On a proof of the collapse conjecture for a diagonal Bianchi type-IX vacuum space-time

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Abstract
It is given a simple proof of the collapse conjecture for a diagonal Bianchi type-IX vacuum space-time. It is shown that the codimension of the infinity stable attractor, restricted to the anisotropy plane, is not zero, thus proving that “escape along a channel” is impossible.

1 Introduction
We present a simple proof of the collapse conjecture for a diagonal Bianchi type-IX vacuum space-time. The collapse conjecture for a Bianchi type-IX space-time has been proved by Lin and Wald (1989-90) [1, 2]. We shall consider here the simple case of a diagonal Bianchi type-IX universe in vacuum. The demonstration presented here differs significantly from the one given in [1]. The collapse conjecture states that an initially expanding Bianchi type-IX space-time can not expand for an infinite time. It should, in a finite time, reach a maximum of expansion and then begin to contract. The proof given in [1] that there do not exist solutions which expand forever is divided in two steps. In the first step it is proved that for any solution which expands forever the dynamical trajectory in the anisotropy plane must “escape to infinity” along one of the channels of the Bianchi type-IX potential. The second step uses detailed properties of the equations of motion to show that such “escape along a channel” is impossible.

The Bianchi type-IX model (dubbed mixmaster) has been intensively study over the years. This model was investigated by Belinskii, Khalatnikov and Lifshitz [3, 4] and Misner [5]. One of the most appealing properties is its chaotic motion dynamics. The emergence of chaos helped the understanding of the singularities in general relativity [5, 6, 7, 8] and the debate is still raging (see [7] for a clear insight).

For historical reasons the approach to the study of the Bianchi model was to rewrite Einstein equations to a Hamiltonian form (with a constrain). This was intensively pursued by Misner [9, 8], and several authors, for which a nice Hamiltonian system was constructed, although this specific form of Hamiltonian was not used in the proof of the collapse conjecture by Lin and Wald [1, 2].

Following Lin and Wald [1, 2], we argue in this paper in favour of a alternative form of writing the equations of motion for the mixmaster universe, namely, the form of a canonical dissipative system [10] (pg. 115, and references there in). Dissipative systems are common in general relativity [12, 11] and in particular in Bianchi type models. Under some conditions, for instance if one consider expanding universes solutions, one can show that there exists functions that are monotonous decreasing along the solutions of the Einstein equations of the model under consideration due to the monotonous behaviour of the determinant of the metric (see, for instance, [13] for particular relevant examples).

The basic idea in this work is to write the equations of motion for the mixmaster model in terms of quantities relative to the overall rate of expansion of the universe, or equivalent, to
the Hubble scalar [11]. This latter form is a natural way of writing the equations of motion and clearly reveals the dissipative nature of the Einstein equations. In fact one can look for the Bianchi type-IX equations as a system of non-linear coupled and non-linear damped oscillators, where the non-linear damping is proportional to the square root of the energy of an equivalent mechanical system. This very specific form of dissipation is a clear signature of its dissipative nature. This is the source of the simplicity of the proof given in this work. By studying the dynamics relatively to the overall rate of expansion of the universe one can reduce the problem of studying the “escape along a channel” to the dynamics of a planar dynamical system in the anisotropy plane, and show that the codimension of the infinity stable attractor, restricted to the anisotropy plane, is not zero, thus proving that “escape along a channel” is impossible.

Let us start by stating some general definitions.

The general Bianchi type-IX space-time has topology $\mathbb{R} \times S^3$, with a simply transitive action of the isometry group $SU(2)$ on $S^3$ spatial slices.

It is described by the following metric [14],

$$ds^2 = -d\tau^2 + e^{2\alpha} \sum_{i,j=1}^{3} \left[ e^{[\beta]} \right]_{ij} d\sigma^i d\sigma^j.$$  \hspace{1cm} (1)

Here $\sigma^i, i = 1, 2, 3,$ are isometry invariant one-forms on the three-sphere $S^3$, $\alpha$ is a scalar, and $[\beta]$ is a traceless $3 \times 3$ matrix. Both $\alpha$ and $[\beta]$ are functions of the proper time $\tau$ only.

One should not forget that there is a particular relevant example of a vacuum spatially-homogeneous model which also has topology $\mathbb{R} \times S^3$, with a simply transitive action of the isometry group $SU(2)$ on $S^3$ spatial slices, the Taub-NUT space-time. This model, as was shown by Misner [16], does not possess a diagonal metric and although non-singular cannot be extended (for details see, [14], pg. 138). The Taub-NUT space-time represents a universe which evolves from an open universe, to closed and an open universe again. The dynamical behaviour of this model for the local rotation symmetric case can be studied by determining the explicit solution of the Einstein equations (same reference, pg. 139) which show that the determinant of the metric is always non-null.

For Bianchi type-IX case the picture is not so simple. For a diagonal space-time, let $\beta_i, i = 1, 2, 3$, denote the diagonal elements of the matrix $[\beta]$. Only two of these quantities are independent since $\beta_1 + \beta_2 + \beta_3 = 0$ on account of the tracelessness of $[\beta]$. We choose the independent variables to be

$$b_1 = -\frac{1}{2\sqrt{6}} \beta_3,$$  \hspace{1cm} (2)

$$b_2 = \frac{1}{6\sqrt{2}} (\beta_1 - \beta_2),$$  \hspace{1cm} (3)

which measure the departure from isotropy. In this case the Einstein vacuum equations takes the form [14],

$$3\dot{\alpha}^2 = \frac{1}{2} \left( \dot{b}_1^2 + \dot{b}_2^2 \right) + e^{-2\alpha} V(b_1, b_2),$$  \hspace{1cm} (4)

$$\ddot{b}_i + 3\dot{\alpha} \dot{b}_i + e^{-2\alpha} \frac{\partial V}{\partial b_i} = 0, \hspace{0.5cm} i = 1, 2,$$  \hspace{1cm} (5)

\footnote{Note that Lin-Wald [1] considered the variables $\beta_+ = \sqrt{6} b_1$ and $\beta_- = \sqrt{6} b_2$.}
with
\[ V(b_1, b_2) = -e^{-\sqrt{2}b_1} \cosh(\sqrt{2}b_2) + \frac{1}{4}e^{-4\sqrt{2}b_1} \left[ \cosh(2\sqrt{2}b_2) - 1 \right], \]
(6)
and
\[ 3\dot{\alpha} + 3\dot{\alpha}^2 - \left( \dot{b}_1^2 + \dot{b}_2^2 \right) = 0. \]
(7)
Note that only two of the three sets of equations (4), (5) and (7) are independent.

2 Proof of the collapse conjecture

Consider the new time variable \( \frac{dt}{d\tau} = e^\alpha \) monotonically related to \( \tau \) then, equations (4), (5) and (7), become,
\[ 3\alpha'' + 3\alpha' = \frac{1}{2} \left( \dot{b}_1^2 + \dot{b}_2^2 \right) + V(b_1, b_2), \]
(8)
\[ \ddot{b}_i + 2\alpha' \dot{b}_i + \frac{\partial V}{\partial b_i} = 0 \quad i = 1, 2, \]
(9)
and
\[ \alpha'' = -\frac{1}{3} \left( \dot{b}_1^2 + \dot{b}_2^2 \right). \]
(10)
Equation (8) is a first integral for the full six order system defined by equations (9) and (10). Let us define the total energy function for the anisotropic variables by
\[ E(b_1, b_2, b_1', b_2') = \frac{1}{2} \left( \dot{b}_1^2 + \dot{b}_2^2 \right) + V(b_1, b_2), \]
(11)
and note that \( E(0, 0, 0, 0) < 0 \) because \( V(b_1, b_2) \) has a negative global minimum at \( (b_1, b_2) = (0, 0) \). In a small neighborhood of the minimum \( (b_1, b_2) = (0, 0) \) the potential (6) takes the form
\[ V(b_1, b_2) = -\frac{3}{4} + \left( b_1^2 + b_2^2 \right) + O_3(b_1, b_2). \]
(12)

Let us see that the rate of expansion of the universe can not be positive for all time. Consider the initial conditions \( \alpha'(t_0) > 0 \) (initially expanding universe), \( \alpha(t_0) \) and \( (b_1(t_0), b_2(t_0)) \) arbitrary. Let us consider that \( \alpha' > 0 \) for all time in order to have a contradiction. Because equations (12) are the equations of two non-linear damped oscillators with damping factor \( \alpha' > 0 \) it follows that the energy \( E \), see (11), should be asymptotically negative, because \( V(b_1, b_2) \) has a negative global minimum at \( (b_1, b_2) = (0, 0) \), which is a contradiction because \( 3E = \alpha'^2 \). Then there exists an instant where \( \alpha' = 0 \) and thereafter \( \alpha' < 0 \).

Then assume that \( \alpha' < 0 \) and consider the variables, which are the Hubble normalised shear variables (11),
\[ x_i = -\frac{b_i'}{\alpha'}, \quad i = 1, 2, \]
(13)
and the time variable change $d\xi = -\alpha' d\tau$, then equations (9) read

$$
\frac{dx_i}{d\xi} = -2x_i + \frac{1}{3}x_i(x_1^2 + x_2^2) - \frac{1}{V} \frac{\partial V}{\partial b_i} \left( 3 - \frac{1}{2}(x_1^2 + x_2^2) \right), \\
\frac{db_i}{d\xi} = x_i, \ i = 1, 2.
$$

In order to study the escape along a channel it is useful to introduce the auxiliary variable $z_i = 1/b_i, \ i = 1, 2$. We obtain

$$
\frac{dz_i}{d\xi} = -z_i^2 x_i.
$$

Because the potential $V$ is invariant under rotations of $2\pi/3$ in the anisotropic plane, see (6), it suffices to study the escape in one of the channels. We chose the one defined by $(b_1, b_2) = (b_1, 0)$.

Asymptotically, for large $b_1$ and $b_2 \neq 0$ one has

$$
\frac{1}{V} \frac{\partial V}{\partial b_i} \sim 2 \sqrt{\frac{2}{3}}.
$$

In this limit the $x_1$ dynamics decouples from the $x_2$ dynamics and the problem reduces to the study of a planar dynamical system of the form

$$
\frac{dx_1}{d\xi} = 2\sqrt{6} - 2x_1 - \sqrt{\frac{2}{3}} x_1^2 + \frac{1}{3}x_1^3, \\
\frac{dz_1}{d\xi} = -z_1^2 x_1,
$$

which shows that the infinity manifold $z_1 = 0$ is invariant $^2$. Equation (19) shows that the stability of the invariant manifold depends on the sign of $x_1$ in a small neighborhood of $x_1 = 0$, and thus the codimension of the stable manifold is not zero thus proving that escape along this channel is impossible.

3 Conclusions

It was given a simple demonstration of the collapse conjecture for vacuum diagonal Bianchi type-IX space-time by studying the dynamics relatively to the overall rate of expansion of the universe. Reducing the equations of motion in the anisotropic plane to a planar dynamical system, it was shown that the infinity stable attractor does not have codimension zero, and thus the “escape along a channel” of the Bianchi-IX potential is impossible.

This demonstration, for this case, is simple and concise and it provides a new understanding of this classical cosmological model.

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$^2$For a similar argument applied to the study of scaling solutions in scalar fields models see [15].
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