POSITIVE SOLUTIONS FOR KIRCHHOFF-SCHRÖDINGER-POISSON SYSTEMS WITH GENERAL NONLINEARITY

DENGFENG LÜ

School of Mathematics and Statistics, South-Central University for Nationalities
Wuhan, 430074, China
School of Mathematics and Statistics, Hubei Engineering University
Xiaogan, 432000, China

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Abstract. In the present paper the following Kirchhoff-Schrödinger-Poisson system is studied:

\[
\begin{align*}
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + \mu \phi(x) u &= f(u) \quad \text{in } \mathbb{R}^3, \\
- \Delta \phi &= \mu u^2 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \( a > 0, b \geq 0 \) are constants and \( \mu > 0 \) is a parameter. Without assuming the Ambrosetti-Rabinowitz type condition and monotonicity condition on \( f \), we establish the existence of positive radial solutions for the above system by using variational methods combining a monotonicity approach with a delicate cut-off technique. We also study the asymptotic behavior of solutions with respect to the parameter \( \mu \). In addition, we obtain the existence of multiple solutions for the nonhomogeneous case corresponding to the above problem. Our results improve and generalize some known results in the literature.

1. Introduction and main results. In this paper, we consider the following Kirchhoff-Schrödinger-Poisson system:

\[
\begin{align*}
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + \mu \phi(x) u &= f(u) \quad \text{in } \mathbb{R}^3, \\
- \Delta \phi &= \mu u^2 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \( a > 0, b \geq 0 \) are constants and \( \mu > 0 \) is a parameter. We assume that nonlinearity \( f \) satisfies the following hypotheses:

\((f_1)\) \( f \in C(\mathbb{R}, \mathbb{R}) \), \( f(0) = 0 \) and \(-\infty < \lim \inf_{t \to 0^+} \frac{f(t)}{t} \leq \lim \sup_{t \to 0^+} \frac{f(t)}{t} = -\nu < 0; \)

\((f_2)\) \(-\infty \leq \lim \sup_{t \to +\infty} \frac{f(t)}{t} \leq 0; \)

\((f_3)\) there exists \( \xi > 0 \) such that \( F(\xi) := \int_0^\xi f(t) \, dt > 0. \)

We remark that this kind of hypotheses was first introduced by Berestycki and Lions [10] in the study of a nonlinear scalar field equation. In [10], the authors...
showed that these hypotheses are almost necessary to get an existence result for the scalar field equation
\[-\Delta u = f(u), \quad u \in H^1(\mathbb{R}^N)(N \geq 3).\] (1.1)

Under these hypotheses, they obtained the existence of a ground state solution for (1.1) by using minimizing arguments.

Problem \((P)\) is often referred to as being nonlocal because the term \((\int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u\) depends not only on the pointwise value of \(\Delta u\), but also on the integral of \(|\nabla u|^2\) over the domain. In this sense, the system \((P)\) is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties which make the study of such problems particularly interesting. Besides, such problems arise in various models of physical and biological systems. Indeed, when \(\mu = 0\) and \(\mathbb{R}^3\) is replaced by a smooth bounded domain \(\Omega\), the system \((P)\) reduces to the following Kirchhoff type equation
\[-\left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(u), \quad x \in \Omega.\] (1.2)

Equation (1.2) is related to the stationary analogue of the equation
\[\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \, dx\right) \frac{\partial^2 u}{\partial x^2} = 0,\] (1.3)

which was proposed by Kirchhoff in [25], where \(\rho\) is the mass density, \(P_0\) is the initial tension, \(h\) is the area of cross section, \(E\) is the Young modulus of the material and \(L\) is the length of the string. Equation (1.3) extends the classical d’Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. Moreover, nonlocal problems also appear in other fields as biological systems, where \(u\) describes a process which depends on the average of itself (for example, population density). After the pioneer work of J.L. Lions [28], where a functional analysis approach was proposed, Kirchhoff type problems began to call attention of several researchers. P. D’Ancona and S. Spagnolo [12] obtained the existence of a global classical periodic solution for a degenerate Kirchhoff equation with real analytic data. In particular, in [12], Kirchhoff’s equation is an example of a quasilinear hyperbolic Cauchy problem that describes the transverse oscillations of a stretched string.

In recent years, the following Kirchhoff type equation
\[
\begin{cases}
-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u + V_0 u = f(u) & \text{in } \mathbb{R}^3, \\
u \in H^1(\mathbb{R}^3),
\end{cases}
\] (1.4)

has been extensively studied by many researchers using variational methods, where \(a, b, V_0 > 0\) are constants. For the case \(f(u) = |u|^{p-2}u, 3 < p < 6\), Li and Ye in [27] used the constrained minimization on a suitable manifold obtained by combining the Nehari manifold and the corresponding Pohozaev identity to get a positive ground state solution for (1.4). Later, in [18], He and Li considered the critical case \(f(u) = |u|^4u + \lambda |u|^{p-2}u, 2 < p \leq 4\), by using a scaling technique, they also obtained a positive ground state solution for (1.4). In [17], He and Zou studied (1.4) under the conditions that \(f \in C^1(\mathbb{R}^+, \mathbb{R}^+)\) such that
\[\lim_{u \to 0} \frac{f(u)}{u^3} = 0, \quad \lim_{|u| \to \infty} \frac{f(u)}{|u|^q} = 0 \text{ for some } q \in (3, 5),\]
and
\[ f'(u)u^2 - 3f(u)u \geq M_0 u^\sigma \] for some \( \sigma \in (4, 6) \) and \( M_0 > 0 \).
Moreover, \( f \) satisfies the Ambrosetti-Rabinowitz type condition ((AR) type condition in short) and the monotonicity condition, namely
\[ \exists \theta > 4 \text{ such that } 0 < \theta F(u) \leq f(u)u \text{ for all } u > 0, \quad \frac{f(u)}{u^3} \text{ is increasing for } u > 0. \]
They proved that (1.4) has a positive ground state solution by using the Nehari manifold method and the mountain-pass theorem. This result has been subsequently improved by Wang et al. [32] to the critical nonlinearity case: \( f(u) \) is replaced by \( |u|^4 u + \lambda f(u) \), \( \lambda > 0 \) is a large parameter, where \( \lim_{u \to 0} \frac{f(u)}{u^3} = 0 \), \( \frac{f(u)}{u^3} \) is strictly increasing for \( u > 0 \) and \( |f(u)| \leq C(1 + |u|^q) \) for some \( q \in (3, 5) \). Using the similar arguments as in [17] they obtained the existence of a positive ground state solution for (1.4). For more related results concerning Kirchhoff type equation we refer the interested readers to [7, 16, 26] and the references therein.

When \( a = 1, b = 0 \), the system \((P)\) reduces to the following electrostatic nonlinear Schrödinger-Poisson system
\[
\begin{align*}
-\Delta u + \mu \phi(x) u &= f(u) & \text{in } \mathbb{R}^3, \\
-\Delta \phi &= \mu u^2 & \text{in } \mathbb{R}^3.
\end{align*}
\]
In [9], V. Benci and D. Fortunato firstly introduced it as a model to describe the interaction of a charged particle with the electrostatic field. In fact, system (1.5) is a variant of the following Schrödinger-Poisson problem
\[
\begin{align*}
\frac{\hbar^2}{2m} \Delta u - u + \omega \phi(x) u + h(u) &= 0 & \text{in } \mathbb{R}^3, \\
\Delta \phi + 4\pi \omega u^2 &= 0 & \text{in } \mathbb{R}^3,
\end{align*}
\]
where \( m, \hbar, \omega > 0 \) are constants. One such system is also known as the Schrödinger-Maxwell system. In (1.6), \( m \) denotes the mass of the particle, \( \omega \) denotes the electric charge and \( \hbar \) is the Planck constant. The unknowns of the system are the wave function \( u \) associated to the particle and the electric potential \( \phi \). The presence of the nonlinear term \( h(u) \) simulates the interaction effect among many particles and the coupled term \( \omega \phi \) concerns the interaction with the electric field. We refer the reader to [9] and the references cited therein for more detailed physical background.

We note that system (1.5) is also a nonlocal problem. Indeed, as we will see in Section 2, for any \( u \in H^1(\mathbb{R}^3) \), by using the Lax-Milgram theorem, there exists a unique \( \phi_u \in D^{1,2}(\mathbb{R}^3) \) such that \( -\Delta \phi_u = \mu u^2 \), inserted \( \phi_u \) into the first equation, then system (1.5) can be reduced to a nonlinear Schrödinger equation with a nonlocal term \( \phi_u \). In the last decade, Schrödinger-Poisson systems like (1.5) have been widely studied by many researchers. Here we would like to cite some related results, for example, for the case \( f(u) = |u|^{p-2}u - u \), D’Aprile and Mugnai [13] studied the existence of a nontrivial radial solution to (1.5) for \( p \in [4, 6) \). In [14], a related Pohozaev type identity is found and with this identity, the authors proved that system (1.5) does not have nontrivial solution for \( p \leq 2 \) or \( p \geq 6 \). On the other hand, d’Avenia [15] obtained the existence of non-radial solution for (1.5) when \( p \in (4, 6) \). Azzollini was concerned with (1.5) under the conditions \( (f_1) - (f_3) \) in [4], he proved the existence of a nontrivial non-radial solution to system (1.5). Azzollini et al. [5] studied the Schrödinger-Poisson system (1.5) and obtained the existence of at least a radial positive solution. In [6], they also proved the multiplicity of radial symmetry solutions for (1.5) by using the symmetric mountain pass theorem with
a truncation argument. Later, Zhang [33] considered the critical nonlinearity case and obtained the existence of ground state solutions for (1.5). We remark that, in [33], the (AR) type condition is required. More recently, the (AR) type condition in [33] has been removed by Zhang et al. [34]. We should mention that, Ruiz [29] considered the following Schrödinger-Poisson system

\[
\begin{cases}
-\Delta u + u + \lambda \phi(x)u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

(1.7)

where \( \lambda > 0 \) is a positive parameter and \( p \in (2, 6) \). It was shown in [29] that when \( p \in (2, 3) \) (respectively \( p = 3 \), (1.7) has at least two (respectively one) positive solutions for small \( \lambda \) by using the mountain-pass theorem and Ekeland’s variational principle. For the case \( p \in (3, 6) \), D. Ruiz proved that there is a positive radial solution for (1.7) by using the constrained minimization method on a new manifold which is obtained by combining the usual Nehari manifold and the Pohozaev type identity. Subsequently, Ambrosetti and Ruiz [1] obtained the existence of multiple bound state solutions for (1.7) when certain conditions on the parameters \( p \) and \( \lambda \) are satisfied. In [3], Azzollini and Pomponio proved the existence of ground state solutions for \( p \in (3, 6) \) by using the concentration compactness principle.

Motivated by the papers mentioned above, in the present paper, we study the Kirchhoff-Schrödinger-Poisson system (\( P \)) on the whole space \( \mathbb{R}^3 \) involving the Berestycki-Lions type nonlinearity. This topic was recently investigated in [6, 8, 35]. As far as we know, problem (\( P \)) has not been considered before. From a mathematical point of view, problem (\( P \)) involves two kinds of nonlocal terms and a more general nonlinear term \( f \), it becomes quite complicated and interesting. It is noticed that in [13, 18, 27, 29], since \( \int_{\mathbb{R}^3} |\nabla u|^2 \) and \( \int_{\mathbb{R}^3} \phi u^2 \) in the corresponding energy functional are homogeneous of degree 4, the scaling techniques (let \( u_t(x) = t^\beta u(t^\gamma x) \) for some suitable choice of \( \beta, \gamma \in \mathbb{R} \) and \( t > 0 \)) play an essential role in dealing with the single nonlocal term \( \int_{\mathbb{R}^3} |\nabla u|^2 \) or \( \int_{\mathbb{R}^3} \phi u^2 \) for \( p \leq 4 \), respectively. However, these scaling techniques are invalid when the two kinds of nonlocal terms are combined in our case. The existence of positive radial solutions of system (\( P \)) is established via variational methods relying upon a monotonicity approach and a delicate cut-off technique. Moreover, the asymptotic of solutions with respect to the parameter \( \mu \) is also explored.

The first result of this paper is the following:

**Theorem 1.1.** Assume that \( (f_1) - (f_3) \) hold, then there exists \( \mu^* > 0 \) such that, for any \( \mu \in (0, \mu^*) \), problem (\( P \)) has a positive radial solution \((u_\mu, \phi_\mu) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\). Moreover, up to a subsequence, \((u_\mu, \phi_\mu)\) converges to \((u_0, 0)\) strongly in \( H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) as \( \mu \to 0 \), where \( u_0 \) is a ground state solution to the limit problem

\[
- (a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u = f(u), \quad u \in H^1(\mathbb{R}^3).
\]

(1.8)

**Remark 1.2.** The existence of the ground states solution to the limit problem (1.8) was obtained in [7] by using minimizing arguments on a suitable natural constraint. More precisely, if \( (f_1) - (f_3) \) hold, then problem (1.8) has a radially symmetric ground state solution.

**Remark 1.3.** (i) If we let \( a = 1, b = 0, \mu = 0 \) and replace the three-dimensional space \( \mathbb{R}^3 \) with \( \mathbb{R}^N (N \geq 3) \), then system (\( P \)) is reduced to the scalar field equation (1.1). Hence our Theorem 1.1 can be regarded as an extension of the classical result in [10] for (1.1). (ii) Note that when \( a = 1, b = 0 \), system (\( P \)) is reduced
to the Schrödinger-Poisson system (1.5), thus our results cover the case for the Schrödinger-Poisson system (1.5).

The Theorem 1.1 also can be regarded as a generalization of the results from [29] which considered the special case that \( a = 1, b = 0 \) and \( f(u) = |u|^{p-2}u - u \) with \( p \in (2,6) \). In fact, an easy calculation shows that \( (f_1) - (f_3) \) are satisfied in the model case \( f(u) = |u|^{p-2}u - \nu u \) for \( \nu > 0 \) and the full subcritical range of \( p \in (2,6) \).

Thus Theorem 1.1 allows us to ensure the next corollary.

**Corollary 1.4.** Consider the following problem

\[
\begin{cases}
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + \nu u + \mu \phi(x)u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\
- \Delta \phi = \mu u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( a > 0, b \geq 0 \) and \( \nu > 0 \) are constants, \( p \in (2,6) \), then there exists \( \mu_* > 0 \) such that, for any \( \mu \in (0, \mu_*) \), the problem (1.9) has a positive radial solution \( (\bar{u}_\mu, \bar{\phi}_\mu) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \). Moreover, up to a subsequence, \( (\bar{u}_\mu, \bar{\phi}_\mu) \) converges to \( (\bar{u}_0, 0) \) in \( H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) as \( \mu \to 0 \), where \( \bar{u}_0 \) is a ground state solution to the limit problem

\[
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + \nu u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^3).
\]

Finally, we also consider the nonhomogeneous case corresponding to the problem (\( \mathcal{P} \)). To be precise, we consider the following system

\[
\begin{cases}
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + \mu \phi(x)u = f(u) + g(x) & \text{in } \mathbb{R}^3, \\
- \Delta \phi = \mu u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( a > 0, b \geq 0 \) are constants, \( \mu > 0 \) is a parameter, and \( g(x) \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \) is a nonnegative function satisfying \( (g_0) g(x) = g(|x|) \neq 0 \), and \( 0 \leq \langle \nabla g(x), x \rangle \in L^2(\mathbb{R}^3) \).

For problem (\( \mathcal{P}_g \)), we would like to cite some related results. Salvatore [30] considered the following nonhomogeneous Schrödinger-Poisson system:

\[
\begin{cases}
- \Delta u + u + \lambda \phi(x)u = |u|^{p-2}u + g(x) & \text{in } \mathbb{R}^3, \\
- \Delta \phi = u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( \lambda > 0 \) is a parameter and \( p \in (2,6) \). In [30], three radially symmetric solutions of (1.10) were obtained for \( p \in (4,6) \) and \( |g|_{2} \) is small enough. In [11], the authors considered problem (1.10) with certain potential and the existence of multiple solutions is established for \( p \in (4,6) \). Recently, Jiang et al. [23] proved that problem (1.10) has at least two solutions for all \( p \in (2,6) \) and \( |g|_{2} \) is small enough.

To the author’s knowledge, it is still open whether the problem (\( \mathcal{P}_g \)) has multiple solutions under the conditions \( (f_1) - (f_3) \) and \( (g_0) \). Another aim of this paper is to prove that problem (\( \mathcal{P}_g \)) has at least two positive radial solutions for \( \mu \) and \( |g|_{2} \) is suitably small, and we have the following result:

**Theorem 1.5.** Assume that \( (f_1) - (f_3) \) and \( (g_0) \) hold, then there exist \( \mu_0, m_0 > 0 \) such that, for all \( \mu \in (0, \mu_0) \) and \( |g|_{2} < m_0 \), problem (\( \mathcal{P}_g \)) has at least two positive radial solutions \( (u_i, \phi_i) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \), \( i = 1, 2 \), with property \( \Psi_{\mu}(u_1, \phi_1) < 0 < \Psi_{\mu}(u_2, \phi_2) \), where \( \Psi_{\mu}(u, \phi) \) is the energy functional of problem (\( \mathcal{P}_g \)) defined by

\[
\Psi_{\mu}(u, \phi) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + b \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{b}{2} \int_{\mathbb{R}^3} \phi u^2 - \frac{\mu}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 - \int_{\mathbb{R}^3} F(u) - \int_{\mathbb{R}^3} g(x)u.
\]
Remark 1.6. The result in Theorem 1.5 is not really surprising since problem \((P)\) also admits two solutions: one is found in Theorem 1.1 and the other one is trivial solution \(u \equiv 0\). Thus one can think that for \(|g|_2\) small the two solutions are perturbations of the two solutions already present in Theorem 1.1 and the trivial solution \(u \equiv 0\).

In order to obtain our results, we have to overcome several difficulties. First, in our general assumptions, we do not assume two standard conditions already present in many papers on the subject of this paper, namely \(f(t)\) is increasing in \((0, +\infty)\) and the \((AR)\) type condition: \(\exists \theta > 4\) such that \(0 < \theta F(t) \leq f(t)t\) for all \(t \neq 0\), which bring about two obstacles to the standard mountain-pass arguments both in checking the geometrical assumptions in the corresponding energy functional and in proving the boundedness of its Palais-Smale\((PS)\) sequences. Second, the competing effect of the two nonlocal terms with the nonlinear term \(f\) gives rise to some difficulties, for example, when we rescale the variables, the behavior of the two kinds of nonlocal terms \(\int_{\mathbb{R}^3} |\nabla u|^2\) and \(\int_{\mathbb{R}^3} \phi u^2\) prevents us from using the variational methods in a standard way. On the other hand, since \(f\) does not have any homogeneity property, we cannot use the usual arguments as in the pure power case. The methods in \([13, 18, 27, 29]\) can not be applied in this paper.

In order to overcome these difficulties, we will use a monotonicity approach developed by L. Jeanjean \([20]\) to deal with our problems, but this approach cannot be used here directly, and we need some crucial modifications for our proof because the two kinds of nonlocal terms \(\int_{\mathbb{R}^3} |\nabla u|^2\) and \(\int_{\mathbb{R}^3} \phi u^2\) are involved in the corresponding energy functional and the combined effect of the two nonlocal terms brings additional difficulties to the problems \((P)\) and \((P_g)\). We will use a delicate cut-off technique to modify the corresponding energy functional, where a cut-off function is introduced to control the nonlocal term \(\int_{\mathbb{R}^3} \phi u^2\).

The rest of this paper is organized as follows. In Section 2, we introduce some notations, set the variational framework for problem \((P)\) and introduce a modified functional by using a cut-off technique. In Section 3, we obtain the existence of positive radial solutions to problem \((P)\), including the asymptotic behavior of solutions as \(\mu \to 0\). Section 4 is devoted to dealing with the nonhomogeneous problem \((P_g)\) and the proof of Theorem 1.5.

2. Variational setting and the modified functional. First, we give some notations:

- \(C, \overline{C}, C_1, C_2, \cdots\) denote various positive constants whose exact values are not important.
- For simplicity, we write \(\int_{\mathbb{R}^3} g\) to mean the Lebesgue integral of \(g(x)\) over \(\mathbb{R}^3\).
- \(\to\) (respectively \(\rightharpoonup\)) denotes strong (respectively weak) convergence.
- \(H^1(\mathbb{R}^3)\) is the usual Sobolev space endowed with the standard inner product
  \[ \langle u, v \rangle_{H^1} = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv), \]
  and the associated norm \(\|u\|_{H^1} = \sqrt{\langle u, u \rangle_{H^1}}\). For fixed \(a > 0\), we also use the notation \(\|u\| = (\int_{\mathbb{R}^3} (a|\nabla u|^2 + u^2))^{1/2}\), which is a norm equivalent to \(\|u\|_{H^1}\).
- \(H^1_r(\mathbb{R}^3)\) is the subspace of \(H^1(\mathbb{R}^3)\) containing only the radial functions.
- For any \(1 \leq s < \infty\), \(L^s(\mathbb{R}^3)\) is the usual Lebesgue space with the norm \(|u|_s := (\int_{\mathbb{R}^3} |u|^s)^{1/s}\).
\( D^{1,2}(\mathbb{R}^3) \) is the completion of \( C_0^\infty(\mathbb{R}^3) \) with respect to the norm \( \|u\|_D = (\int_{\mathbb{R}^3} |\nabla u|^2)^{\frac{1}{2}} \), induced by the scalar product \( \langle u, v \rangle_D = \int_{\mathbb{R}^3} \nabla u \nabla v \).

It is well known that system \((P)\) can be reduced to a single equation. Actually, for each \( u \in H^1(\mathbb{R}^3) \), denoting \( L_u(v) \) the linear functional in \( D^{1,2}(\mathbb{R}^3) \) by

\[
L_u(v) = \int_{\mathbb{R}^3} u^2 v,
\]

then using the Hölder inequality and the Sobolev embedding theorem, there holds

\[
|L_u(v)| \leq \left( \int_{\mathbb{R}^3} (u^2)^{\frac{4}{3}} \right)^{\frac{3}{4}} \left( \int_{\mathbb{R}^3} |v|^6 \right)^{\frac{1}{3}} \leq C \|u\|_{L^{\frac{12}{5}}(\mathbb{R}^3)} \|v\|_D \quad \text{for all } v \in D^{1,2}(\mathbb{R}^3).
\]

Hence, by the Lax-Milgram theorem, there exists a unique \( \phi_u \in D^{1,2}(\mathbb{R}^3) \) such that

\[
\int_{\mathbb{R}^3} \nabla \phi_u \nabla v = \mu \int_{\mathbb{R}^3} u^2 v \quad \text{for all } v \in D^{1,2}(\mathbb{R}^3),
\]

so \( \phi_u \) satisfies the Poisson equation

\[-\Delta \phi = \mu u^2 \quad \text{in } \mathbb{R}^3\]

and \( \phi_u \) can be represented by

\[
\phi_u(x) = \frac{\mu}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy. \tag{2.1}
\]

Moreover, \( \phi_u \) has the following properties (see for example [29]).

**Lemma 2.1.** For any \( u \in H^1(\mathbb{R}^3) \), the following hold:

(i) \( \phi_u \geq 0 \) and \( \|\phi_u\|^2_D = \mu \int_{\mathbb{R}^3} \phi_u u^2; \)

(ii) \( \phi_u(x) = \theta^2 \phi_u(\frac{x}{\theta}) \) for any \( \theta > 0 \), where \( u_0(x) = u(\frac{x}{\theta}); \)

(iii) there exist \( C_1, C_2 > 0 \) independent of \( u \in H^1(\mathbb{R}^3) \) such that

\[
\|\phi_u\|_D \leq C_1 \mu |u|^\frac{12}{5} \quad \text{and} \quad \int_{\mathbb{R}^3} \phi_u u^2 \leq C_2 \mu |u|^\frac{4}{5};
\]

(iv) if \( u \) is a radial function then \( \phi_u \) is radial and has the following expression

\[
\phi_u(r) = \frac{1}{r} \int_0^{+\infty} u^2(s) s \min\{r, s\} \, ds;
\]

(v) if \( u_n \to u \) in \( H^1(\mathbb{R}^3), \) then \( \phi_{u_n} \to \phi_u \) in \( D^{1,2}(\mathbb{R}^3) \) and \( \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \phi_u u^2. \)

Substituting \((2.1)\) into system \((P)\), we can rewrite \((P)\) in the following equivalent equation

\[- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + \mu \phi_u u = f(u), \quad u \in H^1(\mathbb{R}^3). \tag{\bar{P}}\]

The corresponding energy functional of problem \((\bar{P})\) is defined on \( H^1(\mathbb{R}^3) \) by

\[
I_{\mu}(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} F(u),
\]

where \( F(u) := \int_0^u f(s) \, ds \). It is standard to show that \( I_{\mu} \in C^1(\mathbb{H}^1(\mathbb{R}^3), \mathbb{R}) \) under the conditions \((f_1) - (f_3)\). Moreover, by using standard variational arguments as those in [9], the following result can be easily proved.

**Lemma 2.2.** The following statements are equivalent:
(i) \( (u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) is a solution of the system \((P)\).

(ii) \( u \in H^1(\mathbb{R}^3) \) is a critical point of the functional \( I_\mu \) and \( \phi = \phi_u \).

By Lemma 2.2, in order to get solutions of \((P)\), we could look for critical point of \( I_\mu \). But a delicate problem in getting the existence of critical points is the fact that \( I_\mu \) presents the lack of compactness due to its invariance under the translation group. In order to avoid the problem, it is standard to only consider the set of radial functions, so we will look for critical point of \( I_\mu \) on \( H^1_0(\mathbb{R}^3) \). In this way, if \( u \in H^1_0(\mathbb{R}^3) \), then also \( \phi_u \) is radial by (iv) of Lemma 2.1. This choice is justified by the fundamental fact that the embedding \( H^1_0(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3) \) is compact for any \( s \in (2, 3) \) by Strauss’ Lemma (see [31]).

Since we are concerned with the positive solutions of \((P)\), similar as that in [10] (see also [4]), we modify our problem first. Without loss of generality, we assume that
\[
0 < \xi = \inf \{ \tau \in (0, +\infty) : F(\tau) > 0 \},
\]
where \( \xi \) is given in \((f_1)\). Let \( \tau_0 = \min \{ \tau \in (\xi, +\infty) : f(\tau) = 0 \} \) \( (\tau_0 = +\infty \) if \( f(\tau) > 0 \) for any \( \tau \geq \xi \)), define \( \hat{f} : \mathbb{R} \to \mathbb{R} \),
\[
\hat{f}(\tau) = \begin{cases} f(\tau), & \text{if } 0 \leq \tau \leq \tau_0, \\ 0, & \text{if } \tau \geq \tau_0, \\ -\hat{f}(-\tau), & \text{for } \tau < 0. \end{cases}
\]
Observe that \( \hat{f} \) satisfies the same conditions as \( f \).

**Remark 2.3.** If \( u \in H^1(\mathbb{R}^3) \) is a nontrivial solution of \((P)\) with \( \hat{f} \) in the place of \( f \), then by Lemma 2.1(i) and the maximum principle we get that \( u \) is positive and \( u(x) \leq \tau_0 \) for any \( x \in \mathbb{R}^3 \), i.e., \( u \) is a solution of the original problem \((P)\) with \( f \).

By Remark 2.3, from now on, we can replace \( f \) by \( \hat{f} \), but still use the same notation \( f \). With this modification, \( f \) satisfies the stronger condition
\[
\lim_{\tau \to \pm \infty} \frac{f(\tau)}{\tau^5} = 0.
\]
In addition, for \( \tau \geq 0 \), define
\[
f_1(\tau) = \max \{ f(\tau + \nu \tau, 0) =: (f(\tau) + \nu \tau)^+, \quad f_2(\tau) = f_1(\tau) - f(\tau), \quad (2.2)\]
and extend \( f_1, f_2 \) as odd functions for \( \tau < 0 \). Then by (2.2) we have that \( f = f_1 - f_2 \) with \( f_1, f_2 \geq 0 \) on \( \mathbb{R}^+ \), and
\[
f_2(\tau) \geq \nu \tau, \quad \forall \tau \geq 0, \quad (2.3)
\]
and the following limits hold:
\[
\lim_{\tau \to 0} \frac{f_1(\tau)}{\tau} = 0, \quad \lim_{\tau \to +\infty} \frac{f_1(\tau)}{\tau^5} = 0. \quad (2.4)
\]
Moreover, by some computations, we have that for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
f_1(\tau) \leq \varepsilon f_2(\tau) + C_\varepsilon \tau^5, \quad \tau \geq 0. \quad (2.5)
\]
Let \( F_i(u) = \int_0^u f_i(\tau) \, d\tau, i = 1, 2 \), then by (2.4) and (2.5) for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[
F_1(u) \leq \varepsilon F_2(u) + \frac{C_\varepsilon u^6}{6}, \quad \forall u \in \mathbb{R}. \quad (2.6)
\]
In view of [10], it follows from (f3) that there exists a function \( w \in H^1_\text{rad}({\mathbb R}^3) \) such that
\[
\int_{\mathbb R^3} F_1(w) - \int_{\mathbb R^3} F_2(w) = \int_{\mathbb R^3} F(w) > 0.
\]
Then there exists \( \delta_0 \in (0, 1) \) such that
\[
\delta_0 \int_{\mathbb R^3} F_1(w) - \int_{\mathbb R^3} F_2(w) > 0. \tag{2.7}
\]
Since the energy functional \( I_\mu \) involves two nonlocal terms (\( \int_{\mathbb R^3} |\nabla u|^2 \) and \( \int_{\mathbb R^3} \phi_u u^2 \)), in order to obtain the mountain-pass geometry and the bounded (PS) sequences for the functional \( I_\mu \), following [21, 24], we introduce the cut-off function \( \psi \in C^1([0, \infty), \mathbb R) \) satisfying
\[
\psi(t) = 1 \text{ if } 0 \leq t \leq 1; \quad 0 \leq \psi(t) \leq 1 \text{ if } 1 < t < 2; \quad \psi(t) = 0 \text{ if } t \geq 2; \quad \|\psi\|_\infty \leq 2. \tag{2.8}
\]
For convenience, we denote \( \alpha = \frac{12}{7} \), and for every \( \ell > 0 \), we define
\[
k_\ell(u) = \psi \left( \frac{|u|_\alpha}{\ell^\alpha} \right).
\]
We will study the following modified functional \( I^\ell_\mu : H^1({\mathbb R}^3) \to \mathbb R \) defined by
\[
I^\ell_\mu(u) = \frac{a}{2} \int_{\mathbb R^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb R^3} |\nabla u|^2 \right)^2 + \frac{\mu}{4} k_\ell(u) \int_{\mathbb R^3} \phi_u u^2 - \int_{\mathbb R^3} F(u).
\]
We note that, if \( u \) is a critical point of \( I^\ell_\mu \) with \( |u|_\alpha \leq \ell \), then \( u \) is also a critical point of \( I_\mu \).

Since we do not assume that \( f \) satisfies the (AR) type condition, and under our general assumptions (f1)–(f3), we are not able to directly obtain the boundedness of the (PS) sequences. In order to overcome this difficulty, we will apply the following abstract theorem introduced by L. Jeanjean in [20].

**Theorem 2.4.** ([20], Theorem1.1). Let \( X \) be a Banach space equipped with a norm \( \| \cdot \|_X \) and \( J \subset \mathbb R^+ \) be an interval. Consider a family \( \{\Phi_\lambda\}_{\lambda \in J} \) of \( C^1 \) functionals on \( X \) of the form
\[
\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J,
\]
where \( B(u) \geq 0 \) of all \( u \in X \) and either \( A(u) \to +\infty \) or \( B(u) \to +\infty \) as \( \|u\|_X \to \infty \) and such that \( \Phi_\lambda(0) = 0 \). Assume that there are two points \( v_1, v_2 \in X \) such that
\[
c_\lambda = \inf_{\gamma \in Y} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}, \quad \forall \lambda \in J,
\]
where
\[
Y = \{ \gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2 \}.
\]
Then for almost every \( \lambda \in J \) there is a sequence \( \{u_n\} \subset X \) such that
(i) \( \{u_n\} \) is bounded; (ii) \( \Phi_\lambda(u_n) \to c_\lambda \); (iii) \( \Phi'_\lambda(u_n) \to 0 \) in the dual \( X^{-1} \) of \( X \).
Moreover, the map \( \lambda \mapsto c_\lambda \) is non-increasing and continuous from the left.

In our case, let \( X = H^1_\text{rad}({\mathbb R}^3) \) and \( J = [\delta_0, 1] \), where \( \delta_0 \in (0, 1) \) is given in (2.7),
\[
A(u) = \frac{a}{2} \int_{\mathbb R^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb R^3} |\nabla u|^2 \right)^2 + \frac{\mu}{4} k_\ell(u) \int_{\mathbb R^3} \phi_u u^2 + \int_{\mathbb R^3} F_2(u),
\]
\[
B(u) = \int_{\mathbb R^3} F_1(u).
\]
Obviously, \( A(u) \to +\infty \) as \( \|u\| \to +\infty \) and \( B(u) \geq 0, \forall u \in H^1_t(\mathbb{R}^3) \), so that the perturbed functional we study is

\[
I_{\mu,\lambda}^\varepsilon(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{\mu k_\varepsilon(u)}{4} \int_{\mathbb{R}^3} \phi_\varepsilon u^2 + \int_{\mathbb{R}^3} F_2(u) - \lambda \int_{\mathbb{R}^3} F_1(u).
\]

Then \( I_{\mu,\lambda}^\varepsilon \in C^1(H^1_t(\mathbb{R}^3), \mathbb{R}) \) and for every \( u, \varphi \in H^1_t(\mathbb{R}^3) \), one has

\[
\langle (I_{\mu,\lambda}^\varepsilon)'(u), \varphi \rangle = \int_{\mathbb{R}^3} a \nabla u \nabla \varphi + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla u \nabla \varphi + \mu k_\varepsilon(u) \int_{\mathbb{R}^3} \phi_\varepsilon u \varphi + \frac{\mu \alpha}{4 \ell^\alpha} \psi \left( \frac{|u|^\alpha}{\ell^\alpha} \right) \int_{\mathbb{R}^3} \phi_\varepsilon u^2 \int_{\mathbb{R}^3} |u|^{\alpha-2} u \varphi + \int_{\mathbb{R}^3} f_2(u) \varphi - \lambda \int_{\mathbb{R}^3} f_1(u) \varphi.
\]

The following lemma ensures that \( I_{\mu,\lambda}^\varepsilon \) has the mountain-pass geometry.

**Lemma 2.5.** Assume that \((f_1) - (f_5)\) hold. Then for all \( \lambda \in [\delta_0, 1] \), we have

(i) there exist constants \( \rho, \eta > 0 \) such that \( I_{\mu,\lambda}^\varepsilon(u) \geq \eta \) for all \( u \in H^1_t(\mathbb{R}^3) \) with \( \|u\| = \rho \);

(ii) there exists \( e \in H^1_t(\mathbb{R}^3) \setminus \{0\} \) such that \( I_{\mu,\lambda}^\varepsilon(e) < 0 \);

(iii) there holds

\[
e_{\mu,\lambda} := \inf_{\gamma \in \Gamma} \max_{\gamma(t) \in [0,1]} I_{\mu,\lambda}^\varepsilon(\gamma(t)) \geq \eta > \max \{ I_{\mu,\lambda}^\varepsilon(0), I_{\mu,\lambda}^\varepsilon(e) \},
\]

where

\[
\Gamma = \{ \gamma \in C([0,1], H^1_t(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = e \}.
\]

**Proof.** (i) Observe that for any \( u \in H^1_t(\mathbb{R}^3) \) and \( \lambda \in [\delta_0, 1] \), using (2.3) and (2.6) for \( \varepsilon < 1 \), we have

\[
I_{\mu,\lambda}^\varepsilon(u) \geq I_{\mu,1}^\varepsilon(u)
\]

\[
\geq \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{\mu k_\varepsilon(u)}{4} \int_{\mathbb{R}^3} \phi_\varepsilon u^2 + \int_{\mathbb{R}^3} F_2(u) - \int_{\mathbb{R}^3} F_1(u)
\]

\[
\geq \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{\mu}{4} \left( 1 - \varepsilon \right) \int_{\mathbb{R}^3} u^2 - \frac{C_\varepsilon}{6} \int_{\mathbb{R}^3} u^6.
\]

Then by Sobolev embeddings, there exist \( \rho, \eta > 0 \) such that \( I_{\mu,\lambda}^\varepsilon(u) \geq \eta > 0 \) for \( \|u\| = \rho \) small enough and any \( \lambda \in [\delta_0, 1] \).

(ii) Set \( \gamma(t) = w_t(\tau) \), where \( t > 0 \). Define \( \gamma : [0,1] \to H^1_t(\mathbb{R}^3) \) as

\[
\gamma(\tau) = \begin{cases} 
  w_t(\tau), & \text{if } 0 < \tau \leq 1, \\
  0, & \text{if } \tau = 0.
\end{cases}
\]

It is easy to see that \( \gamma \) is a continuous path from 0 to \( w_t \). Moreover, since \( \lambda \geq \delta_0 \), there holds

\[
I_{\mu,\lambda}^\varepsilon(\gamma(1)) \leq I_{\mu,\lambda}^\varepsilon(\gamma(1))
\]

\[
= \frac{at}{2} \int_{\mathbb{R}^3} |\nabla w|^2 + \frac{bt^2}{4} \left( \int_{\mathbb{R}^3} |\nabla w|^2 \right)^2 + \frac{\mu t^5}{4} \psi \left( \frac{t^3 |w|^{\alpha}}{\ell^\alpha} \right) \int_{\mathbb{R}^3} \phi_\varepsilon w^2 - t^3 \left( \delta_0 \int_{\mathbb{R}^3} F_1(w) - \int_{\mathbb{R}^3} F_2(w) \right),
\]

and then, by (2.10), (2.7) and (2.8) we can get that \( I_{\mu,\lambda}^\varepsilon(\gamma(1)) < 0 \) if \( t \) is sufficiently large. Hence (ii) holds by taking \( e = \gamma(1) \).

(iii) Now fix \( \lambda \in [\delta_0, 1] \) and \( \gamma \in \Gamma \). Since \( \gamma(0) = 0 \) and \( I_{\mu,\lambda}^\varepsilon(\gamma(1)) < 0 \), certainly \( \|\gamma(1)\| > \rho \). By continuity, we deduce that there exists \( t_\gamma \in (0,1) \) such
that \(|\gamma(t,\gamma)| = \rho\). Therefore, there exists a constant \(\eta > 0\) such that \(c_{\mu,\lambda} \geq \inf_{\gamma \in \Gamma} I_{\mu,\lambda}(\gamma(\gamma)) \geq \eta > 0 = \max\{I_{\mu,\lambda}(0), I_{\mu,\lambda}(\gamma(1))\}\) holds for any \(\lambda \in [\delta_0, 1]\). \(\square\)

By Lemma 2.5 and Theorem 2.4, we know that for almost every \(\lambda \in [\delta_0, 1]\) there exists a bounded sequence \(\{u_n^\lambda\} \subset H^1_\mu(\mathbb{R}^3)\) such that,

\[
I_{\mu,\lambda}(u_n^\lambda) \to c_{\mu,\lambda} \quad \text{and} \quad (I_{\mu,\lambda})'(u_n^\lambda) \to 0.
\]

To complete this section, we present a compactness result due to Strauss [31], which will be used in the proof of Lemma 3.1 below.

**Theorem 2.6.** Let \(P\) and \(Q : \mathbb{R} \to \mathbb{R}\) be two continuous functions satisfying

\[
\lim_{|t| \to +\infty} \frac{P(t)}{Q(t)} = 0.
\]

Let \(\{v_n\}\) be a sequence of measurable functions: \(\mathbb{R}^N \to \mathbb{R}\) such that

\[
\sup_n \int_{\mathbb{R}^N} |Q(v_n)| < +\infty \quad \text{and} \quad P(v_n(x)) \to v(x) \quad \text{for a.e.} x \in \mathbb{R}^N.
\]

Then for any bounded Borel set \(B\) one has

\[
\lim_{n \to \infty} \int_B |P(v_n(x)) - v(x)| = 0.
\]

If one further assumes that

\[
\lim_{t \to 0} \frac{P(t)}{Q(t)} = 0 \quad \text{and} \quad \lim_{|x| \to +\infty} \sup_n |v_n(x)| = 0,
\]

then \(P(v_n)\) converges to \(v\) in \(L^1(\mathbb{R}^N)\) as \(n \to \infty\).

3. **Proof of Theorem 1.1.** In this section, we will prove the existence of positive radial solutions to problem (\(P\)). We also consider the asymptotic behavior for the solutions with respect to the parameter \(\mu \to 0\). We start with the following lemma.

**Lemma 3.1.** Assume that (\(f_1\)) – (\(f_3\)) hold. Let \(\{u_n\} \subset H^1_\mu(\mathbb{R}^3)\) be a bounded sequence such that

\[
I_{\mu,\lambda}(u_n) \leq C \quad \text{and} \quad (I_{\mu,\lambda})'(u_n) \to 0.
\]

Then, up to a subsequence, \(\{u_n\}\) has a strongly convergent subsequence for any \(\lambda \in [\delta_0, 1]\).

**Proof.** Since \(\{u_n\}\) is bounded, up to a subsequence, we can assume that there exists \(u \in H^1_\mu(\mathbb{R}^3)\) such that

\[
\begin{align*}
    u_n &\to u, \quad \text{in} \ H^1_\mu(\mathbb{R}^3), \\
    u_n &\to u, \quad \text{a.e. in} \ \mathbb{R}^3, \\
    u_n &\to u, \quad \text{in} \ L^s(\mathbb{R}^3), \ 2 < s < 6.
\end{align*}
\]

Moreover, there exists \(A \geq 0\), such that

\[
\int_{\mathbb{R}^3} |\nabla u_n|^2 \to A \quad \text{as} \ n \to \infty.
\]

If we apply Theorem 2.6 to \(P(t) = f_i(t), \ i = 1, 2, Q(t) = |t|^5, v_n = u_n, v = f_i(u), i = 1, 2\) and \(\varphi \in C_0^{\infty}(\mathbb{R}^3)\), by (2.4) and (3.2) we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} f_i(u_n)\varphi = \int_{\mathbb{R}^3} f_i(u)\varphi, \ i = 1, 2.
\]
Moreover, by (3.2) and Lemma 2.1 in [29], we obtain that
\[ k_\ell(u_n) \int_{\mathbb{R}^3} \phi_{u_n} u_n \varphi \to k_\ell(u) \int_{\mathbb{R}^3} \phi_u u \varphi, \quad \text{as } n \to \infty, \quad (3.4) \]
\[ \psi\left(\frac{|u_n|^\alpha}{\ell^\alpha}\right) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \int_{\mathbb{R}^3} |u_n|^{\alpha-2} u_n \varphi \to \psi\left(\frac{|u|^\alpha}{\ell^\alpha}\right) \int_{\mathbb{R}^3} \phi_{u} u^2 \int_{\mathbb{R}^3} |u|^{\alpha-2} u \varphi, \quad \text{as } n \to \infty. \quad (3.5) \]

Then, by (3.1)-(3.5) we deduce that
\[ \int_{\mathbb{R}^3} a \nabla u \nabla \varphi + b A \int_{\mathbb{R}^3} \nabla u \nabla \varphi + \mu k_\ell(u) \int_{\mathbb{R}^3} \phi_{u} u^2 + \frac{\mu \alpha}{4 \ell^\alpha} \psi\left(\frac{|u|^\alpha}{\ell^\alpha}\right) \int_{\mathbb{R}^3} \phi_{u} u^2 \int_{\mathbb{R}^3} |u|^{\alpha-2} u \varphi \]
\[ + \int_{\mathbb{R}^3} f_\ell(u) \varphi - \lambda \int_{\mathbb{R}^3} f_1(u) \varphi = 0, \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^3), \quad (3.6) \]
which yields \((\overline{I}_{\mu, \lambda})'(u) = 0\), where
\[ \overline{I}_{\mu, \lambda}(u) = \frac{a + b A}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{\mu}{4} k_\ell(u) \int_{\mathbb{R}^3} \phi_{u} u^2 + \int_{\mathbb{R}^3} F_2(u) - \lambda \int_{\mathbb{R}^3} F_1(u). \]

By weak lower semicontinuity we can get that
\[ \int_{\mathbb{R}^3} |\nabla u|^2 \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2. \quad (3.7) \]

Again by (3.2) and Lemma 2.1(v), we have
\[ k_\ell(u_n) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to k_\ell(u) \int_{\mathbb{R}^3} \phi_{u} u^2, \quad \text{as } n \to \infty, \quad (3.8) \]
\[ \psi\left(\frac{|u_n|^\alpha}{\ell^\alpha}\right) |u_n|^\alpha \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \psi\left(\frac{|u|^\alpha}{\ell^\alpha}\right) |u|^\alpha \int_{\mathbb{R}^3} \phi_{u} u^2, \quad \text{as } n \to \infty. \quad (3.9) \]

If we apply Theorem 2.6 to \(P(t) = f_1(t) t, \ Q(t) = t^2 + t^6, \ v_n = u_n, \ v = f_1(u)\), by (2.4) and (3.2), we deduce that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^3} f_1(u_n) u_n = \int_{\mathbb{R}^3} f_1(u) u. \quad (3.10) \]

Moreover, by (3.2) and Fatou’s lemma, we get
\[ \int_{\mathbb{R}^3} f_2(u) u \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} f_2(u_n) u_n. \quad (3.11) \]

Notice that \((\overline{I}_{\mu, \lambda})'(u_n) \to 0 \) and (3.3), we can get that \(\lim_{n \to \infty} (\overline{I}_{\mu, \lambda})'(u_n), u_n = 0\).

Then by (3.8)-(3.11) and (3.6), we have
\[ \limsup_{n \to \infty} (a + b A) \int_{\mathbb{R}^3} |\nabla u_n|^2 \]
\[ = \limsup_{n \to \infty} \left( \lambda \int_{\mathbb{R}^3} f_1(u_n) u_n - \int_{\mathbb{R}^3} f_2(u_n) u_n - \mu k_\ell(u_n) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \right. \]
\[ - \frac{\mu \alpha}{4 \ell^\alpha} \psi\left(\frac{|u_n|^\alpha}{\ell^\alpha}\right) |u_n|^\alpha \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \left) \right. \]
\[ \leq \lambda \int_{\mathbb{R}^3} f_1(u) u - \int_{\mathbb{R}^3} f_2(u) u - \mu k_\ell(u) \int_{\mathbb{R}^3} \phi_{u} u^2 - \frac{\mu \alpha}{4 \ell^\alpha} \psi\left(\frac{|u|^\alpha}{\ell^\alpha}\right) |u|^\alpha \int_{\mathbb{R}^3} \phi_{u} u^2 \]
\[ = (a + b A) \int_{\mathbb{R}^3} |\nabla u|^2, \]
which implies that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 \leq \int_{\mathbb{R}^3} |\nabla u|^2.
\] (3.12)
By (3.7) and (3.12), we obtain that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 = \int_{\mathbb{R}^3} |\nabla u|^2,
\] (3.13)
and hence
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} f_2(u_n) u_n = \int_{\mathbb{R}^3} f_2(u) u.
\] (3.14)
Recall that \( f \) is odd on \( \mathbb{R} \) and by (2.2), we have
\[
f_2(t) = (f(t) + \nu t)^+ - f(t) = \nu t + (f(t) + \nu t)^-,
\]
thus \( f_2(t) t = \nu t^2 + k(t) \), where \( k(t) = t(f(t) + \nu t)^- \) is a nonnegative and continuous function, by Fatou’s lemma again we get that
\[
\int_{\mathbb{R}^3} k(u) \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} k(u_n), \quad \int_{\mathbb{R}^3} u^2 \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} u_n^2.
\]
By above two inequalities and (3.14), up to a subsequence, we obtain
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} u_n^2 = \int_{\mathbb{R}^3} u^2,
\]
which combining with (3.13) implies that \( u_n \to u \) in \( H^1_r(\mathbb{R}^3) \).

Then by Theorem 2.4 we have the following result.

**Lemma 3.2.** Assume that \((f_1) - (f_3)\) hold. Then for almost every \( \lambda \in [\delta_0, 1] \), there exists \( u_{\lambda} \in H^1_r(\mathbb{R}^3) \setminus \{0\} \), such that \( I_{\mu,\lambda}^L(u_{\lambda}) = c_{\mu,\lambda} \) and \( (I_{\mu,\lambda}^L)'(u_{\lambda}) = 0 \).

**Proof.** By Theorem 2.4, for almost every \( \lambda \in [\delta_0, 1] \), there exists a bounded sequence \( \{u_n^\lambda\} \subset H^1_r(\mathbb{R}^3) \) (for simplicity, we denote \( \{u_n\} \) instead of \( \{u_n^\lambda\} \)) such that
\[
I_{\mu,\lambda}^L(u_n) \to c_{\mu,\lambda} \quad \text{and} \quad (I_{\mu,\lambda}^L)'(u_n) \to 0.
\]
Up to a subsequence, by Lemma 3.1, we can suppose that there exists \( u_{\lambda} \in H^1_r(\mathbb{R}^3) \) such that \( u_n \to u_{\lambda} \) in \( H^1_r(\mathbb{R}^3) \). Moreover, \( u_{\lambda} \) is nontrivial. Indeed, if \( u_{\lambda} = 0 \), then we have \( u_n \to 0 \) in \( H^1_r(\mathbb{R}^3) \), which implies \( I_{\mu,\lambda}^L(u_n) \to 0 \). This contradicts with the fact that \( I_{\mu,\lambda}^L(u_n) \to c_{\mu,\lambda} \geq \eta > 0 \) (by Lemma 2.5(iii)). Therefore, \( u_{\lambda} \neq 0 \) and we complete the proof.

From the above Lemma 3.2, we see that for almost every \( \lambda \in [\delta_0, 1] \), \( I_{\mu,\lambda}^L \) has a nontrivial critical point \( u_{\lambda} \). In general, it is not known whether it is true for \( \lambda = 1 \). However, motivated by [20], we can choose a sequence \( \{\lambda_n\} \subset [\delta_0, 1] \) and \( \{u_n\} \subset H^1_r(\mathbb{R}^3) \setminus \{0\} \) such that
\[
\lambda_n \nearrow 1, \quad I_{\mu,\lambda_n}^L(u_n) = c_{\mu,\lambda_n} \quad \text{and} \quad (I_{\mu,\lambda_n}^L)'(u_n) = 0.
\] (3.15)
In the following, we need the following Pohozaev type identity. The proof can be done similarly as Appendix A in [5] and Lemma 2.6 in [26], the details are hence omitted.
Lemma 3.3. For \( \lambda \in [\delta_0, 1] \), if \( u \) is a critical point of \( I_{\mu,\lambda}^\ell \), then we have the following Pohozaev type identity:

\[
\frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{5\mu}{4} k_\ell(u) \int_{\mathbb{R}^3} \phi_u u^2 + \frac{3\mu}{4} \psi'\left( \frac{|u|_\alpha}{\ell^\alpha} \right) |u|_\alpha^\alpha \int_{\mathbb{R}^3} \phi_u u^2 = 3\lambda \int_{\mathbb{R}^3} F_1(u) - 3 \int_{\mathbb{R}^3} F_2(u). \]

In order to prove our main results, we present a key result.

Lemma 3.4. Assume \( f \) satisfies \((f_1)-(f_3)\). Let \( u_n \) be a critical point for \( I_{\mu,\lambda_n}^\ell \) at level \( c_{\mu,\lambda_n} \). Then, for \( \ell > 0 \) sufficiently large, there exists \( \mu^* = \mu^*(\ell) \) such that for any \( \mu \in (0,\mu^*) \), up to a subsequence, \( |u_n|_\alpha \leq \ell \), for any \( n \in \mathbb{N} \).

Proof. Since \( u_n \) is a critical point for \( I_{\mu,\lambda_n}^\ell \), at level \( c_{\mu,\lambda_n} \), we have

\[ I_{\mu,\lambda_n}^\ell(u_n) = c_{\mu,\lambda_n} \quad \text{and} \quad (I_{\mu,\lambda_n}^\ell)'(u_n) = 0, \]  

then by Lemma 3.3 we know that, \( u_n \) satisfies the following Pohozaev type identity

\[
\frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \frac{5\mu}{4} k_\ell(u_n) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \\
+ \frac{3\mu}{4} \psi'\left( \frac{|u_n|_\alpha}{\ell^\alpha} \right) |u_n|_\alpha \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = 3\lambda \int_{\mathbb{R}^3} F_1(u_n) - 3 \int_{\mathbb{R}^3} F_2(u_n). \quad (3.17)
\]

Moreover, combining (3.17) with (3.16), we get

\[
a \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 - \frac{\mu}{2} k_\ell(u_n) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \\
- \frac{3\mu}{4} \psi'\left( \frac{|u_n|_\alpha}{\ell^\alpha} \right) |u_n|_\alpha \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \\
= 3c_{\mu,\lambda_n}. \]

Now by Lemma 2.1(iii), we have

\[
a \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 \\
= 3c_{\mu,\lambda_n} + \frac{\mu}{2} k_\ell(u_n) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \frac{3\mu}{4} \psi'\left( \frac{|u_n|_\alpha}{\ell^\alpha} \right) |u_n|_\alpha \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \\
\leq 3c_{\mu,\lambda_n} + C_1 \mu \psi(\mu \psi^2) |u_n|_\alpha^4 + C_2 \psi'\left( \frac{|u_n|_\alpha}{\ell^\alpha} \right) \frac{\mu^2}{\ell^\alpha} |u_n|_\alpha^{4+\alpha}. \quad (3.18)
\]

We are going to estimate the right part of the inequality (3.18). By the min-max definition of the mountain-pass level, we have

\[
c_{\mu,\lambda_n} \leq \max_{\ell > 0} I_{\mu,\lambda_n}^\ell \left( w(\frac{1}{\ell}) \right) \\
\leq \max_{\ell > 0} \left( \frac{\mu t}{2} \int_{\mathbb{R}^3} |w|^2 + \frac{\mu t^2}{4} \left( \int_{\mathbb{R}^3} |\nabla w|^2 \right)^2 + t^3 \left( \int_{\mathbb{R}^3} F_2(w) - \delta_0 \int_{\mathbb{R}^3} F_1(w) \right) \right) \\
+ \max_{\ell > 0} \left( \frac{\mu t^5}{4} \psi\left( \frac{|w|_\alpha}{\ell^\alpha} \right) \int_{\mathbb{R}^3} \phi_{w} w^2 \right) \\
= : d_1 + d_2(\ell). \quad (3.19)
\]

If \( t^3 \geq \frac{2\epsilon}{|w|_\alpha^4} \), then \( d_2(\ell) = 0 \), otherwise \( t^3 < \frac{2\epsilon}{|w|_\alpha^4} \), then by Lemma 2.1(iii) and notice that \( \alpha = \frac{12}{5} \), we have

\[
d_2(\ell) \leq \frac{\mu}{4} \left( \frac{2\epsilon}{|w|_\alpha^4} \right)^{\frac{5}{2}} \int_{\mathbb{R}^3} \phi_{w} w^2 \leq \frac{\mu}{4} \left( \frac{2\epsilon}{|w|_\alpha^4} \right)^{\frac{5}{2}} \cdot C_2 \mu |w|_2^{\frac{4}{12}} = C_3 \mu^2 \ell^4. \quad (3.20)
\]
Similarly, we can also obtain that
\[
\begin{align*}
C_1 \mu^2 k_\ell(u_n)|u_n|^{4\alpha} & \leq C_4 \mu^2 \ell^4, \\
C_2 \psi'(\frac{|u_n|^{\alpha}}{\ell^\alpha}) \frac{\mu^2}{\ell^\alpha} |u_n|^{4+\alpha} & \leq C_5 \mu^2 \ell^4.
\end{align*}
\tag{3.21}
\]
By (3.18)-(3.21) we get that
\[
a \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 \leq 3d_1 + C_6 \mu^2 \ell^4.
\tag{3.22}
\]
On the other hand, since \( \langle I_{\mu,\lambda}^\ell \rangle'(u_n), u_n \rangle = 0 \), by (2.5) we have
\[
\begin{align*}
&\int_{\mathbb{R}^3} a|\nabla u_n|^2 + b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \mu k_\ell(u_n) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \\
+ \frac{\mu \alpha}{4 \ell^\alpha} \psi'(\frac{|u_n|^{\alpha}}{\ell^\alpha}) |u_n|^{\alpha} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \int_{\mathbb{R}^3} f_2(u_n) u_n \\
= \lambda_n \int_{\mathbb{R}^3} f_1(u_n) u_n \leq C_\epsilon \int_{\mathbb{R}^3} |u_n|^6 + \epsilon \int_{\mathbb{R}^3} f_2(u_n) u_n.
\end{align*}
\tag{3.23}
\]
Then by (2.3), (3.23) and (3.22), we deduce that
\[
\begin{align*}
\nu (1 - \epsilon) \int_{\mathbb{R}^3} u_n^2 & \leq (1 - \epsilon) \int_{\mathbb{R}^3} f_2(u_n) u_n \\
& \leq C_\epsilon \int_{\mathbb{R}^3} |u_n|^6 + \frac{\mu \alpha}{4 \ell^\alpha} \psi'(\frac{|u_n|^{\alpha}}{\ell^\alpha}) |u_n|^{\alpha} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \\
& \leq C_\epsilon \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^3 + C \mu^2 \ell^4 \\
& \leq C(d_1 + \mu^2 \ell^4)^{\frac{3}{2}} + C \mu^2 \ell^4.
\end{align*}
\tag{3.24}
\]
Next we show that, if \( \ell \) is sufficiently large, then \( \limsup_{n} |u_n|_{\alpha} \leq \ell \). We suppose by contradiction that there exists a certain \( n_0 \) such that
\[
|u_n|_{\alpha} > \ell, \quad \forall \ n \geq n_0.
\tag{3.25}
\]
Without loss of generality, we may assume that (3.25) is true for any \( u_n \). Therefore, by (3.22) and (3.24), we conclude that
\[
\ell^2 < |u_n|_{\alpha}^2 \leq C|u_n|^2 \leq C_7 + C_9 \mu^2 \ell^4 + C_10 \mu^3 \ell^6,
\]
which is not true for \( \ell \) large and \( \mu \) small enough. Indeed, we can find \( \ell_0 > 0 \) such that \( \ell_0^2 > C_7 + 1 \) and \( \mu^* = \mu^*(\ell_0) \) such that \( C_8 \mu^2 \ell^2 + C_9 \mu^2 \ell^4 + C_10 \mu^3 \ell^6 < 1 \) for any \( \mu \in (0, \mu^*) \), and then we get a contradiction. This completes the proof of Lemma 3.4. \( \square \)

**Remark 3.5.** By the process of the proof of Lemma 3.4, we can obtain the following fact:
there exists \( M > 0 \) independent of \( \mu \) and \( \lambda \) such that \( c_{\mu, \lambda} \leq M \) for all \( \lambda \in [\delta_0, 1] \) and \( \mu \in (0, \mu^*) \). Indeed, by (3.18) and (3.19) we know that \( c_{\mu, \lambda} \leq d_1 + C \mu^2 \ell^4 \leq d_1 + C (\mu^*)^2 \ell^4 =: M \).

Now we give the proof of Theorem 1.1.
Proof of Theorem 1.1. Let $\ell, \mu^*$ be obtained in Lemma 3.4 and fix $\mu \in (0, \mu^*)$. In order to find the critical points of $I_\mu$, by (3.15) we can take a sequence $\{\lambda_n \} \subset [\delta_0, 1]$ satisfying $\lambda_n \not\nearrow 1$, and there exists a sequence $\{u_n\} \subset H^1_\ell(\mathbb{R}^3) \setminus \{0\}$ such that

$$I^\ell_{\mu, \lambda_n}(u_n) = c_{\mu, \lambda_n} \quad \text{and} \quad (I^\ell_{\mu, \lambda_n})'(u_n) = 0.$$  

From Lemma 3.4 we may assume that $|u_n|_\alpha \leq \ell$. Then by (2.8) we have that

$$I^\ell_{\mu, \lambda_n}(u_n) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \int_{\mathbb{R}^3} F_2(u_n) - \lambda_n \int_{\mathbb{R}^3} F_1(u_n).$$

Moreover, following from arguments such as those in (3.22) and (3.24), we can obtain that $\{u_n\}$ is bounded in $H^1_{\ell}(\mathbb{R}^3)$.

Next, we claim that $\{u_n\}$ is a (PS) sequence for $I_\mu$ at the level $c_{\mu, 1}$. Indeed, the boundedness of $\{u_n\}$ implies that $\{I_\mu(u_n)\}$ is bounded. By (3.26), note that

$$\langle I'_\mu(u_n), \varphi \rangle = (I^\ell_{\mu, \lambda_n})'(u_n), \varphi + (\lambda_n - 1) \int_{\mathbb{R}^3} f_1(u_n) \varphi \to 0 \text{ as } n \to \infty.$$  

Thus $I'_\mu(u_n) \to 0$ as $n \to \infty$. Moreover, by using the fact that the map $\lambda \mapsto c_{\mu, \lambda}$ is continuous from the left and Lemma 2.5(iii), we have that

$$\lim_{n \to \infty} I_\mu(u_n) = \lim_{n \to \infty} \left( I^\ell_{\mu, \lambda_n}(u_n) + (\lambda_n - 1) \int_{\mathbb{R}^3} F_1(u_n) \right) = \lim_{n \to \infty} c_{\mu, \lambda_n} = c_{\mu, 1}.$$  

Therefore, $\{u_n\}$ is a bounded (PS) sequence of $I_\mu$. Then by Lemma 3.1, we know that $\{u_n\}$ has a convergent subsequence, assume that $u_n \rightharpoonup u_\mu$ strongly in $H^1_{\ell}(\mathbb{R}^3)$. Consequently, $I'_\mu(u_\mu) = 0$ and $I_\mu(u_\mu) = c_{\mu, 1}$, by Lemma 2.5(iii) we know that $c_{\mu, 1} \geq \eta > 0$, which implies $u_\mu \neq 0$. Hence $u_\mu$ is a positive solution by Remark 2.3.

In the following, we consider the asymptotic behavior of $u_\mu$ as $\mu \to 0$. For any sequence $\{\mu_n\} \subset (0, \mu^*)$ with $\mu_n \not\nearrow 0$ as $n \to \infty$. Let $(u_{\mu_n}, \phi_{\mu_n}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ be a sequence positive solutions obtained in Theorem 1.1. Then by Lemma 2.2, we know that $u_{\mu_n}$ is a critical point of the functional $I_{\mu_n}$ and $\phi = \phi_{\mu_n}$. Hence we have that

$$I_{\mu_n}(u_{\mu_n}) = c_{\mu_n, 1} \quad \text{and} \quad (I_{\mu_n})'(u_{\mu_n}) = 0.$$  

For convenience, we write $\mu$ instead of $\mu_n$. First, we claim that $u_\mu := u_{\mu_n}$ is bounded as $\mu \to 0$. By (3.27) we know that $u_\mu$ satisfies

$$\frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_\mu|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_\mu|^2 \right)^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu^2 - \int_{\mathbb{R}^3} F(u_\mu) = c_{\mu, 1}$$

and the following Pohozaev type identity

$$\frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_\mu|^2 + \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u_\mu|^2 \right)^2 + \frac{5\mu}{4} \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu^2 = 3 \int_{\mathbb{R}^3} F(u_\mu).$$

From (3.28),(3.29) and Remark 3.5, we have

$$a \int_{\mathbb{R}^3} |\nabla u_\mu|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_\mu|^2 \right)^2 \leq 3M \frac{\mu}{2} \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu^2.$$  

Note that by Lemma 2.1, $\frac{b}{4} \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu^2 \to 0$ as $\mu \to 0$. Then (3.30) implies that $\int_{\mathbb{R}^3} |\nabla u_\mu|^2$ is bounded. On the other hand, since $\langle (I'_\mu)'(u_\mu), u_\mu \rangle = 0$, by (2.5) with
\[ \varepsilon = \frac{1}{2} \] we have that
\[
\int_{\mathbb{R}^3} a|\nabla u_\mu|^2 + b \left( \int_{\mathbb{R}^3} |\nabla u_\mu|^2 \right)^2 + \mu \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu^2 + \int_{\mathbb{R}^3} f_2(u_\mu) u_\mu \\
= \int_{\mathbb{R}^3} f_1(u_\mu) u_\mu \leq C \int_{\mathbb{R}^3} |u_\mu|^6 + \frac{1}{2} \int_{\mathbb{R}^3} f_2(u_\mu) u_\mu.
\]
(3.31)

Then by (2.3) and (3.31), we obtain
\[
\frac{\nu}{2} \int_{\mathbb{R}^3} u_\mu^2 \leq \frac{1}{2} \int_{\mathbb{R}^3} f_2(u_\mu) u_\mu \leq C \int_{\mathbb{R}^3} |u_\mu|^6 \leq \tilde{C} \left( \int_{\mathbb{R}^3} |\nabla u_\mu|^2 \right)^3,
\]
(3.32)
so by (3.30) and (3.32) \( u_\mu \) is bounded as \( \mu \to 0 \) and we may assume \( u_\mu \to u_0 \) in \( H^1_\mu(\mathbb{R}^3) \). Moreover, by a similar process of the proof of Lemma 3.1, we can check that \( u_\mu \to u_0 \) in \( H^1_\mu(\mathbb{R}^3) \) as \( \mu \to 0 \).

Next, we observe that
\[
I_\mu(u_\mu) = E(u_\mu) + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu^2,
\]
(3.33)
where \( E(u) \) is the energy functional of the limit problem (1.8) defined as
\[
E(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^3} F(u).
\]

Note that, for any \( \varphi \in C_0^\infty(\mathbb{R}^3) \), there holds
\[
\langle I'_\mu(u_\mu), \varphi \rangle = \langle E'(u_\mu), \varphi \rangle + \mu \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu \varphi,
\]
and by Lemma 2.1, we have that \( \mu \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu \varphi \to 0 \) as \( \mu \to 0 \). Then by the fact that \( \langle I'_\mu(u_\mu), \varphi \rangle = 0 \) we get that \( E'(u_\mu) \to 0 \) as \( \mu \to 0 \). Since \( u_\mu \to u_0 \) in \( H^1_\mu(\mathbb{R}^3) \), then \( E'(u_0) = 0 \) and \( u_0 \) is a solution of (1.8). Moreover, \( u_0 \neq 0 \). In fact, if \( u_\mu \to 0 \) in \( H^1(\mathbb{R}^3) \), which leads to \( I_\mu(u_\mu) = 0 \), this contradicts with \( I_\mu(u_\mu) = c_{\mu, 1} > 0 \). Finally, we claim that \( u_0 \) is a ground state solution of problem (1.8). Define
\[
m^* = \inf \{ E(u) : u \text{ is a nontrivial solution of (1.8)} \}.
\]

Let \( \omega_0(x) \) be a ground state solution of (1.8), then \( E(u_0) \geq E(\omega_0) = m^* \). Following the arguments as in [22], we can obtain that there exists a path \( \gamma(t) \in C([0, \infty), H^1_\mu(\mathbb{R}^3)) \) such that
\[
\begin{cases}
\gamma(0) = 0, & \gamma(1) = \omega_0, \quad E(\gamma(t)) < 0 \text{ for large } t > 1; \\
E(\gamma(t)) \leq E(\gamma(1)) = E(\omega_0) \text{ for all } t \in [0, \infty).
\end{cases}
\]

Moreover, we can use the argument similar as the proof of Lemma 6.1 in [19] to show that \( \{u \in H^1_\mu(\mathbb{R}^3) : E(u) < 0\} \) is path connected, then by a suitable change of variable, \( \gamma(t) \) can be used to find a path \( \tilde{\gamma}(t) \) with the following properties:
\[
\begin{cases}
\tilde{\gamma}(0) = 0, & \tilde{\gamma}(\frac{1}{2}) = \omega_0, \quad \tilde{\gamma}(1) = e, \text{ where } e \text{ is given in Lemma 2.5(ii);} \\
E(\tilde{\gamma}(t)) \leq E(\tilde{\gamma}(\frac{1}{2})) = E(\omega_0) = m^* \text{ for all } t \in [0, 1].
\end{cases}
\]

It is easy to check that \( \tilde{\gamma}(t) \in \Gamma \), where \( \Gamma \) is defined in (2.9), by (3.33) we obtain
\[
I_\mu(\tilde{\gamma}(t)) = E(\tilde{\gamma}(t)) + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_{\tilde{\gamma}(t)} \tilde{\gamma}(t)^2.
\]
Thus
\[ c_{u,1} \leq \max_{t \in [0,1]} I_{\mu}(\gamma(t)) = \max_{t \in [0,1]} \left( E(\gamma(t)) + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_{\gamma(t)}^2 \gamma(t)^2 \right) \leq m^* + \frac{\mu}{4} \max_{t \in [0,1]} \int_{\mathbb{R}^3} \phi_{\gamma(t)}^2 \gamma(t)^2 \to m^*, \quad \text{as } \mu \to 0, \]
from which we get that \( c_{u,1} \leq m^* \). On the other hand, by (3.33) we obtain that \( c_{u,1} \geq E(u_0) \geq m^* \), so we conclude that \( E(u_0) = m^* \). This completes the proof of Theorem 1.1. \( \square \)

4. Proof of Theorem 1.5. In this section, we consider the nonhomogeneous problem \((P_g)\) and give the proof of Theorem 1.5.

For \( u \in H^1(\mathbb{R}^3) \), we define the energy functional
\[ J_{\mu}(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi u^2 - \int_{\mathbb{R}^3} F(u) - \int_{\mathbb{R}^3} g(x)u, \quad (4.1) \]
where \( \phi \) is defined in (2.1). It is easy to check that \( J_{\mu} \in C^1(H^1(\mathbb{R}^3), \mathbb{R}) \) under the conditions \((f_1)-(f_3)\) and \((g_0)\). By Lemma 2.2, in order to get solutions of \((P_g)\), we would look for critical point of \( J_{\mu} \).

We first prove that system \((P_g)\) has a positive radial solution with negative energy. With the aid of Ekeland’s variational principle, this solution can be obtained by seeking a local minimum of the energy functional \( J_{\mu} \) constrained in a neighborhood of zero. First of all, we give a preliminary result.

**Lemma 4.1.** Assume that \((f_1)-(f_3)\) and \((g_0)\) hold. Then for the energy functional \( J_{\mu} \) defined by \((4.1)\), there exist constants \( \rho_0, \eta_0, m_0 > 0 \) such that \( J_{\mu}(u) \geq \eta_0 > 0 \) for all \( u \in H^1_\rho(\mathbb{R}^3) \) with \( \|u\| = \rho_0 \) and \( |g|_2 < m_0 \).

**Proof.** For any \( u \in H^1_\rho(\mathbb{R}^3) \), using (2.2), (2.3) and (2.6) for \( \varepsilon = \frac{1}{2} \), we have
\[
J_{\mu}(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi u^2 - \int_{\mathbb{R}^3} F(u) - \int_{\mathbb{R}^3} F_1(u) - \int_{\mathbb{R}^3} g(x)u \\
\geq \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi u^2 - C_0 \parallel u \parallel^6 - |g|_2 \parallel u \parallel \\
\geq \min \left\{ \frac{1}{2}, \frac{\mu}{4} \right\} \parallel u \parallel^2 - C_0 \parallel u \parallel^6 - |g|_2 \parallel u \parallel. \quad (4.2)
\]
Set \( h(t) = \min \left\{ \frac{1}{2}, \frac{\mu}{4} \right\} t - C_0 t^5 \) for \( t \geq 0 \). By direct calculations, we obtain that
\[
\max_{t \geq 0} h(t) = h(\rho_0) = \frac{4 \min \left\{ \frac{1}{2}, \frac{\mu}{4} \right\}}{5} \left( \frac{\min \left\{ \frac{1}{2}, \frac{\mu}{4} \right\}}{5C_0} \right)^{\frac{1}{4}} =: m_0,
\]
where \( \rho_0 = \left( \frac{\min \left\{ \frac{1}{2}, \frac{\mu}{4} \right\}}{5C_0} \right)^{\frac{1}{4}} \). Then it follows from (4.2) that if \( |g|_2 < m_0 \), there exists \( \eta_0 := \rho_0 (h(\rho_0) - |g|_2) > 0 \) such that \( J_{\mu}(u) \geq \eta_0 > 0 \) for \( \|u\| = \rho_0 \). \( \square \)

**Lemma 4.2.** Assume that \((f_1)-(f_3)\) and \((g_0)\) hold. If \( |g|_2 < m_0 \), where \( m_0 \) is given by Lemma 4.1, then there exists \( u_1 \in H^1_\rho(\mathbb{R}^3) \) such that
\[ J_{\mu}(u_1) = \inf_{u \in H^1(\mathbb{R}^3)} \{ J_{\mu}(u) : \parallel u \parallel \leq \rho_0 \} < 0. \]
Moreover, \( J'_{\mu}(u_1) = 0 \), namely \( (u_1, \phi_{u_1}) \) is a solution of problem \((P_g)\).
Proof. Since $g(x) = g(|x|) \in L^2(\mathbb{R}^3)$, $g(x) \geq 0$ and $g(x) \neq 0$, we can choose a function $v \in H^1_0(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} g(x)v > 0$. Then for $t > 0$ small enough, we have

$$J_\mu(tv) = \frac{at^2}{2} \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{bt^4}{4} \left( \int_{\mathbb{R}^3} |\nabla v|^2 \right)^2 + \frac{\mu t^4}{4} \int_{\mathbb{R}^3} \phi_v v^2 - \int_{\mathbb{R}^3} F(tv) - t \int_{\mathbb{R}^3} g(x)v$$

$$\leq \frac{at^2}{2} \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{bt^4}{4} \left( \int_{\mathbb{R}^3} |\nabla v|^2 \right)^2 + \frac{\mu t^4}{4} \int_{\mathbb{R}^3} \phi_v v^2 - t \int_{\mathbb{R}^3} g(x)v < 0.$$ 

This implies that $\tilde{c}_1 := \inf \{ J_\mu(u) : u \in \overline{B}_{\tilde{\rho}_0} \} < 0$, where $\overline{B}_{\tilde{\rho}_0} = \{ u \in H^1_0(\mathbb{R}^3) : \| u \| \leq \tilde{\rho}_0 \}$. By Ekeland’s variational principle, there is a minimizing sequence $\{ u_n \} \subset \overline{B}_{\tilde{\rho}_0}$ such that

(i) $\tilde{c}_1 \leq J_\mu(u_n) \leq \tilde{c}_1 + \frac{1}{n}$;
(ii) $J_\mu(\vartheta) \geq J_\mu(u_n) - \frac{1}{n} \| \vartheta - u_n \|$ for all $\vartheta \in \overline{B}_{\tilde{\rho}_0}$.

By a standard procedure, we can prove that $\{ u_n \}$ is a bounded (PS) sequence for $J_\mu$. Furthermore, by $(f_1)$, $(f_3)$ and $(g_0)$, using the compactness of the embedding $H^1_0(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)(2 < p < 6)$, we know that $\{ u_n \}$ has a strongly convergent subsequence, and there exists $u_1 \in H^1_0(\mathbb{R}^3)$ such that up to a subsequence $u_n \rightharpoonup u_1$ in $H^1_0(\mathbb{R}^3)$. Hence $J_\mu(u_1) = \tilde{c}_1 < 0$ and $J_\mu'(u_1) = 0$. Moreover, $J_\mu(u_1) = \tilde{c}_1 < 0$ implies that $u_1 \neq 0$. This completes the proof of Lemma 4.2. □

In the following, we discuss the existence of positive energy solution for problem $(\mathcal{P}_g)$. As is known, it is not easy to show that a (PS) sequence of the functional $J_\mu$ is bounded in this case. In order to overcome this difficulty, similar as dealing with previous problem $(\mathcal{P})$, we consider the following perturbed functional

$$J_{\mu,\lambda}^\varepsilon(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_u u^2$$

$$+ \int_{\mathbb{R}^3} F_2(u) - \int_{\mathbb{R}^3} g(x)u - \lambda \int_{\mathbb{R}^3} F_1(u).$$

Then similar as Lemma 2.5 and Lemma 3.1, we can list some properties of $J_{\mu,\lambda}^\varepsilon$.

**Lemma 4.3.** Assume that $(f_1), (f_3)$ and $(g_0)$ hold. Then for any $\lambda \in [\delta_0, 1]$ and $|g|_2 < m_0$, the following hold:

(i) there exist constants $\varrho, \xi_0 > 0$ such that $J_{\mu,\lambda}^\varepsilon(u) \geq \xi_0$ for all $u \in H^1_0(\mathbb{R}^3)$ with $\| u \| = \varrho$;
(ii) there exists $\varepsilon_0 \in H^1_0(\mathbb{R}^3) \setminus \{ 0 \}$ such that $J_{\mu,\lambda}^\varepsilon(\varepsilon_0) < 0$;
(iii) $\tilde{c}_{\mu,\lambda} := \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} J_{\mu,\lambda}^\varepsilon(\gamma(t)) \geq \xi_0 > \max \{ J_{\mu,\lambda}(0), J_{\mu,\lambda}(\varepsilon_0) \}$, where $\Gamma_0 = \{ \gamma \in C([0,1], H^1_0(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = \varepsilon_0 \}$;
(iv) each bounded (PS) sequence for $J_{\mu,\lambda}^\varepsilon$ has a strongly convergent subsequence.

**Lemma 4.4.** Assume that $(f_1), (f_3)$ and $(g_0)$ hold. If $|g|_2 < m_0$, then for almost every $\lambda \in [\delta_0, 1]$, there exists $\bar{u}_\lambda \in H^1_0(\mathbb{R}^3) \setminus \{ 0 \}$, such that $J_{\mu,\lambda}(\bar{u}_\lambda) = \tilde{c}_{\mu,\lambda}$ and $(J_{\mu,\lambda})'(\bar{u}_\lambda) = 0$.

**Proof.** By Lemma 4.3 and Theorem 2.4, for almost every $\lambda \in [\delta_0, 1]$, there exists bounded sequence $\{ \bar{u}_n \} \subset H^1_0(\mathbb{R}^3)$ (for simplicity, we denote $\{ \bar{u}_n \}$ instead of $\{ \bar{u}_n \}$) such that

$$J_{\mu,\lambda}(\bar{u}_n) \to \tilde{c}_{\mu,\lambda}$$

and $(J_{\mu,\lambda})'(\bar{u}_n) \to 0$.

By Lemma 4.3(iv), up to a subsequence, we can suppose that there exists $\bar{u}_\lambda \in H^1_0(\mathbb{R}^3)$ such that $\bar{u}_n \rightharpoonup \bar{u}_\lambda$ in $H^1_0(\mathbb{R}^3)$. Therefore, $(J_{\mu,\lambda})'(\bar{u}_\lambda) = 0$ and $J_{\mu,\lambda}(\bar{u}_\lambda) = \tilde{c}_{\mu,\lambda}$.
\[ \tilde{c}_{\mu, \lambda}. \] Moreover, we can check that \( \tilde{u}_\lambda \neq 0 \). Indeed, if \( \tilde{u}_\lambda = 0 \), then \( \tilde{u}_n \to 0 \) in \( H^1_r(\mathbb{R}^3) \), which implies \( J^\ell_{\mu, \lambda}(\tilde{u}_n) \to 0 \). This contradicts with the fact that \( J^\ell_{\mu, \lambda}(\tilde{u}_n) \to \tilde{c}_{\mu, \lambda} \geq \xi_0 > 0 \) (by Lemma 4.3(iii)). Hence, \( \tilde{u}_\lambda \neq 0 \). Similar as Remark 2.3 we know that \( \tilde{u}_\lambda \) is positive.

Let us now give the proof of Theorem 1.5.

**Proof of Theorem 1.5.** By Lemma 4.4, we can choose a sequence \( \{\lambda_n\} \subset [\delta_0, 1] \) and \( \{\tilde{u}_n\} \subset H^1_r(\mathbb{R}^3) \setminus \{0\} \) with \( \tilde{u}_n \geq 0 \) such that

\[ \lambda_n \nearrow 1, \quad J^\ell_{\mu, \lambda_n}(\tilde{u}_n) = \tilde{c}_{\mu, \lambda_n} \quad \text{and} \quad (J^\ell_{\mu, \lambda_n})'(\tilde{u}_n) = 0. \tag{4.3} \]

We divide the proof of Theorem 1.5 into three steps.

**Step 1:** We claim that, for \( \ell > 0 \) sufficiently large, there exists \( \mu_0 = \mu_0(\ell) \) such that for any \( \mu \in (0, \mu_0) \), up to a subsequence, \( |\tilde{u}_n|_\alpha \leq \ell \), for any \( n \in \mathbb{N} \).

Indeed, by (4.3) we know that, \( \tilde{u}_n \) satisfies the following Pohozaev type identity

\[
\begin{align*}
\frac{a}{2} \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 \right)^2 + \frac{5\mu}{4} k_\ell(\tilde{u}_n) \int_{\mathbb{R}^3} \phi_{\tilde{u}_n} \tilde{u}_n^2 \\
+ \frac{3\mu}{|\alpha|} \frac{\psi'}{|\psi|} \left( \frac{|\tilde{u}_n|_\alpha}{|\tilde{u}_n|_\alpha} \right) \int_{\mathbb{R}^3} \phi_{\tilde{u}_n} \tilde{u}_n^2
\end{align*}
\]

\[= 3\lambda_n \int_{\mathbb{R}^3} F_1(\tilde{u}_n) - 3 \int_{\mathbb{R}^3} F_2(\tilde{u}_n) + \int_{\mathbb{R}^3} (3g(x) + \langle \nabla g(x), x \rangle) \tilde{u}_n. \tag{4.4} \]

Moreover, combining \( J^\ell_{\mu, \lambda_n}(\tilde{u}_n) = \tilde{c}_{\mu, \lambda_n} \) with (4.4), and notice that \( \langle \nabla g(x), x \rangle \geq 0 \), we deduce that

\[
\begin{align*}
\frac{a}{2} \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 \right)^2 &- \frac{\mu}{2} k_\ell(\tilde{u}_n) \int_{\mathbb{R}^3} \phi_{\tilde{u}_n} \tilde{u}_n^2 \\
&- \frac{3\mu}{|\alpha|} \frac{\psi'}{|\psi|} \left( \frac{|\tilde{u}_n|_\alpha}{|\tilde{u}_n|_\alpha} \right) \int_{\mathbb{R}^3} \phi_{\tilde{u}_n} \tilde{u}_n^2 \\
&\leq 3\epsilon_{\mu, \lambda_n} \leq 3\epsilon_{\mu, \lambda_n} + C_1 \epsilon_{\mu, \lambda_n} \leq 3\epsilon_{\mu, \lambda_n} + C_2 \epsilon_{\mu, \lambda_n}.
\end{align*}
\]

Then by Lemma 2.1(iii), we have

\[
\begin{align*}
\frac{a}{2} \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 \right)^2 &- \frac{\mu}{2} k_\ell(\tilde{u}_n) \int_{\mathbb{R}^3} \phi_{\tilde{u}_n} \tilde{u}_n^2 \\
&- \frac{3\mu}{|\alpha|} \frac{\psi'}{|\psi|} \left( \frac{|\tilde{u}_n|_\alpha}{|\tilde{u}_n|_\alpha} \right) \int_{\mathbb{R}^3} \phi_{\tilde{u}_n} \tilde{u}_n^2 \\
&\leq 3\epsilon_{\mu, \lambda_n} + C_1 \epsilon_{\mu, \lambda_n} |\tilde{u}_n|_\alpha^4 + C_2 |\tilde{u}_n|_\alpha^4 + \epsilon_{\mu, \lambda_n}^4 + \epsilon_{\mu, \lambda_n}.
\end{align*}
\]

Now following the same arguments as those in Lemma 3.4, we can find \( \ell > 0 \) large enough, and there exists \( \mu_0 = \mu_0(\ell) \) such that up to a subsequence, \( |\tilde{u}_n|_\alpha \leq \ell \) for any \( \mu \in (0, \mu_0) \) and \( n \in \mathbb{N} \).

**Step 2:** We claim that \( \{\tilde{u}_n\} \) is a (PS) sequence for \( J^\ell_{\mu} \) at the level \( \tilde{c}_{\mu, 1} \).

Indeed, let \( \ell, \mu_0 \) be obtained in Step 1 and fix \( \mu \in (0, \mu_0) \). From Step 1 we may assume that \( |\tilde{u}_n|_\alpha \leq \ell \), then by (2.8) we have that

\[
J^\ell_{\mu, \lambda_n}(\tilde{u}_n) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 \right)^2 + \frac{\mu}{4} \int_{\mathbb{R}^3} \phi_{\tilde{u}_n} \tilde{u}_n^2 \\
+ \int_{\mathbb{R}^3} F_2(\tilde{u}_n) - \int_{\mathbb{R}^3} g(x) \tilde{u}_n - \lambda_n \int_{\mathbb{R}^3} F_1(\tilde{u}_n). \tag{4.5} \]
Moreover, following from arguments such as that in (3.22) and (3.24), we can obtain that \( \{ \tilde{u}_n \} \) is bounded in \( H_0^1(\mathbb{R}^3) \), and the boundedness of \( \{ \tilde{u}_n \} \) implies that \( \{ J_\mu(\tilde{u}_n) \} \) is bounded. By (4.5), for any \( \varphi \in C_c^\infty(\mathbb{R}^3) \), there holds

\[
\langle J'_\mu(\tilde{u}_n), \varphi \rangle = \langle (J^\ell_{\mu,\lambda})(\tilde{u}_n), \varphi \rangle + (\lambda_n - 1) \int_{\mathbb{R}^3} f_1(\tilde{u}_n)\varphi \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus \( J'_\mu(\tilde{u}_n) \to 0 \) as \( n \to \infty \). Moreover, by using the fact that the map \( \lambda \mapsto \bar{c}_{\mu, \lambda} \) is continuous from the left and Lemma 4.3(iii), we obtain that

\[
\lim_{n \to \infty} J_\mu(\tilde{u}_n) = \lim_{n \to \infty} \left( J^\ell_{\mu,\lambda_n}(\tilde{u}_n) + (\lambda_n - 1) \int_{\mathbb{R}^3} f_1(\tilde{u}_n) \right) = \lim_{n \to \infty} \bar{c}_{\mu,\lambda_n} = \bar{c}_{\mu,1}.
\]

Therefore, \( \{ \tilde{u}_n \} \) is a bounded (PS) sequence for \( J_\mu \) at the level \( \bar{c}_{\mu,1} \).

Step 3: Conclusion.

By Step 2 and Lemma 4.3(iv), we know that \( \{ \tilde{u}_n \} \) has a convergent subsequence, without loss of generality, we may assume that \( \tilde{u}_n \to u_2 \) strongly in \( H_0^1(\mathbb{R}^3) \). Consequently, \( J'_\mu(u_2) = 0 \) and \( J_\mu(u_2) = \bar{c}_{\mu,1} \), by Lemma 4.3(iii) we know that \( \bar{c}_{\mu,1} \geq \xi_0 > 0 \), which implies \( u_2 \neq 0 \). Hence \( u_2 \) is a positive solution by applying the similar instruction as in Remark 2.3. Thus, using Lemma 4.2, we complete the proof of Theorem 1.5.

\[\square\]

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