The Number of Singularities in the Intersections of Convex Planar Translates

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Math 518 Undergraduate Thesis Winter 2020
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Acknowledgments

I would like to acknowledge two people in particular, Professor Karoly Bezdek and Ilya Ivanov, for giving me guidance throughout the research and production of this paper. The ability to discuss ideas for a proof with more experienced individuals is invaluable, not only for the proof or the paper’s sake, but also in stimulating my own mathematical thinking and learning. I am grateful to them for supporting me throughout this semester.
Abstract

This purpose of this paper is to prove the following result: let \( \phi \) be a strictly convex, smooth, convex body in \( \mathbb{E}^2 \), if the intersection of \( n \) translates of \( \phi \) has a non-empty interior, and all of the translates contribute to the intersection, then the intersection of these \( n \) translates will have exactly \( n \) points of singularity along its boundary. Furthermore this result is sharp, in the sense that, removing any one of the assumptions from our statement will render the result unable to hold in general.
1 Introduction

In this paper I strive to guide the reader first through the basics of convexity, and then incrementally advancing to the more difficult/complex aspects of convexity. If the reader is already well adept in the field of convex geometry, then skipping right to section 3 would be of no concern. Through section 2 I try to not assume any prior knowledge of convexity, and attempt to develop the theory naturally with an emphasis on examples. The most important part of section 2 is by far subsection 2.4, which is about convex bodies. As mentioned in the abstract, the main result of this paper focuses on this class of sets. Section 2.4 also contains some basics regarding the Gauss mapping of a convex body, something that is heavily used in proving the new results.

Section 3 outlines in more detail the main result of this paper. Also contained in this section is the justification of why the main result is sharp. I give explicit counterexamples to each assumption we take away, and then finish the section with an exposition on the limitations of this theorem in higher dimensions, specifically in $\mathbb{E}^3$.

Section 4 is where I prove the main result of the theorem by induction on the number of translates we are intersecting. Also in this section I develop what I am calling “Chord theory” which focuses on chords, or line segments whose end points belong to the boundary of a convex body. Various lemmas are proven in this section to prove the main result; in particular, I consider Theorem 4.8 and Lemma 4.9 to be the most notable proven results besides the main result.

1.1 A Note on Notations

Before concluding the introduction I want to introduce some of the notation I will be using throughout this paper: $A \subseteq B$ will denote being a subset or equal to, where as $A \subseteq B$ will denote a proper subset, $A \setminus B$ will denote set minus, and $\text{int}(A)$, $\text{bd}(A)$ will denote the interior and boundary respectively, of the set $A$. One thing I will assume of the reader is that they know the basics of analysis, such as knowing what an open set compared to a closed set is, along with other basics as converging sequences, compactness, and disjointedness. If the reader is unfamiliar with these terms, any introductory analysis text will suffice in familiarizing the reader.
2 Basic Convexity

2.1 The Basics, Developing an Intuition

Convexity is a relatively new field in mathematics, while some contributions to the field date back to Archimedes and Euclid. The subject really didn’t take on its own identity until the turn of the 20th century, thanks to the help of great mathematicians such as Hermann Brunn and Hermann Minkowski. They studied convexity in small dimensions which guided the field to generalize convexity to higher dimensions with the efforts of countless other mathematicians.

I want to emphasize that we stand on top of the hill of convex geometry, built up by thousands of hours of work done by our predecessors. Looking down, hoping to give future mathematicians a higher view.

With this perspective, we start climbing the hill from the bottom, which leads us to start with the definition of the space where convexity was first developed.

Definition 2.1. \( \mathbb{E}^n \) will denote the \( n \)-dimensional Euclidean space, that is

\[
\mathbb{E}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R} \}
\]

and we equip this space with the 2-norm

\[
\| (x_1, x_2, \ldots, x_n) \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}
\]

The Euclidean space is full of great mathematics to be learned, some of which I’m sure the reader has encountered before. However, for the purposes of this paper we restrict our view of this space to a special family of subsets of \( \mathbb{E}^n \) called convex sets.

Definition 2.2. We say a set \( \phi \subseteq \mathbb{E}^n \) is convex if for every two points in \( \phi \), the line segment joining them is contained in \( \phi \).

Before diving in to the more mathematically complex aspects of convexity, I want to give the reader some examples of convex and non-convex sets.

First we start with some trivial examples, the empty set \( \emptyset \in \mathbb{E}^n \), and the whole space \( \mathbb{E}^n \subseteq \mathbb{E}^n \) are both convex sets. The former is convex as we cannot select two points in \( \emptyset \), rendering the condition classifying convexity vacuously true. The latter on the other hand is convex as the line segment between any two points in \( \mathbb{E}^n \) is always inside of our space \( \mathbb{E}^n \).

A more typical example of a convex set is a closed ball in any dimension:

Definition 2.3. A closed ball in \( \mathbb{E}^n \) centered at a point \( x \in \mathbb{E}^n \) with radius \( r \in \mathbb{R} \), is denoted by and defined as:

\[
B^n(x, r) = \{ y \in \mathbb{E}^n \mid \| y - x \| \leq r \}
\]

This set is convex as if we take two points in the closed ball, \( y_1, y_2 \in B^n(x, r) \), we know that \( \| y_1 - x \| \leq r \) and \( \| y_2 - x \| \leq r \). From this we know any point in the line segment between \( y_1 \) and \( y_2 \), call it \( z \), will satisfy \( \| z - x \| \leq r \); thus \( z \) will be in \( B^n(x, r) \) also. This can be seen with the help of Figure 1, which shows the 2-dimensional case:
To give the reader a better understanding about the condition classifying convexity outlined in Definition 2.2, I want to ask a question about closed balls in $\mathbb{E}^2$, the choice of dimension here is arbitrary, but chosen as it is the probably the most familiar with the reader: “If we remove a point from $B^2(x, r)$, call it $z$, will $B^2(x, r) \setminus \{z\}$ be convex?” Please take a moment and think about this.

Naively, we are most likely inclined to say no; for example if we remove the center point of our ball, $x$, $B^2(x, r) \setminus \{x\}$ is certainly not convex, we can see this by taking two points in our ball whose line segment passes through the center of our ball. Then the two chosen points will be in our set, $B^2(x, r) \setminus x$, but the line segment between them is not contained in our set (as $x$, which is in the line segment, is missing). Similar reasoning will show that if we remove any interior point of our ball then the remain set will not be convex.

However, what happens if, instead of removing an interior point, we remove a boundary point, call it $v$, of $B^2(x, r)$? The remaining set will surprisingly be convex, as there is no way to pick two points in our ball such that the line segment between those two points passes through $v$. This implies that if we remove $v$ from $B^2(x, r)$ we will not have a gap, created by removing $v$, in any line segments between two points in $B^2(x, r)$. I urge the reader to verify this using the following diagram:

The point $v$ is an example of a point called an extreme point, which is characterized as follows:

**Definition 2.4.** Let $\phi$ be a convex set, then a point $x \in \phi$ is called an extreme point of $\phi$ if $\phi \setminus \{x\}$ remains to be convex.

We generalize our previous example, which showed that interior points of closed balls are not extreme points, with the following lemma:

**Lemma 2.5.** Let $\phi$ be a convex set, then, if $x \in \text{int}(\phi)$, then $\phi \setminus \{x\}$ is not convex. In other words, no interior point of $\phi$ is an extreme point of $\phi$. 

---

Figure 1: Demonstrating that a closed ball is a convex set

![Diagram](image1)

Figure 2: Removing a boundary point of a closed ball remains to be a convex set

![Diagram](image2)
This lemma is quite easy to prove, since $x$ is an interior point of $\phi$, there is some ball centered at $x$ which is contained in $\phi$. Then choosing two points from that ball whose line segment passes through $x$ we obtain two points in $\phi \setminus \{x\}$ whose line segment is not contained in $\phi \setminus \{x\}$.

In our ball example, we saw that an arbitrary boundary point of $B^2(x, r)$ will be an extreme point. This is not a general phenomenon though, and is somewhat unique to closed balls. One may see this through first recognizing that the filled-in square, $S = \{(x, y) | -1 \leq x, y \leq 1\}$, is a convex set, but there exists only few extreme points in the boundary of $S$, specifically only the 4 corners of the square are extreme points of $S$. The following figure shows an example of removing a boundary point, call it $z$, which is not an extreme point. The reader should verify that this set is not convex. That is to say, the reader should find two points in $S \setminus \{z\}$ such that the line segment between the two points is not contained in $S$.

Figure 3: Removing a boundary point of a square can produce a non-convex set

With the very basics of convexity covered, we are a few steps up the hill of convex geometry. I want to leave the reader with first some examples of convex sets which are a bit more exotic than the convex sets we have previously encounter. Following these example I would like to present some examples of non-convex sets as well. All of the examples will be planar, or in other words, in $E^2$, and I recommend the reader go through each example and verify why the set is either convex or not. If the reader wishes, finding which boundary points of our convex sets will be extreme, is a great exercise to develop an intuition for convexity.

Figure 4: Examples of convex sets

Figure 5: Examples of non-convex sets
2.2 Building Up Equivalent Characterizations of Convexity

With the introduction to convexity concluded we may now get into the more mathematically rigorous aspects of convexity. We start with a more formal definition of convexity, which really is just a rephrasing of Definition 2.2 where \( \overline{pq} \) denotes the line segment between \( p \) and \( q \).

**Lemma 2.6.** \( \phi \subseteq \mathbb{E}^n \) is convex \( \iff \) for all \( p, q \in \phi \),

\[
\overline{pq} = \{(1 - t)p + tq \mid t \in [0, 1]\} \subseteq \phi
\]

The following theorem tells us that convexity behaves well under intersections.

**Theorem 2.7.** The intersection of an infinite or finite family of convex sets is a convex set.

A proof sketch of this theorem is as follows: Let \( C \) be the intersection of a family of convex sets, then if we take two points in \( C \), those two points will belong to every member of the family of convex sets. Since all these sets are convex, the line segment between the two chosen points will also be contained in every member of the family, and hence the line segment will be in the intersection, \( C \).

Note that since the empty set is convex, even intersections of disjoint convex sets will remain convex. Also one should note that replacing Theorem 2.7 with unions does not hold in general. Any point by itself will be a convex set, but the union of two disjoint points is easily seen to be not convex.

One of the many fascinating aspects about convexity is the fact that we can characterize the condition that defines convexity in many ways. Lemma 2.6 is the standard classification, but there are many others which on their surfaces seem like they are not related to each other. One classification is based on the idea of a hyperplane, which is defined as follow:

**Definition 2.8.** Let \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{E}^n \), with \( c \in \mathbb{R} \). Then a hyperplane defined by this vector and constant, \( H_{a,c} \subseteq \mathbb{E}^n \), is the set

\[
H_{a,c} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{E}^n \mid a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = c\}
\]

More intuitively, a hyperplane in \( \mathbb{E}^n \) is a \((n - 1)\)-dimensional slice of \( \mathbb{E}^n \). In \( \mathbb{E}^2 \) hyperplanes are lines, and in \( \mathbb{E}^3 \) hyperplanes are planes. Connected to the definition of a hyperplane is the idea of a half space, which is the set of all points below, or above a hyperplane.

**Definition 2.9.** Let \( H_{a,c} \) be a hyperplane, then the positive half-space of \( H_{a,c} \), \( H^+_{a,c} \), is the set of all points above or on \( H_{a,c} \), which can be described by:

\[
H^+_{a,c} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{E}^n \mid a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq c\}
\]

The negative half-space of \( H_{a,c} \), \( H^-_{a,c} \), is the set of all points below or on \( H_{a,c} \) which is defined as:

\[
H^-_{a,c} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{E}^n \mid a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq c\}
\]

A positive (resp. negative) open half-space is the interior of a positive (resp. negative) half-space. Which is simply the half-space with the hyperplane bounding it missing. To obtain an open half-space from a half-space, we turn the inequality defining our half-space into a strict inequality.

With hyperplanes and half-spaces defined, we can now state a property that will be used in our equivalent classification of convexity that we have been building towards.
**Definition 2.10.** Two closed sets $A$ and $B$ can be separated by a hyperplane $H$ if and only if $A$ and $B$ lie in different open halfspaces defined by $H$

With this notion of separated we can now state our first equivalent condition of convexity.

**Theorem 2.11.** A closed set $\phi$ is convex if and only if for every point $p \in \mathbb{E}^n \setminus \phi$, $p$ can be separated by $\phi$ by a hyperplane.

One might wonder why we have only restricted Theorem 2.11 to closed sets, the reason is that this theorem does not hold for open convex sets, a counterexample is as follows: Let $U = \{(x_1, x_2) \in \mathbb{E}^n | x_1^2 + x_2^2 < 1\}$ be the open unit ball, which is just to closed unit ball with its boundary missing. This is easily seen to be convex. But the point $(1, 0) \notin U$ cannot be separated from $U$. The following diagram illustrates the situation:

![Figure 6: A counterexample to Theorem 2.11 if we remove the assumption of $\phi$ being closed](image)

The proof of this theorem is omitted, but I encourage the reader to take a look at our previous examples, which have all been closed sets: for the convex ones, verify Theorem 2.11 holds, and for the non-convex ones, try to find a point, not in the non-convex set, which cannot be separated from this non-convex set (by Theorem 2.11 there must be at least one such point).

Like I have alluded to before there are many other classifications of convexity besides Definition 2.2 and Theorem 2.11. In fact, there is another classification of convexity based on the notion of hyperplanes. The following definition plays a crucial role in establishing our third classification of convexity.

**Definition 2.12.** A hyperplane $H$ is called a support hyperplane for a closed set $\phi$ if $H$ contains points in $\phi$ and $\phi$ is contained in either the positive or negative half-space of $H$.

It is important to note that if $H$ is a support plane of a closed set $\phi$, then the points of $\phi$ which are contained in $H$, must be boundary points of $\phi$. The reason is that if an interior point of $\phi$ was in $H$, the ball around this interior point that is contained in $\phi$ will not lie in either the positive nor negative half-space of $H$, which would imply since this ball is contained in $\phi$ that $\phi$ is not contained in either the positive nor negative half-spaces of $H$, which would contradict that $H$ is a support plane of $\phi$.

With this notion of a support hyperplane we can now state our third classification of convexity:

**Theorem 2.13.** A closed set $\phi \subseteq \mathbb{E}^n$ with non-empty interior is convex if and only if a support hyperplane passes through each point of its boundary.

An open non-convex set will serve as a good counterexample to this theorem, if we were to remove the assumption that $\phi$ is closed. The reason is since this set is open, it has no boundary points which
will vacuously satisfy the property that every boundary point has a support plane. Also the condition of non-empty interior is an important one, as two distinct points will satisfy the condition that each boundary point has a support hyperplane, but a set containing only two distinct points is clearly not convex.

2.3 The Convex Hull

In this subsection I hope to give the reader a brief look into an aspect of convexity called the convex hull. One common mathematical practice, to better understand a specific notion about sets, is to find an operation to take a arbitrary set, and turn it into another set that satisfies our notion. A familiar example of this is the closure of a set. This operator take an arbitrary set, and from it, produces a closed set.

The next definition is another example of this practice, but instead of turning our sets into closed ones, we define an operator which can take an arbitrary set in $\mathbb{E}^n$ and turn it into a convex one.

**Definition 2.14.** The convex hull of a set $A \subseteq \mathbb{E}^n$, $cc(A)$, is the smallest convex set that contains $A$.

Which is equivalent to the following lemma.

**Lemma 2.15.** for $A \subseteq \mathbb{E}^n$

$$cc(A) = \bigcap_{A \subseteq X} X \text{ for } X \text{ a convex set}$$

In other words, the convex hull of a set $A$, is the intersection of all possible convex sets that contain $A$. Since we showed before that the intersection of convex sets are convex, we immediately see that $cc(A)$ will produce a convex set. The convex hull of sets has some interesting properties, a couple are given below.

**Proposition 2.16.** If $A \subseteq B \subseteq \mathbb{E}^n$ then $cc(A) \subseteq cc(B)$

This follows directly from Lemma 2.15 by noticing that the collection of convex sets that contain $B$ is a subset of the collection of convex sets that contain $A$.

**Proposition 2.17.** The convex hull of a set $A$ is the intersection of all half-spaces that contain $A$.

One common use of the convex hull of a set is the construction of polytopes, there definition is given below. But first we need to define what it means for a set to be convexly independent, which the reader should see has some analogies to the notion of linearly independent.

**Definition 2.18.** A set of points $\{x_1, x_2, \ldots, x_k\} \subseteq \mathbb{E}^n$ is convexly independent if for all $i \in \{1, \ldots k\}$

$$x_i \notin cc(\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\})$$

The idea behind this definition is if $x_i \in cc(\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\})$ then

$$cc(\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\}) = cc(\{x_1, \ldots, x_k\})$$

Which means we really did not need $x_i$ in our set to get the same convex hull. To rephrase this definition, a set of points is convexly independent if removing any point of the set we are taking the convex hull of, makes the convex hull a smaller set.

**Definition 2.19.** A polytope $P$ in $\mathbb{E}^n$ is the convex hull of finitely many, but greater or equal to $n + 1$, convexly independent points.

The reason we require greater or equal to $n + 1$ points is that we want our $n$-dimensional polytope to have a non-empty interior.
2.4 Convex Bodies

We have now gone over the basics of convexity. The three different classifications of convexity we derived gives us a strong foundation of understanding these sets. I would like to finish off this section with a discussion of a special class of convex sets, the class of convex bodies. This class of convex shapes is quite well-known, and has been studied to great depth.

**Definition 2.20.** A set $\phi \subseteq \mathbb{E}^n$ is a convex body if and only if $\phi$ is convex, compact, and has non-empty interior.

Since convex bodies are closed, convex, and have non-empty interior, by Theorem 2.13 every boundary point of a convex body will have at least one support hyperplane. From this we can characterize the boundary points of a convex body by how many support hyperplanes that boundary point has.

**Definition 2.21.** Let $\phi \subseteq \mathbb{E}^n$ be a convex body, a point $p \in \text{bd} (\phi)$ is a singular point if and only if more than one hyperplane support $\phi$ at $p$. If only one hyperplane supports $\phi$ at $p$, we call $p$ a regular point.

Note that by Theorem 2.13, since convex bodies are closed, convex, and have non-empty interior, each boundary point must have at least one supporting hyperplane, which means that each boundary point of a convex body will either be regular or singular.

**Definition 2.22.** The set of regular points of a convex body, $\phi$, is denoted $\text{reg}(\phi)$, where the set of singular points of $\phi$ is denoted $\text{sing}(\phi)$.

From our discussion in the previous paragraph we have the following lemma:

**Lemma 2.23.** Let $\phi$ be a convex body, then:

$$\text{bd}(\phi) = \text{reg}(\phi) \cup \text{sing}(\phi)$$

Figure 8 is an example of a convex body with 4 singular points, I urge the reader to verify this, and the following remark.

**Remark 2.24.** If a boundary point $p \in \phi$ is singular, or equivalently $p$ has more than one hyperplane that supports it, then $\phi$ has infinitely many support hyperplanes supporting it at $p$.

To fully explain this remark we will go into a slight detour of spherical convexity and the Gauss image of a boundary point of a convex body. To start I will state the definition of a spherical space (this space we will soon show has a notion of convexity in it) and define what it means for a boundary point of a convex body in $\mathbb{E}^n$ to have a normal vector.
Figure 8: An example of a convex shape with 4 singularities, all of them having infinitely many support planes

**Definition 2.25.** The spherical space inside $\mathbb{E}^n$, $S^{n-1}$ is defined as

$$S^{n-1} = \{x \in \mathbb{E}^n | \|x\| = 1\}$$

Equipped with the metric, where the distance between two points $x, y \in S^1$, is

$$|x - y| = \angle(x, y) = \text{angle between } x \text{ and } y$$

This space is easily seen to be the boundary of the closed unit ball $B^n(o, 1)$, where $o$ denotes the origin. With this we can now state what a normal of a boundary point of a convex body is:

**Definition 2.26.** Let $\phi$ be a convex body, then a vector $u \in S^{n-1}$ is a normal at $p \in bd(\phi)$, if the hyperplane through $p$, orthogonal to $u$, is a support hyperplane of $\phi$, and if the vector $u$ placed at the boundary point $p$ belongs to the half-space not containing $\phi$.

Due to Theorem 2.13 we know that each boundary point of a convex body has at least one normal, since each boundary point has at least one support plane. This relation between boundary points and their normals is the Gauss map. The formal definition is as follows:

**Definition 2.27.** The Gauss map of a convex body $\phi \subseteq \mathbb{E}^n$ is $\Gamma_\phi : bd(\phi) \rightarrow S^{n-1}$ which send each boundary point, $p \in \phi$ to its set of normal vectors.

Note that the Gauss map is not a function! This is due to the fact that one boundary point can be mapped to multiple normal vectors, which implies elements in the domain of the Gauss map can be mapped to multiple elements in the co-domain. Hence the Gauss map in general is not well-defined, but it is functional (i.e. every element in the domain is mapped to something). The Gauss map of a convex body is also easily seen to be surjective on its co-domain, since each vector in $S^1$ will have at least one boundary point in $\phi$ that has a support hyperplane with this normal vector (imagine sliding the hyperplane, normal to an arbitrary vector, across our convex body. Since our convex body is closed we must have a point that touches it “first”).

**Definition 2.28.** Let $\phi$ be a convex body, the Gauss image of a boundary point $p$ of $\phi$ is :

$$\Gamma_\phi(p) = \{u \in S^{n-1} | u \text{ is normal to } p\}$$
Of course if $p$ is regular, $\Gamma_\phi(p)$ is a singleton, Remark 2.24 claims that if $p$ is singular then $\Gamma(p)$ is infinite in size, in fact one can show that its size is uncountable infinite. To see this suppose $p \in \text{bd}(\phi)$ and $p$ has more than one support plane, and hence has more than one normal vector. It can be easily seen that the normal vectors in between our two distinct normal vectors will also be normal’s of $\phi$ at $p$. The figure below considers the 2-dimensional case, $\phi \subseteq \mathbb{E}^2$, where $n_1$ and $n_2$ are the two distinct normals of $\phi$ at $p$. The normal vector $n$ represents how an arbitrary vector in between $n_1$ and $n_2$ will also be a normal of $\phi$ at $p$.

Looking at the diagram we see that any vector $u \in S^{n-1}$ in between the two normal vectors will also be a normal vector at $p$. As the number of vectors in between the two normal vectors is uncountably infinite, the remark is justified.

Figure 9: If a boundary point has two normal’s, it also has all normal’s in between

In our previous paragraph we justified that the Gauss images of boundary points will have the property that for any two points in the Gauss image, the points in between the two points will also be in our Gauss image. Hopefully this property seem familiar to the reader! This is in fact the property of spherical convexity. The formal definition is as follows:

**Definition 2.29.** A set $\phi \subseteq S^{n-1}$ is spherically convex if for any two points in $\phi$ the shortest spherical arc between the two points is also in $\phi$.

From our discussion about Remark 2.12 before we immediately see that the Gauss image of any boundary point of a convex body is a spherically convex set.

To finish off this section I would like to discuss two properties about convex bodies that will be crucial for the main result of this paper: namely, strict convexity, and smoothness. Strict convexity can be classified a couple ways, my favourite way is as follows:

**Definition 2.30.** A convex body $\phi \subseteq \mathbb{E}^n$ is strictly convex, if its boundary contains no line segments.

An equivalent characterization is given below:

**Lemma 2.31.** A convex body $\phi \subseteq \mathbb{E}^n$ is strictly convex if and only if every support hyperplane intersects $\phi$ at only a single point.

Note that this implies that the Gauss map is injective, as no two boundary points can share a normal. Smoothness, on the other hand, ensures our Gauss map will be well defined; that is, each boundary point will at most 1 normal. The formal definition of smoothness is as follows:
Definition 2.32. A convex body $\phi \subseteq \mathbb{E}^n$ is smooth if it has no singular points, or equivalent every boundary point of $\phi$ is regular.

From this definition we may derive the following theorem; note however that we know the Gauss map of a convex body is a functional surjective map. Which implies by our discussion above that if our convex body is strictly convex, the Gauss map will be a functional, surjective, and injective mapping. Furthermore if we suppose our convex body is smooth then we have a functional, well defined, injective and surjective mapping (i.e. a bijection). The reason why smoothness implies our Gauss map will be well defined is because for a smooth convex body, each boundary point is regular which means it only has one normal vector (i.e. every point in our domain is mapped to at most one point). This justifies the following theorem.

Theorem 2.33. Let $\phi$ be a smooth strictly convex, convex body, then $\Gamma_\phi$ is a bijective function.

This theorem is heavily used in the new results proven in this paper, as we will see soon.
3 New results

3.1 The main result

The main result of this paper is as follows:

**Theorem 3.1.** Let \( \phi \) be a strictly convex, smooth, convex body in \( \mathbb{E}^2 \). Let \( \tau_i : \mathbb{E}^2 \to \mathbb{E}^2 \) be a translation for each \( i \in \{1, 2, \ldots, n\} \) with \( n \geq 2 \). Let \( \tau_i(\phi) = \phi_i \) for each \( i \in \{1, 2, \ldots, n\} \). Then if

\[
\text{int} \left( \bigcap_{i=1}^{n} \phi_i \right) \neq \emptyset
\]

and if for all \( j \in \{1, 2 \ldots n\} \)

\[
\bigcap_{i=1}^{n} \phi_i \subsetneq \left( \phi_1 \cap \phi_2 \cap \cdots \cap \phi_{j-1} \cap \phi_{j+1} \cap \cdots \cap \phi_n \right)
\]

Then

\[
|\text{sing} \left( \bigcap_{i=1}^{n} \phi_i \right) | = n
\]

Informally this theorem says that given \( n \) translates of a 2-dimension strictly convex, smooth, convex body, where the intersection of these \( n \) translates has non-empty interior, and removing anyone of these translates from the intersection produces a larger set. Then the intersection of these \( n \) translates will have exactly \( n \) singular points. It should be noted that this theorem is strictly a 2-dimensional statement and does not hold even in \( \mathbb{E}^3 \). However before going into the details of why we cannot generalize the dimension, I want to give the reader some intuition about the problem in \( \mathbb{E}^2 \) by explaining why we need each of the assumptions in Theorem 3.1: namely \( n \geq 2 \), intersection of translates, strictly convex, smooth, intersection has non-empty interior, and why each translate must make the intersection smaller.

Before justifying the assumption I made, I want to first give the reader an example of the general phenomenon I wish to imply from Theorem 3.1. Below is an example of the intersection of three translates of a closed ball in \( \mathbb{E}^2 \). This intersection satisfies all conditions laid out in Theorem 3.1 and as we see, satisfies the conclusion of having exactly three points of singularities.

**Figure 10:** An ideal example showing Theorem 3.1

I developed Theorem 3.1 first by considering the intersection of finitely many balls, with non-empty interior, and with no redundancies in the intersection; these shapes are know as ball polytopes. After proving this I proceeded to generalize the case to the largest class of convex bodies I could find. This then led to Theorem 3.1.
3.2 Justifying assumptions

Theorem 3.1 is sharp in the sense that if you remove any of the assumptions the result in general no longer holds. To show why this is true, I shall examine the six assumptions of Theorem 3.1 individually, and give a counterexample for each, that satisfies all the other assumptions in Theorem 3.1 besides the one.

3.2.1 Why we need to consider the intersection of at least two translates

This is the easiest assumption to justify, if we only have the intersection of one translate, φ₁, then there is no singularities as φ is smooth. We also can not have the intersection of zero translates because we require our intersection to have non-empty interior, so the number of translates, n, must be greater or equal to two.

3.2.2 Why we require the φᵢ’s to be translates

Below is a counterexample of Theorem 3.1 if we loosen the assumption that φᵢ’s are translates of φ, to the assumption that the φᵢ’s are merely images of isometries of φ.

Figure 11: Counterexample to Theorem 3.1 if we remove the assumption of translates

3.2.3 Why φ need to be strictly convex

Below is a counterexample of Theorem 3.1 if we take out the strictly convex assumption.

Figure 12: Counterexample to Theorem 3.1 if we remove the assumption of strict convexity

Here we see the intersection of two translates, φ₁ and φ₂ of φ: a square whose corners have been rounded. We see that this example satisfies all the assumptions besides strict convexity, the square with rounded corners is a smooth convex body and the intersection of φ₁ and φ₂ has non-empty interior and removing φ₁ or φ₂ from this intersection produces a larger set. But we see that the intersection of φ₁ ∩ φ₂ has no singular points on its boundary (if our theorem were to hold for non-strictly convex bodies we would need exactly two singular points).
3.2.4 Why \( \phi \) needs to be smooth

The reason is that requiring smoothness of \( \phi \) will imply there are no singular points of \( \phi \), and then in the intersection we will not deal with extra singularities. Below are two counterexamples: the intersection of two triangles, and the intersection of two squares, where the edges have been adjusted to be round (so that we still satisfy our strict convexity assumption).

Figure 13: Counterexample to Theorem 3.1 if we remove the assumption of smoothness

In both cases all assumptions are satisfied expect for smoothness and we see that the intersection of the two triangles has 3 singularities, and the intersection of the two squares has 4 singularities; if our theorem were to hold for non-smooth bodies we must have 2 singularities in each cases.

3.2.5 Why the intersection must have a non-empty interior.

The reason is that since we need the translates to be strictly convex, the only way our translates can intersect and have an empty interior is if the intersection is just a dot, in which case the intersection has only one point of singularity. Below is an example of the intersection of two closed balls with non-empty interior.

Figure 14: Counterexample to Theorem 3.1 if we remove the assumption of non-empty interior

We see from this example that the intersection of two translates of a closed ball can just have one singular point if we do not require the intersection to have non-empty interior.

3.2.6 Why we require our intersection to not have any redundancies.

First, I will give the formal definition of a set being redundant.
Definition 3.2. Given an intersection,
\[ \bigcap_{i=1}^{n} A_i, \]
we say that a set in this intersection, \( A_j \), is redundant if taking out this set from our intersection still results in the same set, that is to say:
\[ \bigcap_{i=1}^{n} A_i = (A_1 \cap \ldots \cap A_{j-1} \cap A_{j+1} \cap \ldots \cap A_n) \]
We say a set \( A_j \) is not redundant if
\[ \bigcap_{i=1}^{n} A_i \neq (A_1 \cap \ldots \cap A_{j-1} \cap A_{j+1} \cap \ldots \cap A_n) \]
Since the following subset relation always holds
\[ \bigcap_{i=1}^{n} A_i \subseteq (A_1 \cap \ldots \cap A_{j-1} \cap A_{j+1} \cap \ldots \cap A_n) \]
With this we can rephrase the definition that \( A_j \) is not redundant if
\[ \bigcap_{i=1}^{n} A_i \subsetneq (A_1 \cap \ldots \cap A_{j-1} \cap A_{j+1} \cap \ldots \cap A_n) \]
which is exactly the last assumption in Theorem 3.1. The reason we require that none of our translates are redundant is demonstrated by the following example. Consider the intersection of 3 circles:

Figure 15: Counterexample to Theorem 3.1 if we remove the assumption that none of our translates are redundant

With this example we see that the intersection of our three circles only has two singularities, of course if our theorem were to hold for redundant intersection we must have three singularities in this intersection.

Another way to think about this assumption is to realize if we have the intersection of \( n \) sets and one of our sets is redundant, then we do not really have an intersection of \( n \) sets. What we have can be formed by the intersection of \( n - 1 \) sets (since one of our \( n \) sets is redundant) and therefore the intersection is required to have no redundant sets.
3.3 Counterexample to Theorem 3.1 in higher dimensions

With all the assumptions in Theorem 3.1 justified, we can now discuss why this theorem does not hold in higher dimensions. Consider the intersection of two balls in $\mathbb{E}^3$, which is depicted in Figure 16:

We can understand this intersection by imagining it as gluing two tops of a ball together. The cross section of this intersection made up from the middle solid line, and the "back" dashed line will form a perfect circle, whereas the cross section made up of the left and right solid lines will be a Reuleaux 2-gon, or in other words the intersection of two balls in $\mathbb{E}^2$. This intersection actually has an uncountable amount of singularities which make up the boundary of the perfect circle cross section. This shows that Theorem 3.1 does not hold in three dimensions.

4 Proof of Theorem 3.1

Proof. We prove Theorem 3.1 by induction on the number of translates in our intersection starting with two. Let $\phi$ be a strictly convex, smooth, convex body in $\mathbb{E}^2$.

4.1 Base Case

I aim to prove that the intersection of two translates of $\phi$ will have exactly two singularities, provided the intersection have non-empty interior and neither one of our translates are redundant.

Let $\phi_1$ and $\phi_2$ be two translates of $\phi$, given by the two translations $\phi_1 = \tau_1(\phi)$ and $\phi_2 = \tau_2(\phi)$. Suppose further that $\text{int}(\phi_1 \cap \phi_2) \neq \emptyset$ and $\phi_1$ and $\phi_2$ are both non-redundant. Since $\phi_1$ and $\phi_2$ are translates of $\phi$, this implies that there is a translation $\tau$ which takes $\phi_1$ to $\phi_2$. Specifically, $\tau$ is defined by $\phi_2 = \tau(\phi_1) = \tau_2(\tau_1^{-1}(\phi_1))$. Note that $\tau$ is a non-identity translation as if it was the identity translation we would obtain $\phi_1 = \phi_2$ which would imply $\phi_1 \cap \phi_2 = \phi_2$ and by extension, $\phi_1$ will be redundant a contradiction, so $\tau$ must not be the identity translation. Let $u \in \mathbb{S}^1$ be defined by:

$$u = \frac{\tau(0,0)}{\|\tau(0,0)\|}$$

Note that since $\tau$ is a non-identity translation, $\|\tau(0,0)\| \neq 0$ which makes $u$ properly defined. Another way to describe $u$ is the normalized vector in the direction of the translation $\tau$.

Before showing that $\phi_1 \cap \phi_2$ has exactly two singularities, two lemmas are needed in order to classify when a boundary point of $\phi_1 \cap \phi_2$ is a singularity.

Lemma 4.1. Let $\phi_1$ and $\phi_2$ be translates of a strictly convex, smooth, convex body $\phi \subseteq \mathbb{E}^2$, whose intersection $\phi_1 \cap \phi_2$ has a non-empty interior, and where neither $\phi_1$ nor $\phi_2$ are redundant in the intersection $\phi_1 \cap \phi_2$. Let $p \in \text{bd}(\phi_1) \cap \text{bd}(\phi_2)$ and let $H_1$ and $H_2$ be the supporting hyperplanes at $p$ of $\phi_1$ and $\phi_2$, with normal unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively. Then $H_1 \parallel H_2$ or equivalently $\mathbf{n}_1 \neq \pm \mathbf{n}_2$. 20
Proof. I will prove this by contradiction. Suppose \( n_1 = \pm n_2 \) we have two cases:

1. Suppose \( n_1 = n_2 \), which is equivalent to \( \Gamma_{\varphi_1}(p) = \Gamma_{\varphi_2}(p) \), then since \( \varphi \) is a smooth, strictly convex body, we know that the Gauss map \( \Gamma : bd(\varphi) \to S^1 \) is an injective function. Let \( c_1 \) and \( c_2 \) be the points in the boundary of \( \varphi \) such that \( \tau_1(c_1) = p = \tau_2(c_2) \), note that the Gauss images of \( \tau_1(c_1) \) and \( \tau_2(c_2) \) of \( \varphi_1 \) and \( \varphi_2 \) respectively, will be the same as the Gauss images of \( c_1 \) and \( c_2 \) respectively in \( \varphi \). The reason is that since \( \tau_1 \) and \( \tau_2 \) are translations, the support plane at \( \tau_1(c_1) = p \) of \( \varphi_1 \) will be the image, under \( \tau_1 \), of the support plane at \( c_1 \) of \( \varphi \), and the image of a hyperplane under a translation will always have the same normal vector as its pre-image.

\[
n_1 = n_2 \implies \Gamma_{\varphi_1}(p) = \Gamma_{\varphi_2}(p) \implies \Gamma_{\varphi}(\tau_1^{-1}(p)) = \Gamma_{\varphi}(\tau_2^{-1}(p)) \implies \Gamma_{\varphi}(c_1) = \Gamma_{\varphi}(c_2)
\]

But \( \Gamma_{\varphi} \) is injective which implies \( c_1 = c_2 \). This gives us the following two equations:

\[
\tau_1(c_1) = p = \tau_2(c_2) \text{ and } c_1 = c_2
\]

But if two translations \( \tau_1 \) and \( \tau_2 \) agree on a point, this implies \( \tau_1 = \tau_2 \), however this would lead to

\[
\varphi_1 = \tau_1(\varphi) = \tau_2(\varphi) = \varphi_2
\]

a contradiction as \( \varphi_1 \) would be redundant.

2. Suppose \( n_1 = -n_2 \), then let \( H_1 \) and \( H_2 \) be the support planes at \( p \) of \( \varphi_1 \) and \( \varphi_2 \) respectively. Since \( H_1 \) and \( H_2 \) both go through \( p \), and their normal’s are opposite, \( H_1 \) must equal \( H_2 \). Also, \( \varphi_1 \) and \( \varphi_2 \) must be bound to the positive and negative closed half-spaces that are generated by \( H_1 = H_2 \). Since \( \varphi_1 \) and \( \varphi_2 \) are strictly convex, only \( p \) can be contained in \( H_1 = H_2 \), and the rest of \( \varphi_1 \setminus \{p\} \), and \( \varphi_2 \setminus \{p\} \) are bound to the opposite open half-spaces determined by \( H_1 = H_2 \). This implies that \( \varphi_1 \cap \varphi_2 = p \), a contradiction as we assumed \( \varphi_1 \cap \varphi_2 \) has non-empty interior. The following diagram illustrates the situation:

Figure 17: Two translates of \( \varphi \) cannot have a boundary intersection with opposite normal vectors

[Diagram of two translates of \( \varphi \) with opposite normal vectors]

Lemma 4.1 tells us that if two translates intersect with non-empty interior, and the translates are not equal to each other; then where the boundary of these two translates intersect, we must have non-parallel support planes for each translate. The following lemma shows that these points, where the boundaries of two translates intersect, are exactly our points of singularities.

**Lemma 4.2.** Let \( \varphi_i \) for \( i \in \{1, 2, \ldots, n\} \) be a translate of a strictly convex, smooth, convex body in \( \mathbb{R}^2 \), Suppose \( p \in bd(\bigcap_{i=1}^{n} \varphi_i) \), then:

\[
p \text{ is a singular point of } \bigcap_{i=1}^{n} \varphi_i \iff p \in bd(\varphi_j) \cap bd(\varphi_k) \text{ for some } j \neq k \in \{1, 2, \ldots, n\}
\]
Proof. Suppose \( p \in \text{bd}(\bigcap_{i=1}^{n} \phi_i) \), to prove this statement I shall show that:

1. If \( p \in \text{bd}(\phi_j) \cap \text{bd}(\phi_k) \) for some \( j \neq k \in \{1, 2, \ldots, n\} \) then \( p \) is singular.
2. If \( p \notin \text{bd}(\phi_j) \cap \text{bd}(\phi_k) \) for all \( j \neq k \in \{1, 2, \ldots, n\} \) then \( p \) is not singular.

We start with statement 1. Suppose \( p \in \text{bd}(\phi_j) \cap \text{bd}(\phi_k) \) for some \( j \neq k \in \{1, 2, \ldots, n\} \). Then from our previous lemma, we know \( H_j \parallel H_k \), for support planes of \( \phi_j \) and \( \phi_k \) at \( p \). Since \( p \in \text{bd}(\bigcap_{i=1}^{n} \phi_i) \) and \( \bigcap_{i=1}^{n} \phi_i \subseteq (\phi_j \cap \phi_k) \) we see that \( H_j \) and \( H_k \) will also be support planes of \( \bigcap_{i=1}^{n} \phi_i \) respectively at \( p \), which implies that \( p \) is a singular point of \( \bigcap_{i=1}^{n} \phi_i \). The following figure illustrates this:

![Figure 18: The intersection of the boundary of two translates will produce a singular point](image)

Moving on to the second statement 2. Suppose \( p \notin \text{bd}(\phi_j) \cap \text{bd}(\phi_k) \) for all \( j \neq k \in \{1, 2, \ldots, n\} \). Since \( p \in \text{bd}(\bigcap_{i=1}^{n} \phi_i) \) this suggests that \( p \in \text{bd}(\phi_l) \) for some \( l \in \{1, 2, \ldots, n\} \) and \( p \notin \text{bd}(\phi_i) \) for all \( l \neq i \in \{1, 2, \ldots, n\} \). Since \( \bigcap_{i=1}^{n} \phi_i \subseteq \phi_l \) and \( p \) is in the boundary of both, the unique support plane at \( p \) of \( \phi_l \) will be a support plane of \( \bigcap_{i=1}^{n} \phi_i \) at \( p \). This support plane will actually be the only support plane of \( \bigcap_{i=1}^{n} \phi_i \) at \( p \), because the only way we can have another support plane at \( p \) is if the boundary of another translate intersects the translate of \( \phi_l \) at \( p \) in the intersection \( \bigcap_{i=1}^{n} \phi_i \); this would imply that \( p \) belong to the boundary of two translates, a contradiction as we suppose it only belonged to the boundary of one translate. Hence, we must only have one support plane of \( \bigcap_{i=1}^{n} \phi_i \) at \( p \), this implies that \( p \) is regular which is equivalent to \( p \) being not singular. The following figure shows the situation:

![Figure 19: If a boundary point belongs to the boundary of only one translate, then it will be regular](image)

By the previous lemma we know that \( p \) is a singularity of \( \phi_1 \cap \phi_2 \) if and only if \( p \in \text{bd}(\phi_1) \cap \text{bd}(\phi_2) \). To prove our base case I will show that only two points belong to the intersection of the boundary of these two translates. However, in order to show this we must first take a brief interlude in what I am calling, “Chord theory.”

4.2 A Brief Interlude in 'Chord Theory

**Definition 4.3.** A chord of a convex body \( \phi \subseteq \mathbb{E}^2 \) is a non-empty set:

\[
\kappa = \phi \cap H
\]

For some hyperplane \( H \).
A chord of a convex body can be equivalently described as a line segment contained in \( \phi \) whose boundary points belong to the boundary of \( \phi \) (note that here we are not excluding the possibility of a single point being called a chord). Here is an example of some chords in various convex bodies we have seen.

Figure 20: Examples of chords in various convex bodies

![Chord Examples](image)

With this definition we form a function that takes a chord and maps it to its length. But instead of the domain of this function being all chords in a convex body \( \phi \), we restrict the domain to just be chords perpendicular to a given direction \( w \in \mathbb{S}^1 \).

**Definition 4.4.** Let \( w \in \mathbb{S}^1 \), and let \( K \) be the set of all chords in a convex body perpendicular to the direction \( w \). The chord function of \( \phi \) in the direction \( w \) is \( \theta_w^\perp : K \to \mathbb{R}^+ \cup \{0\} \), which is the function that takes a chord in \( K \) to its length, a non-negative real number.

For intuition purposes, the choice of \( \theta \) to represent the chord function is because \( \theta \) looks like a chord inside an ellipse. Definition 4.4 is a fine definition of a function, but the chord function will be more useful to us if we parameterize it in the following way:

Let \( H_w \) and \( H_{-w} \) be the support hyperplanes for \( \phi \) with normal vectors \( w \) and \(-w\) respectively, at the points \( h_w \) and \( h_{-w} \) respectively (in general the choice of points \( h_w \) and \( h_{-w} \) are not unique here, unless \( \phi \) is strictly convex, regardless of the point choices, the following still holds). Let \( h_wh_{-w} \) be the line segment (or in this case a chord of \( \phi \)) connecting the points \( h_w \) and \( h_{-w} \). With this line segment, we form an alternative definition of the chord function of \( \phi \) in the direction \( w \):

\[
\theta_w^\perp : h_wh_{-w} \to \mathbb{R}^+ \cup \{0\}
\]

which is defined by

\[
\theta_w^\perp(a) = \{\text{the length of the chord segment in } K \text{ that passes through } a\}
\]

Figure 21 shows the situation with some convex bodies: the first couple are arbitrary convex bodies, chosen to see how the chord length function can vary over different sets. The last example is a convex body satisfying all the assumptions laid out in Theorem 3.1. I urge the reader to think about some properties that we might expect the chord function of convex bodies to have, and then think about what extra properties the chord function might have, if we impose some of our assumptions in Theorem 3.1, such as smoothness or strict convexity; as we will see soon, this is exactly out next step in this interlude.

One general trend in all of these examples is that the chord function is always continuous, this is proven in the following lemma:

**Lemma 4.5.** For any convex body \( \phi \), and any \( w \in \mathbb{S}^1 \) the chord function of \( \phi \) in direction \( w \), \( \theta_w^\perp : h_wh_{-w} \to \mathbb{R}^+ \cup \{0\} \) is continuous.

**Proof.** We prove this by contradiction, suppose \( \theta_w^\perp \) is not continuous, then there exists a strictly increasing/decreasing (in regards to \( h_wh_{-w} \)) converging sequence of points \( a_i \in h_wh_{-w} \) for \( i \in \mathbb{N} \) where \( \lim_{i \to \infty} a_i = a \), and such that:

\[
\lim_{i \to \infty} \theta_w^\perp(a_i) \neq \theta_w^\perp(a)
\]
Instead of thinking of the $a_i$’s as points in the line segment $h_w h_{-w}$, we can equivalently treat them as chords (since every chord in $K$ uniquely corresponds with an $a \in h_w h_{-w}$ and vice versa). We have two cases to consider with the assumption $\lim_{i \to \infty} \theta_{-w}(a_i) \neq \theta_{-w}(a)$.

1. Suppose $\lim_{i \to \infty} \theta_{-w}(a_i) < \theta_{-w}(a)$, then let $\kappa = H \cap \phi$ be the chord corresponding to the point $a$ and let $\kappa_i$ be the chord corresponding to the point $a_i$, for each $i \in \mathbb{N}$. Since $\lim_{i \to \infty} a_i = a$, the chords, $\kappa_i$, must converge to a chord that also intersects $a$ and is perpendicular to the direction $w$ (since all of the chords in our sequence are perpendicular to $w$). This means both $\kappa$ and $\lim_{i \to \infty} \kappa_i$ will both be contained in the hyperplane $H$. Since the length $\lim_{i \to \infty} \kappa_i$ is strictly less than $\kappa$. One of $\kappa$’s end points must not be contained in $\lim_{i \to \infty} \kappa_i$. The following diagram illustrates the situation:

Figure 22: The limit of the chords $\kappa_i$ being smaller than the chord at the limit, $\kappa$

With this diagram we see that the end points of the converging sequence of chords of $\kappa_i$, will converge to the end point of $\lim_{i \to \infty} \kappa_i$, which will be contained in $\kappa$. However, from our diagram we see that there must exists a $k \in \mathbb{N}$ sufficiently large such that the line segment between the end point of $\kappa$ and the end point of $\kappa_k$ is not contained in $\phi$.

2. $\lim_{i \to \infty} \theta_{-w}(a_i) > \theta_{-w}(a)$. This case is quite analogous to the previous one, instead of our chord $\kappa$, corresponding to the point $a$, poking outside of limit chord, we instead have that the limit chord will be poking outside of $\kappa$ (in this case “poking outside” refers to having a boundary point that is not contained in the other chord in the hyperplane $H$). The following diagram illustrates the situation:

Figure 23: The limit of the chords $\kappa_i$, being larger than the chord of the limit $\kappa$
From the diagram we see that since the limit chord pokes outside of $\kappa$, the boundary of $\phi$ where the limit chord pokes out must be open, but this is a contradiction as $\phi$ is a convex body and in particular closed.

Knowing that $\theta^{\perp w}$ is continuous, I will prove another property that $\theta^{\perp w}$ will have; but instead of considering an arbitrary convex body, for this property, we must restrict our attention to strictly convex, convex bodies. Considering the chord function of a square, which is not strictly convex, will give a straightforward counterexample to the following statement.

**Lemma 4.6.** Let $\phi \subseteq \mathbb{E}^2$ be a strictly convex convex body, and let $w \in S^1$. Then the chord function of $\phi$ in the direction $w$ will have a unique element in the domain which obtains the maximum value of the chord function; in other words, $\phi$ has a unique chord of maximum length perpendicular to $w$.

**Proof.** We prove this by contradiction, suppose $\phi$ had two chords of maximum length, $\kappa$ and $\kappa'$, both perpendicular to $w$. Immediately, we obtain that the parallelogram formed by these two chords must be contained in $\phi$ as $\phi$ is convex. Figure 24 illustrates this:

![Figure 24: The convex hull of the two chords $\kappa$ and $\kappa'$](image)

Since these chords have endpoints that belong to the boundary of a convex body, there must be two segments of the boundary of $\phi$ that connects the endpoints of each chord of maximum length. Since the parallelogram must be contained in $\phi$, these boundary segments must not pierce the interior of the parallelogram. I claim that the boundary segments also cannot be outside of the parallelogram. This is because if a boundary segment did go outside the parallelogram we would obtain a chord with greater length then our maximum length chords, the diagram below shows this:

![Figure 25: If the $bd(\phi)$ is outside the parallelogram](image)

Since the boundary segments connecting the end points of our chords can neither go outside nor inside the parallelogram, it must be the case that the boundary segment of $\phi$ connecting the end points, is the line segment connecting the end points. But this is a contradiction as $\phi$ was assumed to be strictly convex and hence cannot contain a line segment in its boundary.

We are almost done our interlude in chord theory, we need only one more lemma which will lead to the main theorem we are trying to derive from this section:
Lemma 4.7. Let \( \phi \subseteq \mathbb{E}^2 \) be a strictly convex convex body, and let \( w \in S^1 \). Let \( \theta_{\perp w} : \overline{hw h_{-w}} \to \mathbb{R}^+ \cup \{0\} \) be the chord function for \( \phi \) in the direction \( w \), and let \( \kappa_{\max} \) be the unique chord of maximum length with the corresponding point \( m \in \overline{hw h_{-w}} \) being the point contained in this maximum chord. Then \( \theta_{\perp w} \) will be a strictly increasing function on the restricted domain \( h_w m \) and will be a strictly decreasing function on the domain \( mh_{-w} \).

Proof. The first part to this proof is the realization that \( \theta_{\perp w} \) cannot stagnate over either of the restricted domains. The reason is that if \( \theta_{\perp w} \) did stagnate, this would imply that the boundary of our convex body that is spanned by the end points of these chords would have to form a line segment, a contradiction as we assumed \( \phi \) is strictly convex.

The last thing I need to show for this lemma is that we cannot have a local maximum on \( \theta_{\perp w} \) besides at the point \( m \). I will prove this by contradiction, suppose there was a local maximum of \( \theta_{\perp w} \) at a point \( m' \), this implies the chord that goes through \( m' \) which is perpendicular to \( w \), call it \( \kappa_{m'} \), will be the local maximum in terms of the length of chords around it. Let \( P \) be the convex hull of the chords \( \kappa_{\max} \) and \( \kappa_{m'} \) the following diagram depicts the situation:

![Figure 26: The convex hull of \( \kappa_{m'} \) and \( \kappa_{\max} \)](image)

It can be easily shown that since the chords around the chord \( \kappa_{m'} \), have strictly smaller length than \( \kappa_{m'} \), there must be an end point of a chord in \( \phi \) that belongs to the interior of \( P \). This is a contradiction: as the end point of this curve is a boundary point of \( \phi \), which because \( \phi \) is a convex body we know that it has a support plane at this end point, but any support plane through this point will not contain \( P \) which would imply \( P \) is not be contain in \( \phi \) a contradiction.

With these lemmas completed we can now state the main theorem of this interlude, which practically implies our base case:

Theorem 4.8. Let \( \phi \) be a strictly convex, convex body, and let \( w \in S^1 \) and let \( \eta \) be the length of the maximum chord in \( \phi \) perpendicular to \( w \). Then given a real number \( r \in (0, \eta) \) there exists exactly two chords in \( \phi \) that are perpendicular to \( w \) with length \( r \).

Proof. The first thing to note is that since \( \phi \) is strictly convex, the chord function \( \theta_{\perp w} \) will be zero at the point \( hw \) and \( h_{-w} \). Then since the chord function is strictly increasing from zero to \( \eta \) and then strictly decreasing from \( \eta \) to zero, we must pass the value \( r \) exactly twice. Figure 27 shows this in action:

![Figure 27: Example of \( \theta_{\perp w} \) for a strictly convex, convex body](image)
4.3 Back to the base case

With our interlude completed we can now finish up the proof of our base case. Recall that $\phi_1$ and $\phi_2$ are translates of a strictly convex, smooth, convex body $\phi$, I aim to show that the intersection of $\phi_1$ and $\phi_2$ has exactly two singularities. To this end I will prove that $bd(\phi_1) \cap bd(\phi_2)$ will have only two points in it.

Recall that since $\phi_1$ and $\phi_2$ are translations of $\phi$ there exists a translation $\tau$ that takes $\phi_1$ to $\phi_2$ and we let $u \in S^1$ be the normalized direction of this translation. Let $w \in S^1$ be the vector perpendicular to $u$ and consider the set $K_1$ of all chord perpendicular to $w$ that are in $\phi_1$.

For an arbitrary chord $\kappa \in K_1$ where $\kappa = \{H \cap \phi\}$ for some hyperplane $H$, since the direction of our translation, $\tau$, is perpendicular to the normal of the hyperplane $H$, we see that

$$\tau(\kappa) \subseteq H$$

It is also easily seen that since

$$\bigcup_{\kappa \in K_1} \kappa = \phi_1$$

we must have that

$$\phi_2 = \tau(\phi_1) = \tau\left( \bigcup_{\kappa \in K_1} \kappa \right)$$

But

$$\tau\left( \bigcup_{\kappa \in K_1} \kappa \right) = \bigcup_{\kappa \in K_1} \tau(\kappa)$$

This implies

$$bd(\phi_1) \cap bd(\phi_2) = bd\left( \bigcup_{\kappa \in K_1} \kappa \right) \cap bd\left( \bigcup_{\kappa \in K_1} \tau(\kappa) \right)$$

Since $\kappa \in K_1$ is contained in the hyperplane $H$, the only points in $\phi_2$ that can intersect $\kappa$ will be points that also belong to $H$. However, this is easily seen from our discussion above that the only points in $\phi_2$ that are also in $H$ will be $\tau(\kappa)$. Figure 28 illustrates this:

Figure 28: Orange chords of $\phi_1$ being mapped to the blue chords of $\phi_2$

Figure 28 shows that there are only two points of intersection between these two translates because by Theorem 4.8 there is only two chords that have the length of the distance of our translation (the distance of our translation must be strictly between zero and length of the chord of maximum length perpendicular to $w$, as $\tau$ is not the identity and $int(\phi_1 \cap \phi_2) \neq \emptyset$) and the boundaries of $\phi_1$ and $\phi_2$ can only intersect when one end point of a chord is mapped to its other endpoint; this suggests that the chord has a length equalling to the distance of the translation, $\tau$. This concludes our base case. 

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4.4 Inductive Step

Our inductive hypothesis is that the intersection of \(n\) translates of \(\phi\), \(\Phi' = \bigcap_{i=1}^{n}\phi_i\) (assuming the intersection has no redundancies and has non-empty interior) will have exactly \(n\) singular points. I want to show that this implies that the intersection of \(n+1\) translates of \(\phi\), \(\Phi = \bigcap_{i=1}^{n+1}\phi_i\) (assuming the intersection has no redundancies and has non-empty interior) has exactly \(n+1\) singular points.

Let \(\Phi = \bigcap_{i=1}^{n+1}\phi_i\) be the intersection of \(n+1\) translates of \(\phi\), where the intersection is non-redundant and has non-empty interior. Let \(\Phi' = \bigcap_{i=1}^{n}\phi_i\) which is easily seen to be an intersection of \(n\) translates, with no redundancies and having non-empty interior. From our inductive hypothesis we have that:

\[
\left| \text{sing}(\Phi') \right| = \left| \text{sing}(\bigcap_{i=1}^{n}\phi_i) \right| = n
\]

I will show that intersecting \(\phi_{n+1}\) with \(\Phi' = \bigcap_{i=1}^{n}\phi_i\) (this intersection is just \(\Phi\)) will add exactly one singularity.

\textbf{Proof.} I will prove this by contradiction. We have two cases:

Case 1: Suppose \(\phi_{n+1}\) adds no singularities, in other words \(\left| \text{sing}(\Phi' \cap \phi_{n+1}) \right| = n\)

This case is relatively easy to prove via contradiction, since \(\Phi' \cap \phi_{n+1}\) has \(n\) singularities, it will have exactly \(n\) “edges.” For our purposes an edge, in this intersection, will be the set of boundary points strictly in between two adjacent singularities. This begs the definition: we call two singularities adjacent in \(\Phi'\) if there exists a boundary segment of \(\Phi'\) connecting these singularities with only regular points. Since all points on this edge are regular, each point must only belong to the boundary of one translate by Lemma 4.2; in fact all points along an edge will belong to the boundary of the same translate, as if two points in an edge belonged to the boundary of different translates, then in the boundary arc between these two points (which is contained in the edge) we must have a point of cross-over, that is a point that belongs to the boundary of two translates, which would imply that there is a singularity between the two points, which would imply that these two points do not belong to the same edge. With this, since we supposed \(\Phi' \cap \phi_{n+1}\) has only \(n\) edges, each edge can only correspond with one translate, this implies that at most \(n\) out of the \(n+1\) translates are responsible for all the edges of \(\Phi' \cap \phi_{n+1} = \Phi\), but this implies that there is at least one edge which is redundant, a contradiction as we supposed our intersection, \(\Phi\), to not be redundant.

Case 2: Suppose \(\phi_{n+1}\) adds 2 or more singularities, in other words \(\left| \text{sing}(\Phi' \cap \phi_{n+1}) \right| \geq n + 2\). In order to derive our contradiction for this case we need the following lemma:

\textbf{Lemma 4.9.} Let \(\phi_1\) and \(\phi_2\) be two distinct translates of a strictly convex, smooth, convex body \(\phi\). Then

\[
\Gamma_{\phi_1}(bd(\phi_1) \setminus \{\phi_2\}) \subseteq S^1
\]

Will have spherical measure greater than \(\pi\). Let \(m\) denote the spherical measure, which allows us to rephrase this to:

\[
m(\Gamma_{\phi_1}(bd(\phi_1) \setminus \{\phi_2\})) > \pi
\]

Before proceeding to the proof, to give the reader an idea of what Lemma 4.9 is saying, consider Figure 29, which illustrates the statement:

\textbf{Proof.} Let \(\tau\) be the translation that takes \(\phi_1\) to \(\phi_2\), in the normalized direction \(u \in S^1\). Let \(w, -w \in S^1\) be the vectors perpendicular to \(u\), and let \(H_w\) and \(H_{-w}\) be the support planes for \(\phi_1\) at the points \(h_w\) and \(h_{-w}\) with normal vectors \(w\) and \(-w\) respectively. The Figure 30 allows us to visualizes the labels:
As we saw from our interlude in chord theory \( \phi_2 = \tau(\phi_1) \) will be contained in the hyperplanes \( H_w \) and \( H_{-w} \) and will support \( \phi_2 \) at the point \( h'_w \) and \( h'_{-w} \) respectively. Figure 31 depicts this situation:

I claim the only place \( \text{bd}(\phi_2) \) can intersect \( \text{bd}(\phi_1) \) is on the “right side” of \( \text{bd}(\phi_1) \). By the right side of \( \phi_1 \) I mean

\[
\Gamma_{\phi_1}^{-1}(H_u) = \{ x \in \text{bd}(\phi_1) | \Gamma_{\phi_1}(x) \in H_u \}
\]

where \( H_u \subseteq S^1 \) is the open hemisphere of \( S^1 \) with center \( u \). This can equivalently be described more intuitively as the set of boundary points of \( \phi_1 \) strictly between \( h_w \) and \( h_{-w} \) that lie in direction \( u \) from \( h_w \) and \( h_{-w} \).

Why it is that the only place \( \text{bd}(\phi_2) \) can intersect \( \text{bd}(\phi_1) \) is on the right side of \( \text{bd}(\phi_1) \)? Well suppose the left side of \( \text{bd}(\phi_2) \) intersected the left side of the \( \text{bd}(\phi_1) \) (the left side here is the same definition as the right side but replacing \( u \) with \( -u \)). Then we must have more than two points of boundary intersection, a contradiction to our base case where we proved that the intersection of two translates will only have two
spots of boundary intersection. This can be seen with the help of the following diagram (the blue dots represent the boundary intersections):

Figure 32: The left side of $\phi_2$ cannot intersect the left side of $\phi_1$

So $bd(\phi_2)$ may only intersect the right side of $bd(\phi_1)$ which implies that

$$m(\Gamma_{\phi_1}(bd(\phi_1) \cap \phi_2)) \leq \pi$$

Since $bd(\phi_1) \cap \phi_2$ is contained in the right side of $\phi_1$. It is easily seen that $bd(\phi_1) \cap \phi_2$ will be a closed boundary segment of $\phi_1$ which implies the Gauss image of this boundary segment will be closed as well. This together with the fact that a closed set inside an open hemisphere of $S^1$ will have measure strictly less than $\pi$ gives us the following inequality:

$$m(\Gamma_{\phi_1}(bd(\phi_1) \cap \phi_2)) < \pi$$

which implies since $bd(\phi_1) = (bd(\phi_1) \cap \phi_2) \cup (bd(\phi_1) \setminus \phi_2)$ and $m(\Gamma_{\phi_1}(bd(\phi_1))) = 2\pi$ that

$$m(\Gamma_{\phi_1}(bd(\phi_1) \setminus \phi_2)) > \pi$$

This completes the proof of this lemma.

**Remark 4.10.** The Gauss images of each edge in $\Phi'$ will have spherical measure strictly less than $\pi$.

**Proof.** The lemma implies this immediately if we consider only the intersection of two translates. To see this is true for the intersection of $n$ translates, take an edge $E_i$ with corresponding translate $\phi_i$, then we know that $bd(\phi_i) \cap \phi_j$ for some $\phi_j$, will have a Gauss image with spherical measure less than $\pi$. Noting that $E_i \subseteq bd(\phi_i) \cap \phi_j$ completes the justification why the Gauss image of $E_i$ will have spherical measure less than $\pi$.

With this lemma we can now derive our contradiction to our second case, where $\phi_{n+1}$ adds two or more singularities to $\Phi'$. We separate this case into two cases: either $sing(\Phi') \subseteq sing(\Phi)$ or $sing(\Phi') \not\subseteq sing(\Phi)$.

2.a) Suppose $sing(\Phi') \subseteq sing(\Phi)$. This implies that $sing(\Phi') \subseteq \Phi$ which implies that $sing(\Phi') \subseteq \phi_{n+1}$.

Claim: If $sing(\Phi') \subseteq \phi_{n+1}$ then $\Phi' \subseteq \phi_{n+1}$ (which would make $\phi_{n+1}$ redundant in the intersection of $\Phi$).

**Proof.** I will prove this by contradiction. Suppose $sing(\Phi') \subseteq \phi_{n+1}$ but $\Phi' \not\subseteq \phi_{n+1}$, then since $\Phi'$ contains points in $\phi_{n+1}$ but is not contained in $\phi_{n+1}$, there must be a boundary segment of $\Phi'$ that passes outside of $\phi_{n+1}$, since all the singular points are contained inside of $\phi_{n+1}$ the only points that can pass outside of $\phi_{n+1}$ must belong to a singular edge. Figure 33 shows the situation:

Let $E_i$ be an edge of $\Phi'$ that passes outside of $\phi_{n+1}$ where $\phi_i$ is the translate corresponding to that edge. By our previous remark we know that the Gauss image of $E_i$ must have spherical measure less than
π, which would imply the boundary segment that passes outside of \( \phi_{n+1} \) has measure less than \( \pi \). This leads to a contradiction as this would imply that

\[
m(\Gamma_{\phi_i}(bd(\phi_i) \setminus \{\phi_{n+1}\})) < \pi
\]

Which is a contradiction to Lemma 4.9.

2.b) Suppose \( \text{sing}(\Phi') \not\subseteq \text{sing}(\Phi) \) then there exists a singularity \( s_i \in \text{sing}(\Phi') \) such that \( s_i \notin \text{sing}(\Phi) \). I claim there must be another singularity \( s_j \in \text{sing}(\Phi') \) such that \( s_j \notin \text{sing}(\Phi) \). Why? Well if there was only one singularity \( s_i \) that was not in \( \text{sing}(\Phi) \), then all \( n - 1 \) other singularities will be contained in \( \text{sing}(\Phi) \) which in particular means every singularity besides \( s_i \) is contained in \( \phi_{n+1} \). Which implies all other edges, that are not adjacent to \( s_i \), will be contained in \( \phi_{n+1} \) (one can easily see this implication looking back to part 2.a, which showed that an edge of \( \Phi' \) can not be the only thing outside of \( \phi_{n+1} \)). Since \( s_i \) is not contained in \( \Phi \) it will also not be contained in \( \phi_{n+1} \) (since \( \Phi = \Phi' \cap \phi_{n+1} \) and \( s_i \in \Phi' \)). Hence \( \phi_{n+1} \) must intersect both edges adjacent to \( s_i \), “cutting off” the singularity \( s_i \), this is shown in Figure 34.

Figure 34: \( \phi_{n+1} \) cutting off the singularity \( s_i \), implies \( |\text{sing}(\Phi)| = n + 1 \)

However, we see that \( \phi_{n+1} \) has removed exactly one singularity, but has only added two (each one corresponding to the two edges \( \phi_{n+1} \) intersects) this is a contradiction as this would imply that

\[
|\text{sing}(\Phi)| = |\text{sing}(\Phi' \cap \phi_{n+1})| = n + 1
\]

which contradicts our assumption \( |\text{sing}(\Phi)| \geq n + 2 \).

Thus, there must be at least two singularities \( s_i, s_j \in \text{sing}(\Phi') \) such that both \( s_i \) and \( s_j \) are not contain in \( \text{sing}(\Phi) \). Figure 35 is a diagram showing our situation:

It should be noted that in order for \( \phi_{n+1} \) to cut off two singularities, it must intersect both adjacent edges for each singularity (if it didn’t then either \( \phi_{n+1} \) has cut off the entire adjacent edge, which is a
contradiction, or it will not cut of the singularity, a contradiction). Also note that \( s_i \) and \( s_j \) cant be adjacent singularities as this would imply either \( \phi_{n+1} \) cuts off the entire edge in between \( s_i \) and \( s_j \), or we have \( \phi_{n+1} \) intersecting the edge in between them twice (which is easily seen to be a contradiction using Lemma 4.9).

With this stated we see there are at least four points of intersection in \( bd(\phi_{n+1}) \cap bd(\Phi') \). Where the four points of intersection we are considering is in the 4 adjacent edges of \( s_i \) and \( s_j \). Label these four point \( n_1, n_2, n_3, n_4 \) going around the boundary of \( \Phi' \) clockwise, where \( n_1 \) is the intersection in the edge in the clockwise direction of \( s_i \). The following diagram depicts this:

Figure 36: \( bd\phi_{n+1} \) intersects \( bd(\Phi') \) in at least 4 points

Since all four of these points belong to the boundary of a strictly convex, smooth, convex body, \( \phi_{n+1} \), and are distinct, all four Gauss images of these points will be distinct:

\[
\Gamma_{\phi_{n+1}}(n_i) \neq \Gamma_{\phi_{n+1}}(n_j)
\]

For all \( i \neq j \in \{1, 2, 3, 4\} \). Since we have 4 distinct points, ordered clockwise on \( S^1 \) it must be the case that either:

\[
\angle_{cw}(\Gamma_{\phi_{n+1}}(n_1), \Gamma_{\phi_{n+1}}(n_2)) < \pi
\]

or

\[
\angle_{cw}(\Gamma_{\phi_{n+1}}(n_3), \Gamma_{\phi_{n+1}}(n_4)) < \pi
\]

Where \( \angle_{cw} \) stands for the clockwise angle between the two points on \( S^1 \). Figure 37 should give the reader an idea why this is the case:

WLOG suppose the former case:

\[
\angle_{cw}(\Gamma_{\phi_{n+1}}(n_1), \Gamma_{\phi_{n+1}}(n_2)) < \pi
\]

Let \( E_i \) be the edge of \( \Phi' \) adjacent to \( s_i \) in the clockwise direction of \( s_i \), that is to say \( E_i \) is one of the edges between \( s_i \) and \( s_j \). Let \( \phi_i \) be the translate responsible for the edge \( E_i \). By the construction of \( \Phi' \) we obtain that \( \Phi' \subseteq \phi_i \). Now consider Figure 38:
Imagine starting from the point $n_1$ in the diagram above, and going up along the boundary of $\phi_{n+1}$ (which is initially outside of both $\phi_i$ and $\Phi'$). Let $w$ be the second point of intersection in $bd(\phi_{n+1}) \cap bd(\phi_i)$ (the first being $n_1$). Since $\Phi' \subseteq \phi_i$, as we move along the boundary of $\phi_{n+1}$ we must make contact with $w$ before $n_2$ which implies that:

$$\angle_{cw}(\Gamma_{\phi_{n+1}}(n_1), \Gamma_{\phi_{n+1}}(w)) \leq \angle_{cw}(\Gamma_{\phi_{n+1}}(n_1), \Gamma_{\phi_{n+1}}(n_2)) < \pi$$

But this is a contradiction as

$$\angle_{cw}(\Gamma_{\phi_{n+1}}(n_1), \Gamma_{\phi_{n+1}}(w)) = m(\Gamma_{\phi_{n+1}}(bd(\phi_{n+1} \setminus \{\phi_i\})))$$

and by Lemma 4.9, this must be greater than $\pi$. Which completes the proof. \qed
5 Concluding Remarks

Throughout this paper three results stand out to me the most: Theorem 3.1, Theorem 4.8, and Lemma 4.9. All of these statements are about the intersection of translates of a strictly convex, smooth, convex body $\phi$ in $\mathbb{E}^2$. Theorem 4.8 and Lemma 4.9 are statements about the properties the intersection of two translates of $\phi$ will have, and as we saw in the paper, we used both of these statements in order to derive our main result Theorem 3.1.

The main challenges I faced while proving Theorem 3.1 was trying to use the properties that the intersection of two translates of $\phi$ will have, to imply properties that the intersection of an arbitrary number of translates of $\phi$ will have. The key to this implication was Theorem 4.8 and Lemma 4.9. The other significant challenge I faced was deriving the sharp assumptions of Theorem 3.1. It may seem obvious, in hindsight, to consider the intersection of strictly convex, smooth, convex bodies, since their Gauss mappings will be bijective; however, this took a considerable amount of time to realize.

The most prevalent research question that arises from this paper is, if Theorem 3.1 has any analogs in higher dimensions. As subsection 3.3 showed, the intersection of two closed balls in $\mathbb{E}^3$ can have an uncountable amount of singularities, which may make generalizing Theorem 3.1 seem implausible in higher dimension, but we need to realize that all of the singularities we derived from this intersection were not “full” singularities, which is to say that the Gauss image of these singularities do not contain an open subset of $S^2$ (the Gauss images of these singularities can be easily seen to be curves in $S^2$). From this, we see that in fact, that the intersection of two closed balls contains no full singularities. My hope is that one could derive a number for the amount of full singularities in the intersection of $n$ translates for some class of convex bodies in $\mathbb{E}^n$. However, only time will tell if this is possible.

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