Q-MANIFOLDS AND MACKENZIE THEORY: AN OVERVIEW

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Abstract. “Mackenzie theory” stands for the rich circle of notions that have been put forward by Kirill Mackenzie (solo or in collaboration): double structures such as double Lie groupoids and double Lie algebroids, Lie bialgebroids and their doubles, non-trivial dualities for double and multiple vector bundles, etc. “Q-manifolds” are (super)manifolds — I normally omit the prefix “super” — with a homological vector field, i.e., a self-commuting odd vector field. They may have an extra $\mathbb{Z}$-grading (called weight) not necessarily linked with the $\mathbb{Z}_2$-grading (parity). I will speak about double Lie algebroids (discovered by Mackenzie) and explain how this quite complicated fundamental notion is equivalent to a very simple one if the language of $Q$-manifolds is used. In particular, it shows how the two seemingly different notions of a “Drinfeld double” of a Lie bialgebroid due to Mackenzie and Roytenberg respectively, turn out to be the same thing if properly understood.

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1. Introduction

This text is meant to be a brief overview of the topics announced in the title. It does not contain new results (except probably for the remark concerning $Q$-manifold homology, which we wish to elaborate elsewhere). The original exposition of Mackenzie’s constructions is in

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his papers [10, 21, 11, 12, 13, 14, 15, 16, 17, 18]. See also the book [19]. There is a paper [20], which is now in the process of publication and which is a substantially reworked version of the earlier paper [16]. The main point of this text is to give an introduction to the approach to double Lie algebroids based on (graded) $Q$-manifolds. Details can be found in [29], where the result (the equivalence between Mackenzie’s double Lie algebroids and a certain class of $Q$-manifolds, see below) was formulated and proved for the first time. Some background information can also be found in [27].

Double Lie algebroids first appeared as the infinitesimals corresponding to the double Lie groupoids introduced by Mackenzie [10, 11] as double objects in the sense of Ehresmann (Lie groupoid objects in the category of Lie groupoids). The abstract notion came about later [12, 13, 14] and turned out to be quite complicated in formulation. (One of the reasons is that a Lie algebroid is not defined diagrammatically, so it is not possible to follow a categorical approach, which works well for the groupoid case.) Although there was absolutely no doubt that this was the ‘right’ notion — one justification was in the appearance of double Lie algebroids as ‘Drinfeld doubles’ of Lie bialgebroids [14] — their application was somewhat hindered by the complexity of the definition. It had been of a considerable interest for experts to give an alternative simpler description for them. This was achieved in [29].

This paper is an outcome of my participation in the program on “Poisson sigma models, Lie algebroids, deformations, and higher analogues” at the Erwin Schrödinger Institute in Vienna in August–September 2007, where it was written. It is a pleasant duty to express my gratitude to the Institute for the hospitality and the wonderful atmosphere for research and communications, and to the organizers of the program (Thomas Strobl, Henrique Bursztyn and Harald Grosse) for the invitation. The first announcement of the results of [29] was made in July 2006 at the XXV Białowieża meeting.

2. Lie algebroids and Lie bialgebroids

First, let us recall some well-known facts concerning **Lie algebroids**.

A **Lie algebroid** is a vector bundle $E \rightarrow M$ over a manifold $M$ endowed with a vector bundle map $a: E \rightarrow TM$ (called the anchor) and a Lie algebra structure on the space of sections $C^\infty(M, E)$ satisfying

$$[u, f v] = a(u) f v + (-1)^\tilde{a} f [u, v]$$

$$a([u, v]) = [a(u), a(v)]$$

for all $u, v \in C^\infty(M, E)$ and all $f \in C^\infty(M)$.

This notion allows three other equivalent descriptions. Recall that a $Q$-manifold is a supermanifold endowed with a homological vector field (often denoted by $Q$). We shall also consider Poisson manifolds and odd
Poisson (or Schouten) manifolds, sometimes using the abbreviations $P$- and $S$-manifolds respectively. ($S$ for Schouten.)

By a graded manifold we shall understand a supermanifold with a privileged class of atlases where the coordinates are assigned weights taking values in $\mathbb{Z}$ and the coordinate transformations are polynomial in coordinates with nonzero weights respecting the total weight. It is also assumed that the coordinates with nonzero weights run over the whole $\mathbb{R}$ (no restriction on range). Note that in general there is no relation between weight and parity. A natural example of a graded manifold is the total space of a vector bundle, for which we assume by default the following graded structure: the coordinates on the base have zero weight, the linear coordinates on fibers are assigned weight 1. (Any graded manifold having only non-negative weights decomposes into a tower of affine fibrations, the first level being a vector bundle.) See more in [27].

**Proposition 2.1.** There is a one-to-one correspondence between the following objects:

- Lie algebroids;
- vector bundles with a Poisson bracket of weight $-1$ on the total space;
- vector bundles with a Schouten bracket of weight $-1$ on the total space;
- vector bundles with a homological vector field of weight $+1$ on the total space.

Indeed, consider a vector bundle $E \to M$ and its three neighbors: the dual bundle $E^*$, the opposite bundle $\Pi E$ and the antidual $\Pi E^*$. (Here $\Pi$ denotes the parity reversion functor.)

\[
\begin{array}{c}
E \\
\Pi E \\
E^* \\
\Pi E^*
\end{array}
\]

A Lie algebroid structure for the bundle $E \to M$ is equivalent to either of the following: an even Poisson bracket of weight $-1$ on the manifold $E^*$; an odd Poisson bracket of weight $-1$ on the manifold $\Pi E^*$; and a homological vector field of weight 1 on $\Pi E$. It is worth mentioning that a Lie algebroid structure on a bundle is introduced in terms of its sections, while each of the three other descriptions gives a structure on the total space.

Let me show the explicit formulas. Let $x^a$ be local coordinates on the base $M$ and $e_i$ make a local frame for $E \to M$. Then the corresponding
coordinates on the fibers of $E^*$, $\Pi E$ and $\Pi E^*$ will be denoted by $u_i$, $\xi^i$ and $\eta_i$, respectively. (Here $\xi^i e_i$ and similar expressions — in this order — are assumed to be invariant.) The correspondence between structures is as follows.

The anchor and Lie bracket for the basis sections of $E$:

$$a(e_i) = Q_i^a(x) \frac{\partial}{\partial x^a},$$

and

$$[e_i, e_j] = (-1)^{ij} Q_{ij}^k(x) e_k.$$

The (nonzero) even Poisson brackets of the coordinates on the manifold $E^*$ (I am skipping the precise signs):

$$\{x^a, u_i\} = \pm Q_i^a, \quad \{u_i, u_j\} = \pm Q_{ij}^k u_k.$$

Similarly, the odd brackets on $\Pi E^*$:

$$\{x^a, \eta_i\} = \pm Q_i^a, \quad \{\eta_i, \eta_j\} = \pm Q_{ij}^k \eta_k.$$

The homological field on $\Pi E$:

$$Q = \xi^i Q_i^a(x) \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ij}^k(x) \frac{\partial}{\partial \xi^k}.$$

(This is analogous to the similar statement for Lie algebras. For a Lie algebra $\mathfrak{g}$ we have the coalgebra $\mathfrak{g}^*$ with a linear Poisson structure, the anticoalgebra $\Pi \mathfrak{g}^*$ with a linear Schouten structure, and the antialgebra $\Pi \mathfrak{g}$ with a quadratic homological field. Note that when we pass to vector bundles, we have to use the invariant notion of weight instead of speaking of something being ‘linear’ or ‘quadratic’ in coordinates.)

We call a vector bundle with a homological vector field of weight +1 on the total space, a Lie antialgebroid.

(An arbitrary graded $Q$-manifold with $w(Q) = 1$, may therefore be called a generalized Lie antialgebroid.)

**Remark 2.1.** If we take a Lie antialgebroid $\Pi E$ as a primary object, then the anchor and the Lie bracket for $E$ can be expressed by coordinate-free formulas using the derived bracket construction:

$$a(u)f := [[Q, i(u)], f] \quad \text{(3)}$$

and

$$i([u, v]) := (-1)^{\bar{u}} [[Q, i(u)], i(v)]. \quad \text{(4)}$$

Here the map $i: C^\infty(M, E) \to \mathfrak{X}(\Pi E)$ has the following appearance in coordinates:

$$i(u) = (-1)^{\bar{u}} u^i(x) \frac{\partial}{\partial \xi_i}. \quad \text{(5)}$$

(See, for example, [27]. See [6, 7] for the derived bracket construction. See [28] for higher derived brackets.)
Although the four descriptions are equivalent in the sense that they all carry the same information, it turns out that the description in the language of Lie antialgebroids is the most efficient of all. As an example let us discuss the notion of a Lie algebroid morphism. If $E_1 \to M_1$ and $E_2 \to M_2$ are Lie algebroids over different bases, the definition of a morphism is not obvious. One first has to define a pull-back Lie algebroid $\varphi^n E_2$ w.r.t. a map of bases $\varphi: M_1 \to M_2$ as the ‘vector bundle’ $TM_1 \times_{TM_2} E_2 \to M_1$ with a certain anchor and Lie bracket, which is not, strictly speaking, a genuine Lie algebroid, because it is not in general a true vector bundle. Still, by this the problem reduces to defining a morphism of Lie algebroids over the same base $M_1$, which is straightforward. (The precise definitions of the pull-back construction and of Lie algebroid morphisms over different bases can be found in the original papers [9, 2] and the book [19].) The power of the language of $Q$-manifolds can be illustrated by the following

**Proposition 2.2.** Consider Lie algebroids $E_1 \to M_1$ and $E_2 \to M_2$. Let $\Pi E_1$ and $\Pi E_2$ be the corresponding antialgebroids defined by the vector fields $Q_1 \in \mathfrak{x}(\Pi E_1)$ and $Q_2 \in \mathfrak{x}(\Pi E_2)$ respectively. A vector bundle map given by the horizontal arrows of

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\Phi} & E_2 \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{\varphi} & M_2
\end{array}
$$

is a morphism of Lie algebroids if and only if the vector fields $Q_1$ and $Q_2$ are $\Phi^H$-related, where

$$
\Phi^H: \Pi E_1 \to \Pi E_2
$$

is the induced map of the opposite vector bundles.

(This characterization was given by Vaintrob [26].)

Shortly: a map of vector bundles is a Lie algebroid morphism if the induced map of antialgebroids is a morphism of $Q$-manifolds (which is much easier to handle).

Another illustration of the usefulness of the approach based on $Q$-manifolds is an analog of homology for Lie algebroids. More generally, for a $Q$-manifold $M$, the standard cochain complex is $(C^\infty(M), Q)$. The dual complex or the chain complex can be defined as $(\text{Vol}(M), L_Q)$. Here $\text{Vol}(M)$ stands for the Berezin volume forms and $L_Q$, for the Lie derivative w.r.t. the vector field $Q$. Note that ‘chains’ are defined as cochains with certain “dualizing” coefficients. We are not looking at the $\mathbb{Z}$-grading here; it can be taken care of when necessary. There is a natural isomorphism (a “Poincaré duality”) between the cochain complex $(C^\infty(M), Q)$ and the chain complex $(\text{Vol}(M), L_Q)$ if there is an invariant non-vanishing volume form $\rho$. One can easily see that the
necessary and sufficient condition for this is that the cohomology class \([\text{div}_\rho Q]\) (which is independent of \(\rho\)) in the standard complex equals zero. We call it the modular class of a \(Q\)-manifold. In the Lie algebroid case it is the modular class introduced in [1]. The point of considering \((\text{Vol}(M), L_Q)\) as the chain complex (i.e., giving the homology of a \(Q\)-manifold) is that it has a natural pairing with the cochain complex (provided certain conditions of compactness and orientability are fulfilled, otherwise one has to consider compactly-supported densities). It is not to be confused with the mentioned Poincaré duality. Most important, it possesses the correct functorial behavior: for a map of \(Q\)-manifolds \(\phi\) there is a push-forward chain map \(\phi_*\), hence there is an induced push-forward map on homology. This is applicable to the particular case of Lie algebroids, for which we obtain a homology theory with a proper behavior under morphisms. For a Lie algebroid \(E \to M\), the chain complex \((\text{Vol}(\Pi E), L_Q)\) coincides with the complex \(C(E, Q_E)\) introduced in a different language by Evens, Lu and Weinstein [1]. \((Q_E\) is their notation has nothing to do with homological vector fields and stands for a representation of \(E\) defined in [1].) (For a very particular case of the morphism \(a: E \to TM\) given by the anchor, it is possible to show that the induced map of chain complexes \(a_*\) coincides with a map to differential forms on \(M\) introduced in the top degree in [1] in connection with a version of Poincaré duality. The fact that it should be viewed as a special case of a push-forward, which is a chain map in all degrees, was not known in the literature. We hope to elaborate it elsewhere.)

Now let us turn to Lie bialgebroids. A Lie bialgebroid over a supermanifold \(M\) is a Lie algebroid \(E \to M\) such that the dual bundle \(E^* \to M\) is also endowed with a structure of a Lie algebroid and a compatibility condition is satisfied. There are several equivalent ways of expressing this compatibility condition. The original condition was given by Mackenzie and Xu [21], who first introduced Lie bialgebroids; it was then replaced by a more efficient description due to Kosmann-Schwarzbach [5], which in its turn can be reformulated using \(Q\)-manifolds. Consider \(\Pi E\). It is a Lie antialgebroid because \(E\) is a Lie algebroid, and it also carries a Schouten bracket of weight \(-1\) because \(E^*\) is a Lie algebroid. The condition is that the homological vector field is a derivation of the bracket. Shortly: \(\Pi E\) is a \(QS\)-manifold of weights \((1, -1)\). One can show that this condition is symmetric in \(E, E^*\) (see later), though it is not manifestly symmetric. Hence, Lie bialgebroids is a self-dual notion. They are often denoted as pairs \((E, E^*)\).

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1This remark resulted from discussions with Vladimir Roubtsov at ESI in August 2007.
3. Double and multiple Lie algebroids

What is a double Lie algebroid? Such objects naturally appear as infinitesimals, i.e., by application of the Lie functor, for double Lie groupoids (taken in the sense of Ehresmann, as groupoid objects in the category of groupoids). Double Lie algebroids of double Lie groupoids were introduced by Mackenzie in [10]. The corresponding abstract notion [11] appeared later and turned out, in its original form, to be quite complicated and non-obvious. The main reason for this is that properties of brackets for Lie algebroids are not expressed diagrammatically, so one cannot approach double objects for them by methods of category theory as one does for double groupoids. There is no doubt, however, that it is a correct notion; this was shown, in particular, by the fact that double Lie algebroids in the sense of Mackenzie naturally appear also in the constructions of a ‘Drinfeld double’ for Lie bialgebroids [12, 14] (see later). Only quite recently an alternative simplifying description in terms of Q-manifolds was obtained [29], which also helped to simplify, in the hindsight, Mackenzie’s original approach (see [20]).

A double Lie algebroid is, first of all, a double vector bundle. Let us recall this notion. A double vector bundle with base $M$ is a fiber bundle $D \rightarrow M$ with a special structure. A local model for it (a trivial double vector bundle) has the appearance $U \times V_1 \times V_2 \times V_{12}$ where $U \subset M$ is an open set and $V_i, V_{ij}$ are vector spaces. Admissible transformations are maps $V_1 \times V_2 \times V_{12} \rightarrow V_1 \times V_2 \times V_{12}$, depending on points of $U$ as parameters, which for each $V_i$ are linear and for $V_{12}$ linear in $V_{12}$ plus an extra term bilinear in $V_1 \times V_2$. In other words, if we use linear coordinates $u^i$ for $V_1$, $w^\alpha$ for $V_2$, and $z^\mu$ for $V_{12}$, we have a transformation law of the form

$$u^i = u^i T^i(x'),$$

$$w^\alpha = w^\alpha T_\alpha^\alpha(x'),$$

$$z^\mu = z^\mu T^i_\mu T^\mu_\alpha (x') + w^\alpha u^i T^\mu_\alpha T^i, (x').$$

(all coefficients are functions of $x \in U$). In particular, it follows that there is a diagram

$$\begin{array}{ccc}
D & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & M
\end{array}$$

for which each side is a vector bundle. Here $V_1$ is the standard fiber for $A \rightarrow M$; $V_2$, for $B \rightarrow M$; $V_1 \times V_{12}$, for $D \rightarrow B$; and $V_2 \times V_{12}$, for $D \rightarrow A$. There is also a vector bundle $K \rightarrow M$ with the standard fiber $V_{12}$. It is called the core of the double vector bundle $D \rightarrow M$.

(There is an immediate generalization giving the notion of a $k$-fold vector bundle. Such a bundle has the standard fiber of the form $\prod V_i \times \prod_{i<j} V_{ij} \times \ldots \times V_{12...k}$ and the transformation law similar to the above.
By picking a subset $i_1 < \ldots < i_l$ one obtains a face, which is an $l$-fold vector bundle. A 1-fold vector bundle is an ordinary vector bundle.

From the definition it is clear that the total space of a double vector bundle $D \to M$ is naturally a bigraded manifold, and the total space of a $k$-fold vector bundle $P \to M$ is a $k$-graded manifold. One can speak about the total weight as the sum of all (partial) weights.

There are constructions that naturally lead to multiple vector bundles.

Example 3.1. If $E \to M$ is a vector bundle, then the tangent $TE$ has the structure of a double vector bundle:

$$
\begin{array}{ccc}
TE & \longrightarrow & E \\
&T_p \downarrow & \downarrow p \\
TM & \longrightarrow & M
\end{array}
$$

To check that it is indeed a double vector bundle, one has to write down the changes of coordinates. If

$$v^i = v'^i T^i_v$$

is the transformation law for the fiber coordinates on $E$, then for $TE$ we obtain the fiber coordinates $v^i, \dot{x}^a, \dot{v}^i$ where $x^a$ are local coordinates on the base $M$, and clearly

$$\dot{x}^a = \dot{x}^{a'} \frac{\partial x^a}{\partial x^{a'}} \tag{10}$$

$$\dot{v}^i = \dot{v}'^i T^i_v + v'^i x^a \frac{\partial T^i_v}{\partial x^a}, \tag{11}$$

which satisfies the definition of a double vector bundle. Here as the core we have a vector bundle with the transition functions obtained from (11) by setting $v^i$ and $\dot{x}^a$ to zero; therefore it is a copy of $E \to M$.

Example 3.2. The cotangent $T^*E$, for a vector bundle $E \to M$, is also a double vector bundle. Here we have the diagram

$$
\begin{array}{ccc}
T^*E & \longrightarrow & E \\
\downarrow & & \downarrow \\
E^* & \longrightarrow & M
\end{array}
$$

The core bundle in this case is $T^*M \to M$. We shall return to this example later.

(See book [19].)

Both examples can be generalized to the case when we already start from a multiple vector bundle; taking the tangent or cotangent gives a multiple vector bundle of the multiplicity increased by one.
Now the definition of a double Lie algebroid with base $M$ is as follows: it is a double vector bundle

\[ D \longrightarrow B \]
\[ \downarrow \quad \downarrow \]
\[ A \longrightarrow M \]

such that each side (which is a vector bundle) carries the structure of a Lie algebroid and certain compatibility conditions are satisfied. The main problem happens to be to formulate these compatibility conditions. To this we are going to proceed, but we will have to make a further digression to constructions with multiple vector bundles.

Double (and multiple) vector bundles possess a non-trivial duality theory. Let $D \to M$ be a double vector bundle. There is a diagram of ordinary vector bundles (9). Consider $D$ as a vector bundle over $A$ and take the dual, which we denote $D_A^*$ explicitly indicating the base.

**Proposition 3.1.** $D_A^*$ is again a double vector bundle over $M$, given by a square

\[ D_A^* \longrightarrow K^* \]
\[ \downarrow \quad \downarrow \]
\[ A \longrightarrow M \]

where $K^* \to M$ is the dual to the core $K \to M$.

(Of course, the same is true if we interchange $A$ and $B$.)

This can be explained using coordinates (or local trivializations) as follows. In the notation $x^a, u^i, w^\alpha, z^\mu$ for coordinates on $D$ used above, for $D_A^*$ we must have local coordinates $x^a, u^i, w^\alpha, z^\mu$ so that the form $w^\alpha w_\alpha + z^\mu z_\mu$ be invariant. The invariance condition together with the transformation law (7) and (8) imply the transformation law

\[ w^\alpha' = T^\alpha_{\alpha'} w_\alpha + u^{i'} T^{\mu}_{\nu\alpha'} z_\mu; \]  
(13)

\[ z^\mu' = T_{\mu'}^\mu z_\mu. \]  
(14)

We see that a double vector bundle is indeed obtained, and the new core is $B^* \to M$, the dual for $B \to M$. If we apply duality to the double vector bundle $D_A^*$, then, depending in which direction we dualize, we either come back to $D$ or arrive at the other dual $D_B^*$. Hence there are three double vector bundles in duality, making a ‘corner’:

\[ D_A^* \longrightarrow K^* \longrightarrow D_B^* \]
\[ \downarrow \quad \downarrow \]
\[ A \longrightarrow M \longrightarrow B \]

(15)
This amazing duality between the ‘horizontal’ and ‘vertical’ duals of a double vector bundles was discovered by Mackenzie \cite{15} and independently by Konieczna–Urbański \cite{3}. The canonical pairing between $D_A^*$ and $D_B^*$ as vector bundles over $K^*$ is given by the invariant bilinear form

$$u^i u_i - w^\alpha w_\alpha. \tag{16}$$

(Recall that $u^i, w^\alpha$ and $u_i, w_\alpha$ are precisely the fiber coordinates on $D_A^*$ and $D_B^*$ over $K^*$, so that $x^a, z_\mu$ are the base coordinates.) The main statement is the invariance of the form \(\text{(16)}\), for which the minus sign is absolutely essential (the two terms being not separately invariant).

Similar statements hold for triple and multiple vector bundles. (See Mackenzie \cite{18} for a detailed analysis.)

Besides duality functors another type of operation that can be applied to multiple vector bundles is changing parity in fibers. As the parity reversion can be applied only to linear objects, there are $k$ partial parity reversion functors $\Pi_i$ for a $k$-fold vector bundle. One can show that

$$\Pi_i \Pi_j = \Pi_j \Pi_i,$$

meaning natural isomorphism.

Now we can proceed to double algebroids. Consider the double vector bundle given by \(\text{(12)}\). The obvious part of the definition is that all sides should carry a Lie algebroid structure. Now, the original definition due to Mackenzie contained three extra conditions: the so-called Conditions I, II, and III, which are rather complicated in their formulation. We shall not reproduce them here, because it would require a different level of technical detail than we are using. Instead we shall give an informal idea of their content.

Condition I is the easiest for understanding. It basically requires that each of the algebroid structures respects the linear structure in the other direction. It turns out, in particular, that the Lie algebroid structures on $D \rightarrow A$ and $D \rightarrow B$ induce those on $B$ and $A$. Besides this, if we consider homological vector fields $Q_{DA}$ on $\Pi_A D$ and $Q_{DB}$ on $\Pi_B D$ corresponding to these Lie algebroid structures, they must have zero weights in the other direction, i.e.,

$$w(Q_{DA}) = (1, 0), \quad w(Q_{DB}) = (0, 1).$$

Note that weight zero means that a vector field generates linear transformations. This, in particular, allows to dualize and to reverse parity.

Now instead of giving a technical description of Mackenzie’s conditions II and III, let us turn to what I call a big picture. The lesson that we learned from considering ordinary Lie algebroids is that this notion has different (and equivalent) manifestations, which we see by looking at all the neighbors of a given vector bundle. Let us do the same for the double vector bundle \(\text{(12)}\). In the list below we indicate which structures correspond to the two Lie algebroid structures on $D \rightarrow A$
and $D \to B$. Altogether there are 12 different neighbors (counting the original double vector bundle).

\[ \begin{array}{ccc}
D_A^* & \longrightarrow & K^* \\
\downarrow & & \downarrow \\
A & \longrightarrow & M \\
\end{array} \]

P. bracket on $D_A^*$, $(-1, -1)$; L. algd. on $D_A^* \to K^*$

(17)

\[ \begin{array}{ccc}
D_B^* & \longrightarrow & B \\
\downarrow & & \downarrow \\
K^* & \longrightarrow & M \\
\end{array} \]

P. bracket on $D_B^*$, $(-1, -1)$; L. algd. on $D_B^* \to K^*$

(18)

\[ \begin{array}{ccc}
\Pi_A D & \longrightarrow & \Pi B \\
\downarrow & & \downarrow \\
A & \longrightarrow & M \\
\end{array} \]

H. v. field on $\Pi_A D$, $(1, 0)$; L. algd. on $\Pi_A D \to \Pi B$

(19)

\[ \begin{array}{ccc}
\Pi_B D & \longrightarrow & B \\
\downarrow & & \downarrow \\
\Pi A & \longrightarrow & M \\
\end{array} \]

H. v. field on $\Pi_B D$, $(0, 1)$; L. algd. on $\Pi_B D \to \Pi A$

(20)

\[ \begin{array}{ccc}
\Pi^2 D & \longrightarrow & \Pi B \\
\downarrow & & \downarrow \\
\Pi A & \longrightarrow & M \\
\end{array} \]

Two h. v. fields on $\Pi^2 D$, weights $(0, 1)$ and $(1, 0)$

(21)

\[ \begin{array}{ccc}
\Pi_A D_A^* & \longrightarrow & \Pi K^* \\
\downarrow & & \downarrow \\
A & \longrightarrow & M \\
\end{array} \]

S. bracket on $\Pi_A D_A^*$, $(-1, -1)$; L. algd. on $\Pi_A D_A^* \to \Pi K^*$

(22)

\[ \begin{array}{ccc}
\Pi_B D_B^* & \longrightarrow & B \\
\downarrow & & \downarrow \\
\Pi K^* & \longrightarrow & M \\
\end{array} \]

S. bracket on $\Pi_B D_B^*$, $(-1, -1)$; L. algd. on $\Pi_B D_B^* \to \Pi K^*$

(23)
Now, the general philosophy is: if we do not know how to define compatibility of the two structures for some of these pictures, look at the neighbors for which a compatibility condition comes about naturally. In our list there are the exactly five double vector bundles given by diagrams (21), (24), (25), (26), and (27), where both structures are defined on the total space. It is absolutely clear which condition should be considered in each of these cases: the commutativity of the vector fields for (21) and the derivation property w.r.t. the bracket for (24), (25), (26), and (27).

**Proposition 3.2.** The compatibility conditions for (24), (25), (26) and (27) are equivalent, and are different ways of saying that \((D_A^*, D_B^*)\) is a Lie bialgebroid over \(K^*\).

**Proof.** Indeed, for any Lie bialgebroid \((E, E^*)\) the compatibility can be stated in terms of either \(E\) or \(E^*\) (as a \(QS\)-structure on either \(\Pi E\) or \(\Pi E^*\), respectively). This corresponds to (24) or (25) in our case. In our special case there is also an extra option of changing parity in the second direction, which adds (26) and (27) to the picture. □

On the other hand, to say that \((D_A^*, D_B^*)\) is a Lie bialgebroid over \(K^*\) is a natural compatibility condition for (17) and (18). And it is precisely Mackenzie’s Condition III. It was found in [29] and then proved by
Mackenzie in his own framework \cite{20} that his original Condition II is subsumed by Condition III (though it is not obvious).

**Theorem.** A double vector bundle $D \to M$ is a double Lie algebroid if and only if on the double vector bundle $\Pi^2 D \to M$ the two induced homological vector fields (corresponding to the two Lie algebroid structures) commute.

**Proof.** Consider one of the manifestations of the bialgebroid condition, say, for concreteness, \cite{24}. The derivation property means that the flow of the vector field preserves the bracket. On the other hand, the commutativity condition for \cite{21} means that the flow of one field preserves the other. Now the claim follows from functoriality: notice that a linear transformation preserves a Lie bracket if and only if the adjoint map preserves the corresponding linear Poisson bracket and if and only if the ‘$\Pi$-symmetric’ map preserves the corresponding homological vector field.

We may define a **double Lie antialgebroid** as a double vector bundle endowed with two commuting homological vector fields of weights $(1,0)$ and $(0,1)$. In a similar way one defines a **$k$-fold Lie antialgebroid**. It is clear that the structures of double Lie algebroids and double Lie antialgebroids are equivalent. This also gives an efficient general notion of a **multiple Lie algebroid**.

4. **Example: Drinfeld double for Lie bialgebroids**

Recall that Drinfeld’s classical double of a Lie bialgebra is again a Lie bialgebra with “good” properties. An analog of this construction for Lie algebroids turned out to be a puzzle. Three constructions of a ‘double’ of a Lie bialgebroid have been suggested. Suppose $(E, E^*)$ is a Lie bialgebroid over a base $M$. Liu, Weinstein and Xu \cite{8} suggested to consider as its double a structure of a Courant algebroid on the direct sum $E \oplus E^*$. Mackenzie in \cite{12,13,14,16} and Roytenberg in \cite{24} suggested two different constructions based on cotangent bundles. Though they look very different (in particular, Roytenberg’s double is a supermanifold, and Mackenzie stays in the classical world), we shall show now that they are essentially the same.

Roytenberg previously showed \cite{24} that the Liu–Weinstein–Xu double is recovered from his own construction as a derived bracket, generalizing the results of C. Roger \cite{23} and Y. Kosmann-Schwarzbach \cite{1,6} for Lie bialgebras. Therefore, proving that the Mackenzie and Roytenberg pictures are equivalent or, actually, the same, if understood properly, shows conclusively that this ‘cotangent double’ is fundamental, and should be regarded as the correct extension of Drinfeld’s double of Lie bialgebras to Lie bialgebroids.

Both Roytenberg’s and Mackenzie’s construction use the statement that the cotangent bundles of dual vector bundles are isomorphic \cite{21},
an extension of \([25]\); see also \([17, 24, 27]\)). Hence there is a double vector bundle

\[
\begin{array}{ccc}
T^* E & \longrightarrow & E^* \\
\downarrow & & \downarrow \\
E & \longrightarrow & M
\end{array}
\]

(28)

Mackenzie shows that it is a double Lie algebroid. He calls it the \textit{cotangent double} of a Lie bialgebroid \((E, E^*)\). Note that the canonical symplectic structure on \(T^* E\) corresponds to the invariant scalar product on Drinfeld’s double \(\mathfrak{d}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{b}^*\) of a Lie bialgebra \(\mathfrak{b}\).

On the other hand, Roytenberg uses the description of Lie algebroids via homological vector fields. He considers the double vector bundle

\[
\begin{array}{ccc}
T^* \Pi E & \longrightarrow & \Pi E^* \\
\downarrow & & \downarrow \\
\Pi E & \longrightarrow & M
\end{array}
\]

(29)

and homological vector fields \(Q_E \in \mathfrak{X}(\Pi E)\) and \(Q_{E^*} \in \mathfrak{X}(\Pi E^*)\) defining Lie algebroid structures on \(E \to M\) and \(E^* \to M\), respectively. Recall that vector fields on a manifold correspond to fiberwise linear functions (Hamiltonians) on the cotangent bundle so that the commutator maps to the Poisson bracket. Denote the functions corresponding to \(Q_E\) and \(Q_{E^*}\) by \(H_E\) and \(H_{E^*}\), respectively. Roytenberg shows that under the natural symplectomorphism \(T^* \Pi E = T^* \Pi E^*\) the linear function \(H_{E^*}\) on \(T^* \Pi E^*\) corresponding to the vector field \(Q_{E^*}\) transforms precisely into the fiberwise quadratic function \(S_E\) on \(T^* \Pi E\) specifying the Schouten bracket on \(\Pi E\) induced by the Lie structure on \(E^*\). The derivation property of \(Q_E\) w.r.t. the Schouten bracket on \(\Pi E\) is one of the equivalent definitions of a Lie bialgebroid \([5]\), and the most convenient. Hence, Roytenberg’s statement means that it is also equivalent to the commutativity of the Hamiltonians \(H_E\) and \(H_{E^*}\) under the canonical Poisson bracket. They generate commuting homological vector fields \(X_{H_E}\) and \(X_{H_{E^*}}\) on the cotangent bundle \(T^* \Pi E\). In our language, \(X_{H_E}\) and \(X_{H_{E^*}}\) make \((29)\) a Lie antialgebroid. (One can see that the conditions for weights are satisfied.) Roytenberg \([24]\) calls the supermanifold \(T^* \Pi E = T^* \Pi E^*\) together with the homological vector field \(Q = X_{H_E} + X_{H_{E^*}}\) on it, the \textit{Drinfeld double} of \((E, E^*)\).

If we slightly refine Roytenberg’s picture, considering the double Lie antialgebroid given by \(X_{H_E}\) and \(X_{H_{E^*}}\), rather than a single \(Q\)-manifold, we can immediately see that by our theorem his picture becomes identical to that of Mackenzie.

Indeed, apply the complete reversion of parity to \((28)\). Notice that \(\Pi^2 T^* E = \Pi^2 T^* E^*\) coincides with \(T^* \Pi E = T^* \Pi E^*\) (easily checked in coordinates). By the theorem, the double vector bundle \((28)\) is a double
Lie algebroid if and only if the corresponding double vector bundle
\[
\Pi^2 T^* E = \Pi^2 T^* E^* \longrightarrow \Pi E^* \\
\downarrow \quad \downarrow \\
\Pi E \longrightarrow M
\]
which is identical with (29), is a double Lie antialgebroid. It remains to identify the respective homological vector fields on the ultimate total space, which can be achieved by a direct inspection.

We have arrived at the following statement.

**Proposition 4.1.** Roytenberg’s and Mackenzie’s pictures give the same notion of a double of a Lie bialgebroid (up to a change of parity).

We can now identify the two constructions and speak simply of the (cotangent) double of a Lie bialgebroid as a double Lie algebroid, most efficiently described in the “anti-” language of diagram (29).

**More on doubles.**

Recall that Drinfeld’s classical double of a Lie bialgebra is not just a Lie algebra, but also a coalgebra, and in fact a Lie bialgebra again. This gives a direction in which to look in the case of Lie bialgebroids. Note that this second structure (for doubles of Lie bialgebroids) has not been discovered previously.

However, for many people including the author there was absolutely no doubt that such a structure exists. The following conjectured statement was put down in my notes of 2002 as a guideline for (not yet obtained at that time) alternative description of double Lie algebroids.

**General principle.** Taking the double of an n-fold Lie bialgebroid should give an \((n + 1)\)-fold Lie bialgebroid, with an additional property, such as a symplectic structure.

Of course it involved new notions yet to be defined. As for “double Lie bialgebroids” (or “bi-double Lie algebroids”), and the further multiple “bi-” case, this notion is properly defined in our joint work with Kirill Mackenzie, and is a subject of our forthcoming paper [22], where precise definitions and statements can be found.

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