Weighted weak group inverse for Hilbert space operators

Dijana MOSIĆ¹, Daochang ZHANG²

¹ Faculty of Sciences and Mathematics, University of Niš, P. O. Box 224, 18000 Niš, Serbia
² College of Sciences, Northeast Electric Power University, Jilin 132012, China

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Abstract We present the weighted weak group inverse, which is a new generalized inverse of operators between two Hilbert spaces, and we extend the notation of the weighted weak group inverse for rectangular matrices. Some characterizations and representations of the weighted weak group inverse are investigated. We also apply these results to define and study the weak group inverse for a Hilbert space operator. Using the weak group inverse, we define and characterize various binary relations.

Keywords Weak group inverse, weighted core-EP inverse, Wg-Drazin inverse, Hilbert space
MSC 47A62, 47A05, 15A09

1 Introduction

Throughout this paper, let \( \mathcal{B}(X,Y) \) be the set of all bounded linear operators from \( X \) to \( Y \), where \( X \) and \( Y \) are infinite-dimensional complex Hilbert spaces. In the case that \( X = Y \), we set \( \mathcal{B}(X) = \mathcal{B}(X,X) \). For \( A \in \mathcal{B}(X,Y) \), \( A^* \), \( N(A) \), \( R(A) \), and \( \sigma(A) \) represent the adjoint of \( A \), the null space, the range, and the spectrum of \( A \), respectively. We call \( P \in \mathcal{B}(X) \) an idempotent if \( P^2 = P \), and the orthogonal projector if \( P^2 = P = P^* \). If \( L \) and \( M \) are closed subspaces, we denote by \( P_{L,M} \) an idempotent on \( L \) along \( M \), and by \( P_L \) the orthogonal projector onto \( L \).

Let \( A \in \mathcal{B}(X,Y) \setminus \{0\} \). There always exists \( B \in \mathcal{B}(Y,X) \setminus \{0\} \) such that \( BAB = B \), which is not unique in general and it is called an outer inverse of \( A \). The outer inverse is uniquely determined if we fix its range and kernel. For a subspace \( T \) of \( X \) and a subspace \( S \) of \( Y \), the outer inverse \( B \) of \( A \) with the prescribed range \( T \) and the null space \( S \) is unique, if it exists, and denoted by \( A_{T,S}^{(2)} \). We now present some special classes of outer inverses.
For a fixed operator $W \in \mathcal{B}(Y,X)\setminus\{0\}$, an operator $A \in \mathcal{B}(X,Y)$ is called Wg-Drazin invertible \[4\] if there exists a unique operator $B \in \mathcal{B}(X,Y)$ (denoted by $A^{d,w}$) such that

$$AWB = BWA, \quad BWAWB = B, \quad A - AWBW A \text{ is quasinilpotent.}$$

If $X = Y$ and $W = I$, then $A^{d} = A^{d,W}$ is the generalized Drazin inverse (or the Koliha-Drazin inverse) of $A$ \[10\]. We use $\mathcal{B}(X,Y)^{d,W}$ and $\mathcal{B}(X)^{d}$, respectively, to denote the sets of all Wg-Drazin invertible operators in $\mathcal{B}(X,Y)$ and generalized Drazin invertible operators in $\mathcal{B}(X)$.

The W-weighted Drazin inverse is a particular case of the Wg-Drazin inverse for which $A - AWBW A$ is nilpotent. The Drazin inverse is a special case of the generalized Drazin inverse for which $A - A^{2}B$ is nilpotent, or equivalently, $A^{k+1}B = A^{k}$ for some non-negative integer $k$. The smallest such $k$ is called the index of $A$ and it is denoted by $\text{ind}(A)$. In the case that $\text{ind}(A) \leq 1$, $A$ is group invertible and the group inverse $A^{\#}$ of $A$ is a special case of a Drazin inverse. The Drazin inverse is very useful, and its applications in automatics, probability, statistics, mathematical programming, numerical analysis, game theory, econometrics, control theory, and so on, can be found in \[2,3\]. For more recent results related to generalized Drazin inverse, W-weighted Drazin inverse, and Drazin inverse, see \[17,20–23\].

Prasad and Mohana \[16\] introduced the core-EP inverse for a square matrix of arbitrary index, as a generalization of the core inverse restricted to a square matrix of index one \[1\]. The core-EP inverse was presented for generalized Drazin invertible operators on Hilbert spaces in \[15\].

As a generalization of the core-EP inverse of a square matrix to a rectangular matrix, the weighted core-EP inverse was given in \[6\]. In \[14\], the weighted core-EP inverse was defined for a Wg-Drazin invertible bounded linear operator between two Hilbert spaces, extending the concept of the weighted core-EP inverse for a rectangular matrix \[6\].

Let $W \in \mathcal{B}(Y,X)\setminus\{0\}$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible. Then there exists the unique operator $B$ which satisfies conditions

$$WAWB = P_{R((WA)^{d})}, \quad R(B) \subseteq R((AW)^{d}),$$

and it is called the W-weighted core-EP inverse of $A$, denoted by $A^{\circ,w}$. If $X = Y$ and $W = I$, then $A^{\circ} = A^{\circ,W}$ is the core-EP inverse of $A$ \[15\]. In the case that $A \in \mathcal{B}(X)$ and $\text{ind}(A) \leq 1$, the core-EP inverse of $A$ is the core inverse of $A$, denoted by $A^{\circ}$. Recently, many results concerning the weighted core-EP and core-EP inverse appeared in papers \[5,8,9,12,18,24\].

In \[19\], the weak group inverse was recently defined for square matrices of an arbitrary index and presented as a generalization of the group inverse. A weak group inverse for rectangular matrices, called the weighted weak group inverse, was given in \[7\].

We extend the definition of the weighted weak group inverse of a rectangular matrix to a Wg-Drazin invertible bounded linear operator between
two Hilbert spaces. We obtain some properties of the weighted weak group inverse, in particular, an operator matrix representation, characterizations and representations of the weighted weak group inverse. As an application of these results, we present and characterize the weak group inverse of a generalized Drazin invertible bounded linear operator on a Hilbert space. Using the weak group inverse, we define and study several binary relations.

2 Weighted weak group inverse

In order to define the weighted weak group inverse of a Wg-Drazin invertible bounded linear operator between two Hilbert spaces as an extension of the weighted weak group inverse of a rectangular matrix, we need the following auxiliary result. We use the notations $\mathcal{B}(X,Y)^{-1}$ and $\mathcal{B}(X,Y)^{\text{qnil}}$ for the sets of all invertible and quasinilpotent operators of $\mathcal{B}(X,Y)$, respectively.

**Lemma 1** [14] Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and $A \in \mathcal{B}(X,Y)^{d,W}$. Then

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \rightarrow \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix}$$

(1)

and

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix} : \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix} \rightarrow \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix},$$

(2)

where

$$A_1 \in \mathcal{B}(R((WA)^d), R((AW)^d))^{-1}, \quad W_1 \in \mathcal{B}(R((AW)^d), R((WA)^d))^{-1},$$

$$A_3W_3 \in \mathcal{B}(N[((AW)^d)^*])^{\text{qnil}}, \quad W_3A_3 \in \mathcal{B}(N[((WA)^d)^*])^{\text{qnil}}.$$  

In addition,

$$A^{d,W} = \begin{bmatrix} (W_1A_1W_1)^{-1} & W_1^{-1}U \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \rightarrow \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix},$$

(3)

$$A^{\oplus,W} = \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \rightarrow \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix},$$

(4)

$$\left( AW \right)^d = \begin{bmatrix} (A_1W_1)^{-1} & T \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix} \rightarrow \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix},$$

$$\left( WA \right)^d = \begin{bmatrix} (W_1A_1)^{-1} & U \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \rightarrow \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix},$$

where

$$T = \sum_{n=0}^{\infty} (A_1W_1)^{-(n+2)}(A_1W_2 + A_2W_3)(A_3W_3)^n$$
and
\[ U = \sum_{n=0}^{\infty} (W_1 A_1)^{-(n+2)} (W_1 A_2 + W_2 A_3)(W_3 A_3)^n. \]

We first give the algebraic definition of a new outer inverse.

**Theorem 1** Let \( W \in \mathcal{B}(Y, X) \backslash \{0\} \) and \( A \in \mathcal{B}(X, Y)^{d,W} \). Then the system of equations
\[ AWBWB = B \quad \text{and} \quad AWB = A^@^{,W}WA \] is consistent and it has the unique solution given by
\[ B = \begin{bmatrix} (W_1 A_1 W_1)^{-1} (A_1 W_1)^{-2} (A_2 + W_1^{-1} W_2 A_3) & 0 \\ 0 & 0 \end{bmatrix}; \]
\[ \begin{bmatrix} R((WA)^d) \\ N[(WA)^d] \end{bmatrix} \rightarrow \begin{bmatrix} R((AW)^d) \\ N[(AW)^d] \end{bmatrix}, \] (6)

where \( A \) and \( W \) are represented as in (1) and (2), respectively.

**Proof** Using (1), (2), (4), and (6), we get
\[ AWB = \begin{bmatrix} W_1^{-1} (A_1 W_1)^{-1} A_2 + (W_1 A_1 W_1)^{-1} W_2 A_3 \\ 0 \end{bmatrix} = A^@^{,W}WA \]
and
\[ AWBWB = B, \]
that is, \( B \) is a solution of system (5).

If an operator \( B \) satisfies (5), then
\[ B = (AWB)WB = A^@^W(AWB) = A^@^{,W}WA^@^{,W}WA = (A^@^{,W}W)^2 A, \]
and \( B \) is the unique solution of system (5). \( \square \)

For \( X = Y \) and \( W = I \) in Theorem 1, we get the following consequence.

**Corollary 1** Let \( A \in \mathcal{B}(X)^{d} \). Then the system of equations
\[ AB^2 = B \quad \text{and} \quad AB = A^@^A \] is consistent and it has the unique solution given by
\[ B = \begin{bmatrix} A_1^{-1} & A_1^{-2} A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(A^d) \\ N[(A^d)^*] \end{bmatrix} \rightarrow \begin{bmatrix} R(A^d) \\ N[(A^d)^*] \end{bmatrix}, \] (8)

where
\[ A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} R(A^d) \\ N[(A^d)^*] \end{bmatrix} \rightarrow \begin{bmatrix} R(A^d) \\ N[(A^d)^*] \end{bmatrix}, \] (9)
\( A_1 \in \mathcal{B}(R(A^d)) \) is invertible, and \( A_3 \in \mathcal{B}[N((A^d)^*]) \) is quasinilpotent.
**Definition 1** Let $W \in \mathcal{B}(Y, X) \setminus \{0\}$ and $A \in \mathcal{B}(X, Y)^{d, W}$. The W-weighted weak group inverse of $A$ is defined as

$$A^{\otimes, W} = (A^{\otimes, W} W)^2 A.$$ 

**Definition 2** Let $A \in \mathcal{B}(X)^{d}$. The weak group inverse of $A$ is defined as

$$A^\otimes = (A^\otimes)^2 A.$$ 

Remark that, by Lemma 1 and Theorem 1, the W-weighted weak group inverse is different from the Wg-Drazin inverse and W-weighted core-EP inverse. Hence, the W-weighted weak group and weak group inverses provide new classes of outer inverses for operators. If $X$ and $Y$ are finite dimensional, then, for $W \in \mathcal{B}(Y, X) \setminus \{0\}$, every operator $A \in \mathcal{B}(X, Y)$ has the W-weighted weak group inverse. For $A \in \mathcal{B}(X)$ and $\text{ind}(A) \leq 1$, the weak group inverse of $A$ is the group inverse of $A$.

We also have a definition of the weighted weak group inverse from a geometrical point of view.

**Theorem 2** Let $W \in \mathcal{B}(Y, X) \setminus \{0\}$ and $A \in \mathcal{B}(X, Y)^{d, W}$. The system of conditions

$$WAWB = P_{R(WA^{d, W})}, N(A^{\otimes, W} W A) \text{ and } R(B) \subseteq R(A^{d, W})$$

(10) is consistent and it has the unique solution $B = (A^{\otimes, W} W)^2 A$.

**Proof** For $B = (A^{\otimes, W} W)^2 A$, we have

$$WAWB = WA^{\otimes, W} WA$$

is a projector onto

$$R(WA^{\otimes, W}) = R(WA^{d, W})$$

along

$$N(WA^{\otimes, W} WA) = N(A^{\otimes, W} WA) \text{ and } R(B) \subseteq R(A^{\otimes, W}) = R(A^{d, W}),$$

i.e., $B$ satisfies conditions (10).

Assume that two operators $B_1$ and $B_2$ satisfy conditions (10). First,

$$WAW(B_1 - B_2) = P_{R(WA^{d, W}), N(A^{\otimes, W} WA)} - P_{R(WA^{d, W}), N(A^{\otimes, W} WA)} = 0$$

implies

$$R(B_1 - B_2) \subseteq N(WAW) \subseteq N(A^{d, W} WAW).$$

Furthermore,

$$R(B_1) \subseteq R(A^{d, W}) = R(A^{d, W} WAW) \text{ and } R(B_2) \subseteq R(A^{d, W} WAW)$$

give

$$R(B_1 - B_2) \subseteq R(A^{d, W} WAW) \cap N(A^{d, W} WAW) = \{0\}.$$
Therefore, $B_1 = B_2$ and only $B$ satisfies (10).

Consequently, the geometrical approach is given now for the weak group inverse.

**Corollary 2** Let $A \in \mathcal{B}(X)^d$. Then the system of conditions

$$AB = P_{R(A^d), N(A^{\ominus}A)} \quad \text{and} \quad R(B) \subseteq R(A^d)$$

(11)

is consistent and it has the unique solution $B = (A^{\ominus})^2 A$.

We consider some idempotents determined by the weighted weak group inverse and observe that the weighted weak group inverse is an outer inverse.

**Lemma 2** Let $W \in \mathcal{B}(Y, X) \setminus \{0\}$ and $A \in \mathcal{B}(X, Y)^{d,W}$. Then

(i) $AWA^{\ominus,W}W$ is a projector onto $R(A^{d,W})$ along $N(A^{\ominus,W}WAW)$;

(ii) $WAWA^{\ominus,W}$ is a projector onto $R(WA^{d,W})$ along $N(A^{\ominus,W}WA)$;

(iii) $A^{\ominus,W}WAW$ is a projector onto $R(A^{d,W})$ along $N(A^{\ominus,W}W(AW)^2)$;

(iv) $WA^{\ominus,W}WA$ is a projector onto $R(WA^{d,W})$ along $N(A^{\ominus,W}(WA)^2)$;

(v) $A^{\ominus,W} = (WAW)^{(2)}_{R(A^{d,W}), N(A^{\ominus,W}WA)}$.

**Proof** (i) Notice that $AWA^{\ominus,W}W = A^{\ominus,W}WAW$ is a projector onto $R(A^{\ominus,W}) = R(A^{d,W})$ along $N(A^{\ominus,W}WAW)$.

(v) Since

$$A^{\ominus,W} = (A^{\ominus,W}W)^2 A, \quad AWA^{\ominus,W} = A^{\ominus,W}WA,$$

we obtain

$$A^{\ominus,W}WAWA^{\ominus,W} = (A^{\ominus,W}W)^2 AW A^{\ominus,W}WA = (A^{\ominus,W}W)^2 A = A^{\ominus,W}.$$  

Hence, $A^{\ominus,W}$ is an outer inverse of $WAW$ with

$$R(A^{\ominus,W}) = R((A^{\ominus,W}W)^2 A), \quad N(A^{\ominus,W}) = N((A^{\ominus,W}W)^2 A).$$

From

$$A^{d,W} = A^{\ominus,W}WAWA^{d,W}WA$$

$$= A^{\ominus,W}WA^{\ominus,W}WA^{d,W}WAWA$$

$$= (A^{\ominus,W}W)^2 (AW)^2 A^{d,W},$$

we get

$$R(A^{d,W}) \subseteq R((A^{\ominus,W}W)^2 A) \subseteq R(A^{d,W}),$$

which yields $R(A^{\ominus,W}) = R(A^{d,W})$. By (1), (2), and (4), we show that

$$A(WA^{\ominus,W})^2 = A^{\ominus,W}.$$

Now, we have

$$A^{\ominus,W}WA = AW(A^{\ominus,W}WA^{\ominus,W}WA) = AW A^{\ominus,W},$$
which gives
\[ N(A^{\otimes, W}) \subseteq N(A^{\otimes, W} WA) \subseteq N(A^{\otimes, W}), \]
i.e.,
\[ N(A^{\otimes, W}) = N(A^{\otimes, W} WA). \]
Thus, the desired result holds.

Similarly, we verify (ii)–(iv). □

By Lemma 2, notice that the weak group inverse of \( A \) is an outer inverse of \( A \).

**Corollary 3** Let \( A \in B(X^d) \). Then
(i) \( AA^{\otimes} \) is a projector onto \( R(A^d) \) along \( N(A^{\otimes} A) \);
(ii) \( A^{\otimes} A \) is a projector onto \( R(A^d) \) along \( N(A^{\otimes} A^2) \);
(iii) \( A^{\otimes} = A^{(2)}_{R(A^d), N(A^{\otimes} A)} \).

Several characterizations of the weighted weak group inverse are presented now.

**Theorem 3** Let \( W \in B(Y, X) \backslash \{0\} \) and \( A \in B(X, Y)^{d,W} \). Then, for \( B \in B(X, Y) \), the following statements are equivalent:
(i) \( B \) is the \( W \)-weighted weak group inverse of \( A \);
(ii) \( B \) satisfies
\[ A^{\otimes, W} WAWB = B, \quad AWB = A^{\otimes, W} WA; \]
(iii) \( B \) satisfies
\[ BWAWB = B, \quad AWB = A^{\otimes, W} WA, \quad BWA^{\otimes, W} = A^{\otimes, W} WA^{\otimes, W}; \]
(iv) \( B \) satisfies
\[ BWAWB = B, \quad AWB = A^{\otimes, W} WA, \quad BWA^{d,W} = A^{d,W} W A^{d,W}. \]

**Proof**
(i) \( \Rightarrow \) (ii) and (iii). The equality \( B = (A^{\otimes, W} W)^2 A \) gives
\[ B = A^{\otimes, W} WAW(A^{\otimes, W} W)^2 A = A^{\otimes, W} WAWB \]
and
\[ BWA^{\otimes, W} = (A^{\otimes, W} W)^2 AW A^{\otimes, W} = A^{\otimes, W} W A^{\otimes, W}. \]
The rest is clear.
(ii) \( \Rightarrow \) (i). It follows by \( B = A^{\otimes, W} W (AWB) = A^{\otimes, W} WA^{\otimes, W} WA = A^{\otimes, W} \).
(iii) \( \Rightarrow \) (i). We have
\[ B = BW (AWB) = (BWA^{\otimes, W}) WA = A^{\otimes, W} W A^{\otimes, W} WA = A^{\otimes, W}. \]
(iii) ⇔ (iv). Using
\[ A^{\otimes,W} = A^{d,W} W A W A^{\oplus,W}, \quad A^{d,W} = A^{\oplus,W} W A W A^{d,W}, \]
we obtain these equivalences. □

Applying Theorem 3, we can characterize the weak group inverse in the following way.

**Corollary 4** Let \( A \in \mathcal{B}(X)^d \). Then, for \( B \in \mathcal{B}(X) \), the following statements are equivalent:

(i) \( B \) is the weak group inverse of \( A \);

(ii) \( B \) satisfies
\[ A^{\otimes} AB = B, \quad AB = A^{\otimes} A; \]

(iii) \( B \) satisfies
\[ BAB = B, \quad AB = A^{\otimes} A, \quad BA^{\otimes} = A^{\otimes} A^{\otimes}; \]

(iv) \( B \) satisfies
\[ BAB = B, \quad AB = A^{\otimes} A, \quad BA^{d} = (A^{d})^{2}. \]

In the case that \( X \) and \( Y \) are finite dimensional, the condition
\[ BW A^{d,W} = A^{d,W} W A^{d,W} \quad \text{(resp.,} \quad BA^{d} = (A^{d})^{2} \text{)} \]
of Theorem 3 (iv) (resp., Corollary 4 (iv)) can be replaced with the equivalent condition
\[ BW (AW)^{k+1} = (AW)^{k} \quad (k = \text{ind}(AW)) \]
\[ \text{(resp.,} \quad BA^{k+1} = A^{k} \quad (k = \text{ind}(A)). \]

Using idempotents and orthogonal projectors, we present some representations of the weighted weak group inverse of \( A \) in the next theorem.

**Theorem 4** Let \( W \in \mathcal{B}(Y, X) \setminus \{0\} \) and \( A \in \mathcal{B}(X,Y)^{d,W} \). Then the following statements holds:

(i) \( A^{\otimes,W} = A^{\oplus,W} P_{R(W A^{d,W},N(A^{\oplus,W} W A))}; \)

(ii) \( A^{\otimes,W} = A^{d,W} P_{R(W A^{d,W},N(A^{\oplus,W} W A))}; \)

(iii) \( WP_{R((AW)^d)} \) is Moore–Penrose invertible, \( WA(WA)^{\otimes} W A \) is group invertible, and
\[ A^{\otimes,W} = (WP_{R((AW)^d)})^\dagger (WA(WA)^{\otimes} W A)^\#; \]
\[ = (WP_{R((AW)^d)})^\dagger (P_{R((AW)^d)} W A)^\#; \]

(iv) \( A^{\otimes,W} = [(AW)^{2}]^{\otimes} A W A^{\otimes,W} W A = [(AW)^{2}]^{\otimes} A P_{R(W A^{d,W},N(A^{\oplus,W} W A)); \} \)
(v) $WP_{R((AW)^d)}$ is Moore–Penrose invertible and
\[ A^{\otimes,W} = [(AW)^d]^2 (WP_{R((AW)^d)})^\dagger A; \]

(vi) $WP_{R((AW)^d)}$ is Moore–Penrose invertible and
\[ A^{\otimes,W} = (WP_{R((AW)^d)})^\dagger [(WA)^2]^{\otimes} WA; \]

(vii) $(AW)^3(AW)^d$ is group invertible and
\[ A^{\otimes,W} = [(AW)^3(AW)^d]^{\otimes} AW A^{\otimes,W} WA. \]

Proof (i) It follows by
\[ A^{\otimes,W} = (A^{\otimes,W} W)^2 A = A^{\otimes,W} P_{R(W,A,W),N(A^{\otimes,W} WA)}. \]

(ii) We get
\[ A^{\otimes,W} = (A^{\otimes,W} W)^2 A = A^{d,W} WAW (A^{\otimes,W} W)^2 A = A^{d,W} WAW^{\otimes,W} WA = A^{d,W} P_{R(W,A,W),N(A^{\otimes,W} WA)}. \]

(iii) Let $A$ and $W$ be represented as in (1) and (2), respectively. Notice that the orthogonal projector $P_{R((AW)^d)}$ has the following representation:
\[ P_{R((AW)^d)} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}: \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix} \rightarrow \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix}. \]

We observe that
\[ WP_{R((AW)^d)} = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} \]

is Moore–Penrose invertible and
\[ (WP_{R((AW)^d)})^\dagger = \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \]

Since $A$ is Wg-Drazin invertible, $WA$ is generalized Drazin invertible, and
\[ WA = \begin{bmatrix} W_1 A_1 & W_1 A_2 + W_2 A_3 \\ 0 & W_3 A_3 \end{bmatrix}, \quad (WA)^{\otimes} = \begin{bmatrix} (W_1 A_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \]

We now have
\[ WA(WA)^{\otimes} WA = \begin{bmatrix} W_1 A_1 & W_1 A_2 + W_2 A_3 \\ 0 & 0 \end{bmatrix}. \]
is group invertible, and
\[
(WA(WA)^\oplus WA)\# = \begin{bmatrix}
(W_1A_1)^{-1} & (W_1A_1)^{-2}(W_1A_2 + W_2A_3) \\
0 & 0
\end{bmatrix}.
\]

Therefore,
\[
(WP_{R(AW)^d})^\dagger (WA(WA)^\oplus WA)\# = \begin{bmatrix}
(W_1A_1W_1)^{-1} & (A_1W_1)^{-2}(A_2 + W_1^{-1}W_2A_3) \\
0 & 0
\end{bmatrix}
= A^{\oplus,W}.
\]

(iv) Since \(A\) is \(W_g\)-Drazin invertible, \(AW\) is generalized Drazin invertible and so \((AW)^2\) is generalized Drazin invertible too. Hence, \([(AW)^2]^{\oplus}\) exists and, using (1) and (2), we have
\[
[(AW)^2]^{\oplus} = \begin{bmatrix}
(A_1W_1)^{-2} & 0 \\
0 & 0
\end{bmatrix}.
\]

By (4), we obtain
\[
[(AW)^2]^{\oplus}AWA^{\oplus,W}WA = \begin{bmatrix}
(A_1W_1)^{-2} & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
A_1 & A_2 + W_1^{-1}W_2A_3 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
(W_1A_1W_1)^{-1} & (A_1W_1)^{-2}(A_2 + W_1^{-1}W_2A_3) \\
0 & 0
\end{bmatrix}
= A^{\oplus,W}.
\]

Similarly, we verify parts (v) and (vi).

We have new representations for the weak group inverse by Theorem 4.

**Corollary 5** Let \(A \in \mathcal{B}(X)^d\). Then the following statements hold:

(i) \(A^{\oplus} = A^{\oplus}P_{R(A^d),N(A^{\oplus})}\);
(ii) \(A^{\oplus} = A^dP_{R(A^d),N(A^{\oplus})}\);
(iii) \(AA^{\oplus}A\) is group invertible and \(A^{\oplus} = (AA^{\oplus}A)^\# = (P_{R(A^d)}A)^\#\);
(iv) \(A^{\oplus} = (A^2)^{\oplus}AA^{\oplus}A = (A^2)^{\oplus}AP_{R(A^d),N(A^{\oplus})}\);
(v) \(A^{\oplus} = (A^d)^2P_{R(A^d)}A\);
(vi) \(A^{\oplus} = (A^2)^{\oplus}A\);
(vii) \(A^3A^d\) is group invertible and \(A^{\oplus} = (A^3A^d)^{\oplus}AA^{\oplus}A = (A^3A^d)^{\oplus}A\).

**Proof** Because \(P_{R(A^d)}\) is the orthogonal projector, notice that \(P_{R(A^d)}\) is Moore–Penrose invertible and \((P_{R(A^d)})^\dagger = P_{R(A^d)}\). Then we easily verify the result. \(\square\)

Recall that \(A \in \mathcal{B}(X,Y)^{d,W}\) with \(A^{d,W} = B \in \mathcal{B}(X,Y)\) if and only if \(AW \in \mathcal{B}(Y)^d\) with \((AW)^d = BW\), if and only if \(WA \in \mathcal{B}(X)^d\) with \((WA)^d =
WB [4]. For the W-weighted core-EP inverse, the situation is similar in the case of the operator \( WA \), but it is a little different for \( AW \) [14]. We see now that the W-weighted weak group inverse acts as the W-weighted core-EP inverse.

**Theorem 5** Let \( W \in \mathcal{B}(Y, X) \setminus \{0\} \) and let \( A \in \mathcal{B}(X, Y) \).

(a) The following statements are equivalent:
   (i) \( A \) is W-weighted weak group invertible with \( A^{\otimes, W} = B \);
   (ii) \( WA \) is weak group invertible with \( (WA)^{\otimes} = WB \).

In addition,
\[
A^{\otimes, W} = A[(WA)^{\otimes}]^2.
\]

(b) If \( A \) is Wg-Drazin invertible, \( A \) and \( W \) are represented by (1) and (2), respectively, then
   (i) \( (AW)^{\otimes} = A^{\otimes, W}W \) if and only if \( W_2A_3W_3 = 0 \);
   (ii) \( A^{\otimes, W} = [(AW)^{\otimes}]^2A \) if and only if \( A_2W_3A_3 = 0 \).

**Proof** (a) (i) \( \Rightarrow \) (ii). By [14, Theorem 2.4], \( WA^{\otimes, W} = (WA)^{\otimes} \), which implies
\[
WA^{\otimes, W} = W(A^{\otimes, W}W)^2A = [(WA)^{\otimes}]^2WA = (WA)^{\otimes}.
\]

(ii) \( \Rightarrow \) (i). We observe that
\[
A[(WA)^{\otimes}]^2 = A[(WA)^{\otimes}]^2WA[(WA)^{\otimes}]^2WA
= A[(WA)^{\otimes}]^3WA
= A(WA^{\otimes, W})^3WA
= (A^{\otimes, W}W)^2A
= A^{\otimes, W}.
\]

(b) (i) It follows from the equalities
\[
(WA)^{\otimes} = \begin{bmatrix}
(A_1W_1)^{-1} & (A_1W_1)^{-2}(A_1W_2 + A_2W_3) \\
0 & 0
\end{bmatrix}
\]
and
\[
A^{\otimes, W}W = \begin{bmatrix}
(A_1W_1)^{-1} & (A_1W_1)^{-2}(A_1W_2 + A_2 + W_1^{-1}W_2A_3)W_3 \\
0 & 0
\end{bmatrix}.
\]

(ii) Using (6) and
\[
[(AW)^{\otimes}]^2A = \begin{bmatrix}
(W_1A_1W_1)^{-1} & (A_1W_1)^{-2}(A_2 + W_1^{-1}W_2A_3 + (A_1W_1)^{-1}A_2W_3A_3) \\
0 & 0
\end{bmatrix},
\]
we get that (ii) holds. \( \square \)
Using the weak group inverses of $AW$ and $WA$, we obtain the following formula for the W-weighted weak group inverses of $A$.

**Theorem 6** Let $W \in \mathcal{B}(Y, X) \setminus \{0\}$ and $A \in \mathcal{B}(X, Y)^{d,W}$. Then

$$A^{\otimes,W} = (AW)^{\otimes}A(WA)^{\otimes}.$$  

**Proof** Let $A$ and $W$ be given by (1) and (2), respectively. Then

$$(AW)^{\otimes}A(WA)^{\otimes} = \begin{bmatrix}
W_1^{-1} (A_1W_1)^{-1}A_2 + (A_1W_1)^{-2}(A_1W_2 + A_2W_3)A_3 \\
0
\end{bmatrix}
\times \begin{bmatrix}
(W_1A_1)^{-1} (W_1A_1)^{-2}(W_1A_2 + W_2A_3) \\
0
\end{bmatrix}
= A^{\otimes,W}. \quad \Box$$

We also investigate the necessary and sufficient conditions for $AWA^{\otimes,W} = A^{\otimes,W}WA$ holding.

**Theorem 7** Let $W \in \mathcal{B}(Y, X) \setminus \{0\}$ and $A \in \mathcal{B}(X, Y)^{d,W}$. If $A$ and $W$ are represented by (1) and (2), respectively, then the following statements are equivalent:

(i) $AWA^{\otimes,W} = A^{\otimes,W}WA$;
(ii) $(W_1A_2 + W_2A_3)W_3A_3 = 0$;
(iii) $[(WA)^{\otimes}]^2 = [(WA)^2]^{\otimes}$.

In this case, $A^{\otimes,W} = A^{d,W}$.

**Proof** (i) $\Leftrightarrow$ (ii). The equalities

$$AWA^{\otimes,W} = \begin{bmatrix}
W_1^{-1} (A_1W_1)^{-1}A_2 + (W_1A_1W_1)^{-1}W_2A_3 \\
0
\end{bmatrix}$$

and, for $E = (A_2 + W_1^{-1}W_2A_3)W_3A_3$,

$$A^{\otimes,W}WA = \begin{bmatrix}
W_1^{-1} (A_1W_1)^{-1}A_2 + (W_1A_1W_1)^{-1}W_2A_3 + (A_1W_1)^{-2}E \\
0
\end{bmatrix}$$

give $AWA^{\otimes,W} = A^{\otimes,W}WA$ if and only if

$$(A_2 + W_1^{-1}W_2A_3)W_3A_3 = 0,$$

which is equivalent to

$$(W_1A_2 + W_2A_3)W_3A_3 = 0.$$

(ii) $\Leftrightarrow$ (iii). First, we observe that

$$[(WA)^{\otimes}]^2 = \begin{bmatrix}
(W_1A_1)^{-2} (W_1A_1)^{-3}(W_1A_2 + W_2A_3) \\
0
\end{bmatrix}.$$
Furthermore,
\[(WA)^2 = \begin{bmatrix} (W_1A_1)^2 & W_1A_1(W_1A_2 + W_2A_3) + (W_1A_2 + W_2A_3)W_3A_3 \\ 0 & (W_3A_3)^2 \end{bmatrix} \]
is generalized Drazin invertible and, by Corollary 1,
\[\left[(WA)^2\right]^\circ = \begin{bmatrix} (W_1A_1)^{-2} & (W_1A_1)^{-3}(W_1A_2 + W_2A_3) + (W_1A_1)^{-4}(W_1A_2 + W_2A_3)W_3A_3 \\ 0 & 0 \end{bmatrix}.\]
Thus, \[\left[(WA)^2\right]^\circ = \left[(WA)^2\right]^\circ\] is equivalent to \((W_1A_2 + W_2A_3)W_3A_3 = 0\).

By \((W_1A_2 + W_2A_3)W_3A_3 = 0\) and (3), we obtain \(A^{d,W} = A^\circ \backslash W\).

Applying Theorem 7 and Theorem 5 (b) (ii), we get the equivalent

\[\text{conditions for } AA^\circ = A^\circ A \text{ to be satisfied. In the following corollary, we observe that conditions (i)–(iii) appeared for the weak group inverse of a square matrix, but condition (iv) is new even for matrix case.} \]

**Corollary 6** Let \(A \in \mathcal{B}(X)^d\). If \(A\) is represented by (9), then the following statements are equivalent:

(i) \(AA^\circ = A^\circ A\);

(ii) \(A_2A_3 = 0\);

(iii) \((A^\circ)^2 = (A^2)^\circ\);

(iv) \(A^\circ = (A^\circ)^2A\).

In this case, \(A^\circ = A^{d}\).

Similarly, as Theorem 7, we verify the next result.

**Theorem 8** Let \(W \in \mathcal{B}(Y,X)\{0\}\) and \(A \in \mathcal{B}(X,Y)^{d,W}\). If \(A\) and \(W\) are represented by (1) and (2), respectively, then the following statements are equivalent:

(i) \([(AW)^\circ]^2 = [(AW)^2]^\circ\);

(ii) \((A_1W_2 + A_2W_3)A_3W_3 = 0\).

### 3 Weighted weak group relations

Various types of partial orders and pre-orders were defined based on various types of generalized inverses \([11,13]\). We first introduce a new binary relation using the weak group inverse.

**Definition 3** Let \(B \in \mathcal{B}(X)\) and \(A \in \mathcal{B}(X)^d\). Then we say that \(A\) is below \(B\) under the weak group relation (denoted by \(A \leq^\circ B\)) if

\[AA^\circ = BA^\circ, \quad A^\circ A = A^\circ B.\]
Remark that the relation ‘$\leq\otimes$’ is not a partial order, because it is not anti-symmetric. Indeed, if $A, B \in \mathcal{B}(X)$ are quasinilpotent and $A \neq B$, then $A^d = 0$ and $B^d = 0$ imply $A^{\otimes} = 0$ and $B^{\otimes} = 0$. Hence,

$$A \leq^{\otimes} B, \quad B \leq^{\otimes} A,$$

but $A \neq B$.

By the following example, we see that the relation ‘$\leq^{\otimes}$’ is not transitive and so it is not a pre-order.

**Example 1** Consider $3 \times 3$ block matrices

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.
\]

Then

$$A^{\otimes} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B^{\otimes} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The equalities

$$AA^{\otimes} = BA^{\otimes}, \quad A^{\otimes}A = A^{\otimes}B, \quad BB^{\otimes} = CB^{\otimes}, \quad B^{\otimes}B = B^{\otimes}C,$$

give

$$A \leq^{\otimes} B, \quad B \leq^{\otimes} C.$$

Because $A^{\otimes}A \neq A^{\otimes}C$, the relation $A \leq^{\otimes} C$ is not satisfied and thus ‘$\leq^{\otimes}$’ is not transitive.

**Lemma 3** Let $B \in \mathcal{B}(X)$ and $A \in \mathcal{B}(X)^d$. Then

(i)

$$AA^{\otimes} = BA^{\otimes} \iff A^{\otimes}A = B(A^{\otimes})^2A$$

$$\iff A^{\otimes} = B(A^{\otimes})^2$$

$$\iff A^d = BA^{\otimes}A^d$$

$$\iff AA^d = BA^d,$$

(ii) $A^{\otimes}A = A^{\otimes}B \iff A^{\otimes}A^2 = A^{\otimes}AB$.

We can get characterizations of the weak group relation combining conditions of Lemma 3 (i) and (ii).

For operators between two Hilbert spaces, we consider a weighted operator and define the following binary relations.

**Definition 4** Let $A, B \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)\setminus\{0\}$. If $A$ is Wg-Drazin invertible, then we say that

(i) $A \leq^{\otimes, W, r} B$ if $AW \leq^{\otimes} BW$, 


(i) \( A \leq_{\otimes, W, l} B \) if \( WA \leq_{\otimes} WB \),

(ii) \( A \leq_{\otimes, W} B \) if \( A \leq_{\otimes, W, r} B \) and \( A \leq_{\otimes, W, l} B \),

where \( \leq_{\otimes} \) is adequately considered on \( \mathcal{B}(X) \) or \( \mathcal{B}(Y) \).

Several characterizations of the relation \( \leq_{\otimes, W, r} \) are presented now.

**Theorem 9** Let \( W \in \mathcal{B}(Y, X) \setminus \{0\} \), \( A \in \mathcal{B}(X, Y)^{d, W} \), and \( B \in \mathcal{B}(X, Y) \). Then the following statements are equivalent:

(i) \( A \leq_{\otimes, W, r} B \);

(ii) \( AW(AW)^{\otimes} = BW(AW)^{\otimes} \) and \( (AW)^{\otimes} AW = (AW)^{\otimes} BW \);

(iii) the following matrix representations with respect to the orthogonal sums

\[
X = R((WA)^{d}) \oplus N[(WA)^{d}]^{*} \quad \text{and} \quad Y = R((AW)^{d}) \oplus N[(AW)^{d}]^{*}
\]

hold:

\[
A = \begin{bmatrix} A_{1} & A_{2} \\ 0 & A_{3} \end{bmatrix}, \quad W = \begin{bmatrix} W_{1} & W_{2} \\ 0 & W_{3} \end{bmatrix}, \quad B = \begin{bmatrix} A_{1} & B_{2} \\ 0 & B_{3} \end{bmatrix},
\]

where

\[
A_{1} \in \mathcal{B}(R((WA)^{d}), R((WA)^{d}))^{-1}, \quad W_{1} \in \mathcal{B}(R((WA)^{d}), R((WA)^{d}))^{-1},
\]

\[
(A_{2} - B_{2})W_{3} + (A_{1}W_{1})^{-1}(A_{1}W_{2} + A_{2}W_{3})(A_{3} - B_{3})W_{3} = 0,
\]

\[
A_{3}W_{3} \in \mathcal{B}(N[(AW)^{d}]^{*})^{\text{qnil}}, \quad W_{3}A_{3} \in \mathcal{B}(N[(WA)^{d}]^{*})^{\text{qnil}}.
\]

**Proof**

(i) \( \Leftrightarrow \) (ii). By the definition of the relation \( \leq_{\otimes, W, r} \), it is clear.

(ii) \( \Rightarrow \) (iii). Let \( A \) and \( W \) be given by (1) and (2), respectively. Suppose

\[
B = \begin{bmatrix} B_{1} & B_{2} \\ B_{4} & B_{3} \end{bmatrix}, \quad \begin{bmatrix} R((WA)^{d}) \\ N[(WA)^{d}]^{*} \end{bmatrix} \rightarrow \begin{bmatrix} R((AW)^{d}) \\ N[(AW)^{d}]^{*} \end{bmatrix}.
\]

Then

\[
BW = \begin{bmatrix} B_{1}W_{1} & B_{1}W_{2} + B_{2}W_{3} \\ B_{4}W_{1} & B_{4}W_{2} + B_{3}W_{3} \end{bmatrix},
\]

\[
AW(AW)^{\otimes} = \begin{bmatrix} I & (A_{1}W_{1})^{-1}(A_{1}W_{2} + A_{2}W_{3}) \\ 0 & 0 \end{bmatrix},
\]

and

\[
BW(AW)^{\otimes} = \begin{bmatrix} B_{1}W_{1}(A_{1}W_{1})^{-1} & B_{1}W_{1}(A_{1}W_{1})^{-2}(A_{1}W_{2} + A_{2}W_{3}) \\ B_{4}W_{1}(A_{1}W_{1})^{-1} & B_{4}W_{1}(A_{1}W_{1})^{-2}(A_{1}W_{2} + A_{2}W_{3}) \end{bmatrix}.
\]

Therefore, \( AW(AW)^{\otimes} = BW(AW)^{\otimes} \) is equivalent to \( B_{1} = A_{1} \) and \( B_{4} = 0 \). Furthermore, the equalities

\[
(AW)^{\otimes} AW
\]

\[
= \begin{bmatrix} I & (A_{1}W_{1})^{-1}(A_{1}W_{2} + A_{2}W_{3}) + (A_{1}W_{1})^{-2}(A_{1}W_{2} + A_{2}W_{3})A_{3}W_{3} \\ 0 & 0 \end{bmatrix},
\]
and $A$ where:

Then the following statements are equivalent:

Corollary 7

Let $X = (A_1 W_1)^{-1}(A_1 W_2 + B_2 W_3) + (A_1 W_1)^{-2}(A_1 W_2 + A_2 W_3)B_3 W_3$, and $(AW) \otimes AW = (AW) \otimes BW$ give $(A_2 - B_2)W_3 + (A_1 W_1)^{-1}(A_1 W_2 + A_2 W_3) \times (A_3 - B_3)W_3 = 0$.

(iii) $\Rightarrow$ (ii). This part can be checked by direct computations. □

In an analogy way, we characterize the relation $\leq \otimes, W, l$.

Theorem 10

Let $W \in \mathcal{B}(Y, X) \backslash \{0\}$, $A \in \mathcal{B}(X, Y)^{d, W}$, and $B \in \mathcal{B}(X, Y)$. Then the following statements are equivalent:

(i) $A \leq \otimes, W, l B$;

(ii) $WA(WA)^{\otimes} = WB(WA)^{\otimes}$ and $(WA)^{\otimes}WA = (WA)^{\otimes}WB$;

(iii) $WA W A^{\otimes, W} = W B W A^{\otimes, W}$ and $W A^{\otimes, W} W A = W A^{\otimes, W} W B$;

(iv) the following matrix representations with respect to the orthogonal sums $X$ and $Y$ as in (12) hold:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix}, \quad B = \begin{bmatrix} A_1 - W_1^{-1}W_2 B_4 & B_2 \\ B_4 & B_3 \end{bmatrix},$$

where $A_1$ and $W_1$ satisfy (14), $W_3 B_4 = 0$,

$$B_2 = A_2 + W_1^{-1}W_2(A_3 - B_3) + (W_1 A_1 W_1)^{-1}(W_1 A_2 + W_2 A_3) W_3 (A_3 - B_3),$$

and $A_3 W_3$ and $W_3 A_3$ satisfy (15).

Combining Theorems 9 and 10, we get the following results.

Corollary 7

Let $W \in \mathcal{B}(Y, X) \backslash \{0\}$, $A \in \mathcal{B}(X, Y)^{d, W}$, and $B \in \mathcal{B}(X, Y)$. Then the following statements are equivalent:

(i) $A \leq \otimes, W, l B$;

(ii) the matrix representations (13) with respect to the orthogonal sums $X$ and $Y$ as in (12) hold, where $A_1$ and $W_1$ satisfy (14), $B_2$ satisfies (16),

$$W_2 A_3 W_3 (A_3 - B_3) W_3 = 0,$$

and $A_3 W_3$ and $W_3 A_3$ satisfy (15).

Corollary 8

Let $A \in \mathcal{B}(X)^{d}$ and $B \in \mathcal{B}(X)$. Then the following statements are equivalent:

(i) $A \leq \otimes B$;

(ii) the following matrix representations with respect to the orthogonal sum $X = R(A^d) \oplus N[(A^d)^*]$ hold:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad B = \begin{bmatrix} A_1 & A_2 + A_1^{-1}A_2 (A_3 - B_3) \\ 0 & B_3 \end{bmatrix},$$
where $A_1 \in \mathcal{B}(R(A^d))$ is invertible and $A_3 \in \mathcal{B}(N[(A^d)^*])$ is quasinilpotent.

Acknowledgements  The authors are grateful to referees for careful reading of the paper. The first author was supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 174007 (451-03-68/2020-14/200124). The second author was supported by the National Natural Science Foundation of China (Grant Nos. 11901079, 61672149, 11601211) and the Scientific and Technological Research Program Foundation of Jilin Province, China (Grant Nos. JJKH20190690KJ, 20190201095JC, 20200401085GX.)

References

1. Baksalary O M, Trenkler G. Core inverse of matrices. Linear Multilinear Algebra, 2010, 58(6): 681–697
2. Campbell S L. Generalized Inverses of Linear Transformations. London: Pitman, 1979
3. Campbell S L. Singular Systems of Differential Equations II. Research Notes in Math, Vol 61. San Francisco: Pitman, 1982
4. Dajić A, Koliha J J. The weighted g-Drazin inverse for operators. J Aust Math Soc, 2007, 82: 163–181
5. Ferreyra D E, Levis F E, Thome N. Maximal classes of matrices determining generalized inverses. Appl Math Comput, 2018, 333: 42–52
6. Ferreyra D E, Levis F E, Thome N. Revisiting the core EP inverse and its extension to rectangular matrices. Quaest Math, 2018, 41(2): 265–281
7. Ferreyra D E, Orquera V, Thome N. A weak group inverse for rectangular matrices. Rev R Acad Cienc Exactas Fís Nat Ser A Math RACSAM, 2019, 113(4): 3727–3740
8. Gao Y, Chen J. Pseudo core inverses in rings with involution. Comm Algebra, 2018, 46(1): 38–50
9. Gao Y, Chen J, Patrício P. Representations and properties of the W-weighted core-EP inverse. Linear Multilinear Algebra, 2018, https:/doi.org/10.1080/03081087.2018.1535573
10. Koliha J J. A generalized Drazin inverse. Glasg Math J, 1996, 38: 367–381
11. Mitra S K, Bhimasankaram P, Malik S B. Matrix Partial Orders, Shorted Operators and Applications. Series in Algebra, Vol 10. Singapore: World Scientific, 2010
12. Mosić D. Core-EP pre-order of Hilbert space operators. Quaest Math, 2018, 41(5): 585–600
13. Mosić D. Generalized inverses. Niš: Faculty of Sciences and Mathematics, University of Niš, 2018
14. Mosić D. Weighted core-EP inverse of an operator between Hilbert spaces. Linear Multilinear Algebra, 2019, 67(2): 278–298
15. Mosić D, Djordjević D S. The gDMP inverse of Hilbert space operators. J Spectr Theory, 2018, 8(2): 555–573
16. Prasad K M, Mohana K S. Core-EP inverse. Linear Multilinear Algebra, 2014, 62(6): 792–802
17. Robles J, Martínez-Serrano M F, Dopazo E. On the generalized Drazin inverse in Banach algebras in terms of the generalized Schur complement. Appl Math Comput, 2016, 284: 162–168
18. Wang H. Core-EP decomposition and its applications. Linear Algebra Appl, 2016, 508: 289–300
19. Wang H, Chen J. Weak group inverse. Open Math, 2018, 16: 1218–1232
20. Wang X, Ma H, Stanimirović P S. Recurrent neural network for computing the W-weighted Drazin inverse. Appl Math Comput, 2017, 300: 1–20
21. Wang X, Yu A, Li T, Deng C. Reverse order laws for the Drazin inverses. J Math Anal Appl, 2016, 444(1): 672–689
22. Zhang D, Mosić D, Guo L. The Drazin inverse of the sum of four matrices and its applications. Linear Multilinear Algebra, 2020, 68(1): 133–151
23. Zhang D, Mosić D, Tam T. On the existence of group inverses of Peirce corner matrices. Linear Algebra Appl, 2019, 582: 482–498
24. Zhou M, Chen J, Li T, Wang D. Three limit representations of the core-EP inverse. Filomat, 2018, 32(17): 5887–5894