Consistency of $\ell_1$ Penalized Negative Binomial Regressions

Fang Xie$^1$

School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, P.R. China.

Zhijie Xiao

Department of Economics, Boston College, Chestnut Hill, MA 02467

Abstract

We prove the consistency of the $\ell_1$ penalized negative binomial regression (NBR). A real data application about German health care demand shows that the $\ell_1$ penalized NBR produces a more concise but more accurate model, comparing to the classical NBR.

Keywords: negative binomial regression, $\ell_1$ consistency, penalized maximum likelihood, high-dimensional regression

1. Introduction

Count data is an important type of statistical data in which the observation takes non-negative integer values. Count data naturally arises in many areas such as health care demand (Riphahn et al., 2003), consumer credit behaviors (Greene, 1994), vehicle crash (Wei and Lovegrove, 2013), psychology (Gardner et al., 1995) and so on. Poisson distribution is a counting measure extensively used to model count data (Cameron and Trivedi, 1998), and the Poisson regression has been an important generalized linear model that is widely used in applications (Cupal et al., 2015; Stefany et al., 2009). Moreover, penalized Poisson regressions have been extensively studied and used to model high-dimensional count data (Algamal and Lee, 2015; Ivanoff S, 2016; Li and Cevher, 2015). However, a major lim-

$^1$Corresponding author

Email addresses: fangxie219@foxmail.com (Fang Xie), zhijie.xiao@bc.edu (Zhijie Xiao)
itation of the Poisson regression is its restrictive assumption that the variance equals the mean. In practice, more and more applications are found to have an overdispersion feature that sample variance is much larger than sample mean (Cameron and Trivedi, 1998; Hilbe, 2011), which violates the assumptions of Poisson regression. For this reason, a more general and flexible regression model, the negative binomial regression, has attracted a great deal of research attention and become a popular model in analyzing count data.

The NBR, as a generalization of the Poisson regression, loosens the highly restrictive assumption that the variance is equal to the mean made by the Poisson model. The negative binomial distribution has two parameters, the mean parameter $\mu$, and the over-dispersion parameter $r$. dispersion property. When $r \to \infty$, the negative binomial distribution converges to a Poisson distribution with the parameter $\mu$ (Cameron and Trivedi, 1998; Hilbe, 2011).

Nowadays, negative binomial distribution is becoming more and more important in modeling real data in health care science (Lu et al., 2013; Riphahn et al., 2003), biology (Mi et al., 2015), psychology (Walters, 2007), medicine (Aeberhard et al., 2014; An et al., 2016), ecology (Lindén and Mäntyniemi, 2011), finance (Cameron and Trivedi, 1996) and so on. As the dimension of data increases, variable selection is very important and necessary to simplify the fitting models. Stimulated by the great success of many penalized regressions such as lasso (Tibshirani, 1996), the NBRs with a penalty have recently been proposed to analyze high dimensional data, for example, the data of the association between multiple biomarkers and prolonged hospital length of stay (Wang et al., 2016). However, there are a few literature about the penalized negative binomial regression and hence less statistical theories. So, the first and main goal of this paper is to rigorously prove the consistency property of $\ell_1$ penalized NBR.

In addition to the theoretical analysis, we also apply the $\ell_1$ penalized NBR for analyzing real data about German health care demand. The data, supplied by the German Socioeconomic Panel (GSOEP), consist of 27326 samples observed in seven year. There are two dependent variables and 23 variables. For the brevity of analysis, we only consider one
dependent variable, the number of doctor visit within the last quarter prior to the survey (DOCVIS). We compare the $\ell_1$ penalized NBR with the classical NBR. The results show the $\ell_1$ penalized NBR can produce a more simple and efficient model, which exhibits the most effective variables and gives a much smaller prediction error.

The rest of the paper is organized as follows. In Section 1.1, we give the notations throughout the paper. In Section 2.1, we introduce the model and the $\ell_1$ penalized NBR method. The theoretical results are shown in Section 2.2. Sections 3 and 4 show the numerical results of $\ell_1$ penalized NBR based on synthetic data and real data. We finally conclude in Section 5.

1.1. Notations

For any vector $\mathbf{v} = (v_1, \cdots, v_d) \in \mathbb{R}^d$, $\|\mathbf{v}\|_q$ denotes its $l_q$-norm with $0 < q < \infty$. When $q = \infty$, $\|\mathbf{v}\|_\infty = \max_{i \in [d]} |v_i|$ where the notation $[d] = \{1, 2, \cdots, d\}$. Let $\{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1}$ be two real sequences. The notation $b_n = o(a_n)$ means that $\lim_{n \to \infty} b_n/a_n = 0$. The notation $b_n = O(a_n)$ means that $\lim_{n \to \infty} b_n/a_n = C$ with $C$ being some constant.

2. Model and Main theorems

In this section, we first introduce the negative binomial model and the corresponding $\ell_1$ regularized method. Then, we give our theoretical results on the consistency of this method.

2.1. Model and penalized negative binomial regression

The negative binomial regression model assumes that the response variable $Y$ has a negative binomial distribution, and the logarithm of its expected value can be modeled by a linear combination of unknown parameters. Thus, given observed data $(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_n, y_n)$, we assume that

$$y_i | \mathbf{x}_i \sim \text{NB}(r, \mu_i) \text{ with } \mu_i = e^{\mathbf{x}_i^\top \beta} \text{ and } r > 0,$$

with $y_i \in \mathbb{R}$, $\mathbf{x}_i \in \mathbb{R}^p$ and NB$(r, \mu_i)$ signifies a negative binomial distribution with parameters $(r, \mu_i)$, where the mean $\mu_i$ is modeled as $e^{\mathbf{x}_i^\top \beta}$, and $\beta^* = (\beta^*_1, \cdots, \beta^*_p)^\top \in \mathbb{R}^p$ is an unknown parameter to be estimated.
By definition, the negative binomial probability density function has the form (Washington et al., 2010)

\[ P(y_i | x_i) = \left[ \frac{r}{r + \mu_i} \right]^r \frac{\Gamma(r + y_i)}{\Gamma(r)} \left[ \frac{\mu_i}{r + \mu_i} \right]^{y_i}, \]

with \( \mu_i = e^{x_i^\top \beta^*} \), for \( i = 1, 2, \ldots, n \), where \( \Gamma(\cdot) \) is the gamma function.

Denote \( x_i = (x_{i1}, \ldots, x_{ip})^\top \). Without loss of generality, we assume that

\[ \frac{1}{n} \sum_{i=1}^{n} x_{ij} = 0 \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} x_{ij}^2 = 1. \]

The estimator \( \hat{\beta} \) for negative binomial regression obtained by \( \ell_1 \) penalized maximum log-likelihood method is defined by

\[ \hat{\beta} = \arg \min_{\beta \in \mathbb{R}^p} \{ L(\beta) + \lambda \| \beta \|_1 \}, \quad (2.1) \]

where

\[ L(\beta) = -\frac{1}{n} \sum_{i=1}^{n} \left( y_i (x_i^\top \beta - \ln(r + e^{x_i^\top \beta})) - r \ln(r + e^{x_i^\top \beta}) \right). \]

From Karush-Kuhn-Tucker conditions, we know that \( \lambda \geq \| \nabla L(\beta) \|_\infty \). So, if the \( \hat{\beta} \) is close to \( \beta^* \), the event \( \{ \lambda \geq c \| \nabla L(\beta^*) \|_\infty \} \) will hold with high probability with some constant \( c > 1 \). In fact, we indeed prove this event holds with probability approaching to 1 as \( n, p \to \infty \), see Lemmas 2.3 and 2.2 below.

2.2. Theoretical results

We aim to prove the consistency of estimator \( \hat{\beta} \) and obtain the convergence rate of estimation error.

Firstly, we denote \( V = \| \nabla L(\beta^*) \|_\infty \) where

\[ \nabla L(\beta^*) = -\frac{1}{n} \sum_{i=1}^{n} x_i \times \sqrt{\frac{r e^{x_i^\top \beta^*}}{e^{x_i^\top \beta^*} + r}} \times \frac{(y_i - e^{x_i^\top \beta^*})}{\sqrt{e^{x_i^\top \beta^*} (r + e^{x_i^\top \beta^*})}}. \]

Notice that the last term on the right-hand side of equation above is the normalized form of \( y_i \). This part will play an important role later in evaluating the probability of event \( \{ \lambda \geq c \| \nabla L(\beta^*) \|_\infty \} \). For each \( 1 \leq i \leq n \), denote

\[ v_i = \sqrt{\frac{r e^{x_i^\top \beta^*}}{e^{x_i^\top \beta^*} + r}} \quad \text{and} \quad v_n = \max_{1 \leq i \leq n} v_i. \]
For the negative binomial distribution, it is easy to check that its mean and variance have the following relationship:

$$\text{Var}(y_i | x_i) = \mu_i \left(1 + \frac{\mu_i}{r}\right),$$

where $\mu_i$ is the mean and the parameter $r$ describes the dispersion of the negative binomial distribution. If there exists some positive constant $B > 1$ such that $r \leq \mu_i/B - 1$ for all $i$, we obtain that $\text{Var}(y_i | x_i)/\mu_i \in [B, +\infty)$ for all $i$. Hence, $v_i \leq \sqrt{\mu_i/B}$ for each $i$ and $v_n \leq \max_{1 \leq i \leq n} \sqrt{\mu_i/B}$. This implies that the value of $v_n$ depends on the maximum of means $\mu_i$.

For notation ease, we itemize the conditions throughout the paper:

**C1.** Sparsity of $\beta^*$: $s = |S| < n$ where $S = \{j \in [p]: \beta_j^* \neq 0\}$.

**C2.** There exists a positive constant $r$ such that $\sup_{i \in [n], j \in [p]} |x_{ij}| \leq R < \infty$.

**C3.** $n, p$ satisfy that $\sqrt{n} < p \leq o(e^{n^{1/\alpha}})$ and $p/\alpha > 8$ for all $\alpha \in (0, 1)$.

**C4.** For any $\delta \in \mathbb{R}^p$ satisfying $\|\delta_{S^c}\|_1 \leq \gamma \|\delta_S\|_1$ with some $\gamma > 1$, there exists a positive constant $\phi_0$ such that $\langle \delta, \nabla^2 L(\beta^*) \delta \rangle \geq \phi_0^2 \|\delta_S\|_2^2$.

Conditions **C1** and **C2** are conventional conditions that have been widely used in literature. Condition **C3** is assumed since we focus on the high-dimensional regression problems including $n > p$ and $n < p$. Condition **C4** is the popular restricted eigenvalue assumption, see also in (Bickel et al., 2009; Jia et al., 2019).

The following theorem gives the bounds of $|L(\hat{\beta}) - L(\beta^*)|$ and $\ell_1$ estimation error of $\hat{\beta}$.

**Theorem 2.1.** Let $\hat{\beta}$ be defined in (2.1). Suppose that the conditions **C1**, **C2**, and **C4** hold. If $\lambda \geq cV$ with some $c > 1$ and $\lambda s \leq (c - 1)^2 \phi_0^2/6cR(c + 1)$ with $s, R, \phi_0$ defined in **C1**, **C2**, and **C4** respectively, we have

$$\|\hat{\beta} - \beta^*\|_1 \leq \frac{C\lambda s}{\phi_0^2}, \tag{2.2}$$

$$|L(\hat{\beta}) - L(\beta^*)| \leq \frac{C\lambda^2 s}{\phi_0^2}, \tag{2.3}$$

where $C = 2c_1c(c + 1)/(c - 1)^2$ with some constant $c_1 \in (2, 3)$. 

The proof of Theorem 2.1 is given in the Supplementary Material (see Appendix A). From inequalities (2.2) and (2.3), we can know that \( \|\hat{\beta} - \beta^*\|_1 = O(\lambda s) \) and \( |L(\hat{\beta}) - L(\beta^*)| = O(\lambda^2 s) \) which are satisfactory. But these two orders are obtained based on the event \( \{ \lambda \geq cV \} \) with some \( c > 1 \) and the condition \( \lambda s \leq (c - 1)^2 \phi_0^2 / 6cR(c + 1) \). So, we need to have further studies on the event and the order of \( s \). The order of \( s \) is discussed in Remark 2.6.

We first study the event \( \{ \lambda \geq cV \} \) with some \( c > 1 \). Recall \( V = \|\nabla L(\beta^*)\|_\infty \) and denote by \( V(1 - \alpha|X) \) the \( (1 - \alpha) \)-quantile of \( V \). We give two choices of \( \lambda \) as follows. Given any \( \alpha \in (0, 1) \) and some \( c > 1 \), define

- **exact choice**: \( \lambda = cV(1 - \alpha|X) \),
- **asymptotic choice**: \( \lambda = cv_n(\sqrt{n})^{-1}\Phi^{-1}(1 - \alpha/2p) \),

where \( v_n \) is defined in (2.2) and \( \Phi \) is the cumulative distribution function of standard normal distribution.

We will show in the following two lemmas that under these two choices of \( \lambda \), the probabilities of the event \( \{ \lambda \geq cV \} \) will approach to 1 as \( n, p \to \infty \).

**Lemma 2.2.** If \( \lambda = cV(1 - \alpha|X) \) with some \( \alpha \in (0, 1) \), then we have

\[
P(\lambda \geq cV) \geq 1 - \alpha.
\]

Lemma 2.2 can be easily proved by the definition of the quantile.

**Lemma 2.3.** If \( \lambda = cv_n(\sqrt{n})^{-1}\Phi^{-1}(1 - \alpha/2p) \) with some \( \alpha \in (0, 1) \), then we have

\[
P(\lambda \geq cV) \geq 1 - \alpha \left( 1 + O(1)(\sqrt{2 \log (2p/\alpha)} - \sqrt{nm})^3 n^{-1/2}(3w_1 \log p + m) \right) \\
\times \left( 1 + \frac{1}{\log (p/\alpha)} \right) \frac{\exp\{(n \log (p/\alpha))^{1/2}m - nm^2/2\}}{1 - \sqrt{nm/(\log (p/\alpha))^{1/2}}} + C_1n/p^2,
\]

where \( m = 6C_1w_1 \log p/p^3 \) with some positive constants \( C_1 \) and \( w_1 \). In particular, as \( n, p \to \infty \), we have

\[
P(\lambda \geq cV) \geq 1 - \alpha(1 + o(1)).
\]
The proof of Lemma 2.3 is given in the Supplementary Material (see Appendix B). The key technique is Cramér type moderate deviation theorem.

Combing Lemmas 2.2 and 2.3 with Theorem 2.1, we can obtain the following propositions 2.4 and 2.5 respectively, which give the bounds of $\ell_1$ estimation error under two choices of $\lambda$.

**Proposition 2.4.** Let $\hat{\beta}$ be defined in (2.1). Suppose that the conditions $C_1, C_2, C_3,$ and $C_4$ hold. If $\lambda = cV(1 - \alpha|X)$ with some $\alpha \in (0, 1)$ and $c > 1$ and $\lambda s \leq (c - 1)^2 \phi_0^2 6cR(c+1)$ with $s, R, \phi_0$ defined in $C_1, C_2,$ and $C_4$, then with probability at least $1 - \alpha$, the inequalities (2.2) and (2.3) hold.

**Proposition 2.5.** Let $\hat{\beta}$ be defined in (2.1). Suppose that the conditions $C_1, C_2, C_3,$ and $C_4$ hold. If $\lambda = cvn^{-1}\Phi^{-1}(1 - \alpha/2p)$ with some $\alpha \in (0, 1)$ and $c > 1$ and $\lambda s \leq (c - 1)^2 \phi_0^2 / 6cR(c+1)$ with $s, R, \phi_0$ defined in $C_1, C_2,$ and $C_4$, then with probability at least

$$1 - \alpha \left(1 + O(1)(\sqrt{2\log (2p/\alpha)} - \sqrt{nm})n^{-1/2}(3w_1 \log p + m)\right)$$

$$\times \left(1 + \frac{1}{\log (p/\alpha)} \exp\left\{-2(n \log (p/\alpha))^{1/2}m + nm^2\right\}\right)^1_{1 - \sqrt{nm}/(\log (p/\alpha))^{1/2}} + C_1 n/p^2,$$

with $m = 6C_1 w_1 \log p/p^2$ with some positive constants $C_1$ and $w_1$, the inequalities (2.2) and (2.3) hold.

**Remark 2.6.** Notice that the asymptotic choice of $\lambda$ is order of $\sqrt{(\log p)/n}$. Then, $\|\hat{\beta} - \beta^*\|_1 = O(s \sqrt{(\log p)/n})$ and $|L(\hat{\beta}) - L(\beta^*)| = O(s \log p)/n)$. If $s$ satisfies $s = o(\sqrt{n/\log p})$, then $\|\hat{\beta} - \beta^*\|_1 = o(1)$ and $|L(\hat{\beta}) - L(\beta^*)| = o(\sqrt{(\log p)/n})$.

3. Simulations

In this section, we conduct simulations to show the performances of $\ell_1$ penalized NBR in the perspective of the variation of the estimation error of estimator $\hat{\beta}$, true positive rate and true negative rate.

We generated the data $(y_i, x_i)$ from the negative binomial distribution $NB(r, \mu_i)$ with the following setting: $r$ varies in $\{2, 1, 0.5, 0.25\}$, $\mu_i = e^{x_i^\top \beta^*}$ for $i \in [n]$, $p$-dimensional
observations \( x_i \sim N(0, \Sigma) \) with \( \Sigma_{jk} = \rho^{|j-k|} \) for \( j, k \in [p] \), \( p \)-dimensional true parameter vector \( \beta^* \) has 5 nonzero components taking value in \([-1,1]\) randomly. We set \( p = 30, n \in \{100, 200, 400, 800\} \), and \( \rho = 0.5 \). The smaller \( r \) means that the data \( y_i \) are more over-disperse.

We will compare the estimation error of estimator \( \hat{\beta} \), true positive rate (sensitivity) and true negative rate (specificity). We define the estimation error by \( \| \hat{\beta} - \beta^* \|_1 \). Sensitivity is the fraction of the number of correctly selected predictors in all the effective predictors, and specificity is the fraction of the number of correctly unselected predictions in all the ineffective predictors. An ideal estimator should have the estimation error close to 0, and sensitivity and specificity close to 1. We repeat each realization 100 times to obtain the mean and standard deviation (SD) of the three criteria above. The results are shown in Table 1. It indicates that for fixed \((r, p, \rho)\), the estimation error decreases with \( n \) increasing. At the same time, sensitivity increases towards 1 as we expect, but this pays a price on specificity. So, we could choose an appropriate sample size to make a trade-off between sensitivity and specificity based on the actual situation.
4. An Application on German health care demand

We will apply the NBR with the $\ell_1$-penalized MLE method to a real dataset on German health care demand in this section. The data is a part of the German Socioeconomic Panel (GSOEP) data which was employed in (Riphahn et al., 2003). The data source can be downloaded on http://qed.econ.queensu.ca/jae/2003-v18.4/riphahn-wambach-million/. The data consist of 27326 observations from 7293 individuals observed one or several times during years \{1984, 1985, 1986, 1987, 1988, 1991, 1994\}. The number of observations for each year above are \{3874, 3794, 3792, 3666, 4483, 4340, 3377\}. In the original data, there are two dependent variables and 23 variables. The two dependent variables are DOCVIS (number of doctor visits within the last quarter prior to the survey) and HOSPVIS (number of hospital visits in the last calendar year). But in the interest of brevity, we just study on DOCVIS in this paper. We list all the variable and its mean and standard deviation in Table S1 in the Supplementary Material.

Consider the data in each observed year, we build models for DOCVIS by the $\ell_1$ penalized NBR method and classical NBR method via maximum likelihood estimation. We randomly choose 500 samples to train and 500 samples to test for each year’s data. For the $\ell_1$ penalized NBR, we use the 10-fold cross validation to select the penalty level $\lambda$. We exhibit the prediction errors (PE) and regression coefficients of both two NBR methods in Table 2. The "P-NBR" is short for $\ell_1$ penalized NBR. The results show that the $\ell_1$ penalized NBR not only simplifies the model but also produces more accurate PE than the classical NBR. So we can conjecture that the true model is sparse, in which the most important variables to effect DOCVIS are FEMALE, HSAT and HHKIDS. The classical NBR used all variables including some uncorrelated ones and made a misleading prediction.

5. Conclusion

We studied on two theoretical choices of penalty level, under which we proved that the $\ell_1$ penalized negative binomial estimator is $\ell_1$ consistent. We then conduct a simulation, whose results further confirm the convergence tendency of estimation errors. Finally, an real
Table 2: $\ell_1$-penalized NBR vs general NBR

| Variables  | 1984 | 1985 | 1986 | 1987 | 1988 | 1991 | 1994 |
|------------|------|------|------|------|------|------|------|
| P-NBR      | 44.391 | 2394.968 | 54.382 | 2905.62 | 33.194 | 3173.689 | 75.535 | 1981.128 | 21.60 | 42156.603 | 56.350 | 2095.221 | 39.373 | 2927.628 |
| NBR        |      |      |      |      |      |      |      |
| PE         | 44.391 | 2394.968 | 54.382 | 2905.62 | 33.194 | 3173.689 | 75.535 | 1981.128 | 21.60 | 42156.603 | 56.350 | 2095.221 | 39.373 | 2927.628 |
| Intercept  | 2.193 | 3.208 | 1.669 | 0.217 | 1.945 | -0.663 | 2.705 | 3.243 | 2.053 | 0.343 | 2.051 | 1.152 | 2.182 | -0.503 |
| FEMALE     | 0.161 | 0.331 | 0.066 | 0.265 | 0.049 | 0.355 | 0.061 | 0.165 | 0.039 | 0.246 | 0.033 | 0.175 | 0.354 | 0.636 |
| AGE        | -0.332 | 0.395 | 0.731 | 0.005 | 0.274 |      | -0.117 |      | -0.244 |      | 0.175 | 0.494 |      | -0.097 |
| HSAT       | -1.599 | -1.8 | -1.241 | -1.434 | -1.207 | -1.464 | -1.086 | -1.362 | -1.115 | -1.310 | -1.549 | -1.762 | -1.717 | -2.180 |
| HANDDUM    | 0.079 | -0.021 | 0.182 | 0.168 |      | -0.214 | -0.455 |      | 0.032 | 0.052 | 0.010 | 0.092 | 0.092 | 0.577 |
| HANDPER    | 0.048 | 0.091 | 0.148 | 0.054 |      |      | -0.041 |      | 0.092 | 0.142 |      |      | -0.060 | -0.479 |
| HHNINC     | 0.019 | -0.177 |      |      |      |      |      |      |      |      |      |      |      |      |
| HHKIDS     | -0.028 | -0.214 | -0.128 | -0.031 | -0.159 | -0.038 | -0.136 | -0.167 | -0.007 | -0.088 | -0.040 |      |      |      |
| EDUC       | -0.344 | 1.425 | 2.872 | -0.111 | 0.028 | 2.123 | 0.407 | 1.700 |      |      |      |      |      |      |
| MARRIED    | 0.014 | 0.24 | 0.000 | -0.081 | -0.210 | 0.140 | 0.057 |      |      |      |      |      |      |      |
| HAUPTS     | 0.13 | -0.257 | -0.394 |      | 0.006 | 0.431 | 0.075 |      |      |      |      |      |      |      |
| REALS      | 0.082 | -0.102 | -0.334 | 0.043 | 0.128 | 0.070 | 0.445 |      |      |      |      |      |      |      |
| FACHHS     | 0.058 | -0.165 | -0.270 | -0.014 | -0.156 | -0.061 | -0.002 |      |      |      |      |      |      |      |
| ABITUR     | -0.039 | -0.176 | -0.549 | 0.022 | -0.040 | -0.200 | 0.016 | 0.136 |      |      |      |      |      |      |
| UNIV       | 0.132 | -0.217 | -0.062 | -0.319 | -0.160 | -0.179 | -0.049 | -0.026 |      |      |      |      |      |      |
| WORKING    | 0.155 | 0.327 | -0.256 | -0.214 | -0.038 | -0.448 | -0.025 | 0.112 | -0.070 |      |      |      |      |      |
| BLUES      | -0.104 | -0.117 | 0.309 | -0.131 | 0.369 | -0.058 | 0.049 |      |      |      |      |      |      |      |
| WHITEC     | -0.092 | -0.155 | 0.291 | -0.111 | -0.006 | 0.277 | -0.035 | -0.313 | 0.125 |      |      |      |      |      |
| SELF       | -0.129 | -0.02 | -0.219 | 0.016 | 0.238 | -0.063 | -0.212 | 0.135 | -0.087 | 0.005 |      |      |      |      |
| BEAMT      | -0.228 | -0.171 | 0.267 | -0.096 | 0.184 | -0.051 | 0.107 |      |      |      |      |      |      |      |
| PUBLIC     | -0.407 | 0.095 | 0.177 | 0.520 | 0.080 | 0.328 | -0.240 | 0.297 | 0.107 |      |      |      |      |      |
| ADDON      | -0.149 | 0.02 | -0.110 | 0.008 | -0.020 | -0.067 | -0.044 |      |      |      |      |      |      |      |

Application shows that the $\ell_1$ penalized NBR can produce a concise model that only contains key variables, and it has much smaller prediction errors than the classical NBR.

Appendix A. Proof of Theorem 2.1

Let $\delta = \hat{\beta} - \beta^*$. Recall that $S = \{ j : \beta^*_j \neq 0 \}$. By definition of $\hat{\beta}$ and the convexity of $L(\beta)$, we have

$$L(\hat{\beta}) - L(\beta^*) \leq \lambda (||\beta^*||_1 - ||\hat{\beta}||_1)$$

$$= \lambda (||\beta^*_S||_1 - ||\hat{\beta}_S||_1) + (||\beta^*_S^c|| - ||\hat{\beta}_S^c||_1)$$

$$\leq \lambda (||\delta_S||_1 - ||\delta_S^c||_1), \quad (A.1)$$

and

$$L(\hat{\beta}) - L(\beta^*) \geq \delta^\top \nabla L(\beta^*) \geq -V ||\delta||_1 \geq -\frac{\lambda}{c} ||\delta||_1, \quad (A.2)$$
where $V = \|\nabla L(\beta^*)\|_\infty$, and the last inequality utilizes the condition $\lambda > cV$. Combining (A.1) and (A.2), we obtain that
\[
\|\delta_S\|_1 \leq \frac{c+1}{c-1} \|\delta_S\|_1,
\]
which makes the condition C4 hold with $\gamma = \frac{c+1}{c-1} > 1$.

For any $\beta, u, v \in \mathbb{R}^p$, we have
\[
\nabla^2 L(\beta)[v, v] = \frac{1}{n} \sum_{i=1}^n (x_i^T v)^2 \exp \{x_i^T \beta\} (y_i + r) (r + \exp \{x_i^T \beta\})^2
\]
\[
\nabla^3 L(\beta)[u, v, v] = \frac{1}{n} \sum_{i=1}^n x_i^T u (x_i^T v)^2 r \exp \{x_i^T \beta\} (y_i + r) (r - \exp \{x_i^T \beta\}) (r + \exp \{x_i^T \beta\})^3.
\]

Under the assumption (C2), it is easy to verify that
\[
|\nabla^3 L(\beta)[u, v, v]| \leq \sup_{i \in [n]} |x_i^T u| \nabla^2 L(\beta)[v, v]
\]
\[
\leq \sup_{i \in [n], j \in [p]} |x_{ij}| \|u\|_1 \nabla^2 L(\beta)[v, v]
\]
\[
\leq R \|u\|_1 \nabla^2 L(\beta)[v, v].
\]

Setting $u = \delta = \hat{\beta} - \beta^*$, we have
\[
|\nabla^3 L(\beta)[u, v, v]| \leq R(1 + \frac{c+1}{c-1}) \|\delta_S\|_1 \nabla^2 L(\beta)[v, v]
\]
\[
\leq 2cR\sqrt{s} \frac{R}{c-1} \|\delta_S\|_2 \nabla^2 L(\beta)[v, v]. \quad \text{(A.3)}
\]

Denoting $\tilde{R} = \frac{2cR\sqrt{s}}{c-1}$, (A.3) becomes $|\nabla^3 L(\beta)[u, v, v]| \leq \tilde{R} \|\delta_S\|_2 \nabla^2 L(\beta)[v, v]$.

Thus, $L(\cdot)$ is a self-concordant like function with parameter $\tilde{R}$ with respect to $l_2$-norm. By Theorem 6.1 of [13] and condition C4, we have
\[
L(\hat{\beta}) - L(\beta^*) \geq \delta^T \nabla L(\beta^*) + \frac{\delta^T \nabla^2 L(\beta^*)\delta}{R^2 \|\delta_S\|_2^2} (\exp \{-\tilde{R}\|\delta_S\|_2\} + \tilde{R}\|\delta_S\|_2 - 1)
\]
\[
\geq -\|\nabla L(\beta^*)\|_\infty \|\delta\|_1 + \frac{\delta^T \nabla^2 L(\beta^*)\delta}{R^2 \|\delta_S\|_2^2} (\exp \{-\tilde{R}\|\delta_S\|_2\} + \tilde{R}\|\delta_S\|_2 - 1) \quad \text{(A.4)}
\]
\[
\geq -\frac{\lambda}{c} \|\delta\|_1 + \frac{\delta^T \nabla^2 L(\beta^*)\delta}{R^2 \|\delta_S\|_2^2} (\exp \{-\tilde{R}\|\delta_S\|_2\} + \tilde{R}\|\delta_S\|_2 - 1)
\]
Combining (A.1) and (A.4), we have
\[
\frac{\delta^\top \nabla^2 L(\beta^*) \delta}{R^2 \|\delta_S\|_2^2} \left(\exp \{-\tilde{R}\|\delta_S\|_2\} + \tilde{R}\|\delta_S\|_2 - 1\right) \leq \lambda \|\delta_S\|_1 + \frac{\lambda}{c} \|\delta\|_1 \leq \frac{c+1}{c-1} \lambda \|\delta_S\|_1 \leq \frac{c+1}{c-1} \lambda \sqrt{s} \|\delta_S\|_2.
\] (A.5)

By condition C4 and (A.5), we have
\[
\exp \{-\tilde{R}\|\delta_S\|_2\} + \tilde{R}\|\delta_S\|_2 - 1 \leq \left(\frac{c}{c+1}\right) \lambda \sqrt{s} \tilde{R}^2 \left(\frac{c-1}{c} \phi_0^2\right).
\] (A.6)

Set
\[
h = \frac{(c+1)\lambda \sqrt{s} \tilde{R}}{(c-1)\phi_0^2} = \frac{2c(c+1)\lambda Rs}{(c-1)^2\phi_0^2},
\] (A.7)

then according to the condition on \(\lambda\) such that \(\lambda s \leq \frac{(c-1)^2\phi_0^2}{6cR(c+1)}\), we have \(h \leq \frac{1}{3}\). Denote \(w = \tilde{R}\|\delta_S\|_2\), then to solve (A.6) is equivalent to solve the inequality \(\exp \{-w\} + w - 1 \leq hw\) which implies \(\{w : \exp \{-w\} + w - 1 \leq hw, h \leq \frac{1}{3}\} \subseteq \{w : \frac{w^2}{2} - \frac{w^3}{6} \leq hw, h \leq \frac{1}{3}\}\). Since under the condition \(h \leq \frac{1}{3}\), the solution of inequality \(\frac{w^2}{2} - \frac{w^3}{6} \leq hw\) is \(w \leq c_1 h\) for some constant \(c_1 \in (2,3]\), then
\[
\{w : \exp \{-w\} + w - 1 \leq hw, h \leq \frac{1}{3}\} \subseteq \{w \leq c_1 h\ \text{with some constant} \ c_1 \in (2,3]\}.
\]

So, from (A.6), we obtain
\[
\tilde{R}\|\delta_S\|_2 \leq \frac{c_1 \lambda \sqrt{s}(c+1)}{(c-1)\phi_0^2} \tilde{R},
\]
that is,
\[
\|\delta_S\|_2 \leq \frac{c_1 \lambda \sqrt{s}(c+1)}{(c-1)\phi_0^2}.
\] (A.8)

Hence, notice the relationship \(\|\delta\|_1 \leq (1 + \frac{c+1}{c}) \sqrt{s} \|\delta_S\|_2 = \frac{2\sqrt{s}}{c-1} \|\delta_S\|_2\), by (A.8) we have
\[
\|\delta\|_1 \leq \frac{2c_1(c+1)}{(c-1)^2\phi_0^2} \lambda s.
\] (A.9)

Then, (2.2) is obtained. Furthermore, by (A.1) and (A.2), we obtain
\[
|L(\hat{\beta}) - L(\beta^*)| \leq \lambda \|\delta\|_1 \leq \frac{2c_1(c+1)}{(c-1)^2\phi_0^2} \lambda^2 s,
\] (A.10)

which implies (2.3). We finish the proof. \(\square\)
Appendix B. Proof of Lemma 2.3

Recall the gradient of $L(\beta)$ at the point $\beta = \beta^*$

$$\nabla L(\beta^*) = -\frac{1}{n} \sum_{i=1}^{n} x_i \times v_i \times \frac{(y_i - \exp \{x_i^\top \beta^*\})}{\sqrt{\exp \{x_i^\top \beta^*\}(r + \exp \{x_i^\top \beta^*\})}},$$

where

$$v_i = \sqrt{\frac{r \exp \{x_i^\top \beta^*\}}{\exp \{x_i^\top \beta^*\} + r}}.$$

Denote $\epsilon_i = (y_i - \exp \{x_i^\top \beta^*\})/\sqrt{\exp \{x_i^\top \beta^*\}(r + \exp \{x_i^\top \beta^*\})}/r$, then

$$V = \|\nabla L(\beta^*)\|_{\infty} = \max_{j \in [p]} \left| \frac{1}{n} \sum_{i=1}^{n} x_{ij} \epsilon_i \right|.$$

Recall $v_n = \max_{1 \leq i \leq n} v_i$ and denote $t_{p,\alpha} = \Phi^{-1}(1 - \frac{\alpha}{2p})$, then $\lambda = cv_n(\sqrt{n})^{-1}t_{p,\alpha}$. Hence

$$\mathbb{P}(cV > \lambda) = \mathbb{P}(\max_{j \in [p]} \left| \frac{1}{n} \sum_{i=1}^{n} x_{ij} \epsilon_i \right| > v_n(\sqrt{n})^{-1}t_{p,\alpha})$$

$$\leq \mathbb{P}(\max_{j \in [p]} \left| \frac{1}{n} \sum_{i=1}^{n} x_{ij} \epsilon_i \right| > (\sqrt{n})^{-1}t_{p,\alpha})$$

$$(B.1) \leq p \max_{j \in [p]} \mathbb{P}(\left| \sum_{i=1}^{n} x_{ij} \epsilon_i \right| > \sqrt{nt_{p,\alpha}}).$$

Since $y_i | x_i \sim \text{NB}(r, \mu_i)$ with $\mu_i = \exp \{x_i^\top \beta^*\}$, then

$$\mathbb{E}(\exp \left\{ \theta \epsilon_i \right\}) = \exp \left\{ -\theta \sqrt{\frac{r \mu_i}{\mu_i + r}} \left( 1 + \frac{\mu_i}{r} \left( 1 - \exp \left\{ \theta \sqrt{\frac{r}{\mu_i}} \right\} \right) \right) \right\}^{r^{-}\theta}$$

is a positive constant for all $\theta < \sqrt{\frac{\mu_i(\mu_i + r)}{r}} \ln\left(\frac{\mu_i}{r} + 1\right)$. By the exponential Chebyshev’s inequality, we have

$$\mathbb{P}(|\epsilon_i| > A) < \exp \{-A/w_1\} \mathbb{E}(\exp \{\epsilon_i/w_1\}) = C_1 \exp \{-A/w_1\}$$

(B.2)
with some constant $C_1 = \mathbb{E}(\exp\{\epsilon_i/w_1\}) > 0$ and $w_1 > \left(\frac{\sqrt{\mu_i(\mu_i + r)}}{r} \ln\left(\frac{r}{\mu_i}\right) + 1\right)^{-1}$. Denote $\hat{\epsilon}_i = \epsilon_i 1_{|\epsilon_i| \leq A}$ and $\tilde{\epsilon}_i = \epsilon_i 1_{|\epsilon_i| > A}$. Taking $A = 3w_1 \log p$, we have

$$
\mathbb{P}(\sum_{i=1}^n x_{ij} \epsilon_i > \sqrt{nt_{p,\alpha}}) = \mathbb{P}(\sum_{i=1}^n x_{ij} (\hat{\epsilon}_i + \tilde{\epsilon}_i) > \sqrt{nt_{p,\alpha}}, \sup_{i \in [n]} |\epsilon_i| \leq A) + \mathbb{P}(\sum_{i=1}^n x_{ij} \hat{\epsilon}_i > \sqrt{nt_{p,\alpha}}, \sup_{i \in [n]} |\epsilon_i| > A)
$$

where

$$
\mathbb{P}(\sum_{i=1}^n \hat{\epsilon}_i > \sqrt{nt_{p,\alpha}}) = \mathbb{P}(\sup_{i \in [n]} |\epsilon_i| > A).
$$

Denote $P_1 = \mathbb{P}(\sum_{i=1}^n x_{ij} \hat{\epsilon}_i > \sqrt{nt_{p,\alpha}})$ and $P_2 = \mathbb{P}(\sup_{i \in [n]} |\epsilon_i| > A)$, then the above inequality can be written as

$$
\mathbb{P}(\sum_{i=1}^n x_{ij} \epsilon_i > \sqrt{nt_{p,\alpha}}) \leq P_1 + P_2. \tag{B.3}
$$

By inequality (B.2) with $A = 3w_1 \log p$, we obtain that

$$
P_2 \leq \sum_{i=1}^n \mathbb{P}(|\epsilon_i| > A) \leq C_1 n \exp \{-3 \log p\} = C_1 n/p^3. \tag{B.4}
$$

To estimate the $P_1$, we need the following Sakhanenko type moderate deviation theorem of (Sakhanenko, 1991), i.e.

**Lemma Appendix B.1.** Let $\eta_1, \cdots, \eta_n$ be independent random variables with $\mathbb{E}\eta_i = 0$ and $|\eta_i| < 1$ for all $i \in [n]$. Denote $\sigma_n^2 = \sum_{i=1}^n \mathbb{E}\eta_i^2$ and $T_n = \sum_{i=1}^n \mathbb{E}|\eta_i|^3/\sigma_n^3$. Then there exists a positive constant $D$ such that for all $x \in [1, \frac{1}{D} \min\{\sigma_n, L_n^{-1/3}\}]$

$$
\mathbb{P}(\sum_{i=1}^n \eta_i > x\sigma_n) = (1 + O(1)x^3T_n)\tilde{\Phi}(x),
$$

where $\tilde{\Phi}(x) = 1 - \Phi(x)$ and $\Phi(x)$ is the cumulative distribution function of standard normal distribution.
Since \( \mathbb{E}(\epsilon_i) = \mathbb{E}(\hat{\epsilon}_i) + \mathbb{E}(\hat{\epsilon}_i) = 0 \), then it is easy to obtain that

\[
|\mathbb{E}\hat{\epsilon}_i| = |\mathbb{E}\hat{\epsilon}_i| \leq \mathbb{E}|\hat{\epsilon}_i| = \mathbb{E}|\epsilon_i| 1_{|\epsilon_i| > A} = \int_A^{+\infty} z dF(z) + \int_{-\infty}^{-A} -zdF(z)
\]

\[
= \left\{ z(F(z) - 1)\right\} |\mathcal{A}^{+\infty} - \int_A^{+\infty} (F(z) - 1) dz \right\} + \left\{ \int_{-\infty}^{-A} F(z) dz - zF(z)|^{-A}_{-\infty} \right\}
\]

\[
\leq A(1 - F(A)) + \int_A^{+\infty} C_1 \exp \{-z/w_1\} dz + \int_{-\infty}^{-A} C_1 \exp \{z/w_1\} dz + AF(-A)
\]

\[
\leq C_1(A + 2w_1) \exp \{-A/w_1\} \leq 2C_1 \exp \{-A/w_1\} = \frac{6C_1w_1 \log p}{p^3},
\]

where the last second and third inequalities utilize the relations \( F(a) = \mathbb{P}(\epsilon_i > a) \leq C_1 \exp \{-a/w_1\} \) and \( F(-a) = \mathbb{P}(\epsilon_i < -a) \leq C_1 \exp \{-a/w_1\} \) for any \( a > 0 \). Denote \( m = 6C_1w_1 \log p/p^3 \), then \( |\mathbb{E}\hat{\epsilon}_i| \leq m \) and \( m = o(n^{-2}) \).

Since

\[
P_1 = \mathbb{P}(\sum_{i=1}^{n} x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i) > \sqrt{nt_{p,a}})
\]

\[
\leq \mathbb{P}(\sum_{i=1}^{n} x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)) > \sqrt{nt_{p,a}} - |\sum_{i=1}^{n} x_{ij}\mathbb{E}\hat{\epsilon}_i|,
\]

we need to estimate \( |x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)| \) and \( |\sum_{i=1}^{n} x_{ij}\mathbb{E}\hat{\epsilon}_i| \). By condition C2,

\[
|x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)| \leq (\sup_{i[n], j[p]} |x_{ij}|)(|\hat{\epsilon}_i| + |\mathbb{E}\hat{\epsilon}_i|) \leq R(A + m).
\]

By Cauchy-Schwarz inequality,

\[
|\sum_{i=1}^{n} x_{ij}\mathbb{E}\hat{\epsilon}_i| \leq \sqrt{(\sum_{i=1}^{n} x_{ij}^2)(\sum_{i=1}^{n} |\mathbb{E}\hat{\epsilon}_i|^2)} \leq nm.
\]

Denoting \( \eta_{ij} = x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)/R(A + m) \), we have \( \mathbb{E}\eta_{ij} = 0 \) and \( |\eta_{ij}| < 1 \). Notice that \( \mathbb{E}\hat{\epsilon}_i^2 = \mathbb{E}\epsilon_i^2 = 1 \). Denoting \( \sigma_{nj}^2 = \sum_{i=1}^{n} \mathbb{E}\eta_{ij}^2 \) and \( T_{nj} = \sum_{i=1}^{n} \mathbb{E}|\eta_{ij}|^3/\sigma_{nj}^3 \), we have
\[ \sigma_{nj}^2 = \frac{1}{R^2(A + m)^2} \sum_{i=1}^{n} \mathbb{E}(x_{ij}^2(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)^2) \leq \frac{1}{R^2(A + m)^2} \sum_{i=1}^{n} x_{ij}^2 \mathbb{E}\hat{\epsilon}_i^2 \leq \frac{1}{R^2(A + m)^2} \sum_{i=1}^{n} x_{ij}^2 \]

\[ = \frac{n}{R^2(A + m)^2}, \]

\[ T_{nj} \leq \sum_{i=1}^{n} \mathbb{E}|\eta_{ij}|^2/\sigma_{nj}^3 = \frac{1}{\sigma_{nj}}. \]

Hence, \( \sigma_{nj}^2 = O\left(\frac{n}{(A+m)^2}\right) \) and \( L_{nj} = O\left(\frac{A+m}{\sqrt{n}}\right) \). By inequality (B.5) and Lemma Appendix B.1, for large enough \( n, p \) such that \( n \leq p \leq o(\exp\{n^{1/5}\}) \) (condition C3), we have

\[ P_1 \leq \mathbb{P}\left(|\sum_{i=1}^{n} x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)| > \frac{\sqrt{n}}{R(A + m)}(t_{p,\alpha} - \sqrt{nm})\right) \]

\[ \leq \mathbb{P}\left(|\sum_{i=1}^{n} \eta_{ij}| > \sigma_{nj}(t_{p,\alpha} - \sqrt{nm})\right) \]

\[ = 2 \left(1 + O(1)(t_{p,\alpha} \cdot \sqrt{nm})^3 T_{nj}\right) \tilde{\Phi}(t_{p,\alpha} - \sqrt{nm}) \]

with \( t_{p,\alpha} - \sqrt{nm} \) uniformly in \([1, O(n^{1/6}(\log p)^{-1/3})]\). Next, we estimate \( O(1)(t_{p,\alpha} - \sqrt{nm})^3 T_{nj} \) and \( \tilde{\Phi}(t_{p,\alpha} - \sqrt{nm}) \) respectively. Notice that \( \log (p/\alpha) < t_{p,\alpha}^2 < 2 \log (2p/\alpha) \) when \( p/\alpha > 8 \).

Then, under condition C3, we have

\[ O(1)(t_{p,\alpha} - \sqrt{nm})^3 T_{nj} = O(1)(\sqrt{2 \log (2p/\alpha) - \sqrt{nm}})^3 n^{-1/2}(3w_{1} \log p + m). \]

Furthermore, by the fact that for all \( a > 0 \) the inequality \( \frac{a}{1+a} \phi(a) \leq \tilde{\Phi}(a) \leq \frac{\phi(a)}{a} \) holds

16
where \( \phi(\cdot) \) is the density function of standard normal distribution, we have

\[
\Phi(t_{p,\alpha} - \sqrt{nm}) \leq \frac{\Phi(t_{p,\alpha} - \sqrt{nm})}{t_{p,\alpha} - \sqrt{nm}} = \phi(t_{p,\alpha}) \frac{1 + t_{p,\alpha}^2}{t_{p,\alpha}(t_{p,\alpha} - \sqrt{nm})} \exp\{t_{p,\alpha}\sqrt{nm} - nm^2/2\} \\
\leq \Phi(t_{p,\alpha}) \frac{1 + t_{p,\alpha}^2}{t_{p,\alpha}(t_{p,\alpha} - \sqrt{nm})} \exp\{t_{p,\alpha}\sqrt{nm} - nm^2/2\} \\
= \frac{\alpha}{2p} \left(1 + \frac{1}{\log (p/\alpha)}\right) \frac{1}{1 - \sqrt{nm}/(\log (p/\alpha))^{1/2}} \exp\{(n \log (p/\alpha))^{1/2}m - nm^2/2\}.
\]

Combining (B.6), (B.7) and (B.8), we have

\[
P_1 \leq \frac{\alpha}{p} \left(1 + O(1)(\sqrt{2 \log (2p/\alpha)} - \sqrt{nm})^3 n^{-1/2}(3w_1 \log p + m)\right) \\
\times (1 + \frac{1}{\log (p/\alpha)}\frac{1}{1 - \sqrt{nm}/(\log (p/\alpha))^{1/2}} \exp\{(n \log (p/\alpha))^{1/2}m - nm^2/2\}).
\]

Hence, combining (B.1), (B.3), (B.4) and (B.9), we have

\[
P(\lambda < cV) \leq p(P_1 + P_2) \\
\leq \alpha \left(1 + O(1)(\sqrt{2 \log (2p/\alpha)} - \sqrt{nm})^3 n^{-1/2}(3w_1 \log p + m)\right) \\
\times (1 + \frac{1}{\log (p/\alpha)}\frac{1}{1 - \sqrt{nm}/(\log (p/\alpha))^{1/2}} \exp\{(n \log (p/\alpha))^{1/2}m - nm^2/2\} + C_1 n/p^2,
\]

where \( C_1 \) and \( w_1 \) are some positive constants.

So, the probability of event \( \{\lambda \geq cV\} \) is

\[
P(\lambda \geq cV) \geq 1 - \alpha \left(1 + O(1)(\sqrt{2 \log (2p/\alpha)} - \sqrt{nm})^3 n^{-1/2}(3w_1 \log p + m)\right) \\
\times (1 + \frac{1}{\log (p/\alpha)}\frac{1}{1 - \sqrt{nm}/(\log (p/\alpha))^{1/2}} \exp\{(n \log (p/\alpha))^{1/2}m - nm^2/2\} \right) - C_1 n/p^2.
\]

Additionally, notice that \( m, \sqrt{nm} \) and \( nm^2 \) are \( o(n^{-2}) \). As \( n, p \to \infty \) with \( n \leq p \leq o(\exp \{n^{1/5}\}) \), it is easy to obtain that

\[
P(\lambda \geq cV) \leq 1 - \alpha(1 + o(1)).
\]
Appendix C. Table

References

Aeberhard, W.H., Cantoni, E., Heritier, S., 2014. Robust inference in the negative binomial regression model with an application to falls data. Biometrics 70, 920 – 931.

Algamal, Z., Lee, M., 2015. Penalized poisson regression model using adaptive modified elastic net penalty. Electronic Journal of Applied Statistical Analysis 8, 236–245.

An, Q., Wu, J., Fan, X., Pan, L., Sun, W., 2016. Using a negative binomial regression model for early warning at the start of a hand foot mouth disease epidemic in dalian, liaoning province, china. PLoS One 11, e0157815.

Bickel, P.J., Ritov, Y., Tsybakov, A.B., 2009. Simultaneous analysis of lasso and dantzig selector. The Annals of Statistics 37, 1705–1732.

Cameron, A.C., Trivedi, P.K., 1996. 12 count data models for financial data. Handbook of Statistics 14, 363 – 391.

Cameron, A.C., Trivedi, P.K., 1998. Regression Analysis of Count Data. Cambridge University Press, Cambridge, U.K.

Cupal, M., Deev, O., Linnertova, D., 2015. The poisson regression analysis for occurrence of floods. Procedia Economics and Finance 23, 1499–1502.

Gardner, W., Mulvey, E.P., Shaw, E.C., 1995. Regression analyses of counts and rates: Poisson, overdispersed poisson, and negative binominal models. Psychological Bulletin 118, 392–404.

Greene, W.H., 1994. Accounting for Excess Zeros and Sample Selection in Poisson and Negative Binominal Regression Models. Working Papers.

Hilbe, J.M., 2011. Modeling Count Data. Springer Berlin Heidelberg, Berlin, Heidelberg.
Ivanoff S, Picard F, R.V., 2016. Adaptive lasso and group-lasso for functional poisson regression. Journal of Machine Learning Research 17, 1–46.

Jia, J., Xie, F., Xu, L., 2019. Sparse poisson regression with penalized weighted score function. Electronic Journal of Statistics 13, 2898–2920.

Li, Y.H., Cevher, V., 2015. Consistency of \( \ell_1 \)-regularized maximum-likelihood for compressive poisson regression, in: 2015 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pp. 3606–3610.

Lindén, A., Mäntyniemi, S., 2011. Using the negative binomial distribution to model overdispersion in ecological count data. Ecology 92, 1414–1421.

Lu, H.X., Wong, M.C.M., Lo, E.C.M., McGrath, C., 2013. Risk indicators of oral health status among young adults aged 18 years analyzed by negative binomial regression. BMC Oral Health 13, 40.

Mi, G., Di, Y., Schafer, D.W., 2015. Goodness-of-fit tests and model diagnostics for negative binomial regression of rna sequencing data. PLoS One 10, 1–16.

Riphahn, R.T., Wambach, A., Million, A., 2003. Incentive effects in the demand for health care: a bivariate panel count data estimation. Journal of Applied Econometrics 18, 387–405.

Sakhanenko, A.I., 1991. Berry-esseen type estimates for large deviation probabilities. Siberian Mathematical Journal 32, 647–656.

Stefany, C., Stephen, G.W., Leona, S.A., 2009. The analysis of count data: A gentle introduction to poisson regression and its alternatives. Journal of Personality Assessment 91, 121–136.

Tibshirani, R., 1996. Regression shrinkage and selection via the lasso. Journal of the Royal Statistical Society. Series B (Methodological) 58, 267–288.
Walters, G.D., 2007. Using poisson class regression to analyze count data in correctional and forensic psychology. Criminal Justice and Behavior 34, 1659–1674.

Wang, Z., Ma, S., Zappitelli, M., Parikh, C., Wang, C.Y., Devarajan, P., 2016. Penalized count data regression with application to hospital stay after pediatric cardiac surgery. Statistical Methods in Medical Research 25, 2685–2703.

Washington, S.P., Karlaftis, M.G., F., M., 2010. Statistical and econometric methods for transportation data analysis. CRC press.

Wei, F., Lovegrove, G., 2013. An empirical tool to evaluate the safety of cyclists: Community based, macro-level collision prediction models using negative binomial regression. Accident Analysis & Prevention 61, 129–137.
| Variables | Description | Mean | SD   |
|-----------|-------------|------|------|
| DOCVIS    | number of doctor visits in last three months | 3.184 | 5.69 |
| HOSPVIS   | number of hospital visits in last calendar year | 0.138 | 0.884 |
| ID        | person - identification number, 1, · · · , 7293 |      |      |
| FEMALE    | female = 1; male = 0 | 0.479 | 0.5  |
| 1984      | Year = 1984 (0/1) | 0.142 | 0.349 |
| 1985      | Year = 1985 (0/1) | 0.139 | 0.346 |
| 1986      | Year = 1986 (0/1) | 0.139 | 0.346 |
| 1987      | Year = 1987 (0/1) | 0.134 | 0.341 |
| 1988      | Year = 1988 (0/1) | 0.164 | 0.37  |
| 1991      | Year = 1991 (0/1) | 0.159 | 0.366 |
| 1994      | Year = 1994 (0/1) | 0.124 | 0.329 |
| AGE       | age in years | 43.526 | 11.33 |
| HSAT      | health satisfaction, coded 0 (low) - 10 (high) | 6.785 | 2.294 |
| HANDDUM   | handicapped = 1; otherwise = 0 | 0.214 | 0.41  |
| HANDPER   | degree of handicap in percent (0 - 100) | 7.012 | 19.265 |
| HHNINC    | household nominal monthly net income in German marks / 1000 | 3.521 | 1.769 |
| HHKIDS    | children under age 16 in the household = 1; otherwise = 0 | 0.403 | 0.49  |
| EDUC      | years of schooling | 11.321 | 2.325 |
| MARRIED   | married = 1; otherwise = 0 | 0.759 | 0.428 |
| HAUPTS    | highest schooling degree is Hauptschul degree = 1; otherwise = 0 | 0.624 | 0.484 |
| REALS     | highest schooling degree is Realschul degree = 1; otherwise = 0 | 0.197 | 0.398 |
| FACHHS    | highest schooling degree is Polytechnical degree = 1; otherwise = 0 | 0.041 | 0.198 |
| ABITUR    | highest schooling degree is Abitur = 1; otherwise = 0 | 0.117 | 0.321 |
| UNIV      | highest schooling degree is university degree = 1; otherwise = 0 | 0.072 | 0.258 |
| WORKING   | employed = 1; otherwise = 0 | 0.677 | 0.468 |
| BLUEC     | blue collar employee = 1; otherwise = 0 | 0.244 | 0.429 |
| WHITEC    | white collar employee = 1; otherwise = 0 | 0.3  | 0.458 |
| SELF      | self employed = 1; otherwise = 0 | 0.062 | 0.241 |
| BEAMT     | civil servant = 1; otherwise = 0 | 0.075 | 0.263 |
| PUBLIC    | insured in public health insurance = 1; otherwise = 0 | 0.886 | 0.318 |
| ADDON     | insured by add-on insurance = 1; otherwise = 0 | 0.019 | 0.136 |
SUPPLEMENTARY MATERIAL TO "CONSISTENCY OF $\ell_1$ PENALIZED NEGATIVE BINOMIAL REGRESSIONS"

FANG XIE AND ZHIJIE XIAO

APPENDIX A. PROOF OF THEOREM 2.1

Let $\delta = \tilde{\beta} - \beta^*$. Recall that $S = \{ j : \beta^*_j \neq 0 \}$. By definition of $\tilde{\beta}$ and the convexity of $L(\beta)$, we have

$$L(\tilde{\beta}) - L(\beta^*) \leq \lambda(\|\beta^*\|_1 - \|\tilde{\beta}\|_1)$$

(A.1)

$$= \lambda[(\|\beta^*_S\|_1 - \|\tilde{\beta}_S\|_1) + (\|\beta^*_S^c\| - \|\tilde{\beta}_S^c\|_1)]$$

$$\leq \lambda(\|\delta_S\|_1 - \|\delta_{S^c}\|_1),$$

and

(A.2)

$$L(\tilde{\beta}) - L(\beta^*) \geq -V\|\delta\|_1 \geq \frac{\lambda}{c}\|\delta\|_1,$$

where $V = \|\nabla L(\beta^*)\|_\infty$, and the last inequality utilizes the condition $\lambda > cV$. Combining (A.1) and (A.2), we obtain that

$$\|\delta_{S^c}\|_1 \leq \frac{c+1}{c-1}\|\delta_S\|_1,$$

which makes the condition C4 hold with $\gamma = \frac{c+1}{c-1} > 1$.

For any $\beta, u, v \in \mathbb{R}^p$, we have

$$\nabla^2 L(\beta)[v, v] = \frac{1}{n} \sum_{i=1}^{n} (x_i^T v)^2 r \exp\{x_i^T \beta\}(y_i + r) (r + \exp\{x_i^T \beta\})^3$$

$$\nabla^3 L(\beta)[u, v, v] = \frac{1}{n} \sum_{i=1}^{n} x_i^T u (x_i^T v)^2 r \exp\{x_i^T \beta\}(y_i + r)(r - \exp\{x_i^T \beta\}) (r + \exp\{x_i^T \beta\})^3.$$

Under the assumption (C2), it is easy to verify that

$$|\nabla^3 L(\beta)[u, v, v]| \leq \sup_{i \in [n]} |x_i^T u| |\nabla^2 L(\beta)[v, v]|$$

$$\leq R\|u\|_1 |\nabla^2 L(\beta)[v, v]|$$

$$\leq R |\nabla^2 L(\beta)[v, v]|.$$

Setting $u = \delta = \tilde{\beta} - \beta^*$, we have

(A.3)

$$|\nabla^3 L(\beta)[u, v, v]| \leq R(1 + \frac{c+1}{c-1})\|\delta\|_1 |\nabla^2 L(\beta)[v, v]|$$

$$\leq \frac{2cR\sqrt{s}}{c-1} |\delta_S^2| |\nabla^2 L(\beta)[v, v]|.$$
Denoting $\tilde{R} = \frac{2\sqrt{R\lambda}}{c-1}$, (A.3) becomes $|\nabla^3L(\beta)[u, v, v]| \leq \tilde{R}\|\delta_s\|_2^2\nabla^2L(\beta)[v, v]$. Thus, $L(\cdot)$ is a self-concordant like function with parameter $\tilde{R}$ with respect to $l_2$-norm. By Theorem 6.1 of [13] and condition C4, we have

$$L(\hat{\beta}) - L(\beta^*) \geq \delta^T \nabla L(\beta^*) + \frac{\delta^T \nabla^2 L(\beta^*) \delta}{\tilde{R}^2\|\delta_s\|_2^2} (\exp \{-\tilde{R}\|\delta_s\|_2\} + \tilde{R}\|\delta_s\|_2 - 1)$$

(A.4)

$$\geq -\|\nabla L(\beta^*)\|_\infty \|\delta\|_1 + \frac{\delta^T \nabla^2 L(\beta^*) \delta}{\tilde{R}^2\|\delta_s\|_2^2} (\exp \{-\tilde{R}\|\delta_s\|_2\} + \tilde{R}\|\delta_s\|_2 - 1)$$

$$\geq -\frac{\lambda}{c}\|\delta\|_1 + \frac{\delta^T \nabla^2 L(\beta^*) \delta}{\tilde{R}^2\|\delta_s\|_2^2} (\exp \{-\tilde{R}\|\delta_s\|_2\} + \tilde{R}\|\delta_s\|_2 - 1)$$

Combining (A.1) and (A.4), we have

$$\frac{\delta^T \nabla^2 L(\beta^*) \delta}{\tilde{R}^2\|\delta_s\|_2^2} (\exp \{-\tilde{R}\|\delta_s\|_2\} + \tilde{R}\|\delta_s\|_2 - 1)$$

(A.5)

$$\leq \lambda\|\delta_s\|_1 + \frac{\lambda}{c}\|\delta\|_1 \leq \frac{c+1}{c-1} \lambda\|\delta_s\|_1 \leq \frac{c+1}{c-1} \lambda \sqrt{s}\|\delta_s\|_2.$$  

By condition C4 and (A.5), we have

$$\exp \{-\tilde{R}\|\delta_s\|_2\} + \tilde{R}\|\delta_s\|_2 - 1 \leq \frac{(c+1)\lambda \sqrt{s}\tilde{R}}{(c-1)\phi_0^2} \|\delta_s\|_2.$$  

Set

$$h = \frac{(c+1)\lambda \sqrt{s}\tilde{R}}{(c-1)\phi_0^2} = \frac{2c(c+1)\lambda R \sqrt{s}}{(c-1)^2\phi_0^2},$$

then according to the condition on $\lambda$ such that $\lambda s \leq \frac{(c-1)^2\phi_0^2}{6cR(c+1)}$, we have $h \leq \frac{1}{3}$. Denote $w = \tilde{R}\|\delta_s\|_2$, then to solve (A.6) is equivalent to solve the inequality $\exp \{-w\} + w - 1 \leq hw$. By Taylor formula, we have $w^2 - \frac{w^3}{6} \leq \exp \{-w\} + w - 1 \leq hw$ which implies $\{w : \exp \{-w\} + w - 1 \leq hw, h \leq \frac{1}{3}\} \subseteq \{w : \frac{w^2}{2} - \frac{w^3}{6} \leq hw, h \leq \frac{1}{3}\}$. Since under the condition $h \leq \frac{1}{3}$, the solution of inequality $\frac{w^2}{2} - \frac{w^3}{6} \leq hw$ is $w \leq c_1h$ for some constant $c_1 \in (2, 3]$, then

$$\{w : \exp \{-w\} + w - 1 \leq hw, h \leq \frac{1}{3}\} \subseteq \{w \leq c_1h \text{ with some constant } c_1 \in (2, 3]\}.$$  

So, from (A.6), we obtain

$$\tilde{R}\|\delta_s\|_2 \leq \frac{c_1\lambda \sqrt{s}(c+1)}{\phi_0^2(c-1)} \tilde{R},$$

that is,

$$\|\delta_s\|_2 \leq \frac{c_1\lambda \sqrt{s}(c+1)}{\phi_0^2(c-1)}.$$  

(A.8)

Hence, notice the relationship $\|\delta\|_1 \leq (1 + \frac{c_1c+c+1}{c-1}) \sqrt{s}\|\delta_s\|_2 = \frac{2c\sqrt{s}}{c-1}\|\delta_s\|_2$, by (A.8) we have

$$\|\delta\|_1 \leq \frac{2c_1c(c+1)}{(c-1)^2\phi_0^2} \lambda s.$$  

(A.9)
Then, (2.2) is obtained. Furthermore, by (A.1) and (A.2), we obtain

\[
|L(\hat{\beta}) - L(\beta^*)| \leq \lambda \|\delta\|_1 \leq \frac{2c_1c(c + 1)}{(c - 1)^2\phi_0^2} \lambda^2 s,
\]

which implies (2.3). We finish the proof. □

**APPENDIX B. PROOF OF LEMMA 2.3**

Recall the gradient of \( L(\beta) \) at the point \( \beta = \beta^* \)

\[
\nabla L(\beta^*) = -\frac{1}{n} \sum_{i=1}^{n} x_i \times v_i \times \frac{(y_i - \exp\{x_i^\top \beta^*\})}{\sqrt{\exp\{x_i^\top \beta^*\}(r + \exp\{x_i^\top \beta^*\})}},
\]

where

\[
v_i = \sqrt{\frac{r \exp\{x_i^\top \beta^*\}}{\exp\{x_i^\top \beta^*\} + r}}.
\]

Denote \( \epsilon_i = (y_i - \exp\{x_i^\top \beta^*\})/\sqrt{\exp\{x_i^\top \beta^*\}(r + \exp\{x_i^\top \beta^*\})}/r \), then

\[
V = \|\nabla L(\beta^*)\|_{\infty} = \max_{j \in [p]} \left| \frac{1}{n} \sum_{i=1}^{n} x_{ij} v_i \epsilon_i \right|.
\]

Recall \( v_n = \max_{1 \leq i \leq n} v_i \) and denote \( t_{p,\alpha} = \Phi^{-1}(1 - \frac{\alpha}{2p}) \), then \( \lambda = cv_n(\sqrt{n})^{-1}t_{p,\alpha} \). Hence

\[
P(cV > \lambda) = P(\max_{j \in [p]} \left| \frac{1}{n} \sum_{i=1}^{n} x_{ij} v_i \epsilon_i \right| > v_n(\sqrt{n})^{-1}t_{p,\alpha})
\]

\[
\leq P(\max_{j \in [p]} \left| \frac{1}{n} \sum_{i=1}^{n} x_{ij} \epsilon_i \right| > (\sqrt{n})^{-1}t_{p,\alpha})
\]

\[
\leq p \max_{j \in [p]} P(\left| \sum_{i=1}^{n} x_{ij} \epsilon_i \right| > \sqrt{nt_{p,\alpha}}).
\]

(B.1)

Since \( y_i|x_i \sim NB(r, \mu_i) \) with \( \mu_i = \exp\{x_i^\top \beta^*\} \), then

\[
E(\exp\{\theta \epsilon_i\}) = \exp\left\{ -\theta \sqrt{\frac{r \mu_i}{\mu_i + r}} \right\} \left( 1 + \frac{\mu_i}{r} \left( 1 - \exp\left\{ \theta \sqrt{\frac{r}{\mu_i(\mu_i + r)}} \right\} \right) \right)^{-r}
\]

is a positive constant for all \( \theta < \sqrt{\frac{\mu_i(\mu_i + r)}{r \mu_i}} \ln(\frac{r}{\mu_i} + 1) \). By the exponential Chebyshev’s inequality, we have

\[
P(|\epsilon_i| > A) < \exp\left\{ -A/w_1 \right\} E(\exp\{\epsilon_i/w_1\}) = C_1 \exp\left\{ -A/w_1 \right\}
\]

(B.2)
with some constant $C_1 = \mathbb{E}(\exp \{ \epsilon_i/w_1 \}) > 0$ and $w_1 > \left( \sqrt{\frac{\mu_1(\mu_1 + r)}{\mu_1}} \ln(\frac{\mu_1}{\mu_1} + 1) \right)$. Denote $\hat{\epsilon}_i = \epsilon_i 1_{|\epsilon_i| \leq A}$ and $\tilde{\epsilon}_i = \epsilon_i 1_{|\epsilon_i| > A}$. Taking $A = 3w_1 \log p$, we have

$$\mathbb{P}(\sum_{i=1}^n x_{ij} \epsilon_i > \sqrt{n} t_{p,\alpha}) = \mathbb{P}(\sum_{i=1}^n x_{ij} (\hat{\epsilon}_i + \tilde{\epsilon}_i) > \sqrt{n} t_{p,\alpha}, \sup_{i \in [n]} |\epsilon_i| \leq A)$$

$$+ \mathbb{P}(\sum_{i=1}^n x_{ij} (\hat{\epsilon}_i + \tilde{\epsilon}_i) > \sqrt{n} t_{p,\alpha}, \sup_{i \in [n]} |\epsilon_i| > A)$$

$$\leq \mathbb{P}(\sum_{i=1}^n x_{ij} \hat{\epsilon}_i > \sqrt{n} t_{p,\alpha}) + \mathbb{P}(\sup_{i \in [n]} |\epsilon_i| > A).$$

Denote $P_1 = \mathbb{P}(\sum_{i=1}^n x_{ij} \hat{\epsilon}_i > \sqrt{n} t_{p,\alpha})$ and $P_2 = \mathbb{P}(\sup_{i \in [n]} |\epsilon_i| > A)$, then the above inequality can be written as

(B.3) \[ \mathbb{P}(\sum_{i=1}^n x_{ij} \epsilon_i > \sqrt{n} t_{p,\alpha}) \leq P_1 + P_2. \]

By inequality (B.2) with $A = 3w_1 \log p$, we obtain that

(B.4) \[ P_2 \leq \sum_{i=1}^n \mathbb{P}(|\epsilon_i| > A) \leq C_1 n \exp \{ -3 \log p \} = C_1 n/p^3. \]

To estimate the $P_1$, we need the following Sakhanenko type moderate deviation theorem of (2), i.e.

**Lemma B.1.** Let $\eta_1, \ldots, \eta_n$ be independent random variables with $\mathbb{E}\eta_i = 0$ and $|\eta_i| < 1$ for all $i \in [n]$. Denote $\sigma_n^2 = \sum_{i=1}^n \mathbb{E}\eta_i^2$ and $T_n = \sum_{i=1}^n \mathbb{E}|\eta_i|^3/\sigma_n^3$. Then there exists a positive constant $D$ such that for all $x \in [1, 1/D \min\{\sigma_n, L_n^{-1/3}\}]$

$$\mathbb{P}(\sum_{i=1}^n \eta_i > x\sigma_n) = (1 + O(1)x^3T_n)\Phi(x),$$

where $\Phi(x) = 1 - \Phi(x)$ and $\Phi(x)$ is the cumulative distribution function of standard normal distribution.

Since $\mathbb{E}(\epsilon_i) = \mathbb{E}(\hat{\epsilon}_i) + \mathbb{E}(\tilde{\epsilon}_i) = 0$, then it is easy to obtain that

$$|\hat{\epsilon}_i| = |\mathbb{E}(\hat{\epsilon}_i)| \leq \mathbb{E}(|\epsilon_i| 1_{|\epsilon_i| > A}) = \int_A^{+\infty} zdF(z) + \int_{-\infty}^{-A} -zdF(z)$$

$$= \left\{ z(F(z) - 1) \right\}_{A}^{+\infty} - \int_A^{+\infty} (F(z) - 1)dz + \left\{ \int_{-\infty}^{-A} F(z)dz - zF(z) \right\}_{-\infty}^{-A}$$

$$\leq A(1 - F(A)) + \int_A^{+\infty} C_1 \exp \{-z/w_1\}dz + \int_{-\infty}^{-A} C_1 \exp \{z/w_1\}dz + AF(-A)$$

$$\leq C_1(A + 2w_1) \exp \{-A/w_1\} \leq 2C_1 A \exp \{-A/w_1\} = \frac{6C_1 w_1 \log p}{p^3},$$
where the last second and third inequalities utilize the relations $F(a) = \mathbb{P}(\epsilon_i > a) \leq C_1 \exp \{-a/w_1\}$ and $F(-a) = \mathbb{P}(\epsilon_i < -a) \leq C_1 \exp \{-a/w_1\}$ for any $a > 0$. Denote $m = 6C_1 w_1 \log p/p^3$, then $|\mathbb{E}\hat{e}_i| \leq m$ and $m = o(n^{-2})$.

Since
\[ P_1 = \mathbb{P}(\sum_{i=1}^{n} x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i + \mathbb{E}\hat{\epsilon}_i)) > \sqrt{n}t_{p,\alpha} \]
(B.5)
\[ \leq \mathbb{P}(\sum_{i=1}^{n} x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)) > \sqrt{n}t_{p,\alpha} - \sum_{i=1}^{n} x_{ij}\mathbb{E}\hat{\epsilon}_i|, \]
we need to estimate $|x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)|$ and $\sum_{i=1}^{n} x_{ij}\mathbb{E}\hat{\epsilon}_i|$. By condition C2,
\[ |x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)| \leq (\sup_{i\in[n], j\in[p]} |x_{ij}|) (|\hat{\epsilon}_i| + |\mathbb{E}\hat{\epsilon}_i|) \leq R(A + m). \]
By Cauchy-Schwarz inequality,
\[ \left| \sum_{i=1}^{n} x_{ij}\mathbb{E}\hat{\epsilon}_i \right| \leq \sqrt{\left( \sum_{i=1}^{n} x_{ij}^2 \right) \left( \sum_{i=1}^{n} |\mathbb{E}\hat{\epsilon}_i|^2 \right)} \leq nm. \]

Denoting $\eta_{ij} = x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)/R(A + m)$, we have $\mathbb{E}\eta_{ij} = 0$ and $|\eta_{ij}| < 1$. Notice that $\mathbb{E}\hat{\epsilon}_i^2 = \mathbb{E}\epsilon_i^2 = 1$. Denoting $\sigma_{nj}^2 = \sum_{i=1}^{n} \mathbb{E}\eta_{ij}^2$ and $T_{nj} = \sum_{i=1}^{n} \mathbb{E}|\eta_{ij}|^3/\sigma_{nj}^3$, we have
\[ \sigma_{nj}^2 = \frac{1}{R^2(A + m)^2} \sum_{i=1}^{n} \mathbb{E}(x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i)^2) \]
\[ \leq \frac{1}{R^2(A + m)^2} \sum_{i=1}^{n} x_{ij}^2 \mathbb{E}\hat{\epsilon}_i^2 \leq \frac{1}{R^2(A + m)^2} \sum_{i=1}^{n} x_{ij}^2 \]
\[ = \frac{n}{R^2(A + m)^2}, \]
\[ T_{nj} \leq \sum_{i=1}^{n} \mathbb{E}|\eta_{ij}|^2/\sigma_{nj}^3 = \frac{1}{\sigma_{nj}^2}. \]
Hence, $\sigma_{nj}^2 = O\left(\frac{n}{(A+m)^2}\right)$ and $L_{nj} = O\left(\frac{A+m}{\sqrt{n}}\right)$. By inequality (B.5) and Lemma B.1, for large enough $n, p$ such that $n \leq p \leq o(\exp\{n^{1/3}\})$(condition C3), we have
\[ P_1 \leq \mathbb{P}(\frac{\sum_{i=1}^{n} x_{ij}(\hat{\epsilon}_i - \mathbb{E}\hat{\epsilon}_i))}{R(A + m)} > \frac{\sqrt{n}}{R(A + m)}(t_{p,\alpha} - \sqrt{nm})) \]
(B.6)
\[ \leq \mathbb{P}(\sum_{i=1}^{n} \eta_{ij}) > \sigma_{nj}(t_{p,\alpha} - \sqrt{nm})) \]
\[ = 2 \left( 1 + O(1) (t_{p,\alpha} - \sqrt{nm})^3 T_{nj} \right) \Phi(t_{p,\alpha} - \sqrt{nm}) \]
with $t_{p,\alpha} - \sqrt{nm}$ uniformly in $[1, O(n^{1/6}(\log p)^{-1/3})]$. Next, we estimate $O(1)(t_{p,\alpha} - \sqrt{nm})^3 T_{nj}$ and $\Phi(t_{p,\alpha} - \sqrt{nm})$ respectively. Notice that $\log (p/\alpha) < t_{p,\alpha}^2 < 2 \log (2p/\alpha)$ when $p/\alpha > 8$. 
Then, under condition \( C3 \), we have
\[
O(1)(t_{p,\alpha} - \sqrt{nm})^3 T_{nj} = O(1)(\sqrt{2 \log (2p/\alpha)} - \sqrt{nm})^3 n^{-1/2}(3w_1 \log p + m).
\] (B.7)

Furthermore, by the fact that for all \( a > 0 \) the inequality \( \frac{a}{1+a}\phi(a) \leq \Phi(a) \leq \frac{\phi(a)}{a} \) holds where \( \phi(\cdot) \) is the density function of standard normal distribution, we have
\[
\Phi(t_{p,\alpha} - \sqrt{nm}) \leq \frac{\phi(t_{p,\alpha} - \sqrt{nm})}{t_{p,\alpha} - \sqrt{nm}} = \phi(t_{p,\alpha}) \frac{1 + t_{p,\alpha}^2}{1 + t_{p,\alpha}^2 \phi(t_{p,\alpha})} \exp\{t_{p,\alpha} \sqrt{nm} - nm^2/2\}
\]
\[
= \Phi(t_{p,\alpha}) \frac{1 + t_{p,\alpha}^2}{t_{p,\alpha}(t_{p,\alpha} - \sqrt{nm})} \exp\{t_{p,\alpha} \sqrt{nm} - nm^2/2\}
\]
\[
\leq \Phi(t_{p,\alpha}) \left(1 + \frac{1}{t_{p,\alpha}}\right) \frac{1}{1 - \sqrt{nm}/(\log (p/\alpha))^{1/2}} \exp\{(n \log (p/\alpha))^{1/2}m - nm^2/2\}
\] (B.8)

Combining (B.6), (B.7) and (B.8), we have
\[
P_1 \leq \frac{\alpha}{p}(1 + O(1)(\sqrt{2 \log (2p/\alpha)} - \sqrt{nm})^3 n^{-1/2}(3w_1 \log p + m))
\]
\[
\times (1 + \frac{1}{\log (p/\alpha)}) \frac{\exp\{(n \log (p/\alpha))^{1/2}m - nm^2/2\}}{1 - \sqrt{nm}/(\log (p/\alpha))^{1/2}}.
\] (B.9)

Hence, combining (B.1), (B.3), (B.4) and (B.9), we have
\[
P(\lambda < cV) \leq p(P_1 + P_2)
\]
\[
\leq \alpha \left(1 + O(1)(\sqrt{2 \log (2p/\alpha)} - \sqrt{nm})^3 n^{-1/2}(3w_1 \log p + m))
\]
\[
\times (1 + \frac{1}{\log (p/\alpha)}) \frac{\exp\{(n \log (p/\alpha))^{1/2}m - nm^2/2\}}{1 - \sqrt{nm}/(\log (p/\alpha))^{1/2}} + C_1 n/p^2,
\]

where \( C_1 \) and \( w_1 \) are some positive constants.

So, the probability of event \( \{\lambda \geq cV\} \) is
\[
P(\lambda \geq cV) \geq 1 - \alpha \left(1 + O(1)(\sqrt{2 \log (2p/\alpha)} - \sqrt{nm})^3 n^{-1/2}(3w_1 \log p + m))
\]
\[
\times (1 + \frac{1}{\log (p/\alpha)}) \frac{\exp\{(n \log (p/\alpha))^{1/2}m - nm^2/2\}}{1 - \sqrt{nm}/(\log (p/\alpha))^{1/2}} - C_1 n/p^2.
\]

Additionally, notice that \( m, \sqrt{nm} \) and \( nm^2 \) are \( o(n^{-2}) \). As \( n, p \to \infty \) with \( n \leq p \leq o(\exp\{n^{1/5}\}) \), it is easy to obtain that
\[
P(\lambda \geq cV) \leq 1 - \alpha(1 + o(1)).
\]

**Appendix C. Table**
### Table S1. list of variables

| Variables | Description | Mean  | SD    |
|-----------|-------------|-------|-------|
| DOCVIS    | number of doctor visits in last three months | 3.184 | 5.69  |
| HOSPVIS   | number of hospital visits in last calendar year | 0.138 | 0.884 |
| ID        | person - identification number, 1, · · · , 7293 |     |       |
| FEMALE    | female = 1; male = 0 | 0.479 | 0.5   |
| 1984      | Year = 1984 (0/1) | 0.142 | 0.349 |
| 1985      | Year = 1985 (0/1) | 0.139 | 0.346 |
| 1986      | Year = 1986 (0/1) | 0.139 | 0.346 |
| 1987      | Year = 1987 (0/1) | 0.134 | 0.341 |
| 1988      | Year = 1988 (0/1) | 0.164 | 0.37  |
| 1991      | Year = 1991 (0/1) | 0.159 | 0.366 |
| 1994      | Year = 1994 (0/1) | 0.124 | 0.329 |
| AGE       | age in years | 43.526 | 11.33 |
| HSAT      | health satisfaction, coded 0 (low) - 10 (high) | 6.785 | 2.294 |
| HANDDUM   | handicapped = 1; otherwise = 0 | 0.214 | 0.41  |
| HANDPER   | degree of handicap in percent (0 - 100) | 7.012 | 19.265 |
| HHNINC    | household nominal monthly net income in German marks / 1000 | 3.521 | 1.769 |
| HHKIDS    | children under age 16 in the household = 1; otherwise = 0 | 0.403 | 0.49  |
| EDUC      | years of schooling | 11.321 | 2.325 |
| MARRIED   | married = 1; otherwise = 0 | 0.759 | 0.428 |
| HAUPTS    | highest schooling degree is Hauptschul degree = 1; otherwise = 0 | 0.624 | 0.484 |
| REALS     | highest schooling degree is Realschul degree = 1; otherwise = 0 | 0.197 | 0.398 |
| FACHHS    | highest schooling degree is Polytechnical degree = 1; otherwise = 0 | 0.041 | 0.198 |
| ABITUR    | highest schooling degree is Abitur = 1; otherwise = 0 | 0.117 | 0.321 |
| UNIV      | highest schooling degree is university degree = 1; otherwise = 0 | 0.072 | 0.258 |
| WORKING   | employed = 1; otherwise = 0 | 0.677 | 0.468 |
| BLUEC     | blue collar employee = 1; otherwise = 0 | 0.244 | 0.429 |
| WHITEC    | white collar employee = 1; otherwise = 0 | 0.3  | 0.458 |
| SELF      | self employed = 1; otherwise = 0 | 0.062 | 0.241 |
| BEAMT     | civil servant = 1; otherwise = 0 | 0.075 | 0.263 |
| PUBLIC    | insured in public health insurance = 1; otherwise = 0 | 0.886 | 0.318 |
| ADDON     | insured by add-on insurance = 1; otherwise = 0 | 0.019 | 0.136 |