Asymptotic zero distribution of multiple orthogonal polynomials associated with Macdonald functions

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Abstract

We study the asymptotic zero distribution of type II multiple orthogonal polynomials associated with two Macdonald functions (modified Bessel functions of the second kind). Based on the four-term recurrence relation, it is shown that, after proper scaling, the sequence of normalized zero counting measures converges weakly to the first component of a vector of two measures which satisfies a vector equilibrium problem with two external fields. We also give the explicit formula for the equilibrium vector in terms of solutions of an algebraic equation.

1 Introduction

Given a positive measure $\mu$ on the real line for which the support is not finite and all moments exist, there exists a sequence of monic orthogonal polynomials $\{p_k\}$ of degree $k$. Such polynomials satisfy a three-term recurrence relation of the form

$$xp_k(x) = p_{k+1}(x) + b_k p_k(x) + a_k^2 p_{k-1}(x), \quad k \geq 0,$$

(1.1)

with $a_k \geq 0$, $b_k \in \mathbb{R}$ and $p_0 \equiv 1$, $p_{-1} \equiv 0$.

It is well-known that the zeros of $p_k$ are real and simple. We can associate with $p_k(x)$ the normalized zero counting measure

$$\nu(p_k) = \frac{1}{k} \sum_{j=1}^{k} \delta_{x_{j,k}},$$

(1.2)

where $x_{j,k}$, $j = 1, \ldots, k$, are the zeros of $p_k$ and $\delta_x$ denotes the Dirac point mass at $x$. A measure $\nu$ is called the asymptotic zero distribution of $\{p_k\}$ if

$$\lim_{k \to \infty} \int f \, d\nu(p_k) = \int f \, d\nu$$

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for every bounded continuous function \( f \) on \( \mathbb{R} \), i.e., it is the weak limit of the measures \( \nu(p_k) \).

Suppose \( \lim_{k \to \infty} a_k = a \) and \( \lim_{k \to \infty} b_k = b \) with \( a > 0 \) and \( b \in \mathbb{R} \), then the polynomials generated by (1.1) have the asymptotic zero distribution \( w_{[\alpha, \beta]} \) with density

\[
\frac{dw_{[\alpha, \beta]}(x)}{dx} = \begin{cases} \frac{1}{\pi \sqrt{(\beta - x)(x - \alpha)}}, & x \in [\alpha, \beta], \\ 0, & \text{elsewhere,} \end{cases}
\]

where \( \alpha = b - 2a, \beta = b + 2a; \) cf. [16].

This result has been extended in [17] to the case of orthogonal polynomials generated by the recurrence relation

\[ xp_{k,n}(x) = p_{k+1,n}(x) + b_{k,n}p_{k,n}(x) + a_{k,n}^2p_{k-1,n}(x), \quad k, n \in \mathbb{N}, \]

with varying recurrence coefficients \( a_{k,n} > 0 \) and \( b_{k,n} \in \mathbb{R} \) depending on a parameter \( n \). Assume that the recurrence coefficients have continuous limits

\[
\lim_{k/n \to s} a_{k,n} = a(s), \quad \lim_{k/n \to s} b_{k,n} = b(s),
\]

where \( a : (0, \infty) \to [0, \infty), \quad b : (0, \infty) \to \mathbb{R} \) and the notation \( \lim_{k/n \to s} \) means that both \( k, n \to \infty \) with \( k/n \to s > 0 \), it is proved that the asymptotic zero distribution is given by the average

\[
\lim_{k/n \to s} \nu(p_{k,n}) = \frac{1}{s} \int_0^s w_{[\alpha(s), \beta(s)]} ds,
\]

where \( \alpha(s) := b(s) - 2a(s), \beta(s) := b(s) + 2a(s) \) and \( w_{[\alpha, \beta]} \) is defined by (1.3) if \( \alpha < \beta \) and by \( \delta_\alpha \) if \( \alpha = \beta \); see Theorem 1.10 of [17].

A natural generalization of this case consists in considering, for each \( n \in \mathbb{N} \), polynomials \( P_{k,n} \) satisfying an \( m \)-term recurrence relation with varying coefficients:

\[
xP_{k,n}(x) = P_{k+1,n}(x) + b_{k,n}^{(0)}P_{k,n}(x) + b_{k,n}^{(1)}P_{k-1,n}(x) + \cdots + b_{k,n}^{(m-2)}P_{k+2-m,n}(x),
\]

where \( P_0 \equiv 1, P_{-1} \equiv 0, \ldots, P_{-m+2} \equiv 0 \) and the recurrence coefficients have scaling limits

\[
\lim_{k/n \to s} b_{k,n}^{(j)} = b^{(j)}(s), \quad j = 0, \ldots, m - 2,
\]

for certain functions \( b^{(0)}, \ldots, b^{(m-2)} \).

For the simplest case \( b_{k,n}^{(j)} = b^{(j)}(s) \), i.e., we remove the dependence on the parameter \( n \) and the recurrence coefficients in [15] are actually constant, the zeros of \( P_k = P_{k,n} \) are closely related to the spectrum of certain banded Toeplitz matrix. Indeed, if we associate with the functions \( b^{(j)} \) a family of functions

\[
A_s(z) = z + b^{(0)}(s)z^{-1} + \cdots + b^{(m-2)}(s)z^{-m+2},
\]

(1.6)
and the sequence of \( k \times k \) Toeplitz matrices \((T_k(A_s))_k\) with symbol \( A_s \), defined by

\[
(T_k(A_s))_{jl} = \begin{cases} 
1, & \text{if } l = j + 1, \\
b^{(i)}(s), & \text{if } l = j - i, \\
0, & \text{otherwise},
\end{cases}
\]

(1.7)

it is readily seen that \( P_k(\lambda) = 0 \) if and only if \( \lambda \) is an eigenvalue of \( T_k(A_s) \). Hence, the investigation of limiting zero distribution of \( P_k \) is equivalent to the study of the limiting behavior of the spectrum of \( T_k(A_s) \) as \( k \to \infty \).

The limiting behavior of the spectrum of \( T_k(A_s) \) as \( k \to \infty \) is characterized by the solutions of the algebraic equation \( A_s(z) = x \); see [3]. For every \( x \in \mathbb{C} \), there exist exactly \( m - 1 \) solutions of the equation \( A_s(z) = x \) (assume that \( b^{(m-2)}(s) \neq 0 \)), which we denote by \( z_j(x, s) \), \( j = 1, \ldots, m-1 \) and label these solutions by their absolute value so that

\[
|z_1(x, s)| \geq |z_2(x, s)| \geq \cdots \geq |z_{m-1}(x, s)| > 0.
\]

(1.8)

We put

\[
\Gamma_1(s) = \{ x \in \mathbb{C} | |z_1(x, s)| = |z_2(x, s)| \},
\]

(1.9)

which is a finite union of analytic arcs.

It was shown by Schmidt and Spitzer [20] that the eigenvalues of \( T_k(A_s) \) accumulate on the contour \( \Gamma_1(s) \) as \( k \) tends to \( \infty \). Moreover, Hirschman [12] proved that the sequence of normalized counting measures of the eigenvalues of \( T_k(A_s) \) converges weakly to a Borel probability measure \( \mu_1^s \) supported on \( \Gamma_1(s) \) as \( k \to \infty \); see also [3, Chapter 11]. The precise form of \( \mu_1^s \) is given by

\[
d\mu_1^s(x) = \frac{1}{2\pi i} \left( \frac{z'_{1-}(x, s)}{z_{1-}(x, s)} - \frac{z'_{1+}(x, s)}{z_{1+}(x, s)} \right) dx,
\]

(1.10)

which is due to the result in [10]. Here, we have that \( ' \) denotes the derivative with respect to \( x \), \( dx \) is the complex line element on \( \Gamma_1(s) \) and \( z_{1\pm}(x, s) \) is the limiting value of \( z_1(\tilde{x}, s) \) as \( \tilde{x} \to x \) from the \( \pm \) side of \( \Gamma_1(s) \). Moreover, the measure \( \mu_1^s \) is also characterized by an equilibrium problem; see Theorem 2.2 below for a statement in the context of the specific example considered in this paper.

Under certain conditions, the polynomials \( P_{k,n} \) satisfying the recurrence (1.5) have a limiting zero distribution as well, which is an average, with respect to the parameter \( s \), of the measures (1.10). More precisely, we have (see Theorem 1.2 in [14]):

**Theorem 1.1.** Let for each \( n \in \mathbb{N}, m - 1 \) sequences \( \{ b^{(j)}_{k,n} \}_{k=0}^{\infty}, j = 0, \ldots, m-2 \), of real coefficients be given and assume that there exist continuous functions \( b^{(j)} : [0, \infty) \to \mathbb{R}, j = 0, \ldots, m-2 \), such that for each \( s \geq 0 \),

\[
\lim_{k/n \to s} b^{(j)}_{k,n} = b^{(j)}(s), \quad j = 0, \ldots, m-2.
\]

(1.11)

Let \( P_{k,n} \) be the monic polynomials generated by the recurrence (1.5) and suppose that
The polynomials $P_{k,n}$ have real and simple zeros $x_{k,n}^1 < \cdots < x_{k,n}^k$ satisfying for each $k$ and $n$ the interlacing property
\[ x_{j+1,n}^{k+1} < x_{j,n}^{k+1} < x_{j+1,n}^{k+1}, \quad \text{for } j = 1, \ldots, k, \]

(b) $\Gamma_1(s) \subset \mathbb{R}$ for every $s > 0$, where $\Gamma_1(s)$ is given by (1.9).

Then the normalized zero counting measures $\nu(P_{k,n}) = \frac{1}{k} \sum_{j=1}^{k} \delta_{x_{j,n}^{k+1}}$ have a weak limit as $k, n \to \infty$ with $k/n \to \xi > 0$ given by
\[
\lim_{k/n \to \xi} \nu(P_{k,n}) = \frac{1}{\xi} \int_0^\xi \mu_1^\alpha ds, \tag{1.12}
\]
where $\mu_1^\alpha$ is the measure (1.10).

It is worth noting that in [4], the authors present a conditional theorem giving the asymptotic zero distribution for polynomials satisfying a specific four-term recurrence relation.

The aim of this paper is to give more insight on the nature of the asymptotic zero distribution in the particular case of type II multiple orthogonal polynomials associated with two Macdonald functions (modified Bessel functions of the second kind) $K_\nu(x)$ ($\nu \geq 0$). A feature of the present case is the appearance of a vector equilibrium problem with two external fields, for which the first component of the unique minimizer is the weak limit of the normalized counting zero measures.

We mainly follow the idea in [14], where the authors consider a model of non-intersecting squared Bessel paths and derive a vector equilibrium problem for the limiting zero distribution of type II multiple orthogonal polynomials associated with the modified Bessel functions of the first kind [6, 7]. In that case, the vector equilibrium problem involves two measures supported on the positive real line and the negative real line, respectively, with an external field acting on the first measure and a constraint acting on the second measure [14, Theorem 1.7].

2 Statement of results

2.1 Multiple orthogonal polynomials associated with Macdonald functions

Assuming $x > 0$, we define the scaled Macdonald function $\rho_\nu$ by
\[
\rho_\nu(x) = 2x^\nu/2 K_\nu(2\sqrt{x}), \tag{2.1}
\]
and consider two weights
\[
d\mu_1(x) = x^\alpha \rho_\nu(x) dx, \quad d\mu_2(x) = x^\alpha \rho_{\nu+1}(x) dx, \quad \alpha > -1, \quad \nu \geq 0, \tag{2.2}
\]
on the positive real line. For any $k, m \in \mathbb{N}$, the type II multiple orthogonal polynomials $p_{k,m}^a$ for the system of weights $(\mu_1, \mu_2)$ are such that $p_{k,m}^a$ is a monic polynomial of degree $k + m$ and satisfies the following multiple orthogonality conditions:

\begin{align}
\int_0^\infty p_{k,m}^a(x)x^j d\mu_1(x) &= 0, \quad j = 0, 1, \ldots, k - 1, \\
\int_0^\infty p_{k,m}^a(x)x^j d\mu_2(x) &= 0, \quad j = 0, 1, \ldots, m - 1. 
\end{align}

By taking $m = k$, we set $P_{2k}(x) = p_{k,k}^a(x), \quad P_{2k+1}(x) = p_{k+1,k}^a(x)$.

An explicit formula for $P_k$ is given by

\[ P_k(x) = \sum_{j=0}^{k} a_k(j)x^{k-j}, \]  

where

\[ a_k(j) = (-1)^j \binom{k}{j} \frac{(\alpha + 1)_k(\alpha + \nu + 1)_k}{(\alpha + 1)_{k-j}(\alpha + \nu + 1)_{k-j}}, \quad 0 \leq j \leq k; \]

see [5, Theorem 2]. It is shown in [23] that $P_k$ satisfies the following four-term recurrence relation

\[ xP_k(x) = P_{k+1}(x) + b_kP_k(x) + c_kP_{k-1}(x) + d_kP_{k-2}(x) \]

with recurrence coefficients

\begin{align}
\quad b_k &= (k + \alpha + 1)(3k + \alpha + 2\nu) - (\alpha + 1)(\nu - 1), \\
\quad c_k &= k(k + \alpha)(k + \alpha + \nu)(3k + 2\alpha + \nu), \\
\quad d_k &= k(k - 1)(k + \alpha - 1)(k + \alpha)(k + \alpha + \nu - 1)(k + \alpha + \nu). 
\end{align}

These polynomials constitute one of few examples of multiple orthogonal polynomials that are not related to the classical orthogonal polynomials. They are first introduced by Van Assche and Yakubovich in [23], which solve an open problem posed by Prunikov [18]; see also [2, 4] for recent study.

Our goal is to investigate the limiting zero distribution of scaled polynomials $P_k$. Namely, we introduce a new parameter $n \in \mathbb{N}$ and put

\[ P_{k,n}(x) := \frac{P_k(nx^2)}{n^{2k}}. \]

Clearly, $P_{k,n}(x)$ is a polynomial of degree $k$ for each $n$. In view of (2.6)–(2.8), it is readily seen that $P_{k,n}(x)$ satisfies the following recurrence relation

\[ xP_{k,n}(x) = P_{k+1,n}(x) + b_{k,n}P_{k,n}(x) + c_{k,n}P_{k-1,n}(x) + d_{k,n}P_{k-2,n}(x), \]
with recurrence coefficients given by
\[ b_{k,n} = \frac{(k + \alpha + 1)(3k + \alpha + 2\nu) - (\alpha + 1)(\nu - 1)}{n^2}, \]
\[ c_{k,n} = \frac{k(k + \alpha)(k + \alpha + \nu)(3k + 2\alpha + \nu)}{n^4}, \]
\[ d_{k,n} = \frac{k(k - 1)(k + \alpha - 1)(k + \alpha)(k + \alpha + \nu - 1)(k + \alpha + \nu)}{n^6}. \]

As in (1.2), the normalized counting zero measure of \( P_{k,n} \) is defined by
\[ \nu(P_{k,n}) = \frac{1}{k} \sum_{P_{k,n}(x)=0} \delta_x. \]

We will derive a vector equilibrium problem with two external fields and show that the first component of the equilibrium vector is the weak limit of \( \nu(P_{k,n}) \) as \( k, n \to \infty \) with \( k/n \to \xi > 0 \). The equilibrium vector itself can be explicitly given in terms of the solutions of an algebraic equation. Our results are actually rather general in the sense that we also allow the parameters \( \alpha \) and \( \nu \) to increase proportionally to \( n \) as \( n \) increases.

### 2.2 Statement of results

We scale the parameters \( \alpha \) and \( \nu \) in the following way:
\[ \alpha \mapsto pn, \quad \nu \mapsto qn, \]
with \( p, q > 0 \). The results corresponding to \( \alpha \) and \( \nu \) fixed can be obtained by taking the limits as \( p, q \to 0 \), respectively.

Let \( k, n \to \infty \) in such a way that \( k/n \to s \), for some \( s \geq 0 \), we then observe that the recurrence coefficients (2.10) have scaling limits \( b(s) \), \( c(s) \) and \( d(s) \) given by
\[ \lim_{k/n \to s} b_{k,n} = b(s) = 3s^2 + 4sp + 2sq + p^2 + pq, \]
\[ \lim_{k/n \to s} c_{k,n} = c(s) = s(s + p)(s + p + q)(3s + 2p + q), \]
\[ \lim_{k/n \to s} d_{k,n} = d(s) = s^2(s + p)^2(s + p + q)^2. \]

Clearly, these limits depend on \( p \) and \( q \). As in (1.6), we have the associated family of symbols
\[ A_s(z) = z + b(s) + c(s)z^{-1} + d(s)z^{-2}, \]
and the solutions \( z_1(x, s) \), \( z_2(x, s) \) and \( z_3(x, s) \) of the algebraic equation \( A_s(z) = x \). We define \( \Gamma_1(s) \) as in (1.9) and similarly
\[ \Gamma_2(s) = \{ x \in \mathbb{C} \mid |z_2(x, s)| = |z_3(x, s)| \}. \]

The following proposition ensures that the polynomials \( P_{k,n} \) satisfy the hypothesis (a) of Theorem (1.1).
Proposition 2.1. Let $p, q > 0$. Then the polynomials $P_{k,n}$ generated by \eqref{2.9} with recurrence coefficients \eqref{2.10} have real and simple zeros in $(0, \infty)$ with the interlacing property.

Proof. The proof of the proposition follows from the fact that the measures $(d\mu_1, d\mu_2)$ from \eqref{2.2} form an AT system (cf. \cite{17, 22, 4}), which implies all the zeros of $P_k$ are simple, lie in $(0, +\infty)$ \cite{17, 22}, and satisfy the interlacing property \cite{1}. By Proposition 2.3 stated below, it is easily seen that the hypothesis (b) is also satisfied. Therefore, we see from Theorem 1.1 that, with the scaling given in \eqref{2.12}, the probability measure

$$
\nu^\xi_1 = \frac{1}{\xi} \int_0^\xi \mu^s_1 \, ds, \quad \xi > 0, (2.16)
$$

is the weak limit of the normalized zero counting measures. The main result of this paper is that $\nu^\xi_1$ can also be obtained as the first component of a vector of measures $(\nu^\xi_1, \nu^\xi_2)$ that satisfies a vector equilibrium problem with two external fields.

To define $\nu^\xi_2$, we need to introduce the second measure $\mu^s_2$, which is supported on $\Gamma_2(s)$ (see \eqref{2.15}) and given by

$$
d\mu^s_2(x) = \frac{1}{2\pi i} \left( \frac{z^s_2-(x,s)}{z^s_2-(x,s)} - \frac{z^s_2+(x,s)}{z^s_2+(x,s)} \right) \, dx, \quad x \in \Gamma_2(s). (2.17)
$$

It is a positive measure on $\Gamma_2(s)$ with total mass $1/2$. For each $\xi > 0$, we define $\nu^\xi_2$ in a manner similar to the definition of $\nu^\xi_1$ in \eqref{2.16}, i.e.,

$$
\nu^\xi_2 = \frac{1}{\xi} \int_0^\xi \mu^s_2 \, ds. (2.18)
$$

Then $\nu^\xi_2$ is a measure on $\bigcup_{s<\xi} \Gamma_2(s) = \Gamma_2(\xi)$ with total mass $1/2$.

An essential point for the rest of the paper is that the vector of measures $(\mu^s_1, \mu^s_2)$ is characterized by a vector equilibrium problem. In the present context, this is stated in the following theorem.

Theorem 2.2. For each $s > 0$, the vector $(\mu^s_1, \mu^s_2)$ is the unique minimizer for the energy functional

$$
\iint \log \frac{1}{|x-y|} \, d\mu_1(x) \, d\mu_1(y) + \iint \log \frac{1}{|x-y|} \, d\mu_2(x) \, d\mu_2(y) - \iint \log \frac{1}{|x-y|} \, d\mu_1(x) \, d\mu_2(y) (2.19)
$$

among all vectors $(\mu_1, \mu_2)$ satisfying supp$(\mu_j) \subset \Gamma_j(s)$ for $j = 1, 2$, and

$$
\int d\mu_1 = 1, \quad \int d\mu_2 = \frac{1}{2}.
$$
The measures $\mu_1^s$ and $\mu_2^s$ satisfy the following Euler-Lagrange variational conditions:

$$2 \int \log |x-y|d\mu_1^s(y) - \int \log |x-y|d\mu_2^s(y) = \ell^s, \quad x \in \Gamma_1(s), \quad (2.20)$$

for some constant $\ell^s$, and

$$2 \int \log |x-y|d\mu_2^s(y) - \int \log |x-y|d\mu_1^s(y) = 0, \quad x \in \Gamma_2(s), \quad (2.21)$$

We shall obtain the equilibrium problem for the vector of measures $(\nu_1^\xi, \nu_2^\xi)$ by integrating the variational conditions $(2.20)$ and $(2.21)$ with respect to the variable $s$. The main difficulty lies in the fact that $\Gamma_1(s)$ and $\Gamma_2(s)$ are varying with $s$. Hence, it is necessary to study how these contours depend on $s$. The next proposition reveals that $\Gamma_1(s)$ and $\Gamma_2(s)$ are indeed real intervals and actually are increasing as $s$ increases. The monotonicity of $\Gamma_1(s)$ and $\Gamma_2(s)$, as we will see later, plays an important role in the derivation of the equilibrium problem.

**Proposition 2.3.** For each $s > 0$, we have that $\Gamma_1(s) \subset (0, \infty)$ and $\Gamma_2(s) \subset (-\infty, 0)$. More precisely, there exist $\eta(s) < 0 < \beta(s) < \gamma(s)$ so that

$$\Gamma_1(s) = [\beta(s), \gamma(s)], \quad \Gamma_2(s) = (-\infty, \eta(s)]. \quad (2.22)$$

In addition, we have

(a) $\gamma(s)$ is positive and strictly increasing for $s > 0$, with $\lim_{s \to 0^+} \gamma(s) = p(p+q)$ and $\lim_{s \to \infty} \gamma(s) = \infty$,

(b) $\beta(s)$ is positive and strictly decreasing for $s > 0$ with $\lim_{s \to 0^+} \beta(s) = p(p+q)$ and $\lim_{s \to \infty} \beta(s) = 0$,

(c) $\eta(s)$ is negative and strictly increasing for $s > 0$ with $\lim_{s \to 0^+} \eta(s) = -q^2/4$ and $\lim_{s \to \infty} \eta(s) = 0$.

Figure 1 gives an illustrative plot of the functions $\beta(s)$, $\gamma(s)$ and $\eta(s)$.

Now we come to the main result of this paper, i.e., the equilibrium problem for the vector of measures $(\nu_1^\xi, \nu_2^\xi)$. The fact that $\Gamma_1(s)$ and $\Gamma_2(s)$ are increasing as $s$ increases, induces two external fields $V_1(x)$ and $V_2(x)$ acting on $\nu_1^\xi$ and $\nu_2^\xi$, respectively.

**Theorem 2.4.** For every $\xi > 0$, the vector of measures $(\nu_1^\xi, \nu_2^\xi)$ is the unique minimizer for the energy functional

$$\int \int \log \frac{1}{|x-y|}d\nu_1(x)d\nu_1(y) + \int \int \log \frac{1}{|x-y|}d\nu_2(x)d\nu_2(y) - \int \int \log \frac{1}{|x-y|}d\nu_1(x)d\nu_2(y) + \frac{1}{\xi} \int V_1(x)d\nu_1(x) + \frac{1}{\xi} \int V_2(x)d\nu_2(x), \quad (2.23)$$

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over all vectors of measures \((\nu_1, \nu_2)\) such that \(\text{supp}(\nu_1) \subset [0, \infty)\), \(\int d\nu_1 = 1\) and \(\text{supp}(\nu_2) \subset (-\infty, 0]\), \(\int d\nu_2 = 1/2\), where
\[
V_1(x) = \int_0^\infty \log \frac{|z_1(x, s)|}{|z_2(x, s)|} \, ds \quad \text{and} \quad V_2(x) = \int_0^\infty \log \frac{|z_2(x, s)|}{|z_3(x, s)|} \, ds. \tag{2.24}
\]

The measures \(\nu_1^\xi\) and \(\nu_2^\xi\) are characterized by the following variational conditions:
\[
2 \int \log |x - y| d\nu_1^\xi(y) - \int \log |x - y| d\nu_2^\xi(y) - \frac{1}{\xi} V_1(x) \begin{cases} 
= \ell, & \text{for } x \in \text{supp}(\nu_1^\xi), \\
\leq \ell, & \text{for } x \in [0, \infty),
\end{cases} \tag{2.25}
\]
for some \(\ell\), and
\[
2 \int \log |x - y| d\nu_2^\xi(y) - \int \log |x - y| d\nu_1^\xi(y) - \frac{1}{\xi} V_2(x) \begin{cases} 
= 0, & \text{for } x \in \text{supp}(\nu_2^\xi), \\
\leq 0, & \text{for } x \in (-\infty, 0].
\end{cases} \tag{2.26}
\]

Since the usual equilibrium problems provide a powerful tool in the asymptotic study of orthogonal polynomials (cf. [8, 9, 19]), we hope the vector equilibrium problem stated above will be helpful in further investigation of the asymptotics of \(P_k\) in (2.5); see also [11, 13] for a recent applications of vector equilibrium problems in some random models.

Finally, we give the explicit formulas of the external fields \(V_1\) and \(V_2\) defined in (2.24), and the densities of the measures \(\nu_1^\xi\), \(\nu_2^\xi\) given in (2.16) and (2.18), respectively.

**Theorem 2.5.** For every \(p, q > 0\), we have
\[
V_1(x) = \sqrt{q^2 + 4x} - p \log(4x) - q \log(\sqrt{q^2 + 4x} + q) \\
- 2p - q + p \log(4p^2 + 4pq) + q \log(2p + 2q), \tag{2.27}
\]
Figure 2: The densities of the measures \( \nu_1^\xi \) and \( \nu_2^\xi \). (\( \xi=1, \ p=1.7, \ q=8 \))

and

\[
V_2(x) = \begin{cases} 
0, & \text{for } x < -q^2/4, \\
-2\sqrt{q^2 + 4x} + q \log \left( \frac{q + \sqrt{q^2 + 4x}}{q - \sqrt{q^2 + 4x}} \right), & \text{for } -q^2/4 \leq x < 0.
\end{cases}
\] (2.28)

The densities of the measures \( \nu_1^\xi \), \( \nu_2^\xi \) are given by

\[
\frac{d\nu_1^\xi}{dx}(x) = c \frac{(z_{1+}(x,s) - z_{1-}(x,s))}{z_{1+}(x,s)z_{1-}(x,s)}, \quad \text{for } x \in \text{supp}(\nu_1^\xi),
\] (2.29)

\[
\frac{d\nu_2^\xi}{dx}(x) = \begin{cases} 
c \frac{(z_{2+}(x,s)-z_{2-}(x,s))}{z_{2+}(x,s)z_{2-}(x,s)x}, & \text{for } x \in \text{supp}(\nu_2^\xi) \text{ and } -q^2/4 < x < 0, \\
c \frac{(z_{2+}(x,s)-z_{2-}(x,s))}{z_{2+}(x,s)z_{2-}(x,s)x} + \frac{\sqrt{q^2+4x}}{2\pi x}, & \text{for } x \in \text{supp}(\nu_2^\xi) \text{ and } x \leq -q^2/4,
\end{cases}
\] (2.30)

where

\[
c = \frac{\xi(\xi + p)(\xi + p + q)}{2\pi i}.
\]

Figure 2 shows the graph of the densities of \( \nu_1^\xi \) and \( \nu_2^\xi \).

Remark 1. If we take \( p, q \to 0 \), which corresponds to fixed parameters \( \alpha \) and \( \nu \), the symbol (2.13) becomes

\[
A_\nu(z) = z + 3s^2 + \frac{3s^4}{z} + \frac{s^6}{z^2}.
\]

An straightforward calculation using (2.29) gives

\[
\frac{d\nu_1^\xi}{dx}(x) = \begin{cases} 
\frac{4}{2\pi i}h\left(\frac{4x}{27\xi^2}\right), & x \in (0, \frac{27\xi^2}{4}), \\
0, & \text{elsewhere},
\end{cases}
\]

\[
\frac{d\nu_2^\xi}{dx}(x) = \begin{cases} 
c \frac{(z_{2+}(x,s)-z_{2-}(x,s))}{z_{2+}(x,s)z_{2-}(x,s)x} + \frac{\sqrt{q^2+4x}}{2\pi x}, & x \in \text{supp}(\nu_2^\xi) \text{ and } x \leq -q^2/4,
\end{cases}
\]
Figure 3: The densities of the measures $\nu_1^1$ and $\nu_1^2$ in the case $p = q = 0$.

with

$$h(y) = \frac{3\sqrt{3} (1 + \sqrt{1 - y})^{1/3} - (1 - \sqrt{1 - y})^{1/3}}{y^{2/3}},$$

which agrees with Theorem 2.7 in [4]. This case is illustrated in Figure 3.

The rest of this paper is organized as follows. We first prove Proposition 2.3 in Section 3. The proof of Theorem 2.4 is given in Section 4. We conclude this paper with the proof of Theorem 2.5, where we use a nonlinear transformation to evaluate the integrals used to define the external fields and the equilibrium vector.

3 Proof of Proposition 2.3

The symbol (2.14) with the functions $b(s)$, $c(s)$ and $d(s)$ from (2.13) allows for a factorization

$$A_s(z) = \frac{(z + s(s + p))(z + s(s + p + q))(z + (s + p)(s + p + q))}{z^2}. \quad (3.1)$$

By (3.1), it follows that $A_s$ has three negative simple zeros $r_1, r_2, r_3$. We order them so that

$$r_1 < r_2 < r_3 < 0;$$

see Figure 4 for the graph of $A_s(z)$.

The derivative of $A_s(z)$,

$$A_s'(z) = 1 - c(s)z^{-2} - 2d(s)z^{-3},$$

has three roots in the complex plane. From Figure 4 we see that all zeros of $A_s'$ are real. We denote the zeros of $A'(z)$ by $y_1, y_2$ and $y_3$ so that

$$y_1 < y_2 < 0 < y_3, \quad (3.2)$$

as indicated in Figure 4. To emphasize the dependence on $s$, we also write $y_1(s)$, $y_2(s)$ and $y_3(s)$.

Before proving Proposition 2.3 we first need two lemmas. The fact that all the zeros $r_j$ of the symbol $A_s$ are strictly negative plays a key role in these proofs.

**Lemma 3.1.** Assume that $z_1, z_2 \in \mathbb{C}$ are such that $z_1 \neq z_2$, $|z_1| = |z_2|$ and $A_s(z_1) = A_s(z_2) = x$. Then $z_1 = \overline{z_2}$ and $x \in \mathbb{R}$.

*Proof.* This lemma is essentially Lemma 4.1 in [14]. For the convenience of the readers, we repeat the proof here.

Since $z_1 \neq z_2$ and $|z_1| = |z_2|$, we may assume $z_1 = \rho e^{i\theta_1}$, $z_2 = \rho e^{i\theta_2}$ with $\rho > 0$ and $\theta_1 \neq \theta_2$, $\theta_{1,2} \in [-\pi, \pi]$. Since the zeros $r_j$ of $A_s$ are strictly negative, it is easily seen that $f(\theta) := |A_s(\rho e^{i\theta})|$ is an even function on $[-\pi, \pi]$, which is strictly decreasing as $\theta$ increases from 0 to $\pi$. Hence, the equality

$$|A_s(\rho e^{i\theta_1})| = |A_s(\rho e^{i\theta_2})|$$

holds if and only if $\theta_2 = -\theta_1$. It then follows that $z_1 = \overline{z_2}$, and

$$x = A_s(z_1) = A_s(\overline{z_2}) = \overline{x},$$

so that $x \in \mathbb{R}$. $\Box$

**Lemma 3.2.** For each $s > 0$, we have $\Gamma_1(s) \cup \Gamma_2(s) \subset \mathbb{R}$ and $\Gamma_1(s) \cap \Gamma_2(s) = \emptyset$.

*Proof.* To show $\Gamma_1(s) \cup \Gamma_2(s) \subset \mathbb{R}$, we consider two cases, based on whether the equation $A_s(z) - x = 0$ has a double root or not. If $x \in \Gamma_1(s) \cup \Gamma_2(s)$ and $A_s(z) - x = 0$ has a double root, then there exists $z_1 \in \mathbb{C}$ such that $A_s(z_1) = x$ and $A'_s(z_1) = 0$. Since all zeros of

![Figure 4: The graph of $A_s(z)$](image-url)
$A_s'(z)$ are real, it follows that $x \in \mathbb{R}$. On the other hand, suppose $x \in \Gamma_1(s) \cup \Gamma_2(s)$ and $A_s(z) - x = 0$ does not have a double root, then there exist $z_1, z_2 \in \mathbb{C}$ such that $z_1 \neq z_2$, $|z_1| = |z_2|$ and $x = A_s(z_1) = A_s(z_2)$. We then conclude form Lemma 3.1 that $x \in \mathbb{R}$. This proves that $\Gamma_1(s) \cup \Gamma_2(s) \subset \mathbb{R}$.

To show $\Gamma_1(s) \cap \Gamma_2(s) = \emptyset$, we observe that, if $x \in \Gamma_1(s) \cap \Gamma_2(s)$, there exist three solutions of $A_s(z) = x$, one negative solution $z_1 < 0$ and two complex conjugated solutions $z_2$ and $\bar{z}_2$. Moreover $z_1 \neq z_2$ and $z_1 \neq \bar{z}_2$. Now $|z_1| = |z_2|$ ($x \in \Gamma_1(s)$) and $A_s(z_1) = A_s(z_2)$. Again, by Lemma 3.1 we obtain $z_1 = \bar{z}_2$, which is a contradiction. Therefore, $\Gamma_1(s) \cap \Gamma_2(s) = \emptyset$.

**Proof of Proposition 2.3** Since $p$ and $q$ are positive, there are three local extrema of $A_s(z)$, namely $\beta(s), \gamma(s), \eta(s)$, such that

$$\eta(s) < 0 < \beta(s) < \gamma(s).$$

If $x \in (\eta(s), \beta(s)) \cup (\gamma(s), \infty)$, there exist three different real solutions of $A_s(z) = x$. These solutions differ in absolute value, which is obvious if $x \in (\eta(s), \beta(s))$ (cf. Figure 4) and a consequence of Lemma 3.1 if $x \in (\gamma(s), \infty)$. On the other hand, there is one real and two complex conjugated solutions whenever $x \in (\gamma(s), \infty)$. Therefore $A_s(z)$ has a double root at $y_1 = y_2(s) > 0$ and one negative root whose absolute value is less than $y_3(s)$. Thus $\gamma(s) \in \Gamma_1(s)$. By the same argument, we see $\beta(s) \in \Gamma_1(s)$. In view of the fact that $\Gamma_1(s)$ is connected (see 21,3 Theorem 11.19), it then follows that $\Gamma_1(s) = [\beta(s), \gamma(s)]$. Similarly, we notice that $A_s(z) = \eta(s)$ has a double root at $y_2(s) < 0$ and a negative root whose absolute value is larger than $|y_2(s)|$. Therefore $\eta_2(s) \in \Gamma_2(s)$ and $\Gamma_2(s) = (-\infty, \eta(s)]$.

To show that $\gamma(s)$ is an increasing function, we introduce

$$B(z, s) = A_s\left(\frac{s(s + p)(s + p + q)}{z - s}\right) = \frac{z(z + p)(z + p + q)}{z - s}. \quad (3.3)$$

Taking the partial derivative of $B(z, s)$ with respect to $s$, we obtain

$$\frac{\partial B(z, s)}{\partial s} = \frac{z(z + p)(z + p + q)}{(z - s)^2}. \quad (3.4)$$

As a function of $z$, it is easily seen that $B(z, s)$ has a local minimum at $s + s(s + p)(s + p + q)/y_3(s)$ and

$$\gamma(s) = A_s(y_3(s)) = B(s + s(s + p)(s + p + q)/y_3(s), s).$$
This, together with (3.6), implies that

\[
\gamma'(s) = \frac{\partial B(s + s(p + q))}{\partial s} \frac{\partial y_3(s)}{\partial s} + \frac{\partial B(s + s + p)(s + p + q)}{\partial s} \frac{\partial y_3(s)}{\partial s} \frac{\partial (s + s + p)(s + p + q)}{\partial s}
\]

\[
= \frac{\partial B(s + s + p)(s + p + q)}{\partial s} \frac{\partial y_3(s)}{\partial s}
\]

\[
= \frac{\hat{z}_3(s)(\hat{z}_3(s) + p)(\hat{z}_3(s) + p + q)}{\left(\hat{z}_3(s) - s\right)^2},
\]

(3.5)

where \( \hat{z}_3(s) = s + s(s + p)(s + p + q)/y_3(s) \). As \( y_3(s) > 0 \) and \( p, q > 0 \), we have \( \hat{z}_3(s) > 0 \) for all \( s > 0 \). Therefore \( \gamma(s) \) is increasing on \((0, \infty)\) by (3.6). The monotonicity of \( \beta(s) \) and \( \eta(s) \) can be proved in similar manners. Indeed, by the same argument, we have

\[
\eta'(s) = \frac{\hat{z}_2(s)(\hat{z}_2(s) + p)(\hat{z}_2(s) + p + q)}{(\hat{z}_2(s) - s)^2},
\]

(3.6)

\[
\beta'(s) = \frac{\hat{z}_1(s)(\hat{z}_1(s) + p)(\hat{z}_1(s) + p + q)}{(\hat{z}_1(s) - s)^2},
\]

(3.7)

where \( \hat{z}_j(s) = s + s(s + p)(s + p + q)/y_j(s) \), \( j = 1, 2 \). Note that \( y_2(s) \) and \( y_1(s) \) are local extreme points of \( A_s(z) \), whose zeros are \(-s(s + p), -s(s + p + q)\) and \(-(s + p)(s + p + q)\). Therefore, in view of (3.2) (see also Figure 4), we have

\[-s(s + p + q) < y_2(s) \quad \text{and} \quad -(s + p)(s + p + q) < y_1(s) \quad < -s(s + p + q),\]

which implies

\[-(p + q) < \hat{z}_2(s) < -p \quad \text{and} \quad -p < \hat{z}_1(s) < 0.\]

(3.8)

Combining (3.6) – (3.8), it follows that \( \eta'(s) > 0 \) and \( \beta'(s) < 0 \) for all \( s > 0 \), which gives the desired monotonicity of \( \beta(s) \) and \( \eta(s) \).

Finally, we come to the boundary values of \( \gamma(s) \), \( \beta(s) \) and \( \eta(s) \). A straightforward calculation yields that \( y_1(s), y_2(s) \) and \( y_3(s) \) have the following behavior as \( s \to 0 \)

\[
y_1(s) = -\sqrt{p(p + q)(2p + q)s + O(s)},
\]

\[
y_2(s) = -\frac{2p(p + q)}{2p + q} s + O(s^2),
\]

\[
y_3(s) = \sqrt{p(p + q)(2p + q)s + O(s)}.
\]

Hence,

\[
\gamma(s) = A_s(y_3(s)) = p(p + q) + O(s),
\]

\[
\eta(s) = A_s(y_2(s)) = -\frac{1}{4} q^2 + O(s),
\]

\[
\beta(s) = A_s(y_1(s)) = p(p + q) + O(s),
\]

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as \( s \to 0^+ \). This proves the limits

\[
\lim_{s \to 0^+} \beta(s) = p(p + q) \quad \text{and} \quad \lim_{s \to 0^+} \eta(s) = -\frac{1}{4}q^2.
\]

On the other hand, note that \( y_1(s), y_2(s) \) and \( y_3(s) \) have the following behavior as \( s \to \infty \),

\[
\begin{align*}
y_1(s) &= -s^2 + c_1 s + \mathcal{O}(1), \\
y_2(s) &= -s^2 + c_2 s + \mathcal{O}(1), \\
y_3(s) &= 2s^2 + \frac{8p - 4q}{3} - s + \mathcal{O}(1),
\end{align*}
\]

where \( c_1 = \frac{-4p - 2q - \sqrt{p^2 + pq + q^2}}{3} \) and \( c_2 = \frac{-4p - 2q + \sqrt{p^2 + pq + q^2}}{3} \), the limits of \( \gamma(s) \), \( \beta(s) \) and \( \eta(s) \) as \( s \to \infty \) can be obtained in a manner similar to the situation \( s \to 0^+ \), we omit the details and this completes the proof of the Proposition 2.3.

4 Proof of Theorem 2.4

From the definitions of \( \nu^1_\xi \) and \( \nu^2_\xi \) in (2.16) and (2.18), it is clear that \( \text{supp}(\nu^1_\xi) \subset [0, \infty), \int d\nu^1_\xi = 1 \) and \( \text{supp}(\nu^2_\xi) \subset (-\infty, 0], \int d\nu^2_\xi = 1/2 \). Indeed, on account of the fact that the sets \( \Gamma_1(s) = \text{supp}(\mu^1_\xi) \) and \( \Gamma_2(s) = \text{supp}(\mu^2_\xi) \) are increasing as \( s \) increases (see Proposition 2.3), it follows that

\[
\text{supp}(\nu^1_\xi) = \bigcup_{s \leq \xi} \Gamma_1(s) = \Gamma_1(\xi),
\]

and

\[
\text{supp}(\nu^2_\xi) = \bigcup_{s \leq \xi} \Gamma_2(s) = \Gamma_2(\xi).
\]

Thus, in order to show that \( (\nu^1_\xi, \nu^2_\xi) \) is the minimizer of the energy functional (2.23) under the conditions stated in Theorem 2.4 it suffices to prove that the vector \( (\nu^1_\xi, \nu^2_\xi) \) satisfies the variational conditions (2.25) and (2.26).

The basic idea is, as mentioned in Subsection 2.2, to integrate the variational conditions (2.20) and (2.21) with respect to \( s \) from 0 to \( \xi \). Here, we need a more general expression for the variational conditions (2.20) and (2.21), namely

\[
2 \int \log |x - y| d\mu^1_\xi(y) - \int \log |x - y| d\mu^2_\xi(y) = \log \left| \frac{z_1(x, s)}{z_2(x, s)} \right|,
\]

and

\[
2 \int \log |x - y| d\mu^2_\xi(y) - \int \log |x - y| d\mu^1_\xi(y) = \log \left| \frac{z_2(x, s)}{z_3(x, s)} \right|,
\]

for all \( x \in \mathbb{C} \), which are contained in the proof of Theorem 2.3 of [10]. These conditions reduce to (2.20) and (2.21) whenever \( x \in \Gamma_1(s) \) and \( x \in \Gamma_2(s) \), respectively.
Proofs of (2.25) and (2.26). Multiplying both sides of (4.3) by $1/\xi$ and integrating with respect to $s$ from 0 to $\xi$, we obtain from interchanging the order of integration that
\[ 2 \int \log |x - y|d\nu^x_1(y) - \int \log |x - y|d\nu^x_2(y) - \ell = \frac{1}{\xi} \int_0^\xi \log \left| \frac{z_1(x, s)}{z_2(x, s)} \right| ds, \tag{4.5} \]
for all $x \in \mathbb{C}$ and some constant $\ell \in \mathbb{R}$. The measures $\nu^x_1$ and $\nu^x_2$ in (4.5) are defined in (2.16) and (2.18), respectively.

Let $x \geq 0$. Since $|z_1(x, s)| \geq |z_2(x, s)|$ for every $s$, it follows from (4.5) and (2.24) that
\[ 2 \int \log |x - y|d\nu^x_1(y) - \int \log |x - y|d\nu^x_2(y) - \ell \leq \frac{1}{\xi} \int_0^\infty \log \left| \frac{z_1(x, s)}{z_2(x, s)} \right| ds = \frac{1}{\xi} V_1(x). \tag{4.6} \]

Suppose now $x \in \text{supp}(\nu^x_1)$, then $x \in \Gamma_1(\xi)$ by (4.1), and therefore $x \in \Gamma_1(s)$ for every $s \geq \xi$, since the sets are increasing. Thus $|z_1(x, s)| = |z_2(x, s)|$ for every $s \geq \xi$, and equality holds in (4.6) for every $x \in \text{supp}(\nu^x_1)$. This completes the proof of (2.25).

The variational condition (2.26) for $\nu^x_2$ can be proved in a manner similar to (2.25) by using (4.4) and (4.2), we omit the details.

5 Proof of Theorem 2.5

We conclude this paper with the proof of Theorem 2.5. We start with the following lemma that embodies the behavior of three solutions of $A_s(z)$ as $s \to 0+$.

Lemma 5.1. Let $A_s(z)$ be given by (3.1), and let $z_1(x, s)$, $z_2(x, s)$ and $z_3(x, s)$ be the solutions of $A_s(z) = x$, ordered as in (1.8). Then
\[ \lim_{s \to 0^+} z_1(x, s) = x - p(p + q), \]
\[ \lim_{s \to 0^+} s^{-1}z_2(x, s) = \frac{-p(p + q)(2p + q + \sqrt{q^2 + 4x})}{2(p^2 + pq - x)}, \]
\[ \lim_{s \to 0^+} s^{-1}z_3(x, s) = \frac{-p(p + q)(2p + q - \sqrt{q^2 + 4x})}{2(p^2 + pq - x)}. \tag{5.1} \]

Proof. The lemma follows by a straightforward computation.

To prove Theorem 2.5, we need to establish the identities (2.27)–(2.30).

Proof of (2.27). Let $x > 0$. From Proposition 2.3 it follows that there exists a unique $s^*(x) \geq 0$ so that for all $s > 0$,
\[ x \in \Gamma_1(s) \iff s \geq s^*(x). \tag{5.2} \]

Then $\log |z_1(x, s)/z_2(x, s)| = 0$ for all $s \geq s^*(x)$, and so by (2.24)
\[ V_1(x) = \int_0^{s^*(x)} \log \left| \frac{z_1(x, s)}{z_2(x, s)} \right| ds. \tag{5.3} \]
There is a special value

\[ x_0 = p(p + q) = \lim_{s \to 0^+} \beta(s) = \lim_{s \to 0^+} \gamma(s) \]

that belongs to every \( \Gamma_1(s) \) for any \( s > 0 \). Then \( s^*(x_0) = 0 \) and

\[ V_1(x_0) = 0. \]  (5.4)

The derivative of (5.3) is

\[ V'_1(x) = \int_0^{s^*(x)} \left( \frac{1}{z_1(x, s)} \frac{\partial z_1(x, s)}{\partial x} - \frac{1}{z_2(x, s)} \frac{\partial z_2(x, s)}{\partial x} \right) ds. \]  (5.5)

In order to evaluate this integral, we introduce new variables

\[ \tilde{z}_j(x, s) = \frac{s(s + p)(s + p + q)}{z_j(x, s)} + s, \quad j = 1, 2, 3. \]  (5.6)

Since each \( z_j(x, s) \) is a solution of \( A_s(z) = x \), it follows that \( \tilde{z}_j(x, s) \) for \( j = 1, 2, 3 \) is a solution of the equation \( B(z, s) = x \), where

\[ B(z, s) = \frac{z(z + p)(z + p + q)}{z - s}; \]  (5.7)

see (5.3).

Taking partial derivatives with respect to \( s \) and \( x \) on both sides of \( B(\tilde{z}_j(x, s), s) = x \), and applying the chain rule, we obtain

\[ \left( \frac{\partial B}{\partial z}(\tilde{z}_j(x, s), s) \right) \frac{\partial \tilde{z}_j(x, s)}{\partial s} + \frac{\partial B}{\partial s}(\tilde{z}_j(x, s), s) = 0, \]

\[ \left( \frac{\partial B}{\partial z}(\tilde{z}_j(x, s), s) \right) \frac{\partial \tilde{z}_j(x, s)}{\partial x} = 1 \]  (5.8)

for \( j = 1, 2, 3 \). From (5.7), it is elementary to deduce that

\[ \frac{\partial B}{\partial s}(\tilde{z}_j(x, s), s) = \frac{\tilde{z}_j(x, s)(\tilde{z}_j(x, s) + p)(\tilde{z}_j(x, s) + p + q)}{(\tilde{z}_j(x, s) - s)^2} = \frac{x}{\tilde{z}_j(x, s) - s}. \]

Combining this with (5.8), we obtain

\[ \frac{x}{\tilde{z}_j(x, s) - s} \frac{\partial \tilde{z}_j(x, s)}{\partial x} = -\frac{\partial \tilde{z}_j(x, s)}{\partial s}, \quad j = 1, 2, 3. \]  (5.9)

By (5.6), we also note that

\[ \frac{\partial \tilde{z}_j(x, s)}{\partial x} = -\frac{s(s + p)(s + p + q) \partial z_j(x, s)}{z_j^2(x, s)} = -\frac{\tilde{z}_j(x, s) - s \partial z_j(x, s)}{\tilde{z}_j(x, s)} \frac{\partial \tilde{z}_j(x, s)}{\partial s}. \]  (5.10)
Substituting (5.10) into (5.9) gives
\[
\frac{1}{z_j(x, s)} \frac{\partial z_j(x, s)}{\partial s} = \frac{1}{x} \frac{\partial \tilde{z}_j(x, s)}{\partial s}, \quad j = 1, 2, 3. \tag{5.11}
\]

Hence, using (5.11) in (5.5) we get
\[
V'_1(x) = \frac{1}{x} \int_0^{s^*(x)} \left( \frac{\partial \tilde{z}_1(x, s)}{\partial s} - \frac{\partial \tilde{z}_2(x, s)}{\partial s} \right) ds.
\]

It then follows from the fundamental theorem of calculus that
\[
V'_1(x) = \frac{1}{x} \left[ \tilde{z}_1(x, s^*(x)) - \tilde{z}_2(x, s^*(x)) \right] - \lim_{s \to 0^+} \left( \tilde{z}_1(x, s) - \tilde{z}_2(x, s) \right). \tag{5.12}
\]

By definition, \( s^*(x) \) is the smallest value of \( s \geq 0 \) for which \( x \in \Gamma_1(s) \). Then \( x = \gamma(s^*(x)) \) if \( x_0 < x \) and \( x = \beta(s^*(x)) \) if \( 0 < x < x_0 \). We can observe from Figure 4 that \( z_1(\gamma(s), s) = z_2(\gamma(s), s) \) and \( z_1(\beta(s), s) = z_2(\beta(s), s) \). Therefore,
\[
\tilde{z}_1(x, s^*(x)) = \tilde{z}_2(x, s^*(x))
\]
and (5.12) reduces to
\[
V'_1(x) = -\frac{1}{x} \lim_{s \to 0^+} \left( \tilde{z}_1(x, s) - \tilde{z}_2(x, s) \right). \tag{5.13}
\]

From (5.6) and Lemma 5.1 we find that
\[
\lim_{s \to 0^+} \tilde{z}_1(x, s) = 0 \quad \text{and} \quad \lim_{s \to 0^+} \tilde{z}_2(x, s) = -\frac{2(p^2 + pq - x)}{2p + q + \sqrt{q^2 + 4x}}.
\]

Then (5.13) leads to
\[
V'_1(x) = -\frac{1}{x} \frac{2(p^2 + pq - x)}{2p + q + \sqrt{q^2 + 4x}}. \tag{5.14}
\]

We obtain \( V_1(x) \) by integrating (5.14) with respect to \( x \), which gives
\[
V_1(x) = \sqrt{q^2 + 4x} - p \log(4x) - q \log(\sqrt{q^2 + 4x} + q) + C.
\]

The constant of integration \( C \) can be determined by requiring \( V_1(x_0) = 0 \); see (5.4). This gives
\[
C = -2p - q + p \log(4p^2 + 4pq) + q \log(2p + 2q),
\]
and (2.27) is proved. \( \square \)
Proof of (2.28). Let \( x < 0 \). Again, it follows from Proposition 2.3 that there exists a unique \( s^*(x) \geq 0 \) so that for all \( s > 0 \),

\[
x \in \Gamma_2(s) \iff s \geq s^*(x).
\]  

(5.15)

Then \( \log |z_2(x,s)/z_3(x,s)| = 0 \) for all \( s \geq s^*(x) \), and so by (2.24)

\[
V_2(x) = \int_0^{s^*(x)} \log \left| \frac{z_2(x,s)}{z_3(x,s)} \right| \, ds.
\]  

(5.16)

There also exists a special value

\[
\tilde{x}_0 = -q^2/4 = \lim_{s \to 0^+} \eta(s)
\]

so that \( x \in \Gamma_2(s) \) for any \( s > 0 \) if \( x \leq \tilde{x}_0 \). Then, for any \( x \leq \tilde{x}_0 \), we have \( s^*(x) = 0 \) and

\[
V_2(x) = 0.
\]  

(5.17)

If \( -q^2/4 = \tilde{x}_0 \leq x < 0 \), by the same argument in the proof of (2.27), we have

\[
V_2'(x) = -\frac{\sqrt{q^2 + 4x}}{x}.
\]  

(5.19)

By integrating (5.19) with respect to \( x \), we obtain

\[
V_2(x) = -2\sqrt{q^2 + 4x} + q \log \left( \frac{q + \sqrt{q^2 + 4x}}{q - \sqrt{q^2 + 4x}} \right) + \tilde{C},
\]

for \( -q^2/4 \leq x < 0 \). The constant of integration \( \tilde{C} \) can be determined by requiring \( V_2(\tilde{x}_0) = 0 \); see (5.17). This leads to \( \tilde{C} = 0 \) and (2.28) is proved.

Proofs of (2.29) and (2.30). By the definitions of \( \nu_1^\xi \) and \( \nu_2^\xi \) in (2.16) and (2.18), we obtain from (1.10) and (2.17) that

\[
\frac{d\nu_1^\xi}{dx}(x) = \frac{1}{2\pi i \xi} \int_0^\xi \left( \frac{1}{z_1_-(x,s)} \frac{\partial z_1_-(x,s)}{\partial x} - \frac{1}{z_1_+(x,s)} \frac{\partial z_1_+(x,s)}{\partial x} \right) \, ds,
\]

\[
\frac{d\nu_2^\xi}{dx}(x) = \frac{1}{2\pi i \xi} \int_0^\xi \left( \frac{1}{z_2_+(x,s)} \frac{\partial z_2_+(x,s)}{\partial x} - \frac{1}{z_2_-(x,s)} \frac{\partial z_2_-(x,s)}{\partial x} \right) \, ds.
\]
With \( s^*(x) \) given in (5.2) and (5.15), it is readily seen that

\[
\frac{d\nu_1}{dx}(x) = \frac{1}{2\pi i \xi} \int_{s^*(x)}^{\xi} \left( \frac{1}{z_1-(x, s)} \frac{\partial z_{1-}(x, s)}{\partial x} - \frac{1}{z_1+(x, s)} \frac{\partial z_{1+}(x, s)}{\partial x} \right) ds,
\]

\[
\frac{d\nu_2}{dx}(x) = \frac{1}{2\pi i \xi} \int_{s^*(x)}^{\xi} \left( \frac{1}{z_2+(x, s)} \frac{\partial z_{2+}(x, s)}{\partial x} - \frac{1}{z_2-(x, s)} \frac{\partial z_{2-}(x, s)}{\partial x} \right) ds.
\]

Now, by introducing the change of variable (5.6), one can evaluate the above integrals using similar methods as given in the proofs of (2.27) and (2.28). We omit the details and this completes the proof of Theorem 2.5. \( \square \)

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