Support of dS/CFT correspondence from space-time perturbations

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Abstract

We analyse the spectrum of perturbations of the de Sitter space on the one hand, while on the other hand we compute the location of the poles in the Conformal Field Theory (CFT) propagator at the border. The coincidence is striking, supporting a dS/CFT correspondence. We show that the spectrum of thermal excitations of the CFT at the past boundary $I^-$ together with that spectrum at the future boundary $I^+$ is contained in the quasi-normal mode spectrum of the de Sitter space in the bulk.

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1 Introduction

The main goal of string theory nowadays is to prove itself capable of coping with experimental evidences. On the other hand, recently, it has been made almost evident from supernova observations [1] as well as from the data supplied by the COBE satellite [2], that we live in a flat world with a kind of special matter, or possibly a positive cosmological constant, i.e., de Sitter space-time [3]. It is thus annoying that several results in string theory are obtained only in Anti-de Sitter (AdS) space, namely a space with a negative cosmological constant, especially the celebrated correspondence between the bulk AdS space and the Conformal Field Theory (CFT) at the border [4, 5]. It is even deceiving, in view of the above observations, that de Sitter space poses serious questions for the formulation of string theory [6]. It is our aim here to contribute to the extension of the bulk to border correspondence to the case where the bulk cosmological constant is positive, that is, to investigate whether the de Sitter space in the bulk has any relationship with a purported CFT at its border.

In fact, a holographic duality relating quantum gravity on $D$-dimensional de Sitter space ($dS_D$) to a CFT residing on the past boundary ($I^-$) of $dS_D$ has recently been proposed by Strominger in [7], what has motivated several works discussing the setup of this duality (see, for instance, [8]). Moreover, as shown by Gibbons and Hawking [9] the de Sitter space cosmological event
horizon is endowed with entropy and temperature, in analogy with a black hole event horizon. Thus, quantum gravity, as a natural description of black holes, has to be extended to the whole universe [3].

We have already analysed the question raised above in the case of three-dimensional de Sitter space, where both, perturbations of $dS_3$ solution in the bulk, as well as the two-dimensional CFT propagator spectrum at the border, are explicitly computable [10]. There we obtained both spectra, and they turned out to coincide exactly. Indeed, the solution of the Klein-Gordon equation in the bulk of the de Sitter space leads to a hypergeometric equation, which, upon employing the boundary condition imposing the vanishing of the field at the event horizon, leads to a four-fold set of quasi-normal modes. Inspection of the propagator of the CFT at the boundary space, for the physically realizable boundary conditions of the fields, shows a spectrum that exactly coincides with the previously described one.

Nevertheless, three-dimensional de Sitter space-time is far simpler than real systems and the actual problem remains basically unsolved. Furthermore, three-dimensional de Sitter space does not admit a black hole event horizon (there is no black hole solution), but only a cosmological horizon. On the other hand, four-dimensional Schwarzschild-de Sitter solution displays a very peculiar spectrum of perturbations. There are quasi-normal modes at short times, followed by a power law decay typical of the asymptotically flat Schwarzschild solution, and finally by an exponential tail [11, 12]. Our
suspicion is that only this exponential tail is reflected in the CFT at the boundary. For the moment the problem is too complex, and we remain for the time being on perturbations of the empty dS space, which is exactly computable, comparing with the corresponding CFT results at the border. The conclusions should apply for small black holes, in view of the above peculiar spectrum of perturbations.

In summary, a qualitative correspondence between quasi-normal modes in AdS spaces and the decay of perturbations in the dual CFT has been obtained [13, 14, 15, 16, 17]. In the case of an AdS space-time as described by the BTZ black hole [18] a precise agreement between quasi-normal mode frequencies and the location of the poles in the retarded correlation function of the corresponding perturbations in the dual CFT has been obtained in [19], whereas in [10] a similar correspondence was found for three-dimensional dS space. The investigation of the validity of this agreement for four-dimensional dS space, as well as its extension for higher dimensions, is thus an important question. Therefore, our aim is to generalize the previously obtained three-dimensional results, as well as reconfirming the many indications of such a holographic duality for de Sitter space.
2 The Geometry of the de Sitter Space-Time

The $D$-dimensional de Sitter space-time, $dS_D$, can be visualized as the hyperboloid

$$\eta_{AB}X^AX^B \equiv -(X^0)^2 + (X^1)^2 + \ldots + (X^D)^2 = a^2 ,$$  \hspace{1cm} (1)

embedded in $(D + 1)$-dimensional Minkowski space-time with metric

$$ds^2 = \eta_{AB}dX^AdX^B ,$$  \hspace{1cm} (2)

where $\eta_{AB} = diag(-1, 1, \ldots, 1)$ and $A, B = 0, 1, \ldots, D$. The parameter $a$ is the de Sitter radius.

The $dS_D$ metric is induced from the flat metric (2) on the hypersurface (1). It is a solution of the Einstein field equations with a positive cosmological constant $\Lambda$, related to $a$ by $\Lambda = (D - 2)(D - 1)/2a^2$ [20].

Several coordinate systems used to describe the structure of $dS_D$ are discussed in [20]. Among these, the static coordinates are particularly useful, since they make the existence of a (cosmological) event horizon manifest in space-time, being more suitable for the purpose of this work.

As in [21], we fix

$$(X^0)^2 - (X^D)^2 = -a^2V(r) ,$$  \hspace{1cm} (3)

where

$$V(r) = 1 - \frac{r^2}{a^2} .$$  \hspace{1cm} (4)
Then, we use (1) to arrive at the constraint

\[(X^1)^2 + ... + (X^{D-1})^2 = r^2 \]

so that the coordinates \(X^1, ..., X^{D-1}\) range over a \((D - 2)\)-sphere, \(S^{D-2}\), of radius \(r\). By parametrizing the hyperbola (3) by

\[X^0 = \sqrt{r^2 - a^2 \cosh \frac{t}{a}} , \quad (6)\]
\[X^D = \sqrt{r^2 - a^2 \sinh \frac{t}{a}} , \quad (7)\]

the induced metric on the hypersurface (1) is given by

\[ds^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2 d\Omega_{D-2}^2 \quad , \quad (8)\]

where \(d\Omega_{D-2}^2\) is the metric of the unit sphere \(S^{D-2}\). This is the form of the \(dS_D\) metric in static coordinates. From (8) it is easy to see that an observable at \(r = 0\) is surrounded by an event horizon at \(r = a\). The static coordinates do not cover the whole space-time, but only the interior region of the cosmological horizon, which corresponds to the left triangle in the Penrose-Carter diagram shown below. The past \((I^-)\) and future \((I^+)\) event horizons correspond to \(r = \infty\) [9].

In view of the computation of the de Sitter invariant Hadamard two-point function it is convenient to define an invariant \(P(X, X')\) associated to two points \(X\) and \(X'\) in de Sitter space. We define

\[a^2 P(X, X') = \eta_{AB} X^A X'^B \quad , \quad (9)\]
which is related to the geodesic distance $d(X, X')$ between two points by $P = \cos(d/a)$. In static coordinates, we can easily express $P(X, X')$ as

$$a^2 P(X, X') = -\sqrt{r^2 - a^2 \sqrt{r'^2 - a^2 \cosh \frac{t - t'}{a}}} + rr' \cos \Theta,$$

(10)

where $\Theta = \Theta(\Omega, \Omega')$ denotes the geodesic distance between two points on the unit sphere $S^{D-2}$. Later we will restrict our results to the case $D = 4$, where

$$\cos \Theta = \sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta'$$

(11)

holds, as can easily be obtained in terms of the usual polar and azimuthal angles $\theta$ and $\phi$. 

Figure 1: The Penrose-Carter diagram of de Sitter space-time.
3 Scalar Perturbations of the $dS_D$ Space

Scalar perturbations of the $dS_D$ space-time are described by the Klein-Gordon wave equation

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu \nu} \partial_\nu \Phi \right) - \mu^2 \Phi = 0 \quad ,$$

(12)

where $\mu$ is the mass of the perturbing scalar field $\Phi$ and $g_{\alpha \beta}$ is the $dS_D$ metric,

$$g_{\mu \nu} = \text{diagonal}(-V(r), V^{-1}(r), r^2 \hat{1}_{D-2}) \quad ,$$

(13)

where $\hat{1}_{D-2}$ is the identity matrix on $S^{D-2}$. Therefore, $\sqrt{-g} = \sqrt{-\det(g_{\alpha \beta})} = r^{(D-2)\sin^{D-3} \theta_1, \sin^{D-4} \theta_2... \sin \theta_{D-3}}$ and from (12) we have

$$\frac{1}{r^{D-2}} \partial_t \left( r^{D-2}(-V^{-1}(r)) \partial_t \Phi(t, r, \Omega) \right) + \frac{1}{r^{D-2}} \partial_r \left( r^{D-2}V(r) \partial_r \Phi(t, r, \Omega) \right) + \frac{1}{r^2} \partial^2_{D-2} \Phi(t, r, \Omega) - \mu^2 \Phi(t, r, \Omega) = 0 \quad ,$$

(14)

where $\Omega$ denotes the set of angular variables, and $\partial^2_{D-2}$ corresponds to the angular derivatives, that is, the Laplacian on the unit sphere $S^{D-2}$.

The above equation is separable, with the Ansatz

$$\Phi(t, r, \Omega) = e^{-i \omega t} R_\ell(r) Y^m_\ell(\Omega) \quad ,$$

(15)

where $Y^m_\ell(\Omega)$ are the (hyper-)spherical harmonics on $S^{D-2}$, which obey

$$\partial^2_{D-2} Y^m_\ell(\Omega) = -\ell(\ell + D - 3) Y^m_\ell(\Omega) \quad .$$

(16)

In $Y^m_\ell$, $\ell$ is a positive integer and $m$ is a collective index ($m_1, m_2, ..., m_{D-3}$).
Therefore, we have for the radial dependence in $R_\ell$,
\[
\frac{V(r)}{r^{D-2}} \frac{d}{dr} \left( V(r) r^{D-2} \frac{dR(r)}{dr} \right) + \left[ \omega^2 - V(r) \left( \frac{\ell(\ell + D - 3)}{r^2} + \mu^2 \right) \right] R(r) = 0,
\]
(17)
where we have omitted the index $\ell$ in $R(r)$.

Defining
\[
z = \frac{r^2}{a^2},
\]
(18)
equation (17) can be written as
\[
\frac{4}{a^2} z(z - 1)^2 \frac{d^2 R(z)}{dz^2} + \frac{2}{a^2} (z - 1)[z(D + 1) - (D - 1)] \frac{dR(z)}{dz} + \left[ \omega^2 - (1 - z) \left( \frac{\ell(\ell + D - 3)}{a^2 z} + \mu^2 \right) \right] R(z) = 0.
\]
(19)
Now, by setting the Ansatz
\[
R = z^\alpha (1 - z)^\beta F(z),
\]
(20)
equation (19) can be transformed into
\[
z(z - 1) \frac{d^2 F(z)}{dz^2} + \frac{1}{2} [(4\alpha + 4\beta + D + 1)z - (4\alpha + D - 1)] \frac{dF(z)}{dz} + \left( (\alpha + \beta)^2 + \frac{D - 1}{2} (\alpha + \beta) + \frac{\mu^2 a^2}{4} \right) F(z) + \frac{(4\beta^2 + \omega^2 a^2)z - (4\alpha^2 + 2\alpha (D - 3) - \ell(\ell + D - 3))(z - 1)}{4z(z - 1)} F(z) = 0,
\]
(21)
which is a hypergeometric equation, that is,
\[
z(z - 1) \frac{dF(z)}{dz} + [(1 + A + B)z - C] \frac{dF(z)}{dz} + ABF(z) = 0,
\]
(22)
provided we choose the arbitrary constants $\alpha$ and $\beta$ as

$$4\alpha^2 + 2\alpha(D - 3) - l(l + D - 3) = 0,$$
$$4\beta^2 + \omega^2 a^2 = 0,$$

whose solutions are

(i) : $\alpha = \frac{l}{2}, \quad \beta = \frac{ia\omega}{2}$ ;

(ii) : $\alpha = \frac{l}{2}, \quad \beta = -\frac{ia\omega}{2}$ ;

(iii) : $\alpha = -\frac{l + D - 3}{2}, \quad \beta = \frac{ia\omega}{2}$ ;

(iv) : $\alpha = -\frac{l + D - 3}{2}, \quad \beta = -\frac{ia\omega}{2}$.

For case (i), the constants $A$, $B$, and $C$ are given by the set

$$A_\pm = B_\pm = \frac{l + ia\omega}{2} + \frac{1}{4} \left[D - 1 \pm \sqrt{(D - 1)^2 - 4\mu^2 a^2}\right], \quad (23)$$
$$C = \ell + \frac{D - 1}{2}. \quad (24)$$

Therefore we have the two solutions for $R(z)$, that is

$$R(z) = z^{l/2}(1 - z)^{ia\omega/2} F\left(\frac{l + ia\omega + h_-}{2}, \frac{l + ia\omega + h_+}{2}, l + \frac{D - 1}{2}, z\right) \quad (25)$$

and

$$R(z) = z^{l/2}(1 - z)^{ia\omega/2} F\left(\frac{l + ia\omega + h_+}{2}, \frac{l + ia\omega + h_-}{2}, l + \frac{D - 1}{2}, z\right), \quad (26)$$

where $F$ is the usual hypergeometric function \textsuperscript{[23]} and

$$h_\pm = \frac{1}{2} \left[D - 1 \pm \sqrt{(D - 1)^2 - 4\mu^2 a^2}\right]. \quad (27)$$
The corresponding quasi-normal modes are solutions of the equations

\[ C - A = -n \quad , \quad C - B = -n \quad , \quad (28) \]

i.e.,

\[
\begin{align*}
 l + \frac{D - 1}{2} & - \frac{l + ia\omega + h_{\mp}}{2} + n = 0 \quad , \\
 l + \frac{D - 1}{2} & - \frac{l + ia\omega + h_{\pm}}{2} + n = 0 \quad .
\end{align*}
\]

The corresponding quasi-normal frequencies are then given by

\[ i\omega = 2n + l + D - 1 - h_{\pm} \quad (29) \]

\[ = 2n + l + h_{\mp} \quad . \quad (30) \]

This means that we have the two sets of solutions

\[ \pm i\omega_R = 2n + l + h_+ \quad , \quad \pm i\omega_L = 2n + l + h_- \quad , \quad (31) \]

where we have included the complex conjugated solutions.

The case (ii) is obtained from the case (i) by complex conjugation and the quasi-normal frequencies are solutions implying case (ii) as a solution. Further quasi-normal modes are obtained from the symmetry \( \ell \rightarrow -(\ell + D - 3) \), which implies cases (iii) and (iv) above, or the set

\[ \pm i\omega_R = 2n - (l + D - 3) + h_+ \quad , \quad \pm i\omega_L = 2n - (l + D - 3) + h_- \quad . \quad (32) \]
4 Two-Point Correlator for CFT Operators at the Boundary

We define a de Sitter invariant Hadamard two-point function as \[ G(X, X') = \text{const} \langle 0 | \Phi(X) \Phi(X') | 0 \rangle \], which obeys

\[ (\nabla_X^2 - \mu^2)G(X, X') = 0 \quad , \tag{34} \]

where \( \nabla_X^2 \) is the Laplacian on \( dS_D \). The Green function \( G(X, X') \) depends on \( X \) and \( X' \) only through the invariant \( P(X, X') \), the distance between the points \( X \) and \( X' \), given by eq.(9). We can write \( G(X, X') = G(P(X, X')) \), and from (34) we obtain \[ (1 - P^2) \frac{d^2 G}{dP^2} - DP \frac{dG}{dz} - \mu^2 a^2 G(P) = 0 \quad , \tag{35} \]

which, by means of the change of variable \( z = (1 + P)/2 \) becomes a hypergeometric equation

\[ z(1 - z) \frac{d^2 G}{dz^2} + \left( \frac{D}{2} - Dz \right) \frac{dG}{dz} - \mu^2 a^2 G(z) = 0 \quad . \tag{36} \]

The solution is the hypergeometric function \( F \),

\[ G(z) = \text{Re} F \left( h_+, h_-; \frac{D}{2}; z \right) \quad , \tag{37} \]

i.e.,

\[ G(P) = \text{Re} F \left( h_+, h_-; \frac{D}{2}; \frac{1 + P}{2} \right) \quad , \tag{38} \]
where $h_\pm$ is given by (27).

In order to explicitly calculate the two-point correlator for an operator coupled to the bulk field $\Phi$, we restrict now our considerations to the four-dimensional case ($D = 4$). The results can be naturally extended for any dimension. The two-point correlator can be obtained analogously to [7, 21] from

$$
\lim_{r \to \infty} \int dt d\theta d\phi dt' d\theta' d\phi' \sin \theta \sin \theta' \frac{(rr')^2}{a^2} \times \left[ \Phi(t, r, \theta, \phi) \partial_r G(t, r, \theta, \phi; t', r', \theta', \phi') \partial_r \Phi(t', r', \theta', \phi') \right]_{r = r'},
$$

(39)

where $dr_* = (-V(r))^{-1/2}dr$.

The asymptotic behaviour of $G(X, X')$ at the conformal boundary, in static coordinates, is given by (see equation (B.9) of [21])

$$
\lim_{r, r' \to \infty} G(t, r, \theta, \phi; t', r', \theta', \phi') = c_+ (rr')^{-h_+} [\cosh\left(\frac{t - t'}{a}\right) - \cos \Theta]^{-h_+} + (h_+ \leftrightarrow h_-),
$$

(40)

where $c_+$ is a constant and $\cos \Theta$ is given by (11) in four dimensions.

As in [7, 21] we also impose the following boundary condition for $\Phi$ at the boundary $I^-$

$$
\lim_{r \to \infty} \Phi(t, r, \theta, \phi) = r^{-h_-} \Phi_-(t, \theta, \phi).
$$

(41)

However, as stressed by Strominger [7] the boundary condition (41) is not mandatory, and we may also choose $\Phi(t, r, \theta, \phi) = r^{-h_+} \Phi_+(t, \theta, \phi)$ at infinity. This leads to similar correlators, obtained by the change $h_+ \leftrightarrow h_-$. Thus,
from this point onwards we simply use $h$, and at the end we shall obtain the full set of solutions choosing either value of $h$.

The use of (40) and (41) in (39) leads to

$$D(\Phi, \Phi') \equiv \lim_{r \to \infty} \int dt d\theta d\phi dt' d\theta' d\phi' \sin \theta \sin \theta' \left(\frac{rr'}{a^2}\right)^2 \times \left[\Phi(x) \partial_{r} G(x, x') \partial_{r'} \Phi(x')\right]_{r=r'} = \quad (42)$$

$$= \int dt d\theta d\phi dt' d\theta' d\phi' \sin \theta \sin \theta' \Phi_-(t, \theta, \phi) \Phi_-(t', \theta', \phi')$$

$$\left[\cosh\left(\frac{t-t'}{a}\right) - \left(\sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta'\right)\right]^h.$$  

$$\quad (43)$$

Now we substitute above the Ansatz $\Phi_-(t, \theta, \phi) = e^{i\omega t} Y^m_\ell(\theta, \phi)$. We note that up to this point there is a priori no relationship here between the parameter $\omega$ in this Ansatz and that denoting the quasi-normal modes from the bulk perturbations. It follows that

$$D_{\omega \omega'} = \int dt d\theta d\phi dt' d\theta' d\phi' \sin \theta \sin \theta' Y^m_\ell(\theta, \phi) Y^{m'}_{\ell'}(\theta', \phi') e^{i\omega t} e^{i\omega' t'}$$

$$\left[\cosh\left(\frac{\tau-a}{a}\right) - \left(\sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta'\right)\right]^h,$$

$$\quad (44)$$

which can be suitably written in the form (up to a constant factor, $C$, coming from the normalization of the spherical harmonics)

$$D_{\omega \omega'} = C \delta_{mm'} \delta(\omega - \omega') \int d\tau d\theta d\phi d\varphi \sin \theta \sin \theta' P^m_\ell(\cos \theta) P^{m'}_{\ell'}(\cos \theta') \times$$

$$e^{im\varphi} e^{i\omega \tau} \left[\cosh\left(\frac{\tau-a}{a}\right) - \left(\sin \theta \sin \theta' \cos \varphi + \cos \theta \cos \theta'\right)\right]^{-h}, \quad (45)$$

where $\varphi = \phi - \phi'$ and $\tau = t - t'$. 

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It is our aim here to display the spectrum of perturbations, that is, the poles of (45) in \( \omega \). This is not too difficult, since each pole is characterized by the asymptotic behaviour of the integrand for \( \tau \to \infty \). Indeed, a pole \( 1/(\omega^2 - \omega_0^2) \) corresponds to an asymptotic behaviour \( \exp(i\omega_0\tau) \). Thus, by expanding (45) in powers of 

\[
\epsilon = \frac{1}{\cosh(\gamma)} \left( \sin \theta \sin \theta' \cos \varphi + \cos \theta \cos \theta' \right), \tag{46}
\]

we have

\[
D_{\omega\omega'} = C\delta_{mm'}\delta(\omega - \omega') \int \frac{d\tau e^{i\omega\tau}}{\cosh(\gamma)} \left[ a_0 + a_1 \frac{h}{\cosh(\gamma)} + a_2 \frac{h(h + 1)}{2! \cosh^2(\gamma)} + a_3 \frac{h(h + 1)(h + 2)}{3! \cosh^3(\gamma)} + \ldots \right], \tag{47}
\]

where

\[
a_0 = \int d\theta d\theta' \sin \theta \sin \theta' P_{\ell}^m(\cos \theta) P_{\ell'}^m(\cos \theta') \int d\varphi e^{im\varphi}, \tag{48}
\]

\[
a_1 = \int d\theta d\theta' \sin \theta \sin \theta' P_{\ell}^m(\cos \theta) P_{\ell'}^m(\cos \theta') \int d\varphi e^{im\varphi} \times \[ \sin \theta \sin \theta' \cos \varphi + \cos \theta \cos \theta' \] \ldots \tag{49}
\]

\[
a_N = \int d\theta d\theta' \sin \theta \sin \theta' P_{\ell}^m(\cos \theta) P_{\ell'}^m(\cos \theta') \int d\varphi e^{im\varphi} \times \[ \sin \theta \sin \theta' \cos \varphi + \cos \theta \cos \theta' \] \ldots \tag{50}
\]

It is simple to solve the integrals in \( \varphi \) by means of the orthogonality relation of \( \exp(im\varphi) \). This will fix the value of \( m \) at each term in the expansion. The integrals in \( \theta \) and \( \theta' \) are a little more involved, but they can be
handled by using standard integrals (as, e.g., in [23]) involving the $P^m_\ell(\theta)$’s themselves and contributions of sines and cosines. For example, up to the second-order term in the $\epsilon$-expansion we find integrals of the type

$$a_0 = 2\pi \delta_{m,0} \int_{-1}^{1} dx' P^0_\ell(x') \int_{-1}^{1} dx P^0_\ell(x), \quad (51)$$

$$a_1 = \pi \delta_{m,1} \int_{-1}^{1} dx' P^1_\ell(x')(1 - x'^2)^{1/2} \int_{-1}^{1} dx P^1_\ell(x)(1 - x^2)^{1/2} + \int_{-1}^{1} dx' P^0_\ell(x') x' \int_{-1}^{1} dx P^0_\ell(x) x, \quad (52)$$

$$a_2 = \frac{\pi}{2} \delta_{m,2} \int_{-1}^{1} dx' P^2_\ell(x')(1 - x'^2) \int_{-1}^{1} dx P^2_\ell(x)(1 - x^2) + \int_{-1}^{1} dx' P^0_\ell(x') x'^2 \int_{-1}^{1} dx P^0_\ell(x) x^2 + \pi \delta_{m,0} \int_{-1}^{1} dx' P^0_\ell(x')(1 - x'^2) \int_{-1}^{1} dx P^0_\ell(x)(1 - x^2) + \frac{\pi}{2} \delta_{m,-2} \int_{-1}^{1} dx' P^{-2}_\ell(x')(1 - x'^2) \int_{-1}^{1} dx P^{-2}_\ell(x)(1 - x^2), \quad (53)$$

where $x = \cos \theta$ and $x' = \cos \theta'$. The computation of the above integrals fixes the values of $\ell$ and $\ell'$ that leads to a non-vanishing contribution for the coefficients $a_0, a_1,$ and $a_2$. This can be easily done using the orthogonality of the $P^m_\ell(\theta)$’s, for some integrals, and resorting to a table for others. Almost all the integrals appearing in $a_0, a_1,$ and $a_2$ can be expressed as a product of $P^m_\ell$’s with the same index $m$. This allow us to obtain for those integrals

$$\int_{-1}^{1} dx P^0_\ell(x) = \frac{2}{2\ell + 1} \delta_{\ell,0}, \quad (54)$$
\[
\int_{-1}^{1} dx P_\ell^1(x)(1 - x^2)^{1/2} = \frac{2(\ell + 1)!}{(2\ell + 1)(\ell - 1)!} \delta_{\ell,1}, \tag{55}
\]
\[
\int_{-1}^{1} dx P_\ell^0(x)x = \frac{2}{2\ell + 1} \delta_{\ell,1}, \tag{56}
\]
\[
\int_{-1}^{1} dx P_\ell^2(x)(1 - x^2) = \frac{2(\ell + 2)!}{3(2\ell + 1)(\ell - 2)!} \frac{1}{\ell} \delta_{\ell,2}, \tag{57}
\]
\[
\int_{-1}^{1} dx P_\ell^1(x)x(1 - x^2)^{1/2} = \frac{2(\ell + 1)!}{3(2\ell + 1)(\ell - 1)!} \delta_{\ell,2}, \tag{58}
\]
\[
\int_{-1}^{1} dx P_\ell^0(x)x^2 = \frac{4}{3(2\ell + 1)} \delta_{\ell,2} + \frac{4}{3(2\ell + 1)} \delta_{\ell,0}, \tag{59}
\]

from where we see that only \(\ell, \ell' = 0; 1; 2; 2; (0; 2)\) contribute for the above integrals, respectively. The integrals containing \(P^{m}_{\ell}\) can be solved by means of the relation \(P^{m}_{\ell} = (-1)^m \frac{(\ell + m)!}{(\ell + m)!} P^m_{\ell}\).

It remains the integral \(\int_{-1}^{1} dx P_\ell^0(x)(1 - x^2)\), that appears in \(a_{2}\), which cannot be written as a product of the \(P_{\ell}^m\)'s with the same \(m\). In this case, we use [23] to obtain

\[
\int_{-1}^{1} dx' P_{\ell'}^0(x')(1 - x'^2) \int_{-1}^{1} dx P_{\ell}^0(x)(1 - x^2) = \frac{\pi}{\Gamma(2 + \frac{\ell}{2} + 1) \Gamma(2 - \frac{\ell}{2}) \Gamma(\frac{\ell'}{2} + 1) \Gamma(-\frac{\ell'}{2} + 1)} \times (\ell' \leftrightarrow \ell). \tag{60}
\]

We conclude that only \(\ell, \ell' = 0, 2\) contribute with a non-vanishing value for the integrals in (60).

It is easy to verify that the non-vanishing contributions for the integrals in each higher-order term in the expansion (47) are similar to those shown above. In fact, for the third-order term, we have the contributions \(\ell, \ell' = 1, 3\), and as the inspection of a generic term \(a_N\) in the expansion shows, the contributions of \(\ell, \ell'\) will be of the type \(\ell, \ell' = N, N - 2, N - 4, \ldots\). This
allows us to write (47) as

\[ D_{\omega\omega'} = \delta(\omega - \omega') \sum_{N=0} c_N \int d\tau e^{i\omega\tau} \left[ \cosh \frac{\tau}{a} \right]^{-((h+l+2N)}. \quad (61) \]

This integral has poles at

\[ \omega = \pm \frac{\tau}{a} (h_\pm + l + 2n), \quad (62) \]

in view of the behaviour of the integrand for large values of \( \tau \).

The spectrum (62) coincides exactly with the quasi-normal frequencies (31) obtained in perturbations of the bulk of the de Sitter space-time, except only for the values \( \pm \xi \pm 1/2 \), where \( \xi = \sqrt{9 - 4\mu^2 a^2} \). This can be easily verified by inspection of the bulk spectrum, given by (31-32).

5 Concluding remarks

We have shown that the quasi-normal modes arising from a scalar perturbation of the de Sitter space are, with exception of only four of them, contained in the spectrum of two-point function of the corresponding three-dimensional conformal field theory at the boundary.

Although the computation of the two-point correlator has been performed in four-dimensional de Sitter space, the results can be generalized to \( D \) dimensions, where presumably the quasi-normal modes obtained in bulk de Sitter space are, with the exception of a small number of them, contained in
the spectrum of the corresponding \((D - 1)\)-dimensional CFT at the boundary. In fact, this seems to occur, since the form of the two-point correlator as given by (44) can be directly generalized to \(D\) dimensions. In this case, we will have to handle with hyperspherical harmonics and the denominator of (44) takes the form \([\cosh \frac{\tau}{a} - \cos \Theta]^{-h}\), where \(\Theta = \Theta(\Omega, \Omega')\) is the geodesic distance between two points on the unit sphere \(S^{D-2}\).

Our results give directions to build foundations of an extension of the celebrated AdS/CFT correspondence to the de Sitter space. We do not know the interpretation of the small number of states left out of the CFT spectrum, but for extensive magnitudes, such as entropy, they presumably do not matter.

At last, we stress the fact, already well signalized, for example, by Strominger [7], that the whole de Sitter space-time cannot be probed by a single observer, and describing the whole de Sitter space-time corresponds to describing both sides of a black hole’s event horizon. In spite of the discussion involving a region smaller than the full de Sitter space, we have shown a striking evidence that a CFT describes well the holographic projection of the bulk space, thus providing strong support for a dS/CFT correspondence, since bulk eigenmodes are fully described in the region of space probed by a single observer.

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References

[1] B. P. Schmidt *et al.*, Astrophys. J. 507, 46 (1998);
    A. G. Riess *et al.*, Astron. J. 116, 1009 (1998);
    S. Perlmutter *et al.*, Astrophys. J. 517, 565 (1999);
    S. Perlmutter, in *Proc. of the 19th Intl. Symp. on Photon and Lepton Interactions at High Energy LP99*, ed. J.A. Jaros and M. E. Peskin, Int. J. Mod. Phys. A 15S1, 715 (2000).

[2] P. de Bernardis *et al.*, Nature 404, 955 (2000);
    R. Stompor *et al.*, Astrophys. J. 561, L7 (2001).

[3] R. Bousso, [hep-th/0205177](http://arxiv.org/abs/hep-th/0205177).

[4] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998);
    S. S. Gubser, I. Klebanov, and A. Polyakov, Phys. Lett. B 428, 105 (1998);
    E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998).

[5] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. 323, 183 (2000).
[6] E. Witten, hep-th/0106109.

[7] A. Strominger, J.H.E.P 0110, 034 (2001), hep-th/0106113.

[8] S. Nojiri and S. D. Odintsov, Phys. Lett. B 531, 143 (2002); S. Nojiri, S. D. Odintsov, and S. Oguchi, Phys. Rev. D 66, 023522 (2002).
H. W. Lee and Y. S. Myung, Phys. Lett. B 537, 117 (2002);
B. McInnes, Nucl. Phys. B 627, 311 (2002);
M. Cvetic, S. Nojiri, and S. D. Odintsov, Nucl. Phys. B 628, 295 (2002);
Z. Chang and C.-B. Guan, hep-th/0204014;
B. C. da Cunha, hep-th/0208018.

[9] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2738 (1977).

[10] E. Abdalla, B. Wang, A. Lima-Santos, and W. G. Qiu, Phys. Lett. B 538, 435 (2002).

[11] P. Brady, C. M. Chambers, W. Krivan, and P. Laguna, Phys. Rev. D 55, 7538 (1997).

[12] E. Abdalla, C. Molina, and A. Saa (in preparation).

[13] J. S. F. Chan and R. B. Mann, Phys. Rev. D59, 064025 (1999);
J. S. F. Chan and R. B. Mann, Phys. Rev. D55, 7546 (1997).

[14] G. T. Horowitz and V. E. Hubeny, Phys. Rev. D62, 024027 (2000);
G. T. Horowitz, Class. Quant. Grav. 17, 1107 (2000).
[15] B. Wang, C. Y. Lin, and E. Abdalla, Phys. Lett. B481, 79 (2000);
   B. Wang, C. Molina, and E. Abdalla, Phys. Rev. D63, 084001 (2001);
   J. M. Zhu, B. Wang, and E. Abdalla, Phys. Rev. D63, 124004 (2001).

[16] V. Cardoso and J. P. S. Lemos, Phys. Rev. D63, 124015 (2001).

[17] B. Wang, E. Abdalla, and R. B. Mann, Phys. Rev. D65, 084006 (2002).

[18] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).

[19] D. Birmingham, I. Sachs, and S. N. Solodukhin, Phys. Rev. Lett. 88, 151301 (2002).

[20] M. Spradlin, A. Strominger, and A. Volovich, Les Houches Lectures on de Sitter Space, hep-th/0110007.

[21] D. Klemm, Nucl. Phys. B 625, 295 (2002).

[22] Higher Transcendental Functions, V. II (Bateman Manuscript Project),
ed. by A. Erdélyi (McGraw-Hill, New York, 1953).

[23] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products, Academic Press (2000).

[24] P. Candelas and D. J. Raine, Phys. Rev. D 12, 965 (1975).