Symmetries and Decoherence of Two-Component Confined Atomic Clouds:
Study of the Atomic Echo in the two-component Bose-Einstein Condensate.

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Implications of the internal symmetries on the dynamics of the trapped two-component atomic vapors are discussed. In the cases of \textsuperscript{87}Rb (bosons) as well as of \textsuperscript{40}K (fermions) trapped in the two hyperfine states, the intrinsic \textit{su}(2) symmetry can be realized with a very good precision. Such a symmetry protects the global operators, which are the generators of the symmetry, from any decoherence. The case of boson-fermion mixture is discussed as well. The role of external factors, breaking the symmetry, in inducing the decoherence of the global operators is considered.

It is shown that, the loss of the correlations is not faster than the rates of the induced heating or losses, provided the noise is characterized by short correlation length. The case of the extrinsic long-ranged fluctuations is also considered. Intrinsic mechanisms of decoherence of the correlators of the condensate operators of the two-component condensate are analyzed. The atomic echo is discussed as a test for the reversibility of the phase diffusion effect. The intensity and the profile of the multiple echo are calculated numerically as well as analytically.

I. INTRODUCTION

Successful achievements of confinement and cooling to the state of degeneracy of Bose \textsuperscript{1} and Fermi \textsuperscript{2} atomic gases have created unprecedented variety of possibilities for studies of the many-body effects. The fundamental issue of emergence of irreversibility can now fully be addressed experimentally and theoretically in the wide range of situations.

Studies of the dissipation of the normal modes of the confined Bose-Einstein condensates (BEC) \textsuperscript{3} began immediately after BEC has been realized. Very recently, the data on the damping in the Fermi mixtures has been reported as well \textsuperscript{4}.

The Ramsey spectroscopy study of the temporal correlations of the relative phase of the two-component \textsuperscript{87}Rb BEC has been conducted by the JILA group \textsuperscript{5}. No decay of the global relative phase has been detected on the time scale of the experiment \(\leq 100\text{ms}\). The question was then posed in \textsuperscript{6} as to why the phase correlations are so robust despite an apparent fast relaxation of other degrees of freedom such as, e.g., the relative motion of the two condensates.

Many intriguing questions are associated with the finiteness of the number of bosons \(N\) forming BEC in the trap. It has been realized that the global phase may experience collapses and revivals as long as \(N\) is finite \textsuperscript{7}.

Symmetries play a very special role in the many-body dynamics. As was discovered by Pitaevskii and Rosch \textsuperscript{8}, a 2D gas trapped by the oscillator potential and characterized by the contact two-body interaction can be described by the dynamical symmetry \(SO(2,1)\), and this results in the non-decaying breathing mode. The symmetry protects the mode from the decay. In our recent work \textsuperscript{9}, it has been found that, in the case of the \textsuperscript{87}Rb mixture characterized by the proximity to the intrinsic \textit{su}(2) symmetry \textsuperscript{10}, the dynamics of the global generators is also protected by this symmetry from the intrinsic decoherence. In the case of the broken symmetry, the rate of the decoherence is controlled by the deviations from the \textit{su}(2), and, accordingly, the decoherence is slow as long as the deviations are small. This answers the question \textsuperscript{11} about the robustness of the phase dynamics.

In another our work \textsuperscript{12}, the nature of relaxation of the condensate operators was discussed. I has been shown that, if the number of bosons \(N_0\) in the 3D equilibrium condensate is macroscopic, the decoherence of the condensate time-correlators is rather a reversible dephasing than a true irreversible decay. In some sense, the Bose permutation symmetry, representing the indistinguishability of particles which populate the lowest (condensate) single particle state, protects the condensate operators from the irreversible loss of memory. It was suggested to test this by the echo-type experiment. Thus, even though the intrinsic \textit{su}(2) symmetry can be broken, the dynamics of the condensate operators remains reversible.

In the present work, we, first, extend our analysis \textsuperscript{13} to the case of the two-component fermion mixture \textsuperscript{14}, and will show that, if the trapping potentials for the components are identical, the global quantities - the generators of the symmetry formed by the Fermi operators - do not decay in spite of the presence of the interaction between the components. We also analyze the case of the boson-fermion mixture, which is close to the \textit{su}(2) symmetry with respect to the mutual (formal) transformation of bosons and fermions into each other. In this situation, the decoherence of certain global operators, which, however, are not the generators, is suppressed as well.

Second, we analyze the problem of the decoherence in the case of deviations from the \textit{su}(2) symmetry in the two-component BEC. Extrinsic and intrinsic mechanisms of the decoherence are considered. For the extrinsic factors,
we concentrate on the interaction with the hot background gas, which induces heating and losses (or deposition). The heating is modeled by introducing a random white noise potential, which is characterized by some spatial correlation length $L_p$. We show that the ratio of the phase decay rate $\tau^{-1}_{\text{irr}}$ to the heating rate $\tau^{-1}_H$ is essentially given by the factor $L_p^2/L_c^2$, where $L_c$ stands for the thermal length of the trapped atoms at the BEC transition temperature $T_c$. Obviously, $L_p$ is given by the thermal length of the background gas, which is $\ll L_c$. This implies that the phase memory loss turns out to be not faster than the heating rate. Similarly, incoherent losses from the trap are unable to produce the phase decay faster than the rate of the losses.

We also analyze the role of the instrumental effects, which may produce the long range noise breaking the intrinsic symmetry. In this situation, the phase decay rate can be significantly enhanced, if compared with the corresponding rate of disrupting the thermal equilibrium within the atomic cloud. However, we estimate that the phase decay rate is still too small in the current traps, provided the instrumental effects are on the level of the equilibrium room temperature thermal noise.

Regarding the intrinsic factors, we extend our analysis \(^1\) to the case of the two-component BEC, and show that the intrinsic factors do not induce the irreversible decay of the correlators of the condensate operators. At finite temperatures $T$, the phase exhibits reversible phase diffusion effect similar to that analyzed previously at $T = 0$ by many authors \(^2\). In fact, in any exact many-body eigenstate, the evolution of the condensate operators is given by the relative chemical potential. Averaging over the thermal ensemble produces dephasing, whose rate is determined by the fluctuations of the chemical potential. At $T = 0$, these fluctuations are produced by the averaging over the initial coherent state and/or shot noise, and, at $T \neq 0$, the thermal averaging induces an additional contribution. Thus, at finite $T$ the nature of the global relative phase diffusion remains, practically, the same as that at $T = 0$, provided intrinsic factors only contribute to the decoherence.

Finally, we analyze the atomic echo effect, suggested previously for testing the reversibility of the phase diffusion \(^3\), in the context of the JILA experiment \(^4\). The strength of the echo is evaluated. When the dephasing is dominated by the shot noise in the total number of bosons in the trap, the echo can be as large as $100\%$, for the $\pi$-pulse employed for the time reversal \(^3\). When the phase diffusion is dominated by the self-interaction effects at $T = 0$, the echo about $50\%$ can be achieved by a weak time-reversal pulse, which transfers coherently a (relatively) small number of bosons between the components. An analysis of the echo at finite temperature is presented as well. Our conclusion is that the echo, although suppressed, can still be observed at finite temperatures, provided the population of the condensate state is dominant.

II. INTRINSIC SYMMETRIES AND THE DYNAMICS OF THEIR GENERATORS IN THE TWO-COMPONENT ATOMIC MIXTURES

Here we consider simplest cases of the two-component mixtures: bosons of sort 1 + bosons of sort 2; fermions of sort 1 + fermions of sort 2; and one component bosons + one component fermions.

A. Two-component Bose gas

First, for the purpose of consistency, we will briefly outline the results \(^3\) for the two-component Bose mixture. We employ the following two-component Hamiltonian

$$H = \int d\mathbf{x} \left\{ \Psi^\dagger_1 (H_1 - \epsilon_0/2) \Psi_1 + \Psi^\dagger_2 (H_2 + \epsilon_0/2) \Psi_2 + \left( \frac{\hbar \Omega(t)}{2} \Psi_2^\dagger \Psi_1 + H.c. \right) + \right.$$ 

$$\left. + \frac{2\pi \hbar^2}{m} \left[ a_1 \Psi^\dagger_1 \Psi^\dagger_1 \Psi_1 \Psi_1 + a_2 \Psi^\dagger_2 \Psi^\dagger_2 \Psi_2 \Psi_2 + 2a_{12} \Psi^\dagger_1 \Psi^\dagger_2 \Psi_2 \Psi_1 \right] \right\},$$

(1)

where $\Psi_i$ is the second quantized Bose field of the $i$-the component ($i = 1, 2$); $H_j = -\hbar^2 \nabla^2/m_j + U_j(\mathbf{x})$ stands for the one-particle part which includes the kinetic energy and the trapping potential; the quantity $\epsilon_0 = \text{const}$ is the detuning of the external field $\hbar \Omega(t)$ (taken in the rotating wave approximation) from the resonance between the components; $\Omega(t)$ stands for the corresponding Rabi frequency treated as an envelope of the rf-pulses; The binary collision terms in (1) are taken in the contact form, with $a_1$, $a_2$, $a_{12}$ being the corresponding scattering lengths.

The intrinsic $su(2)$ symmetry corresponds to the situation $H_1 = H_2$, $a_1 = a_2 = a_{12}$. The generators of this symmetry...
\[ I_z = \frac{1}{2} \int dx (\Psi_2^\dagger \Psi_2 - \Psi_1^\dagger \Psi_1); \quad I_+ = \int dx \Psi_2^\dagger \Psi_1; \quad I_- = \int dx \Psi_1^\dagger \Psi_2 \]

(2)

represent the \( su(2) \) algebra of the angular momentum operators. Indeed, it is easy to see that these operators obey the standard \( su(2) \) commutation relations

\[ [I_z, I_+] = I_+, \quad [I_z, I_-] = -I_-, \quad [I_+, I_-] = 2I_z, \]

(3)

provided the field operators obey the Bose commutation rule \( [\Psi(x), \Psi_j^\dagger(x')] = \delta_{ij} \delta(x - x') \). The operator \( I_z = (N_2 - N_1)/2 \), with \( N_{1,2} \) standing for the total number of bosons in the first and the second components, respectively, can be viewed as the \( z \)-component of the angular momentum operator; \( I_\pm = (I_+ + I_-)/2, \quad I_y = (I_+ - I_-)/2i \) are the \( x,y \)-components, respectively.

If the condensate is present, the operators \( I_\pm \) carry the information about the relative global phase \( \varphi \) (we may call it simply "phase"). Indeed, in this case, \( \Psi \approx \sqrt{N_j/V} e^{i\varphi_j} \), where \( \varphi_j \) is the global phase of the \( j \)th component, and \( V \) stands for the effective volume occupied by the condensate. Thus, \( I_\pm \approx \sqrt{N_1 N_2} e^{i\varphi}, \quad \varphi = \varphi_1 - \varphi_2 \). Accordingly, the correlator \( \langle I_+(t)I_-(0) \rangle \), where the mean is taken with respect to some set of initial states, carries the information about the time correlation properties of the phase.

In the case of the exact intrinsic symmetry, the generators (2) obey linear Heisenberg equations

\[ i\hbar \dot{I}_z = \frac{\hbar \Omega^*(t)}{2} I_+ - \frac{\hbar \Omega(t)}{2} I_-, \quad i\hbar \dot{I}_+ = -\epsilon_0 I_+ + \hbar \Omega(t) I_z, \quad i\hbar \dot{I}_- = \epsilon_0 I_- - \hbar \Omega^*(t) I_z \]

(4)

where the \( su(2) \) commutation relations (3) have been employed (8). If \( \Omega(t) \) and \( \epsilon_0 \) are not subjected to any noise, the solution for the correlator \( \langle I_+(t)I_-(0) \rangle \) exhibits no decoherence despite the presence of the two-body interaction in the Hamiltonian (8). It should also be noted that Eq.(8) are exact, and they do not require a presence of the BEC.

The decoherence will arise, if either the one-particle parts \( H_{1,2} \) are different or the scattering lengths deviate from the condition \( a_1 = a_2 = a_{12} \). The nature of the decoherence in the case of the BEC will be discussed later.

B. Two-component Fermi mixture

In the case of the two-component Fermi gas, the \( su(2) \) intrinsic symmetry can hold as well. For example, trapping \(^{40}\text{K} \) in the optical trap results in the identical potentials for both components (4). This, in fact, implies that the intrinsic \( su(2) \) symmetry holds for any value of the inter-component scattering length. Indeed, in the Hamiltonian (8), the Bose operators \( \Psi_{1,2} \) are to be replaced by the Fermi operators \( F_{1,2} \) obeying the anti-commutation rule

\[ \{F_i(x), F_i^\dagger(x')\} = \delta_{ij} \delta(x - x'), \]

(5)

where \{\ldots,\} denotes the anti-commutator. Accordingly, the terms \( \sim a_1, a_2 \) vanish in Eq.(8). Introducing the \( su(2) \) generators (2), where the Bose-operators are replaced by the Fermi-operators, the \( su(2) \) commutation relations (3) follow. Finally, the equations of motion for the generators turn out to be Eq.(8). Thus, in the case of the two-component fermion gas, the dynamics of the generators is always linear regardless of the value of the scattering length \( a_{12} \), as long as the one-particle Hamiltonians \( H_{1,2} \) are the same. We should, however, note that the higher order scattering harmonics change this situation, and the corresponding restrictions should be imposed on the scattering amplitudes of the higher orders (p-,f- waves etc). These processes, however, are insignificant at low temperatures and at low densities.

C. Bose-Fermi mixture

In the case of the mixture of, e.g., one-component Bose and Fermi gases, the respective fields \( \Psi(x), F(x) \) are introduced, and the Hamiltonian takes the form

\[ H = \int dx \{ \Psi^\dagger (H_1 - \frac{\epsilon_0}{2}) \Psi + F^\dagger (H_2 + \frac{\epsilon_0}{2}) F + \frac{2\pi \hbar^2}{m} [a_1 \Psi^\dagger \Psi F^\dagger F \Psi + 2a_{12} \Psi^\dagger F^\dagger F \Psi] \}, \]

(6)

where it was taken into account that no interconversion between bosons and fermions is possible, and that the self-interaction between the one-component fermions vanishes in the s-wave approximation.
Let us analyze a special situation — $H_1 = H_2$, $a_1 = a_{12}$. This corresponds to the (accidental) intrinsic $su(2)$ symmetry between the bosons and the fermions. Indeed, the interaction term can be rewritten as $a_1(\Psi^\dagger \Psi)^2 + 2a_{12}\Psi^\dagger \Psi F^\dagger F = a_1[\Psi^\dagger \Psi + F^\dagger F]^2$ in this case, because $[F^\dagger F]^2$ vanishes (apart from a trivial term $\sim F^\dagger F$, which simply renormalizes the constant $\epsilon_0$ in Eq. (6)). Now, if $H_1 = H_2$, the formal $su(2)$ transformation acting on the "spinor" made of $\Psi$, $F$ retains the Hamiltonian intact.

We introduce the operators

$$I_z = \frac{1}{2} \int d\mathbf{x}(\Psi^\dagger \mathbf{F} - \mathbf{F}^\dagger \Psi), \quad I_+ = \int d\mathbf{x}\Psi^\dagger \mathbf{F}, \quad I_- = \int d\mathbf{x}\mathbf{F}^\dagger \Psi, \quad I = \int d\mathbf{x}(\Psi^\dagger \Psi + \mathbf{F}^\dagger \mathbf{F}). \tag{7}$$

It should be noted that these operators do not form a closed algebra, as can easily be verified by commuting them with each other. Instead, they obey the following relations

$$[I_z, I_+] = I_+, \quad [I_z, I_-] = -I_-, \quad \{I_+, I_-\} = I, \quad [I_z, I] = [I_\pm, I] = 0. \tag{8}$$

It can be verified that the Heisenberg equations of motion for the above operators are also linear. Specifically,

$$\dot{I}_z = 0, \quad \dot{I} = 0, \quad i\hbar I_\pm = \mp \epsilon_0 I_\pm, \tag{9}$$

as long as the above condition of the intrinsic $su(2)$ symmetry holds. Thus, the situation of the Bose-Fermi mixtures can also exhibit the dissipationless dynamics of certain global quantities (which are not, however, the generators of the symmetry). It should be mentioned that any deviations from the intrinsic symmetry will result in the decoherence of these global quantities, and the rate of such a decoherence will also be controlled by the proximity to the symmetry.

III. THE NATURE OF DECOHERENCE IN THE CASE OF THE BROKEN $SU(2)$ SYMMETRY IN THE TWO-COMPONENT BOSE-MIXTURE

Here we will discuss various factors which break the intrinsic $su(2)$, and thereby introduce the decoherence in, e.g., the correlator

$$\rho_{ij}(\mathbf{x}, \mathbf{x}', t, t') = \langle \Psi_i^\dagger(\mathbf{x}, t)\Psi_j(\mathbf{x}', t') \rangle, \quad t > t'. \tag{10}$$

Hereafter we consider the Bose statistics only.

First, we address the decay induced by extrinsic factors such as: 1) collisions with hot background gas, which result in heating of the confined cloud and in losses from the trap; 2) Thermal noise of the trapping potential, which corresponds to temperatures much higher than the confined gas.

We raise a general question: Under what conditions can the loss of the memory, produced by the extrinsic factors, be much faster than the corresponding time of disrupting the equilibrium in BEC? In other words, if $\tau_h$ is the time of the heating induced by some extrinsic factor, and $\tau_{irr}$ stands for the time of the irreversible loss of the time correlations in (10) due to the same factor, can it be that the condition

$$\tau_{irr} \ll \tau_h \tag{11}$$

holds? In the same sense, if $\tau_L$ stands for a typical life-time of the confined cloud, which is being subjected to losses, can the condition (11) hold, where the role of $\tau_h$ is replaced by $\tau_L$?

Below we will show that the range of the spatial correlations of the extrinsic factors plays a crucial role in fulfilling the condition (11). In fact, the case 2) turns out to be the most efficient in destroying the time correlations without introducing significant disturbances into the system.

A. Rates of disrupting the equilibrium, and the loss of the phase memory

The purpose of the following is to outline the framework within which the effects of environment on the confined cloud can be analyzed, and to obtain general criterion allowing to compare the rates of disrupting the equilibrium and of the decoherence. In this section, we will consider the effects of interaction with the background (hot) gas.

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1 We believe that the conclusion ensuing from this analysis is general
order to do this consistently, the corresponding interaction term should be added to the Hamiltonian \(H\). We choose it in the form

\[
H_L = \sum_{ij,kl} \int d\mathbf{x}g_{ij,kl}\Psi^\dagger_i B_k^\dagger B_l\Psi_j,
\]

where the summation runs over all the components of the confined cloud and of the background gas described by the fields \(B_i\); \(g_{ij,kl}\) are the corresponding interaction constants.

The term \(H_L\) describes two basic effects [12]: a) a fast particle strikes a confined boson and ejects it from the trap; b) the background particle does not eject any boson from the cloud and, instead, imparts a substantial energy producing heating in the trap, and, possibly, exchange between the components. The term \(H_L\) can be employed in order to derive the equation for the one-particle density matrix (OPDM) \(\rho_{ij}(\mathbf{x},\mathbf{x}',t) = \langle \Psi^\dagger_i(\mathbf{x},t)\Psi_j(\mathbf{x}',t) \rangle = \rho_{ij}(\mathbf{x},\mathbf{x}',t,t)\) as well as for the correlator \(\langle \mathbf{1} \rangle\). The OPDM equation becomes a set of the kinetic equations in the Wigner representation. Here we will not implement this to full extent. Instead, we will analyze an exactly solvable model which captures basic features of the effect b), and then we will present results for the effect a).

The field \(\xi_{ij} = g_{ij,kl}B_k^\dagger B_l\) in Eq. (12) can be viewed as some random mean field potential acting on the confined cloud. As long as the background gas is hot, this effective random potential can be considered as a white noise. As it will be seen later, this assumption allows obtaining the OPDM equation exactly in the case when collisions between the confined bosons are ignored.

In order to simplify further analysis, we ignore the exchange of the components induced by the collisions\(^2\), and consider just a one-component random potential \(\xi\) which satisfies the condition

\[
\langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = \delta(t-t')C(|\mathbf{x}-\mathbf{x}'|),
\]

where \(C(|\mathbf{x}-\mathbf{x}'|)\) stands for some smooth function which decays to zero on some typical length \(L_p\). This function describes spatial correlations of the noise.

It should be noted that the above model is not suitable for treating a thermal equilibrium between the background and the cloud. Indeed, the external time dependent potential may produce unlimited heating, similarly to the case of the Brownian motion, if no friction term counterbalancing the external noise is introduced. In order to treat the equilibrium, a fully Hamiltonian analysis based on Eq. (12) should be implemented. This, however, is not required because the confined cloud is essentially out of the equilibrium with the background, and we are estimating the rate of disrupting the equilibrium in the cloud.

The Hamiltonian is taken in the form \(H\), where the noisy potential should be added. We choose \(a_1 = a_2 = a_{12} = 0\). Furthermore, for the sake of simplicity, we consider no trapping potential and set without the loss of generality \(\epsilon_0 = 0, \Omega = 0\), so that the Hamiltonian takes a form

\[
H = \int d\mathbf{x}\{\Psi^\dagger_1 H_0\Psi_1 + \Psi^\dagger_2 H_0\Psi_2 + \xi_1\Psi^\dagger_1\Psi_1 + \xi_2\Psi^\dagger_2\Psi_2\}, \quad \xi_1 = -\xi_2 = \xi,
\]

where \(H_0 = -\hbar^2/(2m)\nabla^2\), and we have included the interaction with the random potential \(\xi\) through the term breaking the intrinsic \(su(2)\) symmetry, because, according to the preceding analysis, no decoherence of the global operators can be induced in the case of the intrinsic symmetry (\(\xi_1 = \xi_2\) in the considered situation).

In the presence of the noise, the meaning of the averaging in Eq. (10) should be specified. Let us assume that some initial state \(|t = 0\rangle\) was created at the time moment \(t = 0\), and the noise affects the following evolution at later times \(t > 0\). Thus, the averaging should be performed, first, over the initial state (or a set of states with some weight), and, then, the averaging over the gaussian noise is to be done.

The OPDM characterizes the rate of the heating. Indeed, the mean kinetic energy of the particles is defined as

\[
K = \sum_{i=1,2} \int d\mathbf{x}H_{0\mathbf{x}}\rho_{ii}(\mathbf{x},\mathbf{x}',t), \quad \mathbf{x}' \rightarrow \mathbf{x},
\]

where, the coordinate dependence \(H_{0\mathbf{x}}\) in the kinetic energy operator \(H_0\) indicates that this operator acts on the coordinate \(\mathbf{x}'\), and, then, one should set \(\mathbf{x}' = \mathbf{x}\) and integrate over \(\mathbf{x}\).

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\(^2\) while complicating the calculations, inclusion of the random exchange does not change the main conclusion.
Equations for the OPDM follow from the Heisenberg equations \( i\hbar \dot{\Psi}_a = (H_0 + \xi_a)\Psi_a \). Employing the Furutzu-Novikov theorem [13] (see details in Appendix A), we obtain the exact equations for the OPDM as

\[
 i\hbar \dot{\rho}_{ii}(x, x', t) = [-H_{0x} + H_{0x'}] \rho_{ii}(x, x', t) - \frac{2i\hbar}{C(0)} \rho_{ii}(x, x', t); \quad (16)
\]

\[
 i\hbar \dot{\rho}_{12}(x, x', t) = [-H_{0x} + H_{0x'}] \rho_{12}(x, x', t) - \frac{2i\hbar}{C(0)} \rho_{12}(x, x', t). \quad (17)
\]

It should be noted that reinstating the inter-particle interaction will result in a chain of coupled equations for the correlators, where the mean involves averaging over the noise. On the each next step, the decoupling of the noise term can be done by simply adding the collision integrals obtained with no noise present. Inclusion of such integrals, practically, it is enough to neglect such corrections. Accordingly, Eqs.(16), (17) can be done by means of employing the Furutzu-Novikov theorem [13]. This will lead to finding the corrections to the correlators, where the mean involves averaging over the noise. On the each next step, the decoupling of the noise term.

We choose the initial state as a uniform condensate characterized by finite correlations between the components. Accordingly \( \rho_{ij}(x, x', t=0) = \rho_{ij}^{(0)} = \text{const} \) for all the components. As time proceeds, this state becomes destroyed. Indeed, the solution of Eqs.(16), (17) corresponding to the chosen initial condition is

\[
 \rho_{ii}(x, x', t) = e^{-2[C(0)-C(|x-x'|)]t/\hbar^2} \rho_{ii}^{(0)}, \quad (18)
\]

\[
 \rho_{12}(x, x', t) = e^{-2[C(0)+C(|x-x'|)]t/\hbar^2} \rho_{12}^{(0)}. \quad (19)
\]

The increase of the kinetic energy, then, follows from Eq.(13) as

\[
 K = K(t) = -\frac{N}{m} \nabla_{x=0}^2 C(|x|) t, \quad K(t) = \frac{NC(0)}{mL_p^2}, \quad (21)
\]

where the relations \( \rho_{ii}(x, x, t) = \rho_{ii}^{(0)}, \quad (\rho_{12}^{(0)} + \rho_{22}^{(0)}) V = N, \quad \text{with} \quad V \quad \text{being the total volume of the system, have been employed. We have also defined} \quad \text{the noise correlation length} \quad L_p \quad \text{through the relation} \quad \nabla_{x=0}^2 C(|x|) = -C(0)/L_p^2, \quad \text{and imposed the condition} \quad \nabla_{x=0} C(|x|) = 0. \]

We note that Eq.(21) accounts for the loss of the correlations between the components with the effective rate \( \tau_{rr}^{-1} \approx 2(C(0) + C(x-x'))/\hbar^2 \). This rate can be obtained more accurately from the equation for the correlator \( \rho_{12}(x, x', t, t' = 0) \), which describes the decay of these correlations and carries the information about the relative phase (below \( T_c \)). Applying the Furutzu-Novikov theorem [13] again, and assuming that the initial state is independent of the noise, we find

\[
 \rho_{12}(x, x', t, t' = 0) = e^{-C(0)t/\hbar^2} \rho_{12}^{(0)}, \quad t > 0. \quad (22)
\]

Thus, the rate of the loss of the phase memory is

\[
 \tau_{rr}^{-1} = \frac{C(0)}{k}, \quad (23)
\]

which is a factor of 2-4 different from the estimate obtained from the OPDM.

We define the time \( \tau_h \) as the time when the kinetic energy per particle becomes comparable to the BEC transition temperature \( T_c \) (so that the BEC is destroyed). Accordingly, employing Eq.(22),

\[
 \tau_h^{-1} \approx \frac{C(0)}{mT_c L_p^2}, \quad (24)
\]

Taking the ratio, we find

\[
 \frac{\tau_{rr}^{-1}}{\tau_h^{-1}} \approx \frac{mT_c L_p^2}{k} = \left( \frac{L_p}{L_c} \right)^2, \quad (25)
\]

where we have introduced the thermal length \( L_c = \hbar/\sqrt{mT_c} \) at the transition temperature.

Thus, the loss of the phase memory can occur faster than the destruction of the condensate, only if the correlation length \( L_p \) is longer than \( L_c \) [14]. From general physical considerations, it is clear that this condition is not realized.
in the actual traps, if the main source of the noise is the background gas. The correlation length \( L_p \) is simply the thermal length of the hot gas, which is obviously much shorter than the thermal length of the confined cloud at \( T_c \). It is also worth noting that similar conclusion can be reached in the general case, when the random field produces exchange between the components. Rigorous proof of this is also based on the Furutzu-Novikov theorem.

The losses from the trap do not change the situation. The rate of the decay of the phase memory is practically comparable to the rate of the escape from the trap. Physically, this is so because the losses do not introduce any long range correlations which may disrupt the global relative phase faster than the loss of particles. This conclusion is based on the analysis of the corresponding Wigner equations following from the OPDM, where the collision integral is derived from the term \((\ref{eq:collint})\)). Detailed account of this will be presented elsewhere.

It should be noted that a special attention should be paid to the situation of the quasi-equilibrium, when, e.g., the deposition to the trap and the evaporative cooling occur simultaneously, so that the trap remains at constant temperature and retains a fixed on average number of atoms. In this case, both rates of the deposition and of the losses can be large. Accordingly, the decoherence will be fast despite that no significant changes in the temperature and in the population are observed. The analysis of this situation will be presented elsewhere. Thus, if all the individual rates of disrupting the equilibrium are kept low on the experimental time scale, the correlations between the BEC components will persist on this scale.

Below we will analyze the situation when the white noise is long ranged. As it will be seen, the above conclusion does not hold in this case any more.

### B. The phase decoherence induced by noisy external long ranged potential

The noisy potential \( \xi \) could be induced by some external macroscopic fields. For example, some magnetic or electric fields can be imposed by extended macroscopic wires or by lasers. These fields fluctuate due to thermal and quantum effects. The magnetic trapping potential itself can fluctuate due to the fluctuations of, e.g., electric fields can be imposed by extended macroscopic wires or by lasers. These fields fluctuate due to thermal

Employing Eqs.\((\ref{eq:collint}), (\ref{eq:rhoint})\), and applying the Furutzu-Novikov theorem (see Appendix A ), we find the equations for the OPDM

\[
\hat{h} \hat{\rho}_{ii}(x, x', t) = \left\{ \frac{\hbar^2 (\nabla_x^2 - \nabla_{x'}^2)}{2m} - U(x) + U(x') - i \frac{\Delta}{\hbar} \right\} \rho_{ii}(x, x', t); \\
\hat{h} \hat{\rho}_{12}(x, x', t) = \left\{ \frac{\hbar^2 (\nabla_x^2 - \nabla_{x'}^2)}{2m} - U(x) + U(x') - i \frac{\Delta}{\hbar} \right\} \rho_{12}(x, x', t).
\]

We will employ the above equations for estimating the magnitude of the ratio of interest \( \tau_{\text{irr}}/\tau_h \) akin to Eqs.\((\ref{eq:irr}), (\ref{eq:tau})\). The loss of the correlations between the components is determined by the term \( \sim \Delta \) in Eq.\((\ref{eq:rho})\), so that the rate \( \tau_{\text{irr}}^{-1} \) can be estimated as

\[
\tau_{\text{irr}}^{-1} \approx \frac{\Delta}{\hbar} U^2
\]

where the bar denotes the effective averaging over the volume of the BEC.

The heating rate can be found from Eqs.\((\ref{eq:heating})\). We calculate the rate of the increase of the total energy in the trap as
\[ \mathcal{E} = \sum_i \int dx' \hat{x}' \frac{-\hbar^2}{2m} \nabla^2 x' + U(x') \hat{\rho}_{ii}(x, x', t) = \]
\[ \sum_i \int dx' \hat{x}' \frac{\Delta}{2m} \nabla^2 x' (U'(x) - U'(x'))^2 \hat{\rho}_{ii}(x, x', t) \approx N \frac{\Delta}{m} \left[ \nabla U(x)^2 \right]^2. \] (31)

Accordingly, we find from Eq. (31) \( \tau_h^{-1} \approx \frac{\Delta}{\Delta t \left[ \nabla U(x)^2 \right]^2} \). Then, the ratio \( \tau_{irr}^{-1} / \tau_h^{-1} \) becomes
\[ \frac{\tau_{irr}^{-1}}{\tau_h^{-1}} \approx \frac{mT \Delta U^2}{\hbar^2 \left[ \nabla U(x)^2 \right]^2}. \] (32)

If the potential is long ranged as, e.g., \( U' \sim r^\delta \) with some \( \delta > 0 \), and the cloud has a typical size \( r_c \), then, \( \Delta U^2 / [\nabla U(x)^2]^2 \approx r_c^2 \). Hence, we arrive at the estimate Eq. (22), where the role of the noise correlation length \( \mathcal{L}_n \) is played by the condensate typical size \( r_c \). Now it can be seen that, under the given circumstances, \( \tau_{irr}^{-1} / \tau_h^{-1} \gg 1 \), because \( L_c \ll r_c \). In other words, fluctuations of the trapping potential (or any other extended potential) destroy the inter-component coherence much faster than producing any substantial heating effect. This, however, does not mean that the rate of the decoherence is a significant issue in the actual traps, if the noise of the trapping potential is kept on the level of the thermal noise even at room temperature. Indeed, let us estimate the absolute value of the decoherence rate (30). For this purpose, we assume that the fluctuations \( \eta U' \) are of the same order as those of the trapping potential \( U \) caused by the thermal noise \( \delta I^{(c)} \) of the electric currents \( I^{(c)} \) which create the magnetic trap. Then, we set \( \eta \approx \delta I^{(c)} / I^{(c)} \), and impose the condition of the white noise: \( \langle \delta I^{(c)}(t) \delta I^{(c)}(t') \rangle = 2T \delta(t - t') \), following from the classical limit of the fluctuation-dissipation theorem. Here \( R \) stands for the coil resistance. Then we find \( \Delta = 2T / R I^{(c)}^2 \) in Eq. (23). For the mean in Eq. (30) we employ \( \Delta U^2 \approx \overline{U^2} \approx \mu_0^2 \), where \( \mu_0 \) stands for the mean chemical potential in the trap. Thus, Eq. (30) yields
\[ \tau_{irr}^{-1} \approx \frac{k_B T \mu_0^2}{\hbar^2 I^{(c)}^2 R}, \] (33)

where we have reinstated the Boltzmann constant \( k_B \). Eq. (33) gives \( \tau_{irr}^{-1} = 10^{-15} - 10^{-13} \text{s}^{-1} \), for typical values \( T = 300K, \mu / \hbar = 10^3 - 10^4 \text{s}^{-1} \), and for \( I^2 R = 1 \text{W} \). Thus, the instrumental thermal noise does not produce any significant decoherence.

C. Intrinsic decoherence of the global relative phase of the two-component BEC

In this section we will discuss mechanisms of the decoherence induced by self-interaction in the trap. In principle, the two-component condensate can be viewed as a system in which certain information is imprinted into the relative phase between the components. Any quantum phase memory device, which could be employed in quantum information processing, should be able to retain the phase information for long enough time. Thus, it is important to understand the fundamental mechanisms of the relative phase memory loss.

A well studied mechanism of the global phase decay at \( T = 0 \) - the phase diffusion [1] can be understood in terms of fluctuations of the chemical potential induced by binary interactions and by a finite variance of the number of particles in the quantum coherent state. These fluctuations produce collapse of the global phase on some time \( \tau_c \). It has also been found that the PD exhibits spontaneous revivals [1] on some revival time \( \tau_R \sim N \gg \tau_c \). The time \( \tau_R \), which is too long on any experimental time scale, can be considered as a duration of the Poincare cycle. The spontaneous revivals indicate that the dynamics is reversible in a sense that it repeats itself. This, however, does not necessarily mean that the dynamics of \( N \gg 1 \) particles can be physically reversed in time by imposing some physical external action. Thus, it is important to understand the physical reversibility of the quantum dynamics of the condensate on much shorter time scale \( \tau_c < \tau_R \). This question will be addressed later. We will show that the time \( \tau_c \) is not the time after which the phase memory is lost irreversibly. The phase information can be recovered (partially) by imposing the time reversal pulse.

Questions we are addressing here are: Can the evolution of the global relative be viewed as the PD process at \( T \neq 0 \)? If it is the PD, is this process physically reversible?

Before we start, it is worth noting that two conceptually very different situations should be clearly distinguished – i) the dynamics of the relative phase in the presence of the permanent exchange of bosons between the components, and ii) the phase dynamics with no exchange. In the case i), the relative global phase is characterized by some particular equilibrium value. Deviations from it increase the energy, and thereby activate irreversible processes of relaxation to the equilibrium. On the contrary, in the case ii), if once prepared due to a short exchange of bosons, the global
relative phase cannot be viewed as a normal mode, if no exchange of bosons exists during the following evolution (akin to the situation realized by the JILA group [5]). We consider the case ii) in this paper.

Here we will extend our analysis Ref. [10] of the problem of the global phase time-correlations to the two-component case. The central element of this analysis is the concept of the **projected** Hamiltonian and the **projected many body eigenstates**, which will be often called just "projected states", of this Hamiltonian. The total numbers of bosons in the each component play a role of parameters (not the operators) in such a Hamiltonian, if no losses or exchange between the components exist.

We represent an exact many-body eigenstate |m, N, M⟩, which is characterized by the following quantum numbers - the total number of bosons N = N_2 + N_1 and the half of the population difference M = (N_2 - N_1)/2 as well as by a set of the quantum numbers m referring to the normal excitations, as an expansion

$$|m, N, M⟩ = \sum_{n_{a1}, n_{a2},...} C_{n_{a1}, n_{a2},...}(m, N, M)|n_{10}⟩|n_{20}⟩|n_{11}⟩|n_{21}⟩..., \tag{34}$$

in the Fock space |n_{10}⟩|n_{20}⟩|n_{11}⟩|n_{21}⟩... of the population numbers n_{aj} of some set of single particle states. Here the Greek index refers to the component α = 1, 2, and the Latin index labels the single particle states in the corresponding component, so that n_{a0} stands for the population of the αth condensate component.

The expansion coefficients C_{n_{a1}, n_{a2},...}(m, N, M) form the Fock representation of the eigenstates. We call these coefficients the **projected states**, and introduce a short notation |m, N, M⟩ for them [10].

It is important to note that, while being orthogonal with respect to the set m for given N, M, the projected states are not orthogonal for different N, M. In other words, (M, N, m°|m, N, M⟩ = δ_m°_m and (M', N', m|m, N, M⟩ ≠ δ_m°_M δ_N°_N. Furthermore, in the case of the exact su(2) symmetry, the exact relation (M', N, m|m', N, M⟩ = δ_m°_m holds regardless of the values M, M'.

In 3D systems, close to equilibrium, the population factor of the condensate state in each component is macroscopic, and is characterized by relatively small fluctuations. If ∆n_{a0} stands for the fluctuation of the condensate population, and \(\overline{n_{a0}}\) is the mean population, the relative value ∆n_{a0}/\(\overline{n_{a0}}\) ≪1 [14]. This circumstance can be employed to express matrix elements of any condensate operator in terms of the overlap of the projected states. Specifically, let us say one wishes to find the matrix element (\(a^+_m a_{10}\) |m', N, m⟩ = (M, N, m|a^+_m a_{10}|m', N, M - 1) of the condensate operators. Employing the representation [14] and the smallness of the fluctuations of the condensate state populations, one finds

$$\langle a^+_m a_{10}\rangle_{m', m'} = \sum_{n_{a1}, n_{a2},...} \sqrt{n_{10}(n_{20} + 1)} C^*_{n_{a1}, n_{a2},...}(m, N, M)C_{n_{a1}, n_{a2},...}(m', N, M - 1) \equiv \sqrt{n_{10}n_{20}} \langle M, N, m|m', N, M - 1⟩ + o(\frac{\Delta n_{a0}}{\overline{n_{a0}}}). \tag{35}$$

Thus, the matrix elements of the condensate operators are given by the overlap of the corresponding projected eigenstates, with different values of the population difference. This simplifies considerably calculations of the correlators of the condensate operators.

Indeed, the global relative phase time-correlation properties are completely described by the correlator ρ_{12}(t) = ⟨a^+_m a_{10}(t)a^+_m a_{10}(0)⟩. Employing Eq. (35), one obtains

$$\rho_{12}(t) = \sqrt{n_{10}n_{20}} e^{iH(N, M)t} e^{-iH(N, M - 1)t} \tag{36}$$

within the chosen accuracy (here and below we employ units in which ℏ = 1). H(N, M) stands for the **projected** Hamiltonian defined as H(N, M) = \(\sum_m |m, N, M⟩⟩ E_m(N, M)|m, N, M⟩⟩, with E_m(N, M) being the exact many body eigenenergies [10]. The averaging is performed over the thermal ensemble. Explicitly, ⟨...⟩ = \(\sum_{m, N, M} P_m(N, M)|m, N, m⟩⟨m, N, M⟩, where P_m(N, M) stands for the canonical Boltzmann factor.

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3 The meaning of the projected Hamiltonian can be understood from the analogy with motion in a central potential, when the total angular momentum (and its z-component) is conserved. Due to the separation of the angular variables, the equation for the radial part can be formulated in terms of the Hamiltonian with some effective potential, which depends on the angular momentum as a parameter. This Hamiltonian for the radial part is the **projected** Hamiltonian.
The evolution operator \( e^{-iH(N,M-1)t} \) can be expressed as \( e^{-iH(N,M-1)t} = e^{-iH(N,M)t} e^{\text{Exp}(-i \int_{0}^{t} dt' H'(t'))} \), where the zeroth-order part is \( H(N,M) \) and the perturbation \( H' = H(N,M-1) - H(N,M) \) is taken in the interaction representation with respect to the zeroth-order part \( H'(t) = e^{iH(N,M)t} H' e^{-iH(N,M)t} \). Finally, Eq.(36) takes a form

\[
\rho_{12}(t) = \sqrt{\mu_{10}^{2} \langle \text{Exp}(-i \int_{0}^{t} dt' H'(t')) \rangle},
\]

\[
H'(t) = e^{iH(N,M)t} \{ -\frac{\partial H(N,M)}{\partial M} + \frac{1}{2} \frac{\partial^2 H(N,M)}{\partial M^2} + ... \} e^{-iH(N,M)t}
\]

within the chosen accuracy. It is clear that any correlator of the condensate operators can be represented in the form similar to Eq.(37), (38).

In fact, such an expansion is valid as long as a macroscopically populated BEC is present. Otherwise, the projected Hamiltonian is a very sharp function of \( M \), because a transfer of even a single atom between the thermal components inevitably corresponds to creating excitations in the case of the broken \( su(2) \) symmetry. In the presence of the BEC, the each derivative term in the expansion of \( H'(t) \) introduces an additional factor \( 1/N \). Accordingly, it is enough to keep the first term only, so that \( H' = \frac{\partial H(N,M)}{\partial M} \). \( H' \) can be viewed as an operator of the relative chemical potential \( \mu_m = E_m(N,M) - E_m(N,M-1) \approx \partial E_m(N,M)/\partial M \) in the given eigenstates. The diagonal matrix elements of \( H' \) give the exact values of the \( \mu_m \). The ensemble mean of it, then, becomes \( \overline{\mu} = \sum_{m,N,M} P(m,N,M) \mu_m = \langle \partial H(N,M)/\partial M \rangle \). It is important that \( \mu_m = H' = 0 \), if the intrinsic symmetry holds. Thus, another interpretation, \( H' \) is proportional to the change of the interaction energy per one transferred particle. As discussed in Ref. [10], the proportionality factor is of the order of unity in the one-component BEC. In the two-component situation, this factor should contain the combination of the scattering lengths which vanishes at the point of the intrinsic \( su(2) \) symmetry.

We employ the cumulant expansion of \( \langle M,N,m| \text{Exp}(-i \int_{0}^{t} dt' H'(t'))|m,N,M \rangle \). Then, within the accuracy \( 1/N \), we reproduce Beliaev’s result [17], \( \rho_{12}(t) \sim \exp(i\overline{\mu}t) \). We are, however, interested in the next non-vanishing effect of large but finite \( N \). Therefore, the next cumulant should be retained as well, and the higher ones can safely be neglected within the \( 1/N^2 \) accuracy. Accordingly, we write

\[
\langle M,N,m| \text{Exp}(-i \int_{0}^{t} dt' H'(t'))|m,N,M \rangle = \exp\{i\mu_m t - \int_{0}^{t} dt' \int_{0}^{t'} d\tau \langle M,N,m| \delta H'(|m,N,M\rangle + o(1/N^2)\}
\]

where we have introduced the notation \( \delta H'(|m,N,M\rangle \) for the off-diagonal part of the total matrix \( H'(|m,N,M\rangle \). The operator \( \delta H'(|m,N,M\rangle \) can be viewed as fluctuation of \( H' \).

As it has been discussed in Ref. [14], the double integral in the exponent determines the processes when adding or removing one boson to or from the BEC disturbs the normal component. In principle, a situation can be contemplated when such a disturbance is so strong that this term dominates, and rapidly suppresses the overlap of the projected states (differing in the total number of bosons by 1). This situation is similar to the orthogonality catastrophe (OC) occurring in Fermi liquids [15]. In Beliaev’s work [17], however, it has been proven that such processes are insignificant at \( T = 0 \). Thus, no the OC is to be anticipated in the BEC at \( T = 0 \). This implies that the double integral in Eq.(39) is finite and small in the limit \( t \rightarrow \infty \).

At finite \( T \), without a rigorous proof, the Beliaev’s result is widely applied to higher eigenstates [18]. Below, we will employ Eq.(39) and give semi-quantitative general arguments which support the Beliaev’s result at finite \( T \). First, we note that the correlator in Eq.(38) can be calculated within the perturbation theory with respect to interaction between the quasiparticles. This has been done in the case of the single-component BEC in Ref. [19]. The result of such calculations can be represented as

\[
\sum_{m,N,M} P(m,N,M) \langle M,N,m| \text{Exp}(-i \int_{0}^{t} dt' H'(t'))|m,N,M \rangle = e^{\overline{\mu}t - (\tau_d)^2 - t/t_{OC}},
\]

where

\[
\tau_d^{-2} = \frac{1}{2}(\mu - \overline{\mu})^2, \quad t_{OC}^{-1} \approx \int_{0}^{\infty} dt \langle \delta H'(|m,N,M\rangle \delta H'(|m,N,M\rangle \rangle.
\]

The gaussian factor in Eq.(40) is a result of applying the central limit theorem for the averaging of \( e^{i\mu_m t} \) over the ensemble. The quantity \( t_{OC} \) has been called the OC time [10]. It determines how rapidly the overlap decays. Its decay is controlled by excitations. The rate \( t_{OC}^{-1} \) can be estimated as
where $\tau_n$ stands for a typical relaxation time of the correlator $\langle \delta H'(\tau)\delta H'(0) \rangle$ which is taken in the exponential form $\sim \exp(-t/\tau_n)$. In 3D system the time $\tau_n$ is finite because of very small statistical weight of the low energy modes. This time is given by a typical relaxation time of the normal modes characterized by the eigenenergies around the chemical potential. Obviously, $\tau_n \ll \tau_d$ (11). Thus, taking into account the nature of the operator $H'$ discussed above (see below Eq.(23)), we conclude that $t_{oc}^{-1} \sim 1/N$, because $\langle \delta H'(0)\delta H'(0) \rangle \sim 1/N$ as fluctuation of any extensive quantity taken per one particle. It should, however, be noted that the form (11) is not valid in the limit $t \to \infty$. In Ref. [4], we have discussed that the double integral in Eq.(23) should remain constant and small in the limit $t = \infty$ ($t \geq t_{oc}$) even at $T \neq 0$. The reason for this is the following: the spectral weight of the correlator $\langle \delta H'(t)\delta H'(0) \rangle$ in the limit of zero frequencies is effectively collected from the lowest order processes of scattering of the quasiparticles with almost zero energy transfer. The term $H'$, which is actually proportional to the interaction energy, removes the degeneracy between the corresponding quasiparticle states. This suppresses the spectral weight in a very similar manner to the effect of the level repulsion studied in the Random Matrix Theory [20]. This is essentially a non-perturbative effect, and it becomes important at frequencies smaller than a typical value of the interaction matrix element, which is scaled effect of the level repulsion studied in the Random Matrix Theory [20]. This is essentially a non-perturbative effect, and it becomes important at frequencies smaller than a typical value of the interaction matrix element, which is scaled as $\sim 1/\sqrt{N}$ [10]. Thus, the exponential factor $e^{-t/t_{oc}}$ in Eq.(10) is correct only for times shorter than some time $t' \sim \sqrt{N}$, and it levels off to a constant $\sim \exp(-t'/t_{oc}) \approx 1 + o(1/\sqrt{N})$ (note that $t_{oc} \sim N$) as $t$ becomes longer than $t'$.

The quantity $\tau_d$ is determined by the ensemble fluctuations of the chemical potential, and it describes the rate of the phase diffusion at finite temperature. We will call it the thermal PD in order to distinguish it from the quantum PD.

The evolution of the condensate operators can be viewed as a result of the ensemble averaging of the correlators (36) simply becomes

$$\rho_{12}(t) = \sqrt{\tau_{d20}} \sum_{m,N,M} P(m,N,M) e^{i\mu_m t} = \sqrt{\tau_{d20}} e^{-t/\tau_d}$$

within the central limit approximation.

The dephasing time $\tau_d$ can be calculated in any order with respect to the interparticle interaction. For this purpose, in the weakly interacting system, it is enough to consider the Bogolubov gas of the non-interacting quasiparticles as a model of the excited states. Specifically, the eigenenergies are represented as $E_m(N,M) = E_0(N,M) + \sum_a \epsilon_a(N,M)n_a$, where $E_0(N,M)$ is the ground state energy and $\epsilon_a(N,M)$ stands for the spectrum of the quasiparticles characterized by the population factors $n_a$. Then, Eq.(11) yields

$$\tau_d^{-2} = \frac{1}{2} \left( \sum_a \frac{\partial \epsilon_a(N,M)}{\partial M} n_a - \left( \sum_a \frac{\partial \epsilon_a(N,M)}{\partial M} n_a \right)^2 \right) + \tau_c^{-2},$$

$$\tau_c^{-2} = \frac{1}{2} \left( \frac{\partial E_0(N,M)}{\partial M} - \left( \frac{\partial E_0(N,M)}{\partial M} \right)^2 \right),$$

where $\tau_c$ stands for the quantum phase diffusion (at $T = 0$) collapse time. The quantum PD has been analyzed by many authors in Refs. [1] in the case when the initial state is the coherent state with well defined global phase. We note that, in the two-component BEC, the shot noise in the initial deposition of $N$ should also contribute to the dephasing of the condensate correlators at $T = 0$ even in the situation when no initial global relative phase existed. Furthermore, in actuality, the shot noise is most likely to wash out completely the effect of the quantum PD in Eq.(13). In our work [8], we discussed this in detail, and have found that in $^{87}$Rb the collapse time $\tau_c$ can be as short as 30 ms, if the shot noise reaches values about 10%. This should be compared with typical estimates of the quantum PD collapse times $\sim 1 - 10^8$ [6].
It should also be noted that \( \tau_d^{-1} = 0 \) in the case of the exact \( su(2) \) symmetry. Here we will not focus on calculating the dephasing rates for the specific geometries \( \mathbb{R}^3 \), and will use it as a parameter.

Summarizing, the evolution of the condensate operators does not disturb the normal component, and, as a consequence, the correlators of the condensate operators exhibit reversible dephasing determined by the ensemble fluctuations of the relative chemical potential. In the next section we will discuss how such a reversibility can be tested in the atomic echo experiment.

IV. ATOMIC ECHO

As discussed above, the evolution of the condensate operators in 3D system including finite number of bosons does not disturb significantly the normal component. The decoherence of the condensate operators is a result of the ensemble averaging, with no decoherence occurring in any eigenstate. In this section, we will address the question of physical reversibility, so that an initial phase information can be recovered on times longer than the dephasing time \( \tau_d \).

In general, the echo effect (spin echo, photon echo \( \mathbb{R}^2 \), plasma echo \( \mathbb{R}^3 \)) – stimulated revival of some quantity – is widely employed as a test for reversibility of dynamics. Once initially prepared, the quantity may exhibit an apparent decay, which is a sort of inhomogeneous broadening without actual loss of the memory of the initial state. Then, if some time-reversal pulse is imposed at later time \( T \), which can be much longer than the dephasing time \( \tau_d \), the quantity may revive completely or partially at time \( t \approx 2T \); and, if the time reversal pulse is strong, at integers of \( T \) (multiple echoes).

Generally speaking, any Hamiltonian dynamics is reversible in time. This means that the formal time-reversal of the evolution will result in the restoration of the initial state. This, however, cannot always be done physically, that is, by applying some physical disturbance to the system. Thus, a proper criterion for the physical reversibility of the dynamics should be introduced. We will employ the criterion which relies on the strength of the echo. Specifically:

If an external pulse can induce the echo whose strength is significantly larger than fluctuations (quantum or statistical) of the measured quantity, the evolution of this quantity can be considered (partially) reversible.

It is important to realize that the echo in the condensate can be observed even though the time delay \( T \) is much longer than typical relaxation times of the excitations in the system and/or in the bath. The time delay should be shorter than the irreversibility time \( \tau_{irr} \). As discussed in Sec. III, this time is determined by the strength of the extrinsic fluctuations, which can even be treated within the \( \delta \)-correlated approximation, that is, within the assumption of infinitely short relaxation times.

In our work \( \mathbb{R}^3 \), we have presented the echo solution in the two-component BEC in the case of the shot noise dominated dephasing. In this situation, the echo strength reaches 100\%, and is achieved by imposing the \( \pi \)-type time reversal pulse. Here we will analyze the echo in the case of the intrinsic dephasing, that is, when either the quantum or thermal PD takes place.

A. Atomic echo at \( T = 0 \)

Let us consider the echo experiment in the context of the three pulses scheme \( \mathbb{R}^3 \). We take \( U_1 \) as the evolution operator which corresponds to the \( \pi/2 \) pulse which converts the initial state with \( N_2 = N_1 = 0 \) into a coherent state of the two condensates \( \mathbb{R}^3 \). Then, some time-reversal pulse \( U' \) is imposed at time \( T > 0 \) later, and, finally, the read out \( \pi/2 \) pulse \( \mathbb{R}^3 \) is imposed at time \( t > T \).

We choose, as a measured quantity, the population difference represented by the operator \( I_z \) in Eq.(2) and measured by the JILA group \( \mathbb{R}^3 \). A state vector is now labeled as \( |j, M\rangle \) by the conserving total angular momentum \( j = N/2 \) and by its \( z \)-projection \( M \). At \( T = 0 \), the excitation label \( "m" \) can be suppressed. Then, the mean after the read out pulse becomes

\[
\langle I_z \rangle = \langle j, j| U_1 e^{iTH(U')}\exp[i(t-T)H]U_1^\dagger J_z U_1 e^{-i(t-T)H}U' e^{-iTH}U_1^\dagger |j, j\rangle .
\] (47)

The pulse operators \( U_1, U' \) can be found in the sudden approximation, if the duration of the pulses is short. Then, a general pulse operator is \( U(|v|, \phi) = \exp(-i(v^* I_z + H.c.)) \), where \( v_+ = \int dt \Omega(t) = |v| \exp(i\phi) \) is the time integral (over the pulse duration) of the Rabi frequency \( \Omega(t) \) in Eq.(2), and the operators \( I_{\pm} \) are represented in Eq.(2). The value \( |v| \) is the magnitude of the pulse, and \( \phi \) for its phase. Specifically, for the \( \pi/2 \)-pulse, one can choose \( U_1 = U(\pi/4, \phi = -\pi/2) = \exp(i\frac{\pi}{4} I_y) \). For the time reversal pulse, we will take \( U' = \exp(i\beta I_y) \), where \( \beta \) stands for the pulse strength. The dependence of the echo strength on \( \beta \) will be investigated below.
It is worth noting that the operator $e^{iβI_θ}$ is the operator of finite rotations, and $β$ stands for the corresponding Euler angle. An explicit form for the matrix elements $d_j^{M|j}(θ) = \langle M', j|e^{iβI_θ}|j, M \rangle$ is well known for any arbitrary dimension of the $su(2)$ representation (see, e.g., in Ref. [23]). For example,

$$d_j^{M}(θ) = \langle j, j|e^{iθI_θ}|j, M \rangle = \sqrt{\frac{(2j)!}{(j+M)!(j-M)!}} \left( \cos \frac{θ}{2} \right)^{j+M} \left( \sin \frac{θ}{2} \right)^{j-M}$$

(48)

where $M = -j, -j + 1, ..., j - 1, j$. The expression for arbitrary $M, M'$ involves Jacobi polynomials, and can be found in Ref. [23].

Thus, the mean $⟨z⟩$ can be calculated as a finite product of the matrices $N + 1$ by $N + 1$. In our work [8] we have analyzed the exact echo solution, when the main source of the dephasing is the shot noise. In what follows, we study the echo effect in the situation of the quantum phase method. As can be seen, $⟨24⟩$. This approximation simplifies calculations of the mean $⟨47⟩$ considerably. On Fig.1, we have compared the results $J_2$, which yields both the limiting representation of the Bessel functions, and the Jacobi polynomials $\hypergeometric{3}{3}{j, M, K}$.

The complete representation of the mean $⟨47⟩$ becomes

$$⟨z⟩ = 2Re \sum_{M, M', K} d_j^{(M')M} d_j^{(M)(M')} (\frac{z}{2})^{(M')M} J_{M'-1}(βj) J_{M-1}(βj) \times$$

$$e^{i(l(E_0(M)-E_0(M'))+E_0(M'-1))(t-T)},$$

$$E_0(M) = (ε_0 + b_1 N) M + \frac{b_2}{2} M^2 + \frac{b_3}{2} N^2,$$
where $(I_x)_{K-1} = (I_x)_{K-1}$ stands for the matrix elements of the operator $I_x$; we have introduced the ground state eigenvalue $E_0(M)$ of the Hamiltonian [1] as a function of the half of the population difference $M$ and of the total number of bosons $N$, with $b_{1,2,3}$ being the coefficients which can be obtained from Eq.(1) within, e.g., the two-mode approximation [23]. The results of the numerical calculations of Eq.(52) are presented on Fig.2 (for $b_1 = 0$). As can be seen, the multiple echoes occur at times $t \approx kT$, $k = 2, 3, 4,...$.

The multiple echo solution can be found, practically, exactly in the limit $T \gg \tau_e = 2(\sqrt{b_2})^{-1}$. As it will be seen later, the maximum echo is achieved when $\beta \approx \tau_e/(T\sqrt{J})$, so that the argument of the Bessel functions is $\approx \sqrt{\tau_e/T} \ll \sqrt{J}$. The Bessel function $J_{n-m}(z)$ becomes essentially zero when its order $|n-m| > |z|$ [24]. Thus, in Eq.(52), while the index $M$ runs over the range $\approx [\sqrt{\tau_e/T}, \sqrt{J}]$, the other indices stay relatively close to $M$. Accordingly, in the sum (52), we approximate $\sqrt{(j + K)/(j - K + 1)} \approx j$. Then, the sum over $K$ can formally be extended from $-\infty$ to $+\infty$. This, allows to calculate this sum in a closed form by employing Eq.(53) and by implementing the identity (9.1.79) of Ref. [24]. Accordingly, Eq.(52) acquires the form

$$
(I_x) = \sum_{L=1}^{\infty} J_{L+1}(\beta jb_2(t - T))e^{-(t-\tau(L+1)^2)/\tau^2} \cos[\epsilon_0(t - T(L+1)) + \pi(L+1)/2],
$$

where we have introduced the notation $L = M' - M$, and have eliminated all the terms with $L \leq 0$ because their contributions become significant only at negative times. We also employed the central limit theorem and made the replacement $\sum_M d_M^j d_M^- j^j/\sqrt{2\pi} \approx \exp[ib_2(M + L/2)(t - T(L+1))] \approx \exp[-(t - T(L+1))^2/\tau^2]$, which is valid for $L \ll \sqrt{J}$.

Eq.(54) describes the stimulated revivals – the echoes. They occur at the times $t \approx T(L+1)$, $L \geq 1$. The analytical solution (54) is shown on Fig.2 together with the numerical one, both calculated within the quantum phase method. The agreement is quite good.

It is interesting to note that the magnitude of the echoes exhibits non-monotonous dependence on the strength $\beta$ of the time-reversal pulse. Another feature is that the maximum echo can be created by the weak time-reversal pulse. Let us consider this in detail for the first echo ($L = 1$) at the maximum of the gaussian, which occurs at $t = 2T$. Then, the relative echo magnitude becomes

$$
|I_x| = J_2(\beta jb_2T) = J_2(2\beta \sqrt{J}T/\tau_e),
$$

where we have expressed $b_2$ in terms of the collapse time, which follows from Eqs.(10), (53) as $\tau_e^{-1} = \sqrt{J}b_2/2$ (we ignore the shot noise). Taking into account that the first maximum $J_2(z) \approx 0.5$ occurs at $z \approx 3$ [24], we find the magnitude of the strength of the time-reversal pulse

$$
\beta_{\text{max}} \approx 1.5 \frac{\tau_e}{T\sqrt{J}}
$$

required to achieve the maximum echo. Conversely, $J_2(z) = 0$ for $z \approx 5$ for the first time, and the echo vanishes for $\beta \approx 1.7\beta_{\text{max}}$. Taking the higher value of $\beta$ results in the oscillatory dependence of the echo strength on $\beta$, with the total amplitude slowly diminishing as $\sim 1/\sqrt{\beta}$. Specifically, for $\beta \sim 1$, Eq.(55) yields $|I_x|/j \sim j^{-1/4} \ll 1$. As it will be seen later, at $T \neq 0$ this dependence can cross over to $|I_x|/j \sim j^{-1/2} \ll 1$, which makes the echo practically zero because it is on the level of the statistical noise.

The above result indicates that, in order to revive the quantum phase, which exhibits the quantum phase diffusion [8], it is enough to exchange coherently the number of bosons $\approx \beta j \sim \sqrt{N}$ between the components.

In the next section, we will consider the effect of finite temperature on the echo.

**B. Echo at $T \neq 0$**

As discussed in Sec.IIIC, the condensate operators acting on the eigenstates do not mix them with other eigenstates. This is stated by Eq.(13), which is an extension of the Beliaev’s result [17] obtained for $T = 0$, and which is widely employed at $T \neq 0$ [13]. Nevertheless, a special attention should be focused on calculating the matrix elements of the time reversal pulse $e^{i\beta b_2}$ at finite $T$, when the normal component is present. Indeed, a formal separation of the operator $J_y \sim N$ into the condensate and the normal parts could be expected to produce a sort of the Debye-Waller thermal factor $\exp(-\beta^2 N')$, where $N'$ stands for the number of particles in the thermal cloud. This logic, however, does not take into account the symmetry considerations.
We start our analysis of this problem by noting that, in the case of the exact $su(2)$ symmetry, the eigenstates $|m, N, M\rangle_{(0)}$ of the symmetric Hamiltonian $H_{(0)}$ form the $su(2)$ representations for every value from the set of the excitation quantum numbers $m$. This set should contain also the value $j$ characterizing the dimension of the corresponding representation. To emphasize this, we will use the notation $|m, j, N, M\rangle$ for the state $|m, N, M\rangle_{(0)}$ characterized by some value $j$. If in the initial state, all the bosons belong to one component (as in the JILA experiment \[5\]), this value is $j = N/2$, and it remains unchanged during following evolution. This is so because the Hamiltonian forms a closed algebra with the generators $[6]$. Accordingly,

$$\langle M, N, j, m | e^{i\beta I_3} | m', j, N, M' \rangle = d^{(j)}_{M,M'}(\beta) \delta_{m,m'}$$

(57)

for the initial set of states characterized by some $j$.

The situation is different if the intrinsic symmetry is broken, so that the Hamiltonian does not form a closed algebra any more. Then, it is natural to expect that the value $j$ is not a good quantum number under this circumstance. Furthermore, it is clear that, at $T > T_c$, this value should relax to $j \sim \sqrt{N}$, even though initially it was $j = N/2$. Indeed, the square of the total angular momentum operator (2) is $I_3^2 = I_+ I_- + I_- I_+$. At the point of equal populations, the means $\langle I_\pm \rangle = 0$, $\langle I_+ I_- \rangle \sim N$ and $\langle I_+ I_- \rangle \sim N$, where we have employed Eq. (3), and have taken into account that no long range order exists above $T_c$. This should be contrasted with the situation at $T = 0$, when practically all bosons occupy the condensate states, and $\langle I_+ I_- \rangle \sim N^2$.

What happens at $T \neq 0$ and $T < T_c$? The dimension of the dominant representation is given by the numbers of the condensate atoms. Indeed, keeping in mind the explicit form (2), one can write $\langle I_+ I_- \rangle = \int d\Psi_1(x)\Psi_1(x) d\tilde{\Psi}_1(x')\tilde{\Psi}_1(x') = N \sum_{n=1} N \sum_{n=1} (1 + o(1/N)) = (N/2)^2$ for equal populations in the uniform case. Furthermore, the correlator $\langle I_+ I_- (t) I_+ I_- (0) \rangle$ remains time independent as long as the BEC is present. Indeed, employing Eq. (43), it is easy to show that this correlator does not depend on time in the main $\sim N^2$ limit. Thus, the dimension of the representation is practically selected by the initial condition.

Now let us consider how Eq. (47) changes when the $su(2)$ symmetry is broken intrinsically, provided the population difference $M$ is still conserved (due to the absence of the exchange between the components). We employ Eq. (57) in calculating the matrix element and find $\langle M', N, m' | e^{i\beta I_3} | m, N, M \rangle = \sum_{n,j} d^{(j)}_{M,M'}(\beta) \langle M', N, m' | n, j \rangle \langle n, j | M, m \rangle$. This is equal to $\sum_n d^{(j)}_{M,M'}(\beta) \langle M', N, m' | n, j \rangle \langle n, j | M, m \rangle \times \langle M, N, j, n | m, N, M \rangle$ due to the property of the projected states that their overlaps are equal to the overlaps of the corresponding eigenstates for the same values of $M, N$. Finally, due to the orthonormality of the projected states in the subspace with given $j$, we find

$$\langle M', N, m' | e^{i\beta I_3} | m, N, M \rangle = \sum_{n,j} \langle M', N, m' | \mathcal{P}_j | m, N, M \rangle \times \langle M, N, j, n | m, N, M \rangle$$

(58)

where $\mathcal{P}_j$ projects the eigenstates to the space of functions with angular momentum $j$, and the lower limit is either 0 or 1/2, depending on whether $N$ is even or odd, respectively. Eq. (58) is a generalization of Eq. (47). This equation gives the representation of the matrix elements of the pulse operator in terms of the universal values $d^{(j)}_{M,M'}(\beta)$ and the overlaps of the exact projected eigenstates. Employing Eq. (58) as well as the orthonormality of the projected eigenstates for the same $M$, we find the following representation for Eq. (47)

$$\langle I_\pm \rangle = \sum_{j_1,j_2,j_3} \sum_{M_1, M_2, j} d^{(j_1)}_{M_1, M_2} \frac{\pi}{2} d^{(j_2)}_{M_2, M_3} (-\beta) \langle I_\pm \rangle^{(j_2)}_{M_2, M_3} d^{(j_3)}_{M_3, M_4} (-\beta) d^{(j_3)}_{M_4,j} \times G(M_1, M_2, M_3, M_4, t)$$

(59)

where $j = N/2$; $\langle I_\pm \rangle^{(j_2)}_{M_2, M_3}$ stands for the matrix elements of the operator $I_\pm$ calculated in the space of the angular momentum $j_2$; and

$$G \left( M_1, M_2, M_3, M_4, t \right) = \sum_m P_m(N) \langle j, N, m | e^{iTH(M_1)} P_{j_1} e^{i(t-T)H(M_2)} P_{j_2} e^{-i(t-T)H(M_3)} P_{j_3} e^{-iTH(M_4)} | m, N, j \rangle$$

(60)

It is important to note that the projected eigenstates $|m, j, N, M\rangle$ do not depend on $M$. To continue the parallel between the projected states and the radial part of the wave function in a central potential (see Sec. III C), we note that this radial part does not depend on the $z$-projection $M$ of the angular momentum.
where we have chosen the initial state characterized by \( j = N/2 \) (initial populations \( N_2 = N, \ N_1 = 0 \)) and have suppressed the parameter \( N \) in the notation \( H(N, M) \) of the projected Hamiltonian: \( \hat{P}_m(N) \) denotes the canonical normalized thermal distribution of the initial states (characterized by \( M = j, \ j = N/2 \)).

We note that Eqs. (52), (60) are exact. They, however, can conveniently be employed only if the BEC is present. As discussed above, in the case of the dominant population of the BEC state, the dimension \( 2j+1 \) of the \( su(2) \) representation is dictated by the initial condition \( N_2 = N, \ N_1 = 0 \). Accordingly, one can choose \( j_1 = j_2 = j_3 = j = N/2 \) (that is, omit the external summation in Eq.(59)), and set \( \mathcal{P}_j = 1 \) in Eq.(58). At \( T = 0 \), Eq.(59) naturally transforms into Eq.(52), if one chooses the only term \( m = 0 \) , that is the ground state, in the sum (60).

In what follows, we will only estimate the factor (60) for the first echo within the cumulant expansion, which was employed in Sec.IIIC and is outlined in Appendix B. We consider temperatures low enough, so that the fluctuations of the condensate populations \( \xi_j(M) \) can be ignored.

The suppression of the echo at \( t = 2T \) is described by the factor (60), where we set \( t = 2T \). Employing Eq.(39) as shown in Appendix B, we find

\[
(I_2(2T)) \approx j \sum_{M,M'} d_{j,M}^{(j)}(\pi/2)d_{M+1,j}^{(j)}(-\pi/2)J_{M-M}(\beta_j)J_{M-M-2}(\beta_j) \times \sum_m \hat{P}_m(M)e^{iT\{E_m(M-1) - E_m(M) + E_m(M') - E_m(M'-1) - [2(M-M'-\frac{1}{2})^2 + \frac{3}{2}t_{OC}^{-1}\}},
\]

(61)

where we have taken into account that the effective values of \( M, M' \) are much less than \( j = N/2 \); the definition of the effective distribution function \( \hat{P}_m(M) \) created by the first pulse is given in Appendix B, Eq.(B3). We have approximated the matrices of finite rotations by the Bessel functions similarly to how it was done in Eq.(52). The matrix \( I_2 \) has also been replaced by \( j/2 \) in this space, and, accordingly, the terms \( M_1 = M_2 \pm 1 \) have been selected.

We note that Eq.(52) (with \( t = 2T \)) is, in fact, the limiting case \( t_{OC} \to \infty \) of Eq.(51). Thus, we have actually assumed that the limit \( t_{OC} \to \infty \) corresponds to \( T = 0 \), and, conversely, \( t_{OC} \) becomes small enough to include its effect at high \( T \). Here we will not calculate a specific expression for \( t_{OC} \) as a function of \( T \).

In order to obtain a closed form expression for Eq.(51), we employ the smallness \( |M/N| \ll 1, \ |M'/N| \ll 1 \), and, accordingly, expand the eigenenergies as

\[
E_m(M) = E_m(0) + \epsilon_m M + \frac{1}{2} b_{2m} M^2 + o(1/N^2),
\]

(62)

\[
\epsilon_m = \frac{\partial E_m(M)}{\partial M} \bigg|_{M=0}, \quad b_{2m} = \frac{\partial^2 E_m(M)}{\partial M^2} \bigg|_{M=0},
\]

(63)

around the point \( M = 0 \) of equal populations. Then, Eq.(51) becomes

\[
(I_2(2T)) \approx j \sum_K J_K(\beta_j) J_{K-2}(\beta_j) e^{i T \left( \frac{b_2}{N} K \right) - \frac{3}{2} t_{OC}^{-1}}
\]

(64)

where we have ignored the fluctuations of \( b_{2m} \sim 1/N \) and wrote \( \sum_m \hat{P}_m(M) e^{i T(M - M')} b_{2m} = e^{i T(M - M') \bar{b}_2} \), with \( \bar{b}_2 = \sum_m \hat{P}_m(0) b_{2m} \). It is worth noting, that, in the limit \( t_{OC} \to \infty \), Eq.(54) yields the first echo magnitude (55) (where the value \( b_2 = b_{2m=0} \) is replaced by \( \bar{b}_2 \)). We have also taken into account that

\[
\sum_M d_{j,M}(\pi/2) d_{M+1,j}(\pi/2) = 1 \quad \text{with a very good accuracy in the limit} \quad j \gg 1.
\]

The exponential factor in Eq.(54), containing \( t_{OC}^{-1} \), produces some suppression of the echo strength. As can be verified numerically, the sum (54) at its maximum decreases from the value about 0.45, which corresponds to \( t_{OC} = \infty \), to the value about 0.06, corresponding to \( t_{OC} \approx 2T \). A change is also exhibited by the value of \( \beta_j \) at the maximum of the echo. It shifts from the value given by Eq.(54) \( \beta \approx 1/\sqrt{T} \) (in the limit \( t_{OC} = \infty \)) to \( \beta \approx 4.5/j \) for \( t_{OC} \approx 2T \). This can be understood as follows, as \( t_{OC} \to 2T \), only few Bessel functions depending on the product \( \beta_j \) contribute to the sum (54), and \( \beta_j \approx 4.5 \) corresponds to the maximum of this contribution. As we discussed in Sec.IIIC, \( t_{OC} \sim N \) as long as the population of the condensate is comparable to \( N \). Thus, practically, \( t_{OC} \gg T \approx \tau_4 \sim \sqrt{N} \). This implies that the thermal effects do not suppress the echo substantially, if compared with the zero \( T \) limit. It should, however, be noted that at temperatures high enough, the fluctuations of the condensate population \( \xi_j(M) \) would lead to the necessity of including the summation over \( j_{1,2,3} \) in Eq.(54). This problem will be analyzed elsewhere.

---

6 Otherwise, the complete form (59) with the summation over \( j_{1,2,3} \) should be analyzed.
We also note that external factors, which introduce irreversible decoherence, may suppress the echo significantly. If a typical time of the loss of the coherence due to these factors is $\tau_{irr}$, as given by, e.g., Eq. (24), the first echo intensity will acquire the exponential factor $\sim e^{-2T/\tau_{irr}}$, and, accordingly, no echo will be practically observed in the situation $T > \tau_{irr}$.

**V. SUMMARY**

We have shown that the intrinsic symmetry in the multi-component atomic mixtures has critical impact on the decoherence of the Ramsey fringes. This symmetry should be broken in one way or another in order to induce damping. The extrinsic factors, such as, e.g., the interaction with the background gas, can produce the decoherence of the inter-component correlator. This decoherence, however, is not faster than the rates of the heating and of the induced losses.

We have shown that, in the two-component BEC, the intrinsic decoherence is a result of the ensemble fluctuations of the relative chemical potential. In other words, the evolution of the condensate operators occurs as though the normal component is not affected by it. Therefore, this evolution can be considered reversible.

We have analyzed the atomic echo effect in the Ramsey spectroscopy of the two-component BEC. The strength of the echo is obtained in the case of the phase diffusion at zero and finite temperatures. We have found that the echo survives finite temperatures, as long as the population of the condensate state remains dominant.

We also point out that the echo effect depends essentially on the nature of the many-body correlations, and as well as on the deviations from the intrinsic $su(2)$ symmetry. Accordingly, its measurement would provide valuable information on the nature of the many body correlations in the trapped multi component BEC. Because of this, we suggest that the atomic echo effect should be studied experimentally.

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**APPENDIX A: EQUATION FOR OPDM IN THE PRESENCE OF WHITE NOISE**

Furutzu-Novikov [3] theorem states that

$$\langle \xi(x, t)Z(x', t') \rangle = \int \, dy \int \, d\tau \langle \xi(x, t)\xi(y, \tau) \rangle \frac{\delta Z(x', t')}{\delta \xi(y, \tau)},$$

(A1)

where the averaging is performed over the gaussian noise $\xi(x, t)$; $Z(x, t)$ is an arbitrary smooth functional of $\xi$. This relation can be verified by expanding $Z(x, t)$ in the functional powers of $\xi$ and by comparing the l.h.s. and the r.h.s. term by term.

We apply (A1) to the Heisenberg equations, following from Eq.(14) and written for the operators $\hat{\rho}_{ij}(x, x', t) = \Psi_i^\dagger(x, t)\Psi_j(x', t)$.

$$i\hbar \partial_t \hat{\rho}_{ij}(x, x', t) = [-H_0x - \xi_i(x, t) + H_{0x} + \xi_j(x', t)]\hat{\rho}_{ij}(x, x', t), ~ \xi_1 = -\xi_2 = \xi,$$

(A2)

where no two-body interaction is taken into account. We average Eq.(A2) over the initial condition and the noise, and obtain for the OPDM $\rho_{ij}(x, x', t) = \langle \rho_{ij}(x, x', t) \rangle$:

$$i\hbar \partial_t \rho_{ij}(x, x', t) = [-H_0x + H_{0x}^\dagger] \rho_{ij}(x, x', t) + \langle [-\xi_i(x, t) + \xi_j(x', t)]\rho_{ij}(x, x', t) \rangle. $$

(A3)

Then, we represent the mean $\langle [-\xi_i(x, t) + \xi_j(x', t)]\rho_{ij}(x, x', t) \rangle$ in accordance with Eq.(A1), and write down the equation for the functional derivative as

$$i\hbar \frac{\partial}{\partial t} \frac{\delta \rho_{ij}(x, x', t)}{\delta \xi(y, \tau)} = [-H_0x + H_{0x}^\dagger] \frac{\delta \rho_{ij}(x, x', t)}{\delta \xi(y, \tau)} + \langle [-\xi_i(x, t) + \xi_j(x', t)]\frac{\delta \rho_{ij}(x, x', t)}{\delta \xi(y, \tau)} \rangle$$

(A4)

$$+ [-\delta(x - y) + \delta(x' - y)] \langle \rho_{ij}(x, x', t) \rangle \delta(t - \tau).$$

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We take into account the causality, so that \( \delta \rho_{ij}(x, x', t)/\delta \xi(y, \tau) = 0 \) for \( t < \tau \). Then, we find

\[
\frac{\delta \rho_{ij}(x, x', t)}{\delta \xi(y, \tau = t - 0)} = \frac{i}{\hbar} \left[ -\delta(x - y) + \delta(x' - y) \right] \langle \rho_{ij}(x, x', t) \rangle
\]

from Eq.(A4). We also take advantage of the white noise structure \([13]\). Finally, substituting Eq.(13) and Eq.(A5) into Eq.(A4), we obtain Eqs.(16), (17).

**APPENDIX B: THERMAL SUPPRESSION OF THE ECHO**

To simplify the notations, we will be suppressing \( N \) – the total number of bosons. We take \( j_{1,2,3} = j \), and represent the factor \([30]\) at \( t = 2T \), and for \( M_3 = M_2 - 1 \) (the term with \( M_3 = M_2 + 1 \) is just the complex conjugate) as

\[
G(M_1, M_2, M_2 - 1, M_4, 2T) = \sum_{m'} \frac{\langle \hat{m}', \hat{m}|m, \hat{M}_4\rangle \langle \hat{M}_4, n|\hat{m}', j \rangle}{e^{i\hat{H}(M_1)} e^{i\hat{H}(M_2)} e^{-i\hat{H}(M_2 - 1)} e^{-i\hat{H}(M_4)}} ,
\]

and select the diagonal terms (with respect to the excitation label) only \( m = n \). In fact, the evolution of the off-diagonal terms is controlled by the excitations, and, therefore, they decay rapidly on the chosen time scale \( \tau_d \). Then, Eq.(B1) acquires the form:

\[
G(M_1, M_2, M_2 - 1, M_4, 2T) = \sum_m \frac{\hat{P}_m(M_4)}{e^{i\hat{H}(M_1)} e^{i\hat{H}(M_2)} e^{-i\hat{H}(M_2 - 1)} e^{-i\hat{H}(M_4)}} ,
\]

The quantity \( \hat{P}_m(M_4) \) can be considered as an effective distribution function created by the first pulse for the given population difference \( 2M_4 \). As it can be seen, it satisfies the normalization condition \( \sum_m \hat{P}_m(M_4) = 1 \).

As seen from Eqs.(B2), (B3), the first echo is dominated by the term \( M_4 = M_1 + 1 \). Accordingly, in what follows we will consider the term \( G(M - 1, M', M' - 1, M, 2T) \) only. We apply the cumulant expansion in order to evaluate it. Thus,

\[
G(M - 1, M', M' - 1, M, 2T) = \sum_m \hat{P}_m(M) \langle \hat{M}, \hat{m}|\hat{m}, \hat{M}_4\rangle e^{-i\hat{H}(M)} e^{i\hat{H}(M - 1)} \times e^{i\hat{H}(M') e^{-i\hat{H}(M' - 1)}} = G_1 c G_{2c} .
\]

The first cumulant

\[
G_{1c} = \sum_m \hat{P}_m(M) e^{i\hat{H}[E_m(M - 1) - E_m(M') - E_m(M' - 1)]} (B41)
\]

corresponds to the assumption \( \langle \hat{M}, \hat{m}|\hat{m}', \hat{M}' \rangle = \delta_{m, m'} \) for any value of the difference \( M - M' \) akin to Eq.(13). This, however, is not completely the case. In fact, for large values of \( M - M' \), a sort of the Debye-Waller factor enters this product. The second cumulant \( G_{2c} \) takes care of this effect.

We employ the identity

\[
e^{-i\hat{H}(M') T} = e^{-i\hat{H}(M) T} \text{Exp}[-i(M' - M) \int_0^T dt' H'(t')],
\]

\[
H'(t) = e^{i\hat{H}(M) t} \left\{ \frac{\partial H(M)}{\partial M} + o(1/N) \right\} e^{-i\hat{H}(M) t},
\]

where we have followed the same reasoning which lead us to Eq.(B8) – keeping the first term of the expansion of the projected Hamiltonian.

We are interested in evaluating the effect of suppression of the diagonal matrix elements (with different \( M \) and the same \( m \)) on the time evolution. Then, we write

\[
\langle \hat{M}, \hat{m}|e^{-i\hat{H}(M) T} e^{i\hat{H}(M - 1)} e^{i\hat{H}(M')} e^{-i\hat{H}(M' - 1)}|\hat{m}, \hat{M} \rangle \approx
\]

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\[ \langle \hat{M}, M \rangle | e^{-itH(M-1)} | \hat{M}, M \rangle \langle \hat{M}, M | e^{-itH(M)} | \hat{M}, M \rangle \langle \hat{M}, M | e^{itH(M') \ast} | \hat{M}, M \rangle \langle \hat{M}, M | e^{itH(M'-1)} | \hat{M}, M \rangle, \]

which after employing Eqs. (B6)-(B7) for the each mean, and substituting back to Eq. (B4), yields the first \((B5)\) and second cumulant \((B5)\), and second cumulant \((B6)\):

\[
G_{2c} = \exp \left\{ -[2(M - M') - \frac{1}{2}]^2 + \frac{3}{2} \right\} \times 
\int_0^\tau dt' \int_0^{t'} dt \sum_m \hat{P}_m(M) \langle \hat{M}, N, m | \delta H'(t) \delta H'(0) | m, \hat{N}, M \rangle, \tag{B8}
\]

where \(\delta H'(t)\) is defined below Eq. (B1). We employ the definition of \(t_{OC}\) in Eq. (B1), and rewrite Eq. (B4) as

\[
G(M - 1, M', M'-1, M, 2\tau) = 
\sum_m \hat{P}_m(M) e^{\tau \left( E_m(M-1) - E_m(M) + E_m(M') - E_m(M'-1) \right) - \left[ (M - M')^2 - \frac{1}{2} \right] t_{OC}}, \tag{B9}
\]

which is valid in the limit \( \tau_n \leq \tau \ll t_{OC} \). This expression is to be employed in Eq. (B3).

[1] Anderson, M.H., Ensher, J.R., Matthews, M.R., et al., 1995, Science 269, 198; Davis, K.B., Mewes, M.-O., Andrews, M.R., et al., 1995, Phys. Rev. Lett. 75, 3969; Bradley, C.C., Sackett, C.A., Tollett, J.J., and Hulet, R.G., 1995, Phys. Rev. Lett. 75, 1687.

[2] DeMarco, B., Jin, D.S., 1999, Science 285, 1703.

[3] Dalfovo, F., Giorgini, S., Pitaevskii, L.P., Stringari, S., 1999, Rev. Mod. Phys. 71, 463.

[4] Genser, S.D., Jin, D.S., 2001, cond-mat/0105441.

[5] Hall, D.S., Matthews, M.R., Wieman, C.E., Cornell, E.A., 1998, Phys. Rev. Lett. 81, 1543.

[6] Leggett, A., Sols, F., 1991, Foundations of Phys. 21, 353; Sols, F., 1994, Physica B 194-196, 1389; 1998, Phys. Rev. Lett. 81, 1344; Wright, E.M., Walls, D.F., Garrison, J.C., 1996, Phys. Rev. Lett. 77, 2158; Lewenstein, M., and You, L., 1996, Phys. Rev. Lett. 77, 3489; Javanainen, J., Wilkens, M., 1997, Phys. Rev. Lett. 78, 4675.

[7] Pitaevskii, L.P., Roach, A., 1997, Phys. Rev. A 55, R 835.

[8] Kuklov, A.B., Birman, Joseph L., 2000, Phys. Rev. Lett. 85, 5488.

[9] Matthews, M.R., Anderson, B.P., Haljan, P.C., et al, 1999, Phys. Rev. Lett. 83, 3358.

[10] Kuklov, A.B., Birman, Joseph L., 2001, Phys. Rev. A 63, 013609.

[11] Jin D.S., private communication.

[12] Cornell, E.A., Ensher, J.R., Wieman, C.E., 1999, cond-mat/9903109.

[13] Furutzu, K., 1963, J. Res. NBS D 667, 303; Novikov, E.A., 1965, Sov. Phys.- JETP 20, 1290.

[14] Kuklov, A.B., Birman, Joseph L., 2001, Bulletin of APS, 46, #3, Abstracts Q1 6, Q1 7.

[15] Abdulaev, F. Kh., Baizakov, B.B., Konotop, V.V., 2001, cond-mat/0105563.

[16] Giorgini, S., Pitaevskii, L.P., Stringari, S., 1998, Phys. Rev. Lett. 80, 5040.

[17] Beliaev, S.T., 1958, JETP 34, 417; ibid. 34, 433 (1958).

[18] Anderson, P.W., 1967, Phys. Rev. Lett. 18, 1049; Mahan, G.D. Many-Particle Physics, (Plenum Press, New York and London, 1993), Ch.8, Sec.8.3.

[19] Lifshitz, E.M., Pitaevskii, L.P., Statistical Physics, Part 2, Pergamon, Oxford, 1981.

[20] Mehta, M.L., Random Matrices, Academic Press, Boston (1991); Brody, T.A., Flores, J., French, J.B., et al, 1981, Rev. Mod. Phys. 53, 385.

[21] Koji Kajimura in Physical Acoustics, ed. W.P. Mason, V.XVI, Academic Press, NY, Toronto 1982; Wessbluth, M., Photon-Atom Interactions, Boston - Toronto, Ch. 3.5, (1988).

[22] Lifshitz, E.M., Pitaevskii, L.P., Physical Kinetics, Pergamon, 1981.

[23] Landau, L.D., Lifshitz, E.M., Quantum Mechanics, Ch. VIII, Pergamon Press, Oxford - Frankfurt, (1977).

[24] Handbook of Mathematical Functions, ed. by Abramovitz, M., and Stegun, I.A., Dover, Inc., NY 1972.

[25] Milburn, G.J., Corney, J., Wright, E.M., Walls, D.F., 1997, Phys. Rev. A, 55, 4318.
FIG. 1. Comparison of the results of the exact calculations of Eq. (47) with that obtained within the quantum phase method. Solid line corresponds to the quantum phase method. Dotted line – the exact calculations. Parameters chosen: $j = 45$ (total number of particles $N = 2j = 90$), $\beta = 0.22$, $T = 10$, $b_2 = 0.05$, $\epsilon_0 = 2$. 
FIG. 2. The echo effect in a confined two-component Bose-Einstein condensate at $T = 0$. Vertical axis is labeled by the relative magnitude of the population difference after the read-out pulse: $I_z$ is the half of the population difference; $j$ stands for the half of the total number of bosons. Solid line – the analytical solution (54); Dotted line – the numerical solution of Eq.(52).

Parameters chosen: $j = 2000$ (total number of particles $N = 2j = 4000$), $T = 2$, $b_2 = 0.1$, $\epsilon_0 = 40$; $\beta = 0.0005$, $\beta j = 1$. 

FIG. 2a
FIG. 3. Same as Fig 2a except the values $\beta = 0.0025$, $\beta j = 5$. 

Fig. 2b