Tomographic transform on a sphere and topological insulators

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The tomographic transform was first introduced in the field theory literature long ago. It is closely related to the Radon transform. In this paper we show how the tomographic transform can be implemented on a sphere and apply this result to study surface excitations of a spherical topological insulator with a single Dirac cone on the surface.

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Tomographic transform finds its origin in field theory in Alan Luther’s multidimensional bosonization construction, but explicitly it was written down by Charles Sommerfeld and he also coined the term. It is also closely related to the well known Radon transform. In this paper we show how the tomographic transform can be implemented on a sphere and apply this result to study surface excitations of a spherical topological insulator having the topology of a sphere.

The aim of this paper is to implement tomographic transform on a sphere and apply the developed formalism to study surface excitations of a spherical topological insulator. Since the sphere has no boundary, this is more appealing, because one does not need to worry about boundary conditions which are simply ignored in the case when the boundary is infinite plane. It is also obvious that condensed matter systems are always of finite dimension.

The relation between quantum spin Hall effect and strong topological insulators is well known and discussed in detail in the literature. For example, in the topological band theory one uses the analogy with quantum spin Hall effect to count topological invariants of 3D insulators, and in the topological field theory both 3D topological insulators and quantum spin Hall systems are descendants of 4D quantum Hall effect. In this paper we show that surface excitations of a spherical topological insulator are the sum of edge states of quantum spin Hall disks of the same radius as the sphere.

The spherical topological insulator was already studied in the Refs. however our approach is completely different and it will turn out that one can still learn some interesting facts studying this case. We also briefly discuss a strong topological insulator having the topology of a torus and make rigorous connection of hydrodynamic theory of surface excitations with the topological band theory.

We discuss some technical details first. In the case of infinite plane the tomographic transform of a function $h(x, y)$ is

$$h_{\theta}(\xi) = \int_0^\infty \left( \frac{k}{2\pi^3} \right)^{1/2} dk \int \cos k(\xi - \xi') h(x', y')dx'dy'$$

(1)

where $\xi = \hat{k} \cdot \mathbf{x}$, $\xi' = \hat{k} \cdot \mathbf{x}'$, $\hat{k} = (\cos \theta, \sin \theta)$. It satisfies the equation

$$\int h^2(x, y)dx dy = \int_\mathcal{R} d\theta \int h^2_{\theta}(\xi)d\xi$$

(2)

where $\mathcal{R} = \{-\pi/2 \leq \theta \leq \pi/2\}$. This is slightly modified version of the tomographic transform. We need to find something like this on a sphere.

The starting point is the analogy with Funk transform. The general reference on this topic is . The Funk transform of a function $f$ on a 2-sphere is defined as

$$Ff(x) = \int_{x \in C(x)} f(u)ds(u)$$

(3)

where $x$ is a unit vector and the integration is over the arclength $ds$ of the great circle $C(x)$ consisting of all vectors perpendicular to $x$. This can be viewed as Radon transform on a sphere. In the case of the infinite plane $C(x)$ are rays perpendicular to two-dimensional vector $x$.

This suggests that Euler angles could be the suitable choice as parameter space of the tomographic representation. Indeed, there are three angles $\alpha, \beta, \gamma$ needed to parametrize rotations in three-dimensions and the measure in the space of these angles is given by $d\omega = \sin \beta d\alpha d\beta d\gamma$. Two angles $\alpha$ and $\beta$ can be chosen as the direction of $x$ and the third angle $\gamma$ as the angle denoting the position on the circle $C(x)$.

Now we pose the problem. Given the function of two variables $h(\theta, \phi)$ we need to find a function $h_{\alpha, \beta}(\gamma)$ which is real and satisfies the equation

$$\int_{S^2} h^2(\theta, \phi)d\Omega = \int_\mathcal{U} \sin \beta d\alpha d\beta \int_0^{2\pi} d\gamma h^2_{\alpha, \beta}(\gamma)$$

(4)

The pair $(\alpha, \beta)$ is an index, $\gamma$ is an argument, and $u$ is the upper hemisphere.

To get further insight into the problem we need to study the Dirac Hamiltonian on a sphere, since multidimensional bosonization and tomographic transform are closely related to each other. According to the Ref.
Dirac Hamiltonian on a curved surface of a topological insulator is
\[ H = \frac{\hbar}{2} [\nabla \cdot \mathbf{n} + \mathbf{n}(\hat{\rho} \times \hat{\sigma}) + (\hat{\rho} \times \hat{\sigma} \mathbf{n})] \tag{5} \]
where \( \mathbf{n} \) is a unit vector normal to the surface and \( v = 1 \) is the electron velocity. On a unit sphere this is \((\hat{\rho} \hat{\sigma} \hat{I})\). There is a constant term, \( \frac{\hbar}{2} \), which was dropped. It will be inserted in the final result.

Then one can observe that the operator \((\hat{\rho} \hat{\sigma} \hat{I})\) and the operator \(\hat{\sigma}_z \hat{I}_z\) (which is Dirac Hamiltonian on a unit circle) have the same set of eigenvalues, i.e. integers. In (1) we used Fourier analysis. Therefore we can try to use spherical harmonic analysis in the case (4) and expand the function \( h \) first
\[ h(\theta, \varphi) = \sum_{l m} h_{lm} Y_{lm}(\theta, \varphi) \tag{6} \]

There are a set of orthogonal functions of three Euler angles \(D^l_{m m'}(\alpha, \beta, \gamma) = e^{i m' \gamma} d^l_{m m'}(\beta) e^{i m \alpha} \). Their explicit form can be found in textbooks.\(^{20}\) We will need only the orthogonality relation
\[ \int D^l_{m m'}(\alpha, \beta, \gamma) D^{l'}_{m' m''}(\alpha, \beta, \gamma) d\omega = \frac{1}{2j_{l} + 1} \delta_{m, m'} \delta_{m', m''} \tag{7} \]
and transformation properties under inversion and complex conjugation
\[ d^l_{m m'}(\pi - \beta) = (-1)^l d^l_{m', m}(\beta) \tag{8} \]
\[ d^l_{m m'}(\beta) = (-1)^{m - m'} d^l_{m', -m}(\beta) \tag{9} \]
These functions are wavefunctions of a symmetric rigid rotator. The matrix \( \hat{D}^l \) gives unitary irreducible representations of rotation group. Therefore one can try to use them to construct the tomographic transform.

Indeed, one can check that the function
\[ h_{\alpha \beta}(\gamma) = \sum_{l m} \sqrt{\frac{2l + 1}{8\pi^2}} [D^l_{l m}(\alpha, \beta, \gamma) h_{lm} + D^l_{l - m}(\alpha + \pi, \pi - \beta, \pi - \gamma) h_{l,-m}] \tag{10} \]
is real and satisfies Eq. (4). It has the symmetry \( h_{\alpha \beta}(\gamma) = h_{\pi + \alpha, \pi - \beta}(\pi - \gamma) \). Further, if \( \int h(\theta, \varphi) d\Omega = 0 \), i.e. \( h \) describes surface excitations of some 3D incompressible fluid, then \( \int h_{\alpha \beta}(\gamma) d\Omega = 0 \), which means that \( h_{\alpha \beta}(\gamma) \) can be viewed as edge excitations of some 2D incompressible fluid.

The next step is to rewrite the Dirac Hamiltonian in terms of some spinors \( \psi_{\alpha \beta}(\gamma) \). One can expand the initial spinor \( \psi = (\psi^\uparrow, \psi^\downarrow) \) as
\[ \psi(\theta, \varphi) = \sum_{l m} a_{l m} Y_{l m}(\theta, \varphi) \]
\[ \psi(\theta, \varphi) = \sum_{l m} \tilde{a}_{l m} Y_{l, m+1}(\theta, \varphi) \]
Then the Hamiltonian becomes
\[ H = \sum_{l m} \left[ m a^\dagger_{l m} a_{l m} - (m + 1) a^\dagger_{l m} a_{l m} \right] + \sqrt{(l - m)(l + m + 1)} \left( a^\dagger_{l m} a_{l m+1} + a^\dagger_{l m+1} a_{l m} \right) \tag{11} \]
Defining new Fermi fields
\[ \psi_{\alpha \beta}(\gamma) = \sum_{l m m'} a_{l m} D^l_{m' m}(\alpha, \beta, \gamma) \tag{12} \]
\[ \psi_{\alpha \beta}(\gamma) = \sum_{l m m'} \tilde{a}_{l m} D^l_{m' m+1}(\alpha, \beta, \gamma) \tag{13} \]
one can write
\[ H = \int \psi^\dagger_{\alpha \beta}(\gamma)(\hat{\sigma} \hat{J}) \psi_{\alpha \beta}(\gamma) d\omega \tag{14} \]
where
\[ J_+ = J_x + i J_y = e^{i \alpha} \left( \frac{\partial}{\partial \beta} + i \cot \beta \frac{\partial}{\partial \alpha} - \frac{1}{\sin \beta} \frac{\partial}{\partial \gamma} \right) \]
\[ J_- = J_x - i J_y = e^{-i \alpha} \left( -\frac{\partial}{\partial \beta} + i \cot \beta \frac{\partial}{\partial \alpha} - \frac{1}{\sin \beta} \frac{\partial}{\partial \gamma} \right) \]
\[ \hat{J}_z = -\frac{\partial}{\partial \alpha} \]
This is due to relations
\[ \hat{J}_- D^l_{m' m+1} = \sqrt{(l - m)(l + m + 1)} D^l_{l m} \tag{15} \]
\[ \hat{J}_+ D^l_{m' m} = \sqrt{(l - m)(l + m + 1)} D^l_{l+1 m} \tag{16} \]
\( \hat{J} \) is the space fixed rigid rotator angular momentum operator. We want to replace the initial problem of a Dirac electron on a sphere with another problem: coherent states of a symmetric rigid rotator with spin \( 1/2 \) (\( \alpha \) and \( \beta \) are polar angles that determine the direction of the symmetry axis and \( \gamma \) describes rotations around this axis). This could be done retaining only \( m' = \pm l, \pm (l-1) \) in the sums (12, 13) and multiplying various terms with proper coefficients. Now, when one passes to the body fixed frame, the action of \((\hat{\sigma} \hat{J})\) on the rotated spinors becomes diagonal (we will do this in the reverse order; see (24)). To do this one should perform Euler rotation \( U(\alpha, \beta, \gamma) \). This is easy to see in the quasiclassical approximation when \( l \) is large. In this limit the operator \((\hat{\sigma} \hat{J})\) becomes \((\hat{\sigma} \mathbf{n}) \hat{J} \), because angular momentum is directed along \( \mathbf{n} \), and is diagonalized by rotation to a frame in which \( z \) axis points towards \( \mathbf{n} \). Under rotations of the coordinate frame, the spin part of the wave function and its angular part transform independently. The inverse of the operator which acts on spin part is
\[ \hat{U}(\alpha, \beta, \gamma)^{-1} = \left( \begin{array}{cc} \cos \frac{\beta}{2} e^{-i(\alpha - \gamma)/2} & -\sin \frac{\beta}{2} e^{-i(\alpha + \gamma)/2} \\ \sin \frac{\beta}{2} e^{i(\alpha - \gamma)/2} & \cos \frac{\beta}{2} e^{i(\alpha + \gamma)/2} \end{array} \right) \tag{17} \]
The transformation of angular part in the sector with momentum \( l \) is described by the D-function

\[
\psi_{lm} = \sum_{m'} D_{m'm}^{l} \psi_{lm'}
\]  

(18)

There is the following formula

\[
\hat{U}^{-1} \hat{\sigma}_z \hat{U} = (n \hat{\sigma})
\]  

(19)

where \( n = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta) \), which will be used later.

Spinors in the rotated frame \( \psi_{\alpha \beta}(\gamma) \) can be bosonized introducing boson fields

\[
\phi^{(1)}_{\alpha \beta}(\gamma) = \sqrt{2} \sum_{l \geq 0} \frac{e^{-il} \sqrt{l} [b_{\alpha \beta}(l) \gamma + h.c.]}{l!}
\]  

(20)

and \( \phi^{(2)}_{\alpha \beta}(\gamma) = \sqrt{2} \sum_{l \geq 0} \frac{e^{-il} \sqrt{l} [b_{\alpha \beta}^*(l) \gamma + h.c.]}{l!} \)  

(21)

\( \epsilon \) is needed for regularization of the theory and one should take the limit \( \epsilon \to 0 \) in the final result. Boson creation and annihilation operators satisfy the commutation relations

\[
[b_{\alpha}^{(i)}(l), b_{\beta}^{(k)^\dagger}(j)] = \delta_{ik} \delta_{lj} \delta_{\alpha \beta}
\]  

(22)

and their time dependence is \( e^{-it} \). Correlators of bosonic exponents are

\[
\langle \exp[-\phi_{\alpha}^{(1)}(t, \gamma)] \exp[\phi_{\beta}^{(1)}(0, \gamma')]) \rangle = \frac{\epsilon^2}{[1 - e^{-i(t-\gamma+\gamma')}]^2} \delta_{\alpha \beta}
\]  

(23)

and for \( \phi^{(2)} \) one should change the sign of \( \gamma \) and \( \gamma' \).

The usual Fermi field is given by

\[
\psi(\theta, \varphi) = \frac{1}{\sqrt{2\pi e}} \sum_{lm} \int d\Omega \int_0^{2\pi} \frac{d\nu}{2\pi} (-1)^{m} Y_{lm}(\theta, \varphi) \hat{U}^{-1} (\nu) \times \left\{ e^{-i(t-\gamma)/2} e^{\phi^{(1)}} \begin{pmatrix} D_{l-1,m} + \sqrt{2l+1} D_{l-1,-m} & 0 \\ 0 & \sqrt{2l+1} D_{l-1,-m} \end{pmatrix} \right\} + e^{-i(t+\gamma)/2} e^{\phi^{(2)}} \begin{pmatrix} 0 & \sqrt{2l+1} D_{l-1,m} \\ D_{l-1,m} & 0 \end{pmatrix}
\]  

(24)

Here \( \hat{U} \) stands for \( \hat{U}(\alpha, \beta, \gamma) \), the arguments of D-functions are \( \alpha, \beta, \gamma \), \( \phi^{(i)} \) is the shortening for \( \phi^{(i)}_{\alpha \beta}(\gamma) \) and \( d\Omega = \sin \beta d\alpha d\beta \). This expression seems complicated, but it has a clear meaning. We will demonstrate this by deriving equations of motion for this field. First, we compute the derivative \( id/dt \). Due to simple equations of motions for \( \phi^{(1)} \) and \( \phi^{(2)} \)

\[
\partial_t \phi^{(1)} + \partial_t \phi^{(2)} = 0, \quad \partial_t \phi^{(1)} - \partial_t \phi^{(2)} = 0
\]  

(25)

one can replace this derivative by \( \mp id/d\gamma \) and then integrate by parts. \( d/d\gamma \) now is multiplication by a constant. On the other hand, action of \( (\sigma \dot{J}) \) on rho of \( \phi \) can be replaced by \( -i \sigma J \), \( J \) acting on D-functions only, not on \( \hat{U} \) or bosonic exponents (this is due to formulas \( 15 \) and \( 16 \)). Using \( (\sigma \dot{J}) \hat{U}^{-1} = \hat{U}^{-1}(\sigma \dot{P}) \) (\( J \) does not act on \( \hat{U} \)) where \( \dot{P} \) is the body fixed rigid rotator angular momentum operator

\[
\hat{P}_z = \hat{P}_1 - i \hat{P}_2 = e^{\gamma} \left( \frac{\partial}{\partial \beta} - i \cot \beta \frac{\partial}{\partial \gamma} + i \frac{\partial}{\sin \beta \partial \alpha} \right)
\]  

\[
\hat{P}_z = \hat{P}_1 + i \hat{P}_2 = e^{-\gamma} \left( \frac{\partial}{\partial \beta} - i \cot \beta \frac{\partial}{\partial \gamma} + i \frac{\partial}{\sin \beta \partial \alpha} \right)
\]  

\[
\hat{P}_z = -i \frac{\partial}{\partial \gamma}
\]  

which has anomalous commutation relations \([\hat{P}_z, \hat{P}_j] = -\varepsilon_{ijk} \hat{P}_k \) and acts on the D-functions according to

\[
\hat{P}_z D_{m'+1,m}^{l} = \sqrt{(l-m')(l+m'+1)} D_{m',m}^{l}
\]  

\[
\hat{P}_z D_{m,m}^{l} = \sqrt{(l-m')(l+m'+1)} D_{m',m}^{l}
\]  

and the fact that spinors in square brackets are eigenfunctions of \( (\sigma \hat{P}) \) with eigenvalues \(-l \) and \( l+1 \), one can show that

\[
\frac{d\psi}{dt} = \left[ \frac{1}{2} + (\sigma \hat{J}) \right] \psi
\]  

This is correct equation of motion.

Now we will outline the derivation pairwise correlation functions \( \langle \psi \psi^{\dagger} \rangle \) and \( \langle \psi^{\dagger} \psi \rangle \) from \( 24 \) (details of the calculations can be found in appendix A). Using \( 23 \) one can see that integration over \( \gamma \) and \( \gamma' \) in the fermionic correlator pick up the sector with momentum \( l \) from the bosonic correlator. First of the two spinors in the square brackets in \( 24 \) are important for the calculation of the correlator \( \langle \psi \psi^{\dagger} \rangle \), and the second ones for the correlator \( \langle \psi^{\dagger} \psi \rangle \) only. There are two independent contributions to the fermionic correlator emerging from two different bosonic exponents. The part obtained from the terms in the second square brackets in \( 24 \) can be converted to the the part obtained from the first square brackets, but where the integration is over the lower hemisphere, by using transformation properties of D-functions \( 5, 6 \).
Calculating the emerging integrals of the product of three D-functions (cos $\beta = d_0^0(\beta)$, sin $\beta = \sqrt{2}d_0^1(\beta)$) which are now over the entire sphere, using formulas in [20], it can be shown that one obtains correct expressions for fermionic correlators. For example

$$
\langle \psi(t, \theta, \varphi) \psi^\dagger(0, \theta', \varphi') \rangle = \sum_{lm} e^{-i(t' + 1/2)} \frac{1}{2l + 1} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')
$$

which coincides with the expression obtained by conventional means using eigenfunctions of the operator $\hat{d} l$ with eigenvalue $l$

$$
\frac{1}{\sqrt{l + 1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} Y_{lm}\left| Y_{l,m+1} \right|
$$

and eigenvalue $-l - 1$

$$
-\frac{1}{\sqrt{l + 1}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} Y_{lm}\left| Y_{l,m+1} \right|
$$

The necessity of introducing additional factors $\sqrt{l / (l + 1)}$ in Eq. (24) can be seen by explicit calculations (see appendix A).

The operators (24) do not anticommute. Therefore one should insert Klein factors in (24) that ensure anticommutation relations. In the continuum case one uses the limit $k \to 0$ of some boson creation operators for this purpose, where $k$ is momentum. This is not possible in the discrete case we are dealing with. We will introduce some finite but very small compressibility of the liquid and static bosonic field $b_{\alpha\beta}^{(i)}$ instead. Then if one adds a term $b_{\alpha\beta}^{(i)} \hat{d}^{1/2}$ to $\phi_{\alpha\beta}^{(i)}$ and inserts the operator

$$
\hat{O}_{\Omega} = e^{\pi i \sum_{\gamma \leq \Omega} [b_{\alpha\beta}^{(1)} + b_{\alpha\beta}^{(2)}] \hat{d}^{1/2}}
$$

into the integrand in (24), then (24) will satisfy the required anticommutation relations

$$
\{ \psi_i(t, \mathbf{x}), \psi_k(t, \mathbf{x}') \} = \delta_{ik} \delta(\mathbf{x} - \mathbf{x}')
$$

We assume that $\delta$ is small. Introducing the operator $b_{\alpha\beta}^{(i)}$ into the theory modifies the electronic density operator: there will be an extra term proportional to $\delta^{1/2}$ which tends to 0 in the limit $\delta \to 0$. This is consistent with the incompressibility of the liquid.

One can trace the analogy between the scheme developed in this paper and that of Luther’s almost in every step. Luther also showed that bosonic exponents in his scheme lead to correct expressions for general correlation functions. His analysis relies only on the two-point correlation functions, equations of motion for the fermi fields and anticommutation relations. One can directly extend this analysis to the present case and there is no need to repeat them here.

The formula (24) has no useful applications. It was derived only to show that bosonization on a sphere can be consistently carried out.

Though we mentioned the relation of our analysis to topological insulators throughout the paper several times, it was quite abstract. Now we will make this relation more concrete. Following the Ref. [8], one can describe low lying excitations of a strong topological insulator as surface deformations of a two-component 3D incompressible liquid confined by a smooth potential well with the Hamiltonian

$$
H = \frac{1}{2} \rho_0 e E R^2 \int \left[ \frac{\partial_\gamma^2}{\Omega} \psi_1^\dagger(\theta, \varphi) + \frac{\partial_\gamma^2}{\Omega} \psi_2^\dagger(\theta, \varphi) \right] d\Omega
$$

where $E$ is the electric field of the confining potential on the surface, $\rho_0$ is the density of the 3D electronic liquid, $\int h_i(\theta, \varphi) d\Omega = 0$, $i = 1, 2$ and $R$ is the radius of the sphere. This theory is a modification of the hydrodynamic theory of the edge excitations of fractional quantum Hall systems. Using (14) and (10) one can rewrite (28) as follows

$$
H = \frac{1}{2} n e E R \int_u d\Omega \int_0^{2\pi} \left[ h_1^2(\gamma) + h_2^2(\gamma) \right] d\gamma
$$

where $h_{\alpha\beta}$ can be viewed as surface deformations of a 2D incompressible fluid with the density $n = C \rho_0 R$ and having a shape of a disk of radius $R$, $C$ is a numeric constant ($h_{\alpha\beta}$ has been rescaled by a factor $C^{-1/2}$ compared to (10)). We have assumed in (29) that electric field of the potential which confines this liquid is $E$. In analogy with the paper [8] we assume that $h_{\alpha\beta}(\gamma)$ are edge states of quantum spin Hall system, i.e. they have equations of motion

$$
\partial_\gamma h_{\alpha\beta}(\gamma) = (-1)^v v \partial_\gamma h_{\alpha\beta}(\gamma)
$$

where $v = eE/\hbar n$. Thus the constant $C$ is not arbitrary: $C = eE/\hbar \rho_0 v R$. The theory (24), (30) can be easily quantized

$$
H = \int_u d\Omega \sum_{i > 0} \sum_{i > 0} v b_{\alpha\beta}^{(i)}(l) b_{\alpha\beta}^{(i)}(l)
$$

where the boson operators $b_{\alpha\beta}^{(i)}$ were introduced earlier (20,22).

Thus we have found that surface excitations of a spherical topological insulator with a single Dirac cone on the surface are the sum of quantum spin Hall edge states. Since the transform (10) is non-local, every edge state that has a fixed location in the tomographic representation is spread over the entire surface of the topological insulator.

Now we will briefly discuss a toric topological insulator for which one can make a direct connection with the topological band theory. This case is simpler than a spherical case.

Suppose that there is an ideal crystal of a strong topological insulator with atomic spacing $a$ having $N \gg 1$ unit cells along $x$ and $y$ directions and periodic boundary conditions imposed in these directions. $z = 0$ is one of the
surfaces of this crystal. The function $h(x, y)$ defined on this surface and having the property $\int h(x, y) dx dy = 0$ can be expanded as ($L = Na$)

$$h(x, y) = \sum_{mn} h_{mn} e^{2\pi i (mx + ny) / L}$$

(32)

and there can be defined a new function

$$h_{(mn)}(\xi) = (m^2 + n^2)^{1/4} \sum_{l=1}^{\infty} [h_{lm, ln} e^{2\pi il\xi / L_{(mn)}} + h_{-lm, -ln} e^{-2\pi il\xi / L_{(mn)}}]$$

(33)

where

$$L_{(mn)} = Na_{(mn)}, \quad a_{(mn)} = \frac{a}{\sqrt{m^2 + n^2}}$$

and $(mn)$ denotes a pair of coprime integers $m$ and $n$ such that $m > 0$ and $n$ arbitrary or one of the pairs $m = 0, n = 1$ or $m = 1, n = 0$. $(mn)$ are in fact Miller indices and $l$ has the meaning of a winding number. It is easy to see now that

$$\int h^2(x, y) dx dy = L \sum_{(mn)} \int_0^{L_{(mn)}} h^2_{(mn)}(\xi) d\xi$$

(34)

where the sum is over all possible pairs $(mn)$. This means that we have rewritten the 2D hamiltonian as a sum of 1D hamiltonians over closed loops of length $L_{(mn)}$.

One can see that if $h(x, y)$ describes surface excitations of 3D incompressible fluid, $h_{00} = 0$, then $h_{(mn)}(\xi)$ describes edge excitations of a 2D incompressible fluid, $\int h_{(mn)}(\xi) d\xi = 0$.

The above procedure applies also to 2D fermionic hamiltonian

$$\frac{2\pi}{L} \sum_{l, z} \hat{\psi}_{lm, ln}^\dagger \psi_{lm, ln} (m\hat{\sigma}_z + n\hat{\sigma}_y)$$

(35)

The operator $m\hat{\sigma}_z + n\hat{\sigma}_y$ can be diagonalized by a rotation. Diagonalized hamiltonian is $2\pi \hat{\sigma}_z \sqrt{m^2 + n^2} / L$. Eq. (35) describes surface excitations of a strong topological insulator for small momenta $l\sqrt{m^2 + n^2} \ll N$. Surface deformations of two incompressible liquids $h_{i(mn)}(t, \xi), i = 1, 2$ will satisfy equations of motion

$$\partial_t h_{i(mn)}(t, \xi) = -(1)^i \partial_\xi h_{i(mn)}(t, \xi)$$

These equations are consistent with the spectrum $2\pi v \sqrt{m^2 + n^2} / L$ of the fermionic hamiltonian (35) and the exponent $e^{2\pi i l\xi / L_{(mn)}}$ we have chosen in (33). It is clear that one can construct bosonization scheme in analogy with the Ref. [1].

Now it is easy to make parallel with the $Z_2$ invariants of 2D band structures. Let $H(k_x, k_y, k_z)$ be a tight binding Hamiltonian of a strong topological insulator (see, e.g. the Ref. [1]) and $H_{(mn)}(2\pi l / L_{(mn)}, k_z) = H(2\pi l / L, 2\pi l / L, k_z)$.

There is one to one correspondence between $H_{(mn)}(2\pi l / L_{(mn)}, z)$ and the edge state hamiltonians (35) for fixed direction $(mn)$. The hamiltonian $H_{(mn)}(2\pi l / L_{(mn)}, k_z)$ is periodic with the period $\frac{2\pi}{a_{(mn)}}$ along $k_z$ direction.

This means that the 2D lattice has Brillouin zone $\{ -\frac{\pi}{a_{(mn)}} \leq k_\xi \leq \frac{\pi}{a_{(mn)}}, -\frac{\pi}{a} \leq k_z \leq \frac{\pi}{a} \}$. It follows from the topological band theory that all hamiltonians $H_{(mn)}$ for fixed $(mn)$ are hamiltonians supporting quantum spin Hall effect. The easiest way to see this is to use the simple counting argument due to Roy [24], or the approach of the papers [12,25]. In other words $Z_2$ invariants of the planes $(mn)$ are all equal to 1 in the case of a strong topological insulator. This means that the fields $h_{i(mn)}, i = 1, 2$ (in the hydrodynamic theory) are the edge states of quantum spin Hall systems.

There were some difficulties in the continuous case [2]. It is clear now that these difficulties can be overcome by considering a discrete version of the tomographic transform.

The assumption of equal number of sites along $x$ and $y$ directions is not essential. If there are different number of sites, then this would only lead to unnecessary complication of the analysis.

In summary, we have shown how the tomographic transform can be implemented on a sphere. Using this transform, we constructed Luther’s version of the bosonization on a sphere and showed that surface excitations of a spherical topological insulator with a single Dirac cone on the surface are the sum of quantum spin Hall edge states. However, this is not true for the entire topological insulator: strong topological insulator can not be presented as a sum of two-dimensional topological insulators. We also constructed discrete version of the tomographic transform on a torus and rigorously showed how the connection can be made between hydrodynamic theory of surface excitations of a non-interacting topological insulator and the topological band theory.

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Appendix A: Calculation of correlation functions
In this appendix, we will show how to compute two-point correlation functions starting from the Eq. (24). After calculation of bosonic exponents and integration over $\gamma$ and $\gamma'$ one obtains

$$
\langle \hat{\psi}(t, \theta, \varphi) \hat{\psi}(0, \theta', \varphi') \rangle = \frac{1}{2\pi} \sum_{lm} (l+1) e^{-ilt(l+1/2)} (-1)^{m_1 - m_2} Y_{lm_1}(\theta, \varphi) Y^*_{lm_2}(\theta', \varphi') 
\times \left[ \int d\Omega \frac{1 + \hat{\sigma} \hat{n}}{2} D^*_{l-m_1}(\alpha, \beta, 0) D^*_{l-m_2}(\alpha, \beta, 0) + \int d\Omega \frac{1 - \hat{\sigma} \hat{n}}{2} D^*_{l-m_1}(\alpha, \beta, 0) D^*_{l-m_2}(\alpha, \beta, 0) \right]
$$

Using Eqs. (8) and (9) it can be shown that the second integral in square brackets equals

$$
\int d\Omega \frac{1 + \hat{\sigma} \hat{n}}{2} D^*_{l-m_1}(\alpha, \beta, 0) D^*_{l-m_2}(\alpha, \beta, 0)
$$

where ‘$L$’ means that integration is over the lower hemisphere. The integral

$$
\int d\Omega \frac{1 + \hat{\sigma} \hat{n}}{2} D^*_{l-m_1}(\alpha, \beta, 0) D^*_{l-m_2}(\alpha, \beta, 0)
$$

can be calculated using the following integral of the product of three D-functions

$$
\int D_{l_1 m_1}^j(\omega) D_{l_2 m_2}^{j_2}(\omega) D_{l_3 m_3}^{j_3}(\omega) \frac{d\omega}{8\pi^2} = \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \quad (A1)
$$

and the fact that

$$
\frac{1 \pm \cos \beta}{2} = d_{l, \pm 1}^1(\beta), \quad \frac{1}{\sqrt{2}} \sin \beta = d_{l, 0}^0(\beta)
$$

Here \( \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \) is a 3j-symbol which is non-zero only when \( m_3 = -m_1 - m_2 \). For \( j_1 = j_2 \) and \( j_3 = 1 \), 3j-symbols can be easily calculated using tables in \( 20 \). For example,

$$
\left( \begin{array}{ccc} l & l & 1 \\ m & -m & 0 \end{array} \right) = (-1)^{l-m} \frac{2m}{2l(2l+1)(2l+2)^{1/2}}, \quad \left( \begin{array}{ccc} l & l & 1 \\ m & -m & 1 \end{array} \right) = (-1)^{l-m} \left[ \frac{2(l-m)(l+m+1)}{2l(2l+1)(2l+2)} \right]^{1/2}
$$

The result coincides with the one obtained by using the eigenfunction \( |20\rangle \) of the operator \( \hat{\alpha} \hat{\bar{\alpha}} \).

For the other correlator \( \langle \hat{\psi}^\dagger \hat{\psi} \rangle \), after integration over $\gamma$ and $\gamma'$, one has

$$
\langle \hat{\psi}^\dagger(0, \theta', \varphi') \hat{\psi}(t, \theta, \varphi) \rangle = \frac{1}{2\pi} \sum_{lm} \frac{l}{2l+1} e^{ilt(l+1/2)} (-1)^{m_1 - m_2} Y_{lm_1}(\theta, \varphi) Y^*_{lm_2}(\theta', \varphi') \quad (A2)
\times \int d\Omega \left\{ 2l D^*_{l-m_1}(\alpha, \beta, 0) D^*_{l-m_2}(\alpha, \beta, 0) \frac{1 + \hat{\sigma} \hat{n}}{2} + D^*_{l-m_1}(\alpha, \beta, 0) D^*_{l-m_2}(\alpha, \beta, 0) \frac{1 - \hat{\sigma} \hat{n}}{2} 
+ \hat{\sigma}_+ (\alpha, \beta) \sqrt{2} D^*_{l-m_1}(\alpha, \beta, 0) D^*_{l-m_2}(\alpha, \beta, 0) + \hat{\sigma}_- (\alpha, \beta) \sqrt{2} D^*_{l-m_1}(\alpha, \beta, 0) D^*_{l-m_2}(\alpha, \beta, 0) \right\} \quad (A3)
$$

where

$$
\hat{\sigma}_\pm (\alpha, \beta) = \frac{1}{2} \hat{U}^{-1}(\alpha, \beta, 0) \hat{\sigma}_\pm \hat{U}(\alpha, \beta, 0)
$$

or explicitly

$$
\hat{\sigma}_+(\alpha, \beta) = \hat{\sigma}_-^T(-\alpha, -\beta) = \frac{1}{2} \left( \begin{array}{cc} \sin \beta & e^{-i\alpha}(1 + \cos \beta) \\ e^{i\alpha}(1 - \cos \beta) & \sin \beta \end{array} \right)
$$
Let $\hat{A}$ be the expression in the first square brackets in (A2) and $\hat{B}$ in the second. Then using the Eqs. (8) and (9) it can be shown that

$$\int_L \hat{A} d\Omega = \int_u \hat{B} d\Omega$$

The integral $\int \hat{A} d\Omega$ can be calculated using (A1). For example,

$$\int A_{11} d\Omega = \frac{1 - m_1}{l} \delta_{m_1 m_2}$$

and

$$\langle \psi_{\uparrow}^\dagger (0, \theta', \varphi') \psi_{\uparrow} (t, \theta, \varphi) \rangle = \sum_{l m} \frac{1 - m}{2l + 1} e^{i(t+1/2)} Y_{l m}^* (\theta, \varphi) Y_{l m}^* (\theta', \varphi')$$

which coincides with the expression obtained from (27) by conventional means.

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