Voting Power of Teams Working Together

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Abstract
Voting power determines the “power” of individuals who cast votes; their power is based on their ability to influence the winning-ness of a coalition. Usually each individual acts alone, casting either all or none of their votes and is equally likely to do either. This paper extends this standard “random voting” model to allow probabilistic voting, partial voting, and correlated team voting. We extend the standard Banzhaf metric to account for these cases; our generalization reduces to the standard metric under “random voting”. This new paradigm allows us to answer questions such as “In the 2013 US Senate, how much more unified would the Republicans have to be in order to have the same power as the Democrats in attaining cloture?”

Keywords. power indices, generating function, voting power, Banzhaf voting power, Congress, cloture

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1 Introduction

In a weighted voting game there are players who cast votes. Analysis of the players’ voting power has a long history with many models in use [4, 7]. The simplest voting model, “random voting”, is when each voter is equally likely to support or oppose a motion. Even in this simple case there are multiple ways to define the players’ power, the most common are the Shapley–Shubik power [10] and the Banzhaf power [8, 11].

Forming coalitions is a way for voters to influence their voting power [5, 6]. Interestingly, Gelman [5] proves “under the random voting model, this average voting power is maximized under simple popular vote (majority rule) and is lower under any coalition system” and makes the observation “Joining a coalition is generally beneficial to those inside the coalition but hurts those outside.”. Not only can voters form coalitions, they can also vote probabilistically. Some papers [6] create stochastic models for coalitions of voters.

While using generating functions to compute the Shapley–Shubik power or Banzhaf power is well known, it has usually been in the context of counting combinatorial possibilities for the “random voting” model. We have generalized the standard generating function approach to allow more sophisticated models of voting to be analyzed.

The main results presented in this paper are the following:

1. We review how Banzhaf power is defined and then illustrate the well-known process of determining Banzhaf power using generating functions for the “random voting” model. By generalizing to (simple) weighted generating functions we show how to directly compute the Banzhaf power; we do not determine it indirectly via the usually obtained combinatorial counting.

2. Still using the “random voting” model we introduce Influence Polynomials; these are a proxy for a player’s weighted generating function when used to compute the Banzhaf power.

3. We introduce a model of voting in which players have probabilities corresponding to the number of votes they cast. This is represented by a (general) weighted generating function, which we call a voting structure. We show how to determine players’ Influence Polynomials from their voting structures. These Influence Polynomials allow a generalized Banzhaf power to be determined; this reduces to the usual Banzhaf power when the voting structure represents the “random voting” model.

4. We create and analyze voting structures for a coalition represented by a leader. In these coalitions each member follows the guidance of the leader probabilistically; not with certainty.

Several examples are given, including an example related to the US Senate.

2 Banzhaf power

In the usual way a weighted voting game is represented by the vector \([q; w_1, w_2, \ldots, w_n]\) where:

1. There are \(n\) players.
2. Player \(i\) has \(w_i\) votes (with \(w_i > 0\)).
3. A coalition is a subset of players.
4. A coalition $S$ is **winning** if $\sum_{i \in S} w_i \geq q$, where $q$ is the **quota**.

5. A game is proper if $\frac{1}{2} \sum w_i < q$.

To define the Banzhaf power consider all $2^n$ possible coalitions of players. For each coalition, if player $i$ can change the winning-ness of the coalition, by either entering or leaving the coalition, then player $i$ is marginal. The Banzhaf power index ($\beta$) of a player is proportional to the number of times that a player is marginal; hence the total power of all players is 1.

As a continuing example consider the $[6;4,3,2,1]$ weighted voting game where the the players are named $\{A, B, C, D\}$ and “random voting” is used. There are $2^4 = 16$ subsets (or coalitions) of four players; the following enumeration shows all coalitions (left) and the marginal players for each (right):

1. $\{ \}$ → $\{ \}$
2. $\{ A \}$ → $\{ B, C \}$
3. $\{ B \}$ → $\{ A \}$
4. $\{ C \}$ → $\{ A \}$
5. $\{ D \}$ → $\{ \}$
6. $\{ A, B \}$ → $\{ A, B \}$
7. $\{ A, C \}$ → $\{ A, C \}$
8. $\{ A, D \}$ → $\{ B, C \}$
9. $\{ B, C \}$ → $\{ A, D \}$
10. $\{ B, D \}$ → $\{ A, C \}$
11. $\{ C, D \}$ → $\{ A, B \}$
12. $\{ A, B, C \}$ → $\{ A \}$
13. $\{ A, B, D \}$ → $\{ A, B \}$
14. $\{ A, C, D \}$ → $\{ A, C \}$
15. $\{ B, C, D \}$ → $\{ B, C, D \}$
16. $\{ A, B, C, D \}$ → $\{ \}$

Player $A$ is marginal 10 times, players $B, C$ are each marginal 6 times, and player $D$ is marginal 2 times. The total number of times that players are marginal is $24 = 10 + 6 + 6 + 2$. Hence player $A$ has Banzhaf power $\beta(A) = \frac{10}{24} = \frac{5}{12}$. The other players have the powers: $\beta(B) = \beta(C) = \frac{6}{24} = \frac{1}{4}$ and $\beta(D) = \frac{2}{24} = \frac{1}{12}$. These powers can be determined by hand as shown above or using an online tool such as [9].

### 2.1 Banzhaf Power via Generating Functions

Imagine that each player in the $[6;4,3,2,1]$ game can choose, with equal likelihood, to either be in the coalition or to not be in the coalition. Using generating functions [3, 9] we represent the votes that player $A$ casts (i.e., 0 or 4) by the polynomial:

$$G_A = \frac{a^0 x^0}{2} + \frac{a^4 x^4}{2} = \frac{1}{2} + \frac{a^4 x^4}{2}$$

Each term of this polynomial has the form $\omega a^n x^n$ where $n$ represents the number of votes cast (e.g., $a^4 x^4$ means that $A$ casts 4 votes) and $\omega$ (e.g., $\frac{1}{2}$ for each term here) represents the probability of casting that many votes for a coalition. Note that the probabilities sum to one: $G_A|_{a=x=1} = 1$.

While previous authors used generating functions to determine voting power, they did not include the $\omega$ factor – they were counting the number of coalitions, not determining the probability of each. In this paper we call a generating function of this type a “voting structure”.

Similarly, the votes cast by players $\{B, C, D\}$ can be represented as

$$G_B = \frac{1}{2} + \frac{b^3 x^3}{2}, \quad G_C = \frac{1}{2} + \frac{c^2 x^2}{2}, \quad G_D = \frac{1}{2} + \frac{dx}{2}.$$  

The letters $\{a, b, c, d\}$ are used in order to understand the upcoming intermediate computations; later all these variables will be given the numerical value one. Multiplying all four generating
functions together yields

\[
G_A G_B G_C G_D = \frac{1}{16} \left[ (a^4 b^3 c^2 d)x^{10} + (a^4 b^3 c^2)x^9 + (a^4 b^3 d)x^8 + (a^4 b^3 + a^4 c^2 d)x^7 \\
+ (a^4 c^2 + b^3 c^2 d)x^6 + (a^4 d + b^3 c^2)x^5 + (a^4 + b^3 d)x^4 \\
+ (b^3 + c^2 d)x^3 + (c^2)x^2 + (d)x + 1 \right]
\]

Each term in this expression represents a coalition: the power of \( x \) indicates the total votes in that coalition; the letters \( \{a, b, c, d\} \) indicate the coalition composition; and the numerical coefficient (\( \frac{1}{16} \) for each term) is the probability of that coalition. For example, the \( x^7 \) terms shows that there are two 7 vote coalitions: \( \{A, B\} \) and \( \{A, C, D\} \); each has probability \( \frac{1}{16} \) of occurring. Similarly there are two 6 vote coalitions: \( \{A, C\} \) and \( \{B, C, D\} \); each also has a probability \( \frac{1}{16} \) of occurring.

Let’s focus on Player A. While all coalitions with 6 or more votes is a winning coalition, they are not necessarily coalitions that A made winning. For example, if the \( \{A, B, C, D\} \) coalition (with 10 votes) were to lose player A then it would still have 6 votes and would still be a winning coalition. To identify the coalitions that A can make winning, we need to start with coalitions not involving A that are not winning, add player A’s votes to them, and see which ones are then winning.

To find the non-winning coalitions not involving player A multiply the generating functions for just the players \( \{B, C, D\} \):

\[
G_B G_C G_D = \frac{1}{8} \left[ (b^3 c^2 d)x^6 + (b^3 c^2)x^5 + (b^3 d)x^4 + (b^3 + c^2 d)x^3 + (c^2)x^2 + (d)x + 1 \right]
\]

The coalitions that have a power \( x^k \) with \( k \leq q-1 \) are the coalitions that are not winning. Introduce the following notation

**Definition:** For the polynomial \( Z(x) = \sum_i \delta_i x^i \) define \( \{Z(x)\}_\alpha^\beta = \sum_{\alpha \leq k \leq \beta} \delta_k x^k \).

This extracts a set of consecutive terms in a polynomial.

so that the non-winning coalitions without A are:

\[
\left\{G_B G_C G_D\right\}_0^{q-1} = \left\{G_B G_C G_D\right\}_0^5 = \frac{1}{8} \left( b^3 c^2 x^5 + (b^3 d)x^4 + (b^3 + c^2 d)x^3 + (c^2)x^2 + (d)x + 1 \right)
\]

\[ (3) \]

To determine which coalitions \( A \) can make winning, multiply Equation (3) by \( G_A \) and extract the winning coalitions, these are the \( x^k \) terms with \( k \geq q = 6 \):

\[
\left\{G_A \left\{G_B G_C G_D\right\}_0^{q-1}\right\}_q = \left\{G_A \left\{G_B G_C G_D\right\}_0^5\right\}_6 \]

\[
= \frac{1}{16} \left[ (a^4 b^3 c^2)x^9 + (a^4 b^3 d)x^8 + (a^4 b^3 + a^4 c^2 d)x^7 + (a^4 c^2)x^6 \right]
\]

\[ (4) \]

This shows 5 coalitions that \( A \) has made winning; the first two are \( \{A, B, C\} \) and \( \{A, B, D\} \). The probability of these winning coalitions involving \( A \) is the numerical coefficient of each coalition.
While the variables \( \{a, b, c, d\} \) in Equations (11) and (2) are useful for identifying coalitions, they are not needed in the following. Replacing \( \{a, b, c, d\} \) with the value one in Equation (4) results in

\[
\left\{ G_A \left\{ G_B G_C G_D \right\}^{q-1}_0 \right\}^\infty_{q=0} = \frac{1}{16} \left( x^9 + x^8 + 2 x^7 + x^6 \right)
\]

That is, among the coalitions that \( A \) made winning there are: 2 with 7 votes and 1 with each of 6, 8, or 9 votes. Summing the above numerical coefficients (i.e., setting \( x = 1 \)) determines the probability that \( A \) has made any coalition winning:

\[
\text{Prob}_A \equiv \text{Probability}[\text{Player } A \text{ has made a coalition winning}]
\]

\[= \left\{ G_A \left\{ G_B G_C G_D \right\}^{q-1}_0 \right\}^\infty_{q=0} \bigg|_{a=b=c=d=1}^{x=1} = \frac{5}{16} \tag{5}\]

This can be interpreted as follows: if one of the 16 possible coalitions not involving \( A \) were selected (uniformly) at random then \( \frac{5}{16} \)th of the time that coalition is one for which \( A \) is marginal.

Similarly, by focusing on each of the other players one at a time, we can compute (the subscript “\( V \)” is used at mean “when \( a = b = c = d = x = 1^v \)):

\[
\begin{align*}
\text{Prob}_B & = \left\{ G_B \left\{ G_A G_C G_D \right\}^{q-1}_0 \right\}^\infty_{q=0} \bigg|_{a=b=c=d=1}^{x=1} = \frac{3}{16} \\
\text{Prob}_C & = \left\{ G_C \left\{ G_A G_B G_D \right\}^{q-1}_0 \right\}^\infty_{q=0} \bigg|_{a=b=c=d=1}^{x=1} = \frac{3}{16} \\
\text{Prob}_D & = \left\{ G_D \left\{ G_A G_B G_C \right\}^{q-1}_0 \right\}^\infty_{q=0} \bigg|_{a=b=c=d=1}^{x=1} = \frac{1}{16}
\end{align*}
\]

Computing the relative weights of these probabilities we recover the Banzhaf powers found earlier:

\[
\beta(A) = \frac{\text{Prob}_A}{\text{Prob}_A + \text{Prob}_B + \text{Prob}_C + \text{Prob}_D} = \frac{5}{12} = \frac{5}{12}
\]

\[
\beta(B) = \frac{\text{Prob}_B}{\text{Prob}_A + \text{Prob}_B + \text{Prob}_C + \text{Prob}_D} = \frac{3}{12} = \frac{1}{4} = \beta(C)
\]

\[
\beta(D) = \frac{\text{Prob}_D}{\text{Prob}_A + \text{Prob}_B + \text{Prob}_C + \text{Prob}_D} = \frac{1}{12}
\]

Careful inspection reveals that the probabilistic computation in this section is identical to the enumerative computation; just expressed differently. The probabilities found here \((\frac{5}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{16})\) are proportional to the counts \((10, 6, 6, 2)\) found earlier, so the voting powers are the same.

\[\text{The intermediate computation is:}
\]

\[
\begin{align*}
G_B \left\{ G_A G_C G_D \right\}^{q-1}_0 & = \frac{1}{16} \left[ (b^3 d) x^8 + (a^4 b^3) x^7 + (b^3 c^2 d) x^6 \right] \\
G_C \left\{ G_A G_B G_D \right\}^{q-1}_0 & = \frac{1}{16} \left[ (a^4 c^2 d) x^7 + (a^4 c^2 + b^3 c^2 d) x^6 \right] \\
G_D \left\{ G_A G_B G_C \right\}^{q-1}_0 & = \frac{1}{16} \left[ (b^3 c^2 d) x^6 \right]
\end{align*}
\]
2.2 Banzhaf Power via Influence Polynomials

Rewrite the computation appearing in Equation (5) as

\[
\text{Prob}_A = \left\{ G_A \left( \left\{ G_B G_C G_D \right\}_{0}^{q-1} \right) \right\}_{q}^{\infty} \bigg|_{x=1} = \left\{ G_A R(x) \right\}_{q}^{\infty} \bigg|_{x=1}
\]

where \( R(x) = \sum_{j=0}^{q-1} r_j x^j \) is a polynomial of degree no more than \( q - 1 \). (We assume now that the \{a, b, c, d\} terms all have the value one.) The constant part of \( G_A \) cannot contribute to raising an exponent of \( x \) to change a non-winning coalition into a winning coalition, as needed for the \( \cdot \}_{q}^{\infty} \) computation, so it can be neglected and \( \text{Prob}_A \) can be written as:

\[
\text{Prob}_A = \left\{ (\text{Non-constant part of } G_A) \ R(x) \right\}_{q}^{\infty} \bigg|_{x=1}
\]

\[
= \left\{ \left( \frac{1}{2} x^4 \right) R(x) \right\}_{q}^{\infty} \bigg|_{x=1}
\]

\[
= \left\{ \left( \frac{1}{2} x^4 \right) \left( \sum_{j=0}^{q-1} r_j x^j \right) \right\}_{q}^{\infty} \bigg|_{x=1}
\]

\[
= \sum_{j=q-4}^{q-1} \frac{1}{2} r_j
\]

\[
= \left( \sum_{j=q-4}^{q-1} \frac{1}{2} x^j \right) \otimes \left( \sum_{j=0}^{q-1} r_j x^j \right)
\]

\[
= I_A(x) \otimes R(x)
\]

where we have defined the Influence Polynomial for \( A \), \( I_A(x) \) with degree \( q - 1 \), and we have introduced the following notation:

**Definition:** For two polynomials \( R(x) = \sum_{j} r_j x^j \) and \( S(x) = \sum_{j} s_j x^j \) define the sum of product coefficients to be \( R(x) \otimes S(x) = \sum_{j} r_j s_j \). That is, the coefficients of common powers are multiplied together and then added.
The representation in Equation (7) is exactly equivalent to the expression in Equation (5).

Similarly

\[
\text{Prob}_B = I_B(x) \otimes \left\{ G_A G_C G_D \right\}_0^{q-1} \quad I_B(x) = \sum_{j=q-3}^{q-1} \frac{1}{2} x^j
\]

\[
\text{Prob}_C = I_C(x) \otimes \left\{ G_A G_B G_D \right\}_0^{q-1} \quad I_C(x) = \sum_{j=q-2}^{q-1} \frac{1}{2} x^j
\]

\[
\text{Prob}_D = I_D(x) \otimes \left\{ G_A G_B G_C \right\}_0^{q-1} \quad I_D(x) = \sum_{j=q-1}^{q-1} \frac{1}{2} x^j
\]

This section has used influence polynomials to compute the voting probabilities for the simplest voting structure, when a voter is equally likely to cast all or none of their votes (“random voting”). The paradigm of using influence polynomials also works for votes distributed partially or non-uniformly. The next section shows how to compute the influence polynomial in these cases.

3 Non-Uniform Probabilities

The generating function in Equation (1) represents the votes that player \( A \) can cast for a coalition and represents two equally likely situations, that “none” or “all” of the available votes were cast. In more complex situations, weighted generating functions can capture how players distribute their votes in ways that are not all or nothing and to vote with non-uniform probabilities. For example, we might choose

\[
G_A = \frac{1}{10} a^0 x^0 + \frac{4}{10} a^2 x^2 + \frac{3}{10} a^3 x^3 + \frac{2}{10} a^4 x^4
\]

which we interpret as follows: Player \( A \) will contribute 0 votes to a coalition \( 1/10 \) of the time, 2 votes \( 4/10 \) of the time, 3 votes \( 3/10 \) of the time, and 4 votes \( 2/10 \) of the time.

Now we must interpret what it means for a player to be marginal when that player can exercise non-uniform and partial voting. It is no longer adequate to merely multiply the vote structures (i.e., generating functions) together as in Equation (5), as we now indicate. Imagine that player \( A \) has the voting structure \( G_A = x^4 \); that is, they give all 4 of their votes to every coalition. Blindly using Equation (5) would give Prob\(_A\) = \( \frac{5}{2} \). This is a larger value than what was obtained in the random voting model, and must be wrong. If player \( A \) always give 4 votes to every coalition, then we claim that player \( A \) has no power. This is because player \( A \) has lost the ability to influence any coalition; the other players always know what player \( A \) will do, in any circumstance. Think of this in a political context: if a politician has already decided to vote for (or against) a piece of legislation then they cannot influence that legislation. The framers of the legislation will only modify the legislation to influence undecided voters.

In general, if a player always casts all, or none, of their votes then that player cannot ever be marginal. Stated differently, whenever a player cannot influence others by having the ability to change the winning-ness of coalitions, then that player has no power.

Let’s work through an example. Assume, as usual, that \( q \) votes are needed for a coalition to be winning. Suppose that a coalition not including player \( A \) already has \( Z \) votes with \( Z < q \) and that player \( A \) has the vote structure in Equation (9). Then there is a probability that each coalition without player \( A \) will become, after player \( A \) votes, winning \( (v) \) or losing \( (1-v) \).
Consider, for example, what this means when \( v = 99\% \). While player A is nearly always giving enough votes to make the coalition winning, the other players know that only 1% of time will player A keep the coalition from being winning. Hence, player A will get little attention from the other players – there is little of player A’s behavior that can be influenced. Now consider instead what \( v = 60\% \) means; more than half the time player A gives enough votes for the coalition to be winning but a large fraction of the time (40%) player A is not giving enough votes for a coalition to be winning. In this case player A is much more influential in determining whether or not a coalition is going to be winning.

We define player A’s ability to be marginal to be equal to the percentage of votes that are “in play”, the minimum of \( v \) and \( 1 - v \); define \( \gamma = \min(v, 1 - v) \). When \( v = 99\% \) then there is only \( \gamma = 1\% \) that is “in play” and player A’s influence is small; when \( v = 60\% \) then \( \gamma = 40\% \) and player A’s votes need to be negotiated by the other players – player A is more of a “swing voter” in this case.

With this thinking the Influence Polynomial for any vote structure is determined as follows:

1. Assume the vote structure for a player is: \( G = \sum_{j=0}^{q-1} g_j x^j \) where some \( \{g_j\} \) may be zero
2. Define the partial sums: \( v_Z = \sum_{j=q-Z}^{q-1} g_j \) and \( \gamma_Z = \min(v_Z, 1 - v_Z) \) for \( Z = 1, 2, \ldots, q - 1 \)
3. Then the Influence Polynomial for that player is \( I(x) = \sum_{Z=1}^{q-1} \gamma_Z x^Z \)

This definition is consistent with the evaluations given earlier, for “random voting”, as shown in the next section. Table 1 shows the Influence Polynomial computations for the vote structure in Equation (9); the result is

\[ I_A(x) = 0x^1 + \frac{2}{10}x^2 + \frac{5}{10}x^3 + \frac{1}{10}x^4 + \frac{1}{10}x^5 \]

Using the Influence Polynomial \( I_A(x) \) we define the Influence of A, \( I(A) \), to be:

\[ I(A) = I_A(x) \otimes \left\{ G_B G_C G_D \right\}^{q-1} \]

which is a generalization of the probability defined in Equations (7) and (8). This becomes the probability shown in those equations when a player is using “random voting”. Once the influences have been determined for each player, they are normalized as in Equation (6) to determine what we define to be the Generalized Banzhaf power; for player A this is denoted \( \beta'(A) \).

The Generalized Banzhaf power is a generalization of the Banzhaf power that accounts for arbitrary voting structures. For random voting, the Generalized Banzhaf power is the Banzhaf power.

### 3.1 Influence Polynomial for Random Voting

The Influence Polynomials as defined algorithmically in the last section is consistent with the values given in Equation (8), as we now show. Assume use of random voting, that is:

\[ G_N = \frac{1}{2} + \frac{X^N}{2} = \frac{1}{2} \sum_{j=0}^{q-1} (\delta_{j0} + \delta_{jN}) x^j \]
| Number of votes coalition has without player A | Probability of coalition winning with A’s votes: \( v_Z \) | Probability of coalition not winning with A’s votes: \( 1 - v_Z \) | Fraction of A’s votes that are “in play”: \( \gamma_Z = \min(v_Z, 1 - v_Z) \) | \( x^Z \) |
|---|---|---|---|---|
| \( Z = 1 \) | 0 | 1 | 0 | \( x^1 \) |
| \( Z = 2 \) | \( \frac{2}{10} \) | \( \frac{8}{10} \) | \( \frac{2}{10} \) | \( x^2 \) |
| \( Z = 3 \) | \( \frac{5}{10} = \frac{2}{10} + \frac{3}{10} \) | \( \frac{5}{10} \) | \( \frac{5}{10} \) | \( x^3 \) |
| \( Z = 4 \) | \( \frac{9}{10} = \frac{2}{10} + \frac{3}{10} + \frac{4}{10} \) | \( \frac{1}{10} \) | \( \frac{1}{10} \) | \( x^4 \) |
| \( Z = 5 \) | \( \frac{9}{10} = \frac{2}{10} + \frac{3}{10} + \frac{4}{10} \) | \( \frac{1}{10} \) | \( \frac{1}{10} \) | \( x^5 \) |

Table 1: A coalition without player A has \( Z \) votes and a winning coalition needs \( q = 6 \) votes; player A votes using the vote structure in Equation (9).

where \( \delta_{ij} \) is the usual Kronecker delta and \( N \leq q - 1 \). Using the procedure for determining the Influence Polynomial in the last section (recall \( Z \leq q - 1 \)), we compute

\[
v_Z = \sum_{j=q-Z}^{q-1} g_j = \frac{1}{2} \sum_{j=q-Z}^{q-1} (\delta_{j0} + \delta_{jN}) = \begin{cases} \frac{1}{2} & Z \geq q - N \\ 0 & Z < q - N \end{cases}
\]

\[
\gamma_Z = \min(v_Z, 1 - v_Z) = \begin{cases} \frac{1}{2} & Z \geq q - N \\ 0 & \text{otherwise} \end{cases}
\]

\[
I(x) = \sum_{Z=1}^{q-1} \gamma_Z x^Z = \frac{1}{2} \sum_{Z=q-N}^{q-1} x^Z = \frac{1}{2} \left( x^{q-N} + x^{q-N+1} + \cdots + x^{q-1} \right)
\]

If, for example, \( q = 6 \) and \( N = 3 \) then \( I(x) = \frac{1}{2} \left( x^3 + x^4 + x^5 \right) \) as shown in Equation (8) for player B.

3.2 Example: \([6;4,3,2,1]\) game with one player having non-uniform votes

We assume the voting structures appearing in Equations (2) and (9)

\[
G_A = \frac{1}{10} + \frac{4}{10}x^2 + \frac{3}{10}x^3 + \frac{2}{10}x^4,
\]

\[
G_B = \frac{1}{2} + \frac{1}{2}x^3,
\]

\[
G_C = \frac{1}{2} + \frac{1}{2}x^2,
\]

\[
G_D = \frac{1}{2} + \frac{1}{2}x
\]

for which the Influence Polynomials have been determined to be:

\[
I_A(x) = \frac{2}{10}x^2 + \frac{5}{10}x^3 + \frac{1}{10}x^4 + \frac{1}{10}x^5
\]

\[
I_B(x) = \frac{1}{2} \left( x^3 + x^4 + x^5 \right)
\]

\[
I_C(x) = \frac{1}{2} \left( x^4 + x^5 \right)
\]

\[
I_D(x) = \frac{1}{2} \left( x^5 \right)
\]
Using Equation (10) and its analogues we find the influences \( \{I(A), I(B), I(C), I(D)\} \). Normalizing the influences by their sum gives the Generalized Banzhaf powers \( \{\beta'(A), \beta'(B), \beta'(C), \beta'(D)\} \):

\[
\begin{align*}
I(A) &= \frac{7}{30}, & I(B) &= \frac{13}{30}, & I(C) &= \frac{3}{10}, & I(D) &= \frac{1}{10}, \\
\beta'(A) &= \frac{7}{30}, & \beta'(B) &= \frac{13}{30}, & \beta'(C) &= \frac{6}{30}, & \beta'(D) &= \frac{4}{30},
\end{align*}
\]

3.3 Example: [6;4,3,2,1] game with one player voting parametrically

For the [6;4,3,2,1] game suppose that players B, C, and D vote as before; that is, using random voting (each is equally likely to give no votes or all votes). Suppose now that player A gives 0 votes with probability \(1 - p\) and gives 4 votes with probability \(p\); that is player A has the parametric vote structure (with \(0 \leq p \leq 1\))

\[
G_A = (1 - p) + px^4
\]

For this voting structure, \( I_A(x) = (x^2 + x^3 + x^4 + x^5) \min(p, 1 - p) \) and

\[
\beta'(A) = \frac{5 \min(1 - p, p)}{\Delta_4}, \quad \beta'(B) = \beta'(C) = \frac{1 + p}{\Delta_4}, \quad \beta'(D) = \frac{1 - p}{\Delta_4}
\]

where \( \Delta_4 = 3 + p + 5 \min(1 - p, p) \). These results are shown in Figure 1. Observe that:

1. Player A has a Generalized Banzhaf power of zero when \(p = 0\) or \(p = 1\). This is expected, player A has no power when there are no votes “in play”.
2. Player A has a maximal Generalized Banzhaf power when \(p = \frac{1}{2}\). This is expected, this is when player A has the most votes “in play”.
3. Players B and C always have the same Generalized Banzhaf power.
4. When \(p = 0\) (player A casts no votes) the game is the same as [6;3,2,1] for the players \(\{B, C, D\}\). In this case players B, C, and D all have equal Generalized Banzhaf power of \(\frac{1}{3}\), which is the same as their Banzhaf power.
5. When \( p = 1 \) (player \( A \) casts 4 votes) the game is the same as \([2; 3, 2, 1]\) for the players \( \{B, C, D\} \); this is an improper game, but the meaning is clear. In this case players \( B \) and \( C \) have equal Generalized Banzhaf power of \( \frac{1}{2} \) and player \( D \) has a Generalized Banzhaf power of zero.

In the \([6; 4, 3, 2, 1]\) game a player other than player \( A \) could vote parametrically. In the following three examples player \( B \), \( C \), or \( D \) gives 0 votes with probability \( 1 - p \) and gives all its votes with probability \( p \); in each case the other players use random voting. Figure 2 shows the results graphically.

1. The voting structures and Generalized Banzhaf powers when player \( B \) votes parametrically:

\[
G_A = \frac{1}{2} (1 + x^4) \quad G_B = (1 - p) + px^3 \quad G_C = \frac{1}{2} (1 + x^2) \quad G_D = \frac{1}{2} (1 + x)
\]

\[
\beta'(A) = \frac{2 + p}{\Delta_3} \quad \beta'(B) = \frac{3 \min(1 - p, p)}{\Delta_3} \quad \beta'(C) = \frac{2 - p}{\Delta_3} \quad \beta'(D) = \frac{p}{\Delta_3}
\]

where \( \Delta_3 = 4 + p + 3 \min(1 - p, p) \)

2. The voting structures and Generalized Banzhaf powers when player \( C \) votes parametrically:

\[
G_A = \frac{1}{2} (1 + x^4) \quad G_B = \frac{1}{2} (1 + x^3) \quad G_C = (1 - p) + px^2 \quad G_D = \frac{1}{2} (1 + x)
\]

\[
\beta'(A) = \frac{2 + p}{\Delta_2} \quad \beta'(B) = \frac{2 - p}{\Delta_2} \quad \beta'(C) = \frac{3 \min(1 - p, p)}{\Delta_2} \quad \beta'(D) = \frac{p}{\Delta_2}
\]

where \( \Delta_2 = 4 + p + 3 \min(1 - p, p) \)

3. The voting structures and Generalized Banzhaf powers when player \( D \) votes parametrically:

\[
G_A = \frac{1}{2} (1 + x^4) \quad G_B = \frac{1}{2} (1 + x^3) \quad G_C = \frac{1}{2} (1 + x^2) \quad G_D = (1 - p) + px
\]

\[
\beta'(A) = \frac{3 - p}{\Delta_1} \quad \beta'(B) = \frac{1 + p}{\Delta_1} \quad \beta'(C) = \frac{1 + p}{\Delta_1} \quad \beta'(D) = \frac{\min(1 - p, p)}{\Delta_1}
\]

where \( \Delta_1 = 5 + p + \min(1 - p, p) \)
4 Teams and Leaders

Another generalization of traditional voting is to consider “teams” (or coalitions) of players that work together, although not with complete unanimity. For example, for the $[6;4,3,2,1]$ game assume that player $A$ (with 4 votes) represents a team (shown as $A^{\text{team}}$) of 3 members $\{a_1, a_2, a_3\}$ with the first two members having 1 vote each and the last member having 2 votes.

Suppose the following:

1. $A^{\text{team}}$ has a leader who influences how the $A^{\text{team}}$ members cast their votes. We define the leader’s power to the same as their team’s power.

2. The $A^{\text{team}}$ leader wants each individual $A^{\text{team}}$ member to cast their votes with probability $L$ and to not cast their votes with probability $(1 - L)$.

3. Each individual $A^{\text{team}}$ member follow their leader’s desire with probability $p$ and each member does so independently of other team members.

In this case the appropriate generating function representation of $A^{\text{team}}$’s votes is

$$G_{A^{\text{team}}} = L \left[ \left( (1 - p) + ap \right) (1 - p) + ap \right] \left( (1 - p) + a^2 p^2 \right)$$

$$+ (1 - L) \left[ \left( p + a(1 - p)x \right) \left( p + a(1 - p)x \right) \left( p + a^2 (1 - p)x^2 \right) \right]$$

The first term (with the $L$ coefficient) represents the votes cast if the leader wishes $A^{\text{team}}$ to be part of a coalition, the second term (with the $(1 - L)$ coefficient) represents the votes cast if the leader wishes $A^{\text{team}}$ to not be part of a coalition. The generating functions for each member are multiplied together, in each sub-expression, since each team member acts independently.

As before, this generating function has $x$ exponents of 0, 1, . . . , 4 representing the number of votes that $A^{\text{team}}$ can cast. Note that the expression is correctly normalized; $G_{A^{\text{team}}}\big|_{a=x=1} = 1$ for any value of $p$. Table 2 interprets $G_{A^{\text{team}}}$ for specific values of $L$ and $p$.

4.1 Teams whose members each have one vote

An important special case is a team whose members each have one vote. For example, this could represent Congress where each Congressperson has one vote for their team; and the teams are called Democrats, Republicans, or Independents. The voting structure for a team ($G_{\text{uniform team}}$) of $n$ members, where each member has a single vote is:

$$G_{\text{uniform team}} = L \left( (1 - p) + px \right)^n + (1 - L) \left( p + (1 - p)x \right)^n$$

Special cases of this are:

- If $p = \frac{1}{2}$ then $G_{\text{uniform team}} = \left( \frac{1}{2} + \frac{1}{2}x \right)^n$ independent of $L$.
  (This is reasonable, team members are not influenced by their leader’s choice.)
Parameter values | Value of $G_{\text{A team}}$ | Interpretation |
|------------------|-------------------------|------------------|
| $p = 1$          | $L \left( a^4 x^4 \right) + (1 - L)$ | All players vote exactly as their leader wishes. Structurally this has the form of one player voting parametrically. |
| $p = 1, L = \frac{1}{2}$ | $\frac{1}{2} + \frac{a^4 x^4}{2}$ | Players vote exactly as the leader wishes and the leader is equally likely to support or oppose joining a coalition. $G_{\text{A team}}$ is the same as $G_A$ in Equation (1). |
| $p = 1, L = 1$ | $a^4 x^4$ | Players vote exactly as the leader wishes and the leader wants to join a coalition. All 4 votes are cast. |
| $p = 1, L = 0$ | 1 | Players vote exactly as the leader wishes and the leader is opposed to joining a coalition. No votes are cast. |
| $p = \frac{1}{2}$ | $\frac{1}{8} (a x + 1)^2 (a^2 x^2 + 1)$ | Players vote randomly and are not following their leader. $G_{\text{A team}}$ does not depend on $L$. |
| $p = 0$ | $L + (1 - L) a^4 x^4$ | Players do the exact opposite of what their leader wants. Structurally this has the form of one player voting parametrically. |
| $p \rightarrow 1 - p$ and $L \rightarrow 1 - L$ | $G_{\text{A team}}$ | If the leader switches their desire to join a coalition and the players switch their likelihood of following their leader, the result is the same. |

Table 2: Interpretation of $G_{\text{A team}}$ from Equation (15) for selected parameter values.

- If $p = 1$ then $G_{\text{uniform team}} = L x^n + (1 - L)$ (This is reasonable, with complete unanimity the team acts like one voter who distributes all the votes or none of the votes.)

Figure 3 shows the coefficients of $G_{\text{A team}}$ when $n = 50$ for various values of $L$ and $p$. Since $n$ is large the coefficients closely approximate either a Gaussian (when $p = \frac{1}{2}$) of the sum of two Gaussians. Figure 4 shows the coefficients of the Influence Polynomials for $G_{\text{A team}}$ when $n = 50$ for the same values of $L$ and $p$.

4.2 Example: [6;4,3,2,1] game when first player is a team

Consider a voting structure where $A_{\text{team}}$ has 3 members (using Equation (15) with individual weights of $\{2,1,1\}$) while the other players use uniform voting

$$G_{A_{\text{team}}} = L \left[ \left( (1 - p) + px \right)^2 \left( (1 - p) + px^2 \right) \right] + (1 - L) \left[ \left( p + (1 - p)x \right)^2 \left( p + (1 - p)x^2 \right) \right]$$

$$G_B = \frac{1}{2} + \frac{1}{2} x^3, \quad G_C = \frac{1}{2} + \frac{1}{2} x^2, \quad G_D = \frac{1}{2} + \frac{1}{2} x$$

the Generalized Banzhaf power for team $A$ is shown in Figure 5 (left) as a function of $L$ and $p$.

Now consider a voting structure where $A_{\text{team}}$ has 4 identical members (using Equation (16),

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each team $A$ member has 1 vote) while the other players use uniform voting

$$G_{A\text{team}} = \frac{1}{2} + \frac{1}{2}x, \quad G_B = \frac{1}{2} + \frac{1}{2}x^3, \quad G_C = \frac{1}{2} + \frac{1}{2}x^2, \quad G_D = \frac{1}{2} + \frac{1}{2}x$$

(18)

the Generalized Banzhaf power for team $A$ is shown in Figure 5 (right) as a function of $L$ and $p$.

In both of these cases:

1. The symmetry represented by $\{p, L\} \rightarrow \{1 - p, 1 - L\}$ is apparent
2. For any value of $L$ the maximum power for each $A^{\text{team}}$ is attained when $p = \frac{1}{2}$.
3. When $p = \frac{1}{2}$ the maximum power for each $A^{\text{team}}$ is attained when $L = 0$ or $L = 1$.
4. When $p$ is zero or one and $L$ is zero or one then $A^{\text{team}}$ has zero power.

In each case, $A^{\text{team}}$ has the most power when the members are least predictable ($p = \frac{1}{2}$) and the leader is decisive (either $L = 0$ or $L = 1$)

### 4.3 The US Senate

The techniques developed in this paper can be applied to political voting. Consider the 113th Congress, 1st Session (started January 2013) where there were 52 Democrats, 46 Republicans, and
Figure 5: Contour plots of $\beta'(A)$ when team $A$ has 3 members (left, Equation (17)) or 4 members (right, Equation (18)). The $L$ axis (0 to 1) is horizontal, the $p$ axis ($\frac{1}{2}$ to 1) is vertical. For all the contour plots in this paper the color scale goes from 0 (blue) to $\frac{1}{2}$ (red).

2 Independents in the Senate [1]. To obtain cloture\footnote{“Cloture is a motion or process in parliamentary procedure aimed at bringing debate to a quick end.” [12]} in the Senate 60 votes are sometimes needed; this naturally leads to the [60; 53, 45, 2] game. We assume a voting structure in which the Democrats and Republican teams have members each casting a single vote according to Equation (16) and the Independents use “random voting” (are equally likely to give 0 or 2 votes to any coalition). That is:

$$G_{D\text{team}} = L_D \left( (1 - p_D) + p_D x \right)^{53} + (1 - L_D) \left( p_D + (1 - p_D) x \right)^{53}$$

$$G_{R\text{team}} = L_R \left( (1 - p_R) + p_R x \right)^{45} + (1 - L_R) \left( p_R + (1 - p_R) x \right)^{45}$$

$$G_I = \frac{1}{2} + \frac{1}{2} x^2$$

(19)

where $p_D$ (resp. $p_R$) represents the probability that an individual Democrat (resp. Republican) votes the way their leader desires as indicated by $L_D$ (resp. $L_R$).

The Washington Post [2] lists the frequency with which Democratic and Republican senators voted with their party for the 112\textsuperscript{th} Congress. For the Democrats the average value was 94\% while for the Republicans it was 84\%; we refer to this as the cohesion value. For the 113\textsuperscript{th} Congress, we assume the values $p_R = 0.94$ and $p_D = 0.84$ for the Democratic and Republican cohesion.

When both the Democratic leader and the Republican leaders agree on an issue then there is little contention. Voting power becomes interesting when one team is in favor of an action ($L = 1$) and the other team is opposed ($L = 0$). Hence, consider two cases:

1. The Democratic leader wants to obtain cloture ($L_D = 1$) while the Republican leader is opposed to it ($L_R = 0$). The Generalized Banzhaf power for the teams at the cohesion value are: Democrats 0.35, Republicans 0.35, Independents 0.30.

   It is somewhat surprising that the Democrats, Republicans, and Independents all have similar power, especially since the Independents have only two members!

2. The Republican leader wants to obtain cloture ($L_R = 1$) while the Democratic leader is
opposed to it ($L_D = 0$). The Generalized Banzhaf power for the teams at the cohesion value are: Democrats 0.41, Republicans 0.31, Independents 0.28.

The Generalized Banzhaf power for the three teams as $p_D$ and $p_R$ are varied, is shown in Figure 6.

Partial derivatives indicate how the Generalized Banzhaf values change as the cohesion value changes. At the cohesion point, $(p_R, p_D) = (0.94, 0.84)$, we numerically compute:

1. When $(L_D = 1)$ and $(L_R = 0)$: \[ \frac{\partial \beta'(\text{Dem})}{\partial p_D} = 0.04, \quad \frac{\partial \beta'(\text{Dem})}{\partial p_R} = -0.36, \]
   \[ \frac{\partial \beta'(\text{Rep})}{\partial p_D} = 0.06, \quad \frac{\partial \beta'(\text{Rep})}{\partial p_R} = -0.37. \]
   In this case, interestingly, both the Democrats and the Republicans increase their power if either the Democratic cohesion increases or the Republican cohesion decreases.

2. When $(L_R = 1)$ and $(L_D = 0)$: \[ \frac{\partial \beta'(\text{Dem})}{\partial p_D} = -1.1, \quad \frac{\partial \beta'(\text{Dem})}{\partial p_R} = 0.25, \]
   \[ \frac{\partial \beta'(\text{Rep})}{\partial p_D} = 0.44, \quad \frac{\partial \beta'(\text{Rep})}{\partial p_R} = -0.08. \]
   In this case, the Democrats’ power increases if either the Democratic cohesion decreases or the Republican cohesion increases. Just the opposite is true for the Republicans; their power increases if either the Democratic cohesion increases or the Republican cohesion decreases.

In each of these cases the Republicans can adopt the same strategy to increase their power: increase Democratic cohesion or decrease Republican cohesion.

5 Summary

We have shown how to determine voting power when each player in a weighted voting game has a “voting structure”, a weighted generating function representing probabilities of them contributing any number of their votes to a coalition. The resulting Generalized Banzhaf values can be computed with polynomial arithmetic and reduce to the usual Banzhaf values when random voting is used.

Voting structures can also be used to represent voter coalitions. In this case each coalition who tries to influence the voting of each coalition member. This model was applied to the US Senate to show who has (Democrats, Republicans, or Independents) more power in attaining cloture. When the Democrats are in favor of cloture and the Republicans are not then, surprisingly, all three parties have similar power.

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Figure 6: The US Senate \([60, 53, 45, 2]\) game. The left column has \((L_D = 0, L_R = 1)\); the right column has \((L_R = 1, L_D = 0)\). The top row shows Democrats’ power; middle row shows Republicans’ power; bottom row shows Independents’ power. The dots show the cohesion point \((p_R, p_D) = (94\%, 84\%)\). For each plot the horizontal axis is \(p_D\) and the vertical axis is \(p_R\), both varying from \(\frac{1}{2}\) to 1.
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