Spinorial Field and Lyra Geometry

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Abstract

The Dirac field is studied in a Lyra space-time background by means of the classical Schwinger Variational Principle. We obtain the equations of motion, establish the conservation laws, and get a scale relation relating the energy-momentum and spin tensors. Such scale relation is an intrinsic property for matter fields in Lyra background.

1 Introduction

After Einstein’s approach to gravitation [1], several others theories have been developed, as part of efforts to cure problems arising when the gravitational field is coupled to matter fields. Thus, as soon as Einstein presented the General Relativity, Weyl [2] proposed a new geometry in which a new scalar field accompanies the metric field and changes the scale of length measurements. The aim was to unify gravitation and electromagnetism, but this theory was briefly refuted by Einstein because the non-metricity had direct consequences over the spectral lines of elements which has never been observed.

The study of the spin coupling to gravitation has been and it is a central problem. The principal path to incorporate spin in geometrical theories of gravitation is the use of so called Riemann-Cartan geometry [3] which has a nonsymmetric connection, in such a way that a new geometrical concept enters in scene: the torsion. However, analyzing the Cauchy data, one can proof that the torsion is a non-propagating entity and therefore must be different of zero only in the interior of matter.

Some years after Weyl, Lyra [4] proposed a new geometry which removes the non-integrability of the length transfer of a vector under parallel transport (the metricity condition is restored) by introducing a gauge or scale function into the modified Riemannian geometry. The study of this geometry was completed by Scheibe [4] and, it was analyzed by Sen [5] and several others as an alternative to describe the gravitational field, and more recently it has been applied to study viscous and higher dimensional [6] cosmological models, domain walls [7], and several others applications. The attractive of Lyra’s geometry resides in the fact that the torsion is propagating which in the context of spin-gravitational coupling is interesting.

Following the spirit of finding a solution to the difficulties encountered in the quantization of Einstein’s theory of gravitation, many endeavours have been made to generalize and modify such theory. Among them some may prove to be of crucial importance in the formulation of a future and successful quantum gravity as for example the torsion [8], conformal gauge symmetry [2, 10] and supersymmetry [11].

On the other hand, in the early days of quantum field theory, many calculations were undertaken in which the electromagnetic field was considered as a classical background field interacting with quantized matter. Such semiclassical approach readily gives results in complete accordance with the fully quantized theory. And, since the quantization of gravity is still a controversial matter, semiclassical behavior of quantum matter in non-Euclidean manifolds appears as natural way to achieve some results about the fully quantized theory again. In this context, the role of symmetries on the manifold is of special importance in the definition of the particle content [12].

In this paper we want to use the Lyra Geometry, a curved and torsioned manifold where scale symmetry is implemented in a non-usual way, to study the behavior of the spinorial (Dirac) field in presence of a classical background. The main tool for this study will be the Schwinger Variational Principle, which is able to construct, in a unique and direct proceeding, equations of motion, canonical generators and conservation laws. Such properties were studied in the context of Riemann-Cartan geometry in [13].
In the next section the basics of Lyra Geometry\cite{1,2} are introduced, followed by the definition of covariant differentiation of spinors in this manifold. Next, the classical Schwinger Action Principle \cite{13,14} is presented and then applied in section 5. Finally, we make some comments on the results obtained.

## 2 Lyra Geometry

Lyra geometry is a generalization of the usual Riemannian one, constructed by attaching to each coordinate system a \textit{scale function} $\phi(x)$. Let be $M \subseteq \mathbb{R}^N$ and $U$ an open set of $\mathbb{R}^n$, also let be $\chi : U \cap M$. Since $\chi$ is injective ($\cap$), is settled a univalent equivalence relation between points in the domain $U$ and the image $\chi(U)$. The couple $(U, \chi)$ is called a coordinate system. A \textit{reference system} is defined by the triple $(U, \chi, \phi)$ such that

$$\bar{\phi}(\bar{x}) = \phi(x(\bar{x})), \quad \frac{\partial \phi}{\partial \phi} \neq 0.$$  \hspace{1cm} (1)

Therefore, the scale function does not transform like a scalar function, but like a diffeomorphism in the same way as a coordinate. Then, by general changes of the reference system a Lyra vector transforms as

$$\bar{A}^\mu(\bar{x}) = \frac{\bar{\phi}(\bar{x})}{\phi(x)} \partial x^\mu A^\nu(x).$$  \hspace{1cm} (2)

In differential geometry, parallel transport defined as a \textit{linear} map from vectors in a given point to vectors in another infinitesimal neighbor point, on the same manifold,

$$\delta V^\mu \equiv L^\mu_\alpha(V) \, dx^\alpha.$$  \hspace{1cm} (3)

Besides the linearity,

$$\delta (\lambda V_1^\mu + V_2^\mu) = \lambda L^\mu_\alpha(V_1) \, dx^\alpha + L^\mu_\alpha(V_2) \, dx^\alpha,$$  \hspace{1cm} (4)

one also demands a Leibnitz composition rule,

$$\delta (V_1^\mu V_2^\nu) = L^\mu_\alpha(V_1) V_2^\nu \, dx^\alpha + L^\nu_\alpha(V_2) V_1^\mu \, dx^\alpha.$$  \hspace{1cm} (5)

Notwithstanding, the crucial point in the definition of the parallel transport is on the linearity, dictated by

$$L^\mu_\alpha (\lambda V) = \lambda L^\mu_\alpha (V)$$  \hspace{1cm} (6)

which, in conjunction with the infinitesimal nature of the transformation, conducts to conclusion the map is proportional to the transported vector itself\footnote{It is important to note this is a general expression, and $\Gamma$ is the connection on the manifold under study, and not just the Lyra connection defined below.}, i.e.,

$$\delta V^\mu = -\Gamma^\mu_{\alpha\beta} V^\beta dx^\alpha.$$  \hspace{1cm} (7)

Actually, it is simple to proof that \textit{all} linear infinitesimal application has this property.

With these considerations, the Lyra parallel transportation is defined by

$$\delta A^\mu(x) \equiv -\Gamma^\mu_{\alpha\beta} A^\beta \phi(x) \, dx^\alpha,$$  \hspace{1cm} (8)

where,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{\phi} \Gamma^\rho_{\mu\nu} + \frac{1}{\phi} \left[ \partial_\nu \ln \left( \frac{\phi}{\bar{\phi}} \right) \delta^\rho_\mu - \partial_\sigma \ln \left( \frac{\phi}{\bar{\phi}} \right) g_{\nu\sigma} g^{\rho\alpha} \right],$$  \hspace{1cm} (9)

with

$$\bar{\Gamma}^\rho_{\mu\nu} \equiv \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right).$$  \hspace{1cm} (10)

Under a general reference system transformation, $\Gamma$ transforms as

$$\Gamma^\rho_{\alpha\beta} = \frac{\bar{\phi}}{\phi} \Gamma^\rho_{\lambda\mu} \frac{\partial x^\rho}{\partial x^\lambda} \frac{\partial x^\nu}{\partial x^\mu} + \frac{1}{\phi} \frac{\partial x^\rho}{\partial x^\nu} \frac{\partial^2 x^\nu}{\partial x^\mu \partial x^\sigma} + \frac{1}{\phi} \frac{\partial}{\partial x^\alpha} \ln \left( \frac{\phi}{\bar{\phi}} \right) \delta^\rho_\beta.$$  \hspace{1cm} (11)
Using the standard definitions, one finds for the Lyra geometry the covariant derivative

\[ \nabla_\mu F^\lambda_{\sigma...} \equiv \frac{1}{\phi} \partial_\mu F^\lambda_{\sigma...} + \Gamma^\lambda_{\mu\alpha} F^\alpha_{\sigma...} + \cdots - \Gamma^\alpha_{\mu\sigma} F^\lambda_{\alpha...} - \cdots, \] (12)

for a general Lyra tensor. We also find the Lyra curvature tensor

\[ R^\rho_{\beta\alpha\sigma} \equiv \frac{1}{\phi^2} \left( \partial_\rho (\phi \Gamma^\mu_{\alpha\beta}) - \partial_\beta (\phi \Gamma^\rho_{\alpha\sigma}) + \phi \Gamma^\rho_{\beta\lambda} \phi \Gamma^\lambda_{\alpha\sigma} - \phi \Gamma^\rho_{\alpha\lambda} \phi \Gamma^\lambda_{\beta\sigma} \right), \] (13)

and the Lyra torsion tensor

\[ \tau^\mu_{\alpha\beta} \equiv 2 \left( \Gamma^\mu_{[\alpha\beta]} + \frac{1}{\phi} \delta^\mu_{[\beta} \partial_{\alpha]} \ln \phi \right). \] (14)

The antisymmetrization is defined by \( \Gamma^\mu_{[\alpha\beta]} \equiv \frac{1}{2} (\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) \), and the metricity condition \( \nabla_\lambda g_{\mu\nu} = 0 \) imply

\[ \Gamma^\rho_{\mu\nu} = \frac{1}{\phi} \hat{\Gamma}^\rho_{\mu\nu} + \Gamma^\rho_{[\mu\nu]} - \Gamma^\alpha_{[\mu\sigma]} g_{\nu\alpha} g^\sigma_{\rho} - \Gamma^\alpha_{[\nu\sigma]} g_{\alpha\mu} g^\sigma_{\rho}. \] (15)

In the next section we introduce the behavior of spinorial fields in the Lyra geometry.

3 Covariant Differentiation of Spinors

To construct the covariant derivative of Dirac field in Lyra geometry, we follow the standard procedure of analyzing the behavior of the field under local Lorentz transformations,

\[ \psi (x) \to \psi' (x) = U (L (x)) \psi (x), \] (16)

where \( U \) is an one-half spin representation of Lorentz group.

What we want is to define a spin connection \( S_\mu \) in a such way that the object

\[ \nabla_\mu \psi \equiv \psi' (x) = U (x) \psi (x), \] (17)

transforms like a spinor,

\[ \nabla_\mu \psi \to (\nabla_\mu \psi)' = U (x) \nabla_\mu \psi \] (18)

and therefore \( S \) transforms as

\[ S'_\mu = U (x) S_\mu U^{-1} (x) - \frac{1}{\phi} \left( \partial_\mu U \right) U^{-1} (x). \] (19)

If we remember the Dirac matrices algebra

\[ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu} \] (20)

and use the requirement of compatibility among spin connection and metric, we can found

\[ \nabla_\mu g^{\alpha\beta} = 0 \Rightarrow \nabla_\mu \{ \gamma^\alpha, \gamma^\beta \} = 0. \] (21)

From the covariant derivative of the fermion field (14) and remembering that \( \bar{\psi} \psi \) must be a scalar under the transformation (14), it follows that

\[ \nabla_\mu \bar{\psi} = \frac{1}{\phi} \partial_\mu \bar{\psi} - \bar{\psi} S_\mu. \] (22)

Then, we use the covariant derivative of the fermionic current

\[ \nabla_\mu (\bar{\psi} \gamma^\nu \psi) = \frac{1}{\phi} \partial_\mu (\bar{\psi} \gamma^\nu \psi) + \gamma^\nu_{\mu\lambda} (\bar{\psi} \gamma^\lambda \psi) = (\nabla_\mu \bar{\psi}) \gamma^\nu \psi + \bar{\psi} (\nabla_\mu \gamma^\nu) \psi + \bar{\psi} \gamma^\nu (\nabla_\mu \psi) \]
to get the following expression for the covariant derivative\(^2\) of \(\gamma^\nu\)

\[
\nabla_\mu \gamma^\nu = \frac{1}{\phi} \partial_\mu \gamma^\nu + \Gamma^\nu_{\mu \lambda} \gamma^\lambda + S_\mu \gamma^\nu - \gamma^\nu S_\mu .
\]

(23)

A sufficient condition compatible with (21) is

\[
\nabla_\mu \gamma^\nu = \frac{1}{\phi} \partial_\mu \gamma^\nu + \Gamma^\nu_{\mu \lambda} \gamma^\lambda + S_\mu \gamma^\nu - \gamma^\nu S_\mu = 0 .
\]

(24)

To solve the equation above we introduce the tetrad field \(e^\mu_a\) and its inverse \(e^a_\mu\) related to the space-time metric by the following equations

\[
g^{\mu\nu}(x) = \eta_{ab} e^\mu_a(x) e^{\nu b}(x),
\]

\[
g_{\mu\nu}(x) = \eta^{ab} e^{\mu a}_a(x) e^{\nu b}_b(x),
\]

\[
e^{a}_\mu(x) = g^{\mu\nu}(x) \eta^{ab} e^{\nu b}_b(x),
\]

(25)

where \(\text{det}(e^a_\mu) = e = \sqrt{-g}\).

Therefore, the metricity condition can be expressed as

\[
\nabla_\mu e^{\nu a}_a = \frac{1}{\phi} \partial_\mu e^{\nu a}_a + \Gamma^\nu_{\mu \alpha} e^{\alpha a}_a + \omega_{\mu a}^{\phantom{\mu a} b} e^{\nu b}_b = 0 .
\]

(26)

Expressing \(\gamma^\nu\) in terms of the tetrad fields, \(\gamma^\nu = e^{\nu a}_a \gamma^a\), in the equation (24) we get

\[
\omega_{\mu a}^{\phantom{\mu a} b} e^{\nu b}_b \gamma^a = S_\mu \gamma^\nu - \gamma^\nu S_\mu = [S_\mu, \gamma^\nu] ,
\]

(27)

from which we found that

\[
S_\mu = \frac{1}{2} \omega_{\mu a b} \Sigma^{a b} ,
\]

(28)

where we define

\[
\Sigma^{a b} = \frac{1}{4} \left[ \gamma^a, \gamma^b \right] .
\]

(29)

4 Schwinger Action Principle

In following we use a classical version of the Schwinger Action Principle such as it was used in the context of Classical Mechanics by Sudarshan and Mukunda [14]. The Schwinger Action Principle is the most general version of the usual variational principles. It was proposed originally at the scope of the Quantum Field Theory [15], but its application goes beyond this area. In the classical context, the basic statement of the Schwinger Principle is

\[
\delta S = \delta \int_{\Omega} dxe^{\phi} L = \int_{\partial \Omega} d\sigma_\mu G^\mu ,
\]

where \(S\) is the classical action and \(G^\mu\) are the generators of the canonical transformations. The Schwinger Principle can be employed, choosing suitable variations in each case, to obtain commutation relations in the quantum context or canonical transformations in the classical one, as well as equations of motion or still perturbative expansions.

Here, we will apply the Action Principle to derive equations of motion of the Dirac field in an external Lyra background and its conservations laws associated with translations and rotations in such space.

\(^2\)This defines the general covariant derivative of an object with both tensorial and spinorial indexes,

\[
\nabla_\mu F^\lambda_B^A = \frac{1}{\phi} \partial_\mu F^\lambda_B^A + \Gamma^\lambda_{\mu \alpha} F^\alpha_B^A + S_\mu C F^\lambda_B^C - S_\mu C B F^\lambda_C^A .
\]
5 Dynamics of the Dirac Field Coupled to the Lyra Manifold

With the definitions of section 3 the Dirac Lagrangian in Minkowski spacetime, expressed by

$$\mathcal{L}^{M_4} = \frac{i}{2} \left( \bar{\psi} \gamma^a \partial_a \psi - \partial_a \bar{\psi} \gamma^a \psi \right) - m \bar{\psi} \psi,$$

(30)

can be generalized by the minimal coupling procedure to the Lyra space-time as

$$S = \int_\Omega dx^4 \phi \left\{ \frac{i}{2} \bar{\psi} \gamma^\mu \nabla_\mu \psi - \frac{i}{2} \nabla_\mu \bar{\psi} \gamma^\mu \psi - m \bar{\psi} \psi \right\}.$$

(31)

To derive in a direct way the equations of motion, energy-momentum and spin tensor for the Dirac field coupled to Lyra manifold we will use the Schwinger action principle. Thus, making the variation of the action integral we get

$$\delta S = \int_\Omega dx^4 \phi \left\{ \frac{4}{\phi} \mathcal{L} - \frac{i}{2\phi} \bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{i}{2\phi} \partial_\mu \bar{\psi} \gamma^\mu \psi \right\} \delta \phi + \int_\Omega dx^4 \phi \left\{ \frac{\delta \mathcal{L}}{\delta \psi} \right\} \delta \psi + \int_\Omega dx^4 \phi \left\{ \frac{\delta \mathcal{L}}{\delta \bar{\psi}} \right\} \delta \bar{\psi} +$$

$$+ \int_\Omega dx^4 \phi \left\{ \frac{i}{2} \bar{\psi} \left( \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \right) \right\} + \int_\Omega dx^4 \phi \left\{ \frac{i}{2} \bar{\psi} \left( \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \right) \right\} +$$

$$+ \int_\Omega dx^4 \phi \left\{ \frac{i}{2} \bar{\psi} \left( \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \right) \right\} +$$

(32)

where $\mathcal{L}$ is the Lagrangian density in (31).

Choosing different specializations of the variations, one can easily obtain, for instance, the equations of motion and the energy-momentum and spin tensors.

5.1 Equations of Motion

We choose to make functional variations only in the Dirac field thus we set $\delta \phi = \delta e^b = \delta \omega_{ab} = 0$ in the general variation (32) and after some manipulations such as integrations by parts, we obtain

$$\delta S = \int_\Omega dx^4 \partial_\mu \left[ \phi^4 e^{i/2} \left( \bar{\psi} \gamma^\mu \left( \frac{1}{\phi} (\delta \psi) - \frac{1}{\phi} (\delta \bar{\psi}) \right) \right) \gamma^\mu \psi \right] +$$

$$+ \int_\Omega dx^4 e \left( \delta \bar{\psi} \right) \left[ i \gamma^\mu \left( \nabla_\mu \psi + \frac{1}{2} \tilde{\tau}_\mu \psi \right) - m \bar{\psi} \right] +$$

$$+ \int_\Omega dx^4 e \left( \delta \psi \right) \left[ -i \left( \nabla_\mu \bar{\psi} + \frac{1}{2} \tilde{\tau}_\mu \bar{\psi} \right) \right] \gamma^\mu - m \bar{\psi} \right] \delta \psi,$$

(33)

where $\tilde{\tau}_\mu$ is the trace torsion and, it is given by

$$\tilde{\tau}_\mu = \tilde{\tau}_{\mu \rho} = \frac{3}{\phi} \partial_\mu \ln \phi.$$

(34)

Now, by the action principle we get the generator of the variations of the spinorial field

$$G_{\delta \psi} = \int_{\delta \Omega} d\sigma_\mu \left[ \phi^4 e^{i/2} \left( \bar{\psi} \gamma^\mu \left( \frac{1}{\phi} (\delta \psi) - \frac{1}{\phi} (\delta \bar{\psi}) \right) \right) \gamma^\mu \psi \right],$$

(35)

and its equations of motion,

$$i \gamma^\mu \left( \nabla_\mu \psi + \frac{1}{2} \tilde{\tau}_\mu \psi \right) - m \bar{\psi} = 0,$$

$$i \left( \nabla_\mu \bar{\psi} + \frac{1}{2} \tilde{\tau}_\mu \bar{\psi} \right) \gamma^\mu + m \bar{\psi} = 0.$$

(36)
5.2 Energy-Momentum Tensor and Spin Density Tensor

Now, we vary only the background manifold and we assume that $\delta \omega_{\mu ab}$ and $\delta e^\mu_a$ are independent variations, the general variation \[ (32) \] reads
\[
\delta S = \int_\Omega dx \phi^4 e \left\{ \left[ \frac{i}{2} \left( \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right) - e^\mu_a L \right] \delta e^\mu_a + \frac{i}{2} \left( \bar{\psi} \gamma^\mu \Sigma^{ab} \psi + \bar{\psi} \Sigma^{ab} \gamma^\mu \psi \right) \frac{1}{2} \delta \omega_{\mu ab} \right\},
\]
where we have used $\delta e = -ee^\mu_a \delta e^\mu_a$. First, holding only the variations in the tetrad field, $\delta \omega_{\mu ab} = 0$, we find
\[
\delta S = \int_\Omega dx \phi^4 e \left\{ \frac{i}{2} \left( \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right) - e^\mu_a L \right\} \delta e^\mu_a,
\]
the expression between brackets defines the energy-momentum density tensor,
\[
T^\mu_a \equiv \frac{1}{2} \phi^4 e \delta S \delta e^\mu_a = \frac{i}{2} \bar{\psi} \left\{ \gamma^\mu, \Sigma^{ab} \right\} \psi = \frac{1}{2} \varepsilon^{abc} e^\mu_c \bar{\psi} \gamma^5 \gamma^d \psi,
\]
which can be written in coordinates as $T^\mu_\nu \equiv e^\nu_a T^\mu_a$.

5.3 Functional Scale Variations and the Trace Relation

Under pure infinitesimal variation of the scale function we get from the equation \[ (32) \]
\[
\delta S = - \int_\Omega dx \phi^4 e \left[ \left( \frac{i}{2} \bar{\psi} \gamma^\mu \nabla_\mu \psi - \frac{i}{2} \nabla_\mu \bar{\psi} \gamma^\mu \psi - 4L \right) - \frac{i}{2} \bar{\psi} \left\{ \gamma^\mu, S^{ab}_\mu \right\} \psi \right] \frac{\delta \phi}{\phi},
\]
and using the definition of the energy-momentum \[ (39) \] and the spin tensor \[ (41) \] we can write
\[
\delta S = - \int_\Omega dx \phi^4 e \left( T^\mu_\nu e^\mu_a - \frac{1}{2} S^{\mu ab} \omega_{\mu ab} \right) \frac{\delta \phi}{\phi},
\]
from which we get the following identity:
\[
T^\mu_\nu e^\mu_a - \frac{1}{2} S^{\mu ab} \omega_{\mu ab} = 0,
\]
which is the so called trace relation. Such identity can be used to constraint the form of the connection $\omega$ in a given content of matter.

5.4 Conservation Laws

As a final application of the Schwinger Action Principle, let us derive the conservation laws associated to local Lorentz transformations and infinitesimal general coordinate transformations.
5.4.1 Spin Conservation

Under local Lorentz transformations, the functional variations of the tetrad and the spin connection are given by

\[ \delta e^\mu_a = \delta \varepsilon^b_a e^\mu_b \]  

\[ \delta \omega_{\mu ab} = \omega^c_{\mu} \delta \varepsilon_{ac} - \omega^c_{\mu a} \delta \varepsilon_{cb} - \frac{1}{\phi} \partial_\mu \delta \varepsilon_{ab} , \]

with \( \delta \varepsilon_{ab} = -\delta \varepsilon_{ba} \), where the first variation expresses the vectorial character of the tetrad in Minkowsky spacetime, and the second one comes from (19) with

\[ U = 1 + \frac{1}{2} \delta \varepsilon_{ab} \Sigma^{ab} . \]

The general expression (37) can be written using the definitions of energy-momentum and spin tensors as being

\[ \delta S = \int_\Omega d\sigma^4 \phi ( T^a_{\mu} \delta e^\mu_a + S^{\mu ab} \frac{1}{2} \delta \omega_{\mu ab} ) . \]

Substituting variations (44) and (45) and after some integration by parts and algebraic simplifications, we get

\[ \delta S = - \int_{\partial \Omega} d\sigma^4 \left( \phi^3 e S^{\mu ab} \frac{1}{2} \delta \varepsilon_{ab} \right) + \int_\Omega d\sigma^4 \left( \nabla^\mu S^{\mu ab} + \tilde{\tau}^\mu S^{\mu ab} - T^{ab} + T^{ba} \right) \frac{1}{2} \delta \varepsilon_{ab} , \]

Thus, from the Action Principle, we learn

\[ G_{\delta \varepsilon} = - \int_{\partial \Omega} d\sigma^4 \left[ \phi^3 e S^{\mu ab} \frac{1}{2} \delta \varepsilon_{ab} \right] , \]

\[ \nabla^\mu S^{\mu ab} + \tilde{\tau}^\mu S^{\mu ab} = T^{ab} - T^{ba} . \]

\( G_{\delta \varepsilon} \) is the generator of infinitesimal changes in the spinorial field under local Lorentz transformations, and (50) gives us the conservation law for the spinning content of the theory. The most important aspect here is the coupling among the spin density tensor and the propagating torsion.

5.4.2 Energy-Momentum Consonervation

Now we will concentrate our attention on the more complicated case of a general coordinate transformation. From Lyra’s transformation rule for vectors and the infinitesimal displacement \( \bar{x}^\mu = x^\mu + \delta x^\mu \), we have,

\[ \bar{e}^\mu_a (\bar{x}) = \frac{\phi(x)}{\phi(x)} \frac{\partial \bar{x}^\mu}{\partial x^\nu} e^\nu_a (x) , \]

with its functional variation \( \delta e^\mu_a (x) \equiv \bar{e}^\mu_a (x) - e^\mu_a (x) \) given by

\[ \delta e^\mu_a (x) = \frac{\phi(x)}{\phi(x)} \left[ e^\mu_a (x) + e^\nu_a (x) \partial_\nu \delta x^\mu - \delta x^\nu \partial_\nu e^\nu_a (x) + e^\mu_a (x) \delta x^\nu \partial_\nu \ln \phi (x) \right] - e^\mu_a (x) . \]

While for the covariant Lyra vector defined by the spin connection we have

\[ \bar{\omega}_{\mu ab} (\bar{x}) = \frac{\phi(x)}{\phi(x)} \frac{\partial x^\nu}{\partial x^\mu} \omega_{\nu ab} (x) , \]

and its functional variation is

\[ \delta \omega_{\mu ab} (x) = \frac{\phi(x)}{\phi(x)} \left[ \omega_{\mu ab} (x) - \omega_{\nu ab} (x) \partial_\nu \delta x^\mu - \delta x^\nu \partial_\nu \omega_{\mu ab} (x) - \omega_{\mu ab} (x) \delta x^\nu \partial_\nu \ln \phi (x) \right] - \omega_{\mu ab} (x) . \]
Substituting the expressions \( \Phi_2 \) and \( \Phi_3 \) in \( \Phi_7 \) and, after some calculations we obtain for the variation of the action the following expression

\[
\delta S = \int_{\Omega} d\sigma^\mu e^\phi \left[ \frac{\tilde{\phi}}{\phi} T^\mu - \frac{\phi}{\phi} S^{\mu ab} \frac{1}{2} \omega_{vab} \right] d\mathbf{x}^\nu + \int_{\Omega} d\sigma^\mu e^\phi \left( \frac{\tilde{\phi}}{\phi} - 1 \right) \left( T^\mu_a e^\mu_a - \frac{\phi}{\phi} \frac{1}{2} \omega^{\mu ab} S^{\mu ab} \right) \delta x^\nu + \int_{\Omega} d\sigma^\mu e^\phi \left[ \frac{\tilde{\phi}}{\phi} \left( \nabla^a T^\mu_a + \bar{\tau}^{\mu ab} T^\mu_a + \bar{\tau}^a T^\mu_a \right) + \frac{\phi}{\phi} \left( \partial_\nu \ln \frac{\tilde{\phi}}{\phi} \right) T^\nu_a + \right. \\
+ \left( \frac{\tilde{\phi}}{\phi} - \frac{\phi}{\phi} \right) \omega_{vab} T^\mu_a + \frac{1}{\phi} \left( \partial_\nu \ln \frac{\tilde{\phi}}{\phi} \right) \frac{1}{2} \omega_{vab} S^{ab} + \frac{1}{2} \frac{\phi}{\phi} S^{\mu ab} R_{vab} \right] \delta x^\nu.
\]

According to the Schwinger Principle, from the surface integral we obtain the generator of the infinitesimal changes in the Dirac field under general coordinate transformations

\[
G_{\delta x} = \int_{\Omega} d\sigma^\mu e^\phi \left[ \frac{\tilde{\phi}}{\phi} T^\mu - \frac{\phi}{\phi} S^{\mu ab} \frac{1}{2} \omega_{vab} \right] \delta \mathbf{x}^\nu.
\]

Next, from the third integral in \( \Phi_9 \) and due to the invariance of the action under general coordinate transformations we obtain the conservation law of energy-momentum.

\[
\frac{\tilde{\phi}}{\phi} \left( \nabla^a T^\mu_a + \bar{\tau}^{\mu ab} T^\mu_a + \bar{\tau}^a T^\mu_a \right) + \frac{\phi}{\phi} S^{\mu ab} R_{vab} + \\
+ \left( \frac{\tilde{\phi}}{\phi} - \frac{\phi}{\phi} \right) \omega_{vab} T^\mu_a + \frac{1}{\phi} \left( \partial_\nu \ln \frac{\tilde{\phi}}{\phi} \right) \frac{1}{2} \omega_{vab} S^{ab} + \frac{1}{2} \frac{\phi}{\phi} S^{\mu ab} R_{vab} = 0.
\]

By imposing the invariance of the action under general coordinate transformations, the second integral in \( \Phi_9 \) allows to get a new relation, we named it as the trace symmetry. Such identity for \( \phi \equiv \phi \) can be written as

\[
T^\mu_a e^\mu_a - \frac{\phi}{\phi} \frac{1}{2} S^{\mu ab} \omega_{vab} = 0,
\]

it can be considered as the generalization of the trace relation shown in \( \Phi_8 \). Formally, for \( \phi \equiv \phi \) it reduces to \( \Phi_8 \).

If we select the special case \( \tilde{\phi} \equiv \phi \), we get the following equations

\[
G_{\delta x} = \int_{\Omega} d\sigma^\mu e^\phi \left( T^\mu_a - S^{\mu ab} \frac{1}{2} \omega_{vab} \right) \delta \mathbf{x}^\nu,
\]

\[
\nabla^a T^\mu_a + \bar{\tau}^{\mu ab} T^\mu_a + \bar{\tau}^a T^\mu_a + \frac{1}{2} S^{\mu ab} R_{vab} = 0,
\]

6 Final Remarks

We have constructed the coupling of spinorial fields to Lyra background and found the general laws governing the motion of the spinor and the conservation of energy-momentum and spin, using several specializations of the Schwinger Action Principle which were applied in a classical context.

It is possible to establish the Riemannian limit of the Lyra geometry setting the scale functions to be \( \phi = \phi \equiv 1 \), thus, the conservation laws for the Dirac fields reads

\[
\nabla^a S^{\mu ab} = T^{ab} - T^{ba}.
\]

and

\[
\nabla^a T^\mu_a + \frac{1}{2} S^{\mu ab} R_{vab} = 0.
\]

The conservation laws \( \Phi_9 \) and \( \Phi_10 \) for the Dirac fields, in the case \( \phi = \phi \), are similar to the laws founded when the background is the Riemann-Cartan geometry. Such identification can be done by replacing \( \bar{\tau}^{\mu \lambda} \rightarrow 2 \bar{Q}^{\mu \beta \nu} \), where \( Q^{\mu \beta \nu} = \Gamma^{\mu \beta \nu} \) is the RC torsion, thus we have spin conservation law

\[
\nabla^a S^{\mu ab} + 2 \bar{Q}_\mu S^{\mu ab} = T^{ab} - T^{ba}
\]
and the energy-momentum law

\[ \nabla_\mu T^\mu_\nu + 2Q^\mu_\beta T^\beta_\nu + 2Q_\mu T^\mu_\nu + \frac{1}{2} S^{\mu \nu a b} R^a b \mu = 0 \quad (64) \]

where \( Q_\mu = Q^\nu \mu \nu \). Such as obtained in [13].

On the other hand, the trace symmetry [58] and the trace relation [13] are new properties provided by the scale invariance of the Lyra geometry. The results here obtained are still preliminaries, but it seems to indicate that Lyra Geometry would be a useful tool to implement a kind of conformal symmetry even in the case of massive fields. To make precise this reason, it is necessary a deep study of the relation among conformal invariance and Lyra scale transformations. The investigation of the back reaction of the one-half spin field upon the geometry would show us how a propagation equation for the scale function can be obtained. These studies are currently under discussion.

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