Generalization of the Fermi pseudopotential

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Abstract

Introduced eighty years ago, the Fermi pseudopotential has been a powerful concept in multiple fields of physics. It replaces the detailed shape of a potential by a delta-function operator multiplied by a parameter giving the strength of the potential. For Cartesian dimensions \(d > 1\), a regularization operator is necessary to remove singularities in the wave function. In this study, we develop a Fermi pseudopotential generalized to \(d\) dimensions (including non-integer) and to non-zero wavenumber, \(k\). Our new fractional calculus approach has the advantage of circumventing singularities that occur in the wave function at integer values of \(d\) while being valid arbitrarily close to integer \(d\). In the limit of integer dimension, we show that our generalized pseudopotential is equivalent to previously derived \(s\)-wave pseudopotentials. Our pseudopotential generalizes the operator to non-integer dimension, includes energy \((k)\) dependence, and simplifies the dimension-dependent coupling constant expression derived from a Green’s function approach. We apply this pseudopotential to the problem of two cold atoms \((k \to 0)\) in a harmonic trap and extend the energy expression to arbitrary dimension.

Keywords: pseudopotential, fractional calculus, arbitrary dimension, two cold trapped atoms, ultra-cold atoms

1. Introduction

We present a generalization of the Fermi pseudopotential to all values of the spatial dimension, both even and odd integer, as well as non-integer dimension, and to non-zero wavenumber. As an initial application, we use this pseudopotential to generalize the energy for two ultra-cold atoms in a trap to arbitrary dimensions. The Fermi pseudopotential has had a profound influence since its introduction in 1936 [1], having been used extensively in nuclear, condensed matter, atomic and chemical physics. Pseudopotentials replace the details of short-range interaction potentials with shape-independent potentials that reproduce the physics often through a single parameter. When the range of interaction is small compared to the interparticle distance, the details of the interaction become unimportant allowing the use of pseudopotentials.

Early in the use of the Fermi pseudopotential in the context of nuclear physics [2–5], it was recognized that in three dimensions, a regularizing operator must be combined with the delta function to remove singularities in the wave function. The shape independence of the Fermi pseudopotential has been embodied in an effective range expansion [4] that is the basis of far-reaching applications in multiple fields of physics. Applications of the Fermi pseudopotential to many-body systems was initiated by Huang and Yang and applied to systems of hard-core bosons [6–9].

The Fermi pseudopotential, also referred to as a zero-range or contact potential, has seen a resurgence of use in the current literature of ultra-cold atom gases [10–15]. The experimental realization of atomic gas Bose–Einstein condensates led to the study of ultra-cold systems of atoms with long de Broglie wavelengths that permitted the use of simple pseudopotentials since the details of the interaction potential were not relevant.

Although the great majority of work using pseudopotentials has been for three-dimensional systems, the Fermi
Pseudopotential has also been used in other integer dimensions, including one, two and five dimensions. The one-dimensional Fermi pseudopotential (i.e., the delta function) has been used extensively as an exactly solvable model problem of many-body theory [16]. It has also been applied to quark tunneling [17], hard photodetachment and multiphoton ionization [18–20], periodic lattices [21], in the theory of transport phenomena [22] and two cold atoms in a harmonic trap [23]. The five-dimensional Fermi pseudopotential has also been used in the problem of narrow resonances in infinite, uniform chains [24].

Two dimensions is of particular interest since several interesting collective phenomena can emerge in reduced dimensionality including the Berezinsky–Kosterlitz–Thouless transition [25–30], quantum magnetism [31], superconductivity [32, 33], superfluidity [34] and topological insulators [35]. Two-dimensional systems have been suggested as potential platforms for topological quantum computation [36]. Studies using two-dimensional pseudopotentials include quantum gases [37] as well as for two cold atoms in a harmonic trap [23].

More recently the unprecedented control of ultra-cold systems has resulted in highly anisotropic trapping configurations yielding quasi-one and quasi-two dimensional systems [38, 39]. When these ultra-cold particles are tightly confined along two directions and weakly confined along the third, a quasi-one-dimensional system is created. The fractional quantum Hall effect in quasi-two dimensions may be possible for bosons confined tightly along one direction and weakly along the other two. These quasi-one- and two-dimensional configurations can be combined with a tunable scattering length to lead to largely unexplored physical regimes. These configurations have been studied by several theoretical approaches [29, 30, 40–42] including renormalizing the coupling strength [43], but an interesting alternate approach would be to use a Fermi pseudopotential for non-integer dimensions between one and two, perhaps specified by the ratio of the confinement strengths.

The Fermi pseudopotential has previously been extended to all the odd integers [44]. General expressions for all the even integers \(d \geq 2\) have not been explicitly given due to a complicated singularity structure which results in expressions that are not generally useful [44]. Our method has the advantage of circumventing singularities that occur at integer values of \(d\), simplifying the dimension-dependent coupling constant expression derived from an earlier Green’s function approach [44], generalizing the pseudopotential to non-integer dimension and including energy (\(k\)) dependence. This work thus offers a broader approach to obtaining valid pseudopotentials in any dimension of interest.

This paper is organized in the following way. In section 2 we discuss the singularities that occur at integer values of \(d\) and our strategy to circumvent these singularities by staying at non-integer values of dimension and using non-integer or ‘fractional’ derivatives to develop a general pseudopotential. Appendix A reviews the relevant mathematical properties of fractional calculus that are used in our formulation of the generalized pseudopotential operator.

In section 3, we construct the generalized pseudopotential for two cases: \(k \to 0\) and finite \(k\). In appendix B, we construct the general \(d\) pseudopotential for \(k \to 0\). This development closely parallels the 3D development found in Huang’s Statistical Mechanics [45]. With the help of appendices C and D, we construct the general-dimension pseudopotential with non-zero wavenumber \(k\) (equation (16)). We derive this general pseudopotential by introducing an arbitrary-dimension ‘fractional’ Frobenius series solution to the \(d\)-dimensional relative-motion Schrödinger free-wave equation with a hard-sphere boundary condition. In section 4 and appendix E, we show that integer limits of the general pseudopotential, with zero and non-zero \(k\), agree with previous generalizations. Our pseudopotential simplifies the expression for the dimension-dependent coupling constant in an earlier Green’s function derivation [44], and includes energy (\(k\)) dependence. In section 5 along with appendices G and H, we apply our general pseudopotential to the problem of two cold atoms in a harmonic trap [23] and generalize the Busch energy relationships to arbitrary \(d\) (equations (40) and (41)). Section 6 contains discussion and possible future directions.

2. Pseudopotentials and singularities

Pseudopotentials are shape-independent (contact) potentials that approximate the effect of detailed short-range interaction potentials. The simplest pseudopotential is the \(\delta\)-function, introduced by Fermi [1]. Later, it was noted that, for three dimensions a regularizing operator of the form \(\left(\frac{\partial}{\partial r}\right)^7\) is required to remove singularities in the wave function [2–5]. For odd \(d \geq 3\), not only must the leading singularity of the form \(1/r^{d-2}\) be removed by the pseudopotential, but it must also remove other lower-order singularities in \(1/r\). For example, when \(d = 7\), singularities \(1/r^5\), \(1/r^3\), and \(1/r\) arise and must be removed. When \(d\) is an even integer, \(\ln(r)\) singularities arise in the series solution resulting in expressions that become quite complicated as even \(d\) values increase. The Green’s function approach used by Wódkiewicz [44] to derive the arbitrary odd integer-\(d\) pseudopotential removes all singularities in higher dimensions but does not include energy dependence. In more strongly interacting regimes, energy dependence becomes important, and it has been shown, for example, that using an energy-dependent scattering length improves the description of two-atom systems [43, 46]. Our generalized pseudopotential removes singularities in higher dimensions and includes energy dependence.

In the current study, we consider non-integer dimensions and demonstrate agreement with previous results in the limit of integer dimensionality. Our strategy to circumvent this singularity structure is to work only with non-integer dimensions. This approach bypasses these singularities because \(d\) is never allowed to be integer, although it may be arbitrarily close. Developing the formalism for non-integer dimension pseudopotentials requires the use of non-integer calculus, specifically the ability to take derivatives at non-integer order. This branch of mathematics has developed several alternative definitions for non-integer or ‘fractional’ derivatives. In appendix A we review the properties of the
fractional derivative that we have chosen based on the physics to develop our pseudopotentials. After deriving our pseudopotential, we demonstrate agreement with previous results in the limit of integer dimensionality.

3. Construction of the generalized pseudopotential

The purpose of the regularized Fermi pseudopotential is to obtain a Schrödinger equation with an inhomogeneous term that reproduces the effect of the hard-sphere boundary condition on the scattered wave function. That is, we wish to derive a pseudopotential that places a node in the wave function at \( r = \bar{a} \), where \( \bar{a} \) is small compared to the interatomic spacing. For \( r > \bar{a} \), the system follows a non-interacting \( d \)-dimensional relative-motion Schrödinger equation for two atoms interacting via a hard sphere of radius \( \bar{a} \):

\[
(\nabla_d^2 + \bar{a}^2)\psi(r) = 0, \quad r > \bar{a},
\]

where \( \nabla_d^2 = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} \frac{\partial}{\partial r} \right) \). The quantity \( \bar{a} \) is a radius in \( d \) dimensions. For notational simplicity, we write \( \bar{a} \) instead of \( \bar{a}_d \) when the dimension is clear from the context. When the dimension is not obvious, we use a subscript (e.g. \( \bar{a}_3 \)). In the following sections, we derive the pseudopotential for \( k \to 0 \) from the boundary condition of the Schrödinger equation (section 3.1) and for arbitrary \( k \) by solving the Schrödinger equation (equation (1)) for \( \psi(r) \) (section 3.2).

3.1. General \( d \) pseudopotential with \( k \to 0 \) from the boundary condition

We derive the \( k \to 0 \) general \( d \) (non-integer) pseudopotential in some detail because many of the techniques used are similar to the more complicated arbitrary-\( k \) solution. Similar to [45] (Huang pp 276–279 for \( d = 3 \)), we integrate the \( d \)-dimensional Schrödinger equation (equation (1)) when \( k \to 0 \) and the wave function satisfies the boundary condition:

\[
\psi(r) \to \left(1 - \frac{\bar{a}^d - 2}{r^d}\right)B \quad \text{as} \quad r \to 0,
\]

where \( B \) is an integration constant that depends on the boundary condition as \( r \to \infty \), \( d \) is the spatial dimensionality and \( k \) is the magnitude of the relative wave vector defined by \( E = \hbar^2 k^2 / 2\mu \), with \( E \) being the relative-motion energy and \( \mu = m/2 \) the reduced mass of the system.

To isolate \( B \) in terms of \( \psi \) as \( r \to 0 \), one can operate on equation (2) with \( r^{d-2} \) and then a derivative \( c_d \delta_d^{d-2} \) of order \( (d - 2) \), which may be non-integer (equation (A.5)). In appendix B and with this choice for non-integer derivative, we obtain the following expression for the generalized pseudopotential as \( k \to 0 \):

\[
c_d \delta_d^{(d)(0)}(r) = \frac{\Omega(d) \bar{a}^{d-2}}{\Gamma(d - 2)} \delta^{(d)}(r) c_d \delta_d^{d-2} r^{d-2},
\]

where \( \Omega(d) = 2\pi^{d/2} / \Gamma(d/2) \) is the solid angle parameterized by \( d \). The delta function, \( \delta^{(d)}(r) \), is well defined in \( d \) dimensions including non-integer and has the expected behavior when integrated inside a \( d \)-dimensional integral. Similarly, the order \( (d - 2) \) derivative with respect to \( r \), \( c_d \delta_d^{d-2} \), is well defined in \( d \) dimensions. When \( d \) is integer, the derivative has the expected behavior of standard calculus. For non-integer \( d \), the derivative becomes an integral (equation (A.5)); however, there are closed-form expressions for non-integer derivatives of standard functions used in this study, like powers and Gaussians. This arbitrary \( d \) pseudopotential (equation (3)) will be the starting point for the ultra-cold two trapped atom derivation in section 5. This pseudopotential is the limiting case \( (k \to 0) \) of the more general \( k \)-dependent pseudopotential (equation (16)) derived in the next section 3.2.

3.2. General \( d \) pseudopotential for arbitrary energy \( k \)

When \( k \) is non-zero, one cannot simply integrate the Schrödinger equation. To derive the pseudopotential we first develop the power series solution of the Schrödinger equation to identify the singularities that need to be removed. Solving the differential equation of equation (1) and developing a series solution for \( \psi \) using a non-integer \( d \) Frobenius method (see equations (C.1)–(C.9) for details), we find a regular solution

\[
\psi_{reg}(r) = \sum_{n=0}^{\infty} \alpha_n r^{2n}
\]

and an irregular solution

\[
\psi_{irreg}(r) = c \ln(r) \psi_{reg} + \frac{1}{r^{d-2}} \sum_{n=0}^{\infty} \beta_n r^{2n}.
\]

In general, the coefficient \( c \) depends on the dimension \( d \) (see appendix C for all cases). However, since we assume non-integer \( d \), we can allow \( c = 0 \) and avoid the \( \ln(r) \) term (see discussion near equations (C.23) and (C.24)). The non-integer-\( d \) series solution can be made arbitrarily close to integer, and we will find that the limit for odd \( d \) is smooth while the even \( d \) limit must be handled more carefully.

Thus, for non-integer dimension, we may then combine equations (4) and (5) to obtain the non-integer \( d \) solution to the Schrödinger equation:

\[
\psi(r) = \sum_{n=0}^{\infty} \alpha_n r^{2n} + \frac{1}{r^{d-2}} \sum_{n=0}^{\infty} \beta_n r^{2n}.
\]

Similar to the approach for \( k \to 0 \) (section 3.1 and appendix B), we integrate the left side of the Schrödinger equation (1) over a small sphere of infinitesimal radius \( \epsilon \) about the origin. However, now \( k \) is allowed to be non-zero and we use \( \psi \) in equation (6). We find that as \( \epsilon \to 0 \) (appendix D)

\[
\int (\nabla_d^2 + k^2) \psi(r) dV = \Omega(d) \beta_0 (2 - d) \int \delta^{d}(r) dV
\]

or (from equation (D.16))

\[
(\nabla_d^2 + k^2) \psi = \Omega(d) \beta_0 (2 - d) \delta^{d}(r).
\]
Now our goal is to find an explicit expression for $\beta_0$ from the series solution. We do this by choosing an operator that will give $\beta_0$ in terms of the full wave function as $r \to 0$. We first operate on equation (6) with $r^{d-2}$ and a derivative of order $d - 2$ to remove singularities when $r \to 0$:

$$[c_0 D^{d-2}_r r^{d-2} \psi]_{r=0} = \left[ c_0 D^d - \sum_{n=0}^{\infty} \alpha_2 n r^{2n} + \sum_{n=0}^{\infty} \beta_2 n r^{2n} \right]_{r=0}$$

$$= \left[ \sum_{n=0}^{\infty} \Gamma(2n + d - 1) \alpha_2 n r^{2n} + \sum_{n=0}^{\infty} \Gamma(2n + 1) \beta_2 n r^{2n} \right]_{r=0}$$

$$= \sum_{n=0}^{\infty} \left[ \frac{\Gamma(2n + 1)}{\Gamma(2n + 1)} \beta_2 n r^{2n} + \frac{\Gamma(2n + 1)}{\Gamma(2n - d + 3)} \beta_2 n r^{2n-d+2} \right]_{r=0},$$

where in the second line we used a fractional derivative equation for powers (See equation (A.4)). When $r \to 0$, we note the summation on the right does not contribute because of the lower limit $\left\lfloor \frac{d}{2} \right\rfloor$ (floor of $d/2$) and only the $n = 0$ term contributes from the left summation so that

$$[c_0 D^{d-2}_r r^{d-2} \psi]_{r=0} = \Gamma(d - 1) \beta_0.$$

We wish to rewrite $\beta_0$ in terms of the wave function $\psi$ to substitute into equation (8). Recall the relationship between $\alpha_0$ and $\beta_0$ (equation (C.26)) derived from imposing the boundary condition (node at $r = \hat{a}$) for the Schrödinger equation:

$$\alpha_0 = -\frac{\beta_0}{\bar{a}^{d-2}} \frac{\psi(2 - d - \frac{\bar{k}^2 \bar{a}^2}{4})}{\psi(\frac{d}{2}; -\frac{\bar{k}^2 \bar{a}^2}{4})},$$

where $\psi(\cdot; \cdot)$ is the confluent hypergeometric function [47]. Using this expression for $\alpha_0$ to write equation (10) in terms of $\beta_0$, we find

$$[c_0 D^{d-2}_r r^{d-2} \psi]_{r=0} = -\Gamma(d - 1) \beta_0 \frac{\psi(2 - d - \frac{\bar{k}^2 \bar{a}^2}{4})}{\psi(\frac{d}{2}; -\frac{\bar{k}^2 \bar{a}^2}{4})},$$

and we can isolate $\beta_0$:

$$\beta_0 = \frac{\bar{a}^{d-2} \psi(2 - d - \frac{\bar{k}^2 \bar{a}^2}{4})}{\bar{a}^{d-1} \frac{\psi(\frac{d}{2}; -\frac{\bar{k}^2 \bar{a}^2}{4})}{\psi(\frac{d}{2}; -\frac{\bar{k}^2 \bar{a}^2}{4})}} [c_0 D^{d-2}_r r^{d-2} \psi]_{r=0}.$$

Finally, substituting this $\beta_0$ in equation (8), we obtain the finite $d$, finite $k$ pseudopotential

$$\left( \nabla^2_d + \bar{k}^2 \right) \psi(r) = \frac{\Omega(d) (d - 2) \bar{a}^{d-2}}{\Gamma(d - 1)} T(d, \bar{k} \hat{a}) \delta^{(d)}(r) c_0 D^{d-2}_r r^{d-2} \psi(r),$$

where we define

$$T(d, \bar{k} \hat{a}) = \frac{\partial F_{1} \left( \frac{2 - d}{2}; -\frac{\bar{k}^2 \hat{a}^2}{4} \right)}{\partial F_{1} \left( \frac{d}{2}; -\frac{\bar{k}^2 \hat{a}^2}{4} \right)} \prod_{n=0}^{\infty} \frac{\Gamma(2n + 1)}{\Gamma(2n + 1)} \beta_2 n r^{2n} + \sum_{n=0}^{\infty} \frac{\Gamma(2n + 1)}{\Gamma(2n - d + 3)} \beta_2 n r^{2n-d+2} \right]_{r=0},$$

which is related to the scattering phase shift in $d$ dimensions for $s$-wave. The case $T(3, \bar{k} \hat{a}) = \tan(k\hat{a})/(k\bar{a})$ leads to the correct $s$-wave scattering phase shift for a hard sphere with radius $\hat{a}$ in $d = 3$. Higher dimensions have more complex forms, which we discuss along with other limits below. We note that $T(d, \bar{k} \hat{a}) = 0$ for even $d > 2$ ($T(1 = 1$ for $d = 2$) and requires a perturbation expansion of equation (15) near even $d > 2$.

Simplifying the operator on the right side of equation (14), we obtain the general $k$ and $d$ pseudopotential,

$$c_0 V_k^{(d)} = \frac{\Omega(d) \bar{a}^{d-2}}{\Gamma(d - 2)} T(d, \bar{k} \hat{a}) \delta^{(d)}(r) c_0 D^{d-2}_r r^{d-2}.$$

In the next section (section 4), we explore cases of the generalized pseudopotential and test its validity in various limits of $k$ and $d$. This pseudopotential was also designed to be valid for non-integer dimensions, which is relevant for trapped systems that are highly anisotropic and behave as quasi-one or quasi-two dimensional. Similar to the $k \to 0$ pseudopotential (equation (3)), this general pseudopotential is parameterized by $d$ and is well defined when $d$ is non-integer. In section 5, we demonstrate the mathematical techniques for applying our generalized pseudopotential to systems of non-integer dimension. Specifically, we derive the energy for two ultracold atoms ($k \to 0$) in a harmonic trap in arbitrary dimension (equation (38)).

**4. Cases of the generalized pseudopotential for specific limits of $d$ and $k$**

Here we demonstrate some specific cases of the $s$-wave, arbitrary-$d$ pseudopotential (equation (16)). We show the analytical forms of the pseudopotential for $d = 3$ and $d = 5$ with zero and non-zero $k$. We also derive the relationship between the simple coupling constant in our generalized pseudopotential with the more complicated expression in [44].

**4.1. Non-zero $k$ with $d = 3$ and $d = 5$**

When the non-integer dimension approaches $d \to 3$, the finite $k$ generalized pseudopotential (equation (16)) takes the form

$$c_0 V_k^{(3)} = \frac{\Omega(3) \bar{a}}{\Gamma(1)} T(3, \bar{k} \hat{a}) \delta^{(3)}(r) c_0 D_r r,$$

where the phase-shift related function (equation (15)) becomes

$$T(3, \bar{k} \hat{a}) = \frac{\tan(k \hat{a})}{k \hat{a}}$$

and yields

$$c_0 V_k^{(3)} = \frac{4\pi}{k \cot(k \hat{a})} \delta^{(3)}(r) \frac{d}{dr} r,$$
which is consistent with the $s$-wave pseudopotential derived by Huang (equation (13.12) p 277 in [45]). For $d \to 3$, the prefactor that comes from $T$ is positive and increases with $k\bar{a}$ and diverges as $k\bar{a} \to \pi/2$. Different behavior emerges for $d \to 5$ next.

Similarly, when the non-integer dimension approaches $d \to 5$, the finite $k$ generalized pseudopotential (equation (16)) takes the form

$$c_0 V^{(5)}_k = \frac{\Omega(5)s^3}{\Gamma(3)} T(5, k\bar{a}) \delta^{(5)}(r) \sqrt{|c_0 D_3 r|},$$

(20)

where the phase-shift related function (equation (15)) is

$$T(5, k\bar{a}) = \frac{3}{(k\bar{a})^3} \left( \frac{\tan (k\bar{a}) - k\bar{a}}{k\bar{a} \tan (k\bar{a}) - 1} \right)$$

(21)

and results in

$$c_0 V^{(5)}_k = \frac{4\pi^2}{k^3} \left( \frac{\tan (k\bar{a}) - k\bar{a}}{k\bar{a} \tan (k\bar{a}) - 1} \right) \delta^{(5)}(r) \frac{d^3}{dr^3} r^3.$$

(22)

As $k\bar{a}$ increases, the expression in parentheses for $d \to 5$ above is a negative and decreasing function of $k\bar{a}$ and diverges as $k\bar{a} \to 0.8603$ (first positive numerical solution).

4.2. The $k\to0$ limit of the arbitrary $d$ pseudopotential

For $d \to 3$, the series expansion about $k \to 0$ of the phase-shift related function (equation (15)) gives the usual result for the hard-sphere potential:

$$T(3, k\bar{a}) = \frac{\tan (k\bar{a})}{k\bar{a}} \approx 1 + \frac{k^2 a^2}{3} + O(k^4).$$

(23)

Thus, in the $k \to 0$ (cold atom) $d \to 3$ limit, the generalized pseudopotential,

$$c_0 V^{(3)}_k = \frac{4\pi \bar{a} k^{3}(r)}{\bar{a} d^3} \frac{d}{dr},$$

(24)

agrees with Huang (equation (13.11) p 277 in [45]).

For non-even $d$, the series expansion about $k \to 0$ of the phase-shift related function (equation (15)) is

$$T(d, k\bar{a}) \approx 1 - \frac{(d - 2) k^2 a^2}{(d - 4)} \frac{d}{dr} + O(k^4)$$

(25)

and the generalized pseudopotential (equation (16)) approaches

$$c_0 V^{(d)}_k \approx \frac{\Omega(d) d^{d-2}}{\Gamma(d - 2)} \delta^{(d)}(r) c_0 D_r d^{d-2} r^{d-2} - 2,$$

(26)

which is the $k \to 0$ pseudopotential (equation (3)) previously derived from the boundary condition of the Schrödinger equation.

4.3. Relationship between the $k \to 0$ generalized pseudopotential $c_0 V^{(d)}_k$ and the Wódkiewicz pseudopotential (Green's function derivation) $V^{(d)}_W$

The operator in equation (26) has the same form as the operators derived by Wódkiewicz for odd $d$ (equations 5.4–5.5 in [44]). However, it is not obvious how the prefactors are related because of the complicated summation (equation 5.5 [44]) and the way the coupling constant $a_d$ is defined in [44]. The coupling $a_d$ is not simply the hard sphere radius in $d$ dimensions like our $\bar{a}_d$. For each dimension, $a_d$ is related to the $d\bar{a}$ scattering length $a_d$. Here we show how Wódkiewicz’s $a_d$ is related to $\bar{a}_d$ and how the Wódkiewicz prefactors (equation 5.5 [44]) can be simplified.

First, in appendix E.1 we show that the complicated regularizing coefficient $\gamma_{2n+1}$ (equation 5.5 in [44])

$$\gamma_{2n+1} = \frac{\pi^{-1/2} \Gamma \left( \frac{1}{2} - n \right)}{\sum_{l=0}^{n-1} (-1)^l + 2^{2n} + \pi \left( \frac{1}{2} + \frac{1}{2n} \right) \left( \frac{1}{2} + \frac{1}{2n} \right) \left( \frac{1}{2} + \frac{1}{2n} \right) \left( \frac{1}{2} + \frac{1}{2n} \right)}$$

under specific assumptions simplifies to

$$\gamma_{2n+1} = \frac{1}{\Gamma(2n)} = \frac{1}{\Gamma(d - 1)}.$$

(27)

Next, we show how Wódkiewicz’s $a_d$ is related to $\bar{a}_d$. The coupling constant or area of the potential $a_d$ in [44] is defined in terms of the scattering amplitude,

$$f_k^{(d)} = f_k^{(2n+1)} = -\frac{e^{-k a_d}}{k^{2n-1} + \pi e^{-k a_d} + \pi \left( \frac{1}{2} + \frac{1}{2n} \right) \left( \frac{1}{2} + \frac{1}{2n} \right) \left( \frac{1}{2} + \frac{1}{2n} \right) \left( \frac{1}{2} + \frac{1}{2n} \right)}$$

(29)

(equation 3.1 in [44]). The scattering amplitude equation in [44] contains a typographical error, which we note in appendix I. The area of the potential $a_d$ is determined by finding the pole of the scattering amplitude $f_k^{(d)}$ in the physical $k$ half plane while keeping the energy at each single bound state in $d$-dimensions constant. In [44], expressions for $a_d$ are only derived for $d = 1, 3,$ and 5, but here we extend to all odd $d$ the relationship between $a_d$ and the one-dimensional area of the potential $a_1$ (see equation (E.17)):

$$a_d = \frac{(2\pi)^{(d-1)/2} (d - 2)!}{a_1^{d-2} m^{d-1}}, \quad d \text{ odd}.$$

(30)

Substituting our general formula above for the strength or ‘area of potential,’ $a_d$, into Wódkiewicz’s definition of the pseudopotential in terms of $a_d$ (equation (1.2) in [44]),

$$V_W^{(d)}(r) = -a_d \delta^{(d)}(r) \bar{R}_d,$$

(31)

where $\bar{R}_d$ is the regularizing operator (equation (5.4) in [44]), we produce an explicit general expression for the Wódkiewicz Fermi pseudopotential for odd $d > 1$ (see equation (E.19) derivation):

$$V_W^{(d)}(r) = -\frac{\hbar^2 (d - 2) \pi^{(d-1)/2} (d - 4)!}{m^{d-1} a_d^{d-2}} \times \frac{1}{\Gamma(d - 2)} \delta^{(d-2)} \frac{d^{d-2}}{\partial r^{d-2}}, \quad d > 1 \text{ odd},$$

(32)

where we remind the reader that $(-1)! = 1$.

In the approach in [44], $a_1$ is the one-dimensional area of potential, and the pseudopotential coupling constant in other dimensions is related to $a_1$. This causes $a_1$ to appear in the denominator of equation (32) for $d > 1$. In contrast, we use the characteristic length $\bar{a}_d$ of the potential, which causes the repulsive strength of $c_0 V^{(d)}_k(r)$ (equation (26) limit of equation (16)) to be directly proportional to $\bar{a}_d$ for $d > 1$. We verified (table 1) that the limit of $c_0 V^{(d)}_k(r)$ when $d$
approaches an odd integer agrees with the pseudopotential results listed in [44] ($d = 1, 3, \text{ and } 5$). Equating equations (26) and (32) we find that our wave function node position $\tilde{a}_d$ and the constant $a_1$ used by Wódkiewicz [44] are related by the following expression

$$\tilde{a}_d^{-2} = \left(\frac{2i\hbar^2}{ma_1}\right)^{d-2} \left(\frac{\hbar^2}{2m(d-2)}\right). \tag{33}$$

The two columns of table 1 match (and all odd dimensions match) when using the above conversion equation (33). We note that, when $d = 1, 3, \text{ and } 5$, the relationship between $\tilde{a}_d$ and $a_1$ becomes

$$a_1 = \frac{2}{a_3}, \quad a_3 = \frac{\hbar^4}{2m^2a_1}, \quad \text{and} \quad a_5 = \frac{\hbar^2}{ma_1} \sqrt{\frac{3\hbar^2}{2m}}. \tag{34}$$

As $d \to 1$ the generalized pseudopotential (26) prima facie approaches zero. However, taking the $d \to 1$ limit in the non-integer (‘fractional’) derivative part of equation (26) and using the non-integer derivative (A.4), the 1$d$ pseudopotential approaches the non-zero result shown in table 1. See appendix E for the 1$d$ derivation of $c_a V_1^{(k=0)} = -\frac{2}{a_1} \delta^{(1)}(r)$. For even dimensions, the pseudopotential takes on a different form involving a logarithmic term inside the regularization operator. Wódkiewicz did not present an explicit expression for the pseudopotential in the case of general even dimension. In the current work, we derived an expression for the Wódkiewicz pseudopotential for all odd dimension and validated the equivalence of our non-integer operator. It may be useful for future study to produce a general even dimension expression for the Wódkiewicz pseudopotential.

5. Energy solution of two cold atoms in a harmonic trap in arbitrary dimension

In this section, we extend the Busch derivation in [23] of the energy for two cold atoms in a trap for $d = 1, 2, 3$ to arbitrary dimension including non-integer. The Schrödinger equation for the relative motion of two cold atoms in a harmonic trap is defined by:

$$(H^{(d)}_{\text{osc}} + \tilde{V}^{(d)}(r))\Psi(r) = E^{(d)}\Psi(r). \tag{35}$$

For the three-dimensional case, Busch used $\sqrt{2} \pi a_0 \delta^{(3)}(r) \frac{\partial}{\partial r}$ in place of $\tilde{V}^{(d)}(r)$ where $a_0$ is the $s$-wave scattering length (equation (3) of [23]). Note that the Busch $a_0$ is a radius in the dimension of space being considered and is not related to the Wódkiewicz coupling constant $a_d$. In this section, we solve the two trapped atom energy for arbitrary $d$ with our generalized pseudopotential $c_a V^{(d)}_{k=0}$ (equation (26)) in place of $\tilde{V}^{(d)}(r)$ in equation (35).

5.1. Exact energy solution

The energy for two cold trapped atoms with the generalized pseudopotential may be solved by expanding $\Psi$ in equation (35) in terms of the $d$-dimensional harmonic oscillator basis wave functions, $\phi_n^{(d)}$, satisfying $H^{(d)}_{\text{osc}} \phi_n^{(d)} = E^{(d)}\phi_n^{(d)}$. In appendix F, we derive the solution of the $d$-dimensional harmonic oscillator, which is needed for the two cold atom trap derivation below and appendix G. Recalling our ultra-cold ($k \to 0$) general $d$ pseudopotential from equation (26):

$$c_a V^{(d)}_{k=0} = \frac{\Omega(d)\hbar^{d-2}}{\Gamma(d-2)} \delta^{(d)}(r) \ c_a D_r^{d-2} \rho^{d-2}. \tag{36}$$

and expanding $\Psi(r) = \sum_{n=0}^\infty c_n \phi_n^{(d)}(r)$, equation (35) becomes:

$$\sum_{n=0}^\infty c_n E_n^{(d)} \phi_n^{(d)} + \frac{\Omega(d)\hbar^{d-2}}{\Gamma(d-2)} \delta^{(d)}(r) \left[ c_a D_r^{d-2} \rho^{d-2} \sum_{n=0}^\infty c_n \phi_n^{(d)}(r) \right]_{r=0} = \sum_{n=0}^\infty c_n E_n^{(d)} \phi_n^{(d)}, \tag{37}$$

where $c_a D_r^{d-2}$ is the Caputo fractional derivative. We follow the integral procedure in Busch with special considerations for non-integer dimension (appendix G) to reduce the energy to the following implicit equation for the energy

$$\frac{\pi}{\Gamma^2\left(\frac{d}{2}\right)} \left[ \Gamma\left(-\frac{E^{(d)}}{2} + \frac{d}{2}\right) + \frac{\sin\left(d\frac{dz}{2}\right)}{d^{d-2}} \right] = \frac{\pi}{\Gamma^2\left(\frac{d}{2}\right)} \left[ \Gamma\left(-\frac{E^{(d)}}{2} + \frac{d}{2}\right) \right]. \tag{38}$$

In order to compare with the three-dimensional results in Busch et al we convert our $a_d$ scattering length in $d$ dimensions to relative motion units ($a_d$, used in [23]):

$$a_d^{-2} = \frac{a_d^{d-2}}{2^{d/2}}, \tag{39}$$

where the $2^{d/2}$ arises from relative motion units in the $\delta^{(d)}(r)$. 
Using this conversion our energy functional becomes

\[
\frac{\pi (d - 2)}{\Gamma^2 \left( \frac{d}{2} \right)} \frac{\Gamma \left( -\frac{E_n}{\Gamma} + \frac{4}{d} \right)}{\Gamma \left( -\frac{E_n}{\Gamma} + \frac{4}{d} - \frac{n}{d} \right)} = -\sin \left( \frac{\pi d}{2} \right) a_o^{d - 2} / 2^{d/2}.
\]

(40)

5.2. Approximate analytical solution for the energy

Using a Taylor series expansion about \( \delta \) of equation (38) (see appendix \( H \)) and using the equation (39) unit conversion \( a_o \), we obtain a weakly interacting analytical solution for \( E^{(d)} \).

\[
E^{(d)} \approx 2n + \frac{d}{2} + \frac{2\pi (d - 2)}{\sin \left( \frac{\pi d}{2} \right) \Gamma^3 \left( \frac{d}{2} \right)} \frac{\Gamma \left( n + \frac{4}{d} \right)}{\Gamma(n + 1)\Gamma(1 - \frac{d}{2})} a_o^{d - 2}.
\]

(41)

When \( d \to 3 \),

\[
E^{(3)} \approx 2n + \frac{3}{2} + \frac{2^{3/2} \Gamma \left( n + \frac{3}{d} \right)}{\pi \Gamma(n + 1)} a_o^{d - 2}.
\]

(42)

one can verify that the expression agrees with the Busch solution for \( d = 3 \) (replacing the factorials in the binomial coefficient with gamma functions in equation (18) of [23]).

When \( d \to 1 \),

\[
E^{(1)} \approx 2n + \frac{1}{2} - \frac{2^{1/2} \Gamma \left( n + \frac{1}{d} \right)}{\pi \Gamma(n + 1)} a_o^{d - 2},
\]

(43)

one can verify that the expression agrees with the Busch solution for \( d = 1 \) (replacing the factorials in the binomial coefficient with gamma functions in equation (20) of [23]).

To find energies near \( d = 2 \), we expand equation (40) about \( d = 2 \). To first order in \( d - 2 \), equation (40) yields (see equation (H.16))

\[
\psi \left( \frac{1}{2} - \frac{E^{(2)}}{2} \right) = \ln \left( \frac{1}{a_o^2} \right),
\]

(44)

where \( \psi(\cdot) \) is the logarithmic derivative of Euler’s \( \Gamma \)-function.

6. Discussion and conclusions

We reviewed the fractional (non-integer order) calculus required to derive our non-integer pseudopotential in appendix \( A \) and justified our choice of the particular non-integer derivative used in the derivation. In a further validation, we applied our generalized pseudopotential to the problem of two cold atoms in a harmonic trap and obtained a general \( d \) relationship for the energy (equations (40) and (41)) which agreed with a previous solution in \( d = 1, 2 \), and 3 dimensions [23].

For the two-atom trapped energy in arbitrary dimension, we used the ultra-cold \(( k \to 0 \), i.e. energy-independent) limit of the generalized pseudopotential. In the strongly interacting regime, it may become important to include energy dependence in the interaction, for example, by using an energy-dependent scattering length [43, 46]. Thus, an important future extension of our arbitrary-dimension trapped two-atom energy will be to obtain a solution that has energy dependence \(( k > 0 \) from the pseudopotential [48].

An interesting application of current interest in atomic physics is the creation of highly anisotropic trapping configurations of cold atoms yielding quasi-one and quasi-two dimensional systems [38, 39]. Quasi-one and two-dimensional systems can be created by using tight confinement along one or two directions with weak confinement elsewhere. These quasi-one- and two-dimensional configurations can be explored by tuning both the confinement and the interaction parameters to explore physical regimes. Our work offers the interesting approach of using a Fermi pseudopotential for non-integer dimensions between one and two, characterized by the ratio of the different confinement strengths [49, 50].

While our pseudopotential is non-integer dimensional, it is valid arbitrarily close to integer dimension and gives correct limits for integer dimension. The limit for odd dimension is smooth. In even dimension, a term involving \( \ln(r) \) arises in the series solution. This logarithmic behavior was also found in our \( d = 2 \) two cold-atom energy functional. Deriving an explicit regularization operator that removes the logarithmic singularities for even dimension remains a future project. Thus, the current study opens new opportunities for mathematical development in fractional calculus and theoretical study of the pseudopotential and anisotropic trapped systems.

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Appendix A. Fractional calculus

We take advantage of so-called ‘fractional calculus’ to solve the \( d \)-dimensional pseudopotential and the energy for two cold atoms in a trap in \( d \) dimensions. Fractional calculus is the standard term used in the literature even though the order is valid beyond the domain of rationals [51]. Fractional calculus has a long history of mathematical development [51, 52] and has been used in domains to describe systems such as
anomalous diffusion [53] or the dynamics of neuron cluster response that may adapt on multiple time scales [54].

The fractional calculus approach has some advantages in circumventing singularities in our series solution of the wave function in appendix C, but fractional derivatives must be applied carefully. There are various practical definitions of fractional order derivatives that depend on a lower integration limit (equation (A.1)). Each definition must behave the same in the limit of integer order; however, the properties of each derivative may lead to different, and sometimes counterintuitive, properties away from integer order. For our derivation of the non-integer dimension pseudopotential, we are primarily concerned with derivatives of powers of \( r \) (equation (A.4)) because we develop a ‘fractional’ Frobenius series solution of the differential equation and we need to ensure that the general pseudopotential operator will converge near \( r = 0 \). The Caputo definition (equation (A.5)) with the Reimann–Liouville integration limit [55] helps ensure that our pseudopotential is well behaved for non-integer \( d \) when \( r \to 0 \). However, our \( d \)-dimensional derivation of two cold atoms in a trap also applies fractional derivatives to Gaussian functions (\( e^{-r^2/2} \)). We use the generalized Leibniz product rule (equation (A.7)) to ensure consistent derivatives of Gaussians for all \( d \), including non-integer.

Most definitions of fractional derivatives differ by the choice of the lower limit of integration in the definition. For a general lower integration limit \( x_0 \), the definition of a fractional derivative, \( x_0 D_\alpha^\beta f(x) \), of order \( \alpha \in \mathbb{R}^+ \) (the positive real numbers) with respect to \( x \) is given as follows (see [51]):

\[
x_0 D_\alpha^\beta f(x) = \frac{1}{\Gamma(-\alpha)} \int_{x_0}^{x} (x-x')^{-\alpha-1}f(x')dx'. \tag{A.1}
\]

The order \( \alpha \) may be real or complex and the arbitrary lower limit of integration \( x_0 \) arises from iterated integration. Any selections for \( x_0 \) will yield the results of conventional calculus when \( \alpha \) is integer, but different selections for the lower limit \( x_0 \) will result in the emergence of different properties away from integer \( \alpha \). There are many fractional derivatives defined in the literature, the most common being Reimann–Liouville \(( x_0 = 0 \) [56], Weyl \(( x_0 \to -\infty \)), and Caputo [57]. Fractional differentiation is non-local and collapses to a local calculation for integer-order derivatives. For fractional differentiation in the time domain, this non-local behavior can be interpreted as taking into account the history of the function. The value of the lower bound determines the duration of the memory of the differentiation.

To show the effect of the lower integration limit \( x_0 \), one may derive from equation (A.1) the following formula for the \( \alpha \)th fractional derivative of a power, \( x^\alpha (\alpha > 0, \beta > 0) \):

\[
x_0 D_\rho^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} - \frac{x_0^{\beta+1}}{(\beta+1)\Gamma(\alpha)} x^{-\alpha-1} F_2 \left[ \begin{array}{c} 1+\alpha, 1+\beta; 2+\beta; x_0 \end{array} \right] (x). \tag{A.2}
\]

where  \( F_2 \) is the Gaussian hypergeometric function. The first term on the right hand side of equation (A.2) would be the only term if factorials were simply replaced by gamma functions in integer-order differentiation. The second term goes to 0 when the order \( \alpha \) is integer, but when \( \alpha \) is not integer, the second term depends on the lower integration limit \( x_0 \) from equation (A.1). Letting \( x_0 = 0 \) in the definition is a reasonable choice (Reimann–Liouville), but other familiar properties may change away from integer order differentiation, such as the derivative of \( e^x \) not being itself, the derivative of a constant not being zero, or the usual product rule not being valid. In the remainder of this section, we describe some of the issues of common fractional derivative definitions and the rationale behind our choice of Caputo.

A.1. Riemann–Liouville (RL) derivative

Selecting \( x_0 = 0 \) yields the RL derivative, defined and denoted as follows:

\[
x_0 D_\rho^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^{x} (x-x')^{-\alpha-1}f(x')dx'. \tag{A.3}
\]

For RL derivatives, setting \( x_0 = 0 \) in equation (A.2), we see that the derivative of a power of \( x \) has the following form:

\[
x_0 D_\rho^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{A.4}
\]

which is a simple extension of the pattern that emerges from differentiation of integer orders. From the above equation, it can be seen that the RL derivative of a constant is, in general, non-zero, which poses a fundamental problem for our series solution and the \( r \to 0 \) limit of the pseudopotential derivation. However, the Caputo derivative addresses this problem, next.

A.2. Caputo derivative

An alternative formulation of the fractional derivative that retains the utility of the RL derivative with respect to power functions while allowing the additional property that the derivative of a constant is zero is the Caputo fractional derivative (CFD) with the RL lower integration limit \( x_0 = 0 \). In general, the lower limit in the CFD may be \( x_0 \) as above, but we focus on the \( x_0 = 0 \) RL limit because of our extensive use of powers in the series solution of the Schrödinger equation. Proposed by Caputo for the theory of viscoelasticity [55], the CFD is defined as follows:

\[
x_0 C_\rho^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^{x} (x-x')^{m-\alpha-1} f^{(m)}(x')dx'. \tag{A.5}
\]

with \( m = \lceil \alpha \rceil \) (the ceiling of \( \alpha \)) so that \( (m-1 < \alpha < m) \) and \( f^{(m)} \) is thus an integer order derivative. The subscript \( C_0 \) is meant to indicate the Caputo derivative and the lower integration limit \( x_0 = 0 \). When \( \alpha \to m \) the CFD becomes a conventional \( m \)-th order derivative. The CFD is defined for \( \alpha \in \mathbb{R}^+ \), though it can be extended to complex orders. This CFD possesses the RL pattern for powers (equation (A.4)) with the additional property that

\[
x_0 C_\rho^\alpha x^\beta = 0 \quad \text{for} \quad \beta \in \mathbb{Z}^+ \quad \text{and} \quad \alpha > \beta, \tag{A.6}
\]

where \( \mathbb{Z}^+ \) is the set of all positive integers. The property in equation (A.6) is essential for the series solution approach we
adopt in section 3.2 and appendix C. This property is satisfied by equation (A.5) because the integer part of a real-order derivative is applied first \( f^{(m)} \) and then the remaining non-integer \( (m - \alpha) \) derivative is taken.

For the derivation of the implicit function for the energy of two cold atoms in a trap in fractional dimension (section 5 and appendix G), we take the fractional derivative of Gaussian functions from the harmonic oscillator basis functions \( e^{-r^2/2} \). To calculate these derivatives, we use the Leibniz product rule for Caputo derivatives, which is extended to fractional order (see p 34 of [58]):

\[
D^\alpha(f(r), g(r)) = \sum_{p=0}^{\infty} \left( \begin{array}{c} \eta \\ p \end{array} \right) (D^{\alpha-p} f(r)) g^{(p)}(r) - \sum_{p=0}^{m-1} \frac{r^{p-\eta}}{\Gamma(p + 1 - \eta)} (f g^{(p)})(0),
\]

(A.7)

where \( m = [\eta] \). We also use the generalized Leibniz rule to take the \( d \to 1 \) limit of our generalized pseudopotential in appendix E.3.

**Appendix B. Derivation of pseudopotential for \( k \to 0 \)**

Assuming that \( B \) is finite and independent of \( r \), we can construct the following from equation (2) when \( r \) is small:

\[
[c_d \tilde{D}^{d-2}_r r^{d-2} \psi(r)]_{r=0} = B c_d \tilde{D}^{d-2}_r (r^{d-2} - \tilde{a}^{d-2}) = B c_d \tilde{D}^{d-2}_r r^{d-2} - B \Gamma(d-1),
\]

(B.1)

where we allow non-integer \( d \) and adopt Caputo’s definition (equation (A.5)) for non-integer (‘fractional’) order derivatives, \( c_d \tilde{D}^{d-2} \), which follows the RL formulas for differentiating power functions (equation (A.4)) with the additional important property of equation (A.6). It follows that

\[
B = \frac{1}{\Gamma(d-1)} [c_d \tilde{D}^{d-2}_r r^{d-2} \psi(r)]_{r=0}.
\]

(B.2)

On the other hand, we can also construct the following from equation (2), as \( r \to 0 \):

\[
\frac{d^{d-1} \psi}{dr^{d-1}} \to r^{d-1} B(d-2) \tilde{a}^{d-2} = B(d-2) \tilde{a}^{d-2}.
\]

(B.3)

We now integrate both sides of equation (B.3) over the full solid angle. The left side of equation (B.3) becomes (see [5] p 75 for \( d = 3 \) case)

\[
\int r^{d-1} \frac{d \psi}{dr} dr \Omega = \int \nabla r \psi \cdot n \, dS = \int \nabla_r^2 \psi \, dv,
\]

(B.4)

where \( n \) is a unit vector normal to the surface of the \( d \)-dimension hypersphere, and the right side of equation (B.3) becomes

\[
\int B(d-2) \pi^{d-2} d\Omega = B(d-2) \pi^{d-2} \int d\Omega = B(d-2) \pi^{d-2} \Omega(d),
\]

(B.5)

where \( \Omega(d) = 2\pi^{d/2}/\Gamma(d/2) \). Equating the right hand sides of equations (B.4) and (B.5), we obtain

\[
\int \nabla_r^2 \psi \, dv = B(d-2) \tilde{a}^{d-2} \Omega(d),
\]

(B.6)

and interposing an integral of a delta function on the right side, we can equate integrands:

\[
\int \nabla_r^2 \psi \, dv = \int B(d-2) \tilde{a}^{d-2} \Omega(d) \delta^{(d)}(r) \, dv.
\]

(B.7)

Thus, as \( r \to 0 \), substituting \( B \) from equation (B.2) into equation (B.7), we obtain the limit of the integrand:

\[
\nabla_r^2 \psi \to \frac{\Omega(d)(d-2) \tilde{a}^{d-2}}{\Gamma(d-1)} \delta^{(d)}(r) c_d \tilde{D}^{d-2}_r r^{d-2} \psi.
\]

(B.8)

Simplifying the operator on the right side of equation (B.8), we obtain the pseudopotential for small \( k \),

\[
c_d V^{(d)}_{k \to 0}(r) = \frac{\Omega(d)(d-2) \tilde{a}^{d-2}}{\Gamma(d-2)} \delta^{(d)}(r) c_d \tilde{D}^{d-2}_r r^{d-2}.
\]

(B.9)

Equation (3) will be the limiting case \( (k \to 0) \) of the more general \( k \)-dependent pseudopotential (equation (16)), and it will be the starting point for the ultracold two trapped-atom derivation in section 5.

**Appendix C. Fractional-Frobenius method for series solution of the wave function**

Before constructing the pseudopotential we first perform a series solution to identify the singularities to be removed. We use the Frobenius method to construct the series solution, but in our derivation of the pseudopotential we assume that \( d \) is fractional. Non-integer \( d \) avoids the ln(\( r \)) singularities that arise for all integer \( d \) except for 1 and 3, and because \( d \) is allowed to be arbitrarily close to integer, the pseudopotential can be calculated for integer \( d \) as a limit.

We rewrite the Laplacian of the relative motion for \( r > \tilde{a} \) (equation (1)):

\[
\left( \frac{1}{r^{d-1}} \frac{d}{dr} r^{d-1} \frac{d}{dr} + k^2 \right) \psi(r) = 0,
\]

(C.1)

as a second order linear differential equation that can be solved by the Frobenius method:

\[
\psi'' + \frac{d - 1}{r} \psi' + k^2 \psi = 0.
\]

(C.2)

Substituting \( \psi(r) = \sum_{\lambda=0}^{\infty} a_\lambda r^{j+\lambda} \) and simplifying, we are left with

\[
\sum_{\lambda=0}^{\infty} a_\lambda (j + \lambda)(j + \lambda + d - 2) r^{j+\lambda-2} + k^2 \sum_{\lambda=0}^{\infty} a_\lambda r^{j+\lambda},
\]

(C.3)

yielding the indicial equation:

\[
j(j + d - 2) = 0
\]

(C.4)
and a recurrence relation:

$$\alpha_{n+2} = \frac{-k^2\alpha_n}{(n + 2 + j)(n + d + j)}.$$  \hfill (C.5)

The roots of the indicial equation are $j = 0$ and $j = 2 - d$, where the first root yields the regular solution:

$$\psi_{reg}(r) = \sum_{n=0}^\infty \alpha_n r^n,$$  \hfill (C.6)

with a recursive coefficient function:

$$\alpha_{n+2} = \frac{-k^2}{(n + 2)(n + d)} \alpha_n.$$  \hfill (C.7)

where $\alpha_1 = 0$, which sets all $\alpha_{odd} = 0$, and $\alpha_0$ is an arbitrary, non-zero constant. By simple algebraic manipulation, we can rewrite the formula for $\alpha_n^r$:

$$\alpha_{2n} = \alpha_0 \cdot \frac{(-1)^n k^{2n}(d - 2)^n!!}{(2n)!!(2n + d - 2)^n!!}, \quad n \in \mathbb{Z}^+. \hfill (C.8)$$

With the second indicial root $j = 2 - d$, the second solution becomes

$$\psi_{reg}(r) = c \ln(r) \psi_{reg} + \frac{1}{\rho^{d-2}} \sum_{n=0}^\infty \beta_n r^n.$$  \hfill (C.9)

For completeness, we solve for the coefficients $c$ and $\beta_n$ for all cases of $d$. However, since our approach uses ‘fractional’ $d$ we only need case 2, $d \not\in \mathbb{Z}$. In this case, $c = 0$ and the $\ln(r)$ term is removed (see discussion near equations (C.23) and C.24). The non-integer-$d$ series solution can be made arbitrarily close to integer, and we will find that the limit for odd $d$ is smooth while the even $d$ limit must be handled more carefully.

We now solve for the coefficients $c$ and $\beta_n$ by substituting the irregular solution $\psi_{reg}(r)$ from equation (C.9) into the original equation (C.2):

$$\sum_{n=0}^\infty c \alpha_n (2n - 2 + d) r^{n-2} + \beta_1 (3 - d) r^{1-d}$$

$$+ \sum_{n=0}^\infty [\beta_{n+2}(n + 4 - d)(n + 2)$$

$$+ k^2 \beta_n] r^{n+2-d} = 0.$$  \hfill (C.10)

Taking note that the first series is a sum of powers of $r$ starting at $r^{-2}$ and that the other terms are series of powers of $r$ that depend on dimension $d$, we solve for the constants $c$ and $\beta_n$ for the following cases.

(1) $d \in \mathbb{Z}$:

(a) $d = 2$:

equation (C.10) can be rewritten as:

$$\sum_{n=1}^\infty c \alpha_n (2n) r^{n-2} + \beta_1 r^{-1}$$

$$+ \sum_{n=0}^\infty [\beta_{n+2}(n + 2) + k^2 \beta_n] r^n = 0.$$  \hfill (C.11)

or

$$\sum_{n=2}^\infty [2nc\alpha_n + \beta_n n^2 + k^2\beta_{n-2}] r^{n-2}$$

$$+ (2c\alpha_1 + \beta_1) r^{-1} = 0,$$  \hfill (C.12)

which implies $\beta_1 = 0$ (since $\alpha_1 = 0$) and $2nc\alpha_n + \beta_n n^2 + k^2\beta_{n-2} = 0$. Hence,

$$c = \frac{-k^2\beta_{n-2} + \beta_n n^2}{2n\alpha_n} = -\frac{3\beta_1}{2\alpha_3}.$$  \hfill (C.13)

(b) $d = 3$:

equation (C.10) can now be rewritten as:

$$\sum_{n=0}^\infty c \alpha_n (2n + 1) r^{n-2}$$

$$+ \sum_{n=0}^\infty [\beta_{n+2}(n + 1)(n + 2)$$

$$+ k^2\beta_n] r^{n-1} = 0.$$  \hfill (C.14)

The first sum in equation (C.10) requires $c = 0$; the second term is automatically 0, making $\beta_0$ and $\beta_1$ arbitrary; and the third term yields the recursive coefficient function:

$$\beta_{n+2} = \frac{-k^2\beta_n}{(n + 1)(n + 2)},$$  \hfill (C.15)

or

$$\beta_n = \beta_0 \left(\frac{-1}{n} \left[\frac{(d-1)!!}{n!!(n-1)!!}\right]^k\right),$$  \hfill (C.16)

where $\bar{n}_2$ denotes the common residue of $n$ mod 2.

(c) $d \geq 4$:

equation (C.10) can be rewritten as:

$$\beta_1(3 - d) r^{-1} + \sum_{n=2}^{d-3} (n\beta_n + \beta_{n-2}k^2) r^{n-d}$$

$$+ \sum_{n=d-2}^\infty [c \alpha_{n+d-2}(2n + 2 - d)$$

$$+ \beta_n (n + 2 - d)$$

$$+ k^2\beta_{n-2}] r^{n-d} = 0.$$  \hfill (C.17)

The first term implies $\beta_1 = 0$. The second term reveals the recurrence relation:

$$\beta_n = \frac{-k^2\beta_{n-2}}{n}, \quad n = 2, 3, \ldots, d - 3$$  \hfill (C.18)
and hence $\beta_{2j+1} = 0$ for $j < d/2 - 2$. The third term requires
\[
c = -\beta_{n}(n + 2 - d) + k^2\beta_{n-2},
\]
\[
n = d - 3, d - 2, \ldots.
\]
(C.19)

We note that in the special case of $d = 4$, the second term in equation (C.17) disappears, and we do not have the second recurrence relation of $\beta$ described in equation (C.18). Therefore, in this case, most values of the $\beta$’s (except $\beta_1$) are arbitrary.

(d) Otherwise: When $d < 1$, we can rewrite equation (C.10) as:
\[
\sum_{n=0}^{d-3} c_n(2n - 2 + d)r^{n-2} + \beta_1(3 - d)r^{1-d}
\]
\[
+ \sum_{n=0}^{d-4} f_n(2n - 2 + d)
\]
\[
+ \beta_{n+d-2}(n + d - 2) + k^2\beta_{n+d-4})r^{n-2} = 0,
\]
(C.20)

which leads to $c = 0$, $\beta_1 = 0$, and $\beta_0$ is arbitrary, yielding the recursive coefficient function from equation (C.5):
\[
\beta_n = \frac{-k^2\beta_{n-2}}{n(n + 2 - d)}, \quad n = 2, 3, \ldots
\]
(C.21)

which sets all $\beta_{\text{odd}}$ to zero, and generalizes $\beta_0$ to
\[
\beta_n = \beta_0 \cdot \frac{(-1)^n k^2n(d - 2)!!}{n !!(n + 2 - d)!!}, \quad n \in \mathbb{Z}_{\text{even}}.
\]
(C.22)

(2) $d \notin \mathbb{Z}$:

The first sum in equation (C.10) does not have common powers of $r$ with the second and third term, forcing $c = 0$. Also, when $d \notin \mathbb{Z}$, $3 - d = 0$, which implies $\beta_1 = 0$. The last sum yields the recursive relation:
\[
\beta_{n+2} = \frac{-k^2\beta_n}{(n + 4 - d)(n + 2)},
\]
or
\[
\beta_{2n} = \beta_0 \cdot \frac{(-1)^n k^2n(2 - d)!!}{(2n)!!(2n + 2 - d)!!}, \quad n \in \mathbb{Z}^+.
\]
(C.24)

In conclusion, for fractional dimension $d \notin \mathbb{Z}$, the solution to the Schrödinger equation has the form:
\[
\psi(r) = \sum_{n=0}^{\infty} \alpha_n r^n + \frac{1}{r^{d-2}} \sum_{n=0}^{\infty} \beta_n r^n.
\]
(C.25)

Applying the boundary condition of the Schrödinger equation, we have:
\[
0 = \sum_{n=0}^{\infty} \alpha_n a^{2n} + \sum_{n=0}^{\infty} \beta_n a^{2n+2-d}
\]
\[
= \alpha_0 \sum_{n=0}^{\infty} (-1)^n k^2n(2 - d)!! a^{2n}
\]
\[
+ \beta_0 \sum_{n=0}^{\infty} (-1)^n k^2n(2 - d)!! a^{2n+2-d},
\]

Hence,
\[
\alpha_0 = -\frac{\beta_0 \sum_{n=0}^{\infty} (-1)^n k^2n(2 - d)!! a^{2n}}{\sum_{n=0}^{\infty} (-1)^n k^2n((2n + 2 - d)!!)
\]
\[
= -\frac{\beta_0 \psi_1(2 - d, \frac{k^2}{2})}{\psi_1(\frac{d}{2}, \frac{k^2}{4})}.
\]
(C.26)

where $\psi_1(\cdot;\cdot)$ is the confluent hypergeometric function [47].

**Appendix D. Integration over a sphere for general $d$ pseudopotential**

Here we provide details of the derivation of equation (8) in section 3.2. This involves the integration of the Schrödinger equation (equation (1)) about a $d$-dimensional sphere with infinitesimal radius $\epsilon$. We begin with the non-integer-$d$ solution $c = 0$, equation (6):
\[
\psi(r) = \sum_{n=0}^{\infty} \alpha_n r^{2n} + \frac{1}{r^{d-2}} \sum_{n=0}^{\infty} \beta_n r^{2n}.
\]
(D.1)

The regular part of $\psi$ (right hand side of equation (D.1)), $\psi_{\text{reg}} = \sum_{n=0}^{\infty} \alpha_n r^{2n}$, does not contain singularities, so as $r \to 0$, $(\nabla_d^2 + k^2)\psi_{\text{reg}} \to 0$. Thus, we only need to consider the irregular solution (left hand side of equation (D.1)) when integrating the Schrödinger equation over the infinitesimal sphere:
\[
\int (\nabla_d^2 + k^2)\psi dV = \int (\nabla_d^2 + k^2)\psi_{\text{irreg}} dV.
\]
(D.2)

The integral involving $k^2$ is zero:
\[
\int \nabla_d^2 \psi_{\text{irreg}} dV = \int \psi_{\text{irreg}} dV
\]
(D.3)

The integral involving $k^2$ is zero:
\[
\left[ \int k^2 \psi_{\text{irreg}} dV \right]_{r=0}^{r=\epsilon} = \left[ \int k^2 \left( \sum_{n=0}^{\infty} \beta_n r^{2n-d+2} \right) dV \right]_{r=0}^{r=\epsilon} = 0
\]
(D.4)

\[
\left[ \int_0^\epsilon k^2 \left( \sum_{n=0}^{\infty} \beta_n r^{2n-d+2} \right) dr \right]_{r=0}^{r=\epsilon} = 0
\]
(D.5)

\[
\left[ \int_0^\epsilon k^2 \left( \sum_{n=0}^{\infty} \beta_n r^{2n+1} \right) dr \right]_{r=0}^{r=\epsilon} = 0
\]
(D.6)

\[
\left[ k^2 \sum_{n=0}^{\infty} \beta_n (2n + 2) r^{2n+2} \right]_{r=0}^{r=\epsilon} = 0
\]
(D.7)

\[
\left[ k^2 \sum_{n=0}^{\infty} \beta_n (2n + 2) \right]_{r=0}^{r=\epsilon} = 0
\]
(D.8)
Returning to equation (D.3):
\[
\int (\nabla^2 + k^2) \psi dV = \int \nabla^2 \psi_{\text{reg}} dV + \int k^2 \psi_{\text{reg}} dV^0
\]
\[= \int \nabla^2 \left( \sum_{n=0}^{\infty} \beta_{2n} r^{2n-d+1} \right) dV \]
\[= \int dS \cdot \nabla \left( \sum_{n=0}^{\infty} \beta_{2n} r^{2n-d+1} + 2 \right) dV \]
\[= \Omega(d) \epsilon d^{-1} \sum_{n=0}^{\infty} \beta_{2n}(2n - d + 2) e^{2n-d+1} \]
\[= \Omega(d) \sum_{n=0}^{\infty} \beta_{2n}(2n - d + 2) e^{2n}; \]
We take the limit \( \epsilon \to 0 \) and interpose the integral of a delta function:
\[
\int (\nabla^2 + k^2) \psi dV \to \Omega(d) \beta_0(2 - d)
\]
\[\int (\nabla^2 + k^2) \psi dV = \int \Omega(d) \beta_0(2 - d) \delta^{(d)}(r) dV. \]
Equating the integrands gives the following and, thence, equation (8)
\[
(\nabla^2 + k^2) \psi = \Omega(d) \beta_0(2 - d) \delta^{(d)}(r). \]

Appendix E. Relationship between the generalized pseudopotential and the Wódkiewicz Fermi pseudopotential (Green’s function approach) regularizing factor \( \gamma_{2n+1} \) and coupling constant \( a_d \)

E.1. The regularization factor

At first glance, equation (26) derived by the non-integer calculus approach appears not to agree with the regularization operator derived in [44] by Wódkiewicz for odd dimension \( d = 2n + 1 \) using the Green’s function approach (equation (5.4) in [44]),
\[
R_0^{(2n+1)} = \gamma_{2n+1} \frac{\partial^{2n-1}}{\partial k^{2n-1}} f^{(2n-1)}. \]

because of the complicated looking analytical form of the regularization coefficient (equation (5.5) in [44]):
\[
\gamma_{2n+1} = \frac{\pi^{-1/2} \Gamma \left( \frac{1}{2} - n \right)}{\sum_{l=0}^{n-1} (-1)^{l-n} \frac{(l-n)!^{2n-1}}{l!(n-l)!^{2n-1}}}. \]

However, as we show next, the result of our non-integer calculus approach (equation (26)) and the Green’s function calculation for odd integer dimension (equations (E.1) and (E.2)) are in agreement, and we obtain the simplifying result for equation (E.1) taken from [44] for positive integer \( n \):
\[
\gamma_{2n+1} = \frac{1}{\Gamma(2n)} = \frac{1}{\Gamma(d-1)}. \]

We now verify by direct means the relationship in equation (E.3) for \( n \in \mathbb{Z}^+ \), which we predicted based on consistency between the non-integer calculus and Green’s function derivations of the regularizing operator. We first note that the formula in equation (E.2) is obfuscated by the summation in the denominator, but this sum can be simplified to \((-1)^n 2^{2n-1}(n - 1)! \) or \((-1)^n 2^{2n-1} \Gamma(n) \), which we use to replace the summation in equation (E.2):
\[
\gamma_{2n+1} = \frac{\Gamma \left( \frac{1}{2} - n \right)}{\sqrt{\pi} \left( -1 \right)^n 2^{2n-1} \Gamma(n)}. \]

In the denominator of equation (E.4), we use the identity\( \Gamma(z + \frac{1}{2}) = \frac{\pi^{1/2}}{\sin \pi(z + \frac{1}{2})} \), which when rearranged we may replace \( 2^{2n-1} \Gamma(n) \) in the denominator of equation (E.4) to give
\[
\gamma_{2n+1} = \frac{\Gamma \left( \frac{1}{2} - n \right) \Gamma \left( \frac{1}{2} + n \right)}{\pi \left( -1 \right)^n \Gamma(2n)}. \]

Substitution of \( z = n + \frac{1}{2} \) into the identity \( \Gamma(1 - z) \Gamma(z) = \frac{\pi}{\sin \pi(z + \frac{1}{2})} \) gives
\[
\Gamma \left( \frac{1}{2} - n \right) \Gamma \left( \frac{1}{2} + n \right) = \frac{\pi}{\sin \pi \left( n + \frac{1}{2} \right)} \]

and substituting this into equation (E.5) yields
\[
\gamma_{2n+1} = \frac{1}{\Gamma(2n)} \frac{1}{(-1)^n \sin \left( \pi \left( n + \frac{1}{2} \right) \right)}. \]

One can now see from equation (E.6) that equation (E.3) holds for positive and negative integers \( n \) and 0, since \( (-1)^n \sin \pi \left( n + \frac{1}{2} \right) = 1 \) for \( n \in \mathbb{Z} \), and thus we demonstrate the consistency between the Green’s function and non-integer calculus approaches. The positive and negative domains for \( n \) correspond to positive and negative odd dimensionality, respectively. The equation (E.3) coefficient goes to 0 for negative (integer) \( n \) or, equivalently, negative integer dimensionality. We show in section E.3 below that the generalized pseudopotential is finite and non-zero when \( d \to 1 \).

E.2. The general (odd \( d > 1 \)) expression for Wódkiewicz pseudopotential

Wódkiewicz derived the Fermi pseudopotential for \( d = 1, 3 \) and 5. We use the same procedure to obtain the general result for all odd \( d \). Starting with the formula for the scattering amplitude (Corrected version of equation (3.1), [44]), which contains a typographical error as we note in appendix I:
\[
f_k^{(d)} = -\frac{i^{-n} k^{n-1} (2n - 1)!!}{k^{2n-1} + i^{n-1} k^{2(n-1)+1} a_0^{1/n}} \]

we obtain the solution for the single pole of \( f_k^{(d)} \) in the physical \( k \) half plane:
\[
k^{2n-1} = -i 2^n \pi^{n}(2n - 1)!! \frac{\beta}{a_{2n} + \alpha}. \]
Now, using the following $d = 1$ single bound state energy (equation (3.3) in [44]):
\[ E_0 = -\frac{m}{2\hbar^2}a_1^2 \]  
(8.9)
and setting the energy constant across dimensions, we have:
\[ E = E_0 \]  
(8.10)
\[ \Rightarrow \frac{\hbar^2 k^2}{2m} = -\frac{m}{2\hbar^2}a_1^2 \]  
(8.11)
\[ \Rightarrow k^2 = -\frac{m}{\hbar^2}a_1^2 \]  
(8.12)
\[ \Rightarrow k = \pm \frac{m}{\hbar^2}a_1 \]
(8.13)
\[ \Rightarrow k^{2n-1} = \left(\frac{m}{\hbar^2}a_1\right)^{2n-1} \]  
(8.14)
\[ \Rightarrow \frac{-i2^{n-1} \pi^n (2n - 1)!! \hbar}{a_{2n+1}} = \pm \frac{2^{n-1} m^{2n-1}}{\hbar^{2n-2}} a_1^{2n-1} \]  
(8.15)
from which we can clearly solve for $a_{2n+1}$:
\[ a_{2n+1} = \left(-\frac{\hbar^4}{m^4} \pi^{2n} (2n)!! (2n - 1)!!\right) a_1 a_1^{2n-1} \]  
(8.16)
or, since $d = 2n + 1$,
\[ a_d = \left(-\frac{\pi^5}{m^5} (2\pi)^4 (2n)!! (2n - 2)!!\right) \frac{\hbar^{2d-2}}{a_1 a_1^{2d-2}} \]  
(8.17)
We can now substitute our general formula above for the strength or ‘area of potential,’ $a_d$, into the following relation (equation (1.2) in [44])
\[ V^{(d)}(r) = -a_d \delta^{(d)}(r) \hat{R}_d \]  
(8.18)
and we obtain the general formula of Wódkiewicz Fermi pseudopotential for odd $d > 1$:
\[ V^{(d)}_{W}(r) = -\frac{\hbar^{2d-2} (2\pi)^{4d} (2d - 2)!!}{m^{d-1} a_1^{d-2}} \delta^{(d)}(r) \frac{1}{\Gamma(d - 1)} \frac{\partial^{d-2} r^{d-2}}{\partial r^{d-2}} \]  
(8.19)
\[ = -\frac{\hbar^{2d-2} (2\pi)^{4d} (2d - 4)!!}{m^{d-1} a_1^{d-2}} \delta^{(d)}(r) \frac{1}{\Gamma(d - 2)} \frac{\partial^{d-2} r^{d-2}}{\partial r^{d-2}}. \]
(8.20)
We rewrote $V^{(d)}_{W}(r)$ in the second form so that it more closely resembles our general-dimension pseudopotential.

### E.3. $d \rightarrow 1$ Comparison of the Wódkiewicz and generalized pseudopotentials

When $d = 1$ in the Wódkiewicz pseudopotential, the regularization factor $\frac{1}{\Gamma(d - 1)} \frac{\partial^{d-2} r^{d-2}}{\partial r^{d-2}}$ is not required due to the lack of singularities in the limit $r \rightarrow 0$ of the Green’s functions [44]; hence, dropping the regularizing operator in equation (8.19) gives
\[ V^{(1)}_{W}(r) = -\frac{\hbar^{2-2} (2\pi)^{2-1} (1 - 2)!!}{m^{1} a_1^{1}} \delta^{(1)}(r) = -a_1 \delta^{(1)}(r), \]
(8.21)
where recall $(-1)!! = 1$. The $d \rightarrow 1$ limit of our generalized pseudopotential (equation (26) from the $k \rightarrow 0$ limit of equation (16)) agrees with the above result without the need to drop the regularizing operator, as we now show. Indeed, operating on a wave function $\Psi$ with our generalized pseudopotential (equation (26)), the generalized Leibniz product rule (equation (A.7)) gives
\[ c_0 V^{(d)}_{k=0}(\Psi) = \frac{\Omega(d) \delta^{d-2}}{\Gamma(d - 2)} c_0 D^{2-2} r^{2-2} \Psi \]
(8.22)
\[ = \frac{\Omega(d) \delta^{d-2}}{\Gamma(d - 2)} \delta^{(d)}(r) \sum_{k=0}^{\infty} \frac{(d - 2)}{k} (D^{d-2 - k} r^{d-2}) D^k \Psi \]
(8.23)
where the second sum from the Leibniz rule (equation (A.7)) does not appear above because the upper bound of the sum is $m - 1 = [d - 2] - 1 = -2 < 0$. Next, applying the RL fractional derivative of powers (equation (A.4)) gives
\[ c_0 V^{(d)}_{k=0}(\Psi) = \frac{\Omega(d) \delta^{d-2}}{\Gamma(d - 2)} \delta^{(d)}(r) \times \sum_{k=0}^{\infty} \frac{\Gamma(d - 1)}{\Gamma(k + 1)} \frac{\Gamma(d - k - 1)}{\Gamma(k + 1 - 1)} r^k D^k \Psi \]
(8.24)
\[ = \Omega(d) \delta^{d-2} \delta^{(d)}(r) \sum_{k=0}^{\infty} \frac{(d - 2)(d - 1)}{\Gamma(k + 1)} \Gamma(d - k - 1) D^k \Psi. \]
(8.25)
When $r \rightarrow 0$, the only non-zero term in equation (8.25) is $k = 0$. Hence,
\[ c_0 V^{(d)}_{k=0}(\Psi) = \Omega(d) \delta^{d-2} \delta^{(d)}(r)(d - 2) \Gamma(d - 1) D^0 \Psi \]
(8.26)
\[ = \Omega(d) \delta^{d-2} \delta^{(d)}(r)(d - 2) \Psi. \]
(8.27)
Thus, when $d \rightarrow 1$, we obtain
\[ c_0 V^{(1)}_{k=0}(\Psi) = -2 \delta^{(1)}(r) \Psi. \]
(8.28)
Converting our scattering length $\tilde{a}_1$ to the Wódkiewicz constant $a_1 (\tilde{a}_1 = \frac{\tilde{a}_1}{2})$ by equation (34), we verify that our pseudopotentials agree for one dimension:
\[ c_0 V^{(1)}_{k=0}(\Psi) = -a_1 \delta^{(1)}(r). \]
(8.29)

### Appendix F. Derivation of the s-wave d-dimensional isotropic harmonic oscillator wave function and energy

Here we derive the solution of the $d$-dimensional harmonic oscillator Schrödinger equation. We obtain energies and the wave function in terms of the Laguerre polynomial. These non-interacting results are needed as a basis for the trapped energy for two atoms interacting via the generalized pseudopotential, derived in appendix G. We begin with the $d$-dimensional harmonic oscillator Schrödinger equation in
oscillator units:
\[
\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{d - 1}{d} \frac{d}{dr} + \frac{1}{2} r^2\right] \phi_n^{(d)}(r) = E_n^{(d)} \phi_n^{(d)}. \tag{F.1}
\]

Transforming this differential equation into the Laguerre form requires two steps. First we transform the wave function as
\[
\phi_n^{(d)} = e^{-\frac{1}{2} r^2} \phi_n^{(d)}, \tag{F.2}
\]
which leads to
\[
-\frac{1}{2} \frac{d^2}{dr^2} + \frac{d - 1 - 2r^2}{d} \frac{d}{dr} - d \phi_n^{(d)}(r) = E_n^{(d)} \phi_n^{(d)}. \tag{F.3}
\]

Second we let \( u = r^2 \) to obtain
\[
\left[u \frac{d^2}{du^2} + \left(\frac{d}{2} - u \right) \frac{d}{du} + \left(\frac{E_n^{(d)}}{2} - \frac{d}{4}\right)\right] \phi_n^{(d)}(u) = 0, \tag{F.4}
\]
and in order for the above equation to match the associated Laguerre differential equation:
\[
\left[u \frac{d^2}{du^2} + \left(\alpha - u + 1 \right) \frac{d}{du} + n\right] L_n^{(\alpha)}(u) = 0, \tag{F.5}
\]
we require \( \alpha = \frac{d - 2}{2} \) and the following relationship to hold
\[
E_n^{(d)} = 2n + \frac{d}{2} \tag{F.6}
\]

As an aside, we note that the Laguerre polynomial can be further generalized to \( \alpha \in \mathbb{C} \) [59]. Using this value above for \( \alpha \) and equation (F.2), we can write the \( d \)-dimensional harmonic oscillator wave function in terms of the Laguerre polynomial:
\[
\phi_n^{(d)}(r) = A e^{-r^2/2} L_n^{d-2}(r^2), \tag{F.7}
\]
where \( A \) is a normalization constant. We use the normalization condition \( \int_0^{\infty} |\phi_n^{(d)}|^2 dr = 1 \) to find \( A \), which leads to the following integral to solve
\[
A^2 \Omega(d) \int_0^{\infty} e^{-r^2} \left(L_n^{d-2}(r^2)\right)^2 r^{d-1} dr = 1. \tag{F.8}
\]

We can solve equation (F.8) using the similar known relationship for Laguerre polynomials [60]:
\[
\int_0^{\infty} e^{-x}(L_n(x))^2 dx = \frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)}. \tag{F.9}
\]

If we let \( x = r^2 \) and \( \lambda = \frac{d - 2}{2} \), then
\[
\int_0^{\infty} e^{-r^2} \left(L_n^{d-2}(r^2)\right)^2 r^{d-1} dr = \frac{\Gamma(n+d/2)}{\Gamma(n+1)}, \tag{F.10}
\]
and substituting this for the integral in equation (F.8), we find the normalization constant is
\[
A = \left[\frac{2\Gamma(n+1)}{\Omega(d)\Gamma(n+d/2)}\right]^{1/2}. \tag{F.11}
\]

and finally the \( d \)-dimensional harmonic oscillator wave function is
\[
\phi_n^{(d)}(r) = \left[\frac{2\Gamma(n+1)}{\Omega(d)\Gamma(n+d/2)}\right]^{1/2} e^{-r^2/2} L_n^{d-2}(r^2). \tag{F.12}
\]

Because of the involvement of \( \phi_n^{(d)}(0) \) in equation (G.5) from the derivation of the solution of two trapped atoms interacting via the generalized pseudopotential, it proves useful to write the normalization constant in equation (F.12) in terms of \( L_n^{d-2}(0) \). We do this by noting that
\[
L_n^{d-2}(0) = \frac{\Gamma(n+d/2)}{\Gamma(n+1)}, \tag{F.13}
\]
which can be used in equation (F.12) to obtain
\[
\phi_n^{(d)}(r) = \left[\frac{2}{\Omega(d)\Gamma(d/2)}\right]^{1/2} L_n^{d-2}(0)^{-1/2} e^{-r^2/2} L_n^{d-2}(r^2). \tag{F.14}
\]

Appendix G. Details of the construction of the implicit energy equation \( E^{(d)} \) for two cold atoms \((k \rightarrow 0)\) in a harmonic trap with pseudopotential interaction in non-integer dimension (equation (40))

In this appendix we generalize the Busch derivation in [23] to arbitrary dimension, including non-integer. We start at equation (37) and rearrange the terms to obtain:
\[
\sum_{n=0}^{\infty} c_n (E_n^{(d)} - E^{(d)}) \phi_n^{(d)} + \frac{\Omega(d)\alpha^{d-2}}{\Gamma(d-2)} \phi^{(d)}(r) \left[\sum_{n=0}^{\infty} c_n \phi_n^{(d)}\right]_{r \rightarrow 0} = 0, \tag{G.1}
\]
where \( c_n D_r^{d-2} \) is the Caputo fractional derivative and we are using the \( k \rightarrow 0 \) generalized pseudopotential (equation (26)). Projecting both sides onto \( \phi_n^{(d)}(r) \), we have
\[
c_n (E_n^{(d)} - E^{(d)}) + \frac{\Omega(d)\alpha^{d-2}}{\Gamma(d-2)} \phi^{(d)}(0) \times \left[\sum_{n=0}^{\infty} c_n \phi_n^{(d)}\right]_{r \rightarrow 0} = 0, \tag{G.2}
\]
and \( c_n = A \frac{\phi_n^{(d)}(0)}{E_n^{(d)} - E^{(d)}} \), yielding:
\[
A \phi_n^{(d)}(0) + \frac{\Omega(d)\alpha^{d-2}}{\Gamma(d-2)} \phi^{(d)}(0) \times \left[\sum_{n=0}^{\infty} c_n \phi_n^{(d)}\right]_{r \rightarrow 0} = 0, \tag{G.3}
\]
or, dividing by \( A \phi_n^{(d)}(0) \),
\[
1 + \frac{\Omega(d)\alpha^{d-2}}{\Gamma(d-2)} \left[\sum_{n=0}^{\infty} \frac{\phi_n^{(d)}(0)}{E_n^{(d)} - E^{(d)}}\right]_{r \rightarrow 0} = 0. \tag{G.4}
\]
which results in

\[
\frac{\Omega(d)}{\Gamma(d-2)} \left[ \sum_{n=0}^{\infty} \frac{\phi^{(d)}(0)}{E_n^{(d)} - E_n^{(d)}} \right] = -\frac{1}{d^{d-2}}. 
\]  

(G.5)

In order to simplify equation (G.5), we insert the representation of the harmonic oscillator s-wave functions for arbitrary dimension (equation (F.14)):

\[
\phi_n^{(d)}(r) = \frac{2}{\Omega(d)\Gamma(d/2)} \left[ c_d r^{d-2} e^{-r^2/2} \sum_{n=0}^{\infty} E_n^{(d)} L_n^{d-2}(r^2) \right] = -\frac{1}{d^{d-2}}. 
\]

where \(L_n^{d-2}(r^2)\) is the associated Laguerre polynomial, yielding:

\[
\frac{2}{\Gamma(d-2)\Gamma(d/2)} \left[ c_d r^{d-2} e^{-r^2/2} \sum_{n=0}^{\infty} \frac{L_n^{d-2}(r^2)}{n!} \right] = -\frac{1}{d^{d-2}}. 
\]

Recalling the unperturbed harmonic oscillator energies (equation (F.6)), \(E_n^{(d)} = 2n + \frac{d}{2}\), we introduce the variable

\[
\nu = \frac{E_n^{(d)}}{2} - \frac{d}{4}
\]

as the non-integer equivalent of quantum number \(n\) and obtain:

\[
\frac{1}{\Gamma(d-2)\Gamma(d/2)} \left[ c_d r^{d-2} e^{-r^2/2} \sum_{n=0}^{\infty} \frac{L_n^{d-2}(r^2)}{n!} \right] = -\frac{1}{d^{d-2}}. 
\]

Using the identity

\[
\int_0^{\infty} \frac{dy}{1+y^2} (\frac{y}{1+y})^{n-\nu-1} = 1, \quad n - \nu > 0, 
\]

the summation becomes

\[
\sum_{n=0}^{\infty} \frac{L_n^{d-2}(r^2)}{n!} = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{dy}{1+y^2} \left( \frac{y}{1+y} \right)^{n-\nu-1} L_n^{d-2}(r^2). 
\]

(G.11)

We now use the generating functions of the Laguerre Polynomials [60],

\[
\sum_{n=0}^{\infty} L_k^{d-2}(x) z^n = (1 - z)^{(k+1)} e^{-z/(1-z)}, 
\]

(G.12)

letting \(k = \frac{d}{2} - \frac{1}{2}\), \(x = r^2\), and \(z = \frac{y}{1+y}\), to rewrite equation (G.11):

\[
\sum_{n=0}^{\infty} \frac{L_n^{d-2}(r^2)}{n!} = \int_0^{\infty} dy \left( \frac{1}{1+y} \right)^{\nu-1} e^{-r^2/(1+y)^{\frac{d}{2}}}. 
\]

(G.13)

which simplifies to

\[
\Gamma(a) U(a, b, z) = \int_0^{\infty} e^{-z t} t^{b-1} (1 + t)^{a-1} dt, 
\]

(G.15)

where \(a = -\nu\), \(b = \frac{d}{2}\), \(z = r^2\), and \(t = y\), equation (G.11) becomes

\[
\sum_{n=0}^{\infty} \frac{L_n^{d-2}(r^2)}{n!} = \Gamma(-\nu) U\left(-\nu, \frac{d}{2}, r^2\right). 
\]

(G.16)

We further examine the behavior of equation (G.16) as \(r \to 0\) by using the identity for \(U\) in terms of Kummer’s function, \(M\) (13.1.2 and 13.1.3 from [47]) to get:

\[
\Gamma(-\nu) U\left(-\nu, \frac{d}{2}, r^2\right) = \frac{\pi \Gamma(-\nu)}{\sin \left( \frac{\pi}{2} \right)} \frac{M\left(-\nu, \frac{d}{2}, r^2\right)}{\Gamma\left(1 + \frac{d}{2} - \nu\right)\Gamma\left(\frac{d}{2}\right)} 
\]

\[
- r^{2-d} M\left(1 - \nu - \frac{d}{2}, 2 - \frac{d}{2}, r^2\right) \Gamma\left(-\nu\right) \Gamma\left(\frac{d}{2}\right) 
\]

\[
\frac{\pi \cos \left( \frac{\pi}{2} \right)}{2} \left( \frac{\Gamma(-\nu)}{\Gamma\left(1 + \frac{d}{2} - \nu\right)\Gamma\left(\frac{d}{2}\right)} \right) 
\]

\[
- r^{2-d} \left[ \frac{(-\nu)^{\frac{d}{2}}}{M\left(1 - \nu - \frac{d}{2}, 2 - \frac{d}{2}, r^2\right)} \right] \sum_{i=0}^{\infty} \frac{r^{2i}}{(2 - \frac{d}{2})^{2i}} 
\]

(G.17)

where \(\Gamma(\nu)\) is the Pochhammer symbol. Note the Kummer’s series above are truncated because they do not contribute following differentiation and the \(r \to 0\) limit is applied in equation (G.9). From here, the \([\cdot]_{r \to 0}\) limit in equation (G.9)

Eq. 18: We are interested in this derivative as \( r \to 0 \). The derivative in parentheses in equation (G.21) is
\[
D^\nu e^{-r^2/2} = 2\pi e^{-r^2/2}(-1)^\nu U\left(-\nu, \frac{1}{2}; \frac{r^2}{2}\right)
\]

We will now show that the summation above (G.21) is 0 as \( r \to 0 \). The derivative in parentheses in equation (G.21) is
\[
D^\nu e^{-r^2/2} = 2\pi e^{-r^2/2}(-1)^\nu U\left(-\nu, \frac{1}{2}; \frac{r^2}{2}\right)
\]

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\]

We will now show that the summation above (G.21) is 0 as \( r \to 0 \). The derivative in parentheses in equation (G.21) is
\[
D^\nu e^{-r^2/2} = 2\pi e^{-r^2/2}(-1)^\nu U\left(-\nu, \frac{1}{2}; \frac{r^2}{2}\right)
\]
Finally, using the relation for $\nu$ (equation (G.8)),

$$\frac{-\sin\left(\frac{d\pi}{2}\right)}{\bar{a}^{d-2}} = \frac{\pi(d-2)}{\Gamma^2\left(\frac{d}{2}\right)} \frac{\Gamma\left(-\frac{E_{\text{ho}}}{2} + \frac{d}{2}\right)}{\Gamma\left(-\frac{E_{\text{ho}}}{2} + \frac{4-d}{4}\right)}. \quad (G.28)$$

In section 5 we convert our $\bar{a}_d$ to the relative motion units $a_\nu$ used in [23] via equation (39) to make comparison easier.

**Appendix H. Weak interaction perturbation solution for the $d$-dimension energy equation for two cold atoms in a harmonic trap**

Starting with equation (38) (derived in equation (G.28)), we have

$$\frac{-\sin\left(\frac{d\pi}{2}\right)}{\bar{a}^{d-2}} = \frac{\pi(d-2)}{\Gamma^2\left(\frac{d}{2}\right)} \frac{\Gamma\left(-\frac{E_{\text{ho}}}{2} + \frac{d}{2}\right)}{\Gamma\left(-\frac{E_{\text{ho}}}{2} + \frac{4-d}{4}\right)}. \quad (H.1)$$

For notational simplicity, below we drop the $d$ subscript in the dimension-dependent radius, $\bar{a}_d$.

First, we start with the substitution $\epsilon = -\nu = -\frac{E}{2} + \frac{d}{4}$ and take the reciprocal of both sides to yield

$$-\csc\left(\frac{d\pi}{2}\right)\frac{\pi(d-2)\bar{a}^{d-2}}{\Gamma^2\left(\frac{d}{2}\right)} = \frac{\Gamma\left(\epsilon + 1 - \frac{d}{2}\right)}{\Gamma(\epsilon)}. \quad (H.2)$$

In order to expand about $\bar{a} \to 0$, we first observe that the LHS also goes to zero when the gamma function in the denominator approaches poles, when $\epsilon = -n$ for $n \in \mathbb{Z}$. Therefore, we will expand $\epsilon$ about $\epsilon = -n$ to obtain $\epsilon = -n + \bar{a}^{d-2}\epsilon_1$. In effect, we are trying to evaluate

$$\left[\frac{\Gamma\left(\epsilon + 1 - \frac{d}{2}\right)}{\Gamma(\epsilon)}\right]_{\epsilon \to -n}. \quad (H.3)$$

Using the multiplicative identity, $(\epsilon + n)/\epsilon + n)$, allows us to write

$$\left[\frac{\Gamma\left(\epsilon + 1 - \frac{d}{2}\right)}{\Gamma(\epsilon)}\right]_{\epsilon \to -n} = \left[\frac{\Gamma\left(\epsilon + 1 - \frac{d}{2}\right)(\epsilon + n)}{\Gamma(\epsilon)(\epsilon + n)}\right]_{\epsilon \to -n} = \frac{\Gamma\left(-n + 1 - \frac{d}{2}\right)\bar{a}^{d-2}\epsilon_1}{\Gamma(\epsilon)(\epsilon + n)}_{\epsilon \to -n}. \quad (H.4)$$

Using $\text{Res}(f, z_0) = \lim_{\epsilon \to z_0} (\epsilon - z_0)f(\epsilon)$ and that $\Gamma(-n)$ is a pole with $\text{Res}(\Gamma(\epsilon), -n) = (-1)^n/\Gamma(n + 1)$, we have

$$\left[\frac{\Gamma\left(\epsilon + 1 - \frac{d}{2}\right)}{\Gamma(\epsilon)}\right]_{\epsilon \to -n} = \frac{\Gamma\left(-n + 1 - \frac{d}{2}\right)\bar{a}^{d-2}\epsilon_1}{(-1)^n/\Gamma(n + 1)}. \quad (H.5)$$

which simplifies to

$$\left[\frac{\Gamma\left(\epsilon + 1 - \frac{d}{2}\right)}{\Gamma(\epsilon)}\right]_{\epsilon \to -n} = (-1)^n\Gamma(n + 1)\Gamma\left(-n + 1 - \frac{d}{2}\right)\bar{a}^{d-2}\epsilon_1. \quad (H.6)$$

Exploiting the gamma function identity

$$(-1)^n\Gamma(z - n) = \frac{\Gamma(z)\Gamma(1 - z)}{\Gamma(n + 1 - z)} \quad (H.7)$$

where in this instance, $z = 1 - \frac{d}{2}$, our approximation becomes

$$\left[\frac{\Gamma\left(\epsilon + 1 - \frac{d}{2}\right)}{\Gamma(\epsilon)}\right]_{\epsilon \to -n} \approx \frac{\Gamma(n + 1)\Gamma\left(1 - \frac{d}{2}\right)}{\Gamma\left(n + \frac{d}{2}\right)}\bar{a}^{d-2}\epsilon_1. \quad (H.8)$$

Substituting the above $\epsilon \to -n$ limit for the right hand side of equation (H.2), we find

$$-\csc\left(\frac{d\pi}{2}\right)\frac{\pi(d-2)\bar{a}^{d-2}}{\Gamma^2\left(\frac{d}{2}\right)} = \left[\frac{\Gamma\left(\epsilon + 1 - \frac{d}{2}\right)}{\Gamma(\epsilon)}\right]_{\epsilon \to -n} = \frac{\Gamma(n + 1)\Gamma\left(1 - \frac{d}{2}\right)}{\Gamma\left(n + \frac{d}{2}\right)}\bar{a}^{d-2}\epsilon_1. \quad (H.9)$$

which we can solve for $\epsilon_1$:

$$\epsilon_1 = -\csc\left(\frac{d\pi}{2}\right)\frac{\pi(d-2)}{\Gamma^2\left(\frac{d}{2}\right)} \frac{\Gamma(n + \frac{d}{2})}{\Gamma(n + 1)\Gamma\left(1 - \frac{d}{2}\right)}. \quad (H.10)$$

Our approximation for $\epsilon = -n + \bar{a}^{d-2}\epsilon_1$ becomes

$$\epsilon = -n - \bar{a}^{d-2} \frac{\pi(d-2)}{\sin\left(\frac{d\pi}{2}\right)\Gamma^2\left(\frac{d}{2}\right)} \frac{\Gamma(n + \frac{d}{2})}{\Gamma(n + 1)\Gamma\left(1 - \frac{d}{2}\right)}. \quad (H.11)$$

and finally using our earlier definition, $\epsilon = -\frac{E}{2} + \frac{d}{4}$, and solving for $E$, our first order approximation for energy $E$ about small $\bar{a}$ is

$$E^{(d)} = 2n + \frac{d}{2} + \frac{2\pi(d-2)}{\sin\left(\frac{d\pi}{2}\right)\Gamma^2\left(\frac{d}{2}\right)} \frac{\Gamma(n + \frac{d}{2})}{\Gamma(n + 1)\Gamma\left(1 - \frac{d}{2}\right)} \bar{a}^{d-2}. \quad (H.12)$$
In section 5 we verify that the $d = 1, 2, 3$ results match [23] by replacing their factorials in the binomial coefficient with gamma functions and converting our $a_d$ to their units $a_o$ via equation (39).

H.1. Expansion about $d = 2$

The $d \to 2$ limit for the energy requires special consideration. Using the scattering length unit conversion equation (39), our exact energy functional in terms of Busch’s $a_o$ is

$$\frac{\pi (d - 2)}{\Gamma^2 \left( \frac{d}{2} \right)} \left[ \frac{E^{(2)}_0}{2} + \frac{d}{4} \right] = -\sin \left( \frac{\pi}{d} \right) a_o^{d-2} / 2^{d/2}. \quad (H.13)$$

Letting $d = 2$ in $\Gamma^2 \left( \frac{d}{2} \right)$ and $2^{d/2}$ and rearranging, the above equation becomes

$$-\frac{\sin \left( \frac{\pi}{d} \right)}{d - 2} \left[ \frac{E^{(2)}_0}{2} + \frac{1 - d}{4} \right] = \frac{\pi}{2} a_o^{d-2} / 2. \quad (H.14)$$

Expanding both sides about $d = 2$, to first order in $d - 2$ we find

$$\frac{\pi}{2} + \frac{\pi}{4} (d - 2) \ln a_o + \ldots$$

$$= \frac{\pi}{2} - \frac{\pi}{4} (d - 2) \psi \left( \frac{1}{2} - \frac{E^{(2)}_0}{2} \right) + \ldots, \quad (H.15)$$

where $\psi(\cdot)$ is the logarithmic derivative of Euler’s gamma function. Finally, the first-order terms yield the first-order approximation to the $2d$ energy:

$$\psi \left( \frac{1}{2} - \frac{E^{(2)}_0}{2} \right) = \ln \left( \frac{1}{a_o} \right). \quad (H.16)$$

### Appendix I. Typographical errors in Wódkiewicz [44]

1. In calculating the scattered wave (equation (2.8)), the complex parameter $z$ should be evaluated at $z = e^{-i\pi/2k}/(4\alpha)$, not $z = e^{i\pi/2k}/4\alpha$.
2. Derived from equation (2.8), equation (2.10) should have no negative sign in the first exponential term:

$$\lim_{r \to \infty} \psi^{(z)}(r) = e^{ikr} + f^{(d)}_k e^{ikr} / r^{1/2-1/2}.$$ 

This correction matches equations (3.2a), (3.5a) and (3.9a).
3. equation (3.1) should read

$$f^{(d)}_k = -\frac{i^n k^{n-1} (2n - 1)!!}{k^{2n-1} + i2^{n-1} n^{2n-1} (2n - 1)!! / \pi}.$$ 

This formula is the result of the derivation described in the paper and also matches with the rest of the paper’s equations (3.2b), (3.5b) and (3.9b).
4. The symbol $k$ in equation (5.5) should be $l$.

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