REPRESENTABILITY OF PI-ALGEBRAS

BE’ERI GREENFELD AND LOUIS ROWEN

Abstract. This note concerns the still open question of representability of Noetherian PI-algebras. Extending a result of Rowen and Small (with an observation of Bergman) that every finitely generated module over a commutative Noetherian ring containing a field is representable, we provide a representability machinery for a Noetherian PI-algebra \( R \) containing a field, which includes the case that \( R \) is finite (as a module) over a commutative subalgebra isomorphic to \( R/N \). We construct a family of non-representable PI-algebras demonstrating the sharpness of these results, as well as of some well known previous representability results.

1. Introduction

One of the major problems in the theory of algebras satisfying polynomial identities (PI-algebras) is to determine whether a given PI-algebra is representable, namely, whether it embeds into a matrix ring over a field, or more generally a commutative ring. To distinguish between the two cases, we call the latter case weakly representable (after [8]).

We produce a representability machinery, allowing one to deduce that, under certain conditions, a given Noetherian PI-algebra is representable, including the case that \( R \) is finite over a commutative subalgebra isomorphic to \( R/N \). The significance of this result is in that there is an affine PI-algebra satisfying ACC on ideals satisfying this property, which is not weakly representable (§3.1).

In another direction, a ring \( R \) with nilradical \( \text{Nil}(R) \triangleleft R \) is semiprimary if \( R/\text{Nil}(R) \) is semisimple Artinian. Amitsur-Rowen-Small proved that a semiprimary PI-algebra whose radical squared zero is weakly representable (appears in [8, Section 6]). We present a semiprimary PI-algebra with radical cubed zero, which is non-weakly representable (§3.2).

Even the question of whether arbitrary Artinian PI-algebras are representable still seems to be open [1, Question 5 in page 388].

2. Representability

2.1. A representability machinery. We write \( N \) for \( \text{Nil}(R) \), which is nilpotent when \( R \) is left Noetherian. We present this result in a fair amount of generality.

Theorem 2.1. Let \( R \) be a left Noetherian algebra over a field, which contains a weakly Noetherian subalgebra \( W \subseteq R \) such that \( R/N \) is a finitely generated left module over \( W \), the reduction of \( W \) modulo \( N \), satisfying the condition:

\[ W[c^{-1}] \text{ is finite over its center, for some } c \text{ of } C := \text{Cent}(W) \text{ which is regular in } R. \]

2010 Mathematics Subject Classification. Primary: 16R20, 16P20, 16P40 Secondary: 16P60, 16S50.

Key words and phrases. PI-algebra, Noetherian algebra, representable algebra, universal derivations.

The authors thank Lance Small for many helpful insights.

This work was supported by the U.S.-Israel Binational Science Foundation (grant no. 2010149).
Then $R$ is representable.

Proof. Let $d$ be the nilpotency index of $N$. Consider $R$ as a (left) $W$-module via the natural action of $W$, being a subalgebra of $R$.

Since $R/N$ is finite over $W$, we can write:

$$R/N = \overline{Wv_1} + \cdots + \overline{Wv_q}$$

Pick arbitrary lifts $r_1, \ldots, r_q \in R$ such that $W_r = R_0 + N$. Therefore:

$$R = Wr_1 + \cdots + Wr_q + N.$$  

We now prove by induction on $e$ that $N^{d-e}$ is a finitely generated $W$-module. First take $e = 1$. Since $R$ is left Noetherian, we can write $N^{d-1}$ as a finitely generated $R$-module:

$$N^{d-1} = \sum_{i=1}^{k} Ru_i = \sum_{i=1}^{k} (Wr_1 + \cdots + Wr_q + N)u_i = \sum_{i=1}^{k} (Wr_1 + \cdots + Wr_q)u_i$$

where the third equality follows since $Nu_1, \ldots, Nu_k \subseteq N^{d} = 0$.

Now assume that the claim was proved for $e' < e$. Again, since $R$ is left Noetherian we can write $N^{d-e}$ as a finitely generated $R$-module:

$$N^{d-e} = \sum_{i=1}^{l} Rv_i = \sum_{i=1}^{l} (Wr_1 + \cdots + Wr_q + N)v_i = \sum_{i=1}^{l} (Wr_1 + \cdots + Wr_q)v_i + N^{d-e+1}$$

But by the induction hypothesis, $N^{d-e+1}$ is a finitely generated $W$-module, so $N^{d-e}$ is finitely generated as well. In particular, we get that $N$ itself is a finitely generated $W$-module, and therefore so is $R$.

We can localize $R[c^{-1}]$ to be a finite module over $W[c^{-1}]$, and thus over $C[c^{-1}]$, which is Noetherian by the Eakin-Formanek theorem [7, Theorem 5.1.12] since $W[c^{-1}]$ is weakly Noetherian. But then $\text{End}_{C[c^{-1}]} R[c^{-1}]$ is representable by [3, Corollary 3.4]. Now $R^{op}$ acts by right multiplication on itself, and this action is $C$-equivariant. Therefore $R^{op}$ is representable; considering the transpose of this representation, we obtain that $R$ is representable as well. \hfill \Box

Note that the condition satisfied in either of the following situations:

(i) $W$ already is finite over its center, in particular if $W$ is commutative;

(ii) $W$ is a semiprime PI-algebra and any regular central element $c$ of $W$ is regular in $R$, since there is such $c$ as seen via [7, Theorem 1.8.48]. Note that if $R$ is uniform as a $W, C$-bimodule then by Fitting’s lemma, $\text{Ann}_R(c^k) = 0$ for some $k$ (since $Rc^k \neq 0$), so $\text{Ann}_R(c) = 0$.

**Corollary 2.2.** Suppose $R$ is a left Noetherian algebra over a field, and $R/N$ is finite over a commutative polynomial ring $F[x]$. Then $R$ is representable.

**Proof.** Lift $x$ to an element $c$ of $R$. Then $W := F[c]$ is a commutative algebra satisfying the hypothesis of the theorem, since $W \approx F[x]$ is Noetherian. \hfill \Box

**Corollary 2.3.** Let $R$ be a left Noetherian algebra with nilpotent radical $N$, such that $R/N$ is affine of linear growth. Then $R$ is representable.

**Proof.** Since $R/N$ is affine of linear growth, by [9] it is a finitely generated module over a central polynomial ring in one variable $F[t] \subseteq R/N$. \hfill \Box
3. Examples

3.1. A non-representable affine PI-algebra with ACC on ideals. We give an example of a non-weakly representable affine PI-algebra satisfying ACC on ideals. Moreover, the quotient of our example by its nilpotent radical is a polynomial ring in one variable, thus emphasizing the sharpness of Corollary 2.2. (Compare with [6].) Let $A$ be an $F$-algebra and $M$ an $A$-bimodule. Given an $F$-linear map $B : M \otimes_A M \rightarrow F$, we can define an $F$-algebra:

$$R = \begin{pmatrix} F & M & F \\ 0 & A & M \\ 0 & 0 & F \end{pmatrix}$$

whose multiplication is given by:

$$\left( \begin{array}{ccc} \alpha_1 & v & \lambda \\ 0 & f & w \end{array} \right) \left( \begin{array}{ccc} \alpha'_1 & v' & \lambda' \\ 0 & f' & w' \end{array} \right) = \left( \begin{array}{ccc} \alpha_1 \alpha'_1 + v f' & \alpha_1 \lambda' + \alpha_2 \lambda + B(v, w') \\ 0 & f f' & f w' + \alpha_2 \alpha'_2 \end{array} \right)$$

Since $B$ is $A$-equivariant, namely $B$ is defined over $M \otimes_A M$, this multiplication law endows $R$ with a well-defined $F$-algebra structure. To check associativity, note that the only difficulty would be checking the products involving the 1, 2 and 3 positions, but

$$(\alpha_1 e_{1,1} \cdot v' e_{1,2}) \cdot w'' e_{2,3} = B(\alpha_1 v', w'') = \alpha_1 B(v', w') = (\alpha_1 e_{1,1} \cdot (v' e_{1,2} \cdot w'' e_{2,3})).$$

We now specify $A$ and $M$. Let $A = F[t]$ and let $M = Fu_1 + Fu_2 + \cdots$ be a countable dimensional $F$-vector space. We consider $M$ as an $F[t]$-bimodule through:

$$tu_i = ut_i = u_{i+1}$$

We now specify $B$, writing it as a bilinear form $B$. Set:

$$B(u_i, u_j) = \begin{cases} 1, & \exists t \geq 1: i + j = 2^t \\ 0, & \text{otherwise} \end{cases}$$

It is easy to verify that:

$$B(u_i v^j, u_k) = B(u_{i+j}, u_k) = B(u_i, u_{j+k}) = B(u_i, v^j u_k)$$

So $B$ is a well-defined $F$-linear map defined over $M \otimes_F[t]M$ and thus $R$ is a well defined $F$-algebra:

$$R = \begin{pmatrix} F & M & F \\ 0 & F[t] & M \\ 0 & 0 & F \end{pmatrix}$$

Proposition 3.1. The algebra $R$ is an affine PI-algebra whose radical cubed zero and $R/N \cong F \times F[t] \times F$.

Proof. The algebra $R$ is generated over $F$ by the following elements:

$$E_{1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{2,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{3,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
Observe that the nilpotent radical of $R$ is:

$$N = \begin{pmatrix} 0 & M & F \\ 0 & 0 & M \\ 0 & 0 & 0 \end{pmatrix}$$

So $N^3 = 0$ and $R/N \cong F \times F[t] \times F$. Moreover, $R$ satisfies the polynomial identity:

$$[X_1, X_2][X_3, X_4][X_5, X_6] = 0,$$

as claimed.

**Proposition 3.2.** The algebra $R$ does not satisfy ACC on (left) annihilators.

**Proof.** For a vector $w \in M$ denote: $w^\perp = \{v \in M \mid B(v, w) = 0\}$. Notice that:

$$\text{Ann} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u_1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} F & u_1^\perp & F \\ 0 & 0 & M \\ 0 & 0 & F \end{pmatrix}$$

Consider the sets:

$$S_n = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u_{2n} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u_{2n+1} \\ 0 & 0 & 0 \end{pmatrix}, \ldots \right\}$$

It is easy to see that:

$$\begin{pmatrix} 0 & u_{2n} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \text{Ann}(S_{n+1}) \setminus \text{Ann}(S_n)$$

since $2^n + 2^m$ is not a power of 2 for $m > n$. Therefore we have an infinite strictly increasing ascending chain of annihilators:

$$\text{Ann}(S_1) \subset \text{Ann}(S_2) \subset \cdots$$

So $R$ does not satisfy ACC on left annihilators. \qed

We now consider two-sided ideals of $R$.

**Proposition 3.3.** The algebra $R$ satisfies ACC on two sided ideals.

**Proof.** Let $I \triangleleft R$, fix some $r \in I$, and write:

$$r = \begin{pmatrix} \alpha_1 & v & \lambda \\ 0 & f(t) & w \\ 0 & 0 & \alpha_2 \end{pmatrix}$$

Then:

$$E_{1,1}re_{1,1} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{2,2}re_{2,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
\[ \begin{bmatrix} E_{3,3}rE_{3,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta \end{pmatrix}, \quad E_{1,1}rE_{3,3} = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{bmatrix} \]

\[ \begin{bmatrix} (E_{2,2} - 1)rE_{2,2} = \begin{pmatrix} 0 & v \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{2,2}r(E_{2,2} - 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \end{bmatrix} \]

Moreover:
\[ \begin{bmatrix} 0 & v & 0 \\ 0 & 0 & 0 \\ 0 & p(t) & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & p(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & vp(t) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & p(t) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & p(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & p(t)w \\ 0 & 0 & 0 \end{bmatrix} \]

Then:
\[ r \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & t^i & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & vt^i & 0 \\ 0 & f(t)t^i & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

And:
\[ \begin{bmatrix} 0 & p(t) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot r \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & p(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & vp(t) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & f(t)q(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

It follows that \( I \) contains a subspace of the form:
\[ \begin{pmatrix} E & V & L \\ 0 & J & W \\ 0 & 0 & K \end{pmatrix} \]

where \( E, K, L \in \{0, F\}, J \subset F[t] \) and \( V, W \) are \( F[t] \)-sub-bimodules of \( M \). Moreover, \( V, W \) are non-zero if and only if \( v, w \) are non-zero, respectively. Notice that any non-zero ideal of \( F[t] \) is finite codimensional, and any non-zero sub-bimodule of \( M \) is finite codimensional. Therefore we have that \( I \) is finite codimensional inside a subspace of the form:
\[ \begin{pmatrix} E & V & L \\ 0 & J & W \\ 0 & 0 & K \end{pmatrix} \]

where \( E, K, L \in \{0, F\}, J \in \{0, F[t]\} \) and \( V, W \in \{0, M\} \). It follows that any ascending chain of ideals of \( R \) stabilizes. \( \square \)

**Remark 3.4.** We can modify the algebra \( R \) constructed above to be moreover irreducible with \( R/N \cong F[t] \), since an algebra with \( \text{ACC} \) on ideals is a subdirect product of finitely many irreducible algebras, and a subdirect product of irreducible (weakly) representable algebras is again (weakly) representable.

### 3.2. A non weakly-representable semiprimary PI-algebra.

We now extend the construction from Subsection 3.1. We take \( A = F(t) \) and \( M = V \) a 1-dimensional \( F(t) \)-vector space, which we naturally identify with \( F(t) \). Then \( V \otimes_{F(t)} V \cong V \) as an \( F \)-vector space by \( v \otimes w \mapsto vw \). We fix an \( F \)-linear basis for \( F(t) \), say, \( \mathfrak{B} \) containing \( 1, t, t^2, \ldots \) and define \( \tilde{B} : V \otimes_{F(t)} V \to F \) on basis elements as follows:
\[
\tilde{B}(1,v) = \begin{cases} 
1, & \text{if } \exists k \geq 1 : v = t^{2^k} \\
0, & \text{otherwise}
\end{cases}
\]

We can therefore form, in the same manner as of Subsection 3.1, the following \(F\)-algebra:

\[
S = \begin{pmatrix}
F & V & F \\
0 & F(t) & V \\
0 & 0 & F
\end{pmatrix}
\]

Notice that the \(F\)-algebra \(R\) constructed in Subsection 3.1 embeds into \(S\):

\[
R = \begin{pmatrix}
F & M & F \\
0 & F[t] & M \\
0 & 0 & F
\end{pmatrix} \hookrightarrow \begin{pmatrix}
F & V & F \\
0 & F(t) & V \\
0 & 0 & F
\end{pmatrix} = S
\]

Notice that, if \(N \lhd S\) is the nilpotent radical of \(S\), then \(S/N \cong F \times F(t) \times F\), so \(S\) is a semiprimary PI-algebra. Since \(R\) is non-weakly representable, we get that so is \(S\).

References

[1] A. Ya. Belov, Y. Karasik, L. H. Rowen, *Computational Aspects of Polynomial Identities: Volume I, Kemer’s Theorem, 2nd Edition*, Chapman & Hall/CRC Monographs and Research Notes in Mathematics (2016).

[2] G. M. Bergman, W. Dicks, *On universal derivations*, Journal of Algebra 36 (2), 193–211 (1975).

[3] J. Cuntz, D. Quillen, *Algebra Extensions and Nonsingularity*, Journal of the American Mathematical Society 8 (2), 251–289 (1995).

[4] R. S. Irving, *Affine PI-algebras not embeddable in matrix rings*, Journal of Algebra 82 (1), 94–101 (1983).

[5] J. Lewin, *On some infinitely presented associative algebras*, Journal of the Australian Mathematical Society 16 (3), 290–293 (1973).

[6] Markov, V.T., *On the representability of finitely generated PI-algebras by matrices. (Russian)* Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1989, no. 2, 17-20, 104; translation in Moscow Univ. Math. Bull. 44 (1989), no. 2, 23-27.

[7] L. Rowen, *Polynomial Identities in Ring Theory*, Academic Press (Pure and Applied Math. Series), 384 pp., March, 1980.

[8] L. Rowen, L. W. Small, *Representability of algebras finite over their centers*, Journal of Algebra 442, 506–524 (2015).

[9] L. W. Small, J. T. Stafford, R. Warfield, *Affine algebras of Gelfand Kirillov dimension one are PI*, Math. Proc. Cambridge. Phil. Soc. 97 (3), 407–414 (1985).

[10] R. Vale, *Notes on quasi-free algebras*, available online: pi.math.cornell.edu/ rvale/ada.pdf.