I dedicate this paper to my wife Inna and my daughter Sabina-Stefany.

ON TITS BUILDINGS OF TYPE $A_n$

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Abstract. Let $P$ and $P'$ be projective spaces having the same dimension, this dimension is denoted by $n$ assumed to be finite. Denote by $\mathcal{F}$ and $\mathcal{F}'$ the sets of maximal flags of the spaces $P$ and $P'$, respectively. A subset of $\mathcal{F}$ ($\mathcal{F}'$) is said to be an apartment if it is the intersection of $\mathcal{F}$ ($\mathcal{F}'$) and an apartment of the $A_n$-building associated with the projective space $P$ ($P'$). We show that any mapping $f : \mathcal{F} \rightarrow \mathcal{F}'$ sending apartments to apartments is induced by a strong embedding of $P$ to $P'$ or to the dual space $P^{**}$. Moreover this embedding is a collineation if $f$ is surjective.

1. Introduction

According to J. Tits [25], a building is a chamber complex together with a distinguished family of subcomplexes called apartments and satisfying certain collection of axioms. In the present paper we will consider so-called $A_n$-buildings (buildings of type $A_n$); these buildings are associated with projective spaces. We study mappings of the chamber sets of $A_n$-buildings which send apartments to apartments. The main result of the paper (Theorem 3.1) says that these mappings are induced by strong embeddings of the corresponding projective spaces.

1.1. Chamber complexes. Let $X$ be a set and $\Delta$ be a set of proper subsets of $X$. We say that $\Delta$ is a complex over $X$ if the following conditions hold true:

- (C1) $\Delta$ contains each one-element subset of $X$,
- (C2) if $A \in \Delta$ then any proper subset of $A$ belongs to $\Delta$.

For this case elements of $\Delta$ are known as simplexes, one-element simplexes (elements of the set $X$) are said to be vertices of the complex $\Delta$.

Recall that the rank of a complex is the maximum of the cardinal numbers of simplexes belonging to this complex. In this paper we will always require that the rank of a complex is finite.

Let $\Delta$ be a complex over a set $X$, let also $X'$ be a subset of $X$ and $\Delta'$ be a complex over $X'$. If $\Delta'$ is contained in $\Delta$ then we say that $\Delta'$ is a subcomplex of the complex $\Delta$.

Two complexes $\Delta$ and $\Delta'$ over sets $X$ and $X'$ (respectively) are said to be isomorphic if there exists a bijection $f : X \rightarrow X'$ such that $f(\Delta) = \Delta'$, this bijection is called an isomorphism.

Example 1.1 (Flag complex). Let $(X, \leq)$ be a partially ordered set. A set $F \subseteq X$ is said to be a flag if for any two elements $x, y \in F$ we have $x \leq y$ or $y \leq x$. The set of all flags is a complex over $X$, it is known as the flag complex of the partially ordered set $(X, \leq)$.

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Now consider a complex $\Delta$ which rank is finite and denoted by $n$. Then each simplex is contained in a maximal simplex; maximal simplexes are called chambers. We will also require that our complex satisfies the following condition: any two chambers have the same cardinal number (clearly, this number is equal to $n$). Simplexes spanned by $n - 1$ vertices are known as panels. Two chambers are said to be adjacent if their intersection is a panel. We say that $\Delta$ is a chamber complex if for any two chambers $C$ and $C'$ there is a sequence of chambers
\[ C = C_0, C_1, \ldots, C_m = C' \]
such that $C_{i-1}$ and $C_i$ are adjacent for each number $i = 1, \ldots, m$.

A chamber complex is called thick if each panel is contained in at least three chambers. For the case when any panel is contained in exact two chambers we say that our complex is thin.

**Example 1.2** ($A_n$-complex). Set $I_n := \{1, \ldots, n+1\}$ and consider the set of all proper subsets of $I_n$ ordered by the relation $\subset$. The corresponding flag complex is denoted by $A_n$. It is a thin chamber complex containing $(n+1)!$ distinct chambers. The rank of this complex is equal to $n$.

1.2. **Buildings.** Let $\Delta$ be a complex together with a family of subcomplexes $\aleph$ called apartments and satisfying the following axioms:

(TB1) All apartments are isomorphic thin chamber complexes.

(TB2) For any two simplexes there exists an apartment containing them.

Then $\Delta$ is a chamber complex which rank is equal to the rank of the apartments. The pair $(\Delta, 8)$ is said to be a building [25] if the chamber complex $\Delta$ is thick and the following axiom holds true:

(TB3) Let $\Sigma$ and $\Sigma'$ be apartments; let also $S$ and $U$ be simplexes belonging to both $\Sigma$ and $\Sigma'$. Then there exists an isomorphism of the complex $\Sigma$ to the complex $\Sigma'$ which leaves fixed all simplexes contained in $S$ and $U$.

All buildings of finite rank $\geq 3$ with finite apartments were completely determined by J. Tits [26]. A lot of information related with buildings and their applications can be found in [7], see also [9] and [24].

In this paper we will consider only $A_n$-buildings, $n \geq 2$. By the definition, a building $(\Delta, 8)$ has type $A_n$ if its apartments are isomorphic to the complex $A_n$ (Example 1.2). It is well-known that for this case $\Delta$ is the flag complex of certain $n$-dimensional projective space [25].

1.3. **Main result.** Let $(\Delta, 8)$ and $(\Delta', 8')$ be buildings of type $A_n$. Let also $P$ and $P'$ be the associated projective spaces (i.e. $\Delta$ and $\Delta'$ are the flag complexes of these spaces). Put $\mathcal{F}$ and $\mathcal{F}'$ for the sets of all chambers of the complexes $\Delta$ and $\Delta'$, respectively (it is trivial that $\mathcal{F}$ and $\mathcal{F}'$ consist of all maximal flags of $P$ and $P'$). We consider mappings of $\mathcal{F}$ to $\mathcal{F}'$ which send apartments to apartments (a subset of $\mathcal{F}$ or $\mathcal{F}'$ is said to be an apartment if it consists of all chambers belonging to certain apartment of the corresponding building) and show that these mappings are induced by strong embeddings of $P$ to $P'$ or to the dual projective space $P'^*$ (Theorem 3.1).

**Remark 1.1.** Adjacency preserving bijections of $\mathcal{F}$ to itself were studied by H. Havlicek, K. List and C. Zanella [14]. The background for the investigation of adjacency preserving mappings can be found in [3] and [20].
2. Preliminaries: Basic properties of projective spaces

2.1. Projective spaces. A projective space is a set of points \( P \) together with a family of proper subsets called lines and satisfying certain collection of axioms (see, for example, [29]). One of these axioms says that for any two distinct points \( p, q \in P \) there is unique line containing them, we will denote this line by \( pq \).

A set \( S \subset P \) is said to be a subspace if for any two distinct points \( p \) and \( q \) belonging to \( S \) the line \( pq \) is contained in \( S \). By this definition, the empty set and one-point sets are subspaces (since they do not contain two distinct points). The intersection of any collection of subspaces is a subspace. For any subset \( X \subset P \) the minimal subspace containing \( X \) (the intersection of all subspaces containing \( X \)) is called spanned by \( X \) and denoted by \( \langle X \rangle \).

We say that a subspace is \( n \)-dimensional if \( n + 1 \) is the smallest number of points spanning this subspace; a subspace is infinity-dimensional if it is not spanned by a finite set of points. The empty set is unique \((-1)\)-dimensional subspace, points and lines are 0-dimensional and 1-dimensional subspaces, respectively. The dimensions of other subspaces are greater than 1. Note also that the dimension of a projective space is not less than 2 (otherwise, there is only one line which coincides with the space).

A set \( X \subset P \) is said to be independent if the subspace \( \langle X \rangle \) is not spanned by a proper subset of \( X \). An independent set \( B \) contained in a subspace \( S \) is called a base for \( S \) if \( B = S \); for the case when \( S \) coincides with \( P \) we will say that \( B \) is a base for our projective space.

It is well-known that for any subspace \( S \subset P \) the following statements hold true:

1. If \( S \) is \( n \)-dimensional then any base for \( S \) consists of \( n + 1 \) points.
2. Each base for \( S \) can be extended to a base for any subspace containing \( S \).

By (2), for any two subspaces \( S \) and \( U \) the inclusion \( S \subset U \) implies that the dimension of \( S \) is not greater than the dimension of \( U \) and these subspaces have the same dimension only for the case when \( S = U \).

Example 2.1. Let \( P \) be an \( n \)-dimensional projective space, \( n \geq 3 \) and \( \mathcal{L} \) be the set of lines of this space. Let also \( p \in P \). Denote by \( \mathcal{L}_p \) the set of all lines containing the point \( p \). A subset of \( \mathcal{L}_p \) is said to be a line if it consists of all lines \( L \in \mathcal{L}_p \) contained in certain plane (2-dimensional subspace) of \( P \). The set \( \mathcal{L}_p \) together with the family of lines defined above is an \( (n - 1) \)-dimensional projective space. A subset of \( \mathcal{L}_p \) is a \( k \)-dimensional subspace if and only it consists of all lines \( L \in \mathcal{L}_p \) contained in certain \((k + 1)\)-dimensional subspace of \( P \).

Example 2.2. Let \( V \) be a left vector space over a division ring. Denote by \( P(V) \) the set of all 1-dimensional subspaces of \( V \). A subset of \( P(V) \) is a line if it consists of all 1-dimensional subspaces contained in certain 2-dimensional subspace of \( V \). The set \( P(V) \) together with the family of lines defined above is a projective space if the dimension of \( V \) is not less than 3. This projective space is \( n \)-dimensional if the dimension of \( V \) is equal to \( n + 1 \).

Let \( P \) and \( P' \) be projective spaces, let also \( \mathcal{L} \) and \( \mathcal{L}' \) be the sets of lines. These spaces are called isomorphic if there exists a bijection \( f : P \to P' \) such that

\[ f(\mathcal{L}) = \mathcal{L}' ; \]

this bijection is said to be a collineation.
Remark 2.1. Projective spaces isomorphic to the projective spaces associated with vector spaces are known as Desarguesian. Any projective space the dimension of which is not less than 3 is Desarguesian (see [2] or [29]). However, non-Desarguesian planes (2-dimensional projective spaces) exist.

2.2. Dual space. Let $P$ be a projective space. The dimension of this space is assumed to be finite and denoted by $n$. Put $P^*$ for the set of all $(n-1)$-dimensional subspaces of $P$. We say that a subset of $P^*$ is a line if it consists of all $(n-1)$-dimensional subspaces containing certain $(n-2)$-dimensional subspace of $P$. The set $P^*$ and the family of all lines defined above form an $n$-dimensional projective space; it is called dual to the projective space $P$. We will exploit the following well-known properties of the dual space:

1. A subset of $P^*$ is a $k$-dimensional subspace if and only if it consists of all $(n-1)$-dimensional subspaces containing certain $(n-k-1)$-dimensional subspace of $P$. Hence there is a one-to-one correspondence between the set of all $k$-dimensional subspaces of $P^*$ and the set of all $(n-k-1)$-dimensional subspaces of $P$. In particular, the second dual space $P^{**}$ coincides with $P$.

2. Points $p_1^*, \ldots, p_{n+1}^* \in P^*$ form a base for the dual space if and only if there is a base for $P$ such that each $p_i^*$ is spanned by points of this base.

Example 2.3. Let $n \geq 3$ and $p \in P$. The set $\mathcal{L}_p$ (consisting of all lines of $P$ passing through the point $p$) has the natural structure of an $(n-1)$-dimensional projective space (Example 2.1). The dual projective space consists of all $p^* \in P^*$ containing $p$; it is an $(n-1)$-dimensional subspace of $P^*$. We will denote this subspace by $P^*_p$.

Example 2.4. Let $V$ be a left vector space over a division ring, the dimension of $V$ is assumed to be finite and not less than 3. Then $P(V)^*$ (the projective space dual to $P(V)$) is isomorphic to $P(V^*)$ ($V^*$ is the vector space dual to $V$) [2].

2.3. Embeddings of projective spaces. Let us consider projective spaces $P$ and $P'$ and denote by $\mathcal{L}$ and $\mathcal{L}'$ the sets of lines of these spaces. We say that an injection $f : P \to P'$ is an embedding if it is collinearity preserving and maps each triple of non-collinear points to non-collinear points, in other words, if the following two conditions hold true:

1. For any line $L \in \mathcal{L}$ there exists a line $L' \in \mathcal{L}'$ such that $f(L) \subset L'$.
2. For each line $L' \in \mathcal{L}'$ there is at most one line $L \in \mathcal{L}$ such that $f(L) \subset L'$.

It is trivial that any bijective embedding is a collineation.

An embedding is said to be strong if it transfers independent subsets to independent subsets.

Remark 2.2. Non-strong embeddings exist (see, for example, [3] or [10]).

Remark 2.3. Assume that our projective spaces have the same dimension. Then any strong embedding of $P$ to $P'$ sends bases to bases. Inversely, W.-l. Huang and A. Kreuzer [15] (see also [18]) have shown that any base preserving surjection of $P$ to $P'$ is a collineation. It is natural to ask are non-surjective base preserving mappings of $P$ to $P'$ strong embeddings?

Let $f : P \to P'$ be a strong embedding. Then for any subspace $S \subset P$ the dimension of the subspace $\overline{f}(S)$ is equal to the dimension of $S$. Assume also that our projective spaces have the same dimension which is finite. Then the embedding
$f$ induces the mapping

$$f^* : P^* \to P'^*$$

which transfers $p^* \in P^*$ to $f(p^*) \in P'^*$. This mapping is a strong embedding of $P^*$ to $P'^*$; it is said to be dual to $f$. A direct verification shows that the second dual embedding coincides with $f$, in other words, $f^{**} = f$.

**Example 2.5.** If the dimension of our spaces is not less than 3 then for any point $p \in P$ the embedding $f$ defines the strong embedding of $L^p$ to $L'^{f(p)}$ such that the dual embedding is the restriction of $f^*$ to $P^*_p$.

**Example 2.6.** Let $V$ and $V'$ be left vector spaces over division rings $R$ and $R'$ (respectively), the dimensions of $V$ and $V'$ are assumed to be not less than 3. Then $P(V)$ and $P(V')$ are projective spaces (Example 2.2). A mapping $l : V \to V'$ is called **semilinear** if

$$l(x + y) = l(x) + l(y) \quad \forall x, y \in V$$

and there exists a homomorphism $\sigma : R \to R'$ such that

$$l(ax) = \sigma(a)l(x) \quad \forall x \in V, a \in F;$$

it is easy to see that $\sigma$ is a monomorphism if the mapping $l$ is non-zero. Any semilinear injection $l : V \to V'$ induces the mapping

$$P(l) : P(V) \to P(V')$$

which sends each 1-dimensional subspace $Rx, x \in V - \{0\}$ to the subspace $R'l(x)$. Note that the mapping $P(l)$ is non-injective, for example, if $l$ is the semilinear injection of $\mathbb{R}^{2n}$ to $\mathbb{C}^n$ defined by the formula

$$l(x_1, y_1, \ldots, x_n, y_n) := (x_1 + iy_1, \ldots, x_n + iy_n).$$

However, if $l$ preserves the linear independence (maps any set of linearly independent vectors to a set of linearly independent vectors) then $P(l)$ is a strong embedding. Inversely, any strong embedding of $P(V)$ to $P(V')$ is induced by a semilinear injection of $V$ to $V'$ preserving the linear independence; it is a simple consequence of Faure-Frölicher-Havlicek’s version of the Fundamental Theorem of Projective Geometry [10], [11] and [12]. The classical version of this theorem (see [1], [2]) says that any collineation of $P(V)$ to $P(V')$ is induced by a semilinear isomorphism of $V$ to $V'$; this statement was first proved by O. Veblen [28] for projective spaces over finite fields (more historical information can be found in [17]).

3. $A_n$-buildings and their morphisms

3.1. Buildings associated with projective spaces. Let $P$ be an $n$-dimensional projective space. Denote by $\Delta$ the flag complex of $P$ (the flag complex of the set of all proper subspaces of $P$ ordered by $\subset$). Let $B$ be a base for $P$ and $\mathcal{B}$ be the set of all subspaces spanned by points of this base. The flag complex of $(B, \subset)$ is isomorphic to $A_n$ and called the apartment associated with the base $B$. The complex $\Delta$ together with the family of all apartments defined above is a building of type $A_n$.

It was noted above that there are not other non-trivial building of this type: each $A_n$-building ($n \geq 2$) is associated with certain $n$-dimensional projective space [25].
Denote by $\mathfrak{F}$ the set of all maximal flags (chambers of the complex $\Delta$). Since the dimension of the projective space is equal to $n$, any maximal flag consists of $n$ proper subspaces; it is a chain of subspaces

$$p \in S_1 \subset \cdots \subset S_{n-2} \subset p^*$$

where $p \in P$, $p^* \in P^*$ and each $S_i$ is an $i$-dimensional subspace. Recall that a subset $\mathfrak{A} \subset \mathfrak{F}$ is called an apartment in $\mathfrak{F}$ if it is the intersection of $\mathfrak{F}$ with an apartment of $\Delta$; in other words, there is a base for our projective space such that $\mathfrak{A}$ consists of all maximal flags which components are spanned by points of this base.

Let $\mathfrak{X} \subset \mathfrak{F}$. The set consisting of all subspaces contained in flags belonging to $\mathfrak{X}$ will be called the trace of $\mathfrak{X}$ and denoted by $T(\mathfrak{X})$. If $B$ is a base for $P$ then the apartment of $\mathfrak{F}$ associated with $B$ is the maximal subset of $\mathfrak{F}$ which trace coincides with the set of all subspaces spanned by points of $B$.

**Example 3.1.** For a subspaces $S \subset P$ we denote by $\mathfrak{F}(S)$ the set of all maximal flags containing $S$. Let $p \in P$. If $n \geq 3$ then $\mathfrak{F}(p)$ is the set of all maximal flags of the projective space $L_p$ (Example 2.1); apartments of $\mathfrak{F}(p)$ are the intersections of $\mathfrak{F}(p)$ with apartments $\mathfrak{A} \subset \mathfrak{F}$ such that $p \in T(\mathfrak{A})$.

### 3.2. Morphisms of $A_n$-buildings.

Now assume that $P'$ is another $n$-dimensional projective space. The set of all maximal flags of this space will be denoted by $\mathfrak{F}'$.

Let $f : P \to P'$ be a strong embedding. Since for any subspace $S \subset P$ the dimension of the subspace $f(S)$ is equal to the dimension of $S$, the embedding $f$ induces the mapping of $\mathfrak{F}$ to $\mathfrak{F}'$ which sends any maximal flag

$$F = (p, S_1, \ldots, S_{n-2}, p^*)$$

to the maximal flag

$$f(F) = (f(p), f(S_1), \ldots, f(S_{n-2}), f^*(p^*))$$

This mapping is injective and transfers apartments to apartments, besides it is bijective only for the case when $f$ is a collineation.

The dual principles of Projective Geometry (Subsection 2.3) show that the building of any finite-dimensional projective space coincides with the building associated with the dual projective space. Therefore, any strong embedding of $P$ to $P^{**}$ induces certain injection of $\mathfrak{F}$ to $\mathfrak{F}'$ which maps apartments to apartments.

**Theorem 3.1.** Any mapping of $\mathfrak{F}$ to $\mathfrak{F}'$ sending apartments to apartments is the injection induced by a strong embedding of $P$ to $P'$ or $P^{**}$: in particular, any surjection of $\mathfrak{F}$ to $\mathfrak{F}'$ transferring apartments to apartments is induced by a collineation of $P$ to $P'$ or $P^{**}$.

**Remark 3.1.** It was established by H. Havlicek, K. List and C. Zanella [14] that if $n = 3$ and $f$ is a bijective transformation of $\mathfrak{F}$ preserving the adjacency relation in both directions then $f$ is induced by a collineation of $P$ to itself or to the dual projective space.

**Remark 3.2.** For each number $k = 1, \ldots, n-2$ denote by $G_k$ and $G'_k$ the Grassmann spaces of $k$-dimensional subspaces of our projective spaces. If $B$ is a base for $P$ then the set consisting of all $k$-dimensional subspaces spanned by points of $B$ is known as the base subset of $G_k$ associated with $B$. Any strong embedding of $P$ to $P'$ induces certain injective mapping of $G_k$ to $G'_k$. Similarly, strong embeddings of $P$ to $P^{**}$ induce injections of $G_k$ to $G'_{n-k-1}$: these are injections to $G'_k$ if $n = 2k + 1$. It is not
difficult to see that these mappings send base subsets to base subsets. M. Pankov [23] (see also [21] and [22]) has shown that for the case when \( n \geq 3 \) there are not other mappings of \( G_k \) and \( G'_k \) satisfying this condition; i.e. if \( n \geq 3 \) and \( f : G_k \rightarrow G'_k \) maps base subsets to base subsets then \( f \) is induced by a strong embedding of \( P \) to \( P' \) or \( P^{**} \) (the second possibility can be realized only for the case when \( n = 2k+1 \)); this embedding is a collineation if \( f \) is surjective. Geometrical transformations of Grassmann spaces were studied by many authors, see, for example, [3], [4], [8], [13], [16], [19], [27].

4. Proof of Theorem 3.1

Let \( f : \mathfrak{G} \rightarrow \mathfrak{G}' \) be a mapping which transfers apartments to apartments. First of all note that \( f \) is injective (for any two elements of \( \mathfrak{G} \) there exists an apartment containing them; by our hypothesis, the restriction of \( f \) to any apartment is injective). We will show that \( f \) is induced by a strong embedding of \( P \) to \( P' \) or \( P^{**} \).

4.1. Main Lemma. Let \( \mathfrak{A} \) be an apartment in \( \mathfrak{G} \). Then \( \mathfrak{A}' := f(\mathfrak{A}) \) is an apartment in \( \mathfrak{G}' \). Let also

\[
B = \{p_1, \ldots, p_{n+1}\} \quad \text{and} \quad B' = \{p'_1, \ldots, p'_{n+1}\}
\]

be the bases associated with \( \mathfrak{A} \) and \( \mathfrak{A}' \), respectively. We set

\[
p^*_i := B - \{p_i\} \quad \text{and} \quad p^{*_i} := B' - \{p'_i\}
\]

for each number \( i \in I_n \) and denote by \( \mathfrak{A}_i \) and \( \mathfrak{A}^i \) (\( \mathfrak{A}'_i \) and \( \mathfrak{A}'^i \)) the sets of all flags belonging to \( \mathfrak{A} \) (\( \mathfrak{A}' \)) and containing \( p_i \) or \( p^*_i \) (\( p'_i \) or \( p^{*_i} \)), respectively. It is easy to see that these sets contain \( n! \) elements.

Note also that \( \mathfrak{A} \) can be presented as the disjoint union of \( \mathfrak{A}_1, \ldots, \mathfrak{A}_n \) and the disjoint union of \( \mathfrak{A}'_1, \ldots, \mathfrak{A}'^n \). The intersection of \( \mathfrak{A}_i \) and \( \mathfrak{A}'_i \) will be denoted by \( \mathfrak{A}'^i \). This set is not empty if \( i \neq j \), for this case it contains \((n-1)!\) elements. Each \( \mathfrak{A}_{ij} \) (\( \mathfrak{A}'_{ij} \)) is the disjoint union of all \( \mathfrak{A}'_i \) (\( \mathfrak{A}'^i_j \)) such that \( j \neq i \).

Lemma 4.1. There exists a permutation \( \sigma_f \) of the set \( I_n \) such that one of the following possibilities is realized:

1. \( f(\mathfrak{A}_i) = \mathfrak{A}'_{\sigma_f(i)} \) and \( f(\mathfrak{A}'_i) = \mathfrak{A}'^{\sigma_f(i)} \) for all \( i \in I_n \),
2. \( f(\mathfrak{A}_i) = \mathfrak{A}'_{\sigma_f(i)} \) and \( f(\mathfrak{A}'_i) = \mathfrak{A}'_{\sigma_f(i)} \) for all \( i \in I_n \).

This statement will be proved in a few steps (Subsection 4.2 – 4.5).

4.2. Maximal inexact subsets of apartments. We say that a subset \( \mathfrak{X} \) of the apartment \( \mathfrak{A} \) is exact if \( \mathfrak{A} \) is a unique apartment containing \( \mathfrak{X} \); otherwise, \( \mathfrak{X} \) is said to be inexact. It is easy to see that \( \mathfrak{X} \) is exact if and only if for each \( i \in I_n \) the intersection of all subspaces \( S \in T(\mathfrak{X}) \) containing \( p_i \) coincides with \( p_i \).

Take two distinct numbers \( i, j \in I_n \) and denote by \( \mathfrak{X}_{ij} \) the set of all subspaces \( S \in T(\mathfrak{A}) \) such that

\[
p_ip_j \subset S \quad \text{or} \quad p_i \notin S.
\]

Put \( \mathfrak{X}_{ij} \) for the set of all flags spanned by subspaces belonging to \( \mathfrak{X}_{ij} \); in other words, \( \mathfrak{X}_{ij} \) is the maximal subset of \( \mathfrak{G} \) which trace is \( \mathfrak{X}_{ij} \). If \( k \neq i \) then the intersection of all subspaces belonging to \( \mathfrak{X}_{ij} = T(\mathfrak{X}_{ij}) \) and containing \( p_k \) coincides with \( p_k \). Any flag \( F \in \mathfrak{A} - \mathfrak{X}_{ij} \) contains a subspace intersecting the line \( p_ip_j \) by \( p_i \). Therefore, \( \mathfrak{X}_{ij} \cup \{F\} \) is an exact subset of \( \mathfrak{A} \) and \( \mathfrak{X}_{ij} \) is a maximal inexact subset of \( \mathfrak{A} \).

\footnote{recall that \( I_n = \{1, \ldots, n+1\} \), see Example 1.2}
Now we show that there are not other maximal inexact subsets of apartments.

**Lemma 4.2.** If \( \mathfrak{X} \) is a maximal inexact subset of \( \mathfrak{A} \) then there exist \( i, j \in I_n \) such that \( \mathfrak{X}_{ij} \) coincides with \( \mathfrak{X} \).

**Proof.** Since \( \mathfrak{X} \) is inexact, there is a number \( i \in I_n \) such that the intersection of all subspaces \( S \in T(\mathfrak{X}) \) containing \( p_i \) does not coincide with \( p_i \); we denote this intersection by \( U \). Then one of the following possibilities is realized:

1. the dimension of \( U \) is greater than 0,
2. \( U = \emptyset \).

For the case (1) the inclusion \( p_i p_j \subset U \) holds for certain number \( j \in I_n - \{i\} \). For the second case we can take any number \( j \in I_n - \{i\} \). It is trivial that for each of these cases \( T(\mathfrak{X}) \) is contained in \( \mathfrak{X}_{ij} \). These are both maximal inexact subsets and the inverse inclusion holds true. \( \square \)

In particular, Lemma 4.2 implies that all maximal inexact subsets of apartments of \( \mathfrak{F} \) and \( \mathfrak{F}' \) have the same cardinal number.

**Lemma 4.3.** The mapping \( f \) transfers inexact subsets of \( \mathfrak{A} \) to inexact subsets of \( \mathfrak{A}' \); moreover, maximal inexact subsets go over to maximal inexact subsets.

**Proof.** Let \( \mathfrak{X} \) be an inexact subset of \( \mathfrak{A} \). Then \( \mathfrak{X} \) is contained in at least two distinct apartments of \( \mathfrak{F} \). The \( f \)-images of these apartments are distinct apartments of \( \mathfrak{F}' \) containing \( f(\mathfrak{X}) \) (since \( f \) is injective). Hence \( f(\mathfrak{X}) \) is an inexact subset of \( \mathfrak{A}' \).

Now assume that the inexact subset \( \mathfrak{X} \) is maximal and consider a maximal inexact subset \( \mathfrak{Y} \subset \mathfrak{A}' \) containing \( f(\mathfrak{X}) \). Since \( f \) is injective, the remark given after Lemma 4.2 shows that \( f(\mathfrak{X}) \) and \( \mathfrak{Y} \) have the same cardinal number. Then \( f(\mathfrak{X}) \) coincides with \( \mathfrak{Y} \). \( \square \)

4.3. **Complements to maximal inexact subsets.** In this subsection we will study the structure of the complement set

\[ \mathfrak{C}_{ij} := \mathfrak{A} - \mathfrak{X}_{ij}. \]

Any permutation \( \sigma \) of the set \( I_n \) defines the flag

\[ \mathcal{F}_\sigma = (p_{\sigma(1)}, S_1, \ldots, S_{n-2}, p_{\sigma(n+1)}) \in \mathfrak{A} \]

where each \( S_j \) is spanned by the points \( p_{\sigma(1)}, \ldots, p_{\sigma(j+1)} \). The mapping

\[ \sigma \rightarrow \mathcal{F}_\sigma \]

is a bijection of the set of all permutations of \( I_n \) to \( \mathfrak{A} \). For any flag \( \mathcal{F} \in \mathfrak{A} \) the corresponding permutation of \( I_n \) will be denoted by \( \sigma_{\mathcal{F}} \); if

\[ \mathcal{F} = (p_{i_1}, S_1, \ldots, S_{n-2}, p_{n+1}) \]

and each \( S_j \) is the subspace spanned by \( p_{i_1}, \ldots, p_{i_{j+1}} \), then \( \sigma_{\mathcal{F}}(j) = i_j \) for all \( j \), hence \( \sigma_{\mathcal{F}}^{-1}(j) \) is the order number of the minimal component of the flag \( \mathcal{F} \) containing \( p_j \) and if this point does not belong to any component of \( \mathcal{F} \) then \( \sigma_{\mathcal{F}}^{-1}(j) = n + 1 \).

Denote by \( \mathfrak{R}_{ij} \) the set of all flags \( \mathcal{F} \in \mathfrak{A} \) satisfying the following condition:

\[ 1 < \sigma_{\mathcal{F}}^{-1}(i) < \sigma_{\mathcal{F}}^{-1}(j) < n + 1. \]

If \( n = 2 \) then this set is empty.
Lemma 4.4. The set $\mathcal{C}_{ij}$ is the disjoint union of $\mathfrak{A}_i \cup \mathfrak{A}_j$ and $\mathfrak{R}_{ij}$.

Proof. It is trivial that the sets $\mathfrak{A}_i \cup \mathfrak{A}_j$ and $\mathfrak{R}_{ij}$ are disjoint. Since $p_i$ and $p_j^*$ do not belong to $X_{ij}$, $\mathfrak{A}_i \cup \mathfrak{A}_j$ does not intersect $X_{ij}$ and we have

$$\mathfrak{A}_i \cup \mathfrak{A}_j \subset \mathcal{C}_{ij}.$$ 

For any flag

$$\mathcal{F} = (p_k, S_1, \ldots, S_{n-2}, p_m^*) \in \mathfrak{R}_{ij}$$

the subspace $S_{\sigma_{\mathcal{F}}^{-1}(i)}$ (the minimal subspace of $\mathcal{F}$ containing $p_i$) does not contain the point $p_j$; thus $S_{\sigma_{\mathcal{F}}^{-1}(i)}$ is not an element of $X_{ij}$ and $\mathcal{F} \notin X_{ij}$; this implies that

$$\mathfrak{R}_{ij} \subset \mathcal{C}_{ij}.$$ 

Inversely, let us consider a flag $\mathcal{F} \in \mathcal{C}_{ij}$ which does not belong to $\mathfrak{A}_i \cup \mathfrak{A}_j$. Then

$$\mathcal{F} \notin \mathfrak{A}_i \Rightarrow \sigma_{\mathcal{F}}(1) \neq i \text{ and } \mathcal{F} \notin \mathfrak{A}_j \Rightarrow \sigma_{\mathcal{F}}(n+1) \neq j.$$ 

If $\sigma_{\mathcal{F}}^{-1}(i) > \sigma_{\mathcal{F}}^{-1}(j)$ then each component of the flag $\mathcal{F}$ belongs to $X_{ij}$ and we get $\mathcal{F} \in X_{ij}$, this contradicts to the condition $\mathcal{F} \in \mathcal{C}_{ij}$. Therefore, $\sigma_{\mathcal{F}}^{-1}(i) < \sigma_{\mathcal{F}}^{-1}(j)$ and $\mathcal{F}$ is an element of the set $\mathfrak{R}_{ij}$. \[\square\]

For any subspace $S \in T(\mathfrak{A})$ there exists unique subspace $S^c \in T(\mathfrak{A})$ non-intersecting $S$ and such that our projective space is spanned by $S$ and $S^c$. For any flag

$$\mathcal{F} = (p_i, S_1, \ldots, S_{n-2}, p_j^*) \in \mathfrak{A}$$

the flag

$$\mathcal{F}^c := (p_j = (p_j)^c, (S_{n-2})^c, \ldots, (S_1)^c, p_i^* = (p_i)^c) \in \mathfrak{A}$$

is said to be the complement. The transformation of $\mathfrak{A}$ sending each flag to its complement is bijective; moreover,

$$\langle \mathfrak{A}_i \rangle^c = \mathfrak{A}_i^t, \langle \mathfrak{A}_j \rangle^c = \mathfrak{A}_i, \langle \mathfrak{R}_{ij} \rangle^c = \mathfrak{R}_{ji}$$

and Lemma 4.4 shows that

$$(\mathcal{C}_{ij})^c = \mathcal{C}_{ji}.$$ 

Lemma 4.5. Let $n \geq 3$ and $i, j, k, m$ be distinct elements of $I_n$. Then

$$|\mathfrak{A}_m^k \cap \mathfrak{R}_{ij}| = \frac{(n-1)!}{2}$$

and

$$|\mathfrak{A}_k \cap \mathfrak{R}_{ij}| = |\mathfrak{A}_k \cap \mathfrak{R}_{ij}| = \frac{(n-2)(n-1)!}{2}.$$ 

Proof. Let us consider the bijective transformation $d_{ij}$ of $\mathfrak{A}$ defined by the formula

$$d_{ij}(\mathcal{F}) := \mathcal{F}_{\sigma_{ij}} \sigma_{\mathcal{F}}$$

where $\sigma_{ij}$ is the permutation sending $i$ and $j$ to $j$ and $i$ (respectively) and leaving fixed all elements of the set $I_n - \{i,j\}$. This bijection maps $\mathfrak{A}_m^k \cap \mathfrak{R}_{ij}$ to $\mathfrak{A}_m^k - \mathfrak{R}_{ij}$; thus these sets have the same cardinal number. Since

$$\mathfrak{A}_m^k = (\mathfrak{A}_m^k \cap \mathfrak{R}_{ij}) \cup (\mathfrak{A}_m^k - \mathfrak{R}_{ij})$$

contains $(n-1)!$ elements, we get the first equality. Then the second equality is a consequence of the following fact: $\mathfrak{A}_k \cap \mathfrak{R}_{ij}$ ($\mathfrak{A}_k \cap \mathfrak{R}_{ij}$) is the disjoint union of all $\mathfrak{A}_l^k \cap \mathfrak{R}_{ij}$ ($\mathfrak{A}_k \cap \mathfrak{R}_{ij}$) such that $l \in I_n - \{i,j,k\}$. \[\square\]
4.4. Dispositions of two complement sets. Let \( \mathcal{C}_{ij} \) and \( \mathcal{C}_{km} \) be distinct complement sets. Set \( J := \{i, j, k, m\} \).

Then \( 2 \leq |J| \leq 4 \) and there are the following possibilities for the disposition of \( \mathcal{C}_{ij} \) and \( \mathcal{C}_{km} \):

1. \( i = m \) and \( j = k \);
2. \( |J| = 3 \) and \( i = k \);
3. \( |J| = 3 \) and \( j = m \);
4. \( |J| = 3 \) and \( i = m \);
5. \( |J| = 3 \) and \( j = k \);
6. \( |J| = 4 \).

For each number \( l \in \{1, \ldots, 6\} \) denote by \( n_l \) the cardinal number of the intersection of \( \mathcal{C}_{ij} \) and \( \mathcal{C}_{km} \) for the case \((l)\). If \( n = 2 \) then the case \((6)\) is not realized and the number \( n_6 \) is not defined.

**Lemma 4.6.** \( n_1 = 0 \).

**Proof.** By Lemma 4.4,

\[
\mathcal{C}_{ij} = (\mathcal{A}_i \cup \mathcal{A}_j) \cup \mathcal{R}_{ij},
\]

\[
\mathcal{C}_{ji} = (\mathcal{A}_j \cup \mathcal{A}_i) \cup \mathcal{R}_{ji};
\]

we have also

\[
(\mathcal{A}_i \cup \mathcal{A}_j) \cap (\mathcal{A}_j \cup \mathcal{A}_i) = \emptyset,
\]

\[
(\mathcal{A}_i \cup \mathcal{A}_j) \cap \mathcal{R}_{ji} = (\mathcal{A}_j \cup \mathcal{A}_i) \cap \mathcal{R}_{ij} = \emptyset.
\]

and get the equality

\[
\mathcal{C}_{ij} \cap \mathcal{C}_{ji} = \mathcal{R}_{ij} \cap \mathcal{R}_{ji}.
\]

However,

\[
\mathcal{F} \in \mathcal{R}_{ij} \Rightarrow \sigma_{1}^{-1}(i) < \sigma_{1}^{-1}(j)
\]

contradicts to

\[
\mathcal{F} \in \mathcal{R}_{ji} \Rightarrow \sigma_{1}^{-1}(j) < \sigma_{1}^{-1}(i).
\]

Therefore, \( \mathcal{R}_{ij} \cap \mathcal{R}_{ji} = \emptyset \) and the intersection of our complement sets is empty. \( \square \)

**Lemma 4.7.** \( n_2 = n_3 \) and \( n_4 = n_5 \).

**Proof.** The equality (4.1) shows that the bijection

\[
\mathcal{F} \rightarrow \mathcal{F}^c
\]

transfers pairs of complement sets satisfying the condition (2) or (4) to pairs of complement sets satisfying the condition (3) or (5), respectively. \( \square \)

**Lemma 4.8.** If \( n \geq 4 \) then

\[
n_2 = n! + (n - 2)(n - 1)! + \frac{(n - 2)(n - 3)(n - 1)!}{3}.
\]

We have also

\[
n_2 = n! + (n - 2)(n - 1)!
\]

if \( n = 2, 3 \).
Proof. Since
\[ \mathcal{C}_{ij} = (\mathcal{A}_i \cup \mathcal{A}_j) \cup \mathcal{R}_{ij}, \]
\[ \mathcal{C}_{im} = (\mathcal{A}_i \cup \mathcal{A}_m) \cup \mathcal{R}_{im}, \]

(Lemma 4.4) and

\[ (\mathcal{A}_i \cup \mathcal{A}_j) \cap (\mathcal{A}_i \cup \mathcal{A}_m) = \mathcal{A}_i, \]
\[ (\mathcal{A}_i \cup \mathcal{A}_j) \cap \mathcal{R}_{im} = \mathcal{A}_j \cap \mathcal{R}_{im}, \]
\[ (\mathcal{A}_i \cup \mathcal{A}_m) \cap \mathcal{R}_{ij} = \mathcal{A}_m \cap \mathcal{R}_{ij}; \]

the intersection of \( \mathcal{C}_{ij} \) and \( \mathcal{C}_{im} \) is the disjoint union of the following sets:

\[ \mathcal{A}_i, \mathcal{A}_j \cap \mathcal{R}_{im}, \mathcal{A}_m \cap \mathcal{R}_{ij}, \mathcal{R}_{ij} \cap \mathcal{R}_{im}. \]

If \( n = 2 \) then the sets \( \mathcal{R}_{ij} \) and \( \mathcal{R}_{im} \) are empty and the required equality is trivial. For the case when \( n \geq 3 \) we have

\[ n_2 = |\mathcal{A}_i| + |\mathcal{A}_j \cap \mathcal{R}_{im}| + |\mathcal{A}_m \cap \mathcal{R}_{ij}| + |\mathcal{R}_{ij} \cap \mathcal{R}_{im}| = n! + (n - 2)(n - 1)! + |\mathcal{R}_{ij} \cap \mathcal{R}_{im}| \]

(Lemma 4.5). We will find the cardinal number of the set \( \mathcal{R}_{ij} \cap \mathcal{R}_{im} \).

First of all note that this set is empty if \( n = 3 \), since for this case for any flag \( \mathcal{F} \in \mathcal{A} \) one of the numbers

\[ \sigma_{\mathcal{F}}^{-1}(i), \sigma_{\mathcal{F}}^{-1}(j), \sigma_{\mathcal{F}}^{-1}(m) \]

is equal to 1 or \( n + 1 \). Let \( n \geq 4 \). Then \( I_n - J \) contains at list two elements and we take distinct numbers \( l \) and \( s \) belonging to this set. If \( \mathcal{F} \in \mathcal{A}_l \) then there are exact 6 possibilities for the disposition of the numbers (4.2) in the set \( \sigma_{\mathcal{F}}^{-1}(J) \); only for two of these cases \( \sigma_{\mathcal{F}}^{-1}(i) \) is less than both \( \sigma_{\mathcal{F}}^{-1}(j), \sigma_{\mathcal{F}}^{-1}(m) \) and the flag \( \mathcal{F} \) belongs to \( \mathcal{R}_{ij} \cap \mathcal{R}_{im} \). Therefore,

\[ |\mathcal{A}_l^+ \cap \mathcal{R}_{ij} \cap \mathcal{R}_{im}| = \frac{2|\mathcal{A}_l^+|}{6} = \frac{(n - 1)!}{3} \]

The set \( \mathcal{R}_{ij} \cap \mathcal{R}_{im} \) is the disjoint union of all \( \mathcal{A}_l^+ \cap \mathcal{R}_{ij} \cap \mathcal{R}_{im} \) such that \( l \) and \( s \) do not belong to \( J \). It is easy to see that there are \( (n - 2)(n - 3) \) distinct \( \mathcal{A}_l^+ \) with \( s, l \in I_n - J \). Thus

\[ |\mathcal{R}_{ij} \cap \mathcal{R}_{im}| = \frac{(n - 2)(n - 3)(n - 1)!}{3}. \]

\[ \square \]

**Lemma 4.9.** If \( n \geq 4 \) then

\[ n_4 = (n - 1)! + (n - 2)(n - 1)! + \frac{(n - 2)(n - 3)(n - 1)!}{6} \]

For the case when \( n = 2 \) or 3 we have

\[ n_4 = (n - 1)! + (n - 2)(n - 1)!. \]

**Proof.** By Lemma 4.4,

\[ \mathcal{C}_{ij} = (\mathcal{A}_i \cup \mathcal{A}_j) \cup \mathcal{R}_{ij}, \]
\[ \mathcal{C}_{ki} = (\mathcal{A}_k \cup \mathcal{A}_i) \cup \mathcal{R}_{ki}. \]

A direct verification shows that

\[ (\mathcal{A}_i \cup \mathcal{A}_j) \cap (\mathcal{A}_k \cup \mathcal{A}_l) = \mathcal{A}_k^l, \]
\[ (\mathcal{A}_i \cup \mathcal{A}_j) \cap \mathcal{R}_{ki} = \mathcal{A}_i^l \cap \mathcal{R}_{ki}, \]
\[ (\mathcal{A}_k \cup \mathcal{A}_i) \cap \mathcal{R}_{ij} = \mathcal{A}_k \cap \mathcal{R}_{ij}. \]
Proof. Lemma 4.4 says that
\[ n_4 = |A_i| + |A_i \cap R_{ki}| + |A_k \cap R_{ij}| + |R_{ij} \cap R_{ki}| = (n-1)! + (n-2)(n-1)! + |R_{ij} \cap R_{ki}|. \]

The set $R_{ij} \cap R_{ki}$ is empty if $n \leq 3$; indeed, for any flag $F \in A$ one of the numbers
\[ \sigma_F^{-1}(i), \sigma_F^{-1}(j), \sigma_F^{-1}(k) \]

is equal to 1 or $n+1$. Assume that $n \geq 4$ (for this case $I_n - J$ contains at least two elements) and take two distinct numbers $l$ and $s$ belonging to $I_n - J$. It was noted above (the proof of Lemma 4.8) that for any flag $F \in A_i$ there are exactly $6$ possibilities for the disposition of the numbers (4.3) in the set $\sigma_F^{-1}(J)$. The inequality
\[ \sigma_F^{-1}(k) < \sigma_F^{-1}(i) < \sigma_F^{-1}(j) \]

holds only for one of these cases. Then we have
\[ |A_i \cap R_{ij} \cap R_{ki}| = \frac{|A_i|}{6} = \frac{(n-1)!}{6} \]

and the arguments used to prove Lemma 4.8 show that
\[ |R_{ij} \cap R_{ki}| = \frac{(n-2)(n-3)(n-1)!}{6}. \]

Lemma 4.10. If $n \geq 5$ then
\[ n_6 = n! + (n-1)(n-1)! + \frac{(n-3)(n-4)(n-1)!}{4}. \]

We have also
\[ n_6 = n! + (n-1)(n-1)! \]

if $n = 3, 4$.

Recall that for the case when $n = 2$ the number $n_6$ is not defined.

Proof. Lemma 4.4 says that
\[ C_{ij} = (A_i \cup A^j) \cup R_{ij}, \]
\[ C_{km} = (A_k \cup A^m) \cup R_{km}. \]

Since
\[ (A_i \cup A^j) \cap (A_k \cup A^m) = A_i^{m} \cup A_k^{j}, \]
\[ C_{ij} \cap C_{km} \]

is the union of the following non-intersecting sets:
\[ A_i^m, A_k^j, (A_i \cup A^j) \cap R_{km}, (A_k \cup A^m) \cap R_{ij}, R_{ij} \cap R_{km}. \]

The set $A_i \cup A^j$ is the disjoint union of all $A_i^l$, $l \neq i$ and all $A^j_s$, $s \neq j$. Each of these collections contains $n$ sets, but the set $A_i^l$ belongs to the both collections; note also that $A_i^m$ and $A_k^{j}$ do not intersect $R_{km}$. In other words, $(A_i \cup A^j) \cap R_{km}$ is the disjoint union of $2n - 3$ distinct sets containing $\frac{(n-1)!}{2}$ elements (Lemma 4.5) and
\[ |(A_i \cup A^j) \cap R_{km}| = \frac{(2n-3)(n-1)!}{2}. \]
Similarly, 
\[ |(A_{ij} \cup A)^{m} \cap R| = \frac{(2n-3)(n-1)!}{2}. \]

Therefore, 
\[ n_{6} = |A^{m}| + |A^{2}| + |(A_{i} \cup A^{1}) \cap R| + |(A_{k} \cup A^{m}) \cap R_{ij}| + |R_{ij} \cap R_{km}| = 2(n-1)! + (2n-3)(n-1)! + |R_{ij} \cap R_{km}| = n! + (n-1)(n-1)! + |R_{ij} \cap R_{km}|. \]

Thus we have to find the cardinal number of the set \( R_{ij} \cap R_{km} \).

If \( n \leq 4 \) then \( R_{ij} \cap R_{km} \) is empty, since for this case for any flag \( F \in A \) at least one of the numbers
\[ \sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k), \sigma^{-1}(m) \]
is equal to 1 or \( n + 1 \). Now assume that \( n \geq 5 \). Then \( I_{n} - J \) contains at least two elements and we take two distinct numbers \( l \) and \( s \) belonging to this set. The bijection \( d_{ij} \) (see the proof of Lemma 4.5) transfers
\[ (A_{ij} \cap R_{km} \cap R_{ij}) \rightarrow (A_{ij} \cap R_{km} - R_{ij}) \]
Hence these sets have the same cardinal number. Since
\[ (A_{ij} \cap R_{km}) = (A_{ij} \cap R_{km} \cap R_{ij}) \cup ((A_{ij} \cap R_{km}) - R_{ij}) \]
contains \( \frac{(n-1)!}{2} \) elements, we have
\[ |A_{ij} \cap R_{ij} \cap R_{km}| = \frac{(n-1)!}{4}. \]
The set \( R_{ij} \cap R_{km} \) is the disjoint unions of all \( A_{ij} \cap R_{ij} \cap R_{km} \) such that \( s, l \in I_{n} - J \).

There are exactly \( (n-3)(n-4) \) distinct sets satisfying this condition and we get the equality
\[ |R_{ij} \cap R_{km}| = \frac{(n-3)(n-4)(n-1)!}{4}. \]

Theorem 4.11. The number \( n_{2} < n_{4} \) is not equal to \( n_{1}, n_{4} = n_{5} \) and \( n_{6} \).

Proof. Clearly, \( n_{2} > 0 = n_{1} \). It is easy to see that \( n_{2} > n_{4} \). We need to establish that \( n_{2} \) is not equal to \( n_{6} \) if \( n \geq 3 \).

If \( n = 3 \) or 4 then
\[ n_{6} - n_{2} = (n-1)! = 2 \]
or
\[ n_{6} - n_{2} = (n-1)! - \frac{(n-2)(n-3)(n-1)!}{3} = 2, \]
respectively. For the case when \( n \geq 5 \) we have
\[ n_{2} - n_{6} = \frac{(n-2)(n-3)(n-1)!}{3} - (n-1)! - \frac{(n-3)(n-4)(n-1)!}{4} = (n-1)! \frac{4(n-2)(n-3) - 3(n-3)(n-4) - 12}{12} = (n-1)! \frac{n^2 + n - 24}{12} > 0. \]

We will say that the complement sets \( C_{ij} \) and \( C_{km} \) are adjacent if one of the cases (2) or (3) is realized, i.e. \( i = k \) or \( j = m \).
Lemma 4.12. The mapping $f$ transfers complement subsets of $\mathfrak{A}$ to complement subsets of $\mathfrak{A}'$; moreover, adjacent complement subsets go over to adjacent complement subsets.

Proof. The first statement follows from Lemma 4.3. Since $f$ is injective, the $f$-images of two adjacent complement subsets of $\mathfrak{A}$ are complement subsets of $\mathfrak{A}'$ which intersection contains $n_2 = n_3$ elements. By Lemma 4.11, these complement subsets are adjacent. □

4.5. Proof of Lemma 4.1. First of all we give two simple lemmas.

Lemma 4.13. For each number $i \in I_n$ we have

$$\bigcap_{j \in I_n - \{i\}} C_{ij} = \mathfrak{A}_i$$

and

$$\bigcap_{j \in I_n - \{i\}} C_{ji} = \mathfrak{A}'_i.$$

Proof. It is trivial that the intersection of all $C_{ij}, j \in I_n - \{i\}$ contains $\mathfrak{A}_i$. If $F \in \mathfrak{A}_k$ and $k \neq i$ then $F$ does not belong to $C_{ik}$. Therefore, our intersection coincides with $\mathfrak{A}_i$ and we get the first equality. The transformation $F \to F^c$ sends the first equality to the second equality. □

Lemma 4.14. For any collection of $n$ mutually adjacent complement subsets of $\mathfrak{A}$ there exists a number $i \in I_n$ such that this collection is consisting of all $C_{ij}$ or all $C_{ji}$ with $j \in I_n - \{i\}$.

Proof. Easy verification. □

Now we can prove Lemma 4.1.

By Lemma 4.13,

$$f(\mathfrak{A}_i) = \bigcap_{j \in I_n - \{i\}} f(C_{ij})$$

for any $i \in I_n$. Lemma 4.12 shows that $\{f(C_{ij})\}_{j \in I_n - \{i\}}$ is a collection of $n$ mutually adjacent complement subsets of $\mathfrak{A}'$. Then Lemma 4.13 and 4.14 guarantee the existence a number $\sigma(i) \in I_n$ such that $f(\mathfrak{A}_i)$ coincides with $\mathfrak{A}'_{\sigma(i)}$ or $\mathfrak{A}_{\sigma(i)}$.

We want to establish that the set

$$J := \{ i \in I_n \mid f(\mathfrak{A}_i) = \mathfrak{A}'_{\sigma(i)} \}$$

is empty or coincides with $I_n$. In particular, this means that the mapping $\sigma : I_n \to I_n$ is bijective (since the restriction of $f$ to $\mathfrak{A}$ is a bijection to $\mathfrak{A}'$).

Proof. Assume that $J$ is not empty and does not coincide with $I_n$. There exist $k \in J$ and $m \in I_n - J$ such that $\sigma(k) \neq \sigma(m)$ (otherwise, the mapping $\sigma : I_n \to I_n$ is constant and $f(\mathfrak{A})$ is contained in certain $\mathfrak{A}' \cup \mathfrak{A}''$). Then

$$f(\mathfrak{A}_k \cap \mathfrak{A}_m) = f(\mathfrak{A}_k) \cap f(\mathfrak{A}_m) = \mathfrak{A}'_{\sigma(k)} \cap \mathfrak{A}'_{\sigma(m)} = \mathfrak{A}'_{\sigma(k)} = \emptyset$$

contradicts to $\mathfrak{A}_k \cap \mathfrak{A}_m = \emptyset$. □

If $J$ coincides with $I_n$ then $f(\mathfrak{A}^i) = \mathfrak{A}'_{\sigma(i)}$ for each $i \in I_n$. 
Proof. The arguments given above imply the existence of a permutation \( \omega \) of \( I_n \) such that
\[
f(\mathfrak{A}_i) = \mathfrak{A}_{\omega(i)} \quad \forall \ i \in I_n
\]
or
\[
f(\mathfrak{A}_i) = \mathfrak{A}_{\omega'(i)} \quad \forall \ i \in I_n.
\]
Clearly, for the first case the mapping \( f \) is not injective. Thus the second equality holds true. Since \( \mathfrak{A}_i \cap \mathfrak{A}_i \) is empty,
\[
f(\mathfrak{A}_i \cap \mathfrak{A}_i) = f(\mathfrak{A}_i) \cap f(\mathfrak{A}_i) = \mathfrak{A}_{\omega(i)} \cap \mathfrak{A}_{\omega'(i)} = \emptyset;
\]
this means that \( \sigma(i) = \omega(i) \). \( \square \)

If \( J \) is empty then the similar arguments show that \( f(\mathfrak{A}_i) = \mathfrak{A}_{\omega'(i)} \) for all \( i \in I_n \).

4.6. The final part of the proof of Theorem 3.1. Recall that (Example 3.1) for any subspaces \( S \subset P \) and \( S' \subset P' \) we denote by \( \mathfrak{F}(S) \) and \( \mathfrak{F}'(S') \) the sets of all maximal flags containing these subspaces.

Let \( p \in P \). Assume that \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are distinct flags belonging to the set \( \mathfrak{F}(p) \). Since apartments containing \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) exist, Lemma 4.1 shows that the flags \( f(\mathcal{F}_1) \) and \( f(\mathcal{F}_2) \) have the same point or the same \((n-1)\)-dimensional subspace.

Now take \( \mathcal{F}_0 \in \mathfrak{F}(p) \) and denote by \( \mathcal{X} \) (or \( \mathcal{Y} \)) the set of all flags \( \mathcal{F} \in \mathfrak{F}(p) \) such that \( f(\mathcal{F}_0) \) and \( f(\mathcal{F}) \) have the same point (or the same \((n-1)\)-dimensional subspace, respectively). Then
\[
(4.4) \quad \mathcal{X} \cup \mathcal{Y} = \mathfrak{F}(p).
\]

We want to show that one of these sets coincides with \( \mathfrak{F}(p) \).

Proof. If
\[
(4.5) \quad \mathcal{X} \neq \mathfrak{F}(p) \ \text{and} \ \mathcal{Y} \neq \mathfrak{F}(p)
\]
then (4.4) implies the existence of
\[
\mathcal{F}_1 \in \mathcal{X} - \mathcal{Y} \ \text{and} \ \mathcal{F}_2 \in \mathcal{Y} - \mathcal{X}.
\]
The points and the \((n-1)\)-dimensional subspaces contained in the flags \( f(\mathcal{F}_1) \) and \( f(\mathcal{F}_2) \) are different. Hence one of the conditions (4.5) does not hold. \( \square \)

Thus we have established that for any point \( p \in P \) there exists a point \( g(p) \) belonging to \( P' \) or \( P'^* \) and such that
\[
f(\mathfrak{F}(p)) \subset \mathfrak{F}'(g(p)).
\]
Since any two points of \( P \) are contained in some base, Lemma 4.1 guarantees that one of the following two possibilities is realized:
\[
(A) \ g(p) \in P' \ \forall p \in P,
(B) \ g(p) \in P'^* \ \forall p \in P.
\]
Similarly, for any point \( p^* \in P^* \) there is a point \( h(p^*) \) of \( P' \) or \( P'^* \) such that
\[
f(\mathfrak{F}(p^*)) \subset \mathfrak{F}'(h(p^*)).
\]
Moreover, it follows from Lemma 4.1 that for the cases (A) and (B) we have
\[
h(p^*) \in P'^* \ \forall p^* \in P^* \ \text{and} \ h(p^*) \in P' \ \forall p^* \in P^*,
\]
respectively.
Lemma 4.15. If $p \in P$ and $p^* \in P^*$ contains $p$ then $g(p)$ and $h(p^*)$ are incident subspaces \(^2\).

Proof. Consider a maximal flag $\mathfrak{F}$ containing $p$ and $p^*$. Then $g(p)$ and $h(p^*)$ are contained in the flag $f(\mathfrak{F})$. This gives the required. □

The mapping $g$ is a strong embedding of $P$ to $P'$ or $P'^*$.

Proof. It is trivial that $g$ maps bases to bases. Hence it transfers independent subsets to independent subsets (a subset of a projective space is independent if and only if it is contained in certain base for this space). In particular, this means that $g$ is injective. Thus we have to show that $g$ is collinearity preserving. We prove this statement for the case (A), the case (B) is similar.

Any two distinct points $p_1, p_2 \in P$ are contained in some base $B = \{p_i\}_{i=1}^{n+1}$ for $P$. The points

$$p'_i := g(p_1), \ldots, p'_{n+1} := g(p_{n+1})$$

form a base for $P'$; in what follows this base will be denoted by $B'$. For each $i \in I_n$ we set

$$p^*_i := B - \{p_i\}, \; p'^*_i := B' - \{p'_i\}$$

and denote by $\mathfrak{A}$ and $\mathfrak{A}'$ the apartments in $\mathfrak{F}$ and $\mathfrak{F}'$ associated with the bases $B$ and $B'$, respectively. Then $f(\mathfrak{A}) = \mathfrak{A}'$ and Lemma 4.1 shows that $h(p^*_i) = p'^*_i$ for all $i \in I_n$. Since

$$p_1p_2 = \bigcap_{i=3}^{n+1} p^*_i,$$

we have

$$g(p_1p_2) \subset \bigcap_{i=3}^{n+1} h(p^*_i) = \bigcap_{i=3}^{n+1} p'^*_i = p'_1p'_2$$

by Lemma 4.15. □

Lemma 4.15 states that the $g$-image of any $p^* \in P^*$ is contained in $h(p^*)$. Since $g$ is a strong embedding to $P'$ or $P'^*$, $g(p^*)$ is an $(n-1)$-dimensional subspaces of $P'$ or $P'^*$ (respectively). The similar holds for $h(p^*)$. Hence the subspaces $h(p^*)$ and $g(p^*)$ are coincident. This means that $h = g^*$ (Subsection 2.3). In particular, Theorem 3.1 is proved for the case when $n = 2$.

Now assume that $n \geq 3$ and prove Theorem 3.1 by induction. Take an arbitrary point $p \in P$ and consider the restriction of $f$ to $\mathfrak{F}(p)$. It is an injection to $\mathfrak{F}'(g(p))$ sending apartments to apartments (Example 3.1). By the inductive hypothesis, this mapping is induced by a strong embedding $t$ of $\mathcal{L}_p$ to $\mathcal{L}'_{g(p)}$ or $\mathcal{L}'^*_{g(p)}$ (Example 2.5). It is easy to see that the dual embedding $t^*$ coincides with the restriction of $h$ to $P^*_p$; denote this restriction by $h'$. Then $t = h^*$ is defined by $g$ (Example 2.5). This implies the required: the mapping $f$ is induced by the embedding $g$.

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\(^2\)recall that two subspaces are incident if one of them is contained to the other.
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