Generalized hyperbolic functions, circulant matrices and functional equations

by

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Abstract

There is a contrast between the two sets of functional equations

\begin{align*}
  f_0(x + y) &= f_0(x)f_0(y) + f_1(x)f_1(y), \\
  f_1(x + y) &= f_1(x)f_0(y) + f_0(x)f_1(y),
\end{align*}

and

\begin{align*}
  f_0(x - y) &= f_0(x)f_0(y) - f_1(x)f_1(y), \\
  f_1(x - y) &= f_1(x)f_0(y) - f_0(x)f_1(y)
\end{align*}

satisfied by the even and odd components of a solution of \( f(x + y) = f(x)f(y) \). J. Schwaiger and, later, W. Förg-Rob and J. Schwaiger considered the extension of these ideas to the case where \( f \) is sum of \( n \) components. Here we shorten and simplify the statements and proofs of some of these results by a more systematic use of matrix notation.

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1 Introduction

By a more systematic use of matrix notation, we shorten the statements and proofs of some results of J. Schwaiger [16] on generalized hyperbolic functions and their characterization by functional equations. We also discuss some stability results for these equations; our results here are motivated by those of W. Förg–Rob and J. Schwaiger [9]; see also [10].

The results discussed here can be readily understood in the case $n = 2$. A function $f : C \to C$ can be written as a sum of even and odd components in a standard way:

$$f(x) = f_0(x) + f_1(x),$$

where $f_0(x) = [f(x) + f(-x)]/2, \; f_1(x) = [f(x) - f(-x)]/2$. In case $f$ is an exponential function, i.e., if $f$ satisfies

$$f(x + y) = f(x)f(y), \quad (1.1)$$

for all $x$ and $y$, then the components satisfy

$$f_0(x + y) = f_0(x)f_0(y) + f_1(x)f_1(y),$$
$$f_1(x + y) = f_1(x)f_0(y) + f_0(x)f_1(y), \quad (1.2)$$

and

$$f_0(x - y) = f_0(x)f_0(y) - f_1(x)f_1(y),$$
$$f_1(x - y) = f_1(x)f_0(y) - f_0(x)f_1(y). \quad (1.3)$$

In case $f$ is the usual exponential function, then $f_0$ and $f_1$ are the cosh and sinh functions, respectively, and (1.2), (1.3) are the familiar sum and difference relations for these functions. But, of course, (1.1) also has many discontinuous solutions [1]. Chapters 2 and 3].

Conversely, the general solution of (1.3) is expressible in terms of a single arbitrary exponential function, i.e., it is known [20] that if $f_0$ and $f_1$ satisfy (1.3), then $f_0$ and $f_1$ are the even and odd components of a single exponential function: $f_0(x) = [g(x) + g(-x)]/2, \; f_1(x) = [g(x) - g(-x)]/2$. In fact, $g(x) = f_0(x) + f_1(x)$. On the
other hand \[19\], the general solution of (1.2) depends on two exponential functions: 
\[f_0(x) = \frac{[g_1(x) + g_2(x)]}{2}, \quad f_1(x) = \frac{[g_1(x) - g_2(x)]}{2}\]. We could think of \(g_1(x)\) as \(e^x\), for example, and \(g_2\) as the identically 0 function.

The reason that (1.3) has fewer solutions than (1.2) is that (1.3) implies (1.2) but not vice-versa. To see that (1.3) implies (1.2), we first interchange \(x\) and \(y\) in (1.3) to see that \(f_0\) must be even and \(f_1\) odd. Then replacing \(y\) by \(-y\) in (1.3) we get (1.2). On the other hand, \(f_0(x) = f_1(x) = e^x/2\) satisfies (1.2) but not (1.3).

We get “higher order generalized hyperbolic functions” by considering a function written as a sum of \(n\) components, forming a natural extension of even and odd components in the case \(n = 2\). The present work discusses the appropriate generalizations of (1.2), (1.3) and their solutions in this setting. Theorem 2.8 gives the systems of equations analogous to (1.2), (1.3) and Theorem 3.1 gives the general solutions of these systems. Theorem 5.2 discusses the stability of these systems and makes it clear why the system (1.3) is stable but the system (1.2) is not.

The name “higher order generalized hyperbolic functions” is intended to convey that the functions discussed here generalise the hyperbolic functions in two ways. First, there is the general order \(n\) rather than the special order 2. Second, we do not confine attention to continuous solutions of the relevant functional equations.

The main results reported in §2 and §3 are proved in [16], essentially by the same methods, but with a less systematic use of matrix methods. An attempt to understand [16] and [9] is what led to the present paper. In [16] and [9], \(n\)-tuples of functions were used; here we find it more convenient to base the development more explicitly on circulant matrices formed from these \(n\)-tuples; this is partially done in [16] and [9]. The question of stability is further considered in [10].
We note that (1.2) and (1.3) can be written
\[
\begin{bmatrix} f_0(x + y) & f_1(x + y) \\ f_1(x + y) & f_0(x + y) \end{bmatrix} = \begin{bmatrix} f_0(y) & f_1(y) \\ f_1(y) & f_0(y) \end{bmatrix} \cdot \begin{bmatrix} f_0(x) & f_1(x) \\ f_1(x) & f_0(x) \end{bmatrix}
\] (1.4)
and
\[
\begin{bmatrix} f_0(x - y) & f_1(x - y) \\ f_1(x - y) & f_0(x - y) \end{bmatrix} = \begin{bmatrix} f_0(y) & -f_1(y) \\ -f_1(y) & f_0(y) \end{bmatrix} \cdot \begin{bmatrix} f_0(x) & f_1(x) \\ f_1(x) & f_0(x) \end{bmatrix},
\] (1.5)
so our task is to find higher-dimensional analogues of (1.4) and (1.5).

Our results are stated for complex-valued functions of a complex variable, but, as in [9], [10] and [16], they can be extended to complex-valued functions on an abelian group \( G \) with an automorphism \( \sigma \) satisfying \( \sigma^n = id_G \).

F. Zorzitto [21] has considered a far-reaching generalization, based on group theory, of the results of [9] and [16].

## 2 Preliminary results

### 2.1 Special matrices

The \( n \times n \) Fourier matrix [7, p. 32] \( \mathcal{F}_n \) is given by

\[
\mathcal{F} = \mathcal{F}_n = \frac{1}{\sqrt{n}} \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\
1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2}
\end{bmatrix}.
\] (2.1)

\( \mathcal{F}_n \) is a symmetric matrix whose complex-conjugate is given by

\[
\mathcal{F}^* = \mathcal{F}_n^* = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2}
\end{bmatrix}.
\]
Sometimes the Fourier matrix is defined as in (2.1) but without the factor $1/\sqrt{n}$. The present definition has the advantage that the matrix $\mathcal{F}_n$ is unitary: $\mathcal{F}_n^*\mathcal{F}_n = \mathcal{F}_n\mathcal{F}_n^* = I$.

We use the notation \[ \circ(a^T) = \circ(a_0, a_1, \ldots, a_{n-1}) \] for the circulant matrix:

$$A = \circ(a^T) = \circ(a_0, a_1, \ldots, a_{n-1}) = \begin{bmatrix}
a_0 & a_1 & \cdots & a_{n-1} \\
a_{n-1} & a_0 & \cdots & a_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \cdots & a_0
\end{bmatrix}. \quad (2.2)$$

We also use the diagonal matrix

$$\Omega = \Omega_n = \text{diag}[1, \omega, \ldots, \omega^{n-1}] .$$

We have [7, Theorem 3.2.1]

$$\mathcal{F}^*\Omega\mathcal{F} = \pi, \quad (2.3)$$

or

$$\Omega = \mathcal{F}\pi\mathcal{F}^*, \quad (2.4)$$

where $\pi$ is the $n \times n$ permutation matrix

$$\pi = \pi_n = \circ(0,1,\ldots,0) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix} .$$

In fact all circulants are polynomials in $\pi$; we have [7, p. 68]

$$\circ(a_0, a_1, \ldots, a_{n-1}) = p_\gamma(\pi), \quad (2.5)$$

where

$$p_\gamma(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}. \quad (2.6)$$

Every circulant matrix can be diagonalized:
Lemma 2.1. If $A$ is a circulant matrix given by (2.2), then

$$
\mathcal{F} A \mathcal{F}^* = \sqrt{n} \text{diag}[\mathcal{F}^* a].
$$

(2.7)

Here we use the usual boldface notation for column vectors: $a = [a_0 \ a_1 \ \ldots \ a_{n-1}]^T$ and diag[a] denotes the square matrix whose diagonal elements are $a_0, a_1, \ldots, a_{n-1}$.

Proof. This is shown in ([12, Theorem 3.2.2]) in the form

$$
\mathcal{F} A \mathcal{F}^* = \text{diag}(p_\gamma(1), p_\gamma(\omega), \ldots, p_\gamma(\omega^{n-1})),
$$

(2.8)

which is equivalent to (2.7) on using (2.6).

By conjugation and iteration of (2.3) we get, since $\pi^T = \pi^{-1}$ and $\mathcal{F}^T = \mathcal{F}$,

$$
\mathcal{F} \Omega^{-m} \mathcal{F}^* = \pi^m,
$$

(2.9)

for $m = 0, 1, \ldots$ and, taking inverses, this also holds for $m = -1, -2, \ldots$. This gives

$$
\mathcal{F}[\Omega^{-m} \pi^k \Omega^m] \mathcal{F}^* = \mathcal{F} \Omega^{-m} \mathcal{F}^*[\mathcal{F} \pi^k \mathcal{F}^*] \mathcal{F} \Omega^m \mathcal{F}^* = \pi^m \mathcal{F} \pi^k \mathcal{F}^* \pi^{-m},
$$

(2.10)

for integers $m$ and nonnegative integers $k$. Hence, in view of the representation (2.5), we get:

Lemma 2.2. For circulant $A$, and $m = 0, 1, \ldots$,

$$
\mathcal{F}[\Omega^{-m} A \Omega^m] \mathcal{F}^* = \pi^m \mathcal{F} A \mathcal{F}^* \pi^{-m}.
$$

(2.11)

2.2 Components of a function

Throughout the rest of this paper, we let $n$ be a fixed integer $\geq 2$. Following J. Schwaiger [16], we make the following definition:
Definition 2.3. A function $h : C \to C$ is of type $j$ ($j = 0, \ldots, n - 1$), if $h(\omega x) = \omega^j h(x)$ where $\omega = \omega_n = e^{2\pi i/n}$.

In case $n = 2$, the type 0 and type 1 functions are the even and odd functions, respectively. In the cases $n = 3$ and $n = 4$, examples of functions of types 0, 1, 2 and 0, 1, 2, 3, respectively, are given in (4.1) and (4.2). It is easy to generate examples in the case of analytic functions: if $n = 3$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in a disk centred at 0, then $f_j(z) = \sum_{k=0}^{\infty} a_{3k+j} z^{3k+j}$ is of type $j$, $j = 0, 1, 2$. But we have no need of analyticity in the discussion.

Lemma 2.4. Every function $f : C \to C$ can be expressed uniquely as a sum of functions $f_j$ of type $j$, $j = 0, \ldots, n - 1$, called the components of $f$. In fact,

$$f = \sum_{j=0}^{n-1} f_j$$

(2.12)

where

$$\begin{bmatrix}
  f_0(x) \\
  f_1(x) \\
  \vdots \\
  f_{n-1}(x)
\end{bmatrix} = \frac{1}{\sqrt{n}} \mathcal{F} \begin{bmatrix}
  f(x) \\
  f(\omega x) \\
  \vdots \\
  f(\omega^{n-1} x)
\end{bmatrix} = \frac{1}{\sqrt{n}} \mathcal{F}^* \begin{bmatrix}
  f(x) \\
  f(\omega^{-1} x) \\
  \vdots \\
  f(\omega^{-n+1} x)
\end{bmatrix}.$$  (2.13)

Proof. Equation (2.12) with $f_j$ of type $j$, $j = 0, \ldots, n - 1$, implies that

$$\begin{bmatrix}
  f(x) \\
  f(\omega x) \\
  \vdots \\
  f(\omega^{n-1} x)
\end{bmatrix} = \sqrt{n} \mathcal{F}^* \begin{bmatrix}
  f_0(x) \\
  f_1(x) \\
  \vdots \\
  f_{n-1}(x)
\end{bmatrix}$$

(2.14)

which is obtained from (2.12) by replacing $x$ by $x, \omega x, \ldots, \omega^{n-1} x$ in order. Also, of course, (2.14) implies (2.12). But (2.14) obviously has a unique solution for the $f_i$ given by the first equality in (2.13), and each of the $f_i$ is of type $i$.

The second equation in (2.13) is obtained similarly.
Definition 2.5. The circulant matrix

\[
F(x) = \text{circ}(f_0, f_1, \ldots, f_{n-1}) = \begin{bmatrix}
f_0 & f_1 & \cdots & f_{n-1} \\
f_{n-1} & f_0 & \cdots & f_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
f_1 & f_2 & \cdots & f_0
\end{bmatrix},
\]

(2.15)

whose first row is formed from the components of \( f \) in order will be called the circulant matrix corresponding to \( f \).

Lemma 2.6. If \( f : \mathbb{C} \to \mathbb{C} \) is any function, the corresponding circulant matrix function is given by

\[
F(x) = \mathcal{F}^* \text{diag} [f(x), f(\omega x), \ldots, f(\omega^{n-1}x)] \mathcal{F}.
\]

(2.16)

Proof. This comes from Lemma 2.1.

2.3 Exponential functions

Lemma 2.7. If \( f \) is an exponential function, the corresponding circulant matrix function \( F \) satisfies \( F(x+y) = F(y)F(x) \).

Proof. This is a fairly immediate consequence of Lemma 2.6. \( \mathcal{F}F(x+y)\mathcal{F}^* \) is a diagonal matrix with entries \( f(\omega^i(x+y)) = f(\omega^if(x))f(\omega^iy), \ i = 0, \ldots, n-1 \). But \( f(\omega^ix), f(\omega^iy), i = 0, \ldots, n-1 \), are the respective entries of the diagonal matrices \( \mathcal{F}F(x)\mathcal{F}^* \) and \( \mathcal{F}F(y)\mathcal{F}^* \).

Lemma 2.8. For any function \( f \), the corresponding circulant matrix function \( F \) satisfies the equation \( F(\omega y) = \Omega^{-1}F(y)\Omega \).

Proof. This is a fairly immediate consequence of Definition 2.3.

Successive application of Lemmas 2.7 and 2.8 give

Theorem 2.9. If \( f \) is an exponential function, the corresponding circulant matrix
function $F$ satisfies

$$F(x + \omega^m y) = \Omega^{-m} F(y) \Omega^m F(x), \quad (2.17)$$

for $m = 0, 1, \ldots, n - 1$.

Theorem 2.9 is a matrix form of [16, equation(4.4)], which is written in the notation

$$f_j(x + \omega^t y) = \sum_{l=0}^j \omega^{(j-l)t} f_l(x) f_{j-l}(y) + \sum_{l=j+1}^{n-1} \omega^{(n+j-l)t} f_l(x) f_{n+j-l}(y), \quad t = 0, 1, \ldots, n - 1.$$  

In case $n = 2$, (2.17) reduces to (1.4) for $m = 0$ and to (1.5) for $m = 1$.

3 A converse result

We now consider equation (2.17) where $F(x)$ is circulant, but no assumption is made about its entries $f_i$. It will come as no surprise that if (2.17) holds for all $m = 0, 1, \ldots, n - 1$, then the $f_i$ are the components of an exponential function and $F(x) = \text{circ}(f_0, f_1, \ldots, f_{n-1})$. But what if we have (2.17) for only a single value of $m$? Then it is clear that (2.17) holds also for integer multiples of this value. Thus if g.c.d. $(n, m) = 1$, and (2.17) holds for a fixed $m \in \{0, 1, \ldots, n - 1\}$, then it holds for all $m \in \{0, 1, \ldots, n - 1\}$. On the other hand, if (2.17) holds for a fixed $m \in \{0, 1, \ldots, n - 1\}$ with g.c.d. $(n, m) = d > 1$, then we can assert only that it holds for values of $m$ which are multiples of $d$, modulo $n$, and it turns out that the general solution of (2.17) depends on $d$ arbitrary exponential functions:

**Theorem 3.1.** Let a circulant matrix function $F(x)$ satisfy

$$F(x + \omega^m y) = \Omega^{-m} F(y) \Omega^m F(x), \quad (3.1)$$
for a fixed value of $m \in \{0, 1, \ldots, n-1\}$, and let $d = \text{g.c.d.}(n, m)$. Then $F(x) = \text{circ}(f(x)^T)$ where

$$f(x) = \frac{1}{\sqrt{n}} F \begin{bmatrix} h(x) \\ h(\omega^m x) \\ \vdots \\ h(\omega^{m(n/d-1)} x) \end{bmatrix},$$

(3.2)

$h(x)$ being an arbitrary $d$–tuple of exponential functions. In particular, when $d = 1$,

$$f(x) = \frac{1}{\sqrt{n}} F \begin{bmatrix} g(x) \\ g(\omega x) \\ \vdots \\ g(\omega^{n-1} x) \end{bmatrix},$$

(3.3)

where $g$ is an arbitrary exponential function.

To prove Theorem 3.1, we use the ideas of [16]. We write (3.1) in the form

$$FFF(x + \omega^m y)F^* = [F\Omega^{-m} F(y) \Omega^m F^*][FF(x)F^*],$$

(3.4)

and using Lemmas 2.1 and 2.2, this is

$$G(x + \omega^m y) = \pi^m G(y) \pi^{-m} G(x),$$

(3.5)

where $G(x)$ is the diagonal matrix

$$G(y) = \sqrt{n} \left[ \text{diag} F^* f(y) \right].$$

Using the notation

$$G_m(x) = \pi^m G(x) \pi^{-m},$$

(3.6)

we have

$$G_m(x) = \text{diag}[g_m(x), g_{m+1}(x), \ldots, g_{m+n-1}(x)],$$

(3.7)

where suffixes are taken modulo $n$, since

$$\pi \text{diag}[\lambda_1, \ldots, \lambda_n] \pi^{-1} = \text{diag}[\lambda_2, \ldots, \lambda_n, \lambda_1],$$

so the effect of the transformation $G(x) \to \pi^m G(x) \pi^{-m}$ is to replace $g_j$ by $g_{j+m}$. 
The easiest case is $m = 0$ and hence $d = \text{g.c.d.}(m, n) = n$. In this case (3.5) is

$$G(x + y) = G(x)G(y),$$

(3.8)

so, since $G$ is diagonal, its entries must be exponential functions. In case $m = 0$ this is all we can say about the $g$’s; they are independent.

The other extreme case is where $d = 1$. In this case the set of multiples $\{km, \ k = 0, 1, \ldots, n - 1\}$ modulo $n$ is precisely the set $\{1, 2, \ldots, n\}$ so (3.5) can be written as a system of $n$ equations

$$g_0(x + \omega^my) = g_m(y)g_0(x),$$
$$g_m(x + \omega^my) = g_{2m}(y)g_m(x),$$
$$\cdots$$
$$g_{(n-1)m}(x + \omega^my) = g_0(y)g_{(n-1)m}(x).$$

(3.9)

There are two possibilities. One is that $g_0(0) = 0$; hence from (3.9), all of the $g_i$ are identically 0 and the assertion of the Theorem holds trivially. The other possibility is that $g_0(0) \neq 0$. Putting $x = y = 0$ in the first of equations (3.9) then gives $g_m(0) = 1$. Then the remaining equations, with $y = 0$, give $g_{2m}(0) = g_{3m}(0) = \cdots = g_0(0) = 1$ or $g_2(0) = g_3(0) = \cdots = g_0(0) = 1$. Then putting $x = 0$ in (3.9) we see that the $g_k$ are given by the formula

$$g_{km}(y) = g_0(\omega^{km}y), \ k = 0, 1, \ldots, n - 1.$$ 

(3.10)

The second of these equations gives

$$g_m(y) = g_0(\omega^my),$$

(3.11)

which, together with the first of equations (3.9) shows that $g_0$ is an exponential function. Now (3.10) is equivalent to

$$g_k(y) = g_0(\omega^ky), \ k = 0, 1, \ldots, n - 1.$$ 

(3.12)

Thus $G$ is determined by the single exponential function $g_0$ and $f$ is determined by (3.3). This completes the proof in the case $m = 1$. 

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In case \( d = \text{g.c.d.}(n, m) > 1 \), the equations (3.9) are replaced by \( d \) sets of \( n/d \) equations corresponding to the cosets of the additive subgroup of \( \{0, 1, \ldots, n-1\} \) generated by \( \{0, 1, \ldots, d-1\} \). The \( r \)th such set of equations (where \( r \) is one of the numbers \( 0, 1, \ldots, d-1 \)), reads

\[
\begin{align*}
g_r(x + \omega^m y) &= g_{r+m}(y)g_r(x), \\
g_{r+m}(x + \omega^m y) &= g_{r+2m}(y)g_{r+m}(x), \\
&\quad \ldots \\
g_{r+(n/d-1)m}(x + \omega^m y) &= g_{r+(n/d-1)m}(y)g_{r+(n/d-1)m}(x).
\end{align*}
\] (3.13)

Each such set of equations can be dealt with as in the case \( m = 1 \). That is either all of the \( g_{r+km} \) (\( k = 0, 1, \ldots, n/d-1 \)) are identically 0 or they are given by the formula

\[
g_{r+km}(x) = g_r(\omega^k x), \quad k = 0, 1, \ldots, n/d-1,
\] (3.14)

where \( g_r \) is an exponential function. Thus \( G \) is determined by the \( d \)-tuple \( h(x)^T = (g_0(x), g_1(x), \ldots, g_{d-1}(x)) \) of exponential functions and we get the main assertion of Theorem 3.1.

### 4 Continuous solutions

In the case \( n = 1 \), a continuous solution of \( f(x+y) = f(x)f(y) \), is given by \( f(x) = e^{ax} \).

For \( n > 1 \), \( e^x \) has components

\[
F_{n,k}^{(1)}(x) = \frac{1}{n} \sum_{j=0}^{n} \omega^{jk} e^{\omega^j x}, \quad k = 0, \ldots, n-1,
\]
in a minor variation of the notation of \[17\], \[18\]. For further references and history of these functions, see \[14\]. It should be noted that the term “generalized hyperbolic functions” as used in \[14\], \[17\] and \[18\], corresponds to some of the continuous solutions of the system of functional equations considered in the present paper. The three continuous generalized hyperbolic functions of order 3 are

\[
\begin{align*}
F_{3,0}^1(x) &= \frac{1}{3} \left[ e^x + 2e^{-x/2} \cos \frac{\sqrt{3}x}{2} \right]; \\
F_{3,1}^1(x) &= \frac{1}{3} \left[ e^x - 2e^{-x/2} \cos \left( \frac{\sqrt{3}x}{2} + \frac{\pi}{3} \right) \right]; \\
F_{3,2}^1(x) &= \frac{1}{3} \left[ e^x - 2e^{-x/2} \cos \left( \frac{\sqrt{3}x}{2} - \frac{\pi}{3} \right) \right].
\end{align*}
\] (4.1)
In the case \( n = 4 \), we get the compact formulas
\[
\begin{align*}
F_{4,0}^1(x) &= (1/2)(\cosh x + \cos x), \\
F_{4,1}^1(x) &= (1/2)(\sinh x + \sin x), \\
F_{4,2}^1(x) &= (1/2)(\cosh x - \cos x), \\
F_{4,3}^1(x) &= (1/2)(\sinh x - \sin x),
\end{align*}
\]
(4.2)
given by Battioni \[4\], a special case of a more general result
\[
\begin{align*}
F_{2m,r}^1(x) &= \frac{[F_{m,r}^1(x) + F_{m,r}^{-1}(x)]}{2}, \quad r = 0, 1, \ldots, m - 1, \\
F_{2m,r+m}^1(x) &= \frac{[F_{m,r}^1(x) - F_{m,r}^{-1}(x)]}{2}, \quad r = 0, 1, \ldots, m - 1,
\end{align*}
\]
given in a different notation in \[8\] (33), p. 216].

The decomposition of various special functions into sums of functions of order \( j, j = 0, \ldots, n - 1 \) has been considered by Y. Ben Cheikh; see \[5\] and \[6\].

5 Stability

We use the usual 1-norm for square matrices:
\[
\|A\| = \|A\|_1 = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|,
\]
If \( B \) is a diagonal matrix, we have
\[
\|AB\| \geq \|A\| \min_{1 \leq i \leq n} |b_{ii}|. 
\]

**Definition 5.1.** (See, e.g., \[8\], \[13\].) For fixed \( m \), we say that the equation
\[
F(x + \omega^m y) = \Omega^{-m} F(y) \Omega^m F(x),
\]
(5.2)
for circulant \( F \), is **stable** if the condition
\[
\|F(x + \omega^m y) - \Omega^{-m} F(y) \Omega^m F(x)\| < \varepsilon, \quad \text{for some } \varepsilon > 0 \text{ and all } x, y,
\]
(5.3)
implies that either \( F \) is bounded or \( F \) satisfies (5.2).
It has been noted that in the case \( n = 2 \), the system (1.3) is stable \([15]\) but (1.2) is not \([12]\). An appropriate generalization is

**Theorem 5.2.** Let \( F \) be a circulant \( n \times n \) matrix function and let \( m \) be a fixed integer in the set \( \{0, 1, \ldots, n - 1\} \). The equation

\[
F(x + \omega^m y) = \Omega^{-m}F(y)\Omega^m F(x),
\]

(5.4)

is stable if and only if \( \text{g.c.d.}(n, m) = 1 \).

**Proof.** The idea of the proof is based on the proof of \([9, \text{Proposition 2}]\). Suppose that \( \text{g.c.d.}(n, m) = 1 \) and let \( F \) be unbounded and satisfy (5.3). We need to show that \( F \) satisfies (5.4). We introduce the diagonal matrix

\[
G(x) = \mathcal{F}F(x)\mathcal{F}^* = \sqrt{n} \text{diag} [\mathcal{F}^*f(x)]
\]

as in Lemma 2.1. Clearly \( G \) is also unbounded, since \( \|\mathcal{F}\| = \|\mathcal{F}^*\| = \sqrt{n} \). These relations, together with Lemma 2.2, show that (5.3) is equivalent to

\[
\|G(x + \omega^m y) - G_m(y)G(x)\| \leq n\varepsilon = \varepsilon_1,
\]

(5.5)

with the notation of (3.6), (3.7). This leads to

\[
\|G(x + (\omega^m y + \omega^m z)) - G_m(y + z)G(x)\| \leq \varepsilon_1
\]

and

\[
\|G((x + \omega^m z) + \omega^m y) - G_m(y)G(x + \omega^m z)\| \leq \varepsilon_1
\]

so

\[
\|G_m(y + z)G(x) - G_m(y)G(x + \omega^m z)\| \leq 2\varepsilon_1,
\]

for all \( x, y, z \). Also, from (5.5) and the norm property \( \|AB\| \leq \|A\|\|B\| \), we get

\[
\|G_m(y)G(x + \omega^m z) - G_m(y)G_m(z)G(x)\| \leq \varepsilon_1\|G_m(y)\|.
\]
Combining the last two inequalities, we get

\[ \| G_m(y + z)G(x) - G_m(y)G_m(z)G(x) \| \leq \varepsilon_1(2 + \| G_m(y) \|). \]

Since \( G \) is diagonal, we may use (5.1) to get

\[
\| G_m(y + z) - G_m(y)G_m(z) \| \min_{1 \leq i \leq n} |g_i(x)| \leq \| [G_m(y + z) - G_m(y)G_m(z)]G(x) \| \\
\leq \varepsilon_1(2 + \| G_m(y) \|). \tag{5.6}
\]

We see from (5.5) that \( \| G(\omega^n y) - G_m(y)G(0) \| \leq \varepsilon_1 \); hence the unboundedness of one diagonal entry in \( G \) implies the unboundedness of all diagonal entries in \( G \). (Here it is crucial that g.c.d.\((n, m) = 1\); otherwise integers multiples of \( m \) will not cover all \( j = 0, \ldots, n - 1 \).) This means that, given \( y \) and \( z \), we can choose \( x \) such that

\[ \min_{1 \leq i \leq n} |g_i(x)| \geq 2 + \| G_m(y) \|. \]

Hence, from (5.6), \( \| G_m(y + z) - G_m(y)G_m(z) \| \leq \varepsilon_1 \). Since the elements of \( G_m \) are the same as those of \( G \), in a different order, we finally get

\[ \| G(y + z) - G(y)G(z) \| \leq \varepsilon_1. \tag{5.7} \]

Thus each element \( g_i \) in the diagonal matrix \( G(y) \) is unbounded and satisfies the inequality \( |g_i(y + z) - g_i(y)g_i(z)| \leq \varepsilon_1 \). By [2, Theorem 1], we must therefore have \( g_i(y + z) - g_i(y)g_i(z) = 0 \), so

\[ G(y + z) - G(y)G(z) = 0 \tag{5.8} \]

and, as in the proof of Theorem 2.9, \( F \) is a solution of (5.2).

To prove the "only if" assertion of the Theorem, we note that if g.c.d.\((n, m) > 1\), the unboundedness of one diagonal entry in \( G \), does not imply the unboundedness of all entries there. So, for example, suppose that \( g_i(x) = e^x \), for \( i \) a multiple of
\[ d = \text{g.c.d.}(n, m) > 1 \text{ and } g_i(x) = 2, \text{ when } i \text{ is not a multiple of } d. \] Then \( G \) is an unbounded solution of (5.7), with \( \varepsilon_1 = 2 \), which does not satisfy (5.8). Thus we can have an unbounded solution of (5.3) which does not satisfy (5.4). This, of course, is exemplified by the non-stability of the equations (1.2).

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**References**

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989.

[2] J. A. Baker, The stability of the cosine equation, *Proc. Amer. Math. Soc.* 80: 411–416 (1980).

[3] J. A. Baker, J. Lawrence and F. Zorzitto, The stability of the equation \( f(x + y) = f(x) + f(y) \), *Proc. Amer. Math. Soc.* 74: 242–246 (1979).

[4] G. Battioni, Su una generalizzazione delle funzioni iperboliche e delle funzioni circolari, *Riv. Mat. Univ. Parma (2)* 10: 39–48 (1969).

[5] Y. Ben Cheikh, Decomposition of the Bessel functions with respect to the cyclic group of order \( n \), *Le Matematiche* 52: 365–378 (1997).

[6] Y. Ben Cheikh, Decomposition of Laguerre polynomials with respect to the cyclic group of order \( n \), *J. Comp. Appl. Math.* 99: 55–66 (1998).

[7] P. J. Davis, *Circulant Matrices*, 2nd ed., Chelsea Publishing, New York, 1994.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, vol. 3, McGraw-Hill, New York, 1955.
[9] W. Förg–Rob and J. Schwaiger, On the stability of a system of functional equations characterizing generalized hyperbolic and trigonometric functions, *Aequationes Math.* 45: 285–296 (1993).

[10] W. Förg–Rob and J. Schwaiger, On the stability of some functional equations for generalized hyperbolic functions and for the generalized cosine equation, *Results Math.* 26: 274–280 (1994).

[11] G. L. Forti, Hyers-Ulam stability of functional equations in several variables, *Aequationes Math.* 50: 143–190 (1995).

[12] R. Ger, Stability of addition formulae for trigonometric mappings, *Zeszyty Nauk. Politech. Ślask. mat.-Fiz.* 64: 75–84 (1990).

[13] D. H. Hyers, G. Isac and T. M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications 34, Birkhäuser Boston, Inc., Boston, MA, 1998.

[14] M. E. Muldoon and A. Ungar, Beyond sin and cos, *Math. Mag.* 69: 3–14 (1996).

[15] J. Schwaiger, Remark 15, Report on 28th International Symposium on Functional Equations, *Aequationes Math.* 41: 289–290 (1991).

[16] J. Schwaiger, On generalized hyperbolic functions and their characterization by functional equations, *Aequationes Math.* 43: 198–210 (1992).

[17] A. Ungar, Generalized hyperbolic functions, *Amer. Math. Monthly* 89: 688-691 (1982).

[18] A. Ungar, Higher order alpha-hyperbolic functions, *Indian J. Pure Appl. Math.* 15: 301-304 (1984).
[19] L. Vietoris, Zur Kennzeichnung des Sinus und verwandter Funktionen durch Funktionalgleichungen, *J. Reine Angew. Math.* 186: 1–15 (1944).

[20] W. H. Wilson, On certain related functional equations, *Bull. Amer. Math. Soc.* 26: 300–312 (1919–1920).

[21] F. Zorzitto, Groups acting on circulant matrices, *Aequationes Math.* 48: 294–305 (1994).