STABILITY OF THE CHARI-LOKTEV BASES
FOR LOCAL WEYL MODULES OF $\mathfrak{sl}_{r+1}[t]$

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Abstract. We prove stability of the Chari-Loktev bases with respect to the inclusions of local Weyl modules of the current algebra $\mathfrak{sl}_{r+1}[t]$. This is conjectured in [8] and the $r = 1$ case is proved in [7]. Local Weyl modules being known to be Demazure submodules in the level one representations of the affine Lie algebra $\hat{\mathfrak{sl}}_{r+1}$, we obtain, by passage to the direct limit, bases for the level one representations themselves.

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be its current algebra. Local Weyl modules, introduced by Chari and Pressley [3] are important finite-dimensional $\mathfrak{g}[t]$-modules. These modules are characterized by the following universal property: any finite-dimensional $\mathfrak{g}[t]$-module generated by a one-dimensional highest weight space, is a quotient of a local Weyl module. Corresponding to every dominant integral weight $\lambda$ of $\mathfrak{g}$, there is one local Weyl module denoted by $W(\lambda)$.

In [3], for $\mathfrak{g} = \mathfrak{sl}_2$, Chari and Pressley also produced monomial bases for local Weyl modules. Later Chari and Loktev [2] extended the construction of these bases to $\mathfrak{g} = \mathfrak{sl}_{r+1}$. Using these bases they also showed that the local Weyl modules are $\mathfrak{g}[t]$-stable Demazure modules occurring in a level one representations of the affine Lie algebra $\hat{\mathfrak{g}}$. As a consequence, we get an embedding of local Weyl modules $W(\lambda) \hookrightarrow W(\lambda + k\theta)$, where $\theta$ is the long root and $k$ is a non-negative integer.

It is important to note that for every non-negative integer $k$, the local Weyl module $W(\lambda + k\theta)$ can be realized as $\mathfrak{g}[t]$-stable Demazure module occurring in a fixed level one representation of $\hat{\mathfrak{g}}$; we shall denote this level one representation here by $V$. Thus we have a chain of inclusions:

$$W(\lambda) \hookrightarrow W(\lambda + \theta) \hookrightarrow \cdots \hookrightarrow W(\lambda + k\theta) \hookrightarrow W(\lambda + (k+1)\theta) \hookrightarrow \cdots (\hookrightarrow V) \quad (1.1)$$

such that the union of the modules in the chain equals $V$.

For $\mathfrak{g} = \mathfrak{sl}_2$, it is shown in [7] that after a suitable normalization, the Chari-Pressley bases behave well with respect to the inclusions in (1.1). Moreover in the limit, these bases stabilize and give a nice monomial basis for $V$. For $\mathfrak{g} = \mathfrak{sl}_{r+1}$, we consider the Chari-Loktev (CL) bases of local Weyl modules. In [8], an elegant combinatorial description for their parameterizing set is given: namely, as the set of partition overlaid patterns (POPs). Moreover a weight preserving injective map between the parameterizing sets of the bases of $W(\lambda + k\theta)$ and $W(\lambda + (k+1)\theta)$ is given. Using this it is conjectured that after a suitable normalization the CL bases also have the

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stability property with respect to the inclusions in (1.1). The purpose of this paper is to prove this conjecture.

More precisely, let $\mathbb{P}_\lambda$ denote the parametrizing set of the CL basis for $W(\lambda)$: the elements of $\mathbb{P}_\lambda$ are POPs with bounding sequence $\underline{k}$, where $\underline{k}$ is an integer sequence corresponding to $\lambda$. In [3], for each non-negative integer $k$, a weight preserving embedding from $\mathbb{P}_\lambda \hookrightarrow \mathbb{P}_{\lambda+k\theta}$ into $\mathbb{P}_{\lambda+(k+1)\theta}$ is given. Thus we have a chain $\mathbb{P}_\lambda \hookrightarrow \mathbb{P}_{\lambda+\theta} \hookrightarrow \mathbb{P}_{\lambda+2\theta} \hookrightarrow \cdots$. Given an element $\mathfrak{P}$ of $\mathbb{P}_\lambda$ and a non-negative integer $k$, let $\mathfrak{P}^k$ be its image in $\mathbb{P}_{\lambda+k\theta}$ and let $v_{\mathfrak{P}^k}$ be the corresponding normalized CL basis element. Consider the sequence $v_{\mathfrak{P}^k}, k = 0, 1, 2, \ldots$, of elements in $V$. We prove that this sequence stabilizes for large $k$ (see Theorem 3.4). Passing to the direct limit, we obtain a basis for $V$ consisting of the stable CL basis elements (see [3,3]).

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2. Notation and Preliminaries

Throughout the paper, $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{Z}$ the set of integers, $\mathbb{N}$ the set of positive integers, $\mathbb{Z}_{\geq 0}$ the set of non-negative integers, $\mathbb{C}[t]$ the polynomial ring, $\mathbb{C}[t, t^{-1}]$ the ring of Laurent polynomials, and $\mathfrak{U}(\mathfrak{a})$ the universal enveloping algebra corresponding to a complex Lie algebra $\mathfrak{a}$.

2.1. The Lie algebra $\mathfrak{sl}_{r+1}$. Let $\mathfrak{g} = \mathfrak{sl}_{r+1}$, the Lie algebra of $(r + 1) \times (r + 1)$ trace zero matrices over the field $\mathbb{C}$ of complex numbers. Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g}$ consisting of trace zero diagonal matrices. Let $R \subset \mathfrak{h}^*$ denote the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For $\alpha \in R$, let $\mathfrak{g}_\alpha$ be the root space corresponding to $\alpha$. Let $\mathfrak{b}$ be the standard Borel subalgebra of $\mathfrak{g}$ consisting of upper triangular matrices. For $1 \leq i \leq r + 1$, let $\varepsilon_i \in \mathfrak{h}^*$ be the projection to the $i^{th}$ co-ordinate. Set $I = \{1, 2, \ldots, r\}$. Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i \in I,$ be the set of simple roots and $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j = \varepsilon_i - \varepsilon_{j+1}, 1 \leq i \leq j \leq r,$ be the set of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{b}$. Let $\theta = \alpha_{1,r}$ be the highest root of $\mathfrak{g}$. For $1 \leq i, j \leq r + 1$, let $E_{i,j}$ be the $(r + 1) \times (r + 1)$ matrix with 1 in the $(i, j)^{th}$ position and 0 elsewhere. Set

$$x_{\varepsilon_i-\varepsilon_j} = E_{i,j}, \quad h_{\varepsilon_i-\varepsilon_j} = E_{i,i} - E_{j,j}, \quad \forall 1 \leq i \neq j \leq r + 1,$$

$$x_{i,j}^+ = x_{\alpha_{i,j}} = x_{\varepsilon_i-\varepsilon_{j+1}}, \quad x_{i,j}^- = x_{\alpha_{i,j}} = x_{\varepsilon_{j+1}-\varepsilon_i}, \quad \forall 1 \leq i \leq j \leq r.$$ Define subalgebras $\mathfrak{n}^\pm$ of $\mathfrak{g}$ by

$$\mathfrak{n}^\pm = \bigoplus_{1 \leq i \leq j \leq r} \mathbb{C} x_{i,j}^\pm.$$

Now we have the following decomposition: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. For $x, y \in \mathfrak{g}$, let $(x|y) := \text{trace}(xy)$ be the normalized invariant bilinear form on $\mathfrak{g}$. Let $W$ denote the Weyl group of $\mathfrak{g}$.

Let $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i, i \in I$, be the set of fundamental weights of $\mathfrak{g}$. The weight lattice $P$, the set $P^+$ of dominant integral weights, and the root lattice $Q$ of $\mathfrak{g}$ are defined as follows:

$$P = \sum_{i \in I} \mathbb{Z} \omega_i, \quad P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \omega_i, \quad Q = \sum_{i \in I} \mathbb{Z} \alpha_i.$$
For \( \lambda = m_1 \varpi_1 + \cdots + m_r \varpi_r \in P^+ \), we associate an integer sequence \( \lambda \) by \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = 0) \), where \( \lambda_i := m_i + \cdots + m_r \). Given an integer sequence \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = 0) \), we associate an element \( \lambda \) of \( P^+ \) by \( \lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_r \varepsilon_r = (\lambda_1 - \lambda_2) \varpi_1 + \cdots + (\lambda_r - \lambda_{r+1}) \varpi_r \).

2.2. The affine Lie algebra \( \widehat{\mathfrak{g}} \). Let \( \widehat{\mathfrak{g}} \) be the (untwisted) affine Lie algebra corresponding to \( \mathfrak{g} \) defined by

\[
\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,
\]

where \( c \) is central and the other Lie brackets are given by

\[
[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\delta_{m,-n}(x|y)c,
\]

\[
[d, x \otimes t^m] = m(x \otimes t^m),
\]

for all \( x, y \in \mathfrak{g} \) and integers \( m, n \). The Lie subalgebras \( \widehat{\mathfrak{h}} \) and \( \widehat{\mathfrak{b}} \) of \( \widehat{\mathfrak{g}} \) are given by

\[
\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \widehat{\mathfrak{b}} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}c \oplus \mathbb{C}d.
\]

We regard \( \mathfrak{h}^* \) as a subspace of \( \widehat{\mathfrak{h}}^* \) by setting \( \langle \lambda, c \rangle = \langle \lambda, d \rangle = 0 \) for all \( \lambda \in \mathfrak{h}^* \). Let \( \delta, \Lambda_0 \in \widehat{\mathfrak{h}}^* \) be given by

\[
\langle \delta, \mathfrak{h} + \mathbb{C}c \rangle = 0, \quad \langle \delta, d \rangle = 1, \quad \langle \Lambda_0, \mathfrak{h} + \mathbb{C}d \rangle = 0, \quad \langle \Lambda_0, c \rangle = 1.
\]

There is a non-degenerate, symmetric, \( \widehat{W} \)-invariant, bilinear form \( (\cdot \mid \cdot) \) on \( \widehat{\mathfrak{h}}^* \), given by requiring that \( \mathfrak{h}^* \) be orthogonal to \( \mathbb{C}\delta + \mathbb{C}\Lambda_0 \), together with the relations

\[
(\alpha_i | \alpha_i) = 2, \quad (\alpha_i | \alpha_j) = -\delta_{i,j}, \quad \forall 1 \leq i < j \leq r, \quad (\delta | \delta) = (\Lambda_0 | \Lambda_0) = 0, \quad \text{and} \quad (\delta | \Lambda_0) = 1.
\]

The elements \( \alpha_0 = \delta - \theta, \alpha_1, \ldots, \alpha_r \) are the simple roots of \( \widehat{\mathfrak{g}} \) and the corresponding coroots are \( \alpha_0^\vee = c - h_\theta, \alpha_1^\vee = h_\alpha_1, \ldots, \alpha_r^\vee = h_\alpha_r \). Set \( \widehat{I} = I \cup \{0\} \). Let \( e_i, f_i \) (\( i \in \widehat{I} \)) denote the Chevalley generators of \( \widehat{\mathfrak{g}} \):

\[
e_0 = x_1^+_s \otimes t, \quad f_0 = x_1^+ \otimes t^{-1}, \quad e_i = x_i^+, \quad f_i = x_i^-, \quad \forall i \in I.
\]

For \( \alpha \in R \) and \( s \in \mathbb{Z} \), set \( x_{\alpha + s \delta} = x_\alpha \otimes t^s \). The weight lattice (resp. the set of dominant integral weights) of \( \widehat{\mathfrak{g}} \) is defined by

\[
\widehat{P} \ (\text{resp. } \widehat{P}^+) = \{ \Lambda \in \widehat{\mathfrak{h}}^* : \langle \Lambda, \alpha_p^\vee \rangle \in \mathbb{Z} \ (\text{resp. } \mathbb{Z}_{\geq 0}) \}, \quad \forall p \in \widehat{I}.
\]

For an element \( \Lambda \in \widehat{P} \), the integer \( \langle \Lambda, c \rangle \) is called the level of \( \Lambda \).

2.3. The Weyl group of \( \widehat{\mathfrak{g}} \). For each \( p \in \widehat{I} \), the fundamental reflection \( s_{\alpha_p} \) (or \( s_p \)) is given by

\[
s_p(\Lambda) = \Lambda - (\Lambda, \alpha_p^\vee) \alpha_p, \quad \forall \Lambda \in \widehat{\mathfrak{h}}^*.
\]

The subgroup \( \widehat{W} \) of \( GL(\widehat{\mathfrak{h}}^*) \) generated by all fundamental reflections \( s_p, p \in \widehat{I} \) is called the affine Weyl group. We regard \( W \) naturally as a subgroup of \( \widehat{W} \). Given \( \alpha \in \mathfrak{h}^* \), let \( t_\alpha \in GL(\widehat{\mathfrak{h}}^*) \) be defined by

\[
t_\alpha(\Lambda) = \Lambda + (\Lambda | \delta) \alpha - (\Lambda | \alpha) \delta - \frac{1}{2} (\Lambda | \delta) (\alpha | \alpha) \delta, \quad \text{for } \Lambda \in \widehat{\mathfrak{h}}^*.
\]

It is easy to see that

\[
t_\alpha t_\beta = t_{\alpha + \beta} \quad \text{and} \quad w t_\alpha w^{-1} = t_{w \alpha}, \quad \forall \alpha, \beta \in \mathfrak{h}^*, w \in W.
\]

The translation subgroup \( T_Q \) of \( \widehat{W} \) is defined by

\[
T_Q := \{ t_\alpha \in GL(\widehat{\mathfrak{h}}^*) : \alpha \in Q \}.
\]
The following proposition gives the relation between \( W \) and \( \hat{W} \).

**Proposition 2.1.** \([6]\) Let \( \hat{W} = W \times T_Q \).

The extended affine Weyl group \( \hat{W} \) is the semi-direct product

\[
\hat{W} := W \times T_P,
\]

where \( T_P = \{ t_\alpha \in GL(\mathfrak{h}^\ast) : \alpha \in P \} \). For \( i \in I \), consider the element \( \sigma_i = t_{w_0,i}w_0 \in \hat{W} \), where \( w_0 \) is the longest element in \( W \) and \( w_0,i \) is the longest element in \( W_{\infty,i} \), the stabilizer of \( \varpi_i \) in \( W \). It is an automorphism of the Dynkin diagram of \( \hat{g} \):

\[
\sigma_i \alpha_p = \alpha_{i+p(\mod r+1)}, \quad \forall \ p \in \mathcal{I}, \quad \text{and} \quad \sigma_i \rho = \rho.
\]

Here, \( \rho \in \mathfrak{h}^\ast \) is the Weyl vector, defined by \( \langle \rho, \alpha \rangle = 1 \), \( \forall \ p \in \mathcal{I} \), and \( \langle \rho, d \rangle = 0 \). Let \( \Sigma \) be the subgroup generated by \( \{ \sigma_i : i \in I \} \). Now we also have \( \hat{W} = \hat{W} \times \Sigma \) (see [1, Chapter VI]).

### 2.4. Irreducible modules of \( \hat{g} \)

Given \( \Lambda \in \hat{P}^+ \), let \( L(\Lambda) \) be the irreducible \( \hat{g} \)-module with highest weight \( \Lambda \). It is the cyclic \( \hat{g} \)-module generated by \( v_\Lambda \), with defining relations:

\[
\begin{align*}
h v_\Lambda &= \langle \Lambda, h \rangle v_\Lambda, \quad \forall \ h \in \mathfrak{h}, \\
e_p v_\Lambda &= 0, \quad \forall \ p \in \mathcal{I}, \\
f_p^{(\Lambda, \alpha_p^\vee)+1} v_\Lambda &= 0, \quad \forall \ p \in \mathcal{I}.
\end{align*}
\]

It has the weight space decomposition: \( L(\Lambda) = \bigoplus_{\mu \in \widehat{h}} L(\Lambda, \mu) \). The \( \mu \) for which \( L(\Lambda, \mu) \neq 0 \) are the weights of \( L(\Lambda) \).

The following two results are well-known:

**Proposition 2.2.** \([6]\) Let \( \Lambda \in \hat{P}^+ \) is of level 1. Then

1. the set of weights of \( L(\Lambda) \) is \( \{ t_\alpha (\Lambda) - m \delta : \alpha \in Q, m \in \mathbb{Z}_{\geq 0} \} \),
2. for \( \alpha \in Q \) and \( m \in \mathbb{Z}_{\geq 0} \), we have

\[
\dim L(\Lambda)_{t_\alpha (\Lambda) - m \delta} = \text{the number of } r \text{-colored partitions of } m \text{ (see [2.9])}.
\]

**Theorem 2.3.** \([5]\) Given \( m \in \mathbb{Z}_{\geq 0} \), every element of the weight space of \( L(\Lambda_0) \) of weight \( \Lambda_0 - m \delta \) can be written as \( g_m v_{\Lambda_0} \) for some polynomial \( g_m \) in \( \alpha_1^\vee \cdots \alpha_r^\vee \), \( i \in I, j \in \mathbb{N} \).

We let \( \Lambda_i := \sigma_i \Lambda_0 \) for \( i \in I \). Then, \( \Lambda_0, \Lambda_1, \ldots, \Lambda_r \) are (a choice of) fundamental weights corresponding to the coroots \( \alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_r^\vee \), i.e., \( \langle \Lambda_p, \alpha_q^\vee \rangle = \delta_{p,q} \) for \( p, q \in \mathcal{I} \). Let \( v_{\Lambda_p} \) denote a highest weight vector of \( L(\Lambda_p) \) for \( p \in \mathcal{I} \).

### 2.5. The current algebra and its Weyl modules

The current algebra \( \mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t] \) is a Lie algebra with Lie bracket is obtained from that of \( \mathfrak{g} \) by extension of scalars to \( \mathbb{C}[t] \):

\[
[x \otimes t^m, y \otimes t^n] := [x, y] \otimes t^{m+n}, \quad \forall \ x, y \in \mathfrak{g}, \ m, n \in \mathbb{Z}_{\geq 0}.
\]

**Definition 2.4.** (see [2, §1.2.1]) Given \( \lambda \in P^+ \), the local Weyl module \( W(\lambda) \) is the cyclic \( \mathfrak{g}[t] \)-module with generator \( w_\lambda \) and relations:

\[
(n^+ \otimes t \mathbb{C}[t]) w_\lambda = 0, \quad (h \otimes t^s) w_\lambda = \langle \lambda, h \rangle \delta_{s,0}, \quad \forall \ h \in \mathfrak{h}, \ s \in \mathbb{Z}_{\geq 0}, \quad f_i^{(\lambda, \alpha_i^\vee)+1} w_\lambda = 0, \quad \forall \ i \in I.
\]
2.6. Weyl modules as Demazure modules. Given \( w \in \widehat{W} \) and \( \Lambda \in \widehat{P}^+ \), define a \( \hat{b} \)-submodule \( V_w(\Lambda) \) of \( L(\Lambda) \) by

\[
V_w(\Lambda) := U(\hat{b}) L(\Lambda)_{w\Lambda}.
\]

We call the \( \hat{b} \)-module \( V_w(\Lambda) \) as the Demazure module of \( L(\Lambda) \) associated to \( w \). More generally, given an element \( w \) of the extended affine Weyl group \( \widehat{W} \), we write \( w = u_\tau \) with \( u \in \widehat{W}, \tau \in \Sigma \), and define, following [4], the associated Demazure module by \( V_w(\Lambda) := V_u(\tau(\Lambda)) \).

The following theorem identifies the local Weyl modules with the \( \mathfrak{g}[t] \)-stable Demazure modules. **Theorem 2.5.** [2, 4] Given \( \lambda \in P^+ \), the local Weyl module \( W(\lambda) \) is isomorphic to the \( \mathfrak{g}[t] \)-stable Demazure module \( V_{t_{w_0}(\lambda)}(\Lambda_0) \), as modules for the current algebra \( \mathfrak{g}[t] \).

2.7. Inclusions of Weyl modules. Let \( \lambda \in P^+ \) and \( \Lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = 0) \) be its corresponding sequence. Let \( i_\lambda \in \widehat{I} \) be the remainder when \( \sum_{i=1}^{r+1} \lambda_i \) is divided by \( r + 1 \). Set \( \varpi_0 \) as the zero element in \( \mathfrak{h}^* \). It is easy to see that \( \lambda - \varpi_i \in Q \). Since \( w\Lambda_0 = \Lambda_0 \) for all \( w \in W \), using (2.5), we have

\[
t_{w_0(\lambda)} \Lambda_0 = t_{w_0(\lambda - \varpi_i)} t_{w_0 \varpi_i} (\Lambda_0) = t_{w_0(\lambda - \varpi_i)} w_0 i_\lambda (\Lambda_0).
\]

Thus from Theorem 2.5 we get

\[
W(\lambda) \cong \mathfrak{g}[t] V_{t_{w_0(\lambda - \varpi_i)} w_0} (\Lambda_i) \subset L(\Lambda_i).
\]

For every \( k \geq 0 \), it is important to note that \( i_{\lambda+k\theta} = i_\lambda \) and hence \( W(\lambda + k\theta) \) is also a Demazure submodule of \( L(\Lambda_i) \).

For every \( w \in W \), it is well-known that

\[
t_{w_0(\lambda - \varpi_i)} w \leq t_{w_0(\lambda + \theta - \varpi_i)} w \leq \cdots \leq t_{w_0(\lambda + k\theta - \varpi_i)} w \leq t_{w_0(\lambda + (k+1)\theta - \varpi_i)} w \leq \cdots,
\]

where \( \leq \) is the Bruhat order on the affine Weyl group. Hence using (2.6), we get a chain of Demazure submodules of \( L(\Lambda_i) \):

\[
W(\lambda) \hookrightarrow W(\lambda + \theta) \hookrightarrow \cdots \hookrightarrow W(\lambda + k\theta) \hookrightarrow W(\lambda + (k+1)\theta) \hookrightarrow \cdots \hookrightarrow L(\Lambda_i) \]

such that union of the modules in the chain equals \( L(\Lambda_i) \).

2.8. Partitions. A *partition* is a non-increasing sequence of non-negative integers that is eventually zero. The non-zero elements of the sequence are called the *parts* of the partition. If the sum of the parts of a partition \( \pi : \pi_1 \geq \pi_2 \geq \cdots = m \), then the partition is said to be a *partition of* \( m \), and we write \(|\pi| = m\).

2.8.1. *Partition fits into a rectangle.* Let \( d, d' \) be non-negative integers. We say that a partition fits into a rectangle \((d, d')\), if the number of parts is at most \( d \) and every part is at most \( d' \).

2.8.2. *Complement of a partition.* Let \( \pi \) be a partition fits in a rectangle \((d, d')\). The complement \( \pi^c \) of \( \pi \) is given by \((d' - \pi_d \geq \cdots \geq d' - \pi_1)\). Note that \( \pi^c \) also fits into the rectangle \((d, d')\).
2.9. Colored partitions. Let $r$ be a positive integer. An $r$-colored partition is a partition in which each part is assigned an integer between 1 and $r$. The number assigned to a part is its color. We may think of an $r$-colored partition as an ordered $r$-tuple $(\underline{\pi}^1, \ldots, \underline{\pi}^r)$ of partitions: the partition $\underline{\pi}^i$ consists of all parts of color $i$ of the $r$-colored partition. An $r$-colored partition of a non-negative integer $m$ is an $r$-colored partition $(\underline{\pi}^1, \ldots, \underline{\pi}^r)$ with $|\underline{\pi}^1| + \cdots + |\underline{\pi}^r| = m$.

2.10. Gelfand-Tsetlin patterns. A Gelfand-Tsetlin (GT) pattern (or just pattern) $\mathcal{P}$ is an array of integral row vectors $\underline{\lambda}^1, \ldots, \underline{\lambda}^r, \underline{\lambda}^{r+1}$ (where $\underline{\lambda}^j = (\lambda_{1}^j, \ldots, \lambda_{r}^j)$):

\[
\begin{array}{ccc}
\lambda_{1}^1 & & \\
\lambda_{2}^1 & \lambda_{2}^2 & \\
& \cdots & \cdots \\
\lambda_{1}^r & & \\
& \cdots & \lambda_{r}^r \\
\lambda_{1}^{r+1} & \lambda_{2}^{r+1} & \cdots \\
& & \lambda_{r+1}^{r+1}
\end{array}
\]

subject to the following conditions:

$\lambda_{i}^{j+1} \geq \lambda_{i}^{j} \geq \lambda_{i+1}^{j+1}, \quad \forall 1 \leq i \leq j \leq r.$

The last sequence $\underline{\lambda}^{r+1}$ of the pattern $\mathcal{P}$ is its bounding sequence.

Fix $\lambda \in P^+$ and a pattern $\mathcal{P} : \underline{\lambda}^1, \ldots, \underline{\lambda}^r, \underline{\lambda}^{r+1} = \underline{\lambda}$ with bounding sequence $\underline{\lambda}$.

2.10.1. The weight of a pattern. The weight $\text{wt} \mathcal{P} \in \mathfrak{h}^*$ of $\mathcal{P}$ is defined by

$$\text{wt} \mathcal{P} := a_1 \epsilon_1 + a_2 \epsilon_2 + \cdots + a_{r+1} \epsilon_{r+1},$$

where $a_j = \sum_{i=1}^{j} \lambda_{i}^j - \sum_{i=1}^{j-1} \lambda_{i}^{j-1}$.

Note that $a_1 + a_2 + \cdots + a_{r+1} = \lambda_1 + \lambda_2 + \cdots + \lambda_r$.

2.10.2. Differences of a pattern. For $1 \leq i \leq j \leq r$, the differences $d_{i,j}(\mathcal{P})$ and $d'_{i,j}(\mathcal{P})$ (or just $d_{i,j}$ and $d'_{i,j}$ if $\mathcal{P}$ is clear from the context) of $\mathcal{P}$ are given by

$$d_{i,j}(\mathcal{P}) := \lambda_{i}^{j+1} - \lambda_{i}^{j}, \quad \text{and} \quad d'_{i,j}(\mathcal{P}) := \lambda_{i}^{j} - \lambda_{i+1}^{j+1}.$$ 

2.10.3. Area of a pattern. The triangular area or just area $\Delta(\mathcal{P})$ of $\mathcal{P}$ is defined by

$$\Delta(\mathcal{P}) := \sum_{1 \leq i \leq j \leq r} d_{i,j} d'_{i,j}.$$ 

2.10.4. Trapezoidal area of a pattern. The trapezoidal area $\square(\mathcal{P})$ of $\mathcal{P}$ is defined by

$$\square(\mathcal{P}) := \sum_{1 \leq i \leq j \leq r} d_{i,j} \left( \sum_{p=i}^{j} d'_{p,j} \right).$$
2.10.5. **Shift of a pattern.** For \( k \in \mathbb{Z}_{\geq 0} \), the shift \( P^k \) of \( P \) by \( k \) is a pattern with bounding sequence \( \lambda + k\mathbb{Z} \): suppose that \( \eta^1, \ldots, \eta^r, \eta^{r+1} \) be the row vectors of \( P^k \), then
\[
\eta^i_j := \begin{cases} 
\lambda_i^j + 2k, & i = 1 \text{ and } 1 < j \leq r + 1, \\
\lambda_i^j, & 1 < i \leq r + 1, \\
\lambda_i^j + k, & \text{otherwise.}
\end{cases}
\]
We observe that
\[
d_{i,j}(P^k) = d_{i,j}(P) + \delta_{i,j}k, \quad d'_{i,j}(P^k) = d'_{i,j}(P) + \delta_{i,j}k, \quad \forall 1 \leq i \leq j \leq r, \quad \text{and} \quad \text{wt } P^k = \text{wt } P.
\]

2.11. **Partition overlaid patterns (POPs).** A *partition overlaid pattern (POP)* consists of a GT pattern \( P \), and for every pair \((i, j)\) of integers with \( 1 \leq i \leq j \leq r \), a partition \( \pi(j)^i \) that fits into the rectangle \((d_{i,j}(P), d'_{i,j}(P))\) (see [5] for more details). For \( \lambda \in P^+ \), let \( \mathbb{P}_\lambda \) denote the set of POPs with bounding sequence \( \lambda \).

The bounding sequence, area \( \Delta(\mathbb{P}) \), trapezoidal area \( \square(\mathbb{P}) \), weight \( \text{wt } \mathbb{P} \), and the differences \( d_{i,j}(\mathbb{P}), d'_{i,j}(\mathbb{P}) \) (or just \( d_{i,j} \) and \( d'_{i,j} \) if \( \mathbb{P} \) is clear from the context), \( 1 \leq i \leq j \leq r \), of a POP \( \mathbb{P} \) are just the corresponding notions attached to the underlying pattern.

Fix \( \lambda \in P^+ \) and a POP \( \mathbb{P} \) with bounding sequence \( \lambda \). Let \( \lambda^1, \ldots, \lambda^r, \lambda^{r+1} = \lambda \) be the underlying pattern of \( \mathbb{P} \) and \( \pi(j)^i \), \( 1 \leq i \leq j \leq r \), be the partition overlay.

2.11.1. **Restriction of a POP.** For \( 1 \leq i \leq j \leq r + 1 \), define \( \lambda^j_i := \lambda^j_1, \lambda^j_{i+1}, \ldots, \lambda^j_j \). Observe that \( \lambda^j_1 = \lambda_1^j \). For \( s \in I \cup \{r + 1\} \), the *restriction* \( \mathbb{P}_s \) or \( \text{res}_s(\mathbb{P}) \) of \( \mathbb{P} \) to \( s \) is a POP with bounding sequence \( \lambda^s_{r+1} \); the row vectors of \( \mathbb{P}_s \) are \( \lambda^s_1, \lambda^s_{s+1}, \ldots, \lambda^s_r \) and \( \pi(j)^s \), \( s \leq i \leq j \leq r \), be the partition overlay. Observe that \( \mathbb{P}_1 = \mathbb{P} \).

2.11.2. **Depth of a POP.** The depth \( d(\mathbb{P}) \) of \( \mathbb{P} \) is defined by
\[
d(\mathbb{P}) := \sum_{1 \leq i \leq j \leq r} d^j_i(\mathbb{P}), \quad \text{where} \quad d^j_i(\mathbb{P}) := d_{i,j}(\sum_{p=i+1}^{j} d^j_p) + |\pi(j)^i|.
\]
For \( s \in I \), we observe that
\[
d(\mathbb{P}_s) = \sum_{s \leq i \leq j \leq r} d^j_i(\mathbb{P}_s) = d(\mathbb{P}_{s+1}) + \sum_{j=s}^{r} d^j_s(\mathbb{P}) = d(\mathbb{P}_{s+1}) + \sum_{j=s+1}^{r} d^j_s(\mathbb{P}) + |\pi(s)^s|.
\]  \hspace{1cm} (2.8)
From [5] Corollary 3.4, we have the following:
\[
\square(\mathbb{P}) = \Delta(\mathbb{P}) + d(\mathbb{P}) - \sum_{1 \leq i \leq j \leq r} |\pi(j)^i| = \frac{1}{2}((\lambda|\lambda) - (\text{wt } \mathbb{P}|\text{wt } \mathbb{P})).
\]  \hspace{1cm} (2.9)

2.11.3. **Shift of a POP.** For \( k \in \mathbb{Z}_{\geq 0} \), the *shift* \( \mathbb{P}^k \) of \( \mathbb{P} \) by \( k \) is a POP with bounding sequence \( \lambda + k\mathbb{Z} \); the underlying pattern of \( \mathbb{P}^k \) is \( P^k \) and \( \pi(j)^i \), \( 1 \leq i \leq j \leq r \), be the partition overlay. Note that the underlying partition overlay for \( \mathbb{P}^k \) and \( \mathbb{P} \) is same. It is easy to observe that
\[
\text{wt } \mathbb{P}^k = \text{wt } \mathbb{P} \quad \text{and} \quad d(\mathbb{P}^k) = d(\mathbb{P}).
\]  \hspace{1cm} (2.10)
2.11.4. Shift and then restrict. For \( k \in \mathbb{Z}_{\geq 0} \) and \( s \in I \cup \{r+1\} \), set \( \Psi^k_s := \text{res}_s(\Psi^k) \).

2.11.5. Invariant set of a POP. The invariant set \( \mathcal{I}(\Psi) \) of \( \Psi \) is a set consists of the partition overlay of \( \Psi \) and the differences of \( \Psi \) which are invariant under the shift. More precisely,

\[
\mathcal{I}(\Psi) := \{ \pi(d_i) : 1 \leq i < j \leq r \} \cup \{ \pi(d'_i) : 1 \leq i \leq j \leq r \} \cup \{ \pi(j) : 1 \leq i \leq j \leq r \}.
\]

For \( s \in I \cup \{r+1\} \), note that

\[
\mathcal{I}(\Psi_s) = \{ \pi(d_i) : 1 \leq i < j \leq r \} \cup \{ \pi(d'_i) : 1 \leq i \leq j \leq r \} \cup \{ \pi(j) : 1 \leq i \leq j \leq r \}.
\]

Set \( \mathcal{I}_s^j(\Psi) := \{ \pi(d_j) \} \cup \{ \pi(j)_s : 1 \leq s < j \leq r \} \). Now we have

\[
\mathcal{I}(\Psi_s) = \mathcal{I}(\Psi_{s+1}) \cup ( \cup_{s\leq j \leq r} \mathcal{I}_s^j(\Psi)) \quad \forall \ s \in I.
\]

(2.11)

3. The main result

3.1. The Chari-Loktev bases for local Weyl modules in type A. In this subsection, we recall the bases given by Chari and Loktev [2] in terms of POPs (see [8]). Fix notation and terminology as in [2].

3.1.1. Let \( d, d' \) be non-negative integers and \( \pi \) be a partition that fits into the rectangle \((d, d')\). For \( \alpha \in \mathbb{R}^+ \), the monomial \( x^\pm_\alpha(d, d', \pi) \) corresponding to the complement of \( \pi \) is given by

\[
x^\pm_\alpha(d, d', \pi) := \left( \prod_{i=1}^d x^\pm_\alpha \right) \cdot t^{d_\pi}.
\]

Set \( x^\pm_{i,j}(d, d', \pi) := x^\pm_{\alpha_{i,j}}(d, d', \pi) \) for all \( 1 \leq i \leq j \leq r \).

3.1.2. Let \( \lambda \in \mathbb{P}^+ \) and \( \Psi \) be a POP with bounding sequence \( \lambda \). Let \( d_{i,j}, d'_{i,j} \), \( 1 \leq i < j \leq r \), be the differences and \( \pi(j)_s \), \( 1 \leq i \leq j \leq r \), be the partition overlay of \( \Psi \). Define \( \rho_\Psi \in \text{U}(\mathfrak{n}^- \otimes \mathbb{C}[t]) \) as follows:

\[
\rho_\Psi := x^-_{1,1}(d_{1,1}, d'_{1,1}, \pi(1)^1) \left( \prod_{i=1}^2 x^-_{i,2}(d_{i,2}, d'_{i,2}, \pi(2)^1) \right) \cdots \left( \prod_{i=1}^r x^-_{i,r}(d_{i,r}, d'_{i,r}, \pi(r)^1) \right).
\]

(3.1)

The order of the factors matters in the expression for \( \rho_\Psi \). Since \( [x^-_{i,j}, x^-_{p,q}] = 0, \forall \ 1 \leq i \leq p \leq q \leq j \leq r \), it is easy to see that

\[
\rho_\Psi = \left( \prod_{j=1}^r x^-_{1,j}(d_{1,j}, d'_{1,j}, \pi(j)^1) \right) \left( \prod_{j=2}^r x^-_{2,j}(d_{2,j}, d'_{2,j}, \pi(j)^2) \right) \cdots \left( \prod_{j=r}^r x^-_{r,r}(d_{r,r}, d'_{r,r}, \pi(r)^r) \right).
\]

(3.2)

Set \( \rho_{\Psi_{r+1}} := 1 \). We observe that

\[
\rho_\Psi_s = \left( \prod_{j=s}^r x^-_{s,j}(d_{s,j}, d'_{s,j}, \pi(j)^s) \right) \rho_\Psi_{s+1}, \quad \forall \ s \in I.
\]

Define \( v_\Psi := \epsilon_\Psi \rho_\Psi w_\lambda \), where \( \epsilon_\Psi \in \{ \pm 1 \} \) is defined in [1.4].
Since (Lemma 3.3. Let $v_\Psi, \Psi$ belongs to the set $\mathbb{P}_\lambda$ of POPs with bounding sequence $\Lambda$, form a basis for the local Weyl module $W(\lambda)$.

We shall call the bases given in the last theorem as the Chari-Loktev (or CL) bases.

3.2 The main theorem: stability of the CL bases. We wish to study for $\lambda \in \mathcal{P}^+$ and $k \in \mathbb{Z}_{\geq 0}$, the compatibility of CL bases with respect to the embeddings $W(\lambda) \hookrightarrow W(\lambda + k\theta)$ in $L(\Lambda_{i,\lambda})$ (see [2,7]). We first recall the weight preserving embedding from $\mathbb{P}_\lambda$ into $\mathbb{P}_{\lambda+k\theta}$ given in [8, Corollary 5.13] at the level of the parametrizing sets of these bases: for $\Psi \in \mathbb{P}_\lambda$, the shift $\Psi^k$ of $\Psi$ by $k$ be its image in $\mathbb{P}_{\lambda+k\theta}$.

For every $\lambda \in \mathcal{P}^+$, we will fix the following choice of $w_\lambda$ in $L(\Lambda_{i,\lambda})$:

$$w_\lambda := T_{\lambda} v_{\Lambda_0},$$

where $T_{\lambda}$ is the linear isomorphism from $L(\Lambda_0) \rightarrow L(\Lambda_{i,\lambda})$ defined in [4.3].

Lemma 3.2. Let $\lambda \in \mathcal{P}^+$ and $\Psi \in \mathbb{P}_\lambda$. Then the weight of $v_\Psi$ in $L(\Lambda_{i,\lambda})$ is

$$t_{\Psi, -\Xi_{i,\lambda}}(\Lambda_{i,\lambda}) - d(\Psi)\delta,$$

where $\Xi_{i,\lambda}$ denotes the restriction of $\Lambda_{i,\lambda}$ to $\mathfrak{h}$.

Proof. It is clear from the definition of $v_\Psi$ that its weight in $L(\Lambda_{i,\lambda})$ is

$$t_{\lambda}(\Lambda_0) - \sum_{1 \leq i \leq r} d_{i,j} \alpha_{i,j} + (\Delta(\Psi) - \sum_{1 \leq i \leq r} |\pi(j)|)\delta$$

$$= \Lambda_0 + \text{wt} \Psi - \left( \frac{1}{2} (\lambda|\lambda) - \Delta(\Psi) + \sum_{1 \leq i \leq r} |\pi(j)|\right)\delta$$

$$= \Lambda_0 + \text{wt} \Psi - \left( \frac{1}{2} (\text{wt} \Psi|\text{wt} \Psi) + d(\Psi)\right)\delta$$

(3.3)

where the last equality follows from (2.9). Since $\Lambda_{i,\lambda}$ is of level 1, we obtain using [6] (6.5.3) that

$$t_{\Psi, -\Xi_{i,\lambda}}(\Lambda_{i,\lambda}) = \Lambda_0 + \text{wt} \Psi + \frac{1}{2}((\Lambda_{i,\lambda}|\Lambda_{i,\lambda}) - (\text{wt} \Psi|\text{wt} \Psi))\delta.$$  

(3.4)

Since $(\Lambda_{i,\lambda}|\Lambda_{i,\lambda}) = 0$, we get the result from (3.3)–(3.4). □

The following is immediate from Lemma 3.2 and (2.10).

Lemma 3.3. Let $\lambda \in \mathcal{P}^+$, $\Psi \in \mathbb{P}_\lambda$, and $k \in \mathbb{Z}_{\geq 0}$. Then the basis vectors $v_\Psi \in W(\lambda)$ and $v_{\Psi^k} \in W(\lambda + k\theta)$ lie in the same weight space of $L(\Lambda_{i,\lambda})$.

It is not true that $v_\Psi$ and $v_{\Psi^k}$ are equal as elements of $L(\Lambda_{i,\lambda})$ (see [7, Example 1]). We will however see below that $v_\Psi = v_{\Psi^k}$ for all stable $\Psi$. More precisely, let

$$\mathbb{P}^{\text{stab}}(\lambda) := \{ \Psi \in \mathbb{P}_\lambda : d_{\ell,\ell}(\Psi) \geq d(\Psi), \forall 1 \leq \ell \leq r \}$$

(see §§2.10–2.11).

The following theorem is the main result of this paper.
Theorem 3.4. Let \( g = sl_{r+1} \). Let \( \lambda \in P^+ \) and \( \mathfrak{P} \in \mathbb{P}_{\text{stab}}(\lambda) \). Then
\[
v_{\mathfrak{P} k} = v_{\mathfrak{P}} \quad \text{for all } k \in \mathbb{Z}_{\geq 0},
\]
i.e., they are equal as elements of \( L(\Lambda_{i\lambda}) \).

This theorem is proved in [4.4].

Remark 3.5. Theorem 3.4 is conjectured in [8, Conjecture 6.1] and the \( r = 1 \) case is proved in [4, Theorem 6] under the additional assumption that
\[
d(\mathfrak{P}) \leq \begin{cases} \min\{d_{1,1}(\mathfrak{P}), d'_{1,1}(\mathfrak{P})\}, & \lambda_1 \text{ even,} \\ \min\{d_{1,1}(\mathfrak{P}), (d'_{1,1}(\mathfrak{P}) - 1)\}, & \lambda_1 \text{ odd.} \end{cases}
\]

3.3. Bases for level one representations of \( \hat{g} \). Fix \( i \in \hat{I}, \gamma \in Q, \) and \( d \in \mathbb{Z}_{\geq 0} \). Consider the irreducible module \( L(\Lambda_i) \) and its weight space of weight \( t_\gamma(\Lambda_i) - d\delta \). Set \( \mu = \varpi_i + \gamma \), the restriction of \( t_\gamma(\Lambda_i) - d\delta \) to \( h^* \). Let \( \lambda \in P^+ \) such that \( \mu \) is a weight of the corresponding irreducible representation \( V(\lambda) \) of \( g \). Note that \( i_\lambda = i \).

For \( k \in \mathbb{Z}_{\geq 0} \), from Lemma 3.2, we get that the CL basis indexing set for \( W(\lambda + k\theta)t_\gamma(\Lambda_i) - d\delta \) is the set \( \mathbb{P}_{\lambda,\mu}^k(d) \) of \( \mathbb{P} \)-OPs with bounding sequence \( \lambda + k\theta \) with weight \( \mu \) and depth \( d \). From [8, Theorem 5.10], for \( k \geq d \), there exist a bijection from the set \( \mathcal{P}_r(d) \) of all \( r \)-colored partitions of \( d \) onto \( \mathbb{P}_{\lambda,\mu}^k(d) \). Since this bijection is produced by the “shift by \( k \)” operator, we have
\[
d_{\ell,\ell}(\mathfrak{P}) \geq k, \quad \forall 1 \leq \ell \leq r, \quad \text{for every } \mathfrak{P} \in \mathbb{P}_{\lambda,\mu}^k(d).
\] 
(3.5)
For \( k \geq d \), by Proposition 2.2, we now have
\[
W(\lambda + k\theta)t_\gamma(\Lambda_i) - d\delta = L(\Lambda_i)t_\gamma(\Lambda_i) - d\delta,
\]
and the set \( B_{\gamma,d} := \{ v_{\mathfrak{P}} : \mathfrak{P} \in \mathbb{P}_{\lambda,\mu}^k(d) \} \) is a basis for \( L(\Lambda_i)t_\gamma(\Lambda_i) - d\delta \). By Theorem 3.4, using (3.5), the set \( B_{\gamma,d} \) is independent of the choice of \( k \) for any \( k \geq d \).

Finally, to obtain a basis for \( L(\Lambda_i) \), we take the disjoint union over the weights of \( L(\Lambda_i) \):
\[
B := \bigsqcup_{\gamma,d} B_{\gamma,d}.
\]
We may view \( B \) as a direct limit of the CL bases for the Demazure submodules of \( L(\Lambda_i) \).

4. Proof of the main result

4.1. Frenkel-Kac translation operators. We recall the necessary facts from [5]. Let \( (V, \pi) \) be an integrable representation of \( \hat{g} \) with weight space decomposition \( V = \bigoplus_{\nu \in h^*} V_\nu \). For a real root \( \gamma = \alpha + s\delta \) (\( \alpha \in R, s \in \mathbb{Z} \)) of \( \hat{g} \), we define
\[
r_\gamma^{\pi} := e^{-\pi(x_\gamma)}e^{\pi(x_{-\gamma})}e^{-\pi(x_\gamma)}.
\] 
(4.1)
The operator \( r_\gamma^{\pi} \) is a linear automorphism of \( V \) such that \( r_\gamma^{\pi}(V_\nu) = V_{s_\gamma(\nu)} \), where \( s_\gamma \in \hat{W} \) is the reflection defined by \( \gamma \). Given \( w \in \hat{W} \) and its reduced expression \( w = s_{i_1}s_{i_2}\cdots s_{i_q} \), define
\[
r_{w}^{\pi} := r_{\alpha_{i_1}}^{\pi}r_{\alpha_{i_2}}^{\pi}\cdots r_{\alpha_{i_q}}^{\pi}.
\]
Note that $r^\pi_\nu(V_\nu) = V_{\nu\pi}\nu$.

For each $\beta \in Q$, there exists a translation operator $T^\pi_\beta$ on $V$ such that

$$T^\pi_\alpha = r_{\beta - \alpha} r_\alpha, \quad \alpha \in R \quad \text{and} \quad T^\pi_\beta T^\pi_\beta' = \epsilon(\beta, \beta') T^\pi_{\beta + \beta'} \quad \beta, \beta' \in Q,$$

where $\epsilon$ is a 2-cocycle of $Q$ with values in $\{\pm 1\}$ (see [5, §2.3]). These operators satisfy $T^\pi_\beta(V_\nu) = V_{\nu\beta}(\nu)$ for all $\nu \in \hat{h}^*$, $\beta \in Q$.

We will only need these operators in two cases, namely when $(V, \pi)$ is either the adjoint representation or the basic representation of $\mathfrak{g}$. We note that $T^\pi_\beta$ is in fact a Lie algebra automorphism of $\mathfrak{g}$. For ease of notation, we will denote the translation operators corresponding to the basic representation simply by $T_\beta$, suppressing the $\pi$ in the superscript.

The key properties of the translation operators are given in [5, Propositions 1.2 and 2.3]. We summarize them for our context below:

**Proposition 4.1.** Let $\mu \in Q$. Then

1. $T_\mu T_{-\mu} = \text{id}_{L(\Lambda_0)}$.
2. $T_{\mu - d\alpha} T_{d\alpha} = \epsilon(\mu - d\alpha, d\alpha) T_\mu$, $\forall \alpha \in R^+, d \in \mathbb{Z}_{\geq 0}$.
3. $T_\mu X T_{-\mu} v = T^\text{ad}_\mu(X) v$, $\forall X \in \mathfrak{g}$, $v \in L(\Lambda_0)$.
4. $T^\text{ad}_\mu(x^-_\alpha \otimes t^s) = (x^-_\alpha \otimes t^{s + [\mu(\alpha)]})$, $\forall \alpha \in R^+$, $s \in \mathbb{Z}$.
5. $T_\mu (h \otimes t^s) v = T^\text{ad}_\mu(h \otimes t^s) T_\mu v = (h \otimes t^s) T_\mu v$, $\forall h \in \mathfrak{h}$, $v \in L(\Lambda_0)$, $s \in \mathbb{Z} \setminus \{0\}$.

4.2. The goal of this subsection is to define a translation operator $T_{\varpi_i}$ associated to a fundamental weight $\varpi_i$ ($i \in I$) of $\mathfrak{g}$.

4.2.1. Let $\tau$ be an automorphism of $\hat{\mathfrak{g}}$ such that $\tau \hat{\mathfrak{h}} = \hat{\mathfrak{h}}$. We have the induced action of $\tau$ on $\hat{\mathfrak{h}}$ by $\langle \tau \lambda, h \rangle = \langle \lambda, \tau^{-1}h \rangle$. Given an $\hat{\mathfrak{g}}$-module $V$, let $V^\tau$ denote the module with the twisted action

$$x \circ v = \tau^{-1}(x)v, \quad \text{for} \quad x \in \hat{\mathfrak{g}}, v \in V.$$

Observe that for automorphisms $\tau_1, \tau_2$, we have $V^{\tau_1 \tau_2} \simeq (V^{\tau_2})^{\tau_1}$.

For $i \in I$, we now study the twisted actions on $L(\Lambda_0)$ by two specific automorphisms $\tilde{\sigma}_i, \tilde{\phi}_{w_0, w_{0,i}}$ of $\hat{\mathfrak{g}}$. First, recall from [2] that $\sigma_i = t_{\varpi_i, w_{0,i}w_{0}}$ is an automorphism of the Dynkin diagram of $\hat{\mathfrak{g}}$:

$$\sigma_i \alpha_p = \alpha_{i + p (\text{mod } r + 1)}, \quad \forall p \in \hat{I}, \quad \text{and} \quad \sigma_i \rho = \rho.$$

Consider the Lie algebra automorphism $\tilde{\sigma}_i$ of $\hat{\mathfrak{g}}$ given by the relations

$$\tilde{\sigma}_i(e_p) = e_{i + p (\text{mod } r + 1)}, \quad \tilde{\sigma}_i(f_p) = f_{i + p (\text{mod } r + 1)}, \quad \tilde{\sigma}_i(\alpha^\vee_p) = \alpha^\vee_{i + p (\text{mod } r + 1)}, \quad \forall p \in \hat{I}, \quad \text{and} \quad \tilde{\sigma}_i(\rho^\vee) = \rho^\vee. \quad (4.2)$$

Here $\rho^\vee \in \hat{\mathfrak{h}}$ is the unique element for which $\langle \alpha_p, \rho^\vee \rangle = 1$, $\forall p \in \hat{I}$, and $\langle \Lambda_0, \rho^\vee \rangle = 0$. Observe that $\tilde{\sigma}_i$ leaves $\hat{\mathfrak{h}}$ invariant, and its induced action on $\mathfrak{h}$ coincides with $\sigma_i$, i.e.,

$$\langle \sigma_i \Lambda, h \rangle = \langle \Lambda, \tilde{\sigma}_i^{-1}h \rangle, \quad \forall h \in \hat{\mathfrak{h}}, \Lambda \in \hat{\mathfrak{h}}. \quad (4.3)$$

Given $w \in W$, define the map $\tilde{\phi}_w : g \rightarrow g$ by

$$\tilde{\phi}_w(x_\alpha) = x_{w_\alpha}, \quad \tilde{\phi}_w(h_\alpha) = h_{w_\alpha}, \quad \forall \alpha \in R.$$
It is easy to see that $\phi_w$ is an automorphism of $\mathfrak{g}$ and it can be extended to an automorphism $\tilde{\phi}_w$ of $\mathfrak{g}$ by defining

$$\tilde{\phi}_w(c) = c, \quad \tilde{\phi}_w(d) = d, \quad \tilde{\phi}_w(x \otimes t^s) = \phi_w(x) \otimes t^s, \quad \forall x \in \mathfrak{g}, s \in \mathbb{Z}. $$

Clearly $\tilde{\phi}_w$ leaves $\mathfrak{h}$ invariant, and its induced action on $\mathfrak{h}^*$ coincides with $w$, i.e.,

$$\langle w\Lambda, h \rangle = \langle \Lambda, \tilde{\phi}_w^{-1}h \rangle, \quad \forall h \in \mathfrak{h}, \Lambda \in \mathfrak{h}^*. \quad (4.4)$$

For $i \in I$, set $\tilde{t}_{\alpha_i} := \sigma_i \tilde{\phi}_{w_0w_0,i}$. Observe that $\tilde{t}_{\alpha_i}$ leaves $\mathfrak{h}$ invariant, and $\tilde{t}_{\alpha_i}(g_\alpha) = g_{\tilde{t}_{\alpha_i}(\alpha)}$ for all $\alpha \in R$. From (4.3)–(4.4), we have

$$\langle t_{\alpha_i}\Lambda, h \rangle = \langle \Lambda, \tilde{t}_{\alpha_i}^{-1}h \rangle, \quad \forall h \in \mathfrak{h}, \Lambda \in \mathfrak{h}^*. \quad (4.5)$$

For $p \in \tilde{I}$, note that

$$w_0w_0,i(\alpha_p) = \begin{cases} \alpha_{r+1+p-i}, & p < i, \\ \alpha_{p-i}, & p > i, \\ -\theta, & p = i, \\ \alpha_{r+1-i} + \delta, & p = 0. \end{cases} \quad \text{and} \quad w_0w_0(\alpha_p) = \begin{cases} \alpha_{p+i-r-1}, & r + 1 - p < i, \\ \alpha_{p+i}, & r + 1 - p > i, \\ -\theta, & r + 1 - p = i, \\ \alpha_i + \delta, & p = 0. \end{cases} \quad (4.6)$$

Using (4.2) and (4.3), for $p \in \tilde{I}$, we get

$$\tilde{t}_{\alpha_i}(e_p) = \begin{cases} e_p, & p \neq 0, i, \\ e_i \otimes t^{-1}, & p = i, \\ x_\theta \otimes t^2, & p = 0, \end{cases} \quad \tilde{t}_{\alpha_i}(f_p) = \begin{cases} f_p, & p \neq 0, i, \\ f_i \otimes t, & p = i, \\ x_\theta \otimes t^{-2}, & p = 0, \end{cases} \quad (4.7)$$

and

$$\tilde{t}_{\alpha_i}^{-1}(e_p) = \begin{cases} e_p, & p \neq 0, i, \\ e_i \otimes t, & p = i, \\ x_\theta, & p = 0, \end{cases} \quad \tilde{t}_{\alpha_i}^{-1}(f_p) = \begin{cases} f_p, & p \neq 0, i, \\ f_i \otimes t^{-1}, & p = i, \\ x_\theta, & p = 0. \end{cases} \quad (4.8)$$

Thus

$$\tilde{t}_{\alpha_i}(x_\alpha \otimes t^s) = (x_\alpha \otimes t^{s-(\alpha_i|\alpha)}) \quad \text{and} \quad \tilde{t}_{\alpha_i}^{-1}(x_\alpha \otimes t^s) = (x_\alpha \otimes t^{s+(\alpha_i|\alpha)}), \quad \forall \alpha \in R, s \in \mathbb{Z}. \quad (4.9)$$

**Proposition 4.2.** With notation as above, for $i \in I$, we have $L(\Lambda_0)^{t_{\alpha_i}} \simeq L(\Lambda_i)$.

**Proof.** We consider the $U(\mathfrak{g})$-linear map $L(\Lambda_i) \to L(\Lambda_0)^{t_{\alpha_i}}$ which sends $v_{\Lambda_i}$ to $v_{\Lambda_0}$. To show this is well defined, we only need to check that $v_{\Lambda_0} \in L(\Lambda_0)^{t_{\alpha_i}}$ satisfies the relations in (2.2)–(2.4) for $\Lambda = \Lambda_i$. Since $w_0w_0,\Lambda_0 = \Lambda_0$, the relations in (2.2) follows from (4.3)–(4.4). The relations in (2.3) are immediate from (4.8) by using Proposition 2.2. Since $f_p v_{\Lambda_0} = 0$ for $p \neq 0, i$, and $x_\theta v_{\Lambda_0} = 0$, to prove the relations in (2.4), we only need to show that $(f_i \otimes t^{-1})^2 v_{\Lambda_0} = 0$ in $L(\Lambda_0)$. But this follows easily by a standard $\mathfrak{sl}_2$ argument using the $\mathfrak{sl}_2$ copy spanned by $e_i \otimes t, f_i \otimes t^{-1}$, and $\alpha_i^+ + c$. Now, this map is a surjection, since $v_{\Lambda_0}$ generates $L(\Lambda_0)^{t_{\alpha_i}}$. Since $L(\Lambda_i)$ is irreducible, it must be an isomorphism. \( \square \)
For $i \in I$, let $T_{\varpi_i}$ be the isomorphism from $L(A_0)^{I_{\varpi_i}}$ onto $L(A_i)$. Observe that

$$T_{\varpi_i} L(A_0)^\nu = L(A_i)^{I_{\varpi_i}(\nu)}, \quad \forall \nu \in \hat{h}^s.$$ 

Set $T^{-1}_{\varpi_i} := T^{-1}_{\varpi_i}$. The isomorphism $T^{-1}_{\varpi_i} : L(A_i) \to L(A_0)^{I_{\varpi_i}}$ maps $v_{A_i} \mapsto v_{A_0}$. It is then determined on all of $L(A_i)$ by $\hat{h}$-linearity, i.e., by the relation

$$T^{-1}_{\varpi_i}(x \cdot v) = \hat{t}^{-1}_{\varpi_i}(x) T^{-1}_{\varpi_i}(v), \quad \forall x \in \hat{h}, \; v \in L(A_i).$$ 

(4.10)

Now using (4.9), we get

$$T^{-1}_{\varpi_i}(x \alpha \otimes t^s) T_{\varpi_i} v = (x^\alpha \otimes t^{s-(\varpi_i(\alpha))}) v, \quad \forall \alpha \in R^+, \; v \in L(A_0), \; s \in \mathbb{Z}.$$ 

(4.11)

4.3. For $\lambda \in P^+$ and $\beta \in Q$, we define a linear isomorphism $T_{\lambda-\beta} : L(A_0) \to L(A_{i_\lambda})$ as follows:

$$T_{\lambda-\beta} := T_{\varpi_{i_\lambda}} T_{\lambda-\varpi_{i_\lambda}}.$$ 

Suppose there is $\lambda' \in P^+$ and $\beta' \in Q$ such that $\lambda-\beta = \lambda'-\beta'$. Then it is easy to see that $i_\lambda = i_{\lambda'}$. Hence the definition is well defined. Observe that

$$T_{\lambda-\beta} L(A_0)^\nu = L(A_{i_\lambda})^{I_{\lambda-\beta}(\nu)}, \quad \forall \nu \in \hat{h}^s.$$ 

Set

$$T^{-1}_{\lambda-\beta} := T^{-1}_{\lambda-\beta} \quad \text{and} \quad \epsilon(\beta + w_{i_\lambda}, \beta') := \epsilon(\beta, \beta'), \quad \forall \lambda \in P^+, \; \beta, \beta' \in Q.$$ 

Proposition 4.3. Let $\lambda \in P^+$ and $\beta \in Q$. Then

1. $T_{\lambda-\beta} - d\alpha = \epsilon(\lambda - \beta - d\alpha, d\alpha) T_{\lambda-\beta}, \quad \forall \alpha \in R^+, \; d \in \mathbb{Z}_{\geq 0}.$

2. $T_{\epsilon(\lambda-\beta)} (x^\alpha \otimes t^s) T_{\lambda-\beta} v = (x^\alpha \otimes t^{s-(\alpha+a)}(\lambda(\alpha))) v, \quad \forall \alpha \in R^+, \; v \in L(A_0), \; s \in \mathbb{Z}.$

Proof. The proof is immediate from (4.11) and Proposition 4.1(2)-(1). □

4.4. Given $\lambda \in P^+$, $P \in \mathbb{P}_\lambda$, $k \in \mathbb{Z}_{\geq 0}$, and $s \in I \cup \{r+1\}$, define $\epsilon_{P_k} \in \{\pm 1\}$ as follows:

$$\epsilon_{P_k} := \prod_{p=s}^{r} (-1)^{\left| d_{p,p+k} \right|} \prod_{j=p}^{r} \epsilon(\lambda + k \alpha_1, \lambda - \sum_{p<i \leq j \leq r} d_{i,j} \alpha_{i,j} - \sum_{u=j}^{r} d_{p,u} \alpha_{p,u}, \; d_{p,j} \alpha_{p,j}).$$ 

Here, $[x]$ denotes the greatest integer less than or equal to $x$.

We are now in a position to state the main result of this section.

Theorem 4.4. Let $\lambda \in P^+$, $P \in \mathbb{P}_\lambda$, $k \in \mathbb{Z}_{\geq 0}$, and $s \in I \cup \{r+1\}$. If $d_{\ell,\delta}(P) \geq d(P, \delta)$ for all $s \leq \ell \leq r$, then

$$\epsilon_{P_k} \rho_{P_k} T_{\lambda+k\theta} v_{A_0} = T_{\lambda+k \alpha_{i,s} - \sum_{i \leq j \leq r} d_{i,j}(P) \alpha_{i,j}} f_{\mathcal{I}(P_s)} v_{A_0},$$ 

(4.12)

where $f_{\mathcal{I}(P_s)}$ is a polynomial in $t_j$, $i \in I, j \in \mathbb{N}$, depends only on the elements of the set $\mathcal{I}(P_s)$ such that the weight of $f_{\mathcal{I}(P_s)} v_{A_0}$ in $L(A_0)$ is $\Lambda_0 - d(P, \delta).$
4.4.1. **Proof of Theorem 3.4 from Theorem 4.4.** We observe that the expression on the left hand side of (4.4) depends on $k$, the one on the right hand side, when $s = 1$, is independent of it. The fact that these two expressions are equal when $d_{k,\ell}(\mathcal{P}) \geq d(\mathcal{P}_\ell)$ for all $1 \leq \ell \leq r$, what leads to the stability properties of interest. Thus Theorem 4.4 for $s = 1$, proves Theorem 3.4.

The rest of the paper is devoted to proving Theorem 4.4.

4.5. In this subsection, for $x \in \mathfrak{g}$, $s \in \mathbb{Z}$, and $m \in \mathbb{N}$, we set $xt^s := x \otimes t^s$ and $[m] := \{1, 2, \ldots, m\}$. The following two lemmas are elementary.

**Lemma 4.5.** Let $\alpha \in R^+$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, $h_1, \ldots, h_n \in \mathfrak{h}$, and $p, q_1, \ldots, q_n \in \mathbb{Z}_{\geq 0}$. Then

$$(y_\alpha t^p) \left( \prod_{i=1}^{n} h_i^{-q_i} \right) = \sum_{0 \leq k \leq n} \sum_{A \subseteq [n]} \left( \prod_{i \in A} \langle \alpha, h_i \rangle \right) \left( \prod_{i \in [n] \setminus A} h_i^{-q_i} \right) \left( y_\alpha t^{p-\sum_{i \in A} q_i} \right).$$

**Proof.** Proceed by induction on $n$. In case $n = 1$, we have

$$(y_\alpha t^p) (h_1 t^{-q_1}) = (h_1 t^{-q_1}) (y_\alpha t^p) + [y_\alpha t^p, h_1 t^{-q_1}] = (h_1 t^{-q_1}) (y_\alpha t^p) + \langle \alpha, h_1 \rangle (y_\alpha t^{p-q_1}),$$

and the result is obvious. Now suppose that $n \geq 2$. Since $[y_\alpha t^p, h_n t^{-q_n}] = \langle \alpha, h_n \rangle (y_\alpha t^{p-q_n})$, we have

$$(y_\alpha t^p) \left( \prod_{i=1}^{n} h_i^{-q_i} \right) = \left( \prod_{i=1}^{n-1} h_i^{-q_i} \right) (y_\alpha t^p) + \langle \alpha, h_n \rangle (y_\alpha t^{p-q_n}) \left( \prod_{i=1}^{n-1} h_i^{-q_i} \right).$$

Using the induction hypothesis the right hand side of the last equation becomes

$$\left( \prod_{i=1}^{n-1} h_i^{-q_i} \right) \sum_{0 \leq k' \leq n-1} \sum_{A' \subseteq [n-1]} \left( \prod_{i \in A'} \langle \alpha, h_i \rangle \right) \left( \prod_{i \in [n-1] \setminus A'} h_i^{-q_i} \right) \left( y_\alpha t^{p-\sum_{i \in A'} q_i} \right) + \langle \alpha, h_n \rangle \sum_{0 \leq k'' \leq n-1} \sum_{A'' \subseteq [n-1]} \left( \prod_{i \in A''} \langle \alpha, h_i \rangle \right) \left( \prod_{i \in [n-1] \setminus A''} h_i^{-q_i} \right) \left( y_\alpha t^{p-q_n-\sum_{i \in A''} q_i} \right).$$

This completes the proof. Indeed, for any $A \subseteq [n]$, there exists $B \subseteq [n-1]$ such that either $A = B$ or $A = B \cup \{n\}$. \hfill \Box

**Lemma 4.6.** Let $\alpha \in R^+$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, $h_1, \ldots, h_n \in \mathfrak{h}$, and $p_1, \ldots, p_m, q_1, \ldots, q_n \in \mathbb{Z}_{\geq 0}$. Then

$$\left( \prod_{i=1}^{m} y_\alpha t^{p_i} \right) \left( \prod_{i=1}^{n} h_i^{-q_i} \right) = \sum_{0 \leq k_1 \leq n} \sum_{A_1 \subseteq [n]} \left( \prod_{i \in A_1} \langle \alpha, h_i \rangle \right) \left( \prod_{i \in [n] \setminus A_1} h_i^{-q_i} \right) \left( \prod_{j=1}^{m} y_\alpha t^{p_j-\sum_{i \in A_j} q_i} \right).$$
Proof. Proceed by induction on \( m \). In case \( m = 1 \), we have the result from Lemma 4.5. Now suppose that \( m \geq 2 \). Using Lemma 4.5 we have

\[
\left( \prod_{i=1}^{m} y_{\alpha} t^{p_i} \right) \left( \prod_{i=1}^{n} h_{t^{-q_i}} \right) = \left( \prod_{i=1}^{m-1} y_{\alpha} t^{p_i} \right) \sum_{0 \leq k_m \leq n} \sum_{A_m \subseteq [n] \atop |A_m| = k_m} \left( \prod_{\alpha \in A_m} \langle \alpha, h_t \rangle \right) \left( \prod_{i \in [n]\setminus A_m} h_{t^{-q_i}} \right) \left( y_{\alpha} t^{p_m-\sum_{i \in A_m} q_i} \right).
\]

Using the induction hypothesis, we have

\[
\left( \prod_{i=1}^{m-1} y_{\alpha} t^{p_i} \right) \left( \prod_{i \in [n]\setminus A_m} h_{t^{-q_i}} \right) = \sum_{0 \leq k_m \leq n-\sum_{j=i+1}^{m} k_j} \sum_{A_m \subseteq [n] \atop |A_m| = k_m} \left( \prod_{i \in [n]\setminus \cup_{j=1}^{m-1} A_j} \langle \alpha, h_t \rangle \right) \left( \prod_{i \in [n]\setminus \cup_{j=1}^{m-1} A_j} h_{t^{-q_i}} \right) \left( \prod_{j=1}^{m-1} y_{\alpha} t^{-q_j} \right).
\]

Substituting (4.14) in the right hand side of (4.13), we get the result. \( \Box \)

4.6. The following result follows from [7, Theorem 10 and Proposition 13 (1)].

**Theorem 4.7.** [7] Let \( \alpha \in R^+ \). Let \( d \in \mathbb{Z}_{\geq 0} \) and \( \pi \) be a partition such that \( d \geq |\pi| \). Then

\[
x_{\alpha}^{-}(d, d, \pi) T_{\mu} g_{m} v_{\Lambda_{0}} = (-1)^{|\pi|} f_{\pi} v_{\Lambda_{0}},
\]

where \( f_{\pi} \) is a polynomial in \( \alpha^{i} t^{-j}, j \in \mathbb{N}, \) depends only on \( \pi \) and not on \( d \) such that the weight of \( f_{\pi} v_{\Lambda_{0}} \) in \( L(\Lambda_{0}) \) is \( \Lambda_{0} - |\pi| \delta \).

**Proposition 4.8.** Let \( d, d', m \in \mathbb{Z}_{\geq 0}, \alpha \in R^+, \) and \( \pi \) be a partition. Let \( \lambda \in P^+ \) and \( \beta \in Q \) with \( (\lambda - \beta|\alpha) = d + d' \), and set \( \mu = \lambda - \beta \). Let \( g_{m} \) be a polynomial in \( \alpha^{i} t^{-j}, i \in I, j \in \mathbb{N}, \) such that the weight of \( g_{m} v_{\Lambda_{0}} \) in \( L(\Lambda_{0}) \) is \( \Lambda_{0} - m \delta \). Then

1. the weight of \( x_{\alpha}^{-}(d, d', \pi) T_{\mu} g_{m} v_{\Lambda_{0}} \) in \( L(\Lambda_{1}) \) is \( t_{\mu-\alpha} (\Lambda_{0} - (|\pi| + m) \delta) \).
2. If \( d \geq |\pi| + m \), we have

\[
x_{\alpha}^{-}(d, d', \pi) T_{\mu} g_{m} v_{\Lambda_{0}} = (-1)^{|\pi|} \epsilon(\mu - d \alpha, d \alpha) T_{\mu-\alpha} f_{\pi} g_{m} v_{\Lambda_{0}},
\]

where \( f_{\pi} g_{m} \) is a polynomial in \( \alpha^{i} t^{-j}, i \in I, j \in \mathbb{N}, \) depends only on \( \pi, g_{m} \) and not on \( d, d' \), such that the weight of \( f_{\pi} g_{m} v_{\Lambda_{0}} \) in \( L(\Lambda_{1}) \) is \( \Lambda_{0} - (|\pi| + m) \delta \).

**Proof.** Since the weight of \( T_{\mu} g_{m} v_{\Lambda_{0}} \) is \( t_{\mu}(\Lambda_{0} - m \delta) \) and \( (\mu|\alpha) = d + d' \), we have the weight of \( x_{\alpha}^{-}(d, d', \pi) T_{\mu} g_{m} v_{\Lambda_{0}} \) is \( t_{\mu}(\Lambda_{0} - m \delta) - d \alpha + (dd' - |\pi|) \delta = t_{\mu-\alpha} (\Lambda_{0} - (|\pi| + m) \delta) \). Hence part \( \Box \).
Proposition 4.9. Let \( g \) be a polynomial in \( \alpha^t, i \in I, j \in \mathbb{N} \), and positive integers \( \eta, q \), \( 1 \leq i \leq d \), depend on \( g \) such that \( m \geq \sum_{i=1}^{d} \eta_i q \). Part (2) now follows from Theorem 4.7 and part (1).

We now prove part (2). Using Propositions 4.1 and 4.3 we have

\[
\left( \prod_{i=1}^{d} x^{-i} \otimes t^{d-i} \right) T_{\mu} g_m v_{\Lambda_0} = \epsilon(\mu - d\alpha, d\alpha) T_{\mu - d\alpha} \left( \prod_{i=1}^{d} T_{-d\alpha} \left( x^{-i} \otimes t^{d-i} \right) T_{-d\alpha} \right) T_{d\alpha} g_m v_{\Lambda_0}
\]

\[
= \epsilon(\mu - d\alpha, d\alpha) T_{\mu - d\alpha} \left( \prod_{i=1}^{d} x^{-i} \otimes t^{d-i} \right) g_m T_{d\alpha} v_{\Lambda_0}
\]

\[
(4.15)
\]

Using Lemma 4.6, the right hand side of the last equation becomes

\[
\epsilon(\mu - d\alpha, d\alpha) T_{\mu - d\alpha} \sum_{q} f_{g_m}^{q} \left( \prod_{i=1}^{d} x^{-i} \otimes t^{d-i} \right) T_{d\alpha} v_{\Lambda_0},
\]

for some polynomials \( f_{g_m}^{q} \) in \( \alpha^t, i \in I, j \in \mathbb{N} \), and positive integers \( \eta, q \), \( 1 \leq i \leq d \), depend on \( g \) such that \( m \geq \sum_{i=1}^{d} \eta_i q \).

Proposition 4.9. Let \( \lambda \in \mathbb{P}, k \in \mathbb{P}, \mu \in \mathbb{P}, k \in \mathbb{P} \), \( m \in \mathbb{Z}_{\geq 0} \), and \( s \in I \). Set

\[
\mu = \lambda + k\alpha_{1,s} - \sum_{s < i \leq j \leq r} d_{i,j} \alpha_{1,j}.
\]

Let \( g_m \) be a polynomial in \( \alpha^t, i \in I, j \in \mathbb{N} \), such that the weight of \( g_m v_{\Lambda_0} \) in \( L(\Lambda_0) \) is \( \Lambda_0 - m\delta \).

Then for every \( s < q \leq r + 1 \), we have

\[
\left( \prod_{j=q}^{r} x^{-j} (d_{s,j}, d_{s,j} + \delta_{1,s} k, \pi(j)^s) \right) T_{\mu} g_m v_{\Lambda_0}
\]

\[
= \left( \prod_{j=q}^{r} \left( \epsilon(\mu - \sum_{u=j}^{r} d_{s,u} \alpha_{s,u}, d_{s,j} \alpha_{s,j}) \right) T_{\mu - \sum_{j=q}^{r} d_{s,j} \alpha_{s,j}} f_{g_m}^{q} v_{\Lambda_0},
\]

\[
(4.16)
\]

where \( f_{g_m}^{q} \) is a polynomial in \( \alpha^t, i \in I, j \in \mathbb{N} \), depends only on \( g_m \) and the elements from the sets \( T_s(\mathbb{P}) \), \( q \leq j \leq r \), such that the weight of \( f_{g_m}^{q} v_{\Lambda_0} \) in \( L(\Lambda_0) \) is \( \Lambda_0 - (m + \sum_{j=q}^{r} d_{s}^{j}(\mathbb{P}))k \).

Proof. Proceed by induction on \( q \). In the case \( q = r + 1 \), by taking \( f_{g_m}^{r+1} = g_m \), both sides are equal to \( T_{\mu} g_m v_{\Lambda_0} \). Now suppose that \( q \leq r \). By the induction hypothesis, we have

\[
\left( \prod_{j=q+1}^{r} x^{-j} (d_{s,j}, d_{s,j} + \delta_{1,s} k, \pi(j)^s) \right) T_{\mu} g_m v_{\Lambda_0}
\]

\[
= \left( \prod_{j=q+1}^{r} \left( \epsilon(\mu - \sum_{u=j}^{r} d_{s,u} \alpha_{s,u}, d_{s,j} \alpha_{s,j}) \right) T_{\mu - \sum_{j=q+1}^{r} d_{s,j} \alpha_{s,j}} f_{g_m}^{q+1} v_{\Lambda_0},
\]

\[
(4.17)
\]

where \( f_{g_m}^{q+1} \) is a polynomial in \( \alpha^t, i \in I, j \in \mathbb{N} \), depends only on \( g_m \) and the elements from the sets \( T_s(\mathbb{P}) \), \( q < j \leq r \), such that the weight of \( f_{g_m}^{q+1} v_{\Lambda_0} \) in \( L(\Lambda_0) \) is \( \Lambda_0 - (m + \sum_{j=q+1}^{r} d_{s}^{j}(\mathbb{P}))k \).
Acting both sides of (4.17) with $x_{s,q}^{-}(d_{s,q}, d'_{s,q} + \delta_{1,s}k, \pi(q)^{s})$, we get

$$
\left( \prod_{j=q+1}^{r} \epsilon(\mu - \sum_{u=j}^{r} d_{s,u} \alpha_{s,u}, d_{s,j} \alpha_{s,j}) \right) \left( \prod_{j=q}^{r} x_{s,j}^{-}(d_{s,j}, d'_{s,j} + \delta_{1,s}k, \pi(j)^{s}) \right) T_{\mu} g_{m} \nu_{\Lambda_{0}}
$$

$$
= x_{s,q}^{-}(d_{s,q}, d'_{s,q} + \delta_{1,s}k, \pi(q)^{s}) T_{\mu - \sum_{j=q+1}^{r} d_{s,j} \alpha_{s,j}} f_{g_{m}}^{-1} \nu_{\Lambda_{0}}.
$$

Set $\nu = \mu - \sum_{j=q}^{r} d_{s,j} \alpha_{s,j}$. We observe that

$$
(\nu|\alpha_{s,q}) = (\lambda_{s}^{r+1} - \lambda_{q+1}^{r+1}) + \delta_{1,s}k + \sum_{j=q+1}^{r} d_{q+1,j} - \sum_{i=s+1}^{q} d_{i,q} - \sum_{j=q}^{r} d_{s,j} - d_{s,q}
$$

$$
= (\lambda_{s}^{r+1} - \lambda_{q+1}^{r+1}) + \delta_{1,s}k + (\lambda_{q+1}^{r+1} - \lambda_{q+1}^{q+1}) - (\lambda_{s}^{q+1} - \sum_{i=s}^{q} d'_{i,q} - \lambda_{q+1}^{q+1}) - (\lambda_{s}^{r+1} - \lambda_{s}^{q}) - d_{s,q}
$$

$$
= \sum_{i=s}^{q} d'_{i,q} + \delta_{1,s}k - d_{s,q}.
$$

Using Proposition 4.3 and (4.19), the right hand side of (4.18) becomes

$$
e(\nu, d_{s,q} \alpha_{s,q}) T_{\nu} \left( \prod_{p=0}^{d_{s,q}} T_{-\nu} \left( x_{s,q}^{-} \otimes t^{d'_{s,q} + \delta_{1,s}k - \pi(q)^{s}} \right) T_{\nu} \right) T_{d_{s,q} \alpha_{s,q}} f_{g_{m}}^{-1} \nu_{\Lambda_{0}}
$$

$$
= e(\nu, d_{s,q} \alpha_{s,q}) T_{\nu} \left( \prod_{p=0}^{d_{s,q}} \left( x_{s,q}^{-} \otimes t^{d'_{s,q} - \pi(q)^{s} - \sum_{i=s+1}^{q} d'_{i,q}} \right) \right) T_{d_{s,q} \alpha_{s,q}} f_{g_{m}}^{-1} \nu_{\Lambda_{0}}.
$$

From Theorem 2.3, it is easy to see that

$$
\left( \prod_{p=0}^{d_{s,q}} \left( x_{s,q}^{-} \otimes t^{d'_{s,q} - \pi(q)^{s} - \sum_{i=s+1}^{q} d'_{i,q}} \right) \right) T_{d_{s,q} \alpha_{s,q}} f_{g_{m}}^{-1} \nu_{\Lambda_{0}} = f_{g_{m}}^{q} \nu_{\Lambda_{0}},
$$

where $f_{g_{m}}^{q}$ is a polynomial in $\alpha_{i,j} t^{-j}$, $i \in I, j \in \mathbb{N}$, depends only on $f_{g_{m}}^{q+1}$ and the elements from the set $T^{q}_{s}(\mathcal{P})$ such that the weight of $f_{g_{m}}^{q} \nu_{\Lambda_{0}}$ in $L(\Lambda_{0})$ is

$$
\Lambda_{0} - \left( m + \sum_{j=q+1}^{r} d_{s,q}^{j}(\mathcal{P}) + d_{s,q}^{2} - d_{s,q}(d_{s,q} - \sum_{i=s+1}^{q} d'_{i,q}) + |\pi(q)^{s}| \right) \delta = \Lambda_{0} - \left( m + \sum_{j=q}^{r} d_{s,q}^{j}(\mathcal{P}) \right) \delta.
$$

Substituting (4.21) into (4.20), we get the result. \qed

4.7. Proof of Theorem 4.4. Proceed by induction on $s$. In the case $s = r+1$, by taking $f_{I(P_{r+1})} = 1$, both sides of (4.12) are equal to $T_{\lambda+k\theta} \nu_{\Lambda_{0}}$. Now suppose that $s \leq r$. Set

$$
\mu = \lambda + k \alpha_{1,s} - \sum_{s<i,j\leq r} d_{i,j} \alpha_{i,j}.
$$
By the induction hypothesis, we have

$$\rho_{\Psi_{s+1}} T_{\lambda+k\theta} v_{\Lambda_0} = \epsilon_{\Psi_{s+1}} T_{\mu} f_{\mathcal{I}(\Psi_{s+1})} v_{\Lambda_0},$$

(4.22)

where $f_{\mathcal{I}(\Psi_{s+1})}$ is a polynomial in $\alpha_i t^{-j}, i \in I, j \in \mathbb{N}$, depends only on the elements from the set $\mathcal{I}(\Psi_{s+1})$ such that the weight of $f_{\mathcal{I}(\Psi_{s+1})} v_{\Lambda_0}$ in $L(\Lambda_0)$ is $\Lambda_0 - d(\Psi_{s+1}) \delta$. Since

$$\rho_{\Psi_{s}} = \prod_{j=s}^{r} x_{s,j} (d_{s,j} + \delta_{s,j} k, d'_{s,j} + \delta_{1,s} k, \pi(j)^s) \rho_{\Psi_{s+1}},$$

we get from (4.22) that

$$\rho_{\Psi_{s}} T_{\lambda+k\theta} v_{\Lambda_0} = \epsilon_{\Psi_{s+1}} \left( \prod_{j=s}^{r} x_{s,j} (d_{s,j} + \delta_{s,j} k, d'_{s,j} + \delta_{1,s} k, \pi(j)^s) \right) T_{\mu} f_{\mathcal{I}(\Psi_{s+1})} v_{\Lambda_0}.$$

Now using Proposition 4.9 with $q = s + 1$, we get

$$\epsilon_{\Psi_{s+1}} \left( \prod_{j=s}^{r} \epsilon(\mu - \sum_{u=j}^{r} d_{s,u} \alpha_{s,u}, d_{s,j} \alpha_{s,j}) \right) \rho_{\Psi_{s}} T_{\lambda+k\theta} v_{\Lambda_0}$$

$$= x_{s,s} (d_{s,s} + k, d'_{s,s} + \delta_{1,s} k, \pi(s)^s) T_{\mu} - \sum_{j=s+1}^{r} d_{s,j} \alpha_{s,j} f_{s+1} v_{\Lambda_0},$$

(4.23)

where $f_{s+1}$ is a polynomial in $\alpha_i t^{-j}, i \in I, j \in \mathbb{N}$, depends only on the elements from the sets $\mathcal{I}(\Psi_{s+1})$ and $\mathcal{I}_d(\Psi)$, $s < j \leq r$, such that the weight of $f_{s+1} v_{\Lambda_0}$ in $L(\Lambda_0)$ is $\Lambda_0 - (d(\Psi_{s+1}) + \sum_{j=s+1}^{r} d'(\Psi_j)) \delta$. Since

$$d'_{s,s} + \delta_{1,s} k - \pi(s)_i^s - (\mu - \sum_{j=s+1}^{r} d_{s,j} \alpha_{s,j} - (d_{s,s} + k) \alpha_{s,s})$$

$$= d'_{s,s} + \delta_{1,s} k - \pi(s)_i^s - (\lambda_{s+1}^{r+1} - \lambda_{s+1}^{r+1} + k + \delta_{1,s} k + \sum_{j=s+1}^{r} d_{s+1,j} - \sum_{j=s+1}^{r} d_{s,j} - 2(d_{s,s} + k))$$

$$= d'_{s,s} - \pi(s)_i^s - (\lambda_{s+1}^{r+1} - \lambda_{s+1}^{r+1} - k - d_{s,s} + \sum_{j=s+1}^{r} d_{s+1,j} - \sum_{j=s}^{r} d_{s,j})$$

$$= d_{s,s} + k - \pi(s)_i^s,$$

for all $1 \leq i \leq s$, and

$$d_{s,s} \geq d(\Psi_s) = |\pi(s)^s| + d(\Psi_{s+1}) + \sum_{j=s+1}^{r} d'_j(\Psi),$$

we get the result from (4.23) by using Proposition 4.8 and (2.11).
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