REGULAR POINTS IN AFFINE SPRINGER FIBERS

MARK GORESKY, ROBERT KOTTWITZ, AND ROBERT MACPHERSON

1. Introduction

Let $G$ be a connected reductive group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. We put $F = \mathbb{C}(\!(\epsilon)\!)$ and $\mathcal{O} = \mathbb{C}[\![\epsilon]\!]$. Let $X = X_G$ denote the affine Grassmannian $G(F)/G(\mathcal{O})$. For $u \in \mathfrak{g}(F)$ we write $X^u$ for the affine Springer fiber

$$X^u = \{ g \in G(F)/G(\mathcal{O}) : \text{Ad}(g^{-1})(u) \in \mathfrak{g}(\mathcal{O}) \}. $$

studied by Kazhdan and Lusztig in [KL88].

For $x = gG(\mathcal{O}) \in X^u$ the $G(\mathcal{O})$-orbit (for the adjoint action) of $\text{Ad}(g^{-1})(u)$ in $\mathfrak{g}(\mathcal{O})$ depends only on $x$, and its image under $\mathfrak{g}(\mathcal{O}) \rightarrow \mathfrak{g}(\mathbb{C})$ is a well-defined $G(\mathbb{C})$-orbit in $\mathfrak{g}(\mathbb{C})$. We say that $x \in X^u$ is regular if the associated orbit is regular in $\mathfrak{g}(\mathbb{C})$. (Recall that an element of $\mathfrak{g}(\mathbb{C})$ is regular if the nilpotent part of its Jordan decomposition is a principal nilpotent element in the centralizer of the semisimple part of its Jordan decomposition.) We write $X^u_{\text{reg}}$ for the (Zariski open) subset of regular elements in $X^u$.

From now on we assume that $u$ is regular semisimple with centralizer $T$, a maximal torus in $G$ over $F$. Assume further that $u$ is integral, by which we mean that $X^u$ is non-empty. Kazhdan and Lusztig [KL88] show that $X^u$ is then a locally finite union of projective algebraic varieties, and in Cor. 1 of §4 of [KL88] they show that the open subset $X^u_{\text{reg}}$ of $X^u$ is non-empty (and hence dense in at least one irreducible component of $X^u$). The action of $T(F)$ on $X$ clearly preserves the subsets $X^u$ and $X^u_{\text{reg}}$. Bezrukavnikov [Bez96] proved that $X^u_{\text{reg}}$ forms a single orbit under $T(F)$. (Actually Kazhdan-Lusztig and Bezrukavnikov consider only topologically nilpotent elements $u$, but the general case can be reduced to their special case by using the topological Jordan decomposition of $u$.)

The goal of this paper is to characterize regular elements in $X^u$ (for integral regular semisimple $u$ as above). When $T$ is elliptic (in other words, $F$-anisotropic modulo the center of $G$) the characterization gives no new information. At the other extreme, in the split case, the characterization gives a clear picture of what it means for a point in $X^u$ to be regular.

We will now state our characterization in the split case, leaving the more technical general statement to the next section (see Theorem 1). Fix a split maximal torus $A \subset G$ over $\mathbb{C}$ and denote by $\mathfrak{a}$ its Lie algebra. We identify the affine
Grassmannian $A(F)/A(O)$ for $A$ with the cocharacter lattice $X_*(A)$, the cocharacter $\mu$ corresponding to the class of $\mu(\epsilon)$ in $A(F)/A(O)$. For any Borel subgroup $B = AN$ containing $A$ ($N$ denoting the unipotent radical of $B$) there is a well-known retraction $r_B : X \to X_*(A)$ defined using the Iwasawa decomposition $G(F) = N(F)A(F)G(O)$: the fiber of $r_B$ over $\mu \in X_*(A)$ is $N(F)\mu(\epsilon)G(O)/G(O)$. The family of cocharacters $r_B(x)$ ($B$ ranging through all Borel subgroups containing $A$) has been studied by Arthur [Art76, Lemma 3.6]; it is the volume of the convex hull of these points that arises as the weight factor for (fully) weighted orbital integrals for elements in $A(F)$. In particular Arthur shows that for $x \in X$ and any pair $B, B'$ of adjacent Borel subgroups containing $A$, there is a unique non-negative integer $n(x, B, B')$ such that

\[(1.0.1) \quad r_B(x) - r_{B'}(x) = n(x, B, B') \cdot \alpha_{B,B'}^\vee,\]

where $\alpha_{B,B'}$ is the unique root of $A$ that is positive for $B$ and negative for $B'$.

The main result of this paper (in the split case) is that for $x \in X^u$

\[(1.0.2) \quad n(x, B, B') \leq \text{val} \alpha_{B,B'}(u)\]

for every pair $B, B'$ of adjacent Borel subgroups containing $A$, and that $x \in X^u$ is regular if and only if all the inequalities (1.0.2) are actually equalities.

2. Statements

2.1. Notation. We write $g$ for the Lie algebra of $G$ and follow the same convention for groups denoted by other letters.

Choose an algebraic closure $\overline{F}$ of $F$ and let $\Gamma = \text{Gal}(\overline{F}/F)$. We write $G_F$ for the $F$-group obtained from $G$ by extension of scalars from $C$ to $F$.

As before we use $\mu \mapsto \mu(\epsilon)$ to identify the cocharacter group $X_*(A)$ with $A(F)/A(O)$. By means of this identification the canonical surjection $A(F) \to A(F)/A(O)$ can be viewed as a surjection

\[(2.1.1) \quad A(F) \to X_*(A).\]

Let $\Lambda = \Lambda_G$ denote the quotient of the coweight lattice $X_*(A)$ by the coroot lattice (the subgroup of $X_*(A)$ generated by the coroots of $A$ in $G$). Up to canonical isomorphism $\Lambda$ is independent of the choice of $A$; moreover when defining $\Lambda$ we could replace $A$ by any maximal torus $T$ in $G_F$. There is a canonical surjective homomorphism

\[(2.1.2) \quad G(F) \to \Lambda,\]

characterized by the following two properties: it is trivial on the image of $G_{sc}(F)$ in $G(F)$ (where $G_{sc}$ denotes the simply connected cover of the derived group of $G$), and its restriction to $A(F)$ coincides with the composition of \((2.1.1)\) and the canonical surjection $X_*(A) \to \Lambda$.

Recall that $X$ denotes the affine Grassmannian $G(F)/G(O)$ for $G$. The homomorphism \((2.1.2)\) is trivial on $G(O)$ and hence induces a canonical surjection

\[(2.1.3) \quad \nu_G : X \to \Lambda,\]

whose fibers are the connected components of $X$. 

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2.2. **Parabolic subgroups.** We will concerned with parabolic subgroups \( P \) of \( G \) containing \( A \). Such a parabolic subgroup has a unique Levi subgroup \( M \) containing \( A \), and we refer to \( M \) as the Levi component of \( P \).

As usual, by a Levi subgroup of \( G \), we mean a Levi subgroup of some parabolic subgroup of \( G \). Let \( M \) be a Levi subgroup of \( G \) containing \( A \). We write \( \mathcal{P}(M) \) for the set of parabolic subgroups of \( G \) that contain \( A \) and have Levi component \( M \). Thus any \( P \in \mathcal{P}(M) \) can be written as \( P = MN \) where \( N = N_P \) denotes the unipotent radical of \( P \). As usual there is a notion of adjacency: two parabolic subgroups \( P = MN \) and \( P' = MN' \) in \( \mathcal{P}(M) \) are said to be adjacent if there exists (a unique) parabolic subgroup \( Q = LU \) containing both \( P \) and \( P' \) such that the semisimple rank of \( L \) is one greater than the semisimple rank of \( M \). Thus \( U = N \cap N' \), and, moreover, if \( L \) is chosen so that \( L \supset A \), then

\[
I = m \oplus (n \cap \tilde{n}') \oplus (n' \cap \tilde{n})
\]

where \( \tilde{N} \) denotes the unipotent radical of the parabolic subgroup \( \tilde{P} = MN \) opposite to \( P \) (and where \( \tilde{N}' \) is opposite to \( N' \)).

Given adjacent \( P, P' \) in \( \mathcal{P}(M) \) we define an element \( \beta_{P,P'} \in \Lambda_M \) (the coweight lattice for \( A \) modulo the coroot lattice for \( M \)) as follows. Consider the collection of elements in \( \Lambda_M \) obtained from coroots \( \alpha' \) where \( \alpha \) ranges through the set of roots of \( A \) in \( n \cap n' \). We define \( \beta_{P,P'} \) to be the unique element in this collection such that all other members in the collection are positive integral multiples of \( \beta_{P,P'} \). Note that although \( \Lambda_M \) may have torsion elements, the elements in our collection lie in the kernel of the canonical map from \( \Lambda_M \) to \( \Lambda_G \), and this kernel is torsion-free. Thus any member of our collection can be written uniquely as a positive integer times \( \beta_{P,P'} \). Note also that \( \beta_{P',P} = -\beta_{P,P'} \). In case \( M = A \), so that \( P, P' \) are Borel subgroups, \( \beta_{P,P'} \) is the unique coroot of \( A \) that is positive for \( P \) and negative for \( P' \).

2.3. **Retractions from \( X \) to \( X_M \).** The inclusion of \( M(F) \) into \( G(F) \) induces an inclusion of the affine Grassmannian \( X_M \) for \( M \) into the affine Grassmannian \( X \) for \( G \). Let \( P \in \mathcal{P}(M) \) and let \( X_P \) denote the set \( P(F)/P(O) \). The canonical inclusion of \( P \) in \( G \) induces a bijection \( i \) from \( X_P \) to \( X \), and the canonical surjection \( P \to M \) induces a canonical surjective map \( p \) (of sets) from \( X_P \) to \( X_M \). We define the retraction \( r_P = r_P^G : X \to X_M \) as the composed map \( p \circ i^{-1} \). Given \( x \in X \) we often denote by \( x_P \) the image of \( x \) under the retraction \( r_P \).

These retractions satisfy the following transitivity property. Suppose that \( L \supset M \) are Levi subgroups containing \( A \), and suppose further that \( P \in \mathcal{P}(M) \) and \( Q \in \mathcal{P}(L) \) satisfy \( Q \supset P \). Let \( P_L \) denote the parabolic subgroup \( P \cap L \) in \( L \). Then

\[
r_P^Q = r_{P_L}^L \circ r_P^G.
\]

Moreover, for any \( x \in X \) the element \( \nu_M(x_P) \) maps to \( \nu_L(x_Q) \) under the canonical surjection \( \Lambda_M \to \Lambda_L \), and in particular \( \nu_M(x_P) \to \nu_G(x) \) under \( \Lambda_M \to \Lambda_G \).

2.4. **Definition of \( n(x,P,P') \).** A point \( x \in X \) determines points \( \nu_M(x_P) \) in \( \Lambda_M \), one for each \( P \in \mathcal{P}(M) \). This family of points arises in the definition of the weighted orbital integrals occurring in Arthur’s work. A basic fact about this family of points is that whenever \( P, P' \) are adjacent parabolic subgroups in \( \mathcal{P}(M) \), there is a (unique) non-negative integer \( n(x,P,P') \) such that

\[
n(x,P,P') = \nu_M(x_{P'}) - \nu_M(x_P) = n(x,P,P') \cdot \beta_{P,P'}.
\]
2.5. Fixed point sets \( X^u \). Let \( u \in g(F) \). Define a subset \( X^u \) of \( X \) by

\[
X^u = \{ g \in G(F)/G(O) : \text{Ad}(g^{-1})(u) \in g(O) \}.
\]

2.6. Conjugacy classes associated to fixed points. Let \( u \in g(F) \). Suppose that the coset \( x = gG(O) \) lies in \( X^u \). The image of \( \text{Ad}(g^{-1})(u) \) under the canonical surjection \( g(O) \to g(C) \) gives a well-defined \( G(C) \)-conjugacy class \( \bar{u}_G(x) \) (for the adjoint action) in \( g(C) \).

As above let \( M \) be a Levi subgroup of \( G \) and let \( P \in \mathcal{P}(M) \). Now suppose that \( u \in m(F) \) and that \( x \in X^u \). Choose \( p \in P(F) \) such that \( x = pG(O) \); thus \( x_P \) is the coset \( mM(O) \), where \( m \) denotes the image of \( p \) under the canonical homomorphism from \( P \) onto \( M \). Of course \( \text{Ad}(p^{-1})(u) \) lies in \( p(O) \), and its image in \( p(C) \) gives a well-defined \( P(C) \)-conjugacy class \( \bar{u}_P(x) \) in \( p(C) \). It follows that \( x_P \) lies in \( X^u_M \) (as was first noted by Kazhdan-Lusztig [KLS]), and also that \( \bar{u}_P(x) \) maps to \( \bar{u}_G(x) \) (respectively, \( \bar{u}_M(x_P) \)) under the map on conjugacy classes induced by \( p(C) \to g(C) \) (respectively, \( p(C) \to m(C) \)).

2.7. Review of regular elements. An element \( u \in g(C) \) is regular if the nilpotent part of its Jordan decomposition is a principal nilpotent element in the centralizer of the semisimple part of its Jordan decomposition, or, equivalently, if the set of Borel subalgebras containing \( u \) is finite. It is well-known that the set of regular elements in \( g(C) \) is open.

As above let \( M \) be a Levi subgroup of \( G \) and let \( P \in \mathcal{P}(M) \). Suppose that \( u \) is a regular element in \( g(C) \) that happens to lie in \( p(C) \). Then the image \( u_M \) of \( u \) in \( m(C) \) is regular in \( m(C) \).

2.8. Regular points in \( X^u \). We say that \( x \in X^u \) is regular if the associated conjugacy class \( \bar{u}_G(x) \in g(C) \) consists of regular elements. We denote by \( X^u_{\text{reg}} \) the set of regular elements in \( X^u \); the subset \( X^u_{\text{reg}} \) is open in \( X^u \).

As above let \( M \) be a Levi subgroup of \( G \) and let \( P \in \mathcal{P}(M) \). Suppose that \( u \in m(F) \). We have already seen that \( r_P \) maps \( X^u \) into \( X^u_M \), and that the conjugacy class in \( g(C) \) associated to \( x \in X^u \) is compatible with the conjugacy class in \( m(C) \) associated to the retracted point \( x_P \in X^u_M \), compatible in the sense that there is a conjugacy class in \( p(C) \) that maps to both of them. Therefore \( x_P \) is regular in \( X^u_M \) if \( x \) is regular in \( X^u \).

2.9. Set-up for the main result. As before let \( M \) denote a Levi subgroup of \( G \) containing \( A \). We now assume that \( u \) is an integral regular semisimple element of \( g(F) \) that happens to lie in \( m(F) \). (It is equivalent to assume that the centralizer \( T \) of \( u \) is contained in \( M_{F'} \).) For each pair \( P = MN, P' = MN' \) of adjacent parabolic subgroups in \( \mathcal{P}(M) \) we are going to define a non-negative integer \( n(u, P, P') \). This collection of integers measures how far \( X^u \) sticks out from \( X^u_{\text{reg}} \).

As before we need the parabolic subgroups \( \tilde{P} = MN \) and \( \tilde{P}' = MN' \) opposite to \( P \) and \( P' \) respectively. Let \( \alpha \) be a root of \( T \) in \( N \cap \tilde{N}' \). Since \( T, N \) and \( N' \) are defined over \( F \), the group \( \text{Gal}(\overline{F}/F) \) preserves the set of roots of \( T \) in \( N \cap \tilde{N}' \). Let \( F_\alpha \) denote the field of definition of \( \alpha \), so that \( \text{Gal}(\overline{F}/F_\alpha) \) is the stabilizer of \( \alpha \) in \( \text{Gal}(\overline{F}/F) \). For any finite extension \( F' \) of \( F \) (e.g. \( F_\alpha \)) we normalize the valuation \( \text{val}_{F'} \) on \( F' \) so that a uniformizing element in \( F' \) has valuation 1, or, equivalently, so that \( \epsilon \) has valuation \( [F' : F] \). There exists a unique positive integer \( m_\alpha \) such that the image of the element \( \alpha^\vee \) in \( \Lambda_M \) is equal to \( m_\alpha \cdot \beta_{P,P'} \), where \( \beta_{P,P'} \) is the element of \( \Lambda_M \) defined above. Note that \( m_\alpha \) depends only on the orbit of \( \alpha \) under
the Galois group; here we use that the Galois group acts on the cocharacter group of $T$ through the Weyl group of $M$, so that any two elements in the Galois orbit of $\alpha^\vee$ have the same image in $\Lambda_M$. Finally we define $n(u, P, P')$ as the sum

$$n(u, P, P') = \sum \text{val}_F(\alpha(u)) \cdot m_\alpha,$$

where the sum is taken over a set of representatives $\alpha$ of the orbits of $\text{Gal}(\overline{F}/F)$ on the set of roots of $T$ in $N \cap \tilde{N}'$. In the special case that $M = A$ (and hence $T = A$) $n(u, P, P')$ is equal to $\text{val}_F(\alpha(u))$, where $\alpha$ is the unique root of $A$ that is positive for $P$ and negative for $P'$.

**Theorem 1.** Let $M$ and $u$ be as above, and let $x \in X^u$. Recall that $x_P \in X^u_M$ for all $P \in \mathcal{P}(M)$.

(a) For every pair $P, P' \in \mathcal{P}(M)$ of adjacent parabolic subgroups

$$n(x, P, P') \leq n(u, P, P').$$

(b) The point $x$ is regular in $X^u$ if and only if the following two conditions hold:

(i) the point $x_P$ is regular in $X^u_M$ for all $P \in \mathcal{P}(M)$, and

(ii) for every pair $P, P' \in \mathcal{P}(M)$ of adjacent parabolic subgroups

$$n(x, P, P') = n(u, P, P').$$

3. Proofs

3.1. **The case of $SL(2)$**. The key step in proving our main theorem is to verify it for $SL(2)$, where it reduces to a computation that can be found in [Lan80]. To keep things self-contained we reproduce the calculation here. Let $A, B, \tilde{B}$ denote the diagonal, upper triangular and lower triangular subgroups of $SL(2)$ respectively, and let $\alpha$ be the unique root of $A$ that is positive for $B$. Of course $\beta_{B, \tilde{B}} = \alpha^\vee$. Let

$$x \in X$$

and let $u = \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix}$ for non-zero $c \in \mathcal{O}$. Note that $n(u, B, \tilde{B}) = \text{val}_F(c)$.

We will show that $x \in X^u$ if and only if $n(x, B, \tilde{B}) \leq n(u, B, \tilde{B})$, and that $x \in X^u_{\text{reg}}$ if and only if $n(x, B, \tilde{B}) = n(u, B, \tilde{B})$.

The difference $\nu_A(x_B) - \nu_A(x_{\tilde{B}})$ and the sets $X^u$ and $X^u_{\text{reg}}$ are invariant under the action of $A(F)$ on $X$, so it is enough to consider $x$ of the form $x = gG(\mathcal{O})$ with $g = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$. (Note that for this reason our calculations apply just as well to any group whose semisimple rank is 1.) For such $x$ we have $\nu_A(x_B) = 0$. If $t \in \mathcal{O}$, then $\nu_A(x_B) = 0$. If $t \notin \mathcal{O}$, then $\begin{bmatrix} 0 & -1 \\ 1 & t^{-1} \end{bmatrix} \in G(\mathcal{O})$ and thus

$$\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} t^{-1} & 1 \\ 0 & t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & t^{-1} \end{bmatrix} \in \begin{bmatrix} t^{-1} & 1 \\ 0 & t \end{bmatrix} \cdot G(\mathcal{O}),$$

which shows that $\nu_A(x_B) = \text{val}_F(t^{-1}) \cdot \alpha^\vee$. We conclude that $n(x, B, \tilde{B})$ equals 0 if $t \in \mathcal{O}$ and equals $\text{val}_F(t^{-1})$ if $t \notin \mathcal{O}$. In any case $n(x, B, \tilde{B})$ is a non-negative integer.

For $x, u$ as above we have

$$\text{Ad}(g^{-1})u = \begin{bmatrix} c & 0 \\ -2ct & -c \end{bmatrix}.$$
Therefore \( x \in X^u \iff ct \in \mathcal{O} \iff n(x, B, \bar{B}) \leq n(u, B, \bar{B}). \) Moreover \( x \in X^u_{\text{reg}} \iff ct \in \mathcal{O}^\times \) or \((c \in \mathcal{O}^\times \text{ and } t \in \mathcal{O}) \iff n(x, B, \bar{B}) = n(u, B, \bar{B}).\)

3.2. Review of \( n(x, P, P'). \) We need to review Arthur’s proof of the existence of the non-negative integers \( n(x, P, P'). \) We begin with the case \( M = A. \) Let \( x \in X. \) We must check that for any two adjacent Borel subgroups \( P, P' \in \mathcal{P}(A) \) there is a (unique) non-negative integer \( n(x, P, P') \) such that

\[
\nu_A(x_P) - \nu_A(x_{P'}) = n(x, P, P') \cdot \alpha^\vee,
\]

where \( \alpha \) is the unique root of \( A \) that is positive for \( P \) and negative for \( P'. \) For this we consider the unique parabolic subgroup \( Q \) containing \( P \) and \( P' \) whose Levi component \( L \) has semisimple rank 1. By transitivity of retractions we have

\[
(3.2.1) \quad \nu_A(x_P) - \nu_A(x_{P'}) = \nu_A(y_B) - \nu_A(y_{\bar{B}})
\]

where \( y = x_Q \) and \( B = L \cap P, \ \bar{B} = L \cap P'. \) This reduces us to the case in which \( G \) has semisimple rank 1, which has already been done. For future use we note that (3.2.1) can be reformulated as the equality

\[
n(x, P, P') = n(y, B, \bar{B}).
\]

Again let \( x \in X. \) Now we check that for any Levi subgroup \( M \supset A \) and any adjacent parabolic subgroups \( P = MN, \ P' = MN' \) in \( \mathcal{P}(M) \) there is a (unique) non-negative integer \( n(x, P, P') \) such that

\[
\nu_M(x_P) - \nu_M(x_{P'}) = n(x, P, P') \cdot \beta_{P, P'}.
\]

Fix a Borel subgroup \( B_M \) in \( M \) and let \( B \) (respectively, \( B' \)) be the inverse image of \( B_M \) under \( \mathcal{O} \to M \) (respectively, \( \mathcal{O}' \to M \)); thus \( B, B' \) are Borel subgroups containing \( A. \)

Now choose a minimal gallery of Borel subgroups \( B = B_0, B_1, B_2, \ldots, B_l = B' \) joining \( B \) to \( B' \), and for \( i = 1, \ldots, l \) let \( \alpha_i \) be the unique root of \( A \) that is positive for \( B_{i-1} \) and negative for \( B_i. \) Then

\[
\nu_A(x_B) - \nu_A(x_{B'}) = \sum_{i=1}^{l} n(x, B_{i-1}, B_i) \cdot \alpha_i^\vee.
\]

Note that \( \{\alpha_1, \ldots, \alpha_l\} \) is precisely the set of roots of \( A \) in \( n \cap n' \) and that for each \( i \) there exists a (unique) positive integer \( m_i \) such that the image of \( \alpha_i^\vee \) in \( \Lambda_M \) is equal to \( m_i \cdot \beta_{P, P'} \). Applying the canonical surjection \( \Lambda_A \to \Lambda_M \) to the previous equation, we find that (3.2.3) gives

\[
\nu_M(x_P) - \nu_M(x_{P'}) = n(x, P, P') \cdot \beta_{P, P'}.
\]

where \( n(x, P, P') \) is the non-negative integer

\[
\sum_{i=1}^{l} m_i \cdot n(x, B_{i-1}, B_i).
\]
3.3. **Proof of part of the main theorem in case** $A = T$. Let $u \in \mathfrak{a}(O)$ and assume that $u$ is regular in $\mathfrak{g}(F)$. Let $x \in X^u$.

Let $M$ be a Levi subgroup of $G$ containing $A$. We are going to prove the first main assertion in our theorem, namely that for any pair of adjacent $P, P' \in \mathcal{P}(M)$ there is an inequality

$$n(x, P, P') \leq n(u, P, P').$$

Let $B, B', B_0, \ldots, B_l$ and $\alpha_i, m_i \ (i = 1, \ldots, l)$ be as in 3.2 Then by definition

$$n(u, P, P') = \sum_{i=1}^{l} m_i \cdot \text{val}_F(\alpha_i(u)).$$

Let $M_i$ be the Levi subgroup containing $A$ whose root system is $\{ \pm \alpha_i \}$, and let $B'_i, B''_i$ denote the Borel subgroups in $M_i$ obtained by intersecting $B_{i-1}, B_i$ with $M_i$. Let $Q_i$ be the unique parabolic subgroup in $\mathcal{P}(M_i)$ such that $Q_i$ contains $B_{i-1}$ and $B_i$. We showed in 3.2 that

$$n(x, P, P') = \sum_{i=1}^{l} m_i \cdot n(x, B_{i-1}, B_i)$$

and that

$$n(x, B_{i-1}, B_i) = n(y_i, B'_{i-1}, B''_i),$$

where $y_i = x_{Q_i} \in X^u_{M_i}$. Since $M_i$ has semisimple rank 1, we know that

$$n(y_i, B'_{i-1}, B''_i) \leq \text{val}_F(\alpha_i(u)).$$

This completes the proof of the first main assertion.

Now suppose that $x$ is regular in $X^u$. Then each point $y_i \in X^u_{M_i}$ above is regular in $X^u_{M_i}$, and therefore from the rank 1 case (see 3.1 we know that

$$n(y_i, B'_{i-1}, B''_i) = \text{val}_F(\alpha_i(u)).$$

We conclude that if $x$ is regular in $X^u$, then

$$n(x, P, P') = n(u, P, P'),$$

which is another of the assertions in our theorem.

3.4. **Proof of the rest of the main theorem in case** $M = A = T$. We continue with $u \in \mathfrak{a}(O)$ and $x \in X^u$ as before, but for the moment we only consider the case $M = A$. We assume that

$$(3.4.1) \quad n(x, P, P') = \text{val}_F(\alpha_{P, P'}(u))$$

for all adjacent Borel subgroups $P, P' \in \mathcal{P}(A)$, where $\alpha_{P, P'}$ is the unique root of $A$ that is positive for $P$ and negative for $P'$. We want to prove that $x$ is regular in $X^u$. To do so we must first select a suitable Borel subgroup $B \in \mathcal{P}(A)$.

Let $u_0 \in \mathfrak{a}(C)$ denote the image of $u$ under $\mathfrak{a}(O) \rightarrow \mathfrak{a}(C)$, and let $M$ denote the centralizer of $u_0$ in $G$. Thus $M$ is a Levi subgroup of $G$ containing $A$, and we choose $P \in \mathcal{P}(M)$. Then we obtain a suitable Borel subgroup by taking any $B \in \mathcal{P}(A)$ such that $B \subset P$. For any $B$-simple root $\alpha$ we denote by $B_\alpha$ the unique Borel subgroup in $\mathcal{P}(A)$ that is adjacent to $B$ and for which $\alpha$ is negative, and we write $P_{\alpha}$ for the unique parabolic subgroup containing $B$ and $B_\alpha$ such that the semisimple rank of the Levi component $M_{\alpha}$ of $P_{\alpha}$ is 1. Consider the element (well-defined up to $B(C)$-conjugacy) $v := \bar{u}_B(x) \in \mathfrak{b}(C)$ defined in 2.6. The equation (3.4.1) plus the semisimple rank 1 theory implies that the points $x_{P_{\alpha}} \in X^u_{M_{\alpha}}$ are
regular, and this in turn implies (see 2.6) that for every $B$-simple root $\alpha$ the image of the element $v$ under $b(C) \rightarrow p_\alpha(C) \rightarrow m_\alpha(C)$ is regular in $m_\alpha(C)$. Moreover it is evident that the image of $v$ under the canonical surjection $b(C) \rightarrow a(C)$ is equal to $u_0$. Using only these facts, we now check that $v$ is regular in $g(C)$ (and hence that $x$ is regular in $X^u$).

Let $v = v_s + v_n$ be the Jordan decomposition of $v$, with $v_s$ semisimple and $v_n$ nilpotent. Since it is harmless to replace $v$ by any $B(C)$-conjugate, we may assume without loss of generality that $v_s \in a(C)$. Then, since $v_s \mapsto u_0$ under $b(C) \rightarrow a(C)$, it follows that $v_s = u_0$. Since $v_n$ commutes with $v_s = u_0$, it lies in $m(C)$, and we must check that $v_n$ is a principal nilpotent element in $m(C)$. As $v_n$ lies in the Borel subalgebra $(b\cap m)(C)$ of $m(C)$, it is enough to check that the projection of $v_n$ into each simple root space of $(b\cap m)(C)$ is non-zero, and this follows from the statement (proved above) that the image of $v$ under $b(C) \rightarrow p_\alpha(C) \rightarrow m_\alpha(C)$ is regular in $m_\alpha(C)$ for every simple root $\alpha$ of $A$ in $M$.

3.5. End of the proof of the main theorem in case $A = T$. We continue with $u \in a(\mathcal{O})$ and $x \in X^u$ as above. Let $M$ be any Levi subgroup containing $A$. It remains to prove that if $x_P$ is regular in $X^u_M$ for all $P \in \mathcal{P}(M)$ and if

\[(3.5.1) \quad n(x, P, P') = n(u, P, P')\]

for every adjacent pair $P, P' \in \mathcal{P}(M)$, then $x$ is regular in $X^u$. We have already proved this in case $M = A$, and now we want to reduce the general case to this special case.

The equality (3.5.1) is equivalent to the equality

\[(3.5.2) \quad \nu_M(x_P) - \nu_M(x_{P'}) = n(u, P, P') \cdot \beta_{P, P'}\]

Fix $P \in \mathcal{P}(M)$ and sum (3.5.2) over the set of neighboring pairs in a minimal gallery joining $P$ to its opposite $P \in \mathcal{P}(M)$. Doing this yields the equality

\[(3.5.3) \quad \nu_M(x_P) - \nu_M(x_{P'}) = \sum_{\alpha \in R_N} \varphi_F(\alpha(u)) \cdot \pi_M(\alpha^\vee),\]

where $\pi_M : X(A) \rightarrow \Lambda_M$ is the canonical surjection and $R_N$ is the set of roots of $A$ in $n$.

Fix a Borel subgroup $B_M$ in $M$ containing $A$ and let $B$ (respectively, $B_1$) be the Borel subgroups in $\mathcal{P}(A)$ obtained as the inverse image of $B_M$ under $P \rightarrow M$ (respectively, $P \rightarrow M$). Then (3.5.3) implies (see 2.6) that

\[\nu_A(x_B) - \nu_A(x_{B_1}) \equiv \sum_{\alpha \in R_N} \varphi_F(\alpha(u)) \cdot \alpha^\vee\]

modulo the coroot lattice for $M$. Since $R_N$ is also the set of roots that are positive on $B$ and negative on $B_1$, it follows that

\[\nu_A(x_B) - \nu_A(x_{B_1}) = \sum_{\alpha \in R_N} j_\alpha \cdot \alpha^\vee\]

for some integers $j_\alpha$ such that $0 \leq j_\alpha \leq \varphi_F(\alpha(u))$. (To prove this pick a minimal gallery joining $B$ to $B_1$ and use the inequality stated in the main theorem for each neighboring pair in the gallery.) Comparing this equality with the congruence, we see that the linear combination

\[(3.5.4) \quad \sum_{\alpha \in R_N} (\varphi_F(\alpha(u)) - j_\alpha) \cdot \alpha^\vee\]
maps to 0 in $\Lambda_M$.

We get a basis for $\Lambda_M \otimes \mathbb{R}$ by taking the elements $\beta_{P,P'}$ as $P'$ varies through the set of parabolic subgroups in $\mathcal{P}(M)$ adjacent to $P$. Moreover for any $\alpha \in R_N$ the image $\pi_M(\alpha^\vee)$ of $\alpha^\vee$ in $\Lambda_M$ is a non-negative linear combination of basis elements $\beta_{P,P'}$ (with at least one non-zero coefficient). Therefore the fact that $\nu_{A}(3.5.6)$ maps to 0 in $\Lambda_M$ means that

$$\nu_{A}(x_{B}) - \nu_{A}(x_{B_1}) = \sum_{\alpha \in R_N^+} \text{val}_F(\alpha(u)) \cdot \alpha^\vee,$$

By hypothesis $x_\beta$ is regular. Therefore (transitivity of retractions plus the part of our theorem we have already proved) for all adjacent Borel subgroups $B_1, B_2 \in \mathcal{P}(A)$ such that $B_1, B_2 \subset \bar{P}$ we have

$$\nu_{A}(x_{B_1}) - \nu_{A}(x_{B_2}) = \text{val}_F(\alpha_{B_1,B_2}(u)) \cdot \alpha_{B_1,B_2}^\vee,$$

where $\alpha_{B_1,B_2}$ denotes the unique root that is positive on $B_1$ and negative on $B_2$. Summing these equalities over neighboring pairs in a minimal gallery joining $B_1$ to $B$, we find that

$$\nu_{A}(x_{B_1}) - \nu_{A}(x_{B_0}) = \sum_{\alpha \in R_N^+} \text{val}_F(\alpha(u)) \cdot \alpha^\vee,$$

where $R_N^+$ denotes the set of roots of $A$ in $B_M$. Adding this last equality to $\nu_{A}(3.5.6)$, we see that

$$\nu_{A}(x_{B}) - \nu_{A}(x_{B_0}) = \sum_{\alpha \in R_N^+} \text{val}_F(\alpha(u)) \cdot \alpha^\vee.$$

Now consider any minimal gallery $B = B_0, B_1, \ldots, B_l = \bar{B}$ joining $B$ to $B$. Then

$$\nu_{A}(x_{B}) - \nu_{A}(x_{\bar{B}}) = \sum_{i=1}^{l} n(x, B_{i-1}, B_i) \cdot \alpha_i^\vee,$$

where $\alpha_i$ is the unique root that is positive for $B_{i-1}$ and negative for $B_i$. We know that $n(x, B_{i-1}, B_i) \leq \text{val}_F(\alpha_i(u))$ for all $i$. Subtracting $\nu_{A}(3.5.7)$ from $\nu_{A}(3.5.6)$, we find that 0 is a non-negative linear combination of positive roots; therefore each coefficient in this linear combination is 0, which means that

$$n(x, B_{i-1}, B_i) = \text{val}_F(\alpha_i(u))$$

for $i = 1, \ldots, l$.

Now consider any pair $B', B''$ of adjacent Borel subgroups in $\mathcal{P}(A)$. After reversing the order of $B', B''$ if necessary we can find a minimal gallery as above and an index $i$ such that $(B_{i-1}, B_i) = (B', B'')$. Therefore

$$\nu_{A}(x_{B'}) - \nu_{A}(x_{B''}) = \text{val}_F(\alpha(u)),$$

where $\alpha$ is the unique root that is positive on $B'$ and negative on $B''$. Since both sides of $\nu_{A}(3.5.8)$ remain unchanged when $B', B''$ are switched, we see that $\nu_{A}(3.5.8)$ holds for any adjacent pair $B', B''$. By what we have already done, it follows that $x$ is regular in $X^u$. 

\[\text{REGULAR POINTS IN AFFINE SPRINGER FIBERS 9}\]
3.6. Proof of the main theorem in general. Now let $M$ be any Levi subgroup of $G$ containing $A$, and let $u$ be an integral regular semisimple element of $\mathfrak{g}(F)$ that happens to lie in $\mathfrak{m}(F)$. Let $T = \text{Cent}_{G,F}(u)$, a maximal torus in $M_F$. We choose a finite extension $F'/F$ that splits $T$.

We normalize the valuation $\text{val}_{F'}$ on $F'$ so that uniformizing elements in $F'$ have valuation 1. Thus $\text{val}_{F'}(\epsilon) = [F' : F]$. We write $X'$ for the set $G(F')/G(O_{F'})$. The inclusion $G(F) \hookrightarrow G(F')$ induces a canonical injection $X \hookrightarrow X'$.

For any $P \in \mathcal{P}(M)$ the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{r_P} & X_M \\
\downarrow & & \downarrow \\
X' & \xrightarrow{r_P'} & X'_M
\end{array}
$$

commutes, where the horizontal maps are retractions and the vertical maps are the canonical injections. Moreover the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\nu_G} & \Lambda_G \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\nu_G'} & \Lambda_G
\end{array}
$$

commutes, where the left vertical map is the canonical injection and the right vertical map is multiplication by $e := [F' : F]$.

For any $x \in X^u$ the image of $x$ in $X'$ lies in $(X')^u$, and $x$ is regular in $X^u$ if and only if $x$ is regular in $(X')^u$. Indeed the conjugacy class $\bar{u}_G(x)$ attached to $u$ and $x$ is the same for $X$ and $X'$.

The torus $T$ is conjugate under $M(F')$ to $A$, so our theorem is true for $T$ over $F'$. Therefore for $x \in X^u$ and adjacent $P = MN, P' = MN' \in \mathcal{P}(M)$

$$
(3.6.1) \quad e \cdot n(x, P, P') \leq \sum_{\alpha \in R_N \cap R_{N'}} \text{val}_{F'}(\alpha(u)) \cdot m_\alpha,
$$

and $x$ is regular in $X^u$ if and only if all of these inequalities are equalities. (As before $R_N$ denotes the set of roots of $A$ in $n$; the positive integers $m_\alpha$ were defined in 2.3.) Dividing by $e$, and noting that the term indexed by $\alpha$ depends only on the $\Gamma$-orbit of $\alpha$, we find that (3.6.1) is equivalent to the inequality

$$
n(x, P, P') \leq n(u, P, P').
$$

This completes the proof of the theorem.

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