

**LOCALLY-FINITE EXTENSIVE CATEGORIES, THEIR SEMI-RINGS, AND DECOMPOSITION TO CONNECTED OBJECTS**

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**Abstract.** Let $\mathcal{C}$ be the category of finite graphs. Lovász shows that the semi-ring of isomorphism classes of $\mathcal{C}$ (with coproduct as sum, and product as multiplication) is embedded into the direct product of the semi-ring of natural numbers. Our aim is to generalize this result to other categories. For this, one crucial property is that every object decomposes to a finite coproduct of connected objects. We show that a locally-finite extensive category satisfies this condition. Conversely, a category where any object is decomposed into a finite coproduct of connected objects is shown to be extensive. The decomposition turns out to be unique. Using these results, we give some sufficient conditions that the semi-ring (the ring) of isomorphism classes of a locally finite category embeds to the direct product of natural numbers (integers, respectively). Such a construction of rings from a category is a most primitive form of Burnside rings and Grothendieck rings.

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1. **Introduction and main results**

Let $\mathbb{N}$ denote the semi-ring of non-negative integers. Here, we mean by semi-ring a set $S$ equipped with two commutative binary operators $+, \times$, where both operators give monoid structures on $S$, with unit denoted by 0, 1, respectively, such that the operators satisfy the distributive laws and $0 \times x = 0$ for every $x \in S$. The
same object is often called a rig, because it is similar to the ring without the inverse with respect to + (thus, lacking “negative,” hence called “ring − negative = rig”). In this paper, a monoid means a commutative monoid.

In Lovász’s seminal work [6] (3.6)Theorem, he gives an injective semi-ring homomorphism of the “Burnside semi-ring” of the category of finite directed graphs (and more generally, of finite relational structures) to an infinite product of the copies of the semi-ring \( \mathbb{N} \).

The motivation of this study is to generalize this result in terms of category theory. We shall show that the notion of extensive categories [1] plays a central role. We shall define terminologies and state our main results, while observing the Lovász’s work.

**Definition 1.1.** A category \( \mathcal{C} \) is locally finite, if for any two objects, the corresponding hom-set is a finite set.

For a category \( \mathcal{C} \), we denote by \( \operatorname{Dec}(\mathcal{C}) \) the set of isomorphism classes of \( \mathcal{C} \). (\( \operatorname{Dec} \) means the de-categorification.) Let \( \mathcal{C} \) be a category with finite coproduct. This requires the existence of an initial object 0. We denote by \( A + B \) the coproduct. Then, \( \operatorname{Dec}(\mathcal{C}) \) has a monoid structure inherited from +, denoted by \( (\operatorname{Dec}(\mathcal{C}), +) \). This kind of constructions is well-known, in the context of Burnside rings and Grothendieck groups. Here we deal with most primitive cases, arising from the coproducts and products in a category.

**Definition 1.2.** Let \( \mathcal{C} \) be a locally finite category, and \( D \) its object. We define

\[
   h_D : \operatorname{Dec}(\mathcal{C}) \to \mathbb{N}, \quad [A] \mapsto \# \operatorname{Hom}(D, A),
\]

where \( \# \) denotes the cardinality of the set.

The following gives the definition of connectedness. \footnote{There seems no decisive name of this classical construction of semi-ring, e.g., appearing in the construction of Burnside rings. The definition is given shortly after as \( \operatorname{Dec}(\mathcal{C}, +, \times) \).}

**Definition 1.3.** An object \( C \in \mathcal{C} \) is said to be connected, if the natural mapping

\[
   \operatorname{Hom}(C, A) \coprod \operatorname{Hom}(C, B) \to \operatorname{Hom}(C, A + B)
\]

is bijective.

By definition, the following holds.

**Proposition 1.4.** Let \( \mathcal{C} \) be a locally finite category with finite coproducts. Let \( C \) be a connected object. Then,

\[
   h_C : (\operatorname{Dec}(\mathcal{C}), +) \to (\mathbb{N}, +)
\]

is a monoid morphism.

Suppose that \( \mathcal{C} \) has finite direct products (and hence a terminal object). It induces a monoid structure \( (\operatorname{Dec}(\mathcal{C}), \times) \). By definition,

\[
   \operatorname{Hom}(D, A \times B) \to \operatorname{Hom}(D, A) \times \operatorname{Hom}(D, B)
\]

is a bijection for any object \( D \), and we have the following.

\footnote{This definition requires less than the standard that requires preservation of infinite coproducts \footnote{\textsuperscript{2} §5.2, P.453}, but appropriate in the present context where infinite coproducts may not exist. Both become equivalent if the category is infinitary extensive, which is called “extensive” in \footnote{2} 5.1.1 Definition, P.449, see the footnote on Proposition \footnote{2} for the equivalence.}
Proposition 1.5. Let $C$ be a locally finite category with finite products. Let $D$ be any object. Then,

$$h_D : (\text{Dec}(C), \times) \to (\mathbb{N}, \times)$$

is a morphism of monoids.

The following definition is according to Carboni et. al., see [1, Definition 3.1 and Proposition 3.2].

Definition 1.6. Let $C$ be a category with finite coproducts and finite products. If the natural morphism

$$A \times B + A \times C \to A \times (B + C)$$

is an isomorphism, then $C$ is said to be distributive. In a distributive category, it holds that

$$A \times 0 \cong 0.$$

As a consequence, we have

Proposition 1.7. Let $C$ be a distributive category. Then, $(\text{Dec}(C), +, \times)$ is a semi-ring. If $C$ is moreover locally finite and $C$ is a connected object, then

$$h_C : (\text{Dec}(C), +, \times) \to (\mathbb{N}, +, \times)$$

is a semi-ring homomorphism.

For example, the category $\text{FinSets}$ of finite sets is distributive, and the semi-ring $(\text{Dec}(\text{FinSets}), +, \times)$ is naturally isomorphic to $(\mathbb{N}, +, \times)$. This might be considered as a definition of $\mathbb{N}$. In this case, if we take $C$ as a singleton, $h_C$ gives an isomorphism.

In a category, it holds that

$$A \cong B \Rightarrow \# \text{Hom}(X, A) = \# \text{Hom}(X, B)$$

for all objects $X$. A category where the converse holds is said to be combinatorial.

Definition 1.8. ([7, 1.7 Definition]) A locally finite category $C$ is said to be combinatorial, if for all objects $X$

$$\# \text{Hom}(X, A) = \# \text{Hom}(X, B)$$

hold then $A$ is isomorphic to $B$.

Lovász [5] proved that the categories of operations with finite structures (including the category of finite graphs) are combinatorial. Various sufficient conditions for combinatoriality are known. See Pultr [7], Isbell [5], Dawar, Jakl, and Reggio [3], Reggio [8], and Fujino and Matsumoto [4].

If any object $X$ of $C$ is a finite coproduct of connected objects $C_1, \ldots, C_n$, then

$$\text{Hom}(X, -) \cong \prod_{i=1}^{n} \text{Hom}(C_i, -)$$

follows, and to show $A \cong B$, it suffices to show $h_{C_i}(A) = h_{C_i}(B)$ for every connected $C$ if $C$ is combinatorial. Then $h_C$ is a monoid morphism, and hence $\text{Dec}(C, +)$ embeds into a direct product of copies of $(\mathbb{N}, +)$. Thus we have the following immediate consequence, which is a straightforward generalization of Lovász’s arguments.
Proposition 1.9. Let \( \mathcal{C} \) be a locally finite category with finite coproducts. Suppose that \( \mathcal{C} \) is combinatorial, and any object is a coproduct of a finite number of connected objects. Choose a representative system \( C_i \ (i \in I) \) from the set of isomorphism classes of connected objects. Then,

\[
\prod_{i \in I} h_{C_i} : (\text{Dec}(\mathcal{C}), +) \to \prod_{i \in I} (\mathbb{N}, +)
\]

is an injective homomorphism of monoids. If, moreover, \( \mathcal{C} \) is distributive, then

\[
\prod_{i \in I} h_{C_i} : (\text{Dec}(\mathcal{C}), +, \times) \to \prod_{i \in I} (\mathbb{N}, +, \times)
\]

is an injective homomorphism of semi-rings.

Thus, it becomes important whether each object is decomposed as a coproduct of a finite number of connected objects. The following are main results of this paper.

Theorem 1.10.

1. Let \( \mathcal{C} \) be a locally-finite extensive category. Then, every object is a coproduct of a finite number of connected objects.
2. Let \( \mathcal{C} \) be a category with finite coproducts such that every object \( D \) is a finite coproduct of connected objects. Then, \( \mathcal{C} \) is an extensive category.
3. Let \( \mathcal{C} \) be an extensive category. If an object \( D \) decomposes to a finite coproduct of connected objects, then the decomposition is unique, up to ordering and isomorphisms of each component.

The notion of extensive categories is established by Carboni et. al. \([1]\), and widely accepted as a natural generalization of distributive categories. For example, any topos is extensive. We shall give a definition of extensive categories in the next section. The following proposition is known.

Proposition 1.11. \([1, \text{Proposition 4.5}]\) An extensive category with finite products is distributive.

Thus, our main theorem yields the following theorem.

Theorem 1.12. Let \( \mathcal{C} \) be a locally finite, extensive, and combinatorial category. Let \( C_i \ (i \in I) \) be representatives of the set of isomorphism classes of connected objects. Then,

\[
\prod_{i \in I} h_{C_i} : (\text{Dec}(\mathcal{C}), +) \to \prod_{i \in I} (\mathbb{N}, +)
\]

is an injective morphism of monoids. If, moreover, \( \mathcal{C} \) has finite products, then

\[
\prod_{i \in I} h_{C_i} : (\text{Dec}(\mathcal{C}), +, \times) \to \prod_{i \in I} (\mathbb{N}, +, \times)
\]

is an injective morphism of semi-rings.

There is a universal way to obtain a group from a monoid (the left adjoint to the forgetful functor), called the Grothendieck group of the monoid. This makes a monoid into a group, here denoted by

\[(\text{Dec}(\mathcal{C}), +) \mapsto (\text{Dec}(\mathcal{C}), +, -)\].\]
The monoid \( \mathbb{N} \) is transferred to the additive group \( \mathbb{Z} \). The same construction makes a semi-ring into a ring, which we denote

\[
(\text{Dec}(\mathcal{C}), +, \times) \mapsto (\text{Dec}(\mathcal{C}), +, -).
\]

Theorem 1.10 implies the following

**Corollary 1.13.** Under the conditions of Theorem 1.10, we have a canonical monoid injection

\[
(\text{Dec}(\mathcal{C}), +) \hookrightarrow (\text{Dec}(\mathcal{C}), +, -),
\]

and if moreover \( \mathcal{C} \) is combinatorial, then we have an injective group homomorphism

\[
\prod_{i \in I} h_{\mathcal{C}_i} : (\text{Dec}(\mathcal{C}), +, -) \rightarrow \prod_{i \in I} (\mathbb{Z}, +).
\]

(Note that \( \prod_{i \in I} h_{\mathcal{C}_i} \) here is the extension to the Grothendieck group.) In this case, if \( \mathcal{C} \) has finite products, then (1.2) is an injection of semi-rings

\[
(\text{Dec}(\mathcal{C}), +, \times) \hookrightarrow (\text{Dec}(\mathcal{C}), +, -, \times),
\]

and (1.3) is an injective ring homomorphism

\[
(\text{Dec}(\mathcal{C}), +, -, \times) \rightarrow \prod_{i \in I} (\mathbb{Z}, +, \times).
\]

For example, consider the functor categories \( \text{FinSets}^D \) for a finite category \( D \). It is locally finite, and is shown to be combinatorial, by methods in for example [4]. It is an elementary topos, and hence is extensive [1], and has finite products. By Theorems 1.10, 1.12 and Corollary 1.13, \( \text{Dec}((\text{FinSets}^D, +, -, \times)) \) embeds into a direct product of copies of \( \mathbb{Z} \). In particular, there are no nilpotent elements. By the same argument, a similar embedding is obtained for the category of \( G \)-finite sets where \( G \) is a group, which is known as a theorem by Burnside.

2. Proof of main results

2.1. Extensive categories. This section follows Carboni et. al. [1]. For a coproduct diagram

\[
A \rightarrow A + B \leftarrow B,
\]

we call \( A \rightarrow A + B \) the coprojection, and denote by \( i_A \).

**Definition 2.1.** A category \( \mathcal{E} \) is an extensive category, if it has finite coproducts and satisfies the following conditions.

1. For any morphism \( f : A \rightarrow X_1 + X_2 \), there is a pullback \( X_1 \prod_{X_1 + X_2} A \) along the coprojection.
2. Suppose that the following diagram commutes.

\[
\begin{array}{ccc}
A_1 & \rightarrow & A & \leftarrow & A_2 \\
\downarrow & & \downarrow f & & \downarrow \\
X_1 & \rightarrow & X_1 + X_2 & \leftarrow & X_2.
\end{array}
\]

(2.1)

Then, the top row is a coproduct, if and only if the both squares are pullbacks.
Proposition 2.2. [1, Proposition 2.6]

In an extensive category, the following three squares are pullbacks. In particular, the coprojections are mono:

\[
\begin{array}{ccc}
A & \longrightarrow & A \\
\downarrow & & \downarrow \\
A & \longrightarrow & A + B
\end{array}
\quad
\begin{array}{ccc}
B & \longrightarrow & B \\
\downarrow & & \downarrow \\
B & \longrightarrow & B + A
\end{array}
\quad
\begin{array}{ccc}
0 & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & A + B
\end{array}
\]

Proof. This follows from the following commutative diagram with rows being co-products, and the second condition of Definition 2.1:

\[
\begin{array}{ccc}
0 & \longrightarrow & A_2 \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & A_1 + A_2
\end{array}
\quad
\begin{array}{ccc}
A_2 & \longleftarrow & A_2 \\
\downarrow & & \downarrow \\
A_1 + A_2 & \longleftarrow & A_2
\end{array}
\]

\[\square\]

Proposition 2.3. [1, Proposition 2.8]

In an extensive category, any morphism \(A \to 0\) is an isomorphism.

Proof. Take \(\alpha : A \to 0\). In the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\text{id}_A} & A \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
0 & \xleftarrow{\text{id}_0} & 0
\end{array}
\quad
\begin{array}{ccc}
A & \xleftarrow{\text{id}_A} & A \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
0 & \xrightarrow{\text{id}_0} & 0
\end{array}
\]

the bottom row is a direct product, the two squares are pullbacks, and hence the second condition of Definition 2.1 shows that the top row is a coproduct. Then, we have \(A \xrightarrow{\alpha} A\) and \(A \xrightarrow{\alpha} 0\to A\), and by the universality of the top row as a coproduct implies that these two coincide. Thus \(A \xrightarrow{\alpha} 0\to A\) is the identity, and \(0\to A \xrightarrow{\alpha} 0\) is the identity, hence \(A \cong 0\).

Lemma 2.4. In an extensive category, if \(f : X \to A\) and \(g : X \to B\) satisfy \(i_A \circ f = i_B \circ g : X \to A + B\), then \(X \cong 0\).

Proof. Proposition 2.2 implies that \(f\) and \(g\) factor through the pullback of \(A \to A + B \leftarrow B\), which is 0, hence the claim follows from Proposition 2.3. \(\square\)

The following proposition is a slight variant of (2, 5.2.1 Theorem, P.453).

Proposition 2.5. Let \(X\) be an object of an extensive category \(\mathcal{E}\). The following are equivalent.

1. \(X\) is not connected.
2. \(X \cong 0\), or, there are two objects \(U \not\cong 0\), \(V \not\cong 0\) such that \(U + V \cong X\).

\[\text{To be precise, (2) deals with infinitary extensive categories. This proposition proves that the notion of connectedness here coincides with that in (2, 5.2.1, P.453) for infinitary extensive categories, since the contraposition of this proposition gives the equivalent condition (vi) to the connectedness in (2, 5.2.1 Theorem, P.453).}\]
Proof. Suppose (2). If \( X \cong 0 \), then
\[
\text{Hom}(0, A) \coprod \text{Hom}(0, B) \to \text{Hom}(0, A + B)
\]
is not injective, hence by Definition 1.3 \( X \) is not connected. If \( X \cong U + V \), in
\[
\text{Hom}(U + V, U) \coprod \text{Hom}(U + V, V) \to \text{Hom}(U + V, U + V),
\]
we shall show that \( \text{id}_{U + V} \) in the right-hand side does not come from the left-hand side, hence \( U + V \) is not connected. If it does, we may assume that it comes from \( \text{Hom}(U + V, V) \), i.e., \( U + V \to V \) is an identity. Then \( i_U \) is a split epi, and mono by Proposition 2.2, hence is an isomorphism. This implies that in the right most diagram of Proposition 2.2, the bottom arrow is an isomorphism, and so is the top, which implies \( V \cong 0 \), contradicting (2). Thus, (2) implies (1).

Suppose (1). (1.1) for \( C = X \) in Definition 1.3 is not bijective. Suppose that it is not injective. Since coprojections are mono, this implies that there are \( X \to A \) and \( X \to B \), which give one same morphism \( X \to A + B \) after composing with coprojections. By Lemma 2.4, \( X \cong 0 \), which proves the claim in this case. Suppose that (1.3) is not surjective. Take an \( f : X \to A + B \) which does not come from the left. We take pullbacks
\[
\begin{array}{ccc}
X_A & \longrightarrow & X \\
\downarrow & & \downarrow f \\
A & \longrightarrow & A + B
\end{array}
\]
\[
\begin{array}{ccc}
& & \leftarrow X_B \\
& & \downarrow & \downarrow \\
& & B
\end{array}
\]
By definition of an extensive category, we have \( X \cong X_A + X_B \). Suppose that \( X_A \cong 0 \). Then, \( X \cong 0 + X_B \cong X_B \) implies that \( f \) is in the image of the composition \( X \to X_B \to B \) in \( \text{Hom}(X, B) \), which contradicts the assumption that \( f \) is not in the image. Thus, \( X_A \not\cong 0 \). Similarly, \( X_B \not\cong 0 \), and hence (2) holds.

2.2. Decomposition to connected objects. This section proves (1) and (3) in Theorem 1.10

Proposition 2.6. Suppose that an extensive category \( E \) is locally finite. Then, every object \( X \) is a coproduct of a finite number of connected objects.

Proof. Suppose that \( X \cong X_1 + X_2 + \cdots + X_n \) with \( X_k \not\cong 0 \). Let \( i_1, i_2 \) be the coprojections from \( X \) to \( X + X \) (to the left component, to the right component, respectively). By Lemma 2.4, \( i_1 X_k \neq i_2 X_k \) follows, and hence
\[
\# \text{Hom}(X_k, X + X) \geq 2.
\]
Thus
\[
\# \text{Hom}(X, X + X) = \prod_{i=1}^{n} \# \text{Hom}(X_i, X + X) \geq 2^n.
\]
By locally finiteness, \( 2^n \) is bounded above, and so is \( n \). If \( X \) is connected, then it is a coproduct of one connected object. If \( X \) is not connected, then Proposition 2.6 implies that \( X \cong 0 \) or \( X \cong U, V, U \neq 0, V \neq 0 \). In the former case, \( X \) is a coproduct of zero of connected objects, hence the claim holds. In the latter case, if both \( U \) and \( V \) are connected, then the claim holds. Otherwise, by the same procedure, we may decompose \( U \) or \( V \) into a coproduct of two non-initial objects. This procedure
stops after a finite iteration, since the number $n$ is bounded above. Thus, we have shown that $X$ is a coproduct of a finite number of connected objects. □

This proves (1) in Theorem 1.10. To show the uniqueness (3), we prepare some lemmas.

**Lemma 2.7.** Let $A, B, X,$ and $C$ be objects of an extensive category, and assume $C$ connected. Suppose that $f : C + X \to A + B$ is an isomorphism. By connectedness of $C$, we may assume that there is a $g : C \to A$ such that $fi_C = i_Ag$ (by symmetry between $A$ and $B$). Then, there exists an object $Y$ such that

$$A \cong C + Y, \ X \cong Y + B. \quad (2.4)$$

**Proof.** By $fi_C = i_Ag$ we have a commutative diagram

$$
\begin{array}{ccc}
C & \overset{g}{\longrightarrow} & A \\
\downarrow{\text{id}_C} & & \downarrow{f^{-1}i_A} \\
C & \overset{i_C}{\longrightarrow} & C + X.
\end{array}
$$

Since $f^{-1}i_A$ is mono and the left vertical arrow is an isomorphism, this diagram is a pullback. We consider the following diagram:

$$
\begin{array}{ccc}
C & \overset{g}{\longrightarrow} & A & \leftarrow Y \\
\downarrow{\text{id}_C} & & \downarrow{f^{-1}i_A} & & \downarrow{\text{id}_A} \\
C & \overset{i_C}{\longrightarrow} & C + X(\cong A + B) & \leftarrow X \\
\uparrow{f^{-1}i_B} & & \uparrow{\text{id}_B} & & \uparrow{\text{id}_B} \\
Z' & \longrightarrow & B & \longrightarrow Z.
\end{array}
$$

The left top square is the observed pullback. The right top $Y$ is defined by the pullback. By the axiom of extensive categories, we have

$$A \cong C + Y.$$

The bottom row is the pullback along $f^{-1}i_B$, giving $Z'$ and $Z$. At the left bottom square, $fi_C = i_Ag$ implies that $Z'$ is the pullback of $C \overset{g}{\longrightarrow} A \overset{i_A}{\longrightarrow} A + B$ and $B \overset{i_B}{\longrightarrow} A + B$, and is 0 by Lemma 2.4. Thus $B \cong Z' + Z \cong Z$ holds. The extensivity gives

$$X \cong Y + Z \cong Y + B.$$ □

A connected object is cancellable.

**Lemma 2.8.** For objects $X, X'$ and a connected object $C$ in an extensive category, $C + X \cong C + X'$ implies $X \cong X'$.

**Proof.** Let $f : C + X \to C + X'$ be an isomorphism. By the connectedness of $C$, either one of the following holds.

1. $fi_C = i_Cg$ holds for some $g : C \to C$.
2. $fi_C = i_Xg$ holds for some $g : C \to X$.
In the first case, by Lemma 2.7 for \( A = C \) and \( B = X' \), we have
\[ C \cong C + Y \quad \text{and} \quad X \cong Y + X', \]
where the connectedness of \( C \) and Proposition 2.5 imply that \( Y \cong 0 \), thus \( X \cong X' \).

In the second case, by Lemma 2.7 for \( A = X' \) and \( B = C \), we have
\[ A = X' \cong C + Y \quad \text{and} \quad X \cong Y + B = Y + C, \]
and hence \( X' \cong X \).

The following is an analogue to the unique factorization theorem, which proves (3) of Theorem 1.10.

Corollary 2.9. In an extensive category, suppose that an object is decomposed in two ways as
\[ f : \sum_{i=1}^{s} C_i \sim \rightarrow \sum_{i=1}^{t} D_i, \quad (2.5) \]
for connected objects \( C_1, \ldots, C_s \) and \( D_1, \ldots, D_t \). Then \( s = t \), and by changing the ordering, \( C_i \cong D_i \) for \( i = 1, \ldots, s \).

Proof. Suppose that \( s = 0 \). If \( t \geq 1 \), then there is a morphism \( D_1 \rightarrow 0 \), which is impossible by connectivity of \( D_1 \) and by Proposition 2.5. Thus \( t = 0 \), and the claim follows. Hence, we may assume that \( s, t \geq 1 \). By connectivity of \( C_1 \), there exists a \( k \) and \( g : C_1 \rightarrow D_k \) such that \( f_i C_1 = i_{D_k} g \). By Lemma 2.7 we have a \( Y \) such that
\[ C_1 \cong D_k + Y. \]
By Proposition 2.5 \( Y \cong 0 \) and \( C_1 \cong D_k \). By changing the ordering, we may assume \( C_1 \cong D_1 \). By Lemma 2.8 we have
\[ \sum_{i=2}^{s} C_i \cong \sum_{i=2}^{t} D_i. \quad (2.6) \]
By iterating the argument, we will have \( s = 0 \) or \( t = 0 \). Then, the argument at the beginning of this proof shows that \( s = t \), and \( C_i \cong D_i \) for \( i = 1, \ldots, s \).

This completes the proofs of (1) and (3) in Theorem 1.10.

2.3. Decomposability implies extensivity. This section proves (2) in Theorem 1.10. Throughout this section, we assume that \( \mathcal{C} \) is a category whose object is a finite coproduct of connected objects. We shall prove that \( \mathcal{C} \) is extensive. Every argument is about \( \mathcal{C} \).

Lemma 2.10. A coprojection
\[ X_1 \xrightarrow{i \times j} X_1 + X_2 \]
is mono.

Proof. We consider \( D \rightrightarrows X_1 \rightarrow X_1 + X_2 \). To check the mono property, note that \( D \cong \sum_i D_i \) with \( D_i \) connected, and since
\[ \text{Hom}(D, -) \cong \prod_i \text{Hom}(D_i, -), \]
it suffices to check the mono-property for each \( D_i \). In other words, we may assume that \( D \) is connected, and it suffices to show that if \( f, g : D \to X_1 \) satisfy \( i_{X_1} f = i_{X_1} g \), then \( f = g \). By connectivity of \( D \),

\[
\text{Hom}(D, X_1) \to \text{Hom}(D, X_1 + X_2), \quad f \mapsto i_{X_1} f
\]
is injective. Thus \( f = g \) follows. \( \square \)

**Lemma 2.11.** Let \( A, X \) and \( Y \) be objects, and \( f : A \to X + Y \) be a morphism. We may assume

\[
A \cong \sum_{j \in J} A_j,
\]

with \( A_j \) connected and \( J \) a finite set. Then, by Definition 1.3, for each \( j \), either one of the following two holds.

1. There exists \( g_j : A_j \to X \) such that \( f \circ i_{A_j} = i_X \circ g_j \) holds.
2. There exists \( g_j : A_j \to Y \) such that \( f \circ i_{A_j} = i_Y \circ g_j \) holds.

Let \( J_X \) be the set of \( j \) satisfying (1), and \( J_Y \) the set of \( j \) satisfying (2). Thus, \( J_X \coprod J_Y = J \).

**Proposition 2.12.** Let \( A, X \) and \( Y \) be objects, and \( f : A \to X + Y \) be a morphism, as above. Define

\[
A_X := \sum_{j \in J_X} A_j \quad \text{and} \quad A_Y := \sum_{j \in J_Y} A_j.
\]

Then, \( A \cong A_X + A_Y \) holds, and the following diagram commutes:

\[
\begin{array}{ccc}
A_X & \xrightarrow{i_{A_X}} & A \\
g_X & & \downarrow f \\
X & \xrightarrow{i_X} & X + Y
\end{array}
\quad \begin{array}{ccc}
& & \\
& & \\
& & Y
\end{array}
\]

\[
\text{(2.7)}
\]

**Proof.** The claim \( A \cong A_X + A_Y \) follows from \( J = J_X \coprod J_Y \). For commutativity, if \( j \in J_X \), then

\[
\begin{array}{ccc}
A_j & \xrightarrow{i_{A_j}} & A \\
g_j & & \downarrow f \\
X & \xrightarrow{i_X} & X + Y
\end{array}
\]

\[
\text{(2.8)}
\]

commutes. Taking the coproduct over \( J_X \), we have the left commutative square of \( \text{(2.7)} \). The commutativity at the right square follows similarly. \( \square \)

We shall prove the first condition of Definition 2.1.

**Proposition 2.13.** The left and right squares in \( \text{(2.7)} \) are pullbacks.
Proof. Take $D$, $h_1$, and $h_2$ in the following diagram with $fh_1 = i_X h_2$, and we shall show the unique existence of $k$ that makes two triangles commute.

By the same reason as in the proof of Lemma 2.10, we may assume that $D$ is connected. By connectivity of $D$ and $A = \sum_{j \in J} A_j$, there exists a unique $j \in J$ such that

\[ h_1 = i_{A_j} s, \quad s : D \to A_j. \]  

We claim that $j \in J_X$. By connectivity of $D$, we have a bijection

\[ \text{Hom}(D, X) \coprod \text{Hom}(D, Y) \to \text{Hom}(D, X + Y). \]  

By $fh_1 = i_X h_2$, $fh_1 \in \text{Hom}(D, X + Y)$ lies in the image of $\text{Hom}(D, X)$. Suppose that $j \in J_Y$. Then by Lemma 2.11

\[ fh_1 = fi_{A_j} g j s : D \to A_j \xrightarrow{g j} Y \xrightarrow{i Y} X + Y. \]

This implies that $fh_1 \in \text{Hom}(D, X + Y)$ also comes from $\text{Hom}(D, Y)$, which is absurd. Thus, $j \in J_X$. Consequently, we have $g j : A_j \to X$, which makes the diagram (2.8) commute, namely,

\[ fi_{A_j} = i_X g j. \]

We define $k : D \to A_X$ as a composition

\[ D \xrightarrow{k} A_j \xrightarrow{i_{A_j}} A_X. \]

By (2.10), this $k$ satisfies

\[ i_{A_X} k = i_{A_X} i_{A_j} s = i_{A_j} s = h_1, \]

which makes the upper triangle in (2.9) commute. Since $i_{A_X}$ is mono by Lemma 2.10 and Proposition 2.12, such a $k$ is unique. By the commutativity of the square in the diagram (2.9), $i_X h_2 = fh_1 = f i_{A_X} k = i_X g k$. By Lemma 2.10, $i_X$ is mono and hence $h_2 = g k$. This shows that $A_X$ is the pullback. The same argument shows that $A_Y$ is the pullback.

This proves the first condition of Definition 2.1. We shall prove the second condition (2.1). Assume that the both squares are pullbacks in (2.1). By Proposition 2.13 we have $A_{X_1} := \sum_{j \in J_{X_1}} A_j$ is a pullback and hence isomorphic to $A_1$, and similarly $A_{X_2} := \sum_{j \in J_{X_2}} A_j \cong A_2$. Thus,

\[ A_1 \cong A_{X_1} \to A \leftarrow A_{X_2} \cong A_2 \]

is a coproduct. Conversely, suppose that the top row of (2.1) is a coproduct. Let $A_1 \cong \sum_{j \in J} B_j$ with connected $B_j$, and $A_2 \cong \sum_{k \in K} C_k$ with connected $C_k$, with $J \cap K = \emptyset$. Then we have $A \cong \sum_{j \in J} B_j + \sum_{k \in K} C_k$. The commutativity of the left square of (2.1) implies that $\sum_{j \in J} B_j$ is constructed from $A$ by the method described in Proposition 2.12 hence $A_1$ is a pullback by Proposition 2.13. Similarly,
$A_2$ is a pullback. This proves that $C$ is extensive, and completes the proof of (2) in Theorem 1.10.

2.4. **Injectivity preserved by Grothendieck groups.** This section proves Corollary 1.13. We show the injectivity of (1.2). Under the condition of Theorem 1.10, every object of $C$ is a finite coproduct of connected objects in a unique way. This shows that $(\text{Dec}(C), +)$ is a free (commutative) monoid generated by the representatives $C_i (i \in I)$ of the isomorphism classes of connected objects, hence a (possibly infinite) direct sum of copies of $\mathbb{N}$. It is easy to see that its Grothendieck group is the direct sum of copies of $\mathbb{Z}$, to which the free monoid injects. Thus (1.2) is injective. Note that $\prod_{i \in I} \mathbb{N}$ injects to its Grothendieck group $\prod_{i \in I} (\mathbb{Z}, +)$, hence

$$(\text{Dec}(C), +) \to \prod_{i \in I} (\mathbb{Z}, +)$$

is injective. The injectivity of (1.3) follows from the next general lemma.

**Lemma 2.14.** Let $M$ be a free commutative monoid and $g : M \to G$ a monoid injection to a commutative group $G$. By the universality, $g$ extends to a group homomorphism from the Grothendieck group $(M, +, -)$ to $G$,

$$f : (M, +, -) \to G.$$ 

Then, $f$ is injective.

**Proof.** Let $c_j (j \in J)$ be a free generator of the free monoid $M$, and suppose that $f$ maps $\sum_{j \in J} a_j c_j \in (M, +, -)$ to $0 \in G$ with $a_j \in \mathbb{Z}$. This implies that $f\left(\sum_{a_j > 0} a_j c_j - \sum_{a_j < 0} (-a_j) c_j\right) = 0$, hence $g\left(\sum_{a_j > 0} a_j c_j\right) = g\left(\sum_{a_j < 0} (-a_j) c_j\right)$, and by the injectivity of $g$, we have $\sum_{a_j > 0} a_j c_j = \sum_{a_j < 0} (-a_j) c_j$, hence the both sides are zero because of the definition of free generators, and the injectivity of $f$ follows.

It is well-known that taking the Grothendieck group of a semi-ring gives a ring, and is a functor. Thus, the rest of Corollary 1.13 follows.

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