Some New Inequalities for Beta Distributions

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Abstract

This note provides some new inequalities and approximations for beta distributions, including exponential tail inequalities, inequalities of Hoeffding and Bernstein type, and Gaussian tail inequalities and approximations. Some of the results are extended to Gamma distributions.

Keywords: Beta distribution; tail inequalities; density ratios; Gaussian approximation

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1 Introduction

Beta distributions play an important role in statistics and probability theory (Gupta and Nadarajah, 2004). A frequent obstacle in problems involving beta distributions is the lack of analytic expressions for their distribution function, the normalized incomplete beta function. Therefore one often has to resort to inequalities and approximations.

This note provides some new inequalities for the beta distribution $\text{Beta}(a, b)$ with parameters $a, b > 0$, its distribution function $B_{a,b}$ and its density function $\beta_{a,b}$ on $[0, 1]$, where

$$\beta_{a,b}(x) := B(a, b)^{-1}x^{a-1}(1-x)^{b-1}, \quad x \in (0, 1).$$

Here $B(a, b) := \int_0^1 x^{a-1}(1-x)^{b-1} dx = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, and $\Gamma(\cdot)$ denotes the gamma function.

2 Exponential tail inequalities

For $p \in (0, 1)$ and $x \in [0, 1]$ let

$$K(p, x) := p \log \left( \frac{p}{x} \right) + (1-p) \log \left( \frac{1-p}{1-x} \right) \in [0, \infty].$$

This function $K(p, \cdot)$ is strictly convex with $K(p, p) = 0$. Our starting point is the following exponential inequality (cf. Dümbgen 1998, Proposition 2.1):

Proposition 1. Let $p := a/(a+b)$. Then for arbitrary $x \in [0, 1]$,

$$\frac{x^a(1-x)^b}{p^a(1-p)^b} = \exp(-(a+b)K(p, x)) \geq \begin{cases} B_{a,b}(x) & \text{if } x \leq p, \\ 1 - B_{a,b}(x) & \text{if } x \geq p. \end{cases}$$
In case of \(a \geq 1\) or \(b \geq 1\), this inequality can be refined as follows:

**Lemma 2.** Suppose that \(a \geq 1\). With \(p_r := (a - 1)/(a + b - 1) < p\),

\[
1 - B_{a,b}(x) \begin{cases} \leq \frac{x^{a-1}(1-x)^b}{p_r^{a-1}(1-p_r)^b} & = \exp\left(- (a + b - 1) K(p_r, x) \right) \text{ for } x \in [p_r, 1], \\ \geq \frac{x^{a-1}(1-x)^b}{bB(a,b)} & = \frac{\beta_{a,b+1}(x)}{a+b} \text{ for } x \in [0, 1]. \end{cases}
\]

Suppose that \(b \geq 1\). With \(p_\ell := a/(a + b - 1) > p\),

\[
B_{a,b}(x) \begin{cases} \leq \frac{x^a(1-x)^{b-1}}{p_\ell^a(1-p_\ell)^{b-1}} & = \exp\left(- (a + b - 1) K(p_\ell, x) \right) \text{ for } x \in [0, p_\ell], \\ \geq \frac{x^a(1-x)^{b-1}}{aB(a,b)} & = \frac{\beta_{a+1,b}(x)}{a+b} \text{ for } x \in [0, 1]. \end{cases}
\]

**Remark 3.** At first glance, the upper bounds in Lemma 2 seem to be weaker than the ones in Proposition 1 at least in the tail regions, because the factor \(a + b - 1\) is strictly smaller than \(a + b\). But elementary algebra reveals that in case of \(a \geq 1\),

\[
(a + b - 1) K(p_r, x) - (a + b) K(p, x) = \log \left( \frac{x}{p} \right) + (a + b - 1) \log \left( 1 + \frac{1}{a + b - 1} \right) - (a - 1) \log \left( 1 + \frac{1}{a - 1} \right) > 0 \text{ for } x \in [p, 1],
\]

because \(h(y) := y \log(1 + 1/y)\) (with \(h(0) := 0\)) is strictly increasing in \(y \geq 0\). Analogously, if \(b \geq 1\), then

\[
(a + b - 1) K(p_\ell, x) - (a + b) K(p, x) = \log \left( \frac{1-x}{1-p} \right) + (a + b - 1) \log \left( 1 + \frac{1}{a + b - 1} \right) - (b - 1) \log \left( 1 + \frac{1}{b - 1} \right) > 0 \text{ for } x \in (0, p].
\]

Thus the bounds in Lemma 2 are strictly smaller than the bounds in Proposition 1. This is illustrated in Figure 1 for \((a,b) = (4,8)\).

**Remark 4.** The upper bound for the right tail in Lemma 2 can be improved substantially if \(1 \leq a \leq b\). Indeed, the proof Lemma 2 shows that for arbitrary \(0 < x_o \leq x \leq 1\),

\[
1 - B_{a,b}(x) \leq \left( 1 - B_{a,b}(x_o) \right) \frac{x^{a-1}(1-x)^b}{x_o^{a-1}(1-x)^b}.
\]

Specifically, it is well-known that \(\text{Median}(\text{Beta}(a,b)) \leq p\), see Groeneveld and Meeden (1977), so \(1 - B_{a,b}(p) \leq 1/2\) and

\[
1 - B_{a,b}(x) \leq \frac{x^{a-1}(1-x)^b}{2p^{a-1}(1-p)^b} \text{ for } x \in [p, 1].
\]

This upper bound is strictly smaller than the bound in Lemma 2 (restricted to \(x \in [p, 1]\)), provided that \(2p^{a-1}(1-p)^b > p_r^{a-1}(1-p_r)^b\), and this is equivalent to \(h(a - 1) > h(a + b - 1) - \log(2)\) with the increasing function \(h(y) = y \log(1 + 1/y), y > 0\). Since \(h(a + b - 1) < \lim_{y \to \infty} h(y) = 1\), a sufficient condition is that \(h(a - 1) \geq 1 - \log(2)\), which is fulfilled for \(a \geq 1.152\).
Figure 1: Exponential tail inequalities for \( \text{Beta}(a, b) \) when \((a, b) = (4, 8)\). In the upper panel, the green line shows \(1 - B_{a,b}\), the black line is its upper bound from Lemma 2, and the blue line is its upper bound from Proposition 1. One also sees the distribution function \(B_{a,b}\) and its bounds as dotted lines. The additional red line is the upper bound from Remark 4. The lower panel shows these bounds on a log-scale.
Proof of Lemma 2. Since $B_{a,b}(x) = 1 - B_{b,a}(1-x)$ and $K(q,x) = K(1-q, 1-x)$ for $q \in (0, 1)$ and $x \in [0,1]$, it suffices to prove the upper bound for $1 - B_{a,b}(x)$, $x \in [p_r, 1]$.

In case of $a = 1$, the asserted bounds are sharp, because $B(a,b) = 1/b$, $p_r = 0$ and $B_{a,b}(x) = 1 - (1-x)^b$. In case of $a > 1$, the ratio

$$\frac{1 - B_{a,b}(x)}{x^{a-1}(1-x)^b} = \frac{B(a,b)-1}{1-x} \int_{x}^{1} \left( \frac{u}{x} \right)^{a-1} \left( \frac{1-u}{1-x} \right)^{b-1} du$$

is strictly decreasing in $x \in (0,1)$. Indeed, with $w(u) := (1-u)/(1-x) \in (0,1)$ for $u \in (x,1)$, we have $dw(u)/du = -1/(1-x)$, and $u = 1 - (1-x)w(u)$, so the ratio in question may be rewritten as

$$B(a,b)^{-1} \int_{0}^{1} \left( w + \frac{1-w}{x} \right)^{a-1} w^{b-1} dw,$$

which is clearly strictly decreasing in $x \in (0,1)$ with limit $b^{-1}B(a,b)^{-1}$ as $x \to \infty$. Consequently, on the one hand, $1 - B_{a,b}(x)$ is greater than $x^{a-1}(1-x)^b/(bB(a,b)) = \beta_{a,b+1}(x)/(a+b)$. On the other hand, for $0 < x_o < x \leq 1$,

$$1 - B_{a,b}(x) \leq (1 - B_{a,b}(x_o)) \frac{x^{a-1}(1-x)^b}{a_{a-1}(1-x_o)^b} \leq \frac{x^{a-1}(1-x)^b}{a_{a-1}(1-x_o)^b}.$$ 

Since $x^{a-1}(1-x)^b$ is increasing in $x \in [0, p_r]$ and decreasing in $x \in [p_r, 1]$, the latter bound is trivial for $x \leq p_r$, whereas for $x > p_r$, it becomes minimal for $x_o = p_r$. \hfill \Box

The inequalities in Lemma 2 imply Hoeffding type exponential inequalities. Indeed, it follows from $K(q,q) = 0$,

$$\frac{\partial}{\partial x} K(q,x) = \frac{x-q}{x(1-x)}$$

and $x(1-x) \leq 1/4$ that

$$K(q,x) \geq 2(x-q)^2.$$

This yields the following inequalities:

**Corollary 5.** For arbitrary $a, b > 0$ and $x \in [0,1]$,

$$B_{a,b}(x) \leq \exp(-2(a+b-1)(x-p_r)^2) \quad \text{if } b \geq 1 \text{ and } x \leq p_r,$$

$$1 - B_{a,b}(x) \leq \exp(-2(a+b-1)(x-p_r)^2) \quad \text{if } a \geq 1 \text{ and } x \geq p_r.$$ 

Further tail and concentration inequalities for the Beta distribution have been derived by Marchal and Arbel (2017), Bobkov and Ledoux (2019, Appendix B) and Skorski (2021). Marchal and Arbel (2017) prove that Beta$(a,b)$ is subgaussian with a variance parameter that is the solution of an equation involving hypergeometric functions. An analytic upper bound for the variance parameter is $(4(a+b+1))^{-1}$, which implies the tail inequalities

$$\exp(-2(a+b+1)(x-p)^2) \geq \begin{cases} B_{a,b}(x) & \text{if } x \leq p, \\ 1 - B_{a,b}(x) & \text{if } x \geq p. \end{cases}$$

These bounds are weaker than the one-sided bounds in Corollary 5. For the right tails, the difference

$$(a+b-1)(x-p_r)^2 - (a+b+1)(x-p)^2$$
is strictly concave in \( x \) with value \( b^2/[(a + b)^2(a + b - 1)] > 0 \) for \( x \in \{p, 1\} \). Analogously, for the left tails, the difference

\[
(a + b - 1)(x - p)^2 - (a + b + 1)(x - p)^2
\]

is strictly concave in \( x \) with value \( a^2/[(a + b)^2(a + b - 1)] > 0 \) for \( x \in \{0, p\} \). Proposition B.10 of Bobkov and Ledoux (2019) states that for \( X \sim \text{Beta}(a, b) \) and \( \varepsilon \geq 0 \),

\[
P(|X - p| \geq \varepsilon) \leq 2 \exp(-(a + b)\varepsilon^2/8).
\]

This inequality is weaker than the result by Marchal and Arbel (2017). Skorski (2021) derives a Bernstein type inequality. With the parameters

\[
v^2 := \frac{p(1-p)}{a + b + 1}, \quad c := \max\left(\frac{|1 - 2p|}{a + b + 2}, \frac{\sqrt{p(1-p)}}{a + b + 2}\right),
\]

he shows that

\[
P(\pm(x - p) \geq \varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{2(v^2 + c\varepsilon)}\right).
\]

The next result shows that our bounds imply a stronger Bernstein type inequality.

**Lemma 6.** Let \( a, b \geq 1 \). Then for \( x \in [p, 1] \),

\[
1 - B_{a,b}(x) \leq \exp\left(-\frac{(a + b + 1)(x - p)^2}{2p(1-p) + (4/3)(1-2p)(x - p)}\right),
\]

and for \( x \in [0, p] \),

\[
B_{a,b}(x) \leq \exp\left(-\frac{(a + b + 1)(x - p)^2}{2p(1-p) + (4/3)(2p - 1)(p - x)}\right).
\]

With the notation of Skorski (2021), our upper bounds read

\[
P(\pm(x - p) \geq \varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{2(v^2 + c\varepsilon)}\right)
\]

with \( v^2 \) as before and \( c = (2/3)(1 - 2p)/(a + b + 1) \). In particular,

\[
|c| = \frac{2(a + b + 2)}{3(a + b + 1)} \frac{|1 - 2p|}{a + b + 2}.
\]

Since \( a, b \geq 1 \), the factor \( 2(a + b + 2)/[3(a + b + 1)] \) is at most \( 8/9 \) and converges to \( 2/3 \) as \( a + b \to \infty \).

**Proof of Lemma 6.** For symmetry reasons, it suffices to derive the upper bound for \( 1 - B_{a,b}(x) \). In view of Lemma 2, it suffices to show that for \( x \in [p, 1] \),

\[
(a + b - 1)K(p_r, x) \geq \frac{(a + b + 1)(x - p)^2}{2p(1-p) + (4/3)(1-2p)^+(x - p)}.
\]

To this end we use Lemma S.12 of Dümbgen and Wellner (2022) which implies that

\[
(a + b - 1)K(p_r, x) \geq \frac{(a + b - 1)(x - p_r)^2}{2(x + (p_r - x)/3)(1 - x - (p_r - x)/3)} \frac{(a + b - 1)(x - p_r)^2}{2(p_r + (2/3)(x - p_r))(1 - p_r - (2/3)(x - p_r))}.
\]
To simplify notation, we write \( m := a + b, y := x - p \in [0, 1 - p] \) and \( \delta := p - p_r = (1 - p)/(m - 1) \).

Then our lower bound for \((a + b - 1)K(p_r, x)\) reads

\[
\frac{(m - 1)(y + \delta)^2}{2(p + (2/3)y - \delta/3)(1 - p - (2/3)y + \delta/3)},
\]

and we want to show that this is greater than or equal to

\[
\frac{(m + 1)y^2}{2[p(1 - p) + (2/3)(1 - 2p)y]},
\]

Note first that since \((m - 1)\delta = (1 - p)\),

\[
(m - 1)(y + \delta)^2 = (m - 1)y^2 + 2(1 - p)y + (1 - p)\delta
\]

(1)

\[
= (m + 1)y^2 + 2(1 - p - y)y + (1 - p)\delta
\]

\[
> (m + 1)y^2.
\]

Furthermore,

\[
(p + (2/3)y - \delta/3)(1 - p - (2/3)y + \delta/3)
\]

(2)

\[
= p(1 - p) + (2/3)(1 - 2p)y + (2p - 1)\delta/3 - (2y - \delta)^2/9,
\]

and in case of \(0 < p \leq 1/2\), the right hand side is not larger than \(p(1 - p) + (2/3)(1 - 2p)y\). This proves our assertion in case of \(0 < p \leq 1/2\) already, and it remains to treat the case \(1/2 < p < 1\). To this end, we have to show that the ratio of (1) and \((m + 1)y^2\) is not smaller than the ratio of (2) and \(p(1 - p) + (2/3)(1 - 2p)y\). This assertion is equivalent to the inequality

\[
\frac{2(1 - p - y)y + (1 - p)\delta}{(m + 1)y^2} \geq \frac{(2p - 1)\delta/3 - (2y - \delta)^2/9}{p(1 - p) + (2/3)(1 - 2p)y}.
\]

With \(\lambda := (m - 1)^{-1}\) and \(z := y/(1 - p) \in [0, 1]\), the latter inequality is equivalent to

\[
\frac{2(1 - z)z + \lambda}{(\lambda^{-1} + 2)z^2} \geq \frac{(2p - 1)\lambda - (1 - p)(2z - \lambda)^2/3}{3p + 2(1 - 2p)z}.
\]

Since the left-hand side is strictly positive and \(3p + 2(1 - 2p)z \geq 3p + 2(1 - 2p) = 2 - p\), it suffices to show that

\[
\frac{(1 - p)(2z - \lambda)^2}{3(2 - p)} + \frac{2(1/z - 1) + \lambda/z^2}{1 + 2\lambda} \lambda \geq \frac{(2p - 1)}{2 - p}\lambda.
\]

If \(z \leq 2/3\), the second summand on the left-hand side is at least

\[
\frac{2(3/2 - 1) + \lambda(3/2)^2}{1 + 2\lambda} \lambda = \frac{1 + (9/4)\lambda}{1 + 2\lambda} \lambda > \frac{2p - 1}{2 - p}\lambda.
\]

Thus it suffices to consider \(z \geq 2/3\). It follows from \(1 \leq b = (1 - p)m\) that \(m \geq 1/(1 - p)\), whence \(\lambda \leq (1 - p)/p\). Thus \(2z - \lambda \geq 2z - (1 - p)/p > 0\) and \(1 - p \geq p\lambda\). Consequently, it suffices to verify that

\[
p(2z - (1 - p)/p)^2 + \frac{2(1/z - 1) + \lambda/z^2}{1 + 2\lambda} \geq \frac{(2p - 1)}{2 - p}.
\]
The second summand on the left-hand side equals
\[
\frac{1}{2z^2} \frac{4z(1-z) + 2\lambda}{1 + 2\lambda} = \frac{1}{2z^2} \left( \frac{1 - 4z(1-z)}{1 + 2\lambda} \right),
\]
an increasing function of \(\lambda > 0\) for any fixed \(z \in (0, 1]\). Consequently, inequality (4) would be a consequence of
\[
p(2z - (1 - p)/p)^2 + 2(1/z - 1) \geq \frac{(2p - 1)}{2 - p}.
\]
Since \(1/z - 1 = (1 - z)/z \geq 1 - z\), it even suffices to show that
\[
(p/3)(2z - (1 - p)/p)^2 + 2(2 - p)(1 - z) \geq 2p - 1.
\]
The minimiser of the left-hand side, as a function of \(z \in \mathbb{R}\), is given by \(z_o = 2/p - 5/4 > 3/4\).

If \(p \leq 8/9\), then \(z_o \geq 1\), so it suffices to verify (5) for \(z = 1\). Indeed,
\[
(p/3)(2 - (1 - p)/p)^2 = (4/3)(2p - 1) + (1 - p)^2/(3p) > 2p - 1.
\]
If \(8/9 \leq p < 1\), then (5) is equivalent to
\[
2p - 1 \leq (p/3)(2z_o - (1 - p)/p)^2 + 2(2 - p)(1 - z_o) = 10 - 5/p - (15/4)p.
\]
But this inequality is equivalent to
\[
(6) \quad (23/4)p + 5/p \leq 11.
\]
The left-hand side is convex in \(p\), so it suffices to verify it for \(p = 8/9\) and \(p = 1\). The left-hand side of (6) equals \(46/9 + 45/8 = 10 + 1/9 + 5/8 < 11\) if \(p = 8/9\), and \(10 + 3/4 < 11\) if \(p = 1\). This concludes our proof of Lemma 6.

3 Gaussian tail inequalities

Now suppose that \(a, b > 1\). With \(\mu := (a - 1)/(a + b - 2) \in (0, 1)\), the density \(\beta_{a,b}\) may be written as
\[
\log \beta_{a,b}(x) = -\log B(a, b) + (a + b - 2)(\mu \log(x) + (1 - \mu) \log(1 - x)).
\]
Note that \(g_\mu(x) := \mu \log(x) + (1 - \mu) \log(1 - x)\) satisfies
\[
g_\mu'(x) = -\frac{x - \mu}{x(1 - x)},
\]
so the inequality \(x(1 - x) \leq 1/4\) implies that
\[
\frac{d}{dx}(2(x - \mu)^2 + g_\mu(x)) = (x - \mu) \left( 4 - \frac{1}{x(1 - x)} \right) \begin{cases} 
\leq 0 & \text{for } x \geq \mu, \\
\geq 0 & \text{for } x \leq \mu.
\end{cases}
\]
Consequently, if $\phi_{\mu,\sigma}$ denotes the probability density of $N(\mu,\sigma^2)$ with $\sigma := (4(a + b - 2))^{-1/2}$, then

$$
\rho(x) := \frac{\beta_{a,b}}{\phi_{\mu,\sigma}}(x)
$$

is monotone decreasing in $x \geq \mu$ and monotone increasing in $x \leq \mu$, where $\beta_{a,b} := 0$ on $\mathbb{R}\setminus(0,1)$. Consequently, for $x \geq \mu$,

$$
1 - B_{a,b}(x) \leq \frac{1 - B_{a,b}(x)}{1 - B_{a,b}(\mu)} \leq \frac{\int_0^x \rho(t)\phi_{\mu,\sigma}(t) \, dt + \int_x^\infty \rho(t)\phi_{\mu,\sigma}(t) \, dt}{\rho(x)\int_x^\infty \phi_{\mu,\sigma}(t) \, dt + \rho(x)\int_x^\infty \phi_{\sigma}(t) \, dt} = \frac{N(\mu,\sigma^2)([x,\infty))}{N(\mu,\sigma^2)([\mu,\infty))} = 2\Phi(-2\sqrt{a+b-2(x-\mu)}).
$$

Analogous arguments apply to $B_{a,b}(x)$ for $x \leq \mu$, and we obtain the following bounds.

**Lemma 7.** For $a, b > 1$ and $\mu = (a - 1)/(a + b - 2)$,

$$
1 - B_{a,b}(x) \leq 2\Phi(-2\sqrt{a+b-2(x-\mu)}) \quad \text{for } x \geq \mu,
$$

$$
B_{a,b}(x) \leq 2\Phi(2\sqrt{a+b-2(x-\mu)}) \quad \text{for } x \leq \mu.
$$

Figures 2 and 3 illustrate these bounds in case of $a = 4$ and $b = 8$. For $x \leq \mu = 0.3$, the bounds from Lemma 2 are stronger, but for $x \geq \mu$, the bounds from Lemma 7 are sometimes substantially smaller than those from Lemma 2.

### 4 Gaussian approximation of Beta($a, a$)

Inspired by Dümbgen et al. (2021), we want to compare the densities $\beta_{a,a}$ with the density $\phi_{1/2,\sigma}$ of $N(1/2,\sigma^2)$ for various choices of $\sigma > 0$, where $a > 1$. Precisely, we want to determine

$$
R(\sigma) := \max_{x \in (0,1)} \frac{\beta_{a,a}}{\phi_{1/2,\sigma}}(x),
$$

because for arbitrary Borel sets $S \subset \mathbb{R}$,

$$
\text{Beta}(a,a)(S) \leq R(\sigma) N(1/2,\sigma^2)(S)
$$

and

$$
|\text{Beta}(a,a)(S) - N(1/2,\sigma^2)(S)| \leq 1 - R(\sigma)^{-1}.
$$

Moreover, we want to find $\sigma > 0$ such that this quantity is minimal.
Figure 2: Tail inequalities for Beta$(a, b)$ when $(a, b) = (4, 8)$. The green line shows $1 - B_{a,b}$, the black line is its upper bound from Lemma 2, while the red line is the upper bound from Lemma 7. In addition one sees the distribution function $B_{a,b}$ and its bounds as dotted lines.

Figure 3: Ratios of the upper bounds from Lemma 2 (black) and Lemma 7 (red) to the true tail probabilities of Beta$(a, b)$ when $(a, b) = (4, 8)$. 
To determine \( R(\sigma) \), note first that for fixed \( a \) and \( \sigma \),

\[
\log \frac{\beta_{a,a}}{\phi_{1/2,\sigma}}(x) = \log \sqrt{2\pi\sigma^2} - \log B(a, a) + \frac{(x - 1/2)^2}{2\sigma^2} + (a - 1) \log(x(1-x))
\]

\[
= \log \sqrt{2\pi\sigma^2} - \log B(a, a) + \frac{(x - 1/2)^2}{2\sigma^2} + (a - 1) \log(1 - (x - 1/2)^2)
\]

\[
= \text{const}(a, \sigma) + \frac{y}{8\sigma^2} + (a - 1) \log(1 - y),
\]

where \( y := (2x - 1)^2 \in [0, 1] \). Since

\[
\frac{d}{dy} \left( \frac{y}{8\sigma^2} + (a - 1) \log(1 - y) \right) = \frac{1}{8\sigma^2} - \frac{a - 1}{1 - y},
\]

the maximum of \( \log(\beta_{a,a}/\phi_{1/2,\sigma}) \) is attained at \( x \in (0, 1) \) such that \( y = (1 - 8\sigma^2(a - 1))^+ \), and the resulting value of \( \log R(\sigma) \) is

\[
\log R(\sigma) = \log \sqrt{2\pi} - \log B(a, a) + (a - 1) \log(1/4)
\]

\[
+ \log(\sigma^2)/2 + ((8\sigma^2)^{-1} - a + 1)^+ + (a - 1) \log \min\{8\sigma^2(a - 1), 1\}
\]

\[
= \log \sqrt{2\pi} - \log B(a, a) - (2a - 1/2) \log(2)
\]

\[
+ \log(8\sigma^2)/2 + ((8\sigma^2)^{-1} - a + 1)^+ + (a - 1) \log \min\{8\sigma^2(a - 1), 1\}.
\]

This is strictly monotone increasing in \( 8\sigma^2 \geq (a - 1)^{-1} \), so we restrict our attention to values \( \sigma \) in \( (0, (8(a - 1))^{-1/2}] \). Then,

\[
(7) \quad \log R(\sigma) = \log \sqrt{2\pi} - \log B(a, a) - (2a - 1/2) \log(2)
\]

\[
+ (8\sigma^2)^{-1} + (a - 1/2) \log(8\sigma^2) - a + 1 + (a - 1) \log(a - 1).
\]

Note also the Stirling type approximation

\[
\log \Gamma(y) = \log \sqrt{2\pi} + (y - 1/2) \log(y) - y + r(y),
\]

where \( r(y) = 1/(12y) + O(y^{-2}) \) as \( y \to \infty \) (cf. Dümbgen et al. 2021, Lemma 10). Consequently,

\[
\log \sqrt{2\pi} - \log B(a, a) = \log \sqrt{2\pi} + \log \Gamma(2a) - 2 \log \Gamma(a)
\]

\[
= (2a - 1/2) \log(2a) - 2(a - 1/2) \log(a) + r(2a) - 2r(a)
\]

\[
= (2a - 1/2) \log(2) + \log(a)/2 + \tilde{r}(a),
\]

with \( \tilde{r}(a) := \hat{r}(2a) - 2\tilde{r}(a) \). This leads to

\[
(8) \quad \log R(\sigma) = \tilde{r}(a) + \log(a)/2
\]

\[
+ (8\sigma^2)^{-1} + (a - 1/2) \log(8\sigma^2) - a + 1 + (a - 1) \log(a - 1).
\]

For the particular choice of \( \sigma \), there are at least three possibilities:

**Moment matching.** A first candidate for \( \sigma \) would be the standard deviation of Beta\((a, a)\),

\[
\sigma_1(a) := (8(a + 1/2))^{-1/2}.
\]
Figure 4: The density $\beta_{a,a}$ (black) for $a = 5$ and its Gaussian approximation $\phi_{1/2, \sigma}$ for $\sigma = (8(a + 1/2))^{-1}$ (green), $\sigma = (8(a - 1))^{-1/2}$ (red) and $\sigma = (8(a - 2))^{-1/2}$ (blue).

**Local density matching.** Since $\log \beta_{a,a}(x) - \log \beta_{a,a}(1/2)$ equals $-4(a - 1)(x - 1/2)^2$ plus a remainder of order $O((x - 1/2)^4)$ as $x \to 1/2$, another natural choice would be $\sigma_2(a) := (8(a - 1))^{-1/2}$.

**Minimizing $R(\sigma)$.** Note that $\log R(\sigma) = \text{const}(a) + (a - 1/2) \log(8\sigma^2) + (8\sigma^2)^{-1}$. Since

$$\frac{d}{dy} \left((a - 1/2) \log(y) + y^{-1}\right) = \frac{a - 1/2}{y} - \frac{1}{y^2} = \frac{(a - 1/2)\left(y - (a - 1/2)^{-1}\right)}{y^2},$$



the optimal value of $\sigma$ equals

$$\sigma_3(a) := (8(a - 2))^{-1/2}.$$

**Numerical example.** Figure 4 shows for $a = 5$ the beta density $\beta_{a,a}$ and the Gaussian approximations $\phi_{1/2, \sigma}$, where $\sigma = \sigma_1(a), \sigma_2(a), \sigma_3(a)$. Figure 5 depicts the corresponding log-density ratios $\log(\beta_{a,a}/\phi_{1/2, \sigma})$. The values of $R(\sigma)$, rounded to four digits, are $R(\sigma_1(a)) = 1.1660$, $R(\sigma_2(a)) = 1.0905$ and $R(\sigma_3(a)) = 1.0582$.

Our specific values $\sigma_j(a)$ are of the type $\sigma = (8(a + \delta))^{-1/2}$ for some $\delta \geq -1$. The next lemma provides two important properties of the resulting value $\log R(\sigma)$.

**Lemma 8.** Let $\sigma(a) := (8(a + \delta))^{-1/2}$ for $a > 1$ with a fixed number $\delta \geq -1$. Then $\log R(\sigma(a))$ is strictly decreasing in $a > 1$, and

$$\log R(\sigma) = \frac{\delta(\delta + 1) + 3/4}{2a} + O(a^{-2}).$$
Figure 5: The log-density ratios \( \log(\beta_{a,a}/\phi_{1/2,a}) \) for \( a = 5 \), where \( \sigma = \sigma_1(a) \) (green), \( \sigma_2(a) \) (red) or \( \sigma_3(a) \) (blue).

For our specific standard deviations \( \sigma_j(a) \) we obtain the limits

\[
\lim_{a \to \infty} a \log R(\sigma_j(a)) = \begin{cases} 
3/4 & \text{if } j = 1, \\
3/8 & \text{if } j = 2, \\
1/4 & \text{if } j = 3.
\end{cases}
\]

**Remark 9.** Similarly as in Section 3 we may conclude that for arbitrary \( x > 1/2 \) and \( \sigma = (8(a + \delta))^{-1/2} \),

\[
1 - B_{a,a}(x) \leq R(\sigma) \Phi \left( -\frac{x - 1/2}{\sigma} \right) \leq \frac{R(\sigma)}{2} \exp \left( -4(a + \delta)(x - 1/2)^2 \right).
\]

Even the latter bound is stronger than the bound \( \exp \left( -4(a + 1/2)(x - 1/2)^2 \right) \) by Marchal and Arbel (2017), as soon as \( \delta \geq 0.5 \) and \( R(\sigma) \leq 2 \). For \( \delta = 0.5 \), this is the case for \( a \geq 1.4 \), and for \( \delta = 1 \), we only need \( a \geq 1.9 \).

**Proof of Lemma.** At first we analyze \( \tilde{r}(a) \). We use Binet’s formula \( r(y) = \int_0^\infty e^{-yt}w(t) \, dt \) with a certain function \( w \) satisfying \( 12^{-1} e^{-t/12} < w(t) < 12^{-1} \), see Dümbgen et al. (2021, Lemma 10). Consequently, \( 2r(a) - r(2a) = \int_0^\infty (2e^{-at} - e^{-2at})w(t) \, dt \), and since \( 2e^{-at} - e^{-2at} = e^{-at}(2 - e^{-at}) > 0 \), we conclude that

\[
2r(a) - r(2a) \begin{cases} < \frac{1}{12} \int_0^\infty (2e^{-at} - e^{-2at}) \, dt = \frac{1}{8a}, \\
> \frac{1}{12} \int_0^\infty (2e^{-(a+1/12)t} - e^{-(2a+1/12)t}) \, dt = \frac{a + 1/36}{8(a + 1/12)(a + 1/24)}. \end{cases}
\]
In particular, as \( a \to \infty \),

\[
\tilde{r}(a) = -\frac{1}{8a} + O(a^{-2}).
\]

Moreover,

\[
\frac{d}{da} \tilde{r}(a) = 2 \int_0^\infty (e^{-at} - e^{-2at})w(t) \, dt < \frac{1}{6} \int_0^\infty (e^{-at} - e^{-2at}) \, dt = \frac{1}{8a^2}.
\]

Next we verify that \( \log R(\sigma(a)) \) is strictly decreasing in \( a > 1 \). It follows from representation \((8)\) that

\[
\log R(\sigma(a)) = \tilde{r}(a) + \log(a) / 2 + \delta - (a - 1/2) \log(a + \delta) + 1 + (a - 1) \log(a - 1).
\]

According to \((10)\), the derivative of this with respect to \( a \) is not greater than

\[
\frac{1}{8a^2} + \frac{1}{2a} - \frac{a - 1/2}{a + \delta} - \log(a + \delta) + \log(a - 1) + 1.
\]

For fixed \( a > 1 \), the derivative of this bound with respect to \( \delta \geq 1 \) equals \(- (\delta + 1/2)/(a + \delta)^2\), so it is maximal for \( \delta = -1/2 \). This leads to

\[
\frac{d}{da} \log R(\sigma(a)) \leq \frac{1}{8a^2} + \frac{1}{2a} + \log\left(\frac{a - 1}{a - 1/2}\right) = \frac{1}{8a^2} + \frac{1}{2a} + \log\left(1 - \frac{1}{2(a - 1/2)}\right)
\]

\[
< \frac{1}{8a^2} + \frac{1}{2a} + \log\left(1 - \frac{1}{2a}\right) = -\sum_{k \geq 3} (2a)^{-k} / k < 0.
\]

It remains to prove the expansion of \( \log R(\sigma(a)) \) as \( a \to \infty \). To this end, we rewrite \((11)\) as

\[
\log R(\sigma(a)) = \tilde{r}(a) + \delta - (a - 1/2) \log(1 + \delta/a) + (a - 1) \log(1 - 1/a).
\]

Since \( \log(1 + y) = y + O(y^2) = y - y^2/2 + O(y^3) \) as \( y \to 0 \),

\[
\delta - (a - 1/2) \log(1 + \delta/a) = \delta - \frac{(a - 1/2)\delta}{a} + \frac{(a - 1/2)\delta^2}{2a^2} + O(a^{-2})
\]

\[
= \frac{\delta(\delta + 1)}{2a} + O(a^{-2}),
\]

\[
1 + (a - 1) \log(1 - 1/a) = 1 - \frac{a - 1}{a} - \frac{a - 1}{2a^2} + O(a^{-2})
\]

\[
= \frac{1}{2a} + O(a^{-2}).
\]

Combining these expansions with \((9)\) leads to the desired expansion of \( \log R(\sigma(a)) \).

\[\square\]

## 5 Extensions to gamma distributions

If \( X \sim \text{Beta}(a, b) \), then \( bX \) has approximately a gamma distribution \( \text{Gamma}(a) \) with shape parameter \( a > 0 \) and scale parameter one. Denoting the distribution function of \( \text{Gamma}(a) \) with \( G_a \), the bounds in Lemma 2 and Remark 4 lead to the following bounds:
Lemma 10. For arbitrary \( a > 0 \),

\[
G_a(x) \leq \frac{x^a e^{-x}}{a^a e^{-a}} \quad \text{for} \quad 0 \leq x \leq a.
\]

For \( a \geq 1 \),

\[
1 - G_a(x) \leq \begin{cases} 
\frac{x^a e^{-x}}{(a-1)^{a-1} e^{-1}} & \text{for} \quad x \geq a - 1, \\
\frac{2^{a-1} e^{-x}}{2 a^a e^{-a}} & \text{for} \quad x \geq a,
\end{cases}
\]

Instead of the approximation argument, one could verify directly, that \( G_a(x)/[x^a e^{-x}] \) is monotone increasing in \( x > 0 \), and that in case of \( a \geq 1 \), \( [1 - G_a(x)]/[x^{a-1} e^{-x}] \) is monotone decreasing in \( x > 0 \). Note also that in case of \( a \geq 1 \), the median of Gamma\( (a) \) is contained in \((a - 1, a)\), see Groeneveld and Meeden (1977).

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