THE PLURICANONICAL SYSTEMS OF A
PRODUCT-QUOTIENT VARIETY

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Abstract. We give a method for the computation of the plurigenera of a
product-quotient manifold. We give two different types of applications to it:
to the construction of Calabi-Yau threefolds and to the determination of the
minimal model of a product-quotient surface of general type.

Contents

Introduction 2
1. Minimal models of quotients of product of two curves 5
2. Product quotient varieties birational to Calabi-Yau threefolds 7
3. Examples of numerical Calabi-Yau product-quotient threefolds 8
4. The sheaves of ideals $I_d$ on a smooth projective variety with a finite
group action 11
5. The sheaves of ideals $I_d$ for cyclic quotient singularities 13
6. A Calabi-Yau 3-fold 19
7. A fake Calabi-Yau 3-fold 22
8. Some minimal surfaces of general type 25
References 29

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PLURICANONICAL SYSTEMS

INTRODUCTION

The product-quotient varieties are the varieties obtained by taking a minimal resolution of the singularities of a quotient \( X := (\prod_1^n C_i) / G \), the quotient model, where \( G \) is a finite group acting diagonally, i.e. as \( g(x_1, \ldots, x_n) = (gx_1, \ldots, gx_n) \). Usually the genera of the curves \( C_i \) are assumed to be at least 2: for sake of simplicity we will implicitly assume it from now on.

They have been introduced in [BP12] in the first nontrivial case \( n = 2 \), as generalization of the varieties isogenous to a product of unmixed type, where the action is assumed to be free.

The product-quotient varieties have proved in the last decade to form a very interesting class, because even if they are relatively easy to construct, there are several objects with interesting properties among them. Indeed the class has been a fruitful source of examples with applications in different areas of algebraic geometry.

For example [GP15] constructs in this way several K3 surfaces with automorphisms of prime order that are not symplectic. A completely different application is the construction of rigid not infinitesimally rigid compact complex manifolds obtained in [BP18], answering a question about 50 years old.

A classical problem is the analysis of the possible behaviours of the canonical map of a surface of general type. [Bea79] provides upper bounds for both the degree of the map and the degree of its image, but very few examples realizing values near those bounds are in literature. The current best values have been recently attained respectively in [GPR18] and [Cat18] with this technique.

For what concern the classification of the surfaces of general type [LP16] study the asymptotic behaviour of the number of connected components of the Gieseker moduli space of the surfaces of general type with fixed deformation invariant \( K^2 \).

It has been proved in [Cat92] that this number is upper bounded by \((K^2)^{77K^2}\); [LP16] constructs examples that show that this bound cannot be much improved, in the sense that they contradict any upper bound asymptotically better than \( C(K^2)^{\sqrt{K^2}} \) (\( C \) constant).

Last but not the least, product-quotients surfaces have been useful to construct several new examples of surfaces of general type \( S \) with \( \chi(O_S) = 1 \), the minimal possible value, see [Pig15] and the references therein. Restricting for sake of simplicity to the regular case, the minimal surfaces of general type with geometric genus \( p_g = 0 \), whose classification is a long standing problem known as Mumford’s dream, we have now dozens of families of them constructed as product-quotient surfaces, see [BC04, BCG08, BCGP12, BP12, BP16], a huge number if compared with the examples constructed with other techniques, see [BCP11]. Concerning higher dimensions, we shall point out that a complete classification of threefolds isogenous to a product with \( \chi(O_X) = -1 \), the maximal possible value, has been achieved recently see [FG16, Gle17].
It is very likely that the list of product-quotient surfaces of general type with $p_g = 0$ in [BP16] is complete. Anyhow we are not able to prove it. The main obstruction to get a full classification is in the fact that it is very difficult to determine the minimal model of a regular product-quotient variety. Indeed that list was produced by a computer program able to classify all regular product-quotient surfaces $S$ with $p_g = 0$ and a given value of $K^2$. The minimal surfaces of general type $S$ with $p_g = 0$ have, by standard inequalities, $1 \leq K^2_S \leq 9$, but we do not know a lower bound for the minimal resolution of the singularities, possibly not minimal, of the quotient model of a product-quotient surface of general type $S$ with $p_g = 0$. Detecting the rational curves with self-intersection $-1$ in one of these surfaces may be very difficult, see for example the fake Godeaux surface in [BP12, Section 5].

More generally, in birational geometry one would like to know, given an algebraic variety, one of the “simplest” variety in its birational class, a “minimal” one. This is the famous Minimal Model Program, producing a variety with nef canonical system and at worse terminal singularities, or a Mori fiber space. We are not able at the moment to run a minimal model program explicitly for a general product-quotient variety even in dimension 2. Anyhow, knowing all plurigenera $h^0(dK)$ of an algebraic variety gives a lot of informations on its minimal models. Actually, this is the main motivation for this paper, whose main result is a method for computing all plurigenera of a product-quotient variety.

Another motivation for this paper was to investigate methods to construct systematically Calabi-Yau threefolds. Indeed, most of the known Calabi-Yau threefolds are constructed by taking the resolution of a generic anticanonical section of a toric Fano fourfold. The idea stem from Batyrev’s seminal paper [Bat94] but the complete list of the topologically different Calabi-Yau threefolds which one can obtain with this method was obtained with the help of the computer (see [KS00]) with the classification of the 473,800,776 reflexive polytopes in dimension 4. Apart from these Calabi-Yau threefolds, very few examples are known and their construction involves ad hoc methods such as quotients by group actions (see, for example, [BF12, BFNP14, BF16]). Hence, the idea of using the well-known machinery of the product-quotient varieties could prove to be really effective in the direction of finding a lot of new examples of Calabi-Yau threefolds as minimal models of product-quotient threefolds.

The paper is organized as follows.

The first two sections are devoted to discuss some possible applications of a formula for the plurigenera of product-quotient manifolds.

In section 1 we discuss under which assumptions we know that a product-quotient variety is minimal. Then we concentrate in the case of dimension 2, writing explicitly a formula for the number of curves contracted by the morphism onto the minimal model in terms of the plurigenera.

In section 2 we move to dimension 3, trying to generalize the constructions in [GP15]. We first prove that no product-quotient varieties can be Calabi-Yau.
Still, as in [GP15] for dimension 2, they may be birational to a Calabi-Yau. We then introduce the concept of numerical Calabi-Yau variety, that is a variety whose Hodge numbers are compatible with a possible Calabi-Yau minimal model. Then we show that a numerical Calabi-Yau product-quotient threefold is birational to a Calabi-Yau threefold if and only if all its plurigenera are equal to 1.

In section 3 we produce, with the help of the computer program MAGMA, 12 families of numerical Calabi-Yau threefolds.

In section 4 we prove our main result, Theorem 4.5, that reduces the computation of the plurigenus $P_d(X)$ of a minimal resolution of the singularities of a quotient $X := Y/G, Y$ smooth, to the computation of a certain ideal sheaf $I_d$ on $Y$. Then, in section 5, we show how to compute $I_d$ when all stabilizers are cyclic, like in the case of product-quotient varieties.

In section 6 and 7 we apply our theorem to two of the numerical Calabi-Yau threefolds produced in section 3, showing that one is birational to a Calabi-Yau threefold and the other is not. Finally, in section 8, we apply our theorem to show the minimality of a bunch of product-quotient surfaces whose quotient model has several singularities that are not canonical, that makes this result difficult to achieve with other techniques.

Notation. All algebraic varieties in this note are complex, quasi-projective and integral, so irreducible and reduced.

A curve is an algebraic variety of dimension 1, a surface is an algebraic variety of dimension 2.

For every projective algebraic variety $X$ we consider the cohomological dimensions of its structure sheaf $q_i(X) := h^i(X, \mathcal{O}_X), \forall 1 \leq i \leq \dim X$. For $i = n$ this is the geometric genus $p_g(X) := q_n(X)$, for $i < n$ they are called irregularities. If $X$ is smooth, by Hodge Theory $q_i(X) = h^{i,0}(X) := h^0(X, \Omega^i_X)$. If $X$ is a curve, there are no irregularities and the geometric genus is the usual genus $g(X)$. If $S$ is a surface the unique irregularity $q_1(S)$ is usually denoted by $q(S)$.

If $X$ is normal the dualizing sheaf $\omega_X$ ([Har77, III.7]) is a Weil divisorial sheaf and we denote by $K_X$ a canonical divisor, so $\omega_X \cong \mathcal{O}_X(K_X)$. A normal variety is Gorenstein if $\omega_X$ is a line bundle, i.e. if the divisor $K_X$ is Cartier. A normal variety is $\mathbb{Q}$-Gorenstein if $K_X$ is $\mathbb{Q}$-Cartier i.e. if there exists $d \in \mathbb{N}$ such that $dK_X$ is Cartier. A normal variety is factorial, resp. $\mathbb{Q}$-factorial if every integral Weil divisor is Cartier, resp. $\mathbb{Q}$-Cartier.

We use the symbols $\sim_{lin}$ for linear equivalence of Cartier divisors, $\sim_{num}$ for numerical equivalence of $\mathbb{Q}$-Cartier divisors.

For every normal projective variety $X$ we consider its $d$-th plurigenus $P_d(X) = h^0(X, \mathcal{O}_X(dK_X))$. By Serre duality $P_1(X) = p_g(X)$.

We write $\mathbb{Z}_m$ for the cyclic group of order $m$, $\mathcal{D}_m$ for the dihedral group of order $2m$, $\mathcal{S}_m$ for the symmetric group in $m$ letters.

For $a, b, c \in \mathbb{Z}$, $a \equiv_b c$ means that $b$ divides $a - c$. 


1. MINIMAL MODELS OF QUOTIENTS OF PRODUCT OF TWO CURVES

Since we are assuming that all curves $C_i$ have genus at least 2, $K_{\prod C_i}$ is ample and therefore, if $G$ is a finite group acting freely in codimension 1 on $\prod C_i$, as for product quotient varieties (of dimension at least 2), $K_X$ is ample too.

In particular, if $G$ acts freely then $X$ is smooth and $K_X$ is ample, so $X$ is a smooth minimal variety of general type.

If, slightly more generally, the action of $G$ is free in codimension 1 and $X$ has at worse canonical singularities, then we can take a terminalization of $X$, a crepant resolution $\hat{X} \to X$ of the canonical singularities of $X$ such that $\hat{X}$ has terminal singularities. Then $K_{\hat{X}}$ is automatically nef and therefore $\hat{X}$ is a minimal model of $X$.

The first example appeared in literature of a quotient $X := (\prod_{i=1}^n C_i) / G$ of general type such that a minimal resolution of the singularities $\hat{X}$ of $X$ is not a minimal variety is the product-quotient surface studied in [MP10, 6.1], where its minimal model is determined by studying the Albanese morphism of $\hat{X}$. Indeed, in this case $q(\hat{X}) = 1$, so the Albanese morphism of $\hat{X}$ is a fibration onto an elliptic curve, and therefore all rational curves, including all curves contracted on the minimal model, are contained in its fibres.

More generally, if $q(\hat{X}) \neq 0$, the Albanese morphism of $\hat{X}$ gives some obstructions to the existence of $K_{\hat{X}}$-negative curves, since it contracts every rational curve. Indeed, in dimension $n = 2$, if $G$ contains an automorphism that exchange the factors $C_i$ (in particular $C_1 \cong C_2$) we have the following results:

**Theorem 1.1** ([Pig17, Theorem 3]). Let $C$ be a curve of genus at least two and let $G$ be a group of automorphisms of $C \times C$ that contains an automorphism exchanging the factors.

Let $\hat{X}$ be the minimal resolution of the singularities of the quotient $(C \times C) / G$. If $q(\hat{X}) \geq 3$ then $\hat{X}$ is minimal.

**Theorem 1.2** ([FP15, Theorem 4.5]). Let $C$ be a curve of genus at least two and let $G$ be a group of automorphisms of $C \times C$ that contains an automorphism exchanging the factors. Assume moreover the action of $G$ to be free in codimension 1. Let $\hat{X}$ be the minimal resolution of the singularities of the quotient $(C \times C) / G$. If $q(\hat{X}) \neq 0$, then $\hat{X}$ is minimal.

Both results are sharp.

If $q(\hat{X}) = 0$ we have no Albanese morphism and then determining the minimal model is much more difficult. The first example in literature of a product-quotient variety $\hat{X}$ that is not minimal with $h^0(\Omega^1_{\hat{X}}) = 0$ is the fake Godeaux surface in [BP12, Section 5], whose minimal model is determined by a complicated ad hoc argument.

See also [BP16, Section 6] for some conjectures and partial results about some sufficient conditions for the minimality of $\hat{X}$ when $h^0(\Omega^1_{\hat{X}}) = 0$. 
On the other hand, several informations on the birational class of $X$ can be obtained without running an explicit minimal model program for it, by computing some of the birational invariants of $X$.

The geometric genus and the irregularities of $\hat{X}$ are its simplest birational invariants. They are not difficult to compute for product-quotient varieties.

The next natural birational invariants to consider are the plurigenera $P_d(\hat{X})$, $\forall d \in \mathbb{N}$. They determine a very important birational invariant, the Kodaira dimension $\text{Kod}(\hat{X})$, that equals the rate of growth of $P_d$. This is the fundamental invariant used by the Enriques-Kodaira classification of surfaces and its higher dimensional analogues.

If $X$ is of general type, i.e. $\text{Kod}(\hat{X}) = \dim(\hat{X})$, an important role in the classification theory is played by the volume $\text{vol}(K_{\hat{X}})$ of its canonical divisor, that is also determined by the plurigenera.

Indeed, let us now restrict for sake of simplicity to the case $n = 2$. If $\hat{X}$ is a surface of general type then the natural map of $\hat{X}$ on its minimal model is the composition of $r$ elementary contractions, where $r = \text{vol}(K_{\hat{X}}) - K_{\hat{X}}^2$, and $\text{vol}(K_{\hat{X}})$ equals the self intersection of a canonical divisor of the minimal model.

By [BHPVdV04, Proposition 5.3] a surface of general type $S$ is minimal if and only if $h^1(\mathcal{O}_S(dK_S)) = 0$ for all $d \geq 2$. Then, by Riemann-Roch, $P_d(S) = \chi(\mathcal{O}_S) + (\frac{d^2}{2})K_S^2$, and therefore

$$ (1.1) \quad \left(\frac{d}{2}\right)K_S^2 = P_d(S) + q(S) - p_g(S) - 1. $$

Since the right-hand side of (1.1) is a birational invariant it follows that if $\hat{X}$ is of general type, then

$$ \text{vol}(K_{\hat{X}}) = \frac{P_3(\hat{X}) - P_2(\hat{X})}{2} = P_3(\hat{X}) + q(\hat{X}) - p_g(\hat{X}) - 1. $$

By the Enriques-Kodaira classification and Castelnuovo rationality criterion, every surface $\hat{X}$ with $K^2_{\hat{X}} > 0$ and $P_2(\hat{X}) \neq 0$ is of general type. We have proved the following.

**Proposition 1.3.** Assume $\hat{X}$ is a surface with $K^2_{\hat{X}} > 0$ and $P_2(\hat{X}) \neq 0$.

Then $\hat{X}$ is a surface of general type and

$$ \text{vol}(K_{\hat{X}}) = P_2(\hat{X}) + q(\hat{X}) - p_g(\hat{X}) - 1 = \frac{P_3(\hat{X}) - P_2(\hat{X})}{2}. $$

Similarly, we can compute the volume of the canonical divisor of $\hat{X}$ if we know any pair of plurigenera $P_d$, $d \geq 2$, or one of its plurigenera, geometric genus and all irregularities. Once we compute $K^2_{\hat{X}}$, that is an easy computation, we immediately deduce if $\hat{X}$ is minimal and more generally the number $r$ of irreducible curve of $X$ contracted on the minimal model.
2. **Product quotient varieties birational to Calabi-Yau threefolds**

An important class of varieties is the class of the Calabi-Yau varieties. The smooth Calabi-Yau varieties are one of the three building blocks of the Beauville-Bogomolov decomposition [Bea83] of smooth projective varieties (and more generally, of compact Kähler manifolds) with trivial first Chern class. The Beauville-Bogomolov theorem has been recently extended to the singular case [HP17], requiring an extension of the notion of Calabi-Yau to minimal models. The following is the natural definition, a bit more general than the one necessary for the Beauville-Bogomolov decomposition in [HP17].

**Definition 2.1.** A complex projective variety $Z$ with at most terminal singularities is called Calabi-Yau if it is Gorenstein, $K_Z \sim_{\text{lin}} 0$ and $q_i(Z) = 0$ $\forall 1 \leq i \leq \dim Z - 1$.

There are no Calabi-Yau varieties of dimension 1. Calabi-Yau varieties of dimension 2 are usually called $K3$ surfaces.

We show now that there is no Calabi-Yau product-quotient variety.

**Proposition 2.2.** Let $X = (C_1 \times \ldots \times C_n)/G$ be the quotient model of a product-quotient variety and let $\rho: \hat{X} \to X$ be a partial resolution of the singularities of $X$ such that $\hat{X}$ has at most terminal singularities.

Then $K_{\hat{X}} \not\sim_{\text{num}} 0$.

**Proof.** Let $\pi: \prod C_i \to X$ be the quotient map. Then $\pi$ is unramified in codimension 1. So since $K_{\prod C_i}$ is ample, then $K_X$ is ample too, so it has strictly positive intersection with every curve of $X$. Since codim $\text{Sing } X \geq 2$ one can easily find a curve $C$ in $X$ not containing any singular point of $X$: for example a general fibre of the projection $X = (C_1 \times \ldots \times C_n)/G \to (C_2 \times \ldots \times C_n)/G$. Set $\hat{C} = \rho^* C$. Then $K_X \hat{C} = K_X C \neq 0$ and therefore $K_{\hat{X}} \not\sim_{\text{num}} 0$. \hfill $\Box$

So there is no hope to construct a Calabi-Yau variety directly as partial resolution of the singularities of a product-quotient variety, but one can still hope to get something birational to a Calabi-Yau variety. [GP15] constructed several $K3$ surfaces that are birational to product-quotient variety. Their method starts by constructing product-quotient surfaces with the right first birational invariants $p_g = q = 1$.

We follow a similar approach for constructing Calabi-Yau threefolds. This leads to the following definition:

**Definition 2.3.** A normal threefold $\hat{X}$ is a numerical Calabi-Yau if $p_g(\hat{X}) = 1$, $q_i(\hat{X}) = 0$ for $i = 1, 2$.

Product-quotient threefolds that are birational to a Calabi-Yau threefold are numerical Calabi-Yaus.
**Proposition 2.4.** Let \( \hat{\mathcal{X}} \) be a product-quotient threefold. Assume that \( \hat{\mathcal{X}} \) is birational to a Calabi-Yau threefold. Then \( \hat{\mathcal{X}} \) is a numerical Calabi-Yau.

**Proof.** Let \( Z \) be a Calabi-Yau threefold birational to \( \hat{\mathcal{X}} \). To prove that \( \hat{\mathcal{X}} \) is a numerical Calabi-Yau, we take a common resolution \( \hat{\mathcal{Z}} \) of the singularities of \( Z \) and of \( \hat{\mathcal{X}} \). Since \( \hat{\mathcal{X}} \) and \( Z \) have terminal singularities, and terminal singularities are rational (see \( [\text{Elk81}] \)), it holds

\[
p_g(Z) = p_g(\hat{\mathcal{Z}}) = p_g(\hat{\mathcal{X}}) \quad \text{and} \quad q_i(Z) = q_i(\hat{\mathcal{Z}}) = q_i(\hat{\mathcal{X}})
\]

by the Leray spectral sequence. \( \square \)

**Remark 2.5.** It follows that the quotient model of a numerical Calabi-Yau product-quotient threefold has at least a singular point that is not canonical. Indeed, since the quotient map \( \pi: C_1 \times C_2 \times C_3 \to X \) is quasi-étale and the curves \( C_i \) have genus at least two, then \( K_X \) is ample. If \( X \) had only canonical singularities, then \( \hat{\mathcal{X}} \) would be of general type, and so would be \( Z \), a contradiction.

**Remark 2.6.** Let \( X \) be the quotient model of a numerical Calabi-Yau product-quotient threefold and \( \rho: \hat{\mathcal{X}} \to X \) be a resolution, then \( p_g(\hat{\mathcal{X}}) = 1 \Rightarrow \kappa(\hat{\mathcal{X}}) \neq -\infty \).

Now we run a Minimal Model Program on \( \hat{\mathcal{X}} \). Assume that it ends with a Mori fibre space, then \( \kappa(\hat{\mathcal{X}}) = -\infty \) according to \( [\text{Mat02}, \text{Theorem 3-2-3}] \) which is impossible. Therefore, the Minimal Model Program ends with a threefold \( Z \) with terminal singularities and \( K_Z \) nef.

We close this section with its main result, a criterion to compute if a numerical Calabi-Yau product-quotient threefold is birational to a Calabi-Yau threefold.

**Proposition 2.7.** Let \( \hat{\mathcal{X}} \) be a numerical Calabi-Yau product-quotient threefold. If \( P_d(\hat{\mathcal{X}}) = 1 \) for all \( d \geq 1 \), then \( \hat{\mathcal{X}} \) is birational to a Calabi-Yau threefold.

**Proof.** Let \( Z \) be a minimal model of \( \hat{\mathcal{X}} \). It suffices to show that \( K_Z \) is trivial.

According to Kawamatas abundance for minimal threefolds \( [\text{Kaw92}] \), some multiple \( m_0K_Z \) is base point free. By assumption \( h^0(m_0K_Z) = h^0(m_0K_{\hat{\mathcal{X}}}) = 1 \), which implies that \( m_0K_Z \) is trivial. In particular \( m_0K_{Z^0} \) is trivial, where \( Z^0 = Z \setminus Sing(Z) \) is the smooth locus. Since \( Z \) has terminal singularities \( h^0(K_{Z^0}) = h^0(K_{\hat{\mathcal{X}}}) = 1 \) and it follows that \( K_{Z^0} \) is trivial. Therefore \( K_Z \), being the closure of \( K_{Z^0} \), is also trivial. \( \square \)

3. **Examples of numerical Calabi-Yau product-quotient threefolds**

In this section we present an algorithm that allow us to systematically search for numerical Calabi-Yau threefolds. We use a MAGMA implementation of this algorithm to produce a list of examples of such threefolds. For a detailed account about classification algorithms and the language of product quotients, we refer to \( [\text{Gle16}] \).
To describe the idea of the algorithm, suppose that the quotient model of a numerical Calabi-Yau threefold

$$X = (C_1 \times C_2 \times C_3)/G$$

is given. Then $C_i/G \cong \mathbb{P}^1$ and we have three $G$-covers:

$$f_i: C_i \to \mathbb{P}^1.$$ 

Let $b_{i,1}, \ldots, b_{i,r_i}$ be the branch points of $f_i$ and denote by $T_i := [m_{i,1}, \ldots, m_{i,r_i}]$ the three unordered lists of branching indices, that will be called the types in the sequel. They inherit a lot of combinatorial properties:

**Proposition 3.1.** For the type $T_i$ and their entries, it holds:

i) $m_{i,j} \leq 4g(C_i) + 2$,

ii) $m_{i,j}$ divides the order of $G$,

iii) $r_i \leq \frac{4(g(C_i) - 1)}{n} + 4$,

iv) $2g(C_i) - 2 = |G|\left(-2 + \sum_{j=1}^{r_i} \frac{m_{i,j} - 1}{m_{i,j}}\right)$

**Proof.** ii) follows from the fact that the $m_{i,j}$ are the orders of the stabilizers of the points above the branch points $b_{i,j}$.

i) is an immediate consequence of the classical bound of Wiman for the order of an automorphism of a curve of genus at least 2, since the stabilizers are cyclic.

iv) is the Riemann-Hurwitz formula. iii) follows from iv) and $m_{i,j} \geq 2$. \qed

**1st Step:** The first step of the algorithm is based on the proposition above. As an input value we fix an integer $g_{\text{max}}$. The output is a full list of numerical Calabi-Yau product-quotient threefolds, such that the genera of the curves $C_i$ are bounded from above by $g_{\text{max}}$.

According to Hurwitz bound on the automorphism group, it holds

$$|\text{Aut}(C_i)| \leq 84(g_{\text{max}} - 1).$$

Consequently there are only finitely many possibilities for the order $n$ of the group $G$. On the other hand, for fixed $g_i \leq g_{\text{max}}$ and fixed group order $n$, there are only finitely many possibilities for integers $m_{i,j} \geq 2$ fulfilling the constraints from the proposition above. We wrote a MAGMA code, that returns all admissible combinations

$$[g_1, g_2, g_3, n, T_1, T_2, T_3].$$

**2nd Step:** For each tuple $[g_1, g_2, g_3, n, T_1, T_2, T_3]$ determined in the first step, we search through the groups $G$ of order $n$ and check if we can realize three $G$ covers $f_i: C_i \to \mathbb{P}^1$ with branching indices $T_i := [m_{i,1}, \ldots, m_{i,r_i}]$. By Riemann's existence
Theorem such covers exists if and only if there are elements $h_{i,j} \in G$ of order $m_{i,j}$, which generate $G$ and fulfill the relations

$$
\prod_{j=1}^{r_i} h_{i,j} = 1_G \quad \text{for each } 1 \leq i \leq 3.
$$

Let $X$ be the quotient of $C_1 \times C_2 \times C_3$ by the diagonal action of $G$. The singularities

$$
\frac{1}{n}(1, a, b)
$$

of $X$ can be determined using the elements $h_{i,j}$ cf. [BP12, Proposition 1.17]. The same is true for the invariants $p_g$ and $q_i$ of a resolution cf. [FG16, Section 3], since they are given as the dimensions of the $G$-invariant parts of $H^0(\Omega^i_{C_1 \times C_2 \times C_3})$, which can be determined using the formula formula of Chevalley-Weil see [FG16, Theorem 2.8]. The threefolds with only canonical singularities are discarded as well as those with invariants different from $p_g = 1$, $q_1 = q_2 = 0$. As an output we return the following data of $X$: the group $G$, the types $T_i$, the set of canonical singularities $S_c$ and the set of non-canonical singularities $S_{nc}$.

We run our MAGMA implementation of the algorithm for $g_{\text{max}} = 6$ and the additional restriction that the $f_i: C_i \to \mathbb{P}^1$ are branched in only three points i.e. $r_i = 3$. The output is in Table 1.

They may be birational to a Calabi-Yau threefold or not. Both cases occur, as we will see in Sections 6 and 7.

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**Table 1.** Some numerical Calabi-Yau product-quotient threefolds. Each row corresponds to a threefold, each column to one of the data of the construction: from left to right the group $G$, its $\text{Id}$ in the MAGMA database of finite groups, the three types, the canonical singularities and the singularities that are not canonical. The symbol $(a,b)^\lambda$ used in the last two columns of the table denotes $\lambda$ cyclic quotient singularities of type $\frac{1}{n}(1, a, b)$. We recall the definition in Section 5.
4. The sheaves of ideals $I_d$ on a smooth projective variety with a finite group action

**Proposition 4.1.** Let $Y$ be a smooth quasi-projective variety, let $G$ be a finite subgroup of $\text{Aut}(Y)$ and let $\psi: \hat{X} \rightarrow Y/G$ be a resolution of the singularities. Then there exists a normal variety $\tilde{Y}$, a birational and proper morphism $\phi: \tilde{Y} \rightarrow Y$ and a finite surjective morphism $\epsilon: \tilde{Y} \rightarrow \hat{X}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\epsilon} & \hat{X} \\
\phi \downarrow & & \psi \downarrow \\
Y & \rightarrow & Y/G \\
\end{array}
\]

Up to isomorphism $\tilde{Y}$ is the normalisation of the fibre product $Y \times_{Y/G} \hat{X}$ and $\phi$ and $\epsilon$ are the natural maps.

**Proof.** Note that the assumptions imply the irreducibility of the fibre product $Y \times_{Y/G} \hat{X}$. Since the quotient map $\pi$ is finite and surjective the base change $p: Y \times_{Y/G} \hat{X} \rightarrow \hat{X}$ is also finite and surjective. By definition, $\epsilon$ is the composition of $p$ with the normalisation and therefore finite and surjective, too. Similarly $\phi$ is proper and birational and the existence is proven.

Let $Y'$ be a normal variety, $\phi'$ proper and birational and $\epsilon'$ finite and surjective such that $\pi \circ \phi' = \psi \circ \epsilon'$. Then, by the universal properties of the fibre product and the normalisation, there exists a unique morphism $f: Y' \rightarrow \tilde{Y}$ such that the following diagram commutes

\[
\begin{array}{ccc}
Y' & \xrightarrow{\epsilon'} & \hat{X} \\
\phi' \downarrow & & \psi \downarrow \\
\tilde{Y} & \xrightarrow{\epsilon} & \hat{X} \\
\phi \downarrow & & \psi \downarrow \\
Y & \rightarrow & Y/G \\
\end{array}
\]

Note that $f$ is finite and of degree 1, because $\epsilon'$ and $\epsilon$ are finite and $\deg(\psi') = 1$. Since $\tilde{Y}$ is normal, Zariski’s Main Theorem implies that $f$ is an isomorphism.

**Remark 4.2.** Note that $\tilde{Y}$ fails to be smooth in general cf. [Kol07, Example 2.30].

**Proposition 4.3.** The $G$ action on $Y$ lifts to an action on $\tilde{Y}$ such that $\hat{X}$ is the quotient.

**Proof.** Consider the natural $G$ action on $Y \times_{Y/G} \hat{X}$. By the universal property of the normalisation it lifts to an action on $\tilde{Y}$. The birational map $\tilde{Y}/G \rightarrow \hat{X}$ induced by $\epsilon$ has finite fibres and is therefore an isomorphism by Zariski’s Main theorem.
Remark 4.4. Let $Y$ be a normal quasi-projective variety and $G < \text{Aut}(Y)$ be a finite group. Then the quotient map $\pi: Y \to X := Y/G$ induces an isomorphism

$$\pi^* : H^0(X, L) \simeq H^0(Y, \pi^* L)^G$$

for any line bundle $L$ on $X$.

The quotient $X := Y/G$ is a normal $\mathbb{Q}$-factorial quasi projective variety, in particular each canonical divisor $K_X$ is $\mathbb{Q}$-Cartier. Let $\psi: \tilde{X} \to X$ be a resolution of singularities and let $K_{\tilde{X}}$ be a canonical divisor. Let $d$ be an integer such that $dK_{\tilde{X}}$ is Cartier and define the pullback of $K_X$ in the usual way:

$$\psi^* K_X := \frac{1}{d} \psi^*(d K_X).$$

By [Mat02, Remark 4-1-2] there exists a canonical divisor $K_{\tilde{X}}$ such that

$$K_{\tilde{X}} = \psi^* K_X + E,$$

where the $\mathbb{Q}$-divisor $E$ is supported on the exceptional locus $\text{Exc}(\psi)$.

The quotient map $\pi: Y \to X$ is finite and the singular locus $\text{Sing}(X) \subset X$ has codimension $\geq 2$. The restricted map

$$\pi: Y \setminus \pi^{-1}(\text{Sing}(X)) \to X \setminus \text{Sing}(X)$$

is a finite morphism between smooth varieties and the Hurwitz formula

$$K_{Y \setminus \pi^{-1}(\text{Sing}(X))} = \pi^* K_{X \setminus \text{Sing}(X)} + R^0$$

holds. Since $\pi^{-1}(\text{Sing}(X)) \subset Y$ has codimension at least $\geq 2$ the pullback $\pi^* K_{X \setminus \text{Sing}(X)}$ yields a unique divisor on $Y$, that coincides with $\pi^* K_X := \frac{1}{d} \psi^*(d K_X)$.

Therefore, the Hurwitz formula extends uniquely

$$K_Y = \pi^* K_X + R.$$

We point out that $Y$ is smooth, and then $R$ is Cartier.

**Theorem 4.5.** Let $Y$ be a smooth projective variety, let $G$ be a finite subgroup of $\text{Aut}(Y)$, let $\psi: \tilde{X} \to X := Y/G$ be a resolution of the singularities and $\pi: Y \to X$ be the quotient map. We write

$$\pi^* K_X + R = K_Y \quad \text{and} \quad \psi^* K_X + E = K_{\tilde{X}}$$

as above and $E = P - N$, where $P, N$ are effective without common components. Consider the sheaf of ideals

$$\mathcal{I}_d := \mathcal{O}_Y(-dR) \otimes \phi_* \mathcal{O}_{\tilde{X}}(\epsilon^* dE) \cong \mathcal{O}_Y(-dR) \otimes \phi_* \mathcal{O}_{\tilde{X}}(-\epsilon^* dN).$$

Then, for all $d \geq 1$ there is a natural isomorphism

$$H^0 \left( \tilde{X}, \mathcal{O}(dK_{\tilde{X}}) \right) \simeq H^0 \left( Y, \mathcal{O}(dK_Y) \otimes \mathcal{I}_d \right)^G,$$

where $\phi$ and $\epsilon$ are the morphisms from Proposition 4.1.
Proof. Using Remark 4.4, we compute
\[ \epsilon^*dK = \epsilon^*(\psi^*dK_X + dE) \]
\[ = \epsilon^*\psi^*dK_X + \epsilon^*dE \]
\[ = \phi^*\pi^*dK_X + \epsilon^*dE \]
\[ = \phi^*(dK_Y - dR) + \epsilon^*dE. \]

Since the divisors \( \epsilon^*dK \) and \( \phi^*(dK_Y - dR) \) are Cartier, the divisor \( \epsilon^*dE \) is also Cartier and we obtain the isomorphism of line bundles
\[ \mathcal{O}_{\tilde{Y}}(\epsilon^*dK) \cong \mathcal{O}_{\tilde{Y}}(\phi^*(dK_Y - dR)) \otimes \mathcal{O}_{\tilde{Y}}(\epsilon^*dE) \]

According to Proposition 4.3 \( \tilde{X} \) is the quotient of \( \tilde{Y} \) by \( G \), by Remark 4.4
\[ H^0(\tilde{X}, \mathcal{O}_X(dK)) \cong H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\phi^*(dK_Y - dR)) \otimes \mathcal{O}_{\tilde{Y}}(\epsilon^*dE))^G \]

Using the projection formula:
\[ H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\phi^*(dK_Y - dR)) \otimes \mathcal{O}_{\tilde{Y}}(\epsilon^*dE))^G = H^0(Y, \mathcal{O}_Y(dK_Y - dR) \otimes \phi_*\mathcal{O}_{\tilde{Y}}(\epsilon^*dE))^G, \]

since \( \phi_*\mathcal{O}_{\tilde{Y}}(\epsilon^*dE) \cong \phi_*\mathcal{O}_{\tilde{Y}}(-\epsilon^*dN) \) the proof is finished. \( \square \)

Theorem 4.5 allows to compute the plurigenus \( P_d(\tilde{X}) \) if one may determine in an effective way the sheaf of ideals \( \phi_*\mathcal{O}(\epsilon^*dE) \cong \phi_*\mathcal{O}(-\epsilon^*dN) \), which is in general very hard to do.

However, we are interested in product-quotient varieties, whose singularities are cyclic quotient singularities. In the next section we provide a strategy for computing these ideal for manifolds having only this type of singularities restricting, for sake of simplicity, to the case of isolated singularities.

5. THE SHEAVES OF IDEALS \( I_d \) FOR CYCLIC QUOTIENT SINGULARITIES

In this section we specialize to the case of a \( G \)-action, where the fixed locus of every automorphism \( g \in G \) is isolated and the stabilizer of each point \( y \in Y \) is cyclic. Under this assumption, each singularity of \( Y/G \) is an isolated cyclic quotient singularity
\[ \frac{1}{m}(a_1, \ldots, a_n), \]
i.e. locally in the analytic topology, around the singular point the variety \( Y/G \) is isomorphic to a quotient \( \mathbb{C}^n/H \), where \( H \simeq \mathbb{Z}_m \) is a cyclic group generated by a diagonal matrix
\[ \text{diag} (\xi^{a_1}, \ldots, \xi^{a_n}), \quad \text{where} \quad \xi := \exp \left( \frac{2\pi \sqrt{-1}}{m} \right) \quad \text{and} \quad \gcd(a_i, m) = 1. \]

Since cyclic quotient singularities fit into the category of affine toric varieties, we can use toric geometry to construct a resolution \( \tilde{X} \) of the quotient \( Y/G \) and give a local description of the variety \( \tilde{Y} \) in Proposition 4.1. We collect some basics about
cyclic quotient singularities from the toric point of view, that will be used in the sequel. For details we refer to [CLS11].

**Remark 5.1.**

- As an affine toric variety, the singularity \( \frac{1}{m}(a_1, \ldots, a_n) \) is given by the lattice \( N := \mathbb{Z}^n + \mathbb{Z}[a_1, \ldots, a_n] \) and the cone \( \sigma := \text{cone}(e_1, \ldots, e_n) \), where the vectors \( e_i \) are the euclidean unit vectors. We denote this affine toric variety by \( U_\sigma \).
- The inclusion \( i: (\mathbb{Z}^n, \sigma) \to (N, \sigma) \) induces the quotient map \( \pi: \mathbb{C}^n \to \mathbb{C}^n/\mathbb{Z}_m \).
- There exists a subdivision of the cone \( \sigma \), yielding a fan \( \Sigma \) such that the toric variety \( \widetilde{X}_\Sigma \) is smooth and the morphism \( \psi: \widetilde{X}_\Sigma \to U_\sigma \) induced by the identity map of the lattice \( N \) is a resolution of \( U_\sigma \) i.e. birational and proper.

Now, the local construction of \( \widetilde{Y} \) as a toric variety is straightforward. Observe that the fan \( \Sigma \) is also a fan in the lattice \( \mathbb{Z}^n \). We define \( \widetilde{Y}_\Sigma \) to be the toric variety associated to \( (\mathbb{Z}^n, \Sigma) \). The commutative diagram

\[
\begin{array}{ccc}
(\mathbb{Z}^n, \Sigma) & \longrightarrow & (N, \Sigma) \\
\downarrow & & \downarrow \\
(\mathbb{Z}^n, \sigma) & \longrightarrow & (N, \sigma)
\end{array}
\]

of inclusions induces a commutative diagram of toric morphisms, which is the local version of the diagram from Proposition 4.1:

\[
\begin{array}{ccc}
\widetilde{Y}_\Sigma & \xrightarrow{\epsilon} & \widetilde{X}_\Sigma \\
\phi \downarrow & & \psi \downarrow \\
\mathbb{C}^n & \xrightarrow{\pi} & \mathbb{C}^n/\mathbb{Z}_m
\end{array}
\]

**Proposition 5.2.** The map \( \epsilon: \widetilde{Y}_\Sigma \to \widetilde{X}_\Sigma \) is finite and surjective and \( \phi: \widetilde{Y}_\Sigma \to \mathbb{C}^n \) is birational and proper.

**Proof.** We need to show that \( \mathbb{C}[N^\vee \cap \tau^\vee] \subset \mathbb{C}[\mathbb{Z}^n \cap \tau^\vee] \) is a finite ring extension for all cones \( \tau \) in \( \Sigma \). Clearly, any element of the form \( c \chi^q \in \mathbb{C}[\mathbb{Z}^n \cap \tau^\vee] \) is integral over \( \mathbb{C}[N^\vee \cap \tau^\vee] \), because \( mq \in N^\vee \cap \tau^\vee \) and \( c \chi^q \) solves the monic equation \( x^m - c^m \chi^{mq} = 0 \). The general case follows from the fact that any element in \( \mathbb{C}[\mathbb{Z}^n \cap \tau^\vee] \) is finite sum of elements of the form \( c \chi^q \) and finite sums of integral
elements are also integral. Since $\Sigma$ is a refinement of $\sigma$ the morphism $\phi$ is birational and proper according to [CLS11, Theorem 3.4.11]. □

For the next step, we describe how to determine the discrepancy divisor in $\hat{X}$ over each singular point of the quotient $Y/G$ and its pullback under the morphism $\epsilon$.

**Proposition 5.3** ( [CLS11, Proposition 6.2.7 and Lemma 11.4.10]).

- The exceptional prime divisors of the birational morphisms $\psi: \hat{X}_\Sigma \rightarrow U_\sigma$ and $\phi: \tilde{Y}_\Sigma \rightarrow \mathbb{C}^n$

  are in one to one correspondence with the rays $\rho \in \Sigma \setminus \sigma$.

- Write $v_\rho \in \mathbb{N}$ for the primitive generator of the ray $\rho$ and $E_\rho \subset \hat{X}_\Sigma$ for the corresponding prime divisor, then it holds

  $$K_{X_\Sigma} = \psi^* K_{U_\sigma} + E$$

  where

  $$E := \sum_{\rho \in \Sigma \setminus \sigma} (\langle v_\rho, e_1 + \ldots + e_n \rangle - 1) E_\rho.$$

- Write $w_\rho \in \mathbb{Z}^n$ for the primitive generator of the ray $\rho$ and $F_\rho \subset \tilde{Y}_\Sigma$ for the corresponding prime divisor, then

  $$\epsilon^* E_\rho = \lambda_\rho F_\rho$$

  where $\lambda_\rho > 0$ such that $w_\rho = \lambda_\rho v_\rho$.

  In particular

  $$\epsilon^* E = \sum_{\rho \in \Sigma \setminus \sigma} \lambda_\rho (\langle v_\rho, e_1 + \ldots + e_n \rangle - 1) F_\rho = \sum_{\rho \in \Sigma \setminus \sigma} (\langle w_\rho, e_1 + \ldots + e_n \rangle - 1) F_\rho.$$

It remains to determine the pushforward $\phi_* \mathcal{O}_{\tilde{Y}_\Sigma}(\epsilon^* dE)$ for $d \geq 1$. We provide a recipe for computing $\phi_* \mathcal{O}_{\tilde{Y}_\Sigma}(\epsilon^* D)$ a general Weil divisor $D$ supported on the exceptional locus of $\phi: \tilde{Y}_\Sigma \rightarrow \mathbb{C}^n$.

**Proposition 5.4.** Let $\phi: \tilde{Y}_\Sigma \rightarrow \mathbb{C}^n$ be the birational morphism from above and

$$D = \sum_{\rho \in \Sigma \setminus \sigma} u_\rho F_\rho, \quad u_\rho \in \mathbb{Z}$$

be a Weil-divisor, supported on the exceptional locus of $\phi$. For each integer $k \geq 1$, we define the sheaf of ideals $I_{kD} := \phi_* \mathcal{O}(kD)$, then:

1) The ideal of global sections $I_{kD} \subset \mathbb{C}[x_1, \ldots, x_n]$ is given by

$$I_{kD} = \bigoplus_{\alpha \in P_D} \mathbb{C} \cdot \chi^\alpha,$$

where $P_D := \{ u \in \mathbb{R}^n \mid u_i \geq 0, \langle u, w_\rho \rangle \geq -u_\rho \}$

is the polyhedron associated to $D$. 

ii) Let $l = (l_1, \ldots, l_n)$ be a tuple of positive integers such that $l_i \cdot e_i \in P_D$ and define  
\[ \square_l := \{ y \in \mathbb{R}^n \mid 0 \leq y_i \leq l_i \}. \]
Then, the set of monomials $\chi^\alpha$, where $\alpha$ is a lattice point in the polytope $k(\square_l \cap P_D)$ generate $I_{kD}$.

Proof. i) By definition of the pushforward and the surjectivity of $\phi$, it holds  
\[ I_{kD} = \phi_* O_{\widetilde{Y}_\Sigma}(kD)(\mathbb{C}^n) = H^0(\widetilde{Y}_\Sigma, O_{\widetilde{Y}_\Sigma}(kD)). \]
According to [CLS11, Proposition 4.3.3], we have  
\[ H^0(\widetilde{Y}_\Sigma, O(kD)) = \bigoplus_{\alpha \in P_{kD} \cap \mathbb{Z}^n} \mathbb{C} \cdot \chi^\alpha \]
and the claim follows since $kP_D = P_{kD}$. Note that the inequalities $u_i \geq 0$ imply  
\[ \chi^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \quad \text{for all} \quad \alpha \in P_{kD} \cap \mathbb{Z}^n. \]

ii) Let $\chi^\alpha$ be a monomial, such that the exponent $\alpha = (\alpha_1, \ldots, \alpha_n) \in P_{kD} \cap \mathbb{Z}^n$ is not contained in the polytope  
\[ k(\square_l \cap P_D) = \square_{kl} \cap P_{kD}, \]
say $kl_1 < \alpha_1$. Then we define $\beta_1 := \alpha_1 - kl_1$ and write $\chi^\alpha$ as a product  
\[ \chi^\alpha = \chi^{(\beta_1, \alpha_2, \ldots, \alpha_n)} \chi^{kl_1 e_1}. \]

Remark 5.5.

- Note that the inequalities $\langle u, w_\rho \rangle \geq -ku_\rho$ in the definition of the polyhedron  
\[ P_{kD} = \{ u \in \mathbb{R}^n \mid u_i \geq 0, \quad \langle u, w_\rho \rangle \geq -ku_\rho \} \]
are redundant if $u_\rho \geq 0$.
- In particular, for $D = \epsilon^* E$ we have $u_\rho = \lambda_\rho (\langle v_\rho, e_1 + \ldots + e_n \rangle - 1)$. This integer is, according to Proposition 5.3, equal to the discrepancy of $E_\rho$ multiplied with $\lambda_\rho > 0$. For example, if the singularity is canonical i.e. all $u_\rho \geq 0$, then the ideal $I_{\epsilon^* kE}$ is trivial.

Remark 5.6. If we perform the star subdivision of the cone $\sigma$ along all rays generated by a primitive lattice point $v_\rho$ with  
\[ \langle v_\rho, e_1 + \ldots + e_n \rangle - 1 < 0 \]
we obtain a fan $\Sigma'$ that is not necessarily smooth. However, there is a subdivision of $\Sigma'$ yielding a smooth fan $\Sigma$. Since the new rays $\rho \in \Sigma \setminus \Sigma'$ do not contribute to the polyhedra of $\epsilon^* kE$, there is no need to compute $\Sigma$ explicitly.

From the description of the ideal $I_{kD}$, it follows that $(I_D)^k \subset I_{kD}$ for all positive integers $k$. However, this inclusion is in general not an equality. The reason is that the polytope $\square_l \cap P_D$ may not contain enough lattice points. We can solve this problem by replacing $D$ with a high enough multiple:
Proposition 5.7. Let $D$ be a divisor as in Proposition 5.4. Then, there exists a positive integer $s$ such that

$$(I_{sD})^k = I_{skD} \quad \text{for all} \quad k \geq 1.$$ 

Proof. Let $l = (l_1, \ldots, l_n)$ be a tuple of positive integers such that $l_i \cdot e_i \in P_D$. According to Proposition 5.4 ii) the monomials $\chi^\alpha$ with

$$\alpha \in k(\square_l \cap P_D) \cap \mathbb{Z}^n$$

generate $I_{kD}$ for all $k \geq 1$. Since the vertices of the polytope $\square_l \cap P_D$ have rational coordinates, there is a positive integer $s'$ such that $s'(\square_l \cap P_D)$ is a lattice polytope i.e. the convex hull of finitely many lattice points. We define $s := s'(n - 1)$, then $s(\square_l \cap P_D)$ is a normal lattice polytope (see [CLS11, Theorem 2.2.12]), which means that

$$(ks'(\square_l \cap P_D)) \cap \mathbb{Z}^n = k(s'(\square_l \cap P_D) \cap \mathbb{Z}^n) \quad \text{for all} \quad k \geq 1.$$ 

Clearly, this implies $(I_{sD})^k = I_{skD}$ for all $k \geq 1$. \hfill \square

Remark 5.8. According to its proof we can take in Proposition 5.7 $s = (n - 1)s'$ where $s'$ is the smallest positive integer such that all the vertices of $s'P_D$ have integral coordinates.

Listing 1. Computation of the ideal $I_{ke^*E}$ for the singularity $1/n(1,a,b)$

```plaintext
// The first function determines the lattice points that we need to blow up according 1 // to Computational Rem 5.6. It returns the primitive generators "w_\rho" of these points 2 // according to Prop 5.3 and the discrepancy of the pullback divisor eps^{\ast}E.

Vectors:=function(n,a,b)
7 Ve:={};
for i in [1..n−1] do
x:=i/n;
y:=(i*a mod n)/n;
z:=(i*b mod n)/n;
d:=x+y+z−1;
if d lt 0 then
   lambda:=Lcm([Denominator(x),Denominator(y),Denominator(z)]);
   Include(~Ve,[lambda*x,lambda*y,lambda*z,lambda*d]);
end if;
end for;
return Ve;
end function;
```

// The function "IntPointsPoly" determines a basis for the monomial ideal,
// according to Proposition 5.4. However, this basis is not necessarily minimal.
// The subfunction "MinMultPoint" is used to determine the cube in ii) of Proposition 5.4.

MinMultPoint:=function(P,v)
7 n:=1;
while n*v notin P do
   n:= n+1;
end while;
```
end while;
return n;
end function;

IntPointsPoly:=function(n,a,b,k)
L:=ToricLattice(3);
La:=Dual(L);
e1:=L![1,0,0]; e2:=L![0,1,0]; e3:=L![0,0,1];
P:=HalfspaceToPolyhedron(e1,0) meet
    HalfspaceToPolyhedron(e2,0) meet
    HalfspaceToPolyhedron(e3,0);
Vec:=Vectors(n,a,b);
for T in Vec do
  w:=L![T[1],T[2],T[3]];
  u:=T[4];
  P:= P meet HalfspaceToPolyhedron(w,−k∗u);
end for;
multx:=MinMultPoint(P,La![1,0,0]);
multy:=MinMultPoint(P,La![0,1,0]);
multz:=MinMultPoint(P,La![0,0,1]);
P:=P meet HalfspaceToPolyhedron(L![−1,0,0],−k∗multx) meet
    HalfspaceToPolyhedron(L![0,−1,0],−k∗multy) meet
    HalfspaceToPolyhedron(L![0,0,−1],−k∗multz);
return Points(P);
end function;

// The next functions are used to find the (unique) minimal monomial basis of the ideal.

IsMinimal:=function(Gens)
test:=true; a:=0;
for a1 in Gens do
  for a2 in Gens do
    if a1 ne a2 then
      d:=a2−a1;
      if d.1 ge 0 and d.2 ge 0 and d.3 ge 0 then
        a:=a1; test:=false;
        break a1;
      end if;
    end if;
  end for;
end for;
return test, a;
end function;

SmallerGen:=function(Gens,a)
Set:=Gens;
for b in Gens do
  if d.1 ge 0 and d.2 ge 0 and d.3 ge 0 and b ne a then
    Exclude(~Set,b);
  end if;
end for;
return Set;
end function;

MinBase:=function(n,a,b,k)
F:=RationalField();
PL<\text{x1},\text{x2},\text{x3}>:=\text{PolynomialRing}(\text{F},3);

test:=false;

\text{Gens}:=\text{IntPointsPoly}(n,a,b,k);

\textbf{while} test \text{ eq false} \textbf{do}

\hspace{1em} test, a:=\text{IsMinimal}(\text{Gens});

\hspace{1em} \textbf{if} test \text{ eq false} \textbf{then}

\hspace{3em} \text{Gens}:=\text{SmallerGen}(\text{Gens},a);

\hspace{1em} \textbf{end if};

\textbf{end while};

MB:=\{\};

\textbf{for} g \textbf{in} \text{Gens} \textbf{do}

\hspace{1em} Include(\sim \text{MB},\text{PL.1}^g.1\ast\text{PL.2}^g.2\ast\text{PL.3}^g.3);

\textbf{end for};

\textbf{return} MB;

\textbf{end function};

6. A Calabi-Yau 3-fold

In this section we apply Theorem 4.5 to the first numerical Calabi-Yau threefold listed in Section 3.

We start by giving an explicit description of the threefold writing the canonical ring of the curve \( C := C_1 \cong C_2 \cong C_3 \) and the group action on it.

We consider the hyperelliptic curve

\[
C := \{ y^2 = x_0^6 + x_1^6 \} \subset \mathbb{P}(1,1,3)
\]

of genus 2, together with the \( \mathbb{Z}_6 \)-action generated by the automorphism \( g \) defined by

\[
g((x_0 : x_1 : y)) = (x_0 : \omega x_1 : y), \quad \text{where} \quad \omega := e^{2\pi i/6}.
\]

By standard adjunction there is an isomorphism of graded rings among \( R(C, K_C) := \bigoplus d H^0(C, O_C(dK_C)) \) and \( \mathbb{C}[x_0, x_1, y]/(y^2 - x_0^6 - x_1^6) \), where \( \deg x_i = 1 \) and \( \deg y = 2 \).

\textbf{Lemma 6.1.} The action of \( g \) on \( R(C, K_C) \) induced by the pull-back of holomorphic differential forms is

\[
x_0 \mapsto \omega x_0 \quad \quad x_1 \mapsto \omega^2 x_1 \quad \quad y \mapsto \omega^3 y = -y
\]

\textbf{Proof.} Consider the smooth affine chart \( x_0 \neq 0 \) with local coordinates \( u := \frac{x_0}{x_1} \) and \( v := \frac{x_1}{x_0} \). In this chart \( C \) is the vanishing locus of \( f := v^2 - u^6 - 1 \). By adjunction the monomial \( x_0, x_1, y \in R(C, K_C) \) correspond respectively to the forms that, in this chart, are

\[
x_0 \mapsto \frac{du}{dv} = \frac{du}{2v} \quad \quad x_1 \mapsto u \frac{du}{2v} \quad \quad y \mapsto v \left( \frac{du}{2v} \right)^3
\]

The statement follows since \( g \) acts on the local coordinates as \( (u, v) \mapsto (\omega u, v) \). □
Proposition 6.2. The threefold $X := C^3/\mathbb{Z}_6$, where the group $\mathbb{Z}_6$ acts as above on each copy of $C$, is a numerical Calabi-Yau threefold.

The singular points of $X$ that are not canonical are 8, all of type $\frac{1}{6}(1, 1, 1)$.

Proof. The points on $C$ with non-trivial stabilizer subgroup of $\mathbb{Z}_6$ are the four points $p_0, p_1, p_2, p_3$ with the following weighted homogeneous coordinates $(x_0 : x_1 : y)$:

\[ p_0 = (1 : 0 : 1) \quad p_1 = (1 : 0 : -1) \quad p_2 = (0 : 1 : 1) \quad p_3 = (0 : 1 : -1) \]

We give in the table below, for each point $p_j$, a generator of its stabilizer, and its action on a local parameter of the curve $C$ near $p_j$.

| point   | $p_{0/1} = (1 : 0 : \pm 1)$ | $p_{2/3} = (0 : 1 : \pm 1)$ |
|---------|-----------------------------|-----------------------------|
| generator of the stabilizer | $g$ | $g^2$ |
| local action | $x \mapsto \omega x$ | $x \mapsto \omega^4 x$ |

$p_0$ and $p_1$ are then stabilized by the whole group $\mathbb{Z}_6$, forming then two orbits of cardinality 1, whereas $p_2$ and $p_3$ are stabilized by the index two subgroup of $\mathbb{Z}_6$, and form a single orbit.

Consequently the points with nontrivial stabilizer are the 64 points $p_{i_1} \times p_{i_2} \times p_{i_3}$ forming 8 orbits of cardinality 1, the points $p_{i_1} \times p_{i_2} \times p_{i_3}$ with $i_j \in \{0, 1\}$, and and 28 of cardinality 2. So $C^3/\mathbb{Z}_6$ has 36 singular points:

- 8 singular points of type $\frac{1}{6}(1, 1, 1)$, the classes of the points $p_{i_1} \times p_{i_2} \times p_{i_3}$ with $i_j \in \{0, 1\}$: these singular points are not canonical;
- 4 singular points of type $\frac{1}{3}(1, 1, 1)$, the classes of the points $p_{i_1} \times p_{i_2} \times p_{i_3}$ with $i_j \in \{2, 3\}$: these singular points have a crepant resolution;
- 24 singular points of type $\frac{1}{3}(1, 1, 2)$, the classes of the remaining points $p_{i_1} \times p_{i_2} \times p_{i_3}$: these are terminal singularities.

We prove now that a resolution $\rho: \hat{X} \to X = C^3/\mathbb{Z}_6$ has invariants $p_g(\hat{X}) = 1, q_1(\hat{X}) = q_2(\hat{X}) = 0$ using representation theory and the fact that

\[ H^0(\hat{X}, \Omega^i_{\hat{X}}) \simeq H^0(C^3, \Omega^i_{C^3})^G. \]

By Lemma 6.1 the character of the natural representation $\varphi: \mathbb{Z}_6 \to GL(H^0(K_C))$ is $\chi_\varphi = \chi_\omega + \chi_\omega^2$. By Künneth’s formula the characters $\chi_i$ of the $\mathbb{Z}_6$ representations on $H^0(C^3, \Omega^i_{C^3})$ are respectively

\[ \chi_3 = \chi_\omega^3 \quad \chi_2 = 3\chi_\omega^2 \quad \chi_1 = 3\chi_\omega. \]

The claim follows, since $\chi_3$ contains exactly one copy of the trivial character whereas $\chi_2$ and $\chi_1$ do not contain the trivial character at all. \qed

We write coordinates

\[ ((x_{01} : x_{11} : y_1), (x_{02} : x_{12} : y_2), (x_{03} : x_{13} : y_3)) \]
on \( \mathbb{P}(1, 1, 3)^3 \), so that \( C^3 \) is the locus defined by the ideal \((y_j^2 - x_{ij}^6 - x_{ij}^6)\).

Künneth’s formula yields a basis for \( H^0(dK_{C^3}) \):

\[
\left\{ \prod_{i=1}^{3} x_{0i}^{a_i} x_{1i}^{b_i} y_{i}^{c_i} \mid a_i + b_i + 3c_i = d, \quad c_i = 0, 1 \right\}
\]
on which \( g \) acts as

\[
\prod_{i=1}^{3} x_{0i}^{a_i} x_{1i}^{b_i} y_{i}^{c_i} \mapsto \omega \sum_i (a_i + 2b_i + 3c_i) \prod_{i=1}^{3} x_{0i}^{a_i} x_{1i}^{b_i} y_{i}^{c_i}
\]

By the proof of Proposition 6.2, writing \( p_i = (1 : 0 : (-1)^i) \in \mathbb{P}(1, 1, 3) \) for \( i = 0, 1, \)
the eight points

\( p_{i1} \times p_{i2} \times p_{i3}, \quad i_j = 0, 1 \)

are precisely those that descend to the eight singularities of type \( \frac{1}{6}(1, 1, 1) \).

To determine the plurigenera of \( X \) we need the following lemma.

**Lemma 6.3.** \( \forall d \geq 1 \), the sheaf of ideals \( \mathcal{I}_d \) equals \( P^d \) where \( P \) is the ideal of the reduced scheme \( \{ p_{i1} \times p_{i2} \times p_{i3} | i_j = 0 \} \).

**Proof.** As already mentioned, all non-canonical singularities are of type \( \frac{1}{6}(1, 1, 1) \).
These singularities are resolved by a single toric blowup along the ray \( \rho \) generated by \( v := \frac{1}{6}(1, 1, 1) \). The polyhedron associated to the divisor \( \epsilon^*dE = -3dF_\rho \) is

\[
P_{-3dF_\rho} = \{ u \in \mathbb{R}^3 \mid u_i \geq 0, \quad u_1 + u_2 + u_3 \geq 3d \},
\]
so the corresponding ideal is just the \( 3d \)-th power of the maximal ideal. \( \square \)

Then we can prove

**Proposition 6.4.** \( X = C^3/\mathbb{Z}_6 \) is birational to a Calabi-Yau threefold.

**Proof.** By Proposition 2.7 we only need to prove that all plurigenera are equal to 1, so, by Theorem 4.5, that, \( \forall d \geq 1 \),

\[
H^0 \left( C^3, \mathcal{O}_{C^3}(dK_{C^3}) \otimes \mathcal{I}_d \right) \cong \mathbb{C}
\]

The vector space \( H^0 \left( C^3, \mathcal{O}_{C^3}(dK_{C^3}) \right) \) is contained in the multigraded ring

\[
R := \mathbb{C}[x_{01}, x_{11}, y_1, x_{02}, x_{12}, y_2, x_{03}, x_{13}, y_3] / \left( y_i^2 - x_{0i}^6 - x_{1i}^6 \right)
\]
with gradings

\[
\begin{align*}
\deg x_{01} &= (1, 0, 0) & \deg x_{11} &= (1, 0, 0) & \deg y_1 &= (2, 0, 0) \\
\deg x_{02} &= (0, 1, 0) & \deg x_{12} &= (0, 1, 0) & \deg y_2 &= (0, 2, 0) \\
\deg x_{03} &= (0, 0, 1) & \deg x_{13} &= (0, 0, 1) & \deg y_3 &= (0, 0, 2)
\end{align*}
\]
as the subspace $R_{d,d,d}$ of the homogeneous elements of multidegree $(d,d,d)$. By Lemma 6.1 the natural action of $G$ on $H^0(C^3,\mathcal{O}_{C^3}(dK_{C^3}))$ is induced by the restriction of the following action of its generator $g$ on $R$:
\[
\begin{align*}
    x_0i &\mapsto \omega x_0i \\
    x_1i &\mapsto \omega^2 x_1i \\
    y_i &\mapsto \omega^3 y_i
\end{align*}
\]
By Lemma 6.3, since the elements of $R$ vanishing on the reduced scheme $\{p_i \times p_2 \times p_3| i;j = 0\}$ form the ideal $(x_{11}, x_{21}, x_{31})$
\[
H^0(C^3,\mathcal{O}_{C^3}(dK_{C^3}) \otimes \mathcal{I}_d) = R_{d,d,d} \cap (x_{11}, x_{12}, x_{13})^{3d} = \langle (x_{11}x_{12}x_{13})^d \rangle
\]
is one dimensional.
Since its generator $x_{11}x_{12}x_{13}$ is $G$–invariant, the proof is complete. \hfill \square

7. A fake Calabi-Yau 3-fold

We consider the hyperelliptic curves
\[
C_2 := \{y^2 = x_0x_1(x_0^4+x_1^4)\} \subset \mathbb{P}(1,1,3) \quad \text{and} \quad C_3 := \{y^2 = x_0^3+x_1^3\} \subset \mathbb{P}(1,1,4)
\]
of respective genus two and three, together with the $\mathbb{Z}_8$-actions $g(x_0 : x_1 : y) = (x_0 : \omega^2x_1 : \omega y)$ on $C_2$ and $g(x_0 : x_1 : y) = (x_0 : \omega x_1 : y)$ on $C_3$, where $\omega = e^{2\pi i/8}$.

**Proposition 7.1.** The threefold $X = (C_2^3 \times C_3)/G$, where $G = \mathbb{Z}_8$ acts diagonally, is a numerical Calabi-Yau threefold. $X$ has exactly 44 singular points and more precisely
\[
6 \times \frac{1}{8}(1,1,3), \quad 2 \times \frac{1}{8}(1,1,1), \quad 3 \times \frac{1}{4}(1,1,3), \quad 1 \times \frac{1}{4}(1,1,1). \quad \text{32} \times \frac{1}{2}(1,1,1).
\]

**Proof.** The points with non-trivial stabilizer on $C_2$ are $q_0 := (0 : 1 : 0)$ and $q_1 := (1 : 0 : 0)$ with the full group as stabilizer and the points
\[
p_i := (1 : x_i : 0), \quad \text{where} \quad x_i^4 = -1
\]
with stabilizer $\langle g^4 \rangle \cong \mathbb{Z}_2$.

Next, we compute the local action around the points $p_i$ and $q_i$.

The points $q_1$ and $p_i$ are contained in the smooth affine chart $x_0 \neq 0$ of $\mathbb{P}(1,1,3)$, with affine coordinates $u = \frac{x_0}{y}$ and $v = \frac{x_1}{y}$. Here, the curve is the vanishing locus of the polynomial $f := v^2 - u^5 - u$ and the group acts via $(u, v) \mapsto (\omega^2u, \omega v)$.

Since $\frac{\partial f}{\partial u}(q_1) = -1$ and $\frac{\partial f}{\partial v}(p_i) = 4$, by the implicit function theorem, $v$ is a local parameter for $C_2$ near these points. In particular $g$ acts around $q_1$ as the multiplication by $\omega$ and $g^4$ acts around $p_i$ as the multiplication by $\omega^4 = -1$.

A similar computation on the affine chart $x_1 \neq 0$ shows that $g$ acts around $q_0$ as the multiplication by $\omega^3$. The table below summarizes our computation.

| point      | $q_0 = (0 : 1 : 0)$ | $q_1 = (1 : 0 : 0)$ | $p_i = (1 : x_i : 0)$ |
|------------|---------------------|---------------------|-----------------------|
| Stab       | $\langle g \rangle$ | $\langle g \rangle$ | $\langle g^4 \rangle$ |
| local action | $x \mapsto \omega^4x$ | $x \mapsto \omega x$ | $x \mapsto -x$ |
Similarly, for $C_3$, we obtain:

| Points | $s_1 = (1 : 0 : 1)$, $s_3 = (0 : 1 : 1)$, $s_2 = (1 : 0 : -1)$, $s_4 = (0 : 1 : -1)$ |
|--------|------------------------------------------------------------------|
| Stab   | $\langle g \rangle$ $\langle g^2 \rangle$                        |
| Local action | $x \mapsto \omega x$ $x \mapsto \omega^b x$                        |

Then the diagonal action on $C_2^2 \times C_3$ admits $6 \cdot 4 \cdot 4 = 144$ points with non-trivial stabilizer. The 8 points of the form

$q_i \times q_j \times s_k, \quad \text{where} \quad i, j \in \{0, 1\} \quad \text{and} \quad k \in \{1, 2\}$

are stabilized by the full group. Therefore, they are mapped to 8 singular points on the quotient. These singularities are:

$$
\begin{cases}
2 \times \frac{1}{5}(1, 1, 1) & \text{for } i = j = 0 \\
4 \times \frac{1}{5}(1, 1, 3), & \text{for } i \neq j \\
2 \times \frac{1}{5}(1, 3, 3) & \text{for } i = j = 1
\end{cases}
$$

The 8 points

$q_i \times q_j \times s_k, \quad \text{where} \quad i, j \in \{0, 1\} \quad \text{and} \quad k \in \{3, 4\}$

have $\langle g^2 \rangle \cong \mathbb{Z}_4$ as stabilizer. They map then to 4 singular points on the quotient:

$$
\begin{cases}
1 \times \frac{1}{5}(1, 1, 1) & \text{for } i = j = 0 \\
3 \times \frac{1}{5}(1, 1, 3), & \text{else}
\end{cases}
$$

The remaining 128 points have stabilizer $\mathbb{Z}_2$. These points yield 32 terminal singularities of type $\frac{1}{2}(1, 1, 1)$ on the quotient.

To show that $X$ is numerical Calabi-Yau, we verify that

$$p_1(\hat{X}) = 1, \quad \text{and} \quad q_2(\hat{X}) = q_1(\hat{X}) = 0$$

for a resolution $\hat{X}$ of $X$ like in the proof of Proposition 6.2. □

This example is not birational to a Calabi-Yau threefold.

**Proposition 7.2.** Let $\rho: \hat{X} \to X$ be a resolution of the singularities of $X$ and $Z$ be a minimal model of $\hat{X}$.

Then $Z$ is not Calabi-Yau.

**Proof.** We show that $P_2(\hat{X}) \geq 3$. A monomial basis of $H^0(2K_{C_2^2 \times C_3})$ is

$$
\left\{ \prod_{i=1}^{3} x_i^a x_i^b y_i^c \mid a_1 + b_1 + 3c_1 = a_2 + b_2 + 3c_2 = 2, \quad a_3 + b_3 + 4c_3 = 4 \right\}.
$$
The table below displays all points on $\mathbb{C}_2^2 \times \mathbb{C}_3$ with non-trivial stabilizer, that descend to a non-canonical singularity and the germ of the plurisection
\[
\prod_{i=1}^{3} x_{0i}^{a_i} \cdot x_{1i}^{b_i} \cdot y_i^{c_i}
\]
in local coordinates up to a unit as well as the stalks of the ideal $\mathcal{I}_2$ in these points.

| point          | singularity | germ                      | stalk                        |
|----------------|-------------|---------------------------|------------------------------|
| $(q_0, q_0, s_{1/2})$ | $\frac{1}{2}(1, 1, 3)$ | $y_1^{2a_1+c_1} y_2^{2a_2+c_2} x_3^{b_3}$ | $((y_1, y_2)^2 + (x_3))^2$ |
| $(q_1, q_1, s_{1/2})$ | $\frac{1}{2}(1, 1, 1)$ | $y_1^{2b_1+c_1} y_2^{2b_2+c_2} x_3^{b_3}$ | $(y_1, y_2, x_3)^{10}$      |
| $(q_0, q_1, s_{1/2})$ | $\frac{1}{2}(1, 3, 3)$ | $y_1^{2a_1+c_1} y_2^{2b_2+c_2} x_3^{b_3}$ | $((y_2, x_3)^3 + (y_1))^2$  |
| $(q_1, q_0, s_{1/2})$ | $\frac{1}{2}(1, 3, 1)$ | $y_1^{2b_1+c_1} y_2^{2a_2+c_2} x_3^{b_3}$ | $((y_1, x_3)^3 + (y_2))^2$  |
| $(q_0, q_0, s_{3/4})$ | $\frac{1}{2}(1, 1, 1)$ | $y_1^{2a_1+c_1} y_2^{2a_2+c_2} x_3^{a_3}$ | $(y_1, y_2, x_3)^2$         |

With the help of MAGMA we found the following three monomial sections of $H^0(2K_{\mathbb{C}_2^2 \times \mathbb{C}_3} \otimes \mathcal{I}_2)$:
\[
x_{11}^2 x_{12}^2 x_{03}^2 x_{13}^2, \quad x_{01} x_{11}^2 x_{12}^4 x_{13}^4 \quad \text{and} \quad x_{11}^2 x_{02} x_{12} x_{13}^4.
\]
Using the same argument as in Lemma 6.1, we obtain the action on the canonical ring $R(C_2, K_{C_2})$ as
\[
x_0 \mapsto \eta^2 x_0, \quad x_1 \mapsto \eta^6 x_1, \quad y \mapsto \eta^8 y, \quad \text{where} \quad \eta^2 = \omega
\]
and on $R(C_3, K_{C_3})$ as
\[
x_0 \mapsto \eta x_0, \quad x_1 \mapsto \eta^3 x_1, \quad y \mapsto \eta^4 y.
\]
We conclude that the three sections above are also $\mathbb{Z}_8$ invariant, in particular $P_2(\hat{X}) \geq 3$. □

Note that each of the three monomials in the proof of Proposition 7.2 contains a variable that do not appear in the other two. This implies that the subring of the canonical ring of $\hat{X}$ generated by them is isomorphic to the ring of polynomials in three variables. In particular $\text{kod}(\hat{X}) \geq 2$. 
8. Some minimal surfaces of general type

We construct in this section some product-quotient surfaces with several singular points, and investigate their minimality with our technique.

The construction is as follows.

**Definition 8.1.** Let $a, b \in \mathbb{N}$ such that $\gcd(ab, 1 - b^2) = 1$, $ab \geq 4$ and $b \geq 3$. Define $n = ab$ and let $1 \leq e \leq n - 1$ be the only integer such that $e \cdot (1 - b^2) \equiv n \mod 1$ (i.e., $e$ represent the inverse modulo $n$ of $1 - b^2$). For example, one can take $a = b \geq 3$. Define

$$\omega = e^{2\pi i/n} \quad \text{and} \quad \lambda = e^{2\pi i/(n - b)}.$$ 

Consider the Fermat curve $C$ of degree $n$ in $\mathbb{P}^2$, i.e., the plane curve

$$x_0^n + x_1^n + x_2^n = 0,$$

where $x_i$ are projective coordinates on $\mathbb{P}^2$. Consider the natural action $\rho_1$ of $G := \mathbb{Z} \oplus \mathbb{Z}$ on $C$ generated by

$$(8.1) \quad g_1 \cdot (x_0 : x_1 : x_2) = (\lambda x_0 : \lambda x_1 : \lambda x_2) \quad h_1 \cdot (x_0 : x_1 : x_2) = (\lambda x_0 : \lambda x_1 : \lambda x_2).$$

Define

$$(8.2) \quad g_2 := g_1 h_1^b \quad h_2 := h_1^{-b} h_1^{-1} \quad (\text{and} \quad k_2 = g_2^{-1} h_2^{-2} = g_1^{-1} h_1^{1-b}).$$

Under the assumptions made, $g_2$ and $h_2$ are generators of $G$, inducing a second $G$-action $\rho_2$ on $C$ by

$$g_2 \cdot (x_0 : x_1 : x_2) := g_1 \cdot (x_0 : x_1 : x_2) \quad h_2 \cdot (x_0 : x_1 : x_2) := h_1 \cdot (x_0 : x_1 : x_2)$$

The diagonal action $\rho_1 \times \rho_2$ on $C \times C$ induces a product quotient surface $X_{a,b}$ with quotient model $X_{a,b}$.

The action $\rho_1$ has 3 orbits where the action is not free:

$$\text{Fix}(g_1) = \{ (1 : 0 : -\eta) \mid \eta^n = 1 \}$$

$$\text{Fix}(h_1) = \{ (1 : -\eta : 0) \mid \eta^n = 1 \}$$

$$\text{Fix}(k_1) = \{ (0 : 1 : -\eta) \mid \eta^n = 1 \}$$

respectively stabilized by $\langle g_1 \rangle$, $\langle h_1 \rangle$ and $\langle k_1 := g_1^{-1} h_1^{-1} \rangle$. Notice $g_1 = g_2^b h_2^b$ and $h_1 = g_2^{-eb} h_2^{-e}$. The following relations hold:

$$\langle g_1 \rangle \cap \langle g_2 \rangle = \langle g_1 \rangle \cap \langle h_2 \rangle = \langle 1 \rangle \quad \langle g_1 \rangle \cap \langle k_2 \rangle = \langle 1 \rangle$$

$$\langle h_1 \rangle \cap \langle h_2 \rangle = \langle h_1 \rangle \cap \langle k_2 \rangle = \langle 1 \rangle \quad \langle k_1 \rangle \cap \langle k_2 \rangle = \begin{cases} \langle 1 \rangle & \text{if } n \text{ is odd} \\ \langle (g_1 h_1)^{n/2} \rangle & \text{if } n \text{ is even} \end{cases}.$$

The only points of $C \times C$ with non trivial stabilizer are

| Fixed points | #Points | Stabilizer | Type of singularity on $X$ |
|--------------|---------|------------|-----------------------------|
| Fix($g_1)^2$ | $n^2$   | $\langle g_1^n \times g_2^2 \rangle \simeq \mathbb{Z}_b$ | $\frac{1}{2}(1, 1)$ | Any $n$ |
| Fix($h_1)^2$ | $n^2$   | $\langle h_1^n \times h_2^{-2} \rangle \simeq \mathbb{Z}_b$ | $\frac{1}{2}(1, b - 1)$ | Any $n$ |
| Fix($k_1)^2$ | $n^2$   | $\langle k_1^{n/2} \times k_1^{-n/2} \rangle \simeq \mathbb{Z}_2$ | $\frac{1}{2}(1, 1)$ | $n \text{ even}$ |
In particular, the only non canonical singularities of \( X_{a,b} \) are \( b \) points of type \( \frac{1}{b}(1,1) \).

Since \( C/G \simeq \mathbb{P}^1 \) then \( q(\tilde{X}_{a,b}) = 0 \). Moreover, we have, by the formulas in [BP12],

\[
K_{X_{a,b}}^2 = \frac{8(g(C) - 1)^2}{\#G} = 2(n - 3)^2
\]

and, as we have exactly \( b \) singular points of type \( \frac{1}{b}(1,1) \),

\[
r^*K_{X_{a,b}} = K_{\tilde{X}_{a,b}} + \frac{b - 2}{b}(E_1 + \cdots + E_b)
\]

where \( E_i \) are the exceptional divisors introduced by the resolution over the non canonical points. These are disjoint rational curves with selfintersection \(-b\) so

\[
(8.3) \quad K_{\tilde{X}_{a,b}}^2 = K_{X_{a,b}}^2 - b^2 \frac{(b - 2)^2}{b^2} = 2(n - 3)^2 - (b - 2)^2.
\]

**Remark 8.2.** Notice that \( K_{\tilde{X}_{a,b}}^2 \geq 2 \) for all \((a,b)\) satisfying our assumptions, unless \((a,b) \in \{(1,4),(1,5)\}\).

There is an isomorphism among \( H^0(\omega_C) \) and \( H^0(\mathcal{O}_C(n-3)) = H^0(\mathcal{O}_{\mathbb{P}^2}(n-3)) = \mathbb{C}[x_0,x_1,x_2]_{n-3} \). \( \rho_1 \) induces then, via pull-back of holomorphic forms on \( C \), a \( G \)-action on it. We wrote \( \rho_1 \) so that this action coincides with the natural action induced by (8.1) on monomials of degree \( n-3 \). Explicitly, if \( m_0 + m_1 + m_2 = n - 3 \) we have

\[
g_1 \cdot x_0^{m_0} x_1^{m_1} x_2^{m_2} = (g_1^{-1})^* (x_0^{m_0} x_1^{m_1} x_2^{m_2}) = \\
\quad = \lambda^{-m_0-m_1-m_2} \omega^{-m_1} x_0^{m_0} x_1^{m_1} x_2^{m_2} = \omega^{-m_1-1} x_0^{m_0} x_1^{m_1} x_2^{m_2}
\]

and

\[
h_1 \cdot x_0^{m_0} x_1^{m_1} x_2^{m_2} = (h_1^{-1})^* (x_0^{m_0} x_1^{m_1} x_2^{m_2}) = \\
\quad = \lambda^{-m_0-m_1-m_2} \omega^{-m_2} x_0^{m_0} x_1^{m_1} x_2^{m_2} = \omega^{-m_2-1} x_0^{m_0} x_1^{m_1} x_2^{m_2}.
\]

The canonical action induced by \( \rho_2 \) and the bicanonical action follow accordingly. Working as in the previous sections we computed \( p_g(\tilde{X}_{a,b}) = h^0(\omega_{C \times C})^G \) and \( h^0(\omega_{C \times C}^\otimes)^G \) for the case \( a = b \). The values are all in Table 8. We stress that for \( a = b \geq 3 \) we always get \( K_{\tilde{X}_{a,b}}^2 > 0 \) and \( p_g(\tilde{X}_{a,b}) > 0 \) so \( \tilde{X}_{a,b} \) is of general type.

As the only non canonical singular points are of type \( \frac{1}{b}(1,1) \) we have

\[
P_2(\tilde{X}_{b,b}) = H^0(\omega_{\tilde{X}_{b,b}}^\otimes) \simeq H^0(\omega_{C \times C}^\otimes) \otimes I_{R_{nc}}^{2b-4})^G
\]

where \( I_{R_{nc}} \) is the ideal sheaf of functions vanishing at order at least \( 2b - 4 \) in all the points of

\[
R_{nc} = \text{Fix}(g_1)^2 = \{(1 : 0 : -\eta_1) \times (1 : 0 : -\eta_2) \mid \eta_1^n = \eta_2^n = 1\}.
\]
Using the embedding of \( C \times C \) in \( \mathbb{P}^2 \times \mathbb{P}^2 \) we have \( H^0(\omega_{C \times C}^{\otimes 2}) = H^0(\mathcal{O}_{C \times C}(2n - 6, 2n - 6)) \). To simplify the computation, we just look for the invariant monomials with the right vanishing order on \( R_{nc} \): in principle their number is only a lower bound for \( P_2(\hat{X}_{b,h}) \); the vanishing order of \( x_0^{m_0} x_1^{m_1} x_2^{m_2} y_0^{n_0} y_1^{n_1} y_2^{n_2} \) with \( 0 \leq m_1, n_1, m_2, n_1 + n_2 \leq 2n - 6 \) and \( 0 \leq m_2, n_2 \leq n - 1 \) equals \( m_1 + n_1 \).

We prove

**Proposition 8.3.** Assume \( a = b \geq 3 \). Then \( H^0(\omega_Y^{\otimes 2} \otimes \mathcal{I}_{R_{nc}})^G \) is generated by invariant monomials. Moreover, the codimension of \( H^0(\omega_Y^{\otimes 2} \otimes \mathcal{I}_{R_{nc}})^G \) in \( H^0(\omega_Y^{\otimes 2})^G \) is \( b(b - 3) \).

**Proof.** We give only a sketch of the proof.

The invariants bicanonical monomials are those of the form \( x_0^{m_0} x_1^{m_1} x_2^{m_2} y_0^{n_0} y_1^{n_1} y_2^{n_2} \) with

\[
\begin{cases}
(I) : & m_1 + 4 + 2b + n_1 + bn_2 \equiv n_0 0 \\
(II) : & m_2 - 2b - bn_1 - n_2 \equiv n_0 0 \\
0 \leq m_1, n_1, m_1 + m_2, n_1 + n_2 \leq 2n - 6 \\
0 \leq m_2, n_2 \leq n - 1 \\
m_0 + m_1 + m_2 = n_0 + n_1 + n_2 = 2b - 6
\end{cases}
\]

We prove that if \( \nu := m_1 + n_1 \leq 2b - 4 \) then \( b \geq 4 \) and \( \nu = b - 4 \). Hence the Proposition is true for \( b = 3 \) and we can assume \( b \geq 4 \). In this case we solve (8.4) under the assumption \( \nu = b - 4 \), finding \( b - 3 \) possibilities for the pair \((m_1, n_1)\). We denote by \( W_b \) the vector space generated by the invariant bicanonical monomials and with \( W_b^{(m_1,n_1)} \) its subspace generated by monomials with have assigned exponents for the variables \( x_1 \) and \( y_1 \). Monomials in \( W_b^{(m_1,n_1)} \) satisfy \( m_2, n_2 \in \{-3 + kb \mid 1 \leq k \leq b\} \) and if \( m_2 = -3 + k_m b \) and \( n_2 = -3 + k_n b \) then \( k_m \equiv b \), \( k_n \equiv n_1 + 2 \). Using these observations, we prove that the dimension of \( W_b^{(m_1,n_1)} \) is \( b \) which implies then that \( W_b \) has codimension \( b(b - 3) \) in \( H^0(\omega_Y^{\otimes 2})^G \).

What remains to prove is that a polynomial whose monomials have \( \nu = b - 4 \) cannot vanish in all the points of \( R_{nc} \) with order at least \( 2b - 4 \). A polynomial \( p \) in \( W_b \) is a linear combination of polynomials living in \( W_b^{(m_1,n_1)} \). In affine coordinates \( z_i = x_i/x_0, w_i = y_i/y_0 \) we have

\[
p = \sum_{j=0}^{b-4} z_i^j w_1^{b-4-j} p_j(z_2, w_2)
\]

with \( z_i^j w_1^{b-4-j} p_j(z_2, w_2) \in W_b^{(m_1,n_1)} \). In the second part of the proof we prove that if a polynomial \( p \in W_b \) vanish at order at least \( 2b - 4 \) in the points of \( R_{nc} \) than, necessarily \( p_j(\eta, -\mu) = 0 \) for all pairs of \( n \)-th root of 1. This is obtained by implicit differentiation and by keeping track of the order of vanishing of the various
In the third and final part of the proof we prove that if \( z_1^j w_1^{b-4-j} q(z_2, w_2) \in W_b^{(m_1, n_1)} \) and \( q(-\eta, -\mu) = 0 \) for enough pairs of \( n \)-th roots of 1 then \( q \) is actually 0. More precisely, we prove that if \( q(-1, \mu_i) = 0 \) for \( \mu_1, \ldots, \mu_b \) with \( \mu^n \) is 1 and \( \mu_i \neq \mu_j \) for \( i \neq j \), then \( q = 0 \). This can be seen as follows. If \( \text{Mod}(x, b) \) is the only representative of \( x \) modulo \( b \) in the range \([1, b]\) one can chose the monomials \( f_k = z_1^{m_1} w_1^{b-4-m_1} q_k \) with

\[
q_k = (-1)^{b(k+\text{Mod}(k+2+n_1,b))} z_2^{-3+b \text{Mod}(k+2+n_1,b)} w_2^{3+kb} \quad 1 \leq k \leq b
\]

as basis for \( W_b^{(m_1, n_1)} \). The coefficient is simply to get easier computations. If \( q = \sum_{k=1}^b \lambda_k q_k \) then

\[
q(-1 - \mu) = \mu^{-3} \sum_k \lambda_k (\mu^b)^k.
\]

We know that \( q(-1, -\mu_i) = 0 \) for \( \mu_1, \ldots, \mu_b \). Then either \( \lambda_k = 0 \) for all \( k \) or the matrix \( A = ( (\mu^b)^k)_{1 \leq i, k \leq b} \) has determinant 0. But \( A \) is a Vandermonde-type matrix associated to \( \{ \mu_1^b, \ldots, \mu_b^b \} \) and its determinant is zero if and only if there is a pair \( (i, j) \) with \( i \neq j \) such that \( \mu_i^b = \mu_j^b \). But this is against the hypothesis so we have, finally, \( p = 0 \). \( \square \)

Having a way to compute \( P_2 \) means also that we have a way to determine whether our surfaces are minimal or not. Indeed, by Proposition 1.3, we have

\[
\text{vol}(K_{\hat{X}_{a,b}}) = P_2(\hat{X}_{a,b}) - \chi(O_S) \geq K_{\hat{X}_{a,b}}^2
\]

with equality if and only if \( S \) is already minimal. Here we summarize the invariants for the product-quotient surfaces obtained for \( 3 \leq a = b \leq 12 \).

| \( b \) | \( g(C) \) | \( K^2_{\hat{X}_{b,b}} \) | \( K^2_{\hat{X}_{a,b}} \) | \( p_g(\hat{X}_{a,b}) \) | \( \chi(O_{\hat{X}_{b,b}}) \) | \( h^0(\omega^2_{\hat{X}_{a,b}}) \) | \( h^0(\omega^2_{C \times C}) \) | \( P_2(\hat{X}_{a,b}) \) | \( \text{vol} K_{\hat{X}_{a,b}} \) | \( \text{vol} K_{\hat{X}_{a,b}} - K_{\hat{X}_{a,b}}^2 \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 28 | 72 | 71 | 9 | 10 | 81 | 81 | 71 | 0 |
| 4 | 105 | 338 | 334 | 43 | 44 | 382 | 378 | 334 | 0 |
| 5 | 276 | 968 | 959 | 122 | 123 | 1092 | 1082 | 959 | 0 |
| 6 | 595 | 2178 | 2162 | 274 | 275 | 2455 | 2437 | 2162 | 0 |
| 7 | 1128 | 4232 | 4207 | 531 | 532 | 4767 | 4739 | 4207 | 0 |
| 8 | 1953 | 7442 | 7406 | 933 | 934 | 8380 | 8340 | 7406 | 0 |
| 9 | 3160 | 12168 | 12119 | 1524 | 1525 | 13698 | 13644 | 12119 | 0 |
| 10 | 4851 | 18818 | 18754 | 2356 | 2357 | 21181 | 21111 | 18754 | 0 |
| 11 | 7140 | 27848 | 27767 | 3485 | 3486 | 31341 | 31253 | 27767 | 0 |
| 12 | 10153 | 39762 | 39662 | 4975 | 4976 | 44746 | 44638 | 39662 | 0 |

Hence we can conclude

**Proposition 8.4.** For all \( 3 \leq b \leq 12 \), \( \hat{X}_{b,b} \) is a regular minimal surface of general type.
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