Quantum group approach to steady states of boundary-driven open quantum systems

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Abstract. We present a systematic approach for constructing steady state density operators of Markovian dissipative evolution for open quantum chain models with integrable bulk interaction and boundary incoherent driving. The construction is based on fundamental solutions of the quantum Yang–Baxter equation pertaining to quantum algebra symmetries and their quantizations ($q$-deformations). In particular, we facilitate a matrix-product state description, by resorting to generic spin-$s$ infinite-dimensional solutions associated with non-compact spins, serving as ancillary degrees of freedom. After formally deriving already known steady states for the anisotropic spin-1/2 Heisenberg chain from first symmetry principles, we obtain a class of solutions belonging to interacting quantum gases with $SU(N)$-symmetric Hamiltonians, using a restricted set of incoherent boundary jump processes, and point out how new non-trivial generalizations emerge from twists of quantum group structures. Finally, we discuss the possibilities for the analytical calculation of observables by employing the algebraic properties of associated auxiliary vertex operators.

Keywords: algebraic structures of integrable models, integrable spin chains (vertex models), quantum integrability (Bethe ansatz), rigorous results in statistical mechanics

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1. Introduction

The intrinsic complexity of quantum configuration space, typically preventing accurate numerical calculation or efficient classical simulation of strongly correlated many-body systems, is one of the main reasons why exact analytical solutions, despite their scarcity, will always have a mesmerizing role among theoretical physicists. Before the formulation of the theory of integrability with a rigid algebraic framework, we witnessed several remarkable solutions of bona fide many-body quantum systems: from the revolutionary Bethe’s coordinate ansatz for solutions of magnetic chains in one dimension, with Onsager’s ongoing work on the Ising model, Baxter’s solvable 2D classical vertex models...
Quantum group approach to steady states and the theory of factorizable scattering matrices. A significant breakthrough was initiated in the late 1960s with the solution of the Korteweg–de Vries equation, soon formalized by Lax and others via an $L - A$ matrix pair formulation or analogous zero-curvature representation, which matured with a Hamiltonian formalism ($r$-matrix approach) in the early 1970s. That result gave birth to the classical theory of solitons (known as the classical inverse scattering method), predominantly developed in the Leningrad School of Mathematical Physics [1], which has been able to resolve a few other paradigmatic nonlinear partial differential equations (nonlinear Schrödinger equation, sine-Gordon model etc) within a unified algebraic framework, whose ‘quantization’ has finally led to the quantum inverse scattering method (QISM), often simply called the algebraic Bethe ansatz (ABA) [2], which is nowadays one of the main pillars of modern theoretical physics, finding its applications in condensed matter physics, quantum information theory and quantum field theories.

Since objects from QISM are merely a special (symmetry-enhanced) form of matrix-product states, the latter playing a profound role in an efficient description of quantum states in cutting edge numerical simulations, it is of no surprise that Baxter’s and Faddeev’s work strongly influenced onward progress in the area of exact solvability. Few of the most prominent solutions belong in the realm of classical exclusion processes and reaction–diffusion processes in the context of classical master equations, and, for example, valence-bond states in the quantum domain [3, 4]. However, in spite of all those major developments in the area of quantum integrability, until very recently, surprisingly, no exact many-body solutions of nonequilibrium quantum dynamics have appeared in the literature.

In the seminal work of Prosen, a general method for exact diagonalization of quadratic (quasi-free) Liouville operators in terms of normal decay modes within the Markovian quantum master equation framework, by performing operator quantization over Liouville–Fock space, has been introduced [5], and soon further developed in [6, 7]. Interacting systems are much harder to deal with, and at the moment we still lack exact many-body solutions of the full Liouvillian spectrum for a genuinely interacting many-body system. Therefore, it is reasonable to start addressing first the simplest and perhaps physically most interesting cases of Liouville eigenmodes, namely the steady state density operators, i.e. time-asymptotic states (fixed points) of Liouvillian dynamics. A first solution in this direction has been presented in a matrix-product formulation for a non-interacting spin chain with non-trivial bulk dephasing noise [8]. Ultimately, Prosen devised an exact ansatz for the axially anisotropic Heisenberg spin chain in one spatial dimension driven away from the equilibrium regime via two boundary incoherent jump processes acting on the leftmost and rightmost particles only [9]–[11]. The steady state density operator has been cast in a matrix-product operator (MPO) form via an auxiliary hopping process, reminiscent of previously known matrix-product solutions of asymmetric classical exclusion processes (ASEP) [12]–[14]. Since at first glance the presented solutions appeared rather mystical, predominantly as they were obtained in a crude ad hoc fashion and thus provided only little insight into the structure of the problem, they naturally called for a deeper theoretical understanding. Significant progress in this respect has been made by Karevski et al [15], demonstrating that the auxiliary process admits a symmetry of deformed spin algebra, explaining a peculiar bulk cancellation mechanism which appeared in [9], originally cast in

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terms of homogeneous cubic algebraic relations. In spite of the authors in [15] succeeding in generalizing solutions via the inclusion of ultra-local boundary coherent fields into the Hamiltonian, the central algebraic condition, encapsulated in a suitable local operator-divergence form, has been employed without resorting to any underlying (fundamental) algebraic principles. In particular, no reference to the powerful tools of the algebraic framework of the existing theory of quantum integrability has been made or pointed out. Finally, undeniable evidence that constructed steady state solutions belong to the Yang–Baxter integrability paradigm has been unfolded just recently [16], serving as the prevailing motivation for the origin of this work.

The main goal of the present paper is to elaborate on a direct relation of all presently known steady state solutions of anisotropic spin-1/2 Heisenberg spin chains to the formalism of quantum integrability, i.e. providing a link to solutions of the famous Yang–Baxter equation with associated quantum group structure. The presented approach allows for a unified treatment of the models with integrable bulk interaction pertaining to so-called fundamental integrable models.

The paper is structured as follows. For the sake of completeness, we start in section 2 by briefly introducing basic and essential concepts from QISM. We proceed in section 3 by outlining our setup for studying nonequilibrium states of quantum chains exhibiting boundary dissipation processes, which allow for an elegant description of some non-trivial steady states. In section 4 we present our main result, a general construction of nonequilibrium steady states of Markovian quantum evolution with unitary integrable bulk evolution and ultra-local boundary dissipation—subsequently applying it to simple models. In particular, a complete derivation of the steady state density operator for anisotropic spin-1/2 chains with an unrestricted form of the dissipator is presented in a compact algebraic formulation. We, moreover, argue how the only compatible dissipator is the known maximally polarizing incoherent driving. Additionally, we supplement those results with analogous solutions for the isotropic integrable higher-spin chains or, equivalently, multicomponent particle-preserving quantum gases admitting a global SU(N) symmetry, using a restricted set of Lindblad operators. In section 5 we discuss the possibility of analytic evaluation of local physical observables in terms of vertex operators imposed on the auxiliary spaces, establishing a relation between the particle current density and the ratio of nonequilibrium partition functions pertaining to systems whose sizes differ by one particle. We conclude by stressing some intriguing connections to recently discovered pseudo-local conserved operators and discuss some possible further improvements.

2. Brief overview: quantum inverse scattering method

With the aim of making a rather abstract forthcoming discussion comprehensive enough, we begin by introducing the main objects and concepts of QISM. Their relationship to the central algebraic structure of quantum groups is elaborated in appendix A.

The principal goal of QISM is to establish a sufficient condition that a (quantum) system has to be compliant with in order to possess a countably infinite number of local integrals of motion (conserved charges). Imposing periodic boundary conditions
for a system consisting of \( n \) copies of local \( N \)-level quantum spaces \( \mathfrak{h} \cong \mathbb{C}^N \), i.e. \( \mathfrak{h}_s = \mathfrak{h}_1 \otimes \cdots \otimes \mathfrak{h}_n = \mathfrak{h}^\otimes n \), a set of mutually commuting local operators—Hamiltonians \( \{ H_m \} \) from \( \mathfrak{h}_s \)—arise from analytic series expansion of a parameter-dependent quantum transfer operator \( \tau(\lambda) \), at the so-called shift point (or regular point) \( \lambda_0 \), as a consequence of its involutive property

\[
[\tau(\lambda), \tau(\mu)] = 0, \quad \lambda, \mu \in \mathbb{C}.
\]  

(2.1)

Explicitly, a desired set of local conserved charges, with \( H_2 \) being usually interpreted as a Hamiltonian, is recovered upon taking logarithmic derivatives of \( \tau(\lambda) \) with respect to the complex spectral parameter \( \lambda \) around \( \lambda_0 \):

\[
H_m \sim \left( \frac{d}{d\lambda} \right)^m \ln \tau(\lambda) \bigg|_{\lambda_0}.
\]  

(2.2)

The transfer operator \( \tau(\lambda) \) is canonically defined by means of a ‘larger’ operator—the monodromy matrix \( T_{\lambda}(\lambda) \) operating in \( \mathfrak{h}_a \) (being a Hilbert space associated with auxiliary degrees of freedom) with elements from \( \mathfrak{h}_s \) as a partial trace over auxiliary space \( \mathfrak{h}_s \), \( \tau(\lambda) = \text{Tr}_a T_{\lambda}(\lambda) \). Throughout the paper we shall write boldface symbols whenever objects operate non-trivially on both the auxiliary space \( \mathfrak{h}_a \) and the spin space \( \mathfrak{h}_s \).

The key property of monodromy matrices is the existence of a similarity transformation \( R(\lambda, \mu) \) which intertwines a product of two monodromy matrices \( T(\lambda)T(\mu) \) for different values of spectral parameters via the RTT equation,

\[
R_{12}(\lambda, \mu)T_{1}(\lambda)T_{2}(\mu) = T_{2}(\mu)T_{1}(\lambda)R_{12}(\lambda, \mu).
\]  

(2.3)

The latter can be thought of as the definition of an abstract homogeneous associative quadratic algebra composed of non-commuting elements \( \{ T_j(\lambda) \} \), with the \( R \)-matrix prescribing the algebra’s ‘structure constants’. In equation (2.3) we employed a standard index notation where subscript indices refer to auxiliary spaces in which \( R \) and \( T \) matrices operate non-trivially, namely \( R_{12}(\lambda, \mu) \) is a linear map in \( \mathfrak{h}_1 \otimes \mathfrak{h}_2 \) (thus its elements are scalars with respect to \( \mathfrak{h}_s \)) and \( T_j(\lambda) \) operates non-trivially in \( \mathfrak{h}_j \otimes \mathfrak{h}_s \). The intertwining \( R \)-matrix must satisfy an additional compatibility-type condition,

\[
R_{12}(\lambda, \mu)R_{13}(\lambda, \eta)R_{23}(\mu, \eta) = R_{23}(\mu, \eta)R_{13}(\lambda, \eta)R_{12}(\lambda, \mu),
\]  

(2.4)

known as the (quantum) Yang–Baxter equation (YBE), in order to ensure the associativity of the RTT algebra (2.3). Notice how, in contradistinction to (2.3), the condition (2.4) extends over the three-fold product of auxiliary spaces \( \mathfrak{h}_1 \otimes \mathfrak{h}_2 \otimes \mathfrak{h}_3 \), with \( R \)-matrices (with elements in \( \mathbb{C} \)) equipped with subscript indices pertaining to spaces where they operate non-trivially, i.e. non-identically.

The YBE can be regarded as a generalization of the permutation group. Introducing a permutation map \( \Pi \) of two tensor factors,

\[
\Pi(a \otimes b) \equiv P(a \otimes b)P = b \otimes a,
\]  

(2.5)

with \( a \otimes b \) from \( \mathfrak{h}_1 \otimes \mathfrak{h}_2 \), we essentially obtain a constant solution \( P_{12}P_{13}P_{23} = P_{23}P_{13}P_{12} \). Equivalently, one can define a braid \( R \)-matrix, \( \check{R}(\lambda, \mu) := PR(\lambda, \mu) \), which fulfils a braid-associativity condition,

\[
\check{R}_{12}(\lambda, \mu)\check{R}_{23}(\mu, \eta)\check{R}_{12}(\lambda, \mu) = \check{R}_{23}(\mu, \eta)\check{R}_{12}(\lambda, \mu)\check{R}_{23}(\mu, \eta).
\]  

(2.6)
The RTT equation (2.3) is, in fact, due to the locality principle, i.e. factorization of the space $\mathcal{H}_s$ on local physical spaces $\mathcal{H}_j$, implied by the existence of a local unit, denoted by an operator $L_j(\lambda) \equiv L_{a,j}(\lambda)$—usually referred to as quantum Lax operator—by means of a spatially ordered product of the form
\[ T_a(\lambda) = L_1(\lambda)L_2(\lambda)\cdots L_n(\lambda) = \prod_{j=1}^{n} L_j(\lambda). \] (2.7)

Here all the operators $L_j(\lambda)$ are over an $n$-fold product physical space $\mathcal{H}_s$, with matrix elements from $\mathcal{H}_a$, and index $j$ designates local physical spaces $\mathcal{H}_j$ where $L_j$ operate non-trivially. The local Yang–Baxter equation, simply referred to as the RLL relation,
\[ R_{12}(\lambda,\mu)L_{1,j}(\lambda)L_{2,j}(\mu) = L_{2,j}(\mu)L_{1,j}(\lambda)R_{12}(\lambda,\mu) \] (2.8)
then implies (2.3). With the help of the latter property, it is easy to establish the commutativity of the transfer matrices by tracing over spaces $\mathcal{H}_1, \mathcal{H}_2$,
\[ \tau(\lambda)\tau(\mu) = \text{Tr}_{12}(T_1(\lambda)T_2(\mu)) = \text{Tr}_{12}(\tilde{R}_{12}(\lambda,\mu)T_1(\lambda)T_2(\mu)\tilde{R}_{12}(\lambda,\mu)^{-1}) = \text{Tr}_{12}(T_1(\mu)T_2(\lambda)) = \tau(\mu)\tau(\lambda). \] (2.9)

Constructing an integrable system eventually boils down to finding solutions of the fundamental commutation relation (2.3). The most important class of solutions of the RTT equations actually arise as particular continuous deformations of Lie algebraic structures and define a quadratic associative algebras referred to as quantum groups.

In order to keep the debate as little technical as possible and preserve the coherence of the presentation, we shall nonetheless omit the details of such a construction at this point and rather refer the reader to appendix A, where we state the definition and essential characteristics of a quantum group and provide an explicit example for the $q$-deformation of the $\mathfrak{sl}(2, \mathbb{C})$ Lie algebra as well. We should emphasize, however, that the reference to the Faddeev–Reshetikhin–Takhtajan (FRT) definition of a quantum group is crucial for our construction, since it offers an adequate form of the Lax matrix which needs to be utilized in order to satisfy the required algebraic condition for a solution of our nonequilibrium problem (defined in section 3).

3. Lindblad master equation with boundary dissipation

Let us now switch gears and discuss the seemingly unrelated problem of dissipative evolution of a many-body open quantum system. Note that the word ‘open’ is here reserved for the non-unitary quantum evolution instead of a type of boundary condition in accordance to standard terminology of integrable systems. In any case, the particular setup we shall study here naturally lacks cyclic invariance since it models a chain of quantum particles coupled to macroscopic baths at its ends.

We begin by introducing a setup of a boundary-driven Markovian open quantum system and present recent findings of steady states for the Heisenberg XXZ spin-1/2 chain, to set the stage for the general construction of steady states by means of the integrability structures presented in section 2.
3.1. Steady state solution of Markovian quantum master equation

Our aim is to address the idea of the analytic construction of exact steady states for nonequilibrium quantum many-body evolution for some paradigmatic one-dimensional models of strongly interacting particles. By facilitating a master equation description, i.e. using the most general completely positive trace-preserving dynamical map [17] for the system’s density matrix with a time-independent generator, and focusing solely on the steady state density operators, we look for fixed-point solutions of the Markovian evolution,

$$\rho(t) = \hat{V}(t)\rho(0), \quad \hat{V}(t) = e^{\hat{L}t}. \quad (3.1)$$

The flow is generated by the Liouville operator $\hat{L}$ of the Lindblad form,

$$\frac{d}{dt}\rho(t) = \hat{L}\rho(t), \quad \hat{L}(\rho) = -i \hat{a}d_H(\rho) + \hat{D}(\rho). \quad (3.2)$$

We introduced linear super-operator maps

$$\hat{a}d_H(\rho) := [H, \rho], \quad \hat{D}(\rho) := \sum_{\mu} A_{\mu}\rho A_{\mu}^\dagger - \frac{1}{2}\{A_{\mu}^\dagger A_{\mu}, \rho\}, \quad (3.3)$$

where $\{A, B\} := AB + BA$ denotes the anti-commutator. Operators from the set $\{A_{\mu}\}$, governing the dissipative processes given by linear operator $\hat{D}$, are called the Lindblad operators and fully describe the incoherent (non-unitary) part of quantum evolution. In our setup, their role will mainly be to establish the gradient of a ‘chemical potential’, i.e. to introduce forces inducing current-carrying states, thus allowing one to study genuine far-from-equilibrium situations. At this point we would like to emphasize that using Lindblad operators to model a sort of realistic reservoir or thermal bath is outside the scope in our setting, as essentially we care only about providing simple enough conditions for an analytic treatment to drive a system out of equilibrium. Consequently, one has to understand that the steady state under discussion will be generically far from the linear-response regime. To this end, we adopt a setup where dissipative processes affect only the first and the last particle in a chain, namely we take the dissipator to be of the form

$$\hat{D}(\rho) = \hat{D}_L(\rho) + \hat{D}_R(\rho), \quad (3.4)$$

with the dissipation super-operator $\hat{D}_{L,R}$ operating non-trivially only in the first/last local quantum space, $h_1$ and $h_n$, respectively. The nonequilibrium steady state density operator (NESS) is therefore a fixed-point solution $\rho_\infty$ of (3.2), defined by $\hat{L}(\rho_\infty) = 0$, or equivalently

$$i \hat{a}d_H(\rho_\infty) = \hat{D}_L(\rho_\infty) + \hat{D}_R(\rho_\infty). \quad (3.5)$$

3.2. Exact solution for a boundary-driven anisotropic Heisenberg spin-1/2 chain

In this section we briefly discuss the steady state for the anisotropic Heisenberg spin-1/2 chain, found previously in [9, 10]. The solution to (3.5) with a global Hamiltonian $H$ operating over $n$-body Hilbert space $\mathcal{H}_n \cong (\mathbb{C}^2)^\otimes n$

$$H^{XXZ} = \sum_{j=1}^{n-1} h_j^{XXZ}, \quad h_j^{XXZ} := 2\sigma_j^+\sigma_{j+1}^{-} + 2\sigma_j^-\sigma_{j+1}^{+} + \Delta\sigma_j^z\sigma_{j+1}^z. \quad (3.6)$$

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where $\Delta \equiv \cos (\gamma) \in [-1, 1]$ defines the anisotropy parameter, and maximally polarizing left/right channels\(^3\), given by

$$A_1 = \sqrt{\Gamma} \sigma_1^+, \quad A_n = \sqrt{\Gamma} \sigma_n^-,$$

with coupling strength parameter $\Gamma \in \mathbb{R}$, can be cast as a Cholesky-type factorized (non-normalized) density matrix $\rho_\infty = SS^\dagger$. The ‘Cholesky’ factor $S$ (also called simply the $S$-operator), an element of $\mathcal{H}_s$, is defined in a matrix-product operator (MPO) form expanded in the standard basis in $\mathcal{H}_s$ using binary vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ for $\alpha_j, \beta_j \in \{0, 1\}$,

$$S = \sum_{\alpha, \beta} \langle l | L^{\alpha_1 \beta_1} L^{\alpha_2 \beta_2} \cdots L^{\alpha_n \beta_n} | r \rangle \prod_{j=1}^n E^{\alpha_j \beta_j},$$

with a set of auxiliary matrices $\{L^{ij}\}$ and a pair of boundary vectors $|l\rangle, |r\rangle$. Throughout this paper we interchangeably use two complete basis sets of linear operators on a local qubit space $\mathcal{H} \cong \mathbb{C}^2$: the canonical Pauli matrices

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

along with $\sigma^0 = 1_2$, and the standard unit matrices (Weyl basis) $\{E^{ij}\}$, for $i, j = \{0, 1\}$, satisfying $E^{ij} E^{kl} = E^{il} \delta_{jk}$ and $\mathfrak{gl}(n, \mathbb{C})$ (ultra-local) commutation relations

$$[E^{ij}_m, E^{kl}_p] = (E^{ii}_m \delta_{jk} - E^{kj}_m \delta_{il}) \delta_{mp}.$$  

(3.10)

The strategy to find a steady state density matrix is based on a compact formulation via homogeneous MPO of the form (3.8), by means of a spatially ordered product of local matrices $L_j(\Gamma)$,

$$S(\Gamma) = \langle l | L_1(\Gamma) L_2(\Gamma) \cdots L_n(\Gamma) | r \rangle = \langle l | \prod_{j=1}^n L_j(\Gamma) | r \rangle.$$  

(3.11)

The elements of $L_j$ are non-commuting operators (matrices) acting on an auxiliary space,

$$L_j(\Gamma) = \sum_{k,l=1}^2 E^{kl}_j \otimes L^{lk}(\Gamma) = \begin{pmatrix} L^{11}(\Gamma) & L^{21}(\Gamma) \\ L^{12}(\Gamma) & L^{22}(\Gamma) \end{pmatrix}_j.$$  

(3.12)

An ansatz of this type has been proposed in [15] and is in fact inspired directly from the analogy to known matrix-product realizations of partition functions for classical asymmetric exclusion processes [12, 14, 18], lifted to quantum spaces (see also [19]).

Notice that the structure of (3.5), with dissipators acting ultra-locally only on the boundary sites, imposes a condition on the action of the adjoint global Hamiltonian,

$$i \hat{\text{ad}}_H(S) = \sum_{k=(0,\pm,z)} \left( w^{k}_{L} \sigma^{k}_{1} W^{k}_{R} + w^{k}_{R} W^{k}_{L} \sigma^{k}_{n} \right).$$

(3.13)

\(^3\) Here the notion of ‘maximal driving’ refers to a situation where, at a particular boundary, only one type of incoherent quantum jump may occur, i.e. either upward flips $\sigma^+$ or downward flips $\sigma^-$. If in addition the rates of the two processes are equal, we speak of symmetric driving.

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with $W^k_R, W^k_L$ operating in $\mathfrak{h}_2 \otimes \cdots \otimes \mathfrak{h}_n$ and $\mathfrak{h}_1 \otimes \cdots \otimes \mathfrak{h}_{n-1}$, respectively. Operators from (3.13) are essentially MPOs with the same local unit $L$, only contracted differently. In other words, $ad_L$ has to non-trivially modify the density operator only in the boundary physical spaces $\mathfrak{h}_1$ and $\mathfrak{h}_n$. Consequently, a sufficient requirement to ensure this property at the level of MPO is to satisfy the following form of the local operator-divergence condition with respect to the interaction density $h^{XXZ}_{j,j+1}$,

$$[h^{XXZ}_{j,j+1}, L_j L_{j+1}] = B_j L_{j+1} - L_j B_{j+1}. \quad (3.14)$$

After multiplying this expression by $\prod_{m=1}^{j-1} L_m$ from the left and by $\prod_{m=j+2}^{n} L_m$ from the right, and contracting with respect to given left and right auxiliary boundary vectors, one arrives at the equation formally equivalent to (3.13).

An explicit form of the local matrix $L$ and the boundary auxiliary matrix $B$ can be found in [15], implemented by a set of $\mathcal{U}_q(\mathfrak{so}(2, \mathbb{C}))$ generators acting in an infinitely dimensional auxiliary Hilbert space $\mathcal{H}_a$. Nonetheless, no insight has been presented on why matrices with deformed symmetry are of central interest in such a setup and how solutions of (3.14) emerge from more fundamental (algebraic) principles. The purpose of the forthcoming discussion is therefore to make use of the presented formalism of quantum groups to (i) generate solutions of (3.14) pertaining to fundamental integrable models and derive a canonical representation of solutions with deformed symmetries, and (ii) to formulate and solve a system of compatibility (boundary) equations for a class of models with $SU(N)$-invariant interaction.

4. Exact matrix-product form solutions for fundamental spin chains

Although there originally existed no transparent interrelation between the presented NESS solutions and the existing theory of quantum integrability, the algebraic formulation via objects with deformed continuous symmetry in [15] has given a first hint in this direction. Shortly after, a remarkable transfer-operator property has been revealed [16] upon noticing that the ‘ground state’ contraction of the monodromy operator, $S(\Gamma) = T_0(\Gamma)$, defining the solution of the maximally driven anisotropic Heisenberg chain with the dissipation coupling strength parameter $\Gamma$, forms a commuting family,

$$[S(\Gamma_1), S(\Gamma_2)] = 0, \quad \Gamma_{1,2} \in \mathbb{C}. \quad (4.1)$$

This finding, which called for the existence of the underlying Yang–Baxter structure, has also been explained in reference [16] by means of an infinite-dimensional $R$-matrix.

That being said, the purpose of this section is to unveil the connection between the QISM and the steady state solutions for boundary-driven dissipative one-dimensional models within the framework introduced in section 3, and supplement the presentation with a simple rigorous construction of NESS for XXZ $s = 1/2$ Heisenberg chains by using elementary ingredients of the FRT construction (see the appendix A) and the QISM (see the summary in section 2). We furthermore elaborate on the proposed Cholesky-type factorization of the density matrix and the role of infinite-dimensional representations (Verma modules) for the Lax matrix and their role in the construction of the steady states. Finally, the procedure is presented on some simple examples of multicomponent quantum models exhibiting full rotational symmetry of higher-rank spin algebras.
4.1. Sutherland equation: local divergence condition

First, let us demonstrate how the solutions of (3.14) emerge from an infinitesimal form of the fundamental solutions of the YBE (2.4). Essentially, the equation (3.14) is nothing but Sutherland’s local condition, which is sufficient for establishing commutativity between the Hamiltonian density and a transfer matrix obtained as a partial trace over the monodromy matrix for a given Lax matrix, assuming periodic boundary conditions [20]. The same fact has been elaborated on in Sklyanin’s lecture notes [21].

Our starting point is a parametrized local Yang–Baxter equation—the so-called RLL relation—of the difference form, defined in a two-fold product of auxiliary spaces $H_1 \otimes H_2$,

$$R_{12}(\lambda - \mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda - \mu),$$

(4.2)

obtained by Baxterization [22] of the quantum group defining relations (see (A.6) in appendix A) with the spectral parameter $\lambda \in \mathbb{C}$, via variable $x = q^{-1} = e^{\gamma \lambda}$,

$$R^q_{12}(\lambda) = xR^q_{12} - x^{-1}R_{12}^{-1},$$

$$L^q_j(\lambda) = xL^q_j - x^{-1}L^{-1}_j, \quad j = 1, 2.$$  

(4.3)

Note that choosing $\lambda = \mu$ (or equivalently $x = 1$) in the equation (4.2) defines a shift point, where the $R$-matrix becomes proportional to the permutation operator in $H_1 \otimes H_2$,

$$R^q_{12}(0) = (q - q^{-1})P_{12},$$

(4.4)

implying $\tilde{R}^q_{12}(0) = (q - q^{-1})I_{12}$. Taking the derivative of (4.2) with respect to $\lambda$ at $\lambda = \mu$, and applying the permutation operator from the left therefore results in the infinitesimal RLL equation (or the differential local Yang–Baxter equation)

$$[\partial_\lambda \tilde{R}^q_{12}(\lambda)|_{\lambda=0}, L_1(\lambda)L_2(\lambda)] = -\tilde{R}^q_{12}(0)(\partial_\lambda L_1(\lambda))L_2(\lambda) + L_1(\lambda)(\partial_\lambda L_2(\lambda))\tilde{R}^q_{12}(0),$$

(4.5)

which is after using (4.4) equivalent (up to trivial rescaling of operators $\tilde{R}^q_{12}(\lambda)$ and $L_j(\lambda)$) to the Sutherland equation

$$[h_{12}, L_1(\lambda)L_2(\lambda)] = B_1(\lambda)L_2(\lambda) - L_1(\lambda)B_2(\lambda),$$

(4.6)

after interpreting the two-body interaction operator as $h_{12} \sim \partial_\lambda \tilde{R}_{12}(\lambda)|_{\lambda=0}$ and identifying $B_j(\lambda) \sim \partial_\lambda L_j(\lambda)$. The precise derivation follows in the upcoming presentation.

Finally, a comparison with the previously stated form of the Sutherland equation (3.14) indicates that we need to formally identify auxiliary spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ with local physical spaces, $h_j$ and $h_{j+1}$, respectively, whereas matrix elements of $L$-operators, i.e. the spin-algebra generators now operate in the ‘third space’, which we interpret as the auxiliary space $\mathcal{S}_a$. Thus, we can identify local auxiliary matrices $L$ in the ansatz (3.8) with Lax operators of the corresponding integrable bulk Hamiltonian, automatically satisfying the local divergence condition (3.14).

As the next step we shall construct an ansatz for the NESS by using the bulk cancellation properties for the MPO $S$. 

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4.2. Cholesky-factorized form and boundary equations

As we have learned, an ansatz for the NESS density matrix can be formulated as Cholesky-type\(^4\) decomposition with an MPO factor \(S\). Despite being an at least intuitively meaningful candidate for the density operator as it represents a manifestly positive operator, it seems difficult to justify such an ansatz in a rigorous way\(^5\). One could for instance propose a simpler ansatz, where the NESS operator \(\rho_\infty\) is sought as a linear expression in MPOs, i.e. as \(\rho_\infty \sim S \pm S^\dagger\) with an \(S\)-operator of the form (3.11). Nevertheless, for spin-1/2 chains, no solutions exist in this case, as becomes evident after realizing that the algebraic condition (3.14) is insufficient to provide traceless operators in boundary local spaces, after partially tracing out the auxiliary space \(\mathcal{H}_a\). Accordingly, at this stage we shall simply restrict ourselves to the Cholesky form of solutions \(\rho_\infty = SS^\dagger\), which we argue encodes a minimal ansatz within a single-MPO description for the NESS density operator. More precisely, as the operator \(SS^\dagger\) can indeed be represented alternatively as contraction of a ‘doubled’ monodromy operator operating in \(\mathcal{H}_a \otimes \mathcal{H}_a\), with a product auxiliary space \(\mathcal{H}_a = \mathcal{H}_{a_1} \otimes \mathcal{H}_{a_2}\), we assume the auxiliary \(\mathcal{H}_a\) of our Lax operator is irreducible.

Therefore, expanding \(\hat{\text{ad}}_H(\rho_\infty)\) by means of the Leibniz rule we have
\[
\hat{\text{ad}}_H(\rho_\infty) = -iS(\hat{\text{ad}}_H(S))^\dagger + i\hat{\text{ad}}_H(S)S^\dagger = \tilde{\mathcal{D}}_L(SS^\dagger) + \mathcal{D}_R(SS^\dagger). \tag{4.7}
\]
By employing the MPO form (3.11), in conjunction with (4.6), we observe that \(\hat{\text{ad}}_H\) operating on the \(S\)-operator leaves behind a telescoping sum,
\[
\hat{\text{ad}}_H(S) = \langle l|[H, L_1 \cdots L_n]|r\rangle
= \sum_{j=1}^{n-1} \langle l|[L_1 \cdots L_{j-1}[h_{j,j+1}, L_j L_{j+1}]L_{j+2} \cdots L_n]|r\rangle
= \langle l|[B_1 L_2 \cdots L_n]|r\rangle - \langle l|[L_1 \cdots L_{n-1} B_n]|r\rangle =: S^{(L)} - S^{(R)}, \tag{4.8}
\]
resulting solely in the two boundary defect-operators \(S^{(L,R)}\). With aid of this result, we rewrite (4.7) in the form
\[
i(S^{(L)} S^\dagger - SS^{(L)^\dagger}) + i(SS^{(R)^\dagger} - S^{(R)} S^\dagger) = \tilde{\mathcal{D}}_L(SS^\dagger) + \mathcal{D}_R(SS^\dagger). \tag{4.9}
\]
The key idea now is to use locality of the dissipation and solve (4.9) by imposing a stronger condition via identification of two separate boundary equations, written out in fully expanded form as
\[
\langle l \otimes l \left( iB_1^{(1)} - iB_1^{(2)} - \mathcal{H}_L(L_1) \right) \left( \prod_{k=2}^n L_k \right) |r \otimes r\rangle = 0, \tag{4.10}
\]
\[
\langle l \otimes l \left( \prod_{k=1}^{n-1} L_k \right) \left( iB_n^{(1)} - iB_n^{(2)} + \mathcal{H}_R(L_n) \right) |r \otimes r\rangle = 0,
\]
\(^4\) In order to deal with standard Cholesky decomposition, \(S\) has to be in addition a triangular operator in a quantum many-body basis. However, as a consequence of non-unitarity of representations for auxiliary spin and vacuum (lowest weight) contraction, this is always the case for the solutions we discuss in this paper.

\(^5\) It is perhaps noteworthy that solutions of a similar type, however not strictly of Cholesky form, which lack the transfer matrix property, have already been constructed for the anisotropic Heisenberg \(s = 1/2\) model with asymmetric boundary incoherent driving via a model-specific generalized ansatz [11].

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with the shorthand notation $L_j := L_{1,j} L_{2,j}$, $B_j := B_{1,j} B_{2,j}$, where the superscript $t$ denotes the transposition of the local physical space $h_j$ and the overlined symbol denotes complex-conjugated operators. In fact, we may factor out the ‘free parts’ $\prod L_j$, and instead require annihilation of the partially contracted expressions, i.e.

$$
\langle l | \otimes \langle l | \left( i B_1^{(1)} - i B_1^{(2)} - \hat{D}_L(L_1) \right) = 0,
$$

$$(i B_n^{(1)} - i B_n^{(2)} + \hat{D}_R(L_n)) | r \rangle \otimes | r \rangle = 0.
$$

Thus, the task has been reduced to finding a parametrization of a set of auxiliary matrices $L^j_{ik}$ and appropriate auxiliary boundary bra-vector $\langle l|$ and ket-vector $| r \rangle$ which solve (4.11) for a specific set of dissipative channels $\{A_n\}$.

We have to stress at this stage that it is not a priori clear whether the symmetric dissipators considered in [10] exhaust all possibilities which yield a solution of (4.11) for the model discussed in section 3. To this end, one has to incorporate the most general form of a local dissipator over $h_m$, which is of the Lindblad form,

$$
\hat{D}(\rho) = \sum_{i,j} \sum_{k,l} G_{ij}^{kl} (E_{m}^{ij} \rho (E_{m}^{ij})^\dagger - \frac{1}{2} \{(E_{m}^{kl})^\dagger (E_{m}^{ij}), \rho \}),
$$

where $G = G^\dagger$ is a positive rate-matrix, $G \geq 0$.

Evaluating the boundary equations (4.11) with the general form of the left/right dissipation tensors $G_L$ and $G_R$ from (4.12), and projecting the result onto the (Weyl) basis elements in the boundary physical spaces, i.e. $E_1^{ij}$ on the left and $E_n^{ij}$ on the right, results in a system of vector-valued equations for each $i,j = 1,2,\ldots,N$. Finding a solution to the problem which displays $U_q(sl(N,\mathbb{C}))$ symmetry thus boils down, accounting for hermicity of $\rho_\infty$, to simultaneously satisfying $N(N+1)$ (possibly linearly dependent) equations, for the general spin-$s$ Lax operator $\hat{L} = \hat{L}(\lambda,s)$.

### 4.3. Lowest weight representations (Verma modules)

It is now clear that in order to solve the boundary conditions (4.11) we have to be able to freely vary the representation parameters of the Lax and boundary operators, since equations (4.11) may be obeyed only for particular representations from the continuum of solutions to the bulk divergence condition. Therefore, we have to find a solution of the Sutherland equation (4.10) (e.g. by using the FRT construction outlined in appendix A) in terms of abstract objects, in particular $\hat{L}$, which can be evaluated in generic representation spaces. It is crucial, however, not to exclude infinitely dimensional representations from consideration. In other words, in order to guarantee a continuous family of solutions, it is essential to allow for (spin) representations going beyond standard finite-dimensional unitary representations. For this purpose we shall employ Verma modules (see e.g. [23])—those are, for generic set of representation parameters (i.e. weight vector), infinite-dimensional irreducible non-unitary representations, admitting a lowest weight vector—with no associated simply connected compact group. Therefore, since Verma representations describe spaces of our ancillary particles we might, loosely speaking, proclaim the latter as non-compact spins.

\[ \text{doi:10.1088/1742-5468/2014/01/P01001} \]
Example: representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Focusing for the moment on our paradigmatic example, the XXZ spin-1/2 chain exhibiting $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ symmetry, we define a representation space $\mathfrak{S}_s$, spanned by an infinite basis of state vectors $\{v_k\}_{k=0}^\infty$, and labeled by a complex (spin) representation parameter $s \in \mathbb{C}$ designating the lowest weight. A convenient realization of such a module is a space $\mathbb{C}[x]$ of polynomials in variable $x$ with canonical monomial basis $\{v_k = x^k = |k\rangle\}$ and lowest weight state $v_0 = 1$ [24]. Spin generators are given by $q$-differential operators (denoting $\partial = \partial/\partial x$),

$$S_q^z = x\partial - s, \quad S_q^+ = x[2s - x\partial]_q, \quad S_q^- = x^{-1}[x\partial]_q.$$  \hspace{1cm} (4.13)

In the undeformed case $q \to 1$ we recover $\mathfrak{sl}(2,\mathbb{C})$ generators as simple first-order differential operators

$$S^z = x\partial - s, \quad S^+ = -x^2\partial + 2sx, \quad S^- = \partial,$$  \hspace{1cm} (4.14)

and the quadratic central element (Casimir operator) $C = (S^z)^2 - S^+S^-$. The basis vectors of $\mathfrak{S}_s$ can be constructed by means of the generating function, a coherent state vector,

$$e^{\xi S^+}v_0 = \sum_{k=0}^\infty \frac{\xi^k}{k!}(S^+)^kv_0 = \sum_{k=0}^\infty \frac{\xi^k}{k!}(2s)_k v_k,$$  \hspace{1cm} (4.15)

using the falling factorial $(2s)_k = (2s)(2s - 1) \cdots (2s - k + 1)$, which produces an infinite sequence of states for generic $s \neq \ell/2$ ($\ell = \mathbb{Z}_+ \equiv \{0, 1, 2, \ldots\}$), whereas for exceptional cases ($s = \ell/2$) there exists also a highest weight state and a module reduces to finite ($\ell + 1$)-dimensional invariant subspaces $\mathfrak{S}_\ell \subset \mathfrak{S}_s$ spanned by basis elements $\{x^k\}_{k=0}^n$. The latter ones are equivalent to unitary representations of $\mathfrak{su}(2,\mathbb{C})$. For instance, $s = \frac{1}{2}$ corresponds to $\mathfrak{S}_f := \mathfrak{S}_{1/2} \cong \mathbb{C}^2$, being the fundamental spin representation with associated standard (Pauli) generators $S^+ = \sigma^+, S^- = \sigma^-$ and $S^z = \frac{1}{2}\sigma^z$. Another type of finite-dimensional representation, with no classical correspondence, occurs after algebra quantization for $q$, being a root of unity $q = e^{\gamma i}, \gamma = \pi l/m \ (l, m \in \mathbb{N})$, referred to as cyclic representations\(^6\).

The dual vector space (bra-vectors) is defined via basis $\{|l\rangle\}$, such that the bi-orthogonality relation $\langle l|k \rangle = \delta_{l,k}$ holds.

Product representations. An $R$-matrix associated with the algebra of $\{T_j^i(s)\}$ operates in a product space $\mathfrak{S}_{s_1} \otimes \mathfrak{S}_{s_2}$. A generic product representation (with spin parameters $s_1, s_2 \not\in \frac{1}{2}\mathbb{Z}_+$) is a space of polynomials in two variables $\mathbb{C}[x, y]$ and is simply decomposable as a direct sum of infinite-dimensional lowest weight modules,

$$\mathfrak{S}_{s_1} \otimes \mathfrak{S}_{s_2} = \bigoplus_{\nu=0}^\infty \mathfrak{S}_{s_1+s_2-\nu},$$  \hspace{1cm} (4.16)

labeled by the index $\nu \in \mathbb{Z}_+$. For spin parameters being both positive half-integers, $s_1, s_2 \in \frac{1}{2}\mathbb{Z}_+$, we have a finite number of factors in decomposition (4.16), labeled as $\nu = 0, 1, \ldots, 2\min(s_1, s_2)$. The lowest weight vectors $v^{(\nu)}_\nu(x, y) = (x - y)^\nu$ are destroyed upon action

\(^6\) In cases when the representation becomes reducible we implicitly consider restriction to an invariant subspace only.

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of the global lowering generator $S^-(s_1, s_2) = S_1^-(s_1) + S_2^-(s_2)$,

$$S^-(s_1, s_2)\psi^0_\nu(x, y) = 0,$$

$$S^+(s_1, s_2)\psi^0_\nu(x, y) = (s_1 + s_2 - \nu)\psi^0_\nu(x, y),$$  \hspace{1cm} (4.17)

whereas the infinite tower of states $\{\psi^m_\nu(x, y)\}$ spanning $\mathfrak{S}_{s_1+s_2-\nu}$ is given by repeated action of the raising operator $S^+(s_1, s_2) = S_1^+(s_1) + S_2^+(s_2)$,

$$\psi^m_\nu(x, y) = (S^+(s_1, s_2))^m\psi^0_\nu(x, y).$$  \hspace{1cm} (4.18)

$\mathcal{U}_q(\mathfrak{sl}(N, \mathbb{C}))$ representations. For higher-dimensional quantum spaces we approach à-la Dobrev [25] and, similarly as in the $N = 2$ case, consider realization of algebra generators in terms of linear differential operators and their quantized counterparts—now operating in a space of polynomials in $N(N-1)/2$ commuting variables $x_i^k$, $1 \leq i \leq k-1, 2 \leq k \leq N$—with associated number operators $N_i^k$, defined such that $N_i^k x_j^k = \delta_{ij} \delta_{kl} x_j^l$, and corresponding $q$-differential operators $D_i^k = (x_i^k)^{-1}[N_i^k]_q$. We shall work with the generic $N$-dimensional representation parameter (weight) vector $\mathbf{r} = (r_0, r_1, \ldots, r_{N-1})$, $r_k \in \mathbb{C}$, chosen with the convention that a representation is irreducible iff all $r_i$ are non-negative integers. In this picture, a value of the central element $\lambda$ w.r.t. $\mathcal{U}_q(\mathfrak{sl}(N, \mathbb{C}))$ pertains to the combination $\lambda = \sum_{i=0}^{N-1} (N - i)r_i$. Further details on the recursive construction of $\mathcal{U}_q(\mathfrak{sl}(N, \mathbb{C}))$ representations, compliant with the parametrization we use whenever $N \geq 3$, can be found in [25].

For instance, in the simplest $N = 2$ case, which we shall explicitly treat in a moment, we deal with a space of polynomials in one variable $x \equiv x_1^1$, a single spin representation parameter $r_1 = 2s$ and a spectral parameter which reads $\lambda = 2r_0 + r_1$. In the undeformed $q \rightarrow 1$ limit, the number operator becomes $N_x \equiv N_x^2 = x \partial_x$ and $D_x \equiv D_x^2$ reduces to an ordinary differential operator, $D_x = \partial_x$. For further details and canonical construction of $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{C}))$ Verma modules we refer the reader to [25].

In the following we shall use the approach outlined above to find steady states of models with $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ deformed symmetry for spin-1/2 Hamiltonians, as well as $SU(N)$-symmetric hopping Hamiltonians with local physical space of dimension $N$.

4.4. Solution with quantized rotational symmetry

We begin by writing down the Baxterized $R$-matrix, defined as in (4.3), only rescaled by $\gamma^{-1}$,

$$R_{12}^q(\lambda) = \gamma^{-1}(q - q^{-1}) \begin{pmatrix} \left[ -i\lambda + 1 \right]_q & \left[ -i\lambda \right]_q & e^{-\gamma \lambda} & \left[ i\lambda \right]_q \\ e^{\gamma \lambda} & \left[ -i\lambda \right]_q & \left[ -i\lambda + 1 \right]_q \end{pmatrix}. \hspace{1cm} (4.19)$$

After taking the derivative with respect to $\lambda$ at $\lambda = 0$, and subsequently left-multiplying by $P_{12}$, we readily extract the interaction
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\[ h_{12} := \partial_\lambda \hat{R}_{12}^\lambda |_{\lambda = 0} \]

\[ = \begin{pmatrix}
    q + q^{-1} & -(q - q^{-1}) & 2 \\
    -(q - q^{-1}) & 2 & q - q^{-1} \\
    2 & q - q^{-1} & q + q^{-1}
\end{pmatrix}. \tag{4.20} \]

Throughout our discussion we strictly restrict ourselves to \( q \)-parameter from the unit circle,

\[ q = e^{i\gamma}, \quad \gamma \in \mathbb{R}, \tag{4.21} \]

producing XXZ interaction in the massless regime \( |\Delta| < 1 \). With this definition, employing the spectral parameter is prescribed by \( \gamma \), i.e. \( 2 \cos (\gamma) = q + q^{-1} \) and \( 2i \sin (\gamma) = q - q^{-1} \), we recast this result in terms of the Pauli spin variables, and obtain the 2-body \( \mathcal{U}_q(\mathfrak{su}(2)) \)-covariant interaction

\[ h_{12} = 2 \sigma_1^+ \sigma_2^- + 2 \sigma_1^- \sigma_2^+ + \cos \gamma (\sigma_1^0 \sigma_2^0 + \sigma_1^0 \sigma_2^0) - i \sin \gamma (\sigma_1^0 \sigma_2^0 - \sigma_1^0 \sigma_2^0). \tag{4.22} \]

Plugging the interaction (4.22) into the differential YBE (4.5), using the regularity \( \hat{R}_{12}(0) = c(\gamma) \mathbb{I}_{12} \), \( c(\gamma) = 2i \gamma - 1 \sin (\gamma) \), we finally obtain the Sutherland equation (4.6), with the Lax operator

\[ L_j(\lambda) = \begin{pmatrix}
    [(-i\lambda + S^z)]_q & e^{\gamma \lambda} S^- \\
    e^{-\gamma \lambda} S^+ & [-i\lambda - S^z]_q
\end{pmatrix} \tag{4.23} \]

and the boundary operator \( B(\lambda) \), which admits the form

\[ B_j(\lambda) := -c(0) \partial_\lambda L_j(\lambda) = -2i \gamma \sin (\gamma) \partial_\lambda L_j(\lambda) \\
= -2 \begin{pmatrix}
    \cos \gamma (-i\lambda + S^z) & i \sin \gamma e^{\gamma \lambda} S^- \\
    -i \sin \gamma e^{-\gamma \lambda} S^+ & \cos \gamma (-i\lambda - S^z)
\end{pmatrix}. \tag{4.24} \]

Beware that the spin algebra generators in the matrix element of the operators (4.23) and (4.24) also depend on the deformation parameter \( q \). Moreover, in the \( q \to 1 \) (\( \gamma \to 0 \)) undeformed limit we recover the \( \mathfrak{sl}_2 \)-covariant Lax operator (writing \( u := -i\lambda \))

\[ \lim_{\gamma \to 0} L_j(\lambda) = \begin{pmatrix}
    u + S^z & S^- \\
    S^+ & u - S^z
\end{pmatrix} = u \cdot \sigma_j^0 \otimes 1 + \bar{\sigma}_j \otimes \bar{S}, \tag{4.25} \]

compliant with the \( SU(2) \)-invariant interaction, \( \lim_{\gamma \to 0} h_{12} = 2 P_{12} \).

It is important to stress that the interaction \( h_{j,j+1} \) is not Hermitian, therefore not of the right form for the application we have in mind. Nevertheless, we show how the problem can easily be circumvented at the level of the Sutherland equation, resting on the fact that the hermicity is violated only by an anti-Hermitian surface term, namely the anisotropic Heisenberg interaction can be expressed as

\[ h_{12}^{\text{XXZ}} = h_{12} - i \sin (\gamma) (\sigma_1^0 \sigma_2^0 - \sigma_1^0 \sigma_2^0). \tag{4.26} \]
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Consequently, we expand the commutator on the right side of (4.6),

$$[h_{12}, L_1 L_2] = [h_{12}^{XXZ}, L_1 L_2] + i ([b_1, L_1] L_2 - L_1 [b_2, L_2]),$$

(4.27)

by means of a local operator $b_j : = -\sin(\gamma)\sigma^z_j$, and absorb the surface term $B_j^\theta(\lambda) : = [b_j, L_j(\lambda)]$ into redefinition of the boundary operator,

$$B_j^{XXZ}(\lambda) : = B_j(\lambda) + B_j^\theta(\lambda)
= -2 \begin{pmatrix} \cos(\gamma(-i\lambda + S^z)) & 0 \\ 0 & \cos(\gamma(-i\lambda - S^z)) \end{pmatrix}_j.$$

(4.28)

The $L$-operator remains in principle intact in this case, however, for convenience we may remove the exponential scalar factors $e^{\pm\gamma\lambda}$ from the off-diagonal elements as they are merely expressing $\lambda$-dependent spin-algebra automorphism, i.e. we employ the Lax operator in the form of

$$L_j^{XXZ}(\lambda) : = \begin{pmatrix} [-i\lambda + S^z]_q & S^- \\ S^+ & [-i\lambda - S^z]_q \end{pmatrix}_j.$$

(4.29)

An alternative (more formal, but equivalent) procedure to produce Hermitian interaction involves twisting the Hopf algebra objects, i.e. generating a new pair of $R$- and $L$-operators satisfying the local Yang–Baxter equation (4.2), which is briefly outlined in appendix B.

For the sake of clarity, we shall consistently omit the XXZ superscript on the operators $L_j^{XXZ}(\lambda), B_j^{XXZ}(\lambda)$ in the subsequent discussion. We hope this choice will not lead to confusion with $U_q(\mathfrak{sl}(2, \mathbb{C}))$-symmetric objects associated with the non-Hermitian interaction $h_{12}$ from (4.22). Thus, we proceed by evaluating the non-vanishing action of the Lax matrix $L$ on the boundary vacuum states. We have,

$$L_{11}^{11}(0) = [-i\lambda - s]_q|0\rangle, \quad \langle 0|L_{11}^{11} = [-i\lambda - s]_q\langle 0|,
L_{11}^{22}(0) = [-i\lambda + s]_q|0\rangle, \quad \langle 0|L_{11}^{22} = [-i\lambda + s]_q\langle 0|,
L_{21}^{11}(0) = 0, \quad \langle 0|L_{21}^{11} = \langle 1|,
L_{12}^{11}(0) = [s]_q|1\rangle, \quad \langle 0|L_{12}^{11} = 0,$$

(4.30)

whereas the boundary matrix $B$ gives

$$B_{11}^{11}|0\rangle = -2 \cos(\gamma(-i\lambda - s))|0\rangle,
B_{11}^{22}|0\rangle = -2 \cos(\gamma(-i\lambda + s))|0\rangle,$$

(4.31)

operating equally on the bra-vector $|0\rangle$.

In the case of the adopted (i.e. fixed) boundary vectors ($|r\rangle = |0\rangle$ and $|l\rangle = \langle 0|$) facilitated in (4.30), (4.31) the incoherent maximal driving, implemented via left channel $A_1 = \sqrt{\Gamma}\sigma^+_1$ and right channel $A_n = \sqrt{\Gamma}\sigma^-_n$, determines the only dissipator $\hat{D}$ yielding a solution, given by the spin parameter $s$ related to the coupling via

$$\Gamma = 4 \sin(\gamma) \coth(\gamma \text{Im}(s)) = 4i\frac{\cos(\gamma \text{Im}(s))}{[i\text{Im}(s)]_q},$$

(4.32)

and $\text{Re}(s) = 0$, in accordance with the solutions [10, 15]. The isotropic solution takes the simpler form of $s = 4i\Gamma^{-1}$. 

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4.5. Integrable chains with \( SU(N) \) global symmetry

The presented formalism can be immediately facilitated to address certain isotropic quantum integrable models and their \( q \)-deformed counterparts. Let us initially treat the isotropic chains exhibiting global \( SU(N) \) symmetry (i.e. higher-spin generalization of the \( S = 1 \) Lai–Sutherland model), \( N = 2s + 1 \) \( (N = 2, 3, \ldots) \), with the local quantum (spin) space \( \mathfrak{h}_j \cong \mathfrak{S}_s \cong \mathbb{C}^N \). In this case we prefer to avoid the use of spin variables \( S^k_j \) \( (k = x, y, z) \) over \( \mathfrak{h}_j \), since the interaction can be expressed via permutation operators \( \mathcal{P}^{(N)}_{j,j+1} \) over two adjacent quantum spaces \( \mathfrak{h}_j \otimes \mathfrak{h}_{j+1} \) more elegantly as

\[
H^{(N)} := \sum_{j=1}^{n-1} 2P^{(N)}_{j,j+1} = \sum_{j=1}^{n-1} \sum_{k,l=1}^{N} 2E_{jk}^{kl}E_{j+1}^{kl}.
\]  

The factor of two was introduced in order to have \( H^{(N=2)} = H^{XXZ}(\Delta = 1) + \text{const.} \)

\( U_q(\mathfrak{gl}(N, \mathbb{C})) \)-invariant Lax operator. We could in principle proceed along the lines of the preceding discussion, however, since we address the isotropic models, i.e. interactions related to non-deformed (Lie) symmetries, we rather employ an instructive shortcut to the solution of the Sutherland equation, entirely avoiding an explicit use of algebraic objects from the FRT construction.

We start by proposing the following expansions for constant operators \( \mathbf{L} \) and \( \mathbf{B} \),

\[
\mathbf{L}_m = \sum_{i,j=1}^{N} E_{ij}^{m} \otimes L_{ji}^{i}, \quad \mathbf{B}_m = \sum_{i,j=1}^{N} E_{ij}^{m} \otimes B_{ji}^{i}.
\]  

After considering interaction \( h^{(N)}_{j,j+1} = P^{(N)}_{j,j+1} \) over \( \mathbb{C}^N \otimes \mathbb{C}^N \) we derive

\[
[P^{(N)}_{m,m+1}, \mathbf{L}_m \mathbf{L}_{m+1}] = \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} E_{ij}^{m} E_{m+1}^{kl} \otimes [L_{jk}^{i}, L_{li}^{i}],
\]

\[
\mathbf{B}_m \mathbf{L}_{m+1} - \mathbf{L}_m \mathbf{B}_{m+1} = \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} E_{ij}^{m} E_{m+1}^{kl} \otimes (B_{ji}^{m} L_{lk}^{i} - L_{ji}^{m} B_{lk}^{i}),
\]

which gives rise to quadratic algebraic relations for the matrix elements

\[
[L_{jk}^{i}, L_{li}^{i}] = B_{ji}^{m} L_{lk}^{i} - L_{ji}^{m} B_{lk}^{i},
\]

obviously reducing upon restricting \( \mathbf{B} = -\mathbb{I} \), i.e. \( B_{ij} = \delta_{ij} \cdot (-\mathbb{I}) \), to the defining relations of the algebra \( \mathfrak{gl}(N, \mathbb{C}) \). Moreover, the solution can be readily extended by including a central element (of the form \( u \cdot \mathbb{1}, u \in \mathbb{C} \)) to the operator \( \mathbf{L} \), enabling us to work with the \( \mathfrak{gl}(N, \mathbb{C}) \)-covariant Lax matrix reading

\[
\mathbf{L}_j(u) = u \cdot \mathbb{1} \otimes \mathbb{1} + \sum_{k,l=1}^{N} E_{jk}^{kl} \otimes L_{lk}^{i}.
\]
This result is of course of no surprise since it is implied by the underlying YBE, with the fundamental $\mathfrak{g}(N, \mathbb{C})$ $R$-matrix of the form

$$R_{12}(u) = u \cdot 1_{12} + \sum_{i,j=1}^{N} E_{i}^{ij} E_{2}^{ji}, \quad (4.38)$$

which agrees (up to additive constant) with the $R$-matrix from the FRT construction (see (A.5) in appendix A). Notice also that the $N = 2$ case is consistent with the previously obtained result (4.22) upon identifying $h^{XXZ}_{12}(\Delta = 1) = h_{12}(\gamma = 0) \sim 2P_{12}$, for $P_{12}$ in $\mathcal{S}_{1/2} \otimes \mathcal{S}_{1/2} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$.

**Solution of the boundary equations.** Having facilitated the Lax matrix (4.37), we can proceed as discussed above and look for a solution of the boundary equations (4.11). Since a general problem of characterizing all integrable steady states is hard even in the simplest case $N = 2$, we shall restrict ourselves in the forthcoming discussion by fixing the boundary vectors and assuming a restricted form of the dissipation, determined by the *primitive* set of $2N^2$ Lindblad operators acting at the ends of the chain, defined as

$$A_{1}^{(i,j)} = \sqrt{\Gamma_{i}^{ij}} E_{1}^{ij}, \quad A_{n}^{(i,j)} = \sqrt{\Gamma_{n}^{ij}} E_{n}^{ij}, \quad i,j = 1,2,\ldots,N. \quad (4.39)$$

As in the $N = 2$ case, we expect that the left and the right boundary contraction vectors are in fact the left and the right auxiliary vacua which, in the representation for the generators $L_{ij}$ used in [25], satisfy

$$\langle 0 | x^{k}_{i} | 0 \rangle = 0. \quad (4.40)$$

With this choice of vacua one discovers that the compatibility conditions (4.11) further restrict the weight vector $r$, imposing $r_{1} = r_{2} = \cdots = r_{N-2} = 0$, hence leaving $r_{0}$ and $r_{N-1}$ as the only non-vanishing representation parameters. Further, we find that almost all rates vanish except $N - 1$ equal rates on each side of the chain,

$$\Gamma_{k}^{NN} = \Gamma_{k}^{ij} = 0, \quad i,j = 1,\ldots,N-1, \quad k = 1,n \quad \Gamma_{i}^{NN} = \Gamma^{k}, \quad \Gamma_{n}^{NN} = \Gamma^{R}, \quad j = 1,\ldots,N-1. \quad (4.41)$$

Since $E^{ij}$ for $i < j$ represent (by convention) ‘particle-raising channels’ (generalizing $\sigma^{+}$ kicking from the $s = 1/2$ case), the induced particle currents flow in the direction of increasing lattice site numbers. Due to similarity with the $s = 1/2$ case we shall refer to the set of primitive Lindblad operators defined by (4.41) as the *maximal incoherent driving*. The Hamiltonian (4.33) and the Lindblad operators (4.39) generate, by multiplication and addition, the entire algebra of operators, which in turn implies *uniqueness* of the steady state density operator $\rho_{\infty}$ [26].

It is perhaps instructive to point out that simply switching the left and the right subset of Lindblad dissipators (4.41) *does not* produce a solution with reversed current. Even worse—and at first glance somewhat surprisingly—the boundary system of equations admits no solution in this case. Instead, one has to simultaneously transpose the physical space $\mathcal{H}_{s}$ and reverse the sign of the vector $r$ in the definition of the Lax operator (which
is equivalent to a spin-flip algebra automorphism on the generators). The origin of such asymmetry is in the use of non-unitary representations of $U_\theta (\mathfrak{sl}(N, \mathbb{C}))$.

The restriction (4.41) further simplifies the Lax matrix to

$$L_m = \sum_{j=1}^{N-1} \left( E_{m}^{jj}(x_j^N \partial_{x_j^N} + r_0) + E_{m}^{Nj}(x_j^N \partial_{x_j^N} + r_N) - E_{m}^{NN}(x_j^N \partial_{x_j^N}) \right)$$

$$+ \sum_{j \neq k=1}^{N-1} E_{m}^{kj}(x_j^N \partial_{x_k^N} + E_{m}^{NN}(r_0 + r_{N-1}).$$

(4.42)

By using the properties (4.40) we can evaluate the matrices appearing in the boundary equations (4.11) explicitly. On the left side we obtain

$$\langle 0, 0 \vert \hat{B}_L(L_0) = -(N - 1) \Gamma^L |r_0 + r_{N-1}|^2 E_{1}^{NN} \langle 0, 0 \vert$$

$$+ \Gamma^L \sum_{j=1}^{N-1} (|r_0 + r_{N-1}|^2 E_{1}^{jj} \langle 0, 0 \vert$$

$$- \frac{1}{2} (N - 1)(|r_0 + r_{N-1}| E_{1}^{Nj} \langle 0, 1_j \vert + (\bar{r}_0 + \bar{r}_{N-1}) E_{1}^{jN} \langle 1_j, 0 \vert),$$

(4.43)

and similarly on the right side,

$$\otimes (4.44)$$

$$\langle 0, 0 \vert \hat{D}_R(L_n) = |0, 0 \rangle (N - 1)|r_0|^2 \Gamma^R E_{n}^{NN} - \Gamma^R \sum_{j=1}^{n-1} (|r_0|^2 \langle 0, 0 \vert E_{jj}$$

$$- \frac{1}{2} (1_{j}, 0 \rangle r_0 r_{N-1} E_{n}^{Nj} + |0, 1_j \rangle r_0 \bar{r}_{N-1} E_{n}^{jN})).$$

(4.45)

We introduced a bra–ket notation to denote vectors in a doubled auxiliary space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, i.e. we identify $|0, 0 \rangle \equiv 1, |1_k, 0 \rangle \equiv x_k^N$, and $|0, 1_k \rangle \equiv y_k^N$ and similarly for the bra-vectors.

By solving (4.11) with explicit decompositions (4.43) and (4.44) we arrive at the ‘integrable set’ of the dissipators parametrized by (4.41) and $\Gamma^L = \Gamma, \Gamma^R = (N - 1)^2 \Gamma$ with the representation parameters

$$r_0 = -\frac{4i}{(N - 1)^2 \Gamma}, \quad r_{N-1} = -N r_0.$$
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determining the (non-normalized) steady state density matrix \( \rho_\infty = \mathcal{S} \mathcal{S}^\dagger \) via the Lax matrix \( \mathbf{L}_m(r_0, r_{N-1}) \) by virtue of the \( \mathcal{S} \)-operators

\[
\mathcal{S}(\Gamma) = \langle 0 | \prod_{j=1}^n \mathbf{L}_j(r_0(\Gamma), r_{N-1}(\Gamma)) | 0 \rangle. \tag{4.46}
\]

At this point we should make a subtle but however very important remark. The left system of boundary equations does not admit (for \( N \geq 3 \)) any solutions by itself, i.e. in a sense of solutions for the partially contracted expressions (4.11). Let us take a look at the \( N = 3 \) case: by facilitating the representation parameters \( r \) as given by (4.45) (which do fulfil the right boundary system (4.11) though), the contributions from the unitary part \( i(\mathcal{B}_1^{(1)} - \mathcal{B}_1^{(2)}) \) entering in the \( E_{12}^{12} \) and \( E_{21}^{21} \) physical components in fact cannot be eliminated with the form of the left dissipator \( \mathcal{D}_L \), as provided by (4.41). But they need not be! The only quantum number which is lifted in this case corresponds to auxiliary ‘color’ \( \chi_2 \), which evidently cannot be generated in the free process (i.e. generated by a product of Lax matrices only) starting from the right vacuum \( 1 \equiv |0, 0\rangle \) due to the constraint \( r_1 = 0 \), and consequently is a valid solution of the necessary requirement encapsulated by the fully contracted boundary equations (4.10). Similar logic applies for \( N > 3 \). These solutions of non-local type suggest that considering the local boundary system (4.11) may be an overly restrictive requirement when searching for solutions of the global condition (4.10).

Physical interpretation and possible generalizations. It is worth noticing that the just obtained class of solutions is a particular generalization of the previously known one, namely the maximally driven isotropic spin-1/2 Heisenberg model. In fact it can be viewed as an embedding of the \( N = 2 \) solution to higher-dimensional quantum spaces, since it only involves a pair of non-zero representation parameters in combination with a vanishing spectral parameter \( \lambda \equiv Nr_0 + r_{N-1} = 0 \) (see (4.45)). Hence solutions are determined by a single independent representation parameter, which for \( N = 2 \) coincides with the spin parameter \( s \). Thus, physically speaking, the full rotational invariance of the interaction along with incoherent quantum jumps, which for our choice (4.41) occur only between a particular extreme state and all remaining states with the same rates, indicates that the entire (stationary) process somewhat ‘sees’ the \( N - 1 \) levels as being part of one composite particle. Consequently, there are no currents among the levels disconnected by the dissipation, and asymptotic particle density profiles are found to agree with the \( N = 2 \) case (see section 5 for further details).

Unlike in the \( N = 2 \) case, it appears difficult already for \( N = 3 \) to solve (4.11) in its full generality, especially ultimately verifying if solutions pertain to positive rate matrices \( \mathcal{G}_{L,R} \), thus representing realistic noisy process.

On the other hand, there is an issue with \( q \)-deformation when \( N > 2 \); it turns out that hermicity violation of the interaction is not of surface type, hence disallowing a simple amendment of the Sutherland equation (or alternatively applying a twist on a level of Hopf algebra), and consequently rendering the corresponding higher anisotropic models, i.e. the \( N \)-component Perk–Schultz models,

\[
H_{\text{PS}}^q = \sum_{j=1}^{n-1} \sum_{\alpha=1}^{N-1} \sum_{\beta=\alpha+1}^{N} E_{j+1}^{\alpha\beta} E_j^{\beta\alpha} + E_j^{\beta\alpha} E_{j+1}^{\alpha\beta} - qE_j^{\alpha\alpha} E_{j+1}^{\beta\beta} - q^{-1}E_j^{\beta\beta} E_{j+1}^{\alpha\alpha} \tag{4.47}
\]

not applicable for the description of boundary-driven open quantum systems.
Non-fundamental models. We have to stress that the presented technique for constructing solutions which are compliant with the bulk divergence algebraic condition (Sutherland equation) from the universal $R$-matrix is only possible when the so-called fundamental models are addressed, namely the integrable models whose Lax operators are equivalent to the corresponding $R$-matrices. The reason is primarily that the auxiliary spin label from a Lax operator is reserved for a generic non-compact spin (needed to be tuned with the dissipation parameters in our application), whereas the remaining two spin labels from the YBE have to be the same in order to discuss homogeneous models. The latter two spin labels define the auxiliary indices with respect to the FRT construction, which are eventually, as we have seen (see section 4.1), interpreted as local physical indices of two adjacent quantum spaces. For non-fundamental models on the other hand, the interaction cannot simply be deduced from regularity properties through the derivative of the $R$-matrix, but instead a pathway via analytic properties of a monodromy matrix needs to be taken (see e.g. [27]).

5. Expectation values of observables

An important aspect in regard to analytic exact solutions for the NESS is to be able to efficiently calculate expectation values of local physical observables. For this purpose we may define generic local vertex operators $X_N$ as elements over $\mathcal{S}_s \otimes \mathcal{S}_s$, for a system of size $n$ with local space $H \cong \mathbb{C}^N$, associated with some local physical operator $X_j$ supported on a contiguous sub-lattice (consisting of say $k$ physical sites between positions $j$ and $j + k - 1$). The steady state expectation values can then be expressed entirely in terms of the amplitudes in the auxiliary space $H_a$ as

$$\langle X_j \rangle := \frac{\text{Tr}(X_j SS^\dagger)}{\text{Tr}(SS^\dagger)} = (Z^{(n)}_N)^{-1}(0, 0)|(T_N)^{j-1}X_N(T_N)^{n-j-k+1}|0, 0\rangle,$$

(5.1)

where $T_N$, denoting the transition vertex operator, and the $n$-particle nonequilibrium partition function

$$Z^{(n)}_N := \langle 0, 0|(T_N)^n|0, 0\rangle$$

(5.2)

were introduced.

Vertex operators. Particle density profiles are calculated by means of the on-site elementary vertex operators $\{V_N^{(k)}\}$, $k = 1, 2, \ldots, N$, which we define as

$$V_N^{(k)} = \sum_{j=1}^{N} L_j^k \otimes \mathcal{T}_j^k,$$

(5.3)

enabling a compact definition of the transition operator,

$$T_N = \sum_{k=1}^{N} V_N^{(k)}.$$  

(5.4)

For brevity we shall drop the index $N$ from here on.
To each pair of local density operators $E^{kk}_j$ we may associate a current density operator $J^{(k,l)}_j$, defined in terms of a local continuity equation at site $j$,

$$\frac{d}{dt}(E^{kk}_j - E^{kl}_j) = \langle J^{(k,l)}_{j-1,j} - J^{(k,l)}_{j,j+1} \rangle = 0, \quad k, l = 1, 2, \ldots, N,$$

explicitly reading

$$J^{(k,l)}_{j,j+1} = i(E^{kl}_j E^{kl}_{j+1} - E^{lk}_j E^{lk}_{j+1}).$$

Introducing a corresponding two-site vertex operator $\mathbb{J}^{(k,l)} = i(\mathbb{J}^{(k,l)}_+ - \mathbb{J}^{(k,l)}_-)$ and taking into account that the current expectation value must not depend on position (by virtue of continuity equation), we arrive at

$$\langle J^{(k,l)} \rangle = (\mathcal{Z}_N^{(n)})^{-1} \langle 0, 0 | (\mathbb{T}_N)^{n-2} \mathbb{J}^{(k,l)} | 0, 0 \rangle,$$

with

$$\mathbb{J}^{(k,l)} = i \sum_{i,j=1}^{N} [L^{ij} L^{jk} \otimes \overline{L}^{ik} \overline{L}^{jl} - L^{ik} L^{jl} \otimes \overline{L}^{ij} \overline{L}^{jk}].$$

Note that products of elements $L^{ij}$ above cannot be simplified since they do not consist of spin generators from the fundamental representation $\mathfrak{S}_f$, for which (3.10) holds.

5.1. Vertex operator algebra for spin-1/2 chain

Spin current. In this section we state some explicit results for the simplest $N = 2$ case. It is helpful to notice that the transition vertex $\mathbb{T}$ preserves the subspace spanned by diagonal states $|k, k\rangle \equiv (x^2)^k(y^2)^k =: z^k \ (k \in \mathbb{Z}_+)$, which allows us to reduce the expression (5.7) by virtue of the reduced vertex (hatted) operators $\mathbb{T}$ and $\mathbb{J}$, obtained through orthogonal projections onto the subspace $\mathfrak{H}_{\text{diag}} = \text{span}\{z^k \equiv |k, k\rangle; k \in \mathbb{Z}_+\}$,

$$\langle J^{(k,l)} \rangle = (\hat{\mathcal{Z}}^{(n)})^{-1} \langle 0 | \hat{\mathbb{T}}^{n-2} \hat{\mathbb{J}} | 0 \rangle.$$

By expressing how $\mathbb{T}$ and $\mathbb{J}$ operate in $\mathfrak{H}_{\text{diag}}$,

$$\mathbb{T} = \left( [k - s + i\lambda]_q[k - \overline{s} - i\overline{\lambda}]_q + [k - s - i\overline{\lambda}]_q[k - \overline{s} + i\lambda]_q \right) [k] \langle k |$$

$$+ \ [k + 1]_q^2 [k + 1]_q \langle k + 1 | + [k - 2s]_q[k - 2\overline{s}]_q [k + 1] \langle k |,$$

$$\mathbb{J} = \left( i[k - s + i\lambda]_q[k - s - i\overline{\lambda}]_q[k + 1]_q[k - 2\overline{s}]_q \right) [k] \langle k |$$

$$+ i[k - \overline{s} - i\overline{\lambda}]_q[k - \overline{s} + i\lambda]_q[k - 1 - 2s]_q$$

$$+ i[k - \overline{s} - i\overline{\lambda}]_q[k - \overline{s} + i\lambda]_q[k + 1]_q[k - 2s]_q$$

$$- i[k - s + i\lambda]_q[k - s - i\overline{\lambda}]_q[k - 1 - 2s]_q [k] \langle k |$$

$$+ i[k + 1]_q^2 \left( [k - s + i\lambda]_q[k + 1 - \overline{s} - i\overline{\lambda}]_q \right) [k] \langle k |$$

$$+ i[k + 1]_q^2 \left( [k - \overline{s} - i\overline{\lambda}]_q[k + 1 - \overline{s} + i\lambda]_q \right) [k] \langle k |$$

$$+ i[k + 1]_q^2 \left( [k - \overline{s} - i\overline{\lambda}]_q[k + 1 - \overline{s} + i\lambda]_q \right) [k] \langle k |.$$
one may observe that $\hat{T}$ and $\hat{J}$ are in fact proportional to each other provided $\text{Re} (\lambda) = 0$ ($s = \text{Re} (s) + i \text{Im} (s)$ can still be arbitrary though),

$$\hat{J} = \rho_\gamma (\text{Im} (s)) \hat{T}, \quad \rho_\gamma (s) := i [2 i \text{Im} (s)]_q.$$

With aid of this result we find (as noted in [10]) for the expectation value of the spin current density,

$$\langle J^{(k,l)} \rangle = \rho_\gamma (\text{Im} (s)) \frac{\hat{Z}^{(n-1)}_{\text{diag}}}{\hat{Z}^{(n)}_{\text{diag}}},$$

which interestingly coincides with an analogous expression known from exact solutions of classical ASEP [12], despite the latter being governed by so-called reaction–diffusion (quadratic) algebras [28, 18]. Essentially, equation (5.13) is implied by the local continuity equation and a weaker requirement, namely

$$\hat{T} |0\rangle \sim \hat{\nabla} |0\rangle, \quad \langle 0| \hat{T} \sim \langle 0| \hat{\nabla},$$

which is important in the $SU(N)$ case where the proportionality relation (5.12) breaks down while the partial contractions (5.14) remain valid, with the $N$-dependent proportionality factor of $-8/((N-1)^2)$. For exceptional (i.e. reducible) cases, pertaining to deformation parameters $\gamma = \pi (l/m)$ with $l < m$ and $l, m \in \mathbb{Z}^+$, one could in principle access a dense set of anisotropy parameters in the massless regime $| \cos (\gamma) | < 1$ by means of diagonalization of the reduced transition operator $\hat{T}$, acting in an $m$-dimensional invariant subspace in $S_{\text{diag}}$ spanned by states $\{|k, k\}; k = 0, \ldots, m - 1\}$, as long as $m$ is sufficiently small. In this sense, the asymptotic behavior $n \to \infty$ of the expression (5.13) is dominated by the largest eigenvalue. For the undeformed (critical) $q = 1$ case, however, $\hat{T}$ is an irreducible infinitely dimensional operator, hence the eigenproblem for $\hat{T}$ seems to require analytic treatment in the spirit of the approaches employed in studies of algebras associated with ASEP [13, 18].

We shall abstain from this technical aspect at the moment, going beyond the scope of this paper, thus leaving it open for future analysis.

Particle density profiles. Computation of particle density profiles in the thermodynamic ($n \to \infty$) limit can on the other hand be assisted with the help of closed algebraic relations among vertex operators $\hat{T}$ and $\{\hat{\nabla}^{(k)}\}$, which we have found with the assistance of computer symbolic algebra. In particular, by defining the magnetization density vertex operator $\hat{\nabla}^{(z)} := \hat{\nabla}^{(0)} - \hat{\nabla}^{(1)}$, the following third-order relations from the free algebra of vertex operators $\{\hat{T}, \hat{\nabla}^{(z)}\}$ can be found:

$$[\hat{T}, [\hat{T}, \hat{\nabla}^{(z)}]] = \kappa_0^0 (\text{Im} (s)) \hat{\nabla}^{(z)} + \kappa_1^1 (\hat{T}, \hat{\nabla}^{(z)}),$$

\footnote{There is nonetheless one apparent difference in comparison to ASEP, namely in our case the transition operator takes place in the product representation of quantum algebra.}
with coefficient functions
\[
\begin{align*}
\kappa_0^0 (\text{Im}(s)) &= 8 [i \text{Im}(s)]^2 q \cos (2\gamma) \cosh (2\gamma \text{Im}(s)), \\
\kappa_1^0 &= -2 \cos (2\gamma),
\end{align*}
\]
reducing in the undeformed limit \(\gamma \to 0\) to \(\kappa_0^0 = -8 (\text{Im}(s))^2\), \(\kappa_1^0 = -2\), as stated previously in \([10]\).

It remains another appealing problem how to understand those types of relations (and eventually their counterparts for higher \(N\)) from the algebraic standpoint, whereas from the practical perspective it is desirable to devise a method which would allow the evaluation of particle density profiles for finite chains of size \(n\) and/or access to closed-form results in the thermodynamic regime based on first symmetry principles (i.e. without resorting to model-specific asymptotic observations, unlike in \([10]\)).

**Density profiles.** By means of explicit symbolic contractions for finite system sizes \(n\) where the effective sizes of the auxiliary space \(\mathcal{H}_s\) become finite, we have computed the particle density profiles for each particle species for the \(N\)-dimensional case, and conjectured the following thermodynamic (asymptotic in \(n\)) forms,
\[
\begin{align*}
\langle E_{kk}^{jj} \rangle &= \frac{1}{2(N - 1)} \left[ 1 + \cos \left( \frac{j\pi}{n} \right) \right], \quad k = 1, 2, \ldots, N - 1, \\
\langle E_{NN}^{jj} \rangle &= -(N - 1) \langle E_{kk}^{jj} \rangle, \quad k \neq N.
\end{align*}
\]
For \(N = 2\), i.e. spin-1/2 case, these correctly reduce to the previously found cosine-shaped magnetization profiles \([10]\).

### 6. Conclusions and discussion

We provided an explicit connection between exact MPO solutions of nonequilibrium steady states for boundary-driven dissipative quantum evolution with integrable bulk Hamiltonians, which have appeared just recently in the literature, and rudimentary concepts of quantum theory of integrability, namely the Yang–Baxter equation and the representation theory of quantum groups. The use of quantum algebras enables us to realize a suitable cancellation mechanism (known as the Sutherland equation) for the action of the unitary part of a Liouville super-operator, whose global action consequently affects the boundary quantum spaces only. In the second stage, the remaining surface-like terms enter into a system of boundary compatibility conditions which are subsequently treated independently of the bulk structure in combination with a general form of a dissipation rate-matrix, and with suitably chosen boundary states. Although we were able to find solutions belonging to \(SU(N)\)-invariant quantum gases within a restricted set of dissipative channels by adopting boundary vacuum states, it still remains an open problem to systematically classify all possible solutions for a given bulk interaction.

Moreover, we asserted how lowest weight transfer matrices, defined in terms of non-unitary \(\mathfrak{sl}(N, \mathbb{C})\) realizations of ancillary spaces, in a sense provide a natural setup for...
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a description of nonequilibrium states of open systems in the absence of translational invariance. As has been recently pointed out [16], the presented exact solutions offer an interesting perspective into the theory of quantum transport [29]. In the first place, they open a possibility for analytic studies of the paramount physical properties (e.g. transport behavior, nonequilibrium phase transitions [30], large deviation theory [31] and counting statistics [32]) of quantum out-of-equilibrium processes and their steady states by means of well-understood underlying symmetry principles, along the lines of their classical counterparts. Remarkably, in spite of the intrinsic far-from-equilibrium character of the problem, the solutions at hand also provide a valuable insight on transport theory within the linear-response regime, as they essentially (by construction) represent a continuous family of pseudo-local conserved charges [33], complementing an infinite sequence of local (standard) conservation laws, having a profound influence on anomalous transport behavior in quantum spin chains [9, 34]. Quite strikingly, the most recent numerical evidence [35] reveals that quantum integrability structure, giving birth to pseudo-local integrals of motion, most likely survives a passage to the classical integrable continuous models and corresponding integrable lattice regularizations, indicating that part of the presented integrability structures indeed have well-defined classical counterparts.

Finally, we give some remarks on limitations of the presented framework, apart from the inability of extending the formalism to accommodate for the non-fundamental models, as we have already argued. Requiring solutions exhibiting an underlying quantum group symmetry seems to be inessential from purely algebraic requirements imposed by the bulk and the boundary conditions for NESS density operators, since merely a weaker condition imposed by the Sutherland type of equation is in fact sufficient in the bulk in order to implement the desired cancellation property. In this regard, the boundary matrix need not necessarily be related to the derivative of the Lax matrix with respect to the spectral parameter. On the other hand, we could of course seek for algebraic constructions not strictly of quantum group type, e.g. some other quadratic algebras admitting infinite-dimensional realizations. That being said, one could address elliptic (eight-vertex) solutions of the YBE, with no associated quasi-triangular Hopf algebra instead, one can construct algebraic objects called elliptic quantum groups [36], which are two-parametric deformations of $U(g(N,C))$. 

Besides that, a constrained Cholesky-type factorized form of the density operator in terms of an abstract (non-Hermitian) transfer matrix does presumably not exhaust all possibilities within the boundary-driven setup. Eventually, we could have additionally included coherent (ultra-local) boundary fields [15], which might turn out to have an important role in the bulk-boundary compatibility condition. Exploring these directions could be an interesting option for further developments.

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Appendix A. Faddeev–Reshetikhin–Takhtajan’s construction of quantum groups

One of the indisputably most celebrated and influential results of the Leningrad School of
Mathematical Physics is Faddeev–Takhtajan–Reshetikhin’s (FRT) work on ‘quantization’
of Lie algebras [39], which is formally speaking a realization of algebraic objects arising
as suitable continuous deformations of Lie algebraic structures. These quantized algebras,
simply called (somewhat misleadingly) quantum groups, materialize as solutions of the
celebrated Yang–Baxter equation, and represent a cornerstone of the quantum inverse
scattering method. Subsequently, those algebraic structures have been given a more formal
mathematical meaning independently by Drinfeld [40] and Jimbo [41]. In particular,
quantum groups can be understood as non-trivial (so-called quasi-triangular) Hopf
algebras.

There exists an extensive literature on the subject [42]–[45]. Here, mainly for the sake
of completeness, we shall follow the FRT formulation [39]. For our purpose, a quantum
group is understood as a one-parametric deformation of a universal enveloping algebra
\( \mathcal{U}(g) \) for a given Lie algebra \( g \). We shall restrict ourselves entirely to \( q \)-deformations of
the algebra \( \mathcal{U}(\mathfrak{sl}(N, \mathbb{C})) \), merely for its importance in one-dimensional quantum models.

We first attempt to justify the role of the RTT equation in the language of quantum
groups. Imagine an associative quadratic algebra \( A(R) \) (over \( \mathbb{C} \)) generated by a unit \( 1 \)
and a set of \( N^2 \) elements \( \{T_{ij}\} \), \( i, j = 1, \ldots, N \), with \( R \)-matrix \( R \). The algebra admits a
bialgebra structure, with coproduct \( \Delta : A \to A \otimes A \) prescribed as
\[
\Delta(1) = 1 \otimes 1, \quad \Delta(T_{ij}^k) = \sum_k T_{ik}^j \otimes T_{jk}^i, \tag{A.1}
\]
and counit \( \epsilon : A \to \mathbb{C} \), \( \epsilon(T_{ij}^k) = \delta_{ij}^k \). Denoting \( A^* \) as a dual space of \( A \), \( A^* \) is generated
by \( 1' \) and \( \{l_{ij}^\pm\} \), such that \( 1'(T_{ij}^k) = \delta_{i,j} \). A coproduct in \( A \) is induced by the product
(multiplication) in \( A^* \), namely for two elements \( l_1, l_2 \in A^* \) and \( a \in A \) we have in particular
\[
(l_1 l_2, a) \equiv l_1 l_2(a) = (l_1 \otimes l_2)(\Delta(a)). \tag{A.2}
\]
By focusing on deformations of \( g \cong \mathfrak{gl}(N, \mathbb{C}) \), one can define a subalgebra \( U(R) \equiv U_q(g) \)
of \( A \) by means of the relations
\[
(L_{ij}^\pm, T) = R_q^\pm, \quad (1', T) = 1, \tag{A.3}
\]
expressing the duality of bialgebras in a compact fashion by introducing two triangular
matrix-functionals \( L_{ij}^\pm \) with elements from \( A^* \), i.e. \( [L_{ij}^\pm]_{kj} = l_{ij}^\pm \), and triangular (invertible)
matrices \( R_q^\pm \),
\[
R_q^+ = PR_qP, \quad R_q^- = R_q^{-1}, \tag{A.4}
\]
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prescribed by means of the lower-triangular matrix $R_q$ operating in $\mathbb{C}^N \otimes \mathbb{C}^N$ ($q \in \mathbb{C}$),

$$R_q = \sum_{i \neq j=1}^N E^{ii} \otimes E^{jj} + q \sum_{i=1}^N E^{ii} \otimes E^{ii} + (q - q^{-1}) \sum_{1 \leq j \leq i \leq N} E^{ij} \otimes E^{ij}. \quad (A.5)$$

The independent set of equations, given as

$$R_q^+ L_1^+ L_2^+ = L_2^+ L_1^+ R_q^+, \quad R_{q}^+ L_1^- L_2^- = L_2^- L_1^- R_{q}^+, \quad (A.6)$$

serves as an abstract definition of a $q$-deformation of a ‘classical’ Lie algebra $\mathfrak{gl}(N, \mathbb{C})$, which is facilitated in our construction of a steady state density operator in section 4. The induced coproduct in the dual bialgebra reads

$$\delta(1') = 1' \otimes 1', \quad \delta(l_{ij}^+) = \sum_k l_{ik}^+ \otimes l_{kj}^+. \quad (A.7)$$

In addition to coproduct and counit maps we also have an antipode (a coinverse) $\eta$, which is based on the existence of the inverse $T^{-1}$, $\eta(T_j) = [T^{-1}]_j$, completing a bialgebra $\mathcal{A}$ to a Hopf algebra.

Universal $\mathcal{R}$-matrix. Referring back to equation (2.3), the quadratic algebra of monodromy matrix elements $\{T_j\}$, written out component-wise using Einstein convention,

$$R^{ij}_{ab} T^a_k T^b_l = T^l_i T^a_k R^{ab}_{kl}, \quad (A.8)$$

is prescribed via $R$-matrix, a constant solution of the quantum Yang–Baxter equation (2.4). The latter has the role of controlling non-commutativity of elements (operators) $\{T_j\}$.

On abstract level, non-cocommutativity of a coproduct in bialgebra $\mathcal{A}$ can be formally given by means of a universal $\mathcal{R}$-matrix, $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$, governing the similarity of a coproduct and an opposite coproduct,

$$\mathcal{R} \Delta(a) = (\Pi \circ \Delta)(a) \mathcal{R}. \quad (A.9)$$

The non-trivial condition of the latter type endows a Hopf algebra with a quasi-triangular structure. The associativity of a coproduct over triple-product algebra $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ requires the two expressions,

$$(\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12} \quad (A.10)$$

to be equivalent, hence

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}. \quad (A.11)$$

Thus, we have finally established the connection of a quantum group to the (parameter-independent) Yang–Baxter equation. Evaluating a universal element $\mathcal{R}_{12}$ from (A.9) for our choice of deformed Lie algebras $\mathfrak{gl}(N, \mathbb{C})$ in $\mathbb{C}^N \otimes \mathbb{C}^N$ explicitly yields (A.5). Notice that in QISM we employ parameter-dependent solutions (2.4) which are formally obtained via Baxterization [22] of quantum group objects.

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The reader interested in a more detailed and concise presentation of quantum groups is referred, for example, to [42, 39]. Here we shall only explicitly treat the $q$-deformed $\mathfrak{sl}(2, \mathbb{C})$ algebra, defining the main object (in particular the form of the Lax matrix) needed to construct solutions for the $s = 1/2$ anisotropic Heisenberg chain.

**Example: deformation of $\mathfrak{sl}(2, \mathbb{C})$ algebra.** The simplest example is to consider the algebra of spin-1/2 generators, i.e. the $\mathfrak{sl}(2, \mathbb{C})$-triple $\{S^+, S^z\}$ with canonical (Lie algebraic) commutation relations,

$$[S^z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z. \quad (A.12)$$

The associated universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$ is a tensor algebra freely generated by $\{S^z, S^\pm\}$, i.e. a tensor algebra of words formed from the algebra generators, modulo their commutation relations. One can now introduce the $q$-deformed universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$, with deformation parameter $q \in \mathbb{C}$, adopting the standard $q$-calculus definition for the deformed number ($q$-number),

$$[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}, \quad (A.13)$$

as a quantum group with elements generated by the identity and the four quantized generators $\{S^\pm, K^\pm\}$, where $K^\pm = q^{\pm S^z}$, subjected to the defining commutation relations

$$[S^+, S^-] = [2S^z]_q = \frac{(K^+)^2 - (K^-)^2}{q - q^{-1}}, \quad K^\pm S^\pm = q^{\pm 1} S^\pm K^\pm. \quad (A.14)$$

The operator $q$-analogs are defined via analytic series expansion in accordance to (A.13). It is noteworthy that the undeformed limit corresponds to the value of the deformation parameter $q = 1$, however, to properly restore ‘classical’ Lie algebra one has to take the differential form of (A.14). We remark that this particular form of ‘quantization’ is of main relevance when considering **axial deformations** of $SU(N)$-symmetric quantum chain models.

In the language of FRT algebra, a quasi-triangular structure for the $\mathcal{U}_q(\mathfrak{sl}(N, \mathbb{C}))$ is explicitly prescribed by the universal element

$$\mathcal{R} = q^{2(S^z \otimes S^z)} \sum_{k=0}^{\infty} \frac{(1 - q^{-2})^k}{[k]!} (q^{S^z} S^+ \otimes q^{-S^z} S^-)^k q^{k(k-1)/2}. \quad (A.15)$$

Evaluating this expression in the fundamental (spin-1/2) representation, and considering that generators $S^\pm$ square to zero, immediately yields (up to a prefactor) the previously stated result (A.5). Additionally, the explicit form for $L$-matrices (A.17) is expressed in a canonical way element-wise as

$$(L^+_{i j})^k = R^k_{i j}, \quad (L^-_{i j})^k = (R^{-1})^k_{i j}. \quad (A.16)$$

In words, $L$-matrices are obtained from the $R$-matrix by associating block elements of the second tensor factor with algebra generators. This in turn leads to the following pair of $L$-operators (with elements proportional to the generators of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$),

$$L^+ = \begin{pmatrix} K & (q - q^{-1})S^- \\ 0 & K^{-1} \end{pmatrix}, \quad L^- = \begin{pmatrix} K^{-1} & 0 \\ -(q - q^{-1})S^+ & K \end{pmatrix}. \quad (A.17)$$
and triangular components of the $R$-matrix,

$$R^+ = \begin{pmatrix} q & q^{-1} - 1 \\ q^{-1} & 1 \end{pmatrix}, \quad R^- = P(R^+)^{-1}P. \quad (A.18)$$

One can now readily check that (A.6) are consistent with the defining relations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ (A.14). We therefore offered a convenient algebraic expression for quantum group relations in an abstract fashion: the functions of generators appearing in the matrix elements of $L^\pm$ can readily be taken from any representation, i.e. the space in which the spin generators operate can be chosen arbitrarily. As we demonstrated in the main text (see section 4) this freedom is of crucial importance for the application we have in mind, as it turns out that FRT algebra (A.6) materializes as a prerequisite bulk cancellation mechanism—a sort of quantum version of an analogous condition in the context of classical exclusion processes [12].

Ultimately, the complete quantum group structure of the $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ is explicitly provided by the Hopf algebra co-structures, namely the coproduct $\Delta$ and the counit $\epsilon$,

$$\Delta(S^\pm) = S^\pm \otimes K^+ + K^- \otimes S^\pm, \quad \Delta(K^\pm) = K^\pm \otimes K^\pm, \quad (A.19)$$
$$\epsilon(S^\pm) = 0, \quad \epsilon(K^\pm) = 1, \quad (A.20)$$

along with the antipode map $\zeta$, prescribing the inverse operation by virtue of $\zeta(T) = T^{-1}$,

$$\zeta(S^\pm) = -q^{\pm1}S^\pm, \quad \zeta(K^\pm) = K^\mp, \quad (A.21)$$

which completes the bialgebra to the Hopf algebra. At last, a quasi-triangular structure is given by the $R$-matrix in the sense of (A.9).

**Appendix B. Twisting solutions**

Here we show how non-Hermitian terms in the Hamiltonian density (4.22) can be removed by twisting the Hopf algebra structure, formally expressed as a similarity transformation on a coproduct while preserving the coassociativity property,

$$\Delta(\xi) \rightarrow \Delta_t(\xi) = F_{12}\Delta(\xi)F_{12}^{-1}, \quad F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}. \quad (B.1)$$

The universal $\mathcal{R}$-matrix then transforms as

$$\mathcal{R} \rightarrow \mathcal{R}_t = F_{21}\mathcal{R}F_{12}^{-1}. \quad (B.2)$$

Recall how non-heromicity arose as a consequence of Baxterization of the parameter-less universal $\mathcal{R}$-matrix, with the introduction of the $\lambda$-dependent factors in its off-diagonal elements, thus producing extra unwanted terms in the interaction $h \sim \partial_\lambda \tilde{R}(\lambda)|_{\lambda=0}$. Following the procedure proposed in [46], the $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ fundamental $\mathcal{R}$-matrix
transforms via the $\beta$-dependent diagonal universal twisting element $K_\lambda = \exp(\beta \lambda S^z)$ in the space $\mathfrak{S}_s$

$$R_\beta(\lambda) = (\mathbb{I} \otimes K_\beta(\lambda)) R(\lambda) (\mathbb{I} \otimes K_\beta(\lambda))^{-1}.$$  \hfill (B.3)

Evaluating (B.3) in the fundamental representation $\mathfrak{S}_f$, i.e. using $K_\beta(f)(\lambda) = \exp(\gamma \lambda / 2)\sigma^z$, with the choice $\beta = \gamma$, yields

$$R_{\gamma}^{(f,f)}(\lambda) = (q - q^{-1}) \begin{pmatrix} [-i\lambda + 1]_q & [-i\lambda]_q \\ [-i\lambda]_q & [-i\lambda + 1]_q \end{pmatrix}.$$  \hfill (B.4)

Accordingly, the $L$-operator can be modified by twisting in the quantum space $\mathfrak{S}_f$ only,

$$L_\beta(\lambda) = e^{\beta \lambda S^z} L(\lambda) e^{-\beta \lambda S^z},$$  \hfill (B.5)

which is an automorphism of $U_q(\mathfrak{sl}(2,C))$ algebra, $S^\pm \to e^{\pm \beta \lambda} S^\pm$, $S^z \to S^z$, yielding at $\beta = \gamma$ the standard $U_q(\mathfrak{sl}(2,C))$-covariant $L$-operator of the XXZ Heisenberg model

$$L_{XXZ}(\lambda) = \begin{pmatrix} [-i\lambda + S^z]_q & S^- \\ S^+ & [-i\lambda - S^z]_q \end{pmatrix}.$$  \hfill (B.6)

It is noteworthy that the applied $\beta$-twist breaks the $U_q(\mathfrak{sl}(2,C))$ symmetry of the (non-Hermitian) interaction (4.22) down to $U(1)$.

**B.1. Twisted Heisenberg model: $\Theta$-XXZ model**

Twist transformations allow one to introduce an extra parameter dependence into solutions of the YBE (see e.g. [27]). As a simple example we show how to transform the trigonometric six-vertex $R$-matrix by means of a diagonal matrix depending on the angle parameter $\Theta \in [0, 2\pi)$. In order to generate the asymmetric (Wu–McCoy) anisotropic Heisenberg model with vector-like (Dzyaloshinskii–Moriya) interaction we have to choose an Abelian (Reshetikhin) twist [47, 48],

$$F_\Theta = e^{-(i\Theta/2)(S^z \otimes \mathbb{1} - \mathbb{1} \otimes S^z)},$$  \hfill (B.7)

evaluated in the product of fundamental representations $\mathfrak{S}_f \otimes \mathfrak{S}_f$,

$$F_{\Theta}^{(f,f)} = \text{diag}(1, e^{-i\Theta/2}, e^{i\Theta/2}, 1),$$  \hfill (B.8)

which produces the following ‘colored’ trigonometric six-vertex $R$-matrix ($\varphi := -i\gamma \lambda$)

$$\tilde{R}_{\Theta}^{(f,f)}(\lambda) = P_{12} R_{\Theta}^{(f,f)}(\lambda) = P_{12} F_{\Theta}^{(f,f)} R_{\Theta}^{(f,f)}(\lambda) F_{\Theta}^{(f,f)}$$  \hfill (B.9)

$$= \frac{2i}{\gamma} \begin{pmatrix} \sin (\varphi + \gamma) & 0 & 0 & 0 \\ 0 & \sin \gamma & e^{i\Theta} \sin \varphi & 0 \\ 0 & e^{-i\Theta} \sin \varphi & \sin \gamma & 0 \\ 0 & 0 & 0 & \sin (\varphi + \gamma) \end{pmatrix}. $$  \hfill (B.10)
Taking the derivative with respect to $\lambda$ at $\lambda = 0$, we extract the interaction with the $\Theta$-modified hopping term,

$$h_{12}^\Theta = (e^{i\Theta} \sigma_1^+ \sigma_2^- + e^{-i\Theta} \sigma_1^- \sigma_2^+) + 2 \cos(\gamma) \sigma_1^z \sigma_2^z + \text{const.},$$

(B.11)

whereas the Lax operator gets transformed into its $\Theta$-twisted correspondent via a universal twist $F^\Theta_{fs}$ evaluated in $S_f \otimes S_s$, i.e.

$$F^\Theta_{fs}(\lambda) = e^{-i(\Theta/4)(\sigma^- \otimes 1)} e^{-i(\Theta/2)(\sigma^0 \otimes S^z)},$$

yielding

$$L^\Theta(\lambda) = F^\Theta_{fs} L(\lambda) F^\Theta_{fs} = \left( \begin{bmatrix} e^{i\Theta(S^z-1/2)} & e^{i\Theta(S^z+1/2)} \\ S^z e^{i\Theta(S^z+1/2)} & e^{i\Theta(S^z+1/2)} \end{bmatrix} \right).$$

(B.12)

The generalized anisotropic Hamiltonian (B.11) which introduces an electric-field term (meanwhile reducing hopping interaction) brings a possibility of studying an interplay between spin currents contributed from the (coherent) bulk interaction and incoherent processes caused by an external magnetization bias, avoiding the use of twisted boundary fields [49, 50]. Curiously, regardless of the parameter $\Theta$, solutions to the boundary equations (4.11) remain unaffected, therefore extending a continuous family of solutions for spin-1/2 chains with maximal incoherent driving provided in section 4.

Appendix C. Commuting transfer matrices

As we already argued in section 4, the introduced $S$-operators factorizing NESS density operators $\rho_\infty$ are in fact, unlike the operators $\rho_\infty$ themselves, abstract non-Hermitian quantum transfer matrices subjected to open boundary conditions. This property is attributed to the existence of the universal $R$-matrix for the Lax operator pertaining to general non-compact spin representations. We shall briefly outline their origin in this section. In this regard, we might (loosely speaking) proclaim the type of solutions presented in the preceding discussion as integrable steady states, based on the fact that they can be thought of as two ‘fused’ lowest weight transfer matrices. Pretty remarkably, those infinite-dimensional spin representations have previously found their places and applications in supersymmetric quantum field theories, in particular in the integrable sector of the high-energy QCD [51]. Here, in contrast, we emphasize their importance in paradigmatic models of (non-relativistic) quantum statistical mechanics, i.e. in the realm of non-canonical mixed-states associated with driven open (one-dimensional) quantum systems.

C.1. Universal $R$-matrix for arbitrary complex-valued spin

We begin by acknowledging a general type of solution of the YBE over triple-product space $S_{s_1} \otimes S_{s_2} \otimes S_{s_3}$,

$$R^{(s_1,s_2)}_{12}(\lambda - \mu) R^{(s_2,s_3)}_{13}(\lambda) R^{(s_3,s_1)}_{23}(\mu) = R^{(s_3,s_1)}_{23}(\mu) R^{(s_2,s_3)}_{13}(\lambda) R^{(s_1,s_2)}_{12}(\lambda - \mu),$$

(C.1)

investigated, for example, in [52]–[55], [24]. We shall refrain from using our boldface convention henceforth. For instance, setting all three representation parameters to...
s_j = \frac{1}{2} = f, i.e. evaluating (C.1) in the three-fold fundamental space $\mathfrak{g}^3_f$, yields the fundamental Yang–Baxter equation

$$R^{(f,f)}_{12}(\lambda - \mu) R^{(f,f)}_{13}(\lambda) R^{(f,f)}_{23}(\mu) = R^{(f,f)}_{23}(\mu) R^{(f,f)}_{13}(\lambda) R^{(f,f)}_{12}(\lambda - \mu),$$

(C.2)

with the well-known $\mathfrak{sl}(2, \mathbb{C})$-invariant solution $R^{(f,f)}_{12}(u) = u + P_{12}$, where the permutation operator $P_{12}$ is over a two-fold product of fundamental spaces $\mathfrak{g}_f \otimes \mathfrak{g}_f \cong \mathbb{C}^2 \otimes \mathbb{C}^2$. Next, choosing $s_1 = s_2 = f$ and $s_3 = s$ we have, besides the standard rational $4 \times 4$ $R$-matrix $R^{(f,f)}(u)$, another operator $R^{(f,s)}(u)$, acting in $\mathfrak{g}_f \otimes \mathfrak{g}_s$, which reads explicitly

$$R^{(f,s)}(u) = (u + \frac{1}{2}) I + \vec{s} \otimes \vec{S} = \left( \frac{(u + \frac{1}{2})I + S^z}{S^+} \left( u + \frac{1}{2} \right)I + S^z \right).$$

(C.3)

The latter is essentially nothing but a standard Lax operator for spin-$s$ (with $\mathfrak{g}_f \cong \mathbb{C}^2$ auxiliary space) from the RLL-relation, modulo an additive constant and shift in the spectral parameter,

$$L^{(f,s)}(u) = R^{(f,s)}(u - \frac{1}{2}),$$

(C.4)

being equivalent to the defining relations of spin-$s$ generators. The third form is obtained by realizing only one space in the fundamental representation, say $s_3 = f$. In this case we find YBE in the form

$$R^{(s_1,s_2)}_{12}(\lambda - \mu) R^{(s_1,f)}_{13}(\lambda) R^{(s_2,f)}_{23}(\mu) = R^{(s_2,f)}_{23}(\mu) R^{(s_1,f)}_{13}(\lambda) R^{(s_1,s_2)}_{12}(\lambda - \mu),$$

(C.5)

which involves a general $\mathfrak{sl}(2, \mathbb{C})$-invariant $R$-operator,

$$[\vec{S}, R^{(s_1,s_2)}(u)] = 0,$$

(C.6)

over the product space $\mathfrak{g}_{s_1} \otimes \mathfrak{g}_{s_2}$ of two arbitrary representations. Those general solutions, addressed from a representation-theoretic point of view, for example, in [56, 57], ensure mutual commutativity for general-type transfer matrices, obtained as (regularized) traces over generic spin representations [58], and are thus related to non-compact spin chains [51, 59]. In parallel to those, solutions to (C.5) have also been constructed for generic representations (i.e. assuming $s_1, s_2 \notin \frac{1}{2} \mathbb{Z}_+$, for otherwise the finite-dimensional representation theory applies [57, 53, 54]) by means of an eigenspace decomposition

$$R^{(s_1,s_2)}(u) \sim \sum_{\nu=0}^{\infty} r_{\nu}(u) P_{\nu}, \quad R^{(s_1,s_2)}(u)\psi^0_{\nu} = r_{\nu}(u)\psi^0_{\nu},$$

(C.7)

where $P_{\nu}$ is a projector from $\mathfrak{g}_{s_1} \otimes \mathfrak{g}_{s_2}$ to subspace $\mathfrak{g}_{s_1+s_2+s_3}$, via the recurrence relation

$$(u + s_1 + s_2 - \nu) r_{\nu+1}(u) = -(u + s_1 + s_2 - \nu) r_{\nu}(u),$$

(C.8)

with explicit solution,

$$r_{\nu}(u) = (-1)^\nu \frac{\Gamma(u + s_1 + s_2) \Gamma(-u + s_1 + s_2 - \nu)}{\Gamma(-u + s_1 + s_2) \Gamma(u + s_1 + s_2 - \nu)},$$

(C.9)

using normalization such that $R^{(s_1,s_2)}(u)|0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle$, where $|0\rangle \otimes |0\rangle = \psi^0_{\nu} = 1$. 

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On the other hand, in the absence of cyclic invariance, our $S$-operator, which is defined as a lowest weight contraction of the monodromy matrix $T(\lambda)$ rather than a partial trace $\text{Tr}_a$ over $\mathcal{H}_a \cong \mathfrak{S}_{s_3}$, also displays the transfer matrix property. This intriguing result, being observed only recently [16], is based on preservation of left/right lowest weight states $\langle 0, 0 | \equiv \langle 0 | \otimes \langle 0 |$ and $| 0 \rangle \equiv | 0 | \otimes | 0 \rangle$,

$$\hat{R}^{(s_1, s_2)}(u)|0, 0\rangle = |0, 0\rangle, \quad \langle 0, 0|\hat{R}^{(s_1, s_2)}(u) = \langle 0, 0|, \quad (C.10)$$

implying

$$S(\lambda_1, s_1)S(\lambda_2, s_2) = \langle 0, 0|T_1^{(s_1)}(\lambda_1)T_2^{(s_2)}(\lambda_2)|0, 0\rangle = \langle 0, 0|\hat{R}_{12}^{(s_1, s_2)}(\lambda_1 - \lambda_2)T_1^{(s_1)}(\lambda_1)T_2^{(s_2)}(\lambda_2)|0, 0\rangle = \langle 0, 0|T_1^{(s_2)}(\lambda_2)T_2^{(s_1)}(\lambda_1)\hat{R}_{12}^{(s_1, s_2)}(\lambda_1 - \lambda_2)|0, 0\rangle = S(\lambda_2, s_2)S(\lambda_1, s_1). \quad (C.11)$$

Although we focused entirely on the non-deformed case, the entire consideration above is applicable to $q$-analogs as well [24], and of course generalizable to the Lie algebra $\mathfrak{sl}(N, \mathbb{C})$ with $N$ complex parameters (or even superalgebra), namely a central element associated with the spectral parameter and the remaining $(N - 1)$ $\mathfrak{sl}(N, \mathbb{C})$ representation parameters.

C.2. Coherent boundary vectors

Clearly, an extension to generalized transfer matrices contracted with left and right coherent states,

$$|\phi_R(s)\rangle := e^{\phi_R S^+(s)}|0\rangle, \quad \langle \phi_L(s)| := \langle 0|e^{\phi_L S^-(s)}, \quad \phi_{L, R} \in \mathbb{C}, \quad (C.12)$$

is straightforward, i.e. a family of $S$-operators

$$S(\lambda, s|\phi_L, \phi_R) \sim \langle \phi_L(s)|T^{(s)}(\lambda)|\phi_R(s)\rangle, \quad (C.13)$$

retains the commutativity property,

$$[S(\lambda, s_1|\phi_L, \phi_R), S(\mu, s_2|\phi_L, \phi_R)] = 0, \quad (C.14)$$

simply as a consequence of the $\mathfrak{sl}(2, \mathbb{C})$ invariance (C.6). Hence, the argument from (C.11) still applies.

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