Exact results for the Spectra of Bosons and Fermions with Contact Interaction

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Abstract

An $N$-body bosonic model with delta-contact interactions projected on the lowest Landau level is considered. For a given number of particles in a given angular momentum sector, any energy level can be obtained exactly by means of diagonalizing a finite matrix: they are roots of algebraic equations. A complete solution of the three-body problem is presented, some general properties of the $N$-body spectrum are pointed out, and a number of novel exact analytic eigenstates are obtained. The FQHE $N$-fermion model with Laplacian-delta interactions is also considered along the same lines of analysis. New exact eigenstates are proposed, along with the Slater determinant, whose eigenvalues are shown to be related to Catalan numbers.

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1 Introduction

There has recently been considerable interest in Bose-Einstein condensates, following their experimental discovery in atomic vapors [1 2 3]. Particular attention has been devoted to fast-rotating condensates, since such systems were demonstrated experimentally to form vortices [4 5], like in superfluid $^4$He; systems with ever bigger numbers of vortices

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were subsequently observed \cite{6,7,8}. A plausible theoretical model describing the system is that of two-dimensional (perpendicular to the axis of rotation) trapped bosons in a harmonic well with delta-function repulsion. The fast rotation implies that the particles are in a strong effective magnetic field, thereby allowing for a lowest Landau level (LLL) analysis. States with a given number of vortices correspond to the sector of fixed angular momentum $L$ in the $N$-body problem.

Apart from its application to Bose-Einstein condensates, the problem is interesting per se, as an $N$-body quantum-mechanical model which can be solved exactly, at least in part. This model has a clear relationship with another well-known two-dimensional setup in which the classical interaction force is also zero at nonzero distance — the anyon model \cite{9}. In the latter, only a small subset of the spectrum can be found exactly \cite{10}, whereas within the lowest Landau level, all eigenstates and eigenvalues are explicitly known \cite{11}. One might expect that the delta-interaction LLL problem at hand would be solvable in the same way, too. But this does not appear to be the case. An obvious explanation for this difficulty lies in the very definition of the model itself: delta interactions are known to be ill-defined in 2 dimensions, since the delta term is too singular with respect to the kinetic term. This singular interaction has to be regularized in some way. In fact, the anyon model itself can be viewed as a regularized version of the delta-interaction model: the long-distance quantum Aharonov-Bohm interaction, at the heart of the anyonic intermediate statistics, smoothes out the perturbative divergences due to the short-range contact interactions also present in the anyon model (exclusion of the diagonal of the configuration space).

Here, in the model considered, the regularization used is the LLL projection itself. It can be checked that restricting to the LLL subspace allows for a finite contact interaction to all orders in perturbation theory.

It was conjectured, based on numerical results, that the ground state for $L \geq N$ has a simple analytical form \cite{12}; it was proved later \cite{13,14} that the state in question is indeed an eigenstate. A hierarchy of $N$-body exact eigenstates for $L = 4$ was discovered \cite{14}. The other results are mainly numerical: the collective vs single-particle nature of the excited states was explored, good single-particle approximations in the $L \ll N$ regime discovered \cite{14}; very recently, a composite-fermion ansatz was suggested, and it was shown that the CF wave functions at $L = N$ become exact in the thermodynamic limit \cite{15}.

In this paper, we concentrate on exact results. In Sec. 2, we formulate the problem, perform the projection onto the lowest Landau level, and show that the interaction can be diagonalized separately in each given angular momentum sector. This means that any level can be obtained by diagonalizing a finite matrix; no basis truncation is necessary. In Sec. 3, we discuss some general properties of the spectrum and review a number of cases.
where the eigenstates have a simple polynomial form and the corresponding eigenvalues are rational numbers. In the next section it is shown, by direct counting, that those simple eigenvalues exhaust the whole spectrum of the three-body problem, which is thus completely solved. Section 5 deals with the $N$-body problem. Given the nature of the wave functions involved, it is convenient to consider states with a given value of $L$ and varying $N$. At $L = 2, 3$, all the $N$-body eigenstates fall into the category of the simple states of Sec. 3. For each of $L = 4$ and $L = 5$, apart from the simple ones, there is one extra $N$-body eigenstate: for $L = 4$, it is the one found in Ref. [14]; for $L = 5$, it is new. Starting with $L = 6$, most eigenvalues are irrational, but all of them are roots of algebraic equations. An algorithm is built that lets one find them all.

Finally, in Sec. 6, a fermionic model introduced so that the Laughlin FQHE wave-functions are its ground states is considered, again in the LLL. Via a mapping onto an equivalent bosonic model, it can be analyzed following the same approach as in the bosonic case. A few eigenstates are given as illustrations of the procedure. The $N$-body Slater determinants, with any $N$, have eigenvalues which are directly related to Catalan numbers.

2 Formulation and method of solution

Consider $N$ bosons in two dimensions in a harmonic well with pairwise contact interaction: 
\[ \hat{V} = V \sum_{i<j} \delta(r_i - r_j). \]
In complex coordinates ($z = x + iy$), the Hamiltonian is
\[
H = -2 \sum_{i=1}^{N} \partial_i \bar{\partial}_i + \frac{\omega^2}{2} \sum_{i=1}^{N} z_i \bar{z}_i + V \sum_{i<j=1}^{N} \delta^2(z_i - z_j).
\] (1)

Projection on the lowest Landau level. As said above, the delta interaction is too singular in two dimensions and as such it is not properly defined. One way to regularize the Hamiltonian is by projecting it on the “lowest Landau level” (LLL) subspace. This means that upon extracting the long-distance harmonic damping exponential factor one restricts to the set of $N$-body wave functions
\[
\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \exp \left( -\frac{\omega}{2} \sum_{i=1}^{N} z_i \bar{z}_i \right) f(z_1, z_2, \ldots, z_N),
\] (2)
where $f(z_1, z_2, \ldots, z_N)$ is analytic.

Indeed, if a magnetic field were added, the single-particle eigenstates corresponding to the LLL (Landau level number $n = 0$, angular momentum $l \geq 0$) would be in the LLL-harmonic eigenstates basis.
\[ \langle z, \bar{z} | 0, l \rangle = \left( \frac{\omega(t + 1)}{\pi l!} \right)^{1/2} z^l e^{-\frac{1}{2} \omega_t z \bar{z}}, \]

where \( \omega_t = \sqrt{\omega_c^2 + \omega^2}, \) \( \omega_c \) being half the cyclotron frequency. Coming back to the purely harmonic problem, consider now the projector on the “LLL” Hilbert space \( P_0 = \sum_{l=0}^{\infty} |0, l\rangle \langle 0, l| \)

\[ \langle z, \bar{z} | P_0 | z', \bar{z}' \rangle = \frac{\omega}{\pi} e^{-\frac{1}{2} \omega (z \bar{z} + z' \bar{z}' - 2z \bar{z}')}, \]

A generic state belonging to the LLL, \( |\psi\rangle = \sum_{l=0}^{\infty} a_l |0, l\rangle, \) is analytic up to the Gaussian factor,

\[ \langle z | \psi \rangle = f(z) e^{-\frac{1}{2} \omega z \bar{z}}, \]

with

\[ f(z) = \sum_{l=0}^{\infty} a_l z^l. \]

Projecting a one-body Hamiltonian on the LLL amounts to

\[ \langle z, \bar{z} | P_0 H P_0 | \psi \rangle = \langle z, \bar{z} | P_0 H | \psi \rangle = \int dz' dz'' e^{\frac{i}{\pi} (z z' + z'' z'') \omega} H(z') f(z') e^{-\frac{1}{2} \omega z' \bar{z}'}, \]

that is to say, for a one-body potential \( V(z, \bar{z}) \), to the eigenvalue equation

\[ \int dz' dz'' e^{\frac{i}{\pi} (z z' + z'' z'')} \left[ \omega + \omega z' \delta' + V(z', \bar{z}') \right] f(z') = E f(z). \]

In the \( N \)-body case with \( \hat{V} = V \sum_{i<j} \delta^2(z_i - z_j) \) one obtains a product of \( N \) integrals of \( f(z_1, ..., z_N) \). Using the Bargmann identity

\[ \frac{\omega}{\pi} \int dz' dz'' e^{-\omega (z' z'' - z z')} h(z') = h(z), \]

after integrations, the projected Hamiltonian reads (for simplicity, \( \omega = 1 \) and \( \frac{V}{2\pi} = 1 \))

\[ H_{LLL} = \hat{L} + \hat{V}, \]

where the angular momentum operator is

\[ \hat{L} = \sum_{i=1}^{N} z_i \partial_i \]

and the LLL-projected contact interaction

\[ \hat{V} = \sum_{i<j=1}^{N} P_{ij}, \]
\( P_{ij} \) being the operator\(^5\) that replaces both \( z_i \) and \( z_j \) in the wave function with the coordinate of the center of mass of the pair \((ij)\), keeping all the other coordinates intact:

\[
P_{ij}f(\ldots, z_i, \ldots, z_j, \ldots) = f\left(\ldots, \frac{z_i + z_j}{2}, \ldots, \frac{z_i + z_j}{2}, \ldots\right). \tag{13}
\]

A generic bosonic eigenstate of \( \hat{L} \) is a symmetrized monomial in the \( z_i \)'s:

\[
f_{l_1 \ldots l_N} = S\left[ \prod_{i=1}^{N} z_i^{l_i} \right], \tag{14}
\]

with

\[
L = \sum_{i=1}^{N} l_i, \quad l_i \geq l_{i+1} \geq 0. \tag{15}
\]

Here the symmetrization operator acts as

\[
S[f(z_1, z_2, \ldots, z_N)] = \frac{1}{N!} \sum_{\mathcal{P}} f(z_{p_1}, z_{p_2}, \ldots, z_{p_N}), \tag{16}
\]

the sum being over all \( N! \) permutations \( \mathcal{P} \) of \((1, \ldots, N)\).

The multiplicity \( G(N, L) \) of a level with given values of \( N \) and \( L \) is the number of ways that \( L \) can be represented as a sum \((15)\), i.e., the number of unordered partitions of \( L \) into \( N \) nonnegative addends. [E.g., for \( N = 3, L = 6 \), the \( G(3, 6) = 7 \) partitions are \((6, 0, 0), (5, 1, 0), (4, 2, 0), (4, 1, 1), (3, 3, 0), (3, 2, 1), (2, 2, 2)\).] It is directly connected to a well-studied quantity \( p_N(M) \), the number of unordered partitions of \( M \) into \( N \) positive addends: By adding 1 to each addend in each partition contributing to \( G(N, L) \), one sees that \( G(N, L) = p_N(L + N) \). This is a polynomial of degree \((N - 1)\) in \( L \): \( G(N, L) = \frac{L^{N-1}}{[N!(N-1)!]} + O(L^{N-2}) \). The generating function is

\[
\prod_{k=1}^{\infty} \frac{1}{1 - x^k z} = 1 + \sum_{M,N=1}^{\infty} G(N, M - N)x^M z^N. \tag{17}
\]

\(^5\)For \( \hat{V} = V \sum_{i<j<k=1}^{N} \delta^2(z_i - z_j)\delta^2(z_i - z_k) \), when the Pfaffian states are known to play a central role, one would have in the LLL

\[
\hat{V} = \sum_{i<j<k=1}^{N} P_{ijk},
\]

where \( P_{ijk} \) is the operator that replaces \( z_i, z_j \) and \( z_k \) in the wave function \( f(\ldots, z_i, \ldots, z_j, \ldots, z_k, \ldots) \) with the coordinate of the center of mass of the triplet \((ijk)\), keeping all the other coordinates intact:

\[
P_{ijk}f(\ldots, z_i, \ldots, z_j, \ldots, z_k, \ldots) = f\left(\ldots, \frac{z_i + z_j + z_k}{3}, \ldots, \frac{z_i + z_j + z_k}{3}, \ldots, \frac{z_i + z_j + z_k}{3}, \ldots\right).\]
The problem reduces to the diagonalization of $\hat{V}$ in the $G(N, L)$-dimensional subspace spanned by $\{f_{l_1...l_N}\}$, for any given $N$ and $L$. A generic state belonging to that subspace is

$$f(z_1, z_2, ..., z_N) = \sum_{\{l_k\}} a_{l_1...l_N} f_{l_1...l_N}$$  \hspace{1cm} (18)

with $f_{l_1...l_N}$ defined by Eq. (14), $a_{l_1...l_N}$ arbitrary coefficients, and the sum being over all $G(N, L)$ possible sets $\{l_k\}$ defined by Eq. (15). The eigenvalue equation is

$$\hat{V} f = E f$$  \hspace{1cm} (19)

[the total energy, according to (10), is $L + E$]. One has

$$\hat{V} f = \sum_{i<j=1}^N P_{ij} f = \sum_{\{l_k\}} a_{l_1...l_N} \sum_{i<j=1}^N P_{ij} f_{l_1...l_N} ,$$  \hspace{1cm} (20)

and upon expanding all the powers of $\frac{z_i + z_j}{2}$ in $P_{ij} f_{l_1...l_N}$, this becomes a polynomial in the $z_i$’s. In Eq. (18), with account for (14), $a_{l_1...l_N}$ is the coefficient at $\prod_k z_k^{l_k}$. Hence, in the polynomial (20), the coefficient at $\prod_k z_k^{l_k}$ must be equated to $E a_{l_1...l_N}$, for each $\{l_k\}$; one obtains $G(N, L)$ equations for as many coefficients $a_{l_1...l_N}$. Every level can thus be found exactly by diagonalization of a finite matrix, without the usual errors induced by basis truncation.

3 General properties

Center of mass. Introduce the center-of-mass coordinate

$$Z = \frac{1}{N} \sum_{i=1}^N z_i$$  \hspace{1cm} (21)

and the relative coordinates

$$\tilde{z}_i = z_i - Z ;$$  \hspace{1cm} (22)

obviously,

$$\sum_{i=1}^N \tilde{z}_i = 0 .$$  \hspace{1cm} (23)

For any function of the center-of-mass coordinate, $P_{ij} F(Z) = F(Z)$, hence

$$\hat{V} F(Z) = \frac{N(N-1)}{2} F(Z) .$$  \hspace{1cm} (24)

Moreover, for any eigenfunction $f(z_1, z_2, ..., z_N)$ of the interaction Hamiltonian,

$$\hat{V} f(z_1, z_2, ..., z_N) = E f(z_1, z_2, ..., z_N) ,$$  \hspace{1cm} (25)
also
\[ \hat{V}F(Z)f(z_1, z_2, ..., z_N) = EF(Z)f(z_1, z_2, ..., z_N) \] (26)
— a center-of-mass excitation does not change the interaction energy (although it does affect the total energy because the angular momentum changes). Thus, for each \((N, L)\) state, there is a “tower” of CM excitations above it: \((N, L + 1)\), \((N, L + 2)\), etc. It is enough to find the “pure relative” states, whose number is
\[ \tilde{G}(N, L) = G(N, L) - G(N, L - 1) . \] (27)

**Known exact eigenstates.** In the \(N\)-body case, denote
\[ \tilde{f}_{l_1...l_N} = \mathcal{S} \left[ \prod_{i=1}^{N} \tilde{z}_{l_i}^{l_i} \right] . \] (28)

One has
\[ \hat{V} \tilde{f}_{1...1} = \sum_{i<j=1}^{N} \left( \tilde{z}_i + \tilde{z}_j \right) \frac{2}{2} \prod_{k=1 \atop k \neq i,j}^{N} \tilde{z}_k = \frac{1}{4} \sum_{i<j=1}^{N} \frac{(\tilde{z}_i + \tilde{z}_j)^2}{\tilde{z}_i \tilde{z}_j} \prod_{k=1}^{N} \tilde{z}_k \]
\[ = \frac{1}{4} \left[ N(N-1) - N + \sum_{i,j=1}^{N} \frac{\tilde{z}_i}{\tilde{z}_j} \right] \prod_{k=1}^{N} \tilde{z}_k = N(N-2) \tilde{f}_{1...1} \] (29)
[at the very last step, Eq. (23) was used].

More generally, for \(1 < L \leq N\),
\[ \hat{V} \tilde{f}_{L00} = \frac{N}{2} \left( N - \frac{L}{2} - 1 \right) \tilde{f}_{1...0} \] (30)
(see Refs. [13, 14] for two different proofs). It has been conjectured numerically that this is the ground state for any \(L \leq N\) [12]. Obviously, for \(L = N\) it reduces to (29). Note also that \(\tilde{f}_{100...0} \equiv 0\) because of Eq. (23).

Further, in the 3-body case, for any \(L > 1\), one has
\[ \hat{V} \tilde{f}_{L00} = \sum_{i<j=1}^{3} P_{ij} \frac{1}{3} \sum_{k=1}^{3} \tilde{z}_k \]
\[ = \frac{1}{3} \sum_{i<j=1}^{3} \left[ 2 \left( \tilde{z}_i + \tilde{z}_j \right)^2 - \tilde{z}_i^L - \tilde{z}_j^L \right] + 3 \tilde{f}_{L00} \]
\[ = \left[ 1 + \frac{(-1)^L}{2^{L-1}} \right] \tilde{f}_{L00} , \] (31)
since \( \tilde{z}_i + \tilde{z}_j = -\tilde{z}_k \) for \( i \neq j \neq k \).

In the 4-body case, for any odd \( L \),
\[
\hat{V}\tilde{f}_{L000} = 3\tilde{f}_{L000} ,
\]
(32)
since in this case \( \tilde{z}_i + \tilde{z}_j = -\tilde{z}_k - \tilde{z}_l \) for \( i \neq j \neq k \neq l \).

In the \( N \)-body case, for \( L = 2 \) and \( L = 3 \), respectively, \( \tilde{f}_{200} \) and \( \tilde{f}_{300} \) are eigenstates of the form (30) since, trivially, \( \sum_{k=1}^{N} \tilde{z}_k^2 = -\sum_{k,l} \tilde{z}_k \tilde{z}_l \) and, less trivially, \( \sum_{k=1}^{N} \tilde{z}_k^3 = \sum_{k,l,m} \tilde{z}_k \tilde{z}_l \tilde{z}_m \).

“Slater excitations”. Consider the square of the \( N \)-particle Slater determinant,
\[
S_N^2 = \prod_{i<j=1}^{N} (z_i - z_j)^2 = \prod_{i<j=1}^{N} (\tilde{z}_i - \tilde{z}_j)^2 .
\]
(33)
Obviously, \( P_{ij}S_N^2 f = 0 \) for any \( f(z_1, \ldots, z_N) \). Hence, a “Slater excitation” of any function is an eigenfunction of \( \hat{V} \) with zero eigenvalue:
\[
\hat{V}S_N^{2n} f(z_1, z_2, \ldots, z_N) = 0 .
\]
(34)
(The presence of a Slater determinant nullifies the probability for the positions of any two particles to coincide, hence the delta interaction has no effect.) An even power of the determinant is required in order to preserve the symmetry of the function. This is somewhat reminiscent of the idea of “composite fermions” [15].

4 Three-body problem

For \( N = 3 \), it turns out that the simple eigenstates described in the previous section exhaust the whole spectrum.

The number of pure relative eigenstates for any \( L \geq 0 \) is
\[
\tilde{G}(3, L) = \left[\frac{L}{6}\right] + 1 - \delta_{L\text{ mod } 6, 1} ,
\]
i.e., \( \tilde{G}(3, 0) = 1, \tilde{G}(3, 1) = 0, \tilde{G}(3, 2) = \ldots = \tilde{G}(3, 5) = 1, \tilde{G}(3, 6) = 2, \) and so on with period 6. Here is the systematics of these states. For any \( L \neq 1 \), there is a state \( \tilde{f}_{L00} \) with the eigenvalue \( 1 + (-1)^L/2^{L-1} \), Eq. (31). These make up all the states for \( L \leq 5 \). [Recall that \( \tilde{f}_{110} \), Eq. (30), is the same as \( \tilde{f}_{200} \).] Now, upon any \( (3, L) \) state \( \tilde{f} \) there is a tower of \( (3, L + 6n) \) Slater excitations \( S_N^{2n} \tilde{f} \), \( n = 1, 2, \ldots \), with zero eigenvalues. These excitations account for the fact that \( \tilde{G}(3, L + 6) = \tilde{G}(3, L) + 1 \) [at \( (3, L + 6) \) there is one new state, \( \tilde{f}_{L+600} \), plus Slater excitations of all the \( (3, L) \) states] and complete the relative spectrum.

As an illustration, all the pure relative \( (3, L) \) eigenstates for up to \( L = 17 \) are enumerated below:
Adding a tower of center-of-mass excitations to each of these, one gets the complete 3-body (LLL) spectrum.

In the spirit of Eq. (34), one could also consider “generalized Slater determinants”, of the form

\[ T_{2n_1,2n_2,2n_3} = \mathcal{S} \left[ (\tilde{z}_1 - \tilde{z}_2)^{2n_1} (\tilde{z}_1 - \tilde{z}_3)^{2n_2} (\tilde{z}_2 - \tilde{z}_3)^{2n_3} \right] ; \quad (36) \]

obviously, \( T_{2n_1,2n_2,2n_3} \) times any symmetric function is a bosonic state annihilated by \( \hat{V} \). However, this does not yield new states. E.g.,

\[ T_{422} = 3S_3^2 \tilde{f}_{200} ; \quad T_{642} = \frac{243}{22} S_3^2 \tilde{f}_{600} - \frac{1}{11} S_3^4 . \quad (37) \]

5 \hspace{1em} N\text{-}body problem

As already stated, the diagonalization of the interaction Hamiltonian can be performed separately in each \((N, L)\) sector. An improvement can be made by excluding the center-of-mass excitations \textit{a priori}, i.e., diagonalizing in the pure relative basis — thereby reducing the dimension of the subspace involved from \(G(N, L)\) to \(\tilde{G}(N, L)\). Since the maximum possible number of nonzero addends in a partition of \(L\) is \(L\) itself, \(G(N, L)\) ceases to grow with \(N\) at \(N = L\). Hence, for any \(N \geq L\) also \(\tilde{G}(N, L) = \tilde{G}(L, L) \equiv \mathcal{G}(L)\). Now, by definition (27), \(\mathcal{G}(L) = G(L, L) - G(L, L - 1)\). In each of the \(G(L, L - 1)\) unordered partitions of \(L - 1\) into \(L\) nonnegative addends, at least one addend is equal to zero. Replacing that zero with one, we see that \(G(L, L - 1)\) is also the number of unordered partitions of \(L\) into \(L\) nonnegative addends of which at least one is equal to one. Hence, \(\mathcal{G}(L)\) is the number of unordered partitions of \(L\) into \(L\) nonnegative addends of which \textit{none} is equal to one. E.g., \(\mathcal{G}(6) = 4\), which corresponds to the four partitions (zeroes dropped for brevity): (6), (4, 2), (3, 3), (2, 2, 2). The generating function is

\[ \prod_{k=1}^{\infty} \frac{1}{1 - x^k} = 1 + \sum_{L=1}^{\infty} \mathcal{G}(L)x^L . \quad (38) \]
Thus, all the pure relative eigenstates for any $N$ at a given $L$ can be found by diagonalizing no more than a $G(L) \times G(L)$ matrix. The basis is formed by the set of states \{\(\tilde{f}_{l_1...l_N} : \sum_{i=1}^{N} l_i = L, \ l_i \geq l_{i+1} \geq 0, \ l_i \neq 1\)\}. Indeed, any state with any number of $l_i$'s equal to 1 is a linear combination of the basis states: Since $\tilde{z}_i = -\sum_{j\neq i}^{N} \tilde{z}_j$,

\[
\tilde{f}_{l_1...l_{i-1},1,l_{i+1}...l_N} = -\left(\tilde{f}_{l_1+1...l_{i-1},0,l_{i+1}...l_N} + \ldots + \tilde{f}_{l_1...l_{i-1}+1,0,l_{i+1}...l_N} + \ldots + \tilde{f}_{l_1...l_{i-1},0,l_{i+1}...l_N+1}\right), \quad (39)
\]

and if some $l_k = 0$, then the addend involving $l_k + 1$ is equal to the LHS, due to symmetry. A single $l_i = 1$ is thus eliminated, and by repeating, one eliminates them all.

One pure relative eigenstate for any $L$ and $N \geq L$ is already known, Eq. (30). For $L = 2$ and $L = 3$, as evidenced by the table above, there are no more. For $L = 4$, apart from the state (30), which in this case is

\[
\tilde{f}_{N,4}^{(1)} = \sum_{i\neq j \neq k \neq l}^{N} \tilde{z}_i \tilde{z}_j \tilde{z}_k \tilde{z}_l, \quad (40)
\]

with the eigenvalue

\[
E_{N,4}^{(1)} = \frac{N(N-3)}{2}, \quad (41)
\]

there is the state

\[
\tilde{f}_{N,4}^{(2)} = \sum_{i \neq j}^{N} (\tilde{z}_i - \tilde{z}_j)^4, \quad (42)
\]

whose eigenvalue is

\[
E_{N,4}^{(2)} = \frac{N(N-1)}{2} - \frac{7}{8}N + \frac{3}{4}, \quad (43)
\]

as found in Ref. [14].

For $L = 5$, the eigenstate

\[
\tilde{f}_{N,5}^{(1)} = \sum_{i\neq j \neq k \neq l \neq m}^{N} \tilde{z}_i \tilde{z}_j \tilde{z}_k \tilde{z}_l \tilde{z}_m, \quad (44)
\]

\[
E_{N,5}^{(1)} = \frac{N(2N-7)}{4}, \quad (45)
\]

is complemented by a new eigenstate

\[
\tilde{f}_{N,5}^{(2)} = \sum_{i \neq j \neq k}^{N} \left(\frac{\tilde{z}_i + \tilde{z}_j}{2} - \tilde{z}_k\right)^5 \quad (46)
\]
with
\[
E_{N,5}^{(2)} = \frac{N(N - 1)}{2} - \frac{15}{16} N + \frac{3}{4}.
\] (47)

All the \(N\)-body eigenstates with \(L \leq 5\) have thus been found exactly. Equations (43) and (47) seem to suggest a systematic pattern — which, however, does not exist.

For \(L = 6\), the basis states are
\[
\begin{align*}
h_1 &= \sum_{i} \tilde{z}_i^6, \\
h_2 &= \sum_{i \neq j} \tilde{z}_i^4 \tilde{z}_j^2, \\
h_3 &= \sum_{i < j} \tilde{z}_i^3 \tilde{z}_j^3, \\
h_4 &= \sum_{i \neq j \neq k} \tilde{z}_i^2 \tilde{z}_j^2 \tilde{z}_k^2;
\end{align*}
\] (48, 49, 50, 51)

defining a matrix \(V_{ij}\) such that
\[
\hat{V} h_i = \sum_j V_{ij} h_j,
\] (52)

one obtains
\[
||V_{ij}|| = \begin{pmatrix}
\frac{16N^2-47N+25}{32} & \frac{15}{32} & \frac{5}{8} & 0 \\
\frac{N+9}{32} & \frac{16N^2-60N+39}{32} & -\frac{3}{8} & \frac{3}{8} \\
\frac{N-7}{64} & -\frac{33}{64} & \frac{8N^2-32N+13}{16} & 0 \\
0 & \frac{3N+6}{8} & 3 & \frac{4N^2-16N+9}{8}
\end{pmatrix}.
\] (53)

One eigenvalue is
\[
E_{N,6}^{(1)} = \frac{N(N - 4)}{2},
\] (54)

which is nothing but Eq. (31) with \(L = 6\); the other three eigenvalues are the roots of the equation
\[
512E^3 + (-768N^2 + 2736N - 2016)E^2 \\
+ (384N^4 - 2736N^3 + 6850N^2 - 7116N + 2664)E \\
+ (-64N^6 + 684N^5 - 2921N^4 + 6378N^3 - 7527N^2 + 4554N - 1080) = 0.
\] (55)

For \(N < L\), there are fewer than \(\mathcal{G}(L)\) states, but the general scheme of calculation remains the same. At \(L = 6\), the rational eigenvalue (54) does not exist for \(N < 6\), but the irrational ones do for all \(N \geq 4\). The table below lists all the pure relative eigenvalues for \(4 \leq N \leq 6\) and \(L \leq 6\).
There appear a few more rational eigenvalues beyond $L = 6$. In the 4-body problem, we already know about the eigenvalue 3 for any odd $L$, Eq. (32); besides, one of the (4,7) eigenvalues is $\frac{11}{8}$, while at (4,12), there duly appears the Slater determinant squared, $S_4^2 = (\tilde{z}_1 - \tilde{z}_2)^2(\tilde{z}_1 - \tilde{z}_3)^2(\tilde{z}_2 - \tilde{z}_3)^2(\tilde{z}_2 - \tilde{z}_4)^2(\tilde{z}_3 - \tilde{z}_4)^2$, with eigenvalue 0, Eq. (34). Most of the values, however, appear to be irrational.

### 6 A model of interacting fermions

Consider now a Fermi gas with the interaction

$$\hat{V} = V \sum_{i<j} \Delta_i \delta(r_i - r_j),$$

which is well-known to have a trivial ground state — the Laughlin FQHE wave functions (with the vanishing eigenvalue) [16].

Let us again consider the model projected on the “LLL”: it transforms into (again, $\omega = 1$ and $V/(2\pi) = 1$)

$$\hat{V} = \sum_{i<j}(z_i - z_j) \left[ \frac{\partial}{\partial z_i} f(z_1, \ldots, z_N) \right]_{z_i, z_j \rightarrow (z_i + z_j)/2}.$$  \hspace{1cm} (57)

Necessarily, one has $f(z_1, \ldots, z_N) = S_N h(z_1, \ldots, z_N)$, where $h(z_1, \ldots, z_N)$ is a regular $N$-body bosonic wave function. In turn, (57) acting on $h(z_1, \ldots, z_N)$ takes the form

$$\sum_{i<j} \left[ \frac{S_N h}{z_i - z_j} \right]_{z_i, z_j \rightarrow (z_i + z_j)/2}.$$  \hspace{1cm} (58)

Therefore, one is back to a situation quite reminiscent of the bosonic problem addressed above, and the same machinery applies, but with the new Fermi LLL-interaction term given by Eq. (58).
One almost obvious $N$-body eigenstate is $S_N = \prod_{i<j}^N (z_i - z_j)$ [i.e., $h(z_1, \ldots, z_N) = 1$]. Indeed, the interaction term maps a fermionic state with a given angular momentum on a fermionic state with the same angular momentum without introducing any singularity. It follows that, in the lowest possible angular momentum sector $L = N(N-1)/2$, the single state $S_N$ is necessarily mapped onto itself. This is expressed in the identity

$$\sum_{i<j} \left[ \frac{S_N}{z_i - z_j} \right] \frac{z_i z_j \rightarrow (z_i + z_j)/2}{S_N} = E_N,$$

or

$$\sum_{i<j} \prod_{k \neq i,j}^N \frac{[(z_i - z_k) + (z_j - z_k)]^2}{(z_i - z_k)(z_j - z_k)} = D_{N-1},$$

where

$$E_N = \frac{D_{N-1}}{2^{2(N-2)}}$$

and

$$D_N = \frac{N + 1}{2} \left[ 4^N - \binom{2N + 1}{N} \right]$$

is related to the Catalan integer sequence $C_j$:

$$D_N = \sum_{j=0}^{N-1} C_j (N-j) 4^{N-j-1}, \quad C_j = \frac{1}{j+1} \binom{2j}{j}.$$

The identity can be shown to hold in general by relating $D_{N-1}$ to the double contour integral in the complex plane

$$I_N = \oint dw\,dz \frac{1}{(2\pi i)^2} \frac{1}{(z-w)^2} \prod_{k=1}^N \frac{(z + w - 2z_k)^2}{(z-z_k)(w-z_k)}$$

such that

$$I_N = 2D_{N-1} + 2^{2(N-1)} NI_1$$

and computing $I_N$ by expanding the contours of integration to infinity.

The eigenvalue grows slightly slower than linearly with $N$:

| $N$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|----|
| $E_N$ | 1 | $\frac{4}{15}$ | $\frac{29}{86}$ | $\frac{325}{128}$ | $\frac{843}{128}$ | $\frac{4165}{512}$ | $\frac{9949}{1024}$ | $\frac{185517}{16384}$ | $\frac{424415}{32768}$ |

The asymptotics is $E_N \to 2N - 4\sqrt{\frac{N}{\pi}}$.

Finally, as examples, here are a few other eigenstates. For all $N$:

$$\tilde{f}_N^{(1)} = S_N \sum_{i=1}^N z_i^2$$
with eigenvalue
\[ E^{(1)}_N = E_N + \frac{1}{2^{2(N-1)}} \sum_{i} z_i^2 \oint \frac{dw \, dz}{(2\pi i)^2} \prod_{k=1}^{N} \frac{(z + w - 2z_k)^2}{(z - z_k)(w - z_k)} = E_N + \frac{1}{2^{2N-2}} \binom{2N}{N-1} \] (67)
and
\[ \tilde{f}^{(2)}_N = S_N \sum_{i=1}^{N} \tilde{z}_i^3 \] (68)
with eigenvalue
\[ E^{(2)}_N = E_N + \frac{3}{2^{2(N-1)}} \frac{1}{2} \sum_{i} z_i^3 \oint \frac{dw \, dz}{(2\pi i)^2} \prod_{k=1}^{N} \frac{(z + w - 2z_k)^2}{(z - z_k)(w - z_k)} = E_N + \frac{1}{2^{2N-2}} \left[ 8 \binom{2N-2}{N-1} - \binom{2N+1}{N-1} \right]. \] (69)

Both (67) and (69) have been obtained by integration in the complex plane, following the procedure used in [17].

For \( N = 4, \, L = 10 \), one eigenstate is found to be a linear combination of \( h_1 = \sum_{i<j} (z_i - z_j)^4 \) and of the Pfaffian state \( h_2 = (z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4) - (z_1 - z_2)(z_1 - z_4)(z_2 - z_3)(z_3 - z_4) + (z_1 - z_2)(z_1 - z_3)(z_2 - z_4)(z_3 - z_4) \). Both \( h_1 \) and \( h_2 \) can be expanded in monomials of \( \tilde{z}_i \) as explained in the bosonic case. Denoting \( f_1 = S_4 h_1 \) and \( f_2 = S_4 h_2 \) and introducing a matrix \( V_{ij} \) such that
\[ \hat{V} f_i = \sum V_{ij} f_j, \] (70)
on one obtains
\[ ||V_{ij}|| = \begin{pmatrix} 43 & 63 & 32 \\ 16 & 32 & 227 \\ 64 & 227 & 128 \end{pmatrix}. \] (71)
The eigenvalues are
\[ E^{(1,2)}_{4,10} = \frac{571 \pm 9\sqrt{393}}{256}. \] (72)

Clearly, the machinery developed above in the bosonic case can be thoroughly used for the fermionic model considered here. It follows that the \( N \)-body problem is solvable in any given angular momentum sector. However, as in the bosonic case, complicated irrational coefficients and eigenvalues are expected, as illustrated in a particular case (72).

7 Conclusion

Considering the quantum-mechanical model of bosons with a delta-function coupling projected on the lowest Landau level, we have completely solved the three-body problem,
identified some analytic eigenstates for \( N \geq 4 \) which belong to two hierarchies (the \( L = 5 \) one is new), and worked out an algorithm through which all other eigenstates can be obtained by means of diagonalizing finite matrices (i.e., they are solutions of algebraic equations of finite power). An exact analytic solution of the \( N \)-body problem is evidently out of reach, but a numerical solution to any precision is quite straightforward. A model of fermions whose ground state is known to be the Laughlin FQHE wave function, has been analyzed along the same lines. We have shown that the Slater determinants, for any number of particles, are eigenstates with rational eigenvalues related to Catalan numbers, and identified a few excited states. Here too, all the levels are solutions of algebraic equations.

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