Joint Feature Selection and Time Optimal Path Parametrization for High Speed Vision-Aided Navigation

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Abstract—We study a problem in vision-aided navigation in which an autonomous agent has to traverse a specified path in minimal time while ensuring extraction of a steady stream of visual percepts with low latency. Vision-aided robots extract motion estimates from the sequence of images of their on-board cameras by registering the change in bearing to landmarks in their environment. The computational burden of the latter procedure grows with the range of apparent motion undertaken by the projections of the landmarks, incurring a lag in pose estimates that should be minimized while navigating at high speeds. This paper addresses the problem of selecting a desired number of landmarks in the environment, together with the time parametrization of the path, to allow the agent execute it in minimal time while both (i) ensuring the computational burden of extracting motion estimates stays below a set threshold and (ii) respecting the actuation constraints of the agent. We provide two efficient approximation algorithms for addressing the aforementioned problem. Also, we show how it can be reduced to a mixed integer linear program for which there exist well-developed optimization packages. Ultimately, we illustrate the performance of our algorithms in experiments using a quadrotor.

I. INTRODUCTION

Time-critical missions are a key aspect of numerous robotic applications. Examples include disaster response, search and rescue, and aid delivery. In order to execute specified tasks precisely, and, in particular, avoid fatal collisions with objects in their environment, autonomous systems require the ability to recover accurate estimates of their pose by inverting percepts from their on-board sensors. However, the path from raw sensor data to localized estimates via limited on-board computational resources incurs a latency. These delays can preclude rapid maneuvers that lie beyond the perception-actuation limits of the agent. Therefore, planning maneuvers with consideration of such constraints allows us to maximize the effectiveness of robotic platforms while ensuring their safety.

A common assortment of on-board sensors includes a combination of an inertial measurement unit (IMU) and a camera. An IMU supplies measurements of acceleration and angular velocity that can be used to perform dead reckoning over short time intervals. However, over longer horizons, such estimates accrue large drift that can be significantly mitigated using measurements supplied by the vision system. The latter typically take the form of differential bearings to landmarks, visually salient regions in the environment, whose projections are captured as features on the pixel array of the on-board camera. Extracting accurate bearing measurements efficiently is a problem in its own right, and some regimes of robot motion render it more challenging than others.

In order to measure the change in bearing to a particular landmark between successive images, the agent must solve the problem of data association. In visual navigation, the task is commonly solved using an iterative search method such as the KLT feature tracker [1], with the area of search constrained by some understanding of the maximum distance between projections of landmarks in consecutive frames. As a result, the computational burden of data association grows with the range of motion underwent by tracked features. The latter, in turn, depends on both the geometry of the scene as well as the trajectory executed by the robot. Tracking a larger set of landmarks or performing more rapid maneuvers increases the cost of data association. However, a greater volume of visual measurements yields more accurate state estimates, whereas faster motion increases the efficacy of the robot. This motivates us to consider a problem at the intersection of visual feature selection and motion planning: which subset of landmarks of desired cardinality should the agent track, so it can traverse a given geometric path in minimum time subject to maintaining projection speeds of selected landmarks below a specified threshold.

Fig. 1: The figure shows a quadrotor navigating a typical indoor environment. The selection of landmarks in the environment that allow the fastest execution of the path is shown in green.
coupled from motion planning. The goal of attention and anticipation [2] is to select landmarks that will remain within field of view throughout the traversal of a segment of the trajectory. This procedure aids the visual navigation process by greedily selecting features ranked by an information gain metric formed by forward simulating the dynamics as a scoring function. [3] propose a stochastic greedy algorithm for feature subset selection for the least squares pose estimation problem. They leverage the submodularity of the problem, to select an approximately optimal subset of features using the maxlogdet metric in low running time.

There are also numerous works planning trajectories with perception constraints. [4] plan locally optimal dynamically feasible trajectories for quadrotors that minimize execution time and keep a specified target within the on-board camera’s field of view. [5] present an algorithm that minimizes the integral of the magnitude of reprojection error of a specified set of landmarks along the planned path from the current position to a given final position. [6] incorporate the perception constraint in their model predictive control framework which minimizes the speed of a particular set of landmarks’ projections. [7] formulate an objective function that combines a co-visibility constraint, which ensures landmarks remain visible in between keyframes, and trajectory execution time that can leverage differential flatness of the quadrotor. [8] plan time optimal paths for maintaining a specified set of landmarks along the planned path from the current position. It consists of first planning a collision-free geometric trajectory. This procedure aids the visual navigation process anticipation [2] is to select landmarks that will remain within field of view throughout the traversal of a segment of the trajectory.

In this work, we consider the task of selecting a desired number of landmarks, together with planning an optimal time parametrization of a specified path, that is both dynamically feasible and ensures computationally efficient extraction of changes in bearing of selected landmarks. Our contributions are: (1) efficient approximation algorithms for addressing the problem of joint feature selection and time optimal path parametrization, together with an analysis of their performance guarantees, and (2) a reduction of the problem to a mixed integer linear program, for which existing optimization packages supply the optimal solution within a matter of minutes for the size of problems that occur in practice.

II. PROBLEM FORMULATION

A. Coordinate Frame Conventions

We denote coordinates of point \( p \in A^3 \) with respect to reference frame \( A \) by \( A^p \in R^3 \). Particularly relevant will be the world frame, the body-fixed frame of the agent, and its camera-fixed frame, denoted by \( W, B, \) and \( C \), respectively. For a generic pair of reference frames \( A \) and \( \tilde{A} \), the transformation from \( A \) to \( \tilde{A} \) will be denoted by \( (A\tilde{A}) \in R^3 \times SO(3) \); the equation \( A^p = \tilde{A}^p + A\tilde{A} \) relates coordinates of the same physical point with respect to the two frames.

B. Problem Description

The problem under consideration consists of the following: Given an agent \( A \) with known dynamics \( q = f(q, u) \in R^n \), where \( q \in R^{2n} \) and \( u \in C(q) \in R^{nu} \) encode its state and actuation constraints, respectively, a set of landmarks with known static positions with respect to the world frame \( L = \{W_{l_1},...,W_{l_n}\} \), the desired number \( k \in \{1,2,...,n\} \) of landmarks to track, and a regularly parametrized path \((\gamma, R) : [0,S_{end}] \rightarrow R^3 \times SO(3)\), find the shortest execution time of \( \gamma \) by \( A \) that is (i) dynamically feasible, and (ii) maintains the projection speed of each landmark in some fixed subset of \( L \), of cardinality \( k \), onto the canvas of the camera on-board \( A \) below a specified threshold \( \pi_{max} \).

We assume the relative pose between the agent and its on-board camera (WLOG with unit focal length), given by the transformation \((B^C, B^R)\) is fixed. When the pose of the agent is \((W_B, W_R) = (\gamma(s), R(s))\), landmark \( W_{l_i} \) has coordinates

\[
C_{l_i}(s) = C_B^R B^R W_{l_i}(s) (W_{l_i} - \gamma(s) - R(s) B^C)
\]

with respect to \( C \), and is captured as the point \( R_{l_i}(s) \) on the canvas of the camera. Equation 1 readily implies that for all \( 1 \leq i \leq n \), the agent maintains the projection speed of landmark \( i \) below threshold \( \pi_{max} \) if and only if

\[
\frac{ds}{dt}(s) \leq U_i(s) \triangleq \max_{\| \frac{d}{ds} (C_{l_i}(s)) \|_2} \forall s \in [0,S_{end}].
\]

Note that \( \frac{ds}{dt} \) represents the rate of increase of the parameter of the path \((\gamma(\cdot), R(\cdot))\); it coincides with the translational speed of the agent when \( s \) corresponds to arc length of \( \gamma \).

Methods in this paper are applicable to a broad class of agents with second order dynamics [14], including models of space vehicles, cars [15], [18], and Dubins’ planes [19]. As a running example, we consider a fully actuated multicopter, whose dynamics are given by [20]

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\dot{\omega}_x \\
\dot{\omega}_y \\
\dot{\omega}_z \\
\dot{\mu}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-g}{R} & 0 & 0 & 0 & 0 \\
0 & \frac{J_1}{J_0} \cdot \omega \times J_0 \omega & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\dot{\tau} \\
\end{bmatrix},
\]

where effective input \( u \) is
where the effective input \( \mathbf{u} \in \mathbb{R}^6 \) is subject to convex membership constraints. An example of such a platform is a hexacopter with tilted motors.

### C. Problem parametrization

In addition to selecting a subset of landmarks for tracking, we will seek the optimal square speed profile \([21] \), \( h : [0, S_{end}] \to [0, \infty) \) defined, with slight abuse of notation, via \( h(s) = (\frac{d}{ds} f)^2 \forall s \in [0, S_{end}] \). The efficacy of such a parametrization of TOPP in motion planning was first captured in \([12] \). One of the main insights in \([12] \) uncovered that upper bounds on norms of acceleration, given by \( a = \frac{1}{2} \gamma' h + \gamma'' h \), induce convex constraints on \( h \), while the execution time \( f_{s_{end}}^S = \frac{d}{ds} h(s) \), is a convex function of \( h \). Furthermore, in our current problem, the constraint of successfully tracking feature \( i \) amounts to a bound on the speed profile \( h(s) \leq M_i(s) \triangleq U_i(s)^2 \forall s \in [0, S_{end}] \).

In general, the optimal \( h \) need not be everywhere differentiable. To see this, consider minimizing the traversal time of a straight line segment by an acceleration-constrained agent (encoded by \( |h'| \leq 2a_{max} \)). The optimal profile involves switching acceleration instantaneously from \( +a_{max} \) to \( -a_{max} \). To address such behaviour, we enforce actuation constraints in the form \( D^+ h(s) \leq f^+(s, h(s)) \) and \( D^- h(s) \geq f^-(s, h(s)) \) \([22] \), where \([23] \)

\[
D^+ h(s) \triangleq \lim_{\delta \to 0} \sup_{\delta \geq 0} \frac{h(s + \delta) - h(s)}{\delta} \\
D^- h(s) \triangleq \lim_{\delta \to 0} \inf_{\delta \geq 0} \frac{h(s + \delta) - h(s)}{\delta}
\]

for suitably chosen functions \( f^+ \) and \( f^- \). For instance, the example above would amount to setting \( f^\pm = \pm 2a_{max} \). Therefore, we solve problem \( P(f^+, f^-) \triangleq \{ (W_1)_j \}_{j=0}^n \):  

\[
\minimize_{h_i \mid [a, b] \to [0, \infty), \forall j = k} \int_a^b \frac{ds}{h(s)}
\]

subject to  
\[
D^+ h(s) \leq f^+(s, h(s)), \; s \in [a, b), \\
D^- h(s) \geq f^-(s, h(s)), \; s \in [a, b) \\
0 \leq h(s) \leq \min_{j \in I} M_j(s), \; s \in [a, b].
\]

### III. METHOD

In this section, we describe three related methods for solving Problem 5. A solution consists of two components: a subset of features of cardinality \( k \), and a dynamically feasible square speed profile which allows tracking selected features. We represent the optimal square speed profile by its values at a set of specified discretization points of the path. In particular, a discretization of interval \([a, b] \), denoted by \( D = D([a, b], (s_i)_{i=0}^N) \), comprises an increasing sequence of discretization points \( a = s_0 < s_1 < \cdots < s_N = b \), at which we compute the sequence \( \{ \hat{h}_i \}_{i=0}^N \approx \{ h_i^{(f)} \}_{i=0}^N \), where \( h^* \) is the optimal square speed profile. We numerically approximate the execution time of a profile \( \{ h_i \}_{i=0}^N \) by \( \sum_{i=0}^{N-1} 2(s_{i+1} - s_i) \).
The task of optimally selecting k features as well as determining the optimal speed profile can be recovered from the solution of the following binary mixed integer linear program:

\[
\begin{align*}
\min & \quad \sum_{i=1}^{N} (s_i - s_{i-1}) y_i \\
\text{s.t.} & \quad y_i \geq \frac{1}{\sqrt{h_j(s_i)}} \quad \forall \ 0 \leq i \leq N \\
& \quad \sum_{j=1}^{n} \zeta_j \geq k \\
& \quad \zeta_i \in \{0, 1\}^n, \quad y \in \mathbb{R}^N.
\end{align*}
\]

Denoting the optimal value of the solution by \((\zeta^{*}_i, y^{*}_i)\), the selected features correspond to indices \(i\) such that \(\zeta^{*}_i = 1\), whereas \(y^{*}_j\) yields the amount of time allocated by the optimal profile for the segment \([s_j, s_{j+1}]\) for every \(1 \leq j \leq N - 1\).
characterization of $\overline{H}_M(1+\varepsilon)$ as the pointwise supremum of the set of feasible profiles that do not exceed the latter bound.

The inequity on the right hand side may be argued as follows. The convexity of the set of feasible square speed profiles implies that

$$\varepsilon + \frac{1}{1+\varepsilon} \overline{H}_M(1+\varepsilon)$$

is a feasible square speed profile that does not exceed $M$. The extremal characterization of $\overline{H}_M$ as the square speed profile that is the pointwise supremum of feasible square speed profiles not exceeding $M$ settles the claim (the inequalities for $\tau$ follow from remarks preceding the claim).

**Theorem 1** Let $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_n\}$ be two sets of square speed profiles, referred to as $p$-profiles and $q$-profiles, respectively. Let $\varepsilon > 0$ be a positive real number such that

$$\left| \frac{p_i}{q_i} \right|, \left| \frac{q_i}{p_i} \right| \leq 1 + \varepsilon$$

for all $1 \leq i \leq n$. In addition, assume that $Q$ can be partially ordered $q_1 \geq q_2 \geq \ldots \geq q_n$: if $i < j$ then $q_i(s) \geq q_j(s)$ for all $s \in [0, S_{end}]$. Defining $T_{p_{\kappa}}$ and $T_{q_{\kappa}}$ to be the execution times of the pointwise minimum of the optimal subset of $k$ $p$-profiles and the subset of $k$ individually fastest $p$-profiles, respectively, we have:

$$T_{p_{\kappa}} \leq (1+\varepsilon)^2 T_{q_{\kappa}}. \quad (6)$$

**Proof**: By assumption, the profiles $(q_i)_{i=1}^n$ have been sorted in order of execution time: $\tau(q_1) \leq \tau(q_2) \leq \ldots \leq \tau(q_n)$. Let $\{p_1, p_2, \ldots, p_k\}$ be the set of $k$ $p$-profiles of shortest duration. Let $m = \arg \max_{1 \leq j \leq k} \tau(q_j)$, and define $\tilde{q} = q_{m}$. In other words, $\tilde{q}$ is the slowest $q$-profile matched to one of the profiles $p_1, p_2, \ldots, p_k$. Thus, we have

$$\tau(p_1, p_2, \ldots, p_k) \leq \tau\left(\left(\frac{q_1}{1+\varepsilon}, \frac{q_2}{1+\varepsilon}, \ldots, \frac{q_k}{1+\varepsilon}\right)\right) \leq \sqrt{1+\varepsilon} \tau(q_1, q_2, \ldots, q_k) = \sqrt{1+\varepsilon} \tau(\tilde{q}), \quad (7)$$

where the last equality follows from the fact that the profile $\tilde{q}$ is dominated by profiles $q_1, q_2, \ldots, q_k$.

We now claim that $\tau(\tilde{q}) \leq (1+\varepsilon)\tau(q_k)$. Indeed, assume for a contradiction $\tau(\tilde{q}) > (1+\varepsilon)\tau(q_k)$. Then,

$$\tau(p_m) \geq \tau((1+\varepsilon)q_m) = \tau((1+\varepsilon)\tilde{q}) = \frac{1}{\sqrt{1+\varepsilon}} \tau(\tilde{q}) > (1+\varepsilon)^{1/2}\tau(q_k) \geq (1+\varepsilon)^{1/2}\tau(q_j) \quad \forall \ 1 \leq j \leq k,$$

where the last line follows from the fact that $q$-profiles have been sorted in order of increasing execution time.

On the other hand, for all $j$ we have

$$\tau(p_j) \leq \tau\left(\frac{q_j}{1+\varepsilon}\right) = \sqrt{1+\varepsilon} \tau(q_j). \quad (9)$$

Hence, $\tau(p_m) > \tau(p_j)$ for all $1 \leq j \leq k$, and so $p_m$ cannot be one of the $k$ $p$-profiles with the smallest execution time, a contradiction. Thus,

$$\tau(p_1, p_2, \ldots, p_k) \leq (1+\varepsilon)^{3/2} \tau(q_k). \quad (10)$$

Similarly, for any set of profiles $p_1, p_2, \ldots, p_k$ we have

$$\tau(p_1, p_2, \ldots, p_k) \geq \frac{1}{\sqrt{1+\varepsilon}} \max_{1 \leq j \leq k} \tau(q_j), \quad (11)$$

where the final inequality follows from the ordering of the $q$-profiles. Hence, $\tilde{T}_{p_{\kappa}} = \tau(p_1, p_2, \ldots, p_k) \leq (1+\varepsilon)^2 T_{q_{\kappa}}$, as desired.

**Theorem 2** Consider the task in which the agent has to traverse a path $\gamma$ with constant orientation $R$ with respect to the world frame. For this theorem, we assume $\gamma$ is a segment of length $L$ that belongs to line $p$. Assume the distance of all $n$ landmarks which remain within field of view of the on-board camera with field of view $\theta$, are at a distance of at least $XL$ from $\gamma$. Then, Algorithm 2 produces a feasible square speed profile whose execution time is within a factor of $((1+\frac{L}{2})/\cos(\theta))^4$ of the optimum.

**Proof**: Consider a landmark with coordinates $\mathbf{W}$. To avoid excessive notation, assume the origin of the reference frame of the camera coincides with the origin of the body frame of the agent. Thus,

$$C\mathbb{I}(s) = R^{-1}(\mathbf{W} - \gamma(s)). \quad (12)$$

Given the bounded field of view of the camera, the magnitude of the velocity of the projection of the landmark is up to an absolute constant (namely $\frac{1}{\cos^2 \theta}$) the rate of change of its bearing vector. Hence, the rate of change of the projection with respect to distance the agent has travelled along the path satisfies

$$\left| \frac{d}{ds} \left(\frac{C\mathbb{I}(s)}{||C\mathbb{I}(s)||_2}\right) \right|_2 \leq \frac{1}{\cos^2 \theta} \left| \frac{d}{ds} \left(\frac{C\mathbb{I}(s)}{||C\mathbb{I}(s)||_2}\right) \right|_2 \leq \frac{1}{\cos^2 \theta} \left| \frac{C\mathbb{I}(s) - (C\mathbb{I}(s))' \times (C\mathbb{I}(s))'}{||C\mathbb{I}(s)||_2^3} \right|_2 = \frac{1}{\cos^2 \theta} \left| \frac{(C\mathbb{I}(s))' \times (C\mathbb{I}(s))}{||C\mathbb{I}(s)||_2^3} \right|_2 \quad (13)$$

The magnitude of the rate of change of the bearing vector may be expressed as follows. Since $R$ is assumed constant, we have

$$C\mathbb{I}(s)' = -R^{-1}\gamma'(s) \quad (14)$$
implying that
\[
\frac{||\mathbf{c}(s) \times \mathbf{c}(s)||_2}{||\mathbf{c}(s)||_2^2} = \frac{||\mathbf{y}'(s) \times (\mathbf{W} - \mathbf{y}(s))||_2}{||\mathbf{W} - \mathbf{y}(s)||_2^2} = \frac{||\sin \angle (\mathbf{y}(s), \mathbf{W} - \mathbf{y}(s))||_2}{||\mathbf{W} - \mathbf{y}(s)||_2} = \frac{\text{dist}(\mathbf{W}, \mathbf{\rho})}{||\mathbf{W} - \mathbf{y}(s)||_2}.
\]

where \( \text{dist}(\mathbf{W}, \mathbf{\rho}) \) denotes the orthogonal distance from point \( \mathbf{W} \) to line \( \mathbf{\rho} \). Denoting the minimal and maximal distance between points of \( \mathbf{y} \) and \( \mathbf{W} \) by \( \bar{d}(W_1, \mathbf{\gamma}) \) and \( \hat{d}(W_1, \mathbf{\gamma}) \) respectively, we have
\[
\left( \frac{\bar{d}(W_1, \mathbf{\gamma})}{\hat{d}(W_1, \mathbf{\gamma})} \right) \leq \frac{\text{dist}(\mathbf{W}, \mathbf{\rho})}{\text{dist}(\mathbf{W}, \mathbf{\rho})} = \left( \frac{\hat{d}(W_1, \mathbf{\gamma})}{\bar{d}(W_1, \mathbf{\gamma})} \right)
\]

From the lower bound on \( \bar{d}(W_1, \mathbf{\gamma}) \), we get that \( 1 \leq \frac{\bar{d}(W_1, \mathbf{\gamma})}{\hat{d}(W_1, \mathbf{\gamma})} \leq (1 + \frac{1}{\xi}) \). Putting the latter relations together, we get that the profile \( \pi_{\max}^2/||\mathbf{c}(s)\times\mathbf{c}(s)||_2 \) is within a factor \( \eta := ((1 + \frac{1}{\xi})^2 \cdot \frac{1}{\cos \theta}) \) of the constant profile \( \pi_{\max}^2/||\mathbf{c}(s)\times\mathbf{c}(s)||_2 \). Letting \( p_j \) and \( q_j \) be the time optimal feasible profiles not exceeding the former and latter profiles for every landmark \( W_1 \in \mathbf{L} \), respectively, we can derive three things. Firstly, the set of \( (q_j)_{j=1}^n \) may be partially ordered as in the definition of Theorem 1. Second, Claim 3 implies that for all \( j \), profiles \( p_j \) and \( q_j \) are within a factor of \( \eta \) of each other. Ultimately, the latter two observations set the stage for an application of Theorem 1 which yields the desired result.

**B. Computational Complexity Analysis**

We split the computational complexity analysis of the three proposed methods into two parts. First, we will state the complexity of the first stage of the algorithm(s), the details of which can be found in [16, and [17]. Then we will analyze the complexity of the second stage of each method.

The first stage consists of executing Algorithm 1, whose running time is of order \( O(NE) \), where \( E \) is the complexity of solving the one-dimensional optimization problem in every iteration of the for loops, and \( N \) is the number of discretization points of the path. Typically, \( f^+(s, \cdot) = \min_{1 \leq j \leq m} f^+_j(s, \cdot) \) is specified as the minimum (maximum) of \( m \) concave (convex) functions, where \( m \) denotes the number of constraints. In this case, \( E = O(Cm \log m) \) [17], where \( C \) is the complexity of obtaining lower and upper bounds of sublevel sets of \( \pm f^+_j(s, \cdot) \). Computing the optimal profile for tracking each individual landmark can be done in parallel, so that the complexity of the first stage of the algorithm is \( O(p^{-1}nNCm \log m) \), where \( p \) denotes the number of threads.

The second stage of the first method proceeds by selecting the \( k \) fastest out of \( n \) features, requiring \( O(k \log n) \) time. Hence, the running time of the first method is \( O(p^{-1}nNCm \log m + O(k \log n)) \).

The second stage of the second method iteratively selects from the set of unchosen features the one which incurs the smallest increase in execution time. Calculating the increase for a single feature has complexity \( O(N) \), yielding a total of \( O(nN) \) for \( O(n) \) such features. Thus, selecting \( k \) features has running time \( O(knN) \), yielding a \( O(p^{-1}nNCm \log m + O(knN))-time algorithm.

The second stage of the third method has running time exponential in \( k \). Although theoretically slower than the first two methods, and as of yet unsuitable for use in real time on embedded systems, it yields the ground truth optimum within minutes for problems of the size we consider.

**V. Experiments**

A. Experimental Setup

For simplicity we assume that the quadrotor used in the experiments is modeled as a double integrator with a camera attached on a 2-axis stabilized gimbal. The camera is assumed to be oriented to face the direction of heading of the quadrotor. The navigation environment is generated using the FlightGoggles photorealistic simulator described in [25]. The chosen environment is a living room (see Figure 2) which would be a typical scenario for autonomous robots. For the experiments we generate 4 trajectory segments shown in Figure 3.

B. Algorithm Comparison

In this experiment, we consider the performance of all of the algorithms mentioned in Section III. The key performance indices measured in this experiment are the execution time and the computation time. We report these indices for a selection problem of selecting 50 features from an available 500 features in the environment. For all the timing experiments, the speed profiles of the features are pre-computed and the timing only includes the selection of the features. All experiments for computation time are performed using an intel i9 7900X computer using a python implementation of the algorithms presented here and the optimization package [24] was allowed to use all available 40 threads. The results of this evaluation are summarized in Table I. As can be seen in the table, the execution times generated by the algorithms are nearly identical but the K-fasted algorithm is orders of magnitude faster to compute.

| Algorithm                  | Execution Time [s] | Computation Time [s] |
|----------------------------|--------------------|----------------------|
| Trajectory                 | Arc                | Slalom               |
| K-fasted                   | 2.3465             | 2.9094               | 0.0006               | 0.0006               |
| Incremental algorithm      | 2.3565             | 2.8889               | 2.1481               | 3.7412               |
| Convex optimization        | 2.2828             | 2.8669               | 258.4733             | 48.97485             |

**TABLE I:** Results of the algorithm comparison.

C. Feature Tracking experiment

For this experiment, the point cloud is generated using stereo matching and the pixels with valid depth values are then used for corner extraction using GoodFeaturesToTrack.
The extracted corners form the candidate point cloud for feature selection. In the first set of experiments, we evaluate the effect on feature tracking. For this experiment, the trajectory and features are specified by the K-Fastest algorithm. We compare the number of features tracked by a standard KLT tracker for the same number of features selected by the quality metric and the K-Fastest method. An example of the selected features is shown in Figure 2. The results are summarized in Table II. As can be seen in the table K-Fastest consistently tracks more features.

D. Robot experiments

For each of the trajectories, we generate a speed profile for an allowed selection of 50, 100 and 150 landmarks in the environment with a maximum feature speed of 288 pixels s\(^{-1}\) and a selection of 100 with 432 pixels s\(^{-1}\). In this experiment, feature selection is performed only using the K-Fastest algorithm. The trajectories are then flown using a custom built quadrotor executing a state of the art INDI controller described in [26] and the ground truth pose and reference command provided to the quadrotor are recorded. The ground truth pose of the quadrotor is then used to project the chosen landmarks back onto the image plane and the consecutive feature velocity along the trajectory segment is measured. Figure 4 shows a qualitative comparison of the feature speeds along the trajectory segment assuming both a gimbaled and non-gimbaled camera. In this case, the trajectory is executed 4 times and the periodic behavior can be observed in the plots. The quadrotor is successfully able to keep the feature speed under the desired threshold and a few features violate the threshold for short periods of time. This can be attributed to the imperfect nature of the trajectory tracking on the real quadrotor. The feature speeds are noticeably higher for the non-gimbaled case as expected but the optimization does compress the speed of the features. The summarization of the execution times for the 16 trajectories flown is shown in Table III. Table IV shows the mean of the maximum feature speed profiles over the trajectory segments.

VI. CONCLUSIONS

The main contribution of this paper is an efficient approximation algorithm for jointly selecting the desired number of features and recovering the time optimal trajectory that allows the agent to track them. We provide performance
Table IV: This table shows the mean of the maximum feature speed profile across all the experiments.

| Trajectory | Feature speed [pixel s⁻¹] | ≤ 288 pixels s⁻¹ | k = 50 | k = 100 | k = 150 | ≤ 432 pixels s⁻¹ | k = 100 |
|------------|---------------------------|-----------------|--------|---------|---------|-----------------|---------|
| Arc        |                           | 23.0246         | 31.0976| 29.5065 | 30.2752 | 23.0246         | 31.0976 |
| Oval       |                           | 98.9926         | 108.0196| 99.8715 | 101.3292| 98.9926         | 108.0196|
| Figure 8   |                           | 76.9707         | 70.5656| 79.7371 | 101.3292| 76.9707         | 70.5656 |
| Slalom     |                           | 24.7777         | 43.6830| 63.5860 | 31.7051 | 24.7777         | 43.6830 |

Fig. 4: Qualitative comparison of the feature speed profiles assuming both a gimbaled camera and non-gimbaled camera.

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