OPTIMALITY OF THE REARRANGEMENT INEQUALITY WITH APPLICATIONS TO LORENTZ-TYPE SEQUENCE SPACES

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Abstract. We characterize the sequences $(w_i)_{i=1}^\infty$ of non-negative numbers for which

$$\sum_{i=1}^\infty a_i w_i$$

is of the same order as

$$\sup_n \sum_{i=1}^n a_i w_{1+n-i}$$

when $(a_i)_{i=1}^\infty$ runs over all non-increasing sequences of non-negative numbers. As a by-product of our work we settle a problem raised in [1] and prove that Garling sequences spaces have no symmetric basis.

1. Introduction

The rearrangement inequality states that, for $n \in \mathbb{N}$, if $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ are a pair of non-increasing $n$-tuples of non-negative scalars then we have

$$\sum_{i=1}^n a_i b_{1+n-i} \leq \sum_{i=1}^n a_i b_{\sigma(i)} \leq \sum_{i=1}^n a_i b_i$$

for every permutation $\sigma$ of the set $\{1, \ldots, n\}$ (see [3, Theorem 368]). Consequently, if $(a_i)_{i=1}^\infty$ and $(w_i)_{i=1}^\infty$ are non-increasing sequences of non-negative scalars,

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^n a_i w_{1+n-i} \leq \sum_{i=1}^\infty a_i w_i.$$

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In this note we wonder about which are the non-increasing sequences \((w_i)_{i=1}^{\infty}\) of non-negative scalars that verify a reverse inequality, i.e., in which cases there is a constant \(C < \infty\) such that
\[
\sum_{i=1}^{\infty} a_i w_i \leq C \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} a_i w_{1+n-i} \tag{1.1}
\]
for every sequence \((a_i)_{i=1}^{\infty}\) of non-negative scalars. For the time being some simple answers can be given. Indeed, on the one hand, if \(w_\infty := \inf_i w_i > 0\) then
\[
\sum_{i=1}^{\infty} a_i w_i \leq w_1 \sum_{i=1}^{\infty} a_i = w_1 \sup_{n} \sum_{i=1}^{n} a_i \leq \frac{w_1}{w_\infty} \sup_{n} \sum_{i=1}^{n} a_i w_{1+n-i}.
\]
On the other hand, if we consider \(W := \sum_{i=1}^{\infty} w_i < \infty\) and let \(w_1 > 0\) (the case \(w_1 = 0\) is trivial) then
\[
\sum_{i=1}^{\infty} a_i w_i \leq a_1 \sum_{i=1}^{\infty} w_i = \frac{W}{w_1} a_1 w_1 \leq \frac{W}{w_1} \sup_{n} \sum_{i=1}^{n} a_i w_{1+n-i}.
\]
In fact, as we will show below, these two cases are the only ones for which (1.1) holds. This will be our main result as far as inequalities is concerned:

**Theorem 1.1 (Main Theorem).** Let \((w_i)_{i=1}^{\infty}\) be a non-increasing sequence consisting of non-negative scalars. The following are equivalent:

(i) There is a constant \(C < \infty\) such that
\[
\sum_{i=1}^{\infty} a_i w_i \leq C \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} a_i w_{1+n-i}
\]
for every sequence \((a_i)_{i=1}^{\infty}\) of non-negative scalars.

(ii) Either \(\sum_{i=1}^{\infty} w_i < \infty\) or \(\inf_{i \in \mathbb{N}} w_i > 0\).

Section 2 is devoted to proving Theorem 1.1. In Section 3 we use Theorem 1.1 to give some functional analytic properties of a recently introduced class of Lorentz-type spaces, called Garling sequence spaces. In particular, Theorem 1.1 is applied to show that Garling sequence spaces have no symmetric basis, answering thus a problem that was recently posed in [1].

Throughout this note we use standard terminology and notation in Banach space theory. As it is customary, we denote by \(\ell_q\), \(1 \leq q \leq \infty\), the Banach space consisting of all \(q\)-summable sequences (bounded sequences in the case \(q = \infty\)) and by \(c_0\) the subspace of \(\ell_\infty\) consisting of all sequences converging to zero. For background on bases in Banach spaces we refer the reader to [2].
2. Proof of the Main Theorem

Proof of Theorem 1.1. As explained in the Introduction, we must only prove that (i) implies (ii).

Assume that (ii) does not hold, that is, \( w = (w_i)_{i=1}^{\infty} \in c_0 \setminus \ell_1 \). Let us denote by \( \mathcal{D} \) the set of (nonzero) non-increasing sequences of non-negative integers. For \( f = (a_i)_{i=1}^{\infty} \in \mathcal{D} \) and \( n \in \mathbb{N} \) we put

\[
A(f, w) = \sum_{i=1}^{\infty} a_i w_i, \quad \text{and} \quad B(f, w, n) = \sup_{n \in \mathbb{N}} B(f, w, n),
\]

where, for \( n \in \mathbb{N} \),

\[
B(f, w, n) = \sum_{i=1}^{n} a_i w_1+ n-i.
\]

With this notation we must prove that

\[
S(w) := \sup_{f \in \mathcal{D}} \frac{A(f, w)}{B(f, w)} = \infty.
\]

We will use the convention that \( \sum_{i=1}^{0} c_i = 0 \) for all sequences of scalars \( (c_i)_{i=1}^{\infty} \).

For \( n \in \mathbb{N} \) put \( W(n) = \sum_{i=1}^{n} w_i \). Since \( w \notin \ell_1 \) we have

\[
\lim_{n \to \infty} W(n) = \infty.
\]

Moreover, since \( w \in c_0 \),

\[
\lim_{n \to \infty} (W(s+n) - W(n)) = 0
\]

for any non-negative integer \( s \). We use these properties to recursively construct an increasing sequence \( (d_k)_{k=0}^{\infty} \) of non-negative integers with \( d_0 = 0 \) verifying

\[
\begin{align*}
(\text{i}) & \quad W(\sum_{j=1}^{k-1} d_j) \leq 2^{-1} W(d_k), \quad \text{and} \\
(\text{ii}) & \quad W(d_{k-1} + d_k) - W(d_k) \leq 2^{1-k} W(d_{k-1})
\end{align*}
\]

for \( k = 1, 2, \ldots. \)

For every integer \( k \geq 0 \) put \( n_k = \sum_{j=1}^{k} d_j \). For each \( r \in \mathbb{N} \) we define a sequence \( f^{(r)} = (a_{i, r})_{i=1}^{\infty} \) by

\[
a_{i, r} = \begin{cases} 1/W(d_k) & \text{if, for some } 1 \leq k \leq r, \ n_{k-1} < i \leq n_k \\ 0 & \text{if } i > n_r. \end{cases}
\]
It is clear that $f^{(r)} \in \mathcal{D}$ for all $r \in \mathbb{N}$. Taking into account the inequality in (i) we obtain

$$A(f^{(r)}, w) = \sum_{k=1}^{r} \frac{1}{W(d_k)} \sum_{i=1+n_{k-1}}^{n_k} w_i$$

$$= \sum_{k=1}^{r} \frac{W(n_k) - W(n_{k-1})}{W(d_k)}$$

$$\geq \sum_{k=1}^{r} \frac{W(d_k) - 2^{-1}W(d_k)}{W(d_k)}$$

$$= \sum_{k=1}^{r} \frac{1}{2}$$

$$= \frac{r}{2}.$$

Let $n \in \mathbb{N}$. In case that $n > n_r$ we have

$$B(f^{(r)}, w, n) = \sum_{i=1}^{n_r} a_{i,r} w_{1+n-i} \leq \sum_{i=1}^{n_r} a_{i,r} w_{1+n_r-i} = B(f^{(r)}, w, n_r).$$

In case that $n \leq n_r$, pick $1 \leq k \leq r$ with $n_{k-1} < n \leq n_k$. We have

$$B(f^{(r)}, w, n) = \frac{W(n - n_{k-1})}{W(d_k)} + \sum_{j=1}^{k-1} \frac{W(n - n_{j-1}) - W(n - n_j)}{W(d_j)}$$

$$\leq \frac{W(n_k - n_{k-1})}{W(d_k)} + \sum_{j=1}^{k-1} \frac{W(n - n_{j-1}) - W(n - n_j)}{W(d_j)}$$

$$= 1 + \sum_{j=1}^{k-1} \frac{W(n - n_{j-1}) - W(n - n_j)}{W(d_j)}.$$

If $k = 1$ we get $B(f^{(r)}, w, n) \leq 1$. Assume that $k \geq 2$. Taking into account inequality (ii) and that, since $w$ is non-increasing, the sequence $(W(n + t) - W(n + s))_{n=1}^{\infty}$ is non-increasing for any $s \leq t$, we obtain

$$B(f^{(r)}, w, n) \leq 1 + \frac{W(n - n_{k-2}) - W(n - n_{k-1})}{W(d_{k-1})}$$

$$+ \sum_{j=1}^{k-2} \frac{W(n - n_{j-1}) - W(n - n_j)}{W(d_j)}$$

$$\leq 1 + \frac{W(n_{k-1} - n_{k-2}) - W(n_{k-1} - n_{k-1})}{W(d_{k-1})}.$$
Therefore \( B(f^{(r)}, w) \leq 3 \). Thus

\[
S(w) \geq \sup_{r \in \mathbb{N}} A(f^{(r)}, w) \geq \sup_{r \in \mathbb{N}} \frac{r}{6} = \infty,
\]

and the proof is over. □

3. Applications to Garling sequence spaces

Let \( 1 \leq p < \infty \) and let \( w = (w_n)_{n=1}^{\infty} \) be a non-increasing sequence of positive scalars. Given a sequence of (real or complex) scalars \( f = (b_k)_{k=1}^{\infty} \) we put

\[
\|f\|_{g(w, p)} = \sup_{\phi \in \mathcal{O}} \left( \sum_{i=1}^{\infty} |b_{\phi(i)}|^p w_i \right)^{1/p}
\]

where \( \mathcal{O} \) denotes the set of increasing functions from \( \mathbb{N} \) to \( \mathbb{N} \). The Garling sequence space \( g(w, p) \) is the Banach space consisting of all sequences \( f \) with \( \|f\|_{g(w, p)} < \infty \).

Notice that if in (3) we replace "\( \phi \in \mathcal{O} \)" with "\( \phi \) is a permutation of \( \mathbb{N} \)" we obtain the norm that defines the weighted Lorentz sequence space

\[
d(w, p) := \left\{ (b_k)_{k=1}^{\infty} \in c_0 : \left( \sum_{i=1}^{\infty} (b_i^*)^p w_i \right)^{1/p} < \infty \right\},
\]

where \( (b_i^*)_{i=1}^{\infty} \) denotes the decreasing rearrangement of \( (b_k)_{k=1}^{\infty} \). So, the Garling sequence space \( g(w, p) \) can be regarded as a variation of the weighted Lorentz sequence space \( d(w, p) \).

Imposing the further conditions \( w \in c_0 \) and \( w \notin \ell_1 \) will prevent us, respectively, from having \( g(w, p) = \ell_p \) or \( g(w, p) = \ell_\infty \). We will assume as well that \( w \) is normalized, i.e., \( w_1 = 1 \). Thus, we put

\[
W := \{ (w_i)_{i=1}^{\infty} \in c_0 \setminus \ell_1 : 1 = w_1 \geq w_2 \geq \cdots \geq w_i \geq w_{i+1} \geq \cdots > 0 \}\]
and we restrict our attention to weights $w \in W$.

For $n \in \mathbb{N}$, we will denote $e_n = (\delta_{i,n})_{i=1}^{\infty}$, where $\delta_{i,n} = 1$ if $n = i$ and $\delta_{i,n} = 0$ otherwise. We have (see [1, Theorem 3.1]) that the canonical sequence $(e_n)_{n=1}^{\infty}$ is a Schauder basis for $g(w,p)$. A question posed and partially solved in [1] is to determine the weights $w \in W$ and the indices $p \in [1, \infty)$ for which $(e_n)_{n=1}^{\infty}$ is a symmetric basis of $g(w,p)$.

Here we provide a complete intrinsic solution to this problem, in the sense that our approach is entirely based on Theorem 1.1.

**Lemma 3.1.** The canonical sequence $(e_n)_{n=1}^{\infty}$ is not a symmetric basis for $g(w,p)$ for any $w \in W$ and any $1 \leq p < \infty$.

**Proof.** Assume that $(e_n)_{n=1}^{\infty}$ is a symmetric basis for $g(w,p)$. Then, there is a constant $C$ so that

$$\|g\|_{g(w,p)} \leq C\|f\|_{g(w,p)}$$

whenever the sequence $g$ is a permutation of the sequence $f$.

Given $r \in \mathbb{N}$ and $\phi \in \mathcal{O}$ let $n(r, \phi)$ be the largest integer $n$ such that $\phi(n) \leq r$. We have $\phi(i) \leq i + r - n(r, \phi)$ for $1 \leq i \leq n(r, \phi)$. Given a non-increasing sequence $(a_i)_{i=1}^{\infty}$ of non-negative numbers we have

$$\sum_{i=1}^{\infty} a_i w_i = \sup_r \sum_{i=1}^{r} a_i w_i$$

$$\leq \sup_{r \in \mathbb{N}} \left\| \sum_{i=1}^{r} a_i^{1/p} e_i \right\|_{g(w,p)}$$

$$\leq C \sup_{r \in \mathbb{N}} \left\| \sum_{i=1}^{r} a_i^{1/p} e_i \right\|_{g(w,p)}$$

$$= C \sup_{r \in \mathbb{N}} \sum_{i=1}^{n(r, \phi)} a_{1+r-\phi(i)} w_i$$

$$\leq C \sup_{r \in \mathbb{N}, \phi \in \mathcal{O}} \sum_{i=1}^{n(r, \phi)} a_{1+n(r, \phi)-i} w_i$$

$$= C \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} a_{1+n-i} w_i$$

$$= C \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} a_i w_{1+n-i}.$$ 

Theorem 1.1 yields the absurdity $w \in \ell_1$ or $w \notin c_0$.  \qed
Now we are ready to establish the advertised structural properties of Garling sequence spaces.

**Theorem 3.2.** Let \( w \in \mathcal{W} \) and \( 1 \leq p < \infty \).

(i) There is no symmetric basis for \( g(w, p) \).

(ii) \( d(w, p) \subsetneq g(w, p) \).

(iii) No subspace of \( d(w, p) \) is isomorphic to \( g(w, p) \).

(iv) Let \( I_{d,g} : d(w, p) \to g(w, p) \) be the natural inclusion map, and let \( T : g(w, p) \to d(w, p) \) be a bounded linear operator. Then (despite the fact that \( I_{d,g} \) is not a strictly singular operator) \( T \circ I_{d,g} \) does not preserve a copy of \( d(w, p) \), i.e., if \( X \) is a subspace of \( d(w, p) \) isomorphic to \( d(w, p) \) then \( T \circ I_{d,g}|_X \) is not an isomorphism.

**Proof.** It follows using Lemma 3.1 in combination with [1, Theorem 5.1].

**References**

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