DISTRIBUTION OF MOMENTS OF HURWITZ CLASS NUMBERS IN ARITHMETIC PROGRESSIONS AND HOLOMORPHIC PROJECTION

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ABSTRACT. In this paper, we study moments of Hurwitz class numbers associated to imaginary quadratic orders restricted into fixed arithmetic progressions. In particular, we fix \( t \) in an arithmetic progression \( t \equiv m \pmod{M} \) and consider the ratio of the \( 2k \)-th moment to the zeroeth moment for \( H(4n - t^2) \) as one varies \( n \). The special case \( n = p^r \) yields as a consequence asymptotic formulas for moments of the trace \( t \equiv m \pmod{M} \) of Frobenius on elliptic curves over finite fields with \( p^r \) elements.

1. Introduction and statement of results

Class numbers of imaginary quadratic orders and the corresponding class numbers of integral binary quadratic forms have a long and storied history going back at least to Gauss, who asked for a classification of all binary quadratic forms with a given class number. Dirichlet’s class number formula relates the class numbers with \( L \)-functions, another central area of study, and the growth of class numbers is in turn closely related to the generalized Riemann hypothesis; see [5] for a detailed history and an introduction to the development of the subject.

There are many identities and relations between sums of class numbers and interesting arithmetic functions. To state one such interesting identity that motivates our study in this paper, let \( \mathcal{Q}_D \) denote the set of integral binary quadratic forms of discriminant \( D < 0 \). The \( |D| \)-th Hurwitz class number is defined by

\[
H(|D|) := \sum_{Q \in \mathcal{Q}_D/\text{SL}_2(\mathbb{Z})} \frac{1}{|\omega_Q|},
\]

where \( \omega_Q \) is half the size of the stabilizer group \( \Gamma_Q \) of \( Q \) in \( \text{SL}_2(\mathbb{Z}) \). By convention, we set \( H(0) := \frac{1}{12} \) and \( H(r) := 0 \) for \( r \notin \mathbb{N}_0 \) or \( r \equiv 1, 2 \pmod{4} \). For a prime \( p \), a famous identity relating sums of these class numbers with an elementary function is given by (see [18, p. 154])

\[
\sum_{t \in \mathbb{Z}} H(4p - t^2) = 2p.
\]

In [9], similar identities were studied in the case when \( t \) is restricted to a given arithmetic progression, leading to further investigation of such sums of class numbers. For example,
identities such as
\[
\sum_{t \in \mathbb{Z} \pmod{5}} H(4p - t^2) = \begin{cases} 
\frac{1}{2}(p+1) & \text{if } p \equiv 1, 2 \pmod{5}, \\
\frac{1}{2}(p-1) & \text{if } p \equiv 3 \pmod{5}, \\
\frac{5}{12}(p+1) & \text{if } p \equiv 4 \pmod{5}
\end{cases}
\]
were proven in [7] and [9]. In this vein, we further consider sums of moments of these Hurwitz class numbers defined by
\[
H_{\kappa,m,M}(n) := \sum_{t \in \mathbb{Z} \pmod{M}} t^\kappa H(4n - t^2).
\] (1.1)

In the following we drop the condition \( t \in \mathbb{Z} \) in the summation and throughout we omit \( \kappa \) from the notation whenever \( \kappa = 0 \). Moments of class numbers have occurred throughout the literature (for example, see [3, 6, 33, 47, 48, 49]). The main result of this paper is an asymptotic formula for the even moments in an arbitrary arithmetic progression, given in terms of the zeroeth moment.

**Theorem 1.1.** Let \( m, M, k \in \mathbb{N} \) be given. As \( n \to \infty \), we have
\[
\frac{H_{2k,m,M}(n)}{n^k H_{m,M}(n)} = C_k + O_{k,M,\varepsilon} \left( n^{-\frac{1}{k}+\varepsilon} \right).
\]

Due to a well-known relation between class numbers and the trace of Frobenius of elliptic curves from the influential works of Deuring [15], Birch [6], and Schoof [40], Theorem 1.1 yields a corollary about moments related to the trace of Frobenius on elliptic curves over finite fields. For an elliptic curve \( E \) defined over the finite field \( \mathbb{F}_{p^r} \) with \( p^r \) elements (\( p \) prime, \( r \in \mathbb{N} \)), the trace of Frobenius is given by
\[
\text{tr}(E) = \text{tr}_{p^r}(E) := p^r + 1 - \# E(\mathbb{F}_{p^r}).
\]

Here \( E(\mathbb{F}_{p^r}) \) is the set of points on the elliptic curve over the finite field \( \mathbb{F}_{p^r} \). Our restriction on \( t \) in (1.1) corresponds to restricting the elliptic curve \( E/\mathbb{F}_{p^r} \) to the set
\[
E_{m,M,p^r} := \{ E/\mathbb{F}_{p^r} : \text{tr}(E) \equiv m \pmod{M} \}.
\]

The weighted \( \kappa \)-th moment with respect to \( \text{tr}(E) \) (for \( \kappa \in \mathbb{N}_0 \)) is then given by
\[
S_{\kappa,m,M}(p^r) := \sum_{E \in E_{m,M,p^r}} \frac{\text{tr}(E)^\kappa}{\# \text{Aut}_{\mathbb{F}_{p^r}}(E)} = \sum_{E \in E_{m,M,p^r}} \frac{\text{tr}(E)^\kappa}{\# \text{Aut}_{\mathbb{F}_{p^r}}(E)}.
\] (1.2)

Before stating our first result, we discuss how \( S_{\kappa,m,M}(p^r) \) is expected to grow as \( p^r \to \infty \). For simplicity, we consider the case \( n = p \) prime in this heuristic argument. By the Hasse bound [25], conjectured by Artin [2] in his thesis, we have that \( |\text{tr}(E)| \leq 2\sqrt{p} \). Taking \(-1 \leq a \leq b \leq 1\) and a fixed elliptic curve \( E \) over \( \mathbb{Q} \) and considering the reduction of \( E \) to elliptic curves over finite fields, it was independently conjectured by Sato and Tate (see e.g. [23]) that if \( E \) does not have complex multiplication, then
\[
\lim_{N \to \infty} \frac{\#{\{p \leq N : 2a\sqrt{p} \leq \text{tr}(E) \leq 2b\sqrt{p}\}}}{\#{\{p \leq N\}}} = \frac{2}{\pi} \int_a^b \sqrt{1-x^2}dx.
\]
This was later proven in a series of collaborations between Barnet-Lamb, Clozel, Geraghty, Harris, Shepherd-Barron, and Taylor [4, 11, 24]. Motivated by the Sato–Tate conjecture,
Birch instead fixed a prime $p$ and varied $E/\mathbb{F}_p$, leading to an investigation of the sums $S_{\kappa,0,1}(p)$. If there is no cancellation (this holds automatically for $\kappa$ even because every term in (1.2) is non-negative), then one expects that for some constant $D_\kappa$

$$S_{\kappa,m,M}(p^r) \sim D_\kappa \sum_{E/\mathbb{F}_{p^r}} \frac{p^{\frac{r\kappa}{2}}}{\# \text{Aut}_{\mathbb{F}_{p^r}}(E)} = D_\kappa p^{\frac{r\kappa}{2}} S_{m,M}(p^r).$$

Birch [6] proved that these constants $D_\kappa$ exist for $M = 1$ and $r = 1$ and moreover showed that they agree with the moments of the Sato–Tate distribution, proving the Sato–Tate conjecture for this family of elliptic curves. This work was influential, leading to a number of generalizations (for example, see [1, 8, 13, 17, 20, 21, 30, 35, 45]).

In an orthogonal direction, Castryck and Hubrechts [10] studied the probability that $t$ lies in a given arithmetic progression, which is closely related to $S_{m,M}(p^r)$. Motivated by [10] and [6], we generalize Birch’s result by restricting the family of elliptic curves to $E_{m,M,p^r}$.

Specifically, we show that for $\kappa = 2k \in 2\mathbb{N}$ the constant $D_{2k}$ does indeed exist and equals the $k$-th Catalan number $C_k$ if $p$ is fixed and $r \to \infty$ or if $r \leq 2$ is fixed and $p \to \infty$.

**Theorem 1.2.** Let $m \in \mathbb{Z}$, $M \in \mathbb{N}$ and $\varepsilon > 0$ be given.

1. For primes $p \to \infty$, we have

$$\frac{S_{2k,m,M}(p)}{p^k S_{m,M}(p)} = C_k + O_{k,M,\varepsilon} \left( p^{-\frac{1}{2}+\varepsilon} \right), \quad \frac{S_{2k,m,M}(p^r)}{p^{rk} S_{m,M}(p^r)} = C_k + O_{k,M,r,\varepsilon} \left( p^{-1+\varepsilon} \right) \quad (r \geq 2).$$

2. Let $p > 3$ be a prime for which $p \nmid \gcd(m,M)$ and $k \in \mathbb{N}$. As $r \to \infty$, we have

$$\frac{S_{2k,m,M}(p^r)}{p^{rk} S_{m,M}(p^r)} = C_k + O_{k,p,M,\varepsilon} \left( p^{-\frac{1}{2}+\varepsilon} r \right).$$

**Remarks.**

1. For $M = 1$, these sums were studied by Birch [6] and implicitly appear in the work of Ihara [27] (see also [28, Theorem 1, Theorem 2]). They obtained a formula for these sums in terms of the trace of Hecke operators that yields the asymptotic obtained in Theorem 1.2. For $M = 2$, formulas for $S_{2k,m,2}$ were obtained by Kaplan and Petrow (see for example [29, Theorem 8]).

2. The implied constant in the error term is ineffective due to an ineffective lower bound in Lemma 3.7 that uses Siegel’s ineffective lower bound [43] for the class numbers of imaginary quadratic fields (see Lemma 2.4 below). Using Littlewood’s conditional effective bound for the class numbers [34], it can be made effective under the Generalized Riemann Hypothesis. Moreover, for fixed $M$ one should in principle be able to obtain an effective version of Lemma 3.7 by computing the Eisenstein series components of certain modular forms; this was carried out for primes $M \leq 7$ in [7, Section 4] and [36, Corollary 7.3].

3. A number of papers have investigated sums where the elliptic curves are restricted to those elliptic curves containing specified subgroups (see [32, Section 6.2] and [28]). For $M$ squarefree, the special case $m = p^r + 1$ overlaps with such results because $\text{tr}(E) \equiv p^r + 1 \pmod{M}$ is equivalent to $M \mid \#E(\mathbb{F}_{p^r})$, which in turn implies that the set of $M$-torsion points

$$E[M] := \{ P \in E : \text{ord}(P) \mid M \}$$
is non-empty. Here \( \text{ord}(P) \) means the order of the point under the group law defined on elliptic curves. To state the special case of Theorem 1.2 in this case, we denote the subset of torsion points of precise order \( M \) by
\[
E^*[M] := \{ P \in E : \text{ord}(P) = M \}
\]
and define
\[
S^*_k,M(p^r) := \sum_{\substack{E/F \mapsto E^*[M] \neq \emptyset \\in E/F \mapsto E^*[M] \neq \emptyset}} \frac{\text{tr}(E)^{\kappa}}{\# \text{Aut}_F(p^r)(E)}.
\]
Then Theorem 1.2 implies the following.

1. As \( p \to \infty \), we have
\[
S^*_{2k,M}(p) = C_k + O_{k,M}(p^{1/2 + \varepsilon}), \quad S^*_{2k,M}(p^r) = C_k + O_{k,M,r}(p^{1/2 + \varepsilon}) (r \geq 2).
\]

2. If \( p > 3 \) is a prime, then as \( r \to \infty \) we have
\[
S^*_{2k,M}(p^r) = C_k + O_{k,p,M}(p^{3/2 + \varepsilon}) (r \geq 2).
\]

After some preliminary setup in Section 2, we begin by investigating Hurwitz class numbers and then the moments in Section 3, proving Theorem 1.1. We then return to the application of these moments to elliptic curves in Section 4, proving Theorem 1.2.

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2. Preliminaries

2.1. Holomorphic and non-holomorphic modular forms. We give a brief overview of the theory of modular forms here; for details, see [31, 38]. For \( d \) odd, we set
\[
\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases}
\]
Let \( \Gamma \) be a congruence subgroup containing \( T := \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \) and if \( \kappa \in \mathbb{Z} \), then we also require that \( \Gamma \subseteq \Gamma_0(4) \). A function \( F : \mathbb{H} \to \mathbb{C} \) satisfies modularity of weight \( \kappa \in \frac{1}{2} \mathbb{Z} \) on \( \Gamma \subseteq \text{SL}_2(\mathbb{Z}) \) with character \( \chi \) if for every \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \) we have
\[
F|\kappa \gamma = \chi(d)F.
\]
Here the weight \( \kappa \) slash operator is defined by
\[
F|\kappa \gamma(\tau) := \left( \frac{c}{d} \right)^{2\kappa} \varepsilon_d^{2\kappa}(c\tau + d)^{-\kappa} F(\gamma \tau),
\]
where \( \left( \cdot \right) \) denotes the extended Legendre symbol. We call \( F \) a (holomorphic) modular form if \( F \) is holomorphic on \( \mathbb{H} \) and \( F(\tau) \) grows at most polynomially in \( v \) as \( \tau = u + iv \to \mathbb{Q} \cup \{ i\infty \} \).

To define certain non-holomorphic modular forms, let \( \Delta_\kappa := -v^2(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}) + i\kappa v(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}) \) be the weight \( \kappa \) hyperbolic Laplace operator. A smooth function \( F \) transforming modular of
weight $\kappa$ is a harmonic Maass form of weight $\kappa$ if $\Delta_\kappa(F) = 0$ and there exists $a \in \mathbb{R}$ such that

$$F(\tau) = O\left(e^{a|\tau|}\right)$$
as $\tau \rightarrow \infty$ and

$$F(u + iv) = O\left(e^{\frac{\pi}{v}}\right)$$
for $u \in \mathbb{Q}$ as $v \to 0^+$.

If $F$ is moreover holomorphic on $\mathbb{H}$, then we call $F$ a weakly holomorphic modular form. For a harmonic Mass form $F$ of weight $\kappa$, $\xi_\kappa(F)$ with $\xi_\kappa := 2i\nu^\kappa \frac{\partial}{\partial \tau}$ is a weakly holomorphic modular form of weight $2 - \kappa$.

Suppose that $\kappa \neq 1$. Letting $\Gamma(\alpha, x)$ denote the incomplete gamma function, a harmonic Maass form of weight $\kappa$ on $\Gamma$ has a Fourier expansion of the form

$$F(\tau) = F^+(\tau) + F^-(\tau)$$

with (for $q := e^{2\pi i \tau}$)

$$F^+(\tau) = \sum_{n \gg -\infty} c^+_F(n)q^n$$

$$F^-(\tau) = c^-_F(0)v^{1-\kappa} + \sum_{0 \neq n < \infty} c^-_F(n)\Gamma(1 - \kappa, -4\pi nu)q^n.$$  

Another type of non-holomorphic modular form that naturally occurs is an almost holomorphic modular form, which is a function $F : \mathbb{H} \to \mathbb{C}$ satisfying weight $\kappa$ modularity on $\Gamma$ for which there exist holomorphic functions $F_j$ $(0 \leq j \leq \ell)$ such that $F(\tau) = \sum_{j=0}^{\ell} F_j(\tau)v^{-j}$. We call $F_0$ a quasimodular form.

There are natural operators that preserve modularity. In particular, suppose that $F(\tau) = \sum_{n \geq n_0} c_{F,v}(n)q^n$ satisfies weight $\kappa$ modularity with Nebentypus character $\chi$ (of modulus $N$) on $\Gamma_0(N) \cap \Gamma_1(M)$ with $M \mid N$. We have that

$$F|_V_0(\tau) := F(\delta \tau)$$

satisfies weight $\kappa$ modularity on $\Gamma_0(\text{lcm}(4, \delta N)) \cap \Gamma_1(M)$ with Nebentypus $\chi \cdot (\frac{\delta}{\cdot})^k$, and

$$F|_U_0(\tau) := \sum_{n \geq n_0} c_{F,\frac{\cdot}{\cdot}}(\delta n)q^n$$

satisfies weight $\kappa$ modularity on $\Gamma_0(\text{lcm}(4, N, \delta)) \cap \Gamma_1(M)$ with Nebentypus $\chi \cdot (\frac{\delta}{\cdot})^k$.

2.2. Rankin-Cohen brackets. For $F_1$, $F_2$ transforming like modular forms of weight $\kappa_1, \kappa_2 \in \frac{1}{2}\mathbb{Z}$, respectively, define for $k \in \mathbb{N}_0$ the $k$-th Rankin-Cohen bracket

$$[F_1, F_2]_k := \frac{1}{(2\pi i)^k} \sum_{j=0}^{k} (-1)^j \binom{\kappa_1 + k - 1}{k - j} \binom{\kappa_2 + k - 1}{j} F_1^{(j)} F_2^{(k-j)}$$

with $\binom{\alpha}{j} := \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)}$. Then $[F_1, F_2]_k$ transforms modular form of weight $\kappa_1 + \kappa_2 + 2k$.

2.3. Elliptic curves and trace of Frobenius. A good introduction to elliptic curves is [44]. For an elliptic curve $E$ defined over $\mathbb{F}_r$, we define the Frobenius endomorphism $\text{Fr}$ from $E$ to itself via the $p$-th power map. Namely, for a point $P = (X, Y) \in E$, we set

$$\text{Fr}(P) := (X^p, Y^p).$$

The trace of Frobenius is given by $\text{tr}(E)$. For $r = 1$, Hasse [25] showed that

$$|\text{tr}(E)| \leq 2\sqrt{p}.$$
The distribution of $\frac{1}{\sqrt{p}} \tr(E)$ has been well-studied and it is natural to group those elliptic curves whose trace of Frobenius agree. For $t \in \mathbb{Z}$, we hence define

$$\mathcal{E}_{p^r,t} := \{ E / \mathbb{F}_{p^r} : \tr(E) = t \}.$$ 

As is well-known, there is a group law defined on elliptic curves and the automorphisms of the group we denote by $\text{Aut}_{\mathbb{F}_{p^r}}(E)$. The automorphism group gives a natural weighting on the elliptic curves in $\mathcal{E}_{p^r,t}$, leading to the definition

$$N_A(p^r; t) := \sum_{E \in \mathcal{E}_{p^r,t}} \frac{1}{\# \text{Aut}_{\mathbb{F}_{p^r}}(E)}.$$ 

These sums naturally occur when investigating $S_{\kappa,m,M}(p^r)$ because

$$S_{\kappa,m,M}(p^r) = \sum_{t \equiv m \pmod{M}} t^\kappa N_A(p^r; t).$$

### 2.4. The class number generating function.

Let $H(\tau) := \sum_{n \in \mathbb{Z}} H(n) q^n$ be the generating function for the Hurwitz class numbers. Its modular properties follow by [26, Theorem 2].

**Theorem 2.1.** The function

$$\widehat{H}(\tau) := H(\tau) + \frac{1}{8\pi \sqrt{v}} + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n \Gamma \left( \frac{1}{2}, 4\pi n^2 v \right) q^{-n^2}$$

is a harmonic Maass form of weight $\frac{3}{2}$ on $\Gamma_0(4)$.

### 2.5. Elliptic curves and class numbers.

For $m \in \mathbb{Z}$, $n, M \in \mathbb{N}$, and $\kappa \in \mathbb{N}_0$, we next relate $S_{m,M,\kappa}(n)$ to certain sums of Hurwitz class numbers given by

$$\mathcal{H}_{\kappa,m,M}(p^r; n) := \sum_{t \equiv m \pmod{M}, p^r \nmid t} t^\kappa H(4n - t^2).$$

We are mostly interested in the case $n = p^r$ with $r \in \mathbb{N}$, which we abbreviate by $\mathcal{H}_{\kappa,m,M}(p^r) := \mathcal{H}_{\kappa,m,M}(p^r; p^r)$. To state the result, we set

$$E_{\kappa,m,M}(p^r) := \delta_{M|m} \delta_{\kappa=0} \delta_{2|r} H(4p) + \delta_{M|m} \delta_{\kappa=0} \delta_{2|r} \frac{1}{2} \left( 1 - \left( -\frac{1}{p} \right) \right) + \frac{1}{3} \left( 1 - \left( -\frac{3}{p} \right) \right) p^\frac{r\kappa}{2} \varrho_{\kappa,m,M}(p^r) + \frac{1}{3} (p-1) 2^{\kappa-2} p^{\frac{r}{2} \kappa} \sigma_{\kappa,m,M}(p^r)$$

with

$$\varrho_{\kappa,m,M}(p^r) := \sum_{t \equiv m \pmod{M}, t^2 \neq p^r} \text{sgn}(t)^\kappa, \quad \sigma_{\kappa,m,M}(p^r) := \sum_{t \equiv m \pmod{M}, t^2 = 4p^r} \text{sgn}(t)^\kappa.$$
Here and throughout $\delta_S := 1$ if a statement $S$ is true and $\delta_S := 0$ otherwise. We note that if $\kappa \in 2\mathbb{N}_0$, then $\varrho_{\kappa,m,M}(p^r) = \varrho_{m,M}(p^r)$, where

$$
\varrho_{m,M}(p^r) := \# \{ t = \pm 2p^\frac{r}{2} \in \mathbb{Z} : t \equiv m \pmod{M} \},
$$

$$
\sigma_{m,M}(p^r) := \# \{ t = \pm p^\frac{r}{2} \in \mathbb{Z} : t \equiv m \pmod{M} \}.
$$

**Lemma 2.2.** For a prime $p > 3$, $\kappa \in \mathbb{N}_0$, and $r \in \mathbb{N}$ we have

$$
2S_{\kappa,m,M}(p^r) = \mathcal{H}_{\kappa,m,M}(p^r) + E_{\kappa,m,M}(p^r).
$$

*Proof.* The claim easily follows, using that by [29, Theorem 3], for a prime $p > 3$ and $r \in \mathbb{N}$ we have

$$
2N_A(p^r; t) = \begin{cases} 
H(4p^r - t^2) & \text{if } t^2 < 4p^r, \ p \nmid t, \\
H(4p) & \text{if } t = 0 \text{ and } r \text{ is odd,} \\
\frac{1}{2} \left(1 - \left(\frac{-1}{p}\right)\right) & \text{if } t = 0 \text{ and } r \text{ is even,} \\
\frac{1}{3} \left(1 - \left(\frac{-3}{p}\right)\right) & \text{if } t^2 = p^r, \\
\frac{1}{12} (p - 1) & \text{if } t^2 = 4p^r, \\
0 & \text{otherwise.}
\end{cases}
$$

The sums $\mathcal{H}_{\kappa,m,M}(p; n)$ are related to $H_{\kappa,m,M}(n)$, as a direct calculation shows.

**Lemma 2.3.** For $m \in \mathbb{Z}$, $M \in \mathbb{N}$, $p$ prime, and $\kappa \in \mathbb{N}_0$, we have

$$
\mathcal{H}_{\kappa,m,M}(p; n) = \sum_{\ell \pmod{p}} H_{\kappa,m+M\ell,Mp}(n).
$$

2.6. Generating functions for sums of moments of class numbers. Taking the generating function of (1.1), for $m \in \mathbb{Z}$, $M \in \mathbb{N}$, and $\kappa \in \mathbb{N}_0$, we study sums of the type

$$
\mathcal{H}_{\kappa,m,M}(\tau) := \sum_{n=0}^{\infty} H_{\kappa,m,M}(n) q^n = \sum_{n=0}^{\infty} \sum_{t \equiv m \pmod{M}} t^\kappa H(4n - t^2) q^n.
$$

We directly see that

$$
\mathcal{H}_{\kappa,m,M} = (\mathcal{H}_{\theta_{\kappa,m,M}}) \mid_{U_4},
$$

where

$$
\theta_{\kappa,m,M}(\tau) := \sum_{n \equiv m \pmod{M}} n^\kappa q^n^2.
$$

2.7. Properties of class numbers. We use the following bounds that follow from results of Siegel [43].

**Lemma 2.4.** For a discriminant $D < 0$ and $\varepsilon > 0$, we have

$$
|D|^\frac{1}{2} - \varepsilon \ll \varepsilon H(|D|) \ll \varepsilon |D|^\frac{1}{2} + \varepsilon.
$$

If $D = \Delta f^2$ with $\Delta < 0$ a fundamental discriminant and $f \in \mathbb{N}$, then (see [12, p. 273])

$$
H(|\Delta|f^2) = H(|\Delta|) \sum_{d \mid f} \mu(d) \left(\frac{\Delta}{d}\right) \sigma \left(\frac{f}{d}\right),
$$

(2.1)
where $\sigma(n) := \sum_{d|n} d$. Setting $D_p := \frac{D}{p^{2\alpha}}$ for an odd prime $p$ such that $2\alpha \leq \text{ord}_p(D) \leq 2\alpha + 1$, this leads to the following useful relation.

**Lemma 2.5.** If $D < 0$ is a discriminant and $p$ is an odd prime, then

$$H(\lfloor D\rfloor p^2) = pH(\lfloor D\rfloor) + \left(1 - \left(\frac{D_p}{p}\right)\right) H(D_p).$$

*Proof.* Let $D = \Delta f^2$ with $\Delta < 0$ a fundamental discriminant and $f \in \mathbb{N}$. Since the sum in (2.1) is multiplicative, if $f = p^\alpha g$ with $\alpha \in \mathbb{N}_0$ and $p \nmid g$, then we can write

$$H(\lfloor D\rfloor) = H(\lfloor \Delta |g|^2\rfloor) \left(\sigma(p^\alpha) - \delta_{\alpha \geq 1} \left(\frac{\Delta}{p}\right) \sigma(p^{\alpha - 1})\right),$$

(2.2)

$$H(\lfloor D\rfloor p^2) = H(\lfloor \Delta |g|^2\rfloor) \left(\sigma(p^{\alpha + 1}) - \left(\frac{\Delta}{p}\right) \sigma(p^\alpha)\right).$$

(2.3)

Note that for $\ell \geq 0$ we have

$$\sigma(p^\ell) = 1 + \delta_{\ell \geq 1} p\sigma(p^{\ell - 1}).$$

Comparing (2.3) with (2.2) yields

$$H(\lfloor D\rfloor p^2) = pH(\lfloor D\rfloor) + \left(1 - \left(\frac{\Delta}{p}\right)\right) H(\lfloor \Delta |g|^2\rfloor).$$

Since $p \nmid g$, we have $(\Delta^2/p) = 1$ and hence $(\Delta/p) = (\Delta^2/p)$. The claim follows from the fact that $\Delta g^2 = D_p$. \hfill $\square$

### 2.8. Holomorphic projection.

Let $F(\tau) = \sum_{n \in \mathbb{Z}} c_{F,v}(n) q^n$ be a (not necessarily holomorphic) function satisfying weight $\kappa \geq 2$ modularity. Suppose furthermore that $F(\tau) - P_{1_\infty}(q^{-1})$ has moderate growth, where $P_{1_\infty} \in \mathbb{C}[v]$ and that a similar condition holds as $\tau \to Q$. We define (see [22, Proposition 5.1, p. 288] for the general statement and [37] for it written in this generality) the *holomorphic projection* of $F$

$$\pi_{\text{hol}}^{\text{reg}}(F)(\tau) := P_{1_\infty}(q^{-1}) + \sum_{n=1}^{\infty} c_F(n) q^n.$$

Here for $n \in \mathbb{N}$

$$c_F(n) := \frac{(4\pi n)^{\kappa-1}}{\Gamma(\kappa-1)} \lim_{s \to 0} \int_0^\infty c_{F,v}(n) v^{\kappa-2-s} e^{-4\pi n v} dv.$$

Note that for a non-holomorphic modular form $F$, there exists a unique cusp form $f$ satisfying $\langle F, g \rangle = \langle f, g \rangle$ for every cusp form $g$, where $\langle \cdot, \cdot \rangle$ denotes the Petersson inner product.

The function $f$ is Sturm’s [46] original definition for the holomorphic projection of $F$ and Gross and Zagier showed in [22, Proposition 5.1, p. 288] that Sturm’s definition matches the definition given here if one additionally assumes that $F$ decays polynomially towards all cusps. We have the following properties of $\pi_{\text{hol}}^{\text{reg}}(F)$ (see [22, Proposition 5.1 and Proposition 6.2] as well as [36, (4.6)]).

**Lemma 2.6.** Suppose that $F$ is continuous and transforms modular of weight $\kappa \geq 2$ on $\Gamma_1(N)$. Then the following hold.

1. If $F$ is holomorphic, then $\pi_{\text{hol}}^{\text{reg}}(F) = F$.  

8
(2) If $\kappa > 2$ and $F$ is bounded towards all cusps, then $\pi^{\text{reg}}_{\text{hol}}(F)$ is a holomorphic modular form. If $\kappa = 2$, then $\pi^{\text{reg}}_{\text{hol}}(F)$ is a quasimodular form of weight two.

(3) If $F_1$ is a weight $\kappa_1 \in \frac{1}{2}\mathbb{N}$ harmonic Maass form and $f_2$ is a weight $\kappa_2 \in \frac{1}{2}\mathbb{N}$ holomorphic modular form, and $k \in \mathbb{N}_0$ with $\kappa := \kappa_1 + \kappa_2 + 2k \geq 2$, then

$$
\pi^{\text{reg}}_{\text{hol}}([F_1, f_2]_k) = [F_1^+, f_2]_k + \pi^{\text{reg}}_{\text{hol}}([F_1^-, f_2]_k).
$$

**Remark.** Lemma 2.6 (3) appears in a slightly different form in [36, (4.6)] because Mertens did not include the constant term in the definition of $F^-$ in [36, (3.1)].

Rearranging Lemma 2.6 (3) yields a formula for $[F_1^+, f_2]_k$. In [36, Theorem 1.2], Mertens considered the special case that $\kappa_1 = \frac{3}{2}$, $\kappa_2 = \frac{1}{2}$, and $\xi_{\frac{3}{2}}(F_1)$ and $F_2$ are both of weight $\frac{1}{2}$ holomorphic modular forms on $\Gamma_1(N)$. Serre and Stark showed in [41, Theorem A] that this space is spanned by unary theta functions. For a character $\chi$ and $a \in \mathbb{N}$, these are given by $\theta_{\chi} | V_a$, where

$$
\theta_{\chi}(\tau) := \sum_{n \in \mathbb{Z}} \chi(n)q^n.
$$

Using Serre and Stark’s classification, one may assume without loss of generality that $F_2 = \theta_{\chi}|V_a$ and $\xi_{\frac{3}{2}}(F_1) = \theta_\psi|V_b$ for some characters $\chi, \psi$ and $a, b \in \mathbb{N}$. In the proof of [36, Theorem 1.2], Mertens related the second term on the right-hand side of Lemma 2.6 (3) to

$$
\Lambda_{\ell, a, b}^{\chi, \psi}(\tau) := 2 \sum_{n=1}^{\infty} \lambda_{\ell, a, b}(n)q^n,
$$

where $\lambda_{\ell, a, b}(n) := \sum_{\substack{\ell^2 - bs^2 = n \\ t \geq 1, s \geq 0}} \chi(t)\psi(s)\left(t\sqrt{a} - s\sqrt{b}\right)^\ell$.

where here and throughout $\sum^*$ means that the terms in the sum with $s = 0$ are weighted by $\frac{1}{2}$. Using the fact that $\pi^{\text{reg}}_{\text{hol}}([F_1, F_2]_k)$ is a quasimodular form by Lemma 2.6 (2), one then obtains the following.

**Lemma 2.7.** Suppose that $\chi$ and $\psi$ are characters of conductors $N_\chi$ and $N_\psi$, respectively, and $N, a, b \in \mathbb{N}$ with $bN_\psi^2 | N$. If $F$ is a harmonic Maass form of weight $\frac{3}{2}$ on $\Gamma_1(4N)$ that grows at most polynomially towards all cusps and satisfies $\xi_{\frac{3}{2}}(F) = \theta_\psi|V_b$, then

$$
\left([F^+, \theta_\chi|V_a]_k - 2^{3-2k}\pi\left(\frac{2k}{k}\right)\Lambda_{2k+1, a, b}^{\chi, \psi}\right)|U_4
$$

is a holomorphic cusp form of weight $2k+2$ on $\Gamma_1(\text{lcm}(4N, 4aN_\chi^2))$ if $k > 0$ and a quasimodular form of weight two if $k = 0$.

**Remark.** The statement of Lemma 2.7 corrects an error in [36, (5.2)] where the constant in front of the second term differs by a factor of $-2\sqrt{\pi}$.

### 3. Holomorphic Projection and the Proof of Theorem 1.1

In this section, we prove Theorem 1.1.
3.1. **Fourier coefficients of certain Rankin–Cohen brackets.** Define (compare with [36, (7.7)], although the notation is different there)

\[ G_{k,m,M}(n) := \sum_{t \equiv m \pmod{M}} p_{2k}(t, n) H \left( 4n - t^2 \right), \]

where \( p_{2k}(t, n) \) denotes the \((2k)\)-th coefficients in the Taylor expansion of \((1 - tX + nX^2)^{-1}\). These appear in the Fourier expansion of \([H, \theta_{m,M}]_k|U_4\) as a direct calculation using the results of Cohen [12] shows.

**Lemma 3.1.** The \(n\)-th Fourier coefficient of \([H, \theta_{m,M}]_k|U_4\) equals \(\frac{(2k)!}{2^k} G_{k,m,M}(n)\).

Using the explicit evaluation ([19, (3), page 29])

\[ p_{2k}(t, n) = \binom{2k}{k} (-1)^{k-m} \frac{1}{(2k-2\mu)!} n^\mu \]

we directly obtain the following lemma.

**Lemma 3.2.** For \(m \in \mathbb{Z}\) and \(k, M \in \mathbb{N}\), we have

\[ H_{2k,m,M}(n) = \frac{k!}{(2k)!} G_{k,m,M}(n) - \sum_{\mu=1}^{k} (-1)^{\mu} \frac{(2k-\mu)!}{\mu!(2k-2\mu)!} n^\mu H_{2k-2\mu,m,M}(n). \]

In order to investigate \(H_{2k,m,M}(n)\), it suffices by Lemma 3.1 and Lemma 3.2 to study the coefficients of \([H, \theta_{m,M}]_k|U_4\). In [36, Theorem 1.2] Mertens applied holomorphic projection on functions related to \([\hat{H}, \theta_{m,M}]_k|U_4\). To state his result (noted two paragraphs before [36, Proposition 7.2]), we set

\[ \Lambda_{\ell,m,M}(\tau) := \sum_{n=1}^{\infty} \lambda_{\ell,m,M}(n)q^n, \quad \text{where} \quad \lambda_{\ell,m,M}(n) := \sum_{\pm \sum_{t>s} \geq 0 \sum_{t \equiv \pm s \pmod{M}}} (t - s)^\ell. \]

**Lemma 3.3.** For \(k \in \mathbb{N}_0\), \(m \in \mathbb{Z}\), and \(M \in \mathbb{N}\), the function

\[ \left( [H, \theta_{m,M}]_k + 2^{-1-2k} \binom{2k}{k} \Lambda_{2k+1,m,M} \right)|U_4 \]

is a holomorphic cusp form of weight \(2 + 2k\) on \(\Gamma_0(4M^2) \cap \Gamma_1(M)\) (resp. \(\Gamma_0(4M^2)\)) if \(M \nmid m\) (resp. \(M \mid m\)) if \(k > 0\) and quasimodular on that group if \(k = 0\).

**Remark.** Lemma 3.3 was stated in a different form in [36]. Specifically, Mertens assumed that \(M\) is prime and an error appears in the constant in front of \(\Lambda_{2k+1,m,M}\) in [36]. The proof in [36] goes through without the assumption that \(M\) is prime, however.

**Proof of Lemma 3.3.** Using Lemma 2.7 and relating the functions \(\Lambda_{\ell,a,b}^{\chi,\psi}\) (resp. \([H, \theta\chi|V_a|)_k\)) to \(\Lambda_{\ell,m,M}\) (resp. \([H, \theta_{m,M}]_k\)) via orthogonality of characters yields the claim. \(\square\)

In light of Lemma 3.2, it is natural to recursively define \(C_0 := 1\) and

\[ C_k := -\sum_{\mu=1}^{k} (-1)^{\mu} \frac{(2k-\mu)!}{\mu!(2k-2\mu)!} C_{k-\mu}. \]
As we show in the next lemma, $C_k$ are the Catalan numbers

$$C_k := \frac{1}{k+1} \binom{2k}{k}.$$

**Lemma 3.4.** We have $C_k = C_k$.

**Proof.** For $k = 0$ the claim holds directly by definition. Assume inductively that the claim holds for all $\ell < k$. Then we have

$$C_k = -\sum_{\mu=1}^{k} (-1)^{\mu} \binom{2k-\mu}{\mu} C_{k-\mu}.$$

It remains to show that

$$-\sum_{\mu=1}^{k} (-1)^{\mu} \binom{2k-\mu}{\mu} C_{k-\mu} = \begin{cases} C_k & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases} \quad (3.1)$$

To show the claim, we take the generating function of the left-hand side of (3.1):

$$L(X) := -\sum_{k=1}^{\infty} \sum_{\mu=1}^{k} (-1)^{\mu} \binom{2k-\mu}{\mu} C_{k-\mu} X^k \quad (3.2)$$

$$= -\frac{1}{1+X} \sum_{k=0}^{\infty} C_k \left( \frac{X}{(1+X)^2} \right)^k + \sum_{k=0}^{\infty} C_k X^k,$$

using the binomial series expansion. We then recall the evaluation of the generating function (see [16, 26.5.2])

$$F(Z) := \sum_{k=0}^{\infty} C_k Z^k = \frac{1 - \sqrt{1 - 4Z}}{2Z},$$

valid for $|Z| < \frac{1}{4}$. Therefore, for $0 < X < \frac{1}{4}$, we obtain

$$F \left( \frac{X}{(1+X)^2} \right) = 1 + X.$$

Plugging this into the first sum in (3.2) yields $L(X) = \sum_{k=1}^{\infty} C_k X^k$. This gives the claim. \( \Box \)

### 3.2. Asymptotic growth of the moments.

In order to obtain the asymptotic growth of $G_{k,m,M}$ and $H_{2k,m,M}$, we require the following straightforward estimates.

**Lemma 3.5.** We have

$$\lambda_{\ell,m,M}(n) \leq n^{\frac{\ell}{2}} \lambda_{m,M}(n) \ll \varepsilon n^{\frac{\ell}{2}+\varepsilon}.$$

We are now ready to prove an asymptotic formula for $H_{2k,m,M}(n)$.

**Proposition 3.6.** We have

$$H_{2k,m,M}(n) = C_k n^k H_{m,M}(n) + O_{k,M,\varepsilon} \left( n^{k+\frac{1}{2}+\varepsilon} \right).$$
Proof. We argue by induction using Lemma 3.2. Since $C_0 = 1$, the claim holds trivially for $k = 0$. For $k \geq 1$, Lemma 3.1 and Lemma 3.3 imply that

$$G_{k,m,M}(n) + \frac{1}{2^{2k} \cdot k!} \lambda_{2k+1,m,M}(4n)$$

is the $n$-th coefficient of a weight $2k + 2$ cusp form. By Deligne’s bound [14] it thus may be bound against $O_{k,M,\varepsilon}(n^{k+\frac{1}{2}+\varepsilon})$. The implied constant in the error term a priori depends on $m$ as well, but by taking the maximum over all of the choices of $m \pmod{M}$, we may drop the dependence on $m$ throughout. Using Lemma 3.5, we obtain

$$G_{k,m,M}(n) \ll_{k,m,M,\varepsilon} n^{k+\frac{1}{2}+\varepsilon}.$$ 

Plugging this into Lemma 3.2, using the inductive hypothesis and the fact that $C_k$ satisfies the recurrence defining $C_k$ by Lemma 3.4, we obtain the claim.

$$\square$$

3.3. The main term. In this subsection we investigate the growth of $H_{m,M}(n)$.

Lemma 3.7. For $m, M \in \mathbb{N}$ fixed, we have, as $n \to \infty$,

$$n^{1-\varepsilon} \ll_{\varepsilon,M} H_{m,M}(n) \ll_{\varepsilon} n^{1+\varepsilon}.$$

Proof. We begin with the lower bound. Since for $4n-t^2 \neq 0$, $H(4n-t^2) \geq 0$ and since $H(0) = -\frac{1}{12}$, we obtain

$$H_{m,M}(n) \geq \sum_{t \equiv m \pmod{M}} \frac{H(4n-t^2)}{t \leq \sqrt{n}} - \frac{1}{6}. \quad (3.3)$$

Using the lower bound in Lemma 2.4, we obtain

$$\sum_{t \equiv m \pmod{M}} \frac{H(4n-t^2)}{t \leq \sqrt{n}} \gg_{\varepsilon} \sum_{t \equiv m \pmod{M}} \frac{(4n-t^2)^{\frac{1}{2}-\varepsilon}}{t \leq \sqrt{n}} \gg_{\varepsilon,M} n^{1-\varepsilon}.$$ 

Plugging this into (3.3) yields the lower bound.

For the upper bound, we again use the fact that every term is non-negative except $4n-t^2 = 0$ to bound

$$H_{m,M}(n) \leq \sum_{|t| < 2\sqrt{n}} \frac{H(4n-t^2)}{t \leq \sqrt{n}}.$$

The upper bound in Lemma 2.4 then yields

$$\sum_{|t| < 2\sqrt{n}} \frac{H(4n-t^2)}{t \leq \sqrt{n}} \ll_{\varepsilon} \sum_{|t| < 2\sqrt{n}} (4n-t^2)^{\frac{1}{2}+\varepsilon} \ll_{\varepsilon} \sum_{|t| < 2\sqrt{n}} n^{\frac{3}{2}+\varepsilon} \ll n^{1+\varepsilon}. \quad \square$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 3.6, we have

$$H_{2k,m,M}(n) = C_k + O_{k,M,\varepsilon} \left( \frac{n^{\frac{1}{2}+\varepsilon}}{H_{m,M}(n)} \right).$$

The claim now follows by using the lower bound in Lemma 3.7 to bound the error term. $\square$
4. Proof of Theorem 1.2

The next lemma relates $\mathcal{H}_{2k,m,M}(p^r)$ to linear combinations of $H_{2k,m,M}(p^j)$ with $0 \leq j \leq r$.

**Lemma 4.1.** Suppose that $p > 3$, $m \in \mathbb{Z}$, $M \in \mathbb{N}$, and $k \in \mathbb{N}_0$.

1. If $r \leq 1$ or both $p \mid M$ and $p 
\not\mid m$, then

   $\mathcal{H}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) - \delta_{M|m} \delta_{k=0} H(4p)$.

2. If $p \nmid M$ and $r \geq 2$, then

   
   \begin{align*}
   \mathcal{H}_{2k,m,M}(p^r) &= H_{2k,m,M}(p^r) - p^{2k+1} H_{2k,m,M}(p^{r-2}) - \delta_{M|m} \delta_{k=0} r \frac{1}{2} \left( 1 - \left( \frac{-1}{p} \right) \right) \\
   &\quad - \delta_{M|m} \delta_{k=0} \delta_{2|\alpha} H(4p) - \frac{k}{12} p^{r+1} \left( 1 - \frac{1}{p} \right) \vartheta_{m,M}(p^r) - \frac{1}{3} \left( 1 - \left( \frac{-3}{p} \right) \right) p^r \sigma_{m,M}(p^r).
   \end{align*}

**Proof.** (1) By definition, we have

   \[ \mathcal{H}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) - \sum_{t \equiv m \pmod{M}} \sum_{p\nmid t} t^{2k} H(4p^r - t^2). \] (4.1)

   It is not hard to see that under the assumptions of (1) the only possible term in the second sum is $t = 0$, which only occurs if $M \mid m$ and $k = 0$, giving the claim.

   (2) Letting $t \mapsto pt$, the second summand in (4.1) equals (we assume that $r \geq 2$)

   \[ p^{2k} \sum_{pt \equiv m \pmod{M}} t^{2k} H((4p^{r-2} - t^2) \cdot p^2) = p^{2k} \sum_{t \equiv m \pmod{M}} t^{2k} H((4p^{r-2} - t^2) \cdot p^2). \] (4.2)

   We next use Lemma 2.5 with $D = t^2 - 4p^{r-2}$ to rewrite the terms in (4.2) with $|t| < 2p^{5-1}$, where this condition is required to assure that $D < 0$. If $0 < |t| < 2p^{5-1}$ and $p > 3$, then $\text{ord}_p(t^2 - 4p^{r-2}) = \text{ord}_p(t^2)$ and hence $\alpha = \text{ord}_p(t)$ in Lemma 2.5, yielding

   \[ H((4p^{r-2} - t^2) \cdot p^2) = pH(4p^{r-2} - t^2) \]

   \[ + \left( 1 - \left( \frac{t}{p^\alpha} \right)^2 - 4p^{r-2-2\alpha} \right) \frac{1}{p} H \left( 4p^{r-2-2\alpha} - \left( \frac{t}{p^\alpha} \right)^2 \right). \] (4.3)

   For $t = 0$, the choice of $\alpha$ in Lemma 2.5 is $\alpha = \frac{r}{2} - 1$ if $r$ is even and $\alpha = \frac{r+3}{2}$ if $r$ is odd, so in this case Lemma 2.5 gives

   \[ H(4p^r) = pH(4p^{r-2}) + \delta_{2|r} \left( 1 - \left( \frac{-1}{p} \right) \right) H(4) + \delta_{2|r} H(4p). \]

   Noting that the Legendre symbol in (4.3) equals 1 unless $\text{ord}_p(t) = \frac{r}{2} - 1$, plugging back into (4.2) for the $|t| < 2p^{5-1}$ terms and using the evaluations $H(0) = \frac{1}{12}$, $H(3) = \frac{1}{3}$, and $H(4) = \frac{1}{2}$ easily yields the claim.

We next bound $|E_{2k,m,M}(p^r)|$.

**Lemma 4.2.** For $k \in \mathbb{N}_0$ and $\varepsilon > 0$ we have

   \[ E_{2k,m,M}(p^r) = \frac{1}{3} 2^{2k-r} p^{k+1} \vartheta_{m,M}(p^r) + O_{k,\varepsilon} \left( p^{r-k-\frac{r-3}{2}+\varepsilon} \right). \]


Proof. The first term in the definition of $E_{2k,m,M}(p^r)$ only occurs if $k = 0$ and we use the upper bound in Lemma 2.4 to conclude that
\[ H(4p) \ll_{\epsilon} p^{r_{1/2} + \epsilon} = p^{r + \frac{\delta_{k=0} H(4p)}{2} + \epsilon}. \]
The second term is clearly $O(1)$ and the third term is $O(p^{r_{k}})$ because $0 \leq \varTheta_{m,M}(p^r) \leq 2$. We finally use $0 \leq \sigma_{m,M}(p^r) \leq 2$ to split the last term in the definition of $E_{2k,m,M}(p^r)$ as
\[ \frac{1}{3}(p-1) \delta_{m,M}(p^r) = \frac{1}{3} p^{r_{k} - 2} r_{k} \sigma_{m,M}(p^r) + O_k(p^{r_{k}}), \]
yielding the claim.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. (1) By Lemma 2.2, we have
\[ \frac{S_{2k,m,M}(p^r)}{p^r S_{m,M}(p^r)} = \frac{\mathcal{H}_{2k,m,M}(p^r) + E_{2k,m,M}(p^r)}{p^r (\mathcal{H}_{m,M}(p^r) + E_{m,M}(p^r))}. \]
We first consider $r = 1$. By Lemma 4.1 (1) the right-hand side of (4.4) equals
\[ \frac{H_{2k,m,M}(p) + E_{2k,m,M}(p)}{p^k (\mathcal{H}_{m,M}(p) - \delta_{k=0} H(4p) + E_{m,M}(p))} = \frac{H_{2k,m,M}(p) + E_{2k,m,M}(p)}{1 - \delta_{k=0}^H(4p) + E_{m,M}(p)}. \]
Using Lemma 4.2 and Lemma 3.7 together with Lemma 2.4, we obtain (note that $\varTheta_{m,M}(p^r) = 0$ for $r$ odd)
\[ \frac{H(4p)}{H_{m,M}(p)} \ll_{M,\epsilon} p^{\frac{1}{2} + \epsilon}, \quad \frac{|E_{2k,m,M}(p)|}{p^k H_{m,M}(p)} \ll_{k,M,\epsilon} p^{\frac{1}{2} + \epsilon}. \]
Using Theorem 1.1 yields the case $r = 1$.

For $r \geq 2$, we use Lemma 4.1 (2) to rewrite
\[ \mathcal{H}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) - p^{2k+1} H_{2k,m,M}(p^{r-2}) + O_k(M)(p^{r+1}). \]
By Proposition 3.6 and the upper bound in Lemma 3.7, we have
\[ p^{2k+1} H_{2k,m,M}(p^{r-2}) = C_k p^{r+1} + O_k(M)(p^{r+1}) \ll_{k,M,\epsilon} p^{r(k+1) + \epsilon - 1} + p^{r(k+1) + \epsilon - 1} \ll_{k,M,\epsilon} p^{r(k+1) + \epsilon - 1}, \]
where in the last bound we use the fact that $r - 1 \geq \frac{k}{2}$ for $r \geq 2$. Plugging this back into (4.5) yields
\[ \mathcal{H}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) + O_{k,M,\epsilon}(p^{r(k+1) + \epsilon - 1}). \]
By Lemma 4.2, we have
\[ E_{2k,m,M}(p^r) = O_k(p^{r(k+1) + \epsilon - 1}). \]
Plugging (4.6) and (4.7) into (4.4) yields
\[ \frac{S_{2k,m,M}(p^r)}{p^r S_{m,M}(p^2)} = \frac{H_{2k,m,M}(p^r) + O_{k,M,\epsilon}(p^{r(k+1) + \epsilon - 1})}{1 + O_{k,M,\epsilon}(p^{r(k+1) + \epsilon - 1})}. \]
The lower bound in Lemma 3.7 now yields the claim.
We then insert Lemma 2.3 into the right-hand side of (4.8) to obtain
\[ S_{2k,m,M}(p^{r}) = \frac{H_{2k,m,M}(p^{r}) + E_{2k,m,M}(p^{r})}{p^{rk}S_{m,M}(p^{r})} = \frac{H_{2k,m,M}(p^{r})}{p^{rk}H_{m,M}(p^{r})} + \frac{E_{2k,m,M}(p^{r})}{p^{rk}H_{m,M}(p^{r})} + \frac{H_{2k,m,M}(p^{r})}{p^{rk}H_{m,M}(p^{r})} + \frac{E_{2k,m,M}(p^{r})}{p^{rk}H_{m,M}(p^{r})}. \]

By Lemma 4.2 and Lemma 3.7 we obtain
\[ \frac{E_{2k,m,M}(p^{r})}{p^{rk}H_{m,M}(p^{r})} \ll_{p,M,\varepsilon} p^{r(-\frac{1}{2}+\varepsilon)} \]
and the proof follows as in the case \( r = 1 \).

Next suppose that \( p \nmid M \). We first rewrite the numerator of (4.4). Plugging Theorem 1.1 into the right-hand side of Lemma 2.3 yields
\[ \mathcal{H}_{2k,m,M}(p^{r}) = \sum_{\ell \equiv (m \mod p) \text{ and } \ell (\mod p)} p^{rk}H_{m+M\ell,Mp}(p^{r}) \left( C_{k} + O_{k,p,M,\varepsilon} \left( p^{r(-\frac{1}{2}+\varepsilon)} \right) \right). \] (4.8)

We then insert Lemma 2.3 into the right-hand side of (4.8) to obtain
\[ \mathcal{H}_{2k,m,M}(p^{r}) = p^{rk}\mathcal{H}_{m,M}(p^{r}) \left( C_{k} + O_{k,p,M,\varepsilon} \left( p^{r(-\frac{1}{2}+\varepsilon)} \right) \right). \]

Plugging back into (4.4) yields
\[ \frac{S_{2k,m,M}(p^{r})}{p^{rk}S_{m,M}(p^{r})} = \frac{C_{k} + O_{k,p,M,\varepsilon} \left( p^{r(-\frac{1}{2}+\varepsilon)} \right) + O \left( \frac{E_{2k,m,M}(p^{r})}{p^{rk}\mathcal{H}_{m,M}(p^{r})} \right)}{1 + O \left( \frac{E_{m,M}(p^{r})}{\mathcal{H}_{m,M}(p^{r})} \right)}. \] (4.9)

By Lemma 2.3 and Lemma 3.7, for any choice \( \lambda \equiv (m \mod p) \) such that \( p \nmid (m+M\lambda) \)
\[ \mathcal{H}_{m,M}(p^{r}) = \sum_{\ell \equiv (m \mod p) \text{ and } \ell (\mod p)} H_{m+M\ell,Mp}(p^{r}) \geq H_{m+M\lambda,Mp}(p^{r}) \gg_{p,M,\varepsilon} p^{r(1-\varepsilon)}. \]

Hence by Lemma 4.2 we have
\[ \frac{E_{2k,m,M}(p^{r})}{p^{rk}\mathcal{H}_{m,M}(p^{r})} \ll_{k,p,M,\varepsilon} \frac{p^{r(k+\frac{1}{2}+\varepsilon)}}{p^{r(k+1-\varepsilon)}} \ll_{p,M} p^{r(-1+\varepsilon)}. \]

The claim now follows from (4.9). \( \square \)

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