Abstract

The Dirac masses of neutrinos can exhibit a strong hierarchy even if the Majorana masses of the right-handed neutrinos are degenerate and the hierarchy of the mass scales governing the oscillations of solar and atmospheric neutrinos is rather small. This phenomenon results from the see-saw mechanism and the algebraic structure of the effective mass operator for the active neutrinos. The large hierarchy of the Dirac masses is drastically modified by a symmetric unitary matrix $R$ acting in the flavor space. A realistic pattern of neutrino masses and mixing is obtained. Maximal mixing for atmospheric neutrinos is attributed to the charged lepton sector. Large mixing in solar neutrino oscillations is due to the neutrino sector. Small $U_{e3}$ is a natural consequence of the model. The masses of the active neutrinos are given by $\mu_3 \approx \sqrt{\Delta m^2_{\odot}}$ and $\mu_1/\mu_2 \approx \tan^2 \theta_{\odot}$. 
1 Introduction

After four decades of heroic experimental [1, 2, 3, 4, 5] and theoretical [6] efforts, see [7, 8] and references therein, the problems of solar neutrinos have finally been solved. A unique solution of these problems exists corresponding to LMA MSW flavor oscillations of active neutrinos. By active neutrinos we understand those experimentally observed, $\nu_e$, $\nu_\mu$ and $\nu_\tau$. After recent results from SuperKamiokande [2] and SNO [3] all other solutions of the solar neutrino problems are not allowed at 3$\sigma$ level [9]. Simultaneously it becomes more and more clear that the oscillations of atmospheric neutrinos are due to $\nu_\mu \leftrightarrow \nu_\tau$ transitions [10, 11]. The third important piece of information is the CHOOZ limit [12] indicating that the element $U_{e3}$ of the Maki-Nakagawa-Sakata (MNS) lepton mixing matrix [13] is small. In this article we show that all these experimental facts can be nicely accommodated by assuming the see-saw mechanism [14] and a large hierarchy of the Dirac masses for three neutrinos. One can consider the results presented in the present paper as a realistic realization of the ideas proposed in a recent paper by one of us [15]. It should be mentioned that there is no room in this model to explain LSND results [16] in terms of oscillations. This model cannot also accommodate the large value of the Majorana mass parameter $\langle m_{\nu_e} \rangle$ which follows from a recent interpretation of the Heidelberg-Moscow data on neutrinoless double beta decays of $^{76}$Ge [18].

Let us start with a brief summary of the ideas and results presented in [15]. It has been shown in [15] that a rather small hierarchy of the observed low energy masses of the active neutrinos can follow from a large hierarchy at the more fundamental level of the Dirac masses. This can happen even in the case when the Majorana masses of the right-handed neutrinos are degenerate. Therefore large hierarchies in the Dirac masses of all other fundamental fermions, i.e. up quarks, down quarks and charged leptons, can be accompanied by a large hierarchy of the Dirac masses for the neutrinos. This large hierarchy is drastically modified by a symmetric unitary operator $R$ related to unitary transformations of the right-handed neutrinos. In some sense the underlying hierarchy of the Dirac masses is hidden and only mildly reflected in the effective low energy masses. This phenomenon is due to the see-saw mechanism and the algebraic structure of the low energy effective mass operator $\mathcal{N}$ describing the masses of the active neutrinos.

Therefore, in the lepton sector there are two operators acting in the flavor space which affect the low energy physics in a most important way. One is the MNS lepton mixing matrix $U_{\text{MNS}}$ [13] which plays a role similar to that of the Cabibbo-Kobayashi-Maskawa mixing matrix for quarks [19]. The other is the matrix $R$ which is imprinted in the observable mass spectrum of the active neutrinos. Up to our knowledge this remarkable matrix $R$ and its role in the low energy physics of neutrinos was not discussed in the literature before [15]. In [15] a form of $R$ has been considered leading to a significant reduction of the mass hierarchy and a form of $U_{\text{MNS}}$ nicely accommodating the small value of its $U_{e3}$ element. It has also been argued that the mass ratio for the two lighter active neutrinos is given by $\tan^2 \theta_\odot$ with $\theta_\odot$ being the mixing angle for solar neutrinos. In the present paper we show that for strongly hierarchical Dirac masses the form of $R$ considered in [15] is up to complex phase factors the only one leading
to acceptable low energy mass spectrum and lepton mixing matrix. We show also that
the smallness of $U_{e3}$ and the mass relations are preserved in the realistic model.

2 Hidden hierarchy

In this section the see-saw mechanism for three families of leptons is discussed. We
repeat arguments given in [15] and show that a small hierarchy of low energy masses
can be obtained from a large hierarchy in the Dirac masses. We show that the observed
small hierarchy for the active neutrinos fixes the form of the matrix $R$ mentioned in
the Introduction up to some complex phase factors and small sub-leading terms. The
mass matrix for the charged leptons can be written as

$$ L = V_R \text{diag}(m_e, m_\mu, m_\tau) \ V_L \equiv V_R \ m^{(l)} \ V_L $$

The matrix $V_R$ multiplying $m^{(l)}$ from the left side can be made equal to one by an
appropriate redefinition of the right-handed charged leptons. This has no observable
consequences because at our low energies only left-handed weak charged currents can
be studied. The corresponding Dirac mass matrix for the neutrinos is

$$ N = U_R \ m^{(\nu)} \ U_L $$

with

$$ m^{(\nu)} = \text{diag}(m_1, m_2, m_3). $$

We choose the reference frame in the flavor space such that the Majorana mass matrix
of the right-handed neutrinos is diagonal. In general in this frame $M_R$ is described
by its three different eigenvalues. However, as we demonstrate in the following, even
taking the simplest case of

$$ M_R = M \cdot 1 $$

we are able to build a phenomenologically successful model. The Majorana masses of
the right-handed neutrinos are equal to $M$, which is huge. This leads to the masses of
active neutrinos much smaller than the masses of other fundamental fermions (leptons
and quarks).

Of course there is an additional freedom for the right-handed Majorana masses not
equal to each other but one can describe the neutrino oscillation data without any
problem assuming that they are all equal. In our present purely phenomenological
approach we do not have at our disposal a sufficient amount of information to study
the mass spectrum of $M_R$. This may be possible in a more complete theory of flavor,
but in this work we assume degenerate Majorana masses. It should be noted, however,
that the arguments which we present can be trivially generalized and applied to the
case of non-equal right-handed Majorana masses.

The form of the matrix $M_R$ depends on the reference frame in a non-trivial way.
In the frame employed in the present paper this matrix is assumed to be proportional
to the unit matrix. However it may have a different form in another reference frame.
In particular it need not be proportional to the unit matrix in the frame where the matrix of neutrino Dirac masses is diagonal and so exhibits a strong hierarchy.

Once the reference frame in the flavor space is fixed by the condition that $M_R$ is diagonal the matrix $U_R$ in eq.(3) cannot be gauged away by a redefinition of the right-handed neutrino fields. Let us assume that $m_1$, $m_2$, and $m_3$ are hierarchical

$$m_1 \ll m_2 \ll m_3$$

which means that the Dirac masses of the neutrinos exhibit the same strongly ordered hierarchical pattern as the Dirac masses of all other fundamental fermions. The mass spectrum of the active neutrinos is dictated by an effective mass operator $N$ of dimension five

$$N = N^T M_R^{-1} N = U_L^T m^{(\nu)} U_R^T M_R^{-1} U_R m^{(\nu)} U_L.$$  

(6)

The algebraic structure of $N$ implies that the resulting mass spectrum is extremely sensitive to the form of

$$U_R^T M_R^{-1} U_R = \frac{1}{M} U_R^T U_R$$

(7)

where the simplifying assumption (4) has been used. The mass spectrum obtained from the matrix $N$ in eq.(6) is now seen to depend crucially on the following matrix $R$,

$$R = U_R^T U_R,$$

(8)

which is symmetric and unitary. It is remarkable that to some extent the condition (4) also fixes the reference frame in the flavor space. Unitary transformations of the right-handed fields are in general complex and cannot be gauged away even if $M_R$ is proportional to the unit matrix. The matrix $R$ can drastically reduce the hierarchy of the mass spectrum for the active neutrinos. So, $R$ is observable, in principle at least, if a large hierarchy of the Dirac masses is the common feature of all quarks and leptons. In this sense $R$ is a physical object which is imprinted in low energy physical quantities, namely the masses of the active neutrinos. Unlike the quark sector with its Cabibbo-Kobayashi-Maskawa mixing matrix [19] the lepton sector has therefore two important matrices in the flavor space. One is the lepton mixing matrix $U_{MNS}$ [13] which affects the form of the weak charged current. Another is the matrix $R$ defined in eq.(8). $R$ affects the form of $U_{MNS}$. Moreover, it is also reflected in the low energy neutrino mass spectrum. In our phenomenological approach we use the experimental input to fix the form of $R$. One may hope that this is a first step towards an underlying theory of flavor.

What can be said about the matrix $R$? In [15] it has been shown that its element $(R)_{33}$ must be equal to zero at the leading order. By this we mean that $(R)_{33}$ cannot be a number of order 1. It can be a small number suppressed by some power of the mass ratio of, say, $m_1$ and $m_3$. Perhaps the easiest way to demonstrate that it must be so is to take $R = 1$ which is representative for the whole class of matrices with $(R)_{33} = \mathcal{O}(1)$. Then for the hierarchical mass spectrum (3) a much stronger hierarchy is obtained for the masses of the active neutrinos

$$\mu_1 = \frac{m_1^2}{M} \ll \mu_2 = \frac{m_2^2}{M} \ll \mu_3 = \frac{m_3^2}{M}.$$  

(9)

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It should be stressed that in eq. (9) much greater means a few orders of magnitude rather than factors below 10. Then for the ratio

$$\rho = \frac{\Delta m^2_{\odot}}{\Delta m^2_{\text{at}}} = \frac{\mu_2^2 - \mu_1^2}{\mu_3^2 - \mu_2^2}$$

(10)

values between \(10^{-4}\) and \(10^{-8}\) are obtained if the mass ratio \(m_3/m_2\) is taken of the order of the corresponding mass ratios for other fundamental fermions, i.e. \(m_b/m_s \sim 30\), \(m_\tau/m_\mu \approx 17\) or \(m_t/m_c \sim 100\). These values of \(\rho\) are two to six orders smaller than the experimental value

$$\rho_{\text{exp}} \approx 5 \cdot 10^{-5} \text{eV}^2 = 2 \cdot 10^{-2}.$$  

(11)

In the above equation the value of \(\Delta m^2_{\odot}\) from a combined fit to SuperKamiokande and K2K data [21] and a recent SNO result [3] for \(\Delta m^2_{\odot}\) are used. Before SNO, much smaller values of \(\Delta m^2_{\odot}\) were also allowed for the oscillations of the solar neutrinos. As we see, if \((R)_{33}\) is not equal to zero, then an unacceptably large hierarchy appears in the mass spectrum of the active neutrinos. This means that \((R)_{33} = 0\) must be assumed.

What about \((R)_{23} = (R)_{32}\)? It turns out that the resulting mass spectrum for the active neutrinos is acceptable from the phenomenological point of view if \((R)_{23} = \mathcal{O}(1)\) is assumed. This spectrum corresponds to the case of the so-called inverted hierarchy. However in the following section it will be seen that the resulting structure of the lepton mixing matrix does not resemble the experimentally observed one. More precisely, one can obtain a realistic mixing matrix assuming a rather artificial form of the charged lepton contribution \(V_L\) and some sort of conspiracy and accidental cancellations. For esthetic reasons we do not consider such a situation acceptable and conclude that up to small sub-leading terms \((R)_{23} = 0\) must be assumed.

The only remaining case is \(R_{33} = R_{23} = 0\) which implies

$$R = \begin{pmatrix} 0 & 0 & \exp i\phi_1 \\ 0 & \exp i\phi_2 & 0 \\ \exp i\phi_1 & 0 & 0 \end{pmatrix}$$

(12)

The complex phase factors in eq. (12) can be of crucial importance for lepton number violating processes like neutrinoless double beta decays. However, these phase factors do not affect our discussion which concentrates on neutrino oscillations. So, for the sake of simplicity, in the following considerations we take the same form of \(R\) as in [13]:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \equiv \mathcal{P}_{13}$$

(13)

It turns out that for strongly hierarchical Dirac masses eq. (13) is a necessary condition for a realistic mixing and mass spectrum. Therefore we assume that some symmetry underlying flavor dynamics forces \(U_R\) to fulfill eq. (13).

Unlike \(V_R\) in the charged lepton sector which at low energies can be gauged away, the matrix \(U_R\) is imprinted in low energy physics and its structure affects the mass
spectrum of the active neutrinos in a most spectacular way. The general solution of eq. (13) is

\[ U_R = R(\alpha, \beta, \gamma) \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & 1 & \omega^* \\ 1 & 1 & 1 \\ \omega^* & 1 & \omega \end{pmatrix} \]  

where \( R(\alpha, \beta, \gamma) \) is an arbitrary 3-dimensional rotation and \( \omega = \exp(2\pi i/3) \). The particular solution in (14) has been chosen to be a symmetric matrix. One may speculate if this interesting matrix plays some role in the theory of flavor.

### 3 Lepton mixing matrix

In this section we study the MNS mixing matrix. Following [15] we show that for the matrix \( R \) given in eq. (13) the so-called bi-maximal mixing is obtained [21]. It is known [9] that after SNO [5] the bi-maximal mixing is excluded as a realistic description of the data on neutrino oscillations. Moreover, the mass spectrum of the active neutrinos following from eq. (13) is also not realistic because two masses are degenerate implying \( \Delta m^2 \simeq 0 \). Both problems are cured in the following section by adding appropriate sub-leading terms. Importantly, the key observation which we present here survives in the realistic model. We also show that assuming \( (R)_{23} = O(1) \) one obtains another structure of \( U_{MNS} \) which cannot be made realistic without very artificial assumptions.

The lepton mixing matrix \( U_{MNS} \) relates two sets of neutrino eigenstates: gauge interaction eigenstates \( \nu_e, \nu_\mu \) and \( \nu_\tau \) and mass eigenstates \( \nu_1, \nu_2, \nu_3 \) of masses \( \mu_1, \mu_2, \) and \( \mu_3 \):

\[
\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U_{MNS} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}.
\]

Let \( \mathcal{O} \) be a unitary matrix such that

\[ \mathcal{O}^T \mathcal{N} \mathcal{O} = \text{diag}(\mu_1, \mu_2, \mu_3). \]  

Eq. (11) implies that \( M^2_L = L^\dagger L \) is diagonalized by \( V_L \), i.e. \( V_L M^2_L V_L^\dagger = \text{diag}(m^2_e, m^2_\mu, m^2_\tau) \). Then from eqs. (4)-(8) one derives

\[ U_{MNS} = V_L \mathcal{O} = V_L U_L^{-1} \mathcal{O}' \]  

where the unitary matrix \( \mathcal{O}' \) is such that

\[ \frac{1}{M} \mathcal{O}'^T m(\nu)^T R m(\nu) \mathcal{O}' = \text{diag}(\mu_1, \mu_2, \mu_3) \]  

with \( R = P_{13} \), c.f. eq. (13). For \( m(\nu) \) given in (3) one obtains, see the following section for a step by step derivation,

\[ \mathcal{O}' = P_{23} U_{12}(\pi/4) \]
with
\[ P_{23} = P_{23}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \] (20)
and
\[ U_{12}(\pi/4) = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (21)

The lepton mixing matrix is, cf. (17),
\[ U_{\text{MNS}} = V_L U_L^{-1} P_{23} U_{12}(\pi/4). \] (22)

The presence of \( P_{23} \) in eq. (22) may be considered very embarrassing. Multiplying any matrix from the right by \( P_{23} \) results in exchanging its second and third columns. Such an operation may perfectly ruin the structure of this matrix. Only for rotations \( O_{23}(\pm \pi/4) \) by \( \pm \pi/4 \) in the 2-3 plane or unitary matrices \( U_{23}(\pm \pi/4) \) analogous to \( U_{12}(\pi/4) \) in eq. (21) the exchange of the second and third column can be easily compensated for by some innocent change of conventions. But these are exactly the matrices which describe the oscillations of atmospheric neutrinos! If
\[ V_L U_L^{-1} = O_{23}(\pm \pi/4) \quad \text{or} \quad V_L U_L^{-1} = U_{23}(\pm \pi/4), \] (23)
the resulting \( U_{\text{MNS}} \) can be cast in the form
\[ U_{\text{MNS}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (24)
by appropriate redefinitions of fields. This form automatically guarantees that \( U_{e3} \) is small. It is quite encouraging that the condition (23) has been realized in many published models, for a review see [22]. A particularly attractive option is to assume
\[ U_L \approx 1 \] (25)
and to attribute the whole unitary transformation in the 2-3 plane to the lepton sector. Then (23) can be obtained from the lopsided form of \( L \) [23]. Eq. (25) nicely agrees with the idea that the analogous matrices for up and down quarks are close to the unit matrix leading to the CKM matrix also close to one.

In grand unified theories like \( SU_5 \), the mass matrices \( L \) for leptons and \( D \) for down quarks are closely related:
\[ L = D^T \] (26)
This relation originates from the fact that in \( SU_5 \) the charged leptons are in fact closely related to charge conjugated fundamental fermions. So, for \( SO_{10} \) GUTs, eq. (23) would imply that \( V_R \approx 1 \). However, in such theories there should be also a close relation between \( U_R \) and \( V_L \). Comparing \( V_L \) following from (23) for \( U_L = 1 \) and \( U_R \) from
we do not find such a relation. From (23) we can get $U_{13} (\pm \pi/4)$ rather than $U_{23} (\pm \pi/4)$. So, we do not know how to realize the scenario described in this paper in $SO_{10}$ unified theories.

After having observed the rather miraculous way the permutation $P_{23}$ leaves intact the pattern of mixing, it is easier to see why an alternative matrix $N'$ for $(R)_{23} = O(1)$ is not acceptable phenomenologically. We can take $R = P_{23}$, see eq.(20), as a representative for the whole corresponding class of matrices. For this choice of $R$ we would obtain

$$N' = \frac{1}{M} U_L^T \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & 0 & m_2 m_3 \\ 0 & m_2 m_3 & 0 \end{pmatrix} U_L.$$  \hspace{1cm} (27)

The corresponding matrix $O'$, c.f. eq.(18) would be $O' = P_{13} U_{12} (\pi/4)$, leading to the lepton mixing matrix

$$U'_{MNS} = V_L U_L^{-1} P_{13} U_{12} (\pi/4).$$ \hspace{1cm} (28)

As opposed to the expression (22), the emerging mixing pattern is hard to reconcile with data. Unlike that of the innocuous matrix $P_{23}$ in (22), the effect of the matrix $P_{13}$ in (28) is deleterious to the structure of $V_L U_L^{-1}$ since it is now columns 1 and 3 that are swapped. To make $U'_{MNS}$ realistic, one would have to resort to replacing the charged lepton matrix $V_L$ with a much less plausible structure.

4 Towards a realistic model

The central idea of the preceding sections is that the fundamental hierarchy of the Dirac masses is drastically deformed by the matrix $R$. The form of $R$ given in eq.(13) is particularly effective in reducing the original hierarchy. For the same reason the role of the sub-leading terms becomes much more prominent. In this section we show that realistic models of lepton mixing and neutrino mass spectrum can be easily obtained when these sub-leading terms are taken into account. Of course there is a problem of the large number of free parameters in such purely phenomenological models. Once again we face the fact that the amount of experimental information is very limited. So, one might argue that only more constrained models based on some underlying symmetry principle can be sufficiently constrained and predictive to go beyond a mere parametrisation of the data. We show that such a conclusion is too strong. In fact the large hierarchy of the Dirac masses helps to reduce the number of relevant sub-leading terms. Then the masses of the active neutrinos can be related to $\Delta m^2_\odot$, $\Delta m^2_\odot$, and $\tan^2 \theta_\odot$. These relations, in principle at least, can provide an experimental test of the picture presented here.

Sub-leading terms are introduced as small off-diagonal elements in the matrix $m^{(v)}$, see eq.(34). As we want to preserve the hierarchy built in the Dirac mass spectrum we

\footnote{An equivalent but less transparent way would be to preserve the diagonal form of $m^{(v)}$ and add some sub-leading terms in $U_R$, c.f. eq.(2). Small sub-leading terms in $U_L$ do not affect the mass spectrum of the active neutrinos. Those terms may be very important for the phenomenology of neutrinos because they can affect the value of $U_{e3}$. However we have little to say about them.}
assume that all non-zero off-diagonal elements of \( m(\nu) \) are small and of order \( m_1 \). Then the structure of the mass operator \( \mathcal{N} \), see eq.(6), selects the matrix elements \( m_{13} \) and \( m_{12} \) whose contributions to \( \mathcal{N} \) are multiplied by \( m_3 \) and thus enhanced. Let us assume that

\[
m_{13} = a m_1 \quad \text{and} \quad m_{12} = b m_1.
\]

As explained above the contributions to \( \mathcal{N} \) from all other off-diagonal elements of \( m(\nu) \) are even smaller and we neglect them in the following discussion. The mass spectrum and the lepton mixing matrix can be computed numerically for non-zero values of \( a \) and \( b \). It is instructive, however, to consider the case \( b = 0 \) which can be solved analytically in terms of reasonably simple expressions. Moreover, it is worth mentioning that the calculation of the leading contribution to the lepton mixing matrix, which was only sketched in Sec.3, can be obtained from the following considerations for \( a = 0 \). For \( b = 0 \)

\[
m(\nu) = \begin{pmatrix} m_1 & 0 & am_1 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}
\]

and

\[
\mathcal{M} = P_{23}U^*_L\mathcal{N}U^{-1}_L P_{23} = \mu \begin{pmatrix} 0 & r & 0 \\ r & 2ar & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

with \( r = m_1 m_3 / m_2^2 \) and \( \mu = m_2^2 / \mathcal{M} \), where \( P_{23} \) defined in eq.(20) exchanges the second and third axes in the flavor space. For \( a > 0 \) the matrix \( \mathcal{M} \) in (31) is diagonalized by

\[
U_{12}^T \mathcal{M} U_{12} = \text{diag} (\mu_1, \mu_2, \mu_3)
\]

with

\[
U_{12} (\alpha) = O_{12} (\alpha) \text{ diag}(i, 1, 1) = \begin{pmatrix} i \cos \alpha & \sin \alpha & 0 \\ -i \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

and \( \tan 2\alpha = 1/a \), which implies that

\[
\mathcal{O}' = P_{23}U_{12} (\alpha).
\]

and the lepton mixing matrix is, cf.(17),

\[
U_{\text{MNS}} = V_L U^*_L P_{23} U_{12} (\alpha).
\]

It should be clear from the discussion in Sec.3 that the angle \( \alpha \) in eq.(33) is equal to the solar mixing angle \( \theta_\odot \). The eigenvalues of the matrix \( \mathcal{M} \) are

\[
\mu_1 = \mu r (\cosh t - \sinh t) \\
\mu_2 = \mu r (\cosh t + \sinh t) \\
\mu_3 = \mu
\]

(36)
with \( a = \sinh t \). The ordering of the eigenvalues is such that e.g. the second column of \( U_{12} \) is equal to the normalized eigenvector of \( M \) for the eigenvalue \( \mu_2 \). This implies that

\[
\cos \alpha = (\cosh t + \sinh t) \sin \alpha
\]

and

\[
\mu_1/\mu_2 = \tan^2 \theta_\odot.
\]

The mass splittings are

\[
\Delta m_\odot^2 = \mu_1^2 - \mu_2^2 = 2\mu_2^2r^2 \sinh 2t,
\]

\[
\Delta m_\odot^2 \approx \mu^2,
\]

which yields the formula for the parameter \( \rho \), see eq.(10),

\[
\rho \approx 2r^2 \sinh 2t.
\]

Comparison with the allowed range for LMA MSW solar neutrino oscillations \[5, 9\] leads to the following ranges for the values of \( a \) and \( r \):

\[
0.35 \leq a \leq 0.75 \quad \text{and} \quad 0.05 \leq r \leq 0.25
\]

with the best fits corresponding to \( a \) between 0.46 and 0.57 and \( r \) between 0.09 and 0.10.\footnote{One might be tempted to assume in \( N \) the same hierarchy as in \( L \). Then the parameter \( r = m_e/m_\tau/\mu_1^2 \) is about 0.08, which is in the range indicated in eq.(42). We are very much indebted to the Referee of PLB for pointing this observation out to us.}

Eqs.(38)-(40) lead to the following approximate formulae for the masses of the active neutrinos:

\[
\mu_1 \approx \sqrt{\Delta m_\odot^2 \tan^2 \theta_\odot / \sqrt{1 - \tan^4 \theta_\odot}}
\]

\[
\mu_2 \approx \sqrt{\Delta m_\odot^2 / \sqrt{1 - \tan^4 \theta_\odot}}
\]

\[
\mu_3 \approx \sqrt{\Delta m_\odot^2}
\]

The errors due to approximations leading to these expressions are small for the interesting range of \( a \) and \( r \).

Examine now the consequences of other perturbations of the Dirac mass matrix. Variation of the second important off-diagonal element, \( m_{12} \), shows that for non-zero values of the parameter \( b \), see eq.(29), the mass relations (43)-(45) are only weakly affected. In Fig.1 we show the test of the mass relation (38) for a few values of \( b \). The ratio plotted in Fig.1 is one if the formula (38) is exact. It is seen that corrections for non-zero values of \( b \) are small. In Fig.2 we show the result of the analogous test of the formula (43). It is clear that it also works quite well. Non-zero \( b \) affects the atmospheric mixing angle, yielding roughly \( \tan^2 \theta_\odot \approx 1 + b \). Thus we must require that \(|b| < 1\) if \( 0.95 \leq \sin^2 2\theta_\odot \leq 1 \) is to be obtained without artificial conspiracy of
Figure 1: Test of the relation (38) between the lighter neutrino masses and the tangent of the sun mixing angle for non-zero values of $b$. The plot shows the dependence of the ratio $\mu_1/(\mu_2 \tan^2 \theta_\odot)$ on the parameter $a$ in eq. (29) for different values of the parameter $b$. For $b = 0$ the value is exactly 1.

contributions from the charged lepton and neutrino sectors. We think that maximal or nearly maximal mixing for the atmospheric neutrinos is an experimental fact which is too striking to assume that it is simply accidental and results from adding two contributions of similar size.

It is worth noting that even though we have succeeded in keeping $U_{e3}$ zero, which is a non-trivial consequence of the structure of the lepton and neutrino mass matrices, it is nevertheless possible to introduce such a perturbation as to obtain any value of $U_{e3}$ being in accordance with the present experimental data, i.e. $U_{e3} \leq 0.1$. This is effected by varying another off-diagonal element of the Dirac matrix, $m_{21}$. One gets an approximate relation $U_{e3} \approx m_{21}/m_2$.

Let us close this section with two remarks on the numerical values of the mass parameters in our picture. The mass of the lightest neutrino mass eigenstate, see eq. (43), is about 3 meV for $\tan^2 \theta_\odot \approx 0.4$. This mass range can be probed by the 10t version of the GENIUS project [24] if the Majorana phases are not too small and there are no strong cancellations between contributions to the mass parameter $\langle m_\nu \rangle$. As a final remark let us note that the mass scale of the Majorana masses is between $10^{10}$ and $10^{11}$ GeV if $m_2 \sim m_c$ is assumed. It has been pointed out in [24] that this is exactly the range of Majorana masses which may be important for baryogenesis; see [24] and references therein.
Figure 2: Test of the formula (43) for $\mu_1$. Shown is the dependence of the ratio $\mu_1 \sqrt{1 - \tan^4 \theta_{\odot}} / \left( \sqrt{\Delta m^2_{\odot}} \tan^2 \theta_{\odot} \right)$ on the parameter $a$ in eq. (29) for different values of the parameter $b$.

5 Summary

It has been shown that the Dirac masses of all fundamental fermions can exhibit a strong hierarchy. A rather small hierarchy of the low energy neutrino masses is due to the symmetric unitary operator $R$ acting in the flavor space. This phenomenon results from the see-saw mechanism and the algebraic structure of the dimension five effective mass operator $N$ describing the masses of the active neutrinos. In the leptonic sector there are two operators acting in the flavor space and observable at low energies: the $U_{\text{MNS}}$ lepton mixing matrix and the matrix $R$. The matrix $R$ affects the form of $U_{\text{MNS}}$ and, moreover, also the low energy mass spectrum. The form of $R$ proposed in [15] leads to a realistic description of neutrino mixing and masses. The mass of the heaviest neutrino is related to the mass scale $\sqrt{\Delta m^2_{\odot}}$ governing the oscillations of atmospheric neutrinos. The masses of the two lighter neutrinos are related to $\sqrt{\Delta m^2_{\odot}}$, the mass scale for oscillations of solar neutrinos and to the solar neutrino mixing angle through the relation $\mu_1 / \mu_2 \approx \tan^2 \theta_{\odot}$. The structure of the lepton mixing matrix $U_{\text{MNS}}$ strongly suggests that the maximal or nearly maximal mixing for the atmospheric neutrinos results from the charged lepton sector. Small $U_{e3}$ is obtained as a consequence of the model. The mass $\mu_1$ of the lightest neutrino mass eigenstate is about 3 meV which can be observed by 10t GENIUS detector if Majorana phases are not too small and there are no strong cancellations between contributions to the mass parameter $\langle m_{\nu_e} \rangle$. 

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