Contactomorphism groups and Legendrian flexibility

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Abstract

We explain a connection between the algebraic and geometric properties of a group of contact transformations, open book decompositions, and flexible Legendrian embeddings. The main result is that, if a closed contact manifold \((V,\xi)\) has a supporting open book whose pages are flexible Weinstein manifolds, then the connected component \(G\) of the identity in its automorphism group is a uniformly simple group: for every non-trivial element \(g\), every other element is a product of at most \(128(\dim V + 1)\) conjugates of \(g\). In particular any conjugation invariant norm on this group is bounded. We also prove the later statement still holds for the universal cover of \(G\).

Introduction

In this paper, \((V,\xi)\) will always denote a connected manifold equipped with a cooriented contact structure. We denote by \(\mathcal{D}(V,\xi)\) the group of compactly supported diffeomorphisms of \(V\) preserving \(\xi\), equipped with the strong \(C^\infty\)-topology. This paper is about the connected component \(\mathcal{D}_0(V,\xi)\) of the identity in \(\mathcal{D}(V,\xi)\), and about its universal cover \(\tilde{\mathcal{D}}(V,\xi)\). See the introduction of Gironella 2017 for information about the complementary question of studying the mapping class group \(\mathcal{D}(V,\xi)/\mathcal{D}_0(V,\xi)\).

Klein’s Erlangen program suggests to study \((V,\xi)\) through its automorphism group, which could be any of the above three groups. Nothing is lost by this perspective according to Banyaga and McInerney 1995 (see also Banyaga 1997, Section 7.5) which proved that every group isomorphism \(\Phi: \mathcal{D}_0(V_1,\xi_1) \to \mathcal{D}_0(V_2,\xi_2)\) (not necessarily continuous) is induced by a contact isomorphism: there exists a diffeomorphism \(\varphi: V_1 \to V_2\) such that \(\varphi_*\xi_1 = \xi_2\) and \(\Phi(g) = \varphi g \varphi^{-1}\) for every \(g\) in \(\mathcal{D}_0(V_1,\xi_1)\).

The main known result about the algebraic structure of these groups is proved in Rybicki 2010: both \(\mathcal{D}_0(V,\xi)\) and its universal cover are perfect groups (every element is a product of commutators), see also Tsuboi 2008a for the non-smooth case. Combined with the results of Epstein 1970 this implies that \(\mathcal{D}_0(V,\xi)\) is simple (any non-trivial normal subgroup is the full group), see
Note that $\widetilde{D}_0(V,\xi)$ is not simple in general due to the exact sequence:

$$1 \rightarrow \pi_1D_0(V,\xi) \rightarrow \widetilde{D}_0(V,\xi) \rightarrow D_0(V,\xi) \rightarrow 1.$$  

Seemingly independently of this algebraic structure studies, one can seek an interesting geometry on these groups. Inspired by the Hofer and Viterbo distances in symplectic geometry, there have been several recent papers on invariant norms on contact transformation groups, see Sandon 2010; Fraser, Polterovich, and Rosen 2017; Colin and Sandon 2015; Borman and Zapolsky 2015; Granja, Karshon, Pabiniak, and Sandon 2017, and the survey Sandon 2015. A conjugation invariant norm on a group $G$ is a function $\nu: G \rightarrow [0, \infty)$ satisfying the following properties:

- $\nu(\text{Id}) = 0$ and $\nu(g) > 0$ for all $g \neq \text{Id}$.
- $\nu(gh) \leq \nu(g) + \nu(h)$ for all $g, h \in G$.
- $\nu(g^{-1}) = \nu(g)$ for all $g \in G$.
- $\nu(hgh^{-1}) = \nu(g)$ for all $g, h \in G$.

Bi-invariant distances are another point of view on the same objects. Such a distance $d$ defines a norm $\nu = d(\cdot, \text{Id})$ and, starting from a norm $\nu$, one gets a distance $d(f, g) = \nu(fg^{-1})$. Invariant norms can arise from quasi-morphisms, see Bavard 1991, Section 1.1.

Given that the groups we consider are huge (they remember everything about the contact manifold), such a geometric structure is disappointing if the norm is bounded or, equivalently, if the associated metric space has finite diameter. Fraser, Polterovich, and Rosen 2017 proved that this always happens for norms on $D_0(S^{2n+1},\xi_0)$ or its universal cover. By contrast, based on work of Givental 1990, Colin and Sandon 2015 proved that the universal cover of $D_0(\mathbb{RP}^{2n+1},\xi_0)$ has an unbounded invariant norm. Similar puzzling differences between spheres and projective spaces were observed in the study of orderability of the corresponding groups in Eliashberg, Kim, and Polterovich 2006. A direct link between invariant norms and orderability is provided by Fraser, Polterovich, and Rosen 2017 which proved that orderable contact manifolds with a periodic Reeb flow have an unbounded invariant norm on the universal cover of $D_0$, and by Colin and Sandon 2015 which defines a norm assuming orderability.

Eliashberg, Kim, and Polterovich 2006, Corollary 1.17 explains how to deduce non-orderability of spheres from the existence of a 2-subcritical Weinstein filling $W$. But the proof uses only the existence of a splitting $W = W' \times C^2$, not the flexibility of $W$, and not directly the flexibility of the relevant Legendrian submanifolds (attaching spheres or stable manifolds). Hence one can argue that there was no previously known direct link between Gromov’s h-principle dichotomy and the existence of unbounded norms on transformation groups.

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1Here and elsewhere, we assume $n \geq 1$. 

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Before describing such a link, we make contact with the algebraic discussion of simplicity. A group being simple means concretely that for every element $f$ and every non-trivial element $g$, one can write $f$ as a product of $N(f, g)$ conjugates $h_i g h_i^{-1}$ or $h_i g^{-1} h_i^{-1}$, for some (finite) number $N(f, g)$. Independently of simplicity, if this holds for some $f$ and $g$ then, by definition of invariant norms, $\nu(f) \leq N(f, g) \nu(g)$ for any invariant norm $\nu$. Hence, invariant norms are all bounded as soon as there is some $g$ such that $N(f, g)$ can be bounded independently of $f$. A group is called uniformly simple if the number $N(f, g)$ can be bounded independently of $f$ and $g$.

**Theorem A.** If a closed connected contact manifold $(V, \xi)$ has a supporting open book whose pages are flexible Weinstein manifolds, then for every non-trivial $g \in D_o(V, \xi)$, every other element is a product of at most $128(\dim V + 1)$ conjugates of $g^{\pm 1}$. The same holds for the universal cover $\tilde{D}_o(V, \xi)$ as soon as $g$ does not lie above the identity. In particular, $D_o(V, \xi)$ is uniformly simple and all invariant norms on $D_o(V, \xi)$ or $\tilde{D}_o(V, \xi)$ are bounded.

The relevant definitions will be recalled in Section 3, including the special case of 4-dimensional pages (see Remark 3.3). Note that using Tsuboi 2008a instead of Rybicki 2010 allows to prove the same result about the group of $C^r$ contact transformations with $1 \leq r < 1 + \dim(V)/2$.

The proof of Theorem A explicitly goes through flexibility of loose Legendrian embeddings. From the point of view of this theorem, the observed difference between spheres and projective spaces is not so much related to fillings, that are external to the contact manifold at hand, but rather to pages of open books, that are internal symplectic manifolds. The standard contact structure on $\mathbb{S}^{2n+1}$ has a well known open book whose pages are $\mathbb{C}^n$, the most extreme example of a flexible Weinstein manifold (hence Theorem A includes the result of Fraser, Polterovich, and Rosen 2017 about spheres). On the other hand, the most well known open book supporting the standard contact structure on $\mathbb{R}P^{2n+1}$ has page $T^* \mathbb{R}P^n$. Since we don’t know any other way to obstruct existence of supporting open books with flexible pages, we state the following corollary (where projective space could be replaced by any manifold where an unbounded norm is known to exist).

**Corollary.** The standard contact structure on projective space has no supporting open book with flexible Weinstein pages.

The known examples of contact manifolds satisfying the hypotheses of Theorem A are obtained as the ideal contact boundary of $W \times \mathbb{C}$ for a flexible Weinstein manifold $W$ (this corresponds to the case where the monodromy is the identity). This includes the class of contact manifolds that Eliashberg, Kim, and Polterovich 2006, Corollary 1.17 prove to be non-orderable. Hence one can ask whether this result extends to all contact manifolds supported by open books with flexible pages.

The proof of Theorem A goes through the following result, where the dimension of the manifold does not appear, and which allows to compute the Colin-Sandon norms.
Theorem B. Let \((V, \xi)\) be a closed connected contact manifold. Let \(\psi_t\) be a positive (or negative) contact isotopy. If \(\xi\) is supported by an open book with flexible page, and \(\epsilon > 0\) is small enough, then every element of \(\mathcal{D}_0(V, \xi)\) or its universal cover is a product of at most 32 conjugates of \(\psi_\pm^\epsilon\). In particular, the oscillation and discriminant (pseudo-)norms of Colin and Sandon are bounded by 32.

Another important point is that the definition of invariant norms asserts no relation between \(\nu\) and any a priori given topology on the group. The norm can be defined purely in the algebraic world. A popular algebraic example is the commutator length on perfect groups, which is defined as the minimal number of factors required to express an element as a product of commutators (under the flexible page assumption, the proof of Theorem B will show that eight factors is always enough). On the symplectic side, the topology induced by the Hofer distance is very different from the smooth topology, and related to the \(C^0\) topology. Here we prove the following general result, without any open book assumption.

Theorem C. Let \((V, \xi)\) be any closed connected contact manifold. Let \(\psi_t\) be a positive or negative contact isotopy. For \(\epsilon > 0\) small enough, there exists a \(C^0\) neighborhood \(U\) of \(\text{Id}\) in \(\mathcal{D}_0(V, \xi)\) or its universal cover such that every element of \(U\) is a product of at most 16 conjugates of \(\psi_\pm^\epsilon\). In particular all invariant norms are bounded on \(U\).

In the above result, belonging to a \(C^0\) neighborhood of identity in the universal cover means being represented by a path which is \(C^0\)-close to identity for all time. A priori this is more restrictive than the pull back of \(C^0\) topology from \(\mathcal{D}_0(V, \xi)\).

The proofs of all these results rely on geometric decompositions of contact transformations following the strategies of Burago, Ivanov, and Polterovich 2008; Tsuboi 2008b, 2009 which proved analogous theorems for diffeomorphism groups (without any flexibility assumption). The decomposition statement is in terms of Giroux’s contact handlebodies, ie open contact manifolds that retract by contact isotopy into arbitrarily small neighborhoods of isotropic complexes. The relevance of such kinds of compressions was explicitly pointed out by Burago, Ivanov, and Polterovich 2008 and already imported into contact topology by Fraser, Polterovich, and Rosen 2017, with a seemingly more general definition of so called portable contact manifolds, but it seems that contact handlebodies are the only known examples.

Theorem D. Let \((M, \xi)\) be a closed connected contact manifold supported by an open book with flexible pages. Every contact isotopy is homotopic with fixed end-points to the composition of four contact isotopies with compact support in the interior of contact handlebodies.

Outline  Section 1 is an expository section recalling elementary but fundamental tools in the algebraic study of transformation groups, with special care
devoted to the universal cover case. Section 2 gathers the statements we need from Murphy’s study of Legendrian flexibility, and carefully proves a folklore stability result: the symplectization of a loose Legendrian embedding is loose in the contactization of the symplectization of the ambient manifold. Section 3 recalls somewhat under-documented aspects of Giroux’s theory of convexity in contact topology and open book decompositions, relating Giroux 1991 and Giroux 2002. Section 4 proves some general transversality theorem for multiple jets of families of contact transformations. Section 5 uses this general result to show that generic contact isotopies satisfy a list of conditions that are helpful in the proof of the above decomposition theorem. Section 6 proves the decomposition theorem by reduction to the generic case and using Murphy’s result. Section 7 combines the decomposition theorem, algebraic tools from Section 1 and Rybicki’s theorem to prove the main results (Giroux’s existence of supporting open books is also used for Theorem C).

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1 Displacement, compression and conjugates

In this mostly expository section, we review definitions and observations in the algebraic study of transformation groups. These observations are all elaborations on the fundamental observation that transformations with disjoint supports commute. They originate at least as far as Anderson 1962 and play a key role in Burago, Ivanov, and Polterovich 2008; Tsuboi 2008b, 2009, 2012, which are our main sources for this section. In the following exposition, we will pay somewhat more attention to the universal cover of the relevant group (expliciting uses of Lemma 1.1 below), and bypass some technical definitions that are not necessary for us.

In this section we fix a smooth manifold $M$ and a connected and locally contractible subgroup $G$ of the group $D_c(M)$ of smooth compactly supported diffeomorphisms of $M$. Other regularity classes and weaker topological assumptions would work as well, and we will only use $G = D_c(V;\xi)$ in later sections, but we want to emphasize that the current section involves no contact geometry. An isotopy in $G$ is a smooth path $t \mapsto \varphi_t$ starting from $\text{Id}$ in $G$, where smooth means that $(t,x) \mapsto \varphi_t(x)$ is smooth. We denote by $\widetilde{G}$ the space of smooth homotopy classes of isotopies in $G$, with fixed end-points. This means that two isotopies $\varphi^0$ and $\varphi^1$ are homotopic if there is a smooth map $\Phi: M \times [0,1] \times [0,1] \to M$
such that, for all $t$, $s$ and $x$, $\Phi(x, t, 0) = \varphi_t^0(x)$, $\Phi(x, t, 1) = \varphi_t^1(x)$, $\Phi(x, 0, s) = x$, $\Phi(x, 1, s) = \varphi_t^0(x) = \varphi_t^1(x)$. In particular this implies $\varphi_t^0 = \varphi_t^1$ and this common value provides a map $\pi: \tilde{G} \to G$ which is a universal cover for $G$.

Time-wise composition gives a group law on the set of isotopies which descends to a group law on $\tilde{G}$ such that $\pi$ is a group morphism. Time reparametrization of isotopies act trivially on $\tilde{G}$ hence one can also define the group law on $\tilde{G}$ by concatenation of isotopies and suitable time reparametrization.

A subset $A \subset M$ is displaced by a diffeomorphism $g$ if $g(A) \cap A = \emptyset$. The support $\text{supp}(g)$ of an element $g$ in $G$ is the closure of the set of points $x$ in $M$ that are displaced by $g$. We will also write, somewhat abusing terminology, that an element $g$ of $G$ is supported in some subset $U$ if it can be represented by an isotopy $g_t$ such that $\text{supp}g_t \subset U$ for all $t$.

We say that a flow $\varphi$ in $G$, i.e. a group homomorphism $t \mapsto \varphi_t$ from $\mathbb{R}$ to $G$, compresses an open set $M'$ onto a subset $L \subset M$ if, for every neighborhood $U$ of $L$ and every compact $K \subset M'$, there is some $T$ such that $\varphi_t(K) \subset U$ for all $t \geq T$.

The conjugate of an element $h$ of $G$ or $\tilde{G}$ by another element $g$ is $c_g(h) = ghg^{-1}$. It is seen as “$h$ transported by $g$”. In particular, $\text{supp}(c_g(h)) = g(\text{supp}h)$. Our first lemma will be useful to study $\tilde{G}$.

**Lemma 1.1** (Homotopies for conjugates and commutators). Let $f$ and $g$ be two isotopies in $G$. The following are isotopies in $G$ which are homotopic:

$$ (t \mapsto c_{f_t}(g_t)) \sim (t \mapsto c_{f_t}(g_t)). $$

The same is true with:

$$ (t \mapsto [f_t, g_t]) \sim (t \mapsto [f_t, g_t]) \sim (t \mapsto [f_t, g_t]). $$

**Proof.** The second part of the statement follows from the first one since $[f, g] = c_f(g)g^{-1} = f c_g(f^{-1})$.

In order to prove the first part, first notice that all paths indeed start at $\text{Id}$. One possible homotopy between the two isotopies is:

$$(s, t) \mapsto c_{f_s(1-t), t}(g_t).$$

All required properties can be checked directly. For instance, for all $s$, $(s, 0) \mapsto c_f(\text{Id}) = \text{Id}$ while $(s, 1) \mapsto c_{f_t}(g_t)$ which is indeed the common end-point of both isotopies.

The key to uniform simplicity is a result relating displacement and compression to conjugation.

**Proposition 1.2.** Let $M'$ and $M''$ be open sets in $M$, $M'' \subset M'$, let $L$ be a compact subset in $M''$, and let $g$ be an element of $G$ or $\tilde{G}$ such that $g(L) \subset M'' \setminus L$ and $L \subset g(M'')$. If there exist flows $\varphi$ and $\theta$ in $G$ compressing $M'$ and $M''$ respectively onto $L$, and such that $\theta$ has compact support in $M'$, then every element $f$ of $G$ or $\tilde{G}$ that is a product of commutators of elements with support in $M'$ is a product of at most eight conjugates of $g^{\pm 1}$. 

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Everything else in this section consists of internal details of the proof. The first magic trick turns commutators into conjugates of a displacing isotopy. It is an easy adaptation to $\tilde{G}$ of Tsuboi 2008b, Remark 6.6. Very close considerations also appear in Burago, Ivanov, and Polterovich 2008, Section 2.

**Lemma 1.3** (Commutators trading). Let $a, b$ and $g$ be three isotopies in $G$. If $\text{supp}(a_t) \cap g_1(\text{supp}(b_t))$ is empty then $[a, b]$ is homotopic, with fixed end-points, to a product of two conjugates of $g$ and two conjugates of $g^{-1}$. In particular $[a_1, b_1]$ is a product of two conjugates of $g_1$ and two conjugates of $g_1^{-1}$.

**Proof.** We set $\alpha_t = c_{g_1^{-1}}(a_t)$. By assumption and transport of support, $b$ and $c$ have disjoint support hence $\alpha_t b_t = b_t \alpha_t$ and:

$$a_t b_t a_t^{-1} b_t^{-1} = (g_1 \alpha_t g_1^{-1}) b_t (g_1 \alpha_t^{-1} g_1^{-1}) b_t^{-1} = g_1 \alpha_t g_1^{-1} (\alpha_t^{-1} \alpha_t) b_t g_1 \alpha_t^{-1} (b_t^{-1} b_t) g_1^{-1} b_t^{-1} = b_t \alpha_t$$

$$= g_1 c_{\alpha_t} (g_1^{-1}) c_{b_t \alpha_t} (g_1) c_{b_t} (g_1^{-1}).$$

Hence $[a_t, b_t]$ is homotopic, through

$$(s, t) \mapsto g_{1+s(t-1)} c_{\alpha_t} (g_1^{-1}) c_{b_t \alpha_t} (g_1) c_{b_t} (g_1^{-1})$$

to the isotopy

$$t \mapsto g_t c_{\alpha_t} (g_t^{-1}) c_{b_t \alpha_t} (g_t) c_{b_t} (g_t^{-1}).$$

and we get the announced decomposition in $\tilde{G}$. \qed

The next step is to explain, still following Burago, Ivanov, and Polterovich 2008; Tsuboi 2008b, a sufficient condition allowing to turn a product of any number of commutators into a product of two commutators.

**Lemma 1.4** (Commutator crunching). Let $U$ be a subset of $M$. Assume there exists an element $\varphi \in G$ such that the subsets $\varphi^i(U)$ for $i \geq 0$ are pairwise disjoint. Then any product of commutators of elements of $G$ or $\tilde{G}$ with support in $U$ is a product of two commutators.

**Proof.** We prove it for $\tilde{G}$, the case of $G$ is the same except that the invocation of Lemma 1.1 at the end is not needed. Let $\varphi_t$ be path between Id and $\varphi$ in $G$. Let $f$ be an isotopy in $G$. Assume there is some integer $N$ and isotopies $a_i, b_i$, in $G$, supported in $U$, $1 \leq i \leq N$, such that, for all $t$,

$$f_t = \prod_{i=1}^{N} [a_i, t, b_i, t].$$

The goal is to prove that $f$ is a product of two commutators. We set

$$f_{i, t} := \prod_{j=1}^{i} [a_j, t, b_j, t]$$

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so \( f_0 = \text{Id}, \ f_1 = [a_1, b_1], \ldots, f_N = f \). Consider the isotopy

\[
F_t = \prod_{i=1}^N c_{\varphi_1^{-i}}(f_{i,t})
\]

where each \( f_i \) is transported inside \( \varphi_1^{N-i}(U) \), as shown in the first line of Fig. 1. In this picture and the following computations, we drop the subscript \( t \) for clarity. Next consider \( c_{\varphi_1}(F) \) where each piece is shifted by one iterate of \( \varphi_1 \)

\[
\begin{array}{cccc}
U & \varphi_1(U) & \ldots & \varphi_1^{N-1}(U) & \varphi_1(U) \\
F & f = f_N & f_{N-1} & \ldots & f_1 & f_0 = \text{Id} \\
c_{\varphi_1}(F) & \text{Id} & f_N & \ldots & f_2 & f_1 \\
F^{-1} c_{\varphi_1}(F) & f^{-1} & [a_N, b_N] & \ldots & [a_2, b_2] & [a_1, b_1]
\end{array}
\]

Figure 1: Proof of Lemma 1.4

(Second line in Fig. 1). Hence, in \( [F^{-1}, \varphi_1] = F^{-1} c_{\varphi_1}(F) \) we get \( f^{-1} \) in \( U \) and exactly one commutator \([a_i, b_i]\) in each copy of \( U \) since \( f_{k-1}^{-1} f_k = [a_k, b_k] \) (third line in Fig. 1). In formula:

\[
[F^{-1}, \varphi_1] = f^{-1} \prod_{j=0}^{N-1} c_{\varphi_1^{N-j}}([a_{j+1}, b_{j+1}])
\]

\[
= f^{-1} \prod_{j=0}^{N-1} [c_{\varphi_1^{N-j}}(a_{j+1}), c_{\varphi_1^{N-j}}(b_{j+1})].
\]

In the above product, each term has support in its own copy of \( U \) hence we can rewrite by commutation:

\[
[F^{-1}, \varphi_1] = f^{-1} \left[ \prod_{j=0}^{N-1} c_{\varphi_1^{N-j}}(a_{j+1}), \prod_{j=0}^{N-1} c_{\varphi_1^{N-j}}(b_{j+1}) \right] = A = [A, B] = B
\]
and, reintroducing $t$, we get the final formula $f_t = [A_t, B_t][\varphi_1, F_t^{-1}]$. The homotopies from the second part of Lemma 1.1 finish the proof.

In the above result, the hypothesis of disjointness of the $\varphi^i(U)$ is easier to check than it may seem. The following lemma ensures it.

**Lemma 1.5** (Burago, Ivanov, and Polterovich 2008). Let $\varphi$ be a transformation of a set $X$. Let $U$ and $W$ be disjoint subsets of $X$. If $\varphi(U \cup W) \subset W$ then all iterates $\varphi^i(U)$, $i \geq 0$ are pairwise disjoint.

*Proof.* From $\varphi(U \cup W) \subset W$, we learn that $\varphi^i(U) \subset W$ and $\varphi^{i+1}(U) \subset \varphi^i(W)$, for all $i \geq 1$ and $l \geq 0$. First note that $U \cap W = \emptyset$ whereas all $\varphi^i(U)$, $i > 0$ are contained in $W$, hence we can assume $i \geq 1$ when proving the lemma. Next note that $\varphi^i(U) \subset \varphi^{i-1}(W) \setminus \varphi^i(W)$, for every $i \geq 1$. Indeed $\varphi^i(U) = \varphi^{i-1}(\varphi(U)) \subset \varphi^{i-1}(W)$, but $U$ and $W$ are disjoint hence $\varphi^i(U)$ and $\varphi^i(W)$ are disjoint. Since $\varphi^{i+p}(W) \subset \varphi^i(W)$, this can be improved to $\varphi^i(U) \subset \varphi^{i-1}(W) \setminus \varphi^{i+p}(W)$ for all $p \geq 0$. In particular $\varphi^i(U) \subset \varphi^{i-1}(W) \setminus \varphi^{i+p+1}(U)$ for all $p \geq 0$, so $\varphi^i(U) \cap \varphi^j(U) = \emptyset$ for all $j > i$.

We are now ready to prove the main result of this section.

*Proof of Proposition 1.2.* Let $U \subset M''$ be a compact neighborhood of $L$ small enough to ensure $W := g(U) \subset M'' \setminus U$ and $U \subset g(M'')$. The conjugated flow $\tilde{\varphi} = c_g(\varphi)$ compresses $g(M'')$ into $g(L)$. We fix $T$ such that $\tilde{\varphi}_T(U \cup W) \subset W$. By Lemma 1.5, all iterates $\tilde{\varphi}_T(U)$, $i \geq 0$, are pairwise disjoint in $g(M'')$.

Let $f$ be any element of $G$ or $G$ which is a product of commutators of elements with compact support in $M'$. Up to conjugation by some $\varphi_t$, we can assume these elements have support in $U$, hence in $g(M')$. Lemma 1.4 then proves that $f$ is a product of two commutators of elements with compact support in $g(M')$. After conjugating by $g^{-1}$ and then by some $\varphi_t$, we get elements with support in $U$. Lemma 1.3 then finishes the proof since $g$ displaces $U$. 

2 **Loose Legendrian submanifolds**

2.1 **Loose charts**

In this section, we recall the main definitions from Murphy 2012 that we will need. On $\mathbb{R}^3$, we consider the contact form $\alpha_3 = dz - pdq$, its Reeb vector field $\partial_z$, and the front projection $(p, q, z) \mapsto (q, z)$. A Legendrian stabilization is a Legendrian arc $\gamma$ in $\mathbb{R}^3$ whose front projection has a single transverse self intersection, a single cusp singularity, and a single Reeb chord. The action of a stabilization is the action of its Reeb chord.

Let $n \geq 2$ and equip $\mathbb{R}^3 \times \mathbb{R}^{2n-2}$ with the contact form $\alpha = \alpha_3 - \sum y_idx_i$. For any contact manifold $(V, \xi)$ and any subset $U \subset V$, a contact embedding $\varphi: U \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^{2n-2}$ is a *loose chart* for a Legendrian submanifold $L \subset V$ if there is a convex ball $B \subset \mathbb{R}^3$, and a stabilization $\gamma$ with action $a$ contained in $B$, such that $(\varphi(U), \varphi(L)) = (B \times [-p, p]^{2n-2}, \gamma \times 0 \times [-p, p]^{n-1})$ and $a/p^2 < 2$. 

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Definition 2.1. A Legendrian submanifold $L$ of dimension at least 2 is called loose if for every connected component $\Lambda$ of $L$, $\Lambda$ admits a loose chart in $V \setminus (L \setminus \Lambda)$.

Remark 2.2. We reserve the word loose for Legendrian submanifolds of dimension at least 2. However the above definition also makes sense for $n = 1$, with no quantitative condition (the constant $\rho$ disappears). We will use the word stabilized for the corresponding condition on 1-dimensional Legendrian submanifolds.

Remark 2.3. Any connected Legendrian submanifold can be made loose by performing a smooth (not Legendrian) isotopy supported in an arbitrary neighborhood of one of its point.

2.2 Stability of Loose Legendrian embeddings

Given a manifold $V$ with a cooriented contact structure $\xi$, its symplectization $SV$ is the submanifold of covectors in $T^*V$ which define $\xi$, with its coorientation. Given a Legendrian submanifold $\Lambda \subset V$, its preimage under the projection $SV \to V$ is an exact Lagrangian submanifold $S\Lambda \subset SV$. Given a manifold $B$ with a 1-form $\lambda$ such that $d\lambda$ is symplectic, its contactization is the manifold $CB = B \times \mathbb{R}$ equipped with the contact form $\lambda + dt$. An exact Lagrangian submanifold $i : L \to B$ can be lifted to a Legendrian submanifold $CL \subset CB$ as the graph of a primitive of $-i^*\lambda$ (if $L$ is connected, such a Legendrian lift is unique up to a translation in the $\mathbb{R}$ direction). One may repeat these operations and consider for example the Legendrian submanifold $CS\Lambda$ of $CSV$.

Proposition 2.4. If $\Lambda$ is a loose (or stabilized if $\dim \Lambda = 1$) Legendrian submanifold in a contact manifold $(V, \xi)$ then $CS\Lambda$ is loose in $CSV$.

Related observations appear in Murphy and Siegel 2015, Lemma 3.5 and Eliashberg 2017, Propositions 4.3 and 4.4. To prove Proposition 2.4, we shall use the following three lemmas.

Lemma 2.5. Let $(V, \xi = \ker \alpha)$ be a contact manifold. Assume that the Reeb flow of $\alpha$ is complete. Let $\lambda$ be a Liouville form on a manifold $B$. For any function $f$ from $B$ to $\mathbb{R}$, there is an isotopy $\varphi$ of $V \times B$ which moves in the Reeb flow direction, is relative to $V \times \{ f = 0 \}$, and such that $\varphi^*(\alpha + \lambda) = \alpha + \lambda + tdf$.

Proof. Let $R$ be the Reeb vector field of $\alpha$. Let $\varphi$ be the flow of $fR$ on $V \times B$, this flow is complete by assumption on $R$. The announced formula follows from $d(\varphi^*\alpha)/dt = \varphi^* \mathcal{L}_{fR} \alpha = df$. \[\square\]

Lemma 2.6. Let $\lambda$ be a Liouville form with complete Liouville flow on a manifold $B$ and let $(V, \xi)$ be a contact manifold. Given two contact forms $\alpha$ and $\alpha'$ for $\xi$, there is an isotopy $\varphi$ of $V \times B$ which moves in the Liouville flow direction, is relative to the set where $\alpha' = \alpha$, and such that $\ker(\varphi^*(\alpha + \lambda)) = \ker(\alpha' + \lambda)$.
Proof. Let \( f = \alpha'/\alpha : V \to \mathbb{R}_+ \) and \( X \) be the Liouville vector field of \( \lambda \). We set \( g = \ln(1/f) \). Since \( \lambda(X) = 0 \) and \( X \hookrightarrow d\lambda = \lambda, L_g X = g\lambda \). Let \( \psi \) be the flow of \( X \) on \( B \). Let \( \varphi \) be the flow of \( gX \) on \( V \times B \): \( \varphi_t(v, b) = (v, \psi_t(g(v))(b)) \). We have \( d/dt(\varphi_t^*(\alpha + \lambda)) = \varphi_t^*(g\lambda) = ge^tg\lambda \). Hence \( \varphi_t^*(\alpha + \lambda) = \alpha + e^tg\lambda \), and \( \varphi_t^*(\alpha + \lambda) = \ker(\alpha + 1/f\lambda) = \ker(\alpha' + \lambda) \).

In the preceding lemma, the completeness assumption on \( \lambda \) is crucial. Indeed, the discussion of large neighborhoods of contact submanifolds in Niederkrüger and Presas 2010 proves that, when \( W = \mathbb{D}^2 \) and \( \lambda = r^2d\theta \), even a constant rescaling of a contact form is enough to get different contact structures on \( V \times B \). By contrast, the next lemma states that the completeness assumption in Lemma 2.5 does not cost much.

Lemma 2.7. Any contact manifold \((V, \xi)\) admits a complete Reeb vector field.

Proof. Let \( f : V \to \mathbb{R} \) be a proper and positive function, and \( \alpha \) a contact form for \( \xi \). Let \( g = df(R_\alpha) \) and pick a function \( \psi : (0, +\infty) \to (0, +\infty) \) decreasing sufficiently fast so that \( g\psi \circ f \leq 1 \). Then the contact vector field \( X \) with contact Hamiltonian \( \psi \circ f \) with respect to \( \alpha \) writes \( X = \psi \circ f R_\alpha + Y \) where \( Y \in \xi \) and \( df(Y) = 0 \). Hence \( df(X) = g\psi \circ f \leq 1 \) and \( X \) is a complete Reeb vector field.

Proof of Proposition 2.4. Let \( \alpha \) be a contact form for \( \xi \) whose Reeb flow is complete as provided by Lemma 2.7. This induces a trivialization \( \Phi : V \times \mathbb{R} \times \mathbb{R} \to CSV \) in which the contact structure reads \( \ker(e^\epsilon\alpha + ds) \). The diffeomorphism \( g : (v, t, s) \mapsto (v, t, -e^\epsilon s) \) is relative to \( \{s = 0\} \) and pulls back the above contact structure to \( \ker(\alpha - sdt - ds) \). Lemma 2.5 then gives a diffeomorphism \( \Psi \) relative to \( \{s = 0\} \) which pulls back this contact structure to \( \ker(\alpha - sdt) \).

Let \( B \) be a closed ball around the origin in \( \mathbb{R}^3 \), \( \alpha_3 = dz - pdq \), and \( \gamma \) a stabilization in \( B \) with action. We set \( U_\rho = B \times [-\rho, \rho]^{n-1} \times [-\rho, \rho]^{n-1} \subset \mathbb{R}^{2n+1} \). By definition of loose Legendrian embeddings, for each component \( \Lambda_0 \) of \( \Lambda \), there are some \( \rho \) and \( a \) with \( a/\rho^2 < 2 \), and some contact embedding \( \iota : (U_\rho, \ker(\alpha_3 - \sum y_idx_i)) \to (V, \xi) \) such that \( \iota^{-1}(\Lambda) = \gamma \times 0 \times [-\rho, \rho]^{n-1} \). Let \( \beta \) be a contact form on \( V \) such that \( \iota^*\beta = \alpha_3 - \sum y_idx_i \).

Lemma 2.6 gives a self-diffeomorphism \( \Theta \) of \( U_\rho \times \mathbb{R}^2 \) which pulls back \( \ker(\alpha - sdt) \) to \( \ker(\beta - sdt) \), and is relative to \( \{s = 0\} \) since the relevant Liouville vector field is \( s\partial_s \). The map \( \tilde{\iota} = \Phi \circ \Phi \circ \Theta \circ (\iota \times \text{Id}_\mathbb{R}^2) : U_\rho \times [-\rho, \rho]^2 \to CSV \) is a loose chart for CSA.

2.3 Murphy’s h-principle

Recall that when \( M \) and \( N \) are manifolds with boundary, an embedding \( k : M \to N \) is called neat if \( k^{-1}(\partial N) = \partial M \), and \( k \) is transverse to \( \partial N \) along \( \partial M \). A formal Legendrian embedding in a contact manifold \((V, \xi)\) is a couple \((f, F)\) where \( f : L \to V \) is an embedding and \( F : TL \to TV \) is a family of monomorphisms covering \( f \) such that \( F_0 = df \) and \( F_1(TL) \) is Legendrian (i.e. Lagrangian in the contact structure). When \( F_s = df \) for all \( s \), it is a
genuine Legendrian embedding. We will usually suppress $F_s$ from the notation for shortness.

The following statement is a parametric, relative, and folkloric version of Murphy’s h-principle for Loose Legendrian embeddings.

**Theorem 2.8** (Murphy 2012). Let $(V,\xi)$ be a contact manifold with boundary of dimension at least 5, $L$ a connected compact manifold with boundary and $(k_t)_{t \in D^p} : L \to V$ a family of neat formal Legendrian embeddings such that

- $k_t$ is genuine near $\partial L$,
- $k_t$ is genuine for $t \in \partial D^p$,
- there is a fixed loose chart $U \subset \text{Int}(V)$ for all $k_t$, i.e. $k_t^{-1}(U) = \Lambda \subset \text{Int} L$ is independent of $t$, $k_t$ is independent of $t$ on $\Lambda$ and the pair $(U, k_t(\Lambda))$ is a loose chart.

Then there exists a homotopy $(k_{t,s})_{(t,s) \in D^p \times [0,1]}$ of neat formal Legendrian embeddings such that

- $k_{t,0} = k_t$,
- $k_{t,s}$ has a fixed loose chart $U' \subset U$,
- $k_{t,s} = k_t$ near $\partial L$,
- $k_{t,s} = k_t$ for $t \in \partial D^p$.
- $k_{t,1}$ is a genuine Legendrian embedding.

3 Giroux’s convexity theory

3.1 Convexity from symplectic to contact

A Weinstein manifold, as defined in Eliashberg and Gromov 1991, is a tuple $(W,\lambda, f)$ where $\lambda$ is a 1-form whose differential is symplectic and $f$ is an exhausting Morse function which is Lyapounov for the vector field $Z$ defined by $Z \cdot d\lambda = \lambda$. The union of all stable manifolds of critical points of $f$ form the so-called isotropic skeleton $L$, it is a union of isotropic submanifolds diffeomorphic to euclidean space, on which $\lambda$ vanishes. One may lift $L$ to the contactization $(W \times \mathbb{R}, \lambda + dt)$ as $L \times \{0\}$ which is a union of isotropic submanifolds. As soon as $f$ has finitely many critical points and $Z$ is Morse-Smale, this satisfies the following tameness condition which is very weak, but sufficient for most of our purposes.

**Definition 3.1.** A subset $L$ of a contact manifold of dimension $2n+1$ is called an isotropic complex if it admits a filtration:

$$\emptyset = L^{(-1)} \subset L^{(0)} \subset \cdots \subset L^{(n)} = L$$

such that for all $i \in \{0, \ldots, n\}$, $L^{(i)} \setminus L^{(i-1)}$ is an isotropic submanifold without boundary of dimension $i$, and near each point of $L^{(n)} \setminus L^{(n-1)}$, $L = L^{(n)} \setminus L^{(n-1)}$. 

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Based on Murphy’s work, Cieliebak and Eliashberg 2012, Definition 11.29 defined the notion of a flexible Weinstein manifold as follows.

**Definition 3.2.** \((W, \lambda, f)\) is flexible if there is an increasing sequence of real numbers \((c_i)_{i \geq 0}\) such that:

- \(c_0 < \min f\),
- \(c_i \to +\infty\),
- \(c_i\) is a regular value of \(f\),
- there are no \(\mathbb{Z}\)-trajectories joining critical points in \(\{c_i < f < c_{i+1}\}\),
- the link of Legendrian spheres in \(\{f = c_i\}\) corresponding to attaching spheres of index \(n\) critical points lying in \(\{c_i < f < c_{i+1}\}\) is loose (see Definition 2.1).

**Remark 3.3.** By extension, we also use the word flexible in the case where \(W\) is 4-dimensional and the attaching spheres are stabilized, see Remark 2.2. Note however that, although this assumption is enough for our purposes, it is a priori not enough for the purpose of deforming Weinstein structures.

Using Proposition 2.4 we see that when \(W\) is flexible (including the 4-dimensional case), the lift of its isotropic skeleton to the contactization \(CW\) is loose in the following sense.

**Definition 3.4.** An isotropic complex \(L\) is loose if \(L^{(n)} \setminus L^{(n-1)}\) is loose in \(V \setminus L^{(n-1)}\).

Eliashberg and Gromov 1991 also proposed, as a contact analogue of Weinstein manifolds, to study contact manifolds equipped with a Morse function \(f\) and a contact pseudo-gradient of \(f\). Exploration of this definition began in Giroux 1991, and eventually became Giroux’s open book theory. In particular it follows from the existence of supporting open books that every closed contact manifold is convex, in contrast to the symplectic case. Since this implication is not documented, and we need a rather precise statement, we will explain it in this section.

The following proposition is everything we will need. Recollections about open books, including the precise meaning of “exact symplectomorphic to pages” will be in Section 3.2.

**Proposition 3.5** (Giroux ∼ 2001). Let \((V, \xi)\) be a contact manifold supported by an open book \((K, \theta)\) with Weinstein pages. Let \((W_{\pm}, \lambda_{\pm}, f_{\pm})\) be two Weinstein manifolds that are exact symplectomorphic to the pages. There exist:

- contact embeddings \(j_\pm: (W_\pm \times \mathbb{R}, \ker(\lambda + dt)) \hookrightarrow (V \setminus K, \xi)\) with disjoints images
- a Morse function \(F: V \to \mathbb{R}\)
a pseudo-gradient $X$ for $F$ whose flow preserves $\xi$

such that, denoting $L_\pm$ the image of the skeleton of $W_\pm$ by the embedding $j_\pm$
restricted to $W_\pm \times \{0\}$, we have for every neighborhood $U_\pm$ of $L_\pm$ in $V$, the flow
$\varphi_t$ of $X$ satisfies:

$$\bigcap_{t \geq 0} \varphi_t (V \setminus U_-) = L_+ \quad \bigcap_{t \leq 0} \varphi_t (V \setminus U_+) = L_-.$$ 

The proof of the above proposition actually gives more information about
the relation between $(F, X)$ and the Weinstein data:

**Remark 3.6.** In the context of Proposition 3.5, let $M$ be a real number bigger
than all critical values of $f_-$ and $f_+$. One can construct $F$ such that:

- $F$ and $X$ extend the restrictions of $f_-$ and $Z_-$ to $\{f_- \leq M\}$
- $F$ and $X$ extend the restrictions of $4M - f_+$ and $-Z_+$ to $\{f_+ \leq M\}$
- critical points of $F$ are exactly critical points of $f_-$ and $f_+$; with the same
  index for points coming from $f_-$, and index augmented by one for points
  coming from $f_+$
- $F|_K = 2M$

### 3.2 Contact structures and open book decompositions

An open book decomposition of a manifold $V$ is a codimension 2 submanifold $V$
with trivial normal bundle together with a fibration $\theta : V \setminus K \to \mathbb{R}/2\pi\mathbb{Z}$ which
corresponds to the angular coordinate of $D^2$ in some tubular neighborhood
$K \times D^2$ of $K$ in $V$. $K$ is called the binding and the closure of the fibers of
$\theta$ are called the pages. The pages are always cooriented using the canonical
orientation on $\mathbb{R}/2\pi\mathbb{Z}$. If $V$ is oriented, this orients the pages and the binding is
then oriented as the boundary of pages. Recall the following important definition
due to Giroux which provides a link between contact structures and open book
decompositions.

**Definition 3.7.** A contact structure $\xi$ on $V$ is carried by an open book $(K, \theta)$
if there exists a contact form $\alpha$ for $\xi$ such that

- $\alpha$ induces a contact form on $K$,
- $d\alpha$ induces a symplectic form on the interior of each page,
- the orientation of $K$ induced by $\alpha$ agrees with the boundary orientation of
  the pages (oriented by $d\alpha$).

Let us call any contact form as in definition 3.7 a Giroux form adapted to
$(K, \theta)$. It will be more convenient here to work with the following definition,
which is tiny variation on the setup discussed in Giroux 2017.
**Definition 3.8.** Let $(K, \theta)$ be an open decomposition of a manifold $V$. An ideal Giroux form adapted to $(K, \theta)$ is a contact form $\alpha$ on $V \setminus K$ such that:

- $d\theta(R_\alpha) > 0$,
- there is some positive contact form $\beta$ on $K$ and a tubular neighborhood $i : K \times D^2 \to V$ of $K$ such that $i^* \theta = \varphi$ and $i^* \alpha = d\varphi + \beta/r^2$ where $(r, \varphi)$ are polar coordinates on $D^2$.

The two notions are essentially equivalent according to the following lemma.

**Lemma 3.9.** Let $V$ be a closed manifold with an open book decomposition $(K, \theta)$ and $\xi$ a contact structure on $V$.

1. If $\xi$ is defined by an ideal Giroux form adapted to $(K, \theta)$ then it is carried by $(K, \theta)$.
2. If $\xi$ is carried by $(K, \theta)$, then after a small deformation of $\theta$ near $K$, it is defined by an ideal Giroux form adapted to $(K, \theta)$.

**Proof.** Let $\alpha$ be an ideal Giroux form adapted to $(K, \theta)$ and $i : K \times D^2 \to V$ a tubular neighborhood of $K$ as in definition 3.8. We modify $\alpha = \frac{\beta}{r^2} + d\theta$ in this neighborhood by replacing it by $\alpha' = f(r)(\beta + r^2d\varphi)$ with $f(r) = \frac{1}{r^2}$ near $\partial D^2$. The condition on $f$ for $\alpha'$ to be a Giroux form simply writes $f' > 0$ for $r > 0$ which can be easily achieved.

Conversely, let $\alpha$ be a Giroux form adapted to $(K, \theta)$. Pick a tubular neighborhood $i : K \times D^2 \to V$ of $K$ such that $i^* \theta = \varphi$. The conformal symplectic normal bundle $\nu$ of $K$, i.e. the $d\alpha$-orthogonal to $TK$ in $\xi$, inherits a trivialization from $i$. The tubular neighborhood theorem for contact submanifolds then allows to deform $i$ without changing its differential along $K \times \{0\}$ to achieve $i^* \alpha = f(\beta + r^2d\varphi)$ for some positive function $f$ and a positive contact form $\beta$ for $K$. One issue is that $i^* \theta$ is no longer equal to $\varphi$. However, $i^* \theta$ is the angular coordinate of coordinates which only differ by a diffeomorphism tangent to the identity along $K \times \{0\}$. Hence we may deform $i^* \theta$ near $K \times \{0\}$ so that it agrees with $\varphi$ near $K \times \{0\}$, simply by replacing it with $\rho(r)\varphi + (1 - \rho(r))i^* \theta$ for some cutoff function $\rho(r)$ supported in a small neighborhood of $0$. To get an ideal Giroux form it remains to replace the function $f$ by a function equal to $\frac{1}{r^2}$ near $K \times \{0\}$, equal to $f$ away from a neighborhood of $K \times \{0\}$ and still satisfying the condition $\frac{\partial f}{\partial r} < 0$ for $r > 0$.

Here are some nice features of ideal Giroux forms.

- The contact structure $\ker \alpha$ extends smoothly over $K$ since $\ker(d\theta + \beta/r^2) = \ker(\beta + r^2d\theta)$.
- The interior of each page equipped with the restriction of $\alpha$ is a finite type complete Liouville manifold and its contact boundary at infinity is identified with $K$. 

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Let Lemma 3.10.

on any finite collection of pages, and to get disjoint copies of contactizations of allows to fix specific Liouville forms (not only up to exact symplectomorphism)

adapted to an open book $\alpha_c \subset \mathcal{I}\lambda$.

By assumption, there exists a diffeomorphism $\phi$ guarantees by the following trick: denoting by $\gamma$ near $K$.

is possible if (and only if) $k_j \mid \phi$ for some compactly supported function $k_j$.

$\lambda$ and $\gamma$ everywhere, and $\phi \gamma$ satisfies $\phi \gamma = \phi + df$ with $f > 0$ and $f = 2\pi$ outside of a compact set.

In particular, it makes sense to speak of an open book decomposition with page a given complete finite type Liouville manifold $(W, \lambda)$. The following lemma allows to fix specific Liouville forms (not only up to exact symplectomorphism) on any finite collection of pages, and to get disjoint copies of contactizations of these Liouville manifolds inside our contact manifold.

Lemma 3.10. Let $V$ be a closed manifold equipped with an ideal Giroux form $\alpha$ adapted to an open book $(K, \theta)$. Let $(W, \lambda)$ be a finite type Liouville manifold exact symplectomorphic to the pages. For every $c \in S^1$, and every open interval $I \subset S^1$ containing $c$, there is a family of fibrations $\theta_s^c: V \setminus K \to S^1$ such that

- $\theta_s = \theta$ near $K$ and $\theta^{-1}(S^1 \setminus I)$, and everywhere when $s = 0$,
- $\alpha$ is adapted to $(K, \theta_s)$ for all $s$,
- there is an embedding $g: W \times \mathbb{R} \to \theta^{-1}(I)$ such that $g^* \gamma = f(\lambda + dt)$ everywhere, and $f = 1$ and $g^* \theta_1 = c + t$ (mod 2) on $W \times (-\epsilon, \epsilon)$, for some positive $\epsilon$.

Proof. By assumption, there exists a diffeomorphism $i: W \to \theta^{-1}(\epsilon)$ such that $i^* \alpha = \lambda + dk$ for some compactly supported function $k$. This diffeomorphism combines with the flow $\varphi$ of the Reeb field $R$ of $\alpha$ to give an immersion $j_0: W \times \mathbb{R} \to V \setminus K$ sending $(x, t)$ to $\varphi(t)(x)$. We compute $j_0^* \alpha = i^* \varphi_t^* \alpha + (i(R))^t dt = \lambda + dk + dt$. Precomposing with $(x, t) \mapsto (x, t - k(x))$ gives a new immersion $j$ such that $j^* \alpha = \lambda + dt$, but now the page $\theta^{-1}(c)$ is the image of the graph of $\{t = k(x)\}$ in $W \times \mathbb{R}$. Let $f_\pm(x)$ (resp. $f_\pm(x)$) be the time taken by the flow $\varphi$ to travel (resp. travel back) from $i(x)$ to $\theta^{-1}(S^1 \setminus I)$. Both these functions are bounded below by a positive constant thanks to the model behavior of $R$ near $K$. The restriction of $j$ to $Y = \{x, t\} : k(x) - f_\pm(x) < t < k(x) + f_\pm(x)\}$ is a diffeomorphism onto $\theta^{-1}(I)$.

What we know about $j^* \theta$ is that its level set $\{j^* \theta = c\}$ is $\{t = k(x)\}$. We would like to deform the function $j^* \theta$ in a compact of $Y$, among increasing functions of $t$, so that we have $j^* \theta = c + t$ (mod 2) whenever $|t| < 2\epsilon$. This is possible if (and only if) $k - f_- < 0$ and $k + f_+ > 0$. This condition can be guaranteed by the following trick: denoting by $\psi$ and $\psi'$ the Liouville flows of $\lambda$ and $\alpha|_{\theta^{-1}(c)}$ respectively, replacing $i$ by $\psi'_- \circ i \circ \psi_s$ replaces $k$ by $e^{-s} k \circ \psi_s$.

Since $k$ was already compactly supported, it can be made arbitrarily $C^0$-small (note that this trick would not allow to get a $C^1$-small $k$), without changing $f_\pm$. 

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It remains only to replace $Y$ by the full $W \times \mathbb{R}$ without losing control of $j^*\theta$. Let $\rho : \mathbb{R} \to (-2\epsilon, 2\epsilon)$ be diffeomorphism equal to the identity on $(-\epsilon, \epsilon)$ and $h = \text{Id} \times \rho : W \times \mathbb{R} \to W \times (2\epsilon, 2\epsilon)$. We have $h^*(\lambda + dt) = \lambda + \rho^*(t)dt$.

Since the Liouville flow of $\lambda$ is complete, Lemma 2.6 provides a diffeomorphism $\psi : W \times \mathbb{R} \to W \times \mathbb{R}$ fibered over $\mathbb{R}$ such that $\text{ker}(\psi^*(\lambda + \rho^*(t)dt)) = \text{ker}(\lambda + dt)$, and $\psi = \text{Id}$ on $W \times (\epsilon, \epsilon)$. The embedding $g = j \circ h \circ \psi$ has all the required properties.

\[ \square \]

### 3.3 From open book decompositions to convex Morse functions

The goal of this section is to prove Proposition 3.5 and the companion Remark 3.6.

Let $\alpha$ be an ideal Giroux form adapted to $(K, \theta)$. Lemma 3.10 gives disjoint contact embeddings $i_\pm : W_\pm \times \mathbb{R} \leftrightarrow V \times K$ such that, on $W_\pm \times (-\epsilon, \epsilon)$, $i_\pm^*\theta = t \pmod{2\pi}$, $i_\pm^*\theta = \pi + t \pmod{2\pi}$, and $i_\pm^*\alpha = \lambda_\pm + dt$. In particular $i_\pm^*\lambda = 0$.

The starting observation is that, in the contactization of $W_\pm$, the vector field $\mp(Z_\pm + t\partial_t)$ is contact and pseudo-gradient of $\mp(f_\pm + i^2)$, with the required dynamics. For the purpose of fitting both contactizations together, it is more convenient to use the variant $\cos i_\pm^*\theta Z_\pm + \sin i_\pm^*\theta R_\alpha$ (which is the same thing up to order one when $t$ goes to zero, i.e. when $\theta$ is close to $0$ of $\pi$).

Since $i_\pm^*\alpha (\cos i_\pm^*\theta Z_\pm + \sin i_\pm^*\theta R_\alpha) = \sin i_\pm^*\theta$, we can patch these two vector fields to the contact vector field $X'$ on $V \setminus K$ with contact Hamiltonian $\sin \theta$ with respect to $\alpha$ (this is almost the required $X$, but will need some tweaking to extend over $K$). We have $X' = \sin \theta R_\alpha + Y$ where $Y$ is in $\text{ker} \, d\sin \theta$, hence in $\text{ker} \, d\theta$, and in $\xi$. In particular $d\theta(X') = \sin \theta d\theta(R_\alpha)$. On the image of $W_\pm$, $Y = \mp Z_\pm$ by construction.

We now begin to construct the Morse Lyapounov function $F$. In $i_-((W_- \times (-\epsilon, \epsilon))$, we set $F_- = f_- + \lambda (1 - \cos \theta)$, for some large positive $\lambda$ to be specified later. This is a Morse function which extends $f_-$ with critical points set $\{ (w, 0) ; w \in \text{Crit } f_- \}$, and the same Morse index than in $W_-$. We reduce $\epsilon$ to make sure $\epsilon < \pi/2$, hence $\cos \epsilon > 0$. We then have:

\[ \text{Crit } F_- \subset \{ F_- \leq M \} \subset W_- \times (-\epsilon/2, \epsilon/2) \]

as soon as $\lambda \geq (M - \min f_-)/(1 - \cos(\epsilon/2))$ (remember $f_-$ is bounded below by definition of a Weinstein structure). This has the desired pseudo-gradient because $dF_-(X') = \cos t \, df_-(Z_-) + \lambda \sin^2 t$ in $W_- \times (-\epsilon, \epsilon)$. We set $A_- = \{ F_- \leq M \}$. Our final $F$ will extend $F_-$ from $A_-$ to $V$.

In $i_+(W_+ \times (-\epsilon, \epsilon)$ we have almost the same situation if we set $F_+ = 4M - f_+ - \lambda (1 + \cos \theta)$. One difference is that $-\cos \theta = -\cos(t + \pi) = \cos t$ so the index of critical points goes up by one. The required estimates are now

\[ \text{Crit } F_+ \subset \{ F \geq 3M \} \subset W_+ \times (-\epsilon/2, \epsilon/2) \]

which hold whenever $\lambda \geq (M - \min f_+)/(1 - \cos(\epsilon/2))$. 

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Before bridging the gap between $A_-$ and $A_+ := \{F_+ \geq 3M\}$, we need to modify $X'$ near the binding $K$. In the tubular neighborhood $K \times \mathbb{D}^2$, the contact Hamiltonian of $X$ with respect to $r^2 \alpha = \beta + r^2 d\theta$ is equal to $r^2 \sin \theta = ry$ (in particular, $X$ does not extend smoothly on $K$). We replace this Hamiltonian by $\rho(r)y$ where $\rho$ is a smooth non-decreasing function interpolating between a positive constant near 0 and $r$. The corresponding contact vector field is our final $X$. It is smooth everywhere (since its Hamiltonian is) and coincides with $X'$ away from a neighborhood of $K$. A computation shows that:

$$X = \frac{y}{2}(\rho - r^2\rho')R_\beta - \frac{\rho}{2} \cos \theta \partial_\nu + \frac{y}{2r^2} (r\rho' + \rho) \partial_\theta.$$ 

In particular $d\theta(X)$ has the sign of $\sin(\theta)$ on all $V \setminus K$, and $dx(X) = \cos \theta dr(X) - r \sin \theta d\theta(X) = -(\rho + r \sin^2 \theta \rho')/2$ is negative in $K \times \mathbb{D}^2$. Also $X$ is tangent to $\{\theta = 0\} \cup \{\theta = \pi\}$.

We already saw that $X'$, hence $X$, goes transversely out of $A_-$ and into $A_+$. The above computations also allow to check that $X$ has no zero outside $A_- \cup A_+$ and every point of $\partial A_-$ flows to $\partial A_+$ in finite time. Hence one can extend $F_-$ and $F_+$ to a Morse function $F$ with no critical point outside $A_- \cup A_+$, admitting $X$ as a pseudo-gradient, and such that that $F = 2M$ on $K \cup \{\theta = \pm \pi/2\}$.

By construction, stable (resp. unstable) manifolds of critical points belonging to $\{\cos \theta > 0\}$ (resp. $\{\cos \theta < 0\}$) are the corresponding stable manifolds of $Z_-$ (resp. $Z_+$), so this announced dynamics is ensured.

### 3.4 Two lemmas about isotropic complexes

**Proposition 3.11.** Let $V$ be a manifold, $\xi$ a cooriented hyperplane field, $L$ a compact submanifold with conical singularities (in the sense of Laudenbach 1992) whose strata are integral submanifolds of $\xi$, and $\phi_t$ an isotopy generated by a vector field $X_t$ which is positively transverse to $\xi$. Then there exists $\epsilon > 0$, such that $\phi_t(L) \cap L = \emptyset$ for all $t \in ]0, \epsilon].$

**Proof.** If this is wrong, we find sequences $x_n, y_n \in L$ and $t_n > 0$ converging to 0 such that $\phi_{t_n}(x_n) = y_n$. By compactness of $L$ we may assume that $x_n$ and $y_n$ converge, necessarily to the same point $z \in L$. In local coordinates centered at $z$, we have $\phi_t(x) = x + tX_0(0) + o(|t, x|)$. Hence, $\frac{y_n - x_n}{t_n}$ converges to $X_0(0)$ as $n$ goes to $+\infty$. It is therefore enough to prove that at each point $z \in L$, the subset, denoted $C_zL$, of $T_zV$ consisting of all accumulation points of sequences $\frac{y_n - x_n}{t_n}$ with $x_n, y_n \in L$ converging to $z$ and $t_n$ converging to 0, is included in $\xi_z$ (and therefore does not contain $X_0(z)$). Let us now prove this fact, by induction on the dimension of $L$. If $L$ is a finite number of point, it is clear. Assume we have proved it when dim $L < k$ and let $L$ be $k$-dimensional. At a point $z$ in a stratum of dimension $i > 0$, we have locally a product situation $(V, L) \simeq (D^i \times D^{2n+1-i}, D^i \times L')$ where $L'$ is a $(k-i)$-dimensional complex, and thus $C_zL = \mathbb{R}^i \times C_zL'$. By induction hypothesis, we have $C_zL' \subset \xi_z$ and thus $C_zL \subset \xi_z$. Now if $z$ is a 0-dimensional stratum of $L$, we have a $C^1$-chart $\varphi : (D^{2n+1}, 0) \to (V, z)$ such that $\varphi^{-1}(L)$ is the cone over a compact
submanifold with conical singularities $L' \subset S^{2n}$. For $x \in \varphi^{-1}(L) \setminus \{0\}$, the ray $\{tx, t \in (0, 1)\}$ is contained in a single stratum of $L$ and is hence tangent to $\xi$ at each point. Since $\varphi^*\alpha$ is continuous at the origin, we obtain that this ray is in fact entirely contained in $(\varphi^*\xi)_0$. Hence $\varphi^{-1}(L)$ is included in the hyperplane $(\varphi^*\xi)_0$ and we get $C_zL \subset \xi_z$ ∎

Remark 3.12. For any Weinstein manifold $(W, \omega, X)$, according to Cieliebak and Eliashberg 2012, Proposition 12.12, we may deform $X$ near the critical points so that it is the gradient vector field with respect to a flat metric there. When this property is achieved, Laudenbach 1992, Proposition 2 guarantees that the skeleton is a submanifold with conical singularities. Proposition 3.11 will ensure that, when applying Proposition 3.5, we may assume that the skeletons $L_\pm$ are displaced by any small positive contact isotopy.

Proposition 3.11 does not hold for a general isotropic complex without some taming condition (in $\mathbb{R}^3$, think of Legendrian curves whose Lagrangian projection spiral around the origin).

Lemma 3.13. Let $\phi_i, t \in [0, 1]$, be a contact isotopy of $(V, \xi)$ and $L$ an isotropic complex such that for each $i \in \{0, \ldots, n\}$, there exists a basis of open neighborhoods $U_i$ of $L^{(i)}$ such that $L^{(i+1)} \setminus U_i$ is diffeomorphic to a finite disjoint union of disks. Then there exists a collection of contact isotopies $\phi_{i,t}$ supported in the interior of Darboux balls $B_i$ such that, for $t$ near 0, $\phi_i = \phi_{0,t} \circ \cdots \circ \phi_{n,t}$ near $L$. 

Proof. We proceed by induction. By assumption $L^{(0)}$ is a finite union of points and it is thus contained in the interior of some Darboux ball $B_0$. Multiply the section of $TV/\xi \to V \times [0, 1]$ corresponding to $\phi_i$ by a function equal to 1 on some neighborhood $V_0$ of $L^{(0)}$, and supported in the interior of $B_0$. This generates a contact isotopy $\phi_{0,t}$ supported in Int $B_0$ such that $\phi_{0,t} = \phi_i$ near $L^{(0)}$ and for $t \leq \epsilon_0$ where $\epsilon_0$ is a positive number less than half the time needed for $\phi$ to move $L^{(0)}$ outside $V_0$.

Assume we have constructed $\phi_{i,t}$ and $B_i$ up to $i = k$ for some $k \geq 0$. Let $U_k$ be an open neighborhood of $L^{(k)}$ such that $\phi_t = \phi_{0,t} \circ \cdots \circ \phi_{k,t}$ on $U_k$ for $t \leq \epsilon_k$ and such that $L^{(k+1)} \setminus U_k$ is diffeomorphic to a finite disjoint union of disks. Then $L^{(k+1)} \setminus U_k$ is contained in the interior of a Darboux ball $B_{k+1}$ (each isotropic disk is, and then one can connect the disjoint Darboux balls along transverse arcs). The section of $TV/\xi \to V \times [0, 1]$ corresponding to the isotopy $\phi_{k,t}^{-1} \circ \cdots \circ \phi_{0,t}^{-1} \circ \phi_t$ vanish on $U_k \times [0, \epsilon_k]$ by assumption. Multiply this section by a function equal to 1 on a neighborhood $V_{k+1}$ of $L^{(k+1)} \setminus U_k$ and supported in the interior of $B_{k+1}$ to get a contact isotopy $\phi_{k+1,t}$. We have $\phi_{k+1,t} = \phi_{k,t}^{-1} \circ \cdots \circ \phi_{0,t}^{-1} \circ \phi_t$ near $L^{(k+1)}$ and for $t \leq \epsilon_{k+1}$, where $\epsilon_{k+1}$ is positive, less than $\epsilon_k$, and less than half the time needed for $\phi_{k+1}$ to move $L^{(k+1)} \setminus U_k$ outside $V_{k+1}$. The result is now proved by induction. ∎

Remark 3.14. If $(W, \omega, Z, f)$ is a Weinstein manifold of finite type such that $Z$ is Morse-Smale, then its skeleton $L$ as well as its lift in the contactization satisfies the tameness assumption of Lemma 3.13. Indeed, one may first deform the
function $f$ (or $f + t^2$ in the contactization $W \times \mathbb{R}$ where $t$ is the $\mathbb{R}$-coordinate) without changing $Z$, so that $f$ is ordered and for each $i$, the critical points of index $i$ all lie in the same level set $\{ f = i \}$. Now given any neighborhood $V_i$ of $L^{(i)}$, we may deform $f$ without changing critical values so that the sublevel set $U_i = \{ f < i + 1/2 \}$ is contained in $V_i$. Then $L^{(i+1)} \cap U_i = L^{(i+1)} \cap \{ i + 1/2 \leq f \leq i + 1 \}$ is diffeomorphic to a finite disjoint union of disks.

4 Transversality for contact transformations

In this section, we prove Theorem 4.2, a general transversality theorem for multi-jets of families of contact diffeomorphisms. In Section 5, it will ensure that certain properties of contact isotopies hold after an arbitrarily small perturbation.

Let $(V, \xi)$ be a contact manifold, and let $B$ be any manifold. We denote by $\mathcal{D}_B(V, \xi)$ the space of families of contact transformations of $(V, \xi)$ parametrized by $B$, i.e. maps $f: B \times V \to V$ such that each $f_b := f(b, \cdot): V \to V$ is a contact transformation: $(f_b)\ast \xi = \xi$ for all $b$. This space is equipped with the strong $C^\infty$ topology, which makes it a Baire space: a residual subset (i.e. a countable intersection of dense open subsets) is dense. Note that $f_b$ is automatically a local diffeomorphism if $f$ is in $\mathcal{D}_B(V, \xi)$, and the subset of such maps where $f_b$ is a global diffeomorphism for all $b$ is open.

Inside the space $J^k_l(B \times V; V)$ of $k$-jets of germs of maps from $B \times V$ to $V$ at $l$ distinct points, we consider the subspace $J^k_l(B \times V; V; \xi)$ coming from such contact families. Since all contact structures are locally isomorphic, we have $J^0_l(B \times V; V) = J^0_l(B \times V; V)$. However for $k \geq 1$, one has a strict inclusion. These subsets are still nice according to the following proposition, whose proof is postponed to the end of the section.

**Proposition 4.1.** For each $k$ and $l$, $J^k_l(B \times V; V; \xi)$ is a submanifold of $J^k_l(B \times V; V)$ and the projection $J^{k+1}_l(B \times V; V; \xi) \to J^k_l(B \times V; V; \xi)$ is a submersion.

Our version of the Thom-Mather transversality theorem for families of contact diffeomorphisms can now be stated as follows.

**Theorem 4.2.** The families of contact diffeomorphisms whose multijet extension is transverse to a given submanifold $\Sigma$ of $J^k_l(B \times V; V; \xi)$ form a residual subset of $\mathcal{D}_B(V, \xi)$.

**Remark 4.3.** Since a countable intersection of residual subsets is still residual, one may impose simultaneously countably many transversality conditions possibly with varying $(k,l)$. In particular, it applies to a stratified subset. Openness of the set of families satisfying the transversality condition is more subtle, and requires additional properties of the stratification. Since we do not need this property in our application, we will not discuss it further, and content ourselves with residual subsets.
Proof. Pick a contact form \( \alpha \) for \( \xi \). To any family \( f: B \times V \to V \) of contact diffeomorphisms, we associate its Legendrian graph as follows\(^2\). We have, for \((b,v) \in B \times V\), \((f^*\alpha)_{(b,v)} = \mu_v + c(\xi)_{v(b,v)}\) where \( \mu_v \) is a 1-form on \( B \) smoothly depending on \( v \) and \( g \) is a function on \( B \times V \). With these notations we define \( \Lambda_f : B \times V \to M := T^*B \times V \times V \times \mathbb{R} \) by \( \Lambda_f(b,v) = (\mu_v, v, f(b,v), g(b,v)) \) and compute that it is Legendrian for the contact form \( \lambda_B + e^i\alpha_1 - \alpha_2 \) where \( \lambda_B \) is the canonical 1-form on \( T^*B \) and \( \alpha_i \) is the pullback of \( \alpha \) under the \( i \)-th projection to \( V \). Let \( \tau \) be the projection of \( M \) onto the second \( V \) factor. The same computation shows any Legendrian section \( \sigma \) of \( M \) \( B \times V \) gives rise to a family \( \tau \circ \sigma \) of local contact diffeomorphisms. Hence the space of families of contact diffeomorphisms now sits as an open set in the space of Legendrian sections of \( M \). The statement of the theorem is therefore equivalent to the fact that the set of Legendrian sections of \( M \) whose corresponding family has its multijet transverse to \( \Sigma \) is a residual subset.

Let us now have a look at the Legendrian graph construction at the level of multijets. Define \( M_{l}^{(k)} \) to be the \((k,l)\)-multijet extension of \( M \to B \times V \), i.e. the space of \( k \)-jets of sections at \( l \) distinct points. Let \( S_{l}^{k} \subset M_{l}^{(k)} \) be the differential relation corresponding to multijets of Legendrian sections of \( M \). We set \( X = B \times V \times V \) and pick a contact embedding of a \( \beta \) such that \( \beta \) is the canonical 1-form on \( \mathbb{R} \) and compute that it is Legendrian for the contact form \( \lambda_B + e^i\alpha_1 - \alpha_2 \) where \( \lambda_B \) is the canonical 1-form on \( T^*B \) and \( \alpha_i \) is the pullback of \( \alpha \) under the \( i \)-th projection to \( V \). Let \( \tau \) be the projection of \( M \) onto the second \( V \) factor. The same computation shows any Legendrian section \( \sigma \) of \( M \) \( B \times V \) gives rise to a family \( \tau \circ \sigma \) of local contact diffeomorphisms. Hence the space of families of contact diffeomorphisms now sits as an open set in the space of Legendrian sections of \( M \). The statement of the theorem is therefore equivalent to the fact that the set of Legendrian sections of \( M \) whose corresponding family has its multijet transverse to \( \Sigma \) is a residual subset.

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\(^2\)The intrinsic definition, without using \( \alpha \), would be less convenient here.
multijet extensions $M^k_l$ of $M$, which maps $J^{k+1}_l(B \times V)$ to $S^k_l$. This shows that $S^k_l$ is a submanifold, since $J^{k+1}_l(B \times V)$ is a submanifold of $(J^1(B \times V))^k_l$. Moreover, the classical Thom-Mather theorem, applied at order $k + 1$, implies that the space of functions $B \times V \to V$ whose $(k + 1, l)$-multijet is transverse to some submanifold of $J^{k+1}_l(B \times V)$ is a residual subset. Hence so is the space of Legendrian sections of $M$ whose $(k, l)$-multijet extension is transverse to the corresponding submanifold of $S^k_l$.

It remains to prove Proposition 4.1. For $p \in \mathbb{N}$, $n \in \mathbb{N}$ and $k \in \mathbb{N}$, we define $G_{p,2n+1}^k$ to be the set of $k$-jets at the origin of maps $f : \mathbb{R}^p \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ such that $f(0,0) = 0$ and $f_0 = f(0, \cdot) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is a local diffeomorphism. This is a Lie group for the parameterwise composition. Note that $G_{0,p,2n+1}^k$ is the trivial group. Moreover, we have projections $G_{p,2n+1}^k \to G_{p,2n+1}^{k-1}$ which are surjective Lie group homomorphisms, and hence submersions.

Now adding the constraint that for all $b \in \mathbb{R}^p$, $f_b : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is a local contact diffeomorphism (for the contact structure $\xi$) defines a subgroup $H_{p,2n+1}^k$ of $G_{p,2n+1}^k$. The following lemma provides an explicit description of $H_{p,2n+1}^k$, which implies in particular that it is a closed subgroup\(^3\), hence a submanifold by Cartan’s closed subgroup theorem.

**Lemma 4.4.** For $i \geq 0$, let $E_i$ be the bundle $\Lambda^i(\mathbb{R}^{2n+1})$ pulled back by the projection $\mathbb{R}^p \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$. Each map $f : \mathbb{R}^p \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ determines sections $\omega_2(f)$ and $\omega_3(f)$ respectively of $E_2$ and $E_3$ by the formulas

- $\omega_2(f)_{(b,v)} = (f_b^* \alpha \wedge \alpha)_v$,
- $\omega_3(f)_{(b,v)} = (f_b^* d\alpha \wedge \alpha - f_b^* \alpha \wedge d\alpha)_v$,

where $f_b = f(b, \cdot) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$. The subgroup $H_{p,2n+1}^k$ consists of the $k$-jets at the origin of maps $f$ such that $f(0,0) = 0$ and the $(k - 1)$-jets at the origin of the corresponding sections $\omega_2(f)$ and $\omega_3(f)$ vanish (observe that these depend only on the $k$-jet of $f$ at the origin).

**Proof.** If $f : \mathbb{R}^p \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is a family of local contact diffeomorphisms then $\omega_2(f)$ and $\omega_3(f)$ vanish identically.

Conversely if the $k$-jet of $f$ is such that $\omega_2(f)$ and $\omega_3(f)$ vanish at order $k - 1$, we will prove that we can turn $f$ into a family of local contact diffeomorphisms without changing its $k$-jet at the origin. For this we follow the path method.

In the following, the differential forms always depend on the parameter $b$, i.e. are seen as sections of $E^3$, though we often drop the subscript $b$ for notational convenience. We set $\alpha_t = f_t^\ast \alpha$, $\alpha_t = (1 - t)\alpha + t\alpha_1$ and $\xi_t = \ker \alpha_t$. Note that $\xi_t$ is a contact structure near the origin for all $t$ and all $b$. Indeed, at the origin, we have $\alpha_t \wedge \alpha = 0$, hence $\ker \alpha_t = \xi$, and then $d(\alpha_t \wedge \alpha) = 0$ implies that $d\alpha_t|_{\xi}$

\(^3\)This fact is not obvious because a sequence $f_n$ of families of contact transformations whose $k$-jet converges at a point does not necessarily converges near that point.
is a multiple of $d\alpha|_{\xi}$, hence symplectic. The vanishing condition on $\omega_2(f)$ and $\omega_3(f)$ imply, after differentiating with respect to $t$,

$$\hat{\alpha}_t \wedge \alpha_t = o(|b, v|^{k-1}) \quad (1)$$

$$\hat{\alpha}_t \wedge d\alpha_t - \alpha_t \wedge d\hat{\alpha}_t = o(|b, v|^{k-1}). \quad (2)$$

We will construct a local isotopy $\Phi_t$ (fibered over $Id_{\mathbb{R}^2}$) such that, denoting by $\phi_t$ the restriction of $\Phi_t$ to some unspecified slice $\{b\} \times \mathbb{R}^{2n+1}$, $\phi_t^* \xi_t = \xi_0$. It will be generated by a vector field $X_t$ that we decompose as $X_t = f_t R_t + Y_t$ where $R_t$ is the Reeb vector field of $\alpha_t$ and $\alpha_t(Y_t) = 0$. The usual discussion of the path method in this context (see e.g. Geiges 2008, Page 60) ensures that $\phi_t$ will pull back $\xi_t$ onto $\xi_0$ as soon as $(df_t + \alpha_t + Y_t \wedge d\alpha_t) \wedge \alpha_t = 0$. This is equivalent to $(Y_t \wedge d\alpha_t)|_{\xi_t} = - (df_t + \alpha_t)|_{\xi_t}$ and, since $d\alpha_t|_{\xi_t}$ is non-degenerate, this uniquely defines $Y_t$. What is specific to our situation is that we need to ensure that $f_t$ and $Y_t$ are both $o(|b, v|^k)$, so that $\Phi_t(b, v) = (b, v + o(|b, v|^k))$. Per the above discussion, the estimate on $Y_t$ is equivalent to $(df_t + \alpha_t) \wedge \alpha_t = o(|b, v|^k)$.

We set $\gamma_t = \alpha_t - \alpha_t(R_t)\alpha_t$. Plugging $R_t$ into (1) gives $\gamma_t = o(|b, v|^{k-1})$. In addition, (2) ensures that $d\gamma_t \wedge \alpha_t = \gamma_t \wedge d\alpha_t + o(|b, v|^{k-1})$, hence $d\gamma_t \wedge \alpha_t = o(|b, v|^{k-1})$. Recall the de Rham homotopy formula, for any $h: M \times [0, 1] \to M$, $h_t^* - h_0^* = H \circ d + d \circ H$ where $H_t = \int_0^t \partial_s \int h^* \eta$.

Darboux’s theorem, with parameters, ensures the existence of coordinates, smoothly varying in $t$ (and $b$), such that $\alpha_t = dz + \lambda$ where $\lambda$ is the radial Liouville form on $\mathbb{R}^{2n}$: $\lambda = \Sigma(x_i dy_i - y_i dx_i)/2$. In these coordinates, we will use the Liouville homotopy $h: (x, y, z, s) \mapsto (sx, sy, z)$. The corresponding operator $H$ on differential form satisfies $H o (|b, v|^2) = o(|b, v|^2)$ for every $j$, because $dh/\partial s = O(|v|)$. So we can set $f_t = -H \gamma_t$, and have $f_t = o(|b, v|^k)$. In addition, since $\gamma_t(\partial_s) = \gamma_t(R_t) = 0$, we have $h_0^* \gamma_t = 0$. Hence the homotopy formula gives $df_t = -\gamma_t + H d\gamma_t$. In the following computation, we will use this, the observation $h^* \alpha_t = \alpha_t + (s^2 - 1)\lambda$, and its consequence $\partial_s \int h^* \alpha_t = 0$.

$$(df_t + \alpha_t) \wedge \alpha_t = H d\gamma_t \wedge \alpha_t$$

$$= \int (\partial_s \int h^* d\gamma_t) \wedge \alpha_t$$

$$= \int (\partial_s \int h^* d\gamma_t) \wedge (h^* \alpha_t - (s^2 - 1)\lambda)$$

$$= \int (1 - s^2)(\partial_s \int h^* d\gamma_t) \wedge \lambda.$$ 

Since $d\gamma_t \wedge \alpha_t = o(|b, v|^{k-1})$, the first term is $o(|b, v|^k)$. In the second term, we have $\gamma_t = o(|b, v|^{k-1})$, hence $d\gamma_t = o(|b, v|^{k-2})$, and then $\partial_s \int h^* d\gamma_t = o(|b, v|^{k-1})$. Since $\lambda = O(|v|)$, everything is $o(|b, v|^k)$ as needed. \( \square \)

**Proof of Proposition 4.1.** We use the same notations as in the proof of Theorem 4.2. We want to prove that each $R_t^k$ is a submanifold of $X^k_t$ and $R_t^{k+1} \to R_t^k$ is a submersion. It suffices to prove it for $l = 1$, since for $l \geq 2$, $R_t^k$ is an open set in the $l$-fold product of $R_t^k$. 

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Let $p = \dim B$ and $2n + 1 = \dim V$. From Lemma 4.4, we see that $H^k_{p,2n+1}$ is a closed subgroup of $G^k_{p,2n+1}$, and hence a Lie subgroup according to É. Cartan. The Lie group homomorphism $H^k_{p,2n+1} \rightarrow H^{k-1}_{p,2n+1}$ is surjective and is thus a submersion.

To complete the proof, we only need to find local trivializations $X^{(k)} \simeq \mathbb{R}^p \times \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times G^k_{p,2n+1}$ which maps $R^k$ to $\mathbb{R}^p \times \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times H^k_{p,2n+1}$ and commute with the projections $X^{(k)} \rightarrow X^{(k-1)}$ and $\mathbb{R}^p \times \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times H^k_{p,2n+1} \rightarrow \mathbb{R}^p \times \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times H^{k-1}_{p,2n+1}.$

For this it will be convenient to use the Heisenberg group structure to choose, for each $y$ in $\mathbb{R}^{2n+1}$, a contactomorphism depending smoothly on $y$ and sending the origin to $y$. Indeed consider $\lambda = 1/2 \sum (p_i dq_i - q_i dp_i)$, $\alpha = dz + \lambda$ the standard contact form on $\mathbb{R}^{2n+1}$, and $\xi = \ker \alpha$. To this we associate the Heisenberg Lie group structure on $\mathbb{R}^{2n+1}$ where $(p_1, q_1, z_1) \cdot (p_2, q_2, z_2) = (p_1 + p_2, q_1 + q_2, z_1 + z_2 + d\lambda((p_1, q_1), (p_2, q_2)))$. Then $\xi$ is invariant under right translation $R_y: x \mapsto x \cdot y$.

We now build the trivializations. Fix a point $(b, v, w) \in B \times V \times V$, pick local charts $\psi: (U_b, b) \simeq (\mathbb{R}^p, 0)$, Darboux charts $\phi: (U_v, v) \simeq (\mathbb{R}^{2n+1}, 0)$ and $\theta: (U_w, w) \simeq (\mathbb{R}^{2n+1}, 0)$ (having images of charts that are whole spaces is convenient, and easily arranged since an open ball in standard contact space is isomorphic to the whole space). The $k$-jet at $(b', v')$ in $U_b \times U_v$ of a map $f: B \times V \rightarrow V$ with $f(b', v') = v''$ in $U_w$ is sent to $(\psi(b'), \phi(v'), \theta(w'), j^k g(0, 0))$ where $g: \mathbb{R}^p \times \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ is given by

$$g(x, y) = (R_{\theta^{-1}} \circ \theta \circ f) \left( \psi^{-1} (x + \psi(b')) \right), \phi^{-1} \circ R_{\phi^{-1}}(y).$$

Which is indeed a family of contact diffeomorphisms sending $(0, 0)$ to $0$. This map to $\mathbb{R}^p \times \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times H^k_{p,2n+1}$ is indeed a diffeomorphism: the inverse map sends $(x, y, z, j^k g(0, 0))$ to $j^k f(b', v')$ where $b' = \psi^{-1}(x), v' = \phi^{-1}(y)$, and

$$f(b'', v'') = (\theta^{-1} \circ R_y \circ g) \left( \psi(b'') - x, \phi(v'') \cdot y^{-1} \right).$$

\section{Cleaning contact isotopies}

The proof of the decomposition theorem in Section 6 will be reduced to contact isotopies satisfying technical hypotheses. The goal of this section is to prove that these hypotheses can be ensured by perturbation.

\textbf{Definition 5.1.} Let $(V, \xi)$ be a contact manifold, $L_-$ and $L_+$ be isotropic submanifolds of $V$. A contact isotopy $f: I \times V \rightarrow V$ is $(L_-, L_+)$-clean if its restriction $g: I \times L_- \rightarrow V$ satisfies:

(C-1) $g$ is transverse to $L_+$,
(C-2) $\forall (t, t', x) \in I \times I \times L_-$, if $g(t, x) \in L_+$ and $g(t', x) \in L_+$ then $t = t'$,
(C-3) $\forall (t, t', x) \in I \times I \times L_-$, if $g(t, x) \in L_+$, then $g$ is an immersion at $(t', x)$,
\((C.4)\) \(\forall (t, t', t'', x, x') \in I \times I \times I \times L_+ \times L_-, \) if \(g(t, x) \in L_+\) and \(g(t', x') = g(t'', x)\) then \(t' = t''\) and \(x' = x,\)

\((C.5)\) \(\forall (t, x) \in I \times L_-\), if \(g(t, x) \in L_+\) then for any equation \(\alpha\) of \(\xi\) the function \(\phi(s) = \alpha((\delta^I_{0I}(s, x))\) vanishes transversely, i.e. \(\phi(s) = 0 \Rightarrow \phi'(s) \neq 0.\)

If \(L_-\) and \(L_+\) are isotropic complexes (see definition 3.1) rather than submanifolds, we say that \(f\) is \((L_-, L_+)-\)clean if \(f\) is \((L_-^{(i)} \subset L_+^{(i-1)}, L_+^{(j)} \subset L_+^{(j-1)})\)-clean for all \(i, j \in \{0, . . . , n\}.\)

Note that if \(L_-\) is subcritical, that is \(L_- = L_-^{(n-1)},\) the conditions above simply reduce to \(f_t(L_-) \cap L_+ = \emptyset\) for all \(t.\)

**Proposition 5.2.** Let \((V, \xi)\) be a contact manifold of dimension \(2n + 1\) with \(2n + 1 \geq 5,\) and \(L_-, L_+\) disjoint isotropic complexes. The set of \((L_-, L_+)-\)clean isotopies is residual in \(\mathcal{D}_1(V, \xi).\)

**Proof.** For short, we will write \(\mathcal{D}\) for the space \(\mathcal{D}_1(V, \xi)\) of contact isotopies. Because of Remark 4.3, it is enough to prove the result in the case where \(L_-\) and \(L_+\) are submanifolds (of constant dimension). In all codimensions computations below, we will assume that \(L_-\) and \(L_+\) are of dimension \(n.\) If not then all codimensions would be higher, and the conclusion even stronger than needed. All the relevant submanifolds in jet spaces will be defined by independent conditions whose codimensions will thus add up to the submanifold codimension. A condition asking for some \(x \in X\) to be in a submanifold \(X'\) has codimension \(\text{codim}(X').\) A condition asking that \(x = x'\) in \(X\) means that \((x, x')\) is in the diagonal \(\Delta_X \subset X \times X\) hence has codimension \(\text{codim}(\Delta_X) = \dim X.\)

**Item (C-1) for \(f \in \mathcal{D}\)** is implied by (in fact equivalent to) the transversality of \(j^0 f\) to

\[
\Sigma_1 = \{I \times L_- \times L_+\}
\]

Indeed, assume \(j^0 f\) is transverse to \(\Sigma_1.\) Let \(\pi\) be the projection of \(T(I \times V)\) onto the normal bundle \(\nu \Sigma_1 = \{0\} \times \nu L_- \times \nu L_+.\) At any \((t, x)\) such that \(j^0 f(t, x)\) is in \(\Sigma, \pi \circ T_{(t, x)} f\) is surjective. In particular it is surjective onto \(\{0\} \times \{0\} \times \nu L_+.\) Hence \(T_{(t, x)} f|_{T_{t,I} \times T_{t,L_+}}\) is surjective onto \(\nu L_+.\)

**Item (C-2) is equivalent to asking that** \(j_2^0 f\) avoids:

\[
\Sigma_2 = \{(t, x, y), (t', x', y') \in (I \times V \times V)^2, x \in L_-, y \in L_+, x = x, y' = y' \\in L_+\}.
\]

The codimension of \(\Sigma_2\) in \(J_2^0(I \times V, V)\) is \(\text{codim}(L_-) + \text{codim}(L_+) + \dim(V) + \text{codim}(L_+) = 5n + 4.\) For \(n \geq 1, 5n + 4 \geq 4n + 4 = \text{dim}(I \times V)^2,\) so \(j^0 f\) avoiding \(\Sigma_2\) is equivalent to \(j_2^0 f \cap \Sigma_2.\)

In order to discuss conditions involving \(j^1 f,\) we will identify \(\text{Hom}(T_t I, T_y V)\) with \(T_y V\) by \(\varphi \mapsto \varphi(\partial_t),\) so that \(T_{(t, x)} f\) will be represented by some \((A, b)\) in \(\text{Hom}(T_{(t, x)} V, T_f_{(t, x)} V) \times T_f_{(t, x)} V.\)
Observe that $g$ being an immersion at $(t', x)$ is equivalent to $T_{t', x}g(\partial_t) \not\in T_{g(t, x)}\partial_t(L_-)$. Hence condition (C-3) is equivalent to asking that $j_2^1f$ avoids

$$\Sigma_3 = \{((t, x, y, A, b), (t', x', y', A', b')) : x \in L_-, y \in L_+, x' = x, b' \in A(T_{x'}L_-)\}.$$ 

This $\Sigma_3$ is a submanifold of $J_2^1(I \times V, V; \xi)$ whose codimension is $\text{codim}(L_-) + \text{codim}(L_+) + \text{dim}(V) + \text{dim}(L_-) = 5n + 4$. For $n \geq 1$, $5n + 4 > 4n + 4 = \text{dim}(I \times V)^2$, so $j_2^1f$ avoiding $\Sigma_3$ is equivalent to $j_2^1f \cap \Sigma_3$.

Item (C-4) is equivalent to $j_2^0f$ avoiding

$$\Sigma_4 = \{((t, x, y), (t', x', y'), (t'', x'', y'')) : x \in L_-, y \in L_+, x' = x, x'' \in L_-, y' = y''\}.$$ 

This $\Sigma_4$ is a submanifold of $J_2^0(I \times V, V)$ of codimension $\text{codim}(L_-) + \text{codim}(L_+) + \text{dim}(V) + \text{dim}(L_-) + \text{dim}(V) = 7n + 5$. For $n \geq 2$, $7n + 5 > 6n + 6 = \text{dim}(V \times I)^2$ so $j_2^0f$ avoiding $\Sigma_4$ is equivalent to $j_2^0f \cap \Sigma_4$.

Item (C-5) asks that if $g(t, x) \in L_+$ for some $(t, x)$ then the function $t \mapsto \alpha(T_{t, x}g(\partial_t))$ vanishes transversely. Consider the following submanifold of $J_2^1(I \times V, V; \xi)$:

$$\Sigma_5 = \{((t, x, y, A, b), (t', x', y', A', b')) : x \in L_-, y \in L_+, x' = x, \alpha(b') = 0\}.$$ 

We claim that transversality of $j_2^1f$ to $\Sigma_5$ implies condition (C-5) (it is actually equivalent under condition (C-1) but we will not need this equivalence). Assume $j_2^1f$ is transverse to $\Sigma_5$ for some $f$ in $D$. Let $((t, x), (t', x'))$ be a point in $(I \times V)^2 \setminus \Delta$ sent to $\Sigma_5$ by $j_2^1f$. The normal space of $\{x = x', x \in L_-\}$ in $V \times V$ at $(x, x)$ is isomorphic to $\nu_x L_- \times T_x V | [(u, v)] \mapsto [(u), u - v]$. Also, choosing a contact form $\alpha$ allows to identify $TV/\xi$ with $V \times \mathbb{R}$, and we set $\phi_f(t, x) = \alpha(T_{t, x}f(\partial_t))$. Using these identifications and notations, transversality of $j_2^1f$ at $p$ becomes surjectivity of the map

$$T_{t, x}V \times T_{t, x}V \to \nu_x L_- \times \nu_y L_+ \times T_x V \times T_0 \mathbb{R},$$

$$(\tau, u, \tau', u') \mapsto \left(\left[u\right], \left[T_{t, x}f(\tau, u)\right], u - u', T_{t, x}'\phi_f(\tau', u')\right).$$

In particular, this map is surjective onto $\{0\} \times \nu_y L_+ \times \{0\} \times \{0\}$, so $T_{t, x}f$ induces an isomorphism from $T_t I \times T_x L_- \to \nu_{f(t, x)} L_+$. Next we use surjectivity onto $\{0\} \times \{0\} \times \nu_0 \mathbb{R}$ to get existence of $(\tau, u, \tau', u')$ such that $u$ is in $T_x L_-$, $T_{t, x}f(\tau, u)$ is in $T_{f(t, x)}L_+$, $u' = u$, and $T_{t, x}'\phi_f(\tau', u')$ is not zero. The first two conditions imply $u = 0$ by the isomorphism above. Hence $u' = 0$ by the third condition and the last condition becomes $\partial_{x'}\phi_f(t', x') \neq 0$ as desired.

Transversality to $\Sigma_i$ for all $i \in \{1, \ldots, 5\}$ determines a residual subset of $D$ by the transversality theorem, Theorem 4.2, and we have seen that it implies cleanness.
6 The decomposition theorem

The goal of this section is to prove the next result, which implies Theorem D from the introduction, and is our main geometrical ingredient.

Theorem 6.1. Let \((V, \xi)\) be a contact manifold and \(L_-, L_+\) be two disjoint compact isotropic complexes such that \(L_-\) is loose in the complement of \(L_+\). Any contact isotopy is homotopic, with fixed end-points, to a composition \(f_t = g_t \circ f_t^- \circ f_t^+ \circ g'_t\), where \(g_t\) and \(g'_t\) have support in Darboux balls \(B\) and \(B'\), and \(f^\pm\) has compact support away from \(\varphi_1(L_\pm)\) for some contact isotopy \(\varphi\).

Remark 6.2. Carefully reading the proof of the above theorem reveals that \(\varphi\) can be chosen arbitrarily small in \(C^1\) topology, and with support in an arbitrarily small neighborhood of \(L_- \cup L_+\). Since we have no use for this refinement, we neither include it in the statement, nor follow the size of \(\varphi\) along various reductions.

The discussion of Theorem 6.1 will use the following technical definition.

Definition 6.3. Let \(L_-, L_+\) be two subsets of a closed contact manifold \((V, \xi)\). An \((L_-, L_+)-decomposition\) for a contact isotopy \(f\) is a factorization for all \(t\)

\[ f_t = g_t \circ f_t^- \circ f_t^+ \circ g'_t \]

where \(g\) and \(g'\) have compact support in Darboux balls \(B\) and \(B'\), and each \(f^\pm\) has compact support outside of \(L_\pm\).

Using the above definition, the conclusion of Theorem 6.1 is that \(f\) is homotopic to an isotopy admitting an \((\varphi_1(L_-), \varphi_1(L_+))-decomposition\) for some contact isotopy \(\varphi\).

The proof of the decomposition theorem splits into two independent parts. First we explain in Section 6.1 that the approximation result, Proposition 5.2, can be used to reduce to clean isotopies, as introduced in Definition 5.1. Then the crucial part, where the flexibility assumption appears, is Proposition 6.4 below, which will be proved in Section 6.2.

Proposition 6.4. Let \((V, \xi)\) be a contact manifold of dimension \(2n + 1\) with \(2n + 1 \geq 5\), and \(L_-, L_+\) two disjoint compact isotropic complexes such that \(L_-\) is loose in \(V \setminus L_+\). If a contact isotopy is \((L_-, L_+)-clean\) then it has an \((L_-, L_+)-decomposition\).

6.1 Reduction to the clean case

The goal of this section is to prove that Theorem 6.1 follows from Proposition 6.4. The first crucial lemma is a simple consequence of how contact Hamiltonian allow to cut-off contact isotopies.

Lemma 6.5. Let \(K_-\) and \(K_+\) be two compact subsets in a contact manifold \((V, \xi)\). Let \(f\) be a contact isotopy of \((V, \xi)\). If, for all \(t\), \(f_t(K_-)\) is disjoint
from $K_+$ then one can decompose $f_t$ as $f_t^- \circ f_t^+$ where each $f_t^\pm$ has support away from $K_+$. Alternatively, one can decompose $f_t$ as $f_t^+ \circ f_t^-$, with the same support constraints.

**Proof.** By assumption, $f([0,1] \times K_-)$ and $K_+$ are disjoint compact subsets in $V$. Hence there exists a cut-off function $\rho$ with compact support which equals one on a neighborhood of $f([0,1] \times K_-)$ and vanishes on a neighborhood of $K_+$.

Let $X_t$ be the time-dependent vector field generating $f$ and let $H_t$ be the Hamiltonian function corresponding to $X_t$ (either using an auxiliary contact form or seeing $H_t$ as a section of $TV/\xi$). Let $Y_t$ be the time-dependent contact vector field corresponding to the Hamiltonian $\rho H_t$. Because $Y_t$ vanishes outside the support of $\rho$, its flow $f_t^\rho$ is defined for all time $t$ in $[0,1]$ and $f_t^\rho$ is the identity on a neighborhood of $K_+$.

In addition, $f_t^+ = f_t$ on $K_-$. We set $f_t^- = (f_t^+)^{-1} \circ f_t$ or $f_t^- = f_t \circ (f_t^+)^{-1}$, depending on the desired decomposition order. □

**Proof of Theorem 6.1.** Let $f$ be any contact isotopy of $(V, \xi)$. Then Proposition 5.2 gives a contact isotopy $\tilde{f}$ which is an arbitrarily small perturbation of $f$ and is $(L_-, L_+)$-clean. We choose it small enough to make sure that, for all $t$, $\tilde{f}_t^{-1} \circ f_t(L_-)$ is disjoint from $L_+$. Then Lemma 6.5 constructs contact isotopies $\delta^-$ and $\delta^+$, with support disjoint from $L_-$ and $L_+$ respectively, such that, for all $t$,

$$\tilde{f}_t^{-1} \circ f_t = \delta_t^+ \circ \delta_t^-.$$

Proposition 6.4 gives a $(L_-, L_+)$-decomposition of $\tilde{f}$: $\tilde{f}_t = \bar{g}_t \circ \tilde{f}_t^- \circ \tilde{f}_t^+ \circ \bar{g}_t'$, where $g_t$ (resp. $g_t'$) has support in some Darboux ball $B$ (resp. $B'$), and $\tilde{f}_t^\pm$ has support away from $L_\pm$. This can be rewritten as:

$$f_t = \bar{g}_t \circ \tilde{f}_t^- \circ \delta_t^- \circ (\delta_t^-)^{-1} \circ (\tilde{f}_t^+ \circ \delta_t^+) \circ \delta_t^- \circ (\delta_t^+ \circ \delta_t^-)^{-1} \circ \bar{g}_t' \circ (\delta_t^+ \circ \delta_t^-).$$

Lemma 1.1 ensures this isotopy is homotopic to:

$$t \mapsto \bar{g}_t \circ f_t^- \circ c_{(\delta_t^-)^{-1}}(\tilde{f}_t^+ \circ \delta_t^+) \circ c_{(\delta_t^+ \circ \delta_t^-)^{-1}}(\bar{g}_t').$$

where $f_t^\pm$ is relative to $(\delta_t^-)^{-1}(L_+)$, and $g_t'$ has support in the Darboux ball $B' := (\delta_t^+ \circ \delta_t^-)^{-1}B'$, so we set $\varphi_t = (\delta_t^-)^{-1}$ (note that $\varphi_t(L_-) = L_-$ for all $t$). □

### 6.2 Decomposition of clean isotopies

In this section, we prove Proposition 6.4. Remember clean isotopies were introduced in Definition 5.1. Numbered conditions like Item (C-1) in the proof below refer to items in this definition.
The proof is organized into a sequence of steps. Each step uses the statements and notations of previous steps but not their proofs, so step proofs can be checked independently.

Let \( f : [0, 1] \times V \to V \) be a \((L_-, L_+)\)-clean contact isotopy. The first step sets the stage without modifying \( f \), essentially unpacking consequences of the definition of clean isotopies, but also using an engulfing argument relying on the \( h\)-principle for transverse arcs. The second step composes \( f \) with Darboux ball-supported isotopies on both sides to get \( f' \) with convenient fixed loose charts, using Murphy’s flexibility theorem. The third step uses these loose charts, and five invocations of Murphy’s theorem, to deform the isotropic isotopy \( f'|_{L^-} \) until there is no more collision with \( L^+ \). The fourth steps lifts this deformation to a deformation \( f'' \) of \( f' \) by post-composition with a Darboux ball-supported isotopy. The conclusion applies Lemma 6.5 to \( f'' \).

In this proof, the word ball, without adjective, always mean a closed codimension 0 ball in \( V \) with smooth boundary. The word disk will always mean a closed codimension 0 ball in \( L_- \) with smooth boundary.

**Step 1. (See Fig. 2)**

\(1.a\) There exists a finite collection of distinct points \( x_i \in L_+^{(n)} \setminus L_-^{(n-1)} \) and times \( t_i \in (0, 1] \) such that \( f^{-1}(L_+) \cap ([0, 1] \times L_-) = \{(t_i, x_i)\} \).

\(1.b\) There exists a collection of pairwise disjoint disks \( D_i \subset L_+^{(n)} \setminus L_-^{(n-1)} \) centered at \( x_i \), whose union is denoted by \( D \), such that \( f : [0, 1] \times D \to V \) is an embedding.

\(1.c\) There exists a collection of pairwise disjoint balls \( C_i \subset V \), whose union is denoted by \( C \), such that, \( f^{-1}(C_i) \cap ([0, 1] \times L_-) = [0, 1] \times D_i \), and each \( f_t : D_i \to C_i \) is a neat embedding for all \( t \).

\(1.d\) There is a Darboux ball \( B \) containing \( C \) in its interior.

![Figure 2](image)

The set \( X = f^{-1}(L_+) \cap ([0, 1] \times L_-) \) is closed in \([0, 1] \times L_- \) (since \( L_+ \) is closed) and hence compact (since \( L_- \) is compact). Item (C-1) in Definition 5.1 implies that \( X = f^{-1}(L_+) \cap ([0, 1] \times (L_+^{(n)} \setminus L_-^{(n-1)})) \) and that \( X \) is discrete. Hence \( X \) is a finite collection of points \((t_i, x_i)\). Item (C-2) implies that the \( x_i \)
are pairwise distinct. Item (C-4) implies that the arcs \( \gamma_i := f([0,1] \times \{x_i\}) \) are pairwise disjoint and simple. Item (C-3) implies they are embedded. Item (C-5) implies that they are transverse to \( \xi \) except at finitely many points. We claim that any neighborhood \( U \) of an arc \( \gamma \) with this property contains a Darboux ball containing \( \gamma \) in its interior. This then allows us to construct a collection of pairwise disjoint Darboux balls \( B_i \) containing \( \gamma_i \) in its interior and the required Darboux ball \( B \) by a connect sum operation along transverse arcs joining north and south poles of the balls \( B_i \). To prove the claim, pick an open ball \( W \) containing \( \gamma \) in its interior and contained in \( U \) and let \( \{p_j\} \) be the set of points \( x \) where either \( \gamma \) is not transverse to \( \xi \) or \( x \) is an end-point of \( \gamma \). Pick disjoint Darboux balls \( A_j \) centered at the points \( p_j \) and contained in \( W \) and connect the balls \( A_j \) through transverse arcs disjoint from \( \gamma \) to get a single Darboux ball \( A \) contained in \( W \). Then \( \gamma \cap (W \setminus A) \) consists of finitely many transverse arcs with boundary on \( \partial A \), which can be pushed inside of \( A \) by a smooth isotopy since \( \pi_1(W,A) = 0 \). By \( h \)-principle for transverse arcs, there is a transverse isotopy of \( \gamma \) in \( W \) which takes \( \gamma \) in the interior of \( A \). Lift this isotopy to a contact isotopy \( \theta_t \) for \( t \in [0,1] \) supported in \( W \), and the ball \( \theta_t^{-1}(A) \) contains the initial arc \( \gamma \) in its interior and is contained in \( W \) and hence in \( U \).

We now construct the disks \( D_i \) and balls \( C_i \). Note first that \( L^{(n)}_\epsilon \setminus L^{(n-1)}_\epsilon \) is supported in \( \text{Int} \{ \gamma_i \} \) according to Definition 3.1. According to Item (C-4), we have \( f^{-1}(\gamma_i) \cap ([0,1] \times L_-) = [0,1] \times \{x_i\} \). Moreover, \( f \) immerses \([0,1] \times L_- \) near \((t,x_i)\) for all \( t \). A compactness argument then shows that (1.b) holds and \( f^{-1}(f([0,1] \times D_i)) = [0,1] \times D_i \) as soon as the radius of \( D_i \) is sufficiently small. We may then extend \( f([0,1] \times D_i) \) to a ball \( C_i \) such that \([0,1] \times D_i \cap f^{-1}(\partial C_i) = [0,1] \times \partial D_i \) and \( f([0,1] \times D_i) \) is transverse to \( \partial C_i \). Property (1.c) then holds as soon as \( C_i \) is sufficiently thin. Since \( \gamma \subset \text{Int} \; B \), we may assume that \( C_i \subset \text{Int} \; B \) in the above construction.

**Step 2.** (See Figs. 3 and 4) There exist a contact isotopy \( g' \) supported in a Darboux ball \( B' \), a contact isotopy \( h \) supported in \( B \), a collection of closed disks \( D'_i \subset \text{Int} \; D_i \setminus \{x_i\} \) and \( \epsilon \in (0, \min_i t_i) \) such that the contact isotopy \( f'_i := h_t^{-1} \circ f_t \circ (g'_t)^{-1} \) satisfies:

1. \( (f')^{-1}(L_+) \cap ([0,1] \times L_-) = \bigcup_i \{(t_i, x_i)\} \),
2. \( f_t = f'_t \) on \( L_- \setminus D''_i \) for \( t \geq \epsilon \) and \( f'([\epsilon,1] \times D''_i) \cap f'([\epsilon,1] \times (L_- \setminus D''_i)) = \emptyset \)
3. \( f'([\epsilon,1] \times D''_i) \subset C_i \),
4. for \( t \geq \epsilon \), the diffeomorphisms \( f'_t \) restrict to neat Legendrian embeddings \( k_t : D_i \to C_i \) admitting a fixed loose chart \( U_t \subset (\text{Int} \; C_i \setminus (\gamma_i \cup L_{\pm})) \) (fixed means that \( k_t^{-1}(U_t) \) and \( k_t(k_t^{-1}(U_t)) \) are independent of \( t \)).

We pick a collection of disjoint loose charts \( W_i \) in \( V \setminus (L_{\pm} \cup L^{(n-1)}_{\pm}) \) for the connected components of \( L^{(n)}_\epsilon \setminus L^{(n-1)}_\epsilon \) containing \( x_i \) as granted by Definition 3.4 (the fact that the loose charts can be assumed pairwise disjoint is not obvious but follows from Remark 2.3 together with Theorem 2.8). Pick a collection
of pairwise disjoint arcs $\gamma'_i$ in $L^{(n)} \setminus L^{(n-1)}$ connecting $\partial W_i$ to $\partial D_i$. After pushing $W_i$ along a contact vector field tangent to $\gamma'_i$ and $L_-$, and supported in a small neighborhood of $\gamma'_i$, we obtain a new loose chart $W'_i$ for $L_-$ such that $W'_i \cap L_+ = \emptyset$ and $D'_i := W'_i \cap L_-$ intersects $\text{Int} D_i$. A Darboux ball $B'$ containing all the balls $W'_i$ in its interior can be obtained by a connect sum operation along transverse arcs.

Pick a disk $D''_i \subset ([\text{Int} D'_i \cap \text{Int} D_i) \setminus \{x_i\}]$. We claim that there is a Legendrian isotopy $g''_u : D'_i \to W'_i$ such that

- $g''_0$ is the inclusion,
- $g''_u = g''_0$ near $\partial D'_i$,
- $g''_u = g''_0$ in $D'_i \setminus D''_i$,
- $g''_u(D'_i)$ has a loose chart $U_i$ such that $f([0, 1] \times U_i) \subset (\text{Int} C_i) \setminus (L_+ \cup f([0, 1] \times (D_i \setminus D''_i)))$,
- $f([0, 1] \times g''_u(D''_i)) \subset \text{Int} C_i \setminus (L_+ \cup f([0, 1] \times (D_i \setminus D''_i)))$.

Using Remark 2.3, we may first construct a formal Legendrian isotopy supported in a small neighborhood of a point of $D''_i$ satisfying the above properties.

It is a priori genuine only at $u = 0$ and $u = 1$. But using looseness, Theorem 2.8 (with $p = 1$, relative to $\partial D'_i$ in the source and to $\partial W'_i$ in the target) allows us to deform it into a genuine Legendrian isotopy with fixed end-points. Extend this Legendrian isotopy to a contact isotopy, still denoted $g''$, supported in the union of all $W'_i$, and thus in $B'$. Since $g''$ has support in this union, $g''([0, 1] \times L_-) \cap L_+$ is empty. Hence there exists a positive $\epsilon$ such that $f_t(g''([0, 1] \times L_-)) \cap L_+$ is empty for all $t \leq \epsilon$.

We set $g'_t = (g''_t/\epsilon)^{-1}$ for $t \leq \epsilon$ and $g'_t = (g''_t)^{-1}$ for $t \geq \epsilon$ (or rather a version smoothed at $t = \epsilon$). Our choice of $\epsilon$ ensures that $f_t \circ (g'_t)^{-1}(L_-) \cap L_+ = \emptyset$ for $t \leq \epsilon$. In addition $U_i$ is a loose chart for $(g'_t)^{-1}(L_-)$ for all $t \geq \epsilon$. 

Figure 3
But \( f_t(U_i) \) is moving, so the last thing to do is to fix this loose chart using an isotopy \( h \). For this, we cut off the contact vector field generating the isotopy \( f_t \) by multiplying the corresponding section of \( TV/\xi \) by a function \( \rho \): \([0, 1] \times V \rightarrow \mathbb{R} \) equal to 1 on the image of \([0, 1] \times U_i \) by the embedding \((t, x) \mapsto (t, f_t(x))\), and supported in \([0, 1] \times (\text{Int } C_i \smallsetminus (\gamma_i \cup L_+))\). We obtain a contact isotopy \( h_t \) such that \( h_t = f_t \) on \( U_i \) for all \( t \) and hence the family of isotropic embeddings \( h_t^{-1} \circ f_t \circ (g_t')^{-1}((L_+)) \) has a fixed loose chart \( U_i \) for all \( t \geq \epsilon \), ensuring Property (2.d). It is straightforward to check Properties (2.a) to (2.c) using the properties of \( g_t'' \) listed above. The support of \( h \) is in \( B \) thanks to Property (1.d).

**Step 3.** (See Fig. 5) There is a family \( k_{t,s} : D \rightarrow C \), \((t, s) \in [\epsilon, 1] \times [0, 1] \) of neat Legendrian embeddings such that

(3.a) \( k_{t,0} = k_t \),

(3.b) \( k_{t,s} = k_t \) for \( t \) near \( \epsilon \),

(3.c) \( k_{t,s} = k_t \) near \( \partial D \),

(3.d) \( k_{t,1}^{-1}(L_+) = \emptyset \).

We fix \( \eta \) in \((\epsilon, \min_t t_i)\) and \( \theta_s : [\epsilon, 1] \times D \rightarrow [\epsilon, 1] \times D \), \( s \in [0, 1/2] \), a family of embeddings, starting with \( \text{Id} \), supported in \([\eta, 1] \times (D \smallsetminus D'')\) and such that \( \theta_{1/2}([\epsilon, 1] \times D) \cap ([t_i, 1] \times \{x_i\}) = \emptyset \). We set \( k'_{t,s}(x) = k(\theta_s(t, x)) \). Thanks to (1.b) and (2.b), we see that for each \( t \in [\epsilon, 1] \) and \( s \in [0, 1/2] \), \( k'_{t,s} : D \rightarrow C \) is a neat embedding. This satisfies Properties (3.a) to (3.d) but has no reason to be Legendrian for \( s > 0 \). However, it inherits from \( k_{t,0} \) the structure of a formal Legendrian embedding in a homotopically unique way. We will extend the set of \((t, s)\) for which \( k'_{t,s} \) is defined to \([\epsilon, 2] \times [0, 1] \), and then deform the extended
family into a family whose restriction to $[\epsilon, 1] \times [0, 1]$ will be genuinely Legendrian without destroying Properties (3.a) to (3.d). In particular it is important that, for all $s$, $k_{t,s}'|_{\partial D}$ will be $k_{t}'|_{\partial D}$ if $t \leq 1$ and $k_{t}'|_{\partial D}$ if $t \geq 1$. For this, we will apply Theorem 2.8 several times using the fact that the embeddings $k_{t,s}'$ have a fixed loose chart $U_\iota$ by (2.d). Note that we can guarantee that a fixed loose chart remains during each application of Theorem 2.8 (the idea is that a loose chart contains two disjoint loose charts, so we may use one and let the other fixed), we will not repeat it each time. We first apply Theorem 2.8 with $p = 0$, to the embedding $k_{2,1}$ in the contact manifold $C_i \setminus L$ to find a family of formal Legendrian embeddings $k_{t,1}$ for $t \in [1, 2]$ so that $k_{2,1}$ is genuine and $k_{t,1}(D_i) \cap L = \emptyset$. Then we apply Theorem 2.8 with $p = 2$ to the family $k_{t,s}$ with $t \in [1, 2]$ and $s \in [0, 1]$ in the contact manifold $C_i$ to find a family of neat Legendrian embeddings $k_{t,s}$ satisfying Properties (3.a) to (3.d).

**Step 4.** There exists a contact isotopy $l_t$ supported in $B$ such that $(l_t \circ f'_t)(L_-) \cap L_+ = \emptyset$ for all $t$.

For $t \in [\epsilon, 1]$ and $s \in [0, 1]$, we consider the family of vector fields $\nu_{t,s} = \frac{dk_{t,s}}{ds}$ along the embedding $k_{t,s}$. Since $k_{t,s}$ is Legendrian for all $t$ and $s$, it can be extended to a smooth family of contact vector fields $\tilde{\nu}_{t,s}$ supported in an arbitrarily small neighborhood of the support of $\nu_{t,s}$. In particular, we may assume using (3.c) that the support of $\tilde{\nu}_{t,s}$ is contained in the interior of $C_i$. Moreover, we may assume that $\tilde{\nu}_{t,s} = 0$ near $t = \epsilon$ thanks to (3.b). The family
of vector fields $\tilde{\nu}_{t,s}$ then integrates uniquely into a smooth family of contact diffeomorphisms $\psi_{t,s}$ such that $\psi_{t,0} = \text{Id}$ and $\frac{d\psi_{t,s}}{dt} = \tilde{\nu}_{t,s} \circ \psi_{t,s}$. By construction and using (3.a), we get $k_{t,s} = \psi_{t,s} \circ k_t$, and $\psi_{t,s} = \text{Id}$ for $t$ near $\epsilon$. We may thus smoothly extend the family $\psi_{t,s}$ by defining $\psi_{t,s} = \text{Id}$ for $t \leq \epsilon$. We now set $l_t = \psi_{t,1}$, which is supported in $C$ hence in $B$, and get $(l_t \circ f')(L_{-}) \cap L_{+} = \emptyset$ for all $t$.

**Conclusion** Define $g_t := h_t \circ (l_t)^{-1}$ which is a contact isotopy supported in $B$. We have achieved $g_t^{-1} \circ f_t \circ (g_t')^{-1}(L_{-}) \cap L_{+} = \emptyset$ for all $t$. The claim now follows from Lemma 6.5.

### 7 Commutators, fragments and conjugates

In this section we prove all theorems stated in the introduction, except Theorem D which was already subsumed in Theorem 6.1. We first need a lemma which is used in several of them.

**Lemma 7.1.** Let $(V, \xi)$ be a contact manifold and let $G$ denote either $\mathcal{D}_0(V, \xi)$ or its universal cover. Let $B$ be a Darboux ball inside $V$, $p$ a point in the interior of $B$, and $g$ an element of $G$. If $g(p)$ is in $B \setminus \{p\}$ then every element of $G$ with support in the interior of $B$ is a product of eight conjugates of $g^\pm 1$.

**Proof.** The interior of $B$ is isomorphic to $\mathbb{R}^{2n+1}$ equipped with $\ker(dz + \lambda)$ where $\lambda$ is the radial Liouville form on $\mathbb{R}^{2n}$, and $p$ gets mapped to the origin. Let $f$ be an element of $G$ with compact support in the interior of $B$. We consider the Heisenberg dilatation flow $\delta_t: (x, y, z) \mapsto (e^{-t}x, e^{-t}y, e^{-2t}z)$. Note that each ball $B_r := \{ \|x\|^4 + \|y\|^4 + \|z\|^2 < r \}$ is preserved by $\delta_t$, $t \geq 0$. We fix $R$ large enough to make sure that both $g(p)$ and the support of $f$ are in $B_R$, and that $p$ is in $g(B_R)$. We cut $\delta_t$ between radius $4R$ and $5R$ to ensure that it extends to a global flow $\varphi$ of $G$ which compresses $V' := B_{4R}$ onto $\{p\}$. We also cut it between radius $2R$ and $3R$ to get a flow $\theta$ with support in $V'$ and compressing $V'' := B_{2R}$ onto $\{p\}$. It only remain to apply Rybicki’s theorem and Proposition 1.2. \[ \square \]

**Proof of Theorems B and C.** Giroux 2002 ensures that $(V, \xi)$ has a supporting open book with Weinstein pages. Then Proposition 3.5 gives isotropic complexes $L_{-}$ and $L_{+}$, and a contact flow $\varphi_t$ which retracts the complement of one complex onto the other.

Let $G$ be either $\mathcal{D}_0(V, \xi)$ or its universal cover. According to Lemma 6.5, there is a $C^0$-neighborhood of the identity in $G$ or $\tilde{G}$ such that all elements can be written as a product $f^- \circ f^+$ where $f^\pm$ has compact support in the complement of $L_{\pm}$. Alternatively, if we assume the existence of an open book with flexible pages, then Theorem 6.1 decomposes any element of $G$ or $\tilde{G}$ as $g \circ f^- \circ f^+ \circ g'$ where $g$ and $g'$ have support in Darboux balls $B$ and $B'$, and each $f^\pm$ is relative to $\varphi_t(L_{\pm})$ for some contact isotopy $\varphi_t$. Up to conjugating $f^\pm$ by this contact isotopy $\varphi$, we may assume that $f^\pm$ is relative to $L_{\pm}$.
According to Proposition 3.11 (see also Remark 3.12), if \( \psi_\epsilon \) is a positive or negative contact isotopy then \( \psi_\epsilon \) displaces \( L_- \cup L_+ \) for all \( \epsilon \) in some interval \((0, \epsilon_0]\). In the flexible page case, after reducing \( \epsilon_0 \) if needed, we can also assume there exists \( p \) in \( B \) (resp. \( p' \) in \( B' \)) such that \( \psi_\epsilon(p) \) is in \( B \setminus \{p\} \) (resp. \( \psi_\epsilon(p') \) is in \( B' \setminus \{p'\} \)). Such a small \( \epsilon \) is now fixed until the end of the proof.

In both cases we have a decomposition into \( k + 2 \) contact transformations, where \( k = 0 \) in the first case and \( k = 2 \) in the second case. We only need to prove that each piece is a product of eight conjugates of \( \psi_{\pm 1} \). This property is invariant under conjugation so we are free to conjugate each piece (separately). We apply Proposition 1.2 to each piece. Pieces with support in Darboux balls are handled by Lemma 7.1.

We now explain how to deal with \( f^- \), the case of \( f^+ \) being completely symmetric. Let \( V'' = V \setminus N \) where \( N \) is a compact neighborhood of \( L_- \) so small that \( \text{supp}(f^-) \subset V'' \), \( \psi_\epsilon(L_+) \subset V'' \) and \( L_+ \subset \psi_\epsilon(V'') \). We then cut the Hamiltonian defining \( X_t \) near \( L_- \) in order to obtain a flow \( \theta_t \) which has compact support in \( V'' \) but agrees with \( \phi_t \) in \( V'' \). In particular \( \theta \) compresses \( V'' \) onto \( L_+ \). It only remain to apply Rybicki’s theorem and Proposition 1.2. □

Proof of Theorem A. Let \( G \) be either \( D_0(V, \xi) \) or its universal cover. Let \( L_- \) and \( L_+ \) be isotropic complexes associated to an open book with flexible pages as in the proof of Theorem B. We claim there is some \( \psi_\epsilon \) in \( G \) which displaces \( L := L_- \cup L_+ \) and is a product of \( n = (\dim V - 1)/2 \) elements with support in Darboux balls. Indeed, let \( \phi_t \) be a small Reeb flow displacing \( L \) for all positive \( t \) (see Proposition 3.11). Lemma 3.13 (see also Remark 3.14) gives contact isotopies \( \phi_0, \ldots, \phi_n \) such that \( \psi_\epsilon := \phi_0 \circ \cdots \circ \phi_n \) coincides with \( \phi_t \) near \( L \) for small \( t \). Hence there is some \( \epsilon \) such that \( \psi_\epsilon \) displaces \( L \).

The proof of Theorem B proves more generally that every element \( f \) of \( G \) is a product of 32 conjugates of \( \psi_\epsilon \), hence of \( 32(n+1) \) elements \( f_i \) with support in Darboux balls \( B_i \). Let \( g \) be any element of \( G \) displacing some point \( p \) (this means any non-trivial element in the case of \( D_0(V, \xi) \), or any element not lying over the identity in the universal cover case). Let \( B \) be a Darboux ball containing both \( p \) and \( g(p) \) in its interior. Because \( G \) acts transitively on the set of Darboux balls (see e.g. Geiges 2008, Theorem 2.6.7) each \( f_i \) is conjugated to an element with support in \( B \) hence to a product of eight conjugates of \( g_{\pm 1} \) by Lemma 7.1. So \( f \) itself is conjugated to a product of at most \( 256(n+1) \) conjugates of \( g_{\pm 1} \). □

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