Phase Transition in Lattice Surface Systems with Gonihedric Action

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Abstract

We prove the existence of an ordered low temperature phase in a model of soft-self-avoiding closed random surfaces on a cubic lattice by a suitable extension of Peierls contour method. The statistical weight of each surface configuration depends only on the mean extrinsic curvature and on an interaction term arising when two surfaces touch each other along some contour. The model was introduced by F.J. Wegner and G.K. Savvidy as a lattice version of the gonihedric string, which is an action for triangulated random surfaces.

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1 Introduction

The gonihedric string was introduced by Savvidy et. al. [1, 2, 3] as a new action for random surfaces. For triangulated surfaces, the action reads

\[ S = \frac{1}{2} \sum_{\langle ij \rangle} |\vec{X}_i - \vec{X}_j| \theta(\alpha_{ij}), \] (1)

where

\[ \theta(\alpha_{ij}) = |\pi - \alpha_{ij}| \] (2)

and \( \alpha_{ij} \) is the angle between the neighbouring triangles with common link \( \langle ij \rangle \). This string model can be considered as a natural extension of the Feynman integral over paths to an integral over surfaces in the sense, that both amplitudes coincide in cases, when the surface degenerates into a single particle world line. The simulation of (1) shows flat surfaces [4], although some problems arise from the failure to suppress the wanderings of vertices [5].

One possibility to regularize the gonihedric action is to formulate the model on the euclidean lattice, which has been done by Wegner and Savvidy. There are two essentially distinct cases which correspond to non-self-avoiding surfaces [7] and to soft-self-avoiding ones [6]. In the former case self-inter"-sections of the surface do not produce any additional energy, while in case of soft-self-avoiding surfaces, the energy contribution of a link, where self-intersection occurs was defined by \( \theta(\pi/2) \) times the number of pairs of plaquettes which meet under a right angle. It was shown, that in both cases a \( \mathbb{Z}_2 \) Ising model can be constructed, which reproduce the same surface dynamics and an extension to \((d - n)\)-dimensional hypersurfaces on a \( d \)-dimensional lattice was also given.

In this paper we consider the case of \((d - 1)\)-dimensional soft-self-avoiding surfaces on a \( d \)-dimensional cubic lattice. The statistical weight of each surface configuration is given by \( E = l_2 + 4l_4 \), where \( l_2 \) is the number of links, where two \((d - 1)\)-dimensional plaquettes meet under a right angle (i.e. the mean extrinsic curvature) and \( l_4 \) is the number of plaquettes, where four plaquettes meet. This additional term arises, when two surfaces touch each other and is responsible for the soft-self-avoidance. The equivalent spin systems contain just ferromagnetic nearest neighbour and antiferromagnetic next nearest neighbour couplings. If one wants to allow arbitrary self-intersection coupling \( k > 0 \), i.e. \( E = l_2 + 4kl_4 \), the corresponding spin-hamiltonian contains also a plaquette term. The two dimensional model for \( k = 1 \) does not seem to show a phase phase transition [10], whereas in three dimensions a second order phase transition occurs at \( \beta_c \approx 0.44 \) which is close two the critical temperature of the two dimensional Ising model. These results were obtained by
numerical simulations. The three dimensional gonihedric Ising system was also discussed for arbitrary $k > 0$ in [9] using mean field methods and simulations. The authors find a second order phase transition for $k > 0$ and a qualitatively different behaviour for the case $k = 0$. The critical temperature for $k > 0$ was shown to increase with increasing $k$.

In this work we prove the existence of an ordered low temperature phase for $k > 0$ in $d \geq 3$ dimensions using a suitable extension of Peierls contour method [11]. After recapitulating the model, we adapt Peierls argument to the case $k = 1$. An upper bound for $\beta_c$ for the three dimensional model is found, which is in agreement with [8, 9]. Finally we show, how to generalize the argument for arbitrary $k > 0$.

2 The Model

Consider a $d$-dimensional euclidean lattice, with lattice points $r \in \mathbb{Z}^d$. We define a $(d - n)$-dimensional hyperplaquette $\Omega_{\alpha_1,\ldots,\alpha_n}(r)$ (all $\alpha_i$ different) by

$$\Omega_{\alpha_1,\ldots,\alpha_n}(r) = \left\{ x \in \mathbb{R}^d \mid x_{\alpha_i} = r_{\alpha_i}, r_{\alpha} \leq x_{\alpha} \leq r_{\alpha} + 1 \text{ for all } \alpha \neq \alpha_i \right\}$$

(3)

For convenience we call a $(d - 1)$-dimensional hyperplaquette simply a plaquette, a $(d - 2)$-dimensional hyperplaquette a link and a $(d - 3)$-dimensional hyperplaquette a vertex.

A closed surface on the lattice is a collection of plaquettes, where at each link an even number of plaquettes meet. In [3] a hamiltonian for closed surfaces $M$ on the lattice is defined in the following way:

Attach plaquette variables $U_P$ to each plaquette $P$ of the lattice and define

$$U_P = \begin{cases} -1 & \text{if } P \in M \\ +1 & \text{if } P \notin M \end{cases}$$

(4)

The energy of $M$ is given as a sum over all links

$$H = \sum_{\text{all links}} H_{\text{link}},$$

(5)

where $H_{\text{link}}$ contributes $4J$ if four plaquettes meet at the corresponding link, $J$ if two plaquettes meet perpendicular and zero in all other cases. Note that 4 is always the maximum number of plaquettes, which can meet at a link,
independent of the dimension of the lattice. In terms of the link variables this can be written as

\[ H_{\text{link}} = \frac{1}{4} J (2 - U_1 - U_{-1}) (2 - U_2 - U_{-2}). \]  

(6)

\[ U_1, U_{-1}, U_2, U_{-2} \] can not be chosen independently since the surface is closed. This condition requires

\[ U_1 U_{-1} U_2 U_{-2} = 1. \]  

(7)

To resolve this constraint, spin variables \( \sigma = \pm 1 \) were attached to the sites of the dual lattice.

The plaquette variables can then be represented as

\[ U_1 = \sigma_1 \sigma_2, \quad U_{-1} = \sigma_{-1} \sigma_{-2}, \quad U_2 = \sigma_1 \sigma_{-2}, \quad U_{-2} = \sigma_{-1} \sigma_2, \]  

(8)

and the full hamiltonian becomes equivalent to

\[ H = J \sum_r \left[ \frac{d}{2} (d-1) - (d-1) \sum_{\alpha} \sigma(r) \sigma(r - e_\alpha) \right. \]

\[ \left. + \frac{1}{2} \sum_{\alpha < \beta} \left( \sigma(r - e_\alpha) \sigma(r - e_\beta) + \sigma(r) \sigma(r - e_\alpha - e_\beta) \right) \right]. \]  

(9)
In this expression \( r \) runs over all sites of the dual lattice, where the spins are located and \( e_\alpha, e_\beta \) denote unit vectors parallel to the \( d \) cubic axes. One sees that the equivalent spin system contains ferromagnetic nearest neighbour and antiferromagnetic next nearest neighbour couplings, where the ratio between them is fixed and given by

\[
\frac{J_{\text{ferromagnet}}}{J_{\text{antiferromagnet}}} = 2(d - 1).
\]  

(10)

For arbitrary self-intersection coupling \( k > 0 \), the energy contribution of a link, where four plaquettes meet is equal to \( 4Jk \). In this case equation (10) must be replaced by

\[
H_{\text{link}} = \frac{1}{4} Jk(2 - U_1 - U_{-1})(2 - U_2 - U_{-2}) + \frac{1}{4} J(1 - k)(2 - U_1 U_{-1} - U_2 U_{-2})
\]

(11)

and the corresponding spin-hamiltonian becomes

\[
H = J \sum_r \left[ \frac{d}{2}(d - 1)\frac{k + 1}{2} - (d - 1)k \sum_\alpha \sigma(r)\sigma(r - e_\alpha) \right.
\]

\[
+ \frac{1}{2} k \sum_{\alpha < \beta} \left( \sigma(r - e_\alpha)\sigma(r - e_\beta) + \sigma(r)\sigma(r - e_\alpha - e_\beta) \right)
\]

\[
- \frac{1}{2} \sum_{\alpha < \beta} \left( \sigma(r)\sigma(r - e_\alpha)\sigma(r - e_\alpha - e_\beta)\sigma(r - e_\beta) \right). \]

(12)

This hamiltonian also contains a plaquette term as long as \( k \neq 1 \).

Because in the plaquette variables were represented as products of nearest neighbour spins, we can restore the original surface \( M \) from the spin configuration by simply choosing all plaquettes between spins of opposite sign. In this sense, the surface \( M \) can be considered as a domain wall, which separates spins of positive from spins of negative sign. In contrast to the ordinary ferromagnetic Ising model, the energy of a spin configuration is not given by the total number of plaquettes of \( M \), i.e. the surface, but by \( E = (l_2 + 4k l_4)J \), where \( l_2 \) is the number of links contained in \( M \), where two plaquettes meet perpendicular and \( l_4 \) is the number of links, where 4 plaquettes meet. Because of this, one can always insert plain domain walls by swapping a whole ((\( d - 1 \))-dimensional) spin layer without changing the total energy, provided the inserted walls do not cross any existing surface. In particular the ground state of the finite lattice system containing \( N^d \) spins (\( d \)-dimensional cubus) is \((d(2^N - 2) + 2)\)-fold degenerate.
3 Existence of a phase transition for $d \geq 3$

In this section we will prove, that the model defined in the previous section shows a phase transition, provided $d \geq 3$. This will be done with a method which is due to Peierls [11]. The argument proceeds in two steps. (i) We show that the spontaneous magnetization $\hat{M}_N$ for a finite system with special boundary conditions is bounded from below, if the temperature is lower than a critical temperature $T_c$, i.e.

$$\hat{M}_N \geq \alpha > 0 \quad \text{for } T < T_c. \quad (13)$$

The hat indicates the special boundary conditions and the index $N$ refers to the finite system containing $N$ spins. $\alpha$ is independent of $N$, but sensitive to the special boundary conditions we impose. (ii) The free energy $\hat{f}_N(T, H)$ is concave as a function of the magnetic field $H$, i.e.

$$-\frac{\hat{f}_N(T, H) - \hat{f}_N(T, 0)}{H} \geq \hat{M}_N(H = 0) \geq \alpha > 0,$$  \quad (14)

for $T < T_c$ and $H \geq 0$. The thermodynamic limit $\lim_{N \to \infty} \hat{f}_N(T, H)$ exists in the sense of van Hove and is independent of the boundary conditions. The limiting free energy is concave as well. This implies that equation (14) remains valid in the thermodynamic limit. Since the magnetization is antisymmetric in $H$, it follows that $f(T, H)$ is not analytic in $H = 0$ if $T < T_c$.

To get an idea how equation (13) can be shown, consider a 3-dimensional finite volume with $N$ spins and fix the spins at the boundary to +1. The ground state of this system is given by the spin configuration, in which all spins are positive, since then no domain walls are present and hence the total energy is zero. If we now swap a little ‘island’ of spins inside the volume to -1, we need an amount of energy $E$, which is essential proportional to the total number of edge elements of the domain wall, which was established by swapping the spins. Therefore, the probability of occurrence of a domain wall with $l$ edge elements is of order $\nu(l) e^{-\beta Jl}$, where $\nu(l)$ is the total number of possible edge configurations with $l$ edge elements. Now for $d = 3$, the total number of negative spins inside a domain wall is bounded by $(\frac{l}{12})^3$ (volume of a cube), hence the mean number of negative spins will be bounded by

$$\langle N_- \rangle \leq C \sum_l \left(\frac{l}{12}\right)^3 \nu(l) e^{-\beta Jl}, \quad (15)$$

where $C$ is some constant. If $\nu(l)$ does not increase too rapidly for increasing $l$, the right hand side of (15) will be smaller than $\frac{N}{2}$ for large enough $\beta > \beta_c$. 
Thus the spontaneous magnetisation will be strictly non zero, if $\beta > \beta_c$. The argument can easily be extended to $d \geq 3$, if $(\frac{1}{4d})^3$ is replaced by $(\frac{1}{2d(d-1)})^{d-2}$. It does not work in two dimensions, since in this case, the edge elements are vertices and the total number of negative spins inside a domain wall with a given number of vertices can be arbitrary large. Indeed, in two dimensions, the model does not seem to show a phase transition [10, 8]. We now proceed with the details for the case $k = 1$.

Consider a finite volume of the dual lattice containing $N$ spins in its interior $V \setminus \partial V$ and fix the spins at the boundary:

$$\sigma(\mathbf{r}) = +1 \quad \text{if} \quad \mathbf{r} \in \partial V$$

(16)

If a certain configuration $\sigma = (\sigma_1, \ldots, \sigma_N)$ of the $N$ spins is given we denote by $M[\sigma]$ the corresponding closed surface on the original lattice which separates regions of negative from regions of positive spins. We call a link contained in $M[\sigma]$ an edge element, if one of the following two conditions is fulfilled:

- Four plaquettes of $M[\sigma]$ meet at the given link.
- Two plaquettes of $M[\sigma]$ meet perpendicular at the given link.

The set $C^{\text{tot}}$ of all edge elements of $M[\sigma]$ can be divided into connected edge diagrams $C_1, \ldots, C_n$ in a unique way

$$C^{\text{tot}} = C_1 \cup C_2 \cup \ldots \cup C_n$$

(17)

The energy $E[\sigma]$ of the spin configuration can then easily be expressed in terms of connected edge diagrams since $M[\sigma]$ is the original interacting surface and therefore only the edge elements of $M[\sigma]$ contribute to $E[\sigma]$. We define the energy contribution $E[C]$ of a connected edge diagram $C$ as

$$E[C] = J(l_2 + 4l_4),$$

(18)

where $l_2$ denotes the number of edge elements of $C$, where two plaquettes meet and $l_4$ the number of edge elements, where four plaquettes meet. The total energy of a spin configuration can then be written as

$$E[\sigma] = \sum_C E[C].$$

(19)

Clearly, the number of all connected edge diagrams with a given energy contribution $E$ that fit into the finite volume $V$ is finite. We denote this number
by \( g(E) \). Next we attach variables \( \chi^E_i[\sigma], i = 1, \ldots, g(E) \) to all these connected edge diagrams \( C^E_i \). \( \chi^E_i[\sigma] \) assume the value +1 if the corresponding edge diagram appears in \( \sigma \) and 0 otherwise.

The partition function \( Z_N \) for the \( N \) spins is given by

\[
Z_N = \sum_{\sigma} e^{-\beta E[\sigma]} \tag{20}
\]

We are interested in the thermodynamical expectation value \( \langle \chi^E_i \rangle_N \), that is the probability of appearance of \( C^E_i \):

\[
\langle \chi^E_i \rangle_N = \frac{1}{Z_N} \sum_{\sigma} \chi^E_i[\sigma] e^{-\beta E[\sigma]} \tag{21}
\]

In this expression only spin configurations which contain \( C^E_i \) contribute. Numbering them by \( \sigma_1, \ldots, \sigma_k \), (21) reads:

\[
\langle \chi^E_i \rangle_N = \frac{1}{Z_N} \sum_{j=1}^k e^{-\beta E[\sigma_j]} \tag{22}
\]

Consider a particular spin configuration \( \sigma_j \) in which \( C^E_i \) appears. The connected edge diagram \( C^E_i \) belongs to a domain wall, which encloses a certain number of spins. By reversing all these spins, the domain wall disappears and the resulting configuration \( \sigma_j^* \) will not contain \( C^E_i \). Therefore

\[
E[\sigma_j] = E[\sigma_j^*] + E[C^E_i]. \tag{23}
\]

From this we obtain the following estimate:

\[
Z_N \geq \sum_{j=1}^k e^{-\beta E[\sigma_j^*]} \geq e^{\beta E[C^E_i]} \sum_{j=1}^k e^{-\beta E[\sigma_j]} \tag{24}
\]

Together with (23) this implies:

\[
\langle \chi^E_i \rangle_N \leq e^{-\beta E[C^E_i]} \tag{25}
\]

We can now find an upper bound for the mean number of negative spins inside \( V \) as follows:

Consider a given configuration \( \sigma \) and denote its connected edge diagrams by \( C_1, \ldots, C_n \). Each diagram belongs to a domain wall, which contains a
number $n_j$ of negative spins. If $l_j \leq E[C_j]/J$ denotes the number of edge elements in $C_j$, $n_j$ is bounded by

$$n_j \leq \left( \frac{l_j}{2d(d-1)} \right)^{\frac{d}{d-2}} \leq \left( \frac{E[C_j]}{2d(d-1)J} \right)^{\frac{d}{d-2}}. \quad (26)$$

This relation becomes an equation, if the domain wall is a simple cube. The total number of negative spins $N_\sigma$ is therefore bounded by

$$N_\sigma \leq \sum_{j=1}^n \left( \frac{E[C_j]}{2d(d-1)J} \right)^{\frac{d}{d-2}}. \quad (27)$$

Using the variables $\chi^E_\sigma$, we can write

$$N_\sigma \leq \sum_E \sum_{i=1}^n \chi^E_\sigma \left( \frac{E}{2d(d-1)J} \right)^{\frac{d}{d-2}}, \quad (28)$$

where the first summation is over all possible energies $E$, a connected edge diagram can assume. From (25) we get an upper bound for the thermodynamical expectation value of $N_\sigma$:

$$\langle N_- \rangle \leq \sum_E \left( \frac{E}{2d(d-1)J} \right)^{\frac{d}{d-2}} g(E) e^{-\beta E} \quad (29)$$

Note that the right hand side of (29) is not defined for $d = 2$. To proceed with the argument, we need an upper bound for $g(E)$. Consider the following construction method, which can be used to build up every possible diagram:

1. First we number all the vertices ($d-3$)-dimensional hyperplaquettes) of the lattice and keep this numbering once for all fixed.

2. Next we choose one of the $N$ lattice points and attach 3 edge elements, pointing in three given directions. For example if $p = (x_1, \ldots, x_d)$ denotes the chosen point, we can attach the edge elements

$c_1 = (x_1, x_2, \lambda_3, \ldots, \lambda_d)$,
$c_2 = (x_1, \lambda_2, x_3, \lambda_4, \ldots, \lambda_d)$,
$c_3 = (\lambda_1, x_2, x_3, \lambda_4, \ldots, \lambda_d)$,
$x_i \leq \lambda_i \leq x_{i+1}$. This can be done since such an edge configuration must occur at least once in any possible connected edge diagram. We think of this vertex as a corner of the closed surface. This fixes the 8 spins surrounding the vertex $p$. 

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3. Imagine that a connected subdiagram exists already. Now we choose the vertex with the lowest number, which occurs in the construction so far, i.e. where at least one edge element ends and where the configuration of the edge elements surrounding this vertex has not been specified yet. Next we do this specification by choosing the values of not more than four of the eight spins, surrounding the vertex, since we know that at least one edge element ends at this vertex already.

4. Finally by repeating construction step 3., all possible connected edge diagrams can be constructed.

To find an upper bound for $g(E)$, we estimate the maximum number of possible outcomes of the construction procedure just given. If the considered finite lattice volume contains $N$ spins, we have $N$ choices for construction step 2. Each time we perform construction step 3, we have different numbers of choices how to specify the remaining spins. This depends on how many edge elements and what kind of edge elements are already ending at the vertex under consideration. The maximum number of choices arises, if just one edge element, surrounded by two or four plaquettes, is already present at the vertex, leaving four spins to specify, i.e. 16 possibilities. These possibilities are shown in figure 1, together with the corresponding energy contribution of the specified vertex. Edge elements where two plaquettes meet are indicated by a simple edge. Edge elements, where four plaquettes meet are indicated as double edges. The left column shows all cases, where one simple edge coming from the left was already present and correspondingly the right column shows all cases, where one double edge coming from the left was already present. From this picture we read of the maximum numbers of choices $n(E)$, how to specify the spins, such that a vertex of energy $E$ arises:

| maximum number of choice $n(E)$ | $x^{E/J}$          |
|---------------------------------|--------------------|
| 1                               | $x^2$              |
| 4                               | $x^3$              |
| 2                               | $x^4$              |
| 2                               | $x^6$              |
| 4                               | $x^7$              |
| 3                               | $x^8$              |
| 4                               | $x^{10}$           |
| 4                               | $x^{15}$           |
| 1                               | $x^{24}$           |
The function $G_n(x) = N x^3 f^n(x)$, where $f(x)$ is defined by

$$f(x) = \sum_{E} n(E) x^{E/J}$$

$$= x^2 + 4 x^3 + 2 x^4 + 2 x^6 + 4 x^7 + 3 x^8 + 4 x^{10} + 4 x^{15} + x^{24} \quad (30)$$

can now be interpreted as a generating function which counts the number of all connected edge diagrams with energy $E$ at least once, which can be constructed by repeating construction step 3 $n$-times, i.e. the coefficient of $x^{2(d-2) E/J}$ in an expansion of $G_n(x)$ is an upper bound for the number of those diagrams. Since all edge elements were counted $2(d-2)$ times, we have to calculate the coefficient of $x^{2(d-2) E/J}$ and not simply $x^{E/J}$. Summing $G_n(x)$ over all possible numbers of steps

$$G(x) = \sum_{n=0}^{\infty} G_n(x) = N \frac{x^3}{1 - f(x)}, \quad (31)$$

we get a generation function, which counts all connected edge diagrams at least once. Therefore

$$g(E) \leq \frac{1}{(2(d-2) E/J)!} \left. \frac{\partial^2 (d-2) E/J}{\partial x^{2(d-2) E/J}} G(x) \right|_{x=0} \quad (32)$$

provides an upper bound for $g(E)$. We can write $G(x)$ as

$$G(x) = N \sum_{k=1}^{24} \frac{a_k}{x - x_k} \quad (33)$$

Putting this expression in equation (32), we obtain

$$g(E) \leq N \sum_{k=1}^{24} \left( -\frac{a_k}{x_k} \right) \left( \frac{1}{2(d-2) x_k} \right)^{E/J} \quad (34)$$

If we denote the smallest $|x_k|$ by $x_{\min}$, which clearly dominates the increase of $g(E)$ for large $E$, we can further approximate this expression as follows:

$$g(E) \leq N \left( \sum_{k=1}^{24} \left| \frac{a_k}{x_k} \right| \right) \left( \frac{1}{2(d-2) x_{\min}} \right)^{E/J} =: N a c^{(d-2) E/J}, \quad (35)$$

where

$$a \leq 0.625$$
$$c \leq 3.882$$

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Substituting this result into equation (29), we find an upper bound for the density of negative spins:

\[
\frac{\langle N^- \rangle}{N} \leq \sum_E \left( \frac{E}{2d(d-2)J} \right)^{\frac{d-2}{4}} a \frac{e^{(d-2)E/J}}{e^{-\beta E}}
\]

This inequality remains valid in the thermodynamical limit. The sum on the right hand side converges, if \( \beta \) is larger than \( \frac{d-2}{J} \ln(c) \) and approaches zero, if \( \beta \) goes to infinity. Therefore for \( \beta > \beta_c \), it will be smaller than \( \frac{1}{2} \), which implies a non zero spontaneous magnetisation. This proves the existence of a phase transition.

Equation (29) together with an upper bound for \( g(E) \) given in (34) also provides a lower bound for the critical temperature \( T_c \). We find

\[
T_c \geq 0.67J \quad \text{for } d = 3
\]

\[
T_c \geq 0.36J \quad \text{for } d = 4
\]

The value for \( d = 3 \) is in agreement with the critical temperature found in [8, 9], which was \( T_c \approx 2.28J \).

The method used above to prove the existence of an ordered low temperature phase for \( k = 1 \) can easily be generalized to arbitrary \( k = \frac{p}{q} > 0 \), where \( p, q \in \mathbb{N} \). We have

\[
\frac{E}{J} = l_2 + 4k l_4 \geq l = l_2 + l_4 \quad \text{if } k \geq \frac{1}{4}
\]

\[
\frac{E}{4kJ} = \frac{l_2}{4k} + l_4 \geq l = l_2 + l_4 \quad \text{if } k < \frac{1}{4}
\]

Therefore equation (29) remains unchanged if \( k \geq \frac{1}{4} \), whereas in case \( k < \frac{1}{4} \) we have to replace it by

\[
\langle N^- \rangle \leq \sum_E \left( \frac{E}{2d(d-1)4kJ} \right)^{\frac{d-2}{4}} g(E) e^{-\beta E}
\]

An upper bound for \( g(E) \) can be found by expanding the generating function

\[
G(x) = N \frac{x^3}{1 - f(x)},
\]

\[
f(x) = x^2 + 4x^3 + 2x^4 + 2x^6 + 4x^{3+4q} + \frac{2x^{4+4q}}{q} + x^{8q} + 4x^{2+8q} + 4x^{3+12q} + x^{24q}
\]
in powers of $x^q$, i.e.

$$G(x) = N \sum_j \frac{a_j}{x^q - x_j} = \sum_{n=0}^\infty \left( N \sum_j \left( \frac{a_j}{x_j} \right) \left( \frac{1}{x_j} \right)^n \right) x^\frac{n}{q} \tag{44}$$

From this we obtain

$$g(E) \leq N \sum_j \left( \frac{a_j}{x_j} \right) \left( \frac{1}{x^{2(d-2)}_j} \right)^{\frac{qE}{J}} \tag{45}$$

The estimate for $g(E)$ becomes smaller for increasing $k$ and diverges for $k \to 0$. For $k \to \infty$ the behaviour of $g(E)$ for large $E$ is dominated by the smallest zero of the function $1 - x^2 - 4x^3 - 2x^4 - 2x^6$, which gives

$$g(E) \leq \text{const} \ (3.74)^{(d-2)E/J} \quad \text{for} \quad k \to \infty \tag{46}$$

## 4 Summary

We have proven the existence of an ordered low temperature phase for the spin systems defined in section 2, which can be considered as the lattice regularization of the gonihedric string. Each spin configuration corresponds to a particular random surface which is simply the domain wall separating regions of spin with opposite sign. This observation was crucial, since it allowed us to apply Peierls contour method. However the naive application of the argument would yield an inequality of the following form in $d = 3$ dimensions:

$$\langle N_- \rangle_N \leq \sum_b \left( \frac{b}{6} \right)^{\frac{3}{2}} \nu(b) e^{-\beta E_{\min}(b)}, \tag{47}$$

where $\langle N_- \rangle_N$ is the expectation value of the number of negative spins, calculated for the finite system with boundary condition $\sigma(r) = +1, r \in \partial V$. The factor $(b/6)^{\frac{3}{2}}$ is the maximum number of negative spins that a closed surface with $b$ plaquettes can enclose and $\nu(b)$ denotes the number of closed surfaces with $b$ plaquettes. This number is known to grow exponentially, in fact

$$\nu(b) \leq N 3^{b-1}. \tag{48}$$

For $E_{\min}(b)$ we have to put the minimum energy of a closed domain wall with $b$ plaquettes which is

$$E_{\min}(b) = 12 \sqrt[3]{\frac{b}{6}} \quad (d = 3) \tag{49}$$
This shows, that the right hand side of (47) actually diverges for every $\beta$ due to the exponential growth of $\nu(b)$. We therefore had to modify the argument as discussed in section 3. This was possible since the energy of a closed surface is essentially proportional to the number $l$ of edges and the number of negative spins inside is bounded by $(\frac{1}{12})^3$. Instead of (47) we arrived at

$$\langle N_\nu \rangle_N \leq \sum_E \left( \frac{E}{2d(d - 1)4kJ} \right)^{d/4} g(E) e^{-\beta E}$$

for $0 < k < \frac{1}{4}$ and

$$\langle N_\nu \rangle_N \leq \sum_E \left( \frac{E}{2d(d - 1)J} \right)^{d/4} g(E) e^{-\beta E}$$

for $k \geq \frac{1}{4}$. Finally we proved that the number of connected edge diagrams with given energy $g(E)$ does not grow faster than exponentially for any $k > 0$. As pointed out before, the argument does not work in two dimensions, because in this case the (0-dimensional) edges are not connected and therefore do not give a restriction for the number of negative spins.

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References

[1] R.V. Ambartzumian, G.K. Savvidy, K.G. Savvidy and G.S. Sukiasian, Phys. Lett. B275 (1992) 99.
[2] G.K. Savvidy and K.G. Savvidy, Int. J. Mod. Phys. A8 (1993) 3393.
[3] G.K. Savvidy and K.G. Savvidy, Mod. Phys. Lett. A8 (1993) 2963.
[4] C.F. Baillie and D.A. Johnston, Phys. Rev. D45 (1992) 3326.
[5] B. Durhuus and T. Jonsson, Phys. Lett. B297 (1992) 271.
[6] G.K. Savvidy and F.J. Wegner, Nucl. Phys. B413 (1994) 605.
[7] G.K. Savvidy, K.G. Savvidy and F.J. Wegner, Nucl. Phys. B433 (1995) 565.
[8] G.K. Bathas, K.G. Floratos, G.K. Savvidy, K.G. Savvidy ”Two Dimensional and Three Dimensional Spin Systems with Gonihedric Action”, hep-th/9504054

[9] D.A. Johnston and Ranasinghe P.K.C. Malmini, ”Gonihedric 3D Ising Actions”, hep-lat/9508026

[10] D.P. Landau and K. Binder, Phys. Rev. B31 (1985) 5946

[11] R. Peierls, Proc. Cambridge Phil. Soc. 32 (1936) 477; R.B. Griffiths, Phys. Rev. 136A (1964) 437
| left and right spin layer | visual picture | $x_{\text{energy}/J}$ | left and right spin layer | visual picture | $x_{\text{energy}/J}$ |
|--------------------------|----------------|-----------------------|--------------------------|----------------|-----------------------|
| $L_y$                    | $L_x^y$        | $x^3$                 | $L_y$                    | $L_x^y$        | $x^8$                 |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ |                       | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ |                       |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^3$                 | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 |
| $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 | $+$ $+$ $+$ $+$          | $+$ $+$ $+$ $+$ | $x^8$                 |

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figure 1