A static scission configuration in cold ternary fission has been considered in the framework of two mean field approaches. The virial theorems have been suggested to investigate correlations in the phase space, starting from a kinetic equation. The inverse mean field method is applied to solve single-particle Schrödinger equation, instead of constrained selfconsistent Hartree–Fock equations. It is shown, that it is possible to simulate one-dimensional three-center system via inverse scattering method in the approximation of reflectless single-particle potentials.

1 Motivation

Ternary fission involving the emission of α-particle was first observed more than fifty years ago. Emission of α-particles in the spontaneous fission of 252\textsuperscript{Cf} has also a long history of investigation experimentally and theoretically. A renewed interest in these processes arose in connection with

- modern experimental techniques (\(\gamma-\gamma-\gamma\) and \(x-\gamma-\gamma\) triple coincidence, (Gammasphere with 110 Compton suppressed Ge detectors), which allow the fine resolution of the mass, charge and angular momentum content of the fragments.\)

- Recently direct experimental evidence was presented for the cold (neutronless) ternary spontaneous fission of 252\textsuperscript{Cf} in which the third particle is an α-particle, or \(^{10}\text{Be}\). This confirms that a large variety of nuclear large-amplitude collective motions such as bimodal fission, cold binary fission, heavy cluster radioactivity, and inverse processes, such as subbarrier fusion, could belong to the general phenomenon of cold nuclear fragmentation.

- Cluster like models were used successfully to reproduce general features of the cold ternary fragmentation. However the scission configuration has been built in fact by hands.

Therefore it is actual to develop microscopical or semi-microscopical approach to this scission-point concept of nuclear fragmentation. There are
well developed methods to calculate, in the framework of many-body self-consistent approach, static properties of a well isolated nucleus in its ground state. There also exists a well developed two-center shell model \(^1\). However, a three-center shell model has not been developed yet, except for very early steps \(^2\). Three-center shapes are practically not investigated, in comparison with the two-center ones. There exists the generalizations of mean-field models to the case of two-centers \(^3\), but a ternary configuration is out of consideration, because of uncertainties to select a peculiar set of constraints.

There exist a number of calculations for nucleus-nucleus collisions in the frame of time-dependent mean-field methods, but an evolution of the cold fragmentation has not been investigated yet. Therefore, although the principal way to describe nuclear fragmentation in the framework of many-body self-consistent approach exists, it is interesting to develop other mean-field approaches to analyse these phenomena from different points of view.

In this Contribution we suggest two methods to analyse a static scission configuration in cold ternary fission in the framework of mean field approach. In Section 2, starting from kinetic equation the virial theorems has been suggested to investigate correlations in the phase space. It gives possibility to formulate in future generalised set of constraints in momentum and cartesian spaces. In Section 3 the inverse mean field method is applied to solve single-particle Schrödinger equation, instead of constrained selfconsistent Hartree–Fock equations. It makes it possible to simulate one-dimensional three-center system via inverse scattering method.

2 Virial theorems

Within the mean-field approximation, we analyze the evolution of one-body Wigner phase-space distribution function of the full many-body wave function, following the well developed scheme using as the starting point the Vlasov equation for the Wigner phase-space distribution function \(^4\)

\[
\frac{\partial f}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial f}{\partial \vec{r}} - \frac{\partial V}{\partial \vec{r}} \cdot \frac{\partial f}{\partial \vec{p}} = I_{rel},
\]

with a "relaxation term" \(I_{rel}\) is added to the kinetic equation to describe dissipation effects. The quantity \(V(\vec{r}, t)\) is the self-consistent single-particle potential which is assumed here to be local, \(m\) is the mass of nucleon. Out of the Wigner distribution function, virials at different orders can help to extract useful physical information from the total phase space dynamics. Integrating the initial kinetic equation \(^5\) over the momentum space with different polynomial weighting functions of the \(\vec{p}\)-variable one comes, as well
to an infinite chain of equations for local collective observables including the density, collective velocity, pressure and an infinite set of tensorial functions of the time and space coordinates, which are defined as moments of the distribution function in the momentum space:

- the particle \( n(\vec{r}, t) \equiv \int d\vec{p} f(\vec{r}, \vec{p}, t) \), and the mass \( \rho(\vec{r}, t) = m \ n(\vec{r}, t) \) densities,

- the collective current and velocity of nuclear matter
  \( \rho(\vec{r}, t) \vec{u}(\vec{r}, t) = \int d\vec{p} \vec{p} f(\vec{r}, \vec{p}, t) \),

- the pressure tensor and the energy and momentum transfer tensors of different orders
  \( P_{ij}(\vec{r}, t) = \frac{1}{m} \int d\vec{p} q_i q_j f(\vec{r}, \vec{p}, t), \quad q_i \equiv p_i - m u_i, \)
  \( P_{ij..k}^{n}(\vec{r}, t) = \frac{1}{m^{n-1}} \int d\vec{p} q_i q_j..q_k f(\vec{r}, \vec{p}, t), \)

- and the integrals related to relaxation terms
  \( \int d\vec{p} I_{rel} = 0, \quad \int d\vec{p} \vec{p} I_{rel} = 0, \)
  \( \mathbb{R}_{ij} = \frac{1}{m} \int d\vec{p} q_i q_j I_{rel}, \quad \ldots \)

Truncating this chain at order two in \( \vec{q} \) one arrives at the "fluid dynamical" level of description of nuclear processes.

\[
\frac{\partial \rho}{\partial t} + \sum_k \frac{\partial}{\partial x_k} (u_k \rho) = 0,
\]

\[
\rho \frac{Du_i}{Dt} + \sum_k \frac{\partial P_{ik}}{\partial x_k} + \frac{\rho}{m} \frac{\partial V}{\partial x_i} + \rho (\Omega_i \sum_k \Omega_k x_k - \Omega^2 x_i) + \rho \sum_s \varepsilon_{isj} (2 \Omega_s u_j + \frac{d\Omega_s}{dt} x_j) = 0,
\]

\[
\frac{DP_{ij}}{Dt} + \sum_k \left( P_{ik} \frac{\partial u_j}{\partial x_k} + P_{jk} \frac{\partial u_i}{\partial x_k} + P_{ij} \frac{\partial u_k}{\partial x_k} \right)
\]
\[
+ 2 \sum_{s,m} \Omega_m (\varepsilon_{jms} P_{is} + \varepsilon_{ism} P_{js}) \\
+ \sum_k \frac{\partial}{\partial x_k} P_{ijk} = \left( \frac{\partial P_{ij}}{\partial t} \right)_{\text{rel}},
\]

where the usual notation \( \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \sum_k u_k \frac{\partial}{\partial x_k} \) is introduced for the operator giving the material derivative, or the rate of change at a point moving locally with the fluid. The hydrodynamical set of Eqs. (2-5) describes an evolution of a rotating nuclear system. We consider two frames of reference with a common origin: an inertial frame, \((X_1, X_2, X_3)\), and a moving frame, \((x_1, x_2, x_3)\). Let \(x_i = \sum_{j=1}^3 T_{ij} X_j\) be the linear transformation that relates the coordinates, \(\vec{X}\) and \(\vec{x}\), of a point in two frames. The orientation of the moving frame, with respect to the inertial frame, will be assumed to be time dependent. Since \(T_{ij}(t)\) must represent an orthogonal transformation, the vector

\[
\Omega_i = \frac{1}{2} \sum_{j,k,m} \varepsilon_{ijk} \left( \frac{dT}{dt} \right)_{jm} T_{mk}^+,
\]

represents a general time-dependent rotation of the \(\vec{x}\)-frame with respect to the inertial frame.

Let us define integral collective "observables" (the integrals over the whole phase space of one nucleon containing the distribution function appropriately weighted), namely an inertia tensor \(\mathbb{J}_{ij}(t)\), the dynamical part of the angular momentum \(L_i(t)\), the integral pressure tensor \(\Pi_{ij}(t)\) defined as

\[
\mathbb{J}_{ij} \equiv \int d\vec{x} x_i x_j \rho, \quad \Pi_{ij} \equiv \int d\vec{x} P_{ij}, \\
L_k \equiv \sum_{i,j} \varepsilon_{kij} \int d\vec{x} \rho x_i u_j,
\]

The dynamics in terms of the latter "observables" is expressed by a set of virial equations in the rotating frame

\[
\frac{d^2}{dt^2} \mathbb{J}_{ij} + \sum_k \Omega_k (\Omega_j \mathbb{J}_{jk} + \Omega_j \mathbb{J}_{ik}) - 2\Omega^2 \mathbb{J}_{ij} \\
+ 2 \sum_{s,k} \Omega_s \int d\vec{r} \rho u_k (\varepsilon_{isk} x_j + \varepsilon_{jsk} x_i)
\]
\[ + 2\mathcal{V}_{ij} - 2\mathbb{K}_{ij} - 2\Pi_{ij} + \sum_{s,k} \frac{d\Omega_s}{dt} (\varepsilon_{isk}\mathbb{J}_{kj} + \varepsilon_{jsk}\mathbb{J}_{ki}) = 0, \]

\[
\frac{dL_k}{dt} + \sum_{i,j,m} \varepsilon_{kji} \Omega_i \Omega_m \mathbb{J}_{jm} - 2 \sum_s \Omega_s \int d\vec{r} \rho u_k x_s
\]

\[
- \sum_s \frac{d\Omega_s}{dt} \mathbb{J}_{ks} + \frac{d}{dt} (\Omega_k \sum_j \mathbb{J}_{jj}) = 0, \]

\[
\frac{d}{dt} \Pi_{ij} + \mathbb{F}_{ij} + 2 \sum_{s,k} \Omega_s (\varepsilon_{isk} \Pi_{kj} + \varepsilon_{jsk} \Pi_{ki}) = \mathbb{R}_{ij}, \]

\[
\mathbb{F}_{ij} \equiv \sum_k \int d\vec{r} \left( P_{ik} \frac{\partial u_j}{\partial x_k} + P_{jk} \frac{\partial u_i}{\partial x_k} \right), \]

where the tensors of collective kinetic and potential energies, and the relaxation tensor are

\[
\mathbb{K}_{ij} = \int d\vec{x} u_i u_j \rho, \quad \mathcal{V}_{ij} = \int d\vec{x} x_j \frac{\partial V}{\partial x_i} n,
\]

\[
\mathbb{R}_{ij} \equiv \int d\vec{x} \left( \frac{\partial\mathbb{F}_{ij}}{\partial t} \right)_{\text{rel}} x_i x_j.
\]

The above equations constitute a formal framework within which the coupling of the deformations in the \(\vec{r}\)-space and in the \(\vec{p}\)-space can be explicitly worked out. It may help to formulate in a future a generalised set of constrains in the phase space.

3 Inverse mean field method

3.1 The framework

Methods of nonlinear dynamics gave yet the possibility to derive for nuclear physics unexpected collective modes, which can not be obtained by traditional methods of perturbation theory, near some equilibrium state (see e.g. review \cite{27} and \cite{28} for the recent refs.). The most important reason is that the fragmentation and clusterization is a very general phenomenon. There are cluster objects in subnuclear and macro physics. Very different theoretical methods were developed in these fields. However, there are only few basic physical ideas, and most of the methods deal with nonlinear partial differential equations. One of the most important part of soliton theory is the inverse scattering method \cite{29,30,31} and its applications to the integration of...
nonlinear partial differential equations. The inverse methods to integrate nonlinear evolution equations are often more effective than a direct numerical integration. Let us demonstrate this statement for the following simple case. The type of systems under consideration are slabs of nuclear matter, which are finite in the $z$ coordinate and infinite and homogeneous in two transverse directions. The wave function for the slab geometry is

$$\psi_{k_{\perp}n}(x) = \frac{1}{\sqrt{\Omega}} \psi_n(z) \exp(i k_{\perp} \cdot r), \quad \epsilon_{k_{\perp}n} = \frac{\hbar^2 k_{\perp}^2}{2m} + e_n,$$

where $r \equiv (x,y)$, $k_{\perp} \equiv (k_x,k_y)$, and $\Omega$ is the transverse normalization area.

$$-\frac{\hbar^2}{2m} \frac{d^2}{dz^2} \psi_n(z) + U(z) \psi_n(z) = e_n \psi_n(z),$$

A direct method to solve the single-particle problem is to assign a functional of interaction $\mathcal{E}$ (usually an effective density dependent Skyrme force), to derive the ansatz for the one-body potential, as the first variation of a functional of interaction in density $U(z) = U[\rho(z)] = \delta \mathcal{E}/\delta \rho$. Then to solve the Hartree-Fock problem under the set of constraints, which define the specifics of the nuclear system. In the simplest case of a ground state, one should conserve the total particle number of nucleons ($A$), which is related to the "thickness" of a slab, via $A \rightarrow A = (6A \rho_N^2/\pi)^{1/3}$, which gives the same radius for a three-dimensional system and its one-dimensional analogue. As a result, one obtains the energies of the single particle states $e_n$, their wave functions $\psi_n(z)$, the density profile $\rho(x) \rightarrow \rho(z)$

$$\rho(z) = \sum_{n=1}^{N_0} a_n \psi_n^2(z), \quad A = \sum_{n=1}^{N_0} a_n, \quad a_n = \frac{2m}{\pi \hbar^2} (e_F - e_n),$$

and the corresponding single-particle potential. $a_n$ are the occupation numbers, $N_0$ is the number of occupied bound orbitals. The Fermi-energy $e_F$ controls the conservation of the total number of nucleons. The energy (per nucleon) of a system is given by

$$\frac{E}{A} = \frac{\hbar^2}{2mA} \left( \sum_{n=1}^{N_0} a_n \int_{-\infty}^{\infty} (d\psi_n/dz)^2 dz + \frac{\pi}{2} \sum_{n=1}^{N_0} a_n^2 \right) + \frac{1}{A} \int_{-\infty}^{\infty} \mathcal{E}[\rho(z)] dz.$$

Finally, the set of formulas completely defines the direct self-consistent problem. Following the inverse scattering method, one reduces the main differential Schrödinger equation to the integral Gel'fand-Levitan-Marchenko
\[ K(x, y) + B(x + y) + \int_0^\infty B(y + z)K(x, z)dz = 0. \]  
(9)

for a function \( K(x, y) \). The kernel \( B \) is determined by the reflection coefficients \( R(k) = \frac{\hbar^2 k^2}{2m} \), and by the \( N \) bound state eigenvalues

\[ B(z) = \sum_{n=1}^N C_n^2(\kappa_n) + \frac{1}{\pi} \int_{-\infty}^\infty R(k) \exp(ikz)dk, \quad e_n = -\hbar^2 \kappa_n^2 / 2m. \]

\( N \) is the total number of the bound orbitals. The coefficients \( C_n \) are uniquely specified by the boundary conditions and the symmetry of the problem under consideration. The general solution, \( U(z) = -(\hbar^2 / m)(\partial K(z, z)/\partial z) \), should naturally contain both, contributions due to the continuum of the spectrum and to its discrete part. There seems to be no way to obtain the general solution \( U(z) \) in a closed form. Eqs. (6),(9) have to be solved only numerically.

In Ref. [1] we used the approximation of reflectless \( (R(k) = 0) \), symmetrical \( (U(-z) = U(z)) \) potentials. This gave the possibility to derive the following set of relations

\[ U(z) = -\frac{\hbar^2}{m} \frac{\partial^2}{\partial z^2} \ln(\|M\|) = -\frac{2\hbar^2}{m} \sum_{n=1}^N \kappa_n \psi_n^2(z), \]

\[ \psi_n(z) = \sum_{n=1}^N (M^{-1})_{nl} \lambda_l(z), \quad \lambda_n(z) = C_n(\kappa_n) \exp(-\kappa_n z), \]

\[ M_{nl}(z) = \delta_{nl} + \frac{\lambda_n(z) \lambda_l(z)}{\kappa_n + \kappa_l}, \quad C_n(\kappa_n) = \left(2\kappa_n \prod_{l \neq n} \frac{\kappa_n + \kappa_l}{\kappa_n - \kappa_l}\right)^{1/2}. \]  
(10)

Consequently, in the approximation of reflectless potentials \( (R(k) = 0) \), the wave functions, the single-particle potential and the density profiles are completely defined by the bound state eigenvalues via formulas (8),(10).

3.2 Results and Discussion

In Ref. [1] a series of calculations for the different layers, imitating nuclear systems in their ground state was provided. For a direct part of the calculations by the Hartree-Fock method, the interaction functional was chosen in the form of effective Skyrme forces. The calculated spectrum of bound states was fed into the scheme of the inverse scattering method, and the relations were used to recover the wave functions of the states, the single-particle potentials, and the corresponding densities. In this note, we generalize this
method to the case of fragmented configuration, trying to imitate two- and three-center nuclear systems. We use here only the inverse mean-field scheme (10). The details of the approach and systematic calculations of fragmented nuclear systems will be provided in a forthcoming publication. In Fig.1 we present the results of the calculations of three-level \( (N = 3, \ N_0 = 3) \) light \( (A \approx 20, \ A \approx 1.0) \) model system simulating the ground state (Fig. 1(a) solid line), and fragmentation of the system into two fragments (Fig. 1(a), the dotted and dashed lines). In the same figure (Fig. 1(b)) we present the fragmentation of the system into three fragments (solid and dotted lines). One
can see that it is possible to simulate a one-dimensional three-center system via inverse scattering method. The following conclusions can be drawn.

- The density profiles, calculated in the framework of inverse method, are practically identical to the results of calculation by SHF method. These results are valid for the ground state and for the system in the external potential field.

- The global properties of single-particle potentials (the depth and an effective radius) have been reproduced quite well, but the inverse method yields the quite strongly pronounced oscillations of the potential distributions within the inner region, and slightly different asymptotic tails of potential. In the framework of inverse scattering method, all bound states are taken into account in the calculation of the potential (10), but for the density distribution only the occupied states are taken into account (see Eqs. (7)). Therefore, the slope of the tails of the potential and of the density distributions will be different.

- We used, the approximation of reflectionless potentials, which gave us the possibility to obtain a simple closed set of relations (10), to calculate wave functions, density distributions and single particle potentials. The omitted reflection terms (\( R(k) = 0 \)) are not important for the evaluation of the density distributions, due to the fact that only the deepest occupied states are used to evaluate density distribution (see Eq. (7)). The introduction of these reflection terms will lead to a smoothing of the oscillations in the inner part of the potential and to a correction of its asymptotic behaviour.

- The presented method gives a tool to simulate the various sets of the static excited states of the system. This method could be useful to prepare in a simple way an initial state for the dynamical calculations in the frame of mean-field methods.

4 Conclusions

Recent experimental progress in the investigation of cold nuclear fragmentation has made the development of theoretical many-body methods highly desirable. Modern variants of self-consistent Hartree-Fock and relativistic mean-field models give the principal way to describe nuclear fragmentation in the framework of many-body self-consistent approach. However, the generalization of these approaches to three-center case is not provided yet because of existing difficulties to select a suitable set of constraints.
We suggest two methods to analyse a static scission configuration in cold ternary fission in the framework of mean field approach. The virial theorems has been suggested to investigate correlations in the phase space, starting from a kinetic equation. The inverse mean field method is applied to solve single-particle Schrödinger equation, instead of constrained selfconsistent Hartree–Fock equations. It is shown, that it is possible to simulate one-dimensional three-center system via inverse scattering method in the approximation of reflectless single-particle potentials.

These models may be useful as a guide to understand the general properties of fragmented systems and to formulate the suitable set of constraints for the realistic three-dimensional mean field calculations of the three-center nuclear system.

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