Stability Analysis of Adaptive Control Systems with Event-triggered Try-once-discard Protocol

Masashi Wakaiki

Graduate School of System Informatics, Kobe University, 1-1 Rokkodai, Nada, Kobe, 657-8501, Japan

Abstract

This paper addresses the stability analysis of adaptive control systems with the try-once-discard protocol. At every sampling time, an event trigger evaluates errors between the current value and the last released value of each measurement and determines whether to transmit the measurements and which measurements to transmit, based on the try-once-discard protocol and given lower and upper thresholds. For both gain-scheduling controllers and switching controllers that are adaptive to the maximum error of the measurements, we obtain sufficient conditions for the closed-loop stability in terms of linear matrix inequalities.

Keywords: Event-triggered control; try-once-discard protocol; gain-scheduling control; switching control.

1. Introduction

In standard sampled-data control, measurements are periodically transmitted to controllers. For this time-triggered control, many techniques of designing controllers have been developed during the last decades such as the lifting approach [1, 2], the fast sample/fast hold approach [3, 4], and the frequency response operator approach [5, 6]. However, time-triggered control may lead to redundant transmissions, which waste energy and network resources.

As an alternative control paradigm, event-triggered control [7, 8] has been developed. In the event-triggered approach, transmission intervals are determined by a predefined condition on the measurements, and energy and network resources are used only if the measurements have to be transmitted. The effectiveness of event-triggered techniques has been illustrated, e.g., through security in networked control systems in [9, 10, 11] and an experiment of mobile robots in [12, 13], and there are many researches on the analysis and synthesis of event-triggered control as surveyed in [14, 15, 16].

On the other hand, to avoid network congestion, medium access protocols such as the round-robin (RR) protocol and the Try-Once-Discard (TOD) protocol govern the access of nodes to networks. The RR protocol assigns transmissions to the nodes in a certain prefixed order. In contrast, the TOD protocol determines which nodes should transmit their data by evaluating the largest error between the current value and the last released value of the node signals; see [17, 18, 19, 20, 21, 22] and references therein for control techniques based on the TOD protocol.

In this paper, we study the stability analysis of adaptive control systems with the TOD protocol. As the periodic event-triggered control proposed in [23, 24], an event trigger periodically evaluates errors between the current value
and the last released value of the measurements and determine whether to transmit the measurements and which measurements to transmit, based on the TOD protocol and given lower and upper thresholds. We use two types of adaptive controllers: gain-scheduling controllers and switching controllers, both of which change their feedback gains depending on the maximum error between the current value and the last received value of the measurements. We obtain sufficient conditions for closed-loop stability in terms of linear matrix inequalities (LMIs) by incorporating the event-triggered condition into the stability analysis through the so-called S-procedure [25].

This paper is organized as follows. The closed-loop system and the event-triggered TOD protocol we consider is described in Section 2. In Section 3, we provide sufficient conditions for closed-loop stability under gain scheduling control and switching control. An illustrative example is presented in Section 4. We draw concluding remarks and future work in Section 5.

Notation

For a vector \( v = [v_1 \cdots v_n]^\top \in \mathbb{R}^n \), the inequality \( v \geq 0 \) means that every element \( v_i \) satisfies \( v_i \geq 0 \). On the other hand, for a square matrix \( P \), the notation \( P > 0 \) means that \( P \) is symmetric and positive definite. For simplicity, we write a partitioned symmetric matrix \( \begin{bmatrix} Q & W \\ W^\top & R \end{bmatrix} \) as \( \begin{bmatrix} Q & W \\ \ast & R \end{bmatrix} \). For a vector \( v = [v_1 \cdots v_n]^\top \in \mathbb{R}^n \), its Euclidean norm is denoted by \( ||v|| \) and its maximum norm by \( ||v||_\infty = \max(|v_1|, \ldots, |v_n|) \). We define \( J_n := \{ (\ell_1, \ldots, \ell_n) : \ell_p \in \{1, -1 \} (1 \leq p \leq n) \} \).

2. Problem Statement

Consider a linear time-invariant discrete-time system:

\[
x[k+1] = Ax[k] + Bu[k],
\]

(1)

where \( x[k] \in \mathbb{R}^n \) and \( u[k] \in \mathbb{R}^m \) are the state and the input of the plant, respectively. Let \( h > 0 \) be a sampling period. The measurement \( y \) is sampled at time \( t = kh \) for every \( k \geq 0 \):

\[
y[k] = x[k] + w[k],
\]

where \( w[k] \in \mathbb{R}^n \) is the measurement noise.

We assume that the measurement noise is bounded.

Assumption 2.1. \( ||w[k]||_\infty < \bar{w} \) for every \( k \geq 0 \).

Let \( y = [y_1, \ldots, y_n]^\top \) and set parameters \( \lambda_i > 0, \delta_i \geq 0 \) for all \( i = 1, \ldots, n \). We denote by \( \ell_i \) the latest time (\( \leq k \)) at which the \( i \)-th measurement \( y_i \) is transmitted. The event trigger stores information on the last released data \( y_i[\ell_{i-1}^k] \) and calculates the error \( e_i[k] := y_i[\ell_{i-1}^k] - y_i[k] \) and

\[
e_i[k] := \frac{|e_i[k]| - \delta_i}{\lambda_i||y_i[k]||}
\]

for every \( i = 1, \ldots, n \). If \( E'_i \) satisfies \( E'_i \leq \Delta_{\min} \) for every \( i \), then the event trigger discards all the measurements \( y_1[k], \ldots, y_n[k] \). If \( E'_i \) satisfies \( E'_i \geq \Delta_{\max} \) for some \( i \), then the event trigger sends all the measurements \( y_i[k] \) satisfying this inequality. Otherwise, the event trigger sends the measurement \( y_i[k] \) that achieves the maximum of \( E'_i \), as the TOD protocol. If several measurements achieve the maximum, then either one of these measurements can be chosen. In this way, the last released data is updated from \( y_i[\ell_{i-1}^k] \) to \( y_i[\ell_i^k] \) at every step \( k \). Hence the error \( e_i[k] := y_i[\ell_i^k] - y_i[k] \) satisfies

\[
|e_i[k]| \leq \lambda_i E[k] \cdot ||y_i[k]|| \delta_i
\]

(2)

for every \( i = 1, \ldots, n \), where

\[
E[k] :=
\begin{cases} 
\Delta_{\min} & \text{if } \max_i E'_i \leq \Delta_{\min} \\
\Delta_{\max} & \text{if } \max_i E'_i \geq \Delta_{\max} \\
\max_i E'_i & \text{otherwise.}
\end{cases}
\]
We call this algorithm an *event-triggered TOD protocol*. Note that \( E[k] \) can be computed from the transmitted data at the controller side. In fact, the controller receives no measurement, then \( E[k] = \Delta_{\text{min}} \). If the number of the received measurements is more than one, then \( E[k] = \Delta_{\text{max}} \). Otherwise, the controller receives a single measurement, and let it be \( y_i \). Then we have

\[
E[k] = \min \left\{ \Delta_{\text{max}}, \frac{|y_i[l_i] - y_i[l_i - 1]| - \delta_i}{\lambda_i |y_i[l_i]|} \right\}.
\]

Defining the last transmitted measurement \( \bar{y}[k] \) by

\[
\bar{y}[k] := \begin{bmatrix} y_1[l_1] \\ \vdots \\ y_n[l_n] \end{bmatrix},
\]

we can therefore use the following adaptive controller:

\[
u[k] = K(E[k])\bar{y}[k].\]

(3)

Note that even when the released data is not updated, the control input \( u(t) \) in (3) may be changed because of the update of an error bound \( E[k] \).

Fig. 1 illustrates the closed-loop system. The objective of this paper is to study closed-loop stability under the event-triggered TOD protocol and the adaptive controller.

**Remark 2.2.** If the control objective is just stability, it is enough to use a constant feedback gain for the worst case \( E[k] = \Delta_{\text{max}} \). However, here we focus on closed-loop stability under the adaptive control (3) to achieve better system performances as in [26, 27]. The motivation for this problem is the control of complex systems where conflicting requirements make a single controller unsuitable. In particular, as shown in [13, 28], the noise in the sensors increases the number of transmissions and degrades the performances of event-based control strategies.

### 3. Main Results

#### 3.1. Gain scheduling control

Define \( \Delta_1 := \Delta_{\text{min}} \) and \( \Delta_2 := \Delta_{\text{max}} \). Given two feedback gains \( K_1 \) and \( K_2 \), we set an adaptive gain \( K[k] \) to be

\[
K[k] := \frac{\Delta_2 - E[k]}{\Delta_2 - \Delta_1} K_1 + \frac{E[k] - \Delta_1}{\Delta_2 - \Delta_1} K_2.
\]

(4)

Inspired by the stability analysis of piecewise affine systems [29, 30, 31], we obtain the following result based on the so-called S-procedure:

**Theorem 3.1.** Let Assumption [27] hold. The closed-loop system satisfies

\[
\limsup_{k \to \infty} ||x[k]|| \leq \rho \cdot \max_{i=1, ..., n} (\Delta_{\text{max}} \bar{w}_i \lambda_i + \delta_i)
\]

(5)
for some constant $\rho > 0$ if there exist a positive definite matrix $P_p$, a symmetric matrix with nonnegative entries

\[
\begin{bmatrix}
L_{p,t} & N_{p,t} \\
\star & M_{p,t}
\end{bmatrix},
\]

and a symmetric matrix $Z_{p,t}$ ($p = 1, 2, \ell = (\ell_1, \ldots, \ell_n) \in J_n$) such that the following LMI

\[
\begin{bmatrix}
\Gamma_{p,q,t} & \Phi_{p,q,t} \\
\star & \Xi_{q,t}
\end{bmatrix} > 0
\]

is feasible for every $p, q = 1, 2$ and $\ell \in J_n$, where

\[
\Gamma_{p,q,t} := \begin{bmatrix} P_p & 0 \\ \star & N_{q,t} + N_{q,t}^T + Z_{q,t} \end{bmatrix}, \quad \Xi_{q,t} := \begin{bmatrix} P_q & 0 \\ \star & R_{q,t} \end{bmatrix}, \quad \Phi_{p,q,t} := \begin{bmatrix} A + BK_p & BK_p^T \\ \Delta_p \Lambda_{\ell} & 0 \end{bmatrix},
\]

\[
\Lambda_{\ell} := \text{diag}(\lambda_1 \ell_1, \ldots, \lambda_n \ell_n), \quad R_{q,t} := \begin{bmatrix} L_{q,t} + M_{q,t} + N_{q,t}^T + N_{q,t} \\ \star \end{bmatrix},
\]

\[
L_{q,t} := \begin{bmatrix} L_{q,t} & M_{q,t} & N_{q,t}^T & N_{q,t} \end{bmatrix}.
\]

**Proof:** First we consider the case $\vec{w} = 0$ and $\delta_i = 0$ for every $i = 1, \ldots, n$. For positive definite matrices $P_1, P_2$, we define a time-dependent Lyapunov function $V(k, x)$ by

\[
V(k, x) := x^T P[k] x,
\]

where $P[k] := \alpha_1[k] P_1 + \alpha_2[k] P_2$

with

\[
\alpha_1[k] := \frac{\Delta_2 - E[k]}{\Delta_2 - \Delta_1}, \quad \alpha_2[k] := \frac{E[k] - \Delta_1}{\Delta_2 - \Delta_1}.
\]

The dynamics of the closed-loop system is given by

\[
x[k + 1] = A x[k] + BK[k] q[k] = (A + BK[k]) x[k] + BK[k] e[k],
\]

where

\[
e[k] := \vec{y}[k] - x[k] = \begin{bmatrix} e_1[k] \\ \vdots \\ e_n[k] \end{bmatrix}
\]

and every $e_i[k]$ satisfies (3). Setting $\gamma < 1$, we have

\[
\gamma V(k, x[k]) - V(k + 1, x[k + 1]) = \begin{bmatrix} x[k]\T \\ e[k]\T \end{bmatrix} \begin{bmatrix} \gamma P[k] & 0 \\ \star & 0 \end{bmatrix} \begin{bmatrix} x[k]\T \\ e[k]\T \end{bmatrix} - \begin{bmatrix} x[k]\T \\ e[k]\T \end{bmatrix} \begin{bmatrix} (A + BK[k])\T \\ (BK[k])\T \end{bmatrix} P[k + 1] \begin{bmatrix} A + BK[k] & BK[k] \end{bmatrix} \begin{bmatrix} x[k] \\ e[k] \end{bmatrix}.
\]

Since

\[-\lambda \cdot E[k] \cdot |x[k]| \leq e_i[k] \leq \lambda \cdot E[k] \cdot |x[k]|\]

under the event-triggered TOD protocol, $x[k]$ and $e[k]$ satisfy

\[
W[k] \begin{bmatrix} x[k] \\ e[k] \end{bmatrix} \geq 0,
\]

where

\[
W[k] := \begin{bmatrix} E[k] \Lambda_{\ell[k]} & I \\ E[k] \Lambda_{\ell[k]} & -I \end{bmatrix} = \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} E[k] \Lambda_{\ell[k]} & 0 \\ 0 & 1 \end{bmatrix}
\]

with some $\ell[k] = (\ell_1[k], \ldots, \ell_n[k]) \in J_n$. Every $\ell_i[k]$ is chosen such that $|x_i[k]| = \ell_i[k] |x_i[k]|$. Hence if the statement

\[
W[k] \begin{bmatrix} x[k] \\ e[k] \end{bmatrix} \geq 0 \Rightarrow \begin{bmatrix} x[k]\T \\ e[k]\T \end{bmatrix} \begin{bmatrix} \gamma P[k] & 0 \\ \star & 0 \end{bmatrix} \begin{bmatrix} x[k]\T \\ e[k]\T \end{bmatrix} - \begin{bmatrix} x[k]\T \\ e[k]\T \end{bmatrix} \begin{bmatrix} (A + BK[k])\T \\ (BK[k])\T \end{bmatrix} P[k + 1] \begin{bmatrix} A + BK[k] & BK[k] \end{bmatrix} \begin{bmatrix} x[k] \\ e[k] \end{bmatrix} > 0
\]
holds, then we obtain \(\gamma V(k, x[k]) > V(k + 1, x[k + 1])\). For matrices with nonnegative entries
\[
\begin{bmatrix}
L_{1, \ell} & N_{1, \ell} \\
\ast & M_{1, \ell}
\end{bmatrix}, \quad \begin{bmatrix}
L_{2, \ell} & N_{2, \ell} \\
\ast & M_{2, \ell}
\end{bmatrix} (\ell \in \mathcal{J}_n),
\]
define
\[
\begin{bmatrix}
L[k + 1] & N[k + 1] \\
\ast & M[k + 1]
\end{bmatrix}
:= \alpha_1[k + 1] \begin{bmatrix}
L_{1, \ell[k]} & N_{1, \ell[k]}
\end{bmatrix} + \alpha_2[k + 1] \begin{bmatrix}
L_{2, \ell[k]} & N_{2, \ell[k]}
\end{bmatrix}
\]
for each \(k \geq 0\). Then it follows from the S-procedure \(25\) that the matrix inequality
\[
\begin{bmatrix}
\gamma P[k] & 0 \\
\ast & 0
\end{bmatrix} \geq \begin{bmatrix}
(A + BK[k])^T & (BK[k])^T \\
(BK[k]) & 0
\end{bmatrix} P[k + 1] \begin{bmatrix}
A + BK[k] & BK[k]
\end{bmatrix} - W[k]^T \begin{bmatrix}
L[k + 1] & N[k + 1] \\
\ast & M[k + 1]
\end{bmatrix} W[k] > 0
\]
holds for every \(k \geq 0\), then \(V(k, x[k])\) exponentially decreases.

Using arbitrary symmetric matrices \(Z_{1, \ell}\) and \(Z_{2, \ell}\) \((\ell \in \mathcal{J}_n)\), we define \(Z[k + 1] := \alpha_1[k + 1]Z_{1, \ell[k]} + \alpha_2[k + 1]Z_{2, \ell[k]}\) for all \(k \geq 0\) and set
\[
R[k] := \begin{bmatrix}
L[k + 1] + M[k] + N[k]^T & L[k] - M[k] + N[k]^T - N[k] \\
\ast & L[k] + M[k] + Z[k]
\end{bmatrix},
\]
\[
T[k] := \begin{bmatrix}
0 & N[k] + N[k]^T + Z[k]
\ast & 0
\end{bmatrix}
\]
for all \(k \geq 1\). Since
\[
W[k]^T \begin{bmatrix}
L[k + 1] & N[k + 1] \\
\ast & M[k + 1]
\end{bmatrix} W[k] = \begin{bmatrix}
E[k]\Lambda_{\ell[k]} & 0 \\
0 & I
\end{bmatrix} R[k + 1] \begin{bmatrix}
E[k]\Lambda_{\ell[k]} & 0 \\
0 & I
\end{bmatrix} - T[k + 1],
\]
it follows that the matrix inequality \(9\) is feasible if and only if
\[
\begin{bmatrix}
\gamma P[k] & 0 \\
\ast & 0
\end{bmatrix} + T[k + 1] - \begin{bmatrix}
(A + BK[k])^T & (BK[k])^T \\
(BK[k]) & 0
\end{bmatrix} P[k + 1] \begin{bmatrix}
A + BK[k] & BK[k]
\end{bmatrix} - \begin{bmatrix}
E[k]\Lambda_{\ell[k]} & 0 \\
0 & I
\end{bmatrix} R[k + 1] \begin{bmatrix}
E[k]\Lambda_{\ell[k]} & 0 \\
0 & I
\end{bmatrix} > 0.\tag{9}
\]
Moreover, it follows from the Schur complement formula that the matrix inequality \(9\) is feasible if
\[
\begin{bmatrix}
\Gamma[k] & \Phi[k] \\
\ast & \Xi[k]
\end{bmatrix} > 0,
\]
where
\[
\Gamma[k] := \begin{bmatrix}
\gamma P[k] & 0 \\
\ast & 0
\end{bmatrix} + T[k + 1], \quad \Xi[k] := \begin{bmatrix}
P[k + 1] & 0 \\
\ast & R[k + 1]
\end{bmatrix}, \quad \Phi[k] := \begin{bmatrix}
A + BK[k] & BK[k] \\
E[k]\Lambda_{\ell[k]} & 0
\end{bmatrix}^T \Xi[k].
\]
Note that \(K[k]\) and \(E[k]\) also satisfy
\[
K[k] = \alpha_1[k]K_1 + \alpha_2[k]K_2, \quad E[k] = \alpha_1[k]\Lambda_1 + \alpha_2[k]\Lambda_2.
\]
Since
\[
\begin{bmatrix}
\Gamma[k] & \Phi[k] \\
\ast & \Xi[k]
\end{bmatrix} = \sum_{p=1}^{2} \sum_{q=1}^{2} \alpha_p[k]\alpha_q[k + 1] \begin{bmatrix}
\Gamma_{p,q,\ell[k]} & \Phi_{p,q,\ell[k]} \\
\ast & \Xi_{p,q,\ell[k]}
\end{bmatrix},
\]
the inequality \(10\) holds for every \(k \geq 0\) if the LMI \(5\) is feasible for every \(p, q = 1, 2\) and \(\ell \in \mathcal{J}_n\). This completes the proof of the state convergence \(5\) in the case \(\dot{\psi} = 0\) and \(\delta_i = 0\) for every \(i = 1, \ldots, n\).
We next consider the case \( \hat{w} \neq 0 \) or \( \delta_i \neq 0 \). Define
\[
\delta := \max_{i=1,\ldots,n} (\Delta_{\text{max}} \hat{w} \lambda_i + \delta_i).
\]
From (11), we have
\[
|e_i[k]| \leq \lambda_i E[k] \cdot |x_i[k]| + \delta.
\]
Therefore, for every \( i = 1,\ldots,n \), there exist \( f[k], g[k] \in \mathbb{R} \) such that
\[
e_i[k] = f[k] + g[k], \quad |f[k]| \leq \lambda_i E[k] \cdot |x_i[k]|, \quad |g[k]| \leq \delta. \tag{11}
\]
Define \( f[k] := [f_1[k],\ldots,f_n[k]]^T \) and \( g[k] := [g_1[k],\ldots,g_n[k]]^T \). Then
\[
V(k+1, x[k+1]) = ((A + BK[k])x[k] + BK[k]e[k])^T P[k+1] ((A + BK[k])x[k] + BK[k]e[k])
= ((A + BK[k])x[k] + BK[k]f[k])^T P[k+1] ((A + BK[k])x[k] + BK[k]f[k])
+ 2((A + BK[k])x[k] + BK[k]f[k])^T P[k+1] BK[k]g[k]
+ (BK[k]g[k])^T P[k+1] BK[k]g[k].
\]
From (11), there exist constants \( \rho_1 > 0 \) and \( \rho_2 > 0 \) such that
\[
2((A + BK[k])x[k] + BK[k]f[k])^T P[k+1] BK[k]g[k] \leq 2\rho_1 \|x[k]\| \delta \tag{12}
\]
and
\[
(BK[k]g[k])^T P[k+1] BK[k]g[k] \leq 2\rho_2 \delta^2.
\]
Since we have from Young’s inequality that
\[
2\rho_1 \|x[k]\| \delta \leq \rho_1(\theta \|x[k]\|^2 + \delta^2/\theta)
\]
for every \( \theta > 0 \) and since the LMI (6) leads to
\[
\gamma V(k, x[k]) - ((A + BK[k])x[k] + BK[k]f[k])^T P[k+1] ((A + BK[k])x[k] + BK[k]f[k]) > 0,
\]
it follows that
\[
\gamma V(k, x[k]) - V(k+1, x[k+1]) > -\rho_1(\theta \|x[k]\|^2 - (\rho_1/\theta + \rho_2)\delta^2) \geq -\rho_1(\theta V(k, x[k]) - (\rho_1/\theta + \rho_2)\delta^2
\]
for some \( \rho_1' > 0 \). Hence if we choose sufficiently small \( \theta > 0 \) satisfying \( \gamma' := \gamma + \rho_1'/\theta < 1 \), then we have
\[
V(k+1, x[k+1]) < \gamma' V(k, x[k]) + (\rho_1/\theta + \rho_2)\delta^2,
\]
and hence the state convergence (5) holds.

Remark 3.2. As the dimension of the state, \( n \), increases, the number of the LMIs in (6) grows with order \( 2^n \). To avoid this exponential growth, one may employ the following statement:
\[
|e_i[k]| \leq \lambda_i E[k] \cdot |x_i[k]| \quad \text{for all} \quad i = 1,\ldots,n \quad \Rightarrow \quad \|e[k]\| \leq \sqrt{\sum_{i=1}^n \lambda_i^2 \cdot E[k] \cdot \|x[k]\|}. \tag{13}
\]
It follows from (13) that when we apply the S-procedure for obtaining (8), we can use the matrix
\[
\sigma[k+1] = \begin{bmatrix} \sum_{i=1}^n \lambda_i^2 \cdot E[k] \cdot I & 0 \\ \star & -I \end{bmatrix}
\]
with \( \sigma[k+1] > 0 \), instead of the matrix
\[
W[k+1] = \begin{bmatrix} L[k+1] & N[k+1] \\ M[k+1] & 0 \end{bmatrix} W[k].
\]
Although this leads to the decrease of the number of LMIs, the derived sufficient condition becomes conservative. Therefore, we do not proceed along this line.
3.2. Switching control

Define $\Delta_{\text{min}} =: \Delta_0 < \Delta_1 < \cdots < \Delta_r := \Delta_{\text{max}}$ and $\mathcal{E}_p := [\Delta_0, \Delta_1], \mathcal{E}_r := (\Delta_{p-1}, \Delta_p)$ for all $p = 2, \ldots, r$. Given a feedback gains $K_p (p = 1, \ldots, r)$, we define $K(E[k])$ by

$$K(E[k]) := K_p \quad \text{if} \quad E[k] \in \mathcal{E}_p.$$  

(14)

Similarly to the gain-scheduling control (14), we obtain a sufficient condition for closed-loop stability with the switching control (14).

**Theorem 3.3.** Let Assumption\textsuperscript{2} hold. The closed-loop system satisfies the state convergence \textsuperscript{5} for some constant $\rho > 0$ if there exist a positive definite matrix $P$, a symmetric matrix with nonnegative entries

$$\begin{bmatrix}
L_{p,t} & N_{p,t} \\
0 & M_{p,t}
\end{bmatrix},$$

and a symmetric matrix $Z_{p,t} (p = 1, \ldots, r, t \in \mathcal{J}_n)$ such that

$$\begin{bmatrix}
\Gamma_{p,t} & \Phi_{p,t} \\
0 & \Xi_{p,t}
\end{bmatrix} > 0$$

(15)

for all $p = 1, \ldots, r$ and $t = (t_1, \ldots, t_n) \in \mathcal{J}_n$, where

$$\Gamma_{p,t} := \begin{bmatrix} P & 0 \\ 0 & N_{p,t} + N_{p,t}^T + Z_{p,t} \end{bmatrix}, \quad \Xi_{p,t} := \begin{bmatrix} P & 0 \\ 0 & R_{p,t} \end{bmatrix}, \quad \Phi_{p,t} := \begin{bmatrix} A + BK_p & BK_p^T \\ \Delta_p \Lambda_t & 0 \end{bmatrix}, \quad \Xi_{p,t} := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

$$\Lambda_t := \text{diag}(\lambda_1 t_1, \ldots, \lambda_n t_n), \quad R_{p,t} := \begin{bmatrix} L_{p,t} + M_{p,t} + N_{p,t} + N_{p,t}^T \\ L_{p,t} + M_{p,t} + N_{p,t}^T - N_{p,t} \end{bmatrix}.$$

**Proof:** Instead of the time-dependent Lyapunov function $V(k, x)$ in (7), we employ a common Lyapunov function:

$$V(x) := x^T P x.$$

The rest of the proof follows the same lines as that of Theorem\textsuperscript{4} it is therefore omitted.

4. Numerical Example

Consider the unstable batch reactor studied in \textsuperscript{32}, whose continuous-time model has the following system matrix $A_c$ and input matrix $B_c$:

$$A_c := \begin{bmatrix}
1.38 & -0.2077 & 6.715 & -5.676 \\
-0.5814 & -4.29 & 0 & 0.675 \\
1.067 & 4.273 & -6.654 & 5.893 \\
0.048 & 4.273 & 1.343 & -2.104
\end{bmatrix}, \quad B_c := \begin{bmatrix}
0 & 0 \\
5.679 & 0 \\
1.136 & -3.146 \\
1.136 & 0
\end{bmatrix}.$$  

This model is widely used as a benchmark example, and for reasons of commercial security, the data were transformed by a change of basis and of time scale \textsuperscript{32}. Here we discretize this system with a sampling period $h = 0.01$ and use $A := e^{A_c h}$ and $B := \int_0^h e^{A_c t} B_c dt$ for the plant (1).

For the gain-scheduling control and the switching control, we design the following two linear quadratic regulators:

$$K_1 := \begin{bmatrix}
-0.5466 & -0.3732 & -0.4618 & -0.0421 \\
1.7706 & 0.1643 & 1.2380 & -0.7572
\end{bmatrix}, \quad K_2 := \begin{bmatrix}
3.0966 & -6.2406 & -0.8145 & -7.4480 \\
10.8086 & -0.2883 & 8.7038 & -4.2906
\end{bmatrix}.$$  

Fix the lower bound $\Delta_{\text{min}} = 0.05$ and the weighting constants $\lambda_i = 1 (i = 1, \ldots, n)$ of the event-triggered TOD protocol. We see from Theorem\textsuperscript{53} of the case with a single gain that the low gain $K_1$ and the high gain $K_2$ allow
\[ \Delta_{\text{max}} \leq 0.20 \quad \text{and} \quad \Delta_{\text{max}} \leq 0.64 \] without compromising closed-loop stability, respectively. Since we can regard \( E[k] \) as system variation, high gain control stabilizes the closed-loop system with large upper bound \( \Delta_{\text{max}} \) in general. As we see later, however, high gain control is sensitive to measurement noise. Hence we here design the gain scheduling control \( 4 \) and the switching control \( 14 \) so that a low gain is chosen for small \( E[k] \) and a high gain for large \( E[k] \).

Such adaptive control exploits another advantage of high gain control, a fast response, because \( E[k] \) is large at the transient stage. The gain scheduling control \( 4 \) allows \( \Delta_{\text{max}} \leq 0.35 \) from Theorem 3.1 and the switching control \( 14 \) with \( \Delta_{\text{min}} = \Delta_1 < \Delta_2 = 2\Delta_{\text{min}} < \Delta_3 = \Delta_{\text{max}} \) allows \( \Delta_{\text{max}} \leq 0.28 \) from Theorem 3.3.

For the simulation of time responses, we set the parameters \( \Delta_{\text{max}}, \Delta_{\text{min}}, \delta_i \) and \( \delta_i \) of the event-triggered TOD protocol to be \( \Delta_{\text{max}} = 0.28, \Delta_{\text{min}} = 0.05, \delta_i = 1, \) and \( \delta_i = 0.01 \) for all \( i = 1, \ldots, n \). Figures 2a and 2b illustrate the time response of each control under the noise \( w[k] \) that is chosen from the uniform distribution on the interval \([-10^{-3}, 10^{-3}]\) for every \( k \geq 0 \). Both of the gain scheduling control and the switching control exhibit a faster response than the low gain control with \( K_1 \) and are more robust against measurement noise than the high gain control with \( K_2 \).

We can also observe the effectiveness of the adaptive control methods from Table 1, which shows the total number of measurements transmitted in the time interval \([0, 4] \) for each control. We calculate the average of the total number for 100 samples, and if all the measurements are transmitted at every sampling time, then the total number is \( 4n/h = 1600 \). The gain scheduling control and the switching control require less data than the control with a single gain because of their fast convergence and robustness against measurement noise.

\[
\begin{array}{cccc}
\text{Gain scheduling} & \text{Switching} & \text{Low gain} K_1 & \text{High gain} K_2 \\
48.56 & 50.65 & 71.76 & 113.79
\end{array}
\]

5. Conclusion

We studied the stability analysis of adaptive control systems with the event-triggered TOD protocol. For each case of gain scheduling control and switching control, we obtained a sufficient LMI condition for closed-loop stability under bounded noise and illustrated the effectiveness of the adaptive control methods through a numerical example. Future work will focus on addressing the \( L^2 \)-gain analysis of systems with the event-triggered TOD protocol and generalizing the proposed stability analysis to systems with packet losses and modeling uncertainty.

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