Non-Adiabatic Solution to the Time Dependent Quantum Harmonic Oscillator

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Abstract

Using Schwinger Variational Principle we solve the problem of quantum harmonic oscillator with time dependent frequency. Here, we do not take the usual approach which implicitly assumes an adiabatic behavior for the frequency. Instead, we propose a new solution where the frequency only needs continuity in its first derivative or to have a finite set of removable discontinuities.
I. INTRODUCTION

The study of parametric time-dependent systems like time-dependent quantum harmonic oscillator is one of the most useful models for Modern Quantum Mechanics. Its applications cover areas from Quantum Optics to Cosmology.

For instance, in atomic physics, the problem of a charged particle in an electromagnetic time-dependent field was dealt in the analysis of a Paul Trap ([1, 2], and the references therein).

This kind of device brings a special attention to the simple quantum mechanical systems. Similar time-dependent problems were investigated by many authors, like Kulsrud [3] in the study of system with adiabatic behavior of the frequency and Kruskal [4] who solved the same problem using canonical transformations of time dependent systems with slowly changing of the parameters.

Lewis Jr. [5–7] et al. found a kind of dynamical invariant which leads to deal in a more general form systems like time-dependent harmonic oscillators. In 1969 Malkin, Man’ko and Trifonov in [8] solved the problem using the theory of dynamical invariants, proposing a new kind of linear invariants.

Notwithstanding, analytic solutions for the quantum parametric harmonic oscillator are known only in some sorts of adiabatic regimes. Here, we develop a new solution which does not need a slow variation of the frequency, opening a way to new applications in fast varying frequency regimes.

In order to do it we use the Schwinger Quantum Action Principle [9–14], which explores the deep connection between quantum and classical mechanics. Schwinger Principle has been used recently in several applications as field theory in curved and torsioned spaces [15–17], gauge fixing in quantum field theories [18] or even to construct a quaternionic version for the quantum theory [19]. Such approach have many advantages both from theoretical or practical points of view. Any model based in variational principles has deeper fundamental basis and at same time variational techniques can give rise to new exact and approximated solutions.

For instance, in the case of the quantum harmonic oscillator with variable frequency, the usual approach is to study the classical problem and to adapt its solutions to quantum amplitude probability by means an ansatz where the phase can be varied only adiabatically.
However, using a variational approach we are able to solve the same problem for quickly variations of the frequency. In order to see how it proceeds let us to start presenting the usual approach.

II. THE CONVENTIONAL ANSATZ

The equation of motion for the time dependent harmonic oscillator is

$$\frac{d^2 q(t)}{dt^2} + \omega^2(t) q(t) = 0. \quad (1)$$

In [3, 5] was proposed the following ansatz,

$$q(t) = S(t) e^{i\gamma(t)}, \quad (2)$$

This give us a solution of Eq. (1) if, and only if, the differential equations

$$\ddot{S}(t) + \omega^2(t) S(t) = \frac{1}{S^3(t)},$$

$$\dot{\gamma}(t) - \frac{3}{2} \ddot{\gamma}(t) - 2(\omega^2(t) - \dot{\gamma}^2(t)) \dot{\gamma}(t) = 0,$$

are satisfied, implying a very soft (adiabatic) behavior for the phase $\gamma(t)$.

From (2) one can find the propagator for the quantum system [20, 21], but the applications are restricted by the adiabatic assumption [22].

III. SCHWINGER VARIATIONAL PRINCIPLE

The Schwinger Variational principle was conceived to settle a reformulation of Quantum Mechanics without using the correspondence principle. This formulation establishes that any infinitesimal variation of a transformations function $\langle a(t_1) | b(t_0) \rangle$ can be obtained as the matrix element of a single infinitesimal generator: the quantum action operator [23, 24],

$$\delta \langle a(t_1) | b(t_0) \rangle = i \langle a(t_1) | \delta \dot{S}_{t_1, t_0} | b(t_0) \rangle = i \langle a(t_1) | \left( \hat{p} \delta \dot{q} - \hat{H} \delta t \right) \Bigg|_{t_0}^{t_1} | b(t_0) \rangle, \quad (3)$$

where $\delta \dot{S}_{t_1, t_0} = \delta \left[ \hat{S}_{t_1, t_0} \right]$ and $\dot{S}_{t_1, t_0} = \int_{t_0}^{t_1} \hat{L}(t) dt$. 

One essential apparatus of this formalism is the generator $\hat{G}$ defined by

$$\hat{G} = \hat{p} \delta \dot{q} - \hat{H} \delta t.$$
Fixing the boundary conditions on the states in (3), leads to the Schrödinger equation and fixing the boundary conditions on the operators results in the Heisenberg picture.

In order to solve (3) and to obtain the transformation function, it is necessary to order the action function such that,
\[
\delta \langle a(t_1) | b(t_0) \rangle = i \langle a(t_1) | \delta \hat{S}_{t_1,t_0} | b(t_0) \rangle = i \delta \mathcal{W}_{t_1,t_0} \langle a(t_1) | b(t_0) \rangle
\]

obtaining,
\[
\langle a(t_1) | b(t_0) \rangle = e^{i \mathcal{W}_{t_0,t_1}}. \tag{4}
\]

IV. THE NON-ADIABATIC SOLUTION

Instead of using (2) we propose a new solution given by
\[
q(t) = A(t) \exp \left[ \pm i \int_{t_0}^{t} \omega(\tau) d\tau \right] \tag{5}
\]
which imply a second order differential equation
\[
\ddot{A}(t) + 2i \omega(t) \dot{A}(t) + i \dot{\omega}(t) A(t) = 0, \tag{6}
\]
for the amplitude. Here, one needs only a frequency function \( \omega(t) \) with first derivative continuous or having a finite set of removable discontinuities.

On this way, the general solution for the quantum analog problem given in (1) is
\[
\hat{q} = \frac{\hat{q}_0}{C(t_0)} F_1(t) + \frac{\hat{p}_0}{m C(t_0)} F_0(t), \tag{7}
\]
where
\[
F_0(t) = A_0 A^*(t) \exp \left[ -i \int_{t_0}^{t} \omega(\tau) d\tau \right] - \text{c.c.}, \]
\[
F_1(t) = \left[ \dot{A}_0^* - i \omega_0 A_0^* \right] A(t) \exp \left[ i \int_{t_0}^{t} \omega(\tau) d\tau \right] - \text{c.c.},
\]
\[
C(t) = A^*(t) \dot{A}(t) - A(t) \dot{A}^*(t) + 2i \omega(t) |A(t)|^2
\]

V. QUANTUM TRANSITION AMPLITUDE

Taking the non-adiabatic solution for \( \hat{q}(t) \) we can find the canonical momentum
\[
\dot{\hat{p}} = -m \dot{\hat{q}} \frac{C(t)}{F_0(t)} + \hat{q} \frac{\dot{F}_0(t)}{F_0(t)}, \tag{8}
\]
The commutator of $\hat{q}$ in different times can be reached using the canonical relation $[\hat{q}, \hat{p}] = i\hbar$,

$$\hat{q}_0 \hat{q} = \frac{i\hbar F_0(t)}{mC(t)} + \hat{q}_0.$$  \hfill (9)

The quantum Hamiltonian has the form:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{q}^2 = \frac{m}{2}\left[q^2\frac{\dot{F}_0^2(t)}{F_0^2(t)} + \hat{q}_0^2\frac{C^2(t)}{F_0^2(t)} - 2C(t)\frac{\dot{F}_0(t)}{F_0^2(t)}\hat{q}_0 - \frac{i\hbar}{2}\hat{F}_0(t) + \frac{m}{2}\omega^2(t)\hat{q}^2\right].$$

Therefore, the transition amplitude $\langle q, t\mid q_0, t_0 \rangle$ in the Schwinger formulation is given for

$$\langle q, t\mid q_0, t_0 \rangle = A(q, q_0) \exp \left\{-\frac{im}{2\hbar} \int_{t_0}^{t} H(t) dt \right\} = A(q, q_0) \exp \left\{\frac{im}{2\hbar F_0(t)} \left(q^2\frac{\dot{F}_0(t)}{F_0(t)} + \hat{q}_0^2F_1(t) - q_0C(t)\right)\right\}$$

where one can recognize the classical action

$$S(q, q_0, t) = \frac{m}{2F_0(t)} \left(q^2\frac{\dot{F}_0(t)}{F_0(t)} + \hat{q}_0^2F_1(t) - q_0C(t)\right).$$

We can verify this expression, recovering the action for the harmonic oscillator with constant frequency taking the regime $\omega(t) \to \omega_0$, and the amplitudes $A(t)$ constants, then

$$\lim_{\omega(t) \to \omega_0} S(q, q_0, t) = \frac{m}{2\sin \left[\omega_0(t - t_0)\right]} \left(q^2 + \hat{q}_0^2\right) \cos \left[\omega_0(t - t_0)\right] - q_0.$$

In order to fix the explicit form of $A(q, q_0)$ we use

$$\frac{\partial \langle q, t\mid q_0, t_0 \rangle}{\partial q} = \frac{i}{\hbar} \langle q, t\mid \hat{p}\mid q_0, t_0 \rangle,$$

obtaining

$$\frac{\partial \langle q, t\mid q_0, t_0 \rangle}{\partial q} = \left(\frac{1}{A(q, q_0)} \frac{\partial A(q, q_0)}{\partial q} + \frac{i}{\hbar} q \frac{\dot{F}_0(t)}{F_0(t)} - \frac{i}{\hbar} q_0 m C(t) - \frac{i}{\hbar} q_0 m C(t) \right) \langle q, t\mid q_0, t_0 \rangle.$$  \hfill (10)

and comparing with the expressions for momentum $\hat{p}$ and position $\hat{q}$ we obtain in both of cases.
\[
\frac{\partial A(q, q_0)}{\partial q} = \frac{\partial A(q, q_0)}{\partial q_0} = 0,
\]
then the function \(A(q, q_0) = A(t)\) only depends on time and

\[
\langle q, t|q_0, t_0 \rangle = \frac{A(t)}{\sqrt{F_0(t)}} \exp \left\{ \frac{i}{\hbar} \frac{m}{2F_0(t)} \left( q^2 \dot{F}_0(t) + q_0^2 F_1(t) - 2qq_0 C(t) \right) \right\},
\]
and the explicit form of \(A\) can be obtained from \(\lim_{t_0 \to t_1} \langle q|q_0 \rangle = \delta(q - q_0)\), comparing with a suitable sequence of functions converging to the Dirac delta, like the following

\[
\lim_{t_0 \to t_1} \langle q, t|q_0, t_0 \rangle = \lim_{t_0 \to t_1} \frac{K}{\sqrt{F_0(t)}} \exp \left\{ -\frac{i}{\hbar} \frac{mC(t_0)}{2F_0(t)} (q - q_0)^2 \right\}.
\]
then one finds \(A(t) = \sqrt{\frac{imC(t)}{2\pi\hbar}}\), and finally the final form for the transformation function:

\[
\langle q, t|q_0, t_0 \rangle = \sqrt{\frac{imC(t)}{2\pi\hbar F_0(t)}} \exp \left\{ \frac{i}{\hbar} \frac{m}{2F_0(t)} \left( q^2 \dot{F}_0(t) + q_0^2 F_1(t) - 2qq_0 C(t) \right) \right\}.
\]

VI. CONCLUSIONS AND PERSPECTIVES

The last result can be applied in many areas as the study of electromagnetic cavities or any other quantum particle interacting with a classical variable harmonic potential. When we deal with a more realistic behavior for the external fields it is common to find a non-adiabatic variation of its parameters. Therefore, the presented solution provides a more accurate description in this case, with the advantage of to reproduce the know results for adiabatic regimes.

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[1] W. Paul, H. Steinwedel, Zeitschrift für Naturforschung A 8 (1953) 448-450.
[2] Wolfgang Paul, Rev. Mod. Phys 62, (1990) 531.
[3] Russell M. Kulsrud, Phys.Rev. 106, (1957) 205.
[4] Martin Kruskal, *Jour. Math. Phys.* 3, (1962) 4.

[5] H. R. Lewis, Jr., *Phys. Rev. Lett.* 18, (1967) 510.

[6] H. R. Lewis, Jr., *Jour. Math. Phys.* 9, (1968) 1976.

[7] H. R. Lewis, Jr. and W. B. Riessenfeld, *Jour. Math. Phys.* 10, (1969) 1458.

[8] I. A. Malkin, V. I. Man'ko and D. A. Trifonov, *Phys. Lett.* 30A, (1969) 7.

[9] J. S. Schwinger, *Phys. Rev.* 82, (1951) 914.

[10] J. S. Schwinger, *Phys. Rev.* 91, (1953) 713.

[11] J. S. Schwinger, *Phys. Rev.* 91, (1953) 728.

[12] J. S. Schwinger, *Phys. Rev.* 92, (1953) 1283.

[13] J. S. Schwinger, *Phys. Rev.* 93, (1954) 615.

[14] J. S. Schwinger, *Phys. Rev.* 94, (1954) 1362.

[15] R. Casana, C. A. de Melo and B. M. Pimentel, *Class. Quant. Grav.* 24, (2007) 723.

[16] R. Casana, C. A. de Melo and B. M. Pimentel, *Astrophys. Sp. Sci.* 305, (2006) 125.

[17] R. Casana, C. A. de Melo and B. M. Pimentel, *Braz. J. Phys.* 35, (2005) 1151.

[18] C. A. M. de Melo, B. M. Pimentel and P. J. Pompeia, *Il Nuovo Cim.* B121, (2006) 193.

[19] C. A. M. de Melo and B. M. Pimentel, *Adv. App. Clif. Alg.* DOI: 10.1007/s00006-010-0234-8 (2010). Available online.

[20] D. C. Khandekar and S. V. Lawande, *Jour. Math. Phys.* 16 (1975) 384.

[21] C. Farina and A. J. S. Santoja, *Phys. Lett.* A184 (1993) 23.

[22] W. Dietrich and M. Reuter, *Classical and Quantum Dynamics: From Classical paths to Path Integrals*, Chap. 8, 3rd edition (Springer-Verlag, 2001).

[23] J. S. Schwinger, Quantum Kinematics and Dynamics, (W.A. Benjamin Publishers, 1970);

[24] J. S. Schwinger, *Quantum Mechanics: Symbolism of Atomic Measurements* (Springer, 2001).

[25] V. V. Dodonov, A. B. Klimov and D. E. Nikonov, *Phys. Rev.* A47 (1993) 4442.

[26] V. V. Dodonov, V. I. Man’ko and D. E. Nikonov, *Phys. Rev.* A51 (1995) 3328.

[27] V.V. Dodonov and A.V. Dodonov, *J. Rus. Laser Research* 26 (2005) 6.