ON PHASE–FIELD EQUATIONS
OF PENROSE—FIFE TYPE WITH
THE NON–CONSERVED ORDER PARAMETER
UNDER FLUX BOUNDARY CONDITION.
I: GLOBAL–IN–TIME SOLVABILITY
A. Tani

Abstract. We study the initial-boundary value problem for the non-conserved phase-field model proposed by Penrose and Fife in 1990 [1] under the flux boundary condition for the temperature field in higher space dimensions, which is obliged to overcome additional difficulties in the mathematical treatment. In all the existing works concerning this problem, only one due to Horn et al. [2] was discussed under the correct form of the flux boundary condition. Here we prove that the same correctly formulated problem as theirs is well-posed globally-in-time in Sobolev–Slobodetski˘ı spaces. Moreover, it is shown that the temperature keeps positive through the time evolution.

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Dedicated to Professor Ivan E. Egorov
on his seventieth birthday

1. Introduction

In this paper we are concerned with the non-conserved phase-field equation proposed by Penrose and Fife [1,3] which is a continuum model for the description of dynamics of order-disorder phase transition taking into account of both the relaxation and the balance laws based on the second law of thermodynamics:

\[
\frac{\partial \varphi}{\partial t} = M_1 \left( K \Delta \varphi - \frac{1}{\theta} \frac{\partial f}{\partial \varphi} \right), \quad \frac{\partial e}{\partial t} = -M_2 \Delta \frac{1}{\theta} + g, \quad (1.1)
\]

where \( \varphi \) is an order parameter, \( \theta \) is an absolute temperature, \( M_1, M_2 \) and \( K \) are positive constants, \( e = e(\varphi, \theta) \) is the internal energy defined by

\[
e = f - \theta \frac{\partial f}{\partial \theta} \quad (1.2)
\]

with the free energy density \( f = f(\varphi, \theta) \), and \( g = g(x, t) \) is a heat supply. The free energy density \( f(\varphi, \theta) \) assumed in [1,3] is of the form

\[
f(\varphi, \theta) = -C_V(\theta \ln \theta + \delta) - \frac{a}{2} \varphi^2 - b \varphi + c - \theta s_0(\varphi) \quad (1.3)
\]

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with given constants \( C_V > 0, a \geq 0, \delta \geq 0, b, c \) and a concave function \( s_0(\varphi) \). The most commonly used form of \( s_0 \) is the double equal-well potential,

\[
S_0(\varphi) = -\frac{1}{4}(1 - \varphi^2)^2.
\]  

(1.4)

Inserting (1.2)–(1.4) into (1.1), we have

\[
\frac{\partial \varphi}{\partial t} =\frac{M_1}{\Delta \varphi} \left(K \Delta \varphi - \varphi^3 + \varphi + \frac{a\varphi + b}{\theta}\right), \quad C_V \frac{\partial \theta}{\partial t} - (a\varphi + b) \frac{\partial \varphi}{\partial t} = -M_2 \Delta \frac{1}{\theta} + g.
\]  

(1.5)

We consider (1.5) in a bounded domain \( \Omega \subset \mathbb{R}^N (N = 2, 3) \), with a smooth boundary \( \Gamma \). The boundary conditions imposed on \( \Gamma \) are:

\[
\frac{\partial \varphi}{\partial n} = 0, \quad M \frac{\partial}{\partial n} \frac{1}{\theta} = \beta (\theta - \theta_e),
\]

where \( \mathbf{n} = \mathbf{n}(x) \) is an outward unit normal vector to \( \Gamma \) at \( x \in \Gamma \), \( \beta \) is a heat conductivity on the boundary \( \Gamma \) and \( \theta_e (> 0) \) is an external temperature. The latter condition for \( \theta \) is the so-called flux boundary condition.

A phase transition in two-phase system has been theorized as a continuum in which two phases may coexist, so that the transition between them is considered to occur smoothly within an appropriate layer or diffuse interface. The use of diffuse interface models in describing the phase transition is traced back to van der Waals [4, 5], Landau [6] for the second order phase transitions by introducing the notion of order parameter, and Devonshire [7] for the first order transitions. Then, the theory has been extended as continuum models for the description of dynamics of order-disorder phase transition by Cahn [8–10] (Models A and B in [11]; see also [12, 13], Caginalp [14], Penrose and Fife [1, 3] and so on.

In the sequel, without loss of generality, we assume that \( C_V = M_1 = 1, M_2 = M \) and \( \beta \) is a positive constant. And it is more convenient to use \( u = 1/\theta \) instead of \( \theta \), so that our problem is formulated as follows:

\[
\begin{cases}
\frac{\partial \varphi}{\partial t} = K \Delta \varphi - \varphi^3 + \varphi + (a\varphi + b)u, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u}{\partial t} + (a\varphi + b)u^2 \frac{\partial \varphi}{\partial t} = M u^2 \Delta u - gu^2, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial \varphi}{\partial n} = 0, \quad M \frac{\partial u}{\partial n} = \beta \left( \frac{1}{\theta} - \theta_e \right), \quad x \in \Gamma, \quad t > 0, \\
(\varphi, u)|_{t=0} = (\varphi_0, u_0)(x), \quad x \in \Omega.
\end{cases}
\]  

(1.6)

Concerning the mathematical results related to problem (1.6) with \( \beta = 0 \) the existence of a unique strong solution was proved by Zheng [15] and Sprekels and Zheng [16] (see also [17]). In the framework of weak solutions of (1.6) and its generalization there are many papers (see, [18–22] and the references therein). However, for problem (1.6) with \( \beta > 0 \), as was pointed out in [13], all results except [2] were reported under the physically incorrect flux conditions, for example, in [21]

\[
M \frac{\partial}{\partial n} \frac{1}{\theta} = \beta \left( \frac{1}{\theta} - \theta_e \right),
\]

and in [22]

\[
M \frac{\partial \theta}{\partial n} = \beta (\theta - \theta_e).
\]
The aim of the present paper is to show the unique existence of a strong solution in Sobolev–Slobodetskii spaces in higher space dimensions, which are different from [2]. Moreover, a boundedness of the solution is shown up to an arbitrary finite time.

In what follows, we focus our study on the most important case, $N = 3$.

Let us describe our results.

**Theorem 1.1.** Let $T$ be any positive number, and $Ω$ be a bounded domain in $\mathbb{R}^3$ with a boundary $Γ$ belonging to $W^{3/2,1}(Γ)$, $l > 1/2$. Suppose that $g ∈ W^{1,1/2}(Γ)$, $(Q_Γ \equiv Ω \times (0, T))$, $θ_ε ∈ W^{1/2,1/4+l/2}(Γ)$ $(Γ_T := Γ × (0, T))$, and $φ_0, u_0 ∈ W^{1+l}(Ω)$ satisfy

$$\inf_{x ∈ Ω} u_0(x) \equiv u_0 > 0,$$

and the compatibility conditions up to order max$\{[l − 3/2], 0\}$. Then problem (1.6) has a unique solution $(φ, u) ∈ W^{2+l, (2+l)/2}(Q_T) \times W^{2+l, (2+l)/2}(Q_T)$ for some $T^* ∈ (0, T]$ such that

$$\|θ_ε\|_{W^{2+l, (2+l)/2}(Q_T)} < β, \frac{∂θ_ε}{∂t} ∈ L_2(Γ; W^{1,0}_0(0, T))$$

and

$$u(x, t) ≥ \frac{1}{2} u_0 \text{ for any } (x, t) ∈ Q_T.$$  

**Theorem 1.2.** Assume that the hypotheses in Theorem 1.1 with $l = 2$, and

$$\inf_{(x,t) ∈ Γ_T} θ_ε(x, t) \equiv θ_ε > 0, \sup_{t ∈ [0, T]} \|g(t)\|_{\mathcal{V}(Ω)} \leq β, \frac{∂θ_ε}{∂t} ∈ L_2(Γ; W^{1}_0(0, T))$$

hold, where $|Ω|$ is a volume of $Ω$. Then the solution in Theorem 1.1 is extended on $[0, T]$:

$$\|(φ, u)\|_{W^{2+l, (2+l)/2}(Q_T)} ≤ C(T), \ u(x, t) ≥ C_*(T) \text{ for any } (x, t) ∈ Q_T.$$  

Here $C(T)$ and $C_*(T)$ are positive constants depending non-decreasingly on both the data and $T$.

**Remark 1.3.** Our proof is applicable to the similar problems with more general $f$, for example,

$$f(φ, θ) = -C_ν(\theta \ln θ + δ_1 θ + δ_2) + \frac{α_1}{2} (θ − θ^*)φ^2 − \frac{α_3}{3} φ^3 + \frac{α_4}{4} φ^4 + \frac{α_6}{6} φ^6$$

with positive constants $C_ν$, $α_1$, $θ^*$ and nonnegative constants $δ_1$, $δ_2$, $α_3$, $α_4$, $α_6$ satisfying $α_4 + α_6 > 0$ proposed by Alt and Pawlow [23]. The corresponding results to Theorems 1.1 and 1.2 still hold for such an $f$.

Throughout this paper we use the standard Sobolev–Slobodetskii spaces $W^r_s(Ω)$ with norms $\|\cdot\|_r (\|\cdot\|_r = \|\cdot\|_0)$ and the anisotropic ones $W^{r,s}_ε(Q_T) = L_2(0, T; W^{r,s}_ε(Ω)) ∩ L_2(Ω; W^{r,s}_ε(0, T))$ for non-negative real numbers $r$, $s$ and $T(\leq +∞)$. By $W^r_s(0, T; W^{r,s}_ε(Ω))$ we denote the function space of $u(t)$ defined on $(0, T)$ with values in $W^r_s(Ω)$ such that $u(t) ∈ W^{r,s}_ε(Ω)$. In particular, the case $s = r/2$,

$$W^{r, r/2}_ε(Q_T) = L_2(0, T; W^{r/2}_ε(Ω)) ∩ L_2(Ω; W^{r, r/2}_ε(0, T))$$,
is mainly used. For a smooth manifold $\partial \Omega = \Gamma$, the spaces $W^r_2(\Gamma)$ of functions defined on $\Gamma$ are introduced in a standard manner by means of the local coordinates and the partition of unity, and $W^{r,s}_2(\Gamma_T)$, $\Gamma_T = \Gamma \times (0,T)$, can be defined in the same way as above. The spaces of vector fields whose components belong to, for example, $W^{r,s}_2(Q_T)$ are denoted by the same notation as the scalar case, $W^{r,s}_2(Q_T)$, and their norms are supposed to be equal to the sum of norms of all its components. In detail, see [24–26].

In what follows, we also denote by $c$ the universal positive constants which may vary in different places; by $C_i$ and $C_i(T)$ ($i = 1,2,3,\ldots$) the positive constants depending non-decreasingly on the data but not on $T$, and may depend non-decreasingly on both the data and $T$, respectively; besides some constants depend on the indicated quantities.

We give proofs of Theorems 1.1 and 1.2 in Sections 2 and 3, respectively.

2. Local-in-time existence: Proof of Theorem 1.1

(Step 1)) For simplicity let us assume $1 > l > 1/2$, and consider a linear problem

\[
\begin{aligned}
\frac{\partial v}{\partial t} &= d(x,t) \Delta v + f(x,t) \quad (x \in \Omega, \ t > 0), \\
\frac{\partial v}{\partial n} &= h(x,t) \quad (x \in \Gamma, \ t > 0), \quad v|_{t=0} = 0 \quad (x \in \Omega),
\end{aligned}
\]

(2.1)

where $d(x,t) \in W^{1,1/2}_2(Q_T)$, $f(x,t) \in W^{1,1/2}_2(Q_T)$ and $h(x,t) \in W^{1/2+1/4+1/2}_2(\Gamma_T)$ are given functions satisfying $d(x,t) \geq d = \text{const} > 0$ and the compatibility condition.

The theory of linear partial differential equations of parabolic type [25,27] implies that problem (2.1) has a unique solution $v \in W^{2+1,1/2}_2(Q_T)$ satisfying

\[
\|v\|_{W^{2+1,1/2}_2(Q_T)} \leq C_1(\|f\|_{W^{1,1/2}_2(Q_T)} + \|h\|_{W^{1/2+1/4+1/2}_2(\Gamma_T)}).
\]

(2.2)

Let us denote the extension of $(\varphi_0,u_0)$ by $(\tilde{\varphi}_0,\tilde{u}_0) \in W^{2+l,1+1/2}_2(Q_T)$ satisfying

\[
\|(\tilde{\varphi}_0,\tilde{u}_0)\|_{W^{2+l,1+1/2}_2(Q_T)} \leq c(\|\varphi_0,u_0\|_{W^{2+1}_2(\Omega)}).
\]

When we choose in (2.1)

\[
d = K, \quad h(x,t) = -\frac{\partial \tilde{\varphi}_0}{\partial n} \equiv h_1,
\]

\[
f(x,t) = -\varphi^3 + \varphi + (a\varphi + b)u - \frac{\partial \tilde{\varphi}_0}{\partial t} + K\Delta \tilde{\varphi}_0 \equiv f_1(\varphi,u),
\]

the unique solution $v$ gives $\varphi - \tilde{\varphi}_0$; we choose in (2.1)

\[
d = M\tilde{u}_0^2, \quad h(x,t) = \frac{\beta}{M} \left( \frac{1}{u} - \theta_e \right) - \frac{\partial \tilde{u}_0}{\partial n} \equiv h_2(u),
\]

\[
f(x,t) = -(a\varphi + b)u^2 \frac{\partial \varphi}{\partial t} - gu^2 - \frac{\partial \tilde{u}_0}{\partial t} + M(u^2 - \tilde{u}_0^2)\Delta u + M\tilde{u}_0^2 \Delta \tilde{u}_0 \equiv f_2(\varphi,u),
\]

the unique solution $v$ gives $u - \tilde{u}_0$.

In order to estimate $f_1, f_2, h_1, h_2$, we rely on the following lemma:
Lemma 2.1 [26, Lemma 3.5]. If \( w_1 \in W^{m,m/2}_{2} (Q_T) \), \( w_2 \in W^{m',m'/2}_{2} (Q_T) \) with \( m > 5/2, m \geq m' \geq 0 \), then \( w_1 w_2 \in W^{m',m'/2}_{2} (Q_T) \) and
\[
\| w_1 w_2 \|_{W^{m',m'/2}_{2} (Q_T)} \leq \| w_1 \|_{W^{m,m/2}_{2} (Q_T)} \| w_2 \|_{W^{m',m'/2}_{2} (Q_T)}.
\]

By using Lemma 2.1 and interpolation inequalities it is easy to obtain
\[
\| f_1 \|_{W^{1/2}_{2} (Q)} \leq (\| \varphi \|_{W^{2+1,1+1/2}_{2} (Q)} + a \| u \|_{W^{2+1,1+1/2}_{2} (Q)} + 1) \| \varphi \|_{W^{1/2}_{2} (Q)}
+ |b| \| u \|_{W^{1,1/2}_{2} (Q)} + (K + 1) \| \varphi \|_{W^{1,1+1/2}_{2} (Q)}
\leq C_2 (\| \varphi \|_{W^{2+1,1+1/2}_{2} (Q)}, \| u \|_{W^{2+1,1+1/2}_{2} (Q)}, (\varepsilon_1 + C_\varepsilon t) \| \varphi \|_{W^{2+1,1+1/2}_{2} (Q)}
+ |b| (\varepsilon_2 + C_\varepsilon t) \| u \|_{W^{2+1,1+1/2}_{2} (Q)} + c(K + 1) \| \varphi \|_{W^{1+1}_{2} (Q)}, \quad (2.3)
\]
\[
\| f_2 \|_{W^{1/2}_{2} (Q)} \leq \left( (a) \| \varphi \|_{W^{2+1,1+1/2}_{2} (Q)} + |b| \right) \left( \frac{\partial \varphi}{\partial t} \|_{W^{1/2}_{2} (Q)} + \| g \|_{W^{1/2}_{2} (Q)} \right)
\times \| u \|_{W^{2+1,1+1/2}_{2} (Q)} + M \| \Delta u \|_{W^{1/2}_{2} (Q)} \| u^2 - \tilde{u}^2 \|_{W^{2+1,1+1/2}_{2} (Q)}
+ (M \| \tilde{u} \|_{W^{2+1,1+1/2}_{2} (Q)} + 1) \| \tilde{u} \|_{W^{2+1,1+1/2}_{2} (Q)}
\leq C_3 (\| \varphi \|_{W^{2+1,1+1/2}_{2} (Q)}, \| u \|_{W^{2+1,1+1/2}_{2} (Q)}, \| g \|_{W^{1/2}_{2} (Q)}, \| \tilde{u} \|_{W^{1/2}_{2} (Q)}, \| u \|_{W^{1+1}_{2} (Q)}
\times (\varepsilon_3 + C_\varepsilon t) \| u \|_{W^{2+1,1+1/2}_{2} (Q)} + c(M \| u \|_{W^{2+1,1+1/2} (Q)} + 1) \| u \|_{W^{1+1}_{2} (Q)}, \quad (2.4)
\]
Here \( l' \) is any number, \( l' \in (1/2, l) \); \( C_2 \) and \( C_3 \) are positive constants depending on each indicated argument monotonically non-decreasingly; \( \varepsilon_i \) is any positive number; \( C_\varepsilon \) is a positive constant which increases monotonically as \( \varepsilon_i \) tends to 0 (\( i = 1, 2, 3 \)).

Trace theorem leads to
\[
\| h_1 \|_{W^{2+1,1/4+1/2}_{2} (Q_T)} \leq C_4 \| \varphi \|_{W^{2+1,1+1/2}_{2} (Q)}, \quad (2.5)
\]
\[
\| h_2 \|_{W^{2+1,1/4+1/2}_{2} (Q_T)} \leq \frac{\beta}{M} \| \frac{\partial}{\partial t} W^{2+1,1/4+1/2}_{2} (Q_T) + \frac{\beta}{M} \| \theta \varepsilon W^{2+1,1/4+1/2}_{2} (Q_T) + C_4 \| \tilde{u} \|_{W^{2+1,1+1/2}_{2} (Q)}
\leq C_5 \left( \| u \|_{W^{2+1,1+1/2}_{2} (Q_T)}, \frac{1}{\| u \|_{W^{2+1,1/2}_{2} (Q_T)}} (\varepsilon_4 + C_\varepsilon t) \| u \|_{W^{2+1,1+1/2}_{2} (Q)}
+ \frac{\beta}{M} \| \theta \varepsilon \|_{W^{2+1,1/4+1/2}_{2} (Q_T)} + C_4 \| u_0 \|_{W^{1+1}_{2} (Q)}, \quad (2.6)
\]
where \( C_5, \varepsilon_4 \) and \( C_\varepsilon \) have the same properties as \( C_3, \varepsilon_1 \) and \( C_\varepsilon \), respectively.

Let \( A_1 \) and \( A_2 \) be any positive constants satisfying
\[
A_1 > 2c(1 + C_1 (K + 1 + C_4)) \| \varphi \|_{W^{1+1}_{2} (Q)}
\]
and
\[
A_2 > 2c(1 + C_1 (M \| u_0 \|_{W^{1+1}_{2} (Q)} + 1 + C_4)) \| u_0 \|_{W^{1+1}_{2} (Q)} + \frac{2\beta C_1}{M} \| \theta \varepsilon \|_{W^{2+1,1/4+1/2}_{2} (Q_T)}
\]
respectively.

Now we define the successive approximate solution \( \{(\varphi^{(m)}, u^{(m)})\}_{m=0}^{\infty} \) as follows:

\[
(\varphi^{(0)}, u^{(0)}) = (\tilde{\varphi}_0, \tilde{u}_0),
\]

\( \varphi^{(m)} - \tilde{\varphi}_0 \) is a solution to (2.1) with \( d = K, \ h = h_1, \ f = f_1(\varphi^{(m-1)}, u^{(m-1)}) \),

\( u^{(m)} - \tilde{u}_0 \) is a solution to (2.1) with

\[
d = M\tilde{u}_0, \ h = h_2(u^{(m-1)}), \ f = f_2(\varphi^{(m-1)}, u^{(m-1)}) \quad (m = 1, 2, 3, \ldots),
\]

provided

\[
(\varphi^{(m-1)}, u^{(m-1)}) \in W^{2+l,1+1/2}_2(Q_T) \times W^{2+l,1+1/2}_2(Q_T),
\]

\[
\|\varphi^{(m-1)}\|_{W^{2+l,1+1/2}_2(Q_T)} < A_1, \quad \|u^{(m-1)}\|_{W^{2+l,1+1/2}_2(Q_T)} < A_2.
\]

Estimates (2.2), (2.3), (2.5) imply

\[
\|\varphi^{(m)}\|_{W^{2+l,1+1/2}_2(Q_T)} < \frac{A_1}{2} + C_1C_2(A_1, A_2)A_1(\epsilon_1 + C\epsilon_1 t) + |b|C_1A_2(\epsilon_2 + C\epsilon_2 t),
\]

and (2.2), (2.4), (2.6) imply

\[
\|u^{(m)}\|_{W^{2+l,1+1/2}_2(Q_T)} < \frac{A_2}{2} + C_1C_2(A_1, A_2)\|g\|_{W^{1+1/2}_2(Q_T)} \|u_0\|_{W^{l+1}_2(\Omega)} A_2(\epsilon_3 + C\epsilon_3 t) + A_2C_1C_3(A_2, 1/\omega_0)(\epsilon_4 + C\epsilon_4 t).
\]

First we choose \( \epsilon_1, \epsilon_4 \) so small that

\[
C_1(C_2(A_1, A_2)A_1 |b|A_2 \epsilon_2) < \frac{A_1}{4},
\]

\[
C_1(C_3(A_1, A_2, \|g\|_{W^{1+1/2}_2(Q_T)} \|u_0\|_{W^{l+1}_2(\Omega)}) \epsilon_3 + C_5(A_2, 1/\omega_0) \epsilon_4 < \frac{1}{4}
\]

hold, and then choose \( T_0 \in (0, T] \) to satisfy

\[
C_1(C_2(A_1, A_2)A_1 C\epsilon_1 + |b|A_2 C\epsilon_2)T_0 < \frac{A_1}{4},
\]

\[
C_1 \left( C_3(A_1, A_2, \|g\|_{W^{1+1/2}_2(Q_T)} \|u_0\|_{W^{l+1}_2(\Omega)}) C\epsilon_3 + C_5 \left( A_2, \frac{1}{\omega_0} \right) C\epsilon_4 \right) T_0 < \frac{1}{4}.
\]

Thus, from (2.7) and (2.8) we conclude by induction

\[
\|\varphi^{(m)}\|_{W^{2+l,1+1/2}_2(Q_{T_0})} < A_1, \quad \|u^{(m)}\|_{W^{2+l,1+1/2}_2(Q_{T_0})} < A_2 \quad (m = 0, 1, 2, \ldots).
\]

((Step 2)) We show that \( \{(\varphi^{(m)}, u^{(m)})\}_{m=0}^{\infty} \) is a Cauchy sequence in

\[
W^{2+l,1+1/2}_2(Q_{T^*}) \times W^{2+l,1+1/2}_2(Q_{T^*})
\]

for some \( T^* \in (0, T_0] \). For that we begin with the estimates \( f_1(\varphi, u) - f_1(\varphi', u') \),

\( f_2(\varphi, u) - f_2(\varphi', u') \) and \( h_2(u) - h_2(u') \), where \( \varphi', \varphi \) and \( u, u' \) satisfy (2.9).
Inequalities (2.2) and (2.10) yield
\[
\|f_1(\varphi, u) - f_1(\varphi', u')\|_{W^{2,1}_{2}(Q_t)} \leq (3A_1^2 + aA_2 + 1)(\varepsilon_5 + C\varepsilon_t)\|\varphi - \varphi'\|_{W^{2,1}_{2}(Q_t)}^2 + (aA_1 + |b|)(\varepsilon_6 + C\varepsilon_t)\|u - u'\|_{W^{2,1}_{2}(Q_t)}^2.
\]

(2.10)

\[
\|f_2(\varphi, u) - f_2(\varphi', u')\|_{W^{1,1}_{2}(Q_t)} \leq A_2^2(3aA_1 + |b|)(\varepsilon_7 + C\varepsilon_t)\|\varphi - \varphi'\|_{W^{2,1}_{2}(Q_t)}^2 + (A_2(2aA_1 + 2|b| + 2\|g\|_{W^{1,1}_{2}(Q_t)} + (2 + M)A_2) + Mc\|u_0\|_{W^{2,1}_{2}(\Omega)}) \times (\varepsilon_8 + C\varepsilon_t)\|u - u'\|_{W^{2,1}_{2}(Q_t)}^2.
\]

(2.11)

\[
\|h_2(u) - h_2(u')\|_{W^{1,1}_{2}(\Omega)} \leq C_2 \left( A_2, \frac{1}{\mu_0} \right)^2 (\varepsilon_9 + C\varepsilon_t)\|u - u'\|_{W^{2,1}_{2}(Q_t)}^2.
\]

(2.12)

for any \( t \in (0, T_0] \). Here again \( \varepsilon_i \) is any positive number; \( C_{\varepsilon_i} \) is a positive constant increasing monotonically as \( \varepsilon_i \) tends to 0 (\( i = 5, 6, \ldots, 9 \)).

As in ((Step 1)) \( \varphi^{(m)} - \varphi^{(m-1)} \) satisfies (2.1) with
\[
d = K, \quad h(x, t) = 0, \quad f(x, t) = f_1(\varphi^{(m-1)}, u^{(m-1)}) - f_1(\varphi^{(m-2)}, u^{(m-2)}),
\]
and \( u^{(m)} - u^{(m-1)} \) satisfies (2.1) with
\[
d = M\mu_0^2, \quad h(x, t) = h_2(u^{(m-1)}) - h_2(u^{(m-2)}), \quad f(x, t) = f_2(\varphi^{(m-1)}, u^{(m-1)}) - f_2(\varphi^{(m-2)}, u^{(m-2)}).
\]

Hence, inequalities (2.2) and (2.10) yield
\[
\|\varphi^{(m)} - \varphi^{(m-1)}\|_{W^{2,1}_{2}(Q_t)} \leq C_1 \left( 3A_1^2 + aA_2 + 1 \right)(\varepsilon_5 + C\varepsilon_t)\|\varphi^{(m-1)} - \varphi^{(m-2)}\|_{W^{2,1}_{2}(Q_t)}^2 + C_1(aA_1 + |b|)(\varepsilon_6 + C\varepsilon_t)\|u^{(m-1)} - u^{(m-2)}\|_{W^{2,1}_{2}(Q_t)}^2.
\]

(2.13)

and inequalities (2.2), (2.11) and (2.12) yield
\[
\|u^{(m)} - u^{(m-1)}\|_{W^{2,1}_{2}(Q_t)} \leq C_1 A_2^2(3aA_1 + |b|)(\varepsilon_7 + C\varepsilon_t)\|\varphi^{(m-1)} - \varphi^{(m-2)}\|_{W^{2,1}_{2}(Q_t)}^2 + C_1 \left( A_2(2aA_1 + 2|b| + 2\|g\|_{W^{1,1}_{2}(Q_t)} + (2 + M)A_2) + Mc\|u_0\|_{W^{2,1}_{2}(\Omega)} \right) \times (\varepsilon_8 + C\varepsilon_t) + C_2 \left( A_2, \frac{1}{\mu_0} \right)^2 (\varepsilon_9 + C\varepsilon_t) \times \|u^{(m-1)} - u^{(m-2)}\|_{W^{2,1}_{2}(Q_t)}.
\]

(2.14)
Now we take \( \varepsilon_5, \varepsilon_9 \) first and then \( T_1 \in (0, T_0] \) as

\[
(3A_1^2 + aA_2 + 1)\varepsilon_5 + A_2^2(3aA_1 + |b|)\varepsilon_7 < \frac{1}{4C_1},
\]

\[
(aA_1 + |b|)\varepsilon_6 + (A_2(2aA_1 + 2|b| + 2\|g\|_{W_2^{1,1/2}(Q_T)} + (2 + M)A_2) + Mc\|u_0\|_{W_2^{1,1}(\Omega)}^2)\varepsilon_8 + C_5 \left( A_2, \frac{1}{\varepsilon_0} \right)^2 \varepsilon_9 < \frac{1}{4C_1};
\]

\[
T_1((3A_1^2 + aA_2 + 1)C_{\varepsilon_5} + A_2^2(3aA_1 + |b|)C_{\varepsilon_7}) < \frac{1}{4C_1},
\]

so that (2.13) and (2.14) lead to

\[
\|\varphi^{(m)} - \varphi^{(m-1)}\|_{W_2^{2,1,1+1/2}(Q_{T_1})} + \|u^{(m)} - u^{(m-1)}\|_{W_2^{2,1,1+1/2}(Q_{T_1})} \leq \frac{1}{2} \left[ \|\varphi^{(m-1)} - \varphi^{(m-2)}\|_{W_2^{2,1,1+1/2}(Q_{T_1})} + \|u^{(m-1)} - u^{(m-2)}\|_{W_2^{2,1,1+1/2}(Q_{T_1})} \right].
\]

(2.15)

This means that there exists a limit function \((\varphi, u)\) of \((\varphi^{(m)}, u^{(m)})\) on \([0, T_1]\) in the topology of \(W_2^{2,1,1+1/2}(Q_{T_1}) \times W_2^{2,1,1+1/2}(Q_{T_1})\), which satisfies

\[
\|\varphi\|_{W_2^{2,1,1+1/2}(Q_{T_1})} \leq A_1, \quad \|u\|_{W_2^{2,1,1+1/2}(Q_{T_1})} \leq A_2.
\]

(2.16)

The uniqueness of the solution follows from inequality (2.15) with \((\varphi^{(m)}, u^{(m)}) = (\varphi, u)\) and \((\varphi^{(m-1)}, u^{(m-1)}) = (\varphi', u')\) by assuming that \((\varphi, u)\) and \((\varphi', u')\) are two solutions of problem (1.6).

As for (1.8) it is easy to see that

\[
u(x, t) = u_0(x) + \int_0^t \frac{\partial u}{\partial \tau}(x, \tau) d\tau \geq u_0 - \sqrt{7}A_2 \geq \frac{1}{2} u_0
\]

(2.17)

holds if \( t \geq 0 \) is smaller than \( T_2 = (\frac{u_0}{2A_2})^2 \).

Set \( T^* = \min\{T_1, T_2\} \), so that the proof of Theorem 1.1 is complete.

### 3. Global-in-time existence: Proof of Theorem 1.2

Assume that solution \((\varphi, u)\) to problem (1.6) belongs to \(W_2^{4,2}(Q_T) \times W_2^{4,2}(Q_T)\) and has properties (1.7), (1.8) for any \( T > 0 \). First we derive a priori estimates of \((\varphi, u)\) on \([0, T]\). Some calculations in the following are certainly formal, because the regularity of the solution is not sufficient. However, as usual we can easily derive the rigorous results by using the arguments of mollifiers and passing to the limit.
Lemma 3.1. For any \( t \in [0, T] \)

\[
\| \nabla \varphi(t) \|^2 + \| \varphi(t) \|_{L^4(\Omega)}^4 + \int_\Omega \left( \frac{1}{u} + \log u \right) (x, t) \, dx \\
+ \int_0^t \left( \left\| \frac{\partial \varphi}{\partial \tau} (\tau) \right\|^2 + \| \nabla u(\tau) \|^2 \right) \, d\tau + \int_0^t \int_\Gamma \left( \frac{\theta \epsilon u + 1}{u} \right) (x, \tau) \, d\Gamma \leq C_0(T). \quad (3.1)
\]

Proof. Add (1.6)_1 multiplied by \( \partial \varphi / \partial t \) and (1.6)_2 multiplied by \( 1 / u \), and integrate it over \( \Omega \). Then we have

\[
\frac{d}{dt} \left( \frac{K}{2} \| \nabla \varphi(t) \|^2 + \frac{1}{8} \| \varphi(t) \|_{L^4(\Omega)}^4 - \frac{1}{2} \| \varphi(t) \|^2 + \int_\Omega \log u \, dx \right) \\
+ \left\| \frac{\partial \varphi}{\partial t} (t) \right\|^2 + M \| \nabla u(t) \|^2 + \int_\Omega gu \, dx + \beta \int_\Gamma \theta u \, d\Gamma = \beta |\Gamma|. \quad (3.2)
\]

Integrating (3.2) over \((0, t)\), we easily see that

\[
\frac{K}{2} \| \nabla \varphi(t) \|^2 + \frac{1}{8} \| \varphi(t) \|_{L^4(\Omega)}^4 \, d\tau + \int_\Omega \log u(x, t) \, dx \\
+ \int_0^t \left( \left\| \frac{\partial \varphi}{\partial \tau} (\tau) \right\|^2 + M \| \nabla u(\tau) \|^2 \right) \, d\tau + \int_0^t \int_\Omega gu \, dx + \beta \int_0^t \int_\Gamma \theta u \, d\Gamma \\
\leq \frac{K}{2} \| \nabla \varphi_0 \|^2 + \frac{1}{4} \| \varphi_0 \|_{L^4(\Omega)}^4 + \int_\Omega \log u_0(x) \, dx + \beta |\Gamma| t + \frac{1}{2} |\Omega|. \quad (3.3)
\]

Poincaré inequality

\[
\| u(t) \| \leq c \| \nabla u(t) \| + \frac{\sqrt{|\Omega|}}{\sqrt{|\Gamma|}} \int_\Gamma u(x, t) \, d\Gamma \quad (3.4)
\]

implies

\[
\left| \int_0^t \int_\Omega gu \, dx \right| \leq \int_0^t \| g(\tau) \| \| u(\tau) \| \, d\tau \\
\leq \frac{M}{2} \int_0^t \| \nabla u(\tau) \|^2 \, d\tau + \frac{c^2}{2M} \int_0^t \| g(\tau) \|^2 \, d\tau + \frac{\sqrt{|\Omega|}}{\sqrt{|\Gamma|}} \int_0^t \| g(\tau) \| \, d\tau \int_\Gamma \theta u(x, \tau) \, d\Gamma.
\]
Therefore, we have

\[
\frac{K}{2} \| \nabla \varphi(t) \|^2 + \frac{1}{8} \| \varphi(t) \|_{L^4(\Omega)}^4 + \int_{\Omega} \log u(x, t) \, dx
\]

\[
+ \int_0^t \left( \left\| \frac{\partial \varphi}{\partial \tau}(\tau) \right\|^2 + \frac{M}{2} \| \nabla u(\tau) \|^2 \right) \, d\tau + \beta' \int_0^t \int_{\Gamma} \theta_u \, d\Gamma
\]

\[
\leq \frac{K}{2} \| \nabla \varphi_0 \|^2 + \frac{1}{4} \| \varphi_0 \|_{L^4(\Omega)}^4 + \int_{\Omega} \log u_0(x) \, dx + \beta |\Gamma| t + \frac{1}{2} |\Omega|
\]

\[
+ \frac{c^2}{2M} \| g \|_{L^2(Q_T)}^2 \tag{3.5}
\]

with \( \beta' = \beta - \sup_{0 \leq t \leq T} \| g(t) \| \sqrt{|\Omega|/(|\Gamma|)} > 0 \).

Next, multiply (1.6) by \( u^{-2} \) and integrate it over \( \Omega \). Then one gets

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{u} - \frac{a}{2} \varphi^2 - b \varphi \right) \, dx + \beta \int_{\Gamma} \frac{1}{u} \, d\Gamma = \int_{\Omega} g \, dx + \beta \int_{\Gamma} \theta_u \, d\Gamma,
\]

from which it readily follows

\[
\int_{\Omega} \frac{1}{u}(x, t) \, dx + \beta \int_0^t \int_{\Gamma} \frac{1}{u}(x, \tau) \, d\Gamma \leq \frac{1}{16} \| \varphi(t) \|_{L^4(\Omega)}^4 + \int_0^t \int_{\Omega} |g(x, \tau)| \, dx
\]

\[
+ \beta \int_0^t \int_{\Gamma} \theta_u(x, \tau) \, d\Gamma + \left( a + |b| \right)^2 + \frac{|b|^2}{2} |\Omega| + \int_{\Omega} \frac{1}{u_0} \, dx. \tag{3.6}
\]

Since \( \log u + 1/u \geq 0 \) holds, adding (3.5) to (3.6) leads to (3.1). \( \Box \)

**Lemma 3.2.** For any \( t \in [0, T] \)

\[
C_7(T)^{-1} \leq \int_{\Omega} \frac{1}{u}(x, t) \, dx \leq C_7(T). \tag{3.7}
\]

**Proof.** The upper bound is derived from (3.6) and (3.1) with the elementary inequality \( -\log u \leq 1/u \). Moreover, (3.1) leads to

\[
\int_{\Omega} \log u(x, t) \, dx + \int_{\Omega} \frac{1}{u}(x, t) \, dx \leq C_6(T),
\]

so that the lower bound is derived from Jensen’s inequality

\[
-\log \left( \frac{1}{|\Omega|} \int_{\Omega} \frac{1}{u} \, dx \right) \leq -\frac{1}{|\Omega|} \int_{\Omega} \log \frac{1}{u} \, dx = \frac{1}{|\Omega|} \int_{\Omega} \log u \, dx. \quad \Box
\]

Put

\[
\underline{u}(t) = \min_{x \in \Omega} u(x, t), \quad \overline{u}(t) = \max_{x \in \Omega} u(x, t).
\]
Then (3.7) implies
\[ u(t) \leq C_T(T)|\Omega|, \quad \overline{u}(t) \geq \frac{|\Omega|}{C_T(T)}. \tag{3.8} \]

It is easily seen from Poincaré inequality that for any \( t \in [0, T] \)
\[ \|u(t) - \overline{u}(t)\| + \|u(t) - \overline{u}(t)\| \leq c(\|
abla u(t)\| + 1). \tag{3.9} \]

**Lemma 3.3.** For any \( t \in [0, T] \)
\[ \left\| \frac{\partial \varphi}{\partial t}(t) \right\|^2 + \| \nabla u(t) \|^2 + \int_\Gamma \left( \nabla \varphi \right) \cdot \left( \frac{\partial u}{\partial t} - \log(u \theta_c) \right) \, d\Gamma \
+ \int_0^t \left( \left\| \frac{\partial \varphi}{\partial t}(r) \right\|^2 + \left\| \frac{1}{u \theta_c} \right\|^2 + \left\| \frac{1}{u \theta_c} \right\|^2 \right) \, dr \leq C_T(T). \tag{3.10} \]

**Proof.** Adding \( (\partial/\partial t)(1.6) \) multiplied by \( \partial \varphi/\partial t \) and \( (1.6)_2 \) multiplied by \( u^{-2} \partial u/\partial t \), and integrating it over \( \Omega \), we have
\[ \frac{1}{2} \frac{d}{dt} \left( \left\| \frac{\partial \varphi}{\partial t}(t) \right\|^2 + M \| \nabla u(t) \|^2 + \beta \int_\Gamma \left( u \theta_c - \log(u \theta_c) \right) \, d\Gamma \right) \
= K \left\| \frac{\partial \varphi}{\partial t}(t) \right\|^2 + 3 \int_\Omega \left( \varphi \right)^2 \left\| \frac{\partial \varphi}{\partial t} \right\|^2 + \left\| \frac{1}{u \theta_c} \right\|^2 \, dx \
= a \int_\Omega u \left( \frac{\partial \varphi}{\partial t} \right)^2 \, dx + \int_\Omega \left( \frac{\partial \varphi}{\partial t} \right)^2 \, dx - \int_\Omega g \left( \frac{1}{u \theta_c} \right)^2 \, dx + \beta \int_\Gamma \left( \varphi \right)^2 \, d\Gamma \quad \beta \frac{d}{dt} \int_\Omega \left( \log \theta_c \right) \, dx \
\leq a \overline{u}(t) \left\| \frac{\partial \varphi}{\partial t} \right\|^2 + \left\| \frac{\partial \varphi}{\partial t} \right\|^2 + \frac{1}{2} \left\| \frac{1}{u \theta_c} \right\|^2 + \frac{1}{2} \overline{u}(t)^2 \| g(t) \|^2 \
+ \beta \sqrt{ \| \overline{u} \| } \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^2(\Gamma)} \left( \overline{u}(t) + \frac{1}{2} \right). \tag{3.11} \]

Insert the inequality
\[ \overline{u}(t) \leq \frac{3}{\sqrt{\| \Omega \|}} (\| u(t) - \overline{u}(t) \| + \| u(t) - \overline{u}(t) \| + \overline{u}(t) \sqrt{\| \Omega \|}) \]
\[ \leq \frac{3}{\sqrt{\| \Omega \|}} (c \| \nabla u(t) \| + c + C_T(T) \| \Omega \|^{3/2}) \equiv c \| \nabla u(t) \| + C_T(T), \]
which is derived from (3.8) and (3.9), into the right most hand side of (3.11), and
integrate the rewritten (3.11) over (0, t). We have

\[ \left\| \frac{\partial \varphi}{\partial t}(t) \right\|^2 + M\| \nabla u(t) \|^2 + \beta \int_{\Gamma} (u \theta \psi - \log(u \theta)) \, d\Gamma \]

\[ + 2 \int_{0}^{t} \left( K \left\| \frac{\partial \varphi}{\partial t}(\tau) \right\|^2 + 3 \left\| \frac{\partial \varphi}{\partial \tau} \right\|^2 + \frac{1}{2} \left\| \frac{1}{u} \frac{\partial u}{\partial t}(\tau) \right\|^2 \right) \, d\tau \]

\[ \leq c(a + 1) \int_{0}^{t} \left( \left\| \frac{\partial \varphi}{\partial \tau}(\tau) \right\|^2 + \| \nabla u(\tau) \|^2 \right)^2 \, d\tau + C_9(T) \]

\[ \equiv c(a + 1)Y(t) + C_9(T). \quad (3.12) \]

Solving the differential inequality

\[ \frac{d}{dt} Y(t) \leq \frac{2}{\min\{1, M\}^2} (c^2(a + 1)^2 Y(t)^2 + C_9(T)^2), \]

we obtain

\[ Y(t) \leq \frac{C_9(T)^2}{c(a + 1)} \tan \left( \frac{2c(a + 1)C_9(T)T}{\min\{1, M\}^2} \right). \quad (3.13) \]

Thanks to (3.13), (3.10) immediately follows from (3.12). \( \square \)

From (3.10) it easily follows

\[ \int_{0}^{t} \left\| \frac{\partial u}{\partial \tau}(\tau) \right\|^2 \, d\tau \leq \int_{0}^{t} \left\| \frac{1}{u} \frac{\partial u}{\partial \tau} \right\|^2 \| \nabla u \|^2 \, d\tau \leq C_9(T)u^\ast(T)^2, \quad (3.14) \]

\[ \overline{\nabla}(t) \leq u^\ast(T) \equiv c\sqrt{C_9(T)} + C_9^\ast(T). \]

From (1.6) and its differentiation with respect to \( x \)

\[ K \nabla \Delta \varphi = \frac{\partial \nabla \varphi}{\partial t} + 3\varphi^2 \nabla \varphi - \nabla \varphi - au \nabla \varphi - (a\varphi + b) \nabla u, \]

it is clear to derive by the help of (3.1), (3.10) and (3.14)

\[ \| \varphi(t) \|_{W_2^2(\Omega)} + \int_{0}^{t} \left( \| \varphi(\tau) \|_3 + \left\| \frac{\partial \varphi}{\partial \tau}(\tau) \right\|_1 \right) \, d\tau \leq C_{10}(T) \quad \text{for any } t \in [0, T]. \quad (3.15) \]

Lemma 3.4.

\[ u(t) \geq C_{11}(T) \quad \text{for any } t \in [0, T]. \quad (3.16) \]

Proof. Multiply (1.5) with \( C_V = 1, M_2 = M \) by \( \theta^{p-1} \) (\( p \geq 4 \)) and integrate it over \( \Omega \). By integration by parts and Young’s inequality, we get

\[ \frac{d}{dt} \| \theta(t) \|_{L_p(\Omega)}^p + M_p(p-1) \int_{\Omega} \theta^{p-4} |\nabla \theta|^2 \, dx + p\beta \int_{\Gamma} \theta^p \, d\Gamma \]

\[ = p\beta \int_{\Gamma} \theta^{p-1} \psi \, d\Gamma + p \int_{\Omega} \theta^{p-1} (a\varphi + b) \frac{\partial \varphi}{\partial t} \, dx + p \int_{\Omega} g \theta^{p-1} \, dx \]
Using Nirenberg–Gagliardo inequality, the embedding theorem from \( C \) for any positive constant \( C \)

\[
\int_{\Omega} \left( \frac{p}{p-1} \| \theta(t) \|_{L^p(\Gamma)}^p + \frac{1}{p} \| \theta_t(t) \|_{L^p(\Gamma)}^p \right) + C_{12} M \left( \int_{\Omega} \theta^3(p-2) \, dx \right)^{1/3} + \frac{p^2}{4C_{12} M} \left( \int_{\Omega} \left( a \varphi + b \right) \partial \varphi \partial t \partial \theta^{p/2} \, dx \right)^{6/5} \]

\[
+ C_{12} M \left( \int_{\Omega} \theta^{3(p-2)} \, dx \right)^{1/3} \]

\[
+ \frac{p^2}{4C_{12} M} \left( \int_{\Omega} \left( \| g \theta^{p/2} \|_{6/5} \, dx \right) \right)^{5/3}.
\]

Using Nirenberg–Gagliardo inequality, the embedding theorem from \( W^1_2(\Omega) \) in \( L_6(\Omega) \) with a constant \( C_0 \), \( \| v \|_{L_6(\Omega)} \leq \| v \|_{W^1_2(\Omega)}/C_0 \) (3.10) and (3.15), we find that

\[
\int_{\Omega} \theta^{p-4} |\nabla \theta|^2 \, dx = \left( \frac{2}{p-2} \right)^2 \int_{\Omega} \left| \nabla \theta^{(p-2)/2} \right|^2 \, dx \]

\[
\geq \left( \frac{2}{p-2} \right)^2 \left[ C_0 \left( \int_{\Omega} \theta^3(p-2) \, dx \right)^{1/3} - \int_{\Omega} \theta^{p-2} \, dx \right];
\]

\[
\left( \int_{\Omega} \left( a \varphi + b \right) \partial \varphi \partial t \partial \theta^{p/2} \, dx \right)^{6/5} \]

\[
\leq (aC_{10}(T) + |b|) \left( \int_{\Omega} \left| \partial \varphi \partial t \right|^6 \right)^{1/3} \left( \int_{\Omega} \theta^{3p/4} \, dx \right)^{4/3} \]

\[
\leq \frac{(aC_{10}(T) + |b|)^2}{C_0} \left\| \partial \varphi \partial t \right\|_{W^1_2(\Omega)} \left( \int_{\Omega} \theta^{3p/4} \, dx \right)^{4/3};
\]

\[
\left( \int_{\Omega} \left( \| g \theta^{p/2} \|_{6/5} \, dx \right) \right)^{5/3} \leq \frac{1}{C_0} \left\| g \right\|_{W^1_2(\Omega)} \left( \int_{\Omega} \theta^{3p/4} \, dx \right)^{4/3};
\]

\[
p(p-1) \left( \frac{2}{p-2} \right)^2 \int_{\Omega} \theta^{p-2} \, dx \leq p(p-1) \left( \frac{2}{p-2} \right)^2 \bar{v}(t) \int_{\Omega} \theta^{p-1} \, dx
\]

\[
\leq C_{12} \left( \int_{\Omega} \theta^3(p-2) \, dx \right)^{1/3} + \frac{p^2(p-1)^2}{4C_{12}} \left( \frac{2}{p-2} \right)^4 \bar{v}(t)^2 \left\| \theta \right\|_{L^2(\Omega)} \left( \int_{\Omega} \theta^{3p/4} \, dx \right)^{4/3}
\]

for any positive constant \( C_{12} \).

Therefore, choosing \( C_{12} \leq C_0 \), we obtain

\[
\frac{d}{dt} \| \theta(t) \|_{L^p(\Omega)}^p + (C_0 - C_{12}) M \left( \int_{\Omega} \theta^3(p-2) \, dx \right)^{1/3} + \beta \| \theta \|_{L^p(\Gamma)}^p
\]

\[
\leq \beta \| \theta \|_{L^p(\Gamma)}^p + p^2 C_{13}(t) \left( \int_{\Omega} \theta^{3p/4} \, dx \right)^{4/3},
\]

(3.17)
Then (3.19) becomes

\[ C_{13}(t) = \frac{(aC_{10}(T) + |b|)^2}{4MC_{0}C_{12}} \left\| \frac{\partial \varphi}{\partial t} \right\|^{2}_{W_{2}^{1}(\Omega)} + \frac{1}{4MC_{0}C_{12}} \left\| g(t) \right\|^{2}_{W_{2}^{1}(\Omega)} + \frac{9M|\Omega|^{1/3}}{4C_{12}} \gamma(t)^{2}. \]

First letting \( p = 4 \) in (3.17), we have

\[
\frac{d}{dt} \| \theta(t) \|^{4}_{L_{2}(\Omega)} \leq \beta \| \theta_{e} \|^{4}_{L_{2}(T)} + 4^{2}C_{13}(t) \left( \int_{\Omega} \theta^{3} \, dx \right)^{4/3} \\
\leq \beta \| \theta_{e} \|^{4}_{L_{2}(T)} + 4^{2}C_{13}(t)|\Omega|^{1/3} \| \theta(t) \|^{4}_{L_{2}(\Omega)},
\]

from which by virtue of Gronwall’s lemma, (3.1), (3.10) and (3.12) it follows

\[
\| \theta(t) \|_{L^{4}(\Omega)} \leq C_{14}(T) \quad \text{for any} \quad t \in [0, T].
\]

Now let us introduce the sequence \( \{ p_{k} \}_{k=0}^{\infty} \) defined by \( p_{0} = 4, \ p_{k} = 4p_{k-1}/3 \) \((k = 1, 2, 3, \ldots)\). Integrating (3.17) with \( p = p_{k} \) over \((0, t)\), we obtain

\[
\| \theta(t) \|_{L_{p_{k}}(\Omega)} \leq \| \theta_{0} \|_{L_{p_{k}}(\Omega)} + \beta \int_{0}^{t} \| \theta_{e}(t) \|_{L_{p_{k}}(T)} \, dt \quad + \quad \frac{p_{k}^{2}}{t} \int_{0}^{t} C_{13}(\tau) \| \theta(t) \|_{L_{p_{k}}(\Omega)}^{p_{k}} \, d\tau.
\]

Set

\[
\Theta_{k}(t) = \max \{ 1, \| \theta_{0} \|_{L_{\infty}(\Omega)}, \| \theta_{e} \|_{L_{\infty}(T)}, \sup_{\tau \in (0, t)} \| \theta(\tau) \|_{L_{p_{k}}(\Omega)} \}.
\]

Then (3.19) becomes

\[
\Theta_{k}(t)^{p_{k}} \leq \left( 1 + |\Omega| + \beta |\Gamma| t + \int_{0}^{t} C_{13}(\tau) \, d\tau \right) p_{k}^{2} \Theta_{k-1}(t)^{p_{k}} \leq C_{15}(T)^{1/p_{k}} p_{k}^{2/p_{k}} \Theta_{k-1}(t)^{p_{k}},
\]

and thus by induction

\[
\Theta_{k}(t) \leq C_{15}(T)^{1/p_{k}} p_{k}^{2/p_{k}} \Theta_{k-1}(t) \leq \ldots \leq C_{15}(T)^{\mu_{k}} \prod_{j=1}^{k} p_{j}^{2/p_{j}} \Theta_{0}(t), \quad \mu_{k} = \sum_{j=1}^{k} \frac{1}{p_{j}}.
\]

Since it is easily seen that

\[
\mu_{k} = \sum_{j=1}^{k} \frac{1}{p_{j}} \leq \sum_{j=1}^{\infty} \left( \frac{3}{4} \right)^{j} \frac{1}{p_{0}} = \frac{3}{4},
\]

\[
\prod_{j=1}^{k} p_{j}^{2/p_{j}} = \exp \left( \sum_{j=1}^{k} \frac{2}{p_{j}} \log p_{j} \right) \leq \exp \left( \sum_{j=1}^{\infty} \frac{2}{p_{j}} \log p_{j} \right)
\]

\[
\leq \exp \left( \frac{2}{p_{0}} \sum_{j=1}^{\infty} \left( \frac{3}{4} \right)^{j} \left( j \log \frac{4}{3} + \log p_{0} \right) \right) < +\infty,
\]
one can conclude
\[
\lim_{k \to \infty} \Theta_k(t) \leq C_{16}(T).
\]
This means
\[
\lim_{k \to \infty} \sup_{0 \leq \tau \leq t} \|\theta(\tau)\|_{L^p(\Omega)} \leq C_{16}(T),
\]
or equivalently, (3.16) with \(C_{11}(T) = 1/C_{16}(T)\).

**Lemma 3.5.** For any \(t \in [0, T]\)
\[
\|\nabla \Delta \varphi(t)\|^2 + \left\| \frac{\partial \nabla \varphi}{\partial t}(t) \right\|^2 + \int_0^t \left( \left\| \frac{\partial^2 \varphi}{\partial \tau^2}(\tau) \right\|^2 + \left\| \frac{\partial \Delta \varphi}{\partial \tau}(\tau) \right\|^2 + \left\| \Delta^2 \varphi(\tau) \right\|^2 \right) d\tau
\leq C_{17}(T). \tag{3.20}
\]

**Proof.** Differentiate (1.6)\(_1\) with respect to \(t\), and multiply it by \(\partial^2 \varphi/\partial t^2\). Then by integration by part we have
\[
K \frac{d}{dt} \left\| \frac{\partial \nabla \varphi}{\partial t}(t) \right\|^2 + 2 \left\| \frac{\partial^2 \varphi}{\partial t^2}(t) \right\|^2
= 2 \int_\Omega \frac{\partial^2 \varphi}{\partial t^2} \left( -3 \varphi^2 + 1 + au \frac{\partial \varphi}{\partial t} + (a\varphi + b) \frac{\partial u}{\partial t} \right) dx
\leq \left\| \frac{\partial^2 \varphi}{\partial t^2}(t) \right\|^2 + 2(3\|\varphi\|^2_{W^2(\Omega)} + 1 + au(t))^2 \left\| \frac{\partial \varphi}{\partial t}(t) \right\|^2 + 2(a\|\varphi\|_{W^2(\Omega)} + |b|)^2 \left\| \frac{\partial u}{\partial t}(t) \right\|^2.
\]
Hence, integrating this over \((0, t)\), we have by virtue of (3.10), (3.12), (3.14) and (3.15)
\[
\left\| \frac{\partial \nabla \varphi}{\partial t}(t) \right\|^2 + \int_0^t \left\| \frac{\partial^2 \varphi}{\partial \tau^2}(\tau) \right\|^2 d\tau \leq C_{18}(T).
\]
From this it easily follows that
\[
\|\nabla \Delta \varphi(t)\|^2 + \int_0^t \left( \left\| \frac{\partial \Delta \varphi}{\partial \tau}(\tau) \right\|^2 + \left\| \Delta^2 \varphi(\tau) \right\|^2 \right) d\tau \leq C'_{18}(T)
\]
by using the equations of (1.6)\(_1\) operated by \(\nabla\), \(\partial/\partial t\) and \(\Delta\). For the last estimate in (3.20) we use the inequality
\[
\int_0^t \|\Delta u(\tau)\|^2 d\tau \leq C_{19}(T), \tag{3.21}
\]
which results from integrating (1.6)\(_2\) multiplied by \(u^{-2}\Delta u\) over \(Q_t\) by the help of (3.14)–(3.16). □
Lemma 3.6. For any $t \in [0, T]$

\[
\left\| \frac{\partial u}{\partial t} (t) \right\|^2 + \left\| \frac{\partial \nabla u}{\partial t} (t) \right\|^2 + \left\| \left( \frac{1}{u} \frac{\partial u}{\partial t} + u \frac{\partial \theta}{\partial t} \right) (t) \right\|^2_{L^2(\Gamma)} \\
+ \int_0^t \left( \left\| \frac{\partial \nabla u}{\partial \tau} (\tau) \right\|^2 + \left\| \frac{\partial^2 u}{\partial \tau^2} (\tau) \right\|^2 + \left\| \frac{\partial \Delta u}{\partial \tau} (\tau) \right\|^2 + \left\| \Delta^2 u(\tau) \right\|^2 \\
+ \left( \frac{1}{u} \frac{\partial u}{\partial \tau} + u \frac{\partial \theta}{\partial \tau} \right) (\tau) \right\|^2_{L^2(\Gamma)} \right) d\tau \leq C_{20}(T). \tag{3.22}
\]

**Proof.** Recall the following equation derived from differentiating (1.6)2 multiplied by $u^{-2}$ with respect to $t$:

\[
\frac{1}{u^2} \frac{\partial^2 u}{\partial t^2} = M \frac{\partial \Delta u}{\partial t} + \frac{2}{u^2} \left( \frac{\partial u}{\partial t} \right)^2 - (a \varphi + b) \frac{\partial^2 \varphi}{\partial t^2} - a \left( \frac{\partial \varphi}{\partial t} \right)^2 - \frac{\partial \varphi}{\partial t}. \tag{3.23}
\]

First, multiply (3.23) by $\partial u/\partial t$ and then integrate it over $\Omega$. We get by integration by part and Young’s inequality

\[
\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial u}{\partial t} (t) \right\|^2 + M \left\| \frac{\partial \nabla u}{\partial t} (t) \right\|^2 + \beta \left\| \left( \frac{1}{u} \frac{\partial u}{\partial t} + u \frac{\partial \theta}{\partial t} \right) (t) \right\|^2_{L^2(\Gamma)} \\
= \int_\Omega \left( \frac{1}{u} \frac{\partial u}{\partial \tau} \right)^3 + \left( (a \varphi + b) \frac{\partial^2 \varphi}{\partial t^2} + a \left( \frac{\partial \varphi}{\partial t} \right)^2 + \frac{\partial \varphi}{\partial t} \right) \frac{\partial u}{\partial t} dx + \beta \int_\Gamma u^2 \frac{\partial \omega}{\partial t} \frac{\partial u}{\partial \tau} \right\|^2_{L^2(\Gamma)} \\
\leq \int_\Omega \left( \frac{1}{u} \frac{\partial u}{\partial \tau} \right)^3 dx + \frac{u^*(T)^2}{2} \left( (a \varphi + b) \frac{\partial^2 \varphi}{\partial t^2} + a \left( \frac{\partial \varphi}{\partial t} \right)^2 + \frac{\partial \varphi}{\partial t} \right) \frac{\partial u}{\partial t} dx \\
+ \frac{1}{2} \left\| \frac{\partial u}{\partial \tau} \right\|^2_{L^2(\Gamma)} + \frac{\beta u^*(T)^2}{4} \left\| \frac{\partial \omega}{\partial t} \right\|^2_{L^2(\Gamma)}.
\]

From this, together with the interpolation inequality, Lemmas 3.3-3.6, (3.14) and (3.15), it follows

\[
\frac{d}{dt} \left\| \frac{\partial u}{\partial t} (t) \right\|^2 + 2M \left\| \frac{\partial \nabla u}{\partial t} (t) \right\|^2 + 2\beta \left\| \left( \frac{1}{u} \frac{\partial u}{\partial t} + u \frac{\partial \theta}{\partial t} \right) (t) \right\|^2_{L^2(\Gamma)} \\
\leq \frac{c}{C_{11}(T)^2} \left\| \frac{\partial u}{\partial t} (t) \right\|^2_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial \tau} (t) \right\|^2_{L^2(\Gamma)} \left( \left\| \frac{\partial^2 \varphi}{\partial t^2} (t) \right\|^2 + \left\| \frac{\partial \varphi}{\partial t} (t) \right\|^2 + \left\| \frac{\partial \varphi}{\partial \tau} (t) \right\|^2_{L^2(\Gamma)} \right) \\
\leq \delta \left\| \frac{\partial u}{\partial t} (t) \right\|^2_{L^2(\Gamma)} + \frac{C_{21}(T) \left\| \frac{\partial u}{\partial t} (t) \right\|^4_{L^2(\Gamma)} + \left\| \frac{\partial \theta}{\partial t} (t) \right\|^2_{L^2(\Gamma)} + \frac{\beta u^*(T)^2}{4} \left\| \frac{\partial \omega}{\partial t} \right\|^2_{L^2(\Gamma)} + C_{21}(T) \frac{\partial^2 \varphi}{\partial t^2} \right\|^2 + C_{22}(T) \tag{3.24}
\]

with a constant $\delta > 0$ determined later and $C_{\delta}(T) = cC_{11}(T)^{-6}\delta^{-1}$. 
Second, multiply (3.23) by $\partial^2 u / \partial t^2$ and then integrate it over $\Omega$. We have by integration by part and Young’s inequality
\[
\frac{M}{2} \frac{d}{dt} \left\| \frac{\partial \nabla u}{\partial t}(t) \right\|^2 + \left\| \frac{1}{u} \frac{\partial^2 u}{\partial t^2}(t) \right\|^2 + \frac{\beta}{2} \frac{d}{dt} \left( \frac{1}{u} \frac{\partial u}{\partial t} + u \frac{\partial \theta}{\partial t} \right)(t) \right\|_{L_2(\Gamma)}^2
= \int \left( \frac{2}{u^3} \frac{\partial u}{\partial t} \right)^2 - \left( (a \varphi + b) \frac{\partial^2 \varphi}{\partial t^2} + a \left( \frac{\partial \varphi}{\partial t} \right)^2 + \frac{\partial g}{\partial t} \right) \frac{\partial^2 u}{\partial t^2} \, dx
- \beta \int \frac{1}{u} \left( \frac{\partial u}{\partial t} \right)^3 \, d\Gamma + \beta \int \frac{\partial^2 \theta_x}{\partial t^2} \frac{\partial u}{\partial t} \, d\Gamma + \beta \frac{d}{dt} \int \left( \frac{\partial \theta_x}{\partial t} \right)^2 \, d\Gamma
\leq \frac{1}{2} \left\| \frac{1}{u} \frac{\partial^2 u}{\partial t^2}(t) \right\|^2 + C_{23}(T) \left( \left\| \frac{\partial u}{\partial t} \right\|_{L_4(\Omega)}^4 + \left( a \varphi + b \right) \frac{\partial \varphi}{\partial t} \right) + \frac{\beta}{2} \frac{d}{dt} \left( u \frac{\partial \theta}{\partial t} \right)^2 \, d\Gamma.
\]
Like (3.24), due to trace theorem, the interpolation inequality, and Lemmas 3.3–3.6, this inequality yields
\[
M \frac{d}{dt} \left\| \frac{\partial \nabla u}{\partial t}(t) \right\|^2 + \left\| \frac{1}{u} \frac{\partial^2 u}{\partial t^2}(t) \right\|^2 + \beta \frac{d}{dt} \left( \frac{1}{u} \frac{\partial u}{\partial t} + u \frac{\partial \theta}{\partial t} \right)(t) \right\|_{L_2(\Gamma)}^2
\leq \delta' \left\| \frac{\partial u}{\partial t} \right\|_{L_2(\Gamma)}^2 + c \delta'^{-1} \left( \left\| \frac{\partial \nabla u}{\partial t}(t) \right\|_{L_2(\Gamma)}^4 + \left\| \frac{\partial u}{\partial t}(t) \right\|_{L_2(\Gamma)}^2 \right)
+ C_{24}(T) \left( \left\| \frac{\partial u}{\partial t} \right\|_{L_2(\Gamma)}^4 + \left\| \frac{\partial u}{\partial t}(t) \right\|_{L_2(\Gamma)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L_2(\Gamma)}^2 \right)
+ \left\| \frac{\partial^2 \varphi}{\partial t^2} \right\|_{L_2(\Gamma)}^2 + \left\| \frac{\partial \varphi}{\partial t} \right\|_{L_2(\Gamma)}^2 + \left\| \frac{\partial g}{\partial t} \right\|_{L_2(\Gamma)}^2 \right) \right.
+ \beta \frac{d}{dt} \int \left( u \frac{\partial \theta}{\partial t} \right)^2 \, d\Gamma (3.25)
\]
with $\delta' > 0$ is a constant determined later. Here we used
\[
\left\| u \right\|_{L_3(\Gamma)}^3 \leq c \left\| u \right\|_{L_2(\Gamma)}^{3/2} \left\| u \right\|_{L_2(\Gamma)}^{3/2} \leq \delta' \left\| u \right\|_{L_2(\Gamma)}^2 + c \delta'^{-1} (\left\| \nabla u \right\|^4 + \left\| u \right\|^4).
\]
Third, integrating (3.23) multiplied by $\partial \Delta u / \partial t$ over $\Omega$, one obtains
\[
M \left\| \frac{\partial \Delta u}{\partial t} \right\|_{L_2(\Gamma)}^2 \leq \frac{c}{MC_{11}(T^2)^2} \left\| \frac{1}{u} \frac{\partial^2 u}{\partial t^2} \right\|^2 + \frac{c}{MC_{11}(T)^6} \left( \left\| \frac{\partial \nabla u}{\partial t}(t) \right\|_{L_2(\Gamma)}^4 + \left\| \frac{\partial u}{\partial t}(t) \right\|_{L_2(\Gamma)}^4 \right)
+ C_{25}(T) \left( \left\| \frac{\partial^2 \varphi}{\partial t^2} \right\|_{L_2(\Gamma)}^2 + \left\| \frac{\partial \varphi}{\partial t} \right\|_{L_2(\Gamma)}^4 + \left\| \frac{\partial g}{\partial t} \right\|_{L_2(\Gamma)}^2 \right). (3.26)
\]
Then, add (3.24), (3.25) and (3.26) multiplied by $MC_{11}(T)^2/(2c)$, choose $\delta = \delta' = M^2C_{11}(T)^2/(8c)$ and integrate the resultant inequality with respect to $t$. Con-
sequently, we conclude thanks to Lemmas 3.1–3.5
\[
\left\| \frac{1}{u} \frac{\partial u}{\partial t}(t) \right\|^2 + M \left\| \frac{\partial u}{\partial t}(t) \right\|^2 + \beta \left\| \frac{1}{u} \frac{\partial u}{\partial t} + u \frac{\partial \theta}{\partial t} \right\|_{L_2(\Gamma)}^2 \right.
\]
\[
+ \int_0^t \left( 2M \left\| \frac{\partial u}{\partial \tau}(\tau) \right\|^2 + \frac{1}{2} \left\| \frac{\partial^2 u}{\partial \tau^2}(\tau) \right\|^2 + \frac{M^2 C_{11}(T)^2}{4c} \left\| \frac{\partial u}{\partial \tau}(\tau) \right\|^2 \right.
\]
\[
+ 2\beta \left( \frac{1}{u} \frac{\partial u}{\partial \tau} + \frac{u \partial \theta}{2 \partial \tau} \right) \frac{\partial \varphi}{\partial \tau}(\tau) \right\|_{L_2(\Gamma)}^2 \right) \mathrm{d}\tau
\]
\[
\leq C_{26}(T) \int_0^t \left\| \frac{\partial u}{\partial \tau} \right\|^4_1 \mathrm{d}\tau + C_{27}(T) \equiv C_{26}(T) X(t) + C_{27}(T). \tag{3.27}
\]

Then, (3.27) leads to
\[
\frac{d}{dt} X(t) \leq 2C_{28}(T)^2 X(t)^2 + 2C_{29}(T)^2 \tag{3.28}
\]
where
\[
C_{28}(T) = C_{26}(T)/\min\{u^*(T)^{-2}, M\}, \quad C_{29}(T) = C_{27}(T)/\min\{C_{11}(T)^{-2}, M\}.
\]
Solving (3.28), we find
\[
X(t) \leq \frac{C_{29}(T)}{C_{28}(T)} \tan(2C_{28}(T)C_{29}(T)T) \equiv C_{30}(T).
\]

Therefore the left hand of (3.27) is bounded from above by \( C_{26}(T)C_{30}(T) + C_{27}(T) \).

Finally, recalling
\[
M \Delta^2 u = \Delta \left( \frac{1}{u^2} \frac{\partial u}{\partial t} + (u\varphi + b) \frac{\partial \varphi}{\partial t} - g \right)
\]
from (1.6)$_2$, multiplying this by \( \Delta^2 u \) and integrating it over \( Q_t \), one get by virtue of (3.21), (3.27) and Lemmas 3.1–3.5
\[
\int_0^t \left\| \Delta^2 u(\tau) \right\|^2 \mathrm{d}\tau \leq C_{31}(T). \tag{3.29}
\]

Estimates (3.27) and (3.29) lead to (3.22) with
\[
C_{20}(T) = C_{26}(T)C_{30}(T) + C_{27}(T) + C_{31}(T). \quad \blacksquare
\]

Combining Lemmas 3.1–3.6 with Theorem 1.1 implies the existence and uniqueness of the solution in \( Q_T \) for any \( T > 0 \) to problem (1.6).

Therefore, the assertion of Theorem 1.2 is proved.

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