Representations of reductive normal algebraic monoids

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Dedicated to Lex Renner and Mohan Putcha

Abstract The rational representation theory of a reductive normal algebraic monoid (with one-dimensional center) forms a highest weight category, in the sense of Cline, Parshall, and Scott. This is a fundamental fact about the representation theory of reductive normal algebraic monoids. We survey how this result was obtained, and treat some natural examples coming from classical groups.

Introduction

Let $M$ be an affine algebraic monoid over an algebraically closed field $K$. See [10] [13] [12] for general surveys and background on algebraic monoids. Assuming that $M$ is reductive (its group $G$ of units is a reductive group) what can be said about the representation theory of $M$ over $K$?

Recall that any affine algebraic group is smooth and hence normal (as a variety). The normality of the algebraic group plays a significant role in its representation theory, for instance in the proof of Chevalley’s theorem classifying the irreducible representations. Thus it seems reasonable in trying to extend (rational) representation theory from reductive groups to reductive monoids to look first at the case when the monoid $M$ is normal. Furthermore, even in cases where a given reductive algebraic monoid is not normal, one may always pass to its normalization, which should be closely related to the original object.

Renner [11] has obtained a classification theorem for reductive normal algebraic monoids under the additional assumptions that the center $Z(M)$ is 1-dimensional and that $M$ has a zero element. Renner’s classification theorem depends on an alge-
braic monoid version of Chevalley’s big cell, which holds for any reductive affine algebraic monoid (with no assumptions about its center or a zero). As a corollary of its construction, Renner derives a very useful extension principle \[11\] (4.5) which is a key ingredient in the analysis.

1 Reductive normal algebraic monoids

Let \( M \) be a \textit{linear algebraic monoid} over an algebraically closed field \( K \). In other words, \( M \) is a monoid (with unit element \( 1 \in M \)) which is also an affine algebraic variety over \( K \), such that the multiplication map \( \mu : M \times M \to M \) is a morphism of varieties. We assume that \( M \) is irreducible as a variety. Hence the unit group \( G = M^\times \) (the subgroup of invertible elements of \( M \)) is a connected linear algebraic group over \( K \) and \( G \) is Zariski dense in \( M \).

1.1. Associated with \( M \) is its affine coordinate algebra \( K[M] \), the ring of regular functions on \( M \). There exist \( K \)-algebra homomorphisms
\[
\Delta : K[M] \to K[M] \otimes_K K[M], \quad \varepsilon : K[M] \to K
\]
called comultiplication and counit, respectively. For a given \( f \in K[M] \), we have \( \varepsilon(f) = f(1) \); furthermore, if \( \Delta(f) = \sum_{i=1}^r f_i \otimes f'_i \), then \( f(m_1m_2) = \sum_{i=1}^r f_i(m_1)f'_i(m_2) \), for all \( m_1, m_2 \in M \). The maps \( \Delta, \varepsilon \) make \( K[M] \) into a bialgebra over \( K \). This means that they satisfy the bialgebra axioms:
\[
(1) \quad (\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta,
\]
\[
(2) \quad (\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta
\]
where \( \varphi \otimes \varphi' \) denotes the map \( a \otimes a' \mapsto \varphi(a)\varphi'(a') \).

We note that the commutative bialgebra \( (K[M], \Delta, \varepsilon) \) determines \( M \), as the set \( \text{Hom}_{K^{-\text{alg}}}(K[M], K) \) of \( K \)-algebra homomorphisms from \( K[M] \) into \( K \). The multiplication on this set is defined by \( \varphi \cdot \varphi' = (\varphi \otimes \varphi') \circ \Delta \) and the identity element is just the counit \( \varepsilon \). One easily verifies that this reconstructs \( M \) from its coordinate bialgebra \( K[M] \).

More generally, given any commutative bialgebra \( (A, \Delta, \varepsilon) \) over \( K \), one defines on the set \( M(A) = \text{Hom}_{K^{-\text{alg}}}(A, K) \) an algebraic monoid structure with multiplication \( \mu(\varphi, \varphi') = \varphi \cdot \varphi' = (\varphi \otimes \varphi') \circ \Delta \). This gives a functor
\[
\{\text{commutative bialgebras over } K\} \to \{\text{algebraic monoids over } K\}
\]
which is quasi-inverse to the functor \( M \mapsto K[M] \). Thus the two categories are antiequivalent.

1.2. Since \( G \) is dense in \( M \), the restriction map \( K[M] \to K[G] \) (given by \( f \mapsto f|_G \)) is injective, so we may identify \( K[M] \) with a subbialgebra of the Hopf algebra \( K[G] \) of regular functions on \( G \).
1.3. Assume that $M$ is reductive; i.e., its unit group $G = M^\times$ is reductive as an algebraic group. Fix a maximal torus $T$ in $G$. (Up to conjugation $T$ is unique.) Let $X(T) = \text{Hom}(T,K^\times)$ be the character group of $T$; this is the abelian group of morphisms from $T$ into the multiplicative group $K^\times$ of $K$. Let $X\vee(T) = \text{Hom}(K^\times,T)$ be the abelian group of cocharacters into $T$. Let $R \subset X(T)$ be the root system for the pair $(G,T)$ and $R^\vee \subset X\vee(T)$ the system of coroots. According to the classification of reductive algebraic groups, the reductive group $G$ is uniquely determined up to isomorphism by its root datum $(X(T),R,X\vee(T),R^\vee)$.

1.4. We now add the assumption that $M$ is normal as a variety. Let $D = \overline{T}$ be the Zariski closure of $T$ in $M$. Then $T \subset D$ is an affine torus embedding. Let $X(D) = \text{Hom}(D,K)$ be the monoid of algebraic monoid homomorphisms from $D$ into $K$. The restriction $\chi_T$ of any $\chi \in X(D)$ is an element of $X(T)$, so restriction defines a homomorphism $X(D) \rightarrow X(T)$. Since $T$ is dense in $D$, this map is injective, and thus we may identify $X(D)$ with a submonoid of $X(T)$. Renner has shown that the additional datum $X(D)$ is all that is needed to determine $M$ up to isomorphism, under the additional hypotheses (probably unnecessary) that the center

$$Z(M) = \{ z \in M : zm = mz, \text{ for all } m \in M \}$$

is 1-dimensional and that $M$ has a zero element. (One can always add a zero formally, so the last requirement is insubstantial.)

It turns out that the set $X(D)$ also determines the rational representation theory of the reductive normal algebraic monoid $M$, in a sense made precise in Section 3.

1.5. Note that it is easy to construct reductive algebraic monoids. Start with a rational representation $\rho : G \rightarrow \text{End}_K(V)$ of a reductive group $G$ in some vector space $V$ with $\dim_K V = n < \infty$. The image $\rho(G)$ is a reductive affine algebraic subgroup of $\text{End}_K(V) \cong M_n(K)$, the monoid of all $n \times n$ matrices under ordinary matrix multiplication. Desiring our monoid to have a center of at least dimension 1, we include the scalars $K^\times$ as scalar matrices, defining $G_0$ to be the subgroup of $\text{End}_K(V)$ generated by $\rho(G)$ and $K^\times$. Now we set $M = \overline{G_0}$, the Zariski closure of $G_0$ in $\text{End}_K(V) \cong M_n(K)$. This is a reductive algebraic monoid.

For example, if $G = \text{SL}_n(K)$ and $V$ is its natural representation then $G_0 \cong \text{GL}_n(K)$ and $M = M_n(K)$. (In general, to obtain a monoid $M$ closely related to the starting group $G$, one should pick $V$ to be a faithful representation.) There is no guarantee that this procedure will always produce a normal reductive monoid, but if not then one can always pass to its normalization.

2 Examples: symplectic and orthogonal monoids

The paper [4] considered some more substantial examples of reductive algebraic monoids coming from other classical groups. Let $V = K^n$ with its standard basis $\{e_1, \ldots, e_n\}$. Put $i' = n + 1 - i$ for any $i = 1, \ldots, n$. 


2.1. The orthogonal monoid. Assume the characteristic of $K$ is not 2. Define a symmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $V$ by putting

(a) $\langle e_i, e_j \rangle = \delta_{i,j}^f$ for any $1 \leq i, j \leq n$.

Here $\delta$ is Kronecker’s delta function. Let $J$ be the matrix of $\langle \cdot, \cdot \rangle$ with respect to the basis $\{e_1, \ldots, e_n\}$. Then the orthogonal group $O(V)$ is the group of linear operators $f \in \text{End}_K(V)$ preserving the form:

(b) $O(V) = \{f \in \text{End}_K(V) : \langle f(v), f(v') \rangle = \langle v, v' \rangle, \text{ all } v, v' \in V\}$.

Let $A$ be the matrix of $f$ with respect to the basis $\{e_1, \ldots, e_n\}$. Then we may identify $O(V)$ with the matrix group

(c) $O_n(K) = \{A \in M_n(K) : A^TJA = J\}$.

This is contained in the larger group $GO_n(K)$, the group of orthogonal similitudes (see e.g., [9]) defined by

(d) $GO_n(K) = \{A \in M_n(K) : A^TJA = cJ, \text{ some } c \in K^\times\}$.

Note that $GO_n(K)$ is generated by $O_n(K)$ and $K^\times$. We define the orthogonal monoid $OM_n(K)$ to be

(e) $OM_n(K) = \overline{GO_n(K)}$,

the Zariski closure in $M_n(K)$. These monoids (for $n$ odd) were studied by Grigor’ev [8]. In [4] the following result was proved.

2.2 Proposition. The orthogonal monoid $OM_n(K)$ is the set of all $A \in M_n(K)$ such that $A^TJA = cJ = AJA^T$, for some $c \in K$.

Note that the scalar $c \in K$ in the above is allowed to be zero, and the “extra” condition $cJ = AJA^T$ is necessary. If $c \neq 0$ then it is easy to see that $A^TJA = cJ$ is equivalent to $cJ = AJA^T$, but when $c = 0$ this equivalence fails. The equivalence means that we could just as well have defined $GO_n(K)$ by

$$GO_n(K) = \{A \in M_n(K) : A^TJA = cJ = AJA^T, \text{ some } c \in K^\times\}$$

which is perhaps more suggestive for the description of $OM_n(K)$ given above.

2.3. The symplectic monoid. Assume that $n = \dim_K V$ is even, say $n = 2m$. Define an antisymmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $V$ by putting

(a) $\langle e_i, e_j \rangle = \epsilon_i \delta_{i,j}^f$ for any $1 \leq i, j \leq n$.

where $\epsilon_i$ is 1 if $i \leq m$ and $-1$ otherwise. Let $J$ be the matrix of $\langle \cdot, \cdot \rangle$ with respect to the basis $\{e_1, \ldots, e_n\}$. Then the symplectic group $SP(V)$ is the group of linear operators $f \in \text{End}_K(V)$ preserving the bilinear form:
(b) \( \text{Sp}(V) = \{ f \in \text{End}_K(V) : \langle f(v), f(v') \rangle = \langle v, v' \rangle, \text{ all } v, v' \in V \} \).

Let \( A \) be the matrix of \( f \) with respect to the basis \( \{ e_1, \ldots, e_n \} \). Then we may identify \( \text{Sp}(V) \) with the matrix group

(c) \( \text{Sp}_n(K) = \{ A \in M_n(K) : A^TJA = J \} \).

This is contained in the larger group \( \text{GSp}_n(K) \), the group of symplectic similitudes, defined by

(d) \( \text{GSp}_n(K) = \{ A \in M_n(K) : A^TJA = cJ, \text{ some } c \in K^\times \} \).

Note that \( \text{GSp}_n(K) \) is generated by \( \text{Sp}_n(K) \) and \( K^\times \). As in the orthogonal case, we could just as well have defined \( \text{GSp}_n(K) \) by

\[ \text{GSp}_n(K) = \{ A \in M_n(K) : A^TJA = cJ = AJA^T, \text{ some } c \in K^\times \} \]

thanks to the equivalence of the conditions \( A^TJA = cJ \) and \( cJ = AJA^T \) in case \( c \neq 0 \).

We define the symplectic monoid \( \text{SpM}_n(K) \) to be

(e) \( \text{SpM}_n(K) = \overline{\text{GSp}_n(K)} \),

the Zariski closure in \( M_n(K) \). In [4] the following was proved.

2.4 Proposition. The symplectic monoid \( \text{SpM}_n(K) \) is the set of all \( A \in M_n(K) \) such that \( A^TJA = cJ = AJA^T \), for some \( c \in K \).

Note that the scalar \( c \in K \) in the above is allowed to be zero, and the condition \( cJ = AJA^T \) is necessary, just as it was in the orthogonal case.

2.5. Sketch of proof. I want to briefly sketch the ideas involved in the proof of Propositions 2.2 and 2.4. Full details are available in [4]. The method of proof works for any infinite field \( K \) (except that characteristic 2 is excluded in the orthogonal case). We continue to assume that \( n = 2m \) is even in the symplectic case.

Let \( G = \text{GO}_n(K) \) or \( \text{GSp}_n(K) \) and let \( M = \text{OM}_n(K) \) or \( \text{SpM}_n(K) \), respectively. Let \( T \) be the maximal torus of diagonal elements of \( G \). Then we have inclusions

(a) \( T \subset G \subset M \)

and we desire to prove that the latter inclusion is actually an equality. To accomplish this, we consider the action of \( G \times G \) on \( M \) given by \( (g,h) \cdot m = gmh^{-1} \). Suppose that we can show that every \( G \times G \) orbit is of the form \( GaG \), for some \( a \in T \). Then it follows that

(b) \( M = \bigcup_{a \in T} GaG \subset G \)

and this gives the opposite inclusion that proves Propositions 2.2 and 2.4. In fact, as it turned out, the distinct \( a \in T \) in the above decomposition can be taken to be certain idempotents in \( T \).
This suggests the program that was carried out in \[4\], which in the end leads to additional structural information on \(M\):

(i) Classify all idempotents in \(T\).

(ii) Obtain an explicit description of \(T\).

(iii) Determine the \(G \times G\) orbits in \(M\).

Part (i) is easy. For part (ii) one exploits the action of \(T\) on \(T\) by left multiplication and determines the orbits of that action. Part (iii) involves developing orthogonal and symplectic versions of classical Gaussian elimination.

2.6. The normality question. It is clear from the equalities in Propositions 2.2 and 2.4 that \(\text{OM}_n(K)\) and \(\text{SpM}_n(K)\) both have one-dimensional centers and contain zero. What is not clear, and not addressed in \[4\], is whether or not they are normal as algebraic varieties.

This question was recently settled in \[6\], where it is shown that \(\text{SpM}_n(K)\) is always normal, while \(\text{OM}_n(K)\) is normal only in case \(n\) is even. More precisely, it is shown in \[6\] that when \(n = 2m\) is even, \(\text{OM}_n^+(K)\) and \(\text{OM}_n^-(K)\) are both normal varieties. Here

(a) \(\text{OM}_n(K) = \text{OM}_n^+(K) \cup \text{OM}_n^-(K)\)

is the decomposition into irreducible components, where \(\text{OM}_n^+(K)\) is the component containing the unit element 1.

3 Representation theory

From now on we assume that \(M\) is an arbitrary reductive normal algebraic monoid, with unit group \(G = M^\times\). We wish to describe some results of \[5\]. The main result is that the category of rational \(M\)-modules is a highest weight category in the sense of Cline–Parshall–Scott \[1\].

3.1. We work with a fixed maximal torus \(T \subset G\), and set \(D = T\). We assume that \(\dim Z(M) = 1\) and \(0 \in M\). Recall that restriction induces an injection \(X(D) \to X(T)\), so we may identify \(X(D)\) with a submonoid of \(X(T)\). We fix a Borel subgroup \(B\) with \(T \subset B \subset G\) and let the set \(R^-\) of negative roots be defined by the pair \((B, T)\). We set \(R^+ = -(R^-)\), the set of positive roots. We have \(R = R^+ \cup R^-\). Let

\[X(T)^+ = \{ \lambda \in X(T) : (\alpha^\vee, \lambda) \geq 0, \text{ for all } \alpha \in R^+ \}\]

be the usual set of dominant weights. We define

\[X(D)^+ = X(T)^+ \cap X(D)\]

3.2. By a (left) rational \(M\)-module we mean a linear action \(M \times V \to V\) such that the coefficient functions \(M \to K\) of the action are all in \(K[M]\). This is the same as
having a (right) $K[M]$-comodule structure on $V$. This means that we have a comodule structure map

$\Delta_V : V \to V \otimes_K K[M].$

Since $K[M] \subset K[G]$ the map $\Delta_V$ induces a corresponding map $V \to V \otimes_K K[G]$ making $V$ into a $K[G]$-comodule; i.e., a rational $G$-module. Thus, rational $M$-modules may also be regarded as rational $G$-modules. Any rational $M$-module is semisimple when regarded as a rational $D$-module, with corresponding weight space decomposition

$(b) \quad V = \bigoplus_{\lambda \in X(D)} V_\lambda$

where $V_\lambda = \{ v \in V : d \cdot v = \lambda(d)v, \text{ all } d \in D \}.$

Recall that any rational $G$-module $V$ is semisimple when regarded as a rational $T$-module, with corresponding weight space decomposition

$(c) \quad V = \bigoplus_{\lambda \in X(T)} V_\lambda$

where $V_\lambda = \{ v \in V : t \cdot v = \lambda(t)v, \text{ all } t \in T \}.$ If $V$ is a rational $M$-module then the weight spaces relative to $T$ are the same as the weight spaces relative to $D$. So the weights of a rational $M$-module all belong to $X(D)$. Conversely, we have the following.

**3.3 Lemma.** If $V$ is a rational $G$-module such that

$\{ \lambda \in X(T) : V_\lambda \neq 0 \} \subset X(D)$

then $V$ extends uniquely to a rational $M$-module.

This is proved as an application of Renner’s extension principle, which is a version of Chevalley’s big cell construction for algebraic monoids.

**3.4 Remark.** A special case of the lemma (for the case $M = M_n(K)$) can be found in [7].

**3.5.** Next one needs a notion of induction for algebraic monoids, i.e., a left adjoint to restriction. The usual definition of induced module for algebraic groups does not work for algebraic monoids. Instead, we use the following definition. Let $V$ be a rational $L$-module where $L$ is an algebraic submonoid of $M$. We define $\text{ind}_L^M V$ by

$\text{ind}_L^M V = \{ f \in \text{Hom}(M, V) : f(lm) = l \cdot f(m), \text{ all } l \in L, m \in M \}.$

This is viewed as a rational $M$-module via right translation. One can check that in case $L, M$ are algebraic groups then this is isomorphic to the usual induced module.

It is well known that the Borel subgroup $B$ has the decomposition $B = TU$, where $U$ is its unipotent radical. Given a character $\lambda \in X(T)$ one regards $K$ as a rational $T$-module via $\lambda$; this is often denoted by $K_\lambda$. One extends $K_\lambda$ to a rational $B$-module by letting $U$ act trivially. Similarly, we have the decomposition $\overline{B} = DU$. If $\lambda \in X(D)$
then we have $K_\lambda$ as above, and again we may regard this as a rational $\overline{B}$-module by letting $U$ act trivially.

Now we can formulate the classification of simple rational $M$-modules.

**3.6 Theorem.** Let $M$ be a reductive normal algebraic monoid. Let $\lambda \in X(D)$ and let $K_\lambda$ be the rational $\overline{B}$-module as above. Then

(a) $\text{ind}^M_B K_\lambda \neq 0$ if and only if $\lambda \in X(D)^+$.

(b) If $\text{ind}^M_B K_\lambda \neq 0$ then its socle is a simple rational $M$-module (denoted by $L(\lambda)$).

(c) The set of $L(\lambda)$ with $\lambda \in X(D)^+$ gives a complete set of isomorphism classes of simple rational $M$-modules.

Let $\lambda \in X(T)$. Let $\nabla(\lambda) = \text{ind}^G_B K_\lambda$. It is well known that $\nabla(\lambda) \neq 0$ if and only if $\lambda \in X(T)^+$. The following is a key fact.

**3.7 Lemma.** If $\lambda \in X(D)^+$ then $\text{ind}^M_B K_\lambda = \nabla(\lambda) = \text{ind}^G_B K_\lambda$.

**3.8.** Now we consider truncation. Let $\pi \subset X(T)^+$. Given a rational $G$-module $V$, let $\mathcal{O}_\pi V$ be the unique largest rational submodule of $V$ with the property that the highest weights of all its composition factors belong to $\pi$. The (left exact) truncation functor $\mathcal{O}_\pi$ was defined by Donkin [2].

Recall that $X(T)$ is partially ordered by $\lambda \leq \mu$ if $\mu - \lambda$ can be written as a sum of positive roots; this is sometimes called the dominance order. A subset $\pi$ of $X(T)^+$ is said to be saturated if it is predecessor closed under the dominance order on $X(T)$.

In other words, $\pi$ is saturated if for any $\mu \in \pi$ and any $\lambda \in X(T)^+$, $\lambda \leq \mu$ implies that $\lambda \in \pi$.

In order to show that the category of rational $M$-modules is a highest weight category, we are going to take $\pi = X(D)^+$. We need the following observation.

**3.9 Lemma.** The set $\pi = X(D)^+$ is a saturated subset of $X(T)^+$.

For $\lambda \in X(T)^+$, let $I(\lambda)$ be the injective envelope of $L(\lambda)$ in the category of rational $G$-modules. For $\lambda \in X(D)^+$ let $Q(\lambda)$ be the injective envelope of $L(\lambda)$ in the category of rational $M$-modules. The following records the effect of truncation on various classes of rational $G$-modules.

**3.10 Theorem.** Let $\pi = X(D)^+$. For any $\lambda \in X(T)^+$ we have the following:

(a) $\mathcal{O}_\pi \nabla(\lambda) = \begin{cases} \nabla(\lambda) & \text{if } \lambda \in \pi \\ 0 & \text{otherwise.} \end{cases}$

(b) $\mathcal{O}_\pi I(\lambda) = \begin{cases} Q(\lambda) & \text{if } \lambda \in \pi \\ 0 & \text{otherwise.} \end{cases}$

(c) $\mathcal{O}_\pi K[G] = K[M]$.

Note that $K[M]$ is regarded as a rational $M$-module via right translation. A $\nabla$-filtration for a rational $G$-module $V$ is an ascending series

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = V$$
of rational submodules such that for each \( j = 1, \ldots, r \), the quotient \( V_j/V_{j-1} \) is isomorphic to some \( V(\lambda_j) \). Whenever \( V \) is a rational \( G \)-module with a \( \nabla \)-filtration, let \( (V : \nabla(\lambda)) \) be the number of \( \lambda_j \) for which \( \lambda = \lambda_j \). This number is independent of the filtration.

The proof of the above theorem, which relies on results of [3], also shows the following facts.

3.11 Corollary. (a) Let \( \lambda \in \pi = X(D)^+ \). The module \( Q(\lambda) \) has a \( \nabla \)-filtration. Furthermore, it satisfies the reciprocity property

\[
(Q(\lambda) : \nabla(\mu)) = ([\nabla(\mu) : L(\lambda)]
\]

for any \( \mu \in X(D)^+ \), where \([V : L]\) stands for the multiplicity of a simple module \( L \) in a composition series of \( V \).

(b) The module \( K[M] \) has a \( \nabla \)-filtration. Moreover, \( (K[M] : \nabla(\lambda)) = \dim K \nabla(\lambda) \) for each \( \lambda \in X(D)^+ \).

3.12. From these results one obtains the important fact that the category of rational \( M \)-modules is a highest weight category, in the sense of [1]. In particular, one also sees that \( \dim K Q(\lambda) \) is finite, for any \( \lambda \in X(D)^+ \). (In contrast, it is well known that \( \dim K I(\lambda) \) is infinite.)

3.13. It is also shown in [5], exploiting the assumption that \( Z(M) \) is one-dimensional, that the category of rational \( M \)-modules splits into a direct sum of ‘homogeneous’ subcategories each of which is controlled by a finite saturated subset of \( X(D)^+ \). From the results of [11] it then follows that there is a finite dimensional quasihereditary algebra in each homogeneous degree, whose module category is precisely the homogeneous subcategory in that degree. Details are given in [5], where it is also shown that the quasihereditary algebras in question are in fact generalized Schur algebras in the sense of Donkin [2].

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