Universal Model of Finite-Reynolds Number Turbulent Flow in Channels and Pipes

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In this Letter we suggest a simple and physically transparent analytical model of pressure driven turbulent wall-bounded flows at high but finite Reynolds numbers Re. The model provides an accurate quantitative description of the profiles of the mean-velocity and Reynolds-stresses (second order correlations of velocity fluctuations) throughout the entire channel or pipe, for a wide range of Re, using only three Re-independent parameters. The model sheds light on the long-standing controversy between supporters of the century-old log-law theory of von-Kármán and Prandtl and proponents of a new theory promoting power laws to describe the intermediate region of the mean velocity profile.

An important challenge in wall-bounded Newtonian turbulence is the description of the profiles of the mean velocity and second order correlation functions of turbulent-velocity fluctuations throughout the entire channel or pipe at relatively high but finite Reynolds numbers. To understand the issue, focus on a channel or pipe at relatively high but finite Reynolds numbers. To understand the issue, focus on a channel or pipe at relatively high but finite Reynolds numbers. To understand the issue, focus on a channel or pipe at relatively high but finite Reynolds numbers. To understand the issue, focus on a channel or pipe at relatively high but finite Reynolds numbers. To understand the issue, focus on a channel or pipe at relatively high but finite Reynolds numbers. To understand the issue, focus on a channel or pipe at relatively high but finite Reynolds numbers. To understand the issue, focus on a channel or pipe at relatively high but finite Reynolds numbers. 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idea, allowing us to get, within the traditional (second-order) closure procedure, a quantitative description of the mean shear, $S(z) = dV(z)/dz$, the kinetic energy density (per unit mass), $K(z) \equiv \langle |u|^2 \rangle / 2$, and the tangential Reynolds stress, $W(z) = -\langle u_z u_{\perp} \rangle$, in the entire flow and in a wide region of Re$_r$, using only three Re$_r$-independent parameters, $\kappa$, $B$ and $\ell_s$ ($\ell_s \approx 0.311 L$ for the channel and $\ell_s \approx 0.275 L$ for the pipe).

The closure model should relate three objects: $S^+$, $K^+$ and $W^+$. The first (exact) relation between these objects follows from the Navier-Stokes equation for the mean velocity, being the mechanical balance between the momentum generated at distance $z$ from the wall, i.e. $p'(L-z)$, and the momentum transferred to the wall by kinematic viscosity and turbulent transport. In physical and wall units it has the form:

$$\nu S + W = p'(L-z) \Rightarrow S^+ + W^+ = 1 - \zeta , \quad \zeta \equiv z/L .$$

Already in 1877 Boussinesq attempted to close this equation by introducing the notion of turbulent viscosity $\nu_t$, writing $W = \nu_t S$ [10]. Estimating $\nu_t$ as $\kappa_w L_w \sqrt{K}$, one finishes with the closure $W^+ = \kappa_w L_w \sqrt{K} S^+$. Here $L_w$ is a $\zeta$-dependent characteristic scale of energy containing eddies, determining the nonlinear dissipation of $W$, and $\kappa_w$ is a constant introduced here for convenience. A more careful analysis of the balance equation for $W$ (see Ref. [11] and Appendix) that includes the viscous dissipation of $W$, leads to a somewhat more invloved clo-

FIG. 1: Color online. Left and Right upper panels: comparison of the theoretical mean velocity profiles (red solid lines) at different values of Re$_r$ with the DNS data for the channel flow [2, 3] (Left panel, grey squares; model with $\ell_{bat} = 49$, $\kappa = 0.415$, $\ell_s = 0.311$) and with the experimental Super-Pipe data [4] (middle panel, grey circles; model with $\ell_{bat} = 46$, $\kappa = 0.405$, $\ell_s = 0.275$). In orange dashed line we plot the viscous solution $V^+ = z^+$. In green dashed dotted line we present the von-Kármán log-law. Note that the theoretical predictions with three Re$_r$-independent parameters fits the data throughout the channel and pipe, from the viscous scale, through the buffer layer, the log-layer and the wake. For clarity the plots are shifted vertically by five units. Lower panel: The Reynolds-stress profiles (solid lines) at Re$_r$ from 394 to 2003 (in channel) and from 5050 to 165,000 (in pipe) in comparison with available DNS data (dots) for the channel.
sure for \( W \) in a form involving an additional universal, \( \text{Re}_r \)-independent dimensionless function of \( z^+ \):

\[
r_w W^+ \approx \kappa_w \ell_w^+ \sqrt{K^+ S^+},
\]

\[
r_w(z^+) \equiv \left(1 + \frac{\ell_{\text{buf}}^+}{z^+} \right)^{1/6}.
\] (4)

Here \( \ell_{\text{buf}}^+ \approx 49 \) is a \( \text{Re}_r \)-independent length that plays a role of the crossover scale (in wall units) between the buffer and log-law region. In this form, \( \ell_w(z) \propto z \) near the wall, and the choice \( \kappa_w \approx 0.20 \) ensures that \( \lim_{z \to 0} \ell_w(z) = \zeta \).

A third relation to supplement Eqs. (3) and (4) is obtained by balancing the turbulent energy generated by the mean flow at a rate \( SW \), and the dissipation at a rate \( \varepsilon_k \equiv \nu (|\nabla u|^2) \):

\[
S^+ W^+ \approx \varepsilon^+_k; \quad \varepsilon^+_k = K^{+3/2}/[\kappa_K \ell^+_K].
\] (5)

Here the dissipation is estimated via the energy cascade over scales involving a characteristic scale of energy containing eddies, \( \ell_k(z) \) determining the energy transfer rate. The constant \( \kappa_k \) will be used to ensure that the slope of this function at \( z^+ = 0 \) is unity.

Note that in Eqs. (4) and (5) we used a local-balance approximation, neglecting the spatial energy flux. This approximation is very good in the log-law region but it deteriorates near the wall and near the center-line. Nevertheless for our purposes this has no consequences. Near the wall \( W^+ \ll S^+ \) and the local-balance approximation plays no role in the exact mechanical balance (3) that determines \( S^+ \). For the same reason we also do not need to introduce a correction \( r_w(z^+) \) in Eq. (5) due to the direct viscous dissipation (similar to \( r_w(z^+) \) in Eq. (4)) since the length scale replacing \( \ell_{\text{buf}}^+ \) here will be the dissipative scale \( \ell_{\text{diss}}^+ \approx 5 \) which is entirely buried in the region where \( W \) and \( K \) are small. Near the centerline \( S^+ \) tends to zero and Eq. (3) determines \( W^+ \approx 1 - \zeta \), which allows an accurate determination of \( S^+ \), because we know that \( \ell_w \) and \( \ell_k \) must saturate.

**Profiles of the characteristic length-scales \( \ell_k, \ell_w \):**

Now we show that the source of confusion is the assumption that the relevant length scales can be determined a-priori as \( \ell_{\text{ch}}^+ \propto (z^+)^{\alpha} \) with \( \alpha = 1 \) or \( \alpha \neq 1 \). The actual dependence \( \ell_w \) and \( \ell_k \) on \( z \) and \( L \) can be found from the data provided by the numerical simulations. Consider first \( \ell_w \), defined by Eq. (4). We expect that plotting the scaling function \( \ell_w^+ / \text{Re}_r \) computed for different values of \( \text{Re}_r \) should collapse onto one scaling function. The quality of the data collapse for this scaling function is presented in Fig. 2, demonstrating the expected saturation at the center-line.

The second length-scale, \( \ell_k^+ \), is determined by the second of Eq. (5). We again expect that \( \ell_k^+ / \text{Re}_r \) should collapse the data obtained from different value of \( \text{Re}_r \) onto one scaling function. In Fig. 2 we demonstrate that this scaling function leads to acceptable data collapse throughout the channel and for all the four values of \( \text{Re}_r \) for which the simulation data are available.

**Solution, Velocity Profiles and Final Scaling Function:**

Solving Eqs. (3) together with \( S^+ W^+ = K^{+3/2}/(\kappa_k \ell_k^+) \) that follows from Eq. (5), we find

\[
W^+ = \left(\kappa K \ell_k^+ \right)^2 r_w^{-3/2}.
\] (6)

where we have defined the von-Kármán constant \( \kappa \equiv (\kappa_K^2 \kappa_k^{1/4}) \approx 0.415 \) and the crucial scaling function \( \ell^+/(\zeta) \) as follows

\[
\ell^+ = [\ell_k^+ (\zeta) / \ell_k^+]^{1/4} = 4 \sqrt{W^+ r_w^{3/2}/S^+} \varepsilon^+_k.
\] (7)

Note that if one replaces the energy dissipation rate \( \varepsilon^+_k \) by the rate of energy production \( W^+ S^+ \) and takes \( r_w \) as unity this scaling function becomes the Prandtl mixing length [1]. However the latter suffers from a non-physical divergence at the center-line whereas our length saturates to a constant there as it should.

The convincing data collapse for the resulting function \( \ell^+/(\zeta) / \text{Re}_r \) is shown in Fig. 2, rightmost panel. Substituting Eq. (6) in Eq. (3) we find a quadratic equation for \( S^+ \) with a solution:

\[
S^+ = \sqrt{1 + (1 - \zeta)(2\kappa \ell^+ (\zeta))^{2}/r_w (z^+)^{3/2} - 1 - \frac{2[\kappa \ell^+ (\zeta)]^2}{r_w (z^+)^{3/2}}}. \] (8)

To integrate this equation and find the mean velocity profile for any value of \( \text{Re}_r \) we need to determine the scaling function \( \ell^+/(\zeta) \) from the data. A careful analysis of the DNS data allows us to find a good one-parameter fit for \( \ell^+/(\zeta) \)

\[
\frac{\ell^+ (\zeta)}{\text{Re}_r} = \ell_s \left\{1 - \exp \left[-\frac{\tilde{\zeta}}{\ell_s} \left(1 + \tilde{\zeta} \right) \right]\right\}.
\] (9)

where \( \tilde{\zeta} \equiv \zeta/(1 - \zeta)/2 \) and \( \ell_s \approx 0.311 \). The quality of the fit is obvious from the continuous line in the rightmost panel of Fig. 2. Note that the fit function is exactly constant at mid channel, with zero slope. This is required by symmetry, and will be the reason for our excellent fit of data in the wake region.

Finally the theory for the mean velocity contains three parameters, namely \( \ell_s \) together with \( \ell_{\text{buf}}^+ \) (which determines \( B \) in Eq. (1)) and \( \kappa \). We demonstrate now that with these three parameters we can determine the mean velocity profile for any value \( \text{Re}_r \), throughout the channel, including the viscous layer, the buffer sub-layer, the log-law region and the wake. Examples of the integration of Eq. (8) are shown in Fig. 1. It is worthwhile to reiterate that the excellent fits in the viscous and the wake regions (superior to the fits presented in [11, 12]), which are usually most difficult to achieve, are obtained here due to the correct asymptotics of \( \ell^+/(\zeta) \) at \( \zeta \to 0 \) and \( \zeta \to 1 \). In addition, our theory results also in the kinetic
FIG. 2: Color online. The scaling function $\ell^+(\zeta)/Re_\tau$ (Left upper panel), $\ell^+_K(\zeta)/Re_\tau$ (Right upper panel) and the final scaling function $\ell^+(\zeta)$ (Lower panel), as a function of $\zeta \equiv z/L$, for four different values of $Re_\tau$, computed from the DNS data [2, 3]. Note the data collapse everywhere except at $\zeta \to 1$ where $W^+ \sim S^+ \ll 1$ and accuracy is lost. The green dash line represents $\zeta = (1 - \zeta/2)$ with a saturation level 0.5; in orange solid line we show the fitted function Eq. (9) with $\ell_{sat} = 0.311$.

energy, and Reynolds stress profiles which are in a quantitative agreement with the DNS data; for $W$ profiles see Fig. 1.

**Conclusions and application to experiments:** We discussed turbulent channel flow, demonstrating the existence and usefulness of a scaling function $\ell^+(\zeta)$ which allows us to get the profiles of the mean velocities for all values of $Re_\tau$ and throughout the channel, in a good agreement with DNS. We argued that the controversy between power-laws and log-laws is moot, stemming from a rough estimate of the scaling function $\ell^+(\zeta)$. While this function begins near the wall as $z^+$, it saturates later, and its full functional dependence on $\zeta$ is crucial for finding the correct mean velocity profiles. The approach also allows us to delineate the accuracy of the log-law presentation, which depends on $z^+$ and the value of $Re_\tau$. For asymptotically large $Re_\tau$, the region of the log-law can be very large, but nevertheless it breaks down near the mid channel and near the buffer layer, where correction to the log-law were presented.

To show that the present approach is quite general, we apply it now to the experimental data that were at the center of the controversy [5], i.e. the Princeton University Superpipe data [4]. In Fig. 1 right panel we show the mean velocity profiles as measured in the Superpipe compared with our prediction using the same scaling function $\ell^+(\zeta)$. Note that the data spans values of $Re_\tau$ from 5050 to 165000, and the fits with only three $Re_\tau$-independent
constants are very satisfactory. Note the 2\% difference in the value of $\kappa$ between the DNS and the experimental data; we do not know at this point whether this stems from inaccuracies in the DNS or the experimental data, or whether turbulent flows in different geometries have different values of $\kappa$. While the latter is theoretically questionable, we cannot exclude this possibility until a better understanding of how to compute $\kappa$ from first principles is achieved.

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Appendix: The exact balance equation for the Reynolds shear stress can be found in [1]: $P_{yy}^+ + R_{yy}^+ = \varepsilon_{yy}^+ - T_{yy}^+$. Here $P_{yy}^+$ is the production of $W^+$, $R_{yy}^+$ is the redistribution of $W^+$ between other Reynolds stress components, $\varepsilon_{yy}^+$ is the viscous dissipation of $W^+$ and $T_{yy}^+$ is the turbulent transport of $W^+$. Explicit expressions for these terms are in [1]. Since $\tau_{yy}$ is $O(K)$, we approximate $P_{yy}^+ \propto -K^+ S^+$. $R_{yy}^+ = R_{yy}^{visc} + R_{yy}^{diff}$ [1, 11]. The first term describes the return to isotropy, while the second one is responsible for the isotropization of production. A slightly modified Rotta’s model [13] proposes that $R_{yy}^{visc} \propto \sqrt{K W}/\ell_w$. $R_{yy}^{diff}$ is modeled according to [1, 14], such that $R_{yy}^{diff} \propto K^+ S^+$. The viscous dissipation $\varepsilon_{yy}^+ \equiv \nu \langle \partial_k u_x \partial_k u_x \rangle$ is $O(-\nu W z^{-2})$. As explained in the text, we can neglect the non-local term $T_{yy}$ in the balance for the Reynolds stress with impunity. To compensate for its loss in the viscous range we increase the estimate $(-\nu W z^{-2})$ by a factor $\sqrt{K}/K_w$, where $K_w$ is a dimensional constant [11]. Eventually, $\varepsilon_{yy}^+ \propto -W^+ \sqrt{K^+}/z^{+2}$. Hence, the approximate algebraic balance equation for the Reynolds shear stress reads:

$$-a K^+ S^+ + b W^+ \sqrt{K^+} \ell_w^+ + c K^+ S^+ \approx -d W^+ \sqrt{K^+} / z^{+2}, \quad (10)$$

where $a, b, c, d$ - are positive constants of $O(1)$. The last equation may be rearranged to the form of the fist of Eq. (4) but with $r_w \equiv 1 + \ell_{buf}^+ / z^{+2}$, $\ell_{buf}^+ \equiv d/b$. Since the second term is dominant only near the wall where $\ell_w^+ = z^+$, then $r_w \rightarrow 1 + \ell_{buf}^+ / z^+$. In [12] it was realized that this from, which is an interpolation between the near wall and the bulk physics, can be modeled in a way that reflects better the actual width of the buffer layer, using another interpolation formula that reads

$$r_w \equiv \left[ 1 + \left( \frac{\ell_{buf}^+}{z^+} \right)^n \right]^{1/n} \quad (11)$$

with $n = 2$. Best fit to simulational data which is currently available is obtained with $5 < n < 7$. In this Letter we chose $n = 6$ leading to the second of Eqs. (4). This choice simplifies the appearance of the Eqs. (6)-(8).

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