Saddle Point Approximation for Outage Probability Using Cumulant Generating Functions

Sudarshan Guruacharya, Hina Tabassum, and Ekram Hossain
Department of Electrical and Computer Engineering, University of Manitoba, Canada
Emails: \{guruachs, hina.tabassum, ekram.hossain\}@umanitoba.ca

Abstract—This letter proposes the use of saddle point approximation (SPA) to evaluate the outage probability of wireless cellular networks. Unlike traditional numerical integration-based approaches, the SPA approach relies on cumulant generating functions (CGFs) and eliminates the need for explicit numerical integration. The approach is generic and can be applied to a wide variety of distributions, given that their CGFs exist. We illustrate the usefulness of SPA on channel fading distributions such as Nakagami-$m$, Nakagami-$q$ (Hoyt), and Rice distributions. Numerical results validate the accuracy of the proposed SPA approach.

Index Terms—Saddle point approximation, outage probability, Nakagami-$m$, Nakagami-$q$, Rice distribution.

I. INTRODUCTION

Given the instantaneous signal power $p_0$ and interfering signal powers $p_k$ from $k = 1, \ldots, L$ interferers at a wireless receiver, the signal-to-interference ratio (SIR) at the receiver is defined as $\text{SIR} = p_0/I$, where $I$ is the aggregate interference at the receiver from $L$ interferers, i.e., $I = \sum_{k=1}^{L} p_k$. By definition, the SIR outage occurs when $qI > p_0$, where $q$ is the desired SIR threshold. Introducing a new random variable $\gamma = qI - p_0$, the SIR outage probability can be given as

$$P_{\text{out}} = \Pr(\gamma > 0) = Q_\gamma(0),$$

(1)

where $Q_\gamma$ is the complementary cumulative distribution function (CCDF) of $\gamma$.

Given that $\gamma$ is a linear combination of independent random variables $p_k$, we can use the product of the moment generating functions (MGF) of the random variables to obtain the MGF of $\gamma$. The SIR outage probability in (1) can then be evaluated using Gil-Pelaez inversion theorem as [1]:

$$Q_\gamma(x) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Im}\{M_\gamma(jt)e^{-jt}\} \frac{dt}{t},$$

(2)

where $M_\gamma$ is the MGF of $\gamma$ and $\text{Im}\{z\}$ is the imaginary component of complex variable $z$. This MGF approach using an intermediate variable $\gamma$ is both theoretically elegant over direct evaluation of the CDF of SIR as well as numerically advantageous over computationally intensive Monte-Carlo methods.

The MGF approach was first described in [2] and was applied to Nakagami-$m$ fading channels. Since then the integral in (2) has been investigated in several studies. For some

1Note that signal-to-interference-plus-noise ratio (SINR) outage can also be expressed in a similar manner. Since SINR = $p_0/(I + N_0)$, where $N_0$ is the noise power, the SINR outage can be given in terms of $\gamma$ as $\Pr(\gamma > -qN_0)$.

distributions, it is possible to analytically evaluate (2) using residue calculus of complex analysis. Such an attempt was made in [2] and [3] for Nakagami-$m$ fading channels. For arbitrary distributions, the integral can be evaluated numerically. In [4], (2) was transformed into polar coordinates after which the limits of the integral become finite. This finite form is much more amenable to numerical integration techniques, and it became the basis for subsequent developments in the numerical evaluation of the outage probability. In [4], the use of Gaussian quadrature was suggested. This numerical approach was generalized to other distributions in [5], which championed the use of Gauss-Chebyshev quadrature. In [6], the authors suggested the use of trapezoidal rule and Euler summation. A numerical contour integration method, that approximates the steepest descent path, was proposed in [7]; but the authors missed the Lugannani-Rice formula already well established in the literature.

This letter supplements the aforementioned numerical approaches by proposing the use of saddle point approximation (SPA) for outage evaluations. The SPA approach relies on cumulant generating functions (CGFs) and eliminates the need for explicit numerical integration. The integration problem is instead substituted by a minimization problem. The method is generic and can handle a wide variety of distributions. To the best of our knowledge, the use of SPA has not been explored for outage computations. The possibility to use this approach to compute other metrics of interest such as bit error rate and ergodic capacity is also obvious. While the technique is generally valid, in this letter, we focus on well-known channel fading distributions such as Nakagami-$m$, Nakagami-$q$ (Hoyt), and Rice distributions. These distributions can model the empirical fast fading measurements quite well and reduce to other simple fading distributions such as Rayleigh as their special cases.

II. SADDLE POINT APPROXIMATION

SPA (also known as the method of steepest descent) is a powerful method of obtaining asymptotic approximations to Laplace type integrals of the form

$$I(z) = \int_a^b e^{zf(t)}\, g(t) \, dt$$

as $z \to \infty$. The method was introduced in [8] to approximate the PDF of sum of independent and identically distributed (IID) random variables. Based on Gil-Pelaez’s inversion formula, the Lugannani-Rice formula was derived to approximate

2The ergodic capacity is related to SINR outage by $E[C] = \frac{1}{C} (1 - P_{\text{out}}) dq$, where $C = \log_2(1 + \text{SINR})$. 

"ArXiv:1510.05557v1 [cs.IT] 19 Oct 2015"
the CDF of sum of IID random variables in [9]. As the number of variables increase, the better is the approximation. However, the method satisfactorily approximates the CDF of a single random variable as well, which is the reason for its great appeal. An easier derivation of Lugannani-Rice formula, which is more statistically flavored, was given in [10]. An introductio to these techniques can be found in [11] and [12], while a definitive exposition can be found in [13].

For an arbitrary random variable X, the SPA approach requires its MGF as well as CGF to exist. The MGF is given as
\[ M_X(t) = \mathbb{E}[e^{tX}] \]
which is more statistically flavored, was given in [10]. An introduction to these techniques can be found in [11] and [12], while a definitive exposition can be found in [13].

32] requires its MGF as well as CGF to exist. The MGF is given as 
\[ \text{absolutely,} \]
which is more statistically flavored, was given in [10]. An introduction to these techniques can be found in [11] and [12], while a definitive exposition can be found in [13].

For an arbitrary random variable X, the SPA approach requires its MGF as well as CGF to exist. The MGF is given as 
\[ M_X(t) = \mathbb{E}[e^{tX}] \]
which is more statistically flavored, was given in [10]. An introduction to these techniques can be found in [11] and [12], while a definitive exposition can be found in [13].

Ch. 32], while a definitive exposition can be found in [13].

Great appeal. An easier derivation of Lugannani-Rice formula, which is the reason for its
\[ X \]
\[ \text{number of variables increase, the better is the approximation.} \]

However, the method satisfactorily approximates the CDF of \( X \) as well, which is the reason for its great appeal. An easier derivation of Lugannani-Rice formula, which is more statistically flavored, was given in [10]. An introduction to these techniques can be found in [11] and [12], while a definitive exposition can be found in [13].

For an arbitrary random variable X, the SPA approach requires its MGF as well as CGF to exist. The MGF is given as 
\[ M_X(t) = \mathbb{E}[e^{tX}] \]
which is more statistically flavored, was given in [10]. An introduction to these techniques can be found in [11] and [12], while a definitive exposition can be found in [13].

Ch. 32], while a definitive exposition can be found in [13].

Great appeal. An easier derivation of Lugannani-Rice formula, which is the reason for its
\[ X \]
\[ \text{number of variables increase, the better is the approximation.} \]

However, the method satisfactorily approximates the CDF of \( X \) as well, which is the reason for its great appeal. An easier derivation of Lugannani-Rice formula, which is more statistically flavored, was given in [10]. An introduction to these techniques can be found in [11] and [12], while a definitive exposition can be found in [13].

For an arbitrary random variable X, the SPA approach requires its MGF as well as CGF to exist. The MGF is given as 
\[ M_X(t) = \mathbb{E}[e^{tX}] \]
which is more statistically flavored, was given in [10]. An introduction to these techniques can be found in [11] and [12], while a definitive exposition can be found in [13].

Ch. 32], while a definitive exposition can be found in [13].

Great appeal. An easier derivation of Lugannani-Rice formula, which is the reason for its
\[ X \]
\[ \text{number of variables increase, the better is the approximation.} \]

However, the method satisfactorily approximates the CDF of \( X \) as well, which is the reason for its great appeal. An easier derivation of Lugannani-Rice formula, which is more statistically flavored, was given in [10]. An introduction to these techniques can be found in [11] and [12], while a definitive exposition can be found in [13].
distributed, although this is not a necessary assumption and the SPA approach is valid of non-identical interferers as well.

A. Nakagami-m Channel

For Nakagami-m channel, the PDF of signal power \( p_k \) follows Gamma distribution given by

\[
f_{p_k}(x) = \frac{\lambda_m^m}{\Gamma(m_k)} x^{m_k-1} \exp(-\lambda_k x),
\]

where \( x \geq 0 \). The \( m_k \in [0, \infty) \) is the fading parameter and \( \lambda_k \) is defined as \( \lambda_k = m_k / \bar{p}_k \). Rayleigh fading is obtained when \( m_k = 1 \). For the case with identical interferers, let \( \lambda_k = \lambda \) and \( m_k = m \) for \( k = 1, \ldots, L \).

For the Gamma distributed \( p_k \)'s, the MGF of \( \gamma \) is given by

\[
M_{\gamma}(t) = \left(1 - \frac{gt}{\lambda}\right)^{-Lm} \left(1 + t + \frac{m_0}{\lambda_0}\right)^{-m_0},
\]

such that \( \left|\frac{gt}{\lambda}\right| < 1 \) and \( \left|\frac{m}{\lambda}\right| < 1 \). Thus, the convergence strip of \( M_{\gamma} \) is given by \( |t| < \min(\lambda_0, \lambda_0) \). Taking the logarithm of \( M_{\gamma}(t) \), we have the CGF and its derivative as

\[
K_{\gamma}(t) = -Lm \log \left(1 - \frac{gt}{\lambda}\right) - m_0 \log \left(1 + t + \frac{m_0}{\lambda_0}\right),
\]

\[
K'_{\gamma}(t) = - \frac{Lm t}{\lambda} - \frac{m_0}{\lambda_0}.
\]

For Nakagami-m fading with identical interferers, we can explicitly solve the saddle point equation (6) to obtain

\[
\hat{t}(0) = \frac{m_0 \lambda / \lambda_0 - m L \lambda_0}{m L + m_0}.
\]

Substituting the value of \( \hat{t} \) given by (10) in (4) gives us the required SIR outage. Thus, the SPA for Nakagami-m fading with identical interferers can be given in closed form.

B. Rician Channel

For Rician channel, the PDF of signal power \( p_k \) is given by

\[
f_{p_k}(x) = \frac{1 + r_k}{\bar{p}_k} \exp\left[-r_k - \frac{(1 + r_k) x}{\bar{p}_k}\right] I_0\left[2 \sqrt{\frac{r_k (r_k + 1) x}{\bar{p}_k}}\right],
\]

where \( x \geq 0 \), \( r_k \geq 0 \) is the Rice parameter, and \( I_0(\cdot) \) is the modified Bessel function of the first kind with order zero. Rayleigh fading is obtained when \( r_k = 0 \). The corresponding MGF of \( p_k \) is

\[
M_{p_k}(t) = \frac{1 + r_k}{1 + r_k - \bar{p}_k} \exp\left(\frac{r_k \bar{p}_k t}{1 + r_k - \bar{p}_k}\right),
\]

such that \( \left|\frac{r_k \bar{p}_k t}{1 + r_k - \bar{p}_k}\right| < 1 \). Taking its logarithm, the CGF and its derivatives are given by

\[
K_{p_k}(t) = \log(1 + r_k) - \log(1 + r_k - \bar{p}_k) + \frac{r_k \bar{p}_k t}{1 + r_k - \bar{p}_k t},
\]

\[
K'_{p_k}(t) = \bar{p}_k \frac{(1 + r_k)^2 - \bar{p}_k t}{1 + r_k - \bar{p}_k t^2},
\]

\[
K''_{p_k}(t) = \bar{p}_k^2 2 r_k^2 + 3 r_k - \bar{p}_k t + 1 \quad (1 + r_k - \bar{p}_k t^3).
\]

For the case with identical interferers, let \( r_k = r \) and \( \bar{p}_k = \bar{p} \) for \( k = 1, \ldots, L \). The convergence strip is given by \(|t| < \min\left(\frac{1 + r + 1 r_0}{\bar{p} \bar{p}_0}, \frac{1 + r_0}{\bar{p} \bar{p}_0}\right)\). The saddle point equation (6) then becomes a cubic polynomial. Although it is possible to solve a cubic polynomial analytically, in our context we use the Newton-Raphson method in (7) to obtain the saddle point. The first and second derivatives of \( K_{\gamma} \) can be computed as in (8) and (9), respectively. Finally, substituting the obtained value for \( t \) in the LR formula in (4) gives us the required SIR outage probability.

C. Nakagami-q (or Hoyt) Channel

For Nakagami-q channel, the PDF of signal power \( p_k \) is given by

\[
f_{p_k}(x) = \frac{1}{\bar{p}_k \sqrt{1 - b_k^2}} \exp\left[-\frac{-x}{(1 - b_k^2) \bar{p}_k}\right] I_0\left[ \frac{b_k x}{(1 - b_k^2) \bar{p}_k}\right],
\]

where \( x \geq 0 \) and \(-1 \leq b_k = \frac{1 - m_k^2}{2 m_k^2 \lambda^2} \leq 1 \). The \( m_k \in [0, \infty) \) is the fading parameter. Rayleigh fading occurs when \( b_k = 0 \). The MGF of \( p_k \) is given by

\[
M_{p_k}(t) = \frac{1}{\sqrt{(1 - t \bar{p}_k (1 + b_k)(1 - t \bar{p}_k (1 - b_k)))}},
\]

such that \(|t \bar{p}_k (1 + b_k)| < 1 \) and \(|t \bar{p}_k (1 - b_k)| < 1 \). Taking its logarithm, the CGF and its derivatives are

\[
K_{p_k}(t) = -\frac{1}{2} \log(1 - t \bar{p}_k (1 + b_k)) - \frac{1}{2} \log(1 - t \bar{p}_k (1 - b_k)),
\]

\[
K'_{p_k}(t) = \frac{(1 - b_k) \bar{p}_k}{2(1 - (1 - b_k) \bar{p}_k t)} + \frac{(1 + b_k) \bar{p}_k}{2(1 - (1 + b_k) \bar{p}_k t)},
\]

\[
K''_{p_k}(t) = \frac{(1 - b_k)^2 \bar{p}_k^2}{2(1 - (1 - b_k) \bar{p}_k t^2)} + \frac{(1 + b_k)^2 \bar{p}_k^2}{2(1 - (1 + b_k) \bar{p}_k t^2)}.
\]

For the case with identical interferers, let \( b_k = b \) and \( t \bar{p}_k = \bar{p} \) for \( k = 1, \ldots, L \). The convergence strip is given by \(|t| < \min(t_1, t_1 \bar{p}_0)\), where \( t_1 = \min(\frac{1}{\bar{p}_1 (1 + b)}, \frac{1}{\bar{p}_1 (1 - b)}, \frac{1}{\bar{p}_1 (1 + b)}\). The saddle point equation (6) again becomes a cubic polynomial. Therefore, the procedure mentioned for Rician fading channels can again be applied for Nakagami-q channels.

IV. NUMERICAL ILLUSTRATIONS

In the following, we plot the SIR outage as a function of SIR threshold \( q \). We set \( \bar{p}_0 = 5 \) dBm, \( \bar{p} = 0 \) dBm, and \( L = 5 \) for each plot; and the interferers are identically distributed. The tolerance for the Newton-Raphson method is set to \( 10^{-8} \), which is typically achieved within 6 to 10 iterations. In Fig. 1 we plot the results for Nakagami-m channels. We set \( m = 0.5 \) and \( m_0 \) is varied as \( m_0 = 0.5, 0.75, 1, 1.25, 1.5, 1.75 \). In Fig. 2 we plot the results for Rician channels where we set the Rice parameters to \( r = 0.5 \) while \( r_0 \) is varied as \( r_0 = 0, 1, 2, 3, 4 \). In Fig 3 we plot the results for Nakagami-q channels where we set the fading parameter to \( b = 1 \) and \( b_0 \) is varied as \( b_0 = 0, 1, 2, 5, 10 \). We see that the results from SPA agree with the results obtained from numerical integration of Gil-Pelaez formula. For the case of Nakagami-q channels, we see that the error is large around the transition point at lower threshold values. This error becomes worse as the value of \( b \) is increased.
To demonstrate the general validity of the method, we include Fig. 4 where we plot the outage versus threshold for Nakagami-\(m\) channels when the interferers are non-identical. The number of interferers is \(L = 4\). We set \(\bar{p}_0 = 5\) dBm and \(\bar{p}_k = 0\) dBm for all \(k = 1, \ldots, L\). The fading parameters of the four interferers are assumed to be \(m = [3.7, 3.5, 4.1, 1.7, 2.1]\) while \(m_0\) is varied as \(m_0 = 1, 2, 3, 4\). We see that the results obtained by numerically integrating Gil-Pelaez formula agree with those obtained from SPA.

V. CONCLUSION

We have discussed and demonstrated the utility of saddle point approximation for calculation of outage probability via Lugannani-Rice formula. The Nakagami-\(m\), Rice, and Nakagami-\(q\) channels have been studied in detail.

REFERENCES

[1] J. Gil-Pelaez, “Note on the inversion theorem,” Biometrika, vol. 38, pp. 481–482, 1951.
[2] Q.T. Zhang, “Outage probability of cellular mobile radio in the presence of multiple Nakagami interferers with arbitrary fading parameters,” IEEE Trans. Veh. Technol., vol. 44, no. 3, pp. 661–667, Aug. 1995.
[3] G.V.V. Sharma, “Outage probability of cellular mobile radio in the presence of multiple Nakagami interferers with similar fading parameters: A correction,” in 13th National Conference on Communications (NCC), 2007, 26-28 Jan 2007.
[4] Q.T. Zhang, “Outage probability in cellular mobile radio due to Nakagami signal and interferers with arbitrary parameters,” IEEE Trans. Veh. Technol., vol. 45, no. 2, pp. 364–372, May 1996.
[5] A. Annamalai, C. Tellambura, and V.K. Bhargava, “Simple and accurate methods for outage analysis in cellular mobile radio systems – a unified approach,” IEEE Trans. Commun., vol. 49, no. 2, pp. 303–316, Feb. 2001.
[6] Y.-C. Ko, M.S. Alouini, and M.K. Simon, “Outage probability of diversity systems over generalized fading channels,” IEEE Trans. Commun., vol. 48, no. 11, pp. 1783–1787, Nov. 2000.
[7] D. Senaratne and C. Tellambura, “A general numerical method for computing the probability of outage,” IEEE Wireless Communications and Networking Conference 2009 (WCNC 2009), 5-8 Apr. 2009.
[8] H.E. Daniels, “Saddlepoint approximations in statistics,” Ann. Math. Statist. vol. 25, no. 4, pp. 631–650, 1954.
[9] R. Lugannani and S.O. Rice, “Saddle point approximation for the distribution of the sum of independent random variables,” Advances in Applied Probability, vol. 12, no. 2, pp. 475–490, Jun. 1980.
[10] H.E. Daniels, “Tail probability approximations,” International Statistical Review, vol. 55, no. 1, pp. 37–48, Apr. 1987.
[11] C. Goutis and G. Casella, “Explaining the saddlepoint approximation,” The American Statistician, vol. 53, no. 3, pp. 216–224, Aug. 1999.
[12] J.E. Gentle, W.K. Hardle, and Y. Mori, Handbook of Computational Statistics. 2nd ed., Springer-Verlag, 2012.
[13] R.W. Butler, An Introduction to Saddlepoint Methods. Cambridge University Press, 2007.