We consider state redistribution of a “hybrid” information source that has both classical and quantum components. The sender transmits classical and quantum information at the same time to the receiver, in the presence of classical and quantum side information both at the sender and at the decoder. The available resources are shared entanglement, and noiseless classical and quantum communication channels. We derive one-shot direct and converse bounds for these three resources, represented in terms of the smooth conditional entropies of the source state. Various coding theorems for two-party source coding problems are systematically obtained by reduction from our results, including the ones that have not been addressed in previous literatures.

1 Introduction

Quantum state redistribution is a task in which the sender aims at transmitting quantum states to the receiver, in the presence of quantum side information both at the sender and at the receiver. The costs of quantum communication and entanglement required for state redistribution have been analyzed in [35, 13, 36] for the asymptotic scenario of infinitely many copies and vanishingly small error, and in [7, 10, 2] for the one-shot scenario. Various coding theorems for two-party quantum source coding problems are obtained by reduction from these results as special cases, such as the Schumacher compression [24], quantum state merging [17] and the fully-quantum Slepian-Wolf [1, 9]. However, some of the well-known coding theorems cannot be obtained from those results, such as the (fully-classical) Slepian-Wolf (see e.g. [8]) and the classical data compression with quantum side information [11]. This is because the results in [35, 13, 36, 7] only cover the fully quantum scenario, in which the information to be transmitted and the available resources are both quantum.

In this paper, we generalize the one-shot state redistribution theorem in [7] to a “hybrid” situation. That is, we consider the task of state redistribution in which the information to be transmitted and the side information at the parties have both classical and quantum components. Not only quantum communication and shared entanglement, but also classical communication is available as a resource. Our goal is to derive trade-off relations among the costs of the three resources required for achieving the task within a small error. The main result is that we provide the direct and the converse bounds for the rate triplet to be achievable, in terms of the smooth conditional entropies of the source state and the error tolerance. For most of the special cases that have been analyzed in the previous literatures, the two bounds match in the asymptotic limit of infinitely many copies and vanishingly small error, providing the full characterization of the achievable rate region. Our result can be viewed as a one-shot generalization of the classically-
assisted state redistribution protocol, proposed in [18].

Coding theorems for most of the redistribution-type protocols, not only for quantum or classical information source but also for hybrid one, in one-shot scenario are systematically obtained from our result by reduction. In this sense, our result completes the one-shot capacity theorems of the redistribution-type protocols in a standard setting. As examples, we show that the coding theorems for the fully quantum state redistribution, the fully quantum Slepian-Wolf, quantum state splitting, quantum state merging, classical data compression with quantum side information, quantum data compression with classical side information and the fully classical Slepian-Wolf and quantum state redistribution with classical side information only at the decoder [3] can be recovered. The last one would further lead to the family of quantum protocols in the presence of classical side information only at the decoder, along the same line as the one without classical side information [1, 12]. In addition, our result also covers some redistribution-type protocols that have not been addressed in the previous literatures.

We denote the set of linear operators on a Hilbert space \( \mathcal{H} \) by \( \mathcal{L}(\mathcal{H}) \). For normalized density operators and sub-normalized density operators, we use the following notations, respectively:

\[
\mathcal{S}_+(\mathcal{H}) = \{ \rho \in \mathcal{L}(\mathcal{H}) : \rho \geq 0, \text{Tr}[\rho] = 1 \}, \quad (1)
\]
\[
\mathcal{S}_-(\mathcal{H}) = \{ \rho \in \mathcal{L}(\mathcal{H}) : \rho \geq 0, \text{Tr}[\rho] \leq 1 \}. \quad (2)
\]

A Hilbert space associated with a quantum system \( A \) is denoted by \( \mathcal{H}^A \), and its dimension is denoted by \( d_A \). A system composed of two subsystems \( A \) and \( B \) is denoted by \( AB \). When \( M \) and \( N \) are linear operators on \( \mathcal{H}^A \) and \( \mathcal{H}^B \), respectively, we denote \( M \otimes N \) as \( M^A \otimes N^B \) for clarity. In the case of pure states, we abbreviate \(|\psi\rangle^A \otimes |\phi\rangle^B\) as \(|\psi\rangle^A |\phi\rangle^B\). We denote \(|\psi\rangle \langle \psi|\) simply by \( \psi \).

For \( \rho^{AB} \in \mathcal{L}(\mathcal{H}^{AB}) \), \( \rho^A \) represents \( \text{Tr}_B[\rho^{AB}] \). The identity operator is denoted by \( I \). We denote \((M^A \otimes I^B)|\psi\rangle^{AB}\) as \( M^A |\psi\rangle^{AB} \) and \((M^A \otimes I^B)\rho^{AB} (M^A \otimes I^B)^\dagger\) as \( M^A \rho^{AB} M^A \). When \( \mathcal{E} \)

Figure 1: The task of state redistribution for the classical-quantum hybrid source is depicted. The black dots and the circles represent classical and quantum parts of the information source, respectively. The wavy line represents the entanglement resource.
is a supermap from \( \mathcal{L}(\mathcal{H}^A) \) to \( \mathcal{L}(\mathcal{H}^B) \), we denote it by \( \mathcal{E}^{A\to B} \). When \( A = B \), we use \( \mathcal{E}^A \) for short. We also denote \((\mathcal{E}^{A\to B} \otimes \text{id}^C)(\rho^{AC})\) by \( \mathcal{E}^{A\to B}(\rho^{AC}) \). When a supermap is given by a conjugation of a unitary \( U^A \) or a linear operator \( W^A \), we especially denote it by its calligraphic font such as \( \mathcal{U}^A(X^A) := (U^A)X^A(U^A)\) and \( \mathcal{W}^{A\to B}(X^A) := (W^{A\to B})X^A(W^{A\to B})\).

The maximally entangled state between \( A \) and \( A' \), where \( \mathcal{H}^A \cong \mathcal{H}^{A'} \), is defined by

\[
|\Phi\rangle^{AA'} := \frac{1}{\sqrt{d_A}} \sum_{\alpha=1}^{d_A} |\alpha\rangle^A|\alpha\rangle^{A'}
\] (3)

with respect to a fixed orthonormal basis \( \{|\alpha\rangle\}_{\alpha=1}^{d_A} \). The maximally mixed state on \( A \) is defined by \( \pi^A := I^A/d_A \).

For any linear CP map \( \mathcal{T}^{A\to B} \), there exists a finite dimensional quantum system \( E \) and a linear operator \( W^{A\to BE} \) such that \( \mathcal{T}^{A\to B}(\cdot) = \text{Tr}_E[W_{\mathcal{T}}(\cdot)W_{\mathcal{T}}^+] \). The operator \( W_{\mathcal{T}} \) is called a Stinespring dilation of \( \mathcal{T}^{A\to B} \) [25], and the linear CP map defined by \( \text{Tr}_B[W_{\mathcal{T}}(\cdot)W_{\mathcal{T}}^+] \) is called a complementary map of \( \mathcal{T}^{A\to B} \). With a slight abuse of notation, we denote the complementary map by \( \mathcal{T}^{A\to E} \).

### 2.2 Norms and Distances

For a linear operator \( X \), the trace norm is defined as \( |X|_1 := \text{Tr}[\sqrt{X^*X}] \). For subnormalized states \( \rho, \sigma \in \mathcal{S}_c(\mathcal{H}) \), the trace distance is defined by \( \|\rho - \sigma\|_1 \). The generalized fidelity and the purified distance are defined by

\[
\tilde{F}(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1 + \sqrt{(1 - \text{Tr}[\rho])(1 - \text{Tr}[\sigma])}
\] (4)

and

\[
P(\rho, \sigma) := \sqrt{1 - F(\rho, \sigma)^2},
\] (5)

respectively (see Lemma 3 in [29]). The trace distance and the purified distance are related as

\[
\frac{1}{2}\|\rho - \sigma\|_1 \leq P(\rho, \sigma) \leq \sqrt{2}\|\rho - \sigma\|_1
\] (6)

for any \( \rho, \sigma \in \mathcal{S}_c(\mathcal{H}) \). The epsilon ball of a subnormalized state \( \rho \in \mathcal{S}_c(\mathcal{H}) \) is defined by

\[
B^\epsilon(\rho) := \{\tau \in \mathcal{S}_c(\mathcal{H})| P(\rho, \tau) \leq \epsilon\}.
\] (7)

### 2.3 One-Shot Entropies

For any subnormalized state \( \rho \in \mathcal{S}_c(\mathcal{H}^{AB}) \) and normalized state \( \varsigma \in \mathcal{S}_c(\mathcal{H}^B) \), define

\[
H_{\min}(A|B)_{\rho|\varsigma} := \sup\{\lambda \in \mathbb{R}| 2^{-\lambda}I^A \otimes \varsigma \geq \rho^{AB}\}
\] (8)

and

\[
H_{\max}(A|B)_{\rho|\varsigma} := \log \|\sqrt{\rho^{AB}} \sqrt{I^A \otimes \varsigma}^{1/2} \|_1.
\] (9)

The conditional min- and max- entropies (see e.g. [26]) are defined by

\[
H_{\min}(A|B)_{\rho} := \sup_{\sigma^B \in \mathcal{S}_c(\mathcal{H}^B)} H_{\min}(A|B)_{\rho|\sigma},
\] (10)

\[
H_{\max}(A|B)_{\rho} := \sup_{\sigma^B \in \mathcal{S}_c(\mathcal{H}^B)} H_{\max}(A|B)_{\rho|\sigma},
\] (11)

and the smoothed versions thereof are given by

\[
H_{\min}^\epsilon(A|B)_{\rho} := \sup_{\rho^{AB} \in \mathcal{B}(\rho)} H_{\min}(A|B)_{\rho},
\] (12)

\[
H_{\max}^\epsilon(A|B)_{\rho} := \inf_{\rho^{AB} \in \mathcal{B}(\rho)} H_{\max}(A|B)_{\rho}
\] (13)

for \( \epsilon \geq 0 \). In the case where \( B \) is a trivial (one-dimensional) system, we simply denote them as \( H_{\min}(A)_{\rho} \) and \( H_{\max}(A)_{\rho} \), respectively. We define

\[
H_{\min}^{k}(A|B)_{\rho} := \max\{H_{\min}(A|B)_{\rho}, H_{\max}(A|B)_{\rho}\}
\] (14)

and

\[
\tilde{I}_\epsilon(A : C|B)_{\rho} := H_{\min}^\epsilon(A|B)_{\rho} - H_{\min}(A|BC)_{\rho}.
\] (15)

We will refer to (15) as the smooth conditional min mutual information. For \( \tau \in \mathcal{S}(\mathcal{H}^A) \), we also use the “max entropy” in the version of [23] (see Section 3.1.1 therein). Taking the smoothing into account, it is defined by

\[
H_{\max}^\epsilon(A|B)_\tau := \inf_{\Pi : \text{Tr}[\Pi \tau] \geq 1 - \epsilon} \log \text{rank}[\Pi],
\] (16)

where the infimum is taken over all projections \( \Pi \) such that \( \text{Tr}[\Pi \tau] \geq 1 - \epsilon \). The von Neumann entropies and the quantum mutual information are defined by

\[
H(A)_{\rho} := -\text{Tr}[\rho^A \log \rho^A],
\] (17)

\[
H(A|B)_{\rho} := H(AB)_{\rho} - H(B)_{\rho},
\] (18)

\[
I(A : B)_{\rho} := H(A)_{\rho} - H(A|B)_{\rho}.
\] (19)

The properties of the smooth conditional entropies used in this paper are summarized in Appendix A.
3 Formulation and Results

Consider a classical-quantum source state in the form of
\[
\Psi_s^{ABCXYZXY'Z'} := \sum_{x,y,z} p_{xyz} |x\rangle_x |y\rangle_y |z\rangle_z \otimes |\psi_{xyz}\rangle_{ABCR} |xyz\rangle_{XY'Z'}.
\]
(20)

Here, \(p_{xyz}\) is a probability distribution, \(|\psi_{xyz}\rangle\) are pure states, and \(|x\rangle_x, |y\rangle_y, |z\rangle_z, |xyz\rangle\) are orthonormal bases. The systems \(X', Y'\) and \(Z'\) are assumed to be isomorphic to \(X, Y\) and \(Z\), respectively. For the simplicity of notations, we denote \(AX, BY, CZ, X'Y'Z'\) and \(RX'Y'Z'\) by \(A, B, C, T\) and \(R\), respectively. Accordingly, we also denote the source state by \(\Psi_s^{ABCR}\).

We consider a task in which the sender transmits \(C\) to the receiver (see Figure 1 and 2). The sender and the receiver have access to systems \(A\) and \(B\), respectively, as side information. The system \(R\) is the reference system that is inaccessible to the sender and the receiver. The available resources for the task are the one-way noiseless classical and quantum channels from the sender to the receiver, and an entangled state shared in advance between the sender and the receiver. We describe the communication resources by a quantum system \(Q\) with dimension \(2^c\) and a “classical” system \(M\) with dimension \(2^e\). The entanglement resources shared between the sender and the receiver, before and after the protocol, are given by the maximally entangled states \(\Phi^{FAE_B}_{2^e} \otimes \Phi^{FB}_{2^e}\) with Schmidt rank \(2^{e+e_0}\) and \(2^{e_0}\), respectively.

**Definition 1** A tuple \((c, q, e, e_0)\) is said to be achievable within an error \(\delta\) for \(\Psi_s\), if there exists a pair of an encoding CPTP map \(E^{ABCR}_{\Psi_s} \rightarrow ABQMF_A\) and a decoding CPTP map \(D^{BQME_B} \rightarrow BC\Phi_{\Psi_s}\), such that
\[
\| D \circ E (\Psi_s^{ABCR} \otimes \Phi^{FAE_B}_{2^e}) - \Psi_s^{ABCR} \otimes \Phi^{FB}_{2^e} \|_1 \leq \delta.
\]
(21)

Note that, since \(M\) is a classical message, the encoding CPTP map \(E\) must be such that for any input state \(\tau\), the output state \(\Psi_s^{ABCR} \rightarrow ABQMF_A(\tau)\) is diagonal in \(M\) with respect to a fixed orthonormal basis. Note also that we implicitly assume that \(c, q, e_0 \geq 0\), while the net entanglement cost \(e\) can be negative.

Our goal is to obtain necessary and sufficient conditions for a tuple \((c, q, e, e_0)\) to be achievable within the error \(\delta\) for a given source state \(\Psi_s\). The direct and converse bounds are given by the following theorems:

**Theorem 2** (Direct part.) A tuple \((c, q, e, e_0)\) is achievable within an error \(4\sqrt{12\epsilon + 6\delta} + \sqrt{2\epsilon}\) for \(\Psi_s\) if \(d_C \geq 2\) and it holds that
\[
c + 2q \geq \max \{ H^c_{1/2}(\epsilon/2), H^c_{1/2}(\epsilon/2) \} - \log (\delta^4/2),
\]
(22)
\[
c + q + e \geq H^c_{\max}(CY|BY)\Psi_s - \log (\delta^2/2),
\]
(23)
\[
q + e \geq H^c_{\max}(C|BXYZ)\Psi_s - \log \delta^2,
\]
(24)
\[
e_0 \geq \frac{1}{2} (H^c_{\max}(C)\Psi_s - H^c_{\max}(C|BXYZ)\Psi_s) + \log \delta,
\]
(25)
where
\[ \tilde{H}_I^{(c,\epsilon)} := H_{\text{max}}(C|AXYZ)_{\Psi_s}, \quad \tilde{H}_H^{(c,\epsilon)} := H_{\text{max}}(C|AXZ)_{\Psi_s}, \]
and \( H_{\text{max}}(\cdot) \) is defined by (14).

In the case where \( d_C = 1 \), a tuple \((c,0,0,0)\) is achievable for \( \Psi_s \) within the error \( \delta \) if it holds that
\[ c \geq H_{\text{max}}(Z|BY)_{\Psi_s} - \log \frac{\delta^2}{2}, \]
where we informally denoted \( \psi_{xyz}^{AR} \otimes |xyz\rangle \langle xyz|^T \) by \( \psi_{xyz}^{AR} \).

The proofs of Theorem 2 and Theorem 3 will be provided in Section 5 and Section 6, respectively.

We also consider an asymptotic scenario of infinitely many copies and vanishingly small error. A rate triplet \((c,q,e)\) is said to be asymptotically achievable if, for any \( \delta > 0 \) and sufficiently large \( n \in \mathbb{N} \), there exists \( e_0 \geq 0 \) such that the tuple \((ne,nq,ne,ne_0)\) is achievable within the error \( \delta \) for the one-shot redistribution of the state \( \Psi_s^\otimes n \).

## Theorem 3 (Converse part.)
Suppose that a tuple \((c,q,e,e_0)\) is achievable within the error \( \delta \) for \( \Psi_s \). Then, regardless of the value of \( e_0 \), it holds that
\[
\begin{align*}
&c + 2q \geq \max \{ \tilde{H}_I^{(c,\epsilon)}, \tilde{H}_H^{(c,\epsilon)}, -\Delta^{(c,\epsilon)} \} - 6f(\epsilon), \\
&c + q + e \geq H_{\text{min}}(BYCZ)_{\Psi_s} - H_{\text{min}}^{12+6\sqrt{\delta}}(BY)_{\Psi_s} - f(\epsilon), \\
&q + e \geq H_{\text{min}}(BC|XYZ)_{\Psi_s} - H_{\text{min}}^{11+8\sqrt{\delta}}(B|XYZ)_{\Psi_s} - 2f(\epsilon)
\end{align*}
\]
for any \( \epsilon > 0 \).

Here, \( f(x) := -\log(1 - \sqrt{1 - x^2}) \), and
\[
\begin{align*}
\tilde{H}_I^{(c,\epsilon)} &:= H_{\text{min}}(AC|XYZ)_{\Psi_s} - H_{\text{max}}(A|XYZ)_{\Psi_s}, \\
\tilde{H}_H^{(c,\epsilon)} &:= H_{\text{min}}(AXCZ)_{\Psi_s} - H_{\text{max}}(AXZ)_{\Psi_s} + H_{\text{min}}(BC|XYZ)_{\Psi_s} - H_{\text{min}}^{11+8\sqrt{\delta}}(B|XYZ)_{\Psi_s}
\end{align*}
\]
and
\[
\Delta^{(c,\epsilon)} := \sup_{F} \tilde{H}_{\text{min}}^{12+4\sqrt{\delta}}(G_A : Y' | MAAX'Z')_{F(\Psi_s)},
\]
where we formally denoted \( \psi_{xyz}^{AR} \otimes |xyz\rangle \langle xyz|^T \) by \( \psi_{xyz}^{AR} \).

The supremum in (34) is taken over all CPTP maps \( F : \hat{A}C \rightarrow AG_A \) such that \( F(\tau) \) is diagonal in \( M_A \) with a fixed orthonormal basis for any \( \tau \in \mathcal{S}(H^{\hat{A}C}) \), and
\[
\inf_{\{ \omega_{xyz} \}} \left( \psi_{xyz}^{AR} \otimes \omega_{xyz}^{G_A M_A} \right) \leq 2\sqrt{\delta},
\]
where we formally denoted \( \psi_{xyz}^{AR} \otimes |xyz\rangle \langle xyz|^T \) by \( \psi_{xyz}^{AR} \).

Theorem 4 (Asymptotic limit.) In the asymptotic limit of infinitely many copies and vanishingly small error, the inner and outer bounds for the achievable rate region are given by
\[
\begin{align*}
&c + 2q \geq \max \{ \tilde{H}_I, \tilde{H}_H \}, \\
&c + q + e \geq H(C|Z|BY)_{\Psi_s}, \\
&q + e \geq H(C|BXYZ)_{\Psi_s}, \\
&e_0 \geq \frac{1}{2} I(C : BXYZ)_{\Psi_s}
\end{align*}
\]
and
\[
\begin{align*}
&c + 2q \geq \max \{ \tilde{H}_I, \tilde{H}_H - \tilde{\Delta} \}, \\
&c + q + e \geq H(C|Z|BY)_{\Psi_s}, \\
&q + e \geq H(C|BXYZ)_{\Psi_s},
\end{align*}
\]
respectively. Here,
\[
\tilde{H}_I := H(C|AXYZ)_{\Psi_s} + H(CZ|BY)_{\Psi_s}, \\
\tilde{H}_H := H(C|AXZ)_{\Psi_s} + H(C|BXYZ)_{\Psi_s}
\]
and
\[
\tilde{\Delta} := \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \Delta^{(c,\epsilon)}(\Psi_{s_n}^\otimes n),
\]
where \( \Delta^{(c,\epsilon)} \) is defined in Theorem 3.

Theorem 4 immediately follows from the one-shot direct and converse bounds (Theorem 2 and Theorem 3). This is due to the fully-quantum asymptotic equipartition property [28], which implies that the smooth conditional entropies are equal to the von Neumann conditional entropy in the asymptotic limit of infinitely many copies.
That is, for any $\rho \in \mathcal{S}_{\mathcal{X}}(\mathcal{H}^{PQ})$ and $\epsilon > 0$, it holds that
\[
\lim_{n \to \infty} \frac{1}{n} H_{\min}^n(P^n|Q^n)_{\rho^\otimes n} = \lim_{n \to \infty} \frac{1}{n} H_{\max}^n(P^n|Q^n)_{\rho^\otimes n} = H(P|Q)_{\rho}.
\]

A simple calculation using this relation and the chain rule of the conditional entropy implies that the R.H.S.s of (22)-(25) and (29)-(31) coincide with those of (36)-(39) and (40)-(42), respectively, in the asymptotic limit of infinitely many copies.

Due to the existence of the term $\Delta$ in Inequality (40), the direct and converse bounds in Theorem 4 do not match in general. In many cases, however, it holds that $\Delta = 0$ and thus the two bounds matches. This is due to the following lemma about the property of $\Delta(\epsilon, \delta)$:

**Lemma 5** The quantity $\Delta(\epsilon, \delta)$ defined in Theorem 3 is nonnegative, and is equal to zero if there is no classical side information at the decoder (i.e. $\dim Y = \dim Y' = 1$) or if there is neither quantum message nor quantum side information at the encoder (i.e. $\dim A = \dim C = 1$). The quantity $\Delta$ satisfies the same property due to the definition (45).

A proof of Lemma 5 will be provided in Section 6.4. To clarify the general condition under which $\Delta = 0$ is left as an open problem.

**Remark.** The results presented in this section are applicable to the case where the sender and the receiver can make use of the resource of classical shared randomness. To this end, it is only necessary to incorporate the classical shared randomness as a part of classical side information $X$ and $Y$.

### 4 Reduction to Special Cases

In this section, we apply the results presented in Section 3 to special cases of source coding (see Figure 3 in the next page). In principle, the results cover all special cases where some of the components $A$, $B$, $C$, $X$, $Y$ or $Z$ are assumed to be one-dimensional, and where $c$, $q$ or $e$ is assumed to be zero.

Among them, we particularly consider the cases with no classical component in the source state and with no side information at the encoder, which have been analyzed in previous literatures. We also consider quantum state redistribution with classical side information at the decoder, which has not been addressed before. We investigate both the one-shot and the asymptotic scenarios. The one-shot direct and converse bounds are obtained from Theorem 2 and Theorem 3, respectively, and the asymptotic rate region is obtained from Theorem 4. The analysis presented below shows that, for the tasks that have been analyzed in previous literatures, the bounds obtained from our results coincide with the ones obtained in the literatures. It should be noted, however, that the coincidence in the one-shot scenario is only up to changes of the types of entropies and the values of the smoothing parameters. All entropies are for the source state $\Psi_s$. We will use Lemma 21 in Appendix A for the calculation of entropies.

#### 4.1 No Classical Component in The Source State

First, we consider the case where there is no classical component in the source state. It is described by setting $X = Y = Z = \emptyset$. By imposing several additional assumptions, the scenario reduces to different protocols.

##### 4.1.1 Fully Quantum State Redistribution

Our hybrid scenario of state redistribution reduces to the fully quantum scenario, by additionally assuming that $c = 0$. The one-shot direct part is given by

\[
2q \geq H_{2/2, e/2}(C|A) + H_{\max}^{2/2}(C|B) - \log (\delta^2/2), \quad (48)
\]
\[
q + e \geq H_{\max}(C|B) - \log (\delta^2/2), \quad (49)
\]
\[
e_0 \geq \frac{1}{2}(H_{\max}^{2/8}(C) - H_{\max}^{2/8}(C|B)) + \log \delta. \quad (50)
\]
Figure 3: The relation among special cases of communication scenario analyzed in Section 4 are depicted. “SI” and “SI-D” stand for “side information” and “side information at the decoder”, respectively. See Table 1 below for the notations.

| information source | available resources |
|---------------------|---------------------|
| side information at the encoder | communication |
| side information at the decoder | shared correlation |
| to be transmitted | |
| quantum | A | B | C | q | e |
| classical | X | Y | Z | c | - |

Table 1

An example of the tuple satisfying the above conditions is

\[ q = \frac{1}{2} \left( H_{\text{max}}^{(3\epsilon/2,\epsilon/2)}(C|A) + H_{\text{max}}(C|B) \right) - \log(\delta^4/2), \]  

\[ e = \frac{1}{2} \left( -H_{\text{max}}(C|A) + H_{\text{max}}^{(3\epsilon/2,\epsilon/2)}(C|B) + 1 \right), \]  

\[ e_0 = \frac{1}{2} \left( H_{\text{max}}^{3\epsilon/2}(C) - H_{\text{max}}^{3\epsilon/2}(C|B) \right) + \log\delta. \]  

The achievability of \( q \) and \( e \) given by (51) and (52) coincides with the result of [7] (see also [2]). The one-shot converse bound is represented as

\[ 2q \geq H_{\text{min}}^c(AC) - H_{\text{max}}^c(A) + H_{\text{min}}^e(BC) \]  

\[ - H_{\text{min}}^{12\epsilon/3\epsilon+6\sqrt{7}}(B) - 8f(\epsilon), \]  

\[ q + e \geq H_{\text{min}}^c(BC) - H_{\text{max}}^c(BC) + f(\epsilon). \]  

The condition (54) in the above coincides with Inequality (104) in [7]. The rate region for the asymptotic scenario is obtained from Theorem 4, which yields

\[ 2q \geq H(C|A) + H(C|B), \]  

\[ q + e \geq H(C|B). \]  

A simple calculation implies that the above rate region is equal to the one obtained in Ref. [13, 35].

4.1.2 Fully Quantum Slepian-Wolf

The fully-quantum Slepian-Wolf protocol is obtained by setting \( A = \emptyset, c = 0 \). The one-shot direct part obtained from Theorem 2 reads

\[ 2q \geq H_{\text{max}}^{(3\epsilon/2,\epsilon/2)}(C) + H_{\text{max}}^{(3\epsilon/2,\epsilon/2)}(C|B) \]  

\[ - \log(\delta^4/2), \]  

\[ q + e \geq H_{\text{max}}^{(3\epsilon/2,\epsilon/2)}(C|B) - \log(\delta^4/2), \]  

\[ e_0 \geq \frac{1}{2} \left( H_{\text{max}}^{\epsilon/2}(C) - H_{\text{max}}^{3\epsilon/2}(C|B) \right) + \log\delta. \]
An example of the rate triplet \((q, e, e_0)\) satisfying the above inequalities is

\[
q = \frac{1}{2}(H_s^{(3e/2,c/2)}(C) + H_{\max}^{1/2}(C|B)) - \log(\delta^4/2),
\]

\[
e = \frac{1}{2}(-H_s^{(3e/2,c/2)}(C) + H_{\max}^{1/2}(C|B) + 1),
\]

\[
e_0 = \frac{1}{2}(H_{\max}^{2/8} - H_{\max}^{3/2}(C|B)) + \log \delta.
\]

The result is equivalent to the one given by [9] (see Theorem 8 therein), with respect to \(q\) and \(e\). Note, however, that our achievability bound requires the use of initial entanglement resource of \(e + e_0\) ebits, whereas the one by [9] does not. The one-shot converse bound is obtained from Theorem 3, which yields

\[
2q \geq H_{\min}^e(C) + H_{\min}^e(BC) - H_{\min}^{12e+6\sqrt{3}}(B) - 6f(e),
\]

\[
q + e \geq H_{\min}^e(BC) - H_{\min}^{12e+6\sqrt{3}}(B) - f(e).
\]

From Theorem 4, the two-dimensional achievable rate region for the asymptotic scenario is given by

\[
2q \geq H(C) + H(C|B),
\]

\[
q + e \geq H(C|B),
\]

which coincides with the result obtained in [1]. It should be noted that various coding theorems for quantum protocols are obtained from that for the fully quantum Slepian-Wolf protocol, which is referred to as the family of quantum protocols [1, 12].

### 4.1.3 Quantum State Splitting

The task in which \(B = \emptyset\), \(c = 0\) is called quantum state splitting. The one-shot direct part is represented as

\[
2q \geq H_s^{(3e/2,c/2)}(C|A) + H_{\max}^{1/2}(C) - \log(\delta^4/2),
\]

\[
q + e \geq H_{\max}^{1/2}(C) - \log(\delta^2/2),
\]

\[
e_0 \geq \frac{1}{2}(H_{\max}^{2/8}(C) - H_{\max}^{3/2}(C)) + \log \delta.
\]

Note that if a triplet \((q, e, e_0)\) is achievable, then \((q, e + e_0, 0)\) is also achievable. Thus, an example of an achievable rate pair \((q, e)\) is

\[
q = \frac{1}{2}(H_s^{(3e/2,c/2)}(C|A) + H_{\max}^{1/2}(C) - \log(\delta^4/2)),
\]

\[
e = \frac{1}{2}(-H_s^{(3e/2,c/2)}(C|A) + H_{\max}^{1/2}(C) + 1) + \delta e_0,
\]

where we have denoted the R.H.S. of (70) by \(\delta e_0\). This coincides with Lemma 3.5 in [6], up to an extra term \(\delta e_0\). The one-shot converse bound is given by

\[
2q \geq H_{\min}^e(AC) - H_{\max}^e(A) + H_{\max}^e(C)
\]

\[
+ \log(1 - 22e - 16\sqrt{3}) - 6f(e),
\]

\[
q + e \geq H_{\min}^e(C) + \log(1 - 22e - 16\sqrt{3}) - f(e).
\]

The rate region for the asymptotic scenario yields

\[
2q \geq H(\emptyset) + H(C),
\]

\[
q + e \geq H(C).
\]

An example of a rate pair satisfying this condition is

\[
q = \frac{1}{2}(H(C) + H(C|A)),
\]

\[
e = \frac{1}{2}(H(C) - H(C|A)),
\]

This result coincides with Equality (6.1) in [1], under the correspondence \(|\Psi_s\rangle_{ACR} = U_{N}^{R\rightarrow AC}\varphi_{RR}\) with \(U_{N}^{R\rightarrow AC}\) being some isometry.

#### 4.1.4 Quantum State Merging

Quantum state merging is a task in which \(A = \emptyset\), \(q = 0\). The one-shot direct part is given by

\[
c \geq H_s^{(3e/2,c/2)}(C|A) + H_{\max}^{1/2}(C) - \log(\delta^4/2),
\]

\[
e \geq H_{\max}^{1/2}(C|B) - \log(\delta^2),
\]

\[
e_0 \geq \frac{1}{2}(H_{\max}^{2/8}(C) - H_{\max}^{3/2}(C|B)) + \log \delta.
\]

The achievability of the entanglement cost \((80)\) is equal to the one given by [15] (see Theorem 5.2 therein). The one-shot converse bound is obtained from Theorem 3, which yields

\[
c \geq H_{\min}^e(C) + H_{\min}^e(BC)
\]

\[
- H_{\max}^{12e+6\sqrt{3}}(B) - 6f(e),
\]

\[
e \geq H_{\min}^e(BC) - H_{\min}^{11e+8\sqrt{3}}(B) - 2f(e).
\]
The rate region for the asymptotic setting is obtained from Theorem 4 as
\[
c \geq H(C) + H(C|B), \quad (84)
\]
\[
e \geq H(C|B). \quad (85)
\]
This rate region is equivalent to the results in [16, 17]. Note, however, that the protocols in [16, 17] are more efficient than ours, in that the catalytic use of entanglement resource is not required.

4.2 No Side Information At The Encoder

Next, we consider scenarios in which there is no classical or quantum side information at the encoder. This corresponds to the case where \( A = X = \emptyset \). We consider three scenarios by imposing several additional assumptions.

4.2.1 Classical Data Compression with Quantum Side Information at The Decoder

The task of classical data compression with quantum side information was analyzed in [11]. This is obtained by additionally setting \( Y = C = \emptyset \), \( q = e = e_0 = 0 \). The one-shot direct and converse bounds are given by
\[
c \geq H^e_{\text{max}}(Z|B) - \log \frac{\delta^2}{2}, \quad (86)
\]
\[
c \geq H^e_{\text{min}}(BZ) - H_{\text{min}}^{12\epsilon+6\sqrt{3}}(B) - f(\epsilon), \quad (87)
\]
respectively. This result is equivalent to the one obtained in [21] (see also [27]). In the asymptotic limit, the achievable rate region is given by \( c \geq H(Z|B) \), which coincides with the result by [11].

4.2.2 Quantum Data Compression with Classical Side Information at The Decoder

The task of quantum data compression with classical side information at the decoder was analyzed in [4]. This is obtained by imposing additional assumptions \( Z = B = \emptyset \), \( c = 0 \). In the entanglement "unconsumed" scenario \( (e = 0) \), the direct bounds for the one-shot case is given by
\[
q \geq \frac{1}{2} (H_{\text{max}}^{3\epsilon/2,\epsilon/2}(C) + H_{\text{max}}^{1/2}(C|Y)) - \log \frac{\delta^4}{2}, \quad (88)
\]
\[
e_0 \geq \frac{1}{2} (H_{\text{max}}^{3\epsilon/2}(C) - H_{\text{max}}^{3\epsilon/2}(C|Y)) + \log \delta. \quad (89)
\]
Note that the entanglement is used only catalytically. Thus, in the asymptotic regime, the achievable quantum communication rate in the entanglement unassisted scenario \( (e = e_0 = 0) \) is obtained due to the cancellation lemma (Lemma 4.6 in [14]), which reads
\[
q \geq \frac{1}{2} (H(C) + H(C|Y)). \quad (90)
\]
In the case where the unlimited amount of entanglement is available, the converse bounds on the quantum communication cost in the one-shot and the asymptotic scenarios read
\[
q \geq \frac{1}{2} (H_{\text{min}}^{\epsilon}(C) + H_{\text{min}}^{\epsilon}(C|Y) - \Delta^{\epsilon,\delta}) - 6f(\epsilon), \quad (91)
\]
\[
q \geq \frac{1}{2} (H(C) + H(C|Y) - \tilde{\Delta}) + \frac{1}{\epsilon} \log (1 - 22\epsilon - 16\sqrt{\delta}). \quad (92)
\]
The asymptotic result (90) coincides with Theorem 7 in [4], and (92) is similar to Theorem 5 therein. It is left open, however, whether the quantity \( \tilde{\Delta} \) is equal to the function \( I_{(n,\delta)} \) that appears in Theorem 5 of [4] (see Definition 2 in the literature).

4.2.3 Fully Classical Slepian-Wolf

In the fully classical scenario, the Slepian-Wolf problem is given by \( B = C = \emptyset \) in addition to \( X = A = \emptyset \), and \( q = e = e_0 = 0 \). The one-shot achievability is given by
\[
c \geq H_{\text{max}}^{e}(Z|Y) - \log \frac{\delta^2}{2}, \quad (93)
\]
and the one-shot converse bound reads
\[
c \geq H_{\text{min}}^{e}(YZ) - H_{\text{min}}^{12\epsilon+6\sqrt{3}}(Y) - f(\epsilon), \quad (94)
\]
which are equivalent to the result obtained in [22]. It is easy to show that the well-known achievable rate region \( c \geq H(Z|Y) \) follows from Theorem 4.

4.3 Quantum State Redistribution with Classical Side Information at The Decoder

We consider a scenario in which \( X = Z = \emptyset \) and \( c = 0 \). This scenario can be regarded as a generalization of the fully quantum state redistribution, that incorporates classical side information at the
decoder [3]. The one-shot direct bound is represented by
\[ 2q \geq \max \{ \tilde{H}_I^{(3q/2,q/2)}, \tilde{H}_H^{(q/2)} \} - \log (\delta^4/2), \]
\[ q + e \geq H_{\max}^{(q/2)}(C|BY) - \log (\delta^2/2), \]
\[ e_0 \geq \frac{1}{2}(H_{\max}^{(q/2)}(C) - H_{\max}^{(q/2)}(C|BY)) + \log \delta, \]
where
\[ \tilde{H}_I^{(3q/2,q/2)} := H_{\max}^{(3q/2,q/2)}(C|AY) + H_{\max}^{(q/2)}(C|BY), \]
\[ \tilde{H}_H^{(q/2)} := H_{\max}^{(q/2)}(C|A) + H_{\max}^{(q/2)}(C|BY). \]
The converse bound is also obtained from Theorem 3. The inner and outer bounds for the achievable rate region in the asymptotic limit is given by
\[ 2q \geq \tilde{H}_I, \]
\[ q + e \geq H(C|BY), \]
\[ e_0 \geq \frac{1}{2}I(C : BY), \]
and
\[ 2q \geq \max \{ \tilde{H}_I, \tilde{H}_H - \tilde{\Delta} \}, \]
\[ q + e \geq H(C|BY), \]
respectively, where
\[ \tilde{H}_I := H(C|AY) + H(C|BY), \]
\[ \tilde{H}_H := H(C|A) + H(C|BY). \]
We may also obtain its descendants by further assuming \( A = 0 \) or \( B = 0 \), which are generalizations of the fully quantum Slepian-Wolf and quantum state splitting.

It is expected that various quantum communication protocols with classical side information only at the decoder are obtained by reduction from the above result, similarly to the family of quantum protocols [1, 12]. We, however, leave this problem as a future work.

5 Proof of The Direct Part (Theorem 2)

We prove Theorem 2 based on the following propositions:

Proposition 6 A tuple \( (c,q,e,e_0) \) is achievable within the error \( 4\sqrt{2c + 6e} \) for \( \Psi_c \) if \( d_C \geq 2 \) and it holds that
\[ c + q - e \geq H_{\max}^{(c)}(CZ|AX) \Psi_c - \log \frac{\delta^2}{2}, \]
\[ q - e \geq H_{\max}^{(c)}(CAXYZ) \Psi_c - \log \delta^2, \]
\[ c + q + e \geq H_{\max}^{(c)}(CZ|BY) \Psi_c - \log \frac{\delta^2}{2}, \]
\[ q + e \geq H_{\max}^{(c)}(C|BY) \Psi_c - \log \delta^2, \]
\[ e_0 = \frac{1}{2}(\log d_C - q - e). \]

In the case where \( d_C = 1 \) and \( q = e = e_0 = 0 \), the classical communication rate \( c \) is achievable within the error \( \delta \) if it holds that
\[ c \geq \max \{ H_{\max}^{(c)}(Z|AX) \Psi_c, H_{\max}^{(c)}(Z|BY) \Psi_c \} \]
\[ - \log \frac{\delta^2}{2}. \]

Proposition 7 A tuple \( (c,q,e,e_0) \) is achievable within an error \( 4\sqrt{12c + 6e} \) for \( \Psi_c \) if \( d_C \geq 2 \) and it holds that
\[ c + 2q \geq \max \{ \tilde{H}_I^{(c)}, \tilde{H}_H^{(c)} \} - \log (\delta^4/2), \]
\[ c + q + e \geq H_{\max}^{(c)}(CZ|BY) \Psi_c - \log (\delta^2/2), \]
\[ q + e \geq H_{\max}^{(c)}(C|BY) \Psi_c - \log \delta^2, \]
\[ e_0 \geq \frac{1}{2}(\log d_C - H_{\max}^{(c)}(C|BXYZ) \Psi_c) \]
\[ + \log \delta, \]
where
\[ \tilde{H}_I^{(c)} := H_{\max}^{(c)}(C|AXYZ) \Psi_c, \]
\[ + H_{\max}^{(c)}(CZ|BY) \Psi_c, \]
\[ \tilde{H}_H^{(c)} := H_{\max}^{(c)}(C|AXZ) \Psi_c, \]
\[ + H_{\max}^{(c)}(C|BXY) \Psi_c. \]

and
\[ H_{\rho}^{(c)}(C|AXYZ) := \max \{ H_{\min}^{(c)}(C|AXYZ) \rho, H_{\max}^{(c)}(C|AXYZ) \rho \}. \]

In the case where \( d_C = 1 \), a tuple \( (c,0,0,0) \) is achievable for \( \Psi_c \) within the error \( \delta \) if it holds that
\[ c \geq H_{\max}^{(c)}(Z|BY) \Psi_c - \log \frac{\delta^2}{2}. \]
Proofs of Proposition 6 and Proposition 7 will be given in the following subsections. In Section 5.1, we prove the partial bi-decoupling theorem, which is a generalization of the bi-decoupling theorem [36, 7]. Based on this result, we prove Proposition 6 in Section 5.2. We adopt the idea that a protocol for state redistribution can be constructed from sequentially combining protocols for the (fully quantum) reverse Shannon and the (fully quantum) Slepian-Wolf. In Section 5.3, we extend the rate region in Proposition 6 by incorporating teleportation and dense coding, thereby proving Proposition 7. Finally, we prove Theorem 2 from Proposition 7 in Section 5.4.

5.1 Partial Bi-Decoupling

The idea of the bi-decoupling theorem was first introduced in [36], and was improved in [7] to fit more into the framework of the one-shot information theory. The approach in [7] is based on the decoupling theorem in [15]. In this subsection, we generalize those results by using the direct part of randomized partial decoupling [33] to incorporate the hybrid communication scenario.

5.1.1 Direct Part of Partial Decoupling

We first present the direct part of randomized partial decoupling (Theorem 3 in [33]). Let $\Psi^{\hat{C}\hat{S}}$ be a subnormalized state in the form of

$$\Psi^{\hat{C}\hat{S}} = \sum_{j,k=1}^{J} |j\rangle\langle k|^{Z} \otimes \psi_{jk}^{CS} \otimes |j\rangle\langle k|^{Z'}.$$  \hspace{1cm} (121)

Here, $Z$ and $Z'$ are $J$-dimensional quantum system with a fixed orthonormal basis $\{|j\rangle\}_{j=1}^{J}, \hat{C} \equiv ZC, \hat{S} \equiv Z'S$ and $\psi_{jk} \in \mathcal{L}(\mathcal{H}^{C} \otimes \mathcal{H}^{S})$ for each $j$ and $k$. Note that the positive-semidefiniteness of $\Psi^{\hat{C}\hat{S}}$ implies $\psi_{jj} \geq 0$ for all $j$ and the subnormalization condition implies $\sum_{j=1}^{J} \text{Tr}[\psi_{jj}] \leq 1$. Consider a random unitary $U$ on $\hat{C}$ in the form of

$$U := \sum_{j=1}^{J} |j\rangle\langle j|^{Z} \otimes U_{j}^{C},$$  \hspace{1cm} (122)

where $U_{j} \sim \mathcal{H}_{j}$ for each $j$, and $\mathcal{H}_{j}$ is the Haar measure on the unitary group on $\mathcal{H}^{C}$. The averaged state obtained after the action of the random unitary $U$ on $\hat{C}$ is given in the following subsections. In Section 5.1, we prove the partial bi-decoupling theorem, which is a generalization of the bi-decoupling theorem [36, 7]. Based on this result, we prove Proposition 6 in Section 5.2. We adopt the idea that a protocol for state redistribution can be constructed from sequentially combining protocols for the (fully quantum) reverse Shannon and the (fully quantum) Slepian-Wolf. In this subsection, we generalize those results by using the direct part of the randomized partial decoupling theorem in [15]. In this subsection, we consider the hybrid communication scenario.

5.1.1 Direct Part of Partial Decoupling

We first present the direct part of randomized partial decoupling (Theorem 3 in [33]). Let $\Psi^{\hat{C}\hat{S}}$ be a subnormalized state in the form of

$$\Psi^{\hat{C}\hat{S}} = \sum_{j,k=1}^{J} |j\rangle\langle k|^{Z} \otimes \psi_{jk}^{CS} \otimes |j\rangle\langle k|^{Z'}.$$  \hspace{1cm} (121)

Here, $Z$ and $Z'$ are $J$-dimensional quantum system with a fixed orthonormal basis $\{|j\rangle\}_{j=1}^{J}, \hat{C} \equiv ZC, \hat{S} \equiv Z'S$ and $\psi_{jk} \in \mathcal{L}(\mathcal{H}^{C} \otimes \mathcal{H}^{S})$ for each $j$ and $k$. Note that the positive-semidefiniteness of $\Psi^{\hat{C}\hat{S}}$ implies $\psi_{jj} \geq 0$ for all $j$ and the subnormalization condition implies $\sum_{j=1}^{J} \text{Tr}[\psi_{jj}] \leq 1$. Consider a random unitary $U$ on $\hat{C}$ in the form of

$$U := \sum_{j=1}^{J} |j\rangle\langle j|^{Z} \otimes U_{j}^{C},$$  \hspace{1cm} (122)

where $U_{j} \sim \mathcal{H}_{j}$ for each $j$, and $\mathcal{H}_{j}$ is the Haar measure on the unitary group on $\mathcal{H}^{C}$. The averaged state obtained after the action of the random unitary $U$ is given by

$$\Psi^{\hat{C}\hat{S}} := \mathbb{E}_{U}[U \hat{C} \Psi^{\hat{C}\hat{S}} U^\dagger \hat{C}]$$

$$= \sum_{j=1}^{J} p_{j} |j\rangle\langle j|^{Z} \otimes \pi^{C} \otimes \psi_{jj}^{S} \otimes |j\rangle\langle j|^{Z'},$$  \hspace{1cm} (124)

where $p_{j} := \text{Tr}[\psi_{jj}]$ and $\psi_{jj} := p_{j}^{-1} \psi_{jj}$. Consider also the permutation group $\mathbb{P}$ on $[1, \cdots, J]$, and define a unitary $G_{\sigma}$ for any $\sigma \in \mathbb{P}$ by

$$G_{\sigma} := \sum_{j=1}^{J} |\sigma(j)\rangle\langle j|^{Z}.$$  \hspace{1cm} (125)

We assume that the permutation $\sigma$ is chosen at random according to the uniform distribution on $\mathbb{P}$.

Suppose that the state $\Psi^{\hat{C}\hat{S}}$ is transformed by unitaries $U$ and $G_{\sigma}$, and then is subject to the action of a quantum channel (linear CP map) $\mathcal{T}^{\hat{C}\rightarrow \hat{E}}$ (see Figure 4). The final state is represented as

$$\mathcal{T}^{\hat{C}\rightarrow \hat{E}}((G_{\sigma}^{Z} U^{\hat{C}}) \Psi^{\hat{C}\hat{S}} (G_{\sigma}^{Z} U^{\hat{C}})^{\dagger})$$

$$= \mathcal{T}^{\hat{C}\rightarrow \hat{E}} \circ G_{\sigma}^{Z} \circ U^{\hat{C}} \Psi^{\hat{C}\hat{S}}.$$  \hspace{1cm} (126)

We consider how close the final state is, on average over all $U$, to the averaged final state $\mathcal{T}^{\hat{C}\rightarrow \hat{E}} \circ G_{\sigma}^{Z} (\Psi^{\hat{C}\hat{S}}_{av})$, for typical choices of the permutation $\sigma$. The following theorem is the direct part of the randomized partial decoupling theorem, which provides an upper bound on the average distance between $\mathcal{T}^{\hat{C}\rightarrow \hat{E}} \circ G_{\sigma}^{Z} \circ U^{\hat{C}} (\Psi^{\hat{C}\hat{S}})$ and $\mathcal{T}^{\hat{C}\rightarrow \hat{E}} \circ G_{\sigma}^{Z} (\Psi^{\hat{C}\hat{S}}_{av})$. Although the original version in [33] is applicable to any $J \geq 1$, in this paper we assume that $J \geq 2$.

Lemma 8 (Corollary of Theorem 3 in [33]) Consider a subnormalized state $\Psi^{\hat{C}\hat{S}} \in \mathcal{S}_{\mathcal{H}^{C} \otimes \mathcal{H}^{S}}$ that is decomposed as (121). Let $\mathcal{T}^{\hat{C}\rightarrow \hat{E}}$ be a linear trace non-increasing CP map with the complementary channel $\mathcal{T}^{\hat{C}\rightarrow \hat{E}}$. Let $U$ and $G_{\sigma}$ be random unitaries given by (122) and (125),
Figure 5: The situation of partial decoupling under partial trace is depicted.

respectively, and fix arbitrary \( \epsilon, \mu \geq 0 \). It holds that

\[
\mathbb{E}_{\sigma, U} \left[ \left\| \mathcal{T}^{\mathcal{C} \to \mathcal{E}} \circ \mathcal{O}_{\sigma}^{Z} \circ \mathcal{U}^{C}(\hat{\psi}^{C \mathcal{S}}) - \mathcal{T}^{\mathcal{C} \to \mathcal{E}} \circ \mathcal{O}_{\sigma}^{Z}(\hat{\psi}^{C \mathcal{S}}) \right\|_1 \right] \\
\leq \begin{cases} 2^{-\frac{1}{2} H_I} + 2^{-\frac{1}{2} H_{II}^\epsilon} + 4(\epsilon + \mu + \epsilon \mu) & (d_C \geq 2), \\
2^{-\frac{1}{2} H_I} + 4(\epsilon + \mu + \epsilon \mu) & (d_C = 1),
\end{cases}
\tag{127}
\]

where \( \hat{\psi}^{C \mathcal{S}}_{av} : = \mathbb{E}[U^{\hat{C}}(\hat{\psi}^{C \mathcal{S}})] \). The exponents \( H_I \) and \( H_{II}^\epsilon \) are given by

\[
H_I = \log (J - 1) + H_{min}^{\epsilon}(\hat{C} | \hat{S})_{\psi} - H_{max}^{\mu}(\hat{C} | \mathcal{C}(\tau)),
\tag{128}
\]

\[
H_{II} = H_{min}^{\epsilon}(\hat{C} | \mathcal{S})_{\mathcal{C}(\psi)} - H_{max}^{\mu}(C | \mathcal{F}Z)_{\mathcal{C}(\tau)}. \tag{129}
\]

Here, \( \mathcal{C} \) is the completely dephasing operation on \( Z \) with respect to the basis \( \{|j\}_{j=1}^J \), and \( \tau \) is the Choi-Jamiołkowski state of \( \mathcal{T}^{\mathcal{C} \to \mathcal{F}} \) defined by \( \tau^{\mathcal{C} \to \mathcal{F}} : = \mathcal{T}^{\mathcal{C} \to \mathcal{F}}(\Phi^{\mathcal{C} \mathcal{C}'}) \). The state \( \Phi^{\mathcal{C} \mathcal{C}'} \) is the maximally entangled state in the form of

\[
\Phi^{\mathcal{C} \mathcal{C}'} = \frac{1}{\sqrt{J}} \sum_{j=1}^J |j\rangle_{Z}^Z |j\rangle_{Z'} |\Phi_{\tau}\rangle_{CC'}, \tag{130}
\]

5.1.2 Partial Decoupling under Partial Trace

We apply Lemma 8 to a particular case where the channel \( \mathcal{T} \) is the partial trace (see Figure 5).

**Lemma 9** Consider the same setting as in Lemma 8, and suppose that \( Z = Z_LZ_R, \)
\( C = C_LC_R \). We assume that \( Z_L \) and \( Z_R \) are equipped with fixed orthonormal bases \( \{|z_L\}_{z_L \in L}^J \) and \( \{|z_R\}_{z_R \in \Lambda}^J \), respectively, thus \( J = J_LJ_R \) and the orthonormal basis of \( Z \) is given by \( \{|z_L\}_{z_L \in L}^J \). Fix arbitrary \( \epsilon \geq 0 \). If \( d_C \geq 2 \) and

\[
\log \frac{d_{C_L}^2}{d_{Z_R}d_C} \leq H_{min}(\hat{C} | \hat{S})_{\psi} + \log \frac{\delta^2}{2}, \tag{131}
\]

\[
\log \frac{d_{C_R}^2}{d_{C_L}d_C} \leq H_{min}(\hat{C} | \mathcal{S})_{\mathcal{C}(\psi)} + \log \delta^2, \tag{132}
\]

then it holds that

\[
\mathbb{E}_{\sigma, U} \left[ \left\| \text{Tr}_{Z_RC_R} \circ \mathcal{O}_{\sigma}^{Z} \circ \mathcal{U}^{C}(\hat{\psi}^{C \mathcal{S}}) - \text{Tr}_{Z_RC_R} \circ \mathcal{O}_{\sigma}^{Z}(\hat{\psi}^{C \mathcal{S}}) \right\|_1 \right] \leq 4\epsilon + 2\delta, \tag{133}
\]

where \( \hat{\psi}^{C \mathcal{S}}_{av} : = \mathbb{E}[U^{\hat{C}}(\hat{\psi}^{C \mathcal{S}})] \). The same statement also holds in the case of \( d_C = 1 \), in which case the condition (132) can be removed.

**Proof:** We apply Lemma 8 by the correspondence \( \mu = 0, E = Z_LC_L, F = Z_RC_R, J = d_Z \) and \( \mathcal{T}^{\mathcal{C} \to \mathcal{Z}_LC_L} = \mathcal{I} \otimes \mathcal{T}_{Z_RC_R} \). It follows that Ineq. (133) holds if \( d_C \geq 2 \) and

\[
\log (d_Z - 1) + H_{min}^{\epsilon}(\hat{C} | \hat{S})_{\psi} - H_{max}(\hat{C} | \mathcal{Z}_R^C)_{\mathcal{C}(\tau)} + \log \delta^2 \geq 0, \tag{134}
\]

\[
H_{min}(\hat{C} | \mathcal{S})_{\mathcal{C}(\psi)} - H_{max}(C | \mathcal{Z}_R^CZ)_{\mathcal{C}(\tau)} + \log \delta^2 \geq 0. \tag{135}
\]

Here, \( \tau \) is the Choi-Jamiołkowski state of the complementary channel of \( \mathcal{T}^{\mathcal{C} \to \mathcal{Z}_LC_L} \), and is given by

\[
\tau^{\mathcal{C} \mathcal{Z}_R^C} = \tau^{\mathcal{C} \mathcal{Z}_R^C} = \frac{1}{Z} \sum_{j=1}^J |j\rangle_{Z}^Z |j\rangle_{Z'} \Phi_{\tau}, \tag{136}
\]

Using the additivity of the max conditional entropy (Lemma 15 in Appendix A), the entropies are calculated to be

\[
H_{max}(\hat{C} | \mathcal{Z}_R^C)_{\mathcal{C}(\tau)} = \log d_{Z_L} + \log d_{C_L} - \log d_{C_R}, \tag{137}
\]

\[
H_{max}(C | \mathcal{Z}_R^CZ)_{\mathcal{C}(\tau)} = \log d_{C_L} - \log d_{C_R}. \tag{138}
\]

Thus, Inequalities (134) and (135) are equivalent to

\[
\log (d_Z - 1) + H_{min}^{\epsilon}(\hat{C} | \hat{S})_{\psi} - \log \frac{d_{Z_L}d_{C_L}}{d_{Z_R}d_C} + \log \delta^2 \geq 0, \tag{139}
\]

\[
H_{min}(\hat{C} | \mathcal{S})_{\mathcal{C}(\psi)} - \log \frac{d_{C_L}}{d_{C_R}} + \log \delta^2 \geq 0. \tag{140}
\]

Noting that \( d_Z = d_{Z_L}d_{Z_R} \), \( d_C = d_{C_L}d_{C_R} \) and that \( (d_Z - 1)/d_Z \geq 1/2 \), the above two inequalities follow from (131) and (132), respectively.
Thus, the proof in the case of \(d_C \geq 2\) is done. The proof for the case of \(d_C = 1\) proceeds along the same line.

5.1.3 Partial Bi-Decoupling Theorem

Based on Lemma 9, we introduce a generalization of the “bi-decoupling theorem” [36, 7] that played a crucial role in the proof of the direct part of one-shot fully quantum state redistribution. We consider the case where systems \(C\) and \(S\) are composed of three subsystems. The following lemma provides a sufficient condition under which a single pair of \(\sigma\) and \(U\) simultaneously achieves partial decoupling of a state, from the viewpoint of two different choices of subsystems (see Figure 6 in the next page).

**Lemma 10 (Partial bi-decoupling.)** Consider the same setting as in Lemma 8, assume \(Z = Z_LZ_R\), \(C = C_1C_2C_3\), \(S = S_1S_2S_3\) and fix arbitrary \(\varepsilon \geq 0\). If \(d_C \geq 2\) and

\[
\log \frac{d_{C_1}}{d_C} \leq H^*_{\min}(C|Z'S_2S_3)\psi + \log \frac{\delta^2}{2}, \tag{141}
\]

\[
\log \frac{d_{C_2}}{d_C} \leq H^*_{\min}(C|Z'S_2S_3)\psi + \log \frac{\delta^2}{2}, \tag{142}
\]

\[
\log \frac{d_{C_3}}{d_C} \leq H^*_{\min}(C|Z'S_2S_3)\psi + \log \frac{\delta^2}{2}, \tag{143}
\]

\[
\log \frac{d_{C_1}}{d_C} \leq H^*_{\min}(C|Z'S_2S_3)\psi + \log \frac{\delta^2}{2}, \tag{144}
\]

there exist \(\sigma\) and \(U\) such that

\[
\left\| \text{Tr}_{Z_RC_3}\sigma \circ G^Z_\sigma \circ U^C_\sigma (\Psi^C_{S_2S_3}Z') \right\|_1 \leq 12\varepsilon + 6\delta, \tag{145}
\]

\[
\left\| -\text{Tr}_{Z_RC_3}\sigma \circ G^Z_\sigma \circ U^C_\sigma (\Psi^C_{S_1S_3}Z') \right\|_1 \leq 12\varepsilon + 6\delta. \tag{146}
\]

The same statement also holds if \(d_C = 1\), in which case the conditions (142) and (144) can be removed.

**Proof:** Suppose that \(d_C \geq 2\) and the inequalities (141)-(144) are satisfied. We apply Lemma 9 under the correspondence \(C_R = C_{\alpha}C_3\), \(S = S_{\alpha}S_3\) and \(C_L = C_{\bar{\alpha}}\), where \(\alpha = 1, 2\) and \(\bar{\alpha} = 2, 1\) for each. It follows that

\[
\mathbb{E}_{\sigma, U} \left\| \text{Tr}_{Z_RC_3}\sigma \circ G^Z_\sigma \circ U^C_\sigma (\Psi^C_{S_2S_3}Z') \right\|_1 \leq 4\varepsilon + 2\delta. \tag{147}
\]

Markov’s inequality implies that there exist \(\sigma\) and \(U\) that satisfy both (145) and (146), which completes the proof in the case of \(d_C \geq 2\). The proof in the case of \(d_C = 1\) proceeds along the same line.

5.2 Proof of Proposition 6

To prove Proposition 6, we follow the lines of the proof of the direct part of the fully quantum state redistribution protocol in [36]. The key idea is that a protocol for state redistribution can be constructed from sequentially combining a protocol for the fully quantum reverse Shannon and that for the fully quantum Slepian-Wolf. We generalize this idea to the “hybrid” scenario (see Figure 9 in page 34). We only consider the case where \(d_C \geq 2\). The proof for the case of \(d_C = 1\) is obtained along the same line.

5.2.1 Application of The Partial Bi-Decoupling Theorem

Consider the “purified” source state

\[
|\Psi\rangle^{ABCRXYZT} := \sum_{x,y,z} \sqrt{p_{xy}} |x\rangle^X |y\rangle^Y |z\rangle^Z |\Psi_{xyz}^{ABCR}|xyz\rangle^T, \tag{148}
\]

where we denoted \(X'Y'Z'\) simply by \(T\). Let \(C\) be isomorphic to \(C_1C_2C_3\) and \(Z\) to \(Z_LZ_R\). Fix an arbitrary \(\varepsilon > 0\). We apply Lemma 10 under the following correspondence:

\[
S_1 = \hat{A}, \quad S_2 = \hat{B}, \quad S_3 = RX'Y'. \tag{149}
\]

Note that \(\hat{R} = RX'Y'Z'\). It follows that if the dimensions of \(C_1\) and \(C_2\) are sufficiently small (see the next subsection for the details), there exist \(\sigma\)
We consider the purifying systems to be

\[ \Psi = C_1 C_2 C_3 \]

In addition, we introduce systems \( U \) and \( G_\sigma \) that satisfy

\[ \| \text{Tr}_{\text{Z}_R} C_3 \circ \hat{G}_\sigma \circ U \hat{C}(\hat{\Psi}_{\text{av}}^{CBR}) \|_1 \leq 12 \epsilon + 6 \delta, \]

\[ \| \text{Tr}_{\text{Z}_R} C_3 \circ \hat{G}_\sigma \circ U \hat{C}(\hat{\Psi}_{\text{av}}^{CA R}) \|_1 \leq 12 \epsilon + 6 \delta. \]

Let \( |\Psi_{\sigma,1}\rangle C_1 Z_L \hat{B} \hat{D}_A \) be a purification of \( \text{Tr}_{\text{Z}_R} C_3 \circ \hat{G}_\sigma \circ U \hat{C}(\hat{\Psi}_{\text{av}}^{CBR}) \) with \( D_A \) being the purifying system. Similarly, let \( |\Psi_{\sigma,2}\rangle C_2 Z_L \hat{A} \hat{D}_B \) be a purification of \( \text{Tr}_{\text{Z}_R} C_3 \circ \hat{G}_\sigma \circ U \hat{C}(\hat{\Psi}_{\text{av}}^{CA R}) \) with \( D_B \) being the purifying system. Due to Uhlmann’s theorem (133); see also e.g. Chapter 9 in [34]), there exist linear isometries

\[ V^{D_A} \rightarrow Z_R C_2 C_3 A, \quad W^{Z_R C_3 Z_L \hat{B} \hat{D}_B} \]

such that

\[ \| \hat{G}_\sigma \circ U \hat{C}(\hat{\Psi}_{\text{av}}^{CA R}) - V^{D_A} \rightarrow Z_R C_2 C_3 A \|_1 \leq 2 \sqrt{12 \epsilon + 6 \delta}, \]

\[ \| W^{Z_R C_3 \hat{B} \rightarrow \hat{D}_B} \circ \hat{G}_\sigma \circ U \hat{C}(\hat{\Psi}_{\text{av}}^{CBR}) - W^{Z_R C_3 \hat{B} \rightarrow \hat{D}_B} \|_1 \leq 2 \sqrt{12 \epsilon + 6 \delta}. \]

We particularly choose \( C_1, C_2, C_3 \) and \( Z_R \) so that they satisfy the isomorphism

\[ C_1 \equiv E_B, C_2 \equiv F_A, C_3 \equiv Q, Z_R \equiv M. \]

In addition, we introduce systems \( C'' \), \( Z'' \), \( A_1 \) and \( B_2 \) such that

\[ C'' \equiv C, Z'' \equiv Z, A_1 \equiv E_A, B_2 \equiv F_B. \]

We consider the purifying systems to be \( D_A \equiv Z_R C'' A_1 A_1 \) and \( D_B \equiv Z_R C'' B B_2 \), where \( C'' = C'' Z'' \).

### 5.2.2 Explicit Forms of The Purifications

To obtain explicit forms of the purifications \( \Psi_{\sigma,1} \) and \( \Psi_{\sigma,2} \), we define a state \( \Psi_\sigma \) by

\[ |\Psi_\sigma\rangle^\hat{A} \hat{B} \hat{C} \hat{R} Z := \sum_{x,y,z} \sqrt{p_{xyz}} |x \rangle^X |y \rangle^Y |\sigma(z)\rangle^Z |z\rangle^{Z''} \]

\[ \otimes |\psi_{xyz}\rangle^{ABC'R} |xyz\rangle^T. \]

From the definition (20) of the source state \( \Psi_s \), (148) of the purified source state \( \Psi \) and (157) of the state \( \Psi_\sigma \), it is straightforward to verify that the states are related simply by

\[ |\Psi_\sigma\rangle^\hat{A} \hat{B} \hat{C} \hat{R} Z = G^Z \circ P \hat{Z''} \rightarrow \hat{Z''} Z |\Psi\rangle^\hat{A} \hat{B} \hat{C} \hat{R} \]

and

\[ \text{Tr}_Z \otimes C^T (|\Psi_\sigma\rangle^\hat{A} \hat{B} \hat{C} \hat{R} Z) = |\Psi_\sigma\rangle^\hat{A} \hat{B} \hat{C} \hat{R} \]

Here, let \( P \hat{Z''} \rightarrow \hat{Z''} Z \) be a linear isometry defined by

\[ P \hat{Z''} \rightarrow \hat{Z''} Z := \sum_z |z\rangle^{Z''} |z\rangle^Z \langle z \rangle^{Z''}, \]

and \( C \) be the completely dephasing operation on \( T \) with respect to the basis \( \{xyz\}_{x,y,z} \). The state \( \Psi_\sigma \) is simply represented as

\[ |\Psi_\sigma\rangle^\hat{A} \hat{B} \hat{C} \hat{R} Z = \sum_z \sqrt{p_\sigma(z)}^Z |\psi_z\rangle^\hat{A} \hat{B} \hat{C} \hat{R} X Y Y' |z\rangle^{Z''}. \]

where

\[ |\psi_z\rangle^\hat{A} \hat{B} \hat{C} \hat{R} X Y Y' := \sum_{x,y} \sqrt{p_{xyz}}^Z |x \rangle^X |y \rangle^Y |z\rangle^{Z''} \]

\[ \otimes |\psi_{xyz}\rangle^{ABC'R} |xyz\rangle^T. \]
It is convenient to note that
\[
\psi_z^{ABRXY} = \sum_{x,y} \sqrt{p_{xyz}} \psi_{xy}^{ABR} \otimes |x\rangle^{X'} \otimes |y\rangle^{Y'}.
\]

(164)

Due to (148) and (124), the averaged state in (150) is calculated to be
\[
\psi_{av}^{CBR} = \sum_z p_z |z\rangle^{Z} \otimes \pi^C \otimes \psi_z^{BRX'Y'} \otimes |z\rangle^{Z'},
\]

(165)

where \(p_z = \sum_{x,y} p_{xyz}\). It follows that
\[
\text{Tr}_{Z_R} C_b C_s = \text{Tr}_{Z_R} (\psi_{av}^{CBR}) = \sum_z p_z \text{Tr}_{Z_R} (|\phi(z)\rangle \otimes \pi^C)
\]
\[
\otimes \psi_z^{BRX'Y'} \otimes |z\rangle^{Z'}.
\]

(166)

Thus, a purification \(\Psi_{\sigma,1}\) of this state is given by
\[
|\Psi_{\sigma,1}\rangle = \hat{A}^{\hat{B}C''R} A_1 C_1 Z = |\Psi_{\sigma}\rangle \hat{A}^{\hat{B}C''R} Z|\phi_1\rangle \hat{A}_1 C_1,
\]
where \(\phi_1\) is the maximally entangled state of Schmidt rank \(d_{C_1}\). In the same way, the purification \(\Psi_{\sigma,2}\) is given by
\[
|\Psi_{\sigma,2}\rangle = \hat{A}^{\hat{B}C''R} B_2 C_2 Z = |\Psi_{\sigma}\rangle \hat{A}^{\hat{B}C''R} Z|\phi_2\rangle B_2 C_2,
\]
with \(\phi_2\) being the maximally entangled state of Schmidt rank \(d_{C_2}\). Substituting these to (153) and (154), we arrive at
\[
\left\| \text{Tr}_{Z_R} \hat{W} \circ \hat{C} (\Psi_{\sigma}^{\hat{C}ABR}) \right\|_1 \leq 2\sqrt{12\epsilon + 6\delta},
\]

(169)

\[
\left\| \text{Tr}_{Z_R} \hat{W} \circ \hat{C} (\Psi_{\sigma}^{\hat{C}ABR}) - \Psi_{\sigma}^{\hat{A}B\hat{C}'R} Z \otimes \phi_2 B_2 C_2 \right\|_1 \leq 2\sqrt{12\epsilon + 6\delta}.
\]

(170)

Inequality (169) implies that the operation \((\hat{G} \circ \hat{U})^\dagger \circ \hat{V}\) is a reverse Shannon protocol for the state \(\psi_{\sigma}^{\hat{C}\hat{A}(BR)}\), up to the action of a linear isometry \(G \circ P^{Z'' \rightarrow Z'''}\) by which \(\Psi_{\sigma}\) is obtained from \(\Psi\) as (158). Similarly, Inequality (170) implies that the operation \(W \circ \hat{G} \circ \hat{U}\) is a Slepian-Wolf protocol for the state \(\psi_{\sigma}^{\hat{C}B(AR)}\), up to the action of \(G \circ P^{Z'' \rightarrow Z'''}\) (see Figure 9 in page 34). We combine the two protocols to cancel out \((\hat{G} \circ \hat{U})^\dagger\) and \(\hat{G} \circ \hat{U}\). Due to the triangle inequality, it follows from (169) and (170) that
\[
\left\| \text{Tr}_{Z_R} \hat{W} \circ \hat{C} (\Psi_{\sigma}^{\hat{A}B\hat{C}'R} Z \otimes \phi_1 A_1 C_1) - \Psi_{\sigma}^{\hat{A}B\hat{C}'R} Z \otimes \phi_2 B_2 C_2 \right\|_1 \leq 4\sqrt{12\epsilon + 6\delta}.
\]

(171)

5.2.3 Construction of The Encoding and Decoding Operations

Define a partial isometry
\[
\hat{V}_{\sigma}^{\hat{A}C''R} A_1 \rightarrow \hat{A} Z R C_3
\]
\[
:= \text{Tr}_{Z_R} A_1 C'' R Z R C_3 A \circ G_{\sigma}^{Z} \circ P^{Z'' \rightarrow Z'''} .
\]

(172)

Applying the map \(\text{Tr}_{Z} \otimes C^T\) to Inequality (171), and using (158) and (160), it follows that
\[
\left\| \text{Tr}_{Z} \circ \hat{W} \circ \hat{V}_{\sigma} (\Psi_{\sigma}^{\hat{A}B\hat{C}'R} Z \otimes \phi_1 A_1 C_1) - \Psi_{\sigma}^{\hat{A}B\hat{C}'R} Z \otimes \phi_2 B_2 C_2 \right\|_1 \leq 4\sqrt{12\epsilon + 6\delta}.
\]

(173)

We construct a protocol for state redistribution as follows: In the first step, the sender performs the following encoding operation:
\[
\hat{E} \hat{A} \hat{C''R} A_1 \rightarrow \hat{A} Z R C_3
\]
\[
= \text{Tr}_{Z_L} \circ \hat{V}_{\sigma} (\Psi_{\sigma}^{\hat{A}B\hat{C}'R} Z \otimes \phi_1 A_1 C_1)
\]
\[
\hat{C''} \rightarrow \hat{Z''} \rightarrow \hat{Z}_R B_2 \rightarrow \hat{B} \hat{C}'' .
\]

(174)

where \(\hat{C}''\) is the completely dephasing operation on \(Z''\) with respect to the basis \(\{ |z_L\rangle |z_R\rangle \}_{z_L, z_R}\). The sender then sends the classical system \(Z_R \cong M\) and the quantum system \(C_3 \cong Q\) to the receiver, who performs the decoding operation defined by
\[
\hat{D} \hat{Z}_R C_3 B \rightarrow \hat{B} \hat{C}'' .
\]

(175)

Noting that \(\text{Tr}_{Z} = \text{Tr}_{Z_L} \otimes \text{Tr}_{Z_R}\), we obtain from (173) that
\[
\left\| \hat{D} \circ \hat{E} (\Psi_{\sigma}^{\hat{A}B\hat{C}'R} Z \otimes \phi_1 A_1 C_1) - \Psi_{\sigma}^{\hat{A}B\hat{C}'R} Z \otimes \phi_2 B_2 C_2 \right\|_1 \leq 4\sqrt{12\epsilon + 6\delta}.
\]

(176)

From (172) and (174), it is straightforward to verify that \(\hat{E} (\tau)\) is diagonal in \(Z_R\) for any input state \(\tau\). Thus, the pair \((\hat{E}, \hat{D})\) is a state redistribution protocol for the state \(\psi_{\sigma}\) within the error \(4\sqrt{12\epsilon + 6\delta}\).

5.2.4 Evaluation of Entropies

We analyze conditions on the size of systems \(C_1\) and \(C_2\), in order that inequalities (150) and (151) are satisfied. We use the partial bi-decoupling
theorem (Lemma 10) under the correspondence (149), which reads
\[ S_1 = \hat{A}, \quad S_2 = \hat{B}, \quad S_3 = RX'Y'. \] (177)
It follows that inequalities (150) and (151) are satisfied if it holds that
\[ \log \frac{d_{C_3}}{d_{C_1} d_{C_2}} \leq H_{\min}^e(\hat{C}|B\hat{R})_{\Psi} + \log \frac{\delta^2}{2}, \] (178)
\[ \log \frac{d_{C_1}}{d_{C}} \leq H_{\min}^e(\hat{C}|B\hat{R})_{\hat{C}(\Psi)} + \log \delta^2, \] (179)
\[ \log \frac{d_{C_2}}{d_{C}} \leq H_{\min}^e(\hat{C}|\hat{A}\hat{R})_{\Psi} + \log \frac{\delta^2}{2}, \] (180)
\[ \log \frac{d_{C_2}}{d_{C}} \leq H_{\min}^e(\hat{C}|\hat{A}\hat{R})_{\hat{C}(\Psi)} + \log \delta^2. \] (181)

Using the duality of the smooth conditional entropy (Lemma 12), and noting that \( \Psi^{ABC} = \Psi_s^{ABC} \), the min entropies in the first and the third inequalities are calculated to be
\[ H_{\min}^e(\hat{C}|\hat{B}\hat{R})_{\Psi} = -H_{\max}^e(CZ|AX)_{\Psi}, \] (182)
\[ H_{\min}^e(\hat{C}|\hat{B}\hat{R})_{\hat{C}(\Psi)} = -H_{\max}^e(CZ|BY)_{\Psi}. \] (183)

Similarly, due to Lemma 23 and Lemma 26 in Appendix A, and noting that \( C(\Psi) = \Psi_s \), because of (20) and (148), we have
\[ H_{\min}^e(\hat{C}|\hat{B}\hat{R})_{\hat{C}(\Psi)} = -H_{\max}^e(C|AXY)_{\Psi}, \] (184)
\[ H_{\min}^e(\hat{C}|\hat{B}\hat{R})_{\hat{C}(\Psi)} = -H_{\max}^e(C|BYZ)_{\Psi}. \] (185)

In addition, the isomorphism (155) implies
\[ \log d_{C_1} = e + \epsilon_0, \log d_{C_2} = e_0, \] (190)
\[ \log d_{C_3} = q, \log d_{Z_R} = c. \] (191)

Substituting these relations into (178)-(181), we arrive at
\[ c + q - e \geq H_{\max}^e(CZ|AX)_{\Psi}, \quad \log \frac{\delta^2}{2}, \] (192)
\[ q - e \geq H_{\max}^e(C|AXY)_{\Psi}, \quad \log \delta^2, \] (193)
\[ c + q + e \geq H_{\max}^e(C|BY)_{\Psi}, \quad \log \frac{\delta^2}{2}, \] (194)
\[ q + e \geq H_{\max}^e(C|BYZ)_{\Psi}, \quad \log \delta^2. \] (195)

and \( q + e + 2\epsilon_0 = \log d_C \). Combining these all together, we obtain the set of Ineqs. (107)-(111) as a sufficient condition for the tuple \((c, q, e)\) to be achievable within the error \(4\sqrt{12\epsilon + 6\delta} \).

5.3 Proof of Proposition 7 from Proposition 6
We prove Proposition 7 based on Proposition 6 by (i) modifying the first inequality (107), and (ii) extending the rate region by incorporating teleportation and dense coding.

5.3.1 Modification of Inequalities (107) and (112)
We argue that the smooth conditional max entropy in the R.H.S. of Inequality (107) is modified to be \( H_{\max}^e(C|AXZ)_{\Psi} \). Consider a “modified” redistribution protocol as follows: In the beginning of the protocol, the sender prepares a copy of \( Z \), which we denote by \( \tilde{Z} \). The sender then uses \( X\tilde{Z} \) as the classical part of the side information, instead of \( X \) alone, and apply the protocol presented in Section 5.2.1. The smooth max entropy corresponding to the first term in (107) is then given by (see Lemma 24)
\[ H_{\max}^e(CZ|AX\tilde{Z})_{\Psi_s} = H_{\max}^e(C|AXZ)_{\Psi_s}. \] (196)

For the same reason, the term \( H_{\max}^e(Z|AX)_{\Psi_s} \) in the condition (112) is modified to be \( H_{\max}^e(Z|AXZ)_{\Psi_s} \), which is no greater than zero (see Lemma 21 and Lemma 24). It should be noted that the entropies in the other three inequalities are unchanged by this modification.

5.3.2 Extension of the rate region by Teleportation and Dense Coding
To complete the proof of Theorem 2, we extend the achievable rate region given in Proposition 6 by incorporating teleportation and dense coding. More precisely, we apply the following lemma that follows from teleportation and dense coding (see the next subsection for a proof):

Lemma 11 Suppose that a rate tuple \((\hat{c}, \hat{q}, \hat{e}_0)\) is achievable within the error \(\delta\). Then, for any \(\Delta, \mu \geq 0\) and \(\epsilon_0 \geq 0\) such that
\[ -\frac{\hat{c}}{2} \leq \lambda - \mu - \hat{q}, \quad \hat{e}_0 \leq \epsilon_0, \] (197)
the tuple \((c, q, e, \epsilon_0) := (\hat{c} + 2\lambda - 2\mu, \hat{q} - \lambda + \mu, \hat{e} + \lambda + \mu, \epsilon_0)\) is also achievable within the error \(\delta\).
Proof of Proposition 7: Due to Proposition 6 and Lemma 11, a tuple \((c, q, e, e_0)\) is achievable within the error \(\delta\) if there exists \(\lambda, \mu \geq 0\) and \(\hat{e}_0 \leq e_0\) such that the tuple
\[
(c, \hat{q}, \hat{e}, \hat{e}_0) := (c - 2\lambda + 2\mu, q + \lambda - \mu, e - \lambda - \mu, \hat{e}_0)
\] (198)
satisfies
\[
\hat{c} + \hat{q} - \hat{e} \geq H_1, \quad \hat{q} - \hat{e} \geq H_2, \quad \hat{c} + \hat{q} + \hat{e} \geq H_3, \quad \hat{q} + \hat{e} \geq H_4,
\] (199-202)
and \(\hat{c}, \hat{q}, \hat{e} \geq 0\). Here, we have denoted the R.H.S.s of Inequalities (108)-(110) by \(H_2, H_3\) and \(H_4\), respectively, and that of (196) by \(H_1\). Substituting (198) to these inequalities yields
\[
c + q - e \geq H_1 - 2\mu, \quad q - e \geq H_2 - 2\lambda, \quad c + q + e \geq H_3 + 2\lambda, \quad q + e \geq H_4 + 2\mu
\] (204-207)
and
\[
c - 2\lambda + 2\mu \geq 0, \quad q + \lambda - \mu \geq 0.
\] (209-210)

Thus, it suffices to prove that, for any tuple \((c, q, e, e_0)\) satisfying Inequalities (113)-(116), there exist \(e_0 \leq e_0\) and \(\lambda, \mu \geq 0\) such that the above inequalities hold. This is proved by noting that the inequality (113) is expressed as
\[
c + q + e - H_3 \geq \max\{H_2, H_2'\} - q + e, \quad q + e - H_4 \geq H_1 - c - q + e,
\] (211-212)
where
\[
H_2' := H_{\min}^e(C|AXYZ)_{\Psi_s} - \log \delta^2.
\] (213)
The L.H.S. of (211) and (212) are nonnegative because of Inequalities (114) and (115). Thus, there exists \(\lambda, \mu \geq 0\) such that \(2\lambda\) and \(2\mu\) are in between both sides in (211) and (212), respectively. This implies (204)-(207). We particularly choose
\[
\mu = \frac{1}{2}(q + e - H_4), \quad \hat{e}_0 = \frac{1}{2}(\log d_C - H_4).
\] (214)
A simple calculation leads to (208). Noting that \(H_3 \geq H_4\) by the data processing inequality, it follows from (206) that
\[
c - 2\lambda \geq H_3 - q - e \geq H_4 - q - e = -2\mu,
\] (215)
which implies (209). Inequality (210) is obtained by combining (207) with \(2\lambda \geq \max\{H_2, H_2'\} - q + e\). Note that
\[
H_2' + H_4
= H_{\min}^e(C|AXYZ)_{\Psi_s} + H_{\max}^e(C|BXYZ)_{\Psi_s} - 2\log \delta^2
= H_{\min}^e(C|AXYZ)_{\Psi_s} - H_{\min}^e(C|ARXYZ)_{\Psi_s} - 2\log \delta^2
\geq 0,
\] (216-217)
where the third line follows from Lemma 26. This completes the proof of Theorem 2. 

5.3.3 Proof of Lemma 11 (see also Section IV in [18])

We first consider the case where \(\lambda - \mu \geq 0\), and prove that the tuple \((c, q, e, e_0, \delta) := (\hat{c} + 2\lambda - 2\mu, \hat{q} - \lambda + \mu, \hat{e} + \lambda - \mu, \hat{e}_0, \delta)\) is achievable if a rate tuple \((\hat{c}, \hat{q}, \hat{e}, \hat{e}_0, \delta)\) is achievable and \(\hat{c}, \hat{q}, \hat{e} \geq 0\). Suppose that there exists a protocol \((\mathcal{E}, \mathcal{D})\) with the classical communication cost \(\hat{c}\), the quantum communication cost \(\hat{q}\), the net entanglement cost \(\hat{e}\) and the catalytic entanglement cost \(\hat{e}_0\) that achieves the state redistribution of the state \(\Psi_s\) within the error \(\delta\). We construct a protocol \((\mathcal{E}', \mathcal{D}')\) such that the \(\lambda - \mu\) qubits of quantum communication in the protocol \((\mathcal{E}, \mathcal{D})\) is simulated by quantum teleportation, consuming \(\lambda - \mu\) ebits of additional shared entanglement and \(2\lambda - 2\mu\) bits of classical communication. The net costs of the resources are given by \(\hat{c} + 2\lambda - 2\mu\), \(\hat{q} - \lambda + \mu\), \(\hat{e} + \lambda - \mu\) and the catalytic entanglement cost is \(\hat{e}_0\), which implies achievability of the tuple \((\hat{c} + 2\lambda - 2\mu, \hat{q} - \lambda + \mu, \hat{e} + \lambda - \mu, \hat{e}_0, \delta)\).

Second, we consider the case where \(\lambda - \mu \leq 0\). Suppose that there exists a protocol \((\mathcal{E}, \mathcal{D})\) with the classical communication cost \(\hat{c}\), the quantum communication cost \(\hat{q}\) and the net entanglement cost \(\hat{e}\) that achieves the state redistribution of the state \(\Psi_s\) within the error \(\delta\). We construct a protocol \((\mathcal{E}'', \mathcal{D}'')\) such that the \(2\mu - 2\lambda\) bits of classical communication in \((\mathcal{E}, \mathcal{D})\) is simulated by dense
coding, consuming $\mu - \lambda$ ebits of shared entanglement and $\mu - \lambda$ qubits of quantum communication. The net costs of the resources are given by $\hat{c} - 2\mu + 2\lambda, \hat{q} + \mu - \lambda, \hat{e} + \mu + \lambda, \hat{e}_0, \delta$.

5.4 Proof of Theorem 2 from Proposition 7

The achievability for the case of $d_C = 1$ immediately follows from the condition (120) in Proposition 7. Thus, we only consider the case where $d_C \geq 2$.

Let $\Pi$ be a projection onto a subspace $\mathcal{H}^{CN} \subseteq \mathcal{H}^C$ such that $\dim[\mathcal{H}^{CN}] = 2^{H^{c_{2\beta}}(C)\psi_s}$ and that $\text{Tr}[\Pi \psi_s^C] \geq 1 - e^2/8$. Such a projection exists due to the definition of $H^{c_{2\beta}}$ given by (16). Consider the “modified” source state defined by

$$\psi_{s,\Pi}^{ABCNR} := \Pi(\psi_{s}^{ABC\hat{R}})\Pi.$$  \hspace{1cm} (219)

From the gentle measurement lemma (see Lemma 32 in Appendix B), it holds that

$$P(\psi_s, \psi_{s,\Pi}) \leq \frac{e}{2}, \quad \|\psi_s - \psi_{s,\Pi}\|_1 \leq \frac{e}{\sqrt{2}}.$$  \hspace{1cm} (220)

Thus, due to the definitions of the smooth entropies (12) and (13), we have

$$H^{c_{2\beta}}(CZ|BY) \psi_s \geq H^{c_{2\beta}}(C_{\Pi}Z|BY) \psi_{s,\Pi}$$  \hspace{1cm} (221)

$$\geq H^{3c_{2\beta}/2}(CZ|BY) \psi_s,$$  \hspace{1cm} (222)

$$H^{3c_{2\beta}/2}(C|AXYZ) \psi_s \geq H^{3c_{2\beta}/2}(C_{\Pi}|AXYZ) \psi_{s,\Pi}$$  \hspace{1cm} (223)

and so forth.

Suppose that the tuple $(c, q, e, e_0)$ satisfies Inequalities (22)-(25) in Theorem 2. It follows that

$$c + 2q \geq \max\{H_{\Pi}^{(\psi_s)}, H_{\Pi}^{(\psi_s)}\} \psi_{s,\Pi} - \log(\delta^4/2),$$  \hspace{1cm} (224)

$$c + q + e \geq H^{c_{2\beta}}(C_{\Pi}Z|BY) \psi_{s,\Pi} - \log(\delta^4/2),$$  \hspace{1cm} (225)

$$q + e \geq H^{c_{2\beta}}(C_{\Pi}|BXYZ) \psi_{s,\Pi} - \log(\delta^2),$$  \hspace{1cm} (226)

$$e_0 \geq \frac{1}{2}(\log d_C - H^{c_{2\beta}}(C_{\Pi}|BXYZ) \psi_{s,\Pi}) + \log(\delta).$$  \hspace{1cm} (227)

Thus, due to Proposition 7, the tuple $(c, q, e, e_0)$ is achievable within an error $4\sqrt{12\epsilon + 6\delta}$ for the state $\psi_{s,\Pi}$. That is, there exists a pair of an encoding CPTP map $\mathcal{E}_\Pi^{ABC\hat{R}E_A\rightarrow A\hat{Q}MF_A}$ and a decoding CPTP map $\mathcal{D}_\Pi^{B\hat{Q}ME_B\rightarrow \hat{B}\hat{C}nFB}$, such that

$$\|\mathcal{D}_\Pi \circ \mathcal{E}_\Pi(\psi_{s,\Pi}^{ABC\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}) - \psi_{s,\Pi}^{ABC\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}\|_1 \leq 4\sqrt{12\epsilon + 6\delta}.$$  \hspace{1cm} (228)

Define an encoding map $\mathcal{E}_\Pi^{A\hat{C}E_A\rightarrow A\hat{Q}MF_A}$ and a decoding map $\mathcal{D}_\Pi^{B\hat{Q}ME_B\rightarrow \hat{B}\hat{C}nFB}$ for the state $\psi_s$ by

$$\mathcal{E}_\Pi^{A\hat{C}E_A\rightarrow A\hat{Q}MF_A}(\tau) = \mathcal{E}_\Pi((\Pi_C^{\tau}I_C) - \text{Tr}[(\Pi_C^{\tau} - \Pi_C^{\tau})\psi_s]) \xi_0,$$  \hspace{1cm} (229)

where $\xi_0$ is an arbitrary fixed state on $\hat{A}\hat{Q}MF_A$, and $\mathcal{D} = \mathcal{D}_\Pi$. Note that the system $C_{\Pi}$ is naturally embedded into $C$. By the triangle inequality, we have

$$\|\mathcal{D} \circ \mathcal{E}_\Pi(\psi_s^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}) - \psi_s^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}\|_1 \leq \|\mathcal{D} \circ \mathcal{E}_\Pi(\psi_s^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}) - \mathcal{D} \circ \mathcal{E}_\Pi(\psi_{s,\Pi}^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0})\|_1$$

$$+ \|\mathcal{D} \circ \mathcal{E}_\Pi(\psi_{s,\Pi}^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}) - \psi_{s,\Pi}^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}\|_1$$

$$\leq \|\mathcal{D} \circ \mathcal{E}_\Pi(\psi_{s,\Pi}^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}) - \psi_{s,\Pi}^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}\|_1 + 2\|\psi_{s,\Pi}^{AB\hat{C}\hat{R}} - \psi_{s,\Pi}^{AB\hat{C}\hat{R}}\|_1$$  \hspace{1cm} (231)

$$\leq 4\sqrt{12\epsilon + 6\delta} + \sqrt{2\epsilon},$$  \hspace{1cm} (232)

Here, Inequality (231) follows from $\mathcal{D}_\Pi \circ \mathcal{E}_\Pi(\psi_{s,\Pi}^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0}) = \mathcal{D} \circ \mathcal{E}_\Pi(\psi_{s,\Pi}^{AB\hat{C}\hat{R}} \otimes \Phi_{E_AE_B}^{2\epsilon_0})$ and the monotonicity of the trace distance, and the last line from (220) and (228). Hence, the tuple $(c, q, e, e_0)$ is achievable within an error $4\sqrt{12\epsilon + 6\delta} + \sqrt{2\epsilon}$ for the state $\psi_s$, which completes the proof of Theorem 2.

6 Proof of The Converse Part (Theorem 3 and Lemma 5)

We prove the one-shot converse bound (Theorem 3). The proof proceeds as follows: First, we
We define the "purified" source state $E$ be the Stinespring dilations of the encoding operation $U$. Let $6.1$ Construction of States

Let $U_{E}^{\hat{A}\hat{C}E_{A} \rightarrow \hat{A}QMF_{A}\hat{G}_{A}}$ and $U_{D}^{BQME_{B} \rightarrow \hat{B}\hat{C}F_{B}\hat{G}_{B}}$ be the Stinespring dilations of the encoding operation $E$ and the decoding operation $D$, respectively, i.e.,

$$E = \mathrm{Tr}_{\hat{A}}\hat{G}_{A} \circ U_{E}, \quad D = \mathrm{Tr}_{\hat{B}}\hat{G}_{B} \circ U_{D}. \quad (233)$$

We define the "purified" source state $|\Psi\rangle$ by

$$|\Psi\rangle^{ABCXYZT} := \sum_{x,y,z} \sqrt{p_{xyz}} |x\rangle^{X} |y\rangle^{Y} |z\rangle^{Z} |\psi_{xyz}\rangle^{ABCR} |xyz\rangle^{T}, \quad (234)$$

and consider the states

$$|\tilde{\Psi}\rangle^{\hat{A}QMF_{A}\hat{G}_{A}\hat{B}\hat{RE}_{B}} := U_{E}^{\dagger} |\Psi\rangle^{\hat{A}\hat{B}\hat{C}\hat{R}} |\Phi_{2^{x+y}}^{E_{A}E_{B}}\rangle, \quad (235)$$

$$|\Psi_{f}\rangle^{\hat{A}\hat{B}\hat{C}\hat{R}F_{A}\hat{G}_{A}\hat{G}_{B}} := U_{D}^{\dagger} |\tilde{\Psi}\rangle. \quad (236)$$

The state $\tilde{\Psi}$ is a purification of the state after the encoding operation, and $\Psi_{f}$ is the one after the decoding operation. See Figure 7 for the diagram.

Due to the relation (6) between the trace distance and the purified distance, the condition (21) implies that

$$P \left(C^{T} (\Psi_{f})^{\hat{A}\hat{B}\hat{C}\hat{R}F_{A}\hat{G}_{A}\hat{G}_{B}} \otimes |\Phi_{2^{x+y}}^{F_{A}F_{B}}\rangle \right) \leq 2\sqrt{\delta}, \quad (237)$$

with $C^{T}$ being the completely dephasing operation on $T$ with respect to the basis $\{|xyz\rangle\}$. Due to an extension of Uhlmann’s theorem (see Lemma 30 in Appendix B), there exists a pure state $|\Gamma\rangle^{\hat{A}\hat{B}\hat{C}\hat{G}_{A}\hat{G}_{B}\hat{R}}$, which is represented in the form of

$$|\Gamma\rangle = \sum_{x,y,z} \sqrt{p_{xyz}} |x\rangle^{X} |y\rangle^{Y} |z\rangle^{Z} |\psi_{xyz}\rangle^{ABCR} |xyz\rangle^{T}, \quad (238)$$

such that

$$P \left( (\Psi_{f})^{\hat{A}\hat{B}\hat{C}\hat{R}F_{A}\hat{G}_{A}\hat{G}_{B} \Gamma} \otimes |\Phi_{2^{x+y}}^{F_{A}F_{B}}\rangle \right) \leq 2\sqrt{\delta}. \quad (239)$$

Using this state, we define

$$|\tilde{\Gamma}\rangle^{\hat{A}QMF_{A}\hat{G}_{A}\hat{B}\hat{RE}_{B}} := U_{D}^{\dagger} |\Gamma\rangle^{\hat{A}\hat{B}\hat{C}\hat{G}_{A}\hat{G}_{B} \hat{R} \otimes |\Phi_{2^{x+y}}^{F_{A}F_{B}}\rangle}. \quad (240)$$

Due to the isometric invariance of the purified distance, it follows from (239) and (236) that

$$P \left( (\tilde{\Psi})^{\hat{A}QMF_{A}\hat{G}_{A}\hat{B}\hat{RE}_{B} \Gamma} \otimes |\Phi_{2^{x+y}}^{F_{A}F_{B}}\rangle \right) \leq 2\sqrt{\delta}. \quad (241)$$

Relations among the states defined as above are depicted in Figure 8. Some useful properties of these states are presented in the following, and will be used in the proof of the converse part.
6.1.1 Decomposition of \( U_E \) and \( U_D \)

Since \( M \) is a classical system, we may, without loss of generality, assume that \( U_E \) and \( U_D \) are decomposed as

\[
U_E = \sum_m |m\rangle^M_M |m\rangle^A_C \otimes v_m^{\mathrm{AC} \rightarrow \mathrm{AQF} \rightarrow \mathrm{G}_A},
\]

(242)

\[
U_D = \sum_m |m\rangle^M_M |m\rangle^B_Q \otimes u_m^{\mathrm{BQF} \rightarrow \mathrm{G}_B}. \tag{243}
\]

Here, \( M_A \) and \( M_B \) are quantum systems isomorphic to \( M \) with the fixed orthonormal basis \( \{|m\}_m \), the operators \( u_m \) are linear isometries, and \( \hat{G}_A \) and \( \hat{G}_B \) are such that \( \hat{G}_A = G_A M_A \) and \( \hat{G}_B = G_B M_B \). It follows that

\[
U_D \circ U_E = \sum_m |m\rangle^M_M |m\rangle^M_M \otimes (u_m \circ v_m). \tag{244}
\]

Since \( Z \) is a classical system, we may further assume that \( v_m \) are decomposed as

\[
v_m := \sum_z |z\rangle^Z \otimes v_m^{\mathrm{AC} \rightarrow \mathrm{AQF} \rightarrow \mathrm{G}_A}, \tag{245}
\]

where \( Z'' \) is a system isomorphic to \( Z \) with the fixed orthonormal basis \( \{|z\}_z \) and \( \hat{G}_A = G_A Z'' \).

The operators \( v_{m,z} \) are linear operators such that \( \sum_m v_{m,z}^\dagger v_{m,z} = I \) for all \( z \). It should be noted that \( \hat{G}_A = G_A M_A Z'' \).

6.1.2 Properties of \( \tilde{\Psi} \) and \( \Psi_f \)

Since \( |\tilde{\Psi}\rangle \) is defined as (235) by \( U_E \) that is in the form of (242), it is decomposed into

\[
|\tilde{\Psi}\rangle = \sum_m \sqrt{q_m} |m\rangle^M_M |m\rangle^A_C |\tilde{\Psi}\rangle_m, \tag{246}
\]

with some probability distribution \( \{q_m\}_m \) and pure states \( \{|\tilde{\Psi}\rangle_m\}_m \). Thus, we have

\[
C^M(\Phi) = \sum_m q_m |m\rangle^M_M \otimes |m\rangle^A_C \otimes |\tilde{\Psi}\rangle_m |\tilde{\Psi}\rangle_m, \tag{247}
\]

where \( C^M \) is the completely dephasing operation on \( M \) with respect to the basis \( \{|m\}_m \). Similarly, due to (244), (245) and (236), the state \( |\Psi_f\rangle \) is decomposed into

\[
|\Psi_f\rangle = \sum_m \sqrt{q_m} |m\rangle^M_M |m\rangle^B_Q |\Psi_{f,m}\rangle. \tag{248}
\]

From (245), it holds that \( \langle z|z|^Z |\Psi_f\rangle \propto \delta_{z_1,z_2} \). Thus, the states \( |\Psi_{f,m}\rangle \) are further decomposed into

\[
|\Psi_{f,m}\rangle = \sum_z \sqrt{q_z} |z\rangle^Z |\Psi_{f,m,z}\rangle |\tilde{\Psi}\rangle. \tag{249}
\]

6.1.3 Properties of \( \Gamma \)

From the definition (238), it follows that

\[
C^T(\Gamma) = \sum_{x,y,z} p_{x,y,z} |x y z\rangle \langle x y z|^{X Y Z} \otimes |\psi_{x y z}\rangle |\psi_{x y z}\rangle^{A B C R} \tag{250}
\]

\[
\otimes |\phi_{x y z}\rangle |\phi_{x y z}\rangle^{G_A G_B} \otimes |x y z\rangle^{T \Gamma} \tag{251}
\]

and that

\[
\mathrm{Tr}_T(\Gamma) = \sum_{x,y,z} p_{x,y,z} |x y z\rangle \langle x y z|^{X Y Z} \tag{252}
\]

\[
\otimes |\psi_{x y z}\rangle |\psi_{x y z}\rangle^{A B C R} \otimes |\phi_{x y z}\rangle |\phi_{x y z}\rangle^{G_A G_B}. \tag{253}
\]

Both states are ensembles of pure states on \( A B C R G_A G_B \), classically labelled by \( x y z \) on \( X Y Z \) or \( T \), that are decoupled between \( A B C R \) and \( G_A G_B \). It follows from (250) that

\[
\mathrm{Tr}_{G_A G_B} \otimes C^T(\Gamma) = |\tilde{\Psi}\rangle_{\hat{A} \hat{B} \hat{C} \hat{R}}. \tag{254}
\]

Due to (248), (249) and Lemma 31 in Appendix B, we may, without loss of generality, assume that \( |\phi_{x y z}\rangle \) is in the form of

\[
|\phi_{x y z}\rangle = |\phi_{x y z}\rangle^{G_A M_A G_B} |z\rangle^{Z''} \tag{255}
\]
and 
\[
|\phi'_{xyz}\rangle_{\Gamma A M A G B} := \sum_m \sqrt{p_{m|xyz}} |m\rangle_{MA}|m\rangle_{MB} |\phi_{m,xyz}\rangle_{AGB}.
\]

Substituting this to (250), we have
\[
C^T(\Gamma)^{AGA_M X Y Z T}
= \sum_{x,y,z} p_{xyz} |xyz\rangle \langle \epsilon z|_{X Y Z} \otimes |z\rangle \langle z|_{T'} \otimes |x| \langle x|_{T}.
\]

Thus, the state $C^T(\Gamma)$ given by is classically coherent in $ZZ''$. Denoting $p_{xyz}p_{m|xyz}$ by $p_{m,xyz}$, it follows from (250) that
\[
C^T \circ C^{M_A}(\Gamma^{AGA_M X Y Z T})
= \sum_{x,y,z} p_{m,xyz} \psi'_{xyz} \otimes |m\rangle \langle m|_{MA} \otimes |m\rangle \langle m|_{MB}
\otimes |\phi_{m,xyz}\rangle_{AGB} \otimes |xyz\rangle \langle xyz|_{T}.
\]

with $C^{M_A}$ being the completely dephasing operation on $M_A$ with respect to the basis $\{|m\rangle\}_m$. It should also be noted that
\[
\Gamma^{AGA_M X Y Z T}
= \sum_{x,y,z} p_{m,xyz} |m\rangle \langle m|_{MA} \otimes \psi'_{xyz}
\otimes |\phi_{m,xyz}\rangle_{AGB} \otimes |xyz\rangle \langle xyz|_{T}.
\]

6.2 Inequalities for Proving Theorem 3
As an intermediate goal for the proof of Theorem 3, we prove that the following four inequalities hold for the states $\Psi_s$ and $\Gamma$ defined by (20) and (238), respectively:
\[
ce + e_0 + H_{\min}^e(AXCZ)_{\Psi_s}
\leq c + e_0 + H_{\max}^e(AXCZ)_{\Psi_s} - H_{\min}^{7\epsilon + 2\sqrt{\delta}}(G_A | M A X Z\Gamma) - 4f(\epsilon),
\]

\[
q + e \geq H_{\min}^{e}(AC|XYZ)_{\Psi_s} - H_{\max}^{e}(A|XYZ)_{\Psi_s} - H_{\min}^{4\epsilon + 2\sqrt{\delta}}(G_A M A | XYZ\Gamma) - 3f(\epsilon),
\]

\[
c + q + e \geq H_{\min}^{e}(BYCZ)_{\Psi_s} - H_{\min}^{12\epsilon + 6\sqrt{\delta}}(B)_{\Psi_s} + H_{\max}^{4\epsilon + 2\sqrt{\delta}}(G_A M A | XYZ\Gamma) - f(\epsilon),
\]

\[
q + e \geq H_{\min}^{e}(BC|XYZ)_{\Psi_s} - H_{\min}^{11\epsilon + 8\sqrt{\delta}}(B)_{\Psi_s} + H_{\max}^{7\epsilon + 6\sqrt{\delta}}(G_A | M A X Y Z\Gamma) - 2f(\epsilon),
\]

where $f(x) := -\log(1 - \sqrt{1 - x^2})$. The proof of these inequalities will be given in the following subsections. We will extensively use the properties of the smooth conditional entropies, which are summarized in Appendix A.

6.2.1 Proof of Inequality (258)
We start with
\[
e + e_0 + H_{\min}^e(AXCZ)_{\Psi_s}
= c + e_0 + H_{\max}^e(AXCZ)_{\Psi_s} - H_{\min}^{4\epsilon + 2\sqrt{\delta}}(G_A M A | M\Psi_s)_{\Gamma} + 2f(\epsilon)
\]
\[
\leq c + q + H_{\min}^e(AXF_A G_A | M\Psi_s)_{\Gamma} + 2f(\epsilon),
\]

where (262) follows from $\Psi_s \hat{\hat{\Lambda}} = \hat{\Lambda} \Psi_s$ (263) from the superadditivity of the smooth conditional min entropy for product state (Lemma 16); (264) from the fact that $\langle \Psi_s\rangle$ is obtained from $|\Psi_s\rangle$ by an isometry $U_{\Psi}$ as (235), under which the smooth conditional entropy is invariant (Lemma 14); (265) from the chain rule (360); and (266) from the dimension bound (Lemma 19).

The third term in (266) is further calculated as
\[
H_{\min}^e(AXF_A G_A | M\Psi_s)_{\Gamma}
\leq H_{\min}^{4\epsilon + 2\sqrt{\delta}}(G_A | M A X Z\Gamma) + e_0 + 2f(\epsilon)
\]
\[
H_{\min}^{4\epsilon + 2\sqrt{\delta}}(G_A | M A X Z\Gamma) + e_0 + 2f(\epsilon)
\]

Accepted in Quantum 2021-12-24, click title to verify. Published under CC-BY 4.0.
Here, (268) follows from the monotonicity of the smooth conditional entropy (Lemma 13); (269) from $\hat{G}_A \equiv G_A M_A Z''$; (270) from Lemma 23 and the fact that $M_A$ is a classical copy of $M$ as (247); (271) from the continuity of the smooth conditional entropy (Lemma 20) and the fact that $\hat{\Gamma}$ and $\hat{\Psi}$ are $2\sqrt{\delta}$-close with each other as (241); (272) from the fact that $\hat{\Gamma}$ is converted to $\Gamma$ by $U_D$ as (240), which does not change the reduced state on $AXZ'' F_A G_A M_A$; (273) from the dimension bound (Lemma 19); (274) from the fact that $Z''$ is a classical copy of $Z$, due to (255); (275) from the chain rule (360); (276) from the fact that conditioning reduces the entropy due to the monotonicity of the smooth conditional entropy (Lemma 13); and (277) from the fact that $\Gamma^A X Z = \Psi_s^A X Z$.

Combining these inequalities, we obtain

$$e + \epsilon_0 + H_{\min}^c(AXCZ)\Psi_s \leq c + q + 2f(\epsilon) + H_{\max}^c(AXZ)\Psi_s + H_{\min}^c(\hat{G}_A | M_A AXZ)_{\Gamma} + e_0 + 2f(\epsilon).$$

(278)

which implies (258).

6.2.2 Proof of Inequality (259)

We have

$$e_0 + H_{\min}^{2\epsilon+2\sqrt{\delta}}(\hat{G}_A | T)_{C^T(\Gamma)} = e_0 + H_{\min}^{2\epsilon+2\sqrt{\delta}}(\hat{B} \hat{C} \hat{R} \hat{G}_A | T)_{C^T(\Gamma)} \geq H_{\min}^{2\epsilon+2\sqrt{\delta}}(\hat{B} \hat{C} \hat{R} \hat{G}_A | T)_{C^T(\Gamma) \otimes \Phi_{\hat{\epsilon}} F_0}$$

$$= H_{\min}^{2\epsilon+2\sqrt{\delta}}(\hat{B} \hat{R} \hat{E} \hat{B} | T)_{C^T(\hat{\Gamma})} \geq H_{\min}^{2\epsilon+2\sqrt{\delta}}(\hat{B} \hat{R} \hat{E} \hat{B} | T)_{C^T(\hat{\Gamma})} - q - f(\epsilon).$$

(284)

Here, (280) is from the fact that $\Gamma$ is a pure state on $\hat{A} \hat{B} \hat{C} \hat{R} \hat{G}_A \hat{G}_B$ as (238), which is transformed by $C^T$ to an ensemble of classically-labelled pure states, to which Lemma 27 is applicable; (281) from the dimension bound (Lemma 19); (282) from the fact that $\hat{\Gamma}$ is obtained from $\Gamma \otimes \Phi_{\hat{\epsilon}} F_0$ by an isometry as (240) under which the smooth conditional entropy is invariant (Lemma 14); (283) from the chain rule (359); and (284) from the dimension bound (Lemma 18).

The first term in (284) is further calculated to be

$$H_{\min}^{2\epsilon+2\sqrt{\delta}}(\hat{B} \hat{R} \hat{E} \hat{B} | T)_{C^T(\hat{\Gamma})} \geq H_{\min}^{2\epsilon}(\hat{B} \hat{R} \hat{E} \hat{B} | T)_{C^T(\Psi_s)} \geq H_{\min}^{2\epsilon}(\hat{B} \hat{R} \hat{E} \hat{B} | T)_{C^T(\Psi_s)} \geq H_{\min}^{2\epsilon}(\hat{B} \hat{R} \hat{E} \hat{B} | T)_{C^T(\Psi_s)} + 2f(\epsilon).$$

(285)

Inequality (286) is from the continuity of the smooth conditional entropy (Lemma 20) and the fact that $\hat{\Gamma}$ and $\hat{\Psi}$ are $2\sqrt{\delta}$-close with each other as (241); (287) from Lemma 27 and the fact that $\hat{\Psi}$ is a pure state on $\hat{A} \hat{B} \hat{R} \hat{Q} \hat{M} \hat{F} \hat{A} \hat{G}_A \hat{E}_B$ as (235), which is transformed by $C^T$ to an ensemble of classically-labelled pure states; (288) from $\hat{G}_A \equiv G_A M_A Z''$ and the fact that $M$ is a classical copy of $M_A$ as (247); (289) from the monotonicity of the smooth conditional min entropy under unital maps (Lemma 13); (290) from the isometric invariance of the smooth conditional entropy (Lemma 14) and the fact that $\hat{\Psi}$ is obtained by an isometry $U_E$ from $\Psi$ as (235); (291) from the superadditivity of the smooth conditional entropy (Lemma 16); and (292) from $C^T(\Psi) = \Psi_s$ and the property of the smooth conditional entropy for CQ states (Lemma 23).

The second term in (279) is bounded as

$$H_{\min}^{2\epsilon+2\sqrt{\delta}}(\hat{G}_A | T)_{C^T(\Gamma)} \geq H_{\min}^{2\epsilon}(AG_A M_A | XYZ) \Gamma \geq H_{\max}^{2\epsilon}(AG_A M_A | XYZ) \Gamma + 2f(\epsilon).$$

(293)

Here, (293) follows from $\hat{G}_A \equiv G_A M_A Z''$ and the fact that $C^T(\Gamma)$ is classically coherent in $XX'$ and in $ZZ''$ because of (255); (294) from the chain rule (360); and (295) from $C^T(\Gamma) = \Psi_s^{AXYZ}$ and the fact that the system $A$ in the conditioning part is decoupled from $G_A M_A$ when conditioned by $XYZ$ as (251) in addition to Lemma 25.
Combining these all together, we arrive at
\[ e_0 + H_{\text{max}}(A | XYZ)_{\Psi_s} + H_{\text{min}}^{5e+2\sqrt{\Delta}} (G_A M_A | XYZ)_{\Gamma} + 2f(\epsilon) \]
\[ \geq H_{\text{min}} (AC | XYZ)_{\Psi_s} + e + e_0 - q - f(\epsilon). \]  
(296)

This completes the proof of Ineq. (259).

### 6.2.3 Proof of Inequality (260)

We first calculate
\[
H_{\text{min}}^{s}(BYCZ)_{\Psi_s} = H_{\text{min}}^{s}(BYCZ)_{\Gamma} \]
\[ \leq H_{\text{min}}^{5e+4\sqrt{\Delta}} (BYCZ|F_{B})_{\Gamma} + f(\epsilon) \]
(297)
\[ \leq H_{\text{min}}^{5e+2\sqrt{\Delta}} (BYCZ|F_{B})_{\Gamma} + f(\epsilon) \]
(298)
\[ \leq H_{\text{min}}^{5e+2\sqrt{\Delta}} (BYCZ|F_{B})_{\Gamma} + f(\epsilon) \]
(299)
\[ = H_{\text{min}}^{5e+2\sqrt{\Delta}} (BYCZ|F_{B})_{\Gamma} + f(\epsilon) \]
(300)

Here, (297) follows from \( \Psi_{\text{max}}^{BYCZ} = \Gamma^{BYCZ} \); (298) from the chain rule (359); (299) from the superadditivity of the smooth conditional entropy for product states (Lemma 16); and (300) from the fact that \( \tilde{\Gamma} \) is obtained by an isometry \( U_D^\dagger \) from \( \Gamma \otimes \Phi_{2^0} \) as (240).

The first term in (300) is further calculated to be
\[
H_{\text{min}}^{12e+6\sqrt{\Delta}} (BYE_BQM)_{\tilde{\Gamma}} \]
(301)
\[ \leq H_{\text{min}}^{12e+6\sqrt{\Delta}} (BYE_BQM)_{\Psi} \]
(302)
\[ \leq H_{\text{min}}^{12e+6\sqrt{\Delta}} (BYE_B)_{\Psi} + c + q \]
(303)
\[ = H_{\text{min}}^{12e+6\sqrt{\Delta}} (BYE_B)_{\Psi_s \otimes \Phi_{c+e_0}} + c + q \]
(304)
\[ \leq H_{\text{min}}^{12e+6\sqrt{\Delta}} (BY)_{\Psi_s} + e + e_0 + c + q, \]
(305)

where (302) follows from the continuity of the smooth conditional entropy (Lemma 20) and the fact that \( \tilde{\Gamma} \) and \( \tilde{\Psi} \) are \( 2\sqrt{\Delta} \)-close with each other as (241); (303) from the dimension bound (Lemma 19); (304) from the fact that \( \tilde{\Psi} \) is converted to \( \Psi_s \otimes \Phi_{c+e_0} \) by an operation \( U_C \) by Alice as (235), which does not change the reduced state on \( BYE_B \); and (305) from the dimension bound (Lemma 19) and \( \Psi_s^{BY} = \Psi^{BY} \).

For the third term in (300), we have
\[ H_{\text{min}}^{5e+2\sqrt{\Delta}} (\hat{G}_B | BYCZ)_{\Gamma} \]
(306)
\[ \geq H_{\text{min}}^{5e+2\sqrt{\Delta}} (\hat{G}_B | BCXYZ)_{\Gamma} \]
(307)
\[ = H_{\text{min}}^{5e+2\sqrt{\Delta}} (\hat{G}_B | XYZ)_{\Gamma} \]
(308)
\[ = H_{\text{min}}^{5e+2\sqrt{\Delta}} (\hat{G}_A | XYZ)_{\Gamma} \]
(309)
\[ = H_{\text{min}}^{5e+2\sqrt{\Delta}} (G_A M_A | XYZ)_{\Gamma} \]
(310)

Here, (307) is from the monotonicity of the smooth conditional entropy (Lemma 13); (308) from the fact that \( \Gamma \) is decoupled between \( BC \) and \( \hat{G}_B \) when conditioned by \( XYZ \) as (251), and the property of the smooth conditional entropy (Lemma 25); (309) from Lemma 27 and the fact that \( \Gamma^{G_A \hat{G}_B XYZ} \) is an ensemble of classically-labelled pure states on \( \hat{G}_A \hat{G}_B \) as (251); and (310) from \( \hat{G}_A \equiv G_A M_A Z'' \), Lemma 23 and the fact that \( Z'' \) is a classical copy of \( Z \) due to (255).

Combining these all together, we arrive at
\[ H_{\text{min}}^{s}(BYCZ)_{\Psi_s} \]
\[ \leq H_{\text{min}}^{5e+2\sqrt{\Delta}} (G_A M_A | XYZ)_{\Gamma} \]
(311)

### 6.2.4 Proof of Inequality (261)

We have
\[ e + e_0 + H_{\text{min}}^{11e+8\sqrt{\Delta}} (B | XYZ)_{\Psi_s} \]
(312)
\[ = e + e_0 + H_{\text{min}}^{11e+8\sqrt{\Delta}} (ACR | XYZ)_{\Psi_s} \]
(313)
\[ = e + e_0 + H_{\text{min}}^{11e+8\sqrt{\Delta}} (\hat{A} \hat{C} R | T)_{\Psi_s} \]
(314)
\[ \geq H_{\text{min}}^{11e+8\sqrt{\Delta}} (\hat{A} \hat{C} R | T) + f(\epsilon) \]
(315)
\[ = H_{\text{min}}^{11e+8\sqrt{\Delta}} (\hat{A} \hat{C} R | T) + f(\epsilon) \]
(316)
\[ \geq H_{\text{min}}^{10e+8\sqrt{\Delta}} (\hat{A} \hat{M} F A \hat{G}_A R | T) - f(\epsilon) \]
(317)
\[ + H_{\text{min}}^{(Q) \hat{A} \hat{M} F A \hat{G}_A R | T} - f(\epsilon) \]
(318)
\[ \geq H_{\text{min}}^{10e+8\sqrt{\Delta}} (\hat{B} \hat{E}_B Q | T) + f(\epsilon) \]
(319)
\[ = H_{\text{min}}^{10e+8\sqrt{\Delta}} (\hat{B} \hat{E}_B Q | T) - f(\epsilon) \]
(320)

where (313) follows from Lemma 27; (314) from Lemma 23 and the fact that \( T = X'Y'Z' \) is a classical copy of \( XYZ \); (315) from \( \Psi_s = C^T(\Psi) \), (316)
from the dimension bound (Lemma 19), (317) from the fact that $\tilde{\Psi}$ is obtained from $\Psi \otimes \Phi_{2^{r+\epsilon_0}}$ by applying the isometry $U_E$ as (235), under which the smooth conditional entropy is invariant (Lemma 14), (318) from the chain rule (359), (319) from the dimension bound (Lemma 18), and (320) from Lemma 27 and the fact that $\Psi$ is a pure state on $\hat{A}\hat{B}\hat{R}\hat{F}_A\hat{G}_A Q M E_B$ as (235), which is converted by $C^T$ to an ensemble of classically-labelled pure states. 

The first term in (320) is further calculated to be

$$H_{\min}^{10c+8\sqrt{3}}(\hat{B}EBQ|T)_{C^T(\Psi)}$$

$$= H_{\min}^{10c+8\sqrt{3}}(\hat{B}EBQ|T)_{C^T \otimes C^M(\Psi)}$$

$$\geq H_{\min}^{10c+8\sqrt{3}}(\hat{B}EBQ|TM)_{C^T \otimes C^M(\Psi)}$$

$$= H_{\min}^{10c+8\sqrt{3}}(\hat{B}EBQM(T)_{C^T \otimes C^M(\Psi)})$$

$$= H_{\min}^{10c+8\sqrt{3}}(\hat{B}^{C}F^B \hat{G}_{B}TMA)_{C^T \otimes C^M(\Psi_f)}$$

$$\geq H_{\min}^{10c+6\sqrt{3}}(\hat{B}^{C}F^B \hat{G}_{B}TMA)_{C^T \otimes C^M(\Gamma) \otimes \Phi_{2^{r_0}}}$$

$$\geq H_{\min}^{7c+6\sqrt{3}}(\hat{G}_{B}TMA)_{C^T \otimes C^M(\Gamma) \otimes \Phi_{2^{r_0}}} + H_{\min}^{e}(\hat{B}^{C}F^B T \hat{G}_{B}MA)_{C^T \otimes C^M(\Gamma) \otimes \Phi_{2^{r_0}}} - f(\epsilon).$$

Inequality (322) is due to the fact that $C^M$ does not change the reduced state on $\hat{B}EBQT$; (323) from the monotonicity of the conditional entropy (Lemma 13); (324) from the property of the conditional entropy for classical-quantum states (Lemma 23) and the fact that $M_A$ is a classical copy of $M$ as (247); (325) from the fact that $\Psi_f$ is obtained from $\Psi$ by the isometry $U_D$ as (236), under which the smooth conditional entropy is invariant; (326) from the continuity (Lemma 20) and the fact that $\Gamma \otimes \Phi_{2^{r_0}}$ is $2\sqrt{\delta}$-close to $\Psi_f$ as (239); and (327) from the chain rule (359).

The second term in (327) is further calculated as

$$H_{\min}^{e}(\hat{B}^{C}F^B | T \hat{G}_{B}MA)_{C^T \otimes C^M(\Gamma) \otimes \Phi_{2^{r_0}}}$$

$$\geq H_{\min}^{e}(\hat{B}^{C}T \hat{G}_{B}MA)_{C^T \otimes C^M(\Gamma)} + e_0$$

$$\geq H_{\min}^{e}(\hat{B}^{C}T \hat{G}_{B}MA)_{C^T(\Gamma)} + e_0$$

$$= H_{\min}^{e}(\hat{B}^{C}T)_{C^T(\Gamma)} + e_0$$

$$= H_{\min}^{e}(BC|XYZ)_{\psi_s} + e_0$$

where (329) follows from the superadditivity of the smooth conditional entropy (Lemma 16); (330) from the monotonicity of the smooth conditional entropy (Lemma 13); (331) from Lemma 25 and the fact that the state $C^T(\Gamma)$ is decoupled between $\hat{B}^{C}$ and $\hat{G}_{B}MA$ when conditioned by $T$ as (250); (332) from Equality (252); and (333) from Lemma 23.

The first term in (327) is calculated as

$$H_{\min}^{7c+6\sqrt{3}}(\hat{G}_{B}TMA)_{C^T \otimes C^M(\Gamma)}$$

$$= H_{\min}^{7c+6\sqrt{3}}(G_A|TM)_{\Gamma}$$

$$= H_{\min}^{7c+6\sqrt{3}}(G_A|M_A XYZ)_{\Gamma}$$

$$= H_{\min}^{7c+6\sqrt{3}}(G_A|MA X Y Z)_{\Gamma}$$

where (335) is from $\hat{G}_{B} = G_{B}M_{B}$, Equality (256) and Lemma 27; (336) from Lemma 23 and the fact that $T = XYZ^T$ is a copy of $XYZ$ as (238); and (337) from Lemma 25 and the fact that the state $\Gamma$ is decoupled between $A$ and $G_A$ when conditioned by $M_A X Y Z$ as (257).

Combining these all together, we arrive at

$$c + 2q \geq H_{\min}^{e}(AC|XYZ)_{\psi_s} - H_{\max}^{e}(A|XYZ)_{\psi_s}$$

$$+ H_{\min}^{e}(BYCZ)_{\psi_s} - H_{\min}^{12e+6\sqrt{3}}(BY)_{\psi_s} - 4f(\epsilon)$$

$$= H_{\min}^{e}(\epsilon, \delta) - 4f(\epsilon).$$

This completes the proof of Inequality (261). □

6.3 Proof of Theorem 3 from Inequalities (258)-(261)

Since $\Gamma$ is diagonal in $M_A X Y Z$ as (257), and due to the properties of the smooth conditional entropies for classical-quantum states (Lemma 25), we have

$$H_{\min}^{5e+2\sqrt{3}}(G_A|M_A XY Z)_{\Gamma} \geq 0,$$

$$H_{\min}^{7e+6\sqrt{3}}(G_A|M_A XYZ)_{\Gamma} \geq 0.$$}

Thus, Inequalities (260) and (261) implies Inequalities (30) and (31) in Theorem 3, respectively. Summing up both sides in (259) and (260) yields

$$c + 2q \geq H_{\min}^{e}(AC|XYZ)_{\psi_s} - H_{\max}^{e}(A|XYZ)_{\psi_s}$$

$$+ H_{\min}^{e}(BYCZ)_{\psi_s} - H_{\min}^{12e+6\sqrt{3}}(BY)_{\psi_s} - 4f(\epsilon)$$

$$= H_{\min}^{e}(\epsilon, \delta) - 4f(\epsilon).$$

Accepted in *Quantum* 2021-12-24, click title to verify. Published under CC-BY 4.0.
Similarly, combining Inequalities (258) and (261), we obtain
\[ c + 2q \geq H_{\min}(AXCZ)_{\Psi_s} - H_{\max}(AXZ)_{\Psi_s} + H_{\min}(BC|XYZ)_{\Psi_s} - H_{\min}^{1+8\sqrt{q}}(B|XYZ)_{\Psi_s} - H_{\min}^{q+2\sqrt{q}}(G_A|MAAXZ)_{\Gamma} + H_{\min}^{q+6\sqrt{q}}(G_A|MAAXYZ)_{\Gamma} - 6f(\epsilon) \]
\[ = \hat{H}_{(c,\delta)}^{(e,\delta)} - \Delta^{(e,\delta)} - 6f(\epsilon), \] (342)
where we have defined
\[ \Delta^{(e,\delta)} := H_{\min}^{q+2\sqrt{q}}(G_A|MAAXZ)_{\Gamma} - H_{\min}^{q+6\sqrt{q}}(G_A|MAAXYZ)_{\Gamma}. \] (344)
In the following, we prove that
\[ \Delta^{\epsilon,\delta}_{(c,\delta)} \leq \Delta^{(e,\delta)}. \] (345)
Combining this with (343) in addition to (341), we arrive at Inequality (29) in Theorem 3.

We start by noting that
\[ \Delta^{(e,\delta)} := H_{\min}^{q+2\sqrt{q}}(G_A|MAAXZ)_{\Gamma} - H_{\min}^{q+6\sqrt{q}}(G_A|MAAXYZ)_{\Gamma} \]
\[ \leq \tilde{F}_{\min}^{q+4\sqrt{q}}(G_A:Y'|MAAX'Z')_{\hat{\Psi}} \] (347)
The first line follows from Lemma 23 and the fact that XYZ is a copy of X'Y'Z' as (238), and the second line from the continuity bounds for the smooth conditional entropy (Lemma 20) and the definition of the smooth conditional min mutual information (15). Hence, it suffices to prove that there exists an operation \( \mathcal{F} : \hat{A} \hat{C} \to AG_AM_A \) satisfying
\[ \mathcal{F}(\Psi_{\hat{A} \hat{C} \hat{R}}) = \tilde{\Psi}_{AG_{MA} \hat{R}}, \quad \mathcal{C}_{MA} \circ \mathcal{F} = \mathcal{F} \] (348)
and that \( \tilde{\Psi} \) satisfies the condition
\[ \inf_{\{\omega_{xyz}\}} P \left( \tilde{\Psi}_{AG_{MA} \hat{R}}, \sum_{x,y,z} p_{xyz} \Psi_{\hat{A} \hat{C} \hat{R}} | \omega_{xyz} \right) \leq 2\sqrt{\delta}. \] (349)

Recall that the state |\( \tilde{\Psi} \rangle \) is obtained by an encoding isometry \( \hat{U}_{\tilde{C} A \hat{C} E A \to \hat{A} Q M F A G} \) from |\( \Psi_{4x+g} \rangle \) as (235), where \( \hat{G}_A = G_{MA} \hat{Z}' \). We define an operation \( \mathcal{F} : \hat{A} \hat{C} \to AG_{MA} \) by
\[ \mathcal{F}(\tau) := \text{Tr}_{Q M F A X'Z'} U_{\tilde{C}}(\tau \otimes \pi_{2x+g})_{\hat{E}}. \] (350)
Noting that \( U_{\tilde{C}} \) is in the form of (242), this implies (348). To obtain the decoupling condition (349), we note that, since \( \tilde{\Psi} \) is converted by an operation by Bob to \( \Psi_f \) as (236), it holds that \( \Psi_{AG_{MA} \hat{R}} = \Psi_f_{AG_{MA} \hat{R}} \).
Thus, tracing out \( BC \hat{F}_A \hat{B} \hat{G}_B X'' \) in (239), we obtain
\[ P \left( \tilde{\Psi}_{AG_{MA} \hat{R}}, \Gamma_{AG_{MA} \hat{R}} \right) \leq 2\sqrt{\delta}. \] (351)
Due to (238), the state \( \Gamma \) is in the form of
\[ \Gamma_{AG_{MA} \hat{R}} = \sum_{x,y,z} p_{xyz} \Psi_{\hat{A} \hat{C} \hat{R}} | \omega_{xyz} \rangle |xyz\rangle |xyz|^T. \] (352)
This implies (349) and completes the proof of Inequality (29).

6.4 Property of \( \Delta^{(c,\delta)} \) (Proof of Lemma 5)
Due to the definition of the smooth conditional min mutual information (15) and (34), it is straightforward to verify that \( \Delta^{(c,\delta)} \geq 0 \). The equality holds if \( Y' \cong Y \) is a one-dimensional system, that is, if there is no classical side information at the decoder. In the case where there is neither quantum message nor quantum side information at the encoder, i.e.
\[ d_A = d_C = 1, \quad \hat{A} = X, \quad \hat{C} = Z, \] (353)
the source state \( \Psi_s \) is represented as
\[ \Psi_s^{XZBR} = \sum_{x,y,z} p_{xyz} |x\rangle |x\rangle |y\rangle |y\rangle |z\rangle |z\rangle |\Psi_{xyz}^{BR}. \] (354)
Thus, for any CPTP map \( \mathcal{F} : XZ \to G_{MA} \), we have
\[ \mathcal{F}(\Psi_s)^{G_{MA} \hat{R}} = \sum_{x,y,z} p_{xyz} \omega_{xyz}^{G_{MA} \hat{R}} \otimes \hat{\rho}_{xyz}, \] (355)
where \( \omega_{xyz} := \mathcal{F}(\hat{F}) |x\rangle |x\rangle |y\rangle |y\rangle |z\rangle |z\rangle \). It follows that
\[ \mathcal{F}(\Psi_s)^{G_{MA} X'Y'Z'} = \sum_{x,z} p_{xz} \omega_{xz}^{G_{MA} \hat{R}} \otimes |xz\rangle |xz\rangle X'Z' \]
\[ \otimes \left( \sum_{y} p_{y} |y\rangle |y\rangle Y' \right), \] (356)
and consequently, \( \tilde{F}_{\min}^{q+4\sqrt{q}}(G_A:Y'|MAX'Z') = 0 \). This implies \( \Delta^{(c,\delta)} = 0 \), and completes the proof of Lemma 5.
7 Conclusion

In this paper, we investigated the state redistribution of classical and quantum hybrid sources in the one-shot scenario. We analyzed the costs of classical communication, quantum communication and entanglement. We obtained the direct bound and the converse bound for those costs in terms of smooth conditional entropies. In most of the cases that have been analyzed in the previous literatures, the two bounds coincide in the asymptotic limit of infinitely many copies and vanishingly small error. Various coding theorems for two-party source coding tasks are systematically obtained by reduction from our results, including the ones that have not been analyzed in the previous literatures.

To investigate the protocol that are covered by our result, but have not been addressed in the previous literature, in detail is left as a future work. Another direction is to explore the family of quantum communication protocols in the presence of classical side information only at the decoder. It would also be beneficial to analyze the relation between our results and the one-shot bounds for entanglement-assisted communication of classical and quantum messages via a noisy quantum channel [32].

Acknowledgement

This work was supported by JSPS KAKENHI (Grant No. 18J01329), and by JST, PRESTO Grant Number JPMJPR1865, Japan.

References

[1] Anura Abeyesinghe, Igor Devetak, Patrick Hayden, and Andreas Winter, “The mother of all protocols: restructuring quantum information’s family tree”, Proc. Roy. Soc. A 465(2108):2537–2563 (2009)

[2] Anurag Anshu, Rahul Jain, and Naqueeb Ahmad Warsi, “A one-shot achievability result for quantum state redistribution”, IEEE Trans. Inf. Theory 64(3): 1425–1435 (2018)

[3] Anurag Anshu, Min-Hsiu Hsieh, and Rahul Jain, “Noisy quantum state redistribution with promise and the alpha-bit”, IEEE Trans. Inf. Theory 66(12): 7772–7786 (2020)

[4] Zahra Baghali Khanian and Andreas Winter, “Distributed compression of correlated classical quantum sources or: The price of ignorance”, IEEE Trans. Inf. Theory 66(9): 5620–5633 (2020)

[5] Mario Berta, Matthias Christandl, Roger Colbeck, Joseph M. Renes, and Renato Renner, “The uncertainty principle in the presence of quantum memory”, Nat. Phys. 6(9): 659–662 (2010)

[6] Mario Berta, Matthias Christandl, and Renato Renner, “The quantum reverse shannon theorem based on one-shot information theory”, Comm. Math. Phys. 306(3):579–615 (2011)

[7] Mario Berta, Matthias Christandl, and Dave Touchette, “Smooth entropy bounds on one-shot quantum state redistribution”, IEEE Trans. Inf. Theory 62(3):1425–1439 (2016)

[8] T. M. Cover and J. A. Thomas, Elements of Information Theory (2nd ed.), Wiley InterScience, 2005.

[9] Nilanjana Datta and Min-Hsiu Hsieh, “The apex of the family tree of protocols: optimal rates and resource inequalities”, New J. Phys. 13, 093042 (2011)

[10] Nilanjana Datta, Min-Hsiu Hsieh, and Jonathan Oppenheim, “An upper bound on the second order asymptotic expansion for the quantum communication cost of state redistribution”, J. Math. Phys. 57(5), 052203 (2016)

[11] I Devetak and A Winter, “Classical data compression with quantum side information”, Phys. Rev. A 68(4), 042301 (2003)

[12] I Devetak, AW Harrow, and A Winter, “A family of quantum protocols”, Phys. Rev. Lett. 93(23), 230504 (2004)

[13] Igor Devetak and Jon Yard, “Exact cost of redistributing multipartite quantum states”, Phys. Rev. Lett. 100(23), 230501 (2008)

[14] Igor Devetak, Aram W. Harrow, and Andreas J. Winter, “A resource framework for quantum shannon theory”, IEEE Trans. Inf. Theory 54(10):4587-4618 (2008)

[15] Frederic Dupuis, Mario Berta, Juerg Wullschleger, and Renato Renner, “One-shot decoupling”, Comm. Math. Phys. 328(1):251-284 (2014)
[16] M Horodecki, J Oppenheim, and A Winter, “Partial quantum information”, Nature 436 (7051): 673–676 (2005)

[17] Michal Horodecki, Jonathan Oppenheim, and Andreas Winter, “Quantum state merging and negative information”, Comm. Math. Phys. 269(1):107–136 (2007)

[18] Min-Hsiu Hsieh and Mark M. Wilde, “Trading classical communication, quantum communication, and entanglement in quantum shannon theory”, IEEE Trans. Inf. Theory 56(9): 4705-4730 (2010)

[19] Robert Koenig, Renato Renner, and Christian Schaffner, “The operational meaning of min- and max-entropies”, IEEE Trans. Inf. Theory 55(9): 4337-4347 (2009)

[20] T Ogawa and H Nagaoka, “A new proof of the channel coding theorem via hypothesis testing in quantum information theory”, In ISIT: 2002 IEEE Int. Symp. Info.Theory, Proceedings, page 73

[21] Joseph M. Renes and Renato Renner, “One-shot classical data compression with quantum side information and the distillation of common randomness or secret keys”, IEEE Trans. Inf. Theory 58(3):1985–1991 (2012)

[22] R Renner and S Wolf, “Simple and tight bounds for information reconciliation and privacy amplification”, In Advances in Cryptology - ASIACRYPT 2005, pp.199-216, Springer, Berlin, Heidelberg

[23] Renato Renner, “Security of quantum key distribution”, Int. J. Quant. Info. 6(1): 1–127, (2008)

[24] Benjamin Schumacher, “Quantum coding”, Phys. Rev. A 51(4):2738–2747 (1995)

[25] W Forrest Stinespring, “Positive functions on C* algebras”, Proc. of the Amer. Math. Soc., 6(2): 211–216 (1955)

[26] M. Tomamichel, Quantum Information Processing with Finite Resources, Springer Briefs in Mathematical Physics, 2016

[27] Marco Tomamichel and Masahito Hayashi, “A hierarchy of information quantities for finite block length analysis of quantum tasks”, IEEE Trans. Inf. Theory 59(11):7693–7710 (2013)

[28] Marco Tomamichel, Roger Colbeck, and Renato Renner, “A fully quantum asymptotic equipartition property”, IEEE Trans. Inf. Theory 55(12):5840–5847 (2009)

[29] Marco Tomamichel, Roger Colbeck, and Renato Renner, “Duality between smooth min- and max-entropies”, IEEE Trans. Inf. Theory 56(9):4674-4681 (2010)

[30] Armin Uhlmann, “The “transition probability” in the state space of a C*-algebra”, Rep. Math. Phys., 9(2):273–279 (1976)

[31] Alexander Vitanov, Frederic Dupuis, Marco Tomamichel, and Renato Renner, “Chain rules for smooth min- and max-entropies”, IEEE Trans. Inf. Theory 59(5):2603–2612 (2013)

[32] Eyuri Wakakuwa and Yoshifumi Nakata, “Randomized partial decoupling unifies one-shot quantum channel capacities”, arXiv:2004.12593 (2020)

[33] Eyuri Wakakuwa and Yoshifumi Nakata, “One-shot randomized and nonrandomized partial decoupling”, Comm. Math. Phys. 386(2):589–649 (2021)

[34] Mark Wilde, Quantum Information Theory, Camb. Univ. Press, 2013

[35] Jon T. Yard and Igor Devetak, “Optimal quantum source coding with quantum side information at the encoder and decoder”, IEEE Trans. Inf. Theory 55 (11):5339–5351 (2009)

[36] Ming-Yong Ye, Yan-Kui Bai, and Z. D. Wang, “Quantum state redistribution based on a generalized decoupling”, Phys. Rev. A 78(3) (2008)
A Definitions and Properties of Smooth Entropies

In this appendix, we summarize the properties of the smooth conditional entropies that are used in the main text. For the properties of the purified distance used in some of the proofs, see Appendix B.

A.1 Basic Properties

Lemma 12 (duality: see e.g. [29]) For any subnormalized pure state $|\psi\rangle$ on system $ABC$, and for any $\epsilon > 0$, $H^\epsilon_{\text{max}}(A|B)_\psi = -H^\epsilon_{\text{min}}(A|C)_\psi$.

Lemma 13 (monotonicity: Theorem 18 in [29] and Theorem 6.2 in [26]) For any $\rho^{AB} \in \mathcal{S}_\leq(\mathcal{H}^{AB})$, $0 \leq \epsilon \leq \sqrt{\text{Tr}[\rho]}$, any unital CPTP map $\mathcal{E} : A \rightarrow C$ and any CPTP map $\mathcal{F} : B \rightarrow D$, it holds that $H^\epsilon_{\text{min}}(A|B)_\rho \leq H^\epsilon_{\text{min}}(C|D)_{\mathcal{E} \otimes \mathcal{F}(\rho)}$.

Lemma 14 (isometric invariance: Lemma 13 in [29]) For any $\epsilon \geq 0$, $\rho^{AB} \in \mathcal{S}_\leq(\mathcal{H}^{AB})$ and any linear isometries $U : A \rightarrow C$ and $V : B \rightarrow D$, $H^\epsilon_{\text{min}}(A|B)_\rho = H^\epsilon_{\text{min}}(C|D)_{U \otimes V(\rho)}$.

Lemma 15 (additivity: see Section I C in [19]) For any $\rho \in \mathcal{S}(\mathcal{H}^{AB})$ and $\sigma \in \mathcal{S}(\mathcal{H}^{CD})$, it holds that

$$H^\epsilon_{\text{max}}(AC|BD)_{\rho \otimes \sigma} = H^\epsilon_{\text{max}}(A|B)_\rho + H^\epsilon_{\text{max}}(C|D)_\sigma.$$  \hspace{1cm} (357)

Lemma 16 (superadditivity: Lemma A.2 in [15]) For any states $\rho^{AB}$, $\sigma^{CD}$ and $\epsilon, \epsilon' \geq 0$, it holds that

$$H^{\epsilon' \epsilon''}_{\text{min}}(AC|BD)_{\rho \otimes \sigma} \geq H^\epsilon_{\text{min}}(A|B)_\rho + H^\epsilon_{\text{min}}(C|D)_\sigma.$$  \hspace{1cm} (358)

Lemma 17 (chain rule: see [31]) For any $\epsilon > 0$, $\epsilon', \epsilon'' \geq 0$ and $\rho \in \mathcal{S}_\leq(\mathcal{H}^{ABC})$, it holds that

$$H^{\epsilon + \epsilon' + 2\epsilon''}_{\text{min}}(AB|C)_{\rho} \geq H^\epsilon_{\text{min}}(B|C)_\rho + H^{\epsilon''}_{\text{min}}(A|BC)_\rho - f(\epsilon),$$

$$H^\epsilon_{\text{min}}(AB|C)_\rho \leq H^\epsilon_{\text{max}}(B|C)_\rho + H^{\epsilon + \epsilon' + 2\epsilon''}_{\text{min}}(A|BC)_\rho + 2f(\epsilon),$$

where

$$f(\epsilon) := -\log(1 - \sqrt{1 - \delta^2}).$$  \hspace{1cm} (361)

Lemma 18 (dimension bounds: Corollary of Lemma 20 in [29]) For any state $\rho^{AB}$ and $\epsilon \geq 0$, it holds that

$$H^\epsilon_{\text{min}}(A|B)_\rho \geq -\log d_A,$$

$$H^\epsilon_{\text{max}}(A|B)_\rho \leq \log d_A.$$  \hspace{1cm} (362)

(363)

Lemma 19 (dimension bound: Lemma 21 in [9]) For any state $\rho^{ABC}$ and $\epsilon > 0$, it holds that

$$H^\epsilon_{\text{min}}(AB|C)_\rho \leq H^\epsilon_{\text{min}}(A|C)_\rho + \log d_B.$$  \hspace{1cm} (364)

Lemma 20 (continuity) For any $\epsilon, \delta \geq 0$, any $\rho^{AB}$ and $\sigma^{AB} \in \mathcal{B}^{\delta}(\rho)$, it holds that

$$H^{\epsilon + \delta}_{\text{min}}(A|B)_\rho \geq H^\epsilon_{\text{min}}(A|B)_\sigma.$$  \hspace{1cm} (365)

Proof: Let $\hat{\sigma}^{AB} \in \mathcal{B}^{\delta}(\sigma)$ be such that $H^\epsilon_{\text{min}}(A|B)_\sigma = H^\epsilon_{\text{min}}(A|B)_{\hat{\sigma}}$. Due to the triangle inequality for the purified distance, it holds that

$$P(\rho, \hat{\sigma}) \leq P(\rho, \sigma) + P(\sigma, \hat{\sigma}) \leq \epsilon + \delta,$$

which implies $\hat{\sigma} \in \mathcal{B}^{\epsilon + \delta}(\rho)$. Thus, we obtain Inequality (365) as

$$H^\epsilon_{\text{min}}(A|B)_\sigma = H^\epsilon_{\text{min}}(A|B)_{\hat{\sigma}} \leq \sup_{\hat{\rho} \in \mathcal{B}^{\epsilon + \delta}(\rho)} H^\epsilon_{\text{min}}(A|B)_{\hat{\rho}} = H^{\epsilon + \delta}_{\text{min}}(A|B)_\rho.$$  \hspace{1cm} (367)
Lemma 21 (one-dimensional system.) Suppose that $d_A = 1$. For any $\epsilon \geq 0$ and $\rho \in S(\mathcal{H}^{AB})$, it holds that

\begin{align}
0 & \leq H^\epsilon_{\text{min}}(A|B)_\rho \leq -\log (1-2\epsilon), \quad (368) \\
0 & \geq H^\epsilon_{\text{max}}(A|B)_\rho \geq \log (1-2\epsilon). \quad (369)
\end{align}

Proof: Since $d_A = 1$, there exists a fixed vector $|e\rangle \in \mathcal{H}^A$ such that $I^A = |e\rangle\langle e|$ and that any $\tilde{\rho} \in S(\mathcal{H}^{AB})$ is represented as $|e\rangle\langle e|^A \otimes \tilde{\rho}^B$. Due to the definition of the smooth conditional min entropy, we have

\begin{align}
H^\epsilon_{\text{min}}(A|B)_\rho = H_{\text{min}}(A|B)_{\rho|\sigma} & \geq \sup_{\sigma_B \in S(\mathcal{H}^B)} H_{\text{min}}(A|B)_{\rho^A|\rho^B} \\
& \geq H_{\text{min}}(A|B)_{\rho^A|\rho^B} \\
& = \sup\{ \lambda \in \mathbb{R} | 2^{-\lambda} I^A \otimes \rho^B \geq \rho^{AB} \} \\
& = \sup\{ \lambda \in \mathbb{R} | 2^{-\lambda} |e\rangle\langle e|^A \otimes \rho^B \geq |e\rangle\langle e|^A \otimes \tilde{\rho}^B \} \\
& = 0. \quad (375)
\end{align}

This implies the first inequality in (368). To prove the second inequality in (368), let $\hat{\rho} \in B'(\rho)$ and $\sigma^B \in S(\mathcal{H}^B)$ be such that

\begin{align}
H^\epsilon_{\text{min}}(A|B)_\rho = H_{\text{min}}(A|B)_{\hat{\rho}} = H_{\text{min}}(A|B)_{\rho|\sigma}. \quad (376)
\end{align}

By definition, it holds that

\begin{align}
2^{-H^\epsilon_{\text{min}}(A|B)_\rho} I^A \otimes \sigma^B \geq \hat{\rho}^{AB}, \quad (377)
\end{align}

which is equivalent to

\begin{align}
2^{-H^\epsilon_{\text{min}}(A|B)_\rho} |e\rangle\langle e|^A \otimes \sigma^B \geq |e\rangle\langle e|^A \otimes \hat{\rho}^B. \quad (378)
\end{align}

By taking the trace in both sides, we obtain

\begin{align}
2^{-H^\epsilon_{\text{min}}(A|B)_\rho} \geq \text{Tr}[\hat{\rho}]. \quad (379)
\end{align}

The R.H.S. of the above inequality is evaluated as

\begin{align}
\text{Tr}[\hat{\rho}] = \|\hat{\rho}\|_1 \geq \|\rho\|_1 - \|\rho - \hat{\rho}\|_1 \geq 1 - 2\epsilon, \quad (380)
\end{align}

where the last line follows from (6) and the condition $\hat{\rho} \in B'(\rho)$. This implies the second inequality in (368). Inequality (369) follows due to the duality relation (Lemma 12).

A.2 Classical-Quantum States

Lemma 22 (Lemma A.5 in [15]) For any state $\rho^{ABK} \in S(\mathcal{H}^{ABK})$ in the form of

\begin{align}
\rho^{ABK} = \sum_k p_k \rho^{AB}_k \otimes |k\rangle\langle k|^K, \quad (381)
\end{align}

where $\rho_k \in S(\mathcal{H}^{AB})$, $\langle k|k' \rangle = \delta_{k,k'}$ and $\{p_k\}_k$ is a normalized probability distribution, it holds that

\begin{align}
H_{\text{min}}(A|BK)_\rho = -\log \left( \sum_k p_k \cdot 2^{-H_{\text{min}}(A|B)_{\rho_k}} \right). \quad (382)
\end{align}
Lemma 23 (Lemma A.7 in [15]) For any state $\rho^{ABK_1K_2} \in S_{\leq}^{\epsilon}(H^{ABK_1K_2})$ in the form of
\[\rho^{ABK_1K_2} = \sum_k p_k \rho_k^{AB} \otimes |k\rangle\langle k|^1 \otimes |k\rangle\langle k|^2,\] (383)
where $|k\rangle = \delta_{k,k'}$, and for any $\epsilon \geq 0$, it holds that
\[H_{\min}^{\epsilon}(AK_1|BK_2)_{\rho} = H_{\min}^{\epsilon}(A|BK_2)_{\rho} = H_{\min}^{\epsilon}(A|BK_1)_{\rho}.\] (384)

Lemma 24 (Lemma 29 in [33]) In the same setting as in Lemma 23, it holds that
\[H_{\max}^{\epsilon}(AK_1|BK_2)_{\rho} = H_{\max}^{\epsilon}(A|BK_2)_{\rho} = H_{\max}^{\epsilon}(A|BK_1)_{\rho}.\] (385)

Lemma 25 Consider a state in the form of
\[\rho^{ACK} = \sum_k p_k \rho_k^A \otimes \sigma_k^C \otimes |k\rangle\langle k|^K.\] (386)
For any $\epsilon > 0$, it holds that
\[H_{\min}^{\epsilon}(A|CK)_{\rho} = H_{\min}^{\epsilon}(A|K)_{\rho} \geq 0.\] (387)

Proof: It is straightforward to verify that there exists a quantum operation $E : K \rightarrow CK$ such that $\rho^{ACK} = E(\rho^{AK})$. Due to the monotonicity of the smooth conditional min entropy under operations on the conditioning system, we have
\[H_{\min}^{\epsilon}(A|K)_{\rho^{AK}} \leq H_{\min}^{\epsilon}(A|CK)_{\rho^{ACK}} = H_{\min}^{\epsilon}(A|CK)_{E(\rho^{AK})} \leq H_{\min}^{\epsilon}(A|K)_{\rho^{AK}},\] (388)
which implies $H_{\min}^{\epsilon}(A|CK)_{\rho} = H_{\min}^{\epsilon}(A|K)_{\rho}$. The non-negativity follows due to Lemma 22 as
\[H_{\min}^{\epsilon}(A|K)_{\rho} \geq H_{\min}(A|K)_{\rho} = -\log \left(\sum_k p_k \cdot 2^{-H_{\min}(A)_{p_k}}\right) \geq -\log \left(\sum_k p_k\right) = 0,\] (389)
which completes the proof. \qed

A.3 Classically-labelled Pure States

Lemma 26 Consider a state in the form of
\[\rho^{ABCK} = \sum_k p_k |\psi_k\rangle\langle \psi_k|^ABC \otimes |k\rangle\langle k|^K.\] (390)
For any $\epsilon > 0$, it holds that
\[H_{\max}^{\epsilon}(A|BK)_{\rho} = -H_{\min}^{\epsilon}(A|CK)_{\rho}.\] (391)

Proof: It is straightforward to verify that a purification of the state $\rho$, defined by (390), is given by
\[|\psi_{\rho}\rangle^{ABCKK'} = \sum_k \sqrt{p_k} |\psi_k\rangle^{ABC} |k\rangle^K |k\rangle^{K'}.\] (392)
Due to the duality of the smooth conditional entropies (Lemma 12), we have
\[H_{\max}^{\epsilon}(A|BK)_{\rho} = H_{\max}^{\epsilon}(A|BK)_{\psi_{\rho}} = -H_{\min}^{\epsilon}(A|CK')_{\psi_{\rho}} = -H_{\min}^{\epsilon}(A|CK)_{\rho},\] (393)
which completes the proof. \qed
Lemma 27 Consider the same setting as in Lemma 26. For any \( \epsilon > 0 \), it holds that

\[
H^\epsilon_{\text{min}}(A|K)_\rho = H^\epsilon_{\text{min}}(B|K)_\rho.
\]  

(394)

Proof: To prove (394), let \( \hat{\rho}^{AK} \in B^\rho(\rho) \) and \( \varsigma \in S_\infty(K) \) be such that

\[
H^\epsilon_{\text{min}}(A|K)_\rho = H^\min_{\text{min}}(A|K)_{\hat{\rho}^A} = H^\min_{\text{min}}(A|K)_{\hat{\rho}^{AK}}.
\]

(395)

With \( C \) being the completely dephasing operation on \( K \) with respect to the basis \( \{|k\}\}_k \), it holds that

\[
P(C(\hat{\rho}), C(\rho)) \leq P(\hat{\rho}, \rho) \leq \epsilon.
\]

(396)

In addition, if

\[
2^{-\lambda} I^A \otimes \varsigma^K \geq \hat{\rho}^{AK},
\]

then

\[
2^{-\lambda} I^A \otimes C(\varsigma)^K \geq \text{id}^A \otimes C^K(\hat{\rho}^{AK}).
\]

(398)

Thus, without loss of generality, we may assume that both \( \hat{\rho}^{AK} \) and \( \varsigma \) are diagonal in \( \{|k\}\}_k \). That is, we may assume that \( \hat{\rho}^{AK} \) and \( \varsigma \) are in the form of

\[
\hat{\rho}^{AK} = \sum_k \hat{p}_k \hat{\rho}_k^A \otimes |k\rangle \langle k|^K, \quad \varsigma = \sum_k q_k |k\rangle \langle k|.
\]

(399)

Suppose that the Schmidt decomposition of \( |\psi_k\rangle \) is given by

\[
|\psi_k\rangle = \sum_j \sqrt{\mu_j} |e_j\rangle^A |f_j\rangle^K_B,
\]

(400)

Define linear operators \( v_k : H^A \to H^B \) and \( V : H^A \otimes H^K \to H^B \otimes H^K \) by

\[
v_k := \sum_j |f_j\rangle^B \langle e_j|^A \quad (\forall k)
\]

(401)

and \( V := \sum_k v_k \otimes |k\rangle \langle k|^K \). It is straightforward to verify that \( \rho^{BK} = V \rho^{AK} V^\dagger \). Thus, due to the monotonicity of the purified distance under trace non-increasing CP maps (Lemma 7 in [29]), it holds that

\[
P(\rho^{BK}, V \hat{\rho}^{AK} V^\dagger) \leq P(\rho^{AK}, \hat{\rho}^{AK}) \leq \epsilon.
\]

(402)

Applying \( V \) to the both sides in condition (397), it follows that

\[
2^{-\lambda} V(I^A \otimes \varsigma^K) V^\dagger \geq V \hat{\rho}^{AK} V^\dagger.
\]

(403)

Noting that \( I^B \geq (v_k^\dagger v_k)^B \), this implies that

\[
2^{-\lambda} I^B \otimes \varsigma^K \geq V \hat{\rho}^{AK} V^\dagger.
\]

(404)

Thus, we arrive at

\[
H^\epsilon_{\text{min}}(A|K)_\rho \leq H^\epsilon_{\text{min}}(B|K)_\rho.
\]

(405)

By exchanging the roles of \( A \) and \( B \), we also obtain the converse inequality. This completes the proof of Equality (394).
B Properties of The Purified Distance

We summarize the properties of the purified distance, used in Appendix A to prove the properties of the smooth conditional entropies.

**Lemma 28 (monotonicity: Lemma 7 in [29])** For any subnormalized states $\rho, \sigma \in \mathcal{S}_{\geq}(\mathcal{H})$ and for any completely positive trace non-increasing map $\mathcal{E}$, it holds that $P(\rho, \sigma) \geq P(\mathcal{E}(\rho), \mathcal{E}(\sigma))$. Consequently, for any linear isometry $\mathcal{U}$, it holds that $P(\rho, \sigma) = P(\mathcal{U}(\rho), \mathcal{U}(\sigma))$

**Lemma 29** For any normalized state $\rho$ on system $A$ and any normalized pure state $|\phi\rangle$ on system $AB$, the purified distance satisfies

$$P(\rho^A, \phi^A) = \min_{|\psi\rangle^{AB}} P(|\psi\rangle, |\phi\rangle) = \sqrt{1 - \max_{|\psi\rangle^{AB}} |\langle \psi | \phi \rangle|^2},$$

where the minimum and the maximum are taken over all purifications $|\psi\rangle$ of $\rho$.

**Proof:** Follows from Definition 4 and Lemma 8 in [29].

**Lemma 30** Consider a state $\Gamma$ on $KAB$ and a pure state $|\Psi\rangle$ on $KABC$ in the form of

$$|\Psi\rangle = \sum_k \sqrt{p_k} |k\rangle^K |\psi_k\rangle^{ABCD}, \quad \Gamma = \sum_k p_k |k\rangle^K \otimes |\gamma_k\rangle^{AB}.$$

There exists a set of pure states $\{|\phi_k\rangle\}_k$ on $CD$ such that, for the state

$$|\tilde{\Gamma}\rangle = \sum_k \sqrt{p_k} |\gamma_k\rangle^{AB} |\phi_k\rangle^{CD},$$

it holds that

$$P \left( |\tilde{\Gamma}\rangle, |\Psi\rangle \right) = P \left( \Gamma^{KAB}, \mathcal{C}^K \circ \text{Tr}_{CD}(|\Psi\rangle) \right),$$

where $\mathcal{C}$ is the completely dephasing operation on $K$ with respect to the basis $\{|k\rangle\}_k$.

**Proof:** It is straightforward to verify that a purification of the state $\mathcal{C}^K \circ \text{Tr}_{CD}(|\Psi\rangle)$ is given by

$$|\Psi_p\rangle = \sum_k \sqrt{p_k} |k\rangle^K |\psi_k\rangle^{ABCD} |k\rangle^{K'},$$

and that any purification of the state $\Gamma^{KAB}$ to the system $KABCDK'$ is in the form of

$$|\Gamma_p\rangle = \sum_k \sqrt{p_k} |\gamma_k\rangle^{AB} |\xi_k\rangle^{CDK'},$$

with $\{|\xi_k\rangle\}_k$ being a set of orthogonal states. A simple calculation yields

$$|\langle \Psi_p | \Gamma_p \rangle| = \sum_k p_k |\langle \psi_k^{ABCD} |k\rangle^{K'} |\langle \gamma_k^{AB} |\xi_k^{CDK'} \rangle|.$$

The maximum of the above quantity over all orthogonal $\{|\xi_k\rangle\}_k$ is achieved by $\{|\xi_k\rangle\}_k$ that is decomposed into $|\xi_k\rangle^{CDK'} = |\phi_k\rangle^{CD} |k\rangle^{K'}$. Using this $\{|\phi_k\rangle\}_k$, we define a state $|\tilde{\Gamma}\rangle$ by

$$|\tilde{\Gamma}\rangle := \sum_k \sqrt{p_k} |\gamma_k\rangle^{AB} |\phi_k\rangle^{CD}$$

and a purification of $\Gamma^{KAB}$ by

$$|\Gamma_p^*\rangle := \sum_k \sqrt{p_k} |\gamma_k\rangle^{AB} |\phi_k\rangle^{CD} |k\rangle^{K'}.$$
It follows that

\[
\max_{\{\xi_k\}_k} |\langle \psi_p | \Gamma_p \rangle| = |\langle \psi_p | \Gamma^*_p \rangle| \tag{415}
\]

\[
= \sum_k p_k \langle \psi_k |^{ABC}_k \rangle^{AB} \langle \phi_k |^{CD} \rangle \tag{416}
\]

\[
= |\langle \Psi | \tilde{\Gamma} \rangle|. \tag{417}
\]

In addition, the states |\Psi_p\rangle and |\Gamma^*_p\rangle are obtained by a linear isometry \(P^{K \to K'} := \sum_k |k\rangle^K \langle k|^{K'}\) from |\Psi\rangle and |\tilde{\Gamma}\rangle as

\[
|\Psi_p\rangle = P^{K \to K'} |\Psi\rangle, \quad |\Gamma^*_p\rangle = P^{K \to K'} |\tilde{\Gamma}\rangle \tag{418}
\]

Thus, due to the property of the purified distance (Lemma 29 and Lemma 28), it follows that

\[
P \left( |\tilde{\Gamma}\rangle \langle \tilde{\Gamma}|, |\Psi\rangle \langle \Psi| \right) = P \left( |\Gamma^*_p\rangle \langle \Gamma^*_p|, |\Psi_p\rangle \langle \Psi_p| \right) = P \left( \Gamma^{KAB}, C^K \circ \text{Tr}_{CD}(|\Psi\rangle \langle \Psi|) \right) \tag{419}
\]

which completes the proof. ■

**Lemma 31** Consider the same setting as in Lemma 30, and assume that \(C\) and \(D\) are composite systems \(C_0 M_C\) and \(D_0 M_D\), respectively, where \(M_C\) and \(M_D\) are isomorphic quantum systems with an orthonormal basis \(\{|m\rangle_m\}\). In addition, suppose that the state \(\Psi\) is classically coherent in \(M_C M_D\), i.e., that

\[
\|\langle m |^{M_C} \langle m' |^{M_D} |\Psi\rangle\| \propto \delta_{m,m'}. \tag{420}
\]

Then, without loss of generality, we may assume that the states \(|\phi_k\rangle\) are classically coherent in \(M_C M_D\).

**Proof:** It is straightforward to verify that the state \(\Psi\) is classically coherent in \(M_C M_D\) if and only if all \(\psi_k\) are classically coherent in \(M_C M_D\). Consequently, the maximum of each term in (416) is achieved by \(\phi_k\) that is classically coherent in \(M_C M_D\), which completes the proof. ■

**Lemma 32** (gentle measurement: Lemma 5 in [20] and Corollary of Lemma 7 in [5]) Let \(\epsilon \in (0, 1]\), \(\rho \in S(\mathcal{H})\) and \(\Lambda \in \mathcal{L}(\mathcal{H})\) be such that \(0 \leq \Lambda \leq I\) and \(\text{Tr}[\Lambda \rho] \geq 1 - \epsilon\). It holds that

\[
\|\rho - \sqrt{\Lambda} \rho \sqrt{\Lambda}\| \leq 2\sqrt{\epsilon}, \quad P(\rho, \sqrt{\Lambda} \rho \sqrt{\Lambda}) \leq \sqrt{2\epsilon}. \tag{421}
\]
Figure 9: The construction of encoding and decoding operations in the proof of the direct part is depicted. (i) is obtained by cancelling out $G_\sigma U$ and $(G_\sigma U)^\dagger$, corresponding to Inequality (171) obtained from (153) and (154). (ii) follows from the fact that the state $|\psi_{\sigma}^3\rangle$ is obtained from $|\psi_1\rangle$ by applying $P$ and $G_\sigma$, due to (158). In (iii), we trace out $Z \equiv Z_L Z_R$ and apply the completely dephasing operation $C$ to $X' Y' Z'$. See Inequalities (173) and (176) that are obtained from (171). Note that the source state $\psi_s$ is obtained from $|\Psi\rangle$ and $|\phi_1\rangle$ as (160).