DYNAMIC TRANSITION AND PATTERN FORMATION FOR CHEMOTACTIC SYSTEMS

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Abstract. The main objective of this article is to study the dynamic transition and pattern formation for chemotactic systems modeled by the Keller-Segel equations. We study chemotactic systems with either rich or moderated stimulant supplies. For the rich stimulant chemotactic system, we show that the chemotactic system always undergoes a Type-I or Type-II dynamic transition from the homogeneous state to steady state solutions. The type of transition is dictated by the sign of a non dimensional parameter $b$. For the general Keller-Segel model where the stimulant is moderately supplied, the system can undergo a dynamic transition to either steady state patterns or spatiotemporal oscillations. From the pattern formation point of view, the formation and the mechanism of both the lamella and rectangular patterns are derived.

1. Introduction

Chemotaxis is a remarkable phenomenon occurring in many biological individuals, which involves mobility and aggregation of the species in two aspects: one is random walking, and the other is the chemically directed movement. For example, in the slime mould Dictyostelium discoideum, the single-cell amoebae move towards regions of relatively high concentration of a chemical called cyclic-AAMP which is secreted by the amoebae themselves. Many experiments demonstrate that under some proper conditions a bacterial colony can form a rather regular pattern, which is relatively stable in certain time scale. A series of experimental results on the patterns formed by the bacteria Escherichia coli (E. coli) and Salmonella typhimurium (S. Typhimurium) were derived in [2, 3], where two types of experiments were conducted: one is in semi-solid medium, and the other is in liquid medium. They showed that when the bacteria are exposed to intermediates of TCA cycle, they can form various regular patterns, typically as ringlike and sunflowerlike formations. In all these experiments, the bacteria are known to secrete aspartate, a potent chemoattractant; also see [10, 1].

The most interesting work done by Budrene and Berg are the semi-solid experiments with E. coli and S. typhimurium. A high density bacteria were inoculated in a petri dish containing a uniform distribution of stimulant in the semi-solid medium, i.e. 0.24% soft agar in succinate. The stimulant provides main food source for the bacteria. In a few days, the bacteria spread out from the inoculum, eventually

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Key words and phrases. chemotaxis, Keller-Segel model, rich stimulant two-component system, general three-component Keller-Segel with moderated stimulant supplies, steady state patterns, spatiotemporal oscillatory patterns.

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covering the entire surface of the dish with a stationary pattern where the higher density population is separated by regions of near zero cell density. The S. typhimurium patterns are concentric rings and are either continuous or spotted; see Figure 1.1. The E. coli patterns are more complex with symmetry between individual aggregates. A large number of patterns has been observed. The most typical forms are concentric rings, sunflower type spirals, radial stripes, radial spots and chevrons. In the process of pattern formation, the population of bacteria has gone through many generations.

![Figure 1.1. The black ring line and spot represent high density of bacteria.](image)

The liquid experiments with E. coli and S. typhimurium exhibit relatively simple patterns which appear quickly in a few minutes, and last about half an hour before disappearing. Two types of patterns are observed, and they rely on the initial conditions. The simplest patterns are produced when the liquid medium contains a uniform distribution of bacteria and a small amount of the TCA cycle intermediate. The bacteria collect in aggregates of about the same size over the entire surface of the liquid. The second type of patterns appears when a small amount of TCA is added locally to a special spot in a uniform distribution of bacteria. In this case, the bacteria begin to form aggregates which occur on a ring centered about the special spot, and in a random arrangement inside the ring. In particular, in these liquid experiments, the timescale to form patterns is less than the time required for bacterial birth and death. Therefore, the growth of bacteria does not contribute to the pattern formation process.

Here we have to address that in these experiments, none of the chemicals placed in the petri dish is a chemo-attractant. Hence, the chemoattractants, which play a crucial role in bacterial chemotaxis, are produced and secreted by the bacteria themselves.

In their pioneering work [5], E. F. Keller and L. A. Segel proposed a model in 1970, called the Keller-Segel equations, to describe the chemotactic behaviors of the slime mould amoebae. In their equations, the growth rate of amoeba cells was ignored, i.e., the model can only depict the chemotaxis process in a small timescale, as exhibited in the liquid medium experiments with E. Coli and S. Typhimurium by [2, 3]. However, in the semi-solid medium experiments, the timescale of pattern formation process is long enough to accommodate many generations of bacteria. Therefore, various revised models were presented by many authors, taking into consideration the effects of the stimulant (i.e. food source) and the growth rate of population; see among others [10] and the references therein. Also, there is a vast literature on the mathematical studies for the Keller-Segel model; see among others [13, 4, 12, 11].

The main objective of this article is to study the dynamic transition and pattern formation for chemotactic systems modeled by the Keller-Segel equations. The
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The study is based on the dynamic transition theory developed recently by the authors. The key philosophy for the dynamic transition theory is to search for all transition states. The stability and the basin of attraction of the transition states provide naturally the mechanism of pattern formation associated with chemotactic systems.

Another important ingredient of the dynamic transition theory is the introduction of a dynamic classification scheme of transitions, with which phase transitions are classified into three types: Type-I, Type-II and Type-III. In more mathematically intuitive terms, they are called continuous, jump and mixed transitions respectively. Basically, as the control parameter passes the critical threshold, the transition states stay in a close neighborhood of the basic state for a Type-I transition, are outside of a neighborhood of the basic state for a Type-II (jump) transition. For the Type-III transition, a neighborhood is divided into two open regions with a Type-I transition in one region, and a Type-II transition in the other region.

Two types of Keller-Segel models are addressed in this article. The first is the model for rich stimulant chemotactic systems (with rich nutrient supplies). In this case, the equations are a two-component system, describing the evolution of the population density of biological individuals and the chemoattractant concentration. In this case we show that the chemotactic system always undergoes a Type-I or Type-II dynamic transition from the homogeneous state to steady state solutions. The type of transition is dictated by the sign of a nondimensional parameter $b$. For example, in a non-growth system in a narrow domain, for the spatial scale smaller than a critical number, the system undergoes a Type-I (continuous) transition, otherwise the system undergoes a Type-II (jump) transition, leading to a more complex pattern away from the basic homogeneous state.

The second is a more general Keller-Segel model where the stimulant is moderately supplied. In this case, the model is a three-component system describing the evolution of the population density of biological individuals, the chemoattractant concentration, and the stimulant concentration. In this case, the system can undergo a dynamic transition to either steady state patterns or spatiotemporal oscillation. In both cases, the transition can be either a Type-I or Type-II dictated respectively by two nondimensional parameter $b_0$ and $b_1$.

For simplicity, we consider in this article only the case where the first eigenvalue of the linearized problem around the homogeneous pattern is simple (real or complex), and we shall explore more general case elsewhere. In the case considered, for the Type-I transition, when the linearized eigenvalue is simple, we show that both the lamella and rectangular pattern can form depending on the geometry of the spatial domain. Namely, for narrow domains, the lamella pattern forms, otherwise the rectangular pattern occurs. For Of course, for Type-II transitions, more complex patterns emerge far from the basic homogeneous state.

The paper is arranged as follows. Section 2 introduces the Keller Segel model. The rich stimulant case is addressed in Section 3, and the general three-component system is studied in Section 4. Section 5 explores some biological conclusions of the main theorems.
2. Keller-Segel Model

The general form of the revised Keller-Segel model is given by

$$
\begin{align*}
\frac{\partial u_1}{\partial t} &= k_1 \Delta u_1 - \chi \nabla (u_1 \nabla u_2) + \alpha_1 u_1 \left( \frac{\alpha_2 u_3}{\alpha_0 + u_3} - u_1^2 \right), \\
\frac{\partial u_2}{\partial t} &= k_2 \Delta u_2 + r_1 u_1 - r_2 u_2, \\
\frac{\partial u_3}{\partial t} &= k_3 \Delta u_3 - \delta u_1 u_3 + q(x),
\end{align*}
$$

where $u_1$ is the population density of biological individuals, $u_2$ is the chemoattractant concentration, $u_3$ is the stimulant concentration, $q(x)$ is the nutrient source, and $\chi$ is a chemotactic response coefficient.

Equations (2.1) are supplemented with the Neumann condition:

$$
\frac{\partial (u_1, u_2, u_3)}{\partial n} = 0 \text{ on } \partial \Omega.
$$

For simplicity, we consider in this article the case where the spatial domain $\Omega$ is a two-dimensional (2D) rectangle:

$$
\Omega = (0, L_1) \times (0, L_2) \text{ for } L_1 \neq L_2.
$$

It is convenient to introduce the non-dimensional form of the model. For this purpose, let

$$
\begin{align*}
t &= t'/r_2, \\
x &= \sqrt{k_2/r_2} x', \\
u_1 &= \sqrt{\alpha_2} u_1', \\
u_2 &= k_2 u_2'/\chi, \\
u_3 &= \alpha_0 u_3',
\end{align*}
$$

and we define the following non-dimensional parameters:

$$
\begin{align*}
\lambda &= r_1 \sqrt{\alpha_2 \chi} / r_2 k_2, \\
\alpha &= \alpha_1 \alpha_2 / r_2, \\
\mu &= k_1 / k_2, \\
r &= k_3 / k_2, \\
\delta &= r_3 \sqrt{\alpha_2 / r_2}, \\
\delta_0 &= q / r_2 \alpha_0.
\end{align*}
$$

Then suppressing the primes, the non-dimensional form of the Keller-Segel model is given by:

$$
\begin{align*}
\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - \nabla (u_1 \nabla u_2) + \alpha u_1 \left( \frac{u_3}{1 + u_3} - u_1^2 \right), \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 - u_2 + \lambda u_1, \\
\frac{\partial u_3}{\partial t} &= r \Delta u_3 - \delta u_1 u_3 + \delta_0, \\
\frac{\partial u}{\partial n} \bigg|_{\partial \Omega} &= 0, \\
u(0) &= u_0 \text{ in } \Omega.
\end{align*}
$$

The non-dimensional of $\Omega$ is written as

$$
\Omega = (0, L_1) \times (0, L_2) \text{ with } L_1 \neq L_2.
$$
Often times, the following form of the Keller-Segel equations is discussed in some literatures:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - \nabla (u_1 \nabla u_2) + \alpha u_1 \left( \frac{u_3}{1 + u_3} - u_1^2 \right), \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 - u_2 + \beta_0, \\
\frac{\partial u_3}{\partial t} &= r \Delta u_3 - \delta u_1 u_3 + \delta_0, \\
\frac{\partial u}{\partial n} \bigg|_{\partial \Omega} &= 0, \\
u(0) &= u_0.
\end{align*}
\]

(2.6)

The biological significance of (2.6) is that the diffusion and degradation of the chemoattractant secreted by the bacteria themselves are almost balanced by their production. The main advantage of (2.6) lies in its mathematical simplicity, and as we shall see from the main results of this article, the main characteristics of the pattern formation associated with the model are retained.

3. Dynamic Transitions for Rich Stimulant System

3.1. The model. We know that as nutrient $u_3$ is richly supplied, the Keller-Segel model (2.1) is reduced to a two-component system:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - \nabla (u_1 \nabla u_2) + \alpha u_1 (1 - u_1^2), \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 - u_2 + \lambda u_1, \\
\frac{\partial (u_1, u_2)}{\partial n} \bigg|_{\partial \Omega} &= 0, \\
u(0) &= u_0.
\end{align*}
\]

(3.1)

It is easy to see that $u^* = (1, \lambda)$ is a steady state of (3.1). Consider the deviation from $u^*$:

\[u = u^* + u'.\]

Suppressing the primes, the system (3.1) is then transformed into

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - 2\alpha u_1 - \Delta u_2 - \nabla (u_1 \nabla u_2) - 3\alpha u_1^2 - \alpha u_1^3, \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 - u_2 + \lambda u_1, \\
\frac{\partial (u_1, u_2)}{\partial n} \bigg|_{\partial \Omega} &= 0, \\
u(0) &= u_0.
\end{align*}
\]

(3.2)

3.2. Dynamic transition and pattern formation for the diffusion and degradation balanced case. We start with an important case where the diffusion and degradation of the chemoattractant secreted by the bacteria themselves are almost balanced by their production. In this case, the second equation in (3.2) is given by

\[0 = \Delta u_2 - u_2 + \lambda u_1.\]
With the Newman boundary condition for $u_2$, we have $u_2 = [-\Delta + 1]^{-1}u_1$ and the functional form of the resulting equations are given by

$$\frac{\partial u_1}{\partial t} = \mathcal{L}_\lambda u_1 + G(u_1, \lambda),$$

where the operators $\mathcal{L}_\lambda : H_1 \to H$ and $G : H_1 \times \mathbb{R} \to \mathbb{R}$ are defined by

$$\mathcal{L}_\lambda u_1 = \mu \Delta u_1 - 2\alpha u_1 - \lambda \Delta [-\Delta + I]^{-1} u_1,$$

$$G(u_1, \lambda) = -\lambda \nabla (u_1 \nabla [-\Delta + I]^{-1} u_1) - 3\alpha u_1^2 - \alpha u_1^3.$$

Here the two Hilbert spaces $H$ and $H_1$ are defined by

$$H = L^2(\Omega), \quad H_1 = \{ u_1 \in H^2(\Omega) \mid \frac{\partial u_1}{\partial n} = 0 \text{ on } \Omega \}.$$

To study the dynamic transition of this problem, we need to consider the linearized eigenvalue problem of (3.3):

$$\mathcal{L}_\lambda \beta = \beta(\lambda).$$

Let $\rho_k$ and $e_k$ be the eigenvalues and eigenfunctions of $-\Delta$ with the Neumann boundary condition given by

$$e_k = \cos \frac{k_1 \pi x_1}{L_1} \cos \frac{k_2 \pi x_2}{L_2}, \quad \rho_k = \pi^2 \left( \frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2} \right),$$

for any $k = (k_1, k_2) \in \mathbb{N}_+^2$. Here $\mathbb{N}_+$ is the set of all nonnegative integers. In particular, $e_0 = 1$ and $\rho_0 = 0$.

Obviously, the functions in (3.6) are also eigenvectors of (3.5), and the corresponding eigenvalues $\beta_k$ are

$$\beta_k(\lambda) = -\mu \rho_k - 2\alpha + \frac{\lambda \rho_k}{1 + \rho_k}.$$

Define a critical parameter by

$$\lambda_c = \min_{\rho \in \mathcal{S}} \frac{(\rho_k + 1)(\mu \rho_k + 2\alpha)}{\rho_k}.$$

Let

$$\mathcal{S} = \{ K = (K_1, K_2) \in \mathbb{N}_+^2 \text{ such that } K \text{ achieves the minimization in (3.8).} \}.$$

Then it follows from (3.7) and (3.8) that

$$\beta_k(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_c \\ = 0 & \text{if } \lambda = \lambda_c \\ > 0 & \text{if } \lambda > \lambda_c \end{cases} \quad \forall K = (K_1, K_2) \in \mathcal{S},$$

$$\beta_k(\lambda_c) < 0 \quad \forall k \in \mathbb{Z}^2 \text{ with } k \notin \mathcal{S}.$$

Notice that for any $K = (K_1, K_2) \in \mathcal{S}$, $K \neq 0$, and

$$\lambda_c = \frac{(\rho_K + 1)(\mu \rho_K + 2\alpha)}{\rho_K}.$$

We note that for properly choosing spatial geometry, we have

$$\rho_K = \pi^2 \left( \frac{K_1^2}{L_1^2} + \frac{K_2^2}{L_2^2} \right) = \sqrt{\frac{2\alpha}{\mu}} \quad \forall K = (K_1, K_2) \in \mathcal{S},$$

$$\lambda_c = 2\alpha + \mu + 2\sqrt{2\alpha \mu}.$$
Remark 3.1. From the pattern formation point of view, for the Type-I transition, or rectangular: the patterns described by the transition solutions given in (3.15) are either lamella \( \lambda < \lambda \) in the case where \( b > \lambda \) \( 3.14 \)

Let \( \beta \) be the parameter defined by (3.14). Assume that the eigenvalue \( \beta \) satisfying (3.14) is simple. Then, for the system (3.3) we have the following assertions:

1. The system always undergoes a dynamic transition at \( (u, \lambda) = (0, \lambda_c) \). Namely, the basic state \( u = 0 \) is asymptotically stable for \( \lambda < \lambda_c \), and is unstable for \( \lambda > \lambda_c \).

2. For the case where \( b < 0 \), this transition is continuous (Type-I). In particular, the system bifurcates from \( (0, \lambda_c) \) to two steady state solutions on \( \lambda > \lambda_c \), which can be expressed as

\[
(3.15) \quad u^\pm(x, \lambda) = \pm \sqrt{2 |b|} \frac{\beta(\lambda)}{2} \cos \frac{K_1 \pi x_1}{L_1} \cos \frac{K_2 \pi x_2}{L_2} + o \left( \beta^{1/2} \right),
\]

and \( u^\pm(x, \lambda) \) are attractors.

3. For the case \( b > 0 \), this transition is jump (Type-II), and the system has two saddle-node bifurcation solutions at some \( \lambda^*(0 < \lambda^* < \lambda_c) \) such that there are two branches \( v_1^\lambda \) and \( v_3^\lambda \) of steady states bifurcated from \( (v^*, \lambda^*) \), and there are two other branches \( v_3^\lambda \) and \( v_3^\lambda \) bifurcated from \( (u^*, \lambda^*) \). In addition, \( v_1^\lambda \) and \( v_3^\lambda \) are saddles, \( v_2^\lambda \) and \( v_3^\lambda \) are attractors, with \( v_1^\lambda, v_3^\lambda \to 0 \) as \( \lambda \to \lambda_c \).

Two remarks are now in order.

Remark 3.1. From the pattern formation point of view, for the Type-I transition, the patterns described by the transition solutions given in (3.15) are either lamella or rectangular:

- lamella pattern for \( K_1 K_2 = 0 \),
- rectangular pattern for \( K_1 K_2 \neq 0 \).

In the case where \( b > 0 \), the system undergoes a more drastic change. As \( \lambda^* < \lambda < \lambda_c \), the homogeneous state, the new patterns \( v_2^\lambda \) and \( v_3^\lambda \) are metastable. For
Remark 3.2. If we take the growth term \( f(u) \) as \( f = \alpha u_1(1 - u_1) \) instead of \( f = \alpha u_1(1 - u_1^2) \) in (3.1), (3.2) and (3.3), then Theorem 3.1 still holds true except the assertion on the existence of the two saddle-node bifurcation solutions, and the parameter should be replaced by

\[
b = -\mu \rho_K + \alpha - \frac{(2\mu \rho_K + \alpha)(2\mu \lambda^2_K + 10\alpha \rho_K + \alpha - \mu \rho_K)}{2(1 + \text{sign}K_1 K_2)[(\mu \rho_{K_1} + \alpha)(1 + \rho_{K_2}) - \lambda_c \rho_{K_2}]} \]

(3.18)

3.3. Pattern formation and dynamic transition for the general case. Consider the general case (3.1). In this case, the unknown variable becomes \( \rho \) where

\[
\frac{\partial u}{\partial t} + \sum_{i=1}^{N} e \xi_k \nabla (u_k - u_i) = \mu \Delta u + \alpha u - \mu \rho_K - \alpha \rho_K u
\]

(3.19)

where \( \rho_k \) are the eigenvalues as in (3.6). It is easy to see that all eigenvectors \( \varphi_k \) and eigenvalues \( \beta_k \) of (3.11) can be expressed as follows

\[
\varphi_k = \begin{pmatrix} \xi_{k1} e_k \\ \xi_{k2} e_k \end{pmatrix},
\]

(3.19)

\[
B_k^\lambda \begin{pmatrix} \xi_{k1} \\ \xi_{k2} \end{pmatrix} = \beta_k \begin{pmatrix} \xi_{k1} \\ \xi_{k2} \end{pmatrix},
\]

(3.20)

where \( e_k \) are as in (3.6), and \( \beta_k \) are also the eigenvalues of \( B_k^\lambda \). By (3.18), \( \beta_k \) can be expressed by

\[
\beta_k^\pm(\lambda) = \frac{1}{2} \left[ -B \pm \sqrt{B^2 - 4((\rho_k + 1)(\mu \rho_k + 2\alpha) - \lambda \rho_k)} \right],
\]

(3.21)

\[B = (\mu + 1)\rho_k + 2\alpha + 1.\]
Let $\lambda_c$ be the parameter as defined by (3.8). It follows from (3.21) and (3.8) that
\[
\beta^+_K(\lambda) \begin{cases}
< 0 & \text{if } \lambda < \lambda_c, \\
= 0 & \text{if } \lambda = \lambda_c, \\
> 0 & \text{if } \lambda > \lambda_c,
\end{cases}
\]
\[
\begin{cases}
\text{Re}\beta^+_K(\lambda_c) < 0 & \forall k \in \mathbb{Z}^2, \\
\text{Re}\beta^+_K(\lambda_c) > 0 & \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_K,
\end{cases}
\]
with $K = (K_1, K_2)$ as in (3.11).

Then we have the following dynamic transition theorem.

**Theorem 3.2.** Let $b$ be the parameter defined by (3.14). Assume that the eigenvalue $\beta_K^+$ satisfying (3.22) is simple. Then the assertions of Theorem 3.1 hold true for (3.2), with the expression (3.15) replaced by
\[
u^\pm = \pm \sqrt{a\beta^+_K(\lambda_c)} \left( \frac{\rho_K + 1}{\lambda_c} \right) \cos \frac{K_1\pi x_1}{L_1} \cos \frac{K_2\pi x_2}{L_2} + o(\beta_K^+ [1/2]),
\]
\[
a = \frac{8(\mu \rho_K + \rho_K + 2\alpha + 1)}{\rho_K + 1)^3|b|}.
\]

### 3.4. Proof of Main Theorems.

**Proof of Theorem 3.1.** Assertion (1) follows directly from the general dynamic transition theorem in Chapter 2 of [7]. To prove Assertions (2) and (3), we need to reduce (3.3) to the center manifold near $\lambda = \lambda_c$. We note that although the underlying system is now quasilinear in this general case, the center manifold reduction holds true as well; see [6] for details.

To this end, let $u = xe_k + \Phi$, where $\Phi(x)$ the center manifold function of (3.3). Since $L_\lambda : H_1 \to H$ is symmetric, the reduced equation is given by
\[
dx{t} = \beta_K(\lambda)x + \frac{1}{(e_K, e_K)}(G(xe_K + \Phi, \lambda, e_K),
\]
where $G : H_1 \to H$ is defined by (3.3), and
\[
(e_K, e_K) = \int_\Omega e_K^2 dx = \frac{2 - \text{sign}(K_1K_2)}{4}|\Omega|.
\]

It is known that the center manifold function satisfies that $\Phi(x) = O(x^2)$. A direct computation shows that
\[
< G(xe_K + \Phi, \lambda_c, e_K >
\]
\[
= -\alpha x^3 \int_\Omega e_K^4 dx - 6\alpha x \int_\Omega e_K^2 \Phi dx
\]
\[
+ \lambda_c x \int_\Omega [e_K \nabla e_k \cdot \nabla (\Delta + I)^{-1}\Phi + \Phi \nabla e_k \cdot \nabla (\Delta + I)^{-1}e_K] dx + o(x^3).
\]

It is clear that
\[
(\Delta + I)^{-1}e_K = \frac{1}{\rho_K + 1} e_K, \quad \Delta e_K = -\rho_K e_K.
\]
We infer from (3.26) that

\[
G(xe_K + \Phi, \lambda_c, e_K) = -\alpha x^3 \int_\Omega e_K^3 dx - 6\alpha x \int_\Omega e_K^2 \Phi dx + \lambda_c x \int_\Omega \left[ \frac{1}{\rho_K + 1} |\nabla e_K|^2 \Phi - |\nabla e_K|^2 (-\Delta)^{-1} \Phi + \rho_K e_K^2 (-\Delta + I)^{-1} \Phi \right] dx + o(x^3).
\]

Using the approximation formula for center manifold functions given in (A.11) in [8], \( \Phi \) satisfies the equation

\[
-\mathcal{L} \lambda_c \Phi = G_2(xe_K, \lambda_c) + o(x^2)
\]

\[
= x^2 \left[ \left( \frac{\rho_K \lambda_c}{\rho_K + 1} - 3\alpha \right) e_K^2 - \frac{\lambda_c}{\rho_K + 1} |\nabla e_K|^2 \right] + o(x^2).
\]

In view of (3.28), we find

\[
e_K^2 = \frac{1}{4} [e_0 + e_{2K_1} + e_{2K_2} + e_{2K}],
\]

\[
|\nabla e_K|^2 = \frac{1}{4} [\rho_K e_0 + (\rho_{K_2} - \rho_{K_1}) e_{2K_1} + (\rho_{K_1} - \rho_{K_2}) e_{2K_2} - \rho_K e_{2K}].
\]

Thus, (3.28) is written as

\[
-\mathcal{L} \lambda_c \Phi = \frac{x^2}{4} \left[ -3\alpha e_0 + \left( \frac{2\rho_{K_1} \lambda_c}{\rho_K + 1} - 3\alpha \right) e_{2K_1} + \left( \frac{2\rho_{K_2} \lambda_c}{\rho_K + 1} - 3\alpha \right) e_{2K_2} + \left( \frac{2\rho_K \lambda_c}{\rho_K + 1} e_{2K} - 3\alpha \right) e_{2K} \right] + o(x^2).
\]

Denote by

\[
\Phi = \Phi_0 e_0 + \Phi_{2K_1} e_{2K_1} + \Phi_{2K_2} e_{2K_2} + \Phi_{2K} e_{2K}.
\]

Note that

\[
-\mathcal{L} \lambda_c e_{2K} = \frac{1}{1 + \rho_{2K}} [(1 + \rho_{2K})(\mu_{2K} + 2\alpha) - \lambda_c \rho_{2K}] e_{2K}.
\]

Then, by (3.11) and (3.30)- (3.32) we obtain

\[
\Phi_0 = -\frac{3}{8},
\]

\[
\Phi_{2K_1} = \frac{(1 + \rho_{2K_1})(2\mu_{2K_1} \rho_{2K_1} + 4\rho_{2K_1} - 3\alpha\rho_K)}{4\rho_K [(1 + \rho_{2K_1})(\mu_{2K_1} + 2\alpha) - \rho_{2K_1} \lambda_c]},
\]

\[
\Phi_{2K_2} = \frac{(1 + \rho_{2K_2})(2\mu_{2K_2} \rho_{2K_2} + 4\rho_{2K_2} - 3\alpha\rho_K)}{4\rho_K [(1 + \rho_{2K_2})(\mu_{2K_2} + 2\alpha) - \rho_{2K_2} \lambda_c]},
\]

\[
\Phi_{2K} = \frac{(1 + \rho_{2K})(2\mu_{2K} + \alpha)}{4[(1 + \rho_{2K})(\mu_{2K} + 2\alpha) - \rho_{2K} \lambda_c]}.
\]
Inserting (3.31) and (3.6) into (3.27) we get

\begin{equation}
(3.34) \quad <G(x e_K + \Phi, \lambda_c), e_K> = -\alpha x^3 \int_\Omega e_K^4 dx
\end{equation}

\begin{equation}
- \frac{6\alpha x(2 - \text{sign}(K_1 K_2))}{4} \left[ \int_\Omega \left[ \Phi_0 e_0^2 + \Phi_2 K_1 e_2 K_1 + \Phi_2 K_2 e_2 K_2 + \Phi_2 e_2 K_2 \right] dx \right]
\end{equation}

\begin{equation}
+ \frac{\lambda_c x(2 - \text{sign}(K_1 K_2))}{4(\rho K + 1)} \left[ \int_\Omega \left[ \rho K \Phi_0 e_0^2 + (\rho K - \rho K_1) \Phi_2 K_1 e_2 K_1 \\
+ (\rho K - \rho K_2) \Phi_2 K_2 e_2 K_2 - \rho K \Phi_2 e_2 K_2 \right] dx \right]
\end{equation}

\begin{equation}
- \frac{\lambda_c x(2 - \text{sign}(K_1 K_2))}{4} \left[ \int_\Omega \left[ \rho K \Phi_0 e_0^2 + \frac{\rho K_2}{1 + \rho K} \Phi_2 K_1 e_2 K_1 \\
+ \frac{\rho K}{1 + \rho K} \Phi_2 K_2 e_2 K_2 - \frac{\rho K}{1 + \rho K} \Phi_2 e_2 K_2 \right] dx \right]
\end{equation}

\begin{equation}
+ \frac{\Phi_2}{1 + \rho K} e_2 K_2 \right] dx + o(x^3)
\end{equation}

\begin{equation}
= -\alpha x^3 \frac{1}{4} \int_\Omega e_K^4 dx + \frac{\Omega |x(2 - \text{sign}(K_1 K_2))|}{4}
\end{equation}

\begin{equation}
\times \left\{ (\mu \rho K - 4\alpha) \Phi_0 + \frac{1}{1 + \text{sign}(K_1)} \left( \lambda_c (\rho K - \rho K_1) \right) \Phi_2 K_1 + \frac{2\lambda_c \rho K_1 - 6\alpha}{1 + \rho K} \Phi_2 K_1 \\
+ \frac{1}{1 + \text{sign}(K_2)} \left( \lambda_c (\rho K - \rho K_2) \right) \Phi_2 K_2 + \frac{2\lambda_c \rho K_2 - 6\alpha}{1 + \rho K} \Phi_2 K_2 \\
+ \frac{1}{2(1 + \text{sign}(K_1 K_2))} \left( \frac{-\lambda_c \rho K}{1 + \rho K} + \frac{2\lambda_c \rho K - 6\alpha}{1 + \rho K} \right) \Phi_2 K_1 \right\} dx + o(x^3).
\end{equation}

Also, we note that

\begin{equation}
\int_\Omega e_K^4 = \int_0^{L_1} e_K^4 dx \int_0^{L_2} e_K^4 dx = \frac{24 - 15\text{sign}(K_1 K_2)}{64}.
\end{equation}

Then, putting (3.33) into (3.34) we get

\begin{equation}
(3.35) \quad <G(x e_K + \Phi, \lambda_c), e_K> = \frac{(2 - \text{sign}(K_1 K_2)) \Omega |x|^3}{32} b + o(x^3),
\end{equation}

where \( b \) is the parameter given by (3.11).

By (3.24) and (3.35), we derive the reduced equation on the center manifold as follows:

\begin{equation}
\frac{dx}{dt} = \beta_K(\lambda)x + \frac{b}{8} x^3 + o(x^3).
\end{equation}

Based on the dynamic transition theory developed in Chapter 2 in [7], we obtain Assertions (2) and (3), except that two saddle-node bifurcations occur at the same point \( \lambda = \lambda^* \). To prove this conclusion, we note that if \( u^*(x) \) is a steady state solution of (3.33), then

\begin{equation}
\nu^*(x) = u^*(x + \pi) = u^*(x - \pi)
\end{equation}

is also a steady state solution of (3.33). This is because the eigenvectors (3.6) form an orthogonal base of \( H_1 \). Hence, two saddle-node bifurcations on \( \lambda < \lambda_c \) imply
that they must occur at the same point \( \lambda = \lambda^* \). Thus the proof of the theorem is complete. \(\square\)

**Proof of Theorem 3.2.** Assertion (1) follows from (3.22) and (3.23). To prove Assertions (2) and (3), we need to get the reduced equation of (3.2) to the center manifold near \( \lambda = \lambda_c \).

Let \( u = x \cdot \varphi_K + \Phi \), where \( \varphi_K \) is the eigenvector of (3.17) corresponding to \( \beta_K \) at \( \lambda = \lambda_c \), and \( \Phi(x) \) the center manifold function of (3.2). Then the reduced equation of (3.2) read

\[
\frac{dx}{dt} = \beta^+_K(\lambda)x + \frac{1}{\varphi_K, \varphi^*_K} < G(x \cdot \varphi_K + \Phi), \varphi^*_K > ,
\]

Here \( \varphi^*_K \) is the conjugate eigenvector of \( \varphi_K \).

By (3.19) and (3.20), \( \varphi_K \) is written as

\[
\varphi_K = (\xi_1 e_k, \xi_2 e_k)^T ,
\]

with \( (\xi_1, \xi_2) \) satisfying

\[
\begin{pmatrix}
- (\mu \rho_K + 2 \alpha) \\
\lambda_c \\
- \rho_K \\
- (\rho_K + 1)
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix} = 0 ,
\]

from which we get

\[
(\xi_1, \xi_2) = (\rho_K + 1, \lambda_c) .
\]

Likewise, \( \varphi^*_K \) is

\[
\varphi^*_K = (\xi^*_1 e_k, \xi^*_2 e_k)^T ,
\]

with \( (\xi^*_1, \xi^*_2) \) satisfying

\[
\begin{pmatrix}
- (\mu \rho_K + 2 \alpha) \\
\rho_K \\
\lambda_c \\
- (\rho_K + 1)
\end{pmatrix}
\begin{pmatrix}
\xi^*_1 \\
\xi^*_2
\end{pmatrix} = 0 ,
\]

which yields

\[
(\xi^*_1, \xi^*_2) = (\rho_K + 1, \rho_K) .
\]

By (3.19), the nonlinear operator \( G \) is

\[
G(u_1, u_2) = G_2(u_1, u_2) + G_3(u_1, u_2) ,
\]

\[
G_2(u_1, u_2) = -(\nabla u_1 \nabla u_2 + u_1 \Delta u_2 + 3 \alpha u_1^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,
\]

\[
G_3(u_1, u_2) = -\alpha u^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} .
\]

It is known that the center manifold function

\[
\Phi(x) = (\Phi_1(x), \Phi_2(x)) = O(x^2) .
\]

Then, in view of (3.38) and (3.40), by direct computation we derive that

\[
(G(x \xi_1 e_K + \Phi_1, x \xi_2 e_K + \Phi_2), \varphi_K^*)
\]

\[
= (xG_2(\xi_1 e_K, \Phi_1) + xG_2(\Phi_1, \xi_2 e_K) + x^3 G_3(\xi_1 e_K, \xi_2 e_K), \varphi_K^*) + o(x^3)
\]

\[
= x\xi_1^3 \int_{\Omega} |\nabla e_K|^2 - \frac{1}{2} \xi_1 \Delta e_K^2 - 6 \alpha \xi_1 \Phi_1 e_K^2 |d\Omega
\]

\[
- \alpha\xi_1^3 x^3 \int_{\Omega} e_K^4 + o(x^3) .
\]
Using the approximation formula for center manifold functions given in (A.11) in [8], \( \Phi = (\Phi_1, \Phi_2) \) satisfies

\[
(3.44) \quad -L_{\lambda_c} \Phi = -x^2 G_2(\xi_1 e_k, \xi_2 e_K) + o(x^2) \\
= -x^2 (\xi_1 \xi_2 |\nabla e_K|^2 + (3\alpha \xi_1^2 - \xi_1 \xi_2 \rho_K) e_K^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(x^2).
\]

From (3.46) we see that

\[
\begin{aligned}
\lambda_c^2 &= \frac{1}{4} (1 + e_{2K_1})(1 + e_{2K_2}) = \frac{1}{4} (e_0 + e_{2K_1} + e_{2K_2} + e_{2K}), \\
|\nabla e_K|^2 &= \frac{\rho_{K_1}}{4} (1 - e_{2K_1})(1 + e_{2K_2}) + \frac{\rho_{K_2}}{4} (1 + e_{2K_1})(1 - e_{2K_2}) \\
&= \frac{\rho_K}{4} (e_0 + \frac{\rho_{K_2} - \rho_{K_1}}{4} e_{2K_1} + \frac{\rho_{K_1} - \rho_{K_2}}{4} e_{2K_2} - \frac{\rho_K}{4} e_{2K}).
\end{aligned}
\]

Thus, (3.44) is written as

\[
(3.45) \quad L_{\lambda_c} \Phi = -\frac{\xi_1 x^2}{4} (3\alpha \xi_1 e_0 + (3\alpha \xi_1 - 2\xi_2 \rho_K_1) e_{2K_1} + (3\alpha \xi_1 - 2\xi_2 \rho_K_2) e_{2K_2}) (1) + o(x^3).
\]

Let

\[
(3.46) \quad \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Phi_0_1 \\ \Phi_0_2 \end{pmatrix} e_0 + \begin{pmatrix} \Phi_{2K_1}^2 \\ \Phi_{2K_2}^2 \end{pmatrix} e_{2K_1} + \begin{pmatrix} \Phi_{2K_1}^2 \\ \Phi_{2K_2}^2 \end{pmatrix} e_{2K_2} + \begin{pmatrix} \Phi_{2K}^2 \\ \Phi_{2K}^2 \end{pmatrix} e_{2K}
\]

It is clear that

\[
L_{\lambda} \begin{pmatrix} \Phi_k^2 \\ \Phi_{2k}^2 \end{pmatrix} e_k = B_k^{\lambda} \begin{pmatrix} \Phi_k^2 \\ \Phi_{2k}^2 \end{pmatrix} e_k,
\]

where \( B_k^{\lambda} \) is the matrix given by (3.35). Then by (3.45) and (3.46) we have

\[
\begin{pmatrix} \Phi_{2k}^2 \\ \Phi_{2k}^2 \end{pmatrix} = -\frac{(3\alpha \xi_1^2 - 2\xi_1 \xi_2 \rho_K) x^2}{4 B_{2k}} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

for \( k = K, K_1, K_2, \) and \( B_{2k} = B_{2k}^{\lambda}. \)

Direct computation shows that

\[
(3.47) \quad \begin{pmatrix} \Phi_0_1 \\ \Phi_0_2 \end{pmatrix} = \begin{pmatrix} 3\xi_1^2 x^2 \\ 8 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_c \end{pmatrix},
\]

\[
(3.48) \quad \begin{pmatrix} \Phi_{2K_1}^2 \\ \Phi_{2K_2}^2 \end{pmatrix} = \frac{\xi_1 (3\alpha \xi_1 - 2\xi_2 \rho_K_1)}{4 \det B_{2K_1}} \begin{pmatrix} 1 + \rho_{2K_1} \\ \lambda_c \end{pmatrix},
\]

\[
(3.49) \quad \begin{pmatrix} \Phi_{2K_1}^2 \\ \Phi_{2K_2}^2 \end{pmatrix} = \frac{\xi_1 (3\alpha \xi_1 - 2\xi_2 \rho_K_2)}{4 \det B_{2K_2}} \begin{pmatrix} 1 + \rho_{2K_2} \\ \lambda_c \end{pmatrix},
\]

\[
(3.50) \quad \begin{pmatrix} \Phi_{2K}^2 \\ \Phi_{2K}^2 \end{pmatrix} = \frac{\xi_1 (3\alpha \xi_1 - 2\xi_2 \rho_K)}{4 \det B_{2K}} \begin{pmatrix} 1 + \rho_{2K} \\ \lambda_c \end{pmatrix}.
\]
Inserting (3.47) into (3.43), by (3.40) and (3.42) we get
\[<G(x \varphi_K + \Phi), \varphi^*_K> = \frac{(2 - \text{sign} K_1 K_2)(\rho K + 1)|\Omega|}{8} \]
\[\times \left[ -8 \alpha (\rho K + 1)^3 \int_{\Omega} e^4_K dx \right] + 2(\xi_2 \rho K - 6 \alpha \xi_1) \Phi^0_1 x \]
\[+ \frac{2}{1 + \text{sign} K_1} \left( \xi_2 (\rho K_2 - \rho K_1) - 6 \alpha \xi_1 \right) \Phi^2_{1K_1} x \]
\[+ \frac{2}{1 + \text{sign} K_2} \left( \xi_2 (\rho K_1 - \rho K_2) - 6 \alpha \xi_1 \right) \Phi^2_{2K_2} x \]
\[-\frac{2}{1 + \text{sign} K_1 K_2} \left( \xi_2 \rho K + 6 \alpha \xi_1 \right) \Phi^2_K x \]
\[+ \frac{(\rho K + 1) \rho_2 K_1^2 \Phi^2_{1K_1} x + (\rho K + 1) \rho_2 K_2^2 \Phi^2_{2K_2} x}{1 + \text{sign} K_1} \frac{2}{1 + \text{sign} K_2} \left( 2 - \text{sign} K_1 K_2 \right) \Phi^2_{2K_2} x + o(x^3). \]

By definition, we have
\[\rho K_1 + \rho K_2 = \rho K, \quad \rho_2 K = 4 \rho K \quad \forall K = (K_1, K_2), \]
\[< \varphi, \varphi^*> = \left[ (\rho K + 1)^2 + \rho K \lambda_\epsilon \right] \int_{\Omega} e^2_K dx \]
\[= \frac{2 - \text{sign}(K_1 K_2)}{4} (\rho K + 1)(\mu \rho K + \rho K + 2 \alpha + 1)|\Omega|. \]

In view of (3.37)-(3.40), the reduced equation (3.37) is given by
\[\frac{dx}{dt} = \beta^*_K (\lambda) x + \frac{(\rho K + 1)^3 b x^3}{8(\mu \rho K + \rho K + 2 \alpha + 1)} + o(x^3), \]
where \( b \) is the parameter as in (3.14). Then the theorem follows readily from (3.37). The proof is complete. \( \square \)

4. Transition of Three-Component Systems

4.1. The model. Hereafter \( \delta_0 \geq 0 \) is always assumed to be a constant. Hence, (2.10) has a positive constant steady state \( u^* \) given by
\[(u_1^*, u_2^*, u_3^*) \text{ with } u_1^* = \left( \frac{u_3^*}{1 + u_3^*} \right)^{1/2}, \quad u_2^* = \lambda u_1^*, \quad u_3^* u_1^* = \frac{\delta_0}{\delta}. \]

It is easy to see that \( u_1^* \) is the unique positive real root of the cubic equation
\[x^3 - \left( \frac{\delta_0}{\delta} \right)^2 x - \left( \frac{\delta_0}{\delta} \right)^2 = 0. \]

Consider the translation
\[(u_1, u_2, u_3) \rightarrow (u_1^* + u_1, u_2^* + u_2, u_3^* + u_1). \]
Then equations (2.6) are equivalent to
\[
\frac{\partial u_1}{\partial t} = \mu \Delta u_1 - 2\alpha u_1^2 u_1 - u_1^* \Delta u_2 + \frac{\alpha u_1^*}{(1 + u_3^*)^2} u_3 + g(u),
\]
\[
\frac{\partial u_3}{\partial t} = r \Delta u_3 - \delta u_3^* u_3 - \delta u_3^* u_1 - \delta u_1 u_3,
\]
(4.3)
\[
- \Delta u_2 + u_2 = \lambda u_1,
\]
\[
\frac{\partial (u_1, u_2, u_3)}{\partial n} \bigg|_{\partial \Omega} = 0,
\]
\[
u(0) = u_0,
\]
where \(u = (u_1, u_3), u_2 = \lambda [-\Delta + 1]^{-1} u_1\), and
\[
g(u) = -\nabla (u_1 \nabla u_2) - 3\alpha u_1^2 u_1^* - \alpha u_1^3 + \frac{\alpha (u_1 + u_1^*) (u_3 + u_3^*)}{1 + u_3 + u_3^*}
\]
\[
- \frac{\alpha u_1^* u_3^*}{1 + u_3^*} - \frac{\alpha u_1 u_3}{(1 + u_3^*)^2} - \frac{\alpha u_3 u_1}{1 + u_3^*}.
\]

The Taylor expansion of \(g\) at \(u = 0\) is expressed by
\[
g(u) = -\nabla (u_1 \nabla u_2) - 3\alpha u_1^2 u_1^* + \frac{\alpha u_1 u_3}{(1 + u_3^*)^2} - \frac{\alpha u_1^* u_3^*}{(1 + u_3^*)^2}
\]
\[
- \alpha u_1^3 - \frac{\alpha u_1 u_3^2}{(1 + u_3^*)^3} + \frac{\alpha u_1^* u_3^3}{(1 + u_3^*)^4} + o(3).
\]

Let
\[
H = L^2(\Omega, \mathbb{R}^2),
\]
\[
H_1 = \{ u \in H^2(\Omega, \mathbb{R}^2) | \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \}.
\]

Define the operators \(L_\lambda : H_1 \to H\) and \(G_\lambda : H_1 \to H\) by
\[
L_\lambda u = \begin{pmatrix} \mu \Delta - 2\alpha u_1^2 u_1^* - \lambda u_1^* \Delta [-\Delta + I]^{-1} - \delta u_3^* \Delta [-\Delta + I]^{-1} \frac{\alpha u_1^*}{(1 + u_3^*)^2} - r \Delta - \delta u_3^* \frac{\alpha u_1}{(1 + u_3^*)^2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \end{pmatrix},
\]
(4.5)
\[
G(u, \lambda) = \begin{pmatrix} g(u) \\ -\delta u_3 u_3 \end{pmatrix},
\]

Then the problem (4.3) takes the following the abstract form:
\[
\frac{du}{dt} = L_\lambda u + G(u, \lambda),
\]
(4.6)
\[
u(0) = u_0.
\]

It is known that the inverse mapping
\[
[-\Delta + I]^{-1} : H \to H_1
\]
is a bounded linear operator. Therefore we have
\[
L_\lambda : H_1 \to H \text{ is a sector operator, and}
\]
\[
G_\lambda : H_\theta \to H \text{ is } C^\infty \text{ bounded operator for } \theta \geq \frac{1}{2}.
\]

We note that the transition of (4.3) from \(u = 0\) is equivalent to that of (2.6) from \(u = u^*\).
Theorems 3.1 and 3.2 show that a two-component system undergoes only a dynamic transition to steady states. As we shall see, the transition for the three-component system (2.5) is quite different – it can undergo both steady state and spatiotemporal transitions.

4.2. Linearized eigenvalue of (2.6). The eigenvalue equations of (2.6) at the steady state \((u_1^*, u_2^*, u_3^*)\) given by (4.11) in their abstract form are given by

\[ L_\lambda \varphi = \beta \varphi, \]

where \(L_\lambda : H_1 \rightarrow H\) as defined in (4.5). The explicit form of (4.7) is given by

\[
\begin{pmatrix}
\mu \Delta - 2\alpha u_1^{*2} - \lambda u_1^* \Delta [-\Delta + I]^{-1} \frac{\alpha u_1^*}{1 + u_1^*} \\
-\delta u_3^*
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_3
\end{pmatrix}
= \beta
\begin{pmatrix}
\psi_1 \\
\psi_3
\end{pmatrix}.
\]

As before, let \(\rho_k\) and \(e_k\) be the eigenvalue and eigenvector of \(-\Delta\) with Neumann boundary condition given by (3.6), and let

\[
\psi_k = (\psi_{k1}, \psi_{k3}) = (\xi_{k1} e_k, \xi_{k3} e_k).
\]

Then, it is easy to see that \(\psi_k\) is an eigenvector of (4.7) provided that \((\xi_{k1}, \xi_{k3}) \in \mathbb{R}^2\) is an eigenvector of the matrix \(A_k^\lambda\):

\[ A_k^\lambda \begin{pmatrix} \xi_{k1} \\ \xi_{k3} \end{pmatrix} = \beta_k \begin{pmatrix} \xi_{k1} \\ \xi_{k3} \end{pmatrix}, \]

with

\[ (4.8) \quad A_k^\lambda = \begin{pmatrix} \frac{\lambda \rho_k u_1^*}{1 + \rho_k} - \mu \rho_k - 2\alpha u_1^{*2} & \frac{\alpha u_1^*}{1 + u_1^*} \\ -\delta u_3^* & -\delta u_3^* - \rho_k - \delta u_1^* \end{pmatrix}. \]

The eigenvalues \(\beta_k\) of \(A_k^\lambda\), which are also eigenvalues of (4.7), are expressed by

\[ (4.9) \quad \beta_k^\pm(\lambda) = \frac{1}{2} \left[ a \pm \sqrt{a^2 - 4 \det A_k^\lambda} \right]. \]

\[ a = \text{tr} A_k^\lambda = \frac{\lambda \rho_k u_1^*}{1 + \rho_k} - \mu \rho_k - 2\alpha u_1^{*2} - \rho_k - \delta u_1^*. \]

To derive the PES, we introduce two parameters as follows:

\[ (4.10) \quad \Lambda_c = \min_{\rho_K \neq \rho_K} \left\{ \frac{(\rho_K + 1) \rho_K u_1^*}{\rho_K u_1^*} \left[ \mu \rho_K + 2\alpha u_1^{*2} + \rho_K + \delta u_1^* \right] \right\}, \]

\[ (4.11) \quad \lambda_c = \min_{\rho_K \neq \rho_K} \left\{ \frac{(\rho_K + 1) \rho_K u_1^*}{\rho_K u_1^*} \left[ \mu \rho_K + 2\alpha u_1^{*2} + \frac{\alpha \delta}{(1 + u_1^*)^2(r \rho_K + \delta u_1^*)} \right] \right\}. \]

Let \(K = (K_1, K_2)\) and \(K^* = (K_1^*, K_2^*)\) be the integer pairs such that \(\rho_K\) and \(\rho_K\) satisfy (4.10) and (4.11) respectively.

**Theorem 4.1.** Let \(\Lambda_c\) and \(\lambda_c\) be the parameters defined by (4.10) and (4.11) respectively. Then we have the following assertions:

1. As \(\Lambda_c < \lambda_c\), the eigenvalues \(\beta_k^\pm(\lambda)\) of (4.9) are a pair of conjugate complex numbers near \(\lambda = \Lambda_c\), and all eigenvalues of (4.9) satisfy

\[ (4.12) \quad \text{Re} \beta_k^\pm(\lambda_c) \begin{cases} < 0 & \lambda < \Lambda_c, \\ = 0 & \lambda = \Lambda_c, \\ > 0 & \lambda > \Lambda_c, \end{cases} \]

\[ (4.13) \quad \text{Re} \beta_k^\pm(\lambda_c) < 0, \quad \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_K \]
As $\lambda < \Lambda_c$, the eigenvalue $\beta_k^\pm(\lambda)$ is real near $\lambda = \lambda_c$, and all of (4.9) satisfy

\begin{align}
\beta_k^+(\lambda) &\begin{cases} < 0, & \lambda < \lambda_c, \\ = 0, & \lambda = \lambda_c, \\ > 0, & \lambda > \lambda_c, \end{cases} \\
\beta_k^-(\lambda) &\begin{cases} \Re \beta_k^+(\lambda) < 0, & \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_{K^*}, \\ \Re \beta_k^-(\lambda) < 0, & \forall |k| \geq 0. \end{cases}
\end{align}

Proof. By (4.9) we can see that $\beta_k^\pm(\lambda)$ are a pair of complex eigenvalues of (4.7) near some $\lambda = \lambda^*$, and satisfy

\begin{align}
\Re \beta_k^\pm(\lambda) &\begin{cases} < 0, & \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ > 0, & \lambda > \lambda^*, \end{cases} \\
\text{if and only if} & \begin{cases} \text{tr} A_k^\lambda = 0, & \det A_k^\lambda > 0, \\ \text{det} A_k^\lambda = 0. & \end{cases}
\end{align}

Likewise, $\beta_k^+(\lambda)$ is real near $\lambda = \lambda^*$ and satisfies

\begin{align}
\beta_k^+(\lambda) &\begin{cases} < 0, & \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ > 0, & \lambda > \lambda^*, \end{cases} \\
\text{if and only if} & \begin{cases} \text{tr} A_k^\lambda < 0, & \det A_k^\lambda > 0, \\ \det A_k^\lambda = 0. & \end{cases}
\end{align}

Due to the definition of $\lambda_c$ and $\Lambda_c$, when $\Lambda_c < \lambda_c$ we have

\begin{align}
\text{tr} A_k^{\lambda_c} = 0, \\
\text{tr} A_k^{\Lambda_c} < 0, & \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_{K^*}, \\
\det A_k^{\lambda_c} > 0, & \forall |k| \geq 0,
\end{align}

and when $\lambda_c < \Lambda_c$,

\begin{align}
\det A_k^{\lambda_c} = 0, \\
\det A_k^{\Lambda_c} > 0, & \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_{K^*}, \\
\text{tr} A_k^{\lambda_c} < 0, & \forall |k| \geq 0.
\end{align}

It is known that the real parts of $\beta_k^\pm(\lambda)$ are negative at $\lambda$ if and only if

\begin{align}
\det A_k^{\lambda} > 0, & \text{ tr } A_k^{\lambda} < 0.
\end{align}

Hence, Assertions (1) and (2) follow from (4.16) and (4.17) respectively. The theorem is proved.

4.3. Dynamic transition theorem for (2.6). Based on Theorem 4.1 we immediately get the following transition theorem for (2.6).

Theorem 4.2. Let $\Lambda_c$ and $\lambda_c$ be given by (4.10) and (4.11) respectively. Then the following assertions hold true for (4.3):

(1) As $\lambda_c < \Lambda_c$, the eigenvalue $\beta_k^+(\lambda)$ is real near $\lambda = \lambda_c$, and all of (4.9) satisfy

\begin{align}
\beta_k^+(\lambda) &\begin{cases} < 0, & \lambda < \lambda_c, \\ = 0, & \lambda = \lambda_c, \\ > 0, & \lambda > \lambda_c, \end{cases} \\
\beta_k^-(\lambda) &\begin{cases} \Re \beta_k^+(\lambda) < 0, & \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_{K^*}, \\ \Re \beta_k^-(\lambda) < 0, & \forall |k| \geq 0. \end{cases}
\end{align}

(2) As $\lambda_c < \Lambda_c$, the eigenvalue $\beta_k^+(\lambda)$ is real near $\lambda = \lambda_c$, and all of (4.9) satisfy

\begin{align}
\beta_k^+(\lambda) &\begin{cases} < 0, & \lambda < \lambda_c, \\ = 0, & \lambda = \lambda_c, \\ > 0, & \lambda > \lambda_c, \end{cases} \\
\beta_k^-(\lambda) &\begin{cases} \Re \beta_k^+(\lambda) < 0, & \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_{K^*}, \\ \Re \beta_k^-(\lambda) < 0, & \forall |k| \geq 0. \end{cases}
\end{align}
(1) When $\Lambda_c < \lambda_c$, the system undergoes a dynamic transition to periodic solutions at $(u, \lambda) = (0, \Lambda_c)$. In particular, if the eigenvalues $\beta_K^c$ satisfying (4.14) are complex simple, then there is a parameter $b_0$ such that the dynamic transition is continuous (Type-I) as $b_0 < 0$, and is jump (Type-II) as $b_0 > 0$ with a singularity separation of periodic solutions at some $\lambda^* < \Lambda_c$.

(2) When $\lambda_c < \Lambda_c$, the system undergoes a dynamic transition to steady states at $(u, \lambda) = (0, \lambda_c)$. If $\beta_K^c(\lambda)$ satisfying (4.14) is simple, then there exists a parameter $b_1$ such that the transition is continuous as $b_1 < 0$, and jumping as $b_1 > 0$ with two saddle-node bifurcations at some $\bar{\lambda} < \lambda_c$ from $(u^+, \lambda)$ and $(u^-, \lambda)$.

**Remark 4.1.** By applying the standard procedure used in the preceding sections, we can derive explicit formulas for the two parameters $b_0$ and $b_1$ in Theorem 4.2. However, due to their complexity, we omit the details. Instead, in the following, we shall give a method to calculate $b_0$, and for $b_1$ we refer the interested readers to the proof of Theorem 3.2.

4.4. **Computational procedure of $b_0$.** The procedure to compute the parameter $b_0$ in Assertion (1) of Theorem 1.2 is divided into a few steps as follows.

**Step 1.** The reduced equations of (1.6) to center manifold at $\lambda = \Lambda_c$ are expressed by

\[
\begin{align*}
\frac{dx}{dt} &= -\rho x + \frac{1}{\varphi, \varphi^*} <G(x\varphi + y\psi + \Phi, \Lambda_c), \varphi^*> , \\
\frac{dy}{dt} &= \rho \eta x + \frac{1}{\psi, \psi^*} <G(x\varphi + y\psi + \Phi, \Lambda_c), \psi^*> ,
\end{align*}
\]

(4.18)

where $\varphi$ and $\psi$ are the eigenvectors of $L_\lambda$ at $\lambda = \Lambda_c$, $\varphi^*$ and $\psi^*$ the conjugate eigenvectors, and $L_\lambda, G_\lambda : H_1 \rightarrow H$ the operators defined by (4.5), $\Phi$ is the center manifold function.

**Step 2.** Solving the eigenvectors $\varphi, \psi$ and their conjugates $\varphi^*, \psi^*$. We know that $\psi_i$ and $\psi^*_i$ are

\[
\begin{align*}
\varphi &= (\xi_1 e_K, \xi_2 e_K), & \psi &= (\eta_1 e_K, \eta_2 e_K), \\
\varphi^* &= (\xi_1^* e_K, \xi_2^* e_K), & \psi^* &= (\eta_1^* e_K, \eta_2^* e_K),
\end{align*}
\]

(4.19)

and $\xi_i, \xi_i^*$ satisfy

\[
\begin{align*}
\left(\begin{array}{cc}
\frac{\Lambda_0 \rho K u_i^*}{1+\rho K} - \mu K - 2\alpha u_1^2 & 2\alpha u_i^* u_1^* \\
-\delta u^*_3 & -r\rho K - \delta u^*_1
\end{array}\right) \left(\begin{array}{c}
\xi_1 \\
\xi_2
\end{array}\right) &= \rho \left(\begin{array}{c}
\eta_1 \\
\eta_2
\end{array}\right), \\
\left(\begin{array}{cc}
\frac{\Lambda_0 \rho K u_i^*}{1+\rho K} - \mu K - 2\alpha u_1^2 & 2\alpha u_i^* u_1^* \\
-\delta u^*_3 & -r\rho K - \delta u^*_1
\end{array}\right) \left(\begin{array}{c}
\eta_1 \\
\eta_2
\end{array}\right) &= -\rho \left(\begin{array}{c}
\xi_1 \\
\xi_2
\end{array}\right),
\end{align*}
\]

and

\[
\begin{align*}
\left(\begin{array}{cc}
\frac{\Lambda_0 \rho K u_i^*}{1+\rho K} - \mu K - 2\alpha u_1^2 & 2\alpha u_i^* u_1^* \\
-\delta u^*_3 & -r\rho K - \delta u^*_1
\end{array}\right) \left(\begin{array}{c}
\xi_1 \\
\xi_2
\end{array}\right) &= -\rho \left(\begin{array}{c}
\eta_1^* \\
\eta_2^*
\end{array}\right), \\
\left(\begin{array}{cc}
\frac{\Lambda_0 \rho K u_i^*}{1+\rho K} - \mu K - 2\alpha u_1^2 & 2\alpha u_i^* u_1^* \\
-\delta u^*_3 & -r\rho K - \delta u^*_1
\end{array}\right) \left(\begin{array}{c}
\eta_1^* \\
\eta_2^*
\end{array}\right) &= \rho \left(\begin{array}{c}
\xi_1 \\
\xi_2
\end{array}\right),
\end{align*}
\]
where $\Lambda_\ast$ is as in (4.10), $e_k$ as in (3.6), and

$$\Lambda_\ast \rho_K u^*_1 \frac{1}{1+\rho_K} - \mu\rho_K - 2\alpha u^*_1 u^*_2 = r\rho_K + \delta u^*_1,$$

$$\rho = \det A^\ast_K = \frac{\alpha \delta_0}{(1+u^*_3)^2} - (\gamma\rho_K + \delta u^*_1)^2.$$  

Here, we use that $u^*_1 u^*_3 = \delta_0 / \delta$. From these equations we obtain

$$\xi_1 = -(r\rho_K + \delta u^*_1), \quad \xi_2 = \delta u^*_3,$$

$$\eta_1 = -\rho, \quad \eta_2 = 0,$$

(4.20)

$$\eta^*_1 = \delta u^*_3, \quad \eta^*_2 = \gamma\rho_K + \delta u^*_1.$$

Due to (4.19) and (4.20) we see that

$$\langle \varphi, \varphi^* \rangle = \langle \psi, \psi^* \rangle = (\eta_1 \eta^*_1 + \eta_2 \eta^*_2) \int_{\Omega} e^2_K dx = -\delta \rho u^*_3 \int_{\Omega} e^2_K dx,$$

$$\langle \varphi, \psi^* \rangle = \langle \psi, \varphi^* \rangle = 0.$$  

**Step 3.** We need to calculate

$$\langle G(x\varphi + y\psi + \Phi, \Lambda_\ast), \omega^*_1 \rangle,$$  

with $\omega^*_1 = \varphi^*, \omega_2 = \psi^*$. 

By (4.19) we have $G = G_2 + G_3$, and

$$G_2(\omega, \lambda) = \left( -\lambda \nabla \omega_1 \nabla (-\Delta + I)^{-1} \omega_1 - 3\alpha u^*_1 \omega^2_1 + \frac{\alpha \omega_1 \omega^2_2}{(1+u^*_3)^2} - \frac{\alpha u^*_1 \omega^2_2}{(1+u^*_3)^2} \right),$$  

$$G_3(\omega, \lambda) = \left( -\alpha \omega^3_1 - \frac{\alpha \omega_1 \omega^2_2}{(1+u^*_3)^3} + \frac{\alpha u^*_1 \omega^2_2}{(1+u^*_3)^3} \right),$$  

for $\omega = (\omega_1, \omega_2) \in H_1$. By (4.20) we find

$$\langle G_3(x\varphi + y\psi + \Phi, \Lambda_\ast), \varphi^* \rangle = 0.$$  

Noting that

$$\int_{\Omega} e_K e_J e_I dx = 0, \quad \forall K, J, I \in \mathbb{Z}^2,$$

$$\Phi = (\Phi_1, \Phi_2) = O(x^2),$$  

we have

$$\langle G_2(x\varphi + y\psi + \Phi, \Lambda_\ast), \varphi^* \rangle$$

$$= \int_{\Omega} [\xi^*_1 e_K g_{21} + \xi^*_2 e_K g_{22}] dx$$

$$= (\text{by } \xi^*_1 = 0)$$

$$= \int_{\Omega} \xi^*_2 e_K [\delta(x\xi_1 e_K + y\eta_1 e_K + \Phi_1)(x\xi_2 + y\eta_2 + \Phi_2)] dx$$

$$= \delta \xi^*_2 \left( \xi_2 x \int_{\Omega} \Phi_1 e^2_K dx + \xi_1 x \int_{\Omega} \Phi_2 e^2_K dx + \eta_1 y \int_{\Omega} \Phi_2 e^2_K dx \right).$$
Thus, we get
\begin{equation}
< G(x\varphi + y\psi + \Phi, \Lambda_c), \varphi^* >
= -\delta \xi_2 \left[ \xi_2 x \int_{\Omega} \Phi_1 e_K^2 dx + \xi_1 x \int_{\Omega} \Phi_2 e_K^2 dx + \eta_1 y \int_{\Omega} \Phi_2 e_K^2 dx \right] + o(3).
\end{equation}

In the same fashion, we derive
\begin{equation}
< G(x\varphi + y\psi + \Phi, \Lambda_c), \psi^* >
= \left( \frac{\alpha \xi_2 \eta_1^*}{(1 + u^*)^2} - 6 \alpha u^* \xi_1 \eta_1^* - \delta \xi_2 \eta_2^* \right) x \int_{\Omega} \Phi_1 e_K^2 dx - 6 \alpha u^* \xi_1 \eta_1^* y \int_{\Omega} \Phi_1 e_K^2 dx
+ \left( \frac{\alpha \eta \eta_1^*}{(1 + u^*)^2} - \frac{2 \alpha u^* \xi_2 \eta_1^*}{(1 + u^*)^3} - \delta \xi_1 \eta_2^* \right) x \int_{\Omega} \Phi_2 e_K^2 dx
+ \left( \frac{\alpha \eta \eta_1^*}{(1 + u^*)^2} - \delta \eta \eta_3^* \right) y \int_{\Omega} \Phi_2 e_K^2 dx + \frac{A_c \xi_1 \eta_1^*}{1 + \rho} x \int_{\Omega} \Phi_1 |\nabla e_K|^2 dx
+ \frac{A_c \eta \eta_1^*}{1 + \rho} y \int_{\Omega} \Phi_1 |\nabla e_K|^2 dx - \frac{1}{2} A_c \xi_1 \eta_1^* x \int_{\Omega} e_K^2 \Delta(-\Delta + I)^{-1}\Phi_1 dx
- \frac{1}{2} A_c \xi_1 \eta_1^* y \int_{\Omega} e_K^2 \Delta(-\Delta + I)^{-1}\Phi_1 dx
- \left( \frac{\alpha \eta_1^*}{(1 + u^*)^4} \xi_1^3 - \frac{\alpha \xi_2 \xi_1}{(1 + u^*)^3} - \alpha \xi_3^2 \right) \int_{\Omega} e_K^4 dx \right] x^3
- \left( \frac{\alpha \xi_2 \eta_1^*}{(1 + u^*)^3} + 3 \alpha \xi_1^2 \eta_1^* \right) \int_{\Omega} e_K^4 dx \right] x^2 y
- \left( 3 \alpha \xi_1 \eta_1^* \int_{\Omega} e_K^4 dx \right) x y^2 - \left( \alpha \eta_1^* \int_{\Omega} e_K^4 dx \right) y^3 + o(3).
\end{equation}

**STEP 4.** By the formula of center manifold function in the complex case in Theorem A.1 in [8], we have
\begin{equation}
\Phi = \left( \begin{array}{c}
\Phi_1 \\
\Phi_2
\end{array} \right) = \left( \begin{array}{c}
\Phi_1^1 \\
\Phi_2^1
\end{array} \right) + \left( \begin{array}{c}
\Phi_1^2 \\
\Phi_2^2
\end{array} \right) + \left( \begin{array}{c}
\Phi_1^3 \\
\Phi_2^3
\end{array} \right) + o(3),
\end{equation}
with
\begin{equation}
L_{\lambda_c} \left( \begin{array}{c}
\Phi_1^1 \\
\Phi_2^1
\end{array} \right) = x^2 G_{11} + xy(G_{12} + G_{21}) + y^2 G_{22},
- (L_{\lambda_c} + 4 \rho^2) \left( \begin{array}{c}
\Phi_1^2 \\
\Phi_2^2
\end{array} \right) = 2 \rho^2 \left[ (x^2 - y^2)(G_{22} - G_{11}) - 2xy(G_{12} + G_{21}) \right],
(L_{\lambda_c} + 4 \rho^2) \left( \begin{array}{c}
\Phi_1^3 \\
\Phi_2^3
\end{array} \right) = \rho \left[ (y^2 - x^2)(G_{12} + G_{21}) + 2xy(G_{11} - G_{22}) \right].
\end{equation}
Here \( G_{ij} = G_2(\Psi^i, \Psi^j, \lambda_c) \) with \( \Psi^1 = \varphi \) and \( \Psi^2 = \psi \), and \( G_2 \) is as defined in Step 3. Namely
\begin{equation}
G_{ij} = \left( \begin{array}{c}
-A_c \nabla(\Psi_1^1 \nabla(-\Delta + I)^{-1} \Psi_1^1) \\
0
\end{array} \right) + \left( \begin{array}{c}
\frac{\alpha \Psi_1 \Psi_4}{(1 + u^*)^2} - \frac{2 \alpha u^* \xi_2 \Psi_4}{(1 + u^*)^3} - 3 \alpha u^* \Psi_1 \Psi_1^2 \Psi_1^3 \\
- \delta \Psi_1 \Psi_2^2
\end{array} \right),
\end{equation}
with \( \Psi_1^i = \Gamma_i^2 e_K, 1 \leq i, l \leq 2 \), and
\begin{equation}
\Gamma_1^1 = \xi_1, \quad \Gamma_2^1 = \xi_2, \quad \Gamma_1^2 = \eta_1, \quad \Gamma_2^2 = \eta_2.
\end{equation}
which are given by (4.20).

Direct calculation shows that

\[(4.26)\quad G_{ij} = -\frac{\Lambda_2 \Gamma_1^2 \Gamma_2^j}{1 + \rho_K} \nabla(e_K \nabla e_K) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \varepsilon^2 \left( \frac{\alpha \Gamma_2^j}{(1 + u_3)^2} - \frac{\alpha u_1^i \Gamma_1^j}{(1 + u_3)^2} - 3\alpha u_1^i \Gamma_1^j \right).\]

For simplicity, we only consider the case where \(K = (K_1, 0)\). In this case, by (3.6) we can see that

\[\varepsilon^2 = \frac{1}{2}(e_0 + e_{2K}), \quad \nabla(e_K \nabla e_K) = -\rho_K e_{2K}.\]

Then, by (4.26), we have

\[(4.27)\quad G_{ij} = \left( \begin{array}{c} h_{ij}^0 \\ g_{ij}^0 \end{array} \right) e_0 + \left( \begin{array}{c} h_{ij}^{2K} \\ g_{ij}^{2K} \end{array} \right) e_{2K}, \quad 1 \leq i, j \leq 2,
\]

where

\[h_{ij}^0 = \frac{1}{2} \left[ \frac{\alpha \Gamma_2^j}{(1 + u_3)^2} - \frac{\alpha u_1^i \Gamma_1^j}{(1 + u_3)^2} - 3\alpha u_1^i \Gamma_1^j \right],
\]

\[h_{ij}^{2K} = \frac{\rho_K \lambda_1 \Gamma_1^j}{1 + \rho_K} + h_{ij}^0,
\]

\[g_{ij}^0 = g_{ij}^{2K} = \frac{1}{2} \delta_{ij} \Gamma_2^j.
\]

Let

\[(4.29)\quad \left( \begin{array}{c} \Phi^K_1 \\ \Phi^K_2 \end{array} \right) = \left( \begin{array}{c} \psi^0_{11} \\ \psi^0_{21} \\ \psi^0_{22} \end{array} \right) e_0 + \left( \begin{array}{c} \psi^{2K}_{11} \\ \psi^{2K}_{21} \\ \psi^{2K}_{22} \end{array} \right) e_{2K}, \quad 1 \leq k \leq 3.
\]

Then it follows from (4.24) and (4.27) that

\[(4.30)\quad \left( \begin{array}{c} \varphi^0_{21} \\ \varphi^0_{12} \\ \varphi^{2K}_{21} \\ \varphi^{2K}_{12} \end{array} \right) = \frac{1}{2} B_x \left[ x^2 \left( \begin{array}{c} h_{11}^0 \\ g_{11}^0 \\ h_{21}^{2K} \\ g_{21}^{2K} \end{array} \right) + xy \left( \begin{array}{c} h_{12}^0 + h_{01}^0 \\ g_{12}^0 + g_{01}^0 \\ h_{12}^{2K} + h_{01}^{2K} \\ g_{12}^{2K} + g_{01}^{2K} \end{array} \right) + y^2 \left( \begin{array}{c} h_{22}^0 \\ g_{22}^0 \\ h_{22}^{2K} \\ g_{22}^{2K} \end{array} \right) \right],
\]

\[\left( \begin{array}{c} \varphi_{21}^0 \\ \varphi_{22}^0 \\ \varphi_{21}^{2K} \\ \varphi_{22}^{2K} \end{array} \right) = \frac{1}{2} B_y \left[ x \left( \begin{array}{c} h_{11}^0 \\ g_{11}^0 \\ h_{21}^{2K} \\ g_{21}^{2K} \end{array} \right) + xy \left( \begin{array}{c} h_{12}^0 + h_{01}^0 \\ g_{12}^0 + g_{01}^0 \\ h_{12}^{2K} + h_{01}^{2K} \\ g_{12}^{2K} + g_{01}^{2K} \end{array} \right) + y^2 \left( \begin{array}{c} h_{22}^0 \\ g_{22}^0 \\ h_{22}^{2K} \\ g_{22}^{2K} \end{array} \right) \right],
\]

\[\left( \begin{array}{c} \varphi_{21}^0 \\ \varphi_{22}^0 \\ \varphi_{21}^{2K} \\ \varphi_{22}^{2K} \end{array} \right) = \frac{1}{2} B_x B_y \left[ (x^2 - y^2) \left( \begin{array}{c} h_{22}^0 - h_{01}^0 \\ g_{22}^0 - g_{01}^0 \\ h_{22}^{2K} - h_{01}^{2K} \\ g_{22}^{2K} - g_{01}^{2K} \end{array} \right) + 2xy \left( \begin{array}{c} h_{12}^0 + h_{01}^0 \\ g_{12}^0 + g_{01}^0 \\ h_{12}^{2K} + h_{01}^{2K} \\ g_{12}^{2K} + g_{01}^{2K} \end{array} \right) \right].
\]

where \(B_k = -A_k^\lambda\) with \(A_k^\lambda\) as defined by (4.8). By (4.23) and (4.29) we obtain an explicit expression of \(\Phi\) as follows:

\[(4.31)\quad \Phi_1 = (\varphi_{11}^0 + \varphi_{21}^0 + \varphi_{31})e_0 + (\varphi_{11}^{2K} + \varphi_{21}^{2K} + \varphi_{31}^{2K})e_{2K} + o(2),
\]

\[\Phi_2 = (\varphi_{12}^0 + \varphi_{22}^0 + \varphi_{32})e_0 + (\varphi_{12}^{2K} + \varphi_{22}^{2K} + \varphi_{32}^{2K})e_{2K} + o(2).
\]
Here, by (4.30), (4.28), and (4.25), \( \varphi^k_{ij} \) are 2-order homogeneous functions of \((x, y)\), with the coefficients depending explicitly on the parameters defined in (2.4).

**Step 5.** Finally, inserting (4.31) into (4.21) and (4.22), we can write (4.18) in the following form

\[
\begin{align*}
\frac{dx}{dt} &= -\rho y + a_{11}x^3 + a_{12}x^2 y + a_{13}xy^2 + a_{14}y^3 + o(4), \\
\frac{dy}{dt} &= \rho x + a_{21}x^3 + a_{22}x^2 y + a_{23}xy^2 + a_{24}y^3 + o(4).
\end{align*}
\]

Then, by Theorem 2.4.5 in [7], the parameter \( b_0 \) in Theorem 4.2 is obtained by

\[
b_0 = 3a_{11} + 3a_{24} + a_{12} + a_{23},
\]

where \( a_{11}, a_{24}, a_{12}, a_{23} \) can be explicitly expressed in the terms in (4.20)–(4.22).

4.5. **Transition for the system (2.5).** We are now in a position to discuss the transition of (2.5). With the translation (4.2), the system (2.5) is rewritten in the following form

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - 2\alpha u_1^2 u_1 - u_1^* \Delta u_2 + \frac{\alpha u_1^* u + 3}{(1 + u_3^*)^2} + g(u), \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 - u_2 + \lambda u_1, \\
\frac{\partial u_3}{\partial t} &= r \Delta u_3 - \delta u_1^* u_3 - \delta u_3^* u_1 - \delta u_1 u_3, \\
\frac{\partial u}{\partial n} \bigg|_{\partial \Omega} &= 0, \\
u(0) &= u_0,
\end{align*}
\]

where \( g(u) \) is as in (4.4). Here the notation \( u \) stands for three-component unknown:

\[
u = (u_1, u_2, u_3).
\]

Let

\[
L_\lambda u = \begin{pmatrix}
\mu \Delta - 2\alpha u_1^2 & -u_1^* \Delta & \frac{\alpha u_1^* u + 3}{(1 + u_3^*)^2} \\
\lambda & \Delta - 1 & 0 \\
-\delta u_3^* & 0 & r \Delta - \delta u_1^*
\end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.
\]

Then, all eigenvalues \( \beta^j_k(\lambda) \) and eigenvectors \( \psi^j_k \) of \( L_\lambda \) satisfy

\[
D^\lambda_k \begin{pmatrix} \xi^j_{k1} \\ \xi^j_{k2} \\ \xi^j_{k3} \end{pmatrix} = \beta^j_k(\lambda) \begin{pmatrix} \xi^j_{k1} \\ \xi^j_{k2} \\ \xi^j_{k3} \end{pmatrix}, \quad 1 \leq j \leq 3, \quad k \in \mathbb{Z}^2,
\]

with

\[
\psi^j_k = (\xi^j_{k1} e_k, \xi^j_{k2} e_k, \xi^j_{k3} e_k),
\]

and \( e_k \) as in (3.6), \( D^\lambda_k \) is a 3 \times 3 matrix given by

\[
D^\lambda_k = \begin{pmatrix}
-(\mu \rho_k + 2\alpha u_1^* u_1^2) & u_1^* \rho_k & \frac{\alpha u_1^* u + 3}{(1 + u_3^*)^2} \\
\lambda & -((\rho_k + 1) & 0 \\
-\delta u_3^* & 0 & -r \rho_k + \delta u_1^*
\end{pmatrix}.
\]
We introduce the following three parameters:
\[
\begin{align*}
A_k^\lambda &= -\text{tr} D_k^\lambda = \mu \rho_k + 2\alpha u_1^2 + \rho_k + 1 + r \rho_k + \delta u_1^*, \\
B_k^\lambda &= \det \begin{pmatrix} -\mu \rho_k + 2\alpha u_1^2 & u_1^\ast \rho_k \\ \lambda & -(\rho_k + 1) \end{pmatrix} + \det \begin{pmatrix} -\mu \rho_k + 2\alpha u_1^2 & \frac{\alpha u_1^*}{1 + u_3^*} \\ -\delta u_3^* & -(r \rho_k + \delta u_1^*) \end{pmatrix} + (\rho_k + 1)(r \rho_k + \delta u_1^*), \\
C_k^\lambda &= -\text{det} D_k^\lambda = (\mu \rho_k + 2\alpha u_1^2)(\rho_k + 1)(r \rho_k + \delta u_1^*) - u_1^* \rho_k \lambda (r \rho_k + \delta u_1^*) + \frac{\alpha u_1^*}{1 + u_3^*} \delta u_3^* (r \rho_k + 1).
\end{align*}
\]

By the Routh–Hurwitz theorem, we know that all eigenvalues \( \beta_k^1 \) of \( D_k^1 \) have negative real parts if and only if
\[
(4.33) \quad A_k^\lambda > 0, \quad A_k^\lambda B_k^\lambda - C_k^\lambda > 0, \quad C_k^\lambda > 0.
\]

Let \( \Lambda^c \) and \( K = (K_1, K_2) \) satisfy
\[
(4.34) \quad A_k^\lambda > 0, \quad A_k^\lambda B_k^\lambda - C_k^\lambda = 0, \quad C_k^\lambda > 0, \quad A_k^\lambda > 0, \quad A_k^\lambda B_k^\lambda - C_k^\lambda > 0, \quad C_k^\lambda > 0, \quad \forall k \text{ with } \rho_k \neq \rho_K.
\]

Then \( \Lambda^c \) satisfies that
\[
(4.35) \quad \Lambda^c = \inf_{\rho_k} \frac{1}{\rho_k u_1^*} \left[ (\mu + r) \rho_k + 2\alpha u_1^2 + \delta u_1^* \right] \\
\times \left[ (r + 1) \rho_k + \delta u_1^* + 1 + \frac{\alpha \delta}{(\mu \rho_k + 2\alpha u_1^2 + \rho_k + 1)(1 + u_3^*)^2} \right],
\]

and \( \rho_K \) satisfies (4.35). In particular, under the condition (4.34), there is a pair of complex eigenvalues \( \beta_K^1(\lambda) \) and \( \beta_K^2(\lambda) \) of \( D_K^\lambda \), such that
\[
(4.36) \quad \text{Re } \beta_K^{1,2}(\lambda) \begin{cases} < 0, & \lambda < \Lambda^c, \\ = 0, & \lambda = \Lambda^c, \\ > 0, & \lambda > \Lambda^c, \end{cases}
\]

and the other eigenvalues \( \beta_k^j(\lambda) \) of \( L_\lambda \) satisfy
\[
(4.37) \quad \begin{cases} \text{Re } \beta_k^j(\Lambda^c) < 0, & \forall k \text{ with } \rho_k \neq \rho_K, \text{ and } 1 \leq j \leq 3, \\ \beta_k^j(\Lambda^c) < 0. \end{cases}
\]

Let \( \lambda_c \) and \( K^* = (K_1^*, K_2^*) \) satisfy
\[
(4.38) \quad A_K^{\lambda_c} > 0, \quad A_K^{\lambda_c} B_K^{\lambda_c} - C_K^{\lambda_c} > 0, \quad C_K^{\lambda_c} = 0, \\
A_k^{\lambda_c} > 0, \quad A_k^{\lambda_c} B_k^{\lambda_c} - C_k^{\lambda_c} > 0, \quad C_k^{\lambda_c} > 0, \quad \forall k \text{ with } \rho_k \neq \rho_K^*.
\]

Then \( \lambda_c \) is given by
\[
(4.39) \quad \lambda_c = \inf_{\rho_k} \frac{(\rho_k + 1)}{\rho_k u_1^*} \left[ \mu \rho_k + 2\alpha u_1^2 + \frac{\alpha \delta}{(1 + u_3^*)(r \rho_k + \delta u_1^*)} \right],
\]
and $\lambda_c$ arrives its minimal at $\rho_{K^*}$. From the Routh-Hurwitz criterion (4.33), we deduce that with (4.38) there is a real eigenvalue $\beta_{K^*}'(\lambda)$ of $D_{\lambda c}K^*$ satisfies

$\beta_{K^*}'(\lambda) \begin{cases} < 0, & \lambda < \lambda_c, \\ = 0, & \lambda = \lambda_c, \\ > 0, & \lambda > \lambda_c, \end{cases}$

(4.40)

$\begin{cases} \text{Re}\beta_{K^*}'(\lambda_c) < 0, & j = 2, 3, \\ \forall k \in \mathbb{Z}^2, & \rho_k \neq \rho_{K^*} \text{ and } 1 \leq j \leq 3. \end{cases}$

(4.41)

It is clear that (4.36) and (4.37) hold true as $\Lambda_c < \lambda_c$, and (4.39)-(4.40) hold true as $\lambda_c < \Lambda_c$. Hence, we have the following transition theorem for (4.32).

**Theorem 4.3.** Let $\Lambda_c$ and $\lambda_c$ be given by (4.35) and (4.39) respectively. Then, Assertions (1) and (2) of Theorem 4.2 hold true for the system (4.32).

5. Biological Conclusions

5.1. Biological significance of transition theorems. Pattern formation is one of the characteristics for bacteria chemotaxis, and is fully characterized by the dynamic transitions. Theorems 3.1–4.3 tell us that the nondimensional parameter $\lambda$, given by

$$\lambda = \sqrt{r_1r_2}\chi,$$

(5.1)

plays a crucial role to determine the dynamic transition and pattern formation. Actually, the key factor in (5.1) is the product of the chemotactic coefficient $\chi$ and the production rate $r_1$ which depends on the type of bacteria. When $\lambda$ is less than some critical value $\lambda_c$, the uniform distribution of biological individuals is a stable state. When $\lambda$ exceeds $\lambda_c$, the bacteria cells aggregate to form more complex and stable patterns.

As seen in (3.11), (4.10), (4.11) and (4.35), under different biological conditions, the critical parameter $\lambda_c$ takes different forms and values. But, a general formula for $\lambda_c$ is of the following type:

$$\lambda_c = a_0 + \inf_{\rho_k} \left( a_1\rho_k + \frac{a_2}{\rho_k} + \frac{a_3}{b_1\rho_k + b_0} + \frac{a_4}{\rho_k(b_1\rho_k + b_0)} \right),$$

(5.2)

where $\rho_k$ are taken as the eigenvalues of $-\Delta$ with the Neumann boundary condition. When $\Omega$ is a rectangular region, $\rho_k$ are given by (3.6), and the coefficients $a_j$ ($1 \leq j \leq 4$), $b_0, b_1 \geq 0$ depend on the parameters in (2.4), with $a_0, a_1, a_2, b_0, b_1 > 0$, $a_3, a_4 \geq 0$.

In particular, for the system with rich nutrient supplies, (5.2) becomes

$$\lambda_c = a_0 + \inf_{\rho_k} \left( a_1\rho_k + \frac{a_2}{\rho_k} \right).$$

(5.3)

The eigenvalues $\rho_k$, depending on the geometry of $\Omega$, satisfy

$$\rho_0 < \rho_1 \leq \cdots \leq \rho_k \leq \cdots, \quad \rho_k \to \infty \text{ as } k \to \infty,$$

where $L$ is the length scale of $\Omega$.

We infer from (5.2) and (5.3) that

$$\lambda_c \to \infty \quad \text{as} \quad |\Omega| \to 0 \quad (L \to 0).$$
It implies that when the container $\Omega$ is small, the homogenous state is state and there is no pattern formation of bacteria under any biological conditions.

5.2. Spatiotemporal oscillation. Theorems 4.2 and 4.3 show that there are two critical parameters $\lambda_c$ and $\Lambda_c$, such that if $\lambda_c < \Lambda_c$, the patterns formed by biological organisms are steady, as exhibited by many experimental results, and if $\Lambda_c < \lambda_c$ a spatial-temporal oscillatory behavior takes place.

For the case with rich nutrient, $u_1^* = 1$, $u_3^* = \infty$.

In this situation, $\lambda_c$ in (4.11) is reduced to (3.8), and obviously we have that $\lambda_c < \Lambda_c$ for both (4.10) and (4.35), and the dynamic transition and pattern formation are determined by Theorems 3.1 and 3.2. Hence there is no spatiotemporal oscillations for the rich nutrient case, and the time periodic oscillation of chemotaxis occurs only for the case where the nutrient is moderately supplied.

In particular, if $\mu, r \equiv 0$, and

$$\delta^2u_1^2(1 + u_3^*)^2 < a\delta_0,$$

then for $\Lambda_c$ defined by (4.10) and (4.35), we have

$$\Lambda_c < \lambda_c.$$

In this case, a spatial-temporal oscillation pattern are expected for $\lambda > \Lambda_c$.

5.3. Transition types. One of the most important aspects of the study for phase transitions is to determine the transition types for a given system. The main theorems in this article provide precise information on the transition types. In all cases, types are precisely determined by the sign of some non dimensional parameters; see $b$, $b_0$ and $b_1$ respectively in the main theorems. Hence a global phase diagram can be obtained easily by setting the related parameter to be zero.

For example, when $\Omega = (0, L_1)$ is one-dimensional or when $K = (K_1, 0)$ (resp. $K = (0, K_2)$), the parameter $b$ in (3.14) can be simplified into the following form

$$b = 2 \left[ -3\mu \rho K + 9\alpha - \frac{(2\mu \rho K + \alpha)(2\mu \lambda_2^2 + 28\alpha \rho K + 4\alpha - \mu \rho K)}{(\mu \rho_2 K + 2\alpha)(\rho_2 K + 1) - \rho_2 K \lambda_c} \right].$$

For a non-growth system, $\alpha = 0, K = (1, 0)$, $\lambda_c = \mu(\rho K+1)$. Then, (5.4) becomes

$$b = \frac{\mu}{3}(1 - 20\lambda_1), \quad \lambda_1 = \frac{\pi^2}{L_1^2},$$

and $\lambda = \frac{\alpha r_1}{r_2 k_2}$, with $a = \frac{1}{16\pi} \int_{\Omega} u_1 \, dx$. It follows from (5.5) that

$$b = \begin{cases} < 0 & \text{if } L_1 < 2\sqrt{5}\pi, \\ > 0 & \text{if } L_1 > 2\sqrt{5}\pi. \end{cases}$$

By Theorems 3.1 and 3.2 the phase transition of (3.3) and (3.1) from $(u, \lambda) = (u^*, \lambda_c)$ is continuous if the length scale $L_1$ of $\Omega$ is less than $2\sqrt{5}\pi$, and jump if $L_1$ is bigger than $2\sqrt{5}\pi$.

In addition, when we take

$$\chi(u) = \frac{\chi_1 u_1}{(\beta + u_2)^2}.$$
as the chemotaxis function, by Remark 11, the parameter $b$ of (5.5) is replaced by

$$b_1 = \frac{\mu}{3} \left( 1 - \frac{20\pi^2}{L_1^2} \right) + \frac{4\kappa\mu^2\pi^2}{L_1^2}$$

with

$$\kappa = \frac{k_2}{\beta_{\chi} + k_2\lambda_c}, \quad \text{and} \quad \lambda_c = \mu \left( \frac{\pi^2}{L_1^2} + 1 \right).$$

The above conclusion amounts to saying that for a non-growth system, the parameter

$$\lambda = \frac{r_1\chi}{r_2k_2}a, \quad \text{with} \quad a = \frac{1}{|\Omega|} \int_{\Omega} u_1 dx,$$

is proportional to the average density $a$ of initial condition of $u_1$ ($u_1$ is conservation). Hence, the biological individual is in a homogenous distribution state provided

$$\frac{1}{|\Omega|} \int_{\Omega} \varphi dx < \frac{r_2k_2}{r_1\chi} \mu \left( \frac{\pi^2}{L_1^2} + 1 \right), \quad \varphi = u_1(0),$$

and the bacteria will aggregate to form numbers of high density regions provided

$$\frac{1}{|\Omega|} \int_{\Omega} \varphi dx > \frac{r_2k_2}{r_1\chi} \mu \left( \frac{\pi^2}{L_1^2} + 1 \right).$$

Moreover, under the condition (5.7), if the scale $L_1$ of $\Omega$ is smaller than some critical value $L_c$ (in (5.6) $L_c = 2\sqrt{5\pi}$), i.e. $L_1 < L_c$, the continuous transition implies that there is only one high density region of bacteria to be formed, and if $L_1 > L_c$ then the jump transition expects a large number of high density regions to appear.

5.4. Pattern formation. As mentioned before, the pattern formation behavior is dictated by the dynamic transition of the system. In this article, we studied the formation of two type patterns—the lamella and the rectangular patterns, although the approach can be generalized to study the formation of other more complex patterns.

For a growth system, the critical parameter $\lambda_c$ takes its value at some eigenvalue $\rho_K$ of $-\Delta$ for $K = (K_1, K_2)$, as shown by (3.11) and (4.11). From the pattern formation point of view, for the Type-I transition, the patterns described by the transition solutions in the main theorems are either lamella or rectangular:

- lamella pattern for $K_1K_2 = 0$,
- rectangular pattern for $K_1K_2 \neq 0$.

In the case where $b > 0$, the system undergoes a more drastic change. As $\lambda^* < \lambda < \lambda_c$, the homogeneous state, the new patterns $v_2^+ \text{ and } v_4^+$ are metastable. For $\lambda > \lambda_c$, the system undergoes transitions to more complex patterns away from the basic homogeneous state.

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