Modification of the Formula of Papkovich-Föppl for a Compressed Thin-Walled Rod with an Open Profile

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Abstract. The article is about thin-walled rods of open cross-section compressed by multiparameter load. It is known that the critical multiparameter load factor can be calculated by the formula of Papkovich-Föppl. In this article, the formula is modified so that it takes into account the change in cross-section dimensions of the rod. The modified formula can be used to optimize such dimensions.

1. Introduction
Compressed thin-walled rods of open cross-section can be constructions of buildings. In this case, some requirements are imposed on such rods. One of these requirements is stability.

We can judge about stability of a rod using a critical load factor. It is known that the critical multiparameter load factor \( \lambda_{cr} \) can be calculated by the formula of Papkovich and Föppl [1-4]:

\[
\frac{1}{\lambda_{cr}} = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{i,cr}}
\]

where \( \lambda_i \) is the \( i \)-th load factor; \( \lambda_{i,cr} \) is the critical value of the \( i \)-th load factor if the remaining load factors are zero; \( n \) is the number of load factors.

Optimization of compressed and stable rods was first investigated by Keller J. [5], Tadjbakhsh I. and Keller J. [6], Keller J. and Niordson F. [7], Taylor J. [8], etc. Applying the method of mathematical programming and the finite element method for solving such problems is presented in the research of Zarghamee M. [9].

If we optimize cross-section dimensions of a rod, then these dimensions change. Such changes lead to a change in \( \lambda_{i,cr} \) (\( i = 1, n \)). Calculating absolutely all the changed critical values requires a large number of calculations. Therefore, it is important to reduce the number of such computations.

2. Problem definition, assumptions and solutions
The object of this research is a thin-walled rod of open cross-section.

We accept two kinematic assumptions of Vlasov V. Z. [10]: 1) the configuration of the cross-section in its own plane remains unchanged during deformation; 2) the shear strain vanishes at the middle surface of the cross-section.

Suppose the material of the rod obeys R. Hooke’s law [11].
Suppose the rod is compressed by a quasi-static multi-parameter load. The parameters of this load are load factors \( \lambda_i \) \( (i = 1, n) \) where \( n \) is the number of such factors. The factors form a vector \( \vec{\lambda} \).

It is required to modify formula (1) so as to consider the change in cross-section dimensions of the rod. The modified formula should not require a large number of calculations of all the changed critical values of the load factors.

The critical load factor \( \lambda_{cr} \) can be determined by the finite element method as the smallest positive root \( \lambda \) of the equation (2) with \( \bar{q} \neq 0 \):

\[
(K^{tot} - \lambda K^{tot})\bar{q} = 0
\]

(2)

where \( K^{tot} \) is the stiffness matrix of the rod; \( K^g \) is the geometric stiffness matrix of the rod; \( \bar{q} \) is the vector of the displacements which describe the transition of the system from initial equilibrium state to new equilibrium state.

The matrices \( K^{tot} \) and \( K^g \) are formed from the matrices of individual finite elements.

Formulas for the calculation of the elements of the stiffness matrix of an individual finite element are well known. These formulas are given in the research of Bychkov D. V. [12] and others.

Formulas for the calculation of the elements of the geometric stiffness matrix of an individual finite element are contained in the research of Izhendeev A. V. [13]. These formulæ were obtained on the basis of the energy stability criterion of Bryan G. H. [14] and Trefftz E. [15].

According to the research [13], the formula for the calculation of the element of the geometric stiffness matrix can be written as

\[
k_{6, i, j} = N l^4 \left( a + \sum_{m} b_m \right)
\]

(3)

where \( N \) is the longitudinal force; \( l \) is the length of the finite element; \( k \) is a positive integer; \( a \) is a real number that does not depend on cross-section dimensions of the rod; \( b_m \) is a real number that is directly proportional to one of the fractions (4).

These fractions are expressed as

\[
a_y^2 \frac{l}{l^2}, a_z^2 \frac{l}{l^2}, i_y^2, i_z^2, \frac{(I_a / A)}{l^4}
\]

(4)

where \( a_y \) and \( a_z \) are the coordinates of the flexural centre of the rod cross-section measured along the main central axes \( Y \) and \( Z \) of the rod cross-section; \( i_y \) and \( i_z \) are the radii of inertia of the rod cross-section relative to the main central axes \( Y \) and \( Z \) of the rod cross-section; \( I_a \) is the sectorial moment of inertia of the rod cross-section; \( A \) is the area of the rod cross-section.

Statement 1. Since any dimension of the rod cross-section is small in comparison with the length of the rod [10], then the fractions (4) are small.

The Rayleigh’s relation [16] corresponding to the equation (2) can be written as

\[
r = \frac{q^T K^{tot} \bar{q}}{q^T K^g \bar{q}}
\]

(5)

The inverse value can be written as [17]

\[
\frac{1}{r} = \sum_{i=1}^{n} \lambda_i \frac{q^T K^{tot} \bar{q}}{q^T K^g \bar{q}} \approx \sum_{i=1}^{n} \lambda_i \frac{1}{\lambda_{i,cr}}
\]

(6)

For any \( i \) and \( j \), equation (7) is correct.

\[
\frac{\lambda_{i,cr}}{\lambda_{i,cr}} \approx \frac{q^T K^{tot} \bar{q}}{q^T K^g \bar{q}}, \frac{q^T K^{tot} \bar{q}}{q^T K^g \bar{q}} = \frac{q^T K^{tot} \bar{q}}{q^T K^{tot} \bar{q}}
\]

(7)

Equations (3), (7) and statement 1 lead to statement 2.
Statement 2. If the changes of the cross-section dimensions of the rod are small, then the ratio \( \frac{\lambda_{j,ct}}{\lambda_{i,ct}} \) remains almost constant.

This statement allows to modify the formula of Papkovich-Föppl for a compressed thin-walled rod of open cross-section:

\[
\frac{1}{\lambda_{ct}} = \frac{\lambda_{p,ct,new}}{\lambda_{p,ct}} \sum_{i=1}^{n} \frac{\lambda_{i}}{\lambda_{i,ct,new}} = \sum_{i=1}^{n} \frac{\lambda_{i}}{\lambda_{i,ct,new}}
\]

where \( p \) is any integer from 1 to \( n \); 'new' is the label for the critical value of the load factor after changing the cross-section dimensions of the rod.

This modification of the formula of Papkovich-Föppl implies calculation of the eigenvalue of a pair of matrices one time, not \( n \) number of times.

3. Numerical example

Consider the rod shown in figure 1.

![Figure 1. The compressed rod.](image)

![Figure 2. The cross-sections of the rod.](image)

The supports of the rod are as below:
1) there is no longitudinal displacement of the center of gravity of the lower cross-section of the rod;
2) there are no transverse displacements of the flexural centers of the upper and lower cross-sections of the rod;
3) there are no rotations of the upper and lower cross-sections of the rod around its longitudinal axis.

The end cross-sections of the rod can have a deplanation.

The material of the rod is steel with Young's modulus \( E = 2.0 \cdot 10^5 \) MPa and the shear modulus \( G = 0.8 \cdot 10^5 \) MPa.

The options of the cross-sections of the rod are as below.
Option 1 is shown in figure 2,a where \( h = 300 \) mm, \( b = 300 \) mm, \( \delta_w = 5 \) mm and \( \delta_l = 10 \) mm.
Option 2 is shown in figure 2,a where \( h = 300 \) mm, \( b = 300 \) mm, \( \delta_w = 10 \) mm and \( \delta_l = 5 \) mm.
Option 3 is shown in figure 2,b where \( h = 300 \) mm, \( b = 300 \) mm, \( \delta_w = 10 \) mm and \( \delta_l = 10 \) mm.
Option 4 is shown in figure 2,b where \( h = 300 \) mm, \( b = 50 \) mm, \( \delta_w = 10 \) mm and \( \delta_l = 10 \) mm.

Table 1 contains the values of \( \frac{\lambda_{i,ct}}{\lambda_{4,ct}} \) \( (i = 1, 4) \) for each of these options.
Table 1. The values of $\frac{\lambda_{i,cr}}{\lambda_{4,cr}}$ ($i = 1, 4$).

| Option number | $\frac{\lambda_{1,cr}}{\lambda_{4,cr}}$ | $\frac{\lambda_{2,cr}}{\lambda_{4,cr}}$ | $\frac{\lambda_{3,cr}}{\lambda_{4,cr}}$ | $\frac{\lambda_{4,cr}}{\lambda_{4,cr}}$ |
|---------------|----------------|----------------|----------------|----------------|
| 1             | 0.477          | 0.735          | 0.897          | 1              |
| 2             | 0.473          | 0.734          | 0.894          | 1              |
| 3             | 0.477          | 0.737          | 0.899          | 1              |
| 4             | 0.474          | 0.736          | 0.897          | 1              |

The data in table 1 confirm statement 2.

4. Conclusions
The modification of the formula of Papkovich-Föppl for compressed thin-walled rods of open cross-section was proposed. This modification takes into account the change in cross-section dimensions of the rod. The modification involves calculating the eigenvalue of a matrix pair only once. This is achieved due to the fact that if the changes of the cross-section dimensions of the rod are small, then the ratio of the critical values of the two load factors remains almost constant. The proposed formula can be used to optimize the cross-section dimensions of the rod.

5. References
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