SIMPLICIAL RESOLUTIONS AND GANEA FIBRATIONS

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Abstract. In this work, we compare the two approximations of a path-connected space $X$, by the Ganea spaces $G_n(X)$ and by the realizations $\|\Lambda_n X\|$ of the truncated simplicial resolutions emerging from the loop-suspension cotriple $\Sigma \Omega$. For a simply connected space $X$, we construct maps $\|\Lambda_n X\|_{n-1} \to G_n(X) \to \|\Lambda_n X\|_n$ over $X$, up to homotopy. In the case $n = 2$, we prove the existence of a map $G_2(X) \to \|\Lambda_2 X\|_1$ over $X$ (up to homotopy) and conjecture that this map exists for any $n$.

We use the category Top of well pointed compactly generated spaces having the homotopy type of CW-complexes. We denote by $\Omega$ and $\Sigma$ the classical loop space and (reduced) suspension constructions on Top.

Let $X \in \text{Top}$. First we recall the construction of the Ganea fibrations $G_n(X) \to X$ where $G_n(X)$ has the same homotopy type as the $n$-th stage, $B_n \Omega X$, of the construction of the classifying space of $\Omega X$:

1. The first Ganea fibration, $p_1 : G_1(X) \to X$, is the associated fibration to the evaluation map $ev_X : \Sigma \Omega X \to X$.
2. Given the $n$-th fibration $p_n : G_n(X) \to X$, let $F_n(X)$ be its homotopy fiber and let $G_n(X) \cup C(F_n(X))$ be the mapping cone of the inclusion $F_n(X) \to G_n(X)$. We define now a map $p'_{n+1} : G_n(X) \cup C(F_n(X)) \to X$ as $p_n$ on $G_n(X)$ and that sends the (reduced) cone $C(F_n(X))$ on the base point. The $(n + 1)$-st fibration of Ganea, $p_{n+1} : G_{n+1}(X) \to X$, is the fibration associated to $p'_{n+1}$.
3. Denote by $G_{\infty}(X)$ the direct limit of the canonical maps $G_n(X) \to G_{n+1}(X)$ and by $p_{\infty} : G_{\infty}(X) \to X$ the map induced by the $p_n$’s.

From a classical theorem of Ganea [8], one knows that the fiber of $p_n$ has the homotopy type of an $(n+1)$-fold reduced join of $\Omega X$ with itself. Therefore the maps $p_n$ are higher and higher connected when the integer $n$ grows. As a consequence, if $X$ is path-connected, the map $p_{\infty} : G_{\infty}(X) \to X$ is a homotopy equivalence and the total spaces $G_n(X)$ constitute approximations of the space $X$.

The previous construction starts with the couple of adjoint functors $\Omega$ and $\Sigma$. From them, we can construct a simplicial space $\Lambda_\bullet X$, defined by $\Lambda_n X = (\Sigma \Omega)^{n+1} X$ and augmented by $d_0 = ev_X : \Sigma \Omega X \to X$. Forgetting the degeneracies, we have a facial space (also called restricted simplicial space in [2] 3.13)). Denote by $\|\Lambda_\bullet X\|$ the realization of this facial space (see [7] or Section 4). An adaptation of the proof of Stover (see [8] Proposition 3.5)) shows that the augmentation $d_0$ induces a map $\|\Lambda_\bullet X\| \to X$ which is a homotopy equivalence. If we consider the successive stages of the realization of the facial space $\Lambda_\bullet X$, we get maps $\|\Lambda_\bullet X\|_n \to X$ which constitute a second sequence of approximations of the space $X$. In this work, we study the relationship between these two sequences of approximations and prove the following results.

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Theorem 1. Let $X \in \text{Top}$ be a simply connected space. Then there is a homotopy commutative diagram

$$\begin{array}{ccc}
\|\Lambda \cdot X\|_{n-1} & \xrightarrow{G_n(X)} & \|\Lambda \cdot X\|_n \\
\downarrow^{p_n} & & \downarrow^{G_n(X)} \\
& X & \\
\end{array}$$

The hypothesis of simply connectivity is used only for the map $G_n(X) \rightarrow \|\Lambda \cdot X\|_n$, see Theorem 3 and Theorem 5. In the case $n = 2$, the situation is better.

Theorem 2. Let $X \in \text{Top}$. Then there are homotopy commutative triangles

$$\begin{array}{ccc}
\|\Lambda \cdot X\|_1 & \xrightarrow{G_2(X)} & \|\Lambda \cdot X\|_n \\
\downarrow^{p_2} & & \downarrow^{G_2(X)} \\
& X & \\
\end{array}$$

We conjecture the existence of maps $\|\Lambda \cdot X\|_{n-1} \xrightarrow{G_n(X)}$ over $X$ up to homotopy, for any $n$.

This work may also be seen as a comparison of two constructions: an iterative fiber-cofiber process and the realization of progressive truncatures of a facial resolution. More generally, for any cotriple, we present an adapted fiber-cofiber construction (see Definition 9) and ask if the results obtained in the case of $\Sigma \Omega$ can be extended to this setting.

Finally, we observe that a variation on a theorem of Libman is essential in our argumentation, see Theorem 4. A proof of this result, inspired by the methods developed by R. Vogt (see [9]), is presented in an Appendix.

This program is carried out in Sections 1-8 below, whose headings are self-explanatory:

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### 1. Facial spaces

A facial object in a category $C$ is a sequence of objects $X_0, X_1, X_2, \ldots$ together with morphisms $d_i : X_n \rightarrow X_{n-1}$, $0 \leq i \leq n$, satisfying the facial identities
\[ d_i d_j = d_{j-1} d_i \ (i < j). \]

\[
\begin{array}{c}
X_0 \xleftarrow{d_0} X_1 \xleftarrow{d_0} X_2 \quad \cdots \quad X_{n-1} \xleftarrow{d_0} X_n \xleftarrow{d_n} \cdots
\end{array}
\]

The morphisms \( d_i \) are called face operators. We shall use notation like \( X_\bullet \) to denote facial objects. With the obvious morphisms the facial objects in \( \mathbf{C} \) form a category which we denote by \( \mathbf{dC} \). An augmentation of a facial object \( X_\bullet \) in a category \( \mathbf{C} \) is a morphism \( d_0 : X_0 \to X \) with \( d_0 \circ d_0 = d_0 \circ d_1 \). The facial object \( X_\bullet \) together with the augmentation \( d_0 \) is called a facial resolution of \( X \) and is denoted by \( X_\bullet \xrightarrow{d_0} X \).

1.1. Realization(s) of a facial space. As usual, \( \Delta^n \) denotes the standard \( n \)-simplex of \( \mathbb{R}^{n+1} \) and the inclusions of faces are denoted by \( \delta^i : \Delta^n \to \Delta^{n+1} \). We consider the point \( (0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \) as the base-point of the standard \( n \)-simplex \( \Delta^n \). If \( X \) and \( Y \) are in \( \text{Top} \), we denote by \( X \times Y \) the half smashed product \( X \times Y = X \times Y/\ast \times Y \).

A facial space is a facial object in \( \text{Top} \). The realization of a facial space \( X_\bullet \) is the direct limit

\[ ||X_\bullet||_\infty = \lim_{\longrightarrow} ||X_\bullet||_n \]

where the spaces \( ||X_\bullet||_n \) are inductively defined as follows. Set \( ||X_\bullet||_0 = X_0 \). Suppose we have defined \( ||X_\bullet||_{n-1} \) and a map \( \chi_{n-1} : X_{n-1} \times \Delta^{n-1} \to ||X_\bullet||_{n-1} \) (\( \chi_0 \) is the obvious homeomorphism). Then \( ||X_\bullet||_n \) and \( \chi_n \) are defined by the pushout diagram

\[
\begin{array}{ccc}
X_n \times \partial \Delta^n & \xrightarrow{\varphi_n} & ||X_\bullet||_{n-1} \\
\downarrow & & \downarrow \\
X_n \times \Delta^n & \xrightarrow{\chi_n} & ||X_\bullet||_n
\end{array}
\]

where \( \varphi_n \) is defined by the following requirements, for any \( i \in \{0, 1, \ldots, n\} \),

\[ \varphi_n \circ (X_n \times \delta^i) = \chi_{n-1} \circ (d_i \times \Delta^{n-1}) : X_n \times \Delta^{n-1} \to ||X_\bullet||_{n-1}. \]

It is clear that \( \varphi_1 \) is a well-defined continuous map. For \( \varphi_n \) with \( n \geq 2 \), this is assured by the facial identities \( d_i d_j = d_{j-1} d_i \ (i < j) \).

We also consider another realization of the facial space \( X_\bullet \). The free realization of \( X_\bullet \) is the direct limit

\[ |X_\bullet|_\infty = \lim_{\longrightarrow} |X_\bullet|_n \]

where the spaces \( |X_\bullet|_n \) are inductively defined as follows. Set \( |X_\bullet|_0 = X_0 \). Suppose we have defined \( |X_\bullet|_{n-1} \) and a map \( \tilde{\chi}_{n-1} : X_{n-1} \times \Delta^{n-1} \to |X_\bullet|_{n-1} \) (\( \tilde{\chi}_0 \) is the obvious homeomorphism). Then \( |X_\bullet|_n \) and \( \tilde{\chi}_n \) are defined by the pushout diagram

\[
\begin{array}{ccc}
X_n \times \partial \Delta^n & \xrightarrow{\tilde{\varphi}_n} & |X_\bullet|_{n-1} \\
\downarrow & & \downarrow \\
X_n \times \Delta^n & \xrightarrow{\tilde{\chi}_n} & |X_\bullet|_n
\end{array}
\]

where \( \tilde{\varphi}_n \) is defined by the following requirements, for any \( i \in \{0, 1, \ldots, n\} \),

\[ \tilde{\varphi}_n \circ (X_n \times \delta^i) = \tilde{\chi}_{n-1} \circ (d_i \times \Delta^{n-1}) : X_n \times \Delta^{n-1} \to |X_\bullet|_{n-1}. \]

Again the facial identities \( d_i d_j = d_{j-1} d_i \ (i < j) \) assure that \( \tilde{\varphi}_n \) is a well-defined continuous map. Since \( \tilde{\chi}_{n-1} \) is base-point preserving, so is \( \tilde{\varphi}_n \) and hence \( \tilde{\chi}_n \).
We sometimes consider facial spaces with upper indexes \(X^\bullet\). In such a case, the realizations up to \(n\) are denoted by \(|X^\bullet||n|\) and \(|X^\bullet|^n\).

Let \(X_\bullet \xrightarrow{d_0} X\) be a facial resolution of a space \(X\). We define a sequence of maps \(|X_\bullet||n| \to X\) as follows. The map \(|X_\bullet||0| \to X\) is the augmentation. Suppose we have defined \(|X_\bullet||n−1| \to X\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
X_{n-1} \times \Delta^{n-1} & \xrightarrow{\chi_{n-1}} & |X_\bullet||n-1| \\
\pr & & \\
X_{n-1} & \xrightarrow{(d_0)^{n-1}} & X,
\end{array}
\]

where \((d_0)^n\) denotes the \(n\)-fold composition of the face operator \(d_0\). Consider the diagram

\[
\begin{array}{ccc}
X_n \times \Delta^n & \xrightarrow{\varphi_n} & |X_\bullet||n-1| \\
\pr & & \\
X_n & \xrightarrow{(d_0)^{n+1}} & X.
\end{array}
\]

The upper square is commutative for all \(i\) and so is the outer diagram. It follows that the lower square is commutative. We may therefore define \(|X_\bullet||n| \to X\) to be the unique map which extends \(|X_\bullet||n−1| \to X\) and which, pre-composed by \(\chi_n\), is the composite \(X_n \times \Delta^n \xrightarrow{\pr} X_n \xrightarrow{(d_0)^{n+1}} X\). Similarly, we define a sequence of maps \(|X_\bullet||n| \to X\). We refer to the maps \(|X_\bullet||n| \to X\) and \(|X_\bullet||n| \to X\) as the canonical maps induced by the facial resolution \(X_\bullet \to X\). The next statement relates these two realizations; its proof is straightforward.

**Proposition 1.** Let \(X_\bullet\) be a facial space. Then for each \(n \in \mathbb{N}\), the canonical map \(|X_\bullet||n| \to X\) factors through the canonical map \(|X_\bullet||n| \to X\).

**1.2. Facial resolutions with contraction.** A contraction of a facial resolution \(X_\bullet \xrightarrow{d_0} X\) consists of a sequence of morphisms \(s : X_{n-1} \to X_n\) \((X_{-1} = X)\) such that \(d_0 \circ s = \text{id}\) and \(d_i \circ s = s \circ d_{i-1}\) for \(i \geq 1\).

**Proposition 2.** Let \(X_\bullet \xrightarrow{d_0} X\) be a facial resolution which admits a contraction \(s : X_{n-1} \to X_n\) \((X_{-1} = X)\). For any \(n \geq 0\), \(|X_\bullet||n|\) can be identified with the quotient space \(X_n \times \Delta^n / \sim\) where the relation is given by

\[
(x, t_0, \ldots, t_k, \ldots, t_n) \sim (sdx, 0, t_0, \ldots, \hat{t}_k, \ldots, t_n), \quad \text{if } t_k = 0.
\]

As usual, the expression \(t_k\) means that \(t_k\) is omitted. Under this identification the canonical map \(|X_\bullet||n| \to X\) is given by \([x, t_0, \ldots, t_k, \ldots, t_n] \mapsto (d_0)^{n+1}(x)\) and the inclusion \(|X_\bullet||n| \hookrightarrow |X_\bullet||n+1|\) is given by \([x, t_0, \ldots, t_k, \ldots, t_n] \mapsto [sx, 0, t_0, \ldots, t_k, \ldots, t_n]\).

**Proof.** We first note that the simplicial identities together with the contraction properties guarantee that the relation is unambiguously defined if various parameters are zero and also that the two maps

\[
\begin{array}{ccc}
X_n \times \Delta^n / \sim & \to & X_{n+1} \times \Delta^{n+1} / \sim \\
[x, t_0, \ldots, t_k, \ldots, t_n] & \mapsto & [sx, 0, t_0, \ldots, t_k, \ldots, t_n]
\end{array}
\]
and

\[ X_n \times \Delta^n / \sim \rightarrow X \]

\[ [x, t_0, ..., t_k, ..., t_n] \mapsto (d_0)^{n+1}(x) \]

that we will denote by \( \iota_n \) and \( \varepsilon_n \) respectively are well-defined.

Beginning with \( \xi_0 = \text{id} \), we next construct a sequence of homeomorphisms \( \xi_n : [X_n]_n \rightarrow X_n \times \Delta^n / \sim \) inductively by using the universal property of pushouts in the diagram

\[
\begin{array}{ccc}
X_n \times \partial \Delta^n & \xrightarrow{\bar{\varphi}_n} & [X]_{n-1} \\
\downarrow & & \downarrow \xi_{n-1} \\
X_n \times \Delta^n & \xrightarrow{\bar{\chi}_n} & [X]_n \\
\downarrow q_n & & \uparrow \tau_{n-1} \\
X_n \times \Delta^n / \sim & \xrightarrow{\iota_{n-1}} & X_n \times \Delta^{n-1} / \sim
\end{array}
\]

where \( q_n \) is the identification map. If \( t_k = 0 \), the construction up to \( n - 1 \) implies

\[ \xi_{n-1} \circ \bar{\varphi}_n (x, t_0, ..., t_n) = q_{n-1} \circ (d_k \times \Delta^{n-1}) = [d_k x, t_0, ..., \hat{t}_k, ..., t_n]. \]

Therefore, we see that the diagram

\[
\begin{array}{ccc}
X_n \times \partial \Delta^n & \xrightarrow{\xi_{n-1} \circ \bar{\varphi}_n} & X_{n-1} \times \Delta^{n-1} / \sim \\
\downarrow & & \downarrow \tau_{n-1} \\
X_n \times \Delta^n & \xrightarrow{q_n} & X_n \times \Delta^n / \sim
\end{array}
\]

is commutative and, by checking the universal property, that it is a pushout. Thus \( \xi_n \) exists and is a homeomorphism. Through this sequence of homeomorphisms, \( \iota_n \) corresponds to the inclusion \( [X_n]_{n-1} \hookrightarrow [X_n]_n \) and \( \varepsilon_n \) to the canonical map \( [X_n]_n \rightarrow X \).

**Proposition 3.** Let \( X_n \xrightarrow{d_0} X \) be a facial resolution which admits a natural contraction \( s : X_{n-1} \rightarrow X_n \) \( (X_{-1} = X) \). For any \( n \geq 0 \), the canonical map \( [X_n]_n \rightarrow X \) admits a (natural) section \( \sigma_n : X \rightarrow [X_n]_n \) and the inclusion \( [X_n]_{n-1} \hookrightarrow [X_n]_n \) is naturally homotopic to \( \sigma_n \) pre-composed by the canonical map:

\[
\begin{array}{ccc}
[X_n]_{n-1} & \xrightarrow{\sigma_n} & [X_n]_n \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]

In particular, if the facial resolution \( X_n \rightarrow * \) admits a natural contraction then the inclusions \( [X_n]_{n-1} \hookrightarrow [X_n]_n \) are naturally homotopically trivial.

**Proof.** Through the identification established in Proposition 2 the section \( \sigma_n : X \rightarrow [X_n]_n \) is given by

\[ \sigma_n(x) = [(s)^{n+1}(x), 0, ..., 0, 1]. \]

Using the fact that

\[ sd_n sd_{n-1} \cdots sd_2 sd_1 s = (s)^{n+1}(d_0)^n, \]

\[ sd_n sd_{n-1} \cdots sd_2 sd_1 s = (s)^{n+1}(d_0)^n, \]
we calculate that the (well-defined) map $H : |X|_{n-1} \times I \to |X|_{n-1}$ given by
\[
H([x, t_0, ..., t_{n-1}], u) = [sx, u, (1-u)t_0, ..., (1-u)t_{n-1}]
\]
is a homotopy between the inclusion and $\sigma_n$ pre-composed by the canonical map $|X|_{n-1} \to X$. □

2. First part of Theorem 1: the map $\|\Lambda \cdot X\|_{n-1} \to G_n(X)$

Let $X \in \text{Top}$. We consider the facial resolution $\Lambda \cdot X \to X$ where $\Lambda_n(X) = (\Sigma \Omega)^{n+1}X$, the face operators $d_i : (\Sigma \Omega)^{n+1}X \to (\Sigma \Omega)^nX$ are defined by $d_i = (\Sigma \Omega)^i(ev_{(\Sigma \Omega)^{n-i}}X)$, and the augmentation is $d_0 = ev_X : \Sigma \Omega X \to X$.

Theorem 3. Let $X \in \text{Top}$. For each $n \in \mathbb{N}$, the canonical map $\|\Lambda \cdot X\|_{n-1} \to X$ factors through the Ganea fibration $G_n(X) \to X$.

The proof uses the next result.

Lemma 4. Given a pushout
\[
\begin{array}{ccc}
\Sigma A \times \partial \Delta^n & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\Sigma A \times \Delta^n & \xrightarrow{g} & Y'
\end{array}
\]
where the left-hand vertical arrow is a cofibration, then there exists a cofiber sequence
\[
\Sigma A \wedge \partial \Delta^n \xrightarrow{f} Y' \xrightarrow{g} Y'.
\]

Proof. With the Puppe trick, we construct a commutative diagram
\[
\begin{array}{ccc}
\Sigma A \vee (\Sigma A \wedge \partial \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \partial \Delta^n) \\
\downarrow & & \downarrow \\
\Sigma A \vee (\Sigma A \wedge \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \Delta^n)
\end{array}
\]
from which we obtain a commutative diagram
\[
\begin{array}{ccc}
\Sigma A \vee (\Sigma A \wedge \partial \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \partial \Delta^n) \\
\downarrow & & \downarrow \\
\Sigma A \vee (\Sigma A \wedge \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \Delta^n)
\end{array}
\]
because the left-hand vertical arrow is a cofibration. We form now
\[
\begin{array}{ccc}
\Sigma A \wedge \partial \Delta^n & \xrightarrow{\sim} & \Sigma A \vee (\Sigma A \wedge \partial \Delta^n) \\
\downarrow & & \downarrow \\
\Sigma A \wedge \Delta^n & \xrightarrow{\sim} & \Sigma A \vee (\Sigma A \wedge \Delta^n)
\end{array}
\]
\[
\begin{array}{ccc}
\Sigma A \vee (\Sigma A \wedge \partial \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \partial \Delta^n) \\
\downarrow & & \downarrow \\
\Sigma A \vee (\Sigma A \wedge \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \Delta^n)
\end{array}
\]
\[
\begin{array}{ccc}
(\Sigma A \times \partial \Delta^n) & \xrightarrow{\sim} & Y \\
\downarrow & & \downarrow \\
(\Sigma A \times \Delta^n) & \xrightarrow{\sim} & Y'
\end{array}
\]
where $\bullet_1$ and $\bullet_2$ are built by pushout and the left-hand square is a pushout. The map $\bullet_2 \to Y'$ is a weak equivalence because it is induced between pushouts by the weak equivalence $\bullet_1 \to \Sigma A \times \Delta^n$. □
Proof of Theorem. We suppose that \( \Phi_{n-2} : \| \Lambda_n X \|_{n-2} \to G_{n-1}(X) \) has been constructed over \( X \) and observe that the existence of \( \Phi_0 \) is immediate. We consider the following commutative diagram

\[
(\Sigma \Omega)^n(X) \wedge \partial \Delta^{n-1} \to \Phi_{n-2} \| \Lambda_n X \|_{n-2} \xrightarrow{\lambda_{n-2}} \| \Lambda_n X \|_{n-1} \to G_{n-1}(X) \]

where the left-hand column is a cofibration sequence by Lemma. From the equalities

\[
p_{n-1} \circ \Phi_{n-2} \circ \tilde{v}_{n-2} = \lambda_{n-2} \circ v_{n-2} = \lambda_{n-1} \circ v_{n-2} \circ \tilde{v}_{n-2} \simeq \ast,
\]

we deduce a map \( \hat{\Phi}_{n-2} : (\Sigma \Omega)^n(X) \wedge \partial \Delta^{n-1} \to F_{n-1}(X) \) making the diagram homotopy commutative. From the definition of \( G_n(X) \) as a cofiber, this gives a map \( \Phi_{n-1} : \| \Lambda_n X \|_{n-1} \to G_n(X) \) over \( X \).

Instead of the explicit construction above, we can also observe that the cone length of \( \| \Lambda_n X \|_{n-1} \) is less than or equal to \( n \) and deduce Theorem from basic results on Lusternik-Schnirelman category, see [1].

3. The facial space \( G_\ast(X) \)

For a space \( X \) we denote by \( P'X \) the Moore path space and by \( \Omega'X \) the Moore loop space. Path multiplication turns \( \Omega'X \) into a topological monoid. Given a space \( X \), we define the facial space \( G_\ast(X) \) by \( G_n(X) = (\Omega'X)^n \) with the face operators \( d_i : (\Omega'X)^n \to (\Omega'X)^{n-1} \) given by

\[
d_i(\alpha_1, \ldots, \alpha_n) = \begin{cases} 
(\alpha_2, \ldots, \alpha_n) & i = 0 \\
(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n) & 0 < i < n \\
(\alpha_1, \ldots, \alpha_{n-1}) & i = n.
\end{cases}
\]

The purpose of this section is to compare the free realization of \( G_\ast(X) \) to the construction of the classifying space of \( \Omega'X \).

We work with the following construction of \( B\Omega'X \). The classifying space \( B\Omega'X \) is the orbit space of the contractible \( \Omega'X \)-space \( E\Omega'X \) which is obtained as the direct limit of a sequence of \( \Omega'X \)-equivariant cofibrations \( E_0 \Omega'X \to E_{n+1} \Omega'X \). The spaces \( E_n \Omega'X \) are inductively defined by \( E_0 \Omega'X = \Omega'X \), \( E_{n+1} \Omega'X = E_n \Omega'X \cup \theta (\Omega'X \times CE_n \Omega'X) \) where \( \theta \) is the action \( \Omega'X \times E_n \Omega'X \to E_n \Omega'X \) and \( C \) denotes the free (non-reduced) cone construction. The orbit spaces of the \( \Omega'X \)-spaces \( E_n \Omega'X \) are denoted by \( B_n \Omega'X \). For each \( n \in \mathbb{N} \) this construction yields a cofibration \( B_n \Omega'X \to B\Omega'X \). It is well known that for simply connected spaces this cofibration is equivalent to the \( n \)th Ganea map \( G_n(X) \to X \).
Proposition 5. For each $n \in \mathbb{N}$ there is a natural commutative diagram

\[
B_n \Omega' X \longrightarrow |G_\ast(X)|_n \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
\textbf{Remark.} Note that the upper horizontal map in the diagram of Proposition \ref{prop:contractibility} is not a homotopy equivalence in general. Indeed, for $X = \ast$, $B_1 \Omega' X$ is contractible but $|\gamma_\ast(X)|_1 \simeq S^1$. It can, however, be shown that there also exists a diagram as in Proposition \ref{prop:contractibility} with the horizontal maps reversed.

\section{The Facial Resolution $\Omega' \Lambda_\ast X \to \Omega' X$ Admits a Contraction}

Consider the natural map $\gamma_X : \Omega' X \to \Omega' \Sigma \Omega X$, $\gamma_X(\omega, t) = (\nu(\omega, t), t)$ where $\nu(\omega, t) : \mathbb{R}^+ \to \Sigma \Omega X$ is given by

$$\nu(\omega, t)(u) = \begin{cases} [\omega_t, \frac{u}{t}], & u < t, \\ [c_\ast, 0], & u \geq t. \end{cases}$$

Here, $c_\ast$ is the constant path $u \mapsto \ast$ and $\omega_t : I \to X$ is the loop defined by $\omega_t(s) = \omega(ts)$.

\textbf{Lemma 6.} The map $\gamma_X$ is continuous.

\textbf{Proof.} It suffices to show that the map $\nu^\beta : \Omega' X \times \mathbb{R}^+ \to \Sigma \Omega X$, $(\omega, t, u) \mapsto \nu(\omega, t)(u)$ is continuous. Consider the subspace $W = \{ \omega \in X^{\mathbb{R}^+} : \omega(0) = \ast \}$ of $X^{\mathbb{R}^+}$ and the continuous map $\rho : W \times \mathbb{R}^+ \to X^{\mathbb{R}^+}$ given by

$$\rho(\omega, t)(u) = \begin{cases} \omega(u), & u \leq t, \\ \omega(t), & u \geq t. \end{cases}$$

Note that if $(\omega, t) \in P' X$ then $\rho(\omega, t) = \omega$. Consider the continuous map

$$\phi : W \times \mathbb{R}^+ \times [0, \frac{\pi}{2}] \to \Sigma P' X$$

defined by

$$\phi(\omega, r, \theta) = \begin{cases} [\rho(\omega, r \cos \theta), r \cos \theta, \tan \theta], & \theta \leq \frac{\pi}{4}, \\ [c_\ast, 0, 0], & \theta \geq \frac{\pi}{4}. \end{cases}$$

When $r = 0$, we have $\phi(\omega, r, \theta) = [c_\ast, 0, 0]$ for any $\theta$. Therefore $\phi$ factors through the identification map

$$W \times \mathbb{R}^+ \times [0, \frac{\pi}{2}] \to W \times \mathbb{R}^+ \times \mathbb{R}^+, (\omega, r, \theta) \mapsto (\omega, r \cos \theta, r \sin \theta)$$

and induces a continuous map $\psi : W \times \mathbb{R}^+ \times \mathbb{R}^+ \to \Sigma P' X$. Explicitly,

$$\psi(\omega, t, u) = \begin{cases} [\rho(\omega, t), t, \frac{u}{t}], & u < t, \\ [c_\ast, 0, 0], & u \geq t. \end{cases}$$

Consider the continuous map $\xi : P' X \to PX$ defined by $\xi(\omega, t)(s) = \omega(ts)$. Note that $\xi(\omega, t) = \omega_t$ if $(\omega, t) \in P' X$ and, in particular, that $\xi(c_\ast, 0) = c_\ast$. The restriction of $\Sigma \xi \circ \psi$ to $\Omega' X \times \mathbb{R}^+$ factors through the subspace $\Sigma \Omega X$ of $\Sigma PX$ and the continuous map

$$\Omega' X \times \mathbb{R}^+ \to \Sigma \Omega X, (\omega, t, u) \mapsto (\Sigma \xi \circ \psi)(\omega, t, u)$$

is exactly $\nu^\beta$.

\textbf{Proposition 7.} The maps $s = \gamma_{(\Sigma \Omega)^n X} : \Omega'(\Sigma \Omega)^n X \to \Omega'(\Sigma \Omega)^{n+1} X$ define a contraction of the facial resolution $\Omega' \Lambda_\ast X \to \Omega' X$.

\textbf{Proof.} We have $(\Omega'(ev_X) \circ \gamma_X)(\omega, t) = \Omega'(ev_X)(\nu(\omega, t), t) = (\beta(\omega, t), t)$ where

$$\beta(\omega, t)(u) = \begin{cases} \omega_t(\frac{u}{t}) = \omega(u), & u < t, \\ \ast = \omega(u), & u \geq t. \end{cases}$$

Hence $(\Omega'(ev_X) \circ \gamma_X) = \text{id}_{\Omega' X}$.

In the same way one has $(\Omega'(ev_{\Sigma \Omega})^n X) \circ \gamma_{(\Sigma \Omega)^n X}) = \text{id}_{(\Sigma \Omega)^n X}$. This shows the relation $d_0 \circ s = \text{id}$. It remains to check that $d_j \circ s = s \circ d_{j-1}$, for $j \geq 1$. For
(ω, t) ∈ Ω′(ΣΩ)\n X we have (d_j ∘ s)(ω, t) = (Ω′(ΣΩ)\n j(ev(ΣΩ)\n n−j X) ◦ γ(ΣΩ)\n n X)(ω, t) = (σ(ω, t), t) where

\[\sigma(ω, t)(u) = \left\{ \begin{array}{l}
\Sigma(Ω)^{j}(ev(ΣΩ)^{n−j X})[\omega, \frac{u}{t}] = \left[\Sigma(Ω)^{j−1}(ev(ΣΩ)^{n−j X}) ◦ \omega, \frac{u}{t}\right], \quad u < t, \\
\Sigma(Ω)^{j}(ev(ΣΩ)^{n−j X})[c_*, 0] = [c_*, 0], \quad u ≥ t.
\end{array} \right.\]

On the other hand, (s ∘ d_{j−1})(ω, t) = (γ(ΣΩ)\n n−1 X ◦ Ω′(ΣΩ)\n j−1(ev(ΣΩ)\n n−1 X))(ω, t) = (τ(ω, t), t) where

\[\tau(ω, t)(u) = \left\{ \begin{array}{l}
\left[\Sigma(Ω)^{j−1}(ev(ΣΩ)^{n−j X}) ◦ \omega, \frac{u}{t}\right], \quad u < t, \\
[c_*, 0], \quad u ≥ t.
\end{array} \right.\]

This shows that \(d_j ∘ s = s ∘ d_{j−1}\) (j ≥ 1).

\[\square\]

5. Second Part of Theorem 1

The map \(G_n(X) → \|\Lambda_n X\|_n\)

A bifacial space is a facial object in the category \(d\text{Top}\) of facial spaces. We will use notations like \(Z^*\) to denote bifacial spaces and refer to the upper index as the column index and to the lower index as the row index. In this way, a bifacial space can be represented by a diagram of the following type:

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\partial_0 & \partial_{n+1} & \partial_0 & \partial_{n+1} & \partial_0 & \partial_{n+1} & \partial_0 & \partial_{n+1} \\
Z^0_m & d_0 & Z^1_n & d_1 & Z^2_n & \cdots & Z^{p−1} & d_p & Z^p_n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\partial_0 & \partial_n & \partial_0 & \partial_n & \partial_0 & \partial_n & \partial_0 & \partial_n & \partial_0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\partial_0 & \partial_1 & \partial_0 & \partial_1 & \partial_0 & \partial_1 & \partial_0 & \partial_1 & \partial_0 \\
Z^1_0 & d_0 & Z^1_0 & d_1 & Z^1_0 & \cdots & Z^1_0 & d_p & Z^1_0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\partial_0 & \partial_1 & \partial_0 & \partial_1 & \partial_0 & \partial_1 & \partial_0 & \partial_1 & \partial_0 \\
Z^0_0 & d_0 & Z^0_0 & d_1 & Z^0_0 & \cdots & Z^0_0 & d_p & Z^0_0 \\
\end{array}
\]

As in this diagram we shall reserve the notation \(\partial_i\) for the face operators of a column facial space and the notation \(d_i\) for the face operators of a row facial space. For any \(k, |Z^*_m|^k\) (resp. \(|Z^*_m|^m\)) is the realization up to \(m\) of the \(k\)th column (resp. \(k\)th row) and \(|Z^*_m|^m\) (resp. \(|Z^*_m|^m\)) is the facial space obtained by realizing each column (resp. each row) up to \(m\).

The construction of the map \(G_n(X) → \|\Lambda_n X\|_n\) relies heavily on the following result which is analogous to a theorem of A. Libman [5]. As A. Libman has pointed out to the authors, this result can be derived from [5] (private communication). For the convenience of the reader, we include, in an appendix, an independent proof of the particular case we need.

**Theorem 4.** Consider a facial space \(Z^*_k\) and a facial resolution \(Z^*_k \xrightarrow{d_k} Z^*_k\) such that each row \(Z^*_k \xrightarrow{d_k} Z^*_k\) admits a contraction. Then, for any \(n\), there exists a not necessarily base-point preserving continuous map \(|Z^*_k|^{-1} \rightarrow \||Z^*_m|^n|\) which is a section up to free homotopy of the canonical map \(\||Z^*_m|^n\| \rightarrow \|Z^*_k|^{-1}\).

The second part of Theorem 1 can be stated as follows.
Theorem 5. Let $X \in \text{Top}$ be a simply connected space. For each $n \in \mathbb{N}$ the $n$th Ganea map $G_n(X) \to X$ factors up to (pointed) homotopy through the canonical map $\|\Lambda \bullet X\|_n \to X$.

Proof. Consider the column facial space $Z_{-1}^* = G_\bullet(X)$ and the facial resolution $Z_i^* \leftarrow Z_i^1$ where $Z_i^1 = G_i(\Lambda_j X)$. Each row facial resolution $Z_{-1}^i = G_i(X) \leftarrow Z_i^\bullet = G_i(\Lambda \bullet X)$ admits a contraction. Since $G_0(\Lambda \bullet X) = \ast$, this is clear for $i = 0$. For $i > 0$, $G_i(\Lambda \bullet X) = (\Omega' \Lambda \bullet X)^i$. Indeed, since, by Proposition 7, the facial resolution $\Omega' X \leftarrow \Omega' \Lambda \bullet X$ admits a contraction, its $i$th power also admits a contraction.

For $n \in \mathbb{N}$ consider the commutative diagram

\[
\begin{array}{ccc}
B_n \Omega' X & \longrightarrow & |G_\bullet(X)|_n \\
\downarrow & & \downarrow \\
B \Omega' X & \\ & \searrow f & \searrow \downarrow \\
& X &
\end{array}
\]

in which the left-hand square is the natural square of Proposition 4. Recall that the lower left horizontal map is a homotopy equivalence. Since $X$ is simply connected, $X$ is naturally weakly equivalent to $B \Omega' X$ and hence to $|G_\bullet(X)|_\infty$. It follows that the map $|G_\bullet(\Lambda \bullet X)|_n \to |G_\bullet(X)|_\infty$ is weakly equivalent to the map $|\Lambda \bullet X|_n \to X$. Since this last map factors through the map $|\Lambda \bullet X|_n \to X$ and since, by Theorem 4 the upper right horizontal map of the diagram above admits a free homotopy section, we obtain a diagram

\[
\begin{array}{ccc}
B_n \Omega' X & \longrightarrow & |\Lambda \bullet X|_n \\
\downarrow & & \downarrow \\
B \Omega' X & f & \searrow \downarrow \\
& X &
\end{array}
\]

which is commutative up to free homotopy and in which $f$ is a (pointed) homotopy equivalence. Since the left hand vertical map is equivalent to the Ganea map $G_n(X) \to X$, there exists a diagram

\[
\begin{array}{ccc}
G_n(X) & \longrightarrow & |\Lambda \bullet X|_n \\
\downarrow & & \downarrow \\
X & g & \searrow \downarrow \\
& X &
\end{array}
\]

which is commutative up to free homotopy and in which $g$ is a (pointed) homotopy equivalence. This implies that the Ganea map $G_n(X) \to X$ factors up to free homotopy through the canonical map $|\Lambda \bullet X|_n \to X$. Since $X$ is simply connected and $|\Lambda \bullet X|_n$ is connected, the Ganea map $G_n(X) \to X$ also factors up to pointed homotopy through the canonical map $|\Lambda \bullet X|_n \to X$. □

6. Proof of Theorem 2

Proof. Recall the homotopy fiber sequence

\[
\begin{array}{ccc}
\Omega X * \Omega X & \longrightarrow & \Sigma \Omega X \longrightarrow \Sigma \Omega X \\
\downarrow & & \downarrow \\
h & \longrightarrow & d_0 \\
\Omega X * \Omega X & \longrightarrow & \Sigma \Omega X
\end{array}
\]

where $h$ is the Hopf map. This sequence is natural in $X$ and the space $G_2(X)$ is equivalent to the pushout of $\mathcal{C}(\Omega X * \Omega X) \longrightarrow \Omega X * \Omega X \longrightarrow \Sigma \Omega X$, where $\mathcal{C}(Y)$
denotes the (reduced) cone over a space $Y$. We use the following diagram

$$
\begin{align*}
\text{(2)} & \quad \mathcal{C}(\Omega X * \Omega X) \xrightarrow{d_0} \mathcal{C}(\Omega \Sigma \Omega X * \Omega \Sigma \Omega X) \xrightarrow{d_0} \mathcal{C}(\Omega (\Sigma \Omega)^2 X * \Omega (\Sigma \Omega)^2 X) \\
\text{(1)} & \quad\Omega X * \Omega X \xrightarrow{d_0} \Omega \Sigma \Omega X * \Omega \Sigma \Omega X \xrightarrow{d_0} \Omega (\Sigma \Omega)^2 X * \Omega (\Sigma \Omega)^2 X \\
\text{(0)} & \quad \Sigma \Omega X \xrightarrow{d_0} (\Sigma \Omega)^2 X \xrightarrow{d_0} (\Sigma \Omega)^3 X \\
\text{(-1)} & \quad X \xrightarrow{d_0} \Sigma \Omega X \xrightarrow{d_0} (\Sigma \Omega)^2 X
\end{align*}
$$

We observe that

- the image of Line (-1) by $\Omega$ has a contraction in the obvious sense;
- Line (0) is the image of Line (-1) by $\Sigma \Omega$ therefore Line (0) admits a contraction;
- the face operators of Line (1) are the maps $\Omega d_i * \Omega d_i$ with the face operators $d_i$ of Line (-1), thus Line (1) admits a contraction;
- Line (2) admits a contraction induced by the previous one.

From the expression of the Hopf map $h : \Omega X * \Omega X \to \Sigma \Omega X$, $h([\alpha, t, \beta]) = [\alpha^{-1} \beta, t]$, we observe that the map $H : (\Omega X * \Omega X) \times [0, 1] \to X$, defined by $H([\alpha, t, \beta], s) = \alpha^{-1} \beta(st)$, induces a natural extension of $d_0 \circ h$ to $\mathcal{C}(\Omega X * \Omega X)$. Therefore, we can complete the diagram by maps from Line (2) to Line (-1) which are compatible with face operators.

Denote by $\tilde{G}$ the homotopy colimit of the framed part of the diagram. We have a commutative square:

$$
\begin{array}{c}
G_2(X) \\
\downarrow \\
X \\
\uparrow \\
\|A \cdot X\|_1
\end{array}
\quad \quad \quad
\begin{array}{c}
\tilde{G} \\
\downarrow \\
\|A \cdot X\|_1
\end{array}
$$

Lemma 8 provides a homotopy section of the map $\tilde{G} \to G_2(X)$. Thus we obtain a map

$$
G_2(X) \to \|A \cdot X\|_1
$$

up to homotopy over $X$. $\Box$

**Lemma 8.** We consider the following diagram in $\textbf{Top}$, satisfying $d_0 \circ d_0 = d_0 \circ d_1$ and the obvious commutativity conditions.
Let $\tilde{G}$ be the homotopy colimit of the framed part and $G_{-1}$ be the homotopy colimit of the first column. We denote by $\tilde{d}: \tilde{G} \to G_{-1}$ the map induced by $d_0$. If the lines of the previous diagram admit contractions in the obvious sense, then the map $d$ has a (pointed) homotopy section.

Proof. This is a special case of a dual of a result of Libman in [5]. It is not covered by the proof of the last section but this situation is simple and we furnish an ad-hoc argument for it.

First we construct maps $f: A_{-1} \to \|A\|_1$, $g: B_{-1} \to \|B\|_1$ and $k: C_{-1} \to \|C\|_1$ such that $\|A\|_1 \circ g \simeq f \circ \alpha_{-1}$ and $k \circ \beta_{-1} \simeq \|B\|_1 \circ g$. With the same techniques as in Proposition 2, it is clear that $\|A\|_1$ is homeomorphic to the quotient $A \times \Delta^1$ by the relation $(a, t_0, t_1) \sim (sd_0a, 0, 1)$ if $t_1 = 0$. So, we define $f$, $g$ and $k$ by

$$f(a) = [s_A s_A(a), 0, 1], g(b) = [s_B s_B(b), 0, 1] \text{ and } k(c) = [s_C s_C(c), 0, 1]$$

A computation gives:

$$\|\alpha\|_1 \circ g(b) = [\alpha_1 s_B s_B(b), 0, 1] = [s_A d_0 \alpha_0 s_B s_B(b), 0, 1] = [s_A s s_A s_B s_B, 0, 1]$$

$$f \circ \alpha_1(b) = [s_A s s_A s_B s_B, 0, 1] = [s_A s_A \alpha_0 s_B(b), 0, 1] = [s_A d_1 s_A \alpha_0 s_B(b), 0, 1] = [s_A s_A s_B, 0, 1]$$

the last equality coming from our construction of $\|A\|_1$. These two points, $\|A\|_1 \circ g(b)$ and $f \circ \alpha_1(b)$, are canonically joined by a path that reduces to a point if $b = \ast$. The same argument gives the similar result for $k$. We observe now that these homotopies give a map between the two mapping cylinders which is a section up to pointed homotopy.

7. Open Questions

The main open question after these results concerns the existence of maps over $X$ up to homotopy, $G_n(X) \to \|A_{-1} X\|_{n-1}$ for any $n$. This question is related to the Lusternik-Schnirelman category (LS-category in short) cat $X$ of a topological space $X$. Recall that cat $X \leq n$ if and only if the Ganea fibration $G_n(X) \to X$ admits a section. The truncated resolutions bring a new homotopy invariant $\ell_\Sigma(X)$ defined in a similar way as follows:

$$\ell_\Sigma(X) \leq n \text{ if the map } \|A_{-1} X\|_{n-1} \to X \text{ admits a homotopical section.}$$

From Theorem 1 and Theorem 2 we know that this new invariant coincides with the LS-category for spaces of LS-category less than or equal to 2 and satisfies

$$\text{cat } X \leq \ell_\Sigma(X) \leq 1 + \text{cat } X.$$

Grants to the result in dimension 2, $\ell_\Sigma(X)$ does not coincide with the cone length. We conjecture its equality with the LS-category and the existence of maps $G_n(X) \to \|A_{-1} X\|_{n-1}$ over $X$ up to homotopy.

We now extend our study by considering a cotriple $T$. Recall that a cotriple $(T, \eta, \epsilon)$ on $\text{Top}$ is a functor $T: \text{Top} \to \text{Top}$ together with two natural transformations $\eta_X: T(X) \to X$ and $\epsilon_X: T(X) \to T^2(X)$ such that:

$$\epsilon_F(X) \circ \epsilon_X = F(\epsilon_X) \circ \epsilon_X \text{ and } \eta_{T(X)} \circ \epsilon_X = T(\eta_X) \circ \epsilon_X = \text{id}_{T(X)}.$$
It is well known that $T$ gives a simplicial space $\Lambda^T_n X$ defined by $\Lambda^T_n X = T^{n+1}(X)$. From it, we deduce a facial space and the truncated realizations $\|\Lambda^T_n X\|_n$. If $T$ satisfies $T(*) \sim *$, takes its values in suspensions and $\Omega^l(\Lambda^T_n X)$ admits a contraction, a careful reading of the proofs in this work shows that we get the same conclusions as in Theorem 1 and Theorem 2 with the Ganea spaces $G_n(X)$ and the realizations $\|\Lambda^T_n X\|_i$.

We could also use a construction of the Ganea spaces adapted to the cotriple $T$ as follows.

**Definition 9.** Let $T$ be a cotriple and $X$ be a space, the $n$th fibration of Ganea associated to $T$ and $X$ is defined inductively by:
- $p^T_0: G^T_0(X) \to X$ is the associated fibration to $\eta_X: T(X) \to X$,
- if $p^T_n: G^T_n(X) \to X$ is defined, we denote by $F^T_n(X)$ its homotopy fiber and build a map $p^T_{n+1}: G^T_n(X) \cup C(T(F^T_n(X))) \to X$ as $p^T_n$ on $G^T_n(X)$ and sending the cone $C(T(F^T_n(X)))$ on the base point. The fibration $p^T_{n+1}$ is the associated fibration to $p^T_{n+1}$.

The results of this paper and the questions above have their analog in this setting. New approximations of spaces arise from the truncated realizations $\|\Lambda^T_n X\|_i$, and from the adapted fiber-cofiber constructions. One natural problem is to look for a comparison between them. These questions can also be stated in terms of LS-category. For instance, does the Stover resolution (see [8]) of a space by wedges of spheres give the $s$-category defined in [9]?

8. Appendix: Proof of Theorem 3

The purpose of this appendix is to give a proof of Theorem 3. This proof is contained in the Subsection 8.2 below and uses the constructions and notation of the following subsection.

8.1. $n$-facial spaces and $n$-rectifiable maps. Let $n \geq 0$ be an integer. A facial space $X_\bullet$ is a $n$-facial space if, for any $k \geq n + 1$, $X_k = *$. To any facial space $Y_\bullet$, we can associate an $n$-facial space $T^n_\bullet(Y)$ by setting $T^n_k(Y) = Y_k$ if $k \leq n$ and $T^n_k(Y) = *$ if $k \geq n + 1$. Obviously, for any $k \leq n$, we have $|T^n_k(Y)|_k = |Y_k|_k$.

Let $Y_\bullet$ be a facial space with face operators $\partial_i: Y_k \to Y_{k-1}$. We associate to $Y_\bullet$ two $n$-facial spaces $J^n_\bullet(Y)$ and $J^n_\bullet(Y)$ and morphisms $\eta, \zeta, \pi, \bar{\pi}$ which induce homotopy equivalences between the realizations up to $n$ and such that the following diagram is commutative:

$$
\begin{array}{ccc}
T^n_\bullet(Y) & \cong & J^n_\bullet(Y) \\
\eta & \cong & \zeta \\
\downarrow & & \downarrow \\
\text{id} & & \bar{\pi} \\
T^n_\bullet(Y).
\end{array}
$$

For any integer $k \geq 1$ we denote by $\partial_k$ the set $\{\partial_0, ..., \partial_k\}$ of the $(k + 1)$ face operators $\partial_i: Y_k \to Y_{k-1}$ and, for any integer $l \geq k$, we set $\partial_{k+1} := \partial_k \times \partial_{k+1} \times \ldots \times \partial_l$.

The $n$-facial space $J^n_\bullet(Y)$. For $0 \leq k \leq n$, consider the space:

$$(Y_k \times \Delta^0) \coprod_{m=1}^{n-k} (\partial_{k+1} : k+m \times Y_{k+m} \times \Delta^m).$$
An element of this space will be written \((\partial_1, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m)\) with the convention 
\((\partial_1, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) = (y, 1)\) if \(m = 0\). Set
\[
J_k^n(Y) := \left( (Y_k \times \Delta^n) \coprod \prod_{m=1}^{n-k} (\partial_{k+1}, k+m, Y_k + \Delta^n) \right) / \sim
\]
where the relations are given by
\[
(\partial_1, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) \sim (\partial_1, \ldots, \partial_{i_m-1}, \partial, y, t_0, \ldots, t_m), \quad \text{if } t_m = 0,
\]
and
\[
(\partial_1, \ldots, \partial_{i_p}, \partial_{i_{p+1}}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) \sim (\partial_1, \ldots, \partial_{i_{p-1}}, \partial_{i_p}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m),
\]
if \(t_p = 0\) and \(i_p < i_{p+1}\).

Together with the face operators \(J_\partial_i: J_k^n(Y) \to J_{k-1}^n(Y), 0 \leq i \leq k\), defined by
\[
J_\partial_i(\partial_1, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) = (\partial, \partial_1, \ldots, \partial_{i_m}, y, 0, t_0, \ldots, t_m),
\]
\(J_k^n(Y)\) is a \(n\)-facial space.

**The \(n\)-facial space** \(I_k^n(Y)\). For \(0 \leq k \leq n\), we consider now the space:
\[
(\partial_1, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) \sim (\partial_1, \ldots, \partial_{i_m-1}, \partial_{i_m}, y, t_0, \ldots, t_m), \quad \text{if } t_m+1 = 0,
\]
and
\[
(\partial_1, \ldots, \partial_{i_p}, \partial_{i_{p+1}}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) \sim (\partial_1, \ldots, \partial_{i_{p-1}}, \partial_{i_p}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m),
\]
if \(t_{p+1} = 0\) and \(i_p < i_{p+1}\).

Together with the face operators \(I_\partial_i: I_k^n(Y) \to I_{k-1}^n(Y), 0 \leq i \leq k\), defined by
\[
I_\partial_i(\partial_1, \ldots, \partial_{i_m}, y, t_0, t_1, \ldots, t_m) = (\partial, \partial_1, \ldots, \partial_{i_m}, y, 0, t_1, \ldots, t_m+1),
\]
\(I_k^n(Y)\) is a \(n\)-facial space.

**The morphisms** \(\eta, \zeta, \pi, \pi\). The facial maps \(\eta: T_k^n(Y) \to I_k^n(Y)\), \(\zeta: J_k^n(Y) \to I_k^n(Y)\), \(\pi: I_k^n(Y) \to T_k^n(Y)\) and \(\pi: J_k^n(Y) \to T_k^n(Y)\) are respectively defined (for \(k \leq n\)) by:
\[
\eta_k(y) = (y, 1, 0),
\]
\[
\zeta_k(\partial_1, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) = (\partial_1, \ldots, \partial_{i_m}, y, 0, t_0, \ldots, t_m),
\]
\[
\pi_k(\partial_1, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) = \partial_i \cdots \partial_{i_m} y \quad \text{and} \quad \pi_k(y, t_0, t_1) = y,
\]
\(\pi_k = \pi_k \circ \zeta_k\).
We have $\pi_k \circ \eta_k = \text{id}$ so that the following diagram is commutative:

\[
\begin{array}{c}
T^n(Y) \xrightarrow{\eta} I^n(Y) \xrightarrow{\zeta} J^n(Y) \\
\downarrow \pi \quad \quad \quad \downarrow \pi \\
T^n(Y)
\end{array}
\]

In order to see that these morphisms induce homotopy equivalences between the realizations up to $n$, it suffices to see that, for any $k$, $0 \leq k \leq n$, the maps $\eta_k, \zeta_k, \pi_k, \overline{\pi}_k$ are homotopy equivalences. Thanks to the commutativity of the diagram above we just have to check it for the maps $\pi_k$ and $\overline{\pi}_k$. These two maps admit a section: we have already seen that $\pi_k \circ \eta_k = \text{id}$ and, on the other hand, the map $\varphi_k : T_k^n(Y) \to J_k^n(Y)$ given by $\varphi_k(y) = (y,1)$ (which does not commute with the face operators) satisfies $\overline{\pi}_k \circ \varphi_k = \text{id}$. The conclusion follows then from the fact that the two homotopies

$$H_k : I_k^n(Y) \times I \to I_k^n(Y)$$

$$((\partial_1, \ldots, \partial_{i_m}, y, t_0, \ldots, t_{m+1}), u) \mapsto (\partial_1, \ldots, \partial_{i_m}, y, u + (1-u)t_0, (1-u)t_1, \ldots, (1-u)t_{m+1})$$

$$\overline{H}_k : J_k^n(Y) \times I \to J_k^n(Y)$$

$$((\partial_1, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m), u) \mapsto (\partial_1, \ldots, \partial_{i_m}, y, u + (1-u)t_0, (1-u)t_1, \ldots, (1-u)t_m)$$

satisfy $H_k(-,0) = \text{id}$, $H_k(-,1) = \eta_k \circ \pi_k$ and $\overline{H}_k(-,0) = \text{id}$, $\overline{H}_k(-,1) = \varphi_k \circ \overline{\pi}_k$.

**n-rectifiable map.** We write $\varphi : T^n_\ast(Y) \rightarrow J^n_\ast(Y)$ to denote the collection of maps $\varphi_k : T_k^n(Y) \to J_k^n(Y)$ given by $\varphi_k(y) = (y,1)$. Recall that $\varphi$ is not a morphism of facial spaces since it does not satisfy the usual rules of commutation with the face operators. In the same way we write $\psi : Y_\ast \rightarrow Z_\ast$ for a collection of maps $\psi_k : Y_k \rightarrow Z_k$ which do not satisfy the usual rules of commutation with the face operators and we say that $\psi$ is an $n$-rectifiable map if there exists a morphism of facial spaces $\overline{\psi} : J^n_\ast(Y) \rightarrow T^n_\ast(Z)$ such that $\overline{\psi}_k \circ \varphi_k = \psi_k$ for any $k \leq n$. So, an $n$-rectifiable map $\psi : Y_\ast \rightarrow Z_\ast$ induces a map between the realizations up to $n$ of the facial spaces $Y_\ast$ and $Z_\ast$.

8.2. **Proof of Theorem** Let $Z^n_\ast \xrightarrow{d_0} Z^{n-1}_\ast$ be a facial resolution of a facial space $Z^{n-1}_\ast$ such that each row $Z_k^n \xrightarrow{d_0} Z_k^{n-1}$ admits a contraction and let $n \geq 0$. We first note that the realization of $Z^n_\ast$ up to $p$ along the rows and up to $n$ along the columns leads to two canonical maps:

$$||Z_\ast^n||_n \to |Z^{n-1}_\ast|_n$$

$$||Z_\ast^n||_n \to |Z^{n-1}_\ast|_n.$$

Induction on $p$ and standard colimit arguments show that these two maps are equal (up to homeomorphism). Here we prove that $||Z_\ast^n||_n \to |Z^{n-1}_\ast|_n$ admits a homotopy section.

For any $k$, we denote by $s_k$ the contraction of the $k$th row

$$Z^{-1}_k \xrightarrow{d_0} Z^{0}_k \xrightarrow{d_0} Z^{1}_k \xrightarrow{d_0} X^2_k \xrightarrow{d_0} \cdots \xrightarrow{d_0} Z^{n-1}_k \xrightarrow{d_0} Z^n_k$$

and, in order to simplify the notation we will write $L_k$ for the realization up to $n$ of this facial space. That is, $L_k = |Z^n_\ast|_n$. Recall, from Proposition 2 that the
existence of the contraction permits the following description of $L_k$:

$$L_k = Z_k^n \times \Delta^n / \sim$$

where the relation is given by

$$(z, t_0, ..., t_i, ..., t_n) \sim (s_k d_i z, 0, t_0, ..., \hat{t}_i, ..., t_n) \quad \text{if} \quad t_i = 0.$$  

With respect to this description, the canonical map $L_k \to Z_k^{-1}$ is given by $[z, t_0, ..., t_i, ..., t_n] \mapsto d_0^{n+1} z$ and is denoted by $\varepsilon_n$ (without reference to $k$).

Realizing all the lines, we obtain a facial map:

$$\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\partial_0 & \partial_{n+1} & \partial_0 & \partial_{n+1} & \partial_0 & \partial_0 & \partial_0 & \\
Z_n^{-1} & \xrightarrow{\varepsilon_n} & L_n & \xleftarrow{\varepsilon_n} & Z_1^{-1} & \xrightarrow{\varepsilon_n} & L_1 & \xleftarrow{\varepsilon_n} & Z_0^{-1} & \xrightarrow{\varepsilon_n} & L_0 \end{array}$$

The face operators $\partial_i : L_k \to L_{k-1}$ are given by $\partial_i[z, t_0, ..., t_n] = [\partial_i z, t_0, ..., t_n]$. Our aim is thus to see that the map obtained after realization (and always denoted by $\varepsilon_n$)

$$|Z_n^{-1}| \xrightarrow{\varepsilon_n} \ |L_n|$$

admits a section up to homotopy.

For each $k$, the map $\varepsilon_n : L_k \to Z_k^{-1}$ admits a (strict) section given by $z \mapsto [s_k^{n+1} z, 0, 0, ..., 0, 1]$ which we denote by $\psi_k$. The collection $\psi$ of these maps does not define a facial map since the contraction $s_k$ are not required to commute with the face operators $\partial_i$ of the columns. The key is that $\psi : Z_n^{-1} \to L_n$ is an $n$-rectifiable map. We can indeed consider, for each $k \leq n$, the (well-defined) map $\bar{\psi}_k : J_k^n(Z^{-1}) \to L_k$ given by:

$$\bar{\psi}_k(\partial_{i_1}, ..., \partial_{i_m}, z, t_0, ..., t_m) = [s_k^{n+1-m} \partial_{i_1} s_{k+1} \partial_{i_2} s_{k+2} ... \partial_{i_m} s_{k+m} z, 0, ..., 0, t_0, ..., t_m].$$

Straightforward calculation shows that the maps $\bar{\psi}_k$ commute with the face operators $\partial_i$ so that the collection $\bar{\psi}$ is a facial map. This morphism also satisfies $\bar{\psi}_k \circ \varphi_k = \psi_k$ for any $k \leq n$ (which implies that $\psi$ is an $n$-rectifiable map) and $\varepsilon_n \psi = \pi$. We have hence the following commutative diagram:

$$\begin{array}{ccccccc}
T^n(Z^{-1}) & \xrightarrow{\eta} & J_k^n(Z^{-1}) & \xrightarrow{\zeta} & J_k^n(Z^{-1}) & \xrightarrow{\bar{\psi}} & T^n(L) \\
\text{id} & \xrightarrow{\pi} & \text{id} & \xrightarrow{\varepsilon_n} & \text{id} & \xrightarrow{\varepsilon_n} & T^n(Z^{-1}).
\end{array}$$
Since the morphisms $\eta$, $\zeta$, $\pi$ and $\pi$ induce homotopy equivalence between the realizations up to $n$, we get the following situation after realization:

$$
\begin{align*}
|T_n^\bullet(Z^{-1})|_n & \sim |I_n^\bullet(Z^{-1})|_n \\
& \rightarrow |J_n^\bullet(Z^{-1})|_n \\
& \sim |T_n^\bullet(L)|_n
\end{align*}
$$

Since $|T_n^\bullet(Z^{-1})|_n = |Z_n^{-1}|_n$ and $|T_n^\bullet(L)|_n = |L_n|_n$, we obtain that the map $|L_n| \rightarrow |Z_n^{-1}|_n$ admits a homotopy section. □

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