Moy-Prasad maps for SL(2) over extensions of $\mathbb{Q}_2$

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9/27/2011

Abstract

A parametrization for characters of abelian quotients of compact subgroups of $G := \text{SL}_2(F)$ is constructed for $F$ an algebraic extension of $\mathbb{Q}_2$ that corresponds to the Moy-Prasad maps of fields of odd residual characteristic.

In his Ph.D. thesis, [11], Joseph Shalika established that the representations of Weil ([13]) form an exhaustive list of irreducible, cuspidal representations over $G := \text{SL}_2(F)$ for $F$ a local field of odd residual characteristic. To that end, Shalika employs a now classical filtration of the compact subgroups of $G$ and then lists the possible characters on compact subgroups of $G$ contained in such representations to count the representations of $G$ of a particular “level”. In his 1972 paper, [1], Casselman extended this technique to construct the irreducible cuspidal representations which occur in the construction of Weil for $\text{SL}_2(F)$ with $F$ of even residual characteristic. In 1976, Nobs establish that exactly four “exceptional” representations for $\text{SL}_2(\mathbb{Q}_2)$ existed outside the construction of Weil. Later that year, Nobs along with Wolfart identified these representations in [9] by using tensor products of other representations.

In his 1978 papers, [3] and [4], Kutzko built on the parametrization of Shalika to explicitly construct cuspidal representations of $\text{GL}_2(F)$ for arbitrary residual characteristic. An alternate presentation of this approach can be found in Kutzko’s 1972 Ph.D. Thesis, [5]. His parametrization became know as the theory of cuspidal types.

This technique was modified back to $\text{SL}_2(F)$ of odd residual characteristic by Manderscheid in his 1984 papers, [6] and [7]. In 1994 Moy and Prasad proved that a more general approach for classifying irreducible representations could be employed for arbitrary reductive groups over arbitrary $p$. While their approach differed from that of Kutzko, they rely on the implicit existence of a parametrization of characters corresponding to that used in Shalika’s thesis. [8].
The functions underlying the parametrization of characters employed by Moy and Prasad became known as Moy-Prasad maps. They provide an $G$-isomorphism to certain characters of compact subgroups of $G$ from quotients of fractional ideals in the Lie Algebra corresponding to $G$. Recently they have been employed in work by Yu, [14] and in the exposition the harmonic analysis of $SL_2(F)$ for $p \neq 2$ by Adler, Debacker, Sally and Spice, [2].

As presented in the literature, these $G$-isomorphisms factors through a $p$-adic Killing form on the Lie Algebra in question. As the killing form on the trace zero matrices $\mathbb{Z}/2\mathbb{Z}$ is known to be degenerate, the traditional formulation of Moy-Prasad maps does not freely transfer to $SL_2(F)$ for $F$ of residual characteristic two. In this note I develop a counterpart for Moy-Prasad maps for the case where $p = 2$.

This work is a generalization of a portion of my dissertation, [12] and is the first in a sequence of papers which will develop the representation theory and harmonic analysis of $SL_2(F)$ where $p = 2$, employing the methods of Kutzko and Mandersheid.

**Notation.** I will use $l$, $m$ and $n$ to denote integers. Here is some notation I will employ:

- Let $F$ be an algebraic extension of $\mathbb{Q}_2$ of ramification index $e$.
- Let $G = SL_2(F)$.
- Let $\mathfrak{o}$ denote the ring of integers of $F$.
- Let $p$ be the prime ideal of $\mathfrak{o}$.
- Let $\varpi$ be a local uniformizing parameter that generates $p$.
- Denote the valuation of $x$ in $F^\times$ by $v(x)$.
- Let $K$ denote a maximal compact subgroup of $G$, namely $SL_2(\mathfrak{o})$.
- For $n, m \geq 0$, let $K_n^m$ denote the compact subgroup of $G$ which consists of elements of the form,

$$1_2 + \begin{pmatrix} p^n & p^{n+m} \\ p^{n+m} & p^n \end{pmatrix}.$$  (1)

In this paper I offer the following results.

**Theorem 1.** If $e, n \geq m$, then $K_n^m$ is a normal subgroup of $K$.

**Theorem 2.** If $m \leq e$, then the set of characters of the quotient $K_n^m/K_2n$ as a $G$-set is parametrized by matrices of the form

$$\begin{pmatrix} p^{-2n-e}/p^{-n-e} & p^{-2n-m}/p^{-n-m} \\
 p^{-2n-m}/p^{-n-m} & p^{-2n-e}/p^{-n-e} \end{pmatrix}.$$
Proof of Theorem 1

Notation. In both this proof and the next I will denote \( u = \frac{e^c}{2} \). Note that \( u \in o^\times \).
I recall two famous subgroups of \( K \).

- For \( n \geq 1 \), let \( K_n \) be the kernel in \( K \) of the map induced by the natural homomorphism \( o \to o/p^n \) and let \( K_0 = K \).
- For \( n \geq 1 \), let \( B_n \) be the preimage in \( K \) of the group of upper triangular matrices of the map induced by the natural map \( o \to o/p^n \) and let \( B_0 = K \).

For \( n \geq 0 \), I will also denote by \( \left[ \begin{array}{c} x \\ y \end{array} \right]_n \), the homothety class of \( (x, y) \in o \times o \) in the projective line over \( o/p^n \). For my purposes, the projective line over \( o/p^0 \) is a singleton set.

Note that for the calculations that follow, one may consider \( \left[ \begin{array}{c} x \\ y \end{array} \right]_n \) as a vertex on the Bruhat-Tits tree that is distance \( n \) from the vertex stabilized by \( K \) as in [10]. While this perspective can be insightful with respect to Theorem 1, it is not necessary for purposes of proof.

I consider the transitive action of \( K \) on the projective line over \( o/p^n \):
\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left[ \begin{array}{c} x \\ y \end{array} \right]_n = \left[ \begin{array}{c} ax + by \\ cx + dy \end{array} \right]_n
\]

It is well known and easily verified that the stabilizer of the projective point \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]_n \) is none other than \( B_n \). Consequently, for \( \gamma \in K \) the conjugate \( B_n \gamma \) is the stabilizer of \( \gamma^{-1} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]_n \).

I now consider the group of matrices:
\[
B_{n+m} \cap K_n = 1_2 + \left( \begin{array}{c} p^n \\ p^{n+m} \end{array} \right)
\]

Lemma 1. If \( n, e \geq m \), the group \( K_n^m \) is the intersection of each of the conjugates of \( B_{n+m} \cap K_n \) under \( K \).

Proof. First it is easy to observe that \( K_n^m \) is the intersection of \( B_{n+m} \cap K_n \) with the stabilizer of \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]_{n+m} \), since the latter is precisely the group of determinant-one matrices of the form
\[
\left( \begin{array}{cc} o^\times & p^{m+n} \\ o & o^\times \end{array} \right).
\]

To show that is a subset of the other conjugates of \( B_{n+m} \cap K_n \) and hence the intersection of all conjugates of \( B_{n+m} \cap K_n \), I compute the action of \( K_n^m \) on
an arbitrary point of the projective line, illustrating that it is trivial.

\[
\begin{bmatrix}
1 + \varpi^n a & \varpi^{n+m} b \\
\varpi^{n+m} c & 1 + \varpi^n d
\end{bmatrix}
\begin{bmatrix} x \\ 1 \end{bmatrix}
= \begin{bmatrix}
1 + \varpi^n a & 0 \\
0 & (1 + \varpi^n a)^{-1}
\end{bmatrix}
\begin{bmatrix} x \\ 1 \end{bmatrix}
= \begin{bmatrix}
(1 + \varpi^n a)x \\
(1 + \varpi^n a)^{-1}1
\end{bmatrix}
= \begin{bmatrix}
(1 + 2\varpi^n a + \varpi^{2n} a^2)x \\
1
\end{bmatrix}
= \begin{bmatrix}
(1 + \varpi^{n+c} a + \varpi^{2n} a^2)x \\
1
\end{bmatrix}
\]

because \( e, n \geq m \).

\[ \square \]

Theorem 1 is a direct consequence of the preceding lemma.

**Proof of Theorem 2**

The parametrization of characters referenced implicitly in Theorem 2 is defined as follows.

\[ \chi_A(X) := \chi(\text{tr} \ ((X - 1)A)). \tag{2} \]

The proof of Theorem 2 is a consequence of the following three lemmas.

**Lemma 2.** The pairing \( \langle A, B \rangle = \text{tr} (AB) \) of trace-zero matrices,

\[
\begin{bmatrix}
p^n & p^{n+m} \\
p^{n+l} & p^n
\end{bmatrix}
\times
\begin{bmatrix}
p^{m-n-e} & p^{m-n-l} \\
p^{n-m} & p^{m-n-e}
\end{bmatrix}
\rightarrow \mathfrak{a}, \tag{3}
\]

is bilinear and non-degenerate in the sense that if \( \langle B, A \rangle \in \mathfrak{p} \) for every trace-zero matrix, \( B \in \begin{bmatrix}
p^n & p^{n+m} \\
p^{n+l} & p^n
\end{bmatrix} \), then

\[ A \in \mathfrak{p} \cdot \begin{bmatrix}
p^{m-n-e} & p^{m-n-l} \\
p^{n-m} & p^{m-n-e}
\end{bmatrix} \]

**Proof.** This follows from direct calculation.

\[
\text{tr}
\begin{pmatrix}
\varpi^n & \varpi^{m} a_2 \\
\varpi^{n+m} a_3 & -a_1
\end{pmatrix}
\varpi^n
\begin{pmatrix}
\varpi^{m} b_1 & \varpi^{m} b_2 \\
-\varpi^{m} b_3 & -\varpi^{m} b_1
\end{pmatrix}
= \text{tr}
\begin{pmatrix}
a_1 & \varpi^{m} a_2 \\
\varpi^{n+m} a_3 & -a_1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{u} b_1 & \frac{1}{u} b_2 \\
\frac{1}{u} b_3 & -\frac{1}{u} b_1
\end{pmatrix}
= \frac{1}{2u} a_1 b_1 + a_2 b_2 + a_3 b_3 + \frac{1}{2u} a_1 b_1
= \frac{1}{u} a_1 b_1 + a_2 b_2 + a_3 b_3
\]

4
If \( v(b_i) = 0 \) setting \( a_i = 1 \) and \( a_j = 0 \) for \( j \neq i \) ensures that \( \langle B, A \rangle \not\in p \).

This is a bilinear pairing since \( \text{tr}(xBA) + \text{tr}(yCA) = \text{tr}((xB + yC)A) \) and similarly in the right-hand argument.

**Lemma 3.** The map \( A \mapsto \chi_A \) is a group homomorphism and a map of G-sets. That is,

\[
\chi_A^g(X) := \chi_A(X^g) = \chi_{A^g}(X).
\]

**Proof.** First note that by Lemma 2,

\[
\chi_A(X) \cdot \chi_B(X) = \chi(\langle X, A \rangle) \chi(\langle X, B \rangle)
= \chi(\langle X, A + B \rangle) = \chi_A + \chi_B(X).
\]

Employing the fact that the trace of a matrix is invariant under conjugation, I calculate that

\[
\chi_A^g(X^g) = \chi(\langle X^g - 1, A \rangle)
= \chi(\langle (X - 1)^g, A \rangle)
= \chi(\text{tr}((X - 1)^gA))
= \chi(\text{tr}((X - 1)A^g))
= \chi_{A^g}(X),
\]

which illustrates the preservation of the G-action.

**Lemma 4.** If \( n \geq 1 \) and \( l, m \geq -1 \), the map \( X \mapsto X - 1 \) induces an isomorphism from a multiplicative quotient of determinate-one matrices to an additive quotient of trace-zero matrices:

\[
\frac{(12 + \left(\frac{p^n}{p^{n+l}}, \frac{m^{n+m}}{p^n}\right))/(12 + \left(\frac{p^{2n}}{p^{2n+l}}, \frac{p^{2n+m}}{p^{2n}}\right))}{(12 + \left(\frac{p^n}{p^{n+l}}, \frac{p^{n+m}}{p^n}\right))}
\]

**Proof.** First note that so long as the constraints of \( l, m, \) and \( n \) are respected, the set of determinant one matrices of the form

\[
\left(\frac{12 + \left(\frac{p^n}{p^{n+l}}, \frac{m^{n+m}}{p^n}\right)}{(12 + \left(\frac{p^n}{p^{n+l}}, \frac{p^{n+m}}{p^n}\right))}
\]

is a group. Namely it is the intersection of \( K_n, B_{n+l} \) and the stabilizer of \([\frac{p}{p^{n+m}}]_{n+m} \).

\footnote{Note that the second and third lemmas hold even when \( F \) is an algebraic extension of \( \mathbb{Q}_p \) with \( p \neq 2 \).}
Again I calculate:

\[(1 + \varpi^n (a_1 \varpi^m a_2) + (a_1 \varpi^m b_2) + \varpi^{2n} (b_1 \varpi^m b_3)) \equiv (1 + \varpi^n (a_1 + b_1 \varpi^m (a_2 + b_3)) + \varpi^{2n} (x_1 \varpi^m x_2)) \equiv (1 + \varpi^n (a_1 + b_1 \varpi^m (a_2 + b_3)) + \varpi^{2n} (d - (a_1 + b_1))).\]

Where \(x_i\) are the appropriate linear combinations of the \(a_j, b_k\), \(d\) is the difference between \(a_4 + b_4\) and \(-a_1 - b_1\) and \(r\) is a remainder in \((\varpi^{2n+1} \varpi^{2n+m})\).

I must verify that \(d \in \varpi^{2n}\). To complete the proof I observe that

\[(1 + \varpi^n a_1) + \varpi^n a_4 - \varpi^{2n+m+1} a_2 a_3 = 1,
\]

so

\[a_4 + a_1 = \varpi^{2n}(\varpi^{m+1} a_2 a_3 - a_1 a_4).\]

An identical calculation for \(b_4 + b_1\) places \(d\) clearly within \(\varpi^{2n}\).

\[\square\]

I now can prove a slightly more general theorem than Theorem 2.

**Theorem 2'.** If \(n \geq 1\) and \(-1 \leq l, m \leq e\), then the set of characters of the quotient of determinant-one matrices

\[
\begin{pmatrix}
1 + \left(\frac{\varpi^n}{\varpi^{n+1}} \varpi^{n+m}\right) \\
1 + \left(\frac{\varpi^{2n}}{\varpi^{2n+1}} \varpi^{2n+m}\right)
\end{pmatrix}
\]

as a \(G\)-set is parametrized by trace-zero matrices of the form

\[
\begin{pmatrix}
\varpi^{-2n-e} / \varpi^{-n-e} \\
\varpi^{-2n-m} / \varpi^{-n-m}
\end{pmatrix}
\]

by the map

\[A \mapsto \chi_A.\]

**Proof.** Consider the function \(\chi_A\) factored into three parts:

\[\chi_A = X \mapsto (X - 1) \mapsto \text{tr} ((X - 1)A) \mapsto \chi(\text{tr} ((X - 1)A)).\]

As each factor of \(\chi_A\) is a group homomorphism (Lemmas 2 and 3 and 4), \(\chi_A\) is indeed a character.

By the linearity of the pairing \(\langle B, A \rangle\) (Lemma 2), if \(\chi_A = \chi_A', \) then

\[\chi_A / \chi_A' = \chi_{A - A'} = 1\] (Lemma 5). In such a case \(A - A' \in \left(\varpi^{-n-e} / \varpi^{-n-e}, \varpi^{-2n-m} / \varpi^{-n-m}\right)\),
by the non-degeracy of the pairing $\langle B, A \rangle$ (Lemma 2). Hence, the map is 1-to-1. Since the factor $X \mapsto X - 1$ induces an isomorphism to an abelian groups (Lemma 4), Pontryagin duality implies that the characters of the multiplicative quotient are in one-to-one correspondence with the elements of the quotient itself.

The map is onto since:

$$\left| \begin{pmatrix} p^n/p^{2n} & p^{n+m}/p^{2n+m} \\ p^{n+l}/p^{2n+l} & p^n/p^{2n} \end{pmatrix} \right| = \left| \begin{pmatrix} p^{-2n-e}/p^{-n-e} & p^{-2n-l}/p^{-n-l} \\ p^{-2n-m}/p^{-n-m} & p^{-2n-e}/p^{-n-e} \end{pmatrix} \right| .$$

\[ \square \]

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