Deformation of bichromatic wave groups based on third order side band solution of Benjamin-Bona-Mahony equation

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Abstract. This paper concerns on propagation of Benjamin Bona Mahony (BBM) wave groups. The previous results, experimental, analytical and numerical, show that nonlinear effects will deform wave groups and may lead to large waves with wave heights larger than twice the original input; the deformation may show itself as peaking and splitting. To investigate this, especially at which location the waves will achieve their maximum amplitude, and to determine the amplitude amplification factor, a concept called Maximal Temporal Amplitude (MTA) is applied. This quantity is a tool that can be used to measure the maximum amplitude of the waves over time. In this paper we will use Benjamin-Bona-Mahony (BBM) model and third order side band approximation theory to investigate the peaking and splitting phenomena of the wave groups which is initially in bichromatic signal. The bichromatic signal here is a signal that is described by superposition of two monochromatic signals with the same value in amplitude but slightly different in frequencies. We present that the waves undergo deformation in their propagation.

1. Introduction

This research is motivated by the vast interest in studying the behavior of extreme wave. Investigation on extreme wave, also known as freak, rogue, or giant wave, is an active study in the fluid dynamic realm. A wave is called extreme when its height exceeds the significant wave height of measured wave train by factor more than 2.2 \[1\]. The occurrence of this wave is unpredictable while it is a major threat to the marine transportation and offshore facility due to its impact causing severe damage and loss \[2, 3\]. The study of extreme wave is important in order to investigate its characteristics and to generate it at a certain position in a water tank on its application in the hydrodynamic laboratory activity. The wave is generated to examine the endurance of a model of a marine object in order to prepare it to the worst scenario that may happen in the ocean. Some studies about extreme wave have been conducted experimentally \[4, 5, 6\], analytically \[7, 8, 9\], and numerically \[10, 11, 12\].

Extreme wave can occur in coastal area, shallow water, as well as deep water. One of the physical mechanism of extreme wave generation is modulational instability which is also known as Benjamin-Feir Instability. This phenomenon happens as a result of wave nonlinearity behavior;
the instability produces growing modulation of the envelope. Instability can occur when a linear wave is modulated by other waves with slightly different frequencies. A monochromatic wave being perturbed by a pair of side bands will experience instability and focusing phenomenon which amplify the amplitude during its propagation. This wave is called trichromatic waves, also known as Benjamin-Feir type waves. An equation which yields solutions exhibiting extreme wave is the so-called Nonlinear Schrödinger (NLS) equation. NLS equation is an asymptotic approximation of modulated wave which describes an evolution of propagating waves. The evolution of some wave groups’ envelopes satisfies this equation. Halfiani et al [13] derived the NLS equation as the amplitude equation of Benjamin-Bona-Mahony (BBM) wave when the solution is developed in wave group form. NLS also appears as the envelope equation for Korteweg-de Vries (KdV) [14, 15, 16, 17] and Kadomtsev-Petviashvili (KP) [18] wave group. The equation has been studied for broad applications in fluid dynamic, quantum electronics, optics, and plasma physics [19, 14, 20, 21, 18]. One of NLS’ solutions, Soliton on Finite Background, shows that the waves experience modulational instability during its propagation and is a well approximation to the Benjamin-Feir type waves [22, 23]. The nonlinear evolution of the trichromatic signal derived from Korteweg-de Vries (KdV) equation was also explained in [7].

Instability also happens to the bichromatic wave type, a superposition of two monochromatic waves with same amplitude and slightly different frequencies. This phenomenon had been investigated numerically and experimentally in [6] and [24]. In [25, 26], bichromatic signal is implemented to generate extreme waves by applying third and fifth order theories of KdV equation. The results show that the waves’ height keeps increasing until it reaches an extreme position (a spatial position where the wave height reach its maximum). The higher the order, the higher the maximum amplitude. However, the maximum wave heights are still far below the maximum height of wave generated by numerical software HUBRIS which is governed from Laplace equation [24], whereas the extreme positions conform. The same signal is also applied to the Boussinesq equation, which exhibits an amplification on its amplitude with maximum height and extreme position meet the HUBRIS’ result [8]. The extreme position can be determined by using a quantity called Maximal Temporal Amplitude (MTA). MTA measures the maximum value of the wave height at every spatial position during the observation time [25, 26, 8].

This paper focuses on the propagation of wave group governed from the Benjamin-Bona-Mahony (BBM) equation. The solution will be presented as an asymptotic expansion up to the third order. It will be determined a solution as wave groups which represent bichromatic signal as the boundary condition at \( x = 0 \). The bichromatic signal is implemented to observe the wave deformation along the propagation. The extreme position and amplitude amplification will be investigated through the wave’s MTA.

2. Third Order Theory of BBM Model

Benjamin-Bona-Mahony (BBM) equation is known as the regularized long wave equation. It was proposed as the a revised model of KdV equation, at which BBM can better explain the physical interpretation of the propagating wave. The model is given by the following normalized equation as what stated in [27],

\[
\eta_t + \eta_x + \eta \eta_x - \eta_{xxt} = 0, \tag{1}
\]

where \( \eta \) is elevation, \( x \) and \( t \) are respectively space and time variables.

The solution of (1) will be developed in to the asymptotic expansion up to the third order:

\[
\eta(x, t) = \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \varepsilon^3 \eta^{(3)} + O(\varepsilon^4) \tag{2}
\]
To anticipate the secular terms which may occur, we expand the wave number into the asymptotic series based on Linstedt-Poincare method:

\[ k_p = k_p^{(0)} + \varepsilon k_p^{(1)} + \varepsilon^2 k_p^{(2)} + O(\varepsilon^3) \]  

for each \( p = 1, 2 \). By substituting (2) and (3) into (1), we obtain an equation equivalent to the linear form of equation (1) for the first order equation \( O(\varepsilon) \). For this linear equation, we choose an ansatz

\[ \eta^{(1)}(x, t) = \sum_{p=1}^{2} a_p e^{i\theta_p} + c.c., \]

where \( a_p \) represents amplitude, \( \theta_p = \omega_p t - k_p x + \psi_p \) for \( i = 1, 2 \), \( \omega_p \) is frequency, \( k_p \) is wavenumber, \( \psi_p \) is phase shift, and \( c.c. \) is the complex conjugate. Equation (4) expresses bichromatic signal. Substituting the ansatz into the linear equation of \( O(\varepsilon) \) yields the dispersion relation

\[ \Omega(k_p^{(0)}) = \omega_p = \frac{k_p^{(0)}}{1 + (k_p^{(0)})^2}, \quad p = 1, 2. \]

On the second order \( O(\varepsilon^2) \), there appears a resonant term which yields the same solution as the homogeneous solution of the equation (1). To eliminate this term, we set \( k_p^{(1)} = 0 \). Thus, the solution for \( O(\varepsilon^2) \) is

\[ \eta^{(2)} = \sum_{p=1}^{2} \sum_{q=1}^{2} a_p a_q (A_{+pq} e^{i(\theta_p + \theta_q)} + A_{-pq} e^{i(\theta_p - \theta_q)}) + c.c. \]

where

\[ A_{+pq} = \frac{k_q^{(0)}}{(\omega_p + \omega_q) - (k_p^{(0)} + k_q^{(0)}) + (k_p^{(0)} + k_q^{(0)})^2 (\omega_p + \omega_q)}, \]

\[ A_{-pq} = \begin{cases} \frac{-k_q^{(0)}}{(\omega_p - \omega_q) - (k_p^{(0)} - k_q^{(0)}) + (k_p^{(0)} - k_q^{(0)})^2 (\omega_p - \omega_q)}, & p \neq q \\ 0, & p = q \end{cases} \]

\( p, q \in \{1, 2\} \). On the equation of \( O(\varepsilon^3) \), the value of \( k_p^{(2)} \) needs to be assigned to treat another resonant term. The value of the correction wave number reads

\[ k_p^{(2)} = \frac{-\left( a_p^2 k_p^{(0)} A_{+pp} + 2a_p^2 k_p^{(0)} (2A_{+,pt} + A_{-,pt} + A_{-,tp}) \right)}{1 + 2k_p^{(0)} \Omega_p}, \]

where \( t = (p \mod 2) + 1 \), for \( p = 1, 2 \). Hence, the equation of \( O(\varepsilon^3) \) gives a solution

\[ \eta^{(3)} = \sum_{p=1}^{2} \sum_{q=1}^{2} \sum_{r=1}^{2} a_p a_q a_r B_{+} e^{i(\theta_p + \theta_q + \theta_r)} + \sum_{p=0}^{2} \sum_{q=0}^{2} \sum_{r=0}^{2} a_p a_q a_r B_{-} e^{i(\theta_p + \theta_q - \theta_r)} + c.c. \]

where

\[ B_{+} = \frac{C_{+} A_{+pq}}{D_{+}}, \quad B_{-} = \frac{C_{-} (A_{+pq} + A_{+,pr} + A_{+,rq})}{D_{-}}, \quad C_{\pm} = \left( k_p^{(0)} + k_q^{(0)} \pm k_r^{(0)} \right), \]

\[ D_{\pm} = (\omega_p + \omega_q \pm \omega_r) - (k_p^{(0)} + k_q^{(0)} \pm k_r^{(0)}) + (k_p^{(0)} + k_q^{(0)} \pm k_r^{(0)})^2 (\omega_p + \omega_q \pm \omega_r). \]
Furthermore, in order to acquire the bichromatic signal at \( x = 0 \), the second and third order solution needs to be compensated by their free waves. Free waves for second and third order side band are given by

\[
\eta^{(2)}_{fw} = \sum_{p=1}^{2} \sum_{q=1}^{2} a_p a_q (A_p e^{i\varphi(\theta_p + \theta_q)} + A_- e^{i\varphi(\theta_p - \theta_q)}) + \text{c.c.} \\
\eta^{(3)}_{fw} = \sum_{p=1}^{2} \sum_{q=1}^{2} a_p a_q a_r B_p e^{i\varphi(\theta_p + \theta_q + \theta_r)} + \sum_{p=q}^{2} \sum_{r=1}^{2} a_p a_q a_r B_- e^{i\varphi(\theta_p + \theta_q - \theta_r)} + \text{c.c.}
\]

where \( \varphi(\theta_p) = \omega_p t - \Omega^{-1}(\omega_p)x + \psi_p \). Therefore, the final solution is

\[
\eta(x,t) \approx \eta^{(1)} + \eta^{(2)} - \eta^{(2)}_{fw} + \eta^{(3)} - \eta^{(3)}_{fw}
\]

3. Extreme Position Of Bichromatic Wave Signal

In this section, the extreme position of the initially bichromatic signal will be determined. The maximum amplitude’s position can be analyzed through the wave envelope. Assuming \( a_1 = a_2 = a \), equation (4) can be rewritten as

\[
\eta^{(1)} = q(\cos \theta_1 + \cos \theta_2)
= 2q \cos(\nu t - \kappa x + \Psi) \cos(\omega_t - k t + \vartheta),
\]

where \( q = 2a, \nu = \frac{1}{2}(\omega_1 - \omega_2), \kappa = \frac{1}{2}(k_1 - k_2), \Psi = \frac{1}{2}(\psi_1 - \psi_2), \vartheta = \frac{1}{2}(\omega_1 + \omega_2), k = \frac{1}{2}(k_1 + k_2), \) and \( \vartheta = \frac{1}{2}(\psi_1 + \psi_2) \). Here we assume that \( \omega_1 > \omega_2 \). The value of \( k_p \) is in the form of expansion (3) and \( k_p^{(0)} = \Omega^{-1}(\omega_p) \) for \( p = 1, 2 \). Hence, equation (6) and (9) are equivalent to

\[
\eta^{(2)} = 2a^2 (A_{+,11} \cos(2\theta_1) + A_{+,22} \cos(2\theta_2) + (A_{+,12} + A_{+,21}) \cos(\theta_1 + \theta_2))
+ (A_{-,12} + A_{-,21}) \cos(\theta_1 + \theta_2)),
\]

\[
\eta^{(2)}_{fw} = 2a^2 (A_{+,11} \cos(\varphi(2\theta_1)) + A_{+,22} \cos(\varphi(2\theta_2)) + (A_{+,12} + A_{+,21}) \cos(\varphi(\theta_1 + \theta_2))
+ (A_{-,12} + A_{-,21}) \cos(\varphi(\theta_1 + \theta_2))).
\]

For the third order solution, the equation (8) becomes

\[
\eta^{(3)} = 2a^3 (B_{+,111} \cos(3\theta_1) + B_{+,222} \cos(3\theta_2) + (B_{+,112} + B_{+,121} + B_{+,211}) \cos(2\theta_1 + \theta_2)
+ (B_{+,122} + B_{+,212} + B_{+,221}) \cos(\theta_1 + 2\theta_2) + B_{-,112} \cos(\theta_1 + \Delta \theta) + B_{-,221} \cos(\theta_2 - \Delta \theta)),
\]

where \( \Delta \theta = \theta_1 - \theta_2 \).

Equation (14) is waves consisting frequencies whose value are far from the frequencies of bichromatic waves (4), so are the first four terms in the equation (14). These waves will not be considered since they will not affect the shift of the wave train. Meanwhile, the last two terms of equation (14) have frequencies which are close to the frequencies of bichromatic waves (4). These waves will interact with the bichromatic waves creating a wave train and causing deformation along the propagation. These waves will be denoted as side band waves. The third order side band wave and its free wave are stated as follow,

\[
\eta^{(3)}_{sb} = 2a^3 (B_{-,112} \cos(\theta_1 + \Delta \theta) + B_{-,221} \cos(\theta_2 - \Delta \theta))
\eta^{(3)}_{sb,fw} = 2a^3 (B_{-,112} \cos(\varphi(\theta_1 + \Delta \theta)) + B_{-,221} \cos(\varphi(\theta_2 - \Delta \theta)))
\]
where $\Delta \theta = \theta_1 - \theta_2$

The bichromatic waves $\eta^{(1)}$ and third order side band $\eta^{(3)}_{sb}$ form wave groups which have envelopes with wave number $\kappa$. These waves will have the same maximum amplitude’s position. Therefore, the extreme position may be determined through the superposition of these waves with the side band free wave $\eta^{(3)}_{sb,fw}$. While the envelope of side band $\eta^{(3)}_{sb,fw}$ has wave number $K = \frac{1}{2}(\Omega^{-1}(\omega_1 + \Delta \omega) - \Omega^{-1}(\omega_2 - \Delta \omega))$ where $\Delta \omega = \omega_1 - \omega_2$, the wave length of the envelope is $\lambda = 2\pi/(|\kappa - K|/2)$. Thus, the position of the amplitude of the wave train is $x_{max} = 2\pi/|\kappa - K|$. However, this value only applicable when $\lambda$ is the wave length of the MTA.

4. Maximal Temporal Amplitude and Amplitude Amplification

In this section, the evolution of the wave train will be observed. First, all the variables and parameters need to be transformed to the physical measure. The BBM equation presented in (1) is in nondimensional form with the following scaling factors for the variables as well as the corresponding frequency and wave number:

$$\eta = \frac{3}{2} \eta_{lab}, \quad x = \sqrt{6} x_{lab}, \quad t = \sqrt{\frac{6g}{h}} t_{lab}, \quad \omega = \frac{1}{6} \sqrt{\frac{6h}{g}} \omega_{lab}, \quad k = \sqrt{\frac{6}{6}} k_{lab} h.$$

The extreme position and amplitude amplification will be observed through the Maximal Temporal Amplitude (MTA). MTA is defined as [25]

$$MTA(x) = \max_t \eta(x, t).$$

We consider a water tank with water depth 5 m and gravitational acceleration 9.8 m/s^2. Taking $q = 0.4$ m gives the initial amplitude of the bichromatic signal at 0.8 m, $\omega_{1,lab} = 0.57$ rad/s, $\omega_{2,lab} = 0.53$ rad/s gives corresponding wave numbers $k_{1,lab}^{(0)} = 5.85$ rad/m, $k_{2,lab}^{(0)} = 6.31$ rad/m, and $\psi_1 = \psi_2 = 0$. Hence, the MTA is presented in figure 1.

![Figure 1. The MTA plot of the bichromatic wave with $q_{lab} = 0.4$ m, $\omega_{1,lab} = 0.57$ rad/s, $\omega_{2,lab} = 0.53$ rad/s and $\nu = 0.02$ rad/s.](image)

Figure 1 illustrates that the wave groups experience amplification on its amplitude during the propagation. The wave initially having the amplitude of bichromatic signal 0.8 m at $x = 0$ keeps increasing in height until it reach extreme position at about $x = 99.5$ m with wave height $\eta_{max} = 1.24$ m. It gives the factor of amplification as much as 1.55. Figure 2 describes the wave
Figure 2. Wave signals on time domain at (a) $x = 0$ m (b) $x = 50$ m (c) $x = 99.5$ m (d) $x = 130$ m for parameter values $q_{lab} = 0.4$ m, $\omega_{1,lab} = 0.57$ rad/s, $\omega_{2,lab} = 0.53$ rad/s.

Figure 3. Amplitude spectrum of the wave signals at (a) the initial position $x = 0$ m and (b) the extreme position $x = 99.5$ m for parameter values $q_{lab} = 0.4$ m, $\omega_{1,lab} = 0.57$ rad/s, $\omega_{2,lab} = 0.53$ rad/s.

signals at some position including the extreme position. It can be seen that at $x = 0$ m the signal is in bichromatic wave form. The wave groups undergo deformation and energy focusing which cause the peaking. The amplitude gradually increases until it achieves the maximum amplitude at $x = 107$ m and then it decreases before it starts to increase again. This peaking phenomenon occurs periodically. The signal’s spectrum at the initial position $x = 0$ m and the extreme position $x = 107$ m are displayed in figure 3. At the initial position, the signal is formed by two linear signals with slightly different frequencies $\Delta \omega = 0.04$ rad/s and the same amplitude. Furthermore, at the extreme position, a pair of side bands with smaller amplitudes have appeared. The side bands are contributed by the third order solutions with supporting waves’ frequencies $\omega_1 + \Delta \omega = 0.61$ rad/s and $\omega_2 - \Delta \omega = 0.49$ rad/s. However, although the wave train deforms and peaks, its amplitude amplification factor is not enough to satisfy the definition of an extreme wave as stated in [1].

The maximum position and peaking are affected by the value of frequency difference and the initial amplitude. Table 1 presents the maximum position where the wave achieves the maximum peaking and the amplitude amplification factor (AAF) for some values of frequency difference. The closer the both frequencies, the higher the AAF, but the farther the maximum position. This fact however can be a problem in generating an extreme wave with the highest amplification in a wave tank because the extreme position may exceed the tank’s spatial limitation. Meanwhile, table 2 shows the AAF for some values of initial amplitude. Higher initial amplitude may trigger
higher amplitude amplification and vice versa while the amplitude change does not significantly affect the maximum position.

**Table 1.** Maximum position and AAF for some values of frequency difference with \( q_{lab} = 0.4 \, \text{m} \), \( \psi_1 = \psi_2 = 0 \).

| \( \omega_{1,lab} \) (rad/s) | \( \omega_{2,lab} \) (rad/s) | \( x_{max} \) (m) | \( \eta_{max} \) (m) | AAF  |
|--------------------------|--------------------------|----------------|----------------|-----|
| 0.57                     | 0.51                     | 40             | 0.98           | 1.22|
| 0.57                     | 0.52                     | 60.5           | 1.07           | 1.34|
| 0.57                     | 0.54                     | 195.5          | 1.61           | 2.01|
| 0.57                     | 0.55                     | 518.8          | 2.66           | 3.33|

**Table 2.** AAF for some values of \( q_{lab} \) with \( \omega_{1,lab} = 0.57 \, \text{rad/s}, \omega_{2,lab} = 0.53 \, \text{rad/s} \), \( \psi_1 = \psi_2 = 0 \).

| \( q_{lab} \) (m) | \( x_{max} \) (m) | \( \eta_{max} \) (m) | AAF  |
|----------------|----------------|----------------|-----|
| 0.32           | 100            | 0.87           | 1.35|
| 0.36           | 101            | 1.04           | 1.45|
| 0.46           | 103            | 1.59           | 1.73|
| 0.5            | 104            | 1.86           | 1.86|

**5. Conclusion**

We have derived a solution of BBM equation on the basis of bichromatic wave group to investigate the occurrence of extreme wave. The initially bichromatic signal experiences focusing phenomena which results a peaking on its amplitude. The extreme position of the the maximum peaking can be determined analytically for a certain case. However, the amplification factor and the extreme position can be observed by using the MTA. The experimental results of the MTA suggest that the frequency difference value will affect the maximum position and the amplitude amplification. The closer the frequencies of the bichromatic waves, the higher the amplitude amplification, yet the farther the maximum position. Meanwhile, the initial amplitude change does not cause significant shift in the maximum position but it does affect the amplification factor. Higher amplitude may trigger higher AAF and vice versa.

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**References**

[1] Dean R G 1990 *Freak Waves: A Possible Explanation, Water Wave Kinetics* (Amsterdam: Kluwer) pp. 609-21

[2] Divinsky B V, Levin B V, Lopatikin L I, Pelinovsky E N and Slyungaev A V 2004 A freak wave in the Black Sea, observations and simulation *Oklady Earth Sci.* **395** 438-43

[3] Waseda T, Tamura H and Kinoshita T 2012 Freakish sea index and sea states during ship accidents *J. Marine Sci. Tech.* **17** 305-14

[4] Waseda T, Sinchi M, Kiyomatsu K, Nishida T, Takahashi S, Asaumi S, Kawai Y, Tamura H and Miyazawa Y 2014 Deep water observations of extreme waves with moored and free GPS buoys *Ocean Dyn.* **64** 1269-80
[5] Peric R, Hoffmann N and Chabchoub A 2015 Initial wave breaking dynamics of Peregrine-type rogue waves: a numerical and experimental study Eur. J. Mech. B Fluids 49 71-6
[6] Westhuis J, van Groesen E and Huijsmans R H M 2001 Numerics and numeric of bichromatic wave group J. Waterway Port Coast. Ocean Eng. 127 334-42
[7] Ramli M 2015 Nonlinear evolution of wave group with three frequencies Far East J. Math. Sci. 97 (8) 925-37
[8] Ramli M 2016 Amplitude amplification factor of bichromatic waves propagation in hydrodynamic laboratory IAENG Int. J. App. Math. 46 (1) 29-34
[9] Marwan and Andonowati 2003 Wave deformation on the propagation of bichromatic signal and its effect to the maximum amplitude J. Math. Sci. 8 81-7
[10] Hu Z, Tang W, Xu H and Zhang X 2015 Numerical study of rogue waves as nonlinear Schrödinger breather solutions under finite water depth Wave Motion 52 81-90
[11] Shunyaev A, Sergeeva A and Pelinovsky E 2015 Wave amplification in the framework of forced nonlinear Schrödinger equation: the rogue wave context Phys. D 303 18-27
[12] Onorato M, Residori S, Bortolozzo U, Montina A and Arecchi F T 2013 Rogue waves and their generating mechanisms in different physical contexts Phys. Rep. 528 47-89
[13] Halfiani V, Salmawaty and Ramli M 2017 An Envelope Equation of Benjamin Bona Mahony Wave Group Far East J. Math Sci. 102 (6)
[14] El-Labany S K, Moslem W M, El-Bedwehy N A, Sabry R and Abd El-Razek H N 2012 Rogue Wave in Titan’s Atmosphere Astrophys. Space Sci. 338 3-8
[15] Boyd J P, and Guan Y C 2001 Weakly Nonlinear Wavepackets in The Korteweg-de Vries Equation: The KdV/NLS Connection Mathematics and Computers in Simulation 55 (4-6) 317-328
[16] Bruckner M C, Dull W P and Schneider G 2014 Validity of The KdV Equation for The Modulation of Periodic Traveling Waves in The NLS Equation J. Math. Anal. Appl. 414 166-175
[17] Schneider G 2011 Justification of the NLS Approximation for the KdV Equation Using the Miura Transformation Advances in Mathematical Physics Article ID 854719: 4 pages
[18] El-Wakil S A, Abulwafa E M, Elhanbaly A and El-Shewy E K 2014 Rogue Waves for Kadomstev-Petviashvili Equation in Electron-Positron-Ion Plasma Astrophys. Space Sci. 353 501-506
[19] Akhmediev N N and Ankiewicz A 1997 Solitons: Nonlinear Pulses and Beams (London : Chapman and Hall)
[20] Bacha M, Boukhalfa S and Tribeche M 2012 Ion-Acoustic Rogue Waves in a Plasma with a q-Nonextensive Electron Velocity Distribution Astrophys. Space Sci. 341 591-595
[21] Rahman A U and Ali S 2014 Solitary and Rogue Waves in Fermi-Dirac Plasmas: Relativistic Degeneracy Effects Astrophys. Space Sci. 351 165-172
[22] Ramli M 2009 The deterministic generation of extreme surface water waves based on soliton on finite background in laboratory Int. J. Engineer. 22 (3) 243-49
[23] Karjanto N, van Groesen E and Peterson P 2002 Investigation of the maximum amplitude increase from the Benjamin-Feir Instability Journal of the Indonesian Mathematical Society 8 (4) 39-47
[24] Stansberg C 1997 On the nonlinear behavior of ocean wave groups Proc. WAVES 97 2 1227-41
[25] Marwan 2010 On The Maximal Temporal Amplitude of down stream running nonlinear water waves Tamkang J. Math. 41 (1) 51-69
[26] Ramli M, Munzir S, Khairuman T and Halfiani V 2014 Amplitude increasing formula of bichromatic wave propagation based on fifth order side band solution of Korteweg de Vries equation Far East J. Math Sci. 90 (1) 97-117
[27] Benjamin T B, Bona J L and Mahony J J 1972 Model equations for long waves in nonlinear dispersive systems Philos. Trans. Roy. Soc. London, Ser. A 272 47-78