ON EQUIVARIANT HOMEOMORPHISMS OF BOUNDARIES OF CAT(0) GROUPS

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Abstract. In this paper, we investigate an equivariant homeomorphism of the boundaries $\partial X$ and $\partial Y$ of two proper CAT(0) spaces $X$ and $Y$ on which a CAT(0) group $G$ acts geometrically. We provide a sufficient condition to obtain a $G$-equivariant homeomorphism of the two boundaries $\partial X$ and $\partial Y$ as a continuous extension of the quasi-isometry $\phi : Gx_0 \to Gy_0$ defined by $\phi(gx_0) = gy_0$, where $x_0 \in X$ and $y_0 \in Y$.

1. Introduction

In this paper, we investigate an equivariant homeomorphism of the boundaries of two proper CAT(0) spaces on which a CAT(0) group acts geometrically as a continuous extension of a quasi-isometry of the two CAT(0) spaces.

Definitions and details of CAT(0) spaces and their boundaries are found in [8] and [20]. A geometric action on a CAT(0) space is an action by isometries which is proper ([8, p.131]) and cocompact. We note that every CAT(0) space on which some group acts geometrically is a proper space ([8, p.132]). A group $G$ is called a CAT(0) group, if $G$ acts geometrically on some CAT(0) space $X$.

It is well-known that if a Gromov hyperbolic group $G$ acts geometrically on a negatively curved space $X$, then the natural map $G \to X \ (g \mapsto gx_0)$ extends continuously to an equivariant homeomorphism of the boundaries of $G$ and $X$. Also if a Gromov hyperbolic group $G$ acts geometrically on negatively curved spaces $X$ and $Y$, then the boundaries of $X$ and $Y$ are $G$-equivariant homeomorphic. Indeed the natural map $Gx_0 \to Gy_0 \ (gx_0 \mapsto gy_0)$ extends continuously to a $G$-equivariant homeomorphism of the boundaries of $X$ and $Y$. The boundaries of Gromov hyperbolic groups are quasi-isometric invariant (cf. [8], [11], [20], [21], [22]).

Here in [22], Gromov asked whether the boundaries of two CAT(0) spaces $X$ and $Y$ are $G$-equivariant homeomorphic whenever a CAT(0) group $G$ acts geometrically on the two CAT(0) spaces $X$ and $Y$. In [7], P. L. Bowers and K. Ruane have constructed an example that the natural quasi-isometry $Gx_0 \to Gy_0 \ (gx_0 \mapsto gy_0)$ does not extend continuously to any map between the boundaries $\partial X$ and $\partial Y$ of $X$ and $Y$. Also S. Yamagata [39] has constructed a similar example using a right-angled Coxeter group and its Davis complex. Moreover, there is a research by C. Croke and B. Kleiner [13] on an equivariant homeomorphism of the boundaries $\partial X$ and $\partial Y$.

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Also, C. Croke and B. Kleiner \[12\] have constructed a CAT(0) group $G$ which acts geometrically on two CAT(0) spaces $X$ and $Y$ whose boundaries are not homeomorphic, and J. Wilson \[33\] has proved that this CAT(0) group has uncountably many boundaries. Recently, C. Mooney \[33\] has showed that the knot group $G$ of any connected sum of two non-trivial torus knots has uncountably many CAT(0) boundaries.

Also, it has been observed by M. Bestvina \[5\] that all the boundaries of a given CAT(0) group are shape equivalent, and he has asked the question whether all the boundaries of a given CAT(0) group are cell-like equivalent. This question is an open problem and there are some resent research (cf. \[2\], \[34\]).

The purpose of this paper is to provide a sufficient condition to obtain a $G$-equivariant homeomorphism between the two boundaries $\partial X$ and $\partial Y$ of two CAT(0) spaces $X$ and $Y$ on which a CAT(0) group $G$ acts geometrically as a continuous extension of the natural quasi-isometry $Gx_0 \to Gy_0$ ($gx_0 \to gy_0$), where $x_0 \in X$ and $y_0 \in Y$.

Now we recall the example of Bowers and Ruane in \[7\]. Let $G = F_2 \times \mathbb{Z}$ and $X = Y = T \times \mathbb{R}$, where $F_2$ is the rank 2 free group generated by $\{a, b\}$ and $T$ is the Cayley graph of $F_2$ with respect to the generating set $\{a, b\}$. Then we define the action \(\cdot\) of the group $G$ on the CAT(0) space $X$ by

\[
(a, 0) \cdot (t, r) = (a \cdot t, r),
\]

\[
(b, 0) \cdot (t, r) = (b \cdot t, r),
\]

\[
(1, 1) \cdot (t, r) = (t, r + 1),
\]

for each $(t, r) \in T \times \mathbb{R} = X$, and also define the action \(\ast\) of the group $G$ on the CAT(0) space $Y$ by

\[
(a, 0) \ast (t, r) = (a \cdot t, r),
\]

\[
(b, 0) \ast (t, r) = (b \cdot t, r + 2),
\]

\[
(1, 1) \ast (t, r) = (t, r + 1),
\]

for each $(t, r) \in T \times \mathbb{R} = Y$. Then the group $G$ acts geometrically on the two CAT(0) spaces $X$ and $Y$, and the quasi-isometry $g \cdot x_0 \mapsto g \ast y_0$ (where $x_0 = (1, 0) \in X$ and $y_0 = (1, 0) \in Y$) does not extend continuously to any map from $\partial X$ to $\partial Y$. Indeed, for $g_i = a^i b^i \in F_2$, \(\{g_i^n \mid i \in \mathbb{N}\} \to a^\infty\) as $i \to \infty$ in $\partial T$,

\[
\lim_{n \to \infty} (g_i^n, 0) \cdot x_0 = [g_i^\infty, 0],
\]

\[
\lim_{n \to \infty} (a^n, 0) \cdot x_0 = [a^\infty, 0],
\]

in $X \cup \partial X$, and

\[
\lim_{n \to \infty} (g_i^n, 0) \ast y_0 = [g_i^\infty, \frac{\pi}{4}],
\]

\[
\lim_{n \to \infty} (a^n, 0) \ast y_0 = [a^\infty, 0],
\]

in $Y \cup \partial Y$. Hence any map from $\partial X$ to $\partial Y$ obtained as a continuously extension of the quasi-isometry $G \cdot x_0 \to G \ast y_0$ ($g \cdot x_0 \to g \ast y_0$) must send $[g_i^\infty, 0]$ to $[g_i^\infty, \frac{\pi}{4}]$ and fix $[a^\infty, 0]$. However, this is incompatible with continuously at $[a^\infty, 0]$, because $[g_i^\infty, 0] \to [a^\infty, 0]$ as $i \to \infty$ (\[7\], p.187).

Here in this example, we note that
(a) the point \(a^i \cdot x_0\) is in the geodesic segment from \(x_0\) to \(g^i \cdot x_0\) in \(X\), i.e.,
\[a^i \cdot x_0 \in [x_0, g^i \cdot x_0] \] in \(X\) for any \(i \in \mathbb{N}\) and
\[a^i \cdot y_0 \in [y_0, g^i \cdot y_0] \] in \(X\) for any \(i \in \mathbb{N}\) and
\[g^i \cdot y_0\) is unbounded for \(i \in \mathbb{N}\) in \(Y\), i.e., there does not exist a constant
\(M > 0\) such that \(d(a^i \cdot y_0, [y_0, g^i \cdot y_0]) \leq M\) for any \(i \in \mathbb{N}\) in \(Y\).

Based on this observation, we consider a condition.
We suppose that a group \(G\) acts geometrically on two \(\text{CAT}(0)\) spaces \(X\) and \(Y\).
Let \(x_0 \in X\) and \(y_0 \in Y\). Then we define the condition \((*)\) as follows:
\((*)\) There exist constants \(N > 0\) and \(M > 0\) such that \(GB(x_0, N) = X\),
\(GB(y_0, M) = Y\) and for any \(g, a \in G\), if \([x_0, gx_0] \cap B(ax_0, N) \neq \emptyset\) in \(X\)
then \([y_0, gy_0] \cap B(ay_0, M) \neq \emptyset\) in \(Y\).

In this paper, we prove the following theorem.

**Theorem 1.1.** Suppose that a group \(G\) acts geometrically on two \(\text{CAT}(0)\) spaces \(X\) and \(Y\). Let \(x_0 \in X\) and \(y_0 \in Y\). If the condition \((*)\) holds, then there exists a \(G\)-equivariant homeomorphism of the boundaries \(\partial X\) and \(\partial Y\) as a continuous extension of the quasi-isometry \(\phi : Gx_0 \to Gy_0\) defined by \(\phi(gx_0) = gy_0\).

### 2. \(\text{CAT}(0)\) Spaces and Their Boundaries

Details of \(\text{CAT}(0)\) spaces and their boundaries are found in \([1, 8, 19, 20]\) and \([57]\).

A proper geodesic space \((X, d_X)\) is called a \(\text{CAT}(0)\) space, if the “\(\text{CAT}(0)\)-inequality” holds for all geodesic triangles \(\triangle\) and for all choices of two points \(x\) and \(y\) in \(\triangle\). Here the “\(\text{CAT}(0)\)-inequality” is defined as follows: Let \(\triangle\) be a geodesic triangle in \(X\). A comparison triangle for \(\triangle\) is a geodesic triangle \(\triangle'\) in the Euclidean plane \(\mathbb{R}^2\) with same edge lengths as \(\triangle\). Choose two points \(x\) and \(y\) in \(\triangle\). Let \(x'\) and \(y'\) denote the corresponding points in \(\triangle'\). Then the inequality
\[d_X(x, y) \leq d_{\mathbb{R}^2}(x', y')\]
is called the \(\text{CAT}(0)\)-inequality, where \(d_{\mathbb{R}^2}\) is the natural metric on \(\mathbb{R}^2\).

Every proper \(\text{CAT}(0)\) space can be compactified by adding its “boundary”. Let \((X, d_X)\) be a proper \(\text{CAT}(0)\) space, and let \(\mathcal{R}\) be the set of all geodesic rays in \(X\). We define an equivalence relation \(\sim\) in \(\mathcal{R}\) as follows: For geodesic rays \(\xi, \zeta : [0, \infty) \to X\),
\[\xi \sim \zeta \iff \text{Im} \xi \subset B(\text{Im} \zeta, N) \text{ for some } N \geq 0,\]
where \(B(A, N) := \{x \in X \mid d_X(x, A) \leq N\}\) for \(A \subset X\). Then the boundary \(\partial X\) of \(X\) is defined as
\[\partial X = \mathcal{R}/\sim\]

For each geodesic ray \(\xi \in \mathcal{R}\), the equivalence class of \(\xi\) is denoted by \(\xi(\infty)\).

It is known that for each \(\alpha \in \partial X\) and each \(x_0 \in X\), there exists a unique geodesic ray \(\xi_\alpha : [0, \infty) \to X\) such that \(\xi_\alpha(0) = x_0\) and \(\xi_\alpha(\infty) = \alpha\). Thus we can identify the boundary \(\partial X\) of \(X\) as the set of all geodesic rays \(\xi\) with \(\xi(0) = x_0\).

Let \((X, d_X)\) be a proper \(\text{CAT}(0)\) space and let \(x_0 \in X\). We define a topology on \(X \cup \partial X\) as follows:

1. \(X\) is an open subspace of \(X \cup \partial X\).
2. Let \(\alpha \in \partial X\) and let \(\xi_\alpha\) be the geodesic ray such that \(\xi_\alpha(0) = x_0\) and \(\xi_\alpha(\infty) = \alpha\). For \(r > 0\) and \(\epsilon > 0\), we define
\[U_{\partial X}(\alpha; r, \epsilon) = \{x \in X \cup \partial X \mid x \notin B(x_0, r), \ d_X(\xi_\alpha(r), \xi_\alpha(r)) < \epsilon\}\]
where \( \xi_x : [0, d_X(x_0, x)] \to X \) is the geodesic (segment or ray) from \( x_0 \) to \( x \). Let \( \epsilon_0 > 0 \) be a constant. Then the set

\[
\{ U_{X \cup \partial X}(\alpha; r, \epsilon_0) \mid r > 0 \}
\]

is a neighborhood basis for \( \alpha \) in \( X \cup \partial X \).

Here it is known that the topology on \( X \cup \partial X \) is not dependent on the basepoint \( x_0 \in X \) and \( X \cup \partial X \) is a metrizable compactification of \( X \).

Also for \( \alpha \in \partial X \) and the geodesic ray \( \xi_\alpha \) with \( \xi_\alpha(0) = x_0 \) and \( \xi_\alpha(\infty) = \alpha \) and for \( r > 0 \) and \( \epsilon > 0 \), we define

\[
U'_{X \cup \partial X}(\alpha; r, \epsilon) = \{ x \in X \cup \partial X \mid x \notin B(x_0, r), \ d_X(\xi_\alpha(r), \text{Im} \xi_x) < \epsilon \},
\]

where \( \xi_x : [0, d(x_0, x)] \to X \) is the geodesic (segment or ray) from \( x_0 \) to \( x \). Let \( \epsilon_0 > 0 \) be a constant. Then the set

\[
\{ U'_{X \cup \partial X}(\alpha; r, \epsilon_0) \mid r > 0 \}
\]

is also a neighborhood basis for \( \alpha \) in \( X \cup \partial X \) (cf. [21] Lemma 4.2).

Suppose that a group \( G \) acts on a proper CAT(0) space \( X \) by isometries. For each element \( g \in G \) and each geodesic ray \( \xi : [0, \infty) \to X \), a map \( g \xi : [0, \infty) \to X \) defined by \( (g \xi)(t) := g(\xi(t)) \) is also a geodesic ray. For two geodesic rays \( \xi \) and \( \xi' \), if \( \xi(\infty) = \xi'(\infty) \) then \( g(\xi(\infty)) = g(\xi'(\infty)) \). Thus \( g \) induces a homeomorphism \( \partial X \) and \( G \) acts on \( \partial X \) by homeomorphisms. Here we note that if a sequence \( \{ x_i \mid i \in \mathbb{N} \} \subset X \) converges to \( \alpha \in \partial X \) in \( X \cup \partial X \), then for any \( g \in G \), the sequence \( \{ gx_i \mid i \in \mathbb{N} \} \subset X \) converges to \( ga \in \partial X \) in \( X \cup \partial X \).

**Definition 2.1.** Let \( (X, d_X) \) be a proper CAT(0) space and let \( \{ x_i \mid i \in \mathbb{N} \} \subset X \) be an unbounded sequence in \( X \). In this paper, we say that the sequence \( \{ x_i \mid i \in \mathbb{N} \} \) is a **Cauchy sequence in** \( X \cup \partial X \), if there exists \( \epsilon_0 > 0 \) such that for any \( r > 0 \), there is a number \( i_0 \in \mathbb{N} \) as

\[
x_i \in U_{X \cup \partial X}(x_{i_0}; r, \epsilon_0)
\]

for any \( i \geq i_0 \). Here

\[
U_{X \cup \partial X}(x_{i_0}; r, \epsilon_0) = \{ x \in X \mid x \notin B(x_0, r), \ d_X(\xi_{x_{i_0}}(r), \xi_x(r)) < \epsilon \},
\]

where \( \xi_x \) is the geodesic segment from \( x_0 \) to \( x \) in \( X \).

We show the following lemma which is used later.

**Lemma 2.2.** Let \( (X, d_X) \) be a proper CAT(0) space and let \( \{ x_i \mid i \in \mathbb{N} \} \subset X \) be an unbounded sequence in \( X \). Then the sequence \( \{ x_i \mid i \in \mathbb{N} \} \) is a Cauchy sequence in \( X \cup \partial X \) defined above if and only if the sequence \( \{ x_i \mid i \in \mathbb{N} \} \) converges to some point \( \alpha \in \partial X \) in \( X \cup \partial X \).

**Proof.** We first show that if the sequence \( \{ x_i \mid i \in \mathbb{N} \} \) converges to some point \( \alpha \in \partial X \) in \( X \cup \partial X \), then \( \{ x_i \mid i \in \mathbb{N} \} \) is a Cauchy sequence in \( X \cup \partial X \) defined above.

Suppose that \( \{ x_i \mid i \in \mathbb{N} \} \) converges to \( \alpha \in \partial X \) in \( X \cup \partial X \). Let \( \epsilon_0 > 0 \). Since the set

\[
\{ U_{X \cup \partial X}(\alpha; r, \epsilon_0) \mid r > 0 \}
\]

is a neighborhood basis for \( \alpha \) in \( X \cup \partial X \), for each \( r > 0 \), there exists a number \( i_0 \in \mathbb{N} \) such that

\[
x_i \in U_{X \cup \partial X}(\alpha; r, \frac{\epsilon_0}{2})
\]
for any \( i \geq i_0 \). Then for any \( i \geq i_0 \),
\[
    d_X(\xi_{x_{i_0}}(r), \xi_x(r)) \leq d_X(\xi_{x_{i_0}}(r), \xi_\alpha(r)) + d_X(\xi_\alpha(r), \xi_x(r))
\]
\[
\leq \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2}
\]
\[
= \epsilon_0.
\]
Hence \( x_i \in U_{X \cup \partial X}(x_{i_0}; r, \epsilon_0) \) for any \( i \geq i_0 \). Thus the sequence \( \{x_i\} \) is a Cauchy sequence in \( X \cup \partial X \).

Next, we show that if \( \{x_i\} \) is a Cauchy sequence in \( X \cup \partial X \) defined above, then \( \{x_i\} \) converges to some point \( \alpha \in \partial X \) in \( X \cup \partial X \).

Suppose that \( \{x_i\} \) is a Cauchy sequence in \( X \cup \partial X \). Since the set \( \{x_i\} \) is unbounded in \( X \), there exists a limit point \( \alpha \in \text{Cl}\{x_i\} \cap \partial X \). Here there exists a subsequence \( \{x_{i_j}\} \) which converges to \( \alpha \) in \( X \cup \partial X \).

Then we show that the sequence \( \{x_i\} \) converges to the point \( \alpha \) in \( X \cup \partial X \).

Since \( \{x_i\} \) is a Cauchy sequence in \( X \cup \partial X \), there exists \( \epsilon_0 > 0 \) such that for any \( r > 0 \), there is a number \( i_0 \in \mathbb{N} \) as \( x_i \in U_{X \cup \partial X}(x_{i_0}; r, \epsilon_0) \) for any \( i \geq i_0 \), i.e., \( d_X(\xi_{x_{i_0}}(r), \xi_x(r)) \leq \epsilon_0 \) for any \( i \geq i_0 \). Also since the subsequence \( \{x_{i_j}\} \) converges to \( \alpha \) in \( X \cup \partial X \), there exists \( i_{j_0} \geq i_0 \) such that \( x_{i_{j_0}} \in U_{X \cup \partial X}(\alpha; r, 1) \), i.e., \( d_X(\xi_{x_{i_{j_0}}}(r), \xi_\alpha(r)) \leq 1 \). Then for any \( i \geq i_{j_0} \),
\[
    d_X(\xi_x(r), \xi_\alpha(r)) \leq d_X(\xi_{x_{i_0}}(r), \xi_x(r)) + d_X(\xi_x(r), \xi_{x_{i_{j_0}}}(r)) + d_X(\xi_{x_{i_{j_0}}}(r), \xi_\alpha(r))
\]
\[
\leq \epsilon_0 + \epsilon_0 + 1
\]
\[
= 2\epsilon_0 + 1,
\]
since \( i \geq i_{j_0} \geq i_0 \). Hence for any \( r > 0 \), there exists a number \( i_{j_0} \in \mathbb{N} \) such that for any \( i \geq i_{j_0} \),
\[
    x_i \in U_{X \cup \partial X}(\alpha; \epsilon_0 + 1),
\]
where \( \epsilon_0 + 1 \) is a constant. Thus the sequence \( \{x_i\} \) converges to the point \( \alpha \in \partial X \) in \( X \cup \partial X \).

3. PROOF OF THE MAIN THEOREM

We suppose that a group \( G \) acts geometrically on two \( \text{CAT}(0) \) spaces \((X, d_X)\) and \((Y, d_Y)\). Let \( x_0 \in X \) and \( y_0 \in Y \). Now we suppose that the condition (\( * \)) holds; that is,
\[
    (\ast) \text{ there exist constants } N > 0 \text{ and } M > 0 \text{ such that } GB(x_0, N) = X, \quad GB(y_0, M) = Y \quad \text{and for any } g, a \in G, \text{ if } [x_0, gx_0] \cap B(ax_0, N) \neq \emptyset \text{ in } X \text{ then } [y_0, gy_0] \cap B(ay_0, M) \neq \emptyset \text{ in } Y.
\]

Our goal is to show that the quasi-isometry \( \phi : Gx_0 \to Gy_0 \) defined by \( \phi(gx_0) = gy_0 \) continuously extends to a \( G \)-equivariant homeomorphism of the boundaries \( \partial X \) and \( \partial Y \).

Since the map \( \phi : Gx_0 \to Gy_0 \) defined by \( \phi(gx_0) = gy_0 \) is a quasi-isometry (cf. [11, p.138], [21, 22]), there exist constants \( \lambda > 0 \) and \( C > 0 \) such that
\[
    \frac{1}{\lambda}d_Y(gy_0, hy_0) - C \leq d_X(gx_0, hx_0) \leq \lambda d_Y(gy_0, hy_0) + C
\]
for any \( g, h \in G \).

We first show the following.
Proposition 3.1. Let \( \{g_i\} \subset G \) be a sequence. If \( \{g_ix_0\} \subset X \) is a Cauchy sequence in \( X \cup \partial X \) defined in Section 2, then \( \{g_iy_0\} \subset Y \) is also a Cauchy sequence in \( Y \cup \partial Y \).

Proof. Let \( \{g_i\} \subset G \). Suppose that \( \{g_ix_0\} \subset X \) is a Cauchy sequence in \( X \cup \partial X \).

To prove that \( \{g_iy_0\} \subset Y \) is a Cauchy sequence in \( Y \cup \partial Y \), we show that there exists \( M' > 0 \) such that for any \( R > 0 \), there is \( i_0 \in \mathbb{N} \) as

\[
g_iy_0 \in U_{Y \cup \partial Y} (g_iy_0; R, M')
\]

for any \( i \geq i_0 \).

Let \( M' = \lambda(2N + 1) + 2M + C \) and let \( R > 0 \).

Since \( \{g_ix_0\} \subset X \) is a Cauchy sequence in \( X \cup \partial X \), for \( r = \lambda(R + C + M) + N \), there exists \( i_0 \in \mathbb{N} \) such that

\[
g_ix_0 \in U_{X \cup \partial X} (g_ix_0; r, 1)
\]

for any \( i \geq i_0 \).

Then

\[
d_X(x_0, g_ix_0) \geq r, \quad d_X(x_0, g_ix_0) \geq r, \quad \text{and} \quad d_X(\xi_{g_ix_0}(r), \xi_{g_ix_0}(r)) \leq 1,
\]

where \( \xi_{g_ix_0} \) is the geodesic from \( x_0 \) to \( g_ix_0 \) and \( \xi_{gix_0} \) is the geodesic from \( x_0 \) to \( g_ix_0 \) in \( X \).

Since \( GB(x_0, N) = X \), there exist \( a, b \in G \) such that \( d_X(ax_0, \xi_{gix_0}(r)) \leq N \) and \( d_X(bx_0, \xi_{gix_0}(r)) \leq N \). Then

\[
[x_0, g_ix_0] \cap B(ax_0, N) \neq \emptyset \quad \text{and} \quad [x_0, g_ix_0] \cap B(bx_0, N) \neq \emptyset.
\]

Hence by the condition (*),

\[
[y_0, g_iy_0] \cap B(ay_0, M) \neq \emptyset \quad \text{and} \quad [y_0, g_iy_0] \cap B(by_0, M) \neq \emptyset.
\]

Thus

\[
\xi_{g_iy_0}(r'_i) \in [y_0, g_iy_0] \cap B(ay_0, M) \quad \text{and} \quad \xi_{g_iy_0}(r'_i) \in [y_0, g_iy_0] \cap B(by_0, M)
\]

for some \( r'_i > 0 \) and \( r' > 0 \).

To obtain that for any \( i \geq i_0 \),

\[
g_iy_0 \in U_{Y \cup \partial Y} (g_iy_0; R, M'),
\]

we show that

\[
r'_i \geq R, \quad r' \geq R \quad \text{and} \quad d_Y(\xi_{g_iy_0}(r'_i), \xi_{g_iy_0}(r'_i)) \leq M'.
\]

First,

\[
r'_i = d_Y(y_0, \xi_{g_iy_0}(r'_i))
\geq d_Y(y_0, ay_0) - M
\geq \frac{1}{\lambda}d_X(x_0, ax_0) - C - M
\geq \frac{1}{\lambda}(r - N) - C - M
= R,
\]

because \( d_X(x_0, ax_0) \geq r - N \) and \( r = \lambda(R + C + M) + N \).
By the same argument,
\[ r' = d_Y(y_0, \xi_{g_iy_0}(r')) \]
\[ \geq d_Y(y_0, b_{y_0}) - M \]
\[ \geq \frac{1}{\lambda}d_X(x_0, bx_0) - C - M \]
\[ \geq \frac{1}{\lambda}(r - N) - C - M \]
\[ = R, \]
because \( d_X(x_0, bx_0) \geq r - N \) and \( r = \lambda(R + C + M) + N. \)
Also,
\[ d_Y(\xi_{g_{i+1}y_0}(r_0'), \xi_{g_{i+1}y_0}(r')) \leq d_Y(a_{y_0}, b_{y_0}) + 2M \]
\[ \leq (\lambda d_X(a_{x_0}, b_{x_0}) + C) + 2M \]
\[ \leq \lambda(d_X(\xi_{g_{i+1}x_0}(r), \xi_{g_{i+1}x_0}(r)) + 2N) + C + 2M \]
\[ \leq \lambda(1 + 2N) + C + 2M \]
\[ = M', \]
because \( d_X(\xi_{g_{i+1}x_0}(r), \xi_{g_{i+1}x_0}(r)) \leq 1 \) and \( M' = \lambda(2N + 1) + 2M + C. \)
Thus
\[ r_0' \geq R, \ r' \geq R \]
and \( d_Y(\xi_{g_{i+1}y_0}(r_0'), \xi_{g_{i+1}y_0}(r')) \leq M'. \)
Hence
\[ d_Y(\xi_{g_iy_0}(R), \xi_{g_iy_0}(R)) \leq d_Y(\xi_{g_{i+1}y_0}(r_0'), \xi_{g_{i+1}y_0}(r')) \leq M', \]
since \( Y \) is a CAT(0) space. Also we obtain that
\[ d_Y(y_0, g_{i+1}y_0) \geq R \]
and \( d_Y(y_0, g_iy_0) \geq R. \)
Thus
\[ g_iy_0 \in U_Y(\partial Y, g_iy_0; R, M') \]
for any \( i \geq i_0. \) Hence we obtain that \( \{g_iy_0\} \subset Y \) is a Cauchy sequence in \( Y \cup \partial Y. \) \( \square \)

Then we define a map \( \tilde{\phi} : \partial X \to \partial Y \) as a continuous extension of the quasi-isometry \( \phi : Gx_0 \to Gy_0 \) defined by \( \phi(g_{x_0}) = g_{y_0} \) as follows: For each \( \alpha \in \partial X, \) there exists a sequence \( \{g_\alpha x_0\} \subset Gx_0 \subset X \) which converges to \( \alpha \) in \( X \cup \partial X. \) Then the sequence \( \{g_\alpha x_0\} \subset X \) is a Cauchy sequence in \( X \cup \partial X \) by Lemma 2.2. By Proposition 3.1, the sequence \( \{g_\alpha y_0\} \subset Y \) is also a Cauchy sequence in \( Y \cup \partial Y. \) Hence by Lemma 2.2, the sequence \( \{g_iy_0\} \subset Y \) converges to some point \( \tilde{\alpha} \in \partial Y \) in \( Y \cup \partial Y. \) Then we define \( \phi(\alpha) = \tilde{\alpha}. \)

**Proposition 3.2.** The map \( \tilde{\phi} : \partial X \to \partial Y \) is well-defined.

**Proof.** Let \( \alpha \in \partial X \) and let \( \{g_i x_0\}, \{h_i x_0\} \subset Gx_0 \subset X \) be two sequences which converge to \( \alpha \) in \( X \cup \partial X. \) As the argument above, by Lemma 2.2 and Proposition 3.1, the sequence \( \{g_\alpha y_0\} \subset Y \) converges to some point \( \tilde{\alpha} \in \partial Y \) and the sequence \( \{h_\alpha y_0\} \subset Y \) converges to some point \( \tilde{\beta} \in \partial Y. \) Then we show that \( \tilde{\alpha} = \tilde{\beta}. \)

Here we can consider a sequence \( \{\tilde{g}_j x_0 \mid j \in \mathbb{N}\} \subset Gx_0 \subset X \) such that
\[ \{\tilde{g}_j x_0 \mid j \in \mathbb{N}\} = \{g_i x_0 \mid i \in \mathbb{N}\} \cup \{h_i x_0 \mid i \in \mathbb{N}\} \]
and the sequence \( \{\tilde{g}_j x_0\} \) converges to \( \alpha \) in \( X \cup \partial X. \) Then the sequence \( \{\tilde{g}_j x_0\} \)

is a Cauchy sequence in \( X \cup \partial X \) and the sequence \( \{\tilde{g}_j y_0\} \) is also in \( Y \cup \partial Y \) by
Proposition 3.5. Hence the sequence \( \{ \tilde{g}_j y_0 \} \) converges to some point \( \tilde{y} \in \partial Y \) in \( Y \cup \partial Y \). Here we note that the two sequences \( \{ g_j y_0 \} \) and \( \{ h_i y_0 \} \) are subsequences of \( \{ \tilde{g}_j y_0 \} \). Hence we obtain that \( \tilde{\alpha} = \tilde{\beta} = \tilde{\gamma} \).

Thus the map \( \tilde{\phi} : \partial X \to \partial Y \) defined as above is well-defined. \( \square \)

Next, we show the following.

Proposition 3.3. The map \( \tilde{\phi} : \partial X \to \partial Y \) is surjective.

Proof. Let \( \tilde{\alpha} \in \partial Y \). There exists a sequence \( \{ g_j y_0 \} \subset G y_0 \subset Y \) which converges to \( \tilde{\alpha} \) in \( Y \cup \partial Y \). Then we consider the set \( \{ g_j x_0 \mid i \in \mathbb{N} \} \) which is an unbounded subset of \( X \). Here

\[
\text{Cl}\{g_ix_0 \mid i \in \mathbb{N} \} \cap \partial X \neq \emptyset,
\]

and there exists a subsequence \( \{ g_i, x_0 \mid j \in \mathbb{N} \} \subset \{ g_j x_0 \} \) which converges to some point \( \alpha \in \partial X \). Then the sequence \( \{ g_i y_0 \} \) converges to \( \tilde{\alpha} \) in \( Y \cup \partial Y \), because \( \{ g_j y_0 \} \) is a subsequence of the sequence \( \{ g_i y_0 \} \), which converges to \( \tilde{\alpha} \) in \( Y \cup \partial Y \).

Hence \( \tilde{\phi}(\alpha) = \tilde{\alpha} \) by the definition of the map \( \tilde{\phi} \). Thus the map \( \tilde{\phi} : \partial X \to \partial Y \) is surjective. \( \square \)

Here we provide a lemma.

Lemma 3.4. For any \( \tilde{N} \geq N \), there exists \( \tilde{M} > 0 \) such that \( GB(y_0, \tilde{M}) = Y \) and for any \( g, a \in G \), if \( [x_0, g x_0] \cap B(ax_0, \tilde{N}) \neq \emptyset \) in \( X \) then \( [y_0, g y_0] \cap B(a y_0, \tilde{M}) \neq \emptyset \) in \( Y \).

Proof. For \( \tilde{N} \geq N \), we put \( \tilde{M} = \lambda(N + \tilde{N}) + C + M \).

Let \( g, a \in G \) as \( [x_0, g x_0] \cap B(ax_0, \tilde{N}) \neq \emptyset \) in \( X \). Then there exists a point \( x_1 \in [x_0, g x_0] \cap B(ax_0, \tilde{N}) \). Since \( GB(x_0, N) = X \), there exists \( a' \in G \) such that \( x_1 \in B(a' x_0, N) \). Then \( x_1 \in [x_0, g x_0] \cap B(a' x_0, N) \) and \( [x_0, g x_0] \cap B(a' x_0, N) \neq \emptyset \) in \( X \). By the condition \( (\ast) \), \( [y_0, g y_0] \cap B(a y_0, \tilde{M}) \neq \emptyset \) in \( Y \). Hence \( d_Y(a y_0, [y_0, g y_0]) \leq \tilde{M} \).

Here we note that

\[
d_Y(a y_0, [y_0, g y_0]) \leq d_X(a' x_0, ax_0) + C
\leq \lambda(d_X(a' x_0, x_1) + d_X(x_1, ax_0)) + C
\leq \lambda(N + \tilde{N}) + C.
\]

Hence

\[
d_Y(a y_0, [y_0, g y_0]) \leq d_Y(a y_0, a' y_0) + d_Y(a' y_0, [y_0, g y_0])
\leq \lambda(N + \tilde{N}) + C + M
\leq \tilde{M}.
\]

Thus we obtain that \( [y_0, g y_0] \cap B(a y_0, \tilde{M}) \neq \emptyset \) in \( Y \). \( \square \)

Let \( \tilde{N} = 2N \). By Lemma 3.4, there exists \( \tilde{M} > 0 \) such that \( GB(y_0, \tilde{M}) = Y \) and for any \( g, a \in G \), if \( [x_0, g x_0] \cap B(ax_0, \tilde{N}) \neq \emptyset \) in \( X \) then \( [y_0, g y_0] \cap B(a y_0, \tilde{M}) \neq \emptyset \) in \( Y \).

Here we show the following technical lemma.

Lemma 3.5. Let \( \alpha \in \partial X \) and let \( \xi_\alpha : [0, \infty) \to X \) be the geodesic ray in \( X \) such that \( \xi_\alpha(0) = x_0 \) and \( \xi_\alpha(\infty) = \alpha \). Let \( \{ g_i x_0 \} \subset G x_0 \subset X \) be a sequence which converges to \( \alpha \) in \( X \cup \partial X \) such that \( d_X(g_i x_0, \xi_\alpha(i)) \leq N \) for any \( i \in \mathbb{N} \) (since \( GB(x_0, N) = X \), we can take such a sequence). Then
there exists a neighborhood basis for $\bar{\bar{\alpha}}$

Here $\bar{\bar{\alpha}} = \bar{\bar{\overline{\phi}}}(\alpha)$ and $\xi_{\bar{\bar{\alpha}}} : [0, \infty) \to Y$ is the geodesic ray in $Y$ such that $\xi_{\bar{\bar{\alpha}}} (0) = y_0$ and $\xi_{\bar{\bar{\alpha}}} (\infty) = \bar{\bar{\alpha}}$.

Proof. (1) For any $i, j \in \mathbb{N}$ with $i < j$,
\[
d_X(g_i x_0, [x_0, g_j x_0]) \leq \tilde{N} \quad \text{for any } i, j \in \mathbb{N} \text{ with } i < j,
\]
and let $x_i$ for any $i \in \mathbb{N}$, where we obtain the inequality $d_X(\xi_\alpha(i), [x_0, g_j x_0]) \leq N$, since $d_X(g_j x_0, \xi_\alpha(j)) \leq N$, $i < j$ and $X$ is a CAT(0) space.

(2) By Lemma 3.3 and the definition of $\tilde{M}$, we obtain that $d_Y(g_i y_0, [y_0, g_j y_0]) \leq \tilde{M}$ for any $i, j \in \mathbb{N}$ with $i < j$ from (1).

(3) We note that the sequence $\{g_i y_0\}$ converges to $\bar{\bar{\alpha}}$ by the definition of the map $\bar{\bar{\phi}} : \partial X \to \partial Y$.

Let $i \in \mathbb{N}$ and let $R = d_Y(y_0, g_i y_0)$. Since the sequence $\{g_j y_0\}$ converges to $\bar{\bar{\alpha}}$, there exists $j_0 \in \mathbb{N}$ such that
\[
d_Y(\xi_{\bar{\bar{\alpha}}}(R), \xi_{g_{j_0} y_0}(R)) < 1
\]
for any $j \geq j_0$, because the set
\[
\{U_Y \cup \partial Y(\bar{\bar{\alpha}}; r, 1) \mid r > 0\}
\]
defined in Section 2 is a neighborhood basis for $\bar{\bar{\alpha}}$ in $Y \cup \partial Y$.

Let $j \in \mathbb{N}$ with $j > i$ and $j > j_0$ since $i < j$, we obtain that $d_Y(g_i y_0, [y_0, g_j y_0]) \leq \tilde{M}$ by (2). Hence there exists $r > 0$ such that $d_Y(g_i y_0, \xi_{g_{j_0} y_0}(r)) \leq \tilde{M}$. Here we note that $r \leq R$ by [21] Lemma 4.1 and we can obtain that
\[
d_Y(\xi_\alpha(r), \xi_{g_{j_0} y_0}(r)) < d_Y(\xi_{\bar{\bar{\alpha}}}(R), \xi_{g_{j_0} y_0}(R)) < 1,
\]
since $Y$ is a CAT(0) space. Then
\[
d_Y(g_i y_0, \text{Im } \xi_\alpha) \leq d_Y(g_i y_0, \xi_{g_{j_0} y_0}(r)) + d_Y(\xi_{g_{j_0} y_0}(r), \text{Im } \xi_{\bar{\bar{\alpha}}})
\]
\[
< \tilde{M} + 1.
\]
Hence $d_Y(g_i y_0, \text{Im } \xi_\alpha) \leq \tilde{M} + 1$ for any $i \in \mathbb{N}$.

(4) We obtain that $d_X(g_i x_0, g_{i+1} x_0) \leq 2N + 1$ for any $i \in \mathbb{N}$, because
\[
d_X(g_i x_0, [x_0, g_{i+1} x_0]) \leq d_X(g_i x_0, \xi_\alpha(i)) + d_X(\xi_\alpha(i), [x_0, g_{i+1} x_0])
\]
\[
\leq N + i + N
\]
\[
= 2N + 1,
\]
since $d_X(g_i x_0, \xi_\alpha(i)) \leq N$ for any $i \in \mathbb{N}$ by the definition of the sequence $\{g_i x_0\}$.

(5) Since the map $\phi : Gx_0 \to Gy_0 (gx_0 \mapsto gy_0)$ is a quasi-isometry, we obtain that $d_Y(g_i y_0, g_{i+1} y_0) \leq \lambda(2N + 1) + C$ for any $i \in \mathbb{N}$ by (4).
(6) For each \( i \in \mathbb{N} \), there exists \( r_i > 0 \) such that \( d_Y(g_i y_0, \xi_{\alpha}(r_i)) \leq \tilde{M} + 1 \) by (3). Then by (5),
\[
d_Y(\xi_{\alpha}(r_i), \xi_{\alpha}(r_{i+1})) \leq d_Y(\xi_{\alpha}(r_i), g_i y_0) + d_Y(g_i y_0, g_{i+1} y_0) + d_Y(g_{i+1} y_0, \xi_{\alpha}(r_{i+1}))
\leq (\tilde{M} + 1) + (\lambda(2N + 1) + C) + (\tilde{M} + 1)
= 2(\tilde{M} + 1) + \lambda(2N + 1) + C.
\]
Hence we obtain that
\[
\text{Im} \xi_{\alpha} \subset \bigcup \{ B(g_i y_0, 3(\tilde{M} + 1) + \lambda(2N + 1) + C) \mid i \in \mathbb{N} \}.
\]

\[\square\]

Now we show the following.

**Proposition 3.6.** The map \( \phi : \partial X \to \partial Y \) is injective.

*Proof.* Let \( \alpha, \alpha' \in \partial X \), and let \( \xi_{\alpha} : [0, \infty) \to X \) and \( \xi_{\alpha'} : [0, \infty) \to X \) be the geodesic rays in \( X \) such that \( \xi_{\alpha}(0) = \xi_{\alpha'}(0) = x_0 \), \( \xi_{\alpha}(\infty) = \alpha \) and \( \xi_{\alpha'}(\infty) = \alpha' \). Let \( \{g_i x_0\}, \{g_i' x_0\} \subset G x_0 \subset X \) be sequences such that \( d_X(g_i x_0, \xi_{\alpha}(i)) \leq N \) and \( d_X(g_i' x_0, \xi_{\alpha'}(i)) \leq N \). Then the sequence \( \{g_i x_0\} \) converges to \( \alpha \) and the sequence \( \{g_i' x_0\} \) converges to \( \alpha' \) in \( X \cup \partial X \).

Let \( \tilde{\alpha} = \bar{\phi}(\alpha) \) and \( \tilde{\alpha}' = \bar{\phi}(\alpha') \). Also let \( \xi_{\tilde{\alpha}} : [0, \infty) \to Y \) and \( \xi_{\tilde{\alpha}'} : [0, \infty) \to Y \) be the geodesic rays in \( Y \) such that \( \xi_{\tilde{\alpha}}(0) = \xi_{\tilde{\alpha}'}(0) = y_0 \), \( \xi_{\tilde{\alpha}}(\infty) = \tilde{\alpha} \) and \( \xi_{\tilde{\alpha}'}(\infty) = \tilde{\alpha}' \).

Then by Lemma 3.5,
\[
\begin{align*}
(1) & \quad d_X(g_i x_0, [x_0, g_j x_0]) \leq \tilde{N} \text{ for any } i, j \in \mathbb{N} \text{ with } i < j, \\
(2) & \quad d_Y(g_i y_0, [y_0, g_j y_0]) \leq \tilde{M} \text{ for any } i, j \in \mathbb{N} \text{ with } i < j, \\
(3) & \quad d_Y(g_i y_0, \text{Im} \xi_{\tilde{\alpha}}) \leq \tilde{M} + 1 \text{ for any } i \in \mathbb{N}, \\
(4) & \quad d_X(g_i x_0, g_{i+1} x_0) \leq 2N + 1 \text{ for any } i \in \mathbb{N}, \\
(5) & \quad d_Y(g_i y_0, g_{i+1} y_0) \leq \lambda(2N + 1) + C \text{ for any } i \in \mathbb{N}, \\
(6) & \quad \text{Im} \xi_{\tilde{\alpha}} \subset \bigcup \{ B(g_i y_0, 3(\tilde{M} + 1) + \lambda(2N + 1) + C) \mid i \in \mathbb{N} \}.
\end{align*}
\]

and
\[
\begin{align*}
(1') & \quad d_X(g_i' x_0, [x_0, g_j' x_0]) \leq \tilde{N} \text{ for any } i, j \in \mathbb{N} \text{ with } i < j, \\
(2') & \quad d_Y(g_i' y_0, [y_0, g_j' y_0]) \leq \tilde{M} \text{ for any } i, j \in \mathbb{N} \text{ with } i < j, \\
(3') & \quad d_Y(g_i' y_0, \text{Im} \xi_{\tilde{\alpha}'}) \leq \tilde{M} + 1 \text{ for any } i \in \mathbb{N}, \\
(4') & \quad d_X(g_i' x_0, g_{i+1} x_0) \leq 2N + 1 \text{ for any } i \in \mathbb{N}, \\
(5') & \quad d_Y(g_i' y_0, g_{i+1} y_0) \leq \lambda(2N + 1) + C \text{ for any } i \in \mathbb{N}, \\
(6') & \quad \text{Im} \xi_{\tilde{\alpha}'} \subset \bigcup \{ B(g_i y_0, 3(\tilde{M} + 1) + \lambda(2N + 1) + C) \mid i \in \mathbb{N} \}.
\end{align*}
\]

To prove that the map \( \bar{\phi} : \partial X \to \partial Y \) is injective, we show that if \( \alpha \neq \alpha' \) then \( \tilde{\alpha} \neq \tilde{\alpha}' \).

We suppose that \( \alpha \neq \alpha' \). Then the geodesic rays \( \xi_{\alpha} \) and \( \xi_{\alpha'} \) are not asymptotic. Hence for any \( t > 0 \), there exists \( r_0 > 0 \) such that \( d_X(\xi_{\alpha}(r_0), \text{Im} \xi_{\alpha'}) > t \). Then for \( i_0 \in \mathbb{N} \) with \( i_0 \geq r_0 \),
\[
\begin{align*}
\quad d_X(g_i x_0, \text{Im} \xi_{\alpha'}) & \geq d_X(\xi_{\alpha}(i_0), \text{Im} \xi_{\alpha'}) - d_X(g_{i_0} x_0, \xi_{\alpha}(i_0)) \\
& \geq d_X(\xi_{\alpha}(r_0), \text{Im} \xi_{\alpha'}) - N \\
& > t - N.
\end{align*}
\]
Since $d_X(g'_jx_0, \text{Im} \xi_{\tilde{\alpha}'}) \leq N$ for any $j \in \mathbb{N}$, we obtain that $d_X(g_{i_0}x_0, g'_jx_0) > t - 2N$ for any $j \in \mathbb{N}$. Hence for any $j \in \mathbb{N}$,

$$d_Y(g_{i_0}y_0, g'_jy_0) \geq \frac{1}{\lambda}d_X(g_{i_0}x_0, g'_jx_0) - C$$

$$> \frac{1}{\lambda}(t - 2N) - C.$$

Here by (6'),

$$\text{Im} \xi_{\tilde{\alpha}'} \subset \bigcup\{B(g'_jy_0, 3(\tilde{M} + 1) + \lambda(2N + 1) + C) \mid j \in \mathbb{N}\}.$$

Let $j_0 \in \mathbb{N}$ such that

$$d_Y(g_{i_0}y_0, g'_jy_0) = \min\{d_Y(g_{i_0}y_0, g'_jy_0) \mid j \in \mathbb{N}\}.$$

Then

$$d_Y(g_{i_0}y_0, \text{Im} \xi_{\tilde{\alpha}'}) \geq \min\{d_Y(g_{i_0}y_0, g'_jy_0) \mid j \in \mathbb{N}\} - (3(\tilde{M} + 1) + \lambda(2N + 1) + C)$$

$$= d_Y(g_{i_0}y_0, g'_jy_0) - (3(\tilde{M} + 1) + \lambda(2N + 1) + C)$$

$$> \frac{1}{\lambda}(t - 2N) - C - (3(\tilde{M} + 1) + \lambda(2N + 1) + C),$$

since $d_Y(g_{i_0}y_0, g'_jy_0) > \frac{1}{\lambda}(t - 2N) - C$ for any $j \in \mathbb{N}$ by the argument above.

Thus for any $t > 0$, there exists $i_0 \in \mathbb{N}$ such that

$$d_Y(g_{i_0}y_0, \text{Im} \xi_{\tilde{\alpha}'}) > \frac{1}{\lambda}(t - 2N) - C - (3(\tilde{M} + 1) + \lambda(2N + 1) + C).$$

Here by (3), there exists $R_0 > 0$ such that

$$d_Y(g_{i_0}y_0, \xi_{\tilde{\alpha}}(R_0)) \leq \tilde{M} + 1.$$

Then

$$d_Y(\xi_{\tilde{\alpha}}(R_0), \text{Im} \xi_{\tilde{\alpha}'}) \geq d_Y(g_{i_0}y_0, \text{Im} \xi_{\tilde{\alpha}'}) - d_Y(g_{i_0}y_0, \xi_{\tilde{\alpha}}(R_0))$$

$$> \left(\frac{1}{\lambda}(t - 2N) - C - (3(\tilde{M} + 1) + \lambda(2N + 1) + C) - (\tilde{M} + 1)\right)$$

$$= \left(\frac{1}{\lambda}(t - 2N) - C - (4(\tilde{M} + 1) + \lambda(2N + 1) + C)\right).$$

Since $t > 0$ is an arbitrary large number, the two geodesic rays $\xi_{\tilde{\alpha}}$ and $\xi_{\tilde{\alpha}'}$ are not asymptotic and $\tilde{\alpha} \neq \tilde{\alpha}'$.

Therefore, the map $\tilde{\phi} : \partial X \to \partial Y$ is injective. \hfill \Box

From Propositions 3.3 and 3.6, we obtain that the map $\tilde{\phi} : \partial X \to \partial Y$ is bijective.

We show the following.

**Proposition 3.7.** The map $\tilde{\phi} : \partial X \to \partial Y$ is continuous.

**Proof.** Let $\alpha \in \partial X$ and let $\tilde{\alpha} = \tilde{\phi}(\alpha)$. We put $\tilde{c} = \lambda(2N + 3) + C + 2(\tilde{M} + 1)$ which is a constant.

To prove that the map $\tilde{\phi} : \partial X \to \partial Y$ is continuous at the point $\alpha \in \partial X$, we show that for any $\tilde{r} > 0$, there exists $r > 0$ such that if $\beta \in U_{X \cup \partial X}(\alpha; r, 1)$ then $\tilde{\beta} \in U_{Y \cup \partial Y}(\tilde{\alpha}; \tilde{r}, \tilde{c})$ where $\tilde{\beta} = \tilde{\phi}(\beta)$, because $\{U_{X \cup \partial X}(\alpha; r, 1) \mid r > 0\}$ and $\{U_{Y \cup \partial Y}(\tilde{\alpha}; \tilde{r}, \tilde{c}) \mid \tilde{r} > 0\}$ are neighborhood basis for $\alpha$ and $\tilde{\alpha}$ in $\partial X$ and $\partial Y$, respectively.

For $\tilde{r} > 0$, we take $r = \lambda(\tilde{r} + C + \tilde{M} + 1) + N + 1$.

Let $\beta \in U_{X \cup \partial X}(\alpha; r, 1)$ and let $\tilde{\beta} = \tilde{\phi}(\beta)$.
By Lemma 3.5 there exists a sequence \( \{g_i\} \subset G \) such that

1. the sequence \( \{g_i x_0\} \subset X \) converges to \( \alpha \) in \( X \cup \partial X \),
2. \( d_X(g_i x_0, \text{Im} \xi_\alpha) \leq N \) for any \( i \in \mathbb{N} \),
3. \( \text{Im} \xi_\alpha \subset \bigcup \{ B(g_i x_0, N + 1) \mid i \in \mathbb{N} \} \),
4. the sequence \( \{g_i y_0\} \subset Y \) converges to \( \alpha \bar{\alpha} \) in \( Y \cup \partial Y \), and
5. \( d_Y(g_i y_0, \text{Im} \xi_{\bar{\alpha}}) \leq \bar{M} + 1 \) for any \( i \in \mathbb{N} \).

Here since \( d_X(g_i x_0, \xi_\alpha(i)) \leq N \) and \( d_X(\xi_\alpha(i), \xi_\alpha(i + 1)) = 1 \) for any \( i \in \mathbb{N} \) in Lemma 3.5, we can obtain the statement (3) above. Also, there exists a sequence \( \{h_j\} \subset G \) such that

1. the sequence \( \{h_j x_0\} \subset X \) converges to \( \beta \) in \( X \cup \partial X \),
2. \( d_X(h_j x_0, \text{Im} \xi_\beta) \leq N \) for any \( j \in \mathbb{N} \),
3. \( \text{Im} \xi_\beta \subset \bigcup \{ B(h_j x_0, N + 1) \mid j \in \mathbb{N} \} \),
4. the sequence \( \{h_j y_0\} \subset Y \) converges to \( \beta \bar{\beta} \) in \( Y \cup \partial Y \), and
5. \( d_Y(h_j y_0, \text{Im} \xi_{\bar{\beta}}) \leq \bar{M} + 1 \) for any \( j \in \mathbb{N} \).

Since \( \beta \in U_{X \cup \partial X}(\alpha; r, 1) \),

\[
d_X(\xi_\alpha(r), \xi_\beta(r)) < 1.
\]

By (3) and (3'), there exist \( i_0 \in \mathbb{N} \) and \( j_0 \in \mathbb{N} \) such that

\[
d_X(g_{i_0} x_0, \xi_\alpha(r)) \leq N + 1 \quad \text{and} \quad d_X(h_{j_0} x_0, \xi_\beta(r)) \leq N + 1.
\]

Also by (5) and (5'), there exist \( \tilde{r} > 0 \) and \( \tilde{r}' > 0 \) such that

\[
d_Y(g_{i_0} y_0, \xi_\alpha(\tilde{r})) \leq \tilde{M} + 1 \quad \text{and} \quad d_Y(h_{j_0} y_0, \xi_\beta(\tilde{r}')) \leq \tilde{M} + 1.
\]

Then

\[
d_X(g_{i_0} x_0, h_{j_0} x_0) \leq d_X(g_{i_0} x_0, \xi_\alpha(r)) + d_X(\xi_\alpha(r), \xi_\beta(r)) + d_X(\xi_\beta(r), h_{j_0} x_0)
\]
\[
\leq (N + 1) + 1 + (N + 1)
\]
\[
= 2N + 3.
\]

Hence

\[
d_Y(g_{i_0} y_0, h_{j_0} y_0) \leq \lambda d_X(g_{i_0} x_0, h_{j_0} x_0) + C
\]
\[
\leq \lambda(2N + 3) + C.
\]

Then

\[
d_Y(\xi_\alpha(\tilde{r}), \xi_\beta(\tilde{r}')) \leq d_Y(\xi_\alpha(\tilde{r}), g_{i_0} y_0) + d_Y(g_{i_0} y_0, h_{j_0} y_0) + d_Y(h_{j_0} y_0, \xi_\beta(\tilde{r}'))
\]
\[
\leq (\tilde{M} + 1) + (\lambda(2N + 3) + C) + (\tilde{M} + 1)
\]
\[
= \lambda(2N + 3) + C + 2(\tilde{M} + 1)
\]
\[
= \tilde{c}.
\]
Also,
\[
\tilde{r} = d_Y(y_0, \xi_\alpha(\tilde{r})) \\
\geq d_Y(y_0, g_0 y_0) - d_Y(g_i y_0, \xi_\alpha(\tilde{r})) \\
\geq d_Y(y_0, g_0 y_0) - (\tilde{M} + 1) \\
\geq \frac{1}{\lambda} d_X(x_0, g_0 x_0) - C - (\tilde{M} + 1) \\
\geq \frac{1}{\lambda} (d_X(x_0, \xi_\alpha(r)) - d_X(g_0 x_0, \xi_\alpha(r))) - C - (\tilde{M} + 1) \\
\geq \frac{1}{\lambda} (r - (N + 1)) - C - (\tilde{M} + 1) \\
= \bar{r},
\]
since \( r = \lambda(\bar{r} + C + \tilde{M} + 1) + N + 1 \). Thus we obtain that
\[
d_Y(\xi_\alpha(\bar{r}), \text{Im} \xi_\beta) \leq d_Y(\xi_\alpha(\tilde{r}), \text{Im} \xi_\beta) \leq d_Y(\xi_\alpha(\bar{r}), \xi_\beta(\tilde{r}')) \leq \bar{c}.
\]

Hence \( \bar{\beta} \in U_{Y \cup \partial Y}(\tilde{\alpha}; \bar{r}, \bar{c}). \)

Thus the map \( \tilde{\varphi} : \partial X \to \partial Y \) is continuous. \( \square \)

Finally, we show the following.

**Theorem 3.8.** The map \( \tilde{\varphi} : \partial X \to \partial Y \) is a \( G \)-equivariant homeomorphism.

**Proof.** By the argument above, the map \( \tilde{\varphi} : \partial X \to \partial Y \) is well-defined, bijective and continuous.

From the definition and the well-definedness of \( \tilde{\varphi} \), we obtain that the map \( \tilde{\varphi} : \partial X \to \partial Y \) is \( G \)-equivariant. Indeed for any \( \alpha \in \partial X \) and \( g \in G \), if \( \{g_i x_0\} \subset G x_0 \subset X \) is a sequence which converges to \( \alpha \) in \( X \cup \partial X \), then \( \tilde{\varphi}(\alpha) \) is the point of \( \partial Y \) to which the sequence \( \{g_i y_0\} \subset G y_0 \subset Y \) converges in \( Y \cup \partial Y \). Then \( \{g g_i x_0\} \subset G x_0 \subset X \) is the sequence which converges to \( ga \) in \( X \cup \partial X \) and \( \tilde{\varphi}(ga) \) is the point of \( \partial Y \) to which the sequence \( \{g g_i y_0\} \subset G y_0 \subset Y \) converges in \( Y \cup \partial Y \). Here we note that the sequence \( \{gg_i y_0\} \subset G y_0 \subset Y \) converges to \( g\alpha \) in \( Y \cup \partial Y \) by the definition of the action of \( G \) on \( \partial Y \). Hence \( \tilde{\varphi}(ga) = g \tilde{\varphi}(\alpha) \) for any \( \alpha \in \partial X \) and \( g \in G \) and the map \( \tilde{\varphi} : \partial X \to \partial Y \) is \( G \)-equivariant.

Also, the map \( \tilde{\varphi} : \partial X \to \partial Y \) is closed, since \( \partial X \) and \( \partial Y \) are compact and metrizable.

Therefore, we obtain that the map \( \tilde{\varphi} : \partial X \to \partial Y \) is a \( G \)-equivariant homeomorphism. \( \square \)

4. REMARK

The author thinks that there is a possibility that the main theorem, the condition (\( \ast \)) and some arguments in this paper can be used to investigate boundaries of CAT(0) groups and interesting open problems on

1. (equivariant) rigidity of boundaries of CAT(0) groups;
2. (equivariant) rigidity of boundaries of Coxeter groups;
3. (equivariant) rigidity of boundaries of Davis complexes of Coxeter groups.
(4) (equivariant) rigidity of boundaries of CAT(0) spaces on which Coxeter groups act geometrically by reflections;
(5) (equivariant) rigidity of boundaries of CAT(0) spaces on which right-angled Coxeter groups act geometrically by reflections;
(6) (equivariant) rigidity of boundaries of CAT(0) cubical complexes on which CAT(0) groups act geometrically,

etc.

Here we can find some recent research on CAT(0) groups and their boundaries in [12], [19], [23], [26], [30], [32], [33], [34], [36] and [38]. Details of Coxeter groups and Coxeter systems are found in [6], [9] and [29], and details of Davis complexes which are CAT(0) spaces defined by Coxeter systems and their boundaries are found in [14], [15] and [35]. We can find some recent research on boundaries of Coxeter groups in [10], [16], [17], [18], [28], [31]. Every cocompact discrete reflection group of a geodesic space becomes a Coxeter group (cf. [25]), and we say that a Coxeter group $W$ acts geometrically on a CAT(0) space $X$ by reflections if the Coxeter group $W$ is a reflection group of $X$ (cf. [27]).

References

[1] A. D. Alexandrov, V. N. Berestovskii and I. G. Nikolaev, Generalized Riemannian spaces, Russ. Math. Surveys 41 (1986), 1–54.
[2] F. D. Ancel, C. Guilbault, and J. Wilson, The Croke-Kleiner boundaries are cell-like equivalent, preprint.
[3] W. Ballmann and M. Brin, Orbihedra of nonpositive curvature, Inst. Hautes Études Sci. Publ. Math. 82 (1995), 169–209.
[4] W. Ballmann, M. Gromov and V. Schroeder, Manifolds of Nonpositive Curvature, Progr. Math. vol. 61, Birkhäuser, Boston MA, 1985.
[5] M. Bestvina, Local homology properties of boundaries of groups, Michigan Math. J. 43 (1996), 123–139.
[6] N. Bourbaki, Groupes et Algèbres de Lie, Chapters IV-VI, Masson, Paris, 1981.
[7] P. Bowers and K. Ruane, Boundaries of nonpositively curved groups of the form $G \times \mathbb{Z}^n$, Glasgow Math. J. 38 (1996), 177–189.
[8] M. R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer-Verlag, Berlin, 1999.
[9] K. S. Brown, Buildings, Springer-Verlag, 1980.
[10] P. Caprace and K. Fujiwara, Rank-one isometries of buildings and quasi-morphisms of Kac-Moody groups, arXiv:0808.0470v3, preprint.
[11] M. Coornaert and A. Papadopoulos, Symbolic dynamics and hyperbolic groups, Lecture Notes in Math.1539, Springer-Verlag, 1993.
[12] C. B. Croke and B. Kleiner, Spaces with nonpositive curvature and their ideal boundaries, Topology 39 (2000), 549–556.
[13] C. B. Croke and B. Kleiner, The geodesic flow of a nonpositively curved graph manifold, Geom. Funct. Anal. 12 (2002), 479–545.
[14] M. W. Davis, Groups generated by reflections and aspherical manifolds not covered by Euclidean space, Ann. of Math. 117 (1983), 293–324.
[15] M. W. Davis, Nonpositive curvature and reflection groups, in Handbook of geometric topology (Edited by R. J. Daverman and R. B. Sher), pp. 373–422, North-Holland, Amsterdam, 2002.
[16] A. N. Dranishnikov, On boundaries of hyperbolic Coxeter groups, Topology Appl. 110 (2001), 29–38.
[17] A. N. Dranishnikov, Boundaries of Coxeter groups and simplicial complexes with given links, J. Pure Appl. Algebra 137 (1999), 139–151.
[18] H. Fischer, Boundaries of right-angled Coxeter groups with manifold nerves, Topology 42 (2003), 423–446.
[19] R. Geoghegan and P. Ontaneda, *Boundaries of cocompact proper CAT(0) spaces*, Topology 46 (2007), 129–137.
[20] E. Ghys and P. de la Harpe (ed), *Sur les Groupes Hyperboliques d’après Mikhael Gromov*, Progr. Math. vol. 83, Birkhäuser, Boston MA, 1990.
[21] M. Gromov, *Hyperbolic groups*, Essays in group theory (Edited by S. M. Gersten), pp. 75–263, M.S.R.I. Publ. 8, 1987.
[22] M. Gromov, *Asymptotic invariants for infinite groups*, Geometric Group Theory (G.A. Niblo and M.A. Roller, eds.), LMS Lecture Notes, vol. 182, Cambridge University Press, Cambridge, 1993, pp. 1–295.
[23] U. Hamenstädt, *Rank-one isometries of proper CAT(0)-spaces*, arXiv:0810.3793v1, preprint.
[24] T. Hosaka, *The interior of the limit set of groups*, Houston J. Math. 30 (2004), 705–721.
[25] T. Hosaka, *Reflection groups of geodesic spaces and Coxeter groups*, Topology Appl. 153 (2006), 1860–1866.
[26] T. Hosaka, *On splitting theorems for CAT(0) spaces and compact geodesic spaces of non-positive curvature*, arXiv:math.GR/0405551v1, preprint.
[27] T. Hosaka, *Parabolic subgroups of Coxeter groups acting by reflections on CAT(0) spaces*, arXiv:math/0409472v1, preprint.
[28] T. Hosaka, *On boundaries of Coxeter groups and topological fractal structures*, arXiv:0912.0061v2, preprint.
[29] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.
[30] M. Mihalik and K. Ruane, *CAT(0) groups with non-locally connected boundary*, J. London Math. Soc. (2) 60 (1999), 757–770.
[31] M. Mihalik, K. Ruane and S. Tschants, *Local connectivity of right-angled Coxeter group boundaries*, J. Group Theory 10 (2007), 531–560.
[32] N. Monod, *Superrigidity for irreducible lattices and geometric splitting*, J. Amer. Math. Soc. 19 (2006), 781–814.
[33] C. Mooney, *Examples of non-rigid CAT(0) groups from the category of knot groups*, arXiv:0706.1581v2, preprint.
[34] C. Mooney, *All CAT(0) boundaries of a group of the form H × K are CE equivalent*, arXiv:0705.1510v1, preprint.
[35] G. Moussong, *Hyperbolic Coxeter groups*, Ph.D. thesis, Ohio State University, 1988.
[36] P. Papasoglu and E. L. Swenson, *Boundaries and JSJ decompositions of CAT(0)-groups*, Geom. Funct. Analy. 19 (2009), 558–590.
[37] E. L. Swenson, *A cut point theorem for CAT(0) groups*, J. Differential Geom. 53 (1999), 327–358.
[38] J. M. Wilson, *A CAT(0) group with uncountably many distinct boundaries*, J. Group Theory 8 (2005), 229–238.
[39] S. Yamagata, *On ideal boundaries of some Coxeter groups*, Advanced Studies Pure Math. 55 (2009), 345–352.

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