Correlations in nuclear energy recurrence relations

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Abstract
The excitation energies of states belonging to the ground state bands of heavy even-even nuclei are analysed using recurrence relations. Excellent agreement with experimental data at the 10 keV level is obtained by taking into account strong correlations which emerge in the analysis. This implies that the excitation energies can be written as a polynomial of maximum degree four in the angular momentum.

21.10.-k, 21.10.Hw, 23.20.Lv and 27.20.+n

1 Introduction

Despite, or perhaps because of the uncertainties in representing the strong interaction by a potential, it has been a longstanding quest in nuclear physics to find simple algebraic relationships between the excitation energies of a nucleus wherever possible. Attempts to achieve this have often been inspired by approximate symmetries such as Wigner’s supermultiplets [1] or Elliot’s SU(3) scheme [2]. A textbook example leading to a simple energy formula is provided by the rotational-vibrational model [3,4]. Later, the Interacting Boson Model [5] used linear combinations of Casimir operators of various chains of subgroups of U(6) to generate simple expressions for nuclear spectra. Indeed, investigations into algebraic cluster models of energy levels continue to provide fruitful fields of research to the present day [6].
However, the symmetries invoked by these models are never perfectly realised in real nuclei, and although suggestively close agreement can often be obtained in a number of favourable cases, they generally struggle to achieve precision at the level of a few 10’s of keV. We suggest that it might be profitable to analyse the experimental data more closely without any preconceived ideas about underlying models, to see if this alone suggests any simple relations between excitation energies that could be used for predictive purposes. There have already been such suggestions from our earlier work [7] and encouraged by these findings we examine here the deviations of our previous expressions from measured values with a view to improving their precision without sacrificing their simplicity.

It will turn out that the experimental excitation energies of five consecutive states of the ground state band must be known for our recurrence relations to get started. Whenever this applies, they will provide a useful tool for predicting and/or confirming the energies of the succeeding states of that band, and thus will facilitate the task of nuclear data evaluation. We believe they should be superior to the common practice of seeking to fit the energies with a rotationally inspired formula such as \( E = E_0 + aJ(J + 1) + b[J(J + 1)]^2 \), since they are based purely on empirical values and are completely independent of any underlying model. In particular, they will be helpful in predicting the energy of the next state in a band beyond the highest identified to date.

2 Recurrence relations

We have previously analysed recursion relations involving the angular momenta \( J = 0, 2, 4, \ldots \) and corresponding excitation energies \( E(J) \) of states belonging to the ground state bands of even-even nuclei [7]. Given a pair of angular momenta \( (J, K) \) with \( J > K \) we found empirically that

\[
\frac{E(J - L) - E(K + L)}{E(J) - E(K)} - \frac{(J - L) - (K + L)}{(J - K)} = 0
\]

where \( L \) increases in steps of two from an initial value \( L = 0 \). We note that as the spin values \( (J - L) \) and \( (K + L) \) change with \( L \) their sum remains equal to \( (J + K) \).

In the earlier analysis [7] we found that Eq.(1) was satisfied to within a few percent with the
deviations from that equation strongly correlated. These deviations are given by

\[ d_{L}^{J,K} = \frac{E(J - L) - E(K + L)}{E(J) - E(K)} - \frac{(J - L) - (K + L)}{(J - K)} \] (2)

and here we investigate the nature and extent of the correlations between the various \( d_{L}^{J,K} \), and how they can be used to improve the agreement between theory and data.

3 Application to 87 heavy nuclei

In the database shown in Table 1 [8] we have included all 87 even-even nuclei between Ba and Cm which have more than four protons (or proton-holes) and four neutrons (or neutron-holes) outside closed shells, and for which the ground state bands are accurately known (i.e. with firm \( J^π \) assignments) up to at least \( J^π = 12^+ \).

We concentrate on the typical case defined by \( J = 10, K = 0 \) and \( L_1 = 2, L_2 = 4 \) leaving a more complete analysis to be presented elsewhere [9]. Figure 1 shows that the deviations are surprisingly well described by the simple relation

\[ d_{2}^{10,0} = 2 \ d_{4}^{10,0} \] (3)

Indeed, a least squares straight line fit to all 87 data points yields a gradient of \( 2.007 \pm 0.028 \) and an intercept of \( 0.00006 \pm 0.00032 \). The Figure also indicates that the six points with the highest values of \( d_{2}^{10,0} \) deviate from the straight line in a statistically significant manner. These points correspond to the nuclei \(^{130}\text{Ce}, \ ^{134}\text{Ce}, \ ^{134}\text{Nd}, \ ^{136}\text{Sm}, \ ^{140}\text{Gd} \) and \(^{172}\text{Os} \). The results for \(^{134}\text{Ce} \) are probably problematic due to backbending. There are two \( 8^+ \) states in close proximity at 2.81111 and 3.0176 MeV (we take the first of them as the band member) and similarly three \( 10^+ \) states at 3.2086, 3.7193 and 3.81765 MeV (we take the second of these as the band member).

This band crossing feature is generally expected for nuclei with low deformation at the beginning and end of the rare-Earth region [10]. It therefore comes as no surprise that similar deviations, albeit to a lesser extent, are observed for four other nuclei in the neighbourhood of \(^{134}\text{Ce} \). Table 2 lists the states of the ground state band, and an excited band which can be expected to mix with it, for all six of these outlying nuclei. We regard these few deviations in a positive light as
pinpointing unusual nuclear spectra that are worthy of more detailed scrutiny, and here we would include also $^{150}$Sm, $^{174}$Os, $^{222}$Th and in particular $^{244}$Pu.

The framework introduced above can be used to locate missing members of a band, or to check experimental assignments to the band. We illustrate the procedure for the case where all the band members up to $J = 8$ are known and an estimate of the excitation energy of the $J = 10$ state is required. By Eq.(2)

$$d^{10,0}_2 = \frac{E(8) - E(2)}{E(10) - E(0)} - 0.6 \quad \text{and} \quad d^{10,0}_4 = \frac{E(6) - E(4)}{E(10) - E(0)} - 0.2,$$

which together with Eq.(3) result in

$$\{E(10) - E(0)\} = 5\{E(8) - 2E(6) + 2E(4) - E(2)\}$$

(5)

Table 1 lists the individual ratios $r = d^{10,0}_2/d^{10,0}_4$ as well as the differences $\Delta E$ between the theoretical values of $\{E(10) - E(0)\}$ given by Eq.(5) and their experimental counterparts. Omitting the clearly anomalous case of $^{134}$Ce we find an r.m.s. deviation of $\Delta E = 10.4$ keV, an order of magnitude better than the results obtained when setting $d_{L,K}^{J,0} = 0$ in either of Eqs.(4). Of interest also is that the further analysis undertaken below suggests that Eq.(5) is a particular case of a more general recurrence relation given by

$$\{E(n) - E(n - 10)\} = 5\{E(n - 2) - 2E(n - 4) + 2E(n - 6) - E(n - 8)\}$$

(6)

with $n \geq 10$.

4 Derivation from Taylor series

This energy recurrence relation can also be obtained by assuming that the excitation energy $E(n)$ may be expanded as a Taylor series in angular momentum $n$ which converges sufficiently rapidly that fifth order and higher terms may be neglected. Expanding $E(n_0 \pm 1)$, $E(n_0 \pm 3)$ and $E(n_0 \pm 5)$ about $n_0$ we obtain

$$E(n_0 \pm 1) = E(n_0) \pm E'(n_0) + \frac{E''(n_0)}{2!} \pm \frac{E'''(n_0)}{3!} + \ldots$$

(7)
\[ E(n_0 \pm 3) = E(n_0) \pm 3E'(n_0) + \frac{9E''(n_0)}{2!} \pm \frac{27E'''(n_0)}{3!} + \ldots \]  
(8)

\[ E(n_0 \pm 5) = E(n_0) \pm 5E'(n_0) + \frac{25E''(n_0)}{2!} \pm \frac{125E'''(n_0)}{3!} + \ldots \]  
(9)

By successively subtracting the \((n_0 - 1)\), \((n_0 - 3)\) and \((n_0 - 5)\) terms from their \((n_0 + 1)\), \((n_0 + 3)\) and \((n_0 + 5)\) counterparts in the three equations above we completely eliminate all even powers from the Taylor series expansions obtaining respectively,

\[ [E(n_0 + 1) - E(n_0 - 1)] = 2E'(n_0) + \frac{E''(n_0)}{3} + \ldots \]  
(10)

\[ [E(n_0 + 3) - E(n_0 - 3)] = 6E'(n_0) + 9E''(n_0) + \ldots \]  
(11)

\[ [E(n_0 + 5) - E(n_0 - 5)] = 10E'(n_0) + \frac{250E''(n_0)}{6} + \ldots \]  
(12)

Subtracting twice Eq.(10) from Eq.(11) yields

\[ E(n_0 + 3) - E(n_0 - 3) - 2E(n_0 + 1) + 2E(n_0 - 1) = 2E'(n_0) + \frac{50}{6}E''(n_0) + \ldots \]  
(13)

and the right hand side of Eq.(12) is exactly five times that of Eq.(13). This allows us to write

\[ \{E(n_0 + 5) - E(n_0 - 5)\} = 5\{E(n_0 + 3) - 2E(n_0 + 1) + 2E(n_0 - 1) - E(n_0 - 3)\} \]  
(14)

so that taking \(n_0 = 5\) we obtain Eq.(5) which, taken together with Eq.(4), yields Eq.(3). Also by putting \(n_0 = n - 5\) in Eq.(14) we recover Eq.(6). We thus find that Eq.(6), which with \(n = 10\) generates the excellent agreement with the data shown in Table 1, can be derived from a Taylor series expansion if terms of order five and higher are ignored.

Of course, we have not rigorously proved convergence of the Taylor series, which depends on the (unknown) behaviour of the higher derivates of \(E\) as a function of \(J\). What we have done, starting from a Taylor series, is to completely eliminate all even powers of \(J\), and show that Eqs.(3), (5) and (6) follow if the remaining odd terms of order 5 and higher are ignored. This suggests that the band excitation energies \(E(J)\) considered here take the form of a polynomial of maximum degree four in the angular momentum \(J\), and we note that only cubic forms were considered in earlier work [7, 11].
Since Eq.(6) is satisfied by any E-dependence on $J$ that is of quartic or lower degree, it encompasses a number of models. It is satisfied, for example, by the idealised quadratic form $E_0 + \alpha J(J + 1)$ of a perfect rotor with a constant moment of inertia, or by the anharmonic vibrator (AHV) model for which $E(J) = \epsilon_2 n + \epsilon_4 n(n-1)/2 + \epsilon_6 n(n-1)(n-2)/6$, where $n = J/2$ and $\epsilon_2, \epsilon_4$ and $\epsilon_6$ are parameters fitted to each nucleus in turn \cite{11}. We note, however, that although Eq.(6) is obeyed by these models it is satisfied to the extent that inserting the model quantities on the right hand side of Eq.(6) generates the model quantity on the left hand side. As stated in our introductory paragraph, our emphasis is to extend previous work \cite{7} by finding more accurate direct relations between experimental quantities.

5 Conclusion

In conclusion we have found strong correlations between the deviations $d_{L}^{J,K}$ from our original relations of Eq.(1) above. This has enabled us to write down a highly successful recursion relation, not involving any free parameters, for locating members of ground state bands from a knowledge of the excitation energies of the lower members of the band. For the specific case of $J = 10, K = 0$ and $L_1 = 2, L_2 = 4$ we have shown how a knowledge of the excitation energies of the $J = 2, 4, 6, 8$ band members allows us to predict the excitation energy of the $J = 10$ state with an r.m.s. error of $\sim 10$ keV. We note that this is likely to be a particular case of a more general relation. Although we have restricted attention here to values of $J$ no higher than 10, we have observed that Eq.(6) can be used to predict energies for states with much greater values of $J$ (up to $J \sim 30$ in some cases) and we intend to investigate this further elsewhere.

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Figure 1: Plot of the deviations $d_{2,0}^{10,0}$ against the deviations $d_{4}^{10,0}$ for the 87 nuclei listed in Table 1. A least squares straight line fit to the data indicates a slope of $2.007 \pm 0.028$ and an intercept of $0.00006 \pm 0.00032$. The solid line corresponds to $d_{2,0}^{10,0} = 2d_{4}^{10,0}$. 
Table 1: Database of 87 rare-Earth and Actinide nuclei. We highlight the ratios $r = d_{10}^{10,0}/d_{4}^{10,0}$ from Eq.(4), which are all close to 2.0, and $\Delta E$ the difference between the values of the $10^+$ excitation energy derived using Eq.(5) and that obtained from direct measurement.

| Position | Nucleus | $d_{2}^{10,0}$ | $d_{4}^{10,0}$ | $r$ | $E(10^+)$ (keV) | $\Delta E$ (keV) |
|----------|---------|----------------|----------------|-----|----------------|-----------------|
| 01       | Ba      | 0.017882       | 0.008805       | 2.030 | 3082.3        | 4.199           |
| 02       | Ba      | 0.021973       | 0.010976       | 2.001 | 2044.3        | 0.200           |
| 03       | Ba      | 0.034223       | 0.016736       | 2.044 | 2051.8        | 7.699           |
| 04       | Ce      | 0.022971       | 0.011385       | 2.017 | 2100.9        | 2.099           |
| 05       | Ce      | 0.040477       | 0.018476       | 2.190 | 2809.0        | 49.49           |
| 06       | Ce      | 0.045793       | 0.018965       | 2.414 | 3719.3        | 146.2           |
| 07       | Ce      | 0.025222       | 0.011373       | 1.980 | 1422.6        | -1.60           |
| 08       | Nd      | 0.050431       | 0.024076       | 2.094 | 2816.9        | 32.10           |
| 09       | Nd      | 0.010588       | 0.005401       | 1.960 | 1210.8        | -1.29           |
| 10       | Nd      | 0.008212       | 0.004020       | 2.042 | 1169.0        | 1.000           |
| 11       | Sm      | 0.027561       | 0.014139       | 1.949 | 1952.0        | -7.00           |
| 12       | Sm      | 0.039401       | 0.021568       | 1.826 | 2414.6        | -45.1           |
| 13       | Sm      | 0.017746       | 0.007751       | 2.289 | 2433.2        | 27.30           |
| 14       | Sm      | 0.023663       | 0.011533       | 2.051 | 1609.2        | 4.799           |
| 15       | Sm      | 0.015753       | 0.008027       | 1.962 | 1333.0        | -2.00           |
| 16       | Sm      | 0.008612       | 0.004528       | 1.902 | 1307.4        | -2.89           |
| 17       | Sm      | 0.007835       | 0.003942       | 1.987 | 1227.8        | -0.29           |
| 18       | Gd      | 0.030454       | 0.015726       | 1.936 | 2266.3        | -11.3           |
| 19       | Gd      | 0.047561       | 0.024470       | 1.943 | 2796.8        | -19.2           |
| 20       | Gd      | 0.009676       | 0.005181       | 1.867 | 2300.4        | -7.90           |
| 21       | Gd      | 0.023885       | 0.011789       | 2.025 | 1637.0        | 2.500           |
| 22       | Gd      | 0.018670       | 0.009377       | 1.990 | 1416.1        | -0.59           |
| 23       | Gd      | 0.010814       | 0.006978       | 1.946 | 1350.0        | -2.00           |
| 24       | Gd      | 0.009364       | 0.004735       | 1.977 | 1300.7        | -0.70           |
| 25       | Gd      | 0.009580       | 0.004863       | 1.970 | 1237.9        | -0.89           |
| 26       | Dy      | 0.013211       | 0.006978       | 1.893 | 2304.1        | -8.59           |
| 27       | Dy      | 0.024811       | 0.012289       | 2.018 | 1725.0        | 2.000           |
| 28       | Dy      | 0.021669       | 0.010907       | 1.986 | 1520.1        | -1.10           |
| 29       | Dy      | 0.016316       | 0.008193       | 1.991 | 1428.0        | -0.49           |
| 30       | Dy      | 0.011300       | 0.005657       | 1.997 | 1375.1        | -0.09           |
| 31       | Dy      | 0.010719       | 0.005422       | 1.976 | 1261.3        | -0.79           |
| 32       | Dy      | 0.008053       | 0.003952       | 2.037 | 1341.0        | 0.999           |
| 33       | Er      | 0.013231       | 0.006410       | 2.063 | 2633.1        | 5.400           |
| 34       | Er      | 0.027889       | 0.013799       | 2.020 | 2072.5        | 2.999           |
| 35       | Er      | 0.026597       | 0.013389       | 1.986 | 1761.1        | -1.60           |
| 36       | Er      | 0.020601       | 0.010319       | 1.996 | 1602.8        | -0.29           |
| 37       | Er      | 0.014715       | 0.007496       | 1.963 | 1518.1        | -2.09           |
| 38       | Er      | 0.015441       | 0.007839       | 1.969 | 1349.6        | -1.59           |
| 39       | Er      | 0.007459       | 0.003751       | 1.988 | 1396.8        | -0.29           |
| 40       | Yb      | 0.029203       | 0.014285       | 2.044 | 2373.0        | 7.499           |
| 41       | Yb      | 0.031836       | 0.015799       | 2.015 | 2024.1        | 2.400           |
| 42       | Yb      | 0.027238       | 0.013585       | 2.005 | 1753.4        | 0.600           |
| 43       | Yb      | 0.020150       | 0.010162       | 1.982 | 1605.9        | -1.40           |
| 44       | Yb      | 0.018984       | 0.009555       | 1.986 | 1425.4        | -0.90           |
| 45       | Yb      | 0.011478       | 0.005843       | 1.964 | 1437.5        | -1.50           |
### Database of 87 rare-Earth and Actinide nuclei continued.

| Position | Nucleus | $d_{2}^{\text{PDD}}$ | $d_{4}^{\text{PDD}}$ | $r$ | $E(10^+)$ | $\Delta E$ |
|----------|---------|-----------------|-----------------|-----|------------|----------|
| 46       | Hf 72   | 0.028026        | 0.013743        | 2.039 | 2635.4     | 7.100    |
| 47       | Hf 72   | 0.032790        | 0.016390        | 2.000 | 1971.9     | 0.100    |
| 48       | Hf 72   | 0.027613        | 0.013927        | 1.982 | 1736.1     | -2.09    |
| 49       | Hf 72   | 0.019445        | 0.009768        | 1.990 | 1521.2     | -0.69    |
| 50       | Hf 72   | 0.014786        | 0.007348        | 2.012 | 1570.3     | 0.699    |
| 51       | W 74    | 0.007580        | 0.003815        | 1.987 | 1630.4     | -0.39    |
| 52       | W 74    | 0.036210        | 0.017882        | 2.024 | 2202.1     | 4.899    |
| 53       | W 74    | 0.036210        | 0.017882        | 2.024 | 2202.1     | 4.899    |
| 54       | W 74    | 0.013290        | 0.006719        | 1.968 | 1617.3     | -4.29    |
| 55       | W 74    | 0.026503        | 0.013496        | 1.963 | 1637.5     | -3.99    |
| 56       | W 74    | 0.025659        | 0.013042        | 1.967 | 1648.5     | -3.50    |
| 57       | W 74    | 0.021832        | 0.011060        | 1.973 | 1665.4     | -2.40    |
| 58       | W 74    | 0.021897        | 0.010864        | 2.015 | 1664.1     | 1.400    |
| 59       | W 74    | 0.009929        | 0.005023        | 1.976 | 1712.0     | -0.99    |
| 60       | W 74    | 0.013177        | 0.006470        | 2.036 | 1860.8     | 2.200    |
| 61       | Os 76   | 0.040940        | 0.021503        | 1.903 | 2023.9     | -20.9    |
| 62       | Os 76   | 0.026460        | 0.011808        | 2.240 | 1617.5     | 23.00    |
| 63       | Os 76   | 0.025787        | 0.012349        | 2.088 | 1634.1     | 8.899    |
| 64       | Os 76   | 0.036625        | 0.018658        | 1.963 | 1767.6     | -6.09    |
| 65       | Os 76   | 0.035209        | 0.017273        | 2.038 | 1812.0     | 5.999    |
| 66       | Os 76   | 0.017250        | 0.008636        | 1.997 | 1871.2     | -0.20    |
| 67       | Os 76   | 0.020744        | 0.010251        | 2.023 | 2068.0     | 2.499    |
| 68       | Os 76   | 0.021217        | 0.010393        | 2.041 | 2418.8     | 5.200    |
| 69       | Ra 88   | 0.012720        | 0.007045        | 1.805 | 1342.7     | -9.19    |
| 70       | Ra 88   | 0.026836        | 0.013563        | 1.978 | 959.9      | -1.39    |
| 71       | Th 90   | 0.022955        | 0.012305        | 1.865 | 1461.1     | -12.1    |
| 72       | Th 90   | 0.026852        | 0.013494        | 1.989 | 1173.8     | -0.79    |
| 73       | Th 90   | 0.024531        | 0.012342        | 1.987 | 1040.3     | -0.79    |
| 74       | Th 90   | 0.019324        | 0.009914        | 1.949 | 911.8      | -2.29    |
| 75       | Th 90   | 0.013812        | 0.007063        | 1.955 | 826.8      | -1.30    |
| 76       | U 92    | 0.014621        | 0.007520        | 1.944 | 856.3      | -1.79    |
| 77       | U 92    | 0.012358        | 0.006104        | 2.024 | 805.9      | 0.600    |
| 78       | U 92    | 0.011845        | 0.006017        | 1.968 | 741.2      | -0.69    |
| 79       | U 92    | 0.009868        | 0.004908        | 2.010 | 782.3      | 0.199    |
| 80       | U 92    | 0.008241        | 0.004174        | 1.974 | 747.4      | -0.40    |
| 81       | Pu 94   | 0.009049        | 0.004654        | 1.944 | 773.5      | -1.0     |
| 82       | Pu 94   | 0.006981        | 0.003490        | 1.999 | 773.5      | -3.33    |
| 83       | Pu 94   | 0.008241        | 0.004174        | 1.974 | 747.4      | -0.40    |
| 84       | Pu 94   | 0.008271        | 0.004341        | 1.905 | 778.6      | -1.60    |
| 85       | Pu 94   | 0.011665        | 0.003015        | 3.867 | 802.4      | 22.6     |
| 86       | Pu 94   | 0.007260        | 0.003642        | 1.993 | 818.1      | -0.10    |
| 87       | Cm 96   | 0.006809        | 0.003102        | 2.194 | 760.7      | 2.300    |
Table 2: Excitation energies (keV) of ground state and nearby excited state band members in the six outlying nuclei $^{130}\text{Ce}$, $^{134}\text{Ce}$, $^{134}\text{Nd}$, $^{136}\text{Sm}$, $^{140}\text{Gd}$ and $^{172}\text{Os}$. The superscripts are the band labels used in the Evaluated Nuclear Structure Data File [S].

| $J^\pi$ | $^{130}\text{Ce}$ | $^{134}\text{Ce}$ | $^{134}\text{Nd}$ | $^{136}\text{Sm}$ | $^{140}\text{Gd}$ | $^{172}\text{Os}$ |
|---------|------------------|------------------|------------------|------------------|------------------|------------------|
| 2$^+$   | 253.85$^e$       | 409.20$^k$       | 294.17$^k$       | 254.92$^k$       | 328.6$^a$        | 227.77$^a$       |
| 4$^+$   | 710.37$^e$       | 1048.68$^k$      | 788.92$^k$       | 686.36$^k$       | 836.2$^a$        | 606.17$^a$       |
| 6$^+$   | 1324.1$^e$       | 1863.1$^k$       | 1420.06$^k$      | 1221.4$^k$       | 1464.0$^a$       | 1054.47$^a$      |
| 8$^+$   | 2053.1$^e$       | 2811.1$^k$       | 2126.4$^k$       | 1798.8$^k$       | 2139.7$^a$       | 1524.95$^a$      |
| 10$^+$  | 2809.0$^e$       | 3719.3$^k$       | 2816.9$^k$       | 2414.6$^k$       | 2796.8$^a$       | 2023.87$^a$      |
| 12$^+$  | 3311.9$^e$       | 4183.6$^k$       | 3482.9$^k$       | 3091.8$^k$       | 3267.5$^a$       | 2564.5$^a$       |
| 2$^+$   | 834.55$^f$       | 965.66$^c$       | 793.86$^a$       | 712.88$^c$       | 713.3$^k$        | 810.01$^c$       |
| 4$^+$   | 1322.83$^f$      | 1643.47$^c$      | 1313.03$^a$      | 1170.98$^c$      | 1281.4$^k$       | 1137.88$^c$      |
| 6$^+$   | 1897.6$^f$       | 2303.8$^c$       | 1910.6$^a$       | 1640.96$^c$      | 1881.4$^k$       | 1551.25$^k$      |
| 8$^+$   | 2560.6$^f$       | 3017.6$^c$       | 2467.2$^a$       | 2250.2$^c$       | 2632$^k$         | 2093.63$^c$      |
| 10$^+$  | 3296.5$^f$       | 3208.6$^d$       | 3051.9$^a$       | 2953.7$^c$       | 2926.8$^b$       |                  |
| 12$^+$  | 3985.3$^f$       | 4006.8$^d$       | 3436.5$^a$       | 3682.6$^c$       | 3617$^b$         |                  |