Superdiffusive comb: Application to experimental observation of anomalous diffusion in one dimension

Alexander Iomin
Department of Physics, Technion, Haifa, 32000, Israel
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A possible mechanism of superdiffusion of ultra-cold atoms in a one-dimensional polarization optical lattice, observed experimentally in [Phys. Rev. Lett. 108, 093002 (2012)], is suggested. The analysis is based on a consideration of anomalous diffusion in a fractal comb [Phys. Rev. E 83, 052106 (2011)]. It is shown that the transport exponent is determined by the fractal geometry of the comb due to recoil distributions resulting in Lévy flights of atoms.

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Recently, 1D heavy-tailed distributions were observed in experimental studies of anomalous diffusion of ultracold $^{87}$Rb atoms in a one-dimensional optical lattice [1]. It was found that the initial ensemble of atoms spread superdiffusively, such that the “full widths at half the maximum” (FWHM) increases with time like $t^{\frac{\mu}{2}}$ with diffusion exponent $1 < \mu < 2$. Another important observation was the dependence of the transport exponent on the depth of the lattice potential [1]. The theoretical explanation of this fact, presented within the standard semiclassical treatment of Sisyphus cooling [2], is based on a study of the microscopic characteristics of the atomic motion in optical lattices and recoil distributions resulting in macroscopic Lévy flights in space, such that the Lévy distribution of the flights $g(l)$ depends on the lattice potential depth [3]: $g(l) \sim l^{-1-\nu}$ with $\nu = \frac{1}{\mu} - 1$, where $D = c E_R / U_{\text{trap}}$, while $U_{\text{trap}}$ is the lattice potential depth scaled by recoil energy $E_R$, and $c$ is a dimensionless parameter (adopting notation from Ref. [2]). A relation between the diffusion exponent $\mu$ and the Lévy distribution exponent $\nu$ was established for different regimes of the atomic dynamics, which is described by Fokker-Planck dynamics in an asymptotically logarithmic potential [2, 4].

Here, a relation between the transport exponent and the lattice potential depth is established in the framework of a fractional comb model [5]. We analyze an experimental observation of dependence of a diffusion exponent as a function of the lattice depth $U_{\text{trap}}$, presented in Fig. 3 of Ref. [1] (see here Fig. 1), where the limiting cases correspond to normal diffusion and ballistic motion, and these can be easily modelled by turbulent diffusion on a comb. This anomalous diffusion is described by the 2D distribution function $P = P(x, y, t)$, and a special behavior is that the displacement in the $x$-direction is possible only along the structure axis ($x$-axis at $y = 0$), while any spreading along the $x$ direction inside the fingers (motion along the $y$ direction) is not possible. The Fokker-Planck equation in some dimensionless variables reads (see e.g., [5])

$$\partial_t P = \delta(y) \partial_{x|} P + d \partial_y^n P,$$

where $\partial_{x|}$ is a particular case of the Riesz derivative, which can be defined by its Fourier image (see e.g., [6])

$$\hat{F}[\partial_{x|}^q f(x)] = -|k|^q \hat{f}(k)$$

and $\hat{F}[f(x)] = \hat{f}(k)$ with $q = 1$ in Eq. (1). It should be admitted that the $y$ axis is the auxiliary space, introduced for the trap modelling. In other words, it is introduced to model a non-Markovian process by means of Markovian description. The true distribution is the distribution function along the $x$ axis, which is

$$\overline{P}(x, t) = \int_{-\infty}^{\infty} P(x, y, t) dy.$$

To establish the connection between turbulent diffusion on the comb with anomalous diffusion of cold atoms, we, first, model the limiting cases, shown in Fig.1. An effective constant diffusion coefficient in the $y$ direction
can be considered as a function of the lattice depth \( d = d(U_{\text{trap}}) \). Therefore, Eq. (11) describes the two limiting cases (dashed lines in Fig. 1), where we use that \( d = \Theta(U_{\text{trap}}) \) is the Heaviside function. One easily checks [3] the full width at half maximum (FWHM) of the atom distribution. For \( U_{\text{trap}} = 0 \) one has FWHM = \( \sqrt{(\Delta x)^2} = t \) that corresponds to ballistic motion, while for \( d = 1 \) one finds normal diffusion with FWHM = \( t^{1/2} \). Here we use the definition

\[
\langle x^2(t) \rangle = \int_{-\infty}^{\infty} x^2 \mathcal{P}(x,t) dx . \tag{4}
\]

For the “shallow” trap lattice potentials, we modify Eq. (11) in the following way. From the experimental realization we know that after photon emission due to recoil atoms “fly” on distances distributed by power law and, correspondingly the fingers, as the traps, are distributed by power law with the fractal dimension \( \nu \), related to the Lévy flights. Therefore, this experimental realization can be described by the fractal comb model, developed in [3]. Following this consideration, one can consider this set of the traps as a fractal set \( F_\nu(x) \) of the atom distribution. For \( U_{\text{trap}} = 0 \) one finds normal diffusion with \( \nu = 1 \) one finds normal diffusion with \( \nu = 1 \). Therefore, the effective diffusion coefficient becomes inhomogeneous \( d \to d\chi(x) \), where \( \chi(x) \) is a characteristic function of \( F_\nu(x) \), such that \( \chi(x) = 1 \) for \( x \in F_\nu(x) \) and \( \chi(x) = 0 \) for \( x \notin F_\nu(x) \). Taking this into account, we modify Eq. (11) in the form

\[
\partial_t P = \delta(y) \partial_x^2 P + d\chi(x) \partial_y^2 P . \tag{5}
\]

To arrive at the corresponding modification of Eq. (11), we apply the Fourier transform to Eq. (5) with respect to the \( x \) coordinate. To this end we use the auxiliary identity

\[
\chi(x)f(x) \equiv \partial_x \int_{-\infty}^{x} \chi(y)f(y)dy \equiv -\partial_x \int_{x}^{\infty} \chi(y)f(y)dy
\]

with the boundary conditions \( P(x = \pm \infty) = 0 \). This integration with the characteristic function can be carried out by means of a convolution [3] [10] [11]

\[
\int_{-\infty}^{x} \chi(y)f(y)dy \Rightarrow -\infty \Gamma(\nu)^2 f(x) = \int_{-\infty}^{x} f(y)(x-y)^{\nu-1}dy / \Gamma(\nu)
\]

where \( \Gamma(\nu) \) is the Gamma function and we also use the convenient notations of fractional integration: \( -\infty \Gamma(\nu)^2 f(x) \) [3] [10] [11]. We also used here the following arguments for the characteristic function. Note that

\[
\int_{-\infty}^{\infty} \chi(y)f(y)dy = \sum_{x_j \in F_\alpha} \int_{-\infty}^{\infty} f(y)\delta(y-x_j)dy ,
\]

where

\[
\sum_{x_j \in F_\alpha} \delta(y-x_j) = \mathcal{M}(x) \sim |x|^\alpha - 1
\]

is a local fractal density, such that \( \int_{-\infty}^{\infty} d\mathcal{M}(y) \sim |x|^\alpha \) corresponds to the fractal volume. Therefore, due to Theorem 3.1 in Ref. 9 we have \( \int_{0}^{\infty} f(y)d\mathcal{M}(y) \sim 1 / (\alpha \Gamma) \int_{0}^{\infty} (x-y)^{\alpha-1} f(y)dy \).

Using this fractional integration one obtains from Eq. (5) that the fractional derivative \( \partial_t \Gamma(\nu)^2 P(x,t) \) defined in the Riemann-Liouville form [6] [10] [11]. Performing the Fourier transform and taking the symmetrical form, one obtains the following change in Eq. (11) to \( \partial_t \Gamma(\nu)^2 \). Note that \( \nu = \nu(U_{\text{trap}}) \) is the fractal dimension of the fingers distribution on the \( x \) axis. After this change Eq. (11) reads

\[
\partial_t \hat{P} = -\delta(y)|k|\hat{P} + d|k|^{2-\nu} \partial_x^2 \hat{P} ,
\]

where \( \nu = 1 + \nu(U_{\text{trap}}) \) corresponds to the Lévy distribution related to the lattice potential depth [3]. To satisfy the limiting cases, we have \( \nu = 2 \) and \( d = \Theta(U_{\text{trap}}) \) is the Heaviside function.

Our aim now is to find \( \mathcal{P}(x,t) \), defined in Eq. (4) for “shallow” trap lattice potentials, when \( 1 < \nu < 2 \) \((\nu < D < \nu)\). To this end, an analysis developed in [3] is applied. One carries out the Laplace transform in the time domain \( \hat{\mathcal{L}}[\hat{P}(k,y,t)] = \hat{P}(k,y,s) \). Looking for the solution of the Laplace image in the form

\[
\hat{P}(k,y,s) = \exp[-|y|/\sqrt{k}|^{\nu-2}s/d]f(k,s) ,
\]

one arrives at the intermediate expression in the form of the Laplace and Fourier inversions

\[
P(x,y,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{E}_\alpha(z) \left( -\frac{1}{2} \sqrt{|k|yt/d} \right) dk . \tag{11}
\]

Integration over \( y \) and the inverse Laplace transform yield a solution in the Fourier inversion form:

\[
\mathcal{P}(x,t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{ikx} \hat{E}_\alpha(z) \left( -\frac{1}{2} \sqrt{|k|yt/d} \right) dk . \tag{11}
\]

Here

\[
\hat{E}_\alpha(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{u\alpha-1}e^u du / u^\alpha + z
\]

is the Mittag-Leffler function defined by the inverse Laplace transform with a corresponding deformation of the contour of the integration [12]. First, we admit the scaling variable \( x/t^\nu \) that corresponds to the superdiffusion expansion

\[
\text{FWHM} \sim t^{1/2} .
\]

This also corresponds to the experimental observations, presented in Ref. 3, and to the scaling obtained in Ref. 2 (see Eq. (11) there).

Let us consider an asymptotic behavior of this superdiffusion expansion of the initial packet of ultra-cold
atoms at \( x \to \infty (k \to 0) \). In this case, the argument of the Mittag-Leffler function is small, yielding \( E_{\alpha}(z) \sim \exp \left( -\frac{z}{\Gamma(1+\alpha)} \right) \), and one obtains

\[
P(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \exp \left( -\frac{\sqrt{|k|^\nu t}}{3\sqrt{d\pi}} \right) dk.
\]

This is nothing but the Fourier inversion of the characteristic function of a centered and symmetric Lévy distribution \(^{10,13}\) that describes Lévy flights of atoms. This integration yields the analytical solution in the form of the Fox function with the power-law asymptotics \(^{10,13}\)

\[
P(x, t) \sim \frac{t}{|x|^{1+\nu}}, \quad \nu < 2.
\]

From here one obtains for the first moment \(^{10,13}\)

\[
\langle x(t) \rangle \sim t^\frac{\nu}{\alpha},
\]

which corresponds to the scaling obtained above.

In conclusion, we note that this description is a particular case of a general scheme of the wave-packet spreading described by the fractional Fokker-Planck equation (FFPE) that can be obtained for the true distribution \(^{12}\). Integrating the Laplace image of Eq. \(^{4}\) over \( y \), and taking into account Eq. \(^{9}\), one performs the Laplace and the Fourier inversions that yields the FFPE for the true distribution

\[
\frac{\partial_t}{\partial_t} P(x, t) = \frac{1}{2d} \frac{\partial_{|x|}^\nu}{\partial |x|^\nu} P(x, t).
\]

Here \( \frac{\partial_t}{\partial_t} \) is the Caputo time fractional derivative \(^{14}\) and \( \frac{\partial_{|x|}^\nu}{\partial |x|^\nu} \) is the Riesz fractional derivative \(^{6}\) defined in Eq. \(^{2}\). This equation is a particular case of the FFPE

\[
\frac{\partial_t}{\partial_t} P = K(\alpha, d) \frac{\partial_{|x|}^\nu}{\partial |x|^\nu} P,
\]

where \( K(\alpha, d) \) is a generalized diffusion coefficient and the scaling \( q/\alpha = \nu \) is fulfilled for the initial ensemble of atoms, which spreads superdiffusively with \( \alpha < 1 \) and \( q < 2 \).

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