The matrix product representation for the $q$-VBS state of one-dimensional higher integer spin model

Kohei Motegi*

Okayama Institute for Quantum Physics,
Kyoyama 1-9-1, Okayama 700-0015, Japan

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Abstract

The generalized $q$-deformed valence-bond-solid groundstate of one-dimensional higher integer spin model is studied. The Schwinger boson representation and the matrix product representation of the exact groundstate is determined, which recovers the former results for the spin-1 case or the isotropic limit. As an application, several correlation functions are evaluated from the matrix product representation.

1 Introduction

In one-dimensional quantum systems, a completely different behavior for the integer spin chains from the half-integer spin chains was predicted the Haldane [1, 2]. The antiferromagnetic isotropic spin-1 model introduced by Affleck, Kennedy, Lieb and Tasaki (AKLT model) [3], whose groundstate can be exactly calculated, has been a useful toy model for the deep understanding of Haldane’s prediction of the massive behavior for integer spin chains, such as the discovery of the special type of long-range order [4, 5].

The AKLT model has been generalized to higher spin models, anisotropic models, etc [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The hamiltonians are essentially linear combinations of projection operators with nonnegative coefficients.

In this paper we consider the anisotropic integer spin-$S$ Hamiltonian

$$H = \sum_{k=1}^{L} H(k, k + 1),$$  \hspace{1cm} (1)

$$H(k, k + 1) = \sum_{J=S+1}^{2S} C_J(k, k + 1) \pi_J(k, k + 1),$$  \hspace{1cm} (2)

where $C_J(k, k + 1) \geq 0$, and $\pi_J(k, k + 1)$, which acts on the $k$-th and $(k + 1)$-th site, is the $U_q(su(2))$ projection operator for $V_S \otimes V_S$ to $V_J$ where $V_J$ is the $(2J+1)$-dimensional representation of the quantum group $U_q(su(2))$ [19, 20]. We determine the matrix product representation for the groundstate, which is useful for calculations of correlation functions. For $S = 1$ or $q = 1$ limit, it recovers the known results for the isotropic spin-$S$ model or anisotropic spin-1 model [8, 9, 11, 27]. Several correlation functions are evaluated from the matrix product representation.

This paper is organized as follows. In the next section, we briefly review the quantum group $U_q(su(2))$. By use of the Weyl representation of $U_q(su(2))$, we construct a boson representation for

*E-mail: motegi@gokutan.c.u-tokyo.ac.jp
the valence-bond-solid (VBS) groundstate. The matrix product representation for the VBS state is constructed in section 3, from which several correlation functions are evaluated for $S = 2$ and $S = 3$. Section 4 is devoted to conclusion.

2 Schwinger boson representation of the groundstate

The quantum group $U_q(su(2))$ is defined by generators $X^+, X^-, H$ with relations

$$[X^+, X^-] = \frac{q^{H} - q^{-H}}{q - q^{-1}}, \quad [H, X^\pm] = \pm 2X^\pm.$$  \hfill (3)

The comultiplication is given by

$$\Delta(X^+) = X^+ \otimes q^{H/2} + q^{-H/2} \otimes X^+, \quad \Delta(X^-) = X^- \otimes q^{H/2} + q^{-H/2} \otimes X^-, \quad \Delta(H) = H \otimes 1 + 1 \otimes H.$$  \hfill (4-6)

For convenience, let us define $q$-integer, $q$-factorial and $q$-binomial coefficients as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q, \quad \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!}.$$  \hfill (7)

$U_q(su(2))$ has the Schwinger boson representation [21 22 23]. Introducing two $q$-bosons $a$ and $b$ satisfying

$$aa^\dagger - qa^\dagger a = q^{-N_a}, \quad bb^\dagger - qb^\dagger b = q^{-N_b}, \quad \text{[2]}$$\hfill (8)

$$[N_a, a] = -a, \quad [N_a, a^\dagger] = a^\dagger, \quad [N_b, b] = -b, \quad [N_b, b^\dagger] = b^\dagger,$$  \hfill (9)

$U_q(su(2))$ can be realized through the relations

$$X^+ = a^\dagger b, \quad X^- = b^\dagger a, \quad H = N_a - N_b.$$  \hfill (10)

The basis of $(2j + 1)$-dimensional representation $V_j$ is given by

$$\vert j, m \rangle = \frac{(a^\dagger)^j (b^\dagger)^{-m}(b)^{-m} \vert \text{vac} \rangle}{([j + m]_q! [j - m]_q!)^{1/2}}, \quad (m = -j, \ldots, j).$$  \hfill (11)

We construct the VBS groundstate in terms of Schwinger bosons, following the arguments of [24]. Let us denote the $q$-bosons $a$ and $b$ acting on the $l$-th site as $a_l$ and $b_l$. We utilize the Weyl representation of $U_q(su(2))$ [25 26] for convenience. $a_l^\dagger$ and $b_l^\dagger$ is represented as multiplication by variables $x_l$ and $y_l$ on the space of polynomials $\mathbb{C}[x_l, y_l]$, respectively. $a_l$ and $b_l$ are represented as difference operators

$$a_l = \frac{1}{(q - q^{-1}) x_l} (D^{x_l}_{q} - D^{x_l}_{q^{-1}}), \quad b_l = \frac{1}{(q - q^{-1}) y_l} (D^{y_l}_{q} - D^{y_l}_{q^{-1}}),$$  \hfill (12)

where

$$D^{x_l}_{p} f(x_l, y_l) = f(px_l, y_l), \quad D^{y_l}_{p} f(x_l, y_l) = f(x_l, py_l).$$  \hfill (13)

Then, at the $l$-th site, one has

$$X^+_l = \frac{x_l}{(q - q^{-1}) y_l} (D^{y_l}_{q} - D^{y_l}_{q^{-1}}), \quad X^-_l = \frac{y_l}{(q - q^{-1}) x_l} (D^{x_l}_{q} - D^{x_l}_{q^{-1}}), \quad q^{H_l} = D^{x_l}_{q} D^{y_l}_{q^{-1}}.$$  \hfill (14)
The highest weight vector \( v \) has the following Clebsch-Gordan decomposition:

The tensor product of two irreducible representations has the following decomposition:

\[ V_{S_k} \otimes V_{S_l} = \bigoplus_{J=|S_k - S_l|}^{S_k + S_l} V_J. \]

The highest weight vector \( v_J \in V_J \) has the following form:

\[
v_J = \sum_{m_k + m_l = J} C_{m_k, m_l} x_k^{S_k + m_k} y_l^{-m_l}, \]

Since

\[
X_k^+ v_J = \Delta(X_k^+) \sum_{m_k + m_l = J} C_{m_k, m_l} x_k^{S_k + m_k} y_l^{-m_l} = \sum_{m_k=0}^{J-1} (|S_k - m_k|) q^{J-m_k} C_{m_k, J-m_k} + |S_l - J + m_k + 1| q^{-m_k-1} C_{m_k+1, J-m_k-1}
\]

one has

\[
C_{m_k, J-m_k} = \frac{(-1)^{S_k-m_k}}{(-1)^{S_k}} \begin{bmatrix} S_k + S_l - J \\ S_k - m_k \end{bmatrix} q^{m_k(J+1)} C_{0, J}. \]

Utilizing (19) and

\[
\prod_{j=1}^{m} (1 - z q^{2j-2}) = \sum_{k=0}^{m} (-z)^k q^{k(m-1)} \begin{bmatrix} m \\ k \end{bmatrix},
\]

one gets

\[
v_J = \frac{q^{S_k(J+1)} C_{0, J}}{(-1)^{S_k}} \begin{bmatrix} S_k + S_l - J \\ S_k \end{bmatrix} q \prod_{m=1}^{S} (x_k y_l - q^{2m-2-S_k-S_l} x_k y_l). \]

We are now considering the homogeneous chain, i.e., \( S_k = S \) for all \( k \). The highest weight vector \( v_S \in V_S \subset V_S \otimes V_S \) is divisible by \( \prod_{m=1}^{S} (q^m x_k y_l - q^{-m} y_k x_l) \). Moreover, we conjecture the following:

**Conjecture**

All vectors in \( V_j \subset V_S \subset V_S, \ j = 0, 1, \ldots, S \) are divisible by \( \prod_{m=1}^{S} (q^m x_k y_l - q^{-m} y_k x_l) \).

We have checked this conjecture for several values of \( S \). The vectors for the case \( S = 2 \) are listed in the Appendix. Based on this conjecture and the property of projection operators \( \pi_J w_K = \delta_{JK} w_K, w_K \in V_K \), we have the \( q \)-deformed lemma of Lemma 1 in \[24\].

**Lemma**

All solutions of

\[
\pi_J(k, k+1) |\psi\rangle = 0, \ S + 1 \leq J \leq 2S,
\]

[22]
for fixed $k$ can be represented in the following form

$$|\psi\rangle = f(a_k^\dagger, b_k^\dagger, a_{k+1}^\dagger, b_{k+1}^\dagger) \prod_{m=1}^{S} (q^m a_k^\dagger b_{k+1}^\dagger - q^{-m} b_k^\dagger a_{k+1}^\dagger)|\text{vac}\rangle,$$

(23)

where $f(a_k^\dagger, b_k^\dagger, a_{k+1}^\dagger, b_{k+1}^\dagger)$ is some polynomial in $a_k^\dagger, b_k^\dagger, a_{k+1}^\dagger$ and $b_{k+1}^\dagger$.

From this Lemma, we find the $q$-deformed VBS groundstate is

$$|\Psi\rangle_{PBC} = \prod_{k=1}^{L} \prod_{m=1}^{S} (q^m a_k^\dagger b_{k+1}^\dagger - q^{-m} b_k^\dagger a_{k+1}^\dagger)|\text{vac}\rangle,$$

(24)

where $a_{L+1} = a_1, b_{L+1} = b_1$ for the periodic chain, and

$$|\Psi\rangle_{p_1, p_2} = Q_{\text{left}}(a_1^\dagger, b_1^\dagger; p_1) \prod_{k=1}^{L-1} \prod_{m=1}^{S} (q^m a_k^\dagger b_{k+1}^\dagger - q^{-m} b_k^\dagger a_{k+1}^\dagger) Q_{\text{right}}(a_L^\dagger, b_L^\dagger; p_2)|\text{vac}\rangle,$$

(25)

where

$$Q_{\text{left}}(a_1^\dagger, b_1^\dagger; p_1) = \left[ \begin{array}{c} S \\ p_1 - 1 \end{array} q \right]^{1/2} (a_1^\dagger)^{S-p_1+1} b_{p_1-1}^\dagger, \quad (p_1 = 1, \ldots S + 1),$$

(26)

$$Q_{\text{right}}(a_L^\dagger, b_L^\dagger; p_2) = \left[ \begin{array}{c} S \\ p_2 - 1 \end{array} q \right]^{1/2} (a_L^\dagger)^{p_2-1} b_{p_2+1}^\dagger, \quad (p_2 = 1, \ldots S + 1),$$

(27)

for the open chain, generalizing the results of [6].

3 Matrix product representation

In the last section, we constructed the $q$-VBS states in terms of Schwinger bosons. One can transform them in the matrix product representation as in [11, 27], which are

$$|\Psi\rangle_{PBC} = \text{Tr}[g_1 \otimes g_2 \otimes \cdots \otimes g_{L-1} \otimes g_L],$$

(28)

$$|\Psi\rangle_{p_1, p_2} = [g^\text{start} \otimes g_2 \otimes \cdots \otimes g_{L-1} \otimes g_L]_{p_1, p_2},$$

(29)

where $g_k$ and $g^\text{start}$ are $(S + 1) \times (S + 1)$ matrices whose matrix elements are given by

$$g_k(i, j) = (-1)^{S-i+j} q^{(2i-2-S)(S+1)/2} \times \left( \begin{array}{c} S \\ i-1 \end{array} q \right)^{1/2} (a_k^\dagger)^{S-i+j} (b_k^\dagger)^{S+i-j} |\text{vac}\rangle_k,$$

$$= (-1)^{S-i+j} q^{(2i-2-S)(S+1)/2} \times \left( \begin{array}{c} S \\ i-1 \end{array} q \right)^{1/2} [S-i+j]_q! [S+i-j]_q! |\text{vac}\rangle_k,$$

(30)

$$g^\text{start}(i, j) = \left( \begin{array}{c} S \\ i-1 \end{array} q \right)^{1/2} [S-i+j]_q! [S+i-j]_q! |\text{vac}\rangle_k.$$

(31)

For $q \rightarrow 1$ limit, one recovers the results of [27]. We can also construct the matrix product representation in the following form

$$|\Psi\rangle_{PBC} = \text{Tr}[f_1 \otimes f_2 \otimes \cdots \otimes f_{L-1} \otimes f_L],$$

(32)
where

\[ f_k(i,j) = (-1)^{S-i+1}q^{i+j-2-S}(S+1)/2 \]

\[ \times \left( \begin{bmatrix} S \\ i \end{bmatrix}_q \begin{bmatrix} S \\ j-1 \end{bmatrix}_q \right) \frac{[S-i+j]_q![S+i-j]_q!}{[S+j]_q![S+i]_q!} |S;j-i\rangle_k, \] (33)

which reproduces the result for \( S = 1 \). \[8, 9\].

From the matrix product representation, one can formulate correlation functions. Let \( f_j^f \) be a matrix replacing the ket vectors of the matrix \( f \) by the bra vectors. We define \((S+1)^2 \times (S+1)^2\) matrices \( G \) and \( G^A \) as

\[ G_{(m_j-1,n_j-1;m_j,n_j)} = f_j^f(m_j-1,m_j)f_j(n_j-1,n_j), \] (34)

\[ G^A_{(m_j-1,n_j-1;m_j,n_j)} = f_j^f(m_j-1,m_j)A_jf_j(n_j-1,n_j). \] (35)

Explicitly we have

\[ G_{(a,b,c,d)} = \delta_{a-b,c-d}(-1)^{a+b}q^{(a+b+c+d-2S-4)(S+1)/2} \]

\[ \times \left( \begin{bmatrix} S \\ a \end{bmatrix}_q \begin{bmatrix} S \\ b \end{bmatrix}_q \begin{bmatrix} S \\ c \end{bmatrix}_q \begin{bmatrix} S \\ d \end{bmatrix}_q \right)^{1/2} \]

\[ \times ([S-a+c]_q![S+a-c]_q![S-b+d]_q![S+b-d]_q])^{1/2}. \] (36)

The eigenvalues of \( G \) for \( S = 2 \) are

\[ \lambda_1 = [5]_q[4]_q[2]_q, \] (37)

\[ \lambda_2 = \lambda_3 = \lambda_4 = -[5]_q[2]_q, \] (38)

\[ \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = [2]_q. \] (39)

Moreover, we conjecture that the eigenvalues of \( G \) for general \( S \) is given by

\[ \lambda(l) = (-1)^l \frac{[2S+1]_q!}{[S+1]_q} \left( \begin{bmatrix} S \\ l \end{bmatrix}_q \right), \quad (l = 0, 1, \ldots, S), \] (40)

where the degeneracy of \( \lambda(l) \) is \( 2l + 1 \).

For \( A = S^z \), one has

\[ G^A_{(a,b,c,d)} = \delta_{a-b,c-d}(d-b)(-1)^{a+b}q^{a+b+c+d-2S-4}(S+1)/2 \]

\[ \times \left( \begin{bmatrix} S \\ a \end{bmatrix}_q \begin{bmatrix} S \\ b \end{bmatrix}_q \begin{bmatrix} S \\ c \end{bmatrix}_q \begin{bmatrix} S \\ d \end{bmatrix}_q \right)^{1/2} \]

\[ \times ([S-a+c]_q![S+a-c]_q![S-b+d]_q![S+b-d]_q])^{1/2}. \] (41)

One point function \( \langle A \rangle \) and two point function \( \langle A_1B_r \rangle \) of the periodic chain can be represented as

\[ \langle A \rangle = (\text{Tr } G^L)^{-1} \text{Tr } G^A G^{L-1}, \] (42)

\[ \langle A_1B_r \rangle = (\text{Tr } G^b)^{-1} \text{Tr } G^A G^{r-2} G^B G^{L-r}. \] (43)
Denoting the eigenvalues and the normalized eigenvectors of $L$ as $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_{(S+1)^2}|$ and $|e_1, e_2, \ldots, e_{(S+1)^2}|$, (12) and (13) reduces to
\[
\langle A \rangle = \lambda_1^{-1} \langle e_1 | G^A | e_1 \rangle,
\]
\[
\langle A_1 B_r \rangle = \sum_{n=1}^{(S+1)^2} \lambda_n^{-2} \left( \frac{\alpha_n}{\lambda_1} \right)^r \langle e_1 | G^A | e_n \rangle \langle e_n | G^B | e_1 \rangle.
\]
in the thermodynamic limit $L \to \infty$.

Let us calculate several correlation functions. For $S = 2$, the probability of finding $S^z = m$ value $\langle P(S^z = m) \rangle$ is
\[
\langle P(S^z = 2) \rangle = \langle P(S^z = -2) \rangle = \frac{1}{[5]_q},
\]
\[
\langle P(S^z = 1) \rangle = \langle P(S^z = -1) \rangle = \frac{[2]_q [8]_q}{[5]_q [4]_q^2},
\]
\[
\langle P(S^z = 0) \rangle = \frac{[2]_q}{[5]_q [4]_q} \left( 1 + \frac{[12]_q}{[3]_q [4]_q} \right).
\]
In the $q = 1$ limit, $\langle P(S^z = m) \rangle = 1/5$ for all $m$. As we move away from $q = 1$, $P(S^z = 0)$ increases, i.e., the spins prefer the transverse $x$-$y$ plane. The spin-spin correlation function $\langle S^z_i S^z_r \rangle$ is
\[
\langle S^z_i S^z_r \rangle = \frac{[2]_q [3]_q}{[4]_q} \left( \frac{[2]_q}{[5]_q [4]_q} \right)^r \left\{ (q-q^{-1})(q^2 - q^{-2}) \frac{[6]_q^2}{[3]_q [2]_q^2} + [2]_q^2 (-[5]_q)^r \right\},
\]
which reduces to $-6(-2)^{-r}$ for $q = 1$. $\langle S^z_i S^z_r \rangle$ exhibits exponential decay for large distances, which is a typical behavior of gapful systems.

For $S = 3$, one has
\[
\langle S^z_i S^z_r \rangle = -\frac{[2]_q}{[6]_q [5]_q [3]_q} \left( \frac{[3]_q}{[7]_q [6]_q [5]_q} \right)^r \left\{ (q-q^{-1})^2(q^2 - q^{-2})^2([9]_q - (q^3 - q^{-3})^2)^2 \frac{[4]_q^2}{[2]_q^2} (-[2]_q)^r \right\} + (q^3 - q^{-3})^2 \frac{[8]_q^2 [5]_q [4]_q}{[4]_q} \left( [7]_q [2]_q \right)^r + \frac{[2]_q^4 - 2[3]_q [2]_q [6]_q [2]_q}{[3]_q} \frac{[6]_q [2]_q}{[3]_q} (-[7]_q [6]_q)^r \right\},
\]
which reduces to $-80(-3)^{r-2}(-5)^{r}$ in the $q = 1$ limit.

### 4 Conclusion
In this paper, we considered one-dimensional spin-$S$ $q$-deformed AKLT models. We derived the Schwinger boson representation and the matrix product representation for the valence-bond-solid groundstate. The matrix product representation is practical for calculating correlation functions. The spin-spin correlation functions exhibit exponential decay for large distances.

An interesting problem is to calculate the entanglement entropy of this model, which is a typical quantification of the entanglement of quantum systems. It is interesting to see how the entanglement entropy changes as we move away from the isotropic point $\{28, 29, 30\}$ (see also $\{31, 32\}$ for other VBS states).

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6
Appendix

We list all the vectors in \( v_j \in V_j \subset V_S \otimes V_S, j = 1, 2, \ldots, S \).

\[ S = 2 \]

\[ v_2 \propto x_k^2 x_l^2 (q x_k y_l - q^{-1} x_l y_k)(q^2 x_k y_l - q^{-2} x_l y_k), \]

\[(X_{kl}^{+}) v_2 \propto x_k x_l (q^{-2} x_k y_l + q^2 x_l y_k)(q x_k y_l - q^{-1} x_l y_k)(q^2 x_k y_l - q^{-2} x_l y_k), \]

\[(X_{kl}^{+})^2 v_2 \propto (q - 1)^2 x_k x_l y_k y_l + q^4 x_k^2 y_l^2 (q x_k y_l - q^{-1} x_l y_k)(q^2 x_k y_l - q^{-2} x_l y_k), \]

\[ (X_{kl}^{+})^3 v_2 \propto y_k y_l (q^{-2} x_k y_l + q^2 x_l y_k)(q x_k y_l - q^{-1} x_l y_k)(q^2 x_k y_l - q^{-2} x_l y_k), \]

\[ (X_{kl}^{+})^4 v_2 \propto y_k^2 y_l^2 (q x_k y_l - q^{-1} x_l y_k)(q^2 x_k y_l - q^{-2} x_l y_k), \]

\[ v_1 \propto x_k x_l (x_k y_l - x_l y_k)(q x_k y_l - q^{-1} x_l y_k)(q^2 x_k y_l - q^{-2} x_l y_k), \]

\[ (X_{kl}^{+}) v_1 \propto (q^{-2} x_k y_l + q^2 x_l y_k)(x_k x_l - x_l x_k)(q x_k y_l - q^{-1} x_l y_k)(q^2 x_k y_l - q^{-2} x_l y_k) \]

\[ (X_{kl}^{+})^2 v_1 \propto y_k y_l (x_k y_l - x_l y_k)(q x_k y_l - q^{-1} x_l y_k)(q^2 x_k y_l - q^{-2} x_l y_k), \]

\[ v_0 \propto (q^{-1} x_k y_l - q x_l y_k)(x_k y_l - x_l y_k)(q x_k y_l - q^{-1} x_l y_k)(q^2 x_k y_l - q^{-2} x_l y_k). \]

References

[1] F.D.M. Haldane, Phys. Lett. A 93 (1983) 464.
[2] F.D.M. Haldane, Phys. Rev. Lett. 50 (1983) 1153.
[3] I. Affleck, T. Kennedy, E.H. Lieb and H. Tasaki, Comm. Math. Phys. 115 (1988) 477.
[4] M. de Nijis and K. Rommelse, Phys. Rev. B 40 (1989) 4709.
[5] H. Totsuka, Phys. Rev. Lett. 66 (1991) 798.
[6] D. P. Arovas, A. Auerbach and F.D.M. Haldane, Phys. Rev. Lett. 60 (1988) 531.
[7] A. Klümper, A. Schadschneider and J. Zittartz, J. Phys. A 24 (1991) L955.
[8] A. Klümper, A. Schadschneider and J. Zittartz, Z. Phys. B 87 (1992) 443.
[9] A. Klümper, A. Schadschneider and J. Zittartz, Europhys. Lett. 24 (1993) 293.
[10] M. Oshikawa, J. Phys. Cond. Matt. 4 (1992) 7469.
[11] K. Totsuka and M. Suzuki, J. Phys. A 27 (1994) 6443.
[12] M.T. Batchelor and C.M. Yung, Int. J. Mod. Phys. B 8 (1994) 3645.
[13] M. Greiter and S. Rachel, Phys. Rev. B 75 (2007) 184441.
[14] D. Schuricht and S. Rachel, Phys. Rev. B 78 (2008) 014430.
[15] H-H. Tu, G-M. Zhang and T. Xiang, Phys. Rev. B 78 (2008) 094404.
[16] H-H. Tu, G-M. Zhang, T. Xiang, Z-X. Liu and T-K. Ng, Phys. Rev. B 80 (2009) 014401.
[17] V. Karimipour and L. Memarzadeh, Phys. Rev. B 77 (2008) 094416.
[18] D. P. Arovas, K. Hasebe, X-L. Qi and S-C. Zhang, Phys. Rev. B 79 (2009) 224404.
[19] V. Drinfeld, Sov. Math.-Dokl. 32 (1985) 254.
[20] M. Jimbo, Lett. Math. Phys. 10 (1985) 63.
[21] L.C. Biedenharn, J. Phys. A 22 (1989) L873.
[22] A.J. MacFarlane, J. Phys. A 22 (1989) 4581.
[23] T. Hayashi, Comm. Math. Phys. 127 (1990) 129.
[24] V.E. Korepin and Y. Xu, arXiv:0908.2345
[25] M. Jimbo, Quantum groups and Yang-Baxter equation (Springer-Verlag Tokyo, 1990).
[26] S. Cadransky, Int. J. Th. Phys. 31 (1992) 907.
[27] K. Totsuka and M. Suzuki, J. Phys. Condense. Matter 7 (1995) 1639.
[28] H. Fang, V.E.Korepin and V. Roychowdhury, Phys. Rev. Lett. 93 (2004) 227203.
[29] H. Katsura, T. Hirano and Y. Hatsugai, Phys. Rev. B 76 (2007) 012401.
[30] Y. Xu, H. Katsura, T. Hirano and V.E. Korepin, J. Stat. Phys. 133 (2008) 347.
[31] H. Katsura, T. Hirano and V.E. Korepin, J. Phys. A 41 (2008) 135304.
[32] H. Katsura, N. Kawashima, A. Kirillov, V.E. Korepin and S. Tanaka, arXiv:1003.2007.