Meta Subspace Optimization

Yoni Choukroun, Michael Katz
Huawei
choukroun.yoni@gmail.com, michael.katz@huawei.com

Abstract
Subspace optimization methods have the attractive property of reducing large-scale optimization problems to a sequence of low-dimensional subspace optimization problems. However, existing subspace optimization frameworks adopt a fixed update policy of the subspace, and therefore, appear to be sub-optimal. In this paper we propose a new Meta Subspace Optimization (MSO) framework for large-scale optimization problems, which allows to determine the subspace matrix at each optimization iteration. In order to remain invariant to the optimization problem’s dimension, we design an efficient meta optimizer based on very low-dimensional subspace optimization coefficients, inducing a rule-based agent that can significantly improve performance. Finally, we design and analyze a reinforcement learning procedure based on the subspace optimization dynamics whose learnt policies outperform existing subspace optimization methods.

Introduction
First-order optimization methods are popular in solving optimization problems. Gradient based frameworks are computationally efficient and are proven to converge in various settings ([Noc2006]). However, these methods suffer from inherent slowness due to the poor local linear approximation which is extremely inefficient in ill-conditioned problems.

One popular approach to address these problems is to resort to second-order methods, which can converge extremely fast to stationary points, especially in the proximity of the solution where the problem has a good quadratic approximation. However, when the number of variables is very large, there is a need for optimization algorithms whose storage requirement and computational cost per iteration grow at most linearly with the problem’s dimensions. This constraint has led to the development of a broad family of methods such as variable metric methods and subspace optimization.

Subspace optimization aims at iteratively finding the optimum in a low-dimensional subspace spanned by vectors obtained from a first-order oracle. These directions are generally updated at each new iteration by axiomatically removing the oldest direction of the subspace obtained during the iterative optimization.

Meta optimization is an emerging field of research, focused on learning optimization algorithms by training a parametrized optimizer on a distribution of tasks. These meta optimizers extend the axiomatic update rules of classical first-order methods with high-dimensional parametrized models which provide better updates of the optimizee and have been shown to outperform classical baselines in restricted settings. The subspace optimization paradigm has been proven to be a method of choice for deterministic large scale problems and therefore is the focus of this paper. In this work, we extend the idea of learned optimization to the paradigm of subspace optimization. The meta-optimizer is trained to predict the best direction to be removed from the subspace given a low-dimensional state obtained from previous subspace optimization steps. The main contributions of the paper are as follows.

• We extend the idea of meta optimization to the field of subspace optimization.
• We propose a meta-optimization framework which includes processing of very low-dimensional information obtained from the subspace spanning coefficients, enabling highly efficient training and deployment of the meta optimizer.
• We design a rule-based approach that can outperform popular subspace optimization methods, and propose a state-of-the-art reinforcement learning agent able to determine the optimal subspace matrix.
• Finally, we analyse the agent’s actions throughout the optimization, and demonstrate its ability to learn optimal policies adapted to the given tasks.

Background
Subspace Optimization
The core idea of subspace optimization is to perform the optimization of the objective function in a small subspace spanned by a set of directions obtained from an available oracle. Denoting a function \( f : \mathbb{R}^n \to \mathbb{R}, f \in C^2 \) to be minimized and \( P_k \in \mathbb{R}^{n \times d}, d \ll n \) as the set of directions at iteration \( k \), an iterated subspace optimization method aims at solving the following inner minimization problem

\[
\alpha_k = \arg \min_{\alpha \in \mathbb{R}^d} f(x_k + P_k \alpha), \tag{1}
\]

followed by the update rule

\[
x_{k+1} = x_k + P_k \alpha_k. \tag{2}
\]
The dimensions of the problem are then reduced from the entire optimization space $\mathbb{R}^n$, to a controlled $d$-dimensional subspace of $\mathbb{R}^n$ spanned by the columns of $P_k$.

A direct benefit of subspace optimization is that the low-dimensional optimization task at every iteration can be addressed efficiently using heavier optimization tools such as second-order methods. In this case, the main computational burden in this process is the need to multiply the spanning directions by second-order derivatives, which can be implemented efficiently using Hessian-vector product rules such that $(\partial^2 f) \cdot v = \partial(\partial f \cdot v)$. Alternatively, quasi-Newton methods can be used for the inner optimization.

The subspace structure may vary depending on the chosen optimization technique. Early methods proposed to extend the minimization to a $d$-dimensional subspace spanned by $d$ various previous directions, such as gradients, conjugate directions, previous outer iterations or Newton directions \cite{Cragg and Levy 1969, Miele and Cantrell 1969, Dennis Jr and Turner 1987, Conn et al. 1996}. The Krylov descent method defines the subspace as span$\{H^i \nabla f, \ldots, H^{d-i} \nabla f\}$ for some preconditioning matrix $H$, for example $H = \nabla^2 f$ in \cite{Vinyals and Povey 2012}.

Related to Krylov subspaces, the Conjugate Gradient (CG) method \cite{Hestenes and Stiefel 1952} reduces the search space to current gradient and previous step, i.e. span$\{p_k, \nabla f(x_k)\}$, where $p_k = x_k - x_{k-1}$. CG possesses a remarkable linear convergence rate compared to steepest descent methods in the quadratic case, related to the expanding manifold property \cite{Nocedal 2006}. \cite{Nemirovski 1982} provided optimal worst case complexity in the convex setting with the ORTH-method by defining the subspace as span$\{(\sum_{j=0}^{k} w_j \nabla f(x_j), x_k - x_0, \nabla f(x))\}$, with appropriate weights $\{w_j\}_{j=0}^k$.

The Subsequential Subspace Optimization (SESOP) algorithm \cite{Narkiss and Zibulevsky 2005} extends ORTH by adding the previous search directions $\{p_k\}_{k=0}^d$. This way, the method generalizes the CG method by allowing truncated approximation of the expanding manifold property on non-linear objectives, coupled with the worst case optimality safeguard of the ORTH method. It is important to note that any other (valuable) direction can be further embedded into the subspace structure in order to improve the convergence as in \cite{Conn et al. 1996, Zibulevsky and Elad 2010, Choukroun et al. 2020}.

Meta Optimization

While deep learning has achieved tremendous success by replacing hand-crafted feature engineering with automatic feature learning from large amounts of data, hand-crafted optimization algorithms, such as momentum and adaptive gradient based methods \cite{Sutskever et al. 2013, Kingma and Ba 2014} are still mainly in use for training neural networks. These optimizers typically require careful tuning of hyper parameters and extensive expert supervision in order to be used effectively for different model architectures and datasets \cite{Choi et al. 2019}. Meta optimization attempts to replace hand-designed optimizers by training a parametrized optimizer on a set of tasks and then applying it to the optimization of different tasks.

One meta optimization approach aims at training a controller to automatically adapt the hyper parameters of an existing hand-crafted optimizer based on the training dynamics. \cite{Daniel, Taylor, and Nowozin 2016} exploits features based on the variance of the predicted and observed changes in function value with a linear policy mapping of the learning rate for SGD and RMSProp optimizers. \cite{Xu et al. 2017} learns a small LSTM \cite{Hochreiter and Schmidhuber 1997} network using the current training loss as input in order to predict the learning rate. \cite{Xu et al. 2019} scales the learning rate using an LSTM or an MLP based on features extracted from the training dynamics such as the train and validations losses, the variance of the network predictions, and the last layer’s parameters statistics.

Another thread of research is devoted to the development of more expressive learned optimizers meant to replace existing optimizers entirely. In \textit{learning to learn by gradient descent by gradient descent} \cite{Andrychowicz et al. 2016} the authors propose a learned update rule given by $\theta_{t+1} = \theta_t + g_t(\nabla f(\theta), \phi)$ where $g_t$ is a coordinate-wise LSTM that maps the optimizer’s gradients to the new step vector. \cite{Li and Malik 2017} proposed a similar approach while they rely on policy search to compute the meta-parameters of the optimizer. Some subsequent studies have improved the robustness of these approaches. \cite{Wichrowska et al. 2017} introduces a hierarchical RNN architecture which captures inter-parameter dependencies and uses an ensemble of small tasks with diverse loss landscapes. \cite{Lv, Jiang, and Li 2017} incorporates random scaling of the optimizer when training the RNN-based optimizer which is shown to improve its generalization. In \cite{Metz et al. 2019} the authors improve the meta-training of the learned optimizer by constructing two different unbiased gradient estimators for the variational loss on optimizer performance. Finally, \cite{Metz et al. 2020} construct a dataset of more than a thousand diverse optimization tasks commonly found in machine learning, and show how this large and diverse task distribution is critical for training optimizers able to generalize well.

One of the drawbacks of some of the methods described above is that the meta optimizer has to process extremely high-dimensional data \cite{Li and Malik 2017}, rendering its model complexity, training time and deployment unsustainable. Some works tried to circumvent this difficulty by constraining coordinate-wise meta optimizers, but had difficulty in generalizing to new tasks \cite{Andrychowicz et al. 2016}.

Method

Optimal worst case subspace directions have already been provided by \cite{Nemirovski 1982} for smooth and convex functions. However, for a given (non worst) case and subject to the limitations of numerical optimization, there might exist better schemes for defining the subspace. Combining the ideas of meta learning and subspace optimization, we present in this section the proposed Meta Subspace Optimization (MSO) framework. We first begin by a general formulation of the framework and then constrain it for better computational efficiency.
where some solutions suffer from huge computational cost which maps \( \pi \) at each iteration \( k \) (Hestenes and Stiefel 1952) directly to the subspace matrix \( \pi \). The meta-optimizer \( \pi \) determines the next iteration’s subspace.

**A General Framework**

At each iteration \( k \geq 0 \), let \( S_k = [v_1, v_2, ..., v_L] \) be a list of \( L \) vectors spanning of \( \mathbb{R}^n \), where \( [ \cdot ] \) denotes vector concatenation, \( S_k^n = v_i \in \mathbb{R}^n \) denotes the \( i^{th} \) element of the list, and \( L \) is defined as the memory constraint. The list \( S_k \) comprises first-order information, such as current and previous iterates, steps, and gradients. Let \( \Omega_k \) denote information obtained from the subspace optimizer at iteration \( k-1 \), such as subspace iterates, gradients and any by-product of the subspace optimization. Finally, we define a meta optimizer \( \pi(S_k, \Omega_k) \), which maps \( S_k \) and \( \Omega_k \) to a sequence of \( d \) vectors, spanning another subspace of \( \mathbb{R}^n \), where \( d \leq L \).

Under this framework, the subspace minimization problem in Eq. (1) is defined and performed according to the subspace matrix \( P_k \) obtained from the previous iterate such that \( P_k = \pi(S_{k-1}, \Omega_{k-1}) \in \mathbb{R}^{n \times d} \). Thus, the meta optimizer is used to determine at each step \( k \), based on \( L \) available first-order directions, a smaller set of directions of size \( d \) which is expected to be most beneficial in minimizing the objective in the next subspace iteration. A diagram of the proposed framework is given in Figure 1.

**Meta Subspace Optimization**

Though appealing due to its generality, the meta optimizer previously suggested involves the processing of a potentially large number of very high-dimensional input and output vectors, whose dimensions scale with the problem’s dimension \( n \). This issue is reminiscent of other meta-optimization works where some solutions suffer from huge computational cost (Li and Malik 2017), while others employ ad hoc restrictions on the optimization (Andrychowicz et al. 2016). Contrary to existing works, in order to make our general framework more efficient for both training and inference, we propose to develop a method which scales with the subspace optimization dimensions \( d \), becoming invariant to the size of the original optimization problem.

We first restrict the first-order information available to the meta optimizer to be only \( L \):viewed taken steps, such that \( S_k = [p_{t_1},...,p_{t_L}] \), and we add the mandatory current gradient \( \nabla f(x_k) \) (Nemirovski 1982) and the current step \( p_k \) (Hestenes and Stiefel 1952) directly to the subspace matrix \( P_k \) which we redefine as follows

\[
P_k = [\pi(S_{k-1}, \Omega_{k-1}), p_k, \nabla f(x_k)],
\]

similarly to (Conn et al. 1996, Narkiss and Zibulevsky 2005), where \( t_L < ... < t_1 < k \) and \( \pi : \mathbb{R}^{n \times L} \rightarrow \mathbb{R}^{n \times (d-2)} \).

During the first \( L \) iterations, the list \( S_k \) gets populated with the first \( L \) steps \( p_k \), such that \( S_k = [p_1, ..., p_L] \). In order to maintain a fixed level of complexity (i.e. satisfying the memory constraint \( L \)), once \( S_k \) becomes fully populated, for every \( k > L \), one of its elements must be removed in order to insert the new step \( p_{k+1} \). Thus, to further simplify our scheme, we restrict the memory constraint such that \( L = d - 1 \), reducing the meta subspace optimizer to deciding which direction should be removed from \( S_k \) before adding the new step \( p_{k+1} \) at the end of each subspace optimization iteration. Thus the meta-optimizer is defined as \( \pi : \mathbb{R}^{n \times (d-1)} \rightarrow \mathbb{R}^{n \times (d-2)} \). This formulation becomes more computationally efficient and reduces the effective meta-optimizer’s output space to a single dimension. Under this formulation, by adding the two other ORTH directions, the optimal worst case optimality \( o(\frac{1}{L}) \) of the method can be proved under convexity assumption as in (Narkiss and Zibulevsky 2005). (Empirically, they generally can be omitted). Subspace optimization methods (see (Conn et al. 1996, Narkiss and Zibulevsky 2005, Richardson et al. 2016)) typically implement a FIFO approach where the oldest subspace direction is removed at each iteration, namely

\[
\pi(S_k, \Omega_k) := S_k \setminus \{S_k^1\}. 
\]

This subspace update implicitly assumes that most of the contribution to the next iterates comes from newly taken steps, whereas the importance of more distant steps taken in the past diminishes. However, it turns out, better subspace update strategies exist which are able to identify poor subspace directions and remove them from the subspace at earlier stages.

**Leveraging Optimal Subspace Step Sizes**

According to the subspace optimization paradigm, at each iteration \( k \) a new iterate \( x_{k} \) is updated according to the update rule in Eq. (2), where \( \alpha_k := (\alpha_k^1, ..., \alpha_k^d) \) is a vector whose elements (referenced to henceforth as step sizes) weight each subspace direction. Intuitively, the \( \alpha \) vector can be viewed as a multi-dimensional generalization of the standard one-dimensional line search gradient descent.

Therefore, the step sizes obtained from the subspace optimization convey information about the quality of their associated directions in minimizing the objective, where a subspace direction which contributes more to the update of the next iterate will potentially have a larger associated absolute step size. Thus we redefine the meta-optimizer as

\[
\pi(S_k, \Omega_k) := \pi(S_k, \alpha_k) := S_k \setminus \{S_k^{\pi(\alpha_k)}\}, 
\]

where \( \pi(\alpha_k) \) denotes the effective meta-subspace optimizer. This formulation allows very low-dimensional computations since the meta optimizer’s input dimension scales with the subspace size \( d \). Then, as the optimizer itself, the meta-optimizer remains invariant to the problem’s dimension \( n \).
This last assumption suggests an immediate and simple rule-based (RB) approach for updating the subspace by removing at each iteration the subspace direction whose \( \alpha \) value obtained in the last subspace optimization had the smallest absolute value, such that

\[
\pi(S_k, \alpha_k) = S_k \setminus \{ \exp \min_k |\alpha_k| \} \tag{6}
\]

The proposed rule based approach is both simple and efficient and allows improvement over SOTA subspace optimization methods as shown in Figure 2. This approach consistently outperforms the baselines as demonstrated in the Experiments section.

A Reinforcement Learning Approach

The rule-based approach described above gives rise to the possibility of finding even better decision rules (or meta optimizers) for selecting which direction to remove from the subspace, based on the last step sizes. Moreover, in order to equip the meta optimizer with additional information to make a decision, we allow it to base its decision on the set of vectors \( \alpha(k,h) := \{ \alpha_{k-h+1}, \ldots, \alpha_k \} \) obtained form the last \( h \) previous subspace optimization iterations.

Based on Eq. (5) we focus our attention on rules of the form:

\[
\pi(S_k, \Omega_k) := \pi(S_k, \alpha(k,h)) := S_k \setminus \{ S^\pi(\alpha(k,h)) \} \tag{7}
\]

where \( \pi(\alpha(k,h)) \) is the index of the subspace direction to be removed from the subspace.

In order to find superior policies, we resort to reinforcement learning policy search methods. Specifically, we model the sequence of decisions pertaining to the direction which is to be removed from the subspace as a Markov Decision Process (MDP). The MDP is defined by the tuple \((S, A, p(s_{t+1}|s_t, a_t), r(s, a))\), where \( S \subseteq \mathbb{R}^{d \times d} \) is the state space, \( A \subseteq \{ 1, \ldots, d-1 \} \) is the action space, \( p(s_{t+1}|s_t, a_t) \) is the state transition probability, and \( r(s, a) \) is the reward function.

Under this framework, we represent the meta optimizer as a stochastic policy \( \pi_\theta(a|s) := \pi(a|s; \theta) \) parameterized by \( \theta \), which given a state \( s \in S \), produces a conditional probability distribution over actions \( a \in A \).

The Meta Subspace Optimization algorithm is summarized in Algorithm 1. At the start of each outer iteration \( k \), the subspace \( P_k \) includes at most \( d-1 \) directions, which are all steps taken in the past. \( P_k \) is then extended with the current gradient \( \nabla f(x_k) \) to form a subspace of dimension \( d \). This subspace is used for solving the subspace optimization in Eq. (1) via the BFGS (Noc 2006) algorithm, yielding the optimal coefficient vector \( \alpha_k \). The iterate \( x_{k+1} \) is then updated according to Eq. (2). Next, the gradient \( \nabla f(x_k) \) is removed from the subspace. Then, based on the last \( h \) step vectors \( \{ \alpha_{k-h+1}, \ldots, \alpha_k \} \), the meta optimizer selects which of the \( d-1 \) steps in the subspace \( P_k \) is to be removed. Finally, the current step \( p_{k+1} = x_{k+1} - x_k \) is added to \( P_k \) to form \( P_{k+1} \) for the next iteration.

Assuming the goal is to learn a meta optimizer that minimizes the objective function, the reward \( r(s_k, a_k) \) at step \( k \) of the optimization is computed based on the relative decrease of the objective, following the subspace optimization and subsequent update of the iterate \( x_{k+1} \) such that

\[
r_k := r(s_k, a_k) = \frac{f(x_k) - f(x_{k+1})}{f(x_k)} \tag{8}
\]

In order to train the meta optimizer, we define an episode of \( T \) steps, during which the meta optimizer is used to update the subspace while optimizing an objective function \( f(x) \) according to Algorithm 1. During training, we use the REINFORCE policy search method (Williams 1992; Sutton and Barto 2018) and update the meta optimizer at the end of each episode by the following update rule:

\[
\theta_{k+1} = \theta_k + \eta \frac{1}{m} \sum_{k=1}^{m} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t) (R_t - b(s_t)) \tag{9}
\]

where \( \eta \) is the learning rate, \( m \) is the batch size, i.e. the number of sampled trajectories used to estimate the policy gradient, \( R_t = \sum_{t'=t}^{T} \gamma^{t'-t} r_t \) is the cumulative future reward at time step \( t \), where \( \gamma \in [0, 1] \) is the discount rate, and \( b(s_t) \) is a baseline function which estimates the expected return in state \( s_t \) (Sutton and Barto 2018). As in every meta-learning task, we stress that the trained agent is expected to generalize to other tasks, so that the delay incurred by the agent’s training procedure is amortized over the inference of other related tasks.

Experiments

Experimental Setup

To demonstrate the efficacy of our approach, we train meta subspace optimizers on various non-convex deterministic and stochastic classes of objective functions, namely, the Rosenbrock objective (Rosenbrock 1960), robust linear regression (Li and Malik 2017), and neural network classification of the MNIST dataset (LeCun and Cortes 2010). We then apply the trained optimizers on new objective functions from the respective classes, and compare their convergence performance to several state-of-the-art subspace optimizers.
Algorithm 1: Meta Subspace Optimization (MSO)

**Input:** $f : \mathbb{R}^N \rightarrow \mathbb{R}$, initial point $x_0$, meta subspace optimizer $\pi_\theta : \mathbb{R}^{(d-1) \times h} \rightarrow \{ u \in \mathbb{R}^{d-1} | u^T 1 = 1, u_i \geq 0 \forall i \}$

**Output:** $x_{final}$

**Initialize:** $P_0 = \{ x_0 \}$

for $k = 0, 1, \ldots, K$

if Converged then

return $x_k$

end if

$P_k \leftarrow P_k \cup \{ \nabla f(x_k) \}$

Solve $\alpha_k = \arg \min_{\alpha} f(x_k + P_k \alpha)$ using BFGS

Update $x_{k+1} = x_k + P_k \alpha_k$

$P_k \leftarrow P_k \setminus \{ \nabla f(x_k) \}$

if $\dim(P_k) \geq d - 1$ then

Remove the $a_k^{th}$ direction from $P_k$ where $a_k \sim \pi_\theta(a | \{ \alpha_{k-h+1}, \ldots, \alpha_k \})$

end if

$p_{k+1} = P_k \alpha_k$

$P_{k+1} \leftarrow P_k \cup \{ p_{k+1} \}$

end for

return $x_K$

The trained meta optimizer is composed of a two-layer fully connected network with 128 neurons in each hidden layer and a tanh nonlinearity, followed by an output layer of dimension $d - 1$ and a softmax operation. The meta optimizer is trained according to Eq. (9) using the ADAM optimizer (Kingma and Ba 2014) with a learning rate of $5e^{-3}$. The discount rate $\gamma$ is chosen as 1 to allow consideration of policies whose current decisions regarding the subspace have significant impact on future minimization of the objective. Finally, the baseline function for estimating the expected return in Eq. (9) is implemented by an exponential moving average.

We compare performance with several known subspace optimization baselines, namely, CG (Hestenes and Stiefel 1952), ORTH (Nemirovski 1982), and SESOP (Narkiss and Zibulevsky 2005). We also report the performance of our rule-based method of Eq. (9) which we denote RB. We redirect the reader to existing literature (Narkiss and Zibulevsky 2005) (Conn et al. 1996) for comparison of the baselines with other methods such as Nesterov (Nesterov 1983) or Truncated Newton (Gill, Murray, and Wright 2019). We do not compare with first-order methods since they are not competitive with subspace optimization frameworks in the deterministic setting, while subspace optimization or second-order methods for stochastic problems (batch training) are still an unsolved problem (Bottou, Curtis, and Nocedal 2018) and out of the scope of this paper. In all experiments the subspace dimension is set to $d = 10$ as a good trade-off between the optimization performance and the computational cost (Richardson et al. 2016) for all the relevant methods.

In all experiments, we run the system depicted in Fig. 1 where the function $f(x)$ is the tested objective function in the deterministic case, or the batch dependent objective in the stochastic case, and the meta optimizer module is replaced by the appropriate baseline optimizer where applicable. The auxiliary information $\Omega$ was set in accordance with the chosen optimizer. For instance, in the case of the meta subspace optimizer, $\Omega_k$ is implemented as a matrix whose $i^{th}$ column contains the $h$ last step size values associated with the $i^{th}$ subspace direction. In all experiments we use $h = 5$. Step sizes related to subspace directions at iterations for which these directions are not yet available, are treated as zeros. The BFGS algorithm (Noc 2006) is used with a 1e-5 gradient tolerance threshold.

**Rosenbrock Objective**

In our first experiment, we consider the non-convex Rosenbrock function (Rosenbrock 1960), namely:

$$f(x) := \sum_{i=1}^{n-1} [b(x_{i+1} - x_i^2)^2 + (a - x_i)^2]$$

(10)

where $a, b \in \mathbb{R}$ are constants which modulate the shape of the function and $x_i \in \mathbb{R}$. We train a meta optimizer for optimizing this objective function. The training set of the meta optimizer consists of 50 instances of this objective differing in their initial point $x_0$ which is randomly drawn from a standard Gaussian distribution. In this experiment, the meta optimizer is trained for 10,000 episodes where each episode spans a trajectory of 300 optimization steps.

We evaluate the meta optimizer on 100 instances of this objective, generated using the same procedure as was used for the training set, and compare its convergence to the baselines.
As in (Li and Malik 2017), we consider a linear regression model with a robust loss function. Training the model requires optimizing the following objective:

\[
\begin{align*}
\min_{w, b} f(w, b) & := \frac{1}{n} \sum_{i=1}^{n} \left( y_i - w^T x_i - b \right)^2, \\
& \quad + c (y_i - w^T x_i - b)^2, \tag{11}
\end{align*}
\]

where \( w \in \mathbb{R}^D \) and \( b \in \mathbb{R} \) denote the model’s weight vector and bias respectively, \( x_i \in \mathbb{R}^D \) and \( y_i \in \mathbb{R} \) denote the feature vector and label, respectively, of the \( i \)-th instance, and \( c \in \mathbb{R} \) is a constant that modulates the shape of the loss function. In our experiments, we use \( c = 1 \) and \( D = 100 \). This loss function is not convex in either \( w \) or \( b \).

We train a meta optimizer for optimizing objective functions of this form. The training set (of the meta optimizer) consists of 50 examples of such objective functions, where in each example, we randomly draw a dataset of \( n = 100 \) \((x_i, y_i)\) tuples as follows. We draw 25 random samples from each one of four multivariate Gaussians, each of which has a random mean and an identity covariance matrix. The labels for those points are generated by projecting them along the same random vector, adding a randomly generated bias and perturbing them with i.i.d. Gaussian noise.

The meta optimizer is trained for 2000 episodes. At the start of each episode, the objective function (11) is randomly chosen from the training dataset, and the initial weight vector and bias of the regression model are randomly drawn from a standard Gaussian distribution. Each episode is composed of 400 optimization steps.

We evaluate the meta optimizer on a test set of 100 random objective functions generated using the same procedure, and compare its convergence to the baselines. Figure 3 shows for each algorithm the average convergence performance over all test objectives. Some of the baselines suffer from occasional convergence to local minima, which worsen the average convergence curve while our method remains more robust and presents better convergence rate. Here as well, the meta subspace optimizer outperforms all subspace optimizers throughout the optimization trajectory, even though it was trained only on the first 400 steps. Second best is again the RB method which also demonstrates its benefit over traditional subspace optimizers.

**Neural Network Classification**

In this experiment, we train a meta optimizer for the task of training a small neural network classifier on the MNIST dataset. As in (Andrychowicz et al. 2016), we first train the optimizer to optimize a base network, and then test how well this optimizer generalizes when used to train a neural network with a modified architecture or dataset.

The objective function to be optimized is the cross entropy loss of a fully connected network with one hidden layer of 10 units and a ReLU nonlinearity, (the algorithm remains stable even if the function is not differentiable). In order to test generalization of the meta optimizer across different datasets, we first train the meta optimizer to train the base network on a subset of the MNIST dataset consisting half of the digits. The value of this objective and its gradient are estimated using random minibatches of 8K examples. The large batch setting is important with second-order optimization in order to obtain good estimate of the true gradient (Bottou, Curtis, and Nocedal 2018). The meta optimizer is trained for 1000 episodes. At the start of each training episode, the dataset is shuffled and the weights are randomly drawn. Each episode is run for 14 epochs. We compare the full batch training loss, where the meta optimizer outperforms the baseline optimizers on this task as well during a period of 50 epochs, after being trained to optimize for only 14 epochs.

We test two forms of generalization of the meta optimizer. In a first experiment, we use the trained optimizer to train the base network on the complimentary MNIST digits, which were not used during the meta optimizer’s training. In a second experiment shown in Fig. 5, we apply the meta optimizer to train a modified version of the base network with 20 hidden units, instead of 10. In both cases, the meta optimizer generalizes well, and continues to outperform the hand-crafted baselines even though it is applied in very different conditions than the ones met during its training.

**Analysis**

In this section we analyze the policies learned by the meta optimizer, i.e. which subspace direction is least beneficial for minimizing the objective. In Fig. 3 (right), 4 (left), and 7 (left) we present examples of the meta optimizer’s decision trajectory throughout the optimization of the different
objectives. The y axis represents the action space of the meta optimizer, which in our setting is the set of indices of subspace elements which may be removed from the subspace. For each optimization step, the plots show the probability assigned by the meta optimizer to each of its possible actions. The markedly different policy patterns demonstrate that the meta optimizer is able to adapt its policy to each task as we explain next.

In Fig. 7 (right) we present the meta optimizer’s decision trajectory throughout the optimization of the Rosenbrock objective. It can be observed that in this setting the meta optimizer’s policy is highly sensitive to the optimization coefficients $\alpha_k$ associated with the directions in the subspace. The policy trajectory reveals the following behavior. Once a new direction $p_k$ is added to the subspace and participates in the optimization for the first time, it is either immediately removed from the subspace, or is allowed to stay in the subspace for subsequent optimizations for as long as it is useful or until it becomes oldest. This is observable from the large action probability distribution located at indices 0 and 9, corresponding to the removal of the oldest and newest directions, respectively, and from the smaller non-zero probabilities for removing the intermediate indices.

In Fig. 6 (left) we present the meta optimizer’s decision trajectory throughout the optimization of the robust linear regression objective. In this case, it can be observed that the meta optimizer removes with high probability the most recent direction inserted into the subspace matrix, and keeps all remaining subspace directions intact, letting them participate in the next subspace optimization. Interestingly, this policy is diametrically opposed to existing approaches which axiomatically remove the oldest direction from the subspace. Also, with some (lower) probability, the meta optimizer may decide to remove a more distant subspace direction instead, thus stochastically allowing the new direction $p_k$ to remain longer in the subspace.

In order to further assess the impact of this policy, we compare its performance to a deterministic policy given by $\pi(a|s) = \delta(a,9)$, where $\delta(n,m)$ denotes the Kronecker delta function. Such a singular policy would always remove the most recent direction from the subspace. Fig. 6 (right) shows the average performance of the aforementioned meta optimizer compared to a $\delta(a,9)$ agent’s policy, demonstrating the benefit of the obtained stochastic policy. This finding is reminiscent of the improvement obtained with anchor (fixed) points set into the subspace matrix (Richardson et al. 2016).

Finally, we show that the obtained meta optimizer’s policy to equally divide the subspace between a long term part and a short term part is optimal among all possible 2-way contiguous subspace partitions. Fig. 7 (right) compares the performance of the meta optimizer to the performance of all singular policies, namely, $\delta(a,0), \ldots, \delta(a,9)$. As can be seen, singular policies which are based on extreme index values are substantially worse than those based on the middle indices, where the optimal singular policy is $\delta(a,5)$ which coincides with the meta optimizer’s policy. Apparently, in this setting, devoting half of the subspace for earlier directions in the optimization, and another half for most recent directions gives the best result. We believe this policy is to be linked with the importance of the initial point $x_0$ in the subspace as in the ORTH framework (Nemirovski 1982).

**Conclusion**

We propose for the first time a meta optimization framework applied to the subspace optimization paradigm. We derive from our general framework an efficient reinforcement learning approach where the dimensions of the state and action spaces scale with the subspace dimension such that the agent remains invariant to the size of the optimization problem, enabling efficient training and deployment. We also provide a rule-based method and a reinforcement learning agent able to outperform existing subspace optimization methods setting a new state-of-the-art subspace optimization method. Finally, we analyze the obtained agent in order to understand the optimization dynamics. We believe meta subspace optimization can become a numerical optimization tool of choice for the development of better and more interpretable optimizers.


References

2006. Line Search Methods, 30–65. New York, NY: Springer New York. ISBN 978-0-387-40065-5.

Andrychowicz, M.; Denil, M.; Colmenarejo, S. G.; Hoffman, M. W.; Pfau, D.; Schaul, T.; Shillingford, B.; and de Freitas, N. 2016. Learning to learn by gradient descent by gradient descent. In Proceedings of the 30th International Conference on Neural Information Processing Systems, 3988–3996.

Bottou, L.; Curtis, F. E.; and Nocedal, J. 2018. Optimization methods for large-scale machine learning. Siam Review, 60(2): 223–311.

Choi, D.; Shallue, C. J.; Nado, Z.; Lee, J.; Maddison, C. J.; and Dahl, G. E. 2019. On empirical comparisons of optimizers for deep learning. arXiv preprint arXiv:1910.05446.

Choukroun, Y.; Zibulevsky, M.; Kisilev, P.; and . . . 2020. Primal-Dual Sequential Subspace Optimization for Saddle-point Problems. arXiv preprint arXiv:2008.09149.

Conn, A.; Gould, N.; Sartenaer, A.; and Toint, P. L. 1996. On iterated-subspace minimization methods for nonlinear optimization. Linear and Nonlinear Conjugate Gradient-Related Methods, 50–78.

Cragg, E.; and Levy, A. 1969. Study on a supermemory gradient method for the minimization of functions. Journal of Optimization Theory and Applications, 4(3): 191–205.

Daniel, C.; Taylor, J.; and Nowozin, S. 2016. Learning step size controllers for robust neural network training. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 30.

Dennis Jr, J. E.; and Turner, K. 1987. Generalized conjugate directions. Linear Algebra and its Applications, 88: 187–209.

Gill, P. E.; Murray, W.; and Wright, M. H. 2019. Practical optimization. SIAM.

Hestenes, M. R.; and Stiefel, E. 1952. Methods of conjugate gradients for solving linear systems, volume 49. NBS Washington, DC.

Hochreiter, S.; and Schmidhuber, J. 1997. Long short-term memory. Neural computation, 9(8): 1735–1780.

Kingma, D. P.; and Ba, J. 2014. Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980.

LeCun, Y.; and Cortes, C. 2010. MNIST handwritten digit database.

Li, K.; and Malik, J. 2017. Learning to optimize neural nets. arXiv preprint arXiv:1703.00441.

Lv, K.; Jiang, S.; and Li, J. 2017. Learning gradient descent: Better generalization and longer horizons. In International Conference on Machine Learning, 2247–2255. PMLR.

Metz, L.; Maheswaranathan, N.; Freeman, C. D.; Poole, B.; and Sohl-Dickstein, J. 2020. Tasks, stability, architecture, and compute: Training more effective learned optimizers, and using them to train themselves. arXiv preprint arXiv:2009.11243.

Metz, L.; Maheswaranathan, N.; Nixon, J.; Freeman, D.; and Sohl-Dickstein, J. 2019. Understanding and correcting pathologies in the training of learned optimizers. In International Conference on Machine Learning, 4556–4565. PMLR.

Miele, A.; and Cantrell, J. 1969. Study on a memory gradient method for the minimization of functions. Journal of Optimization Theory and Applications, 3(6): 459–470.

Narkiss, G.; and Zibulevsky, M. 2005. Sequential Subspace Optimization Method for Large-Scale Unconstrained Problems. Technical Report CCIT 559, Technion – Israel Institute of Technology, Faculty of Electrical Engineering.

Nemirovski, A. 1982. Orth-method for smooth convex optimization. Izvestia AN SSSR, Transl.: Eng. Cybern. Soviet J. Comput. Syst. Sci, 2: 937–947.

Nesterov, Y. 1983. A method for unconstrained convex minimization problem with the rate of convergence $O(1/n^2)$ (in Russian). Doklady AN SSSR (the journal is translated to English as Soviet Math. Doicl.), 269(3): 543–547.

Richardson, E.; Herskovitz, R.; Ginsburg, B.; and Zibulevsky, M. 2016. SEBOOST-Boosting Stochastic Learning Using Subspace Optimization Techniques. In Advances in Neural Information Processing Systems, 1534–1542.

Rosenbrock, H. 1960. An automatic method for finding the greatest or least value of a function. The Computer Journal, 3(3): 175–184.

Sutskever, I.; Martens, J.; Dahl, G.; and Hinton, G. 2013. On the importance of initialization and momentum in deep learning. In International conference on machine learning, 1139–1147. PMLR.

Sutton, R. S.; and Barto, A. G. 2018. Reinforcement learning: An introduction. MIT press.

Vinyals, O.; and Povey, D. 2012. Krylov subspace descent for deep learning. In Artificial Intelligence and Statistics, 1261–1268.

Wichrowska, O.; Maheswaranathan, N.; Hoffman, M. W.; Colmenarejo, S. G.; Denil, M.; Freitas, N.; and Sohl-Dickstein, J. 2017. Learned optimizers that scale and generalize. In International Conference on Machine Learning, 3751–3760. PMLR.

Williams, R. J. 1992. Simple statistical gradient-following algorithms for connectionist reinforcement learning. Machine learning, 8(3-4): 229–256.

Xu, C.; Qin, T.; Wang, G.; and Liu, T.-Y. 2017. Reinforcement learning for learning rate control. arXiv preprint arXiv:1705.11159.

Xu, Z.; Dai, A. M.; Kemp, J.; and Metz, L. 2019. Learning an adaptive learning rate schedule. arXiv preprint arXiv:1909.09712.

Zibulevsky, M.; and Elad, M. 2010. L1-L2 optimization in denoising an adaptive learning rate schedule. arXiv:1909.09712.

Zibulevsky, M.; and Elad, M. 2010. L1-L2 optimization in denoising an adaptive learning rate schedule. arXiv:1909.09712.