ON THE MAXIMAL VOLUME OF THREE-DIMENSIONAL HYPERBOLIC COMPLETE ORTHOSCHEMES

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ABSTRACT. A three-dimensional orthoscheme is defined as a tetrahedron whose base is a right-angled triangle and an edge joining the apex and a non-right-angled vertex is perpendicular to the base. A generalization, called complete orthoschemes, of orthoschemes is known in hyperbolic geometry. Roughly speaking, complete orthoschemes consist of three kinds of polyhedra; either compact, ideal or truncated. We consider a particular family of hyperbolic complete orthoschemes, which share the same base. They are parametrized by the “height”, which represents how far the apex is from the base. We prove that the volume attains maximal when the apex is ultraideal in the sense of hyperbolic geometry, and that such a complete orthoscheme is unique in the family.

1. INTRODUCTION

In [Ke], Kellerhals wrote “the most basic objects in polyhedral geometry are orthoschemes”, and she gave a formula to calculate the volumes of complete orthoschemes in the three-dimensional hyperbolic space. What we discuss here is the existence and the uniqueness of the maximal volume of a family of complete orthoschemes parametrized by the “height”.

Consider a family of pyramids in Euclidean space with a fixed base polygon and the locus of apexes perpendicular to the base polygon. The volumes of pyramids strictly increases when the height increases, because pyramids strictly increases as a set. By the same reason, this phenomenon holds true for such a family of pyramids in hyperbolic space. In contrast to the Euclidean case, the volume approaches to a finite value. Furthermore, in hyperbolic space the apex can “run out” the space. Then we can still obtain finite volume hyperbolic polyhedron by truncation with respect to the apex. The volume converges to zero as the vertex goes away from the space. So it is an interesting question when the volume becomes maximum.

As is mentioned above, one of the most fundamental one among all such pyramids is the orthoscheme. An orthoscheme is a kind of simplex which has particular orthogonality among its faces. Let $P_0$, $P_1$, $P_2$ and $P_3$ be the vertices of a simplex $R$ in the three-dimensional hyperbolic space. We denote by $P_iP_j$ the edge spanned by $P_i$ and $P_j$, and by $P_iP_jP_k$ the face spanned by $P_i$, $P_j$, and $P_k$. Such a simplex $R$ is called an orthoscheme (in the ordinary sense) if the edge $P_0P_1$ is perpendicular to the face $P_1P_2P_3$ and the face $P_0P_1P_2$ is orthogonal to $P_2P_3$. In other words, an orthoscheme is a tetrahedron with a right-angled triangle $P_0P_1P_2$ as its base and
an edge joining the apex and a non-right-angled vertex, say $P_2$, is perpendicular to the base. Vertices $P_0$ and $P_3$ are called the principal vertices of $R$. Its precise definition will be given in Section 3.

Though orthoschemes are also considered in Euclidean or spherical spaces, in hyperbolic space the ordinary orthoschemes are extended to the so-called complete orthoschemes. Let $B^3$ be the open unit ball in the three-dimensional Euclidean space $\mathbb{R}^3$ centered at the origin. The set $B^3$ can be regarded as the so-called projective ball model of the three-dimensional hyperbolic space. Any tetrahedron in hyperbolic space appears as a Euclidean tetrahedron in $B^3$. If one or both principal vertices of an orthoscheme $R$ lie in the boundary of $B^3$, the set $R \cap B^3$ is called an ideal polyhedron, which is not bounded in hyperbolic space, while its volume is finite. Take one step further and we allow principal vertices to be in the exterior of $B^3$. The volume of $R \cap B^3$ is no longer finite, but there is a canonical way to delete ends of $R \cap B^3$ with infinite volume so that we obtain a polyhedron of finite volume, called a truncated polyhedron. Complete orthoschemes are, roughly speaking, either compact, ideal or truncated orthoschemes. The precise definitions of complete orthoschemes and truncation will also be given in Section 3.

What we study in this paper is the maximal volume of a family of complete orthoschemes with one parameter. Consider a family of complete orthoschemes that share the same base $P_0P_1P_2$. We allow the vertex $P_3$ to be in the exterior of $B^3$. In this case the base $P_0P_1P_2$ means the truncated polygon obtained from the triangle with vertices $P_0$, $P_1$ and $P_2$. Such a family of complete orthoschemes is parametrized by the hyperbolic length of the edge $P_2P_3$ when $P_3$ is in $B^3$. When the hyperbolic length increases, the orthoscheme strictly increases as a set, which means the volume also increases with respect to the function of the hyperbolic length. This phenomenon holds until the vertex $P_3$ lies in the boundary $\partial B^3$ of $B^3$. The hyperbolic length of $P_2P_3$ is “beyond” the infinity when $P_3$ is in the exterior of $B^3$, but we have a complete orthoscheme with finite volume by truncation. Instead of the hyperbolic length, using the Euclidean length of $P_2P_3$, which we mentioned as “height” in the first paragraph, we can parametrize the family even if $P_3$ is in the complement of $B^3$. The complete orthoscheme approaches the empty set when $P_3$ goes far away from $B^3$. The family thus has maximal volume complete orthoschemes, which arise when $P_3$ lies in the complement of $B^3$.

As a toy model, let us consider the same phenomenon for the two-dimensional complete orthoschemes, namely hyperbolic triangle $P_0P_1P_2$ with right angle at $P_1$. Take a family of complete orthoschemes parametrized by the “height” of $P_1P_2$. The area strictly increases when $P_2$ approaches to the boundary of $B^2$, the projective disc model of the two-dimensional hyperbolic space. The area attains maximal when $P_2$ lies in $\partial B^2$. When $P_2$ is in the exterior of $B^2$, the area decreases, but not necessarily monotonically. These facts are summarized as Theorem 2 in the appendix.

One may expect that the same phenomenon happens for three-dimensional complete orthoschemes. Is the volume attains maximal at least when $P_3$ is in $\partial B^3$? Does the volume decrease when $P_3$ goes far away from $B^3$? Our main result, which is Theorem 1 in Section 5 answers both of the questions negatively.

2. Preliminaries of hyperbolic geometry

There are several models to introduce hyperbolic geometry. Among them we use the hyperboloid model to calculate lengths and angles with respect to the hyperbolic
metric, and use the projective ball model to define complete orthoschemes. Definitions of these two models, together with formulae to calculate hyperbolic lengths and hyperbolic angles, are explained in this section. See [Ra] for basic references on hyperbolic geometry.

As a set, the hyperboloid model $H_T^+$ of the three-dimensional hyperbolic space is defined as a subset of the four-dimensional Euclidean space $\mathbb{R}^4$ by

$$H_T^+ := \{ \mathbf{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1 \text{ and } x_0 > 0 \},$$

where $\langle \cdot, \cdot \rangle$, called the Lorentzian inner product, is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle := -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

for any $\mathbf{x} = (x_0, x_1, x_2, x_3)$ and $\mathbf{y} = (y_0, y_1, y_2, y_3)$ in $\mathbb{R}^4$. The restriction of the quadratic form induced from the Lorentzian inner product to the tangent spaces of $H_T^+$ is positive definite and gives a Riemannian metric on $H_T^+$, which is constant curvature of $-1$. The set $H_T^+$ together with this metric gives the hyperboloid model of the three-dimensional hyperbolic space.

Associated with $H_T^+$, there are two important subsets of $\mathbb{R}^4$:

$$H_S := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1 \}, \quad L^+ := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0 \text{ and } x_0 > 0 \}.$$

Every point $\mathbf{u}$ in $H_S$ corresponds to a half-space

$$R_u := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{u} \rangle \leq 0 \},$$

bounded by a plane

$$P_u := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{u} \rangle = 0 \}.$$

The intersection $P_u \cap H_T^+$ is a geodesic plane with respect to the hyperbolic metric. If $\mathbf{u}$ is taken from $L^+$, the set $R_u$ is defined as

$$R_u := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{u} \rangle \leq -\frac{1}{2} \}.$$

The intersection $R_u \cap H_T^+$ is called a horoball. The intersection of the boundary

$$P_u := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{u} \rangle = -\frac{1}{2} \}$$

of $R_u$ and $H_T^+$ is called a horosphere.

The Lorentzian inner product is also used to calculate distances and angles with respect to the hyperbolic metric. The details of the following results are explained in §3.2 of [Ra]. Let $\mathbf{u}$ be a point in $H_T^+$ and let $\mathbf{v}$ be taken from $H_S$ with $\mathbf{u} \in R_v$, then the hyperbolic distance $\ell$ between $\mathbf{u}$ and the geodesic plane $P_v$ is calculated by

$$\sinh \ell = -\langle \mathbf{u}, \mathbf{v} \rangle.$$  \hspace{1cm} (2.1)

Suppose that $\mathbf{u}$ is in $L^+$ and $\mathbf{v}$ is in $H_S$ with $\mathbf{u} \in R_v$. Let $\ell$ be the signed hyperbolic distance between the horosphere $P_u \cap H_T^+$ and the geodesic plane $P_v \cap H_T^+$. The sign is defined to be positive if the horosphere and the geodesic plane do not intersect, otherwise negative. Then the signed distance $\ell$ is calculated by

$$\frac{e^\ell}{2} = -\langle \mathbf{u}, \mathbf{v} \rangle.$$  \hspace{1cm} (2.2)

If both $\mathbf{u}$ and $\mathbf{v}$ are taken from $H_S$ with $\mathbf{u} \in R_v$ and $\mathbf{v} \in R_u$, then there are three possibilities: $R_u \cap R_v$ intersects $H_T^+$, intersects $L^+$ or does not intersect both.
orthoschemes in the next section. First, every geodesic plane in $B^3$ is given as the intersection of a Euclidean plane and $B^3$. This is because every geodesic plane in $H_T^+$ is defined as the intersection of $H_T^+$ and a linear subspace of $\mathbb{R}^4$ of dimension three, and a geodesic plane in $B^3$ is the image of that in $H_T^+$ by the radial projection. The projection $\mathcal{P}$ thus gives a correspondence between points in the exterior of $B^3$ in $\mathbb{R}P^3$ and the geodesic planes of $B^3$. We call $\mathcal{P}(u)$ for $u \in H_S$ the pole of the plane $\mathcal{P}(P_u)$ or the geodesic plane $\mathcal{P}(P_u \cap H_T^+)$. Conversely, we call $\mathcal{P}(P_u)$ the polar plane of $\mathcal{P}(u)$, and we call $\mathcal{P}(P_u \cap H_T^+)$ the polar geodesic planes of $\mathcal{P}(u)$. If $\mathcal{P}(P_u \cap H_T^+)$ does not pass through the origin of $B^3$, then its pole is given as the apex of a circular cone which is tangent to $\partial B^3$ and has the base circle $P_u \cap \partial B^3$. The second important property is that, for a given geodesic plane, say $P$, in $B^3$, every plane or line which passes through the pole of $P$ is orthogonal to $P$ in $B^3$. This is proved by using Equation (2.3).

### 3. Complete orthoschemes

Following [Ke] we introduce complete orthoschemes. As is mentioned in the introduction, an (ordinary) orthoscheme in the three-dimensional hyperbolic space is a tetrahedron with vertices $P_0$, $P_1$, $P_2$ and $P_3$ which satisfies that $P_0P_1$ is perpendicular to $P_1P_2P_3$ and that $P_0P_1P_2$ is orthogonal to $P_2P_3$. The vertices $P_1$ and $P_3$ are called principal vertices.
Complete orthoschemes are a generalization of ordinary orthoschemes by allowing one or both principal vertices to be ideal or ultraideal. Take $B^3$ as our favorite model of the hyperbolic space in what follows. As a set, any orthoscheme in the ordinary sense are given as a Euclidean tetrahedron in $B^3$. When one or both principal vertices are ideal, the tetrahedron as a set in the hyperbolic space is no more bounded, but still has finite volume. We allow to call such tetrahedra ordinary orthoschemes.

Further generalization of orthoschemes is explained via truncation of ultraideal vertices. Suppose a vertex $v$ of a tetrahedron $R$ is ultraideal. Let $T$ be the half-space bounded by the polar plane of $v$ with $v \notin T$. Truncation of $R$ with respect to $v$ is defined as an operation to obtain a polyhedron $R \cap T$. If $v$ is close enough to $\partial B^3$, then $R \cap T$ is non-empty.

Truncation is also explained by using the hyperboloid model. The inverse image of $v$ for $P$ on $H_S$ consists of two points. Each of them gives a half-space in $R^4$, and one of them corresponds to the inverse image of $T$. In this sense there is a one-to-one correspondence between half-spaces in $B^3$ and points in $H_S$. The point in $H_S$ corresponding to $v$ with respect to $T$ in the sense above is called the proper inverse image of $v$ for truncation of $R$. This correspondence will be used to calculate hyperbolic lengths of edges and hyperbolic dihedral angles between faces of complete orthoschemes.

When one of the principal vertices, say $P_3$, is ultraideal and $P_0$ is not (i.e., ordinal or ideal), we have a polyhedron with finite volume by truncation with respect to $P_3$. Such a polyhedron is called a simple frustum with ultraideal vertex $P_3$. We remark that the vertices $P_0$, $P_1$ and $P_2$ are simultaneously deleted by truncation when $P_3$ is far away from $B^3$, since both the polar geodesic plane of $P_3$ and the triangle $P_0P_1P_2$ are orthogonal to $P_2P_3$ in $B^3$.

Suppose both $P_0$ and $P_3$ are ultraideal. There are three possibilities: the polar planes of $P_0$ and $P_3$ intersect in $B^3$, they are parallel, or they are ultraparallel. In the first case, the polyhedron we obtain by truncation is well known as a Lambert cube. See [Ke] Figure 2 for example. The edge $P_0P_3$ is deleted by truncation. In the third case, on the other hand, the polyhedron obtained by truncation still has the edge induced from $P_0P_3$. We call this polyhedron a double frustum. The second case is the limiting situation of both the first and the third cases. We call polyhedra obtained in the second case double frustum with an ideal vertex.

As a summary, combinatorial types of complete orthoschemes are either

- ordinary orthoschemes, whose principal vertices are either ordinarily points or ideal points,
- simple frustums,
- double frustums possibly with an ideal vertex, or
- Lambert cubes.

4. The Schl"afli differential formula

Kellerhals obtained formulae to calculate volumes of complete orthoschemes in [Ke]; the formula for Lambert cubes is given in Theorem III, and the formula for other kinds of complete orthoschemes is given in Theorem II. In both formulae, they are parametrized by the three non-right hyperbolic dihedral angles. Under the same setting used in Section 3 we denote by $\theta_{i,j}$ the hyperbolic dihedral angle between faces opposite to $P_i$ and $P_j$. When a complete orthoscheme is a Lambert
cube, the geodesic planes containing faces opposite to $P_1$ and $P_2$ are ultraparallel. In this case $θ_{1,2}$ is defined to be the hyperbolic dihedral angle between the polar geodesic planes of the vertex $P_0$ and $P_3$. In this sense the formulae are parametrized by $θ_{0,1}$, $θ_{1,2}$ and $θ_{2,3}$.

Kellerhals used the Schlaffi differential formula to obtain these volume formulae. The volume formulae are not used directly in our arguments; what we will use is the fact that the formulae are parametrized by the three non-right hyperbolic dihedral angles. On the other hand, the Schlaffi differential formula itself plays an important role in our arguments.

The Schlaffi differential formula gives an expression of the differential form of the volume function with respect to the hyperbolic lengths of edges and hyperbolic dihedral angles between faces. As is given in Theorem I in [Ke], the differential form $dV$ of the volume function $V$ of any complete orthoschemes is expressed as

$$dV = -\frac{1}{2} (\ell_{0,1} dθ_{0,1} + \ell_{1,2} dθ_{1,2} + \ell_{2,3} dθ_{2,3}),$$

where $ℓ_{i,j}$ is the hyperbolic length of the edge $P_iP_j$ if both $P_i$ and $P_j$ are points in $B^3$, $ℓ_{i,j}$ is the hyperbolic distance between $P_i$ and the polar geodesic plane of $P_j$ if $P_i$ is a point in $B^3$ and $P_j$ lies in the exterior of $B^3$, and $ℓ_{i,j}$ is the hyperbolic distance between the polar geodesic planes of $P_i$ and $P_j$ if both $P_i$ and $P_j$ lie in the exterior of $B^3$. If a complete orthoscheme is a Lambert cube, then $ℓ_{0,3}$ is taken as the hyperbolic length of the edge obtained as the intersection of the polar geodesic planes of $P_0$ and $P_3$. If one of $P_0$ and $P_3$ is ideal, then the edges with the ideal vertex as an endpoint have infinite hyperbolic lengths. In this case we take any horosphere centered at the ideal vertex, and each infinite length is replaced by the signed hyperbolic distance between the other endpoint and the horosphere. As is mentioned in the concluding remarks in [Mi], the Schlaffi differential formula is still valid by this treatment.

As a result, the Schlaffi differential formula is applicable to any kind of complete orthoschemes. We use the formula as the equation

$$\frac{∂V}{∂θ_{i,j}} = -\frac{1}{2} ℓ_{i,j}$$

for $(i, j) = (0, 1), (1, 2), (2, 3)$. This equation plays a key role in the proof of Theorem 1.

5. Main result

Suppose $B^3$ lies in the $xyz$-coordinate space of $\mathbb{R}^3$. By the action of an isometry, any ordinary orthoscheme can be put as the vertex $P_0$ is in the positive quadrant of the $xy$-plane, the vertex $P_1$ is on the positive part of the $y$-axis, the vertex $P_2$ is the origin, and the vertex $P_3$ is on the positive part of the $z$-axis. Such orthoschemes are parametrized by $(h, r, θ)$, where $h$ is the $z$-coordinate of $P_3$, i.e., the Euclidean distance between $P_2$ and $P_3$, $r$ is the Euclidean distance between $P_0$ and $P_2$, and $θ$ is the Euclidean angle between edges $P_0P_2$ and $P_1P_2$.

When we regard such an orthoscheme as a tetrahedron with base $P_0P_1P_2$, the $z$-coordinate $h$ of $P_3$ is the “height” of the tetrahedron. What we study in this paper is a family of complete orthoschemes parametrized by the “height”. For fixed $r$ and $θ$, we have a one-parameter family $\{R_{r,θ}(h)\}_{0 < h \leq 1}$ of ordinary orthoschemes
parametrized by \( h \). This family is extended even when \( h \geq 1 \) and/or \( r \geq 1 \) with \( r \cos \theta < 1 \), if we mean \( R_{r,\theta}(h) \) a complete orthoscheme.

Let \( V_{r,\theta}(h) \) be the hyperbolic volume of \( R_{r,\theta}(h) \). By the volume formulae, the function \( V_{r,\theta} \) is continuous on \([0, +\infty)\) and piecewise differentiable on the intervals each of which corresponds to a combinatorial type of complete orthoschemes given at the end of Section 3. When \( h \) increases in value approaching 1, the orthoscheme also increases as a set. So \( V_{r,\theta}(h) \) strictly increases in value approaching \( V_{r,\theta}(1) \) as \( h \) approaches 1 from below. When \( h \) approaches positive infinity \(+\infty\), the sequence \( R_{r,\theta}(h) \) of complete orthoschemes converges to the base \( P_0P_1P_2 \); the complete orthoschemes are always ordinary ones when \( 0 < r \leq 1 \), and the complete orthoschemes changes into Lambert cubes from double frustums when \( r > 1 \). In any case \( V_{r,\theta}(h) \) converges to 0 as \( h \) approaches \(+\infty\).

Based on these observations, we have set the following questions. For a given one-parameter family \( \{R_{r,\theta}(h)\}_{h>0} \) of complete orthoschemes, does the function \( V_{r,\theta} \) attain maximal when \( P_3 \) is in \( \partial B^3 \)? Is \( V_{r,\theta} \) strictly decreasing on \((1, +\infty)\)? The next theorem, which is the main result of this paper, answers both of the questions negatively.

**Theorem 1.** For any \( r > 0 \) and \( 0 < \theta < \pi/2 \) with \( r \cos \theta < 1 \), the volume \( V_{r,\theta}(h) \) of \( R_{r,\theta}(h) \) attains maximal for some \( h \in (1, +\infty) \). Furthermore, the maximal volume is unique for any \( r \) and \( \theta \), and it is given before \( R_{r,\theta}(h) \) becomes a Lambert cube.

The outline of the proof is as follows. Using the Schlafli differential formula, we can calculate \( dV_{r,\theta}(h)/dh \) for each combinatorial types of \( R_{r,\theta}(h) \). Since \( V_{r,\theta} \) is a strictly increasing function on \([0, 1]\), proving \( \lim_{h \to 1} dV_{r,\theta}(h)/dh > 0 \) tells us that the function \( V_{r,\theta} \) attains maximal for some \( h \in (1, +\infty) \). The uniqueness of such \( h \) is induced from the uniqueness of the solution of the equation \( dV_{r,\theta}(h)/dh = 0 \) on \((1, +\infty)\).

6. **Proof of the main result**

Our proof of Theorem 1 is organized as follows. After confirming the correspondence between combinatorial types of complete orthoschemes and conditions of parameters \( h, r \) and \( \theta \), we first obtain suitable inverse images of vertices of \( R_{r,\theta}(h) \) for \( \mathcal{P} \). These are used to calculate hyperbolic lengths and hyperbolic dihedral angles appearing in the Schlafli differential formula. Under each of conditions of parameters, we prove that the volume function \( V_{r,\theta} \) with respect to \( h \) attains maximal on \((1, +\infty)\), and that such \( h \) is unique. For \( r > 1 \), we also prove that \( V_{r,\theta} \) does not attain maximal if \( R_{r,\theta}(h) \) is a Lambert cube.

6.1. **Proper inverse images of the vertices.** By the definition of \( R_{r,\theta}(h) \), the coordinates of the vertices are

\[
\begin{align*}
P_0 &= (r \sin \theta, r \cos \theta, 0), & P_1 &= (0, r \cos \theta, 0), \\
P_2 &= (0, 0, 0), & P_3 &= (0, 0, h),
\end{align*}
\]

where \( 0 < \theta < \pi/2 \). As is mentioned after Theorem 1, it is enough to assume that \( h > 1 \) in what follows. A complete orthoscheme \( R_{r,\theta}(h) \) is a simple frustum if \( 0 < r < 1 \), and a simple frustum with ideal vertex \( P_0 \) if \( r = 1 \). When \( r > 1 \), we always assume \( r \cos \theta < 1 \) so that \( P_1 \) is in \( B^3 \). Under these assumptions, a complete orthoscheme \( R_{r,\theta}(h) \) with \( r > 1 \) is either a double frustum, a double frustum with an ideal vertex, or a Lambert cube. These are distinguished via the Euclidean
distance between the origin of $\mathbb{R}^3$ and the edge $P_0P_3$: $R_{r,\theta}(h)$ is a double frustum, a double frustum with an ideal vertex, or a Lambert cube if and only if the Euclidean distance is less than, equal to, or greater than 1 respectively. Since the Euclidean distance is $hr/\sqrt{r^2 + h^2}$, we have that these are equivalent to $h < r/\sqrt{r^2 - 1}$, $h = r/\sqrt{r^2 - 1}$, or $h > r/\sqrt{r^2 - 1}$ respectively. The inequality $hr/\sqrt{r^2 + h^2} < 1$ is also equivalent to $(1 - r^2)h^2 + r^2 > 0$ without the assumption that $r > 1$. We note that this inequality always holds for any $h > 0$ and $0 < r \leq 1$.

As a summary, complete orthoschemes $R_{r,\theta}(h)$ are parametrized by $(h, r, \theta)$, and with $h > 1$ and $0 < \theta < \pi/2$, and

- when $0 < r \leq 1$, complete orthoschemes $R_{r,\theta}(h)$ are simple frustums with $(1 - r^2)h^2 + r^2 > 0$,
- when $r > 1$ with $r \cos \theta < 1$ and $h \leq r/\sqrt{r^2 - 1}$, complete orthoschemes $R_{r,\theta}(h)$ are double frustums (possibly with an ideal vertex), and
- when $r > 1$ with $r \cos \theta < 1$ and $h > r/\sqrt{r^2 - 1}$, complete orthoschemes $R_{r,\theta}(h)$ are Lambert cubes.

We next give the proper inverse images of these vertices for $\mathcal{P}$. When a vertex is in $B_3$, its inverse image for $\mathcal{P}$ must be chosen in $H_+^*$, which is uniquely determined. When a vertex is in the exterior of $B^3$, its inverse image is chosen to be proper inverse image in the sense of truncation. Finally, when a vertex is in $\partial B^3$, we choose its proper inverse image as any element in the inverse for $\mathcal{P}$, which is a subset in $L^*$. Let $p_i$ be the proper inverse image of $P_i$ in this sense. The coordinates of $p_i$ are then as follows:

(1) When $0 < r < 1$, we have

$$ p_0 = \frac{1}{\sqrt{1 - r^2}} (1, r \sin \theta, r \cos \theta, 0), $$

$$ p_1 = \frac{1}{\sqrt{1 - r^2 \cos^2 \theta}} (1, 0, r \cos \theta, 0), $$

$$ p_2 = (1, 0, 0, 0), $$

$$ p_3 = \frac{1}{\sqrt{h^2 - 1}} (1, 0, 0, h). $$

(2) When $r = 1$, the coordinates of $p_1$, $p_2$ and $p_3$ are the same as in the first case and

$$ p_0 = (1, \sin \theta, \cos \theta, 0). $$

(3) When $r > 1$, the coordinates of $p_1$, $p_2$ and $p_3$ are the same as in the first case, and

$$ p_0 = \frac{1}{\sqrt{r^2 - 1}} (1, r \sin \theta, r \cos \theta, 0). $$

The inverse image of the pole of a geodesic plane in $B^3$ consists of two points in $H_S$. For each (ordinary) face of an orthoscheme $R_{r,\theta}(h)$, we choose the inverse image of the pole in $H_S$ so that the half-space defined by this inverse image contains $R_{r,\theta}(h)$. Let $u_i$ be the inverse image of the pole of the face $P_jP_kP_l$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$ in this sense. In other words, $u_i$ is a point in $H_S$ where $R_{u_i}$ contains $R_{r,\theta}(h)$ and
\( P_u \) contains \( P_jP_kP_l \). For any \( r \), the coordinates of \( u_i \) are as follows:
\[
\begin{align*}
u_0 &= (0, -1, 0), \\
u_1 &= (0, \cos \theta, -\sin \theta, 0), \\
u_2 &= \frac{1}{\sqrt{(1 - r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}}(h r \cos \theta, 0, h, r \cos \theta), \\
u_3 &= (0, 0, 0, -1).
\end{align*}
\]

6.2. The maximal value of \( V_{r,\theta} \) and its uniqueness with respect to \( h \). We focus on the derivative \( dV_{r,\theta}(h)/dh \) to prove that \( V_{r,\theta} \) attains maximal on \((1, +\infty)\), as well as its uniqueness.

We first confirm that the function \( V_{r,\theta} \) is piecewise differentiable with respect to \( h \) in general.

We first suppose that \( r \leq 1 \). By the Schl"afli differential formula, the function \( V_{r,\theta} \) is differentiable with respect to the hyperbolic dihedral angles \( \theta_{0,1}, \theta_{1,2} \) and \( \theta_{2,3} \). By the expression of the coordinates of \( u_i \) and \( p_i \) for \( i = 0, 1, 2, 3 \) together with Equation (2.3), these angles are given as smooth functions with respect to \( h \). By the chain rule, \( V_{r,\theta} \) is thus differentiable with respect to \( h \). In particular \( V_{r,\theta} \) is continuous on \([0, +\infty)\).

If \( r > 1 \), then there are two combinatorial types of \( R_{r,\theta}(h) \): a double frustum or a Lambert cube. The function \( V_{r,\theta} \) is not only continuous but also piecewise differentiable on \([0, +\infty)\), for \( V_{r,\theta} \) is differentiable on the intervals corresponding to each combinatorial types of \( R_{r,\theta}(h) \) by the same argument used for \( r \leq 1 \).

Recall that the function \( V_{r,\theta} \) is continuous on \([0, +\infty)\), strictly increasing on \([0, 1]\) and has its limit 0 as \( h \) approaches \(+\infty\). So, to prove that \( V_{r,\theta} \) attains maximal on \((1, +\infty)\), it is enough to prove that the limit of \( dV_{r,\theta}(h)/dh \) is positive as \( h \) approaches 1 from above. The uniqueness of the maximal value of \( V_{r,\theta} \) is induced from the fact that the solution of \( dV_{r,\theta}(h)/dh = 0 \) is at most one on \((1, +\infty)\).

Applying the chain rule and we have
\[
\frac{dV_{r,\theta}(h)}{dh} = \frac{\partial V_{r,\theta}(h)}{\partial \theta_{0,1}} \frac{d\theta_{0,1}}{dh} + \frac{\partial V_{r,\theta}(h)}{\partial \theta_{1,2}} \frac{d\theta_{1,2}}{dh} + \frac{\partial V_{r,\theta}(h)}{\partial \theta_{2,3}} \frac{d\theta_{2,3}}{dh}
\]
The parameter \( \theta_{i,j} \) defined in Section 4 is the hyperbolic dihedral angle between the polar geodesic planes of \( P(u_i) \) and \( P(u_j) \). In other words, \( \theta_{i,j} \) is the hyperbolic dihedral angle along the edge \( P_kP_l \) for \( \{i, j, k, l\} = \{0, 1, 2, 3\} \). As is mentioned in the first paragraph of Section 4 if \( R_{r,\theta}(h) \) is a Lambert cube, then \( \theta_{1,2} \) is taken as the hyperbolic dihedral angle between the polar geodesic planes of \( P_0 \) and \( P_3 \).

The Schl"afli differential formula are used to calculate partial derivatives appeared in the equation above. By Equation (4.1) we have
\[
\frac{\partial V_{r,\theta}(h)}{\partial \theta_{1,2}} = \frac{1}{2} \ell_{0,3}, \quad \frac{\partial V_{r,\theta}(h)}{\partial \theta_{2,3}} = -\frac{1}{2} \ell_{0,1},
\]
where \( \ell_{i,j} \) is the hyperbolic length with respect to the edge \( P_iP_j \) defined in Section 4. Furthermore, the hyperbolic dihedral angle \( \theta_{0,1} \), which coincides with the Euclidean angle \( \theta \) by the definition of \( R_{r,\theta}(h) \), is constant with respect to \( h \), meaning that \( d\theta_{0,1}/dh = 0 \). We thus have
\[
(6.1) \quad \frac{dV_{r,\theta}(h)}{dh} = -\frac{1}{2} \left( \ell_{0,3} \frac{d\theta_{1,2}}{dh} + \ell_{0,1} \frac{d\theta_{2,3}}{dh} \right).
\]

We divide the remaining argument into three cases according to the value of \( r \).
Case (1): single frustums with ordinary vertex $P_0$, i.e., $0 < r < 1$. By Equations (2.1) we have

$$\ell_{0,3} = \arcsinh (-\langle p_0, p_3 \rangle)$$

$$= \arcsinh \frac{1}{\sqrt{1 - r^2} \sqrt{h^2 - 1}}$$

$$= \log \left( \frac{1}{\sqrt{1 - r^2} \sqrt{h^2 - 1}} + \sqrt{\left( \frac{1}{\sqrt{1 - r^2} \sqrt{h^2 - 1}} \right)^2 + 1} \right)$$

$$= \log \frac{(1 - r^2) h^2 + r^2 + 1}{\sqrt{1 - r^2} \sqrt{h^2 - 1}},$$

(6.2)

and by Equation (2.3) we have

$$\theta_{1,2} = \arccos (-\langle u_1, u_2 \rangle)$$

$$= \arccos \frac{h \sin \theta}{\sqrt{(1 - r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}}$$

$$\theta_{2,3} = \arccos (-\langle u_2, u_3 \rangle)$$

$$= \arccos \frac{r \cos \theta}{\sqrt{(1 - r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}}$$

Derivatives of hyperbolic dihedral angles with respect to $h$ are obtained as follows:

$$\frac{d\theta_{1,2}}{dh} = \frac{-r^2 \sin \theta \cos \theta}{\sqrt{(1 - r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}}$$

(6.3)

$$\frac{d\theta_{2,3}}{dh} = \frac{r \sqrt{1 - r^2 \cos^2 \theta} \cos \theta}{(1 - r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}$$

(6.4)

Substitute Equations (6.2), (6.3) and (6.4) to Equation (6.1) and we have

$$\frac{dV_{r,\theta}(h)}{dh} = \frac{1}{2} \left( -\frac{d\theta_{1,2}}{dh} \right) \left( F(h) - \frac{1}{2} \log(1 - r^2) \right),$$

where

$$F(h) := \log \frac{(1 - r^2) h^2 + r^2 + 1}{\sqrt{h^2 - 1}} - C \sqrt{(1 - r^2) h^2 + r^2}$$

(6.5)

and $C := \ell_{0,1} \sqrt{1 - r^2 \cos^2 \theta} / (r \sin \theta)$.

Since

$$\lim_{h \downarrow 1} \left( -\frac{d\theta_{1,2}}{dh} \right) = r^2 \sin \theta \cos \theta,$$

we have

$$\lim_{h \downarrow 1} \frac{dV_{r,\theta}(h)}{dh} = \frac{1}{2} \left( r^2 \sin \theta \cos \theta \right) \left( +\infty - \frac{1}{2} \log(1 - r^2) \right)$$

$$= +\infty,$$

which implies that $V_{r,\theta}$ attains maximal for some $h \in (1, +\infty)$.

This result together with $\lim_{h \uparrow +\infty} V_{r,\theta}(h) = 0$ implies that the uniqueness of the maximal value of the function $V_{r,\theta}$ with respect to $h$ is proved by showing that the
equation \( dV_{r,\theta}(h)/dh = 0 \) has at most one solution on \((1, +\infty)\). Since \( d\theta_{1,2}/dh \neq 0 \) on \((1, +\infty)\) by Equation (6.3), we have

\[
\{ h \in (1, +\infty) \mid \frac{dV_{r,\theta}(h)}{dh} = 0 \} = \{ h \in (1, +\infty) \mid F(h) - \frac{1}{2} \log(1 - r^2) = 0 \}.
\]

Since

\[
\frac{d}{dh} \left( F(h) - \frac{1}{2} \log(1 - r^2) \right) = -\frac{h}{(h^2 - 1) \sqrt{(1 - r^2) h^2 + r^2}} G(h),
\]

where \( G(h) := C (1 - r^2) (h^2 - 1) + 1 \), is negative on \((1, +\infty)\), the function \( F(h) - (1/2) \log(1 - r^2) \) is strictly monotonic with respect to \( h \). This implies that the number of elements in the right-hand side set of Equation (6.6) is at most one, so is the left-hand side.

**Case 2**: single frustums with ideal vertex \( P_0 \), i.e., \( r = 1 \). Using Equation (2.2), we have

\[
\ell_{0,3} = \log \left( -2 \langle p_0, p_3 \rangle \right)
= \log \frac{2}{\sqrt{h^2 - 1}}.
\]

By Equations (6.3) and (6.4) with \( r = 1 \) and we have

\[
- \frac{d\theta_{1,2}}{dh} = \frac{d\theta_{2,3}}{dh} = \frac{\sin \theta \cos \theta}{h^2 \sin^2 \theta + \cos^2 \theta}.
\]

Substitute these equations to Equation (6.1) and we have

\[
\frac{dV_{r,\theta}(h)}{dh} = \frac{1}{2} \left( - \frac{d\theta_{1,2}}{dh} \right) \left( \log \frac{2}{\sqrt{h^2 - 1}} - \ell_{0,1} \right)
= \frac{1}{2} \left( - \frac{d\theta_{1,2}}{dh} \right) \left( -\frac{1}{2} \log(h^2 - 1) + \log 2 - \ell_{0,1} \right).
\]

Since

\[
\lim_{h \downarrow 1} \left( - \frac{d\theta_{1,2}}{dh} \right) = \sin \theta \cos \theta, \quad \lim_{h \downarrow 1} \log(h^2 - 1) = -\infty,
\]

we have \( \lim_{h \downarrow 1} dV_{r,\theta}(h)/dh = +\infty \) in this case.

The uniqueness of the maximal value of \( V_{r,\theta} \) with respect to \( h \) is obtained by the facts that \( \log(h^2 - 1) \) is a strictly monotonic function and that \( d\theta_{1,2}/dh \neq 0 \) on \((1, +\infty)\).

**Case 3**: double frustums or Lambert cubes, i.e., \( r > 1 \). Since our strategy of proving that \( V_{r,\theta} \) attains maximal on \((1, +\infty)\) is to prove that the limit of \( dV_{r,\theta}(h)/dh \) is positive as \( h \) approaches to 1 from above, it is enough to consider the case that \( h \) is close enough to 1, meaning that \( R_{r,\theta}(h) \) are double frustum, not Lambert cubes.
Under this assumption, use Equation (2.4) and we have

\[ \ell_{0,3} = \arccosh (-\langle p_0, p_3 \rangle) \]

\[ = \arccosh \left( \frac{1}{\sqrt{r^2 - 1 \sqrt{h^2 - 1}}} \right) \]

\[ = \log \left( \frac{1}{\sqrt{r^2 - 1 \sqrt{h^2 - 1}}} + \sqrt{\left( \frac{1}{\sqrt{r^2 - 1 \sqrt{h^2 - 1}}} \right)^2 - 1} \right) \]

\[ = \log \left( \frac{\sqrt{(1 - r^2) h^2 + r^2} + 1}{\sqrt{r^2 - 1 \sqrt{h^2 - 1}}} \right). \]

Substitute this equation together with Equations (6.3) and (6.4) to Equation (6.1) and we have

\[ (6.8) \quad \frac{dV_{r,\theta}(h)}{dh} = \frac{1}{2} \left( -\frac{d\theta_{1,2}}{dh} \right) \left( F(h) - \frac{1}{2} \log(r^2 - 1) \right), \]

where \( F \) is the function defined in Case (1).

By the same reason explained in Case (1), we have \( \lim_{h \to 1} dV_{r,\theta}(h)/dh = +\infty \) in this case as well.

We next prove that \( V_{r,\theta} \) does not attain maximal when \( R_{r,\theta}(h) \) is a Lambert cube, i.e., \( h \in (r/\sqrt{r^2 - 1}, +\infty) \). What we actually prove is that \( V_{r,\theta} \) is strictly decreasing, using Equation (6.1). Recall that \( \ell_{0,3} \) is the hyperbolic distance between the polar geodesic plane of \( P(u_1) \) and \( P(u_2) \), and \( \theta_{1,2} \) is the hyperbolic dihedral angle between the polar geodesic planes of \( P_0 \) and \( P_3 \), while \( \ell_{0,1} \) and \( \theta_{2,3} \) are the same as in other cases. Using Equations (2.4) and (2.3), we have

\[ \ell_{0,3} = \arccosh (-\langle u_1, u_2 \rangle) \]

\[ = \arccosh \left( \frac{h \sin \theta}{\sqrt{(1 - r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}} \right) \]

\[ = \log \left( \frac{h \sin \theta + \sqrt{(r^2 - 1) h^2 - r^2 \cos \theta}}{\sqrt{(1 - r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}} \right). \]

\[ \theta_{1,2} = \arccos (-\langle p_0, p_3 \rangle) \]

\[ = \arccos \left( \frac{1}{\sqrt{r^2 - 1 \sqrt{h^2 - 1}}} \right). \]

\[ \frac{d\theta_{1,2}}{dh} = \frac{h}{(h^2 - 1) \sqrt{(r^2 - 1) h^2 - r^2} }. \]

The value \( d\theta_{1,2}/dh \) is positive on \((r/\sqrt{r^2 - 1}, +\infty)\) by this expression, so is \( d\theta_{2,3}/dh \) by Equation (6.4). The value \( \ell_{0,1} \) is positive, for it is the hyperbolic length of an edge. By substituting these results to Equation (6.1), if we can prove that \( \ell_{0,3} > 0 \), then we have \( dV_{r,\theta}/dh < 0 \), namely \( V_{r,\theta} \) is strictly decreasing, on \((r/\sqrt{r^2 - 1}, +\infty)\).

The inequality \( \ell_{0,3} > 0 \) is equivalent to

\[ \frac{h \sin \theta + \sqrt{(r^2 - 1) h^2 - r^2 \cos \theta}}{\sqrt{(1 - r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}} > 1. \]
Calculating
\[
\left( h \sin \theta + \sqrt{(r^2 - 1) h^2 - r^2 \cos \theta} \right)^2 - 1
\]
and we have an inequality
\[
\sqrt{(r^2 - 1) h^2 - r^2} h \sin \theta \geq -\left\{(r^2 - 1) h^2 - r^2\right\} \cos \theta,
\]
which is equivalent to the previous one. This inequality holds on \((r/\sqrt{r^2 - 1}, +\infty)\), for the right-hand side is negative while the left hand side is positive. We have thus proved that \(V_{r,\theta}\) does not attain maximal when \(R_{r,\theta}(h)\) is a Lambert cube.

Since \(V_{r,\theta}\) does not attain maximal when \(R_{r,\theta}(h)\) is a Lambert cube, for the proof of the uniqueness of the maximal value of \(V_{r,\theta}\), we can assume that \(h \in (1, r/\sqrt{r^2 - 1}]\). Under this assumption together with the fact that \(d\theta_{1,2}/dh \neq 0\) on \((1, r/\sqrt{r^2 - 1}]\) by Equation (6.3), what we need to prove is that the number of elements in the set
\[
\left\{ h \in (1, \frac{r}{\sqrt{r^2 - 1}}) \middle| \frac{dV_{r,\theta}(h)}{dh} = 0 \right\}
\]
is at most one, where \(dV_{r,\theta}(h)/dh\) is calculated in Equation (6.5) and the function \(F\) is given in Equation (6.5).

By Equation (6.7), we have
\[
\frac{d}{dh} \left( F(h) - \frac{1}{2} \log(r^2 - 1) \right) = -\frac{h}{(h^2 - 1) \sqrt{(1 - r^2) h^2 + r^2}} G(h),
\]
where we recall that \(G(h) = C \left(1 - r^2\right) \left(h^2 - 1\right) + 1\). Unlike Case (1), the sign of the function \(G\) is not expected to be constant on \((1, r/\sqrt{r^2 - 1}]\), for \(1 - r^2 < 0\).

Since
\[
\frac{h}{(h^2 - 1) \sqrt{(1 - r^2) h^2 + r^2}} \neq 0
\]
on \((1, r/\sqrt{r^2 - 1}]\), we have
\[
\left\{ h \in (1, \frac{r}{\sqrt{r^2 - 1}}) \middle| F'(h) = 0 \right\} = \left\{ h \in (1, \frac{r}{\sqrt{r^2 - 1}}) \middle| G(h) = 0 \right\}.
\]
The function \(G\) is quadratic with respect to \(h\), the coefficient of \(h^2\) is negative and \(G(1) > 0\). These imply that the number of elements in the set of the right-hand side of the equation above is at most one, so is the set of the left-hand side of the equation.

Suppose that the number of elements in the set
\[
\left\{ h \in (1, \frac{r}{\sqrt{r^2 - 1}}) \middle| F(h) - \frac{1}{2} \log(r^2 - 1) = 0 \right\}
\]
is more than 1. By the mean-value theorem together with the fact that the limit of \(F(h) - (1/2) \log(r^2 - 1)\) is 0 as \(h\) approaches \(r/\sqrt{r^2 - 1}\) from below, the set \(\left\{ h \in (1, r/\sqrt{r^2 - 1}) \middle| F'(h) = 0 \right\}\) must contain at least two elements, which contradicts the result obtained above.

We have thus proved Theorem (1). \(\square\)
Appendix A. The maximal area of two-dimensional hyperbolic complete orthoschemes

By the definition of orthoscheme, a triangle $P_0P_1P_2$ in the two-dimensional hyperbolic space is orthoscheme if the edge $P_0P_1$ is perpendicular to the edge $P_1P_2$, namely $P_0P_1P_2$ is a right-angled triangle with the right angle at $P_1$. Without loss of generality, we suppose that $P_0P_1P_2$ lies in the projective disc model $B^2$ with the coordinates $P_0 = (r,0), P_1 = (0,0), P_2 = (0,h)$.

For a given $r > 0$, we consider a family $\{R_r(h)\}_{h>0}$ of complete orthoschemes, where $R_r(h)$ is a complete orthoscheme with vertices $P_0, P_1$ and $P_2$. What we discuss is the maximal area for this family.

**Theorem 2.** The maximal area for $\{R_r(h)\}_{h>0}$ is obtained as follows:

1. For any $r < 1$, the area of $R_r(h)$ attains maximal just for $h = 1$. The maximal area is $\pi/2 - a(1)$, where $a(1)$ is the hyperbolic angle at $P_0$ of $R_r(1)$.

2. The area of $R_1(h)$ attains maximal for any $h \in [1 + \infty)$. The maximal area is $\pi/2$.

3. For any $r > 1$, the area of $R_r(h)$ attains maximal for any $h \in [1, r/\sqrt{r^2 - 1}]$. The maximal area is $\pi/2$.

**Proof.** We start by recalling a formula to calculate the area $A$ of a hyperbolic convex $n$-gon with hyperbolic angles $\alpha_1, \alpha_2, \ldots, \alpha_n$;

$$A = (n - 2)\pi - (\alpha_1 + \alpha_2 + \cdots + \alpha_n).$$

See Theorem 3.5.5 of [Ra] for the proof when $n = 3$.

Let $A_r(h)$ be the area of $R_r(h)$. For any $r > 0$, a complete orthoscheme $R_r(h)$ increases as a set when $h$ approaches 1 from below, which implies that $A_r(h)$ also increases. So, to prove the theorem, it is enough to assume that $h \geq 1$. Using this formula, we obtain the area of $R_r(h)$ for each case.

(1) Suppose $r < 1$. Let $a(h)$ be the hyperbolic angle at $P_0$ of $R_r(h)$.

When $h = 1$, $R_r(1)$ is a triangle with ideal vertex $P_2$. Since the hyperbolic angle at $P_2$ is 0, the area is

$$A_r(1) = \pi - (a(1) + \frac{\pi}{2} + 0) = \frac{\pi}{2} - a(1).$$

When $h > 1$, $R_r(h)$ is a quadrilateral. The hyperbolic angles at the vertices constructed by truncation with respect to $P_2$ are right angles. The area is

$$A_r(h) = 2\pi - (a(h) + \frac{\pi}{2} + \left(\frac{\pi}{2} + \frac{\pi}{2}\right)) = \frac{\pi}{2} - a(h).$$

When $h$ approaches $+\infty$, the corner at $P_0$ increases as a set, so is the angle $a(h)$. This implies that $A_r(h)$ is a strictly decrease function on $[1, +\infty)$.

As a result, $A_r(h)$ attains maximal if and only if $h = 1$ in this case.
(2) Suppose $r = 1$. The hyperbolic angle at $P_0$ is 0 in this case. Use the argument in (1) with $a(h) = 0$ for any $h \geq 1$ and we have the desired conclusion.

(3) Suppose $r > 1$.

When $h = 1$, $R_r(1)$ is a quadrilateral with angle 0 at $P_2$ and three right angles. The area is

\[ A_r(1) = 2\pi - \left(\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + \frac{\pi}{2} + 0\right) = \frac{\pi}{2}. \]

When $h > 1$, there are two kinds of $R_r(h)$, which correspond to double frustums and Lambert cubes of three-dimensional complete orthoschemes.

- If $h < r/\sqrt{r^2 - 1}$, then $R_r(h)$ is a right-angled pentagon. The area is
  \[ A_r(h) = 3\pi - \frac{\pi}{2} \times 5 = \frac{\pi}{2}. \]

- If $h \geq r/\sqrt{r^2 - 1}$, then $R_r(h)$ is a quadrilateral, whose edges consists of $P_0P_1$, $P_1P_2$ and polar lines of $P_0$ and $P_2$. Let $b$ be the hyperbolic angle between these polar lines. Then the area is
  \[ A_r(h) = 2\pi - \left(\frac{\pi}{2} \times 3 + b\right) = \frac{\pi}{2} - b. \]

The maximal area arises when $b = 0$, which occurs if and only if the polar planes of $P_0$ and $P_2$ are parallel, namely $h = r/\sqrt{r^2 - 1}$.

Summarizing these results, we have completed the proof. \qed

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References

[Ke] R. Kellerhals, On the volume of hyperbolic polyhedra, Mathematische Annalen 285 (1989), 541–569.

[Mi] J. Milnor, The Schlafli differential equality, John Milnor Collected Papers Volume 1 Geometry (1994), 281–295, Publish or Perish, Inc., Houston.

[Ra] J. G. Ratcliffe, Foundations of Hyperbolic Manifolds Second Edition, Graduate Texts of Mathematics 149 (2006), Springer-Verlag, New York.

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