A parton picture of de Sitter space during slow-roll inflation

David Seery

Department of Applied Mathematics and Theoretical Physics, Wilberforce Road, Cambridge, CB3 0WA, U.K.

E-mail: djs61@cam.ac.uk

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Abstract. It is well-known that expectation values in de Sitter space are afflicted by infra-red divergences. Long ago, Starobinsky proposed that infra-red effects in de Sitter space could be accommodated by evolving the long-wavelength part of the field according to the classical equations of motion plus a stochastic source term. I argue that — when quantum-mechanical loop corrections are taken into account — the separate universe picture of superhorizon evolution in de Sitter space is equivalent, in a certain leading-logarithm approximation, to Starobinsky’s stochastic approach. In particular the time evolution of a box of de Sitter space can be understood in exact analogy with the DGLAP evolution of partons within a hadron, which describes a slow logarithmic evolution in the distribution of the hadron’s constituent partons with the energy scale at which they are probed.

Keywords: inflation, initial conditions and eternal universe, physics of the early universe, cosmological perturbation theory

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1 Introduction

In the years since the WMAP satellite provided the first precise measurement of the angular power spectrum of the cosmic microwave background (CMB), a consensus has emerged in which inflation is the most likely candidate for the origin of the primordial density perturbation. If this is true it will almost certainly require the presence of new degrees of freedom at a scale associated with the inflationary era, perhaps in the region $10^{12} - 10^{16}$ GeV, and would represent a remarkable discovery of new physics at an energy at least $10^8$ times the accessible limit at the Large Hadron Collider and far beyond the reach of any terrestrial experiment.

The prospect of discoveries such as these has led to significant investment in microwave background experiments, which are now approaching the sensitivity required to detect effects taking place at subleading order in perturbation theory [1, 2]. These effects are expected to play a crucial role in discriminating among the competing theories which could account for the gross Gaussian, adiabatic and scale-invariant character of the inflationary density perturbation. Indeed, one immediate consequence of our imminent ability to measure such tiny contributions has been a strong pressure to develop and refine the theory of inflationary
perturbations, which underlies the interpretation of all CMB observations. At a practical level this has led to the availability of full predictions for the non-linearity in the three- and four-point correlation functions of the primordial curvature perturbation in single-field inflation [3–6] and partial predictions in multi-field models [7–12]. At a more conceptual level, interesting new approaches have been developed by borrowing the idea of an effective field theory from particle physics [13–20]. These enable us to ask fundamental questions about theories of inflation without making a commitment to any specific model.

An example of such a fundamental question concerns the validity of perturbation theory. During inflation there are at least two interesting perturbative scales. The first is a measure of the deceleration of the Hubble rate, defined by $\epsilon \equiv -\dot{H}/H^2$, which provides an indication of the time scale over which the vacuum expectation values of background fields are coherently evolving due to macroscopic classical effects. The second is the ratio of the Hubble scale to the Planck scale, $H/M_P$, which determines the importance of higher orders in perturbation theory associated with “quantum” corrections. Our ability to extract predictions from any theory of inflation is usually limited to the lowest few orders in $\epsilon$ and $H/M_P$, or other related small quantities. Unfortunately, it has been known for a long time that such predictions can be afflicted with infra-red divergences which compensate for the smallness of these expansion parameters and spoil our ability to perform meaningful calculations [21, 22]. These divergences have been explored in a series of papers by Woodard and collaborators: see, for example, refs. [23–26] which contain references to the earlier literature.

We would like to be sure that when we make predictions which are to be compared with precision CMB data, we obtain the right answer for the right reason. To achieve this confidence in our quantitative predictions, and the methods used to obtain them, it is important to arrive at a clear understanding of infra-red issues. Accordingly, they have been subject to investigation by many authors [27–46]. The presence of large infra-red effects is symptomatic of computing an observable defined on some particular length scale within a much larger patch of de Sitter space [37, 39, 47, 48], leading to the existence of a large hierarchy of scales. It is the logarithm of this hierarchy which enters in conjunction with the scales $\epsilon$ and $H/M_P$, and spoils naïve perturbation theory. Whenever large logarithms of this sort play a significant role in quantum field theory, it is usually the case that their resummation can be described by the renormalization group equation. We should therefore expect that significant infra-red effects, if they exist, can be accommodated within this framework [29, 30, 37, 39, 49–51]. On the other hand, it has been suggested by many authors that the stochastic approach to inflation, originally pioneered by Starobinsky [52, 53], functions as an infra-red regulator and leads to infra-red finite predictions [23, 24, 47, 48, 54, 55]. In this interpretation, the presence of large infra-red terms should be understood to reflect the potential for large fluctuations to build up between widely separated points in de Sitter space, and eventually a sensitivity to the onset of eternal inflation. This prescription was suggested earlier in refs. [23, 24].

This resolution of the problem of infra-red divergences, if correct, would be quite remarkable. However, it presents a number of puzzles. Firstly, we are used to applications in which the renormalization group equation leads to screening of masses and couplings constants in the ultra-violet, rather than some form of stochastic dynamics. It is not so easy to see how the two approaches could be related. Secondly, although one can check, once a stochastic formulation is available, that it correctly reproduces correct infra-red behaviour, it would be nice to have an argument which begins with the existence of large infra-red logarithms and arrives at the Langevin equation which describes Starobinsky’s stochastic dynamics. For this purpose, one can seek analogies in other examples of field theory where infra-red effects
play an important role. The key example of this type occurs in hadron physics, where the fundamental degrees of freedom belonging to the gauge theory of colour — the quarks and gluons — are confined by soft QCD effects.

In this paper I would like to suggest that although a large hierarchy of scales in de Sitter space can certainly be understood in terms of renormalization group flow, an even better analogy might be with the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation \cite{56–60} which describes the evolution of parton distribution functions within a hadron. These distribution functions satisfy an equation — itself a manifestation of the renormalization group — which endows them with a slow logarithmic variation as one changes the energy scale at which the target hadron is to be probed. In this analogy, one can show that the DGLAP equation precisely reproduces the Fokker-Planck equation obtained by Starobinsky. This equation describes the diffusion of probability in time as a light scalar field in de Sitter space evolves in the presence of quantum effects. This Fokker-Planck equation could therefore be interpreted as the de Sitter renormalization group equation.

The structure of this paper is as follows. In section 2 the parton picture of hadron structure is briefly recalled, in a form which will allow easy generalization to de Sitter space. This generalization is performed in sections 3–4. Finally, I conclude with a discussion in section 5. Two appendices supplement the discussion of the stochastic formulation of inflation which occurs in the main text. Starobinsky’s original argument, based on a recursive coarse-graining of the Heisenberg field and the introduction of a Langevin equation, is recalled in appendix A. In addition, to aid comparison of the argument given in the present paper with that of other authors, appendix B gives two derivations of the corresponding Fokker-Planck equation, one based on Itô’s stochastic calculus and the other on a conventional path integral. Some issues tangential to the main discussion are briefly summarized in appendix C.

Throughout this paper, units are chosen so that $\hbar = c = 1$ and the reduced Planck mass (defined by $M_p^{-2} = 8\pi G$, where $G$ is Newton’s gravitational constant) is set to unity. The metric is chosen with sign convention $(-,+,+,+)$. 

2 Hadrons and the parton picture

2.1 Deep inelastic scattering and parton distribution functions

The parton picture of hadron structure grew out of the analysis of so-called deep inelastic scattering experiments (see figure 1), which involved collisions between electrons and protons in which a large invariant momentum $Q^2$ was transferred from the $e^-$ beam to the target proton. In these experiments it was observed that the scattering rate behaved as if the electrons were interacting with an elementary electromagnetically charged fermion, whereas such hard scattering behaviour was virtually absent in direct proton-proton collisions. To explain these observations, Bjorken and Paschos \cite{61} and Feynman \cite{62} proposed that hadrons could be understood as a loosely bound collection of pointlike constituents, called partons. In this picture, the electron (which does not feel the strong force) does not interact with the hadron as a whole but rather scatters incoherently from one of its constituents.

In modern terms the partons can be identified with quarks and gluons, which are the fundamental degrees of freedom of QCD. At low energies these are confined into colour-neutral states by soft processes, giving hadrons a characteristic size of order $M_p^{-1}$ where $M_p \sim \Lambda_{\text{QCD}}$ is the pion mass and $\Lambda_{\text{QCD}} \approx 200 \text{MeV}$ is the QCD scale. At energies much greater than $\Lambda_{\text{QCD}}$ the quarks and gluons behave roughly like a plasma of free particles.
Figure 1. Kinematics of deep inelastic electron-hadron scattering. A high energy electron impinges on the target hadron and interacts electromagnetically with one of its constituent partons. After the interaction, the original hadron is disrupted and the ejected quark materializes as a jet of hadrons collinear with the motion of the initial electron.

The parton model can only be expected to supply a good approximation if the impinging electron is sufficiently energetic to resolve the internal structure of the target hadron. This internal structure is described by so-called parton distribution functions, labelled $f_i$ (but varying from hadron to hadron) for each species $i$ which can be present, and defined so that $f_i(x)\,dx$ is the probability of finding a constituent parton of species $i$ carrying a fraction $x$ ($0 \leq x \leq 1$) of the parent hadron’s total momentum. These can be thought of as coarse-grainings over the hadron wavefunction. Since the partons are supposed to be bound within the hadron, their momenta transverse to its direction of propagation must all be small and to a good approximation the partons can be taken to move collinearly. The leading corrections to this picture will be suppressed by powers of $k_\perp/P$, where $k_\perp \equiv |k_\perp|$ is of order the typical transverse 3-momentum (of order $\sim M_\pi$) and $P = |P|$ is the 3-momentum of the parent hadron.

Once the parton distribution functions are at our disposal, and taking into account the assumption that a probe scatters incoherently off a single constituent parton, it is clear how we can write the cross-section for a deep inelastic scattering event. In the case described in figure 1, where an electron of momentum $k$ disrupts a hadron of momentum $P$, it follows that

$$\sigma\left\{e^{-}(k) + p(P) \rightarrow e^{-}(k') + Y\right\} = \int_0^1 dx \sum_j f_j(x)\sigma\left\{e^{-}(k) + q_j(xP) \rightarrow e^{-}(k') + q_j(p')\right\}, \quad (2.1)$$

for any hadronic final state $Y$, where the sum over $j$ includes all quark flavours, with quarks and antiquarks contributing separately. Similar formulae can be obtained for any desired partonic interaction: the general scheme is always parallel to eq. (2.1), which factorizes into a hard subprocess describing interactions among the partons and a soft parton distribution function. The hard subprocess is independent of the hadron in which the partons are contained, and the parton distribution function is independent of the interaction which takes place. In particular, the distributions $f_i$ depend only on $x$ and are independent of the momentum transfer $Q^2$ which is carried by the intermediate boson in figure 1. This behaviour is known as Bjorken scaling [63].

A cross-section such as eq. (2.1) is called “inclusive,” because it does not distinguish between the various allowed final hadronic states $Y$. It is sometimes possible to measure more “exclusive” (or “semi-inclusive”) rates which discriminate among the hadrons which can be
present in $Y$. To describe such exclusive rates one can introduce fragmentation functions $D^h_i(z)$, which are the final-state analogues of the parton distribution functions $f_i(x)$: these give the probability for a parton of species $i$ to produce a hadron $h$ in the final state which carries a fraction $z$ of the parent parton’s momentum. An exclusive cross-section can be written in a form analogous to eq. (2.1), using the fragmentation functions to sum the final-state products of the hard subprocess into final-state products of the overall hadronic process.

The parton picture was introduced as a phenomenological model, with some basis in an intuitive understanding of hadron physics. As such it is independent of QCD itself, or more generally the existence of an underlying gauge theory which exhibits confinement in the infra-red. It is only the identification of partons with the fundamental excitations of an asymptotically free non-Abelian field theory which invests the model with real meaning. Indeed, the parton model can be formally derived from QCD [64].

2.2 Parton evolution and the DGLAP equation

In the simplest parton model, Bjorken scaling is exact. However, if QCD processes are taken into account this is no longer true; instead, a tower of infra-red divergences appears which can compensate for the smallness of the QCD coupling constant at high energy. When resummed, these divergences predict slow violations of Bjorken scaling and give rise to evolution equations which control how the parton distributions functions $f_i$ (and fragmentation functions $D^h_i$) evolve with the momentum scale $Q^2$ at which the hadron is to be probed.

What is the origin of these QCD effects? Consider any process with outgoing quarks. At the point where these leave the diagram, any such process must contain a vertex of the form

\[ \text{---} \quad , \]

where the dashed line is connected to the rest of the diagram. However, nothing can prevent the outgoing quarks from radiating gluons, which means that at leading order in the strong coupling constant, $\alpha_s$, we must also include diagrams of the form

\[ \text{---} \quad , \quad \text{---} \quad , \quad \text{---} . \]

The first of these diagrams is a loop correction, which can produce a divergence when the momentum carried by the circulating gluon approaches zero. The remaining two diagrams account for gluon radiation from the outgoing quarks, and include soft gluons in the final state. If $\alpha_s$ is small, these diagrams will be suppressed compared to the undressed diagram without gluon radiation. Unfortunately, however, these gluon-corrected diagrams may be singular when the momenta carried by the final-state gluons approaches zero, or where they are emitted collinearly with the outgoing quark. In some cases the divergences produced by the combination of these diagrams cancel, but there is no reason of principle for this to occur. Similar corrections must be taken into account for ingoing particles. Where cancellation does not occur, any left-over divergences may overcome the smallness of $\alpha_s$ and cause the parton content encountered by an impinging hard particle to evolve with $Q^2$.

The physical meaning of this evolution is simple to understand. An isolated quark propagating according to the usual rules of quantum field theory is not alone, but rather is accompanied by a cloud of virtual particles which are constantly being emitted and re-absorbed by the physical quark. Any projectile which strikes the cloud with sufficiently high
energy has an opportunity to resolve and interact with one of the virtual particles in its interior, rather than the parent quark. As $Q^2$ increases, such a projectile can resolve vacuum fluctuations with increasingly short lifetimes. It follows that the composition of the virtual cloud must exhibit a slow variation with $Q^2$. Indeed, we can imagine it to be governed by a system of equations of Boltzmann type, which describe a sort of equilibrium among the various species of particle which can exist within the cloud. These are the DGLAP equations, sometimes known (especially in the older literature) as the Altarelli-Parisi equations. In the context of QCD they were obtained by Altarelli & Parisi [59] using the method of the operator product expansion. Their interpretation as an approximate Boltzmann system was given later by Collins & Qiu [65].

Consider any parton which is destined to interact with some impinging projectile $X$. If $X$ is sufficiently energetic, it is possible to resolve processes by which the parton brakes into (or out of) the collision through emission — bremsstrahlung — of soft quanta, as described in figure 2. If the parton is moving increasingly off-shell as it passes from the outside of the emission chain towards the collision region, then each radiation event can be accompanied by a large infra-red logarithm. It follows that to obtain a meaningful picture of this cascade of emission events, we must resum all diagrams with the form of figure 2 which describe radiation of an arbitrary number of soft quanta.

Any diagram of this type, involving radiation of $N$ quanta, can be built by sewing together $N$ copies for the diagram for emission of a single quantum. To deal with this, one can write the probability for an incoming parton (say of species $i$) to resolve into an outgoing parton of species $j$ with a fraction $z$ of its original momentum, accounting for radiation of a single soft particle with transverse energy $\delta \ln Q^2$, in terms of the so-called Altarelli-Parisi splitting function, $P_{j \leftarrow i}(z)$, which satisfies

$$P_{j \leftarrow i}(z) = \delta(1-z) + \alpha_s p(z)\delta \ln Q^2 \quad (2.2)$$

for some $p(z)$ which can be computed by studying S-matrix elements according to the usual rules of quantum field theory.\(^1\) In writing eq. (2.2) and in what follows, it has been assumed that $\delta \ln Q^2 \ll 1$ and that terms of order $(\delta \ln Q^2)^2$ or smaller are negligible. One can

\(^1\)Note that $p(z)$ may not be a pure function; in order to arrive at a properly normalized $P_{ii}$, it may be necessary to interpret $p(z)$ as a distribution in its own right by adding some admixture of $\delta(1-z)$. 

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**Figure 2.** A parton brakes as it enters into a collision with some incoming particle $X$, before scattering into a final hadronic state $Y$. As it brakes, it radiates an arbitrary number of soft gluons $\{ \cdots , 3, 2, 1 \}$ and moves increasingly off-shell. The impinging $X$ projectile can resolve this emission cascade if it is sufficiently energetic.
understand the form of this equation by observing that if \( \delta \ln Q^2 = 0 \), so that no soft particle is radiated, then the original parton must remain at its initial momentum. This accounts for the leading term \( \delta (1 - z) \). On the other hand, if a particle of momentum \( \delta \ln Q^2 \) is radiated, it is possible for the original parton to downgrade its momentum to some fraction \( z \neq 1 \). The potential for this degradation is suppressed by the coupling \( \alpha_s \), but can be enhanced if the available phase space \( \delta \ln Q^2 \) is large.

To avoid complications, let us restrict attention to a single parton species \( i \) which does not undergo transmutation to or from any other species. The parton distribution function \( f_i(x) \) evolves according to the master equation

\[
 f_i(x; Q^2 + \delta Q^2) = \int_0^1 dx' \int_0^1 dz P_{i \rightarrow i}(z)f_i(x'; Q^2)\delta(x - x'z). \tag{2.3}
\]

This is a principle of detailed balance, or Chapman-Kolmogorov equation, in which we account for all the ways a parton could arrive at momentum fraction \( x \) at the probe scale \( Q^2 + \delta Q^2 \) by summing over an intermediate step at a probe scale \( Q^2 \). Passing to the continuum limit, one arrives at an integro-differential equation which describes the evolution of \( f_i \) with \( \ln Q^2 \),

\[
 \frac{\partial f_i(x; Q^2)}{\partial (\alpha_s \ln Q^2)} = \int_x^1 \frac{dz}{z} p(z)f_i(x/z; Q^2). \tag{2.4}
\]

This is the prototype DGLAP equation. It has the characteristic form of a renormalization group equation in \( \ln Q^2 \). If other species of parton, say of type \( j \), can evolve into \( i \)-partons by soft emission then one must include splitting functions of the form \( P_{j \rightarrow i} \) in eqs. (2.3)–(2.4) which couple \( f_i(x) \) to the relevant \( f_j(x) \). By this process one arrives at a system of equations which can be thought of as an approximate Boltzmann hierarchy with collision integrals given by the right-hand sides of eqs. (2.3)–(2.4). Accounts of all these complexities can be found in the literature; see, for example, refs. \([66–68]\).

Eq. (2.4) must be supplemented with an appropriate boundary condition which determines the starting point for the DGLAP evolution. For partons confined within hadrons, such as quarks and gluons, a boundary condition cannot be obtained from first principles because the distribution functions \( f_i \) depend on soft QCD effects which are presently incalculable. Instead, a set of distribution functions at some given scale must be extracted from experiment and the DGLAP evolution of these distribution functions can be compared with the observed distributions at a different scale. At present, this approach gives reasonable agreement between theory and observation. On the other hand, the foregoing discussion applies just as well for particles which can exist in isolation, such as an electron. For such particles, it is easy to find an appropriate boundary condition. For example, when probed at a scale corresponding to its own Compton wavelength, an electron should resolve only into itself and not any member of its surrounding cloud. Therefore we should set \( f_e(x, Q^2) = \delta(1 - x) \) at \( Q^2 \sim m_e^2 \), where \( m_e \) is the electron mass.

A similar discussion can be given for the fragmentation functions \( D^h_i \). To leading order in \( \alpha_s \) the splitting functions \( P_{j \rightarrow i} \) are the same for both, although differences occur at higher orders in perturbation theory. In particular, the fragmentation functions also evolve according to a DGLAP equation which has the form of eq. (2.4).

### 2.3 Leading logarithms

What has been achieved by using eq. (2.4) to evolve the parton distribution functions between two widely separated probe scales (say \( Q^2 \) and \( Q^2_0 \))? Although eq. (2.3) gives a correct
accounting of all powers of $\delta \ln Q^2$ terms in the continuum limit, it is not necessarily exact because the probability function $p(z)$ must be computed by assembling Feynman diagrams into S-matrix elements. It is therefore a perturbative expansion in powers of the strong coupling constant, $\alpha_s$, and perhaps other small quantities. If we work to leading order in $\alpha_s$, the solution of the DGLAP equation will account correctly for all terms of the form $(\alpha_s \ln Q^2/Q_0^2)^n$. However, it will not give useful information regarding terms which are suppressed by higher powers of $\alpha_s$, such as $\alpha_s^n \ln^{n-1} Q^2/Q_0^2$. This level of precision is known as the leading-logarithm approximation, sometimes abbreviated as “LLA”. Better approximations, which account for terms suppressed by extra powers of $\alpha_s$, can be found by retaining higher powers of the coupling constant in $p(z)$.

The leading-logarithm approximation has a particular interpretation in terms of the emission chain depicted in figure 2. Terms which contribute at leading logarithmic order arise from the region of phase space where the cascade of emitted quanta is strongly ordered [66–68] in the sense that

$$Q_0^2 \ll Q_2^2 \ll \cdots \ll Q_n^2 \ll Q_2^2 \ll Q_1^2 \ll Q^2.$$ (2.5)

where $Q_n^2$ is the invariant transverse momentum carried away by the $n$th quantum radiated in the cascade, remembering that the label ‘1’ is attached to the quantum closest to the collision region whereas quanta labelled by higher $n$ are increasingly distant. Other configurations of radiative cascade are possible, such as emission of two or more particles at roughly equal $Q^2$, but since there is only a single logarithm for each hierarchy in $Q^2$ and each emission costs a factor $\alpha_s$ these configurations contribute at the next-to-leading logarithm approximation (“NLLA”) or lower.

The leading logarithm resummation only gives a meaningful approximation if all significant effects actually arise from the large logarithms under consideration. In hadron physics, large effects can also arise from evolution at small Bjorken-$x$, where $\ln x$ logarithms can make a dominant contribution to the physics. In this region, the DGLAP equation must be replaced by another renormalization group equation, the so-called JIMWLK equation [69]. I shall return to this problem in section 5.

### 3 A parton picture of de Sitter space

Now let us apply these ideas to a box of de Sitter space which is undergoing slow-roll inflation. Remarkably, it will be possible to find analogues for all the concepts which played a role in the discussion of hadrons and parton evolution outlined in section 2, including the parton distribution and fragmentation functions, and the strongly-ordered radiative cascade. The most significant difference is that the parton distribution is no longer probed by an impinging projectile such as an electron, because in calculating the density perturbation generated during inflation there is no analogue of a scattering event. Nevertheless, the concept of a probe is still implicitly present in the guise of sampling the density fluctuation smoothed on Hubble-sized regions. Indeed, the wavenumber corresponding to the Hubble scale satisfies $k = aH$ and it will transpire that $k$ plays the role of the probe momentum $\sqrt{Q^2}$, whereas the expectation values of the scalar fields which characterize the de Sitter phase play the role of the Bjorken variable $x$.

These ideas will be developed in this section, in the course of which it will be possible to transcribe eqs. (2.1)–(2.4) and the strong ordering condition (2.5) to the theory of density fluctuations from slow-roll inflation. In section 3.1 the problem of secular infra-red effects
in de Sitter correlation functions is outlined, before proceeding to the leading-logarithm resummation of these terms in section 3.2.

3.1 Infra-red divergences in de Sitter space

The calculation of $n$-point correlation functions during a phase of quasi-de Sitter inflation, and in particular those for $n \geq 3$, has been studied intensively over the last several years. The aim is to compute the expectation value, at time $t$, of some product of fluctuations $\delta \phi^a$ which begin in the vacuum state, for which the appropriate tool is Schwinger’s formulation of expectation values in terms of a closed contour of integration over time \cite{Schwinger}. Correlation functions in this formalism can be represented by the usual Feynman diagrams, except that vertices now exist in “+” and “−” varieties and different propagators are used to describe contractions between $(+,+)$, $(+,-)$, $(-,+)$ and $(-,-)$ vertices. Accounts of this formalism applied to cosmology can be found in refs. \cite{Grishchuk, Balasubramanian, Polarski}. The low order correlation functions therefore correspond to $\delta \phi^a$, $\delta \phi^b \delta \phi^c$, and $\delta \phi^a \delta \phi^b \delta \phi^c$. (3.1)

The first of these is the leading contribution to the two-point expectation value; the second is the leading contribution to the three-point expectation value; and the third and fourth contribute comparably to the four-point expectation value at leading order. The correlation functions have a simple structure in which diagrams with $n$ external legs at approximately equal 3-momenta $k_*$ typically enter proportional to $H^2*$ and a power of $\sqrt{\epsilon_*}$ for odd $n$ \cite{Finelli}, where $H_* \ll 1$ is the Hubble parameter at the time the mode $k_*$ left the horizon. The quantities $H_* \sim 10^{-5}$ and $\epsilon_* \sim 10^{-2}$, whose precise values vary from model to model and depend on the inflationary dynamics, play the role of coupling constants such as the strong coupling $\alpha_s$.

There are several sources of large infra-red effects. One source can be found in the second diagram above, which gives a contact contribution to the three-point expectation value, and is found to contain a term which scales as $\xi_* N^+$ \cite{Zaldarriaga, Carr, Easther, Fasto, Li, Sonoda}, where $\xi_* \sim \epsilon_*$ is an auxiliary slow-roll parameter and $N^+ \equiv \ln|k_\star \eta|$ is a potentially large logarithm which describes by how many e-foldings the mode $k_*$ is outside the horizon at conformal time $\eta$. (The conformal time is related to cosmic time $t$ by a quadrature, $\eta = \int_0^\infty dt/a(t)$.) After a time of order $\xi^{-1}_*$ e-folds, this logarithm will overwhelm its slow-roll prefactor. It follows that all terms of the form $\sim (\epsilon_* N^+)^n$ become comparably large, after which any such expansion will require resummation \cite{Easther}. This was interpreted in terms of a coherent time evolution of the background field by Zaldarriaga \cite{Zaldarriaga}, which can be understood very simply. Once a fluctuation has passed outside the horizon its time evolution becomes approximately classical, even accounting for the inclusion of non-linear effects \cite{Easther}. To leading order in gradients, the fluctuation simply amounts to shifting the background field by some amount and must therefore evolve coherently with it. More generally, each of the diagrams in 3.1 is typically calculated by working to leading order in quantities of order $\epsilon$, and higher powers in the expansion should be expected to be accompanied by powers of $N^+$ which will invalidate the use of perturbation theory after $N^+ \sim \epsilon_*^{-1}$ e-folds \cite{Easther, Easther2}.

Loop diagrams contribute different infra-red effects. These are systematic corrections to each of the $n$-point graphs which account for processes by which an external particle radiates into new quanta, which later coalesce and exit the diagram by a different external leg. Certain corrections to the two- and three-point correlation functions are now known \cite{Grishchuk, Balasubramanian, Polarski, Finelli, Carr, Easther, Fasto, Li, Sonoda}, of the form
where the diagram on the left describes a scalar loop and the diagram on the right a circulating graviton. For each loop, these diagrams typically enter suppressed by one power of $H^2$ and involve one unconstrained integral over momentum space. In some circumstances this integral reduces to $\int d\ln k$ and — neglecting ultra-violet effects, which can be accommodated separately [80, 81] — where such divergences overlap the loop expansion is effectively a power series in $(H^2 \ln k_*/k_0)^n$, where $k_0$ is an infra-red cutoff of order the comoving Hubble length at the onset of inflation. The duration of the inflationary era when the scale $k_*$ corresponds to the Hubble scale is therefore $N^- \equiv \ln k_*/k_0$ e-folds, and terms of this sort will spoil the convergence of perturbation theory at horizon exit for $k_*$-scale modes whenever $N^- \sim H^{-2}$.

More precise estimates can be found in refs. [33, 36, 38, 80]. The quantities $N^+$ and $N^-$ therefore play different but complementary roles.

How are we to understand the meaning of such infra-red divergences? It is clear that their structure is similar to those encountered in computing corrections to the parton distribution functions of hadrons, with $\ln k_*$ playing the role of the probe scale $\ln Q^2$; this scale appears with different hierarchies in $N^-$ and $N^+$, corresponding to different “resolutions” for the ingoing and outgoing particles in the hadron picture. It is reasonable to suppose that the explanation of these large effects is similar to the large infra-red effects of QCD, with $N^-$ effects resummed into distribution functions and $N^+$ effects resummed into fragmentation functions. This leads to a highly intuitive picture. Indeed, in the case of hadrons, resummation entails considering dressed diagrams with extra soft particles in the initial or final states. In a box of de Sitter space this would correspond to dressing the diagrams of 3.1 with soft quanta belonging to all fields which are light during inflation. However, on much smaller scales these soft quanta are tantamount to no more than a redefinition of the background fields, and it is easy to guess that summing over them corresponds to averaging over the different spacetime backgrounds which could have been produced by quantum fluctuations during the evolution of the box.

3.2 The inflationary DGLAP equation

We are now in a position to discuss an analogue of the DGLAP equation for de Sitter space. In this section, we focus on the $N^-$ divergences which would lead to evolution of the parton distribution functions, before moving on to the issue of $N^+$ divergences in section 4.

**Light-cone coordinates.** In the case of hadron physics, the fields whose correlation functions exhibit secular infra-red effects are spin-1/2 fermions interacting by mediation of spin-1 gauge bosons. Let us imagine a deep inelastic scattering event which takes place in a $(d + 1)$-dimensional spacetime, with coordinates $(t, x, z)$ where $t$ is coordinate time, $z$ labels a particular spatial dimension, and $x$ is a $(d - 1)$-dimensional vector. There is a preferred axis associated with the collision of probe and hadron, commonly taken to be the $z$ axis, which permits the introduction of light-cone coordinates $x^\pm$ satisfying $x^\pm \equiv (t \pm z)/\sqrt{2}$. If one shifts to the frame in which the target hadron has infinite momentum longitudinal to the collision axis, the colour field of the hadron is is shrunk to have support only at $x^- = 0$. At the same time, the infinite time dilation observed by the probe relative to the rapidly moving hadron implies that the colour field is independent of $x^+$. For each species $i$, we introduce a field $x_i(x)$ (which for simplicity I will refer to as a Bjorken field) representing the amplitude for a parton of species $i$ to manifest itself with momentum fraction $x_i$ at location
x. These fields describe spatial correlations among the partons, and are determined by the loop expansion of correlation functions in the underlying gauge theory. After coarse-graining the modulus-square of the wavefunctional $\Psi_i(x_i)$ associated with the $x_i$ over $x$, we can recover the parton distribution function $f_i(x)$.

Consider a patch of de Sitter spacetime sourced by the potential energy of some number of scalar fields $\phi^\alpha$ which are labelled by Greek indices $\{\alpha, \beta, \ldots\}$. We identify these fields with the Bjorken fields, and associate the spatial dependence they carry with the coordinates $x$ which were transverse to the light-cone coordinates $x^\pm$. The renormalization group coordinate $x^+$ is associated with time evolution. It may also be necessary to introduce extra Bjorken fields to describe the physical polarizations of other light particles such as gravitons or gauge bosons. There will be a coarse-grained parton distribution function $f_\alpha(\phi)$ for each of these — in principle obtained by coarse-graining over the appropriate wavefunctional, as above — or one can work with a joint parton distribution function $f(\phi^\alpha)$ which describes all the parton species simultaneously. I will choose to focus on the latter, since it contains more information and is required for applications to inflation. The parton distribution functions will suffice for resummation of some large infra-red effects, although to deal with all large terms it will later be necessary to introduce analogues of the fragmentation functions.

Initial conditions. In analogy with the DGLAP evolution for an isolated electron, we can imagine an initial Hubble-sized region characterized by a set of uniform scalar expectation values $\bar{\phi}^\alpha$. This initial box can be compared to an initial parton — such as an electron — at momentum fraction $x = 1$ and yields an initial condition analogous to the electron, $f(\phi^\alpha) = \prod_\alpha \delta(\phi^\alpha - \bar{\phi}^\alpha)$ at the initial time. In the same way that the details of the hadron wavefunction depend on unknown soft QCD effects, the wavefunction for the de Sitter state depends on the unknown details of quantum gravity which are likewise presently incalculable.

If we probe the box at the original Hubble scale we must find that the scalars take the values $\bar{\phi}^\alpha$ everywhere. The analogue of strict Bjorken scaling would correspond to these initial values remaining fixed everywhere in the box, no matter at which scale we probe it. However, as with the case of parton evolution within a hadron, strict Bjorken scaling is violated — in this case, by terms such as $\xi N^-$ and $H^2 N^-$. The former type account for simple coherent time evolution within the box, whereas the latter type account for the possibility of an emission cascade analogous to figure 2. In the case of a de Sitter box, this emission cascade corresponds to gravitational particle production on scales much larger than the probe.

In this section we are only aiming to study the distribution of Bjorken variables encountered by a mode leaving the horizon at an arbitrary point during inflation, which will account for large $N^-$-type terms and corresponds to predicting the distribution of partons encountered by an incoming hard particle. To account for all large infra-red terms one must also accommodate the process by which the debris from a hard scattering event is unwound into final state particles, which absorbs large $N^+$-type terms. The analysis of these terms is deferred to section 4.

Splitting functions. The first step in writing a DGLAP equation for the parton distribution functions is to identify the relevant splitting functions, which give the probability for some parton characterized by the Bjorken variables $\varphi^\alpha$ to split into a parton characterized by $\varphi^\alpha + \delta\varphi^\alpha$ together with other soft quanta. In the hadron case one performs this calculation using the formalism of cut diagrams, in which one effectively calculates $x^-$-ordered correlation functions with lines crossing the surface $x^- = 0$ put on-shell. This bears a strong formal relationship with the Schwinger formalism employed in de Sitter calculations.
Figure 3. The analogue of Altarelli-Parisi splitting functions for de Sitter space. A de Sitter parton, represent by the hatched region, evolves in time from left to right and radiates soft quanta which materialize in the final state. These radiated quanta are fluctuations which are instantaneously drawn over the de Sitter horizon and classicalize. The solid lines represent scalar particles whereas the wavy line represents gravitons; the diagram with three scalar particles in the final state represents the leading non-Gaussian correction, although this turns out not to contribute in a leading-logarithm approximation. In principle, radiation into any light states is permitted.

One can represent these processes by the diagrams of figure 3, in which a non-perturbative de Sitter parton represented by the hatched area evolves in time from left to right. As it does so, it can radiate into any quanta which are light on the Hubble scale. Radiation of a single particle is forbidden by the tadpole condition, so in the leading diagram two quanta materialize in the “final state.” However, in principle any number of particles may be produced, corresponding to inclusion of any of the diagrams described by 3.1 or their higher $n$ generalizations; for example, the leading non-Gaussian correction corresponds to a final state in which three quanta are present. In practice we will see that — as in the hadron case — final states with more than the minimum number of radiated quanta do not contribute in the leading-logarithm approximation.

In de Sitter space there is no analogue of an S-matrix, so a different rule is needed to convert correlation functions into splitting probabilities. The appropriate prescription can be found by reconstructing the wavefunction on configurations of the background fields, leading to [82]

$$
P[\varphi^\alpha(x)] \propto \int \prod_\alpha [d\eta_\alpha] \times \exp \left( \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \prod_{j=1}^{n} d^3 x_j \eta_\alpha(x_1) \cdots \eta_\beta(x_n) \langle \delta \varphi^\alpha(x_1) \cdots \delta \varphi^\beta(x_n) \rangle \right) \times \exp \left( -i \int d^3 x \eta_\alpha(x) \varphi^\alpha(x) \right), \tag{3.2}$$

where $P[\varphi^\alpha]$ is a functional measuring the relative probabilities of the field configurations.

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2 This terminology is intuitive but imprecise, because in the de Sitter case we cannot interpret a correlation function dressed by soft quanta to represent a shift in the initial or final state experienced by some hard subprocess. In the de Sitter case, the state experienced by the hard subprocess is produced by gravitational expansion of the Minkowski vacuum on small scales, and indeed may contain an indefinite number of particles. What we are measuring are the correlations among certain configurations of particles in this state. Nevertheless, I will use this terminology freely because it is familiar from the hadron case.

3 Recall that in 3.1, all external legs are evaluated at the same time, and in the time-ordered diagrams of figure 3 they therefore appear with all legs on the right-hand side. One can think of the Schwinger-formalism diagrams as being rather more like instantons, in comparison with Feynman-formalism diagrams where particles pass through the diagram in a definite direction.
\( \varphi^\alpha(x) \) and \([d\eta]\) denotes functional integration over the field \( \eta(x) \). Note that eq. (3.2) contains another source of infra-red divergences, in the form of each integral over \( x_j \).

**Resummation of large logarithms.** We would like to obtain the probability for a region within the de Sitter box with approximately constant background scalar expectation values on a scale \( k^{-1} \) to evolve into a slightly smaller region, over which the scalars may take different vevs. This will be the de Sitter analogue of the Altarelli-Parisi splitting function. Suppose that the de Sitter region evolves for a short time, amounting to roughly \( \delta N \approx \delta \ln k \) e-folds, during which a fluctuation is imprinted in the each scalar field — or, more generally, each light degree of freedom — over a narrow range of scales \( \delta k \). The Fourier transform of a constant field configuration \( \varphi^\alpha(x) \approx \sigma^\alpha \) on the scale \( (k + \delta k)^{-1} \) is roughly

\[
\varphi^\alpha(k) \approx \frac{2\pi^2 \sigma^\alpha}{k^3 \delta \ln k},
\]

to leading order in \( \delta k \). It follows that the probability for quantum fluctuations to generate a region of size \( (k + \delta k)^{-1} \) in which the vev of a scalar species \( \alpha \) is offset by an amount \( \sigma^\alpha \) satisfies

\[
P(\sigma^\alpha) \propto \int \prod_\beta d\eta_\beta \left( 1 - \frac{1}{2} \eta_\gamma \eta_\delta P^\gamma_\delta(k) \delta \ln k + O(\delta \ln k)^2 \right) \exp(-i\eta_\alpha \sigma^\alpha)
\]

\[
= \prod_\delta \delta(\sigma^\delta) + \frac{1}{2} (\delta \ln k) P^\gamma_\delta \frac{\partial}{\partial \sigma^\beta} \frac{\partial}{\partial \sigma^\gamma} \prod_\delta \delta(\sigma^\delta) + O(\delta \ln k)^2,
\]

where the \( \eta \) integrals in the first line are now one-dimensional and run from \(-\infty \) to \( \infty \). The quantity \( P_\sigma(k) \) is the so-called “dimensionless” power spectrum imprinted in the range \( \delta k \), and is defined by the rule

\[
(\delta \phi^\alpha(k_1) \delta \phi^\beta(k_2))_* = (2\pi)^3 \delta(k_1 + k_2) \frac{2\pi^2}{k^3} P^\alpha_\beta(k),
\]

where \( |k_1| = |k_2| = k \). In principle the coefficient of the leading \( \delta \)-function could be modified by a term of order \( \delta \ln k \) after correctly normalizing this distribution, but since \( P_\sigma \) is independent of the \( \sigma^\alpha \) this does not occur and eq. (3.4) is correct as it stands. Eq. (3.4) has the same interpretation as the hadronic Altarelli-Parisi function, eq. (2.2). If \( \delta \ln k = 0 \), then there is no phase space for splitting to occur and the parton must remain with the same scalar vacuum expectation values. This is described by the leading \( \delta \)-function, which enforces \( \sigma^\alpha = 0 \). For finite \( \delta \ln k \) there is a small phase space for the vacuum expectation values to shift, which is described by the second term, proportional to \( (\delta \ln k) \delta^{\alpha}(\sigma) \). In principle this series could be carried to higher orders in \( \delta \ln k \), generating a Kramers-Moyal expansion.

Comparison of eqs. (2.2) and (3.4) highlights an interesting difference between the hadron and de Sitter cases. When we calculate splitting functions for partons, it is possible to work non-perturbatively in the Bjorken variable, \( x \). In the de Sitter case, our answer is not only perturbative in the “coupling constant” \( H_2^2 \), but is also an expansion in the shifts of the Bjorken variables. In principle this could lead to significant difficulties. Had we directly integrated out the auxiliary variable \( \eta^\alpha \) in eq. (3.4), we would have obtained a Gaussian distribution in \( \sigma^\alpha \) valid only for \( |\sigma^\alpha|^2 \lesssim H_2^2 \delta \ln k \) (for each \( \alpha \)). To construct the master equation, however, we must integrate over the entire range of the Bjorken variable and for the vast majority of this range our perturbative formula is invalid. One must then enquire why we should imagine the final formula in eq. (3.4) to be correct. The reason is that
we expect the random walk executed by the Bjorken variables to be almost-local, in the sense that transitions only occur between approximately neighbouring states. For such transitions eq. (3.4) gives an accurate representation. It is not important that it may give a very poor approximation for non-local transitions where the Bjorken variables jump by a macroscopic amount in a single step, because such transitions are exponentially unlikely. Eq. (3.4) can be thought of as a formal, analytic regularization which captures this concept of locality in the random walk. It will lead to a solution with the character of a diffusion process.

It is important that neither $P$ nor any of the correlation functions in eq. (3.2) can contain large infra-red logarithms, because they are all evaluated within a box of size $\delta \ln k$ e-folds. The potentially large terms $H^2 N^{-} \epsilon^* N^{-}$ are therefore all very small, and indeed will disappear in the continuum limit where we take $\delta \ln k \to 0$.

Let us return to the DGLAP equation. The probability that, when probed at a scale $(k + \delta k)^{-1}$, a region within the de Sitter box resolves to approximately constant scalar expectation values $\varphi^\alpha$ can be written

$$ f(\varphi^\alpha; N + \delta \ln k) = \int_{-\infty}^{\infty} \left( \prod_{\beta, \gamma} d \rho^\beta d \sigma^\gamma \right) f(\rho^\alpha; N) P(\sigma^\alpha) \delta(\rho^\alpha + \sigma^\alpha + \delta \varphi^\alpha - \varphi^\alpha), \quad (3.6) $$

where $\delta \varphi^\alpha \propto \delta \ln k$ is a function of the $\rho^\alpha$, accounting for the coherent time evolution of the background fields, which is the de Sitter analogue of renormalization-group evolution sourced by self-energy diagrams (eg. see pp. 28–30 of ref. [69]). We will assume that the time evolution is given to a good approximation by the separate universe formula

$$ 3H \dot{\varphi}^\alpha = -\delta^{\alpha\beta}V_{,\beta}, \quad (3.7) $$

where $V = V(\varphi^\alpha)$ is an arbitrary potential supporting a phase of slow-roll evolution. Eq. (3.6) is the de Sitter master equation, which is an exact analogue of the hadron master equation, eq. (2.3). As in the case of hadrons, it is of Chapman-Kolmogorov type and describes a Markov process in which the Bjorken variable executes a random walk as the probe scale is varied. When interpreted as a collisional Boltzmann equation, eq. (3.6) can also be related to Polyakov’s discussion of stability in de Sitter space in the presence of particle production (compare section 5 of ref. [83]).

In the limit $\delta \ln k \to 0$ one obtains its continuum counterpart, which exactly generalizes the DGLAP equation,

$$ \frac{\partial f}{\partial \ln k} = \frac{1}{2} \partial_\alpha \partial_\beta (P^{\alpha\beta} f) - \partial_\alpha \left( f \frac{d \varphi^\alpha}{d \ln k} \right), \quad (3.8) $$

where $\partial_\alpha \equiv \partial / \partial \varphi^\alpha$, and $P^{\alpha\beta}$ is a function of the $\varphi^\alpha$. Eq. (3.8) can immediately be recognized as Starobinsky’s diffusion (“Fokker-Planck”) equation — originally obtained by interpreting $f$ as a probability density associated with solutions to a Langevin equation. The generalization to multiple-fields was studied in refs. [84–86]. Specializing to the case of single-field inflation, it follows that

$$ \frac{1}{H} \frac{df}{dt} = \frac{\partial}{\partial \varphi} \left( \frac{Vf}{3H^2} \right) + \frac{H^2}{8\pi^2} \frac{\partial^2 f}{\partial \varphi^2}, \quad (3.9) $$

which is the form of this equation obtained in ref. [52] provided $H$ is taken to be constant. If desired, one could now reverse the argument of appendix B and obtain a Langevin equation for each of the fields $\varphi^\alpha$, in terms of which the analysis may simplify in practice [87].
However, the physical content of the Langevin and Fokker-Planck equations is the same and moreover one can move freely from one representation to the other, so it is also possible to think of the Langevin equation as a de Sitter renormalization group equation. In this version, the importance of the slow-roll approximation in reducing the de Sitter evolution to a renormalization group flow is clear: this description is only valid when the $\dot{\varphi}$ term in eq. (3.7) is irrelevant.

### 3.3 Leading logarithms

Eq. (3.8) is not exact unless $\mathcal{P}_\ast$ can be evaluated precisely. In most applications, it will only be possible to obtain $\mathcal{P}_\ast$ perturbatively in terms of order $H_*^2$ or $\epsilon_*$. In these cases, eq. (3.8) should give a correct resummation of all terms of the form $(H_*^2 N^{-})^n$ and $(\epsilon_* N^{-})^n$, but will not account for terms which are suppressed compared to these by extra powers of $H_*^2$ or $\epsilon_*$. Therefore, one should think of the Starobinsky-DGLAP equation (3.8) as a leading-logarithm resummation. The possibility of this type of resummation was raised by Weinberg [35].

One can formulate a spacetime picture (see figure 4) of the leading-logarithm resummation by analogy with the emission cascade of figure 2. In this picture, a de Sitter parton evolves as one changes the scale on which it is probed by radiating soft quanta belonging to all light fields. The region of phase space corresponding to terms of leading-logarithm order arises from splitting into a single pair of quanta where the emission chain satisfies a strong ordering criterion,

$$k_0 \ll \cdots \ll k_3 \ll k_2 \ll k_1 \ll k,$$

where $k$ is the scale at which one wishes to compute a correlation function of fluctuations, and $k_0$ is the scale of the initial de Sitter box. These soft quanta from intermediate splittings propagate to the right of the diagram, where they dress the process of interest — depicted in figure 4 as a correlation function with three quanta in the final state, although any process of interest can be substituted at this stage — with extra final-state particles. Eq. (3.10) is exactly analogous to the hadron strong ordering condition (2.5), and has the same interpretation. Although other configurations of emission cascade exist in which the de Sitter parton splits into three or more quanta at an intermediate stage, these splittings cost an extra factor of $H_*^2$ for which there is no compensating large logarithm. These terms therefore

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**Figure 4.** The spacetime interpretation of a leading-logarithm resummation in $H_*^2$. 
contribute formally at next-to-leading logarithmic order or below and, if desired, could be included to find the next-order corrections to the DGLAP equation. On the other hand, this line of reasoning immediately suggests that the Starobinsky-DGLAP equation no longer gives a useful resummation in the strongly coupled region where $H_\star$ approaches the Planck mass $M_P$. In this regime there is apparently nothing to suppress splitting into an arbitrary number of quanta at each step in the DGLAP evolution and the situation rapidly escapes beyond any sort of perturbative control.

4 Predictions for inflationary correlation functions

The DGLAP framework provides a framework within which to understand the question of infra-red divergences in inflationary correlation functions, and in addition — at least in the context of the leading-logarithm approximation — it provides a physical picture of their effects. The de Sitter master equation, eq. (3.6), only requires correlation functions to be computed within boxes of size $\delta N \sim \delta \ln k$ e-folds, cutting off the possibility of large infra-red logarithms. However, it is still necessary to give a rule which allows this formalism to be used for the purpose of obtaining predictions which can be compared to observation.

4.1 The core hard subprocess

By analogy with the hadronic case, we expect to identify three distinct phases in the calculation of an observable quantity. In the first phase, a radiative cascade described by DGLAP evolution brings the initial conditions (which are set at some low infra-red scale) up to an energy where the core hard subprocess can be calculated perturbatively. The details of this hard subprocess itself constitute the second phase. Finally, in the third phase (described by fragmentation functions), the radiative cascade is unwound to obtain an observable low-energy final state. In the case of QCD, this final state is composed of well-separated showers of colourless, hadronized degrees of freedom.

The DGLAP evolution appropriate to the first phase was studied in section 3.2. We must also expect to find de Sitter analogues of the second and third phases: resummation of large logarithms, however necessary, is not usually sufficient to capture every process of interest. Let us suppose that the distribution of Bjorken variables within the original de Sitter box has been evolved to some scale $k_F$ according to the DGLAP prescription. Quantities which can be observed in the temperature anisotropy of the CMB, or the distribution of galaxies, are controlled by correlations between modes which exit the horizon over a very short space of e-foldings — typically $\Delta N \lesssim 10$, but at present almost certainly $\Delta N < 10^2$. Note, however, that the range of observable e-foldings grows with time, and is therefore not subject to any restriction as a matter of principle. We can imagine choosing $k_F$ to be characteristic of the horizon size at the epoch when scales of interest are leaving the Hubble radius during inflation. The detailed correlations among the modes which leave over the next $\Delta N \lesssim 10$ e-folds — controlled by calculation within a small box — constitute the analogue of the hard subprocess in deep inelastic scattering, in which large logarithms were under control in virtue of the large momentum transfer, $Q^2$, in eq. (2.1).

The hard subprocess is characterized by time- and length scales which are short compared with those which lead to significant enhancement of infra-red terms. Accordingly, the subprocess should be calculated using fixed-order perturbation theory to whatever precision is necessary. For example: if the aim is to study non-Gaussian features in the statistics of the CMB, then to compute the bispectrum it is necessary to carry the calculation to order
$H^4$, and to compute the trispectrum it is necessary to work to order $H^6$. These terms are not enhanced by large infra-red effects, and consequently are not captured by resummation in powers of logarithms. This effect was observed in a concrete calculation of non-Gaussian correlation functions by Rigopoulos & Shellard [88].

In calculating the details of the hard sub-process, the so-called factorization scale $k_F$ plays a special role; it acts as a cut-off on the momenta of those quanta which can dress the final state, or circulate within loops in its interior. One therefore finds infra-red terms which are at most of order $H^2 \ln k_*/k_F$ or $\epsilon_*/k_F$ and are highly suppressed by a judicious choice of $k_F$. The factorization scale acts as a division between the perturbative and non-perturbative parts of the calculation. In choosing it, orders of magnitude are important but small factors can be reshuffled between the various orders of perturbation theory; in general, we can think of the correct prescription to be a choice of the factorization scale to be at the Hubble scale of interest.

Can the factorization scale be assigned a physical interpretation? It corresponds to the coarse-graining scale in the conventional formulation of stochastic inflation. In a correct calculation all dependence on $k_F$ drops out, but from the point of view of the subprocess within the $k_F$-sized box it is possible to think of it as the generation of a non-perturbative mass on very large scales, as originally pointed out by Starobinsky & Yokoyama [53]. Indeed, the existence of a non-perturbative mass of this type for the photon was suggested by Davis et al. [89], who used the Hartree approximation to study back-reaction from a light scalar field (see also ref. [33]). This result was subsequently reproduced in ref. [25], using a more detailed analysis. A similar non-perturbative mass of order the Hubble scale was recently encountered by Riotto & Sloth, who used the Fokker-Planck equation (3.9) to resum diagrams containing large infra-red effects in an $O(N)$-symmetric scalar field theory [55]. Riotto & Sloth interpreted this mass in terms of screening of fluctuations on scales much larger than the Hubble distance. In the language of the present paper this is equivalent to choosing the factorization scale at the horizon. It is very interesting that this phenomenon is a striking parallel of a key aspect of hadron phenomenology, the existence of a saturation scale $Q_s$ of order the inverse coupling, $\alpha_s^{-1}$ (see, eg., ref. [90, 91]), which defines a correlation length below which a parton observes a coherent background colour field. In both cases the interpretation is the same.

### 4.2 Fragmentation functions and superhorizon evolution

The outcome of the hard subprocess is not observable, in the hadron and de Sitter cases equally, but must be evolved to observable scales using the fragmentation functions associated with the third phase of the “collision.” This evolution is described by a process analogous to the radiative cascade, which in the hadronic case leads to a DGLAP equation with the form of eq. (2.4) but with the “spacelike” splitting functions appropriate for initial state radiation replaced by “timelike” splitting functions appropriate for final state radiation. As remarked in section 2, these two sets of splitting functions are equal to leading order in $\alpha_s$ but differ at higher orders.

In the de Sitter case it is not presently necessary to be so sophisticated and the discussion can be simplified considerably. In particular, since we are observing at most $\Delta N \lesssim 10^2$ e-folds after the hard subprocess has taken place we can ignore processes associated with the perturbative scale $H^2$, which is typically tiny, and concentrate on those associated with the slow-roll scale $\epsilon_* \sim 10^{-2}$. Moreover, to avoid complications, let us agree that correlation functions of the fundamental fields are “observable” when evaluated at the end of inflation. In practice, the input to standard cosmological perturbation theory would be the correlation
functions, evaluated at horizon re-entry, for the curvature perturbation and some number of isocurvature modes, but such issues play no role in the resummation of infra-red effects and can easily be accommodated by standard methods.

With these considerations in mind it is possible to give a simple formula, exactly analogous to eq. (2.1), for a correlation function of operators $\mathcal{O}_j$, each carrying a momentum $|k_j| \gtrsim k_F$, within a large box of de Sitter space of size $k_0^{-1}$

$$\langle \mathcal{O}_1(k_1) \cdots \mathcal{O}_n(k_n) \rangle_{k_0} = \int_{-\infty}^{\infty} \left( \prod_\alpha d\varphi^\alpha \right) f_{k_F}(\varphi^\alpha) \langle \mathcal{O}_1(k_1) \cdots \mathcal{O}_n(k_n) \rangle_{k_F} \bigg|_{\varphi^\alpha},$$

(4.1)

where the explicit $k_F$-dependence of $f(\varphi^\alpha)$ has been displayed, and the subscripts attached to expectation values denote the box size in which they are calculated. The correlation functions on the left- and right-hand sides of eq. (4.1) are to be calculated at a conformal time of order $\eta \sim -k_F^{-1}$, just after the time of horizon exit corresponding to the modes, which are assumed to be of order $k_F$. However, eq. (4.1) does not yet describe an observable. To do so one must take account of the two processes described in figure 5, corresponding to parton evolution and recombination.

The most important of these processes is parton evolution, corresponding to resummation of $(\epsilon, N^+)$ terms, which has already been argued by many authors to correspond to classical time evolution \[74, 75, 78\] and is depicted on the left-hand side of figure 5. In the usual approximation, the “parton” corresponding to the hatched area plus each independent perturbative quantum in the final state is taken to evolve like a separate universe. The other possible process involves recombination, where two or more quanta radiated during the hard subprocess recombine to form only a single quantum in the observable final state; this process is depicted on the right-hand side of figure 5. If $k_F \ll k_*$ then this process could receive large infra-red corrections, but will suppressed by powers of $H_k^2$ for $k_F \sim k_*$. The fragmentation functions, which describe how different final states resulting from the hard sub-process contribute to the observable of interest, can evidently be computed using the usual formulae of the separate universe picture of which the $\delta N$ formula is the most common example. In this picture, the process of recombination is described using so-called “$\delta N$ loops” whereas the

\[ \text{Figure 5. Parton evolution and recombination in de Sitter, with time evolving from left to right in the diagram. Hard subprocess quanta, represented by the solid lines, materialize above the horizon and form the initial condition for the subsequent “third phase” evolution, represented by the dashed lines. For modest evolution subsequent to horizon crossing, this “third phase” evolution corresponds to a mixture of time dependence and recombination. If the correlation functions are observed following only a modest number of e-folds subsequent to horizon crossing, these are the only effects which must be taken into account. In general the recombination process will be very strongly suppressed, together with any other quantum processes associated with the scale $H_2^*$. In this approximation, the non-perturbative de Sitter region, represented by the hatched region at the bottom of the diagram, undergoes only coherent time evolution from left to right.} \]
simpler process of time evolution without recombination is described by the $\delta N$ tree-level terms, although both sets of diagrams may involve large $\epsilon_* N^+$ contributions.

There will be corrections to this procedure suppressed by extra powers of $H^2$ or $\epsilon_*$ compared to the leading-logarithm approximation, but these are usually difficult to compute. An example where it is possible to obtain one such correction explicitly was given by Maldacena [3], who studied the correlation function $\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle$ and gave an argument by which it could be computed in the limit $|k_3| \to 0$. Maldacena framed his discussion in the context of a single field model of inflation, where $\zeta$ is conserved on superhorizon scales and large logarithms containing $N^+$ are absent. In such a model, there is no “phase 3” in the collision process and we observe the bare outcome of the core subprocess (or more precisely, the effect of evolution and recombination is trivial), although the “phase 1” radiative cascade is still operative. In the language of the cascade, a three-point function with one momentum squeezed to zero corresponds at leading order to the forbidden process of tadpole emission, and therefore vanishes.

However, it must be remembered that the hard sub-process, here $\langle \zeta(k_1)\zeta(k_2) \rangle$, was computed in the wrong background. This can be accounted for by expanding in powers of operators carrying much softer momenta [47], which is formally the operator product expansion (OPE) but here is just a Taylor series. By this process the leading term in the emission cascade — corresponding to radiation of two quanta — can be isolated. The result is a correction to $\langle \zeta(k_1)\zeta(k_2) \rangle$ of order $\epsilon H^2$, which is not enhanced by large logarithms, where these quantities are evaluated at the moment of horizon crossing corresponding to the wavenumber $|k_3|$. This argument has been generalized in refs. [92, 93]. In the hadron case it is known that the stochastic interpretation becomes less clear beyond the leading-logarithm approximation, and it is easier to work in terms of the OPE. A similar approach could presumably be constructed in the case of de Sitter space.

5 Discussion

In the foregoing sections it has been shown how the evolution of a box of de Sitter space much larger than the Hubble size can be given a description in the language of partons. In this description the vacuum expectation values of any light fields in the theory play the role of the Bjorken variable $x$, and the Hubble scale plays the role of a probe energy scale $Q^2$. In the hadron picture, $x$ characterizes the momentum fraction of the parton and the probe is an impinging hard particle which communicates a hard momentum transfer of order $Q^2$ to the parton. The analogue of strict Bjorken scaling in this picture occurs when the scalar field expectation values remain the same, irrespective of the scale on which the box is probed. Violations of strict Bjorken scaling arise from two sources: one is the coherent background time evolution of the box, but the real analogue of those processes by which Bjorken scaling is violated in hadrons occur when the parton radiates soft quanta. These can be resolved if the probe scale is sufficiently small. The DGLAP equation which describes the radiation process at leading-logarithm order is exactly Starobinsky’s diffusion equation for the probability density function in stochastic inflation. It follows that on superhorizon scales one can think of time evolution in a slow-roll phase of de Sitter expansion as an increasing refinement of the scale on which features can be resolved. As we anticipated at the outset, this increasing refinement can be understood as a form of renormalization group flow.

How can we reconcile the stochastic nature of the Starobinsky-DGLAP equation with our experience of the renormalization group in the ultra-violet, which ordinarily leads to
When we use the rules of quantum field theory to compute scattering amplitudes or decay rates of point particles in an interacting theory, we imagine the bare degrees of freedom to be surrounded by an unresolved cloud of virtual fluctuations. An impinging particle which probes this cloud at some energy scale cannot distinguish short-lived higher-energy fluctuations within the cloud, so we must replace the bare degrees of freedom by the average effect of scattering off the unresolved cloud. When we integrate the renormalization group from the ultra-violet towards the infra-red we add new modes to the unresolved cloud and average over their contributions, replacing our original point particle description by a new one. As modes are added to the cloud, the masses and couplings of the point particle are slowly screened as increasing numbers of unresolvable quanta contribute incoherently to the averaging procedure. In the DGLAP equation, for hadrons and de Sitter equally, we are driving the renormalization group flow in the opposite direction. In these cases we begin with the existence of an unresolved cloud, and attempt to understand its composition as the probe moves to higher energy scales: in other words, whether we obtain screening or stochastic dynamics is a question of the boundary condition we adopt. The stochastic character of the DGLAP equation is a proxy for the inherent quantum mechanical uncertainty with which quanta become resolved from the cloud, but in either case the physical picture is the same.

In the renormalization group picture, time evolution becomes the direction of the renormalization group flow for the simple reason that it is time evolution which causes the probe scale — the Hubble scale — to vary. It is not ordinarily possible to understand time evolution in these terms, because evolution equations are typically second order differential equations and do not admit a renormalization group interpretation. The crucial ingredient is the slow-roll approximation, which allows the relevant evolution equations to be truncated to first order. One is immediately led to suspect that the underlying structure which allows such a description is connected to a holographic interpretation of de Sitter space, or the proposed “dS/CFT correspondence.” There are reasons to believe that a dS/CFT correspondence may not exist in the same way as the well-established AdS/CFT correspondence, but many important features of the AdS/CFT holographic renormalization group flow are known to have analogues in de Sitter space. These flows have already been shown to reproduce many aspects of the standard theory of inflationary perturbations [94–97].

There appear to be significant obstructions to interpreting a phase of de Sitter evolution directly in terms of the parton evolution of some gauge field theory. Although some aspects are quite analogous — the occurrence of a DGLAP regime, or the apparent existence of saturation scales associated the inverse coupling — some are quite different. Most strikingly, there are no degrees of freedom in the de Sitter picture which could play the role of Bjorken variables describing the gluons of an underlying gauge theory. The gluons are expected to play a dominant role at very high energies, resulting in the formation of the so-called colour glass condensate [69, 90, 91]. At these energies, multiple gluon splittings tend to fill any hadron wavefunction with a universal cascade of soft gluons carrying deeply degraded momenta. In this regime large effects contributed by \(\ln x\) logarithms require the DGLAP equation to be replaced by the so-called Balitsky-JIMWLK equation or its relatives [98, 99]. In the case of an \(O(N)\)-symmetric scalar field theory in de Sitter space one could define variables more precisely analogous to the Bjorken \(x\) by setting \(y^\alpha = \phi^\alpha/\|\phi\|\), where \(\|\phi\|^2 = \phi^\alpha \phi_\alpha\), and for which \(0 \leq y^\alpha \leq 1\).\(^4\) A field at sufficiently small \(y\) has a small expectation value in

\(^4\)This definition was suggested to me by Dmitry Podolsky.
comparison with other fields whose expectation values are approaching the Planck scale, and in this regime one would also expect significant corrections in the de Sitter case.

Although an underlying gauge theory is lacking in de Sitter, it is interesting that one can find a partial, qualitative analogue of the confinement phase transition by which coloured degrees of freedom are softly bound into colourless states at an energy scale around $\Lambda_{\text{QCD}} \sim 200$ MeV. This is the phase transition between eternal inflation and slow-roll inflation towards a terminal vacuum [100, 101], which Starobinsky called the “useful” part of inflation in his original paper on the stochastic formalism [52]. The division between these phases is marked by the self-reproduction scale, which is accessible to the degree that it sits below the Planck mass in many models. In the eternal phase, which we could perhaps loosely imagine as a sort of confined phase for a box of de Sitter space, the partons are strongly interacting and the trajectories of Hubble-sized regions in field space mix randomly. In the terminal phase the partons are weakly interacting and trajectories become collimated, moving on parallel paths in field space. There is an analogy, too, with the asymptotic freedom of QCD by which the theory becomes non-interacting at high energies: as the coupling $H^2$ decreases, interactions can be described increasingly well by a free scalar field theory without dynamical gravity [100]. However, one should remember that all these observations are purely qualitative.

What have we learned from the study of infra-red effects? There seem to be two clear conclusions, both of which have already been emphasized in the literature by authors working with different methods.

Firstly, in making a prediction for what can be observed in the CMB one should carry out the calculation within a box that is in the neighbourhood of a terminal vacuum, by which is meant that the box as a whole (up to fluctuations described by the curvature perturbation) is evolving towards the hypersurface on which inflation ends. One can certainly get a different answer by calculating within a much larger box, but the difference arises because one includes in the average the possibility that some regions of the box are characterized by scalar vevs which are some way from the terminal vacuum [47, 48]. Therefore one should replace eq. (4.1) by a similar equation in which the distribution function $f$ is replaced by the probability that, given inflation is about to end, the scalar vevs take particular values. For single field inflation this probability is simply a $\delta$-function and the entire infra-red structure decouples from the theory [28, 46–48]. Where more than one field is present, there can be a non-trivial effect if inflation can end in different ways at different points in field space, or if one can obtain slightly different predictions by rolling into the same terminal vacuum from different directions.

A special case of these observations applies to the calculation of non-Gaussian effects. Although it has been known for a long time that one can obtain significant non-Gaussianity from large scale stochastic fluctuations [102, 103], such long wavelength phenomena cannot be the source of any non-Gaussian effects presently observed in the CMB. As new modes fall within our Hubble volume there is the prospect that we may eventually be able to interact with non-Gaussian fluctuations on very long wavelengths, but if this possibility exists it would seem to do so only in our long-term future. In making predictions for non-Gaussian fluctuations based on selection of specific trajectories in field space, one should therefore be careful that the trajectory does not intersect a phase of eternal inflation, and can be smoothly glued on to a prescription for exiting inflation. Likewise, if large infra-red loop corrections occur in making predictions for any correlation function, then these will generally require resummation before such correlators can be interpreted as observables.

Secondly, since what we can observe is conditioned upon proximity to the end of inflation, it is clear that we cannot presently probe the large-scale structure generated by
eternal inflation [37, 47–49]. This is like universality in the ultra-violet renormalization group flow, which prevents us from extrapolating the details of quantum gravity near the Planck mass from observations made at more pedestrian scales.\footnote{This point has been emphasized elsewhere by Dmitry Podolsky; see, in particular, the discussion at http://www.nonequilibrium.net/124-talk-munich-regularizing-correlators-curvature-perturbation/.} For example, if we wish to obtain top-down predictions from an \textit{a priori} model, perhaps obtained by construction from string theory or some other model of high-energy physics, then there may exist a landscape of vacua in which inflation can end, and the situation becomes very complicated. One naive way to find a measure on this landscape would be to allow the scalar expectation values to diffuse over the landscape, and record the frequency with which they fall into terminal vacua. Unfortunately, even this naive approach is a difficult undertaking in its own right, and could easily be complicated by other landscape effects [104, 105]. These issues were discussed concretely in refs. [51, 54]. The role of infra-red divergences in this context is to reproduce the measure problem of eternal inflation.

Stochastic formalisms for the purpose of computing the non-Gaussianity from inflation were in use throughout the 90s [106, 107] and more recently have been revived by Rigopoulos, Shellard \& van Tent [88, 108, 109] as a means to account for non-linear time evolution of the curvature perturbation in models where isocurvature fluctuations may be present (see also refs. [110, 111]). They therefore constitute an alternative to the popular $\delta N$ formula, although both are based on a separate universe picture [4, 112]. The predictions of the stochastic formalism reproduce those of the $\delta N$ formula for time evolution, since they both amount to a resummation of terms of the form $(\epsilon_\ast N)^n$. It is now easy to see that the argument of section 3–4 requires resummation of the $(H_\ast^2 N)^n$ terms in the $\delta N$ formula to reproduce the stochastic component of refs. [88, 108, 109]. We conclude that these two methods for computing non-Gaussian correlation functions can be regarded as completely equivalent. In particular, this apparently implies that the inclusion of a stochastic source does not lead to large non-Gaussian signals unless one has implicitly passed to a box in which infra-red effects are important, either because of its large size or because of an interaction with eternal inflation.

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**A Starobinsky’s theory**

As originally formulated, Starobinsky’s proposal of stochastic inflation can be thought of as a means to account for the back-reaction of fluctuations in a theory of a scalar field coupled to gravity in de Sitter space. Consider the Heisenberg operator, $\Phi$, corresponding to any
light scalar degree of freedom. In the stochastic prescription, this operator is coarse-grained according to the prescription\[52\]

\[
\Phi(t, \mathbf{x}) = \bar{\Phi}(t, \mathbf{x}) + \int \frac{d^3k}{(2\pi)^3} \vartheta(k - \varepsilon aH) \left[ a_k \varphi_k(t) e^{-ik \cdot \mathbf{x}} + a_k^\ast \varphi_k^\ast(t) e^{ik \cdot \mathbf{x}} \right] + \delta \phi, \tag{A.1}
\]

where \(\bar{\Phi}\) is a long wavelength mean field, assumed to evolve according to the classical equations of motion, which contains fluctuations much longer than the Hubble scale (that is, \(k \ll aH\)), and \(\delta \phi\) accounts for higher order corrections which are neglected. The short-wavelength integral term is treated according to the usual rules of quantum field theory, in such a way that \(\{a_k, a_k^\ast\}\) are annihilation and creation operators for modes of wavenumber \(k\) and the wavefunctions \(\{\varphi_k e^{-ik \cdot \mathbf{x}}, \varphi_k^\ast e^{ik \cdot \mathbf{x}}\}\) are solutions to the equation of motion in the interaction picture. The quantity \(\vartheta(z)\) is Heaviside’s step function, equal to unity for \(z > 0\) and zero for \(z < 0\), and with an indeterminate value at \(z = 0\) whose meaning will become clear below. Eq. (A.1) is written in terms of a coarse-graining parameter \(\varepsilon < 1\) (not to be confused with the slow-roll parameter \(\epsilon \equiv -\dot{H}/H^2\)) which can be interpreted in the DGLAP picture as a measure of the factorization scale, according to the rule \(k_F = \varepsilon k_s\). The aim is to obtain a probability distribution for the long wavelength field \(\bar{\Phi}\), after which correlation functions formed out of \(\Phi\) reproduce the prescription given in eq. (4.1). Such correlation functions do not depend on \(\varepsilon\) if contributions from both \(\bar{\Phi}\) and the short wavelength quantum mechanical part are kept.\[6\]

An equation of motion for \(\bar{\Phi}\) can be determined by substituting the full Heisenberg field \(\Phi\) into its equation of motion, and using the slow-roll condition to delete the double-derivative term, which yields

\[
3H \dot{\bar{\Phi}} = -\frac{\partial V(\bar{\Phi})}{\partial \bar{\Phi}} + f, \tag{A.2}
\]

where \(H \equiv \dot{a}/a\) is the Hubble parameter, \(V = V(\phi)\) is the potential. In eq. (A.2) \(\alpha\) is a stochastic term, which (after accounting for the finite volume of space and time) is subject to the rule

\[
\langle \alpha(t) \alpha(t') \rangle = 3H^2 \left( \frac{H}{2\pi} \right)^2 \delta(t - t'). \tag{A.3}
\]

The stochastic source means that the long-wavelength field does not evolve homogeneously, but rather develops fluctuations from place to place. From eq. (A.3) one can show that the large scale distribution of \(\phi\) is governed by a probability distribution \(f(\bar{\Phi})\) which evolves according to a Fokker-Planck equation,

\[
\frac{\partial f}{\partial t} = \frac{1}{3H} \frac{\partial (V' f)}{\partial \bar{\Phi}} + \frac{H^3}{8\pi^2} \frac{\partial^2 f}{\partial \bar{\Phi}^2}. \tag{A.4}
\]

\[6\]In applications of the stochastic formalism it is often the case that one wishes to identify large contributions from the mean field \(\bar{\Phi}\), in which case it may be reasonable to neglect contributions from the hard subprocess. However, if this is done then the resulting correlation functions will contain a sensitivity to the coarse-graining scale and in view of the very large scales on which these correlation functions apply one should be wary of interpreting them as observable quantities. This is the counterpart of the principle, within the DGLAP framework, that the distribution function \(f\) is not itself observable.

Indeed, in eq. (4.1) all information about presently observable modes is contained within the hard subprocess and the infra-red resummation in the distribution function \(f\) has the comparatively limited role of averaging over different vacua. It would seem that a similar interpretation should be applied to the coarse-graining scale.
B Starobinsky’s Fokker-Planck equation

In this appendix, the derivation of the Fokker-Planck equation eq. (A.4) is briefly recalled. The process of obtaining a Fokker-Planck equation from a Langevin equation [such as eq. (A.2)] has generated a very large literature. (See, for example, Zinn-Justin [113].) Here we contrast two especially useful approaches, the first of which is based on the stochastic calculus of Itô and Stratonovitch, and an alternative which is based on a path integral.

B.1 Derivation from stochastic calculus

Consider the Langevin equation, eq. (A.2), which can be rewritten in terms of a random variable \( \theta(t) \) which is normally distributed,

\[
\dot{\phi} = -\frac{V'}{3H} + \frac{H^{3/2}}{2\sqrt{\pi}} \theta,
\]

where \( \langle \theta(t) \rangle = 0 \) and \( \langle \theta(t) \theta(t') \rangle = \delta(t-t') \). Equivalently,

\[
d\phi = -\frac{V'}{3H} dt + \frac{H^{3/2}}{2\sqrt{\pi}} d\Theta,
\]

where \( d\Theta \equiv \theta dt \) is a so-called Wiener process. In this form, eq. (B.2) is an example of a stochastic differential equation, and \( d\phi \) is sometimes referred to as an Itô process, or a generalized Wiener process. It is an important theorem in the study of stochastic processes that \( (d\Theta)^2 = dt \) with probability one. This remarkable result can be justified using Itô’s theory of integration with respect to martingales, which allows the construction of solutions to stochastic differential equations such as eq. (B.2) by quadrature (as in conventional calculus). However, the physical content of the statement that \( (d\Theta)^2 = dt \) is considerably clearer in the path integral context to be described in section B.2 below.

We are interested in the probability density for \( \phi(t) \) to take a value in some prescribed range \( (\varphi, \varphi + d\varphi) \) at time \( t \). This probability density is labelled \( f[\phi(t) = \varphi] \), The probability that \( \phi(t) \) lies in any set \( \mathcal{B} \) can be written as an expectation over the indicator function \( I[\phi(t) \in \mathcal{B}] \),

\[
f[\phi(t) \in \mathcal{B}] \equiv \mathbb{E}\{ I[\phi(t) \in \mathcal{B}] \},
\]

which is sometimes known as the Feynman-Kac formula. In the present case, this simply reads \( f[\phi(t) = \varphi] = \mathbb{E}\delta[\phi(t) - \varphi] \). Now consider how the probability density varies with time. It follows immediately that during a small interval \( dt \) during which \( \phi \) undergoes some shift \( d\phi \), the change induced in \( f \) must satisfy

\[
df[\phi(t) = \varphi] = \mathbb{E}\left\{ \delta'[\phi(t) - \varphi] d\phi + \frac{1}{2} \delta''[\phi(t) - \varphi] (d\phi)^2 + \cdots \right\}.
\]

(B.4)

Using eq. (B.2) to substitute for \( d\phi \), recalling that \( \langle d\Theta \rangle = 0 \) and \( (d\Theta)^2 = dt \) with probability one, it follows that

\[
\frac{\partial f}{\partial t} = \int_{-\infty}^{\infty} d\varphi \ f \cdot \left\{ -\delta'[\phi(t) - \varphi] \frac{V'}{3H} + \delta''[\phi(t) - \varphi] \frac{H^3}{8\pi^2} \right\}.
\]

(B.5)

One now integrates by parts. Any boundary terms that are generated are zero, since they involve evaluation of \( f(\varphi) \) at \( |\varphi| = \infty \) and the field cannot reach infinity in finite time.
Once integration by parts has been performed, the Fokker-Planck equation (A.4) is obtained immediately, giving Starobinsky’s equation

$$\frac{df}{dt} = \frac{1}{3H} \frac{\partial (fV')}{\partial \varphi} + \frac{H^3}{8\pi^2} \frac{\partial^2 f}{\partial \varphi^2}. \quad (B.6)$$

Note that we have assumed that $H$ is a constant. Although this approximation is reasonable, one would nevertheless like to remove it and instead study the theory of a scalar field coupled self-consistently to gravity, in which case $H$ would become a function of $\phi$. In this case the noise term in the Langevin equation is said to become “multiplicative,” since the noise depends on the field $\phi$ for which we are trying to solve. Such a self-consistent theory was studied by Salopek & Bond [102, 103], who also give references to the earlier literature.

### B.2 Derivation from a path integral

As an alternative to the stochastic calculus, one may represent the operation of taking expectation values by a path integral. This is directly analogous to the way one may represent a quantum or thermal average using path integrals, and has been explored by many authors; a textbook treatment can be found in the book by Zinn-Justin [113].

Our point of departure for the path integral is the same Feynman-Kac formula, $f(\varphi) = \mathbb{E}\delta[\phi(t) - \varphi]$, given that $\phi(t_0) = \varphi_0$. In terms of a path integral, this reads

$$f[\phi(t) = \varphi | \phi(t_0) = \varphi_0] = \int [d\theta][dE] \delta[\phi(t) - \varphi] \delta[E(\varphi)] \exp \left( -\frac{1}{2} \int_0^t dt' \theta(t')^2 \right), \quad (B.7)$$

where the field equation for $\phi(t)$ is enforced via the constraint $E(\varphi) = 0$,

$$E \equiv \frac{\dot{\phi}}{3H} + \frac{V'}{2\pi} - \frac{H^{3/2}}{2\pi} \theta(t). \quad (B.8)$$

This expression can be compared to the method of ref. [54]. After changing variable from $E$ to $\varphi$ and writing $\delta(E)$ with the aid of an auxiliary field $\lambda$, this is the same as

$$f(\varphi | \varphi_0) = \int [d\theta][d\phi] \delta(\phi(t) = \varphi) [d\lambda] \left| \frac{\delta E}{\delta \phi} \right| \times \exp \left\{ -\frac{1}{2} \int_0^t dt' \left[ \theta^2 - 2i\lambda \left( \frac{\dot{\phi}}{3H} + \frac{V'}{2\pi} - \frac{H^{3/2}}{2\pi} \theta \right) \right] \right\}. \quad (B.10)$$

The determinant $\delta E/\delta \phi$ can be written

$$\det \frac{\delta E}{\delta \phi} = \exp \text{tr} \ln \frac{d}{dt} \left\{ \delta(t - t') + \theta(t - t') \frac{V''}{2H} \right\} = \exp \theta(0) \int_0^t dt' \frac{V''}{2H}, \quad (B.9)$$

in which the determinant of $d/dt$ has been factored out and discarded, since it leads only to an infinite but irrelevant field-independent constant. The overall coefficient depends on the arbitrary value which is assigned to the step function at zero argument, $\theta(0)$, for which one could naturally pick any number in the range $[0,1]$. For the present we make the “forward” choice $\theta(0) = 1$, before returning to this question at the end the section.

After integrating out $\theta$ and $\lambda$, we are left with

$$f(\varphi | \varphi_0) = \int [d\phi] \delta(\phi(t) = \varphi) \exp \left\{ -\frac{1}{2} \int_0^t dt' \left[ \frac{4\pi^2}{H^3} \left( \frac{\dot{\phi}}{3H} \right)^2 - \frac{2V''}{3H} \right] \right\}. \quad (B.10)$$
As in quantum mechanics this path integral has an equivalent representation in terms of a Schrödinger equation, which can be obtained by passing to the Hamiltonian picture. To do so, one interprets the argument of the exponential as an action $S$ and defines a canonical momentum, $p$, via the rule $p \equiv \delta S/\delta \dot{\phi}(t)$. One obtains
\[ p \equiv \frac{4\pi^2}{H^3} \left( \dot{\phi}(t) + \frac{V'}{3H} \right). \] (B.11)

The Hamiltonian which follows from this can be written
\[ H(p, \phi) \equiv \frac{H^3}{8\pi^2} p^2 - \left( \frac{V'}{3H} \right) p + \frac{V''}{3H}, \] (B.12)
and the Schrödinger equation is $\partial f/\partial t = H(-\partial/\partial \phi, \phi)f$, where $H$ is interpreted as an operator with all $p$s to the right of all $\phi$s, giving
\[ \frac{\partial f}{\partial t} = \frac{1}{3H} \frac{\partial (fV')}{\partial \phi} + \frac{H^3}{8\pi^2} \frac{\partial^2 f}{\partial \phi^2}, \] (B.13)
which is again Starobinsky’s equation, eq. (3.9).

C The Starobinsky-DGLAP equation

In this appendix, I briefly mention some subtleties in the derivation of the Starobinsky-DGLAP equation, eq. (3.8), which were omitted in the main text.

Firstly, note that eqs. (3.8)–(3.9) suffer from an operator ordering ambiguity in the term which involves $\partial^2 V$, and (as discussed in B) is present no matter which method is chosen to obtain the Starobinsky-DGLAP equation. It corresponds to an arbitrary choice in discretizing any stochastic process, equivalent to the distinction between the Itô and Stratonovich integrals. In eq. (3.8) this ambiguity requires a prescription for integrating out the $\delta$-function, after which some fraction of the coherent background evolution $\delta \phi^a$ can be included in $f$ and the remainder in $P$. eq. (3.9) is written in the usual operator ordering convention originally chosen by Starobinsky, which corresponds to including all of this term in $P$ and none in $f$. This ambiguity is not expected to have significant consequences for inflationary predictions [114].

Secondly, should we expect corrections to the form of the diffusion equation in a more detailed treatment? To answer this question, it is useful to observe that in the continuum limit the Starobinsky-DGLAP equation receives no corrections from higher-order connected correlation functions. Therefore the diffusion equation is unaffected by non-Gaussian corrections in the splitting functions at any order. Such corrections are present in loop corrections to $P_*$, but since these loops are suppressed by extra powers of $H^2$ it is clear that non-Gaussian effects do not contribute to the leading-logarithm approximation. Another source of corrections to the DGLAP equation could have come from disconnected correlation functions, proportional to two or more powers of a momentum-conservation $\delta$-function. Correlation functions of this sort could contribute in a leading-logarithm approximation because the extra $\delta$-functions each remove one power of $\delta \ln k$. Their effect would be to introduce higher-derivative terms into the Starobinsky-DGLAP equation which correct the diffusion term $\partial^2 f$, completely changing the character of the solutions. These corrections account for non-local transitions which are not between approximately nearest-neighbour Bjorken variables, and would lead to many complications. Such terms can also be encountered in diffusion-type approximations to the hadron DGLAP equation [115].
If the tadpole condition \( \langle \delta \phi^a \rangle = 0 \) is enforced then disconnected diagrams cannot contribute, because to reach leading-logarithm order such a diagram must contain at least one tadpole. It is fairly clear how this is to be interpreted. Recall that multiplication by a product of disconnected diagrams can be thought of as shifting the vacuum in which we calculate any correlation function. (Indeed, it is precisely to account for dressing by soft quanta, which arrange themselves into disconnected diagrams, for which we wish to use the DGLAP equation itself.) Inclusion of a background of disconnected diagrams in the splitting functions is tantamount to an admission that further vacuum redefinitions — described in the parton picture by shifts among the Bjorken variables — must be accounted for, which does not occur if the splitting functions are computed in the correct background. Accordingly, we can conclude that higher-derivative corrections to the DGLAP equation are absent in the leading-logarithm approximation.

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