Exact solution of the $\Phi^3_2$ finite matrix model

Naoyuki Kanomata and Akifumi Sako

1 Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162-8601, Japan

Abstract

We find the exact solutions of the $\Phi^3_2$ finite matrix model (Grosse-Wulkenhaar model). In the $\Phi^3_2$ finite matrix model, multipoint correlation functions are expressed as $G_{|a_1|\ldots|a_{N_1}||a_{B}|\ldots|a_{N_B}|}$. The sum over all Feynman diagrams (ribbon graphs) on Riemann surfaces with $B$-boundaries, and each $|a_1|\ldots|a_{N_1}|$ corresponds to the Feynman diagrams having $N_i$-external lines from the $i$-th boundary. It is known that any $G_{|a_1|\ldots|a_{N_1}||a_{B}|\ldots|a_{N_B}|}$ can be expressed using $G_{|a_1|\ldots|a_{N_1}|}$ type $n$-point functions. Thus we focus on rigorous calculations of $G_{|a_1\ldots|a_{N_1}|}$. The formula for $G_{|a_1\ldots|a_{N_1}|}$ is obtained, and it is achieved by using the partition function $Z[J]$ calculated by the Harish-Chandra-Itzykson-Zuber integral. We give $G_{|a_1|}$, $G_{|ab|}$, $G_{|a_b|}$, and $G_{|a_{b}|c|}$ as the specific simple examples. All of them are described by using the Airy functions.

1 Introduction

Matrix models were well studied in the 1980s and 1990s in the context of non-critical string theories. Perturbative expansions of matrix models can be interpreted by using random simplicial decompositions of two-dimensional surfaces. Each Feynman diagram in perturbative expansions of path integrals represents a corresponding simplicial decomposition of a two-dimensional surface. In particular, Feynman diagrams of $\Phi^3$ matrix models can be regarded as triangulations of two-dimensional surfaces. The sum over two-dimensional surfaces corresponds to path integrals of two-dimensional quantum gravity theories. The $1/N$ expansions of the matrix models of size $N$ are equivalent to the genus expansion of the two-dimensional surface. The double scaling limit is a limit that incorporates contributions from any genus $g$ surfaces. Calculating the genus expansion of the two-dimensional surfaces in this limit gives a fully non-perturbative solution for the two-dimensional gravity theory.

Fukuma, Kawai, and Nakayama proved that the Virasoro constraint condition is equivalent to the condition that the solution of the KdV hierarchy satisfies the string equation. Dijkgraaf, Verlinde, and Verlinde also derived the similar results independently. Witten showed that the Witten-Kontsevich $\tau$-function satisfies the string equation. Furthermore, Witten conjectured that the Witten-Kontsevich $\tau$-function is the $\tau$ function of the KdV hierarchy. Using the $\Phi^3$ matrix model, the proof of this conjecture was done by Kontsevich.

We focus on the Grosse-Wulkenhaar $\Phi^3$ model (the Kontsevich’s $\Phi^3$ matrix model). We refer to the Grosse-Wulkenhaar $\Phi^4$ model simply as the $\Phi^4$ model in the following. Quantum field theories on noncommutative spaces such as Moyal spaces have given a new perspective to matrix models. The problem of UV/IR-mixing generally arises when considering the quantum field theories on noncommutative spaces. Grosse and Wulkenhaar modified the $\Phi^4$ theory on the four-dimensional Moyal space by adding harmonic oscillator potentials to the action. Grosse and Wulkenhaar also proved that this model is renormalizable at all orders of perturbation theories. The $\Phi^3$ matrix model on Moyal spaces was studied by Grosse and Steinacker. In particular, the $n$-point functions of the $\Phi^3$ matrix model in the large $N, V$ limit were calculated in the previous studies by Grosse, Wulkenhaar, and one of the authors of this paper. Any $n$-point function of the $\Phi^3$ matrix model was calculated by solving the Schwinger-Dyson equation exactly by using the Ward-Takahashi identity. The Swiss cheese limit picks up only the Genus 0 contribution while preserving the boundary components. The Schwinger-Dyson equation obtained in this limit is the integral equation corresponding to the Riemann-Hilbert problem.
The Schwinger-Dyson equation for the 1 point function in the Swiss cheese limit coincides with the one in Makkeenko-Semenoff[26]. Using the hierarchy of the Schwinger-Dyson equation and this exact solution, any $n$-point function in the large $N, V$ limit was obtained[16,17]. Afterward, the planar 2-point function of the Grosse-Wulkenhaar type $\Phi^4$ model in large $N, V$ limit was solved exactly by Grosse, Wulkenhaar, and Hock[18], $n$-point functions were solved by Wulkenhaar, and Hock[22]. Wulkenhaar, Branahl, and Hock found blobbed topological recursion of the $\Phi^4$ model[3,2].

In this paper, we find exact solutions of the $\Phi^3$ finite matrix model. In calculating the partition function $Z[J]$, the integration of the off-diagonal elements of the Hermitian matrix is done by using the Airy functions. We use the result to calculate $G_{|a|...|a^n|}$ type $n$-point functions. In Section 6, we succeed to find the exact solutions for $G_{|a|...|a^n|}$ type $n$-point functions. Since $n$-point functions $G_{|a_1...a_{N_1}|...|a^n|}$ can be expressed by using $G_{|a|...|a^n|}$ type $n$-point functions, we can obtain all the exact solutions of the $\Phi^3$ finite matrix model. We also give $G_{|a|}, G_{|a|b|}, G_{|a|b|c|}$ as concrete functions as examples.

This paper is organized as follows. In Section 2 we review the $\Phi^3$ matrix model. In Section 3 we carry out the path integral of the partition function $Z[J]$. In Section 4 using the result in Section 3 we calculate the exact solutions of $G_{|a|}, G_{|a|b|}$. In Section 5 we derive the exact solutions of the $n$-point functions. In Section 6 $G_{|a|b|}$ is given as the simple examples. In Section 7 we summarize this paper.

## 2 Setup of $\Phi^3_2$ Matrix Model

In this section, we review the $\Phi^3$ matrix model based on the previous studies[16,17,21], and we determine the notation in this paper.

Let $\Phi = (\Phi_{ij})$ be a Hermitian matrix for $i, j = 1, 2, \ldots, N$ and $E_{m-1}$ be a discretization of a monotonously increasing differentiable function $e$ with $e(0) = 0$,

$$E_{m-1} = \mu^2 \left( \frac{1}{2} + e \left( \frac{m - 1}{\mu^2 V} \right) \right),$$

where $\mu^2$ is a squared mass and $V$ is a real constant. Let $E = (E_{m-1}\delta_{mn})$ be a diagonal matrix for $m, n = 1, \ldots, N$. Let us consider the following action:

$$S[\Phi] = iV \text{tr} \left( E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3 \right),$$

where $\kappa$ is a renormalization constant (real), $\lambda$ is a coupling constant that is non-zero real, and $i = \sqrt{-1}$. Compared to the paper[16], the difference is that $V$ is replaced with $iV$. By the diagonal matrix $E$ that is not proportional to the unit matrix in general, there is no symmetry for the unitary transformation in $\Phi \rightarrow U\Phi U^*$. Here $U$ is a unitary matrix, and $U^*$ is its Hermitian conjugate.

Let $J = (J_{mn})$ be a Hermitian matrix for $m, n = 1, \ldots, N$ as an external field. Let $D\Phi$ be the integral measure,

$$D\Phi := \prod_{i=1}^{N} d\Phi_{ii} \prod_{1 \leq i < j \leq N} d\text{Re}\Phi_{ij} d\text{Im}\Phi_{ij},$$

where each variable is divided into real and imaginary parts $\Phi_{ij} = \text{Re}\Phi_{ij} + i\text{Im}\Phi_{ij}$ with $\text{Re}\Phi_{ij} = \text{Re}\Phi_{ji}$ and $\text{Im}\Phi_{ij} = -\text{Im}\Phi_{ji}$. Let us consider the following partition function:

$$Z[J] := \int D\Phi \exp \left( -S[\Phi] + iV \text{tr}(J\Phi) \right)$$

$$= \int D\Phi \exp \left( -iV \text{tr} \left( E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3 \right) \right) \exp (iV \text{tr}(J\Phi)).$$

This is given as the simple examples. In Section 7 we summarize this paper.
Using \( \log \frac{Z[J]}{Z[0]} \), the \( \sum_{i=1}^{B} N_i \)-point function \( G_{a_1|a_{N_1}|...|a_{p}...a_{N_B}} \) is defined as

\[
\log \frac{Z[J]}{Z[0]} := \sum_{B=1}^{\infty} \sum_{N_1 \leq \cdots \leq N_B} \sum_{p} (iV)^2-B \frac{G_{[p]...[p]_B}}{S_{(N_1,...,N_B)}} \prod_{\beta=1}^{B} \frac{J_{p}...J_{p}}{\beta_{\beta}},
\]

where \( N_i \) is the identical valence number for \( i = 1, \ldots, B \), \( \prod_{j=1}^{B} J_{p_{ji+1}} \) with \( N_i + 1 \equiv 1 \), \((N_1,...,N_B) = (N^i_1,...,N^i_1,...,N^i,...,N^i)\), and \( S_{(N_1,...,N_B)} = \prod_{i=1}^{N} \nu_i \).

3 Calculation of Partition Function \( Z[J] \)

In this section, we perform the integration of the partition function \( Z[J] \) by dividing the Hermitian matrix into its diagonal and off-diagonal elements. The off-diagonal elements of the Hermitian matrix in the partition function \( Z[J] \) are integrated using the Harish-Chandra-Izykson-Zuber integral \[23, [28, [20, [31, and the integration of the diagonal elements of the Hermitian matrix in the partition function \( Z[J] \) is performed by using Airy functions. The calculations are essentially in the line with the calculations of Kontsevich \[24\]. We write without omitting details because the results are different due to the presence of external fields \( J \) and the renormalization term \( \kappa \).

We introduce a Hermitian matrix \( \tilde{E} = (\tilde{E}_{m,n,m,n}) = \frac{1}{\lambda} E = \left( \frac{E_{m-1}}{\lambda} \right)_{m,n=1} \) for \( m,n = 1,\ldots,N \) and \( \frac{\kappa}{\lambda} = \kappa \). Note that the indices are shifted i.e. \( \tilde{E} = \text{diag} (\tilde{E}_1,\ldots,\tilde{E}_N) \) and \( E = \text{diag} (E_0,\ldots,E_{N-1}) \). Then \( Z[J] \) is written as

\[
Z[J] = \int D\Phi \exp \left( -2i\lambda V \text{tr} \left( \frac{\tilde{E}\Phi^2}{2} + \frac{\kappa\Phi}{2} + \frac{1}{3!}\Phi^3 \right) \right) \exp (iV \text{tr} (J\Phi)).
\]

We introduce a new variable \( X = X - \tilde{E} \). Here \( X = (X_{m,n}) \) is a Hermitian matrix, too. We do a change of variables of the integral measure \( D\Phi \) as

\[
d\Phi_{ij} = \sum_{m,n=1}^{N} \frac{\partial \Phi_{ij}}{\partial X_{mn}} dX_{mn} = dX_{ij}.
\]

By the variable transformation

\[
\text{tr} \left( \frac{\tilde{E}\Phi^2}{2} + \frac{\kappa\Phi}{2} + \frac{1}{3!}\Phi^3 \right) = \text{tr} \left( \frac{(X)^3 - 3(\tilde{E})^2 X + 2(\tilde{E})^3 + 3\tilde{E}X - 3\tilde{E}}{6} \right),
\]

then \( Z[J] \) is given as

\[
Z[J] = \int DX \exp \left( -2i\lambda V \text{tr} \left( \frac{(X)^3 - 3(\tilde{E})^2 X + 2(\tilde{E})^3 + 3\tilde{E}X - 3\tilde{E}}{6} \right) \right) \exp \left( iV \text{tr} (JX - J\tilde{E}) \right)
\]

\[
= \exp \left( -i\lambda V \text{tr} \left( \frac{2}{3}(\tilde{E})^3 - 3\tilde{E} + \frac{1}{3}(J\tilde{E}) \right) \right) \int DX \exp \left( +i\lambda V \text{tr} \left( -\tilde{E}X + \frac{1}{3}JX - \frac{1}{3}X^3 + (\tilde{E})^2 X \right) \right)
\]

\[
= \exp \left( -i\lambda V \text{tr} \left( \frac{2}{3}(\tilde{E})^3 - 3\tilde{E} + \frac{1}{3}(J\tilde{E}) \right) \right) \int DX \exp \left( -i\lambda V \text{tr} (X^3) \right) \exp (i\lambda V \tilde{E} \text{tr} ((M - I + K)X)).
\]

Here \( M = \frac{(\tilde{E})^2}{\kappa} = \frac{E^2}{\lambda \kappa} \), \( K = \frac{J}{\kappa} \), and \( I \) is the unit matrix. Note that

\[
DX = \left( \prod_{i=1}^{N} dx_i \right) \left( \prod_{1 \leq k < \ell \leq N} (x_i - x_k)^2 \right) dU,
\]

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where $x_i$ is the eigenvalues of $X$ for $i = 1, \cdots, N$, $dU$ is the Haar probability measure of the unitary group $U(N)$, and $U$ is the unitary matrix which diagonalize $X$. Then (3.4) can be rewritten as the following:

$$Z[J] = \exp \left( -i\lambda V \text{tr} \left( \frac{2}{3} (E^3) - \tilde{E} + \frac{1}{\lambda} J E \right) \right)$$

$$\int \left( \prod_{i=1}^{N} dx_i \exp \left( -i \frac{\lambda V}{3} x_i^3 \right) \right) \left( \prod_{1 \leq k < l \leq N} (x_l - x_k)^2 \right) \int dU \exp \left( i\lambda V \text{tr} \{ (M - I + K) U \tilde{X} U^* \} \right),$$

(3.5)

where $\tilde{X}$ is the diagonal matrix $X = U^* X U$. We use the following formula.

Fact. The Harish-Chandra-Izykson-Zuber integral \cite{23, 28, 20, 31} for the unitary group $U(n)$ is

$$\int_{U(n)} \exp (t \text{tr} (AUBU^*)) \, dU = c_n \frac{\det_{1 \leq i,j \leq n} (\exp (t \lambda_i(A) \lambda_j(B)))}{t^{\frac{n^2-1}{2}} \Delta(\lambda(A)) \Delta(\lambda(B))}.$$  

(3.6)

Here $A = (A_{ij})$, and $B = (B_{ij})$ are some Hermitian matrices whose eigenvalues denoted by $\lambda_i(A)$ and $\lambda_j(B)$ ($i = 1, \cdots, n$), respectively. $t$ is the non-zero complex parameter, $\Delta(\lambda(A)) := \prod_{1 \leq i < j \leq n} (\lambda_i(A) - \lambda_j(A))$ is the Vandermonde determinant, and $c_n := \prod_{i=1}^{n} \frac{i!}{\pi^{\frac{n(n+1)}{2}}}$ is the constant. $(\exp (t \lambda_i(A) \lambda_j(B)))$ is the $n \times n$ matrix with the $i$-th row and the $j$-th column being $\exp (t \lambda_i(A) \lambda_j(B))$.

Applying the Harish-Chandra-Izykson-Zuber integral (3.6) to $\int dU \exp \left( i\lambda V \text{tr} \{ (M - I + K) U \tilde{X} U^* \} \right)$ in (3.5), the result is

$$\int dU \exp \left( i\lambda V \text{tr} \{ (M - I + K) U \tilde{X} U^* \} \right) = C \prod_{1 \leq i,j \leq N} \frac{\det_{1 \leq i,j \leq n} (x_j - x_i) \prod_{i,j}(s_j - s_i)}{\prod_{1 \leq i \leq N} (s_i - s_t)},$$

(3.7)

where $s_t$ is the eigenvalues of the matrix $M - I + K$ for $t = 1, \cdots, N$ and $C = \left( \prod_{p=1}^{N} p! \right) \times \left( \prod_{i,j}(x_j - x_i) \right)^{\frac{N(N-1)}{2}}$. $(\exp (i\lambda V \tilde{r}_i x_i s_j))$ denotes the $N \times N$ matrix with the $i$-th row and the $j$-th column being $\exp (i\lambda V \tilde{r}_i x_i s_j)$. Then the partition function $Z[J]$ is described as

$$Z[J] = \frac{C}{N!} \exp \left( -i\lambda V \text{tr} \left( \frac{2}{3} (E^3) - \tilde{E} + \frac{1}{\lambda} J E \right) \right) \prod_{1 \leq t < u \leq N} \frac{1}{(s_u - s_t)}$$

$$\int \left( \prod_{i=1}^{N} dx_i \exp \left( -i \frac{\lambda V}{3} x_i^3 \right) \right) \left( \prod_{1 \leq k < l \leq N} (x_l - x_k) \right) \det_{1 \leq m,n \leq N} (x_m - x_n) \prod_{1 \leq i \leq N} (s_i - s_t)$$

$$= C \exp \left( -i\lambda V \text{tr} \left( \frac{2}{3} (E^3) - \tilde{E} + \frac{1}{\lambda} J E \right) \right) \prod_{1 \leq t < u \leq N} \frac{1}{(s_u - s_t)}$$

$$\int \left( \prod_{i=1}^{N} dx_i \exp \left( -i \frac{\lambda V}{3} x_i^3 \right) \exp (i\lambda V \tilde{r}_i x_i s_t) \right) \prod_{1 \leq k < l \leq N} (x_l - x_k).$$

(3.8)
Here we use the following result at the second equality in (3.8):

\[
\int \left( \prod_{i=1}^{N} dx_i \exp \left( -i \frac{\lambda V}{3} x_i^3 \right) \right) \left( \prod_{1 \leq i < j \leq N} (x_i - x_j) \right) \det_{1 \leq m, n \leq N} \exp (i \lambda V \kappa x_m s_n) \]

\[= \sum_{\sigma \in S_N} \int \left( \prod_{i=1}^{N} dx_i \exp \left( -i \frac{\lambda V}{3} x_i^3 \right) \right) \left( \prod_{1 \leq i < j \leq N} (x_i - x_j) \right) (-1)^\sigma \left( \prod_{j=1}^{N} e^{i \lambda V \kappa x_j s_j} \right)\]

\[= \sum_{\sigma \in S_N} \int \left( \prod_{i=1}^{N} dx_i \exp \left( -i \frac{\lambda V}{3} x_i^3 \right) \right) \left( \prod_{1 \leq i < j \leq N} (x_i - x_j) \right) (-1)^\sigma \left( \prod_{j=1}^{N} e^{i \lambda V \kappa x_j s_j} \right) \]

\[= N! \int \left( \prod_{i=1}^{N} dx_i \exp \left( -i \frac{\lambda V}{3} x_i^3 \right) \right) \exp (i \lambda V \kappa x_i s_i) \prod_{1 \leq k < l \leq N} (x_l - x_k). \quad (3.9)\]

In the transformation of the above equation from the second line to the third line, we changed variables as \(x_{s(i)} \mapsto x_i (i = 1, \cdots, N)\).

Using \( \prod_{1 \leq k \neq l \leq N} (x_l - x_k) = \det_{1 \leq k, l \leq N} (x_{l-1}^k) \), we calculate the remaining integral in the right hand side in (3.8) as

\[
\int_{-\infty}^{\infty} \left( \prod_{i=1}^{N} dx_i \exp \left( -i \frac{\lambda V}{3} x_i^3 \right) \right) \exp (i \lambda V \kappa x_i s_i) \det_{1 \leq k, l \leq N} (x_{l-1}^k) \]

\[= \sum_{\sigma \in S_N} \sgn \sigma \prod_{i=1}^{N} \phi_{\sigma(i)}(s_i) \]

\[= \det_{1 \leq i, j \leq N} (\phi_i(s_j)), \quad (3.10)\]

where \(\phi_k(z)\) is defined by

\[
\phi_k(z) = \int_{-\infty}^{\infty} dx \; x^{k-1} \exp \left( -i \frac{\lambda V}{3} x^3 + i \lambda V x z \right). \quad (3.11)\]

and \((\phi_i(s_j))\) is the \(N \times N\) matrix with the \(i\)-th row and the \(j\)-th column being \(\phi_i(s_j)\). Summarizing the results (3.8) and (3.10), we obtain the following:

**Proposition 3.1.** Let \(Z[J]\) be the partition function of the \(\Phi_3^2\) matrix model given by (2.4). Then, \(Z[J]\) is given as

\[
Z[J] = C \exp \left( -i \lambda V tr \left( \frac{2}{3} (\overline{E})^3 - \kappa \overline{E} + \frac{1}{\lambda} J \overline{E} \right) \right) \frac{\det_{1 \leq i, j \leq N} (\phi_i(s_j))}{\prod_{1 \leq i \neq u \leq N} (s_u - s_i)}.
\]

Note that \(\phi_k(z)\) is expressed as

\[
\phi_k(z) = \left( \frac{1}{i \lambda V \kappa} \right)^{k-1} \left( \frac{d}{dz} \right)^{k-1} \int_{-\infty}^{\infty} dx \exp \left( -i \frac{\lambda V}{3} x^3 + i \lambda V x z \right). \quad (3.12)
\]

We use Airy function:

\[
Ai(\gamma L) = \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \exp \left[ i \left( Lx + \frac{x^3}{3 \gamma} \right) \right] dx.
\]

Here \(\gamma \in \mathbb{R}\setminus \{0\}\) and \(L \in \mathbb{R}\). Substituting (3.13) for (3.12), \(\phi_k(z)\) is calculated as follows:

\[
\phi_k(z) = \left( \frac{i}{(\lambda V)^{\frac{1}{3}}} \right)^{k-1} \left( \frac{-2 \pi}{(\lambda V)^{\frac{1}{3}}} \right) \left( \frac{d}{dy} \right)^{k-1} \left. Ai[y] \right|_{y = -\frac{x}{(\lambda V)^{\frac{1}{3}}}}.
\]

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Proposition 3.2. ([24]).

Let \( \left( \text{Ai}^{(j-1)}(y_i) \right) \) be the \( N \times N \) matrix with the \( i \)-th row and the \( j \)-th column being \( \text{Ai}^{(j-1)}(y_i) = \left( \frac{d}{dy_i} \right)^{j-1} \text{Ai}(y_i) \). We then obtain the following:

\[
\det \left( \text{Ai}^{(j-1)}(y_i) \right) = \left( \prod_{1 \leq i < j \leq N} (\partial_{y_i} - \partial_{y_j}) \right) \text{Ai}(y_1) \cdots \text{Ai}(y_N).
\]

The proof is omitted in [24], so it is appended for the reader’s convenience.

Proof. We calculate \( \det \left( \text{Ai}^{(j-1)}(y_i) \right) \) according to the definition of the determinant.

\[
\det \left( \text{Ai}^{(j-1)}(y_i) \right) = \sum_{\sigma \in S_N} \text{sgn} \sigma \prod_{k=0}^{N-1} \text{Ai}^{(k)}(y_{\sigma(k+1)})
= \sum_{\sigma \in S_N} \text{sgn} \sigma \prod_{k=0}^{N-1} \partial_{y_{\sigma(k+1)}}^{k} \text{Ai}(y_{\sigma(k+1)}),
\]

where \( S_N \) is the \( N \)-th order symmetry group. For \( \prod_{k=0}^{N-1} \partial_{y_{\sigma(k+1)}}^{k} \) using similar calculation to the Vandermonde determinant, \( \det \left( \text{Ai}^{(j-1)}(y_i) \right) \) is as follows:

\[
\det \left( \text{Ai}^{(j-1)}(y_i) \right) = \sum_{\sigma \in S_N} \text{sgn} \sigma \left( \prod_{k=0}^{N-1} \partial_{y_{\sigma(k+1)}}^{k} \right) \text{Ai}(y_1) \cdots \text{Ai}(y_N)
= \left( \prod_{1 \leq i < j \leq N} (\partial_{y_i} - \partial_{y_j}) \right) \text{Ai}(y_1) \cdots \text{Ai}(y_N).
\]

We introduce

\[
A_N(y_1, \cdots, y_N) = \left( \prod_{1 \leq i < j \leq N} (\partial_{y_i} - \partial_{y_j}) \right) \text{Ai}(y_1) \cdots \text{Ai}(y_N),
\]

where \( y_j = -\frac{V_{ks_j}}{(\lambda \nu)^{2}} \) for \( j = 1, \ldots, N \). From this, \( \det \left( \phi_i(s_j) \right) \) is calculated as follows:

\[
\det_{1 \leq i, j \leq N} \left( \phi_i(s_j) \right) = \left( \frac{i (N(N-1))}{(\lambda \nu)^N} \right) A_N(y_1, \cdots, y_N).
\]

Summarizing above results, we obtain the following:

Proposition 3.3. Let \( Z[J] \) be the partition function of the \( \Phi^4 \) matrix model given by [24]. Then, \( Z[J] \) is given as

\[
Z[J] = \int \mathcal{D} \Phi \exp \left( -i V \text{tr} \left( E \Phi^2 + \kappa \Phi + \frac{\lambda}{3} \Phi^3 \right) \right) \exp (i V \text{tr} (J \Phi))
= C' \left( \prod_{1 \leq t < u \leq N} (s_u - s_t) \right) A_N(y_1, \cdots, y_N),
\]

where \( C' = \exp \left( -i V \text{tr} \left( \frac{2}{3} E^3 - \lambda \kappa E \right) \right) \left( \prod_{p=1}^{N} p! \right) \left( \frac{(-2)^N \pi^{N(N+1)/2}}{\lambda^{N(N+1)/2} \kappa^{N(N-1)/2}} \right), s_t \) is the eigenvalues of the matrix \( \frac{E^2}{\lambda} - I + \frac{J}{\kappa} \) for \( t = 1, \cdots, N \), and \( y_j = -\frac{V_{ks_j}}{(\lambda \nu)^{2}} \) for \( j = 1, \cdots, N \).
4 Calculation of 1-Point Function $G_{[a]}$ and 2-Point Functions $G_{[ab]}$

In the calculation of the 1-point function $G_{[a]}$, the external field $J$ can be treated as the diagonal matrix $J = \text{diag} (J_1, \cdots, J_N)$. Then the eigenvalues $s_i$ in (3.19) is given $s_i = \frac{\lambda(\bar{E}_i)^2 + J_{ii}}{\kappa} - 1$. Then, the 1-point function $G_{[a]}$ is calculated as follows:

$$G_{[a]} = \frac{1}{\text{Vol}} \frac{\partial \log Z[J]}{\partial J_{aa}} \bigg|_{J=0}$$

$$= \left( \frac{1}{\text{Vol}} \frac{\partial}{\partial J_{aa}} \left( \prod_{1 \leq u < w \leq N} \left( \frac{\lambda(E_u)^2 - \lambda(E_v)^2 + (J_{uu} - J_{ww})}{\kappa} \right) \right) \right) \bigg|_{J=0}$$

$$= \frac{A_N(y_1, \cdots, y_N)}{\prod_{1 \leq p < q \leq N} \left( \frac{\lambda(E_p)^2 - (E_q)^2}{\kappa} \right)}$$

where $y_j = -\frac{VE_{j-1}}{(\lambda \text{Vol})^2} + \frac{VJ_{jj}}{(\lambda \text{Vol})^2}$ for $j = 1, \cdots, N$. Note that

$$\frac{\partial}{\partial J_{aa}} \left( e^{-iV \text{tr}(J\bar{E})} A_N(y_1, \cdots, y_N) \right) = -iV E_{aan} e^{-iV \text{tr}(J\bar{E})} A_N(y_1, \cdots, y_N) + e^{-iV \text{tr}(J\bar{E})} \left( -\frac{V}{(\lambda \text{Vol})^2} \right) (a a_n A_N(y_1, \cdots, y_N)),$$

(4.2)

where $\frac{\partial}{\partial a_n} A_N(y_1, \cdots, y_N) = \frac{\partial}{\partial y_a} A_N(y_1, \cdots, y_N)$. Next, we use the following formula. Let $v_n = v_n(\vec{x}_n) = \det_{1 \leq i, j \leq N} (x_j)^{i-1}$ be the Vandermonde determinant for $\vec{x}_n = (x_1, \cdots, x_n) \in \mathbb{R}^n$. For any $1 \leq k \leq n$

$$\frac{\partial v_n}{\partial x_k} = \sum_{i=1, i \neq k}^n \frac{v_n(\vec{x}_n)}{x_k - x_i}.$$

(4.3)

(See for example [23].) Using this formula, we get

$$\frac{\partial}{\partial J_{aa}} \left( \frac{1}{\kappa} \left( \frac{N(N-1)}{2} \right) \det_{1 \leq i, j \leq N} \left( \frac{\lambda(E_i)^2 + J_{ij}}{(\lambda \text{Vol})^2} \right)^{i-1} \right)$$

$$= \left( \frac{1}{\kappa} \right) \sum_{i=1, i \neq a}^N \left( \frac{\lambda(E_i)^2 + J_{aa}}{(\lambda \text{Vol})^2} \right)^{i-1} \det_{1 \leq i, j \leq N} \left( \frac{(\lambda(E_i)^2 + J_{ii})}{(\lambda \text{Vol})^2} \right).$$

(4.4)

Substituting (4.2) and (4.4) into (4.1), finally $G_{[a]}$ is expressed as

$$G_{[a]} = -\frac{E_{a-1}}{\lambda} - \frac{1}{\kappa} \sum_{i=1, i \neq a}^N \frac{1}{E_{a-1}^2 - E_i^2} + \left( \frac{1}{i} \right) \left( -\frac{1}{(\kappa \text{Vol})^2} \right) \frac{\partial}{\partial a_n} \log A_N(z_1, \cdots, z_N),$$

(4.5)

where $z_j = \frac{VE_{j-1}}{(\lambda \text{Vol})^2} + \frac{VJ_{jj}}{(\lambda \text{Vol})^2}$ for $j = 1, \cdots, N$, and $\frac{\partial}{\partial a_n} = \frac{\partial}{\partial a_n}$.

Next, let us consider 2-point functions $G_{[ab]}$ ($a \neq b$, $a, b \in \{1, 2, \cdots, N\}$). For the calculation, we put $J$ as the matrix all components without $J_{ab}, J_{ba}$ are zero. Note that $\text{tr} J E = \text{tr} J \bar{E} = 0$ for this $J$.
At first, we estimate eigenvalues \( s_t \) for \( t = 1, \ldots, N \) of the matrix \( M - I + K \). The eigenequation is

\[
0 = \det (sI - (M - I + K)) = \left( \prod_{i=1, i \neq a, b}^{N} \left( s - \frac{E_{t-1}^{2}}{\kappa} + 1 \right) \right)^{2} + \left( \frac{E_{t-1}^{2}}{\kappa} \right)^{2} + \left( \frac{E_{t-1}^{2}}{\kappa} - \frac{E_{t-2}^{2}}{\kappa} + 2 \right) s
\]

(4.6)

Eigenvalues of the matrix \( M - I + K \) are labeled as \( s_t = \frac{E_{t-1}^{2}}{\kappa} - 1 \) for \( t \neq a, b, \)

\[
s_a = \frac{E_{a-1}^{2}}{\kappa} + \frac{E_{b-1}^{2}}{\kappa} - 2 + \sqrt{\left( \frac{E_{a-1}^{2}}{\kappa} - \frac{E_{b-1}^{2}}{\kappa} \right)^{2} + 4 \times \frac{J_{ab}J_{ba}}{\kappa^{2}}},
\]

\[
s_b = \frac{E_{b-1}^{2}}{\kappa} + \frac{E_{a-1}^{2}}{\kappa} - 2 - \sqrt{\left( \frac{E_{a-1}^{2}}{\kappa} - \frac{E_{b-1}^{2}}{\kappa} \right)^{2} + 4 \times \frac{J_{ab}J_{ba}}{\kappa^{2}}},
\]

(4.7)

(4.9)

and

\[
s_a|_{J=0} = \frac{E_{a-1}^{2}}{\kappa} - 1
\]

(4.8)

\[
s_b|_{J=0} = \frac{E_{b-1}^{2}}{\kappa} - 1.
\]

(4.10)

Let us calculate \( G_{[ab]} \) by using these \( s_t \).

\[
G_{[ab]} = \frac{1}{iV} \frac{\partial^{2} \log Z(J)}{\partial J_{ab} \partial J_{ba}} \bigg|_{J=0} = \frac{\partial^{2}}{\partial J_{ab} \partial J_{ba}} \bigg|_{J=0} \frac{A_{N}(y_{1}, \ldots, y_{N})}{\prod_{1 \leq t < u \leq N} (s_u - s_t)} \bigg|_{J=0} = \frac{1}{iV} \frac{A_{N}(y_{1}, \ldots, y_{N})|_{J=0}}{\prod_{1 \leq t < u \leq N} (s_u - s_t)|_{J=0}}
\]

(4.11)

Here we use \( \frac{\partial A_{N}(y_{1}, \ldots, y_{N})}{\partial J_{ab}} \bigg|_{J=0} = \frac{\partial \det_{1 \leq k, l \leq N} (s_t)^{k-1}}{\partial J_{ab}} \bigg|_{J=0} = 0 \), since \( s_a \) and \( s_b \) are functions of \( (J_{ab}J_{ba}) \) as we see in (4.7) and (4.9), then \( \frac{\partial A_{N}(y_{1}, \ldots, y_{N})}{\partial J_{ab}} \) and \( \frac{\partial \det_{1 \leq k, l \leq N} (s_t)^{k-1}}{\partial J_{ab}} \) are of the form \( J_{ab} \times \) \( (\ldots) \).

Recall that \( y_{k} = -\frac{V_{K}S_{k}}{(\lambda V)^{2}} \). Using the fact that \( \frac{\partial y_{k}}{\partial J_{ab}} \bigg|_{J=0} = 0 \) and \( \frac{\partial^{2} y_{a}}{\partial J_{ab} \partial J_{ba}} = -\frac{V_{\lambda}}{(\lambda V)^{2}} \frac{1}{E_{a-1}^{2} - E_{b-1}^{2}} = -\frac{\partial^{2} y_{b}}{\partial J_{ab} \partial J_{ba}} \), we obtain

\[
\frac{\partial^{2} A_{N}(y_{1}, \ldots, y_{N})}{\partial J_{ab} \partial J_{ba}} \bigg|_{J=0} = \frac{V_{\lambda}}{(\lambda V)^{2}} \frac{1}{E_{a-1}^{2} - E_{b-1}^{2}} (\partial_{s} A_{N}(z_{1}, \ldots, z_{N}) - \partial_{a} A_{N}(z_{1}, \ldots, z_{N})),
\]

(4.12)

8
where \( z_j = -\frac{VE_{j-1}}{(\lambda V)^2} + \frac{Vk}{(\lambda V)^7} \) for \( j = 1, \ldots, N \). Similarly, we get

\[
\frac{\partial^2}{\partial J_{ab} \partial J_{a'b'}} \left\{ \prod_{1 \leq i < u \leq N} (s_u - s_i) \right\}_{J=0} = \frac{\lambda^2}{E_{a-1}^2 - E_{b-1}^2} \left( \sum_{i=1, i \neq a}^{N} \frac{1}{E_{a-1}^2 - E_{i-1}^2} - \sum_{i=1, i \neq b}^{N} \frac{1}{E_{b-1}^2 - E_{i-1}^2} \right) \det_{1 \leq k, l \leq N} (s_k - s_l)^{i-1},
\]

(4.13)

where we use the formula (4.13), again. Substituting (4.12) and (4.13) into (4.11), \( G_{ab} \) (b < a, and \( E_b < E_a \)) is finally obtained as

\[
G_{ab} = \frac{\lambda^2}{i(V)^2} \sum_{i=1, i \neq a}^{N} \frac{1}{(E_{a-1}^2 - E_{i-1}^2)} \left( E_{a-1}^2 - E_{b-1}^2 \right) \left( \frac{1}{E_{i-1}^2 - E_{b-1}^2} \right) + \frac{\lambda^2}{i(V)} \sum_{i=1, i \neq b}^{N} \frac{1}{(E_{b-1}^2 - E_{i-1}^2)} \left( E_{b-1}^2 - E_{a-1}^2 \right) \left( \frac{1}{E_{i-1}^2 - E_{a-1}^2} \right)
\]

\[
- \frac{\lambda}{i(V)^2} \frac{1}{E_{a-1}^2 - E_{b-1}^2} \partial_a \log A_N(z_1, \ldots, z_N) + \frac{\lambda}{i(V)^2} \frac{1}{E_{b-1}^2 - E_{a-1}^2} \partial_b \log A_N(z_1, \ldots, z_N).
\]

(4.14)

We now refer to the Schwinger-Dyson equation

\[
G_{ab} = \frac{1}{E_{a-1}^2 - E_{b-1}^2} \left( 1 + \lambda \left( \frac{G_{ab} - G_{ba}}{E_{a-1}^2 - E_{b-1}^2} \right) \right)
\]

(4.15)

in reference [16]. Substituting (4.14) for the left side of (4.15) and (4.15) for the right side of (4.15) shows that Schwinger-Dyson equation (4.15) is indeed satisfied.

5 Calculation of n-Point Functions \( G_{[a_1^a][a_2^a] \ldots [a_n^a]} \)

The goal of this section is to obtain the explicit formula of the n-point function \( G_{[a_1^a][a_2^a] \ldots [a_n^a]} \). Here \( a^\beta \) is the pairwise different indices for \( \beta = 1, \ldots, n \). From the definition in (2.5), the n-point function \( G_{[a_1^a][a_2^a] \ldots [a_n^a]} \) is given by

\[
G_{[a_1^a][a_2^a] \ldots [a_n^a]} = (iV)^{n-2} \frac{\partial^n}{\partial J_{a_1} \partial J_{a_2} \ldots \partial J_{a_n}} \log \left( \frac{Z[f]}{Z[0]} \right)_{J=0}.
\]

(5.1)

We use the formula in (19):

\[
\frac{\partial^n}{\partial x_1 \cdot \cdot \cdot \partial x_n} f(y) = \sum_{\pi} f^{[\pi]}(y) \prod_{B \in \pi} \frac{\partial^{[j]}}{\partial x_j},
\]

(5.2)

where \( f(y) \) is the differentiable function of the variable \( y = y(x_1, x_2, \ldots, x_n) \), \( \sum_{\pi} \) means the sum over all partitions \( \pi \) of the set \( \{1, \ldots, n\} \), \( \prod_{B \in \pi} \) is the product over all the parts \( B \) of the partition \( \pi \), and \( |S| \) denotes the cardinality of any set \( S \). Applying (5.2) to (5.1) n-point functions \( G_{[a_1^a][a_2^a] \ldots [a_n^a]} \) is expressed as follows:

\[
G_{[a_1^a][a_2^a] \ldots [a_n^a]} = (iV)^{n-2} \sum_{\pi} \left\{ \left( \frac{d}{dx} \right)^{|\pi|} \log x \right\} \prod_{B \in \pi} \frac{\partial^{[j]}}{\partial J_{a_j} a_j} \left|_{J=0} \right.
\]

(5.3)

After the calculation of

\[
\prod_{j \in B} \frac{\partial^{[j]}}{\partial J_{a_j} a_j} \left|_{J=0} \right.
\]

we get the following result.
Lemma 5.1. We introduce \( C = \exp\left( \frac{\lambda V}{\lambda V} + \frac{2}{3} \lambda^3 \right) \left( \prod_{j=1}^N \left( \frac{2}{3} \lambda^3 \right) \right) \left( -2 \right)^N \frac{\lambda^{N+1}}{V^{N(N+1)}/\lambda^{N(N+1)}} \). Then

\[
\left. \frac{\partial |B| Z[J]}{\prod_{j \in B} \partial J_{a_0,j}} \right|_{J=0} = C \sum_{S \subseteq B} \left( \prod_{i \in S} \left( -i V E_{a_0}^l \right) \right) \sum_{M \subseteq S} \left( \left\{ \prod_{k \in M} \left( -\frac{V}{\lambda \lambda V} \right) \partial_{a^k} \right\} A_N(z_1, \ldots, z_N) \right) \left( \left\{ \prod_{q \in M} \frac{\partial}{\partial a_q} \right\} \frac{1}{\det_{1 \leq l \leq N} \left( \lambda_{l}^{N-1} \right)} \right),
\]

(5.4)

where \( z_j = -\frac{VE_{j-1}^2}{\lambda \lambda V} + \frac{V \kappa}{\lambda \lambda V} \) for \( j = 1, \ldots, N \), \( \partial_{a^k} = \frac{\partial}{\partial z_{a^k}} \) (\( k \in M \)), \( t_l = \frac{(E_l - 1)^2}{\lambda} \) for \( l = 1, \ldots, N \), \( S \) runs through the set of all subsets of \( B \), \( \overline{S} \) is the complement of \( S \) in \( B \), \( M \) runs through the set of all subsets of \( \overline{S} \), and \( \overline{M} = \overline{S} \backslash M \).

Proof. For the calculation of \( G_{[a^1 \ast [a^2 \cdots \ast [a^n \ast] \]}, J_{1, \ldots, J_N} \), we can choose \( J \) as a diagonal matrix \( \text{diag}(J_{1, \ldots, J_N}) \). Then, \( s_t = \frac{\lambda(E_t)^2 + J_t}{\kappa} - 1 \). To calculate

\[
\left. \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \right|_{x=x_0} (uv) = \sum_S \frac{\partial^{|S|}_u}{\prod_{j \in S} \partial x_j} \frac{\partial^{(n-|S|)}_v}{\prod_{j \in S} \partial x_j},
\]

(5.6)

we use the formula in [19]:

\[
\left. \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \right|_{x=x_0} (uv) = \sum_S \frac{\partial^{|S|}_u}{\prod_{j \in S} \partial x_j} \frac{\partial^{(n-|S|)}_v}{\prod_{j \in S} \partial x_j},
\]

where \( u \) and \( v \) are differentiable functions of the variable \( x = (x_1, x_2, \ldots, x_n) \), and \( S \) runs through the set of all subsets of \( \{1, \ldots, n\} \).

Using the formula (5.4) twice for (5.3), we obtain the following:

\[
\left. \frac{\partial^{|B|} Z[J]}{\prod_{j \in B} \partial J_{a_0,j}} \right|_{J=0} = C \sum_{S \subseteq B} \left( \prod_{i \in S} \left( -i V E_{a_0}^l \right) \right) \sum_{M \subseteq S} \left( \left\{ \prod_{k \in M} \left( -\frac{V}{\lambda \lambda V} \right) \partial_{a^k} \right\} A_N(z_1, \ldots, z_N) \right) \left( \left\{ \prod_{q \in M} \frac{\partial}{\partial a_q} \right\} \frac{1}{\det_{1 \leq l \leq N} \left( \lambda_{l}^{N-1} \right)} \right),
\]

(5.7)

For the diagonal \( J, y_k = \left( -\frac{VE_{k-1}^2}{\lambda \lambda V} - 1 + \frac{J_{kk}}{\kappa} \right) \), then the above is rewritten as

\[
\left. \frac{\partial^{|B|} Z[J]}{\prod_{j \in B} \partial J_{a_0,j}} \right|_{J=0} = C \sum_{S \subseteq B} \left( \prod_{i \in S} \left( -i V E_{a_0}^l \right) \right) \sum_{M \subseteq S} \left( \left\{ \prod_{k \in M} \left( -\frac{V}{\lambda \lambda V} \right) \partial_{a^k} \right\} A_N(z_1, \ldots, z_N) \right) \times \left( \left\{ \prod_{q \in M} \frac{\partial}{\partial a_q} \right\} \frac{1}{\det_{1 \leq l \leq N} \left( \lambda_{l}^{N-1} \right)} \right),
\]

(5.8)

where \( z_j = -\frac{VE_{j-1}^2}{\lambda \lambda V} + \frac{V \kappa}{\lambda \lambda V} \) for \( j = 1, \ldots, N \), \( t_l = \frac{(E_l - 1)^2}{\lambda} \) for \( l = 1, \ldots, N \), \( S \) runs through the set of all subsets of \( B \), \( \overline{S} \) is the complement of \( S \) in \( B \), \( M \) runs through the set of all subsets of \( \overline{S} \), and \( \overline{M} = \overline{S} \backslash M \). \( \square \)
Note the cases that each set is an empty set,

$$\prod_{i \in \emptyset} \left( -iV \frac{E_{a_{i-1}}}{\lambda} \right) = 1,$$

$$\left\{ \prod_{k \in \emptyset} \left( - \frac{V}{(\lambda V)^{1/2}} \right) \partial_{a_k} \right\} \Delta_N(z_1, \ldots, z_N) = \Delta_N(z_1, \ldots, z_N),$$

and

$$\left\{ \prod_{q \in \emptyset} \frac{\partial}{\partial \partial_{a_q}} \right\} \frac{1}{\det_{1 \leq i, j \leq N} \left( t_i^{l-1} \right)} = \frac{1}{\det_{1 \leq i, j \leq N} \left( t_i^{l-1} \right)}.$$

Summarizing and the result in Lemma 5.1, we obtain the following:

**Theorem 5.2.** We suppose the partition function $Z[J]$ of the $\Phi^2_3$ matrix model is defined by (2.4) and $G_{[a_1^{a_2} \cdots |a_n]}$ is defined by (5.1). In this case,

$$G_{[a_1^{a_2} \cdots |a_n]} = (iV)^{n-2} C \sum_\pi \left\{ \frac{d}{dx} \right\}_{x \in \mathbb{Z}[0]} \prod_{B \in \pi} \sum_{S \subset B} \left( \prod_{i \in S} \left( -iV \frac{E_{a_{i-1}}}{\lambda} \right) \right) \times \sum_{M \subset S} \left( \prod_{k \in M} \left( - \frac{V}{(\lambda V)^{1/2}} \right) \partial_{a_k} \right) \Delta_N(z_1, \ldots, z_N) \left( \prod_{q \in M} \frac{\partial}{\partial \partial_{a_q}} \right) \frac{1}{\prod_{1 \leq i, j \leq N} \left( t_i^{l-1} \right)} \right), \quad (5.9)$$

where $\sum_\pi$ means the sum over all partitions $\pi$ of the set $\{1, \ldots, n\}$, $\prod_{B \in \pi}$ is over all of the parts $B$ of the partition $\pi$, $|S|$ denotes the cardinality of any set $S$, $\sum_{S \subset B}$ means the sum over all subsets of $B$, $\sum_{M \subset S}$ means the sum over all subsets of $S = B \setminus S$, and $t_1 = \left( \frac{E_{a-1}}{\lambda} \right)^2$.

Now we refer to the formula in Section 5 in [16].

**Theorem 5.3.** We suppose $G_{[a_1^{a_2} \cdots |a_n]}$ is defined by (5.1). In this case,

$$G_{[a_1^{a_2} \cdots |a_n]} = \lambda^{N_1 + \cdots + N_B - B} \sum_{k_1=1}^{N_1} \cdots \sum_{k_B=1}^{N_B} G_{[a_1^{k_1} \cdots |a_B^{k_B}]} \left( \prod_{1 \leq i_1 \neq k_1} N_{1} \right) \left( \prod_{1 \leq i_{1,1} \neq k_{1,1}} P_{a_{k_{1,1}}} \right) \cdots \left( \prod_{1 \leq i_{B,1} \neq k_{B,1}} P_{a_{k_{B,1}}} \right), \quad (5.10)$$

where $2 \leq B$, $N_i > 1$ for $i = 1, \ldots, B$, $\lambda$ is the coupling constant(real), and $P_{ab} := \frac{1}{E_{a-1} - E_{b-1}}$.

Substituting (5.9) into (5.10), all the exact solutions of the $\Phi^2_3$ finite matrix model is obtained.

For the later convenience, we introduce a function $F(S, M, \overline{M})$. Let $B$ be a subset of $\{1, \ldots, n\}$. For $S \subset B$, $\overline{S}$ denotes the complement $B \setminus S$.

$$F(S, M, \overline{M}) := \left( \prod_{i \in S} \left( -iV \frac{E_{a_{i-1}}}{\lambda} \right) \right) \left( \prod_{k \in M} \left( - \frac{V}{(\lambda V)^{1/2}} \right) \partial_{a_k} \right) \Delta_N(z_1, \ldots, z_N) \left( \prod_{q \in \overline{M}} \frac{\partial}{\partial \partial_{a_q}} \right) \frac{1}{\det_{1 \leq i, j \leq N} \left( t_i^{l-1} \right)}, \quad (5.11)$$

where $B = S \cup \overline{S}$, $\overline{S} = M \cup \overline{M}$, and $\partial_{a_k} = \frac{\partial}{\partial z_{a_k}}$. Using this $F(S, M, \overline{M})$, $G_{[a_1^{a_2} \cdots |a_n]}$ is expressed as

$$G_{[a_1^{a_2} \cdots |a_n]} = (iV)^{n-2} C \sum_\pi \left\{ \frac{d}{dx} \right\}_{x \in \mathbb{Z}[0]} \prod_{B \in \pi} \sum_{S \subset B} \sum_{M \subset \overline{S}} F(S, M, \overline{M}). \quad (5.12)$$
6 Calculation of Two-Point Functions $G_{[a|b]}$

The formula (6.2) is used to obtain $G_{[a|b]}$ concretely. We use $a = a_1$ and $b = a_2$ below. At first we estimate the case of $\pi = \{1, 2\}$. In this case $|\pi| = 1$, and it is enough to calculate $F(S, M, \nabla M)$ for $B = \{1, 2\}$. In the context of Theorem 5.2 it corresponds to the part:

$$\left. \frac{d}{dx} \right|_{x=2[q]} (\log x) \prod_{0 \leq j < B} \left. \frac{\partial^{|B|} Z[J]}{\partial J_{a_j^{|a_j|}}} \right|_{J=0} = \frac{1}{Z[0]} \left. \frac{\partial^2 Z[J]}{\partial J_{a_1} \partial J_{a_2}} \right|_{J=0}. \quad (6.1)$$

Calculating all cases for sets $S$, $M$, and $\nabla M$, we obtain the following results. In the case of $F(\{1, 2\}, 0, 0)$,

$$F(\{1, 2\}, 0, 0) = \left( -iV \frac{E_{a_1-1}}{\lambda} \right) \left( -iV \frac{E_{b-1}}{(\lambda V)^{\frac{t}{2}}} \right) \frac{1}{\det_{1 \leq l, j \leq N} (t_i^{l-1})} A_N(z_1, \ldots, z_N). \quad (6.2)$$

In the case of $F(\{1\}, \{2\}, 0)$,

$$F(\{1\}, \{2\}, 0) = \left( -iV \frac{E_{a_1-1}}{\lambda} \right) \left( -V \frac{E_{b-1}}{(\lambda V)^{\frac{t}{2}}} \right) \frac{1}{\det_{1 \leq l, j \leq N} (t_i^{l-1})} \partial_b A_N(z_1, \ldots, z_N). \quad (6.3)$$

$F(\{2\}, \{1\})$ can be calculated in the same way (6.3). The letters $a$ and $b$ in (6.3) are interchanged. In the case of $F(\emptyset, \{1, 2\}, 0)$,

$$F(\emptyset, \{1, 2\}, 0) = \left( -iV \frac{E_{a_1-1}}{\lambda} \right) \left( -V \frac{E_{b-1}}{(\lambda V)^{\frac{t}{2}}} \right) \frac{1}{\det_{1 \leq l, j \leq N} (t_i^{l-1})} \partial_a A_N(z_1, \ldots, z_N). \quad (6.4)$$

In the case of $F(\{1\}, \emptyset, \{2\})$,

$$F(\{1\}, \emptyset, \{2\}) = \left( -iV \frac{E_{a_1-1}}{\lambda} \right) A_N(z_1, \ldots, z_N) \frac{-1}{\det_{1 \leq l, j \leq N} (t_i^{l-1})} \sum_{i=1, i \neq b}^N \frac{1}{t_b - t_i} \quad (6.5)$$

$F(\{2\}, \emptyset, \{1\})$ can be calculated in the same way (6.5). The letters $a$ and $b$ in (6.5) are interchanged. In the case of $F(\emptyset, \{1\}, \{2\})$,

$$F(\emptyset, \{1\}, \{2\}) = \left( -V \frac{E_{a_1-1}}{(\lambda V)^{\frac{t}{2}}} \right) \partial_a A_N(z_1, \ldots, z_N) \frac{-1}{\det_{1 \leq l, j \leq N} (t_i^{l-1})} \sum_{i=1, i \neq b}^N \frac{1}{t_b - t_i} \quad (6.6)$$

$F(\emptyset, \{2\}, \{1\})$ can be calculated in the same way (6.6). The letters $a$ and $b$ in (6.6) are interchanged. In the case of $F(\emptyset, \emptyset, \{1, 2\})$,

$$F(\emptyset, \emptyset, \{1, 2\}) = A_N(z_1, \ldots, z_N) \frac{1}{\det_{1 \leq l, j \leq N} (t_i^{l-1})} \left( \sum_{i=1, i \neq a}^N \frac{1}{t_a - t_i} \sum_{j=1, j \neq b}^N \frac{1}{t_b - t_j} - \frac{1}{(t_a - t_b)^2} \right). \quad (6.7)$$

From this, (6.1) can be calculated as follows:

$$\left. \frac{1}{Z[0]} \frac{\partial^2 Z[J]}{\partial J_{a_1} \partial J_{a_2}} \right|_{J=0} = \left( \frac{\det_{1 \leq l, j \leq N} (t_i^{l-1})}{A_N(z_1, \ldots, z_N)} \right) \left\{ F(\{1, 2\}, \emptyset, 0) + F(\emptyset, \{1, 2\}, 0) + F(\emptyset, \emptyset, \{1, 2\}) \right\} + \sum_{l, n=1, l \neq n}^2 \left( F(\{l\}, \{n\}, 0) + F(\{l\}, \emptyset, \{n\}) + F(\emptyset, \{l\}, \{n\}) \right). \quad (6.8)$$
Next step, let us consider the case \( \pi = \{1, 2\} \), \( |\pi| = 2 \), \( B = \{1\} \), or \( \{2\} \).

The corresponding term \( \left( \frac{d}{dx} \right)^{|\pi|} (\log x) \left|_{x=x=Z[0]} \prod_{B \in \pi} \left[ \frac{\partial \partial J }{ \partial J_{a_j a_j} } \right]_{J=0} \right. \) in Theorem 5.2 is as follows:

\[
\left( \frac{d}{dx} \right)^{|\pi|} (\log x) \left|_{x=x=Z[0]} \prod_{B \in \pi} \left[ \frac{\partial \partial J }{ \partial J_{a_j a_j} } \right]_{J=0} \right. = - \frac{1}{Z[0]^2} \frac{\partial Z[J]}{\partial J_{a}} \left|_{J=0} \right. - \frac{1}{Z[0]^2} \frac{\partial Z[J]}{\partial J_{b}} \left|_{J=0} \right. . \quad (6.9)
\]

Calculating all cases for sets \( S \), \( M \), and \( M \) of \( B = \{1\} \), we obtain the following results.

\[
F(\{1\}, \emptyset, \emptyset) = - i V E_{\alpha-1} \frac{1}{\lambda} A_N(z_1, \ldots, z_N) \frac{1}{\det_{1 \leq i, j \leq N} (t_i^j)} . \quad (6.10)
\]

\[
F(\emptyset, \{1\}, \emptyset) = \left( - \frac{V}{(AV)^{\frac{1}{2}}} \right) \frac{1}{A_N(z_1, \ldots, z_N)} \frac{1}{\det_{1 \leq i, j \leq N} (t_i^j)} . \quad (6.11)
\]

\[
F(\emptyset, \emptyset, \{1\}) = A_N(z_1, \ldots, z_N) \frac{-1}{\det_{1 \leq i, j \leq N} (t_i^j)} \sum_{i=1, i \neq a}^N 1 = \frac{1}{t_a - t_i} . \quad (6.12)
\]

These results can be summarized as follows:

\[
\frac{1}{Z[0]} \frac{\partial Z[J]}{\partial J_{a}} \bigg|_{J=0} = \left( \frac{\det_{1 \leq i, j \leq N} (t_i^j)}{A_N(z_1, \ldots, z_N)} \right) \left\{ F(\{1\}, \emptyset, \emptyset) + F(\emptyset, \{1\}, \emptyset) + F(\emptyset, \emptyset, \{1\}) \right\} . \quad (6.13)
\]

The same calculation is performed for \( B = \{2\} \) as for \( B = \{1\} \):

\[
\frac{1}{Z[0]} \frac{\partial Z[J]}{\partial J_{b}} \bigg|_{J=0} = \left( \frac{\det_{1 \leq i, j \leq N} (t_i^j)}{A_N(z_1, \ldots, z_N)} \right) \left\{ F(\{2\}, \emptyset, \emptyset) + F(\emptyset, \{2\}, \emptyset) + F(\emptyset, \emptyset, \{2\}) \right\} . \quad (6.14)
\]

Note that (6.13) and (6.14) coincide with \( iV \) multiples of the one-point function \( G_{[a]} \) and \( G_{[b]} \) in Section 3. Substituting (6.13) and (6.14) into (6.9) gives the result:

\[
\frac{1}{Z[0]} \frac{\partial Z[J]}{\partial J_{a}} \bigg|_{J=0} - \frac{1}{Z[0]} \frac{\partial Z[J]}{\partial J_{b}} \bigg|_{J=0} = - \left( \frac{\det_{1 \leq i, j \leq N} (t_i^j)}{A_N(z_1, \ldots, z_N)} \right)^2 \sum_{i=1}^2 \prod_{l=1} F(\{l\}, \emptyset, \emptyset) + F(\emptyset, \{l\}, \emptyset) + F(\emptyset, \emptyset, \{l\}) . \quad (6.15)
\]

Finally, adding (6.8) and (6.11) the result of the two point functions \( G_{[a][b]} \) is obtained by

\[
G_{[a][b]} = \frac{1}{Z[0]} \frac{\partial^2 Z[J]}{\partial J_{a,b} \partial J_{b}} \bigg|_{J=0} - \frac{1}{Z[0]} \frac{\partial^2 Z[J]}{\partial J_{a,b} \partial J_{b}} \bigg|_{J=0} = \left( \frac{\det_{1 \leq i, j \leq N} (t_i^j)}{A_N(z_1, \ldots, z_N)} \right)^2 \left\{ F(\{1, 2\}, \emptyset, \emptyset) + F(\emptyset, \{1, 2\}, \emptyset) + F(\emptyset, \emptyset, \{1, 2\}) \right\} + \sum_{l, n=1, l \neq n}^2 \left( F(\{l\}, \{n\}, \emptyset) + F(\{l\}, \emptyset, \{n\}) + F(\emptyset, \{l\}, \{n\}) \right) . \quad (6.16)
\]

For a more complex example, we carry out the calculation for \( G_{[a_1][a_2][a_3]} \) in Appendix A.
7 Summary

In this paper, we found the exact solutions of the $\Phi^3$ finite matrix model (Grosse-Wulkenhaar model). In the $\Phi^3$ finite matrix model, multipoint correlation functions were expressed as $G_{a_1 \cdots a_N | a_1 \cdots a_N}$.

It is known that any $G_{a_1 \cdots a_N | a_1 \cdots a_N}$ can be expressed using $G_{a_1 \cdots a_N}$ type n-point functions as $(5.10)$. Thus we focused on rigorous calculations of $G_{a_1 \cdots a_N}$.

In Section 3, the integration of the off-diagonal elements of the Hermitian matrix was calculated using the Harish-Chandra-Itzykson-Zuber integral $(23, 28, 29, 31)$ in calculating the partition function $Z[J]$. Next the integral of the diagonal elements of the Hermitian matrix was calculated using the Airy functions as similar to $(24)$. In Section 4 and 5, we used the obtained partition function $Z[J]$ to calculate $G_{a_1 \cdots a_N}$ type n-point functions and $G_{a_i}$ type n-point functions. The exact solutions of $G_{a_1 \cdots a_N}$ type n-point functions were found by calculating the n-th derivative $\partial^n / \partial J_{a_1} \cdots \partial J_{a_N}$ of log $Z[J]$ with the external field $J$ as a diagonal matrix. The result of the calculations for $G_{a_1 \cdots a_N}$ was described in Theorem 5.2.

In the formula for $G_{a_1 \cdots a_N}$ in Theorem 5.2 no integral remains. More concretely, the n-point function was determined by a function $F(S, M, M)$ whose variables are $S \subset B$, $M$, and $M$ $(B \setminus S = M \cup M)$ as formula $(5.12)$, where $B$ is an element of a partition of $\{1, \cdots, n\}$. Since the algorithm for finding the exact solutions of $G_{a_1 \cdots a_N}$ type n-point functions is explicitly determined in the formula of Theorem 5.2 the exact solutions can be obtained automatically. Indeed, the calculations for $G_{a_i}$ and $G_{a_i}$ were carried out in Section 6 and Appendix A respectively. Since a general $(N_1 + \cdots + N_B)$-point function $G_{a_1 \cdots a_N}$ is expressed by using $G_{a_1 \cdots a_N}$ type n-point functions, we can obtain all the exact solutions of the $\Phi^3$ finite matrix model.

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A Calculation of the three-point functions $G_{a_1 a_2 a_3}$

We calculate the three point functions $G_{a_1 a_2 a_3}$ using the formula $(5.9)$ or $(5.12)$. $i, l, k \in \{1, 2, 3\}$ and $i \neq l \neq k \neq i$ below.

i). We consider the case $\pi = \{(1, 2, 3)\}, |\pi| = 1$, and $B = \{1, 2, 3\}$, then

\[
\left( \frac{d}{dx} \right)^{|\pi|} \left( \log x \right) \prod_{x \in \mathbb{Z}[0]} \prod_{B \in \pi} \frac{\partial^{|B|} Z[J]}{\partial J_{a_i a_j}} \bigg|_{J=0} = \frac{1}{Z[0]} \prod_{B \in \pi} \frac{\partial^{|B|} Z[J]}{\partial J_{a_i a_j}} \bigg|_{J=0}.
\]

(A.1)

The calculations required to calculate $\frac{1}{Z[0]} \prod_{B \in \pi} \frac{\partial^{|B|} Z[J]}{\partial J_{a_i a_j}} \bigg|_{J=0}$ are written below :

\[
F(\{1, 2, 3\}, \emptyset, \emptyset) = -iV \frac{E_{a_1} - 1}{\lambda} \left( -iV \frac{E_{a_2} - 1}{\lambda} \right) \left( -iV \frac{E_{a_3} - 1}{\lambda} \right) A_N(z_1, \ldots, z_N) \frac{1}{\det_{1 \leq p, q \leq N} (t_p^{n-1})},
\]

(A.2)

\[
F(\{i, l\}, \{k\}, \emptyset) = -iV \frac{E_{a_i} - 1}{\lambda} \left( -iV \frac{E_{a_l} - 1}{\lambda} \right) \left( -\frac{V}{(\lambda V)^2} \right) \partial_{a_i} A_N(z_1, \ldots, z_N) \frac{1}{\det_{1 \leq p, q \leq N} (t_p^{n-1})},
\]

(A.3)

\[
F(\{i\}, \{l, k\}, \emptyset) = -iV \frac{E_{a_i} - 1}{\lambda} \left( -\frac{V}{(\lambda V)^2} \right)^2 \partial_{a_i} \partial_{a_k} A_N(z_1, \ldots, z_N) \frac{1}{\det_{1 \leq p, q \leq N} (t_p^{n-1})},
\]

(A.4)
\[
F(\emptyset, \{1, 2, 3\}, \emptyset) = \left(- \frac{V}{(\lambda V)^{\frac{3}{2}}} \right)^3 \partial_{a_1} \partial_{a_2} \partial_{a_3} \det_{1 \leq p, q \leq N} \left( \frac{t_p^{(q)}}{t_p} \right) \left( z_1, \ldots, z_N \right), \quad (A.5)
\]

\[
F(\{i\}, \{l\}, \{k\}) = \left(- i V \frac{E_{a_i - 1}}{\lambda} \right)^3 \left(- \frac{V}{(\lambda V)^{\frac{3}{2}}} \right) \partial_{a_i} \det_{1 \leq p, q \leq N} \left( \frac{t_p^{(q)}}{t_p} \right) \left( z_1, \ldots, z_N \right) \sum_{r=1, r \neq a_k}^{N} \frac{1}{t_{a_k} - t_r}. \quad (A.6)
\]

\[
F(\{i, l\}, \emptyset, \{k\}) = \left(- i V \frac{E_{a_i - 1}}{\lambda} \right) \left(- \frac{V}{(\lambda V)^{\frac{3}{2}}} \right)^2 \partial_{a_i} \partial_{a_l} \det_{1 \leq p, q \leq N} \left( \frac{t_p^{(q)}}{t_p} \right) \left( z_1, \ldots, z_N \right) \sum_{r=1, r \neq a_k}^{N} \frac{1}{t_{a_k} - t_r}. \quad (A.7)
\]

\[
F(\emptyset, \{i, l\}, \{k\}) = \left(- i V \frac{E_{a_i - 1}}{\lambda} \right)^2 \left(- \frac{V}{(\lambda V)^{\frac{3}{2}}} \right) \partial_{a_i} \partial_{a_l} \det_{1 \leq p, q \leq N} \left( \frac{t_p^{(q)}}{t_p} \right) \left( z_1, \ldots, z_N \right) \sum_{r=1, r \neq a_k}^{N} \frac{1}{t_{a_k} - t_r}. \quad (A.8)
\]

\[
F(\{i\}, \emptyset, \{l, k\}) = \left(- i V \frac{E_{a_i - 1}}{\lambda} \right)^2 \left(- \frac{V}{(\lambda V)^{\frac{3}{2}}} \right)^2 \partial_{a_i} \det_{1 \leq p, q \leq N} \left( \frac{t_p^{(q)}}{t_p} \right) \left( z_1, \ldots, z_N \right) \sum_{r=1, r \neq a_k}^{N} \frac{1}{t_{a_k} - t_r}. \quad (A.9)
\]

\[
F(\emptyset, \{i\}, \{l, k\}) = \left(- \frac{V}{(\lambda V)^{\frac{3}{2}}} \right)^2 \partial_{a_i} \det_{1 \leq p, q \leq N} \left( \frac{t_p^{(q)}}{t_p} \right) \left( z_1, \ldots, z_N \right) \sum_{r=1, r \neq a_k}^{N} \frac{1}{t_{a_k} - t_r}. \quad (A.10)
\]

\[
F(\emptyset, \{1, 2, 3\}) = - \det_{1 \leq p, q \leq N} \left( \frac{t_p^{(q)}}{t_p} \right) \sum_{r, w, f=1, r \neq a_3, w \neq a_3, f \neq a_3}^{N} \frac{1}{t_{a_3} - t_r} \frac{1}{t_{a_2} - t_w} \frac{1}{t_{a_1} - t_f} \det_{1 \leq p, q \leq N} \left( \frac{t_p^{(q)}}{t_p} \right) \left( z_1, \ldots, z_N \right) \sum_{r=1, r \neq a_k}^{N} \frac{1}{t_{a_k} - t_r} + \frac{1}{t_{a_k} - t_r} + \frac{1}{t_{a_k} - t_r} + \frac{1}{t_{a_k} - t_r} \quad (A.11)
\]
If we sum up all the cases for sets $S$, $M$, and $M$ that we have calculated so far and multiply by \( \frac{1}{Z[0]} \), we get the result of (A.1):

\[
\frac{1}{Z[0]} \left. \frac{\partial^3 Z[J]}{\partial J_{i^1} \partial J_{i^2} \partial J_{i^3}} \right|_{J=0} = \frac{1}{\det_{1 \leq p, q \leq N} \left( t_p^{q-1} \right)} A_N(z_1, \ldots, z_N) \prod_{a \in B} \left. \frac{\partial^{|\pi|} Z[J]}{\partial J_{a^1} \cdots \partial J_{a^{|\pi|}}} \right|_{J=0},
\]

\[= \left. \frac{\partial Z[J]}{J_{a^1} \cdots \partial J_{a^{|\pi|}}} \right|_{J=0} \left( \frac{\partial^2 Z[J]}{\partial J_{a^1} \partial J_{a^2} \partial J_{a^3} \cdots} \right)_{J=0}. \tag{A.12}\]

The calculations required to calculate \( \left( \frac{1}{Z[0]^{2}} \right) \left( \frac{\partial Z[J]}{\partial J_{a^1} \cdots \partial J_{a^{|\pi|}}} \right)_{J=0} \left( \frac{\partial^2 Z[J]}{\partial J_{a^1} \partial J_{a^2} \partial J_{a^3} \cdots} \right)_{J=0} \) are written below:

\[F(\{i\}, \emptyset, \emptyset) = -iV \frac{E_{a^1}^{i-1}}{\lambda} A_N(z_1, \ldots, z_N) \frac{1}{\det_{1 \leq p, q \leq N} \left( t_p^{q-1} \right)} \tag{A.14}\]

\[F(\emptyset, \{i\}, \emptyset) = -V \left( \frac{\partial a^1 A_N(z_1, \ldots, z_N)}{\partial J} \right) \frac{1}{\det_{1 \leq p, q \leq N} \left( t_p^{q-1} \right)} \tag{A.15}\]

\[F(\emptyset, \emptyset, \{i\}) = A_N(z_1, \ldots, z_N) \frac{-1}{\det_{1 \leq p, q \leq N} \left( t_p^{q-1} \right)} \sum_{r=1, r \neq a^i}^N \frac{1}{t_{a^i} - t_r} \tag{A.16}\]

\[F(\{i, k\}, \emptyset, \emptyset) = \left( -iV \frac{E_{a^i}^{i-1}}{\lambda} \right) \left( -iV \frac{E_{a^k}^{i-1}}{\lambda} \right) A_N(z_1, \ldots, z_N) \frac{1}{\det_{1 \leq p, q \leq N} \left( t_p^{q-1} \right)} \tag{A.17}\]

\[F(\{l\}, \{k\}, \emptyset) = \left( -iV \frac{E_{a^i}^{i-1}}{\lambda} \right) \left( -V \left( \frac{\partial a^1 A_N(z_1, \ldots, z_N)}{\partial J} \right) \right) \frac{1}{\det_{1 \leq p, q \leq N} \left( t_p^{q-1} \right)} \tag{A.18}\]

\[F(\emptyset, \{l, k\}, \emptyset) = \left( -V \left( \frac{\partial a^1 A_N(z_1, \ldots, z_N)}{\partial J} \right) \right)^2 \frac{1}{\det_{1 \leq p, q \leq N} \left( t_p^{q-1} \right)} \tag{A.19}\]

\[F(\{l\}, \emptyset, \{k\}) = \left( -iV \frac{E_{a^i}^{i-1}}{\lambda} \right) A_N(z_1, \ldots, z_N) \frac{-1}{\det_{1 \leq p, q \leq N} \left( t_p^{q-1} \right)} \sum_{r=1, r \neq a^k}^N \frac{1}{t_{a^k} - t_r} \tag{A.20}\]
iii). We consider the case $\pi = \{1\}$. We have calculated so far and multiply by
\[-\frac{1}{Z[0]^2}\] we get the result of (A.13):
\[
\frac{1}{Z[0]^2} \left( \frac{\partial Z[J]}{\partial J_{a_1 a_1}} \right)_{J=0} \left( \frac{\partial^2 Z[J]}{\partial J_{a_1 a_1} \partial J_{a_2 a_2}} \right)_{J=0} \sum_{i,j,k=1, i \neq j \neq k} \left\{ F(\{i\}, \{j\}, \{k\}) + F(\{i\}, \{j\}, \{k\}) + F(\{j\}, \{i\}, \{k\}) + F(\{j\}, \{k\}, \{i\}) + F(\{k\}, \{j\}, \{i\}) + F(\{k\}, \{i\}, \{j\}) \right\}.
\]

(A.23)

If we sum up all the cases for sets $S$, $M$, and $\bar{M}$ that we have calculated so far and multiply by $-\frac{1}{Z[0]^2}$, we get the result of (A.13):

\[
\frac{1}{Z[0]^2} \left( \frac{\partial Z[J]}{\partial J_{a_1 a_1}} \right)_{J=0} \left( \frac{\partial^2 Z[J]}{\partial J_{a_1 a_1} \partial J_{a_2 a_2}} \right)_{J=0} \sum_{i,j,k=1, i \neq j \neq k} \left\{ F(\{i\}, \{j\}, \{k\}) + F(\{i\}, \{j\}, \{k\}) + F(\{j\}, \{i\}, \{k\}) + F(\{j\}, \{k\}, \{i\}) + F(\{k\}, \{j\}, \{i\}) + F(\{k\}, \{i\}, \{j\}) \right\}.
\]

(A.23)

iii). We consider the case $\pi = \{1\} \cup \{2\} \cup \{3\}$, $|\pi| = 3$, and $B = \{1\} \cup \{2\} \cup \{3\}$, then
\[
\left( \frac{d}{dx} \right)^{|\pi|} (\log x) = \sum_{B \in \pi} \prod_{j \in B} \left( \frac{\partial^{|B|} Z[J]}{\partial J_{a_1 a_1}^{j}} \right)_{J=0}
\]
\[
= \frac{2}{Z[0]^3} \left( \frac{\partial Z[J]}{\partial J_{a_1 a_1}} \right)_{J=0} \left( \frac{\partial Z[J]}{\partial J_{a_2 a_2}} \right)_{J=0} \left( \frac{\partial Z[J]}{\partial J_{a_3 a_3}} \right)_{J=0}.
\]

(A.24)

The calculations required to calculate
\[
\frac{2}{Z[0]^3} \left( \frac{\partial Z[J]}{\partial J_{a_1 a_1}} \right)_{J=0} \left( \frac{\partial Z[J]}{\partial J_{a_2 a_2}} \right)_{J=0} \left( \frac{\partial Z[J]}{\partial J_{a_3 a_3}} \right)_{J=0}
\] are written below:

\[
F(\{i\}, \{0\}, \{0\}) = -iV \frac{E_{a_1} - \lambda}{\lambda} A_N(z_1, \ldots, z_N) \frac{1}{\det_{l \neq k \neq 0 \leq N} (t_{l k}^{(l)})}.
\]

(A.25)

\[
F(\{0\}, \{i\}, \{0\}) = \left( -\frac{V}{(\lambda V)^{\frac{1}{2}}} \right) \frac{1}{\det_{l \neq k \neq 0 \leq N} (t_{l k}^{(l)})}.
\]

(A.26)

\[
F(\{0\}, \{0\}, \{i\}) = A_N(z_1, \ldots, z_N) \frac{1}{\det_{l \neq k \neq 0 \leq N} (t_{l k}^{(l)})} \sum_{l=1, r \neq l}^N \frac{1}{t_{l r} - t_{l r}}.
\]

(A.27)

If we sum up all the cases for sets $S$, $M$, and $\bar{M}$ that we have calculated so far and multiply by $\frac{2}{Z[0]^3}$, we get the result of (A.24):

\[
\frac{2}{Z[0]^3} \left( \frac{\partial Z[J]}{\partial J_{a_1 a_1}} \right)_{J=0} \left( \frac{\partial Z[J]}{\partial J_{a_2 a_2}} \right)_{J=0} \left( \frac{\partial Z[J]}{\partial J_{a_3 a_3}} \right)_{J=0} \sum_{i=1}^3 \left\{ F(\{i\}, \{0\}, \{0\}) + F(\{0\}, \{i\}, \{0\}) + F(\{0\}, \{0\}, \{i\}) \right\}.
\]

(A.28)
From i), ii) and iii), all results up to now are combined to obtain the calculation result of the three point functions $G_{a_1^2[a_2^2|a_3^2]}$.

\[
G_{a_1^2[a_2^2|a_3^2]} = \frac{iV}{Z[0]} \frac{\partial^2 Z[J]}{\partial J_{a_1^2} \partial J_{a_2^2} \partial J_{a_3^2}}|_{J=0} - \frac{iV}{Z[0]^2} \left( \frac{\partial Z[J]}{\partial J_{a_1^2}}|_{J=0} \right) \left( \frac{\partial^2 Z[J]}{\partial J_{a_2^2} \partial J_{a_3^2}}|_{J=0} \right) \\
- \frac{iV}{Z[0]^2} \left( \frac{\partial Z[J]}{\partial J_{a_3^2}}|_{J=0} \right) \left( \frac{\partial^2 Z[J]}{\partial J_{a_1^2} \partial J_{a_2^2}}|_{J=0} \right) + \frac{2iV}{Z[0]^3} \left( \frac{\partial Z[J]}{\partial J_{a_1^2}}|_{J=0} \right) \left( \frac{\partial Z[J]}{\partial J_{a_2^2}}|_{J=0} \right) \left( \frac{\partial Z[J]}{\partial J_{a_3^2}}|_{J=0} \right)
\]

\[=(iv) \left( \frac{\det_{1 \leq p,q \leq N} (t_p^{g-1})}{A_N(z_1, \ldots, z_N)} \right)^2 \sum_{i,l,k=1, i \neq l \neq k}^3 \left\{ F(\{i,l,k\}, \emptyset, \emptyset) + F(\emptyset, \{i,l,k\}, \emptyset) + F(\emptyset, \emptyset, \{i,l,k\}) \right\}
\]

\[= (iv) \left( \frac{\det_{1 \leq p,q \leq N} (t_p^{g-1})}{A_N(z_1, \ldots, z_N)} \right)^2 \sum_{i,l,k=1, i \neq l \neq k}^3 \left\{ F(\{i,l\}, \emptyset, \emptyset) + F(\emptyset, \{i,l\}, \emptyset) + F(\emptyset, \emptyset, \{i,l\}) \right\}
\]

\[+ (2iv) \left( \frac{\det_{1 \leq p,q \leq N} (t_p^{g-1})}{A_N(z_1, \ldots, z_N)} \right)^3 \prod_{i=1}^3 \left\{ F(\{i\}, \emptyset, \emptyset) + F(\emptyset, \{i\}, \emptyset) + F(\emptyset, \emptyset, \{i\}) \right\} \] (A.29)

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