
\|B^\alpha (f, g)\|_{L^r(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}
\end{equation}
holds for $\alpha > \gamma(p, q, r)$, where $\gamma(p, q, r)$ is defined as follows:
\begin{equation}
\gamma(p, q, r) = \begin{cases}
\beta_*(1/p) + \beta_*(1/q), & \text{if } \frac{1}{r} \leq \frac{1}{r_1} + \frac{1}{r_2}, \\
\beta_*(1/p) + \beta_*(1/q) - \frac{d^2 - d - 1}{d(d + 1)} + \frac{d}{2r}, & \text{if } \frac{1}{r_1} + \frac{1}{r_2} \leq \frac{1}{r} \leq \frac{2}{r_1}.
\end{cases}
\end{equation}
Further estimates are possible if we interpolate the estimates in the above with those in Theorem 1.2.

Recall (1.2) and note that the operator $B^\alpha$ is well-defined for $\alpha > -1$. For $\alpha \leq -1$, $B^\alpha (f, g)/\Gamma(\alpha + 1)$ is defined by analytic continuation. $L^p-L^q$ estimates for the classical Bochner-Riesz operator of negative order have been studied by several authors [1, 6, 34, 12] and its connection to the Bochner-Riesz conjecture is now well
understood. It also seems to be an interesting problem to characterize $L^p \times L^q \to L^r$ boundedness of $B^\alpha$ of negative order, but such attempt might be premature in view of current state of art.

We deduce the estimates in Theorem 4.2 from easier $L^2 \times L^2 \to L^r$ estimates. For the purpose we make use of the following localization lemma.

**Lemma 4.3.** Let $1 \leq p, q, r, p_0, q_0, r_0 \leq \infty$ satisfy $1/p + 1/q \geq 1/r$ and $p_0 \leq p, q_0 \leq q, r \leq r_0$, and let $g \in [1/2, 2]$. Suppose $\|B^\alpha \|_{r_0} \leq C\delta B \|f\|_{p_0} \|g\|_{q_0}$ holds uniformly provided that $\phi_1$ and $\phi_2$ in $C_N([-1, 1])$, then for any $\epsilon > 0$, there are constants $C_\epsilon$ and $N'$, such that

$$
\|B^\alpha \|_{r_0} \leq C_\epsilon \delta B \|f\|_{p_0} \|g\|_{q_0}
$$

holds uniformly whenever $\phi_1$ and $\phi_2$ in $C_N([-1, 1])$.

By further refinement of the argument below it is possible to remove $\epsilon > 0$. This lemma can be obtained by adapting the localization argument used for the proof of Proposition 2.8. Hence, we shall be brief.

**Proof.** Let $\epsilon' > 0$. As in the proof of Proposition 2.8, we localize $f$ and $g$ into $3 \times \delta^{-1-\epsilon'}$-balls as follows: set $f_l = f\chi_{B(l,3\delta-1-\epsilon')}$ and $g_l = g\chi_{B(l,3\delta-1-\epsilon')}$ for $l \in \delta^{-1}Z^d$. Then for $x \in B(l, \delta^{-1-\epsilon'})$

$$
|S^\phi_{\rho, \delta}(f - f_l)(x)| \lesssim \delta E \|f\|_r \quad \text{and} \quad |S^\phi_{\rho, \delta}(g - g_l)(x)| \lesssim \delta E \|g\|_r
$$

for all $\rho \in [0, 1]$ where $E(x) = \delta^K(1 + \delta|x|)^{d-1}$ for any $K > 0$ and the implicit constant depends on $K$. Also note from Lemma 2.2 that the convolution kernels of $S^\phi_{\rho, \delta}$, $S^\phi_{\rho, \delta}$ are bounded by $\phi(x) := C\delta(1 + \delta|x|)^{-N}$ for any $N$. Thus, writing

$$
S^\phi_{\rho, \delta}(f - f_l)g = S^\phi_{\rho, \delta}(f - f_l)g + S^\phi_{\rho, \delta}f S^\phi_{\rho, \delta}(g - g_l) + S^\phi_{\rho, \delta}f S^\phi_{\rho, \delta}(g - g_l)
$$

and using the above we see that, if $x \in B(l, \delta^{-1-\epsilon'})$,

$$
|B^\alpha(f, g)(x)| \lesssim \|B^\alpha(f_l, g_l)(x) + (E \|f\|_r)(x) + (E \|g\|_r)(x) + (E \|f\|_r)(x) + (E \|g\|_r)(x).
$$

Since we can take $K$ arbitrary large, the contribution from the last two terms in the right hand side is negligible. Thus, it is sufficient to show

$$
\left( \sum_{l \in \delta^{-1}Z^d} \|B^\alpha(f_l, g_l)\|_{r_0}^r \right)^{\frac{1}{r}} \lesssim \delta B \|f\|_{p_0} \|g\|_{q_0}
$$

Using the assumption $\|B^\alpha(f, g)\|_{r_0} \lesssim \delta B \|f\|_{p_0} \|g\|_{q_0}$ and Hölder's inequality give

$$
\|B^\alpha(f_l, g_l)\|_{r_0} \lesssim \delta^{C_\epsilon} \delta B \delta^{d(\frac{1}{p} + \frac{1}{q} - \frac{1}{p_0} - \frac{1}{q_0} + \frac{1}{r_0})}\|f_l\|_r \|g_l\|_{q_0}.
$$

Since $1/p + 1/q \geq 1/r$, by Hölder inequality again for summation along $l$,

$$
\left( \sum_{l \in \delta^{-1}Z^d} \|B^\alpha(f_l, g_l)\|_{r_0}^r \right)^{\frac{1}{r}} \lesssim \delta^{C_\epsilon} B \|f\|_{p_0} \|g\|_{q_0} \lesssim \delta^{C_\epsilon} B \|f\|_{p_0} \|g\|_{q_0} \lesssim \delta^{C_\epsilon} B \|f\|_{p_0} \|g\|_{q_0}.
$$

This gives the desired bound if we take $\epsilon' = \epsilon/C$ with large enough $C$. \qed

The following is a bilinear version of Proposition 2.8.
Lemma 4.4. Let $0 < \delta \ll 1$, $g \in [1/2, 2]$, and $r \geq \frac{d-1}{d+1}$. Then, for $\epsilon > 0$ there is $N = N(\epsilon)$ such that

\begin{equation}
\|B_{k, g}^{\phi_1, \phi_2}(f, g)\|_r \lesssim \begin{cases} \\
\delta^{1-\epsilon}\|f\|_2\|g\|_2 & \text{if } \frac{1}{r} \leq \frac{1}{r_1} + \frac{1}{r_2}, \\
\delta^{\frac{d+1}{2d} - \frac{1}{r}}\|f\|_2\|g\|_2 & \text{if } \frac{1}{r_1} + \frac{1}{r_2} \leq \frac{1}{r} \leq \frac{2}{r_1}\n\end{cases}
\end{equation}

holds uniformly in $\delta, g$, and $\phi_1, \phi_2$, whenever $\phi_1, \phi_2 \in \mathcal{C}_N([-1, 1])$.

Proof. We start with observing the following: For $2 \leq s \leq r \leq \frac{2(d+1)}{d-1}$ and for any $\epsilon > 0$ and $0 < \delta \ll 1$,

\begin{equation}
(\sum_{\rho \in \mathbb{Z}^n\setminus \{0, 1\}} |S_{\rho, g}^\phi f(x)|^2)^{1/2} \lesssim \begin{cases} \\
\delta^{-\beta_s(\frac{1}{r} - \frac{1}{s})}\|f\|_s & \text{if } \frac{d-1}{r} > \frac{1}{s} + \frac{1}{r}, \\
\delta^{-\beta_s(\frac{1}{r} - \frac{1}{s})}\|f\|_s & \text{if } \frac{d-1}{r} \leq \frac{1}{s} + \frac{1}{r}.
\end{cases}
\end{equation}

Indeed, by (4.5) and Proposition 2.8 we have that, for $2 \leq s \leq r \leq \frac{2(d+1)}{d-1}$ and $k \geq 0$ and $\epsilon > 0$,

\begin{equation}
(\sum_{\rho \in \mathbb{Z}^n\setminus \{0, 1\}} |S_{\rho, g}^\phi f(x)|^2)^{1/2} \lesssim C\delta^{-\beta_s(\frac{1}{r} - \frac{1}{s})} \|f\|_s.
\end{equation}

Note that $\beta_s(\frac{1}{r} - \frac{1}{s}) + \frac{d-1}{2} - \frac{d-1}{r} > 0$ if $\frac{d-1}{d} < \frac{1}{s} + \frac{1}{r}$, and $\beta_s(\frac{1}{r} - \frac{1}{s}) + \frac{d-1}{2} > 0$ if $\frac{d-1}{d} \geq \frac{1}{s} + \frac{1}{r}$. Taking sum over $k$, we have (4.8). Particularly, with $s = 2$ we have

\begin{equation}
(\sum_{\rho \in \mathbb{Z}^n\setminus \{0, 1\}} |S_{\rho, g}^\phi f(x)|^2)^{1/2} \lesssim \begin{cases} \\
\delta^{-\epsilon}\|f\|_2 & \text{if } \frac{d-2}{d} > \frac{1}{r}, \\
\delta^{-\epsilon}\|f\|_2 & \text{if } \frac{d-2}{d} \leq \frac{1}{r} \leq \frac{d-1}{2(d+1)}.
\end{cases}
\end{equation}

Let us set $I_1 = [0, 2^{-4}] \cap [0, 1], I_2 = [2^{-4}, g-2^{-4}] \cap [0, 1], I_3 = [g-2^{-4}, g+2^{-4}] \cap [0, 1], I_4 = [g+2^{-4}, \infty) \cap [0, 1]$. Depending on $g$, $I_3$ and $I_4$ can be an empty set. For $i = 1, \ldots, 4$, we set

$B_i(f, g) = \sum_{\rho \in \mathbb{Z}^n\setminus I_i} |S_{\rho, g}^\phi f S_{\rho, g}^{\phi_2}|$.

Thus, we have

$|B_{k, g}^{\phi_1, \phi_2}(f, g)| \leq \sum_{i=1}^{4} B_i(f, g)$.

Note that if $\rho \in I_4$ then $g - \rho \leq -2^{-4}$, hence the Fourier support of $S_{\rho, g}^{\phi_2}$ is an empty set and $B_4(f, g) \equiv 0$ if $0 < \delta < 2^{-4}$. Thus it is enough to deal with $B_1, B_2,$ and $B_3$. $B_2$ can be handled by using the estimates in Proposition 2.8. In fact,

$B_2(f, g) \leq D_a(f)D_b(g)$

where

$D_a(f) := \left( \sum_{\rho \in \delta \mathbb{Z}^n \cap [2^{-4}, 1]} |S_{\rho, g}^\phi f|^2 \right)^{\frac{1}{2}}, D_b(g) := \left( \sum_{\rho \in \delta \mathbb{Z}^n \cap [2^{-4}, \delta-2^{-4}]} |S_{\rho, g}^{\phi_2}|^2 \right)^{\frac{1}{2}}$.

Since all the radii appearing in $D_a(f)$ and $D_b(g)$ are $\sim 1$, from a slight modification of proof of Proposition 2.8, it is easy to see that $D_a(f)$ and $D_b(g)$ satisfy the same estimate for $D_\delta^\phi f$ which is in Proposition 2.8. Thus, using $L^2 \to L^r$, $r \geq \frac{2(d+1)}{d-1}$
estimates for $\mathcal{D}_a(f)$ and $\mathcal{D}_b(f)$ and Hölder’s inequality we see that, for $r \geq \frac{d+1}{d-1}$ and $\epsilon > 0$,
\[
\|B_2(f,g)\|_r \lesssim \|f\|_2\|g\|_2.
\]
This estimate is acceptable in view of the desired estimate. Hence, we are reduced
to handling $B_1, B_3$ which are of similar nature.

We only handle $B_3$ since $B_1$ can be handled similarly. Now we note that
\[
B_3(f,g) \leq \mathcal{D}_c(f)\mathcal{D}_d(g),
\]
where
\[
\mathcal{D}_c(f) := \left( \sum_{\rho \in \delta Z \cap [2^{-3},1]} |S_{\rho,\delta}^a f|^2 \right)^{\frac{1}{2}}, \quad \mathcal{D}_d(g) := \left( \sum_{\rho \in \delta Z \cap [e^{-2^{-8}},e^{2^{-8}}]} |S_{\rho,\delta}^b g|^2 \right)^{\frac{1}{2}}.
\]
$\mathcal{D}_c(f)$ enjoys the same estimates for $\mathcal{D}_a(f)$ and $\mathcal{D}_b(f)$ since the associated radii are
\[\sim 1,\]
whereas there are small radii in $\mathcal{D}_d(g)$. It is easy to see that the estimate (4.9)
also holds for $\mathcal{D}_d(g)$. If $1/r_1 \leq 1/r_1 + 1/r_2$, then we may choose $\tilde{r}_1, \tilde{r}_2$ such that
\[1/\tilde{r}_1 + 1/\tilde{r}_2 = 1/r,\]
and $\tilde{r}_1 \geq r_1, \tilde{r}_2 \geq r_2$. So using $L^2 \to L'$, $r > \frac{2(d+1)}{d-1}$ estimates
for $\mathcal{D}_c(f)$ and the first estimate in (4.9) for $\mathcal{D}_d(g)$, we get
\[
\|B_3(f,g)\|_r \lesssim \|\mathcal{D}_c(f)\|_{\tilde{r}_1}\|\mathcal{D}_d(g)\|_{\tilde{r}_2} \lesssim \delta^{-r}\|f\|_2\|g\|_2.
\]
If $1/r_1 + 1/r_2 < 1/r \leq 2/r_1$, we take $\tilde{r}_1 = r_1$ and $\tilde{r}_2$ such that $1/\tilde{r}_2 = 1/r - 1/r_1$. Thus
\[r_1 \leq \tilde{r}_2 < r_2.\]
Similarly as before, using both cases in (4.9) we obtain
\[
\|B_3(f,g)\|_r \lesssim \|\mathcal{D}_c(f)\|_{r_1}\|\mathcal{D}_d(g)\|_{\tilde{r}_2} \lesssim \delta^{-r}\|f\|_2\|g\|_2.
\]
By the same argument as before it is easy to see that the same estimates also hold
for $B_1$. This completes the proof. □

Finally we prove Theorem 4.2 by making use of Lemma 4.3 and Lemma 4.4.

**Proof of Theorem 4.2.** It is easy to see that $\gamma(p,q,r) \geq -1$ for $2 \leq p, q \leq \infty$
and $r \geq \frac{d+1}{d-1}$. Thus, combining Lemma 4.3 and Lemma 4.4, we have that, for
\[2 \leq p, q \leq \infty \text{ and } r > \frac{d+1}{d-1},\]satisfying $1/p + 1/q \geq 1/r$,
\[
\|B_{\delta, \epsilon}^{p, q}(f,g)\|_r \lesssim \delta^{-\gamma(p,q,r)-\epsilon}\|f\|_p\|g\|_q.
\]
Since $\gamma(p,q,r) \geq -1$, we use Lemma 3.1 and (3.1) to obtain all the estimates in
Theorem 4.2. □

### 4.3. Lower bound for smoothing order $\alpha$

Similarly, as in case of linear multiplier operator, bilinear multiplier operator also have kernel expressions. We write $\mathcal{B}^\alpha$ as
\[
\mathcal{B}^\alpha(f,g)(x) = \int \int K^\alpha(x-y, x-z)f(y)g(z)dydz, \quad f, g \in \mathcal{S}(\mathbb{R}^d),
\]
where $K^\alpha = \mathcal{F}^{-1}(1-\epsilon|\xi|^2 - |\eta|^2)e^{\epsilon \alpha}$. Note that $K^\alpha$ is the kernel of the Bochner-Riesz operator $\mathcal{R}_1^\alpha$ in $\mathbb{R}^{2d}$. From the estimate for $K^\alpha$ in $\mathbb{R}^{2d}$ and duality, the necessary
condition for $\mathcal{R}_1^\alpha$ was obtained. Similar idea was used in [5] to find some necessary
conditions on $p, q$ for the boundedness of the operator $\mathcal{B}^\alpha$. □
Proposition 4.5. [5, Proposition 4.2] Let \( 1 \leq p, q \leq \infty \) and \( 0 < r \leq \infty \) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \).

(i) If \( \alpha \leq d(\frac{1}{r} - 1) - \frac{1}{2} \), then \( \mathcal{B}^\alpha \) is unbounded from \( L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \) to \( L^r(\mathbb{R}^d) \).

(ii) If \( \alpha \leq d(\frac{1}{r} - 1) - \frac{1}{2} \), then \( \mathcal{B}^\alpha \) is unbounded from \( L^p(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \) to \( L^r(\mathbb{R}^d) \), from \( L^\infty(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \) to \( L^r(\mathbb{R}^d) \), and also from \( L^p(\mathbb{R}^d) \times L^r(\mathbb{R}^d) \) to \( L^1(\mathbb{R}^d) \) for each \( 1 \leq p \leq \infty \).

The first result in Proposition 4.5 follows from the decay of kernel, since \( \mathcal{B}^\alpha(f, g) = K^\alpha \) if \( \hat{f} = \hat{g} = 1 \) on \( B(0, 2) \). The second one is a simple consequence of the linear theory. Unfortunately, these results do not give meaningful necessary condition for the Banach case, since \( d(\frac{1}{r} - 1) - \frac{1}{2} < 0 \) when \( r \geq 1 \). The following gives better lower bound.

Proposition 4.6. Let \( 1 \leq p, q, r \leq \infty \). If \( \mathcal{B}^\alpha \) is bounded from \( L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \) to \( L^r(\mathbb{R}^d) \), then

\[
\alpha \geq \max \left\{ \frac{d - 1}{2} - \frac{d}{p} - \frac{d - 1}{q}, \frac{1}{2} - \frac{d}{2p}, 0 \right\}.
\]

Proof. Let \( \psi(\xi, \eta) = \phi_1(\xi/\epsilon)\phi_2(\eta/\epsilon)\phi_3((1 - \eta_d)/\epsilon) \) where \( \eta' = (\eta_1, \ldots, \eta_{d-1}) \) and \( \phi_1, \phi_2, \phi_3 \) are nontrivial smooth functions supported in \( B(0, 1) \). Then \( L^p \times L^q \to L^r \) boundedness of \( \mathcal{B}^\alpha \) implies \( L^p \times L^q \to L^r \) boundedness of \( \mathcal{B}^\alpha \) defined by

\[
\mathcal{B}^\alpha(f, g) = \int e^{2\pi i \xi(\xi - n)} \psi(\xi, n)(1 - |\xi|^2 - |\eta|^2)^\alpha \hat{f}(\xi)\hat{g}(\eta) d\xi d\eta.
\]

We first note that

\[
\int \mathcal{B}^\alpha(f, g)(x)\phi(x)dx = \iint \int \psi(\xi, \eta)(1 - |\xi|^2 - |\eta|^2)^\alpha \phi^\vee(\xi + \eta)e^{-2\pi i (y \cdot \xi + z \cdot \eta)} d\xi d\eta f(y)g(z)dydz.
\]

Choosing a Schwartz function \( \phi \) such that \( \hat{\phi} = 1 \) on \( B(0, \sqrt{2}) \), it follows that

\[
\int \mathcal{B}^\alpha(f, g)(x)\phi(x)dx = \iint K^\alpha(y, z)f(y)g(z)dydz,
\]

where \( K^\alpha = F^{-1}(\psi(\xi, n)(1 - |\xi|^2 - |\eta|^2)^\alpha) \). Hence, \( L^p \times L^q \to L^r \) boundedness of \( \mathcal{B}^\alpha \) implies

\[
\left| \iint K^\alpha(y, z)f(y)g(z)dydz \right| \lesssim \|f\|_p\|g\|_q.
\]

We choose a small enough \( \epsilon > 0 \). By making use of stationary phase method (in fact, Fourier transform of measure supported in sphere, for example, see [35, p.68]), for \( w = (y, z) \) in a narrow conic neighborhood \( \mathcal{C} \) of \((0, \epsilon_d) \in \mathbb{R}^d \times \mathbb{R}^d \), to say \( \mathcal{C} = \{(y, z) : \sqrt{|y|^2 + |z|^2} \leq \epsilon_0 \epsilon_d \} \) for a small enough \( \epsilon_0 \),

\[
K^\alpha(w) = e^{i|w|a(w)|w|^{-\frac{2d+1-\alpha}{2}}},
\]

where \( a \) is radial and \( |a(w)| \geq c > 0 \) if \( |w| \) is large enough. Let \( R \gg \epsilon_0^{-100} \) and set \( A_R = \{x : (\epsilon_0/10)R^{1/2} \leq |x| < (\epsilon_0/5)R^{1/2}\} \) and \( B_R = \{x : (\epsilon_0/10)R \leq |x| \leq (\epsilon_0/5)R, \ |x| \leq (\epsilon_0/10)|x_d|\} \). Then \( A_R \times B_R \subset \mathcal{C} \). We now set

\[
f(y) = \chi_{A_R}(y), \quad g(z) = \chi_{B_R}(z)e^{-i|z|}.
\]
Thus,
\[ \left| \int K_\alpha(y,z)f(y)g(z)\,dy\,dz \right| = \left| \int_{AR} \int_{BR} e^{i(|w|-|z|)}a(w)|w|^{-\frac{2d+1-\alpha}{2}}\,dy\,dz \right| \gtrsim R^{\frac{d+\alpha}{2}}. \]

because \(|u| - |z| = O(|y|^2/|z|) \leq 1/4\) for large \(R\). Moreover, by (4.11) we get
\[ \left| \int K_\alpha(y,z)f(y)g(z)\,dy\,dz \right| \lesssim R^{\frac{d+\alpha}{4}}. \]

Combining the above two estimates and (4.11), the inequality \(R^{-\alpha + \frac{d+1}{2}} \lesssim R^{\frac{d+\alpha}{4}}\) should hold for any \(R \gg \epsilon_0^{100}\). Letting \(R \to \infty\) gives \(\alpha \geq \frac{d-1}{2} - \frac{d}{p} - \frac{d}{q}\). If we exchange the role of \(\xi\) and \(\eta\) in the function \(\psi_\epsilon\) (i.e., \(\psi_\epsilon(\eta,\xi)\) instead of \(\psi_\epsilon(\xi,\eta)\)), we have the other condition \(\alpha \geq \frac{d-1}{2} - \frac{d}{p} - \frac{d}{2q}\).

\[ \square \]

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