Relativistic theory of di-Holeums - quantized gravitational bound states of two micro black holes

A. L. Chavda\textsuperscript{a}, L. K. Chavda\textsuperscript{b}  

\textsuperscript{a} Physics Department, V.N. South Gujarat University, Udhna-Magdalla Road, Surat - 395007, Gujarat, India. \textsuperscript{b} 49 Mahatma Gandhi Society, City Light Road, Surat 395007, Gujarat, India.  

(Dated: February 14, 2014)

Abstract

The Klein-Gordon equation is solved for di-Holeums (gravitational bound states of two micro black holes) for scalar and vector gravity in its static limit. The relativistic models confirm the predictions of the nonrelativistic Newtonian gravity model, correct to about six significant figures over almost the entire sub-Planck domain. All three models possess a mass range devoid of physics. This is interpreted as evidence that the universe must have more than four dimensions. We show that the formation of Holeums is feasible both in the sub-Planck mass and above-Planck mass ranges.

PACS numbers: 03.65.Ge, 03.65.Pm, 04.60.-m, 04.30.-w

\textsuperscript{*}Electronic address: \textsuperscript{a}a01.l.chavda@gmail.com, \textsuperscript{b}holeum@gmail.com
## Contents

I. Introduction 3

II. SCALAR GRAVITY 4
   A. The Klein-Gordon Equation for Scalar Gravity 4
   B. Energy Eigenvalues of a Holeum 8
   C. Binding Energy and Mass of a Holeum 9
   D. The Radius and the Characteristic Function of the Ground State 10
   E. Asymptotics 11
   F. Classes of Holeum 12
   G. Quantized Gravitational Radiation 14

III. VECTOR GRAVITY 14
   A. The Klein-Gordon Equation with Vector Interaction 14
   B. The Energy Eigenvalues 15
   C. Mass and Binding Energy of Holeum 17
   D. The Radius and the Characteristic Function of the Ground State 18
   E. Asymptotics 19
   F. Gravitational Radiation 21

IV. SUMMARY OF NEWTONIAN GRAVITY 21

V. COMPARISON OF THE MODELS 23
   A. Concordance between Newtonian gravity, relativistic scalar gravity, and relativistic vector gravity 23
   B. No physics up to $10^{10}$ GeV/c², followed by a mass range containing interesting physics 24
   C. Contrast between relativistic scalar gravity and relativistic vector gravity 24

VI. DISCUSSIONS AND CONCLUSIONS 24

References 27
I. INTRODUCTION

The nature of dark matter is one of the most profound mysteries of our times. Numerous candidates for dark matter particles have been proposed, which include Standard Model neutrinos [1, 2], Sterile neutrinos [1, 3], WIMPs [4], MACHOs [5], Supersymmetric particles [6], Kaluza-Klein matter [7], Axions [8], Cryptons [9], and primordial black holes (PBHs) [10], among others.

In 2002 we proposed another type of dark matter particle consisting of stable atoms of PBHs, which we called the Holeum [11]. Since then we have shown that Holeums can be microscopic (such as a di-Holeum consisting of two micro black holes) as well as macroscopic (such as a stellar-mass macro Holeum consisting of $k$ micro black holes, where $k \gg 2$), and can give rise to quantized gravitational radiation [12]. We have also proposed that Holeums can give rise to cosmic rays of all observed energies, including UHECR [13]. Al Dallal has shown that the observations on the very short duration Gamma Ray Bursts are well explained by the Holeum model [14]. Al Dallal has also shown that the Holeum model predicts diffuse Gamma Ray Halos around supernova remnants [15].

Exactly solvable problems have a special niche in theoretical physics. With their help we can grasp the physics of a phenomenon much more clearly and easily. Here we are trying to develop an effective model of quantum gravity for the bound state problem of the di-Holeum. To this end, we investigated the problem of a bound state of two micro black holes (MBHs) in quantized Newtonian gravity using the non-relativistic Schrödinger equation, and obtained order of magnitude, ball-park values of the bound state parameters in our 2002 paper [11]. In this paper, we include special relativity in the model. This is necessary because we are dealing with ultra-compact MBHs and ultra-short distances. We solve the Klein-Gordon equation for the di-Holeum, assuming gravitation to be either a scalar or a vector interaction. Both of these cases are exactly solvable for a static $1/r$ potential.

In § II we present scalar gravity. We calculate the energy eigenvalues, the mass, the binding energy and the radius of the ground state and the characteristic function of the Holeum which determines whether the bound state is a stable Holeum or an unstable one. In § III we present similar calculations for the case of vector gravity. In § IV we present a summary of corresponding results for the case of Newtonian gravity to facilitate their comparison with the relativistic models. In § V we compare the predictions of these three
models. Discussions and conclusions are presented in § VI. In the following treatment, as in [11], we use the term "Holeum" to refer to the di-Holeum for the sake of simplicity.

II. SCALAR GRAVITY

A. The Klein-Gordon Equation for Scalar Gravity

The free particle relativistic Klein-Gordon equation is obtained from

\[ E^2 = \tilde{p}^2 c^2 + \mu^2 c^4 \]  

by letting

\[ E \to i\hbar \frac{\partial}{\partial t}, \tilde{p} \to -i\hbar \tilde{\nabla} \]

Here \( E \) and \( \tilde{p} \) are the total energy and the linear momentum of a particle of reduced mass \( \mu \), respectively and \( \tilde{\nabla} \) is the three dimensional gradient operator. We are considering two identical black holes of mass \( m \) each and having no charge and spin. \( \hbar \) and \( c \) are the Planck’s constant reduced by \( 2\pi \) and the speed of light in vacuum, respectively. Thus, we get the free particle Klein-Gordon equation

\[ -\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left( -\hbar^2 c^2 \tilde{\nabla}^2 + \mu^2 c^4 \right) \psi \]  

Here \( \psi \) is the wave function of the system. The static gravitational interaction potential is given by

\[ V (r) = -\hbar c \frac{\alpha_g}{r} \]  

where

\[ \alpha_g = \left( \frac{m}{m_P} \right)^2 \]  

and

\[ m_P = \left( \frac{\hbar c}{G} \right)^{\frac{1}{2}} \]  

Here \( m_P \) is the Planck mass \( 1.22101 \times 10^{19} \text{ GeV}/c^2 \) and \( G \) is Newton’s universal constant of gravity. \( \alpha_g \) is the dimensionless gravitational coupling constant. To treat \( V (r) \), given by Eq. (1), as a scalar interaction we make the following substitutions in the free-particle
Klein-Gordon equation:

\[ \tilde{p} \to \tilde{p} \quad (7) \]
\[ E \to E \quad (8) \]
\[ \mu c^2 \to \mu c^2 + V (r) \quad (9) \]

Thus we have

\[ -\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left[ -\hbar^2 c^2 \tilde{\nabla}^2 + (\mu c^2 + V (r))^2 \right] \psi \quad (10) \]

This is the Klein-Gordon equation for scalar gravity. To obtain the stationary state solution we let

\[ \psi(\tilde{r}, t) = \psi(\tilde{r}) e^{\left(-\frac{iEt}{\hbar}\right)} \quad (11) \]

Substituting this into Eq. (10) we have

\[ E^2 \psi = \left[ -\hbar^2 c^2 \tilde{\nabla}^2 + (\mu c^2 + V (r))^2 \right] \psi \quad (12) \]

We assume the separability of the wave function

\[ \psi(\tilde{r}) = \frac{R(r)}{r} P(\theta) Q(\phi) \quad (13) \]

The solution for \( Q \) is given by

\[ Q = e^{i m \phi}, \quad m = 0, \pm 1, \pm 2, \ldots \quad (14) \]

\( P(\theta) \) satisfies the associated Legendre differential equation

\[ (P \sin \theta)^{-1} \left( \frac{d}{d\theta} \right) \left( \sin \theta \frac{dP}{d\theta} \right) + l (l + 1) - \frac{m^2}{\sin^2 \theta} = 0 \quad (15) \]

where

\[ l = 0, 1, 2, \ldots \quad (16) \]

The radial part \( R(r) \) satisfies the equation

\[ \frac{R''(r)}{R(r)} + \left[ \frac{E^2 - (\mu c^2 + V (r))^2}{\hbar^2 c^2} - \frac{l (l + 1)}{r^2} \right] = 0 \quad (17) \]

Substituting for \( V (r) \) from Eq. (4) we have

\[ R''(r) + \left[ \frac{E^2 - \mu^2 c^4}{\hbar^2 c^2} + \frac{\alpha_g}{\lambda r} + \frac{C}{r^2} \right] R(r) = 0 \quad (18) \]
where
\[ \lambda = \frac{\hbar}{mc} \]  
(19)
and
\[ C = -\alpha_g^2 - l(l + 1) = \frac{1}{4} - q^2 \]  
(20)
Let
\[ \kappa^2 = \frac{\mu^2 c^4 - E^2}{\hbar^2 c^2} \]  
(21)
\[ B = \frac{\alpha_g}{\lambda} \]  
(22)
\[ q^2 = \alpha_g^2 + \left( l + \frac{1}{2} \right)^2 \]  
(23)
Substituting Eqs. 20 to 23 into Eq. 18 we get
\[ R''(r) + \left[ -\kappa^2 + \frac{B}{r} + \frac{1}{4} - q^2 \right] R(r) = 0 \]  
(24)
Define
\[ \rho = 2\kappa r \]  
(25)
Substituting this into Eq. 24 we have
\[ R''(\rho) + \left[ -\frac{1}{4} + \frac{\lambda_0}{\rho} + \frac{1}{4} - q^2 \right] R(\rho) = 0 \]  
(26)
where
\[ \lambda_0 = \frac{\mu c^2 \alpha_g}{(\mu^2 c^4 - E^2)^{\frac{1}{2}}} \]  
(27)
Let
\[ R(\rho) = \rho^\beta e^{(-\rho/2)F(\rho)} \]  
(28)
This takes Eq. 26 to the form
\[ \frac{F''(\rho)}{F(\rho)} + \left( \frac{2\beta - 1}{\rho} - 1 \right) \frac{F'(\rho)}{F(\rho)} + \frac{(\lambda_0 - \beta)}{\rho} + \frac{(\frac{1}{4} - q^2)}{\rho^2} + \frac{\beta(\beta - 1)}{\rho^2} = 0 \]  
(29)
To cancel the last two terms let
\[ \frac{1}{4} - q^2 = -\beta(\beta - 1) = \frac{1}{4} - \left( \beta - \frac{1}{2} \right)^2 \]  
(30)
\[ \beta = \frac{1}{2} \pm q \]  
(31)
We take the positive square root to avoid the singularity at the origin in Eq. 28. Thus we have
\[
\beta = \frac{1}{2} + \left[ \alpha_g^2 + \left( l + \frac{1}{2} \right)^2 \right]^{\frac{1}{2}}
\]  
(32)

Then we can rewrite Eq. 29 as
\[
\rho F''(\rho) + (2\beta - \rho) F'(\rho) + (\lambda_0 - \beta) F(\rho) = 0
\]  
(33)

We compare this with the differential equation of the confluent hypergeometric function given by
\[
xy''(x) + (b - x)y'(x) - ay(x) = 0
\]  
(34)

The general solution to this equation is given by
\[
y(x) = c_1 F_1(a, b; x) + d' U(a, b; x)
\]  
(35)

Here \( c_1 \) and \( d' \) are constants. The \( U(a, b; x) \) is singular at \( x = 0 \). Therefore we discard it by choosing \( d' = 0 \). The other solution is the confluent hypergeometric function given by
\[
y(x) = F_1(a, b; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}
\]  
(36)

where
\[
(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}
\]  
(37)

This solution is nonsingular at the origin but it blows up at infinity. To avoid this we require that it become a polynomial for large \( x \). Then we see from Eq. 28 that it represents a localized wave function suitable for a bound state. This is true only if
\[
a = -v, \ v = 0, 1, 2, \ldots \infty
\]  
(38)

Comparing Eqs. 33 and 34 we have
\[
a = \beta - \lambda_0
\]  
(39)
\[
b = 2\beta
\]  
(40)

From Eqs. 38, 39 and 40 we have
\[
\lambda_0 = \beta + v = v + \frac{1}{2} + q
\]  
(41)
B. Energy Eigenvalues of a Holeum

In the weak coupling limit $\alpha_g \ll 1$, we have

$$q = \left[ \alpha_g^2 + \left( l + \frac{1}{2} \right)^2 \right]^{1/2} \approx l + \frac{1}{2} + \frac{\alpha_g^2}{2l + 1}$$  \hspace{1cm} (42)

Substituting this into Eq. 41 we have

$$\lambda_0 = n + \frac{\alpha_g^2}{2l + 1}$$  \hspace{1cm} (43)

where

$$n = v + l + 1 = 1, 2, 3, \ldots$$  \hspace{1cm} (44)

This is the principal quantum number of the bound state as we shall see below. Substituting this into Eq. 41, we have

$$\lambda_0 = n + \epsilon_l$$  \hspace{1cm} (45)

where

$$\epsilon_l = \left[ \alpha_g^2 + \left( l + \frac{1}{2} \right)^2 \right]^{1/2} - \left( l + \frac{1}{2} \right)$$  \hspace{1cm} (46)

Now from Eqs. 27 and 45 we have

$$E_{nl} = \mu c^2 \left[ 1 - \frac{\alpha_g^2}{(n + \epsilon_l)^2} \right]^{1/2}$$  \hspace{1cm} (47)

where we have shown only the positive sign of the square root for convenience. Note that the negative sign, too, is equally admissible. With the advent of the relativistic Klein-Gordon and the Dirac equations, there arose the need to take the negative energy solutions seriously. As is well-known, the negative energy solution was assigned to an antiparticle having the same mass as the particle but having opposite quantum numbers. In our case, we have a Holeum consisting of two PBHs. But an anti-Holeum consisting of two anti-PBHs is also equally possible. This is because we are considering the very early universe which was matter-antimatter symmetric before the decoupling of gravity. This will lead to the well-known problem of explaining the absence of antimatter in the present universe. We will not address it here. In the following we will consider mainly the positive energy solutions. But we may also indicate the results for the anti-Holeums. Now for $\alpha_g \ll 1$ and $n \gg \epsilon_l$ we have

$$E_{nl} = \mu c^2 - \frac{\mu c^2 \alpha_g^2}{2n^2}$$  \hspace{1cm} (48)
where we have kept only the first two terms in the expansion in powers of $\alpha_g^2$. The first term is the rest energy and the second one is the usual formula for the energy eigenvalues of the hydrogen atom. Therefore the $n$ that occurs in Eq. 48 and that is given by Eq. 44 is identified as the principal quantum number of the Holeum which is the gravitational analogue of the hydrogen atom.

C. Binding Energy and Mass of a Holeum

The interaction energy is defined by

$$W_{nl} = E_{nl} - \mu c^2$$  \hspace{1cm} (49)$$

The binding energy is given by

$$B_{nl} = |W_{nl}| = \mu c^2 \left| 1 - \frac{\alpha_g^2}{(n + \epsilon_l)^2} \right|^\frac{1}{2} - 1$$  \hspace{1cm} (50)$$

Note that the binding energy for the negative energy solution is also given by the same expression.

For the $1s$ state the binding energy is given by

$$B_{1s} = \frac{mc^2}{2} \left| 1 - \left( \frac{2}{p_0 + 1} \right)^\frac{1}{2} \right|$$  \hspace{1cm} (51)$$

$$p_0 = (1 + 4\alpha_g^2)^\frac{1}{2}$$  \hspace{1cm} (52)$$

For $\alpha_g \ll 1$ the binding energy is given by

$$B_{1s} = \left( \frac{mc^2\alpha_g^2}{4} \right) \left| 1 - \frac{7\alpha_g^2}{4} \right| + o \left( \alpha_g^{13} \right)$$  \hspace{1cm} (53)$$

The mass of the Holeum is given by

$$M_n = 2m + \frac{W_{nl}}{c^2}$$  \hspace{1cm} (54)$$

Substituting from Eqs. 47 and 49 into Eq. 54 we have

$$M_n = \frac{m}{2} \left[ 3 + \left\{ 1 - \frac{\alpha_g^2}{(n + \epsilon_l)^2} \right\}^\frac{1}{2} \right]$$  \hspace{1cm} (55)$$
For the 1s state this is given by

\[ M_{1s} = \frac{m}{2} \left[ 3 + \left( \frac{2}{p_0 + 1} \right) \right]^{\frac{1}{2}} \]  

(56)

For \( \alpha_g \ll 1 \) this is given by

\[ M_{1s} = 2m \left( 1 - \frac{\alpha_g^2}{8} \right) + o \left( \alpha_g^2 \right) \]  

(57)

The first term on the right side of this equation is the same as the mass of a Holeum in Newtonian gravity as we shall see later. Note that the last term in this equation is less than \( 10^{-6} \) even for \( m = 10^{18} \text{ GeV}/c^2 \).

D. The Radius and the Characteristic Function of the Ground State

The normalized radial part of the wave function is given by

\[ \psi(r) = \left[ \frac{8\kappa^3 \Gamma(n-l)}{\Gamma(n-l+2q)(2n-2l+2q-1)} \right]^{\frac{1}{2}} \rho^{n-\frac{1}{2}} e^{-\frac{\rho}{2}} L_n^{2q}(\rho) \]  

(58)

where \( L_n^{2q}(x) \) is an associated Laguerre polynomial and \( n' = n - l - 1 \). The probability density is given by

\[ P = r^2 |\psi(r)|^2 \]  

(59)

Substituting Eq. 58 into Eq. 59 we have

\[ P = \frac{2\kappa \Gamma(n-l)p^{2q+1}e^{-\rho} \left( L_n^{2q}(\rho) \right)^2}{\Gamma(n-l+2q)(2n-2l+2q-1)} \]  

(60)

where \( \kappa \) and \( q \) are given by Eqs. 21 and 42 respectively. From Eq. 60 one can show that the most probable radius of the ground state \( n = 1, l = 0 \) is given by

\[ r_{1s} = \left( \frac{\lambda}{2\alpha_g} \right) (1 + p_0)^2 \]  

(61)

where \( \lambda \) is given by Eq. 19. The mass of the ground state of Holeum is given by Eq. 56 and the Schwarzschild radius of the ground state is given by

\[ R_{1s} = \frac{2M_{1s}G}{c^2} \]  

(62)

The characteristic function of a Holeum is defined by

\[ f_{1s} = \frac{R_{1s}}{r_{1s}} \]  

(63)
One can show that

\[ f_{1s} = (p_0 - 1) \frac{3 + \left( \frac{2}{p_0 + 1} \right)^{\frac{1}{2}}}{2 (p_0 + 1)} \]  

(64)

If \( f_{1s} \geq 1 \) then the Schwarzschild radius of the bound state is greater than the physical radius of the bound state. In this case the Holeum is a black hole. We call it a Black Holeum (BH). It will emit Hawking radiation and evaporate away. But if \( f_{1s} < 1 \), then the Holeum is not a black hole. It will not emit Hawking radiation even though it contains two black holes. It is a stable bound state, as stable as a hydrogen atom of which it is a gravitational analogue.

One can show numerically that all Holeums with constituent masses \( m < m_c(1s) \) are stable but those not satisfying this condition are BHs which will evaporate away. \( m_c(1s) \) is given by

\[ m_c(1s) = 1.2722 m_P \]  

(65)

E. Asymptotics

The asymptotic form of the associated Laguerre polynomial is given by

\[ L_n^{\alpha}(x) = \pi^{-\frac{1}{2}} e^x x^{-\frac{n+1}{2}} \left[ 2 (n+\alpha) - \pi + \frac{\pi}{4} \right] n^{\frac{\alpha-\frac{1}{2}}{4}} + O \left( n^{\frac{\alpha-\frac{3}{4}}{4}} \right) \]  

(66)

where \( n \gg 1 \). Substituting this into Eq. 60 we have

\[ P = \left( \frac{\alpha_q}{2 \pi \chi} \right) \rho^{\frac{1}{2}} n^{2q-\frac{3}{2}} \cos^2 \phi \]  

(67)

\[ \phi = 2 (n \rho)^{\frac{1}{2}} - q \pi - \frac{\pi}{4} \]  

(68)

where we have taken \( l = 0 \) for the s-states for simplicity. For \( n \gg 2q \) one can show that

\[ \frac{d}{d \rho} \ln P = - (2 \tan \phi) \left( \frac{n}{\rho} \right)^{\frac{1}{2}} \]  

(69)

This vanishes for \( \phi = k \pi, k = 0, \pm 1, \pm 2, \ldots \). Now for \( n \gg 2q \), from Eq. 68 we have

\[ \phi = 2 (n \rho)^{\frac{1}{2}} = k \pi \]  

(70)

This gives us

\[ \rho = \frac{k^2 \pi^2}{4n} \]  

(71)
Now $\psi_n(r)$ can have $n$ maxima. Therefore we have $k \leq n$. At the $k^{th}$ maximum the probability is given by

$$P_k = \frac{k\alpha_g n^{2q-3}}{4\lambda} \quad (72)$$

This rises linearly with $k$. These maxima overlap and the peak with the highest probability has $k = n$. Taking this value of $k$ we have

$$r_n = \frac{\rho}{2\kappa} = \frac{n^2\pi^2\lambda}{4\alpha_g} \quad (73)$$

where we have used

$$\kappa = \frac{\alpha_g}{2n\lambda} \quad (74)$$

We also have

$$\lambda = \frac{R}{2\alpha_g} \quad (75)$$

where $R$ is the Schwarzschild radius of the constituent PBH. Substituting this into Eq. (73) we have

$$r_n = \frac{n^2\pi^2R}{8\alpha_g^2} \quad (76)$$

This formula was first derived in the framework of Newtonian gravity. In the next subsection we will derive the same formula for the case of vector gravity.

F. Classes of Holeum

From Eq. (55), with $n \gg 1$, we have

$$M_n = 2m \left(1 - \frac{\alpha^2_g}{8n^2}\right) \quad (77)$$

From this we can calculate the Schwarzschild radius:

$$R_n = \frac{2M_n G}{c^2} \quad (78)$$

Substituting Eq. (77) into Eq. (78) we have

$$R_n = 2R \left(1 - \frac{\alpha^2_g}{8n^2}\right) \quad (79)$$

The characteristic function is given by

$$f_{ns} = \frac{R_n}{r_n} = \frac{16\alpha^2_g}{\pi^2n^2} \left(1 - \frac{\alpha^2_g}{8n^2}\right) \quad (80)$$
where we have used Eqs. [76] and [79]. For further analysis we need the following identity [16]:

\[
\frac{16X}{\pi^2} \left(1 - \frac{X}{8}\right) = 1 - \frac{32X_1X_2}{\pi^2} \tag{81}
\]

where

\[
X = \frac{\alpha_g^2}{n^2} \tag{82}
\]

\[
X_1 = \frac{X}{4} - 1 + \Delta \tag{83}
\]

\[
X_2 = \frac{X}{4} - 1 - \Delta \tag{84}
\]

\[
\Delta = \left(\frac{1 - \pi^2}{32}\right)^\frac{1}{2} = 0.83161 \tag{85}
\]

Substituting Eqs. [81] through [84] into Eq. [80] we have

\[
f_{ns} = 1 - \frac{32X_1X_2}{\pi^2} \tag{86}
\]

From Eq. [86] it is clear that if \(X_1X_2 > 0\), then \(f_{1s} < 1\). This will lead to stable Holeums. This gives rise to two cases: (i) \(X_1 < 0\) and \(X_2 < 0\) which we denote as the mass range \(b\). This leads to the inequality

\[
0 < m < 0.905929 \, m_p \tag{87}
\]

and (ii) \(X_1 > 0\) and \(X_2 > 0\). We shall denote this as the mass range \(a\). This leads to the inequality

\[
1.645217 \, m_p < m < 1.681793 \, m_p \tag{88}
\]

On the other hand if \(X_1X_2 < 0\), then \(f_{1s} > 1\). This will lead to bound states which are black holes. We have called them BHs above. These are unstable and will evaporate away. Since the stable Holeums in the mass range \(a\) are more massive than the Plank mass we have called them Hyper Holeums (HHs). Up to now this classification of Holeums follows that of the Newtonian gravity case exactly. However, from Eq. [55] it is clear that \(M_n\) cannot be zero for real values of \(\alpha_g\). Thus, the Lux Holeum with mass \(M_n = 0\) [16] is ruled out in relativistic scalar gravity. From the discussion following Eq. [64] it is seen that in the case of relativistic scalar gravity for small values of \(n\) there are only two mass ranges corresponding to the H and BH cases whereas for large \(n\) we have just seen three mass ranges: H, BH, HH.
G. Quantized Gravitational Radiation

From the foregoing formulae we can show that when a Holeum undergoes an atomic transition from a higher \((n_2, l_2)\) state to a lower \((n_1, l_1)\) state it emits gravitational radiation of frequency given by

\[
\nu = \frac{mc^2}{4\hbar} \left[ \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right) - 2\alpha_g^2 \left\{ \frac{1}{n_1^3(2l_1 + 1)} - \frac{1}{n_2^3(2l_2 + 1)} \right\} \right]
\]

(89)

where the first term gives the non-relativistic contribution and the second one gives the relativistic contribution. Since \(m\) varies up to \(m_c\) this will be a band spectrum. For \(m\) in the range \(10^{10} \text{ GeV}/c^2\) to \(10^{12} \text{ GeV}/c^2\) we can show that the Holeums have sizes in the atomic and the nuclear domains, respectively. They emit gravitational radiation in the low and medium frequency domains, respectively.

III. VECTOR GRAVITY

A. The Klein-Gordon Equation with Vector Interaction

The relativistic Klein-Gordon equation for a particle subject to a vector interaction \(A^\mu = (V, -\tilde{A})\) is obtained by making the substitution \(\tilde{p} \rightarrow \tilde{p} - \tilde{A}, E \rightarrow E - V\) in the free particle relativistic Klein-Gordon equation

\[
E^2 = \tilde{p}^2c^2 + \mu^2c^4
\]

(90)

where \(\mu\) is the reduced mass of the particle. \(E, \tilde{p}\) and \(c\) have their meanings as in § II. In the static limit \(\tilde{A} \rightarrow 0\) we have

\[
(E - V)^2 \psi(\tilde{r}) = (\tilde{p}^2c^2 + \mu^2c^4) \psi(\tilde{r})
\]

(91)

Here

\[
E \rightarrow i\hbar \frac{\partial}{\partial t}, \tilde{p} \rightarrow -i\hbar \tilde{\nabla}
\]

(92)

This leads to

\[
\left( i\hbar \frac{\partial}{\partial t} - V \right)^2 \psi(\tilde{r}) = \left( \mu^2c^4 - \hbar^2c^2\tilde{\nabla}^2 \right) \psi(\tilde{r})
\]

(93)

The stationary state equation is obtained by letting

\[
\psi(\tilde{r}, t) = \psi(\tilde{r}) e^{-iEt/c} \]

(94)
Substituting Eq. 94 into Eq. 93 we have

\[(E - V)^2 \psi (\tilde{r}) = \left( \mu^2 c^4 - \hbar^2 c^2 \tilde{\nabla}^2 \right) \psi (\tilde{r})\]  

(95)

This is the relativistic Klein-Gordon equation with vector interaction.

**B. The Energy Eigenvalues**

Eq. 95 may be rewritten as

\[-\hbar^2 c^2 \tilde{\nabla}^2 \psi (\tilde{r}) = \left[ (E - V)^2 - \mu^2 c^4 \right] \psi (\tilde{r})\]  

(96)

We assume the separability of the wave function

\[\psi (\tilde{r}) = \frac{R(r)}{r} P (\theta) Q (\phi)\]  

(97)

As before we have

\[Q = e^{i m \phi}, \ m = 0, \pm 1, \pm 2, \ldots \]  

(98)

and

\[P (\theta) = P^m_l (\theta), \ l = 0, 1, 2, \ldots \]  

(99)

is the associated Legendre polynomial. It can be shown that the radial part of the wave function satisfies

\[\frac{R''(r)}{R(r)} + \left[ \frac{(E - V)^2 - \mu^2 c^4}{\hbar^2 c^2} - \frac{l(l + 1)}{r^2} \right] = 0\]  

(100)

The static Newtonian potential is given by

\[V (r) = -\frac{\hbar c \alpha_g}{r}\]  

(101)

as before. Substituting Eq. 101 into Eq. 100 we have

\[R''(r) + \left[ \frac{E^2 - \mu^2 c^4}{\hbar^2 c^2} + \frac{2 \alpha_g E}{\hbar c r} + \frac{\alpha_g^2 - l(l + 1)}{r^2} \right] R(r) = 0\]  

(102)

Let

\[\kappa^2 = \frac{\mu^2 c^4 - E^2}{\hbar^2 c^2}\]  

(103)

\[\rho = 2 \kappa r\]  

(104)
\[ \lambda = \frac{\alpha_g E}{\hbar c k} \]  \hspace{1cm} (105)

Substituting these into Eq. 102 we have

\[ R''(\rho) + \left[ -\frac{1}{4} + \frac{\lambda}{\rho} + \frac{\alpha_g^2 - l(l + 1)}{\rho^2} \right] R(\rho) = 0 \]  \hspace{1cm} (106)

Let

\[ R(\rho) = \rho^{s+1} e^{-\frac{\rho}{2}} w(\rho) \]  \hspace{1cm} (107)

Substituting this into Eq. 106 and dividing by \( R \) we have

\[ \frac{w''}{w} + \frac{s(s + 1)}{\rho^2} + \frac{\lambda - s - 1}{\rho} + \left( \frac{2s + 2}{\rho} - 1 \right) \frac{w'}{w} + \frac{\alpha_g^2 - l(l + 1)}{\rho^2} = 0 \]  \hspace{1cm} (108)

Let us choose

\[ s (s + 1) = l (l + 1) - \alpha_g^2 \]  \hspace{1cm} (109)

By completing the squares in \( s \) and \( l \) we find that the solution to this equation is given by

\[ s = -\frac{1}{2} \pm \left[ \left( \frac{l + 1}{2} \right)^2 - \alpha_g^2 \right] \]  \hspace{1cm} (110)

For a non-singular behavior at \( \rho = 0 \) we take the positive sign on the right hand side of Eq. 110. This simplifies Eq. 108 as follows

\[ \rho w'' + (2s + 2 - \rho) w' + (\lambda - s - 1) w = 0 \]  \hspace{1cm} (111)

We compare it with Kummer's differential equation:

\[ xy''(x) + (b - x) y'(x) - ay(x) = 0 \]  \hspace{1cm} (112)

with

\[ a = s + 1 - \lambda, \; b = 2s + 2 \]  \hspace{1cm} (113)

Kummer's differential equation has the general solution

\[ y(x) = c_1 F_1(a, b; x) + c_2 U(a, b; x) \]  \hspace{1cm} (114)

where \( c_1 \) and \( c_2 \) are constants. \( U(a, b; x) \) blows up at \( x = 0 \). Therefore we choose \( c_2 = 0 \). Now \( F_1(a, b; x) \) blows up at \( x = \infty \) unless \( a = -v \) where \( v \) is a positive integer. In the latter case \( F_1(a, b; x) \) becomes a polynomial and \( \psi(r) \) gets a localized form fit for a bound state. Eq. 111 has a solution given by

\[ w(\rho) = F_1(-\lambda + s + 1, 2s + 2; \rho) \]  \hspace{1cm} (115)
\[ v = \lambda - s - 1 = 0, 1, 2, \ldots \infty \]  
\[ \lambda = s + 1 + v = l + v + 1 + \left[ (l + \frac{1}{2})^2 - \alpha_g^2 \right]^{\frac{1}{2}} - \left( l + \frac{1}{2} \right) \]  
\[ (116) \]

We define the principal quantum number
\[ n = v + l + 1, \ v, l = 0, 1, 2, \ldots \infty \]  
\[ (118) \]

Substituting Eqs. 117 and 118 into Eq. 115 we have
\[ w(\rho) = _1 F_1 (-v, 2s + 2; \rho) \]  
\[ (119) \]

We may rewrite Eq. 117 as
\[ \lambda = n + \epsilon_l, \ \epsilon_l = \left[ (l + \frac{1}{2})^2 - \alpha_g^2 \right]^{\frac{1}{2}} - \left( l + \frac{1}{2} \right) \]  
\[ (120) \]

From Eqs. 120 and 105 we have
\[ (\mu^2 c^4 - E^2)^{\frac{1}{2}} = \frac{\alpha_g E}{n + \epsilon_l} \]  
\[ (121) \]

This may be rewritten as
\[ E_{nl} = \frac{\mu c^2}{\left[ 1 + \left( \frac{\alpha_g}{n+\epsilon_l} \right)^2 \right]^{\frac{1}{2}}} \]  
\[ (122) \]
where we have taken only the positive square root for convenience. These are the energy eigenvalues for vector gravity.

C. Mass and Binding Energy of Holeum

The interaction energy is given by
\[ W_{nl} = E_{nl} - \mu c^2 \]  
\[ (123) \]

The mass of a Holeum is given by
\[ M_{nl} = 2m + \frac{W_{nl}}{c^2} \]  
\[ (124) \]

Substituting from Eqs. 122 and 123 into Eq. 124 we have
\[ M_{nl} = \frac{m}{2} \left[ 3 + \frac{1}{\left[ 1 + \left( \frac{\alpha_g}{n+\epsilon_l} \right)^2 \right]^{\frac{1}{2}}} \right] \]  
\[ (125) \]
In particular, the $n=1$, $l=0$ or the $1s$ state has the mass

$$M_{1v} = \frac{m}{2} \left[ 3 + \left( \frac{p_1 + 1}{2} \right)^{\frac{3}{2}} \right]$$

(126)

For $\alpha_g^2 \ll 1$ we may expand this in powers of $\alpha_g^2$ to get

$$M_{1v} = 2m \left( 1 - \frac{\alpha_g^2}{8} \right) + o \left( \alpha_g^2 \right)$$

(127)

In Eq. (126) $p_1$ is given by

$$p_1 = (1 - 4\alpha_g^2)^{\frac{1}{2}}$$

(128)

The binding energy for the $1s$ state is given by

$$B_{1v} = \frac{mc^2}{2} \left| 1 - \left( \frac{p_1 + 1}{2} \right)^{\frac{1}{2}} \right|$$

(129)

For $\alpha_g^2 \ll 1$ we may expand this in powers of $\alpha_g^2$ to get

$$B_{1v} = \frac{mc^2\alpha_g^2}{4} \left( 1 + \frac{5\alpha_g^2}{4} \right) + o \left( \alpha_g^{13} \right)$$

(130)

**D. The Radius and the Characteristic Function of the Ground State**

The radius of the $1s$ state is given by

$$r_{1v} = R_P \left( \frac{p_1 + 1}{2\alpha_g} \right)^{\frac{3}{2}}$$

(131)

For $\alpha_g^2 \ll 1$ we may expand this in powers of $\alpha_g^2$ to get

$$r_{1v} = \frac{R_P}{\alpha_g^{\frac{1}{2}}} \left( 1 - \frac{3\alpha_g^2}{2} - \frac{9\alpha_g^4}{8} \right) + O \left( \alpha_g^{13} \right)$$

(132)

In Eqs. (131) and (132) $R_P$ is the Schwarzschild radius of a Planck-mass PBH, i.e.

$$R_P = \frac{2m_P G}{c^2}$$

(133)

The characteristic function of the Holeum is given by

$$f_{1v} = \frac{R_{1v}}{r_{1v}} = 4\alpha_g^2 \left[ 3 + \left( \frac{p_1 + 1}{2} \right)^{\frac{1}{2}} \right] \left[ 2 \left( \frac{p_1 + 1}{2} \right)^{\frac{1}{2}} \right]$$

(134)
For $\alpha_g^2 \ll 1$ we may expand this in powers of $\alpha_g^2$ to get

$$f_{1v} = 2\alpha_g^2 \left( 1 + \frac{11\alpha_g^2}{8} \right) + o(\alpha_g^6)$$

(135)

From Eq. [128] it is clear that if

$$\alpha_g \geq \frac{1}{2}$$

(136)

there will be no bound states at all. For the $1s$ state, numerically it is found that we will have stable Holeums if $0 < m < 0.70107 \, m_P$. But there are only unstable BHs for $0.70107 \, m_P < m < 0.7071 \, m_P$.

**E. Asymptotics**

The normalized radial wave function $\psi(r) = \frac{R(r)}{r}$ is given by

$$\psi(r) = N \rho^p e^{-\rho} L_{n'}^p\left(\rho\right)$$

(137)

where

$$p = 2 \left[ \left( l + \frac{1}{2} \right)^2 - \alpha_g^2 \right]^\frac{1}{2}$$

(138)

$$n' = n - l - 1$$

(139)

$$N^2 = \frac{8\kappa^3 \Gamma(n - l)}{\Gamma(n - l + p)(2n - 2l + p - 1)}$$

(140)

$$\kappa = \frac{\alpha_g}{2\lambda \left[ \alpha_g^2 + (n + \epsilon_l)^2 \right]^\frac{1}{2}}$$

(141)

$$\epsilon_l = \left[ \left( l + \frac{1}{2} \right)^2 - \alpha_g^2 \right]^\frac{1}{2} - \left( l + \frac{1}{2} \right)$$

(142)

$$\lambda = \frac{\hbar}{mc}$$

(143)

$$\rho = 2\kappa r$$

(144)

The probability density defined by

$$P = R^2$$

(145)

is given by

$$P = \Theta \rho^{p+1} e^{-\rho} \left[ L_{n'}^p\left(\rho\right)\right]^2$$

(146)
where
\[ \Theta = \frac{2\kappa \Gamma (n - 1)}{\Gamma (n - l + p) (2n - 2l + p - 1)} \] (147)

The asymptotic behavior of \( L_{n'}^p (\rho) \) for \( n' \gg 1 \) is given by
\[ L_{n'}^p (\rho) = \pi^{\frac{1}{2}} \rho^{-\frac{p}{2}} e^{\frac{\rho}{2}} (n')^{\frac{p}{2} - \frac{1}{4}} \cos \Phi \] (148)

where
\[ \Phi = 2 (n' \rho)^{\frac{1}{2}} - \frac{p \pi}{2} - \frac{\pi}{4} \] (149)

Substituting Eqs. 148 and 149 into Eq. 146 and assuming \( n' \gg 1 \) we have
\[ P = \frac{\Theta}{\pi} \rho^{\frac{1}{2}} (n')^{p - \frac{1}{4}} \cos^2 \Phi \] (150)

For \( n' \gg 1 \) the logarithmic derivative of \( P \) is given by
\[ \frac{1}{P} \frac{dP}{d\rho} = -2 \left( \frac{n'}{\rho} \right)^{\frac{1}{2}} \tan \Phi \] (151)

This vanishes for
\[ \Phi = k\pi, \; k = 0, \pm 1, \pm 2, \ldots \] (152)

As argued earlier we have \( k \leq n \). From Eq. 149 for \( n' \gg 1 \) we have
\[ \Phi = 2 (n \rho)^{\frac{1}{2}} \] (153)

where we have replaced \( n' \) by \( n \) for sufficiently large \( n \). From the last two equations we have
\[ \rho_{\text{max}} = \frac{k^2 \pi^2}{4n} \] (154)

at the position of the maximum value of \( P \). Substituting from Eqs. 141 and 144 into Eq. 154 we have
\[ r_{kn} = \frac{k^2 \pi^2 \lambda}{4\alpha_g} \] (155)

Here we have replaced \( r \) by \( r_{kn} \) at the position of a maximum. Now for \( n \gg 1 \) we can show from Eq. 141 that
\[ \kappa = \frac{\alpha_g}{2n \lambda} \] (156)

We can also show that
\[ \lambda = \frac{R}{2\alpha_g} \] (157)
Substituting Eq. 157 into Eq. 155 we get

$$r_{kn} = \frac{n^2 \pi^2 R}{8 \alpha_g^2}$$

(158)

As we have seen above, for \(n \gg 1\) there is an overlapping of peaks and the largest peak with the highest probability occurs at \(k = n\). Therefore for large \(n\) and with \(k = n\), we have

$$r_n = \frac{n^2 \pi^2 R}{8 \alpha_g^2}$$

(159)

This is the formula we had derived for the case of Newtonian gravity [11]. It is also true for the relativistic scalar gravity case as we have already seen above. Thus, we have a universal, model-independent, result for large \(n\). For \(n \gg 1\) Eq. 125 reduces to

$$M_n = 2m \left(1 - \frac{\alpha_g^2}{8n^2}\right)$$

(160)

This is the same as the corresponding equation for Newtonian gravity, see Eq. 163 below.

On carrying out the analysis for the classification of Holeums, given in § II-F above, we can show that the BHs and the HH classes are unphysical because for \(\alpha_g > \frac{1}{2}\) the various quantities such as mass, binding energy, radius etc. become purely imaginary. Thus we are left with only the Holeums having masses in the range given by \(0 < m < 0.7071 \ m_P\).

F. Gravitational Radiation

When a Holeum in an excited state with a principal quantum number \(n_2\) and energy eigenvalue \(E_2\) makes a transition to lower state with the corresponding quantities \(E_1\) and \(n_1\), with \(n_2 - n_1 = 2\), it emits quantized gravitational radiation of frequency given by

$$\nu = \frac{E_2 - E_1}{h}$$

(161)

where \(E_1\) and \(E_2\) are given by Eq. 122.

IV. SUMMARY OF NEWTONIAN GRAVITY

In this paper we have investigated scalar gravity and vector gravity in the framework of the relativistic Klein-Gordon equation. Earlier we have investigated Newtonian gravity in the framework of the nonrelativistic Schrodinger equation [11]. In order to compare these
three models and to arrive at useful conclusions we present below the corresponding results of Newtonian gravity:

If we solve the Schrödinger equation with the potential given in Eq. 4, we get the following energy eigenvalues:

\[
E_n = -\frac{\mu c^2 \alpha_g^2}{2n^2}
\]  

(162)

The mass of a Holeum is given by

\[
M_n = 2m \left( 1 - \frac{\alpha_g^2}{8n^2} \right)
\]  

(163)

The binding energy is given by

\[
B_n = \frac{mc^2 \alpha_g^2}{4n^2}
\]  

(164)

The most probable radius of the ground state is given by

\[
r_1 = R \frac{R}{\alpha_g^2}
\]  

(165)

where

\[
R = \frac{2mG}{c^2}
\]  

(166)

For \( n \gg 1 \) the most probable radius is given by

\[
r_n = \frac{\pi^2 n^2 R}{8\alpha_g^2}
\]  

(167)

From Eq. 161 we can get an expression for the Schwarzschild radius of the \( n^{th} \) excited state of a Holeum:

\[
R_n = 2R \left( 1 - \frac{\alpha_g^2}{8n^2} \right)
\]  

(168)

where

\[
R_n = \frac{2M_n G}{c^2}
\]  

(169)

where \( M_n \) is given by Eq. 163. From Eqs. 167 and 168 one can calculate the characteristic function \( f_{nN} \) for the Newtonian gravity case. It is identical to that for the scalar gravity case given in § II, starting with Eq. 81 and ending with Eq. 86. In other words, in the asymptotic range \( n \gg 1 \) the results of the relativistic scalar case are identical to those of the Newtonian gravity case.
V. COMPARISON OF THE MODELS

A. Concordance between Newtonian gravity, relativistic scalar gravity, and relativistic vector gravity

In Table 1 we present the predictions of the binding energy of a Holeum by three models, namely Newtonian gravity, relativistic scalar gravity, and relativistic vector gravity. A study of the table reveals an interesting concordance: Three different mathematical functions representing the binding energy of a Holeum in three models under consideration give the same numerical value of the binding energy correct to about six significant figures even as the independent variable \( m \) varies over seventeen orders of magnitude. Similar statements can be made about the mass and other properties of the Holeum predicted by the three models. This must be one of the rarest concordances in theoretical physics. Note, however, that the predictions of the relativistic models begin to disagree with those of the Newtonian gravity case above \( 10^{17} \text{ GeV}/c^2 \).

| \( m \text{ GeV}/c^2 \) | Relativistic Scalar Gravity | Relativistic Vector Gravity | Newtonian Gravity |
|-------------------------|----------------------------|-----------------------------|------------------|
| \( 1 \times 10^{10} \)  | \( 1.12476844 \times 10^{-27} \) | \( 1.12476844 \times 10^{-27} \) | \( 1.12476844 \times 10^{-27} \) |
| \( 1 \times 10^{11} \)  | \( 1.12476844 \times 10^{-22} \) | \( 1.12476844 \times 10^{-22} \) | \( 1.12476844 \times 10^{-22} \) |
| \( 1 \times 10^{12} \)  | \( 1.12476844 \times 10^{-17} \) | \( 1.12476844 \times 10^{-17} \) | \( 1.12476844 \times 10^{-17} \) |
| \( 1 \times 10^{13} \)  | \( 1.12476844 \times 10^{-12} \) | \( 1.12476844 \times 10^{-12} \) | \( 1.12476844 \times 10^{-12} \) |
| \( 1 \times 10^{14} \)  | \( 1.12476844 \times 10^{-7} \) | \( 1.12476844 \times 10^{-7} \) | \( 1.12476844 \times 10^{-7} \) |
| \( 1 \times 10^{15} \)  | \( 1.12476844 \times 10^{-2} \) | \( 1.12476844 \times 10^{-2} \) | \( 1.12476844 \times 10^{-2} \) |
| \( 1 \times 10^{16} \)  | \( 1.12476844 \times 10^{3} \) | \( 1.12476844 \times 10^{3} \) | \( 1.12476844 \times 10^{3} \) |
| \( 1 \times 10^{17} \)  | \( 1.12476844 \times 10^{8} \) | \( 1.12476844 \times 10^{8} \) | \( 1.12476844 \times 10^{8} \) |
| \( 1 \times 10^{18} \)  | \( 1.12467990 \times 10^{13} \) | \( 1.12483171 \times 10^{13} \) | \( 1.12476844 \times 10^{13} \) |
| \( 1 \times 10^{19} \)  | \( 6.75175221 \times 10^{17} \) | — | \( 1.12476844 \times 10^{18} \) |
| \( 1.5 \times 10^{19} \) | \( 2.31197 \times 10^{18} \) | — | \( 8.54121 \times 10^{18} \) |

Table 1: Ground state binding energy in \( GeV \) of a Holeum in relativistic scalar gravity, relativistic vector gravity and Newtonian gravity

23
B. No physics up to $10^{10}$ GeV/$c^2$, followed by a mass range containing interesting physics

A remarkable fact that emerges from the Table 1 is that there is absolutely no physics up to the constituent mass $10^{10}$ GeV/$c^2$. We emphasize that this is the common feature of all the three models studied here. This is true not only for the binding energy but also for the other properties of Holeums. This is a "great desert". This is is followed by a mass range containing many potentially very interesting physics phenomena: As shown in [17] and [16] the Holeums of constituent masses between $10^{10}$ GeV/$c^2$ and $10^{11}$ GeV/$c^2$ have roughly atomic dimensions and they emit quantized gravitational radiation of low frequencies in the kHz range accessible to LIGO (Laser Interferometer Gravity-wave Observatory) and other gravity wave detectors. Holeums of masses between $10^{11}$ GeV/$c^2$ and $10^{12}$ GeV/$c^2$ have roughly nuclear dimensions. They emit gravitational waves of higher frequencies. Holeums of masses up to $10^{14}$ GeV/$c^2$ have low binding energies up to about 112 eV. In the Holeum model, Holeums of all masses accumulate overwhelmingly in galactic halos as cold dark matter [16]. They are prone to break up due to collisions, which results in the emission of Hawking radiation whose two components we have previously identified with cosmic rays and gamma-ray bursts [16].

C. Contrast between relativistic scalar gravity and relativistic vector gravity

In contrast to relativistic scalar gravity, relativistic vector gravity imposes a sharp cut-off on the mass of the PBHs that can form stable Holeums. This follows from the factor $(1 - 4a_0^2)^{\frac{3}{2}}$ appearing in the expressions for the energy eigenvalues, mass, binding energy and the radius of the Holeum. This restricts Holeum formation to strictly sub-Planck $m < 0.70107 m_P$. The unstable BHs occur in the mass range $0.70107 m_P < m < 0.707107 m_P$. This is for the ground state. For the highly excited states $n \gg 1$ the stable Holeum formation is restricted to $m < 0.707107 m_P$ and there are no BHs and HHs.

VI. DISCUSSIONS AND CONCLUSIONS

In this paper we have studied the consequences of the Holeum conjecture in the relativistic domain. We have solved the relativistic Klein-Gordon equation in the limit of static gravity
treating the latter either as a scalar or a vector interaction. Our most interesting finding is that the relativistic models confirm the predictions of the non-relativistic Newtonian gravity model within about six significant figures over almost the entire sub-Planck domain. This strongly validates our earlier results based on Newtonian gravity.

The comparison of the two relativistic models with Newtonian gravity reveals two very striking results: (1) As a function of the mass $m$ of the constituent PBHs there is a great gap in the formation of viable Holeums as reflected in the values of the parameters such as the binding energy. This gap stretches over ten orders of magnitude in $m$. (2) This gap is followed by an interval stretching over four orders of magnitude in $m$ containing potentially very interesting physics such as quantized gravitational radiation, cosmic rays, gamma ray bursts, etc.

The presence of such a great gap in a physical theory is spurious and unnatural. It points to an inadequacy in the model. Since gravity and the geometry of space-time are intrinsically inseparable the presence of the gap in the three and the four dimensional models strongly suggests that our universe must have more than four dimensions. Since our model is exclusively a model of gravity this finding has a more direct relevance to the dimensionality of the space-time of the universe.

The presence of a “great desert” is well-known in particle physics. The electroweak unification in particle physics occurs at 300 GeV but the next unification, called the Grand Unification of the strong and the electro-weak interactions, occurs at $10^{16}$ GeV. This is the "great desert" of particle physics. A remedy is suggested by Kaluza-Klein theories with two extra dimensions [18]. They bring down the Planck energy scale to one TeV when the two extra dimensions are compactified. This removes the “great desert” in particle physics.

Kaluza-Klein theory may have a two-fold effect on the Holeum model: (a) It may remove our gap. (b) In such theories it is found that the dark matter particles such as PBHs and Holeums are confined to a manifold different from the one in which the particles of the standard model are confined [17]. This may have interesting implications for Holeum formation in the domain of interesting physics.

Now let us address the issue of the verification of the Holeum conjecture. We first list the criteria for the formation of stable, non-radiating bound states of black holes, namely, Holeums. These are: (1) The rate of the gravitational interactions $\Gamma$ must be greater than $H$, the rate of the expansion of the universe at the temperature of the formation of the
bound state. (2) The binding energy of the would-be bound state must be much greater than the kinetic energy of the primordial brew at the temperature under consideration. (3) The ratio, $f_n$ of the Schwarzschild radius of the bound state, $R_n$ to the most probable radius, $r_n$ of the bound state must be less than unity. That is, $f_n < 1$.

The first criterion simply says that the primordial brew is instantaneously in thermal equilibrium. That is, it expands adiabatically. The first criterion is a rule of the thumb. It is found to give surprisingly good results in general, even though one must use the Boltzmann equation for more precise results in actual practice. The second criterion ensures that the dissociation due to the collisions is negligible. The third one says that the bound state of two black holes is not itself a black hole. Otherwise it will be destroyed by the Hawking radiation it must emit.

Now it is well-known [19] that perturbative interactions mediated by massless gauge bosons such as the photons and the gravitons are incapable of thermalizing the universe above the temperature of $10^{16}$ GeV. But they can thermalize the universe below the latter temperature. That is, the condition (1), $\Gamma > H$, is satisfied in the temperature range less than $10^{16}$ GeV as mentioned above. For the Relativistic Scalar Gravity case, we see from Eq. (65) that the condition $f_{1s} < 1$ is satisfied for $m < 1.2722 \times 10^{19} \text{GeV}/c^2$. This complies with the condition (3). From Table 1 we see that the binding energies of the Holeums with constituent mass in the range

$$10^{19} \text{GeV}/c^2 < m < 1.55 \times 10^{19} \text{GeV}/c^2$$

(170)

satisfy the condition (2) reasonably well. Since these Holeums have masses greater than the Planck mass we call them the Hyper Holeums (HH) [16]. Thus the formation of HHs is feasible as in [16]. Note that the mass range in [16] is somewhat different from the scalar relativistic case.

Now let us consider the domain of interesting physics lying between $10^{10} \text{GeV}/c^2$ and $10^{14} \text{GeV}/c^2$. In particular consider the case $m = 10^{14} \text{GeV}/c^2$ at a temperature of, say, 1 eV. As noted above the interactions mediated by massless gauge bosons, such as the graviton in this case, are capable of thermalizing the universe below the temperature of $10^{16}$ GeV. Thus the condition (1) mentioned above is satisfied. The binding energy of a Holeum made of two black holes each of mass $10^{14} \text{GeV}/c^2$ is about 112 eV which is much greater than the kinetic energy at the temperature of 1 eV. Therefore at a temperature of the order of 1 eV
the condition (2) on the binding energy is satisfied quite well. From Eq. [65] we see that the condition (3) $f_{1s} < 1$ is also satisfied in the case of the scalar gravity under consideration here. Thus, we conclude that the formation of Holeums of mass $10^{14}$ GeV/$c^2$ is quite feasible at the temperature of the order of 1 eV or less. It is clear that a good case for the formation of Holeums in the domain of interesting physics is at hand.

[1] Gianfranco Bertone, Dan Hooper, Joseph Silk, Phys.Rept. 405, 279-390; arXiv:hep-ph/0404175v2 (2005).
[2] L. Bergstrom, Rept. Prog. Phys. 63, 793; arXiv:hep-ph/0002126v1 (2000).
[3] S. Dodelson and L. M. Widrow, Phys. Rev. Lett. 72, 17; arXiv:hep-ph/9303287v1 (1994).
[4] R. Bernabei et al, Riv.Nuovo Cim. 26N1, 1-73; arXiv:astro-ph/0307403v1 (2003).
[5] The MACHO collaboration: C. Alcock, R.A. Allsman, D.R. Alves, T.S. Axelrod, A.C. Becker, D.P. Bennett, K.H. Cook, N. Dalal, A.J. Drake, K.C. Freeman, M. Geha, K. Griest, M.J. Lehner, S.L. Marshall, D. Minniti, C.A. Nelson, B.A. Peterson, P. Popowski, M.R. Pratt, P.J. Quinn, C.W. Stubbs, W. Sutherland, A.B. Tomaney, T. Vandehei, D. Welch, Astrophys.J. 542 (2000) 281-307; arXiv:astro-ph/0001272 (2000).
[6] Stephen P. Martin, A Supersymmetry Primer, arXiv:hep-ph/9709356v6 (2011).
[7] Hsin-Chia Cheng, Jonathan L. Feng, Konstantin T. Matchev, Phys.Rev.Lett. 89, 211301; arXiv:hep-ph/0207125v2 (2002).
[8] Masahiro Kawasaki, Kazunori Nakayama, Axions : Theory and Cosmological Role; arXiv:1301.1123v1 (2013).
[9] Karim Benakli, John Ellis, Dimitri V. Nanopoulos, Phys.Rev.D59:047301,1999; arXiv:hep-ph/9803333v1 (1998).
[10] Paul H. Frampton, Masahiro Kawasaki, Fuminobu Takahashi, Tsutomu T. Yanagida, Primordial Black Holes as All Dark Matter; arXiv:1001.2308v2 (2010).
[11] Chavda L K and Chavda A L, Class. Quantum Grav 19, 2927, 2002; arXiv:gr-qc/0308054 (2003).
[12] Chavda A L and Chavda L K, Quantized Gravitational Radiation from Black Holes and other Macro Holeums in the Low Frequency Domain, arXiv:0903.0703 (2009).
[13] Chavda A L and Chavda L K, Ultra High Energy Cosmic Rays from decays of Holeums in
Galactic Halos, arXiv:0806.0454 (2008).

[14] Dallal Shawqi Al, Adv. Space Res 40, 1236, (2007).

[15] Dallal Shawqi Al, Adv. Space Res 46, 468, (2010).

[16] Chavda L K and Chavda A L, *Holeum, enigmas of cosmology and gravitational waves*, arXiv:gr-qc/0309044 (2003).

[17] Dallal Shawqi Al and Azzam W J, J. Math. Phys. 3 1131, (2012).

[18] Arkani-Hamed N, Dimopoulos S and Dvali G, *The Hierarchy Problem and New Dimensions at a Millimeter*, arXiv:hep-ph/9803315 (1998).

[19] Kolb E W and Turner M S. The Early Universe, Addison-Wesley, Reading, Massachusetts, 1990.