Multipartite Entanglement under Stochastic Local Operations and Classical Communication

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Abstract

Stochastic local operations and classical communication (SLOCC), also called local filtering operations, are a convenient, useful set of quantum operations in grasping essential properties of entanglement. We give a quick overview about the characteristics of multipartite entanglement in terms of SLOCC, illustrating the 2-qubit and the rest ($2 \times 2 \times n$) quantum system. This not only includes celebrated results of 3-qubit pure states, but also has implications to 2-qubit mixed states.

Keywords: multipartite entanglement; stochastic LOCC (SLOCC).

1 Introduction

Quantum information science would offer us remarkably new information society. There, quantum correlation called entanglement enables us to process informational tasks which are less efficient or impossible in the classical manner. Entanglement is expected to be especially intriguing and valuable in the multi-party setting, since the network nature (i.e., interactions of many elements) is essence of information society.

Entanglement is quantum correlation, which is never increasing on average by local operations and classical communication (LOCC) \( \mathbb{1} \). In other words, LOCC will preserve or destroy entanglement. For the bipartite pure states, these monotone properties of entanglement under LOCC have been successfully characterized both in the single copy case and in the asymptotic (infinitely many, identical copies) case. In particular, for the single copy situation, the LOCC equivalence class of entanglement turns out to be reduced to the equivalence class under local unitary operations (LU), in other words, the equivalence class of states with same coefficients in the Schmidt decomposition (the normal form in terms of a biorthonormal product basis) \( \mathbb{2} \mathbb{6} \). Then, monotone properties of different entangled states under LOCC are characterized by a majorization principle of the Schmidt coefficients \( \mathbb{4} \mathbb{5} \).

However, the studies of multipartite entanglement have been found very challenging. Since the useful techniques in the bipartite situation, such as the Schmidt
decomposition, cannot be generalized to the multipartite situation straightforward, it is difficult to capture even the LOCC equivalence class, and certainly the monotonicity of entanglement. Alternatively, the LU equivalence class of entanglement, which is by definition finer than the LOCC equivalence class, has been studied for the 3-qubit pure states \cite{6}. Still, the convertibility (monotonicity) between these LU equivalence classes has not yet been unveiled. The situation partly motivates the introduction of our SLOCC equivalence. As another motivation, it should be noted that we are very interested in the genuine "multipartite" phenomena (such as the limited shareability of multipartite entanglement), which cannot be fully captured by dividing multiparticle into bipartite subgroups.

Here, we will address the stochastic LOCC (SLOCC) \cite{3,7} classification of entanglement, which is a coarse-grained classification of entanglement under LOCC. An advantage of the SLOCC classification lies in the fact that we can grasp the essential nature of entanglement, by successfully characterizing the equivalence class (conservation) of entanglement and monotone (irreversible) properties of entanglement under LOCC.

## 2 Stochastic LOCC

Let us consider the single copy of a multipartite pure state $|\Psi\rangle$ on the finite dimensional, $l$-partite Hilbert space $\mathcal{H} = \mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_l}$ (precisely, we consider a ray on its complex projective Hilbert space $\mathcal{P}(\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_l})$),

$$|\Psi\rangle = \sum_{i_1,\ldots,i_l = 0}^{k_1-1,\ldots,k_l-1} \psi_{i_1\ldots i_l} |i_1\rangle \otimes \cdots \otimes |i_l\rangle,$$  \hspace{1cm} (1)

where a set of $|i_1\rangle \otimes \cdots \otimes |i_l\rangle$ constitutes the standard computational basis and is often abbreviated to $|i_1\ldots i_l\rangle$. In SLOCC, we identify two states $|\Psi\rangle$ and $|\Psi'\rangle$, which are interconvertible with nonvanishing probabilities (the success probability of the conversion may differ in back and forward directions), as equivalent entangled states. On the other hand in LOCC, we ordinarily identify two states interconvertible deterministically (with probability 1) as equivalent ones, as seen in the Section 1. Mathematically, two SLOCC equivalent entangled states $|\Psi\rangle$ and $|\Psi'\rangle$ are connected by invertible local operations \cite{7},

$$|\Psi'\rangle = M_1 \otimes \cdots \otimes M_l |\Psi\rangle, \quad |\Psi\rangle = M_1^{-1} \otimes \cdots \otimes M_l^{-1} |\Psi'\rangle,$$  \hspace{1cm} (2)

where $M_i$ is any local operation having a nonzero determinant on the $i$-th party, i.e., $M_i$ is an element of the general linear group $GL_{k_i}(\mathbb{C})$ (we do not care about the overall normalization and phase so that we can take its determinant 1, i.e., $M_i \in SL_{k_i}(\mathbb{C})$). The point is a nice group structure of SLOCC; i.e., SLOCC would practically consist of successive rounds of measurements and communication of their outcomes, but the whole is given simply by one "large" local operation of Eq. 2. It can be also said that SLOCC are a set of trace decreasing, completely positive maps by the postselection of a sequence of successful outcomes. In short, the SLOCC classification of multipartite entanglement means clarifying the moduli space under the SLOCC equivalence, i.e., the structure of orbits generated by a direct product of special linear groups $SL_{k_1}(\mathbb{C}) \times \cdots \times SL_{k_l}(\mathbb{C})$. 2
Afterwards, we will address the question of noninvertible local operations (at least one of the ranks of $M_i$ in Eq. (2) is not full, so that no inverse operations exist), in order to characterize the order between different SLOCC equivalence classes. By definition, the probabilistic conversion between them would be possible only in one direction, or would be impossible in both directions. The set of invertible and noninvertible local operations are often called local filtering operations \[8, 9\].

In the bipartite (for simplicity, $k_1 = k_2 = k$) case, the SLOCC classification means just classifying the whole states by the so-called Schmidt local rank (equivalently, the rank of the coefficient matrix $\Psi = (\psi_{ij})$ of Eq. (1)). We readily see that the rank is the SLOCC invariant, because $\Psi$ is transformed as

$$\Psi' = M_1 \Psi M_2^T,$$

under an invertible local operation $G = M_1 \otimes M_2 \in SL_k \times SL_k$ (the superscript $T$ stands for the transposition). Although, in this bipartite case, the observation immediately suggests that there exist $k$ different entangled classes distinguished by SLOCC-invariant rank $r_s$, let us put the problem on the following picture, for later convenience. A set $\overline{O_{r_s}}$ of states of the local rank less than or equal to $r_s$ is a closed subset under SLOCC, and $\overline{O_{r_s-1}}$ is the singular locus of $\overline{O_{r_s}}$. This is how the local rank leads to an "onion" structure (mathematically the stratification) of Fig. 1.

$$\overline{O_k} \supset \overline{O_{k-1}} \supset \cdots \supset \overline{O_1} \supset \overline{O_0} = \emptyset,$$

and the orbits $O_{r_s} = \overline{O_{r_s}} - \overline{O_{r_s-1}} (r_s = k, \ldots, 1)$ give $k$ classes of entangled states.

Now we discuss the relationship between these classes under noninvertible local operations. Since the local rank only decreases by noninvertible local operations \[4\], we find that $k$ classes are totally ordered in such a way that an entangled class
of the larger rank is more entangled than that of the smaller rank. In particular, the outermost generic set \( O_k \) can be called the class of maximally entangled states, since this class can convert to all classes by LOCC but other classes never convert to it. On the other hand, the innermost set \( O_1 \) is indeed the class of separable (non-entangled) states, since this class never convert to other classes by LOCC but any other classes can convert to it. Note that, in the multipartite situation, a set of the SLOCC-invariant local ranks, as a straightforward generalization of the Schmidt local rank, is not sufficient to classify entangled classes completely. Still, the onion structure of SLOCC orbit closures, as described above, will be indeed utilized in the generalization to the multipartite situation.

### 3 Characteristics of multipartite entanglement

Let us start with accounting for why we are interested in multipartite entanglement in the 2-qubit and the rest (\( 2 \times 2 \times 2 \times \cdots \times 2 \)) quantum system here.

1. This case includes 4-qubit entanglement distributed over 3 parties, for example. We readily see that two Einstein-Podolsky-Rosen (EPR)-Bell pairs over 3 parties can create the so-called Greenberger-Horne-Zeilinger (GHZ) 3-qubit entangled state, and in turn the GHZ state can create one Bell pair under LOCC. Observing that these multi-party LOCC protocols suggest the transformation between bipartite entanglement and tripartite entanglement, we can see that our situation gives a minimal setting for multi-party LOCC protocols. Then, we would ask, what kinds of essentially different multipartite entanglement there exist in this situation?

2. Our case has implications to 2-qubit mixed states.

3. As a bit mathematical remark, although there are infinitely many orbits in general even under the stochastic LOCC \([7, 10]\), our case will turn out to be finite so that it is amenable to treat. The reason is as follows; the dimension of the Hilbert space expands exponentially as the number of parties \( l \) increases, while the dimension of the local operations we can perform increases just linearly. Precisely, the number of nonlocal complex parameters remained in the generic position of the projective Hilbert space (the space of rays) is given by

\[
\dim \frac{P(\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_l})}{SL_{k_1} \times \cdots \times SL_{k_l}} = (k_1 \times \cdots \times k_l - 1) - (k_1^2 - 1) + \cdots + (k_l^2 - 1 - \delta),
\]

where \( \delta \) is the dimension of the stabilizer of the SLOCC operations \( SL_{k_1} \times \cdots \times SL_{k_l} \) there. So, if Eq. (5) is strictly positive then there are infinitely many orbits in the generic (maximal dimensional) class, otherwise there are just finitely many orbits. For example, in the 4-qubit (\( 2 \times 2 \times 2 \times 2 \)) case, Eq. (5) is \( 15 - 12 = 3 > 0 \) so that 3 nonlocal parameters remain in the generic class \([11, 10]\). In contrast, in the \( 2 \times 2 \times 4 \) case, Eq. (5) is 0 and this case consists of finite orbits.

We utilize two, algebraic and geometric ideas to achieve our goal. Although we illustrate \( 2 \times 2 \times n \) cases here (see also Ref. [12] in detail), it can be seen that these ideas are useful to other cases.

1. Observing an homomorphism \( SL_2(\mathbb{C}) \otimes SL_2(\mathbb{C}) \simeq SO_4(\mathbb{C}) \), and concatenating the indices of Alice and Bob as the "flattened" (2-dimensional) matrix
\( \tilde{\Psi} = (\psi_{i_1i_2i_3}), \) let us consider \( R \), defined as

\[
R = T \tilde{\Psi}, \quad T = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & i & i & 0 \\
0 & -1 & 1 & 0 \\
i & 0 & 0 & -i
\end{pmatrix},
\]

in stead of the "3-dimensional" matrix \( \Psi = (\psi_{i_1i_2i_3}) \) in Eq. (1). According to Eq. (2), our problem is found to be algebraically equivalent to looking for normal forms of the "ordinary" matrix \( R \) under left multiplication of an element \( O \) in \( SO_4(\mathbb{C}) \) and right multiplication of an element \( M^T \) in \( SL_n(\mathbb{C}) \), i.e.,

\[
R' = ORM^T.
\]

2. Geometrically speaking, we focus on a duality, a generalization of the Legendre transformation, between the original state space and its conjugate state space. Associated with the duality between the set of separable states and its dual set of degenerate entangled states, hyperdeterminants appear as key SLOCC invariants in the classification of multipartite entanglement (see Ref. [10] in detail). Indeed, in the bipartite \((k_1 = k_2 = k)\) case (cf. Fig. 1), the set of separable states is given by the set of states of the rank 1, i.e., by \( O^1 \). On the other hand, its dual set is given by the set of all degenerate entangled states where their rank is not full \((\det \Psi = 0)\), i.e., by \( O^k \). Remarkably, the duality also holds for the multipartite case, and the dual set is given, in analogy, by the zero hyperdeterminant: \( \det \Psi = 0 \). Note that the hyperdeterminant for the 3-qubit system has been also known as the 3-tangle [13], and represents the genuine tripartite component in the 3-qubit entanglement sharing (cf. Appendix A). Generally, the absolute value of a hyperdeterminant is the entanglement measure (precisely, entanglement monotone) which represents the amount of generic (non-degenerate) entanglement for the given format (cf. Section 4).

Using these ideas, we obtain the classification of \( 2 \times 2 \times n \) multipartite entanglement, drawn in Fig. 2. For \( n \geq 4 \), the whole space is divided into nine entangled classes (orbits) by invertible local operations, while for \( n = 3 \) or 2, it is divided into eight [10] or six [7] classes, respectively. These essentially different entangled classes constitute a five-graded partial order under noninvertible local operations for \( n \geq 4 \), while a four or three-graded one for \( n = 3 \) or 2, respectively. So, as the third party Clare's subsystem becomes large, the number of SLOCC entangled classes increases and the partial order gets higher. Representative states of the GHZ and W classes are famous in their different physical properties and applications to quantum information processing (QIP) [7]. The GHZ state \(|000\rangle + |111\rangle\) has the maximal amount of generic 3-qubit entanglement measured by the hyperdeterminant of the format \( 2 \times 2 \times 2 \) (cf. Eq. [11] and Appendix A). It violates the Bell's inequality maximally, and enable us to extract one Bell pair between any two parties out of three with probability 1. On the other hand, the W state \(|001\rangle + |010\rangle + |100\rangle\) has the maximal amount of average pairwise entanglement distributed over 3 parties (cf. Appendix A). So, the states in the W class can be utilized in optimal quantum cloning [14].

Remarkably, in the \( 2 \times 2 \times n \) \((n \geq 4)\) cases, the generic (maximal dimensional) class is a unique "maximally entangled" class, lying on the top of the partial order.
Figure 2: (Left) The onion-like classification of SLOCC orbits in the 2-qubit and the rest \((2 \times 2 \times n (n \geq 4))\) quantum system. There are nine different SLOCC entangled classes. (Right) The five-graded partially ordered structure of nine entangled classes. Every class is labeled by its representative, its set of local ranks, and its name. Noninvertible local operations, indicated by dashed arrows, degrade higher entangled classes into lower entangled ones. Both figures partly include the cases for \(n = 3\) and \(2\).

So this class serves as an almighty resource that can create any state probabilistically. The situation is in contrast with the 3-qubit \((n = 2)\) case, where there are two inequivalent entangled classes on the top of the hierarchy. In addition, among the generic class, its representative state (two Bell pairs distributed over 3 parties),

\[
|2 \text{ Bell}\rangle_{ABC12} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AC1} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{BC2} \\
= \frac{1}{2}(|00(00)| + |01(01)| + |10(10)| + |11(11)|)_{ABC12},
\]

is most powerful in the sense that it can create any state with probability 1 by the Clare’s positive operator valued measure (POVM) followed by recovery unitary operations of Alice and Bob. Generally, we find that when one of multiparty holds at least a half of the total Hilbert space, the representative state of the generic class can create any state with certainty. So, the situation is somehow analogous to the bipartite cases.

We briefly discuss an interpretation of Fig. 2 in QIP. Let us observe that the initially prepared state in entanglement swapping \cite{15} is two Bell pairs (of Eq. 8) over 3 parties, included in the generic class. Clare’s local collective measurement by the Bell basis \(|\Phi_\mu\rangle_{C12} (\mu = 1,\ldots,4)\) yields with probability 1/4 a Bell pair \(|\Phi_\mu\rangle_{AB}\), which depends on her outcome \(\mu\), due to \(|2 \text{Bell}\rangle_{ABC12} = \frac{1}{4} \sum_{\mu=1}^4 |\Phi_\mu\rangle_{AB} \otimes |\Phi_\mu\rangle_{C12}\), and any Bell pair \(|\Phi_\mu\rangle_{AB}\) created between Alice and Bob is LOCC equivalent to \(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}\). Thus, we can say that the entanglement swapping is a LOCC protocol creating the isolated (maximal) bipartite entanglement between Alice and Bob from generic entanglement. In other words, entanglement swapping is given by a downward (noninvertible) flow in Fig. 2 from the generic class to the biseparable...
class $B_3$. We readily see that any state containing tripartite entanglement (i.e., all its local ranks are greater than 1) is powerful enough to create probabilistically one Bell pair in one of the biseparable classes $B_1$, $B_2$, or $B_3$. On the other hand, two Bell pairs can create two different kinds of the 3-qubit entanglement, GHZ and W. These LOCC protocols are given by downward flows from the generic class to the GHZ and W class, respectively. That is how we see that important multi-party LOCC protocols in QIP are given as downward flows in the partially ordered structure, such as Fig. 4 of multipartite entangled classes. So, the development of SLOCC classification of multipartite entanglement can be expected to give us an insight in looking for new LOCC protocols.

4 Measures for multipartite entanglement

We give a convenient criterion to distinguish these nine SLOCC classes. A set $(r_1, r_2, r_3)$ of the local ranks of the reduced density matrices, defined by

$$r_i = \text{rk } \rho_i, \quad \rho_i = \text{tr}_{\forall j \neq i}(\langle \Psi | \Psi \rangle),$$

is a straightforward generalization of the Schmidt local rank in the bipartite case, and indeed is a good set of SLOCC invariants in the multipartite case, too. In the bipartite case, this local rank is necessary and sufficient to distinguish SLOCC classes, however in the multipartite case, we need additional SLOCC invariants in order to complete the classification. As we can see in the Table 1, the cases where the local ranks are $(2, 2, 3)$ or $(2, 2, 2)$ are not classified yet.

An option as the additional SLOCC invariants is the rank of $R^T R$ of Eq. (6), associated with the 2-qubit substructure, i.e., with the homomorphism $SL_2(\mathbb{C}) \otimes SL_2(\mathbb{C}) \simeq SO_4(\mathbb{C})$. We will discuss this invariant later in the Section 5. Alternative option is the series of hyperdeterminants adjusted to the smaller formats. These invariants reflect the nature of the onion structure, where the smaller format is embedded in the larger format. In order to classify the $(2, 2, 3)$ case further, we can utilize the hyperdeterminant of the format $2 \times 2 \times 3$,

$$\text{Det}_{223} \Psi = \begin{vmatrix}
\psi_{000} & \psi_{001} & \psi_{002} \\
\psi_{010} & \psi_{011} & \psi_{012} \\
\psi_{100} & \psi_{101} & \psi_{102}
\end{vmatrix}
\begin{vmatrix}
\psi_{010} & \psi_{011} & \psi_{012} \\
\psi_{100} & \psi_{101} & \psi_{102} \\
\psi_{110} & \psi_{111} & \psi_{112}
\end{vmatrix}
- \begin{vmatrix}
\psi_{000} & \psi_{001} & \psi_{002} \\
\psi_{010} & \psi_{011} & \psi_{012} \\
\psi_{110} & \psi_{111} & \psi_{112}
\end{vmatrix},$$

after adjusting $\psi_{i_1i_2i_3}$ to the $2 \times 2 \times 3$ format (i.e., choosing $\forall i_1i_2i_3 = 0 (i_3 \geq 3)$) by Clare’s local operation. Likewise, in order to classify the $(2, 2, 2)$ case, we can utilize the hyperdeterminant of the format $2 \times 2 \times 2$, also called the 3-tangle [13],

$$\text{Det}_{222} \Psi = \psi_{000}^2 \psi_{111}^2 + \psi_{001}^2 \psi_{110}^2 + \psi_{010}^2 \psi_{101}^2 + \psi_{100}^2 \psi_{011}^2
- 2(\psi_{000} \psi_{001} \psi_{110} \psi_{111} + \psi_{000} \psi_{010} \psi_{101} \psi_{111} + \psi_{000} \psi_{100} \psi_{011} \psi_{111}
+ \psi_{001} \psi_{010} \psi_{101} \psi_{110} + \psi_{001} \psi_{100} \psi_{011} \psi_{110} + \psi_{010} \psi_{100} \psi_{011} \psi_{101})
+ 4(\psi_{000} \psi_{011} \psi_{101} \psi_{110} + \psi_{001} \psi_{010} \psi_{100} \psi_{111}).$$
This polynomial invariant distinguishes the GHZ class and the W class. Since these hyperdeterminants represent the genuine component in multipartite entanglement of the given formats, they are also expected to play key roles in entanglement sharing in the similar manner as the 3-qubit case (For a short explanation of entanglement sharing, see the Appendix A).

Due to their importance as measures of entanglement, we show that the absolute value of the hyperdeterminant of format $k_1 \times \cdots \times k_l$ becomes an entanglement monotone, i.e., is decreasing on average under LOCC [16, 2]. This nice property is essentially due to the arithmetic mean-geometric mean inequality for (hyper)determinants, as outlined in Refs. [10]. The proof is similar to the argument of Ref. [7], where the 3-tangle (hyperdeterminant of format $2 \times 2 \times 2$) was proved to be an entanglement monotone.

Any LOCC protocol is decomposed into POVMs, each of which is unilocal operations of just one party. Moreover, we restrict ourselves to treat two-outcome POVMs, since any (local) POVM can be implemented as a sequence of two-outcome POVMs. Along with the permutation invariance, up to sign, of hyperdeterminants over parties, we assume, without loss of generality, that the two-outcome POVM is performed by Alice. Let $M(1)$ and $M(2)$ be the Alice’s two POVM elements, satisfying

$$M^\dagger(1)M(1) + M^\dagger(2)M(2) = 1. \quad (12)$$

By using the singular value decomposition for a $k_1 \times k_1$ matrix $M(\mu) \ (\mu = 1, 2)$, we can write

$$M(1) = U(1)D(1)V = U(1) \begin{pmatrix} \alpha_0 & O \\ \vdots & \ddots \\ O & \alpha_{k_1-1} \end{pmatrix} V, \quad (13)$$

$$M(2) = U(2)D(2)V = U(2) \begin{pmatrix} \beta_0 & O \\ \vdots & \ddots \\ O & \beta_{k_1-1} \end{pmatrix} V, \quad (14)$$

where $U(\mu) \ (\mu = 1, 2)$ and $V$ are unitary matrices, and $D(\mu) \ (\mu = 1, 2)$ are diagonal matrices with $0 \leq \alpha_i \leq 1$ and $0 \leq \beta_i \leq 1$ satisfying $\alpha_i^2 + \beta_i^2 = 1$ for $i = 0, \ldots, k_1 - 1$. Note that according to Eq. (12), we can choose $V$ to be same for $M(1)$ and $M(2)$.
Let an initial pure state before Alice’s POVM be \(|\Psi\rangle\) with the hyperdeterminant \(\text{Det}\Psi\). After the POVM, we obtain two outcome (normalized) states \(|\Psi'(\mu)\rangle\) \((\mu = 1, 2)\) for \(M(\mu)\) with the probability \(p_\mu\). In terms of

\[
V|\Psi\rangle = \begin{pmatrix}
z_0 \\
\vdots \\
z_{k_1-1}
\end{pmatrix},
\]

with the normalization condition \(|z_0|^2 + \cdots + |z_{k_1-1}|^2 = 1\), these outcome states are written as

\[
|\Psi'(1)\rangle = \frac{1}{\sqrt{p_1}} M(1)|\Psi\rangle = \frac{1}{\sqrt{p_1}} U(1) \begin{pmatrix}
\alpha_0 z_0 \\
\vdots \\
\alpha_{k_1-1} z_{k_1-1}
\end{pmatrix},
\]

\[
|\Psi'(2)\rangle = \frac{1}{\sqrt{p_2}} M(2)|\Psi\rangle = \frac{1}{\sqrt{p_2}} U(2) \begin{pmatrix}
\beta_0 z_0 \\
\vdots \\
\beta_{k_1-1} z_{k_1-1}
\end{pmatrix},
\]

where the occurring probabilities are

\[
p_1 = \alpha_0^2 |z_0|^2 + \cdots + \alpha_{k_1-1}^2 |z_{k_1-1}|^2, \quad p_2 = \beta_0^2 |z_0|^2 + \cdots + \beta_{k_1-1}^2 |z_{k_1-1}|^2,
\]

and satisfies \(p_1 + p_2 = 1\). Now, we show that the following inequality, i.e., the defining property of entanglement monotones,

\[
|\text{Det}\Psi| \leq p_1|\text{Det}\Psi'(1)| + p_2|\text{Det}\Psi'(2)|,
\]

holds for all possible POVM elements \(\{M(1), M(2)\}\). Suppose the degree of homogeneity of the hyperdeterminant is \(d\) (say, \(d = 6\) in Eq. (14) and \(d = 4\) in Eq. (11)). Since the hyperdeterminant is \(SL_{k_1} \times \cdots \times SL_{k_1}\) invariant, also immediately \(SU_{k_1} \times \cdots \times SU_{k_1}\) invariant, for the normalized pure states, we have

\[
\text{Det}\Psi'(1) = \text{Det} \frac{1}{\sqrt{p_1}} U(1) \begin{pmatrix}
\alpha_0 z_0 \\
\vdots \\
\alpha_{k_1-1} z_{k_1-1}
\end{pmatrix} = \frac{\alpha_0^{d/k_1} \cdots \alpha_{k_1-1}^{d/k_1}}{\sqrt{p_1^d}} \text{Det}\Psi,
\]

\[
\text{Det}\Psi'(2) = \text{Det} \frac{1}{\sqrt{p_2}} U(2) \begin{pmatrix}
\beta_0 z_0 \\
\vdots \\
\beta_{k_1-1} z_{k_1-1}
\end{pmatrix} = \frac{\beta_0^{d/k_1} \cdots \beta_{k_1-1}^{d/k_1}}{\sqrt{p_2^d}} \text{Det}\Psi.
\]

According to Eq. (19), all we should show is that

\[
1 \geq \frac{\alpha_0^{d/k_1} \cdots \alpha_{k_1-1}^{d/k_1}}{\alpha_0^2 |z_0|^2 + \cdots + \alpha_{k_1-1}^2 |z_{k_1-1}|^2} + \frac{\beta_0^{d/k_1} \cdots \beta_{k_1-1}^{d/k_1}}{\beta_0^2 |z_0|^2 + \cdots + \beta_{k_1-1}^2 |z_{k_1-1}|^2}
\]

\[
\geq \frac{1}{k}(q_1 + \cdots + q_k) \geq \sqrt[q_1 \cdots q_k]{q_1 \cdots q_k},
\]

is fulfilled for any set of \(\alpha_i\) and \(\beta_i = \sqrt{1 - \alpha_i^2}\) \((i = 0, \ldots, k_1 - 1)\). Remember the arithmetic mean-geometric mean inequality: for \(q_1, \ldots, q_k \geq 0\),

\[
\frac{q_1 + \cdots + q_k}{k} \geq \sqrt[q_1 \cdots q_k]{q_1 \cdots q_k},
\]
with the equality if and only if \( q_1 = \cdots = q_k \). Applying it to the denominators of the right hand side of Eq. (22), we find that

\[
\text{r.h.s.} \leq \frac{(\alpha_0 \cdots \alpha_{k_1-1})^{\frac{k_1}{k_1}} + (\beta_0 \cdots \beta_{k_1-1})^{\frac{k_1}{k_1}}}{(|z_0| \cdots |z_{k_1-1}|)^{\frac{k_1}{k_1}}} \leq 1,
\]

with both the equalities attained for

\[
|z_0| = \cdots = |z_{k_1-1}| = \frac{1}{\sqrt{k_1}},
\]

\[
\alpha_0 = \cdots = \alpha_{k_1-1}.
\]

This completes the proof.

5 Implications to 2-qubit mixed states

Let us briefly discuss the implications to 2-qubit mixed states. If the third subsystem is assumed as an ”environment” for 2-qubit mixed states, we can intuitively expect that the 2-qubit substructure of Alice and Bob is more controllable in the \( 2 \times 2 \times n \) pure case, than in the 2-qubit mixed case. Indeed, our result implies that the intuition is correct, and we will see that even when the third ”fairy” Clare helps Alice and Bob by controlling her environment as they desire, there remains a limitation in conversion of the 2-qubit substructure.

The \( 2 \times 2 \times n \) mixed states can be transformed to the Bell diagonal forms under the invertible local operations \[9, 17\]. Let us associate \( \rho \) with its purified state \( |\Psi\rangle \) in the \( 2 \times 2 \times n \) format, i.e.,

\[
\rho = \text{tr}_3(|\Psi\rangle\langle\Psi|) = \tilde{\Psi}^\dagger \tilde{\Psi}.
\]

Corresponding to three relative weights of the four Bell components of (normalized) \( \rho \) after invertible local operations, three relative ratios of the singular values \( s_i (i = 0, \ldots, 3) \) of

\[
R^T R = \tilde{\Psi}^T (i\sigma_y \otimes i\sigma_y) \tilde{\Psi}
\]

in Eq. (28) are the SLOCC invariants of 2-qubit mixed states (we do not consider the mixing here) \[9\]. On the other hand, in our \( 2 \times 2 \times n \) pure states, we are just concerned about whether some of singular values \( s_i \) are zero or not, i.e., about the rank of \( R^T R \) as a SLOCC invariant, in addition to local ranks (cf. the Section 4).

For the degenerate 2-qubit mixed states, their SLOCC conversion is a bit complicated. There are 2-qubit mixed states, often called quasi-distillable states \[18, 17\], that can not convert to the Bell diagonal forms under invertible local operations, but can approach arbitrarily close to them. Still, since we can suitably define the singular values for these states, the above argument for generic mixed states holds true. The exotic property of the quasi-distillable mixed states survives in the ”limit” to the situation as \( 2 \times 2 \times n \) pure states. An example is 2-qubit mixed states associated with the \( 2 \times 2 \times n \) pure states of the W class. In turn, even when Alice and Bob are helped by Clare, they can not convert to the major region associated with the GHZ class although they can do arbitrarily close to it. Finally, we would say, this is why there are more than one classes in the middle grades of the partial order of \( 2 \times 2 \times n \) multipartite entanglement in Fig. 2.
6 Conclusion

Mysteries in the fundamental of quantum theory as well as remarkable potential applications in quantum information processing would remain rooted in the rich properties of multipartite entanglement. Then, SLOCC can be utilized as a convenient and useful set of quantum operations. We have presented a brief overview about several characteristics of multipartite entanglement, illustrating the 2-qubit and the rest \((2 \times 2 \times n)\) quantum system. We have shown that its five-graded partially ordered structure of nine SLOCC entangled classes causes various multipartite phenomena, which cover entanglement sharing as fundamental and multi-party LOCC protocols, for example entanglement swapping, as applications.

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A Shareability of multipartite entanglement

The classifications of entanglement are quite relevant and complementary to the studies of entanglement sharing. The shareability of entanglement is qualitatively a monogamous nature of entanglement [19]. For example, if a 2-qubit system is the maximally entangled pure state (a Bell pair) then neither of 2 qubits can entangle with the other quantum system. Since there is no such a limitation for correlations in classical multipartite systems, this nature is the very quantum effect, and is also expected to have many potential applications to QIP.

However, the quantitative understanding can be said to be still limited despite a lot of efforts. Exceptionally, when the whole quantum system is a 3-qubit pure state \(|\Psi\rangle\), a constraint of the shareability [15] is expressed as

\[
C_{(12)3}^2 = C_{13}^2 + C_{23}^2 + 4|\text{Det}_{222}|. 
\] (29)

The concurrence \(C\) is the measure of entanglement of formation for 2-qubit mixed states \(\rho\) [20], and is given by

\[
C = \max(s_0^1 - s_1^1, -s_2^1 - s_3^1, 0), 
\] (30)

where \(s_i^1 (i = 0, \ldots, 3)\) are the square root of decreasing-ordered eigen values of a non-Hermitian positive operator \(\rho(i\sigma_y \otimes i\sigma_y)\rho^* (i\sigma_y \otimes i\sigma_y)\) (the superscript * stands for the complex conjugate), or equivalently, the decreasing-ordered singular values of \(R^T R\) of Eq. (28).

Although Eq. (29) was first considered as the definition of a residual entanglement, called the 3-tangle \(\tau = 4|\text{Det}_{222}|\), it can be seen as an identity between different kinds of entanglement. Eq. (30) suggests that when we consider its left
hand side represents the entire entanglement between Clare and the rest, there is a trade-off between the amount of genuine tripartite entanglement, given by $4|\text{Det}_{222}\Psi|$, and the amount of pairwise entanglement, given by $C_{13}$ and $C_{23}$. The GHZ and W representative states appear as the extremal states of the trade-off constraint in a straightforward manner. The GHZ state $|000\rangle + |111\rangle$ maximizes the genuine tripartite component 3-tangle ($\tau = 4|\text{Det}_{222}\Psi|$), and accordingly has no pairwise component $C_{jj'} (j, j' = 1, 2, 3)$ at all. On the other hand, the W state $|001\rangle + |010\rangle + |100\rangle$ has no genuine tripartite component, but maximizes the total amount of pairwise component $C^2_{12} + C^2_{23} + C^2_{31}$ [7, 21]. As a nontrivial fact, it is interesting to observe that the intermediate entangled states are equivalent to the GHZ state under the SLOCC equivalence.

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