On some non-conformal fractals

Michał Rams

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-950 Warszawa, Poland
E-mail: rams@impan.gov.pl

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Abstract
This paper presents a simple method of calculating the Hausdorff dimension for a class of non-conformal fractals.

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1. Introduction

An iterated function scheme acting on a complete metric space \(X\) is a finite family of contracting maps \(\mathcal{F} = \{f_k\}_{k=1}^n; f_k : X \to X\). As noted by Hutchinson [Hu], the related multimap

\[
\mathcal{F}(\cdot) = \bigcup_{k=1}^n f_k(\cdot)
\]

(acting on the space \(B(X)\) of nonempty compact subsets of \(X\), considered with the Hausdorff metric) is also a contraction. Hutchinson proved that if \(X\) is complete, so is \(B(X)\). Hence, by the Banach fixed point theorem, there exists a unique nonempty compact set \(\Lambda\) satisfying

\[
\Lambda = \mathcal{F}(\Lambda) = \lim_{n \to \infty} \mathcal{F}^n(A).
\]

The limit does not depend on the choice of \(A \in B(X)\). \(\Lambda\) is called the limit set of the iterated function scheme \(\mathcal{F}\).

By similar reasoning, if we have a finite number of iterated function schemes \(\{\mathcal{F}_i\}_{i=1}^m\) acting on \(X\) and apply them in any order, the pointwise limit

\[
\Lambda_\omega = \lim_{n \to \infty} \mathcal{F}_{\omega_1} \circ \ldots \circ \mathcal{F}_{\omega_n}(A)
\]

exists for all \(\omega \in \Omega = \{1, \ldots, m\}^N\) and does not depend on \(A \in B(X)\).

The question we want to answer (motivated by [Lu], see also [N, GL, GL2] and the incoming paper [Re]) is when the iterated function schemes \(\mathcal{F}_i\) are of some special class (for which we can calculate the Hausdorff dimension of the limit set of any deterministic
iterated function scheme from this class) and the sequence \( \omega \) is chosen, what will be the value of the Hausdorff dimension of \( \Lambda_\omega \)?

We will present a simple method of dealing with this question, working for Lalley–Gatzouras maps [LG], Barański maps [B] and higher dimensional affine-invariant sets of Kenyon and Peres [KP]. The only assumption about \( \omega \) we need is that each symbol \( i \) has a limit frequency of appearance. For simplicity, we will only present the proof for an example: a class of iterated function schemes considered by Lalley and Gatzouras.

We refer the reader interested in other non-conformal random iterated constructions to [F, GL2] and references therein.

2. Lalley–Gatzouras schemes

The Lalley–Gatzouras scheme \( \mathcal{F} \) is a self-affine IFS given by a family of maps \( f_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2 \), \((i, j) \in A\):

\[
f_{i,j}(x, y) = (a_{ij} x + c_{ij}, b_{ij} y + d_{ij}),
\]

where the alphabet \( A \) of allowed symbols is \( 1 \leq i \leq m_1, 1 \leq j \leq m_2(i) \). We will assume that for all \((i, j) \in A\), \( b_{ij} \geq a_{ij} \) (that is, the contraction in the horizontal direction is not weaker than the contraction in the vertical direction for all maps).

We will also assume that for all \((i, j) \in A\), \( 0 < a_{ij} < 1 \) and \( 0 \leq c_{ij} < \ldots < c_{im_2(i)} \leq 1 - a_{im_2(i)} \), \( c_{ij+1} \geq a_{ij} + c_{ij} \) and that \( 0 < b_{ij} < 1 \) and \( 0 \leq d_{ij} < \ldots < d_{im_1} \leq 1 - b_{im_1} \), \( d_{ij+1} \geq b_{ij} + d_{ij} \). We will say that the \textit{separation condition} holds if we actually have \( c_{ij+1} > a_{ij} + c_{ij} \) and \( d_{ij+1} > b_{ij} + d_{ij} \).

The main result of [LG] is the formula for the Hausdorff dimension of the limit set \( \Lambda_1 \):

\[
dim_H(\Lambda) = \max \left\{ \sum_{i} \sum_{j} p_{ij} \log p_{ij} + \sum_{i} q_{i} \log q_{i} \left( \frac{1}{\sum_{i} q_{i} \log b_{i}} - \frac{1}{\sum_{i} \sum_{j} p_{ij} \log a_{ij}} \right) \right\},
\]

where \( \{p_{ij}\} \) is a probability distribution on \( A \), \( q_{i} = \sum_{j} p_{ij} \) and the maximum is over all possible \( \{p_{ij}\} \).

Consider now a family of Lalley–Gatzouras schemes \( \{\mathcal{F}_{k}\}_{k=1}^m \) with alphabets \( A_k \) and maps \( f_{i,j}^{(k)} \). As mentioned above, we can apply them in any order \( F_{\omega_1} \circ F_{\omega_2} \circ \ldots \), \( \omega = \omega_1 \omega_2 \ldots \in \Omega = \{1, \ldots, m\}^\mathbb{N} \) and obtain some limit set \( \Lambda_\omega \). We will assume that the limits

\[
P_k = \lim_{n \to \infty} \frac{1}{n} \sum_{1 \leq l \leq n; \omega_l = k} \epsilon
\]

exist and are positive. We will ask what is the value of \( \dim_H(\Lambda_\omega) \).

Before formulating the answer, let us note that any finite product \( F_{\omega_1} \circ \ldots \circ F_{\omega_n} \) is again a Lalley–Gatzouras scheme. It follows that we can calculate the Hausdorff dimension of \( \Lambda_\omega \) for any periodic sequence \( \omega \). Given a \textit{rational} probabilistic vector \( Q = (Q_1, \ldots, Q_m) \), we can choose a periodic sequence \( \omega(Q) \) in which the frequency of symbol \( k \) is \( Q_k \). Let us write

\[
\mathcal{L}(Q) = \dim_H \Lambda_{\omega(Q)}.
\]

Our main result is as follows.

**Theorem 2.1.** The function \( \mathcal{L}(Q) \) is well defined, does not depend on the choice of \( \omega(Q) \).

We can extend it by continuity to the whole simplex of probabilistic vectors (we will keep the notation \( \mathcal{L}(Q) \) for the extended function). We have

\[
\dim_H(\Lambda_\omega) = \mathcal{L}(P).
\]
3. Proof of theorem 2.1

Let us start by presenting a more detailed description of $\Lambda_\omega$ (compare [Hu]). Let $A_\omega = A_{\omega_1} \times A_{\omega_2} \times \ldots$. We define a projection $\pi_\omega : A_\omega \to \Lambda_\omega$ by the formula

$$\pi_\omega((i_1, j_1), (i_2, j_2), \ldots) = \lim_{n \to \infty} f_{i_1, j_1}^{(\omega)} \circ \ldots \circ f_{i_n, j_n}^{(\omega)}(0, 0),$$

$(i_k, j_k) \in A_k$. We get

$$\Lambda_\omega = \pi_\omega(A_\omega).$$

Because of the nonconformality of the system, the most natural class of subsets of $A_\omega$ to study are not cylinders but rectangles (in particular, approximate squares). The rectangle is defined as follows: given a sequence $(i, j) \in A_\omega$ and two natural numbers $n_1 \leq n_2$ we define $R_{n_1, n_2}(i, j) = \{(i', j') \in A_\omega; i'_k = i_k \forall k \leq n_2, j'_k = j_k \forall k \leq n_1\}$.

We will call $d_1(R_{n_1, n_2}(i, j)) = \prod_{k=1}^{n_1} a_{i_k, j_k}$ the width and

$$d_2(R_{n_1, n_2}(i, j)) = \prod_{k=1}^{n_2} b_{i_k, j_k}$$

the height of the rectangle $R_{n_1, n_2}(i, j)$. Indeed, the projection of a rectangle under $\pi_\omega$ is the intersection of $\Lambda_\omega$ with a geometric rectangle of the same width and of the same height. The rectangle of approximately (up to a constant) equal width and height is called an approximate square.

Our main step is the following proposition.

**Proposition 3.1.** For $Q$ a rational probabilistic vector sufficiently close to $P$ and for any choice of $\omega(Q)$, there exists $K > 0$ and for every $d > 0$ there exists $\varepsilon > 0$ with $\varepsilon(d) \to 0$ as $d \to 0$ such that we can construct a bijection $\tau : A_{\omega(Q)} \to A_\omega$ with the following properties. Let $R = R_{\omega(Q)}(i, j)$ be an approximate square in $A_{\omega(Q)}$ of width $d$. Then $\tau(R)$ contains an approximate square in $A_\omega$ of width at least $d_1^1 + K\varepsilon$ and is contained in an approximate square in $A_\omega$ of width at most $d_1^1 - K\varepsilon$, where $\varepsilon = \max |P_k - Q_k|$.

**Proof.** We will need the following simple statement (a reformulation of (2.1)):

**Lemma 3.2.** For every $n$ there exists $\varepsilon(n)$ such that for each $k$ the $n$-th appearance of symbol $k$ in the sequence $\omega$ takes place between positions $n/P_k(1 - \varepsilon(n/P_k))$ and $n/P_k(1 + \varepsilon(n/P_k))$. Moreover, $\varepsilon(n)$ goes monotonically to 0 as $n$ goes to $\infty$.

Consider now the pair of sequences: $\omega$, the sequence we work with, and $\omega(Q)$, a periodic sequence with frequencies $Q$. We will assume that $Q$ is $\delta$-close to $P$ and that both probabilistic vectors are positive. Obviously, in the sequence $\omega(Q)$ the $n$th appearance of symbol $k$ is at position $n/Q_k$, give or take a constant.

We will define $\chi_\omega,\omega(Q)$ as a permutation of $\mathbb{N}$ in the following way: if $l_1$ is the place of $n$th appearance of symbol $k$ in the sequence $\omega$ and $l_2$ is the place of $n$th appearance of symbol $k$ in the sequence $\omega(Q)$, we set $\chi_\omega,\omega(Q)(l_1) = l_2$. We can then construct a bijection $\tau : A_{\omega(Q)} \to A_\omega$ as

$$\tau((i_1, j_1), (i_2, j_2), \ldots) = (i_{\chi_\omega,\omega(Q)(1)}, j_{\chi_\omega,\omega(Q)(1)}), (i_{\chi_\omega,\omega(Q)(2)}, j_{\chi_\omega,\omega(Q)(2)}), \ldots$$

Denote
\[ D_1 = \chi_{\omega, \omega}(\{1, \ldots, n_1\}) \]
and
\[ D_2 = \chi_{\omega, \omega}(\{n_1 + 1, \ldots, n_2\}) \]

We recall that the rectangle \( R \) is defined as the set of sequences \( (i', j') \in A_\omega(Q) \) for which we fix the first \( n_1 \) \((i'_k, j'_k)\) and the following \( n_2 - n_1 \) \(i'_k\). Hence, the set \( \tau(R) \) is the set of sequences \( (i', j') \in A_\omega \) for which we fix \( (i'_k, j'_k) \) for \( k \in D_1 \) and we fix \( i'_k \) for \( k \in D_2 \).

Denote
\[ r_1 = \inf(N \setminus D_1) - 1, \]
\[ r_2 = \inf(N \setminus (D_1 \cup D_2)) - 1, \]
\[ s_1 = \sup(D_1), \]
\[ s_2 = \sup(D_1 \cup D_2). \]

We have
\[ R_{r_1, r_2}^{(\omega)}(\tau(i, j)) \subset \tau(R) \subset R_{s_1, s_2}^{(\omega)}(\tau(i, j)). \]

Assume \( \delta \) is much smaller than any \( P_k \). By lemma 3.2,
\[ r_1 \geq n_1(1 - K_0 \varepsilon(n_1) - K_0 \delta), \]
\[ r_2 \geq n_2(1 - K_0 \varepsilon(n_2) - K_0 \delta), \]
\[ s_1 \leq n_1(1 + K_0 \varepsilon(n_1) + K_0 \delta), \]
\[ s_2 \leq n_2(1 + K_0 \varepsilon(n_2) + K_0 \delta) \]
for some \( K_0 > 0 \) depending only on the iterated schemes.

Consider the width of \( R_{r_1, r_2}^{(\omega)}(\tau(i, j)) \) versus the width of \( R \). The latter is a product of \( n_1 \) numbers \( a_{ij}^{(k)} \), the former it the subproduct of \( r_1 \) of those numbers. As all \( a_{ij}^{(k)} \) are uniformly bounded away from 0 and 1,
\[ d_1(R_{r_1, r_2}^{(\omega)}(\tau(i, j))) \leq d^{1 - K_0 \varepsilon(n_1) - K_0 \delta} \]
for some uniformly chosen \( K \), depending only on the iterated schemes. Similar reasoning proves
\[ d_2(R_{r_1, r_2}^{(\omega)}(\tau(i, j))) \leq d^{1 - K_0 \varepsilon(n_2) - K_0 \delta}. \]

Consider now the width of \( R_{s_1, s_2}^{(\omega)}(\tau(i, j)) \) versus the width of \( R \). The former a product of \( s_1 \) numbers \( a_{ij}^{(k)} \), the latter is the subproduct of \( n_1 \) of those numbers, the same reasoning as before gives us
\[ d_1(R_{s_1, s_2}^{(\omega)}(\tau(i, j))) \geq d^{1 + K_0 \varepsilon(n_1) + K_0 \delta}, \]
\[ d_2(R_{s_1, s_2}^{(\omega)}(\tau(i, j))) \geq d^{1 + K_0 \varepsilon(n_2) + K_0 \delta}. \]

The rectangles \( R_{r_1, r_2}^{(\omega)}(\tau(i, j)) \) and \( R_{s_1, s_2}^{(\omega)}(\tau(i, j)) \) are not necessarily approximate squares, but we can easily replace the former by some slightly larger rectangle which is an approximate square and we can replace the latter by some slightly smaller rectangle which is an approximate square. We are done.

**Remark.** We can introduce a metric on \( A_\omega \), defining the distance between two points as the sum of width and height of the smallest rectangle containing them both. This metric is natural
because if the maps satisfy the separation condition, \( \pi_\omega \) is bi-Lipschitz (without separation condition it will only be a Lipschitz projection). In this metric, the maps \( \tau, \tau^{-1} \) are Hölder continuous with every exponent smaller than 1 (if \( \delta = 0 \)) or with exponent \( 1 - K \delta \) (if \( \delta \) is positive but small).

This proposition basically ends the proof of theorem 2.1. By proposition 3.3 and lemma 5.2 in [LG], for any Lalley–Gatzouras scheme there exists a probabilistic measure \( \mu \) supported on \( A^\omega \) such that

(i) for a \( \mu \)-typical point \((i,j)\) and the decreasing sequence of all approximate squares \( R_k = R_{n_1(k),n_2(k)}(i,j) \),

\[
\frac{\log \mu(R_k)}{\log d_1(R_k)} \to \dim(\Lambda),
\]

(ii) for every point \( x \in A^\omega \) there exists a decreasing sequence of approximate squares \( R_k = R_{n_1(k),n_2(k)}(i,j) \) for which

\[
\frac{\log \mu(R_k)}{\log d_1(R_k)} \to \dim(\Lambda).
\]

We can define such measure \( \mu_Q \) supported on \( A_{\omega(Q)} \) for any rational \( Q \) (because this is again a Lalley–Gatzouras scheme). We can then transport this measure to \( A_\omega \) by the map \( \tau \). We obtain a measure \( \nu_Q \) such that

(i) for a \( \nu_Q \)-typical point \((i,j)\) and the decreasing sequence of all approximate squares \( R_k = R_{n_1(k),n_2(k)}(i,j) \),

\[
\liminf \frac{\log \nu_Q(R_k)}{\log d_1(R_k)} \geq \mathcal{L}(Q)(1 - K \delta),
\]

(ii) for every point \( x \in A_\omega \) there exists a decreasing sequence of approximate squares \( R_k = R_{n_1(k),n_2(k)}(i,j) \) for which

\[
\limsup \frac{\log \nu_Q(R_k)}{\log d_1(R_k)} \leq \mathcal{L}(Q)(1 + K \delta).
\]

It implies that

\[
\mathcal{L}(Q)(1 - K \delta) \leq \dim_H \Lambda_\omega \leq \mathcal{L}(Q)(1 + K \delta),
\]

the proof is as in [LG].

This result has immediate applications for random systems, obtained by choosing \( \omega \) randomly with respect to some Bernoulli measure on \( \Omega \).

On the other hand, this method is not going to work for stochastically selfsimilar systems considered in [F] or [Gal]. For such systems we would not have a single sequence \( \omega \) but instead \( \omega \) would depend on the point in the fractal. While we would still be able to define \( \tau \) almost everywhere, the sequences \( \omega(x) \) at different points \( x \in \Lambda \) would not all satisfy lemma 3.2, and hence \( \tau \) would not everywhere have nice Hölder properties.

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