Precision of inflationary predictions and recent CMB anisotropy data

Dominik J. Schwarz*, 2 Jérôme Martin†, and Alain Riazuelo†

*Institut für Theoretische Physik, Technische Universität Wien, 1040 Wien, Austria
†Institut d’Astrophysique de Paris, 98 boulevard Arago, 75014 Paris, France

Abstract. Inflationary predictions of the cosmic microwave background anisotropy are often based on the slow-roll approximation. We study the precision of these predictions and compare them with the recent data from BOOMERanG and MAXIMA-1.

INTRODUCTION

High quality measurements of anisotropies in the cosmic microwave background (CMB) probe the cosmic fluctuations generated during an inflationary epoch in the very early Universe [1]. Recently, BOOMERanG [2] and MAXIMA [3] teams announced the clear detection of a first acoustic peak at an angular scale \( \approx 1^\circ \), which confirms the most important prediction of inflation: the Universe seems to be spatially flat [4]. Another generic prediction of inflation is that the primordial spectra of density perturbations and gravitational waves are almost scale-invariant. More CMB precision measurements will be available soon.

We argue [5] that CMB predictions on the basis of the simplest inflationary model, slow-roll inflation [6], are not as precise as could be believed from the accuracy of the power spectra [7]. We compare the predictions from the slow-roll approximation [8] with the exact solutions from the model of power-law inflation [9]. We find unacceptable large errors in the predictions of multipole moments. The reason is as follows: The primordial spectrum is best approximated at some pivot scale \( k_* \). A small error in the spectral index gives rise to a large error at wavenumbers that differ significantly from \( k_* \), due to a large lever arm. A natural choice for the pivot scale is the present Hubble scale, but leads to extremely large errors for high multipole moments. A shift of the pivot scale to the scale of the first acoustic peak decreases these errors dramatically (see Figure 1).

1) Financially supported by the Austrian Academy of Sciences.
2) dschwarz@hep.itp.tuwien.ac.at
3) jmartin@iap.fr
4) alain.riazuelo@obspm.fr
FIGURE 1. The error of the scalar multipole moments $C_\ell$ from the slow-roll approximation w.r.t. the exact solution of power-law inflation. On the left side we show the errors with a pivot scale $k_*$ near the quadrupole, on the right side the pivot scale has been set to the scale of the first acoustic peak.

In [10] we compare the improved (optimal pivot scale) slow-roll predictions with recent CMB data (see Figure 2). Most data analysis so far [4] was based on a power-law shape of the primordial spectra. This shape is not predicted by the slow-roll approximation, only the first two terms in a Taylor expansion with respect to the wavenumber coincide.

PRECISION OF SLOW-ROLL PREDICTIONS

Slow-roll inflation is very simple and is an attractor for many inflationary models. Inflation driven by a single field $\varphi$, may be characterized at a given moment of time $t_*$ by the parameters $\epsilon \equiv -[\dot{H}/H^2]_*$, $\delta \equiv -[\dot{\varphi}/(H\dot{\varphi})]_*$, $\xi \equiv [(\dot{\epsilon} - \dot{\delta})/H]_*$, $\ldots$, where $H$ is the Hubble rate. The condition for inflation is $\epsilon < 1$, whereas slow-roll inflation is characterized by $\epsilon \ll 1$, $|\delta| \ll 1$, $\xi = \mathcal{O}(\epsilon^2, \epsilon\delta, \delta^2)$, and negligible higher derivatives. Based on these approximations the power spectrum of the Bardeen potential $\Phi$ and of the amplitude of gravitational waves $h$ reads [8,5]

$$k^3 P_\Phi = \frac{9H^2l^2_{pl}}{25\pi\epsilon} \left[ 1 - 2\epsilon - 2(2\epsilon - \delta) \left( C + \ln \frac{k}{k_*} \right) \right],$$

$$k^3 P_h = \frac{16H^2l^2_{pl}}{\pi} \left[ 1 - 2\epsilon \left( C + 1 + \ln \frac{k}{k_*} \right) \right],$$

where $C \equiv \gamma_E + \ln 2 - 2 \simeq -0.7296$, $\gamma_E \simeq 0.5772$ being the Euler constant. The pivot scale is defined as $k_* = (aH)_*$. Fixing $k_*$ corresponds to a choice of the time $t_*$ during inflation. The spectral indices can be obtained from $n_S - 1 \equiv d\ln(k^3P_\Phi)/d\ln k = -4\epsilon + 2\delta$ and $n_T \equiv d\ln(k^3P_h)/d\ln k = -2\epsilon$. We call this the next-to-leading order of the slow-roll approximation (at the leading order strictly scale-invariant spectra are predicted).

On the other hand, the power spectra may be calculated exactly for power-law inflation, which is characterized by a power-law behavior of the scale factor, i.e.,
For power-law inflation we have \( \epsilon = \delta \) and \( \xi = 0 \) during inflation. We use \( \epsilon \) to parametrize the spectra, i.e. \( p = 1/\epsilon \). The corresponding power spectra then read [9,11]

\[
k^3 P_\Phi = \frac{9H_*^2}{25\pi\epsilon} f(\epsilon) \left( \frac{k}{k_*} \right)^{-\frac{4}{3}}, \quad k^3 P_h = \frac{16H_*^2}{\pi} f(\epsilon) \left( \frac{k}{k_*} \right)^{-\frac{2}{3}},
\]  

(3)

where \( f(\epsilon) = \frac{2}{(1-\epsilon)^{2/3}} \Gamma[1/(1-\epsilon) + 1/2]/\pi \), with \( f(0) = 1 \). For power-law inflation the spectral indices read:

\[
n_S = 1 + n_T = (1 - 3\epsilon)/(1 - \epsilon).
\]

In the limit \( \epsilon \ll 1 \) the power spectra (3) go to (1) with \( \epsilon = \delta \) and to (2), respectively.

We can now calculate the multipole moments \( C_\ell \) for the power-law and slow-roll spectra for \( \epsilon = \delta \). We define the error from the slow-roll approximation as

\[
e_{C_\ell} \equiv \left| \frac{C_{\text{sr}} - C_{\text{pl}}}{C_{\text{pl}}} \right| \times 100%.
\]

(4)

For similar spectra the error (4) depends only weakly on the transfer function. This allows us to neglect the evolution of the transfer function for this purpose and to obtain an analytic result, which is plotted in Figure 1. The values of \( n_S \) refer to the exact power-law solution. In the left figure \( k_* = k_0 \equiv (aH)_0 \) gives the smallest error for the quadrupole and unacceptably large errors at high multipoles. In the right figure the pivot scale has been chosen to minimize the error around the first acoustic peak, \( \ell \sim 200 \). The corresponding condition is \( D_{\ell_{\text{opt}}} = \ln(k_*r_{\text{iss}}) \), where \( r_{\text{iss}} \) is the comoving distance to the last scattering surface and \( D_{\ell} \equiv 1 - \ln 2 + \Psi(\ell) + (\ell + 1)/[\ell(\ell + 1)] \) with \( \Psi(x) \equiv d\ln\Gamma(x)/dx \). For \( \ell_{\text{opt}} \gg 1 \) this gives \( k_* \simeq (e\ell_{\text{opt}})/(2r_{\text{iss}}) \), where \( r_{\text{iss}} \simeq 3.3/(aH)_0 \) for \( \Lambda \text{CDM} \).

**SLOW-ROLL INFLATION AND CMB ANISOTROPY DATA**

Let us now compare [10] the predictions of slow-roll inflation with recent data from BOOMERanG [2] and MAXIMA-1 [3], supplemented with the COBE/DMR dataset [12]. Instead of fitting ten cosmological parameters we fix the values of otherwise measured parameters and assume that slow-roll inflation is the correct theory. In Figure 2 we present the sum of scalar and tensor CMB band power for a \( \Lambda \text{CDM} \) model with \( \Omega = 1, \Omega_\Lambda = 0.7, \Omega_b h^2 = 0.019 \), and \( h = 0.6 \). The Boltzmann code used here was developed by one of us (A.R.). We see without a \( \chi^2 \) analysis that qualitatively different inflationary models are consistent with the observations: Both models with \( \epsilon = 0.02 \) give reasonable fits, one of these models has a flat scalar spectrum (with \( n_S \neq n_T + 1 \)), the other one has a negative tilt (with \( n_S = n_T + 1 \)). Both models have an important contribution of gravitational waves (\( \sim 20\% \)).

We emphasize that the generic slow-roll predictions (1) and (2) do not have a power-law shape. This fact induces large differences to multipole moments that are
predicted under the assumption that the power-law shape (3) is the generic inflationary prediction. Besides using the correct primordial spectrum a clever choice of the pivot scale can hide unavoidable uncertainties of the multipole moments in the cosmic variance on one side and in the instrumental noise on the other side of the spectrum.

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