Cubillages on cyclic zonotopes, membranes, and higher separation

Vladimir I. Danilov∗ Alexander V. Karzanov† Gleb A. Koshevoy‡

Abstract. We study certain structural properties of fine zonotopal tilings, or cubillages, on cyclic zonotopes $Z(n,d)$ of an arbitrary dimension $d$ and their relations to $(d-1)$-separated collections of subsets of a set $\{1, 2, \ldots, n\}$. (Collections of this sort are well known as strongly separated ones when $d = 2$, and as chord separated ones when $d = 3$.)

Keywords: zonotope, cubillage, higher Bruhat order, strongly separated sets, chord separated sets

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1 Introduction

We consider a generalization of the notions of strongly separated and chord separated set-systems. Let $n$ be a positive integer and denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

Definition. Let $r \in [n-1]$. Two sets $A, B \subseteq [n]$ are called $r$-separated (from each other) if there is no sequence $i_0 < i_1 < \cdots < i_{r+1}$ of elements of $[n]$ such that the elements with even indices (namely, $i_0, i_2, \ldots$) and the elements with odd indices ($i_1, i_3, \ldots$) belong to different sets among $A - B$ and $B - A$ (where $A' - B'$ denotes the set difference $\{i: A' \ni i \not\in B'\}$). In other words, one can choose $r' \leq r$ integers (“separating points”) $a_1 \leq a_2 \leq \cdots \leq a_{r'}$ in $[n]$ such that the intervals $[a_i, a_{i+1}]$ with $i$ even cover one of $A - B$ and $B - A$, while the ones with $i$ odd cover the other of these sets, where $a_{r'+1} := n$ and $[a, b]$ denotes $\{a, a+1, \ldots, b\}$. Accordingly, a collection (set-system) $A \subseteq 2^{[n]}$ is called $r$-separated if any two of its members are such.

We denote the set of all inclusion-wise maximal $r$-separated collections $A$ in $2^{[n]}$ as $S_{n,r+1}$, and the maximal size $|A|$ of such an $A$ by $s_{n,r+1}$ (for technical reasons, we prefer to use the subscript pair $(n, r + 1)$ rather than $(n, r)$). When all collections in $S_{n,r+1}$ are of the same size, $S_{n,r+1}$ is said to be pure.

∗Central Institute of Economics and Mathematics of the RAS, 47, Nakhimovskii Prospect, 117418 Moscow, Russia; email: danilov@cemi.rssi.ru.
†Central Institute of Economics and Mathematics of the RAS, 47, Nakhimovskii Prospect, 117418 Moscow, Russia; email: akarzanov7@gmail.com. Corresponding author.
‡The Institute for Information Transmission Problems of the RAS, 19, Bol’shoi Karetnyi per., 127051 Moscow, Russia; email: koshevoyga@gmail.com. Supported in part by grant RSF 16-11-10075.
In particular, $S_{n,n}$ consists of the unique collection $2^{[n]}$ (since any two subsets of $[n]$ are $(n-1)$-separated), giving the simplest purity case.

The concept of 1-separation was introduced, under the name of strong separation, by Leclerc and Zelevinsky [5] who proved the important fact that

$$\text{for any } n \geq 2, \text{ the set } S_{n,2} \text{ is pure (and } s_{n,2} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} = \frac{1}{2}n(n+1)+1).$$

Recently an analogous purity result on 2-separation was shown by Galashin [3]:

$$\text{for any } n \geq 3, \text{ the set } S_{n,3} \text{ is pure (and } s_{n,3} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}).$$

(In [3], 2-separated sets $A, B \subseteq [n]$ are called chord separated, which is justified by the observation that if $n$ points $1, 2, \ldots, n$ are disposed on a circumference $O$, in this order cyclically, then there is a chord to $O$ separating $A - B$ from $B - A$.)

However, a nice purity behavior as above for $r$-separated set-systems with $r = 1, 2$ is not extended in general to larger $r$’s, as it follows from profound results on oriented matroids due to Galashin and Postnikov [4]. Being specified to $r$-separated set-systems, the following property is obtained.

**Theorem 1.1** [4] $S_{n,r+1}$ is pure if and only if $\min\{r, n-r\} \leq 2$.

One purpose of this paper is to give another proof of Theorem 1.1 (which looks rather transparent modulo appealing to (1.1) and (1.2)).

In fact, the content of this paper is wider. In particular, we are going to demonstrate representable cases of extendable and non-extendable set-systems. Here we say that $\mathcal{A} \subseteq 2^{[n]}$ is $(n, r+1)$-extendable if there exists a maximal by size $r$-separated collection in $2^{[n]}$ including $\mathcal{A}$. (So $S_{n,r+1}$ is pure if and only if any $r$-separated set-system in $2^{[n]}$ is $(n, r+1)$-extendable.)

Our study of separated set-systems is based on a geometric approach whose theoretical grounds were originated in the classical work by Manin and Schechtman [6] where higher Bruhat orders, generalizing weak Bruhat ones, were introduced and well studied. (Recall that the higher Bruhat order for $(n, d)$ compares certain (so-called “packet admissible”) total orders on the set $\binom{[n]}{d}$ of $d$-elements subsets of $[n]$, and it turns into the weak one when $d = 1$, which compares permutations on $[n]$.) Subsequently Voevodskij and Kapranov [7] and Ziegler [8] gave nice geometric interpretations and established additional important results.

Based on these sources, we deal with a cyclic zonotope $Z = Z(n, d)$, that is the Minkowski sum of $n$ segments in $\mathbb{R}^d$ forming a cyclic configuration, and consider a fine zonotopal tiling, that is a subdivision $Q$ of $Z$ into paralleloctopes; we call it a cubillage for short. The vertices of $Q$ are associated, in a natural way, with subsets of $[n]$, forming a collection $\text{Sp}(Q) \subseteq 2^{[n]}$, called the spectrum of $Q$. In the special case $d = 2$, $Q$ is viewed as a rhombus tiling on a zonogon, and it is well known due to [5] that the spectra of these are exactly the maximal strongly separated set-systems in $2^{[n]}$. Similarly, the spectra of cubillages on $Z(n, 3)$ are exactly the maximal chord separated set-systems in $2^{[n]}$, as is shown in [3]. It turned out that this phenomenon is extended to an arbitrary
$d$: the cubillages $Q$ on $Z(n,d)$ are bijective to the maximal by size $(d - 1)$-separated set-systems $S$ in $2^n$, with the equality $\text{Sp}(Q) = S$; see [1].

This paper is organized as follows. Section 2 reviews definitions and basic properties of cyclic zonotopes and cubillages. Section 3 gives a short proof of the non-purity of $S_{6,4}$, which is the crucial case in our method of proof of Theorem 1.1. As a by-product, we obtain non-($n,4$)-extendable 3-separated collections consisting of only three sets. To show the other non-purity cases in Theorem 1.1, we need to use additional notions and constructions, which generalize those exhibited in [1] for $d = 3$ and are discussed in Section 4. Here we introduce a membrane in a cubillage $Q$ on $Z(n,d)$, to be a special $(d - 1)$-dimensional subcomplex $M$ in $Q$ (when the latter is regarded as the corresponding polyhedral complex). An important fact is that any cubillage on $Z(n,d - 1)$ can be lifted as a membrane in some cubillage on $Z(n,d)$. Also we describe nice operations on cubillages on $Z(n,d)$ (called contraction and expansion ones) that produce cubillages on $Z(n - 1,d)$ and $Z(n + 1,d)$.

Section 5 utilizes this machinery to show relations between $(n,d)$-, $(n + 1,d)$-, and $(n + 1,d + 1)$-extendable set-systems. As a consequence, we easily prove the remaining non-purity cases in Theorem 1.1 relying on the above result for $(n,d) = (6,4)$. Also we demonstrate in this section one interesting class of extendable set-systems (in Proposition 5.3) and raise two open questions. Section 6 is devoted to additional results involving inversions of membranes. These objects arise as a natural generalization of the classical notion of inversions for permutations, and their definition for an arbitrary $(n,d)$ goes back to Manin and Schechtman [6]. In particular, we show that for two membranes $M$ and $M'$ in the same cubillage on $Z(n,d)$, if any inversion of $M$ is an inversion of $M'$, then $\text{Sp}(M) \cup \text{Sp}(M')$ is $(d - 1)$-separated (see Theorem 6.4).

2 Cyclic zonotopes and cubillages

The objects that we deal with live in the euclidean space $\mathbb{R}^d$ of dimension $d > 1$. A cyclic configuration of size $n \geq d$ is meant to be an ordered set $\Xi$ of $n$ vectors $\xi_1 = \xi(t_1), \ldots, \xi_n = \xi(t_n)$ in $\mathbb{R}^d$ lying on the Veronese curve $\xi(t) = (1, t, t^2, \ldots, t^{d-1})$, $t \in \mathbb{R}$, and satisfying $t_1 < \cdots < t_n$. A useful property of $\Xi$ is that

\begin{equation}
(2.1) \quad \text{any } d \text{ vectors } \xi_{i(1)}, \ldots, \xi_{i(d)} \text{ with } i(1) < \cdots < i(d) \text{ are independent and, moreover, } \det(A) > 0, \text{ where } A \text{ is the matrix whose } j\text{-th column is } \xi_{i(j)}.\end{equation}

In addition, we will also assume that $\Xi$ is $\mathbb{Z}_2$-independent (i.e., all combinations of vectors of $\Xi$ with coefficients $0,1$ are different).

The configuration $\Xi$ generates the (cyclic) zonotope $Z = Z(\Xi)$ in $\mathbb{R}^d$, the polytope represented as the Minkowski sum of line segments $[0, \xi_i], i = 1, \ldots, n$. An object of our interest is a fine zonotopal tiling on $Z$, that is a subdivision $Q$ of $Z$ into $d$-dimensional parallelotopes of which any two intersecting ones share a common face, and each facet (a face of codimension 1) of the boundary $\partial(Z)$ of $Z$ is contained in one of these parallelotopes. For brevity, we liberally refer to these parallelotopes as cubes, and to $Q$ as a cubillage. In fact, depending on the context, we may think of a cubillage $Q$ in two
ways: either as a set of \( d \)-dimensional cubes (and may write \( C \in \mathcal{Q} \) for a cube \( C \) in \( \mathcal{Q} \)) or as the corresponding polyhedral complex. One can see that

\[(2.2) \text{ each cube in } \mathcal{Q} \text{ is viewed as}
\]

\[
\sum_{b \in X} \xi_b + \left\{ \sum (\lambda_{a(i)}\xi_{a(i)}) : 0 \leq \lambda_{a(i)} \leq 1, \ i = 1, \ldots, d \right\}
\]

for some \( a(1) < \cdots < a(d) \) and \( X \subseteq [n] - a(1)a(2)\cdots a(d) \).

Hereinafter, for a subset \( \{a, \ldots, a'\} \) of \([n]\), we use the abbreviated notation \( a \cdots a' \).

When \( X \subset [n] \) and \( a \cdots a' \) are disjoint, their union may be denoted as \( X a \cdots a' \). Also for a set \( S \) and element \( i \in S \), we may write \( S - i \) for \( S - \{i\} \).

For a cube \( C \) in \( \mathcal{Q} \), we say that the set \( a(1) \cdots a(d) \) is the type of \( C \), denoted as \( \tau(C) \). Also, regarding the first coordinate \( x_1 \) of a vector (point) \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) as its height, we denote the lowest point \( \sum_{b \in X} \xi_b \) of \( C \) by \( \text{bt}(C) \), called the bottom of \( C \). The cells of dimensions 0 and 1 in \( \mathcal{Q} \) are called vertices and edges, respectively.

When needed, each edge \( e \) is directed so as to be a parallel translation of corresponding generating vector \( \xi_i \), and we say that \( e \) is an edge of color \( i \), or an \( i \)-edge. This forms a directed graph on the vertices of \( \mathcal{Q} \), denoted as \( G_{\mathcal{Q}} = (V_{\mathcal{Q}}, E_{\mathcal{Q}}) \).

The subsets \( X \subseteq [n] \) are naturally identified with the corresponding points \( \sum_{b \in X} \xi_b \) in \( \mathbb{Z} \) (which are different due to the \( \mathbb{Z}_2 \)-independence of \( \Xi \)). This represents each vertex of \( \mathcal{Q} \) as a subset of \([n]\), and the collection of these subsets is called the spectrum of \( \mathcal{Q} \) and denoted by \( \text{Sp}(\mathcal{Q}) \).

Note that structural properties of cubillages depend on \( n \) and \( d \), but the choice of a cyclic configuration \( \Xi \) for these parameters is not important in essence; so we may speak of cubillages \( \mathcal{Q} \) on a generic cyclic zonotope, denoted as \( Z(n, d) \). There are known a number of nice properties of \( \mathcal{Q} \). Among those, two rather elementary ones are as follows:

\[(2.3) \text{ all types } \tau(C) \text{ of cubes } C \in \mathcal{Q} \text{ are different and range the set } \left( \begin{array}{c} n \\ d \end{array} \right) \text{ of } d \text{-element subsets of } [n] \text{ (so there are exactly } \left( \begin{array}{c} n \\ d \end{array} \right) \text{ cubes in } \mathcal{Q}) \text{; and}
\]

\[(2.4) |\text{Sp}(\mathcal{Q})| = \left( \begin{array}{c} n \\ 0 \end{array} \right) + \left( \begin{array}{c} n \\ 1 \end{array} \right) + \cdots + \left( \begin{array}{c} n \\ d \end{array} \right).
\]

One more useful property of cubillages (which is shown for \( d = 3 \) in \( \mathbb{I} \) Prop. 3.5) and can be straightforwardly extended to an arbitrary \( d \) is:

\[(2.5) \text{ suppose that for disjoint subsets } X, A \text{ of } [n], \text{ a cubillage } \mathcal{Q} \text{ contains the vertices of the form } X \cup A' \text{ for all } A' \subseteq A; \text{ then these vertices belong to a cube in } \mathcal{Q}.
\]

In particular, if \( \mathcal{Q} \) has vertices \( X \) and \( Xi \), where \( i \in [n] - X \), then \( \mathcal{Q} \) contains the edge connecting these vertices.

A less trivial fact is shown in \( \mathbb{I} \); it says that

\[(2.6) \text{ the correspondence } \mathcal{Q} \mapsto \text{Sp}(\mathcal{Q}) \text{ gives a bijection between the set } \mathcal{Q}_{n,d} \text{ of cubillages on } Z(n, d) \text{ and the set } S^*_{n,d} \text{ of maximal by size } (d - 1) \text{-separated collections in } 2^{[n]} \text{ (in particular, } |\text{Sp}(\mathcal{Q})| = s_{n,d}).
\]
In light of (1.1) and (1.2), $S_{n,d}^* = S_{n,d}$ when $d = 2, 3$, and in view of (2.10), all maximal strongly separated and chord separated set-systems in $2^d$ are represented by the spectra of corresponding cubillages; these facts were established for $d = 2$ in [5] (using equivalent terms of pseudo-line arrangements), and for $d = 3$ in [3]. On the other hand, Theorem 1.1 asserts that $S_{n,d}^* \neq S_{n,d}$ when $d \geq 4$ and $n \geq d + 2$.

In our further analysis we will use additional facts on the structure of the boundary $\partial(Z)$ of a zonotope $Z = Z(n,d)$. Let us say that a set $X \subseteq [n]$ is a $k$-pieced cortege if it is the union of $k$ intervals (including the case $k = 0$). As a non-difficult exercise, one can obtain the following description for the collection $\text{Sp}(Z)$ ($= \text{Sp}(\partial(Z))$) of subsets of $[n]$ represented by the vertices of $Z$:

\begin{equation}
(2.7) \text{for } Z = Z(n,d), \text{Sp}(Z) \text{ consists of exactly those sets } X \subseteq [n] \text{ that are } (d-1)\text{-separated from any subset of } [n]; \text{ specifically: when } d \text{ is even, } \text{Sp}(Z) \text{ is formed by all } d/2\text{-pieced corteges containing at least one of the elements } 1 \text{ and } n \text{ and all } k\text{-pieced corteges with } k < d/2, \text{ while when } d \text{ is odd, } \text{Sp}(Z) \text{ is formed by all } (d+1)/2\text{-pieced corteges containing both } 1 \text{ and } n \text{ and all } k\text{-pieced corteges with } k \leq \frac{(d-1)}{2}.
\end{equation}

In particular, $\text{Sp}(Z)$ is included in any collection in $S_{n,d}$.

In case $n = d$, the zonotope $Z$ turns into one cube and the purity of $S_{n,n}$ is trivial. And in case $n = d+1$, one can conclude from (2.7) that there are exactly two subsets of $[n]$ that do not belong to $\text{Sp}(Z)$, one being formed by the odd elements, and the other by the even elements of $[n]$, i.e., the sets $X = 135 \ldots$ and $Y = 246 \ldots$ Clearly they are not $(d-1)$-separated from each other. Therefore, $S_{n,n-1}$ consists of two collections $\text{Sp}(Z_{n,n-1}) \cup \{X\}$ and $\text{Sp}(Z_{n,n-1}) \cup \{Y\}$, implying that $S_{n,n-1}$ is pure. This together with (1.1) and (1.2) gives “only if” part of Theorem 1.1.

The non-purity cases of this theorem (giving “only if” part) are discussed in Sections 3 and 5.

3 Case $(n,d) = (6, 4)$

This case is crucial and will be used as a base to handle the other non-purity cases in Theorem 1.1 (in Sect. 5).

Consider $Z = Z(6, 4)$. By (2.7), $\text{Sp}(Z)$ consists of all intervals and all 2-pieced corteges containing 1 or 6. A direct enumeration shows that the number of these amounts to 52. Therefore, $2^6 - 52 = 12$ subsets of $[6]$ are not in $\text{Sp}(Z)$, namely:

\begin{equation}
(3.1) \quad 24, 245, 25, 235, 35, 135, 1356, 136, 1346, 146, 1246, 246.
\end{equation}

(Recall that $a \cdots b$ stands for $\{a, \ldots, b\}$.) Let $A_i$ denote $i$-th member in this sequence (so $A_1 = 24$ and $A_{12} = 246$). Form the collection

$\mathcal{A} := \text{Sp}(Z) \cup \{A_1, A_5, A_9\}$.

It consists of 52 + 3 = 55 sets, whereas the number $S_{6,4}$ is equal to $\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 57$. Now the non-purity of $S_{6,4}$ is implied by the following
**Lemma 3.1** $\mathcal{A}$ is a maximal 3-separated collection in $2^{[6]}$.

**Proof** By (2.7), any two $X \in \text{Sp}(Z)$ and $Y \in \mathcal{A}$ are 3-separated. Observe that $|A_{i-1} \triangle A_i| = 1$ for any $1 \leq i \leq 12$ (where $A_0 := A_{12}$ and $A \triangle B$ denotes the symmetric difference $(A - B) \cup (B - A)$). Then any $A, A' \in \{A_1, A_5, A_9\}$ satisfy $|A \triangle A'| \leq 4$. This implies that $A$ and $A'$ are 3-separated. Therefore, the collection $\mathcal{A}$ is 3-separated.

The maximality of $\mathcal{A}$ follows from the observation that adding to $\mathcal{A}$ any member of $\{A_i : 1 \leq i \leq 12, \ i \neq 1, 5, 9\}$ would violate the 3-separation. Indeed, a routine verification shows that $A_1$ is not 3-separated from any of $A_6, A_7, A_8$, and similarly for $A_5$ and $\{A_{10}, A_{11}, A_{12}\}$, and for $A_9$ and $\{A_2, A_3, A_4\}$.

**Remark 1.** To visualize a verification in the above proof, one can use the circular diagram illustrated in the picture below where the sets from (3.1) are disposed in the cyclic order. Here the sets $A_1, A_5, A_9$ are drawn in boxes and connected by lines with those sets where the 3-separation is violated. Note that, instead of $A_1, A_5, A_9$, one could take in the lemma any triple of the form $A_i, A_{i+4}, A_{i+8}$ (taking indices modulo 12).

In conclusion of this section recall that a collection in $2^{[n]}$ is called $(n, d)$-extendable if it can be extended to a maximal by size $(d - 1)$-separated collection in $2^{[n]}$, or, equivalently (in view of (2.6)), if there exists a cubillage on $Z(n, d)$ whose spectrum includes the given collection. An immediate consequence of Lemma 3.1 is the existence of small non-extendable 3-separated collections.

**Corollary 3.2** Any 3-separated triple of the form $\{A_i, A_{i+4}, A_{i+8}\}$, e.g. $\{24, 35, 1346\}$, is not $(6,4)$-extendable (defining $A_{i'}$ as above and taking indices modulo 12).

(Here we take into account that any maximal 3-separated collection in $2^{[6]}$ includes $\text{Sp}(Z(6, 4))$.) Note that, in view of Lemma 5.1 in Sect. 5, this corollary implies that $\{A_i, A_{i+4}, A_{i+8}\}$ is not $(n, 4)$-extendable for any $n \geq 6$.

4 **Membranes and pies**

For further purposes, we need additional notions, borrowing terminology from [1].
4.1 Membranes. Let \( \pi : \mathbb{R}^d \to \mathbb{R}^{d-1} \) be the projection along the last coordinate vector, i.e., sending \( x = (x_1, \ldots, x_d) \) to \( (x_1, \ldots, x_{d-1}) \). Then \( \pi(\Xi) = (\pi(\xi_1), \ldots, \pi(\xi_n)) \) is again a cyclic configuration and \( Z(\pi(\Xi)) \) forms a \((d-1)\)-dimensional cyclic zonotope. We may use notation \( Z(n, d-1) \) for \( Z(\pi(\Xi)) \) and call it the projection of \( Z(n, d) \).

**Definition.** Let \( Q \) be a cubillage on \( Z(n, d) \). By a membrane in \( Q \) we mean a subcomplex \( M \) of \( Q \) that is bijectively projected by \( \pi \) to \( Z(n, d-1) \).

In particular, \( M \) has dimension \((d-1)\), each facet of \( M \) is projected to a \((d-1)\)-dimensional “cube” (represented as in (2.2)) where each \( \xi \) should be replaced by \( \pi(\xi) \), and these “cubes” constitute a cubillage of \( Z(n, d-1) \), whence the collection \( \text{Sp}(M) \) is \((d-2)\)-separated (cf. (2.6)). A converse property says that any cubillage can be lifted as a membrane into some cubillage of the next dimension:

\[(4.1) \text{ for any cubillage} \ Q' \text{ on} \ Z(n, d-1) \text{, there exists a cubillage} \ Q \text{ on} \ Z(n, d) \text{ and a membrane} \ M \text{ in} \ Q \text{ such that} \ \pi(M) = Q'. \]

(This fact can be deduced from results on higher Bruhat orders and related aspects in \([6, 7, 8]\). See also \([1, \text{Sec. 5}]\) for a direct construction when \( d = 3 \).)

When the choice of \( Q \) as in (4.1) is not important for us, we may think of the (abstract) membrane \( M \) in \( Z(n, d) \) representing a cubillage \( Q' \) on \( Z(n, d-1) \), saying that \( M \) is obtained by lifting \( Q' \). (To construct such an \( M \), one should take the points in \( Z(n, d) \) representing the same subsets of \( [n] \) as those in \( \text{Sp}(Q') \) and then extend the corresponding \( 2^{d-1} \)-element subsets of these points into \((d-1)\)-dimensional “cubes”; cf. (2.2)).

Two membranes are of an especial interest. For a closed set \( X \) of points in \( Z = Z(n, d) \), let \( X^{\text{fr}} \) (resp. \( X^{\text{rear}} \)) be the subset of points of \( X \) “seen” in the direction of \( d \)-th coordinate vector \( e_d \) (resp. \( -e_d \)), i.e., such that for each \( x' \in \pi(X) \), this subset contains the point \( x \in \pi^{-1}(x') \cap X \) with the minimal (resp. maximal) value \( x_d \). We call it the front (resp. rear) side of \( X \). When \( X = Z \), the sides \( Z^{\text{fr}} \) and \( Z^{\text{rear}} \) (respecting their cell structures) are membranes of any cubillage on \( Z \). The subcomplex \( Z^{\text{fr}} \cap Z^{\text{rear}} \) is called the rim of \( Z \) and denoted as \( Z^{\text{rim}} \).

Borrowing terminology from \([1]\), we refer to \( \pi(Z^{\text{fr}}) \) and \( \pi(Z^{\text{rear}}) \) are the standard and anti-standard cubillages on \( Z(n, d-1) \), denoted as \( Q_{\text{st}}^{n,d-1} \) and \( Q_{\text{ant}}^{n,d-1} \), respectively.

As a refinement of (2.7), one can characterize the spectra of \( Z^{\text{rim}} \), \( Z^{\text{fr}} \) and \( Z^{\text{rear}} \) for \( Z = Z(n, d) \) as follows (a check-up is routine and we omit it here):

\[(4.2) \]

(i) when \( d \) is even, \( \text{Sp}(Z^{\text{rim}}) \) consists of all \( d/2 \)-pieced corteges containing both elements 1 and \( n \), and all \( k \)-pieced corteges with \( k < d/2 \), whereas when \( d \) is odd, \( \text{Sp}(Z^{\text{rim}}) \) consists of all \((d-1)/2\)-pieced corteges containing at least one of 1 and \( n \), and all \( k \)-pieced corteges with \( k < (d-1)/2 \);

(ii) \( \text{Sp}(Z^{\text{fr}}) - \text{Sp}(Z^{\text{rim}}) \) consists of all \( d/2 \)-pieced corteges containing the element 1 but not \( n \) when \( d \) is even, and consists of all \((d-1)/2\)-pieced corteges containing none of 1 and \( n \) when \( d \) is odd;

(iii) \( \text{Sp}(Z^{\text{rear}}) - \text{Sp}(Z^{\text{rim}}) \) consists of all \( d/2 \)-pieced corteges containing the element \( n \) but not 1 when \( d \) is even, and consists of all \((d+1)/2\)-pieced corteges containing both 1 and \( n \) when \( d \) is odd.
4.2 Pies, contraction and expansion. For a cubillage $Q$ on $Z(n, d)$ and $i \in [n]$, let $\Pi_i = \Pi_i(Q)$ be the subcomplex of $Q$ formed by the cubes $C$ having an edge of color $i$, called $i$-pies. We refer to $\Pi_i$ as the $i$-pie in $Q$. It has the following nice properties (which are shown by attracting standard topological reasonings):

\[(4.3) \quad (i) \quad \Pi_i \text{ is representable as the “direct Minkowski sum” } \{ x = \beta + \alpha : \beta \in B_i, \ \alpha \in S_i \}, \text{ where } B_i \text{ is a subcomplex of } Q \text{ homeomorphic to a } (d - 1)\text{-dimensional ball and } S_i \text{ is the segment } [0, \xi_i];
\]

(ii) removing from $Q$ the point set $\Pi_i - (B_i \cup B'_i)$, where $B'_i := B_i + \xi_i$, produces two connected components $R$ and $R'$ containing $B_i$ and $B'_i$, respectively.

(iii) gluing $R$ with $R'$ shifted by $-\xi_i$, we obtain a cubillage on the zonotope $Z(\Xi - \xi_i)$; it is denoted as $Q/i$ and called the $i$-contraction of $Q$;

(iv) $\text{Sp}(Q/i)$ consists of all sets $X \subseteq [n] - i$ such that at least one of $X, Xi$ is in $\text{Sp}(Q)$.

When $i = n$, the pie structure becomes more transparent. Namely, the maximality of color $n$ in the type of each cube in $\Pi_n$ provides that the ball $B_n$ ($B'_n$) is contained in the front (resp. rear) side of $\Pi_n$ (a similar fact is also true for $i = 1$ but need not hold when $1 < i < n$). This implies that

\[(4.4) \quad B_n \text{ is a membrane in the reduced cubillage } (n\text{-contraction}) Q/n.
\]

In other words, the $n$-contraction operation applied to $Q$, defined by (ii),(iii) in (4.3), transforms the pie $\Pi_n$ into a membrane of $Q/n$. A converse operation blows a membrane into an $n$-pie.

More precisely, let $M$ be a membrane in a cubillage $Q'$ on the zonotope $Z' = Z(n - 1, d)$. Define $Z^-(M)$ ($Z^+(M)$) to be the part of $Z'$ between $(Z')^\text{fr}$ and $M$ (resp. between $M$ and $(Z')^\text{rear}$) and define $Q^-(M)$ ($Q^+(M)$) to be the subcubillage of $Q'$ contained in $Z^-(M)$ (resp. $Z^+(M)$). The $n$-expansion operation for $(Q', M)$ consists in shifting $Z^+(M)$ equipped with $Q^+(M)$ by the vector $\xi_n$ and filling the “space between” $M$ and $M + \xi_n$ by the corresponding set of $n$-cubes. More precisely, each $(d - 1)$-dimensional cube $C'$ (having type $\tau(C')$ and bottom vertex $\text{bt}(C')$) in $M$ generates the $n$-dimensional cube $C = C' + [0, \xi_n]$; so $\tau(C) = \tau(C') \cup \{n\}$ and $\text{bt}(C) = \text{bt}(C')$. Then combining $Q^-(M)$ with the shifted subcubillage $Q^+(M) + \xi_n$ and the “blowed membrane” $M + [0, \xi_n]$ (forming an $n$-pie), we obtain a cubillage on $Z(n, d)$. This cubillage is called the $n$-expansion of $Q'$ using $M$ and denoted as $Q_n(Q', M)$.

The $n$-contraction and $n$-expansion operations are naturally related to each other (as a straightforward generalization to an arbitrary $d$ of Proposition 3.4 from [1]):

\[(4.5) \quad (i) \quad \text{the correspondence } (Q', M) \mapsto Q_n(Q', M), \text{ where } Q' \text{ is a cubillage on } Z(n - 1, d) \text{ and } M \text{ is a membrane in } Q', \text{ gives a bijection between the set of such pairs } (Q', M) \text{ in } Z(n - 1, d) \text{ and the set } Q_{n,d} \text{ of cubillages on } Z(n, d);
\]

(ii) under this correspondence, $Q'$ is the $n$-contraction $Q/n$ of $Q = Q_n(Q', M)$ and $M$ is the image of the $n$-pie in $Q$ under the $n$-contraction operation.
5 Other non-purity cases

Return to proving “only if” part of Theorem 5.1. For any \( (n, d) \) with \( \min\{d - 1, n - d + 1\} \geq 3 \), we have \( n - 6 \geq d - 4 \geq 0 \). Therefore, “only if” part of Theorem 5.1 (with \( r \) replaced by \( d - 1 \)) will follow from Lemma 5.1 concerning \( (n, d) = (6, 4) \) and the next two assertions on lifting set-systems in \( 2^{|n|} \).

Lemma 5.1 Let \( \mathcal{A} \subseteq 2^{[n]} \). Then \( \mathcal{A} \) is \( (n, d) \)-extendable if and only if \( \mathcal{A} \) is \( (n + 1, d) \)-extendable.

Lemma 5.2 Let \( \mathcal{A} \subseteq 2^{[n]} \), \( n' := n + 1 \), and \( \mathcal{A}' := \{Xn' : X \in \mathcal{A}\} \). Then \( \mathcal{A} \) is \( (n, d) \)-extendable if and only if \( \mathcal{A} \cup \mathcal{A}' \) is \( (n + 1, d + 1) \)-extendable.

Proof of Lemma 5.1 Clearly \( \mathcal{A} \) is \( (d - 1) \)-separated relative to \( n \) if and only if it is \( (d - 1) \)-separated relative to \( n + 1 \).

If \( \mathcal{A} \) is \( (n, d) \)-extendable, then \( \mathcal{A} \subseteq \text{Sp}(Q) \) for some cubillage \( Q \) on \( Z = Z(n, d) \). Let \( Q' \) be the \( (n + 1) \)-extension of \( Q \) using as a membrane the rear side \( Z_{\text{rear}} \) of \( Z \) (see Sect. 4.2 for definitions). Then \( \mathcal{A} \subseteq \text{Sp}(Q') \), implying that \( \mathcal{A} \) is \( (n + 1, d) \)-extendable.

Conversely, if \( \mathcal{A} \) is \( (n + 1, d) \)-extendable, then \( \mathcal{A} \subseteq \text{Sp}(Q') \) for some cubillage \( Q' \) on \( Z(n + 1, d) \). Let \( Q \) be the \( (n + 1) \)-contraction of \( Q' \) (i.e., \( Q \) is a cubillage on \( Z(n, d) \) obtained by shrinking the \( (n + 1) \)-pie in \( Q' \), cf. (4.3)). Since the sets in \( \mathcal{A} \) do not contain the element \( n + 1 \), their corresponding vertices in \( Q' \) preserve under the \( (n + 1) \)-contraction operation. So \( \mathcal{A} \subseteq \text{Sp}(Q) \) and therefore \( \mathcal{A} \) is \( (n, d) \)-extendable.

Proof of Lemma 5.2 Let \( \mathcal{A} \) be \( (n, d) \)-extendable (in particular, \( \mathcal{A} \) is \( (d - 1) \)-separated). Take a cubillage \( Q \) on \( Z(n, d) \) with \( \mathcal{A} \subseteq \text{Sp}(Q) \). By (4.1), there exist a cubillage \( Q' \) on \( Z(n, d + 1) \) and a membrane \( M \) in \( Q' \) such that \( Q = \pi(M) \), where \( \pi \) is the projection \( \mathbb{R}^{d+1} \to \mathbb{R}^d \) as in Sect. 4.1. Let \( Q'' \) be the \( n' \)-expansion of \( Q' \) using \( M \). Then \( Q'' \) is a cubillage on \( Z(n + 1, d + 1) \). Take the \( n' \)-pie \( \Pi_{n'} \) in \( Q'' \). Then the side \( B_{n'} \) of \( \Pi_{n'} \) contains the vertices of \( M \), whereas the side \( B'_{n'} \) contains the vertices \( Xn' \) for \( X \in \text{Sp}(M) \) (cf. (4.3)(ii)). Since \( \mathcal{A} \subseteq \text{Sp}(M) = \text{Sp}(B_{n'}) \), we have \( \mathcal{A}' \subseteq \text{Sp}(B'_{n'}) \), whence \( \mathcal{A} \cup \mathcal{A}' \subseteq \text{Sp}(Q'') \). Thus, \( \mathcal{A} \cup \mathcal{A}' \) is \( (n + 1, d + 1) \)-extendable.

Conversely, let \( \mathcal{A} \cup \mathcal{A}' \) be \( (n + 1, d + 1) \)-extendable (in particular, it is \( d \)-separated). Then \( \mathcal{A} \cup \mathcal{A}' \subseteq \text{Sp}(Q) \) for some cubillage \( Q \) on \( Z(n + 1, d + 1) \). By (2.5), any \( X \in \mathcal{A} \) must be connected with \( Xn' \in \mathcal{A}' \) by an \( n' \)-edge in \( Q \). This implies that \( \mathcal{A} \cup \mathcal{A}' \) is contained in the \( n' \)-pie of \( Q \) (in which \( \mathcal{A} \) and \( \mathcal{A}' \) lie in the sides \( B_{n'} \) and \( B'_{n'} \) of \( \Pi_{n'} \), respectively). Then the \( n' \)-contraction operation applied to \( Q \) transforms \( Q \) and \( \Pi_{n'} \) into a cubillage \( Q' \) on \( Z(n, d + 1) \) and a membrane \( M \) in \( Q' \), preserving the vertices of \( B_{n'} \). Hence \( \mathcal{A} \subseteq \text{Sp}(M) \), and now taking the cubillage \( Q'' := \pi(M) \) on \( Z(n, d) \), we obtain \( \mathcal{A} \subseteq \text{Sp}(Q'') \), as required.

Note that Lemmas 5.1 and 5.2 can also be used to demonstrate an interesting class of extendable set-systems, as follows.

Proposition 5.3 Let \( k \in [n] \), \( \mathcal{B} \subseteq 2^{[k]} \), \( D \subseteq \{k + 1, \ldots, n\} \), and \( d := |D| \). Let \( \mathcal{C} \) consist of the sets of the form \( B \cup D' \) for all \( B \in \mathcal{B} \) and \( D' \subseteq D \). Suppose that \( \mathcal{B} \) is \( 2 \)-separated. Then \( \mathcal{C} \) is \( (n, d + 3) \)-extendable.
Proof. We use induction on \( n + d \). When \( d = 0 \), \( C \) becomes \( B \), and therefore it is (\( n, 3 \))-extendable (by (1.2)).

So assume that \( d \geq 1 \). Let \( p \) be the maximal element in \( D \). If \( p = n \) then \( C \) is contained in \( 2^{n-1} \). By Lemma [5.2], \( C \) is (\( n, d + 3 \))-extendable if and only if it is (\( n - 1, d + 3 \))-extendable, and we can apply induction.

Now let \( p = n \). Form \( C' := \{ X \in C : n \not\in X \} \) and \( C'' := \{ X \in C : n \in X \} \). Then \( C' \cap C'' = \emptyset \), \( C' \cup C'' = C \), and the construction of \( C \) implies that \( C'' = \{ Xn : X \in C' \} \). Therefore, by Lemma [5.2], \( C \) is (\( n, d + 3 \))-extendable if and only if \( C' \) is (\( n - 1, d + 2 \))-extendable. Since \( C' \) consists of the sets of the form \( B \cup D' \) for all \( B \in B \) and \( D' \subseteq D \cap \{ n - 1 \} \), we can apply induction.

As a special case in this proposition, we obtain the following

**Corollary 5.4** For any \( D \subseteq \{ n \} \) with \( |D| \leq d \), the collection \( \{ X \subseteq D \} \) (forming the vertex set of a “cube”) is (\( n, d \))-extendable.

(In this case one should take as \( B \) the “cube” on three smallest elements of \( D \).)

Thus, one cube of dimension \( \leq d \) within a zonotope \( Z(n, d) \) can always be extended to a cubillage. In contrast, as we have seen earlier (cf. Corollary [3.2]), a triple of (duly separated) “cubes” of dimension 0 need not be extendable. In light of these facts, we can address the following open question:

(O1) : Whether or not any two “cubes” \( C = \{ A \cup X : X \subseteq D \} \) and \( C' = \{ A' \cup Y : Y \subseteq D' \} \), where \( A, A', D, D' \subseteq \{ n \}, |D|, |D'| \leq d \), and \( C \cup C' \) is (\( d - 1 \))-separated, can be extended to a cubillage on \( Z(n, d) \)?

A similar open question concerns a membrane and a cube:

(O2) : Whether or not any pair consisting of a membrane \( M \) in \( Z(n, d) \) and a “cube” \( C = \{ A \cup X : X \subseteq D \} \), where \( A, D \subseteq \{ n \}, |D| \leq d \), and \( \text{Sp}(M) \cup C \) is (\( d - 1 \))-separated, can be extended to a cubillage on \( Z(n, d) \)?

6 Inversions

Inversions discussed in this section arise as a natural generalization of the classical notion of inversions in elements (permutations) of a symmetric group \( \mathcal{S}_n \), inspired by the observation that a permutation of \( \{ n \} \) can be interpreted as a membrane in the zonogon \( Z(n, 2) \). We will use two ways to define inversions, which are shown to be equivalent. The first one is of a geometric flavor, as follows.

**Definitions.** Consider a cubillage \( Q \) on \( Z = Z(n, d) \), a membrane \( M \) in \( Q \), and the corresponding cubillage \( Q' := \pi(M) \) on \( Z(n, d - 1) \). A \( d \)-tuple \( K \in \binom{[n]}{2} \) is called inversive, or an inversion, for \( M \), as well as for \( Q' \), if the cube \( C \) of \( Q \) having type \( K \) lies before \( M \), i.e., in the region \( Z^-(M) \) between \( Z^K \) and \( M \) (see Sect. [4.1] for definitions). Otherwise (when the cube \( C \in Q \) with \( \tau(C) = K \) lies in the region \( Z^+(M) \) between \( M \) and \( Z^{\text{real}} \)), we say that \( K \) is straight for \( M \) (and for \( Q' \)).
An important fact is that the set of inversions for \(M\) does not depend on the choice of a cubillage \(Q\) on \(Z\) that contains \(M\) as a membrane (see Remark 2 below); we denote this set as \(\text{Inv}(M)\), or \(\text{Inv}(Q')\).

In particular, the smallest case \(\text{Inv}(M) = \emptyset\) (the largest case \(\text{Inv}(M) = \binom{[n]}{d}\)) happens when \(M = Z^i\) (resp. \(M = Z^\ast\)), or, in terms of cubillages, when \(Q' = \pi(M)\) becomes the standard cubillage \(Q_{n,d-1}^\ast\) (resp. the anti-standard cubillage \(Q_{n,d-1}^{\text{ant}}\)). (Note that \(\mathbb{N}\) exhibits necessary and sufficient conditions on a collection in \(\binom{[n]}{d}\) to be the set of inversions for a membrane, but we do not use this in what follows.)

The second way to define inversions relies on a natural binary relations on cubes of a cubillage. More precisely, given a cubillage \(\tilde{Q}\) on \(Z(n,\tilde{d})\), we say that a cube \(C \in \tilde{Q}\) immediately precedes a cube \(C' \in \tilde{Q}\) if their sides \(C^\ast\) and \((C')^\ast\) share a facet (a \((d-1)\)-dimensional face). Then for \(C, C' \in \tilde{Q}\), we write \(C \prec \tilde{Q} C'\) and say that \(C\) precedes \(C'\) if there is a sequence \(C = C_0, C_1, \ldots, C_k = C'\) such that \(C_{i-1}\) immediately precedes \(C_i\) for each \(i\). It can be shown rather easily that the relation \(\prec\) is a partial order (arguing in spirit of the proof of Lemma 4.2 in \([I]\) for \(d = 3\)).

A behavior of this order under contraction operations on pies (defined in Sect. \([1,2]\)) is featured as follows.

**Lemma 6.1** Let \(Q^i\) be the \(i\)-contraction of a cubillage \(\tilde{Q}\) on \(Z(n,\tilde{d})\), where \(i \in [\tilde{n}]\), and suppose that \(D, D'\) are cubes in \(Q^i\) such that \(D \prec \tilde{Q} D'\). Then the cubes \(C, C' \in \tilde{Q}\) with \(\tau(C) = \tau(D)\) and \(\tau(C') = \tau(D')\) satisfy \(C \prec \tilde{Q} C'\).

**Proof** Let \(B_i, R, R'\) be as in \([1,3]\)(ii) for the \(i\)-pie \(\Pi_i\) in \(\tilde{Q}\). It suffices to consider the case when \(D\) immediately precedes \(D'\). Then four cases are possible: (a) both \(D, D'\) lie in the part \(R\) of \(Q^i\), (b) both \(D, D'\) lie in the part \(R' - \xi_i\) of \(Q^i\) corresponding to \(R'\) in \(\tilde{Q}\), (c) \(D\) lies in \(R\) and \(D'\) in \(R' - \xi_i\), or (d) \(D\) lies in \(R' - \xi_i\) and \(D'\) in \(R\).

In case (a), we have \(C = D\) and \(C' = D'\), while in case (b), \(C = D + \xi_i\) and \(C' = D' + \xi_i\). So in both cases \(C\) immediately precedes \(C'\). In case (c), we have \(C = D\) and \(C' = D' + \xi_i\). Also \((D')^\ast\) and \((D')^\ast\) share a facet \(F\) contained in \(B_i\). Therefore, the pie \(\Pi_i\) has the \(i\)-cube \(C''\) that is the sum of \(F\) and the segment \([0,\xi_i]\). One can see that \(C^\ast \cap (C')^\ast = F\) and \((C^\ast)^\ast \cap (C')^\ast = F + \xi_i\). Then \(C\) immediately precedes \(C''\), and \(C''\) immediately precedes \(C'\). This implies \(C \prec \tilde{Q} C'\). Finally, in case (d), \(C = D + \xi_i\) and \(C' = D'\). Also \((D')^\ast\) and \((D')^\ast\) share a facet \(F\) in \(B_i\). Again, \(\Pi_i\) has the \(i\)-cube \(C''\) that is the sum of \(F\) and the segment \([0,\xi_i]\). But now we have 
\[
(C^\ast) \cap (C')^\ast = F + \xi_i,
\]
\[
(C^\ast)^\ast \cap (C')^\ast = F.
\]
Then \(C \prec \tilde{Q} C'' \prec \tilde{Q} C'\), implying \(C \prec \tilde{Q} C'\), as required.  

Now return to \(Q, M, Q'\) as above. Fix a cube \(C \in Q\) and let \(K := \tau(C)\). Suppose we apply to \(Q\) the \(i\)-contraction operation with \(i \in [n] - K\). One can see that under this operation \(M\) turns into a membrane \(M'\) in the resulting cubillage \(Q/i\), and comparing the location of \(C\) relative to \(M\) with that of the “image” of \(C\) relative to \(M'\), one can see that the status of \(K\) preserves, i.e., \(K\) is inversive for \(M'\) if and only if so is for \(M\).

By applying, step by step, the \(i\)-contraction operations to all \(i \in [n] - K\), we produce from \(Q\) the cubillage \(\tilde{Q} := Q/([n] - K)\) consisting of a single cube \(\tilde{C}\) having type \(K\). Accordingly, \(M\) and \(Q'\) turn into the membrane \(\tilde{M} := M/([n] - K)\) in \(\tilde{Q}\) and its projection \(\tilde{Q}' := \pi(\tilde{M})\), respectively. Since \(\tilde{Q}\) has exactly two membranes, namely, \(\tilde{C}^\ast\)
and \( \hat{C}^{\text{rear}} \), and since the status of \( K \) preserves during the contraction process, we can conclude that

(6.1) if \( K \) is inversive (straight) for \( M \), then the reduced membrane \( \hat{M} := M/([n]−K) \)

is isomorphic to the rear (resp. front) side of a cube of type \( K \).

Assuming that \( K \) consists of elements \( k_1 < \cdots < k_d \), we will write \( Z(K,d−1) \) for the
zonotope generated by \( \xi'_p := \pi(\xi_{k_p}) \), \( p = 1, \ldots, d \) (where, as before, \( \xi' \) is a generator from \( \Xi \)). In view of [6.1], there exist exactly two cubillages on \( Z' := Z(K,d−1) \), namely, the standard and anti-standard cubillages, denoted as \( Q_{K,d−1}^{\text{st}} \) and \( Q_{K,d−1}^{\text{ant}} \), respectively (which correspond to the standard and anti-standard cubillages in \( Z(d,d−1) \)).

Since \( Q_{K,d−1}^{\text{st}} \) and \( Q_{K,d−1}^{\text{ant}} \) are the projections of \( C^\text{fr} \) and \( C^{\text{rear}} \), respectively, where \( C \) is a cube of type \( K \) (viz. \( C = Z(K,d) \)), (6.1) implies that

(6.2) \( K \in \binom{[n]}{d} \) is an inversion for a membrane \( M \) in \( Z(n,d) \) if and only if \( \pi(M/([n]−K)) \) is the anti-standard cubillage \( Q_{K,d−1}^{\text{ant}} \) on \( Z(K,d−1) \).

This and Lemma 6.1 lead to a description of inversions for \( M \) in terms of the partial order \( \preceq_M \), giving the second ("intrinsic") way to characterize \( \text{Inv}(M) \). Following [6], for \( K \in \binom{[n]}{d} \), define \( \text{Pac}(K) \) to be the set \( \{K−i : i \in K\} \) of \((d−1)\)-element subsets of \( K \), called the packet of \( K \). Then each \( K' \in \text{Pac}(K) \) is the type of some cube in \( M \). For convenience, we use the same notation \( \preceq_M \) for the corresponding types; so if \( C, C' \in M \) and \( C \preceq_M C' \), we may write \( \tau(C) \preceq_M \tau(C') \).

**Proposition 6.2** Let \( K \in \binom{[n]}{d} \) consist of elements \( k_1 < \cdots < k_d \) and let \( M \) be a membrane in \( Z(n,d) \). Then the elements of \( \text{Pac}(K) \) occur in the lexicographic order

\[
(K−k_d) \preceq_M (K−k_{d−1}) \preceq_M \cdots \preceq_M (K−k_1)
\]

if \( K \) is straight for \( M \), and in the anti-lexicographic order

\[
(K−k_1) \preceq_M (K−k_2) \preceq_M \cdots \preceq_M (K−k_d)
\]

if \( K \) is inversive for \( M \).

**Proof** In view of Lemma 6.1, the required relations for \( \preceq_M \) would follow from similar relations for \( \preceq_{M'} \), where \( M' := M/([n]−K) \). Moreover, since the relations \( \preceq_{M'} \) and \( \preceq_{\pi(M')} \) on \( \text{Pac}(K) \) are the same, it suffices to consider cubillages on \( Z(K,d−1) \), or, equivalently, on \( Z(d,d−1) \). In other words, we have to show that for the sets \( A_i := [d]−i, \quad i = 1, \ldots, d: \)

\[
A_d \prec' A_{d−1} \prec' \cdots \prec' A_1, \quad \text{and} \quad A_1 \prec'' A_2 \prec'' \cdots \prec'' A_d,
\]

where \( \prec' \) (\( \prec'' \)) denotes the order in \( Q' := Q_{d,d−1}^{\text{st}} \) (resp. \( Q'' := Q_{d,d−1}^{\text{ant}} \)).

To show this, we first specify the spectra of \( Q' \) and \( Q'' \). Let \( X \) (\( Y \)) be the set of elements \( i \in [d] \) with \( d−i \) odd (resp. even). Note that \( X \) and \( Y \) are not \( (d−2)\)-separated
from each other; so one of $X, Y$ belongs to $\text{Sp}(Q')$, and the other to $\text{Sp}(Q'')$ (taking into account that each of $\text{Sp}(Q')$ and $\text{Sp}(Q'')$ is $(d - 2)$-separated and that $|\text{Sp}(Q')| = |\text{Sp}(Q'')| = \binom{d}{0} + \binom{d}{1} + \cdots + \binom{d}{d-1} = 2^d - 1$; cf. (2.6) and (2.4)). Using $\text{Sp}(Q') = \text{Sp}(Z^h)$ and $\text{Sp}(Q'') = \text{Sp}(Z^{\text{rear}})$ for the “cube” $Z = Z(d, d)$ and considering (1.2) (ii), (iii), one can conclude that

$$X \in \text{Sp}(Q') \text{ and } Y \in \text{Sp}(Q'').$$

Let us prove (6.3). The cubillage $Q'$ is formed by $d$ cubes $C_1, \ldots, C_d$ of types $A_1, \ldots, A_d$, respectively, each of which must contain the unique vertex of $Q'$ lying in the interior of $Z' = Z(d, d - 1)$, namely, the vertex $X$ (viz. $\sum(\pi(\xi_i) : i \in X)$). It follows that in the digraph $G_{Q'}$ (defined in Sect. 2),

(6.5) the vertex $X$ is incident to $d$ edges $a_1, \ldots, a_d$ of $G_{Q'}$, where each $a_i$ is an $i$-edge, and $a_i$ enters (resp. leaves) $X$ if $i \in X$ (resp. $i \in [d] - X$);

(6.6) for $i = 1, \ldots, d$, the cube $C_i$ contains all edges in $E := \{a_1, \ldots, a_d\}$ except for $a_i$.

Consider “consecutive” cubes $C_i, C_{i+1}$ $(1 \leq i < d)$. They share a facet, namely, the one lying in the hyperplane $H_i$ spanned by the edge set $E - \{a_i, a_{i+1}\}$. The required relation $A_{i+1} \prec' A_i$ in (6.3) can be reformulated as:

(*) when seeing in the direction of the last coordinate vector, the cube $C_i$ is located behind $H_i$ (whereas $C_{i+1}$ is located before $H_i$).

To see (*), for $j = 1, \ldots, d$, denote $\pi(\xi_j)$ by $\varphi_j$, and define the vector $\overline{\varphi}_j$ to be $-\varphi_j$ if $j \in X$, and $\varphi_j$ otherwise. We write $D(\beta, \ldots, \beta')$ for the determinant of the matrix formed by a sequence $\beta, \ldots, \beta'$ of $d - 1$ column vectors in $\mathbb{R}^{d-1}$. Note that (*) says that the edge $a_{i+1}$ of $C_i$ is located behind $H_i$ (and the edge $a_i$ of $C_{i+1}$ before $H_i$). One can realize that this location corresponds to the relation

(**) $D := D(\varphi_1, \varphi_2, \ldots, \varphi_{i-1}, \overline{\varphi}_{i+1}, \varphi_{i+2}, \ldots, \varphi_d, \overline{\varphi}_{i+1}) > 0$.

Now validity of (**) follows from

$$D = (-1)^{d-i-1}D(\varphi_1, \ldots, \varphi_{i-1}, \overline{\varphi}_{i+1}, \varphi_{i+2}, \ldots, \varphi_d)$$

$$= D(\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \varphi_{i+2}, \ldots, \varphi_d) > 0$$

(taking into account (2.1) and the fact that $\overline{\varphi}_{i+1} = -\varphi_{i+1}$ if and only if $d - i - 1$ is odd).

To show (6.4), we argue in a similar way, replacing (6.3) by:

(6.7) in the graph $G_{Q''}$, the vertex $Y$ is incident to $d$ edges $b_1, \ldots, b_d$, where each $b_i$ is an $i$-edge, and $b_i$ enters (resp. leaves) $Y$ if $i \in Y$ (resp. $i \in [d] - Y$).

Using this and the fact that $Y$ if formed by elements $i \in [d]$ with $d - i$ even, one shows that for each $i$, the cube $C_i$ of type $A_i$ is located before the hyperplane separated $C_i$ and $C_{i+1}$ (cf. (*)), and (6.4) follows.
Remark 2. The above proposition implies that the “geometric” definition of Inv(M) (given in the beginning of this section) does not depend on the choice of a cubillage containing M as a membrane. Next, for a membrane M in Z(n, d), we alternatively could give a “packet” definition for straight and inversive tuples $\binom{n}{d}$ in a spirit of the statement in this proposition, and then come to the “geometric” characterization by reversing reasonings in the above proof. This alternative way to define Inv(M) matches the classical definition due to Manin and Schechtman (cf. Theorem 3 in [9]). Recall that they introduced a “packet admissible” total order $\prec$ on $\binom{n}{d}$, which means that for each tuple $K \in \binom{\lfloor \frac{n}{d} \rfloor}{d}$, the elements of Pac(K) become ordered by $\prec$ either lexicographically or anti-lexicographically, and in the latter case, K is said to be an inversion for $(\binom{\lfloor n/d \rfloor}{d}, \prec)$. (Compare $\prec$ with $\prec_M$.)

Note also that the method of proof of Proposition 6.2 enables us to reveal one more useful fact.

Proposition 6.3 Let M be a membrane in Z(n, d) and let $K \in \binom{\lfloor n/d \rfloor}{d}$ consist of elements $k_1 < \cdots < k_d$. Then:

(i) K is inversive for M if and only if there is $X \in \text{Sp}(M)$ such that $X \cap K = \{k_i : d - i \text{ odd}\} =: K^{\text{odd}}$;

(ii) K is straight for M if and only if there is $Y \in \text{Sp}(M)$ such that $Y \cap K = \{k_i : d - i \text{ even}\} =: K^{\text{even}}$.

Proof Let $[n] - K = \{j_1, j_2, \ldots, j_{n-d}\}$ and form the sequence $M_0 = M, M_1, \ldots, M_{n-d}$ of membranes in the corresponding zonotopes, where $M_i = (M_{i-1})/j_i$. So $M' := M_{n-d}$ is a membrane in the final zonotope $Z' := Z(K, d)$ (a single cube). We know that if K is inversive for M, then $M' = (Z')^{\text{rear}}$ and $M'$ contains the vertex $K^{\text{odd}}$, whereas if K is straight for M, then $M' = (Z')^{\text{f}}$ and $M'$ contains the vertex $K^{\text{even}}$ (cf. the proof of Proposition 6.2). Now the result follows by observing that for $1 \leq i \leq n - d$, if the membrane $M_i$ has a vertex $A$, then the previous membrane $M_{i-1}$ has a vertex $A'$ of the form $A$ or $A \cup \{j_i\}$.

As a consequence, we obtain the following result.

Theorem 6.4 Let $M_1, \ldots, M_p$ be membranes in Z(n, d) such that Inv($M_1$) $\subset \cdots \subset$ Inv($M_p$). Then the collection $\text{Sp}(M_1) \cup \ldots \cup \text{Sp}(M_p)$ is $(d - 1)$-separated.

Proof Suppose that this is not so. Then for some $i < j$, there exist $X \in \text{Sp}(M_i)$ and $Y \in \text{Sp}(M_j)$ that are not $(d - 1)$-separated from each other. Therefore, there exist elements $i_1 < i_2 < \cdots < i_{d+1}$ of [n] that alternate in $X - Y$ and $Y - X$. Let for definiteness the elements $i_k$ with $k$ odd are contained in $X - Y$ (and the other in $Y - X$). Consider the $d$-element sets $K := \{i_1, \ldots, i_d\}$ and $K' := \{i_2, \ldots, i_{d+1}\}$. Then, by Proposition 6.3 K is straight for one, and inversive for the other membrane among $M_i, M_j$. But the behavior of $M_i, M_j$ relative to $K'$ is opposite. Thus, neither Inv($M_i$) $\subset$ Inv($M_j$) nor Inv($M_j$) $\subset$ Inv($M_i$) is possible; a contradiction.

We finish this section with two applications.
1) Let $M, N$ be two membranes with $\text{Inv}(M) \subset \text{Inv}(N)$ in $Z = Z(n, d)$. By Theorem 6.4, the collection $C := \text{Sp}(M) \cup \text{Sp}(N)$ is $(d - 1)$-separated; so it is tempting to hope that $C$ is extendable to a maximal by size $(d - 1)$-separated set-system, or, equivalently, that there exists a cubillage $Q$ on $Z$ containing both membranes. We can try to construct such a $Q$ by filling the region $Z^-(M)$ (between $Z^r$ and $M$) with a “partial” cubillage $Q'$, and filling the region $Z^+(N)$ (between $N$ and $Z^{\text{rear}}$) with a “partial” cubillage $Q''$ (such $Q', Q''$ exist by (4.1)). But what is about the rest of $Z$ between $M$ and $N$, denoted as $Z(M, N)$? (Note that $\text{Inv}(M) \subset \text{Inv}(N)$ provides that $M$ lies within $Z^-(N)$.)

Let us say that $M, N$ are agreeable if the collection $\text{Sp}(M) \cup \text{Sp}(N)$ is $(n, d)$-extendable, i.e., a cubillage on $Z$ containing both $M, N$ (equivalently, a “partial” cubillage filling $Z(M, N)$) does exist. Ziegler [8] explicitly constructed two membranes $M, N$ in the zonotope $Z(8, 4)$ such that $\text{Inv}(M) \subset \text{Inv}(N)$ but $M, N$ are not agreeable (in our terms). This together with Theorem 6.4 implies that the set system $S_{8,4}$ is not pure (the latter fact was omitted in [3]). (Compare (O2) in the end of the previous section that considers the union of a membrane and a cube.)

2) In light of the above result for $d = 4$, Ziegler asked about the existence of two non-agreeable membranes in dimension 3. Answering this question, Felsner and Weil [2] proved that for an arbitrary $n$, any two membranes $M, N$ with $\text{Inv}(M) \subset \text{Inv}(N)$ in $Z(n, 3)$ are agreeable. Note that the proof in [2] attracted a non-trivial combinatorial techniques. An alternative proof immediately follows from Galashin’s result in [3] (mentioned in (1.2)) and Theorem 6.4.

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