Dirac quantization and baryon intrinsic frequencies in hypersphere soliton model

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(Dated: November 16, 2021)

Quantizing a soliton on a hypersphere, we obtain the first class Hamiltonian, and evaluate the baryon physical quantities which are in good agreement with the corresponding experimental data. In particular, we find that the predicted value for axial coupling constant is comparable to its experimental value. The prediction for delta baryon mass possessing the Weyl ordering correction obtained in the first class Dirac quantization is improved comparing with that in the second class canonical quantization performed on the hypersphere. Making use of the same input parameters associated with the baryon masses, we also investigate the hypersphere soliton and standard Skyrmion models to compare the corresponding predictions for the physical quantities effectively. Next, we evaluate the intrinsic frequencies of the pulsating baryons. We thus find that the intrinsic pulsating frequency of more massive particle is greater than that of the less massive one. We explicitly evaluate the intrinsic frequencies \( \omega_N = 0.87 \times 10^{23} \text{ sec}^{-1} \) and \( \omega_\Delta = 1.74 \times 10^{23} \text{ sec}^{-1} \) of the nucleon and delta baryon, respectively, to yield the identity \( \omega_\Delta = 2\omega_N \).

Keywords: hypersphere soliton; Dirac quantization; frequency of pulsating baryon; axial coupling constant

I. INTRODUCTION

It is well known that, the Dirac Hamiltonian scheme has been developed [1, 2], to convert the second class constraints into the first class ones. Moreover, exploiting the Dirac quantization in the first class formalism, there have been attempts to quantize the constrained systems [2]. The standard Skyrmion model [2–5] which is the constrained Hamiltonian system, has been exploited [5] to investigate baryon kinematics and predict its physical properties. In this model, the three dimensional physical space is assumed to be topologically compactified to \( S^3 \) and the spatial infinity is located on the north-pole of \( S^3 \). Next the intrinsic frequencies of baryons have been naively studied in the standard Skyrmion model [6].

On the other hand, the hypersphere soliton model (HSM) [3, 7] has been proposed to obtain a topological lower bound on soliton energy and a set of equations of motion via the canonical quantization in the second class formalism [7]. Later the baryon physical properties, such as baryon masses, charge radii and magnetic moments, have been newly predicted using the canonical quantization in the HSM, to suggest that a realistic hadron physics can be delineated in terms of this phenomenological soliton [3].

In 1962 Dirac proposed [8] that the electron should be considered as a charged conducting surface and its shape and size should pulsate. Here the surface tension of the electron was supposed to prevent the electron from flying apart under the repulsive forces of the charge. Motivated by his idea, we will investigate pulsating baryons in the first class formalism in the HSM [3, 7], to evaluate the intrinsic frequencies of the baryons, baryon masses with the Weyl ordering correction (WOC) and axial coupling constant.

In this paper, we will explicitly evaluate the intrinsic frequencies of the pulsating baryons. To do this, we will use the baryon Hamiltonian spectrum which can be obtained through the Dirac quantization. Moreover in the HSM we will systematically generalize the predictions [3] for baryon physical quantities obtained in the second class formalism, to those in the first class one.

In Sec. II, we will recapitulate the baryon formalism in the HSM in the second class canonical quantization scheme. In Sec. III, we will newly predict the corrected baryon masses, and the axial coupling constant in the HSM in the first class Dirac quantization. In Sec. IV, we will explicitly evaluate the intrinsic frequencies of the pulsating baryons such as nucleon and delta baryon. Sec. V includes conclusions.
II. SET UP OF BARYON FORMALISM IN HSM

In this section, we recapitulate the baryon formulation in the HSM [3, 7], by introducing the Skyrmion Lagrangian density

\[ \mathcal{L} = \frac{f_\pi^2}{4} \text{tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32e^2} \text{tr}[U^\dagger \partial_\mu U, U^\dagger \partial^\mu U]^2, \]  

(2.1)

where \( U \) is an SU(2) chiral field. Note that the HSM includes two parameters, namely a pion decay constant \( f_\pi \) and a dimensionless Skyrme parameter \( e \). We may fix the parameters by exploiting mesonic processes such as the decay of the pion and the \( \pi-\pi \) scattering. On the other hand we also may regard them as effective parameters for a specific purpose [8], for instance in order to describe the baryons in the HSM. In this paper, we will treat model parameters with which we can predict the physical quantities comparable to the corresponding experimental values as shown in Table I. Next the quartic term is necessary to stabilize the soliton in the baryon sector.

Now we investigate baryon phenomenology by using the hyperspherical three metric

\[ ds^2 = \lambda^2 d\mu^2 + \lambda^2 \sin^2 \mu \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \]  

(2.2)

where the ranges of the hyperspherical coordinates are given by \( 0 \leq \mu \leq \pi, \) \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi \leq 2\pi, \) and \( \lambda \) \( (0 \leq \lambda < \infty) \) is the radius parameter of \( S^3 \). Note that \( S^3 \) is described in terms of the three dimensional hypersphere coordinates \( (\mu, \theta, \phi) \) together with the radius parameter \( \lambda \). Next, we obtain the soliton energy in the HSM

\[ E = \frac{f_\pi}{e} \left[ 2\pi L \int_0^\pi d\mu \sin^2 \mu \left( \left( \frac{df}{d\mu} + \frac{1}{L \sin^2 \mu} \right)^2 + 2 \left( \frac{1}{L \mu} + 1 \right) \frac{1}{\sin^2 \mu} \right) + 6\pi^2 \right], \]  

(2.3)

where \( L = e f_\pi \lambda \) \( (0 \leq L < \infty) \) is a radius parameter expressed in dimensionless units. Here \( f(\mu) \) is the profile function for the hypersphere soliton, and satisfies \( f(0) = \pi \) and \( f(\pi) = 0 \) for unit baryon number. Note that one of the advantages of the HSM is that the nonlinear differential equations to be solved numerically in the standard Skyrmion model can be linearized and solved analytically in \( S^3 \) hypersphere [3, 7].

In the HSM, spin and isospin states can be treated by collective coordinates \( a^\mu = (a^0, \vec{a}) \) \( (\mu = 0, 1, 2, 3) \), corresponding to the spin and isospin collective rotation \( A(t) \in SU(2) \) given by \( A(t) = a^0 + i \vec{a} \cdot \vec{\tau} \). Exploiting the coordinates \( a^\mu \), we obtain the Hamiltonian of the form

\[ H = E + \frac{1}{8L} \pi^\mu \pi^\mu, \]  

(2.4)

where \( \pi^\mu \) are canonical momenta conjugate to the collective coordinates \( a^\mu \). Here the soliton energy lower bound \( E \) and moment of inertia \( I \) are given by

\[ E = \frac{6\pi^2 f_\pi}{e}, \]  

(2.5)

\[ I = \frac{3\pi^2}{e^3 f_\pi}. \]  

(2.6)

Note that, in order to obtain \( E \) in (2.5) which is now the BPS topological lower bound in the soliton energy [3, 7, 10, 11], in (2.3) we have used the identity map \( f(\mu) = \pi - \mu \) with the fixed value \( L = L_B \) where

\[ L_B = e f_\pi \lambda_B = 1. \]  

(2.7)

In addition, the identity map with condition \( L = L_B \) is used to predict the physical quantities such as the moment of inertia \( I \) in (2.6), baryon masses, charge radii, magnetic moments, axial coupling constant \( g_A \) and intrinsic pulsation frequencies \( \omega_I \) in the HSM. Note also that the hypersphere coordinates \( (\mu, \theta, \phi) \) are integrated out in (2.3), and \( E \) in (2.5) is a function of \( \lambda_B = \frac{1}{f_\pi} \) or equivalently \( f_\pi \) and \( e \) only. Note that the integral expressions of \( \langle r^2 \rangle_E, I = 1 \) and \( g_A \) are given by (2.10) and (3.7), below, respectively, and \( \omega_I \) is trivially obtainable in terms of \( I \). Moreover, the above physical quantities except \( \langle r^2 \rangle_E, I = 1 \), \( g_A \) and \( \omega_I \) are also given by integral expressions similar to \( E \) in (2.3) [3]. As a result, after integrating out the hypersphere coordinates \( (\mu, \theta, \phi) \), these physical quantities are formulated in terms of \( f_\pi \) and \( e \) only, as shown in (2.6) and (2.9)–(2.12), 3.3)–(3.7) and (4.11).

After performing the canonical quantization in the second class formalism in the HSM, we now construct the Hamiltonian spectrum

\[ \langle H \rangle = E + \frac{1}{2\pi} I (I + 1), \]  

(2.8)
where $I$ (= 1/2, 3/2, ...) are baryon isospin quantum numbers. Exploiting (2.8) we find the nucleon mass $M_N$ for $I = 1/2$ and delta baryon mass $M_\Delta$ for $I = 3/2$, respectively

$$M_N = e f_\pi \left( \frac{6\pi^2}{e^2} + \frac{e^2}{8\pi^2} \right), \quad M_\Delta = e f_\pi \left( \frac{6\pi^2}{e^2} + \frac{5e^2}{8\pi^2} \right).$$

(2.9)

Next, in the HSM we obtain the magnetic isovector mean square charge radius

$$\langle r^2 \rangle_{M,I=1} = \frac{2e f_\pi}{3\tilde{\Omega}} \int_0^{\infty} dA_3 \sin^2 \mu \sin^2 f \left( 1 + \left( \frac{df}{d\mu} \right)^2 + \frac{\sin^2 f}{\sin^2 \mu} \right) = \frac{5}{6e^2 f_\pi^2},$$

(2.10)

where $\tilde{\Omega}$ is the dimensionless moment of inertia defined as $\tilde{\Omega} = e^3 f_\pi I$ and $dA_3 = \lambda_B^3 \sin^2 \mu \sin \theta \, d\mu \, d\theta \, d\phi$. Similarly, we find the other charge radii in the HSM to yield

$$\langle r^2 \rangle_{M,I=0} = \langle r^2 \rangle_{M,I=1} = \langle r^2 \rangle_{M,P} = \langle r^2 \rangle_{M,n} = \langle r^2 \rangle_{E,I=1} = \sqrt{\frac{5}{6e^2 f_\pi}},$$

(2.11)

$$\langle r^2 \rangle_{E,I=0} = \frac{\sqrt{3}}{2} \frac{1}{ef_\pi}, \quad \langle r^2 \rangle_{E,P} = \frac{19}{24} \frac{1}{(ef_\pi)^2}, \quad \langle r^2 \rangle_{E,n} = -\frac{1}{24} \frac{1}{(ef_\pi)^2},$$

(2.12)

where the subscripts $E$ and $M$ denote the electric and magnetic radii, respectively. Using the charge radii in (2.11) we choose $\langle r^2 \rangle_{M,P} = 0.80$ fm as an input parameter. One can then have

$$ef_\pi = 225.23 \text{ MeV} = (0.876 \text{ fm})^{-1}$$

(2.13)

and, with this fixed value of $ef_\pi$, one can proceed to calculate the other charge radii as shown in Prediction I in Table I.

III. BARYON PHYSICAL PROPERTIES IN THE FIRST CLASS DIRAC QUANTIZATION IN HSM

The model predictions of phenomenological soliton have been shown to include rigorous treatments of geometrical constraints because these constraints affect the predictions themselves. Motivated by this, we consider the first class Hamiltonian of the form (3.4)

$$\tilde{\Omega}_1 = a^\mu a^\mu - 1 \approx 0, \quad \tilde{\Omega}_2 = a^\mu \pi^\mu \approx 0,$$

(3.1)

so that we have the Poisson algebra $\Delta_{kk'} = \{ \Omega_k, \Omega_{k'} \} = 2\epsilon^{kk'} a^\mu a^\mu$. The HSM thus becomes a second class constrained system.

Note that the Dirac quantization should be applied to the hypersphere soliton, in order to predict more rigorously physical quantities by using the first class formalism. Following the Dirac Hamiltonian scheme, we find the first class constraints

$$\tilde{\Omega}_1 = a^\mu a^\mu - 1 + 2\theta = 0, \quad \tilde{\Omega}_2 = a^\mu \pi^\mu - a^\mu a^\mu \pi_\theta = 0,$$

(3.2)

where $(\theta, \pi_\theta)$ are the Stückelberg fields. The first class constraints now satisfy $\{ \tilde{\Omega}_1, \tilde{\Omega}_2 \} = 0$. We next obtain the first class Hamiltonian of the form

$$\tilde{H} = E + \frac{1}{8\tilde{\Omega}} (\pi^\mu - a^\mu \pi_\theta)(\pi^\nu - a^\nu \pi_\theta) \frac{a^\nu a^\nu}{a^\nu a^\nu + 2\theta}.$$  

(3.3)

Note that the first class Hamiltonian is strongly involutive with the first class constraints $\{ \tilde{\Omega}_i, \tilde{H} \} = 0$ ($i = 1, 2$). After performing the Dirac quantization, we obtain the Hamiltonian spectrum with WOC

$$\langle \tilde{H} \rangle = E + \frac{1}{8\tilde{\Omega}} \left( I(I+1) + \frac{1}{3} \right).$$

(3.4)

1 For the experimental data except $\langle r^2 \rangle_n$ we refer to Ref. [3]. The experimental value of $\langle r^2 \rangle_n$ is given by $-(0.341 \text{ fm})^2$ [12].
Next we find the magnetic moments \[ M \] data. In Predictions I and II, the model parameters \((f_\pi = 58 \text{ MeV}, \epsilon = 3.89)\) and \((f_\pi = 64 \text{ MeV}, \epsilon = 4.49)\) are exploited in the first class Dirac quantization with WOC, respectively. In Prediction III, the model parameters \((f_\pi = 65 \text{ MeV}, \epsilon = 5.45)\) are used in the standard Skyrmion model. The input parameters are indicated by \(*\).

| Quantity | Prediction I | Prediction II | Prediction III | Experiment |
|----------|--------------|---------------|----------------|------------|
| \(\langle r^2 \rangle_{M_1=0} \) | 0.80 fm | 0.63 fm | 0.92 fm | 0.81 fm |
| \(\langle r^2 \rangle_{M_1=1} \) | 0.80 fm | 0.63 fm | \(\infty\) | 0.80 fm |
| \(\langle r^2 \rangle_{M_1=0} \) | 0.80 fm* | 0.63 fm | \(\infty\) | 0.80 fm |
| \(\langle r^2 \rangle_{M_1=1} \) | 0.80 fm | 0.63 fm | \(\infty\) | 0.79 fm |
| \(\langle r^2 \rangle_{M_1=0} \) | 0.76 fm | 0.60 fm | 0.59 fm | 0.72 fm |
| \(\langle r^2 \rangle_{M_1=1} \) | 0.80 fm | 0.63 fm | \(\infty\) | 0.88 fm |
| \(\langle r^2 \rangle_{p} \) | (0.780 fm)\(^2\) | (0.61 fm)\(^2\) | \(\infty\) | (0.805 fm)\(^2\) |
| \(\langle r^2 \rangle_{n} \) | -(0.179 fm)\(^2\) | -(0.14 fm)\(^2\) | \(-\infty\) | -(0.341 fm)\(^2\) |
| \(\mu_p \) | 2.98 | 1.88 | 1.87 | 2.79 |
| \(\mu_n \) | -2.45 | -1.32 | -1.31 | -1.91 |
| \(\mu_N \) | 5.69 | 3.72 | 3.72 | 4.7 - 6.7 |
| \(\mu_{N\Delta} \) | 3.84 | 2.27 | 2.27 | 3.29 |
| \(M_N \) | 939 MeV\(^*\) | 939 MeV\(^*\) | 939 MeV\(^*\) | 939 MeV |
| \(M_\Delta \) | 1112 MeV | 1232 MeV\(^*\) | 1323 MeV\(^*\) | 1232 MeV |
| \(g_A \) | 1.30 | 0.98 | 0.61 | 1.23 |

Here, comparing with the canonical quantization spectrum result \((\langle H \rangle \text{ in (2.8)}, \langle \tilde{H} \rangle \text{ obtained via the Dirac quantization with WOC})\) has the additional term \(\frac{2\pi^2}{e^2} \) in (3.3). This additional contribution originates from the first class constraints in (3.2). Using (3.4) we newly obtain the nucleon and delta baryon masses, respectively

\[
M_N = e f_\pi \left( \frac{6\pi^2}{e^2} + \frac{e^2}{6\pi^2} \right), \quad M_\Delta = e f_\pi \left( \frac{6\pi^2}{e^2} + \frac{2e^2}{3\pi^2} \right). \tag{3.5}
\]

Next we find the magnetic moments [3]

\[
\mu_p = \frac{2 M_N}{e f_\pi} \left( \frac{e^2}{48\pi^2} + \frac{\pi^2}{2e^2} \right), \quad \mu_n = \frac{2 M_N}{e f_\pi} \left( \frac{e^2}{48\pi^2} - \frac{\pi^2}{2e^2} \right),
\]

\[
\mu_{\Delta++} = \frac{2 M_N}{e f_\pi} \left( \frac{e^2}{16\pi^2} + \frac{9\pi^2}{10e^2} \right), \quad \mu_{N\Delta} = \frac{2 M_N}{e f_\pi} \left( \frac{\sqrt{2}\pi^2}{2e^2} \right), \tag{3.6}
\]

where \(M_N\) is now given by the nucleon mass with WOC in (3.5).

Now, in Prediction I, we fix the model parameters \((f_\pi = 58 \text{ MeV}, \epsilon = 3.89)\), to reproduce the experimental values for \(M_N = 939 \text{ MeV}\) and \(\langle r^2 \rangle_{M_1=0} = 0.80 \text{ fm}\). Next, exploiting the above values of \(f_\pi\) and \(\epsilon\), we evaluate the delta baryon mass to yield \(M_\Delta = 1112 \text{ MeV}\). This predicted value for delta baryon mass with WOC is improved comparing with the value \(M_\Delta = 1097 \text{ MeV}\) obtained by using the canonical quantization formula in (2.9). Next, in the HSM we newly obtain the axial coupling constant

\[
g_A = \frac{4\pi}{e^2} \int_0^{2\pi} d\mu \sin^2 \mu (1 + \cos \mu) = \frac{2\pi^2}{e^2}, \tag{3.7}
\]

where we also have used the identity map \(f(\mu) = \pi - \mu\) with the fixed value \(L = L_R\). The theoretical prediction of \(g_A\) is given by 1.30, which is comparable to its experimental value 1.23 [3]. The predictions for the physical quantities

\[\text{Note that, in the previous results [3] obtained in the second class canonical quantization in the HSM, we have used the model parameters } f_\pi = 56 \text{ MeV and } \epsilon = 4.03 \text{ so that the physical quantities } M_N, M_\Delta, \mu_p, \mu_n, \mu_{\Delta++}, \text{ and } \mu_{N\Delta} \text{ can fit the corresponding experimental data as well as possible. In this case we have the prediction } M_N = 868 \text{ MeV for instance.}\]
are listed in Prediction I. Note that the predicted values for \( \mu_{\Delta^+} \), \( \langle r^2 \rangle_{M,I=0} \), \( \langle r^2 \rangle_{M,I=1} \), \( \langle r^2 \rangle_{M,n} \) (in addition to the input parameters \( M_N \) and \( \langle r^2 \rangle _{M,p} \)) are almost the same as the corresponding experimental data. Next the predictions for \( g_A \), \( \langle r^2 \rangle _{E,I=0} \) and \( \langle r^2 \rangle _{p} \) are within about 6% of the experimental values, while those for \( M_\Delta \), \( \mu_p \) and \( \langle r^2 \rangle _{E,I=1} \) are within about 10% of the experimental ones.

Next, in Prediction II, we predict the baryon physical quantities using the model parameters \( (\pi = 64 \text{ MeV}, e = 4.49) \) in the Dirac quantization with WOC. These predictions except the axial coupling constant and charge radii are almost same as those in Prediction III \[\text{[3, 5, 13]} \] evaluated in the standard Skyrmion model which uses the model parameters \( (\pi = 65 \text{ MeV}, e = 5.45) \). Note that both in Prediction II and in Prediction III, we exploit the same input parameters \( M_N = 939 \text{ MeV} \) and \( M_\Delta = 1232 \text{ MeV} \) to compare these two predictions effectively. The prediction of the axial coupling constant is improved in Prediction II, compared with that in Prediction III. Note that the charge radii in Predictions I and II have finite values comparable to the corresponding experimental data. In contrast, some predicted values for the charge radii in Prediction III are infinite ones.

Now, it seems appropriate to comment on the hypersurface area \( A_3 \) of the hypersphere \( S^3 \) of radius parameter \( \lambda_B \), and the charge radius \( \langle r^2 \rangle _{M,I=1} \) in (3.10). Exploiting the hyperspherical three metric in (2.2), we find that \( A_3 \) can be analyzed in terms of three arc length elements \( \lambda_B d\mu, \lambda_B \sin \mu d\theta \) and \( \lambda_B \sin \mu \sin \theta d\phi \), from which we readily find the three dimensional hypersurface area \( A_3 = 2\pi^2 \lambda_B^2 \). Here note that, since on the thin bubble-like \( S^3 \) soliton there exists no line element such as \( dr \) defined in spherical coordinates \( (r, \theta, \phi) \), we do not have a volume and instead we obtain the three dimensional area \( A_3 \) for the hypersurface \( S^3 \). We can then define \( \lambda_B \) as the radial distance from the center of \( S^3 \) to the hypersurface \( S^3 \) in \( R^4 \). In fact, inserting the value \( cf \pi = (0.876 \text{ fm})^{-1} \) into (2.13) into the condition \( L_B = 1 \) in (2.7), in our HSM we obtain the fixed radius parameter given by \( \lambda_B = \frac{1}{c f \pi} = 0.876 \text{ fm} \). On the other hand, the charge radius \( \langle r^2 \rangle _{M,I=1} \) is the physical quantity expressed in (3.10). Integrating over a relevant surface density on \( S^3 \) corresponding to the integrand in (2.10), we evaluate \( \langle r^2 \rangle _{M,I=1} \) which is now independent of \( \mu \), to yield a specific value of the magnetic isovector root mean square charge radius. The calculated charge radius then can be defined as the fixed radial distance to the point on a hypersurface manifold which does not need to be located only on the compact manifold \( S^3 \) of radius parameter \( \lambda_B \). This hypersurface manifold is now a submanifold in \( R^4 \) which is located at \( r = 0.80 \text{ fm} \) far from the center of \( S^3 \). Note that \( \langle r^2 \rangle _{M,I=1} \) denotes the radial distance which is a geometrical invariant giving the same value both in \( R^3 \) (for instance in volume \( R^3 \) which contains the center of \( S^3 \) and is described in terms of \( (r, \theta, \phi) \) at \( \mu = \frac{\pi}{2} \)) and in \( R^4 \). Next, the physical quantity \( \langle r^2 \rangle _{M,I=1} \) calculated in \( R^3 \) (and in \( R^4 \)) then can be compared with the corresponding experimental value, similar to other physical quantities such as \( M_N \) and \( M_\Delta \).

Finally it seems appropriate to comment on the Betti numbers associated with the manifold \( S^3 \) in the HSM. First of all, the \( p \)-th Betti number \( b_p(M) \) is defined as the maximal number of \( p \)-cycles on \( M \):

\[
b_p(M) = \dim H_p(M),
\]

(3.8)

where \( H_p(M) \) is the homology group of the manifold \( M \). For the case of \( S^3 \), we obtain

\[
\begin{align*}
H_0(S^3) &= H_3(S^3) = \mathbb{Z}, \\
H_p(S^3) &= 0, \text{ otherwise.}
\end{align*}
\]

(3.9)

The non-vanishing Betti numbers related with \( S^3 \) are thus given by \( b_0(S^3) = b_3(S^3) = 1 \).

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3 As a toy model of soliton embedded in \( R^3 \), we consider a uniformly charged manifold \( S^2 \) described in terms of \( (\theta, \phi) \) and a fixed radius parameter \( \lambda_B \) where we have \( A_2 = 4\pi\lambda_B^2 \). By integrating over a surface charge density residing on \( S^2 \), one can calculate the physical quantity such as the electric potential, at an arbitrary observation point \( \frac{\mu}{2} \) which does not need to be located only on the compact manifold \( S^2 \) of radius parameter \( \lambda_B \). Next, since the \( S^2 \) soliton of fixed radius parameter \( \lambda_B \) is embedded in \( R^3 \), we manifestly define an arbitrary radial distance from the center of the compact manifold to an observation point which is located in \( R^3 = S^2 \times R \). Here \( S^2 \) denotes foliation leaves 14 of spherical shell of radius parameter \( \lambda \) (0 ≤ \( \lambda < \infty \)) and \( R \) is a manifold associated with radial distance. Note that the radial distance itself is a fixed geometrical invariant producing the same value both in \( R^2 \) (for instance on equatorial plane \( R^2 \) which contains the center of \( S^2 \) and is delineated by \( (r, \phi) \) at \( \theta = \frac{\pi}{2} \)) and in \( R^3 \). The same mathematical logic can be applied to \( S^3 \) soliton of fixed radius parameter \( \lambda_B \) embedded in \( R^4 = S^3 \times R \) where \( S^3 \) stands for foliation leaves of hyperspherical shell of radius parameter \( \lambda \) (0 ≤ \( \lambda < \infty \)) and \( R \) is a manifold related with radial distance.
IV. INTRINSIC FREQUENCIES OF PULSATING BARYONS

In this section, we investigate the baryon intrinsic frequencies in the HSM. To do this, we introduce an additional term proportional to the first class constraint $\tilde{\Omega}_2$ into $\tilde{H}$ in (3.3) to obtain the equivalent first class Hamiltonian

$$\tilde{H}' = \tilde{H} + \frac{1}{4L} \pi_\theta \tilde{\Omega}_2,$$  \hspace{1cm} (4.1)

which naturally generates the Gauss law constraints

$$\{ \tilde{\Omega}_1, \tilde{H}' \} = \frac{1}{2L} \tilde{\Omega}_2, \quad \{ \tilde{\Omega}_2, \tilde{H}' \} = 0.$$  \hspace{1cm} (4.2)

Exploiting the Hamiltonian in (4.1), we obtain

$$\langle \tilde{H}' \rangle = E + \frac{1}{2L} \left[ I(I+1) + \frac{1}{4} \right],$$  \hspace{1cm} (4.3)

to yield the predicted values for $M_N$ and $M_\Delta$ which are the same as the previous ones, $M_N = 939$ MeV and $M_\Delta = 1112$ MeV.

For the first class physical variable $\tilde{W}$ governed by the Hamiltonian $\tilde{H}'$ in (4.1), the equation of motion in the Poisson bracket form is given by

$$\dot{\tilde{W}} = \{ \tilde{W}, \tilde{H}' \},$$  \hspace{1cm} (4.4)

where overdot denotes time derivative. Exploiting the equation of motion in (4.4), we end up with

$$\dot{\tilde{a}}^\mu = \{ \tilde{a}, \tilde{H}' \} = \frac{1}{4L} \tilde{\pi}^\mu, \quad \dot{\tilde{\pi}}^\mu = \{ \tilde{\pi}, \tilde{H}' \} = -\frac{1}{4L} \tilde{a}^\rho \tilde{a}_\rho \tilde{\pi}^\mu. \hspace{1cm} (4.5)$$

Here the first class physical fields $\tilde{a}^\mu$ and $\tilde{\pi}^\mu$ are given as follows

$$\tilde{a}^\mu = a^\mu \left( \frac{a^\nu a^\nu + 2\theta}{a^\mu a^\nu} \right)^{1/2}, \quad \tilde{\pi}^\mu = (\pi^\mu - a^\mu \pi_\theta) \left( \frac{a^\nu a^\nu + 2\theta}{a^\mu a^\nu} \right)^{1/2}, \hspace{1cm} (4.6)$$

which fulfill $\{ \tilde{\Omega}_i, \tilde{a}^\mu \} = \{ \tilde{\Omega}_i, \tilde{\pi}^\mu \} = 0 \hspace{0.25cm} (i = 1, 2)$. Here note that we can recover the first class constraint $\tilde{a}^\mu \tilde{a}^\mu = 1$.

Next we have the identities among the physical fields

$$\{ \tilde{a}^\mu, \tilde{\pi}^\nu \} = \delta^{\mu\nu} - \tilde{a}^\mu \tilde{a}^\nu, \quad \{ \tilde{\pi}^\mu, \tilde{\pi}^\nu \} = \tilde{\pi}^\mu \tilde{a}^\nu - \tilde{a}^\mu \tilde{\pi}^\nu, \hspace{1cm}$$

$$\{ \tilde{a}^\mu, \tilde{H} \} = \frac{1}{4L} (\tilde{\pi}^\mu - \tilde{a}^\mu \tilde{\pi}^\nu \tilde{a}^\nu), \quad \{ \tilde{\pi}^\mu, \tilde{H} \} = \frac{1}{4L} (\tilde{a}^\mu \tilde{\pi}^\nu \tilde{a}^\nu - \tilde{a}^\mu \tilde{a}^\nu \tilde{\pi}^\nu), \hspace{1cm}$$

$$\{ \tilde{a}^\mu, \pi_\theta \} = \tilde{a}^\mu, \quad \{ \tilde{\pi}^\mu, \pi_\theta \} = -\tilde{\pi}^\mu. \hspace{1cm} (4.7)$$

Moreover we find

$$\tilde{\dot{a}}^\mu = \{ \tilde{\dot{a}}^\mu, \tilde{H}' \} = \frac{1}{4L} \tilde{\dot{\pi}}^\mu. \hspace{1cm} (4.8)$$

Exploiting (4.3) and (4.8), we proceed to obtain

$$\tilde{\dot{a}}^\mu = -\frac{1}{4L^2} \left[ I(I+1) + \frac{1}{4} \right] \tilde{a}^\mu, \hspace{1cm} (4.9)$$

where $I$ again denotes the isospin quantum number of the baryon. Note that the equations of motion for $\tilde{a}^\mu$ in (4.9) represent those for harmonic oscillators of the form

$$\tilde{\dot{a}}^\mu = -\omega_I^2 \tilde{a}^\mu, \hspace{1cm} (4.10)$$

\textsuperscript{4} Note that $\langle \tilde{H}' \rangle$ with $\tilde{H}' = \tilde{H} + \frac{1}{4L} \pi_\theta \tilde{\Omega}_2$ is the same as $\langle \tilde{H} \rangle$ in (3.3) since $\tilde{\Omega}_2$ term does not affect the Hamiltonian spectrum.
where the intrinsic pulsating frequency of the baryon with the isospin quantum number $I$ is given by

$$\omega_I = \frac{1}{2I} \left[ I(I + 1) + \frac{1}{4} \right]^{1/2}. \quad (4.11)$$

One of the simple solutions for the equation of motion (4.10) is given as follows

$$\tilde{a}^\mu = \frac{1}{\sqrt{2}} \sin \left( \omega_I t + \frac{1}{2} \mu \pi \right), \quad (\mu = 0, 1, 2, 3), \quad (4.12)$$

which satisfy the first class constraint $\tilde{a}^\mu \tilde{a}_\mu = 1$.

It seems appropriate to comment on the Gauss law constraints in (4.2). To derive the equations of motion for the harmonic oscillators in (4.10), we have used (4.1) and (4.4), where $\tilde{H}'$ is exploited instead of $\tilde{H}$. Note that $\tilde{H}'$ satisfies the Gauss law constraints in (4.2). The Gauss law constraints are now physically meaningful since using $\tilde{H}'$ associated with these constraints is crucial to predict $\omega_I$ in (4.11).

Inserting the model parameters $f_\pi = 58$ MeV and $\epsilon = 3.89$ into the formula for $I$ in (2.6), we obtain

$$\mathcal{I} = \text{certain expression}. \quad (4.13)$$

Now exploiting the value for $\mathcal{I}$ in (4.13) and the formula for $\omega_I$ in (4.11), we arrive at

$$\omega_N = 0.87 \times 10^{23} \text{ sec}^{-1}, \quad \omega_\Delta = 1.74 \times 10^{23} \text{ sec}^{-1}. \quad (4.14)$$

Here we observe that the intrinsic pulsating frequency for more massive particle is greater than that for the less massive one, namely $\omega_\Delta = 2\omega_N$.

V. CONCLUSIONS

In summary, we have evaluated the intrinsic pulsating frequencies of the baryons. To do this, we have exploited the HSM, where we have constructed the first class Hamiltonian to quantize the hypersphere soliton. Next, we have evaluated the baryon physical quantities such as baryon masses, magnetic moments, charge radii and axial coupling constant, in Predictions I and II in Table I. These predictions are much better than those in Prediction III which exploits the standard Skyrmion model. In particular, the charge radii in Prediction I are in good agreement with the corresponding experimental data. Finally, the intrinsic frequency for more massive particle has been shown to be greater than that for the less massive one.

Acknowledgments

The author would like to thank the anonymous referee for helpful comments. He was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, NRF-2019R1I1A1A01058449.
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