GENERALIZING THE GAGA PRINCIPLE

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ABSTRACT. This paper generalizes the fundamental GAGA results of Serre [Ser56] in three ways—to the non-separated setting, to stacks, and to families. As an application of these results, we show that analytic compactifications of $M_{g,n}$ possessing modular interpretations are algebraizable.

1. INTRODUCTION

A locally separated and locally of finite type algebraic $\mathbb{C}$-space $X$ may be functorially analytified to an analytic space $X_{an}$. More generally, one may functorially analytify a locally of finite type $\mathbb{C}$-stack with locally separated diagonal. It was shown by B. Conrad and M. Temkin in [CT09, Thm. 2.2.5] that the analytification of a locally of finite type algebraic $\mathbb{C}$-space is an analytic space only if it is locally separated. In particular, for locally of finite type $\mathbb{C}$-stacks $Z$ and $X$ with locally separated diagonals, one obtains an analytification functor

$$\text{Hom}(Z, X) \to \text{Hom}(Z_{an}, X_{an})$$

The first main result of this paper is

**Theorem 1.** For a locally of finite type Deligne-Mumford $\mathbb{C}$-stack $X$ with quasi-compact and separated diagonal, and a proper Deligne-Mumford $\mathbb{C}$-stack $Z$, the analytification functor:

$$\text{Hom}(Z, X) \to \text{Hom}(Z_{an}, X_{an})$$

is an equivalence.

In [Lur04], J. Lurie proves a related result to Theorem 1—the stack $X$ is permitted to be algebraic (as opposed to Deligne-Mumford), but the diagonal is assumed to be affine. Thus Theorem 1 is stronger than what appears in [loc. cit.] when applied to Deligne-Mumford stacks (e.g. schemes and algebraic spaces). Our proof of Theorem 1 will rely on a version of Serre’s GAGA principle for non-separated algebraic stacks.

For a locally of finite type algebraic $\mathbb{C}$-stack $X$, define the category of pseudosheaves on $X$, $\text{PS}_p(X)$, to have objects the pairs $(Z \xrightarrow{s} X, F)$, where $s$ is a representable, quasi-finite, and separated morphism of algebraic stacks, with $Z$ proper, and $F$ a coherent $O_Z$-module. A morphism of pseudosheaves $(Z \xrightarrow{s} X, F) \to (Z' \xrightarrow{s'} X, F')$ is a pair $(Z' \xrightarrow{t} Z, F \xrightarrow{\phi} t_* F')$, where $t$ is a finite $X$-morphism and $\phi$ is a morphism of coherent $O_Z$-modules. The category of pseudosheaves on an algebraic stack $X$ is a non-abelian refinement of the category of coherent sheaves on $X$.

A morphism of analytic stacks $\sigma : \mathcal{Z} \to \mathcal{X}$ is **locally quasi-finite** if it is representable by analytic spaces, and for any $x \in |\mathcal{X}|$ the fiber $\sigma^{-1}(x)$ is a finite set of points. Thus, for an analytic stack $\mathcal{X}$, we may similarly define a category of analytic pseudosheaves $\text{PS}_p(\mathcal{X})$.

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with objects given by pairs $(Z \xrightarrow{\sigma} X, F)$, where $\sigma$ is a locally quasi-finite and separated morphism of analytic stacks, $Z$ is a proper analytic stack, and $F$ is a coherent $O_Z$-module. There is an analytification functor for pseudosheaves:

$$\Psi_X : \mathbf{PS}_p(X) \to \mathbf{PS}_p(X_{an}).$$

Theorem 1 will follow from

**Theorem 2.** For any locally of finite type Deligne-Mumford $\mathbb{C}$-stack $X$ with quasi-compact and separated diagonal, the analytification functor

$$\Psi_X : \mathbf{PS}_p(X) \to \mathbf{PS}_p(X_{an})$$

is an equivalence.

Theorem 2 admits a useful variant. For a locally of finite type algebraic $\mathbb{C}$-stack $X$, define the full subcategory $\mathbf{PS}_p,\text{DM}(X) \subset \mathbf{PS}_p(X)$ to consist of those pseudosheaves $(Z \xrightarrow{\sigma} X, F)$, where the algebraic stack $Z$ is Deligne-Mumford. For an analytic stack $X$, one may similarly define a full subcategory $\mathbf{PS}_p,\text{DM}(X) \subset \mathbf{PS}_p(X)$. There is an induced analytification functor:

$$\tilde{\Psi}_X : \mathbf{PS}_p,\text{DM}(X) \to \mathbf{PS}_p,\text{DM}(X_{an}).$$

**Theorem 3.** For any locally of finite type algebraic $\mathbb{C}$-stack $X$ with quasi-compact and separated diagonal, the analytification functor

$$\tilde{\Psi}_X : \mathbf{PS}_p,\text{DM}(X) \to \mathbf{PS}_p,\text{DM}(X_{an})$$

is an equivalence.

This follows from Theorem 2, since for any locally of finite type algebraic $\mathbb{C}$-stack $X$ with quasi-compact and separated diagonal, there is an open substack $X^0 \subset X$ which is Deligne-Mumford such that the inclusion $X^0 \hookrightarrow X$ induces an equivalence $\mathbf{PS}_p(X^0) \cong \mathbf{PS}_p,\text{DM}(X)$. Also, if an algebraic stack $X$ has affine stabilizers, then $\mathbf{PS}_p(X) \cong \mathbf{PS}_p(X)$. Thus Theorem 2 can be trivially strengthened to show that the functor $\psi_X$ is an equivalence for algebraic stacks with affine stabilizers.

What follows will be made precise in §5. Let $g > 1$ and $n \geq 0$. Take $M_{g,n}$ (resp. $M_{g,n}$) to be the algebraic $\mathbb{C}$-stack (resp. analytic stack) of smooth, $n$-pointed, algebraic (resp. analytic) curves of genus $g$. An algebraic (resp. analytic) modular compactification of $M_{g,n}$ (resp. $M_{g,n}$) is a proper algebraic (resp. analytic) stack $N$ (resp. $N$), with finite stabilizers, possessing a "modular" interpretation, together with a diagram of dense open immersions $M_{g,n} \hookrightarrow V \hookrightarrow N$ (resp. $M_{g,n} \hookrightarrow V \hookrightarrow N$). Theorem 3 implies

**Theorem 4.** Analytic modular compactifications of $M_{g,n}$ are algebraizable to algebraic modular compactifications of $M_{g,n}$.

Using Moishezon techniques (as in [Art70]) for the coarse moduli spaces, previously existing technology demonstrates that it is possible to algebraize the analytic coarse moduli spaces which are modular compactifications of $M_{g,n}$. The Moishezon techniques fail to algebraize the modular interpretation, as well as the moduli stack, however. Describing the modular compactifications of $M_{g,n}$ is an active area of research in moduli theory, referred to as the Hassett-Keel program (cf. [HH08], [HH09], and [AFS10]).

We will now discuss the GAGA principle for families. The Hilbert functor, which parameterizes closed immersions into a fixed space, always fails to be an algebraic space in the non-separated setting. This assertion is due to C. Lundkvist and R. Skjelnes [LS08]. In [HalR10], a generalization of the Hilbert functor, the Hilbert stack, was shown to be an
algebraic stack for non-separated algebraic stacks. For an algebraic stack $Y$, and a locally finitely presented morphism of algebraic stacks with quasi-compact and separated diagonal $X \to Y$, the Hilbert stack $\mathcal{H}_X/Y$ assigns to each $Y$-scheme $T$ the groupoid of quasi-finite and representable morphisms of algebraic stacks $Z \to X \times_Y T$ such that the composition $Z \to X \times_Y T \to T$ is proper, finitely presented, flat, and with finite diagonal.

For a morphism of analytic stacks $X \to Y$, the analytic Hilbert stack $\mathcal{H}_X^a/Y$ assigns to each analytic $Y$-space $T$, the groupoid of locally quasi-finite morphisms of analytic stacks $Z \to X \times_Y T$, such that the composition $Z \to X \times_Y T \to T$ is proper and flat with finite diagonal. Hence, given a morphism $X \to Y$ of locally of finite type algebraic $\mathbb{C}$-stacks with quasi-compact and separated diagonal, there is thus an induced morphism of $Y^a$-stacks:

$$\Phi_{X/Y} : (\mathcal{H}_X/Y)^a \to \mathcal{H}_{X^a/Y^a}^a.$$  

It is important to note that there are no results in the literature for analytic spaces which exert that $\mathcal{H}_{X^a/Y^a}^a$ is an analytic stack in the case that the morphism $X^a \to Y^a$ is non-separated. We also prove the following GAGA result for families.

**Theorem 5.** Let $X \to Y$ be a quasi-separated morphism of locally of finite type algebraic $\mathbb{C}$-spaces. Then the natural map

$$\Phi_{X/Y} : (\mathcal{H}_X/Y)^a \to \mathcal{H}_{X^a/Y^a}^a$$

is an equivalence of $Y^a$-stacks. In particular, $\mathcal{H}_{X^a/Y^a}^a$ is an analytic $Y^a$-stack.

We wish to emphasize that Theorems 1–3 and 5 are completely new, even in the case where $X$ is a scheme.

**Remark 1.1.** The arguments in this paper for Theorem 5 generalize readily to the setting of a morphism of algebraic stacks $X \to Y$. What obstructs a full proof of Theorem 5 in this generality is a lack of foundations for analytic stacks. More precisely, we would require an analog of Corollary 3.3 for analytic stacks, for which no precise reference exists in the current literature.

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1.1. **Notation.** We introduce some notation here that will be used throughout the paper. For a category $\mathcal{C}$ and $X \in \text{Obj} \mathcal{C}$, we have the slice category $\mathcal{C}/X$, with objects the morphisms $V \to X$ in $\mathcal{C}$, and morphisms commuting diagrams over $X$, which are called $X$-morphisms. If the category $\mathcal{C}$ has finite limits and $f : Y \to X$ is a morphism in $\mathcal{C}$, then for $(V \to X) \in \text{Obj}(\mathcal{C}/X)$, define $V_Y := V \times_X Y$. Given a morphism $p : V' \to V$, there is an induced morphism $p_Y : V'_Y \to V_Y$. There is usually an induced functor...
f^* : \mathcal{E}/X \rightarrow \mathcal{E}/Y : (V \rightarrow X) \mapsto (V_Y \rightarrow Y). Given a \( (2,1) \)-category \( \mathcal{C}' \), these notions readily generalize.

Given a ringed space \( U := ([U], \mathcal{O}_U) \), a sheaf of ideals \( J \subset \mathcal{O}_U \), and a morphism of ringed spaces \( g : V \rightarrow U \), we define the pulled back ideal \( \mathcal{J}_V = \text{im}(g^*J \rightarrow \mathcal{O}_V) \subset \mathcal{O}_V \).

Fix a scheme \( S \), then an algebraic \( S \)-space is a sheaf \( F \) on the big étale site of \( S \), \( (\text{Sch}/S)_{\text{ét}} \), such that the diagonal morphism \( \Delta_F : F \rightarrow F \times_S F \) is represented by schemes, and there is a smooth surjection \( U \rightarrow F \) from an \( S \)-scheme \( U \). An algebraic \( S \)-space is a stack \( H \) on \( (\text{Sch}/S)_{\text{ét}} \), such that the diagonal morphism \( \Delta_H : H \rightarrow H \times_S H \) is represented by algebraic \( S \)-spaces and there is a smooth surjection \( U \rightarrow H \) from an algebraic \( S \)-space \( U \). We make no separation assumptions on our algebraic stacks, but all algebraic stacks that are used in this paper will possess quasi-compact and separated diagonals, thus all of the results of [LMB] apply.

Let \( \text{An} \) denote the category of analytic spaces. The étale topology on \( \text{An} \) is the Grothendieck pretopology generated by local analytic isomorphisms. An analytic stack is an étale stack over \( \text{An} \), with diagonal representable by analytic spaces, admitting a smooth surjection from an analytic space.

2. Examples

We feel that it is instructive to provide examples illustrating the necessity of some of the hypotheses of the results contained in this paper.

Example 2.1. Let \( X \) be the Deligne-Mumford stack \( \mathbb{B} \mathbb{Z} \), and fix an elliptic \( \mathbb{C} \)-curve \( E \). Observe that \( H^1(E, \mathbb{Z}) = 0 \) and \( H^1(E, \mathbb{C}) \neq 0 \). Hence, there is a map of analytic stacks \( \mathbb{E}_{\mathbb{an}} \rightarrow \mathbb{B} \mathbb{Z}_{\mathbb{an}} \) which is not algebraizable to a map of algebraic \( \mathbb{C} \)-stacks \( E \rightarrow \mathbb{B} \mathbb{Z} \). In particular, we conclude that the quasi-compact diagonal assumption which appears in Theorem 1 is necessary.

Example 2.2. Fix an elliptic \( \mathbb{C} \)-curve \( E \). Then \( H^1(E, E) = E \otimes \mathbb{C} \) and \( H^1(E, E) = E \otimes \mathbb{C} \). Let \( \gamma \in H^1(E, E) \) \( \gamma \) corresponds to an analytic map \( \mathbb{E}_{\mathbb{an}} \rightarrow \mathbb{B} \mathbb{E}_{\mathbb{an}} \), which is not algebraizable to a map of algebraic \( \mathbb{C} \)-stacks \( E \rightarrow \mathbb{B} \mathbb{E} \). Thus, we see that a strengthening of Theorem 1 to the setting where \( X \) is separated, but not Deligne-Mumford, is not possible.

Note that given a projective \( \mathbb{C} \)-scheme \( X \) and a locally quasi-finite analytic \( X_{\mathbb{an}} \)-space \( Z \), it is easy to produce examples where \( Z \) is non-algebraizable when \( Z \) is not compact. Indeed, analytic open subsets \( X_{\mathbb{an}} \) which do not arise as Zariski open subsets of \( X \) are such examples.

Example 2.3. Consider \( X := \mathbb{P}^1_\mathbb{C} \) and one of its coordinate patches \( t : \mathbb{A}^1_\mathbb{C} \hookrightarrow X \). Let \( U \hookrightarrow \mathbb{A}^1_\mathbb{C} \) be the complement of the closed unit disc. Note that \( U \) is an analytic space, and the image of \( U \) in \( X_{\mathbb{an}} \) is also open. Let \( Z = X_{\mathbb{an}} \Pi U \cap X_{\mathbb{an}} \), taken in the category of ringed spaces. Then \( Z \) is a compact, but non-Hausdorff analytic space. Indeed, \( Z \) is obtained by gluing two analytic spaces along an open analytic subspace, and there is a continuous surjection \( X_{\mathbb{an}} \Pi X_{\mathbb{an}} \rightarrow Z \). There is also a local analytic isomorphism of analytic spaces \( Z \rightarrow X_{\mathbb{an}} \), but clearly \( Z \) is not algebraizable. In particular, we conclude that the separatedness of the map \( Z \rightarrow X_{\mathbb{an}} \) is necessary in the definition of pseudosheaves, if we would like Theorem 2 to hold.

Example 2.4. In [Har77] App. B, there is given an example of a smooth projective surface \( Y \) over \( \mathbb{C} \), with an open subscheme \( t : U \hookrightarrow Y \) which possesses a pair of non-isomorphic line bundles \( \mathcal{L} \) and \( \mathcal{M} \) such that the analytified line bundles \( \mathcal{L}_{\mathbb{an}} \) and \( \mathcal{M}_{\mathbb{an}} \) are isomorphic.
Note that since \( \mathcal{L} \) (resp. \( \mathcal{M} \)) is \( \mathcal{O}_U \)-coherent, there is a coherent \( \mathcal{O}_X \)-module \( \mathcal{L} \) (resp. \( \mathcal{M} \)) together with an isomorphism \( \gamma^* \mathcal{L} \cong \mathcal{L} \) (resp. \( \gamma^* \mathcal{M} \cong \mathcal{M} \)). Define the smooth, universally closed, and finite type \( \mathbb{C} \)-scheme \( X \) by gluing two copies of \( Y \) along \( U \). Let \( p, q : Y \rightrightarrows X \) denote the two different inclusions of \( Y \) into \( X \). On \( Y_{an} \), we have two coherent \( \mathcal{O}_{Y_{an}} \)-modules \( \mathcal{L}_{an} \) and \( \mathcal{M}_{an} \). Note that since there is an induced analytic isomorphism \( j_{an}^* \mathcal{L}_{an} \cong j_{an}^* \mathcal{M}_{an} \), we may obtain a coherent \( \mathcal{O}_{X_{an}} \)-module \( \mathcal{F} \) such that \( p_{an}^* \mathcal{F} \cong \mathcal{L}_{an} \) and \( q_{an}^* \mathcal{F} \cong \mathcal{M}_{an} \). If \( \mathcal{F} \) is algebraizable, then there is a coherent \( \mathcal{O}_Y \)-module \( \mathcal{F} \) together with an analytic isomorphism \( F_{an} \cong \mathcal{F} \). In particular, we see that there are induced analytic isomorphisms of coherent sheaves on \( Y_{an} \):

\[
(p^* \mathcal{F})_{an} \cong p_{an}^* \mathcal{F} \cong \mathcal{L}_{an} \quad \text{and} \quad (q^* \mathcal{F})_{an} \cong q_{an}^* \mathcal{F} \cong \mathcal{M}_{an}.
\]

Since \( Y \) is a projective \( \mathbb{C} \)-scheme, by GAGA [SGA7i, Exp. XII, Thm. 4.4], the induced isomorphism \( \gamma^*(p^* \mathcal{F})_{an} \cong \mathcal{L}_{an} \) (resp. \( \gamma^*(q^* \mathcal{F})_{an} \cong \mathcal{M}_{an} \)) is uniquely algebraizable to an algebraic isomorphism \( p^* \mathcal{F} \cong \mathcal{L} \) (resp. \( q^* \mathcal{F} \cong \mathcal{M} \)). However, this implies an isomorphism:

\[
\mathcal{L} \cong \gamma^* \mathcal{L} \cong \gamma^* p^* \mathcal{F} \cong \gamma^* q^* \mathcal{F} \cong \gamma^* \mathcal{M} \cong \mathcal{M},
\]

which is a contradiction. Hence, \( \mathcal{F} \) is not algebraizable. From \( \mathcal{F} \), we can construct the finite and non-algebraizable \( \mathcal{O}_{X_{an}} \)-algebra \( \mathcal{O}_{X_{an}}[\mathcal{F}] \). Taking the analytic spec of this algebra produces a finite homeomorphism \( \mathbb{Z} \rightarrow X_{an} \) such that \( \mathbb{Z} \) is non-algebraizable. Hence, it is necessary for \( \mathbb{Z} \) to be proper, if we would like any locally quasi-finite and separated map \( \mathbb{Z} \rightarrow X_{an} \) to be algebraizable.

3. Preliminaries

In this section, we will recall and prove some results for separated spaces. The generalizations of the classical GAGA results to separated algebraic stacks will be dealt with in Appendix A. For a morphism of analytic spaces \( X \rightarrow Y \), the analytic Hilbert functor \( \mathrm{Hilb}^m_{X/Y} \) assigns to each analytic \( Y \)-space \( T \), the set of isomorphism classes of closed immersions \( \mathbb{Z} \rightarrow X_T \) such that the composition \( \mathbb{Z} \rightarrow T \rightarrow X \rightarrow Y \rightarrow X_T \) is proper and flat. If \( U \) is another analytic \( Y \)-space, then one may also define the functor \( \mathrm{Hom}_Y(U, X) \), which assigns to each analytic \( Y \)-space \( T \), the set of \( T \)-morphisms \( U_T \rightarrow X_T \).

In the case where the morphism \( X \rightarrow Y \) is separated, then J. Bingener [Bin80] showed that the functor \( \mathrm{Hilb}^m_{X/Y} \) naturally has the structure of a separated analytic \( Y \)-space. If, in addition, \( U \) is proper and flat over \( Y \), Bingener also proved [loc. cit.] that \( \mathrm{Hom}_Y(U, X) \) is a separated analytic \( Y \)-space.

**Remark 3.1.** The analyticity of the absolute Hilbert functor (i.e. when \( Y \) is a point), is due to [Dou66].

The following variant of the Hom-space will be useful in this paper. Again, we fix an analytic \( Y \)-space. Let \( U \rightarrow V \) be a morphism of analytic \( Y \)-spaces. Define \( \mathrm{Sec}_Y(U/V) \) to be the functor which assigns to each \( Y \)-space \( T \), the set of analytic sections to the morphism \( U_T \rightarrow V_T \). Standard arguments involving fiber products prove the following

**Proposition 3.2.** Fix an analytic space \( U \), and let \( U \rightarrow V \) be a separated morphism of analytic \( Y \)-spaces, where \( V \rightarrow Y \) is proper and flat. Then \( \mathrm{Sec}_Y(U/V) \) is a separated analytic \( Y \)-space.

We will record for future reference the following

**Corollary 3.3.** Let \( X \rightarrow Y \) be a morphism of analytic stacks such that the diagonal map \( \Delta_{X/Y} \) is a monomorphism. Then the diagonal map \( \Delta_{\mathbb{Z} \times X/Y} \) is representable by separated analytic spaces.
Proof. Fix an analytic \(\mathcal{Y}\)-space \(\mathcal{T}\). Let \(\mathcal{Z}, \mathcal{Z}'\) be locally quasi-finite and separated analytic \(\mathcal{X} \times \mathcal{Y}\)-stacks which are both \(\mathcal{T}\)-proper and flat. The monomorphism condition on the diagonal of the map \(\mathcal{X} \rightarrow \mathcal{Y}\) ensures that the diagonals of the maps \(\mathcal{Z} \rightarrow \mathcal{T}\) and \(\mathcal{Z}' \rightarrow \mathcal{T}\) are monomorphisms. In particular, the analytic stacks \(\mathcal{Z}\) and \(\mathcal{Z}'\) are thus analytic spaces. It remains to show that \(\text{Isom}_{\mathcal{H}^{\mathrm{an}}}_{\mathcal{Z}\mathcal{Z}'}\) is a separated analytic \(\mathcal{T}\)-space. There is, however, an inclusion of étale \(\mathcal{T}\)-sheaves

\[
\text{Isom}_{\mathcal{H}^{\mathrm{an}}}_{\mathcal{Z}\mathcal{Z}'}(\mathcal{Z}, \mathcal{Z}') \hookrightarrow \text{Sec}_X(\mathcal{Z} \times \mathcal{X} \mathcal{Z}'/\mathcal{Z}),
\]

which, by [Dou66 §10.1, Prop. 1], is representable by open embeddings of analytic spaces. Applying Proposition 3.2, we conclude the result. \(\square\)

Using GAGA for separated and locally of finite type \(\mathcal{C}\)-stacks (cf. Theorem [X.1] and Corollary [A.3], the method of proof employed in Corollary 3.3 readily proves the following

Lemma 3.4. For a locally of finite type algebraic \(\mathcal{C}\)-stack \(X\) with locally separated diagonal, the analytification functor

\[
\Psi_X : \mathbf{PS}_p(X) \rightarrow \mathbf{PS}_p(X_{\text{an}})
\]

is fully faithful.

Given algebraic \(\mathcal{C}\)-spaces \(U\) and \(V\), it is a non-trivial task to determine how the set of analytic morphisms between \(U_{\text{an}}\) and \(V_{\text{an}}\) relates to the set of algebraic morphisms between \(U\) and \(V\). Since we are interested in analytifications of algebraic stacks, which are defined by moduli problems, such a relation will be essential for us to proceed, however.

Lemma 3.5. Fix a locally of finite type algebraic \(\mathcal{C}\)-stack \(V\) with locally separated diagonal and a locally of finite type algebraic \(\mathcal{V}\)-stack \(U\) with locally separated diagonal. Let \(S\) be a local artinian \(\mathcal{V}\)-scheme, then the analytification functor

\[
\text{Hom}_V(S, U) \rightarrow \text{Hom}_{V_{\text{an}}}(S_{\text{an}}, U_{\text{an}})
\]

is an equivalence of categories.

Proof. It suffices to treat the case where \(V = \text{Spec} \mathbb{C}\). Let \(k > 0\) be such that the maximal ideal \(m_S\) of the artinian \(\mathcal{C}\)-algebra \(\Gamma(S, O_S)\) satisfies \(m_S^k = 0\). If \(U\) is a scheme, then a morphism \(S \rightarrow U\) is equivalent to specifying a point \(u \in U(\mathbb{C})\) and a morphism of \(\mathcal{C}\)-algebras \(O_{U, u}/m^k_{U, u} \rightarrow \Gamma(S, O_S)\). Similarly, an analytic morphism \(S_{\text{an}} \rightarrow U_{\text{an}}\) is equivalent to specifying a point \(u \in U_{\text{an}} = U(\mathbb{C})\), as well as a morphism of \(\mathcal{C}\)-algebras \(O_{U_{\text{an}}, u}/m^k_{U_{\text{an}}, u} \rightarrow \Gamma(S, O_S)\). Recalling that there is a bijection of \(\mathcal{C}\)-algebras \(O_{U, u}/m^k_{U, u} \cong O_{U_{\text{an}}, u}/m^k_{U_{\text{an}}, u}\) as well as a bijection of sets \(U_{\text{an}} \rightarrow U(\mathbb{C})\), we conclude that we’re done in the case that \(U\) is a scheme. If \(U\) is an algebraic stack, let \(U' \rightarrow U\) be a smooth cover by a scheme \(U'\). Since \(S\) is local artinian, the lifting criterion for smoothness implies that the analytification functor \(\text{Hom}(S, U) \rightarrow \text{Hom}(S_{\text{an}}, U_{\text{an}})\) is essentially surjective. That the analytification functor is fully faithful is clear. \(\square\)

Lemma 3.6. Fix an analytic stack \(\mathcal{W}\), and consider a morphism of analytic \(\mathcal{W}\)-stacks \(p : U \rightarrow V\). Suppose that for any local artinian \(\mathcal{W}\)-scheme \(S\), the functor

\[
p_* : \text{Hom}_\mathcal{W}(S_{\text{an}}, U) \rightarrow \text{Hom}_\mathcal{W}(S_{\text{an}}, V)
\]

is fully faithful. Then \(p\) is a monomorphism of analytic stacks, and is, in particular, representable by analytic spaces. If, in addition, the functor \(p_* : |S_{\text{an}}|\) is essentially surjective, then \(p\) is an analytic isomorphism.
Proof. First, we assume that the \( p \) is representable by analytic spaces and \( p_\ast |_{S_{an}} \) is an equivalence. The lifting criteria for analytic étale morphisms shows that \( p \) is a bijective étale morphism, hence an analytic isomorphism. In the case where the map \( p \) is not assumed to be representable by analytic spaces, and \( p_\ast |_{S_{an}} \) is fully faithful, we observe that the diagonal map \( \Delta_p \) is representable by analytic spaces and \( (\Delta_p)_\ast |_{S_{an}} \) is an equivalence. By what we have shown already, the diagonal map \( \Delta_p \) is an analytic isomorphism, and so the map \( p \) is a monomorphism of analytic stacks. Noting that a monomorphism of analytic stacks is representable by analytic spaces, then we're done. \( \square \)

To prove Theorem 5, the following strengthening of Lemma 3.6 will be useful.

**Corollary 3.7.** Fix an analytic stack \( Y \). Consider a morphism of étale \( Y \)-stacks \( P : \mathcal{F} \rightarrow \mathcal{G} \). Suppose the following conditions are satisfied:

1. \( \mathcal{F} \) is an analytic stack;
2. the diagonal of \( \mathcal{G} \) is representable by analytic spaces;
3. for any local artinian \( Y \)-scheme \( S \), the functor \( \mathcal{F}(S_{an}) \rightarrow \mathcal{G}(S_{an}) \) is fully faithful (resp. an equivalence).

Then \( P \) is a monomorphism (resp. an equivalence).

**Proof.** Conditions (1) and (2) imply that the morphism \( P \) is representable by analytic stacks. Thus we may assume that \( \mathcal{G} \) is an analytic space, and \( \mathcal{F} \) is an analytic stack. Condition (3) implies Lemma 3.6 applies, and we’re done. \( \square \)

We defer the proof Theorem 2 to §4, and focus on applying this result to the proof of Theorem 5. By Lemma 3.5 and Corollary 3.7, to prove Theorem 5, the next result will be of use.

**Lemma 3.8.** Fix a locally of finite type algebraic \( C \)-space \( Y \), and a locally of finite type algebraic \( Y \)-space \( X \). For any local artinian \( Y \)-scheme \( S \), the functor

\[
\Phi_{X/Y}(S_{an}) : HS_{X/Y}(S) \rightarrow HS_{X_{an}/Y_{an}}(S_{an})
\]

is an equivalence.

**Proof.** First, we show that \( \Phi_{X/Y}(S_{an}) \) is fully faithful. Given quasi-finite and separated maps \( Z, Z' \rightrightarrows X \times_Y S \), where the two compositions \( Z, Z' \rightrightarrows S \) are proper and flat, then regarding \( X \times_Y S \) as a locally of finite type \( C \)-space, as well as \( Z \) and \( Z' \) as proper \( C \)-spaces, we conclude by the full faithfulness of \( \Psi_{X \times Y S} \) that any analytic \( X_{an} \times_Y S_{an} \)-map \( Z_{an} \rightarrow Z'_{an} \) arises from an algebraic \( X \times_Y S \)-map \( Z \rightarrow Z' \) over \( C \). Applying GAGA (cf. Corollary [A.3]), we conclude that \( C \)-maps \( Z \rightarrow Z' \) which analytify to \( S_{an} \)-maps \( Z_{an} \rightarrow Z'_{an} \) are automatically \( S \)-maps—hence we’ve shown that \( \Phi_{X/Y}(S_{an}) \) is fully faithful.

To show that the functor \( \Phi_{X/Y}(S_{an}) \) is essentially surjective, we note that given a locally quasi-finite and separated morphism of analytic spaces \( Z \rightarrow X_{an} \times_Y S_{an} \) such that the composition \( Z \rightarrow S_{an} \) is proper and flat, then again regarding \( Z \) and \( X \times_Y S \) as living over \( C \), then Theorem 2 implies that the morphism \( Z \rightarrow X_{an} \times_Y S_{an} \) algebraizes to a quasi-finite and separated morphism of algebraic \( C \)-spaces \( Z \rightarrow X \times_Y S \) such that \( Z \) is \( C \)-proper. Applying GAGA (cf. Corollary [A.3]), again, we may algebraize the flat morphism \( Z_{an} \rightarrow S_{an} \) to a flat morphism \( Z \rightarrow S \), which proves essential surjectivity of the functor \( \Phi_{X/Y}(S_{an}) \). \( \square \)
Proof of Theorem 3. We use the criteria of Corollary 3.7. For (1), by [HalR10] Thm. 2, we have that \( HS_{X/Y} \) an is an analytic stack. For (2), by Corollary 3.3 we deduce that \( HS_{X/Y} \) an has representable diagonal. For (3), we apply Lemma 3.8. 

4. Non-separated GAGA

To prove Theorem 3, we will utilize two methods of dévissage, which are summarized by the next two propositions.

Proposition 4.1 (Birational dévissage). Suppose that \( X \) is a finite type \( \mathbb{C} \)-scheme. Suppose that for any closed immersion \( V \hookrightarrow X \), there is a proper, schematic, and birational morphism \( V' \rightarrow V \) such that the functor \( \Psi_{V'} \) is an equivalence, then the functor \( \Psi_X \) is an equivalence.

Proposition 4.2 (Generically finite étale dévissage). Suppose that \( X \) is a finite type Deligne-Mumford \( \mathbb{C} \)-stack with quasi-compact and separated diagonal. Suppose that for any closed immersion \( V \hookrightarrow X \), there is a finite and generically étale map \( V' \rightarrow V \) such that the functors \( \Psi_{V'} \) and \( \Psi_{V \times V'} \) are equivalences, then the functor \( \Psi_X \) is an equivalence.

It will also be necessary to have a result on sections to étale morphisms.

Proposition 4.3. Consider a quasi-compact and étale map of locally of finite type algebraic \( \mathbb{C} \)-spaces \( W \rightarrow Z \). Then, the map of sets

\[
\text{Hom}_Z(Z, W) \rightarrow \text{Hom}_{Z_{an}}(Z_{an}, W_{an})
\]

is bijective.

The proofs of Propositions 4.1 and 4.2 will fill the remainder of this section, and we will utilize arguments very similar to those in [HalR10] §4. With the above results at our disposal, however, we can prove Theorem 3 immediately.

Proof of Theorem 3. By Lemma 3.4, we know that the functor \( \Psi_X \) is fully faithful, so it remains to show that the functor \( \Psi_X \) is essentially surjective.

Basic Case. First, we suppose that \( X \) is a finite type \( \mathbb{C} \)-scheme, and the structure morphism factors as the composition \( X \rightarrow Y \rightarrow \text{Spec} \mathbb{C} \), where the morphism \( f \) is quasi-compact étale and the morphism \( g \) is projective. Fix a locally quasi-finite morphism of analytic spaces \( s : Z \rightarrow X_{an} \) such that \( Z \) is proper. Since the analytic space \( Y_{an} \) is separated, one concludes that the composition \( f \circ s : Z \rightarrow Y_{an} \) is a locally quasi-finite and proper morphism of analytic spaces. By [RG71] Thm. XII.4.2 such a morphism is a finite. Since \( Y \) is also projective, by [SGA] Exp. XII, Thm. 4.6, we conclude that there is a finite morphism of \( \mathbb{C} \)-schemes \( Z \rightarrow Y \) such that \( Z \) is proper, and \( Z_{an} \cong \overline{Z} \) over \( Y_{an} \). To complete the proof in this setting, it suffices to produce a quasi-finite morphism of schemes \( t : Z \rightarrow X \) such that \( t_{an} = s \). Note that it is equivalent to produce a section to the quasi-compact étale morphism \( h : X \times_Y Z \rightarrow Z \) which agrees with the analytic section to the morphism of analytic spaces \( h_{an} \) induced by \( s \). This is precisely the content of Proposition 4.3. Now, if we attach to \( Z \) a coherent \( \mathcal{O}_Z \)-module \( \mathcal{F} \), then since \( Z \) is algebraizable, GAGA for proper \( \mathbb{C} \)-schemes [SGA] Exp. XII, Thm. 4.4 implies that \( \mathcal{F} \) is algebraizable. Thus the functor \( \Psi_X \) is an equivalence in this case.

Finite type \( \mathbb{C} \)-schemes. Next, we just assume that \( X \) is a finite type \( \mathbb{C} \)-scheme. Note that any closed subscheme \( V \hookrightarrow X \) is also a finite type \( \mathbb{C} \)-scheme. Thus, by [RG71] Cor. 5.7.13, there is a schematic and birational morphism \( V' \rightarrow V \) such that the structure morphism of \( V' \) over \( \mathbb{C} \) factors as \( V' \xrightarrow{f_V} Y_V \xrightarrow{g_V} \text{Spec} \mathbb{C} \), where \( f_V \) is quasi-compact...
étale and $g_V$ is projective. By the basic case considered above, the functor $Ψ_V.$ is an equivalence and so by Proposition 4.1 we conclude that the functor $Ψ_X$ is an equivalence.

**Finite type Deligne-Mumford $C$-stacks.** Here, we assume that $X$ is a finite type Deligne-Mumford $C$-stack. Note that if $V → X$ is any closed immersion, then $V$ is also a finite type Deligne-Mumford $C$-stack. By [LMB Thm. 16.6], there is a finite and generically étale morphism $V' → V$, where $V'$ is a scheme. Since $V'$ is thus a finite type $C$-scheme, we may conclude by the case previously considered that the functors $Ψ_{V'}$ and $Ψ_{V', X, V'}$ are equivalences. Hence, by Proposition 4.2 we conclude that the functor $Ψ_X$ is an equivalence.

**General case.** Fix a locally quasi-finite and separated morphism of analytic stacks $s : Z → X_{\text{an}}$ where $Z$ is proper with finite diagonal. Let $X^0$ denote the open substack of $X$ which has quasi-finite diagonal. Note that since $Z$ has finite diagonal, then the quasi-finite and separated map $s : Z → X_{\text{an}}$ factors canonically through $X^0_{\text{an}}$. Replacing $X$ by $X^0$, we may henceforth assume that $X$ has quasi-finite and separated diagonal. Next, let $O_X$ denote the category of quasi-compact open subsets of $X$. We note that $\{U| U \in O_X\}$ is an open cover of $X$ and so $(s^{-1}(U)| U \in O_X)$ is an open cover of $Z$. Since $Z$ is a compact topological space, and the exhibited cover is closed under finite unions, there is an open immersion $U → X$ such that the map $Z → X_{\text{an}}$ factors uniquely through $U_{\text{an}}$ and $U_{\text{an}}$ is a finite type $C$-stack with quasi-finite and separated diagonal. By the previous case considered, we conclude that the functor $Ψ_X$ is essentially surjective, thus an equivalence.

Given a morphism of ringed spaces $f : U → V$, then we say that $f$ is Stein if the map of sheaves $f^\sharp : O_V → f_*O_U$ is an isomorphism. By [GR84 §10.6.1], if the morphism $f$ is a proper morphism of analytic spaces, then there is a Stein factorization $U \xrightarrow{A(f)} V$, where the morphism $A(f)$ is proper, Stein, surjective, with connected fibers, and the morphism $B(f)$ is finite. Similarly, if the morphism $f$ is a proper morphism of locally noetherian schemes, then by [EGA III, 4.3.1], the morphism $f$ has a Stein factorization $U \xrightarrow{A(f)} V$, whose formation commutes with flat base change on $V$. What is important here is that by [SGA1 Exp. XII, Thm. 4.2], the formation of the Stein factorization commutes with analytification.

**Lemma 4.4.** Fix a proper morphism of locally of finite type $C$-schemes $π : Y → X$, and a locally quasi-finite and separated morphism of analytic spaces $σ : Z → X_{\text{an}}$ such that $Z_{\text{an}}$ is proper, and the analytic space $ζ_{\text{an}}$ is algebraizable to a quasi-finite and separated $Y$-scheme $W$. In the Stein factorization of the proper morphism $(π_{\text{an}})_* : ζ_{Y_{\text{an}}} → ζ$ of analytic spaces, $ζ_{Y_{\text{an}}} \xrightarrow{A((π_{\text{an}})_*)} A(ζ_{Y_{\text{an}}}) \xrightarrow{B((π_{\text{an}})_*)} ζ$, the analytic space $A(ζ_{Y_{\text{an}}})$ is algebraizable to a quasi-finite and separated $X$-scheme.

**Proof.** By [EGA IV, 18.12.13], there is a finite $Y$-scheme $\mathfrak{F} : \mathbb{W} → Y$ and an open immersion $j : W → \mathbb{W}$. Define the map $ρ := π \circ \mathfrak{F} : \mathbb{W} → X$, which is proper, and consider the Stein factorization $\mathbb{W} \xrightarrow{A(ρ)} A(ξ) \xrightarrow{B(ρ)} X$. We claim that $A(ρ)^{-1}A(ρ)(|W|) = |W|$. In particular, since the map $A(ρ)$ is proper and surjective, it is universally submersive, thus it will follow that $A(ρ)(|W|)$ is an open subset of $A(ξ)$. Since the separated morphism $W → A(ξ)$ has proper fibers, it follows that for any $w ∈ A(ξ)$, the morphism on the geometric fibers over $w$: $W_π → W_π$ is an open and closed immersion. Since $W_π$ is geometrically connected, we conclude that $W_π = W_π$ or $∅$, and the claim is proved. We let the quasi-finite and separated morphism $s' : Z' → X$ denote the induced open subscheme of $A(ρ)(|W|)$ defined by the image of $|W|$. It now remains to construct a unique morphism of analytic spaces $α : Z'_an → Z$ which is compatible with the induced map.
\[ \beta : W_{an} \to Z^*_{an}. \] Indeed, since \( W_{an} \to Z^*_{an} \) is proper and Stein, then \( W_{an} \to Z^*_{an} \to \mathbb{Z} \) is the Stein factorization of \( (\tau_{an})_{\mathbb{Z}} \).

We first define the map \( \alpha \) set-theoretically. For \( w \in W^*_{an} \), the subset \( \beta^{-1}(w) \subset W_{an} \) is closed and connected, thus since \( (\tau_{an})_{\mathbb{Z}} \) is proper, we deduce that \( \nu_w := (\tau_{an})_{\mathbb{Z}} \beta^{-1}(w) \) is a closed and connected subset of \( \mathbb{Z} \). Moreover, \( \nu_w \) is also quasi-finite over the image of \( w \) in \( X_{an} \) and we deduce that \( \nu_w \) is a single point. Hence, we obtain a well-defined map of sets \( \alpha : |W^*_{an}| \to |\mathbb{Z}| \). Clearly, \( \alpha \) is continuous, as the maps \( (\tau_{an})_{\mathbb{Z}} \) and \( \beta \) are surjective and submersive. We obtain the map on functions from the following composition:

\[ \mathcal{O}_{\mathbb{Z}} \to ((\tau_{an})_{\mathbb{Z}})_* \mathcal{O}_{W_{an}} \cong \alpha_* \beta_* \mathcal{O}_{W_{an}} \cong \alpha_* \mathcal{O}_{W^*_{an}}, \]

with the last isomorphism because \( \beta \) is Stein. \( \square \)

We have two easy lemmata.

**Lemma 4.5.** Fix a compact analytic stack \( \mathcal{X} \), a \( \mathcal{O}_{\mathcal{X}} \)-coherent sheaf \( \mathcal{F} \) and a coherent \( \mathcal{O}_{\mathcal{X}} \)-ideal \( \mathcal{J} \).

1. If \( \text{supp } \mathcal{F} \subset |\mathcal{V}(\mathcal{J})| \), then there is a \( k > 0 \) such that \( \mathcal{J}^k \mathcal{F} = (0) \).
2. Given a coherent subsheaf \( \mathcal{F}' \subset \mathcal{F} \) such that \( \mathcal{J} \mathcal{F}' = (0) \), then there is a \( k > 0 \) such that \( (\mathcal{J}^k \mathcal{F}) \cap \mathcal{F}' = (0) \).

**Proof.** For (1), the Rückert Nullstellansatz [GR84, §3.2] implies that for any \( x \in \mathcal{X} \), there is an open neighborhood \( U_x \) and a \( k_x \) such that \( (\mathcal{J}^k \mathcal{F})|_{U_x} = (0) \). Since \( \mathcal{X} \) is compact, we conclude the result. For (2), fix \( x \in \mathcal{X} \) and observe that since the local ring \( \mathcal{O}_{\mathcal{X}, x} \) is noetherian, by [AM69, Cor. 10.10] \( \exists k_x > 0 \) such that \( (\mathcal{J}^k \mathcal{F})|_x \cap \mathcal{F}'|_x = (0) \). Note that for any \( x \in \mathcal{X} \), the sheaf of \( \mathcal{O}_{\mathcal{X}} \)-modules \( \mathcal{G}_x = (\mathcal{J}^k \mathcal{F}) \cap \mathcal{F}' \) is coherent, and has closed support. Thus, as \( (\mathcal{G}_x)|_x = (0) \), then there is an open neighborhood \( U_x \) of \( x \in \mathcal{X} \) such that \( \mathcal{G}_x|_{U_x} = (0) \). Since \( \mathcal{X} \) is compact in the analytic topology, we conclude that there is a finite set of points \( x_1, \ldots, x_n \in \mathcal{X} \) such that \( \mathcal{G}^k|_{U_{x_i}} = (0) \) and \( \{U_{x_i}\}_{i=1}^n \) covers \( \mathcal{X} \). Take \( k = \max_i k_{x_i} \), then \( (\mathcal{J}^k \mathcal{F}) \cap \mathcal{F}' = \bigcap_{i=1}^n \mathcal{G}^k|_{U_{x_i}} = (0) \), as claimed. \( \square \)

**Lemma 4.6.** Fix a proper morphism of finite type \( \mathbb{C} \)-schemes \( \pi : Y \to \mathcal{X} \), and a locally quasi-finite and separated morphism of analytic spaces \( \sigma : \mathcal{Z} \to X_{an} \), such that \( \mathcal{Z} \) is compact. Let \( U \hookrightarrow X \) be an open subscheme such that the induced map \( \pi^{-1}U \to U \) is an isomorphism. Fix a coherent ideal sheaf \( I \subset \mathcal{O}_{\mathcal{X}} \) such that \( |\mathcal{V}(I)| = |X \setminus U| \). The Stein factorization of the proper morphism of analytic spaces \( (\tau_{an})_{\mathcal{Z}} : \mathcal{Z}_{\mathbb{Y}} \to \mathcal{Z} \) induces a map of coherent \( \mathcal{O}_{\mathcal{Z}} \)-algebras \( \mathcal{B}((\tau_{an})_{\mathcal{Z}})^k : \mathcal{O}_{\mathcal{Z}} \to \mathcal{B}((\tau_{an})_{\mathcal{Z}})_* \mathcal{O}_A((\tau_{an})_{\mathcal{Z}}) \). There is a \( k > 0 \) such that the coherent \( \mathcal{O}_{\mathcal{Z}} \)-ideal \( (I_{an})_{\mathcal{Z}} \) annihilates the kernel and cokernel of the map \( \mathcal{B}((\tau_{an})_{\mathcal{Z}})^k \).

**Proof.** Define the open subset \( U' = \sigma^{-1}(U_{an}) \), then certainly we have that the map \( \mathcal{B}((\tau_{an})_{\mathcal{Z}})^k \) is an isomorphism when restricted to the open subset \( U' \). In particular, it follows that the kernel and cokernel of the map \( \mathcal{B}((\tau_{an})_{\mathcal{Z}})^k \) are supported on the analytic set \( |\mathcal{V}(I_{an})_{\mathcal{Z}}| \). Now apply Lemma 4.5(1). \( \square \)

For a morphism of ringed topoi \( f : U \to V \), we denote the induced map on structure sheaves by \( f^* : \mathcal{O}_V \to \mathcal{O}_U \). Define the **conductor** of the morphism \( f \) to be \( \mathcal{C}_f := \text{Ann}_{\mathcal{O}_V}(\text{coker } f^*) \). The result that follows is the main dévissage technique used in proving Propositions 4.1 and 4.2.
Lemma 4.7. Consider a finite type $\mathbb{C}$-stack $X$ with locally separated diagonal. Suppose that we have a commutative diagram of analytic stacks:

$$
\begin{array}{ccc}
Z' & \xrightarrow{f} & Z \\
\downarrow{s'} & & \downarrow{s} \\
X_{\text{an}} & \xrightarrow{\iota} & Z \\
\end{array}
$$

where $Z'$ and $Z$ are compact. Suppose that $Z'$ is algebraizable to a quasi-finite and separated algebraic $X$-stack, $s$ and $s'$ are locally quasi-finite and separated, and $f$ is finite and surjective.

1. If there is a coherent ideal $\mathcal{I} \subset \mathcal{O}_Z$ such that $\mathcal{I} \cap \ker f^\sharp = \{0\}$ and $V(\mathcal{I})$ is algebraizable to a quasi-finite and separated $X$-stack, then there is a factorization of the morphism $f$ as $Z' \to Z'' \xrightarrow{\beta} Z$ such that $\beta$ is finite and surjective, $\beta^\sharp : \mathcal{O}_Z \to \mathcal{O}_Z^\beta$ is injective, $Z''$ is algebraizable to a quasi-finite and separated algebraic $X$-stack, and $\mathcal{E}_\beta \subset \mathcal{E}_\beta$.

2. If there is a coherent ideal $\mathcal{J} \subset \mathcal{O}_X$ such that for any $k$ and any coherent sheaf of $\mathcal{O}_Z$-ideals $\mathcal{J} \supset (\mathcal{J}_n)_{Z}$ the analytic $X_{\text{an}}$-stack $V(\mathcal{J})$ is algebraizable to a quasi-finite and separated algebraic $X$-stack, and $\ker f^\sharp$ and $\operatorname{coker} f^\sharp$ are annihilated by $(\mathcal{J}_n)_{Z}$ for some $j$, then $Z$ is algebraizable to a quasi-finite and separated algebraic $X$-stack.

To prove Lemma 4.7 and Proposition 4.2, it will be necessary to understand certain pushouts for analytic stacks, which we will defer until later in this section. We are, however, ready to prove Proposition 4.7.

Proof of Proposition 4.7. For a Zariski closed subset $|V| \subset |X|$, let $P_{|V|}$ be the statement: for any closed subscheme $V_0 \subset X$ such that $|V_0| = |V|$, the functor $\Psi_{V_0}$ is an equivalence. Since the statement $P_0$ is trivially true, by the principle of noetherian induction and the statement of the Proposition, we will have proven the Lemma if we show the following: the truth of the statement $P_{|V|}$ for all proper closed subsets $|V| \subsetneq |X|$ implies the truth of the statement $P_{|X|}$.

By hypothesis, there is a proper and birational $S$-morphism $p : X' \to X$ such that $\Psi_{X'}$ is an equivalence. Consider a dense open subscheme $U \subset X$ for which $p^{-1}U \to U$ is an isomorphism, and let $I$ be a coherent sheaf of ideals with support $|X| \setminus |U|$. Next, suppose we have a locally quasi-finite and separated morphism of analytic spaces $\sigma : \mathcal{Z} \to X_{\text{an}}$, where $\mathcal{Z}$ is proper. The assumptions on $X'$ ensure that $\mathcal{Z}_{X'_{\text{an}}}$ is algebraizable to a quasi-finite and separated $X'$-scheme. By Lemma 4.4, we conclude that in the Stein factorization of the morphism $(p_{|Z|})_{Z} : \mathcal{Z}_{X'_{\text{an}}} \to Z$, that $\mathcal{A}(\mathcal{Z}_{X'_{\text{an}}})$ is algebraizable to a quasi-finite and separated $X$-scheme. For notational brevity, we set $Z' = \mathcal{A}(\mathcal{Z}_{X'_{\text{an}}})$ and let the morphism $f : Z' \to Z$ be the morphism $\mathcal{B}((p_{|Z|})_{Z})$ from the Stein factorization of the morphism $(p_{|Z|})_{Z}$. By Lemma 4.6, the kernel and cokernel of the map $f^\sharp : \mathcal{O}_Z \to f_*\mathcal{O}_{Z'}$ are annihilated by $(I_n)_Z$ for some $j$. By noetherian induction, $V((I_n)_Z)$ is algebraizable to a quasi-finite and separated $X$-scheme for all $k$. Hence, we may apply Lemma 4.7 to conclude that $Z$ is algebraizable to a quasi-finite and separated $X$-scheme.

Hence, given $(\mathcal{Z}, \mathcal{F}) \in \mathbf{PS}_p(X_{\text{an}})$, by what we have proven, we know that the locally quasi-finite and separated morphism $\mathcal{Z} \to X_{\text{an}}$ is algebraizable to a quasi-finite and separated morphism of schemes $Z \to X$, where $Z$ is $\mathbb{C}$-proper. By GAGA for proper $\mathbb{C}$-schemes [SGA5, Exp. XII, Thm. 4.4], we deduce that the coherent $\mathcal{O}_{X_{\text{an}}}$-module $\mathcal{F}$ is algebraizable. Hence, the functor $\Psi_X$ is essentially surjective. \qed
Next, observe that given a diagram of ringed spaces $E := \{Z^1 \leftarrow Z^3 \rightarrow Z^2\}$, let the topological space $|Z_4|$ be the colimit of the induced diagram $|E| := ||Z_1^i| \leftarrow |Z^3| \rightarrow |Z_2^i||$ in the category of topological spaces. We have induced maps $m^i : |Z_1^i| \to |Z^2|$, and the colimit of the diagram $E$ in the category of ringed spaces is the ringed space $Z^4 := \{Z_|^i|, m^i_1O_{Z^1 \times m^i_2O_{Z^3}}, m^i_2O_{Z^2}\}$. We may promote the morphisms of topological spaces $m^i : |Z^1| \to |Z^4|$ to morphisms of ringed spaces $m^i : Z^1 \to Z^4$. If the ringed spaces $Z^i$ are locally ringed, and the maps $m^i$ are morphisms of locally ringed spaces, then since $|Z^1| \pitchfork |Z^2| \to |Z^4|$ is surjective, it is easy to see that $Z^4$ is the colimit of the diagram $E$ in the category of locally ringed spaces. These observations will be of use when the ringed spaces $Z^i$ are schemes or analytic spaces. Indeed, to show that a scheme (resp. analytic space) is a pushout of some schemes (resp. analytic spaces), it will suffice to show that it is the pushout in the category of ringed spaces, and the maps involved are all maps of schemes (resp. analytic spaces), which will typically be clear.

Now, let $X$ be a locally noetherian algebraic stack, and suppose that we have quasi-finite and separated morphisms $s^i : Z^i \to X$ for $i = 1, 2, 3$. In addition, assume that we have finite $X$-morphisms $t^j : Z^3 \to Z^j$ for $j = 1, 2$. It was shown in [HalR10 Thm. 2.10], that the resulting diagram of algebraic $X$-stacks $[Z^1 \leftarrow Z^3 \rightarrow Z^2]$ has a colimit, $Z^4 := Z^1 \pitchfork Z^2$, in the category of quasi-finite, separated, and representable algebraic $X$-stacks. Let $m^i : Z^i \to Z_4$ denote the resulting $X$-morphisms, which are finite. It was also shown [loc. cit.], that the Zariski topological space $|Z^4|$ was the colimit of the diagram of topological spaces $[[Z^1] \leftarrow [Z^3] \rightarrow [Z^2]]$, and that there was an isomorphism of coherent sheaves $O_{Z^4} \to m_1^iO_{Z^1} \times m_2^iO_{Z^3}$. All of this commutes with flat base change on $Z_4$. Retaining this notation, we have the main result needed to prove Lemma 4.7.

**Lemma 4.8.** If $X$ is a locally of finite type algebraic $C$-stack with locally separated diagonal, then $Z^d_{X|m}$ is the colimit of the diagram $[Z^i_{an} \leftarrow Z^3_{an} \rightarrow Z^2_{an}]$ in the category of locally quasi-finite and separated analytic stacks over $X_{an}$, and remains so after flat base change on $X$.

**Proof.** For $k = 1, 2, 3$ let $m^k : Z^k \to Z^d$ denote the canonical map. First, we assume that $X$ is a scheme. For $l = 1, \ldots, 4$, let $\phi^l : Z^d_{an} \to Z^l$ denote the canonical map of ringed spaces. We note once and for all that by [SGA1 Exp. XII, 1.3.1], the functor $\phi^l$ from $O_{Z^l}$-modules to $O_{Z^d}$-modules is exact. Also, we have a bijection of sets:

$$
\left|Z^1_{an}\right| \pitchfork |Z^3_{an}| \rightarrow Z^1(C) \left|Z^3_{an}\right| \rightarrow Z^2(C) \rightarrow Z^4(C) \rightarrow |Z^4_{an}|.
$$

Thus, we conclude that the canonical, continuous map $\psi : \left|Z^1_{an}\right| \left|Z^3_{an}\right| \rightarrow |Z^4_{an}|$ is a bijection. Since it is also a proper map, we conclude that $\psi$ is a homeomorphism. Also, we have that the natural map $\psi^d : O_{Z^d_{an}} \to (m^l_{an})^1O_{Z^1} \times (m^l_{an})^2O_{Z^3} \times (m^l_{an})^3O_{Z^2}$ factors as the sequence of bijections:

$$
O_{Z^d_{an}} \to \phi^4_1(m^l_{an})^1O_{Z^1} \times m^l_{an}O_{Z^3} \times m^l_{an}O_{Z^2} \\
\to \phi^4_2m^l_{an}O_{Z^1} \times \phi^4_3m^l_{an}O_{Z^3} \times \phi^4_4m^l_{an}O_{Z^2} \\
\to (m^l_{an})^1O_{Z^1} \times (m^l_{an})^2O_{Z^3} \times (m^l_{an})^3O_{Z^2}.
$$

Hence, we conclude that $Z^d_{an}$ is the colimit of the diagram in the category of ringed spaces, and remains so after flat base change on $X$. It is clear that this implies that $Z^d_{an}$ is the colimit in the category of analytic spaces.
Next, we assume that $X$ is an algebraic space. Let $X_1 \to X$ be an étale cover by a scheme, and furthermore take $X_2 = X_1 \times_X X_1$. Take $Z^4_{\mathbf{a}} = Z^4_{\mathbf{a}} \times_X X_i$ for $i = 1, 2$ and $k = 1, 2, 3, 4$. By the case of schemes already considered, we know for $i = 1$ and $2$ that the analytic space $(Z^4_{\mathbf{a}})_{\mathbf{a}}$ is the colimit of the diagram $((Z^4_{\mathbf{a}})_{\mathbf{a}} \leftarrow (Z^4_{\mathbf{a}})_{\mathbf{a}} \to (Z^4_{\mathbf{a}})_{\mathbf{a}})$ in the category of analytic spaces. The universal properties furnish us with an étale equivalence relation of analytic spaces $((Z^4_{\mathbf{a}})_{\mathbf{a}} \Rightarrow (Z^4_{\mathbf{a}})_{\mathbf{a}})$, with quotient in the category of analytic spaces $Z_{\mathbf{a}}^4$. The universal property of the pushouts and étale descent immediately imply that $Z_{\mathbf{a}}^4$ is the colimit in the category of analytic spaces. In the case that $X$ is an algebraic stack, we may argue exactly the same as in the case where $X$ is an algebraic space (but instead use smooth covers), and conclude that $Z_{\mathbf{a}}^4$ is the colimit in the category of locally quasi-finite and separated analytic stacks over $X_{\mathbf{a}}$. □

From here, we may prove a Lemma that is necessary for Proposition 4.2.

**Lemma 4.9.** Fix a finite type Deligne-Mumford $\mathbb{C}$-stack $X$ with quasi-compact and separated diagonal. Let $\pi : X^1 \to X$ be a finite and surjective morphism and consider an open substack $U \subset X$ such that the induced morphism $\pi^{-1}U \to U$ is étale. Fix a coherent $O_X$-ideal $I$ such that $|X \setminus U| = |V(J)|$. Consider a locally quasi-finite and separated morphism of analytic stacks $\sigma : Z \to X_{\mathbf{a}}$, where $Z$ is proper. Let $X^2 := X^1 \times_X X^1$ and for $i = 1$ and $2$ define $Z^1 := Z \times_{X_{\mathbf{a}}} X_{\mathbf{a}}^i$. There is an induced coequalizer diagram $[Z^2 \rightrightarrows Z^1]$, with the maps appearing finite. Suppose that this diagram has a coequalizer $W$ in the category of locally quasi-finite and separated analytic $X_{\mathbf{a}}$-stacks (e.g. if $Z^1$ and $Z^2$ are algebraizable). There is an induced finite and surjective morphism $\eta : W \to Z$ of analytic stacks. Then there is a $k > 0$ such that the kernel and cokernel of the map $\eta^k$ is annihilated by the coherent $O_Z$-ideal $(I_{\mathbf{a}})_Z^k$.

*Proof.* Consider the open analytic substack $U' := \sigma^{-1}(U_{\mathbf{a}})$ of $Z$. Let $g_i : Z^1 \to Z$ be the induced maps, then the induced map $g^{-1}_1U' \to U'$ is finite étale. Hence, we deduce by étale descent that the map $\eta^{-1}U' \to U$ is an analytic isomorphism. In particular, we deduce that $|\text{supp ker } \eta^k|$ and $|\text{supcoker } \eta^k|$ are contained in $|V((I_{\mathbf{a}})_Z)|$. Applying Lemma 4.5, we deduce the result. □

*Proof of Proposition 4.2.* As in Proposition 4.1 we prove the result by noetherian induction on the closed substacks of $X$. Hence, it suffices to assume that the functor $\Psi_V$ is an equivalence for any closed substack $V \to X$ such that $|V| \subseteq |X|$. By assumption, there is a finite and generically étale map $\pi : X^1 \to X$ such that the functors $\Psi_{X^1}$ and $\Psi_{X^1 \times X^1}$ are equivalences. Fix a dense open substack $U \to X$ such that $\pi^{-1}U \to U$ is étale and let $J$ be a coherent $O_X$-ideal such that $|X \setminus U| = |V(J)|$. Let $Z \to X_{\mathbf{a}}$ be a locally quasi-finite and separated morphism with $Z$ proper. Let $X^2 = X^1 \times_X X^1$ and for $i = 1$ and $2$ we set $Z^1 = Z \times_{X_{\mathbf{a}}} X_{\mathbf{a}}^i$. By the hypotheses on $\Psi_{X^1}$, and $\Psi_{X^1 \times X^1}$, the diagram $[Z^2 \rightrightarrows Z^1]$ is algebraizable. Hence, by Lemma 4.8 we conclude that the coequalizer in the category of locally quasi-finite and separated $X_{\mathbf{a}}$-stacks exists, and is algebraizable. We denote this coequalizer by $W$. By Lemma 4.9 we deduce that the induced map $\eta : W \to Z$ has the property that ker$\eta^k$ and coker$\eta^k$ are both annihilated by $(I_{\mathbf{a}})_Z^k$ for some $k > 0$. By Lemma 4.7, we deduce that the morphism $Z \to X_{\mathbf{a}}$ is algebraizable to a quasi-finite, separated, and representable morphism $Z \to X$, with $Z$ $\mathbb{C}$-proper.

Thus, given $(Z, J) \in \mathbf{PS}_p(X_{\mathbf{a}})$, by what we have proven, we know that the locally quasi-finite and separated morphism $Z \to X_{\mathbf{a}}$ is algebraizable. By GAGA for proper $\mathbb{C}$-stacks (cf. Theorem A.1), we deduce that the functor $\Psi_X$ is essentially surjective. □
Let $f : \mathcal{Z}' \to \mathcal{Z}$ be a finite morphism of analytic stacks. If $\mathcal{I} \triangleleft \mathcal{O}_{\mathcal{Z}}$ is a coherent ideal such that $\mathcal{I} \subset \mathcal{E}_r$, then the image of $\mathcal{I}$ in $f_*\mathcal{O}_{\mathcal{Z}'}$, generates a $f_*\mathcal{O}_{\mathcal{Z}'}$-ideal $\mathcal{I}' \subset \mathcal{E}_r$, which lies in the image of $\mathcal{O}_{\mathcal{Z}}$ (this is a general property of conductors). If $f^\sharp$ is an injective map, then it is easily verified that $\mathcal{I}'$ is a coherent $\mathcal{O}_{\mathcal{Z}}$-ideal and $(\mathcal{I})_{\mathcal{Z}'} = \mathcal{I}_{\mathcal{Z}'}$ as $\mathcal{O}_{\mathcal{Z}'}$-ideals. We will use this notation for the remainder of this section.

**Lemma 4.10.** Let $\mathcal{X}$ be an analytic stack. Consider a finite morphism of locally quasi-finite analytic $\mathcal{X}$-stacks $f : \mathcal{Z}' \to \mathcal{Z}$, such that $f^\sharp : \mathcal{O}_{\mathcal{Z}} \to f_*\mathcal{O}_{\mathcal{Z}'}$ is injective. Fix an ideal $\mathcal{I} \triangleleft \mathcal{O}_{\mathcal{Z}}$ such that $\mathcal{I} \subset \mathcal{E}_r$. Then the diagram of analytic $\mathcal{X}$-stacks:

$$
\begin{array}{ccc}
V(\mathcal{I})' & \to & \mathcal{Z}' \\
\downarrow & & \downarrow f \\
V(\mathcal{I}) & \to & \mathcal{Z}
\end{array}
$$

is cocartesian in the category of locally quasi-finite analytic $\mathcal{X}$-stacks.

**Proof.** First, we assume that $\mathcal{X}$ is an analytic space, and we will show that the diagram:

$$
\begin{array}{ccc}
V(\mathcal{I})' & \to & \mathcal{Z}' \\
\downarrow & & \downarrow f \\
V(\mathcal{I}) & \to & \mathcal{Z}
\end{array}
$$

is cocartesian in the category of locally ringed spaces, thus in the category of analytic spaces. This implies that the diagram is cocartesian in the category of analytic $\mathcal{X}$-spaces. We will use the criterion of [Perci05, Sc. 4.3(b)]. Note that from the associated cartesian conductor square for rings, it suffices to show that $\mathcal{Z}$ has the correct topological space. Since $f^\sharp$ is injective and $f$ is finite, then $f$ is surjective and closed, thus subservive. Let $\mathcal{U} = \mathcal{Z} - V(\mathcal{I})$ and $\mathcal{U}' = \mathcal{Z}' - V(\mathcal{I}_{\mathcal{Z}'})$. It remains to show that $f$ induces a bijection of sets $\mathcal{U}' \to \mathcal{U}$. Since $\mathcal{I} \subset \mathcal{E}_r$, then for $u \in \mathcal{U}$ we have that the map $f^\sharp_{\mathcal{I}u} : \mathcal{O}_{\mathcal{Z},u} \to (f_*\mathcal{O}_{\mathcal{Z}'})_u$ is a bijection. Thus, since $f$ is finite, we may conclude that the induced surjective morphism $f^{-1}(\mathcal{U}) \to \mathcal{U}$ has connected fibers—thus is a bijection of sets. Hence, we are reduced to showing that the inclusion $\mathcal{U}' \hookrightarrow f^{-1}(\mathcal{U})$ is surjective. This follows from $(\mathcal{Z} - V(\mathcal{I})) \cap f(V(\mathcal{I}_{\mathcal{Z}'})) = \emptyset$, which is obvious.

In the case where $\mathcal{X}$ is an analytic stack, since all of these constructions commute with smooth base change on $\mathcal{Z}$, we may work smooth locally on $\mathcal{X}$ and deduce the result from the case of analytic spaces already proved.

Finally, we arrive at the proofs of Lemma 4.7 and Proposition 4.3.

**Proof of Lemma 4.7.** For (1), we consider the diagram of analytic $\mathcal{X}_{an}$-stacks $[V(\mathcal{I})] \leftarrow V(\mathcal{I})_{\mathcal{Z}'} \to \mathcal{Z}'$. By hypothesis, these analytic stacks are all algebraizable to quasi-finite and separated algebraic $\mathcal{X}$-stacks and so an application of Lemma 4.8 produces a factorization of the map $f : \mathcal{Z}' \to \mathcal{Z}$ into $\mathcal{Z} \to \mathcal{Z}'' \overset{\beta}{\to} \mathcal{Z}$, where $\mathcal{Z}''$ is algebraizable to a quasi-finite and separated algebraic $\mathcal{X}$-stack. An easy calculation with rings verifies that $\beta$ has the desired properties.

For (2), by Lemma 4.5.2, we may replace $\mathcal{I}$ by some power $\mathcal{I}^k$ such that $(\mathcal{I}_{an})^1_{\mathcal{Z}'} \cap \ker f^\sharp = \{0\}$. We now observe that (1) applies, and we are reduced to the case where $f^\sharp$ is injective. Since there is a $\ell$ such that $\mathcal{E}_r \supset (\mathcal{I}_{an})^1_{\mathcal{Z}'}$, then $V((\mathcal{I}_{an})^1_{\mathcal{Z}'})$ is algebraizable to
a quasi-finite and separated algebraic $X$-stack. Moreover, $\mathcal{V}(\{\beta_{an}\}_{\mathbb{Z}})$ is algebraizable to a quasi-finite and separated algebraic $X$-stack. Since the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{V}(\{\beta_{an}\}_{\mathbb{Z}}) & \xrightarrow{\text{Z'}} & Z' \\
\downarrow & & \downarrow \\
\mathcal{V}(\{\beta_{an}\}_{\mathbb{Z}}) & \xrightarrow{Z} & Z
\end{array}
$$

is cocartesian in the category of locally quasi-finite analytic $X_{an}$-stacks (by Lemma 4.10), then an application of Lemma 4.8 implies that $Z$ is algebraizable to a quasi-finite and separated algebraic $X$-stack. □

**Proof of Proposition 4.3**. The interesting point here is that we permit the map $W \to Z$ to be non-separated. On the small étale site of $Z$, we define the sheaf $H_{W/Z} := \text{Hom}_{Z}(\cdot, W)$. By [SGAIV, Exp. IX, 2.7.1], since $W \to Z$ is quasi-compact and étale, the sheaf $H_{W/Z}$ is constructible. The analytification of the étale sheaf $H_{W/Z}$ is the sheaf $H_{W_{an}/Z_{an}} := \text{Hom}_{Z_{an}}(\cdot, W_{an})$ on the analytic small étale site of $Z_{an}$. We now have the comparison map on global sections

$$
\text{Hom}_{Z}(Z, W) = \Gamma(Z, H_{W/Z}) \to \Gamma(Z_{an}, H_{W_{an}/Z_{an}}) = \text{Hom}_{Z_{an}}(Z_{an}, W_{an}).
$$

which we must show is a bijection. More generally, it suffices to show that if $G$ is a constructible sheaf of sets on $Z$, then the comparison map

$$
\delta_{G} : \Gamma(Z, G) \to \Gamma(Z_{an}, G_{an})
$$

is bijective. This may be checked étale locally on $Z$, so we may assume that $Z$ is separated and of finite type over $\mathbb{C}$. The constructibility of $G$ guarantees that there is an inclusion $G \hookrightarrow G'$, where $G'$ is a constructible sheaf of abelian groups on $Z$. By M. Artin’s general result on the comparison between étale and complex cohomology for constructible sheaves of abelian groups, the comparison map $\delta_{G'}$ is bijective. One now deduces that the comparison map $\delta_{G}$ is bijective. □

5. APPLICATIONS

In this section, we will make Theorem 4 precise. Fix an integer $n \geq 0$. Let $U_{n}$ denote the moduli stack of all $n$-pointed $\mathbb{C}$-curves. That is, a morphism from a $\mathbb{C}$-scheme $T$ to $U_{n}$ is equivalent to a proper, flat, and finitely presented morphism of algebraic $\mathbb{C}$-spaces $C \to T$ with one-dimensional geometric fibers, together with $n$ sections to the map $C \to T$.

It was shown, in [Hal10], which is an appendix to [Smy09], that the stack $U_{n}$ is algebraic, locally of finite presentation over $\mathbb{C}$, with quasi-compact and separated diagonal.

**Definition 5.1.** Fix $g > 1$, and let $M_{g,n}$ denote the moduli stack of smooth curves of genus $g$. An algebraic modular compactification of $M_{g,n}$ is a proper Deligne-Mumford $\mathbb{C}$-stack $N$, fitting into a 2-fiber diagram of algebraic $\mathbb{C}$-stacks:

$$
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{j'} & M_{g,n} \\
\downarrow_{i'} & & \downarrow_{i} \\
N & \xrightarrow{j} & U_{n}
\end{array}
$$

where the map $j$ is an open immersion, and the maps $i'$ and $j'$ have dense image.
On the analytic side, let $\mathcal{U}_n$ denote the $\text{An}$-stack of all analytic curves. That is, a map from an analytic space $\mathcal{T}$ to $\mathcal{U}_n$ is equivalent to a proper and flat morphism of analytic spaces $\mathcal{C} \to \mathcal{T}$ with one-dimensional fibers, together with $n$ sections to the map $\mathcal{C} \to \mathcal{T}$.

**Definition 5.2.** Fix $g > 1$, and let $\mathcal{M}_{g,n}$ denote the analytic moduli stack of smooth, $n$-pointed curves of genus $g$. An **analytic modular compactification** of $\mathcal{M}_{g,n}$ is a proper Deligne-Mumford analytic stack $\mathcal{N}$, fitting into a 2-fiber diagram of $\text{An}$-stacks:

$\begin{array}{ccc}
\mathcal{V} & \xrightarrow{j'} & \mathcal{M}_{g,n} \\
\downarrow{\iota'} & & \downarrow{\iota} \\
\mathcal{N} & \xrightarrow{j} & \mathcal{U}_n
\end{array}$

where the map $j$ is an open immersion, and the maps $\iota'$ and $j'$ are dense open immersions.

Clearly, there is a natural morphism of $\text{An}$-stacks: $(\mathcal{U}_n)_{\text{an}} \to \mathcal{U}_n$. We may no claim of originality for the following result, but we were unable to find a precise reference.

**Theorem 5.3.** The morphism of $\text{An}$-stacks $(\mathcal{U}_n)_{\text{an}} \to \mathcal{U}_n$ is an equivalence; hence, $\mathcal{U}_n$ is an analytic stack. Moreover, this equivalence sends $(\mathcal{M}_{g,n})_{\text{an}}$ to $\mathcal{M}_{g,n}$.

**Proof.** The latter claim follows from the first, since smoothness of a proper algebraic $\mathbb{C}$-space can be tracked by its analytification. We prove the first claim using the criteria of Corollary [5.7] By [Bir80], we know that the diagonal map $\Delta_{\mathcal{U}_0}$ is representable by analytic spaces. Since the forgetful map $\mathcal{U}_n \to \mathcal{U}_0$ is representable by analytic spaces for any $n \geq 0$, we deduce that the diagonal map $\Delta_{\mathcal{U}_n}$ is representable by analytic spaces for all $n \geq 0$. By Lemma [5.5] it remains to verify that the functor $\mathcal{U}_n(S) \to \mathcal{U}_n(S_{\text{an}})$ is an equivalence for any local artinian $\mathbb{C}$-scheme $S$. Note that the GAGA results for proper $\mathbb{C}$-schemes (cf. [SGA4] Exp. XII, Cor. 4.5]) show that this functor is fully faithful. To get essential surjectivity, we note that for a proper and flat analytic curve $\mathcal{C} \to S_{\text{an}}$, $\mathcal{C}$ is a one-dimensional proper analytic space, thus is algebraizable. The flat structure map $\mathcal{C} \to S_{\text{an}}$ also algebraizes using the classical GAGA results [loc. cit.]. The classical GAGA results [loc. cit.] also show that the sections may be algebraized, which gives the desired equivalence. □

**Proof of Theorem 4.** Let $j : \mathcal{N} \hookrightarrow \mathcal{U}_n$ be an analytic modular compactification of $\mathcal{M}_{g,n}$. Theorems 5.3 and 3 imply that the open immersion of analytic stacks $j : \mathcal{N} \hookrightarrow \mathcal{U}_n$ is the algebraizable to an open immersion $j : \mathcal{N} \hookrightarrow \mathcal{U}_n$, where $\mathcal{N}$ is a proper Deligne-Mumford $\mathbb{C}$-stack. Next, form the 2-fiber square:

$\begin{array}{ccc}
\mathcal{V} & \xrightarrow{j'} & \mathcal{M}_{g,n} \\
\downarrow{\iota'} & & \downarrow{\iota} \\
\mathcal{N} & \xrightarrow{j} & \mathcal{U}_n
\end{array}$

Clearly, all maps in the above diagram are open immersions, and it remains to show that $\iota'$ and $j'$ have dense image. This may be checked after passing to the analytifications, and since analytification commutes with 2-fiber products, we’re done. □
APPENDIX A. SEPARATED GAGA

For this paper we required a mild strengthening of the classical GAGA results contained in [SGA1] Exp. XII, for which we could not find a reference for in the existing literature. For a separated and locally of finite type algebraic \(C\)-stack \(X\), define \(\text{Coh}_p(X)\) to be the category of coherent sheaves of \(\mathcal{O}_X\)-modules with proper support. For a separated analytic stack \(X\), define \(\text{Coh}_p(X)\) to be the category of coherent sheaves of \(\mathcal{O}_X\)-modules with proper support. If \(X\) is a separated and locally of finite type algebraic \(C\)-stack, there is an analytification functor

\[ A_X : \text{Coh}_p(X) \to \text{Coh}_p(X_{an}). \]

If the algebraic stack \(X\) is a projective \(C\)-scheme, then the functor \(A_X\) was shown to be an equivalence by Serre [Ser56]. If the algebraic stack \(X\) is assumed to be a proper \(C\)-scheme, then the functor \(A_X\) was shown to be an equivalence by Grothendieck in [SGA1] Exp. XII. We were unable to find a reference proving that the analytification functor \(A_X\) is an equivalence in the case that the algebraic \(C\)-stack \(X\) is a quasi-projective \(C\)-scheme. For the purposes of this paper, we need that the analytification functor \(A_X\) is an equivalence in the case of a separated Deligne-Mumford \(C\)-stack \(X\). Utilizing the work of M. Olsson [Ols05], however, makes it of equal difficulty to prove the following

**Theorem A.1** (Separated GAGA). Let \(X\) be a separated and locally of finite type algebraic \(C\)-stack. Then the analytification functor

\[ A_X : \text{Coh}_p(X) \to \text{Coh}_p(X_{an}) \]

induces an equivalence of categories.

This readily implies the following corollaries.

**Corollary A.2.** Let \(X\) be a separated and locally of finite type algebraic \(C\)-stack. Given a finite morphism of analytic stacks \(Z \to X_{an}\), where \(Z\) is proper, then \(Z\) is uniquely algebraizable to a finite \(X\)-stack \(Z \to X\).

**Corollary A.3.** Let \(X\) be a separated and locally of finite type Deligne-Mumford \(C\)-stack. Suppose that \(Z\) is a proper Deligne-Mumford \(C\)-stack, then the analytification functor

\[ \text{Hom}(Z, X) \to \text{Hom}(Z_{an}, X_{an}) \]

is an equivalence of categories.

Before we prove Theorem A.1, we need the following preliminary result.

**Proposition A.4.** Let \(f : X \to Y\) be a separated morphism of locally of finite type algebraic \(C\)-stacks with locally separated diagonals. Fix a coherent \(\mathcal{O}_X\)-module \(F\), with support proper over \(Y\). Then for each \(i \geq 0\), the analytic comparison map

\[ (R^if_*)_{\text{an}} \to R^i(f_{\text{an}})_* F_{\text{an}} \]

is an isomorphism of coherent \(\mathcal{O}_{Y_{an}}\)-modules. In particular, if \(X\) is a separated and locally of finite type algebraic \(C\)-stack, then for each \(i \geq 0\), the comparison map

\[ H^i(X, F) \to H^i(X_{an}, F_{an}) \]

is an isomorphism of \(C\)-vector spaces.

**Proof.** The statement is local on \(Y\) for the smooth topology, so we may assume henceforth that \(Y\) is a separated and finite type \(C\)-scheme. Next, let \(Z\) denote the stack-theoretic
support of \( F \), and let \( \iota : Z \to X \) denote the inclusion. Let \( g = f \circ \iota \), then since the functors \( \iota_* \) and \( (\iota_*)_{\text{an}} \) are exact, we have a commutative diagram for every \( i \geq 0 \):

\[
\begin{array}{ccc}
(R^i f_* [\iota_! \ast F])_{\text{an}} & \longrightarrow & R^i (f_{\text{an}})_* [(\iota_{\text{an}})_* \ast F]_{\text{an}} \\
\downarrow & & \downarrow \\
(R^i g_* [\iota^* F])_{\text{an}} & \longrightarrow & R^i (g_{\text{an}})_* [(\iota^* {\text{an}})_* \ast F]_{\text{an}}.
\end{array}
\]

The two vertical maps are isomorphisms. In particular, since the natural maps \( \iota_* \ast F \to F \) and \( (\iota_{\text{an}})_* [\iota^* F]_{\text{an}} \) are isomorphisms, then we are reduced to the case where the original morphism \( f \) is proper. In the case that the morphism \( f \) is representable by schemes, the result follows from [SGA4, Exp. XII, Thm. 4.2]. Next, assume that \( f \) is representable by algebraic spaces, then by Chow’s Lemma [Knu71, Thm. 4.3.1], there is a projective and birational \( Y \)-morphism \( g : X' \to X \), such that the composition \( X' \to Y \) is projective. Repeating verbatim the devissage argument given in the proof of [SGA4, Exp. XII, Thm. 4.2(2)] proves the result in the case that the morphism \( f \) is representable by algebraic spaces.

For the general case, we note that standard reductions, combined with noetherian induction and devissage allow us to reduce to the case where \( X \) is reduced, \( \text{supp } F = X \) and the result is proven for all coherent sheaves \( G \) on \( X \) with \( \text{supp } G \mid \subseteq |X| \). Using [Ols05, Thm. 1.1], there is a proper and representable \( Y \)-morphism \( h : V \to X \), where \( V \) is a projective \( Y \)-scheme. Since \( X \) is reduced, then by generic flatness [EGA IV, 6.9.1], there is a dense open \( W \hookrightarrow X \) for which the morphism \( h^{-1}W \to W \) is flat. Let \( h^2 : V \times_X V \to X \) denote the induced morphism, then there is a natural map

\[
\alpha : F \to F' := \text{eq}(h_* h^* F \to h^2_* (h^2)^* F),
\]

and by fppf descent, it is an isomorphism over \( W \). In particular, \( \ker \alpha, \text{im } \alpha, \) and \( \text{coker } \alpha \) are supported on the complement of \( |W| \). Since the map \( h \) is representable, then by what we have already proven, the coherent sheaf \( F' \) satisfies the conclusion of the Proposition. Using the resulting two exact sequences involving all of these terms, and devissage, we deduce the result.

\[\square\]

**Proof of Theorem** [A.4]. That the functor \( A_X \) is fully faithful is exactly the same as [SGA4, Exp. XII, Thm. 4.4], where we use Proposition [A.4] in place of [SGA4, Exp. XII, Thm. 4.2]. We now proceed to show that the functor \( A_X \) is essentially surjective. An easy reduction allows us to immediately reduce to the case where the algebraic stack \( X \) is assumed to be quasi-compact.

**Quasi-projective case:** Consider a compactification \( j : X \hookrightarrow \overline{X} \), where \( \overline{X} \) is a projective \( \mathbb{C} \)-scheme. Let \( F \in \text{Coh}_p(X) \), then \( Z := V(\text{Ann } F) \) is a proper analytic space, which is a closed analytic subspace of \( \overline{X}_{\text{an}} \). By [SGA4, Exp. XII, Thm. 4.4], we conclude that \( Z \) is algebraizable to a projective \( \mathbb{C} \)-scheme \( Z \hookrightarrow \overline{X} \). By passing to analytifications, it is easy to verify that the map \( Z \to \overline{X} \) factors through \( X \). Let \( i : Z \hookrightarrow X \) denote the resulting closed immersion of \( \mathbb{C} \)-schemes, then since analytification preserves annihilators we have that the natural map \( (i_{\text{an}})_* [i^* \ast F]_{\text{an}} \to F \) is an isomorphism; and it suffices to show that \( i_{\text{an}}^* F \in \text{Coh}(Z_{\text{an}}) \) lies in the essential image of \( A_Z \). This is clear from the usual GAGA statements [SGA4, Exp. XII, Thm. 4.4].

**General case:** This is a standard devissage argument, similar to that used in the proof of Proposition [A.4], so we omit it. \( \square \)
REFERENCES

[AFS10] J. Alper, M. Fedorchuk, and D. I. Smyth, Singularities with $\mathbb{G}_m$-action and the log minimal model program for $\overline{\mathcal{M}}_g$, ArXiv e-prints (2010), arXiv:1010.3751.

[AM69] M. F. Atiyah and I. G. Macdonald, Introduction to commutative algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Art70] M. Artin, Algebraization of formal moduli. II. Existence of modifications, Ann. of Math. (2) 91 (1970), 88–135.

[Bin80] J. Bingener, Darstellbarkeitskriterien für analytische Funktoren, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 3, 317–347.

[CT09] B. Conrad and M. Temkin, Non-Archimedean analytification of algebraic spaces, J. Algebraic Geom. 18 (2009), no. 4, 731–788.

[Dou66] A. Douady, Le problème des modules pour les sous-espaces analytiques compacts d’un espace analytique donné, Ann. Inst. Fourier (Grenoble) 16 (1966), no. fasc. 1, 1–95.

[EGA] A. Grothendieck, Eléments de géométrie algébrique. I.H.E.S. Publ. Math. 4, 8, 11, 17, 20, 24, 28, 32 (1960, 1961, 1961, 1963, 1964, 1965, 1966).

[Fer03] D. Ferrand, Conducteur, descente et pincement, Bull. Soc. Math. France 131 (2003), no. 4, 553–585.

[GR84] H. Grauert and R. Remmert, Coherent analytic sheaves, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 265, Springer-Verlag, Berlin, 1984.

[Hal10] J. Hall, Moduli of singular curves, updated and expanded appendix to: [Smy09], 2010, arXiv:1011.6097.

[HalR10] J. Hall and D. Rydh, The Hilbert Stack, arXiv:1011.5484v1. Submitted.

[Har77] R. Hartshorne, Algebraic geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[HH08] B. Hassett and D. Hyeon, Log minimal model program for the moduli space of stable curves: The first flip, ArXiv e-prints (2008), arXiv:0806.3444.

[HH09] B. Hassett and D. Hyeon, Log canonical models for the moduli space of curves: the first divisorial contraction, Trans. Amer. Math. Soc. 361 (2009), no. 8, 4471–4489.

[Knu71] D. Knutson, Algebraic spaces, Lecture Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin, 1971.

[LMB] G. Laumon and L. Moret-Bailly, Champs algébriques, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, Berlin, 2000.

[LS08] C. Lundkvist and R. Skjelnes, Non-effective deformations of Grothendieck’s Hilbert functor, Math. Z. 258 (2008), no. 3, 513–519.

[Lur04] J. Lurie, Tannaka duality for geometric stacks, [arXiv:math0412266].

[Ol05] M. Olsson, On proper coverings of Artin stacks, Adv. Math. 198 (2005), no. 1, 93–106.

[RG71] M. Raynaud and L. Gruson, Critères de platitude et de projectivité. Techniques de “platification” d’un module, Invent. Math. 13 (1971), 1–89.

[Ser56] J. P. Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, Grenoble 6 (1955–1956), 1–42.

[SGA1] A. Grothendieck and M. Raynaud, Revêtements étalés et groupe fondamental, Séminaire de Géométrie Algébrique, I.H.E.S., 1963.

[SGA1V] M. Artin, A. Grothendieck, J. L. Verdier, P. Deligne, and B. Saint-Donat, Théorie des topos et cohomologie étale des schémas, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, 1973.

[Smy09] D. I. Smyth, Towards a classification of modular compactifications of the moduli space of curves, arXiv:0902.3690.

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