SKELETONS OF MONOMIAL IDEALS

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Abstract. In analogy to the skeletons of a simplicial complex and their Stanley–Reisner ideals we introduce the skeletons of an arbitrary monomial ideal \( I \subset S = K[x_1, \ldots, x_n] \). This allows us to compute the depth of \( S/I \) in terms of its skeleton ideals. We apply these techniques to show that Stanley’s conjecture on Stanley decompositions of \( S/I \) holds provided it holds whenever \( S/I \) is Cohen–Macaulay. We also discuss a conjecture of Soleyman-Jahan and show that it suffices to prove his conjecture for monomial ideals with linear resolution.

Introduction

Let \( \Delta \) be a simplicial complex of dimension \( d - 1 \) on the vertex set \( \{1, \ldots, n\} \), \( K \) a field and \( K[\Delta] \) the Stanley–Reisner ring of \( \Delta \). The depth of \( K[\Delta] \) can be expressed in terms of the skeletons of \( \Delta \), as has been shown by D. Smith [6, Theorem 3.7] for pure simplicial complexes, and by Hibi [5, Corollary 2.6] in general. The \( j \)th skeleton of \( \Delta \) is the simplicial subcomplex \( \Delta^{(j)} = \{ F \in \Delta : |F| \leq j \} \) of \( \Delta \). The result is that \( \text{depth } K[\Delta] = \max \{ j : \Delta^{(j)} \text{ is Cohen–Macaulay} \} \).

The purpose of this paper is to generalize this result as follows: first note that we have the following chain of Stanley–Reisner ideals \( I_\Delta = I_{\Delta^d} \subset I_{\Delta^{d-1}} \subset \cdots \subset I_0 \subset S \) with \( \text{dim } S/I_\Delta(j) = j \) for all \( j \). Now for an arbitrary monomial ideal \( I \subset S \) we want to define in a natural way a similar chain of monomial ideals \( I = I_d \subset I_{d-1} \subset \cdots \subset I_0 \subset S \) with \( \text{dim } S/I_j = j \) for all \( j \), and of course this chain should satisfy the condition that \( \text{depth } S/I = \max \{ j : S/I_j \text{ is Cohen–Macaulay} \} \). We show in Section 1 that such a natural chain of monomial ideals with these properties indeed exists. The ideal \( I_j \) is called the \( j \)th skeleton ideal of \( I \).

For the construction of the skeleton ideals of \( I \) we consider the so-called characteristic poset \( P^g_{S/I} \) introduced in [1]. Here \( g \in \mathbb{N}^n \) is an integer vector such that \( g \geq a \) for all \( a \) for which \( x^a \) belongs to the minimal set of monomial generators of \( I \), and \( P^g_{S/I} \) is the (finite) poset of all \( b \in \mathbb{N}^n \) such that \( b \leq g \) and \( x^b \notin I \). Here the partial order on \( \mathbb{N}^n \) is defined as follows: \( a \leq b \) if and only if \( a(i) \leq b(i) \) for \( i = 1, \ldots, n \). In case of a Stanley–Reisner ideal \( I_\Delta \) and \( g = (1, 1, \ldots, 1) \) this poset is just the face poset of \( \Delta \). For

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each \( b \in \mathbb{N}^n \), let \( \rho(b) = \{|j| \colon b(j) = g(j)\} \). It has been shown in [1] Corollary 2.6] that 
\[ \dim S/I = \max\{ \rho(b) : b \in P_S^{S/I} \} \] . We use this integer function \( \rho \) to define the skeleton ideals of \( I \), and let \( I_j \) be the monomial ideal generated by \( I \) and all \( x^b \) with \( \rho(b) > j \).

It is easy to see that \( \dim S/I_j = j \) for all \( j \). The crucial result however is that for all \( j \), \( I_{j-1}/I_j \) is Cohen-Macaulay module of dimension \( j \), see Theorem [1.2]. From this result we easily deduce in Corollary [1.5] a generalization of the result of Hibi, namely that depth \( S/I = \max\{j : S/I_j \text{ is Cohen–Macaulay} \} \).

In Section 2 we apply the results and techniques introduced in Section 1 to deduce some results on Stanley decomposition. Let \( M \) be a finitely generated \( \mathbb{Z}^n \)-graded \( S \)-module, \( m \in M \) be a homogeneous element and \( Z \subset X = \{x_1, \ldots, x_n\} \). We denote by \( mK[Z] \) the \( K \)-subspace of \( M \) generated by all homogeneous elements of the form \( mu \), where \( u \) is a monomial in \( K[Z] \). The \( K \)-subspace \( mK[Z] \) is called a Stanley space of dimension \( |Z| \) if \( mK[Z] \) is a free \( K[Z] \)-module.

A decomposition \( D \) of \( M \) as a finite direct sum of Stanley spaces is called a Stanley decomposition of \( M \). The minimal dimension of a Stanley space in the decomposition \( D \) is called the Stanley depth of \( D \), denoted by \( \text{sdepth} D \). We set

\[ \text{sdepth} M = \max\{ \text{sdepth} D : D \text{ is a Stanley decomposition of } M \} , \]

and call this number the Stanley depth of \( M \). A famous conjecture of Stanley asserts that \( \text{sdepth} M \geq \text{depth} M \).

As one of the main results of Section 2 we show in Corollary [2.2] that for each monomial ideal \( I \) Stanley’s conjecture holds for \( S/I \) if it holds whenever \( S/I \) is Cohen–Macaulay. We also discuss a conjecture of Soleyman-Jahan. His conjecture asserts that we can always find a Stanley decomposition \( M = \bigoplus_{j=1}^r m_j K[Z_j] \) of \( M \) with \( |\deg m_j| \leq \text{reg}(M) \) for all \( j \). Here \( |a| = \sum_{i=1}^n a(i) \) for \( a \in \mathbb{Z}^n \). We show in the case that \( M = I \) is a monomial ideal, it suffices to prove this conjecture when \( I \) has a linear resolution.

1. Characteristic posets and skeletons

Let \( K \) be a field, \( S = K[x_1, \ldots, x_n] \) the polynomial ring in \( n \) variables and \( I \subset S \) a monomial ideal. We denote by \( G(I) \) the unique minimal set of monomial generators of \( I \).

Let \( G(I) = \{u_1, \ldots, u_m\} \) with \( u_i = x^{a_i} \) and \( a_i \in \mathbb{N}^n \). Here, for any \( c \in \mathbb{N}^n \) we denote as usual by \( x^c \) the monomial \( x_1^{c(1)} x_2^{c(2)} \cdots x_n^{c(n)} \).

Observe that \( \mathbb{N}^n \) with the natural partial order introduced in the introduction is a distributive lattice with meet \( a \wedge b \) and join \( a \vee b \) defined as follows: \( (a \wedge b)(i) = \min\{a(i), b(i)\} \) and \( (a \vee b)(i) = \max\{a(i), b(i)\} \). We also denote by \( \varepsilon_j \) the \( j \)th canonical unit vector in \( \mathbb{Z}^n \).

Let \( J \subset S \) be another monomial ideal with \( I \subset J \), minimally generated by \( x^{b_1}, \ldots, x^{b_s} \).

We choose \( g \in \mathbb{N}^n \) such that \( a_i \leq g \) and \( b_j \leq g \) for all \( i \) and \( j \), and let \( P_{g_j/I}^g \) be the set of all \( b \in \mathbb{N}^n \) with \( b \leq g \), \( b \geq b_j \) for some \( j \), and \( b \nmid a_i \) for all \( i \). The set \( P_{g_j/I}^g \) viewed as a
subposet of \( \mathbb{N}^n \) is a finite poset, and is called the \textit{characteristic poset} of \( J/I \) with respect to \( g \), see [4].

For any \( b \in \mathbb{N}^n \) we define subsets \( Y_b = \{x_j: b(j) \neq g(j)\} \) and \( Z_b = \{x_j: b(j) = g(j)\} \) of \( X = \{x_1, \ldots, x_n\} \) and set \( \rho(b) = |\{j: b(j) = g(j)\}| = |Z_b| \). Let \( d = \dim J/I \) be the Krull dimension of \( J/I \). It is shown in [4, Corollary 2.6] that

\[
(1) \quad d = \max \{\rho(b): b \in P^g_{j/I}\}.
\]

As a consequence of (1) we obtain

**Lemma 1.1.** Let \( d = \dim J/I \). Then \( \rho(b) \leq d \) for all \( b \in \mathbb{N}^n \) with \( x^b \in J \setminus I \).

**Proof.** Let \( g' = g \lor b \). Then \( b \in P^g_{j/I} \), and hence \( |\{j: b(j) = g'(j)\}| \leq d \), by (1). Since \( \rho(b) = |\{j: b(j) = g(j)\}| \leq |\{j: b(j) \geq g(j)\}| = |\{j: b(j) = g'(j)\}| \), the assertion follows. \( \square \)

Formula (1) leads us to consider for each \( j \leq d \), the monomial ideal \( I_j \) generated by \( I \) together with all monomials \( x^b \) such that \( \rho(b) > j \). We then obtain a chain of monomial ideals

\[
I = I_d \subset I_{d-1} \subset \cdots \subset I_0 \subset S.
\]

Of course this chain of ideals depends not only on \( I \), but also on the choice of \( g \).

Consider the special case, where \( I = I_\Delta \) is the Stanley–Reisner ideal of a simplicial complex \( \Delta \) on the vertex set \( \{1, \ldots, n\} \). Then for \( g = (1, \ldots, 1) \) we have \( I_j = I_{\Delta(j)} \). This observation justifies to call \( I_j \) the \( j \)-th skeleton ideal of \( I \) (with respect to \( g \)).

The following result is crucial for this note.

**Theorem 1.2.** For each \( 0 \leq j \leq d \), the factor module \( I_{j-1}/I_j \) is a direct sum of cyclic Cohen–Macaulay modules of dimension \( j \). In particular, \( I_{j-1}/I_j \) is a \( j \)-dimensional Cohen–Macaulay module.

**Proof.** Replacing \( I \) by \( I_j \) it suffices to consider the case \( j = d \). Let

\[
J = (I, \{x^b: b \in A\}), \quad \text{where} \quad A = \{b \in P^g_{S/I}: \rho(b) = d\},
\]

then \( I_{d-1}/I_d = J/I \).

Let \( \{Z_1, \ldots, Z_r\} \) be the collection of those subsets of \( X \) with the property that for each \( i = 1, \ldots, r \) there exists \( b \in A \) such that \( Z_i = Z_b \). Let \( A_i = \{b \in A: Z_b = Z_i\} \), and let \( b, b' \in A_i \). Then \( b \cap b' \in A_i \). Thus the meet of all the elements in \( A_i \) is the unique smallest element in \( A_i \). We denote this element by \( b_i \). Then \( Z_i = Z_{b_i} \). Obviously the elements \( f_i = x^{b_i} + I, i = 1, \ldots, r \) generate \( J/I \). We claim that

\[
J/I = \bigoplus_{i=1}^r Sf_i.
\]
The cyclic module $S_{f_i}$ is $\mathbb{Z}^n$-graded with a $K$-basis $x^a + I$ with $a \geq b_i$ and $x^a \not\in I$. Given $c \in \mathbb{N}^n$ with $c \geq b_i$ and $c \geq b_j$ for some $1 \leq i < j \leq r$, then $\rho(c) > d$, and so $x^c \in I$, by Lemma 1.1. This shows that the sum of the cyclic modules $S_{f_i}$ is indeed direct.

Next we notice that if $x^c = x^{c_1}x^{c_2}$ with $x^{c_1} \in K[Z_{b_i}]$ and $x^{c_2} \in K[Y_{b_i}]$ belongs to $\text{Ann}(S_{f_i})$, then $x^{c_2} \in \text{Ann}(S_{f_i})$. Indeed, $x^c = x^{c_1}x^{c_2} \in \text{Ann}(S_{f_i})$ if and only if $a_j \leq b_i + c_1 + c_2$ for some $j$. Since $c_1(k) = 0$ for all $k$ with $x_k \in Y_{b_i}$, it follows that $a_j(k) \leq (b_i + c_2)(k)$ for all $k \in Y_{b_i}$, while for $k$ with $x_k \in Z_{b_i}$ we have $a_j(k) \leq g(k) = b_i(k) = (b_i + c_2)(k)$. Hence $a_j \leq b_i + c_2$, which implies that $x^{c_2} \in \text{Ann}(S_{f_i})$.

It follows that $\text{Ann}(S_{f_i})$ is generated by monomials in $K[Y_{b_i}]$. In other words, there exists a monomial ideal $M_i \subset K[Y_{b_i}]$ such that $\text{Ann}(S_{f_i}) = M_i S$.

For each $k$ with $x_k \in Y_{b_i}$ we have $b_i(k) < g(k)$ and $\rho(b_i + (g(k) - b_i(k))e_k) = d + 1$. Therefore Lemma 1.1 implies that $x^{b_i} x^{g(k) - b_i} \in I$. It follows that $x^{b_i} x^{g(k) - b_i} \in M_i$ for all $k$ with $x_k \in Y_{b_i}$. Hence we see that $\dim K[Y_{b_i}]/M_i = 0$. This implies that $S_{f_i} = S/M_i S$ is Cohen–Macaulay of dimension $d$.

\[\Box\]

**Remark 1.3.** It is also possible to define skeletons of $J/I$ in the same way as for $S/I$, and one obtains a chain of ideals $I = I_d \subset I_{d-1} \subset \cdots \subset I_0 \subset J$. Some of the factor modules $I_{j-1}/I_j$ however may be zero in this generality. But whenever $I_{j-1}/I_j \neq 0$ it follows again that $I_{j-1}/I_j$ is Cohen–Macaulay of dimension $j$, though not always a direct sum of cyclic modules.

**Corollary 1.4.** For $j = 0, \ldots, d - 1$ and $i = 0, \ldots, j - 1$, we have $\text{depth}(I_j/I) \geq j + 1$, and $H^i_m(S/I) \cong H^i_m(S/I_j)$.

**Proof.** We prove the assertion by induction on $d - j$. For $j = d - 1$ the assertion follows from Theorem 1.2. Let $j < d - 1$. Then the exact sequence

$$0 \to I_{j+1}/I \to I_j/I \to I_j/I_{j+1} \to 0$$

implies that

$$\text{depth}(I_j/I) \geq \min\{\text{depth}(I_{j+1}/I), \text{depth}(I_j/I_{j+1})\},$$

see [2] Proposition 1.2.9. By Theorem 1.2 $\text{depth}(I_j/I_{j+1}) = j + 1$ and by induction hypothesis $\text{depth}(I_{j+1}/I) \geq j + 2$. Hence $\text{depth}(I_j/I) \geq j + 1$.

The short exact sequence

$$0 \to I_j/I \to S/I \to S/I_j \to 0$$

yields the long exact sequence

$$\cdots \to H^i_m(I_j/I) \to H^i_m(S/I) \to H^i_m(S/I_j) \to H^{i+1}_m(I_j/I) \to \cdots$$

of local cohomology. By the first part of the statement we have $H^k_m(I_j/I) = 0$ for $k \leq j$. This yields the desired isomorphisms.

\[\Box\]

As an application of Theorem 1.2 we obtain the following characterization of the depth of $S/I$ which generalizes a classical result of Hibi [5, Corollary 2.6].
Corollary 1.5. Let $I \subset S$ be a monomial ideal. Then
\[ \text{depth } S/I = \max\{j : S/I_j \text{ is Cohen–Macaulay}\}, \]
and $S/I_j$ is Cohen–Macaulay for all $j \leq \text{depth } S/I$.

Proof. Let $d = \dim S/I$ and $t = \text{depth } S/I$. Since $I_j = (I_{d-1})_j$ for $j \leq d - 1$, both assertions follow by induction on $d$ once we can show the following:

(i) If $t < d$, then $\text{depth } S/I_{d-1} = t$.
(ii) If $S/I$ is Cohen–Macaulay, then $S/I_{d-1}$ is Cohen–Macaulay.

Proof of (i): The exact sequence
\[ 0 \to I_{d-1}/I \to S/I \to S/I_{d-1} \to 0 \]
implies that
\[ \text{depth } S/I_{d-1} \geq \min\{\text{depth}(I_{d-1}/I) - 1, \text{depth } S/I\}, \tag{2} \]
with equality if $t < d - 1$, see [2, Proposition 1.2.9]. By Theorem 1.2, $\text{depth}(I_{d-1}/I) - 1 = d - 1$. It follows that $\text{depth } S/I_{d-1} = t$, if $t < d - 1$. On the other hand, if $t = d - 1$, then $\text{depth } S/I_{d-1} \geq d - 1$. However, since $\dim S/I_{d-1} = d - 1$, we again get $\text{depth } S/I_{d-1} = d - 1 = t$.

Proof of (ii): If $S/I$ is Cohen–Macaulay, then $\text{depth } S/I_{d-1} \geq d - 1$. Since $\dim S/I_{d-1} = d - 1$, the assertion follows. \qed

The proof of Corollary 1.5 provides the following additional information.

Corollary 1.6. We have $\text{depth } S/I_{j-1} \leq \text{depth } S/I_j \leq \text{depth } S/I$ for all $0 \leq j \leq \dim S/I$.

2. Applications to Stanley decompositions

Let $I \subset S$ be a monomial ideal. In the recent paper [4] it was shown that the Stanley depth of $S/I$ can be computed by means of properties of $P^g_{S/I}$. The result [4, Theorem 2.1] can be summarized as follows: given any poset $P$ and $a, b \in P$, we set $[a, b] = \{c \in P : a \leq c \leq b\}$ and call $[a, b]$ an interval. Of course, $[a, b] \neq \emptyset$ if and only if $a \leq b$. Suppose $P$ is a finite poset. A partition of $P$ is a disjoint union
\[ \mathcal{P} : P = \bigcup_{i=1}^r [c_i, d_i] \]
of non-empty intervals. Let $\mathcal{P} : P^g_{S/I} = \bigcup_{i=1}^r [c_i, d_i]$ be a partition of $P^g_{S/I}$. We set
\[ \rho(\mathcal{P}) = \min\{\rho(d_i) : i = 1, \ldots, r\}. \]
Then
\[ \text{sdepth } S/I = \max\{\rho(\mathcal{P}) : \mathcal{P} \text{ is a partition of } P^g_{S/I}\}. \]
We use this characterization of the Stanley depth and the results of the previous section to prove

**Proposition 2.1.** For all $0 \leq j \leq d = \dim S/I$ we have

$$\sdepth S/I \geq \sdepth S/I_j.$$  

*Proof.* Observe that $P^g_{S/I_j} = \{ a \in P^g_{S/I} : \rho(a) \leq j \}$. Let $t$ be the Stanley depth of $S/I_j$. Then there exists a partition $\mathcal{P} : P^g_{S/I_j} = \bigcup_{i=1}^r [c_i, d_i]$ with $\rho(\mathcal{P}) = t$. We complete the partition of $\mathcal{P}$ to a partition $\mathcal{P}'$ of $S/I$ by adding the intervals $[a, a]$ with $a \in P^g_{S/I} \setminus P^g_{S/I_j}$. Since $\rho(a) > j$ for all $a \in P^g_{S/I} \setminus P^g_{S/I_j}$ it follows that $\rho(\mathcal{P}') = t$. Hence $\sdepth S/I \geq t$, as desired. □

We call an algebra of the form $S/I$ a monomial factor algebra if $I \subset S$ is a monomial ideal. Stanley’s conjecture for a monomial factor algebra $S/I$ says that $\depth S/I \leq \sdepth S/I$.

**Corollary 2.2.** Suppose Stanley’s conjecture holds for all Cohen–Macaulay monomial factor algebras of dimension $t$. Then the conjecture holds for all monomial factor algebras of depth $t$. In particular, Stanley’s conjecture holds for all monomial factor algebras if and only if it holds for all Cohen–Macaulay monomial factor algebras.

*Proof.* Let $S/I$ be a monomial factor algebra with $t = \depth S/I$. Then $S/I_t$ is Cohen–Macaulay of dimension $t$, see Corollary 1.5. Our assumption implies that $\sdepth S/I_t = t$. Thus the assertion follows from Proposition 2.1. □

As a concrete application we have

**Corollary 2.3.** Let $S/I$ be a monomial factor algebra with $\depth S/I \leq 1$. Then $S/I$ satisfies Stanley’s conjecture.

*Proof.* According to Corollary 2.2 it suffices to show that any Cohen-Macaulay $K$-algebra $S/I$ of dimension $\leq 1$ satisfies Stanley’s conjecture. This is trivially the case if $\dim S/I = 0$, and has been shown if $\dim S/I = 1$ in [1, Corollary 3]. □

We now prove a statement which in a certain sense is dual to that of Corollary 2.2. Let $I \subset J$ be monomial ideals and $\mathcal{D} = \bigoplus_{i=1}^r x^{c_i} K[Z_i]$ a Stanley decomposition of $J/I$. The number $\max\{|c_i| : i = 1, \ldots, r\}$ is called the $h$-regularity of $\mathcal{D}$, denoted by $\hreg(\mathcal{D})$.

We set

$$\hreg(J/I) = \min\{ \hreg(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } J/I \}$$

and call this number the $h$-regularity of $J/I$. In [7], the second author conjectured that $\hreg(J/I) \leq \reg(J/I)$.
Let \( g \in \mathbb{N}^n \) with \( g \geq a \) for all minimal monomial generators \( x^a \) of \( I \) and \( J \). For a partition \( \mathcal{P} : P_{j/I}^g = \bigcup_{j=1}^r [c_i, d_i] \), we set
\[
\sigma(\mathcal{P}) = \max\{\sigma_i(\mathcal{P}) : i = 1, \ldots, r\},
\]
where
\[
\sigma_i(\mathcal{P}) = \max\{|c| : c \in [c_i, d_i] \text{ and } c(j) = c_i(j) \text{ for all } j \text{ with } x_j \in Z_{d_i}\}.
\]

**Proposition 2.4.** \( \hreg(J/I) = \min\{\sigma(\mathcal{P}) : \mathcal{P} \text{ is a partition of } P_{j/I}^g\} \).

**Proof.** To each partition \( \mathcal{P} : P_{j/I}^g = \bigcup_{j=1}^r [c_i, d_i] \) belongs a Stanley decomposition \( \mathcal{D}(\mathcal{P}) \), as described in [4, Theorem 2.1(a)]. The assignment is such that \( \hreg(\mathcal{D}(\mathcal{P})) = \sigma(\mathcal{P}) \). This shows that \( \hreg(J/I) \leq \min\{\sigma(\mathcal{P}) : \mathcal{P} \text{ is a partition of } P_{j/I}^g\} \).

In order to prove that equality holds, we need to find a partition \( \mathcal{P} \) with \( \hreg(J/I) = \sigma(\mathcal{P}) \). Let \( \mathcal{D} : J/I = \bigoplus_{i=1}^r x^a K[Z_i] \) be a Stanley decomposition of \( J/I \) with \( \hreg(\mathcal{D}) = \hreg(J/I) \). In [4, Theorem 2.1(b)] it is shown \( \mathcal{P} : P_{j/I}^g = \bigcup_{i, c_i \leq g} [c_i, d_i] \) is a partition of \( P_{j/I}^g \), where \( d_i(j) = c_i(j) \) if \( x_j \notin Z_i \), and \( d_i(j) = g(j) \) otherwise. Thus we see that \( \hreg(J/I) = \max\{|c_i| : i = 1, \ldots, r\} = \sigma(\mathcal{P}) \). \( \square \)

Observe that the preceding proposition implies in particular that \( \hreg(J/I) \) can be computed in a finite number of steps.

For a graded ideal \( I \) we denote by \( I_{\geq j} \) the \( j \)th truncation of \( I \), that is, the ideal generated by all homogeneous elements \( f \in I \) with \( \deg f \geq j \).

**Proposition 2.5.** For all \( j \geq 0 \) we have \( \hreg(I) \leq \hreg(I_{\geq j}) \).

**Proof.** We choose \( g \in \mathbb{N}^n \) such that \( g \geq a \) for all generators \( x^a \) of \( I \) and \( I_{\geq j} \). Let \( \mathcal{P} \) be a partition of \( P_{I_{\geq j}}^g \) with \( \sigma(\mathcal{P}) = \hreg(I_{\geq j}) \). We complete the partition \( \mathcal{P} \) to a partition \( \mathcal{P}' \) of \( P_{I}^g \) by adding the intervals \([a, a]\) with \( a \in P_{I_{\geq j}}^g \setminus P_{I_{\geq j}}^g \). Then Proposition 2.4 implies that \( \hreg(I) \leq \sigma(\mathcal{P}') = \sigma(\mathcal{P}) = \hreg(I_{\geq j}) \). \( \square \)

**Corollary 2.6.** Suppose \( \hreg(I) \leq \reg(I) \) for all ideals with linear resolution. Then this inequality is valid for all graded ideals.

**Proof.** By a result of Eisenbud and Goto [3] (see also [2, Theorem 4.3.1]) one has
\[
\reg(I) = \min\{j : I_{\geq j} \text{ has a linear resolution}\}.
\]
We choose a \( j \) such that \( I_{\geq j} \) has a linear resolution. Then our assumption and Proposition 2.5 imply that \( \hreg(I) \leq \hreg(I_{\geq j}) \leq \reg(I_{\geq j}) = \reg(I) \). \( \square \)

For a monomial ideal \( I \) with linear resolution, minimally generated by \( x^{a_1}, \ldots, x^{a_r} \), Soleyman-Jahan’s conjecture reads as follows: there exist \( b_1, \ldots, b_r \in P_I^g \) such that \( P_I^g = \bigcup_{j=1}^r [a_i, b_i] \). In other words, \( P_I^g \) can be partitioned by intervals whose lower ends correspond to the generators of \( I \).
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