Star-uniform Graphs

Mikio Kano¹, Yunjian Wu² and Qinglin Yu²,³ *

¹ Department of Computer and Information Sciences
Ibaraki University, Hitachi, Ibaraki, Japan
² Center for Combinatorics, LPMC
Nankai University, Tianjin, China
³ Department of Mathematics and Statistics
Thompson Rivers University, Kamloops, BC, Canada

Abstract

A star-factor of a graph $G$ is a spanning subgraph of $G$ such that each of its component is a star. Clearly, every graph without isolated vertices has a star factor. A graph $G$ is called star-uniform if all star-factors of $G$ have the same number of components. To characterize star-uniform graphs was an open problem posed by Hartnell and Rall, which is motivated by the minimum cost spanning tree and the optimal assignment problems. We use the concepts of factor-criticality and domination number to characterize all star-uniform graphs with the minimum degree at least two. Our proof is heavily relied on Gallai-Edmonds Matching Structure Theorem.

Key words: star-factor, Gallai-Edmonds decomposition, factor-criticality, domination number, star-uniform

1 Introduction

Throughout this paper, all graphs considered are simple. We refer the reader to [2] and [6] for standard graph theoretic terms not defined in the paper.

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We call a graph with only one vertex trivial and all other graphs nontrivial. If $S \subset V(G)$, then $G - S$ is the subgraph of $G$ obtained by deleting the vertices in $S$ and all the edges incident with them. Similarly, if $E' \subset E(G)$, then $G - E' = (V(G), E(G) - E')$. The set of vertices adjacent

*Corresponding email: yu@tru.ca
to $S$ in $G$ is denoted by $N_G(S)$. If $G$ is not a forest, then the length of a shortest cycle in $G$ is called its girth, denoted by $g(G)$, and a cycle of order $g(G)$ is called a girth cycle. An odd (or even) cycle (or path) is the one with odd (or even) number of vertices. The union $G_1 \cup G_2$ of the graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. A star is a graph isomorphic to a complete bipartite graph $K_{1,n}$ for some $n \geq 1$, and the vertex of degree $n$ is called the center of the star.

A star-factor of a graph $G$ is a spanning subgraph of $G$ each component of which is a star. It is not hard to see that every graph without isolated vertices admits a star-factor. If one limits the sizes of the stars used, such a star-factor may not always exist. In [1], Amahashi and Kano presented a criterion for the existence of a restricted star-factor, i.e., $\{K_{1,1}, \cdots, K_{1,n}\}$-factor. Yu [9] obtained an upper bound on the maximum number of edges in a graph with a unique star-factor.

A vertex subset $S$ of a graph $G$ is a dominating set if every vertex of $G$ either belongs to $S$ or is adjacent to a vertex of $S$. The cardinality of a smallest dominating set is called the domination number of $G$, denoted by $\gamma(G)$. For extensive bibliographies regarding work on domination in graphs the reader is referred to [5].

A matching $M$ of $G$ is a subset of $E(G)$ such that no two elements of $M$ are adjacent. The number of edges in $M$ is called the size of $M$. A matching $M$ is called a maximum matching if $G$ has no matching $M'$ with $|M'| > |M|$. If every $v \in V(G)$ is incident to some edge in $M$, then $M$ is said to be a perfect matching. A near-perfect matching in a graph $G$ is one covering all but exactly one vertex of $G$. A graph $G$ is said to be factor-critical if $G - v$ has a perfect matching for every $v \in V(G)$ and this concept is first introduced by Gallai [6]. An $M$-alternating path (or $M$-alternating cycle) in $G$ is a path (or cycle) whose edges are alternately in $M$ and $E - M$. Let $M_1$ and $M_2$ be matchings in $G$ and $M_1 \cup M_2$ denote the subgraph formed by the union of the two edge sets, so $V(M_1 \cup M_2) = V(M_1) \cup V(M_2)$ and $E(M_1 \cup M_2) = E(M_1) \cup E(M_2)$. The components of this subgraph are edges, alternating even cycles or alternating paths.

Let $G$ be a graph. Denote by $D(G)$ the set of all vertices in $G$ which are not covered by at least one maximum matching of $G$, $A(G)$ be the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$. Finally let $C(G) = V(G) - A(G) - D(G)$ (see Figure 1).

Gallai [3] and Edmonds (see [6]), independently, obtained the following canonical decomposition theorem for maximum matching in graphs. This result can be considered as a refinement of Tutte’s famous 1-Factor Theorem and it provides a complete structural characterization of the maximum matchings in graphs. It was the foundation for Edmonds’ well-known polynomial algorithm for finding a maximum matching in graphs. Its full power is still waiting to be explored in the future.

**Theorem 1** (Gallai-Edmonds Structure Theorem for Matchings). Let $G$ be a graph and $D(G)$, $A(G)$ and $C(G)$ be the sets defined as above. Then
(a) every component of the subgraph induced by $D(G)$ is factor-critical;
(b) the subgraph induced by $C(G)$ has a perfect matching;
(c) the order of $A(G)$ is less than the number of the components in $D(G)$;
(d) every maximum matching of $G$ consists of a near-perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and a matching which matches all vertices of $A(G)$ with vertices in distinct components of $D(G)$.

Figure 1 The Gallai-Edmonds decomposition of a graph $G$

We say that a graph $G$ is star-uniform if all star-factors of $G$ have the same number of components. This concept is motivated by an open problem posed by Hartnell and Rall [4], which asked to characterize the family of graphs that all its star-factors have the same number of edges and is motivated by the minimum cost spanning tree and the optimal assignment problems. Hartnell and Rall characterized star-uniform graphs with girth at least five. Wu and Yu [8] settled the case of star-uniform graphs with girth three but without leaves.

To prove that a graph $G$ is not star-uniform, we usually show that $G$ contains two star-factors with different numbers of components. In this paper, we use the factor-criticality and domination number to characterize all star-uniform graphs with the minimum degree at least two. Our proof is heavily relied on Gallai-Edmonds Structure Theorem.

2 Main Results

Let $S$ be a star-factor with the maximum number of components among all star-factors of $G$. If we choose one edge from each component of $S$, it yields a matching $M$. Conversely, suppose $M$ is a maximum matching in $G$, then $G - V(M)$ is an independent set, and for each edge $uv$ in $M$, $u$ and $v$ can not be adjacent to distinct vertices of $G - V(M)$ due to the maximality of $M$. For each isolated vertex $x$ in $G - V(M)$, we add an edge $e \in E(G)$ joining $x$ to a vertex in $V(M)$ and obtain a star-factor with $|M|$ components. Hence we have the following proposition.
Proposition 1. Let $G$ be a connected graph. Then the maximum number of components of star-factors in $G$ is equal to the number of edges of a maximum matching in $G$ (e.g., the matching number).

Proposition 1 shows the relationship between the maximum number of components of star-factors and the matching number. Similarly, we have a proposition to relate the minimum number of components of star-factors and the domination number.

Proposition 2. Let $G$ be a connected graph. Then the minimum number of components of star-factors in $G$ is equal to the domination number $\gamma(G)$.

Proof. Let $S$ be a star-factor in $G$ with the minimum number of components. Then all centers in $S$ form a dominating set, so $\gamma(G)$ is not greater than the minimum number of components of star-factors in $G$.

Conversely, suppose $D$ is a dominating set of the minimum order. Then every vertex of $V(G) - D$ has at least one neighbor in $D$ and every vertex of $D$ has at least one neighbor in $V(G) - D$ since $G$ has no isolated vertices. Now we construct a bipartite graph $B$ with bipartition $(V(G) - D) \cup D$ and edge set $E(B) = \{uv \mid u \in V(G) - D, v \in D \text{ and } uv \in E(G)\}$. Then $B$ has a star-factor, which can be regarded as a star-factor of $G$. Since the number of components of a star factor of $B$ is at most $|D|$, it follows that $\gamma(G) = |D|$ is greater than or equal to the minimum number of components of star-factors in $G$. Therefore the proposition is proved.

The following result of Ore [7] provided a bound for domination numbers of graphs without isolated vertices is a corollary of Proposition 2.

Theorem 2. (Ore [7]) If a graph $G$ has no isolated vertex, then $\gamma(G) \leq \lfloor \frac{|V(G)|}{2} \rfloor$.

Every star factor $S$ of a bipartite graph $G$ with bipartition $X \cup Y$ such that $|X| \leq |Y|$ has at most $|X|$ components. So, if $\gamma(G) = |X|$, then, by Proposition 2, each component of $S$ contains exactly one vertex of $X$, and thus we can obtain a matching from $S$ that covers $X$. Therefore the following result follows.

Proposition 3. Let $G$ be a connected bipartite graph with bipartition $X \cup Y$ such that $|X| \leq |Y|$. If $\gamma(G) = |X|$, then $G$ contains a matching of $|X|$ edges.

Combining Propositions 1 and 2 the following theorem is true for all star-uniform graphs.

Theorem 3. A connected graph $G$ is star-uniform if and only if the size of a maximum matching of $G$ is equal to the domination number $\gamma(G)$.

Next is the main result of this paper.

Theorem 4. A connected graph $G$ with $\delta(G) \geq 2$ is star-uniform if and only if $G$ is one of the graphs shown in Figure 2 or a bipartite graph with bipartition $X \cup Y$ such that $g(G) = 4$ and $\gamma(G) = |X| \leq |Y|$.
Hence, there are only nine star-uniform graphs containing odd cycles but there are infinite bipartite star-uniform graphs. Note that the infinite family of bipartite graphs mentioned in Theorem 4 satisfies $2 = |X| \leq |Y|$ or $3 \leq |X| < |Y|$.

**Proof of Theorem 4.** It is easy to check that all graphs shown in Figure 1 are star-uniform. Let $G$ be a bipartite graph with bipartition $X \cup Y$ such that $\gamma(G) = |X| \leq |Y|$. Then $G$ is star-uniform by Proposition 3 and Theorem 3.

Conversely, we shall prove that any connected star-uniform graph with the minimum degree at least two is one of graphs given in the theorem. Let $G$ be a connected star-uniform graph with $\delta(G) \geq 2$. We consider the following cases.

**Case 1.** $G$ has a perfect matching.

Let $M$ be a perfect matching of $G$. Since $G$ has a perfect matching, $|V(G)|$ is even. If $|V(G)| = 4$, then $G$ can only be a 4-cycle as $G$ is star-uniform. If $|V(G)| > 4$, then there exist three distinct edges $\{v_1v_2, v_3v_4, v_5v_6\}$ of $M$ such that $v_2v_3, v_4v_5 \in E(G)$ since $G$ is connected and $\delta(G) \geq 2$. Then we can find two stars $T_1$ and $T_2$ with centers $v_2$ and $v_5$, respectively, which cover $\{v_1v_2, v_3v_4, v_5v_6\}$. Thus $\{T_1, T_2, M - \{v_1v_2, v_3v_4, v_5v_6\}\}$ is a star-factor of $G$ with $|M| - 1$ components, but $M$ is a star-factor with $|M|$ components, a contradiction to star-uniform of $G$.

Therefore $G$ is a 4-cycle, which is one of the bipartite graphs given in the theorem.

**Case 2.** $G$ has no perfect matching.

Let $M$ be a perfect matching of $G$. By Gallai-Edmonds Structure Theorem, we know that each component of the subgraph induced by $D(G)$ is factor-critical and the subgraph induced by $C(G)$ has a perfect matching. Moreover, $M$ consists of a near-perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and a matching which matches all vertices of $A(G)$ to vertices in distinct components of $D(G)$. Since $G$ is connected, for each component $D$ of $D(G)$, there is at least one vertex in $D$ which is adjacent to a vertex in $A(G)$. As $D$ is factor-critical, without loss of generality, we may assume that each isolated vertex in $V(G) - V(M)$ is adjacent to a vertex in $A(G)$. Now we add an edge $e \in E(G)$ between each isolated vertex in $V(G) - V(M)$ and some vertex in $A(G)$, then we
obtain a special star-factor $\mathcal{S}$ with $|M|$ components such that all vertices in $A(G)$ are centers of stars in $\mathcal{S}$.

In the following discussion, we often delete some edges from $\mathcal{S}$ and then add other edges in $E(G) - \mathcal{S}$ to $\mathcal{S}$ to construct another star-factor with different number of components and thus yields a contradiction to that $G$ is star-uniform.

Claim 1. $C(G) = \emptyset$.

Let $C$ be a component of $C(G)$. Since $G$ is connected, then there exists an edge $u_1v_1 \in \mathcal{S}$ in $C$ such that $u_1$ is adjacent to a vertex $x$ in $A(G)$, and $v_1$ is also adjacent to a vertex in $C$ or $A(G)$ since $\delta(G) \geq 2$. If $v_1$ is adjacent to a vertex $y$ in $A(G)$, then deleting edge $u_1v_1$ from $\mathcal{S}$ and adding two edges $u_1x$ and $v_1y$ to $\mathcal{S}$, we obtain another star-factor with $|M| - 1$ components. If $v_1$ is adjacent to a vertex $v_2$ in $C$, then deleting edge $u_1v_1$ from $\mathcal{S}$ and adding two edges $u_1x$ and $v_1v_2$ to $\mathcal{S}$, we also obtain a star-factor with $|M| - 1$ components. Either case yields a contradiction.

Claim 2. $A(G)$ is an independent set.

Suppose $uv$ is an edge in the subgraph induced by $A(G)$ and $T_u$ is a star in $\mathcal{S}$ with center $u$. For each leaf $x$ in $T_u$, where $x$ is in a component $D$ of $D(G)$, we perform the following operation: if $D$ is singleton, then $x$ is adjacent to another vertex $y$ in $A(G)$ (since $\delta(G) \geq 2$) and we delete the edge $ux$ from $\mathcal{S}$ and add the edge $xy$ to $\mathcal{S}$; if $D$ is nontrivial, then $x$ is adjacent to a vertex $z$ in $D$, we delete the edge $ux$ from $\mathcal{S}$ and add the edge $xz$ to $\mathcal{S}$. By adding the edge $uv$ to $\mathcal{S}$, we obtain another star-factor $\mathcal{S}'$ with $|M| - 1$ components since $v$ is a center of some star in $\mathcal{S}$, a contradiction.

Claim 3. $A(G) = \emptyset$ or all components of $D(G)$ are singletons.

Suppose $A(G) \neq \emptyset$ and a component $D$ in $D(G)$ is nontrivial. Then there exists a star $T_v$ in $\mathcal{S}$ such that a leaf $x$ of $T_v$ is contained in $D$ and the center $v$ is in $A(G)$. If $T_v$ has another leaf $x'$, we delete the edge $vx'$ from $\mathcal{S}$ and add another edge $x't$ to $\mathcal{S}$, where $t \in A(G) \cup D(G)$ is the center of a star of $\mathcal{S}$. Now we obtain another star-factor with the same number of components as $\mathcal{S}$. So, without loss of generality, we may assume that $v$ has no other leaves except $x$ in $T_v$. Suppose $e_1, e_2, \ldots, e_m$ are $K_{1,1}$-stars of $\mathcal{S}$ in $D$. Then $x$ is adjacent to a vertex in $D$, but cannot be adjacent to the two vertices of a certain star $e_i$ ($1 \leq i \leq m$), otherwise the four vertices in $vx$ and $e_i$ can be covered by one star with center $x$, a contradiction. So let $e_i = yz$ be a star with only one vertex, say $y$, adjacent to $x$ and $z$ is adjacent to a vertex $u \in D \cup A(G)$ since $\delta(G) \geq 2$. Then by removing $yz$ from $\mathcal{S}$, and adding $yx$ and $zu$ to $\mathcal{S}$, we can obtain another star-factor with $|M| - 1$ components, a contradiction.

Subcase 2.1. $A(G) = \emptyset$.

Since $G$ is connected, $A(G) = \emptyset$ implies that $G$ is factor-critical.

Claim 4. If $G$ is factor-critical, then $|V(G)| \leq 7$.

Since $G$ is factor-critical, then for each edge $xy$, both $G - x$ and $G - y$ have perfect matchings, denoted by $M_x$ and $M_y$, respectively. Let $H = M_x \cup M_y$. Then $H$ is a spanning
subgraph of $G$ and contains an alternating path $P_{xy}$ connecting $x$ and $y$. Without loss of generality, we may assume that all components except $P_{xy}$ are independent multiple edges and the alternating path $P_{xy}$ is a longest path among all pairs $(M_x, M_y)$ of perfect matchings in $G - x$ and $G - y$ over all edges in $E(G)$. The order of $P_{xy}$, denoted by $p$, is odd. In the following, we assume $|V(G)| \geq 9$ and then construct two star-factors of $G$ with different numbers of components, and thus yields a contradiction.

If $p \geq 9$, then $P_{xy} \cup \{xy\}$ has two star-factors $S_1$ and $S_2$ with $\lfloor \frac{p}{2} \rfloor$ stars and $\lfloor \frac{p}{2} \rfloor - 1$ stars, respectively. So $H \cup \{xy\}$ contains two star-factors $S_1 \cup (M_x - P_{xy})$ and $S_2 \cup (M_x - P_{xy})$ with different numbers of components, a contradiction.

If $p = 7$, then there is at least one vertex $u$ in $P_{xy}$ which is adjacent to a vertex $v_1$ in $H - P_{xy}$ since $|V(G)| \geq 9$ and $G$ is connected. Then $P_{xy} \cup \{xy, uv_1, v_1v_2\}$ has two star-factors $S_3$ and $S_4$ with three stars and four stars, respectively. That is, $G$ contains two star-factors $S_3 \cup (M_x - P_{xy} - v_1v_2)$ and $S_4 \cup (M_x - P_{xy} - v_1v_2)$ with different numbers of components.

If $p = 5$, then there exists an edge $v_1v_2$ in $H - P_{xy}$ which is joined to $P_{xy}$ by an edges $v_1v_1$, where $v_1 \in V(P_{xy})$ since $G$ is connected and $|V(G)| \geq 9$. Suppose that $v_2$ is adjacent to another edge $v_3v_4$ in $H - P_{xy} - v_1v_2$. Suppose that $v_2$ is adjacent to $v_3$ in $G$, then $P_{xy} \cup \{xy, v_1v_2, v_3v_4, u_1u_1, v_2v_3\}$ has two star-factors $S_5$ and $S_6$ with three stars and four stars, respectively. So $S_5 \cup \{M_x - P_{xy} - v_1v_2 - v_3v_4\}$ and $S_6 \cup \{M_x - P_{xy} - v_1v_2 - v_3v_4\}$ are two star-factors with different numbers of components in $G$. Thus $v_2$ is not adjacent to any vertex in $H - P_{xy}$. In this case, $v_2$ must be adjacent to a vertex of $P_{xy}$ as $\delta(G) \geq 2$. If $v_1$ and $v_2$ have a common neighbor $u_2$, then $P_{xy} \cup \{xy, v_1v_2, v_1u_2, v_2u_2\}$ can be decomposed into two stars or three stars, so $G$ contains two star-factors with different numbers of components.

If $v_1$ and $v_2$ are adjacent to two distinct vertices $u_3$ and $u_4$ in $P_{xy}$ respectively, then $u_3$ and $u_4$ are nonadjacent in $P_{xy} \cup \{xy\}$ since $P_{xy}$ is a longest alternating path. Then $P_{xy} \cup \{xy, v_1v_2, v_1u_3, v_2u_4\}$ can be decomposed into two stars or three stars, we again obtain two star-factors with different numbers of components in $G$.

If $p = 3$, then there exists an edge $v_1v_2$ in $H - P_{xy}$ such that $v_1u$ is an edge of $G$, where $u \in P_{xy}$. If $v_2$ is incident with another edge $v_3v_4$ in $H - P_{xy} - v_1v_2$, where $v_2v_3$ is an edge of $G$, then $P_{xy} \cup \{xy, v_1v_2, v_3v_4, uv_1, v_2v_3\}$ can be decomposed into two stars or three stars. So $H \cup \{xy, uv_1, v_2v_3\}$ has two star-factors with the different numbers of components. Otherwise both $v_1$ and $v_2$ are only adjacent to vertices in $P_{xy}$. If $v_2$ is adjacent to $u' \not= u$ in $G$, denote $z = P_{xy} - u - u'$, then $M_{v_1} = M_x - P_{xy} - v_1v_2 + v_2u' + uz$ and $M_{v_2} = M_x - P_{xy} - v_1v_2 + v_1u + u'z$ are maximum matching of $G - v_1$ and $G - v_2$, respectively, and the path connecting $v_1$ and $v_2$ in $M_{v_1} \cup M_{v_2}$ is of length 5, which is a contradiction to the choice of $P_{xy}$. Hence $v_2$ is also adjacent to $u$ in $G$. Then $P_{xy} \cup \{xy, v_1v_2, uv_1, uv_2\}$ can be decomposed into one star or two stars, we obtain two star-factors of $G$ with different numbers of components. Consequently Claim 4 is proved.

All factor-critical graphs of order three, five or seven, with $\delta(G) \geq 2$ and $\gamma(G) = \lfloor \frac{V(G)}{2} \rfloor$ are shown in Figure 2.

Subcase 2.2. $A(G) \neq \emptyset$ and all components of $D(G)$ are singletons.
By Claim 1, this assumption implies that $G$ is a bipartite graph with bipartition $A(G) \cup D(G)$.

Claim 5. $g(G) = 4$ and $\gamma(G) = |A(G)|$.

By Gallai-Edmonds Structure Theorem, $|A(G)| < |D(G)|$ and every maximum matching of $G$ covers all vertices in $A(G)$. So $\gamma(G) = |A(G)|$ by Theorem 3.

Since $G$ is a bipartite graph, $g(G) = 4, 6, 8, \cdots$. Suppose that $g(G) \geq 6$. Let $C$ be a girth cycle of $G$. If a vertex not contained in $C$ is adjacent to two distinct vertices of $C$, then there is a cycle shorter than $g(C)$. Hence any vertex in $V(G) - V(C)$ is adjacent to at most one vertex of $V(C)$. Thus by deleting all the edges incident with the cycle $C$ but not on it, we obtain a spanning subgraph $H$ without isolated vertices. But $C$ is a component of $H$ and $C$ can be decomposed into $g(G)/2$ stars or $(g(G)/2) - 1$ stars. This is a contradiction since $H$ is a spanning subgraph of $G$. Hence $g(G) = 4$. Since $G$ has no perfect matching, we have $2 \leq |A(G)| < |D(G)|$.

This completes the proof of Theorem 4.

Next we construct an infinite family of star-uniform connected bipartite graphs. A connected graph that has no cut vertices is called a block. A block of a graph is a subgraph that is a block and is maximal with respect to this property: Every graph is the union of its blocks. Let $G$ be a connected graph such that each block of $G$ is $K_{2,2}$ and each block has a pair of nonadjacent vertices of degree two in $G$, e.g., the graph shown in Figure 3 is such a graph. Then $G$ is star-uniform. To see this, let $u$ and $v$ be two nonadjacent vertices of degree two in a block $B$, then neither $u$ nor $v$ can be a center of a star $K_{1,n}$ ($n \geq 2$) in any star-factor of $G$. For a star $K_{1,1}$, any its vertex can be considered as a center, so we designate other two vertices rather than $u$ and $v$ in $B$ as the centers in any star-factor of $G$. Hence all star-factors of $G$ have the same number of components, i.e., $G$ is star-uniform.

![Figure 3](image)

An edge-weighting of a graph $G$ is a function $w : E(G) \rightarrow \mathbb{N}^+$, where $\mathbb{N}^+$ is a set of positive integers. The weight of a star-factor $S$ in $G$ under $w$ is the sum of all the weight values for edges belonging to $S$, i.e., $w(S) = \Sigma_{e \in E(S)} w(e)$. Now it is nature to ask the following question which is proposed in [4] and still open.
Question 1. For a given graph $G$, does there exist an edge-weighting $w$ of $G$ such that every star-factor of $G$ has the same weights under $w$?

References

[1] A. Amahashi and M. Kano, On factors with given components, *Discrete Math.*, 42(1982), pp. 1-6.

[2] B. Bollobás, *Modern Graph Theory*, 2nd Edition, Springer-Verlag New York, Inc. 1998.

[3] J. Edmonds, Paths, trees, and flowers, *Canad. J. Math.*, 17(1965), pp. 449-467.

[4] B. L. Hartnell and D. F. Rall, On graphs having uniform size star factors, *Australas. J. Combin.*, 34(2006), pp. 305-311.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[6] L. Lovász and M.D. Plummer, *Matching Theory*, North-Holland Inc., Amsterdam, 1986.

[7] O. Ore, *Theory of Graphs*, AMS Publication 38, Providence, RI, 1962.

[8] Y. Wu and Q. Yu, Uniform Star-factors of Graphs with Girth Three, (submitted).

[9] Q. Yu, Counting the number of star-factors in graphs, *J. Combin. Math. Combin. Comput.*, 23(1997), pp. 65-76.