Research Article

Approximation Properties of Generalized $\lambda$-Bernstein–Stancu-Type Operators

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The present study introduces generalized $\lambda$-Bernstein–Stancu-type operators with shifted knots. A Korovkin-type approximation theorem is given, and the rate of convergence of these types of operators is obtained for Lipschitz-type functions. Then, a Voronovskaja-type theorem was given for the asymptotic behavior for these operators. Finally, numerical examples and their graphs were given to demonstrate the convergence of $G_{m,\lambda}^\alpha(f, x)$ to $f(x)$ with respect to $m$ values.

1. Introduction

The Bernstein operators, which are positive linear operators, are of great importance for the theory of approximation. In [1], Bernstein operators were introduced by Bernstein to prove the Weierstrass approximation theorem. For $g \in C[0,1]$, the classical Bernstein operators are given as

$$B_m(g, x) = \sum_{j=0}^{m} g\left(\frac{j}{m}\right) q_{m,j}(x), \quad x \in [0,1],$$

(1)

where the Bernstein basis functions $q_{m,j}(x)$ are defined as

$$q_{m,j}(x) = \binom{m}{j} x^j(1-x)^{m-j}; \quad j = 0, 1, \ldots, m.$$  

(2)

There are many generalizations of the $B_m(g, x)$ operators. In [2], Gadjiev and Ghorbanalizadeh gave Bernstein–Stancu operators with shifted knots:

$$S_{m,\alpha,\beta}(g, x) = \left(\frac{m+\beta_2}{m}\right)^{m} \sum_{j=0}^{m} g\left(\frac{j+\alpha_1}{m+\beta_1}\right) p_{m,j}(x),$$

(3)

where $x \in A_m = [\alpha_2/(m+\beta_2), (m+\alpha_2)/(m+\beta_2)]$ and $p_{m,j}(x) = \binom{m}{j} x^j((m+\alpha_2)/(m+\beta_2) - x)^{m-j}$ for $j = 0, 1, \ldots, m$, and $\alpha_1$ and $\alpha_2$ are positive real numbers, and $0 \leq \alpha_1 \leq \beta_2$ for $i = 1, 2$. In the case of $\alpha_2 = \beta_2 = 0$ for $i = 1, 2$, operator (3) reduces to the classical Bernstein operator (1). In addition, the case of $\alpha_2 = \beta_2 = 0$ was handled and examined by Stancu in [3] and these operators were called the classical Bernstein–Stancu operators. A Dunker variant of the Bernstein–Stancu operators in (3) was studied by Dinlemez Kantar and Ergelen in [4], and a Voronovskaja-type approximation theorem for these operators was given. Several studies were conducted on some approximation properties, and asymptotic-type results were given for these operators in [5–14].

In [5], Cai et al. introduced the Bernstein operators with shape parameter $\lambda \in [-1,1]$ as follows:

$$B_m^\lambda(g, x) = \sum_{j=0}^{m} g\left(\frac{j}{m}\right) q_{m,j}(\lambda, x), \quad m \in \mathbb{N},$$

(4)
where \( x \in [0, 1] \) and \( \overline{q}_{m,j}(\lambda, x) \) are Bézier basis functions with shape parameter \( \lambda \in [-1, 1] \) defined by

\[
\overline{q}_{m,0}(\lambda, x) := q_{m,0}(x) - \left( \frac{\lambda}{m + 1} \right) q_{m+1,1}(x),
\]
\[
\overline{q}_{m,j}(\lambda, x) := q_{m,j}(x) + \lambda \left[ \frac{m - 2j + 1}{m^2 - 1} q_{m+1,j}(x) - \frac{m - 2j - 1}{m^2 - 1} q_{m+1,j+1}(x) \right]; \quad j = 1, 2, \ldots, m - 1,
\]
\[
\overline{q}_{m,m}(\lambda, x) := q_{m,m}(x) - \frac{\lambda}{m+1} q_{m+1,m}(x).
\]

Later in [6], Cai proposed Kantorovich-type \( \lambda \)-Bernstein operators, as well as their Bézier variant, and examined the approximation results. Kantorovich-type \( \lambda \)-Bernstein operators were also studied by Acu et al. in [7], and they considered the approximation properties and asymptotic-type results.

Recently, in [8], Srivastava et al. constructed \( \lambda \)-Bernstein–Stancu operators defined by

\[
B_{m,a,b}^\lambda(g, x) = \sum_{j=0}^{m} g \left( \frac{j + \alpha}{m + \beta} \right) \overline{q}_{m,j}(\lambda, x), \quad m \in \mathbb{N},
\]

where \( \overline{p}_{m,j}(\lambda, x) \) is given by

\[
\overline{p}_{m,0}(\lambda, x) := p_{m,0}(x) - \frac{\lambda}{m + 1} p_{m+1,1}(x),
\]
\[
\overline{p}_{m,j}(\lambda, x) := p_{m,j}(x) + \lambda \left[ \frac{m - 2j + 1}{m^2 - 1} p_{m+1,j}(x) - \frac{m - 2j - 1}{m^2 - 1} p_{m+1,j+1}(x) \right]; \quad j = 1, 2, \ldots, m - 1,
\]
\[
\overline{p}_{m,m}(\lambda, x) := p_{m,m}(x) - \frac{\lambda}{m+1} p_{m+1,m}(x),
\]

where \( p_{m,j}(x) := \binom{m}{j} (x - \alpha_i / (m + \beta_j))^{m-j} ((m + \alpha_j) / (m + \beta_j) - x)^{m-j} \) for \( j = 0, 1, \ldots, m \). \( \alpha_i \) and \( \beta_i \) are positive real numbers, \( 0 \leq \alpha_i \leq \beta_i \) for \( i = 1, 2 \), and \( x \in A_m := [\alpha_2 / (m + \beta_1), (m + \alpha_2) / (m + \beta_2)] \). We give approximation properties and Voronovskaja-type approximation theorem for asymptotic behavior of operator (7). When \( \alpha_i = \beta_i = 0 \) for \( i = 1, 2 \) and \( \lambda = 0 \), operator (7) reduces to the classical Bernstein operator (1). When \( \lambda = 0 \), it reduces to the Bernstein–Stancu operator (3). When \( \alpha_i = \beta_i = 0 \) for \( i = 1, 2 \), it reduces to the Bernstein operator with shape parameter \( \lambda \in [-1, 1] \) (4).

In the present study, we introduce the following generalized \( \lambda \)-Bernstein–Stancu operators with shifted knots for \( g \in C[0, 1] \):

\[
G_{m,\lambda}^{a,b}(g, x) = \left( \frac{m + \beta_2}{m} \right) \sum_{j=0}^{m} g \left( \frac{j + \alpha}{m + \beta} \right) \overline{p}_{m,j}(\lambda, x),
\]

where \( \alpha_i \) and \( \beta_i \) are positive real numbers satisfying \( 0 \leq \alpha_i \leq \beta_i \) for \( i = 1, 2 \) and Bézier basis functions \( \overline{p}_{m,j}(\lambda, x) \) with shape parameter \( \lambda \in [-1, 1] \) are defined by

2. Some Preliminary Results

Lemma 1. For generalized \( \lambda \)-Bernstein–Stancu operators with shifted knots, we have the following equalities:
(i) \( G_{m, \lambda}^\alpha (1, x) = 1, \)

\[
\begin{align*}
(ii) \quad G_{m, \lambda}^{\alpha, \beta} (s, x) &= \frac{m + \beta_1}{m + \beta_1} \left( x - \frac{\alpha_2}{m + \beta_2} \right) + \frac{\alpha_1}{m + \beta_1}, \\
&+ \lambda \left\{ \frac{1}{(m + \beta_1)(m - 1)} \left[ \frac{m}{m + \beta_2} - 2 \left( x - \frac{\alpha_2}{m + \beta_2} \right) + \left( \frac{m + \beta_2}{m} \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right], \\
&+ (\alpha_1 - 1) \left( \frac{m + \alpha_3}{m + \beta_2 - x} \right)^{m+1} \right\} \right\} + \frac{\alpha_1}{m + \beta_1} \left( \frac{m + \beta_2}{m} \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \left( m + \alpha_3 \right) \left( m + \beta_2 - x \right)^m \right\}.
\end{align*}
\]

(iii) \( G_{m, \lambda}^{\alpha, \beta} (s^2, x) = \frac{m + \beta_2}{(m + \beta_1)^2} \left[ \left( m - 1 \right) \left( m + \beta_2 \right) \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 + \left( 1 + 2\alpha_1 \right) \left( x - \frac{\alpha_2}{m + \beta_2} \right) \right] + \frac{\alpha_2}{(m + \beta_1)^2} + \lambda \left\{ \frac{1}{\left( m + \beta_1 \right)^2 (m - 1)} \left[ \frac{m}{m + \beta_2} + 2m \left( x - \frac{\alpha_2}{m + \beta_2} \right) - 4m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 \right] + \left( m + \beta_2 \right)^m \left( 2m + 1 \right) \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right\},
\]

(iv) \( G_{m, \lambda}^{\alpha, \beta} (s^3, x) = \frac{m + \beta_2}{(m + \beta_1)^2} \left[ \left( m - 1 \right) \left( m - 2 \right) \left( m + \beta_2 \right)^2 \left( x - \frac{\alpha_2}{m + \beta_2} \right)^3 \right] + \frac{3(m - 1)(1 + \alpha_1)(m + \beta_2)}{m} \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 + \left( 1 + 3\alpha_1 \right) \left( 1 + \alpha_1 \right) \left( x - \frac{\alpha_2}{m + \beta_2} \right) \right\} \left[ \frac{m}{m + \beta_2} + 2m \left( x - \frac{\alpha_2}{m + \beta_2} \right) + \left( m + \beta_2 \right)^m \left( 2 \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right) \right] + \frac{\alpha_2}{m + \beta_1} \left( m + \beta_2 \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^m \right\}.
\]

\[
\begin{align*}
&= \frac{m + \beta_2}{m + \beta_1} \left[ \frac{3(m - 1)(1 + \alpha_1)(m + \beta_2)}{m} \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 \right] \left( 1 + 3\alpha_1 \right) \left( 1 + \alpha_1 \right) \left( x - \frac{\alpha_2}{m + \beta_2} \right), \\
&+ \frac{\alpha_2}{(m + \beta_1)^3} + \lambda \left\{ \frac{3m}{(m + \beta_1)^3} \left[ \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 - 2 \left( x - \frac{\alpha_2}{m + \beta_2} \right)^3 \right] + \frac{1}{(m + \beta_1)^3} \left[ \frac{m}{m + \beta_2} - 2 \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 \right] + \frac{\alpha_2}{m + \beta_1} \left( m + \beta_2 \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right\} + \frac{\alpha_2}{(m + \beta_1)^3} \left( m + \beta_2 \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^m \right\} + \frac{\alpha_1}{m + \beta_1} \left( \frac{m + \beta_2}{m} \right)^m \left( 3 \alpha_1 \right) \left( m + \beta_2 \right)^{m+1} \left( m + \beta_2 \right), \\
&+ \frac{\alpha_1}{m + \beta_1} \left( m + \beta_2 \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^m \right\} \left( \frac{3(m - 1)}{m + \beta_2} + 6m \left( x - \frac{\alpha_2}{m + \beta_2} \right) \right) + \frac{\alpha_1}{m + \beta_1} \left( \frac{m + \beta_2}{m} \right)^m \left( \alpha_1^2 - 3(\alpha_1 - 1) \right) \left( m + \alpha_3 \right) \left( m + \beta_2 \right)^{m+1} \\
&+ (12m - 6\alpha_1) \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 + \frac{\alpha_1}{m + \beta_1} \left( \frac{m + \beta_2}{m} \right)^m \left( \alpha_1^2 - 3(\alpha_1 - 1) \right) \left( m + \alpha_3 \right) \left( m + \beta_2 \right)^{m+1} \\
&+ 6(m + 1) \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{(m + 1)} \right\} + \frac{3\alpha_1}{(m + \beta_1)^3} \left( m + 1 \right) \left( \frac{m + \beta_2}{m} \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1}.
\end{align*}
\]
\[ G_{n,\alpha}^{\beta}(s^4, x) = \frac{m + \beta_2}{(m + \beta_1)^4} \left[ \frac{(m-1)(m-3)(m+\beta_2)^3}{m^3} \left( x - \frac{\alpha_2}{m + \beta_2} \right)^4 \right. \\
+ \frac{2(m-1)(m-2)(2\alpha_1+3)(m+\beta_2)^2}{m^2} \left( x - \frac{\alpha_2}{m + \beta_2} \right)^3 + \frac{(m-1)(6\alpha_1\alpha_2 + 7)(m+\beta_2)}{m} \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 \\
+ \frac{2(2\alpha_1^3 + 3\alpha_1^2 + 2\alpha_1)}{(m + \beta_1)^4} \left( x - \frac{\alpha_2}{m + \beta_2} \right) \left. \right] \\
+ \frac{\lambda}{(m + \beta_1)^4 (m-1)} \left[ \frac{-m}{m + \beta_2} + 2 \left( x - \frac{\alpha_2}{m + \beta_2} \right) + 44 \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 \\
+ \left( \frac{m + \beta_2}{m} \right)^m \left( 45 \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} - \left( \frac{m + \alpha_2}{m + \beta_2} \right)^{m+1} \right) \right] \\
+ \frac{\alpha_1}{(m + \beta_1)^4 (m-1)} \left[ 12\alpha_1 \left( x - \frac{\alpha_2}{m + \beta_2} \right) + 12m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 - 24m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^3 \right. \\
+ \left( \frac{m + \beta_2}{m} \right)^m \left( 12(m + 2 + \alpha_1) \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} + \alpha_1 \left( x - \frac{\alpha_2}{m + \beta_2} \right) \left( \frac{m + \alpha_2}{m + \beta_2} - x \right)^m \right) \right. \\
+ \frac{\alpha_1}{(m + \beta_1)^4 (m-1)} \left. \left[ 2(2\alpha_1^3 - 3\alpha_1 + 2) \left( \frac{m}{m + \beta_2} - 2 \left( x - \frac{\alpha_2}{m + \beta_2} \right) \right) \right. \\
- 24m(\alpha_1 + 1) \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 + \left( \frac{m + \beta_2}{m} \right)^m \left( 2(2\alpha_1^2 + 9\alpha_1 + 14) \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right. \\
+ \left( \alpha_1^3 - 4\alpha_1^2 + 6\alpha_1 - 4 \right) \left( \frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right] \].
Proof. If we use Bézier basis functions (8) in \( \lambda \)-Bernstein–Stancu operators (7), we obtain

\[
G_{m,\lambda}^{\alpha,\beta}(1, x) = \left( \frac{m + \beta_2}{m} \right) \sum_{j=0}^{m} \left( \frac{m + \beta_2}{m + \beta_1} \right)^{m-j} \left( \frac{j + \alpha_1}{m + \beta_1} \right) \bar{p}_{m,j} (x) = \left( \frac{m + \beta_2}{m} \right)^{m} \sum_{j=0}^{m} \bar{p}_{m,j} (x)
\]

\[
= \left( \frac{m + \beta_2}{m} \right)^{m} \left( \frac{m}{m + \beta_2} \right)^{m} = 1,
\]

\[
G_{m,\lambda}^{\alpha,\beta}(s, x) = \left( \frac{m + \beta_2}{m} \right) \sum_{j=0}^{m} \left( \frac{j + \alpha_1}{m + \beta_1} \right) \bar{p}_{m,j} (\lambda, x)
\]

\[
= \left( \frac{m + \beta_2}{m} \right)^{m} \sum_{j=0}^{m} \left( \frac{j + \alpha_1}{m + \beta_1} \right) \bar{p}_{m,j} (\lambda, x)
\]

\[
= \left( \frac{m + \beta_2}{m} \right)^{m} \left[ \frac{\alpha_1}{m + \beta_1} \left( p_{m,0} (x) - \frac{\lambda}{m+1} p_{m+1,0} (x) \right) \right.
\]

\[
+ \sum_{j=1}^{m-1} \frac{j + \alpha_1}{m + \beta_1} \left( p_{m,j} (x) + \lambda \left( \frac{m - 2j + 1}{m^2 - 1} p_{m+1,j} (x) - \frac{m - 2j - 1}{m^2 - 1} p_{m+1,j+1} (x) \right) \right) \left. \right]\]

\[
+ \frac{m + \alpha_1}{m + \beta_1} \left( p_{m,m} (x) - \frac{\lambda}{m+1} p_{m+1,m} (x) \right) \right]\]

\[
= \left( \frac{m + \beta_2}{m} \right)^{m} \sum_{j=0}^{m} \left( \frac{j + \alpha_1}{m + \beta_1} \right) \bar{p}_{m,j} (x) + \lambda \left\{ Y_{1,m}^{\alpha,\beta} (x) - Y_{2,m}^{\alpha,\beta} \right\},
\]

where

\[
Y_{1,m}^{\alpha,\beta} (x) = \left( \frac{m + \beta_2}{m} \right)^{m} \sum_{j=0}^{m} \left( \frac{j + \alpha_1}{m + \beta_1} \right) m - 2j + 1 m^2 - 1 p_{m+1,j} (x),
\]

\[
Y_{2,m}^{\alpha,\beta} (x) = \left( \frac{m + \beta_2}{m} \right)^{m} \sum_{j=1}^{m-1} \left( \frac{j + \alpha_1}{m + \beta_1} \right) m - 2j - 1 m^2 - 1 p_{m+1,j+1} (x).
\]

Thanks to the linearity of Bernstein–Stancu operators (3), we obtain

\[
\left( \frac{m + \beta_2}{m} \right)^{m} \sum_{j=0}^{m} \frac{j + \alpha_1}{m + \beta_1} \bar{p}_{m,j} (x) = \left( \frac{m + \beta_2}{m + \beta_1} \right)^{m} p_{m+1,0} (x) + \frac{\alpha_1}{m + \beta_1}. \]

Now, we will compute \( Y_{1,m}^{\alpha,\beta} (x) \) and \( Y_{2,m}^{\alpha,\beta} (x) \)

\[
Y_{1,m}^{\alpha,\beta} (x) = \left( \frac{m + \beta_2}{m} \right)^{m} \sum_{j=0}^{m} \left( \frac{j + \alpha_1}{m + \beta_1} \right) m - 2j + 1 m^2 - 1 p_{m+1,j} (x)
\]

\[
= \left( \frac{m + \beta_2}{m} \right)^{m} \frac{1}{(m + \beta_1) (m-1)} \left[ (m + 1) \left( \frac{x - \alpha_2}{m + \beta_2} \right) \sum_{j=0}^{m-1} p_{m,j} + \frac{\alpha_1 m}{m + \beta_1} \sum_{j=0}^{m-1} p_{m+1,j} (x) \right]
\]

\[
- \left( \frac{m + \beta_2}{m} \right)^{m} \frac{2 (m+1)}{(m + \beta_1) (m^2 - 1)} \left[ m \left( \frac{x - \alpha_2}{m + \beta_2} \right) \sum_{j=0}^{m-2} p_{m-1,j} (x) \right]
\]

\[
+ (1 + \alpha_1) \left( \frac{x - \alpha_2}{m + \beta_2} \right) \sum_{j=0}^{m-1} p_{m,j} (x)
\]
\[
\begin{align*}
\chi_{2,m}^{\alpha \beta}(x) &= \frac{1}{m + \beta_1} \left[ \left( x - \frac{\alpha_2}{m + \beta_2} \right) - \left( \frac{m + \beta_2}{m} \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \\
&\quad + \frac{2m}{(m + \beta_1)(m - 1)} \left[ - \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 + \left( \frac{m + \beta_2}{m} \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \\
&\quad + \frac{\alpha_1}{(m + \beta_1)(m - 1)} \left[ -2 \left( x - \frac{\alpha_2}{m + \beta_2} \right) - \left( \frac{m + \beta_2}{m} \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} + \frac{m}{m + \beta_2} \right],
\end{align*}
\]

(13)
Using (12), \( Y_{1,m}^{\alpha\beta}(x) \), and \( Y_{2,m}^{\alpha\beta}(x) \) in \( G_{m,\lambda}^{\alpha\beta}(s, x) \), we obtain the following equation:

\[
G_{m,\lambda}^{\alpha\beta}(s, x) = \frac{m + \beta_2}{m + \beta_1} \left( x - \frac{\alpha_2}{m + \beta_2} \right) + \frac{\alpha_1}{m + \beta_1} \\
\quad + \lambda \left\{ \frac{1}{(m + \beta_1)(m - 1)} \left[ \frac{m + \beta_2}{m + \beta_1} - 2 \left( x - \frac{\alpha_2}{m + \beta_2} \right) + \left( \frac{m + \beta_2}{m} \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \right. \\
\quad \left. + (\alpha_1 - 1) \left( \frac{m + \beta_2}{m + \beta_1} - x \right)^{m+1} \right\} + \frac{\alpha_1}{m + \beta_1} \left( \frac{m + \beta_2}{m} \right)^m \left( x - \frac{\alpha_2}{m + \beta_2} \right) \left( \frac{m + \alpha_2}{m + \beta_2} - x \right)^m. \tag{14} \]

Then, we have the following equality for the third moment by using the linearity of \( G_{m,\lambda}^{\alpha\beta}(g(s), x) \):

\[
G_{m,\lambda}^{\alpha\beta}(s^2, x) = \left( \frac{m + \beta_2}{m} \right)^m \sum_{j=0}^m \left( \frac{j + \alpha_1}{m + \beta_1} \right)^2 \tilde{p}_{m,j}(\lambda, x) \\
= \left( \frac{m + \beta_2}{m} \right)^m \left[ \frac{\alpha_1}{m + \beta_1} \right]^2 \left( \frac{m + \beta_2}{m + \beta_1} - 2 \left( \frac{m + \beta_2}{m} \right)^m \left( \frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right] \\
\quad + \left( \frac{m + \alpha_1}{m + \beta_1} \right)^2 \left( \frac{m + \beta_2}{m + \beta_1} - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \left( \frac{m + \alpha_2}{m + \beta_2} - x \right)^m. \tag{15} \]

where

\[
Y_{3,m}^{\alpha\beta}(x) = \left( \frac{m + \beta_2}{m} \right)^m \sum_{j=0}^m \left( \frac{j + \alpha_1}{m + \beta_1} \right)^2 \frac{m - 2j + 1}{m^2 - 1} \tilde{p}_{m+1,j}(x), \\
Y_{4,m}^{\alpha\beta}(x) = \left( \frac{m + \beta_2}{m} \right)^m \sum_{j=1}^{m+1} \left( \frac{j + \alpha_1}{m + \beta_1} \right)^2 \frac{m - 2j - 1}{m^2 - 1} \tilde{p}_{m+1,j+1}(x). \tag{16} \]

From the linearity of the Bernstein–Stancu operators (3) and Bernstein basis functions, we obtain

\[
\left( \frac{m + \beta_2}{m} \right)^m \sum_{j=0}^m \left( \frac{j + \alpha_1}{m + \beta_1} \right)^2 \tilde{p}_{m,j}(x) \\
= \frac{m + \beta_2}{(m + \beta_1)^2} \left[ \frac{(m - 1)(m + \beta_2)(x - \alpha_2/(m + \beta_2))^2}{m} + (1 + 2\alpha_1) \left( x - \frac{\alpha_2}{m + \beta_2} \right)^2 \right] + \frac{\alpha_1^2}{(m + \beta_1)^2}. \tag{17} \]
Next, we compute $\gamma_{3,m}^{\alpha \beta}(x)$ and $\gamma_{4,m}^{\alpha \beta}(x)$

$$\gamma_{3,m}^{\alpha \beta}(x) = \left(\frac{m + \beta_2}{m}\right)^m \sum_{j=0}^{m} \left(\frac{j + \alpha_1}{m + \beta_1}\right)^2 \frac{m - 2j + 1}{m^2 - 1} \cdot \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-2} \cdot \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-2} p_{m-1,j}(x)$$

$$+ (m + 1)(1 + 2\alpha_1) \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-1} p_{m,j}(x) + \alpha_1 \sum_{j=0}^{m} p_{m+1,j}(x)$$

$$- \left(\frac{m + \beta_2}{m}\right)^m \frac{1}{(m + \beta_1)^4(m - 1)} \left[ m(m + 1) \left(\frac{\alpha_2}{m + \beta_2}\right)^3 \sum_{j=0}^{m-2} p_{m-1,j}(x) \right]$$

$$+ (m + 1)(1 + 2\alpha_1) \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-1} p_{m,j}(x)$$

$$- 2(m + 1) m \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-1} p_{m-1,j}(x)$$

$$+ 2\beta_1 \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-1} p_{m-1,j}(x)$$

$$= \left(\frac{m + \beta_2}{m}\right)^m \sum_{j=0}^{m} \left(\frac{j + \alpha_1}{m + \beta_1}\right)^2 \frac{m - 2j + 1}{m^2 - 1} \cdot \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-2} \cdot \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-2} p_{m-1,j}(x)$$

$$+ (m + 1)(1 + 2\alpha_1) \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-1} p_{m,j}(x) + \alpha_1 \sum_{j=0}^{m} p_{m+1,j}(x)$$

$$- \left(\frac{m + \beta_2}{m}\right)^m \frac{1}{(m + \beta_1)^4(m - 1)} \left[ m(m + 1) \left(\frac{\alpha_2}{m + \beta_2}\right)^3 \sum_{j=0}^{m-2} p_{m-1,j}(x) \right]$$

$$+ (m + 1)(1 + 2\alpha_1) \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-1} p_{m,j}(x)$$

$$- 2(m + 1) m \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-1} p_{m-1,j}(x)$$

$$+ 2\beta_1 \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-1} p_{m-1,j}(x)$$

$$\gamma_{4,m}^{\alpha \beta}(x) = \left(\frac{m + \beta_2}{m}\right)^m \sum_{j=0}^{m} \left(\frac{j + \alpha_1}{m + \beta_1}\right)^2 \frac{m - 2j + 1}{m^2 - 1} \cdot \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-2} \cdot \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-2} p_{m+1,j+1}(x)$$

$$= \left(\frac{m + \beta_2}{m}\right)^m \sum_{j=0}^{m} \left(\frac{j + \alpha_1}{m + \beta_1}\right)^2 \frac{m - 2j + 1}{m^2 - 1} \cdot \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-2} \cdot \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-2} p_{m+1,j}(x)$$

$$- (m + 1)(1 - 2\alpha_1) \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-1} p_{m,j}(x) + (1 - 2\alpha_1 + \alpha_1^2) \sum_{j=0}^{m-1} p_{m+1,j+1}(x)$$

$$- \left(\frac{m + \beta_2}{m}\right)^m \frac{2}{(m + \beta_1)^4(m^2 - 1)} \left[ m(m - 1) \left(\frac{\alpha_2}{m + \beta_2}\right)^3 \sum_{j=0}^{m-3} p_{m-1,j}(x) \right]$$

$$+ 2\alpha_1 \left(\frac{\alpha_2}{m + \beta_2}\right)^2 \sum_{j=0}^{m-2} p_{m-1,j}(x) + (1 - 2\alpha_1 + \alpha_1^2) \sum_{j=0}^{m-1} p_{m,j}(x)$$
Combining (17), $Y^{a,\beta}_{3,m}(x)$, and $Y^{a,\beta}_{4,m}(x)$, we obtain the result for $G^{a,\beta}_{m,\lambda}(s^2, x)$. Using the same technique in the above moments, we obtain $G^{a,\beta}_{m,\lambda}(s^2, x)$ and $G^{a,\beta}_{m,\lambda}(s^4, x)$.

Corollary 1. Using Lemma 1, we obtain the following inequalities of central moments $G^{a,\beta}_{m,\lambda}( (s-x)^n, x)$ for $n=1, 2$ and for fixed $x \in [0,1]$ and $\lambda \in [-1,1]$:

\[
G^{a,\beta}_{m,\lambda}(s-x, x) = \frac{\beta_2 - \beta_1}{m + \beta_1} x + \frac{\alpha_1 - \alpha_2}{m + \beta_1} \\
+ \lambda \left[ -2 + \frac{(m + \beta_2)m}{m + \beta_1} (x - (\alpha_2/(m + \beta_2)))m + (\alpha_1 m - 2 \alpha_1 + 1)((m + \beta_2)/m)((m + \alpha_2)/(m + \beta_2)) - x \right] \\
+ \frac{m + 2 \alpha_2}{(m + \beta_1)(m - 1)(m + \beta_2)} \frac{\alpha_2(m + \beta_2/m)(x - (\alpha_2/(m + \beta_2)))m}{(m + \beta_1)(m - 1)(m + \beta_2)} \\
+ \frac{(\alpha_1 - \alpha_2 - 1)m + 2 \alpha_1 \alpha_2 - \alpha_2}{(m + \beta_1)(m - 1)(m + \beta_2)} \frac{((m + \alpha_2)/(m + \beta_2) - x)m}{(m + \beta_1)(m - 1)(m + \beta_2)} \\
+ \frac{(m + 2 \alpha_2)(\alpha_2 - 1) + (\alpha_1 m - 2 \alpha_1 + 1)((m + \beta_2)/m)((m + \alpha_2)/(m + \beta_2)) - x}{(m + \beta_1)(m - 1)(m + \beta_2)} \right].
\]
\[ C_{m,\lambda}^{\alpha, \beta}(x) = \frac{-m^2 + (\beta_2^2 - 2\beta_2 \beta_1 - 2\beta_2^2) m - \beta_2^2 x^2}{m(m + \beta_1)^2} \]

\[ + \frac{m^2 + (\beta_2 - 2\beta_2 \alpha_2 + 2\alpha_2 + 2\beta_2 \alpha_1 + 2\alpha_2 \beta_1 - 2\beta_1 \alpha_1) m + 2\beta_2 \alpha_2}{m(m + \beta_1)^2} x \]

\[ + \frac{\alpha_2^2 + \alpha_1^2 - \alpha_2 - 2\alpha_1 \alpha_2 m - \alpha_2^2}{m(m + \beta_1)^2} \]

\[ + \lambda \left[ \frac{4\beta_1}{(m + \beta_1)^2 (m - 1)} \right] \]

\[ - \frac{2((m + \beta_2)/m)^{m} (x - \alpha_2/(m + \beta_2))^{m} - (2\alpha_i m - 4\alpha_i + 2) ((m + \beta_2)/m)^{m} ((m + \alpha_2)/(m + \beta_2) - x)^{m}}{(m + \beta_1)(m - 1)} \]

\[ + \frac{(2\beta_2 - 2\beta_1 + 4\alpha_2 - 4\alpha_1) m - 4\alpha_1 \beta_2 - 4\alpha_2 \beta_1}{(m + \beta_1)^2 (m - 1)(m + \beta_2)} \]

\[ + \frac{(2\beta_2 - 2\alpha_2 + 2\alpha_2 \beta_2 + 2\alpha_2 \beta_1)}{(m + \beta_1)^2 (m - 1)(m + \beta_2)} \]

\[ + \frac{(\alpha_2^2 \beta_2 - 2\alpha_2^2 + 2\alpha_2 \alpha_2 \beta_1 - 4\alpha_2 \alpha_2 - 2\alpha_2 \beta_1 + 2\beta_1 + 2\alpha_1 + 2\alpha_2 - 1) m - 2\alpha_2 \beta_2 + 2\alpha_1 \beta_2 - \beta_2 + 2\alpha_2 \beta_1}{(m + \beta_1)^2 (m - 1)(m + \beta_2)} \]

\[ + \frac{(2\alpha_1 \beta_2 - 2\alpha_2 \beta_2 + 4\alpha_2 \alpha_2 - 4\alpha_2^2 - \beta_2 \beta_1 + \beta_2 + 4\alpha_2 \alpha_2 \beta_1)}{(m + \beta_1)^2 (m + \beta_2)(m - 1)} \]

\[ + \frac{(2\alpha_1 \beta_2 - 2\alpha_2 \beta_2 + 4\alpha_2 \alpha_2 - 4\alpha_2^2 - \beta_2 \beta_1 + \beta_2 + 4\alpha_2 \alpha_2 \beta_1)}{(m + \beta_1)^2 (m + \beta_2)(m - 1)} \]

\[ - \frac{(2\alpha_2 m + 2\alpha_2 \alpha_2 + \alpha_2)((m + \beta_2)/m)^{m} (x - \alpha_2/(m + \beta_2))^{m}}{(m + \beta_1)^2 (m + \beta_2)(m - 1)} \]
Using Lemma 1 and the linearity of $G_{m,\lambda}^{\alpha,\beta}(g(s),x)$, we have the following Corollary 2.

**Corollary 2.** We obtain the following equalities:

\( i) \lim_{m \to \infty} G_{m,\lambda}^{\alpha,\beta}(s-x,x) = (\beta_2 - \beta_1)x + \alpha_1x\)

\( ii) \lim_{m \to \infty} G_{m,\lambda}^{\alpha,\beta}((s-x)^2, x) = x - x^2 \)

\( iii) \lim_{m \to \infty} G_{m,\lambda}^{\alpha,\beta}((s-x)^4, x) = 3x^4 - 6x^3 + (24\lambda_2^2 - 24\alpha_2\lambda_1 - 6\lambda_1 + 3)x^2 \)

3. **Convergence Properties of $G_{m,\lambda}^{\alpha,\beta}$**

For the asymptotic behavior of $G_{m,\lambda}^{\alpha,\beta}(g(x),x)$ operators, we give the following Korovkin-type approximation theorem.

**Theorem 1.** If $g \in C[0,1]$ and $\lambda \in [-1,1]$, then $G_{m,\lambda}^{\alpha,\beta}(g, x)$ operators converge uniformly to $g$ on $[0,1]$, where $C[0,1]$ is a Banach space of all continuous functions on $[0,1]$ with norm $\|g\| = \sup_{x \in [0,1]} |g(x)|$.

**Proof.** Using the equalities (i), (ii), and (iii) of Lemma 1, we obtain

\[
\lim_{m \to \infty} \left\| G_{m,\lambda}^{\alpha,\beta}(s^i, x) - x^i \right\| = 0, \quad i = 0, 1, 2.
\]

Therefore, the proof is completed using Korovkin theorem.

For $\epsilon > 0$, the Peetre K-functional is defined as

\[
K_2(g, \epsilon) = \inf_{h \in C^1([0,1])} \left\{ \|g - h\| + \epsilon\|h''\| \right\},
\]
where $C^2[0, 1] = \left\{ h \in C[0, 1] : h, h'' \in C[0, 1] \right\}$. For $g \in C[0, 1]$, the second-order modulus of continuity is defined as

$$\varphi_2(g, \sqrt{\varepsilon}) = \sup_{0 \leq l \leq \sqrt{\varepsilon}} \sup_{h, x \in [l, x+2l] \cap [0, 1]} |g(x+2l) - 2g(x+l) + g(x)|.$$ \hspace{1cm} (22)

By [9, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$K_2(g, \varepsilon) \leq C \varphi_2(g, \sqrt{\varepsilon}).$$ \hspace{1cm} (23)

Also, the usual modulus of continuity of $g \in C[0, 1]$ is defined as $\varphi(g, \varepsilon) = \sup_{0 \leq l \leq \varepsilon} \sup_{x \in [0, 1]} |g(x+l) - g(x)|$. \hfill \square

Theorem 2. We obtain the following inequality for $g \in C[0, 1]$ and $\lambda \in [-1, 1]$:

$$\left| G_{\alpha, \beta, m, \lambda}(g, x) - g(x) \right| \leq 4C\varphi_2\left( g, \sqrt{\left( \frac{\Psi_{\alpha, \beta, m, \lambda}^n(x)}{2} + \Phi_{\alpha, \beta, m, \lambda}^n(x) \right)^2} \right) + \varphi(g, \Psi_{\alpha, \beta, m, \lambda}^n(x)), \hspace{1cm} (24)$$

where $\Psi_{\alpha, \beta, m, \lambda}^n(x)$ and $\Phi_{\alpha, \beta, m, \lambda}^n(x)$ are defined in Corollary 1.

Proof. Let us define the following operators:

$$\overline{G}_{\alpha, \beta, m, \lambda}(g, x) = G_{\alpha, \beta, m, \lambda}(g, x) - g \left( \frac{m + \beta_2}{m + \beta_1} x + \frac{\alpha_1 - \alpha_2}{m + \beta_1} \right) + \lambda \left( \frac{1}{m + \beta_1} \right) \left( \frac{m}{m + \beta_2} \right) \left( x - \frac{\alpha_2}{m + \beta_2} \right) \hspace{1cm} (25)$$

From the linearity of $\overline{G}_{\alpha, \beta, m, \lambda}(g, x)$ and the equalities (i) and (ii) of Lemma 1, we obtain

$$\overline{G}_{\alpha, \beta, m, \lambda}(s-x, x) = 0. \hspace{1cm} (26)$$

Using Taylor’s expansion for $h \in C^2[0, 1]$, we write

$$\overline{G}_{\alpha, \beta, m, \lambda}(h, x) = h(x) + \overline{G}_{\alpha, \beta, m, \lambda}\left( \int_x^s (s-u)h''(u)du, x \right)$$

Applying generalized $\lambda$-Bernstein–Stancu operators to both sides of (27) and using (26), we yield

$$h(s) = h(x) + h'(x)(s-x) + \int_x^s (s-u)h''(u)du. \hspace{1cm} (27)$$

Using Taylor’s expansion for $h \in C^2[0, 1]$, we write

$$\left| \overline{G}_{\alpha, \beta, m, \lambda}(h, x) - h(x) \right| \leq \int_x^s \left| \left( \frac{G_{\alpha, \beta, m, \lambda}^n(s, u) - G_{\alpha, \beta, m, \lambda}^n(s, x)}{2} \right)^2 + \frac{h''}{2} \left( G_{\alpha, \beta, m, \lambda}^n(s - x)^2 \right) \right| du, x \hspace{1cm} (28)$$

$$\leq \frac{h''}{2} \left( G_{\alpha, \beta, m, \lambda}^n(s - x)^2 \right)^2 + \frac{h''}{2} \left( G_{\alpha, \beta, m, \lambda}^n(s - x)^2 \right)^2, x \hspace{1cm} (29)$$

$$\leq \frac{h''}{2} \left( \left( \Psi_{\alpha, \beta, m, \lambda}^n(x) \right)^2 + \Phi_{\alpha, \beta, m, \lambda}^n(x) \right)^2. \hspace{1cm} (30)$$
And, we find

\[
|G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| \leq |G_{m,\lambda}^{\alpha,\beta}(g - h, x) - (g - h)(x)| + |G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| \\
+ |g(G_{m,\lambda}(s, x)) - g(x)|
\]

(29)

Therefore, if we take infimum on the right side of (29), overall \( h \in C^2[0, 1] \), we obtain

\[
|G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| \leq 4K_2 \left( g, \frac{\left( \Psi_{m,\lambda}^{\alpha,\beta}(x) \right)^2 + \Phi_{m,\lambda}^{\alpha,\beta}(x)}{8} \right) + \phi(g, \Psi_{m,\lambda}^{\alpha,\beta}(x)).
\]

(30)

Using inequality (23), we have

\[
|G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| \leq 4C_2 \left( g, \frac{\left( \Psi_{m,\lambda}^{\alpha,\beta}(x) \right)^2 + \Phi_{m,\lambda}^{\alpha,\beta}(x)}{8} \right) + \phi(g, \Psi_{m,\lambda}^{\alpha,\beta}(x)).
\]

(31)

Thus, Theorem 2 is proved.

**Remark 1.** Since \( \lim_{m \to 0} \Psi_{m,\lambda}^{\alpha,\beta}(x) = 0 \) and \( \lim_{m \to 0} \Phi_{m,\lambda}^{\alpha,\beta}(x) = 0 \) for \( \forall x \in [0, 1] \), these limits give us a rate of pointwise convergence of the operators \( G_{m,\lambda}^{\alpha,\beta}(g, x) \) to \( g(x) \).

The space of the Lipschitz-type functions is defined as

\[
\operatorname{Lip}_x(\xi) = \{ g \in C[0, 1]: |g(y) - g(x)| \leq \kappa |y - x|^\xi; \}
\]

\[
x, y \in \mathbb{R},
\]

where \( \kappa > 0 \) and \( 0 < \xi \leq 1 \) [10].

In the following theorem, we obtain the rate of convergence of generalized \( \lambda \)-Bernstein–Stancu operators \( G_{m,\lambda}^{\alpha,\beta}(g, x) \) for functions in \( \operatorname{Lip}_x(\xi) \).

**Theorem 3.** If \( g \in \operatorname{Lip}_x(\xi), x \in [0, 1], \) and \( \lambda \in [-1, 1], \) then we have

\[
|G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| \leq \kappa \left[ \Phi_{m,\lambda}^{\alpha,\beta}(x) \right]^{\xi/2},
\]

(33)

where \( \Phi_{m,\lambda}^{\alpha,\beta}(x) \) is defined in Corollary 1.

**Proof.** Because \( g \in \operatorname{Lip}_x(\xi) \) and \( G_{m,\lambda}^{\alpha,\beta}(g, x) \) are linear positive operators, we obtain the inequality by using Hölder’s inequality:

\[
|G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| \leq G_{m,\lambda}^{\alpha,\beta}(|g(s) - g(x)|, x) = \left( \frac{m + \beta_2}{m} \right)^m \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda, x) \left| \frac{j + \alpha_1}{m} \right|^{\xi/2} - g(x)
\]

(34)

Hence, we proved Theorem 3.
Finally, we give the main result of the article in Theorem 4.

**Theorem 4.** If \( g \in C[0, 1] \), then for every \( x \in (0, 1) \) and \( \lambda \in [-1, 1] \), we obtain

\[
\lim_{m \to \infty} m \left[ G_{m, \lambda}^\alpha (g, x) - g(x) \right] = g'(x) \left[ (\beta_2 - \beta_1) x + \alpha_1 - \alpha_2 \right] + \frac{g''(x)}{2} (x - x^2)
\]

where \( g''(x) \) exists.

**Proof.** Using Taylor’s formula, we can write the following equation for a fixed \( x \in [0, 1] \):

\[
g(s) = g(x) + g'(x)(s - x) + \frac{g''(x)}{2} (s - x)^2 + r(s, x) (s - x)^2,
\]

where \( r(s, x) \) is Peano form of the remainder, \( r(s, x) \in C[0, 1] \), and \( \lim_{x \to x} r(s, x) = 0 \). If we apply \( G_{m, \lambda}^\alpha (g, x) \) to (36), then we have

\[
\lambda = -1,
\quad \alpha_1 = \alpha_2 = 1,
\quad \beta_1 = \beta_2 = 2
\]

\[
\lambda = 1,
\quad \alpha_1 = \alpha_2 = 1,
\quad \beta_1 = \beta_2 = 2
\]

**Figure 1:** Convergence of \( G_{m, -1}^{1, 2} (f, x) \) to \( f(x) \) with respect to \( m \) values.

**Figure 2:** Convergence of \( G_{m, 1}^{1, 2} (f, x) \) to \( f(x) \) with respect to \( m \) values.
\[
G_{m,1}^{\alpha, \beta} (g, x) - g (x) = g' (x) G_{m,1}^{\alpha, \beta} ((s - x), x) + \frac{g'' (x)}{2} G_{m,1}^{\alpha, \beta} ((s - x)^2, x),
\]
\[
+ G_{m,1}^{\alpha, \beta} (r (s, x) (s - x)^2, x),
\]
\[
\lambda = -1,
\alpha_1 = \alpha_2 = 10,
\beta_1 = \beta_2 = 11
\]

Figure 3: Convergence of \( G_{m,1}^{10,11} (f, x) \) to \( f (x) \) with respect to \( m \) values.

\[
\lambda = 1,
\alpha_1 = \alpha_2 = 10,
\beta_1 = \beta_2 = 11
\]

Figure 4: Convergence of \( G_{m,1}^{10,11} (f, x) \) to \( f (x) \) with respect to \( m \) values.
and we obtain

\[ \lim_{m \to \infty} \left[ G_{m,\lambda}^r(g, x) - g(x) \right] = g'(x) \lim_{m \to \infty} m G_{m,\lambda}^r((s-x), x) \]

\[ + \frac{g''(x)}{2} \lim_{m \to \infty} m G_{m,\lambda}^r((s-x)^2, x) + \lim_{m \to \infty} m G_{m,\lambda}^r(r(s-x)(s-x)^2, x). \]  

(38)

Using Cauchy–Schwarz inequality, we obtain

\[ C_m^r \left( (s-x)^2, x \right) \leq \left( C_m^r \left( (s-x)^4, x \right) \right)^{1/2} \]

\[ \cdot \left( G_m^r \left( r(s-x)(s-x)^2, x \right) \right)^{1/2}. \]  

(39)

Because \( \lim_{s \to x} r(s-x) = 0 \) and \( G_m^r((s-x)^4, x) \) are finite operators, we have

\[ \lim_{m \to \infty} G_m^r \left( r(s-x)(s-x)^2, x \right) = 0. \]  

(40)

In the end, by using the equalities (i) and (ii) of Corollary 2 and (40) in (38), we yield

\[ \lim_{m \to \infty} \left[ G_{m,\lambda}^r(g, x) - g(x) \right] = g'(x) \left[ (\beta_2 - \beta_1)x + \alpha_1 - \alpha_2 \right] \]

\[ + \frac{g''(x)}{2} (x-x^2). \]  

(41)

Thus, we proved Theorem 4.

4. Numerical Examples

In this section, we show the theoretical results demonstrated in the previous sections by the following example.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Convergences of \( G_{m,\lambda}^{10,11}(f, x) \), \( B_{m,\lambda}^{20,10,11}(f, x) \), and \( B_{m,\lambda}^{20}(f, x) \) to \( f(x) \).}
\end{figure}

**Example 1.** Let the trigonometric function \( f(x) = 1 - \cos(4e^x) \) for \( x \in A_m = [\alpha_2/m + \beta_2, \ (m \pm \alpha_2/m + \beta_2)] \), \( 0 \leq \alpha_i \leq \beta_i \ (i = 1, 2) \), and \( m = 10, 20, 50, 100 \). First, let us choose \( \alpha_1 = \alpha_2 = 10 \) and \( \beta_1 = \beta_2 = 2 \).

For \( \lambda = -1 \), the graphs of \( G_{m,\lambda}^{10,11}(f, x) \) and \( f(x) \) are shown in Figure 1.

For \( \lambda = 1 \), the graphs of \( G_{m,\lambda}^{10,11}(f, x) \) and \( f(x) \) are shown in Figure 2.

Now, let us choose \( \alpha_1 = \alpha_2 = 10 \) and \( \beta_1 = \beta_2 = 11 \). The graph of \( f(x) \) and the graph of \( G_{m,\lambda}^{10,11}(f, x) \) with \( \lambda = -1 \) and \( \lambda = 1 \) are shown in Figures 3 and 4, respectively.

As a result, Figure 5 reveals that the curve of \( G_{m,\lambda}^{10,11}(f, x) \) with \( \alpha_1 = \alpha_2 = 10 \) and \( \beta_1 = \beta_2 = 11 \) studied in the article approaches the curve of the function \( f(x) \) much better than the curves of the operators \( B_{m,\lambda}^{20}(f, x) \) and \( B_{m,\lambda}^{20,10,11}(f, x) \) defined in [5, 8], respectively.

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
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