Garfinkle-Horowitz-Strominger dilaton black hole: resonant frequency, thermodynamics and Hawking radiation

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We obtain the exact solutions of both angular and radial parts of the Klein-Gordon equation describing a charged massive scalar fields in the Garfinkle-Horowitz-Strominger black hole spacetime, which are given in terms of the spherical harmonics and the confluent Heun functions, respectively. Furthermore, we discuss the physical phenomena related to the resonant frequency associated to this field, and also some aspects related to its thermodynamics and Hawking radiation.

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I. INTRODUCTION

The theoretical studies concerning physical processes which occur in the spacetime associated to black holes are of special interest and, certainly, can help us to understand the physics of these objects predicted by general relativity. Among these studies, we can mention the ones corresponding to resonant frequencies [1–3], scattering of particles and fields of different spins [4–8] and Hawking radiation [9].

During last years, the Heun functions have gained more and more importance due to the large number of applications in different areas of physics (see [10] and references therein) and in special, in the solutions of problems related to a scalar field in gravitational backgrounds [11]. The use of the Heun’s functions permits us, for example, to find the exact solutions of the Klein-Gordon equation in some black hole spacetimes [12]. Otherwise, without the use of these functions, it is not possible to find exact analytical solutions of the Klein-Gordon equation in the whole region of spacetime. In fact, it will be possible to find the solutions for scalar fields only for some specific regions very close and far away from the black hole horizons.

In the present paper, we apply the confluent Heun functions to obtain the solutions of the Klein-Gordon equations for a charged massive scalar field in the Garfinkle-Horowitz-Strominger dilaton black hole (GHS dilaton black hole). These solutions, which are given in terms of the confluent Heun functions, are used to examine the resonant frequencies, and the Hawking radiation. In discussing the Hawking radiation, we study some aspects of the thermodynamics of this black hole.

This paper is organized as follows. In Sec. II we present the solutions of the Klein-Gordon equation for a charged massive scalar field in the GHS dilaton black hole spacetime, for both angular and radial parts. In Sec. III we obtain the resonant frequencies for both massive and massless scalar particles. In Sec. IV we discuss the thermodynamics and the Hawking radiation of scalar waves. Finally, in Sec. V the conclusions are given.

II. KLEIN-GORDON EQUATION IN THE GHS DILATON BLACK HOLE

In this work our background is the solution obtained from the low energy effective action in string theory by dropping all the fields, except the metric $g_{\sigma\tau}$, the dilaton $\phi$ and the Maxwell field $F_{\sigma\tau}$. This corresponds to the static, spherically symmetric and charged dilaton black hole called GHS dilaton black hole [13], whose metric is given by

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r(r - a) d\Omega^2,$$

(1)
with
\[ d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2 . \] (2)

The parameter \( a \) is related to the dilaton field, namely,
\[ a = \frac{Q^2 e^{-2\phi_0}}{M} , \] (3)
where \( \phi_0 \) is the asymptotic value of the dilaton field, such that
\[ e^{-2\phi} = e^{-2\phi_0} \left( 1 - \frac{Q^2 e^{-2\phi_0}}{Mr} \right) , \] (4)
and \( M \) and \( Q \) are the physical mass and the magnetic charge of the GHS dilaton black hole, respectively.

For our purposes, let us focus on \( \phi_0 = 0 \). In this case, we have
\[ a = \frac{Q^2}{M} . \] (5)
Notice that when \( Q = 0 \), which implies \( a = 0 \), we recover the Schwarzschild spacetime.

Now, we want to study the interaction between charged scalar fields and the GHS dilaton black hole. In order to do this, let us consider the Klein-Gordon equation, which can be write as
\[
\left[ \frac{1}{\sqrt{-g}} \partial_\sigma (g^{\sigma\tau} \sqrt{-g} \partial_\tau) - ie (\partial_\sigma A^\sigma) - 2ieA^\sigma \partial_\sigma - \frac{ie}{\sqrt{-g}} A^\sigma (\partial_\sigma \sqrt{-g}) - e^2 A^\sigma A_\sigma - \mu_0^2 \right] \Psi = 0 ,
\] (6)
with
\[ \sqrt{-g} = r(r - a) \sin \theta , \] (7)
where \( \mu_0 \) is the mass of the scalar particle, and \( e \) is the charge of the particle. Note that we have chosen the units where \( G = c = \hbar = 1 \). In this background, the 4-vector electromagnetic potential is given by
\[ A_\sigma dx^\sigma = -\frac{Q}{r} \, dt . \] (8)

Thus, substituting Eq. (11) into Eq. (5), we obtain
\[
\left\{ \begin{array}{l}
- \frac{r^2(r - r_d)}{r - r_h} \frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial r} \left[ (r - r_h)(r - r_d) \frac{\partial}{\partial r} \right] - L^2 \\
- \frac{2ieQ}{r - r_h} (r - r_d) \frac{\partial}{\partial t} + \frac{e^2 Q^2 (r - r_a)}{r - r_h} - r(r - r_d) \mu_0^2 \end{array} \right\} \Psi = 0 ,
\] (9)
where \( r_h = 2M, \ r_d = a, \) and \( L^2 \) is the angular momentum operator given by
\[ L^2 = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} . \] (10)
Due to the spacetime symmetry we can separate the scalar wave function as

$$\Psi(r, t) = R(r)Y_l^m(\theta, \varphi)e^{-i\omega t} ,$$  \hspace{1cm} (11)

where $Y_l^m$ are the spherical harmonics, $l = \{0, 1, 2, \ldots\}$ and $|m| \leq l$ are the orbital and the azimuthal quantum numbers, respectively. The frequency (energy) is taken as $\omega > 0$, which corresponds to the flux of particles at infinity. Therefore, by using this separation of variables, we can write Eq. (9) as

$$\frac{d}{dr} \left[ (r - r_h) (r - r_d) \frac{dR}{dr} \right] + \left[ \frac{r - r_d}{r - r_h} (\omega r - eQ)^2 - \lambda_{lm} - r(r - r_d)\mu_0^2 \right] R = 0 ,$$  \hspace{1cm} (12)

where $\lambda_{lm} = l(l + 1)$.

In what follows, we will obtain the radial solution, $R(r)$, of the Klein-Gordon equation.

A. Radial equation

Nowadays, expressions for the solutions of the radial Klein-Gordon equation in the GHS dilaton spacetime are only known in either the asymptotic regimes, namely, very close to event horizon and far from the black hole [16, 17]. Otherwise, it is possible to know the solution in whole spacetime, but only numerically [18–20].

Thus, in order to solve exactly Eq. (12), we use the procedure developed in our recent papers (see for example [21] and references therein). In this way, the general solution of the radial part of the Klein-Gordon equation for a charged massive scalar field in the GHS dilaton black hole, in the region exterior to the event horizon, can be written as

$$R(x) = e^{\frac{1}{2} \alpha x} x^\frac{1}{2} \beta \{ C_1 \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; x) + C_2 x^{-\beta} \text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta; x) \} ,$$  \hspace{1cm} (13)

with

$$x = \frac{r - r_h}{r_d - r_h} ,$$  \hspace{1cm} (14)

where $C_1$ and $C_2$ are constants, and the parameters $\alpha$, $\beta$, $\gamma$, $\delta$, and $\eta$ are given by

$$\alpha = 2(r_h - r_d)(\mu_0^2 - \omega^2)^\frac{1}{2} ,$$  \hspace{1cm} (15)

$$\beta = 2i(\omega r_h - eQ) ,$$  \hspace{1cm} (16)

$$\gamma = 0 ,$$  \hspace{1cm} (17)
\[ \delta = (r_h - r_d)[2eQ\omega + r_h(\mu_0^2 - 2\omega^2)] , \]  
(18)

\[ \eta = -\lambda_{lm} - \delta . \]  
(19)

Due to the fact that \( \beta \) is not necessarily an integer, these two functions form linearly independent solutions of the confluent Heun differential equation [22], namely,

\[ \frac{d^2U}{dz^2} + \left( \alpha + \frac{\beta + 1}{z} + \frac{\gamma + 1}{z-1} \right) \frac{dU}{dz} + \left( \frac{\mu + \nu}{z} - \frac{\nu}{z-1} \right) U = 0 , \]  
(20)

where \( U(z) = \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z) \) are the confluent Heun functions, with the parameters \( \alpha, \beta, \gamma, \delta \) and \( \eta \), related to \( \mu \) and \( \nu \) by

\[ \mu = \frac{1}{2}(\alpha - \beta - \gamma + \alpha\beta - \beta\gamma) - \eta , \]  
(21)

\[ \nu = \frac{1}{2}(\alpha + \beta + \gamma + \alpha\gamma + \beta\gamma) + \delta + \eta . \]  
(22)

It is worth calling attention to the fact that we have obtained an analytical solution for the Klein-Gordon equation in the background under consideration in the whole space, which means that includes the regions nearby the event horizon and far from the black hole.

Next, we will use this radial solution to investigate two interesting phenomena: the resonant frequency and the Hawking radiation.

**III. RESONANT FREQUENCIES**

In this section, we follow the recently developed technique of Ref. [23] for computing the Resonant Frequencies (RFs) for scalar waves propagating in a GHS dilaton black hole.

The RFs are associated with the solution given by Eq. (13) under certain boundary conditions, i.e., the radial solution should be finite on the exterior event horizon and well behaved at asymptotic infinity. The first condition is completely satisfied as we will see from the wave solution which describes the Hawking radiation in this background. The latter condition requires that \( R(x) \) must have a polynomial form. Indeed, the function \( \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; x) \) becomes a polynomial of degree \( n \) if the following \( \delta \)-condition is satisfied:

\[ \frac{\delta}{\alpha} + \frac{\beta + \gamma}{2} + 1 = -n , \]  
(23)

where \( n = \{0, 1, 2, \ldots\} \) is the principal quantum number.
Substituting Eqs. (15)-(18) into Eq. (23), we find the following expression which involves the RFs associated to charged massive scalar particles in the background under consideration:

\[
\frac{2eQ\omega + r_h(\mu_0^2 - 2\omega^2)}{2(\mu_0^2 - \omega^2)^{\frac{1}{2}}} + i(\omega r_h - eQ) = -(n + 1),
\]

(24)

This is a nontrivial quantization law, because it gives a complex number, that is, we obtain a frequency (energy) spectrum such that \( \omega = \omega_R + i \omega_I \), where \( \omega_R \) and \( \omega_I \) are the real and imaginary parts, respectively. Indeed, the main feature of the RFs corresponds to the decay rate of the oscillation, which is characterized by the imaginary part. We remark that the eigenvalues given by Eq. (24) are not degenerate, since that there is no dependence on the eigenvalue \( \lambda_{lm} \).

Equation (24) is a second order equation for \( \omega \) and hence we can numerically obtain values for the RFs by using the \textit{FindRoot} function in the \textbf{Wolfram Mathematica\textcopyright 9}, such that \( (\omega - \omega_{n}^{(1)})(\omega - \omega_{n}^{(2)}) = 0 \). The RFs for \( n = 0, e = 0.1, \) and \( \mu_0 = 0.6 \) are shown in Tables I and II, where we have chosen the units \( M = 1 \).

\begin{table}[h]
\centering
\begin{tabular}{ccc}
\hline
\( a \) & \text{Re}[\omega_{0}^{(1)}] & \text{Im}[\omega_{0}^{(1)}] \\
\hline
0.00 & 0.00000 & 0.16643 \\
0.01 & 0.00455 & 0.16642 \\
0.04 & 0.00911 & 0.16639 \\
0.09 & 0.01366 & 0.16635 \\
0.16 & 0.01822 & 0.16629 \\
0.25 & 0.02277 & 0.16621 \\
0.36 & 0.02733 & 0.16611 \\
0.49 & 0.03189 & 0.16599 \\
0.64 & 0.03645 & 0.16586 \\
0.81 & 0.04101 & 0.16571 \\
1.00 & 0.04557 & 0.16554 \\
1.21 & 0.05014 & 0.16535 \\
1.44 & 0.05471 & 0.16514 \\
1.69 & 0.05928 & 0.16491 \\
1.96 & 0.06385 & 0.16467 \\
2.25 & 0.06843 & 0.16441 \\
\hline
\end{tabular}
\caption{The scalar resonant frequencies \( \omega_{n}^{(1)} \) of a GHS dilaton black hole for \( e = 0.1 \) and \( \mu_0 = 0.6 \). We focus on the fundamental mode \( n = 0 \). The units are given in multiples of the total mass \( M \).}
\end{table}
TABLE II. The scalar resonant frequencies $\omega_n^{(2)}$ of a GHS dilaton black hole for $e = 0.1$ and $\mu_0 = 0.6$. We focus on the fundamental mode $n = 0$. The units are given in multiples of the total mass $M$.

| $a$    | Re[$\omega_0^{(2)}$] | Im[$\omega_0^{(2)}$] |
|--------|----------------------|----------------------|
| 0.00   | -0.58681             | 0.04178              |
| 0.01   | -0.58659             | 0.04292              |
| 0.04   | -0.58637             | 0.04406              |
| 0.09   | -0.58616             | 0.04522              |
| 0.16   | -0.58595             | 0.04638              |
| 0.25   | -0.58574             | 0.04754              |
| 0.36   | -0.58554             | 0.04872              |
| 0.49   | -0.58535             | 0.04990              |
| 0.64   | -0.58516             | 0.05110              |
| 0.81   | -0.58497             | 0.05230              |
| 1.00   | -0.58479             | 0.05350              |
| 1.21   | -0.58461             | 0.05472              |
| 1.44   | -0.58444             | 0.05594              |
| 1.69   | -0.58427             | 0.05716              |
| 1.96   | -0.58410             | 0.05840              |
| 2.25   | -0.58394             | 0.05964              |

The RFs that we obtained are shown in Figs. (1)-(3), (4)-(6), (7)-(9), and (10)-(12) as a function of $n$, $a$, $e$, and $\mu_0$, respectively.
FIG. 1. The real part of the scalar resonant frequencies of a GHS dilaton black hole as a function of $n$ for $e = 0.1$ and $a = 0.1$. The units are given in multiples of the total mass $M$.

FIG. 2. The imaginary part of the scalar resonant frequencies of a GHS dilaton black hole as a function of $n$ for $e = 0.1$ and $a = 0.1$. The units are given in multiples of the total mass $M$. 
FIG. 3. The scalar resonant frequencies of a GHS dilaton black hole as a function of $n$ for $e = 0.1$ and $a = 0.1$. The units are given in multiples of the total mass $M$.

FIG. 4. The real part of the scalar resonant frequencies of a GHS dilaton black hole as a function of $a$ for $e = 0.1$ and $\mu_0 = 0.4$. The units are given in multiples of the total mass $M$. 
FIG. 5. The imaginary part of the scalar resonant frequencies of a GHS dilaton black hole as a function of $a$ for $e = 0.1$ and $\mu_0 = 0.4$. The units are given in multiples of the total mass $M$.

FIG. 6. The scalar resonant frequencies of a GHS dilaton black hole as a function of $a$ for $e = 0.1$ and $\mu_0 = 0.4$. The units are given in multiples of the total mass $M$. 
FIG. 7. The real part of the scalar resonant frequencies of a GHS dilaton black hole as a function of $e$ for $a = 1.1$ and $\mu_0 = 0.5$. The units are given in multiples of the total mass $M$.

FIG. 8. The imaginary part of the scalar resonant frequencies of a GHS dilaton black hole as a function of $e$ for $a = 1.1$ and $\mu_0 = 0.5$. The units are given in multiples of the total mass $M$. 
FIG. 9. The scalar resonant frequencies of a GHS dilaton black hole as a function of \( e \) for \( a = 1.1 \) and \( \mu_0 = 0.5 \). The units are given in multiples of the total mass \( M \).

FIG. 10. The real part of the scalar resonant frequencies of a GHS dilaton black hole as a function of \( \mu_0 \) for \( e = 0.1 \) and \( a = 1.6 \). The units are given in multiples of the total mass \( M \).
FIG. 11. The imaginary part of the scalar resonant frequencies of a GHS dilaton black hole as a function of $\mu_0$ for $e = 0.1$ and $a = 1.6$. The units are given in multiples of the total mass $M$.

FIG. 12. The scalar resonant frequencies of a GHS dilaton black hole as a function of $\mu_0$ for $e = 0.1$ and $a = 1.6$. The units are given in multiples of the total mass $M$. 
A. Massless scalar fields

In the case where we have a massless scalar field, the expression for the RFs can be exactly solved for $\omega_n$. It is given by

$$\omega_n = \frac{eQ}{r_h} + i \frac{n + 1}{2r_h},$$

where the principal quantum number $n$ is a positive integer or zero.

The eigenvalues given by Eq. (25) are also not degenerate, since that there is no dependence on the eigenvalue $\lambda_{lm}$. The RFs for $n = 1$, $e = 1.1$, and $\mu_0 = 0$ are shown in Table III and in Figs. (13)-(15) and (16)-(18), we present the RFs as function of $a$ and $e$, respectively.

| $a$  | $\text{Re}(\omega_1)$ | $\text{Im}(\omega_1)$ |
|------|----------------------|----------------------|
| 0.00 | 0.000                | 0.500                |
| 0.01 | 0.055                | 0.500                |
| 0.04 | 0.110                | 0.500                |
| 0.09 | 0.165                | 0.500                |
| 0.16 | 0.220                | 0.500                |
| 0.25 | 0.275                | 0.500                |
| 0.36 | 0.330                | 0.500                |
| 0.49 | 0.385                | 0.500                |
| 0.64 | 0.440                | 0.500                |
| 0.81 | 0.495                | 0.500                |
| 1.00 | 0.550                | 0.500                |
| 1.21 | 0.605                | 0.500                |
| 1.44 | 0.660                | 0.500                |
| 1.69 | 0.715                | 0.500                |
| 1.96 | 0.770                | 0.500                |
| 2.25 | 0.825                | 0.500                |
FIG. 13. The real part of the massless scalar resonant frequencies of a GHS dilaton black hole as a function of $a$ for $e = 0.1$. The units are given in multiples of the total mass $M$.

FIG. 14. The imaginary part of the massless scalar resonant frequencies of a GHS dilaton black hole as a function of $a$ for $e = 0.1$. The units are given in multiples of the total mass $M$. 
FIG. 15. The massless scalar resonant frequencies of a GHS dilaton black hole as a function of $a$ for $e = 0.1$. The units are given in multiples of the total mass $M$.

FIG. 16. The real part of the massless scalar resonant frequencies of a GHS dilaton black hole as a function of $e$ for $a = 1.1$. The units are given in multiples of the total mass $M$. 
FIG. 17. The imaginary part of the massless scalar resonant frequencies of a GHS dilaton black hole as a function of $e$ for $a = 1.1$. The units are given in multiples of the total mass $M$.

FIG. 18. The massless scalar resonant frequencies of a GHS dilaton black hole as a function of $e$ for $a = 1.1$. The units are given in multiples of the total mass $M$. 

IV. SOME ASPECTS OF THERMODYNAMICS AND HAWKING RADIATION

In this section we do a brief review about the thermodynamics quantities of the GHS dilaton black hole and then discuss the Hawking radiation.

A. Thermodynamics

The surface area of the exterior event horizon $A_h$ is given by

$$A_h = \int \int \sqrt{-g} \, d\theta \, d\phi \bigg|_{r=r_h} = 4\pi r_h (r_h - a) .$$

(26)

From this result, we can write the entropy at the exterior event horizon $S_h$ as

$$S_h(M, Q) = \frac{A_h}{4} = 4\pi M^2 - 2\pi Q^2$$

(27)

or

$$S_h(M, a) = 4\pi M^2 - 2\pi Ma ,$$

(28)

whose behavior as a function of $M$ for different values of $Q$ and $a$ are shown in Figs. (19) and (20), respectively.

The first law of thermodynamics is given by

$$dM = T_h dS_h + \Phi_h dQ .$$

(29)

From this law, we can obtain the equation of state for the Hawking temperature $T_h$, as

$$T_h = \left. \frac{\partial M}{\partial S_h} \right|_Q = \frac{\kappa_h}{2\pi} ,$$

(30)

where $\kappa_h$ is the gravitational acceleration on the background exterior event horizon surface, namely,

$$\kappa_h = \frac{1}{2r_h} .$$

(31)

The thermodynamic quantity $\beta_h$ is given by

$$\beta_h = \frac{1}{k_B T_h} .$$

(32)

In Figs. (21) and (22) we present the Hawking temperature and the gravitational acceleration, respectively, as a function of $M$.

The Legendre transformation for the free energy is given by

$$F = U - TS = M - TS .$$

(33)
FIG. 19. Entropy of a GHS dilaton black hole as function of $M$, for different values of $Q$.

FIG. 20. Entropy of a GHS dilaton black hole as function of $M$, for different values of $a$. The solid red lines mark the points where the entropy starts from zero to reach positive values.
FIG. 21. Hawking temperature of a GHS dilaton black hole as a function of $M$.

FIG. 22. Gravitational acceleration of a GHS dilaton black hole as a function of $M$. 
Thus, substituting Eqs. (27) and (30) into Eq. (33), we obtain the following expression for the free energy at the exterior event horizon $F_h$:

$$F_h(M, Q) = \frac{M}{2} + \frac{Q^2}{4M}$$

(34)

or

$$F_h(M, a) = \frac{M}{2} + \frac{a}{4}.$$  

(35)

In Figs. (23) and (24) we present the free energy as function of $M$ for different values of $Q$ and $a$, respectively.

The equation of state for the electric potential nearby the exterior event horizon $\Phi_h$ is given by

$$\Phi_h(M, Q) = \frac{\partial M}{\partial Q} \bigg|_{S_h} = \frac{Q}{r_h}$$

(36)

or

$$\Phi_h(M, a) = \sqrt{aM} \frac{1}{r_h}.$$  

(37)

The electric potential as a function of $M$ for different values of $Q$ and $a$ is shown in Figs. (25) and (26), respectively.

The equation of state for the heat capacity at constant charge $C_Q$ is given by

$$C_Q = T_h \frac{\partial S_h}{\partial T_h} \bigg|_Q = -\frac{1}{8\pi T_h^2} = -8M^2\pi,$$

(38)

where we have used the entropy written as

$$S_h(T_h, Q) = \frac{1}{16\pi T_h^2} - 2\pi Q^2.$$  

(39)

In Figs. (27) and (28) we present the modulus of the heat capacity as a function of $M$ and $T_h$, respectively.

In summary, we can plot together the Hawking temperature, the entropy and the heat capacity as a function of $M$ in Fig. (29).

From Figs. (19)-(29), we can get some important conclusions. The free energy has a minimum at $M = Q/\sqrt{2}$, where $F = Q/\sqrt{2}$ at this point. Thus, for masses smaller than this value, the entropy becomes negative and hence the GHS dilaton black hole exhibits stability at this point (free energy minimum), which may suggest that the black hole no longer evaporates. Furthermore, for any value of $M > Q/\sqrt{2}$, the system is thermodynamically unstable because the heat capacity is negative (global instability), and the free energy is positive (local instability). The entropy has a negative region when $M < a/2$. Thus, there is a minimum value of mass, namely, $M = a/2$, necessary to form the GHS dilaton black hole, where $S_h = 0$. 
FIG. 23. Free energy of a GHS dilaton black hole as a function of $M$, for different values of $Q$. The solid red lines mark the points where the entropy starts from zero to reach positive values.

FIG. 24. Free energy of a GHS dilaton black hole as a function of $M$, for different values of $a$. The solid red lines mark the points where the entropy starts from zero to reach positive values.
FIG. 25. Electric potential of a GHS dilaton black hole as a function of $M$, for different values of $Q$. The solid red lines mark the points where the entropy starts from zero to reach positive values.

FIG. 26. Electric potential of a GHS dilaton black hole as a function of $M$, for different values of $a$. The solid red lines mark the points where the entropy starts from zero to reach positive values.
FIG. 27. Heat capacity of a GHS dilaton black hole as function of $M$. 

FIG. 28. Heat capacity of a GHS dilaton black hole as a function of $T_h$. 
FIG. 29. Hawking temperature, entropy, heat capacity, and free energy of a GHS dilaton black hole as a function of $M$ for $Q = 0.5$. 
B. Black hole radiation

Now, we want to study the black body radiation emitted by a GHS dilaton black hole. In order to do this, we need to consider the radial solution near the exterior event horizon, that is, to analyze the radial solution when \( r \to r_h \) which implies that \( x \to 0 \).

In this way, let us consider the expansion in power series of the confluent Heun function with respect to the independent variable \( x \), in a neighborhood of the regular singular point \( x = 0 \) \[24\], which can be written as

\[
\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; x) = 1 + \frac{1}{2} \frac{(-\alpha \beta + \beta \gamma + 2\eta - \alpha + \beta + \gamma)}{(\beta + 1)} x \\
+ \frac{1}{8(\beta + 1)(\beta + 2)} (\alpha^2 \beta^2 - 2\alpha \beta^2 \gamma + \beta^2 \gamma^2) \\
- 4\eta \alpha \beta + 4\eta \beta \gamma + 4\alpha^2 \beta - 2\alpha \beta^2 - 6\alpha \beta \gamma \\
+ 4\beta^2 \gamma + 4\beta \gamma^2 + 4\eta^2 - 8\eta \alpha + 8\eta \beta + 8\eta \gamma \\
+ 3\alpha^2 - 4\alpha \beta - 4\alpha \gamma + 3\beta^2 + 4\beta \delta \\
+ 10\beta \gamma + 3\gamma^2 + 8\eta + 4\beta + 4\delta + 4\gamma)x^2 + \ldots . \tag{40}
\]

Thus, in this limit, the radial solution given by Eq. (13) becomes

\[ R(r) \sim C_1 (r - r_h)^{\beta/2} + C_2 (r - r_h)^{-\beta/2}, \tag{41} \]

where we have only considered contributions of the first term in the expansion, and all constants are included in \( C_1 \) and \( C_2 \). Then, considering the solution of the time dependence, near the exterior event horizon \( r_h \) of the GHS dilaton black hole, we can write

\[ \Psi = e^{-i\omega t} (r - r_h)^{\pm \beta/2}. \tag{42} \]

From Eq. (16), the parameter \( \beta \) can be rewritten as

\[ \frac{\beta}{2} = i(\omega r_h - eQ) = \frac{i}{2\kappa_h} (\omega - \omega_h), \tag{43} \]

where

\[ \omega_h = e\Phi_h. \tag{44} \]

Therefore, in the GHS black hole exterior horizon surface, the ingoing and outgoing wave solutions are given by

\[ \Psi_{in} = e^{-i\omega t} (r - r_h)^{-i/2\kappa_h (\omega - \omega_h)}, \tag{45} \]
\[ \Psi_{\text{out}}(r > r_h) = e^{-i\omega t}(r - r_h) e^{i\beta_h(\omega - \omega_h)} \]  (46)

Next, we obtain the following expression for the relative scattering probability of the scalar wave at the exterior event horizon surface

\[ \Gamma_h(\omega) = \left| \frac{\Psi_{\text{out}}(r > r_h)}{\Psi_{\text{out}}(r < r_h)} \right|^2 = e^{-\frac{2\pi}{\kappa_h}(\omega - \omega_h)} = e^{-\beta_h(\omega - \omega_h)} . \]  (47)

Thus, by using the Damour-Ruffini-Sannan method [25, 26], we get the resulting Hawking radiation spectrum of scalar particles (mean number of particles emitted), which is given by

\[ \bar{N}_\omega = \frac{\Gamma_h}{1 - \Gamma_h} = \frac{1}{e^{\beta_h(\omega - \omega_h)} - 1} . \]  (48)

That is the black body spectrum described by scalar particles which are emitted from the GHS dilaton black hole. Indeed, this is a finite solution for the wave function near the exterior event horizon of the background under consideration.

In the limit where \( \omega \) is a continuous variable, this yields a total rate of particle emission, namely,

\[ \frac{dN}{dt} = \int_0^\infty \frac{\bar{N}_\omega}{2\pi} d\omega = \frac{1}{\beta_h} \ln \left( \frac{1}{1 - e^{\beta_h\omega_h}} \right) . \]  (49)

The mass loss rate can be calculated as

\[ \frac{dM}{dt} = -\frac{1}{2\pi} \int_0^\infty \bar{N}_\omega \omega d\omega = -\frac{1}{2\pi} \frac{1}{\beta_h^2} \text{Li}_2(e^{\beta_h\omega_h}) , \]  (50)

where \( \text{Li}_n(z) \) is the polylogarithm function with \( n \) running from 1 to \( \infty \).

The flux of scalar particles, \( \Phi \), at infinity, i.e., far from the GHS dilaton black hole, is given by

\[ \Phi = \left| \frac{dM}{dt} \right| = \frac{1}{2\pi} \frac{1}{\beta_h^2} \text{Li}_2(e^{\beta_h\omega_h}) . \]  (51)

Finally, according to the canonical assembly theory [27], assuming that the frequency (energy) is continuous, the free energy of the scalar particle, \( F_e \), can be expressed as

\[ F_e = -\int_0^\infty \frac{\Gamma_h(\omega)}{e^{\beta_h(\omega - \omega_h)} - 1} d\omega = \frac{e^{\beta_h\omega_h} + \ln(1 - e^{\beta_h\omega_h})}{\beta_h} . \]  (52)

In Fig. 30 is shown the free energy of the scalar particle as a function of \( M \) for different values of \( Q \). It is worth calling attention to the fact that the free energy of the scalar particle will be a real number only if the magnetic charge \( Q \) or the charge of the scalar particle \( e \) is negative.
FIG. 30. Free energy of the scalar particle in a GHS dilaton black hole as a function of $M$ for $e = -0.1$.

V. CONCLUSIONS

In this work we have considered the interaction between scalar fields and the GHS dilaton black hole, and solved the Klein-Gordon equation and then analyzed some interesting phenomena which correspond to the resonant frequencies and the Hawking radiation.

We obtain a general expression for the resonant frequencies from the boundary conditions imposed to the radial solution and studied the behavior of the oscillations and how fast they disappear. By using a numerical method, we get some values for the resonant frequencies as a function of the involved parameters. We also analyzed the case of massless scalar particles.

Concerning thermodynamics, we examined how the quantities such as entropy, free energy and heat capacity depend on the parameter associated to the dilaton.

The Hawking radiation spectrum was obtained from the asymptotic behavior of the radial solution at the exterior event horizon, where we have used the expansion in power series of the confluent Heun function.

In summary, all results obtained depend on the dilaton parameter $a$, which implies that, in principle, we can use these results to fit some observable datas and hence shed some light on this feature of the quantum gravity theory.
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