Bilinear expansions of lattices of KP $\tau$-functions in BKP $\tau$-functions: a fermionic approach

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Dedicated to Grigori Olshanski, on the occasion of his 70th birthday

Abstract

We derive a bilinear expansion expressing elements of a lattice of KP $\tau$-functions, labelled by partitions, as a sum over products of elements of an associated lattice of BKP $\tau$-functions, labelled by strict partitions. This generalizes earlier results relating determinants and Pfaffians of minors of skew symmetric matrices, with applications to Schur functions and Schur $Q$-functions. It is deduced using the representations of KP and BKP $\tau$-functions as vacuum expectation values (VEV’s) of products of fermionic operators of charged and neutral type, respectively. The lattice is generated by insertion of products of pairs of charged creation and annihilation operators. The result follows from expanding the product as a sum of monomials in the neutral fermionic generators and applying a factorization theorem for VEV’s of products of operators in the mutually commuting subalgebras. Applications include the case of inhomogeneous polynomial $\tau$-functions of KP and BKP type.

1 Introduction

In [2], a bilinear expansion was derived, expressing determinants of submatrices of a skew symmetric matrix as sums over products of Pfaffians of their principal minors. This was shown to follow from the relation between the Plücker [6] and Cartan [11] embeddings of Grassmannians and orthogonally isotropic Grassmannians in the projectivization of an exterior space. Using fermionic methods, a function theoretic realization of this identity was derived in [9], expressing Schur functions $s_{\lambda}$, labelled by integer partitions $\lambda$, as sums over products of pairs $Q_{\mu^+}Q_{\mu^-}$ of Schur $Q$-functions, labelled by pairs $(\mu^+, \mu^-)$ of strict...
partitions \cite{17}. Schur functions and Schur $Q$-functions are the basic building blocks for $\tau$-functions of the KP and BKP integrable hierarchies, and the simplest case of this bilinear relation is the well-known fact that the square of any BKP $\tau$-function is the restriction of a KP $\tau$-function to the odd flow variables \cite{4,11,16}.

The integrable KP and BKP hierarchies are closely related to the $A_\infty$ and $B_\infty$ root lattices, respectively \cite{11}. In this work we define, for every GL($\infty$) group element $\hat{g}$, an $A_\infty$ lattice of KP $\tau$-functions, labelled by partitions, and for every SO($\infty$) group element $\hat{h}$, a $B_\infty$ lattice of BKP lattice $\tau$-functions, labelled by strict partitions. When $\hat{g}$ and $\hat{h}$ are suitably related (eq. (2.58), we show (Theorem 3.3) how a generalization of the bilinear expansions of \cite{2} and \cite{9} follows, expressing the elements of the lattice of KP $\tau$-functions as sums over products of pairs of $\tau$-functions from the associated lattice of BKP type. Varying the choice of infinite orthogonal group element $\hat{h}$ determining the BKP $\tau$-function at the origin of the lattice gives all possible functional realizations of such bilinear relations.

Two key ingredients underlying this result are: the identification of polarizations (Section 3.2) associated with a given partition, which are just pairs $(\mu^+, \mu^-)$ of strict partitions sharing the same intersections and unions, together with the factorization Lemma \cite{2,2} which allows us to express vacuum expectation values (VEV’s) of products of mutually anticommuting sets of neutral fermionic operators $\{\phi_j^\pm\}_{j\in\mathbb{Z}}$ as products of VEV’s of the two separate types. The results of \cite{2} and \cite{9} are recovered as special cases, together with other examples involving more general lattices of KP and BKP $\tau$-functions, in particular, those consisting of inhomogeneous polynomial $\tau$-functions.

2 Fermionic Fock space, creation and annihilation operators, KP and BKP $\tau$-functions

2.1 Charged fermions and KP flows

We recall some definitions and notations for free fermions and KP flows \cite{7,11,18}. Given a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$ and its dual space $\mathcal{H}^*$, with dual basis $\{e^j\}_{j\in\mathbb{Z}}$ satisfying

$$e^j(e_k) = \delta_{jk},$$

the fermionic Fock space $\mathcal{F}$ is defined as the $\mathbb{Z}$-graded semi-infinite wedge product

$$\mathcal{F} := \Lambda^{\infty/2}\mathcal{H} = \sum_{n\in\mathbb{Z}} \mathcal{F}_n,$$

where $\mathcal{F}_n$ denotes the $n$th fermionic charge sector. The elements of $\mathcal{F}$ are denoted by ket vectors $|\cdot\rangle$, while those of the dual space $\mathcal{F}^*$ are denoted by bra vectors $\langle\cdot|$. Let

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, 0, 0 \ldots)$$

be an extended integer partition of length $\ell(\lambda)$, consisting of a weakly decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0$ of $\ell(\lambda)$ positive integers, followed by an infinite sequence
\((\{\ell(\lambda)+j = 0\}_{j=1,2,\ldots} of 0's. An orthonormal basis \{|\lambda; n\rangle\} for \mathcal{F}_n is provided by the elements

\[|\lambda; n\rangle = e_{l_1} \wedge e_{l_2} \wedge \cdots,\]

where \((l_1, l_2, \ldots)\) is a strictly decreasing sequence of integers, called particle positions, defined in terms of the partition \(\lambda\) by

\[l_j = \lambda_j - j + n, \quad j = 1, 2, \ldots\]

which, after the first \(\ell(\lambda)\) elements, saturate to the sequence of all subsequent decreasing integers, \(n - \ell(\lambda), n - \ell(\lambda) - 1, \ldots\). The dual basis vectors are denoted \(\langle \lambda; n|\), and satisfy

\[\langle \lambda; n| \mu; m \rangle = \delta_{\lambda, \mu} \delta_{nm}.\]

The vacuum states in each sector \(\mathcal{F}_n\) correspond to the trivial partition \(\emptyset := (0, 0, \ldots)\) and are denoted

\[|n\rangle := |\emptyset; n\rangle.\]

The irreducible representation

\[\Gamma: \mathcal{C}_{\mathcal{H}+\mathcal{H}^*}(Q) \to \text{End}(\mathcal{F})\]

\[\Gamma: \xi \mapsto \Gamma_\xi\]

of the Clifford algebra \(\mathcal{C}_{\mathcal{H}+\mathcal{H}^*}(Q)\) algebra on \(\mathcal{H} + \mathcal{H}^*\) with respect to the scalar product

\[Q(v + \mu, w + \nu) := \nu(v) + \mu(w), \quad v, w \in \mathcal{H}, \quad \mu, \nu \in \mathcal{H}^*\]

is generated by scalars and linear elements, the latter acting by exterior and interior multiplication:

\[\Gamma_v := v \wedge, \quad \Gamma_\mu := i_\mu.\]

The fermionic creation and annihilation operator are representations of the basis elements:

\[\psi_j := \Gamma_{e_j} = e_j \wedge, \quad \psi_j^\dagger := \Gamma_{e_j} := i_{e_j}\]

and satisfy the anticommutation relations

\[\left[\psi_j, \psi_k\right]_+ = 0, \quad \left[\psi_j^\dagger, \psi_k^\dagger\right]_+ = 0, \quad \left[\psi_j, \psi_k^\dagger\right]_+ = \delta_{jk}.\]

For \(j > 0\), \(\psi_{-j}\) and \(\psi_{j-1}^\dagger\) (resp. \(\psi_{-j}^\dagger\) and \(\psi_{j-1}\)) annihilate the right (resp. left) vacua:

\[\psi_{-j}|0\rangle = 0, \quad \psi_{j-1}^\dagger|0\rangle = 0, \quad \forall j > 0,\]

\[\langle 0|\psi_{-j}^\dagger = 0, \quad \langle 0|\psi_{j-1} = 0, \quad \forall j > 0.\]

The basis states may be expressed by acting with a product of pairs of operators \(\psi_j \psi_k^\dagger\) on the vacuum:

\[|\lambda; 0\rangle := |\lambda\rangle = (-1)^{\sum_{j=1}^{r} \beta_j} \prod_{j=1}^{r} (\psi_{\alpha_j}(t)\psi_{-\beta_{j-1}}^\dagger(t)) |0\rangle,\]

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where \((\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r)\) are the Frobenius indices \([17]\) of the partition \(\lambda\).

The fermionic representation of elements \(g \in \text{GL}_0(\mathcal{H})\) of the connected component of the group \(\text{GL}(\mathcal{H})\) can be expressed as exponentials of \(\mathfrak{gl}(\mathcal{H})\) algebra elements, which are linear combinations of normal ordered products \(:\psi_j \psi^\dagger_k:\) of creation and annihilation operators \([11]\)

\[
\hat{g}(\vec{A}) = e^{\sum_{j,k \in \mathbb{Z}} \hat{A}_{jk} \psi_j \psi^\dagger_k},
\]

where normal ordering
\[
:\hat{L}_1 \hat{L}_2:\ = \hat{L}_1 \hat{L}_2 - \langle 0 | \hat{L}_1 \hat{L}_2 | 0 \rangle.
\]

of a product of linear elements \(\hat{L}_1 \hat{L}_2\) is defined so the vacuum expectation value (VEV) vanishes. There are various conventions on restricting the permissible values of the doubly infinite matrix elements \(\hat{A}_{jk}\), depending on the class of solutions to the KP hierarchy one wishes to include \([11, 25, 26]\).

The KP flow parameters, denoted
\[
\mathbf{t} = (t_1, t_2, \ldots),
\]

may be interpreted as linear exponential coordinates on an infinite abelian subgroup \(\Gamma_+ \subset \text{GL}(\mathcal{H})\) defined by

\[
\Gamma_+ = \{ \gamma_+(\mathbf{t}) := e^{\sum_{j=1}^{\infty} t_j \Lambda^j} \},
\]

where \(\Lambda \in \text{End}(\mathcal{H})\) is the shift operator which, acts on basis elements by

\[
\Lambda(e_j) = e_{j-1}.
\]

The KP flows are generated by the action of \(\Gamma_+\) on the infinite Grassmannian \(\text{Gr}_{\mathcal{H}^+}(\mathcal{H})\) of subspaces \(w \subset \mathcal{H}\) commensurate with the subspace

\[
\mathcal{H}^+ = \text{span}\{e_{-j}\}_{j \in \mathbb{N}^+},
\]

satisfying certain admissibility conditions \([25, 26]\). They are mapped into the projectivization \(\mathbf{P}(\mathcal{F})\) of \(\mathcal{F}\) by the Plücker map:

\[
\mathfrak{P} : \text{span}\{w_j\}_{j \in \mathbb{N}^+} \rightarrow [w_1 \wedge w_2 \wedge \cdots],
\]

where \(\text{span}\{w_j\}_{j \in \mathbb{N}^+}\) is any admissible basis of \(w \in \text{Gr}_{\mathcal{H}^+}(\mathcal{H})\). The image \(\mathfrak{P}(\mathcal{H}^+)\) of \(\mathcal{H}^+ \subset \mathcal{H}\) is the projectivized vacuum state \([|0\rangle]\), and the Plücker map intertwines the action of \(\Gamma_+\) induced on \(\text{Gr}_{\mathcal{H}^+}(\mathcal{H})\) with the (projectivized) fermionic representation of \(\Gamma_+ \subset \text{GL}(\mathcal{H})\) on \(\mathcal{F}\), given by

\[
\Gamma_{\gamma_+}(\mathbf{t}) = \hat{\gamma}_+(\mathbf{t}) := e^{\sum_{j=1}^{\infty} t_j J_j},
\]

where

\[
J_j := \sum_{k \in \mathbb{Z}} \psi_k \psi^\dagger_{k+j}, \quad j = 1, 2, \ldots
\]
are the positive Fourier components of the current operator \([11]\), which mutually commute
\[
[J_n, J_m] = 0, \quad n, m \in \mathbb{N}^+.
\] (2.25)

By (2.13), the current components \(J_n\) annihilate the vacuum \(|0\rangle\):
\[
J_n|0\rangle = 0, \quad n \in \mathbb{N}^+
\] (2.26)
and hence \(\Gamma_+\) stabilizes the vacuum
\[
\hat{\gamma}_+(t)|0\rangle = |0\rangle.
\] (2.27)

For each \(g \in \text{GL}_0(\mathcal{H})\), we define the corresponding KP \(\tau\)-function
\[
\tau^\text{KP}_g(t) = \langle 0|\hat{\gamma}_+(t) \hat{g}|0\rangle,
\] (2.28)
which satisfies the Hirota bilinear residue equations \([11][25]\) of the KP hierarchy. More
generally, \(\hat{g}\) need not be a \(\text{GL}_0(\mathcal{H})\) group element, but any element of the Clifford algebra
satisfying the bilinear commutation relation
\[
[I, \hat{g} \otimes \hat{g}] = 0,
\] (2.29)
where \(I\) is the endomorphism of \(\mathcal{F} \otimes \mathcal{F}\) defined by
\[
I := \sum_{j \in \mathbb{Z}} \psi_j \otimes \psi_j^\dagger.
\] (2.30)

**2.2 Neutral fermions and BKP flows**

Neutral fermions \(\phi_j^+\) and \(\phi_j^-\) are defined \([4][5][11][12][27]\) by
\[
\phi_j^+ := \frac{\psi_j + (-1)^j \psi_j^\dagger}{\sqrt{2}}, \quad \phi_j^- := \frac{i \psi_j - (-1)^j \psi_j^\dagger}{\sqrt{2}}, \quad j \in \mathbb{Z}
\] (2.31)
(where \(i = \sqrt{-1}\), and satisfy
\[
[\phi_j^+, \phi_k^-]_+ = 0, \quad [\phi_j^+, \phi_j^+]_+ = [\phi_j^-, \phi_j^-]_+ = (-1)^j \delta_{j+k,0}.
\] (2.32)
In particular,
\[
(\phi_0^+)^2 = (\phi_0^-)^2 = \frac{1}{2}.
\] (2.33)

Acting on the vacua \(|0\rangle\) and \(|1\rangle\), we have
\[
\phi_j^+|0\rangle = \phi_j^-|0\rangle = \phi_j^+|1\rangle = \phi_j^-|1\rangle = 0, \quad \forall j > 0, \quad \forall j > 0,
\] (2.34)
\[
\langle 0|\phi_j^+ = \langle 0|\phi_j^- = \langle 1|\phi_j^+ = \langle 1|\phi_j^- = 0, \quad \forall j > 0,
\] (2.35)
\[
\phi_0^+|0\rangle = -i \phi_0^-|0\rangle = \frac{1}{\sqrt{2}} \psi_0|0\rangle = \frac{1}{\sqrt{2}}|1\rangle,
\] (2.36)
\[
\langle 0|\phi_0^+ = i \langle 0|\phi_0^- = \frac{1}{\sqrt{2}} \langle 0|\psi_0^\dagger = \frac{1}{\sqrt{2}} \langle 1|.
\] (2.37)
Let $\tilde{\mathcal{H}}^{\pm} \subset \mathcal{H} + \mathcal{H}^*$ be the two mutually orthogonal subspaces defined by

$$
\tilde{\mathcal{H}}^+_0 := \text{span}\{f^+_j := \frac{1}{\sqrt{2}}(e_j + (-1)^{-j}e^{-j})\}_{j \in \mathbb{Z}},
$$

$$
\tilde{\mathcal{H}}^-_0 := \text{span}\{f^-_j := \frac{i}{\sqrt{2}}(e_j - (-1)^{-j}e^{-j})\}_{j \in \mathbb{Z}}.
$$

Restricting the scalar product $Q$ to each of these subspaces, we have the induced scalar products $Q_{\pm}$ on $\tilde{\mathcal{H}}^\pm_0$ defined by

$$
Q_+(f^+_j, f^+_k) := Q(f^+_j, f^+_k) = (-1)^{j} \delta_{j+k,0},
$$

$$
Q_-(f^-_j, f^-_k) := Q(f^-_j, f^-_k) = (-1)^{j} \delta_{j+k,0}.
$$

We thus have two mutually commuting fermionic representations of the orthogonal algebras $\mathfrak{o}(\tilde{\mathcal{H}}^\pm, Q_{\pm})$ defined by the span of the bilinear elements $\{\phi_j^\pm \phi_k^\pm\}_{j,k \in \mathbb{Z}}$.

The fermionic representations of elements $h^\pm \in \text{SO}(\tilde{\mathcal{H}}^\pm, Q_{\pm})$ in the connected component of the corresponding mutually commuting orthogonal groups $\text{SO}(\tilde{\mathcal{H}}^\pm, Q_{\pm})$, are defined by exponentiation $[4,5,16]$

$$
\hat{h}^\pm(A) := \frac{1}{2} \sum_{j,k \in \mathbb{Z}} A_{jk} \phi_j^\pm \phi_k^\pm,
$$

where $\{A_{jk}\}_{j,k \in \mathbb{Z}}$ are the elements of a doubly infinite skew symmetric matrix $A$. Let $\mathcal{F}_{\phi^\pm} \subset \mathcal{F}$ be the two subspaces of $\mathcal{F}$ defined as

$$
\mathcal{F}_{\phi^+} := \text{span}\{|\hat{\alpha}^+\rangle\}, \quad \mathcal{F}_{\phi^-} := \text{span}\{|\hat{\alpha}^-\rangle\}
$$

where

$$
|\alpha^+\rangle := \prod_{j=1}^{r} \phi^+_{\alpha_j} |0\rangle, \quad |\alpha^-\rangle := \prod_{j=1}^{r} \phi^-_{\alpha_j} |0\rangle.
$$

Then $\mathcal{F}_{\phi^+}$ and $\mathcal{F}_{\phi^-}$ are invariant under the fermionic representations (2.40) of the groups $\text{SO}(\tilde{\mathcal{H}}^\pm, Q_{\pm})$.

The positive neutral fermion current components $J^B_+ \, j$ and $\hat{J}^B_- \, j$ are defined as

$$
J^B_+ := \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^{k+1} \phi^+_{\hat{\alpha}_{j-k}} \phi^-_{\hat{\alpha}_{j-k}}, \quad J^B_- := \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^{k+1} \phi^-_{\hat{\alpha}_{j-k}} \phi^+_{\hat{\alpha}_{j-k}}, \quad j \in \mathbb{N}^+.
$$

from which it follows that the even components $\{J^B_{2j} \}$ vanish while the odd ones mutually commute:

$$
[J^B_{2j-1}, J^B_{2j-1}] = 0, \quad [J^B_{2j-1}, J^B_{2k-1}] = 0, \quad [J^B_{2j-1}, J^B_{2k-1}] = 0, \quad j, k \in \mathbb{N}^+.
$$

It also follows, from (2.34), that the neutral current components $J^B_\pm \, j$ annihilate the vacua $|0\rangle$ and $|1\rangle$:

$$
J^B_+ |0\rangle = 0, \quad J^B_- |0\rangle = 0, \quad J^B_+ |1\rangle = 0, \quad J^B_- |1\rangle = 0, \quad j \in \mathbb{N}^+.
$$

and, from (2.24), (2.43), that:
Lemma 2.1. For odd \( n = 2j - 1 \), we have
\[
J_{2p-1} = J_{2j-1}^B + J_{2j-1}^B, \quad j \in \mathbb{N}^+.
\] (2.46)

The BKP flow parameters are denoted
\[
t_B := (t_1, t_3, \ldots),
\] (2.47)
and these may be viewed as determining a subset \( \{t'\} \) of the \( t \)’s , where
\[
t' := (t_1, 0, t_3, 0, t_5, \ldots).
\] (2.48)

Following [4,5,11,16,27], we define two mutually commuting abelian groups of BKP flows
\[
\Gamma^B^+ = \{ \gamma^B^+(t_B^+) \} \quad \text{and} \quad \Gamma^B^- = \{ \gamma^B^-(t_B^-) \},
\] (2.49)
with Clifford representations
\[
\hat{\gamma}^B^+(t_B) := e^{\sum_{j=0}^{\infty} J_B^+ 2 j - 1 t_{2j-1}} \quad \text{and} \quad \hat{\gamma}^B^-(t_B) := e^{\sum_{j=1}^{\infty} J_B^- 2 j - 1 t_{2j-1}}.
\] (2.50)

By (2.45), these both stabilize the vacua \( |0\rangle \) and \( |1\rangle \)
\[
\hat{\gamma}^B^+(t_B) |0\rangle = |0\rangle, \quad \hat{\gamma}^B^+(t_B) |1\rangle = |1\rangle;
\hat{\gamma}^B^-(t_B) |0\rangle = |0\rangle, \quad \hat{\gamma}^B^-(t_B) |1\rangle = |1\rangle.
\] (2.51)

It also follows that
\[
\hat{\gamma}_\pm (t') = \hat{\gamma}_\pm^B+(t_B) \hat{\gamma}_\pm^B^-(t_B).
\] (2.52)

The BKP \( \tau \)-function, defined as
\[
\tau^B_{\hat{h}}(|\hat{\alpha}\rangle) = \langle 0 | \hat{\gamma}^B^+(t_B) \hat{h}^+ |0\rangle = \langle 0 | \hat{\gamma}^B^-(t_B) \hat{h}^- |0\rangle,
\] (2.53)
then satisfies the BKP Hirota bilinear residue equations [4,5,11].

More generally, \( \hat{h}^\pm \) need not be SO(\( \mathcal{H}^\pm \)) group elements for (2.52) to define a BKP \( \tau \)-function. Define \( \mathcal{I}_B^{\pm} \in \text{End} (\mathcal{F}_\phi^\pm \otimes \mathcal{F}_\phi^\pm) \) as
\[
\mathcal{I}_B^{\pm} := \sum_{j \in \mathbb{Z}} (-1)^j \phi_0^\pm \otimes \phi_0^\pm,
\] (2.54)
and let \( P^{\pm} \in \text{End} (\mathcal{F}_\phi^\pm) \) be defined on the bases \( \{|\alpha^\pm\rangle\} \) as
\[
P^{\pm} |\alpha^\pm\rangle := (-1)^{m(\alpha^\pm)} |\alpha^\pm\rangle,
\] (2.55)

where \( m(\alpha^\pm) \) is the length (i.e. the cardinality) of \( \alpha^\pm \). Then (2.52) defines a BKP \( \tau \)-function [4,5,7,11] provided the bilinear relations
\[
\mathcal{I}_B^{\pm} |v^\pm\rangle \otimes |v^\pm\rangle = (\phi_0 \otimes \phi_0) P^{\pm} |v^\pm\rangle \otimes P^{\pm} |v^\pm\rangle.
\] (2.56)

are satisfied by the elements
\[
|v^\pm\rangle = \hat{h}^\pm |0\rangle.
\] (2.57)

We also have the following key factorization lemma [2,11,27].
Lemma 2.2 (Factorization). If $U^+$ and $U^−$ are either even or odd degree elements of the subalgebra generated by the operators $\{\phi_i^+\}_{i \in \mathbb{Z}}$ and $\{\phi_i^-\}_{i \in \mathbb{Z}}$ respectively, the VEV of their product can be factorized as:

$$\langle 0 | U^+ U^- | 0 \rangle = \begin{cases} 
\langle 0 | U^+ | 0 \rangle \langle 0 | U^- | 0 \rangle & \text{if } U^+ \text{ and } U^- \text{ are both of even degree} \\
0 & \text{if } U^+ \text{ and } U^- \text{ have different parity} \\
2i\langle 0 | U^+ \phi_0^+ | 0 \rangle \langle 0 | U^- \phi_0^- | 0 \rangle & \text{if } U^+ \text{ and } U^- \text{ are both of odd degree}
\end{cases}$$

(2.57)

Defining

$$g(h(A)) = h^+(A)h^-(A), \quad \hat{g}(h(A)) := \hat{h}^+(A)\hat{h}^-(A)$$

(2.58)

and using the relations (2.31) between neutral and charged fermions and the mutual commutativity of the algebras $o(\hat{H}^\pm, Q^\pm)$, we can re-express $\hat{g}$ as the fermionic representation of an element of $GL(H)$

$$\hat{g}(A) = \hat{g}(h(A)) = e^{\sum_{j, k \in \mathbb{Z}} (-1)^k A_j, -k^j \psi_j \psi_j^\dagger}$$

(2.59)

where

$$\hat{A} = (-1)^k A_j, -k$$

(2.60)

Comparing formulae (2.28) and (2.52) for the KP $\tau$-function $\tau_{\delta(h)}^{KP}(t')$ and the BKP $\tau$-function $\tau_{\delta(h)}^{BKP}(t_B)$, using (2.51) and (2.58) and Lemma 2.2, we obtain the following well-known result [4, 5]

$$\tau_{\delta(h)}^{KP}(t') = \left(\tau_{\delta(h)}^{BKP}(t_B)\right)^2.$$  

(2.61)

The next section is aimed at obtaining the general bilinear relation expressing a lattice of KP $\tau$-functions, labelled by partitions, associated with $\tau_{\delta(h)}^{KP}(t')$, as sums over products of the elements of a lattice of BKP $\tau$-functions, labelled by strict partitions, similarly associated with $\tau_{\delta(h)}^{BKP}(t_B)$

3 Bilinear expansions of KP $\tau$-function lattices in BKP $\tau$-functions

In this section, we define lattices of KP and BKP $\tau$-functions related to a given pair $\tau_{\delta(h)}^{KP}(t), \tau_{\delta(h)}^{BKP}(t_B)$, labelled by partitions and strict partitions, respectively, and derive a bilinear expansion for the first of these in terms of the second.

3.1 Lattices of KP and BKP $\tau$-functions

In the following, we view the KP and BKP flow variables $t$ and $t_B$, which are the arguments of our $\tau$-functions, as independent coordinates. To make contact with the usual
definitions of Schur functions and Schur Q-functions, as symmetric polynomials, or symmetric functions of a finite or infinite set of variable \( \{x_j\}_{j \in \mathbb{N}^+} \), we recall that the variables \( t \) and \( t_B \) may be restricted to equal the (normalized) power sums in these,

\[
t_i = \frac{1}{i} \sum_{j=1}^{N} x_j^i,
\]

(3.1)

(where \( N \) could be a finite positive integer or \( \infty \)). We then denote the infinite sequence \( t = (t_1, t_2 \ldots) \) as

\[
t = [x] := ([x]_1, [x]_2, \ldots), \quad [x]_i := \frac{1}{i} \sum_{j=1}^{N} x_j^i, \quad i \in \mathbb{N}^+.
\]

(3.2)

However, when we specialize to the values \( t = t' \), in which all the even flow variables vanish, we cannot interpret these as the usual power sums \((3.2)\). Instead, in order to restrict these to symmetric polynomials, we use the *supersymmetric* parametrization \((10)\)

\[
t' = [y] - [-y], \quad t_{2i-1} = \frac{2}{2i-1} \sum_{j=1}^{N} y_j^{2i-1},
\]

(3.3)

where \( y = (y_1, y_2, \ldots) \) is an auxiliary (finite or infinite) sequence of independent variables.

For any KP \( \tau \)-function \( \tau^g_{KP}(t) \), we associate an \( A_\infty \) lattice of KP \( \tau \)-functions

\[
\pi_{(\alpha|\beta)}(g)(t) := \langle 0|\hat{\gamma}_+(t)\hat{g}\rangle \lambda,
\]

(3.4)

labelled by partitions \( \lambda = (\alpha|\beta) \) and, for any BKP \( \tau \)-function \( \tau^h_{BKP}(t_B) \), a \( B_\infty \) lattice of BKP \( \tau \)-functions

\[
\kappa_\alpha(h)(t_B) := \langle 0|\hat{\gamma}^+(t_B)\hat{h}^+|\alpha^+\rangle = \langle 0|\hat{\gamma}^-(t_B)\hat{h}^-|\alpha^-\rangle,
\]

(3.5)

labelled by strict partitions \( \alpha \) of even cardinality \( r \).

**Remark 3.1.** Note that if we change the elements \( \hat{g} \rightarrow \hat{g}_0\hat{g} \) and \( \hat{h}^\pm \rightarrow \hat{h}_0\hat{h} \) in \((3.4)\) and \((3.5)\) by right multiplication by elements \( \hat{g}_0, \hat{h}_0 \) of the stabilizer of the vacuum \( |0\rangle \), this has no effect on the \( \tau \)-functions \( \tau^g_{KP}(t) \) and \( \tau^h_{BKP}(t_B) \), but will change the other elements \( \pi_{(\alpha|\beta)}(g)(t), \kappa_\alpha(h)(t_B) \) of the associated lattices.

By Wick’s theorem, \( \pi_{(\alpha|\beta)}(g)(t) \) is the determinant of the matrix \( \pi_{(\alpha_j|\beta_k)}(g)(t) \) corresponding to hook partitions \((\alpha_j|\beta_k)\)

\[
\pi_{(\alpha|\beta)}(g)(t) = \det \left( \pi_{(\alpha_j|\beta_k)}(g)(t) \right)_{1 \leq j, k \leq r},
\]

(3.6)

and \( \kappa_\alpha(h)(t_B) \) is the Pfaffian of the skew matrix \( \kappa_{(\alpha_j,\alpha_k)}(h)(t_B) \) corresponding to the Frobenius rank \( r = 2 \) strict partitions \( \alpha = (\alpha_j, \alpha_k) \):

\[
\kappa_\alpha(h)(t_B) = \text{Pf} \left( \kappa_{(\alpha_j,\alpha_k)}(h)(t_B) \right)_{1 \leq j, k \leq r}.
\]

(3.7)

where, for \( \alpha_k \geq \alpha_j \), \( \kappa_{(\alpha_j,\alpha_k)}(h)(t_B) \) is defined by

\[
\kappa_{(\alpha_j,\alpha_k)}(h)(t_B) := -\kappa_{(\alpha_k,\alpha_j)}(h)(t_B).
\]

(3.8)
3.2 Polarizations

**Definition 3.1.** A polarization of $(\alpha|\beta)$, is a pair $(\mu^+, \mu^-)$ of strict partitions with cardinalities (or lengths)

$$m(\mu^+) := \#(\mu^+), \quad m(\mu^-) := \#(\mu^-)$$

(3.9)

(including possibly a zero part $\mu^+_{m(\mu^+)} = 0$ or $\mu^-_{m(\mu^-)} = 0$), satisfying

$$\mu^+ \cap \mu^- = \alpha \cap I(\beta), \quad \mu^+ \cup \mu^- = \alpha \cup I(\beta),$$

(3.10)

where

$$I(\beta) := (I_1(\beta), \ldots I_r(\beta))$$

(3.11)

is the strict partition [17] with parts

$$I_j(\beta) = \beta_j + 1, \quad j = 1, \ldots r.$$  

(3.12)

The set of all polarizations of $(\alpha|\beta)$ is denoted $\mathcal{P}(\alpha, \beta)$.

We denote the strict partition obtained by intersecting $\alpha$ with $I(\beta)$ as

$$S := \alpha \cap I(\beta),$$

(3.13)

and its cardinality as

$$s := \#(S).$$

(3.14)

Since both $\alpha$ and $I(\beta)$ have cardinality $r$, it follows that

$$m(\mu^+) + m(\mu^-) = 2r,$$

(3.15)

so $m(\mu^\pm)$ must have the same parity. It is easily verified [9] that the cardinality of $\mathcal{P}(\alpha, \beta)$ is $2^{2r-2s}$. The following was proved in [9].

**Lemma 3.1 (Binary sequence associated to a polarization).** For every polarization $\mu := (\mu^+, \mu^-)$ of $\lambda = (\alpha|\beta)$ there is a unique binary sequence of length $2r$

$$\epsilon(\mu) = (\epsilon_1(\mu), \ldots, \epsilon_{2r}(\mu)),$$

(3.16)

with

$$\epsilon_j(\mu) = \pm, \quad j = 1, \ldots 2r,$$

(3.17)

such that

1. The sequence of pairs

$$((\alpha_1, \epsilon_1(\mu)), \ldots (\alpha_r, \epsilon_r(\mu)), (\beta_1 + 1, \epsilon_{r+1}(\mu)), \ldots, (\beta_r + 1, \epsilon_{2r}(\mu))$$

(3.18)

is a permutation of the sequence

$$((\mu^+_1, +), \ldots (\mu^+_{m(\mu^+)}, +), (\mu^-_1, -), \ldots, (\mu^-_{m(\mu^-)}, -))$$

(3.19)
\begin{align}
\epsilon_j(\mu) = + & \quad \text{if } \alpha_j \in S, \quad \text{and } \epsilon_{r+j}(\mu) = - \quad \text{if } \beta_j + 1 \in S, \quad j = 1, \ldots, r. \tag{3.20}
\end{align}

**Definition 3.2.** The *sign* of the polarization \((\mu^+, \mu^-)\), denoted \(\text{sgn}(\mu)\), is defined as the sign of the permutation that takes the sequence (3.18) into the sequence (3.19).

Denote by
\[
\pi(\mu^\pm) := \#(\alpha \cup \mu^\pm)
\]
the cardinality of the intersection of \(\alpha\) with \(\mu^\pm\). It follows that
\[
\pi(\mu^+) + \pi(\mu^-) = r + s. \tag{3.22}
\]

We then have

**Lemma 3.2.**
\[
|\lambda\rangle = \frac{(-1)^{\frac{1}{2}r(r+1)+s}}{2^{r-s}} \sum_{\mu \in \mathcal{P}(\alpha, \beta)} \text{sgn}(\mu)(-1)^{\pi(\mu^-)} i^{m(\mu^-)} \prod_{j=1}^{m(\mu^+)} \phi_{\mu_j}^+ \prod_{k=1}^{m(\mu^-)} \phi_{\mu_k}^- |0\rangle. \tag{3.23}
\]

**Proof.** In eq. (2.15), reorder the product over the factors \(\psi_{\alpha_j} \psi_{-\beta_j-1}^\dagger\) so the \(\psi_{\alpha_j}\) terms precede the \(\psi_{-\beta_j-1}^\dagger\) ones, giving an overall sign factor \((-1)^{\frac{1}{2}r(r-1)}\). Then substitute
\[
\psi_{\alpha_j} = \frac{1}{\sqrt{2}}(\phi_{\alpha_j}^+ - i\phi_{\alpha_j}^-), \quad \psi_{-\beta_j-1}^\dagger = \frac{(-1)^j}{\sqrt{2}}(\phi_{\beta_j+1}^+ + i\phi_{\beta_j+1}^-), \quad j \in \mathbb{Z}, \tag{3.24}
\]
which follows from (2.31), for all the factors, and expand the product as a sum over monomial terms of the form
\[
\prod_{j=1}^{m(\mu^+)} \phi_{\mu_j}^+ \prod_{k=1}^{m(\mu^-)} \phi_{\mu_k}^- |0\rangle. \tag{3.25}
\]
Taking into account the sign factor \(\text{sgn}(\mu)\) corresponding to the order of the neutral fermion factors, as well as the powers of \(-1\) and \(i\), and noting that there are \(2^s\) resulting identical terms, then gives (3.23). \(\square\)

**Definition 3.3.** *Supplemented partitions.* If \(\nu\) is a strict partition of cardinality \(m(\nu)\) (with 0 allowed as a part), we define the associated *supplemented partition* \(\hat{\nu}\) to be
\[
\hat{\nu} := \begin{cases} 
\nu & \text{if } m(\nu) \text{ is even,} \\
(\nu, 0) & \text{if } m(\nu) \text{ is odd.} 
\end{cases} \tag{3.26}
\]
We denote by \(m(\hat{\nu})\) the cardinality of \(\hat{\nu}\).
3.3 Bilinear expansion theorem

Combining the above, we arrive at our main result.

**Theorem 3.3.** Restricting the group element \( \hat{g} \) to \( \hat{g}(h) = \hat{h}^+ \hat{h}^- \) and the \( t \) values to \( t' \), the lattice of KP \( \tau \)-functions \( \pi_{(\alpha|\beta)}(g(h))(t') \) may be expressed as sums over products of pairs of BKP \( \tau \)-functions \( \kappa_{\mu^+}(h)(t_B) \) and \( \kappa_{\mu^-}(h)(t_B) \) as follows

\[
\pi_{(\alpha|\beta)}(g(h))(t') = (-1)^{\frac{1}{2}}\frac{r}{2r-s} \sum_{\mu \in \mathcal{P}(\alpha|\beta)} \text{sgn}(\mu)(-1)^{\pi(\mu^-) + \frac{1}{2}m(\mu^-)} \kappa_{\mu^+}(h)(t_B) \kappa_{\mu^-}(h)(t_B).
\]

(3.27)

**Proof.** Substituting (3.23) into (3.4), using (3.5) and (2.42) and applying the factorizations (2.58), (2.51) and Lemma 2.2 gives the bilinear expansion (3.27). \( \square \)

4 Examples

The first two examples, 4.1 and 4.2, show how the results of [2] and [9] are recovered by choosing special values for the parameters in Theorem 3.3. Examples (4.3 and 4.4) give families of lattices of polynomial \( \tau \)-functions of KP and BKP type that generalize the Schur functions and Schur \( Q \)-functions [3, 8, 10, 12–14]. These include, as special cases: factorial, shifted and interpolation Schur functions [3, 19, 20] interpolation Schur \( Q \)-functions [10], and multivariate Laguerre polynomials [15, 20].

**Example 4.1 (Determinants of skew matrices as bilinear sums over Pfaffians).** The bilinear identity in [2] follows as a particular cases of Theorem 3.3. Setting \( t = 0 = (0, 0, \ldots) \) in (3.27), the \( \tau \)-function \( \pi_{(\alpha|\beta)}(0)(g) \) becomes the \( (\alpha|\beta) \) Plücker coordinate of the element \( g^T(H) \) of the Grassmannian \( \text{Gr}_H(H + H^*) \) of subspaces of \( H + H^* \) commensurate with the subspace \( H \) which is mapped, under the Plücker map, to the vacuum \( |0\rangle \).

In the big cell of the Grassmannian, which consists of those elements that are the graph of a map \( w : H \to H^* \), it is the determinant of minors of the affine coordinate matrix \( A \) which, for \( g \) of the form (2.58) is skew symmetric. And \( \kappa_{\mu^\pm}(0_B) \) are the Cartan coordinates [2] of the element \( h^T(H) \) of the maximal isotropic Grassmannian \( \text{Gr}^0_H(H + H^*, Q) \), which is the Pfaffian of the principle minors of the affine coordinate matrix \( A \) corresponding to the strict partitions \( \mu^\pm \). This reproduces the expansions of the determinants of minors of a skew matrix as sums over products of Pfaffians of their principal minors given by Corollary 6.4 of [2].

\[
\det(A_{(I|J)}) = (-1)^{\frac{1}{2}(r+1)+s} \sum_{(K,L) \in \mathcal{P}(I,J)} (-1)^{\pi + s/2} \text{sgn}(K,L) \text{Pf}(A_{(K|K)}) \text{Pf}(A_{(L|L)}),
\]

(4.1)

where \( A \) is a skew symmetric \( N \times N \) matrix, \( I \) and \( J \) are the rows and columns

\[
I = (I_1, \ldots, I_r) = I(\alpha), \quad J = (J_1, \ldots, I_r) = I(\beta)
\]

(4.2)
of the \( r \times r \) minor \( A_{[I,J]} \),

\[
K = (K_1, \ldots, K_k) = I(\mu^+), \quad L = (L_1, \ldots, L_l) = I(\mu^-)
\]

are the rows and columns of the pair of principal minors \( A_{[K|K]} \) and \( A_{[L|L]} \), and

\[
\pi := \#(I \cap L), \quad k = m(\mu^+) = \#(K), \quad l = m(\mu^-) = \#(L),
\]

The ordered subsets \( I, J, K, L \subset \{1, \ldots, N\} \) are polarizations related by

\[
I \cap J = K \cap L := S, \quad I \cup J = K \cup L,
\]

which, by the Giambelli identity (which is a particular case of (3.6)), is the determinant of the matrix formed from Schur functions corresponding to hook partitions, whose restriction to \( t' \) equals the skew matrix formed from the Schur \( Q \)-functions \( Q_{jk}(t_B) \) with Frobenius rank 2. Then \( \kappa_{\hat{\mu}}(I) \) becomes the (scaled) Schur \( Q \)-functions

\[
Q_{\hat{\mu}}(t_B) = \kappa_{\hat{\mu}}(I)(t_B) = \langle 0|\hat{\gamma}^+(t)|\lambda \rangle,
\]

which, by (3.7), are the Pfaffians of the principal minors of the same skew symmetric matrix \( \hat{Q}_{\mu,\mu}(t_B) \).

Example 4.2 (Schur functions as bilinear sums over Schur \( Q \)-functions). If we set \( h \) equal to the identity element \( I \in O(\mathcal{H} + \mathcal{H}^*, Q) \), the \( \tau \)-function \( \pi_{\alpha|\beta}(t')(I) \) becomes the Schur function

\[
s_{\alpha|\beta}(t) = \pi_{\alpha|\beta}(I)(t) = \langle 0|\hat{\gamma}^+(t)|\lambda \rangle,
\]

which, by (3.27), therefore reduces to the result obtained in \[9\], expressing Schur functions as sums over products of Schur \( Q \)-functions.

Eq. (3.27) therefore reduces to the result obtained in \[9\], expressing Schur functions as sums over products of Schur \( Q \)-functions.

\[
s_{\alpha|\beta}(t') = \frac{(-1)^{1/2}r(r+1)+s}{2^{2r-s}} \sum_{\mu \in \mathcal{P}(\alpha,\beta)} \text{sgn}(\mu)(-1)^{\pi(\mu^-)+\frac{1}{2}m(\mu^-)} Q_{\mu^+} \left( \frac{1}{2}t_B \right) Q_{\mu^-} \left( \frac{1}{2}t_B \right).
\]

Example 4.3. (Lattices of polynomial KP vs. BKP \( \tau \)-functions). A broad family of KP and BKP \( \tau \)-function lattices consisting of more general (inhomogeneous) polynomials in the flow variables is obtained if special restrictions are placed on the matrices \( \tilde{A} \) relating them to those appearing in (2.16) and (2.40). Such polynomial KP \( \tau \)-functions, their BKP analogs and their fermionic representations were studied in \[8, 10, 13, 14, 24\].
expressible as finite linear combinations of ordinary Schur and Schur \( Q \)-functions. We introduce the following fermionic operators
\[
\psi_i(\tilde{A}) = \hat{g}(\tilde{A}) \psi_i \left( \hat{g}(\tilde{A}) \right)^{-1}, \quad \psi^*_i(\tilde{A}) = \hat{g}(\tilde{A}) \psi^*_i \left( \hat{g}(\tilde{A}) \right)^{-1}
\]  
(4.9)
where \( \hat{g}(\tilde{A}) \) is defined in (2.10), and
\[
\phi^+_i(A) = \hat{h}^+(A) \phi^+_i(\hat{h}^+(A))^{-1},
\]  
(4.10)
where \( \hat{h}^±(A) \) is defined in (2.40).

**Lemma 4.1.** If the infinite skew matrix \( A \) is chosen to satisfy the antidiagonal triangular condition:
\[
A_{j,-k} = 0 \quad \text{if} \quad j > k
\]  
(4.11)
and hence \( \tilde{A} \) is has the upper triangular form
\[
\tilde{A}_{jk} = \begin{cases} 
(-1)^k A_{j,-k} & \text{if} \quad j \leq k \\
0 & \text{if} \quad j > k,
\end{cases}
\]  
(4.12)
then
\[
\hat{g}(\tilde{A}) = \hat{h}^+(A) \hat{h}^-(A)
\]  
(4.13)
and
\[
\psi_j(\tilde{A}) = \frac{1}{\sqrt{2}} (\phi^+_j(A) - i\phi^-_j(A)), \quad \psi^*_j(\tilde{A}) = \frac{(-1)^j}{\sqrt{2}} (\phi^+_j(A) + i\phi^-_j(A))
\]  
(4.14)

**Proof.** This follows from the definitions (2.16) and (2.40) of \( \hat{g}(\tilde{A}) \) and \( \hat{h}^±(A) \), the relation (2.31) between charged and neutral fermionic operators, the skew symmetry of the matrix \( A \), which leads to the cancellation of the sum over mixed terms \( \phi^+_j \phi^-_k \) in the exponent, and the mutual anticommutativity of the two types of neutral fermi operators \( \{\phi^+_j\}_{j \in \mathbb{Z}} \) and \( \{\phi^-_j\}_{j \in \mathbb{Z}} \)[4]

To define a lattice of polynomial KP \( \tau \)-functions that generalize the Schur polynomials we note that, because \( \tilde{A} \) is upper triangular, \( \hat{g}(h(A)) \) stabilizes the vacuum \(|0\rangle\)
\[
\hat{g}(\tilde{A})|0\rangle = |0\rangle.
\]  
(4.15)

It follows from (3.6) that
\[
\pi_{(\alpha|\beta)}(g(h(A)))(t) = \det \left( \langle 0|\hat{\gamma}_+(t)\hat{g}(\tilde{A})\psi_\alpha\psi^*_\beta_{-1}|0\rangle \right)_{i,j=1,...,r} = \langle 0|\hat{\gamma}_+(t)\hat{g}(\tilde{A})|\lambda\rangle.
\]  
(4.16)
Since both \( \hat{\gamma}_+(t) \) and \( \hat{g}(\tilde{A}) \) stabilize the vacuum, we may define
\[
s_\lambda(t|\tilde{A}) := \pi_{(\alpha|\beta)}(g(h(A)))(t)
\]  
= \((-1)^{\sum \beta_j} (-1)^{\frac{r(r-1)}{2}} \langle 0|\hat{\gamma}_+(t)\psi_\alpha_1(\tilde{A}) \cdots \psi_\alpha_r(\tilde{A})\psi^*_\beta_{-1}(\tilde{A}) \cdots \psi^*_\beta_{-1}(\tilde{A})|0\rangle
\]

14
\[= (-1)^{\sum_{j=1}^{r} \beta_j} \det \left( \langle 0 | \hat{\gamma}_+ (t) \psi_{\alpha_i} (\hat{A}) \psi_{-\beta_j - 1}^* (\hat{A}) | 0 \rangle \right)_{i,j=1, \ldots, r} \]  

(4.17)

and these can be interpreted as generalized Schur functions (cf. eqs. (2.15), 4.6).

Note that, as for ordinary Schur functions [17] we have, for any \( p = (p_1, p_2, \ldots) \),

\[ s_{\lambda}(t + p | \hat{A}) = \sum_{\rho \subseteq \lambda} s_{\lambda/\rho} (p | \hat{A}) s_{\rho}(t), \]

(4.18)

where the generalized skew Schur function is defined fermionically as

\[ s_{\lambda/\rho}(t | \hat{A}) := (-1)^{\sum_{j=1}^{r} \beta_j} (\lambda)(\rho \sum_{j=1}^{r} \beta_j - 1)(\hat{A}) \cdot \cdot \cdot (\hat{A}) | 0 \rangle \]

(4.19)

This follows from factorizing

\[ \hat{\gamma}_+ (t + p) = \hat{\gamma}_+ (t) \hat{\gamma}_+ (p) \]

(4.20)

in (4.16) and inserting the projection operator

\[ \sum_{\rho} | \rho; 0 \rangle \langle \rho; 0 | \]

(4.21)

onto the \( n = 0 \) fermionic charge sector between the factors. It follows from (4.19) that \( s_{\lambda/\rho}(t | \hat{A}) \) is, indeed, a polynomial in the variables \( t = (t_1, t_2, \ldots) \) since, setting \( p = 0 \),

\[ s_{\lambda}(t | \hat{A}) = \sum_{\rho \subseteq \lambda} s_{\lambda/\rho} (0 | \hat{A}) s_{\rho}(t). \]

(4.22)

Next, we define the (scaled) generalized Schur \( Q \)-functions to be

\[ Q_{\hat{\mu}}^\pm (t_B | A) := \kappa_\alpha (h(A))(t_B) = \langle 0 | \hat{\gamma}_B^\pm (t_B) \phi_{\hat{\mu}}^\pm (A) \cdot \cdot \cdot \phi_{\hat{\mu}}^\pm (m(\hat{A})) | 0 \rangle \]

\[ = \text{Pf} \left( \langle 0 | \hat{\gamma}_B^\pm (t_B) \phi_{\hat{\mu}}^\pm (A) \cdot \cdot \cdot \phi_{\hat{\mu}}^\pm (m(\hat{A})) | 0 \rangle \right)_{i,j=1, \ldots, m(\hat{A})} \]

\[ := 2^{-\frac{1}{2} m(\hat{\mu})} Q_{\hat{\mu}}^\pm \left( \frac{1}{2} t_B | A \right)\]

(4.23)

(Recall that \( \hat{\mu} \) denotes the \textit{supplemented partition}, of even cardinality, which is obtained from \( \mu \) by adding a 0 part if it has odd cardinality.) These too are polynomials in the BKP flow variables \( t_B \) since, like (4.22), we may express \( Q_{\hat{\mu}}^\pm (t_B | A) \) as a finite linear combination of the Schur \( Q \)-functions \( Q_{\hat{\nu}}(t_B) \).

\[ Q_{\hat{\mu}}^\pm (t_B | A)(p_B)) = \sum_{\nu \subseteq \hat{\mu}} Q_{\hat{\mu}/\nu}(0 | A) Q_{\nu}(t_B), \]

(4.24)

Restricting to this particular case of Theorem 3.3 gives:
Corollary 4.2. The generalized Schur function $s_{(\alpha|\beta)}$ (4.17), evaluated at $t = t'$, may be expressed as the following sum over products $Q_{\hat{\mu}^+}Q_{\hat{\mu}^-}$ of pairs of generalized $Q$ Schur functions (4.23):

$$s_{(\alpha|\beta)}(t'|A) = \frac{(-1)^{\frac{1}{2} r(r+1)+s}}{2^{2s}} \sum_{\mu \in P(\alpha,\beta)} \text{sgn}(\mu)(-1)^{\pi(\mu^-) + \frac{1}{2} m(\hat{\mu}^-)} Q_{\hat{\mu}^+}(\frac{1}{2}t_B|A)Q_{\hat{\mu}^-}(\frac{1}{2}t_B|A).$$

(4.25)

Remark 4.1. To interpret the results in terms of symmetric polynomials in a set of auxiliary variables, and consistently be able to choose the specialization $t = t'$ in (4.25), we must reinterpret them as supersymmetric power sums, as in (3.3).

A particularly simple case consists of the double $D(\alpha)$ of a strict partition $\alpha$, for which the Frobenius indices $(\alpha|\beta)$ are related by $\alpha = I(\beta)$. Since the only polarization in this case is itself, the sum consists of just one term, and we obtain (as for ordinary Schur functions)

$$s_{D(\alpha)}(t'|A) = 2^{-r} \left( Q_{\hat{\alpha}}(\frac{1}{2}t_B|A^\pm) \right)^2.$$  

(4.26)

Example 4.4. (Lattices of polynomial $\tau$-functions parametrized by $r$ and $p$). An interesting subclass of generalized Schur functions and Schur $Q$-functions is obtained if we choose the skew matrix $A$ to be parametrized in terms two sets of infinite parameters $r := \{r_j\}_{j \in \mathbb{Z}}$ and $p = (p_1, p_2, p_3, \ldots)$, where we assume the relations

$$r_j = r_{1-j}, \quad j \in \mathbb{N}.\quad (4.27)$$

Let

$$\hat{A}_j^r := \sum_{k \in \mathbb{Z}} \psi_k \psi_{k+j}^\dagger r_k \cdots r_{k+j}, \quad j = 1, 2, 3, \ldots\quad (4.28)$$

$$\hat{A}_j^r := \sum_{k \in \mathbb{Z}} (-1)^{k+1} \phi_k^\pm \phi_{-k+j}^\pm r_k \cdots r_{k+j}, \quad j = 1, 3, 5, \ldots\quad (4.29)$$

$$\hat{A}^r(p) := \sum_{j>0} p_j \hat{A}_j^r =: \sum_{j,k \in \mathbb{Z}} \hat{A}^r(p)_{jk} \psi_j \psi_k^\dagger,$$  

(4.30)

$$\hat{A}^r(p_B) := \sum_{j>0, \text{odd}} p_j \hat{A}_j^r \pm =: \frac{1}{2} \sum_{j,k \in \mathbb{Z}} A^r(p_B)_{jk} \phi_j^\pm \phi_k^\pm,$$ 

(4.31)

where

$$\hat{A}^r(p)_{jk} = p_{k-j} r_j \cdots r_k, \quad k > j,$$

(4.32)

$$A^r(p_B)_{jk} = \begin{cases} (-1)^k p_{j-k} r_j \cdots r_k, & j + k < 0 \text{ and odd} \\ 0 & \text{if } j + k \text{ is even or } j + k \geq 0. \end{cases}$$

(4.33)
and define the group elements
\[ \hat{g}(\tilde{A}^r(p)) := e^{\tilde{A}^r(p)}, \quad \hat{h}^\pm(A^r(p_B)) := e^{\tilde{A}^r(p_B)}. \] (4.34)

We then have:

**Proposition 4.3.**

\[ s_\lambda(t|\tilde{A}^r(p)) = \pi(\alpha|\beta)(g(\tilde{A}^r(p)))(t) = \sum_{\rho \leq \lambda} r_{\lambda/\rho} s_{\lambda/\rho}(p) s_\rho(t), \] (4.35)

where \( s_{\lambda/\rho}(p) \) is the skew Schur function corresponding to the skew partition \( \lambda/\rho \), and

\[ r_{\lambda/\rho} := \prod_{i=1}^{\ell(\lambda)} \prod_{j=\rho_i+1}^{\lambda_i} r_{j-i} = \frac{r_\lambda}{r_\rho} \] (4.36)

is the content product over nodes of the skew Young diagram of \( \lambda/\rho \), and

\[ \mathcal{Q}_{\hat{\mu}}(t_B|A^r(p_B)) =\kappa_{\hat{\mu}}(g(A^r(p_B)))(t_B) = \sum_{\nu \leq \hat{\mu}} r_{\mu/\nu}^B \mathcal{Q}_{\hat{\mu}/\nu}(p_B) \mathcal{Q}_\nu(t_B), \] (4.37)

where

\[ \mathcal{Q}_{\hat{\mu}/\nu}(p_B) = (\nu^+|\gamma_B^\pm(p_B)|\hat{\mu}^+) \] (4.38)

is the (scaled) skew Schur Q-function corresponding to the skew partition \( \mu^+/\nu \) and

\[ r_{\mu/\nu}^B := \prod_{i=1}^{m(\mu^+)} \prod_{j=\nu_i+1}^{\mu_i^+} r_j = \frac{r_{\mu^+}}{r_{\nu^+}} \] (4.39)

is the content product over nodes of the shifted skew Young diagram [10] of \( \mu^+/\nu \).

**Proof.** This follows along the lines of [21, 23]. Define the operators

\[ \hat{C}^r := \exp \left( \sum_{j<0} T_j^r \psi_j^\dagger \psi_j - \sum_{i \geq 0} T_i^r \psi_i \psi_i^\dagger \right) \] (4.40)

where

\[ r_j = e^{T_{j-1}^r - T_j^r}, \quad T_0^r := 0. \] (4.41)

Imposing the restriction

\[ T_j^r = -T_{-j}^r, \] (4.42)

implies

\[ r_j = r_{1-j}. \] (4.43)

We also define

\[ \hat{C}^{r \pm} := \exp \sum_{j \geq 0} (-1)^{j+1} T_j^r \phi_j^{\pm} \phi_j^{\pm}. \] (4.44)
It then follows from (2.31) that
\[ \hat{C}^r = \hat{C}^{r-} + \hat{C}^{r+}. \] (4.45)

From (2.12) and (2.32) we get
\[ \hat{C}^r \psi_j (\hat{C}^r)^{-1} = e^{-T^r_j \psi_j}, \quad \hat{C}^{r\pm} (\hat{C}^{r\pm})^{-1} = e^{-T^r_j \phi_j^\pm}. \] (4.46)

It follows that the operators defined in (4.28, (4.29), (4.31) are given by
\[ \hat{A}^r_j := \hat{C}^r J^r_j (\hat{C}^r)^{-1}, \quad j = 1, 2, \ldots \] (4.47)
and
\[ \hat{A}^{r\pm}_j := \hat{C}^{r\pm} J^{B\pm}_j (\hat{C}^{r\pm})^{-1}, \quad j = 1, 3, 5, \ldots \] (4.48)

For odd \( j \) we have
\[ \hat{A}^r_i = \hat{C}^r (J^B_i + J^B_{i+2}) (\hat{C}^r)^{-1} = \hat{A}^{r+}_i + \hat{A}^{r-}_i, \quad i = 1, 3, 5, \ldots \] (4.49)

Defining
\[ \hat{\gamma}^r_+(p) := (\hat{C}^r)^{-1} \hat{\gamma}_+(p) \hat{C}^r, \quad \hat{\gamma}^{r,B\pm}_+(p_B) := (\hat{C}^{r\pm})^{-1} \hat{\gamma}^{B\pm}_+(p_B) \hat{C}^{r\pm}, \] (4.50)

it follows from (4.45) and (2.51) that
\[ \hat{\gamma}^r_+(p) = \hat{\gamma}^{r,B+}_+(p_B) \hat{\gamma}^{r,B-}_+(p_B). \] (4.51)

The equalities
\[ s_\lambda (t | A^r(p)) = \langle 0 | \hat{\gamma}_+(t) \hat{\gamma}^r_+(p) | \lambda \rangle \] (4.52)
and
\[ Q_{\mu\pm} (t_B | A^{r\pm}(p_B)) = \langle 0 | \hat{\gamma}_\pm(t_B) \hat{\gamma}^{r,B\pm}(p_B) | \mu \pm \rangle. \] (4.53)

follow from definitions (4.17), (4.23), when the group elements \( \hat{g}(h(A)) \) and \( \hat{h}^\pm(A) \) are chosen as in (4.34) and \( \psi_i(A) \) and \( \phi_j^\pm(A) \) computed by inserting these in (4.9) and (4.10).

Inserting the projection operator
\[ \Pi_0 = \sum_{\rho \in \mathcal{P}} | \rho; 0 \rangle \langle \rho; 0 | \] (4.54)
onto the subspace \( \mathcal{F}_0 \subset \mathcal{F} \) between \( \hat{\gamma}_+(t) \) and \( \hat{\gamma}^r_+(p) \) in (4.52), and using
\[ s_\lambda (t) = \langle 0 | \hat{\gamma}_+(t) | \lambda \rangle \] (4.55)
and the slightly more general relation
\[ r_{\lambda/\rho} s_{\lambda/\rho} (p) = \langle \rho | \hat{\gamma}^r_+(p) | \lambda \rangle \] (4.56)
gives (4.35). (See formula (3.5.7) in [22]).
Similarly, to prove (4.37), insert the projection operators

$$\Pi^\pm = \sum_{\nu^\pm} |\nu^\pm\rangle\langle \nu^\pm|$$  \hspace{1cm} (4.57)

onto the subspaces $\tilde{\mathcal{F}}^\pm \subset \mathcal{F}$ between $\tilde{\gamma}^{B\pm}(t_B)$ and $\gamma^{r\cdot B\pm}(p_B)$ in (4.53) and use

$$Q_{\mu^\pm}(t_B) = \langle 0|\tilde{\gamma}^{B\pm}(t_B)|\mu^\pm\rangle$$  \hspace{1cm} (4.58)

and

$$r_{\mu^\pm/\nu}^{B\pm}Q_{\mu^\pm/\nu}(p_B) = (\nu^\pm|\gamma^{r\cdot B\pm}(p_B)|\mu^\pm).$$  \hspace{1cm} (4.59)

It then follows from (4.51) that the polynomials $s_\lambda(t_B|A_r(p'))$ and $Q_{\mu^\pm}(t_B|A_r^\pm(p_B))$ are related by (4.25), with $\tilde{A} = \tilde{A}_r(p')$ and $A = A_r(p_B)$.

**Remark 4.2.** With the special choice

$$p' = t_0 := (1, 0, 0, \ldots)$$  \hspace{1cm} (4.60)

$$t_j = \frac{1}{j} \sum_{k=1}^{N} x_k^j$$

$$r_j = r_j(z, z') := -(z + j)(z' + j),$$  \hspace{1cm} (4.61)

eq (4.35) gives the multivariate Laguerre polynomial [15, 20]. To see this, substitute (4.60), (4.61) into eq. (4.35) to obtain

$$s_\lambda(t|A_r(z, z')(t_0)) = \sum_{\rho \subseteq \lambda} (-1)^{|\lambda| - |\rho|} (z)_{\lambda/\rho}(z')_{\lambda/\rho}s_{\lambda/\rho}(t_0)s_{\rho}(t),$$  \hspace{1cm} (4.62)

where

$$(z)_{\lambda/\rho} := \prod_{(i, j) \in \lambda/\rho} (z + j - i)$$  \hspace{1cm} (4.63)

is the content product of $\{z + j\}_{j \in \mathbb{Z}}$ over the skew Young diagram $\lambda/\rho$, and

$$s_{\lambda/\rho}(t_0) = \det \left( \frac{1}{(\lambda_i - i - \rho_j + j)!} \right)_{1 \leq i, j \leq \ell(\lambda)}.$$  \hspace{1cm} (4.64)

Eq. (4.62) coincides with formula (4.3) of [20] for multivariate Laguerre polynomials, where the coefficients were also written in terms of integer evaluations of shifted Schur functions [19].

Note that an arbitrary choice of parameters $(z, z')$ in formula (4.62) does not satisfy the relation (2.58), which is needed in order to apply Corollary 4.2. For this, the reduction condition (4.43) must be satisfied, which requires that we impose the constraint $z' = -1 - z$, giving

$$r_j = (z + j)(z + 1 - j) = r_{1-j}.$$  \hspace{1cm} (4.65)
Substituting this into (4.39), we get eq. (4.37), with

\[
R_{\mu/\nu}^B(z) := \prod_{i=1}^{m(\nu)} \prod_{j=\nu_i+1}^{\mu_i} (z + j)(1 + z - j)
\]  

(4.66)

and

\[
Q_{\mu/\nu}(\frac{1}{2}t_0) = \text{Pf}\left(\frac{\mu_i - \mu_j - \nu_i + \nu_j}{(\mu_i + \mu_j - \nu_i - \nu_j)(\mu_i - \nu_i)! (\mu_j - \nu_j)!}\right)_{1 \leq i,j \leq m(\mu)}.
\]

(4.67)

with the understanding that \(\nu_i = 0\) if \(i > m(\nu)\).

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