Abstract

A word labeled oriented graph (WLOG) is an oriented graph $\mathcal{G}$ on vertices $X = \{x_1, \ldots, x_k\}$, where each oriented edge is labeled by a word in $X^{\pm 1}$. WLOGs give rise to presentations which generalize Wirtinger presentations of knots. WLOG presentations, where the underlying graph is a tree are of central importance in view of Whitehead’s Asphericity Conjecture. We present a class of aspherical word labeled oriented graphs. This class can be used to produce highly non-injective aspherical labeled oriented trees and also aspherical cyclically presented groups.

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1 Introduction

A word labeled oriented graph (WLOG) is an oriented graph $\mathcal{G}$ on vertices $X = \{x_1, \ldots, x_k\}$, where each oriented edge is labeled by a non-empty word in $X^{\pm 1}$. Associated with a word labeled oriented graph comes a WLOG-presentation $P(\mathcal{G})$ on generators $X$ and relators in one-to-one correspondence with edges in $\mathcal{G}$. For an edge with initial vertex $x_i$, terminal vertex $x_j$, labeled $w$, we have a relation $x_iw = wx_j$. A WLOG-complex $K(\mathcal{G})$ is the standard 2-complex associated with the WLOG-presentation $P(\mathcal{G})$, and a WLOG-group $G(\mathcal{G})$ is the group defined by the WLOG-presentation. We say a word labeled oriented graph is aspherical if its associated WLOG-complex is aspherical. A word labeled oriented graph is called injective if no edge label or its inverse is a subword of any other edge label. A word labeled oriented tree (WLOT) is a word labeled oriented graph where the underlying graph is a tree.

If every edge label of a word labeled oriented graph consists of a single letter from $X$, we simply speak of a labeled oriented graph (LOG). LOG presentations generalize Wirtinger presentations of knots. Labeled oriented trees (LOTs) are of central importance in view of Whitehead’s Asphericity Conjecture which states that a subcomplex of an aspherical 2-complex is aspherical. LOT presentations which are
Writinger presentations of knots are known to be aspherical by a classical result of Papakyriakopoulos [8]. See Bogley [1] and Rosebrock [9] for surveys on the Whitehead Conjecture. Recently the authors have shown that injective labeled oriented trees are aspherical [4]. Here we show that certain injective word labeled oriented graphs are aspherical (in fact, diagrammatically reducible, a strong combinatorial version of asphericity). We should note that a word labeled oriented graph $G'$ can be subdivided to produce a labeled oriented graph $G$, and $K(G)$ is a subdivision of $K(G')$. See Example 2.2 and [5] for details. In particular, if $G$ is aspherical then so is $G'$. The labeled oriented graph $G'$ is typically highly non-injective, even if one starts with an injective WLOG $G$. Thus our result also gives access to wide classes of non-injective aspherical labeled oriented graphs and trees.

Word labeled oriented trees have appeared before in work of Howie [5], where they were called weakly labeled oriented trees.

Another motivation for this paper comes from the interest in cyclically presented groups. Let $F$ be the free group on generators $\{x_1, \ldots, x_n\}$. Let $\phi$ be the automorphism on $F$ defined by $\phi(x_i) = x_{i+1}$ in case $1 \leq i \leq n - 1$, and $\phi(x_n) = x_1$. According to Howie and Williams [6] for $w \in F$

$$P(n, w) = \langle x_1, \ldots, x_n \mid w, \phi(w), \ldots, \phi^{n-1}(w) \rangle$$

is called a cyclic presentation and the group $G(n, w)$ it defines is called a cyclically presented group. The automorphism $\phi$ induces a shift automorphism of $G(n, w)$. There is a connection between asphericity and the dynamics of the shift automorphism, see Bogley [2]. We present several classes of cyclically presented groups which arise from word labeled oriented graphs and are aspherical by our main result Theorem 2.1.

\section{Main Theorem}

Let $K$ be the standard 2-complex associated with a finite group presentation. The complex $K$ is called diagrammatically reducible (DR) if there does not exist a reduced spherical diagram $f : C \rightarrow K$. See Gersten [3] for definitions and details. The property DR implies asphericity.

Combinatorial curvature plays an important role in the study of asphericity of 2-complexes. Given a closed surface $S$ with a cell structure, one can assign real numbers $\omega(c)$ to the corners $c$ of the 2-cells of $S$, thought of as angles. The curvature $\kappa(v)$ at a vertex is defined as $2 - \sum \omega(w)$, where the sum is taken over the corners at $v$. The curvature $\kappa(d)$ of a 2-cell $d$ is defined as $\sum \omega(c) - (|\partial d| - 2)$, where the sum is taken over all the corners of $d$ and $|\partial d|$ denotes the number of edges in the boundary of $d$. The combinatorial Gauss-Bonnet Theorem asserts that summing up the curvature at all the vertices and 2-cells yields twice the Euler characteristic of the surface $S$. The combinatorial Gauss-Bonnet theorem is the basis for asphericity tests such as Gersten’s weight test (see [3]). More details on combinatorial curvature can be found in Section 14 of McCammond’s survey [7]. The remaining of this section is devoted to the proof of our main result.


Theorem 2.1 Let $P(G) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle$ be a WLOG-presentation coming from an injective word labeled oriented graph $G$. Then each relation $r_i$ is of the form $x_{\alpha(i)}w_i x_{\beta(i)}$ where $w_i$ is a word $w_i = t_{i,1} \ldots t_{i,s_i}$ with $t_{i,j} \in \{x_1, \ldots, x_n\}^{\pm 1}$. We assume $s_i \geq 2$ for $i = 1, \ldots, k$. We further assume that

1. Each relator is cyclically reduced.
2. For each relator $r_i$ the words $x_{\alpha(i)}t_{i,1}^{-1}x_{\alpha(i)}, x_{\beta(i)}t_{i,s_i}^{-1}, t_{i,s_i}x_{\beta(i)}$ are not pieces (i.e. these words and their inverses are not subwords in another relator or in the same relator at another place).
3. No word $w_i$ has the form $x_{\pm m}^{\pm 1}$ for $m \geq 2$.

Then $K(G)$ is DR.

Proof: Let $K = K(G)$. Assume there exists a reduced spherical diagram $f : C \to K$. We will assign weights to the corners of the cell decomposition of $C$ such that the curvature at each vertex and at each 2-cell is non-positive. Thus the total curvature of $C$ is non-positive, contradicting the fact that $C$ is a sphere with total curvature 4, according to Gauss-Bonnet.

Assume $d$ is a 2-cell in $C$ with boundary word $x_iw_qx_j^{-1}w_q^{-1}$. The long sides of $d$ are the sides labeled by $w_q$, the remaining two sides labeled $x_i$ and $x_j$ are called short sides of $d$. The first and last vertices of the long sides of $d$ are called extremal vertices of $d$. Note that every 2-cell in $C$ has exactly four extremal vertices. The other vertices are referred to as non-extremal. If $d_1$ and $d_2$ are 2-cells in $C$ that share a long side, that is the intersection $d_1 \cap d_2$ is a long side for both $d_1$ and $d_2$, then we call $d_1 \cup d_2$ a stacked pair, and the long side in the intersection a stack line. Note that because $G$ is injective and $f : C \to K$ is reduced, if $d_1 \cup d_2$ is a stacked pair, then $d_1$ and $d_2$ are mapped to the same 2-cell in $K$ under $f$, with the same orientations. A stacked pair is shown in Figure 1.

Consider a long side $S$ of a 2-cell $d$ in $C$. We will assign weights to the corners of $d$ along $S$ so that their sum is $\leq |\partial d|/2 - 1$.

Case I: All non-extremal vertices on $S$ have valency 2.

In this case the long side $S$ is a stack line. Assign to the first and the last corner of $S$ weight $1/2$ and assign weight 1 to all other corners (see Figure II). Note that these weights sum up to $|\partial d|/2 - 1$. 

![Figure 1. Weights in case I. The stack line is shown in blue.](image-url)
Case II: There is a non-extremal vertex of $S$ of valency greater or equal to 3.

Assign weight $2/3$ to the corners at the first and last vertex of $S$ in $d$. Assign weight 1 to corners at vertices of valency 2 on $S$. Let $v$ be a non-extremal vertex on $S$ of valency $\geq 3$. If $v$ is the endpoint of a stack line then assign weight 1 to the corner in $d$ at $v$. If $v$ is not the endpoint of a stack line assign weight $2/3$ to the corner in $d$ at $v$.

![Figure 2. Weights in case II. The stack line is shown in blue.](image)

Note that there are at least 3 corners in $d$ along $S$ of weight $2/3$. There are already two weights $2/3$ at the first and last corner of $d$ along $S$. If there were no other corners with weight $2/3$ then all non-extremal vertices along $S$ would have to be the endpoints of stack lines. But then $S$ would be labeled $x_k^{\pm m}$, for some $x_k$ and some $m \geq 2$, contradicting condition 3 in the theorem. Thus, the sum of the weights along a long side is $\leq |\partial d|/2 - 1$.

Consider a 2-cell $d$ in $C$. Using the above process we have assigned weights at the corners along both long sides of $d$. Since the sum of the weights along a long side of $d$ is $\leq |\partial d|/2 - 1$, the sum of the weights of all corners of $d$ is $\leq |\partial d| - 2$, and hence $\kappa(d) \leq 0$.

It remains to check the curvature at the vertices of $C$. Because of condition 1 all vertices in $C$ have valency at least two. Because of condition 2 an extremal vertex has valency at least three. Since the smallest weight assigned is $1/2$ we have $\kappa(v) \leq 0$ for a vertex of valency greater or equal to four. Thus we only need to worry about vertices of valency two and three. The two corners at a vertex $v$ of valency two both have weight 1 which sums up to 2. So $\kappa(v) = 0$.

Now suppose that $v$ is a vertex of valency three. Let $d_1$, $d_2$ and $d_3$ be the three 2-cells in $C$ that share the vertex $v$ and let $c_i$ be the corner at $v$ in $d_i$.

If $v$ is not the end of a stack line, then the weights assigned to the corners $c_i$ are all $2/3$. This is because weight $1/2$ is only assigned to corners at the end of stack lines and weight 1 is assigned only to corners at the the end of stack lines or at corners at vertices of valency 2. Thus we have $\kappa(v) = 0$. 

Next suppose $v$ is the endpoint of a stack line $L$ and assume without loss of generality that $L = d_1 \cap d_2$. Note that $d_1 \cap d_3$ contains a short side of $d_1$ and $d_2 \cap d_3$ contains a short side of $d_2$. Thus $L$ is the only stack line with end vertex $v$. We next argue that $v$ can not be an extremal vertex of $d_3$. Assume this would be the case. The situation is depicted in Figure 3. The stacked 2-cells $d_1$ and $d_2$ have to have the same labels on their short sides, so $t = x_j$ and the boundary word of $d$ is not reduced, contradicting the fact that each relator in $P(G)$ is cyclically reduced.

It follows that the vertex $v$ is a non-extremal vertex of $d_3$ and hence situation is exactly as shown in Figure 2. So the weights at $v$ are $1/2, 1/2$, and $1$. Thus we have $\kappa(v) = 0$. □

**Example 2.2** The WLOT-presentation

\[ P(G) = \langle a, b, c, d \mid a(da) = (da)b, b(db) = (db)c, c(ac) = (ac)d \rangle \]

satisfies the hypothesis of Theorem 2.1 Hence $K(G)$ is DR. Note that $G$ is an interval with four vertices, each edge label has length two. Subdividing $G$ into a labeled oriented interval on seven vertices, introducing a new vertex at every edge midpoint, yields a labeled oriented interval $G'$ with the presentation

\[ P(G') = \langle a, x, b, y, c, z, d \mid ad = dx, xa = ab, bd = dy, yb = bc, ca = az, zc = cd \rangle. \]

Since $K(G')$ is obtained from $K(G)$ by subdivision and subdividing preserves DR (see [3], 6.10), the complex $K(G')$ is also DR. Note that $G'$ is not injective.
3 Cyclically presented groups and further examples

We study the cyclic presentations
\[ C(n, w) = P(n, x_1w^{-1}x_2^{-1}) = \]
\[ = \langle x_1, \ldots, x_n \mid x_1w = wx_2, x_2\phi(w) = \phi(w)x_3, \ldots, x_n\phi^{n-1}(w) = \phi^{n-1}(w)x_1 \rangle \]
where \( \phi \) is the shift automorphism. Note that \( C(n, w) \) is a WLOG-presentation where the underlying graph is a circle on \( n \) vertices.

**Theorem 3.1** Let \( w = x_j^t x_k^m \) where \( t \neq 0 \) and \( m \neq 0 \) are integers and \( j \neq k \) are integers mod \( n \). Assume \( j \neq 1, k \neq 2, 2j - 1 \neq k, j \neq k - 1, j \neq 2k - 2, j + k \neq 3 \). Then \( K(C(n, w)) \) is DR.

**Proof:** We apply Theorem 2.1 to \( C(n, w) \). The word labeled oriented circle that defines the WLOG-presentation \( C(n, w) \) is injective because all words \( w, \phi(w), \phi^2(w), \ldots, \phi^{n-1}(w) \) are different and of the same length. Note that the first relator \( x_1x_j^t x_k^m x_2^{-1}(x_j^tx_k^m)^{-1} \) is cyclically reduced because of \( j \neq 1, k \neq 2, j \neq k \). Since the other relators are obtained by shifting the subscripts on the first relator, all relators are cyclically reduced and hence condition 1 of the assumptions in Theorem 2.1 holds. Condition 3 follows from the fact that \( t \neq 0, m \neq 0 \), and \( j \neq k \). Let \( A = \{ x_1, x_j, x_j^{-1}, x_2, x_k, x_k^{-1} \} \), where \( \epsilon = 1 \) in case \( t > 0 \), \( \epsilon = -1 \) in case \( t < 0 \), and \( \tau = 1 \) in case \( m > 0 \), \( \tau = -1 \) in case \( m < 0 \). Note that an element \( a \in A \) being a piece would mean \( a \) or \( a^{-1} \) is contained in a shift of \( A \), or \( a \) or \( a^{-1} \) is equal to a shift of \( x_j^tx_k^m \). The assumption \( j \neq k - 1 \) implies \( j - 1 \neq k - 2 \), and \( j + k \neq 3 \) implies \( k - 2 \neq 1 - j \), which shows that \( a \) or \( a^{-1} \) is not a shift of an element from \( A \). Now \( j \neq 2k - 2 \) implies \( j - k \neq 2 - k \), thus \( (x_2)^{\pm 1} \) or \( (x_k)^{\pm 1} \) can not be a shift of \( x_j^tx_k^m \). Furthermore \( 2j - 1 \neq k \) implies \( j - 1 \neq k - j \), hence \( (x_1)^{\pm 1} \) or \( (x_j)^{\pm 1} \) can not be a shift of \( x_j^tx_k^m \). We have shown that no element from \( A \) or its inverse can be a piece. Hence no shift of an element from \( A \) or its inverse can be a piece and condition 2 is indeed satisfied. The result follows from Theorem 2.1. \( \square \)

**Example 3.2** The presentations \( C(n, x_3x_1^m), n \geq 5, m \neq 0 \) and \( t \neq 0 \), satisfy the hypothesis of Theorem 3.1 and are therefore DR. One can show that all presentations \( C(n, x_3x_1), n \geq 5 \), do not satisfy the weight test stated in 3. Thus the most obvious test for asphericity fails for these examples.

Theorem 3.1 is a result about word labeled oriented circles. Because of Whitehead’s Asphericity Conjecture we are interested in the asphericity of labeled oriented trees. We can omit one relator of \( C(n, w) \) and obtain a word labeled oriented interval presentation that is also DR as a subpresentation of a presentation that is DR.
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