SETS OF RIGGED PATHS WITH VIRAŠORO CHARACTERS

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Abstract. Let \( \{ M_{r,s}^{(p,p')} \}_{1 \leq r \leq p-1, 1 \leq s \leq p'-1} \) be the irreducible Virasoro modules in the \((p, p')\)-minimal series. In our previous paper, we have constructed a monomial basis of \( \bigoplus_{p=1}^{p-1} M_{r,s}^{(p,p')} \) in the case \( 1 < p'/p < 2 \). By ‘monomials’ we mean vectors of the form \( \phi_{-n_i}^{(r_1, r_{1-L-1})} \cdots \phi_{-n_j}^{(r_j, r_0)} |b(s), s\rangle \), where \( \phi_{-n_i}^{(r_1, r_{1-L-1})} : M_{r,s}^{(p,p')} \to M_{r,s}^{(p,p')} \) are the Fourier components of the \((2,1)\)-primary field and \(|b(s), s\rangle\) is the highest weight vector of \( M_{r,s}^{(p,p')} \). In this article, we introduce for all \( p < p' \) with \( p \geq 3 \) and \( s = 1 \) a subset of such monomials as a conjectural basis of \( \bigoplus_{p=1}^{p-1} M_{r,s}^{(p,p')} \). We prove that the character of the combinatorial set labeling these monomials coincides with the character of the corresponding Virasoro module. We also verify the conjecture in the case \( p = 3 \).

1. Introduction

Consider a representation \( M \) of a vertex operator algebra \( \mathcal{V} \). Let \( \phi^{(i)}(z) \in \mathcal{V} \) be a collection of fields, and let \( \phi_{i}^{(i)} \) be the corresponding Fourier coefficients.

Question. Find a set \( I \) of sequences of pairs \( (i_j, n_j) \) and a vector \( v \in M \) such that vectors \( \{ \phi_{n_i}^{(i_L)} \cdots \phi_{n_j}^{(i_1)} v \ | (i_j, n_j) \in I \} \) form a basis of \( M \).

Vectors of such form are usually called “monomials”. The question of finding a monomial basis, i.e., a basis consisting of monomials, is an important, well-known and old problem, solved in many interesting non-trivial cases. Examples include: integrable \( \mathfrak{sl}_n \) modules in terms of \( e_{11}(z) \) currents [LE] and [EP]: “big” and “small” coinvariants of \( \mathfrak{sl}_2 \) integrable modules in terms of the currents \( e(z), f(z), h(z) \) [FKLMM1] and [FKLMM2]; \((2,2n+1)\)-Virasoro minimal series in terms of the Virasoro current [PF]; and \((3,3n \pm 1)\)-Virasoro minimal modules tensored with Fock spaces in terms of an abelian current [FM].

Following the same philosophy, in [FJMMT] we constructed a basis of the \((p, p')\)-Virasoro minimal series representations \( M_{r,s}^{(p,p')} (1 \leq r \leq p-1, 1 \leq s \leq p'-1) \) with \( 1 < p'/p < 2 \) and \( p \geq 3 \), in terms of the \((2,1)\)-primary field \( \phi(z) \). The basis has the form

\[
\phi_{-n_i}^{(r_1, r_{1-L-1})} \cdots \phi_{-n_j}^{(r_j, r_0)} |b(s), s\rangle
\]

where \( r_0 = b(s), r_L = r, r_i = r_{i+1} + 1, \phi_{-n_i}^{(r, r')} : M_{r,s}^{(p,p')} \to M_{r,s}^{(p,p')} \) are the Fourier coefficients of \( \phi(z) \), and \( n_i \in \Delta_{r,s} - \Delta_{r_{i-1}, s} + \mathbb{Z} \). For each fixed \( s \), \( 1 \leq b(s) \leq p-1 \) is so chosen that the conformal dimension \( \Delta_{r,s} \) of the space \( M_{r,s}^{(p,p')} \) attains the minimum at \( r = b(s) \). We have also the condition

\[
n_{i+1} - n_i \geq w(r_{i+1}, r_i, r_{i-1}) \quad (1 \leq i \leq L - 1)
\]

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where \( w(r'', r', r) \) \((r = r' \pm 1, r'' = r'' \pm 1)\) are certain rational numbers (see (1.2)–(1.5) below). It was shown that the condition (1.3) is a consequence of the quadratic relations in the algebra generated by Fourier coefficients of the primary field \( \phi(z) \).

In this paper we consider the case \( p'/p > 2 \). In this case, in addition to the quadratic relations, the algebra has cubic relations. To our surprise, we found that in many cases (if not all) the cubic relations result in a very simple exclusion rule in addition to (1.2):

\[
(1.3) \quad n_{i+2} - n_i \geq 1.
\]

The condition (1.3) is void in the case \( 1 < p'/p < 2 \) since it follows from (1.2). We conjecture that in the general case \( p < p' \) the monomials satisfying (1.2) and (1.3) form a basis (see Conjecture 2.3 for the precise statement).

The purpose of this paper is to give some evidences for this conjecture. Our results are two-fold. First, we show that the combinatorial set defined by conditions (1.2), (1.3) (which we call the set of ‘rigged paths’) has the same graded character as that of the Virasoro module for any \((p, p')\) minimal theory with \( s = 1 \) (see Theorem 2.2). Second, we prove the conjecture for the \((3, 3n \pm 1)\) minimal theories with \( s = 1 \) (see Theorem 3.5).

The first statement is purely combinatorial. We prove it by showing that the characters of Virasoro modules and the characters of the combinatorial sets enjoy the same recurrence when the parameter changes from \( p' \rightarrow p \) to \( p' \). (See Proposition 3.2 and Proposition 3.3).

The second statement is based on the work \([FJM]\), in which the representation space

\[
V^{(3,p')} = \left( M^{(3,p')}_{1,1} \oplus (\oplus_{n:\text{even}} \mathcal{F}_{n\beta}) \right) \oplus \left( M^{(3,p')}_{2,1} \oplus (\oplus_{n:\text{odd}} \mathcal{F}_{n\beta}) \right)
\]

of the abelian current \( a(z) = \phi(z) \otimes \Phi_\beta(z) \) is studied. Here \( \Phi_\beta(z) : \mathcal{F}_{n\beta} \rightarrow \mathcal{F}_{(n+1)\beta} \) is a certain bosonic vertex operator acting on the bosonic Fock space \( \mathcal{F}_{n\beta} \). It was shown that the abelian current \( a(z) \) satisfies cubic relations, and by exploiting the cubic relations a monomial basis of the space \( W^{(3,p')} \) generated from the vector \( (1, 1) \otimes |0\rangle \), where \( |0\rangle \in \mathcal{F}_0 \) is the highest weight vector, was constructed. From the construction for \( W^{(3,p'-3)} \), we deduce a monomial basis of the space

\[
M^{(3,p')} = M^{(3,p')}_{1,1} \oplus M^{(3,p')}_{2,1}.
\]

The story is as follows. We construct a filtration of the space \( M^{(3,p')} \) by using the operator \( \phi(z) \). This filtration induces a current \( \tilde{a}(z) \) acting on the corresponding graded space, from the operator \( a(z) \) acting on \( V^{(3,p')} \). The correlation functions for the operator \( \tilde{a}(z) \) are equal to those for the operator \( \tilde{a}(z) \) up to simple factors. The latter belong to the space of correlation functions of the operator \( a(z) \) with \( p' \) replaced by \( p' - 3 \) up to simple factors. The spanning property of the monomials (1.1) with (1.2), (1.3) is deduced by using these identifications of correlation functions and the monomial basis of the space \( W^{(3,p'-3)} \) constructed in \([FJM]\).

The plan of our paper is as follows. In Section 2 we define the combinatorial sets of rigid paths and formulate Conjecture 2.3. In Section 3 we show that Virasoro characters satisfy a recurrence relation when the parameter changes from \( p' \rightarrow p \). In Section 4 we show that the combinatorial sets of rigid paths satisfy the same recurrence relation. Section 5 is devoted to the proof of the conjecture in the case \( (p, p') = (3, 3n \pm 1) \).

In the paper by P. Jacob and P. Mathieu \([JM]\), the problem of constructing monomial basis is studied, and combinatorial conditions similar to ours are proposed. Their study is restricted to the
(3, p) case, and in this case the paths \((r_L, \ldots, r_1)\) do not appear in combinatorics. We thank one of the referees for attracting our attentions to this paper.

We also thank another referee for providing us with the simple proof of Proposition 3.4 as given in the below.

2. Rigged paths

2.1. Minimal series. We recall some definitions about minimal conformal field theory. For the details we refer to [DMS]. Let \(Vir\) be the Virasoro algebra with the standard \(\mathbb{C}\)-basis \(\{L_n\}_{n \in \mathbb{Z}}\) and \(c\) satisfying
\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \quad [c, L_n] = 0.
\]

Fix a pair \((p, p')\) of relatively prime positive integers. We set
\[
t = \frac{p'}{p}.
\]

We assume \(p \geq 3\) so that the (2, 1) primary field exists (see below). We consider the minimal series of representations \(M_{r,s}^{(p,p')} (1 \leq r \leq p - 1, 1 \leq s \leq p' - 1)\) of \(Vir\). The module \(M_{r,s}^{(p,p')}\) is generated by a vector \(|r, s\rangle\) called the highest weight vector. It is an irreducible module. The central element \(c\) acts as the scalar
\[
c_{p,p'} = 13 - 6 \left( t + \frac{1}{t} \right).
\]

The highest weight vector satisfies
\[
L_n|r, s\rangle = 0 \text{ if } n > 0, \quad L_0|r, s\rangle = \Delta_{r,s}^{(t)}|r, s\rangle,
\]
where
\[
\Delta_{r,s}^{(t)} = \frac{(rt - s)^2 - (t - 1)^2}{4t}.
\]

We fix \(s\). The (2, 1) primary field
\[
\phi^{(r \pm 1, r)}(z) = \sum_{n \in \mathbb{Z}-\Delta_{r,s}^{(t)} + \Delta_{r \pm 1,s}^{(t)}} \phi_{-n}^{(r \pm 1, r)} z^{-n - \Delta_{r \pm 1,s}^{(t)}}
\]
is a generating series of linear operators \(\phi_{-n}^{(r \pm 1, r)}\) acting as
\[
\phi_{-n}^{(r \pm 1, r)} : M_{r,s}^{(p,p')} \rightarrow M_{r \pm 1,s}^{(p,p')}.
\]

Up to a scalar multiple, they are characterized by the commutation relations with the Virasoro generators:
\[
[L_n, \phi^{(r \pm 1, r)}(z)] = z^n (z \partial + (n + 1)\Delta_{r \pm 1,s}^{(t)}) \phi^{(r \pm 1, r)}(z).
\]

The module \(M_{r,s}^{(p,p')}\) is graded by eigenvalues of the operator \(L_0\):
\[
M_{r,s}^{(p,p')} = \bigoplus_d \left( M_{r,s}^{(p,p')} \right)_d, \quad \left( M_{r,s}^{(p,p')} \right)_d = \{ |v\rangle \in M_{r,s}^{(p,p')} | L_0|v\rangle = d|v\rangle \}.
\]
The character $\chi_{r,s}^{(p,p')} (q)$ is defined by the formula
$$
\chi_{r,s}^{(p,p')} (q) = \text{tr}_{M_{r,s}^{(p,p')}} q^L.
$$

The primary field preserves the grading:
$$
\phi_{-n}^{(r',p')} (M_{r,s}^{(p,p')})_d \subset (M_{r',s}^{(p,p')})_{d+n}.
$$

2.2. Quadratic relations. In [FJMMT] we derived a set of quadratic relations for the Fourier components of the (2,1) primary field of the following form:

$$
(2.1) \quad \sum_{n,n'=d} c_{n,n'}^{(r,r',r''')} \phi_{-n}^{(r,r')} \phi_{-n'}^{(r',r''')} = 0,
$$

where $r, r''$, and $d$ are fixed, and the coefficients $c_{n,n'}^{(r,r',r''')}$ are such that if $n' > N$ for some $N$ they are zero except for finitely many of them. Let us describe a consequence of the quadratic relations in a little bit different way than in [FJMMT]. Define weights $w(t)(r, r', r'') = w(t)(r'', r', r) = w(t)(p - r, p - r', p - r'')$ where $r - r', r' - r'' \in \{-1, 1\}$ by

$$
(2.2) \quad w(t)(r, r \pm 1, r \pm 2) = \frac{t}{2}, \quad \text{if } 1 \leq r, r \pm 1, r \pm 2 \leq p - 1,
$$

$$
(2.3) \quad w(t)(r, r + 1, r) = 2 - \frac{t}{2} + [rt] - rt, \quad \text{if } 2 \leq r, r + 1 \leq p - 1,
$$

$$
(2.4) \quad w(t)(r, r - 1, r) = 1 - \frac{t}{2} - [rt] + rt, \quad \text{if } 1 \leq r - 1, r \leq p - 2,
$$

$$
(2.5) \quad w(t)(1, 2, 1) = w(t)(p - 1, p - 2, p - 1) = 3 - \frac{3t}{2}.
$$

Here $[x]$ is the integer part of $x$.

**Proposition 2.1.** Any monomial of the form

$$
(2.6) \quad \phi_{-n_L}^{(r_L, r_{L-1})} \cdots \phi_{-n_1}^{(r_1, r_0)}
$$

can be written as an infinite linear combination of monomials satisfying

$$
(2.7) \quad n_{i+1} - n_i - w(t)(r_{i+1}, r_i, r_{i-1}) \geq 0.
$$

**Proof.** Essentially, the proof is given in [FJMMT] (see Proposition 3.3 in that paper). Note, however, that in [FJMMT] the weights $w(t)(1, 2, 1)$ and $w(t)(p - 1, p - 2, p - 1)$ are given as special cases of (2.3) and (2.4), respectively. This is because the range of $t$ was restricted to $1 < t < 2$, wherein the two expressions coincide. In [FJMMT], the quadratic relations were derived without the restriction on $t$, and the statement that any monomial of the form (2.6) can be rewritten into an infinite linear combination of those which satisfy (2.7) was, in effect, proved without the restriction $t < 2$ by using (2.5), instead of using the special cases of (2.3) and (2.4).

We give a few more remarks. As stated above, if we rewrite a monomial by using the relations of the form (2.1), we obtain an infinite linear combination. Note, however, that if we fix the degree $n_1 + \cdots + n_L$ and restrict $n_1 \geq N$ for some $N$, there exists only finitely many monomials satisfying the condition (2.7). Therefore, such infinite sums are meaningful after completion.
Let us consider vectors of the form
\[
\phi_{-n_L}^{(r_L,r_{L-1})} \cdots \phi_{-n_1}^{(r_1,r_0)} | r_0, s \rangle,
\]
where \(| r_0, s \rangle\) is the highest weight vector of \(M_{r_0,s}^{(p,p')}\). By abuse of terminology we call such vectors monomials.

Proposition 2.1 implies that any vector of the above form belongs to the linear span (in the sense of finite linear combinations) of those satisfying (2.7) and the highest weight condition
\[
n_1 + \Delta_{r_0,s}^{(t)} - \Delta_{r_1,s}^{(t)} \geq 0.
\]

2.3. Statement of the results. Now we restrict to the case \(t > 1 \) and \((r_0, s) = (1, 1)\).

We say a monomial
\[
\phi_{-n_L}^{(r_L,r_{L-1})} \cdots \phi_{-n_1}^{(r_1,r_0)} | 1, 1 \rangle
\]
is admissible if and only if (2.7), and
\[
n_1 \geq \Delta_{2,1}^{(t)} = \frac{3t - 2}{4},
\]
and
\[
n_{i+2} - n_i \geq 1
\]
when \(r_{i-1}, r_i, r_{i+1} \in \{r, r + 1\}\) for some \(1 \leq r \leq p - 2\), \(1 \leq i \leq L - 2\), are satisfied.

Set
\[
v^{(t)}(r) = 1 - w^{(t)}(r, r + 1, r) - w^{(t)}(r + 1, r, r + 1).
\]

We have
\[
v^{(t)}(r) = \begin{cases} 
p' - 5 & \text{if } p = 3; \\
2t - 3 & \text{if } p > 3 \text{ and } r = 1, p - 2; \\
[(r + 1)t] - [rt] - 2 & \text{if } 1 < r < p - 2. 
\end{cases}
\]

Although \(w^{(t)}\) is not necessarily an integer, \(v^{(t)}\) are integers. Note also that \(v^{(t)}(r) = v^{(t)}(p - 1 - r)\).

A rigged path of length \(L\) is a table of integers of the form
\[
P = \left( \begin{array}{cccc}
r_L & r_{L-1} & \cdots & r_1 \\
\sigma_{L-1} & \cdots & \sigma_1 & \sigma_0
\end{array} \right)
\]
where \(r_0 = 1\), \(1 \leq r_i \leq p - 1\) (\(0 \leq i \leq L\)) and
\[
r_{i+1} - r_i \in \{-1, 1\} \quad (0 \leq i \leq L - 1).
\]

In the usual terminology, a sequence of integers \((r_i)\) satisfying (2.13) is called a path. A rigged path is decorated by the rigging \((\sigma_i)\). For brevity, we often call a rigged path simply a path.

A path is called admissible at level \(t\) if \(\sigma_i \geq 0\) (\(0 \leq i \leq L - 1\)), and
\[
\sigma_i + \sigma_{i+1} \geq v^{(t)}(r)
\]
when \(r_{i-1}, r_i, r_{i+1}, r_{i+2} \in \{r, r + 1\}\) for some \(1 \leq r \leq p - 2\), \(1 \leq i \leq L - 2\).
Note that these conditions correspond to (2.10), (2.7) and (2.11), respectively, if we set
\[ \sigma_0 = n_1 - \Delta_{2,1}, \quad \sigma_i = n_{i+1} - n_i - w^{(t)}(r_{i+1}, r_i, r_{i-1}) \quad (1 \leq i \leq L - 1). \]

If \( t < 2 \), we have \( v^{(t)}(r) \leq 0 \) and the condition (2.14) follows from the positivity of \( \sigma_i \)'s. If \( t > 2 \), we have
\[ v^{(t)}(r) \geq 0 \quad (1 \leq r \leq p - 2), \]
and moreover
\[ v^{(t)}(1) = v^{(t)}(p - 2) \geq 1. \]

We denote by \( C^{(t)}_L \) the set of rigged paths of length \( L \) which are admissible at level \( t \), and by \( C^{(t)}_{L,r} \) the subset of \( C^{(t)}_L \) consisting of paths such that \( r_L = r \). The subset \( C^{(t)}_{L,r} \) is empty unless \( r \equiv L + 1 \mod 2 \).

We define the degree of \( P \in C^{(t)}_L \) by
\[ d(P) = \sum_{i=1}^{L} n_i \]
\[ = L \Delta_{2,1}^{(t)} + \sum_{i=1}^{L-1} (L - i) w^{(t)}(r_{i+1}, r_i, r_{i-1}) + \sum_{i=0}^{L-1} (L - i) \sigma_i, \]
and the character of \( C^{(t)}_{L,r} \) by
\[ \text{ch}_q C^{(t)}_{L,r} = \sum_{P \in C^{(t)}_{L,r}} q^{d(P)}. \]

Our main result is the following identity:

**Theorem 2.2.** We have
\[ \chi^{(p, p')}_{r, 1}(q) = \sum_{L \geq 0} \text{ch}_q C^{(t)}_{L, r}. \]

Theorem 2.2 will be proved at the end of Section 4. This result motivates us to make the following conjecture:

**Conjecture 2.3.** For \( 1 \leq r \leq p - 1 \), the set of admissible monomials of the form (2.8), where \( r_L = r, r_0 = 1 \) and \( s = 1 \), is a basis of \( M_{r, 1}(p, p') \).

The case \( 1 < t < 2 \) of the conjecture has been proved in [FJMM1]. We prove Conjecture 2.3 in the case \( p = 3 \) in Section 5.

3. Recurrence structure

3.1. Recurrence relation for Virasoro characters. Recall the formula for \( \chi^{(p, p')}_{r, s}(q) \) [RC]:
\[ \chi^{(p, p')}_{r, s}(q) = \frac{q^{\Delta_{2,1}^{(t)}}}{(q)_{\infty}} \left( \sum_{n \in \mathbb{Z}} q^{pp'n^2 + (p'r + ps)n} - \sum_{n \in \mathbb{Z}} q^{pp'n^2 + (p'r + ps)n + rs} \right). \]
Here \( (q)_\infty = \prod_{j=1}^{\infty} (1 - q^j) \). In the case \( 1 < t < 2 \), \( \chi_{r,1}^{(p,p')} (q) \) was written in the following form, [FJMMT W]:

\[
\chi_{r,1}^{(p,p')} (q) = q^{\Delta_{r,1}} \sum_{m \equiv r-1 \mod 2} \frac{1}{(q)_m} K_{m,r}^{(p,p'-p)} (q),
\]

where

\[
K_{m,r}^{(p,p')} (q) = \frac{m^2 - (r-1)^2}{4} \sum_{n \in \mathbb{Z}} q^{mpn^2 + prn^2} \left( \left[ \frac{m}{2} + \frac{m}{2} + pn \right] - \left[ \frac{m}{2} + \frac{m}{2} + pn \right] \right).
\]

Here we have used the notation

\[
(q)_n = \prod_{j=1}^{n} (1 - q^j), \quad [m] = \frac{(q)_m}{(q)_n (q)_{m-n}}.
\]

The identity (3.2) can be generalized to the case \( t > 2 \):

**Proposition 3.1.** Set \( k = \lfloor t \rfloor \). Then the following equality holds:

\[
\chi_{r,1}^{(p,p')} (q) = q^{\Delta_{r,1}} \sum_{\substack{m_0, \ldots, m_{k-1} \geq 0 \\ m_0 \equiv r-1 \mod 2}} q^{Q(k)(m_0, \ldots, m_{k-1}) - \frac{k-1}{4} (v^2 - 1)} \frac{(q)_{m_0} \cdots (q)_{m_{k-1}}} {K_{m_0,r}^{(p,p'-kp)} (q)}.
\]

Here

\[
Q(k)(m_0, \ldots, m_{k-1}) = \frac{k-1}{4} m_0^2 + \sum_{j=1}^{k-1} (k-j)m_j^2 + \sum_{j=1}^{k-1} (k-j)m_0m_j + 2 \sum_{1 \leq j < j' \leq k-1} (k-j)m_jm_{j'} + \frac{k-1}{2} m_0 + \sum_{j=1}^{k-1} (k-j)m_j.
\]

Proposition 3.1 will be proved in Section 3.2.

For \( L \in \mathbb{Z}_{\geq 0} \) denote by \( \chi_{r,1}^{(p,p')} (q) \) the right hand side of (3.4) with the sum replaced by the partial sum over \( m_0, \ldots, m_{k-1} \) satisfying \( m_0 + 2(m_1 + \cdots + m_{k-1}) = L \). Then we have

\[
\chi_{r,1}^{(p,p')} (q) = \sum_{L \geq 0} \chi_{r,1;L}^{(p,p')} (q).
\]

It follows from Proposition 3.1 that

**Proposition 3.2.** We have

\[
\chi_{r,1;L}^{(p,p')} (q) = \sum_{m \geq 0} q^{\frac{L^2}{4} + \frac{L}{2}} (q)_m \chi_{r,1;L-2m}^{(p,p'-p)} (q).
\]

In Section 4 we prove (see Theorem 4.13)
Proposition 3.3.

\begin{equation}
\text{ch}_q C^{(t)}_{L,r} = \sum_{m \geq 0} \frac{q^{\frac{t^2}{4} + \frac{t}{2}}}{(q)_m} \text{ch}_q C^{(t-1)}_{L-2m,r}.
\end{equation}

Theorem 2.2 follows from these identities.

3.2. Proof of Proposition 3.1. In the following we set \(1/(q)_n = 0\) for \(n < 0\). To prove Proposition 3.1 we use the following formula (the proof given here is provided by one of the referees):

Proposition 3.4. For \(l \in \mathbb{Z}_{\geq 0}\) and \(\mu \in \mathbb{Z}\), we have

\begin{equation}
\frac{1}{(q)_\infty} = \sum_{N_0, \ldots, N_l \geq 0} \frac{q^{\sum_{j=0}^l N_j^2 + \mu \sum_{j=0}^l N_j}}{(q)_{N_0 + \mu}(q)_{N_1 - N_0} \cdots (q)_{N_l - N_{l-1}}}.
\end{equation}

Proof. Consider the following functions \(g_l(z)\) \((l = 0, 1, \ldots)\):

\[g_l(z) = \sum_{N_0, \ldots, N_l \geq 0} \frac{z^{\sum_{j=0}^l N_j} q^{\sum_{j=0}^l N_j}}{(q)_{N_0 + \mu}(q)_{N_1 - N_0} \cdots (q)_{N_l - N_{l-1}}},\]

where \((z)_n = \prod_{j=0}^{n-1} (1 - zq^j)\). Let us prove that \(g_l(z) = 1/(zq)_\infty\). Then we obtain (3.7) by setting \(z = q^\mu\).

First note that \(g_0(z)\) can be rewritten in terms of the basic hypergeometric series \(\phi_1\) as follows:

\[g_0(z) = \lim_{a,b \to \infty} 2\phi_1(a, b; zq, q, zq/ab),\]

where

\[2\phi_1(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(q)_n} z^n.\]

On the other hand we have the \(q\)-Gauss sum identity (see, e.g., [GR]):

\[2\phi_1(a, b; c; q, c/ab) = (c/a)_\infty (c/b)_\infty (c/ab)_\infty.\]

This implies that \(g_0(z) = 1/(zq)_\infty\).

Next consider the case where \(l > 0\). Rewrite the definition of \(g_l(z)\) as

\[g_l(z) = \sum_{N_1, \ldots, N_l \geq 0} \frac{z^{\sum_{j=1}^l N_j} q^{\sum_{j=1}^l N_j^2}}{(q)_{N_1}(q)_{N_1 - N_0} \cdots (q)_{N_l - N_{l-1}}} f_{N_1}(z),\]

where

\[f_N(z) = (q)^N \sum_{j=0}^{N} \frac{z^j q^j}{(zq)^j(q)_N-j}.
\]

It is easy to see that

\[f_N(z) = \lim_{a \to \infty} 2\phi_1(a, q^{-N}; zq; q, q^{N+1}/a).\]
Using the $q$-Gauss sum formula again, we see that $f_N(z) = 1/(zq)_N$. This implies $g_1(z) = g_{-1}(z)$ and therefore we get $g_1(z) = g_0(z) = 1/(zq)_\infty$.

**Proof of Proposition 3.1**

Substituting (3.3) to the right hand side of (3.4), we obtain two sums, which we refer to as I and II. First consider I:

$$
\sum_{m_0, \ldots, m_{k-1} \geq 0 \atop m_0 \equiv r \mod 2} q^{Q(k)(m_0, \ldots, m_{k-1}) - \mu k}(r^2 - 1) \frac{(q)_{m_0} \cdots (q)_{m_{k-1}}}{(q)_{m_0+1} - pn}.
$$

Set

$$
N_0 = \frac{m_0 - r + 1}{2} - pn, \quad N_j - N_{j-1} = m_j \quad (j = 1, \ldots, k - 1)
$$

and rewrite (3.8) as a summation over $N_0, \ldots, N_{k-1} \geq 0$ and $n \in \mathbb{Z}$. Then (3.8) becomes

$$
\sum_{n \in \mathbb{Z}} q^{pp' n^2 + (p' r - p)n} F_k(r - 1 + 2pn),
$$

where

$$
F_k(\mu) = \sum_{N_0, \ldots, N_{k-1} \geq 0} \frac{q^{\sum_{j=0}^{k-1} N_j^2 + \mu N_0 + (\mu + 1) \sum_{j=1}^{k-1} N_j}}{(q)_{N_0 + \mu}(q)_{N_1 - N_0} \cdots (q)_{N_{k-1} - N_{k-2}}^{r-1} - r}. \quad (3.9)
$$

In the same way rewrite II by setting

$$
N_0 = \frac{m_0 + r + 1}{2} + pn, \quad N_j - N_{j-1} = m_j \quad (j = 1, \ldots, k - 1).
$$

The result is

$$
\sum_{n \in \mathbb{Z}} q^{pp' n^2 + (p' r + p)n} F_k(-r - 1 - 2pn).
$$

Thus we find that the right hand side of (3.3) is equal to

$$
q^{\Delta_r(1)} \sum_{n \in \mathbb{Z}} q^{pp' n^2 + (p' r - p)n} F_k(r - 1 + 2pn) - q^{r + 2pn} F_k(-r - 1 - 2pn).
$$

Set $\mu = r + 2pn$. The last part of (3.9) reads $F_k(\mu - 1) - q^\mu F_k(-\mu - 1)$. In the sum $F_k(-\mu - 1)$, we replace $N_j$ by $N_j + \mu$. Then, we have

$$
F_k(\mu - 1) - q^\mu F_k(-\mu - 1) = (1 - q^\mu) \sum_{N_0, \ldots, N_{k-1} \geq 0} \frac{q^{\sum_{j=0}^{k-1} N_j^2 + \mu \sum_{j=1}^{k-1} N_j}}{(q)_{N_0 + \mu}(q)_{N_1 - N_0} \cdots (q)_{N_{k-1} - N_{k-2}}} = 1 - q^\mu.
$$

Here we used (3.7) in the last equality. Hence (3.9) is equal to the right hand side of the formula (8.1). This completes the proof.
4. The bijection

4.1. Particles. In each path \( P \in C_L^{(t)} \), we locate “particles”.

Recall, that a path \( P \in C_L^{(t)} \) is a table of integers

\[
P = \begin{pmatrix} r_L & r_{L-1} & \ldots & r_1 & r_0 \\ \sigma_{L-1} & \ldots & \sigma_1 & \sigma_0 \end{pmatrix}.
\]

The integers \( r_x \) satisfy the conditions \( r_0 = 1, 1 \leq r_x \leq p - 1 \) \((1 \leq x \leq L)\) and \(|r_x - r_{x-1}| = 1\) \((1 \leq x \leq L)\). The integers \( \sigma_x \) satisfy the conditions \( \sigma_x \geq 0 \) \((0 \leq x \leq L - 1)\) and

\[
\sigma_x + \sigma_{x-1} \geq v^{(t)}(r) \quad (2 \leq x \leq L - 1)
\]

if

\[
r_{x+1}, r_x, r_{x-1}, r_{x-2} \in \{r, r+1\} \quad \text{for some } 1 \leq r \leq p - 2.
\]

It is convenient to define \( \sigma_L = \infty \), and use the convention that \( \infty + 1 = \infty \).

Our aim is to relate the sets \( C_L^{(t-1)} \) \((L = 0, 1, 2, \ldots)\) to the sets \( C_L^{(t)} \) \((L = 0, 1, 2, \ldots)\). There is an injective mapping from \( C_L^{(t-1)} \) to \( C_L^{(t)} \) such that the image of the mapping is equal to the subset of \( C_L^{(t)} \) consisting of paths \( P \) for which the inequality \( 4.2 \) is strict, and \( \sigma_x \geq 1 \) \((1 \leq x \leq L - 1)\), and \( \sigma_x \geq 2 \) if \( r_{x+1} = r_{x-1} = 1 \) or \( p - 1 \). We say the path \( P \) has no particle in such case. When these conditions are violated, we observe the appearance of “particles” in \( P \). We define the number of particles in \( P \). Then, we construct a bijection from \( C_L^{(t-1)} \times \pi_m \) to the subset of paths in \( C_L^{(t)} \) with \( m \) particles. Here we denote by \( \pi_m \) the set of partitions \((\lambda_1, \ldots, \lambda_m)\) of length \( m \), i.e.,

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0.
\]

We now give a precise definition of particles.

Given a path \( P \in C_L^{(t)} \), we define an equivalence relation in the set \( \{1, \ldots, L - 1\} \). We say a neighboring pair of integers \( x, x-1 \in \{1, \ldots, L - 1\} \) is connected if and only if \( 4.3 \) is valid and

\[
\sigma_x + \sigma_{x-1} = v^{(t)}(r) \quad (2 \leq x \leq L - 1).
\]

We say \( x \sim y \), where \( x, y \in \{1, \ldots, L - 1\} \), if and only if all neighboring pairs of integers in the interval between \( x \) and \( y \) are connected.

We call an equivalence class \( B \) for this equivalence relation a block of particles if one of the following is satisfied:

\[
|B| \geq 2,
\]

\[
B = \{x\} \quad \text{and} \quad \sigma_x = 1, (r_{x+1}, r_x, r_{x-1}) = (1, 2, 1) \text{ or } (p - 1, p - 2, p - 1),
\]

\[
B = \{x\} \quad \text{and} \quad \sigma_x = 0, r_{x+1} = r_{x-1}.
\]

We denote by \( \mathcal{B}(P) \) the set of blocks for the path \( P \). We call a block \( B \) an isolated particle if \( B \) consists of one element, \( |B| = 1 \).

For each block \( B \in \mathcal{B}(P) \), let \( \max(B), \min(B) \) be the largest and the smallest integer in \( B \). The blocks in \( P \) are naturally ordered: \( B > B' \) if and only if \( \min(B) > \max(B') \).

If \( x \) belongs to a block \( B \), we have \( r_{x+1} = r_{x-1} \). We denote this number by \( r'_x \).
**Definition 4.1.** Let $P \in C_L^{(t)}$. We define a map $m : \mathcal{B}(P) \to \mathbb{Z}_{\geq 0}$. Namely, for $B \in \mathcal{B}(P)$, we set

$$m(B) = \begin{cases} 
\left\lfloor \frac{|B|}{2} \right\rfloor & \text{if } \sigma_x \geq 2 \text{ or if } \sigma_x = 1 \text{ and } r'_x \neq 1, p - 1; \\
\left\lfloor \frac{|B|+1}{2} \right\rfloor & \text{if } \sigma_x = 0 \text{ or if } \sigma_x = 1 \text{ and } r'_x = 1, p - 1.
\end{cases}$$

(4.8)

where $x = \max(B)$. The number $m(B)$ is called the number of particles in the block $B$. We also set

$$m(P) = \sum_{B \in \mathcal{B}(P)} m(B).$$

(4.9)

The number $m(P)$ is called the number of particles in the path $P$.

If $|B|$ is odd, then we have $\sigma_{\min(B)} = \sigma_{\max(B)}$. Therefore, one can use $x = \min(B)$ in Definition 4.1. Note also that if $m(P) \leq \lfloor L/2 \rfloor$.

4.2. **Propagation of particles.** Roughly speaking, blocks in a path $P$ are the location of particles in the sequence of integers

$$L - 1, L - 2, \ldots, 2, 1.$$

We move the particles in $P$ by changing blocks locally in this sequence without changing the total number of particles. We number the particles from the left to the right. We define the left (resp., right) move of the $j$-th particle, $M_j^+$ (resp., $M_j^-$). These operations change a path $P \in C_L^{(t)}$ to a path $M_j^\pm P \in C_L^{(t)}$. For some paths $P \in C_L^{(t)}$, the path $M_j^\pm P \in C_L^{(t)}$ is not defined. If a block contains more than one particles, i.e., $m(B) \geq 2$, the left move is defined only for the leftmost particle in the block, and the right move is only for the rightmost one. In other words, particles in a block are located without gaps and they cannot pass each other. We start with the definition of the domain of the operations $M_j^\pm$.

We set

$$D^+(P) = \{ \max(B) | B \in \mathcal{B}(P) \},$$

(4.10)

and

$$D^-(P) = \begin{cases} 
\{ \min(B) | B \in \mathcal{B}(P) \} \setminus \{1\} & \text{if } \sigma_0 = \sigma_1 = 0; \\
\{ \min(B) | B \in \mathcal{B}(P) \} & \text{otherwise}.
\end{cases}$$

(4.11)

We define maps $m_j^\pm : D^\pm(P) \to \{1, \ldots, m(P)\}$:

$$m_j^+(x) = 1 + \sum_{B \in \mathcal{B}(P) \atop \max(B) > x} m(B),$$

(4.12)

and

$$m_j^-(x) = \sum_{B \in \mathcal{B}(P) \atop \min(B) \geq x} m(B).$$

(4.13)

Set

$$I_j^\pm(P) = \text{Im}(m_j^\pm) \subset \{1, \ldots, m(P)\}.$$
The map \( m_P^\pm \) is injective, and therefore, on the image we can define the inverse mappings
\[
(4.15) \quad x_p^\pm = (m_p^\pm)^{-1} : I^\pm(P) \to \{1, \ldots, L - 1\},
\]
\[
(4.16) \quad j \mapsto x_p^\pm(j).
\]

The left move \( M_j^+P \) is defined if and only if \( j \in I^+(P) \), and the right move \( M_j^-P \) is defined if and only if \( j \in I^-(P) \). In such cases, we define the position of the \( j \)-th particle by \( x_p^+(j) \) or \( x_p^-(j) \). If \( j \in I^+(P) \cap I^-(P) \), then we have \( x_p^+(j) = x_p^-(j) \). In this case, the corresponding particle is isolated.

In general, for each \( 1 \leq j \leq m(P) \) there exists a unique block \( B_j \in \mathcal{B}(P) \) such that
\[
(4.17) \quad 1 + \sum_{\substack{B \in \mathcal{B}(P) \\mid B > B_j \\}} m(B) \leq j \leq \sum_{\substack{B \in \mathcal{B}(P) \\mid B \geq B_j \\}} m(B).
\]

We say that the \( j \)-th particle is located in the block \( B_j \). However, we do not specify its position except for the leftmost one or the rightmost one.

**Definition 4.2.** Let \( P \in C_L^{(t)} \). Recall our convention \( \sigma_L = \infty \). Fix \( j \in I^\pm(P) \) and set \( x = x_p^\pm(j) \). We define
\[
(4.18) \quad M_j^\pm P = \left( \frac{r^+_L}{\sigma_L - 1} \frac{r^-_{L-1}}{\sigma_{L-1} - 1} \ldots \frac{r^+_1}{\sigma_1 - 1} \frac{r^-_0}{\sigma_0 - 1} \right),
\]
by setting \( r_y^+ = r_y \) (\( 0 \leq y \leq L \)) and \( \sigma_y^+ = \sigma_y \) (\( 0 \leq y \leq L - 1 \)) except for the following.

*Case 1 \( \sigma_x \neq 0 \):
\[
(4.19) \quad \sigma_x^\pm = \sigma_x - 1,
(4.20) \quad \sigma_{x+1}^\pm = \sigma_{x+1} - 1.
\]

*Case 2 \( \sigma_x = 0 \) and \( r_x' = 1, p - 2 \):
\[
(4.21) \quad \sigma_x^\pm = 1,
(4.22) \quad \sigma_{x+1}^\pm = \sigma_{x+1} - 1.
\]

*Case 3 \( \sigma_x = 0 \) and \( r_x' \neq 1, p - 2 \):
\[
(4.23) \quad r_x^\pm = 2r_x' - r_x,
(4.24) \quad \sigma_{x+1}^\pm = \begin{cases} \sigma_{x+1} - v^{(t)}(r) - 1 & \text{if } r_{x+2} = r_x; \\ \sigma_{x+1} + v^{(t)}(r + \varepsilon) & \text{if } r_{x+2} = r_x + 2\varepsilon, \end{cases}
(4.25) \quad \sigma_{x+1}^\pm = \begin{cases} \sigma_{x+1} - v^{(t)}(r) & \text{if } r_{x+2} = r_x; \\ \sigma_{x+1} + v^{(t)}(r + \varepsilon) + 1 & \text{if } r_{x+2} = r_x + 2\varepsilon, \end{cases}
\]

where \( r = \min \{r_x, r_x'\} \) and \( \varepsilon = \pm 1 \).

The maps \( M_i^\pm \) change the degree of a path by \( \pm 1 \) and do not change the number of particles:

**Lemma 4.3.** Let \( P \in C_L^{(t)} \) and \( j \in I^\pm(P) \). We have \( M_j^\pm P \in C_L^{(t)} \), \( m(M_j^\pm P) = m(P) \) and \( d(M_j^\pm P) = d(P) \pm 1 \).

The proof is only case-checking.
4.3. Properties of $M_j^\pm$. We list properties of moves $M_j^\pm$. In most cases, we omit proofs because they are only case-checkings. However, the following remark might help understanding.

If $j \in I^+(P)$ and the $j$-th particle belongs to a block $B$ with more than one particles, after the move $M_j^+$ this particle quits the block and the number of particles in $B$ decreases. The $j$-th particle either becomes an isolated particle or joins in another block to increase the number of particles in that block. The consideration of $M_j^-$ is similar.

The moves $M_j^+$ and $M_j^-$ are the inverse to each other:

**Lemma 4.4.** Let $j \in I^+(P)$. Then, we have $j \in I^+(M_j^+ P)$ and $M_j^+ M_j^P = P$.

The moves $M_i^\pm$ and $M_j^\pm$ are commutative as far as they are defined:

**Lemma 4.5.** Suppose that $i \neq j + 1$. If $M_i^\pm M_j^\pm P$ is defined, then $M_j^\pm M_i^\pm P$ is also defined and $M_j^\pm M_i^\pm P = M_i^\pm M_j^\pm P$.

The moves $M_i^\pm$ and $M_j^\pm$ are commutative as far as they are defined:

**Lemma 4.6.** Suppose that $i \neq j$. If $M_i^\pm M_j^\pm P$ is defined, then $M_j^\pm M_i^\pm P$ is also defined and $M_j^\pm M_i^\pm P = M_i^\pm M_j^\pm P$.

We are particularly interested in the relation between the moves $M_{j+1}^+$ and $M_j^-$:

**Lemma 4.7.** The move $M_{j+1}^+ P$ is defined if and only if the move $M_j^- P$ is defined.

The move $M_{j+1}^+ M_j^- P$ is defined if and only if the move $M_j^- M_{j+1}^+ P$ is defined, and in such a case we have $M_{j+1}^+ M_j^- P = M_j^- M_{j+1}^+ P$.

**Corollary 4.8.** Let $P \in C_L^{(t)}$ and let $l$ be a positive integer. The move $(M_{j+1}^+)^l P$ is defined if and only if the move $(M_j^-)^l P$ is defined.

**Proof.** We use induction on $l$. The case $l = 1$ is proved in Lemma 4.7. Suppose that we have proved the statement for $l - 1$. By Lemma 4.7, $(M_j^-)^l P = M_j^- (M_j^-)^{l-1} P$ is defined if and only if $M_{j+1}^+ (M_j^-)^{l-1} P$ is defined. Again by Lemma 4.7, $M_{j+1}^+ (M_j^-)^{l-1} P$ is defined if and only if $(M_{j+1}^+)^{l-1} M_{j+1}^+ P$ is defined. By induction hypothesis, $(M_{j+1}^-)^{l-1} M_{j+1}^+ P$ is defined if and only if $(M_{j+1}^-)^{l-1} M_{j+1}^+ P$ is defined. Thus, we have proved that $(M_{j+1}^+)^l P$ is defined if and only if the path $(M_j^-)^l P$ is defined.

4.4. Rigging of particles. We define the rigging of particles $\lambda_j(P)$ ($j = 1, \ldots, m(P)$). Let $\pi_m$ be the set of partitions of length $m$. The rigging $(\lambda_1(P), \lambda_2(P), \ldots, \lambda_{m(P)}(P))$ is by definition an element of $\pi_m(P)$.

Let $P \in C_L^{(t)}$. We set formally $\lambda_{m(P)+1}(P) = 0$. Starting from $j = m(P)$ we define $\lambda_j(P)$ inductively by requiring that $\lambda_j(P) - \lambda_{j+1}(P)$ is equal to the maximal number $l$ such that the path $(M_j^-)^l P$ is defined. Thus we obtain a partition. We call $\lambda_j(P)$ the rigging of the $j$-th particle in the path $P$.

Note that by Corollary 4.8 if $j \neq m(P)$ the number $\lambda_j(P) - \lambda_{j+1}(P)$ is also equal to the maximal number $l$ such that path $(M_{j+1}^+)^l P$ is defined. Therefore, using Lemma 4.7 we have
Proposition 4.9. Let $P \in C^{(t)}_{L,r}$.

If $M_j^+ P$ is defined if and only if $\lambda_j(P) < \lambda_{j-1}(P)$ and in such case $\lambda_i(M_j^+ P) = \lambda_i(P) + \delta_{i,j}$.

If $M_j^- P$ is defined if $\lambda_j(P) > \lambda_{j-1}(P)$ and in such case $\lambda_i(M_j^- P) = \lambda_i(P) - \delta_{i,j}$.

Now we describe paths with zero rigging. Fix an integer $m \geq 1$. The following lemma follows from the definitions.

Lemma 4.10. There is a bijection from the set of rigged paths in $C^{(t)}_{L,r}$ which have $m$ particles and zero rigging $\lambda_1 = \cdots = \lambda_m = 0$ to the set of rigged paths in $C^{(t)}_{L-2m,r}$ which have no particles. The bijection maps the path

$$P_m = \begin{pmatrix} r_L & \cdots & r_{2m+1} & 1 & 2 & \cdots & 1 & 2 & 1 & 2 & 1 \\ \cdots & \sigma_{2m+1} & \sigma_{2m} & 0 & \cdots & v^{(t)}(1) & 0 & v^{(t)}(1) & 0 & 0 \end{pmatrix} \in C^{(t)}_{L,r}$$

to the path

$$P_0 = \begin{pmatrix} r_L & \cdots & r_{2m+1} & 1 \\ \cdots & \sigma_{2m+1} & \sigma_{2m} - v^{(t)}(1) \end{pmatrix} \in C^{(t)}_{L-2m,r}.$$ 

Examples. We set $v = v^{(t)}(1)$.

$L = 2, m(P) = 1$:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{M_1^+} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{M_1^+} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{M_1^+} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\cdots}.$$

$L = 3, m(P) = 1, a \geq 0$:

$$\begin{pmatrix} 2 & 1 & 2 & 1 \\ v + a & 0 & 0 \end{pmatrix} \xrightarrow{M_1^+} \begin{pmatrix} 2 & 1 & 2 & 1 \\ v + a - 1 & 1 & 0 \end{pmatrix} \xrightarrow{M_1^+} \begin{pmatrix} 2 & 1 & 2 & 1 \\ v + a - 1 & 0 & 1 \end{pmatrix} \xrightarrow{M_1^+} \begin{pmatrix} 2 & 1 & 2 & 1 \\ v - 1 & 1 & a \end{pmatrix} \xrightarrow{\cdots} \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & v & a \end{pmatrix} \xrightarrow{M_1^+} \begin{pmatrix} 2 & 3 & 2 & 1 \\ 0 & 0 & a \end{pmatrix} \xrightarrow{M_1^+} \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & v + 1 & a \end{pmatrix} \xrightarrow{M_1^+} \begin{pmatrix} 2 & 3 & 2 & 1 \\ 0 & 0 & a \end{pmatrix} \xrightarrow{M_1^+} \cdots.$$

$L = 4, m(P) = 1, a \geq 0, 2 \leq b < v$:

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ b & v + a & 0 & 0 \end{pmatrix} \xrightarrow{M_1^+} \cdots \xrightarrow{M_1^+} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ b & v & a \end{pmatrix} \xrightarrow{M_1^+} \cdots \xrightarrow{M_1^+} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ b & v - b & b & a \end{pmatrix} \xrightarrow{\cdots}.$$

$L = 4, m(P) = 1, a \geq 0, b < v$:

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ b & v + a & 0 & 0 \end{pmatrix} \xrightarrow{M_1^+} \cdots \xrightarrow{M_1^+} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ b & v & 0 & a \end{pmatrix} \xrightarrow{M_1^+} \cdots \xrightarrow{M_1^+} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ b & 0 & v & a \end{pmatrix} \xrightarrow{\cdots}.$$
L = 4, m(P) = 1, a, b ≥ 0: Set \( v' = v^{(t)}(2) \).
\[
\begin{array}{c}
\begin{pmatrix}
3 & 2 & 1 & 2 & 1 \\
b & v + a & 0 & 0
\end{pmatrix}
\xrightarrow{M^+_1} \ldots \xrightarrow{M^+_1} 
\begin{pmatrix}
3 & 2 & 1 & 2 & 1 \\
b & v & a
\end{pmatrix}
\xrightarrow{M^+_1} 
\begin{pmatrix}
3 & 2 & 3 & 2 & 1 \\
v' + b & 0 & 0 & a
\end{pmatrix}
\end{array}
\]
\[
\begin{array}{c}
\begin{pmatrix}
3 & 2 & 1 & 2 & 1 \\
b - 1 & 0 & v + 1 & a
\end{pmatrix}
\xrightarrow{M^+_1} \ldots \xrightarrow{M^+_1} 
\begin{pmatrix}
3 & 2 & 1 & 2 & 1 \\
0 & 0 & v + b & a
\end{pmatrix}
\end{array}
\]
\[
\begin{array}{c}
\begin{pmatrix}
3 & 2 & 3 & 2 & 1 \\
v' & 0 & b & a
\end{pmatrix}
\xrightarrow{M^+_1} \ldots \xrightarrow{M^+_1} 
\begin{pmatrix}
3 & 4 & 3 & 2 & 1 \\
0 & b & a
\end{pmatrix}
\end{array}
\]

\( L = 4, m(P) = 2: \)
\[
\begin{array}{c}
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
0 & v & 0 & 0
\end{pmatrix}
\xrightarrow{M^+_1} 
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
v & 0 & 0 & 0
\end{pmatrix}
\xrightarrow{M^+_1} 
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
1 & v + 1 & 0 & 0
\end{pmatrix}
\xrightarrow{M^+_1} 
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
v + 1 & 0 & 0 & 0
\end{pmatrix}
\end{array}
\]

\( \downarrow M^+_1 \)
\[
\begin{array}{c}
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
v - 1 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow{M^+_1} 
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
v & 1 & 0 & 0
\end{pmatrix}
\xrightarrow{M^+_1} 
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
v & 1 & 0 & 0
\end{pmatrix}
\end{array}
\]

\( \downarrow M^+_1 \)
\[
\begin{array}{c}
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\xrightarrow{M^+_1} 
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 \\
1 & v & 0 & 1
\end{pmatrix}
\end{array}
\]

\( \downarrow M^+_1 \)

4.5. **Bijection.** We are in a position to construct the bijection which leads to Proposition 3.6.

Let \( \bar{P} \in C^{(t-1)}_L \)

\[
\bar{P} = \begin{pmatrix}
\bar{r}_L & \bar{r}_{L-1} & \cdots & \bar{r}_0 \\
\bar{\sigma}_{L-1} & \cdots & \bar{\sigma}_0
\end{pmatrix}.
\]

Define the numbers \( \sigma_0, \ldots, \sigma_{L-1} \) and the path \( P_{(0)} \in C^{(t)}_L \) by the formulas

\[
\begin{align*}
\sigma_0 &= \bar{\sigma}_0, \\
\sigma_i &= \bar{\sigma}_i + 1/2 + w^{(t-1)}(\bar{r}_{i+1}, \bar{r}_i, \bar{r}_{i-1}) - w^{(t)}(\bar{r}_{i+1}, \bar{r}_i, \bar{r}_{i-1}), & (1 \leq i \leq L - 1), \\
P_{(0)} &= \begin{pmatrix}
\bar{r}_L & \bar{r}_{L-1} & \cdots & \bar{r}_0 \\
\sigma_{L-1} & \cdots & \sigma_0
\end{pmatrix}.
\end{align*}
\]

**Lemma 4.11.** We have an inclusion

\[
\iota_0 : C^{(t-1)}_{L,r} \rightarrow C^{(t)}_{L,r},
\]

\[
P \mapsto P_{(0)}.
\]

The image of \( \iota_0 \) coincides with the subset of paths in \( C^{(t)}_{L,r} \) with no particles.
Proof. For $i > 1$ by a simple computation we have 
\[ \sigma_i = \bar{\sigma}_i \text{ if } r_{i-1} - r_{i+1} = \pm 2, \]
\[ \sigma_i = \bar{\sigma}_i + 1 \text{ if } r_{i-1} - r_{i+1} = 0 \text{ and } r_{i-1} \neq 1, p - 1, \]
\[ \sigma_i = \bar{\sigma}_i + 2 \text{ if } r_{i-1} - r_{i+1} = 1 \text{ or } r_{i-1} = r_{i+1} = p - 1. \]
The statement of the lemma follows straightforwardly. \( \square \)

Let $\bar{P} \in \mathcal{C}_{L-2m,r}^{(t-1)}$, and let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \pi_m$ be a partition. The path $P_{(0)} = \iota_0(\bar{P}) \in \mathcal{C}_{L-2m,r}^{(t)}$ is constructed in Lemma 4.11. The path $P_{(m)} \in \mathcal{C}_{L,r}^{(t)}$ is constructed from $P_{(0)}$ in Lemma 4.10.

Lemma 4.12. Notation being as above, for any non-negative integer $m$ we have an inclusion
\[ \iota_m : \mathcal{C}_{L-2m,r}^{(t-1)} \times \pi_m \to \mathcal{C}_{L,r}^{(t)}, \]
\[ (\bar{P}, \lambda) \mapsto (M_1^+)^\lambda_1 \cdots (M_m^+)^\lambda_m \iota_m P_{(m)}. \]
The image of $\iota_m$ coincides with the subset of paths in $\mathcal{C}_{L,r}^{(t)}$ with $m$ particles. We have
\[ (4.26) \quad d(\iota_m(\bar{P}, \lambda)) = d(\bar{P}) + \sum_{i=1}^m \lambda_i + L^2/4 + L/2. \]

Proof. The degree of a path is defined in (2.16). The relation (4.26) follows by a straightforward computation. The rest follows from Lemma 4.9. \( \square \)

We obtain the main result of this section.

Theorem 4.13. The map $\iota := \bigsqcup_{m=0}^{\infty} \iota_m$ defines a bijection
\[ \bigsqcup_{m=0}^{\infty} (\mathcal{C}_{L-2m,r}^{(t-1)} \times \pi_m) \to \mathcal{C}_{L,r}^{(t)}, \]
with the property (4.26).

Proposition 3.3 is a direct consequence of Theorem 4.13.

5. The case $p = 3$

The aim of this section is to deduce Conjecture 2.3 for $p = 3$ from the work [FJM].

Throughout this section, we fix an integer $p' > 3$ coprime to 3. Omitting the upper index we write the $(2, 1)$-field $\phi^{(r', r)}(z)$ as $\phi(z)$, since for $p = 3$ the choice of $r' = r \pm 1$ is uniquely determined from $r$. We set
\[ M^{(3,p')} = M_{1,1}^{(3,p')} \oplus M_{2,1}^{(3,p')} . \]
In the following we deal with bi-graded $\mathbb{C}$-vector spaces of the form $X = \oplus_d X_{d,L}$ with $\dim X_{d,L} < \infty$. We will refer to the index $d$ and $L$ as degree and weight, respectively. We set $X_L = \oplus_d X_{d,L}$ and define the restricted dual space by $X_L^* = \oplus_d \text{Hom}_\mathbb{C}(X_{d,L}, \mathbb{C})$. 
5.1. Extended modules and monomial basis. First we review the results of [FJM] which are relevant to us.

Consider the Heisenberg algebra with generators \( \{ h_n \}_{n \in \mathbb{Z}} \) satisfying \([h_m, h_n] = m \delta_{m+n,0}\). Denote by

\[
\mathcal{F}_\gamma = \mathbb{C}[h_{-1}, h_{-2}, \cdots] |\gamma\rangle
\]

the Fock space with highest weight vector \(|\gamma\rangle \) (\(\gamma \in \mathbb{C}\)), where \(h_n |\gamma\rangle = 0 \quad (n > 0)\) and \(h_0 |\gamma\rangle = \gamma |\gamma\rangle\).

We set

\[
\beta = \sqrt{p' - 2}\frac{2}{2}.
\]

We use the vertex operator which acts on

\[
\Phi_\beta(z) := \exp \left(-\beta \sum_{n \neq 0} \frac{h_n}{n} z^{-n}\right) e^{\beta Q} z^\beta \Phi_\beta(z).
\]

Here \(: \cdots :\) stands for the normal ordering symbol, and \(e^{\beta Q} : \mathcal{F}_\gamma \sim \mathcal{F}_{\gamma + \beta}\) is an isomorphism of vector spaces such that \([h_n, e^{\beta Q}] = 0 \quad (n \neq 0)\), \(e^{\beta Q} |\gamma\rangle = |\gamma + \beta\rangle\). Set

\[
V^{(3,p')} = \oplus_{L \in \mathbb{Z}} V^{(3,p')}_{L} \text{ where } V^{(3,p')}_{L} = \begin{cases} M_{1,1}^{(3,p')} \otimes \mathcal{F}_{L \beta} & \text{if } L \text{ is even;} \\ M_{2,1}^{(3,p')} \otimes \mathcal{F}_{L \beta} & \text{if } L \text{ is odd}; \end{cases}
\]

\(|\text{vac}\rangle = |1,1, \cdots, 0\rangle \in V^{(3,p')}_{0}\).

Introduce a field with coefficients in \(\text{End}(V^{(3,p')}_{L})\),

\[
a(z) = \phi(z) \otimes \Phi_\beta(z).
\]

The parameter \(\beta\) is so chosen that \(a(z)\) has the expansion

\[
a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},
\]

and that the coefficients \(a_n\)'s are mutually commutative,

\[
[a_m, a_n] = 0 \quad (m, n \in \mathbb{Z}).
\]

We have also \(a_n |\text{vac}\rangle = 0\) for \(n > 0\). Later we will use the relation

\[
a(z)^2 = k \cdot \text{id} \otimes : \Phi_\beta(z)^2 : ;
\]

where \(k\) is a nonzero constant. We have

\[
[h_n, a(z)] = \beta z^n a(z).
\]

Let \(A\) be a subalgebra of \(\text{End}(V^{(3,p')})\) generated by \(a_n\)'s. Our main concern is the following subspace of \(V^{(3,p')}\) generated from \(|\text{vac}\rangle\) by acting with \(a_n\)'s:

\[
W^{(3,p')} = A \cdot |\text{vac}\rangle \subset V^{(3,p')}.
\]

We introduce a bi-grading to \(W^{(3,p')}\) by assigning the degree and weight as \(\text{deg } a_n = -n, \text{ wt } a_n = 1\) and \(\text{deg } |\text{vac}\rangle = 0, \text{ wt } |\text{vac}\rangle = 0\).
The restricted dual space \( W^{(3,p')} \) is identified with a subspace of \( \Lambda_L \) by
\[
W^{(3,p')} \rightarrow \Lambda_L, \quad \langle v \rangle \mapsto \langle v|a(x_L) \cdots a(x_1)|\text{vac}\rangle.
\]
If \( n > 0 \), \( h_n|\text{vac}\rangle = 0 \) and from Proposition 5.1, we see that \( W^{(3,p')} \) is invariant by \( h_n \). The right action of \( h_n \) (\( n > 0 \)) on \( W^{(3,p')} \) corresponds to the multiplication by \( \beta \sum_{j=1}^L x_j^n \) on \( \Lambda_L \). Therefore, the image \( I^{(3,p')} \) of (5.4) is an ideal of \( \Lambda_L \).

The linear isomorphism \( \tau = \text{id} \otimes e^{2\beta Q} : V^{(3,p')} \rightarrow V^{(3,p')} \) has an effect of shifting the indices
\[
\tau|\text{vac}\rangle_{2l} = |\text{vac}\rangle_{2l+2}, \quad \tau a_n \tau^{-1} = a_{n-p'+2},
\]
where \( |\text{vac}\rangle_{2l} = |1,1\rangle \otimes |2l\rangle \).

We have an increasing filtration (FJM, Proposition 2.3)
\[
W^{(3,p')} \subset \tau^{-1}(W^{(3,p')}) \subset \tau^{-2}(W^{(3,p')}) \subset \cdots,
\]
\[
V^{(3,p')} = \bigcup_{N \geq 0} \tau^{-N}(W^{(3,p')}).
\]

Let \( \hat{\Lambda}_L = \mathbb{C}[x_1^\pm, \ldots, x_L^\pm] \) be the space of symmetric Laurent polynomials, and let \( \hat{I}_L^{(3,p')} \) be the ideal of \( \hat{\Lambda}_L \) generated by \( I^{(3,p')} \). By Propositions 5.1, 5.5 and 5.7, we see that the subspace of \( \hat{\Lambda}_L \) spanned by all matrix elements
\[
\langle v|a(x_L) \cdots a(x_1)|u\rangle \quad (\langle v\rangle \in (V^{(3,p')})^*, |u\rangle \in V^{(3,p')})
\]
coincides with \( \hat{I}_L^{(3,p')} \). Note also that \( f \in \hat{I}_L^{(3,p')} \) if and only if \( x_1 \cdots x_L f \in \hat{I}_L^{(3,p')} \).

The following fact is proved in (FJM), Theorem 2.8.

**Proposition 5.1.** For each \( L \geq 0 \), the space \( W^{(3,p')} \) has a basis consisting of elements
\[
a_{-\lambda_L} \cdots a_{-\lambda_1}|\text{vac}\rangle,
\]
where \( \lambda = (\lambda_L, \ldots, \lambda_1) \) runs over all \( L \)-tuples of positive integers satisfying
\[
\lambda_L \geq \cdots \geq \lambda_1, \quad \lambda_{i+2} - \lambda_i \geq p'-2 \quad (1 \leq i \leq L-2).
\]

Let us reformulate Proposition 5.1 in a form suitable for later use.

**Proposition 5.2.** For any \( \lambda = (\lambda_L, \ldots, \lambda_1) \in \mathbb{Z}^L \) with \( \lambda_L \geq \cdots \geq \lambda_1 \), there exist unique \( c_{\lambda\mu} \in \mathbb{C} \) such that the following identity holds as operators on \( V^{(3,p')} \):
\[
a_{-\lambda_L} \cdots a_{-\lambda_1} = \sum_{\mu} c_{\lambda\mu} a_{-\mu_L} \cdots a_{-\mu_1},
\]
where the sum in the right hand side is taken over all \( \mu = (\mu_L, \ldots, \mu_1) \in \mathbb{Z}^L \) satisfying (5.5), (5.10) and \( \sum_{i=1}^L \mu_i = \sum_{i=1}^L \lambda_i \). The coefficients \( c_{\lambda\mu} \) are shift invariant:
\[
c_{\lambda+(1,\ldots,1),\mu+(1,\ldots,1)} = c_{\lambda\mu}.
\]
Proposition 5.4. Let $x$ be the diagonal and $\lambda$ holds for all $\lambda$ satisfying (5.9), (5.10) and $\mu$ and $\hat{\lambda}$ is clear. Setting $c_{\lambda\mu} = \lim_{N \to \infty} c_{\lambda\mu}^{(N)}$ and noting (5.7), we obtain (5.11). The uniqueness of $c_{\lambda\mu}$ is clear.

The last statement follows from the commutation relation (5.11). □

Corollary 5.3. A symmetric Laurent polynomial $f = \sum_{\lambda_1, \ldots, \lambda_l \in \mathbb{Z}} f_{\lambda_1, \ldots, \lambda_l} x_{\lambda_1}^1 \cdots x_{\lambda_l}^l$ belongs to $\tilde{I}_L^{(3,p')} \cap \hat{D}_L$ if and only if

$$f_{\lambda_1, \ldots, \lambda_l} = \sum_{\mu} c_{\lambda\mu} f_{\mu_1, \ldots, \mu_l}$$

holds for all $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}^L$ with $\lambda_1 \geq \cdots \geq \lambda_l$. In particular, if $f_{\lambda_1, \ldots, \lambda_l} = 0$ holds for all $\lambda$ satisfying (5.9), (5.10), then $f = 0$.

We quote one more fact which follows immediately from [FJM], Propositions 4.2 and 4.4.

Proposition 5.4. Let $\hat{D}_L$ be the subspace of $\hat{\Lambda}_L$ consisting of Laurent polynomials vanishing on the diagonal $x_i - x_j = 0$ ($i \neq j$). Then

$$\tilde{I}_L^{(3,p')} \cap \hat{D}_L = \tilde{I}_L^{(3,p'-3)} \times \prod_{1 \leq i < j \leq L} (x_i - x_j)^2.$$

When $p' = 4, 5$, we define $\tilde{I}_L^{(3,p'-3)}$ in the right hand side to be $\hat{\Lambda}_L$. 

Proof. The sum in the right hand side of (5.11) is a well defined element of $\text{End}(V^{(3,p')})$ because for a given vector in $V^{(3,p')}$ only finitely many monomials $a_{-\mu_L} \cdots a_{-\mu_1}$ acts non-trivially. For $N \in \mathbb{Z}$ and $\mu = (\mu_L, \cdots, \mu_1) \in \mathbb{Z}^L$, let us write $\mu^{(N)} = (\mu_L^{(N)}, \cdots, \mu_1^{(N)})$, $\mu_i^{(N)} = \mu_i + N(p' - 2)$. Choosing $N$ so that $\lambda_i^{(N)} > 0$, we apply Proposition 5.1 to $\lambda^{(N)}$. There exist unique constants $c_{\lambda\mu}^{(N)} \in \mathbb{C}$ such that

$$\left( a_{-\lambda_L^{(N)}} \cdots a_{-\lambda_1^{(N)}} - \sum_{\mu} c_{\lambda\mu}^{(N)} a_{-\mu_L^{(N)}} \cdots a_{-\mu_1^{(N)}} \right) |_{\text{vac}} = 0,$$

where the sum is taken over $\mu$ satisfying (5.9), (5.10) and $\mu_1 > -N(p' - 2)$. Since $a_\nu$'s commute and $W^{(3,p')}$ is cyclic over $\mathbb{C}[a_{-1}, a_{-2}, \cdots]$, the operator inside the parentheses vanishes on $W^{(3,p')}$. Applying $\tau^{-N}$ on both sides, we deduce that

$$a_{-\lambda_L} \cdots a_{-\lambda_1} = \sum_{\mu} c_{\lambda\mu}^{(N)} a_{-\mu_L} \cdots a_{-\mu_1} \quad \text{on } \tau^{-N}(W^{(3,p')}).$$

If $N' > N$, then $\tau^{-N}(W^{(3,p')}) \subset \tau^{-N'}(W^{(3,p')})$. Therefore we have

$$\sum_{\mu} (c_{\lambda\mu}^{(N)} - c_{\lambda\mu}^{(N')}) a_{-\mu_L} \cdots a_{-\mu_1} = 0 \quad \text{on } \tau^{-N}(W^{(3,p')}),$$

or equivalently

$$\sum_{\mu} (c_{\lambda\mu}^{(N)} - c_{\lambda\mu}^{(N')}) a_{-\mu_L} - N(p' - 2) \cdots a_{-\mu_1} - N(p' - 2) = 0 \quad \text{on } W^{(3,p')}.$$
5.2. Monomial basis for $M^{(3,p')}$. For $p = 3$, the admissibility of the rigged paths is equivalent to the following conditions for the $n_i$'s (see Section 2.3).

\[(5.12)\quad n_1 \in \mathbb{Z}_{\geq 0} + \Delta_{2,1},\]
\[(5.13)\quad n_{i+1} - n_i \geq -\frac{p'}{2} + 3 \quad (1 \leq i \leq L - 1),\]
\[(5.14)\quad n_{i+2} - n_i \geq 1 \quad (1 \leq i \leq L - 2).\]

Our goal in this section is to prove the following result.

**Theorem 5.5.** The set of monomials

$$\phi_{-n_L} \cdots \phi_{-n_1}|1,1\rangle$$

satisfying the conditions (5.12), (5.13) and (5.14) is a basis of $M^{(3,p')}$. We have shown in Theorem 2.2 that the character of this set matches with that of $M^{(3,p')}$. For the proof of Theorem 5.5, it is therefore sufficient to show that this set spans $M^{(3,p')}$. For that purpose, we consider a filtration of $M^{(3,p')}$

$$\{0\} = F_{-1} \subset F_0 \subset \cdots \subset F_i \subset \cdots \subset M^{(3,p')},$$

defined by

\[(5.15)\quad F_0 = \mathbb{C}|1,1\rangle,\]
\[(5.16)\quad F_i = F_{i-1} + \sum_{n \in \mathbb{Z}+(-1)^{i-1}\Delta_{2,1}} \phi_{-n} F_{i-1},\]

where $\Delta_{2,1} = (p' - 2)/4$. In the right hand side of (5.16), we have

$$\phi_{-n} = \begin{cases} \phi_{(2,1)}^{(2,1)} & \text{if } i \text{ is odd;} \\ \phi_{(1,2)}^{(1,2)} & \text{if } i \text{ is even.} \end{cases}$$

If $r \neq r''$ the action of $\phi_{-n}^{(r',r)}$ is zero on $M^{(3,p')}_{r',1}$ by definition. Since the operator product expansion of $\phi(z)$ contains the energy-momentum tensor $T(z)$, we have $\cup_{i \geq 0} F_i = M^{(3,p')}$. Let

$$\tilde{\phi}_n : F_i/F_{i-1} \longrightarrow F_{i+1}/F_i$$

denote the operator induced from $\phi_n$ on the associated graded space $\tilde{M}^{(3,p')} = \oplus_{i \geq 0} F_i/F_{i-1}$. From the remark made above, the proof of Theorem 5.5 reduces to the following statement.

**Lemma 5.6.** Let $\tilde{B}_L^{(3,p')}$ denote the set of vectors

\[(5.17)\quad \tilde{\phi}_{-n_L} \cdots \tilde{\phi}_{-n_1}|1,1\rangle \quad (n_i \in \mathbb{Z} - (-1)^i \Delta_{2,1})\]

satisfying the conditions (5.12), (5.13) and (5.14). Then $\tilde{M}_L^{(3,p')} = \text{span}_\mathbb{C} \tilde{B}_L^{(3,p')}$. The rest of this subsection is devoted to the proof of Lemma 5.6.

We set

$$\tilde{\phi}(z) = \sum_{n \in \mathbb{Z}+(-1)^i\Delta_{2,1}} \tilde{\phi}_{-n} z^{n-\Delta_{2,1}} \quad \text{on } F_i/F_{i-1}.$$
The field \( \tilde{\phi}(z) \) satisfies a quadratic relation following from that of \( \phi(z) \) (see e.g. [FJMMNT]),

\[
\sum_{k \geq 0} c_k \tilde{\phi}_{-n-2-k} \tilde{\phi}_{-n+k} + \sum_{k \geq 0} c_k \tilde{\phi}_{-n+2+k} = 0,
\]

where \( \sum_{k \geq 0} c_k z^k = (1 - z)^{p'/2 - 2} \). This can be rewritten as a relation of the form

\[
(5.18) \quad \tilde{\phi}_{-n_2} \tilde{\phi}_{-n_1} = \sum_{l \geq 1} C_{n_1,n_2,l} \tilde{\phi}_{-n_2-l} \tilde{\phi}_{-n_1+l} \quad (n_2 - n_1 \leq -p'/2 + 2)
\]

with some \( C_{n_1,n_2,l} \in \mathbb{C} \). Given any monomial \((5.17)\), we apply \((5.18)\) to rewrite it as a linear combination of monomials satisfying the condition \((5.13)\). In each step of rewriting, \( \sum_{i=1}^{L} n_i \) is invariant and \( \sum_{i=1}^{L} in_i \) strictly increases. Since \( \sum_{j=1}^{i} n_j > 0 \) for each \( i \), \( \sum_{i=1}^{L} in_i \) is bounded from above. Therefore, the process terminates after a finite number of steps. To prove Lemma 5.6, we must also fulfill the second condition \((5.14)\). To this end, we make use of the knowledge in the previous subsection.

Consider the filtration of \( V^{(3,p')} \) induced from that of \( M^{(3,p')} \).

\[
V^{(3,p')} = \oplus_{L \in \mathbb{Z}} (V^{(3,p')}_L)_{i},
\]

\[
(V^{(3,p')}_L)_{i} = \begin{cases} 
(F_i \cap M^{(3,p')}_{1,1}) \otimes \mathcal{F}_{L\beta} & \text{if } L \text{ is even;} \\
(F_i \cap M^{(3,p')}_{2,1}) \otimes \mathcal{F}_{L\beta} & \text{if } L \text{ is odd.}
\end{cases}
\]

Define

\[
\tilde{a}_n : V^{(3,p')}_{i-1} / V^{(3,p')}_{i-2} \longrightarrow V^{(3,p')}_{i+1} / V^{(3,p')}_{i}
\]

to be the operator induced from \( a_n : V^{(3,p')}_{i} \rightarrow V^{(3,p')}_{i+1} \). Setting \( \tilde{a}(z) = \sum_{n \in \mathbb{Z}} \tilde{a}_n z^{-n-1} \), we have clearly the relation

\[
\tilde{a}(z) = \tilde{\phi}(z) \otimes \Phi_{\beta}(z).
\]

Since the relations \((5.11)\) are homogeneous, they remain valid for \( \tilde{a}_n \)'s. From the relation \((5.2)\), we have also \( \tilde{a}(z)^2 = 0 \).

Now for \( \langle \tilde{v} \rangle \in \tilde{M}^{(3,p')} \) and \( |\tilde{u}\rangle \in \tilde{M}^{(3,p')} \), we consider a matrix element

\[
g(x_1, \ldots, x_L) = \langle \tilde{v} | \tilde{\phi}(x_L) \cdots \tilde{\phi}(x_1) | \tilde{u} \rangle
\]

which is a formal series in \( x_1, \ldots, x_L \).

\textbf{Lemma 5.7.} There exists an element \( f \in \tilde{L}^{(3,p'-3)}_L \) such that

\[
g(x_1, \ldots, x_L) = f(x_1, \ldots, x_L) \times \prod_{1 \leq i < j \leq L} (x_j - x_i)^{-\beta^2+2}.
\]

Here the right hand side means its expansion in the domain \(|x_L| > \cdots > |x_1|\).

\textbf{Proof.} Set \( \langle \tilde{v}_L \rangle = \langle \tilde{v} \rangle \otimes (L\beta) \), \( |\tilde{u}_0\rangle = |\tilde{u}\rangle \otimes |0\rangle \), and consider

\[
F(x_1, \ldots, x_L) = \langle \tilde{v}_L | \tilde{a}(x_L) \cdots \tilde{a}(x_1) | \tilde{u}_0 \rangle = g(x_1, \ldots, x_L) \times \prod_{1 \leq i < j \leq L} (x_j - x_i)^{\beta^2}.
\]
We have $F \in \tilde{I}_L(3,p')$ because the defining relations (5.11) for the operator algebra dual to the ideal $F \in I_L(3,p')$ are equally valid for $\tilde{a}(z)$. Moreover the relation $\tilde{a}(z)^2 = 0$ implies $F \in \tilde{D}_L$. Hence from Proposition 5.4 we can write

$$F(x_1, \cdots, x_L) = f(x_1, \cdots, x_L) \prod_{1 \leq i < j \leq L} (x_j - x_i)^2$$

with some $f \in \tilde{I}_L(3,p'-3)$. This proves the assertion. \(\square\)

**Lemma 5.8.** Let $|\tilde{u}\rangle \in \tilde{M}(3,p')$ be a homogeneous element and let $m = (m_3, m_2, m_1)$ be a triple of integers such that the relations (5.13), (5.14) for $L = 3$ are not satisfied. Then we have a relation

$$(5.19) \quad \phi_{-m_3} \phi_{-m_2} \phi_{-m_1} |\tilde{u}\rangle = \sum_{n} d_{m,n} \phi_{-n_3} \phi_{-n_2} \phi_{-n_1} |\tilde{u}\rangle$$

with some $d_{m,n} \in \mathbb{C}$, where the sum is taken over $n = (n_3, n_2, n_1)$ satisfying the relations (5.13), (5.14) and $\sum_{i=1}^{3} n_i = \sum_{i=1}^{3} m_i$, $\sum_{i=1}^{3} i n_i > \sum_{i=1}^{3} i m_i$.

**Proof.** Given $|\tilde{u}\rangle$, let $X$ be the linear span of all monomials $\phi_{-m_3} \phi_{-m_2} \phi_{-m_1} |\tilde{u}\rangle$, and let $X_0$ be the subspace spanned by those whose indices $m$ satisfy (5.13) and (5.14). We show $X = X_0$.

Let $\langle \tilde{v} \mid \in \tilde{M}(3,p')^{*}$ be an element orthogonal to the space $X_0$. Consider the matrix element

$$g(x_1, x_2, x_3) = \langle \tilde{v} | \phi(x_3) \phi(x_2) \phi(x_1) |\tilde{u}\rangle.$$  

Let $f$ be as in Lemma 5.7 with $L = 3$, and expand them as

$$f(x_1, x_2, x_3) = \sum_{\lambda_3, \lambda_2, \lambda_1 \geq \lambda_1} f_{\lambda_3, \lambda_2, \lambda_1} \phi_{\lambda_3} \phi_{\lambda_2} \phi_{\lambda_1},$$

$$g(x_1, x_2, x_3) = \sum_{n_1, n_2, n_3} g_{n_3, n_2, n_1} x_3^{n_3-\Delta_{2,1}} x_2^{n_2-\Delta_{2,1}} x_1^{n_1-\Delta_{2,1}},$$

where $g_{n_3, n_2, n_1} = \langle \tilde{v} | \phi_{-n_3} \phi_{-n_2} \phi_{-n_1} |\tilde{u}\rangle$.

Suppose $\lambda_3 \geq \lambda_2 \geq \lambda_1$ and $\lambda_3 - \lambda_1 \geq p' - 5$. The coefficient $f_{\lambda_3, \lambda_2, \lambda_1}$ is a linear combination of $g_{n_3, n_2, n_1}$ such that

$$\lambda_i = n_i - \Delta_{2,1} + (i - 1)(\beta^2 - 2) - \alpha_{i-1} + \alpha_i,$$

with some $\alpha_1, \alpha_2 \geq 0$ and $\alpha_0 = \alpha_3 = 0$. In particular

$$n_3 - n_1 = \lambda_3 - \lambda_1 - (p' - 6) + \alpha_1 + \alpha_2 \geq 1.$$  

Using the quadratic relation (5.13), we can further rewrite $g_{n_3, n_2, n_1}$ as linear combinations of those satisfying (5.13). Since $n_3 - n_1 = \sum_{i=1}^{3} i n_i - 2 \sum_{i=1}^{3} i n_i$ does not decrease, the resulting terms all satisfy (5.14) as well. By the choice of $\langle \tilde{v} \mid$, $g_{n_3, n_2, n_1} = 0$ holds for all such $n = (n_3, n_2, n_1)$. It follows that $f_{\lambda_3, \lambda_2, \lambda_1} = 0$ holds for all $\lambda = (\lambda_3, \lambda_2, \lambda_1)$ with $\lambda_3 \geq \lambda_2 \geq \lambda_1$ and $\lambda_3 - \lambda_1 \geq p' - 5$. Applying Corollary 5.4 we conclude that $f = g = 0$. This shows $X = X_0$.

It remains to verify that if $m$ violates (5.13) or (5.14), then $m_3 - m_1 < n_3 - n_1$ for all terms in the right hand side of (5.19). This is evident if (5.14) is violated. Otherwise we can use the quadratic relation (5.18) to rewrite it as a linear combination of those satisfying (5.13). In the process $m_3 - m_1$ strictly increases, and the verification reduces to the first case. \(\square\)
Proof of Lemma 5.6. Given a monomial (5.17), suppose the conditions (5.13)–(5.14) are not valid for a triple \((m_{i+1}, m_i, m_{i-1})\). We apply Lemma 5.8 to reduce it. In the process, \(\sum_{i=1}^{L} n_i\) does not change and \(\sum_{i=1}^{L} in_i\) strictly increases. Therefore the process terminates after a finite number of steps. Proof of Lemma 5.6 is now complete. \(\square\)

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