STAR FACTORIZATION AND NONCROSSING PARTITIONS

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ABSTRACT. We develop the relationship between minimal transitive star factorizations and noncrossing partitions. This gives a new combinatorial proof of a result by Irving and Rattan, and a specialization of a result of Kreweras. It also arises in a poset on the symmetric group whose definition is motivated by the Subword Property of the Bruhat order.

1. Introduction

Let \( S_n \) be the symmetric group on \([n] := \{1, \ldots, n\}\). For any \( k \in [n] \), called the pivot, the group \( S_n \) is generated by the transpositions

\[ \{ (i \, k) : i \in [n] \setminus \{k\} \}. \]

For \( \pi \in S_n \), a decomposition \( \pi = g_1 \cdots g_r \) for \( g_i \in \star_{n,k} \) is a star factorization of \( \pi \). Following terminology of Pak in [7], we say that this star factorization is transitive if \( \{g_1, \ldots, g_r\} = \star_{n,k} \). A minimal transitive star factorization refers to minimality of the length \( r \). If \( \pi \in S_n \) has \( m \) disjoint cycles, then a minimal transitive star factorization of \( \pi \) has length \( n + m - 2 \) [4,7]. Let

\[ \star_k(\pi) \]

denote the minimal transitive star factorizations of \( \pi \) with pivot \( k \). Throughout this work, all “star factorizations” are assumed to be minimal transitive star factorizations.

Irving and Rattan showed in [4] that the number of star factorizations of a permutation \( \pi \in S_n \) depended only on the cycle type of \( \pi \): if \( \pi \) has cycles of lengths \( \ell_1, \ldots, \ell_m \), then

\[ |\star_k(\pi)| = (n + m - 2)_{\ell_1} \cdots \ell_m, \]

where \((a)_b\) is the falling factorial. Perhaps most striking about this result is its independence of the pivot \( k \). In [11], we gave a combinatorial proof of that independence.

In this note, we give a new combinatorial proof of Equation (1), based on an interpretation of star-factorizations in terms of noncrossing partitions. This is indicative of a close relationship between these two classes of objects, which can be used to prove/recover several interesting results. First, using the enumeration given by [11] in Equation (1), we recover a result of Kreweras about enumerating noncrossing partitions with specified (labeled) part sizes [5]. This enumeration turns out to be independent of the part sizes themselves, and depends only on how many parts there are. Second, we show that in a Bruhat-style poset defined in terms of the minimal transitive star factorizations of elements of \( S_n \), the poset’s intervals are built out of noncrossing partition posets. Once again, the pivot value is immaterial, and the posets defined in this way for two different pivots are equal.

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This interplay between star factorizations and noncrossing partitions not only highlights their connection, but also a notion of “independent of . . . ” that certain features of each may share.

Section 2 reviews the main results of Irving and Rattan. These results are used to establish the relationship between star factorizations and labeled noncrossing partitions in Proposition 2.9 and Theorem 2.12. This yields that the number labeled noncrossing partitions of $X$ into $m$ parts of specified sizes is the falling factorial $(X)_{m-1}$ in Corollary 2.13. Section 3 revisits the enumeration of [4] and provides a new combinatorial proof of that result in Corollary 3.4. In Section 4, we define a poset on $S_n$ using star factorizations, motivated by the Subword Property of the Bruhat order. This is equivalent to a poset that can be defined in terms of permutation cycles (Theorem 4.3), and is thus yet another example of pivot independence arising in a property defined by star factorizations (Corollary 4.4). The relationship between this poset and noncrossing partitions is discussed in Lemma 4.5 and Corollary 4.6, and further elucidated in examples and discussions in that section. We conclude with two open questions in Section 5.

2. Star factorizations as labeled noncrossing partitions

Before envisioning star factorizations as labeled noncrossing partitions, we review two results of Irving and Rattan.

Fix positive integers $n \geq k$ and $\pi \in S_n$. Suppose that $\pi$ consists of $m$ disjoint cycles, $C_1, \ldots, C_m$, where $C_i$ has length $\ell_i$. Unless specified otherwise, cycles are index in order of their minimal elements. Let $p$ be the index of the cycle containing the pivot $k$.

Lemma 2.1 ([4]).

(a) If $C_i = (a_1 a_2 \cdots a_t)$, where $i \neq p$, then some transposition $(k a_j)$ appears exactly twice in $\delta$ and all $(k a_h)$ with $h \neq j$ appear exactly once in $\delta$. These appear as $(k a_j)(k a_{j-1}) \cdots (k a_i)(k a_t) \cdots (k a_{j+1})(k a_j)$, from left to right in $\delta$.

(b) If $C_p = (k b_2 \cdots b_t)$, then each transposition $(k b_h)$ appears exactly once in $\delta$. These appear as $(k b_t) \cdots (k b_2)$, from left to right in $\delta$.

We can turn $\delta \in \star_k(\pi)$ into a word

$$\omega(\delta) \in [m]^{n+m-2}$$

recording the index of the cycle containing $i$ for each factor $(k i)$.

Example 2.2. Let $\pi = (13)(285)(4)(67)$ and $k = 6$, with

$$\delta = (6 8)(6 1)(6 3)(6 1)(6 2)(6 5)(6 8)(6 7)(6 4)(6 4).$$

Then $p = 4$ because 6 appears in the fourth cycle, and

$$\omega(\delta) = 2111222433.$$

Irving and Rattan characterize the possible words $\omega(\delta)$ that are formed by $\delta \in \star_k(\pi)$.

Definition 2.3. A word $\omega \in [m]^{n+m-2}$ is valid if

- the symbol $p$ appears $\ell_p - 1$ times,
- for all $j \in [m] \setminus \{p\}$, the symbol $j$ appears $\ell_j + 1$ times,
- for $i \neq j$, there is no subword $ijij$ in $\omega(\delta)$, and
- for $i \neq p$, there is no subword $ipi$ in $\omega(\delta)$. 
Lemma 2.4 ([4]). A word $\omega \in [m]^{n+m-2}$ is valid if and only if $\omega = \omega(\delta)$ for some $\delta \in \mathfrak{s}_k(\pi)$.

The third requirement in Definition 2.3 suggests a kind of noncrossing phenomenon, but the fourth requirement is not quite on target. However, if we add a copy of $p$ to either end of a valid word $\omega$, then we can indeed rephrase this as a noncrossing problem.

**Definition 2.5.** Given a valid word $\omega$ for $\pi$, let $\hat{\omega}$ be the necklace obtained by inserting $p$ between the first and last letters of $\omega$.

For consistency, we will read necklaces in counterclockwise order.

**Example 2.6.** The word $\omega = 2111222433$, valid for $(13)(285)(4)(67)$ with pivot $k = 6$, corresponds to the necklace $\hat{\omega}$ displayed in Figure 1 with the inserted copy of 4 circled.

These necklaces allow us to recharacterize valid words as noncrossing partitions.

**Definition 2.7.** Fix a sequence of positive integers $\ell = (\ell_1, \ldots, \ell_t)$. A labeled noncrossing partition of type $\ell$ is a noncrossing partition of $\{x_1 + \cdots + x_t\}$ in which the parts have sizes $(x_1, \ldots, x_t)$, and the part of size $x_i$ (equivalently, each element in that part) is labeled $i$. Let $L-NC(\ell)$ be the set of labeled noncrossing partitions of type $\ell$. The rotation classes of these objects are the labeled noncrossing necklaces of type $\ell$, denoted $L-NCN(\ell)$.

**Example 2.8.** As illustrated in Figure 2, $|L-NC(2, 2)| = 4$ and $|L-NCN(2, 2)| = 1$.

Throughout this section, we will refer to the sequence $\ell' = (\ell'_1, \ldots, \ell'_m)$, where

$$\ell'_i := \begin{cases} \ell_i + 1 & \text{if } i \in [m] \setminus \{p\}, \\ \ell_p & \text{if } i = p. \end{cases}$$

**Proposition 2.9.** Consider a necklace $\alpha$ with $\ell'_i$ copies of $i$, for all $i \in [m]$. Then $\alpha \in L-NCN(\ell')$ if and only if $\alpha = \hat{\omega}$ for some valid word $\omega \in [m]^{n+m-2}$.
Figure 2. The four labeled noncrossing partitions of type (2, 2), which all are rotationally equivalent.

Proof. If $\alpha \in \text{L-NCN}(\ell')$, then removing any copy of $p$ and reading counterclockwise will produce a string $\omega$ that satisfies all requirements of Definition 2.3. Thus $\alpha = \overline{\alpha}$, with that chosen copy of $p$ being the inserted one. Now suppose that $\alpha$ has a crossing, and consider all $\ell'_p$ words obtained by removing one copy of $p$ from $\alpha$ and reading the remaining letters in counterclockwise order. If the crossing in $\alpha$ involves values other than $p$, then that crossing will also appear in the resulting word, violating the third requirement for validity. On the other hand, if that crossing involves $p$ and some letter $i$, then there is a substring $ipi$ appearing in all $\ell'_p$ words obtained in this way, violating the fourth requirement for validity. \hfill \Box

Proposition 2.9 is illustrated in Figure 3, continuing Example 2.6.

Figure 3. A labeled noncrossing necklace of type $(3, 4, 2, 2)$, marked to match Figure 1.

Definition 2.10. Define the map

$\text{SF}_k : \text{L-NCN}(\ell') \times [\ell_1] \times \cdots \times [\ell_m] \to *_k(\pi)$

$(\alpha, d_1, \ldots, d_m) \mapsto \delta$

as follows. First, fix a presentation of $\alpha$ and remove the $d_p$th appearance of $p$ when reading the necklace in counterclockwise order from the topmost letter. Let $\omega$ be the (valid) word formed by reading the remainder of $\alpha$ in counterclockwise order from this removal (hence $\alpha = \overline{\alpha}$). We know how to interpret each $p$ in this word by Lemma 2.1(a). For each $j \in [m]\{p\}$ let $(k, c_{d_j})$ be the factor that will appear twice in the $\delta$, where $c_{d_j}$ is the $d_j$th smallest value in cycle $C_j$. Thus, by Lemma 2.1(b), we have completely defined $\delta$.

Example 2.11. Let $\pi = (13)(285)(4)(67)$ and $k = 6$, so $\ell' = (3, 4, 2, 2)$. Let $\alpha$ be the necklace depicted in Figure 3. Let $(c_1, c_2, c_3, c_4) = (1, 3, 1, 2)$. This $c_4$ removes the circled
The number of labeled noncrossing partitions of Corollary 2.13. only on the number of specified part sizes, not on the sizes themselves. This can also be shown using a result of Kreweras [5]. It is surprising that this depends as required by Definition 2.10. Form \( \delta_p \), then we could rotate the necklace so that \( d_p = e_p \). In order to have the correct factors appear twice in the factorization output by \( \text{SF}_k \), the values \( d_i \) and \( e_i \) must be equal for all \( i \in [m] \setminus \{p\} \). Thus \( \text{SF}_k \) is injective.

To show that \( \text{SF}_k \) is surjective, we can take any \( \delta = \text{SF}_k(\pi) \) and define \( d_i \) for all \( i \in [m] \setminus \{p\} \) as required by Definition 2.10. Form \( \omega(\delta) \), and let \( d_p \) indicate the letter \( p \) that was added to form this necklace. Thus \( \text{SF}_k(\omega, d_1, \ldots, d_m) = \delta \).

We can use Theorem 2.12 to enumerate labeled noncrossing partitions with specified part sizes. This can also be shown using a result of Kreweras [5]. It is surprising that this depends only on the number of specified part sizes, not on the sizes themselves.

Corollary 2.13. The number of labeled noncrossing partitions of \( X \) into \( m \) parts of specified sizes is the falling factorial \( (X)_{m-1} \).

Proof. If \( m = 1 \) then there is clearly 1 \( = (X)_{1-1} \) labeled noncrossing partition of \( X \). Now assume that \( m > 1 \). Theorem 2.12 and Equation (1) yield

\[
|\text{L-NCN}(\ell')| = (n + m - 2)_{m-2}.
\]

Recall the definition of \( \ell' \) in Equation (2), and write \(|x|\) for the sum of the terms in a sequence. Thus, for any sequence \( x = (x_1, \ldots, x_m) \) of positive integers with at most one \( x_i = 1 \), we have \(|\text{L-NCN}(x)| = (|x| - 1)_{m-2} \). We can expand this to allow more 1s by inducting on \( m \). If \( m = 2 \), then certainly \(|\text{L-NCN}(x)| = 1 = (|x| - 1)_{2-2} \). Now suppose that we have a sequence \( x = (x_1, \ldots, x_m) \) of positive integers, with \( m \geq 2 \). Consider \( x^+ := (x_1, \ldots, x_m, 1) \). (Note, inserting the 1 anywhere else in \( x \) will work similarly.) Then each labeled noncrossing necklace of type \( x^+ \) can be obtained by inserting the label \( m + 1 \) anywhere among the \(|x|\) letters in any element of \( \text{L-NCN}(x) \). Thus

\[
|\text{L-NCN}(x^+)| = |x| \cdot |\text{L-NCN}(x)| = |x| \cdot (|x| - 1)_{m-2} = (|x|)_{m-1} = (|x^+| - 1)_{(m+1)-2};
\]

as desired.

It remains now to convert noncrossing necklaces to noncrossing partitions. If \( m = 1 \), then \(|\text{L-NC}(x)| = 1 = |x|_{1-1} \). On the other hand, if \( m > 1 \), then the \(|x|\) rotations of a necklace of
Define sequences \( a \) factorizations. We can use those partitions as an intermediary between elements of \( \mathcal{L} \) and elements of \( \mathcal{L} - \mathcal{NC} \) type \( \mathcal{L} \).

Fix positive integers \( n \geq k > k' \) and a permutation \( \pi \in \mathcal{S}_n \). As before, suppose that \( \pi \) consists of \( m \) disjoint cycles, \( C_1, \ldots, C_m \), where \( C_i \) has length \( \ell_i \). Let \( p \) be the index of the cycle containing the pivot \( k \) and let \( p' \) be the index of the cycle containing the pivot \( k' \).

Define sequences \( a = (a_1, \ldots, a_m) \) and \( b = (b_1, \ldots, b_m) \) as:

\[
a_i := \begin{cases} 
\ell_i + 1 & \text{if } i \in [m]\{p\}, \text{ and} \\
\ell_p & \text{if } i = p,
\end{cases}
\]

\[
b_i := \begin{cases} 
\ell_i + 1 & \text{if } i \in [m]\{p'\}, \text{ and} \\
\ell_{p'} & \text{if } i = p'.
\end{cases}
\]

Fix \( \delta \in \mathcal{S}_n \). Our goal will be to define a corresponding \( \delta' \in \mathcal{S}_n \). We will do this using \( \mathcal{SF}_k \) from the previous section, and the map \( \text{Shift} \) defined below.

**Definition 3.1.** Define the map

\[
\text{Shift} : \mathcal{L} - \mathcal{NC}(a) \times [\ell_1] \times \cdots \times [\ell_m] \rightarrow \mathcal{L} - \mathcal{NC}(b) \times [\ell_1] \times \cdots \times [\ell_m]
\]

\[
(\alpha, d_1, \ldots, d_m) \mapsto (\beta, d_1, \ldots, d_m)
\]

as follows. If \( p = p' \), then set \( \beta := \alpha \). Otherwise, first fix a presentation of \( \alpha \) and locate the \( d_p \)-th copy of \( p \) when reading the necklace in counterclockwise order from the topmost letter. Counterclockwise from there, locate the first copy of \( p' \), reading a string of the form \( ps_1 \cdots s_t \), where \( s_t = p' \). Let \( h \in [1, t] \) be maximal such that the necklace obtained by replacing the string \( ps_1 \cdots s_{t-1}p' \) by \( ps_1 \cdots s_{h-1} p s_h \cdots s_{t-1} \) would not have any crossings. Let \( \beta \in \mathcal{L} - \mathcal{NC}(b) \) be the necklace obtained by this replacement.

We know that the \( h \) described in Definition 3.1 exists because \( s_1 \) has the desired property.

**Example 3.2.** Let \( n = 8 \), \( k = 6 \), and \( k' = 3 \), and let \( \pi = (13)(285)(4)(67) \). Then \( p = 4 \) and \( p' = 1 \). Let \( \alpha \) be the necklace depicted in Figure 3. Then \( \text{Shift}(\alpha, 1, 3, 1, 2) = (\beta, 1, 3, 1, 2) \), where \( \beta \) is the necklace depicted in Figure 4 with the \( c_1 \)-st counterclockwise-from-top appearance of \( p' \) circled. Thus

**Proposition 3.3.** The map \( \text{Shift} \) is invertible.

**Proof.** Let \( \text{Shift}' : \mathcal{L} - \mathcal{NC}(b) \times [\ell_1] \times \cdots \times [\ell_m] \rightarrow \mathcal{L} - \mathcal{NC}(a) \times [\ell_1] \times \cdots \times [\ell_m] \) be defined exactly as \( \text{Shift} \), except for replacing “counterclockwise from there” by “clockwise from there.” Then \( \text{Shift}' \circ \text{Shift} \) and \( \text{Shift} \circ \text{Shift}' \) are the identity maps on their respective spaces. \( \Box \)

**Corollary 3.4.** The map

\[
\mathcal{SF}_{k'} \circ \text{Shift} \circ \mathcal{SF}_k^{-1} : \mathcal{S}(\pi) \rightarrow \mathcal{S}(\pi)
\]

is a bijection.
4. A Bruhat-style poset on star factorizations

The Subword Property of the Bruhat order says that we can fix a reduced decomposition of a permutation $\pi$, and find all $\sigma \leq \pi$ in the Bruhat order by deleting letters from that reduced decomposition [1]. This suggests a new poset structure to $S_n$ in terms of minimal transitive star factorizations.

Definition 4.1. Fix positive integers $n \geq k$. For $\sigma, \pi \in S_n$, say that $\sigma \leq_k \pi$ if there exists $\gamma \in \star_k(\sigma)$ that is a subword of $\delta \in \star_k(\pi)$. Let $\text{Star}_k(n)$ be the poset defined by $\leq_k$ on $S_n$.

Note that Definition 4.1 is similar to the Subword Property of the Bruhat order, but not identical to it because it does not require the same $\delta \in \star_k(\pi)$ to work for all $\sigma \leq_k \pi$. The poset defined by $\leq_1$ on $S_3$ appears in Figure 5.

Definition 4.2. Let $C = (c_1 \cdots c_i)$ be a cycle. An excerpt $D$ of $C$ is a cycle of the form $D = (c_i c_{i+1} \cdots c_{i+j})$ for some positive integers $i$ and $j$, where indices are taken modulo $t$. The cycle $C$ has been sliced. A nonempty excerpt $D$ is proper if $j < t$.

Characterization of the poset $\text{Star}_k(n)$ follows, essentially, from Lemmas 2.1 and 2.4

Theorem 4.3. $\sigma \leq_k \pi$ if and only if every cycle of $\pi$ is an excerpt of a cycle of $\sigma$.

Proof. Suppose that $\sigma \leq_k \pi$, with $\gamma$ and $\delta$ as in Definition 4.1. Because these are minimal transitive star factorizations, any letter that gets deleted from $\delta$ to form $\gamma$ must have been duplicated in $\delta$, by Lemma 2.1. Using the notation of that lemma, let $(k a_j)$ be a letter
Figure 5. The poset $\text{Star}_1(3)$.

deleted from $\delta$ to form $\gamma$. Suppose first that the factors from the $C_i$ containing $a_j$ had appeared between two factors from some other $C_q = (\cdots xy \cdots)$ in $\delta$, as

$$(k y) \cdots (k a_j)(k a_{j-1}) \cdots (k a_1)(k a_t)(k a_{j+1})(k a_j) \cdots (k x),$$

and let $x$ and $y$ be the nearest neighbors with this property. Then deleting either copy of $(k a_j)$ will merge the two cycles, forming

$$(\cdots x a_j a_{j+1} \cdots a_t a_1 \cdots a_{j-1} y \cdots) \text{ or } (\cdots x a_{j+1} \cdots a_t a_1 \cdots a_{j-1} a_j y \cdots),$$

depending on which $(k a_j)$ factor was deleted. On the other hand, if the factors from this $C_i$ do not appear between two factors from any other $C_q$, then $C_i$ merges with the cycle $C_p$ containing the pivot value, as indicated by Lemma 2.1.

Now suppose that every cycle of $\pi$ is an excerpt of a cycle of $\sigma$. Orient the cycles in each permutation so that if the parentheses are deleted then the two resulting words are identical. (For a small example in $S_3$, we could write $\pi = (1)(32)$ and $\sigma = (132)$.) Construct $\gamma \in \mathcal{C}_k(\sigma)$ using Lemma 2.1, factoring each cycle from left to right. Do likewise to construct $\delta \in \mathcal{C}_k(\pi)$. This $\gamma$ is necessarily a subword of $\delta$, and so $\sigma \leq_k \pi$. □

Reminiscent of the work of [4] and the previous section, Theorem 4.3 gives another situation in which the pivot value itself does not matter.

**Corollary 4.4.** For all $k, k' \in [n]$, $\text{Star}_k(n) = \text{Star}_{k'}(n)$.

In other words, we can call this poset $\text{Star}(n)$ and not specify the pivot $k$. Accordingly, we can write $\leq$ for $\leq_k$. The ordering characterization given in Theorem 4.3 allows us to describe the structure of this poset in detail. For example, the identity $(1)(2) \cdots (n)$ is the unique maximal element, and there are $(n - 1)!$ minimal elements: the $n$-cycles.

**Lemma 4.5.** Fix $\sigma \leq \pi$ in $\text{Star}(n)$. Suppose that exactly one cycle of $\sigma$ has been sliced to form $\pi$, and that it resulted in $d$ proper excerpts in $\pi$. Then $[\sigma, \pi]$ is isomorphic to the noncrossing partition lattice $\text{NC}(d)$.

**Proof.** Definition 4.2 and Theorem 4.3 allow us to reduce this to the problem of the interval $[(12 \cdots d), (1)(2) \cdots (d)]$. Brady showed in [2] that the dual to this poset is isomorphic to $\text{NC}(d)$. The lattice of noncrossing partitions is self-dual, completing the proof. □
Corollary 4.6. Suppose that $\sigma \preceq \pi$ in $\text{Star}(n)$, that the cycles of $\sigma$ that get sliced are $C_1, \ldots, C_s$ (meaning that all other cycles of $\sigma$ appear identically in $\pi$), and that $C_i$ results in $d_i$ proper excerpts in $\pi$. Then $[\sigma, \pi] \cong \text{NC}(d_1) \times \cdots \times \text{NC}(d_s)$.

Proof. This follows from Lemma 4.5 and the fact that disjoint cycles act independently. □

Example 4.7. The interval $[[12345)(678), (15)(23)(4)(67)(8)]$ is isomorphic to $\text{NC}(3) \times \text{NC}(2)$, and is depicted in Figure 6.

Figure 6. The interval $[[12345)(678), (15)(23)(4)(67)(8)] \subset \text{Star}(8)$.

Suppose that $\pi \in \mathcal{S}_n$ consists of $m$ disjoint cycles, $C_1, \ldots, C_m$, where $C_i$ has length $\ell_i$. Additional conclusions we can draw from Theorem 4.3 include:

- $\text{Star}(n)$ is not bounded because it has multiple $((n - 1)!) \, \text{minimal elements}$;
- $\text{Star}(n)$ is graded because all maximal chains have $n$ elements;
- as illustrated in Figure 3, two elements of $\text{Star}(n)$ may not have a greatest lower bound (nor, even, a common lower bound) and thus $\text{Star}(n)$ is not a lattice;
- $\pi$ covers $\sum_{i<j} \ell_i \ell_j$ elements, formed by picking orientations of any two cycles and merging them in those orientations;
- $\pi$ is covered by $\sum_i \binom{\ell_i}{2}$ elements, because a cycle of $\ell$ elements can be sliced into two proper excerpts by choosing two elements to serve as the first letters in each excerpt.

In previous work, we studied boolean elements in the Bruhat order [3, 8, 9, 10]. These had such interesting properties that we are motivated to look for boolean intervals in $\text{Star}(n)$, as well. Note that this is a more general question than was studied for the Bruhat order, since we allow arbitrary intervals to be boolean, not just principal order ideals.

Definition 4.8. An interval $[\sigma, \pi]$ is boolean if it is isomorphic to a boolean algebra.

Due to Lemma 4.5 a study of boolean intervals in $\text{Star}(n)$ amounts to understanding when the Catalan number $C_n$ is equal to $2^{n-1}$, and this happens only when $n \in \{1, 2\}$. This and Corollary 4.6 characterize boolean intervals.

Corollary 4.9. The interval $[\sigma, \pi]$ is boolean if and only if each cycle of $\sigma$ is sliced into at most two excerpts to form $\pi$. 

This characterization allows us to enumerate several interesting things:

- the number of boolean intervals with \( \pi \) as maximal element is
  \[
  \sum_{\text{involutions } \alpha \in S_n,\ i<j \text{ with } \alpha(i)=j} \ell_i \ell_j;
  \]

- the number of boolean intervals with \( \pi \) as minimal element is
  \[
  \prod_i \left(1 + \binom{\ell_i}{2}\right).
  \]

Thus, for example, the number of boolean intervals of the form \([\sigma, (1)\cdots(2)\cdots(n)]\) is the number of involutions in \( S_n \), which is sequence A000085 of [6], and there are \( 1 + 3 \cdot 1 = 4 \) boolean intervals of the form \([\sigma, (123)(4)]\):

\[
[(123)(4), (123)(4)], [(1423), (123)(4)], [(1243), (123)(4)], \text{ and } [(1234), (123)(4)].
\]

5. Future research

Two main avenues for future research emerge from this work. The first it to establish how else star factorizations and noncrossing partitions interact, and to determine if that interaction can be leveraged in some way. Secondly, several of the results above exhibited a property of star factorizations that was independent of the pivot value. Thus we ask, how else can this pivot independence arise, and what does it mean in those settings?

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