On the geometry pervading One Particle States

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Abstract

In this paper, a way is given to obtain explicitly the representations of the Poincaré group as can be prescribed by Geometric Quantization. Thus one obtains some forms of the Space of Quantum States of the different relativistic free particles, and I give explicitly these spaces and the corresponding operators for the usually accepted as realistic physical particles. The general description of the massless particles I obtain, is given in terms of solutions of Penrose equations. In the case of Photon, I also give other descriptions, one in terms of the Electromagnetic Field. Since the results are derived from Geometric Quantization, they are related to certain Contact and Symplectic manifolds, that I study in detail. The symplectic manifold must be interpreted, according with Souriau, as the Movement Space of the corresponding classical particle, and that leads to propose one of the spaces I use as the State Space of the corresponding classical particle. These spaces are also described in each case.

1 Introduction.

The Wave Equations of relativistic elementary particles, Klein-Gordon, Dirac, Weil etc., have been originally derived each one in a independent way. An unification was the discovery of the relation of these equations with the representations of Poincaré group (inhomogeneous Lorenz group). The classification of the representations of Poincaré group made by Wigner [Wig39] with important contributions of Majorana, Dirac and Proca [Maj32, Dir36, Pro36] leads to a group theoretical study of Wave Equations by Bargmann and Wigner [BW48].

On the other hand, in Kirillov-Kostant-Souriau theory (Geometric Quantization), a description of Quantum Systems is given in terms of elements of the dual of the Lie algebra of the Lie group under consideration. Of course, this way of seeing Quantum Mechanics is not completely independent of the preceding one, since it has its origin in a method to obtain representations [Kir62, AK71].

The correspondence of Quantum States in the sense of Geometric Quantization with Wave Functions in the Quantum Mechanical sense is not clear in all cases. In Souriau's book [Sou70], a very general way to do the passage is given, but it doesn’t work in all cases.

In this paper we give a geometrical construction that establishes a one to one correspondence from Quantum States in the sense of Geometric Quantization to Wave Functions in the Quantum Mechanical sense, that is valid for all kinds of relativistic elementary particles. The idea is as follows.
In Geometric Quantization, one begins with a regular contact manifold or its associated hermitian line bundle. Quantum states (in the sense of Geometric Quantification) can be considered as being the collection of those sections of the hermitian line bundle which satisfies "Planck’s condition" (cf. Souriau’s book). In this paper we see that these sections are in a one to one correspondence with the (unrestricted) sections of another hermitian line bundle. Thus, this fibre bundle is a good setting to describe the quantum processes under consideration. The main idea to pass from this description to the usual one in terms of Wave Functions, can be intuitively explained as follows. Quantum States in G.Q. attribute an “amplitude of probability” to each movement of the particle. To obtain the corresponding wave function one must proceed as follows: for each event, the corresponding amplitude of probability is obtained by taking all movements passing through the given event, and then “adding up” (in a suitable sense) the corresponding amplitudes of probability. Of course the concept of “movement passing through an event” is only obvious in the case of the ordinary massive spinless particle, and it is defined in section 3.

The Contact Manifold under consideration, fibers on a Symplectic Manifold that, following Souriau, must be considered as being composed by the "movements of the corresponding classical particle". Then, with the constructions made in this paper, it becomes clear what space must be considered as composed by the "states of the corresponding classical particle", for each kind of relativistic particle.

The spaces of Wave Functions and many relevant isomorphic vector spaces we obtain, are the spaces of the representations of Poincaré group that I describe in section 8.

The preceding constructions are made in a explicit way for the relativistic particles usually considered as having physical sense. A section is devoted to massive particles and other to massless particles.

In section 11 a explicit application of the preceeding constructions is done for massive particles. One obtains solutions of Klein-Gordon and Dirac equations, and also a description of the wave functions for massive particles of higher spin.

Massless particles are studied in section 12. For massless particles of spin 1/2 one obtains solutions of Weyl equations and for general spin this method leads in a natural way to the description of massless particles by means of solutions of Penrose wave equations [Pen73]. This is related to the fact that the Contact and Symplectic Manifolds corresponding to Massless Particles are expressed in a natural way in Twistor Space and its Projective Space, as explained in section 12.4.

In the particular case of Photon, in section 12.5 I give also other forms
for the Wave Functions. One of them is in terms of the Electromagnetic Field, and are similar to the Wave Functions proposed by Bialynicki-Birula.

In section 13 I describe for each particle the concrete space that must be considered to be the "Space of states of the corresponding classical particle".

2 Notation

All differentiable manifolds appearing in this paper are assumed to be $C^\infty$, finite dimensional, Hausdorff and second countable.

The set composed by the differentiable vector fields on a differential manifold, $M$, is denoted by $\mathcal{D}(M)$. The set of differential $k$-forms on $M$ is denoted by $\Omega^k(M)$.

Let $X \in \mathcal{D}(M)$, $\omega \in \Omega^k(M)$. We denote by $i(X)\omega$ the interior product of $X$ by $\omega$ and by $L(X)\omega$ the Lie derivative of $\omega$ with respect to $X$.

Let $G$ be a Lie group. The Lie algebra of $G$ is the set consisting of the left invariant vector fields on $G$, provided with its canonical structure of Lie algebra. It is denoted in this paper by $\mathfrak{g}$. I denote by $\mathfrak{g}^\ast$ the dual of $\mathfrak{g}$. $\mathfrak{g}^\ast$ is canonically identified with the set composed by the left invariant 1-forms on $G$. By identification of each element of $\mathfrak{g}$ or $\mathfrak{g}^\ast$ with its value at the neutral element, $e$, the lie algebra of $G$ is identified to the tangent space at $e$, and $\mathfrak{g}^\ast$ to its dual.

There exist a map, the exponential map, $\text{Exp} : G \to G$, such that: if $X \in \mathfrak{g}$, the integral curve of $X$ with initial value $e$ is $t \mapsto \text{Exp} t X$. This integral curve is a one parameter subgroup of $G$.

The Lie algebra of the circle Lie group, $S^1$, is identified to $R$ in such a way that for all $t \in R$, $\text{Exp} t = e^{2\pi i t}$.

If we have an homomorphism of $G$ in $S^1$, its differential is a linear map from the Lie algebra of $G$ into the Lie algebra of $S^1$, which is $R$. Thus the differential of an homomorphism of $G$ into $S^1$, can be considered as an element of $\mathfrak{g}^\ast$.

In the same way, the Lie algebra of the Lie group $R$ is identified to $R$ in such a way that the exponential map becomes the identity. Thus, also in this case, the differential of an homomorphism of $G$ into $R$, can be considered as an element of $\mathfrak{g}^\ast$.

The coadjoint representation is the homomorphism $\text{Ad}^\ast : g \in G \to \text{Ad}^\ast_g \in \text{Aut}(\mathfrak{g}^\ast)$, given by $(\text{Ad}^\ast_g(\alpha))(X) = \alpha(\text{Ad}_{g^{-1}}(X))$ for all $\alpha \in \mathfrak{g}^\ast$, $X \in \mathfrak{g}$, where $\text{Ad}_{g^{-1}}$ is the differential of the automorphism of $G$ that sends each $h \in G$ to $g^{-1}hg$.

Let $M$ be a differentiable manifold and $G$ a Lie group acting on $M$ on the left (resp. on the right). Given an element, $X$, of $\mathfrak{g}$, we denote by $X_M$
the vector field on $M$ whose flow is given by the diffeomorphisms associated by the action to $\{\text{Exp}(-tX) : t \in \mathbb{R}\}$ (resp. $\{\text{Exp}tX : t \in \mathbb{R}\}$). $X_M$ is called the infinitesimal generator of the action associated to $X$.

A principal fibre bundle having $M$ as total space, $B$ as base and $G$ as structural group, will be denoted by $M(B,G)$.

In all that concerns fibre bundles, we use the notation of [KN63].

Part I

General Method

3 A Particular Classical State Space.

In this section I motivate the physical interpretation of the geometrical constructions that are to be made in this paper. I do not enter in the details of the computations. Most part of this section is an easy consequence of Souriau’s book [Sou70], which has inspired this paper.

Let us consider a classical relativistic free particle without spin in Minkowski space-time, with rest mass $m \neq 0$.

Space-time is interpreted as being an abstract four dimensional manifold, $M$, each inertial observer, $R$, providing us with a global chart, $\phi_R = (x_1^R, ..., x_4^R)$. Changes of these global charts are given by transformations corresponding to elements of the Poincaré (i.e. inhomogeneous Lorentz) group, $\mathcal{P}$. We consider a family of inertial frames such that changes are all given by ortochronous proper Poincaré transformations, i.e. elements of the connected component of the identity, $\mathcal{P}^+_\uparrow$.

A geometrical object in $\mathbb{R}^4$ invariant under the action on $\mathbb{R}^4$, provides us with a well defined object in $M$, whose local expression is the same for all inertial frames. An example is the Minkowski metric, $g$. When provided with this metric, $M$ becomes Minkowski space.

The objects defined in this way would be well defined even if the charts were not global.

Charts $\phi_R$ give rise in the canonical way to charts of $TM$, $\dot{\phi}_R = (x_1^R, \dot{x}_1^R)$, where $\dot{x}_R(v) = v(x_1^R)$, for all $v \in TM$. The charts $\bar{\phi}_R = (x_1^R, P_1^R)$, $P_1^R = m \dot{x}_1^R$, are more natural in Physics. Changes of these global charts are given by the transformations corresponding to elements of the Poincaré group, in its
canonical action on $\mathbb{R}^8$ i.e. the action given by

$$(L, C) \ast \begin{pmatrix} z_1 \\ \vdots \\ z_8 \end{pmatrix} = \begin{pmatrix} L \begin{pmatrix} z_1 \\ \vdots \\ z_8 \end{pmatrix} + C \end{pmatrix}$$

Geometrical objects on $\mathbb{R}^8$ invariant under this action, give us well defined geometrical objects on TM, whose local expressions are equal in the different inertial frames. For example, if $(z^1, \ldots, z^8)$ is the canonical coordinate system of $\mathbb{R}^8$, the 1-form:

$$\omega_0 \equiv z^8 dz^4 - \sum_{i=1}^{3} z^{i+4} dz^i$$

the vector field:

$$X_0 \equiv \frac{1}{m} \sum_{i=1}^{4} z^{i+4} \frac{\partial}{\partial z^i}$$

and the submanifold, $\mathcal{E}_0$, given by:

$$z^8 \equiv \sqrt{m^2 + (z^6)^2 + (z^6)^2 + (z^7)^2}$$

are invariant by the action on $\mathbb{R}^8$.

Thus the following 1-form, vectorfield, and submanifold are well defined on TM:

$$\bar{\omega} = P_R^4 dx_R^4 - \sum_{i=1}^{3} P_R^i dx_R^i$$

$$\bar{X} = \frac{1}{m} \sum_{i=1}^{4} P_R^i \frac{\partial}{\partial x_R^i}$$

$$\mathcal{E} = \left\{ v \in TM : P_R^4(v) = \left( m^2 + \sum_{i=1}^{3} (P_R^i(v))^2 \right)^{\frac{1}{2}} \right\}$$

$$= \left\{ v \in TM : g(v, v) = 1, \dot{x}_R^4(v) > 0 \right\}$$

where R is an arbitrary inertial frame.
The restriction of \( \tilde{\omega} \) to \( \mathcal{E}, \omega \), is a contact form. \( \tilde{X} \) is tangent to \( \mathcal{E} \). The restriction of \( \tilde{X} \) to \( \mathcal{E}, X \), is the unique vectorfield such that

\[
i_X \omega = m, \ i_X d\omega = 0.
\]

Each inertial observer associates to each element, \( v \), of \( \mathcal{E} \) eight numbers,

\[
\phi_R(v) = (x_1^R(v), ..., x_4^R(v), P_1^R(v), ..., P_4^R(v))
\]

which are interpreted as giving position, time and momentum-energy (we take \( c=1 \)). Thus \( \mathcal{E} \) must be interpreted as being state space.

The movements of the free particle under consideration are curves in \( \gamma \), having, for each inertial observer, a constant four velocity i.e.

\[
\dot{\phi}_R \circ \gamma(s) = (a^i + s(\dot{x}_R^i)_0, (\dot{x}_R^i)_0)
\]

where the \((\dot{x}_R^i)_0\) are constant, \((\dot{x}_R^4)_0 > 0, (\dot{x}_R^4)_0^2 - \sum_{i=1}^3 (\dot{x}_R^i)_0^2 = 1\).

Since the preceding relation is equivalent to

\[
\overline{\phi}_R \circ \gamma(s) = (a^i + s \overline{P}_R^i_0/m, (\overline{P}_R^i)_0),
\]

where \((\overline{P}_R^i)_0 = m(\dot{x}_R^i)_0\), one sees that the movements are the integral curves of \( X \).

The parameter \( s \) coincides, since we take \( c=1 \), with proper time (of the trajectory in space time).

The action on \( \mathbb{R}^8 \) gives, by means of any of the inertial observers, \( R \), an action on \( M \) by means of

\[
(L, C) \circ v = (\overline{\phi}_R)^{-1}(L, C) \ast (\overline{\phi}_R(v)),
\]

for all \( v \in \mathcal{E} \), \((L, C) \in \mathcal{P}_+^\dagger \).

This action obviously preserves \( \mathcal{E} \), and \( \omega \) in such a way that \((\mathcal{E}, \omega)\) is a homogeneous contact manifold for the given action, but different inertial observers lead to different actions.

Now, we shall describe \( \mathcal{E}_0 \) in a different way, thus obtaining a picture of state space more suitable for its generalisation to other kind of particles.

Let \( Y_8 \) be the infinitesimal generator of the action on \( \mathbb{R}^8 \) associated to the element \( Y \) of the Lie algebra of \( \mathcal{P}_+^\dagger \).

Since the action on \( \mathbb{R}^8 \) preserves \( \omega_0 \), we have

\[
L_{Y_8} \omega_0 = 0,
\]

so that

\[
i_{Y_8} d\omega_0 = -d(i_{Y_8} \omega_0) = -d(\omega_0(Y_8)).
\]
Thus we define a map, $\mu_0$, called the momentum map, from $\mathbb{R}^8$ into $P^*$, by means of

$$\mu_0(z) \cdot Y = -(\omega_0(Y))(z),$$

for all $Y \in P$, $z \in \mathbb{R}^8$.

Let $\{Y^i_1, Y^i_2, Y^i_3, Y^i_4 : i = 1, 2, 3\}$ be the basis of $P$, composed by the usual generators of Lorenz rotations and space-time translations.

It can be proved that $\mu_0 = z^8 Y^*_\delta - \sum_{i=1}^3 \left(\left[[z^1, z^2, z^3] \times (z^5, z^6, z^7)\right]i Y^i_{\alpha} + \left(z^4 z^{i+4} - z^8 z^i\right) Y^i_{\beta} + z^{i+4} Y^i_{\gamma}\right)$

where $\{Y^1_\alpha, ..., Y^3_\delta\}$ is the dual basis of $\{Y^1_\alpha, ..., Y^3_\delta\}$.

The map $\mu_0$ is equivariant for the given action on $\mathbb{R}^8$ and coadjoint action. Thus, since $(0,...,0,m) \in E_0$ and $\mu_0(0,...,0,m) = m Y^*_\delta$, $\mu_0(E_0)$ is the coadjoint orbit of $m Y^*_\delta$.

We also have $\mu_0^* \Omega = d\omega_0$, where $\Omega$ is the Kirillov symplectic form on the coadjoint orbit.

For each $\alpha$ in the coadjoint orbit, $\mu_0^{-1}\{\alpha\}$ is the image of an integral curve of $X_0$, that we know to be local expression of movements. Then $\mu_0$ gives a one to one correspondence between points of the coadjoint orbit and movements of the particle under consideration.

Now we consider the map

$$f : (a, b) \in E_0 \rightarrow (a, \mu_0(a, b)) \in \mathbb{R}^4 \times P^*.$$

By using the above expression for $\mu_0$, one sees that $f$ is an imbedding. Also, $f$ is equivariant when one considers the given action in $E_0$ and the “product action” in $\mathbb{R}^4 \times P^*$ given by

$$(L, C) \ast (a, \alpha) = (La + C, Ad^*_L(C) \alpha), \quad \forall (L, C) \in P^+_\pm, (a, \alpha) \in f(E_0).$$

Observe that, as a consequence, $f(E_0)$ is an orbit of that action.

Each inertial frame, $R$, enables us to “see” state space, $E$, by means of $f \circ \phi_R$, as being $f(E_0) \subset \mathbb{R}^4 \times P^*$. Changes of inertial frames are now given by transformations coming from the product action.

Since $f \circ \phi_R(e) = (x^1_R(e), ..., x^4_R(e), \mu_0(\phi_R(e)))$, if we denote $\mu_R \equiv \mu_0 \circ \phi_R$

we have

$$f \circ \phi_R = (\phi_R, \mu_R),$$
and $\mu_R$ stablishes a one to one correspondence of trajectories of $X$ \textit{(i.e. movements of the particle parametrized by proper time)} with points of the coadjoint orbit. Thus, coadjoint orbit is to be identified to movement space for all inertial frames, although the identification is different for each $R$.

The picture of a state, $e \in \mathcal{E}$, obtained by $R$ is now $(\phi_R(e), \mu_R(e))$ \textit{i.e.} an event, $\phi_R(e)$, and a movement, $\mu_R(e)$, “passing through” the event.

Each $Y \in \mathcal{P}$ defines a function on $\mathcal{P}^*$, denoted by the same symbol. Thus $Y \circ \pi_2 \circ f \circ \phi_R$ is a function on $\mathcal{E}$ \textit{(i.e. a dynamical variable)} where $\pi_2$ is the canonical projection of $R^4 \times \mathcal{P}^*$ onto $\mathcal{P}^*$.

We have:

$$-Y_1^i \circ \mu_R = P_1^i$$
$$Y_3 \circ \mu_R = P_3^i$$
$$-Y_2^i \circ \mu_R = (\overrightarrow{x}_R \times \overrightarrow{P}_R)^i$$
$$Y_4 \circ \mu_R = (P_4^i \overrightarrow{x}_R - x_4^i \overrightarrow{P}_R)^i$$

where $i = 1, 2, 3$, $\overrightarrow{x}_R = (x_1^R, x_2^R, x_3^R)$, and $\overrightarrow{P}_R = (P_1^R, P_2^R, P_3^R)$.

Thus the function on $\mathbf{R}^4 \times \mathcal{P}^*$ given by $P = (P_1^1, P_2^2, P_3^3, P_4^1) \equiv (-Y_1^1 \circ \pi_2, -Y_2^2 \circ \pi_2, -Y_3^3 \circ \pi_2, Y_4 \circ \pi_2)$, can be considered as an abstraction of \textbf{linear momentum}: each inertial observer obtains $f \circ \phi_R(e)$ as the picture of $e \in \mathcal{E}$, and the linear momentum he measures is $P(f \circ \phi_R(e))$.

In the same way the components of $\overrightarrow{l} = (-Y_1^1 \circ \pi_2, -Y_2^2 \circ \pi_2, -Y_3^3 \circ \pi_2)$ and $\overrightarrow{g} = (Y_3 \circ \pi_2, Y_4 \circ \pi_2, Y_4 \circ \pi_2)$ must be interpreted as being the components of (relativistic) \textbf{angular momentum}.

If we denote $\mathcal{P}^+_\perp$ by $G$, we can summarize what has been said as follows:

\begin{quote}
State space of our free particle is such that each inertial observer establishes a diffeomorphism from it to an orbit of $G$ in $\mathbf{R}^4 \times \mathcal{G}^*$, for the canonical action. Changes of inertial observer are given by the transformations given by the same action.
An inertial observer, $R$, thus sees any state as a pair, the first component is an event and the second a movement containing the event. The values at that point of $P$, $\overrightarrow{l}$, $\overrightarrow{g}$, are the values measured by $R$ of \textbf{linear momentum} and \textbf{angular momentum}.
\end{quote}

In this paper I accept that this is also valid for all relativistic free particles, with or without mass or spin. More precisely, it is assumed that the possible state space of the relativistic free particles are the orbits of $G$ in $\mathbf{R}^4 \times \mathcal{G}^*$,
where now $G$ is the universal (two fold) covering group of $\mathcal{P}_4$. The preceding interpretations of movements, states, linear and angular momentum are also considered as valid.

In what follows, we assume that an inertial observer, $R$, has been fixed. If state space is the orbit $\mathcal{O}$ and $(x, \alpha) \in \mathcal{O}$, we have seen how $\alpha$ can be considered as a movement. In fact, this movement is composed by the elements of $\mathcal{O}$ of the form $(y, \alpha)$. The set composed by such $y$, $M_\alpha$, compose the ordinary portrait of the movement $\alpha$ in $\mathbb{R}^4$, obtained by the inertial observer $R$. Obviously

$$M_\alpha = \{ g \ast x : g \in G_\alpha \}$$  \hspace{1cm} (1)$$

where $G_\alpha$ is the isotropy subgroup at $\alpha$ of the coadjoint representation.

This way of looking at states also enables us to determine all movements containing a given event: if $(x, \alpha) \in \mathcal{O}$, the movements containing the event $x$ compose the set

$$N_x = \{ \text{Ad}_g^* \alpha : g \in G_x \}$$ \hspace{1cm} (2)$$

where $G_x$ is the isotropy group at $x$ of the action of $G$ on $\mathbb{R}^4$.

## 4 Universal covering group of the Poincaré Group.

It is a well known fact that the universal covering group of Poincaré group is a semidirect product of $\text{SL}(2, \mathbb{C})$ by a four dimensional real vector space. In this section, I recall some general facts about this group and establishes the notation.

Let $(x^1, x^2, x^3, x^4)$ be the canonical coordinates in $\mathbb{R}^4$, $I$ the $2 \times 2$ unit matrix and $\sigma_1, \sigma_2, \sigma_3$ the Pauli matrices, i.e.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively. A generic point of $\mathbb{R}^4$ will be denoted $x = (x^1, x^2, x^3, x^4)$.

We define an isomorphism $h$, from $\mathbb{R}^4$ onto the real vector espace, $\mathbb{H}(2)$, of the hermitian $2 \times 2$ matrices, by means of

$$h(x) = x^4 I + \sum_{i=1}^{3} x^i \sigma_i = \begin{pmatrix} x^4 + x^3, & x^1 - ix^2 \\ x^1 + ix^2, & x^4 - x^3 \end{pmatrix}$$
We have $\text{Det } h(x) = < x, x >_m$, where $<, >_m$ is Minkowski pseudo-scalar product

$$< x, y >_m = x^4y^4 - \sum_{i=1}^{3} x^iy^i.$$  

If $x = (x^1, x^2, x^3, x^4)$ we denote $\vec{x} = (x^1, x^2, x^3)$ and $h(\vec{x}, x^4) = h(x)$.

The following formulae are useful for many computations along this paper

$$[h(\vec{k}, k_4), h(\vec{x}, x^4)] \overset{\text{def}}{=} h(\vec{k}, k_4)h(\vec{x}, x^4) - h(\vec{x}, x^4)h(\vec{k}, k_4) = 2ih(\vec{k} \times \vec{x}, 0), \quad (3)$$

where $\times$ means ordinary vector product,

$$\{h(\vec{k}, k_4), h(\vec{x}, x^4)\} \overset{\text{def}}{=} h(\vec{k}, k_4)h(\vec{x}, x^4) + h(\vec{x}, x^4)h(\vec{k}, k_4) = 2h(k_4\vec{x} + x^4\vec{k}, k_4x^4 + (\vec{k}, \vec{x})), \quad (4)$$

where $\langle ., . \rangle$ is the usual scalar product in $\mathbb{R}^3$,

$$h(\vec{k}, k_4)\bar{h}(\vec{x}, x^4)\varepsilon - h(\vec{x}, x^4)\bar{h}(\vec{k}, k_4)\varepsilon = 2[h(k_4\vec{x} - x^4\vec{k}, 0) + ih(\vec{k} \times \vec{x}, 0)]$$

where the bar means complex conjugation, and we denote by $\varepsilon$ the matrix $i\sigma_2$, i.e.

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

Notice that

$$^t A \varepsilon A = (\text{Det } A) \varepsilon,$$

so that

$$A \varepsilon = \varepsilon (^t A)^{-1} \quad (5)$$

if $A \in \text{SL}(2, \mathbb{C})$.

Also we have

$$\frac{1}{2} Tr(h(x)\bar{h}(y)\varepsilon) = - < x, y >_m$$

for all $x, y \in \mathbb{R}^4$. 

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We define an action on the left of the Lie group $\text{SL}(2, \mathbb{C})$ on the abelian Lie group $H(2)$ by means of

$$A * H = AHA^*$$

for all $A \in \text{SL}(2, \mathbb{C})$, $H \in H(2)$, where $A^*$ is the transposed of the complex conjugate of $A$. To this action by automorphisms of $H(2)$ then corresponds a semidirect product, $\text{SL}(2, \mathbb{C}) \oplus H(2)$, whose group law is given by

$$(A, H) * (B, K) = (AB, AKA^* + H)$$

The identity element is $(I, 0)$ and $(A, H)^{-1} = (A^{-1}, -A^{-1}HA^*)$.

This semidirect product acts on the left on $\mathbb{R}^4$ by means of

$$(A, H) * x = h^{-1}(Ah(x)A^* + H).$$

Poincaré group, $\mathcal{P}$, is identified to the closed subgroup of $\text{GL}(5; \mathbb{R})$ composed by the matrices

$$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

where $C \in \mathbb{R}^4$ and $L \in O(3, 1)$ (such a matrix is denoted in the following simply by $(L, C)$).

For all $(A, H) \in \text{SL} \oplus H(2)$ (where $\text{SL}$ stands for $\text{SL}(2; \mathbb{C})$), there exists a unique $(L, C) \in \mathcal{P}$ such that $(A, H) * x = Lx + C$ for all $x \in \mathbb{R}^4$.

The map, $\rho$, from $\text{SL} \oplus H(2)$ into $\mathcal{P}$ defined by sending such a $(A, H)$ to the corresponding $(L, C)$, is a homomorphism of Lie groups, whose kernel consists of $(I, 0)$ and $(-I, 0)$. In fact, $\rho$ is a two fold covering map of the identity component in $\mathcal{P}$, $\mathcal{P}_+^\dag$. Since $\text{SL}$ and $H(2)$ are connected and simply connected it follows that $\text{SL} \oplus H(2)$ is the universal covering group of $\mathcal{P}_+^\dag$.

The differential of $\rho$ is an isomorphism from the Lie algebra of $\text{SL} \oplus H(2)$ onto the Lie algebra of $\mathcal{P}$.

The standard method to handle semidirect products enable us to identify the Lie algebra of $\text{SL} \oplus H(2)$ with $\text{sl}(2, \mathbb{C}) \times H(2)$, the Lie bracket being

$$[(a, k), (a', k')] = ([a, a'], ak' + k'a^* - (a'k + ka^*)).$$
If \((a, h) \in \mathfrak{sl}(2, C) \times \mathbb{H}(2), \ t \in \mathbb{R}\), we have

\[
\text{Exp}[t(a, h)] = \left(e^{ta}, \int_0^t e^{sa} h e^{sa^*} \, ds\right). \tag{6}
\]

In this paper, we use the basis of \(\mathfrak{sl} \times \mathbb{H}(2)\) corresponding by \(d\rho\) to the basis of \(P\) associated to linear momentum and angular momentum in section 3. This basis is composed by the following elements

\[
P^k = (0, -\sigma_k), \quad k = 1, 2, 3,
\]
\[
P^4 = (0, \sigma_4) = (0, I),
\]
\[
l^k = \left(i \frac{\sigma_k}{2}, 0\right),
\]
\[
g^k = \left(\frac{\sigma_k}{2}, 0\right). \tag{7}
\]

An element, \(X\), of \(\mathfrak{sl}(2, C) \times \mathbb{H}(2)\) defines a (linear) function on \((\mathfrak{sl}(2, C) \times \mathbb{H}(2))^*\). Its restriction to movement space (a coadjoint orbit) must be considered as a dynamical variable. But also, if we have a representation on a vector space (resp. an action on a manifold) of \(\text{SL} \oplus \mathbb{H}(2)\), \(X\) gives rise to an infinitesimal generator of the representation (resp. the action), i.e., an endomorphism of the vector space (resp. a vector field on the manifold).

We define vector valued functions on the dual of the Lie algebra as follows

\[
P = (P^1, P^2, P^3, P^4)
\]
\[
\overrightarrow{l} = (l^1, l^2, l^3)
\]
\[
\overrightarrow{g} = (g^1, g^2, g^3)
\]

what will be considered as being linear momentum and angular momentum.

We define a non degenerate scalar product in \(\mathfrak{sl} \times \mathbb{H}(2)\) by means of

\[
< (a, k), (b, l) > = 2\text{Re} \text{Tr} \left(\frac{1}{4} k \varepsilon \ell \varepsilon - ab\right) = \frac{1}{2} \text{Tr} (k \varepsilon \ell \varepsilon) - 2\text{Re} \text{Tr} ab.
\]
This scalar product defines in the standard way an isomorphism from the Lie algebra of $\text{SL} \oplus \text{H}(2)$ onto its dual. The image of $(a, k) \in \text{sl} \times \text{H}(2)$ by this isomorphism, will be denoted by $(a, k) \in (\text{sl} \times \text{H}(2))^*$, and is given by

$$\{a, k\} ((b, m)) = \langle a, k \rangle, (b, m) \rangle.$$

With this notation, the values of $P$, $\vec{l}$ and $\vec{g}$ at $\{a, k\}$ are, when written in terms of its hermitian form

$$h(P(\{a, k\})) = -k,$$

$$h(\vec{l}(\{a, k\}), 0) = i(a^* - a),$$

$$h(\vec{g}(\{a, k\}), 0) = -(a + a^*).$$

so that

$$\{a, k\} = \{-\frac{1}{2}h(\vec{g}(\{a, k\}), 0) + \frac{i}{2}h(\vec{l}(\{a, k\}), 0), -h(P(\{a, k\})))\}.$$ (11)

i.e. $-(1/2)h(\vec{g}(\{a, k\}), 0)$ is the hermitian real part of $a$ and the matrix $(1/2)h(\vec{l}(\{a, k\}), 0)$ its hermitian imaginary part.

A straightforward computation, leads to the following formula for the coadjoint representation

$$\text{Ad}^*_{(A,H)}\{a,k\} = \{ AaA^{-1} + \frac{1}{4}(AkA^* \bar{\epsilon} H \bar{\epsilon} - H \bar{\epsilon} AkA^* \bar{\epsilon}), AkA^* \}. $$ (12)

To end this section, we define other functions on the dual of the Lie algebra of $\text{SL} \oplus \text{H}(2)$.

One of these is $|P|$, defined by

$$|P| (\{a, k\}) = \text{Det} (h(P(\{a, k\}))) = \text{Det} (k),$$

whose physical meaning is mass square.

Then,

$$|P| (\text{Ad}^*_{(A,H)}\{a,k\}) = \text{Det}(-AkA^*) = \text{Det}(k),$$

so that the value of $|P|$, is constant along any coadjoint orbit.
The other is defined in terms of the Pauli-Lubanski fourvector, given by

\[ W = (\vec{W}, W^4) \]  \hspace{1cm} (13)

\[ \vec{W} = P^4 \vec{l} + \vec{P} \times \vec{g}, \]  \hspace{1cm} (14)

\[ W^4 = \langle \vec{P}, \vec{l} \rangle . \]  \hspace{1cm} (15)

Using (11), (3) and (4), the hermitian form of \( W \) is found to be

\[ h(W(\{a, k\})) = i(a k - k^* a^*). \]  \hspace{1cm} (16)

One can prove that

\[ h(W(Ad_{\{A, H\}}\{a, k\})) = A h(W(\{a, k\})) A^*, \]

so that the function

\[ |W|(\{a, k\}) = Det(h(W(\{a, k\}))), \]

is also constant along each coadjoint orbit.

5 Quantizable forms

In this section I recall some of the geometric constructions done in [Sou88, Dia82a, Dia82b, Dia91, Dia96a].

I shall give a way to construct homogeneous contact manifolds that fibers on the coadjoint orbits of Lie groups. This is not possible on an arbitrary orbit, but only on the so called quantizable ones.

The idea is to consider the quantizable coadjoint orbits of \( \text{SL}(2; \mathbb{C}) \oplus \mathbb{H}(2) \) as the classical movement space of relativistic free particles. The corresponding homogeneous contact manifolds are geometrical objects enabling us to construct the Wave Functions, and representations of this group.

Let \( G \) be a Lie group and \( \alpha \) an element of the dual of the Lie algebra of \( G, \mathfrak{g}^* \), where we consider the coadjoint action.
We say that \( \alpha \) is **quantizable** if there exists a surjective homomorphism, \( C_\alpha \), from the isotropy subgroup at \( \alpha \), \( G_\alpha \), onto the unit circle, \( S^1 \), whose differential is \( \alpha \) (cf. section 2 where the way in which the differential is identified to an element of \( G^* \), is detailed).

In a more explicit way, this means that \( \alpha \) is **quantizable** if there exists a surjective homomorphism, \( C_\alpha \), from the isotropy subgroup at \( \alpha \), \( G_\alpha \), onto the unit circle, such that, for all \( X \) in the Lie algebra of \( G_\alpha \) and \( t \) in \( \mathbb{R} \)

\[
C_\alpha(\text{Exp} tX) = e^{2\pi i t \alpha(X)}.
\]

If a form is quantizable, all the elements of its coadjoint orbit are quantizable. These orbits are called **quantizable orbits**.

The form \( \alpha \) is said to be **R-quantizable** if there exists a surjective homomorphism from \( G_\alpha \) onto the usual additive Lie group of reals, \( \mathbb{R} \), whose differential is \( \alpha \).

In other words, \( \alpha \) is **R-quantizable** if there exists a surjective homomorphism from \( G_\alpha \) onto \( \mathbb{R} \), \( H_\alpha \), such that, for all \( X \) in the Lie algebra of \( G_\alpha \), and \( t \) in \( \mathbb{R} \)

\[
H_\alpha(\text{Exp} tX) = t \alpha(X).
\]

To such a \( H_\alpha \), one can associate an homomorphism, \( C_\alpha \), onto \( S^1 \) by means of

\[
C_\alpha : g \in G_\alpha \to e^{2\pi i H_\alpha(g)} \in S^1.
\]

The differential of \( C_\alpha \) is \( \alpha \). Thus when \( \alpha \) is **R-quantizable**, it is quantizable.

All elements of a coadjoint orbit containing a R-quantizable form, are R-quantizable. In this case, the orbit is said to be a **R-quantizable orbit**.

In [Dia96a], a slightly more general concept of quantizability is used, but it is unnecessary for the purposes of the present paper.

In what follows we assume that \( \alpha \) is quantizable and \( C_\alpha \) is a homomorphism from \( G_\alpha \) onto the unit circle, whose differential is \( \alpha \). We identify the coadjoint orbit, \( \mathcal{O}_\alpha \), with \( G/G_\alpha \) by means of the diffeomorphism

\[
gG_\alpha \in \frac{G}{G_\alpha} \to \text{Ad}_g^* \alpha \in \mathcal{O}_\alpha.
\]
We define an action of $S^1$ on $G/Ker\ C_\alpha$ by means of

$$ (g\ Ker\ C_\alpha) * s = gh\ Ker\ C_\alpha $$

where $h$ is any element of $G_\alpha$ such that $C_\alpha(h) = s$.

Actually $(G/Ker\ C_\alpha, (G/G_\alpha, S^1))$ is a principal fibre bundle, the bundle action is given by (17) and the bundle projection, by the canonical map,

$$ \pi : gKer\ C_\alpha \rightarrow gG_\alpha $$

This action of $S^1$, commutes with the canonical action of $G$ on $G/Ker\ C_\alpha$.

We denote by $\pi^c$ and $\pi^s$ the canonical maps

$$ \pi^c : g \in G \rightarrow gKer\ C_\alpha \in \frac{G}{Ker\ C_\alpha} $$

$$ \pi^s : g \in G \rightarrow gG_\alpha \in \frac{G}{G_\alpha}. $$

There exist an unique 1-form, $\Omega$, on the homogeneous space $G/Ker\ C_\alpha$ such that

$$ (\pi^c)^*\Omega = \alpha. $$

The 1-form $\Omega$ is a contact form, invariant under the action of $G$.

Let $Z(\Omega)$ be the vectorfield defined by

$$ i_{Z(\Omega)}\Omega = 1, \ i_{Z(\Omega)}\ d\Omega = 0. $$

All the integral curves of $Z(\Omega)$ have the same period. If we denote by $T(\Omega)$ this period, then $\Omega/T(\Omega)$ is a connexion form.

Since the structural group is abelian, the curvature form is $d\Omega/T(\Omega)$.

There exist an unique 2-form, $\omega$, on $G/G_\alpha$, such that

$$ \pi_*\omega = \frac{d\Omega}{T(\Omega)}. $$
Thus
\[(\pi^s)^*\omega = \frac{d\alpha}{T(\Omega)}.\] (20)

The form \(\omega\) is an invariant symplectic form, and its cohomology class is integral.

Each \(m \in G\), define a function on \(G^*\) so that it does also on the coadjoint orbit. Since we think in the coadjoint orbit as being the movement space of a particle, we can say that \(m\) defines a dynamical variable, \(D_m\). Such a \(m \in G\), also define a infinitesimal generator of the canonical action of \(G\) on \(G/KerC_\alpha\), denoted by \(X^c_m\), and a infinitesimal generator of the canonical action of \(G\) on \(G/G_\alpha\), denoted by \(X^s_m\).

Since \(\Omega\) is left invariant by the action of \(G\), we have
\[L_{X^m_c}\Omega = 0,\]
so that the relation \(L_X = d\imath_X + \imath_X d\) leads to
\[\imath_{X^c_m}d\Omega = -d\text{[}\Omega(X^c_m)\text{]}\]

A computation gives
\[(\Omega(X^c_m) \circ \pi^c)(g) = Ad_g^*\alpha_0(-m) = -D_m(Ad_g^*\alpha_0) =
= -D_m(gG_\alpha) = -D_m \circ \pi(gKerC_\alpha),\]
so that
\[\Omega(X^c_m) = -D_m \circ \pi.\] (21)

Thus (20), (21), and the fact that
\[\pi_s X^c_m = X^s_m,\]
lead to
\[\imath_{X^c_m}\omega = \frac{1}{T(\Omega)}dD_m.\] (22)

Since the flow of the vector field \(X^s_m\) (resp. \(X^c_m\)) preserves \(\omega\) (resp. \(\Omega\)), it is an infinitesimal automorphism of the symplectic (resp. contact) structure,
i.e. a locally hamiltonian vector field. Equation (22) tell us that in fact $X_m$ is globally hamiltonian and $D_m/T(\Omega)$ is the corresponding hamiltonian.

As usual in the Theory of Connections, a map, $f$, into $G/KerC_{\alpha}$ is called horizontal, if $f^*\Omega = 0$. Let $f_0$ be a map into $G/G_{\alpha}$. An horizontal lift of $f_0$ is an horizontal map into $G/KerC_{\alpha}$, such that $\pi \circ f = f_0$.

The horizontal lift of curves can be described as follows. Given a curve, $\gamma$ in $G/G_{\alpha}$, the horizontal lift of $\gamma$ to $gKerC_{\alpha}$ is

$$\tilde{\gamma}(t) = (\overline{\gamma}(t) \ K_{\alpha} \ e^{-2\pi i \int_{[0, \ t]}^{\alpha}})$$  \hspace{1cm} (23)

where $\overline{\gamma}$ is any lifting of $\gamma$ to $G$, such that $\overline{\gamma}(0) = g$, and the vertical bar means restriction.

Associated to this principal fibre bundle and the canonical action of $S^1$ on $\mathbb{C}$, one can consider the 1-dimensional vector bundle whose total space is

$$(G / KerC_{\alpha}) \times S^1 \mathbb{C}.$$  

Let us recall its definition.

Consider in $(G/KerC_{\alpha}) \times \mathbb{C}$, an action of $S^1$ defined by

$$(gKerC_{\alpha}, t) \ast z = ((gKerC_{\alpha}) \ast z, z^{-1}t),$$

for all $z \in S^1$.

The elements of $(G/KerC_{\alpha}) \times S^1 \mathbb{C}$ are the orbits of this action.

Let us denote by $[gKerC_{\alpha}, t]$ the orbit of $(gKerC_{\alpha}, t)$. We have

$$\begin{align*}
[gKerC_{\alpha}, t] &= \left\{ ( (gKerC_{\alpha}) \ast s, s^{-1}t ) : \ s \in S^1 \right\} = \\
&= \left\{ ( ghKerC_{\alpha}, C_{\alpha} (h^{-1}) t ) : \ h \in G_{\alpha} \right\}
\end{align*}$$

We define a map, $\overline{\pi}$, from

$$\frac{G}{KerC_{\alpha}} \times S^1 \mathbb{C}$$

onto the coadjoint orbit by means of

$$\overline{\pi}([gKerC_{\alpha}, t]) = gG_{\alpha}$$
Let \( m \in G/G_{\alpha} \), and \( g \in G \) one of its representatives. Since the action of \( S^1 \) on \( \pi^{-1}(m) \) is transitive, we have

\[
\pi^{-1}(m) = \{[g \text{ Ker} C_{\alpha}, t] : t \in \mathbb{C}\}.
\]

Thus, for each \( g \) we obtain a bijection of \( \pi^{-1}(m) \) onto \( \mathbb{C} \).

We consider in \( \pi^{-1}(m) \) the structure of hermitian one dimensional complex vector space such that this bijection is a unitary isomorphism.

This structure is independent of the representative \( g \).

If \( g \) and \( g' \) are representatives of \( m \), we have for all \( t, t', a \in \mathbb{C} \)

\[
[g \text{ Ker} C_{\alpha}, t] + [g' \text{ Ker} C_{\alpha}, t'] = [g \text{ Ker} C_{\alpha}, t + C_{\alpha}(g^{-1}g') t']
\]

\[
a \cdot [g \text{ Ker} C_{\alpha}, t] = [g \text{ Ker} C_{\alpha}, at],
\]

\[
\langle [g \text{ Ker} C_{\alpha}, t], [g' \text{ Ker} C_{\alpha}, t'] \rangle = \overline{C_{\alpha}(g^{-1}g')} t'.
\]

With these operations, \( \pi \) becomes a complex vector bundle of dimension one with a hermitian product in each fiber i.e. a **hermitian line bundle**.

The sections of the hermitian line bundle are in a one to one correspondence with the functions on \( G/\text{Ker} C_{\alpha} \), \( t \), such that \( f((g \text{ Ker} C_{\alpha}) \ast s) = s^{-1} f(g \text{ Ker} C_{\alpha}) \), for all \( s \in S^1 \). These functions will be called from now on **pseudotensorial functions**. This correspondence is as follows.

If \( f \) is a pseudotensorial function, the corresponding section sends \( m \in G/G_{\alpha} \) to \( [r, f(r)] \) where \( r \) is arbitrary in \( \pi^{-1}(m) \).

If \( \sigma \) is a given section of the hermitian line bundle, the corresponding pseudotensorial function, \( f \), is defined by \( \sigma(\pi(r)) = [r, f(r)] \) for all \( r \in G/\text{Ker} C_{\alpha} \).
The canonical action of $G$ on $(G/KerC_{\alpha})$, leads to an action on 
\[(G/KerC_{\alpha}) \times \mathbb{S}^1 \ C,\]
given by 
\[g \ast [h \ KerC_{\alpha}, t] = [gh \ KerC_{\alpha}, t].\]

This action is well defined, as a consequence of the fact that the action of $\mathbb{S}^1$ commutes with the canonical action of $G$.

In the following, I use the same construction of a hermitian line bundle, for each principal bundle, $G/(Ker C)(G/H, S)$, where $C$ is an homomorphism of $H$ into $\mathbb{S}^1$, whose image is $S$.

6 Geometric Quantum States.

In this section I recall some definitions and results from [Dia96b].

Let $G = SL(2, C) \oplus H(2)$ and $\alpha$ a quantizable form of $G$. We use the notation of section 5.

The sections of the hermitian line bundle whose total space is 
\[(G/KerC_{\alpha}) \times \mathbb{S}^1 \ C\]
are called Prequantum States. We use the same denomination for the corresponding pseudotensorial functions.

Now, let us consider the actions of the abelian subgroup $\{I\} \times H(2)$ on $G/Ker C_{\alpha}$ and $G/G_{\alpha}$, induced by the canonical action.

There exist an unique action of $\{I\} \times H(2)$ on $G/Ker C_{\alpha}$ whose orbits are horizontal and such that $\pi$ becomes equivariant.

This action is called horizontal action and is given by
\[(I, K) \ast ((A, H) \ KerC_{\alpha}) = ((A, H + K) \ KerC_{\alpha}) \ast e^{-i\pi Tr(AkA^*e^\alpha)} \quad (24)\]
for all $K \in H(2)$, $(A, H) \in G$, where $\ast$ in the left hand side stands for the new action and in the right hand one, corresponds to the bundle action. $k$ is given by $\alpha = \{a, k\}$.

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We define **Quantum States** as being the Prequantum States that correspond to pseudotensorial functions left invariant by the horizontal action.

Let $\pi_1$ and $\pi_2$ be the canonical projections of $G$ on $SL(2, \mathbb{C})$ and $H(2)$ respectively. We denote $\pi_1(G_\alpha)$ by $(G_\alpha)_{SL}$ and $\pi_2(G_\alpha)$ by $(G_\alpha)_H$.

In section 4 of [Dia96b] it is proved that the map

$$(C_\alpha)_{SL} : (G_\alpha)_{SL} \longrightarrow S^1,$$

defined by

$$(C_\alpha)_{SL}(g) = C_\alpha(g, h) e^{-i\pi Tr(k \epsilon h \epsilon)},$$ (25)

for all $(g, h) \in G_\alpha$, is well defined and a homomorphism.

As a consequence, we can define

$$\tilde{C}_\alpha : (G_\alpha)_{SL} \oplus H(2) \longrightarrow S^1,$$ (26)

by means of

$$\tilde{C}_\alpha(g, r) = (C_\alpha)_{SL}(g) e^{i\pi Tr(k \epsilon r \epsilon)}.$$

$\tilde{C}_\alpha$ is an extension of $C_\alpha$ to $(G_\alpha)_{SL} \oplus H(2)$, and a homomorphism. Its differential coincides with the restriction of $\alpha$ to the Lie algebra of this group.

The canonical action of $G$ on $G/Ker C_\alpha$ maps horizontal orbits to horizontal orbits, thus defining a transitive action on the space of horizontal orbits, $W_\alpha$.

Let us consider the canonical map

$$\tau : \frac{G}{Ker C_\alpha} \to W_\alpha$$

defined by sending each element of $G/Ker C_\alpha$ onto its horizontal orbit.

The isotropy subgroup at $\tau(Ker C_\alpha)$ is $Ker \tilde{C}_\alpha$. As a consequence, we can identify $W_\alpha$ to $G/Ker \tilde{C}_\alpha$, by means of the bijective map

$$(A, H) \in G/Ker \tilde{C}_\alpha \to \tau((A, H)Ker C_\alpha) \in W_\alpha.$$
With the definitions given in section 5, we have seen that

\[
\frac{G}{\text{Ker} \tilde{\mathcal{C}}_\alpha} \left( \frac{G}{G_\alpha} \mathcal{S}^1 \right)
\]

becomes a principal fibre bundle.

Similar definitions provide

\[
\frac{G}{\text{Ker} \tilde{\mathcal{C}}_\alpha} \left( \frac{G}{(G_\alpha)_{SL} \oplus H(2)} \mathcal{S}^1 \right)
\]

with a structure of principal fibre bundle.

The bundle projection is the canonical map from \(G/\text{Ker} \tilde{\mathcal{C}}_\alpha\) onto
\(G/((G_\alpha)_{SL} \oplus H(2))\). The bundle action is given by

\[
((A, H)\text{Ker} \tilde{\mathcal{C}}_\alpha) \ast s = (A, H)(B, K)\text{Ker} \tilde{\mathcal{C}}_\alpha,
\]

where \((B, K)\) is such that \(\tilde{\mathcal{C}}_\alpha(B, K) = s\).

But the map

\[
(A, H)((G_\alpha)_{SL} \oplus H(2)) \in \frac{G}{(G_\alpha)_{SL} \oplus H(2)} \to A(G_\alpha)_{SL} \in \frac{SL}{(G_\alpha)_{SL}} \quad (27)
\]

is a diffeomorphism.

We identify these homogeneous spaces by means of this map, so that the principal fibre bundle becomes

\[
\frac{G}{\text{Ker} \tilde{\mathcal{C}}_\alpha} \left( \frac{SL}{(G_\alpha)_{SL}} \mathcal{S}^1 \right),
\]

where the bundle projection is

\[
\tau_2 : (A, H)\text{Ker} \tilde{\mathcal{C}}_\alpha \to A(G_\alpha)_{SL}.
\]

The canonical maps define the homomorphism of principal \(\mathcal{S}^1\)-bundles given in Figure [1].

Since Quantum States correspond to pseudotensorial functions left-invariant by the horizontal action,

Quantum States are the pull back by \(\iota_1\) of unrestricted pseudotensorial functions on \(G/\text{Ker} \tilde{\mathcal{C}}_\alpha\).
Now, let us associate, to each Quantum State in the sense of section 6, a Wave Function in the ordinary sense of Quantum Mechanics.

From now on, we assume, unless the contrary is explicitly stated, that the State Space is the orbit of \((0, \alpha)\) in \(H(2) \times G^*\).

This choice do not carry very important consequences for the study of the free particle at the quantum level, in the sense that the other choices lead to isomorphic spaces of Quantum States (in all of its forms), and the same Wave Functions. These facts are proved in Remark 5.2 of [Dia96b].

The isotropy subgroup at \((0, \alpha)\), \(G_{(0,\alpha)}\), is composed by the elements of \(G_{\alpha}\) of the form \((A, 0)\) i.e.

\[
G_{(0,\alpha)} = G_{\alpha} \cap (SL \oplus \{0\}).
\]

Let \(\alpha = \{a, k\}\) and denote

\[
SL_1 = \{A \in SL : AaA^{-1} = a\},
\]

\[
SL_2 = \{A \in SL : AkA^* = k\}.
\]

Then

\[
G_{(0,\alpha)} = (SL_1 \cap SL_2) \oplus \{0\},
\]

Notice that \((G_{\alpha})_{SL} \subset SL_2\) and \(SL_1 \cap SL_2 \subset (G_{\alpha})_{SL}\) so that

\[
SL_1 \cap SL_2 = (G_{\alpha})_{SL} \cap SL_1.
\]
But the map
\[
(A, H)((G_\alpha)_{\text{SL}} \cap SL_1) \oplus \{0\} \in \frac{G}{((G_\alpha)_{\text{SL}} \cap SL_1) \oplus \{0\}} \rightarrow
\]
\[
\rightarrow (H, A((G_\alpha)_{\text{SL}} \cap SL_1)) \in H(2) \times \frac{SL}{(G_\alpha)_{\text{SL}} \cap SL_1}
\]
is a diffeomorphism.

Thus, State Space is the image of the injective map
\[
i : (H, A((G_\alpha)_{\text{SL}} \cap SL_1)) \in H(2) \times \frac{SL}{(G_\alpha)_{\text{SL}} \cap SL_1} \rightarrow (28)
\]
\[
\rightarrow (H, Ad_{(A, H)^{\alpha}}) \in H(2) \times \mathbb{C}^*.
\]

This image need not, a priori, be a proper submanifold of $H(2) \times \mathbb{C}^*$, and we consider it provided with the topology and differentiable structure such that $i$ becomes a diffeomorphism. In the following we identify each $i(X)$ with $X$, so that we can say that
\[
H(2) \times \frac{SL}{(G_\alpha)_{\text{SL}} \cap SL_1}
\]
is State Space.

The canonical map from State Space onto Movement Space, can be generalised to all the homogeneous spaces appearing in the commutative diagram of Figure 1. This will be done with the following geometrical construction.

Let $\mathcal{L}$ be a closed subgroup of $G$, and
\[
\mathcal{S} = \{s \in SL : (s, 0) \in \mathcal{L}\},
\]
so that
\[
\mathcal{S} \oplus \{0\} = \mathcal{L} \cap (SL \oplus \{0\}).
\]

Thus we define the map
\[
(H, A\mathcal{S}) \in H(2) \times \frac{SL}{\mathcal{S}} \xrightarrow{\nu} (A, H)\mathcal{L} \in \frac{G}{\mathcal{L}}.
\]
This map is well defined and, if $H$ is a fixed element in $H(2)$, its restriction to 
$\{H\} \times \frac{SL}{S}$
is injective.

Now we apply this to the cases in Figure 1.

When $\mathcal{L} = \text{Ker} C_\alpha$ we have $S = \text{Ker}(C_\alpha)_{SL} \cap SL_1$, so that we have a map

$$(H, A((G_\alpha)_{SL} \cap SL_1)) \in H(2) \times \frac{SL}{(G_\alpha)_{SL} \cap SL_1} \xrightarrow{\nu_2} (A, H)G_\alpha \in \frac{G}{\text{Ker} C_\alpha}.$$

If $\mathcal{L} = G_\alpha$ we have $S = SL_1 \cap SL_2 = (G_\alpha)_{SL} \cap SL_1$, and we thus obtain the map

$$(H, A((G_\alpha)_{SL} \cap SL_1)) \in H(2) \times \frac{SL}{(G_\alpha)_{SL} \cap SL_1} \xrightarrow{\nu_2} (A, H)G_\alpha \in \frac{G}{G_\alpha}.$$

When $\mathcal{L} = \text{Ker} \tilde{C}_\alpha$ we have $S = \text{Ker}(C_\alpha)_{SL}$, so that we have a map

$$(H, AKer(C_\alpha)_{SL}) \in H(2) \times \frac{SL}{Ker(C_\alpha)_{SL}} \xrightarrow{\nu_1} (A, H)Ker \tilde{C}_\alpha \in \frac{G}{\text{Ker} C_\alpha}.$$

If $\mathcal{L} = (G_\alpha)_{SL} \oplus H(2)$ we have $S = (G_\alpha)_{SL}$, and we thus have a map

$$(H, A(G_\alpha)_{SL}) \in H(2) \times \frac{SL}{(G_\alpha)_{SL}} \xrightarrow{\nu_4} (A, H)((G_\alpha)_{SL} \oplus H(2)) \in \frac{G}{(G_\alpha)_{SL} \oplus H(2)}$$

which, using the identification (27), can be written

$$(H, A(G_\alpha)_{SL}) \in H(2) \times \frac{SL}{(G_\alpha)_{SL}} \xrightarrow{\nu_4}$$
Figure 2: Fibre Bundles for Wave Functions

\[ \nu_1 : A(G_\alpha)_{SL} \subseteq SL_{(G_\alpha)SL}. \]

When we denote by \( \iota_3, \iota_4, \tau_3, \tau_4 \), the canonical maps of homogeneous spaces that appears in Figure 2, we obtain the commutative diagram in that Figure.

The maps \( \tau_i \) are the bundle maps of principal fibre bundles whose structural groups are identified by means of \( (C_\alpha)_{SL}, \tilde{C_\alpha} \), or \( C_\alpha \) to subgroups of \( S^1 \).

We already know the bundle actions of \( S^1 \) on \( G/KerC_\alpha \) and \( G/Ker\tilde{C_\alpha} \). The other are defined in a similar way as follows.

The action for the principal bundle corresponding to \( \tau_4 \),

\[ H(2) \times \frac{SL}{Ker(C_\alpha)_{SL}} \left( H(2) \times \frac{SL}{(G_\alpha)SL}, (C_\alpha)_{SL((G_\alpha)SL)} \right), \]

is given by

\[ (H, AKer(C_\alpha)_{SL}) * s = (H, ABKer(C_\alpha)_{SL}) \]

if \( s \in (C_\alpha)_{SL((G_\alpha)SL)} \) and \( B \in (G_\alpha)_{SL} \) is such that

\[ (C_\alpha)_{SL}(B) = s. \]
In the case of the principal bundle corresponding to $\tau_3$,

$$H(2) \times \frac{SL}{\text{Ker}(C_\alpha)_{SL} \cap SL_1} \left( H(2) \times \frac{SL}{(G_\alpha)_{SL} \cap SL_1} , (C_\alpha)_{SL}((G_\alpha)_{SL} \cap SL_1) \right),$$

the bundle action is defined in the same way but using the restriction of $(C_\alpha)_{SL}$ to $(G_\alpha)_{SL} \cap SL_1$.

The pairs $(\nu_1, \nu_2), (\nu_3, \nu_4), (\iota_1, \iota_2), (\iota_3, \iota_4)$ define homomorphisms of principal fibre bundles. In what concerns the structural groups, we have

$$(C_\alpha)_{SL}((G_\alpha)_{SL} \cap SL_1) \subset (C_\alpha)_{SL}((G_\alpha)_{SL}) \subset S^1$$

and the homomorphism of structural groups is, in all cases, the canonical injection of the group of the first bundle into the group of the second.

As a consequence of (8), the linear momentum, is given on the coadjoint orbit of $\alpha$ by $P(\text{Ad}^*_A H)\alpha = -AkA^*$, so that, with the identification of the coadjoint orbit with $G/G_\alpha$, we can write $P((A, H)G_\alpha) = -AkA^*$.

On the other hand, since $(G_\alpha)_{SL} \subset SL_2$, we can define in $SL/(G_\alpha)_{SL}$ a function, $P_0$, by means of $P_0(A(G_\alpha)_{SL}) = -AkA^*$.

But then, $P_0$ is the projection of $P$ by the canonical map

$$(A, H)G_\alpha \in \frac{G}{G_\alpha} \overset{i_0}{\rightarrow} A(G_\alpha)_{SL} \in \frac{SL}{(G_\alpha)_{SL}},$$

and will be denoted in the following simply by $P$ and also called linear momentum, thus we can write

$$P(A(G_\alpha)_{SL}) = -AkA^*$$

(29)

where $k$ is given by $\alpha = \{a, k\}$.

Let us denote $(C_\alpha)_{SL}(G_\alpha)_{SL}$ by $S$.

In [Dia96b] I prove that
The pull back by $\nu_3$, maps in a one to one way the set of quantum states (considered as pseudotensorial functions on $G/KerC_\alpha$ onto the set composed by the functions on $H(2) \times (SL/Ker(C_\alpha)_{SL}$ having the form

$$\phi_f(H, AKer(C_\alpha)_{SL}) = f(A Ker(C_\alpha)_{SL})e^{i\pi Tr(P(A(G_\alpha)_{SL})\epsilon \overline{\mu}_\epsilon)}$$  \hspace{1cm} (30)$$

where $f$ is a pseudotensorial function on the principal fibre bundle $SL \frac{SL}{Ker(C_\alpha)_{SL}} \left( \frac{SL}{(G_\alpha)_{SL}}, S \right)$. 

In order to be more precise we will use the following definitions.

Let $C$ be the complex vector space composed by the pseudotensorial functions of the bundle $SL \frac{SL}{Ker(C_\alpha)_{SL}} \left( \frac{SL}{(G_\alpha)_{SL}}, S \right)$, and $\tilde{V}$ the complex vector space composed by the pseudotensorial functions on $G/KerC_\alpha$.

For each $f \in C$ we define a pseudotensorial function on $G/KerC_\alpha$, $\Psi_f$, by

$$\Psi_f \circ \nu_3 (H, AKer(C_\alpha)_{SL}) = \phi_f (H, AKer(C_\alpha)_{SL}) = f(A Ker(C_\alpha)_{SL})e^{i\pi Tr(P(A(G_\alpha)_{SL})\epsilon \overline{\mu}_\epsilon)}.$$ \hspace{1cm} (31)$$

The map

$$\Psi : f \in C \rightarrow \Psi_f \in \tilde{V}$$

is an isomorphism.

Also we define

$$\Phi_f = \Psi_f \circ \iota_1.$$ \hspace{1cm} (32)$$

These functions are pseudotensorial on $G/KerC_\alpha$ and invariant by the horizontal action, so that they represent Quantum States in the most primitive sense adopted in this paper.
We denote by $\mathcal{V}$ the complex vector space composed by these $\Phi_f$, so that the map

$$\Phi : f \in \mathcal{C} \to \Phi_f \in \mathcal{V}$$

is an isomorphism.

Thus, we can consider Quantum States as being elements of $\mathcal{C}$ or elements of $\mathcal{V}$ or elements of $\tilde{\mathcal{V}}$.

Now we shall regard Quantum States under another form: Wave Functions.

The differentiable manifold

$$W \overset{\text{def}}{=} (H(2) \times (SL/Ker (C_\alpha)_{SL}) \times_S \mathbb{C})$$

is defined in the same way as we have defined

$$(G/Ker C_\alpha) \times S^1 \mathbb{C}$$

in section 5.

$W$ is the total space of the hermitian line bundle associated to the principal fibre bundle

$$(H(2) \times (SL/Ker (C_\alpha)_{SL})) (H(2) \times (SL/ (G_\alpha)_{SL}) , S)$$

and the canonical action of $S$ on $\mathbb{C}$.

The bundle projection is

$$\eta : [(H, AKer (C_\alpha)_{SL}), c]_S \in W \to (H, A(G_\alpha)_{SL}) \in H(2) \times (SL/ (G_\alpha)_{SL}),$$

where $[(H, AKer (C_\alpha)_{SL}), c]_S$ is the orbit of $((H, AKer (C_\alpha)_{SL}), c)$ under the action of $S$.

Since for each $f \in \mathcal{C}$ the function $\Psi_f \circ \nu_3$ is pseudotensorial in

$$H(2) \times (SL/Ker (C_\alpha)_{SL}),$$

it defines a section of $\eta$. This section can be considered as another description of the Quantum State given by $f$.

To complete our way towards Wave Functions, we need to “represent” Quantum States as functions with values in a fixed complex vector space,
not in a different vector space for each point in the base space, as does the sections of \( \eta \).

If \((C_\alpha)_{SL}(g) = 1, \forall g \in (G_\alpha)_{SL}\), we have \( S = \{1\} \), and \( Ker (C_\alpha)_{SL} = (G_\alpha)_{SL} \) so that \( W \approx (H(2) \times (SL/(G_\alpha)_{SL}) \times C \), and \( \eta \) is the canonical projection onto the first two factors. The section of \( \eta \) corresponding to the function \( \phi \) in (30) is

\[
(H, A(G_\alpha)_{SL}) \rightarrow (H, A(G_\alpha)_{SL}, \phi(H, A(G_\alpha)_{SL})).
\]

Thus, if \((C_\alpha)_{SL} = 1\), our task is accomplished by

\[
\phi_f(H, A Ker (C_\alpha)_{SL}) = f(A Ker (C_\alpha)_{SL}) e^{i \pi Tr(P(A(G_\alpha)_{SL}) \epsilon \Pi \epsilon}
\]

itself as a complex valued function on the base space. In this case, \( \phi \) is called the Prewave Function associated to \( f \), and denoted by \( \psi_f \).

In the case where \((C_\alpha)_{SL} \) is not trivial, our goal will be attained by imbedding the hermitian fibre bundle in a trivial one. We do this in a direct way, but a more geometrical view of the method is exposed in remark 5.1 of [Dia96b].

A key concept in our construction of Wave Functions is the following:

A Trivialization of \( C_\alpha \) is a triple \((\rho, L, z_0)\), where \( L \) is a finite dimensional complex vector space, \( z_0 \in L \) and \( \rho \) is a representation of \( SL(2,C) \) in \( L \) such that

1) \( \rho(A)(z_0) = (C_\alpha)_{SL}(A) z_0, \forall A \in (G_\alpha)_{SL} \).

2) The isotropy subgroup at \( z_0 \) is \( Ker (C_\alpha)_{SL} \).

(34)

In what follows, we assume that a trivialization of \( C_\alpha \) is given.

The homogeneous space \( SL/Ker (C_\alpha)_{SL} \) is identified to the orbit of \( z_0 \), \( \mathcal{B} \), by means of

\[
A Ker (C_\alpha)_{SL} \in \frac{SL}{Ker (C_\alpha)_{SL}} \rightarrow \rho(A)(z_0) \in \mathcal{B}.
\]

The action of \( S \) on \( SL/Ker (C_\alpha)_{SL} \) becomes, with this identification, multiplication in \( L \) of elements of \( S \), as complex numbers, by elements of \( \mathcal{B} \), as elements of \( L \).
The canonical map from $SL/Ker(C_\alpha)_{SL}$ onto $SL/(G_\alpha)_{SL}$ will be denoted by $r$, and is given by

$$r(\rho(A)(z_0)) = A(G_\alpha)_{SL}.$$ 

Each $f \in D$ thus becomes a function on $B$, homogeneous of degree $-1$ under ordinary multiplication by elements of $S$. The functions having these characteristics, will be called **$S$-homogeneous of degree $-1$**. The $S$-homogeneous of degree $T$ functions are defined in a similar way. Let us denote by $C$ the complex vector space composed by the $S$-homogeneous of degree $-1$ functions on $B$.

Now, $W$ is $(H(2) \times B) \times_S C$, and $\eta$ maps $[(H, z), c]$ onto $(H, r(z))$.

The sections of $\eta$ corresponding to the functions having the form of $\phi_f$ in (30) are as follows.

Let $f$ be a $S$-homogeneous of degree $-1$ function. The corresponding section, $\sigma$, maps $(H, m)$ to

$$\sigma(H, m) = [(H, z), \phi_f(H, z)]$$

where $z$ is arbitrary in $r^{-1}(m)$.

We define a map, $\chi$, from $W$ into $H(2) \times (SL/(G_\alpha)_{SL}) \times L$, by sending $[(H, z), c]_S \in (H(2) \times B) \times_S C$ to $(H, r(z), cz)$.

The map $\chi$ is injective. In fact the relation

$$\chi([(H, z), c]_S) = \chi([(H', z'), c']_S)$$

is equivalent to

$$(H, r(z), cz) = (H', r(z'), c'z')$$

so that there exist $e^{i\gamma} \in S$ such that $H' = H$, $z' = e^{i\gamma}z$, and $c'z' = cz$. Thus

$$[(H', z'), c']_S = [(H, ze^{i\gamma}), c e^{-i\gamma}]_S = [(H, z), c]_S.$$ 

The fiber of $W$ on $(H, m)$ is $\{[(H, z), c] : c \in C\}$, where $z$ is any fixed element in $r^{-1}(m)$. Its image under $\chi$ is composed by the $(H, m, y)$ such that $y$ is in the one dimensional subspace of $L$ generated by $z$. 

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We have thus immersed our, in general, non trivial bundle, in a trivial one with fiber $L$. This enable us to identify sections of $\eta$ with functions with values in $L$, as follows.

The section $\sigma$ of $\eta$, that corresponds to $f$ can be identified with

$$\chi \circ \sigma(H, m) = (H, m, \phi_f(z)), \tag{33}$$

where $z$ is arbitrary in $r^{-1}(m)$.

The right hand side in the preceding equation is completely determined by its third component:

$$\psi_f(H, m) = \phi_f(z)z = f(z) e^{i\pi \text{Tr}(P(m)e\Pi e)}z. \tag{35}$$

The function $\psi_f$, with $f$ $S$-homogeneous of degree -1, will be called Prewave Function associated to $f$.

The complex vector space composed by the Prewave Functions is denoted by $PW$, so that the map

$$\psi : f \in C \rightarrow \psi_f \in PW \tag{36}$$

is an isomorphism.

The composition of any Prewave Function with $\iota_4$ is a function in State Space that can be considered as giving an amplitude of probability for each state.

As said in the Introduction, we obtain from this Prewave Function a Wave Function, defined on space-time points (i.e. hermitian matrices), by adding up, for each $H \in H(2)$, the amplitudes of probability corresponding to the states $(H, \beta)$ where $\beta$ is a movement whose portrait in space-time contains $H$.

According to (2), we see that these states are the elements of the set

$$\{(H, \text{Ad}_{(B,H)}^\alpha : B \in SL\},$$

that is identified by $i$ (c.f. (28)) with

$$\{H\} \times \frac{SL}{(G_0)_{SL} \cap SL_1}.$$
whose image by $\iota_4$ is
\[{H} \times \frac{SL}{(G_\alpha)_{SL}}.\]

Then to any given Prewave Function, $\psi_f$, we associate a Wave Function, $\tilde{\psi}_f$, as follows
\[
\tilde{\psi}_f(H) = \int_{SL/(G_\alpha)_{SL}} \psi_f(H, m) \, \omega_m
\]
where $m$ is a generic element in $SL/(G_\alpha)_{SL}$, and $\omega$ is a volume element on $SL/(G_\alpha)_{SL}$, left invariant by the canonical action of $SL$ on $SL/(G_\alpha)_{SL}$. This means that, if we denote by $d_A'$ the diffeomorphism of $SL/(G_\alpha)_{SL}$ defined by sending $B(G_\alpha)_{SL}$ to $AB(G_\alpha)_{SL}$, then $(d_A')^* \omega = \omega$.

In the following, we assume that such an invariant volume element is given.

This definition of Wave Functions, forces us to do a restriction on the class of the functions to be considered: it is necessary that the integral exists.

In what follows I consider only Quantum States corresponding to the functions in $\mathcal{C}$ that are continuous with compact support. The complex vector space composed by these Wave Functions is denoted by $WF$.

Of course, there are other possible conditions that can be imposed on $f$ in order to assure integrability. Also, if one obtains for some choice a Prehilbert space, one can consider its completed, in order to have a Hilbert Space. But in this paper I prefer to maintain the “continuous with compact support” condition.

The Wave Functions are another form of description of Quantum States, in fact it is the most usual in Quantum Mechanics.

In the following I change the notation in such a way that, $\mathcal{C}$ stands for the complex vector space composed by the $S$-homogeneous of degree -1 functions on $\mathcal{B}$ that also are continuous with compact support, and $\mathcal{PW}$, $\mathcal{V}$, $\mathcal{\tilde{V}}$, the corresponding isomorphic spaces.

Remark 7.1 Now, let us assume that we know a section of the map $r$,
defined an open set, $D$, in $SL/(G\alpha)_{SL}$,

$$\sigma : D \rightarrow \mathcal{B}.$$ 

In this case, we can associate a prewave function, and thus a wave function, to each $C^\infty$ function, with compact support contained in $D$, as follows.

Let $f_0$ be a $C^\infty$ function, with compact support contained in $D$. We define a function on $\mathcal{B}$, $f$, by means of

$$f(z) = \begin{cases} t^{-1}f_0(r(z)) & \text{if } z \in r^{-1}(U) \text{ where } t \text{ is such that } z = t\sigma(r(z)) \\ 0 & \text{if } z \notin r^{-1}(U) \end{cases}$$

This function is $S$-homogeneous of degree -1 since, for all $e^{ia} \in S$, $z \in r^{-1}(U)$, we have

$$f(e^{ia}z) = t'^{-1}f_0(r(e^{ia}z)),$$

where $t'$ is such that

$$e^{ia} = t'\sigma(r(e^{ia}z)) = t'\sigma(r(z)) = t't^{-1}z,$$

so that

$$e^{ia} = t't^{-1}$$

and

$$f(e^{ia}z) = e^{-ia}t^{-1}f_0(r(e^{ia}z)) = e^{-ia}t^{-1}f_0(r(z)) = e^{-ia}f(z).$$

If $z \notin r^{-1}(U)$, $f(e^{ia}z) = 0 = f(z) = e^{-ia}f(z)$.

The Prewave Function $\psi_f$ then is

$$\psi_f(H, m) = \begin{cases} f_0(m)e^{i\pi Tr(P(m)\varepsilon\Pi\varepsilon)}\sigma(m) & \text{if } m \in U \\ 0 & \text{if } m \notin U \end{cases}$$

and we have an associated Wave Function, $\tilde{\psi}_f$.

This expression contains no reference to any homogeneous function and suggest directly the usual form of Wave Functions.
8 Representation of $SL(2, \mathbb{C}) \oplus H(2)$ on Quantum States

8.0.1 Representation on $S$-homogeneous functions of degree -1

If $(C_\alpha)_{SL} = 1$, the $S$-homogeneous functions of degree -1 are simply functions on $SL/(G_\alpha)_{SL}$.

In this case we denote by $\mathcal{C}$ the complex vector space composed by the continuous functions on $SL/(G_\alpha)_{SL}$ with compact support.

For all $B \in SL$ we denote by $d_B'$ the canonical diffeomorphism of $SL/(G_\alpha)_{SL}$ given by

$$d_B'(A(G_\alpha)_{SL}) = BA(G_\alpha)_{SL}.$$ 

Then, we define a representation, $\delta$, of $SL$ on $\mathcal{C}$ by

$$\delta(f) = f \circ d_A'^{-1}.$$ 

In $\mathcal{C}$ we also define an hermitian product by

$$\langle f, f' \rangle = \int_{SL/(G_\alpha)_{SL}} f \overline{f'} \omega,$$

where $\omega$ is an invariant volume element on $SL/(G_\alpha)_{SL}$.

With this inner product, $\mathcal{C}$ becomes a prehilbert space.

The invariance of $\omega$ enable us to write

$$\langle \delta(A)(f), \delta(A)(f') \rangle = \langle f, f' \rangle$$

so that the representation $\delta$ is unitary.

In case $(C_\alpha)_{SL} \neq 1$, we assume that we have a trivialization, $(\rho, L, z_0)$, and we define $\mathcal{B}$ and $\mathcal{C}$ as in the preceding section.

Of course, also in case $(C_\alpha)_{SL} = 1$, we can have a trivialization and thus apply all that follows.
In $C$ we define a hermitian product:

$$\langle f, f' \rangle = \int_{SL/(G_\alpha)_{SL}} \overline{f} f' \omega,$$

where by $\overline{f} f'$ we means the function defined on $SL/(G_\alpha)_{SL}$, by

$$\overline{f} f'(m) = f(z) f'(z)$$

for all $m \in SL/(G_\alpha)_{SL}$, where $z$ is arbitrary in $r^{-1}(m)$, and $\omega$ is an invariant volume element on $SL/(G_\alpha)_{SL}$.

Also in this case, $C$ becomes a prehilbert space.

A representation, $\delta$, of $SL$ on $C$, is defined by

$$\delta(A) : f \in C \rightarrow f \circ \rho(A^{-1}) \in C.$$

for all $A \in SL$.

If $A \in SL$ we have

$$\langle \delta(A)(f), \delta(A)(f') \rangle = \int_{SL/(G_\alpha)_{SL}} (\overline{f} f' \circ d_A^{-1}) \omega$$

so that the invariance of $\omega$ enable us to write

$$\langle \delta(A)(f), \delta(A)(f') \rangle = \langle f, f' \rangle$$

We see that the representation $\delta$ is, also in this case, unitary.

Also we have a representation, $\delta'$, of $SL \oplus H(2)$ on $C$ defined by:

$$(\delta'(A, H) \cdot f)(z) = f(\rho(A^{-1}) \cdot z) Exp \left(-i \pi Tr \left( P(r(z)) \varepsilon \overline{H} \varepsilon \right) \right),$$

where $(A, H) \in SL \oplus H(2)$, $f \in C$, and $z \in B$.

To prove that $\delta'$ is a representation, notice that

1. $r$ is equivariant $i.e.$

$$r(\rho(A) \cdot z) = d_A'(r(z))$$

for all $A \in SL$. 

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2. Formula (29), leads to
\[ P(d_B(A(G_\alpha)_{SL})) = P(BA(G_\alpha)_{SL}) = -BAkA^*B^* = BP(A(G_\alpha)_{SL})B^* \]
so that
\[ P(r(\rho(A) \cdot z)) = P(d'_A(r(z))) = AP(r(z))A^*. \] (39)

3. For all \( A \in SL \),
\[ A \varepsilon = \varepsilon^t A^{-1} \] (40)

Now, we can prove that \( \delta' \) is a representation as follows.

If \((B, K), (A, H) \in SL \oplus H(2), z \in B\), then
\[
\delta'(B, K) \cdot ((\delta'(A, H) \cdot f))(z) = (\delta'(A, H) \cdot f)(\rho(B^{-1}) \cdot z)
\]
\[
\text{Exp} \left( -i\pi \text{Tr}[P(r(z))\varepsilon K \varepsilon] \right) = f(\rho(A^{-1}) \cdot (\rho(B^{-1}) \cdot z))
\]
\[
\text{Exp} \left( -i\pi \text{Tr}[P(r(\rho(B^{-1}) \cdot z))\varepsilon H \varepsilon + P(r(z))\varepsilon K \varepsilon] \right) =
\]
\[
= f(\rho((BA)^{-1}) \cdot z) \text{Exp} \left( -i\pi \text{Tr}[P(r(z))\varepsilon (BHB^* + K) \varepsilon] \right) =
\]
\[
= (\delta' ((B, K) (A, H)) \cdot f)(z)
\]

The infinitesimal generator of \( \delta \) associated to \( a \in sl(2, \mathbb{C}) \) is the linear map from \( \mathcal{C} \) into itself, \( d \delta(a) \) given by
\[
(d \delta(a) \cdot f)(z) = \left( \frac{d}{dt} \right)_{t=0} \left( \delta (e^{ta}) \cdot f \right)(z).
\]

Then
\[
(d \delta(a) \cdot f)(z) = \left( \frac{d}{dt} \right)_{t=0} \left( f \left( \rho \left( e^{-ta} \right) \cdot z \right) \right),
\]
and thus we see that \( d \delta(a) \) acts on \( f \) as the vector field, \( X_a \), infinitesimal generator of the action on \( B \), defined by
\[ A \ast z = \rho(A) \cdot z, \]
associated to \( a \).

We can give a more explicit form of \( X_a \) as follows.

Let us fix a basis, \( \beta = \{e_1, \ldots, e_q\} \), of \( L \). By means of \( \beta \) we identify \( L \) with \( \mathbb{C}^q \). Let us denote by \( \{z^1, \ldots, z^q\} \) the matrix of \( z \in L \) in the basis.
\(\beta\) and \((d\rho(a))_i^j\), the element in the file \(j\) column \(i\) of the matrix of \((d\rho(a))\) in the basis \(\beta\). We denote by \(\beta^* = \{w^1, \ldots, w^q\}\) the system of (complex) coordinates associated to \(\beta\) (i.e. \(\beta^*\) is the dual basis of \(\beta\)).

If \(w^j_x\) (resp. \(w^j_y\)) is the real (resp. imaginary) part of \(w^j\), it is usually written
\[
\frac{\partial}{\partial w^j} = \frac{1}{2} \left( \frac{\partial}{\partial w^j_x} - i \frac{\partial}{\partial w^j_y} \right), \\
\frac{\partial}{\partial \overline{w}^j} = \frac{1}{2} \left( \frac{\partial}{\partial w^j_x} + i \frac{\partial}{\partial w^j_y} \right).
\]

Thus, when \(f\) is a complex valued differentiable function on an open subset, \(M\), of \(\mathbb{C}^q\), and \(v(t)\) a differentiable map from an open neighbourhood of 0 in \(\mathbb{R}\) into \(M\), we have, using summation convention
\[
\left(\frac{d}{dt}\right)_0 (f \circ v) = \left(\frac{\partial f}{\partial w^j} v(0)\right)'(0) + \left(\frac{\partial f}{\partial \overline{w}^j} v(0)\right)'(0) = \left(\frac{\partial f}{\partial w^j} v(0)\right)'(0) + \left(\frac{\partial f}{\partial \overline{w}^j} v(0)\right)'(0).
\]

Since
\[
\left(\frac{d}{dt}\right)_{t=0} (f (\rho(e^{-t}a) \cdot z)) = \left(\frac{d}{dt}\right)_{t=0} (f ((e^{-t}d\rho(a)) \cdot z))
\]
it follows that
\[
(X_a)_z(f) = \left(\frac{\partial f}{\partial w^j} \right)_z ((-d\rho(a))_i^j z^i) + \left(\frac{\partial f}{\partial \overline{w}^j} \right)_z ((-\overline{d\rho(a)})_i^j \overline{z}^i),
\]
so that an extension of \(X_a\) from \(B\) to \(\mathbb{C}^q\) is
\[
X_a = -\left( w^i (d\rho(a))_i^j \frac{\partial}{\partial w^j} + \overline{w}^i (\overline{d\rho(a)})_i^j \frac{\partial}{\partial \overline{w}^j} \right). \quad (41)
\]

If \((C_a)_{SL} = 1\), and do not use a trivialization, we have similar results, but \(X_a\) is the infinitesimal generator of the canonical action of \(SL\) on \(SL/(G_a)_{SL}\), associated to \(a\).
Thus, in all cases
\[ d\delta(a) \cdot f = X_a(f), \]
or simply, as linear maps on \( C \),
\[ d\delta(a) = X_a. \]

On the other hand, the infinitesimal generator of \( \delta' \) associated to
\( (a, h) \in sl(2, \mathbb{C}) \oplus H(2) \)
is the endomorphism, \( d\delta'(a, h) \) of \( C \) given by
\[ (d\delta'(a, h) \cdot f)(z) = \left( \frac{d}{dt} \right)_{t=0} (\delta' (\text{Exp}(t(a, h)))) \cdot f) (z), \]
but the exponential map in \( SL \oplus H(2) \) is given by (6), so that
\[ (d\delta'(a, h) \cdot f)(z) = \left( \frac{d}{dt} \right)_{t=0} (\delta' \left( \left( e^{ta}, \int_0^t e^{sa} h e^{sa^*} ds \right) \right)) \cdot f)(z), \]
and a short computation thus leads to
\[ d\delta'(a, h) \cdot f = X_a(f) - i\pi \text{Tr} [P(\cdot) \bar{e} \bar{e}] f = X_a(f) + 2\pi i \langle P(\cdot), h \rangle f, \]
where I have used the same symbol for any hermitian matrix and the corresponding element in \( \mathbb{R}^4 \), and \( \langle , \rangle \) is Minkowski product.

Also, we can write
\[ d\delta'(a, h) = X_a + 2\pi i \langle P(\cdot), h \rangle. \tag{42} \]
where \( X_a \) is the endomorphism given by the vectorfield (41), and
\[ 2\pi i \langle P(\cdot), h \rangle \]
is the endomorphism given by ordinary multiplication by this function.

Instead of the infinitesimal generators \( d\delta'(a, h) \), we can use
\[ (a, h)^\theta \overset{\text{def}}{=} \frac{1}{2\pi i} d\delta'(a, h), \tag{43} \]
so that
\[(a, h)^\theta = \frac{1}{2\pi i} X_a + \langle P(r(\cdot)), h \rangle. \quad (44)\]

The endomorphism \((a, h)^\theta\), is hermitian for our hermitian product and is the **Quantum Operator associated to the Dynamical Variable** \((a, h)\), when Quantum States are represented by \(S\)-homogeneous functions.

When Quantum States are represented by Wave Functions, the Quantum Operators acquires its usual form in Quantum Mechanics (c.f. (53)).

In the particular case of the Linear Momentum (c.f. (7)), one sees that
\[
(P^k)^\theta \cdot f = (P^k \circ r) f, \quad k = 1, 2, 3, 4. \quad (45)
\]
where the \(P^k\) in the left hand side are the components of Linear momentum as a dynamical variable, and the ones in the right hand side, the components of Linear momentum considered as a function on \(SL/(G_a)_{SL}\), given by (29).

In what concerns to the Quantum Operators corresponding to Angular Momentum we have
\[
(l_j)^\theta = \frac{1}{4\pi i} X_{i\sigma_j}, \quad j = 1, 2, 3. \quad (46)
\]
\[
(g_j)^\theta = \frac{1}{4\pi i} X_{\sigma j}, \quad j = 1, 2, 3. \quad (47)
\]

Of course, the representations on \(C\) lead to equivalent representations on the isomorphic vector spaces \(\mathcal{V}, \tilde{\mathcal{V}}, \text{ and } \mathcal{PW}\).

We do that in detail in \(\mathcal{PW}\) and \(\mathcal{V}\) as follows:

### 8.0.2 Representation on Prewave Functions.

In \(\mathcal{PW}\) we define an hermitian product by
\[
\langle \psi_f, \psi_{f'} \rangle \overset{def}{=} \langle f, f' \rangle,
\]
and thus becomes a prehilbert space, isomorphic as inner product spaces to \(C\).
The hermitian product can be given in terms of the prewave functions themselves, as follows.

Let $\beta$ be a sesquilinear form on $L$, nonvanishing on $B$. We define

$$
\psi_f \beta \psi_{f'} : m \in SL/(G_\alpha)_{SL} \mapsto \frac{\beta(\psi_f(H, m), \psi_{f'}(H, m))}{\beta(z, z)}
$$

where $z$ is arbitrary in $r^{-1}(m)$ and $H$ is arbitrary in $H(2)$. Thus

$$
\langle \psi_f, \psi_{f'} \rangle = \int_{SL/(G_\alpha)_{SL}} \psi_f \beta \psi_{f'} \omega.
$$

The representation of $G$ on $\mathcal{PW}$ equivalent to $\delta'$ under $\psi$ is given by

$$
\delta_{\mathcal{PW}}^\psi(A, H) \cdot \psi_f = \psi_{\delta'(A, H)'f}
$$

The infinitesimal generator associated to $(a, h) \in \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{H}(2)$ is

$$
d\delta_{\mathcal{PW}}(a, h) \cdot \psi_f = \psi_{d\delta'(a, h)'f}.
$$

The Quantum Operators for Prewave Functions must be defined as

$$
(a, h)^\psi \overset{def}{=} \frac{1}{2\pi i} d\delta_{\mathcal{PW}}^\psi(a, h),
$$

and we have

$$
(a, h)^\psi \cdot \psi_f = \psi_{(a, h)^\psi'f}.
$$

On the other hand, we have a natural action on $H(2) \times SL/(G_\alpha)_{SL}$ defined by

$$
(B, K) \ast (H, A(G_\alpha)_{SL}) = (BHB^* + K, BA(G_\alpha)_{SL}).
$$

Thus we define an, a priori different, representation, $\delta_{\mathcal{PW}}$, on $\mathcal{PW}$ by means of

$$
\delta_{\mathcal{PW}}(A, H) \cdot \psi_f = \rho(A) \circ \psi_f \circ ((A, H)^*)^{-1},
$$

i.e., for all $(K, m) \in H(2) \times SL/(G_\alpha)_{SL}$

$$
(\delta_{\mathcal{PW}}(A, H) \cdot \psi_f)(K, m) = \rho(A) \cdot \psi_f((A, H)^{-1} \ast (K, m)).
$$

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Using (39) and (40) one can see that

\[ \delta_{pw} = \delta^{pw} \]  

as follows

\[
(\delta^{pw}(A, H) \cdot \psi_f)(K, m) = \psi_{g'(A,H),f}(K, m) = f(\rho(A^{-1}) \cdot z) \\
\text{Exp} \left(-i\pi \text{Tr}[P(m)\varepsilon \overline{H} \varepsilon]\right) \text{Exp} \left(i\pi \text{Tr}[P(m)\varepsilon \overline{K} \varepsilon]\right) z = (*)
\]

where \( z \in r^{-1}(m) \). If we denote \( z' = \rho(A^{-1}) \cdot z \), we have

\[
(*) = f(z') \text{Exp} \left(i\pi \text{Tr}[P(r(\rho(A) \cdot z'))\varepsilon (K-H)\varepsilon]\right) (\rho(A) \cdot z') = \\
f(z') \text{Exp} \left(i\pi \text{Tr}[P(r(z'))\varepsilon (A^{-1}(K-H)(A^{-1})^*)\varepsilon]\right) (\rho(A) \cdot z') = \\
\rho(A)\psi_f((A,H)^{-1} * (K, m)) = (\delta_{pw}(A, H) \cdot \psi_f)(K, m).
\]

8.0.3 Representation on Wave Functions.

Now, let us consider \( \mathcal{WF} \), the complex vector space composed by the Wave Functions \( \tilde{\psi}_f \) such that \( f \in \mathcal{C} \).

A ”translation” of the preceeding representations of \( SL \oplus H(2) \) to \( \mathcal{WF} \), is given by

\[
\delta_w(A, H) \cdot \tilde{\psi}_f = \{\delta_{pw}(A, H) \cdot \psi_f\} = \{\psi_{g'(A,H),f}\},
\]

where \( \{\psi\} \) means \( \tilde{\psi} \).

Then

\[
\delta_w(A, H) \cdot \tilde{\psi}_f(K) = \{\delta_{pw}(A, H) \cdot \psi_f\}(K) \\
= \int_{SL/(G_\alpha)_{SL}} \rho(A) \cdot \psi_f((A,H)^{-1} * (K, m)) \omega_m = \\
= \rho(A) \cdot \int_{SL/(G_\alpha)_{SL}} \psi_f((A,H)^{-1} * K, d'_{A^{-1}}m) \omega_m = \\
= \rho(A) \cdot \int_{SL/(G_\alpha)_{SL}} \psi_f((A,H)^{-1} * K, m) \omega_m
\]
because of the invariance of $\omega$. Thus
\[
\left( \delta_w(A, H) \cdot \tilde{\psi}_f \right)(K) = \rho(A) \cdot \tilde{\psi}_f \left( (A, H)^{-1} * K \right).
\] (52)

Compare to (50).

Let us denote by $d\delta_w \cdot (a, h)$ the infinitesimal generator of the representation $\delta_w$ associated to $(a, h) \in sl(2, C) \oplus H(2)$, and
\[
\widehat{(a, h)} \overset{def}{=} \frac{1}{2\pi i} (d\delta_w \cdot (a, h)).
\]

The endomorphism $\widehat{(a, h)}$ of $\mathcal{W} \mathcal{F}$ is the Quantum Operator corresponding to the Dynamical Variable $(a, h)$ when the Quantum States are represented by Wave Functions.

A straightforward computation leads to the following expressions for the operators corresponding to Linear and Angular Momentum

\[
\begin{align*}
\tilde{P}^k \cdot \tilde{\psi}_f &= \frac{1}{2\pi i} \frac{\partial}{\partial x^k} \tilde{\psi}_f \\
\tilde{P}^4 \cdot \tilde{\psi}_f &= i \frac{\partial}{\partial x^4} \tilde{\psi}_f \\
\tilde{l}^k \cdot \tilde{\psi}_f &= \frac{1}{2\pi i} \left( dp \left( \frac{i\sigma_k}{2} \right) + \sum_{j,r=1}^3 \epsilon_{kjr} x^j \frac{\partial}{\partial x^r} \right) \tilde{\psi}_f \\
\tilde{g}^k \cdot \tilde{\psi}_f &= \frac{1}{2\pi i} \left( dp \left( \frac{\sigma_k}{2} \right) - \left( x^4 \frac{\partial}{\partial x^k} + x^k \frac{\partial}{\partial x^4} \right) \right) \tilde{\psi}_f
\end{align*}
\] (53)

where $\epsilon_{ijk}$ are the components of an antisymmetric tensor such that $\epsilon_{123} = 1$.

8.0.4 Representation on Pseudotensorial Functions in the contact manifold.

Recall the isomorphism $\Phi$:
\[
\Phi : f \in \mathcal{C} \rightarrow \Phi f \in \mathcal{V}
\]
If \( f \in \mathcal{C} \), we have
\[
\Phi_f((A, H)Ker C_\alpha) = f(A Ker(C_\alpha)_{SL}) e^{i\pi Tr(P(A(G_\alpha)_{SL})\varepsilon \bar{\varepsilon})}.
\]

When \((C_\alpha)_{SL} = 1\), we have \(Ker(C_\alpha)_{SL} = (G_\alpha)_{SL}\), so that
\[
\Phi_f((A, H)Ker C_\alpha) = f(A(G_\alpha)_{SL}) e^{i\pi Tr(P(A(G_\alpha)_{SL})\varepsilon \bar{\varepsilon})}.
\]

In the case \((C_\alpha)_{SL} \neq 1\), we assume the existence of a trivialization, \((L, \rho, z_0)\), and we identify \(SL/Ker(C_\alpha)_{SL}\) with \(B\). Then the pseudotensorial function \(\Phi_f\) becomes
\[
\Phi_f ((A, H)Ker C_\alpha) = f(\rho(A) \cdot z_0) e^{i\pi Tr(P(\rho(B)\cdot z_0)\varepsilon \bar{\varepsilon})}.
\] (54)

A natural representation of \(G\) on \(V\) is the \(\delta^c\) given by
\[
\delta^c(A, H) \cdot \Phi_f = \Phi_f \circ ((A, H)^{-1} \cdot)
\] (55)
where \(\cdot\) is the canonical action on \(G/Ker C_\alpha\).

Let us prove that
\[
\delta^c(A, H) \cdot \Phi_f = \Phi_{\delta^c(A, H) \cdot f}
\]
(56)
i.e. that \(\Phi\) is equivariant for the representations \(\delta^c\) in \(\mathcal{C}\) and \(\delta^c\) in \(\mathcal{V}\).

In fact, we have
\[
(\delta^c(A, H) \cdot \Phi_f) ((B, K)Ker C_\alpha) = \Phi_f (A^{-1} B, A^{-1} (K - H)(A^{-1})^*) =
\]
\[
f (\rho(A^{-1}) \cdot (\rho(B) \cdot z_0)) e^{i\pi Tr(P(\rho(A^{-1}) \cdot (\rho(B) \cdot z_0))\varepsilon (A^{-1} (K - H)(A^{-1})^*)\bar{\varepsilon})} =
\]
\[
f (\rho(A^{-1}) \cdot (\rho(B) \cdot z_0)) e^{i\pi Tr(A^{-1} P(\rho(B) \cdot z_0)(A^{-1})^* \cdot (K - H)\varepsilon \bar{\varepsilon})} =
\]
\[
= (\delta^c(A, H) \cdot f) (\rho(B) \cdot z_0) e^{i\pi Tr(\rho(B) \cdot z_0)\varepsilon \bar{\varepsilon})
\]
\[
= \Phi_{\delta^c(A, H) \cdot f} ((B, K)Ker C_\alpha).
\]

As a particular consequence, for all \((a, h) \in G\), we have for the infinitesimal generators of \(\delta^c\)
\[
d\delta^c(a, h) \cdot \Phi_f = \Phi_{d\delta^c(a, h) \cdot f}.
\]

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Equation (55), tell us that the infinitesimal generator of the representation \( \delta^c \) associated to \((a, h) \in G\), acts on \( \Phi_f \) as the (vector field) infinitesimal generator of the action on \( G/\text{Ker}\,C_\alpha \), what will be denoted by \( X^c_{(a, h)} \).

If we denote by \((a, h)^c\), the Quantum Operator

\[
(a, h)^c \overset{\text{def}}{=} \frac{1}{2\pi i} d\delta^c(a, h)
\]

we have

\[
(a, h)^c \cdot \Phi_f = \Phi_{(a, h)^c \cdot f}.
\]

Thus, our results in subsection 8.0.1 on Quantum Operators in that case, gives us results on Quantum Operators in our present case.

**Part II**

**Explicit Construction**

**9 Quantizable forms in \( SL(2, \mathbb{C}) \oplus H(2) \).**

In [Dia96a], I give a classification of the coadjoint orbits of the group under consideration. The orbits are divided into 9 Types, and a canonical representative of each orbit is given, according with its type.

Let \( \alpha = \{a, k\} \) be a nonzero element of the dual of the Lie algebra of \( SL \oplus H(2) \), and denote by \( W \) and \( P \) the Pauli- Lubanski and Linear momentum respectively at \( \alpha \) (cf. section 4).

Figure 3 gives what type of coadjoint orbit is the one of \( \alpha \), according with the values of \( W \) and \( P \).

When one knows the type of \( \alpha \), Figure 4 gives us the canonical representative of the orbit of \( \alpha \).

The \( \mathbb{R} \)-quantizable orbits are all of the types 3, 6, 8, 9 and these of type 5 corresponding to the case \( |W| = 0 \).
\[ \begin{array}{|c|c|c|c|c|c|}
\hline
\text{Type} & P & W & P & W & \text{Det } a \\
\hline
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & \neq 0 \\
3 & 0 & <0 & \neq 0 & \neq 0 & \\
4 & 0 & 0 & \neq 0 & \neq 0 & \\
5 & >0 & \leq 0 & \neq 0 & \\
6 & <0 & 0 & \neq 0 & \neq 0 & \\
7 & <0 & >0 & \neq 0 & \neq 0 & \\
8 & <0 & <0 & \neq 0 & \neq 0 & \\
9 & <0 & 0 & \neq 0 & \neq 0 & \\
\hline
\end{array} \]

Figure 3: Types of coadjoint orbits

| Representative | Conditions |
|----------------|------------|
| 1 \( \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0 \right\} \) | \( Im\sqrt{-\text{Det } a} \in \mathbb{R}^+ \) or \( \sqrt{-\text{Det } a} \in \mathbb{R}^+ \) |
| 2 \( \left\{ \sqrt{-\text{Det } a} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right\} \) | \( \sqrt{-|W|} \in \mathbb{R}^+ \) |
| 3 \( \left\{ \sqrt{-|W|} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, -\text{sign}(\text{Tr}(P)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \) | \( \sqrt{|W|} \in \mathbb{R}^+ \) |
| 4 \( \left\{ \frac{i\pi}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\text{sign}(\text{Tr}(P)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \) | \( W = sP \) |
| 5 \( \left\{ \frac{i\pi}{2} \sqrt{-\frac{|W|}{|P|}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\text{sign}(\text{Tr}(P))\sqrt{|P|} I \right\} \) | \( \sqrt{-\frac{|W|}{|P|}} \in \mathbb{R}^+ \cup \{0\} \) |
| 6 \( \left\{ 0, \sqrt{-|P|} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \) | \( \sqrt{-|P|} \in \mathbb{R}^+ \) |
| 7 \( \left\{ \frac{i\pi}{2} \text{sign}(\text{Tr}(W))\sqrt{-\frac{|W|}{|P|}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sqrt{-|P|} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \) | \( \sqrt{-\frac{|W|}{|P|}} \in \mathbb{R}^+ \) |
| 8 \( \left\{ \sqrt{\frac{|W|}{|P^t|}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sqrt{-|P|} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \) | \( \sqrt{-\frac{|W|}{|P^t|}} \in \mathbb{R}^+ \) |
| 9 \( \left\{ \begin{pmatrix} i\eta & 0 \\ 0 & -i\eta \end{pmatrix}, \sqrt{-|P|} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \) | \( \eta = \pm 1 \) |

Figure 4: Canonical Representatives.
Quantizable but not \( \mathbb{R} \)-quantizable are the orbits whose canonical representatives are:

\[
\left\{ \frac{iT}{8\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right\} \quad \text{(type 2)},
\]

\[
\left\{ \frac{i\chi T}{8\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\text{sign}(\text{Tr}(P)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \quad \text{(type 4)},
\]

\[
\left\{ \frac{iT}{8\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\text{sign}(\text{Tr}(P)) \sqrt{|P|} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \quad \text{(type 5)},
\]

\[
\left\{ \frac{i\chi T}{8\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sqrt{-|P|} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \quad \text{(type 7)}.
\]

where, in all cases, \( T \in \mathbb{Z}^+, \chi = 1, -1 \).

The concrete particles we study in this paper are those corresponding to Type 5 (massive particles) and Type 4 (massless particles such that the Pauly-Lubanski fourvector is proportional to Impulsion-Energy). Results concerning other types of particles will be published elsewhere.

10 Guide to the explicit construction of Wave Functions.

Let \( \alpha \) be a quantizable element of \( \mathcal{G}^* \).

To determine the explicit form of the Wave Functions of the corresponding free particles, one can proceed as follows

1. Evaluate the isotropy subgroup at \( \alpha \), of the coadjoint representation, \( \mathcal{G}_\alpha \).

The coadjoint orbit of \( \alpha \), that is the main symplectic manifold associated to \( \alpha \), is identified to the homogeneous space \( \mathcal{G}/\mathcal{G}_\alpha \).
2. Find a surjective homomorphism, $C_\alpha$, from $G_\alpha$ onto the unit circle $\mathbb{S}^1$, whose differential is $\alpha$.

3. Evaluate $(G_\alpha)_{SL}$, that is composed by the $g \in SL$ such that $(g, h) \in G_\alpha$ for some $h \in H(2)$.

4. Determine $(C_\alpha)_{SL}$, defined in (25), and $S \equiv (C_\alpha)_{SL}((G_\alpha)_{SL})$.

5. Find an action of $SL(2, \mathbb{C})$ on a manifold, such that the isotropy subgroup at some point, $p$, is $(G_\alpha)_{SL}$.

Identify $SL/(G_\alpha)_{SL}$ with the orbit of $p$, $\mathcal{K}$.

6. Determine a volume element on $\mathcal{K}$, $\omega$, invariant under the action of $SL$.

7. If $S = \{1\}$, the Prewave Functions are the

$$\psi_f(H, K) = f(K) e^{i\pi Tr(P(K)eH)}$$

where $(H, K) \in H(2) \times K$, $f$ is a function on $K$, and $P$ is the dynamical variable linear momentum, on $K$, given by (29).

8. If $S \neq \{1\}$, find a Trivialization of $C_\alpha$ (c.f. (31)).

Let $\mathcal{B}$ the orbit of $z_0$, identified to $SL/Ker(C_\alpha)_{SL}$,

$$r: \mathcal{B} \to \mathcal{K}$$

the canonical map from $SL/Ker(C_\alpha)_{SL}$ onto $SL/(G_\alpha)_{SL}$.

The Prewave Functions are given by (c.f. (35))

$$\psi_f(H, K) = f(z) e^{i\pi Tr(P(K)eH)}z$$

where $(H, K) \in H(2) \times K$, $z \in r^{-1}(K)$, and $f$ is a function on $\mathcal{B}$ homogeneous of degree -1 under product by modulus one complex numbers.

9. The Prewave Functions corresponding to continuous with compact support $f$ define Wave Functions by means of (c.f. (37))

$$\tilde{\psi}_f(H) = \int_{\mathcal{K}} \psi_f(H, \cdot) \omega.$$
11 Massive particles.

11.1 Massive particles with $T = 0$.

11.1.1 Wave Functions. Klein-Gordon equation.

Let us consider a particle whose movement space is the coadjoint orbit of

$$\alpha_o = \{0, \eta m I\}, \ m \in \mathbb{R}^+, \eta = \pm 1.$$

This orbit is a $\mathbb{R}$-quantizable orbit of the type 5, in the notation of section 9.

The number $(-\eta)$ is the sign of energy i.e. the sign of the value of the dynamical variable $P^4$ (c.f. section 4) at any point of the orbit. According with the usual interpretation of the determinant of momentum-energy, $|P|$, as mass square, the number $m$ must be interpreted as being the mass of the particle.

By direct computation, one sees that

$$G_{\alpha_o} = \{ (A, hI) : \ A \in SU(2), \ h \in \mathbb{R} \}.$$  

The unique homomorphism onto $\mathbb{R}$ whose differential is $\alpha_o$ is given by

$$C'_{\alpha_o}(A, hI) = -\eta mh.$$  

The unique homomorphism onto $\mathbb{S}^1$ whose differential is $\alpha_o$ is given by

$$C_{\alpha_o}(A, hI) = e^{-2\pi i \eta mh}.$$
Then we have

\[ \text{Ker} C'_{\alpha_0} = \{(A, 0) : A \in SU(2) \} = SU(2) \oplus \{0\}, \]

\[ \text{Ker} C_{\alpha_0} = \{(A, \frac{\eta N}{m} I) : A \in SU(2), \ N \in \mathbb{Z}\}, \]

\[ \tilde{C}_{\alpha_0}(A, H) = e^{-\pi i \eta m \text{Tr} H}, \]

\[ (G_{\alpha_0})_{SL} = SU(2), \]

\[ (C_{\alpha_0})_{SL} = 1, \]

\[ SL_1 = SL_2 = \text{Ker} (C_{\alpha_0})_{SL} = (G_{\alpha_0})_{SL}. \]

Thus, in the commutative diagram of figure 2, section 7, the four spaces on the left are the same. All of them represent State Space,

\[ H(2) \times \frac{SL}{(G_{\alpha_0})_{SL} \cap SL_1} \]

what in our present case becomes

\[ H(2) \times \frac{SL}{SU(2)}, \]

and can be obviously identified to

\[ \frac{G}{SU(2) \oplus \{0\}}, \]

by means of the diffeomorphism

\[(H, ASU(2)) \rightarrow (A, H)(SU(2) \oplus \{0\}).\]

Then, in this case, besides the canonical map from State Space onto Movement Space (c.f. figure 2),

\[ \nu_2 : (H, ASU(2)) \in H(2) \times \frac{SL}{SU(2)} \rightarrow (A, H)G_{\alpha} \in \frac{G}{G_{\alpha_0}} \] (57)

we have a natural map from State Space onto $G/\text{Ker} C_{\alpha_0}$

\[ \nu_1 : (A, H)(SU(2) \oplus \{0\}) \in \frac{G}{SU(2) \oplus \{0\}} \rightarrow (A, H)\text{Ker} C_{\alpha_0} \in \frac{G}{\text{Ker} C_{\alpha_0}}. \] (58)

The diagram of figure 2 becomes that of figure 5.
Figure 5: Diagram for Klein-Gordon particles.
Let $H^m$ be the positive mass hyperboloid

$$H^m = \{ H \in H(2) : \det H = m^2, \text{Tr} H > 0 \}.$$  

In $H^m$ we consider the action of $SL(2, C)$ given by

$$A \ast H = AHA^*,$$

for all $H \in H^m$ and $A \in SL(2, C)$. In these conditions we have $Tr(AHA^*) > 0$ as a consequence of the fact that $SL(2, C)$ is connected.

Since the group of restricted homogeneous Lorenz transformations is transitive on $H^m$, so does $SL(2, C)$. The isotropy subgroup at $mI$ is $SU(2)$. Thus $SL/(G_{\alpha\omega})_{SL}$ is identified to $H^m$, by means of the diffeomorphism

$$ASU(2) \in SL \rightarrow m AA^* \in H^m.$$

If $K = mAA^* \in H^m$, then $K$ is identified to $ASU(2) \in SL$, and the function $P$ thus becomes

$$P(K) = P(ASU(2)) = -A(\eta mI)A^* = -\eta K.$$

The manifold $H^m$ has the global parametrization

$$\varphi : (p^1, p^2, p^3) \in \mathbb{R}^3 \mapsto h(p^1, p^2, p^3, (m^2 + p^2)^{\frac{1}{2}}) \in H^m$$

where

$$p^2 = \sum_{i=1}^{3} (p^i)^2.$$

In particular it is orientable.

On the other hand, the restriction of the pseudoriemannian metric defined on $\mathbb{R}^4$ by Minkowski metric, to $H^m$, is negative definite so that its opposite is a riemannian metric, that provides us with a canonical volume element, $\nu$. A computation of the matrix of the riemannian metric in the chart $\varphi$ leads to

$$\nu = \frac{dp^1 \wedge dp^2 \wedge dp^3}{\sqrt{m^2 + p^2}}.$$
where the $p^i$ are the coordinates in the chart $\varphi$.

The action on $H(2)$ defined as in (59) preserves Minkowski metric, so that the action on $H^m$ preserves $\nu$.

As a consequence, $\nu$ is an invariant volume element under the action (59).

Since $(C_{\alpha})_{SL} = 1$, $C$ is composed by functions on $H^m$.

The prewave functions have the form

$$\psi_f : (H, K) \in H(2) \times H^m \mapsto f(K) e^{-i\pi T r(K \epsilon H \epsilon)}.$$ 

where $f$ is in $C$.

The corresponding wave function is

$$\tilde{\psi}_f (X) = \int_{H^m} \psi_f (h(X), \cdot) \nu$$

By direct computation one sees that the prewave and wave functions satisfy Klein-Gordon equation.

If $f'$ is another continuous with compact support function on $H^m$, the hermitian product of the quantum states corresponding to $\psi_f$ and $\psi_{f'}$, defined in section 8.0.2 can be written

$$\langle \psi_f, \psi_{f'} \rangle = \int_{H^m} \psi_f^* \psi_{f'} \nu.$$ 

11.1.2 The Homogeneous Contact and Symplectic Manifolds for Klein-Gordon particles

Let us consider the action of $G$ on $H^m \times \mathbb{R}^3 \times S^1$ given by

$$(A, H) * (K, \gamma, z) = (AKA^*, \gamma, (A, H, \gamma, K), e^{2\pi i m \ell(A, H, \gamma, K)z})$$

where $\gamma (A, H, \gamma, K)$ and $\ell(A, H, \gamma, K)$ are given by

$$h(\gamma (A, H, \gamma, K), 0) = Ah(\gamma, 0) A^* + H + \ell(A, H, \gamma, K) A \frac{K}{m} A^*.$$
Obviously we have

\[ \ell(A, H, \overrightarrow{y}, K) = -m \frac{\text{Tr}(Ah(\overrightarrow{y}, 0) A^* + H)}{\text{Tr}(AKA^*)}. \]  

This is a transitive action, as can be proved by direct computation.

The isotropy subgroup at \((mI, \overrightarrow{0}, 1)\), is \(\text{Ker} C_\alpha\), so that the homogeneous space \(G/\text{Ker} C_\alpha\), can be identified to \(\mathcal{H}^m \times \mathbb{R}^3 \times S^1\) by means of the map

\[ (A, H)\text{Ker} C_\alpha \rightarrow (A, H)\ast (mI, \overrightarrow{0}, 1). \]  

We know (c.f. section 5) that the left-invariant 1-form on \(G\) whose value at \((I, 0)\) is \(\alpha_o, \tilde{\alpha}_o\), projects in a 1-form \(\Omega\), in \(G/\text{Ker} C_\alpha\), which is a contact form, and is homogeneous in the sense that it is preserved by the diffeomorphisms corresponding to the canonical action of \(G\) on \(G/\text{Ker} C_\alpha\).

One way to give explicitly \(\Omega\), is to use coordinate domains in \(G/\text{Ker} C_\alpha\) that are also domains of sections of the canonical map from the group \(G\) onto \(G/\text{Ker} C_\alpha\). The pull back of \(\tilde{\alpha}_0\) by that section, is the restriction of \(\Omega\) to the domain, and we can give its local expression in the coordinate system.

With our identification, the just cited canonical map becomes

\[ \mu : (A, H) \in G \rightarrow (A, H)\ast (mI, \overrightarrow{0}, 1) \in \mathcal{H}^m \times \mathbb{R}^3 \times S^1. \]  

In \(\mathcal{H}^m \times \mathbb{R}^3 \times S^1\) be define a coordinate system for each \(\tau \in \mathbb{R}\) as the inverse of the parametrization

\[ \phi_\tau : (k_1, k_2, k_3, x^1, x^2, x^3, t) \in \mathbb{R}^6 \times \left( -\frac{1}{2}, \frac{1}{2} \right) \rightarrow \]

\[ \rightarrow (m \ h(k_1, k_2, k_3, k_4), x^1, x^2, x^3, e^{2\pi i(t+\tau)}) \in \mathcal{H}^m \times \mathbb{R}^3 \times S^1 \]

where \(k_4 = \sqrt{1 + k_1^2 + k_2^2 + k_3^2}\).
Now, let us consider the map
\[
\sigma_\tau: \phi_\tau(k_1, k_2, k_3, x^1, x^2, x^3, t) \to \left( \left( \sqrt{\frac{k_4 + k_3}{1 + k_1^2 + k_2^2}}, \sqrt{\frac{k_4 + k_3}{1 + k_1^2 + k_2^2}}(k_1 - ik_2), \frac{h(\vec{x}, 0) - \frac{\eta}{m}(t + \tau))h(\vec{k}, k_4)}{\sqrt{\frac{1 + k_2^2 + k_3^2}{k_4 + k_3}}} \right), h(\vec{x}, 0) - \frac{\eta}{m}(t + \tau))h(\vec{k}, k_4) \right) \in G.
\]

We have
\[
\mu \circ \sigma_\tau(\phi_\tau(R)) = \sigma_\tau(\phi_\tau(R)) \ast (mI, \vec{0}, 1) = \phi_\tau(R),
\]
for all \( R \in \mathbb{R}^6 \times (-1/2, 1/2) \), so that \( \sigma_\tau \) is a section of the canonical map \( \mu \).

Then, on the image of \( \phi_\tau \) we have
\[
\Omega = \sigma_\tau^*\tilde{\alpha}_0.
\]

In order to obtain, for example, the value
\[
(\Omega)(\phi_\tau(R)) \cdot \left( \frac{\partial}{\partial k_1} \right)_{(\phi_\tau(R))},
\]
we must find the value of \( \alpha_0 \) on the tangent vector to the curve
\[
\gamma(s) = (\sigma_\tau(\phi_\tau(R)))^{-1} (\sigma_\tau(\phi_\tau(R) + (s, 0, 0, 0, 0, 0, 0)))
\]
at 0.

Since
\[
\alpha_0 = \{0, \eta m I\},
\]
the first component of the tangent vector at 0 of \( \gamma \) has no incidence in the value of \( \alpha_0 \) on it.

The preceding computation gives us
\[
\langle (0, \eta m I), \gamma_0 \rangle = 0.
\]

A similar computation for each of the other coordinates, or a common reasoning with a curve \( \gamma(s) = \phi_\tau(c(s)) \), leads to
\[
\Omega = dt + \eta m k_i dx^i.
\]
Obviously
\[ d\Omega = \eta m dk_i \wedge dx^i, \]
so that
\[ \Omega \wedge (d\Omega)^3 = \frac{\eta m^3}{16} dt \wedge dk_1 \wedge dk_2 \wedge dk_3 \wedge dx^1 \wedge dx^2 \wedge dx^3 \]
what confirms the fact that \( \Omega \) is a contact form.

The characteristic vectorfield \( Z(\Omega) \) defined by
\[ i(Z(\Omega))\Omega = 1, \quad i(Z(\Omega)) d\Omega = 0, \]
has flow, \( \phi_t \), given by
\[ \phi_t(K, \vec{x}, z) = (K, \vec{x}, e^{2\pi it} z). \]
As a consequence, the period of all its integral curves is \( T(\Omega) = 1 \), so that \( \Omega \) is a connection form, and \( d\Omega \) the corresponding curvature form.

Now, let us consider the action of \( G \) on \( \mathcal{H}^m \times \mathbb{R}^3 \) given by
\[ (A, H) * (K, \vec{y}) = (AKA^*, \vec{x}(A, H, \vec{y}, K)) \] (67)
where
\[ h(\vec{x}(A, H, \vec{y}, K), 0) = Ah(\vec{y}, 0) A^* + H + \ell(A, H, \vec{y}, K) A \frac{K}{m} A^*. \] (68)

This is also a transitive action. The isotropy subgroup at \((mI, \vec{0})\), is \( G_{a_0} \), so that \( G/G_{a_0} \), can be identified to \( \mathcal{H}^m \times \mathbb{R}^3 \) by means of the map
\[ (A, H)G_{a_0} \rightarrow (A, H) * (mI, \vec{0}). \] (69)

The left-invariant 2-form on \( G \) whose value at \((I, 0)\) is \( da_0, d\tilde{a}_0 \), projects in a 2-form, \( \omega \), in \( G/G_{a_0} \) which is a simplectic form, and is homogeneous in the sense that it is preserved by the diffeomorphisms corresponding to the canonical action of \( G \) on \( G/G_{a_0} \). With our identifications, the canonical map
\[ (A, H)Ker C_\alpha \rightarrow (A, H)G_\alpha \]
becomes
\[ (K, \vec{x}, z) \in \mathcal{H}^m \times \mathbb{R}^3 \times S^1 \rightarrow (K, \vec{x}) \in \mathcal{H}^m \times \mathbb{R}^3 \]
and, since $T(\Omega) = 1$, $\omega$ must be a projection of $d\Omega$, so that
\[
\omega = \eta m\,dk_i \wedge dx^i,
\]
where now $\{k_1, \ldots, x^3\}$ are the coordinates corresponding to the following global parametrization of $H^m \times \mathbb{R}^3$
\[
\phi: (k_1, k_2, k_3, x^1, x^2, x^3) \in \mathbb{R}^6 \to (m\ h(k_1, k_2, k_3, k_4), x_1, x_2, x_3) \in H^m \times \mathbb{R}^3
\]
where $k_4 = \sqrt{1 + k_1^2 + k_2^2 + k_3^2}$.

In this case we have a global section of the canonical map
\[
\sigma(\phi(R)) \ast (mI, \overrightarrow{0}) = \phi(R).
\]

The coadjoint orbit of $\alpha_0$, becomes identified to $H^m \times \mathbb{R}^3$ by means of
\[
\text{Ad}^*(A,H) \cdot \alpha_0 \to (A, H) \ast (mI, \overrightarrow{0}),
\]
whose inverse is given by
\[
(K, \overrightarrow{x}) \to \text{Ad}^*_\sigma(K, \overrightarrow{x}) \cdot \alpha_0.
\]

Thus, the $(e, g) \in G$, what are dynamical variables on the coadjoint orbit, becomes functions on $H^m \times \mathbb{R}^3$, denoted by $D_{(e, g)}$ in section 5 defined as follows
\[
D(e, g)(\phi(k_1, \ldots, x_3)) = (\text{Ad}^*_\sigma(\phi(k_1, \ldots, x_3))\alpha_0)(e, g).
\]

In particular, the Linear and Angular Momentum, given in (7), are, as functions on $H^m \times \mathbb{R}^3$,
\[
\begin{align*}
P(m, \overrightarrow{x}) &= -\eta m\ k, \\
\overrightarrow{T}(m, \overrightarrow{x}) &= \eta m\ k \times \overrightarrow{x} = \overrightarrow{x} \times (\overrightarrow{P}(m, \overrightarrow{x})) \\
\overrightarrow{J}(m, \overrightarrow{x}) &= -\eta m\ k^4\overrightarrow{x} = (P^4(m, \overrightarrow{x}))\ \overrightarrow{x},
\end{align*}
\]
where we have denoted $D_{P^k}$, $D_{l_i}$, and $D_{g^j}$ by $P^k$, $l_i$, and $g^j$ respectively.

Also, each $a \in G$ define an infinitesimal generator, $X_a^s$, of the action (67).

These infinitesimal generators are defined by means of its flow, and its local expresions can be obtained directly from the definition, but it is also possible to use formula (22) to obtain the following local expresions in the chart $\Phi^{-1}$

\[
X_{P^j}^s = \frac{\partial}{\partial x^j} \quad (71)
\]

\[
X_{P^4}^s = \sum_{j=1}^{3} \frac{k_j}{k_4} \frac{\partial}{\partial x^j} \quad (72)
\]

\[
X_{P^4}^c = \sum_{j,r=1}^{3} \varepsilon_{kjr} k_j \frac{\partial}{\partial k_r} + \sum_{j,r=1}^{3} \varepsilon_{kjr} x^j \frac{\partial}{\partial x^r} \quad (73)
\]

\[
X_{g^j}^c = x^j X_{P^4}^c - k_4 \frac{\partial}{\partial k_j} \quad (74)
\]

where $\varepsilon_{kjr}$ is as in (53).

Equation (22) proves that these vector fields are globally hamiltonian, and the corresponding hamiltonian is, with our actual notation, the function appearing in the subindex in each case.

The infinitesimal generator corresponding to $a \in G$ for the action in the contact manifold is denoted by $X_a^c$. We have in the charts $\Phi^{-1}$

\[
X_{P^j}^c = \frac{\partial}{\partial x^j} \quad (75)
\]

\[
X_{P^4}^c = \frac{1}{k_4} \left( \sum_{j=1}^{3} k_j \frac{\partial}{\partial x^j} + \eta m \frac{\partial}{\partial t} \right)
\]

\[
X_{P^4}^c = \sum_{j,r=1}^{3} \varepsilon_{kjr} k_j \frac{\partial}{\partial k_r} + \sum_{j,r=1}^{3} \varepsilon_{kjr} x^j \frac{\partial}{\partial x^r} \quad (75)
\]

\[
X_{g^j}^c = x^j X_{P^4}^c - k_4 \frac{\partial}{\partial k_j}.
\]

The Quantum Operators representing Linear and Angular Momentum
for Quantum States on the Contact Manifold are \( \frac{1}{2\pi i} \) times the vector fields in (75).

If \( f \) is a \( C^\infty \) function on \( \mathcal{H}^m \) the corresponding **Quantum State** in the contact manifold is

\[
\Phi_f((A, H) \operatorname{Ker} C_\alpha) = f(A (G_\alpha)_{SL}) \exp[i \pi \text{Tr}(P(A (G_\alpha)_{SL})\varepsilon \overline{\mathcal{H}}\varepsilon)]
\]

or, with the identifications we have made

\[
\Phi_f((A, H) \ast (m I, \overrightarrow{0}, 1)) = f(m AA^*) \exp[i \pi \text{Tr}((-\eta m AA^*)\varepsilon \overline{\mathcal{H}}\varepsilon)]
\]

but

\[
\Phi_f((A, H) \ast (m I, \overrightarrow{0}, 1)) = \Phi_f(m AA^*, h^{-1}(H + \ell AA^*), \exp[2\pi i \eta m \ell]),
\]

where \( \ell \) is such that \( \text{Tr}(H + \ell AA^*) = 0 \). Thus

\[
\Phi_f(K, \overrightarrow{x}, z) = f(K) \exp[i \pi \text{Tr}((-\eta K)\varepsilon(h(\overrightarrow{x}, 0) - \frac{K}{m})\varepsilon)]
\]

where \( \ell \) is such that \( z = \exp[2\pi i \eta m \ell] \). Then

\[
\Phi_f(K, \overrightarrow{x}, z) = f(K) \exp[-i \eta \pi \text{Tr}(K \varepsilon \overline{h(\overrightarrow{x}, 0)}\varepsilon)] \exp[i \ell \eta \pi \text{Tr}(K \varepsilon K \varepsilon \frac{K}{m} \varepsilon)]
\]

and one sees that, if \( K = mh(\overrightarrow{k}, k_4) \),

\[
\Phi_f(K, \overrightarrow{x}, z) = f(K) \exp[-2\pi i \eta m (\overrightarrow{k}, \overrightarrow{x})] \zeta.
\]

In the coordinate system associated to \( \phi_r \), we have

\[
\Phi_f \circ \phi_r(\overrightarrow{k}, \overrightarrow{x}, t) = f(mh(\overrightarrow{k}, k_4)) \exp[-2\pi i(\eta m (\overrightarrow{k}, \overrightarrow{x}) + t + \tau)].
\]

### 11.2 Massive particles with \( T \geq 1 \).

#### 11.2.1 Wave Functions for \( T = 1 \). Dirac equation.

Let us consider a particle whose movement space is the coadjoint orbit of

\[
\alpha_1 = \left\{ \frac{i}{8\pi} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \ \eta m I \right\},
\]

(76)
where \( m \in \mathbb{R}^+ \), \( \eta = \pm 1 \). This orbit is a quantizable, not \( \mathbb{R} \)-quantizable orbit, of the type 5, in the notation of section 9.

In this case we have

\[
G_{\alpha_1} = \{ \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix}, hI : \ z \in S^1, \ h \in \mathbb{R} \}.
\]

The unique homomorphism from \( G_{\alpha_1} \) onto \( S^1 \) whose differential is \( \alpha_1 \) is given by

\[
C_{\alpha_1}\left( \begin{pmatrix} e^{2\pi i\phi} & 0 \\ 0 & e^{-2\pi i\phi} \end{pmatrix}, hI \right) = e^{2\pi i(\phi - \eta mh)}.
\]

Then

\[
Ker \ C_{\alpha_1} = \{ \left( \begin{pmatrix} e^{2\pi i\eta mh} & 0 \\ 0 & e^{-2\pi i\eta mh} \end{pmatrix}, hI \right) : h \in \mathbb{R} \},
\]

\[
(G_{\alpha_1})_{SL} = \{ \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} : z \in S^1 \},
\]

\[
(C_{\alpha_1})_{SL}\left( \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} \right) = z,
\]

\[
SL_1 \cap SL_2 = (G_{\alpha_1})_{SL},
\]

\[
SL_1 \cap Ker (C_{\alpha_1})_{SL} = Ker (C_{\alpha_1})_{SL} = \{ I \}.
\]

Let us denote \( (G_{\alpha_1})_{SL} \) by \([S^1]\), \( G_{\alpha_1} \) by \([S^1] \oplus \mathbb{R}\), and \( Ker C_{\alpha_1} \) by \([R]\).

Then, in the commutative diagram of Figure 2 \( \tau_3 \) and \( \tau_4 \) become identical maps, \( \tau_3 = \tau_4 \) and the diagram becomes, with obvious conventions, that of Figure 6.

The homogeneous space \( SL/[S^1] \) can be characterised as follows.

Let \( P_1(\mathbb{C}) \) be the complex projective space corresponding to \( \mathbb{C}^2 \) (i.e. the space \( P_1(\mathbb{C}) \) consists of the one dimensional complex subspaces of \( \mathbb{C}^2 \)).

In \( \mathcal{H}^m \times P_1(\mathbb{C}) \) we can consider the action of \( SL(2, \mathbb{C}) \) given by

\[
A * (H, [z]) = (AHA^*, [Az])
\]

where \([z] \in P_1(\mathbb{C}) \) is the vector subspace generated by \( z \in \mathbb{C}^2 \setminus \{(0,0)\} \).
We know that the partial action on $\mathcal{H}^m$ is transitive. Now let us see that the complete action on $\mathcal{H}^m \times \mathbb{P}_1(\mathbb{C})$ is also transitive.

Let $(K, [w]) \in \mathcal{H}^m \times \mathbb{P}_1(\mathbb{C})$. Then, there exist $A \in SL$ such that $A \ast (K, [w]) = (mI, [w'])$, for some $w' \in \mathbb{C}^2 - \{(0, 0)\}$, but there exist obviously an element $B$ of $SU(2)$ such that

$$\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} = [w'],$$

and we see that

$$(B^{-1}A) \ast (K, [w]) = \left( mI, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Transitivity follows.

On the other hand, the isotropy subgroup at

$$(mI, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

is $[S^1]$, so that, since the action is transitive, one can identify $SL/[S^1]$ to $\mathcal{H}^m \times \mathbb{P}_1(\mathbb{C})$.

It is a well known fact that $\mathbb{P}_1(\mathbb{C})$ is diffeomorphic to the sphere $S^2$. A diffeomorphism can be described as follows.
Let $C^+$ be the future lightcone, composed by the hermitian matrices having 0 determinant and positive trace. The subset, $S_2$, of $C^+$ composed by the elements whose trace is 2 is

$$S_2 = \{ h(x, y, z, 1) : 1 - x^2 + y^2 + z^2 = 0 \}$$

that can be identified to the sphere $S^2$, by means of

$$\delta : (x, y, z) \in S^2 \subset \mathbb{R}^3 \longleftrightarrow h(x, y, z, 1) \in S_2 \subset C^+.$$

The map

$$\beta_0 : [z] \in \mathbb{P}_1(\mathbb{C}) \longrightarrow \frac{2}{z^* z} z z^* \in S_2,$$

is bijective. Its inverse is

$$\beta_0^{-1} : C \in S_2 \subset C^+ \longrightarrow [w] \in \mathbb{P}_1(\mathbb{C}),$$

where $w$ is an eigenvector of $C$ corresponding to the eigenvalue 2. This is a consequence of the fact that

$$\begin{align*}
\left( \frac{2}{z^* z} z z^* \right) z &= 2z \\
\left( \frac{2}{z^* z} z z^* \right) \varepsilon \bar{\varepsilon} &= 0
\end{align*} \quad (77)$$

Thus

$$\beta \overset{\text{def}}{=} \delta^{-1} \circ \beta_0 : \mathbb{P}_1(\mathbb{C}) \rightarrow S^2, \quad (78)$$

can be described by

$$\beta([z]) = \bar{u} \iff \frac{2}{z^* z} z z^* = h(\bar{u}, 1). \quad (79)$$

that is practical in order to use $\beta$.

More practical in order to use $\beta^{-1}$ is

$$\beta([z]) = \bar{u} \iff h(\bar{u}, -1) z = 0. \quad (80)$$

If $p_s$ and $p_n$ denote the stereographical projections from poles $s = (0, 0, -1)$ and $n = (0, 0, 1)$ respectively, we have

$$p_s \left( \beta \left( \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} \right) \right) = \frac{z^2}{z^1} \quad \text{if } z^1 \neq 0, \quad (81)$$

$$p_n \left( \beta \left( \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} \right) \right) = \frac{z^1}{z^2} \quad \text{if } z^2 \neq 0.$$
This can be used to prove easily the differentiability of $\beta$ and $\beta^{-1}$.

Thus, it is possible to change in all that follows $P_1(\mathbb{C})$ by $S^2$, but, at this moment, I prefer to use $P_1(\mathbb{C})$.

In order to describe Prewave Functions in this case, one can consider the trivialisation $(\rho, \mathbb{C}^4, z_0)$, where $z_0 = (1, 0, 1, 0)$ and $\rho$ is given by

$$\rho(A) = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$$

(82)

The orbit of $z_0$ is

$$\mathcal{B} = \left\{ \begin{pmatrix} w \\ z \end{pmatrix} : w, z \in \mathbb{C}^2, z^*w = 1 \right\}. \quad (83)$$

This can be seen by consideration of the map

$$\phi_0 : \begin{pmatrix} w \\ z \end{pmatrix} \in \mathcal{B} \rightarrow (w| - \varepsilon \overline{z}) \in SL(2, \mathbb{C})$$

(84)

(it is a diffeomorphism), and the fact that

$$\rho(w| - \varepsilon \overline{z}) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} w \\ z \end{pmatrix}.$$ \quad (85)

When one identifies $SL/Ker(C_{\alpha_1})_SL$ to $\mathcal{B}$, and $SL/(G_{\alpha_1})_SL$ to $\mathcal{H}m \times P_1(\mathbb{C})$, by means of the preceeding actions, the canonical map between these homogeneous spaces becomes a map, $r$, from $\mathcal{B}$ onto $\mathcal{H}m \times P_1(\mathbb{C})$.

As a consequence of (85), we have

$$(w| - \varepsilon \overline{z}) * (mI, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = r \begin{pmatrix} w \\ z \end{pmatrix}$$

and this leads easily to

$$r \begin{pmatrix} w \\ z \end{pmatrix} = (m(ww^* - \varepsilon \overline{z}z\varepsilon), [w]). \quad (86)$$
On the other hand, if we define

$$\sigma(K, w) = \frac{1}{\sqrt{mw^*K^{-1}w}} \left( w \right) - \frac{1}{m}K \varepsilon \overline{w},$$  \hspace{1cm} (87)$$

where the vertical bar separates the two columns of the $2 \times 2$ matrix, we have

$$\sigma(K, w) \ast (mI, \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]) = (K, [w]).$$  \hspace{1cm} (88)$$

Notice that if $z$ and $z'$ are representatives of the same element of $P^1(\mathbb{C})$, $\sigma(K, z)$ and $\sigma(K, z')$ are in general different.

Thus, since

$$r^{-1}(mI, \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]) = \{ s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} : s \in S^1 \}$$

we have

$$r^{-1}(K, [w]) = \rho(\sigma(K, w))\{ s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} : s \in S^1 \}$$  \hspace{1cm} (89)$$

which leads to

$$r^{-1}(K, [a]) = \left\{ s \begin{pmatrix} I \\ mK^{-1} \end{pmatrix} \frac{a}{\sqrt{ma^*K^{-1}a}} : s \in S^1 \right\}$$  \hspace{1cm} (90)$$

Also we have

$$P(K, [w]) = P(\sigma(K, w)G(\alpha_1)_{SL}) = -\eta K.$$  

If $f$ is a continuous function with compact support in $\mathcal{B}$, and is $S^1$-homogeneous of degree -1 \textit{(i.e. $f \in \mathcal{C}$ in the notation of section 8.0.1)}, the corresponding prewave function is

$$\psi_f : (H, K, [a]) \in H(2) \times \mathcal{H}^m \times P_1(\mathbb{C}) \mapsto f \left( \begin{pmatrix} w \\ z \end{pmatrix} \right) e^{-i\eta \text{Tr}(K \varepsilon \overline{H})} \left( \begin{pmatrix} w \\ z \end{pmatrix} \right)$$
where \( \begin{pmatrix} w \\ z \end{pmatrix} \) is arbitrary in \( r^{-1}(K, [a]) \).

When one considers the Dirac matrices in the representation
\[
\gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,
\]
a straightforward computation proves that these prewave functions satisfy Dirac Equation
\[
(\gamma^\nu \partial_\nu - 2\pi i \eta m) \psi_f = 0.
\]

A sesquilinear form on \( \mathbb{C}^4 \) whose value on \( \mathcal{B} \) is 1, is defined by
\[
\Phi (Z, Z') = \frac{1}{2} Z^* \gamma^4 Z.
\]
Thus, if \( f, f' \in \mathcal{C} \), the hermitian product of \( \psi_f \) and \( \psi_{f'} \) can be written
\[
\langle \psi_f, \psi_{f'} \rangle = \frac{1}{2} \int_{\mathcal{H}^m \times \mathcal{P}_1(\mathbb{C})} \psi_{f}^* \gamma^4 \psi_{f'} \mu.
\]

To describe Wave Functions we need an invariant volume element on \( \mathcal{H}^m \times \mathcal{P}_1(\mathbb{C}) \).

Let us consider the 5-form in \( \mathcal{H}^m \times (\mathbb{C}^2 - \{(0, 0)\}) \), given by
\[
(\mu_0)(K, z) = \nu \wedge \left( z_1^1 dz_2^2 - z_2^2 dz_1^1 \right) \wedge \left( \overline{z^1} d\overline{z}^2 - \overline{z^2} d\overline{z}^1 \right) \frac{1}{(z^* \epsilon K \epsilon z)^2},
\]
where the \( z^k \) are the two canonical projections of \( \mathbb{C}^2 \) onto \( \mathbb{C} \).

This form is well defined since, for all \( K \in \mathcal{H}^m \), the hermitian product defined in \( \mathbb{C}^2 \) by
\[
\langle x, y \rangle = x^* K y
\]
is positive definite.

This differential form projects to an invariant volume element, \( \mu \), in \( \mathcal{H}^m \times \mathcal{P}_1(\mathbb{C}) \). To prove this fact, one must proceed in many steps.

Let us denote by \( \tau \) the canonical map
\[
\tau : (K, z) \in \mathcal{H}^m \times (\mathbb{C}^2 - \{(0, 0)\}) \to (K, [z]) \in \mathcal{H}^m \times \mathcal{P}_1(\mathbb{C})
\]
The triple $\mathcal{H}^m \times (\mathbb{C}^2 - \{(0,0)\})(\mathcal{H}^m \times P_1(\mathbb{C}), \mathbb{C} - \{0\})$ is a principal fibre bundle with projection $\tau$. To prove that there exist a form, $\mu$, such that

$$\tau^*\mu = \mu_0$$

it is enough to see that $\mu_0$ is invariant under the bundle action, what is obvious, and that $\mu_0$ vanishes on vertical vectors, what can be proved as follows.

The vertical vectors at $(K, z)$ are tangent at $t = 0$ to the curves

$$\gamma(t) = (K, \lambda(t)z)$$

with $\lambda$ a $C^\infty$ map from $\mathbb{R}$ into $\mathbb{C}$ such that $\lambda(0) = 1$.

Let $X_{(K,z)}$ be the tangent vector to $\gamma$ at 0.

We have

$$X_{(K,z)} = \lambda'(0) \left( z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} \right) + \overline{\lambda'(0)} \left( \overline{z}^1 \frac{\partial}{\partial z^1} + \overline{z}^2 \frac{\partial}{\partial z^2} \right),$$

and it is easily seen that

$$i_{X_{(K,z)}}(\mu_0)(K,z) = 0.$$

This finishes the proof of the existence of $\mu$.

Let us denote

$$\|p\|^2 = \sum_{i=1}^{3} (p^i)^2$$

and

$$q_1(p_1, p_2, p_3, z) = (h(p_1, p_2, p_3, (m^2 + \|p\|^2)^{\frac{1}{2}}), \left( \begin{array}{c} 1 \\ z \\ z^1 \end{array} \right)) \quad (91)$$

$$q_2(p_1, p_2, p_3, z) = (h(p_1, p_2, p_3, (m^2 + \|p\|^2)^{\frac{1}{2}}), \left( \begin{array}{c} z \\ 1 \\ 1 \end{array} \right))$$

for all $p_1, p_2, p_3, z \in \mathbb{R}^3 \times \mathbb{C}$.

The maps $q_1$ and $q_2$ are parametrizations, whose inverses are local charts that compose an atlas of $\mathcal{H}^m \times P_1(\mathbb{C})$.  

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The local expressions of \( \mu \) in these charts can be obtained as the reciprocal image of \( \mu_0 \) by the maps

\[
\sigma_1(p_1, p_2, p_3, z) = (h(p_1, p_2, p_3, (m^2 + \sum_{i=1}^{3} (p_i^2)^2)^{\frac{1}{2}}), \begin{pmatrix} 1 \\ z \end{pmatrix}),
\]

and

\[
\sigma_2(p_1, p_2, p_3, z) = (h(p_1, p_2, p_3, (m^2 + \sum_{i=1}^{3} (p_i^2)^2)^{\frac{1}{2}}), \begin{pmatrix} z \\ 1 \end{pmatrix}).
\]

These local expressions are given by

\[
\sigma_1^* \mu_0 = \frac{\nu \wedge dz \wedge d\bar{z}}{2\Re(z(p_1 - ip_2)) + (1 - |z|^2)p_3 - (1 + |z|^2)(m^2 + \|p\|^2)^{\frac{1}{2}}},
\]

\[
\sigma_2^* \mu_0 = \frac{\nu \wedge dz \wedge d\bar{z}}{2\Re(z(p_1 + ip_2)) + (|z|^2 - 1)p_3 - (1 + |z|^2)(m^2 + \|p\|^2)^{\frac{1}{2}}},
\]

As a particular consequence, \( \mu \) is a volume element.

Now, let us consider the action of \( SL \) on \( \mathcal{H}^m \times (\mathbb{C}^2 - \{0,0\}) \) given by

\[
A*(K, z) = (AKA^*, Az).
\]

We already know that \( \nu \) is invariant under the action on \( \mathcal{H}^m \), and it is not a difficult matter to see that \( z^1 dz^2 - z^2 dz^1, \overline{z}^1 d\overline{z}^2 - \overline{z}^2 d\overline{z}^1 \) and \( z^* \overline{K} \overline{\varepsilon} z \) are invariant under this action.

As a consequence, \( \mu_0 \) is invariant under the same action. It follows that \( \mu \) is an invariant volume element on \( \mathcal{H}^m \times \mathbb{P}_1(\mathbb{C}) \).

The wave functions have the form

\[
\tilde{\psi}_f(X) = \int_{\mathcal{H}^m \times \mathbb{P}_1(\mathbb{C})} \psi_f(h(X), \cdot, \cdot) \mu
\]

and also satisfies Dirac Equation

\[
(\gamma^\nu \partial_\nu - 2\pi i \eta m) \tilde{\psi}_f = 0.
\]
11.2.2 Wave functions for $T > 1$.

Now, let us consider a particle whose movement space is the coadjoint orbit of
\[
\alpha_T = \left\{ \frac{i T}{8 \pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \eta m I \right\},
\]
(92)
where $T \in \mathbb{Z}^+, m \in \mathbb{R}^+, \eta = \pm 1$.

The case of the preceding section is the particular one given by $T = 1$.

For all $T$ the orbit is a quantizable, not $\mathbb{R}$-quantizable orbit, of the type 5.

Now we have
\[
G_{\alpha_T} = \left\{ \left( \begin{array}{cc} z & 0 \\ 0 & \frac{1}{z} \end{array} \right), \ hI : z \in S^1, \ h \in \mathbb{R} \right\}.
\]
The unique homomorphism from $G_{\alpha_T}$ onto $S^1$ whose differential is $\alpha_T$ is given by
\[
C_{\alpha_T}(\begin{pmatrix} e^{2 \pi i \phi} & 0 \\ 0 & e^{-2 \pi i \phi} \end{pmatrix}, \ hI) = e^{2 \pi i (\phi T - \eta mh)}.
\]

Then
\[
(G_{\alpha_T})_{SL} = \left\{ \left( \begin{array}{cc} z & 0 \\ 0 & \frac{1}{z} \end{array} \right) : z \in S^1 \right\},
\]
\[
(C_{\alpha_T})_{SL}(\begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix}) = z^T,
\]
\[
SL_1 \cap SL_2 = (G_{\alpha_T})_{SL},
\]
\[
SL_1 \cap \text{Ker}(C_{\alpha_T})_{SL} = \text{Ker}(C_{\alpha_T})_{SL} = \left\{ \left( \begin{array}{cc} z & 0 \\ 0 & \frac{1}{z} \end{array} \right) : z \in \sqrt[\sqrt{T}]{1} \right\}.
\]

where $\sqrt[\sqrt{T}]{1}$ is the subgroup of $\mathbb{C}^*$ composed by the roots of order $T$ of 1.

The homogeneous space $SL/(G_{\alpha_T})_{SL}$, is the same as in the preceding section so that it can be identified to $\mathcal{H}^m \times P_1(\mathbb{C})$, and consider on it the invariant volume element $\mu$.

In order to give the wave functions in the case of arbitrary $T$, the following results are useful.
Let $\beta, \beta'$ be quantizable elements of $G^*$ with $\beta'$ not $R$-quantizable, 
$(G_\beta)_{SL} = (G_{\beta'})_{SL}$, $(C_\beta)_{SL} = ((C_{\beta'})_{SL})^T$, where $T \in \mathbb{Z}^+$. 

Notice that, as a consequence of the fact that $\beta'$ is not $R$-quantizable, 
we must have $(C_{\beta'})_{SL}((G_{\beta'})_{SL}) = S^1$.

If $(\rho, L, z_0)$ is a trivialization of $C_{\beta'}$, we consider the triple 
$(\rho \otimes T, L \otimes T, z_0 \otimes T)$, 
where $L \otimes T = L \otimes \cdots \otimes L$, $z_0 \otimes T = z_0 \otimes \cdots \otimes z_0$, and $\rho \otimes T$ is the representation such that $\rho \otimes T(A)(z_1 \otimes \cdots \otimes z_T) = \rho(A)(z_1) \otimes \cdots \otimes \rho(A)(z_T)$.

In lemma 6.1 of [Dia96b] I have proved that, under these circumstances, 
if $(G_\beta)_{SL}$ is connected, $(\rho \otimes T, L \otimes T, z_0 \otimes T)$ is a trivialization of $C_\beta$.

Let us assume that $(\rho \otimes T, L \otimes T, z_0 \otimes T)$ is a trivialization of $C_\beta$ and let $B_T$ be the orbit of $z_0 \otimes T$. The pullback by $z \in B \mapsto z \otimes T \in B_T$, establishes a one to one map from the set of the $S^1$-homogeneous functions of degree -1 on $B_T$, onto the set of the $S^1$-homogeneous functions of degree -T on $B$. If $f$ is one of these functions, the corresponding prewave function of particles corresponding to $\beta$ has the form

$$
\psi_f(H, m) = f(z) e^{i\pi T \text{Tr}(P(m)z \bar{\eta})} z \otimes T
$$

where $z \in r^{-1}(m)$.

Now we apply these results to the particles of type 5 with $T > 1$.

Let $\beta' = \alpha_1$ and $\beta = \alpha_T$. The remark just made leads to the following 
Prewave Function

$$
\psi_f : (H, K, [a]) \in H(2) \times \mathcal{H}^m \times P_1(C) \mapsto f \left( \begin{array}{c} w \\ z \end{array} \right) e^{-i\pi T \text{Tr}(Kz \bar{\eta})} \left( \begin{array}{c} w \\ z \end{array} \right) \otimes T
$$

where $\left( \begin{array}{c} w \\ z \end{array} \right)$ is arbitrary in $r^{-1}(K, [a])$, and $f$ is a function on $B$, $C^\infty$, with compact support and homogeneous of degree -T under multiplication by complex numbers of modulus one.

The Wave Functions are obtained by integration as usual.
11.2.3 Movement Space for massive particles with $T \geq 1$.

We consider the action of $G$ on $\mathcal{H}^m \times \mathbf{P}_1(\mathbb{C}) \times \mathbb{R}^3$ given by

$$(A, H) * (K, [z], \vec{y}) = (AKA^*, [Az], \vec{x}(A, H, \vec{y}, K))$$  \hspace{1cm} (93)$$

where $\vec{x}(A, H, \vec{y}, K)$ is given by

$$h(\vec{x}(A, H, \vec{y}, K), 0) = Ah(\vec{y}, 0)A^* + \ell(A, H, \vec{y}, K)A\frac{K}{m}A^*$$  \hspace{1cm} (94)$$

$$\ell(A, H, \vec{y}, K) = -\frac{mTr(Ah(\vec{y}, 0)A^* + H)}{Tr(AKA^*)}. \hspace{1cm} (95)$$

This is a transitive action. The isotropy subgroup at $(mI, [(1, 0)], \vec{0})$, is found to be $G_{\alpha_T}$. Thus, the map

$$\lambda: (A, H)G_{\alpha_T} \in G/G_{\alpha_T} \leftrightarrow (A, H) * \left(mI, \left[\begin{array}{c} 1 \\ 0 \end{array}\right], \vec{0}\right) \in \mathcal{H}^m \times \mathbf{P}_1(\mathbb{C}) \times \mathbb{R}^3$$

enables us to identify the coadjoint orbit with $\mathcal{H}^m \times \mathbf{P}_1(\mathbb{C}) \times \mathbb{R}^3$.

A parametrization whose image is a neighborhood of $(mI, [(1, 0)], \vec{0})$, is

$$\Gamma_s: (\vec{k}, \vec{x}, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{C} \rightarrow \left(mh(\vec{k}, k_4), \left[\begin{array}{c} 1 \\ z \end{array}\right], \vec{x}\right) \hspace{1cm} (96)$$

where $k_4 = \sqrt{1 + k_1^2 + k_2^2 + k_3^2}$.

The corresponding local chart is given by

$$\Gamma_s^{-1}: \left(K, \left[\begin{array}{c} z^1 \\ z^2 \end{array}\right], \vec{x}\right) \in \{z^1 \neq 0\} \rightarrow \left(\frac{1}{m}h^{-1}(K - Tr(K)I), \vec{x}, \frac{z^2}{z^1}\right)$$

Another parametrization is

$$\Gamma_n: (\vec{k}, \vec{x}, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{C} \rightarrow \left(mh(\vec{k}, k_4), \left[\begin{array}{c} z \\ 1 \end{array}\right], \vec{x}\right) \hspace{1cm} (97)$$

$$\in \mathcal{H}^m \times \mathbf{P}_1(\mathbb{C}) \times \mathbb{R}^3.$$
with $k_4 = \sqrt{1 + k_1^2 + k_2^2 + k_3^2}$, whose inverse is given by

$$
\Gamma_n^{-1}: \left( K, \left[ \begin{array}{c} z^1 \\ z^2 \end{array} \right], \vec{x} \right) \in \{ z^2 \neq 0 \} \rightarrow \left( \frac{1}{m} h^{-1}(K - Tr(K)I), \vec{x}, \frac{z^1}{z^2} \right)
$$

Obviously, these two charts compose an atlas.

The identification of $\mathcal{H}^m \times \mathbb{P}_1(\mathbb{C}) \times \mathbb{R}^3$ with the coadjoint orbit is given by

$$
(A, H) \ast ((mI, \left[ \begin{array}{c} 1 \\ 0 \end{array} \right], \vec{u})) \leftrightarrow Ad^*_{(A, H)} \alpha_T.
$$

Now, let

$$(K, [z], \vec{x}) \in \mathcal{H}^m \times \mathbb{P}_1(\mathbb{C}) \times \mathbb{R}^3.$$

If $\sigma (K, w)$ is given by (87), then

$$(\sigma (K, z), h(\vec{x}, 0)) \ast (mI, \left[ \begin{array}{c} 1 \\ 0 \end{array} \right], \vec{u}) = (K, [z], \vec{x}),$$

so that $(K, [z], \vec{x})$ must be identified to $Ad^*_{(\sigma(K,z),h(\vec{x},0))} \cdot \alpha_T$.

Now let $F$ be a dynamical variable on the coadjoint orbit. $F$ becomes a function on $\mathcal{H}^m \times \mathbb{P}_1(\mathbb{C}) \times \mathbb{R}^3$.

To give explicit expressions of these functions, is now preferable to use the sphere $S^2$ instead of $bf\mathbb{P}_1(\mathbb{C})$ thus writing

$$F(K, [z], \vec{x}) = F(Ad^*_{(\sigma(K,z),h(\vec{x},0))} \cdot \alpha_T) = F(K, \vec{u}, \vec{x}),$$

where $\vec{u} \in S^2$ is related to $[z] \in \mathbb{P}_1(\mathbb{C})$ by (c.f. (79))

$$\frac{2}{z^*z}zz^* = h(\vec{u}, 1).$$

Let us evaluate

$Ad^*_{(\sigma(K,z),h(\vec{x},0))} \cdot \alpha_T.$

To do that computation we need some of the equations (3) to (4), and

$$(\sigma(K,z))^{-1} = \frac{1}{\sqrt{mz^*K^{-1}z}} \left( \frac{mz^*K^{-1}z}{z\varepsilon} \right),$$

(98)
where the horizontal line separates the two files of the $2 \times 2$ matrix. This leads to

$$Ad^T_{(\sigma(k, z), h(\vec{x}, 0))} \cdot \alpha_T = \left\{ \begin{array}{l} -\frac{T}{8\pi} \frac{1}{\langle k_4 \vec{u} - \vec{k}, \vec{u} \rangle} h(\vec{k} \times \vec{u}, 0) + \frac{\eta m}{2} k_4 h(\vec{x}, 0) \\ + i \left[ \frac{T}{8\pi} \frac{1}{\langle k_4 \vec{u} - \vec{k}, \vec{u} \rangle} h(k_4 \vec{u} - \vec{k}, 0) + \frac{\eta m}{2} h(\vec{k} \times \vec{x}, 0) \right], \end{array} \right.$$ 

where

$$h(\vec{k}, k_4) = \frac{1}{m} K.$$ 

Thus, using (11), we obtain

$$P(\eta mh(\vec{k}, k_4), \vec{u}, \vec{x}) = -\eta mh(\vec{k}, k_4),$$

$$\vec{l}(\eta mh(\vec{k}, k_4), \vec{u}, \vec{x}) = \frac{T}{4\pi} \frac{k_4 \vec{u} - \vec{k}}{k_4 - \langle \vec{k}, \vec{u} \rangle} + \eta m \vec{k} \times \vec{x},$$

$$\vec{g}(\eta mh(\vec{k}, k_4), \vec{u}, \vec{x}) = \frac{T}{4\pi} \frac{\vec{k} \times \vec{u}}{k_4 - \langle \vec{k}, \vec{u} \rangle} - \eta m k_4 \vec{x}. \quad (99)$$

Notice that

$$k_4 - \langle \vec{k}, \vec{u} \rangle = \langle k_4 \vec{u} - \vec{k}, \vec{u} \rangle.$$ 

If we denote $P(K, \vec{u}, \vec{x}), \vec{l}(K, \vec{u}, \vec{x})$ and $\vec{g}(K, \vec{u}, \vec{x})$ simply by $P$, $\vec{l}$ and $\vec{g}$ respectively when no danger of confusion exist, and $\vec{P} = (P^1, P^2, P^3)$, we have the following relations between these dynamical variables

$$\vec{l} = \frac{T}{4\pi} \frac{P^4 \vec{u} - \vec{P}}{P^4 - \langle \vec{P}, \vec{u} \rangle} + \vec{x} \times \vec{P}, \quad (100)$$

$$\vec{g} = \frac{T}{4\pi} \frac{\vec{P} \times \vec{u}}{P^4 - \langle \vec{P}, \vec{u} \rangle} + P^4 \vec{x}. \quad (101)$$

The value of the Pauli-Lubanski fourvector at $(m \eta h(\vec{k}, k_4), \vec{u}, \vec{x})$ can be
calculated using (14), (15), (99), ..., (101) and is found to be

\[ W = P^4 T \frac{P^4 \vec{u} - \vec{P}}{4\pi} = -\eta m k \frac{T}{4\pi} \frac{k_4 \vec{u} - \vec{k}}{\langle k_4 \vec{u} - \vec{k}, \vec{u} \rangle} \]

so that we can also write

\[ \vec{l} = \frac{1}{P^4} \vec{W} + \vec{x} \times \vec{P}. \]  

11.2.4 Contact manifold for Dirac particles

Now we pay attention to the contact manifold in the case \( T = 1 \).

We consider the action of \( G \) on \( B \times \mathbb{R}^3 \) given by

\[ (A, H) \ast (w, z, \vec{y}) = (e^{2\pi i m \ell} A w, e^{2\pi i m \ell} (A^*)^{-1} z, \vec{x} (A, H, w, z, \vec{y})) \]

where

\[ h(\vec{x} (A, H, w, z, \vec{y}), 0) = Ah(\vec{y}, 0) A^* + H + \ell A(w w^* - \varepsilon z z^* \bar{z}) A^*. \]

Obviously, we have

\[ \ell = -\frac{Tr(Ah(\vec{x}, 0) A^* + H)}{Tr(A(w w^* - \varepsilon z z^* \bar{z}) A^*)}. \]

This is a transitive action and the isotropy subgroup at

\[ \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) , \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \vec{0} \right) \]

is \( \text{Ker} \ C_{\alpha_1} \), so that we can identify \( G/\text{Ker} \ C_{\alpha_1} \) with \( B \times \mathbb{R}^3 \) by means of

\[ (A, H) \text{Ker} \ C_{\alpha_1} \leftrightarrow (A, H) \ast \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) , \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \vec{0} \right). \]

When \( w \in \mathbb{C}^2 - \{0\} \), the \( z \) such that \( (w, z) \in B \) compose the set

\[ \left\{ \frac{1}{w^* w} (w + y \bar{w}) : y \in \mathbb{C} \right\}. \]
Thus, the map
\[
\Delta_0 : (w, y) \in (\mathbb{C}^2 - \{0\}) \times \mathbb{C} \longrightarrow (w, z(w, y)) \in \mathcal{B},
\]
where
\[
z(w, y) = \frac{1}{w^* w} (w + y \epsilon w),
\]
is a bijection, and in fact a diffeomorphism. Its inverse is given by
\[
(\Delta_0)^{-1} : (w, z) \in \mathcal{B} \longrightarrow (w, t w \epsilon z) \in (\mathbb{C}^2 - \{0\}) \times \mathbb{C}.
\]

Then, \((\Delta_0)^{-1}\) is a global complex coordinate system of \(\mathcal{B}\), and
\[
\Delta : (w, y, \vec{x}) \in (\mathbb{C}^2 - \{0\}) \times \mathbb{C} \times \mathbb{R}^3 \longrightarrow (w, z(w, y), \vec{x}) \in \mathcal{B} \times \mathbb{R}^3,
\]
is a parametrization of the Contact Manifold defined in an an open subset of \(\mathbb{C}^3 \times \mathbb{R}^3\).

Notice that, since we know a diffeomorphism, \(\phi_0\), from \(\mathcal{B}\) onto \(SL(2, \mathbb{C})\) (c.f. (84)), we obtain also a complex parametrization of this group by means of
\[
\phi_0 \circ \Delta_0 : (w, y) \in (\mathbb{C}^2 - \{0\}) \times \mathbb{C} \longrightarrow \left( w \left| \frac{1}{w^* w} (yw - \epsilon w) \right. \right. \in SL(2, \mathbb{C}).
\]

The inverse of \(\phi_0 \circ \Delta_0\) is the global complex chart of \(SL(2, \mathbb{C})\) given by
\[
\phi : (w|v) \in SL(2, \mathbb{C}) \longrightarrow (w, t w \epsilon \vec{x}) \in (\mathbb{C}^2 - \{0\}) \times \mathbb{C}.
\]

The canonical map of \(G\) onto \(G/Ker C_{\alpha_1}\) becomes
\[
(A, H) \in G \longleftrightarrow (A, H) \ast \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right), 0 \right),
\]
and the map
\[
\Sigma : (w, z, \vec{x}) \in \mathcal{B} \times \mathbb{R}^3 \longrightarrow ((w| - \epsilon \vec{z}), h(\vec{x}, 0)) \in G,
\]
is a section of this canonical map, as a consequence of the fact that
\[
\Sigma(w, z, \vec{x}) \ast \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right), 0 \right) = (w, z, \vec{x}).
\]

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The canonical map from $G/Ker C_\alpha$ onto $G/G_\alpha$ becomes
\[
\tilde{r}: (w, z, \vec{x}) \in \mathcal{B} \times \mathbb{R}^3 \rightarrow \Sigma(w, z, \vec{x}) \ast (mI, \left[ \begin{array}{c} 1 \\ 0 \end{array} \right], \vec{0}) = (m(ww^* - \varepsilon zz^* \varepsilon), [w], \vec{x}) \in \mathcal{H}^m \times \mathbb{P}_1(\mathbb{C}) \times \mathbb{R}^3,
\]
i.e. coincides with $r \times Id_{\mathbb{R}^3}$.

The Linear Momentum, Angular Momentum and Pauli-Lubanski four-vector, whose descriptions in the symplectic manifold are given by (99) and (102), when composed with $\tilde{r}$, give functions $P_c, \vec{l}_c, \vec{g}_c, W_c$, on the Contact Manifold, that are the translation of these dynamical variables to this homogeneous space.

In order to give explicit expressions for these functions (c.f. (114)), let us denote
\[
\tilde{r}(w, z, \vec{x}) = (mh(\vec{k}, k_4), \vec{u}, \vec{x}) \in \mathcal{H}^m \times S^2 \times \mathbb{R}^3
\]
we have
\[
h(\vec{u}, 1) = \frac{2}{w^* w} w^* w^*,
\]
and
\[
h(\vec{k}, k_4) = ww^* - \varepsilon zz^* \varepsilon,
\]
but
\[
-h(\vec{u}, 1) \varepsilon h(\vec{k}, k_4) \varepsilon = h(k_4 \vec{u} - \vec{k}, k_4 - \langle \vec{k}, \vec{u} \rangle) + ih(\vec{k} \times \vec{u}, 0) \quad (110)
\]
and
\[
-h(\vec{u}, 1) \varepsilon h(\vec{k}, k_4) \varepsilon = -\frac{2}{w^* w} ww^* \varepsilon (ww^* - \varepsilon zz^* \varepsilon) \varepsilon = \frac{2}{w^* w} wz^*, \quad (111)
\]
leads to
\[
h(k_4 \vec{u} - \vec{k}, k_4 - \langle \vec{k}, \vec{u} \rangle) + ih(\vec{k} \times \vec{u}, 0) = \frac{2}{w^* w} wz^*,
\]
so that
\[
h(k_4 \vec{u} - \vec{k}, k_4 - \langle \vec{k}, \vec{u} \rangle) = \frac{1}{w^* w} (wz^* + zw^*),
\]
\[
h(\vec{k} \times \vec{u}, 0) = i \frac{1}{w^* w} (zw^* - wz^*) \quad (112)
\]
\[ \langle k_4 u - \vec{k}, u \rangle = k_4 - \langle \vec{k}, u \rangle = \frac{1}{w^* w} \]

\[
\left( k_4 \vec{u} - \vec{k}, \frac{1}{k_4 - \langle \vec{k}, u \rangle} \right) = w z^* + zw^* \\
\left( \frac{\vec{k} \times \vec{u}}{k_4 - \langle \vec{k}, u \rangle}, 0 \right) = i (zw^* - w z^*)
\]

Thus

\[
P_c(w, z, \vec{x}) = -\eta m (ww^* - \varepsilon \varepsilon z z^*)
\]

\[
h(\tilde{l}_c(w, z, \vec{x}), 0) = \frac{T}{4\pi} (zw^* + zw^* - I) + h(\vec{x} \times \vec{P}_c, 0)
\]

\[
h(\tilde{g}_c(w, z, \vec{x}), 0) = \frac{i T}{4\pi} (zw^* - w z^*) + P^4 h(\vec{x}, 0)
\]

\[
h(\tilde{W}_c(w, z, \vec{x}), 0) = P^4 \frac{T}{4\pi} (zw^* + zw^* - I)
\]

\[
W^4_c(w, z, \vec{x}) = -\eta m T \frac{1}{8\pi} (w^* w - z^* z)
\]

The formula for \(W^4_c\) can be obtained as follows

\[
\left( \vec{l}, \vec{P} \right) \circ \vec{r} = \left( \frac{T}{4\pi} \langle k_4 \vec{u} - \vec{k}, \vec{P} \rangle \right) \circ \vec{r} =
\]

\[
= \frac{1}{2} Tr \left( h \left( \frac{T}{4\pi} \langle k_4 \vec{u} - \vec{k}, \vec{P} \rangle, 0 \right) \right) \circ \vec{r} =
\]

\[
= \frac{1}{2} Tr \left( \left( \frac{T}{4\pi} (zw^* + zw^* - I) \right) (-\eta m) (ww^* - \varepsilon \varepsilon z z^*) \right) =
\]

\[
= \frac{-\eta m T}{8\pi} (w^* w - z^* z).
\]

The local expresions of these functions in the chart \(\Delta^{-1}\) are obtained simply by changing in the above expresions \(z\) by the \(z(w, y)\) given in (105).

Let \(\Omega_1\) be the contact form and \(\tilde{\alpha}_1\) the left invariant differential form on \(G\) whose value at \((I, 0)\) is \(\alpha_1\). We have

\[
\Omega_1 = \Sigma^* \tilde{\alpha}_1,
\]

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and this formula enables us, by similar procedures to those of section 11.1.2, to see that

$$
\Omega_1 = \frac{iT}{4\pi} \left( z^1 dw^1 - \bar{z}^1 dw^1 + z^2 dw^2 - \bar{z}^2 dw^2 \right) - P^1 dx^1 - P^2 dx^2 - P^3 dx^3,
$$

where (c.f. (105))

$$
\begin{pmatrix}
z^1 \\
z^2
\end{pmatrix} = z(w, y).
$$

This equation must be interpreted as follows: the right hand side of (116) is a differential 1-form in $$\mathbb{C}^4 \times \mathbb{R}^3$$ referred to coordinates $$w^1, \ldots, x^3$$, and $$\Omega_1$$ is the restriction of this form to the submanifold $$B \times \mathbb{R}^3$$.

Obviously, under the same conditions

$$
d\Omega_1 = \frac{iT}{4\pi} \left( dz^1 \wedge dw^1 - d\bar{z}^1 \wedge d\bar{w}^1 + dz^2 \wedge dw^2 - d\bar{z}^2 \wedge d\bar{w}^2 \right) - dP^1 \wedge dx^1 - dP^2 \wedge dx^2 - dP^3 \wedge dx^3.
$$

Now, to obtain the symplectic form we only need sections of the map $$\tilde{r}$$.

On the domain, $$U_s$$, of the local chart $$\Gamma_s^{-1}$$, the map

$$
\tilde{\sigma}_s : \Gamma_s(\vec{k}, \vec{x}, t) \rightarrow \left( \sigma \left( mh(\vec{k}, k_4), \begin{pmatrix} 1 \\ t \end{pmatrix} \right), h(\vec{x}, 0) \right) \ast \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{0} \right),
$$

where $$\sigma(K, w)$$ is given by (87), is a section. Then

$$
\tilde{\sigma}_s \circ \Gamma_s(\vec{k}, \vec{x}, t) = \left( \begin{pmatrix} 1 \\ t \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ t \end{pmatrix} \right),
$$

$$
\left( \begin{pmatrix} 1 \\ t \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ t \end{pmatrix} \right), \vec{x}
$$

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If \( \omega_1 \) is the symplectic form, we have on \( U_s \)
\[
\omega_1 = \tilde{\sigma}_s^*d\Omega_1. \tag{119}
\]

In the same way, on the domain, \( U_n \), of the chart \( \Gamma_n \), we define a section, \( \tilde{\sigma}_n \), by
\[
\tilde{\sigma}_n \circ \Gamma_n(\vec{k}, \vec{x}, t) = \left( \sigma \left( mh(\vec{k}, k_4), \left( \begin{array}{c} t \\ 1 \end{array} \right), h(\vec{x}, 0) \right) \right) \ast \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right), \vec{0}. \tag{120}
\]

Then
\[
\tilde{\sigma}_n \circ \Gamma_n(\vec{k}, \vec{x}, t) = \left( \frac{1}{\sqrt{(t, 1)(h(\vec{k}, k_4))^{-1}}} \left( \begin{array}{c} t \\ 1 \end{array} \right) \right) \ast \left( \frac{1}{\sqrt{(t, 1)(h(\vec{k}, k_4))^{-1}}} \left( \begin{array}{c} t \\ 1 \end{array} \right), \vec{x} \right).
\]

On \( U_n \) we have
\[
\omega_1 = \tilde{\sigma}_n^*d\Omega_1. \tag{121}
\]

Since \( \{U_s, U_n\} \) is an open covering of the symplectic manifold, \( (119) \) and \( (121) \), determine \( \omega \) everywhere.

### 11.2.5 Contact manifold for \( T > 1 \).

Let us denote by \( r_1, \ldots, r_T \) the elements of \( \sqrt{T} \).

We define a properly discontinuous free action, \( \cdot \), of \( \sqrt{T} \) on \( \mathcal{B} \times \mathbb{R}^3 \) by means of
\[
r_j \cdot (w, z, \vec{x}) = (r_jw, r_jz, \vec{x}).
\]

The quotient space is denoted by \( (\mathcal{B} \times \mathbb{R}^3)/\sqrt{T} \), and the canonical map from \( \mathcal{B} \times \mathbb{R}^3 \) onto \( (\mathcal{B} \times \mathbb{R}^3)/\sqrt{T} \), is denoted by \( \pi_T \). Thus \( \pi_T(w, z, \vec{x}) \) is the orbit of \( (w, z, \vec{x}) \), considered as an element of \( (\mathcal{B} \times \mathbb{R}^3)/\sqrt{T} \).
The map $\pi_T$ is a $T$-fold covering map and, since $\mathcal{B} \times \mathbb{R}^3$ is simply connected, the Universal covering map of the quotient space.

The action of $G$ on $\mathcal{B} \times \mathbb{R}^3$ commutes with that of $\sqrt{T}$. As a consequence, the action on $(\mathcal{B} \times \mathbb{R}^3)/\sqrt{T}$ given by

$$(A, H) \ast ((w, z, \vec{x})) \sqrt{T} = ((A, H) \ast (w, z, \vec{x})) \sqrt{T}$$

is well defined, and obviously makes the covering map $\pi_T$ equivariant.

The isotropy subgroup at

$$\left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \end{pmatrix}, \vec{0} \right) \sqrt{T}$$

is $\text{Ker} C_{\alpha_T}$, so that we can identify $G/\text{Ker} C_{\alpha_T}$ with $(\mathcal{B} \times \mathbb{R}^3)/\sqrt{T}$ by means of

$$(A, H)\text{Ker} C_{\alpha_T} \longleftrightarrow (A, H) \ast \left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \end{pmatrix}, \vec{0} \right) \sqrt{T}.$$

The contact form on $\mathcal{B} \times \mathbb{R}^3$, $\Omega_1$, is invariant under the action “·”, so that there exist an unique contact form, $\Omega_T$, on $(\mathcal{B} \times \mathbb{R}^3)/\sqrt{T}$, such that

$$\pi_T^* \Omega_T = \Omega_1.$$

The action of $G$ on $(\mathcal{B} \times \mathbb{R}^3)/\sqrt{T}$ is transitive and preserves $\Omega_T$.

The manifold $(\mathcal{B} \times \mathbb{R}^3)/\sqrt{T}$, when provided with the contact form $\Omega_T$ and the cited action of $G$, is the homogeneous contact manifold that correspond to massive particles with $T \geq 1$.

12 Massless Type 4 particles.

In this section we consider particles whose movement space is the coadjoint orbit of

$$\alpha = \left\{ \frac{i\chi T}{8\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ \end{pmatrix}, \eta \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \end{pmatrix} \right\}, \chi, \eta \in \{\pm 1\}, T \in \mathbb{Z}^+.$$
where $\eta = -\text{sign}(Tr(P))$, (see Table 2 of section [9]). So, at least at the classical level, $\eta$ must be interpreted as the opposite of the sign of energy. We will see in the following subsections that this interpretation is also exact at the quantum level.

Since $|P|$ is mass square and

$$\text{Det} \left( \eta \left( \begin{array} { c c } { 1 } & { 0 } \\ { 0 } & { 0 } \end{array} \right) \right) = 0$$

these orbits correspond to particles with zero mass.

These are quantizable, not $\mathbb{R}$-quantizable orbits of the type 4.

The group $G_\alpha$ is connected, so that there exists at most one homomorphism from $G_\alpha$ onto $S^1$ whose differential is $\alpha$. In fact we have

$$G_\alpha = \left\{ \left( \begin{array} { c c } { z } & { a } \\ { 0 } & { \bar{z} } \end{array} \right), \left( \begin{array} { c c } { b } & { i \chi T a z / 2 \pi } \\ { i \chi T a z / 2 \pi } & { 0 } \end{array} \right) : z \in S^1, b \in \mathbb{R}, a \in \mathbb{C} \right\}$$

and the homomorphism

$$C_\alpha \left( \left( \begin{array} { c c } { z } & { a } \\ { 0 } & { \bar{z} } \end{array} \right), \left( \begin{array} { c c } { b } & { i \chi T a z / 2 \pi } \\ { i \chi T a z / 2 \pi } & { 0 } \end{array} \right) \right) = z\chi^T$$

has differential $\alpha$.

Other computations give us

$$(G_\alpha)_{SL} = \left\{ \left( \begin{array} { c c } { z } & { a } \\ { 0 } & { \bar{z} } \end{array} \right) : z \in S^1, a \in \mathbb{C} \right\}$$

$$(C_\alpha)_{SL} \left( \left( \begin{array} { c c } { z } & { a } \\ { 0 } & { \bar{z} } \end{array} \right) \right) = z\chi^T$$

$$SL_1 = \left\{ \left( \begin{array} { c c } { a } & { 0 } \\ { 0 } & { 1/a } \end{array} \right) : a \in \mathbb{C} \right\}$$

$$SL_2 = (G_\alpha)_{SL}$$

$$SL_1 \cap SL_2 = \left\{ \left( \begin{array} { c c } { z } & { 0 } \\ { 0 } & { \bar{z} } \end{array} \right) : z \in S^1 \right\}$$
The isotropy subgroup at \((\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})\), for the usual action of \(SL(2,\mathbb{C})\) on \(H(2)\) (that is \(A \ast m = AmA^*\)) is \((G_\alpha)_SL\). Thus \(SL/(G_\alpha)_SL\) will be identified to the orbit of \((\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})\), which is
\[ C^+ = \{H \in H(2) : \ Det H = 0, \ Tr H > 0\}. \]

This set is composed by the hermitian matrices that represent spacetime points in the future lightcone, so that it will be called itself, future lightcone.

When this identification is made, the function \(P\) on \(SL/(G_\alpha)_SL\) becomes \(P(H) = -\eta H\).

An invariant volume element in \(C^+\) is
\[ \omega = \frac{1}{\|\bar{p}\|} \, dp^1 \wedge dp^2 \wedge dp^3, \quad (122) \]
where \((p^1, p^2, p^3)\) is the coordinate system corresponding to the parametrization
\[ \phi : (p^1, p^2, p^3) \in \mathbb{R}^3 - \{0\} \mapsto h(\bar{p}, \|\bar{p}\|) \in C^+ \]
where \(\bar{p} = (p^1, p^2, p^3)\), and
\[ \|\bar{p}\| = +\sqrt{\sum_{i=1}^{3} (p^i)^2}. \]

In this parametrization, the linear momentum is given by
\[ P(\phi(\bar{p})) = -\eta \, h(\bar{p}, \|\bar{p}\|). \]

Before to proceed to the study of the general case, we shall consider two particular ones.

12.1 Massless antineutrino

Let us consider the case \(T = 1, \, \chi = 1, \, \eta = -1\).
A trivialization is given by

$$\left( \rho_+, \mathbb{C}^2, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right),$$

where $\rho_+(A)$ is multiplication by $A$.

The orbit of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is $\mathbb{C}^2 - \{0\}$. This space can thus be identified to $SL/Ker(C_\alpha)_{SL}$ so that the canonical map

$$SL/Ker(C_\alpha)_{SL} \longrightarrow SL/(G_\alpha)$$

becomes a map

$$r_+: \mathbb{C}^2 - \{0\} \mapsto \mathbb{C}^+. $$

This map must be equivariant, so that, for all $A \in SL$

$$r_+ \left( A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A^*.$$

Thus $r_+$ is explicitly given by

$$r_+(z) = zz^* \in \mathbb{C}^+. $$

Since $z$ and $\varepsilon \pi$ are eigenvectors of $zz^*$ corresponding to the eigenvalues $\|z\|^2$ and 0 respectively, $r_+^{-1}(H)$ is composed by the eigenvectors of $H$ corresponding to the positive eigenvalue, whose norm is the square root of that eigenvalue.

The principal $\mathbb{S}^1$-bundle whose projection is $r_+$ is related to the Hopf fibration as follows. The image of the restriction of $r_+$ to the sphere $S^3(R) = \{ z \in \mathbb{C}^2 : \|z\|^2 = R^2 \}$ is composed by the elements of $\mathbb{C}^+$ whose trace is $R^2$. Since the image of this subset by the preceding chart is the sphere of radius $R^2/2$, we obtain maps from spheres $S^3$ onto spheres $S^2$. Each one of these mappings is, up to the radius and a reflection, the Hopf fibration.

For each $S$-homogeneous of degree -1 function on $\mathbb{C}^2 - 0$, $f$, we have the following prewave function

$$\psi_+^f : (N, H) \in \mathbb{C}^+ \times H(2) \mapsto f(z) e^{i \pi Tr(N\varepsilon\pi \varepsilon)} z \in \mathbb{C}^2, \quad (123)$$
where \( z \) is an arbitrary element of \( r_+^{-1}(N) \). Here we use the fact that, since \( \eta = -1 \), the linear momentum is given in \( C^+ \) by \( P(N) = N \).

If \( X \) and \( Q \) are the elements of \( \mathbb{R}^4 \) corresponding to \( H \) and \( N \), one can write
\[
\psi_f^+(Q, X) = f(z) e^{-2\pi i (Q \cdot X)} z, \tag{124}
\]
where \( z \) is arbitrary in \( r_+^{-1}(Q) \). The corresponding wave function, if \( f \) is continuous with compact support, is
\[
\tilde{\psi}_f^+(H) = \int_{C^+} \psi_f^+(\cdot, H) \omega.
\]

By direct computation one can see that
\[
\left( \sigma_1 \frac{\partial}{\partial x^1} + \sigma_2 \frac{\partial}{\partial x^2} + \sigma_3 \frac{\partial}{\partial x^3} + \sigma_4 \frac{\partial}{\partial x^4} \right) \psi_f^+(N, h(x)) = 0
\]

The corresponding wave function thus satisfy the same equation, which is the Weyl equation (positive energy), usually admised as corresponding to the antineutrino.

If \( f \) and \( f' \) are pseudotensorial functions, we have
\[
(\psi_f^+(N, H))^* \psi_{f'}^+(N, H) = f(z) f'(z) \text{Tr} N.
\]

Thus, the hermitian product of the corresponding quantum states (\textit{cf.} section 7) can be written as follows
\[
\frac{1}{2} \int_{C^+} \left( \psi_f^+ \right)^* \psi_{f'}^+ \sum_{i=1}^3 dp^i dp^2 dp^3.
\]

To obtain prewave functions directly from functions on \( C^+ \), we use Remark 7.1 as follows.

Let
\[
U = \phi(\mathbb{R}^3 - \{ p^3 = p^2 = 0, p^3 < 0 \}),
\]
\[
V = \phi(\mathbb{R}^3 - \{ p^3 = p^2 = 0, p^3 > 0 \}).
\]

Then the maps
\[
\sigma_U : \phi(p^1, p^2, p^3) \in U \mapsto \left( \frac{\sqrt{\parallel \vec{p} \parallel + p^3}}{\sqrt{\parallel \vec{p} \parallel + p^3}} \right) \in \mathbb{C}^2 - 0 \quad (125)
\]

\[
\sigma_V : \phi(p^1, p^2, p^3) \in V \mapsto \left( \frac{\sqrt{\parallel \vec{p} \parallel - p^3}}{\sqrt{\parallel \vec{p} \parallel - p^3}} \right) \in \mathbb{C}^2 - 0
\]

are sections of \( r_+ \), so that we obtain prewave functions having the form

\[
\psi(\phi(p^1, p^2, p^3), X) = F(p^1, p^2, p^3) \left( \frac{\sqrt{\parallel \vec{p} \parallel + p^3}}{\sqrt{\parallel \vec{p} \parallel + p^3}} \right)
\]

\[
e^{-2\pi i (\parallel \vec{p} \parallel X^4 - p^1 X^1 - p^2 X^2 - p^3 X^3)}
\]

where \( F \) is a complex valued function continuous on \( \mathbb{R}^3 - \{0\} \), with compact support in \( \mathbb{R}^3 - \{p^1 = p^2 = 0, p^3 < 0\} \), and prewave functions having the form

\[
\theta(\phi(p^1, p^2, p^3), X) = J(p^1, p^2, p^3) \left( \frac{\sqrt{\parallel \vec{p} \parallel - p^3}}{\sqrt{\parallel \vec{p} \parallel - p^3}} \right)
\]

\[
e^{-2\pi i (\parallel \vec{p} \parallel X^4 - p^1 X^1 - p^2 X^2 - p^3 X^3)}
\]

where \( J \) is a complex valued function continuous on \( \mathbb{R}^3 - \{0\} \), with compact support in \( \mathbb{R}^3 - \{p^1 = p^2 = 0, p^3 > 0\} \).

If one is interested in the case \( T = 1, \chi = 1, \eta = +1 \), everything is as in the case \( \eta = -1 \), but the exponent of \( e \) in the prewave functions changes its sign, thus giving wave functions for quantum states of negative energy. Remember that \( \eta \) is the opposite of the sign of energy at the classical level.

### 12.2 Massless neutrino

Now let us consider the case in which \( T = 1, \chi = -1, \eta = -1 \).

A trivialisation in this case is

\[
\left( \rho_-, \mathbb{C}^2, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),
\]

\( \rho_-(A) \) being multiplication by \( (A^*)^{-1} \).
Thus we identify $SL/Ker C_\alpha$ with the orbit of 
\[
\begin{pmatrix}
0 \\
1
\end{pmatrix},
\]
which, also in this case, is $\mathbb{C}^2 - \{0\}$. 

Thus, the canonical map 
\[
SL/Ker (C_\alpha)_{SL} \longrightarrow SL/(G_\alpha)
\]
becomes a map 
\[
r_- : \mathbb{C}^2 - \{0\} \rightarrow \mathbb{C}^+.
\]

Now, using equivariance as in the case of antineutrino, we obtain 
\[
r_-(z) = -\overline{\varepsilon z z^* \varepsilon}.
\]

The prewave functions one obtains have exactly the same form that (123) and (124), but now $z$ is in $(r_-)^{-1}(N)$ or $(r_-)^{-1}(h(Q))$ respectively.

The wave functions one obtains in this case satisfies the Weyl equation that, according to Feynman, corresponds to the neutrino.

The map from the real vector space $\mathbb{C}^2$ onto itself defined by sending $z$ to $\varepsilon \overline{z}$, is a complex structure and its restriction to $\mathbb{C}^2 - \{0\}$, gives us an isomorphism of the principal circle bundle corresponding to $r_-$ (resp. $r_+$) onto the principal circle bundle corresponding to $r_+$ (resp. $r_-$). The isomorphism of the structural group is defined by sending each element to its inverse. Thus if $z \in r_+^{-1}(H)$ then $\varepsilon \overline{z} \in r_-^{-1}(H)$ and conversely.

As a consequence, the sections of the map $r_+$ defined in [12.1] give rise to the following sections of $r_-$ 
\[
\sigma'_U(u) = \varepsilon \sigma_U(u), \quad \sigma'_V(v) = \varepsilon \sigma_V(v),
\]
for all $u \in U$ and $v \in V$, and the prewave functions of antineutrino $\psi$ and $\theta$ give rise to prewave functions of neutrino, $\psi'$ and $\theta'$, by means of 
\[
\psi'(\phi(p^1, p^2, p^3), X) = F(p^1, p^2, p^3) \varepsilon \left( \frac{\sqrt{\|p\|^2 + p^3} + p^3}{\sqrt{\|p\|^2 + p^3}} \right)
\]

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$$e^{-2\pi i (\|p\| X^4 - p^1 X^1 - p^2 X^2 - p^3 X^3)}$$

$$\theta' (\phi(p^1, p^2, p^3), X) = J(p^1, p^2, p^3) \epsilon \left( \frac{p^1 - ip^2}{\sqrt{\|p\| - p^3}} \right)$$

$$e^{-2\pi i (\|p\| X^4 - p^1 X^1 - p^2 X^2 - p^3 X^3)}$$

that leads to

$$\psi'(\phi(p^1, p^2, p^3), X) = F(p^1, p^2, p^3) \left( \begin{array}{c} \frac{p^1 - ip^2}{\sqrt{\|p\| + p^3}} \\ - \sqrt{\|p\| + p^3} \end{array} \right)$$

$$e^{-2\pi i (\|p\| X^4 - p^1 X^1 - p^2 X^2 - p^3 X^3)}$$

$$\theta'(\phi(p^1, p^2, p^3), X) = J(p^1, p^2, p^3) \left( \begin{array}{c} \sqrt{\|p\| - p^3} \\ - \frac{p^1 + ip^2}{\sqrt{\|p\| - p^3}} \end{array} \right)$$

$$e^{-2\pi i (\|p\| X^4 - p^1 X^1 - p^2 X^2 - p^3 X^3)}$$

If one consider the case $T = 1$, $\chi = -1$, $\eta = +1$ one obtain similar wave functions but corresponding to negative energy.

12.3 General case

We proceed as in section [11.2.2]

In the case $\chi = +1$ a trivialization is given by

$$\left( (\rho_+) \otimes^T, \ (C^2) \otimes^T, \ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes^T \right),$$

and if $\chi = -1$ a trivialization is given by

$$\left( (\rho_-) \otimes^T, \ (C^2) \otimes^T, \ \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes^T \right),$$

The prewave functions are given by functions on $C^2 - \{0\}$, which are $C^\infty$ with compact support and homogeneous of degree -T under multiplication by modulus one complex numbers.
Let $f_T$ be one of these functions.

If $\chi = 1$, the corresponding prewave function is given by
\[
\psi^+_f: (N, H) \in \mathbb{C}^+ \times H(2) \mapsto f_T(z) \ e^{-i\pi \eta \text{Tr}(Nz\Pi z)}z \otimes T \in (\mathbb{C}^2)^{\otimes T}
\]
where $z$ is an arbitrary element of $r_+^{-1}(N)$.

In the case $\chi = -1$, the corresponding prewave function is
\[
\psi^-_f: (N, H) \in \mathbb{C}^+ \times H(2) \mapsto f_T(z) \ e^{-i\pi \eta \text{Tr}(Nz\Pi z)}z \otimes T \in (\mathbb{C}^2)^{\otimes T}
\]
but now, $z$ is an arbitrary element of $r_-^{-1}(N)$.

The associated wave functions,
\[
\tilde{\psi}^+_f(H) = \int_{C^+} \psi^+_f(\cdot, H) \omega, \quad (126)
\]
\[
\tilde{\psi}^-_f(H) = \int_{C^+} \psi^-_f(\cdot, H) \omega \quad (127)
\]
satisfies Penrose’s wave equations, that we will describe here for the sake of completeness.

Let us consider in $(\mathbb{C}^2)^{\otimes T}$ the basis $\{e_A \otimes e_B \otimes \cdots: A, B, \cdots \in \{1, 2\}\}$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$.

The prewave functions $\psi^+_f$ and the wave functions $\tilde{\psi}^+_f$, have components in this basis which will be denoted by $\{\psi^{AB\cdots}_\pm\}$ and $\{\tilde{\psi}^{AB\cdots}_\pm\}$, respectively.

Let us consider the vector fields in $\mathbb{R}^4$ given by
\[
\nabla_{11} = \frac{1}{2} \left( \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4} \right)
\]
\[
\nabla_{12} = \frac{1}{2} \left( \frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2} \right)
\]
\[
\nabla_{21} = \frac{1}{2} \left( \frac{\partial}{\partial x^1} + i \frac{\partial}{\partial x^2} \right)
\]
\[
\nabla_{22} = \frac{1}{2} \left( \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^3} \right)
\]
and, for all $A, A' \in \{1, 2\}$, $\nabla^A A' = \varepsilon^A B \varepsilon^{A'} B' \nabla_B B'$ (summation convention), where $\{\varepsilon^A B\}$ are the elements of $-\varepsilon$.

We also define

$$\psi_{A'B\ldots}^\pm = \varepsilon_A A' \varepsilon_B B' \ldots \psi_{A'B\ldots}^\pm$$

where $\{\varepsilon_A B\}$ are the elements of $\varepsilon$.

Thus we have for all $h(x) \in C^+$

$$\nabla^A A' \psi_{A'B\ldots}^+(h(x), \cdot) = 0$$
$$\nabla^{A'} A \psi_{A'B\ldots}^-(h(x), \cdot) = 0$$

so that, by derivation under the integral sign, we see that Penrose wave equations:

$$\nabla^A A' \psi_{A'B\ldots}^+ = 0$$
$$\nabla^{A'} A \psi_{A'B\ldots}^- = 0$$

are satisfied.

12.3.1 Helicity

In formula (53) one sees that, if $(\rho, L, z_0)$ is the trivialization we use, the components of spin operator are given by

$$s^k : \tilde{\psi}_f \mapsto s^k \circ \tilde{\psi}_f,$$

$s^k$ being the following endomorphism of $L$

$$s^k = \frac{1}{2\pi i} \frac{i\sigma_k}{2} = \frac{1}{4\pi i} \tilde{i}\sigma_k,$$

where,

$$\tilde{i}\sigma_k = d\rho(i\sigma_k).$$

Thus, in the case $\chi = 1$,

$$\tilde{i}\sigma_k = d((\rho_+)^T)(i\sigma_k),$$
and in the case $\chi = -1$,

$$i\sigma_k = d \left( (\rho_\cdot)^{\otimes T} \right) (i\sigma_k).$$

As a consequence, in both cases, $\chi = +1$ or $\chi = 1$, when acting on monomials we have

$$i\sigma_k \cdot z_1 \otimes \cdots \otimes z_T = \sum_{j=1}^T z_1 \otimes \cdots \otimes z_{j-1} \otimes i\sigma_k z_j \otimes z_{j+1} \otimes \cdots \otimes z_T.$$

We consider the operators $\hat{s}^k$ as also acting on prewave functions by the same formula that in the case of wave functions, without the tilde.

On the other hand, Linear Momentum is given on $C^+$ by

$$P(K) = -\eta K,$$

for all $K \in C^+$. Then, with the notation $K = h(K^1, K^2, K^3, K^4)$ and $P = h(P^1, P^2, P^3, P^4)$, the $P^k$ are functions on $C^+$, given by

$$P^k(K) = -\eta K^k.$$

We also denote

$$\vec{K} = (K^1, K^2, K^3),$$

$$\|\vec{K}\| = \left( \sum_{k=1}^3 (K^k)^2 \right)^{1/2},$$

$$\vec{P} = (P^1, P^2, P^3),$$

$$\|\vec{P}\| = \left( \sum_{k=1}^3 (P^k)^2 \right)^{1/2}.$$ 

The **Helicity Operator** is defined by

$$\mathfrak{h} = \frac{1}{\|\vec{P}\|} \sum_{k=1}^3 P^k \hat{s}^k,$$

which means

$$(\mathfrak{h} \cdot \psi_{f_T}^\pm) (K, H) = \frac{1}{\|\vec{P}\|(K)} \sum_{k=1}^3 P^k(K) s^k \left( \psi_{f_T}^\pm (K, H) \right).$$
Then, if $z \in r_+^{-1}(K)$ and $z_1 = \cdots = z_T = z$, we have

$$
(h \cdot \psi^\pm_{fr}) (K, H) = \frac{1}{K^4} \sum_{k=1}^{3} P^k(K) s^k \left( f_T(z)e^{-i\eta T r_K e} e^T \right) =
$$

$$
= \frac{1}{4\pi K^4} f_T(z)e^{-i\eta T r_K e} e^T
\left( \sum_{j=1}^{T} z_1 \otimes \cdots \otimes z_{j-1} \otimes \left( \sum_{k=1}^{3} P^k(K) \sigma_k z_j \otimes z_{j+1} \otimes \cdots \otimes z_T \right) \right) =
$$

$$
= \frac{1}{4\pi K^4} f_T(z)e^{-i\eta T r_K e} e^T
\left( \sum_{j=1}^{T} z_1 \otimes \cdots \otimes z_{j-1} \otimes (-\eta)((K - K^4 I)z_j) \otimes z_{j+1} \otimes \cdots \otimes z_T \right).
$$

In the case $\chi = +1$, we have $K = r_+(z) = zz^*$, so that

$$(K - K^4 I)z = (zz^* - \|z\|^2 I)z = K^4 z,$$

and then

$$
(h \cdot \psi^+_{fr}) (K, H) = \frac{-\eta T}{4\pi} \psi^+_{fr}(K, H).
$$

On the other hand, in the case $\chi = -1$, we have $K = -\epsilon zz^* \epsilon$. Thus

$$(K - K^4 I)z = (-\epsilon zz^* \epsilon - K^4 I)z = -K^4 z,$$

so that

$$
(h \cdot \psi^-_{fr}) (K, H) = \frac{\eta T}{4\pi} \psi^-_{fr}(K, H).
$$

Thus $\psi^\pm_{fr}$ is an eigenvector of the helicity operator, corresponding to the eigenvalue $-\eta \chi T/4\pi$.

In particular, the sign of helicity is $-\eta \chi$. 92
12.4 The Homogeneous Contact and Symplectic Manifolds for Massless particles of type 4. Twistors.

Let us denote by $\sqrt{T} = \{r_1, \ldots, r_T\}$, the group composed by the roots of order $T$ of 1, and

$$\nu = -\frac{\eta \chi T}{4\pi}.$$ 

In section [12.3.1] we have seen that all wave functions obtained in section [12.3] are eigenvectors of helicity with eigenvalue $\nu$.

The group $\text{Ker} C_{\alpha}$ has $T$ connected components

$$(\text{Ker} C_{\alpha})_j = \left\{ \left( \begin{array}{cc} r_j & a \\ 0 & r_j \end{array} \right), \left( \begin{array}{cc} b & -2i\nu a r_j \\ -2i\nu a r_j & 0 \end{array} \right) : b \in \mathbb{R}, a \in \mathbb{C} \right\}$$

The component of the identity is

$$(\text{Ker} C_{\alpha})_o = \left\{ \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} b & -2i\nu a \\ 2i\nu a & 0 \end{array} \right) : b \in \mathbb{R}, a \in \mathbb{C} \right\}$$

and $(\text{Ker} C_{\alpha})_j$ is the left translation by

$$\tilde{r}_j \overset{\text{def}}{=} \left( \begin{array}{cc} r_j & 0 \\ 0 & r_j \end{array} \right), 0$$

of $(\text{Ker} C_{\alpha})_o$.

As a consequence, the group $\text{Ker} C_{\alpha}/(\text{Ker} C_{\alpha})_o$ is isomorphic to $\sqrt{T}$.

The canonical map

$$\tilde{c} : (A, H) \text{Ker} C_{\alpha} \in \frac{G}{(\text{Ker} C_{\alpha})_o} \rightarrow \frac{G}{\text{Ker} C_{\alpha}}.$$ 

is a $T$-fold covering map, whose structural group is $\text{Ker} C_{\alpha}/(\text{Ker} C_{\alpha})_o$.

Thus, $G/\text{Ker} C_{\alpha}$ is the quotient of $G/(\text{Ker} C_{\alpha})_o$ by a properly discontinuous with no fixed point action of $\text{Ker} C_{\alpha}/(\text{Ker} C_{\alpha})_o$. But, as we have seen, this group can be changed to $\sqrt{T}$. 

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The action of $\sqrt{T}$ on $G/(\text{Ker}C_\alpha)_o$ corresponding to this construction is:

$$((A, H) (\text{Ker} C_\alpha)_o) \ast r_j = ((A, H) \tilde{r}_j) (\text{Ker} C_\alpha)_o.$$  \hspace{1cm} (128)

In what follows, we identify $G/\text{Ker} C_\alpha$ with the quotient space of $G/(\text{Ker} C_\alpha)_o$ by this action.

We have the following commutative diagram

$$\begin{array}{ccc}
G_{(\text{Ker} C_\alpha)_o} & \overset{e}{\longrightarrow} & G_{\text{Ker} C_\alpha} \\
\downarrow & & \downarrow \\
G_{G_\alpha} & \overset{\sim}{\longrightarrow} & G_{G_\alpha}
\end{array}
$$

where the vertical arrow on the right is the bundle map of the contact manifold onto the symplectic manifold, for general $T$. If $T = 1$ both vertical arrows are the same.

The homomorphism

$$\mu_1 : (A, H) \in SL \oplus H(2) \rightarrow \begin{pmatrix} A & -iHA^{*^{-1}} \\ 0 & A^{*^{-1}} \end{pmatrix} \in GL(4, \mathbb{C}).$$

is a representation in $\mathbb{C}^4$.

The isotropy subgroup at

$$q \overset{\text{def}}{=} \begin{pmatrix} 0 \\ 2\nu \\ 0 \\ 1 \end{pmatrix}$$

is $(\text{Ker} C_\alpha)_o$.

Let us denote by $O_q$ the orbit of $q$ by this representation.

In order to describe $O_q$ in another way, we consider in $\mathbb{C}^4$ the hermitian product

$$< \begin{pmatrix} \tilde{w} \\ \tilde{z} \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix}> = \frac{1}{2} (\tilde{w}^{*} z + \tilde{z}^{*} w) \hspace{0.5cm} \forall \tilde{w}, \tilde{z}, w, z \in \mathbb{C}^2$$
whose signature and quadratic form are respectively \((+ , + , - , -)\) and
\[
\Phi \left( \begin{array}{c} w \\ z \end{array} \right) = \text{Re} \ z^* w.
\]

The complex vector space \(C^4\) provided with this hermitian product is Penrose’s Twistor Space.

The representation \(\mu_1\) preserves \(\Phi\), so that any orbit must be contained in a subset of the form \(\Phi = \text{constant}\). It follows that the orbit of \(q\) is contained in the seven dimensional submanifold, \(\mathcal{O}_\nu\), given by
\[
\Phi = 2\nu.
\]

Let
\[
\left( \begin{array}{c} w \\ z \end{array} \right) \in C^4
\]
be such that
\[
\text{sign} \left( \Phi \left( \begin{array}{c} w \\ z \end{array} \right) \right) = \text{sign}(\nu),
\]
and define
\[
A(w, z) = \left( \frac{2\nu}{\Phi ^{w}{z}} \right)^{1/2} \left( e^z \begin{bmatrix} 1 & \frac{1}{2\nu} \left( \text{Im}(z^*w) \right) z - w \end{bmatrix} \right)
\]
\[
H(w, z) = \frac{\text{Im}(z^*w)}{\|z\|^2} I.
\]

Then we have
\[
\mu_1(A(w, z), H(w, z))q = \left( \frac{2\nu}{\Phi ^{w}{z}} \right)^{1/2} \left( \begin{array}{c} w \\ z \end{array} \right).
\]

This enables us to prove that each element of \(\mathcal{O}_\nu\) is in the orbit, so that \(\mathcal{O}_q = \mathcal{O}_\nu\).
Then, the map from $G/(\text{Ker} C_\alpha)_o$ onto $\mathcal{O}_\nu$, given by

$$\tau_T : (A, H) (\text{Ker} C_\alpha)_o \in \frac{G}{(\text{Ker} C_\alpha)_o} \rightarrow \mu_1(A, H) q \in \mathcal{O}_\nu,$$

is a diffeomorphism.

If we identify these spaces by $\tau_T$, the action of $\sqrt{T}$ on $G/(\text{Ker} C_\alpha)_o$ given by (128), translates to an action on $\mathcal{O}_\nu$.

Since

$$\tau_T ((A, H) (\text{Ker} C_\alpha)_o) * r_j = \mu_1((A, H) \tilde{r}_j)q = \mu_1(A, H) \mu_1(\tilde{r}_j)q = \mu_1(A, H)(\tilde{r}_j q) = \tau_T ((A, H) (\text{Ker} C_\alpha)_o),$$

the action of $\sqrt{T}$ on $\mathcal{O}_\nu$ is given by ordinary product by the conjugated:

$$V * r_j = \overline{\tau_j} V$$

Let us denote by $\mathcal{O}_\nu/\sqrt{T}$ the quotient space of $\mathcal{O}_\nu$ by this action.

It follows from the preceding construction that the contact manifold, $G/\text{Ker} C_\alpha$, is diffeomorphic, and will be identified, to $\mathcal{O}_\nu/\sqrt{T}$. The contact form will be determinated later on.

Let $\{w^1, w^2, z^1, z^2\} \in \mathbb{C}^4\setminus\{0\}$, and let us denote by $[\{w^1, w^2, z^1, z^2\}]$ the complex vector subspace of $\mathbb{C}^4$ generated by $\{w^1, w^2, z^1, z^2\}$. The set composed by these subspaces is the complex projective space of $\mathbb{C}^4$, $\mathbb{P}_3(\mathbb{C})$, and we consider it as provided with its canonical differentiable structure.

Penrose also considers the subsets of twistor space, $T^+$, $T^-$, $T^0$, given by $\Phi > 0$, $\Phi < 0$, $\Phi = 0$, respectively, and the subsets of projective space $P_3^+ = \pi(T^+)$, $P_3^- = \pi(T^-)$, $P_3^0 = \pi(T^0)$, where

$$\pi : \mathbb{C}^4\setminus\{0\} \longrightarrow \mathbb{P}_3(\mathbb{C})$$

is the canonical map.

Since $\Phi$ is preserved by the representation, the subsets $T^+$, $T^-$, $T^0$, are stable under $\mu_1$. 

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The representation $\mu_1$ also defines an action of $G$ on $P_3(C)$ such that $\pi$ is equivariant. Explicitly

\[
(A, H) \star \begin{pmatrix}
w^1 \\
w^2 \\
z^1 \\
z^2
\end{pmatrix} = \begin{pmatrix}
A & -iHA^*-1 \\
0 & A^*-1
\end{pmatrix}
\begin{pmatrix}
w^1 \\
w^2 \\
z^1 \\
z^2
\end{pmatrix}.
\] (134)

As a consequence of formula (131), the open subsets of $P_3(C)$ denoted by $P_3^+$ and $P_3^-$ are orbits of this action.

The point $[q]$ of $P_3(C)$ is obviously in $P_{3}^{\text{sign}(\nu)}$ so that its orbit is this open subset.

The isotropy subgroup at $[q]$ is $G_\alpha$.

As a consequence we have a diffeomorphism from $P_3^{\text{sign}(\nu)}$ onto the coadjoint orbit of $\alpha$, given by

\[
\Pi : \mu_1(A, H)q \in P_3^{\text{sign}(\nu)} \longrightarrow Ad^*_{(A,H)\alpha} \in \mathcal{M}S.
\]

were I have denoted the coadjoint orbit by $\mathcal{M}S$, because of its interpretation as Movement Space.

We have

\[
\Pi \left( \begin{pmatrix}
w \\
z
\end{pmatrix} \right) = Ad^*_{(A(w,z),H(w,z)\alpha} \tag{135}
\]

so that, the formula for coadjoint representation in section 4 (12), thus leads, after some computation, to

\[
\Pi \left( \begin{pmatrix}
w \\
z
\end{pmatrix} \right) = \frac{2\nu\eta}{\Phi(w)} \left\{ \frac{i}{4} (wz^* + \bar{\epsilon}z\bar{w}\epsilon), -\bar{\epsilon}z\bar{w}\epsilon \right\} \tag{136}
\]

We identify $P_3^{\text{sign}(\nu)}$ to $\mathcal{M}S$, by means of $\Pi$.

Section 4 provides us with well defined expresions for linear and angular momentum in $\mathcal{M}S$, c.f. (8), which, when composed with $\Pi$, give expressions for linear and angular momentum in $P_3^{\text{sign}(\nu)}$. In its hermitian form, these
expressions are
\[ h(P \left( \begin{bmatrix} w \\ z \end{bmatrix} \right)) = \frac{2\eta \nu}{\Phi \left( \begin{bmatrix} w \\ z \end{bmatrix} \right)} \epsilon zz^* \epsilon \]  
\[ \text{(137)} \]

\[ h(I \left( \begin{bmatrix} w \\ z \end{bmatrix} \right), 0) = \frac{\eta \nu}{2\Phi \left( \begin{bmatrix} w \\ z \end{bmatrix} \right)} (zw^* + wz^* + \epsilon(zw^* + wz^*)\epsilon) \]  
\[ \text{(138)} \]

\[ h(g \left( \begin{bmatrix} w \\ z \end{bmatrix} \right), 0) = \frac{i\eta \nu}{2\Phi \left( \begin{bmatrix} w \\ z \end{bmatrix} \right)} (zw^* - wz^* - \epsilon(zw^* - wz^*)\epsilon) \]  
\[ \text{(139)} \]

The Pauli-Lubanski four vector, when evaluated according with [16], is found to be
\[ h(W \left( \begin{bmatrix} w \\ z \end{bmatrix} \right)) = -\eta \nu h(P \left( \begin{bmatrix} w \\ z \end{bmatrix} \right)). \]  
\[ \text{(140)} \]

In case T=1, the bundle map of the contact manifold onto the coadjoint orbit becomes the canonical map
\[ \pi_1: \begin{bmatrix} w \\ z \end{bmatrix} \in O_1 \rightarrow \begin{bmatrix} w \\ z \end{bmatrix} \in P_3^{\text{sign}(\nu)}, \]
and in the general case, the bundle map is
\[ \pi_\nu: \begin{bmatrix} w \\ z \end{bmatrix} \sqrt{1} \in O_\nu / \sqrt{1} \rightarrow \begin{bmatrix} w \\ z \end{bmatrix} \in P_3^{\text{sign}(\nu)} . \]

Then, the composition with \( \pi_\nu \) of the canonical dynamical variables, are also given by the right hand sides of (137), (138) and (139), but taking \( \Phi = 2\nu \).

We obtain the following formulae for the components of these dynamical variables
\[ P^1 \left( \begin{bmatrix} w \\ z \end{bmatrix} \sqrt{1} \right) = \frac{\eta}{2}(z^1 \bar{z}^2 + z^2 \bar{z}^1) \]
\[ P^2 \left( \begin{bmatrix} w \\ z \end{bmatrix} \sqrt{1} \right) = \frac{i\eta}{2}(z^1 \bar{z}^2 - z^2 \bar{z}^1) \]
\[ P^3 \left( \begin{bmatrix} w \\ z \end{bmatrix} \sqrt{1} \right) = \frac{\eta}{2}(|z^1|^2 - |z^2|^2) \]
\[ P^4 \left( \begin{bmatrix} w \\ z \end{bmatrix} \sqrt{1} \right) = -\frac{\eta}{2}(|z^1|^2 + |z^2|^2) \]

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We can also consider the functions on $T^{\text{sgn}(\nu)}$ obtained by composing the dynamical variables in $\mathcal{M}_S$ with the canonical projection, $\pi$, from $T^{\text{sgn}(\nu)}$ onto $P^{\text{sgn}(\nu)}$, thus obtaining functions whose expression is as the right hand sides of the preceding formulae multiplied by

$$\frac{2\nu}{\Phi(w)}$$

These expressions coincide, up to notational conventions, with the expressions that R. Penrose gives for its energy - momentum and angular momentum in twistor space. We denote these functions by $\tilde{P}^k$, $\tilde{l}^k$, $\tilde{g}^k$.

In general, for all $(a, h)$ in the Lie algebra of $G$, the function it defines on the coadjoint orbit, identified to $P^{\text{sgn}(\nu)}$, is denoted by the same symbol, $(a, h)$, and its composition with $\pi$, $(a, h)$. In a similar way, the infinitesimal generator of the action on $T^{\text{sgn}(\nu)}$ defined by the representation $\mu_1$, associated to $(a, h)$, i.e. the vector field whose flow is given by $\mu_1(\text{Exp}(-t(a, h)))$, is denoted by

$$\tilde{X}_{(a, h)}.$$

Let us consider the following one form on $T^{\text{sgn}(\nu)}$

$$\omega_0 = \frac{i\nu}{2\Phi}(z^1d\bar{w}^1 + w^1d\bar{z}^1 + z^2d\bar{w}^2 + w^2d\bar{z}^2)$$
\[- z^1 dw^1 - w^1 dz^1 - z^2 dw^2 - w^2 dz^2 \]

A computation lead us to
\[ \omega_0 \left( X_{(a, h)} \right) = - (a, h). \quad (141) \]

Let us denote by \( \omega \) the restriction of \( \omega_0 \) to \( O_\nu \).

The one form \( \omega \) is invariant by the action of \( \sqrt{T} \), so that it projects to a well defined one form on \( O_\nu / \sqrt{T} \), that we denote by \( \omega_\nu \).

We know that \( O_\nu / \sqrt{T} \), represent the homogeneous contact manifold corresponding to the kind of particle under consideration. As a consequence of (141) and (21) is not difficult to prove that \( \omega_\nu \) is the contact form.

As a consequence of the fact that the map of \( O_\nu \) on \( O_\nu / \sqrt{T} \) is a covering map, \( \omega \) is also a contact form on \( O_\nu \), that becomes itself an homogeneous contact manifold.

The two form \( d(\omega_\nu) \) thus projects under \( \pi_\nu \) on the symplectic form on \( P_{3_{\text{sign}(\nu)}} \), \( \Omega_\nu \), that corresponds to Kirillov form in the identification of \( P_{3_{\text{sign}(\nu)}} \) with the coadjoint orbit.

Then, \( d\omega \) also projects on \( \Omega_\nu \), under the restriction of the canonical map \( \pi \) to \( O_\nu \).

But \( d\omega \) is the restriction to \( O_\nu \) of \( d\omega_0 \) and we have
\[
d\omega_0 = \frac{i \eta_\nu}{\Phi} (dz^1 \wedge dw^1 + dw^1 \wedge dz^1 + dz^2 \wedge dw^2 + dw^2 \wedge dz^2) + (d\Phi) \wedge \delta,
\]
where \( \delta \) is a one form.

Since \( d\Phi \) vanishes on \( O_\nu \), it follows that \( d\omega \) is also the restriction to \( O_\nu \) of
\[
\Omega = \frac{i \eta_\nu}{\Phi} (dz^1 \wedge dw^1 + dw^1 \wedge dz^1 + dz^2 \wedge dw^2 + dw^2 \wedge dz^2).
\]

Thus, we can obtain explicit expressions of \( \Omega_\nu \) as follows: for each differentiable section of \( \pi \) with values in \( O_\nu \)
\[
\sigma : U \to O_\nu \subset T^{3_{\text{sign}(\nu)}},
\]
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we have on the open set $U$

$$\Omega_\nu = \sigma^* \Omega_0,$$

Also we have

$$\Omega_\nu = d (\sigma^* \omega_0).$$

### 12.4.1 Local expression of the symplectic form.

In $\mathbb{C}^3$ we define

$$D = \{(t, u, v) : \text{sign}(\nu) \Re(\phi(t, u, v)) > 0\},$$

where $\phi(t, u, v) = u + 7v$ and $\Re$ stands for real part.

In $\mathbb{P}^3_{\text{sgn}(\nu)}$ we define

$$D_1 = \left\{ \begin{bmatrix} w^1 \\ w^2 \\ z^1 \\ z^2 \end{bmatrix} : w^1 \neq 0, \text{sign}(\nu) \Re(\Phi\left( \begin{bmatrix} w^1 \\ w^2 \\ z^1 \\ z^2 \end{bmatrix} \right)) > 0 \right\}$$

$$D_2 = \left\{ \begin{bmatrix} w^1 \\ w^2 \\ z^1 \\ z^2 \end{bmatrix} : w^2 \neq 0, \text{sign}(\nu) \Re(\Phi\left( \begin{bmatrix} w^1 \\ w^2 \\ z^1 \\ z^2 \end{bmatrix} \right)) > 0 \right\}$$

$$D_3 = \left\{ \begin{bmatrix} w^1 \\ w^2 \\ z^1 \\ z^2 \end{bmatrix} : z^1 \neq 0, \text{sign}(\nu) \Re(\Phi\left( \begin{bmatrix} w^1 \\ w^2 \\ z^1 \\ z^2 \end{bmatrix} \right)) > 0 \right\}$$

$$D_4 = \left\{ \begin{bmatrix} w^1 \\ w^2 \\ z^1 \\ z^2 \end{bmatrix} : z^2 \neq 0, \text{sign}(\nu) \Re(\Phi\left( \begin{bmatrix} w^1 \\ w^2 \\ z^1 \\ z^2 \end{bmatrix} \right)) > 0 \right\}.$$

The $D_k$ compose an open cover of $\mathbb{P}^3_{\text{sgn}(\nu)}$. 

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The maps

\[ \psi_1 : (t, u, v) \in D \rightarrow \begin{bmatrix} 1 \\ t \\ u \\ v \end{bmatrix} \in D_1 \]

\[ \psi_2 : (t, u, v) \in D \rightarrow \begin{bmatrix} v \\ 1 \\ t \\ u \end{bmatrix} \in D_2 \]

\[ \psi_3 : (t, u, v) \in D \rightarrow \begin{bmatrix} u \\ v \\ 1 \\ t \end{bmatrix} \in D_3 \]

\[ \psi_4 : (t, u, v) \in D \rightarrow \begin{bmatrix} t \\ u \\ v \\ 1 \end{bmatrix} \in D_4 \]

are such that the \((D_k, (\psi_k)^{-1})\) compose an atlas of \(P_{3}^{\text{sgn}(\nu)}\).

For all of these charts the coordinates will be denoted by \((t, u, v)\).

We also define sections of \(\pi_{\nu}\),

\[ \sigma_k : D_k \rightarrow O_{\nu}, \quad k = 1, \ldots, 4, \]

by means of

\[ \sigma_1 \circ \psi_1 : (t, u, v) \in D \rightarrow F(t, u, v) \begin{bmatrix} 1 \\ t \\ u \\ v \end{bmatrix} \in D_1 \]

\[ \sigma_2 \circ \psi_2 : (t, u, v) \in D \rightarrow F(t, u, v) \begin{bmatrix} v \\ 1 \\ t \\ u \end{bmatrix} \in D_2 \]

\[ \sigma_3 \circ \psi_3 : (t, u, v) \in D \rightarrow F(t, u, v) \begin{bmatrix} u \\ v \\ 1 \\ t \end{bmatrix} \in D_3 \]
\[ \sigma_4 \circ \psi_4 : (t, u, v) \in D \to F(t, u, v) \begin{pmatrix} t \\ u \\ v \\ 1 \end{pmatrix} \in D_4 \]

where

\[ F(t, u, v) = \sqrt{\frac{2\nu}{\Re(\phi(t, u, v))}}. \]

For all \( k \) we have

\[ (\sigma_k \circ \psi_k)^* \omega = \frac{\eta \nu}{\Re(\phi)}(d(\Im(\phi)) + i(v \, dt - \overline{\nu} \, dt)), \]

(142)

Where \( \Im(\phi) \) is the imaginary part of \( \phi \).

Then, the local expression of \( \Omega_\nu \) in all of these coordinate systems can be evaluated by means of

\[ \Omega_\nu \overset{\text{loc}}{=} d((\sigma_k \circ \psi_k)^* \omega). \]

(143)

for all \( k \).

Notice that \( \omega \) is not projectable on \( P^{\text{sgn}(\nu)}_3 \). Thus (142) need not be local expressions of a well defined 1-form on all of \( P^{\text{sgn}(\nu)}_3 \).

### 12.5 Alternative form of Wave Functions for the Photon.

#### 12.5.1 Symmetric Wave Functions of the Photon.

There are many proposals in the literature for Wave Functions of the Photon. Including someones that asserts that such a thing does not exist.

In this paper I have described the Wave Functions of the Massless Type 4 Particles, where the case \( T = 2 \) corresponds to the Photon. These Wave Functions satisfies the Penrose Wave Equations.

In this section I describe other forms of the Wave Functions of Photon. Equivalent representations. These forms enables us to relate directly the Wave Functions with the Electromagnetic Potential and the Electromagnetic Field.
Photon is the name of four kinds of particles: the massless Type 4 particles with $T = 2$, i.e. those corresponding to

$$\gamma = \left\{ \frac{i\chi}{4\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \eta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \chi, \eta \in \{\pm 1\} \quad (144)$$

We already know that the value of $-\eta$ is the sign of energy and that $-\eta\chi$ is the sign of helicity (c.f. section 12.3.1). We denote $\ell = -\eta\chi$.

From section 12 we obtain

$$G_\gamma = \left\{ \left( \begin{pmatrix} z & a \\ 0 & \bar{z} \end{pmatrix}, \begin{pmatrix} b & i\chi a z/\pi \\ i\chi a z/\pi & 0 \end{pmatrix} \right) : z \in \mathbb{S}^1, \ b \in \mathbb{R}, \ a \in \mathbb{C} \right\}$$

and the homomorphism

$$C_\gamma \left( \left( \begin{pmatrix} z & a \\ 0 & \bar{z} \end{pmatrix}, \begin{pmatrix} b & i\chi a z/\pi \\ i\chi a z/\pi & 0 \end{pmatrix} \right) \right) = z^{2\chi}$$

has differential $\gamma$.

$$(G_\gamma)_\mathbb{S} = \left\{ \begin{pmatrix} z & a \\ 0 & \bar{z} \end{pmatrix} : z \in \mathbb{S}^1, \ a \in \mathbb{C} \right\}$$

$$(C_\gamma)_\mathbb{S} \left( \begin{pmatrix} z & a \\ 0 & \bar{z} \end{pmatrix} \right) = z^{2\chi}$$

$$SL_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in \mathbb{C} \right\}$$

$$SL_2 = (G_\gamma)_\mathbb{S}$$

$$SL_1 \cap SL_2 = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} : z \in \mathbb{S}^1 \right\}$$

In section 12 we have identified the space $SL/(G_\gamma)_\mathbb{S}$ to the future lightcone, $\mathbb{C}^+ = \{ H \in H(2) : \ Det H = 0, \ Tr H > 0 \}$, by means of the

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diffeomorphism

\[ A(\gamma)_{SL} \in SL/(G_{\gamma})_{SL} \mapsto A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A^* \in C^+. \]

In this section I give another description of the wave functions of a photon, by means of different trivialisations than the ones I have used in section 12. Of course these trivialisations are “isomorphic”, in an obvious sense.

Let us consider first the cases \( \chi = +1 \) i.e. \( \ell = -\eta \).

In these cases, a trivialisation (c.f. section 7) is

\[ (\mu_+, S, s_0^+) \],

where \( S \) is the vector subspace of \( gl(2, C) \), composed by the symmetric matrices,

\[ s_0^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

and

\[ \mu_+(A) \cdot s = As^tA, \]

for all \( A \in SL(2, C) \), \( s \in S \).

Since

\[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0) \]

the orbit of

\[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

by \( \mu_+ \) is composed by the elements of \( S^2 \) of the form

\[ (A \begin{pmatrix} 1 \\ 0 \end{pmatrix}) A^t(A \begin{pmatrix} 1 \\ 0 \end{pmatrix}), \]

for some \( A \in SL(2, C) \). Then, one sees that the orbit is contained in

\[ B = \{ z \ z : z \in C^2 - \{0\} \}. \]

But every \( z \in C^2 - \{0\} \) has the form

\[ A \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]
for some $A \in SL(2, \mathbb{C})$, so that the orbit coincides with $B$.

For each $s \in S$ such that $s \neq 0$ and $Det s = 0$, there exist exactly two $z \in \mathbb{C}^2 - 0$ such that $s = z^t z$. In fact, if

$$s = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in B$$

the $z$ such that $s = z^t z$ are

$$\pm \begin{pmatrix} \alpha \\ \sigma \delta \end{pmatrix}$$

where $\alpha$ is a square root of $a$, $\delta$ is a square root of $d$, and $\sigma \in \{\pm 1\}$ such that $b = \sigma \alpha \delta$.

As a consequence, we also have

$$B = \{s \in S : s \neq 0, Det s = 0\}.$$  

The homogeneous space $SL/Ker(C_\gamma)_{SL}$ is identified to $B$ by means of

$$A Ker(C_\gamma)_{SL} \in SL/Ker(C_\gamma)_{SL} \rightarrow \mu_+(A) \cdot s_0^+ \in B$$

Then, the canonical map

$$SL/Ker(C_\gamma)_{SL} \rightarrow SL/(G_\gamma)_{SL},$$

denoted in the following by $r_+$, becomes

$$r_+ : (A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})^t A) \in B \rightarrow A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A^* \in C^+,$$

for all $A \in SL$, so that

$$r_+(z^t z) = zz^*$$

for all $z \in \mathbb{C}^2 - 0$.

By elementary operations with matrices, one can prove that the relation

$$r_+(s) = C$$

is equivalent to

$$C = (+Tr s\bar{s})^{-1/2} s\bar{s}$$
and also to

\[ s = (+(Tr C'C)^{-1/2})e^{i\phi}(C'C) \]

for some $\phi \in \mathbb{R}$.

To obtain Wave Functions, we need functions on $\mathcal{B}$ homogeneous of degree -1 under product by modulus one complex numbers. If $f$ is one of such functions the corresponding Prewave Function is

\[ \psi_f^{(-\eta,-\eta)}(H, K) = f(s) s e^{2\pi i \langle h^{-1}(K), h^{-1}(H) \rangle_m}, \]

where $s$ is arbitrary in $r^+_\pm(K)$. In $(-\eta, -\eta)$ the first $-\eta$ stands for the sign of energy and the second by $\ell$, helicity.

Now, let us consider the cases $\chi = -1$ i.e. $\ell = \eta$.

In these cases, a trivialisation is

\[ (\mu_-, S, s_0^-), \]

where $S$ is as in the $\chi = +1$ case,

\[ s_0^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

and

\[ \mu_-(A) \cdot s = (A^*)^{-1} s (\overline{A})^{-1}, \]

for all $A \in SL(2, \mathbb{C}), s \in S$.

The orbit of

\[ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

by $\mu_-$ is, as in case $\chi = +1$, $\mathcal{B}$.

The canonical map

\[ r_- : SL/Ker(C_\gamma)_{SL} \longrightarrow SL/(G_\gamma)_{SL} \]

becomes a map from $\mathcal{B}$ onto $\mathbb{C}^+$, given by

\[ r_-(\mu_-(A) \cdot s_0^-) = A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A^*. \]
The maps $r_+$ and $r_-$ are geometrically related as follows.

Let us consider the map

$$J : s \in S \longrightarrow -\epsilon \bar{\epsilon} \in S.$$  \hfill (146)

This is an antilinear map with

$$J^2 = I.$$

Since

$$J(z^t \bar{z}) = (-\epsilon \bar{z})^t(-\epsilon \bar{z}),$$

we have $J(B) = B$.

On the other hand

$$r_- \circ J(\mu_+ (A) \cdot s_0^+) = r_-(-\epsilon A \bar{s}_0 \ A^* \epsilon) = r_-(-\epsilon (A^*)^{-1} \epsilon \bar{s}_0 \epsilon (A)^{-1}) =$$

$$= r_- (\mu_- (A) \cdot s_0^+) = A \ (G_\gamma)_{SL} = r_+ (\mu_+ (A) \cdot s_0^+),$$

so that

$$r_+ = r_- \circ J.$$ Since $J^2 = I$, we also have

$$r_- = r_+ \circ J.$$

As a consequence, $J$ establishes a principal fibre bundle isomorphism over the identical map of $C^+$, the isomorphism of structural groups being the map defined by sending each element of $S^1$ to its inverse.

Another consequence is

$$r_-(z^t \bar{z}) = -\epsilon \bar{z}z^* \epsilon.$$  \hfill (147)

If $f$ is a function on $B$ homogeneous of degree -1 under product by modulus one complex numbers, it also defines a Prewave Function for this kind of photon by means of

$$\psi_f^{(-\eta, \eta)}(H, K) = f(s) \ s e^{2\pi i \eta (h^{-1}(K), h^{-1}(H))_m},$$  \hfill (147)

where $s$, now, is arbitrary in $r_-^{-1}(K)$. Here, in $(-\eta, \eta)$, $-\eta$ stands for the sign of energy and the second $\eta$, which in the present case coincides with $\ell$, stands for helicity.
Thus, the Prewave Functions corresponding to the four kinds of Photon are different. The sign of energy, $-\eta$, and the sign of helicity, $\ell$, determines the type of Prewave Functions to be used, $\psi_f^{(-\eta,\ell)}$, that are given by (145) or (147).

When $f$ is continuous with compact support, the corresponding Wave Function is

$$\tilde{\psi}_f^{(-\eta,\ell)}(x) = \int_{C^+} \psi_f^{(-\eta,\ell)}(h(x), K) \omega_K,$$

where $\omega$ is the invariant volume element on $C^+$ defined in (122).

These Wave Functions represent states whose energy has sign $-\eta$ and are eigenvectors of helicity (c.f. section 12.3.1) corresponding to the eigenvalues

$$\ell \over 2\pi.$$

12.5.2 Prehilbert Space structure.

Let us denote by $\mathcal{F}$ the vector subspace composed by the functions on $\mathcal{B}$ homogeneous of degree $-1$ under product by modulus one complex numbers, and by $\mathcal{F}_c$ the subspace of $\mathcal{F}$ composed by the continuous elements with compact support.

The space $\mathcal{F}_c$ can be provided with a prehilbert space structure, by means of the general method described in section 17 for each kind of Photon. In more detail we proceed as follows.

We separate the cases $-\eta \ell = \pm 1$.

If $f, f' \in \mathcal{F}_c$, its hermitian product is in each case

$$\langle f, f' \rangle_{(-\eta,\ell)} = \int_{C^+} (\overline{f})^{(-\eta,\ell)} f'(x) \omega,$$

where $(\overline{f})^{(-\eta,\ell)}$ is the function defined on $C^+$ by

$$(\overline{f})^{(-\eta,\ell)}(K) = \overline{f}(s) f'(s)$$

for all $K \in C^+$, where $s \in r^{-1}_{-\eta\ell}(K)$.
With each of these inner products, $\mathcal{F}_c$ becomes a prehilbert space.

For prewave functions we define
\[
\langle \psi_f^{(-\eta,\ell)}, \psi_{f'}^{(-\eta,\ell)} \rangle_{(-\eta,\ell)} = \langle f, f' \rangle_{(-\eta,\ell)}.
\]

A sexquilinear form on $\mathcal{S}$ is given by
\[
\Phi(s, s') = \text{Tr}(ss').
\]

Thus (c.f. section 7), the hermitian product of Prewave Functions can be given in terms of the Prewave Functions themselves instead of the functions $f$, by
\[
\langle \psi_f^{(-\eta,\ell)}, \psi_{f'}^{(-\eta,\ell)} \rangle_{(-\eta,\ell)} = \int_{C^+} \psi_f^{(-\eta,\ell)} \Phi_{(-\eta,\ell)} \psi_{f'}^{(-\eta,\ell)} \omega;
\]

where $\psi_f^{(-\eta,\ell)} \Phi_{(-\eta,\ell)} \psi_{f'}^{(-\eta,\ell)}$ is the function on $C^+$ given by
\[
\psi_f^{(-\eta,\ell)} \Phi_{(-\eta,\ell)} \psi_{f'}^{(-\eta,\ell)}(K) = \frac{\text{Tr}(\psi_f^{(-\eta,\ell)}(H, K) \psi_{f'}^{(-\eta,\ell)}(H, K))}{\text{Tr}(ss)}
\]
for all $K \in C^+$, $H \in H(2)$ and $s \in r_{-\eta\ell}^{-1}(K)$.

The hermitian product of Wave Functions is defined as being the hermitian product of the corresponding Prewave Functions.

### 12.5.3 Electromagnetic Potential.

If $K \in C^+$, the tangent space to $C^+$ at $K$ can be identified to the subspace of $H(2)$ given by
\[
T_KC^+ = \{ M \in H(2) : \text{Tr} MeK = 0 \}.
\]

Then, the complexified tangent space can be identified to
\[
^{\mathbb{C}}T_KC^+ = \{ M \in \mathfrak{gl}(2, \mathbb{C}) : \text{Tr} MeK = 0 \}.
\]

A real vector field on $C^+$ is thus a map
\[
A : C^+ \longrightarrow H(2),
\]
such that
\[ \text{Tr} A(K) e K e = 0, \quad (151) \]
for all \( K \in C^+ \).

A complex vector field on \( C^+ \) is thus given by a function
\[ A : C^+ \to gl(2, C), \]
whose real and imaginary hermitian parts are real vectorfields on \( C^+ \).

Since \( \text{Tr} M e K e \) is real for \( M \) and \( K \) hermitian, we see that \( A \) is a complex vector field on \( C^+ \) if and only if
\[ \text{Tr} A(K) e K e = 0, \quad (152) \]
for all \( K \in C^+ \).

Equation (152) is equivalent to say that \( A(K) e K \) is a symmetric matrix.

Let \( A \) be a complex vector field on \( C^+ \). We denote by \( A_R \) and \( A_I \) the real and imaginary hermitian parts of \( A \) and
\[
\begin{align*}
A_R(K) &= h(A^1_R(K), ..., A^4_R(K)) \\
A_I(K) &= h(A^1_I(K), ..., A^4_I(K)) \\
A^\mu(K) &= A^\mu_R + i A^\mu_I, \quad \mu = 1, ..., 4 \\
\vec{A}_R(K) &= (A^1_R(K), ..., A^3_R(K)) \\
\vec{A}_I(K) &= (A^1_I(K), ..., A^3_I(K)) \\
\vec{A}(K) &= (A^1(K), ..., A^3(K)).
\end{align*}
\]

If \( A^j_i(K) \) is the element of \( A(K) \) in the row \( i \) column \( j \), we also have
\[
\begin{align*}
A^1(K) &= \frac{1}{2}(A^2_1(K) + A^2_2(K)) \\
A^2(K) &= \frac{i}{2}(A^2_1(K) - A^2_2(K)) \\
A^3(K) &= \frac{1}{2}(A^1_1(K) - A^1_2(K)) \\
A^4(K) &= \frac{1}{2}(A^1_1(K) + A^1_2(K))
\end{align*}
\]

If \( A(K) \) is , for example, continuous with compact support, for \( \mu = 1, ..., 4, \ x \in \mathbb{R}^4 \) we define
\[
\vec{A}^\mu(x) = \int_{C^+} A^\mu(K) \text{Exp}(2\pi i \eta (h^{-1}(K), x)) \omega.
\]

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Then
\[ \square \tilde{A}^\mu = 0, \]
for all \( \mu \), and
\[ \sum_{\mu=1}^{4} \frac{\partial \tilde{A}^\mu}{\partial x^\mu} = 0. \]

We thus see that the \( \tilde{A}^\mu(x) \) define a complex electromagnetic potential in the Lorenz gauje.

The corresponding electric field is given by
\[ \vec{E}(x) = -\nabla \tilde{A}^3(x) - \frac{\partial \tilde{A}(x)}{\partial x^4}, \]
where
\[ \vec{A}(x) = (\tilde{A}^1(x), \tilde{A}^2(x), \tilde{A}^3(x)), \]
that can be written
\[ \vec{E}(x) = 2\pi i\eta \int_{C^+} (A^4(K) \vec{K} - K^4 \vec{A}(K)) \exp(2\pi i\eta(h^{-1}(K), x)) \omega. \]

The magnetic field is given by
\[ \vec{B}(x) = \nabla \times \vec{A}(x), \]
that can be written
\[ \vec{B}(x) = 2\pi i\eta \int_{C^+} (\tilde{A}(K) \times \vec{K}) \exp(2\pi i\eta(h^{-1}(K), x)) \omega. \]

This electromagnetic field take in general complex values, the real parts of \( \vec{E} \) and \( \vec{B} \) are real electric and magnetic fields, corresponding to the electromagnetic potential given by the real parts of the \( A^\mu(x) \).

### 12.5.4 Wave Functions and the Electromagnetic Potential.

In section 12.5.3 we have seen that a complex vector field on \( C^+ \), define a complex electromagnetic potential in the Lorenz gauje.
Let $A$ be a complex vector field on $C^+$. Then, the matrix $A(K)eK$ is a symmetric matrix, for all $K \in C^+$.

If $s \in r^{-1}_+(K)$ there exist an unique complex number, $f_A(s)$, such that

$$A(K)eK = f_A(s) \cdot s. \quad (153)$$

In fact, if $z \in C^2 - 0$, is such that $s = z^t z$, we have $K = zz^*$, and, since

$$0 = Tr A(K)eK z = l^t z eA(K)eK,$$

there exist a number, $\lambda$, such that

$$A(K)eK = \lambda \cdot z,$$

so that

$$A(K)eK = \lambda \cdot s.$$

Since $s \neq 0$ such a $\lambda$ is unique and is denoted by $f_A(s)$.

Thus the complex vector field $A$ defines a function, $f_A$, on $B$.

If we take $e^{i\phi} s$ instead of $s$ in $r^{-1}_+(K)$, we can take in the preceding reasoning $e^{i\phi/2} z$ instead of $z$, and thus

$$A(K)e^{i\phi/2} z = \lambda' e^{i\phi/2} z$$

leads to

$$\lambda' = e^{-i\phi} \lambda.$$

We thus see that

$$f_A(e^{i\phi} s) = e^{-i\phi} f_A(s)$$

which proves that $f_A$ is $S^1$-homogeneous of degree -1.

The Prewave Function corresponding to $f_A$ for $-\eta \ell = 1$ can be written as

$$\psi_A^{(-\eta,-\eta)}(H, K) = f_A(s) \cdot s e^{2\pi i \eta(h^{-1}(K), h^{-1}(H))}_m,$$

where $s$ is arbitrary in $r^{-1}_+(K)$.

As a consequence of (153) we have

$$\psi_A^{(-\eta,-\eta)}(H, K) = A(K)eK e^{2\pi i \eta(h^{-1}(K), h^{-1}(H))}_m, \quad (154)$$

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that gives Prewave Functions directly in terms of vectorfields.

If $A(K)$ is continuous with compact support, the corresponding Wave Function is

$$\tilde{\psi}_A^{(-\eta,-\eta)}(x) = \int_{C^+} A(K) \epsilon K e^{2\pi i n(h^{-1}(K),x)} m_\omega K. \quad (155)$$

Now we define

$$\hat{f}_A = f_A \circ J,$$

where $J$ is given by (146).

I shall prove that $\hat{f}_A$ is $S^1$-homogeneous of degree -1 on $B$ and that, for all $s \in r_-(K)$, we have

$$- \epsilon A(K) \epsilon K \epsilon = \hat{f}_A(s). \quad (156)$$

This will enable us to give a Prewave Function of Photons with $\ell = \eta$, directly in terms of $A$.

For all $s \in B$, $a \in S^1$ we have

$$\hat{f}_A(as) = f_A(J(as)) = f_A(\overline{a} J(s)) = a f_A(J(s)) = \overline{a} \hat{f}_A(s),$$

so that $\hat{f}_A$ has the appropriate homogeneity to define a Prewave Function.

On the other hand, if $s \in r_-(K)$, we have

$$\hat{f}_A(s) s = f_A(J(s)) J(J(s)) = J(f_A(J(s)) J(s)) =$$

$$= J(A(r_+(J(s))) e r_+(J(s))) = J(A(r_-(s)) e r_-(s)) =$$

$$= J(A(K) e K) = - \epsilon A(K) \epsilon K \epsilon.$$

Then, the Prewave Function for Photons with $\ell = \eta$ corresponding to $\hat{f}_A$ is

$$\psi_A^{(-\eta,\eta)}(H, K) = - \epsilon A(K) \epsilon K \epsilon e^{2\pi i n(h^{-1}(K),h^{-1}(H)) m}. \quad (157)$$

Obviously

$$\psi_A^{(-\eta,\eta)} = - \epsilon \psi_A^{(\eta,\eta)} \epsilon, \quad \psi_A^{(\eta,\eta)} = - \epsilon \psi_A^{(-\eta,\eta)} \epsilon.$$
If $A(K)$ is continuous with compact support, the corresponding Wave Function is

$$\tilde{\psi}_A^{(-\eta,\eta)}(x) = - \int_{C^+} e^{A(K)}e x e^{2\pi i n h^{-1}(K), x} \omega_K. \tag{158}$$

We have

$$\tilde{\psi}_A^{(-\eta,\eta)} = -\epsilon \tilde{\psi}_A^{(\eta,\eta)} \epsilon, \quad \tilde{\psi}_A^{(\eta,\eta)} = -\epsilon \tilde{\psi}_A^{(-\eta,\eta)} \epsilon.$$

With equations (155) and (158) we see that a single complex vectorfield on $C^+$ gives rise to Wave Functions of the four kinds of Photons: those with positive and negative energy and helicity.

The inner products defined in section 12.5.2 lead to the following results.

If $A$ and $A'$ are vector fields on $C^+$ we have

$$\langle \psi_A^{(-\eta,\eta)}, \psi_{A'}^{(-\eta,\eta)} \rangle^{(-\eta,\eta)} = \langle f_A, f_{A'} \rangle^{(-\eta,\eta)},$$

and

$$\langle \psi_A^{(-\eta,\eta)}, \psi_{A'}^{(-\eta,\eta)} \rangle^{(-\eta,\eta)} = \langle \widehat{f_A}, \widehat{f_{A'}} \rangle^{(-\eta,\eta)}.$$

Then, from (139) and (150) one can obtain

$$\langle \psi_A^{(-\eta,\eta)}, \psi_{A'}^{(-\eta,\eta)} \rangle^{(-\eta,\eta)} = \int_{C^+} A\Phi A' =$$

$$= \langle \psi_{A'}^{(-\eta,\eta)}, \psi_A^{(-\eta,\eta)} \rangle^{(-\eta,\eta)}$$

where

$$A\Phi A'(K) = \frac{Tr(A(K)\epsilon K A'(K)\epsilon K)}{Tr(\epsilon K)}$$

for all $K \in C^+$, and $s \in \mathbb{r}^{-1}(K)$.

12.5.5 Wave Functions in terms of the Electromagnetic Field.

Let $A$ a complex vectorfield on $C^+$ and denote

$$\vec{E}(K) = A^4(K)\epsilon K - K^4\epsilon \vec{A}(K) \tag{159}$$

$$\vec{B}(K) = \frac{\vec{A}(K) \times \vec{K}}{Tr(\epsilon K)} \tag{160}$$
Thus, the Electric and Magnetic Fields can be given by

\[ \vec{E}(x) = 2\pi i \eta \int_{C^+} \vec{E}(K) \exp(2\pi i \eta \langle h^{-1}(K), x \rangle) \omega_K \]

\[ \vec{B}(x) = 2\pi i \eta \int_{C^+} \vec{B}(K) \exp(2\pi i \eta \langle h^{-1}(K), x \rangle) \omega_K. \]

Let us prove that

\[ A(K) e^{\vec{K} \epsilon} = \left( \vec{E}(K) + i \vec{B}(K) \right) \cdot \vec{\sigma}, \tag{161} \]

which means

\[ A(K) e^{\vec{K} \epsilon} = \left( \begin{array}{cc} \left( \vec{E} + i \vec{B} \right)^3 & \left( \vec{E} + i \vec{B} \right)^1 - i \left( \vec{E} + i \vec{B} \right)^2 \\ \left( \vec{E} + i \vec{B} \right)^1 + i \left( \vec{E} + i \vec{B} \right)^2 & - \left( \vec{E} + i \vec{B} \right)^3 \end{array} \right). \tag{162} \]

In fact, we have

\[ A(K) e^{\vec{K} \epsilon} = \frac{1}{2} \left( A(K) e^{\vec{K} \epsilon} + i(A(K) e^{\vec{K} \epsilon}) \right) \epsilon = \]

\[ = \frac{1}{2} \left( (A_R + i A_I) \epsilon - \epsilon (i A_R + i A_I) \epsilon \right) = \]

\[ = \frac{1}{2} (A_R \epsilon \vec{K} \epsilon - K \epsilon A_R \epsilon) + \frac{i}{2} (A_I \epsilon \vec{K} \epsilon - K \epsilon A_I \epsilon) \]

and thus (162) follows from (1).

As a consequence, the Prewave Functions corresponding to \( A \) are

\[ \psi_A(-\eta, -\eta)(H, K) = - \left( \left( \vec{E}(K) + i \vec{B}(K) \right) \cdot \vec{\sigma} \right) \epsilon e^{2\pi i \eta \langle h^{-1}(K), h^{-1}(H) \rangle} m, \tag{163} \]

\[ \psi_A(-\eta, \eta)(H, K) = -\epsilon \left( \left( \vec{E}(K) + i \vec{B}(K) \right) \cdot \vec{\sigma} \right) e^{2\pi i \eta \langle h^{-1}(K), h^{-1}(H) \rangle} m. \tag{164} \]

If \( A \) is continuous with compact support, we have

\[ \tilde{\psi}_A(-\eta, -\eta)(x) = - \int_{C^+} \left( \left( \vec{E}(K) + i \vec{B}(K) \right) \cdot \vec{\sigma} \right) \epsilon e^{2\pi i \eta \langle h^{-1}(K), h^{-1}(H) \rangle} m \omega_K, \tag{165} \]
\[\tilde{\psi}^{(-\eta,-\eta)}_A(x) = -\int_{C^+} \epsilon \left( (\overrightarrow{E}(K) + i\overrightarrow{B}(K)) \cdot k \right) e^{2\pi i \eta (h^{-1}(K), h^{-1}(H)) \omega_K}. \quad (166)\]

so that the Wave Functions are

\[\tilde{\psi}^{(-\eta,-\eta)}_A(x) = \frac{i\eta}{2\pi} \epsilon \left( (\overrightarrow{E}(x) + i\overrightarrow{B}(x)) \cdot k \right), \quad (167)\]

or

\[\tilde{\psi}^{(-\eta,-\eta)}_A(x) = \frac{i\eta}{2\pi} \epsilon \left( (\overrightarrow{E}(x) + i\overrightarrow{B}(x)) \cdot k \right). \quad (168)\]

These Wave Functions are given in terms of the components of \(\overrightarrow{E}(x) + i\overrightarrow{B}(x)\) as (up to constants) the Wave Functions of Bialynicki-Birula [BB94].

**Gauje invariance for Photons.** Let us denote by \(\mathcal{D}\) the complex vector space of complex vector fields on \(C^+\).

The map

\[A \in \mathcal{D} \longrightarrow f_A \in \mathcal{F}\]

is linear and its kernel is composed by the vector fields on \(C^+\) having the form

\[A(K) = \left( \begin{array}{c} \lambda(z) \\ \mu(z) \end{array} \right) z^*\]

where \(z \in \mathbb{C}^2 - 0\) is such that \(K = zz^*\) and \(\lambda, \mu\) are functions on \(\mathbb{C}^2 - 0\) \(S^1\)-homogeneous of degree +1.

The kernel of the map

\[A \in \mathcal{D} \longrightarrow \hat{f}_A \in \mathcal{F}\]

is the same and will be denoted by \(\mathcal{N}\).

In particular, for each map

\[L : K \in C^+ \rightarrow L(K) \in gl(2, \mathbb{C}),\]

the vectorfield

\[A_L(K) = L(K) K \quad (169)\]

is in \(\mathcal{N}\).
If $A \in D$, $N \in \mathcal{N}$ we have

$$f_{A+N} = f_A$$

$$\tilde{f}_{A+N} = \tilde{f}_A,$$

so that the Prewave and Wave Functions are invariant under the changes $A \to A + N$:

$$\tilde{\psi}_{A+N}(-\eta, \ell) = \tilde{\psi}_A(-\eta, \ell).$$

Because of (167) and (168) we see that also the Electric and Magnetic Fields corresponding to Photon are invariant under the changes $A \to A + N$.

In the particular case $N = A_L$ with $L(K) = g(K) I$, where $g$ is a complex valued function on $C^+$, the change $A \to A + N$ lead to the following change in the Electromagnetic Potential

$$(A + N)\mu(x) = \tilde{A}_{\mu}(x) + \partial^\mu \phi(x),$$

where

$$\phi(x) = -\frac{i\eta}{2\pi} \int_{C^+} g(K)e^{2\pi i\eta(h^{-1}(K),x)}m_\omega K$$

and

$$\partial_{\mu} \phi = g^{\mu\nu} \frac{\partial}{\partial x^\nu} \phi$$

where the $g^{\mu\nu}$ are the components of Minkowski metric.

Thus, the invariance of the Electromagnetic Field of Photons under the change $A(K) \to A(K) + g(K)K$ is a particular case of the well known gauge invariance of general Electromagnetic Fields.

13 The Classical State Space

13.1 General remarks

In this paper, State Space for a particle defined by $\alpha \in \mathcal{G}^*$ has been stated as the homogeneous space that correspond to the orbit of $(0, \alpha)$ in $H(2) \times \mathcal{G}^*$. Thus, as we have seen in section [7] it is identified to

$$\frac{G}{G_{(0,\alpha)}}$$
where 
\[ G_{(0, \alpha)} = G_\alpha \cap (SL \oplus \{0\}) , \]
or, equivalently, to
\[ H(2) \times \frac{SL}{(G_\alpha)_{SL} \cap SL_1} , \]
that is the form adopted in Figure 2 of that section.

The equivariant maps
\[ \iota_4 : H(2) \times \frac{SL}{(G_\alpha)_{SL} \cap SL_1} \to H(2) \times \frac{SL}{(G_\alpha)_{SL}} \]
and
\[ \nu_2 : H(2) \times \frac{SL}{(G_\alpha)_{SL} \cap SL_1} \to \frac{G}{G_\alpha} , \]
in that Figure, will be used in this section.

When we characterize
\[ H(2) \times \frac{SL}{(G_\alpha)_{SL} \cap SL_1} \]
in terms of other concrete manifolds or of some parametrization, we need
to physically interpret the geometric objects that appear, and, in order to
do that, the more effective way is, in general, to know the values of the
Canonical Dynamical Variables in terms of these objects.

But the value of these Dynamical Variables is well known on the coad-
joint orbit i.e. on \( G/G_\alpha \). Thus, its values in State Space can be obtained by
composition with \( \nu_2 \).

Since Prewave Functions are defined on
\[ H(2) \times \frac{SL}{(G_\alpha)_{SL}} \]
its composition with \( \iota_4 \) gives another version of Quantum States, now defined
on the Classical State Space.

In the definition of Prewave Functions \( \text{35} \), space-time only appears in
the exponential, whose exponent is
\[ -2\pi i \langle h^{-1}(P), h^{-1}(H) \rangle_m . \]
where $P$ is the “traslation” of momentum-energy to $SL/(G_\alpha)_{SL}$. The exponent in the composition of Prewave Functions with $\iota_4$ must have the same expression, but changing $P$ by $P \circ \iota_4$. The conmutativity of the diagramm in Figure 2 imply that $P \circ \iota_4$ can be obtained as the impulsion-energy in $G/G_\alpha$ composite with $\nu_2$.

In the next sections I give the expressions of the canonical dynamical variables on State Space, for different kind of particles.

### 13.2 Klein-Gordon particles

We have seen in section 11.1.1 that State Space for Klein-Gordon particles can be identified to

$$H(2) \oplus \frac{SL}{SU(2)},$$

and this homogeneous space to $H^m \times H(2)$.

If $(K, H) \in H^m \times H(2)$ and $A \in SL$ is such that $K = mAA^*$, we denote

- $H = h(\vec{x}, x^4)$
- $\vec{x} = (x^1, x^2, x^3)$
- $K = mh(\vec{k}, k_4)$
- $\vec{k} = (k^1, k^2, k^3)$
- $k^4 = \sqrt{1 + \|\vec{k}\|^2}$.

Thus, the map (58) becomes,

$$\nu_1(K, H) = (A, H) \ast (mI, \vec{0}, 1) =$$

$$= (K, \vec{x} - \frac{x^4}{k^4} \vec{k}, e^{-2\pi i n x^4 / k^4})$$

and the map (57),

$$\nu_2(K, H)) = (A, H) \ast (mI, \vec{0}) = (K, \vec{x} - \frac{x^4}{k^4} \vec{k}). \quad (170)$$

The Canonical Dynamical Variables on State Space are obtained as composition of (70) with $\nu_2$. If we denote $V_{KG} = V \circ \nu_2$ for each dynamical
variable, $V$, we have

$$P_{KG}(mh(\vec{k}, k_4), h(\vec{x}, x^4)) = -\eta mh(\vec{k}, k_4),$$

$$\vec{I}_{KG}(mh(\vec{k}, k_4), h(\vec{x}, x^4)) = \vec{P} \times \left( \vec{P}_{KG}(mh(\vec{k}, k_4), h(\vec{x}, x^4)) \right),$$

$$\vec{f}_{KG}(mh(\vec{k}, k_4), h(\vec{x}, x^4)) =$$

$$= P_{KG}^4(mh(\vec{k}, k_4), h(\vec{x}, x^4)) \vec{x} - x^4 \vec{P}_{KG}(mh(\vec{k}, k_4), h(\vec{x}, x^4)).$$

where $k_4 = \sqrt{1 + \|\vec{k}\|^2}.$

The Prewave functions have been defined on a manifold that, in this case, coincides with State Space. The map $\iota_4$ is the identical map.

Another fact is that, in this case, State Space is the Universal Covering of the homogeneous contact manifold.

Let us prove that $\nu_1$ is a covering map.

The local expresion of $\nu_1$ in the charts corresponding to $\phi_{\tau}$ and

$$\phi': (k_1, k_2, k_3, x^1, x^2, x^3, x^4) \in \mathbb{R}^7 \longrightarrow$$

$$\longrightarrow (h(m \vec{k}, mk_4), h(x^1, x^2, x^3, x^4)) \in \mathcal{H}^m \times \mathcal{H}(2),$$

is given by

$$(\phi_{\tau})^{-1} \circ \nu_1 \circ \phi'(k_1, \ldots, x^4) =$$

$$(k_1, k_2, k_3, x^1 - x^4 \frac{x^4}{k_4} k_1, x^2 - x^4 \frac{x^4}{k_4} k_2, x^3 - x^4 \frac{x^4}{k_4} k_3, -\tau - \eta m \frac{x^4}{k_4} + N)$$

where the domain of definition is defined by the condition that

$$\tau + \eta m \frac{x^4}{k_4} \notin \mathbb{Z} + \frac{1}{2}$$

and $N$ is defined by the condition that

$$-\tau - \eta m \frac{x^4}{k_4} + N \in (-\frac{1}{2}, \frac{1}{2}).$$

We have

$$((\phi_{\tau})^{-1} \circ \nu_1 \circ \phi')^{-1}\{(\vec{m}, \vec{y}, t)\} =$$

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\[ \{(\vec{n}, \vec{y} - \frac{\eta}{m}(t + \tau + N)\vec{n}, -\frac{\eta}{m}(t + \tau + N)\sqrt{1 + \|\vec{n}\|^2}) : N \in \mathbb{Z}\}. \]

Let \( d \in \mathcal{H}^m \times \mathbb{R}^3 \times S^1 \), and let \( \tau \) be such that \( d \) is in the image of \((\phi_\tau)^{-1}\). This image, \( U_\alpha \), is an open neighborhood of \( d \), and we shall prove that its antiimage by \( \nu_1 \) is union of disjoint open sets, each of them diffeomorphic to \( U_\alpha \) under the restriction of \( \nu_1 \).

We denote by \( U \) the domain of \( \phi_\tau \).

The domain of \((\phi_\tau)^{-1} \circ \nu_1 \circ \phi' \),

\[ W = ((\phi_\tau)^{-1} \circ \nu_1 \circ \phi')^{-1}(U) \]

is the union of the open sets

\[ W_N = \{(\vec{n}, \vec{y} - \frac{\eta}{m}(t + \tau + N)\vec{n}, -\frac{\eta}{m}(t + \tau + N))\sqrt{1 + \|\vec{n}\|^2} : (\vec{n}, \vec{y}, t) \in U\}. \]

where \( N \in \mathbb{Z} \).

But the \( W_N \) are disjoint. In fact, if

\[ (\vec{n}, \vec{y} - \frac{\eta}{m}(t + \tau + N)\vec{n}, -\frac{\eta}{m}(t + \tau + N))\sqrt{1 + \|\vec{n}\|^2} = (\vec{n}', \vec{y}' - \frac{\eta}{m}(t' + \tau + N')\vec{n}', -\frac{\eta}{m}(t' + \tau + N'))\sqrt{1 + \|\vec{n}'\|^2} \]

where \((\vec{n}, \vec{y}, t), (\vec{n}', \vec{y}', t') \in U\), we obtain first \( \vec{n} = \vec{n}' \) and then \( t + N = t' + N' \). Since \(-1/2 < t, t' < 1/2\), it follows that \( N = N' \).

The restriction of \( \nu_1 \) to each of the \( W_N \) is a diffeomorphism onto \( U \).

Since \( \mathcal{H}^m \times H(2) \) is simply connected, we see that State Space is the universal covering space of the contact manifold \( G/Ker C_\alpha \). By reciprocal image it inherits a contact form that must also be homogeneous: it is the contact structure used in section 3. This homogeneous contact form can also be introduced directly by means of \( C_\alpha' \), in the same way that the homogeneous contact structure has been introduced in \( G/Ker C_\alpha \).

The covering can be also considered in the following way.
We define an action of \( \mathbb{Z} \) on \( H^m \times H(2) \) by

\[
N \ast (K, H) = (K, H - N \frac{\eta}{m^2} K).
\]

With the parametrization \( \phi' \) we have

\[
(\phi')^{-1} \circ (N \ast) \circ \phi'((\overrightarrow{k}, \overrightarrow{x}, x^4)) =
\]

\[
(\overrightarrow{k}, \overrightarrow{x} - N \frac{\eta}{m} \overrightarrow{k}, x^4 - N \frac{\eta}{m} k^4).
\]

We thus have a properly discontinuous action without fixed point of \( \mathbb{Z} \) on State Space and the quotient space is \( G/Ker C_\alpha \).

### 13.3 Massive particles with \( T \neq 0 \).

State Space is \( H(2) \times (\text{SL}/[S^1]) \) or, equivalently, \( G/([S^1] \oplus \{0\}) \).

In section 11.2.1 we have identified \( \text{SL}/[S^1] \) with \( H^m \times P_1(C) \).

Contrarily to the Klein-Gordon case, if \( T \neq 0 \), there exist no equivariant map from State Space onto the contact manifold \( G/\mathbb{R} \).

The canonical map \( \nu_2 \) from State Space onto Movement Space, is such that

\[
\nu_2(A \ast (mI, \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]), H) =
\]

\[
(A, H) \ast \left( mI, \left[ \begin{array}{c} 1 \\ 0 \end{array} \right], \overrightarrow{0} \right)
\]

so that

\[
\nu_2(mAA^*, \left[ A \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right], H) =
\]

\[
\left( mAA^*, \left[ A \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right], (H^1, H^2, H^3) - \frac{H^4}{(AA^*)^3}((AA^*)^1, (AA^*)^2, (AA^*)^3) \right)
\]

\[\in H^m \times P_1(C) \times \mathbb{R}^3.\]

where I have denoted, for each \( L \in H(2) \), \( h^{-1}(L) \overset{\text{def}}{=} (L^1, L^2, L^3, L^4) \). Then

\[
\nu_2(mh((\overrightarrow{k}, k_4), [u], h(\overrightarrow{x}, x^4)) =
\]

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where \( k_4 = \sqrt{1 + \| \vec{k} \|^2} \).

Thus, the portrait in State Space of the movement \((mh(\vec{k}, k_4), [z], \vec{\nu})\) is

\[
(\nu_2)^{-1}\{(mh(\vec{k}, k_4), [z], \vec{\nu})\} = (mh(\vec{k}, k_4), [z], h(\vec{\nu}, 0) + \lambda h(\vec{k}, k_4)) : \lambda \in \mathbb{R}.
\]

These results on Canonical Dynamical Variables are better expressed when we use the identification of \( P_1(C) \) with \( S^2 \) given by (79).

If \((mh(\vec{k}, k_4), \vec{v}, h(\vec{x}, x_4)) \in \mathcal{H}^m \times S^2 \times H(2)\), is interpreted as a state of a particle with mass \( m > 0 \), the values of the canonical dynamical variables in this state are obtained by composition of (99) with the map \( \nu_2 \).

When for each dynamical variable \( V \) we denote \( V \circ \nu_2 \) by \( \nu_{\text{massive}} \) we have

\[
P_{\text{massive}}(mh(\vec{k}, k_4), \vec{v}, h(\vec{x}, x_4)) = -\eta mh(\vec{k}, k_4),
\]

\[
\vec{l}_{\text{massive}}(mh(\vec{k}, k_4), \vec{v}, h(\vec{x}, x_4)) = \frac{T}{4\pi k_4} \frac{k_4 \vec{v} - \vec{k}}{\langle \vec{k}, \vec{v} \rangle} + \vec{x} \times \vec{P}_{\text{massive}},
\]

\[
\vec{g}_{\text{massive}}(mh(\vec{k}, k_4), \vec{v}, h(\vec{x}, x_4)) = \frac{T}{4\pi k_4} \frac{\vec{k} \times \vec{v}}{\langle \vec{k}, \vec{v} \rangle} + P_{\text{massive}}^4 \vec{x} - x_4 \vec{P}_{\text{massive}},
\]

In what concerns the Pauli-Lubanski fourvector, we have

\[
\vec{W}_{\text{massive}} = P_{\text{massive}}^4 \frac{T}{4\pi k_4} \frac{k_4 \vec{v} - \vec{k}}{\langle \vec{k}, \vec{v} \rangle},
\]

\[
W_{\text{massive}} = -\eta m \frac{T}{4\pi} \frac{\langle k_4 \vec{v} - \vec{k}, \vec{k} \rangle}{k_4 - \langle \vec{k}, \vec{v} \rangle}.
\]  

As in \( T = 0 \) case, the prewave functions have been defined directly on State Space, the map \( \iota_4 \) being the identical map. Thus they are exactly as in section 11.2.2.
13.4 Massless particles of type 4

Since \((G_\alpha)_SL \cap SL_1 = [S^1]\), State Space for massless particles, \(H(2) \times (SL/(G_\alpha)_SL \cap SL_1)\), can be identified to State Space for massive particles, \(\mathcal{H}^m \times P_1(C) \times H(2)\). Here \(m\) is an arbitrary positive number, with no physical significance.

On the other hand, Movement Space, \(G/G_\alpha\), is identified to \(P_3^{\text{sign}(\nu)}\) by means of the action (134).

The canonical map \(\nu_2\) from State Space onto Movement Space thus becomes for this kind of particle

\[
\nu_2((A, H) \ast (mI, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}), 0) = (A, H) \ast [q]
\]

where

\[
q = \begin{pmatrix} 0 \\ 2\nu \\ 0 \\ 1 \end{pmatrix}.
\]

Then

\[
\nu_2(mAA^*, \begin{pmatrix} A(1) & 0 \\ 0 & 0 \end{pmatrix}, H) = \begin{pmatrix} 2\nu A \begin{pmatrix} 0 \\ 1 \end{pmatrix} - iH(A^*)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ (A^*)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}.
\]

But \((A^*)^{-1} = -\bar{\epsilon}A\epsilon\), so that

\[
\nu_2(mAA^*, \begin{pmatrix} A(1) & 0 \\ 0 & 0 \end{pmatrix}, H) = \begin{pmatrix} (2\nu AA^* - iH)(A^*)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ (A^*)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (2\nu AA^* - iH)\bar{\epsilon}A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \bar{\epsilon}A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}.
\]
Then
\[ \nu_2 : (K, [u], H) \in \mathcal{H}^m \times P_1(C) \times H(2) \rightarrow \left[ \begin{array}{c} (2\nu_m K - iH) \varepsilon \pi \\ \varepsilon \pi \end{array} \right] \in P_3^{\text{sign}(\nu)}. \] (172)

If we fix, for example, \( m = 1 \), then for all
\[ \left[ \begin{array}{c} \omega \\ \pi \end{array} \right] \in P_3^{\text{sign}(\nu)}, \]
we have
\[ (\nu_2)^{-1}\left\{ \left[ \begin{array}{c} \omega \\ \pi \end{array} \right] \right\} = \{ (K, [\varepsilon \pi], H) \in \mathcal{H}^1 \times P_1(C) \times H(2) : \left( \frac{2\nu}{m} K - iH \right) \pi = \omega \}. \]

This set represent the “portrait” of the movement
\[ \left[ \begin{array}{c} \omega \\ \pi \end{array} \right] \]
in \( \mathcal{H}^1 \times P_1(C) \times H(2) \) considered as State Space of massless particles.

The values of dynamical variables \( P, \vec{l}, \vec{g} \), on \( \mathcal{H}^m \times P_1(C) \times H(2) \) for the massless Type 4 particles can be obtained by composition of its expressions (137), (138), (139), with the map \( \nu_2 \) given in (172).

If \( (m h(\vec{k}, k_4), \vec{v}, h(\vec{x}, x_4)) \in \mathcal{H}^m \times S^2 \times H(2) \), is interpreted as a state of a massless Type 4 particle, where now \( m \) is an arbitrary positive number with no physical significance, the values of the canonical dynamical variables in this state are
\[
\begin{align*}
\vec{P}_{\text{massless}}(m h(\vec{k}, k_4), \vec{v}, h(\vec{x}, x_4)) &= P_{4\text{massless}} \vec{v}, \\
\vec{P}_{4\text{massless}}(m h(\vec{k}, k_4), \vec{v}, h(\vec{x}, x_4)) &= \frac{-\eta}{2(k_4 - \langle \vec{k}, \vec{v} \rangle)} \\
\vec{l}_{\text{massless}}(m h(\vec{k}, k_4), \vec{v}, h(\vec{x}, x_4)) &= \frac{\chi T}{4\pi} \frac{k_4 \vec{v} - \vec{k}}{k_4 - \langle \vec{k}, \vec{v} \rangle} + \vec{x} \times \vec{P}_{\text{massless}}, \\
\vec{g}_{\text{massless}}(m h(\vec{k}, k_4), \vec{v}, h(\vec{x}, x_4)) &= \frac{\chi T}{4\pi} \frac{\vec{k} \times \vec{v}}{k_4 - \langle \vec{k}, \vec{v} \rangle} + P_{4\text{massless}} \vec{x} - x_4 \vec{P}_{\text{massless}}.
\end{align*}
\]
The Pauly-Lubanski four vector for type 4 particles is \((c.f. \text{ section } 9)\)

\[ W_{\text{massless}} = \frac{\chi T}{4\pi} P_{\text{massless}}. \]

We thus see that the expression of \(T\) and \(\overline{g}\) are \textit{formally} almost identical for massive and massless particles. But this is not the case for \(P\).

The point \((m \overrightarrow{k}, k_4), \overrightarrow{v}, h(\overrightarrow{x}, x^4))\) can be interpreted as being the state of a massive particle or the state of a massless particle, but the value in this state of linear momentum, energy and angular momentum is different in the massive or in the massless case.

The map \(\iota_4\), what in the case of massive particles whas the identical map, in the case of massless particles can be found to be

\[
\iota_4 : (m \overrightarrow{k}, k_4), \overrightarrow{v}, h(\overrightarrow{x}, x^4)) \in \mathcal{H}^m \times S^2 \times H(2) \rightarrow \\
\rightarrow \left( \frac{h(\overrightarrow{v}, 1)}{2(k_4 - \langle \overrightarrow{k}, \overrightarrow{v} \rangle)}, h(\overrightarrow{x}, x^4)) \right) \in C^+ \times H(2)
\]

The \(P_{KG}\), \(P_{\text{massive}}\) and \(P_{\text{massless}}\) appears in the exponent of the Prewave Functions in State Space of the corresponding particles.

References

[AK71] L. Auslander and B. Kostant. Polarization and unitary representations of solvable lie groups. \textit{Inventiones math.}, 14:255–354, 1971.

[BB94] I. Bialynicki-Birula. On the wave function of the photon. \textit{Acta Physica Polonica A}, 86:98–116, 1994.

[BW48] V. Bargmann and E. P. Wigner. Group theoretical discussion of relativistic wave equations. \textit{Proc. N. A. S.}, 34:211–223, 1948.

[Dia82a] A. Diaz Miranda. Classification des variétés homogènes de contact. \textit{C.R. Acad. Sc. Paris}, 294:235–238, 1982.

[Dia82b] A. Diaz Miranda. Variétés homogènes de contact et espaces hamiltoniens. \textit{C.R. Acad. Sc. Paris}, 294:517–520, 1982.
[Dia91] A. Diaz Miranda. Sistemas de pfaff homogéneos no degenerados. Memorias de la Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid. Serie de Ciencias Exactas, Tomo 26:7–35, 1991.

[Dia96a] A. Diaz Miranda. Quantizable forms. Journal of Geometry and Physics, 19:47–76, 1996.

[Dia96b] A. Diaz Miranda. Wave functions in geometric quantization. International J. of Theor. Phys, 35(10):2139–2168, 1996.

[Dir36] P. A. M. Dirac. Proc. Roy. Soc. A., 155:447, 1936.

[Kir62] A. A. Kirillov. Unitary representations of nilpotent lie groups. Uspehi. Mat. Nauk., 17:57–110, 1962.

[KN63] S. Kobayashi and K. Nomizu. Foundations of Differential Geometry Vol.1. Interscience Publishers, 1963.

[Maj32] E. Majorana. Nuovo Cim., 9:335, 1932.

[Pen75] R. Penrose. Twistor theory, its aims and achievements. In R. Penrose C. J. Isham and D. W. Sciama, editors, Quantum Gravity, pages 141–193. Oxford Univ. Press, 1975.

[Pro36] A. Proca. J. de Phys. Rad., 7:347, 1936.

[Sou70] J. M. Souriau. Structure des systémes dynamiques. Dunod Université. Dunod, 1970.

[Sou88] J. M. Souriau. Quantification géométrique. In Physique quantique et géométrie, pages 141–193. Travaux en Cours, 32, 1988.

[Wig39] E. P. Wigner. On unitary representations of the inhomogeneous lorenz group. Annals of Math., 40:149–204, 1939.