On Yangian-invariant regularization of deformed on-shell diagrams in $\mathcal{N} = 4$ super-Yang–Mills theory

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Abstract
We investigate Yangian invariance of deformed on-shell diagrams with $D = 4$, $\mathcal{N} = 4$ superconformal symmetry. We find that invariance implies a direct relationship between the deformation parameters and the permutation associated with the on-shell graph. We analyse the connection with deformations of scattering amplitudes in $\mathcal{N} = 4$ super-Yang–Mills theory and the possibility of using the deformation parameters as a regulator preserving Yangian invariance. A study of higher-point tree and loop graphs suggests that manifest Yangian invariance of the amplitude requires trivial deformation parameters.

Keywords: supersymmetry, Yangian invariance, scattering amplitudes, regularisation, conformal symmetry

(Some figures may appear in colour only in the online journal)

1. Introduction
One of the most striking features of the $\mathcal{N} = 4$ super-Yang–Mills (sYM) theory is that its tree-level S-matrix is invariant under the infinite-dimensional Yangian algebra $Y[\text{psu}(2, 2|4)]$ [1]; see also [2, 3]. This symmetry arises as the closure of the ordinary superconformal symmetry together with the hidden dual superconformal symmetry that the theory possesses [4]. The existence of this infinite-dimensional symmetry algebra is further important evidence in support of the conjectured integrability of the planar sector of the theory; see [5]. It is however well known that IR divergences break the (dual) superconformal symmetry of scattering amplitudes at loop level to some extent [4].
In [6], the authors proposed an interesting construction relating Yangian invariants in $\mathcal{N} = 4$ sYM to so-called on-shell graphs (or on-shell diagrams). Such graphs are associated with integrals over suitably defined subspaces of Grassmannian manifolds and provide a direct link to the Grassmannian formulation of scattering amplitudes, introduced and studied in [7–10].

In [12, 13], the authors introduced deformed on-shell graphs, where the deformation consists of the shift of the helicities of the external legs by a complex value. In this article we focus on the conditions for manifest Yangian invariance for such deformed graphs.

A topic which is worth investigating in this framework is the relation between deformed on-shell diagrams and scattering amplitudes, a relation which is currently unclear. We will study the possibility of constructing such a correspondence in the context of the tree-level Britto–Cachazo–Feng–Witten (BCFW) recursion relations and their supersymmetric extension [14–17]. Our starting point will be the formulation of the BCFW recursion relations in terms of undeformed on-shell graphs. We will subsequently focus on one simple example—the six-point NMHV amplitude—and study the possibility of constructing a manifestly Yangian-invariant deformed amplitude as a sum of deformed Yangian invariants.

In [12, 13] it was also proposed that the deformation parameters could be used as regulators for loop amplitudes. As an example, the four-point one-loop amplitude in $\mathcal{N} = 4$ sYM was explicitly computed for a particular choice of deformation which happens to break Yangian invariance. We will address the question of the compatibility of this regulating method with Yangian invariance by computing the four-point one-loop deformed on-shell graph with a Yangian-preserving set of deformation parameters.

The paper is organized as follows. In section 2 we review the concept of Yangian invariance, we define undeformed and deformed on-shell graphs and we discuss the conditions that these deformed on-shell graphs have to satisfy in order to be Yangian invariant. Section 3 deals with the translation from Yangian invariants represented by deformed on-shell graphs to scattering amplitudes. We will focus in particular on tree-level NMHV amplitudes as well as the four-point one-loop amplitude in $\mathcal{N} = 4$ sYM, and discuss their compatibility with the deformation of external helicities. Section 4 contains a summary of the results.

2. Yangian invariance

A mechanism for building Yangian invariants starting from three-point vertices in $\mathcal{N} = 4$ sYM and producing so-called on-shell diagrams was developed in [6]. Yangian invariance, however, allows for more general building blocks: in this section we will use deformed three-point vertices (developed in [12, 13]) and investigate the properties of the resulting deformed on-shell diagrams. The relation to scattering amplitudes in $\mathcal{N} = 4$ sYM theory will be discussed in section 3 below.

2.1. Yangian symmetry and representation on spinor variables

The Yangian algebra $Y := Y[\mathfrak{g}]$ is a quantum algebra based on (half of) the affine extension of the Lie algebra $\mathfrak{g}$. In addition to the generators $\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ obeying the usual Lie-algebra relations

\footnote{In what follows, we will not distinguish between Lie algebras and Lie superalgebras. When considering Lie superalgebras, all expressions should be understood with the appropriate graded symmetrization and factors of $(-1)^{\text{par}}$.}
there are level-1 generators $\tilde{J}^a$ satisfying
\begin{equation}
[\tilde{J}^a, \tilde{J}^b] = f_{ij}^{ab} \tilde{J}^c, \tag{2.2}
\end{equation}
and the Serre relations
\begin{equation}
[[\tilde{J}^a, \tilde{J}^b], \tilde{J}^c] + [[\tilde{J}^b, \tilde{J}^c], \tilde{J}^a] + [[\tilde{J}^c, \tilde{J}^a], \tilde{J}^b] = f_{ij}^{ab} f_{j\ell}^{ci} f_{\ell\alpha}^{bh} \tilde{J}^{ai} \tilde{J}^b \tilde{J}^c \tilde{J}^f, \tag{2.3}
\end{equation}
where $[\ldots]$ means graded symmetrization of indices. The complete Yangian algebra $\mathcal{Y} [\mathfrak{g}]$ is obtained by successively commuting level-0 and level-1 generators. The coproduct $\Delta : \mathcal{Y} \rightarrow \mathcal{Y} \otimes \mathcal{Y}$ of the Yangian algebra is defined on the level-0 and level-1 generators as
\begin{equation}
\Delta (\tilde{J}^a) = 1 \otimes \tilde{J}^a + f^{ab} \tilde{J}^b \otimes \tilde{J}^a, \tag{2.4}
\end{equation}
respectively.

In this article, we will consider $\mathfrak{psu}(2, 2|4)$, the Yangian of the maximal superconformal algebra in four dimensions. A convenient representation of $\mathfrak{psu}(2, 2|4)$ acts on functions defined on the on-shell superspace with spinor-helicity variables $\lambda, \tilde{\lambda}$, and Grassmann variables $\eta$. Here, dotted and undotted Greek spinor indices take the values 1 and 2, and capital italic indices correspond to a four-dimensional fundamental representation of SU(4).

The spinors $\lambda$ and $\tilde{\lambda}$ are related to the massless momentum vector via $p_\mu = \sigma_{\mu \alpha} \lambda^\alpha$, where the $\sigma_{\mu \alpha}$ are generalized Pauli matrices which have to be chosen according to the spacetime signature. Here we work with complexified momenta, which amounts to treating $\lambda$ and $\tilde{\lambda}$ as independent complex variables [18]. For a fixed external momentum $p_\mu$ the spinors $\lambda$ and $\tilde{\lambda}$ are defined up to a rescaling:
\begin{equation}
\lambda \rightarrow a \lambda, \quad \tilde{\lambda} \rightarrow a^{-1} \tilde{\lambda}, \tag{2.5}
\end{equation}
where $a$ is an arbitrary nonzero complex number. In the supersymmetric part of the on-shell space, the Grassmann variables have to scale as
\begin{equation}
\eta \rightarrow a^{-1} \eta \tag{2.6}
\end{equation}
for consistency. Imposing reality of momenta in signature $(1, 3)$ amounts to equating $\tilde{\lambda} = \pm \lambda$, where the sign depends on the sign of the energy. This implies that the freedom of rescaling $\lambda, \tilde{\lambda}$ for real momenta is restricted to multiplication by a phase.

In this representation, the generators of $\mathfrak{psu}(2, 2|4)$ read (see [19])
\begin{align}
\Sigma^{\alpha \beta} &= \lambda^\alpha \frac{\partial}{\partial \lambda^\beta} - \frac{1}{2} \delta^{\alpha \beta} \lambda^{\gamma} \frac{\partial}{\partial \lambda^{\gamma}}, & \Sigma^{\alpha \beta} &= \lambda^\alpha \frac{\partial}{\partial \lambda^\beta} - \frac{1}{2} \delta^{\alpha \beta} \lambda^{\gamma} \frac{\partial}{\partial \lambda^{\gamma}}, \\
\Sigma^{\alpha \beta} &= \lambda^\alpha \eta^\beta, & \Sigma^{\alpha \beta} &= \lambda^\alpha \eta^\beta, \\
\Sigma^{a} &= \frac{\partial^2}{\partial \lambda^a \partial \eta^a}, & \Sigma^{a} &= \frac{\partial^2}{\partial \lambda^a \partial \eta^a}, \\
\Sigma_{\alpha a} &= \eta^a \frac{\partial}{\partial \eta^\alpha} - \frac{1}{2} \delta^{\alpha \beta} \eta^{\gamma} \frac{\partial}{\partial \eta^{\gamma}}, & \Sigma_{\alpha a} &= \eta^a \frac{\partial}{\partial \eta^\alpha} - \frac{1}{2} \delta^{\alpha \beta} \eta^{\gamma} \frac{\partial}{\partial \eta^{\gamma}} + 1, \\
\Sigma^a &= \lambda^a \tilde{\lambda}^a, & \Sigma^a &= \lambda^a \tilde{\lambda}^a. \tag{2.7}
\end{align}
The above representation of the Lie superalgebra $\mathfrak{psu}(2, 2|4)$ can be lifted to an evaluation representation of the Yangian $\mathfrak{psu}_Y (2, 2|4)$, for which the level-1 generators act as $\hat{\mathfrak{g}} \cong u \hat{\mathfrak{g}}$. Here $u$ is called the evaluation parameter of the representation. In the evaluation representation, the left-hand side of the Serre relation, equation (2.3), reduces to the usual Jacobi identity. Accordingly, the right-hand side can be shown to vanish.

With the evaluation representation at our disposal, one can construct the action of the Yangian algebra on functions depending on several variables in the on-shell superspace. This action can be obtained from the maximally iterated coproduct

$$\Delta^{n-1}(\hat{\mathfrak{g}}) = \sum_{k=1}^{n} \hat{\mathfrak{g}}_k + f'_{bc} \sum_{1 \leq i < j \leq n} \hat{\mathfrak{g}}_b \hat{\mathfrak{g}}_c = \sum_{k=1}^{n} u_k \hat{\mathfrak{g}}_k + f'_{bc} \sum_{1 \leq i < j \leq n} \hat{\mathfrak{g}}_b \hat{\mathfrak{g}}_c, \quad (2.8)$$

where the first equality holds due to the definition of the coproduct equation (2.4), whereas the second equality applies to evaluation representations. The evaluation parameters $u_k$ can (and will) be different for each element of the tensor product.

In addition to the generators defined in equation (2.7), there is another important quantity for the discussion to follow: the central charge operator $C$ that appears in the anticommutator $[Q_S, ]$, defined as

$$\{Q^A_L, R^B_R\} = \frac{1}{2} \delta^A_B \delta^B_A + \frac{1}{2} [\Omega + \frac{1}{2} C]. \quad (2.9)$$

It is represented by

$$C := \lambda^a \frac{\partial}{\partial \lambda^a} - \tilde{\lambda}^a \frac{\partial}{\partial \tilde{\lambda}^a} - \tilde{\eta}^A \frac{\partial}{\partial \tilde{\eta}^A} + 2. \quad (2.10)$$

In the algebra $\mathfrak{psu}(2, 2|4)$, the central charge operator $C$ is set to zero. If one however considers the central extension to $\mathfrak{su}(2, 2|4)$, the operator $C$ does not need to vanish any more.

A function $Y(\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}, i = 1, ..., n)$, is called a Yangian invariant if it is annihilated by the maximally iterated coproduct $\Delta^{n-1}(\hat{\mathfrak{g}})$ for all operators $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}^\hat{\mathfrak{g}}$. In particular, for the charge operator the condition reads

$$\sum_{i=1}^{n} C_i Y(\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}) = 0. \quad (2.11)$$

One can understand the eigenvalue of the operator $C_i$ as an additional scaling weight for the $i$th copy of the superspace that the Yangian invariant is defined upon. In particular the variable $\lambda_i$ now scales with a factor $\alpha^{1+c_i}$. With nonzero $c_i$ one is effectively working on a weighted projective space.

### 2.2. Yangian invariants from undeformed building blocks

Yangian-invariant objects for constructing scattering amplitudes in $\mathcal{N} = 4$ sYM theory have been explored in [6] in order to derive an all-loop generalization of the BCFW recursion relations [14–17].

The main tools in the investigation of Yangian invariants in [6] are undeformed on-shell graphs, which are planar graphs obtained by gluing (together) basic Yangian-invariant building blocks of two types. The building blocks used are the MHV and $\mathbf{MHV}$ three-point tree-level superamplitudes in $\mathcal{N} = 4$ sYM theory.

After reviewing the undeformed formalism in the current subsection, we are going to study deformed building blocks and check under which circumstances the corresponding deformed on-shell graphs are Yangian invariants.
Undeformed vertices. Expressed in spinor-helicity and Grassmann variables of the on-shell superspace introduced in subsection 2.1, the two basic building blocks used in [6] read

$$A_{3, \text{MHV}} = \frac{\delta^4(P) \delta^4(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \quad A_{3, \text{MHV}} = \frac{\delta^4(P) \delta^4(\tilde{Q})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \tag{2.12}$$

where $P^{\alpha} := \sum_{i=1}^{4} \tilde{\omega}_i^a \tilde{\omega}_i^a$ is the total 4-momentum, whereas $Q^{\dot{\alpha}} := \sum_{i=1}^{4} \dot{\omega}_i^a \tilde{\omega}_i^a$ and $\tilde{Q}^{\dot{\alpha}} := \langle 12 \rangle \dot{\omega}_1^a \tilde{\omega}_2^a + \langle 23 \rangle \dot{\omega}_3^a \tilde{\omega}_4^a$. Furthermore, $\langle ij \rangle = \tilde{\omega}_i^a \tilde{\omega}_j^a$ and $[ij] = \tilde{\omega}_i^a \dot{\omega}_j^a$, where indices are raised and lowered with dotted and undotted totally antisymmetric 2x2 tensors $\epsilon$, where $\epsilon_{12} = -1$.

Being superamplitudes in $\mathcal{N} = 4$ SYM theory, these two 3-vertices are Yangian invariants, as shown for example in [1, 3]. However, for now we would like to focus on their symmetries without making reference to their amplitude properties.

We will represent $A_{3, \text{MHV}}$ and $A_{3, \text{MHV}}$ with a black and a white dot, respectively:

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      \  \    
     / \   / \
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Undeformed on-shell graphs. In this part we briefly review the formalism of on-shell graphs (or diagrams). In particular we will address how to build higher-point Yangian invariants by gluing the basic building blocks introduced in equation 2.12.

Let us consider two trivalent graphs for simplicity. If we consider their product with one leg identified,

$$F_3(p_1, p_2, p_3, p_4, p_I) := A_{3, \text{MHV}}(p_1, p_2, p_I) A_{3, \text{MHV}}(-p_1, p_3, p_4), \tag{2.14}$$

then it is not clear how to treat the state corresponding to momentum $p_I$. Explicitly, if we consider $I$ as an external state, then the total momentum is $\sum_{i=1}^{4} P_i^{\alpha} + P_I^{\dot{\alpha}}$, which clearly does not annihilate $F_3$. On the other hand, if we consider the external states to be just 1, 2, 3 and 4, the total momentum is $\sum_{i=1}^{4} P_i^{\alpha}$ and it does annihilate $F_3$, but now $p_I$ acts as a ‘preferred’ momentum that breaks Lorentz invariance. Therefore, the mere identification of a leg between two superconformal invariants does not give another superconformal invariant.

In order to render the combination of $A_{3, \text{MHV}}(p_1, p_2, p_I)$ and $A_{3, \text{MHV}}(-p_1, p_3, p_4)$ a superconformal invariant, one has to integrate over the on-shell superspace corresponding to the identified leg with the measure

$$\int \frac{d^2 \lambda_I}{\text{Vol}[\text{GL}(1)]} d^4 \tilde{\eta}_I, \tag{2.15}$$

a procedure which we will call gluing.

Invariance under the total momentum generator $P^{\alpha} = \sum_{i=1}^{4} P_i^{\alpha}$ is ensured by the overall delta $\delta(\sum_{i=1}^{4} P_i^{\alpha})$. The resulting glued object can be checked to be invariant under superconformal as well as dual superconformal transformations provided the building blocks are Yangian invariants, as shown in [10].

Correspondingly, one can introduce a graphical description of Yangian invariants in terms of so-called on-shell graphs: connected planar graphs constructed by gluing trivalent...

\footnote{This can be seen as follows: the spinor products are Lorentz invariant (since the delta functions allow us to substitute $p_I = p_1 + p_2$), but a Lorentz transformation acting only on 1, 2, 3, 4 does not leave the delta functions invariant.}
black and white vertices associated with \( A_{3,\text{MHV}} \) as in equation (2.13) and where each internal line corresponds to an integration over the on-shell phase space. For example, equation (2.16) represents the graph obtained by gluing (together) \( A_{3,\text{MHV}} \) and \( A_{3,\text{MHV}} \):

\[
\text{(2.16)}
\]

In [6] it was shown how the equivalence of different on-shell graphs can be encoded in a purely combinatorial object: the permutation \( \sigma \) associated with the on-shell graph. If two on-shell graphs represent the same Yangian invariant, they will encode the same permutation. The permutation associated with an on-shell graph is a bijective map

\[
\sigma: \{1,...,n\} \to \{1,...,n\}, \tag{2.17}
\]

which is constructed as follows: starting from an external point \( i \) of an on-shell graph, follow the internal lines turning right at each black vertex and left at each white vertex. The image \( \sigma(i) \) is given by the ending of the external line \( j \). In figure 1 there is an example for the permutation obtained from following the rules described above.

The equivalence of two on-shell graphs can be deduced graphically. There are two graphical transformations that do not affect the permutation: merging and square move. Merging changes the way that four lines are connected by two equally coloured vertices, while the square move rotates a subgraph consisting of a square of vertices with alternating colours by 90°. Both operations are depicted in figure 2.

2.3. Yangian invariants from deformed building blocks

**Deformed vertices.** The undeformed building blocks \( A_{3,\text{MHV}} \) and \( A_{3,\text{MHV}} \) defined in equation (2.12) satisfy not only the central charge condition for Yangian invariance equation (2.11), but also the stronger constraint

\[
\mathcal{C}_i \cdot A_{3,\text{MHV}} = 0, \quad \mathcal{C}_i \cdot A_{3,\text{MHV}} = 0, \tag{2.18}
\]

that is, the operator \( \mathcal{C} \) defined in equation (2.10) vanishes for each component of the tensor product individually. Taking into account the gluing procedure above, it is not difficult to see that this statement generalizes to all Yangian invariants represented by undeformed on-shell graphs

\[
\mathcal{C}_i \mathcal{Y} = 0 \tag{2.19}
\]

for all \( i \). Following the analysis of [12, 13], we will exploit the fact that the condition for Yangian invariance equation (2.11) is less restrictive than equation (2.19). In particular, we will start with deformed trivalent objects \( A_\bullet \) and \( A_\circ \), which are not annihilated by the central
charge operators $\mathcal{C}_i$ individually as in equation (2.19). After discussing under which conditions these deformed building blocks are Yangian invariants, we will explore in which way one can combine the deformed building blocks in order to obtain new Yangian invariants.

The deformation of the building blocks $\mathcal{A}_\bullet$ and $\mathcal{A}_\circ$ will be naturally given in terms of nonzero eigenvalues $c_i$ corresponding to each operator $\mathcal{C}_i$. The eigenvalue $c_i$ is referred to as the central charge and is accompanied by an evaluation parameter $u_i$ (see subsection 2.1) for each component $i$ of the tensor product. As stated at the end of subsection 2.1, in this configuration the parameters $u_i$ of the evaluation representation of the Yangian need not be all equal.

In general, we consider the central charge to be ingoing for all external particles. However, for each internal line one has to choose a direction for the flow of central charge. Graphically, this is represented by an arrow, whose reversal amounts to flipping the central charge flowing along the line while keeping the evaluation parameter untouched:

$$c, u \quad \equiv \quad -c, u.$$  \hspace{1cm} (2.20)

It can be seen as follows: considering the line to be a two-point invariant, the action of the level-0 generators must satisfy

$$\overline{J}_1 + \overline{J}_2 = 0,$$

which (for $\overline{J} = \mathcal{C}$) implies $c_1 = -c_2$. For level 1, we have

$$\hat{\overline{J}}_1 + \hat{\overline{J}}_2 = u_1 \overline{J}_1 + u_2 \overline{J}_2 = (u_1 - u_2)\overline{J}_1 = 0,$$

meaning that $u_1 = u_2$.

For the objects $\mathcal{A}_\bullet$ and $\mathcal{A}_\circ$ we choose the following convention:

$$\mathcal{C}_i \mathcal{A}_\bullet = c_i \mathcal{A}_\bullet, \quad \mathcal{C}_i \mathcal{A}_\circ = c_i \mathcal{A}_\circ.$$  \hspace{1cm} (2.24)

The condition for Yangian invariance equation (2.11) requires the total central charge operator to annihilate the vertex

$$\sum_{i=1}^{3} \mathcal{C}_i \cdot \mathcal{A}_\bullet = 0, \quad \sum_{i=1}^{3} \mathcal{C}_i \cdot \mathcal{A}_\circ = 0,$$

which translates into the condition $c_1 + c_2 + c_3 = 0$ immediately. In [12, 13], two building blocks $\mathcal{A}_\bullet$ and $\mathcal{A}_\circ$ satisfying the condition equation (2.24) have been introduced. Represented in spinor-helicity variables, they read
\[ \mathcal{A}_* = \frac{\delta^A(P)\delta^Q(Q)}{[12]^{1+c_1}[23]^{1+c_1}[31]^{1+c_1}}; \quad \mathcal{A}_\circ = \frac{\delta^A(P)\delta^\circ(Q)}{[12]^{1-c_1}[23]^{1-c_1}[31]^{1-c_1}}. \]  

While it is not difficult to check the invariance of \( \mathcal{A}_* \) and \( \mathcal{A}_\circ \) under the superconformal generators equation (2.7), Yangian invariance is ensured only after imposing vanishing under one level-1 generator \( \mathcal{J} \). We choose to consider the action of the operator \( \mathcal{P}^\alpha \) on \( \mathcal{A}_*, \mathcal{A}_\circ \) in order to derive a relation between the central charges\(^3 \) \( c_i \) and the evaluation parameters \( u_i \), which is a necessary condition for Yangian invariance.

The level-1 operator \( \hat{\mathcal{P}}^{\alpha} \) 0. equation (2.8) and reads

\[ \hat{\mathcal{P}}^{\alpha} = \sum_{1 \leq i < j \leq n} [(\delta^\beta_{\bar{\alpha}} \mathcal{L}_{\beta} + \delta^\beta_{\bar{\alpha}} \mathcal{L}_{\bar{\beta}} + \delta^\beta_{\alpha} \mathcal{D}_{\beta}) \hat{\mathcal{P}}^{\beta} + \hat{\mathcal{O}}_{\alpha} \mathcal{D}^{\alpha} - (i \leftrightarrow j)] \\
+ \sum_{k=1}^{n} u_k \hat{\mathcal{P}}^{\alpha k}. \]  

The action of \( \hat{\mathcal{P}} \) on \( \mathcal{A}_* \) is easily computed, as the terms in square brackets in equation (2.27) are all single-derivative operators. Then, a short calculation leads to

\[ \hat{\mathcal{P}}^{\alpha} \cdot \mathcal{A}_* = \{ [c_2 + u_1] \lambda_1^a \lambda_1^a + [c_3 - c_1 + u_2] \lambda_2^a \lambda_2^a \}
+ [ - c_1 - c_2 + u_3] \lambda_3^a \lambda_3^a \} \mathcal{A}_*. \]  

A similar equation can be derived for \( \mathcal{A}_\circ \), the only difference being in the sign with which the central charges appear. Since the \( p_k \) sum to zero, there are just two independent equations, and the solution expressing the \( c \) in terms of the evaluation parameters reads, after imposing \( c_1 + c_2 + c_3 = 0 \),

\[ \mathcal{A}_*: \quad c_1 = u_2 - u_3, \quad c_2 = u_3 - u_1, \quad c_3 = u_1 - u_2; \]  

\[ \mathcal{A}_\circ: \quad c_1 = u_3 - u_2, \quad c_2 = u_1 - u_3, \quad c_3 = u_2 - u_1. \]  

Notice that shifting all the \( u \) by a common quantity \( \alpha \) only has the effect of sending \( \hat{\mathcal{P}} \rightarrow \hat{\mathcal{P}} + \alpha \hat{\mathcal{P}} \), and this shift does not affect equation (2.27). Therefore, the evaluation parameters \( u_i \) are defined up to an overall shift.

This type of analysis—albeit in the language of another set of variables and for a limited set of on-shell graphs—was first done in [12, 13].

### 2.4. Deformed on-shell graphs and the permutation flow

Having introduced the deformed building blocks \( \mathcal{A}_* \) and \( \mathcal{A}_\circ \) in equation (2.26), let us now discuss how to glue them (together) in order to obtain Yangian invariants. In parallel to the previous discussion, a deformed on-shell graph is a connected planar graph composed by gluing the deformed building blocks.

In contrast to the case for the gluing procedure for the undeformed graphs built from undeformed blocks equation (2.12), we will have to take care of the central charge \( c \) and the evaluation parameter \( u \) associated with each internal and external line. Gluing two vertices with a common leg \( l \) is again done by integrating over the on-shell superspace corresponding to this particular leg using the measure equation (2.15). The combined object is a Yangian invariant with the following assignment for the internal central charge \( c_l \) and evaluation parameter \( u_l \):

\[ \text{Our naming conventions differ from those used by the authors of [12, 13]; the ‘spectral parameters’ of these references are called central charges here.} \]
\[
\int \frac{d^2 \lambda_I}{\text{Vol}[GL(1)]} \frac{d^2 \tilde{\lambda}_I}{\text{Vol}[GL(1)]} d^4 \eta_I \, A_\bullet(\lambda_I, \tilde{\lambda}_I, \eta_I, c_I, u_I) A_\circ(\lambda_I, -\tilde{\lambda}_I, -\eta_I, -c_I, u_I),
\] (2.31)

where we omitted the dependence on the on-shell superspace variables of particles 1, 2, 3, 4 in the integrand. In appendix A it is shown that the identifications of \(c_I\) and \(u_I\) in equation (2.31) ensure the Yangian invariance of the resulting object.

An object represented by a deformed on-shell graph is a Yangian invariant if the central charges and evaluation parameters on each internal line are identified as described above and equations (2.29) and (2.30) are satisfied at each vertex. Imposing all those constraints simultaneously leads to a system of linear equations relating the central charges \(c_i\) and evaluation parameters \(u_i\) of the external legs.

Two equally coloured vertices. Let us see how this works for an easy example: imposing all constraints for the graph

\[
\begin{align*}
0 &= c_1 + c_2 + c_3 + c_4, \\
0 &= c_1 + c_2 + u_1, \\
0 &= c_1 + 2c_2 + c_3 + u_1, \\
0 &= c_2 + c_3 + u_1.
\end{align*}
\] (2.33)

It is not difficult to see that the solution to the above system could have been obtained from another set of linear equations, corresponding to the following graph:

\[
\begin{align*}
0 &= c_1 + c_2 + c_3 + c_4, \\
0 &= c_1 + c_2 + u_1, \\
0 &= c_1 + 2c_2 + c_3 + u_1, \\
0 &= c_2 + c_3 + u_1.
\end{align*}
\] (2.34)

This does not come unexpectedly: the merger operation is valid for deformed on-shell graphs as well, as already shown in [12, 13].

A chain of vertices. Let us now look at another trivial configuration: a tree on-shell graph with \(n\) external legs, \(n_V = n - 2\) vertices and \(n_I = n - 3\) internal lines, for example

\[
\begin{align*}
0 &= c_1 + c_2 + c_3 + c_4, \\
0 &= c_1 + c_2 + u_1, \\
0 &= c_1 + 2c_2 + c_3 + u_1, \\
0 &= c_2 + c_3 + u_1.
\end{align*}
\] (2.35)

As discussed in subsection 2.3, there are three free parameters associated with each vertex. This turns the consideration of a tree graph into an easy problem: taking the two conditions for gluing an internal line into account, a simple counting shows that the number of independent quantities for the Yangian invariant in equation (2.35) equals the number \(n\) of
A four-point graph and four vertices. In order to produce more interesting configurations, one will need to build on-shell diagrams containing loops. Let us have a look at the simplest case: a box with alternating white and black dots depicted in figure 3.

Solving the linear system corresponding to the four-point on-shell graph in figure 3 leads to

\[ c_1 = -c_3, \quad c_2 = -c_4 \quad \text{and} \quad u_1 = u_3, \quad u_2 = u_4. \] (2.37)

Thus the values of the evaluation parameters and central charges can be identified between legs 1 and 3 as well as between legs 2 and 4. Note furthermore that the condition in equation (2.11) is trivially satisfied with the above solution.

The construction and study of this four-point deformed on-shell graph were performed in [12, 13], where it was argued that it intertwines the external states. In quantum integrable systems, this is the role played by the R-matrix, see figure 4. It depends on the difference of the evaluation parameters of the external states, which is a natural fact in the context of quantum integrable systems.

Let us then explore the properties of R-matrices in quantum integrable systems a little more closely before translating their properties back to the language of on-shell diagrams. Note that all the following equations featuring on-shell graphs are to be understood as ‘the linear system for the graph on the right-hand side is equivalent to the linear system represented on the left-hand side’.

**Figure 3.** Four-point on-shell graph.

**Figure 4.** The four-point tree-level deformed on-shell graph has a structure resembling the R-matrix $R(u_1 - u_2)$ of quantum integrable systems. Note that in order to match the usual conventions for R-matrices, we have reversed the arrows of the four-point on-shell diagram with respect to figure 3.
R-matrices satisfy the Yang–Baxter equation:
\[ R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2) \]
(2.38)
or, in terms of R-matrix and on-shell diagrams,
\[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1} \\
\includegraphics[width=0.2\textwidth]{diagram2}
\end{array} \]
(2.39)

Iterated application is proportional to the identity
\[ R_{12}(u_1 - u_2)R_{21}(u_2 - u_1) = f(u_1, u_2) \cdot I \]
(2.40)
where the function \( f(u_1, u_2) \) is some function of the two spectral parameters.

After identifying the four-point Yangian invariant in figure 3 with \( R(u_1 - u_2) \), it is worth investigating the diagram further. In particular, one can define new variables
\[ u^\pm = u \pm c, \]
(2.41)
which are convenient for tracking how central charges \( c \) and evaluation parameters \( u \) flow through on-shell diagrams. The quantity \( u^+ \) will be written to the left of a line looking in the direction of its arrow, while \( u^- \) will be placed to the right. This convention is compatible with flipping the direction of the arrow using the rules in subsection 2.3
\[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{double-line1} \\
\includegraphics[width=0.2\textwidth]{double-line2}
\end{array} \]
(2.42)

After solving the linear system, one finds the configuration in figure 5. Tracing the quantities \( u^\pm \) through the diagram and connecting them by lines, one can easily recognize the following rules for traversing:
- at a black vertex, turn right,
- at a white vertex, turn left.

Thus it is suggestive to keep track of this information in terms of the following double-line formalism. Hereby the black and white vertices translate into
\[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{double-line3} \\
\includegraphics[width=0.2\textwidth]{double-line4}
\end{array} \]
(2.43)
and figure 5 becomes figure 6, where we did not draw the vertices and their connecting lines for convenience. Comparing the above rules with the definition of the permutation represented by an on-shell graph introduced in subsection 2.2, it becomes obvious that the
quantities \( u^\pm \) follow exactly the lines determining the permutations. In other words, the permutation map \( \sigma \) defined in equation (2.17) above keeps track of the flow of the \( u^\pm \) between external legs. The two vertices defined above are sufficient to translate any on-shell graph into the linear system of equations ensuring Yangian invariance immediately, as will be shown below.

The four-point diagram discussed here is the subgraph appearing in the square-move transformation of the undeformed on-shell graph depicted in figure 2. Employing the double-line formalism, it is trivially seen that this transformation does not alter the permutation in the deformed set-up either: switching the roles of black and white vertices amounts to switching the path that a quantity takes around the loop (see figure 6), while the solution to the conditions ensuring Yangian invariance remains unchanged.

Five points and seven vertices. In order to check the identification between the flow of \( u^\pm \) and the permutation encoded in an on-shell graph, let us consider another example: the five-point on-shell diagram

![Figure 5](image-url)  
**Figure 5.** Four-point on-shell graph with flows of \( u^\pm \) indicated.

![Figure 6](image-url)  
**Figure 6.** Four-point on-shell graph in the double-line formalism and its square-moved version. The square simply switches the role of the flow: a line going around the loop clockwise will do so anticlockwise after the square move.
Imposing Yangian invariance by solving the linear system yields

\[ \{ u_1^+, u_2^+, u_3^+, u_4^+, u_5^+ \} = \{ u_2^-, u_3^-, u_4^-, u_5^-, u_1^- \}, \]

which corresponds to the permutation \((1, 2, 3, 4, 5) \rightarrow (4, 5, 1, 2, 3)\)—a simple cyclic shift by three places.

The same result is implied by the second diagram in the above picture. With the double-line formalism it is obvious that the linear system obtained by imposing Yangian invariance of the corresponding on-shell diagram will fix at most half of the \(2n\) free variables \(c_i\) and \(u_i\), \(i = 1 \ldots n\). The solution to the linear system can be expressed in the permutation \(\sigma\) defined in subsection 2.2. Considering all external particles as ingoing, the solution simply reads

\[ u_{\sigma(i)}^- = u_i^+. \] (2.46)

### 2.5. Superconformal and Yangian anomalies

In this subsection we would like to comment on the issue of exactness of Yangian invariance and on the uniqueness of the invariants that we have constructed.

A common belief for integrable systems is that the extended symmetries present in such systems determine the observables uniquely. In particular, the S-matrix should be determined uniquely by its invariance.

In the previous parts of this section we have constructed a large collection of Yangian invariants which are—most probably—almost all inequivalent. Here it makes sense to compare only those functions with the same external data (number of legs, MHV degree, central charges and evaluation parameters).

It is quite clear that for a given set of external data there exist several on-shell graphs: at a low number of loops in the on-shell diagram, there will only be few, if any, graphs whose permutation structure matches the external data. However, at higher on-shell loops, there must be further graphs with the same permutation structure because the set of permutation structures for a given number of legs is finite. It is unlikely that all of these graphs reproduce the same function, in particular as they are not all connected by the permissible merger and square moves (see figure 2).

This argument seems to show that there are many inequivalent Yangian invariants for the same external data. However, following the analysis in [20] (see [21] for a review), the flaw of the argument is that the derived functions are not actually Yangian invariant. The reason is that the fundamental three-point functions spoil superconformal and Yangian symmetry in a
They are invariant for almost all external momentum configurations, but invariance breaks down when all three momenta are exactly collinear. This can be observed most easily in the following form of the three-point vertex (2.26):

$$A_\ast = \text{sign}((12)) \int \frac{d\bar{z}_1}{\bar{z}_1 + c_1} \frac{d\bar{z}_2}{\bar{z}_2 + c_2} \delta^2(\bar{z}_1 \lambda_1 + \bar{z}_2 \lambda_2 + \lambda_3) \cdot \delta^2(\bar{z}_1 - \bar{z}_1 \lambda_3) \delta^2(\bar{z}_2 - \bar{z}_2 \lambda_3) \delta^4(\bar{\eta}_1 - \bar{z}_1 \bar{\eta}_3) \delta^4(\bar{\eta}_2 - \bar{z}_2 \bar{\eta}_3).$$ (2.47)

The sign factor multiplying the integral is responsible for the conformal ‘anomaly’ at collinear momenta $\delta A_\ast = \delta \text{sign}((12)) \cdot \ldots$. In particular, the deformations in the weights w. r. t. $\bar{z}_1$ and $\bar{z}_2$ do not interfere with the symmetry breaking.

The anomaly of an on-shell graph is the sum of the anomalies of all constituent vertices:

$$\delta \left( \begin{array}{c} \text{graph} \\ \text{on-shell} \end{array} \right) = \sum_{\text{vertices}} \delta \left( \begin{array}{c} \text{vertex} \\ \text{on-shell} \end{array} \right).$$ (2.48)

This form demonstrates that the overall anomaly depends on the precise structure of the on-shell graph. Two inequivalent graphs with equal external structure have different anomaly structures, and therefore they are expected to have different functional dependences. This is most evident if the graphs have a different number of on-shell loops. Moreover one may wonder whether the merger and square moves are in fact exact equivalences of graphs in the above sense.

3. On-shell graphs and scattering amplitudes

In the preceding section we developed the tools needed for constructing deformed Yangian invariants. In the following section, we want to investigate whether it is possible and meaningful to introduce deformed scattering amplitudes as (sums of) deformed on-shell graphs preserving manifest Yangian invariance.

We start with a short review of the formalism linking undeformed on-shell graphs with ordinary scattering amplitudes; the interested reader can find the complete formalism in [6] and references therein. We then investigate the possibility of deforming tree-level amplitudes using deformed Yangian-invariant on-shell graphs. Finally, following the spirit of [12, 13], we investigate whether it is possible to use the deformation parameters as Yangian-preserving regulators for the four-point one-loop amplitude of $\mathcal{N} = 4$ sYM. We discuss our findings at the end of the section.

3.1. Grassmannian formalism

In this subsection we briefly review the Grassmannian formulation of scattering amplitudes in $\mathcal{N} = 4$ sYM—first introduced in [7]—and its relation to on-shell graphs, which has been developed in [6]. The discussion assumes familiarity with the BCFW recursion relations and the supersymmetric extension [14–17] as well as the concepts of helicity amplitudes.

---

4 The discussion requires a real signature of spacetime. It is most straightforward in $(2, 2)$ split signature where three-point function actually exist; therefore let us assume this signature.

5 This is the behaviour for $(2, 2)$ split signature. For $(3, 1)$ Minkowski signature the anomaly works differently, and requires a suitable combination of contributions to the S-matrix; see [22] for further comments.
Grassmannian manifolds. The Grassmannian manifold $G(k, n)$ (from now on simply ‘Grassmannian’) is the space of complex $k$-planes in $\mathbb{C}^n$ passing through the origin. It is a generalization of the notion of complex projective space; in particular, $\mathbb{C}P^{n-1}$ is the Grassmannian manifold $G(1, n)$. A $k$-plane in $\mathbb{C}^n$ is uniquely determined by a set of $k$ linearly independent complex $n$-tuples; therefore one can identify a point in $G(k, n)$ with an equivalence class of $k \times n$ complex matrices $C$ of rank $k$. In fact, given such a matrix $C$, the right action of $\text{GL}(k)$ maps $C$ to a different matrix $C'$. Since $C$ and $C'$ identify the same $k$-plane in $\mathbb{C}^n$, we can identify the Grassmannian $G(k, n)$ with the space of $k \times n$ rank-$k$ complex matrices modulo a $\text{GL}(k)$ rescaling. The matrix represents the $kn$-vectors that identify the plane in $\mathbb{C}^n$.

Let us introduce two further notions used in the Grassmannian language: a top-cell of the above chart of the Grassmannian $G(k, n)$ is the cell where all sets of $k$ consecutive columns of the matrix $C$ become linearly independent. Its dimension is the full dimension $k (n - k)$ of $G(k, n)$. In contrast, a generic cell is allowed to have sets of linearly dependent consecutive columns in its matrix $C$. The stratification of the Grassmannian is then defined to be the system of nested boundaries of cells, where a boundary is described by the above linearly dependent configuration of columns. The decomposition of cells depends on the choice of coordinates for the Grassmannian $G(k, n)$.

Scattering amplitudes as integrals over Grassmannians. The authors of [6, 7] showed that it is possible to express tree-level amplitudes as well as loop amplitudes in $\mathcal{N} = 4$ sYM as integrals over Grassmannian manifolds. In particular, the tree-level amplitude can be written as the following integral:

$$A_{k,n} := \int \frac{d^{\times n} C_{ra}}{\text{Vol}[\text{GL}(k)]} \frac{1}{(12 \ldots k)(23 \ldots k + 1) \ldots (n - 1 \ldots k - 1)} \prod_{i=1}^{k} \delta^{4\Lambda} \left( \sum_{a=1}^{n} C_{ra} Z_a \right),$$

(3.1)

where $(i \ldots i + k)$ is the $i$th $k$-minor of the matrix $C$ and the $Z = (\lambda, \mu; \eta)$ are supertwistor variables, as introduced in [19]. Most of the integrations in equation (3.1) are fixed by the bosonic delta functions; the number of bosonic integrals left to be evaluated via contour integration is $(k - 2)(n - k - 2)$. Notice that equation (3.1) makes superconformal symmetry manifest, since supertwistor variables $Z$ transform linearly under $\text{PSU}(2, 2|4)$.

In order to proceed to loop amplitudes it is useful to introduce yet another way of expressing on-shell graphs. In [6] it was shown that any on-shell graph with $n_F$ faces can be associated with an integral of an $(n_F - 1)$-form defined on the Grassmannian $G(k, n)$, where the parameters $k$ and $n$ are related to the on-shell graph via

$$k = n_w + 2n_b - n_i \quad \text{and} \quad n = 3(n_b + n_w) - 2n_i.$$

(3.2)

Here, $n_w$ denotes the number of white vertices, $n_b$ the number of black vertices and $n_i$ the number of internal lines.

Each on-shell graph can be associated with a matrix $C$ denoting a cell of the Grassmannian, which in turn is a function of $n_F - 1$ auxiliary face variables $f_i$. Expressed in these face variables, the integral associated with the on-shell graph is obtained by integrating over all internal lines and takes the form

$$\prod f_i.$$

(3.3)

Accordingly, amplitudes are constructed by supplementing the measure equation 3.3 with the appropriate delta functions coming from the gluing of 3-vertices. This leads to the integral
The above integral is a generalization of equation (3.1), which in addition allows one to describe loop amplitudes. The precise connection between undeformed on-shell graphs and (loop) amplitudes is nontrivial. We do not discuss the relation between face variables and the matrix $C$ representing a point in the Grassmannian here. The interested reader may consult [6] for the complete formalism, or [12, 13] for a concise review in the light of the deformations introduced in the following sections.

All-loop BCFW recursion relations in the Grassmannian formalism. In [6] it was pointed out that the BCFW recursion relations in the Grassmannian formalism are obtained as solutions to a formal boundary equation. This equation states that the singularities of an amplitude in $\mathcal{N} = 4$ sYM theory are given solely by factorization channels and forward limits, where:

- a factorization channel refers to the splitting of an amplitude in the special kinematical situation where the sum of consecutive momenta becomes on-shell;
- a forward limit of an on-shell $l$-loop amplitude is an $(l-1)$-loop amplitude with two additional legs with opposite momentum; we refer the reader to [23] for the analysis of the role played by forward limits in the description of loop amplitudes.

The solution to this formal boundary equation is a sum of on-shell diagrams, each of which corresponds to a BCFW channel. In particular, the solution provides the integrand for the associated scattering amplitude. Since each summand is a Yangian invariant, this proves that the integrand for any loop amplitude in $\mathcal{N} = 4$ sYM theory is a Yangian invariant modulo partial integration [8, 9]. However, loop amplitudes in $\mathcal{N} = 4$ sYM theory do not exhibit Yangian invariance, as dealing with IR divergences breaks conformal invariance.

Conveniently, the on-shell graphs representing the BCFW channels for a tree-level amplitude can also be expressed as boundaries of the top-cell of a certain codimension. The boundaries of a cell are obtained graphically by removing specific edges (called removable edges in [6]) from the associated on-shell graph.

The identification of the boundaries of the cell associated with a given on-shell graph together with the BCFW construction allows us to express any tree-level amplitude starting from the top-cell.

For loop amplitudes there is no analogue of a top-cell object, that is, an on-shell graph whose boundaries directly yield the BCFW channels.

### 3.2. Top-cell versus BCFW decomposition, deformation and compatibility

In the following subsection we will investigate the possibility of linking deformed on-shell graphs with ‘deformed’ amplitudes in $\mathcal{N} = 4$ sYM. In order to do so, we will first review the tools developed in [12, 13] that relate a Grassmannian integral to a deformed on-shell graph.

---

6 Upon dimensional regularization, additional terms breaking dual conformal invariance can appear [24]. These terms are not present in purely four-dimensional analysis such as the one-loop amplitude analysis considered below.

7 It is possible to show (see [25]) that an on-shell graph representing the top-cell for a certain $k$ and $n$ corresponds to a permutation which is just a cyclic shift by $k$. Furthermore, in the aforementioned paper it was explained how to construct a representative on-shell graph for the top-cell.
Subsequently we will turn to the interplay among on-shell graphs, BCFW decomposition and Yangian invariance.

The Grassmannian integral for deformed scattering amplitudes. The integral corresponding to a deformed on-shell graph is obtained by deforming the ordinary measure equation (3.3). In terms of the face variables $f_i$, the deformation reads

$$\prod_{i=1}^{n_F} \frac{df_i}{f_i} \rightarrow \prod_{i=1}^{n_F} \frac{df_i}{f_i^{1+c_i}}.$$ (3.5)

Here the face shifts $\zeta$ are dual variables with respect to the central charge $c$; the latter is obtained from the difference of the two adjacent face shifts $8$

$$\zeta_a - \zeta_b = c$$ (3.6)

as depicted in figure 7. It is evident that the resulting deformed integrand is not a meromorphic function any more. This is an important fact, since it questions a clear interpretation of the BCFW decomposition of the deformed amplitudes.

Deformed scattering amplitudes. In the context of $\mathcal{N} = 4$ sYM theory, it makes sense to try to construct Yangian-invariant deformed scattering amplitudes. Naturally, those deformed amplitudes should not violate unitarity, that is, they should have the correct singularity structures. In the undeformed case, this is ensured via the BCFW construction. As pointed out in the previous subsection, the BCFW recursion relations simultaneously maintain Yangian invariance.

In order to define a deformed Yangian-invariant amplitude there are two natural approaches:

- One could start from the collection of on-shell diagrams corresponding to the undeformed BCFW channels. One would need to deform these in a Yangian-invariant way individually and subsequently demand compatibility between the deformations, meaning that the parameters $u_i$ and $c_i$ attached to the external legs must be equal for all diagrams. The compatibility among the diagrams will lead to constrained configurations.
- One could deform the on-shell graphs corresponding to the top-cell. However, we have argued above that the integrand resulting from a deformed on-shell diagram is not meromorphic. In terms of the Grassmannian integral a boundary is equivalent to a pole given by the vanishing of a minor. Thus the nonmeromorphicity of the integrand associated with a deformed top-cell prevents a direct BCFW-like decomposition. It is however possible to constrain the deformation parameters in such a way that the removal of the edges corresponding to BCFW channels amounts to a simple residue integral.

For undeformed tree-level MHV amplitudes the BCFW decomposition of the amplitude consists of a single term and thus a single on-shell graph, which is the top-cell. The deformation of this on-shell graph directly translates into the deformation of the associated

8 For a three-point vertex this identity makes the $u$s have the same relationship to the $\zeta$s as to the $\zeta$s. Therefore, the $\zeta$s must agree locally with the $u$s up to an overall shift. Note that the latter shift depends on the position and hence the $\zeta$s are not equivalent to the $u$s.
amplitude. For example, the on-shell graph depicted in figure 5 corresponds to the deformed four-point tree-level amplitude.

For general tree-level amplitudes, these two procedures could in principle lead to different deformations of the amplitude, since in the second case it is not necessarily true that the resulting channels are Yangian invariant by themselves. Moreover, it could (and will) also happen that the only possible solution is the undeformed BCFW decomposition. In the following subsection we will study an easy example and analyse the results originating from the two approaches.

Considering general loop amplitudes, the definition of a top-cell-like object is not clear. Only in the case of the one-loop four-point amplitude does there seem to be an analogue of a top-cell. We will investigate Yangian-invariant deformations of this particular on-shell graph in subsection 3.4 below.

### 3.3. Deformation of the six-point NMHV amplitude

Let us study the simplest nontrivial example: the six-point NMHV amplitude \( A_{6,3} \). The simplicity of this amplitude originates from the fact that the on-shell graphs representing the BCFW channels are codimension-1 boundaries of the top-cell (as opposed to boundaries of higher codimension for other amplitudes). The top-cell graph associated with \( G(3, 6) \) and its six codimension-1 boundaries are depicted in figure 8. The two possible BCFW decompositions of \( A_{6,3} \) are given by adding either the contributions from graphs 1, 3 and 5 or those from graphs 2, 4 and 6. In the following we will choose the latter option.

We will now investigate how to deform the on-shell graphs corresponding to \( A_{6,3} \) following the two different approaches described at the end of subsection 3.2.

**Deformation of the BCFW terms.** We will follow the first approach and impose Yangian invariance on the deformed on-shell graphs 2, 4 and 6 of figure 8 separately, which amounts to satisfying the conditions implied by the double-line formalism introduced in subsection 2.4.

For our choice of graphs, the permutations and their corresponding deformations are listed in figure 9. Compatibility among the three channels is achieved by imposing all conditions simultaneously. Doing so, there are three remaining degrees of freedom: choosing \((u_1^+, u_2^+, u_3^-) \) as parameters, one finds

\[
(u_1^+, u_2^-) = (c, b), \quad (u_2^+, u_3^-) = (a, b), \quad (u_3^+, u_5^-) = (a, c),
\]

\[
(u_4^+, u_5^-) = (b, c), \quad (u_5^+, u_6^-) = (b, a), \quad (u_6^+, u_3^-) = (c, a).
\]  

(3.7)

We stress that the deformation described above is obtained by imposing Yangian invariance on each on-shell graph individually. We will see in the next part how this result compares to the other approach.

**Residues of the top-cell.** The boundaries of an undeformed on-shell graph can be obtained by removing a single edge from the graph. However, only specific internal edges can be removed consistently [6]. For the top-cell graph of \( A_{6,3} \), the removable edges are highlighted in figure 8.

Taking a codimension-1 boundary of a graph corresponds to calculating a residue of the Grassmannian integral. The lack of meromorphicity of the deformed integrand makes the task of taking the residue nontrivial, since one has to take care of the branch-cut structure. It is, however, always possible to force the integral associated with the removal of a removable edge to be meromorphic: the additional condition that one has to impose is the vanishing of the central charge associated with that particular edge. A rigorous derivation of the sufficiency of this condition can be found in appendix B.
Yangian invariance of the top-cell corresponds to the permutation and linear system reported in figure 10. Enforcing the vanishing of central charges on all three edges to be removed amounts to setting

\[ u_5^+ = u_4^+, \quad u_2^+ = u_3^+, \quad u_1^+ = u_6^+. \]  

(3.8)

Consistently, the combination of the conditions reported in table 10 and in equation (3.8) is equivalent to equation (3.7).

It is possible to check that also the alternative BCFW decomposition—involving graphs 1, 3 and 5 of figure 8, which corresponds to the removal of the green lines—admits a nontrivial deformation compatible with the codimension-1 boundaries of the top-cell. Imposing vanishing of the central charges flowing on the red and green lines figure 8 simultaneously leads to a linear system forcing all evaluation parameters to be equal and all
central charges to be zero: Yangian invariance can only be realized in the undeformed case, as there are not enough degrees of freedom available.

The fact that the different on-shell graphs constituting a particular BCFW decomposition of a given tree-level NMHV amplitude admit a nontrivial common deformation is peculiar to the six-point case. For seven-point NMHV, imposing the compatibility for the deformations of the six on-shell graphs representing the six BCFW channels already gives only the trivial solution: all central charges are forced to vanish and all evaluation parameters are equal. This is to be expected, since for tree amplitudes the number of BCFW channels for fixed \( k \) grows roughly quadratically with \( n \), whereas the number of free deformation parameters is of order \( n \).

Therefore—with the exception of the cases for MHV amplitudes and a few low-multiplicity examples in the non-MHV sector—it is not possible to express a deformed amplitude as a sum of deformed objects which are Yangian invariant individually.

| graph | permutation | \( u \pm c \) |
|-------|-------------|------------|
| ![Graph 1](image1.png) | 1 2 3 4 5 6 4 5 6 2 1 3 | \( u_1^+ = u_4^- \), \( u_2^- = u_5^+ \), \( u_3^+ = u_6^- \), \( u_4^- = u_2^+ \), \( u_5^+ = u_1^- \), \( u_6^- = u_3^+ \) |
| ![Graph 2](image2.png) | 1 2 3 4 5 6 3 5 6 1 2 4 | \( u_1^+ = u_5^- \), \( u_2^- = u_6^+ \), \( u_3^+ = u_4^- \), \( u_4^- = u_1^+ \), \( u_5^+ = u_2^- \), \( u_6^- = u_3^+ \) |
| ![Graph 3](image3.png) | 1 2 3 4 5 6 4 6 5 1 2 3 | \( u_1^+ = u_4^- \), \( u_2^- = u_6^+ \), \( u_3^+ = u_5^- \), \( u_4^- = u_1^+ \), \( u_5^+ = u_2^- \), \( u_6^- = u_3^+ \) |

**Figure 9.** Permutation and parameter assignment for the BCFW decomposition of the six-point NMHV amplitude

**Figure 10.** Permutation and parameter assignment for the top-cell on-shell graph for the six-point NMHV amplitude.
because the constraints arising from demanding compatibility between the Yangian-invariant constituent graphs lead to the trivial deformation: all central charges are fixed to zero and all spectral parameters are equal.

3.4. The central charge as a regulator for the four-point one-loop amplitude

In [12, 13] the possibility of using the deformation parameters as regulators for loop amplitudes was investigated. The authors considered and computed a deformed on-shell graph which—in the undeformed limit—corresponds to the four-point one-loop MHV amplitude. The result was manifestly finite, but it turned out not to preserve Yangian symmetry essentially because the latter was not taken into account in the construction.

Our focus is on investigating the possibility of using a Yangian-preserving deformation as a means to regulate loop amplitudes. The case of study will again be the on-shell graph that corresponds to the four-point one-loop MHV amplitude in the undeformed limit. We evaluate the integral corresponding to a Yangian-invariant deformation and study its finiteness. As we shall see, the result that we find is rather unexpected and curious.

The on-shell graph associated with the four-point one-loop amplitude is drawn in figure 11, where we labelled the face shifts defined above equation (3.6) with \( \zeta_1 \ldots \zeta_9 \). Imposing Yangian invariance forces all external charges to vanish, which corresponds to the equality of all external face shifts \( \zeta_1 \ldots \zeta_4 \). The face shifts are related to the evaluation parameters \( u_i \) of the external legs via

\[
\begin{align*}
\zeta_1 &= \zeta_2 = \zeta_3 = \zeta_4 = \Delta, \\
\zeta_5 &= \Delta + u_3 - u_1, \\
\zeta_6 &= \Delta + u_3 - u_2, \\
\zeta_7 &= \Delta + u_4 - u_2, \\
\zeta_8 &= \Delta + u_4 - u_3, \\
\zeta_9 &= \Delta + u_4 - u_3 - u_2, \\
\end{align*}
\]

as can be easily obtained by following the parameters \( \pm u \) along the permutation lines in figure 11. As already stated, this assignment of face shifts here preserves Yangian invariance and differs from the one studied in [12, 13].

The technique used to construct the loop integral in terms of local variables was described by the authors of [12, 13]. As computed in these papers, the deformed four-point one-loop amplitude can be expressed in terms of a massless box integral with a deformed integrand, whose deformation is reminiscent of analytic regularization:

\[
A_{4,2}^{1\text{loop}}(\{ a_i \}; s, t) = s t A_{4,2}^{\text{tree}} \cdot I_{\text{box}}(\{ a_i \}; s, t),
\]

where \( s = 2k_1 \cdot k_2 \) and \( t = 2k_2 \cdot k_3 \) are Mandelstam variables and the deformed one-loop box integral reads

\[
I_{\text{box}}(\{ a_i \}; s, t) = \int \frac{d^4q}{(q^2)^{1+a_i} [(q + k_1)^2]^{1+a_i} [(q + k_1 + k_2)^2]^{1+a_i} [(q - k_4)^2]^{1+a_i} },
\]

The impossibility of conciliating consistently BCFW decomposition and deformation of on-shell diagrams—investigated in subsection 3.3—does not pose any problem here, since in the undeformed limit this graph is the only graph corresponding to the four-point one-loop amplitude.

It is a curious fact that no spinor quantities arise in the integral after Yangian invariance is imposed, when for general face variables they do. This suggests that the integral may be performed in dimensional regularization, if required.
Here the variables \( a_i \) are the following combinations of the evaluation parameters \( u_k \):
\[
\begin{align*}
  a_1 &= u_4 - u_1, \\
  a_2 &= u_3 - u_4, \\
  a_3 &= u_2 - u_3, \\
  a_4 &= u_1 - u_2,
\end{align*}
\]
which implies \( \delta := \sum a_j = 0 \). However, for the following calculation it is advisable to keep the dependence on \( \delta \) explicit, as it will provide easier access to the singular behaviour later on. Introducing Feynman parameters \( x_k \), the integral can be rewritten as
\[
\begin{align*}
  I_{\text{box}} &= \frac{\Gamma \left( 4 + \sum_i a_i \right)}{\prod_{j=1}^4 \Gamma \left( 1 + a_j \right)} \int d^4 q \int_0^1 \prod_{k=1}^4 \left[ dx_k q^{n_k} \right] \delta \left( 1 - \sum_{i=1}^4 x_i \right) \\
  &\quad \times \left[ \left( \sum_{i=1}^4 x_i \right) q^2 + 2q \cdot (x_2k_1 + x_3k_1 + x_3k_2 - x_4k_4) + x_3 s \right]^{-4+\sum_{i=1}^4 a_i} \\
  &= (-1)^q \frac{\Gamma \left( 1 + a_j \right)}{\prod_{i=1}^4 \Gamma \left( 1 + a_j \right)} \int_0^1 \prod_{i=1}^4 \left[ dx_i q^{n_i} \right] \delta \left( 1 - \sum_{i=1}^4 x_i \right) [x_1x_3s + x_2x_4t]^{-2-\beta}. \tag{3.13}
\end{align*}
\]

In order to evaluate the integral, we employ the Mellin–Barnes representation,\(^\text{11}\) which yields

\(^{11}\) We are grateful to Jan Plefka and Radu Roiban for discussions of this point.
\[ I_{\text{box}} = \frac{1}{2\pi i} \prod_{j=1}^{4} \frac{(-1)^{\delta}}{\Gamma(1 + a_j)} \]
\[ \times \int_{-\infty}^{\infty} dz \int_{0}^{1} \prod_{k=1}^{4} \left[ dx_k \ x_k^{a_k} \right] \delta \left( 1 - \sum_{j} x_j \right) \frac{t^{\delta}}{s^{2+\delta+\epsilon}} \frac{(x_2 x_4)^{\epsilon}}{(x_1 x_3)^{2+2+\delta}} \]
\[ \times \Gamma(-\delta) \Gamma(z + 2 + \delta) \]
\[ = \frac{1}{2\pi i} \prod_{j=1}^{4} \frac{(-1)^{\delta}}{\Gamma(1 + a_j)} \]
\[ \times \int_{-\infty}^{\infty} dz \left( \frac{t}{s} \right)^{\delta} \Gamma(1 + a_2 + z) \Gamma(1 + \delta - a_1 - a_2 - a_3 + z) \]
\[ \times \Gamma(\delta - 1 - \delta - z) \Gamma(\delta - 1 - \delta - z) \Gamma(2 + \delta + z) \Gamma(-z), \] (3.14)

where the contour of integration is drawn in figure 12. It separates the poles originating in the gamma functions with argument \( \Gamma(-\delta) \) (right poles) from the poles of \( \Gamma(z + 2 + \delta) \) (left poles). In equation (3.14) one can see why it was useful to keep the parameter \( \delta \) explicit. Instead of being a regulator, it serves as an auxiliary variable for investigating the convergence of the integral. Yangian invariance requires \( \delta = 0 \); therefore one would naively conclude that the result is zero because of the prefactor \( 1/\Gamma(0) \). However, it is still possible for the integral to be nonvanishing if the integral diverges as \( 1/\delta \). We will investigate this issue.

The Mellin–Barnes integral diverges whenever a left and a right pole become coincident—a situation called pinching of poles. In order to extract the possibly divergent contribution, one has to split the contour of integration such that the right poles giving rise to the pinching are moved to the left. Accordingly, the contour integral splits into a convergent part arising from a new contour parallel to the imaginary axis and the residue integrals around the problematic right poles as drawn in figure 13. Computing the two residues at \( a_1 - 1 - \delta \) and \( a_3 - 1 - \delta \) leads to

\[ I_{\text{box}} = \frac{1}{s^{2+\delta} \Gamma(-\delta)} \]
\[ \times \left\{ \left( \frac{t}{s} \right)^{a_{1,2}-1-\delta} \Gamma(1 + \delta - a_1) \Gamma(-\delta + a_1 + a_2) \Gamma(-a_1 + a_3) \Gamma(-a_2 - a_3) \right\} \]
\[ \times \left\{ \left( \frac{t}{s} \right)^{a_{1,3}-1-\delta} \Gamma(1 + \delta - a_3) \Gamma(-\delta + a_2 + a_3) \right\} \]
\[ \times \left\{ \Gamma(1 + \delta - a_1) \Gamma(-\delta + a_1 - a_2) \Gamma(-a_2 - a_3) \right\}, \] (3.15)

Imposing \( \delta = 0 \) in equation (3.15) clearly makes the integral vanish for generic values of \( a_1, a_2, a_3 \):

\[ I_{\text{box}} = 0. \] (3.16)

If this was the complete result, there would be a contradiction to the usual calculation of an undeformed four-point box integral. In particular, the undeformed box integral suffers from infrared divergences, whereas the above equality would imply it to vanish.

However, a more careful analysis shows that for specific values of \( a_1, a_2, a_3 \) the result may be nonzero. For example, when \( a_1 + a_2 = 0 \) and \( a_2 + a_3 = 0 \), the \( 1/\Gamma(-\delta) \) is cancelled in both terms of equation (3.15). Hence, a double pole appears originating from two \( \Gamma \)’s in the
The result is not identically zero, but rather a distribution with simultaneous support at $a_1 - a_2 - 1$ and $a_1 - a_2 + 1$. The derivation is subtle and requires much care. Using a heuristic derivation we find

$$I_{\text{box}} \bigg|_{\delta=0} = -\delta(a_1 + a_2)\delta(a_2 + a_3) \frac{1}{s-t} \left( \frac{s}{t} \right)^{a_1} \frac{\sin(a_1\pi)}{a_1}.$$ (3.17) 

It is straightforward to check that this result is invariant under the exchange $s \leftrightarrow t$, provided that one also makes the exchanges $a_1 \leftrightarrow a_4, a_2 \leftrightarrow a_3$, as can be deduced directly from the integral of equation (3.11). Moreover, the result in the undeformed case, $a_2=0$, is proportional to $\delta^2(0)$ which could be argued to agree qualitatively with the log divergence of the undeformed integral. We refer the reader to appendix C for more details on our derivation.
3.5. Discussion

Let us briefly summarize our findings: In order to define deformed tree-level scattering amplitudes, there is only one object which can be deformed in a Yangian-invariant way without any further constraints: the top-cell. Integrating over the integrand corresponding to a deformed (tree-level) top-cell will lead to deformed MHV amplitudes.

Non-MHV tree-level amplitudes are built as a sum of Yangian-invariant constituent graphs. While deforming each individual constituent graph works as for the top-cell, imposing compatibility between all graphs constrains the deformation parameters. While a few low-multiplicity examples still allow a deformation, the majority of non-MHV amplitudes cannot be deformed in this framework.

A possible way out would be by the construction of Yangian-invariant scattering tree amplitudes in the non-MHV sector from constituent graphs which violate Yangian invariance individually, but sum up to a Yangian-invariant combination. It is not clear whether this approach can be successful. Clearly, the formalism of deformed on-shell graphs explored in this paper is not applicable.

As regards amplitudes beyond tree level, we restricted our study to the four-point one-loop case. In the undeformed case, this amplitude can be expressed via a single on-shell graph taking the role of the top-cell. The computation of the integral corresponding to the Yangian-invariant deformation of this on-shell graph yields a curious result: naively it would appear to be zero—which is trivially Yangian invariant. A more careful analysis suggests it to be a distribution with singular support.

For amplitudes of higher loop order and higher multiplicity, there is no known notion of an on-shell diagram comparable to the top-cell. Thus the situation is less clear than in the tree-level case.

4. Conclusions

In this paper, we describe a mechanism for constructing invariants of the Yangian algebra $\mathfrak{Y}[\mathfrak{su}(2, 2|4)]$ by combining trivalent invariant building blocks and maintaining invariance during the gluing procedure. The invariants are deformations of the on-shell diagrams put forward in [6] which are invariants of $\mathfrak{Y}[\mathfrak{psu}(2, 2|4)]$. Deformation refers to a continuous shift of the central charges of the external legs introduced in [12, 13].

The construction of deformed tree-level scattering amplitudes from deformed on-shell graphs works straightforwardly only for MHV amplitudes. With a few exceptions, the non-MHV amplitudes cannot be deformed in the framework of (manifestly) Yangian-invariant on-shell graphs.

In [20] it was argued that once one considers the holomorphic anomaly only, the complete S-matrix of $\mathcal{N} = 4$ sYM is an invariant object, in contrast to the constituent scattering amplitudes. For the deformed case we have argued that the interpretation of on-shell amplitudes as constituents of an S-matrix is all but evident. Nevertheless one could hope that the assumption of an exactly invariant composite object may lead the way to the identification of a suitable deformed S-matrix and its constituent on-shell graphs.

Finally, in [12, 13] it was suggested that one could employ the shifted helicities of the external legs as a regulator maintaining superconformal invariance. However, for the example of the one-loop amplitude, Yangian invariance was spoilt. If we stick with a Yangian-invariant deformation of the integrand, the resulting expression for the four-point one-loop amplitude will be zero for generic deformation parameters. For special deformations,
however, the divergent behaviour of the integral is difficult to access. It depends on the particular way in which the deformation parameters are taken to zero.

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Appendix A. Yangian invariance of deformed glued objects

In this appendix we provide the explicit calculation showing that an object obtained by gluing two deformed Yangian invariants is Yangian invariant if evaluation parameters and central charges are identified on the internal legs as in equation (2.31). Let us consider two objects,

\[ A(1,\ldots,m, I), \ B(I, m+1,\ldots,n), \]

which are Yangian invariant, that is

\[ \mathcal{J}^a \cdot A \equiv \left[ \sum_{i=1}^{\ell} \mathcal{J}^a_i \right] \cdot A = 0, \]

\[ \mathcal{J}^a \cdot A \equiv \left[ f^a_{bc} \sum_{i=1}^{\ell} \mathcal{J}^b_i \mathcal{J}^c_i + \sum_{k=1}^{\ell} u_k \mathcal{J}^a_k \right] \cdot A = 0, \]

and analogously for \( B \). Defining

\[ d^{\xi} := \frac{d^2 \lambda_i \ d^2 \lambda_j}{\text{Vol}[\text{GL}(1)]} \]

the glued object reads

\[ \mathcal{Y}(1,\ldots,n) = \int d^{4\xi} \ A(1,\ldots,m, I) B(I, m+1,\ldots,n). \]

It is invariant under the action of the level-0 generators for the same reason as in the undeformed case (see the discussion after equation (2.15)), provided that it is annihilated by the total central charge operator \( \mathcal{C} = \sum_i \mathcal{C}_i \). Combined with the Yangian invariance of \( A \) and \( B \), this implies that the central charges attached to the leg \( I \) of \( A \) must be the same (up to a sign) as the central charge attached to the leg \( I \) of \( B \).

In order to derive the condition for gluing the evaluation parameters, it is useful to note that the invariance of \( \mathcal{Y} \) under the action of level-0 generators implies

\[ \int d^{4\xi} \ [ (\mathcal{J}^a_i A) B + A \ (\mathcal{J}^a_i B)] = 0. \]

However, in order to fix the gluing condition for \( u_i \), we have to consider the action of a level-1 generator on the glued object:
\( \tilde{3}^a \cdot \int d^{4\xi} \mathcal{A}(1, ..., m, I) B(I, m + 1, ..., n) \)

\[ = f_{bc}^a \int d^{4\xi} \left[ \sum_{i=1}^{m} \sum_{j=i+1}^{m} (\tilde{\gamma}^b_i \tilde{\gamma}^c_j) \mathcal{A} + \sum_{i=1}^{m} \sum_{j=m+1}^{n} (\tilde{\gamma}^b_i \mathcal{A} (\tilde{\gamma}^c_j B) \right] + \sum_{k=1}^{n} u_k \int d^{4\xi} \mathcal{A} B \]

\[ = f_{bc}^a \int d^{4\xi} \left[ \sum_{i=1}^{m} \sum_{j=i+1}^{m} (\tilde{\gamma}^b_i \tilde{\gamma}^c_j) \mathcal{A} + \sum_{i=1}^{m} \sum_{j=m+1}^{n} (\tilde{\gamma}^b_i \mathcal{A} (\tilde{\gamma}^c_j B) + \right] \]

\[ - \sum_{i=m+1}^{n} \sum_{j=i+1}^{n} \mathcal{A} (\tilde{\gamma}^b_i \tilde{\gamma}^c_j) B + \sum_{k=1}^{n} u_k \int d^{4\xi} \mathcal{A} B \quad (A.7) \]

In the second and fourth terms in the last line one can move the \( \tilde{\gamma}^c \) from \( \tilde{\gamma} \) to \( \tilde{\gamma}^c \) (paying a minus sign) and vice versa using (A.6). Then the second, third and fourth terms can be combined into \( (\tilde{\gamma}^b \cdot \mathcal{A})(\tilde{\gamma}^c \cdot B) \) which obviously vanishes. The remaining first and fifth terms together with the term containing the evaluation parameters vanish for all external legs, leaving a contribution from the internal line \( I \) which reads

\[ \tilde{3}^a \cdot \int \frac{d^2\lambda}{Vol[GL(1)]} d^4\eta_I \mathcal{A} B = \int \frac{d^2\lambda}{Vol[GL(1)]} d^4\eta_I \times [u_I^a (\tilde{\gamma}^b_I \mathcal{A} B) + u_I^b \mathcal{A} (\tilde{\gamma}^c_I B)]. \quad (A.8) \]

Using the identity (A.6), one can show that the above expression vanishes if \( u_I^a = u_I^b \). Therefore, superconformal and dual superconformal invariance imply that the evaluation parameters on internal legs match.

In a similar way, it is also possible to show that, given a Yangian invariant, the object obtained by identifying two adjacent legs and integrating over the on-shell superspace of that leg is again a Yangian invariant. Specifically, given an \( (n + 2) \)-point Yangian invariant \( \mathcal{Y}(1, ..., n) \), the object obtained via

\[ \mathcal{Y}^\prime(1, ..., n) = \int \frac{d^2\lambda}{Vol[GL(1)]} d^4\eta_I \mathcal{Y}(1, ..., n, I, J) \bigg|_{J=I} \quad (A.9) \]

is again Yangian invariant. The only nontrivial check is that of invariance under level-1 generators; a sketch of the invariance (in the undeformed case) is as follows:

\[ \tilde{3}^a \cdot \int d^{4\xi} \mathcal{Y}(1, ..., n, I, J) \bigg|_{J=I} \]

\[ = f_{bc}^a \int d^{4\xi} \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\tilde{\gamma}^b_i \tilde{\gamma}^c_j \mathcal{Y}(1, ..., n, I, J)) \right] \bigg|_{J=I} \]

\[ = -f_{bc}^a \int d^{4\xi} \left[ \tilde{\gamma}^b_i \tilde{\gamma}^c_j \mathcal{Y} + \sum_{i=1}^{n} (\tilde{\gamma}^b_i \tilde{\gamma}^c_j \mathcal{Y}) \right] \bigg|_{J=I} \]

\[ = f_{bc}^a \int d^{4\xi} \left[ \tilde{\gamma}^b_i \tilde{\gamma}^c_j \mathcal{Y} \right] \bigg|_{J=I} \propto f_{bc}^a f_{dc}^b \int d^{4\xi} [\tilde{\gamma}^d_j \mathcal{Y}] \bigg|_{J=I} \quad (A.10) \]
which vanishes for \( \text{psu}(2, 2|4) \). In the derivation, we implicitly used the invariance under the level-0 generators and the ‘integration-by-parts’ identity

\[
\int d^4\xi \left[ \delta^2 \mathcal{Y}(\ldots, I, J) + \delta^2 \mathcal{Y}(\ldots, I, J) \right] |_{\alpha = J} = 0
\]

(A.11)

which is similar to equation (A.6). The generalization to the deformed case is straightforward and leads to the same conditions for \( cs \) and \( us \) as before.

Appendix B. Grassmannian integrals from on-shell graphs

One can associate a Grassmannian integral with any on-shell graph. The integral is most conveniently expressed in terms of the face variables \( f_i \) introduced in subsection 3.1. As already stated there, the Grassmannian integral reads

\[
I_{\text{graph}} = \int \left[ \prod_{i=1}^{n_f-1} \frac{df_i}{f_i} \right] \delta^{2k} \left( \sum_{a=1}^{n} C_{ia} \left( f_i \right) \lambda_a \right) \delta^{2(n-k)} \left( \sum_{s=1}^{n-k} C_{is} \left( f_i \right) \lambda_s \right) \delta^{4k}
\]

\[
\times \left( \sum_{a=1}^{n} C_{ia} \left( f_i \right) \eta_a \right),
\]

(B.1)

where the \( C_{ia} \) are elements of a \( k \times n \) matrix that represents a point in \( G(k, n) \). The expression for \( C_{ia} \) in terms of the face variables \( f_i \) is known (see [6] for the precise construction).

What is more interesting from our point of view is that the integral can be expressed in terms of a different set of variables \( \alpha \) which have a precise physical interpretation: they are related to the BCFW shift associated with adding a so-called BCFW bridge as depicted in figure 14. The momentum flowing along the internal line is fixed to be proportional to \( \tilde{\lambda}, \tilde{x}_j \) by the deltas of the two additional vertices while the external momenta are modified:

\[
\lambda_i = \lambda_i, \quad \lambda_j = \lambda_j + \alpha \hat{\lambda}_i, \\
\tilde{\lambda}_i = \tilde{\lambda}_i - \alpha \tilde{x}_j, \quad \tilde{\lambda}_j = \tilde{\lambda}_j.
\]

(B.2)

In [6], it was demonstrated that the integral associated with an on-shell graph can be expressed in terms of these edge variables, the measure of integration being \( \prod \frac{df_i}{f_i} \). Moreover, it is also possible to show that the removal of an edge \( I \) corresponds to taking the residue around \( \alpha_I = 0 \). Not all of the edges of a graph are removable, but the ones that are can be identified from the permutation encoded by the graph.

It is possible to relate a face variable to the edge variables of the adjoining edges. In order to do so, one must introduce a specific orientation of the edges of the graph, called perfect orientation. Instead of describing the technicalities of perfect orientation, let us stick with the result: a face variable is given by the product of all the adjacent edge variables with anticlockwise orientation divided by the product of all the adjacent edge variables with clockwise orientation. An example is given in figure 15.

Considering the removal of an edge in the deformed integral, it is evident that the measure in general will not be meromorphic in \( \alpha \). However, as pointed out in equation 3.5, the deformed measure in terms of the face variables reads

\[
\prod \frac{df_i}{f_i} \rightarrow \prod \frac{df_i}{f_i + s_i},
\]

(B.3)
Focusing on one particular edge $I$, the related edge variable will appear in the measure as $d\alpha_I/\alpha_I^{i,j} + \alpha_I^{j,i}$, where $i$ and $j$ are the adjacent faces, since $\alpha_I$ appears in the numerator for one face variable and in the denominator for the other. The measure will be of the form $\alpha_I \alpha_I d\alpha_I$ if the difference of the face shifts of the two adjacent faces is zero: this is equivalent to no central charge flowing along the edge $I$.

### Appendix C. One-loop integral singular support

Here, we would like to argue that the deformed one-loop box integral equation (3.11)

$$I_{\text{box}}(\{a_i\}; s, t) = \int \frac{d^4q}{(q^2)^{1+a_i}} \frac{[q + k_1]^2 + a_i}{[(q + k_1)^2 + a_i][q + k_2]^2 + a_i}$$

in the form equation (3.15) is expressed as a distribution with singular support, as in (3.17).

Let us draw an analogy with the case of a single Dirac delta. One can define the Dirac delta function as the weak limit of the delta sequence

$$\eta_\epsilon(x) := \frac{1}{2\pi i} \left[ \frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon} \right] = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2},$$

where $\eta$ satisfies the following limits:

$$\lim_{\epsilon \to 0} \eta_\epsilon(x) = 0 \quad \text{for } x \neq 0,$$

Figure 14. BCFW bridge construction.

Figure 15. Face variable in terms of edge variables: here $f = \frac{\alpha_1 \alpha_2 \alpha_3 \alpha_4}{\alpha_5 \alpha_6}$. 
\[ \lim_{c \to 0} e^{\eta(c)} = \frac{1}{\pi \left( c^2 + 1 \right)} \text{ for } c \neq 0. \]  
\[ \text{(C.4)} \]

It is possible to show now that the function \( I_{\text{box}} \) defined in (3.15) has a similar behaviour in the limit \( \delta \to 0 \). We already saw that the limit \( \lim_{\delta \to 0} I_{\text{box}} = 0 \) for generic \( a_1, a_2, a_3 \); we want now to probe the region where \( a_1 \sim a_3 \sim (-a_2) \). We therefore consider \( I_{\text{box}} \) with \( \sum a_i \propto \delta \neq 0 \), study the region where

\[
\begin{align*}
  a_1 + a_2 & \sim \delta, \\
  a_1 - a_3 & \sim \delta,
\end{align*}
\]

and then take the limit \( \delta \to 0 \), in analogy with taking the limit in (C.4).

Let us then rewrite the result of equation (3.15) as a function of \( a_1, a_2, a_3, a_4 \), in order to obtain

\[
I_{\text{box}} = \frac{1}{\sqrt[2]{2+a_1+a_2+a_3+a_4} \Gamma(-a_1-a_2-a_3-a_4)} \times \left\{ \left( \frac{t}{s} \right)^{-1-a_1-a_2-a_4} \right. \\
\left. \frac{\Gamma(1+a_2+a_3+a_4) \Gamma(-a_3-a_4) \Gamma(-a_1+a_3) \Gamma(-a_2-a_3)}{\Gamma(1+a_2) \Gamma(1+a_3) \Gamma(1+a_4)} \right. \\
+ \left. \left( \frac{t}{s} \right)^{-1-a_1-a_2-a_4} \frac{\Gamma(1+a_1+a_2+a_4) \Gamma(-a_1-a_4)}{\Gamma(1+a_1) \Gamma(1+a_2) \Gamma(1+a_4)} \right\}. \]  
\[ \text{(C.6)} \]

We subsequently set

\[
a_1 = a + d_1 \delta, \quad a_2 = -a + d_2 \delta, \quad a_3 = a + d_3 \delta, \quad a_4 = -a + d_4 \delta, \]  
\[ \text{(C.7)} \]

with \( \sum a_i = d_{1234} \delta \),\(^{12}\) where \( d_{ij} := (d_i + d_j + ...) \). The limit that we investigate is then

\[
\lim_{\delta \to 0} \delta^2 I_{\text{box}} \left( a + d_1 \delta, -a + d_2 \delta, a + d_3 \delta, -a + d_4 \delta \right). \]  
\[ \text{(C.8)} \]

We stress the logic that we are trying to follow here. We saw that \( I_{\text{box}} \) is vanishing for generic configurations of the \( a_i \); see equation (3.16). We are now treating the parameter \( \delta \) as the \( \epsilon \) of the delta sequence, equation (C.3) and investigating the region where \( a_1 + a_2 \sim \delta, \quad a_1 - a_3 \sim \delta \), in analogy with the study of the delta sequence expressed in (C.4) in the region \( \epsilon \sim \epsilon \).

The result of taking the limit in equation (C.8) is

\[
\lim_{\delta \to 0} \delta^2 I = -\frac{1}{st} \frac{(t/s)^a}{\Gamma(1+a) \Gamma(1-a)} \frac{d_{1234}^2}{d_{12} d_{23} d_{34} d_{41}} \]  
\[ = -\frac{1}{st} \frac{(t/s)^a}{\Gamma(1+a) \Gamma(1-a)} \left[ \frac{1}{d_{12} d_{23}} + \frac{1}{d_{23} d_{34}} + \frac{1}{d_{34} d_{41}} + \frac{1}{d_{41} d_{12}} \right]. \]  
\[ \text{(C.9)} \]

\(^{12}\) \( d_{1234} \) is just an overall scale for the \( d_i \); we can (and will) normalize them such that \( d_{1234} = 1 \).
We can always rescale $\delta$, so we set $d_{1234} = 1$. This leads to

$$\lim_{\delta \to 0} \delta^2 I_{\text{box}} = \frac{1}{s^2 \Gamma(1-a)\Gamma(1+a)} \frac{\left(\frac{t}{s}\right)^a}{(d_{12} + 1)(d_{23} + 1)} \frac{1}{a\pi}, \quad (C.10)$$

a result which is similar to equation (C.4), but not equal; however, the explicit form of the limit should depend on the direction along which the limit of all the $a_i$ being equal (up to a sign) is taken. It is possible to get an expression with the same residues as equation (C.4) if we shift the $a_i$ differently, as

$$\lim_{\delta \to 0} \delta^2 I_{\text{box}}(a + d_1 \delta, -a + (d_2 + \frac{1}{2}) \delta, a + d_3 \delta, -a + (d_4 - \frac{1}{2}) \delta) \quad := \lim_{\delta \to 0} \delta^2 I_{\text{box}}, \quad (C.11)$$

where we have defined a different shift for $a_2$ and $a_4$, keeping the condition $\sum a_i = d_{1234} \delta$ fixed. The above equation leads to

$$\lim_{\delta \to 0} \delta^2 I_{\text{box}} = \frac{1}{s^2 \Gamma(\frac{1}{2})} \frac{16 \sin(a\pi)}{a\pi} \frac{1}{(4d_{12}^2 + 1)(4d_{23}^2 + 1)}, \quad (C.12)$$

which, considering also equation (C.4) (with $c \to c/2$), leads to (cf (3.17))

$$I_{\text{box}} = -\delta(a_1 + a_2)\delta(a_2 + a_3) \frac{1}{s^2 \Gamma(\frac{1}{2})} \frac{\left(\frac{t}{s}\right)^a \sin(a_1\pi)}{a_1}. \quad (C.13)$$

There are however two remarks to be stressed. The first is that the choice of shift in equation (C.11) leads to a nonvanishing result with a clear interpretation, but is ad hoc. The second is that the argument leading to equation (C.13) is rather heuristic, based on the fact that the behaviour of $I_{\text{box}}$ in the limit $\delta \to 0$ (and for a particular choice of direction along with the $a_i$ approach zero) is similar to the behaviour of a delta sequence.

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