Nonsymmetric Generalized Jacobi Petrov-Galerkin Algorithms for Third- and Fifth-Order two Point Boundary Value Problems

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Abstract. Two families of certain nonsymmetric generalized Jacobi polynomials with negative integer indexes are used for solving third- and fifth-order two point boundary value problems subject to homogeneous and nonhomogeneous boundary conditions using a dual Petrov-Galerkin method. The key idea behind our method is to use trial functions satisfying the underlying boundary conditions of the differential equations and the test functions satisfying the dual boundary conditions. The method leads to linear systems with specially structured matrices that can be efficiently inverted. The use of generalized Jacobi polynomials leads to simplified analysis, very efficient numerical algorithms. Numerical results are presented to demonstrate the efficiency of our proposed algorithms.

Keywords: Dual-Petrov-Galerkin method, generalized Jacobi polynomials, nonhomogeneous Dirichlet conditions

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1 Introduction

Spectral method, in the context of numerical schemes, was introduced and popularized by Orszag’s pioneer work in the early seventies. The term spectral was probably originated from the fact that the trigonometric functions $e^{ikx}$ are the eigenfunctions of the Laplace operator with periodic boundary conditions. This fact and the availability of Fast Fourier Transform (FFT) are two main advantages of the Fourier spectral method. Thus, using Fourier series to solve PDEs, with principal differential operator being the Laplace operator (or its power) with periodic boundary conditions, results in very attractive numerical algorithms. However, for problems with rigid boundaries, the eigenfunctions of Laplace operator (with non-periodic boundary conditions), although easily available in regular domains, are no longer good candidates as basis functions due to the Gibbs phenomenon. In such cases, it is well known that one should use the eigenfunctions of the singular Sturm-Liouville operator i.e., Jacobi polynomials with a suitable pair of indexes.

Standard spectral methods are capable of providing very accurate approximations to well-behaved smooth functions with significantly less degrees of freedom when compared with finite difference or finite element methods (cf. [7], [8], [19]).

Classical orthogonal polynomials are used successfully and extensively for the numerical solution of differential equations in spectral and pseudospectral methods (see, for instance, [6], [8, 9], [13], [22] and [23]).

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ play important roles in mathematical analysis and its applications (see [30]). In particular, the Legendre, the Chebyshev, the ultraspherical
polynomials have played important roles in spectral methods for partial differential equations (see, for instance, [7], [18]). It is proven that the Jacobi polynomials are precisely the only polynomials arising as eigenfunctions of a singular Sturm-Liouville problem, (see [8], Sec. 9.2). This class of polynomials comprises all the polynomial solution to singular Sturm-Liouville problems on $[-1,1]$.

Guo et al. [21] extended the definition of the classical Jacobi polynomials with indexes $\alpha, \beta > -1$ to allow $\alpha$ and/or $\beta$ to be negative integers. They showed also that the generalized Jacobi polynomials, with indexes corresponding to the number of boundary conditions in a given partial differential equation, are the natural basis functions for the spectral approximation of this equation. Moreover it is shown that the use of generalized Jacobi polynomials not only simplified the numerical analysis for the spectral approximations of differential equations, but also led to very efficient numerical algorithms.

Abd-Elhameed et al. [2] and Doha et al. [17] used the general parameter generalized Jacobi polynomials to handle third- and fifth-order differential equations.

The majority of books and research papers dealing with the theory of ordinary differential equations, or their practical applications to technology and physics, contain mainly results from the theory of second-order linear differential equations, and some results from the theory of some special linear differential equations of higher even order. However there is only a limited body of literature on spectral methods for dispersive, namely, third- and fifth-order equations. This is partly due to the fact that direct collocation methods for third- and fifth-order boundary problems lead to condition numbers of high order, typically of order $N^6$ and $N^{10}$ respectively, where $N$ is the number of retained modes. These high condition numbers will lead to instabilities caused by rounding errors (see, [24] and [28]). In this paper, we introduce some efficient spectral algorithms for reducing these condition numbers to be of $O(N^2)$ and $O(N^4)$ for third- and fifth-order respectively, based on certain nonsymmetric generalized Jacobi Petrov-Galerkin method.

The study of odd-order equations is of interest, for example, the third order equation is of fundamental mathematical interest since it lacks symmetry. Also, it is of physical interest since it contains a type of operator which appears in many commonly occurring partial differential equations such as the Kortweg-de Vries equation. Monographs like those of Mckelvey [27], which include chapters on oscillation properties of third-order differential equations, are exceptional. The interested reader in applications of odd-order differential equations is referred to the monograph by (Gregus [20]), in which many physical and engineering applications of third-order differential equations are discussed [see, pp. 247-258].

From the numerical point of view, Abd-Elhameed [1], Doha and Abd-Elhameed [12, 14], Doha and Bhrawy [3] and Doha et al. [16] have constructed efficient spectral-Galerkin algorithms using compact combinations of orthogonal polynomials for solving elliptic equations of the second-, fourth-, $2n$th- and $(2n+1)$th-order in various situations. Recently, Doha and Abd-Elhameed [15] have introduced a family of symmetric generalized Jacobi polynomials for solving multidimensional sixth-order two point boundary value problems by the Galerkin method. Also some other studies are devoted to third- and fifth-order differential equations in finite intervals (see, [25, 26]).

The main differential operator in odd-order differential equations is not symmetric, so it is convenient to use a Petrov-Galerkin method. The difference between Galerkin and Petrov-Galerkin methods, is that the test and trial functions in Galerkin method are the same, but for Petrov-Galerkin method, the trial functions are chosen to satisfy the boundary conditions of the differential equation, and the test functions are chosen to satisfy the dual
boundary conditions.

In this paper we are concerned with the direct solution techniques for third- and fifth-order elliptic equations, using the generalized Jacobi Petrov-Galerkin method (GJPGM). Our algorithms lead to discrete linear systems with specially structured matrices that can be efficiently inverted.

We organize the materials of this paper as follows. In Section 2, we give some properties of classical and generalized Jacobi polynomials. In Sections 3 and 4, we are interested in using GJPGM to solve third- and fifth-order linear differential equation with constant coefficients subject to homogenous boundary conditions. In Section 5, we study the structure of the coefficient matrices in the systems resulted from applying GJPGM. In Section 6, we are interested in using GJPGM to solve third- and fifth-order linear differential equation with constant coefficients subject to nonhomogeneous boundary conditions. In Section 7, the condition numbers of the systems resulted from applying GJPGM are discussed. In Section 8, we discuss some numerical results. Some concluding remarks are given in Section 9.

2 Some properties of Classical and generalized Jacobi polynomials

2.1 Classical Jacobi polynomials

The classical Jacobi polynomials associated with the real parameters $\alpha > -1, \beta > -1$ (see, [3], [4] and [30]), are a sequence of polynomials $P_n^{(\alpha, \beta)}(x), x \in (-1, 1)(n = 0, 1, 2, ...)$, each respectively of degree $n$. For our present purposes, it is more convenient to introduce the normalized orthogonal polynomials $R_n^{(\alpha, \beta)}(x) = P_n^{(\alpha, \beta)}(x)/P_n^{(\alpha, \beta)}(1)$. This means that $R_n^{(\alpha, \beta)}(x) = n! \Gamma(\alpha + 1) \Gamma(n + \alpha + 1)/\Gamma(n + \alpha + 1) P_n^{(\alpha, \beta)}(x)$. In such case $R_n^{(-\frac{1}{2}, -\frac{1}{2})}(x)$ is identical to the ultraspherical polynomials $C_n^{(\alpha)}(x)$, and the polynomials $R_n^{(\alpha, \beta)}(x)$ may be generated using the recurrence relation

$$2(n + \lambda)(n + \alpha + 1)(2n + \lambda - 1)R_{n+1}^{(\alpha, \beta)}(x) = (2n + \lambda - 1)\frac{1}{3} x R_n^{(\alpha, \beta)}(x)$$

$$+ (\alpha^2 - \beta^2)(2n + \lambda)R_n^{(\alpha, \beta)}(x) - 2n(n + \beta)(2n + \lambda + 1)R_{n-1}^{(\alpha, \beta)}(x), \quad n = 1, 2, \ldots,$$

starting from $R_0^{(\alpha, \beta)}(x) = 1$ and $R_1^{(\alpha, \beta)}(x) = 1$, or obtained from Rodrigue’s formula

$$R_n^{(\alpha, \beta)}(x) = \left(-\frac{1}{2}\right)^n \frac{n! \Gamma(\alpha + 1) \Gamma(n + \alpha + 1)}{\Gamma(n + \alpha + 1)} (1 - x)^{-\alpha} (1 + x)^{-\beta} D^n \left[(1 - x)^{\alpha + n} (1 + x)^{\beta + n}\right],$$

where

$$\lambda = \alpha + \beta + 1, \quad (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)}, \quad D = \frac{d}{dx},$$

and satisfy the orthogonality relation

$$\int_{-1}^{1} (1 - x)^{\alpha} (1 + x)^{\beta} R_n^{(\alpha, \beta)}(x) R_m^{(\alpha, \beta)}(x) \, dx = \begin{cases} 0, & m \neq n, \\ h_n^{(\alpha, \beta)}, & m = n, \end{cases}$$

(1)
Theorem 1. The qth derivative of the normalized Jacobi polynomial \( R_n^{(\alpha,\beta)}(x) \) is given explicitly by

\[
D^q R_n^{(\alpha,\beta)}(x) = (n + \lambda)_q 2^{-q} n! \sum_{i=0}^{n-q} C_{n-q,i}(\alpha + q, \beta + q, \alpha, \beta) R_i^{(\alpha,\beta)}(x),
\]

where

\[
C_{n-q,i}(\alpha + q, \beta + q, \alpha, \beta) = \frac{(n + q + \lambda)_i (i + q + \alpha + 1)_{n-i-q} \Gamma(i + \lambda)}{(n - i - q)! \Gamma(2i + \lambda) i! (i + \alpha + 1)_{n-i}} \times {}_3F_2 \left( \begin{array}{c} -n + q + i, \ n + i + q + \lambda, \ i + \alpha + 1 \\ i + q + \alpha + 1, \ 2i + \lambda + 1 \end{array}; 1 \right).
\]

(For the proof of Theorem 1, see Doha [10]).

2.2 Generalized Jacobi polynomials

Following [21], we can define a family of generalized Jacobi polynomials/functions with indexes \( \alpha, \beta \in \mathbb{R} \).

Let \( w^{\alpha,\beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \). We denote by \( L^2_{w^{\alpha,\beta}}(-1, 1) \) the weighted \( L^2 \) space with inner product:

\[
(u,v)_{w^{\alpha,\beta}}(x) := \int_I u(x)v(x)w^{\alpha,\beta}(x) \, dx,
\]
and the associated norm \( \|u\|_{w^{\alpha,\beta}} = (u, u)^{\frac{1}{2}}_{w^{\alpha,\beta}} \). We are interested in defining Jacobi polynomials with indexes \( \alpha \) and/or \( \beta \leq -1 \), referred hereafter as generalized Jacobi polynomials (GJPs), in such a way that they satisfy some selected properties that are essentially relevant to spectral approximations. In this work, we shall restrict our attention to the cases when \( \alpha \) and \( \beta \) are negative integers.

Let \( \ell, m \in \mathbb{Z} \) (the set of all integers),

\[
J^{(\ell,m)}_k(x) = \begin{cases} 
(1-x)^{-\ell}(1+x)^{-m} R_{k-k_0}^{(-\ell,-m)}(x), & k_0 = -(\ell + m), \ell, m \leq -1, \\
(1-x)^{-\ell} R_{k-k_0}^{(-\ell,-m)}(x), & k_0 = -\ell, \ell \leq -1, m > -1, \\
(1+x)^{-m} R_{k-k_0}^{(-\ell,-m)}(x), & k_0 = -m, \ell > -1, m \leq -1, \\
R_{k-k_0}^{(\ell,m)}(x), & k_0 = 0, \ell, m > -1.
\end{cases}
\]

An important property of the GJPs is that for \( \ell, m, \in \mathbb{Z} \) and \( \ell, m \geq 1 \),

\[
D^i J^{(-\ell,-m)}_k(1) = 0, \quad i = 0, 1, \ldots, \ell - 1; \\
D^j J^{(-\ell,-m)}_k(-1) = 0, \quad j = 0, 1, \ldots, m - 1.
\]

It is not difficult to verify that

\[
J^{(-2,-1)}_k(x) = \frac{4}{(k-1)(2k-3)} \left[ L_{k-3}(x) - \frac{2k-3}{2k-1} L_{k-2}(x) - L_{k-1}(x) + \frac{2k-3}{2k-1} L_k(x) \right], \\
J^{(-1,-2)}_k(x) = \frac{2}{2k-3} \left[ L_{k-3}(x) + \frac{2k-3}{2k-1} L_{k-2}(x) - L_{k-1}(x) - \frac{2k-3}{2k-1} L_k(x) \right], \\
J^{(-3,-2)}_k(x) = \frac{24}{(2k-5)(2k-7)(k-2)} \left[ L_{k-5}(x) - \frac{2k-7}{2k-3} L_{k-4}(x) - \frac{2(2k-5)}{2k-3} L_{k-3}(x) + \frac{2(2k-7)}{2k-1} L_{k-2}(x) + \frac{2k-7}{2k-3} L_{k-1}(x) - \frac{(2k-5)(2k-7)}{(2k-1)(2k-3)} L_k(x) \right], \quad k \geq 5,
\]

\[
J^{(-2,-3)}_k(x) = \frac{8}{(2k-5)(2k-7)} \left[ L_{k-5}(x) + \frac{2k-7}{2k-3} L_{k-4}(x) - \frac{2(2k-5)}{2k-3} L_{k-3}(x) - \frac{2(2k-7)}{2k-1} L_{k-2}(x) + \frac{2k-7}{2k-3} L_{k-1}(x) + \frac{(2k-5)(2k-7)}{(2k-1)(2k-3)} L_k(x) \right], \quad k \geq 5,
\]

where \( L_k(x) \) is the Legendre polynomial of the \( k \)th degree. \( \{J^{(-\ell,-m)}_k(x)\} \) are natural candidates as basis functions for PDFs with the following boundary conditions:

\[
D^i u(1) = a_i, \quad i = 0, 1, \ldots, \ell - 1; \\
D^j u(-1) = b_j, \quad j = 0, 1, \ldots, m - 1.
\]
3 Dual Petrov-Galerkin algorithms for third-order elliptic linear differential equations

We are interested in using the generalized Jacobi-Petrov-Galerkin method to solve the following third-order elliptic linear differential equation

\[ u^{(3)}(x) - \alpha_1 u^{(2)}(x) - \beta_1 u^{(1)}(x) + \gamma_1 u(x) = f(x), \quad x \in (-1, 1), \tag{7} \]

subject to the homogeneous boundary conditions

\[ u(\pm 1) = u^{(1)}(1) = 0. \tag{8} \]

We define the space

\[ V = \{ u \in H^{(2)}(I) : u(\pm 1) = u^{(1)}(1) = 0 \}, \]

and its dual space

\[ V^* = \{ u \in H^{(2)}(I) : u(\pm 1) = u^{(1)}(-1) = 0 \}. \]

where

\[ H^{(2)}(I) = \{ u : \|u\|_{2,\alpha,\beta} < \infty, \|u\|_{2,\alpha,\beta} = \left( \sum_{k=0}^{2} \|\partial_x^k u\|^2_{w^k,k+k} \right)^{\frac{1}{2}} \}. \]

Let \( P_N \) be the space of all polynomials of degree less than or equal to \( N \). Setting \( V_N = V \cap P_N \) and \( V_N^* = V^* \cap P_N \). We observe that:

\[ V_N = \text{span}\{ J_3^{(-2,-1)}(x), J_1^{(-2,-1)}(x), ..., J_N^{(-2,-1)}(x) \}, \]

\[ V_N^* = \text{span}\{ J_3^{(-1,-2)}(x), J_1^{(-1,-2)}(x), ..., J_N^{(-1,-2)}(x) \}. \]

The dual Petrov-Galerkin approximation of (7)-(8) is to find \( u_N \in V_N \) such that

\[
\begin{align*}
(D^3 u_N(x), v(x)) - \alpha_1 (D^2 u_N(x), v(x)) - \beta_1 (Du_N(x), v(x)) \\
+ \gamma_1 (u_N(x), v(x)) &= (f(x), v(x)), \quad \forall v \in V_N^*. \tag{9}
\end{align*}
\]

3.1 The choice of basis functions

We can construct suitable basis functions and their dual basis by setting

\[ \phi_k(x) = J_{k+3}^{(-2,-1)}(x) = (1 - x^2)(1 - x) R_{k+1}^{(2)}(x), \quad k = 0, 1, \ldots, N - 3, \]

\[ \psi_k(x) = J_{k+3}^{(-1,-2)}(x) = (1 - x^2)(1 + x) R_{k+1}^{(1,2)}(x), \quad k = 0, 1, \ldots, N - 3. \]

It is obvious that \( \{\phi_k(x)\} \) and \( \{\psi_k(x)\} \) are linearly independent. Therefore we have

\[ V_N = \text{span}\{ \phi_k(x) : k = 0, 1, 2, \ldots, N - 3 \}, \]

and

\[ V_N^* = \text{span}\{ \psi_k(x) : k = 0, 1, 2, \ldots, N - 3 \}. \]

Now we state and prove the following two lemmas.

Lemma 1.

\[ D^3 J_{k+3}^{(-2,-1)}(x) = 2(k + 1)(k + 3) R_{k+1}^{(1,2)}(x). \tag{10} \]
Finally, with the aid of the two relations (2) and (3), and after some manipulation, we get

\[ D^3 J_{k+3}^{(-2,-1)}(x) = (1 - x^2)(1 - x)D^3 R_k^{(2,1)}(x) + 3(3x^2 - 2x - 1)D^2 R_k^{(2,1)}(x) + 6(-1 + 3x)DR_k^{(2,1)}(x) + 6R_k^{(2,1)}(x). \]

Making use of the relation

\[ (1 - x^2)(1 - x)D^3 R_k^{(2,1)}(x) = (1 + 6x - 7x^2)D^2 R_k^{(2,1)}(x) + (k - 1)(k + 5)(x - 1)DR_k^{(2,1)}(x), \]

we obtain

\[ D^3 J_{k+3}^{(-2,-1)}(x) = 2(x^2 - 1)D^2 R_k^{(2,1)}(x) + [(k - 1)(k + 5)(x - 1) + 6(3x - 1)]DR_k^{(2,1)}(x) + 6R_k^{(2,1)}(x), \]

which in turn with equation (2), and after some manipulation, yields

\[ D^3 J_{k+3}^{(-2,-1)}(x) = -(k + 1)(k + 3)\left[(1 - x)DR_k^{(2,1)}(x) - 2R_k^{(2,1)}(x)\right]. \]

Making use of the two relations (4) and (6), we have

\[ D^3 J_{k+3}^{(-2,-1)}(x) = \frac{1}{6}(k + 1)(k + 3)k(k - 4)R_{k-1}^{(3,2)}(x) + 12R_k^{(2,1)}(x). \]

Finally, with the aid of the two relations (2) and (3), and after some manipulation, we get

\[ D^3 J_{k+3}^{(-2,-1)}(x) = 2(k + 1)(k + 3)R_k^{(1,2)}(x). \]

\[ \square \]

Lemma 2.

\[ D^2 J_{k+3}^{(-2,-1)}(x) = \frac{2(k + 3)}{(2k + 5)}R_{k+1}^{(2,1)}(x) - \frac{(k + 1)(k + 3)}{(k + \frac{3}{2})^2}R_k^{(2,1)}(x) \]

\[ + \frac{2(k^2)}{(2k + 5)}R_{k-1}^{(1,2)}(x), \]

\[ DJ_{k+3}^{(-2,-1)}(x) = \frac{(k + 3)}{2(k + 2)(k + \frac{3}{2})}R_{k+2}^{(1,2)}(x) - \frac{(k + 3)}{(k + \frac{3}{2})^2}R_{k+1}^{(2,1)}(x) - \frac{(k + 1)(k + 3)}{(k + \frac{3}{2})^2}R_k^{(1,2)}(x) \]

\[ + \frac{(k^2)}{(k + \frac{3}{2})^2}R_{k-1}^{(1,2)}(x) - \frac{(k - 1)}{2(k + 2)(k + \frac{3}{2})}R_{k-2}^{(1,2)}(x), \]

\[ J_{k+3}^{(-2,-1)}(x) = \frac{(k + 4)}{4(k + 2)(k + \frac{3}{2})}R_{k+3}^{(1,2)}(x) - \frac{3(k + 3)}{4(k + 2)(k + \frac{3}{2})}R_{k+2}^{(1,2)}(x) - \frac{3(k + 3)}{4(k + \frac{3}{2})^3}R_{k+1}^{(1,2)}(x) \]

\[ + \frac{3(k + 1)(k + 3)}{2(k + \frac{3}{2})^4}R_{k+1}^{(1,2)}(x) + \frac{3(k + 2)}{4(k + \frac{3}{2})^3}R_{k+2}^{(1,2)}(x) \]

\[ - \frac{3(k - 1)}{4(k + 2)(k - \frac{3}{2})^4}R_{k+2}^{(1,2)}(x) - \frac{(k - 2)}{4(k + 2)(k - \frac{3}{2})^3}R_{k+1}^{(1,2)}(x). \]

\[ \square \]

Proof. The proof of Lemma [2] is rather lengthy and it can be accomplished by following the same procedure used in the proof of Lemma [1] \[ \square \]

Now, based on the two Lemmas [1] and [2] the following theorem can be obtained.
Theorem 2. We have, for arbitrary constants $a_k$,

$$D^3 \left[ \sum_{k=0}^{N-3} a_k J_{k+3}^{(-2,-1)}(x) \right] = \sum_{k=0}^{N-3} b_k R_k^{(1,2)}(x),$$  \hspace{1cm} (11)

where

$$b_k = 2(k+1)(k+3)a_k.$$  \hspace{1cm} (12)

Moreover, if

$$D^2 \left[ \sum_{k=0}^{N-3} a_k J_{k+3}^{(-2,-1)}(x) \right] = \sum_{k=0}^{N-2} e_{k,2} R_k^{(1,2)}(x),$$  \hspace{1cm} (13)

then

$$e_{k,2} = a_{k-1} \alpha_{k-1}^{(2)} + a_k \beta_k^{(2)} + a_{k+1} \gamma_{k+1}^{(2)},$$  \hspace{1cm} (14)

where

$$\alpha_k^{(2)} = \frac{2(k+3)_2}{2(k+5)}, \quad \beta_k^{(2)} = -\frac{(k+1)(k+3)}{(k+\frac{5}{2})_2}, \quad \gamma_k^{(2)} = -\frac{2(k)_2}{(2k+3)}.$$  

Also, if

$$D \left[ \sum_{k=0}^{N-3} a_k J_{k+3}^{(-2,-1)}(x) \right] = \sum_{k=0}^{N-1} e_{k,1} R_k^{(1,2)}(x),$$  \hspace{1cm} (15)

then

$$e_{k,1} = a_{k-2} \alpha_{k-2}^{(1)} + a_{k-1} \beta_{k-1}^{(1)} + a_k \gamma_k^{(1)} + a_{k+1} \delta_{k+1}^{(1)} + a_{k+2} \mu_{k+2},$$  \hspace{1cm} (16)

where

$$\alpha_k^{(1)} = \frac{(k+3)_3}{2(k+2)(k+\frac{5}{2})_2}, \quad \beta_k^{(1)} = -\frac{(k+3)_2}{(k+\frac{3}{2})_3}, \quad \gamma_k^{(1)} = -\frac{(k+1)(k+3)}{(k+\frac{3}{2})_2},$$

$$\delta_k^{(1)} = \frac{(k)_2}{(k+\frac{5}{2})_3}, \quad \mu_k^{(1)} = \frac{(k-1)_3}{2(k+2)(k+\frac{5}{2})_2}.$$  

Finally, if

$$\sum_{k=0}^{N-3} a_k J_{k+3}^{(-2,-1)}(x) = \sum_{k=0}^{N} e_{k,0} R_k^{(1,2)}(x),$$  \hspace{1cm} (17)

then

$$e_{k,0} = a_{k-3} \alpha_{k-3}^{(0)} + a_{k-2} \beta_{k-2}^{(0)} + a_{k-1} \gamma_{k-1}^{(0)} + a_k \delta_k^{(0)} + a_{k+1} \mu_{k+1}^{(0)} + a_{k+2} \eta_{k+2}^{(0)} + a_{k+3} \zeta_{k+3}^{(0)},$$  \hspace{1cm} (18)

where

$$\alpha_k^{(0)} = \frac{(k+4)_3}{4(k+2)(k+\frac{5}{2})_3}, \quad \beta_k^{(0)} = -\frac{3(k+3)_3}{4(k+2)(k+\frac{3}{2})_4}, \quad \gamma_k^{(0)} = -\frac{3(k+3)_2}{4(k+\frac{3}{2})_3},$$

$$\delta_k^{(0)} = -\frac{3(k+1)(k+3)}{2(k+\frac{5}{2})_4}, \quad \mu_k^{(0)} = \frac{3(k)_2}{4(k+\frac{5}{2})_3}, \quad \eta_k^{(0)} = -\frac{3(k-1)_3}{4(k+2)(k-\frac{1}{2})_4},$$

$$\zeta_k^{(0)} = -\frac{(k-2)_3}{4(k+2)(k-\frac{1}{2})_3}.$$
The application of Petrov-Galerkin method to equation (7), gives

\[(D^3 u_N(x) - \alpha_1 D^2 u_N - \beta_1 D u_N + \gamma_1 u_N, \psi_k(x)) = (f(x), \psi_k(x)), \quad (19)\]

where

\[u_N(x) = \sum_{k=0}^{N-3} a_k \phi_k(x), \quad \phi_k(x) = J_k^{(-2,-1)}(x), \quad \psi_k(x) = J_k^{(-1,-2)}(x), \quad k = 0, 1, \ldots, N - 3.\]

Substitution of formulae (11), (13), (15) and (17) into (19) yields

\[\left(\sum_{j=0}^{N-3} b_j R_j^{(1,2)}(x) - \alpha_1 \sum_{j=0}^{N-2} e_{j,2} R_j^{(1,2)}(x) - \beta_1 \sum_{j=0}^{N-1} e_{j,1} R_j^{(1,2)}(x) + \gamma_1 \sum_{j=0}^{N} e_{j,0} R_j^{(1,2)}(x), J_k^{(-1,-2)}(x)\right) = \left(f, J_k^{(-1,-2)}(x)\right), \quad (20)\]

where \(b_k\) and \(e_{k,2-q}, 0 \leq q \leq 2\) are given by (12), (14), (16) and (18) respectively. Eq. (20) is equivalent to

\[\left(\sum_{j=0}^{N-3} b_j R_j^{(1,2)}(x) - \alpha_1 \sum_{j=0}^{N-2} e_{j,2} R_j^{(1,2)}(x) - \beta_1 \sum_{j=0}^{N-1} e_{j,1} R_j^{(1,2)}(x) + \gamma_1 \sum_{j=0}^{N} e_{j,0} R_j^{(1,2)}(x), R_k^{(1,2)}(x)\right) = \left(f, R_k^{(1,2)}(x)\right) \in (20),\]

where \(w = (1 - x^2)(1 + x)\). Making use of the orthogonality relation (1), it is not difficult to show that Eq. (20) is equivalent to

\[f_k = (b_k - \alpha_1 e_{k,2} - \beta_1 e_{k,1} + \gamma_1 e_{k,0}) h_k; \quad k = 0, 1, \ldots N - 3. \quad (21)\]

where

\[f_k = \left(f, R_k^{(1,2)}(x)\right) \in (20).\]

This linear system may be put in the form

\[(b_k^1 - \alpha_1 e_{k,2} - \beta_1 e_{k,1} + \gamma_1 e_{k,0}) = f_k^*, \quad k = 0, 1, \ldots N - 3, \quad (22)\]

where

\[f_k^* = \frac{f_k}{h_k^{1/2}}, \quad h_k^{1/2} = \frac{8}{(k+1)(k+2)(k+3)}.\]

which may be written in the matrix form

\[(B_1 + \alpha_1 E_2 + \beta_1 E_1 + \gamma_1 E_0) a = f^*, \quad (23)\]

where

\[a = (a_0, a_1, \ldots, a_{N-3})^T, \quad f^* = (f_0^*, f_1^*, \ldots, f_{N-3}^*)^T,\]

and the nonzero elements of the matrices \(B, E_2, E_1\) and \(E_0\) are given explicitly in the following theorem.
Theorem 3. The nonzero elements \( b_{k,k}^i \) and \( e_{k,k}^{i,j} \), \( 0 \leq i \leq 2 \) for \( 0 \leq k, j \leq N - 3 \) are given as follows:

\[
\begin{align*}
  b_{k,k}^i &= 2(k + 1)(k + 3), \\
  e_{k+1,k}^{2,1} &= -\frac{2(k + 3)(k + 4)}{2k + 5}, \\
  e_{kk}^{1,1} &= \frac{4(k + 1)(k + 3)}{(2k + 3)(2k + 5)}, \\
  e_{k,k+2}^{1,1} &= \frac{-2(k + 1)(k + 2)(k + 3)}{(k + 4)(2k + 5)(2k + 7)}, \\
  e_{k+2,k}^{1,1} &= \frac{-2(k + 3)(k + 4)(k + 5)}{(k + 2)(2k + 5)(2k + 7)}, \\
  e_{k,k+1}^{0,1} &= \frac{3(k + 1)2}{4(k + 3/2)3}, \\
  e_{k,k+3}^{0,1} &= \frac{3(k + 1)3}{4(k + 5)(k + 5/2)3}, \\
  e_{k+2,k}^{0,1} &= \frac{-3(k + 3)5}{4(k + 5/2)4},
\end{align*}
\]

We define the following two spaces

\[
V = \{ u \in H^3(I) : u(\pm 1) = u^{(1)}(\pm 1) = u^{(2)}(1) = 0 \},
\]

and

\[
V^* = \{ u \in H^3(I) : u(\pm 1) = u^{(1)}(\pm 1) = u^{(2)}(-1) = 0 \}.
\]

where

\[
H^3(I) = \{ u : \| u \|_{3,\omega,\beta} < \infty, \| u \|_{3,\omega,\beta} = \left( \sum_{k=0}^{3} \| \partial_x^k u \|_{\omega^{\alpha+k,\beta+k}}^2 \right)^{1/2} \}
\]

Let \( P_N \) be the space of all polynomials of degree less than or equal to \( N \). Setting \( V_N = V \cap P_N \) and \( V_N^* = V^* \cap P_N \). We observe that:

\[
V_N = \text{span}\{ J_5^{(-3,-2)}(x), J_6^{(-3,-2)}(x), ..., J_N^{(-3,-2)}(x) \},
\]

\[
V_N^* = \text{span}\{ J_5^{(-2,-3)}(x), J_6^{(-2,-3)}(x), ..., J_N^{(-2,-3)}(x) \}.
\]

The dual Petrov-Galerkin approximation of (24)-(25) is to find \( u_N \in V_N^* \) such that

\[
\begin{align*}
- (D^5 u_N(x), v(x)) + \alpha_2 (D^4 u_N(x), v(x)) + \beta_2 (D^3 u_N(x), v(x)) - \gamma_2 (D^2 u_N(x), v(x)) \\
- \delta_2 (D u_N(x), v(x)) + \mu_2 (u_N(x), v(x)) = (f(x), v(x)), \quad \forall v \in V_N^*.
\end{align*}
\]
4.1 The choice of basis functions

We can construct suitable basis functions and their dual basis by setting

\[ \phi_k(x) = J_{k+5}^{(-3,-2)}(x) = (1 - x^2)^2(1 - x) R_k^{(3,2)}(x), \quad k = 0, 1, \ldots, N - 5, \]
\[ \psi_k(x) = J_{k+5}^{(-2,-3)}(x) = (1 - x^2)^2(1 + x) R_k^{(2,3)}(x), \quad k = 0, 1, \ldots, N - 5. \]

It is obvious that \( \{\phi_k(x)\} \) and \( \{\psi_k(x)\} \) are linearly independent. Therefore we have

\[ V_N = \text{span}\{\phi_k(x) : k = 0, 1, 2, \ldots, N - 5\}, \]
and

\[ V_N^* = \text{span}\{\psi_k(x) : k = 0, 1, 2, \ldots, N - 5\}. \]

The following two lemmas are needed.

Lemma 3.

\[ D^5 J_{k+5}^{(-3,-2)}(x) = -3(k + 1)(k + 2)(k + 4)(k + 5) R_k^{(2,3)}(x). \]

Proof. Setting \( \alpha = 2, \beta = 1 \) in relation (5), we get

\[ (1 - x^2) R_k^{(3,2)}(x) = \frac{12}{(2k + 5)^3} \left[ (k + 2)(2k + 7) R_k^{(2,1)}(x) + 2(k + 3) R_{k+1}^{(2,1)}(x) \right. \]
\[ \quad - (k + 4)(2k + 5) R_{k+2}^{(2,1)}(x) \right]. \]

Making use of this relation and with the aid of the two relations (6) (for \( q = 2 \)) and (10), we obtain

\[ D^5 J_{k+5}^{(-3,-2)}(x) = \frac{1}{(2k + 5)^3} \left[ (2k + 7)(k - 1) R_{k-2}^{(3,4)}(x) + 2(k + 3) R_{k-1}^{(3,4)}(x) \right. \]
\[ \quad - (2k + 5)(k + 1) R_k^{(3,4)}(x) \right]. \]

Finally, from the two relations (2) and (3), and after some lengthy manipulation, we get

\[ D^5 J_{k+5}^{(-3,-2)}(x) = -3(k + 1)(k + 2)(k + 4)(k + 5) R_k^{(2,3)}(x). \]

Lemma 4.

\[ D^4 J_{k+5}^{(-3,-2)}(x) = -3(k + 2)(k + 4) \frac{3}{2k + 7} R_{k+1}^{(2,3)}(x) + \frac{3(k + 1) R_k^{(2,3)}(x)}{2(k + \frac{5}{2})} \]
\[ + \frac{3(k) R_{k-1}^{(2,3)}(x)}{2(k + 5)}, \]
\[ D^3 J_{k+5}^{(-3,-2)}(x) = -3(k + 4) \frac{3}{4(k + \frac{5}{2})} R_{k+2}^{(2,3)}(x) + \frac{3(k + 2)(k + 4)}{2(k + \frac{5}{2})} R_{k+1}^{(2,3)}(x) \]
\[ + \frac{3(k + 1) R_k^{(2,3)}(x)}{2(k + \frac{5}{2})} - \frac{3(k + 4)}{2(k + \frac{5}{2})} R_{k-1}^{(2,3)}(x) - \frac{3(k + 1) R_k^{(2,3)}(x)}{4(k + \frac{5}{2})} \]
\[ + \frac{3(k + 1) R_{k-1}^{(2,3)}(x)}{4(k + \frac{5}{2})}. \]
\[ D^2 J_{k+5}^{(-3,-2)}(x) = -\frac{3(k+4)5}{8(k+3)(k+\frac{7}{2})_3} R_{k+3}^{(2,3)}(x) + \frac{9(k+4)4}{8(k+\frac{5}{2})_4} R_{k+2}^{(2,3)}(x) \\
+ \frac{9(k+2)(k+4)3}{8(k+\frac{5}{2})_3} R_{k+1}^{(2,3)}(x) - \frac{9(k+1)(k+4)2}{4(k+\frac{3}{2})_4} R_{k}^{(2,3)}(x) \\
- \frac{9(k+3)(k+4)}{8(k+\frac{3}{2})_3} R_{k-1}^{(2,3)}(x) + \frac{9(k-1)4}{8(k+\frac{1}{2})_4} R_{k-2}^{(2,3)}(x) + \frac{3(k-2)5}{8(k+3)(k+\frac{1}{2})_3} R_{k-3}^{(2,3)}(x), \]  

(29)

\[
D J_{k+5}^{(-3,-2)}(x) = -\frac{3(k+5)5}{16(k+3)(k+\frac{7}{2})_4} R_{k+4}^{(2,3)}(x) + \frac{3(k+4)5}{4(k+3)(k+\frac{5}{2})_5} R_{k+3}^{(2,3)}(x) \\
+ \frac{3(k+4)4}{4(k+\frac{3}{2})_4} R_{k+2}^{(2,3)}(x) - \frac{9(k+2)(k+4)3}{4(k+\frac{1}{2})_5} R_{k+1}^{(2,3)}(x) - \frac{9(k+1)(k+4)2}{8(k+\frac{1}{2})_4} R_{k}^{(2,3)}(x) \\
+ \frac{9(k+3)(k+4)}{4(k+\frac{1}{2})_5} R_{k-1}^{(2,3)}(x) + \frac{3(k-1)4}{4(k+\frac{1}{2})_4} R_{k-2}^{(2,3)}(x) - \frac{3(k-2)5}{4(k+3)(k-\frac{1}{2})_5} R_{k-3}^{(2,3)}(x) \\
- \frac{3(k-3)5}{16(k+3)(k-\frac{1}{2})_4} R_{k-4}^{(2,3)}(x), \]

(30)

and

\[
J_{k+5}^{(-3,-2)}(x) = -\frac{3(k+6)5}{32(k+3)(k+\frac{7}{2})_5} R_{k+5}^{(2,3)}(x) + \frac{15(k+5)5}{32(k+3)(k+\frac{5}{2})_6} R_{k+4}^{(2,3)}(x) \\
+ \frac{15(k+4)5}{32(k+3)(k+\frac{5}{2})_5} R_{k+3}^{(2,3)}(x) - \frac{15(k+4)4}{8(k+\frac{3}{2})_6} R_{k+2}^{(2,3)}(x) - \frac{15(k+2)(k+4)3}{16(k+\frac{3}{2})_5} R_{k+1}^{(2,3)}(x) \\
+ \frac{45(k+1)(k+4)2}{16(k+\frac{1}{2})_6} R_{k+3}^{(2,3)}(x) + \frac{15(k+1)3(k+4)}{16(k+\frac{1}{2})_5} R_{k+2}^{(2,3)}(x) - \frac{15(k-1)4}{8(k-\frac{1}{2})_6} R_{k}^{(2,3)}(x) \\
- \frac{15(k-2)5}{32(k+3)(k-\frac{1}{2})_5} R_{k-1}^{(2,3)}(x) + \frac{15(k-3)5}{32(k+3)(k-\frac{3}{2})_6} R_{k-2}^{(2,3)}(x) + \frac{3(k-4)5}{32(k+3)(k-\frac{5}{2})_5} R_{k-3}^{(2,3)}(x). \]

(31)

Applying Petrov-Galerkin method to (24)-(25) and if we make use of the two Lemmas 3 and 4 then after performing some lengthy manipulation, the numerical solution of (24)-(25) can be obtained. This solution is given in the following Theorem.

**Theorem 4.** If \( u_N(x) = \sum_{k=0}^{N-5} a_k J_{k+5}^{(-3,-2)}(x) \) is the Petrov-Galerkin approximation to (24)-(25), then the expansion coefficients \( \{a_k : k = 0, 1, \ldots, N - 5\} \) satisfy the matrix system

\[
(B_2 + \alpha_2 G_4 + \beta_2 G_3 + \gamma_2 G_2 + \delta_2 G_1 + \mu_2 G_0) a = f^*, \]

(32)

where the nonzero elements of the matrices \( B_2 \) and \( G_i \), \( (0 \leq i \leq 4) \) are given as follows:

\[
b_{kk}^2 = r_k, \quad g_{kk}^{4,2} = \frac{r_k}{2(k+\frac{5}{2})_2}, \quad g_{k,k+1}^{4,2} = \frac{3(k+1)3(k+5)}{2k+7}, \quad g_{k,k+1}^{3,2} = \frac{-3(k+1)3(k+5)}{2(k+\frac{5}{2})_3}, \quad g_{k,k+2}^{3,2} = \frac{-3(k+1)4}{4(k+\frac{3}{2})_2}, \quad g_{k+1,k}^{3,2} = \frac{3(k+2)(k+4)3}{2(k+\frac{5}{2})_3}, \quad g_{k+1,k}^{3,2} = \frac{3(k+4)4}{4(k+\frac{3}{2})_2}, \]

\[
g_{k+1,k}^{3,2} = \frac{3(k+1)4}{4(k+\frac{3}{2})_2}, \quad g_{k+2,k}^{3,2} = \frac{3(k+4)4}{4(k+\frac{3}{2})_2}, \quad g_{k,k+2}^{4,2} = \frac{-3(k+1)4}{4(k+\frac{3}{2})_2}, \quad g_{k+1,k}^{3,2} = \frac{3(k+2)(k+4)3}{2(k+\frac{5}{2})_3}, \quad g_{k,k+1}^{4,2} = \frac{3(k+1)3(k+5)}{2k+7}, \]

\[
g_{k,k+2}^{4,2} = \frac{3(k+4)4}{4(k+\frac{3}{2})_2}, \quad g_{k+1,k}^{3,2} = \frac{3(k+1)4}{4(k+\frac{3}{2})_2}, \quad g_{k,k+1}^{4,2} = \frac{3(k+1)3(k+5)}{2k+7}. \]


\[ g_{kk} = \frac{3r_k}{4(k + \frac{3}{2})^4}, \quad g_{k,k+1}^{2,2} = \frac{9(k+1)(k+5)}{8(k + \frac{3}{2})^3}, \quad g_{k,k+2}^{2,2} = -\frac{9(k+1)}{8(k + \frac{3}{2})^4}, \]

\[ g_{k,k+3}^{2,2} = \frac{-3(k+1)}{8(k + 6)(k + \frac{7}{2})^3}, \quad g_{k+1,k}^{2,2} = \frac{-9(k+2)(k+4)}{8(k + \frac{3}{2})^3}, \quad g_{k+2,k}^{2,2} = -\frac{9(k+4)}{8(k + \frac{3}{2})^4}, \]

\[ g_{k,k+3,k}^{2,2} = \frac{3(k+4)}{8(k + 3)(k + \frac{7}{2})^3}, \quad g_{k,k+1}^{1,2} = \frac{3r_k}{8(k + \frac{3}{2})^4}, \quad g_{k,k+1}^{1,2} = -\frac{9(k+1)(k+5)}{4(k + \frac{3}{2})^5}, \]

\[ g_{k+1,k}^{1,2} = \frac{9(k+2)(k+4)}{4(k + \frac{3}{2})^5}, \quad g_{k,k+2,k}^{1,2} = -\frac{3(k+4)}{4(k + \frac{3}{2})^5}, \quad g_{k,k+3,k}^{1,2} = -\frac{3(k+4)}{4(k + \frac{3}{2})^5}, \]

\[ g_{k+4,k}^{1,2} = \frac{3(k+5)}{16(k + 3)(k + \frac{7}{2})^4}, \quad g_{k,k}^{0,2} = \frac{15r_k}{16(k + \frac{1}{2})^6}, \quad g_{k,k+1}^{0,2} = \frac{15(k+1)(k+5)}{16(k + \frac{3}{2})^5}, \]

\[ g_{k,k+2}^{0,2} = -\frac{-15(k+1)}{8(k + \frac{3}{2})^6}, \quad g_{k,k+3}^{0,2} = \frac{-15(k+1)}{32(k+6)(k + \frac{7}{2})^5}, \quad g_{k,k+4}^{0,2} = \frac{15(k+1)}{32(k+7)(k + \frac{3}{2})^6}, \]

\[ g_{k+5,k}^{0,2} = \frac{3(k+1)}{32(k+8)(k + \frac{3}{2})^5}, \quad g_{k+1,k}^{0,2} = \frac{-15(k+2)(k+4)}{16(k + \frac{3}{2})^5}, \quad g_{k+2,k}^{0,2} = \frac{-15(k+4)}{8(k + \frac{3}{2})^5}, \]

\[ g_{k+3,k}^{0,2} = \frac{15(k+4)}{32(k+3)(k + \frac{7}{2})^5}, \quad g_{k+4,k}^{0,2} = \frac{15(k+5)}{32(k+3)(k + \frac{3}{2})^5}, \quad g_{k+5,k}^{0,2} = \frac{-3(k+6)}{32(k+3)(k + \frac{3}{2})^5}, \]

where \( r_k = 3(k+1)(k+2)(k+4)(k+5). \)

5 Structure of the coefficient matrices in the linear systems (23) and (32)

In this section, we discuss the structure of the coefficient matrices \( B_1 \) and \( E_{3-q} \) (1 \( q \leq 3 \)) in the linear system (23), and the coefficient matrices \( B_2 \) and \( G_{5-q} \) (1 \( q \leq 5 \)) in the linear system (32). Hence, we discuss the structure of the two combined matrices \( D_1 = B_1 + \alpha_1 E_2 + \beta_1 E_1 + \gamma_1 E_0 \) and \( D_2 = B_2 + \alpha_2 G_4 + \beta_2 G_3 + \gamma_2 G_2 + \delta_2 G_1 + \mu_2 G_0. \) Also we discuss the influence of these structures on the efficiency of the two systems (23) and (32).

It is clear that each of the matrices \( B_1 \) and \( B_2 \) is diagonal, so it is worthy to note that the two cases correspond to \( \alpha_1 = \beta_1 = \gamma_1 = 0 \) in (23) and \( \alpha_2 = \beta_2 = \gamma_2 = \delta_2 = \mu_2 = 0 \) in (32) lead to two linear systems with diagonal matrices. The result for these two cases are summarized in the following important two corollaries.

Corollary 1. If \( u_N(x) = \sum_{k=0}^{N-3} a_k J_{k+3}^{(-2,-1)}(x) \) and \( \alpha_1 = \beta_1 = \gamma_1 = 0, \) is the Galerkin approximation to problem (7)-(8), then the expansion coefficients \( \{a_k : \ k = 0, 1, \ldots, N-3\} \) are given explicitly by

\[ a_k = \frac{k + 2}{16} f_k, \quad k = 0, 1, \ldots, N-3, \]

where \( f_k = \int_{-1}^{1} (1 - x^2)(1 + x) f(x) R_k^{(1,2)}(x) dx. \)
Corollary 2. If \( u_N(x) = \sum_{k=0}^{N-5} a_k J_{k+5}^{(-3,-2)}(x) \) and \( \alpha_2 = \beta_2 = \gamma_2 = \delta_2 = \mu_2 = 0 \), is the Petrov-Galerkin approximation to the problem (24)-(25), then the expansion coefficients \( \{a_k : k = 0, 1, \cdots, N-5\} \) are given explicitly by

\[
a_k = \frac{k+3}{384} f_k, \quad k = 0, 1, \ldots, N-5,
\]

where \( f_k = \int_{-1}^{1} (1-x^2)^2(1+x) f(x) R_k^{(2,3)}(x) dx \).

Now, each of the matrices \( E_{3-q} \) (\( 1 \leq q \leq 3 \)) and \( G_{5-q} \) (\( 1 \leq q \leq 5 \)) is a band matrix whose total number of nonzero diagonals upper or lower the main diagonal is \( q \). Thus the coefficient matrices \( D_1 \) and \( D_2 \) are four-band and six-band matrices, respectively at most. These special structures of \( D_1 \) and \( D_2 \) simplify greatly the solution of the two linear systems (23) and (32). These two systems can be factorized by \( LU \)-decomposition and the number of operations necessary to construct these factorizations are of order \( 21(N-2) \) and \( 55(N-4) \) respectively, and the number of operations needed to solve the two triangular systems are of order \( 13(N-2) \) and \( 21(N-4) \) respectively.

Note. The total number of operations mentioned in the previous discussion includes the number of all subtractions, additions, divisions and multiplications. (see, [29]).

6 Nonhomogeneous boundary conditions

In the following we describe how third- and fifth-order problems with nonhomogeneous boundary conditions can be transformed into problems with homogeneous boundary conditions.

Let us consider the one-dimensional third-order equation

\[
u^{(3)}(x) - \alpha_1 u^{(2)}(x) - \beta_1 u^{(1)}(x) + \gamma_1 u(x) = f(x), \quad x \in I = (-1, 1),
\]

subject to the nonhomogeneous boundary conditions:

\[
u(\pm 1) = a_\pm, \quad u^{(1)}(1) = a^1.
\]

In such case we proceed as follows:

Set

\[
V(x) = u(x) + a_0 + a_1 x + a_2 x^2,
\]

where

\[
a_0 = \frac{-a_- - 3a_+ + 2a^1}{4},
\]

\[
a_1 = \frac{a_- - a_+}{2},
\]

\[
a_2 = \frac{-a_- + a_+ - 2a^1}{4}.
\]

The transformation (34) turns the nonhomogeneous boundary conditions (33) into the homogeneous boundary conditions

\[
V(\pm 1) = V^{(1)}(1) = 0.
\]
Hence it suffices to solve the following modified one-dimensional third-order equation:

\[ V^{(3)}(x) - \alpha_1 V^{(2)}(x) - \beta_1 V^{(1)}(x) + \gamma_1 V(x) = f^*(x), \quad x \in I = (-1, 1), \]  

where, \( \gamma_1 = \frac{1}{10} \gamma_1 a_2, \) and \( f^*(x) = f(x) + (-2\alpha_1 a_2 - \beta_1 a_1 + \gamma_1 a_0) + (-2\beta_1 a_2 + \gamma_1 a_0)x + \gamma_1 a_2 x^2. \)

If we apply the Petrov-Galerkin method to the modified equation (36), we get the equivalent system of equations

\[ (B_1 + \alpha_1 E_2 + \beta_1 E_1 + \gamma_1 E_0) a = f^*, \]

where \( B_1, E_2, E_1 \) and \( E_0 \) are the matrices defined in Theorem 3, and \( f^* = (f_0^*, f_1^*, \ldots, f_{N-3}^*), \) and

\[
f_k^* = \begin{cases} 
-2\alpha_1 a_2 - \beta_1 a_1 + \gamma_1 a_0, & k = 0, \\
\frac{6}{5}(-2\beta_1 a_2 + \gamma_1 a_0), & k = 1, \\
\frac{10}{3} \gamma_1 a_2, & k = 2, \\
f_k, & k \geq 3,
\end{cases}
\]

where, \( f_k = \int_{-1}^{1} (1 - x^2)(1 + x) R_k^{(1,2)}(x) f(x) \, dx. \)

We can apply the same procedures to solve the fifth-order equation

\[ -u^{(5)}(x) + \alpha_2 u^{(4)}(x) + \beta_2 u^{(3)}(x) - \gamma_2 u^{(2)}(x) - \delta_2 u^{(1)}(x) + \mu_2 u(x) = f(x), \quad x \in (-1, 1), \]  

subject to the nonhomogeneous boundary conditions

\[ u(\pm 1) = a_\pm, \quad u^{(1)}(\pm 1) = a_\pm, \quad u^{(2)}(1) = a_+. \]

In such case, (37)-(38) can be transformed into

\[ -V^{(5)}(x) + \alpha_2 V^{(4)}(x) + \beta_2 V^{(3)}(x) - \gamma_2 V^{(2)}(x) - \delta_2 V^{(1)}(x) + \mu_2 V(x) = f^*(x), \quad x \in I = (-1, 1), \]

subject to the homogenous boundary conditions

\[ V(\pm 1) = V^{(1)}(\pm 1) = V^{(2)}(1) = 0, \]

where

\[ V(x) = u(x) + a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4, \]

with

\[
\begin{align*}
  a_0 &= \frac{1}{16} \left( -2a_- + 8a_+ - 2a_+^2 - 5a_- - 11a_+ \right), \\
  a_1 &= \frac{1}{4} \left( a_- + a_+ + 3a_- - 3a_+ \right), \\
  a_2 &= \frac{1}{8} \left( -6a_+ + 2a_+^2 - 3a_- + 3a_+ \right), \\
  a_3 &= \frac{1}{4} \left( a_- - a_+ + a_- - a_+ \right), \\
  a_4 &= \frac{1}{16} \left( -2a_+^2 + 4a_+ + 2a_- + 3a_- - 3a_+ \right),
\end{align*}
\]
and

\[ f^*(x) = (\mu_2 a_0 - \delta_2 a_1 - 2\gamma_2 a_2 + 6\beta a_3 + 24\alpha_2 a_4) + (\mu_2 a_1 - 2\delta_2 a_2 - 6\gamma_2 a_3 + 24\beta_2 a_4)x \\
+ (\mu_2 a_2 - 3\delta_2 a_3 - 12\gamma_2 a_4)x^2 + (\mu_2 a_3 - 4\delta_2 a_4)x^3 + \mu_2 a_4 x^4 + f(x). \]

If we apply the Petrov-Galerkin method to the modified equation (39), we get the equivalent system of equations

\[ (B_2 + \alpha G_4 + \beta G_3 + \gamma G_2 + \delta G_1 + \mu G_0)\mathbf{a} = \mathbf{f}^*, \]

where \( B_2, G_i, 0 \leq i \leq 4 \) are the matrices defined in Theorem 4 and \( \mathbf{f}^* = (f_0^*, f_1^*, \ldots, f_{N-5}^*) \),

\[ f_k^* = \begin{cases} 
\mu_2 a_0 - \delta_2 a_1 - 2\gamma_2 a_2 + 6\beta_2 a_3 + 24\alpha_2 a_4, & k = 0, \\
\frac{1}{2}(\mu_2 a_1 - 2\delta_2 a_2 - 6\gamma_2 a_3 + 24\beta_2 a_4), & k = 1, \\
\frac{1}{2}(\mu_2 a_2 - 3\delta_2 a_3 - 12\gamma_2 a_4), & k = 2, \\
\frac{50}{131}(\mu_2 a_3 - 4\delta_2 a_4), & k = 3, \\
\frac{238}{143}\mu_2 a_4, & k = 4, \\
f_k, & k \geq 5,
\end{cases} \]

where, \( f_k = \int_{-1}^{1} (1 - x^2)^2 (1 + x) B_k^{(2,3)}(x) f(x) \, dx \).

7 Condition number of the resulting matrices

For the direct collocation method, the condition numbers behave like \( O(N^6) \) and \( O(N^{10}) \) for third- and fifth-order respectively (\( N \): maximal degree of polynomials). In this paper we obtain improved condition numbers with \( O(N^4) \) and \( O(N^6) \) respectively for third- and fifth-order. The advantages with respect to propagation of rounding errors is demonstrated.

For GJPGM, the resulting systems from the two differential equations \( u^{(3)}(x) = f(x) \) and \( -u^{(5)}(x) = f(x) \) are \( B_1 \mathbf{a}^1 = \mathbf{f}^* \) and \( B_2 \mathbf{a}^2 = \mathbf{f}^* \), where \( B_1 \) and \( B_2 \) are two diagonal matrices whose diagonal elements are given by \( b_{kk}^1 \) and \( b_{kk}^2 \), where

\[ b_{kk}^1 = 2(k+1)(k+3), \quad b_{kk}^2 = 3(k+1)(k+2)(k+4)(k+5). \]

Thus we note that the condition numbers of the matrices \( B_1 \) and \( B_2 \) behave like \( O(k^2) \) and \( O(k^4) \) respectively for large values of \( k \). The evaluation of the condition numbers for the matrices \( B_1 \) and \( B_2 \) are easy because of the special structure of them, since \( B_1 \) and \( B_2 \) are diagonal matrices, so their eigenvalues are their diagonal elements, and the condition number in such case has the definition

\[ \frac{\text{Condition number of the matrix}}{\text{Max (eigenvalue of the matrix)}} = \frac{\text{Min (eigenvalue of the matrix)}}{\text{Min (eigenvalue of the matrix)}}. \]

In Table 1 we list the values of the conditions numbers for the matrices \( B_1 \) and \( B_2 \), respectively, for different values of \( N \).
Table 1
Condition number for the matrix $B_n, n = 1, 2$

| $n$ | $N$ | $\alpha_{\text{min}}$ | $\alpha_{\text{max}}$ | Cond($B_n$) | Cond($B_n$) $/ N^{2n}$ |
|-----|-----|----------------------|---------------------|-------------|------------------|
| 1   | 16  | 448                  | 74.667              | 2.917 . $10^{-1}$ |
|     |     |                      |                     |              |                  |
|     | 20  | 720                  | 120                 | 3.000 . $10^{-1}$ |
|     |     |                      |                     |              |                  |
|     | 24  | 1056                 | 176                 | 3.056 . $10^{-1}$ |
|     |     |                      |                     |              |                  |
|     | 28  | 1456                 | 242.667             | 3.095 . $10^{-1}$ |
|     | 6   |                      |                     |              |                  |
|     | 32  | 1920                 | 320                 | 3.125 . $10^{-1}$ |
|     |     |                      |                     |              |                  |
|     | 36  | 2448                 | 408                 | 3.148 . $10^{-1}$ |
|     |     |                      |                     |              |                  |
|     | 40  | 3040                 | 506.667             | 3.167 . $10^{-1}$ |

Remark 1. If we add $\sum_{q=1}^{3} E_{3-q} (1 \leq q \leq 3)$ and $\sum_{q=1}^{5} G_{5-q} (1 \leq q \leq 5)$, where the matrices $E_{3-q}$ and $G_{5-q}$ are the matrices their nonzero elements are given explicitly in Theorems 3 and 4 respectively, to the matrices $B_1$ and $B_2$ respectively, then we find that the eigenvalues of matrices $D_1 = B_1 + \sum_{q=1}^{3} E_{3-q}$, $D_2 = B_2 + \sum_{q=1}^{5} G_{5-q}$ are all real positive. Moreover, the effect of these additions does not significantly change the values of the condition numbers for the systems. This means that matrices $B_1$ and $B_2$, which resulted from the highest derivatives of the differential equations under investigation, play the most important role in the propagation of the roundoff errors. The numerical results of Table 2 illustrate this remark.

Table 2
Condition number for the matrix $D_n, n = 1, 2$

| $N$ | Cond($D_1$) | $\text{Cond}(D_1) / N^2$ | Cond($D_2$) | $\text{Cond}(D_2) / N^4$ |
|-----|-------------|---------------------------|-------------|------------------------|
| 16  | 55.287      | 2.159 . $10^{-1}$         | 827.262     | 1.262 . $10^{-2}$      |
| 20  | 88.679      | 2.217 . $10^{-1}$         | 2278.4      | 1.424 . $10^{-2}$      |
| 24  | 129.929     | 2.256 . $10^{-1}$         | 5104.45     | 1.539 . $10^{-2}$      |
| 28  | 179.037     | 2.284 . $10^{-1}$         | 9980.18     | 1.624 . $10^{-2}$      |
| 32  | 236.003     | 2.305 . $10^{-1}$         | 17715.3     | 1.689 . $10^{-2}$      |
| 36  | 300.826     | 2.321 . $10^{-1}$         | 2925.4      | 1.742 . $10^{-2}$      |
| 40  | 373.507     | 2.334 . $10^{-1}$         | 45677.4     | 1.784 . $10^{-2}$      |
8 Numerical results

Example 1.

Consider the one dimensional equation

\[ u^{(3)}(x) - \alpha_1 u^{(2)}(x) - \beta_1 u^{(1)}(x) + \gamma_1 u(x) = f(x), \quad u(\pm 1) = u^{(1)}(1) = 0, \quad (40) \]

where \( f(x) \) is chosen such that the exact solution for (40) is \( u(x) = (1 - x^2) x^j \sin(m \pi x) \), \( j, m \in \mathbb{N} \). We have \( u_N(x) = \sum_{k=0}^{N-3} a_k (1 - x^2) (1 - x) R_k^{(2,1)}(x) \) and the vector of unknowns \( a = (a_0, a_1, \ldots, a_{N-3})^T \) is the solution of the system \( (B_1 + \alpha_1 E_2 + \beta_1 E_1 + \gamma_1 E_0) a = f^* \), where the nonzero elements of the matrices \( B_1 \) and \( E_{i,n} (0 \leq i \leq 2) \) are given explicitly in Theorem 4.

Table 3 lists the maximum pointwise error \( E \) for \( u - u_N \) to (40), using GJPGM for various values of \( j, m \) and the coefficients \( \alpha_1, \beta_1 \) and \( \gamma_1 \).

| \( N \) | \( j \) | \( m \) | \( \alpha_1 \) | \( \beta_1 \) | \( \gamma_1 \) | \( E \) |
|-------|----|----|-----|-----|-----|-----|
| 8     | 1  | 1  | 0   | 0   | 0   | 2.558 \cdot 10^{-3} |
| 12    | 1  | 1  | 0   | 0   | 0   | 1.909 \cdot 10^{-6} |
| 16    | 1  | 1  | 0   | 0   | 0   | 4.368 \cdot 10^{-10} |
| 20    | 1  | 1  | 0   | 0   | 0   | 2.811 \cdot 10^{-14} |
| 24    | 1  | 1  | 0   | 0   | 0   | 3.885 \cdot 10^{-16} |

| \( N \) | \( j \) | \( m \) | \( \alpha_1 \) | \( \beta_1 \) | \( \gamma_1 \) | \( E \) |
|-------|----|----|-----|-----|-----|-----|
| 8     | 1  | 2  | 3   | 4   | 0   | 4.472 \cdot 10^{-3} |
| 12    | 1  | 2  | 3   | 4   | 0   | 3.687 \cdot 10^{-6} |
| 16    | 1  | 2  | 3   | 4   | 0   | 6.660 \cdot 10^{-10} |
| 20    | 1  | 2  | 3   | 4   | 0   | 4.529 \cdot 10^{-14} |
| 24    | 1  | 2  | 3   | 4   | 0   | 7.771 \cdot 10^{-16} |

Example 2.

Consider the one dimensional fifth-order equation

\[ -u^{(5)}(x) + \alpha_2 u^{(4)}(x) + \beta_2 u^{(3)}(x) - \gamma_2 u^{(2)}(x) - \delta_2 u^{(1)}(x) + \mu_2 u(x) = f(x), \]

\[ u(\pm 1) = u^{(1)}(\pm 1) = u^{(2)}(1) = 0, \quad (41) \]
where \( f(x) \) is chosen such that the exact solution for (41) is \( u(x) = (1-x^2)^2(1-x) \cosh(mx) \), \( m \in \mathbb{R} \). We have \( u_N(x) = \sum_{k=0}^{N-5} a_k(1-x^2)^2(1-x)R_k^{(3,2)}(x) \) and the vector of unknowns \( a = (a_0, a_1, \ldots, a_{N-5})^T \) is the solution of the system

\[
(B_2 + \alpha_2 G_4 + \beta_2 G_3 + \gamma_2 G_2 + \delta_2 G_1 + \mu_2 G_0) a = f^*,
\]

where the nonzero elements of the matrices \( B_2 \) and \( G_{i,n} \) \((0 \leq i \leq 4)\) are given explicitly in Theorem 4.

Table 4 lists the maximum pointwise error \( E \) for \( u - u_N \) to (41), using GJPGM for various values of \( m \) and the coefficients \( \alpha_2, \beta_2 \) and \( \gamma_2 \) and \( \delta_2 \) and \( \mu_2 \).

| \( N \) | \( m \) | \( \alpha_2 \) | \( \beta_2 \) | \( \gamma_2 \) | \( \delta_2 \) | \( \mu_2 \) | \( E \) |
|---|---|---|---|---|---|---|---|
| 8 | 12 | 16 | 20 | 24 |
| 8 | 2 | 0 | 1 | 0 | 1 | 2 | \( 1.927 \cdot 10^{-2} \) |
| 12 | 2 | 0 | 1 | 0 | 1 | 2 | \( 8.652 \cdot 10^{-6} \) |
| 16 | \( \frac{1}{2} \) | 1 | 2 | 1 | 2 | 1 | \( 6.658 \cdot 10^{-5} \) |
| 20 | | | | | | | \( 6.661 \cdot 10^{-16} \) |
| 24 | | | | | | | \( 6.661 \cdot 10^{-16} \) |

Example 3.

Consider the one dimensional nonhomogeneous equation

\[
u^{(3)}(x) - \alpha_1 u^{(2)}(x) - \beta_1 u^{(1)}(x) + \gamma_1 u(x) = f(x),
\]

\[
u(\pm1) = \pm \sinh(m), \quad u^{(1)}(1) = m \cosh(m), \quad m \in \mathbb{R}, \quad (42)
\]

where \( f(x) \) is chosen such that the exact solution for (42) is \( u(x) = \sinh(mx) \).

setting

\[
V(x) = u(x) - \sinh(m)x + \frac{1}{2}[m \cosh(m) - \sinh(m)](1-x^2),
\]
then the differential equation (42) is equivalent to the differential equation
\[ u^{(3)}(x) - \alpha_1 u^{(2)}(x) - \beta_1 u^{(1)}(x) + \gamma_1 u(x) = f(x), \quad x \in (-1, 1), \quad u(\pm 1) = u^{(1)}(1) = 0. \]

In Table 5 we list the maximum pointwise error \( E \) for \( u - u_N \) to (42), using GJPGM for various values of \( m \) and the coefficients \( \alpha_1, \beta_1 \) and \( \gamma_1 \).

| \( N \) | \( m \) | \( \alpha_1 \) | \( \beta_1 \) | \( \gamma_1 \) | \( E \)     |
|------|------|--------|--------|--------|--------|
| 8    | 2    | 0      | 0      | 0      | 2.804 \( \cdot 10^{-8} \) |
| 12   | 1    | 0      | 0      | 0      | 9.536 \( \cdot 10^{-14} \) |
| 16   |      |        |        |        | 1.110 \( \cdot 10^{-16} \) |
| 8    | 1    | 0      | 1      | 1      | 2.819 \( \cdot 10^{-8} \) |
| 12   | 1    | 1      | 1      | 1      | 9.736 \( \cdot 10^{-14} \) |
| 16   |      |        |        |        | 1.110 \( \cdot 10^{-16} \) |
| 8    | 2    | 0      | 0      | 0      | 1545 \( \cdot 10^{-5} \) |
| 12   | 2    | 0      | 1      | 0      | 8.248 \( \cdot 10^{-10} \) |
| 16   |      |        |        |        | 1.310 \( \cdot 10^{-14} \) |
| 8    | 3    | 1      | 0      | 1      | 6.919 \( \cdot 10^{-4} \) |
| 12   | 3    | 1      | 0      | 1      | 1.808 \( \cdot 10^{-7} \) |
| 16   |      |        |        |        | 1.414 \( \cdot 10^{-11} \) |

### 9 Concluding remarks

In this paper, an algorithm for obtaining a numerical spectral solution for third- and fifth-order differential equations using certain nonsymmetric generalized Jacobi-Galerkin method is discussed. The algorithms are very efficient. We have found that, our choice for a certain family of basis functions to solve third- and fifth-order differential equations always lead to linear systems with band matrices that can be efficiently inverted. These special structures, of course simplifies greatly the numerical computations. In particular, for some particular third- and fifth-order differential equations, the resulting systems of these equations are diagonal. High accurate approximate solutions are achieved using a few number of the generalized Jacobi polynomials. The obtained numerical results are comparing favorably with the analytical ones. Furthermore, we do believe that the proposed technique can be applied to Korteweg-de Vries (KDV) equations.

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