DEGENERATION OF RICCI-FLAT CALABI-YAU MANIFOLDS AND ITS APPLICATIONS

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ABSTRACT. This is a survey article of the recent progresses on the metric behaviour of Ricci-flat Kähler-Einstein metrics along degenerations of Calabi-Yau manifolds.

1. Introduction

A Calabi-Yau manifold $X$ is a simply connected complex projective manifold with trivial canonical bundle $\mathcal{O}_X \cong \mathcal{O}_X$, and a polarized Calabi-Yau manifold $(X, L)$ is a Calabi-Yau manifold $X$ with an ample line bundle $L$. Note that the definition of Calabi-Yau manifold varies according to the documents, and we use this one for our convenience. For example, if $f$ is a generic homogenous polynomial of degree $\deg f = n + 1$, $n \geq 3$, such that the hypersurface

$$X = \{ [z_0, \cdots, z_n] \in \mathbb{CP}^n | f(z_0, \cdots, z_n) = 0 \}$$

is smooth, then $X$ is a Calabi-Yau manifold, and the restriction $\mathcal{O}_{\mathbb{CP}^n}(1)|_X$ of the hyperplane bundle is an ample line bundle. The study of Calabi-Yau manifolds became very important in the last four decades, and readers are referred to the survey articles and books [105, 104, 35, 43] for the complete aspects on Calabi-Yau manifolds. The present paper surveys the recent progresses on metric properties of Calabi-Yau manifolds.

In the 1970’s, S.-T. Yau proved the Calabi’s conjecture in [102], which asserts the existence of Ricci-flat Kähler-Einstein metrics on Calabi-Yau manifolds.

**Theorem 1.1 (Calabi-Yau Theorem).** If $X$ is a Calabi-Yau manifold, then for any Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$, there exists a unique Ricci-flat Kähler-Einstein metric $\omega \in \alpha$, i.e.

$$\text{Ric}(\omega) \equiv 0.$$ 

This theorem is obtained by showing the existence and the uniqueness of solution of Monge-Ampère equation

$$\left( \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi \right)^n = (-1)^{\frac{n^2}{2}} \Omega \wedge \overline{\Omega}, \quad \sup_X \varphi = 0,$$

for a given background Kähler metric $\omega_0 \in \alpha$, where $\Omega$ is a holomorphic volume form, i.e. a nowhere vanishing section of $\mathcal{O}_X$. The Kähler metric

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\[ \omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi \] has vanishing Ricci curvature, and the Riemannian holonomy group of \( \omega \) is a subgroup of \( SU(n) \). Conversely, a simply connected Riemannian manifold with holonomy group a subgroup of \( SU(n) \) is a Kähler manifold with trivial canonical bundle, and if it is projective, it is a Calabi-Yau manifold by our definition. In fact, Calabi-Yau manifolds play some roles in physics, precisely the string theory, because the holonomy group of Ricci-flat Calabi-Yau threefolds is \( SU(3) \) (cf. [43]). It makes perfect sense to understand the interaction between the metric properties of Ricci-flat Kähler-Einstein metrics and the algebro-geometric properties of underlying Calabi-Yau manifolds.

Denote \( \mathcal{M}(\tilde{X}) \) the set of all Ricci-flat Kähler-Einstein metrics on Calabi-Yau \( n \)-manifolds with the same underlying differential manifold \( \tilde{X} \). There are two parameters of \( \mathcal{M}(\tilde{X}) \), one is the complex structure and the other is the Kähler class. On a Calabi-Yau manifold \( X \), the Calabi-Yau theorem shows that Ricci-flat Kähler-Einstein metrics are parameterized by the Kähler cone \( \mathbb{K}_X \subset H^{1,1}(X, \mathbb{R}) \), which is an open cone in \( H^{1,1}(X, \mathbb{R}) \) such that \( \alpha \in \mathbb{K}_X \) if and only if there is a Kähler metric representing \( \alpha \). A deformation of \( X \) means a smooth proper morphism \( \pi : X \to S \) for two complex analytic varieties \( X \) and \( S \) such that there is a distinguished point \( 0 \in S \) satisfying \( X = \pi^{-1}(0) \). A universal deformation \( \pi : X \to S \) is a deformation of \( X \) such that any deformation \( \pi' : X' \to S' \) of \( X \) is isomorphic to the pull-back under a unique morphism \( \phi : S' \to S \). The Bogomolov-Tian-Todorov unobstructedness theorem (cf. [88, 92]) asserts that the local universal deformation space of \( X \) exists, and is smooth with tangent space \( H^1(X, T_X) \cong H^{n-1,1}(X) \).

Thus loosely speaking, \( \mathcal{M}(\tilde{X}) \) is a space of real dimension \( h^{1,1} + 2h^{n-1,1}, \) and the tangent space at \( X \) is \( H^{1,1}(X, \mathbb{R}) \oplus H^{n-1,1}(X) \).

If \( \pi : X \to S \) is a deformation of a Calabi-Yau manifold \( X \), and \( \gamma : [0, 1] \to \bigcup \mathbb{K}_{X_t} \) is a curve where \( X_t = \pi^{-1}(t) \), then we have a family of Ricci-flat Kähler-Einstein Calabi-Yau manifolds \( (X_{\gamma(s)}, \omega_{\gamma(s)}) \), \( s \in [0, 1] \), with the same underlying differential manifold \( \tilde{X} \) by the Calabi-Yau theorem. Clearly, \( \omega_{\gamma(s)} \) is a family of Ricci-flat Kähler-Einstein metrics on \( \tilde{X} \) depending on the parameter \( s \in [0, 1] \) continuously. However, if we degenerate \( X \) to some singular variety by deforming the complex structure on \( X \), or degenerate the Kähler classes to a non-Kähler class, the question arises is how those Ricci-flat Kähler-Einstein metrics behave along such degenerations.

We certainly do not expect that those metrics converge to another Ricci-flat Kähler-Einstein metric on \( \tilde{X} \), since the degeneration limit would not support any Kähler metric in the usual sense anymore, and some dramatic phenomena must occur.

There are several motivations to study this question. Firstly, the Calabi-Yau theorem asserts the existence of Ricci-flat Kähler-Einstein metrics by solving the Monge-Ampère equation, a fully nonlinear partial differential equation, and however, very few of them can be written down explicitly. It is desirable to improve our knowledge of Ricci-flat Kähler-Einstein metrics,
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for example what the manifold with these metrics looks like. One method to achieve this understanding is by studying the limiting behaviour of metrics along degenerations (cf. Section 2.1). Secondly, the conifold transition (or more general extremal transition) provides a way to connect Calabi-Yau threefolds with different topology in algebraic geometry (cf. [75]). It was conjectured by physicists Candelas and de la Ossa that this process is continuous in the space of all Ricci-flat Kähler-Einstein threefolds in [9]. Therefore it is important to study how Ricci-flat Kähler-Einstein metrics change in this process. Thirdly, there is a nice corresponding between the compactifications of the moduli spaces of curves in the algebro-geometric sense and the differential geometric counterpart. It is interesting to generalize such compactification to the case of Calabi-Yau manifolds. Finally, Strominger, Yau and Zaslow proposed a conjecture in [85], so called SYZ conjecture, for constructing mirror Calabi-Yau manifolds via dual special lagrangian fibration. Later, a new version of SYZ conjecture was proposed by Kontsevich, Soibelman, Gross and Wilson (cf. [38, 54, 55]) by using the collapsing of Ricci-flat Kähler-Einstein metrics. In both cases, it is needed to understand the metric behaviour of Ricci-flat Kähler-Einstein metrics when complex structures degenerate to the large complex limit.

Now we have a relatively clear picture of the so called non-collapsing limit behaviour of Ricci-flat Kähler-Einstein metrics due to many recent works by various authors (cf. [70, 73, 74, 91, 80, 81, 23] etc.). We say a sequence $(X_k, g_k)$ of Riemannian manifolds non-collapsed, if there is a constant $\kappa > 0$ such that

$$\text{Vol}_{g_k}(B_{g_k}(p, 1)) \geq \kappa,$$

for any metric 1-ball $B_{g_k}(p, 1) \subset X_k$, and otherwise, we say $(X_k, g_k)$ collapsed. Roughly speaking, our current understanding can be summed up as the following: the non-collapsing of Ricci-flat Kähler-Einstein metrics in the differential geometric sense is equivalent to the degeneration of Calabi-Yau manifolds to certain irreducible normal varieties in some algebro-geometric sense. The goal of this paper is to survey these progresses.

This paper is organized as the following. Section 2 is the preparation for the results reviewed in this paper. In Subsection 2.1, we recall an earlier theorem due to R. Kobayashi on K3 surfaces. Many later works in this paper, more or less, can be regarded as generalizations of this result. In Subsection 2.2, we recall the generalized Calabi-Yau theorem for normal varieties due to Eyssidieux, Guedj, and Zeriahi, and in Subsection 2.3, we review Cheeger-Colding’s theory on the Gromov-Hausdorff convergence of Riemannian manifolds with bounded Ricci curvature. Section 3 studies the convergence of Ricci-flat Kähler-Einstein metrics along degenerations of Calabi-Yau manifolds. In Subsection 3.1, we survey the papers [73, 74], which study the convergence of metrics under the assumption of degenerations, and in Subsection 3.2, we review a theorem due to Donaldson and Sun.
which study the algebro-geometric property of limits in the Gromov-Hausdorff convergence. Subsection 3.3 establishes the equivalence between the Gromov-Hausdorff convergence and the algebro-geometric degeneration in the non-collapsing case. Those results are from [93, 86, 99]. In Section 4, we apply the works reviewed in Section 3 to obtain a completion for the moduli space of polarized Calabi-Yau manifolds, which is in [100]. Section 5 studies the effect of surgeries in algebraic geometry on Ricci-flat Kähler-Einstein metrics. In Subsection 5.1, we recall the theorem in [94] first, which shows the convergence of metrics along degenerations of Kähler classes while keeping the complex structure fixed. Then we recall the result of [73] for the metric behaviour under flops, and apply it to the connectedness property of birational equivalent Calabi-Yau threefolds. Subsection 5.2 surveys the metric behaviour of Ricci-flat Kähler-Einstein metrics under extremal transitions in [73], which proves the conjecture due to Candelas and de la Ossa. In Section 6, we briefly review the collapsing of Calabi-Yau manifolds, where our knowledge is much less complete than the non-collapsing case. Finally, we give an analytic proof of a finiteness theorem for polarized complex manifolds in Appendix A, which is a joint work with Valentino Tosatti.

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2. Preliminaries

2.1. Ricci-flat Kähler-Einstein metrics on K3 surfaces. In this subsection, we recall some earlier works on Ricci-flat Kähler-Einstein metrics on K3 surfaces, which stimulate the recent works surveyed in this paper (at least those works of the author).

Let $T$ be a complex two torus, i.e. $T = \mathbb{C}^2/\Lambda$ where $\Lambda \cong \mathbb{Z}^4$ is a lattice. The Kummer K3 surface $X$ is constructed as the following. There is a $\mathbb{Z}_2$-action on $T$ by $-1 : (z_1, z_2) \mapsto (-z_1, -z_2)$, and the quotient $X = T/\mathbb{Z}_2$ is a complex orbifold with 16 ordinary double points as singularities. The holomorphic 2-form $\Omega = dz_1 \wedge dz_2$ is invariant under this $\mathbb{Z}_2$-action, and hence descends a holomorphic volume form on $X$ in the orbifold sense. We say $X$ a Calabi-Yau orbifold. The crepant resolution $\pi : \bar{X} \to X$ is obtained by replacing the singular points by $(-2)$-curves $E_1, \ldots, E_{16}$, which is a K3 surface called the Kummer K3 surface. Note that there is flat orbifold Kähler metric $\omega$ on $X$ induced by the standard flat metric on the torus $T$, and $\pi^* \omega$
represents a non-Kähler semi-ample class. And

\[ \alpha_s = [\pi^*\omega] - s \sum_{i=1}^{16} a_i E_i, \quad s \in (0, 1], \]

is a Kähler class for \(0 < a_i \ll 1, i = 1, \cdots, 16.\)

The Calabi-Yau theorem shows the existence of Kähler-Einstein metrics for K3 surfaces. In the case of Kummer K3 surface, there is an alternative proof without using partial differential equations due to Topiwala in [97] (See also [53] for the related work), where the twistor theory and the deformation theory of complex structure are used. It was expected that the curvature of the Ricci-flat Kähler-Einstein metric representing \(\alpha_s\) on a Kummer K3 surface would concentrate around the \(16\ (−2)\)-curves (cf. [28, 68]) when \(s \ll 1,\) and it was verified by R. Kobayashi in [48], in which the limit behaviour of Ricci-flat Kähler-Einstein metrics \(\tilde{\omega}_s \in \alpha_s\) is studied when \(s \to 0\) by a gluing argument.

Let’s recall this construction. Locally each singular point of \(X\) is of the form \(\mathbb{C}^2/\mathbb{Z}_2,\) the quadric cone in \(\mathbb{C}^3,\) and the crepant resolution is the total space of \(\mathcal{O}_{\mathbb{CP}^1}(-2),\) which is diffeomorphic to the total space of \(T^*S^2.\) We denote \(E\) the \((−2)\)-curve in \(\mathcal{O}_{\mathbb{CP}^1}(-2),\) and we identify \(\mathcal{O}_{\mathbb{CP}^1}(-2) \setminus E\) with \((\mathbb{C}^2/\mathbb{Z}_2) \setminus \{0\}\) by the resolution map. Let \(\rho = |z_1|^2 + |z_2|^2\) on \(\mathbb{C}^2,\) which descents to a function on \(\mathbb{C}^2/\mathbb{Z}_2\) denoted still by \(\rho.\) For any \(0 < a \leq 1,\) if we let

\[ f_a(\rho) = \rho \sqrt{1 + \frac{a^2}{\rho^2}} + a \log(\frac{\rho}{a + \sqrt{a^2 + \rho^2}}), \]

then \(\omega_a = \sqrt{-1}\partial\bar{\partial}f_a\) defines a complete asymptotically locally Euclidean (ALE) Ricci-flat Kähler-Einstein metric on \(\mathcal{O}_{\mathbb{CP}^1}(-2),\) called the Eguchi-Hanson metric. Note that \([\omega_a] = -2aE,\) and \(\omega_a\) converges smoothly to the flat metric \(\omega_{flat} = \sqrt{-1}\partial\bar{\partial}\rho\) of \(\mathbb{C}^2/\mathbb{Z}_2\) when \(a \to 0,\) and \(\omega_a \sim \omega_{flat}\) for \(\rho \gg 1.\) The curvature of the restriction of \(\omega_a\) on \(E\) is \(a^{-1},\) and the volume is \(4\pi a.\)

In [48], a background Kähler metric \(\omega_s\) is constructed such that \(\omega_s = \omega_{reg}\) on a small neighborhood of \(E_i, i = 1, \cdots, 16,\) and \(\omega_s = \omega\) on a big compact subset of \(X_{reg} = \bar{X} \setminus \bigcup_{i=1}^{16} E_i.\) The metric \(\omega_s\) is an approximate Ricci-flat Kähler metric. Let \(\varphi_s\) be the unique potential function such that \(\sup_{X} \varphi_s = 0,\) and \(\tilde{\omega}_s = \omega_s + \sqrt{-1}\partial\bar{\partial}\varphi_s\) be the Ricci-flat Kähler-Einstein metric. By using Yau’s estimate of the Monge-Ampère equation [103], the explicit bounds are obtained in [48], i.e.

i) \(C^0\)-bound: \(\|\varphi_s\|_{C^0} \leq C|s|^2,\)

ii) \(C^2\)-bound:

\[ (1 - C|s|^2)\omega_s \leq \tilde{\omega}_s \leq (1 + C|s|^2)\omega_s, \]

for a constant \(C > 0\) independent of \(s,\)
\text{iii) higher order bounds: } \| \varphi_s \|_{C^k(K)} \leq C_{k,K} |s|^2, \text{ for constants } C_{k,K} > 0 \\
\text{on any compact subset } K \subset X_{\text{reg}}.

As a consequence, $\bar{\omega}_s$ converges smoothly to the standard flat metric $\omega$ on any compact subset $K \subset X_{\text{reg}}$ when $s \to 0$, and the exceptional divisors $E_i$ shrink to points. It is clear that $(\bar{X}, \bar{\omega}_s)$ converges to $(X, \omega)$ in the Gromov-Hausdorff sense (cf. Subsection 2.3). Furthermore, the bubbling limits are also studied in [48], and it is proved that near any $E_i$, $s^{-1}\bar{\omega}_s$ converges the the Eguchi-Hanson metric $\omega_{\mathbb{CP}^1}(-2)$ in a certain sense when $s \to 0$.

The above convergence result is also proved for more general K3 orbifold $X$ in [48] with a little weaker bounds for the metric by gluing ALE gravitational instantons obtained by Kronheimer (cf. [50]). Moreover, since any Ricci-flat Kähler-Einstein metric on a K3 surface is HyperKähler, this result can also be explained as the convergence of metrics along a degeneration of complex structures by performing the HyperKähler rotation. Denote $I$ the complex structure of $\bar{X}$, and let $\Omega_s$ be the holomorphic volume form on $\bar{X}$ with $\bar{\omega}_s^2 = \Omega_s \wedge \bar{\Omega}_s$. For any $s \in (0,1]$, we have a new complex structure $J_s$ with corresponding Kähler form and holomorphic volume form

$$
\bar{\omega}_{J_s} = \text{Re}\Omega_s, \quad \Omega_{J_s} = \text{Im}\Omega_s + \sqrt{-1}\bar{\omega}_s,
$$

and however the respective Riemannian metric does not change. Thus we obtain the convergence of metrics along the degeneration of complex structures $J_s$. As an application, it is shown in [48] and [49] that one can fill the holes in the moduli space of Kähler K3 surfaces by some Kähler K3 orbifolds.

The analogue of the above construction was also carried out for a rigid Calabi-Yau threefold in [60]. [22] gives a different proof of the result for the Kummer K3 surface with less PDE involved. More importantly, Gross and Wilson used the gluing argument to prove a theorem about the collapsing of Ricci-flat Kähler-Einstein metrics on K3 surfaces in [38]. Here the K3 surface $X$ is assumed to admit an elliptic fibration $f : X \to B$ with 24 singular fibers of type $I_1$. The background Kähler metric is constructed by gluing the semi-flat Kähler-Einstein metric on the part with smooth fibers, and the Ooguri-Vafa metric near the singular fibers. Then Yau’s estimate of the Monge-Ampère equation gives the proof in [38]. See Section 6 for more discussions. Furthermore, the similar gluing arguments were used by D. Joyce to construct compact $G_2$ and $\text{Spin}(7)$ manifolds (cf. [44]).

If we want to generalize the above convergence theorem to higher dimensional Calabi-Yau manifolds, which have more complicated singularities, the gluing arguments fail due to many difficulties. One of them is the lack of local model like the Eguchi-Hanson metric in the above case. However, recent progresses on the Monge-Ampère equation and the Gromov-Hausdorff theory make it possible, if we are willing to sacrifice some explicit estimates.
2.2. Calabi-Yau variety. In this subsection, we recall the generalized Calabi-Yau theorem due to Eyssidieux, Guedj, and Zeriahi, which asserts the existence of singular Kähler-Einstein metrics for certain normal varieties.

Let’s recall some notions first. A normal projective variety $X$ is called 1-Gorenstein if the dualizing sheaf $\mathcal{O}_X$ is an invertible sheaf, i.e. a line bundle, and is called Gorenstein if furthermore $X$ is Cohen-Macaulay. A variety $X$ has only canonical singularities if $X$ is 1-Gorenstein, and for any resolution $\bar{\pi}: \bar{X} \rightarrow X$, $\bar{\pi}_*\mathcal{O}_{\bar{X}} = \mathcal{O}_X$, which is equivalent to that the canonical divisor $K_X$ is Cartier, and

$$K_{\bar{X}} = \bar{\pi}^*K_X + \sum_{E} a_E E,$$

where $E$ are effective exceptional prime divisors. The ordinary double point, i.e. the singularity locally given by $z_0^2 + \cdots + z_n^2 = 0$, in $\mathbb{C}^{n+1}$ for $n \geq 2$, is a simple example of canonical singularity. If $X$ has only canonical singularities, then the singularities are rational, and $X$ is Cohen-Macaulay (cf. (C) of [70, Section 3]), which implies that $X$ is Gorenstein.

A Calabi-Yau variety $X$ is a normal projective Gorenstein variety with trivial dualizing sheaf $\mathcal{O}_X \cong \mathcal{O}_X$, and having at most canonical singularities. A polarized Calabi-Yau variety $(X, L)$ is a Calabi-Yau variety $X$ with an ample line bundle $L$. If $X$ has only finite ordinary double points as singularities, we call $X$ a conifold.

There is a notion of Kähler metric for normal varieties (cf. [24, Section 5.2]). Let $X$ be a normal $n$-dimensional projective variety. For any singular point $p \in X_{\text{sing}}$ and a small neighborhood $U_p \subset X$ of $p$, a pluri-subharmonic function $v$ (resp. strictly pluri-subharmonic, and pluri-harmonic) on $U_p$ is an upper semi-continuous function with value in $\mathbb{R} \cup \{-\infty\}$ ($v$ is not locally $-\infty$) such that $v$ extends to a pluri-subharmonic function $\hat{v}$ (resp. strictly pluri-subharmonic, and pluri-harmonic) on a neighborhood of the image of some local embedding $U_p \hookrightarrow \mathbb{C}^n$. We call $v$ smooth if $\hat{v}$ is smooth. A form $\omega$ on $X$ is called a Kähler metric, if $\omega$ is a smooth Kähler metric in the usual sense on $X_{\text{reg}}$ and, for any $p \in X_{\text{sing}}$, there is a neighborhood $U_p$ and a continuous strictly pluri-subharmonic function $v$ on $U_p$ such that $\omega = \sqrt{-1} \partial \bar{\partial} v$ on $U_p \cap X_{\text{reg}}$. We call $\omega$ smooth if $v$ is smooth in the above sense. Otherwise, we call $\omega$ a singular Kähler metric. If $\mathcal{PH}_X$ denotes the sheaf of pluri-harmonic functions on $X$, then any Kähler metric $\omega$ represents a class $[\omega]$ in $H^1(X, \mathcal{PH}_X)$ (cf. Section 5.2 in [24]). Note that $H^1(X, \mathcal{PH}_X) \cong H^{1,1}(X, \mathbb{R})$ if $X$ is smooth. We also have an analogue of Chern-Weil theory for line bundles on $X$ (see [24] for details).

In [24], a generalized Calabi-Yau theorem is obtained for Calabi-Yau varieties, i.e. the existence and the uniqueness of singular Ricci-flat Kähler-Einstein metrics with bounded potentials. The potential function in the theorem is proved to be continuous in [25].
Theorem 2.1 ([24]). Let $X$ be a Calabi-Yau $n$-variety, and $\omega_0$ be a smooth Kähler form on $X$. Then there is a unique continuous function $\varphi$ on $X$ satisfying the following Monge-Ampère equation

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = (-1)^{\frac{n^2}{2}} \Omega^X \wedge \bar{\Omega}^X, \quad \sup_X \varphi = 0, \quad \text{and} \quad \varphi \geq -C,$$

where $\Omega$ is a holomorphic volume form, i.e. a nowhere vanishing section of the dualizing sheaf $\omega_X$, such that $\int_X \omega_0^n = \int_X (-1)^{\frac{n^2}{2}} \Omega^X \wedge \bar{\Omega}^X$. The restriction of the singular Kähler metric $\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$ on the regular locus $X_{\text{reg}}$ is a smooth Ricci-flat Kähler-Einstein metric, and $\omega \in [\omega_0] \in H^1(X, \mathcal{P}\mathcal{H}_X)$.

The Kähler-Einstein metric $\omega$ gives a metric space structure on $X_{\text{reg}}$, and however, unlike the smooth Calabi-Yau case, it is unclear whether $\omega$ induces a compact metric space structure on $X$. If $X$ has only orbifold singularities, then $\omega$ coincides with the orbifold Ricci-flat Kähler-Einstein metric obtained previously in [49], which induces a metric space structure on $X$.

In the general case of Theorem 2.1 little is known for the asymptotic behaviour of $\omega$ and $\varphi$ near the singular locus $X_{\text{sing}}$. Assume that $X$ is a conifold, i.e. $X$ has only ordinary double points as singularities. For any $p \in X_{\text{sing}}$, there is a neighborhood $U_p \subset X$ isomorphic to an open subset of the quadric $Q = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} | z_0^2 + \cdots + z_n^2 = 0\}$. A complete Ricci-flat Kähler-Einstein metric $\omega_{\text{co}}$ is constructed on $Q$ in [33] and previously in [9] for $n = 3$, which is a cone metric. More precisely, if $\rho = |z_0|^2 + \cdots + |z_n|^2$, then

$$\omega_{\text{co}} = \sqrt{-1} \partial \bar{\partial} \rho^{1-\frac{1}{n}}.$$ (2.1)

It is speculated (cf. [101]) that $\omega$ is asymptotic to $\omega_{\text{co}}$ near the singular point $p$ in a certain sense, for example, $||\omega - \omega_{\text{co}}||_{C^0(U_p, \omega_{\text{co}})} \leq C\rho^\varepsilon$ for some constants $C > 0$ and $\varepsilon > 0$. It is clearly true for the case of $n = 2$, in which $\omega_{\text{co}}$ is the standard flat orbifold metric on $\mathbb{C}^2/\mathbb{Z}_2$, and however, it is still open for the higher dimensional case.

Now we apply Theorem 2.1 to obtain some canonical embeddings of Calabi-Yau varieties. Let $X$ be a Calabi-Yau $n$-variety. If $L$ is an ample line bundle on $X$, there is an $m > 0$ such that $L^m$ is very ample, and $H^i(X, L^m) = \{0\}$ for any $i > 0$ and $m \geq m_0$. A basis $\Sigma = \{s_0, \cdots, s_N\}$ of $H^0(X, L^m)$ gives an embedding $\Phi_\Sigma : X \hookrightarrow \mathbb{CP}^N$ by $x \mapsto [s_0(x), \cdots, s_N(x)]$, which satisfies $L^m = \Phi_\Sigma^* \mathcal{O}_{\mathbb{CP}^N}(1)$, where $N = \dim_{\mathbb{C}} H^0(X, L^m) - 1$. The pullback $\omega_\Sigma = \Phi_{\Sigma}^* \omega_{\text{FS}}$ of the Fubini-Study metric is a smooth Kähler metric in the above sense such that $[\omega_\Sigma] = mc_1(L) \in NS_{\mathbb{R}}(X)$. The Hermitian metric $h_{\text{FS}}$ of $\mathcal{O}_{\mathbb{CP}^N}(1)$, whose curvature is the Fubini-Study metric, restricts to an Hermitian metric $h_\Sigma = \Phi_\Sigma^* h_{\text{FS}}$ on $L^m$, which satisfies that $\omega_\Sigma = -\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log |\theta|^2_{h_\Sigma}$ on $X_{\text{reg}}$ for any local section $\theta$ of $L^m$. We regard $\Phi_\Sigma(X)$ as a point in $\mathcal{H}\mathcal{H}_N^P$, denoted still by $\Phi_\Sigma(X)$, where $\mathcal{H}\mathcal{H}_N$ is the Hilbert scheme parametrizing subschemes of $\mathbb{CP}^N$ with the Hilbert polynomial $P = P(k) = \chi(X, L^{mk})$. 
Let \( \omega = \omega_\Sigma + \sqrt{-1} \partial \overline{\partial} \varphi \) be the singular Ricci-flat Kähler-Einstein metric obtained in Theorem 2.1. Note that \( \omega \) is unique in \( mc_1(L) \), and particularly is independent of the choice of \( \Phi_\Sigma \). By the boundedness of \( \varphi \), we have that \( h = \exp(-\varphi) h_\Sigma \) is a singular Hermitian metric on \( L^m \) whose curvature is \( \omega \). We define an \( L^2 \)-norm \( \| \cdot \|_{L^2(h)} \) on \( H^0(X, L^m) \) by

\[
\| s \|^2_{L^2(h)} = \int_X |s|^2 \omega^n = \int_X e^{-\varphi} |s|^2_{h_\Sigma} \omega^n.
\]

If \( h' \) is another Hermitian metric with the same curvature \( \omega \), then \( \partial \bar{\partial} \log \frac{h}{h'} \equiv 0 \), i.e. \( \log \frac{h}{h'} \) is a pluriharmonic function on a closed normal variety \( X \), and thus \( h = e^\zeta h' \) for a constant \( \zeta \). If \( \Sigma_h = \{ s_0, \cdots, s_N \} \) is an orthonormal basis of \( H^0(X, L^m) \) with respect to \( \| \cdot \|_{L^2(h)} \), then \( \Sigma_{h'} = \{ e^{-\frac{\zeta}{2}} s_0, \cdots, e^{-\frac{\zeta}{2}} s_N \} \) is orthonormal with respect to \( \| \cdot \|_{L^2(h')} \), and therefore \( \Sigma_h \) and \( \Sigma_{h'} \) induce the same embedding \( \Phi_{\Sigma_h} = \Phi_{\Sigma_{h'}} \).

If \( \Sigma_h \) and \( \Sigma_{h'} \) are two orthonormal bases of \( H^0(X, L) \) with respect to \( \| \cdot \|_{L^2(h)} \), there is an \( u \in SU(N + 1) \subset SL(N + 1) \) such that \( [\Sigma_h] = [\Sigma_{h'}] \cdot u \), \( \Phi_{\Sigma_{h'}}(x) = \Phi_{\Sigma_h}(x) \cdot u \) for any \( x \in X \), and thus \( \Phi_{\Sigma_{h'}}(X) = \Phi_{\Sigma_h}(X) \cdot u \) in \( Hilb^P_N \), where \(- \cdots : Hilb^P_N \times SL(N + 1) \to Hilb^P_N \) is the \( SL(N + 1) \)-action on \( Hilb^P_N \) induced by the natural \( SL(N + 1) \)-action on \( \mathbb{C}P^N \). We denote the \( SU(N + 1) \)-orbit

\[
RO(X, L^m) = \{ \Phi_{\Sigma_h}(X) \cdot u | u \in SU(N + 1) \} \subset (Hilb^P_N)_{red},
\]

where \((Hilb^P_N)_{red}\) denotes the reduced variety of the Hilbert scheme \( Hilb^P_N \). Note that \( RO(X, L^m) \) is compact under the analytic topology of \((Hilb^P_N)_{red}\), and depends only on the singular Kähler metric \( \omega \), but not on the choice of \( h \).

The embedding constructed here will be used in Section 4 for the completion of the moduli space of polarized Calabi-Yau manifolds.

2.3. Gromov-Hausdorff topology. This subsection recalls the notion and results of Gromov-Hausdorff topology, which is the other main ingredient of the works surveyed in this paper.

In the 1980’s, Gromov introduced the notion of Gromov-Hausdorff distance \( d_{GH} \) between metric spaces in [33], which provides a frame to study families of Riemannian manifolds. For any two compact metric spaces \( A \) and \( B \), the Gromov-Hausdorff distance of \( A \) and \( B \) is

\[
d_{GH}(A, B) = \inf \{ d_H^2(A, B) | \exists \text{ isometric embeddings } A, B \hookrightarrow Z \},
\]

where \( Z \) is a metric space, \( d_H^2(A, B) \) is the standard Hausdorff distance between \( A \) and \( B \) regarded as subsets by the isometric embeddings, and the infimum is taken for all possible \( Z \) and isometric embeddings. We denote \( Met \) the space of the isometric equivalent classes of all compact metric spaces equipped with the topology, called the Gromov-Hausdorff topology, induced by the Gromov-Hausdorff distance \( d_{GH} \). Then \( Met \) is a complete metric space (cf. [33] [72]).
The Gromov-Hausdorff distance is used to obtain many important progresses of Riemannian geometry in the last three decades, and the readers are referred to the survey articles [20, 72, 12] for these achievements. The following is the Gromov pre-compactness theorem:

**Theorem 2.2** ([29]). Let \((X_k, g_k)\) be a family of compact Riemannian manifolds such that Ricci curvatures and diameters satisfy
\[
\text{Ric}(g_k) \geq -C, \quad \text{diam}_{g_k}(X_k) \leq D,
\]
for constants \(C\) and \(D\) independent of \(k\). Then, a subsequence of \((X_k, g_k)\) converges to a compact metric space \((Y, d_Y)\) in the Gromov-Hausdorff sense.

If we further assume that there is a lower bound of volumes in this theorem, i.e. \(\text{Vol}_{\omega_k}(X_k) \geq v > 0\) for a constant \(v\), then the Bishop-Gromov comparison theorem shows
\[
\text{Vol}_{g_k}(B_{g_k}(p, 1)) \geq \kappa,
\]
for a constant \(\kappa > 0\), and thus \((X_k, g_k)\) is non-collapsed. In the non-collapsing case, the structure of the limit space \(Y\) is studied in [13, 14, 16, 15], and many properties of \(Y\) is obtained, for example, the Hausdorff dimension of \(Y\) is the same with that of \(X_k\).

Now let’s focus on compact Ricci-flat Kähler-Einstein manifolds. The Gromov pre-compactness theorem shows that a family of compact Ricci-flat Kähler-Einstein manifolds with a uniform upper bound of diameters converges to a compact metric space by passing to a subsequence. Regularities of the limit is studied in [15, 16], and it is shown that it is almost a Ricci-flat manifold.

**Theorem 2.3** ([15, 16]). Let \((X_k, \omega_k)\) be a family of compact Ricci-flat Kähler-Einstein \(n\)-manifolds, and \((Y, d_Y)\) be a compact metric space such that
\[
(X_k, \omega_k) \xrightarrow{d_{GH}} (Y, d_Y).
\]
If
\[
\text{Vol}_{\omega_k}(X_k) \geq v > 0, \quad \text{and} \quad c_2(X_k) \cdot [\omega_k]^{n-1} \leq C,
\]
for constants \(v\) and \(C\) independent of \(k\), where \(c_2(X_k)\) is the second Chern-class of \(X_k\), then there is a closed subset \(S \subset Y\) with Hausdorff dimension \(\dim_H S \leq 2n - 4\) such that \(Y \setminus S\) is a Ricci-flat Kähler-Einstein \(n\)-manifold. Furthermore, off a subset of \(S\) with \((2n - 4)\)-dimensional Hausdorff measure zero, \(S\) has only orbifold type singularities \(\mathbb{C}^{n-2} \times \mathbb{C}^2 / \Gamma\), where \(\Gamma\) is a finite subgroup of \(SU(2)\).

Actually, the convergence in this theorem is stronger than the Gromov-Hausdorff convergence. It is in the sense of Cheeger-Gromov on the regular locus, i.e. if we denote \(\omega_\infty\) the Ricci-flat Kähler-Einstein on \(Y \setminus S\) inducing the metric \(d_Y\), and denote \(J_k\) (respectively \(J_\infty\)) the complex structure of
X_k (respectively Y \setminus S), then for any compact subset K \subset Y \setminus S, there are smooth embeddings F_{K,k} : K \to X_k such that
\[ F_{K,k}^* \omega_k \to \omega_\infty, \quad \text{and} \quad F_{K,k}^* J_k = dF_{K,k}^{-1} \circ J_k \circ dF_{K,k} \to J_\infty, \]
in the C^\infty-sense.

The tangent cone \( Y_y \) at a point \( y \in Y \) is defined as a pointed Gromov-Hausdorff limit of \((Y, \mu_i dY, y)\) for a sequence \( \mu_i \to +\infty \). In the case of Theorem 2.3, the tangent cone is proved to exist for any \( y \in Y \) (cf. [13, 14]), and \( S \) is defined as the locus where \( Y_y \) is not isometric to \( \mathbb{C}^n \). More importantly, it is shown that any \( Y_y \) is isometric to \( \mathbb{C}(M_y) \times \mathbb{C}^l \) where \( \mathbb{C}(M_y) \) is the metric cone over a compact metric space \( M_y \) of Hausdorff dimension \( 2n - 2l - 1 \) (cf. [13, 14, 15]).

When \( n = 2 \), Theorem 2.3 was obtained previously in [1, 2, 5, 67, 89] etc., where \( Y \) has only finite isolated orbifold points as singularities, and \( Y \) is clearly a complex analytic variety. Moreover, \( Y \) is a K3 orbifold, and this fact was used to study the moduli space of K3 surfaces in [1, 2]. It is expected that \( Y \) is also a complex analytic variety in the higher dimensional case, which was proved by Donaldson and Sun under the assumption that those Kähler metrics \( \omega_k \) belong to integral Kähler classes (cf. [23]). We will continue the discussion on the Gromov-Hausdorff topology, and review Donaldson-Sun’s theorem in Section 3.2.

3. Convergence of Ricci-flat Kähler-Einstein metrics

In this section, we study the relationship between the degeneration in the algebro-geometric sense and the non-collapsing convergence in the Gromov-Hausdorff sense.

3.1. Degeneration. This subsection study the limit behaviour of Ricci-flat Kähler-Einstein metrics along algebro-geometric degenerations.

A degeneration of Calabi-Yau n-manifolds \( \pi : \mathcal{X} \to \Delta \) is a flat morphism from a variety \( \mathcal{X} \) of dimension \( n + 1 \) to a disc \( \Delta \subset \mathbb{C} \) such that for any \( t \in \Delta^* = \Delta \setminus \{0\} \), \( X_t = \pi^{-1}(t) \) is a Calabi-Yau manifold, and the central fiber \( X_0 = \pi^{-1}(0) \) is singular. If there is a relative ample line bundle \( \mathcal{L} \) on \( \mathcal{X} \), we call it a degeneration of polarized Calabi-Yau manifolds, denoted by \((\pi : \mathcal{X} \to \Delta, \mathcal{L})\). If we further assume that \( X_0 \) is a Calabi-Yau variety, then the total space \( \mathcal{X} \) is normal as any fiber \( X_t \) is reduced and normal. Thus the relative dualizing sheaf \( \omega_{\mathcal{X}/\Delta} \) is defined, i.e. \( \omega_{\mathcal{X}/\Delta} \cong \omega_{\mathcal{X}} \otimes \pi^* \omega_\Delta^{-1} \), and is trivial, i.e. \( \omega_{\mathcal{X}/\Delta} \cong \mathcal{O}_{\mathcal{X}} \), since every fiber is normal, Cohen-Macaulay and Gorenstein. For example, if
\[ \mathcal{X} = \{(z_0, \cdots, z_n), t) \in \mathbb{CP}^n \times \Delta | tf(z_0, \cdots, z_n) + g(z_0, \cdots, z_n) = 0 \}, \]
t \in \Delta \subset \mathbb{C}, and \( \pi : \mathcal{X} \to \Delta \) is the restriction of the projection, where \( f \) and \( g \) are generic homogenous polynomials of degree \( n + 1 \) such that \( X_t \) is smooth for any \( t \in \Delta^* \), and \( X_0 \) is a Calabi-Yau variety, then \( \pi : \mathcal{X} \to \Delta \) is a
degeneration of Calabi-Yau manifolds, and the restriction $O_{\mathbb{C}P^n}(1)|_X$ of the hyperplane bundle is relative ample.

Let $(\pi : X \rightarrow \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds with the central fiber $X_0$ a Calabi-Yau variety, and $\omega_t \in c_1(\mathcal{L})|_{X_t} \in H^{1,1}(X_t, \mathbb{R})$, $t \in \Delta^*$, be the unique Ricci-flat Kähler-Einstein metric on $X_t$. The limit behaviour of $\omega_t$, when $t \to 0$, is studied in [76] and [73]. In [76], it is proved that $\omega_t$ converges to the singular Ricci-flat Kähler-Einstein metric $\omega$ obtained in Theorem 2.1 under the technique assumption that $X_0$ is a Calabi-Yau conifold, or the total space $\mathcal{X}$ is smooth. These additional hypotheses are removed in [73], and more precisely, we obtain the following theorem.

**Theorem 3.1 ([73]).** Let $(\pi : X \rightarrow \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds with the central fiber $X_0$ a Calabi-Yau $n$-variety. If $\omega_t$ denotes the unique Ricci-flat Kähler-Einstein metric in $c_1(\mathcal{L})|_{X_t} \in H^{1,1}(X_t, \mathbb{R})$, $t \in \Delta^*$, and $\omega$ denotes the unique singular Ricci-flat Kähler-Einstein metric on $X_0$ with $\omega \in c_1(\mathcal{L})|_{X_0} \in H^1(X_0, \mathcal{P}\mathcal{H}_{X_0})$, then

$$F_t^*\omega_t \rightarrow \omega,$$

in the $C^\infty$-sense on any compact subset $K \subset X_{0,\text{reg}}$, where $F_t : X_{0,\text{reg}} \to X_t$ is a smooth family of embeddings with $F_0 = \text{Id}_{X_0}$. Furthermore, the diameter of $(X_t, \omega_t)$, $t \in \Delta^*$, satisfies

$$\text{diam}_{\omega_t}(X_t) \leq D,$$

where $D > 0$ is a constant independent of $t$.

This theorem shows a nice convergence behaviour of the Ricci-flat Kähler-Einstein metrics $\omega_t$ away from the singular set of $X_0$.

Now we outline the proof of Theorem 3.1. Let $\pi : \mathcal{X} \rightarrow \Delta$ be a Calabi-Yau degeneration with a relative ample line bundle $\mathcal{L}$, and $\omega_t \in c_1(\mathcal{L})|_{X_t} \in H^{1,1}(X_t, \mathbb{R})$, $t \in \Delta^*$, be the unique Ricci-flat Kähler-Einstein metric on $X_t$. There is an embedding $\Phi_\Delta : \mathcal{X} \hookrightarrow \mathbb{C}P^N \times \Delta$ such that $\mathcal{L}^m \cong \Phi_\Delta^* \mathcal{O}_{\mathbb{C}P^N}(1)$ for integers $m > 0$ and $N > 0$. We denote $\omega_t^\sigma = \Phi_\Delta^* \omega_{FS}|_{X_t}$ for any $t \in \Delta^*$, where $\omega_{FS}$ is the Fubini-Study metric on $\mathbb{C}P^N$. We obtained the following proposition in [73], which has some independent interests.

**Proposition 3.2.** Assume that the relative dualizing sheaf is defined and trivial $\varpi_{\mathcal{X}/\Delta} \cong \mathcal{O}_{\mathcal{X}}$. The diameter of $\omega_t$ satisfies

$$\text{diam}_{\omega_t}(X_t) \leq 2 + D(-1)^{\frac{n^2}{2}} \int_{X_t} \Omega_t \wedge \overline{\Omega}_t,$$

where $\Omega_t$ is a relative holomorphic volume form, i.e. a no-where vanishing section of $\varpi_{\mathcal{X}/\Delta}$.

We sketch the proof of this estimate. Note that $\omega_t$ satisfies the Monge-Ampère equation

$$\omega_t^n = (-1)^{\frac{n^2}{2}} e^{\sigma_t} \Omega_t \wedge \overline{\Omega}_t,$$

where $e^{\sigma_t} = V \left((-1)^{\frac{n^2}{2}} \int_{X_t} \Omega_t \wedge \overline{\Omega}_t\right)^{-1}$,
where \( V = n! \text{Vol}_{\omega_t}(X_t) \) is a constant independent of \( t \). For a point \( p \in X_{0,\text{reg}} \), there are coordinates \( z_0, \ldots, z_n \) on a neighborhood \( U \) of \( p \) in \( \mathcal{X} \) such that \( t = \pi(z_0, \ldots, z_n) = z_0 \) and \( p = (0, \ldots, 0) \). There is a \( r_0 > 0 \) such that \( \Delta \times \Delta^n \subset U \), where \( \Delta = \{ |t| < r_0 \} \subset \Delta \), \( \Delta^n = \{ |z_j| < r_0, j = 1, \ldots, n \} \subset \mathbb{C}^n \), and \( \{t\} \times \Delta^n \subset X_t \). Note that locally \( \omega_t^o \) and \( \omega_t \) are families of Kähler metrics on \( \Delta^n \subset \mathbb{C}^n \), and there is a constant \( C' \) independent of \( t \) such that

\[
C' \omega_t \leq \omega_t^o \leq C' \omega_t,
\]

where \( \omega_t = \sqrt{-1} \partial \bar{\partial} \sum_{i=1}^n |z_i|^2 \) is the standard Euclidean Kähler metric on \( \Delta^n \).

Lemma 1.3 in [20] implies that for any \( \delta > 0 \), and any \( t \in \delta \Omega \), there is an open subset \( U_{t,\delta} \) of \( \Delta^n \) such that

\[
\text{Vol}_{\omega_t^o}(U_{t,\delta}) \geq \text{Vol}_{\omega_t^o}(\Delta^n) - \delta, \quad \text{diam}_{\omega_t}(U_{t,\delta}) \leq \bar{C} \delta^{-\frac{1}{2}},
\]

where \( \bar{C} \) is a constant independent of \( t \).

Let \( \delta_t = \frac{1}{2} \text{Vol}_{\omega_t^o}(\Delta^n) \), and let \( p_t \in U_{t,\delta_t} \). We get \( \delta_t \geq \frac{1}{2} \text{Vol}_{\omega_E}(\Delta^n) = \bar{\delta} \) and thus \( U_{t,\delta_t} \subset B_{\omega_t}(p_t, r) \), where \( r = \max\{1, 2\bar{C} \delta_t^{-\frac{1}{2}}\} \). Since \( U \subset \mathcal{X}\setminus X_{0,\text{sing}} \), there is a constant \( \kappa_U > 0 \) such that

\[
(-1)^{\frac{n^2}{2}} \Omega_t \wedge \overline{\Omega}_t \geq \kappa_U(\omega_t^o)^n
\]
on \( U \cap X_t \). We derive

\[
\text{Vol}_{\omega_t}(B_{\omega_t}(p_t, r)) \geq \text{Vol}_{\omega_t}(U_{t,\delta_t}) = \frac{(-1)^{\frac{n^2}{2}}}{n!} e^{\sigma_t} \int_{U_{t,\delta_t}} \Omega_t \wedge \overline{\Omega}_t \geq \frac{\kappa_U}{n!} e^{\sigma_t} \int_{U_{t,\delta_t}} (\omega_t^o)^n \geq \frac{\kappa_U}{2} \text{Vol}_{\omega_t^o}(\Delta^n) \geq C e^{\sigma_t} \text{Vol}_{\omega_E}(\Delta^n) = C e^{\sigma_t}.
\]

The Bishop-Gromov comparison theorem shows that

\[
\text{Vol}_{\omega_t}(B_{\omega_t}(p_t, 1)) \geq \frac{1}{r^{2n}} \text{Vol}_{\omega_t}(B_{\omega_t}(p_t, r)) \geq \frac{C}{r^{2n}} e^{\sigma_t}.
\]

By Theorem 4.1 of Chapter 1 in [78] or Lemma 2.3 in [69], we obtain

\[
\text{diam}_{\omega_t}(X_t) \leq 2 + 8n \frac{\text{Vol}_{\omega_t}(X_t)}{\text{Vol}_{\omega_t}(B_{\omega_t}(p_t, 1))} \leq 2 + De^{-\sigma_t},
\]

and we obtain the estimate.

Now we return to the proof of Theorem [3.1]. If we further assume that the central fiber \( X_0 \) is a Calabi-Yau variety, then it is shown in Appendix B of [73] that \( (-1)^{\frac{n^2}{2}} \int_{X_t} \Omega_t \wedge \overline{\Omega}_t \leq C \) for a constant \( C \) independent of \( t \), and thus we obtain the diameter estimate in Theorem [3.1].
Let \( \varphi_t \) be the unique potential function obtained by the Calabi-Yau theorem, which satisfies \( \omega_t = \omega_0^t + \sqrt{-1} \Theta \varphi_t \), and the Monge-Ampère equation

\[
(\omega_0^t + \sqrt{-1} \Theta \varphi_t)^n = (-1)^n c^n \Omega_t \wedge \overline{\Omega}_t, \quad \sup_{X_t} \varphi_t = 0.
\]

The standard \( C^0 \)-estimate for the Monge-Ampère equation in [103] gives \( \| \varphi_t \|_{C^0} \leq C \) for a constant \( C > 0 \) independent of \( t \) by reversing the roles of \( \omega_0^t \) and \( \omega_t \) (a trick originally coming from [94]) in Section 3 of [73]. Here we need the priori estimate of diameters to control the Sobolev constant. Then by the \( C^2 \)-estimate in [103] and the Chern-Lu inequality, we have

\[
C^t \omega_0^t \leq \omega_t \leq C^t \omega_0^t,
\]

for constants \( C' > 0 \) and \( C_K > 0 \) independent of \( t \), on \( X_t \cap K \), where \( K \) is a compact subset of \( \mathcal{X} \setminus X_{0,\text{sing}} \), and \( C_K \) depends on \( K \). Furthermore, we obtain the \( C^{2,\alpha} \)-estimate \( \| \varphi_t \|_{C^{2,\alpha}(K \setminus X_t)} \leq C_{2,K} \) by the Evans-Krylov theory (cf. [79]), and the higher order estimates \( \| \varphi_t \|_{C^{t}(K \setminus X_t)} \leq C_{t,K} \) by the standard Schauder estimates (cf. [39]).

The smooth convergence of \( \omega_0^t \) on \( X_t \cap K \) to \( \omega_0 \) on \( X \cap K \) implies that any sequence \( \varphi_{t_k} \) has subsequences convergence to a smooth bounded function \( \varphi_0 \) on \( X_{0,\text{reg}} \), which satisfies that \( \omega_0 = \omega_0^t + \sqrt{-1} \Theta \varphi_0 \) is a singular Ricci-flat Kähler-Einstein metric on \( X_0 \). By the uniqueness in Theorem 2.1, \( \omega_0^t = \omega_t \), and we do not need to pass any sequence, and obtain the smooth convergence of \( \varphi_t \) (respectively \( \omega_t \)) to \( \varphi_0 \) (respectively \( \omega_0 \)) on \( X_{0,\text{reg}} \).

Theorem 3.1 shows few information of the behaviour of \( \omega_t \) approaching to the singularities \( X_{0,\text{sing}} \) of \( X_0 \). We do not expect the smooth convergence since the topology of the underlying manifold is changed, and thus we like to consider the Gromov-Hausdorff topology.

By the Gromov pre-compactness theorem (Theorem 2.2), for any sequence \( t_k \), a subsequence of \( (X_{t_k}, \omega_{t_k}) \) converges to a compact metric space \( (Y, d_Y) \) in the Gromov-Hausdorff sense. Since \( \text{Vol}_{\omega_0}(X_t) = \frac{1}{n!} c^n(\mathcal{L}|_{X_t}) \equiv \text{const} \), Theorem 2.3 shows that there is a closed subset \( S \subset Y \) with Hausdorff dimension \( \dim_H S \leq 2n - 4 \) such that \( Y \setminus S \) is a Ricci-flat Kähler \( n \)-manifold. By the smooth convergence of \( \omega_t \) to \( \omega \) on \( X_{0,\text{reg}} \), we construct a local isometric embedding \( \iota : (X_{0,\text{reg}}, \omega_0) \hookrightarrow (Y \setminus S, d_Y) \) in [74], and show that the Hausdorff co-dimension of \( Y \setminus \iota(X_{0,\text{reg}}) \) is bigger or equal to 2. Then \( d_Y(X_{0,\text{reg}}) \) is almost geodesic convex in \( Y \) by [14], i.e. for any two points \( y_1, y_2 \in \iota(X_{0,\text{reg}}) \), and any \( \epsilon > 0 \), there is curve \( \gamma \subset \iota(X_{0,\text{reg}}) \) such that \( y_1 = \gamma(0), y_2 = \gamma(1) \), and

\[
\text{length}_{d_Y}(\gamma) \leq d_Y(y_1, y_2) + \epsilon.
\]

Thus \( (Y, d_Y) \) is the metric completion of \( (X_{0,\text{reg}}, \omega) \).

In summary, we prove the following theorem in [74].

**Theorem 3.3** ([74]). Let \( \pi : \mathcal{X} \to \Delta \), \( X_0, \omega_t \) and \( \omega \) be the same as in Theorem 2.1. We have

\[
(X_t, \omega_t) \stackrel{d_{GH}}{\to} (Y, d_Y),
\]
when $t \to 0$, in the Gromov-Hausdorff sense, where $(Y, d_Y)$ denotes the metric completion of $(X_{0,\text{reg}}, \omega)$, which is a compact metric space.

Theorem 3.3 still offers little understanding of the metric behaviour near the singularities of $X_0$, and we expect more explicit asymptotic behaviours. Assume that the Calabi-Yau variety $X_0$ in Theorem 3.1 and Theorem 3.3 has only ordinary double points as singularities, i.e. $X_0$ is a conifold. For any $p \in X_{0,\text{sing}}$, there is a neighborhood $U_p \subset X$ such that $U_p \cap X_t$, $t \in \Delta$, is isomorphic to an open subset of

$$Q_t = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} | z_0^2 + \cdots + z_n^2 = t\}.$$ 

Note that $Q_0$ is the quadric cone, and for any $t \in \Delta^*$, $Q_t$ is diffeomorphic to the total space of the cotangent bundle $T^*S^n$ of $S^n$. There is a complete Ricci-flat Kähler-Einstein metric

$$\omega_{co,t} = \sqrt{-1} \partial \bar{\partial} f_t(\rho)$$

obtained in [3] when $n = 3$, and in [33] for the general dimension $n$, where $\rho = |z_0|^2 + \cdots + |z_n|^2$, and $f_t(\rho)$ satisfies the ordinary differential equation

$$\rho(f_t')^n + f_t''(f_t')^{n-1}(\rho^2 - |t|^2) = \left(\frac{n-1}{n}\right)^{n+1}.$$ 

This equation can be solve explicitly by changing the variable $\rho = |t| \cosh \tau$, and integrating

$$\frac{d}{d\tau}(f_t'(\tau))^n = n|t|^{n-1}\left(\frac{n-1}{n}\right)^{n+1}(\sinh \tau)^{n-1}.$$ 

Note that $\omega_{co,0} = \omega_{co} = \sqrt{-1} \partial \bar{\partial} \rho^{1-\frac{1}{n}}$, the Ricci-flat cone metric, on $Q_0$, and when $t \to 0$, $\omega_{co,t}$ converges smoothly to $\omega_{co}$. More geometric properties of $\omega_{co,t}$, for example curvatures, are studied in Appendix A of [27].

It is expected (cf. [101] etc.) that $\omega_{co,t}$ approximates $\omega_t$ of Theorem 3.1 and Theorem 3.3 in a certain sense when $t \to 0$, i.e. for instance,

$$\|\omega_{co,t} - \omega_t\|_{C^0(\omega_{co,t}, U_p \cap X_t)} \leq C|t|^\varepsilon,$$

for constants $C > 0$ and $\varepsilon > 0$. Furthermore, $|t|^{-1}\omega_t$ should converge smoothly to $\omega_{co,t}$ on $Q_{t_0}$ for a $t_0 \in \Delta^*$. If they are true, there would be many interesting applications, for example the construction of special lagrangian spheres in $(X_t, \omega_t)$ for $|t| \ll 1$. Since the zero section $S^n \subset T^*S^n$ is a special lagrangian submanifold respecting to $\omega_{co,t}$ and certain holomorphic volume form on $Q_t$, one can deform $S^n$ in $Q_t$ to a special lagrangian sphere in $X_t$ by the standard deformation technique of special lagrangian submanifolds (cf. [33]). As a matter of fact, this is one goal of the study of metrics along degenerations in [101]. The strategy to achieve such goal is the gluing method similar to those in Section 2.1. However, there are many difficulties even if the solution of Theorem 2.1 has the expected asymptotic behaviour (cf. [11, 101]). Unfortunately, our new approach would not provide any new insight either.
3.2. Gromov-Hausdorff convergence. In this subsection, we continue our discussion in Section 2.2 by considering polarized Calabi-Yau manifolds.

Let \((X_k, L_k)\) be a sequence of polarized Calabi-Yau manifolds, and \(\omega_k \in c_1(L_k)\) be a sequence of Ricci-flat Kähler-Einstein metrics with \(\text{diam}_{\omega_k}(X_k) \leq D\). The Gromov pre-compactness theorem (Theorem 2.2) asserts that a subsequence of \((X_k, \omega_k)\) converges to a compact metric space \((Y, d_Y)\) in the Gromov-Hausdorff sense. Since \(\text{Vol}_{\omega_k}(X_k) = \frac{1}{n!} c_1(L_k)^n \geq \frac{1}{n!}\), \((X_k, \omega_k)\) is non-collapsed, and Theorem 2.3 shows that there is a closed subset \(S \subset Y\) such that Hausdorff dimension \(\text{dim}_H S \leq 2n - 4\), and \(Y \setminus S\) is a Ricci-flat Kähler manifold.

In [23], Donaldson and Sun studied the algebro-geometric properties of the limit \(Y\), and showed that \(Y\) is homeomorphic to a projective variety.

**Theorem 3.4** ([23]). Let \((X_k, L_k)\) be a sequence of polarized Calabi-Yau manifolds of dimension \(n\), and \(\omega_k \in c_1(L_k)\) be the unique Ricci-flat Kähler-Einstein metric. We assume that \(\text{Vol}_{\omega_k}(X_k) = \frac{1}{n!} c_1(L_k)^n = \nu\), \(\text{diam}_{\omega_k}(X_k) \leq D\) for constants \(D > 0\) and \(\nu > 0\), and furthermore, \((X_k, \omega_k) \xrightarrow{d_{GH}} (Y, d_Y)\) in the Gromov-Hausdorff sense. Then we have the follows.

i) \(Y\) is homeomorphic to a Calabi-Yau variety \(X_\infty\).

ii) There are constants \(m > 0\) and \(\overline{N} > 0\) satisfying the following. For any \(k\), there is an orthonormal basis \(\Sigma_k\) of \(H^0(X_k, L_k^m)\) respecting to the \(L^2\)-norm induced by \(\omega_k\), which induces an embedding \(\Phi_{\Sigma_k} : X_k \hookrightarrow \mathbb{P}^\overline{N}\) with \(L_k^m = \Phi_{\Sigma_k}^* \mathcal{O}_{\mathbb{P}^\overline{N}}(1)\). And \(\Phi_{\Sigma_k}(X_k)\) converges to \(X_\infty\) in some Hilbert schemes \(\text{Hit}^{\overline{N}}\).

iii) The metric space structure on \(Y\) is induced by the unique singular Ricci-flat Kähler-Einstein metric \(\omega \in \frac{1}{m!} c_1(\mathcal{O}_{\mathbb{P}^\overline{N}}(1)|_{X_\infty})\).

By Proposition 4.15 of [23], \(Y\) is homeomorphic to a projective normal variety with only log-terminal singularities, denoted by \(X_\infty\). Note that the holomorphic volume forms \(\Omega_k\) are parallel with respect to \(\omega_k\), and converge to a holomorphic volume form \(\Omega_\infty\) on the regular locus \(X_{\infty, \text{reg}}\) along the Gromov-Hausdorff convergence by normalizing \(\Omega_k\) if necessary. Thus the dualizing sheaf \(\omega_{X_\infty}\) is trivial, i.e. \(\omega_{X_\infty} \cong \mathcal{O}_{X_\infty}\), and \(X_\infty\) is 1–Gorenstein. Furthermore, the canonical divisor \(K_{X_\infty}\) is Cartier and trivial, which implies that \(X_\infty\) has at worst canonical singularities. Then \(X_\infty\) has only rational singularities, \(X_\infty\) is Cohen-Macaulay and is Gorenstein. Consequently, \(X_\infty\) is a Calabi-Yau variety.

We remark that Theorem 3.4 holds for polarized Kähler manifolds with bounded Ricci curvature (cf. [23]). Readers are also referred to [90] for the case of Fano manifolds, and [23, 90] for the relevant history and applications.

There are some immediate interesting applications of Theorem 3.4.
First of all, we apply Theorem 3.4 to the finiteness question. There is a conjecture due to S.-T. Yau, which says that there are only finitely many diffeomorphism types of Calabi-Yau threefolds. There are many evidences to support it, and for example, [32] proves a weak version of this conjecture, i.e., up to birational equivalence, elliptic fibred Calabi-Yau threefolds have only finitely many diffeomorphism types. For any constant $D > 0$, we define a set of polarized Calabi-Yau $n$-manifolds

$$\mathcal{N}(n, D) = \{(X, L)| \omega \in c_1(L) \text{ with } \text{Ric}(\omega) \equiv 0, \quad \text{diam}_{\omega}(X) \leq D\}.$$ 

Note that $1 \leq c_1(L)^n = n! \text{Vol}_{\omega}(X) \leq V(n, D)$ by the Bishop-Gromov comparison theorem. Thus Theorem 3.4 implies that any $(X, L) \in \mathcal{N}(n, D)$ can be embedded into a comment $\mathbb{CP}^N_{\eta}$, and the image can be regarded as finite possible many components of Hilbert schemes. Consequently, elements in $\mathcal{N}(n, D)$ have only finite many possible complex deformation classes, and of course, have only finite many possible diffeomorphism types (cf. [7]). We will show the generalization of this finiteness result to polarized projective Kähler manifolds with Ricci curvature bounded from below in Appendix A.

Now, we apply Theorem 3.4 to the situation of Theorem 3.1 and Theorem 3.3, and obtain that the compact metric space $Y$ in Theorem 3.3 is homeomorphic to a Calabi-Yau variety $X_{\infty}$. Actually, it is easy to see that $X_{\infty}$ is isomorphic to $X_0$ of Theorem 3.1 (cf. Lemma 2.2 of [106]). However it is not trivial since the embeddings in Theorem 3.4 are induced by orthonormal basis, which are highly transcendental, and would not coincide with algebraic embedding induced by the relative ample line bundle $L$ from the hypothesis. One corollary is the uniqueness of the filling-in for degenerations of Calabi-Yau manifolds.

**Corollary 3.5.** Let $(\pi : \mathcal{X} \to \Delta, \mathcal{L})$ and $(\pi' : \mathcal{X}' \to \Delta, L')$ be two degenerations of polarized Calabi-Yau manifolds with Calabi-Yau varieties $X_0$ and $X'_0$ as the central fibers respectively. If there is a sequence of points $t_k \to 0$ in $\Delta$, and there is a sequence of isomorphism $\psi_k : X_{tk} \to X'_{tk}$ such that $\psi_k^* L|_{X_{tk}} \cong L'|_{X'_{tk}}$, then $X_0$ is isomorphic to $X'_0$.

Note that $\mathcal{X}$ may not be birational to $\mathcal{X}'$ in this corollary. If we have a stronger assumption that $\mathcal{X} \setminus X_0$ is isomorphic to $\mathcal{X}' \setminus X'_0$, then the conclusion is a direct consequence of [6, Theorem 2.1] and [66, Corollary 4.3].

We finish this section by recalling some aspects of the proof of Theorem 3.4. Let $(X_k, L_k), \omega_k$ and $(Y, dv_y)$ be the same as in Theorem 3.4. The main step in the proof of Theorem 3.4 is to construct uniformly many peak sections of $L_k^\nu$ on $X_k$, for a $\nu \in \mathbb{N}$, which are also sufficiently many, and enough to induce embeddings for not only all of $X_k$, but also the limit $Y$. A peak section $s$ of $L_k^\nu$ means that for a point $p \in X_k$, $|s|_{k,p} \geq 1$, $\|\cdot\|_{k,L_k^\nu} = \int_{X_k} |s|_{k}^2 \sim 1$, and $|s|_{k} \ll 1$ on the complement of a small neighborhood of $p$, where $|\cdot|_{k}$ denotes the Hermitian metric on $L_k^\nu$ with curvature $\nu \omega_k$. The peak sections are constructed by gluing a local model first, and then using the Hörmander $L^2$-estimate.
The local model used in [23] is as following. For any \( y \in Y \), the tangent cone \( Y_y \) at \( y \) is a metric cone \( C(M_y) \) over a compact metric space \( M_y \), and the regular locus \( Y_{y,\text{reg}} \) is an open dense subset of \( Y_y \). Let \( o \in Y_y \) be the distinguished vertex point, and let \( r(q) = \text{dist}(q, o) \), i.e. the distance of \( o \) to the point \( q \in Y_y \). On \( Y_{y,\text{reg}} \), the metric is a Ricci-flat Kähler-Einstein cone metric \( dr^2 + r^2 g_M \), where \( g_M \) is a Sasakian-Einstein metric on the regular locus \( M_{y,\text{reg}} \). It is standard that the Ricci-flat Kähler-Einstein metric can be written as \( \omega_y = \frac{1}{2} \partial \bar{\partial} r^2 \) (cf. [82]). If \( L_o \) is the smooth trivial line bundle on \( Y_{y,\text{reg}} \), i.e. \( L_o \simeq Y_{y,\text{reg}} \times \mathbb{C} \) in the sense of smooth bundles, then we define a \( U(1) \)-connection on \( L_o \)

\[
A_o = \frac{1}{4} (\partial r^2 - \bar{\partial} r^2), \quad \text{whose curvature } F_{A_o} = -\sqrt{-1} \omega_y.
\]

Note that \( \bar{\partial}_{A_o} = \partial + A_{o,1} \) is a Cauchy-Riemann operator, i.e. \( \bar{\partial}^{2}_{A_o} = 0 \), and \( \bar{\partial}_{A_o} \) induces a holomorphic structure on \( L_o \). If we define

\[
\sigma_o = e^{-\frac{r^2}{4}}, \quad \text{then } \bar{\partial}_{A_o} \sigma_o = \bar{\partial} \sigma_o + A_{o,1} \sigma_o = 0,
\]

i.e. \( \sigma_o \) is a holomorphic section of \((L_o, A_o)\). The local model is \((L_o, A_o, \sigma_o)\).

We recall the gluing argument of [23], and assume that \( M_y \) is smooth, i.e. \( o \) is an isolated singular point, and \( H^1(M_y, \mathbb{Z}) \) and \( H^2(M_y, \mathbb{Z}) \) are torsion free for simplicity. Denote \( U = \{ q \in C(M_y) | \delta \leq r(q) \leq R \} \) for constants \( 0 < \delta \ll 1 \) and \( R \gg 1 \), and we identify \( M_y = \{ q \in C(M_y) | r(q) = 1 \} \). For an \( \epsilon > 0 \), let \( \omega \) be a Kähler metric respecting to a complex structure \( J \) on \( U \) such that

\[
\|\omega - \omega_y\|_{C^0} \leq \epsilon, \quad \text{and} \quad \|J - J_y\|_{C^0} \leq \epsilon,
\]

where \( J_y \) is the complex structure on \( Y_{y,\text{reg}} \), and let \( A \) be a \( U(1) \)-connection on a complex line bundle \( L \) with curvature \( F_A = -\sqrt{-1} \omega \), which implies that \( \bar{\partial}_{J,A} \) is a Cauchy-Riemann operator, and induces a holomorphic structure on \( L \). Since \( \omega \) (respectively \( \omega_y \)) represents \( c_1(L) \) (respectively \( c_1(L_o) \)), a integral class, and \( \omega \) is close to \( \omega_y \), we have \( c_1(L) = c_1(L_o) = 0 \), and \( L \) is isomorphic to \( L_o \) as smooth bundles. Hence we regard \( \sigma_o \) as a section of \( L_o \).

For a certain cut off function \( \beta \), we like to prove that \((\omega, J, L, A, \sigma = \beta \sigma_o)\) satisfies the so called Property (H) in [23], which guarantees that the result section after applying the Hörmander \( L^2 \)-estimate to these data is a peak section. Property (H) reads that

i) \( \|\sigma\|_{L^2} < (2\pi)^{\frac{n}{2}} \),

ii) \( |\sigma(p)| > \frac{3}{4} \) for a point \( p \in U' \subset U \),

iii) for any smooth section \( \tau \) of \( L \), \( |\tau|(p) \leq C(\|\bar{\partial}_{J,A} \tau\|_{L^p(U')} + \|\tau\|_{L^2(U')}) \),

iv) \( \|\bar{\partial}_{J,A} \sigma\|_{L^2} \leq \min\{\frac{1}{8C^2}, \frac{(2\pi)^{\frac{n}{2}}}{10C^2}\} \),

v) \( \|\bar{\partial}_{J,A} \sigma\|_{L^p(U')} < \frac{1}{8C} \).

Property (H) can be obtained by direct calculations similar to those in Section 2 of [21], if we further have \( \|A - A_o\|_{C^0} \ll 1 \). However this condition may not be true due to the non-trivialness of the first Betti number of \( U \).
Note that $\alpha = (A - A_0)|_{M_y}$ is a connection on $L \otimes L^{-1}|_{M_y}$ with curvature $F_\alpha = -\sqrt{-1}(\omega - \omega_y)|_{M_y}$. Since $\alpha$ is transformed to $\alpha + d \log u$ under a gauge change $u : M_y \to M_y \times U(1)$, we have $\tilde{\alpha} = \alpha_H + \alpha'$ by the Hodge theory where $\alpha_H$ is a harmonic 1-form, and $\alpha'$ satisfies $d^* \alpha' = 0$, $\omega' = F_\alpha$ and $\|\alpha'\|_{L^2} \leq C\|F_\alpha\|_{L^2}$. The harmonic part $\alpha_H$ is a flat Kähler-Einstein metrics are non-collapsed, and converge to a compact manifold. It is Hausdorff convergence, we have the limit being a Calabi-Yau variety. In [23], it is shown that there is an open subset $W \in H^1(M_y, U(1))$, for example we say $W = \{\tilde{\alpha} \in H^1(M_y)| \|\tilde{\alpha}\|_{L^2} \leq 1\}/H^1(M_y, \mathbb{Z})$, such that if $\alpha_H \in W$, then one can extend the gauge change $\tilde{\alpha}$ on $U$ such that $\tilde{\alpha} = A + d \log \tilde{u}$ satisfies $\|A - A_0\|_{C^0} \leq C'(\epsilon_W + \|F_A - F_{A_0}\|_{L^2})$, and furthermore, $(\omega, J, L, A, \sigma)$ satisfies the Property (H).

Now Dirichlet’s theorem asserts that there is an $m \in \mathbb{N}$ satisfying that for any $\zeta \in H^1(M_y, U(1))$, there is a $1 \leq \nu \leq m$ such that $\nu \zeta = \zeta \mod\{H^1(M_y, \mathbb{Z})\}$ for a $\zeta' \in W$. If the above $\alpha_H$ does not belong to $W$, then we define $\tau(\nu A) \in W$ such that $\nu \alpha_H = \tau(\nu A) \mod\{H^1(M_y, \mathbb{Z})\}$ for a $\nu \leq m$. Let $\tilde{U} = \{q \in C(M_y) | m^{-2}\delta \leq r(q) \leq R\}$, and $\mu_\nu : U \to \tilde{U}$ be the dilation map given by $\mu_\nu(x, r) = (x, \nu^{-\frac{1}{2}}r)$ for any $(x, r) \in C(M_y)$ where $x \in M_y$ and $r = r((x, r))$.

By the convergence of $\omega_k$ and the definition of tangent cone, there are embeddings $\Psi_k : \tilde{U} \to X_k$ such that $\|m_0\Psi_k \omega_k - \omega_y\|_{C^0} \leq \epsilon m^{-1}$, for an $m_0 \in \mathbb{Z}$, and $\|\Psi^*_k J_k - J_y\|_{C^0} \leq \epsilon m^{-1}$, where $J_k$ is the complex structure of $X_k$. Let $A_k$ be the $U(1)$-connection on $L_k$ with curvature $F_{A_k} = -\sqrt{-1}\omega_k$ and compatible with the holomorphic structure of $X_k$. By passing to a subsequence, Dirichlet’s theorem shows as above that there is a $1 \leq \nu \leq m$ such that $\tau(\nu m_0 A_k) \in W$ is defined as $\|\nu m_0\Psi_k^* F_{A_k} - \nu F_{A_0}\|_{C^0} \leq \epsilon$. Then $\nu m_0 A_k$ is gauge equivalent to a $U(1)$-connection $\tilde{A}_k$ of $\Psi_k^{-1} L^{m_0}$ on $\tilde{U}$, and

$$\|\tilde{A}_k - \nu A_0\|_{C^0} \leq C'(\epsilon_W + \|\nu m_0\Psi_k^* F_{A_k} - \nu F_{A_0}\|_{L^2}) \leq C'(\epsilon_W + \epsilon).$$

Since $\nu \mu_\nu^* \omega_y = \omega_y$, $\mu_\nu^* J_y = J_y$ and $\nu \mu_\nu^* A_0 = A_0$, one obtain that $(\mu_\nu^* \Psi_k^* \nu m_0 \omega_k, \mu_\nu^* \Psi_k^* J_k, \mu_\nu^{-1} \Psi_k^* L_{A_k}, \mu_\nu^* \tilde{A}_k, \sigma)$ satisfies the Property (H).

By identifying $U$ with $\Psi_k(\mu_\nu(U))$, the Hörmander $L^2$-estimate shows that there is a smooth section $\tilde{\sigma}$ of $L^{m_0}$ such that

$$\nabla_{J_k, \nu m_0 A_k}(\tilde{\sigma}) = \nabla_{J_k, \nu m_0 A_k}(\sigma), \text{ with } \|\tilde{\sigma}\|_{L^2} \leq \frac{1}{\sqrt{\nu m_0}} \|\nabla_{J_k, \nu m_0 A_k}(\sigma)\|_{L^2}.$$

The peak section is $s = \sigma - \tilde{\sigma}$.

### 3.3. Equivalence

Theorem 3.3 shows that under the algebro-geometric assumption of Calabi-Yau degeneration with Calabi-Yau central fiber, Ricci-flat Kähler-Einstein metrics are non-collapsed, and converge to a compact metric space in the Gromov-Hausdorff sense. On the other hand, Theorem 3.4 says that under the differential geometric assumption of Gromov-Hausdorff convergence, we have the limit being a Calabi-Yau variety. It is
speculated that degenerating to Calabi-Yau varieties should be equivalent to the non-collapsing convergence, which is confirmed by a recent work [86]. Actually, we have a triple equivalence among not only the degeneration to Calabi-Yau varieties, and the Gromov-Hausdorff convergence, but also the finiteness of Weil-Petersson distance.

Let $\pi : \mathcal{X} \to \Delta$ be a flat family of Calabi-Yau $n$-manifolds, and $\mathcal{L}$ be a relative ample line bundle on $\mathcal{X}$. The variation of Hodge structures gives a natural semi-positive form, a possibly degenerated Kähler metric, on $\Delta$, called the Weil-Petersson metric,

$$\omega_{WP} = \frac{-1}{2\pi} \partial \bar{\partial} \log \int_{X_t} (-1)^n \Omega_t^n \wedge \Omega_t^n \geq 0,$$

where $\Omega_t$ is a relative holomorphic volume form, i.e. a no-where vanishing section of $\mathcal{W}_{\mathcal{X}/\Delta}$ (cf. [88]). The Weil-Petersson metric $\omega_{WP}$ is the curvature of the first Hodge bundle $\mathcal{H}^n = R^n \pi_* \mathcal{C} \otimes \mathcal{O}_\Delta$ with a natural Hermitian metric, and describes the deformation of complex structures.

In [86], Candelas, Green and Hübsch found some nodal degenerations of Calabi-Yau 3-folds with finite Weil-Petersson distance. In general, [99] shows that if $(\pi : \mathcal{X} \to \Delta, \mathcal{L})$ is a degeneration of polarized Calabi-Yau manifolds, and if the central fiber $X_0$ is a Calabi-Yau variety, then the Weil-Petersson distance between $\{0\}$ and the interior $\Delta^*$ is finite, i.e. $\omega_{WP}$ is not complete on $\Delta^*$. Conversely, if we assume that the Weil-Petersson distance of $\{0\}$ is finite, then after a finite base change $\pi : \mathcal{X} \to \Delta$ is birational to a degeneration $\pi' : \mathcal{X}' \to \Delta$ such that $\mathcal{X}' \setminus X_0 \cong \mathcal{X} \setminus X_0'$, and $X_0'$ is a Calabi-Yau variety by a paper [93]. As a consequence, the algebro-geometric degenerating Calabi-Yau manifolds to a Calabi-Yau variety is equivalent to the finiteness of the Weil-Petersson distance. In [86], the further equivalence to the Gromov-Hausdorff convergence is also established. In summary, we have the following theorem.

**Theorem 3.6 ([99] [93] [86]).** Let $(\pi : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds. Then the following statements are equivalent.

i) After a finite base change, $\pi : \mathcal{X} \to \Delta$ is birational to a degeneration $\pi' : \mathcal{X}' \to \Delta$ such that $\mathcal{X}' \setminus X_0 \cong \mathcal{X} \setminus X_0'$, and $X_0'$ is a Calabi-Yau variety.

ii) $$(X_t, \omega_t) \xrightarrow{dGH} (Y, dy),$$

when $t \to 0$, in the Gromov-Hausdorff sense, where $\omega_t$ denotes the unique Ricci-flat Kähler-Einstein metric in $c_1(\mathcal{L})|_{X_t} \in H^{1,1}(X_t, \mathbb{R})$, $t \in \Delta^*$, and $(Y, dy)$ is a compact metric space.

iii) $\{0\}$ has finite Weil-Petersson distance to the interior $\Delta^*$.

An application of such equivalence is the completion of the moduli space of polarized Calabi-Yau manifolds, which is surveyed in the next section.
4. Moduli space of polarized Calabi-Yau manifolds

Let $\mathcal{M}^P$ be the moduli space of polarized Calabi-Yau manifolds $(X, L)$ of dimension $n$ with a fixed Hilbert polynomial $P = P(\mu) = \chi(X, L^\mu)$, i.e.

$$\mathcal{M}^P = \{(X, L) | P(\mu) = \chi(X, L^\mu)\} / \sim,$$

where $(X_1, L_1) \sim (X_2, L_2)$ if and only if there is an isomorphism $\psi : X_1 \to X_2$ such that $L_1 = \psi^* L_2$. We denote the equivalent class $[X, L] \in \mathcal{M}^P$ represented by $(X, L)$.

The Bogomolov-Tian-Todorov’s unobstructedness theorem of Calabi-Yau manifolds implies that $\mathcal{M}^P$ is a complex orbifold (cf. [88, 92]). The variation of Hodge structures gives a natural orbifold Kähler metric on $\mathcal{M}^P$, called the Weil-Petersson metric, which is the curvature of the first Hodge bundle with a natural Hermitian metric (cf. [88]). From the algebro-geometric viewpoint, Viehweg proved in [98] that $\mathcal{M}^P$ is a quasi-projective variety, and coarsely represents the moduli functor $\mathfrak{M}^P$ for polarized Calabi-Yau manifolds with Hilbert polynomial $P$.

We like to understand $\mathcal{M}^P$ by considering Ricci-flat Kähler-Einstein metrics. The Calabi-Yau theorem gives a continuous map

$$\mathcal{CY} : \mathcal{M}^P \to \text{Met}, \quad \text{by} \quad [X, L] \mapsto (X, \omega),$$

where $\omega$ is the unique Ricci-flat Kähler-Einstein metric representing $c_1(L)$. However, $\mathcal{CY}$ is not injective in general since $\mathcal{M}^P$ contains the information of complex structures.

The compactifications of moduli spaces were studied in various cases, for example, the Mumford’s compactification of moduli spaces for curves (cf. [65]), the Satake compactification of moduli spaces for Abelian varieties (cf. [77]), and more recently the compact moduli spaces for general type stable varieties of higher dimension (cf. [53]). Because of the importance of Calabi-Yau manifolds in mathematics and physics, it is also desirable to have compactifications of $\mathcal{M}^P$.

For constructing compactifications of $\mathcal{M}^P$, Yau suggested that one uses the Weil-Petersson metric to obtain a metric completion of $\mathcal{M}^P$ first, and then tries to compactify this completion (cf. [61]). In [100], an alternative approach is proposed by using the Gromov-Hausdorff distance, instead of the Weil-Petersson metric, to construct a completion. In a recent paper [106], we constructed a completion of $\mathcal{M}^P$ via Ricci-flat Kähler-Einstein metrics and Gromov-Hausdorff topology, which can be viewed as a partial compactification.

Let $\overline{\mathcal{CY}(\mathcal{M}^P)}$ be the closure of the image $\mathcal{CY}(\mathcal{M}^P)$ in $\text{Met}$. There is a natural metric space structure on $\overline{\mathcal{CY}(\mathcal{M}^P)}$ by restricting the Gromov-Hausdorff distance. Since $(\text{Met}, d_{GH})$ is complete, $\overline{\mathcal{CY}(\mathcal{M}^P)}$ is the completion in the Gromov-Hausdorff sense. However, we do not expect that $\overline{\mathcal{CY}(\mathcal{M}^P)}$ have some algebro-geometric properties because of the non-injectivity of $\mathcal{CY}$. We
obtain an enlarge moduli space $\overline{M}^P$ such that $\mathcal{CY}$ extends to surjection from $\overline{M}^P$ to $\mathcal{CY}(\overline{M}^P)$ in [106].

**Theorem 4.1.** There is a Hausdorff topological space $\overline{M}^P$, and a surjection
\[
\mathcal{CY} : \overline{M}^P \to \mathcal{CY}(\overline{M}^P)
\]
satisfying the follows.

i) $\overline{M}^P$ is an open dense subset of $\overline{M}^P$, and $\mathcal{CY}|_{\overline{M}^P} = \mathcal{CY}$.

ii) For any $p \in \overline{M}^P$, $\mathcal{CY}(p)$ is homeomorphic to a Calabi-Yau variety.

iii) There is an exhaustion
\[
\overline{M}^P \subset M^{m(1)} \subset M^{m(2)} \subset \cdots \subset M^{m(l)} \subset \cdots \subset \overline{M}^P = \bigcup_{l \in \mathbb{N}} M^{m(l)},
\]
where $m(l) \in \mathbb{N}$ for any $l \in \mathbb{N}$, such that $M^{m(l)}$ is a quasi-projective variety, and there is an ample line bundle $\lambda_{m(l)}$ on $M^{m(l)}$.

iv) Let $(\pi : X \to \Delta, L)$ be a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety $X_0$ as the central fiber. Assume that for any $t \in \Delta^*$, there is an ample line bundle $L_t$ on $X_t$ such that $L^k_t \cong L|_{X_t}$ for a $k \in \mathbb{N}$, and $[X_t, L_t] \in \overline{M}^P$. Then there is a unique morphism $p : \Delta \to M^{m(l)}$, for $l \gg 1$, such that $\mathcal{CY}(p(t))$ is homeomorphic to $X_t$ for any $t \in \Delta$, and
\[
\mathcal{CY}(p(t)) \to \mathcal{CY}(p(0)),
\]
when $t \to 0$, in the Gromov-Hausdorff sense. Furthermore, $\rho^*\lambda_{m(l)} = \pi_*\omega_{X/\Delta}$ for a $\nu(l) \in \mathbb{N}$.

When $n = 2$, a Calabi-Yau variety is a K3 orbifold, and a degeneration of K3 surfaces to a K3 orbifold is called a degeneration of type I. It is well-known that one can fill the holes in the moduli space of Kähler polarized K3 surfaces by some Kähler K3 orbifolds, and obtain a complete moduli space (cf. [48, 49]). The relationship between such moduli space and the degeneration of Ricci-flat Kähler-Einstein metrics is also established in [48, 49].

We recall the construction of $\overline{M}^P$. For any $D > 0$, we define a subset $\mathcal{M}^P(D)$ of $\overline{M}^P$ by
\[
\mathcal{M}^P(D) = \{[X, L] \in \overline{M}^P | \text{Ricci-flat metric } \omega \in c_1(L) \text{ with diam}_\omega(X) \leq D\}.
\]
We have that if $D_1 \leq D_2$, then $\mathcal{M}^P(D_1) \subset \mathcal{M}^P(D_2)$, and $\mathcal{M}^P = \bigcup_{D > 0} \mathcal{M}^P(D)$.

Note that for a sequence $[X_k, L_k] \in \mathcal{M}^P(D)$, if $(X_k, \omega_k)$ converges to a compact metric space $Y$ in the Gromov-Hausdorff sense, then by Theorem 3.4, there are embeddings $\Phi_k : X_k \hookrightarrow \mathbb{C}P^N$ for an $N > 0$ independent of $k$ such that $L^m_k \cong \Phi_k^*\mathcal{O}_{\mathbb{C}P^N}(1)$ for an $m > 0$, and $\Phi_k(X_k)$ converges to a Calabi-Yau variety $X_\infty$ in the Hilbert scheme $\mathcal{Hilb}_{N}^{m}$, which is homeomorphic to $Y$. By Matsusaka’s Big Theorem (cf. [63]), we take $m$ large enough such that for
any \( [X, L] \in \mathcal{M}^P, \ L^m \) is very ample, and \( H^i(X, L^m) = \{0\}, \ i > 0 \). Thus we have an embedding \( \Phi_{\Sigma} : X \hookrightarrow \mathbb{CP}^N \), and \( \Phi_{\Sigma}(X) \in \mathcal{H}ilb_{\mathcal{N}}^{P_m} \).

Let \( \pi_{\mathcal{H}} : \mathcal{U}_{\mathcal{N}} \to \mathcal{H}ilb_{\mathcal{N}}^{P_m} \) be the universal family over the Hilbert scheme \( \mathcal{H}ilb_{\mathcal{N}}^{P_m} \) of the Hilbert polynomial \( P_m(\mu) = P(m\mu) \), and \( \mathcal{H}^0_{\mathcal{N}} \subset (\mathcal{H}ilb_{\mathcal{N}}^{P_m})_{red} \) be the Zariski open subset parameterizing smooth varieties. Let \( \mathcal{H}_{\mathcal{N}} \) be the subset of \((\mathcal{H}ilb_{\mathcal{N}}^{P_m})_{red}\) such that \( \mathcal{H}^0_{\mathcal{N}} \subset \mathcal{H}_{\mathcal{N}} \subset \overline{\mathcal{H}^0_{\mathcal{N}}} \) where \( \overline{\mathcal{H}^0_{\mathcal{N}}} \) denotes the Zariski closure of \( \mathcal{H}^0_{\mathcal{N}} \) in \((\mathcal{H}ilb_{\mathcal{N}}^{P_m})_{red}\), and a point \( p \in \mathcal{H}_{\mathcal{N}} \) if \( X_p = \pi_{\mathcal{H}}^{-1}(p) \) is a Calabi-Yau variety.

Recall that for any Calabi-Yau variety \( X_p \) where \( p \in \mathcal{H}_{\mathcal{N}} \), the Ricci-flat Kähler-Einstein metric \( \omega \in c_1(\mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \) induces an \( SU(N+1) \)-orbit \( RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \subset \mathcal{H}ilb_{\mathcal{N}}^{P_m} \) by \( (2.2) \). If we let

\[
\mathcal{R}_{\mathcal{N}} = \bigcup_{p \in \mathcal{H}_{\mathcal{N}}} RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \subset \mathcal{H}_{\mathcal{N}},
\]

then there is a natural \( SU(N+1) \)-action on \( \mathcal{R}_{\mathcal{N}} \), which is induced by the \( SL(N+1) \)-action on \( \mathcal{H}ilb_{\mathcal{N}}^{P_m} \). Let

\[
\mathcal{M}_m = \mathcal{R}_{\mathcal{N}} / SU(N+1).
\]

Note that the reduced Hilbert scheme \( (\mathcal{H}ilb_{\mathcal{N}}^{P_m})_{red} \) is Hausdorff under the analytic topology, and so is the subset \( \mathcal{R}_{\mathcal{N}} \). Thus the quotient \( \mathcal{M}_m \) by the compact Lie group \( SU(N+1) \) is also Hausdorff. It is clear that both \( \mathcal{M}^P(D) \) and \( \mathcal{M}^P \) are subsets of \( \mathcal{M}_m \), and furthermore, if \( X \) is a Calabi-Yau variety homeomorphic to a compact metric space in \( \mathcal{CY}(\mathcal{M}^P(D)) \), then \( X \in \mathcal{M}_m \).

We obtain the enlarged moduli space \( \overline{\mathcal{M}}^P = \cup \mathcal{M}_m \) by taking a increasing sequence of \( D_j \).

However, it is unclear whether \( \mathcal{M}_m \) has any algebrao-geometric property from the above construction. We need the full strength of \( [95] \) to prove that \( \mathcal{M}_m \) is a quasi-projective variety. Firstly, we show that \( \mathcal{H}_{\mathcal{N}} \) can be given an open subscheme structure in Section 3.2 of \( [100] \) by the deformation of varieties with canonical singularities \( [47] \). Secondly, by \( [6, \text{Theorem 2.1}] \) and \( [66, \text{Corollary 4.3}] \), if \( (X_1 \to S, L_1) \) and \( (X_2 \to S, L_2) \) are two flat families of polarized Calabi-Yau varieties over a germ of smooth curve \( (S, 0) \), then any isomorphism of these two families over \( S \setminus \{0\} \) extends to an isomorphism over \( S \), which implies the separateness condition of \( \mathcal{M}_m \). Finally, the construction of Section 8 in \( [98] \) shows that there is a geometric quotient \( \mathcal{H}_{\mathcal{N}} \to \mathcal{M}'_m \) of the natural \( SL(N+1) \)-action on \( \mathcal{H}_{\mathcal{N}} \), and an ample line bundle \( \lambda'_m \) on \( \mathcal{M}'_m \), which implies that \( \mathcal{M}'_m \) is a quasi-projective variety. If \( O(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \) is the \( SL(N+1) \)-orbit of the \( SL(N+1) \)-action for a \( p \in \mathcal{H}_{\mathcal{N}} \), then the uniqueness of the Kähler-Einstein metric \( \omega \) implies that

\[
RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) = O(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \cap \mathcal{R}_{\mathcal{N}}.
\]

Thus

\[
\mathcal{M}_m = \mathcal{R}_{\mathcal{N}} / SU(N+1) = \mathcal{H}_{\mathcal{N}} / SL(N+1)
\]
with the quotient topology induced by the analytic topology of $\mathcal{H}_N$, and is homeomorphic to the underlying variety of $\mathcal{M}'_m$. We obtain the conclusion.

Kähler-Einstein metrics were previously used to construct compactifications of moduli spaces for Kähler-Einstein orbifolds in [62], and it is proved that such compactification coincides with the standard Mumford’s compactification in the case of curves. As a matter of fact, the idea of the above construction is from [62]. The real slice $\mathcal{R}_N$ induced by Kähler-Einstein metrics is an analogue of the zero set of momentum map in the Geometric Invariant Theory (cf. [65, 87]).

In [106], we also proved that the points in $\overline{\mathcal{M}}^P \setminus \mathcal{M}^P$ have finite Weil-Petersson distances, which is a corollary of Theorem 3.6.

**Theorem 4.2.** Let $\overline{\mathcal{M}}^P$ and $\overline{\mathcal{CY}}$ be the same as in Theorem 4.1.

i) For any point $x \in \overline{\mathcal{M}}^P \setminus \mathcal{M}^P$, there is a curve $\gamma$ such that $\gamma(0) = x$, $\gamma((0,1]) \subset \mathcal{M}^P$ and the length of $\gamma$ under the Weil-Petersson metric $\omega_{WP}$ is finite, i.e.

$$\text{length}_{\omega_{WP}}(\gamma) < \infty.$$ 

ii) Let $(\pi : X \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds such that for any $t \in \Delta^*$, $L_t^k \cong \mathcal{L}|_{X_t}$ for a $k \in \mathbb{N}$, and $[X_t, L_t] \in \mathcal{M}^P$, where $L_t$ is an ample line bundle. If the Weil-Petersson distance between $0 \in \Delta$ and the interior $\Delta^*$ is finite, then there is a unique morphism $\rho : \Delta \to \mathcal{M}(t)$, for $l \gg 1$, such that $\mathcal{CY}(\rho(t))$ is homeomorphic to $X_t$, $t \in \Delta^*$.

A simple concrete example for Theorem 4.1 and Theorem 4.2 is the mirror Calabi-Yau threefold of the quintic threefold constructed in [8] (cf. Section 18 in [35]), i.e. $X_t$ is the crepant resolution of the quotient

$$Y_s = \{[z_0, \cdots, z_4] \in \mathbb{CP}^4 | z_0^5 + \cdots + z_4^5 + sz_0 \cdots z_4 = 0\}/(\mathbb{Z}_5^*/\mathbb{Z}_5)$$

of the quintic by $\mathbb{Z}_5^*/\mathbb{Z}_5$, where $s^5 = t \in \mathbb{C}$. By choosing a polarization, $\mathcal{M}^P = \mathbb{C}\setminus\{1\}$, and $0$ is an orbifold point of $\mathcal{M}^P$. When $t = 1$, $X_1$ is a Calabi-Yau variety with only one ordinary double points, and however, $t = \infty$ is a large complex limit point, which implies that $t = \infty$ is the cusp end of $\mathcal{M}^P$ and has infinite Weil-Petersson distance. Thus $\overline{\mathcal{M}}^P = \mathbb{C}$.

5. **Surgeries**

Extremal transitions and flops are algebro-geometric surgeries providing ways to connect two topologically distinct projective manifolds, which are interesting in both mathematics and physics. In the minimal model program, all smooth minimal models in a birational equivalence class are connected by sequences of flops. The famous Reid’s fantasy conjectures that all Calabi-Yau threefolds are connected to each other by extremal transitions, so as
to form a huge connected web. The goal of this section is to study the behaviour of Ricci-flat Kähler-Einstein metrics under these surgeries, extremal transitions and flops.

5.1. Degeneration of Kähler classes. In this subsection, we study the limit behaviour of Ricci-flat Kähler-Einstein metrics when their Kähler classes approach to the boundary of the Kähler cone on a fixed Calabi-Yau manifold, which is parallel to the case of degenerations.

As in the degeneration case, we have a prior estimate for the diameter of Ricci-flat Kähler-Einstein metric obtained in [94] and [108]. More precisely, if \( X \) is a Calabi-Yau manifold, and \( \omega_0 \) is a Ricci-flat Kähler-Einstein metric on \( X \), then for any Ricci-flat Kähler-Einstein metric \( \omega \) on \( X \), we have

\[
\text{diam}_\omega(X) \leq 32n + D \left( \int_X \omega \wedge \omega_0^{n-1} \right)^n,
\]

where \( D > 0 \) is a constant depending on \( \omega_0 \), but independent of \( \omega \).

Let X be a Calabi-Yau variety, and \( \bar{\pi} : \bar{X} \to X \) be a crepant resolution, i.e. a resolution such that \( \bar{\pi}^*\mathcal{K}_X = \mathcal{K}_{\bar{X}} \), which implies that \( \bar{X} \) is a Calabi-Yau manifold. If \( L \) is an ample line bundle on \( X \), then \( \bar{\pi}^*L \) is semi-ample and big, but no longer ample, i.e. \( \bar{\pi}^*c_1(L) \in \partial\mathcal{K}_{\bar{X}} = \mathcal{K}_{\bar{X}} \setminus \mathcal{K}_X \). The following convergence theorem is proved in [93].

Theorem 5.1 ([93]). Let \( X \) be a Calabi-Yau variety, and \( \bar{\pi} : \bar{X} \to X \) be a crepant resolution, and \( L \) be an ample line bundle on \( X \). For any family of Kähler classes \( \alpha_s \in \mathcal{K}_{\bar{X}}, s \in (0, 1], \) with \( \lim_{s \to 0} \alpha_s = \bar{\pi}^*c_1(L) \), if \( \bar{\omega}_s \in \alpha_s \) is the Ricci-flat Kähler-Einstein metric, then \( \bar{\omega}_s \) converges smoothly to \( \bar{\pi}^*\omega \) on any compact subset of \( \bar{\pi}^{-1}(X_{\text{reg}}) \), when \( s \to 0 \), where \( \omega \) is the unique singular Ricci-flat Kähler-Einstein metric representing \( c_1(L) \) on \( X \).

Note that in the above theorem, the diameter of \( \bar{\omega}_s \) is uniformly bounded, i.e.

\[
\text{diam}_{\bar{\omega}_s}(\bar{X}) \leq D,
\]

for constant \( D > 0 \) independent of \( s \). For any sequence \( s_k \to 0 \), the Gromov pre-compactness theorem (Theorem 2.2) says that a subsequence of \( (\bar{X}, \bar{\omega}_{s_k}) \) converges to a compact metric space \( (Y, d_Y) \) in the Gromov-Hausdorff sense, and Theorem 2.3 shows that there is a closed subset \( S \subset Y \) with Hausdorff dimension \( \text{dim}_H S \leq 2n - 4 \) such that \( Y \setminus S \) is a Ricci-flat Kähler \( n \)-manifold. In [73], it is proved that \( Y \) is the metric completion of \( (X_{\text{reg}}, \omega) \). An analogue of Theorem 3.4 is obtained for the case of \( X \) being a conifold in [80], and later is generalized to any Calabi-Yau variety in [81]. Furthermore, the analogue for more general Kähler metrics is also studied in a recent preprint [57].

Theorem 5.2 ([81]). Let \( X, \bar{X} \) and \( \bar{\omega}_s \) be the same as in Theorem 5.1. If \( Y \) is the Gromov-Hausdorff limit a sequence \( (\bar{X}, \bar{\omega}_{s_k}) \), then \( Y \) is homeomorphic to \( X \).
As in the case of degeneration, we also like to know more explicit asymptotical behaviour when metrics are approaching to the singularities. If $X$ is a K3 orbifold, Kobayashi’s theorem in Section 2.1 gives a very satisfactory description. However it is not clear in the higher dimensional case.

We assume that the Calabi-Yau variety $X$ in Theorem 5.1 and Theorem 5.2 has dimension 3, and only ordinary double points as singularities, i.e. for any $p \in X_{\text{sing}}$, there is a neighborhood $U_p \subset X$ isomorphic to an open subset of the quadric $Q = \{(w_1, \ldots, w_4) \in \mathbb{C}^4 | w_1 w_2 = w_3 w_4\}$. The crepant resolution $\tilde{X}$ is locally given by the small resolution of the quadric cone

$$\tilde{Q} = \{(w_1, \ldots, w_4, [y_0, y_1]) \in \mathbb{C}^4 \times \mathbb{CP}^1 | w_1 y_1 = w_3 y_0, y_0 w_2 = y_1 w_4 \} \to Q,$$

where $Q$ is biholomorphic to the total space of the bundle $\mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}(1)$. For any $s > 0$, a complete Ricci-flat Kähler-Einstein metric $\bar{\omega}_{\text{co},s}$ is constructed in [9] by

$$\bar{\omega}_{\text{co},s} = \sqrt{-1} \partial \bar{\partial} (f_s(\rho) + 4s^2 \log(|y_0|^2 + |y_1|^2)),$$

where $\rho = |w_1|^2 + \cdots + |w_4|^2$, and $f_s(\rho)$ satisfies the following ordinary differential equation

$$(\rho f'_s(\rho))^3 + 6s^2 (\rho f''_s(\rho))^2 - \rho^2 = 0,$$

which can be solved explicitly. When $s \to 0$, $\bar{\omega}_{\text{co},s}$ converges smoothly to the Ricci-flat cone metric $\bar{\omega}_{\text{co}}$ on $Q$ given by (2.1). Like the case of degeneration, we expect that those metrics $\bar{\omega}_s$ on $\tilde{X}$ are asymptotic to $\bar{\omega}_{\text{co},s}$ near singularities when $s \to 0$.

Now we consider the flops, which connect distinct Calabi-Yau manifolds in the same birational class. If $X$ is a Calabi-Yau variety admitting two different crepant resolutions $(\tilde{X}_1, \pi_1)$ and $(\tilde{X}_2, \pi_2)$ with both exceptional subvarieties of codimension at least 2, the process of going from $\tilde{X}_1$ to $\tilde{X}_2$ is called a flop, denoted by

$$\tilde{X}_1 \to X \longrightarrow \tilde{X}_2.$$ 

The simplest case is the local flop for the quadric $Q$ of dimension 3. Note that $Q$ has two crepant resolutions, one is $\tilde{Q}$ as above and the other is

$$\tilde{Q}' = \{(w_1, \ldots, w_4, [y_0, y_1]) \in \mathbb{C}^4 \times \mathbb{CP}^1 | w_1 y_1 = w_4 y_0, y_0 w_2 = y_1 w_3 \} \to Q,$$

i.e. we have a flop

$$\tilde{Q} \to Q \longrightarrow \tilde{Q}'.$$

Flops were used by Tian and Yau in [91] for constructing topological distinct Calabi-Yau threefolds.

The following theorem is proved in [73], which asserts that flops are continuous in the Gromov-Hausdorff sense when one considers Ricci-flat Kähler-Einstein metrics.

**Theorem 5.3** ([73]). Let $X$ be an $n$-dimensional Calabi-Yau variety, and $L$ be an ample line bundle on $X$. Assume that $X$ admits two crepant resolutions $(\tilde{X}_1, \pi_1)$ and $(\tilde{X}_2, \pi_2)$. Let $\bar{\omega}_{1,s}$ (resp. $\bar{\omega}_{2,s}$), $s \in (0, 1]$, be a
family of Ricci-flat Kähler metrics on $\bar{X}_1$ (resp. $\bar{X}_2$) with Kähler classes $\lim_{s \to 0} [\bar{\omega}_{\alpha,s}] = \bar{\pi}_e^* c_1(L)$, $\alpha = 1, 2$. Then there exists a compact metric space $(Y, dy)$ such that

$$(\bar{X}_1, \bar{\omega}_{1,s}) \xrightarrow{dGH} (Y, dy) \xrightarrow{dGH} (\bar{X}_2, \bar{\omega}_{2,s}), \quad \text{when } s \to 0.$$ 

Furthermore, $(Y, dy)$ is isometric to the metric completion $(X_{reg}, \omega)$ where $\omega \in c_1(L)$ is the singular Ricci-flat Kähler-Einstein metric on $X$.

In the minimal model program, it is proved that for any two Calabi-Yau $n$-manifolds $X$ and $X'$ birational to each other, there are sequences of flops connecting $X$ and $X'$ in [46], which was obtained previously in [50] for the case of dimension 3, i.e., there is a sequence of varieties $X_1, \ldots, X_k$ such that $X = X_1$, $X' = X_k$, and $X_{j+1}$ is obtained by a flop from $X_j$. Consequently there are normal projective varieties $X_{0,1}, \ldots, X_{0,k-1}$, and small resolutions $\bar{\pi}_j : X_j \to X_0$ and $\bar{\pi}_j : X_j \to X_{0,j-1}$. Hence

$$X_1 \to X_{0,1} \dashrightarrow X_2 \to X_{0,2} \dashrightarrow \cdots \to X_{0,k-1} \dashrightarrow X_k.$$ 

Now we restrict to the case of dimension 3, i.e. $\dim CY_3 = \dim CY_3' = 3$. By [50], $X_j$ has the same singularities as $X$, and thus $X_j$ is smooth. Since the exceptional locus of $\bar{\pi}_j$ and $\bar{\pi}_j^+$ are of co-dimension at least 2, $X_{0,j}$ has only canonical singularities, and the canonical bundle of $X_{0,j}$ is trivial (cf. Corollary 1.5 in [15]). Therefore $X_{0,j}$ is a three-dimensional Calabi-Yau variety, and $X_j$ is a three-dimensional Calabi-Yau manifold.

For a fixed Calabi-Yau manifold $X$, if we denote

$$\mathcal{CY}(X, \mathcal{K}_X) = \{(X, \omega) | \text{all of } \omega \text{ with } \text{Ric}(\omega) \equiv 0, \text{Vol}_\omega(X) = 1 \} \subset (\text{Met}, d_{GH}),$$

then by Theorem 5.3, for any $j > 0$, there is a compact metric space $(Y, dy)$ belonging to both of $\mathcal{CY}(X_j, \mathcal{K}_{X_j})$ and $\mathcal{CY}(X_{j+1}, \mathcal{K}_{X_{j+1}})$, where $\mathcal{CY}(X_j, \mathcal{K}_{X_j})$ denotes the closure of $\mathcal{CY}(X_j, \mathcal{K}_{X_j}) \subset (\text{Met}, d_{GH})$, and furthermore $Y$ is homeomorphic to $X_{0,j}$ by Theorem 5.2. Thus

$$\mathcal{CY}(X_j, \mathcal{K}_{X_j}) \subset \mathcal{CY}(X_{j+1}, \mathcal{K}_{X_{j+1}})$$

is path connected. We obtain the following corollary.

**Corollary 5.4** ([73]). *If $X$ is a three-dimensional Calabi-Yau manifold, then*

$$\bigcup_{\text{all such } X'} \mathcal{CY}(X', \mathcal{K}_{X'})$$

*is path connected in $(\text{Met}, d_{GH})$.*

We expect that this corollary still holds for higher dimensional Calabi-Yau manifolds. One of the difficulties to obtain a proof is the lack of knowledge of singular Ricci-flat Kähler-Einstein metrics on singular Calabi-Yau varieties. More precisely, if a Calabi-Yau variety $X$ does not admit a crepant resolution, or there is no Calabi-Yau degeneration with $X$ as central fiber,
we do not know whether a singular Ricci-flat Kähler-Einstein metric on $X$ induces a metric structure on $X$.

5.2. Extremal transition. This subsection study extremal transitions.

Let $X$ be a singular Calabi-Yau variety. Assume that $X$ admits a crepant resolution $\bar{\pi} : \bar{X} \rightarrow X$, and there is a Calabi-Yau degeneration $\pi : \mathcal{X} \rightarrow \Delta$ with $X_0 = X$. The process of going from $\bar{X}$ to $X_t$, $t \in \Delta^*$, is called an extremal transition, denoted by

$$\bar{X} \rightarrow X \rightsquigarrow X_t.$$ 

We call this process a conifold transition if $X$ is a conifold, i.e. $X$ has only ordinary double points as singularities. Locally a conifold transition exists in the case of dimension 3, i.e.

$$Q \rightarrow Q_0 \rightsquigarrow Q_t,$$

where $Q_t = \{ (z_0, \cdots, z_3) \in \mathbb{C}^4 | z_0^2 + \cdots + z_3^2 = t \}$, $t \in \Delta$, is the quadric, and $Q$ is the small resolution of $Q_0$. It contracts the $\mathbb{CP}^1$ in $\mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}(1)$ to obtain the 3-dimensional quadric cone $Q_0$ first, and then deforms the complex structure of $Q_0$ to get those $Q_t$, which are diffeomorphic to the total space of the cotangent bundle $T^*S^3$. A extremal transition is an algebro-geometric surgery providing ways to connect two topologically distinct Calabi-Yau manifolds, which increases complex moduli and decreases Kähler moduli.

The famous Reid’s fantasy conjectures that all Calabi-Yau threefolds are connected to each other by extremal transitions, possibly including non-Kähler Calabi-Yau threefolds, so as to form a huge connected web (cf. [70, 75]). There is also a projective version of this conjecture, the connectedness conjecture for moduli spaces for Calabi-Yau threefolds (cf. [33, 34, 75]). In physics, extremal transitions are related to the vacuum degeneracy problem in string theory (cf. [10, 17, 31, 30, 75]). The explanation of extremal transition in physics is that type II string theories modeled by topologically distinct Calabi-Yau vacua can be continuously connected via suitable black hole condensations (cf. [31]). Readers are referred to the survey article [75] for topology, algebraic geometry, and physics properties of extremal transitions.

Note that physicists are interested in Calabi-Yau manifolds in the first place, because the holomony group of a Ricci-flat Kähler-Einstein metric is a subgroup of $SU(n)$ despite of many works using only algebro-geometric setting in the mathematical physics literatures. It makes perfect sense to consider extremal transitions with metric.

Physicists P. Candelas and X. C. de la Ossa conjectured in [9] that extremal transitions should be "continuous in the space of Ricci-flat Kähler-Einstein metrics", even though these processes involve topologically distinct Calabi-Yau manifolds. This conjecture is verified in [9] for the local conifold transition, i.e.

$$\bar{\omega}_{co,s} \rightarrow \omega_{co} \leftarrow \omega_{co,t},$$
for a compact metric space \((Y,d)\). Hence there is a conifold transition conjecture says that there is a huge connected web \(\Gamma\) such that nodes of \(\Gamma\) is the metric given by \((5.1)\) (respectively \((3.1)\)). In this case, we can regard the local conifold transition is a surgery with metric, which shrinks a holomorphic \(\mathbb{CP}^1\) to one point, obtains a singular Ricci-flat space, and then replaces the singular point by a special lagrangian sphere \(S^3\).

In the general case, we obtain the following theorem in [73], which can be obtained by combining Theorem 3.1 and Theorem 5.1.

\textbf{Theorem 5.5 (73).} Let \(X\) be a Calabi-Yau \(n\)-variety. Assume that

\(\text{i)}\) \(\text{there is a degeneration} (\pi : X \rightarrow \Delta, \mathcal{L}) \text{ of polarized Calabi-Yau manifolds such that} X_0 = X. \text{For any} t \in \Delta^*, \text{let} \omega_t \text{be the unique Ricci-flat Kähler-Einstein metric on} X_t \text{representing} c_1(\mathcal{L})|_{X_t}.\)

\(\text{ii)}\) \(X\) admits a crepant resolution \(\tilde{\pi} : \tilde{X} \rightarrow X. \text{Let} \tilde{\omega}_s, s \in (0,1), \text{be a family of Ricci-flat Kähler-Einstein metrics with Kähler classes}\)

\(\lim_{s \rightarrow 0} [\tilde{\omega}_s] = \tilde{\pi}^* c_1(\mathcal{L})|_X \text{in} H^{1,1}(X, \mathbb{R}).\)

Then there exists a compact metric space \((Y,d_Y)\) such that

\[(X_t, \omega_t) \xrightarrow{d_{GH}} (Y,d_Y) \xrightarrow{d_{GH}} (\tilde{X}, \tilde{\omega}_s), \text{ when} t \rightarrow 0, \ s \rightarrow 0.\]

Furthermore, \((Y,d_Y)\) is isometric to the metric completion \((X_{\text{reg}}, \omega)\) where \(\omega \in c_1(\mathcal{L})|_X\) is the singular Ricci-flat Kähler-Einstein metric on \(X\).

By Theorem \(3.4\), \(Y\) is homeomorphic to the Calabi-Yau variety \(X\).

The following is a simple example that Theorem 5.5 can apply. Let \(\tilde{X}\) be the complete intersection in \(\mathbb{CP}^4 \times \mathbb{CP}^1\) given by

\[y_0 g(z_0, \cdots, z_4) + y_1 h(z_0, \cdots, z_4) = 0, \quad y_0 z_4 - y_1 z_3 = 0,\]

where \(z_0, \cdots, z_4\) are homogeneous coordinates of \(\mathbb{CP}^4\), \(y_0, y_1\) are homogeneous coordinates of \(\mathbb{CP}^1\), and \(g\) and \(h\) are generic homogeneous polynomials of degree 4. Then \(\tilde{X}\) is a crepant resolution of the quintic conifold \(X\) given by

\[z_3 g(z_0, \cdots, z_4) + z_4 h(z_0, \cdots, z_4) = 0.\]

Hence there is a conifold transition \(\tilde{X} \rightarrow X \leadsto \tilde{X}\) for any smooth quintic \(\tilde{X}\) in \(\mathbb{CP}^4\). Theorem 5.5 implies that there is a family of Ricci-flat Kähler metrics \(\tilde{\omega}_s, s \in (0,1),\) on \(\tilde{X}\) and a family of Ricci-flat smooth quintic \((X_t, \omega_t), t \in \Delta^*,\) such that \(X_1 = \tilde{X}\), and

\[(X_t, \omega_t) \xrightarrow{d_{GH}} (Y, d_Y) \xrightarrow{d_{GH}} (\tilde{X}, \tilde{\omega}_s),\]

for a compact metric space \((Y,d_Y)\) homeomorphic to \(X\).

As an application of Theorem 5.5 and Theorem 5.3 we shall explore the path connectedness properties of certain class of Ricci-flat Calabi-Yau threefolds. Inspired by string theory in physics, some physicists made a projective version of Reid’s fantasy (cf. [110] [30] [73]), the so called connectedness conjecture, which is formulated more precisely in [33] (See also [34]). This conjecture says that there is a huge connected web \(\Gamma\) such that nodes of \(\Gamma\)
consist of all deformation classes of Calabi-Yau threefolds, and two nodes are connected $\mathcal{D}_1 - \mathcal{D}_2$ if $\mathcal{D}_1$ and $\mathcal{D}_2$ are related by an extremal transition, i.e., there is a Calabi-Yau 3-variety $X$ that admits a crepant resolution $\bar{X} \in \mathcal{D}_1$, and there is a Calabi-Yau degeneration $\pi : \mathcal{X} \to \Delta$ such that $X = X_0$, and $X_t \in \mathcal{D}_2$ for any $t \in \Delta^*$. It is shown in [30, 17, 4, 33] that many known Calabi-Yau threefolds are connected to each other in the above sense. By combining the connectedness conjecture and Theorem 5.5 and Theorem 5.3, we reach a metric version of connectedness conjecture as follows: if $\mathcal{CY}_3$ denotes the set of Ricci-flat Calabi-Yau threefolds $(X, \omega)$ with volume 1, then the closure $\overline{\mathcal{CY}_3}$ in $(\text{Met}, d_{GH})$ is path connected, i.e., for any two points $p_1$ and $p_2 \in \overline{\mathcal{CY}_3}$, there is a path

$$\gamma : [0, 1] \to \overline{\mathcal{CY}_3} \subset (\text{Met}, d_{GH})$$

such that $p_1 = \gamma(0)$ and $p_2 = \gamma(1)$.

Given a class of Calabi-Yau 3-manifolds known to be connected by extremal transitions in algebraic geometry, Theorem 5.5 can be used to show that the closure of the class of Calabi-Yau 3-manifolds is path connected in $(\text{Met}, d_{GH})$. In [30], it is shown that all complete intersection Calabi-Yau manifolds (CICY) of dimension 3 in products of projective spaces are connected by conifold transitions. Furthermore, in [4, 17] a large number of complete intersection Calabi-Yau 3-manifolds in toric varieties are verified to be connected by extremal transitions, which include Calabi-Yau hypersurfaces in all toric manifolds obtained by resolving weighted projective 4-spaces. As a corollary of Theorem 5.5, we obtain the following result.

**Corollary 5.6** ([73]).

i) $\bigcup_{\text{all such } X} \{\mathcal{CY}(X, \mathbb{K}_X)|\text{CICY threefold } X \text{ in products of projective spaces}\}$

is path connected in $(\text{Met}, d_{GH})$.

ii) There is a path connected subset of $\overline{\mathcal{CY}_3}$, which contains all of CICY threefolds in products of projective spaces, and Calabi-Yau hypersurfaces in toric 4-manifolds obtained by resolving a weighted projective 4-space.

Finally, we remark that many of the above results are expected to have some analogues for $G_2$ and $\text{Spin}(7)$ manifolds. Like Calabi-Yau threefolds in the string theory, $G_2$ and $\text{Spin}(7)$ manifolds play some roles in the M-theory (cf. [40]). There are also topological changing processes for $G_2$ and $\text{Spin}(7)$ manifolds, for example the $G_2$-flop transition (see [40, 18]), which is an analogue of extremal transition for Calabi-Yau threefolds. Theorem 5.5 provides a trivial example for this situation. Let $X$, $X_t$, $\bar{X}$, $\omega_t$ and $\bar{\omega}$ be the same as in Theorem 5.5 with $\dim_{\mathbb{C}} X = 3$, and $\bar{g}_t$ (respectively $\bar{\omega}_s$) be the corresponding Riemannian metric of $\omega_t$ (respectively $\bar{\omega}_s$). It is standard that the holonomy groups of $\bar{g}_t = g_t + d\theta^2$ and $\bar{\omega}_s = \bar{\omega}_s + d\theta^2$ belong to $G_2$ (cf. [41]). If we denote $\bar{X}_t = X_t \times S^1$ and $\bar{X} = \bar{X} \times S^1$, Theorem 5.5 implies
that
\((\tilde{X}_t, \tilde{g}_t) \xrightarrow{d_{GH}} (\tilde{Y}, d_{\tilde{Y}}) \xleftarrow{d_{GH}} (\tilde{\bar{X}}, \tilde{\bar{g}}_s)\),
when \(t \to 0, s \to 0\),
where \(\tilde{Y}\) is homeomorphic to \(X \times S^1\), and \(d_{\tilde{Y}}\) is induced by a \(G_2\)-metric on \(X_{reg} \times S^1\). However since neither advanced PDE nor algebraic geometry is available in this case, it is difficult to obtain a general result (cf. [40, 18]). For instance, it is still open to construct compact \(G_2\) and Spin\((7)\) spaces with isolated singularities (cf. [40]), which should be an analogue of Theorem 2.1.

6. Collapsing of Calabi-Yau manifolds

We finish this paper with a brief review of the collapsing of Ricci-flat Calabi-Yau manifolds, where our knowledge is more limited comparing to the non-collapsing case.

Let \((X_k, \omega_k)\) be a sequence of Ricci-flat Kähler-Einstein Calabi-Yau manifolds with the normalized volume \(\text{Vol}_{\omega_k}(X_k) \equiv \upsilon\). If the diameter of \(\omega_k\) tends to infinite when \(k \to \infty\), i.e.
\[
\text{diam}_{\omega_k}(X_k) \to \infty,
\]
then the same as in the proof of Proposition 3.2, we obtain
\[
\text{Vol}_{\omega_k}(B_{\omega_k}(1)) \leq \frac{8n \text{Vol}_{\omega_k}(X_k)}{\text{diam}_{\omega_k}(X_k) - 2} \to 0,
\]
for any metric 1-ball \(B_{\omega_k}(1)\), by Theorem 4.1 of Chapter 1 in [78] or Lemma 2.3 in [69], i.e. \((X_k, \omega_k)\) must collapse.

Most of the studies of the collapsing of Ricci-flat Calabi-Yau manifolds are motivated by the following version of SYZ conjecture. Let \((X \to \Delta, L)\) be a degeneration of polarized Calabi-Yau manifolds of dimension \(n\). If \(0 \in \Delta\) is a large complex limit point (cf. [35]), a refined version of SYZ conjecture due to Gross, Wilson, Kontsevich and Soibelman (cf. [38, 54, 55]) says that
\[
\text{diam}_{\omega_t}(X_t) \sim \sqrt{-\log |t|},
\]
and
\[
(X_t, \text{diam}^{-2}_{\omega_t}(X_t) \omega_t) \xrightarrow{d_{GH}} (B, dB)
\]
in the Gromov-Hausdorff sense. If \(h_{i,0}(X_t) = 0, 1 \leq i < n\), then \(B\) is homeomorphic to \(S^n\). Furthermore, there is an open subset \(B_0 \subset B\) with \(\text{codim}_B B \setminus B_0 \geq 2\), \(B_0\) admits a real affine structure, and the metric \(dB\) is induced by a Monge-Ampère metric \(g_B\) on \(B_0\), i.e. under affine coordinates \(x_1, \cdots, x_n\), there is a potential function \(\phi\) such that
\[
g_B = \sum_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i dx_j, \quad \text{and} \quad \det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) = 1.
\]
In [55], it is further conjectured that the Gromov-Hausdorff limit \(B\) is homeomorphic to the Calabi-Yau skeleton of the Berkovich analytic space associated to \(X \times_{\Delta} \Delta^*\) by taking some base change if necessary, which gives an algebro-geometric description of \(B\). This conjecture is clearly true for
Abelian varieties, and is verified by Gross and Wilson for fibered K3 surfaces with only type $I_1$ singular fibers in [38].

Let $f : X \to B$ be an elliptically fibered K3 surface with 24 type $I_1$ singular fibers, and let $\omega_k$ be a sequence of Ricci-flat Kähler-Einstein metrics on $X$ with $\omega_k^2$ independent of $k$. If $\epsilon_k = \int_{f^{-1}(b)} \omega_k \to 0$, for a $b \in B$, then [38] proves that

$$(X_k, \epsilon_k \omega_k) \xrightarrow{d_{GH}} (B, d_B),$$

in the Gromov-Hausdorff sense, where $d_B$ is induced by a special Kähler metric $g_B$ on the complement $B_0$ of the discriminant locus. In a local coordinate chart $U \subset B_0$, if $f^{-1}(U) = (U \times \mathbb{C})/\Lambda$, where $\Lambda = \text{Span}_\mathbb{Z}\{1, \tau\}$ and $\tau$ is a holomorphic function on $U$, i.e. the period of fibers, then $\text{Im}(\tau) > 0$ and $g_B = \text{Im}(\tau)dy \otimes d\bar{y}$. Furthermore, [38] shows very explicit asymptotic behaviour of $\omega_k$ when approaching to the limit. Near singular fibers, $\epsilon_k \omega_k$ is asymptotic to the Ooguri-Vafa metric, and on $f^{-1}(B_0)$, $\epsilon_k \omega_k$ is exponentially asymptotic to the semi-flat metric $\omega_{sf,k}$, i.e.

$$\|\epsilon_k \omega_k - \omega_{sf,k}\|_{C^0_{loc}} \leq Ce^{-\epsilon_k},$$

where

$$\omega_{sf,k} = \frac{\sqrt{-1}}{2} \left(\text{Im}(\tau)dy \wedge d\bar{y} + 2\epsilon_k^2 \partial\bar{\partial}(\text{Im}(z))^2 \text{Im}(\tau)\right),$$

and $z$ is the coordinate on $\mathbb{C}$. By using the HyperKähler rotation, the hypothesis of this result can be explained as a sequence of complex structures approaching to the large complex limit while keeping the polarization fixed. Hence the collapsing version of SYZ conjecture holds for certain K3 surfaces.

There are generalizations of Gross and Wilson's theorem to the higher dimensional case. Let $X$ be a Calabi-Yau $n$-manifold admitting a holomorphic fibration $f : X \to Z$, where $Z$ is a normal projective variety of dimension $m$, $m < n$. If $Z_0 \subset Z$ is the complement of the discriminant locus of $f$, then the fiber $X_b = f^{-1}(b)$, $b \in Z_0$, is a Calabi-Yau manifold of dimension $n - m$. Let $\alpha_Z$ be an ample class on $Z$, $\alpha_X$ be an ample class on $X$, and $\omega_s = \alpha_Z + s\alpha_X$, $s \in (0, 1]$, be the unique Ricci-flat Kähler-Einstein metric. The convergence of $\omega_s$ is studied in [95], and it is shown that $\omega_s$ converges to $f^*\omega$ in the current sense, where $\omega$ is a Kähler metric on $Z$ such that the Ricci curvature of $\omega$ is the Weil-Petersson metric for smooth fibers, i.e. $\text{Ric}(\omega) = \omega_{WP}$ on $Z_0$. The convergence is improved to be smooth in [36] under the further assumption that general fibers are Abelian varieties. Moreover, we obtain that $(Z_0, \omega)$ can be local isometrically embedded in any Gromov-Hausdorff limit of $(X, \omega_s)$, and by using the HyperKähler rotation, we proved a version of SYZ for higher dimensional HyperKähler manifolds in [36]. If the base $Z$ has dimension one, [37] shows that $(X, \omega_s)$ converges to $(Z, \omega)$ in the Gromov-Hausdorff sense, which extends Gross-Wilson's result to all elliptically fibred K3 surfaces. Later, better regularities of the convergence are obtained in [42, 96] under more general assumptions.
All these progresses are either for the case of a fixed Calabi-Yau manifold, or that we can use the HyperKähler rotation to convert the question to that case. The collapsing version of SYZ conjecture is still widely open beyond the HyperKähler setting. In a recent preprint [107], an analogue of this conjecture is obtained for canonical polarized manifolds.

Appendix A. Finiteness theorem for polarized manifolds

by Valentino Tosatti, Yuguang Zhang

We show a finiteness theorem for polarized complex manifolds. There are many previous finiteness theorems about diffeomorphism types in Riemannian geometry. Cheeger’s finiteness theorem asserts that given constants $D$, $\nu$, and $\Lambda$, there are only finitely many $n$-dimensional compact differential manifold $X$ admitting Riemannian metric $g$ such that $\text{diam}_g(X) \leq D$, $\text{Vol}_g(X) \geq \nu$ and the sectional curvature $|\text{Sec}(g)| \leq \Lambda$. This theorem can be proved as a corollary of the Cheeger-Gromov convergence theorem (cf. [26, 72]), which shows that if $(X_k,g_k)$ is a family compact Riemannian manifolds with the above bounds, then a subsequence of $(X_k,g_k)$ converges to a $C^{1,\alpha}$-Riemannian manifold $Y$ in the $C^{1,\alpha}$-sense, and furthermore, $X_k$ is diffeomorphic to $Y$ for $k \gg 1$. In [3], Cheeger’s finiteness theorem is generalized to the case where the hypothesis on the sectional curvature bound is replaced by the weaker bounds of Ricci curvature $|\text{Ric}(g)| \leq \lambda$ and the $L^2$-norm of curvature $\|\text{Sec}(g)\|_{L^2} \leq \Lambda$. Furthermore, if $n = 4$ and $g$ is an Einstein metric, then the integral bound of curvature can be replaced by a bound for the Euler characteristic.

We call $(X,L)$ a polarized $n$-manifold, if $X$ is a compact complex manifold with an ample line bundle $L$. In [51], a finiteness theorem for polarized manifolds is obtained. More precisely, Theorem 3 of [51] asserts that for any two constants $V > 0$ and $\Lambda > 0$, there are finite many polynomials $P_1, \ldots, P_\ell$ such that if $(X,L)$ is a polarized $n$-manifold with $c_1(L)^n \leq V$ and $-c_1(X) \cdot c_1(L)^{n-1} \leq \Lambda$, then one $P_i$ is the Hilbert polynomial of $(X,L)$, i.e. $P_i(\nu) = \chi(X,L^\nu)$. Consequently, polarized $n$-manifolds with the above bounds have only finitely many possible deformation types and finitely many possible diffeomorphism types.

For any constants $\lambda > 0$ and $D > 0$, denote

$$\mathfrak{N}(n,\lambda,D) = \{(X,L) \mid \exists \omega \in c_1(L) \text{ with } \text{Ric}(\omega) \geq -\lambda \omega, \text{ diam}_\omega(X) \leq D\}.$$

Then

$$c_1(L)^n = n!\text{Vol}_\omega(X) \leq V = V(n,\lambda,D)$$

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by the Gromov-Bishop comparison theorem, and
\[ -c_1(X) \cdot c_1(L)^{n-1} = - \int_X \text{Ric}(\omega) \wedge \omega^{n-1} \leq n\lambda V. \]

The following proposition is a corollary of Theorem 3 in [51]. Here we give an analytic proof.

**Proposition A.1.** Polarized manifolds in \( \mathcal{M}(n, \lambda, D) \) have only finitely many possible Hilbert polynomials, and for any \( (X, L) \in \mathcal{M}(n, \lambda, D) \) we have
\[
|\chi(X, L^\nu)| \leq C(n, \lambda, D)\nu^n,
\]
for all \( \nu \geq 1 \), where \( C(n, \lambda, D) \) is a constant depending only on \( n, \lambda \) and \( D \). Furthermore, any \( (X, L) \in \mathcal{M}(n, \lambda, D) \) can be embedded in the same \( \mathbb{CP}^N \) with \( L^m \cong O_{\mathbb{CP}^N}(1)|_X \) for integers \( m = m(n, \lambda, D) > 0 \) and \( N = N(n, \lambda, D) > 0 \). As a consequence, manifolds in \( \mathcal{M}(n, \lambda, D) \) have only finitely many possible deformation types and finitely many possible diffeomorphism types.

**Proof.** Let \( (X, L) \in \mathcal{M}(n, \lambda, D) \), and \( \omega \in c_1(L) \) be a Kähler metric with \( \text{Ric}(\omega) \geq -\lambda \omega \), and \( \text{diam}_{\omega}(X) \leq D \). Fix a Hermitian metric \( h \) on \( L \) with curvature equal to \( \omega \). The Gromov-Bishop comparison theorem gives
\[ 1 \leq c_1(L)^n = n!\text{Vol}_{\omega}(X) \leq V = V(n, \lambda, D). \]

We would like to estimate \( h^{0,p}(L^\nu) = \dim H^{0,p}(X, L^\nu) \), \( 0 \leq p \leq n \), for \( \nu \geq 1 \). We denote by \( \langle \cdot, \cdot \rangle \) the pointwise inner product on \( \Omega^{0,p}(X, L^\nu) \) (smooth \( L^\nu \)-valued \( (0,p) \)-forms on \( X \)) induced by the metric \( h^\nu \) on \( L^\nu \) whose curvature is \( -\sqrt{-1}\nu \omega \), and by \( |\cdot| \) its corresponding norm. For any \( s \in \Omega^{0,p}(X, L^\nu) \) we have
\[ \Delta|s|^2 = g^{ij} \partial_i \partial_j |s|^2 = |\nabla s|^2 + |\overline{\nabla} s|^2 + \langle \Delta s, s \rangle + \langle s, \overline{\Delta} s \rangle, \]
where \( \Delta s = g^{ij} \overline{\nabla}_i \nabla_j s \) is the rough Laplacian and \( \overline{\Delta} s = g^{ij} \nabla_i \overline{\nabla}_j s \) its “conjugate”. Commuting covariant derivatives we get
\[ \overline{\Delta} s = \Delta s - \nu n s - \text{Ric}^\omega(s), \]
where if \( p \geq 1 \) and we write locally \( s = \sum_{i_1 \cdots i_p} s_{i_1 \cdots i_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \) with \( s_{i_1 \cdots i_p} \) local smooth sections of \( L^\nu \), then
\[ \text{Ric}^\omega(s) = \sum_{j=1}^p g^{k\bar{k}} R_{k\bar{k}i_1 \cdots i_p} s_{i_1 \cdots \hat{i_j} \cdots i_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}, \]
while if \( p = 0 \) we let \( \text{Ric}^\omega(s) = 0 \). This gives
\[ \Delta|s|^2 = |\nabla s|^2 + |\overline{\nabla} s|^2 + 2\text{Re}\langle \Delta s, s \rangle - \nu n|s|^2 - \langle s, \text{Ric}^\omega(s) \rangle. \]

Next, we apply the Bochner-Kodaira identity [54] Theorem 6.2, which for any \( s \in \Omega^{0,p}(X, L^\nu) \) gives
\[ \Delta_{\overline{\nabla}} s = -\Delta s + \nu s + \text{Ric}^\omega(s), \]
and so if we assume that $\Delta \partial s = 0$, we obtain
\[
\Delta |s|^2 = |\nabla s|^2 + |\nabla s|^2 + 2\langle \text{Ric}^\sharp(s), s \rangle + 2\nu|s|^2 - \nu n|s|^2 - \langle s, \text{Ric}^\sharp(s) \rangle
\]
\[
= |\nabla s|^2 + |\nabla s|^2 + \langle \text{Ric}^\sharp(s), s \rangle - \nu(n-2)|s|^2,
\]
noting that $\langle \text{Ric}^\sharp(s), s \rangle = \langle s, \text{Ric}^\sharp(s) \rangle$. Using that $\langle \text{Ric}^\sharp(s), s \rangle \geq -\lambda_p|s|^2$,
we finally obtain
\[
\Delta |s|^2 \geq -\nu(n-2) + \lambda_p|s|^2.
\]
A standard Moser iteration argument (see e.g. [7, Lemma 2.4]) applied to this differential inequality gives
\[
\sup_X |s|^2 \leq A(n, V, \lambda, D) \frac{\omega^n}{n!} = A \nu(n-2) + \lambda p^n ||s||_{L^2}^2,
\]
where $A$ depends only on the Sobolev constant of $\omega$ and on $n$. Thus $A = A(n, V, \lambda, D)$ by a result of Croke [19].

Now we use the arguments in Lemma 11 and Theorem 12 of the paper of Li [59]. By the Hodge Theorem, we have an isomorphism $H^{0,p}(X, L^\nu) \cong \mathcal{H}^{0,p}(X, L^\nu)$, the space of $\Delta_{\partial}$-harmonic forms in $\Omega^{0,p}(X, L^\nu)$. Let
\[
\rho = \sum |s_i|^2
\]
for an orthonormal basis $s_i$ of $\mathcal{H}^{0,p}(X, L^\nu)$. The function $\rho$ is easily seen to be independent of the choice of orthonormal basis. Let $x \in X$ such that
\[
\rho(x) = \sup_X \rho > 0.
\]
Then
\[
E_0 = \{ s \in \mathcal{H}^{0,p}(X, L^\nu) | s(x) = 0 \},
\]
is a proper linear subspace of $\mathcal{H}^{0,p}(X, L^\nu)$, with orthogonal complement $E_0^\perp$. We claim that $\dim E_0^\perp \leq \binom{n}{p}$. If $s_1, \ldots, s_r, r > \binom{n}{p}$, is an orthonormal basis of $E_0^\perp$, then there are $a_i, i = 1, \ldots, r$, such that $\sum a_i s_i(x) = 0$. Thus $\sum a_i s_i \in E_0$, which is a contradiction.

Let $s_1, \ldots, s_r \in \mathcal{H}^{0,p}(X, L^\nu)$ be an orthonormal basis of $E_0^\perp$, which we can complete to an orthonormal basis of $\mathcal{H}^{0,p}(X, L^\nu)$ with an orthonormal basis $s_{r+1}, \ldots, s_N$ of $E_0$. We have
\[
h^{0,p}(L^\nu) = \int_X \rho \frac{\omega^n}{n!} \leq V \sup_X \rho = V \sup_X \left( \sum_{i=1}^r |s_i|^2 \right)
\]
\[
\leq \binom{n}{p} V \sup_i ||s_i||_{L^\infty}^2
\]
\[
\leq \binom{n}{p} V A(n, V, \lambda, D).
\]
using (A.2), and thus for any $\nu \geq 1$ we have

$$|\chi(X, L^\nu)| = \left| \sum_p (-1)^p h^{0,p}(L^\nu) \right| \leq \sum_p \binom{n}{p} VA(\nu(n-2)+\lambda p)^n \leq C(n, \lambda, D) \nu^n,$$

thus proving (A.1). Since the Hilbert polynomial $P$ of $(X, L)$ is given by

$$P(\nu) = \chi(X, L^\nu) = \int_X e^{\nu c_1(L)} \text{Todd}_X = a_0 \nu^n + a_1 \nu^{n-1} + \cdots + a_n \in \mathbb{Z},$$

it follows that we have only finitely many possible values for $a_0, \cdots, a_n$ by taking sufficiently many values of $\nu$ and solving the linear equations. Hence $(X, L)$ has only finitely many possible Hilbert polynomials.

Now, by Matsusaka’s Big Theorem (cf. [63]), there is an $m_0 > 0$ depending only on $P$ such that for any $m \geq m_0$, $L^m$ is very ample, and $H^i(X, L^m) = \{0\}$, $i > 0$. By choosing a basis $\Sigma$ of $H^0(X, L^m)$, we have an embedding $\Phi_\Sigma : X \hookrightarrow \mathbb{CP}^N$ such that $L^m = \Phi_\Sigma^* \mathcal{O}_{\mathbb{CP}^N}(1)$. We regard $\Phi_\Sigma(X)$ as a point in the Hilbert scheme $\text{Hilb}_{\nu}^{P_m}$ parametrizing the subschemes of $\mathbb{CP}^N$ with Hilbert polynomial $P_m(\nu) = P(m \nu)$, where $N = h^0(X, L^m) - 1$. Finally, $\Phi_\Sigma(X)$ belongs to finitely many possible components of finitely many possible Hilbert schemes, and thus $\mathfrak{N}(n, \lambda, D)$ has only finitely many possible deformation and diffeomorphism types.

Note that for any polarized manifold $(X, L)$, the volume of $(X, \omega)$ as in the definition of $\mathfrak{N}(n, \lambda, D)$ is bounded below uniformly away from zero. We remark that a similar diffeomorphism finiteness result fails for the family of closed Riemannian manifolds $(M, g)$ of real dimension $m$ with $\text{Ric}(g) \geq -\lambda g, \text{diam}_g(X) \leq D, \text{vol}_g(X) \geq v > 0$. Indeed Perelman [71] constructed Riemannian metrics on $\#_k \mathbb{CP}^2$ for all $k \geq 1$, which have positive Ricci curvature, unit diameter and volume bounded uniformly away from zero.

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