Application of Lagrange inversion to wall-crossing for Quot-schemes on surfaces

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Abstract

Grothendieck’s quot-schemes parametrize quotient sheaves of a fixed coherent sheaf. In some cases, they carry a natural perfect obstruction theory in the sense of Behrend–Fantechi [3] and the invariants resulting from integrating against their virtual fundamental classes have been studied in [1, 10, 11] and [5]. The last reference relied on the wall-crossing method as introduced by Joyce [12] and used previously in [6] while the other references used more geometric techniques. To compare the outcomes of the two approaches and extend them further, we are required to compare two different generating series.

We dedicate this short note to give a combinatorial proof of the identity relating to the two power series relying on Lagrange inversion. A special case of this was proved by Mathoverflow users “Alex Gavrilov” and “esg” answering our enquiry.

1 Introduction

Grothendieck in his lecture [9] introduced Quot-schemes as a solution to a moduli problem which captures quotients of a fixed sheaf on a projective variety $X$. More explicitly: fixing an algebraic K-theory class $\alpha$ and a sheaf $E$ on $X$, then $\text{Quot}_S(E_S, n)(E, \alpha)$ parametrizes the quotients

$$E \to F,$$

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where $F$ of class $\alpha$. When $X$ is a surface, $E$ is a vector bundle and $\alpha$ has 1-dimensional support $\text{Quot}_S(E_S, n)(E, \alpha)$ admits a perfect obstruction theory as in [3] which was observed by Marian–Oprea–Pandharipande [14]. As such, one can integrate cohomology classes on $\text{Quot}_S(E_S, n)(E, \alpha)$ with respect to its virtual fundamental class $\left[\text{Quot}_S(E_S, n)(E, \alpha)\right]^{\text{vir}}$. The resulting invariants can be combined into generating series by summing over the Euler-characteristic of $\chi(\alpha)$ and were studied in [1, 16, 10] and [5].

While the previous authors used methods relying on a virtual localization formula of [8], we have instead relied on the wall-crossing framework of Joyce [12] and the methods of its application developed in [6, 5]. It therefore seemed necessary to give an alternative proof of the identity below relying purely on combinatorics.

**Theorem 1.** Let $R(t) \in \mathbb{K}[t]$ be a power-series over a field $\mathbb{K}$ not involving $q$ and $H_i(q)$ the $e > 0$ different Newton–Puiseux solutions to $H_i^e(q) = q^{R(H_i)}$. Set

$$G_e(R) = \exp \left[ - \sum_{n,m,j>0} j^m \left\{ R^m(z) \right\} \frac{1}{n^e} \left\{ R^n(z) \right\} q^n + m \right]$$

then the following holds:

$$G_e(R) = \prod_{i=1}^e \left( \frac{R(H_i)}{R(0)} \right) \prod_{i=1}^e H_i^e \cdot \prod_{i_1 \neq i_2} \left( \frac{1}{H_{i_1} - H_{i_2}} \right) \prod_{i=1}^e \left( \frac{e}{H_i} - \frac{R'(H_i)}{R(H_i)} \right).$$

**Remark 2.** The difficulty in showing this identity came especially from the sum in $j$ under the exponential going only over strictly positive integers which complicated relating this result to known literature: Knuth [13] studies composition polynomials $\{F_n(a)\}_{n \in \mathbb{Z} \geq 0}$ which can be always obtained from a power-series $f$ as

$$F_n(a) = [z^n] \left\{ f^a(z) \right\}$$

for some generating series $f(z)$. These satisfy useful identities like

$$\sum_{j \in \mathbb{Z}} F_{n+j}(a) F_{m-j}(a) = F_{n+m}(a),$$

where we assume that $F_{-k}(a) = 0$ for $k > 0$.

We are grateful to the Mathoverflow authors “Alex Gavrilov” and “esg” who showed the following special case of this statement (see [15]) which restricts to a unique formal power series $H(q)$ and thus makes the expression considerably simpler.
Corollary 3. Let $R(t) \in \mathbb{K}[t]$ be a power-series over a field $\mathbb{K}$ not involving $q$ and $H(q)$ the unique solution to $H(q) = qR(H(q))$. Set

$$G(R) = \exp \left[ - \sum_{n,m \geq 0} j \frac{1}{m} \left\{ R^m(z) \right\} \frac{1}{n} \left\{ z^{n-j} \right\} \left\{ R^n(z) \right\} q^{n+m} \right]$$

then the following holds:

$$G(R) = \left( \frac{R(H)}{R(0)} \right) \cdot \left( 1 - \frac{R'(H)}{R(H)} H \right).$$

Note that this result additionally makes our work [6] completely independent of the previous results in [1].

2 Lagrange inversion for Newton–Puiseux generating series

The primary tool we will rely on is the following slightly less standard Lagrange inversion formula collecting multiple results of Gessel [7]:

**Theorem 1** (Gessel [7], Thm. 2.1.1, Thm. 2.4.1, eq. (2.2.9)). Let $R(z) = \sum_{n=0}^{\infty} r_n z^n$ be a power-series not involving $q$, then for the unique solution $H(q)$ satisfying $H(q) = qR(H(q))$ and a Laurent series $\phi = \phi(t)$, we have

$$\phi(H(q)) = [t^0] \left\{ \phi(t) \right\} + [t^{-1}] \left\{ \phi'(t) \log(R(t)) \right\} + \sum_{n \neq 0} \frac{1}{n} z^{n-1} \left\{ \phi'(z) R(z)^n \right\} q^n,$$

$$\log(H/q) = \sum_{m>0} \frac{1}{m} [t^m] \left\{ R(t)^m \right\} q^m.$$  

We however need a modification of this result to include multiple Newton–Puiseux solutions. For the definition of Newton–Puiseux solutions for implicit equations see for example [2].

**Corollary 2.** Using the notation from Theorem [7], let $H_i(x)$ for $i = 1, \ldots, e$ be the different Newton–Puiseux series which are solutions to

$$\left( H_i(x) \right)^e = xR(H_i(x))$$

(7)
then for any Laurent series $\phi(t)$, we have

$$\sum_{k=1}^{N} \phi(H_i(x)) = e\phi_0 + [t^{-1}]\left\{\phi'(t)\log(R(t))\right\} \tag{8}$$

$$+ \sum_{n \neq 0} \frac{1}{n} [t^{n-1}]\left\{\phi'(t)R(t)^n\right\}x^n, \tag{9}$$

$$\log\left(\prod_{i=1}^{c} H_i(q)\right)/q = \sum_{m > 0} \frac{1}{m} [t^{me}]\left\{R(t)^m\right\}q^m. \tag{10}$$

**Proof.** Let $g(x^{\frac{1}{e}})$ be the unique Newton–Puiseux series satisfying $g(x^{\frac{1}{e}}) = x^{\frac{1}{e}}R\left(g\left(x^{\frac{1}{e}}\right)\right)$ for a fixed $e$’th root of $R$. We note that $F(y, x) = y^e - xR(y)$ is irreducible in $\mathbb{K}[x, y]$. Then we can write it by Weierstrass preparation theorem (see e.g. [11, Chap. 3.2]) as

$$F(y, x) = \epsilon(y, x) \cdot f(y, x),$$

where $\epsilon \in \mathbb{K}[x, y]$ is a unit and $f \in \mathbb{K}[x][y]$ is an irreducible polynomial. Applying the Newton–Puiseux theorem from [11, Chap. 5.1], every solution of (7) can be expressed as $H_k(x) = g(e^{\frac{2\pi ki}{e}} x^{\frac{1}{e}})$. We obtain

$$\sum_{k=1}^{e} \phi(g_k(x)) = \sum_{k=1}^{e} \left([t^0]\{\phi(t)\} + [t^{-1}]\left\{\frac{\phi(t)}{e} \log\left(R(t)\right)\right\}\right)$$

$$+ \sum_{n \neq 0} \frac{1}{n} [z^{n-1}]\left\{\phi'(z)R_{x}^{n}(z)\right\}x^n e^{\frac{2\pi ik}{e}},$$

which gives us the required result (8). The second equation follows by a similar argument. $\square$

The first version of the proof will only use the above results and some clever tricks with generating series.
Combinatorial proof. Using Corollary [2] we can write

\[
\sum_{j>0} \sum_{n,m>0} j[z^{en+j}]\{R(z)^n\}z^{em-j}\{R(z)^m\} \frac{p^n q^m}{n \ m} = \sum_{j>0} \frac{1}{j} \left( \sum_{i_1=1}^{e} H_{i_1}^j(p) \right) \left( \sum_{i_2=1}^{e} H_{i_2}^j(q) \right)
\]

\[- \sum_{j>0} \sum_{-j \leq ne<0} [z^{en+j}]\{R(z)^n\} \frac{p^n}{n} \left( \sum_{i=1}^{e} H_i^j(q) \right)
\]

\[- \sum_{j>0 \ i=1} e \left\{ t^{-j} \log \left( R(t) \right) \right\} H_i^j(q)
\]

\[
= \sum_{i_1, i_2} \log \left( 1 - \frac{H_{i_1}(p)}{H_{i_2}(q)} \right) + \sum_{n>0 \ j \geq ne} [z^{j-\epsilon n}]\{R^{-n}(z)\} \left( \sum_{i=1}^{e} H_i^j(q) \right) \frac{p^{-n}}{n}
\]

\[- \sum_{i=1}^{e} \log \left( R(H) \right)
\]

\[
= \sum_{i_1, i_2} \log \left( 1 - \frac{H_{i_1}(p)}{H_{i_2}(q)} \right) - \sum_{i=1}^{e} \frac{1}{n} \frac{H_i^{en}(q)}{R^n(H_i(q)) p^n}
\]

\[- \sum_{i=1}^{e} \log \left( R(H_i) \right)
\]

\[
= \sum_{i_1, i_2} \log \left( 1 - \frac{H_{i_1}(p)}{H_{i_2}(q)} \right) - \log \left( 1 - \frac{q}{p} \right)
\]

\[- \sum_{i=1}^{e} \log \left( R(H_i) \right),
\]

where in the last step, we used [7]. Therefore we obtain after taking exponential and taking the limit \( p \to q \):

\[
\lim_{p \to q} \prod_{i \neq j} (H_i(p) - H_j(q)) \prod_{i=1}^{e} \frac{H_i(p) - H_i(q)}{p - q} \frac{1}{R(H_i(q))} \prod_{i=1}^{e} \left( \frac{p}{H_i(p)} \right)^e
\]

\[
= \prod_{i \neq j} (H_i(q) - H_j(q)) q^e \prod_{i=1}^{e} \left( \frac{dq}{dH_i} \right)^{-1} \prod_{i=1}^{e} \frac{1}{H_i^e(q)} \prod_{i=1}^{e} R^{-1}(H_i(q))
\]

\[
= \prod_{i \neq j} (H_i(q) - H_j(q)) \prod_{i=1}^{e} \frac{H_i(q)^e}{R(H_i(q))} \left( \frac{e}{H_i} - \frac{R'(H_i(q))}{R(H_i(q))} \right)^{-1}.
\]
After taking the negative power of the final expression, we obtain exactly the one from (4).

An alternative way of showing this result is to unpack the proof of the Lagrange inversion formula of Theorem 1 which follows from a residue formula and use the same approach in our more general case.

**Analytic proof.** Without loss of generality assume that $R(q)$ is a polynomial in $q$ and let us set the notation $\sum_{i=1}^n H_i(q) =: H(q)$. When $\phi(z) = \sum_{n \in \mathbb{Z}}$ is a Laurent-series, we will write
\[
[q^{>0}]\{\phi(z)\} = \sum_{n > 0} \phi_n z^n.
\]
We then have the following identity:
\[
[q^{>0}]\{H^{-j}(q)\} := \sum_{n > 0} [z^n]\{H^{-j}(z)\} q^n = \frac{1}{2\pi i} \oint_{|z|=R} \sum_{n > 0} \left( \frac{q}{z} \right)^n \frac{1}{z} H^{-j}(z) dz
\]
which uses that
\[
\frac{1}{2\pi i} \oint_{|z|=R} z^j dz = \begin{cases} 1 & \text{if } j = -1, \\ 0 & \text{otherwise}. \end{cases}
\]
Using Corollary 2 we can reexpress the term under the exponential in (3) as:
\[
\sum_{j > 0} \frac{1}{j} H^j(q) [q^{>0}]\{H^{-j}(q)\}
\]
where we are using that $|H_{i_1}(z)| > |H_{i_2}(q)|$ when $R = |z| > |q|$. This is true for $R$ sufficiently small, because $H_i(0) = 0$ and the leading coefficient of $H_i(q)$ is $q^{1/2}$. Note that
\[
\frac{1}{2\pi i} \oint_{|z|=d} \left( \frac{1}{z-q} - \frac{1}{z} \right) \sum_{i_1,i_2} \log \left( \frac{H_{i_1}(z)}{H_{i_1}(z) - H_{i_2}(q)} \right) dz,
\]
where we are using that $|H_{i_1}(z)| > |H_{i_2}(q)|$ when $R = |z| > |q|$. This is true for $R$ sufficiently small, because $H_i(0) = 0$ and the leading coefficient of $H_i(q)$ is $q^{1/2}$. Note that
\[
\frac{1}{2\pi i} \oint_{|z|=d} \left( \frac{1}{z-q} - \frac{1}{z} \right) \log \left( \frac{z}{z-q} \right) dz
\]
\[
= - \int_0^{2\pi} \left( \frac{r e^{2\pi i (\theta - \tau)}}{1 - \frac{r^2}{d} e^{2\pi i (\theta - \tau)}} \right) \log \left( 1 - \frac{r}{d} e^{2\pi i (\theta - \tau)} \right) d\tau = 0,
\]
where we used the substitution $z = de^{2\pi i \tau}, q = re^{2\pi i \theta}$ and the integral vanishes, because it is proportional to the integral of a total derivative of a $2\pi$ periodic function.

Therefore, we may work instead with

$$\frac{1}{2\pi i} \int_{|z|=d} D(z, q) dz = \frac{1}{2\pi i} \oint_{|z|=d} \left( \frac{1}{z-q} - \frac{1}{z} \right) \sum_{i_1, i_2} \log \left( \frac{H_{i_1}(z)(z-q)}{z(H_{i_1}(z) - H_{i_2}(q))} \right) dz.$$ (11)

The integral (11) can be expressed as the sum of the following residues:

$$\text{Res}_{z=0}(D(z, q)) = \lim_{z \to 0} \left\{ \left( \frac{z}{z-q} - 1 \right) \left[ \sum_{i=1}^e \log \left( \frac{H_i(z)(z-q)}{z(H_i(z) - H_i(q))} \right) \right] + \sum_{i_1 \neq i_2} \log \left( \frac{H_{i_1}(z)(z-q)}{z(H_{i_1}(z) - H_{i_2}(q))} \right) \right\}$$

$$= \sum_{i=1}^e \log \left( \frac{H_i^e(q)}{q} \right)$$

$$\text{Res}_{z=q}(D(z, q)) = \lim_{z \to q} \left\{ \left( 1 - \frac{z-q}{z} \right) \left[ \sum_{i=1}^e \log \left( \frac{H_i(z)(z-q)}{z(H_i(z) - H_i(q))} \right) \right] + \sum_{i_1 \neq i_2} \log \left( \frac{H_{i_1}(z)(z-q)}{z(H_{i_1}(z) - H_{i_2}(q))} \right) \right\}$$

$$= \sum_{i=1}^e \log \left( \frac{H_i(q)}{q} \right) \frac{1}{H_i'(q)} + \sum_{i_1 \neq i_2} \log \left[ \frac{H_{i_1}(q)}{(H_{i_1}(q) - H_{i_2}(q))} \right]$$

One can again see that this implies (4). Finally, note that as the equation is polynomial in coefficients in each degree, the statement holds for any $R(q)$.

As a conclusion, we were able to prove in two different ways the comparison of the power-series appearing in the work of the author [4] and [1] making our result independent of the latter and making the proof independent of other external approaches.

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