BPS $\mathbb{Z}_2$ monopoles and $\mathcal{N} = 2 \, SU(n)$ superconformal field theories on the Higgs branch

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Abstract

We obtain BPS $\mathbb{Z}_2$ monopole solutions in Yang-Mills-Higgs theories with the gauge group $SU(n)$ broken to $Spin(n)/\mathbb{Z}_2$ by a scalar field in the $n \otimes n$ representation. We show that the magnetic weights of the so-called fundamental $\mathbb{Z}_2$ monopoles correspond to the weights of the defining representation of the dual algebra $so(n)^\vee$, and the masses of the nonfundamental BPS $\mathbb{Z}_2$ monopoles are equal to the sum of the masses of the constituent fundamental monopoles. We also show that the vacua responsible for the existence of these $\mathbb{Z}_2$ monopoles are present in the Higgs branch of a class of $\mathcal{N} = 2 \, SU(n)$ superconformal field theories. We analyze some dualities these monopoles may satisfy.

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1 Introduction

Electromagnetic duality in Yang-Mills-Higgs theories was initially proposed by Goddard, Nuyts, and Olive (GNO) \cite{1} in gauge theories with gauge group $G$ spontaneously broken to $G_0$ by a scalar field $\phi$ in an arbitrary representation, in such a way that $\pi_2(G/G_0)$ is nontrivial, which allows the existence of monopole solutions. Soon after, Montonen and Olive duality was proposed \cite{2} considering a theory with gauge group $SU(2)$ spontaneously broken to $U(1)$ by a scalar field $\phi$ in the adjoint representation. Since then, monopole solutions and the electromagnetic duality have been studied mainly when the scalar field responsible for the symmetry breaking is in the adjoint representation. In this case, the unbroken gauge group $G_0$ necessarily has a $U(1)$ factor which guarantees that $\pi_2(G/G_0) = \mathbb{Z}$ and that the theory can have monopole solutions which we shall call $\mathbb{Z}$ monopoles. On the other hand, much less is known when $G_0$ is semisimple and therefore, $\phi$ necessarily cannot be in the adjoint representation. In these cases, a nontrivial $\pi_2(G/G_0)$ will be a cyclic group $\mathbb{Z}_n$ or a product of cyclic groups and the monopoles are called $\mathbb{Z}_n$ monopoles. Therefore, $\mathbb{Z}_n$ monopoles are relevant for GNO duality when $G_0$ is semisimple, which has renewed interest in the geometric Langlands program \cite{3}. These $\mathbb{Z}_n$ monopoles were analyzed, for example, in \cite{1,4,5} and more recently in \cite{6}.

One of the main motivations for the study of monopoles and electromagnetic dualities is their possible application to the problem of confinement in QCD. Following the ideas of ’t Hooft and Mandelstam, the formation of chromoelectric flux tubes in QCD must be due to a monopole condensate. However, it is not yet clear if these monopoles are $\mathbb{Z}$ monopoles, $\mathbb{Z}_n$ monopoles, or Dirac monopoles. In the last few years, the ideas of ’t Hooft and Mandelstam were applied to supersymmetric non-Abelian theories with $\mathbb{Z}$ monopoles. In particular the confinement of $\mathbb{Z}$ monopoles by the formation of magnetic flux tubes or $\mathbb{Z}_n$ strings in soft broken $\mathcal{N} = 4$ super Yang-Mills theories with arbitrary simple gauge groups was analyzed in \cite{7,8} and it was shown that in these theories the tensions of these $\mathbb{Z}_n$ strings satisfy the Casimir scaling law in the BPS limit, which is believed to be the behavior that the chromoelectric flux tubes in QCD must satisfy\cite{9}. This result indicates that these $\mathbb{Z}_n$ strings can be dual to QCD chromoelectric strings.

In order to understand the properties of the $\mathbb{Z}_n$ monopoles, in \cite{6} we obtained explicitly the asymptotic form of the $\mathbb{Z}_2$ monopoles in $SU(n)$ Yang-Mills-Higgs theories with the gauge group broken to $Spin(n)/\mathbb{Z}_2$ by a scalar in the $n \otimes n$ representation of $SU(n)$ or its symmetric part. In order to obtain these asymptotic forms, we generalized the construction in \cite{4} using the fact that the magnetic weights of the monopoles in this theory must belong to the cosets $\Lambda_r(Spin(n)^\vee)$ or $\lambda_1^\vee + \Lambda_r(Spin(n)^\vee)$ corresponding to the two topological sectors associated to the group $\mathbb{Z}_2$. It is important to note that the fact that $\mathbb{Z}_2$ monopoles are associated to $\mathbb{Z}_2$ topological sectors does not imply that they carry nonadditive magnetic charges as we will explain in sections 2 and 3. We constructed the monopole solutions considering two symmetry breakings of the algebra $su(n)$ to $so(n)$: one in which $so(n)$ is invariant under outer automorphism and another in which it is invariant under Cartan automorphism. In both cases we associated a $su(2)$ subalgebra, subject to some constraints, to each weight of the defining representation of the dual algebra $so(n)^\vee$ and constructed
explicitly the \( \mathbb{Z}_2 \) monopoles called fundamental monopoles. Using linear combinations of
the generators of these \( su(2) \) subalgebras, we were able to construct other \( su(2) \) subalgebras
and the corresponding \( \mathbb{Z}_2 \) monopoles called nonfundamental.

In this paper we write the vacuum solution and the asymptotic forms for the \( \mathbb{Z}_2 \)
monopoles in terms of singlets and triplets with respect to the corresponding \( su(2) \)
subalgebras. We calculate the masses for the BPS monopoles and obtained that the fundamental
BPS \( \mathbb{Z}_2 \) monopoles have the same masses equal to \( 4\pi v/e \), where \( v \) is the norm of the Higgs
vacuum. On the other hand, the masses of the nonfundamental \( \mathbb{Z}_2 \) monopoles are the
sum of the masses of the constituent fundamental monopoles which is consistent with the
interpretation that the nonfundamental monopoles should be multimonopoles composed
of noninteracting fundamental monopoles, similarly to what happens for the \( \mathbb{Z} \) monopoles
\[10\].

Exact electromagnetic duality is expected to happen in superconformal theories (SCFTs),
with a vanishing \( \beta \) function, like \( \mathcal{N} = 4 \) super Yang-Mills theories\[11\], \( \mathcal{N} = 2 \) \( SU(2) \) super
Yang-Mills theories with \( N_F = 4 \) flavors \[12\], etc. More recently, with the works \[13\][14],
there was some renewed interest with the study of dualities in SCFTs. The \( \mathbb{Z}_2 \) monopoles
cannot exist in \( \mathcal{N} = 4 \) super Yang-Mills theories, where all scalars are in the adjoint repre-
sentation. Therefore, in order to analyze some possible dualities that \( \mathbb{Z}_2 \) monopoles may
satisfy in SCFTs, we consider \( \mathcal{N} = 2 \) \( SU(n) \) super Yang-Mills theories with a hypermulti-
plet in the \( n \otimes n \) representation, which has a vanishing \( \beta \) function and which we will denote
by \( \mathcal{N} = 2' \) SCFTs. We showed that its potential accepts the vacua solutions discussed
in the previous sections. These vacua correspond to certain points of the Higgs branch
where the \( \mathbb{Z}_2 \) monopoles can exist. That is different from the Coulomb branch where the
gauge symmetry is generically broken to the maximal torus \( U(1)^r \) [or to \( K \times U(1) \) in some
specific points] and there are \( \mathbb{Z} \) monopoles/dyons everywhere on the Coulomb branch. It is
interesting to note that the BPS equations for the \( \mathbb{Z}_2 \) monopoles do not result on vanishing
of any supercharges. Therefore, even the BPS \( \mathbb{Z}_2 \) monopoles satisfying the first order BPS
equations are in long \( \mathcal{N} = 2 \) massive supermultiplets, like the massive gauge fields in this
theory. We also showed that this \( \mathcal{N} = 2' \) SCFT can have an Abelian Coulomb phase with
\( \mathbb{Z}_2 \) monopoles and \( \mathbb{Z} \) monopoles. From the results we obtained, we discussed some possible
dualities the \( \mathbb{Z}_2 \) monopoles may satisfy.

This paper is organized as follows: we start in Sec. 2 giving a short review of our
generalized construction of the spherically symmetric \( \mathbb{Z}_n \) monopole’s asymptotic forms.
Then, in Sec. 3 obtain explicitly the asymptotic form of the \( \mathbb{Z}_2 \) monopoles in \( SU(n) \) Yang-
Mills-Higgs theories with the gauge group broken to \( Spin(n)/\mathbb{Z}_2 \) by a scalar in the \( n \otimes n \)
representation. We consider two vacua configurations which break \( su(n) \) to \( so(n) \) where for
the first configuration \( so(n) \) is invariant under Cartan automorphism and for the second
configuration it is invariant under outer automorphism. In Sec. 4 we calculate the BPS
masses for the fundamental and nonfundamental \( \mathbb{Z}_2 \) monopoles. In Sec. 5, we show that
the vacua responsible for the breaking \( SU(n) \) to \( Spin(n)/\mathbb{Z}_2 \) belong to the Higgs branch
of a \( \mathcal{N} = 2 \) \( SU(n) \) SCFT and therefore this theory can have these \( \mathbb{Z}_2 \) monopoles. Finally,
in Sec. 6 we discuss some possible dualities these \( \mathbb{Z}_2 \) monopoles can satisfy.
2 General properties of $\mathbb{Z}_n$ monopoles

In this section we shall recall some of the principal results of $\mathbb{Z}_n$ monopoles and fix some conventions. For more details, see [6]. Let us start by considering a Yang-Mills theory with gauge group $G$ which we shall consider to be simple and simply connected. Let us also consider that the theory has a scalar field $\phi$ in a representation $R(G)$ and $\phi_0$ is a vacuum configuration which spontaneously breaks $G$ to $G_0$ such that $\pi_2(G/G_0)$ is nontrivial, and therefore allows the existence of monopoles. Let us denote by $g$ the algebra formed by the generators of $G$ and $g_0$ the generators of $G_0$. Note that in general, the elements of the Cartan subalgebra (CSA) of $g_0$ do not necessarily belong to the CSA of $g$. Therefore, we shall denote by $H_i$ and $E_\alpha$, respectively, the CSA’s generators and the step operators of $g$, and $h_i$ and $f_\alpha$ the corresponding generators of $g_0$. We shall adopt the convention that in the Cartan-Weyl basis, the commutation relations read

$$\begin{align*}
[H_i, E_\alpha] &= (\alpha)^i E_\alpha, \quad (1) \\
[E_\alpha, E_{-\alpha}] &= \frac{2\alpha \cdot H}{\alpha^2}.
\end{align*}$$

We shall denote by $\alpha_i$, $i = 1, 2, ..., r = \text{rank } g$, the simple roots of $g$ and $\lambda_i$, $i = 1, 2, ..., r$, the fundamental weights of $g$. Moreover, we shall denote

$$\begin{align*}
\alpha_i^\vee &= \frac{2\alpha_i}{\alpha_\alpha} \quad \lambda_i^\vee &= \frac{2\lambda_i}{\alpha_i}, \quad \text{(2)}
\end{align*}$$

as the simple coroots and fundamental coweights, respectively. They are simple roots and fundamental weights of the dual algebra $g^\vee$ and satisfy the relations $\alpha_i \cdot \lambda_j^\vee = \lambda_i^\vee \cdot \lambda_j = \delta_{ij}$.

The asymptotic condition $D_i \phi = 0$ for finite energy configurations implies that asymptotically we can write

$$\phi(\theta, \varphi) = g(\theta, \varphi) \phi_0,$$

where $\theta$ and $\varphi$ are the angular spherical coordinates and $g(\theta, \varphi) \in G$. Then, the asymptotic form of the magnetic field of the monopoles can be written as

$$B_i(\theta, \varphi) = \frac{x_i}{2\epsilon r^2} g(\theta, \varphi) \omega \cdot h g(\theta, \varphi)^{-1}$$

where $\omega$ is a real vector called magnetic weight and $h_i$ belongs to the CSA of $g_0$.

Note that when the gauge group $G$ is broken by a scalar field in the adjoint representation, the unbroken gauge group $G_0$ always has a $U(1)$ factor generated by the scalar field vacuum solution $\Phi_0 = \phi_0 a T_a$ and we can define an Abelian magnetic charge for the monopole associated to this $U(1)$ factor

$$g = \frac{1}{|\Phi_0|} \int_{S^2}\text{Tr} \left( B_i \Phi \right) = \frac{1}{|\Phi_0|} \int_{S^2}\text{Tr} \left( B_i g(\theta, \varphi) \Phi_0 g(\theta, \varphi)^{-1} \right).$$

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$^3$We shall adopt the convention of using capital letters to denote Lie groups and lower letters for Lie algebras.
On the other hand, when $\phi$ is not in the adjoint representation, we cannot define the above charge, but we can define magnetic charges associated to the CSA generators $h_a$ of the unbroken group $G_0$ as

$$g_a = \oint_{S^2_\infty} d S_I \text{Tr} \left( B_I g(\theta, \varphi) h_a g(\theta, \varphi)^{-1} \right) = \frac{2\pi}{e} \omega_a. \quad (5)$$

Therefore, these magnetic charges are proportional to the components of the magnetic weight associated to a monopole.

Considering that $G_0$ is semisimple, it can be written as

$$G_0 = \tilde{G}_0/K(G_0)$$

where $\tilde{G}_0$ is the universal covering group of $G_0$ and $K(G_0)$ is the kernel of the homomorphism $\tilde{G}_0 \to G_0$. One can show that $K(G_0)$ is a discrete subgroup of the center of $\tilde{G}_0$, which we will call $Z(\tilde{G}_0)$. Therefore, when $G_0$ is semisimple, the topological charge sectors of the theory are associated to

$$\pi_2(G/G_0) = \pi_1(G_0) = K(G_0) \subset Z(\tilde{G}_0). \quad (6)$$

Hence, $\pi_2(G/G_0)$ is a cyclic group $\mathbb{Z}_n$, or a product of cyclic groups, and the monopoles are called $\mathbb{Z}_n$ monopoles.

Now, the center of a group $\tilde{G}_0$ is a discrete group isomorphic to the classes

$$Z(\tilde{G}_0) = \{ \exp \left[ 2\pi i \Lambda_r(G_0^\vee) \cdot h \right] , \exp \left[ 2\pi i \left( \lambda_\tau(0) + \Lambda_r(G_0^\vee) \right) \cdot h \right] , \ldots , \exp \left[ 2\pi i \left( \lambda_\tau(0) + \Lambda_r(G_0^\vee) \right) \cdot h \right] \}, \quad (7)$$

where $\Lambda_r(G_0^\vee)$ is the root lattice of $G_0^\vee$, the dual group of $G_0$, and the fundamental coweights $\lambda_\tau(0)$ are associated to the nodes of the extended Dynkin diagram of $G_0$ related to the node 0 by a symmetry transformation, as explained in detail in [16]. The relation (7) is due to the fact that the quotient $\Lambda_w(G_0^\vee)/\Lambda_r(G_0^\vee)$ can be represented by the cosets

$$\Lambda_r(G_0^\vee), \lambda_\tau(0) + \Lambda_r(G_0^\vee), \lambda_\tau(0) + \Lambda_r(G_0^\vee), \ldots , \lambda_\tau(0) + \Lambda_r(G_0^\vee). \quad (8)$$

Since $K(G_0) \subset Z(\tilde{G}_0)$, the topological charge sectors (6) are associated to the elements of (7) which are in the kernel of the homomorphism $\tilde{G}_0 \to G_0$.

The group element $g(\theta, \varphi)$ must satisfy the relation

$$g(\pi, 0)^{-1} g(\pi, 2\pi) = \exp \left[ 2\pi i \omega \cdot h \right] \in K(G_0) \subset Z(\tilde{G}_0) \quad (9)$$

where $\exp$ denotes the exponential mapping in $\tilde{G}_0$. Hence, $\exp \left[ 2\pi i \omega \cdot h \right]$ must be in one of the classes of (7) associated to $K(G_0)$. Therefore, the magnetic weights $\omega$ must be only

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4Remember that when we have a monopole solution, the unbroken group is not fixed but varies with the space direction within $G$ by conjugation $g(\theta, \varphi)G_0 g(\theta, \varphi)^{-1}$. |
in the cosets associated to the kernel \( K(G_0) \) and the \( \mathbb{Z}_n \) monopoles will be in the same topological sector if their associated magnetic weights \( \omega \) are in the same coset [6]. The coset \( \Lambda_r(G_0^r) \) corresponds to the trivial element \( \mathbf{1} \) of the group \( \mathbb{Z}_n \), and monopoles with magnetic weights in this coset belong to the trivial topological sector. Note that two \( \mathbb{Z}_n \) monopoles in the same topological sector, i.e., those associated to magnetic weights \( \omega^A \) and \( \omega^B \) in the same coset, does not imply that they are the same monopole since they have different asymptotic magnetic fields [4]. However, some of these monopole solutions, but not all of them, can be related by gauge transformations. These gauge transformations have the effect to produce Weyl reflections on the magnetic weights of the monopoles [1].

Let us now consider a generator

\[
T_3^\beta = \frac{\beta \cdot h}{2}
\]

such that \( \beta \) is a vector that belongs to one of the cosets associated to \( K(G_0) \), that is, \( \beta \) can be a magnetic weight. Let us also consider that exist two other generators \( T_1^\beta, T_2^\beta \not\in g_0 \) exist, which together with \( T_3^\beta \) form a \( su(2) \) subalgebra

\[
[T_i^\beta, T_j^\beta] = i\epsilon_{ijk}T_k^\beta,
\]

which we shall denote \( su(2)_\beta \). Since \( \exp[2\pi i\beta \cdot h] \in K(G_0) \), then \( \exp[2\pi iq\beta \cdot h] \in K(G_0) \) where \( q \in \mathbb{Z} \). Therefore, \( q\beta \) is also in one of the cosets associated to \( K(G_0) \). Since we are interested in the study of fundamental monopoles, we shall consider solutions with spherically symmetric asymptotic forms. As in [6], from these generators, we shall obtain explicit monopole asymptotic forms with spherical symmetry using a generalization of the construction in [4], writing the group element \( g(\theta, \varphi) \) as

\[
g(\theta, \varphi) = \exp(-i\varphi T_3^\beta) \exp(-i\theta T_2^\beta) \exp(i\varphi q T_3^\beta),
\]

which satisfies

\[
g(\pi, 0)^{-1} g(\pi, 2\pi) = \exp[2\pi iq\beta \cdot h] \in K(G_0).
\]

Therefore, the monopole associated to this group element has magnetic weight \( \omega = q\beta \). Hence, for each integer \( q \) and \( su(2)_\beta \) subalgebra with \( T_3^\beta \) satisfying condition (9), we can associate a \( \mathbb{Z}_n \) monopole. A very important difference from the construction in [4] is that in our construction the monopole topological sectors are associated to the cosets [5] and not to the integer \( q \) and therefore, monopoles associated to magnetic weights with the same integer \( q \) are not necessarily in the same topological sector. As a consequence, from our generalized construction we obtain many more solutions. One can think of the monopoles associated to a \( su(2)_\beta \) and with \( |q| > 1 \) as superpositions of \( |q| \) monopoles with \( |q| = 1 \) associated to the same \( su(2)_\beta \). Similarly to [4], we consider that a monopole associated to a \( su(2)_\beta \) subalgebra with a negative integer \(-q\) is the antimonopole of the monopole with positive integer \( q \) and is associated to the same \( su(2)_\beta \). It is interesting to note that in particular, a \( \mathbb{Z}_2 \) monopole and its antiparticle are in the same topological sector, but if one has magnetic weight \( q\beta \), the other has \(-q\beta \).
Using the identity, for \( i \neq j \),
\[
\exp(i a T_j) T_i \exp(-i a T_j) = (\cos a) T_i + (\sin a) \epsilon_{ijk} T_k,
\]
where \( T_i, i = 1, 2, 3 \) form an arbitrary \( su(2) \) subalgebra, we can rewrite the asymptotic form (10) for the magnetic field with \( \omega = q \beta \) and \( g(\theta, \varphi) \) given by (10) as
\[
B_i^{(a)}(\theta, \varphi) = \frac{g x_i}{e r^3} \left[ T_3^\beta \cos \theta + \sin \theta \left( T_1^\beta \cos q \varphi + T_2^\beta \sin q \varphi \right) \right].
\]

One can obtain this asymptotic form from the gauge field configuration [4]
\[
W_i(\theta, \varphi) = g(\theta, \varphi) W_i^{\text{string}} g(\theta, \varphi)^{-1} - \frac{i}{e} (\partial_i g(\theta, \varphi)) g(\theta, \varphi)^{-1},
\]
with
\[
W_r^{\text{string}} = W_\theta^{\text{string}} = 0,
W_\varphi^{\text{string}} = -\frac{q T_3^\beta}{e r} \left( \frac{1 - \cos \theta}{\sin \theta} \right),
\]
which gives
\[
W_r(\theta, \varphi) = 0, \quad W_\theta(\theta, \varphi) = \frac{1}{e r} \left( T_1^\beta \sin q \varphi - T_2^\beta \cos q \varphi \right), \quad W_\varphi(\theta, \varphi) = \frac{q}{e r} \left[ -T_3^\beta \sin \theta + \cos \theta \left( T_1^\beta \cos q \varphi + T_2^\beta \sin q \varphi \right) \right].
\]

3 \( \mathbb{Z}_2 \) monopoles in \( SU(n) \) Yang-Mills-Higgs theories

Let us consider a Yang-Mills-Higgs theory with gauge group \( SU(n) \) and a scalar field \( \phi \) in the direct product representation \( n \otimes n \) of \( SU(n) \). In order for \( \mathbb{Z}_2 \) monopoles to exist, in [6] we find vacuum solutions \( \phi_0 \) which break
\[
SU(n) \rightarrow \frac{Spin(n)}{\mathbb{Z}_2}
\]
for \( n \geq 3 \), where \( Spin(n) \) is the covering group of \( SO(n) \) and is associated to the algebra \( so(n) \). We consider two different vacua: for one vacuum, the unbroken \( so(n) \) is the subalgebra of \( su(n) \) invariant under Cartan automorphism and for the second vacuum, \( so(n) \) is the subalgebra invariant under outer automorphism, and in this case \( n \) must be odd. In both cases, the kernel \( K(G_0) = \mathbb{Z}_2 \) is associated to the cosets
\[
\Lambda_r(Spin(n)^\vee), \quad \lambda_1^r + \Lambda_r(Spin(n)^\vee),
\]
where \( \lambda_1 \) is a fundamental weight of the \( so(n) \) subalgebra, using the convention of [6]. The first coset is associated to the trivial topological sector. As explained in detail in Sec. 6
of \([\mathbb{Z}_2]\), if we consider two \(\mathbb{Z}_2\) monopoles with magnetic weights \(\omega^{(A)}\) and \(\omega^{(B)}\) belonging to the coset \(\lambda_1^\vee + \Lambda_r(Spin(n)^\vee)\), and therefore belonging to the nontrivial topological sector, then the monopole composed by these two monopoles will have magnetic weight \(\omega^{(A)} + \omega^{(B)}\), which belongs to \(\Lambda_r(Spin(n)^\vee)\) since \(2\lambda_1^\vee \in \Lambda_r(Spin(n)^\vee)\) and hence to the trivial topological sector. It means that the \(\mathbb{Z}_2\) monopole carries an additive magnetic charge, since it is proportional to its magnetic weight, and the \(\mathbb{Z}_2\) topological charge of a monopole is related to the exponential of its magnetic weight by Eq. (7).

Before we consider these two symmetry breakings, let us obtain some Lie algebra results for the \(n \otimes n\) representation, which will be useful in the next sections. Let us denote by \(|e_i\rangle, i = 1, 2, ..., n\), the weight states of the \(n\)-dimensional representation of \(su(n)\). In this representation, the generators of \(su(n)\) can be written in terms of the \(n \times n\) matrices \(E_{ij}\) with components \((E_{ij})_{kl} = \delta_{ik}\delta_{jl}\) or

\[
E_{ij} |e_j\rangle = |e_i\rangle .
\]  

In this case, the basis elements of the CSA of \(su(n)\) correspond to the traceless combinations \(E_{ii} - E_{i+1,i+1}\), for \(i = 1, 2, ..., n - 1\). The generator \(E_{ij}\), \(i \neq j\), is the step operator associated to the root \(e_i - e_j\), where \(e_i\) is an orthonormal vector in the \(n\)-dimensional vector space.

In the representation \(n \otimes n\), the weight states are \(|e_i\otimes |e_j\rangle, i, j = 1, 2, ..., n\), and the generators can be written as

\[
E_{ij} = E_{ij} \otimes 1 + 1 \otimes E_{ij}.
\]

In this representation, for a root \(\beta = e_i - e_j\) of \(su(n)\), we can associate a \(su(2)_\beta\) subalgebra

\[
T_3^\beta = \frac{\beta \cdot H}{2} = \frac{1}{2}(E_{ii} - E_{jj}),
\]

\[
T_1^\beta = \frac{E_\beta + E_{-\beta}}{2} = \frac{1}{2}(E_{ij} + E_{ji}),
\]

\[
T_2^\beta = \frac{E_\beta - E_{-\beta}}{2i} = \frac{1}{2i}(E_{ij} - E_{ji}).
\]

Adopting the notation \(|i, j\rangle \equiv |e_i\otimes |e_j\rangle\), we can define the weight states

\[
|0\rangle_{\beta,1} = \frac{1}{\sqrt{2}}(|j, j\rangle - |i, i\rangle),
\]

\[
|0\rangle_{\beta,2} = \frac{i}{\sqrt{2}}(|j, j\rangle + |i, i\rangle),
\]

\[
|0\rangle_{\beta,3} = \frac{1}{\sqrt{2}}(|i, j\rangle + |j, i\rangle),
\]

where \(|0\rangle_{\beta,i}\) is eigenvector of \(T_1^\beta\) with vanishing eigenvalue and one can check

\[
T_1^\beta |0\rangle_{\beta,j} = i \sum_k \epsilon_{ijk} |0\rangle_{\beta,k}.
\]
Remembering that for an arbitrary Lie algebra, a weight state \( |T_i\rangle \) of the adjoint representation is associated to a generator \( T_i \) through the relation

\[
T_i |T_j\rangle = i \sum_k f_{ijk} |T_k\rangle = | [T_i, T_j] \rangle,
\]

where \( f_{ijk} \) are the structure constants of the algebra. Therefore, from Eq. (21) we can conclude that the weight states (20) form an adjoint or triplet representation of the \( su(2)_{\beta} \) subalgebra (19) and we can associate \( |0\rangle_{\beta,j} \) to \( T^\beta_j \).

3.1 \( so(2m+1) \) invariant under outer automorphism

Let us consider first the case where \( so(2m+1) \) is the invariant subalgebra of \( su(2m+1) \) under outer automorphism. In this case, the CSA of \( so(2m+1) \) is inside the CSA of \( su(2m+1) \), as explained in detail in [6]. The vacuum configuration which breaks \( su(2m+1) \) to this \( so(2m+1) \) subalgebra is [6]

\[
\phi_0 = \frac{v}{\sqrt{2}} \sum_{l=1}^{2m+1} (-1)^{l+1} |l, 2m + 2 - l\rangle,
\]

where \( v \) is a real constant.

Let us now analyze the possible \( \mathbb{Z}_2 \) monopole solutions of the theory. Since for the moment we are interested in the so-called fundamental monopoles, we shall consider that \( q = 1 \). The monopoles associated to the nontrivial topological sector must have magnetic weights \( \beta \) in the coset \( \lambda^\vee_r + \Lambda_r(Spin(n)^\vee) \). This condition is written in terms of coweights and coroots of the subalgebra \( so(2m+1) \). We showed that this condition can be written in terms of roots of \( su(2m+1) \) as

\[
\beta \in \sum_{i=1}^{m-1} c_i (\alpha_i + \alpha_{2m+1-i}) + 2c_m + 1) (\alpha_m + \alpha_{m+1})
\]

(23)

where \( c_i \) are arbitrary integers and \( \alpha_i \) are simple roots of \( su(2m+1) \). On the other hand, the monopoles associated to the trivial topological sector must have magnetic weights \( \beta \) in the coset \( \Lambda_r(Spin(n)^\vee) \). This condition can be written in terms of roots of \( su(2m+1) \) as

\[
\beta \in \sum_{i=1}^{m-1} c_i (\alpha_i + \alpha_{2m+1-i}) + 2c_m (\alpha_m + \alpha_{m+1})
\]

(24)

with \( c_i \) being integers. Therefore, \( \beta \) can only be in the particular subspace of \( \Lambda_r(SU(2m+1)) \) which is the union of the subspaces given by conditions (23) and (24). In order to construct \( su(2)_{\beta} \) subalgebras, we consider that \( \beta \) is a root of \( su(2m+1) \) in this subspace. In this case, we can consider a \( su(2)_{\beta} \) subalgebra of the form of (19) which satisfies all the properties.
discussed before. The only roots of $su(2m + 1)$ which satisfy condition (23) of being in the nontrivial sector, are

$$\pm (\alpha_p + \alpha_{p+1} + \ldots + \alpha_{2m+1-p}), \quad p = 1, 2, \ldots, m.$$  \hfill (25)

On the other hand, there is no root of $su(2m + 1)$ which satisfies condition (24). We constructed other $su(2)_{\beta}$ subalgebras associated to other elements in the cosets (17). However, in all the cases we found, the generators were always linear combinations of the generators of (19). Therefore, we call fundamental $\mathbb{Z}_2$ monopoles, the monopoles associated to the $su(2)_{\beta}$ subalgebras (19) with $\beta$ being one of the $2m$ roots (25), similarly to the nomenclature used in [10] for the $\mathbb{Z}$ monopoles. All of these fundamental monopoles are in the nontrivial topological sector. These $2m$ roots can be written as the weights of the $2m$-dimensional defining representation of $so(2m + 1)^\vee = sp(2m)$.

Using the fact that the simple roots of $su(2m + 1)$ can be written as $\alpha_p = e_p - e_{2m+2-p}$, we can write these $2m$ roots, or magnetic weights, as

$$\beta_p = e_p - e_{2m+2-p},$$

for $p = 1, 2, \ldots, m, m + 2, m + 3, \ldots, 2m + 1$. We can write the generators of the $su(2)_{\beta}$ subalgebra (19) associated to $\beta_p$ in the $n \otimes n$ representation as

$$T^\beta_p = \frac{1}{2}(E_{p,p} - E_{2m+2-p,2m+2-p}),$$

$$T^\beta_p = \frac{1}{2}(E_{p,2m+2-p} + E_{2m+2-p,p}),$$

$$T^\beta_p = \frac{1}{2i}(E_{p,2m+2-p} - E_{2m+2-p,p}),$$

and the corresponding weight vectors

$$|0\rangle_{p,1} = (-1)^{p+1} \frac{1}{\sqrt{2}} (|2m + 2 - p, 2m + 2 - p\rangle - |p, p\rangle),$$

$$|0\rangle_{p,2} = (-1)^{p+1} \frac{i}{\sqrt{2}} (|2m + 2 - p, 2m + 2 - p\rangle + |p, p\rangle),$$

$$|0\rangle_{p,3} = (-1)^{p+1} \frac{1}{\sqrt{2}} (|p, 2m + 2 - p\rangle + |2m + 2 - p, p\rangle),$$

which are in the adjoint representation of the $su(2)_{\beta_p}$ subalgebra (26) and satisfy (21).

We can write the vacuum configuration (22) as

$$\phi_0 = |0\rangle_{p,0} + v|0\rangle_{p,3},$$

where

$$|0\rangle_{p,0} = \frac{v}{\sqrt{2}} \sum_{l\neq p,2m+2-p} (-1)^{l+1}|l, 2m + 2 - l\rangle,$$
is a singlet of $\text{su}(2)_{\beta p}$. Note that the $n \otimes n$ representation decomposes into a direct sum of irreducible representations of $\text{su}(2)_{\beta p}$, and since $T_3^{\beta p} \phi_0 = 0$, $\phi_0$ must be a linear combination of weights with zero eigenvalues, which necessarily belong to odd-dimension irreducible representations of $\text{su}(2)_{\beta p}$.

From relation (21), it follows that, for $i \neq j$,

$$\exp \left( i a T_i^{\beta p} \right) |0\rangle_{p,j} = \cos a |0\rangle_{p,j} - \sin a \sum_k \epsilon_{ijk} |0\rangle_{p,k},$$

where $a$ is an arbitrary constant. Hence, acting with the group element (10) on $|0\rangle_{p,3}$ we obtain

$$g(\theta, \varphi) |0\rangle_{p,3} = \cos \theta |0\rangle_{p,3} + \sin \theta \left\{ \cos q \varphi |0\rangle_{p,1} + \sin q \varphi |0\rangle_{p,2} \right\}$$

and therefore, the asymptotic form (3) for the scalar field of the $\mathbb{Z}_2$ monopole can be written as

$$\phi(q)(\theta, \varphi) = |0\rangle_{p,0} + v \left\{ \cos \theta |0\rangle_{p,3} + \sin \theta \left[ \cos q \varphi |0\rangle_{p,1} + \sin q \varphi |0\rangle_{p,2} \right] \right\}. \tag{29}$$

In particular, for $q = 1$ we get

$$\phi(\theta, \varphi) = |0\rangle_{p,0} + v \sum_{a=1}^{3} \frac{x_a}{r} |0\rangle_{p,a},$$

which has the same form of hedgehog as the $\mathbb{Z}$ monopoles.

From the above asymptotic form, we can propose for the scalar field the ansatz

$$\phi(q)(r, \theta, \varphi) = \phi_{\text{sing}} + \phi(q)(r, \theta, \varphi)_{\text{trip}}, \tag{30}$$

where

$$\phi_{\text{sing}} = |0\rangle_{p,0}, \tag{31a}$$

$$\phi(q)(r, \theta, \varphi)_{\text{trip}} = \sum_{a=1}^{3} \Phi(q)(r, \theta, \varphi)_{a} |0\rangle_{p,a}, \tag{31b}$$

$$\Phi(q)(r, \theta, \varphi)_{a} = f(r) v \left( \sin \theta \cos q \varphi, \sin \theta \sin q \varphi, \cos \theta \right), \tag{31c}$$

with $f(r)$ being a real function such that $f(r \to \infty) = 1$ and $f(r = 0) = 0$, in order to avoid a singularity at the origin.

As usual for the adjoint representation, using the association $|0\rangle_{\beta,j}$ to $T_j^{\beta}$, we can define the scalar field taking values in the algebra

$$\Phi(q)(r, \theta, \varphi) = \sum_{a=1}^{3} \Phi(q)(r, \theta, \varphi)_{a} T_a^{\beta}. \tag{32}$$

Then, using the property that group elements $R(g)_{ij}$ in the adjoint representation satisfy

$$T_i R(g)_{ij} = g T_j g^{-1},$$
and the fact that $\phi(q)(r, \theta, \varphi)_{\text{tr}p} = vf(r)g(\theta, \varphi)|0\rangle_{p,3}$, we obtain that

$$\Phi(q)(r, \theta, \varphi) = f(r)vg(\theta, \varphi)T_3^{\beta_p}g(\theta, \varphi)^{-1}. \quad (33)$$

We can construct the so-called nonfundamental monopoles from the $su(2)_\beta$ subalgebras [6].

$$T_3^{n_p\beta_p} = \sum_p n_p T_3^{\beta_p},$$
$$T_1^{n_p\beta_p} = \sum_p n_p T_1^{\beta_p},$$
$$T_2^{n_p\beta_p} = \sum_p n_p T_2^{\beta_p}, \quad (34)$$

where $n_p = 0, 1$ and the summation is over either to positive or negative roots. Then, the corresponding triplet vectors are

$$|0\rangle_1 = \sum_p n_p |0\rangle_{p,1},$$
$$|0\rangle_2 = \sum_p n_p |0\rangle_{p,2},$$
$$|0\rangle_3 = \sum_p n_p |0\rangle_{p,3}, \quad (35)$$

and the singlet is

$$|0\rangle_0 = \phi_0 + v \sum_p n_p |0\rangle_{p,3}. \quad (36)$$

For nonfundamental monopoles associated to these $su(2)_\beta$ subalgebras, we arrive in the same field configurations [30].

It is interesting to note that, since $\beta_p \cdot \beta_q = 2\delta_{p,q}$ [6], the generators $T_i^{\beta_p}$, $p = 1, 2, \ldots, m$ or $m + 2, \ldots, 2m + 1$; $i = 1, 2, 3$ satisfy

$$\text{Tr} \left( T_i^{\beta_p} T_j^{\beta_q} \right) = \frac{\lambda}{2} \delta_{ij} \delta_{pq}, \quad (37)$$

where $\lambda$ is the Dynkin index of the representation. Therefore, the generators $T_i^{\beta_p}$ form a subset of the basis of orthogonal generators of $su(2m + 1)$.

### 3.2 $so(n)$ invariant under Cartan automorphism

Let us now consider the case where the algebra $su(n)$ is broken to the subalgebra $so(n)$ which is invariant under Cartan automorphism. In this case, the vacuum configuration which produces this symmetry breaking is

$$\phi_0 = \frac{v}{\sqrt{2}} \sum_{k=1}^n |k, k\rangle. \quad (38)$$
We must consider the cases \( n = 2m \) and \( n = 2m + 1 \) separately. A basis for the CSA of these subalgebras \( so(2m) \) or \( so(2m + 1) \), which have rank \( m \), is given by the generators

\[
h_k = -i (E_{\alpha_{2k-1}} - E_{-\alpha_{2k-1}}), \quad k = 1, 2, \ldots, m,
\]

(39)

where \( E_{\alpha} \) are generators of \( su(n) \). The asymptotic forms for the fundamental \( \mathbb{Z}_2 \) monopoles were constructed in Ref. \[6\]. Their magnetic weights \( \beta_p \) are the weights of the defining representation of the dual algebra \( so(n)^\vee \). We know that \( so(2m)^\vee = so(2m) \) and \( so(2m + 1)^\vee = sp(2m) \), and the \( 2m \) weights of the defining representation of \( so(2m) \) and \( sp(2m) \) can be written in the basis of orthonormal vectors as

\[
\pm \beta_p = \pm e_p, \quad p = 1, 2, \ldots, m.
\]

For each weight, we can associate a \( su(2)_{\beta_p} \) subalgebra

\[
T_{3,1}^{\pm \beta_p} = \pm \frac{1}{2} e_p \cdot h = \pm \frac{1}{2} h_p = \frac{1}{2t} (E_{\pm \alpha_{2p-1}} - E_{\mp \alpha_{2p-1}}),
\]

\[
T_{1,2}^{\pm \beta_p} = \pm \frac{1}{2} \alpha_{2p-1} \cdot H,
\]

\[
T_{2,3}^{\pm \beta_p} = \frac{1}{2} (E_{\pm \alpha_{2p-1}} + E_{\mp \alpha_{2p-1}}),
\]

(40)

where \( \alpha_i \) is a simple root of \( su(n) \) for \( n = 2m, 2m + 1 \).

In order to construct the ansatz, we proceed in the same way as in the previous case. In the \( n \otimes n \) representation, these generators can be written as

\[
T_{3,1}^{\pm \beta_p} = \pm \frac{1}{2t} (E_{2p-1,2p} - E_{2p,2p-1}),
\]

\[
T_{1,2}^{\pm \beta_p} = \pm \frac{1}{2} (E_{2p-1,2p-1} - E_{2p,2p}),
\]

\[
T_{2,3}^{\pm \beta_p} = \frac{1}{2} (E_{2p-1,2p} + E_{2p,2p-1}).
\]

The corresponding triplet or adjoint representation weight states are

\[
|0\rangle_{p,1} = \frac{-i}{\sqrt{2}} (|2p - 1, 2p\rangle + |2p, 2p - 1\rangle),
\]

\[
|0\rangle_{p,2} = \frac{+i}{\sqrt{2}} (|2p, 2p\rangle - |2p - 1, 2p - 1\rangle),
\]

\[
|0\rangle_{p,3} = \frac{1}{\sqrt{2}} (|2p - 1, 2p - 1\rangle + |2p, 2p\rangle).
\]

(41)

Then, the vacuum configuration \[38\] can be written as

\[
\phi_0 = |0\rangle_{p,0} + v|0\rangle_{p,3},
\]

\[\text{footnote}5\text{For these generators we changed sign conventions with respect to [6].}\]
where
\[ |0\rangle_{p,0} = \frac{v}{\sqrt{2}} \sum_{k \neq 2p-1,2p} |k;k\rangle, \]
is a singlet of \( su(2)_{\beta_p} \). In order to obtain the fundamental monopoles associated to these \( su(2)_{\beta_p} \) subalgebras, we can repeat all the steps of the previous subsection and arrive in the monopole ansatz (30) or (32) with \( T_i^{\beta_p} \) given by (40).

Similarly to the previous case, we can associate nonfundamental monopoles to the \( su(2)_{\beta_p} \) subalgebras [6]
\[ T_{3}^{\pm n_p \beta_p} = \sum_p n_p T_{3}^{\pm \beta_p}, \]
\[ T_{1}^{\pm n_p \beta_p} = \sum_p n_p T_{1}^{\pm \beta_p}, \]
\[ T_{2}^{\pm n_p \beta_p} = \sum_p n_p T_{2}^{\pm \beta_p}. \] (42)

where \( n_p = 0,1 \), and the summation is over either to positive or negative roots. Then, the corresponding triplet and singlet vectors have the same form of Eqs.(43) and (36) with \( |0\rangle_{p,i} \) given by (41).

Like in the previous case, the generators \( T_i^{\beta_p}, p = 1, 2, ..., m; i = 1, 2, 3 \) satisfy the orthogonality condition (37) and therefore they form a subset of the basis of orthogonal generators of \( su(n) \) with \( n = 2m \) or \( 2m + 1 \).

Therefore, for both symmetry breakings we can conclude that for any of the previous \( su(2)_{\beta} \) subalgebras and arbitrary integer \( q \) satisfying (41), \( \Phi(q)(r,\theta,\varphi) \) can be written as a sum of a singlet (which is constant) and a triplet of this \( su(2)_{\beta} \). Moreover, we can associated to this triplet, a scalar field taking values in the \( su(2)_{\beta} \)
\[ \Phi(q)(r,\theta,\varphi) = f(r)v g(\theta,\varphi) T_{3}^{\beta_p} g(\theta,\varphi)^{-1}, \] (43)
for the fundamental monopoles and
\[ \Phi(q)(r,\theta,\varphi) = f(r)v g(\theta,\varphi) T_{3}^{\beta_p} g(\theta,\varphi)^{-1}, \] (44)
for the nonfundamental ones, where \( f(r \to \infty) = 1 \) and \( f(r = 0) = 0 \).

We have seen in both symmetry breakings that for each \( su(2)_{\beta} \) subalgebra with generators \( T_i^{\beta_p} \), we can use to construct monopole solutions, there is another \( su(2)_{-\beta} \) subalgebra with generators \( T_i^{-\beta} \). Hence, we can construct monopoles with the same magnetic weight \( \omega \) considering either \( q > 0 \) and \( \beta < 0 \) or \( q < 0 \) and \( \beta > 0 \). For instance, for algebra \( SU(2)_{-\beta} \) with generators \( T_i^{-\beta} \) and \( q = +1 \), we have
\[ B_i^{(+)}(\theta,\varphi) = \frac{x_i}{e^{\varphi^3}} \left[ -T_3^{\beta} \cos \theta + \sin \theta \left( T_1^{\beta} \cos \varphi + T_2^{\beta} \sin \varphi \right) \right], \]
\[ \Phi(+) = v \left[ -T_3^{\beta} \cos \theta + \sin \theta \left( T_1^{\beta} \cos \varphi + T_2^{\beta} \sin \varphi \right) \right], \]
while for algebra SU(2)\(\beta\) with generators \(T^\beta_i\) and \(q = -1\) we get, for the case that \(so(n)\) is invariant under outer automorphism,

\[
B_i^{(-)}(\theta, \varphi) = \frac{x_i}{er^3} \left[ -T_3^\beta \cos \theta + \sin \theta \left( -T_1^\beta \cos \varphi + T_2^\beta \sin \varphi \right) \right], \\
\Phi_{(-)}(\theta, \varphi) = v \left[ T_3^\beta \cos \theta + \sin \theta \left( T_1^\beta \cos \varphi - T_2^\beta \sin \varphi \right) \right],
\]

where we have used that \(T^{-\beta}_3 = -T_3^\beta\), \(T^{-\beta}_1 = T_1^\beta\) and \(T^{-\beta}_2 = -T_2^\beta\). This suggests the solution obtained from the subalgebra \(su(2)_{-\beta}\) is not the antimonopole for the one obtained from \(su(2)_{\beta}\). The same result is valid for \(so(n)\) invariant under Cartan automorphism, but in these cases \(T_3^{-\beta} = -T_3^\beta\), \(T_1^{-\beta} = -T_1^\beta\), and \(T_2^{-\beta} = -T_2^\beta\) and the asymptotic fields \(B_i^{(-)}\) and \(\Phi_{(-)}\) will have a different form. Therefore, the set of fundamental \(Z_2\) monopoles with \(q = 1\), together with the set of fundamental monopoles with \(q = -1\), has a behavior analogous to particles in two complex conjugated representations, like the \(N\) and \(\bar{N}\) of \(SU(N)\), where the antiparticle of a particle in a multiplet is not in the same multiplet but in the complex conjugate representation.

4 BPS \(Z_2\) monopoles

Let us analyze now the BPS \(Z_2\) monopoles for this theory\(^6\) In [5], is given a general procedure to obtain the BPS bound for \(Z_n\) monopoles. However, let us consider a similar but different procedure for our ansatz. For both symmetry breakings, the gauge field takes values in a subalgebra \(su(2)_{\beta}\), and \(\phi(r, \theta, \varphi)\) can be written as a sum of a singlet (which is constant) and a triplet of this \(su(2)_{\beta}\) subalgebra. Hence, the action of the covariant derivative on the scalar field gives

\[
\mathcal{D}_\mu \phi = \mathcal{D}_\mu \phi_{\text{sing}} + \mathcal{D}_\mu \phi_{\text{trip}} = \mathcal{D}_\mu \phi_{\text{trip}},
\]

and then,

\[
(\mathcal{D}_i \phi)^\dagger(\mathcal{D}_i \phi) = (\mathcal{D}_i \phi_{\text{trip}})^\dagger(\mathcal{D}_i \phi_{\text{trip}}) = \sum_{a=1}^{3} (\mathcal{D}_i \Phi)_a(\mathcal{D}_i \Phi)_a,
\]

where in the last equality we used the fact that in our ansatz \(\Phi_a\) is real. Therefore, we obtain that the mass of a static \(Z_2\) monopole associated to a magnetic weight \(\beta_p\) and

\(^6\)For simplicity let us abolish the subscript \((q)\) for both the scalar and the magnetic field.
arbitrary $q$, for any of the two symmetry breakings, satisfies

$$M_{\beta_p} = \int \left\{ \frac{1}{2} [(B_{ia})^2 + (D_i\Phi)^d(D_i\Phi)] + V \right\} d^3x,$$

$$= \int \left\{ \frac{1}{2} \sum_{a=1}^{3} [(B_{ia})^2 + (D_i\Phi)_a(D_i\Phi)_a] + V \right\} d^3x,$$

$$\geq \pm \int_{S^2} \sum_{a=1}^{3} (B_{ia}\Phi_a) d^2 S_i,$$

$$= \pm \int_{S^2} \frac{q x_i}{e r^3} d^2 s_i = \pm \frac{4\pi}{e} q v = \frac{4\pi}{e} |q| v,$$  \hspace{1cm} (45)

where we used the plus or minus sign depending on whether $q$ is positive or negative, respectively, since the integral in the first line is greater than zero and in the last line we used the field configurations (13) and (31). Notice that we can obtain the same result using

$$\sum_{a=1}^{3} (B_{ia}\Phi_a) = Tr (B_i\Phi) = v \frac{q x_i}{e r^3}$$

where the above trace is in the triplet of $su(2)_{\beta_p}$ subalgebra and using Eqs. (4) and (43).

In this case the group elements $g(\theta, \varphi)$ cancel and therefore, this bound is valid for more general configurations constructed using group elements $g(\theta, \varphi)$ other than (10), satisfying

$$g(\pi, 0)^{-1} g(\pi, 2\pi) = e^{4\pi i T_{3p}},$$

where $a = 1, 2, 3$ are associated to the three generators of the $su(2)_{\beta_p}$ subalgebra and the fields associated to the other generators vanish. From the expression (46), we can see that it does not depend on $\beta_p$, and therefore all fundamental BPS $Z_2$ monopoles have the same mass. Since the fields take values only in the $su(2)_{\beta}$ subalgebra and these BPS equations are the same as in the theory with gauge group $SU(2)$, it is easy to check that these

$$M_{\beta_p} = \frac{4\pi v}{e},$$  \hspace{1cm} (46)

and they satisfy

$$E_i^a = 0,$$  \hspace{1cm} (47a)

$$(D^0\Phi)_a = 0,$$  \hspace{1cm} (47b)

$$B_i^a = \pm (D_i\Phi)_a,$$  \hspace{1cm} (47c)

$$V(\phi) = 0,$$  \hspace{1cm} (47d)

where $a = 1, 2, 3$ are associated to the three generators of the $su(2)_{\beta_p}$ subalgebra and the fields associated to the other generators vanish.
equations are consistent with the equations of motion. Moreover, from these equations, as for the BPS ‘t Hooft-Polyakov monopole\cite{17}, we obtain that

\[ f(r) = \coth \rho - \frac{1}{\rho}, \]
\[ a(r) = 1 - \frac{\rho}{\sinh \rho}, \]

where \( \rho = erv \).

For the nonfundamental monopoles associated to the \( su(2)_{\beta} \) subalgebra \( (31) \) or \( (42) \), the asymptotic forms of the fields are

\[ \Phi_a = \sum_p n_p \Phi_{a\beta}^p, \]
\[ B_{ia} = q \sum_p n_p B_{ia\beta}^p, \]

where \( B_{ia\beta}^p \) and \( \Phi_{a\beta}^p \) are asymptotic forms of the fields of the fundamental monopoles associated to the \( su(2)_{\beta_p} \) subalgebras with \( q = 1 \). Therefore, remembering that \( n_p = 0, 1 \), we obtain that the BPS limit for the nonfundamental monopoles is

\[ M = \frac{4\pi}{e} |q| v \sum_p n_p, \]

which is consistent with the interpretation that the nonfundamental monopoles should be multimonopoles composed of noninteracting fundamental monopoles, similarly to what happens for the \( \mathbb{Z} \) monopoles \cite{10}.

5 \( \mathbb{Z}_2 \) monopoles at the Higgs branch of \( \mathcal{N} = 2' \) SCFTs

As is well known, there exist some supersymmetric theories with \( \mathbb{Z} \) monopoles and a vanishing \( \beta \) function, like \( \mathcal{N} = 4 \) super Yang-Mills theories, where exact electromagnetic duality is expected to be valid. Let us now analyze a supersymmetric theory with a vanishing \( \beta \) function and \( \mathbb{Z}_2 \) monopoles. Let us then consider \( \mathcal{N} = 2 \) \( SU(n) \) super Yang-Mills theories with a hypermultiplet in the \( n \otimes n \) representation, which we will call \( \mathcal{N} = 2' \) SCFTs. For \( \mathcal{N} = 2 \) super Yang-Mills, the perturbative \( \beta \) function is

\[ \beta(e) = \frac{2e^3}{(4\pi)^2} \left( \sum_i x_i - h^\vee \right), \]

where \( x_i \) is the Dynkin index of the hypermultiplets’ representations and \( h^\vee \) is the dual Coxeter number of the gauge group. For \( G = SU(n) \), \( h^\vee = n \) and \( x(n \otimes n) = n \) since for a representation \( R_1 \otimes R_2 \)

\[ x(R_1 \otimes R_2) = d(R_1)x(R_2) + d(R_2)x(R_1), \]
where \( d(R) \) is the dimension of the representation \( R \) and for the representation \( n \), \( x = 1/2 \). Therefore, \( \mathcal{N} = 2' \) SCFTs have \( \beta(e) = 0 \). We shall show that its potential accepts the vacua solutions discussed in the previous sections and therefore \( \mathbb{Z}_2 \) monopoles can exist. The action of the bosonic sector of the \( \mathcal{N} = 2 \) sector of super Yang-Mills with a hypermultiplet can be written as

\[
S = \int \left[ -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} (D_\mu S)_a (D^\mu S)_a + \frac{1}{2} (D_\mu \phi_\alpha)^\dagger (D^\mu \phi_\alpha) + V(S, \phi) \right] d^4 x, \tag{50}
\]

where \( S \) is a scalar field in the adjoint representation, \( \phi_\alpha, \alpha = 1, 2, \) are complex scalar fields in an arbitrary representation, and \( \sigma^p, p = 1, 2, 3 \), are Pauli matrices. The potential can be written as

\[
V = \frac{e^2}{8} \left[ (S^* i f_{a b c} S_c)^2 + (\phi_\alpha^a \sigma^p_{\alpha \beta} T_\alpha \phi_\beta)^2 + \frac{4\mu^2}{e^2} \phi_\alpha^a \phi_\alpha - \frac{4\mu}{e} \phi_\alpha^a (S + S^\dagger) \phi_\alpha + 2\phi_\alpha^a \{ S^\dagger, S \} \phi_\alpha \right], \tag{51}
\]

where \( \mu \) is a mass parameter which we will show must vanish in order for the configuration given by Eq. (22) to be a vacuum of the theory. We can rewrite the potential as

\[
V = \frac{1}{2} \left[ (d_a^a)^2 + (d_a^a)^2 + (D_a)^2 + F_a^f F_a \right], \tag{52}
\]

where

\[
d_a^a = \frac{e}{2} (\phi_\alpha^a \sigma^p_{\alpha \beta} T_\alpha \phi_\beta), \quad p = 1, 2, 3, \tag{53a}
\]

\[
D_a = \frac{e}{2} (S^* i f_{a b c} S_c) + d_a^3, \tag{53b}
\]

\[
F_1 = e \left( S^\dagger - \frac{\mu}{e} \right) \phi_1, \tag{53c}
\]

\[
F_2 = e \left( S - \frac{\mu}{e} \right) \phi_2. \tag{53d}
\]

### 5.1 Non-Abelian Coulomb phase

In order to produce the gauge symmetry breaking \( SU(n) \rightarrow Spin(n)/\mathbb{Z}_2 \), which corresponds to the so-called non-Abelian Coulomb phase, we shall consider the configuration

\[
\phi_{1\text{vac}} = \phi_0, \\
\phi_{2\text{vac}} = 0, \\
S_{\text{vac}} = 0, \tag{54}
\]

where \( \phi_0 \) is one of the two vacua solutions analyzed in the previous sections. These vacua are in the Higgs branch which does not receive quantum corrections and the beta function does not receive nonperturbative corrections \[19\]. We shall first consider the case of symmetry breaking \( su(2m+1) \rightarrow so(2m+1) \), where \( so(2m+1) \) is invariant under outer automorphism. Therefore, we consider that \( \phi_0 \) is given by (22). Since we shall consider solutions with
\( \phi_2 = 0 \), we shall use \( \phi \) and \( F \) to denote \( \phi_1 \) and \( F_1 \), respectively. From Eq.\(^{52}\) we conclude that in order to obtain \( V = 0 \), we must have that

\[
D = \frac{e}{2} \left[ \sum_a (\phi^\dagger T_a \phi) T_a + [S, S^\dagger] \right] = 0, \tag{55a}
\]

\[
F = e \left( S^\dagger - \frac{\mu}{e} \right) \phi = 0. \tag{55b}
\]

We can write

\[
\sum_a (\phi^\dagger T_a \phi) T_a = \sum_i (\phi^\dagger H_i \phi) H_i + \frac{1}{2} \sum_{\alpha > 0} \alpha^2 (\phi^\dagger E_\alpha \phi) E_{-\alpha}. \tag{56}
\]

Using the fact that the weight state \(|e_p\rangle\) of the \((2m + 1)\)-dimensional representation of \(su(2m + 1)\) has weight

\[
\omega_p \equiv e_p - \frac{1}{2m + 1} \sum_{i=1}^{2m+1} e_i
\]

and \(\langle e_p | e_q \rangle = \delta_{pq}\), we obtain that

\[
\sum_i \langle l, 2m + 2 - l | H_i | p, 2m + 2 - p \rangle H_i = (\omega_l + \omega_{2m+2-l}) \cdot H \delta_{lp},
\]

and therefore

\[
\sum_i (\phi^\dagger \phi) H_i = |v|^2 \sum_{l=1}^{2m+1} \sum_{p=1}^{2m+1} (-1)^{l+1} (-1)^{p+1} \langle l, 2m + 2 - l | H_i | p, 2m + 2 - p \rangle H_i,
\]

\[
= |v|^2 \sum_{l=1}^{2m+1} 2 \omega_l \cdot H = 0,
\]

where we used the fact that \(\sum_l \omega_l = 0\) for the \((2m + 1)\)-dimensional representation of the algebra \(su(2m + 1)\). On the other hand, for the step operators we have:

\[
\langle l, 2m + 2 - l | E_\alpha | p, 2m + 2 - p \rangle E_\alpha = (\langle e_l | E_\alpha | e_p \rangle \langle e_{2m+2-l} | e_{2m+2-p} \rangle + \langle e_l | e_p \rangle \langle e_{2m+2-l} | E_\alpha | e_{2m+2-p} \rangle) E_\alpha = 0
\]

and therefore

\[
\sum_{\alpha > 0} \alpha^2 (\phi^\dagger \phi E_\alpha E_{-\alpha}) = 0. \tag{57}
\]

Therefore, we can conclude that the above configuration satisfies Eq.\(^{55a}\). It also satisfies Eq.\(^{55b}\) if we consider \(\mu = 0\) and therefore it is a vacuum of the theory.

For the case of the breaking of \(su(n)\) to \(so(n)\) invariant under Cartan automorphism, one can perform similar calculations using \(\phi_0\) given by \(^{38}\) and verify that it is also a vacuum of this theory. Hence, the \(\mathbb{Z}_2\) monopoles analyzed in the previous sections can exist in this
phase of this theory. Note that there will be $Z_2$ monopoles associated to certain points on the Higgs branch, differently from the $Z$ monopoles of the Coulomb branch where the gauge symmetry is generically broken to the maximal torus $U(1)^r$ (or to $K \times U(1)$ in some specific points) and there are $Z$ monopoles/dyons everywhere on the Coulomb branch.

It is interesting to note that, from the $\mathcal{N} = 2$ supersymmetric variation of the spinorial fields, it is easy to see that the BPS equations for the $Z_2$ monopoles do not result on vanishing of any supercharges. This result indicates that, even satisfying the first order BPS equations (47), the BPS $Z_2$ monopoles are in a long $\mathcal{N} = 2$ massive supermultiplet and in principle their masses can receive quantum corrections. It is good to remember that in this phase where the gauge symmetry breaking is produced by a scalar which is not in the vector supermultiplet, the gauge fields which become massive will also belong to a long supermultiplet. The reason is that in this phase the scalar field is in the hypermultiplet and is “absorbed” by the gauge fields via Higgs mechanism in order to form a massive supermultiplet. Therefore, the $\mathcal{N} = 2$ massive vector supermultiplet will be the combination of a massless vector supermultiplet with a massless hypermultiplet. Note also that in this phase, the electric and magnetic charges,

\[
q = \frac{1}{|S_{\text{vac}}|} \int dS^2_i G^0 a \text{Re} (S_a),
\]

\[
g = \frac{1}{|S_{\text{vac}}|} \int dS^2_i \tilde{G}^0 a \text{Re} (S_a),
\]

which appear as central charges, vanish since the scalar field $S$ in the adjoint representation vanishes asymptotically in this phase.

### 5.2 Abelian Coulomb phase

Let us now show that in this theory we can also have the symmetry breaking sequence

\[
SU(2m+1) \rightarrow \frac{Spin(2m+1)}{Z_2} \rightarrow U(1)^m,
\]

which is the Abelian Coulomb phase. In this phase, $Z_2$ monopoles and $Z$ monopoles as discussed in [20, 4]. Note that this theory can also have an alternative symmetry breaking (Higgs phase) with confinement of $Z$ monopoles by $Z_2$ strings [21, 22]. Differently from the non-Abelian Coulomb phase, in this phase the vacuum moduli space can receive quantum corrections. For this symmetry breaking, we shall only consider the case where the subalgebra $so(2m+1)$ is invariant under outer automorphism. Therefore, we shall consider that $\phi_{1\text{vac}} = \phi_0$ is given by (22) and $\phi_{2\text{vac}} = 0$. We shall also consider that $S_{\text{vac}} = u \cdot H$ with

\[
u = a\delta = a \sum_{i=1}^{2m} \lambda_i^\nu,
\]

(58)
where $a$ is a nonvanishing real constant and $\delta$ is the Weyl vector of $su(2m + 1)$.

Since $[S_{\text{vac}}, S^+_{\text{vac}}] = 0$, it implies that this new configuration also satisfies Eq. (55a). We now must show that this configuration is also a solution of (55b). Substituting (54) in (55b) we obtain

$$F = ve \left\{ \sum_{l=1}^{2m+1} (-1)^{l+1} \left[u \cdot (\omega_l + \omega_{2m+2-l}) - \frac{\mu}{e}\right] |l, 2m + 2 - l\rangle \right\}.$$  

In order to obtain $F = 0$, we must have that

$$u \cdot (\omega_l + \omega_{2m+2-l}) = \frac{\mu}{e} \text{ for } l = 1, 2, \ldots, 2m + 1.$$  

It is easy to show that, for $u$ given by (58), the lhs of this equation vanishes for any $l$. Therefore, if we once more take $\mu = 0$, (54) with $u$ given by (58) is a vacuum solution.

6 Discussion on duality conjectures

In order to establish the possible dualities these $\mathbb{Z}_2$ monopoles may satisfy, one must determine the gauge multiplet the fundamental monopoles fill, by doing for example semiclassical quantization. For this theory that procedure is not simple, since the unbroken gauge group is non-Abelian, which results in nonnormalizable zero modes [10] and we will leave it for a future work. Let us therefore discuss some possible dualities the BPS $\mathbb{Z}_2$ monopoles may satisfy based on the results we obtained so far. The particles dual to the BPS $\mathbb{Z}_2$ monopoles must be in a representation with the same weights as the magnetic weights of the $\mathbb{Z}_2$ monopoles. We have seen that, at the classical level, for a breaking $SU(n) \rightarrow Spin(n)/\mathbb{Z}_2$, each fundamental $\mathbb{Z}_2$ monopole is associated to a weight of the defining representation of $so(n)^\vee$ and all of them have the same classical mass $M_{\beta_\mu} = 4\pi v/e$ in the BPS case. Let us consider for simplicity the even case, $n = 2m$, where $so(2m)$ must be an invariant subalgebra of $su(2m)$ under Cartan involution. In this case $so(2m)^\vee = so(2m)$ and the defining representation has dimension $2m$. Therefore, there are $2m$ fundamental $\mathbb{Z}_2$ monopoles. Let us consider that the dual theory has the same symmetry breaking pattern

$$SU(2m) \rightarrow Spin(2m)/\mathbb{Z}_2,$$

with the same vacuum (38) and also with $\mathbb{Z}_2$ monopoles. For this symmetry breaking we have the branchings

$$su(2m) \rightarrow so(2m),$$

$$(1, 0, \ldots, 0, 0)_{su} \rightarrow (1, 0, \ldots, 0)_{so},$$

$$(1, 0, \ldots, 0, 1)_{su} \rightarrow (0, 1, 0, \ldots, 0)_{so} + (2, 0, \ldots, 0)_{so},$$

$$(2, 0, \ldots, 0, 0)_{su} \rightarrow (2, 0, \ldots, 0)_{so} + (0, 0, \ldots, 0)_{so}.$$
where \((1, 0, \ldots, 0)_su\) is the representation 2\(m\) of \(su(2m)\), \((1, 0, \ldots, 0, 1)_su\) corresponds to the adjoint representation of \(su(2m)\), and the massive gauge fields are in the representation \((2, 0, \ldots, 0)_so\) of \(so(2m)\). Since the \(\mathbb{Z}_2\) monopoles can also be associated to roots of the “broken” generators of \(su(2m)\), we could think to associate these monopoles with particles in the adjoint representation, similarly to the Montonen-Olive case. Moreover, like for the fundamental BPS \(\mathbb{Z}_2\) monopoles, these massive gauge particles have the same mass equal to \(m = \mathcal{O}v\), for the symmetry breaking caused by the vacuum configuration (3.8). Furthermore, in the \(\mathcal{N}' = 2'\) SCFT in the non-Abelian Coulomb phase, they are in a long supermultiplet like the \(\mathbb{Z}_2\) monopoles, as we explained in the last section. On the other hand, the massive gauge particles are associated to the weights of \((2, 0, \ldots, 0)_so\) which are not in the coset \(\lambda_1 + \Lambda_r(Spin(2m))\), where the magnetic weights of the fundamental \(\mathbb{Z}_2\) monopoles are in the dual theory. However, in principle the multiplet of the \(\mathbb{Z}_2\) monopoles may change at the quantum level.

If by semiclassical analysis the fundamental monopoles remain in the 2\(m\) representation, they cannot be dual to gauge particles but, if we consider a different supersymmetric theory, from the above branchings, the \(\mathbb{Z}_2\) monopoles could be dual to particles in the 2\(m\) of \(su(2m)\), since their weights with respect to the unbroken \(so(2m)\) are exactly equal to the magnetic weights of the fundamental \(\mathbb{Z}_2\) monopoles. If we consider that \(\mathbb{Z}_2\) monopoles are dual to particles in a supermultiplet containing spinors, and if the masses of these spinors are due to the vacuum solution of the scalar \(\phi \in 2m \otimes 2m\), we should consider chiral spinors \(\psi_L \in 2m, \psi_R \in 2\overline{m}\) (and therefore \(\bar{\psi}_R \in 2m\)) of \(su(2m)\). Then, if

\[
\begin{align*}
\phi &= \sum_{p,q=1}^{2m} \phi_{pq} |e_p\rangle \otimes |e_q\rangle, \\
\psi_L &= \sum_{p=1}^{2m} \psi_{Lp} |e_p\rangle, \\
\bar{\psi}_R &= \sum_{p=1}^{2m} \bar{\psi}_{Rp} |e_p\rangle.
\end{align*}
\]

the theory can have the Yukawa term

\[
\lambda' \left(\bar{\psi}_{Lp} \psi_{Rq} \phi_{pq} + H.c.\right)
\]

where \(\lambda'\) is a coupling constant in the dual theory. With the vacuum solution (3.8) with constant \(v'\), all spinors become massive with the same classical mass equal to

\[
m_{\psi} = \lambda' v'.
\]

If we consider \(\lambda' = 4\pi/e\) and \(v' = v\), we obtain exactly the classical masses of the fundamental BPS \(\mathbb{Z}_2\) monopoles of the original theory. In this case, the fundamental BPS \(\mathbb{Z}_2\)
monopoles (with \( q = 1 \)) could be dual to \( \psi_L \in 2m \) and their antimonopoles (with \( q = -1 \)) to \( \psi_R \in 2m \). That would be consistent with the property discussed at the end of Sec. 3 that the set of fundamental \( \mathbb{Z}_2 \) monopoles with \( q = 1 \), together with the set of fundamental monopoles with \( q = -1 \), has a behavior analogous to particles in two complex conjugated representations. It is interesting that the symmetry breaking by a scalar \( \phi \) in the representation \( 2m \otimes 2m \) which gives rise to the \( \mathbb{Z}_2 \) monopoles also gives mass to spinors in the \( 2m \) of \( SU(2m) \). Note that a theory with this field content cannot be embedded in the \( \mathcal{N} = 2 \) super Yang-Mills theory, like the one discussed in the previous section, since \( \psi_R \) and \( \psi_L \) are in different representations. However, it can be embedded for example in a \( \mathcal{N} = 1 \) super Yang-Mills theory. In [23] it is also considered a duality between \( \mathbb{Z}_2 \) monopoles and spinors in \( \mathcal{N} = 1 \) super Yang-Mills theory but in this case if one theory has gauge group \( SU(N) \) with \( N_F \) flavors, the dual theory would have gauge group \( SU(N_F - N + 4) \), similarly to Seiberg duality[24].

7 Conclusions

In this work we constructed explicitly BPS \( \mathbb{Z}_2 \) monopole solutions in theories with the gauge group \( SU(n) \) broken to \( Spin(n)/\mathbb{Z}_2 \) using two different vacua of a scalar field in the \( n \otimes n \) representation. Each \( \mathbb{Z}_2 \) monopole is associated to a \( su(2) \) subalgebra and an integer \( q \). The magnetic weights of the so-called fundamental \( \mathbb{Z}_2 \) monopoles correspond to the weights of the defining representation of the dual algebra \( so(n)^\vee \). We calculated the masses for the BPS monopoles and obtained that the fundamental BPS \( \mathbb{Z}_2 \) monopoles have the same masses and the masses are equal to \( 4\pi v/e \), where \( v \) is the norm of the Higgs vacuum. On the other hand, the masses of the nonfundamental \( \mathbb{Z}_2 \) monopoles are the sum of the masses of the constituent fundamental monopoles. This result is consistent with the interpretation that the nonfundamental monopoles should be multimonopoles composed of noninteracting fundamental monopoles, in the BPS case, similarly to what happens for the \( \mathbb{Z} \) monopoles. We showed that the potential of \( \mathcal{N} = 2 \) \( SU(n) \) super Yang-Mills theories with a hypermultiplet in the \( n \otimes n \) representation, which has a vanishing \( \beta \) function, accepts the vacua solutions which break the gauge group \( SU(n) \) to \( Spin(n)/\mathbb{Z}_2 \). These vacua correspond to certain points of the Higgs branch where the \( \mathbb{Z}_2 \) monopoles can exist. It is interesting to note that the BPS equations for the \( \mathbb{Z}_2 \) monopoles do not result on vanishing of any supercharges. Therefore, even the BPS \( \mathbb{Z}_2 \) monopoles satisfying first order BPS equations are in long \( \mathcal{N} = 2 \) massive supermultiplets, like the massive gauge fields in this theory. We discussed some possible dualities the \( \mathbb{Z}_2 \) monopoles may satisfy.

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