Near-Optimal MNL Bandits Under Risk Criteria

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Abstract

We study MNL bandits, which is a variant of the traditional multi-armed bandit problem, under risk criteria. Unlike the ordinary expected revenue, risk criteria are more general goals widely used in industries and businesses. We design algorithms for a broad class of risk criteria, including but not limited to the well-known conditional value-at-risk, Sharpe ratio and entropy risk, and prove that they suffer a near-optimal regret. As a complement, we also conduct experiments with both synthetic and real data to show the empirical performance of our proposed algorithms.

Introduction

Dynamic assortment optimization is one of the fundamental problems in online learning. It has wide applications in industries, for example retailing and advertisement. To motivate the study of the problem, let us consider e-commerce companies like Amazon and Wish who want to sell products to online users when they visit the websites and search for some type of products, for example headphones. Such companies usually have a variety of products with that type in warehouse to sell. Due to space constraint of a website, it is not possible to exhibit all of the available products. Hence, each time when an online user visits the website, only a limited number of products can be displayed. When an online user buys a product, the company will get some profit. So one natural goal for the company is to display on the website an assortment consisting of several products such that the expected revenue is maximized. However, in practice, a company may have more complex strategies other than simply maximizing its revenue, and general risk criteria may be better choices to serve such goals. For example, in risk management, a very common risk criterion called expected shortfall or conditional value at risk (CVaR) is defined as the expected revenue under a certain percentile. If we only consider the expected revenue, we may lead to focus on recommending some products producing high revenue but purchased only by a small portion of users. If the company wishes to maintain a higher level of active and diversified users, then CVaR is more appropriate. Whether it is still possible for the sales manager of the company to design a near-optimal sales strategy when the goal is changed, for example to a kind of risk criteria, is a very practical problem and to the best of our knowledge, has not been studied before.

Suppose a company has $N$ products of a certain category to sell during a sales season, which can be represented by $[N]$, where $[N] \equiv \{1, 2, \ldots, N\}$ and each product corresponds to an element in $[N]$. Let $T$ be the total number of times such products are searched during a sales season and $S_t$ be the assortment displayed by the website at $t$th time of request. The aforementioned sales activity can be modeled by the following game which runs in $T$ time steps: at time step $1 \leq t \leq T$, when an assortment $S_t \subset [N]$ is displayed by the website, the online user will make a choice, i.e., whether to buy a product in $S_t$ or purchase nothing. Following the previous motivation example, we add a cardinality constraint, which means the number of products in $S_t$ can not exceed a predefined number $K \leq N$. Let $c_t$ denote the choice of the online user at time $t$. When $c_t = i$, it means that the online user buys product $i$. For convenience, we use $c_t = 0$ to represent the situation when the online user does not purchase anything. In general, $c_t$ can be viewed as a random variable and there is no doubt that the multinomial logic model (MNL) (Agrawal et al. 2019) has become the most popular one to model the behavior of the online user, i.e., $c_t$, when $S_t$ is provided. Dynamic assortment optimization with MNL choice model is also called MNL bandits. In this model, each product $i$ is assumed to be related to an unknown preference parameter $v_i$ and the probability that a visiting online user chooses product $i$ given assortment $S_t$ is defined by

$$P(c_t = i) \overset{\text{def}}{=} \frac{v_i}{1 + \sum_{j \in S_t} v_j}, \quad (1)$$
where we set the preference parameter of no-purchase $v_0 = 1$. Note that this assumption does not harm the model too much since one can easily scale $v_i$’s to satisfy this condition. Following the literature, we also assume no-purchase is the most frequent choice i.e., $0 \leq v_i \leq 1$, which is often a reasonable assumption in sales activities.

During the last decade, MNL bandits has attracted much attention (Rusmevichientong, Shen, and Shmoys 2010; Sauré and Zeevi 2013; Agrawal et al. 2017, 2019; Dong et al. 2020). However, all of the previous works consider maximizing the expected revenue, which is not always appropriate for practical applications. In this paper, we are interested in designing algorithms for a general class of risk criteria.

### Problem Formulation

Suppose for each product $i \in [N]$, selling it successfully can make the company a profit of $r_i$, which is known beforehand. W.o.l.g., we assume $r_i \in (0, 1]$. This can always be achieved by proper scaling. Moreover, the profit for no-purchase is $r_0 = 0$. Then at time step $t \geq 1$, when assortment $S_t \subset [N]$ and preference parameter vector $\nu = (v_1, \ldots, v_N)$ are provided, the profit can be represented by a random variable $X(S_t, \nu)$ defined by

$$P(X(S_t, \nu) = r_i) = P(\xi_t = i) = \frac{v_i}{1 + \sum_{j \in S_t} v_j} \quad (2)$$

for $i = 0$ and all $i \in S_t$. In addition, we denote by $F(S_t, \nu)$ the cumulative distribution function to $X(S_t, \nu)$. Given time horizon $T$, one natural goal, as explained in the introduction, is to find a policy equipped by the decision maker such that the expected revenue, i.e.,

$$\sum_{t=1}^{T} R(S_t, \nu) = \sum_{t=1}^{T} \mathbb{E}[X(S_t, \nu)] \quad (3)$$

is maximized, where $R(S_t, \nu)$ represents the expected profit when $S_t$ is served. This has been investigated previously in (Agrawal et al. 2017, 2019).

In this paper, instead of expectation, we consider a general class of risk criteria. Some examples of such risk criteria can be found in (Cassell, Mannor, and Zeevi 2018). Suppose $\mathcal{D}$ is the convex set of cumulative distribution functions. In general, we consider the risk criterion $U$ which is a function from $\mathcal{D}$ to $\mathbb{R}$. In the case of expectation, $U(F) = \int x dF(x)$. In particular, since we assumed that $r_i \in (0, 1]$, we will only need $F \in \mathcal{D}[0, 1]$, where we denote by $\mathcal{D}[0, 1]$ the subspace of $\mathcal{D}$ consisting of $F$ that is the cumulative distribution function of random variable $X$ taking values on $[0, 1]$. The goal of this paper is to find a policy such that the following quantity

$$\mathbb{E}\left[\sum_{t=1}^{T} U(F(S_t, \nu))\right] \quad (4)$$

is maximized. Let $S^*$ be the smallest assortment such that

$$U(F(S^*, \nu)) = \max_{S \subseteq [N], |S| \leq K} U(F(S, \nu)).$$

The regret of the game after $T$ time steps, which is a quantity measuring the difference between the optimal policy and policy $\pi$ used by the decision maker, is defined as

$$\mathcal{R}_T^+(([N], \nu, r)) \equiv T U(F(S^*, \nu)) - \mathbb{E}\left[\sum_{t=1}^{T} U(F(S_t, \nu))\right]. \quad (5)$$

where $r = (r_1, \ldots, r_N)$ and $\nu = (v_1, \ldots, v_N)$.

When it is clear from the context, we usually omit the policy $\pi$ and parameters $([N], \nu, r)$. Without much effort, we can see that maximizing (4) is equivalent to minimizing the regret (5).

### Related Work and Our Contribution

To the best of our knowledge, we are the first to study MNL bandits under general risk criteria.

In the past decade, there have been many works on the MNL bandit problem considering maximizing the expected revenue (3). In (Rusmevichientong, Shen, and Shmoys 2010; Sauré and Zeevi 2013), the authors assumed the gap between the best and the second-to-the-best assortments is known and proposed “Explore-then-Commit” algorithms. Later in (Agrawal et al. 2019), the authors proposed the state-of-the-art UCB-type algorithm with a regret upper bound $O(\sqrt{NT \ln T})$. Authors in (Agrawal et al. 2017) utilized Bayesian method i.e., Thompson Sampling to design an algorithm which performs well in practice. Very recently, for the expected revenue, it is showed in (Chen and Wang 2018) that the lower bound for the regret is $\Omega(\sqrt{NT})$.

There are a lot of previous works studying different risk criteria in Multi-armed Bandits (Sani, Lazaric, and Munos 2012; Maillard 2013; Galichet, Sebag, and Teytaud 2014; Zimina, Ibsen-Jensen, and Chatterjee 2014; Vakili and Zhao 2016). In (Cassell, Mannor, and Zeevi 2018), the authors established a thorough theory to deal with general risk criteria.

#### Our Contribution

Note that directly formatting MNL bandits to Multi-armed bandits and then applying the algorithm proposed in (Cassell, Mannor, and Zeevi 2018) will lead to a regret of order $\Omega\left(\sqrt{\left(\frac{K}{N}\right)T}\right)$, which is far from being optimal. In this paper, we present algorithms dealing with general risk criteria which suffer only a $O(\sqrt{NT})$ regret. Furthermore, we show that to make the algorithms work, $U$ only needs to satisfy mild assumptions, which are easy to verify and satisfied by almost all of the widely used risk criteria (see Table 1).
Assumptions

In this section, we first present the aforementioned three assumptions the risk criterion \( U \) should satisfy.

**Assumption 1 (Quasiconvexity).** \( U \) is quasiconvex on \( D[0, 1] \), i.e., for any \( \lambda \in [0, 1] \) and \( F_1, F_2 \in D[0, 1] \), it satisfies

\[
U(\lambda F_1 + (1 - \lambda)F_2) \leq \max\{U(F_1), U(F_2)\}.
\]

In addition to quasiconvexity, we also make the following two assumptions on \( U \).

**Assumption 2 (Boundedness).** For any \( F \in D[0, 1] \) it holds that \( |U(F)| \leq \gamma_1 \).

**Assumption 3 (One-sided Lipschitz Condition).** For any \( v' \geq v \), i.e., \( v'_i \geq v_i \) for all \( i \in [N] \), and \( S \subset [N] \), it holds that

\[
U(F(S, v')) - U(F(S, v)) \leq \frac{\gamma_2}{1 + \sum_{i \in S} v_i} \left( \sum_{i \in S} (v'_i - v_i) \right).
\]

Note that here \( \gamma_1 \) and \( \gamma_2 \) are universal constants related to the risk criterion \( U \).

It seems restrictive to assume that the risk criteria satisfy quasiconvexity. However, we do not see any widely used risk criteria that is not quasiconvex in the literature. In Table 1, we give a list of the risk criteria considered, which are all quasiconvex as shown in (Cassel, Mannor, and Zeevi 2018). To complement, we also show in Table 1 that whether a risk criterion satisfies Assumption 2 and Assumption 3, and give concrete values of \( \gamma_1 \) and \( \gamma_2 \). It turns out that all the risk criteria listed in Table 1 satisfy all three assumptions except VaR, which does not meet Assumption 3 since it is discontinuous in \( v \).

Algorithms

Due to space constraint, we only show RiskAwareUCB, which is a variant of the UCB-type algorithm proposed in (Agrawal et al. 2019), and its guarantee. In a similar way, we also propose RiskAwareTS, a variant of the Thompson Sampling algorithm proposed in (Agrawal et al. 2017). Please refer to the appendix for its description and near-optimal guarantee.

The concrete steps of the proposed algorithm RiskAwareUCB are as follows. We divide all the time steps i.e., \([T]\) into small episodes. During each episode \( \ell \), the same assortment \( S_\ell \) is repeatedly provided to the online user until a no-purchase outcome is observed. Specifically, in each episode \( \ell \), we are providing the assortment

\[
\arg\max_{S \subset [N], |S| \leq K} U(F(S, \bar{v}^\ell)),
\]

where \( \bar{v}^\ell \) is an optimistic estimate of the real unknown preference parameters before the start of episode \( \ell \).

Let \( t_{i, \ell} \) be the number of times the online users buy product \( i \) in the \( \ell \)th episode and \( T_\ell \) be the collection of episodes for which product \( i \) is served until episode \( \ell \) (exclusive). Define \( T_\ell = \{T_i | T_i \in [T] \} \) and \( v^\ell_i = \frac{\sum_{T_i \in T_\ell} t_{i, T_i}}{T_\ell} \).

The \( \ell \)th component of the optimistic preference parameters \( \bar{v}^\ell \) is given by

\[
\bar{v}^\ell = \min \left\{ \bar{v}^\ell + \sqrt{\frac{48 \ln(\sqrt{T_\ell} + 1)}{T_\ell}}, \frac{48 \ln(\sqrt{T_\ell} + 1)}{T_\ell}, 1 \right\}.
\]

Then we have the following theoretical upper bound for RiskAwareUCB.

**Theorem 4.** Suppose the risk criterion \( U \) satisfies Assumption 1, 2 and 3. The regret (5) incurred by the decision maker using RiskAwareUCB is upper bounded by \( O(\sqrt{NT}) \) after \( T \) time steps, where \( O \) hides polylogarithmic factors in \( N \) and \( T \).

Before proceeding, we first prove the following key lemma, which says that the risk gain of the optimal assortment calculated by an optimistic estimate of the preference parameters is never worse than that of \( S^* \).

**Lemma 5 (Monotone Maximum).** For any \( v' \geq v \), it holds that

\[
\max_{S \subset [N], |S| \leq K} U(F(S, v')) \geq U(F(S^*, v)).
\]

**Proof.** Fix \( S \), we first prove that \( U(F(S, u)) \) is a quasi-convex function with respect to vector \( u \). This statement can be easily verified by noticing that for any \( \lambda \in [0, 1] \) and \( u' \), we have

\[
U(F(S, \lambda u + (1 - \lambda)u')) = U \left( \frac{\lambda(1 + \sum_{i \in S} u_i)}{(1 - \lambda)(1 + \sum_{i \in S} u'_i)} F(S, u) + \frac{(1 - \lambda)(1 + \sum_{i \in S} u'_i)}{\lambda(1 + \sum_{i \in S} u_i)} F(S, u') \right) \leq \max\{U(F(S, u)), U(F(S, u'))\}
\]

where the last inequality is due to quasiconvexity of \( U \) on \( D[0, 1] \).

Next we show the following lemma.

**Lemma 6.** Given a quasi-convex function \( V(u) \) defined on \( [0, 1]^n \), suppose there is a point \( \bar{u} = (\bar{u}_1, \cdots, \bar{u}_n) \in [0, 1]^n \) satisfying that \( V(\bar{u}) \geq V(u) \) for any point \( u \neq \bar{u} \) such that \( u_i = \bar{u}_i \) or \( 0 \) for each \( i \) from 1 to \( n \). Then we have that \( V(u') \geq V(\bar{u}) \) for any \( u' \geq \bar{u} \).
Table 1: Widely Used Risk Criteria

| Risk Criterion (Parameter) | Property    | $\gamma_1$ | $\gamma_2$ |
|---------------------------|-------------|-------------|-------------|
| VaR$_{\alpha}$            | Quasiconvex | 1           | Not Exist   |
| CVaR$_{\alpha}$           | Convex     | 1           | 3/10        |
| $n$th-moment Entropy risk ($\theta$) | Convex | 1           | 2e$^{\theta}/\theta$ |
| Below target semi-variance ($r$) | Linear | $r^2$       | 2$r^2$      |
| Negative variance          | Convex     | $\frac{1}{\gamma}$ | 6           |
| Mean-variance ($\rho$)     | Convex     | $1 + \frac{r}{\rho}$ | 2 + 6$\rho$ |
| Sharpe ratio ($r, \gamma$) | Quasiconvex | $\frac{1}{\sqrt{\gamma}}$ | $2e^{-1/2} + 3e^{-3/2}$ |
| Sortino ratio ($r, c$)     | Quasiconvex | $\frac{1}{\sqrt{c^2}}$ | $2e^{-1/2} + \epsilon^{-3/2}$ |

**Proof.** For the sequence of points $u^{(i)} = (u_1^{(i)}, \ldots, u_n^{(i)})$ with $i = 1, 2, \ldots, n$ such that

$$u_j^{(i)} = \begin{cases} \bar{u}_j & j \neq i \\ 0 & j = i, \end{cases}$$

we have that $V(u^{(i)}) < V(\bar{u})$. For any $u' \geq \bar{u}$ and $u' \neq \bar{u}$, we define

$$\lambda_i = \left( \frac{u'_i - \bar{u}_i}{\bar{u}_i} \right) \left( \sum_{i=1}^{n} \frac{u'_i - \bar{u}_i}{\bar{u}_i} \right)^{-1}.$$  

Here $\lambda_i \in [0, 1]$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \lambda_i = 1$. Then the convex combination of $u^{(i)}$

$$\bar{u} = \sum_{i=1}^{n} \lambda_i u^{(i)}$$

is on the same line as $u'$ and $\bar{u}$. By the quasiconvexity of $V$, we have that $V(\bar{u}) \leq \max_{i=1}^{n} V(u^{(i)}) < V(\bar{u})$. If we define $\lambda = \frac{1}{1 + \sum_{i=1}^{n} \lambda_i}$, then

$$V(\bar{u}) = V(\lambda \bar{u} + (1 - \lambda) u') \leq \max\{V(\bar{u}), V(u')\},$$

which means we must have $V(\bar{u}) \leq V(u')$. \hfill \Box

Let $S^1$ be the smallest assortment such that

$$U(F(S^1, v')) = \max_{s \subset \{N \setminus S\} : |S| \leq \kappa} U(F(S, v')).$$

Together with Lemma 6, we obtain that $U(F(S^1, v') \geq U(F(S^*, v')) \geq U(F(S^*, v), \nu)$, which concludes the proof of this lemma. \hfill \Box

**Lemma 7.** Given any $\ell > 0$ and $C_1, C_2 > 0$, we define event

$$E_\ell = \left\{ \forall i \in [N], v_i \leq \tilde{v}_i \leq v_i + C_1 \sqrt{\frac{v_i \ln(\sqrt{N} \ell + 1)}{T_\ell(\ell) \vee 1}} \right. + C_2 \frac{\ln(\sqrt{N} \ell + 1)}{T_\ell(\ell) \vee 1} \left\}.$$  

There exist real numbers $C_1, C_2 > 0$ such that

$$\mathbb{P}(E_\ell) \geq 1 - \frac{1}{\ell}$$

for any $\ell$.

Lemma 7 can be easily derived from Lemma 4.1 of (Agrawal et al. 2019). So we omit its proof here.

**Proof of Theorem 4.** Before proceeding, we introduce several notations. Let $L$ be the total number of episodes when RiskAwareUCB stops after $T$ steps. Denote $\ell$ by the length of the $\ell$th episode. Moreover, set $n_i = T_i(L)$, which is the total number of episodes product $i$ is served before the $L$th episode.

Using the law of total expectation, we rewrite the regret as

$$\mathfrak{R}_T = \mathbb{E} \left[ \sum_{\ell=1}^{L} \left( U(F(S^*, v)) - U(F(S_\ell, v)) \right) \right]$$

$$= \mathbb{E} \left[ \sum_{\ell=1}^{L} \mathbb{E}[l_\ell(U(F(S^*, v)) - U(F(S_\ell, v))) \mid H_\ell] \right],$$

where $H_\ell$ is the history before episode $\ell$. Since $S_\ell$ is determined by $H_\ell$, there is

$$\mathfrak{R}_T = \mathbb{E} \left[ \sum_{\ell=1}^{L} \mathbb{E}[l_\ell \mid H_\ell](U(F(S^*, v)) - U(F(S_\ell, v))) \right].$$

Given $S_\ell$, we know that $l_\ell$ follows a geometric distribution with parameter $1/(1 + \sum_{i \in S_\ell} v_i)$. Hence we have $\mathbb{E}[l_\ell \mid H_\ell] \leq 1 + \sum_{i \in S_\ell} v_i$. We put inequality here since the last episode may end due to time limit. Using aforementioned inequality, we further derive

$$\mathfrak{R}_T \leq \mathbb{E} \left[ \sum_{\ell=1}^{L} \mathbb{E} \left[ \left( 1 + \sum_{i \in S_\ell} v_i \right) \times (U(F(S^*, v)) - U(F(S_\ell, v))) \right] \right]$$

$$= \mathbb{E} \left[ \sum_{\ell=1}^{L} \mathbb{E} \delta_\ell \right],$$

where we have defined $\delta_\ell = (1 + \sum_{i \in S_\ell} v_i) \times (U(F(S^*, v)) - U(F(S_\ell, v)))$. 


We now focus on bounding \( \mathbb{E} \delta \). By a simple calculation, we get
\[
\mathbb{E} \delta = \mathbb{E} [\delta_t 1_{\mathcal{E}_t}] + \mathbb{E} [\delta_t 1_{\mathcal{E}_t}^c] \\
\leq 2 \gamma_1 (N + 1) P(\mathcal{E}_t^c) + \mathbb{E} [\delta_t | \mathcal{E}_t] P(\mathcal{E}_t) \\
\leq \frac{2 \gamma_1 (N + 1)}{\ell} + \mathbb{E} [\delta_t | \mathcal{E}_t] P(\mathcal{E}_t),
\]
where in the second last inequality, we upper bound \( \delta_t \) using \( v_i \leq 1 \) and Assumption 2, and the last inequality is due to Lemma 7. By Lemma 5 and Assumption 3, we get
\[
\mathbb{E} [\delta_t | \mathcal{E}_t] \\
\leq \mathbb{E} \left[ (1 + \sum_{i \in S_t} v_i) (U(F(S_t, \bar{\theta}^t) - U(F(S_t, \nu))) | \mathcal{E}_t \right] \\
\leq \mathbb{E} \left[ \gamma_2 \sum_{i \in S_t} \left( C_1 \frac{v_i \ln(\sqrt{N} \ell + 1)}{T_i(\ell) \vee 1} \\
+ C_2 \frac{\ln(\sqrt{N} \ell + 1)}{T_i(\ell) \vee 1} \right) \right],
\]
where in the last equality we have used the definition of event \( \mathcal{E}_t \). By (10) and (11), we have
\[
\mathbb{E} \delta_t \leq \frac{2 \gamma_1 (N + 1)}{\ell} + \mathbb{E} \left[ \gamma_2 \sum_{i \in S_t} \left( C_1 \frac{v_i \ln(\sqrt{N} \ell + 1)}{T_i(\ell) \vee 1} \\
+ C_2 \frac{\ln(\sqrt{N} \ell + 1)}{T_i(\ell) \vee 1} \right) \right].
\]
Putting (12) back into (9), we derive
\[
\mathcal{R}_T \leq 2 \gamma_1 (N + 1) \mathbb{E} \left[ \sum_{t=1}^{L} \frac{1}{\ell} \right] \\
+ \gamma_2 C_1 \frac{\ln(\sqrt{N} T + 1)}{\mathbb{E} \left[ \sum_{t=1}^{L} \sum_{i \in S_t} \frac{v_i}{T_i(\ell) \vee 1} \right]} \\
+ \gamma_2 C_2 \frac{\ln(\sqrt{N} T + 1)}{\mathbb{E} \left[ \sum_{t=1}^{L} \sum_{i \in S_t} \frac{1}{T_i(\ell) \vee 1} \right]}.
\]
and the proof is complete.

\[\mathcal{R}_T \leq 2 \gamma_1 (N + 1) (\ln T + \gamma) \]
\[+ \gamma_2 C_1 \ln(\sqrt{N} T + 1) (2\sqrt{NT} + N) \]
\[+ \gamma_2 C_2 \ln(\sqrt{N} T + 1) (N \ln T + N (1 + \gamma)) \]
and the proof is complete.

\[\mathcal{R}_T \leq 2 \gamma_1 (N + 1) (\ln T + \gamma) \]
\[+ \gamma_2 C_1 \frac{\ln(\sqrt{N} T + 1)}{\mathbb{E} \left[ \sum_{t=1}^{L} \sum_{i \in S_t} \frac{v_i}{T_i(\ell) \vee 1} \right]} \\
+ \gamma_2 C_2 \frac{\ln(\sqrt{N} T + 1)}{\mathbb{E} \left[ \sum_{t=1}^{L} \sum_{i \in S_t} \frac{1}{T_i(\ell) \vee 1} \right]}.
\]

### Examples of Risk Criteria

In this section, we show that conditional value-at-risk, Sharpe ratio, and entropy risk all satisfy Assumption 2 and Assumption 3. For the proof of the other risk criteria listed in Table 1, we refer to Appendix A.

For proving the one-sided Lipschitz condition, the following lemma is useful. The proof of Lemma 8 is in Appendix A.

**Lemma 8.** For any \( v' \geq v, \text{i.e., } v'_i \geq v_i \text{ for all } i \in [N], \) and \( S \subset [N], \) it holds that
\[
\sum_{i \in S} \left| \frac{v'_i}{1 + \sum_{i \in S} v'_i} - \frac{v_i}{1 + \sum_{i \in S} v_i} \right| \leq \frac{2}{1 + \sum_{i \in S} v_i} \left[ \sum_{i \in S} (v'_i - v_i) \right].
\]
Conditional Value-at-risk

Given $\alpha \in (0, 1]$, the conditional value-at-risk at $\alpha$ percentile for $F \in \mathcal{D}[0, 1]$ is defined as

$$CVaR_\alpha(F) \triangleq \frac{1}{\alpha} \int_0^\alpha \text{Var}_\beta(F) d\beta.$$ 

An equivalent definition is

$$CVaR_\alpha(F) = \frac{1}{\alpha} \left( \alpha - \int_0^1 (F(x) \land \alpha) dx \right).$$

**Proposition 9.** $CVaR_\alpha$ satisfies Assumption 2 and Assumption 3 with $\gamma_1 = 1$ and $\gamma_2 = 3/\alpha$.

**Proof.** It is easy to see that $|CVaR_\alpha(F(S, v))| \leq 1$, which implies $\gamma_1 = 1$.

We now show the value of $\gamma_2$. W.o.l.g., we can assume that the profit of different products are different since for those items with the same revenue, we can combine them into one product and the corresponding abstraction parameter is the sum of those of the arms. Given assortment $S$, we denote the products in $S$ by $|S|$ in the increasing order of their profit. Then for any $k + 1 \in |S|$ and $x \in [r_k, r_{k+1})$,

$$F(S, v'; x) - F(S, v; x) = \frac{1}{1 + \sum_{i=1}^{\vert S \vert} v_i'} - \frac{1}{1 + \sum_{i=1}^{\vert S \vert} v_i}$$

Note that for any $x \in [r_k, r_{k+1})$, by Lemma 8

$$\left| F(S, v'; x) - F(S, v; x) \right| \leq \left| \frac{1}{1 + \sum_{i=1}^{\vert S \vert} v_i'} - \frac{1}{1 + \sum_{i=1}^{\vert S \vert} v_i} \right|$$

$$+ \sum_{i=1}^{\vert S \vert} \frac{v_i'}{1 + \sum_{i=1}^{\vert S \vert} v_i'} - \frac{v_i}{1 + \sum_{i=1}^{\vert S \vert} v_i}$$

$$\leq \frac{3}{1 + \sum_{i=1}^{\vert S \vert} v_i} \left[ \sum_{i\in S} (v_i' - v_i) \right].$$

Clearly the difference between $F(S, v'; x) \land \alpha$ and $F(S, v; x) \land \alpha$ satisfies the same bound. Hence

$$|CVaR_\alpha(F(S, v')) - CVaR_\alpha(F(S, v))|$$

$$\leq \frac{1}{\alpha} \left( \int_0^1 |F_X(x) \land \alpha - F_Y(x) \land \alpha| dx \right)$$

$$\leq \frac{3/\alpha}{1 + \sum_{i=1}^{\vert S \vert} v_i} \left[ \sum_{i=1}^{\vert S \vert} (v_i' - v_i) \right].$$

Sharpe Ratio

Given a minimum average reward $r \in [0, 1]$ and the regularization factor $\epsilon$, for $F \in \mathcal{D}[0, 1]$ we define

$$Sh_{r, \epsilon}(F) = \frac{U_1(F) - r}{\sqrt{\epsilon + \sigma^2(F)}},$$

where $U_1(F)$ is the mean and $\sigma^2(F)$ is the variance.

**Proposition 10.** $Sh_{r, \epsilon}$ satisfies Assumption 2 and Assumption 3 with $\gamma_1 = \frac{1}{\sqrt{\epsilon}}$ and $\gamma_2 = 2e^{-1/2} + 3e^{-3/2}$.

**Proof.** Since $\sigma^2(F) > 0$, $U_1(F) \in [0, 1]$ and $r \in [0, 1]$, it is easy to see that $\gamma_1 = \frac{1}{\sqrt{\epsilon}}$. For the value of $\gamma_2$, by the Lipschitz property of the mean and variance (see Appendix A), we have

$$|Sh_{r, \epsilon}(F(S, v')) - Sh_{r, \epsilon}(F(S, v))|$$

$$\leq \frac{U_1(F(S, v')) - U_1(F(S, v))}{\sqrt{\epsilon + \sigma^2(F(S, v'))}} + \frac{U_1(F(S, v')) - U_1(F(S, v))}{\sqrt{\epsilon + \sigma^2(F(S, v))}}$$

$$\leq \frac{1}{\sqrt{\epsilon}} |U_1(F(S, v')) - U_1(F(S, v))|$$

$$= \frac{1}{2\epsilon^2} \left| \sum_{i=1}^{\vert S \vert} (v_i' - v_i) \right|.$$
By the convexity of the log function and Lemma 8, we have that
\[
|U^{ent}(F(S, v')) - U^{ent}(F(S, v))| \\
= \frac{1}{\theta} \left| \ln \left( \sum_{i \in S} e^{-\theta r_i v'_i} \right) - \ln \left( \sum_{i \in S} e^{-\theta r_i v_i} \right) \right| \\
\leq \frac{e^\theta}{\theta} \left| \sum_{i \in S} e^{-\theta r_i v'_i} \left( 1 + \sum_{i \in S} v_i \right) - \sum_{i \in S} e^{-\theta r_i v_i} \left( 1 + \sum_{i \in S} v_i \right) \right| \\
\leq \frac{2e^\theta / \theta}{1 + \sum_{i \in S} v_i} \left| \sum_{i \in S} (v'_i - v_i) \right| .
\]

\[\square\]

**Experimental Evaluation**

We evaluate RiskAwareUCB and RiskAwareTS in both synthetic and real data. Please refer to the supplementary for the source code.

**Synthetic Data** In this experiment, we fix the number of products \( N = 10 \), cardinality limit \( K = 4 \), horizon \( T = 10^6 \), and set the goal to be \( U = \text{CVaR}_{0.05} \).

We generate 10 uniformly distributed random input instances where \( v_i \in [0, 1] \) and \( r_i \in [0, 1] \). For each input instance, we run 20 repetitions and compute their average as the regret. Figure 1 shows how the worst regret among all input instances changes with square root of time. From the figure, we can see that both algorithms suffer a \( \sqrt{t} \)-rate regret. Moreover, RiskAwareTS performs better than RiskAwareUCB, which aligns with literature that Thompson Sampling performs better in practice.

**Real Data** In this experiment, we consider the “UCI Car Evaluation Database” dataset from the Machine Learning Repository (Dua and Graff 2017) which contains 6 categorical attributes for \( N = 1728 \) cars and consumer ratings for each car. We fix cardinality limit \( K = 100 \), horizon \( T = 10^6 \), and set the goal to be \( U = \text{CVaR}_{0.05} \).

By transforming each attribute to a one-hot vector, we obtain an attribute vector \( m_i \in \{0, 1\}^{21} \) for each car. There are four different values for customer ratings i.e., “acceptable”, “good”, “very good”, and “unacceptable”. We decode “unacceptable” by 0 and others by 1 to represent whether the customer has intention to buy the car. We use logistic regression to predict whether the customer is likely to buy the car and the probability that the customer buys car \( i \) is modeled by
\[
\frac{1}{1 + \exp(-\theta^T m_i)} ,
\]

where \( \theta \in \mathbb{R}^{21} \) is an unknown parameter. After the model is fit with \( L_2 \) regularization, we set the preference parameter \( v_i \) of car \( i \) to be the same as the probability predicted by logistic regression. Since there is no profit data available for cars in this dataset, we generate uniformly distributed profit \( r_i \) from \([0, 1]\) for each car.

We run the experiment for 40 repetitions and compute the average \( \text{CVaR}_{0.05} \) for every consecutive 1000 revealed profits. To save time, when computing the assortment with the best \( \text{CVaR}_{0.05} \), we do local search, i.e., try to replace a car, add a car or delete a car, and stops if we can not find a better assortment. In Figure 2, we report the performance of RiskAwareUCB and RiskAwareTS against UCB and TS where the last two algorithms are set to maximize the expected revenue. From the experiment, we can see that the obtained \( \text{CVaR}_{0.05} \) under UCB and TS are far from optimal.

**Conclusion**

In this work, we have shown the near-optimal algorithms for a general class of risk criteria, which only need to satisfy three mild assumptions. Experiments with both synthetic and real data are conducted to validate our results and show that the ordinary algorithms suffer a worse performance when the goal is changed.
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Appendix A: Lipschitz

In this section, we prove that the \( n \)th-moments, below target semi-variance, negative variance, mean-variance and Sortino ratio all satisfy Assumption 2 and 3. Firstly, we give the proof to Lemma 8 which is useful for verifying Assumption 3.

**Proof of Lemma 8.** Given \( \nu' \geq \nu \), define the subset

\[
S_0 = \left\{ i \in S : \frac{\nu'_i}{1 + \sum_{i \in S} \nu'_i} \geq \frac{\nu_i}{1 + \sum_{i \in S} \nu_i} \right\}.
\]

Then

\[
\sum_{i \in S} \left| \frac{\nu'_i}{1 + \sum_{i \in S} \nu'_i} - \frac{\nu_i}{1 + \sum_{i \in S} \nu_i} \right| = \sum_{i \in S/S_0} \left| \frac{\nu'_i}{1 + \sum_{i \in S} \nu'_i} - \frac{\nu_i}{1 + \sum_{i \in S} \nu_i} \right| + \left| \frac{1}{1 + \sum_{i \in S} \nu'_i} - \frac{1}{1 + \sum_{i \in S} \nu_i} \right|.
\]

Therefore

\[
\sum_{i \in S} \left| \frac{\nu'_i}{1 + \sum_{i \in S} \nu'_i} - \frac{\nu_i}{1 + \sum_{i \in S} \nu_i} \right| \leq 2 \sum_{i \in S/S_0} \left| \frac{\nu'_i}{1 + \sum_{i \in S} \nu'_i} - \frac{\nu_i}{1 + \sum_{i \in S} \nu_i} \right| \leq 2 \sum_{i \in S/S_0} \left( \frac{\nu'_i}{1 + \sum_{i \in S} \nu'_i} - \frac{\nu_i}{1 + \sum_{i \in S} \nu_i} \right) \leq \frac{2}{1 + \sum_{i \in S} \nu_i} \sum_{i \in S} (\nu'_i - \nu_i).
\]

\[\Box\]

**Remark 12.** For any \( \nu' \geq \nu \), from the proof above, we can see that the following stronger result is true

\[
\sum_{i \in S} \left| \frac{\nu'_i}{1 + \sum_{i \in S} \nu'_i} - \frac{\nu_i}{1 + \sum_{i \in S} \nu_i} \right| \leq \frac{2}{1 + \sum_{i \in S} \nu_i} \sum_{i \in S} (\nu'_i - \nu_i).
\]

**Value-at-risk**

Given \( \alpha \in (0, 1]\), value-at-risk at \( \alpha \) percentile for \( F \in \mathcal{D}[0, 1]\) is defined as

\[
\text{VaR}_\alpha(F) \overset{\text{def}}{=} \inf \{ x : F(x) \geq \alpha \}.
\]

It is easy to see that \( |\text{VaR}_\alpha(F(S, \nu))| \leq 1 \) and hence \( \gamma_1 = 1 \). However, \( \text{VaR}_\alpha(F(S, \nu)) \) is not continuous on \( \nu \), and \( \gamma_2 \) does not exist.

**nth-moment**

For any \( n \in \mathbb{N} \), we can define the \( n \)th-moment about zero for \( F \in \mathcal{D}[0, 1]\)

\[
U^n(F) = \int_0^1 x^n dF(x).
\]

**Proposition 13.** The \( n \)th-moment satisfies Assumption 2 and Assumption 3 with \( \gamma_1 = 1 \) and \( \gamma_2 = 1 \).

**Proof.** It is immediate to check that \( |U^n(F)| \leq 1 \). For the value of \( \gamma_2 \), notice that

\[
U^n(F(S, \nu')) - U^n(F(S, \nu)) = \sum_{i \in S} r^n \left[ \frac{\nu'_i}{1 + \sum_{i \in S} \nu'_i} - \frac{\nu_i}{1 + \sum_{i \in S} \nu_i} \right] \leq \sum_{i \in S} r^n \left[ \frac{\nu'_i}{1 + \sum_{i \in S} \nu'_i} - \frac{\nu_i}{1 + \sum_{i \in S} \nu_i} \right] \leq \frac{1}{1 + \sum_{i \in S} \nu_i} \sum_{i \in S} (\nu'_i - \nu_i).
\]
Moreover, using Lemma 8, we have that

\[
|U^n(F(S, u')) - U^n(F(S, v))| \leq \frac{2}{1 + \sum_{i \in S} v_i} \left[ \sum_{i \in S} (v'_i - v_i) \right].
\]  

(16)

**Remark 14.** It is easy to see that the same argument applies to a large class of risk criteria of the form

\[
U(F) = \int_{-\infty}^{\infty} h(x) dF(x).
\]

**Below target semi-variance**

Given a target \( r \in [0, 1] \), we can define the negative below target semi-variance for any \( F \in D[0, 1] \)

\[
-\text{TSV}_r(F) = -\int_0^1 (x - r)^2 1_{\{x \leq r\}} dF(x).
\]

**Proposition 15.** The negative below target semi-variance satisfies Assumption 2 and Assumption 3 with \( \gamma_1 = r^2 \) and \( \gamma_2 = 2r^2 \).

**Proof.** By the definition of \(-\text{TSV}_r(F)\), it is easy to see that it is bounded by \( r^2 \). Then by Lemma 8, there is

\[
\begin{align*}
| - \text{TSV}_r(F(S, u')) + \text{TSV}_r(F(S, v)) | & = \left| \sum_{i \in S, r_i \leq r} (r_i - r)^2 \left( -\frac{v'_i}{1 + \sum_{i \in S} v'_i} + \frac{v_i}{1 + \sum_{i \in S} v_i} \right) \right| \\
& \leq r^2 \sum_{i \in S} \left| \frac{v'_i}{1 + \sum_{i \in S} v'_i} - \frac{v_i}{1 + \sum_{i \in S} v_i} \right| \\
& \leq \frac{2r^2}{1 + \sum_{i \in S} v_i} \left[ \sum_{i \in S} (v'_i - v_i) \right].
\end{align*}
\]

**Negative variance**

For any \( F \in D[0, 1] \), the negative variance is

\[-\sigma^2(F) = -[U^2(F) - (U^1(F))^2].
\]

**Proposition 16.** The negative variance satisfies Assumption 2 and Assumption 3 with \( \gamma_1 = \frac{1}{4} \) and \( \gamma_2 = 6 \).

**Proof.** It is well known that \(|\sigma^2(F)| \leq \frac{1}{4} \) for any random variable taking values in \([0, 1]\). For the value of \( \gamma_2 \), there is

\[
| - \sigma^2(F(S, u')) + \sigma^2(F(S, v)) | \leq |U^2(F(S, u')) - U^2(F(S, v))| + |(U^1(F(S, u'))) - (U^1(F(S, v)))|^2.
\]

By (16) and

\[
| (U^1(F(S, u')))^2 - (U^1(F(S, v)))^2 | \leq |U^1(F(S, u')) - U^1(F(S, v))||U^1(F(S, v')) + U^1(F(S, v))| \\
\leq 2|U^1(F(S, v')) - U^1(F(S, v))|,
\]

we have that

\[
| - \sigma^2(F(S, u')) + \sigma^2(F(S, v)) | \leq \frac{6}{1 + \sum_{i \in S} v_i} \left[ \sum_{i \in S} (v'_i - v_i) \right].
\]

**Remark 17.** Simply consider the negative variance alone does not provide a good risk criterion to our problem here. But we still discuss it here because it is an important building block for other risk criteria including the Sharpe ratio, Sortino ratio and mean-variance.
Mean-variance
Given a weight $\rho > 0$, we define the mean-variance for $F \in D[0, 1]$ as
\[
U^\text{MV}_\rho(F) = U^1(F) - \rho \sigma^2(F).
\]
By the boundedness and the Lipschitz property of the mean and variance, it is immediate to see that $|U^\text{MV}_\rho(F)| \leq 1 + \frac{\rho}{4}$ and
\[
|U^\text{MV}_\rho(F(S, u')) - U^\text{MV}_\rho(F(S, v))| \leq \frac{2 + 6\rho}{1 + \sum_{i \in S} v_i} \left[ \sum_{i \in S} (v'_i - v_i) \right].
\]

Sortino ratio
Given a minimum average reward $r \in [0, 1]$ and the regularization factor $\epsilon$, for $F \in D[0, 1]$ we define
\[
So_{r, \epsilon}(F) = \frac{U^1(F) - r}{\sqrt{\epsilon + TSV_\epsilon(F)}}.
\]

Proposition 18. $So_{r, \epsilon}(F)$ satisfies Assumption 2 and Assumption 3 with $\gamma_1 = \frac{1}{\sqrt{\epsilon}}$ and $\gamma_2 = 2e^{-1/2} + e^{-3/2}$.

Proof. Following similar argument for the Sharpe ratio, using the boundedness and the Lipschitz property of the mean and the below target semi-variance, we have that $So_{r, \epsilon}(F) \leq \frac{1}{\sqrt{\epsilon}}$ and
\[
|So_{r, \epsilon}(F(S, u')) - So_{r, \epsilon}(F(S, v))| \leq \frac{2e^{-1/2} + e^{-3/2}}{1 + \sum_{i \in S} v_i} \left[ \sum_{i \in S} (v'_i - v_i) \right].
\]

Appendix B: Thompson Sampling
We prove the guarantee of RiskAwareTS in this section.

Before proving the theoretical guarantee, we first present the details of the RiskAwareTS algorithm. At the beginning of the algorithm, there is a warm start stage when every product is repeatedly served until a no-purchase outcome is observed. Similar to RiskAwareUCB, the remaining time steps is divided into small episodes. During each episode $\ell$, the same assortment $S_\ell$ is repeatedly provided to the online user until a no-purchase outcome is observed. Specifically, in each episode $\ell$, we are providing the assortment
\[
\arg\max_{S \subset [N], |S| \leq K} U(F(S, \bar{v}^\ell)),
\]
where $\bar{v}^\ell$ is the virtual preference parameters generated by correlated sampling before the start of episode $\ell$.

Let $t_{i, \ell}$ be the number of times the online users buy product $i$ in the $\ell$th episode and $T_{i, \ell}$ be the collection of episodes for which product $i$ is served until episode $\ell$ (exclusive). Define $T_{i, \ell} \overset{\text{def}}{=} |T_i(\ell)|$, $n_{i, \ell} \overset{\text{def}}{=} \sum_{\ell' \in T_{i, \ell}} t_{i, \ell'}$ and
\[
\bar{v}^\ell_i \overset{\text{def}}{=} \frac{n_{i, \ell}}{T_{i, \ell}},
\]
which is an unbiased estimator of the unknown preference parameter $v_i$. Generate $K$ i.i.d. samples $\{\theta_{i, \ell}^{(j)}\}_{j=1}^K$ from $\mathcal{N}(0, 1)$ and define
\[
\mu_{i, \ell}^{(j)}(\ell) \overset{\text{def}}{=} \bar{v}^\ell_i + \theta_{i, \ell}^{(j)} \hat{\sigma}_{i, \ell},
\]
where
\[
\hat{\sigma}_{i, \ell} = \sqrt{\frac{50\bar{v}^\ell_i (\bar{v}^\ell_i + 1)}{T_{i, \ell}}} + 75 \sqrt{\log(TK)}
\]
\[
\frac{T_{i, \ell}}{T_{i, \ell}}.
\]
The $i$th component of the virtual preference parameters $\bar{v}^\ell$ is given by $\bar{v}^\ell_i = \max_{1 \leq j \leq K} \mu_{i, \ell}^{(j)}(\ell)$.

To prove the guarantee, we will need a stronger version of Assumption 3.

Assumption 19 (One-sided Lipschitz Condition). For any $v' \geq v$, i.e., $v'_i \geq v_i$ for all $i \in [N]$, and $S \subset [N]$, it holds that
\[
U(F(S, v')) - U(F(S, v)) \leq \gamma_2 \frac{\sum_{i \in S} (v'_i - v_i)}{1 + \sum_{i \in S} v'_i} \left[ \sum_{i \in S} (v'_i - v_i) \right].
\]
Lemma 5, we know that \( v \) optimal assortment for maker using

Suppose the risk criterion (Agrawal et al. 2017, Theorem 1) and we include it here for the completeness of the paper.

By the definition of \( u \)

Proof. Let \( u = \max \{ v', v \} \), i.e., \( u_i = \max \{ v_i', v_i \} \) for all \( i \in [N] \). Since \( u \geq v \), by Assumption 1 and the proof of Lemma 5, we know that

\[
U(F(S^*, v)) \leq U(F(S^*, u)).
\]

Therefore, using Assumption 19

\[
U(F(S^*, v)) - U(F(S^*, v')) \leq U(F(S^*, u)) - U(F(S^*, v')) \leq \frac{\tilde{\gamma}_2}{1 + \sum_{i \in S} u_i} \left[ \sum_{i \in S} (u_i - v_i) \right].
\]

By the definition of \( u \), one can conclude the proof.

Now we prove the regret upper bound for RiskAwareTS. The proof below is a mild modification of the proof to (Agrawal et al. 2017, Theorem 1) and we include it here for the completeness of the paper.

Theorem 22. Suppose the risk criterion \( U \) satisfies Assumption 1, 2 and 3. The regret (5) incurred by the decision maker using RiskAwareTS is upper bounded by \( \tilde{O}(\sqrt{NT}) \) after \( T \) time steps, where \( \tilde{O} \) hides poly-logarithmic factors in \( N \) and \( T \).

Proof. For completeness, we first introduce some notations, which are already defined in (Agrawal et al. 2017, Appendix D).

Given assortment \( S \), let \( V(S) = \sum_{i \in S} v_i \). Given \( \ell, \tau \leq L \), define

\[
\Delta R_\ell = (1 + V(S_\ell))|U(F(S_\ell, \bar{v}^\ell)) - U(F(S_\ell, v))|,
\]

\[
\Delta R_{\ell, \tau} = (1 + V(S_\tau))|U(F(S_\tau, \bar{v}^\tau)) - U(F(S_\tau, \bar{v}^\tau))|.
\]

Next, we denote by \( \mathcal{A}_0 \) the probability space \( \Omega \) and

\[
\mathcal{A}_\ell = \left\{ |\bar{v}_i(\ell) - v_i| \geq \frac{24 r_i \log(\ell + 1)}{T_i(\ell)} + \frac{48 \log(\ell + 1)}{T_i(\ell)}, \text{ for some } i = 1, \ldots, N \right\}.
\]

Next we define \( \mathcal{T} = \{ \ell : \bar{v}_i^\ell \geq v_i \text{ for all } i \in S^* \cup S_\ell \} \), which indicates the “optimistic” episodes. Then let \( \text{succ}(\ell) = \min \{ \ell \in \mathcal{T} : \ell > \ell \} \), which is the next optimistic episode after episode \( \ell \). Finally, we define \( \mathcal{E}^A(n) = \{ \tau : \tau \in (\ell, \text{succ}(\ell)) \} \) for all \( \ell \in \mathcal{T} \), which is the collection of “non-optimistic” episodes between two adjacent optimistic episodes.

Now we consider the regret

\[
\mathcal{R}_T = \mathbb{E} \left[ \sum_{\ell=1}^{L} l_\ell (U(F(S^*, v)) - U(F(S_\ell, v))) \right]
\]

\[
= \mathbb{E} \left[ \sum_{\ell=1}^{L} l_\ell (U(F(S^*, v)) - U(F(S_\ell, \bar{v}^\ell))) \right] + \mathbb{E} \left[ \sum_{\ell=1}^{L} l_\ell (U(F(S_\ell, \bar{v}^\ell)) - U(F(S_\ell, v))) \right]
\]

\[
\overset{\text{def}}{=} \mathcal{R}_T^1 + \mathcal{R}_T^2. \tag{18}
\]

Next we will show the upper bounds of \( \mathcal{R}_T^1 \) and \( \mathcal{R}_T^2 \) respectively.

Bounding \( \mathcal{R}_T^2 \): By taking conditional probability with respect to the history \( \mathcal{H}_\ell \), following the proof of (9), we have

\[
\mathcal{R}_T^2 = \mathbb{E} \left[ \sum_{\ell=1}^{L} \Delta R_\ell \right]. \tag{19}
\]
Next we bound $\Delta R_\ell$ in two scenarios

\[
E[\Delta R_\ell] = E[\Delta R_\ell 1_{A_{\ell-1}^c}] + E[\Delta R_\ell 1_{A_{\ell-1}^c}]
\]

\[
\leq 2\gamma_2(K + 1)P(A_{\ell-1}) + E[\Delta R_\ell 1_{A_{\ell-1}^c}] \leq \frac{2\gamma_2(K + 1)}{\ell^2} + E[\Delta R_\ell 1_{A_{\ell-1}^c}],
\]

(20)

where the first inequality follows from Assumption 2 and $V(S) \leq K$, and the second inequality follows from (Agrawal et al. 2017, Lemma 7), i.e., $P(A_{\ell-1}) \leq \frac{1}{\ell^2}$. By Proposition 21, we have

\[
\Delta R_\ell \leq (1 + V(S_\ell)) \left[ -\frac{\gamma_2}{1 + \sum_{i \in S_\ell} \max(\bar{v}_i^\ell, v_i)\sum_{i \in S_\ell} |\bar{v}_i^\ell - v_i|} \leq \gamma_2 \sum_{i \in S_\ell} |\bar{v}_i^\ell - v_i| \right].
\]

Therefore,

\[
E[\Delta R_\ell 1_{A_{\ell-1}^c}] \leq \mathbb{E}\left[ \frac{\gamma_2}{1 + \sum_{i \in S_\ell} \max(\bar{v}_i^\ell, v_i)\sum_{i \in S_\ell} |\bar{v}_i^\ell - v_i|} \right]
\]

\[
\leq \mathbb{E}\left[ \gamma_2 \sum_{i \in S_\ell} |\bar{v}_i^\ell - v_i(\ell)| 1_{A_{\ell-1}^c} \right] + \mathbb{E}\left[ \gamma_2 \sum_{i \in S_\ell} |\bar{v}_i(\ell) - v_i| 1_{A_{\ell-1}^c} \right]
\]

\[
\leq \gamma_2 \mathbb{E}\left[ \sum_{i \in S_\ell} |\bar{v}_i^\ell - v_i(\ell)| \right] + \gamma_2 \mathbb{E}\left[ \sum_{i \in S_\ell} \left\{ \sqrt{v_i(\ell)} + \frac{48\log(\ell + 1)}{T_i(\ell)} + \frac{24v_i\log(\ell + 1)}{T_i(\ell)} \right\} \right],
\]

where the last inequality follows from the definition of $A_{\ell-1}$. By the definition of $\bar{v}_i^\ell$ and $v_i(\ell)$, there is

\[
\mathbb{E}\left[ \sum_{i \in S_\ell} |\bar{v}_i^\ell - v_i(\ell)| \right] = \mathbb{E}\left[ \sum_{i \in S_\ell} \max_{j = 1, \ldots, K} \{ \theta_i^{(j)} \} \right] = \mathbb{E}\left[ \sum_{i \in S_\ell} \max_{j = 1, \ldots, K} \{ \theta_i^{(j)} \} \right].
\]

where $\theta_i^{(j)}$ are independent standard normal distributed random variables given $\ell$. By Theorem 1 in (Kamath 2015), it is easy to verify that

\[
\mathbb{E}\left[ \max_{j = 1, \ldots, K} \{ \theta_i^{(j)} \} \right] \leq 4\sqrt{\log K}.
\]

Since $\ell \leq T$, by the definition of $\hat{\sigma}_i(\ell)$, we have

\[
E[\Delta R_\ell 1_{A_{\ell-1}^c}] \leq \mathbb{E}\left[ \sum_{i \in S_\ell} \frac{v_i(\ell) \log(TK)}{T_i(\ell)} \right] + \mathbb{E}\left[ \sum_{i \in S_\ell} \frac{\log(TK)}{T_i(\ell)} \right].
\]

(21)

Combining (19), (20) and (21) gives

\[
\mathcal{R}_T^2 \leq KE \left[ \sum_{\ell = 1}^L \frac{1}{\ell^2} \right] + (TK)\mathbb{E}\left[ \sum_{\ell = 1}^L \sum_{i \in S_\ell} \sqrt{v_i(\ell)} \right] + \log(TK)\mathbb{E}\left[ \sum_{\ell = 1}^L \sum_{i \in S_\ell} \frac{1}{T_i(\ell)} \right].
\]

(22)

Finally we apply the argument for $\ast$, $\ast \ast$ and $\ast \ast \ast$ in the proof of Theorem 4 to (22) and we obtain $\mathcal{R}_T^2 = \tilde{O}(\sqrt{NT})$.

Bounding $\mathcal{R}_T^1$: It remains to show that $\mathcal{R}_T^1 \leq \tilde{O}(\sqrt{NT})$. Notice that

\[
\mathcal{R}_T^1 = \mathbb{E}\left[ \sum_{\ell = 1}^L 1_{\ell \in T} \sum_{\tau \in \mathcal{E}_{\Lambda_n(\ell)}} l_\tau[U(F(S_\ell, v)) - U(F(S_\ell, \tilde{v}_\tau))] \right]
\]

\[
\leq \mathbb{E}\left[ \sum_{\ell = 1}^L 1_{\ell \in T} \sum_{\tau \in \mathcal{E}_{\Lambda_n(\ell)}} l_\tau[U(F(S_\ell, v)) - U(F(S_\ell, \tilde{v}_\tau))] \right]
\]

\[
\leq \mathbb{E}\left[ \sum_{\ell = 1}^L 1_{\ell \in T} \sum_{\tau \in \mathcal{E}_{\Lambda_n(\ell)}} \Delta R_{\ell, \tau} \right],
\]

(23)
where the first inequality follows from Lemma 5 and the second inequality follows from the optimality of $S_\tau$ under parameter $\tilde{v}^\tau$. Similar to (20), we bound $\Delta R_{\ell,\tau}$ in two scenarios

$$\begin{align*}
E \left[ \sum_{\tau \in E^{An}(\ell)} \Delta R_{\ell,\tau} \right] &= E \left[ \sum_{\tau \in E^{An}(\ell)} \Delta R_{\ell,\tau} \mathbbm{1}_{A_{\ell - 1}} + \Delta R_{\ell,\tau} \mathbbm{1}_{A_{\ell - 1}^c} \right] \\
&\leq 2\gamma_1 (K + 1) E [|E^{An}(\ell)|] + E \left[ \sum_{\tau \in E^{An}(\ell)} \Delta R_{\ell,\tau} \mathbbm{1}_{A_{\ell - 1}} \right].
\end{align*}$$

(24)

By (Agrawal et al. 2017, Lemma 5), i.e.,

$$[E (|E^{An}(\ell)|^2)]^{1/2} \leq \frac{e^{12}}{K} + \sqrt{30},$$

(25)

and (Agrawal et al. 2017, Lemma 7), i.e., $P(A_{\ell - 1}) \leq \frac{1}{12}$, we have

$$(K + 1) E [|E^{An}(\ell)|] \leq (K + 1) \left[ E \left[ (|E^{An}(\ell)|^2) \right]^{1/2} \left[ P(A_{\ell - 1}) \right]^{1/2} \right] \leq \frac{(K + 1)}{\ell}.$$

(26)

For the second term in (24), by Proposition 21

$$\begin{align*}
\Delta R_{\ell,\tau} &\leq (1 + V(S_\ell)) \frac{\tilde{\gamma}_2}{1 + \sum_{i \in S_\ell} \max\{\tilde{v}^\ell_i, \tilde{v}^\ell_i \}} \sum_{i \in S_\ell} |\tilde{v}^\ell_i - \tilde{v}^\ell_i| \\
&\leq \frac{\tilde{\gamma}_2 (K + 1)}{1 + \sum_{i \in S_\ell} \tilde{v}^\ell_i} \sum_{i \in S_\ell} |\tilde{v}^\ell_i - \tilde{v}^\ell_i| \\
&\leq \frac{\tilde{\gamma}_2 (K + 1)}{1 + V(S_\ell)} \sum_{i \in S_\ell} (|\tilde{v}^\ell_i - v_i| + |\tilde{v}^\ell_i - v_i|),
\end{align*}$$

where the last inequality follows from $\tilde{v}^\ell_i \geq v_i$ because $\ell$ is an optimistic episode. Then

$$\begin{align*}
E \left[ \sum_{\tau \in E^{An}(\ell)} \Delta R_{\ell,\tau} \mathbbm{1}_{A_{\ell - 1}^c} \right] &\lesssim (K + 1) E \left[ \sum_{\tau \in E^{An}(\ell)} \frac{\mathbbm{1}_{A_{\ell - 1}^c}}{1 + V(S_\ell)} \sum_{i \in S_\ell} (|\tilde{v}^\ell_i - v_i| + |\tilde{v}^\ell_i - v_i|) \right] \\
&\lesssim (K + 1) E \left[ \frac{\mathbbm{1}_{A_{\ell - 1}^c}}{1 + V(S_\ell)} \sum_{\tau \in E^{An}(\ell)} \sum_{i \in S_\ell} |\tilde{v}^\ell_i - v_i| \right] \\
&+ (K + 1) E \left[ \mathbbm{1}_{A_{\ell - 1}^c} \sum_{\tau \in E^{An}(\ell)} \sum_{i \in S_\ell} |\tilde{v}^\ell_i - v_i| \right] \\
&+ (K + 1) E \left[ \mathbbm{1}_{A_{\ell - 1}^c} \sum_{\tau \in E^{An}(\ell)} \sum_{i \in S_\ell} |\tilde{v}^\ell_i - v_i| \right].
\end{align*}$$

We can bound the first two terms in the same way as we obtain (21), and we can bound the third term by (Agrawal et al. 2017, Lemma 7) to obtain

$$\begin{align*}
E \left[ \sum_{\tau \in E^{An}(\ell)} \Delta R_{\ell,\tau} \mathbbm{1}_{A_{\ell - 1}^c} \right] &\lesssim (K + 1) E \left[ \frac{|E^{An}(\ell)|}{1 + V(S_\ell)} \sum_{i \in S_\ell} \left( \sqrt{\frac{v_i \log(TK)}{T_i(\ell)}} + \sqrt{\frac{\log(TK)}{T_i(\ell)} + \frac{1}{\ell^2}} \right) \right].
\end{align*}$$

Apply the Cauchy-Schwarz inequality and we have

$$\begin{align*}
\mathbb{R}_L^1 &\lesssim E \left[ \sum_{\ell = 1}^{L} \frac{(K + 1)}{\ell^2} \right] + (K + 1) E \left[ \sum_{\ell = 1}^{L} \frac{|E^{An}(\ell)|}{1 + V(S_\ell)} \sum_{i \in S_\ell} \left( \sqrt{\frac{v_i \log(TK)}{T_i(\ell)}} + \sqrt{\frac{\log(TK)}{T_i(\ell)} + \frac{1}{\ell^2}} \right) \right] \\
&\lesssim (K + 1) \left[ 1 + \left( E \left[ \sum_{\ell = 1}^{L} |E^{An}(\ell)|^2 \right] \right)^{1/2} \left( E \left[ \sum_{\ell = 1}^{L} \delta^2(\ell) \right] \right)^{1/2} + \left( E \left[ \sum_{\ell = 1}^{L} \Delta^2(\ell) \right] \right)^{1/2} + \sqrt{K} \right] \right].
\end{align*}$$
where
\[ \delta(\ell) = \frac{1}{1 + V(S_\ell)} \sum_{i \in S_\ell} \sqrt{v_i \log(TK) / T_i(\ell)}, \quad \Delta(\ell) = \frac{1}{1 + V(S_\ell)} \sum_{i \in S_\ell} \log(TK) / T_i(\ell). \]

The bound of \( E^{An}(\ell), \delta(\ell) \) and \( \Delta(\ell) \) follows the same argument as in (Agrawal et al. 2017, Proof of Theorem 1 and Lemma 5), and we can conclude that
\[ R_1^n = \tilde{O}(\sqrt{NT}). \]