Simple prescription for computing the nonrelativistic interparticle potential energy related to dual models

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Following a procedure recently utilized by Accioly et al. to obtain the D-dimensional interparticle potential energy for electromagnetic models in the nonrelativistic limit, and relaxing the condition assumed by the authors concerning the conservation of the external current, the prescription found out by them is generalized so that dual models can also be contemplated. Specific models in which the interaction is mediated by a spin-0 particle described first by a vector field and then by a higher-derivative vector field, are analyzed. Systems mediated by spin-1 particles described, respectively, by symmetric rank-2 tensors, symmetric rank-2 tensors augmented by higher derivatives, and antisymmetric rank-2 tensors, are considered as well.

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I. INTRODUCTION

Currently, a great number of models modeling interactions have appeared in the literature. Nonetheless, all these systems are required to possess a well defined behavior in the nonrelativistic limit, whatever their high energy aspects may be. For instance, models describing electromagnetic interactions must reproduce the Coulomb potential energy plus a possible correction; while gravitational systems must lead to Newtonian gravity sometimes enlarged by a correction. Accordingly, any simple method that could enable us to find the nonrelativistic potential energy without expending much effort, should be welcome. Recently, Accioly et al. developed an straightforward method for addressing this question [1, 2].

In Ref. [1], a suitable prescription for computing the nonrelativistic interparticle potential energy related to D-dimensional electromagnetic models described by vector fields is constructed; whereas in Ref. [2], the interparticle potential energy for D-dimensional gravitational systems was analyzed. In both articles, the authors assumed that the external current is conserved.

In this paper our goal is to extend the prescription built out by Accioly et al. [1, 2] in order to incorporate the possibility of dual theories. Duality, which is the equivalence between different mathematical description of the same physics, has been extensively considered in physics [3]. One of the most interesting features of duality is the possibility of changes in the coupling regimes, for instance, a weak (strong) coupling in one model has an strong (weak) coupling in the corresponding model. Hence, provided that we know the correspondence between the original and the dual model, one can perform perturbative computations in both regimes.

In the context of dual theories, a vector field can be associated with both spin-0 or spin-1 particle and, in the first case, the longitudinal part of the an external current contributes to the potential energy. In addition, a rank-2 tensor field has several representations and, in general, can be split in symmetric \( H_{\mu\nu} \) and antisymmetric \( A_{\mu\nu} \) parts. The symmetric tensor, \( H_{\mu\nu} \), in turn, gives rise to four different sectors: spin-2, spin-1, and two spin-0 sectors. As a result, we may expect different kinds of couplings between the symmetric tensor \( H_{\mu\nu} \) and external currents. In fact, the dual representation of spin-1 and spin-0 particles described by a symmetric tensor admits couplings with tensor currents containing components along the longitudinal direction.

In the aforementioned cases, the prescription developed by Accioly et al. [1, 2] cannot be applied, since the presence of longitudinal components in the external currents is in conflict with the hypothesis of conserved external currents. Therefore, in this paper we intend to extend the prescription in order to incorporate the possibility of external currents with longitudinal components.

The paper is organized as follows. In Sec. II we introduce the generalized prescription for the case of interactions mediated by vector fields. In Sec. III we investigate the issue of interactions mediated by a symmetric rank-2 tensor

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field. In Sec. IV we adapt the prescription developed in the two last sections to the situation where the interaction is mediated by an antisymmetric rank-2 tensor field. To test the efficacy and simplicity of the method we have developed, explicit examples of the computation of the interparticle potential energy concerning the situations dealt with in Secs. II-IV, are presented. Finally, in Sec. VI, we concluded. Technical details are relegated to the Appendices.

Natural units are used throughout, and our Minkowski metric is \( \text{diag}(1, -1, ..., -1) \).

II. PRESCRIPTION FOR COMPUTING THE NON-RELATIVISTIC INTERPARTICLE POTENTIAL ENERGY MEDIATED BY VECTOR FIELDS

It is well known that the \( D \)-dimensional static potential energy associated with an external vectorial current \( J^\mu \) can be obtained from the expression [4]

\[
E_D = \frac{1}{2} \int \int d^D x \, d^D y \, J^\mu(x) J^\nu(y),
\]

where \( D_{\mu\nu}(x - y) \) is the Feynman propagator, \( J^\mu(x) \) is an external current and the parameter \( \tau \) stands for a time interval. Consider then the following decomposition for the external current \( J^\mu(x) \), namely, \( J^\mu(x) = J^\mu_T(x) + J^\mu_L(x) \), where \( J^\mu_T \) and \( J^\mu_L \) are, respectively, the transverse and longitudinal parts of the aforementioned current; of course, only the transverse part of the current must necessarily satisfy the conservation law \( \partial_\mu J^\mu_T(x) = 0 \). Taking the above decomposition into account, we find

\[
E_D = \frac{1}{2\tau} \int \int d^D x \, d^D y \, J^\mu_T(x) D_{\mu\nu}(x - y) J^\nu_T(y) + \frac{1}{2\tau} \int \int d^D x \, d^D y \, J^\mu_L(x) D_{\mu\nu}(x - y) J^\nu_L(y) + \frac{1}{2\tau} \int \int d^D x \, d^D y \, J^\mu_T(x) D_{\mu\nu}(x - y) J^\nu_L(y).
\]

Bearing in mind that

\[
D_{\mu\nu}(x - y) = \frac{1}{(2\pi)^D} \int d^D k \, D_{\mu\nu}(k) e^{ik(x-y)},
\]

we may rewrite the potential energy as follows

\[
E_D = \frac{1}{2\tau} \int \frac{d^D k}{(2\pi)^D} D_{\mu\nu}(k) \int \int d^D x \, d^D y \left( J^\mu_T(x) J^\nu_T(y) + J^\mu_L(x) J^\nu_L(y) + J^\mu_T(x) J^\nu_L(y) + J^\mu_L(x) J^\nu_T(y) \right) e^{ik(x-y)}. \tag{4}
\]

Since we are only interested in time-independent currents, \( i.e., \, J^\mu(x) = J^\mu(x), \) we integrate in the variables \( x^0, \, y^0 \) and \( k^0 \), which leads to

\[
E_D = \int \frac{d^{D-1} k}{(2\pi)^{D-1}} D_{\mu\nu}(k) \left[ \Delta^\mu_\nu(k) + \Delta^\mu_\nu(k) + \Omega^\mu_\nu(k) \right], \tag{5}
\]

where we have used the definitions

\[
\Delta^\mu_\nu(k) = \frac{1}{2} \int \int d^{D-1} x \, d^{D-1} y \, J^\mu_T(x) J^\nu_T(y) e^{ik(y-x)}, \tag{6}
\]

\[
\Delta^\mu_\nu(k) = \frac{1}{2} \int \int d^{D-1} x \, d^{D-1} y \, J^\mu_L(x) J^\nu_L(y) e^{ik(y-x)}, \tag{7}
\]

\[
\Omega^\mu_\nu(k) = \frac{1}{2} \int \int d^{D-1} x \, d^{D-1} y \, J^\mu_T(x) J^\nu_L(y) e^{ik(y-x)} + \frac{1}{2} \int \int d^{D-1} x \, d^{D-1} y \, J^\mu_L(x) J^\nu_T(y) e^{ik(y-x)}. \tag{8}
\]

After the addition of a possible gauge fixing term, the most general free electromagnetic Lagrangian (containing only quadratic terms) that can be constructed may be cast in the form

\[
\mathcal{L}_0 = \frac{1}{2} A^\mu [a(\Box) \theta_{\mu\nu} + b(\Box) \omega_{\mu\nu}] A^\nu, \tag{9}
\]
where \( \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu} \) and \( \omega_{\mu\nu} = \partial_\mu \partial_\nu / \Box \) stand for the usual vectorial projection operators, and the coefficients \( a(\Box) \) and \( b(\Box) \) are polynomial functions of the d’Alembertian operator. To perform the transition from the usual coordinate space to the momentum representation we use the straightforward correspondence: \( \Box \rightarrow -k^2 \) and \( F(\Box) \rightarrow F(-k^2) = \tilde{F}(k) \), for any function \( F(\Box) \). The Feynman propagator in momentum space is then obtained directly from the above Lagrangian, as follows

\[
D_{\mu\nu}(k) = \frac{1}{\tilde{a}(k)} \theta_{\mu\nu} + \frac{1}{b(k)} \omega_{\mu\nu}.
\]

Using the above expression and taking into account that \( \theta_{\mu\nu} J_L^\mu = \omega_{\mu\nu} J_T^\mu = 0 \), we promptly obtain

\[
D_{\mu\nu}(k) \Delta_T^{\mu\nu}(k) = \frac{1}{\tilde{a}(k)} \eta_{\mu\nu} \Delta_T^{\mu\nu}(k),
\]

\[
D_{\mu\nu}(k) \Delta_L^{\mu\nu}(k) = \frac{1}{b(k)} \eta_{\mu\nu} \Delta_L^{\mu\nu}(k),
\]

\[
D_{\mu\nu}(k) \Omega^{\mu\nu}(k) = 0,
\]

where \( \tilde{a}(k) = \bar{a}(k)|_{k^0=0} \), \( \bar{b}(k) = \bar{b}(k)|_{k^0=0} \). Substitution of the above equations into (5), allows to conclude that

\[
E_D = \eta_{\mu\nu} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \left[ \frac{\Delta_T^{\mu\nu}(k)}{\tilde{a}(k)} + \frac{\Delta_L^{\mu\nu}(k)}{\bar{b}(k)} \right].
\]

Accordingly, there are two contributions for the interparticle potential energy, one coming from interactions mediated by spin-0 particles (longitudinal sector) and the other one associated with interactions mediated by spin-1 particles (transverse sector). It is important to highlight that for a model which propagates only one of the sectors (spin-0 or spin-1), the contribution associated with the other sector must be discarded. For clarity’s sake, we use the following notation

\[
E_D = E_D^{(T)} + E_D^{(L)}
\]

where

\[
E_D^{(T)} = \eta_{\mu\nu} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\Delta_T^{\mu\nu}(k)}{\tilde{a}(k)} \leftrightarrow \text{spin-1 contribution},
\]

\[
E_D^{(L)} = \eta_{\mu\nu} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\Delta_L^{\mu\nu}(k)}{\bar{b}(k)} \leftrightarrow \text{spin-0 contribution}.
\]

Now, since we are only interested in the computation of the potential energy associated with two point-like static charges, we specify the transverse and longitudinal currents, respectively, as follows

\[
J_T^\mu(x) = \eta^{\mu0} \left( Q_1^{(T)} \delta^d(x - a_1) + Q_2^{(T)} \delta^d(x - a_2) \right),
\]

\[
J_L^\mu(\vec{x}) = \partial^\mu \left( Q_1^{(L)} \delta^d(x - a_1) + Q_2^{(L)} \delta^d(x - a_2) \right),
\]

where \( d = D - 1 \) is the number of spatial dimensions.

Two comments are in order here.

1. The longitudinal sector may be interpreted as the gradient of a scalar current

2. \( Q^{(T)} \) and \( Q^{(L)} \) do not represent charges of the same nature; as a consequence, there is no reason to expect that \( Q^{(T)} \) and \( Q^{(L)} \) have the same dimension. In fact, the dimension of the charges \( Q^{(T)} \) and \( Q^{(L)} \) depends on the dimension of the field which mediates the interaction. A simple dimensional analysis shows that the charges \( Q^{(T)} \) and \( Q^{(L)} \) have, respectively, mass dimension \( 1 - n \) and \( -n \), where \( n \) stands for the mass dimension of the vector field.
Substitution of Eqs. (18) and (19), respectively, into (6) and (7), leads to
\[
\Delta_T^{\mu\nu}(k) = \eta^{\mu\alpha} \eta^{\nu\beta} Q_1^{(T)} Q_2^{(T)} \cos(k \cdot r),
\]  
(20)
\[
\Delta_L^{\mu\nu}(k) = \delta_\beta^\gamma \eta^{\mu\gamma} Q_1^{(L)} Q_2^{(L)} k^i k^j \cos(k \cdot r),
\]  
(21)
where, \( r = a_2 - a_1 \). Using the last two equations in (16) and (17), we obtain
\[
E_D^{(T)} = \frac{Q_1^{(T)} Q_2^{(T)}}{(2\pi)^{D-1}} \int d^{D-1}k e^{ik \cdot r} \frac{\tilde{a}(k)}{\tilde{a}(k)}.
\]  
(22)
\[
E_D^{(L)} = \frac{Q_1^{(L)} Q_2^{(L)}}{(2\pi)^{D-1}} \int d^{D-1}k k^2 e^{ik \cdot r} \frac{\tilde{b}(k)}{\tilde{b}(k)}.
\]  
(23)
It is worth noting that in the preceding equations we have replaced the cosine function by an exponential since the integrals are invariant under the transformation \( k \rightarrow -k \). We call attention to the fact if the external current is conserved, (22) coincides with the result found in Ref. 1.

Summing up, we have built out a simple prescription for computing the non-relativistic interparticle potential energy mediated by vector fields. To utilize the alluded method we follow the steps below.
1. Write the free Lagrangian as in (9).
2. Determine the coefficients \( \tilde{a}(k) = a(k)|_{k^0=0} \) and \( \tilde{b}(k) = b(k)|_{k^0=0} \).
3. Verify if there is some conservation law related with the external source \( J^\mu \) and, if that is the case, exclude the longitudinal or transverse contribution.
4. Determine what mode contributes to the particle content of the theory and, as a consequence, establish what part of the external current couples with the vector field.
5. Compute the interparticle potential energy using Eqs. (15), (22) and (23).

III. PRESCRIPTION FOR COMPUTING THE NON-RELATIVISTIC INTERPARTICLE POTENTIAL ENERGY MEDIATED BY SYMMETRIC RANK-2 TENSOR FIELDS

Since in Secs. III and IV we will make use of an approach similar to that utilized in the last section, we shall only present in the mentioned sections the main results of the method. In this way, boring repetitions will be avoided.

The potential energy mediated by a tensor field may be calculated from [4]
\[
E_D = \frac{1}{2\pi} \int d^D x d^D y J^{\mu\nu}(x) D_{\mu\nu\alpha\lambda}(x - y) J^{\alpha\lambda}(y).
\]  
(24)
We appeal now to the convenient tensor decomposition \( J^{\mu\nu} = J_T^{\mu\nu} + J_L^{\mu\nu} + J_T^{\mu\nu} \), where \( J_T^{\mu\nu} \) has only transverse components and satisfy the conservation law \( \partial_\mu J_T^{\mu\nu} = 0 \). In addition, \( J_T^{\mu\nu} \) can be written in terms of a transverse vectorial current \( J_T^{\mu\nu} = \partial^\mu J_T^{\nu} + \partial^\nu J_T^{\mu} \), while \( J_L^{\mu\nu} \) has only longitudinal components, which implies that it can be written in terms of a scalar current \( J_L^{\mu\nu} = \partial^\mu \partial^\nu J_L \). It is remarkable that the above decomposition is quite general since it preserves all degrees of freedom of \( J^{\mu\nu} \). Therefore,
\[
E_D = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} D_{\mu\nu\alpha\lambda}(k) \left( \Delta_T^{\mu\nu\alpha\lambda}(k) + \Delta_L^{\mu\nu\alpha\lambda}(k) + \Delta_T^{\mu\nu\alpha\lambda}(k) + \Pi^{\mu\nu\alpha\lambda}(k) \right),
\]  
(25)
where the following suitable definitions were used
\[
\Delta_T^{\mu\nu\alpha\lambda}(k) = \frac{1}{2} \int d^{D-1}x d^{D-1}y J_T^{\mu\nu}(x) J_T^{\alpha\lambda}(y) e^{ik \cdot (y - x)},
\]  
(26)
Taking this result into account, we can split the potential energy as follows

\[ \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) = \frac{1}{2} \int d^{D-1}x \int d^{D-1}y \ J^\mu_L_L_\kappa_\kappa (x) J^\nu_L_L_\kappa_\kappa (y) e^{i (y-x)}, \]

\[ \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) = \frac{1}{2} \int d^{D-1}x \int d^{D-1}y \ J^\mu_T_L_\kappa_\kappa (x) J^\nu_T_L_\kappa_\kappa (y) e^{i (y-x)}, \]

\[ \Upsilon^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) = \frac{1}{2} \int \int d^{D-1}x d^{D-1}y \ e^{i (y-x)} \left( J^\mu_T_T_\kappa_\kappa (x) J^\nu_L_L_\kappa_\kappa (y) + J^\mu_T_L_\kappa_\kappa (x) J^\nu_T_L_\kappa_\kappa (y) \right), \]

\[ \Pi^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) = \frac{1}{2} \int \int d^{D-1}x d^{D-1}y \ e^{i (y-x)} \left\{ J^\mu_T_T_\kappa_\kappa (x) J^\nu_L_L_\kappa_\kappa (y) + J^\mu_T_L_\kappa_\kappa (x) J^\nu_T_L_\kappa_\kappa (y) + J^\mu_T_L_\kappa_\kappa (x) J^\nu_T_L_\kappa_\kappa (y) \right\}. \]

On the other hand, the most general free Lagrangian for a rank-2 symmetric tensor field \( H^\mu_\nu \) can be written as follows

\[ \mathcal{L}_0 = \frac{1}{2} H^\mu_\nu \left( A_1 (\square) P^{(2)}_\mu_\nu_\alpha_\lambda + A_2 (\square) P^{(1)}_\mu_\nu_\alpha_\lambda + A_3 (\square) P^{(0-s)}_\mu_\nu_\alpha_\lambda + A_4 (\square) P^{(0-w)}_\mu_\nu_\alpha_\lambda + A_5 (\square) P^{(0-w)}_\mu_\nu_\alpha_\lambda + A_6 (\square) P^{(0-w)}_\mu_\nu_\alpha_\lambda \right) H^{\alpha_\lambda}, \]

where \( \{ P^{(2)}, P^{(1)}, \ldots, P^{(0-w)} \} \) is the set of Barnes-Rivers operators (see Appendix A). The Feynman propagator in momentum space is given in turn by

\[ D(k) = \frac{1}{A_1(k)} P^{(2)} + \frac{1}{A_2(k)} P^{(1)} + \frac{1}{K} \left( \tilde{A}_4(k) P^{(0-s)} + \tilde{A}_3(k) P^{(0-w)} - \tilde{A}_5(k) P^{(0-w)} - \tilde{A}_5(k) P^{(0-w)} \right), \]

where \( K = \tilde{A}_3(k) \tilde{A}_4(k) - \tilde{A}_5(k)^2 \).

It follows that

\[ D_\mu_\nu_\alpha_\lambda (k) \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) = \frac{1}{D-1} \left[ \frac{\tilde{A}_4(k)}{K} \eta_\mu_\nu_\eta_\alpha_\lambda + \frac{(D-1) \eta_\mu_\nu_\eta_\alpha_\lambda - \eta_\mu_\nu_\eta_\alpha_\lambda}{A_1(k)} \right] \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k), \]

\[ D_\mu_\nu_\alpha_\lambda (k) \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) = \frac{1}{A_2(k)} \eta_\mu_\nu_\eta_\alpha_\lambda \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k), \]

\[ D_\mu_\nu_\alpha_\lambda (k) \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) = \frac{\tilde{A}_3(k)}{K} \eta_\mu_\nu_\eta_\alpha_\lambda \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k), \]

\[ D_\mu_\nu_\alpha_\lambda (k) \Upsilon^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) = \frac{-\tilde{A}_5(k)}{(D-1)K} \eta_\mu_\nu_\eta_\alpha_\lambda \Upsilon^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k), \]

\[ D_\mu_\nu_\alpha_\lambda (k) \Pi^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) = 0. \]

Substitution of these results into (25), furnishes the expression

\[ E_D = \left( \eta_\mu_\nu_\eta_\alpha_\lambda - \eta_\mu_\nu_\eta_\alpha_\lambda \right) \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k)}{A_1(k)} + \eta_\mu_\nu_\eta_\alpha_\lambda \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k)}{A_2(k)} + \]

\[ + \eta_\mu_\nu_\eta_\alpha_\lambda \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\tilde{A}_4(k)}{K} \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) + \eta_\mu_\nu_\eta_\alpha_\lambda \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\tilde{A}_3(k)}{K} \Delta^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k) - \eta_\mu_\nu_\eta_\alpha_\lambda \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\tilde{A}_5(k)}{K} \Upsilon^\mu_\nu_\alpha_\lambda_\kappa_\kappa (k). \]

Taking this result into account, we can split the potential energy as follows

\[ E_D = E^{(2)}_D + E^{(1)}_D + E^{(0-s)}_D + E^{(0-w)}_D + E^{(sw)}_D, \]
where the following definitions were used

\[ E_D^{(2)} = \left( \frac{\eta_{\mu\alpha}\eta_{\nu\alpha}}{D - 1} \right) \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\Delta_{TT}^{\mu\nu\alpha\lambda}(k)}{A_1(k)}, \]  
(40)

\[ E_D^{(1)} = \eta_{\mu\alpha} \eta_{\nu\lambda} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\Delta_{TT}^{\mu\nu\alpha\lambda}(k)}{A_2(k)}, \]  
(41)

\[ E_D^{(0-s)} = \frac{\eta_{\mu\nu}\eta_{\alpha\lambda}}{D - 1} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\tilde{A}_4(k)}{K} \Delta_{TT}^{\mu\nu\alpha\lambda}(k), \]  
(42)

\[ E_D^{(0-w)} = \eta_{\mu\nu} \eta_{\alpha\lambda} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\tilde{A}_5(k)}{K} \Delta_{LL}^{\mu\nu\alpha\lambda}(k), \]  
(43)

\[ E_D^{(sw)} = -\frac{\eta_{\mu\nu}\eta_{\alpha\lambda}}{\sqrt{D - 1}} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\tilde{A}_5(k)}{K} \gamma^{\mu\nu\alpha\lambda}(k), \]  
(44)

The energy \( E_D \) was split in several parts so that the contributions coming from each sector of the propagator \( D_{\mu\nu\alpha\lambda} \) were separated. In our notation, the superscript in \( E_D^{(I)} \) (\( I = 2, 1, 0-s, 0-w, sw \)) identifies this correspondence between the propagator sectors and the energy contribution.

Now, let us particularize this expression for external currents \( J_{TT}^{\mu\nu}, J_{LL}^{\mu\nu} \) and \( J_{LL}^{\mu\nu} \) related to two-point-like charges.

\[ J_{TT}^{\mu\nu}(x) = \frac{1}{\sqrt{2}} \eta^\mu\eta^\nu \left( Q_1^{(TT)} \delta^d(x-a_1) + Q_2^{(TT)} \delta^d(x-a_2) \right), \]  
(45)

\[ J_{TT}^{\mu\nu}(x) = \frac{1}{\sqrt{2}} \eta^\mu\eta^\nu \left( Q_1^{(LL)} \delta^d(x-a_1) + Q_2^{(LL)} \delta^d(x-a_2) \right) + \frac{1}{\sqrt{2}} \eta^\mu\eta^\nu \left( \frac{Q_1^{(TT)}}{Q_2^{(TT)}} \delta^d(x-a_1) + Q_2^{(TT)} \delta^d(x-a_2) \right), \]  
(46)

\[ J_{LL}^{\mu\nu}(x) = \partial^\mu \partial^\nu \left( Q_1^{(LL)} \delta^d(x-a_1) + Q_2^{(LL)} \delta^d(x-a_2) \right), \]  
(47)

where \( d = D - 1 \) is the number of spatial dimensions.

Substitution of Eqs. (45), (46) and (47) into the set of equations (26)-(29), allows to write

\[ \Delta_{TT}^{\mu\nu\alpha\lambda}(k) = \eta^\mu\eta^\rho \eta^\sigma\eta^\lambda Q_1^{(TT)} Q_2^{(TT)} \cos(k \cdot r), \]  
(48)

\[ \Delta_{TT}^{\mu\nu\alpha\lambda}(k) = \left( \eta^\mu\eta^\rho \eta^\sigma\eta^\lambda \delta_{i,j} \delta_{m,n} k^i k^j Q_1^{(TT)} + \eta^\mu\eta^\rho \eta^\sigma\eta^\lambda \delta_{m,n} k^i k^j Q_2^{(TT)} \right), \]  
(49)

\[ \Delta_{TT}^{\mu\nu\alpha\lambda}(k) = \delta_{i,j} \delta_{m,n} k^i k^j Q_1^{(TT)} Q_2^{(TT)} \cos(k \cdot r), \]  
(50)

\[ \gamma^{\mu\nu\alpha\lambda}(k) = -\eta^\mu\eta^\rho \eta^\sigma\eta^\lambda k^i k^j \left( Q_1^{(TT)} Q_2^{(TT)} + Q_1^{(LL)} Q_2^{(LL)} \right) \cos(k \cdot r), \]  
(51)

where \( r = a_2 - a_1 \).

Using the above equations in the set of Eqs. (40)-(44), we come to the conclusion that

\[ E_D^{(2)} = \frac{D - 2}{D - 1} Q_1^{(TT)} Q_2^{(TT)} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{ik \cdot r} A_1(k), \]  
(52)
is remarkable that we can write current and

\[ \mathbf{D} = 4. \]

In the case of interactions mediated by antisymmetric tensors, the external current is an antisymmetric

\[ \mathbf{E} \]

form.

The potential energy is given by

\[ \mathbf{A} \]

Consider now an antisymmetric tensor field \( (A_{\mu\nu} = -A_{\nu\mu}) \). For convenience’s sake, we restrict our discussion to \( D = 4 \). In the case of interactions mediated by antisymmetric tensors, the external current is an antisymmetric rank-2 tensor, which we denote by \( j^\mu{}^\nu \). Since this antisymmetric rank-2 tensor carries two different kinds of spin-1 representations, labeled here by \([18]\) and \([1c]\), we may divide the external current into two parts \( j^\mu{}^\nu = j^\mu{}^\nu_v + j^\mu{}^\nu_a \). It is remarkable that we can write \( j^\mu{}^\nu_v = \partial^\mu j^\nu_v - \partial^\nu j^\mu_v \) and \( j^\mu{}^\nu_a = \varepsilon^{\mu\nu\alpha\lambda} \partial_\alpha j_\lambda^a \), where \( j^\mu_v \) represents a conserved vector current and \( j^\mu_a \) stands for a pseudo-vectorial current.

The potential energy is given by

\[
E_{D=4} = \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu\alpha\lambda}(k) \left( \Delta_{\mu\nu\alpha\lambda}^v(k) + \Delta_{\mu\nu\alpha\lambda}^a(k) \right),
\]

where the following judicious definitions were utilized

\[
\Delta_{\mu\nu\alpha\lambda}^v(k) = \frac{1}{2} \int d^3x \int d^3y \, j^\mu_v(x) j^\nu_v(y) e^{i k \cdot (y - x)},
\]

\[
\Delta_{\mu\nu\alpha\lambda}^a(k) = \frac{1}{2} \int d^3x \int d^3y \, j^\mu_a(x) j^\nu_a(y) e^{i k \cdot (y - x)}.
\]

On the other hand, the most general free Lagrangian for an antisymmetric rank-2 tensor field can be cast in the form

\[
\mathcal{L}_0 = \frac{1}{2} A^\mu{}^\nu \left( B_v(\square) P_{\mu\nu,\alpha\lambda}^{[1c]} + B_a(\square) P_{\mu\nu,\alpha\lambda}^{[1b]} \right) A^\alpha\lambda,
\]

where \( \{P_{[1b]}, P_{[1c]}\} \) is the set of antisymmetric Barnes-Rivers spin projectors (see Appendix A).

The Feynman propagator in momentum space is given in turn by

\[
D_{\mu\nu\alpha\lambda}(k) = \frac{1}{B_v(k)} P_{\mu\nu,\alpha\lambda}^{[1c]} + \frac{1}{B_a(k)} P_{\mu\nu,\alpha\lambda}^{[1b]}.
\]

Substitution of the last expression into (58) and (59), furnishes the result

\[
D_{\mu\nu\alpha\lambda}(k) \Delta_{\mu\nu\alpha\lambda}^v(k) = \frac{1}{B_v(k)} \eta_{\mu\alpha} \eta_{\nu\lambda} \Delta_{\mu\nu\alpha\lambda}^v(k),
\]

\[
D_{\mu\nu\alpha\lambda}(k) \Delta_{\mu\nu\alpha\lambda}^a(k) = \frac{1}{B_a(k)} \eta_{\mu\alpha} \eta_{\nu\lambda} \Delta_{\mu\nu\alpha\lambda}^a(k),
\]
Dividing now the potential energy into two contributions, we find

\[ E_{D=4} = E^{[1c]}_{D=4} + E^{[1b]}_{D=4}, \]

where

\[ E^{[1c]}_{D=4} = \eta_{\mu\alpha} \eta_{\nu\lambda} \int \frac{d^3k}{(2\pi)^3} \frac{1}{B_c(k)} \Delta_{\mu\nu}^{\alpha\lambda}(k), \]

\[ E^{[1b]}_{D=4} = \eta_{\mu\alpha} \eta_{\nu\lambda} \int \frac{d^3k}{(2\pi)^3} \frac{1}{B_b(k)} \Delta_{\mu\nu}^{\alpha\lambda}(k). \]

Following Ref. [5], we consider external currents concentrated along two parallel \( d \)-dimensional branes \( (d \leq 2) \). The case \( d = 0 \) leads to the usual point-like distributions. We chose the following external currents

\[ j^\mu_\epsilon = \frac{\partial^\mu \eta_\epsilon^{\nu0}}{\sqrt{2}} \left( Q_1^\epsilon \delta^{(3-d)}(x_\perp - a_1) + Q_2^\epsilon \delta^{(3-d)}(x_\perp - a_2) \right) \]

\[ j^\mu_b = \frac{\epsilon^{\mu\nu\lambda\alpha} \partial_\lambda}{\sqrt{2}} \left( V_\alpha \delta^{(3-d)}(x_\perp - a_1) + W_\alpha \delta^{(3-d)}(x_\perp - a_2) \right), \]

where \( x_\perp \) stands for spatial coordinates orthogonal to the branes, \( a_1 \) and \( a_2 \) represent the position vector along the branes, \( Q^\epsilon \) is a charge associated with the representation \([1c]\) and \( V_\alpha \) and \( W_\alpha \) are pseudo four-vectors that are constant in the reference frame where the calculations are made. Using Eqs. (67) and (68), respectively, in (65) and (66), we obtain the following expressions for the potential energy mediated by an antisymmetric rank-2 tensor associated with two parallel \( d \)-branes

\[ E^{[1c]}_{D=4} = -Q_1^\epsilon Q_2^\epsilon \int \frac{d^n k_\perp k_\perp^2 e^{ik_\perp \cdot r_\perp}}{(2\pi)^n B_c(k_\perp)}, \]

and

\[ E^{[1b]}_{D=4} = \int \frac{d^n k_\perp (k_\perp \cdot V)(k_\perp \cdot W) + k_\perp^2 W^\mu V_\mu e^{ik_\perp \cdot r_\perp}}{(2\pi)^n B_b(k_\perp)}, \]

where \( n = 3 - d \) and \( r_\perp = (a_2 - a_1)_\perp \) is the projection of the vector \( r \) along the orthogonal direction to the branes.

V. COMPUTING THE INTERPARTICLE POTENTIAL ENERGY FOR SOME SPECIFIC MODELS

To text the simplicity and efficacy of the prescription we have built out, we discuss below specific examples concerning the computation of the nonrelativistic interparticle potential energy related to the cases developed in Secs. II-IV.

A. Spin-0 particle described by a vector field

In the usual approach, spin-0 particles are described by the usual Klein-Gordon scalar field, however other descriptions are possible, for instance, vector field representations [6]. The motivation to consider vectorial representation of spin-0 particles is twofold: (i) Vector field possess a more intricate structure than the scalar one, hence, would be expected the possibilities of new vertex (when interacting with other fields) which are not contemplated by the usual scalar description; (ii) One can use the vectorial description for spin-0 particles as a theoretical laboratory to explore a systematic procedure to construct more elaborated dual theories.

In this section we consider a model that describes a massive spin-0 particle via a vectorial representation, being its dynamics governed by the Lagrangian [6]

\[ \mathcal{L}_A^{(0,m)} = \frac{1}{2} \left[ (\partial_\mu A^\mu)^2 - m^2 A_\mu A^\mu \right], \]
which can be rewritten as
\[
\mathcal{L}_A^{(0,m)} = \frac{1}{2} A^\mu \left[ -m^2 \theta_{\mu\nu} - (\Box + m^2) \omega_{\mu\nu} \right] A^\nu. \tag{72}
\]

Therefore, \( \tilde{a}(k) = -m^2 \) and \( \tilde{b}(k) = (k^2 - m^2)|_{k=0} = -(k^2 + m^2) \). A quick glance at the corresponding Feynman propagator (see Eq. (10)) clearly shows that \( A^\mu \) only propagates in the longitudinal sector; as a consequence, we can get rid of the potential energy contribution coming from the transverse external current. Accordingly, the only contribution to the \( D \)-dimensional potential energy between point-like charges comes from the longitudinal current (i.e., \( E_D = E_D^L \)). Consequently, the \( D \)-dimensional potential energy can be computed via the expression
\[
E_D = \frac{Q_1^L Q_2^L}{(2\pi)^{D-1}} \int d^{D-1} k \frac{k^2 e^{ik \cdot r}}{k^2 + m^2} \tag{73}
\]
Performing the integration we obtain
\[
E_D(r) = -\frac{Q_1^L Q_2^L}{(2\pi)^{(D-1)/2}} \left( \frac{m^{D+1}}{r^{D-3}} \right)^{1/2} K_{D-3}(mr), \tag{74}
\]
where \( K_\nu \) is the modified Bessel function of the second order of the order \( \nu \).
For \( D = 4 \), (74) reduces to
\[
E_{D=4}(r) = -\frac{Q_1^L Q_2^L}{4\pi} \frac{m^2 e^{-mr}}{r}. \tag{75}
\]

**B. Spin-0 particle described by a higher-derivative vector field**

We move now to a system in which a spin-0 particle is described by a higher-derivative vector field. In this case we have, essentially, the same motivation of the previous section. The model is defined by the Lagrangian (see [6])
\[
\mathcal{L}_B^{(0,m)} = -\frac{1}{2} \left[ \partial_\mu B^\nu (\Box + m^2) \partial_\nu B^\mu \right]. \tag{76}
\]
However, in the present model we have a peculiarity that we cannot forget to mention. The presence of a \( \Box^2 \) term in the above Lagrangian may lead to the misconception of assume the presence of a ghost state. However, it was demonstrated in Ref. [6] that the above Lagrangian is ghost-free. In fact, there are other examples in the literature in which the presence of higher derivatives do not imply the presence of ghost states, for instance, the so-called “new massive gravity” provide a ghost-free description for higher derivative gravity in \( D = 2 + 1 \) [7].

Note that the above Lagrangian is gauge invariant under the transformation \( B^\mu \rightarrow B^\mu + \partial_\mu \Lambda^\nu \), with \( \Lambda^{\mu\nu} = -\Lambda^{\nu\mu} \). It is astonishing that this gauge symmetry implies that \( J_\mu^0 = \theta_{\mu\nu} J^\nu = 0 \); as a result, the interparticle potential energy has only contributions coming from the longitudinal part. After the addition of a gauge fixing term (We work with the gauge fixing term \( -\lambda F_{\mu\nu}(B)^2 \)), where \( F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu \), the Lagrangian (76) assumes the form
\[
\mathcal{L}_B^{(0,m)} = \frac{1}{2} B^\mu \left[ \Box (\Box + m^2) \omega_{\mu\nu} + \lambda \Box \theta_{\mu\nu} \right] B^\nu, \tag{77}
\]
where \( \lambda \) is a gauge fixing parameter. Consequently, \( \tilde{a}(k) = -\lambda k^2|_{k=0} = \lambda k^2 \) and \( \tilde{b}(k) = (k^2 - m^2)k^2|_{k=0} = (k^2 + m^2)k^2 \). Thence, using Eq. 23, we get
\[
E_D = -\frac{Q_1^L Q_2^L}{(2\pi)^D} \int d^{D-1} k \frac{e^{ik \cdot r}}{k^2 + m^2} \tag{78}
\]
Performing the integral, we obtain a result identical to that found in the last section, namely
\[
E_D(r) = -\frac{Q_1^L Q_2^L}{(2\pi)^{D-1}} \left( \frac{m}{r} \right)^{(D-3)/2} K_{D-3}(mr). \tag{79}
\]
It is interesting to observe that in this example the vector field must have mass dimension \((D - 4)/2\) (in order to obtain an action with mass dimension zero) and, as a consequence, the charge \(Q^{(L)}\) has mass dimension \((4 - D)/2\). So, despite the apparent difference between Eqs. (74) and (79), both expressions give the correct dimension for the potential energy, \(\text{i.e., } +1\).

For \(D = 4\), (79) leads to the result

\[
E_{D=4} = -Q_1^{(L)} Q_2^{(L)} \frac{e^{-mr}}{4\pi r}.
\]

It is worth noticing that in this case we have a well defined energy as \(m \to 0\).

### C. Spin-1 particle described by a symmetric rank-2 tensor field

In the same sense that spin-0 particles may be described scalar or vector fields, one can use a tensorial representation for spin-1 instead of the usual vector field. The same kind of motivations mentioned in Sec. VA may be applied to the present situation. In fact, since a symmetric rank-2 tensor has a more intricate structure than the vectorial representation, we could expect to new vertex possibilities.

We analyze a model in which a symmetric rank-2 tensor field describes a massive spin-1 particle. The dynamics of the system is governed by the Lagrangian \([6, 8]\)

\[
\mathcal{L}^{(1,m)} = -\left(\partial^\mu W_{\mu\nu}\right)^2 + \frac{m^2}{2} \left( W_{\mu\nu}^2 - \frac{W^2}{D - 1} \right),
\]

where \(W_{\mu\nu}\) is a symmetric tensor with mass dimension \((D - 2)/2\). After some algebraic manipulations, the preceding Lagrangian can be rewritten as

\[
\mathcal{L}^{(1,m)} = \frac{1}{2} W^{\mu\nu} \left[ m^2 P^{(2)}_{\mu\nu,\alpha\lambda} - (m^2 + \Box) P^{(1)}_{\mu\nu,\alpha\lambda} + \left( \frac{D - 2}{D - 1} m^2 + 2\Box \right) P^{(0-w)}_{\mu\nu,\alpha\lambda} - \frac{m^2}{\sqrt{D - 1}} \left( P^{(0-w_s)}_{\mu\nu,\alpha\lambda} + P^{(0-w_s)}_{\mu\nu,\alpha\lambda} \right) \right] W^{\alpha\lambda}. \tag{82}
\]

Comparing this expression with (31), the following correspondences can be made

\[
A_1(\Box) = m^2 \Rightarrow \tilde{A}_1(k) = m^2, \tag{83}
\]
\[
A_2(\Box) = m^2 + \Box \Rightarrow \tilde{A}_2(k) = m^2 + k^2, \tag{84}
\]
\[
A_3(\Box) = 0 \Rightarrow \tilde{A}_3(k) = 0, \tag{85}
\]
\[
A_4(\Box) = \frac{D - 2}{D - 1} m^2 + 2\Box \Rightarrow \tilde{A}_4(k) = \frac{D - 2}{D - 1} m^2 + 2k^2, \tag{86}
\]
\[
A_5(\Box) = -\frac{m^2}{\sqrt{D - 1}} \Rightarrow \tilde{A}_5(k) = -\frac{m^2}{\sqrt{D - 1}}. \tag{87}
\]

It is easy to show that the only propagating mode describes a massive spin-1 particle (see Refs. [6, 8]); consequently, the unique contribution relevant to the interparticle potential energy is \(E_D^{(1)}\), given by Eq. (53). Substitution of \(\tilde{A}_2(k) = m^2 + k^2\) into (53), allows to write

\[
E_D = -Q_1^{(TL)} Q_2^{(TL)} \int \frac{q^{D-1}k}{(2\pi)^{D-1}} \frac{k^2 e^{ikr}}{m^2 + k^2}. \tag{88}
\]

Solving this integral, we obtain

\[
E_D(r) = \frac{Q_1^{(TL)} Q_2^{(TL)}}{(2\pi)^{(D-1)/2}} \left( \frac{m^{D+1}}{r^{D-3}} \right)^{1/2} K_{D-2}(mr). \tag{89}
\]

Note that the charge \(Q^{(TL)}\) has mass dimension \((2 - D)/2\) which implies that the potential has the correct mass dimension +1.

In particular, the case \(D = 4\) leads to the result

\[
E_{D=4}(r) = \frac{Q_1^{(TL)} Q_2^{(TL)}}{4\pi} \frac{m^2 e^{-mr}}{r}. \tag{90}
\]
D. Spin-1 particle described by a symmetric rank-2 tensor field containing higher-derivatives

We discuss now a second example of our prescription for symmetric rank-2 tensor fields: a higher derivative model which describes a spin-1 particle via a tensor representation. Once again the presence of higher derivatives may lead to the wrong conclusion that the present model contains ghost states. In Ref. [6] it was shown that this model is absent of ghosts, hence this is an interesting example of unitary higher derivative theories describing massive spin-1 particles.

The model under consideration is defined by the following Lagrangian [6, 9]

\[
\mathcal{L}_H^{(1,m)} = -\frac{1}{4}F_{\mu\nu}^2\partial[H] + \frac{m^2}{2}(\partial_\mu H_{\mu\nu})^2,
\]

where \( F_{\mu\nu}[\partial H] = \partial_\mu(\partial^\alpha H_{\nu\alpha}) - \partial_\nu(\partial^\alpha H_{\mu\alpha}) \) and \( H_{\mu\nu} \) is a symmetric tensor field with mass dimension \((D-4)/2\).

According with reference [6], after the introduction of a gauge fixing term (We work with the gauge fixing term \( F_{\mu\nu}^2 \)), where \( F_{\mu\nu}(H) = \partial_\mu H_{\nu\alpha} - \partial_\nu H_{\mu\alpha} + \eta_{\mu\nu}\partial^\alpha H_{\alpha\beta} \) and \(\lambda\) is a gauge parameter), this Lagrangian may be recast as below

\[
\mathcal{L}_H^{(1,m)} = \frac{1}{2}H_{\mu\nu} \left( 2\lambda\Box P^{(2)}_{\mu\nu\alpha\lambda} - \Box(\Box + m^2)P^{(1)}_{\mu\nu\alpha\lambda} + 2\lambda\Box^2 P^{(0-s)}_{\mu\nu\alpha\lambda} - 2m^2\Box P^{(0-w)}_{\mu\nu\alpha\lambda} \right) H^{\alpha\lambda}.
\]

Comparing the last equation with 31, we obtain

\[
A_1(\Box) = \lambda\Box^4 \Rightarrow \tilde{A}_1(k) = 2\lambda(k^2)^4,
\]

\[
A_2(\Box) = -\Box(\Box + m^2) \Rightarrow \tilde{A}_2(k) = -k^2(k^2 + m^2),
\]

\[
A_3(\Box) = 2\lambda\Box^4 \Rightarrow \tilde{A}_3(k) = 2\lambda(k^2)^4,
\]

\[
A_4(\Box) = 2m^2\Box \Rightarrow \tilde{A}_4(k) = 2m^2k^2,
\]

and \( A_5(\Box) = 0 \). In ref. [6] is argued that the gauge symmetry presented in this model ensures that the only relevant contribution for the external current is \( J_{TL}^{\mu\nu} \); consequently the unique contribution for the potential energy is \( E_D^{(1)} \). Using Eq. (53), we find

\[
E_D = Q_1^{(TL)} Q_2^{(TL)} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{e^{ikr}}{m^2 + k^2}.
\]

Solving the this integral, we arrive at the conclusion that

\[
E_D(r) = \frac{Q_1^{(TL)} Q_2^{(TL)}}{(2\pi)^{(D-1)/2}} \frac{(m)}{r} \left( \frac{m}{r} \right)^{(D-3)/2} K_{D-3}(mr).
\]

Some remarks regarding the mass dimension of the charge \( Q^{(TL)} \) are in order here: (i) The charge \( Q^{(TL)} \) has mass dimension \((4-D)/2\). (ii) Despite the apparent difference between Eqs. (89) and (98), both equations have the correct mass dimension as far as the the nonrelativistic potential energy is concerned.

In particular, for \( D = 4 \) (98) assumes the form

\[
E_{D=4}(r) = \frac{Q_1^{(TL)} Q_2^{(TL)}}{4\pi} e^{-mr}.
\]

E. Spin-0 particle described by Kalb-Ramond

In order to exemplify our method developed to compute the interparticle potential associated with interactions mediated by antisymmetric rank-2 tensor we first consider the Kalb-Ramond (KR) field [10]. The KR field is an alternative description for spin-0 objects. Since the KR field is described by an antisymmetric rank-2 tensor it allows
a large number of interactions than the usual scalar field representation for spin-0 particles. For instance, the KR field can mediated interactions between extended objects, which is very relevant in the context of string theories.

Let us start from the KR Lagrangian

$$\mathcal{L}_{KR} = \frac{1}{6} G_{\mu\nu\alpha} G^{\mu\nu\alpha}, \quad (100)$$

where $G_{\mu\nu\alpha} = \partial_{\mu} B_{\nu\alpha} + \partial_{\nu} B_{\alpha\mu} + \partial_{\alpha} B_{\mu\nu}$ and $B_{\mu\nu}(-\mu\nu) = G_{\mu\nu\alpha}$ is the antisymmetric KR field. The KR model is invariant under the following gauge transformation

$$\mathcal{B}_{\mu} \rightarrow \mathcal{B}_{\mu} + \partial_{\mu} \Lambda, \quad (101)$$

Comparing the last expression with (60), we promptly find

$$\mathcal{B}_{\mu} = -\frac{1}{\sqrt{2}} \mathcal{L}_{\mu}, \quad (102)$$

Remarkably, the conservation equation $\partial_{\mu} j^{\mu\nu} = 0$, which is a consequence of the gauge invariance of the KR model, exclude the possibility of the contribution $j^{\mu\nu}$ in the external source and, as a consequence, the only relevant part for the interparticle potential is the $E_{D=4}^{[16]}$ contribution. Accordingly, using Eq. (70) along with above expression for $\mathcal{B}_{\mu}(k)$, we are lead to the following result

$$E_{D=4}^{[16]} = - \int \frac{d^n k}{(2\pi)^n} \frac{k^2 V(k) W(k)}{k^2} e^{ik \cdot \mathcal{L}_{\mu}} - \int \frac{d^n k}{(2\pi)^n} \frac{k^2 W^2 V}{k^2} e^{ik \cdot \mathcal{L}_{\mu}}. \quad (104)$$

Evaluating the above integral we are lead to the following result

$$E_{D=4}^{[16]} = \frac{2^{(n-2)/2}}{(2\pi)^{n/2}} \frac{(n/2)!}{\Gamma(n/2)} \left( \frac{V \cdot W}{r_{\perp}^n} - \frac{n (V \cdot r_{\perp})(W \cdot r_{\perp})}{r_{\perp}^{n+2}} \right), \quad (105)$$

where we have discarded the contribution of the second integral, since it would be relevant only for $r = 0$. The result obtained for the interparticle potential associated with de KR field using our prescription is in complete agreement with the result obtained in Ref. [5], where the interparticle potential were computed by means of more standard methods.

F. Spin-1 particle described by an antisymmetric rank-2 tensor

As the last example we study a model in which a massive spin-1 particle is described by an antisymmetric rank-2 tensor. The corresponding Lagrangian is given by

$$\mathcal{L}_{B}^{(m,1)} = - (\partial_{\mu} B_{\mu\nu})^2 + \frac{m^2}{2} B_{\mu\nu} B_{\mu\nu}. \quad (106)$$

This system was analyzed in references [11, 12] within the context of strong interaction in order to provide an effective description of the low-energy regime of QCD.

Here we are interested in finding its nonrelativistic potential energy. After some algebraic manipulations, we may recast the above Lagrangian as follows

$$\mathcal{L}_{B}^{(m,1)} = \frac{1}{2} B_{\mu\nu} \left[ (\square + m^2) P^{[16]}_{\mu\nu\alpha\lambda} + m^2 P^{[16]}_{\mu\nu\alpha\lambda} \right] B^{\alpha\lambda}. \quad (107)$$

Comparing the last equation with (60), we promptly find

$$B_{\mu}(\square) = \square + m^2 \Rightarrow \mathcal{B}_{\mu}(k) = k^2 \pm m^2, \quad (108)$$

$$B_{\mu}(\square) = m^2 \Rightarrow \mathcal{B}_{\mu}(k) = m^2. \quad (109)$$
It is straightforward to conclude that $[1e]$ is the only sector which has a particle content; thence, the relevant contribution for the potential energy is $E_{D=4}^{[1e]}$. Accordingly,

$$E_{D=4}^{[1e]} = - Q_1^e Q_2^e \int \frac{d^3 k_\perp}{(2\pi)^3} \frac{k_\perp^2 e^{i k_\perp \cdot r_\perp}}{k_\perp^2 + m^2}, \quad (110)$$

Solving this integral, we obtain

$$E_{D=4}^{[1e]} = \frac{Q_1^e Q_2^e}{(2\pi)^{(3-d)/2}} \left( \frac{m^5}{r_\perp^{d-4}} \right)^{1/2} K_{d/2} (mr_\perp). \quad (111)$$

For a point-like charge, i.e., a 0-brane, we obtain a Yukawa-like potential

$$E_{D=4} = \frac{Q_1^e Q_2^e m^2 e^{-mr}}{4\pi r^d}, \quad (112)$$

which clearly exhibits a repulsive behavior for like charges, as expected.

VI. CONCLUDING REMARKS

Based on a procedure recently built out by Accioly et al. [1] — which allows a straightforward computation of the $D$-dimensional nonrelativistic interparticle potential energy for electromagnetic models — and relaxing the condition assumed by them concerning the conservation of the external current, a general extension of their method was found which, among other things, contemplates dual models.

The main points we have analyzed are listed below.

1. As far as the interactions mediated by vector fields are concerned, the possibility of propagating modes were considered both in longitudinal (spin-0) and transverse (spin-1) sectors.

2. For interactions mediated by a rank-2 symmetric tensor field, we took into account that a symmetric tensor can be split into spin-2, spin-1 and two spin-0 sectors which, as a result, allowed different couplings between the tensor field and external currents.

3. We extended the method we have constructed so that interactions mediated by rank-2 antisymmetric tensors could be included. Although we have restricted our computations to $D = 4$, a very simple algorithm for computing the potential energy was obtained.

Last but not least, we would like to comment on the limit $m \to 0$ concerning the examples analyzed in this work. Before going on, we remark that interesting discussions regarding the massless limit of spin-1 models can be found in Refs. [8, 9]. Coming back to our main theme, we call attention to the fact that in sections VA, VC and VF there is a global multiplicative factor $m^2$ in the potential energy expressions which, at first sight, signs an apparent inconsistency in the limit $m^2 \to 0$. Nevertheless, there exists no inconsistency in these systems. Indeed, it is easy to prove that in the first and second models there is no propagating mode if $m^2 = 0$; consequently, we may not expect any potential energy in this case. The last example is more sophisticated. Using the master action technique [13] (see Appendix B), it is easy to show that if $m^2 = 0$ the model considered in VF behaves like a spin-0 particle, leading to a finite potential but with a different behavior from that found in the massless limit of the alluded section. Therefore, a kind of DVZ discontinuity [14, 15] occurs in this limit.

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Appendix A: Barnes-Rivers operators

To begin with, we define the transverse and longitudinal vectorial projector operators as follows

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \quad \text{and} \quad \omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box},$$  \hspace{1cm} (A1)

which satisfy the trivial algebra

$$\theta^2 = \theta, \quad \omega^2 = \omega \quad \text{and} \quad \theta\omega = \omega\theta = 0.$$  \hspace{1cm} (A2)

Using these operators, we may write the complete set of the $D$-dimensional Barnes-Rivers operators [1, 16] as

$$P^{(2)}_{\mu\nu,\alpha\lambda} = \frac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\lambda} + \theta_{\mu\lambda}\theta_{\nu\alpha}) - \frac{1}{D-1}\theta_{\mu\nu}\theta_{\alpha\lambda},$$  \hspace{1cm} (A3)

$$P^{(1)}_{\mu\nu,\alpha\lambda} = \frac{1}{2}(\theta_{\mu\alpha}\omega_{\nu\lambda} + \theta_{\nu\lambda}\omega_{\mu\alpha} + \theta_{\nu\alpha}\omega_{\mu\lambda} + \theta_{\mu\lambda}\omega_{\nu\alpha}),$$  \hspace{1cm} (A4)

$$P^{(0-s)}_{\mu\nu,\alpha\lambda} = \frac{1}{D-1}\theta_{\mu\nu}\theta_{\alpha\lambda},$$  \hspace{1cm} (A5)

$$P^{(0-w)}_{\mu\nu,\alpha\lambda} = \omega_{\mu\nu}\omega_{\alpha\lambda},$$  \hspace{1cm} (A6)

$$P^{(0-sw)}_{\mu\nu,\alpha\lambda} = \frac{1}{\sqrt{D-1}}\theta_{\mu\nu}\omega_{\alpha\lambda},$$  \hspace{1cm} (A7)

$$P^{(0-ws)}_{\mu\nu,\alpha\lambda} = \frac{1}{\sqrt{D-1}}\omega_{\mu\nu}\theta_{\alpha\lambda},$$  \hspace{1cm} (A8)

which satisfy a very useful algebra having the following non-vanishing products

$$(P^{(2)})^2 = P^{(2)}, \quad (P^{(1)})^2 = P^{(1)}, \quad (P^{(0-s)})^2 = P^{(0-s)},$$

$$(P^{(0-w)})^2 = P^{(0-w)}, \quad (P^{(0-s)})P^{(0-sw)} = P^{(0-sw)},$$

$$P^{(0-w)}P^{(0-ws)} = P^{(0-ws)}, \quad P^{(0-sw)}P^{(0-w)} = P^{(0-sw)},$$

$$P^{(0-ws)}P^{(0-ws)} = P^{(0-s)}, \quad P^{(0-ws)}P^{(0-s)} = P^{(0-ws)},$$

$$P^{(0-ws)}P^{(0-sw)} = P^{(0-w)}.$$  \hspace{1cm} (A9)

On the other hand, the set of antisymmetric four-dimensional Barnes-Rivers operator is given by [17]

$$P^{[1b]}_{\mu\nu,\alpha\lambda} = \frac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\lambda} - \theta_{\mu\lambda}\theta_{\nu\alpha}),$$  \hspace{1cm} (A10)

$$P^{[1c]}_{\mu\nu,\alpha\lambda} = \frac{1}{2}(\theta_{\mu\alpha}\omega_{\nu\lambda} + \theta_{\nu\lambda}\omega_{\mu\alpha} - \theta_{\mu\lambda}\omega_{\nu\alpha} - \theta_{\nu\alpha}\omega_{\mu\lambda}),$$  \hspace{1cm} (A11)

which satisfy the very simple algebra

$$(P^{[1b]})^2 = P^{[1b]}, \quad (P^{[1c]})^2 = P^{[1c]},$$

$$P^{[1b]}P^{[1c]} = P^{[1c]}P^{[1b]} = 0.$$  \hspace{1cm} (A12)
Appendix B: Master action and the physical content of the example VF

To demonstrate the equivalence between the massive antisymmetric spin-1 model found in section VF and the Maxwell-Proca theory, we appeal to the master action technique [13]. In this vein, the master action related to those models can be written as

\[ S_m = \int d^D x \left( m^2 B_{\mu\nu} B^{\mu\nu} + 2m (\partial^\mu B_{\mu\nu}) A^\nu + \frac{1}{2} m^2 A^\mu A^\nu \right). \]  

(B1)

Performing a Gaussian integration in (B1) with respect to the vector field \( A_\mu \), we obtain an action proportional to (106).

On the other hand, we can also integrate (B1) with respect to the antisymmetric tensor field \( B_{\mu\nu} \). However, before doing this, we integrate by parts the aforementioned action, which leads to the result

\[ S_m = \int d^D x [m^2 B_{\mu\nu} B^{\mu\nu} - m B_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^\nu A^\nu] \]  

(B2)

The \( F^{\mu\nu} \) tensor field that appears in (B2) is the familiar Maxwell electromagnetic tensor \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \).

It is easy to see that an integration with respect to \( B_{\mu\nu} \) of (B2) reproduces the Maxwell-Proca action.

A natural question can then be posed at this point: What is the physical content of the massless limit of the massive spin-1 field dealt with in section VF? To answer this question we use the master action

\[ S_0 = \int d^D x \{ C_\nu C^\nu - 2 C^\nu (\partial^\mu B_{\mu\nu}) \} \]  

(B3)

where the vector \( C_\nu \) is an auxiliary vector field.

Integrating this result with respect to \( C_\nu \), we arrive at a model proportional to the massless limit of that given by Eq. (106). Before integrating over \( B_{\mu\nu} \), we perform an integration by parts to obtain the expression

\[ S_0 = \int d^D x \{ C_\nu C^\nu + F^{\mu\nu}_C B_{\mu\nu} \} \]  

(B4)

where we have defined \( F^{\mu\nu}_C = \partial^\mu C^\nu - \partial^\nu C^\mu \). Now, an integration with respect to \( B_{\mu\nu} \) can be understood as a functional Dirac's delta which generates the constraint \( F^{\mu\nu}_C = 0 \). The general solution of this equation is given by \( C_\nu = \partial_\nu \phi \), where \( \phi \) is a scalar field. Using this solution in (B4), we arrive at the following result

\[ S_0 = \int d^D x \{ \partial_\nu \phi \partial^\mu \phi \} \]  

(B5)

A quick inspection of \( S_m \) and \( S_0 \) allows us to conclude that there is a discontinuity in the degrees of freedom of the massless limit of the antisymmetric rank-2 tensor field used in Section VF. The massive case describes a spin-1 particle and the massless one a spin-0 field. This may be related to the “spin jumping” phenomena discussed in [18].

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