Closed timelike curves and causality violation

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The conceptual definition and understanding of time, both quantitatively and qualitatively is of the utmost difficulty and importance. As time is incorporated into the proper structure of the fabric of spacetime, it is interesting to note that General Relativity is contaminated with non-trivial geometries which generate closed timelike curves. A closed timelike curve (CTC) allows time travel, in the sense that an observer that travels on a trajectory in spacetime along this curve, may return to an event before his departure. This fact apparently violates causality, therefore time travel and its associated paradoxes have to be treated with great caution. The paradoxes fall into two broad groups, namely the consistency paradoxes and the causal loops. A great variety of solutions to the Einstein field equations containing CTCs exist and it seems that two particularly notorious features stand out. Solutions with a tipping over of the light cones due to a rotation about a cylindrically symmetric axis and solutions that violate the energy conditions. All these aspects are analyzed in this review paper.

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I. INTRODUCTION

Providing an explicit definition of time is an extremely difficult endeavor, although it does seem to be intimately related to change, an idea reflected in Aristotle’s famous metaphor: Time is the moving image of Eternity. In fact, one may encounter many reflections and philosophical considerations on time over the ages, culminating in Newton’s notion of absolute time. Newton stated that time flowed at the same rate for all observers in the Universe. But,
in 1905, Einstein changed altogether our notion of time. Time flowed at different rates for different observers, and Minkowski, three years later, formally united the parameters of time and space, giving rise to the notion of a four-dimensional entity, spacetime. Adopting a pragmatic point of view, this assumption seems reasonable, as to measure time a changing configuration of matter is needed, i.e., a swinging pendulum, etc. Change seems to be imperative to have an emergent notion of time. Therefore, time is empirically related to change. But change can be considered as a variation or sequence of occurrences [2]. Thus, intuitively, a sequence of successive occurrences provides us with a notion of something that flows, i.e., it provides us with the notion of time. Time flows and everything relentlessly moves along this stream. In Relativity, we can substitute the above empirical notion of a sequence of occurrences by a sequence of events. We idealize the concept of an event to become a point in space and an instant in time. Following this reasoning of thought, a sequence of events has a determined temporal order. We experimentally verify that specific events occur before others and not vice-versa. Certain events (effects) are triggered off by others (causes), providing us with the notion of causality.

Thus, the conceptual definition and understanding of time, both quantitatively and qualitatively is of the utmost difficulty and importance. Special Relativity provides us with important quantitative elucidations of the fundamental processes related to time dilation effects. The General Theory of Relativity (GTR) provides a deep analysis to effects of time flow in the presence of strong and weak gravitational fields [3]. As time is incorporated into the proper structure of the fabric of spacetime, it is interesting to note that GTR is contaminated with non-trivial geometries which generate closed timelike curves [2, 4–7]. A closed timelike curve (CTC) allows time travel, in the sense that an observer which travels on a trajectory in spacetime along this curve, returns to an event which coincides with the departure. The arrow of time leads forward, as measured locally by the observer, but globally he/she may return to an event in the past. This fact apparently violates causality, opening Pandora’s box and producing time travel paradoxes [8], throwing a veil over our understanding of the fundamental nature of Time. The notion of causality is fundamental in the construction of physical theories, therefore time travel and its associated paradoxes have to be treated with great caution. The paradoxes fall into two broad groups, namely the consistency paradoxes and the causal loops.

The consistency paradoxes include the classical grandfather paradox. Imagine traveling into the past and meeting one’s grandfather. Nurturing homicidal tendencies, the time traveler murders his grandfather, impeding the birth of his father, therefore making his own birth impossible. In fact, there are many versions of the grandfather paradox, limited only by one’s imagination. The consistency paradoxes occur whenever possibilities of changing events in the past arise. The paradoxes associated to causal loops are related to self-existing information or objects, trapped in spacetime. Imagine a time traveler going back to his past, handing his younger self a manual for the construction of a time machine. The younger version then constructs the time machine over the years, and eventually goes back to the past to give the manual to his younger self. The time machine exists in the future because it was constructed in the past by the younger version of the time traveler. The construction of the time machine was possible because the manual was received from the future. Both parts considered by themselves are consistent, and the paradox appears when considered as a whole. One is liable to ask, what is the origin of the manual, for it apparently surges out of nowhere. There is a manual never created, nevertheless existing in spacetime, although there are no causality violations. An interesting variety of these causal loops was explored by Gott and Li [9], where they analyzed the idea of whether there is anything in the laws of physics that would prevent the Universe from creating itself. Thus, tracing backwards in time through the original inflationary state a region of CTCs may be encountered, giving no first-cause.

A great variety of solutions to the Einstein Field Equations (EFEs) containing CTCs exist, but, two particularly notorious features seem to stand out. Solutions with a tipping over of the light cones due to a rotation about a cylindrically symmetric axis; and solutions that violate the Energy Conditions of GTR, which are fundamental in the singularity theorems and theorems of classical black hole thermodynamics [10]. A great deal of attention has also been paid to the quantum aspects of closed timelike curves [11–13].

Throughout this paper, we use the notation $G = c = 1$.

## II. STATIONARY AND AXISYMMETRIC SOLUTIONS GENERATING CTCS

It is interesting to note that the tipping over of light cones seems to be a generic feature of some solutions with a rotating cylindrical symmetry. The general metric for a stationary, axisymmetric solution with rotation is given by [2, 4–7]

$$ds^2 = -F(r) \, dt^2 + H(r) \, dr^2 + L(r) \, d\phi^2 + 2 \, M(r) \, d\phi \, dt + H(r) \, dz^2,$$

where $z$ is the distance along the axis of rotation; $\phi$ is the angular coordinate; $r$ is the radial coordinate; and $t$ is the temporal coordinate. The metric components are only functions of the radial coordinate $r$. Note that the determinant, $g = \det(g_{\mu\nu}) = -(FL + M^2)H^2$, is Lorentzian provided that $(FL + M^2) > 0$. 

Due to the periodic nature of the angular coordinate, $\phi$, an azimuthal curve with $\gamma = \{t = \text{const}, r = \text{const}, z = \text{const}\}$ is a closed curve of invariant length $s^{2}_\gamma \equiv L(r)(2\pi)^2$. If $L(r)$ is negative then the integral curve with $(t, r, z)$ fixed is a CTC. If $L(r) = 0$, then the azimuthal curve is a closed null curve. Now, consider a null azimuthal curve, not necessarily a geodesic nor closed, in the $(\phi, t)$ plane with $(r, z)$ fixed. The null condition, $ds^2 = 0$, implies

\[ 0 = -F + 2M \dot{\phi} + L \dot{\phi}^2, \tag{2} \]

with $\dot{\phi} = d\phi/dt$. Solving the quadratic, we have

\[ \dot{\phi} = \frac{d\phi}{dt} = -\frac{M \pm \sqrt{M^2 + FL}}{L}. \tag{3} \]

Due to the Lorentzian signature constraint, $FL + M^2 > 0$, the roots are real. If $L(r) < 0$ then the light cones are tipped over sufficiently far to permit a trip to the past. By going once around the azimuthal direction, the total backward time-jump for a null curve is

\[ \Delta T = \frac{2\pi |L|}{-M + \sqrt{M^2 - FL}}. \tag{4} \]

If $L(r) < 0$ for even a single value of $r$, the chronology-violation region covers the entire spacetime [4]. Thus, the tilting of light cones are generic features of spacetimes which contain CTCs, as depicted in Fig. 1.

**FIG. 1:** The tipping over of light cones, depicted in the figure is a generic feature of some solutions with a rotating cylindrical symmetry. The dashed curve represents a closed timelike curve.

The present section is far from making an exhaustive search of all the EFE solutions generating CTCs with these features, but the best known spacetimes will be briefly analyzed, namely, the van Stockum spacetime, the Gödel universe, the spinning cosmic strings and the Gott two-string time machine, which is a variation on the theme of the spinning cosmic string.

**A. Van Stockum spacetime**

The earliest solution to the EFEs containing CTCs, is probably that of the van Stockum spacetime, which describes a stationary, cylindrically symmetric solution of a rapidly rotating infinite cylinder of dust, surrounded by vacuum. The centrifugal forces of the dust are balanced by the gravitational attraction. The metric, assuming the respective symmetries, takes the form of Eq. (1), and $t$ is required to be timelike at $r = 0$. The coordinates $(t, r, \phi, z)$ have the following domain

\[ -\infty < t < +\infty, \quad 0 < r < \infty, \quad 0 \leq \phi \leq 2\pi, \quad -\infty < z < +\infty. \tag{5} \]
1. The Interior solution

The metric for the interior solution \( r < R \), where \( R \) is the surface of the cylinder, is given by

\[
ds^2 = -dt^2 + 2\omega^2 r^2 d\phi dt + r^2(1 - \omega^2 r^2) d\phi^2 + \exp(-\omega^2 r^2)(dr^2 + dz^2)
\]

(6)

where \( \omega \) is the angular velocity of the cylinder. It is immediate to verify that CTCs arise if \( \omega r > 1 \), i.e., for \( r > 1/\omega \) the azimuthal curves with \((t, r, z)\) fixed are CTCs. The condition \( M^2 + FL = \omega^2 r^4 + r^2(1 - \omega^2 r^2) = r^2 > 0 \) is imposed.

The causality violation region could be eliminated by requiring that boundary of the cylinder to be at \( r = R < 1/\omega \). The interior solution would then be joined to an exterior solution, which would be causally well-behaved. The resulting upper bound to the “velocity” \( \omega R \) would be 1, although the orbits of the particles creating the field are timelike for all \( r \).

Applying the EFE, the energy density and 4-velocity of the dust are given by

\[
8\pi \rho = 4\omega^2 \exp(\omega^2 r^2) \quad \text{and} \quad U^\mu = (1, 0, 0, 0),
\]

(7)

respectively. The coordinate system co-rotates with the dust. The source is simply positive density dust, implying that all of the energy condition are satisfied.

2. The Exterior solution

Van Stockum developed a procedure which generates an exterior solution for all \( \omega R > 0 \) [15]. Consider the following range:

(i) \( 0 < \omega R < 1/2 \).

The exterior solution is given by the following functions

\[
H(r) = \exp(-\omega^2 r^2) \left( \frac{r}{R} \right)^{1-2\omega^2 r^2}, \quad L(r) = \frac{Rr \sinh(3\varepsilon + \theta)}{2 \sinh(2\varepsilon) \cosh(\varepsilon)},
\]

\[
M(r) = \frac{r \sinh(\varepsilon + \theta)}{\sinh(2\varepsilon)}, \quad F(r) = \frac{r \sin(\varepsilon - \theta)}{R \sinh(\varepsilon)},
\]

with

\[
\theta = \theta(r) = (1 - 4\omega^2 R^2)^{1/2} \ln \left( \frac{r}{R} \right) \quad \text{and} \quad \varepsilon = \varepsilon(r) = \arctanh(1 - 4\omega^2 R^2)^{1/2}.
\]

(ii) \( \omega R = 1/2 \).

\[
H(r) = \exp(-1/4) \left( \frac{r}{R} \right)^{-1/2}, \quad L(r) = \frac{Rr}{4} \left[ 3 + \ln \left( \frac{r}{R} \right) \right],
\]

\[
M(r) = \left( \frac{r}{2} \right) \left[ 1 + \ln \left( \frac{r}{R} \right) \right], \quad F(r) = \frac{r}{R} \left[ 1 - \ln \left( \frac{r}{R} \right) \right].
\]

(iii) \( \omega R > 1/2 \).

\[
H(r) = \exp(-\omega^2 r^2) \left( \frac{r}{R} \right)^{-2\omega^2 r^2}, \quad L(r) = \frac{Rr \sin(3\beta + \gamma)}{2 \sin(2\beta) \cos(\beta)},
\]

\[
M(r) = \frac{r \sin(\beta + \gamma)}{\sin(2\beta)}, \quad F(r) = \frac{r \sin(\beta - \gamma)}{R \sin(\beta)},
\]

with

\[
\gamma = \gamma(r) = (4\omega^2 R^2 - 1)^{1/2} \ln \left( \frac{r}{R} \right) \quad \text{and} \quad \beta = \beta(r) = \arctan(4\omega^2 R^2 - 1)^{1/2}.
\]

As in the interior solution, \( FL + M^2 = r^2 \), so that the metric signature is Lorentzian for \( R \leq r < \infty \).
3. Chronology violation region

One may show that the causality violation is avoided for $\omega R \leq 1/2$, but in the region $\omega R > 1/2$, CTCs appear. The causality violations arise from the sinusoidal factors of the metric components. The first zero of $L(r)$ occurs at

$$r_0 = R \left[ \frac{\pi - 3 \arctan(4\omega^2 R^2 - 1)^{1/2}}{(4\omega^2 R^2 - 1)^{1/2}} \right].$$

(8)

Thus, causality violation occurs in the matter-free space surrounding a rapidly rotating infinite cylinder, as shown in Figure 2. The van Stockum spacetime is not asymptotically flat. But, the gravitational potential of the cylinder’s Newtonian analog also diverges at radial infinity. Shrinking the cylinder down to a “ring” singularity, one ends up with the Kerr solution, which also has CTCs (The causal structure of the Kerr spacetime has been extensively analyzed by de Felice and collaborators [16–20]).

In summary, the van Stockum solution contains CTC provided $\omega R > 1/2$. The chronology-violating region covers the entire spacetime. Reactions to the van Stockum solution is that it is unphysical, as it applies to an infinitely long cylinder and it is not asymptotically flat.

![Diagram of van Stockum spacetime](image)

FIG. 2: Van Stockum spacetime showing the tipping over of light cones close to the cylinder, due to the strong curvature of spacetime, which induce closed timelike curves.

B. The Gödel Universe

Kurt Gödel in 1949 discovered an exact solution to the EFEs of a uniformly rotating universe containing dust and a nonzero cosmological constant [21]. The total energy-momentum is given by

$$T_{total}^{\mu\nu} = \rho U^\mu U^\nu - \frac{\Lambda}{8\pi} g^{\mu\nu}.$$ 

(9)
However, the latter may be expressed in terms of a perfect fluid, with rotation, energy density $\bar{\rho}$ and pressure $\bar{p}$, in a universe with a zero cosmological constant, i.e.,

$$T_{\mu\nu}^{\text{total}} = (\bar{\rho} + \bar{p}) U^\mu U^\nu + \bar{g}^\mu\nu,$$

with the following definitions

$$\bar{\rho} = \rho + \frac{\Lambda}{8\pi} \quad \text{and} \quad \bar{p} = -\frac{\Lambda}{8\pi}. \quad \quad (11)$$

The manifold is $R^4$ and the metric of the Gödel solution is provided by

$$ds^2 = -dt^2 - 2e^{\sqrt{2}\omega x} dtdy - \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + dx^2 + dz^2. \quad \quad (12)$$

The four-velocity and the vorticity of the fluid are, $U^\mu = \delta^\mu_0 = (1, 0, 0, 0)$ and $\omega^\mu = (0, 0, 0, \omega)$, respectively. The Einstein field equations provide the following stress-energy scenario:

$$4\pi \rho = \omega^2 = -\Lambda \quad \text{or} \quad \bar{p} = \bar{\rho} = \frac{\omega^2}{8\pi} > 0. \quad \quad (13)$$

Thus, the null, weak and dominant energy conditions are satisfied, while the dominant energy condition is in the imminence of being violated.

Note that the metric (12) is the direct sum of the metrics $g_1$ and $g_2$. The metric $g_1$ is given by

$$ds_1^2 = -dt^2 - 2e^{\sqrt{2}\omega x} dtdy - \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + dx^2,$$

with the manifold $\mathcal{M}_1 = R^3$ defined by the coordinates $(t, x, y)$. The metric $g_2$ is given by $ds_2^2 = dz^2$, with the manifold $\mathcal{M}_2 = R$, defined by the coordinate $z$.

To analyze the causal properties of the solution, it is sufficient to consider $(\mathcal{M}_1, g_1)$. Consider a set of alternative coordinates $(t', r, \phi)$ in $(\mathcal{M}_1, g_1)$, in which the rotational symmetry of the solution, around the axis $r = 0$, is manifest and suppressing the irrelevant $z$ coordinate, defined by $10$ $21$

$$\sqrt{2}\omega x = \sin \phi \sinh(2r),$$

$$e^{\sqrt{2}\omega x} = \cosh(2r) + \cos \phi \sinh(2r),$$

$$\tan \left[ \left( \phi + \omega t - \sqrt{2} t' \right) / 2 \right] = e^{-2r} \tan(\phi/2),$$

so that the metric (14) takes the form

$$ds^2 = 2w^{-2} \left[ -dt'^2 + dr^2 - (\sinh^2 r - \sin^2 \phi) d\phi^2 + 2(\sqrt{2}) \sinh r d\phi dt \right]. \quad \quad (15)$$

Moving away from the axis, the light cones open out and tilt in the $\phi$-direction. The azimuthal curves with $\gamma = \{t = \text{const}, r = \text{const}, z = \text{const}\}$ are CTCs if the condition $r > \ln(1 + \sqrt{2})$ is satisfied.

It is interesting to note that in the Gödel spacetime, closed timelike curves are not geodesics. However, Novello and Rebouças $22$ discovered a new generalized solution of the Gödel metric, of a shear-free nonexpanding rotating fluid, in which successive concentric causal and noncausal regions exist, with closed timelike curves which are geodesics. A complete study of geodesic motion in Gödel’s universe, using the method of the effective potential was further explored by Novello et al $23$. Much interest has been aroused in time travel in the Gödel spacetime, from which we may mention the analysis of the geodesical and non-geodesical motions considered by Pfarr $21$ and Malament $27$ $26$.

C. Gott Cosmic String time machine

1. Gravitational field of a Cosmic String

The string spacetime is assumed to be static and cylindrically symmetric, with the string lying along the axis of symmetry. The most general static, cylindrically symmetric metric has the form

$$ds^2 = -e^{2\nu(\rho)} dt^2 + e^{2\lambda(\rho)} \left( d\rho^2 + dz^2 \right) + e^{2\phi(\rho)} d\phi^2, \quad \quad (16)$$
where \( \nu, \Phi \) and \( \lambda \) are functions of \( \rho \). \( \phi = 0 \) and \( \phi = 2\pi \) are identified.

Suppose that the string has a uniform density \( \epsilon > 0 \), out to some cylindrical radius \( \rho_0 \). The end results will prove to be independent of \( \rho_0 \), so that the string’s transverse dimensions may be reduced to zero, yielding an unambiguous exact exterior metric for the string.

The stress-energy tensor of the string, in an orthonormal frame, is given by

\[
T_{ii} = -T_{zz} = \epsilon
\]

and all the other components are equal to zero, for \( \rho < \rho_0 \). The resulting EFEs are given by:

\[
-e^{-2\lambda} \left[ \Phi'' + (\Phi')^2 + \lambda'' \right] = 8\pi \epsilon ,
\]

\[
e^{-2\lambda} (\lambda' \Phi' + \nu' \lambda' + \lambda' \Phi') = 0 ,
\]

\[
e^{-2\lambda} \left[ \lambda'' + \nu'' + (\nu')^2 \right] = 0 ,
\]

\[
e^{-2\lambda} \left[ -\lambda' \Phi' - \nu' \lambda' + \Phi'' + (\Phi')^2 + \nu'' + (\nu')^2 + \nu' \Phi' \right] = -8\pi \epsilon ,
\]

where the prime denotes a derivative with respect to \( \rho \). These are non-linear equations for the metric functions, and are easily solved in the case of the uniform density string. Conservation of the stress-energy, \( T^\beta_{\alpha \beta} = 0 \), yields

\[
(\nu' + \lambda')\epsilon = 0 .
\]

This implies that through Eq. (21), \( \nu \) and \( \lambda \) are constant, and may be set to zero by an appropriate rescaling of the coordinates \( t, \rho \) and \( z \). Equation (19) is then satisfied automatically and eqs. (18) and (21) become identical, i.e.,

\[
\Phi'' + (\Phi')^2 = -8\pi\epsilon .
\]

Substituting \( R = e^\Phi \), i.e., \( g_{\phi\phi} = R^2 \), yields

\[
R = A \cos(\rho/\bar{\rho}) + B \sin(\rho/\bar{\rho}) ,
\]

where \( \bar{\rho} = (8\pi\epsilon)^{-1/2} \). The metric on the axis will be flat, i.e., no cone singularity, if \( A = 0 \) and \( B = \bar{\rho} \). Thus, the interior metric of a uniform-density string is then given by

\[
ds^2 = -dt^2 + d\rho^2 + \frac{1}{8\pi\epsilon} \sin^2 \left( \sqrt{8\pi\epsilon} \rho \right) d\phi^2 + dz^2 .
\]

The exterior metric for the string spacetime must be a static, cylindrically symmetric, vacuum solution of the EFEs. The most general solution, discovered by Levi-Civita \[30\] is given

\[
ds^2 = -r^{2m}dT^2 + r^{-2m} \left[ r^{2m^2}(dt^2 + dZ^2) + a^2 r^2 d\phi^2 \right] ,
\]

where \( m \) and \( a \) are freely chosen constants. The string is Lorentz invariant in the \( z \)-direction. Requiring that the metric (26) be Lorentz invariant in the \( z \)-direction restricts the values of \( m \), namely, \( m = 0 \) and \( m = 2 \) \[31\].

One may now join the interior and exterior metrics together along the surface of the string at \( \rho = \rho_0 \) and \( r = r_0 \). The Darmois-Israel junction conditions require that the intrinsic metrics induced on the junction surface by the interior and exterior metrics be identical, and that the discontinuity in the extrinsic curvature of the surface be related to the surface stress-energy. Consider the \( m = 0 \) flat exterior case.

The intrinsic metric can then be matched by requiring \( t = T, \ z = Z \) and \( g^{+}_{\phi\phi} = g^{-}_{\phi\phi} \). The latter condition provides

\[
ar_0 = \bar{\rho} \sin(\rho_0/\bar{\rho}) .
\]

Calculating the extrinsic curvature tensors and equating them to each other, so as to have no surface stress-energy present, one obtains the following relation

\[
a^2 = \frac{\bar{\rho}^2}{\bar{\rho}^2 + r_0^2} .
\]

Combining this with the intrinsic metric constraint, Eq. (27), to eliminate \( r_0 \), yields

\[
a = \cos(\rho_0/\bar{\rho}) .
\]

The exterior metric of the string is then given by Eq. (26) with \( m = 0 \), and \( a \) given by Eq. (29).
The concept of a mass per unit length for a cylindrically symmetric source in general relativity is not unambiguously defined, unlike the case of spherical symmetry. For a static, cylindrically symmetric spacetime, a useful simple definition is to integrate the energy-density, \( \epsilon \) over the proper volume of the source, i.e., the string.

The mass per unit length, or linear energy-density, is given by

\[
\mu = \int_0^{\rho_0} \int_0^{2\pi} \epsilon \bar{\rho} \sin(\rho/\bar{\rho}) \, d\phi \, d\rho = 2\pi \epsilon \bar{\rho}^2 \left[ 1 - \cos(\rho/\bar{\rho}) \right],
\]

which, taking into account \( \bar{\rho} = (8\pi\epsilon)^{-1/2} \), reduces to

\[
4\mu = 1 - \cos(\rho_0/\bar{\rho}).
\]

Thus, the exact exterior metric is given by

\[
ds^2 = -dt^2 + dr^2 + (1 - 4\mu)^2 r^2 \, d\theta^2 + dz^2,
\]

which will be used below in the Gott cosmic string spacetime.

### 2. Gott Cosmic String spacetime

An extremely elegant model of a time-machine was constructed by Gott [32]. The Gott time-machine is an exact solution of the EFE for the general case of two moving straight cosmic strings that do not intersect [32]. This solution produces CTCs even though they do not violate the WEC, have no singularities and event horizons, and are not topologically multiply-connected as the wormhole solution (see below). The appearance of CTCs relies solely on the gravitational lens effect and the relativity of simultaneity. We follow the analysis of Ref. [32] closely throughout this section.

The exterior metric of a straight cosmic string is given by Eq. (32). The geometry of a \( t = \text{const.}, z = \text{const} \) section of this solution is that of a cone with an angle deficit \( D = 8\pi\mu \) in the exterior (vacuum) region. Applying a new coordinate \( \phi' = (1 - 4\mu)\phi \), the exterior metric becomes

\[
ds^2 = -dt^2 + dr^2 + r^2 \, d\phi'^2 + dz^2,
\]

where \( 0 \leq \phi' < (1 - 4\mu)2\pi \). The above metric is the metric for Minkowski space in cylindrical coordinates where a wedge of angle deficit \( D = 8\pi\mu \) is missing, and points with coordinates \( (r, \phi' = 0, z, t) \) and \( (r, \phi' = 2\pi - 8\pi\mu, z, t) \) are identified.

Now, the static solution for two parallel cosmic strings separated by a distance \( 2d \) is constructed in the following manner. Consider the metric (32), by replacing the angular and radial coordinates, \( \phi' \) and \( r \), respectively by the Cartesian coordinates, \( x = r\sin(\phi' + 4\pi\mu) \) and \( y = r\cos(\phi' + 4\pi\mu) + d \). This reduces the metric to \( ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \), with the following restrictions:

\[
x^2 + (y - d)^2 \geq r_b^2, \quad |x| \geq (y - d) \tan(4\pi\mu),
\]

and the points with \( x = \pm(y - d)\tan(4\pi\mu) \) are identified.

The 3-surface \( y = 0 \), has the metric \( ds^2 = -dt^2 + dx^2 + dz^2 \) with zero intrinsic and extrinsic curvature, as it is part of a \( (3 + 1) \)-dimensional Minkowski spacetime. It is thus possible to produce a mirror-image copy of the region \( y \geq 0 \), including the interior solution, by joining it along the three-surface \( y = 0 \). This second solution lies in the \( y \leq 0 \) region. The two copies obey all the matching conditions along the surface \( y = 0 \), because the latter is a \( (2 + 1) \)-dimensional Minkowskian spacetime with zero intrinsic and extrinsic curvature. See Figure 3.3 for details.

Consider now two observers \( A \) and \( B \) at rest with respect to the cosmic strings with world lines given by

\[
x_A^t = (t_A, x_A, y_A, z_A) = (t, x_0, 0, 0) \quad \text{and} \quad x_B^t = (t_B, x_B, y_B, z_B) = (t, -x_0, 0, 0),
\]

respectively. It is possible to prove that observer \( B \) sees three images of the observer \( A \) [33]. The central image is from a geodesic passing through the origin 0, the two outer images, which are displaced from the central image by an angle \( \Delta\theta = 4\pi\mu \) on each side, represent geodesics that pass through events \( E_1 - E_2 \) and \( E_3 - E_4 \). Note that \( E_1 \) and \( E_2 \) are identified, as are \( E_3 \) and \( E_4 \).

Considering the following trigonometric relationship

\[
w^2 = (x_0 - y_0 \sin(4\pi\mu))^2 + (d + y_0 \cos(4\pi\mu))^2,
\]
it is simple to verify that the value of $y_0$ to minimize $w_0$ is $y_0 = x_0 \sin(4\pi \mu) - d \cos(4\pi \mu)$. Thus, we have $w_0 < x_0$ if $d < y_0$, and the light beam going through 0 with a gravitational lensing time delay between the two images of $\Delta t = 2(x_0 - w_0)$. Note that if a light beam traversing through $E_1 - E_2$ can beat a light beam traveling through 0, then so can a spaceship traveling at a high enough velocity, $\beta_r < 1$, relative to the string. The spaceship connects two events in the $y = 0$ $(2 + 1)$-dimensional Minkowski spacetime with a spacelike separation.

Let the spaceship begin at $A$ and end at $B$, given by the following events

$$E_i = (-\beta_r^{-1} w_0, x_0, 0, 0) \quad \text{and} \quad E_f = (\beta_r^{-1} w_0, -x_0, 0, 0), \quad (37)$$

respectively. The time for the spaceship to traverse from $E_i$, through $E_1 - E_2$ to $E_f$ is $t = 2\beta_r w_0$. The separation of $E_i$ and $-E_f$ is spacelike providing that $x_0^2 - \beta_r^{-2} w_0^2 > 0$, which can always be verified for high enough $\beta_r < 1$, for $w_0 < x_0$.

The following step is to give the $y \geq 0$ solution a boost with velocity $\beta_s$ in the $+x$-direction via a Lorentz transformation such that $E_i$ and $E_f$ become simultaneous in the laboratory frame. The velocity for the simultaneity to occur is $\beta_s = w_0 \beta_r^{-1} x_0^{-1}$. Analogously, we give the $y \leq 0$ solution a boost with velocity $\beta_s$ in the $-x$-direction. The two solutions $y \geq 0$ and $y \leq 0$ may still be matched together because the Lorentz transformations do not alter the fact that the boundary surface $t = 0$ in each solution is still a $(2 + 1)$-dimensional Minkowskian spacetime with zero intrinsic and extrinsic curvature.

The spaceship goes from $E_i$ through $E_1 - E_2$ and arrives at $E_f$, which is simultaneous in the laboratory frame. By symmetry, the spaceship travels in the opposite direction past the oppositely moving string through $E_3 - E_4$ and arrives back at event $E_i$, which is also simultaneous with $E_f$ in the laboratory frame. The spaceship has completed a CTC, as it encircles the two parallel cosmic strings as they pass each other in a sense opposite to that of the strings’ relative motion. In principle, it is also possible to find a reference frame in which the spaceship arrives at $E_i$ before its departure.

The events in the laboratory frame have the following coordinates: $E_{i,L} = (0, \gamma^{-1} x_0, 0, 0)$ and $E_{f,L} = (0, -\gamma^{-1} x_0, 0, 0)$ with $\gamma^2 = \frac{x_0^2}{x_0^2 - \beta_r^{-2} w_0^2}$ and since $\beta_r < 1$, we have

$$\gamma^2 > \frac{x_0^2}{x_0^2 - w_0^2} = \frac{y_0^2}{x_0^2 - d^2}, \quad (38)$$
or
\[
\gamma_s^2 > \frac{(\sin(4\pi\mu))^2}{1 - \frac{2d}{x_0 \tan(4\pi\mu)} - \frac{d^2}{x_0^2}}. \tag{39}
\]

Considering the following approximations, \(x_0 \gg d\), we have
\[
\gamma_s > (\sin(4\pi\mu))^{-1}, \tag{40}
\]
or simply
\[
\beta_s > \cos(4\pi\mu). \tag{41}
\]

For \(\mu = 10^{-6}\) expected for grand unified cosmic strings, we have \(\gamma_s > 8 \times 10^4\) in order to produce CTCs.

In the laboratory frame it is clear how the CTC is created. The \(E_1 - E_2\) and \(E_3 - E_4\) identifications allow the particle to effectively travel backward in time twice in the laboratory frame. The identifications of \(E_1 - E_2\) and \(E_3 - E_4\) is equivalent to having a complete Minkowski spacetime without the missing wedges where instantaneous, tachyonic, travel in the string rest frames between \(E_1\) and \(E_2\), \(E_3\) and \(E_4\), is possible.

It is also interesting to verify whether the CTCs in the Gott solution appear at some particular moment, i.e., when the strings approach each other’s neighborhood, or if they already pre-exist, i.e., they intersect any spacelike hypersurface. These questions are particularly important in view of Hawking’s Chronology Protection Conjecture \([34]\). This conjecture states that the laws of physics prevent the creation of CTCs. If correct, then the solutions of the EFE which admit CTCs are either unrealistic or are solutions in which the CTCs are pre-existing, so that the time-machine is not created by dynamical processes. Amos Ori proved that in Gott’s spacetime, CTCs intersect every \(t = \text{const}\) hypersurface \([35]\), so that it is not a counter-example to the Chronology Protection Conjecture.

The global structure of the Gott spacetime was further explored by Cutler \([36]\), and it was shown that the closed timelike curves are confined to a certain region of the spacetime, and that the spacetime contains complete spacelike and achronal hypersurfaces from which the causality violating regions evolve. Grant also examined the global structure of the two-string spacetime and found that away from the strings, the space is identical to a generalized Misner space \([37]\). The vacuum expectation value of the energy-momentum tensor for a conformally coupled scalar field was then calculated on the respective generalized Misner space, which was found to diverge weakly on the chronology horizon, but diverge strongly on the polarized hypersurfaces. Thus, the back reaction due to the divergent behavior around the polarized hypersurfaces are expected to radically alter the structure of spacetime, before quantum gravitational effects become important, suggesting that Hawking’s chronology protection conjecture holds for spaces with a noncompactly generated chronology horizon. Soon after, Laurence \([38]\) showed that the region containing CTCs in Gott’s two-string spacetime is identical to the regions of the generalized Misner space found by Grant, and constructed a family of isometries between both Gott’s and Grant’s regions. This result was used to argue that the slowly diverging vacuum polarization at the chronology horizon of the Grant space carries over without change to the Gott space. Furthermore, it was shown that the Gott time machine is unphysical in nature, for such an acausal behavior cannot be realized by physical and timelike sources \([39-43]\).

### D. Spinning Cosmic String

Consider an infinitely long straight string that lies and spins around the \(z\)-axis. The symmetries are analogous to the van Stockum spacetime, but the asymptotic behavior is different \([4, 44]\). We restrict the analysis to an infinitely long straight string, with a delta-function source confined to the \(z\)-axis. It is characterized by a mass per unit length, \(\mu\); a tension, \(\tau\), and an angular momentum per unit length, \(J\). For cosmic strings, the mass per unit length is equal to the tension, \(\mu = \tau\).

In cylindrical coordinates the metric takes the following form
\[
d{s}^2 = - [d(t + 4J\varphi)]^2 + dr^2 + (1 - 4\mu)^2 r^2 d\varphi^2 + dz^2, \tag{42}
\]
with the following coordinate range
\[
-\infty < t < +\infty, \quad 0 < r < \infty, \quad 0 \leq \varphi \leq 2\pi, \quad -\infty < z < +\infty. \tag{43}
\]

Adopting a new set of coordinates
\[
\tilde{t} = t + 4J\varphi = (1 - 4J)\varphi, \tag{44}
\]
the metric may be rewritten as

\[ ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2 + dz^2, \]  

(45)

with a new coordinate range

\[-\infty < t < +\infty, \quad 0 < r < \infty, \quad 0 \leq \varphi \leq (1 - 4\mu)2\pi, \quad -\infty < z < +\infty, \]  

(46)

subject to the following identifications

\[ (\bar{t}, \bar{r}, \bar{\varphi}, \bar{z}) \equiv \left[ t + 8\pi J, r, \varphi + 2\pi(1 - 4\mu), z \right]. \]  

(47)

Outside the core \( r = 0 \), the metric is locally flat, i.e., the Riemann tensor is zero. The geometry is that of flat Minkowski spacetime subject to a somewhat peculiar set of identifications. On traveling once around the string, one sees that the spatial slices are "missing" a wedge of angle \( 8\pi\mu \), which defines the deficit angle \( \Delta\theta = 8\pi\mu \). On traveling once around the string, one undergoes a backward time-jump of

\[ \Delta\bar{t} = 8\pi J. \]  

(48)

Consider an azimuthal curve, i.e., an integral curve of \( \varphi \). Closed timelike curves appear whenever

\[ r < \frac{4J}{1 - 4\mu}. \]  

(49)

These CTCs can be deformed to cover the entire spacetime, consequently, the chronology-violating region covers the entire manifold.

III. SOLUTIONS VIOLATING THE ENERGY CONDITIONS

The traditional manner of solving the EFEs, \( G_{\mu\nu} = 8\pi T_{\mu\nu} \), consists in considering a plausible stress-energy tensor, \( T_{\mu\nu} \), and finding the geometrical structure, \( G_{\mu\nu} \). But one can run the EFE in the reverse direction by imposing an exotic metric \( g_{\mu\nu} \), and eventually finding the matter source for the respective geometry. In this fashion, solutions violating the energy conditions have been obtained. Adopting the reverse philosophy, solutions such as traversable wormholes, the warp drive, the Krasnikov tube and the Ori-Soen spacetime have been obtained. These solutions violate the energy conditions and with simple manipulations generate CTCs.

A. Conversion of traversable wormholes into time machines

Much interest has been aroused in traversable wormholes since the classical article by Morris and Thorne [45]. A wormhole is a hypothetical tunnel which connects different regions in spacetime. These solutions are multiply-connected and probably involve a topology change, which by itself is a problematic issue.

Consider the following spherically symmetric and static wormhole solution

\[ ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]  

(50)

where \( \Phi(r) \) and \( b(r) \) are arbitrary functions of the radial coordinate \( r \). \( \Phi(r) \) is denoted the redshift function, for it is related to the gravitational redshift, and \( b(r) \) is denoted the shape function, because as can be shown by embedding diagrams, it determines the shape of the wormhole [43]. The coordinate \( r \) is non-monotonic in that it decreases from \( +\infty \) to a minimum value \( r_0 \), representing the location of the throat of the wormhole, where \( b(r_0) = r_0 \), and then it increases from \( r_0 \) to \( +\infty \).

A fundamental property of a wormhole is that a flaring out condition of the throat, given by \( (b - b'/r)/b^2 > 0 \), is imposed [45], and at the throat \( b(r_0) = r = r_0 \), the condition \( b'(r_0) < 1 \) is imposed to have wormhole solutions. It is precisely these restrictions that impose the NEC violation in classical general relativity. Another condition that needs to be satisfied is \( 1 - b(r)/r > 0 \). For the wormhole to be traversable, one must demand that there are no horizons present, which are identified as the surfaces with \( e^{2\Phi} \to 0 \), so that \( \Phi(r) \) must be finite everywhere.

Several candidates have been proposed in the literature, amongst which we refer to solutions in higher dimensions, for instance in Einstein-Gauss-Bonnet theory [46, 47], wormholes on the brane [48]; solutions in Brans-Dicke theory...
wormholes constructed in $f(R)$ gravity; wormhole solutions in semi-classical gravity (see Ref. and references therein); exact wormhole solutions using a more systematic geometric approach were found; wormhole solutions and thin shells; geometries supported by equations of state responsible for the cosmic acceleration; spherical wormholes were also formulated as an initial value problem with the throat serving as an initial value surface; solutions in conformal Weyl gravity were found, and thin accretion disk observational signatures were also explored, etc (see Refs. for more details and for a recent review).

One of the most fascinating aspects of wormholes is their apparent ease in generating CTCs. There are several ways to generate a time machine using multiple wormholes, but a manipulation of a single wormhole seems to be the simplest way. The basic idea is to create a time shift between both mouths. This is done invoking the time dilation effects in special relativity or in general relativity, i.e., one may consider the analogue of the twin paradox, in which the mouths are moving with respect to the other, or simply the case in which one of the mouths is placed in a strong gravitational field.

To create a time shift using the twin paradox analogue, consider that the mouths of the wormhole may be moving with respect to the other in external space, without significant changes of the internal geometry of the handle. For simplicity, consider that one of the mouths $A$ is at rest in an inertial frame, whilst the other mouth $B$, initially at rest practically close by to $A$, starts to move out with a high velocity, then returns to its starting point. Due to the Lorentz time contraction, the time interval between these two events, $\Delta T_B$, measured by a clock comoving with $B$ can be made to be significantly shorter than the time interval between the same two events, $\Delta T_A$, as measured by a clock resting at $A$. Thus, the clock that has moved has been slowed by $\Delta T_A - \Delta T_B$ relative to the standard inertial clock. Note that the tunnel (handle), between $A$ and $B$ remains practically unchanged, so that an observer comparing the time of the clocks through the handle will measure an identical time, as the mouths are at rest with respect to one another. However, by comparing the time of the clocks in external space, he will verify that their time shift is precisely $\Delta T_A - \Delta T_B$, as both mouths are in different reference frames, frames that moved with high velocities with respect to one another. Now, consider an observer starting off from $A$ at an instant $T_0$, measured by the clock stationed at $A$. He makes his way to $B$ in external space and enters the tunnel from $B$. Consider, for simplicity, that the trip through the wormhole tunnel is instantaneous. He then exits from the wormhole mouth $A$ into external space at the instant $T_0 - (\Delta T_A - \Delta T_B)$ as measured by a clock positioned at $A$. His arrival at $A$ precedes his departure, and the wormhole has been converted into a time machine. See Figure 4.

For concreteness, following the Morris et al analysis, consider the metric of the accelerating wormhole given by

$$ds^2 = -(1 + g(l) \cos \theta)^2 e^{2\Phi(l)} dt^2 + dl^2 + r^2(l) (d\theta^2 + \sin^2 \theta d\phi^2),$$

where the proper radial distance, $dl = (1 - b/r)^{-1/2} dr$, is used. $F(l)$ is a form function that vanishes at the wormhole mouth $A$, at $l \leq 0$, rising smoothly from 0 to 1, as one moves to mouth $B$: $g = g(t)$ is the acceleration of mouth $B$ as measured in its own asymptotic rest frame. Consider that the external metric to the respective wormhole mouths is $ds^2 \cong -dt^2 + dX^2 + dY^2 + dZ^2$. Thus, the transformation from the wormhole mouth coordinates to the external Lorentz coordinates is given by

$$T = t, \quad Z = Z_A + l \cos \theta, \quad X = l \sin \theta \cos \phi, \quad X = l \sin \theta \sin \phi,$$

for mouth $A$, where $Z_A$ is the time-independent $Z$ location of the wormhole mouth $A$, and

$$T = T_B + v\gamma l \cos \theta, \quad Z = Z_B + \gamma l \cos \theta, \quad X = l \sin \theta \cos \phi, \quad X = l \sin \theta \sin \phi,$$

for the accelerating wormhole mouth $B$. The world line of the center of mouth $B$ is given by $Z = Z_B(t)$ and $T = T_B(t)$ with $ds^2 = dT_B^2 - dZ_B^2$; $v(t) \equiv dZ_B/dT_B$ is the velocity of mouth $B$ and $\gamma = (1 - v^2)^{-1/2}$ the respective Lorentz factor; the acceleration appearing in the wormhole metric is given $g(t) = \gamma^2 dv/dT$.

Novikov considered other variants of inducing a time shift through the time dilation effects in special relativity, by using a modified form of the metric, and by considering a circular motion of one of the mouths with respect to the other. Another interesting manner to induce a time shift between both mouths is simply to place one of the mouths in a strong external gravitational field, so that times slows down in the respective mouth. The time shift will be given by

$$T = \int_{t_A}^{t_B} (\sqrt{g_{tt}(x_A)} - \sqrt{g_{tt}(x_B)} ) \ dt$$

B. The Ori-Soen time machine

A time-machine model was also proposed by Amos Ori and Yoav Soen which significantly ameliorates the conditions of the EFE’s solutions which generate CTCs. The Ori-Soen model presents some notable features. It was verified that CTCs evolve, within a bounded region of space, from a well-defined initial slice $S$, a partial Cauchy
FIG. 4: Depicted are two examples of wormhole spacetimes with closed timelike curves. The wormholes tunnels are arbitrarily short, and its two mouths move along two world tubes depicted as thick lines in the figure. Proper time $\tau$ at the wormhole throat is marked off, and note that identical values are the same event as seen through the wormhole handle. In Figure (a), mouth A remains at rest, while mouth B accelerates from A at a high velocity, then returns to its starting point at rest. A time shift is induced between both mouths, due to the time dilation effects of special relativity. The light cone-like hypersurface $H$ shown is a Cauchy horizon. Through every event to the future of $H$ there exist CTCs, and on the other hand there are no CTCs to the past of $H$. In Figure (b), a time shift between both mouths is induced by placing mouth B in strong gravitational field. See text for details.

Within the framework of general relativity, it is possible to warp spacetime in a small bubblelike region, in such a way that the bubble may attain arbitrarily large velocities, $v(t)$. Inspired in the inflationary phase of the early Universe, the enormous speed of separation arises from the expansion of spacetime itself. The model for hyperfast travel is to create a local distortion of spacetime, producing an expansion behind the bubble, and an opposite contraction ahead of it (see also \cite{72}).

In the Alcubierre warp drive the spacetime metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + [dz - v(t) f(x, y, z - z_0(t))] dt^2. \quad (54)$$

The form function $f(x, y, z)$ possesses the general features of having the value $f = 0$ in the exterior and $f = 1$ in the interior of the bubble. The general class of form functions, $f(x, y, z)$, chosen by Alcubierre was spherically symmetric.
Consider the following form
\[ f(r) = \frac{\tanh[\sigma(r + R)] - \tanh[\sigma(r - R)]}{2 \tanh(\sigma R)}, \]
(56)
in which \( R > 0 \) and \( \sigma > 0 \) are two arbitrary parameters. \( R \) is the “radius” of the warp-bubble, and \( \sigma \) can be interpreted as being inversely proportional to the bubble wall thickness. If \( \sigma \) is large, the form function rapidly approaches a top hat function, i.e.,
\[ \lim_{\sigma \to \infty} f(r) = \begin{cases} 1, & \text{if } r \in [0, R], \\ 0, & \text{if } r \in (R, \infty). \end{cases} \]
(57)

It can be shown that observers with the four velocity
\[ U^\mu = (1, 0, 0, vf), \quad U_\mu = (-1, 0, 0, 0). \]
(58)
move along geodesics, as their 4-acceleration is zero, i.e., \( a^\mu = U^\nu U'^{\nu, \mu} = 0 \). The spaceship, which in the original formulation is treated as a test particle which moves along the curve \( z = z_0(t) \), can easily be seen to always move along a timelike curve, regardless of the value of \( v(t) \). One can also verify that the proper time along this curve equals the coordinate time, by simply substituting \( z = z_0(t) \) in Eq. (54). This reduces to \( d\tau = dt \), taking into account \( dx = dy = 0 \) and \( f(0) = 1 \).

Consider a spaceship placed within the Alcubierre warp bubble. The expansion of the volume elements, \( \theta = U^\mu;_\mu \), is given by \( \theta = v \left( \partial f / \partial z \right) \). Taking into account Eq. (56), we have (for Alcubierre’s version of the warp bubble)
\[ \theta = v \frac{z - z_0}{r} \frac{df(r)}{dr}. \]
(59)
The center of the perturbation corresponds to the spaceship’s position \( z_0(t) \). The volume elements are expanding behind the spaceship, and contracting in front of it, as shown in Figure 5.

One may consider a hypothetical spaceship immersed within the bubble, moving along a timelike curve, regardless of the value of \( v(t) \). Due to the arbitrary value of the warp bubble velocity, the metric of the warp drive permits superluminal travel, which raises the possibility of the existence of CTCs. Although the solution deduced by Alcubierre by itself does not possess CTCs, Everett demonstrated that these are created by a simple modification of the Alcubierre metric [73], by applying a similar analysis as in tachyons.

The modified metric takes the form
\[ ds^2 = -dt^2 + dx^2 + dy^2 + (dz - vf dt)^2, \]
(60)
with

\[ v(t) = \frac{dz_0(t)}{dt} \quad \text{and} \quad r(t) = \sqrt{(z - z_0)^2 + (y - y_0)^2 + z^2}. \]  

(61)

The spacetime is flat in the exterior of a warp bubble with radius \( R \), but now in the modified version is centered in \((0, y_0, z_0(t))\). The bubble moves with a velocity \( v \), on a trajectory parallel with the \( z \)-axis. One may for simplicity consider the form function given by Eq. \((59)\). We shall also impose that \( y_0 \gg R \), so that the form function is negligible, i.e., \( f(y_0) \approx 0 \).

Now, consider two stars, \( S_1 \) and \( S_2 \), at rest in the coordinate system of the metric \((60)\), and located on the \( z \)-axis at \( t = 0 \) and \( t = D \), respectively. The metric along the \( z \)-axis is Minkowskian as \( y_0 \gg R \). Therefore, a light beam emitted at \( S_1 \), at \( t = 0 \), moving along the \( z \)-axis with \( dz/dt = 1 \), arrives at \( S_2 \) at \( t = D \). Suppose that the spaceship initially starts off from \( S_1 \), with \( v = 0 \), moving off to a distance \( y_0 \) along the \( y \)-axis and neglecting the time it needs to cover \( y = 0 \) to \( y = y_0 \). At \( y_0 \), it is then subject to a uniform acceleration, \( a \), along the \( z \)-axis for \( 0 < z < D/2 \), and \(-a \) for \( D/2 < z < D \). The spaceship will arrive at the spacetime event \( S_2 \) with coordinates \( z = D \) and \( t = 2\sqrt{D/a} \equiv T \). Once again, the time required to travel from \( y = y_0 \) to \( y = 0 \) is negligible.

The separation between the two events, departure and arrival is \( D^2 - T^2 = D^2(1 - 4/(aD)) \) and will be spatial if the following condition is verified

\[ a > \frac{4}{D}. \]  

(62)

In this case, the spaceship will arrive at \( S_2 \) before the light beam, if the latter’s trajectory is a straight line, and both departures are simultaneous from \( S_1 \). Inertial observers situated in the exterior of the spaceship, at \( S_1 \) and \( S_2 \), will consider the spaceship’s movement as superluminal, since the distance \( D \) is covered in an interval \( T < D \). However, the spaceship’s worldline is contained within it’s light cone. The worldline of the spaceship is given by \( z = vt \), while it’s future light cone is given by \( z = (v \pm 1)t \). The latter relation can easily be inferred from the null condition, \( ds^2 = 0 \).

Since the quadri-vector with components \((T, 0, 0, D)\) is spatial, the temporal order of the events, departure and arrival, is not well-defined. Introducing new coordinates, \((t’, x’, y’, z’)\), obtained by a Lorentz transformation, with a boost \( \beta \) along the \( z \)-axis. The arrival at \( S_2 \) in the \((t’, x’, y’, z’)\) coordinates correspond to

\[ T’ = \gamma(2\sqrt{D/a} - \beta D), \quad Z’ = \gamma(D - 2\sqrt{D/a}), \]  

(63)

with \( \gamma = (1 - \beta^2)^{-1/2} \). The events, departure and arrival, will be simultaneous if \( a = 4/(\beta^2 D) \). The arrival will occur before the departure if \( T’ < 0 \), i.e.,

\[ a > 4/(\beta^2 D). \]  

(64)

The fact that the spaceship arrives at \( S_2 \) with \( t’ < 0 \), does not by itself generate CTCs. Consider the metric, Eq. \((63)\), substituting \( z \) and \( t \) by \( \Delta z’ = z’ - Z’ \) and \( \Delta t’ = t’ - T’ \), respectively; \( v(t) \) by \(-v(t)\); \( a \) by \(-a\); and \( y_0 \) by \(-y_0\). This new metric describes a spacetime in which an Alcubierre bubble is created at \( t’ = T’ \), which moves along \( y = -y_0 \) and \( x = 0 \), from \( S_1 \) to \( S_2 \) with a velocity \( v’(t) \), and subject to an acceleration \( a’ \). For observers at rest relatively to the coordinates \((t’, x’, y’, z’)\), situated in the exterior of the second bubble, it is identical to the bubble defined by metric, Eq. \((60)\), as it is seen by inertial observers at rest at \( S_1 \) and \( S_2 \). The only differences reside in a change of the origin, direction of movement and possibly of the value of acceleration. The stars, \( S_1 \) and \( S_2 \), are at rest in the coordinate system of the metric, Eq. \((60)\), and in movement along the negative direction of the \( z \)-axis with velocity \( \beta \), relatively to the coordinates \((t’, x’, y’, z’)\). The two coordinate systems are equivalent due to the Lorentz invariance, so if the first is physically realizable, then so is the second. In the new metric, by analogy with Eq. \((63)\), we have \( dr = dt’ \), i.e., the proper time of the observer, on board of the spaceship, traveling in the center of the second bubble, is equal to the time coordinate, \( t’ \). The spaceship will arrive at \( S_1 \) in the temporal and spatial intervals given by \( \Delta t’ > 0 \) and \( \Delta x’ < 0 \), respectively. As in the analysis of the first bubble, the separation between the departure, at \( S_2 \), and the arrival \( S_1 \), will be spatial if the analogous relationship of Eq. \((63)\) is verified. Therefore, the temporal order between arrival and departure is also not well-defined. As will be verified below, when \( z \) and \( z’ \) decrease and \( t’ \) increases, \( t \) will decrease and a spaceship will arrive at \( S_1 \) at \( t < T \). In fact, one may prove that it may arrive at \( t < 0 \).

Since the objective is to verify the appearance of CTCs, in principle, one may proceed with some approximations. For simplicity, consider that \( a \) and \( a’ \), and consequently \( v \) and \( v’ \) are enormous, so that \( T \ll D \) and \( \Delta t’ \ll -\Delta z’ \). In this limit, we obtain the approximation \( T \approx 0 \), i.e., the journey of the first bubble from \( S_1 \) to \( S_2 \) is approximately instantaneous. Consequently, taking into account the Lorentz transformation, we have \( Z’ \approx \gamma D \) and \( T’ \approx -\gamma \beta D \). To determine \( T_1 \), which corresponds to the second bubble at \( S_1 \), consider the following considerations: since the acceleration is enormous, we have \( \Delta t’ \approx 0 \) and \( \Delta t = T_1 - T \approx T_1 \), therefore \( \Delta z = -D \approx \gamma \Delta z’ \) and \( \Delta t \approx \gamma \beta \Delta z’ \), from which one concludes that

\[ T_1 \approx -\beta D < 0. \]  

(65)
D. The Krasnikov tube and closed timelike curves

Krasnikov discovered an interesting feature of the warp drive, in which an observer in the center of the bubble is causally separated from the front edge of the bubble. Therefore he/she cannot control the Alcubierre bubble on demand. Krasnikov proposed a two-dimensional metric \[74\], which was later extended to a four-dimensional model \[75\]. One Krasnikov tube in two dimensions does not generate CTCs. But the situation is quite different in the 4-dimensional generalization, which we present for self-consistency and self-completeness.

Soon after the Krasnikov two-dimensional solution, Everett and Roman \[75\] generalized the analysis to four dimensions, denoting the solution as the Krasnikov tube. Consider that the 4-dimensional modification of the metric begins along the path of the spaceship, which is moving along the \(x\)-axis, occurring at position \(x\) at time \(t \approx x\), the time of passage of the spaceship. Also assume that the disturbance in the metric propagates radially outward from the \(x\)-axis, so that causality guarantees that at time \(t\) the region in which the metric has been modified cannot extend beyond \(\rho = t - x\), where \(\rho = (y^2 + z^2)^{1/2}\). The modification in the metric should also not extend beyond some maximum radial distance \(\rho_{\text{max}} \ll D\) from the \(x\)-axis. Thus, the metric in the 4-dimensional spacetime, written in cylindrical coordinates, is given by \[72\]

\[
ds^2 = -dt^2 + (1 - k(t, x, \rho))dxdt + k(t, x, \rho)d\rho^2 + \rho^2 d\phi^2,
\]

with

\[
k(t, x, \rho) = 1 - (2 - \delta)\theta_\varepsilon(\rho_{\text{max}} - \rho)\theta_\varepsilon(t - x - \rho)[\theta_\varepsilon(x) - \theta_\varepsilon(x + \varepsilon - D)].
\]

For \(t \gg D + \rho_{\text{max}}\) one has a tube of radius \(\rho_{\text{max}}\) centered on the \(x\)-axis, within which the metric has been modified. This structure is denoted by the Krasnikov tube. In contrast with the Alcubierre spacetime metric, the metric of the Krasnikov tube is static once it has been created.

The stress-energy tensor element \(T_{tt}\) given by

\[
T_{tt} = \frac{1}{32\pi(1+k)^2} \left[ -\frac{4(1+k)}{\rho} \frac{\partial k}{\partial \rho} + 3 \left( \frac{\partial k}{\partial \rho} \right)^2 - 4(1+k) \frac{\partial^2 k}{\partial \rho^2} \right],
\]

can be shown to be the energy density measured by a static observer \[75\], and violates the WEC in a certain range of \(\rho\), i.e., \(T_{\mu\nu}U^\mu U^\nu < 0\).

To verify the violation of the WEC, consider the energy density in the middle of the tube and at a time long after it’s formation, i.e., \(x = D/2\) and \(t \gg x + \rho + \varepsilon\), respectively. In this region we have \(\theta_\varepsilon(x) = 1\), \(\theta_\varepsilon(x + \varepsilon - D) = 0\) and \(\theta_\varepsilon(t - x - \rho) = 1\). With this simplification the form function, Eq. \[69\], reduces to

\[
k(t, x, \rho) = 1 - (2 - \delta)\theta_\varepsilon(\rho_{\text{max}} - \rho).
\]

Consider the following specific form for \(\theta_\varepsilon(\xi)\) \[72\] given by

\[
\theta_\varepsilon(\xi) = \frac{1}{2} \left\{ \tanh \left[ 2 \left( \frac{\varepsilon}{\xi} - 1 \right) \right] + 1 \right\},
\]

so that the form function of Eq. \[69\] is provided by

\[
k = 1 - \left( 1 - \frac{\delta}{2} \right) \left\{ \tanh \left[ 2 \left( \frac{\varepsilon}{\xi} - 1 \right) \right] + 1 \right\}.
\]

Choosing the following values for the parameters: \(\delta = 0.1\), \(\varepsilon = 1\) and \(\rho_{\text{max}} = 100\varepsilon = 100\), it can be shown that the negative character of the energy density is manifest in the immediate inner vicinity of the tube wall.

Now, using two such tubes it is a simple matter, in principle, to generate CTCs. The analysis is similar to that of the warp drive, so that it will be treated in summary.

Imagine a spaceship traveling along the \(x\)-axis, departing from a star, \(S_1\), at \(t = 0\), and arriving at a distant star, \(S_2\), at \(t = D\). An observer on board of the spaceship constructs a Krasnikov tube along the trajectory. It is possible for the observer to return to \(S_1\), traveling along a parallel line to the \(x\)-axis, situated at a distance \(\rho_0\), so that \(D \gg \rho_0 \gg 2\rho_{\text{max}}\), in the exterior of the first tube. On the return trip, the observer constructs a second tube, analogous to the first, but in the opposite direction, i.e., the metric of the second tube is obtained substituting \(x\) and \(t\), for \(X = D - x\) and \(T = t - D\), respectively in Eq. \[66\]. The fundamental point to note is that in three spatial dimensions it is possible to construct a system of two non-overlapping tube separated by a distance \(\rho_0\).
After the construction of the system, an observer may initiate a journey, departing from \( S_1 \), at \( x = 0 \) and \( t = 2D \). One is only interested in the appearance of CTCs in principle, therefore the following simplifications are imposed: \( \delta \) and \( \varepsilon \) are infinitesimal, and the time to travel between the tubes is negligible. For simplicity, consider the velocity of propagation close to that of light speed. Using the second tube, arriving at \( S_2 \) at \( x = D \) and \( t = D \), then travelling through the first tube, the observer arrives at \( S_1 \) at \( t = 0 \). The spaceship has completed a CTC, arriving at \( S_1 \) before it’s departure.

**IV. DISCUSSION**

GTR has been an extremely successful theory, with a well established experimental footing, at least for weak gravitational fields. It’s predictions range from the existence of black holes, gravitational radiation to the cosmological models, predicting a primordial beginning, namely the big-bang. However, it was seen that it is possible to find solutions to the EFEs, with certain ease, which generate CTCs. This implies that if we consider GTR valid, we need to include the possibility of time travel in the form of CTCs. A typical reaction is to exclude time travel due to the associated paradoxes. But the paradoxes do not prove that time travel is mathematically or physically impossible. Consistent mathematical solutions to the EFEs have been found, based on plausible physical processes. What they do seem to indicate is that local information in spacetimes containing CTCs are restricted in unfamiliar ways.

The grandfather paradox, without doubt, does indicate some strange aspects of spacetimes that contain CTCs. It is logically inconsistent that the time traveler murders his grandfather. But, one can ask, what exactly impeded him from accomplishing his murderous act if he had ample opportunities and the free-will to do so. It seems that certain conditions in local events are to be fulfilled, for the solution to be globally self-consistent. These conditions are denominated *consistency constraints* [76]. To eliminate the problem of free-will, mechanical systems were developed as not to convey the associated philosophical speculations on free-will [77, 78]. Much has been written on two possible remedies to the paradoxes, namely the Principle of Self-Consistency [61, 78, 80] and the Chronology Protection Conjecture [34].

One current of thought, led by Igor Novikov, is the Principle of Self-Consistency, which stipulates that events on a CTC are self-consistent, i.e., events influence one another along the curve in a cyclic and self-consistent way. In the presence of CTCs the distinction between past and future events are ambiguous, and the definitions considered in the causal structure of well-behaved spacetimes break down. What is important to note is that events in the future can influence, but cannot change, events in the past. The Principle of Self-Consistency permits one to construct local solutions of the laws of physics, only if these can be prolonged to a unique global solution, defined throughout non-singular regions of spacetime. Therefore, according to this principle, the only solutions of the laws of physics that are allowed locally, reinforced by the consistency constraints, are those which are globally self-consistent.

Hawking’s Chronology Protection Conjecture [34] is a more conservative way of dealing with the paradoxes. Hawking notes the strong experimental evidence in favor of the conjecture from the fact that ”we have not been invaded by hordes of tourists from the future”. An analysis reveals that the value of the renormalized expectation quantum stress-energy tensor diverges in the imminence of the formation of CTCs. This conjecture permits the existence of traversable wormholes, but prohibits the appearance of CTCs. The transformation of a wormhole into a time machine results in enormous effects of the vacuum polarization, which destroys it’s internal structure before attaining the Planck scale. Nevertheless, Li has shown given an example of a spacetime containing a time machine that might be stable against vacuum fluctuations of matter fields [81], implying that Hawking’s suggestion that the vacuum fluctuations of quantum fields acting as a chronology protection might break down. There is no convincing demonstration of the Chronology Protection Conjecture, but the hope exists that a future theory of quantum gravity may prohibit CTCs.

Visser still considers the possibility of two other conjectures [4]. The first is the radical reformulation of physics conjecture, in which one abandons the causal structure of the laws of physics and allows, without restriction, time travel, reformulating physics from the ground up. The second is the boring physics conjecture, in which one simply ceases to consider the solutions to the EFEs generating CTCs. Perhaps an eventual quantum gravity theory will provide us with the answers. But, as stated by Thorne [82], it is by extending the theory to it’s extreme predictions that one can get important insights to it’s limitations, and probably ways to overcome them. Therefore, time travel in the form of CTCs, is more than a justification for theoretical speculation, it is a conceptual tool and an epistemological instrument to probe the deepest levels of GTR and extract clarifying views.
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[1] O. Bertolami and F. S. N. Lobo, NeuroQuantology 1 (2009) 1-15 [arXiv:0902.0559].
[2] F. Lobo and P. Crawford, “Time, Closed Timelike Curves and Causality,” in The Nature of Time: Geometry, Physics and Perception, NATO Science Series II. Mathematics, Physics and Chemistry - Vol. 95, Kluwer Academic Publishers, R. Buccheri et al. eds, pp. 289-296 (2003) [arXiv:gr-qc/0206078].
[3] F. S. N. Lobo, “Nature of time and causality in Physics,” in Psychology of Time, 395-422, Emerald, ISBN 978-0-8046-977-5, (2008) [arXiv:0710.0428].
[4] M. Visser, Loretzian wormholes: from Einstein to Hawking AIP Press (1995).
[5] F. Lobo and P. Crawford, “Weak energy condition violation and superluminal travel,” Current Trends in Relativistic Astrophysics, Theoretical, Numerical, Observational, Lecture Notes in Physics 617, Springer-Verlag Publishers, L. Fernández et al. eds, pp. 277–291 (2003) [arXiv:gr-qc/0204038].
[6] J. P. S. Lemos, F. S. N. Lobo and S. Q. de Oliveira, Phys. Rev. D 68, 064004 (2003).
[7] F. J. Tipler, Phys. Rev. Lett. 37, 879-882 (1976).
[8] P. J. Nahin, Time Machines: Time Travel in Physics, Metaphysics and Science Fiction, Springer-Verlag and AIP Press, New York (1999).
[9] J. R. Gott and L.-X. Li, Phys. Rev. D 58, 023501 (1998).
[10] S. W. Hawking and G.F.R. Ellis, The Large Scale Structure of Spacetime, (Cambridge University Press, Cambridge 1973).
[11] S. W. Kim and K. S. Thorne, Phys. Rev. D 43 3929 (1991).
[12] D. Deutsch, Phys. Rev. D 44, 3197 (1991).
[13] S. V. Krasnikov, Phys. Rev. D 54 7322 (1996).
[14] R. M. Wald, General Relativity, (University of Chicago Press, Chicago, 1984).
[15] F. J. Tipler, Phys. Rev. D 9, 2203 (1974).
[16] M. Calvani, F. de Felice, B. Muchotrzeb and F. Salmistraro, Gen. Rel. Grav. 10 335-342 (1979).
[17] F. de Felice and M. Calvani, Gen. Rel. Grav. 9 155-163 (1978).
[18] C. J. S. Clarke and F. de Felice, J. Phys. A 15 2415-2417 (1978).
[19] F. de Felice, Nuovo Cimento 65B 224-232 (1981).
[20] C. J. S. Clarke and F. de Felice, Gen. Rel. Grav. 16 139-148 (1984).
[21] K. Gödel, Rev. Mod. Phys. 21, 447 (1949).
[22] M. Novello and M. J. Rebouças, Phys. Rev. D 19, 2850 (1979).
[23] M. Novello, I. Damiao Soares and J. Tjonnø, Phys. Rev. D 27, 779 (1983).
[24] J. Pfarr, Gen. Rel. Grav. 13 1073 (1981).
[25] D. B. Malament, J. Math. Phys. 26 774 (1985).
[26] D. B. Malament, J. Math. Phys. 28 2427 (1987).
[27] A. Vilenkin, Phys. Rev. D 23, 852 (1981).
[28] R. Linet, Gen. Rel. Grav. 17 1109 (1985).
[29] W. A. Hiscock, Phys. Rev. D 31 3288 (1985).
[30] T. Levi-Civita, Ren. Acc. Lincei 26 307 (1917); 27 3, 183, 220, 240, 283, 343 (1918); 28 3, 101 (1919).
[31] D. Kramer, H. Stephani, E. Herlt, M. MacCallum and E. Schmutzer, Exact Solutions of Einstein’s Field Equations (Cambridge University Press, Cambridge, 1980).
[32] J. R. Gott, Phys. Rev. Lett. 66 1126 (1991).
[33] J. R. Gott, Astrophys. J. 288 422 (1985).
[34] S. W. Hawking, Phys. Rev. D 46, 603 (1992).
[35] A. Ori, Phys. Rev. D 44, R2214 (1991).
[36] C. Cutler, Phys. Rev. D 45, 487 (1992).
[37] J. D. E. Grant, Phys. Rev. D 47, 2388 (1993); J. D. E. Grant, Phys. Rev. D 47 2388 (1993).
[38] D. Laurence, Phys. Rev. D 50, 4957 (1994).
[39] S. Deser, R. Jackiw and G. ’t Hooft, Phys. Rev. Lett. 68 267 (1992).
[40] S. Deser and R. Jackiw, Comments Nucl. Part. Phys. 20 337 (1992).
[41] S. Deser, Class. Quant. Grav. 10 S67 (1993).
[42] S. M. Carroll, E. Farhi and A. H. Guth , Phys. Rev. Lett. 68, 263-266 (1992).
[43] S. M. Carroll, E. Farhi, A. H. Guth and K. D. Olum , Phys. Rev. D 50, 6190-6206 (1994).
[44] B. P. Jensen and H. H. Soleng, Phys. Rev. D 39 1130 (1989).
[45] M. S. Morris and K. S. Thorne, Am. J. Phys. 56, 395 (1988).
[46] B. Bhawal and S. Kar, Phys. Rev. D 48, 2464-2468 (1992).
[47] G. Dotti, J. Oliva, and R. Troncoso, Phys. Rev. D 75, 024002 (2007); H. Maeda and M. Nozawa, Phys. Rev. D 78, 024005 (2008).
