Linking Classical and Quantum Key Agreement: 
Is There “Bound Information”? 

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Abstract

After carrying out a protocol for quantum key agreement over a noisy quantum channel, the parties Alice and Bob must process the raw key in order to end up with identical keys about which the adversary has virtually no information. In principle, both classical and quantum protocols can be used for this processing. It is a natural question which type of protocols is more powerful. We show that in many cases, for instance when the 4-state or the 6-state protocol is used, the limits of tolerable noise are identical for classical and quantum protocols. More precisely, we prove for general states but under the assumption of incoherent eavesdropping that Alice and Bob share some so-called intrinsic information in their classical random variables, resulting from optimal measurements, if and only if the parties’ quantum systems are entangled. In addition, we provide evidence which strongly suggests that the potentials of classical and of quantum protocols are equal in every situation. It is an important consequence of these parallels that many techniques and results from quantum information theory directly apply to problems in classical information theory, and vice versa. For instance, it was previously believed that two parties can carry out unconditionally secure key agreement as long as they share some intrinsic information in the adversary’s view. The analysis of this purely classical problem from the quantum information-theoretic viewpoint shows that this is true in the binary case, but false in general. More explicitly, bound entanglement, i.e., entanglement that cannot be purified by any quantum protocol, has a classical counterpart. This “bound intrinsic information” cannot be distilled to a secret key by any classical protocol. As another application we propose a measure for entanglement based on classical information-theoretic quantities.
Keywords. Key agreement, quantum cryptography, quantum privacy amplification, purification, entanglement, intrinsic mutual information, secret-key rate, information theory.

1 Introduction

In modern cryptography there are mainly two security paradigms, namely computational and information-theoretic security. The latter is sometimes also called unconditional security. Computational security is based on the assumed hardness of certain computational problems (e.g., the integer-factoring or discrete-logarithm problems). However, since a computationally sufficiently powerful adversary can solve any computational problem, hence break any such system, and because no useful general lower bounds are known in complexity theory, computational security is always conditional and, in addition to this, in danger by progress in the theory of efficient algorithms as well as in hardware engineering (e.g., quantum computing).

Information-theoretic security on the other hand is based on probability theory and on the fact that an adversary’s information is limited. Such a limitation can for instance come from noise in communication channels or from the laws of quantum mechanics.

Many different settings in the classical noisy-channel model have been described and analyzed, such as Wyner’s wire-tap channel [29], Csiszár and Körner’s broadcast channel [7], or Maurer’s key agreement from joint randomness [20], [22].

Quantum cryptography on the other hand lies in the intersection of two of the major scientific achievements of the 20th century, namely quantum physics and information theory. Various protocols for so-called quantum key agreement have been proposed (e.g., [3], [10]), and the possibility and impossibility of purification in different settings has been studied by many authors.

The goal of this paper is to derive parallels between classical and quantum key agreement and thus to show that the two paradigms are more closely related than previously recognized. These connections allow for investigating questions and solving open problems of purely classical information theory with quantum-mechanic methods. One of the consequences is that, in contrast to what was previously believed, there exists a classical counterpart to so-called bound entanglement (i.e., entanglement that cannot be purified by any quantum protocol), namely intrinsic information shared by Alice and Bob which they cannot use for generating a secret key by any
classical protocol.

The outline of the paper is as follows. In Section 2 we introduce the classical (Section 2.1) and quantum (Section 2.2) models of information-theoretic key agreement and the crucial concepts and quantities, such as secret-key rate and intrinsic information on one side, and measurements, entanglement, and quantum privacy amplification on the other. In Section 3, we show the mentioned links between these two models, more precisely, between entanglement and intrinsic information (Section 3.1) as well as between quantum purification and the secret-key rate (Section 3.4). We illustrate the statements and their consequences with a number of examples (Sections 3.2 and 3.5 and Appendix B). In Section 3.6 we define and characterize the classical counterpart of bound entanglement, called bound intrinsic information. Finally we show that not only problems in classical information theory can be addressed by quantum-mechanical methods, but that the inverse is also true: In Section 3.3 we propose a new measure for entanglement based on the intrinsic information measure.

2 Models of Information-Theoretically Secure Key Agreement

2.1 Key Agreement from Classical Information: Intrinsic Information and Secret-Key Rate

In this section we describe Maurer’s general model of classical key agreement by public discussion from common information [20]. In this setting, two parties Alice and Bob who are willing to generate a secret key have access to repeated independent realizations of (classical) random variables $X$ and $Y$, respectively, whereas an adversary Eve learns the outcomes of a random variable $Z$. Let $P_{XYZ}$ be the joint distribution of the three random variables. In addition, Alice and Bob are connected by a noiseless and authentic but otherwise completely insecure channel (see Figure 1 in Appendix A). In this situation, the secret-key rate $S(X; Y || Z)$ has been defined as the maximal rate at which Alice and Bob can generate a secret key that is equal for Alice and Bob with overwhelming probability and about which Eve has only a negligible amount of (Shannon) information. For a detailed discussion of the general scenario and the secret-key rate as well as for various bounds on $S(X; Y || Z)$, see [20], [21], [22].

Bound (1) implies that if Bob’s random variable $Y$ provides more information about Alice’s $X$ than Eve’s $Z$ does (or vice versa), then this
advantage can be exploited for generating a secret key:

\[ S(X; Y|Z) \geq \max \{ I(X; Y) - I(X; Z), I(Y; X) - I(Y; Z) \} \]  

(1)

This is a consequence of a result by Csiszár and Körner [7]. It is somewhat surprising that this bound is not tight, in particular, that secret-key agreement can even be possible when the right-hand side of (1) vanishes or is negative. However, the positivity of the expression on the right-hand side of (1) is a necessary and sufficient condition for the possibility of secret-key agreement by one-way communication: whenever Alice and Bob start in a disadvantageous situation with respect to Eve, feedback is necessary. The corresponding initial phase of the key-agreement protocol is then often called advantage distillation.

The following upper bound on \( S(X; Y|Z) \) is a generalization of Shannon’s well-known impracticality theorem [28] and quantifies the intuitive fact that no information-theoretically secure key agreement is possible when Bob’s information is independent from Alice’s random variable, given Eve’s information: \( S(X; Y|Z) \leq I(X; Y|Z) \). However, this bound is not tight. Because it is a possible strategy of the adversary Eve to process \( Z \), i.e., to send \( Z \) over some channel characterized by \( P_{Z|Z} \), we have for such a new random variable \( \overline{Z} \) that \( S(X; Y|Z) \leq I(X; Y|\overline{Z}) \), and hence

\[ S(X; Y|Z) \leq \min_{P_{Z|Z}} \{ I(X; Y|\overline{Z}) \} =: I(X; Y\downarrow Z) \]  

(2)

holds. The quantity \( I(X; Y\downarrow Z) \) has been called the intrinsic conditional information between \( X \) and \( Y \) given \( Z \) [22]. It was conjectured, and evidence supporting this belief was given, that \( S(X; Y|Z) > 0 \) holds if \( I(X; Y\downarrow Z) > 0 \) does [22]. Some of the results below strongly suggest that this is true if one of the random variables \( X \) and \( Y \) is binary and the other one at most ternary, but false in general.

2.2 Quantum Key Agreement: Measurements, Entanglement, Purification

We assume that the reader is familiar with the basic quantum-theoretic concepts and notations. For an introduction, see for example [24].

In the context of quantum key agreement, the classical scenario \( P_{XYZ} \) is replaced by a quantum state vector \( \Psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E \), where \( \mathcal{H}_A \),

\[^{1}\text{We consider pure states, since it is natural to assume that Eve controls all the environment outside Alice and Bob’s systems.}\]
$H_B$, and $H_E$ are Hilbert spaces describing the systems in Alice’s, Bob’s, and Eve’s hands, respectively. Then, measuring this quantum state by the three parties leads to a classical probability distribution. In the following, we assume that Eve is free to carry out so-called generalized measurements (POVMs) \[24\]. In other words, the set $\{|z\rangle\}$ will not be assumed to be an orthonormal basis, but any set generating the Hilbert space $H_E$ and satisfying the condition $\sum_z |z\rangle\langle z| = 1_{H_E}$. Then, if the three parties carry out measurements in certain bases\[2\] $\{|x\rangle\}$ and $\{|y\rangle\}$, and in the set $\{|z\rangle\}$, respectively, they end up with the classical scenario $P_{XYZ} = |\langle x, y, z | \Psi \rangle|^2$. Since this distribution depends on the chosen bases and set, a given quantum state $\Psi$ does not uniquely determine a classical scenario: some measurements may lead to scenarios useful for Alice and Bob, whereas for Eve, some others may (see Appendix B).

The analog of Alice and Bob’s marginal distribution $P_{XY}$ is the partial state $\rho_{AB}$, obtained by tracing over Eve’s Hilbert space $H_E$. More precisely, let $\Psi = \sum_{xyz} c_{xyz} |x, y, z\rangle$, where $|x, y, z\rangle$ is short for $|x\rangle \otimes |y\rangle \otimes |z\rangle$. We can write $\Psi = \sum_z \sqrt{P_Z(z)} \psi_z \otimes |z\rangle$, where $P_Z$ denotes Eve’s marginal distribution of $P_{XYZ}$. Then $\rho_{AB} = Tr_{H_E}(P_\Psi) := \sum_z P_Z(z) \psi_z$, where $P_\psi_z$ is the projector to the state vector $\psi_z$.

An important property is that $\rho_{AB}$ is pure ($\rho_{AB}^2 = \rho_{AB}$) if and only if the global state $\Psi$ factorizes, i.e., $\Psi = \psi_{AB} \otimes \psi_E$, where $\psi_{AB} \in H_A \otimes H_B$ and $\psi_E \in H_E$. In this case Alice and Bob are independent of Eve: Eve cannot obtain any information on Alice’s and Bob’s states by measuring her system.

After a measurement, Alice and Bob obtain a classical distribution $P_{XY}$. However, in order to obtain a well-defined classical scenario one has to assume that also Eve performs a measurement, i.e., that Eve treats her information on the classical level. Indeed, only then a classical distribution $P_{XYZ}$ is defined. But considering that in practice all $P_{XYZ}$ result from some physical process, the assumption that Eve performs the measurement one would like her to perform is not founded on basic principles\[3\]. For example, Eve’s measurement could be done later and depend on the public discussion between Alice and Bob. Consequently, the common approach which starts from $P_{XYZ}$ to prove the security of a key agreement protocol hides an assumption about Eve’s measurement. As we shall see, avoiding this hidden

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\[2\]We assume all bases to be orthonormal.

\[3\]One could argue that if the system in Eve’s hand is classical, then she has no choice for her measurement. But ultimately all systems are quantum mechanical and the apparent lack of choice might purely be a matter of technology.
assumption and staying in the quantum regime can actually simplify the analysis of the scenario.

When Alice and Bob share many independent systems $\rho_{AB}$, there are basically two possibilities for generating a secret key. Either they first measure their systems and then run a classical protocol (process classical information) secure against all measurements Eve could possibly perform (i.e., against all possible distributions $P_{XYZ}$ that can result after Eve’s measurement). Or they first run a quantum protocol (i.e., process the information in the quantum domain) and then perform their measurements. The idea of quantum protocols is to process the systems in state $\rho_{AB}$ and to produce fewer systems in a pure state (i.e., to purify $\rho_{AB}$), thus to eliminate Eve from the scenario. Moreover, the pure state Alice and Bob end up with should be maximally entangled (i.e., even for some different and incompatible measurements, Alice’s and Bob’s results are perfectly correlated). Finally, Alice and Bob measure their maximally entangled systems and establish a secret key. This way of obtaining a key directly from a quantum state $\Psi$, without any error correction nor classical privacy amplification, is called quantum privacy amplification (QPA for short) \cite{8}, \cite{2}. Note that the procedure described in \cite{8} and \cite{2} guarantees that Eve’s relative information (relative to the key length) is arbitrarily small, but not that her absolute information is negligible. The analog of this problem in the classical case is discussed in \cite{21}.

The precise conditions under which a general state $\rho_{AB}$ can be purified are not known. However, the two following conditions are necessary. First, the state must be entangled or, equivalently, not separable. A state $\rho_{AB}$ is separable if and only if it can be written as a mixture of product states, i.e., $\rho_{AB} = \sum_j p_j \rho_{A_j} \otimes \rho_{B_j}$. Separable states can be generated by purely classical communication, hence it follows from bound (2) that entanglement is a necessary condition. The second condition is more subtle: The matrix $\rho'_{AB}$ obtained from $\rho_{AB}$ by partial transposition must have at least one negative eigenvalue \cite{17}, \cite{16}. The partial transposition of the density matrix $\rho_{AB}$ is defined as $(\rho_{AB})_{i,j,\mu,\nu}^{\dagger} := (\rho_{AB})_{i,\nu,\mu,j}$, where the indices $i$ and $\mu$ [j and $\nu$] run through a basis of $\mathcal{H}_A$ [$\mathcal{H}_B$]. Note that this definition is base-dependent. However, the eigenvalues of $\rho'_{AB}$ are not \cite{25}. The second of these conditions

\footnote{Here we do not consider the possibility that Eve coherently processes several of her systems. This corresponds to the assumption in the classical scenario that repeated realizations of $X$, $Y$, and $Z$ are independent of each other.}

\footnote{The term “quantum privacy amplification” is somewhat unfortunate since it does not correspond to classical privacy amplification, but includes advantage distillation and error correction.}
implies the first one: negative (i.e., at least one eigenvalue is negative) partial transposition implies entanglement.

In the binary case ($\mathcal{H}_A$ and $\mathcal{H}_B$ both have dimension two), the above two conditions are equivalent and sufficient for the possibility of quantum key agreement: all entangled binary states can be purified. The same even holds if one Hilbert space is of dimension 2 and the other one of dimension 3. However, for larger dimensions there are examples showing that these conditions are not equivalent: There are entangled states whose partial transpose has no negative eigenvalue, hence cannot be purified [17]. Such states are called bound entangled, in contrast to free entangled states, which can be purified. Moreover, it is believed that there even exist entangled states which cannot be purified although they have negative partial transposition [4].

3 Linking Classical and Quantum Key Agreement

In this section we derive a close connection between the possibilities offered by classical and quantum protocols for key agreement. The intuition is as follows. As described in Section 2.2, there is a very natural connection between quantum states $\Psi$ and classical distributions $P_{XYZ}$ which can be thought of as arising from $\Psi$ by measuring in a certain basis, e.g., the standard basis. (Note however that the connection is not unique even for fixed bases: For a given distribution $P_{XYZ}$, there are many states $\Psi$ leading to $P_{XYZ}$ by carrying out measurements.) When given a state $\Psi$ between three parties Alice, Bob, and Eve, and if $\rho_{AB}$ denotes the resulting mixed state after tracing out Eve, then the corresponding classical distribution $P_{XYZ}$ has positive intrinsic information if and only if $\rho_{AB}$ is entangled. However, this correspondence clearly depends on the measurement bases used by Alice, Bob, and Eve. If for instance $\rho_{AB}$ is entangled, but Alice and Bob do very unclever measurements, then the intrinsic information may vanish (see Example 7 in Appendix B). If on the other hand $\rho_{AB}$ is separable, Eve may do such bad measurements that the intrinsic information becomes positive, despite the fact that $\rho_{AB}$ could have been established by public discussion without any prior correlation (see Example 6 in Appendix B). Consequently, the correspondence between intrinsic information and entan-
glement must involve some optimization over all possible measurements on all sides.

A similar correspondence on the protocol level is supported by many examples, but not rigorously proven: The distribution $P_{XYZ}$ allows for classical key agreement if and only if quantum key agreement is possible starting from the state $\rho_{AB}$.

We show how these parallels allow for addressing problems of purely classical information-theoretic nature with the methods of quantum information theory, and vice versa.

### 3.1 Entanglement and Intrinsic Information

Let us first establish the connection between intrinsic information and entanglement. Theorem 1 states that if $\rho_{AB}$ is separable, then Eve can “force” the information between Alice’s and Bob’s classical random variables (given Eve’s classical random variable) to be zero (whatever strategy Alice and Bob use). In particular, Eve can prevent classical key agreement.

**Theorem 1** Let $\Psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$ and $\rho_{AB} = \text{Tr}_E (P \Psi)$. If $\rho_{AB}$ is separable, then there exists a generating set $\{|z\rangle\}$ of $\mathcal{H}_E$ such that for all bases $\{|x\rangle\}$ and $\{|y\rangle\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, $I(X;Y|Z) = 0$ holds for $P_{XYZ}(x,y,z) := |\langle x,y,z|\Psi \rangle|^2$.

**Proof.** If $\rho_{AB}$ is separable, then there exist vectors $|\alpha_z\rangle$ and $|\beta_z\rangle$ such that $\rho_{AB} = \sum_{z=1}^{dim} p_z P_{\alpha_z} \otimes P_{\beta_z}$, where $P_{\alpha_z}$ denotes the one-dimensional projector onto the subspace spanned by $|\alpha_z\rangle$.

Let us first assume that $n_z \leq \dim \mathcal{H}_E$. Then there exists a basis $\{|z\rangle\}$ of $\mathcal{H}_E$ such that $\Psi = \sum_z \sqrt{p_z} |\alpha_z, \beta_z, z\rangle$ holds [23], [12], [19].

If $n_z > \dim \mathcal{H}_E$, then Eve can add an auxiliary system $\mathcal{H}_{aux}$ to hers (usually called an **ancilla**) and we have $\Psi \otimes |\gamma_0\rangle = \sum_z \sqrt{p_z} |\alpha_z, \beta_z, z, \gamma_0\rangle$, where $|\gamma_0\rangle \in \mathcal{H}_{aux}$ is the state of Eve’s auxiliary system, and $\{|\gamma_z\rangle\}$ is a basis of $\mathcal{H}_E \otimes \mathcal{H}_{aux}$. We define the (not necessarily orthonormalized) vectors $|z\rangle$ by $|z, \gamma_0\rangle = 1_{\mathcal{H}_E} \otimes P_{\gamma_0} |\gamma_0\rangle$. These vectors determine a generalized measurement with positive operators $O_z = |z\rangle \langle z|$. Since $\sum_z O_z \otimes P_{\gamma_0} = \sum_z |z, \gamma_0\rangle \langle z, \gamma_0| = \sum_z 1_{\mathcal{H}_E} \otimes P_{\gamma_0} |\gamma_z\rangle \langle \gamma_z| \sum_z 1_{\mathcal{H}_E} \otimes P_{\gamma_0} = 1_{\mathcal{H}_E} \otimes P_{\gamma_0}$, the $O_z$ satisfy $\sum_z O_z = 1_{\mathcal{H}_E}$, as they should in order to define a generalized measurement [24]. Note that the first case ($n_z \leq \dim \mathcal{H}_E$) is a special case of the second one, with $|\gamma_z\rangle = |z, \gamma_0\rangle$. If Eve now performs the measurement, then we have $P_{XYZ}(x,y,z) = |\langle x,y,z|\Psi \rangle|^2 = |\langle x,y,\gamma_z|\Psi,\gamma_0\rangle|^2$, and

$$P_{XYZ}(x,y,z) = |\langle x,y|\alpha_z,\beta_z\rangle|^2 = |\langle x|\alpha_z\rangle|^2 |\langle y|\beta_z\rangle|^2 = P_X|Z(x,z)P_Y|Z(y,z)$$
holds for all \(|z\rangle\) and for all \(|x,y\rangle\in \mathcal{H}_A \otimes \mathcal{H}_B\). Consequently, \(I(X;Y|Z) = 0\). 

\[\square\]

Theorem \(2\) states that if \(\rho_{AB}\) is entangled, then Eve cannot force the intrinsic information to be zero: Whatever she does (i.e., whatever generalized measurements she carries out), there is something Alice and Bob can do such that the intrinsic information is positive. Note that this does not, a priori, imply that secret-key agreement is possible in every case. Indeed, we will provide evidence for the fact that this implication does generally not hold.

**Theorem 2** Let \(\Psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E\) and \(\rho_{AB} = \text{Tr}_{\mathcal{H}_E}(P_{\Psi})\). If \(\rho_{AB}\) is entangled, then for all generating sets \(\{|z\rangle\}\) of \(\mathcal{H}_E\), there are bases \(\{|x\rangle\}\) and \(\{|y\rangle\}\) of \(\mathcal{H}_A\) and \(\mathcal{H}_B\), respectively, such that I\((X;Y \downarrow Z) > 0\) holds for \(P_{XYZ}(x,y,z) := |\langle x,y,z|\Psi \rangle|^2\).

**Proof.** We prove this by contradiction. Assume that there exists a generating set \(\{|z\rangle\}\) of \(\mathcal{H}_E\) such that for all bases \(\{|x\rangle\}\) of \(\mathcal{H}_A\) and \(\{|y\rangle\}\) of \(\mathcal{H}_B\), \(I(X;Y \downarrow Z) = 0\) holds for the resulting distribution. For such a distribution, there exists a channel, characterized by \(P_{Z|X,Y}\), such that \(I(X;Y|Z) = 0\) holds, i.e.,

\[P_{XZ|Y}(x,z) = P_{X|Z}(x,z)P_{Z|Y}(z)\, .\]  

(3)

Let now \(\rho_z := (1/p_z) \sum_p p_z P_{Z|X}(z) P_{\Psi}\) with \(p_z = P_{Z}(z)\) and \(p_{\bar{z}} = \sum_z P_{Z|X}(z)z\), and where \(\psi_z\) is the state of Alice’s and Bob’s system conditioned on Eve’s result \(z: \Psi \otimes |\gamma_0\rangle = \sum_z \psi_z \otimes |\gamma_z\rangle\) (see the proof of Theorem 1).

From (3) we can conclude \(\text{Tr}(P_z \otimes P_y \rho_z) = \text{Tr}(P_{\bar{z}} \otimes \mathbb{1} \rho_{\bar{z}}) \text{Tr}(\mathbb{1} \otimes P_y \rho_{\bar{z}})\) for all one-dimensional projectors \(P_x\) and \(P_y\) acting in \(\mathcal{H}_A\) and \(\mathcal{H}_B\), respectively. Consequently, the states \(\rho_{z}\) are products, i.e., \(\rho_{z} = \rho_{\bar{z}} \otimes \rho_{\bar{z}}\), and \(\rho_{AB} = \sum_z p_z \rho_z\) is separable. \(\square\)

Theorem \(2\) can be formulated in a more positive way. Let us first introduce the concept of a set of bases \(\{|x^j\rangle, |y^j\rangle\}\), where the \(j\) label the different bases, as they are used in the 4-state (2 bases) and the 6-state (3 bases) protocols \(3\), \(4\), \(1\). Then if \(\rho_{AB}\) is entangled there exists a set \(\{|x^j\rangle, |y^j\rangle\}\) of \(N\) bases such that for all generalized measurements \(\{|z\rangle\}\), \(I(X;Y \downarrow [Z,j]) > 0\) holds. The idea is that Alice and Bob randomly choose a basis and, after the transmission, publicly restrict to the (possibly few) cases where they happen to have chosen the same basis. Hence Eve knows \(j\), and one has \(I(X;Y \downarrow [Z,j]) = (1/N) \sum_{j=1}^N I(X^j;Y^j \downarrow Z)\). If the set of bases is large enough, then for all \(\{|z\rangle\}\) there is a basis with positive intrinsic information.
hence the mean is also positive. Clearly, this result is stronger if the set of bases is small. Nothing is proven about the achievable size of such sets of bases, but it is conceivable that max{dim $\mathcal{H}_A$, dim $\mathcal{H}_B$} bases are always sufficient.

**Corollary 3** Let $\Psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$ and $\rho_{AB} = \text{Tr}_{\mathcal{H}_E}(P_{\Psi})$. Then the following statements are equivalent:

(i) $\rho_{AB}$ is entangled,

(ii) for all generating sets $\{|z\rangle \}$ of $\mathcal{H}_E$, there exist bases $\{|x\rangle \}$ of $\mathcal{H}_A$ and $\{|y\rangle \}$ of $\mathcal{H}_B$ such that the distribution $P_{XYZ}(x,y,z) := |\langle x,y,z|\Psi \rangle|^2$ satisfies $I(X;Y\downarrow Z) > 0$.

A first consequence of the fact that Corollary 3 often holds with respect to the standard bases (see below) is that it yields, at least in the binary case, a criterion for $I(X;Y\downarrow Z) > 0$ that is efficiently verifiable since it is based on the positivity of the eigenvalues of a $4 \times 4$ matrix (see also Example 5). Previously, the quantity $I(X;Y\downarrow Z)$ has been considered to be quite hard to handle.

### 3.2 Examples I

The following examples illustrate the correspondence established in Section 3.1. They show in particular that very often (Examples 1, 2, and 3), but not always (Examples 6 and 7 in Appendix B), the direct connection between entanglement and positive intrinsic information holds with respect to the standard bases (i.e., the bases physicists use by commodity and intuition).

**Example 1.** Let us consider the so-called 4-state protocol of [3]. The analysis of the 6-state protocol [1] is analogous and leads to similar results [15]. We compare the possibility of quantum and classical key agreement given the quantum state and the corresponding classical distribution, respectively, arising from this protocol. The conclusion is, under the assumption of incoherent eavesdropping, that key agreement in one setting is possible if and only if this is true also for the other.

After carrying out the 4-state protocol, and under the assumption of optimal eavesdropping (in terms of Shannon information), the resulting quan-
The state is

\[ \Psi = \sqrt{F/2}|0,0\rangle\otimes\xi_{00} + \sqrt{D/2}|0,1\rangle\otimes\xi_{01} + \sqrt{D/2}|1,0\rangle\otimes\xi_{10} + \sqrt{F/2}|1,1\rangle\otimes\xi_{11} \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4, \]

where \( D \) (the disturbance) is the probability that \( X \neq Y \) holds if \( X \) and \( Y \) are the classical random variables of Alice and Bob, respectively, where \( F = 1 - D \) (the fidelity), and where the \( \xi_{ij} \) satisfy \( \langle \xi_{00}|\xi_{11} \rangle = \langle \xi_{01}|\xi_{10} \rangle = 1 - 2D \) and \( \langle \xi_{ii}|\xi_{ij} \rangle = 0 \) for all \( i \neq j \). Then the state \( \rho_{AB} \) is (in the basis \( \{|00\rangle,|01\rangle,|10\rangle,|11\rangle\})

\[ \rho_{AB} = \frac{1}{2} \left( \begin{array}{cccc}
D & 0 & 0 & -D(1-2D) \\
0 & 1-D & -(1-D)(1-2D) & 0 \\
0 & -D(1-2D) & 1-D & 0 \\
-D(1-2D) & 0 & 0 & D
\end{array} \right), \]

and its partial transpose

\[ \rho^t_{AB} = \frac{1}{2} \left( \begin{array}{cccc}
D & 0 & 0 & -(1-D)(1-2D) \\
0 & 1-D & -D(1-2D) & 0 \\
0 & -D(1-2D) & 1-D & 0 \\
-(1-D)(1-2D) & 0 & 0 & D
\end{array} \right) \]

has the eigenvalues \((1/2)(D \pm (1-D)(1-2D))\) and \((1/2)((1-D) \pm D(1-2D))\), which are all non-negative (i.e., \( \rho_{AB} \) is separable) if

\[ D \geq 1 - \frac{1}{\sqrt{2}}. \tag{4} \]

From the classical viewpoint, the corresponding distributions (arising from measuring the above quantum system in the standard bases) are as follows. First, \( X \) and \( Y \) are both symmetric bits with \( \text{Prob} [X \neq Y] = D \). Eve’s random variable \( Z = [Z_1,Z_2] \) is composed of 2 bits \( Z_1 \) and \( Z_2 \), where \( Z_1 = X \oplus Y \), i.e., \( Z_1 \) tells Eve whether Bob received the qubit disturbed \((Z_1 = 1)\) or not \((Z_1 = 0)\) (this is a consequence of the fact that the \( \xi_{ii} \) and \( \xi_{ij} \) \((i \neq j)\) states generate orthogonal sub-spaces), and where the probability that Eve’s second bit indicates the correct value of Bob’s bit is \( \text{Prob}[Z_2 = Y] = \delta = (1 + \sqrt{1 - \langle \xi_{00}|\xi_{11} \rangle^2})/2 = 1/2 + \sqrt{D(1-D)} \). We now prove that for this distribution, the intrinsic information is zero if and only if

\[ \frac{D}{1-D} \geq 2\sqrt{(1-\delta)^2} = 1 - 2D \tag{5} \]

holds. We show that if the condition \((\delta)\) is satisfied, then \( I(X;Y|Z) = 0 \) holds. The inverse implication follows from the existence of a key-agreement protocol in all other cases (see Example 1 (cont’d) in Section 3.3). If \((\delta)\)
We study the special case \( (0) \) of the resulting distribution (to be normalized). For instance, the entry \( (16) = 2a/\sqrt{8a+1} \) is sufficient to show the first equality. For \( a_{ij} := P_{XYZ}(i,j,u) \), we get

\[
a_{00} = (1-D)(1-\delta)/2, \quad a_{11} = (1-D)\delta/2, \quad a_{01} = a_{10} = (D(1-\delta)/2+D\delta/2)/2 = D/4.
\]

From equality in (3) we conclude \( a_{00}a_{11} = a_{01}a_{10} \), which is equivalent to the fact that \( X \) and \( Y \) are independent, given \( Z = u \).

Finally, note that the conditions (3) and (4) are equivalent for \( D \in [0,1/2] \). This shows that the bounds of tolerable noise are indeed exactly the same for the quantum and classical scenarios. \( \diamond \)

**Example 2.** We consider the bound entangled state presented in [17]. This example received quite a lot of attention by the quantum-information community because it was the first known example of bound entanglement (i.e., entanglement without the possibility of quantum key agreement). We show that its classical counterpart seems to have similarly surprising properties.

Let \( 0 < a < 1 \) and

\[
\Psi = \sqrt{\frac{3a}{8a+1}} \psi \otimes |0\rangle + \sqrt{\frac{1}{8a+1}} \phi_a \otimes |1\rangle + \sqrt{\frac{a}{8a+1}} (|122\rangle + |133\rangle + |214\rangle + |235\rangle + |326\rangle)
\]

where \( \psi = (|11\rangle + |22\rangle + |33\rangle)/\sqrt{3} \) and \( \phi_a = \sqrt{(1+a)/(2)} |31\rangle + \sqrt{(1-a)/(2)} |33\rangle \).

It has been shown in [17] that the resulting state \( \rho_{AB} \) is entangled.

The corresponding classical distribution is as follows. The ranges are \( X = Y = \{1,2,3\} \) and \( Z = \{0,1,2,3,4,5\} \). We write \( (ijk) = P_{XYZ}(i,j,k) \). Then we have \( (110) = (220) = (330) = (122) = (133) = (214) = (235) = (326) = 2a/(16a+2) \), \( (311) = (1+a)/(16a+2) \), and \( (331) = (1-a)/(16a+2) \). We study the special case \( a = 1/2 \). Consider the following representation of the resulting distribution (to be normalized). For instance, the entry \( ^{(0)}_{1} \), \( (1) \) \( 1/2 \) for \( X = Y = 3 \) means \( P_{XYZ}(3,3,0) = 1/10 \) (normalized), \( P_{XYZ}(3,3,1) = 1/20 \), and \( P_{XYZ}(3,3,z) = 0 \) for all \( z \notin \{0,1\} \).
As we would expect, the intrinsic information is positive in this scenario. This can be seen by contradiction as follows. Assume \( I(X; Y \downarrow Z) = 0 \). Hence there exists a discrete channel, characterized by the conditional distribution \( P_{Z|Z}(i, 0) =: a_i \), \( P_{Z|Z}(i, 1) =: x_i \), \( P_{Z|Z}(i, 6) =: s_i \). Then we must have \( a_i, x_i, s_i \in [0, 1] \) and \( \sum_i a_i = \sum_i x_i = \sum_i s_i = 1 \). Using \( I(X; Y \downarrow Z) = 0 \), we obtain the following distributions \( P_{XY|Z}=i \) (to be normalized):

\[
\begin{array}{|c|c|c|c|}
\hline
X & 1 & 2 & 3 \\
\hline
Y (Z) & 1 & (0) 1 & (4) 1 & (1) 3/2 \\
\hline
2 & (2) 1 & (0) 1 & (6) 1 \\
\hline
3 & (3) 1 & (5) 1 & (0) 1 \\
\hline
\end{array}
\]

By comparing the \((2,3)\)-entries of the two tables above, we obtain

\[
1 \geq \sum_i a_i (a_i + x_i/2)/s_i. \tag{6}
\]

We now prove that (6) implies \( s_i \equiv a_i \) (i.e., \( s_i = a_i \) for all \( i \)) and \( x_i \equiv 0 \). Clearly, this does not lead to a solution and is hence a contradiction. For instance, \( P_{XY|Z=1}(1, 2) = 2a_i s_i / (3x_i) \) is not even defined in this case if \( a_i > 0 \).

It remains to show that (6) implies \( a_i \equiv s_i \) and \( x_i \equiv 0 \). We show that whenever \( \sum_i a_i = \sum_i s_i = 1 \) and \( a_i \neq s_i \), then \( \sum_i a_i^2/s_i > 1 \). First, note that \( \sum_i a_i^2/s_i = \sum_i a_i = 1 \) for \( a_i \equiv s_i \). Let now \( s_{i1} \leq a_{i1} \) and \( s_{i2} \leq a_{i2} \). We show that \( a_i^2/s_i + a_i^2/s_i < a_i^2/(s_{i1} - \varepsilon) + a_i^2/(s_{i2} + \varepsilon) \) holds for every \( \varepsilon > 0 \), which obviously implies the above statement. It is straightforward to see that this is equivalent to \( a_{i1} s_{i2} (s_{i2} + \varepsilon) > a_{i2} s_{i1} (s_{i1} - \varepsilon) \), and holds because of \( a_{i1}^2 s_{i2} (s_{i2} + \varepsilon) > a_{i1}^2 a_{i2}^2 \) and \( a_{i2}^2 s_{i1} (s_{i1} - \varepsilon) < a_{i1}^2 a_{i2}^2 \). This concludes the proof of \( I(X; Y \downarrow Z) > 0 \). \( \diamond \)
As mentioned, the interesting point about Example 2 is that the quantum state is bound entangled, and that also classical key agreement seems impossible despite the fact that \( I(X; Y \downarrow Z) > 0 \) holds. This is a contradiction to a conjecture stated in [22]. The classical translation of the bound entangled state leads to a classical distribution with very strange properties as well! (See Example 2 (cont’d) in Section 3.4).

In Example 3, another bound entangled state (first proposed in [18]) is discussed. The example is particularly nice because, depending on the choice of the parameter \( \alpha \), the quantum state can be made separable, bound entangled, and free entangled.

**Example 3.** We consider the following distribution (to be normalized). Let \( 2 \leq \alpha \leq 5 \).

\[
X \quad \begin{array}{c|c|c|c}
Y (Z) & 1 & 2 & 3 \\
\hline
1 & (0) & (4) & (3) \\
2 & (1) & (0) & (5) \\
3 & (6) & (2) & (0) \\
\end{array}
\]

This distribution arises when measuring the following quantum state. Let \( \psi := (1/\sqrt{3}) (|11\rangle + |22\rangle + |33\rangle) \). Then

\[
\Psi = \sqrt{\frac{2}{7}} \psi \otimes |0\rangle + \frac{a}{21} (|12\rangle \otimes |1\rangle + |23\rangle \otimes |2\rangle + |31\rangle \otimes |3\rangle) + \sqrt{\frac{5-a}{21}} (|21\rangle \otimes |4\rangle + |32\rangle \otimes |5\rangle + |13\rangle \otimes |6\rangle),
\]

and

\[
\rho_{AB} = \frac{2}{7} P_\psi + \frac{a}{21} (P_{12} + P_{23} + P_{31}) + \frac{5-a}{21} (P_{21} + P_{32} + P_{13})
\]
is separable if and only if \( \alpha \in [2, 3] \), bound entangled for \( \alpha \in (3, 4] \), and free entangled if \( \alpha \in (4, 5] \) (see Figure 3 in Appendix A).

Let us consider the quantity \( I(X; Y \downarrow Z) \). First of all, it is clear that \( I(X; Y \downarrow Z) = 0 \) holds for \( \alpha \in [2, 3] \). The reason is that \( \alpha \geq 2 \) and \( 5 - \alpha \geq 2 \) together imply that Eve can “mix” her symbol \( Z = 0 \) with the remaining symbols in such a way that when given that \( \bar{Z} \) takes the “mixed value,” then \( XY \) is uniformly distributed; in particular, \( X \) and \( Y \) are independent. Moreover, it can be shown in analogy to Example 2 that \( I(X; Y \downarrow Z) > 0 \) holds for \( \alpha > 3 \).

Examples 1, 2, and 3 suggest that the correspondence between separability and entanglement on one side and vanishing and non-vanishing intrinsic
information on the other always holds with respect to the standard bases or even arbitrary bases. We show in Appendix B that this is not true in general. More precisely, Examples 6 and 7 demonstrate how Eve as well as Alice and Bob can perform bad measurements. Hence the parallelity between the quantum and classical situation must be as it is stated in Theorems 1 and 2.

3.3 A Classical Measure for Quantum Entanglement

It is a challenging problem of theoretical quantum physics to find good measures for entanglement [24]. Corollary 3 above suggests the following measure, which is based on classical information theory.

**Definition 1** Let for a quantum state $\rho_{AB}$

$$\mu(\rho_{AB}) := \min \left( \max_{\{i\}} \{i(X; Y \downarrow Z)\} \right),$$

where the minimum is taken over all $\Psi = \sum_z \sqrt{p_z} \psi_z \otimes |z\rangle$ such that $\rho_{AB} = \text{Tr}_{H_E}(P_\Psi)$ holds and over all bases $\{|i\rangle\}$ of $H_A$ and $\{|j\rangle\}$ of $H_B$, and where $P_{XYZ}(x, y, z) := |\langle x, y, z |\Psi \rangle|^2$. \hfill $\Box$

Then $\mu$ has all the properties required from such a measure. If $\rho_{AB}$ is pure, i.e., $\rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}|$, then we have in the Schmidt basis (see for example [24]) $\psi_{AB} = \sum_j c_j |x_j, y_j\rangle$, and $\mu(\rho_{AB}) = -\sum_j |c_j|^2 \log_2(|c_j|^2) = -\text{Tr}(\rho_{AB} \log_2 \rho_{AB})$, as it should [24]. It is obvious that $\mu$ is convex, i.e., $\mu(\lambda \rho_1 + (1-\lambda) \rho_2) \leq \lambda \mu(\rho_1) + (1-\lambda) \mu(\rho_2)$.

**Example 4 (based on Werner’s states).** Let $\Psi = \sqrt{\lambda} \psi^{-} \otimes |010\rangle + \sqrt{(1-\lambda)/4} |001+012+013+114\rangle$, where $\psi^{-} = |10-01\rangle / \sqrt{2}$, and $\rho_{AB} = \lambda P_{\psi^{-}} + ((1-\lambda)/4) \mathbb{I}$. It is well-known that $\rho_{AB}$ is separable if and only if $\lambda \leq 1/3$. Then the classical distribution is $(010) = (100) = \lambda/2$ and $(001) = (012) = (103) = (114) = (1-\lambda)/4$.

If $\lambda \leq 1/3$, then consider the channel $P_{Z\mid Z}(0, 0) = P_{Z\mid Z}(2, 2) = P_{Z\mid Z}(3, 3) = 1$, $P_{Z\mid Z}(0, 1) = P_{Z\mid Z}(0, 4) = \xi$, $P_{Z\mid Z}(1, 1) = P_{Z\mid Z}(4, 4) = 1 - \xi$, where $\xi = 2\lambda/(1-\lambda) \leq 1$. Then $\mu(\rho_{AB}) = I(X; Y \downarrow Z) = I(X; Y | Z) = 0$ holds, as it should.

If $\lambda > 1/3$, then consider the (obviously optimal) channel $P_{Z\mid Z}(0, 0) = P_{Z\mid Z}(2, 2) = P_{Z\mid Z}(3, 3) = P_{Z\mid Z}(0, 1) = P_{Z\mid Z}(0, 4) = 1$. Then

$$\mu(\rho_{AB}) = I(X; Y \downarrow Z) = I(X; Y | Z) = P_{Z}(0) \cdot I(X; Y | Z) = 0 = \frac{1 + \lambda}{2} \cdot (1 - q \log_2 q - (1-q) \log_2 (1-q)), $$

where $q = 2\lambda/(1 + \lambda)$. \hfill $\Box$
3.4 Classical Protocols and Quantum Privacy Amplification

It is a natural question whether the analogy between entanglement and intrinsic information (see Section 3.1) carries over to the protocol level. The examples given in Section 3.5 support this belief. A quite interesting and surprising consequence would be that there exists a classical counterpart to bound entanglement, namely intrinsic information that cannot be distilled into a secret key by any classical protocol, if $|X| + |Y| > 5$. In other words, the conjecture in [22] that such information can always be distilled would be proved for $|X| + |Y| \leq 5$, but disproved otherwise.

**Conjecture 1** Let $\Psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$ and $\rho_{AB} = \text{Tr}_{\mathcal{H}_E}(P_\Psi)$. Assume that for all generating sets $\{|z\rangle\}$ of $\mathcal{H}_E$ there are bases $\{|x\rangle\}$ and $\{|y\rangle\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that $S(X;Y||Z) > 0$ holds for the distribution $P_{XYZ}(x,y,z) := |\langle x,y,z|\Psi \rangle|^2$. Then quantum privacy amplification is possible with the state $\rho_{AB}$, i.e., $\rho_{AB}$ is free entangled.

**Conjecture 2** Let $\Psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$ and $\rho_{AB} = \text{Tr}_{\mathcal{H}_E}(P_\Psi)$. Assume that there exists a generating set $\{|z\rangle\}$ of $\mathcal{H}_E$ such that for all bases $\{|x\rangle\}$ and $\{|y\rangle\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, $S(X;Y||Z) = 0$ holds for the distribution $P_{XYZ}(x,y,z) := |\langle x,y,z|\Psi \rangle|^2$. Then quantum privacy amplification is impossible with the state $\rho_{AB}$, i.e., $\rho_{AB}$ is bound entangled or separable.

3.5 Examples II

The following examples support Conjectures 1 and 2 and illustrate their consequences. We consider mainly the same distributions as in Section 3.2, but this time under the aspect of the existence of classical and quantum key-agreement protocols.

**Example 1 (cont’d).** We have shown in Section 3.2 that the resulting quantum state is entangled if and only if the intrinsic information of the corresponding classical situation (with respect to the standard bases) is non-zero. Here, we show that such a correspondence also holds on the protocol level. First of all, it is clear for the quantum state that QPA is possible whenever the state is entangled because both $\mathcal{H}_A$ and $\mathcal{H}_B$ have dimension two. On the other hand, the same is also true for the corresponding classical situation, i.e., secret-key agreement is possible whenever $D/(1 - D) < 2\sqrt{(1 - \delta)\delta}$ holds, i.e., if the intrinsic information is positive. This is shown in Appendix C. There we describe the required protocol, more
precisely, the advantage-distillation phase (called repeat-code protocol [20]), in which Alice and Bob use their advantage given by the authenticity of the public-discussion channel for generating new random variables for which the legitimate partners have an advantage over Eve in terms of the (Shannon) information about each other’s new random variables. For a further discussion of this example, see also [15].

Example 2 (cont’d). The quantum state $\rho_{AB}$ in this example is bound entangled, meaning that the entanglement cannot be used for QPA. Interestingly, but not surprisingly given the discussion above, the corresponding classical distribution has the property that $I(X;Y|Z) > 0$, but nevertheless, all the known classical advantage-distillation protocols [20], [22] fail for this distribution! It seems that $S(X;Y||Z) = 0$ holds (although it is not clear how this fact could be rigorously proven, except by proving Conjecture 1 directly).

Example 3 (cont’d). We have seen already that for $2 \leq \alpha \leq 3$, the quantum state is separable and the corresponding classical distribution (with respect to the standard bases) has vanishing intrinsic information. Moreover, it has been shown that for the quantum situation, $3 < \alpha \leq 4$ corresponds to bound entanglement, whereas for $\alpha > 4$, QPA is possible and allows for generating a secret key [18]. We describe a classical protocol here which suggests that the situation for the classical translation of the scenario is totally analogous: The protocol allows classical key agreement exactly for $\alpha > 4$. However, this does not imply (although it appears very plausible) that no classical protocol exists at all for the case $\alpha \leq 4$.

Let $\alpha > 4$. We consider the following protocol for classical key agreement. First of all, Alice and Bob both restrict their ranges to $\{1, 2\}$ (i.e., publicly reject a realization unless $X \in \{1, 2\}$ and $Y \in \{1, 2\}$). The resulting distribution is as follows (to be normalized):

|   | 1     | 2     |
|---|-------|-------|
| X | (0) 2 | (4) $5 - \alpha$ |
| Y | (2) $\alpha$ | (0) 2 |

Then, Alice and Bob both send their bits locally over channels $P_{X|X}$ and $P_{Y|Y}$, respectively, such that the resulting bits $\overline{X}$ and $\overline{Y}$ are symmetric. The channel $P_{\overline{X}|X}$ [and $P_{\overline{Y}|Y}$] sends $X = 0$ [$Y = 1$] to $\overline{X} = 1$ [$\overline{Y} = 0$] with probability $(2\alpha - 5)/(2\alpha + 4)$, and leaves $X$ [$Y$] unchanged otherwise. The distribution $P_{\overline{X}\overline{Y}|Z}$ is then
It is not difficult to see that for $\alpha > 4$, we have $\text{Prob}[X = Y] > 1/2$ and that, given that $X = Y$ holds, Eve has no information at all about what this bit is. Thus the repeat-code protocol described in Appendix C allows for classical key agreement in this situation. For $\alpha \leq 4$ however, classical key agreement, like quantum key agreement, seems impossible. The results of Example 3 are illustrated in Figure 3 in Appendix A.

Example 5. The following distribution $P_{XYZ}$, with binary $X$ and $Y$, was discussed and analyzed in [22] as an example of a simple distribution for which the equivalence of $I(X; Y | Z) > 0$ and $S(X; Y | Z) > 0$ could not be shown.

Assume that the random variables $X$ and $Y$ are distributed according to $P_{XY}(0, 0) = P_{XY}(1, 1) = (1 - \alpha)/2$, $P_{XY}(0, 1) = P_{XY}(1, 0) = \alpha/2$, and $Z = [Z_X, Z_Y]$, where $Z_X$ and $Z_Y$ are generated by sending $X$ and $Y$ over two independent binary erasure channels with erasure probabilities $\delta_X$ and $\delta_Y$, respectively.

If the conjectured parallels between classical and quantum protocols hold, then $I(X; Y | Z) > 0$ implies $S(X; Y | Z) > 0$ because both $X$ and $Y$ are binary. Moreover, due to the proven connection between intrinsic information and entanglement and hence to the eigenvalues of the partial transpose of the density matrix, the condition for $I(X; Y | Z) > 0$ can be explicitly given, and is very simple. This is surprising since the determination of $I(X; Y | Z)$, as well as the advantage-distillation protocols for this distribution, turned out to be quite complicated [22]. The condition under which all the eigenvalues of the partial transpose of the density matrix of the corresponding quantum state are non-negative is

$$(\alpha - \alpha^2) \left( \frac{(1 - \delta_X)(1 - \delta_Y)}{\delta_X} + 2 \right) \left( \frac{(1 - \delta_X)(1 - \delta_Y)}{\delta_Y} + 2 \right) \geq 1.$$ 

This bound is compatible with (but stronger than) all the bounds painfully derived, by working purely in the classical information-theoretic world, in [22].

\begin{table}
\begin{tabular}{|c|c|c|}
\hline
$X$ & 1 & 2 \\
\hline
$Y (Z)$ & $(0) \cdot \frac{\alpha}{\alpha+4}$ & $(1) \cdot \frac{5-\alpha}{\alpha+4}$ \\
\hline
1 & $(2) \cdot \frac{9}{\alpha+4} \cdot \frac{\alpha-5}{\alpha+4}$ & $(2) \cdot \frac{\alpha}{\alpha+4}^2$ \\
\hline
2 & $(2) \cdot \frac{9}{\alpha+4} \cdot \frac{\alpha-5}{\alpha+4}$ & $(0) \cdot 2 \cdot \frac{8}{\alpha+4}^2$ \\
\hline
\end{tabular}
\end{table}
3.6 Bound Intrinsic Information

Examples 2 and 3 suggest that, in analogy to bound entanglement of a quantum state, bound classical information exists, i.e., conditional intrinsic information which cannot be used to generate a secret key in the classical scenario. We give a formal definition of bound intrinsic information.

**Definition 2** Let $P_{XYZ}$ be a distribution with $I(X; Y \downarrow Z) > 0$. Then if $S(X; Y || Z) > 0$ holds for this distribution, the intrinsic information between $X$ and $Y$, given $Z$, is called **free**. Otherwise, if $S(X; Y || Z) = 0$, the information is called **bound**.

Note that the existence of bound intrinsic information could not be proven so far. However, all known examples of bound entanglement, combined with all known advantage-distillation protocols, do not lead to a contradiction to Conjecture 1! Clearly, it would be very interesting to rigorously prove this conjecture because then, all pessimistic results known for the quantum scenario would immediately carry over to the classical setting (where such results appear to be much harder to prove).

Examples 2 and 3 also illustrate nicely what the nature of such bound information is. Of course, $I(X; Y \downarrow Z) > 0$ implies both $I(X; Y) > 0$ and $I(X; Y | Z) > 0$. However, if $|X| + |Y| > 5$, it is possible that the dependence between $X$ and $Y$ and the dependence between $X$ and $Y$, given $Z$, are “orthogonal.” By the latter we mean that for all fixed (deterministic or probabilistic) functions $f : \mathcal{X} \to \{0, 1\}$ and $g : \mathcal{Y} \to \{0, 1\}$ for which the correlation of $f(X)$ and $g(Y)$ is positive, i.e.,

$$P_{f(X)g(Y)}(0, 0) \cdot P_{f(X)g(Y)}(1, 1) > P_{f(X)g(Y)}(0, 1) \cdot P_{f(X)g(Y)}(1, 0),$$

the correlation between the same binary random variables, given $Z = \bar{z}$, is negative (or “zero”) for all $\bar{z} \in \bar{Z}$, where $Z$ is the random variable generated by sending $Z$ over Eve’s optimal channel $P_Z$.

A complete understanding of bound intrinsic information is of interest also because it automatically leads to a better understanding of bound entanglement in quantum information theory.

4 Concluding Remarks

We have considered the model of information-theoretic key agreement by public discussion from correlated information. More precisely, we have compared scenarios where the joint information is given by classical random
variables and by quantum states (e.g., after execution of a quantum
protocol). We proved a close connection between such classical and quantum
information, namely between intrinsic information and entanglement. As
an application, the derived parallels lead to an efficiently verifiable criterion
for the fact that the intrinsic information vanishes. Previously, this quantity
was considered to be quite hard to handle.

Furthermore, we have presented examples providing evidence for the fact
that the close connections between classical and quantum information ex-
tend to the level of the protocols. As a consequence, the powerful tools and
statements on the existence or rather non-existence of quantum-privacy-
amplification protocols immediately carry over to the classical scenario,
where it is often unclear how to show that no protocol exists. In particular,
many examples (only some of which are presented above due to space limi-
tations) coming from measuring bound entangled states, and for which none
of the known classical secret-key agreement protocols is successful, strongly
suggest that bound entanglement has a classical counterpart: intrinsic in-
formation which cannot be distilled to a secret key. This stands in sharp
contrast to what was previously believed about classical key agreement. We
state as an open problem to rigorously prove Conjectures 1 and 2.

Finally, we have proposed a measure for entanglement, based on classical
information theory, with all the properties required for such a measure.
References

[1] H. Bechmann-Pasquinucci and N. Gisin, Incoherent and coherent eavesdropping in the six-state protocol of quantum cryptography, *Phys. Rev. A*, Vol. 59, No. 6, pp. 4238–4248, 1999.

[2] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wooters, Purification of noisy entanglement and faithful teleportation via noisy channels, *Phys. Rev. Lett.*, Vol. 76, pp. 722–725, 1996.

[3] C. H. Bennett and G. Brassard, Quantum cryptography: public key distribution and coin tossing, *Proceedings of the IEEE International Conference on Computer, Systems, and Signal Processing*, IEEE, pp. 175–179, 1984.

[4] D. Bruss, Optimal eavesdropping in quantum cryptography with six states, *Phys. Rev. Lett.*, Vol. 81, No. 14, pp. 3018–3021, 1998.

[5] V. Bužek and M. Hillery, Quantum copying: beyond the no-cloning theorem, *Phys. Rev. A*, Vol. 54, pp. 1844–1852, 1996.

[6] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Proposed experiment to test local hidden-variable theories, *Phys. Rev. Lett.*, Vol. 23, pp. 880–884, 1969.

[7] I. Csiszár and J. Körner, Broadcast channels with confidential messages, *IEEE Transactions on Information Theory*, Vol. IT-24, pp. 339–348, 1978.

[8] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, Quantum privacy amplification and the security of quantum cryptography over noisy channels, *Phys. Rev. Lett.*, Vol. 77, pp. 2818–2821, 1996.

[9] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and A. V. Thapliyal, Evidence for bound entangled states with negative partial transpose, quant-ph/9910026, 1999.

[10] A. E. Ekert, Quantum cryptography based on Bell’s theorem, *Phys. Rev. Lett.*, Vol. 67, pp. 661–663, 1991. See also *Physics World*, March 1998.
[11] C. Fuchs, N. Gisin, R. B. Griffiths, C. S. Niu, and A. Peres, Optimal eavesdropping in quantum cryptography – I: information bound and optimal strategy, *Phys. Rev. A*, Vol. 56, pp. 1163–1172, 1997.

[12] N. Gisin, Stochastic quantum dynamics and relativity, *Helv. Phys. Acta*, Vol. 62, pp. 363–371, 1989.

[13] N. Gisin and B. Huttner, Quantum cloning, eavesdropping, and Bell inequality, *Phys. Lett. A*, Vol. 228, pp. 13–21, 1997.

[14] N. Gisin and S. Massar, Optimal quantum cloning machines, *Phys. Rev. Lett.*, Vol. 79, pp. 2153–2156, 1997.

[15] N. Gisin and S. Wolf, Quantum cryptography on noisy channels: quantum versus classical key agreement protocols, *Phys. Rev. Lett.*, Vol. 83, pp. 4200–4203, 1999.

[16] M. Horodecki, P. Horodecki, and R. Horodecki, Mixed-state entanglement and distillation: is there a “bound” entanglement in nature?, *Phys. Rev. Lett.*, Vol. 80, pp. 5239-5242, 1998.

[17] P. Horodecki, Separability criterion and inseparable mixed states with positive partial transposition, *Phys. Lett. A*, Vol. 232, p. 333, 1997.

[18] P. Horodecki, M. Horodecki, and R. Horodecki, Bound entanglement can be activated, [quant-ph/9806058], 1998.

[19] L. P. Hughston, R. Jozsa, and W. K. Wootters, A complete classification of quantum ensembles having a given density matrix, *Phys. Lett. A*, Vol. 183, pp. 14–18, 1993.

[20] U. Maurer, Secret key agreement by public discussion from common information, *IEEE Transactions on Information Theory*, Vol. 39, No. 3, pp. 733-742, 1993.

[21] U. Maurer and S. Wolf, Strengthening security of information-theoretic secret-key agreement, to appear in *Proceedings of EUROCRYPT 2000*, Lecture Notes in Computer Science, Springer-Verlag, 2000.

[22] U. Maurer and S. Wolf, Unconditionally secure key agreement and the intrinsic conditional information, *IEEE Transactions on Information Theory*, Vol. 45, No. 2, pp. 499–514, 1999.

[23] N. D. Mermin, The Ithaca interpretation of quantum mechanics, *Prama*, Vol. 51, pp. 549–565, 1998.
[24] A. Peres, *Quantum theory: concepts and methods*, Kluwer Academic Publishers, 1993.

[25] A. Peres, Separability criterion for density matrices, *Phys. Rev. Lett.*, Vol. 77, pp. 1413–1415, 1996.

[26] S. Popescu and D. Rohrlich, Thermodynamics and the measure of entanglement, quant-ph/9610044, 1996.

[27] G. Ribordy, J. D. Gautier, N. Gisin, O. Guinnard, and H. Zbinden, Automated plug and play quantum key distribution, *Electron. Lett.*, Vol. 34, pp. 2116–2117, 1998.

[28] C. E. Shannon, Communication theory of secrecy systems, *Bell System Technical Journal*, Vol. 28, pp. 656–715, 1949.

[29] A. D. Wyner, The wire-tap channel, *Bell System Technical Journal*, Vol. 54, No. 8, pp. 1355–1387, 1975.

[30] H. Zbinden, H. Bechmann, G. Ribordy, and N. Gisin, Quantum cryptography, *Applied Physics B*, Vol. 67, pp. 743–748, 1998.
Appendix A: Figures

Figure 1: Secret-Key Agreement from Common Information

Figure 2: The Channel $P_{Z|Z}$ in Example 1
Quantum World

Measurement

| 2 | 3 | 4 | 5 | \( \alpha \) |
|---|---|---|---|---|
| \( I(X;Y|Z) = 0 \) | \( I(X;Y|Z) > 0 \) | \( I(X;Y|Z) > 0 \) | \( S(X;Y||Z) = 0 \) | \( S(X;Y||Z) > 0 \) |
| \( \rho_{AB} \) is ... |
| separable | bound entangled | free entangled |

Classical World

Figure 3: The Results of Example 3

Appendix B: Measuring in “bad” Bases

In this appendix we show, by two examples, that the statements of Theorems 1 and 2 do not always hold for the standard bases and, in particular, not for arbitrary bases: Alice and Bob as well as Eve can perform bad measurements and give away an initial advantage. Let us begin with an example where measuring in the standard basis is a bad choice for Eve.

**Example 6.** Let us consider the quantum states

\[ \Psi = \frac{1}{\sqrt{5}} (|00+01+10\rangle \otimes |0\rangle + |00+11\rangle \otimes |1\rangle), \quad \rho_{AB} = \frac{3}{5} P_{|00+01+10\rangle} + \frac{2}{5} P_{|00+11\rangle}. \]

If Alice, Bob, and Eve measure in the standard bases, we get the classical distribution (to be normalized)

| \( X \) | \( Y \) | \( 0 \) | \( 1 \) |
|---|---|---|---|
| 0 | (0) | (0) | (1) |
| 1 | (0) | (0) | (1) |

For this distribution, \( I(X;Y|Z) > 0 \) holds. Indeed, even \( S(X;Y||Z) > 0 \) holds. This is not surprising since both \( X \) and \( Y \) are binary, and since the
described parallels suggest that in this case, positive intrinsic information implies that a secret-key agreement protocol exists.

The proof of $S(X; Y || Z) > 0$ in this situation is analogous to the proof of this fact in Example 3. The protocol consists of Alice and Bob independently making their bits symmetric. Then the repeat-code protocol can be applied.

However, the partial-transpose condition shows that $\rho_{AB}$ is separable. This means that measuring in the standard basis is bad for Eve. Indeed, let us rewrite $\Psi$ and $\rho_{AB}$ as

$$
\Psi = \sqrt{\Lambda} |\vec{m}, \vec{m}\rangle \otimes |\tilde{0}\rangle + \sqrt{1-\Lambda} |\vec{m}, -\vec{m}\rangle \otimes |\tilde{1}\rangle , \quad \rho_{AB} = \frac{5 + \sqrt{5}}{10} P_{|\vec{m}, \vec{m}\rangle} + \frac{5 - \sqrt{5}}{10} P_{|\vec{m}, -\vec{m}\rangle},
$$

where $\Lambda = (5 + \sqrt{5})/10$, $|\vec{m}, \vec{m}\rangle = |\vec{m}\rangle \otimes |\vec{m}\rangle$, $|\pm \vec{m}\rangle = \sqrt{(1 \pm \eta)/2} |0\rangle \pm \sqrt{(1 \mp \eta)/2} |1\rangle$, and $\eta = 1/\sqrt{5}$.

In this representation, $\rho_{AB}$ is obviously separable. It also means that Eve’s optimal measurement basis is

$$
|\tilde{0}\rangle = \frac{1}{\sqrt{5\Lambda}} |0\rangle - \frac{1}{\sqrt{5(1-\Lambda)}} |1\rangle , \quad |\tilde{1}\rangle = -\frac{1}{\sqrt{5\Lambda}} |0\rangle + \frac{1}{\sqrt{5(1-\Lambda)}} |1\rangle.
$$

Then, $I(X; Y \downarrow Z) = 0$ holds for the resulting classical distribution.

Not surprisingly, there also exist examples of distributions for which measuring in the standard bases is bad for Alice and Bob. These states are entangled, but $I(X; Y \downarrow Z) = 0$ holds.

**Example 7.** Let the following classical distribution be given:

| X \ Y (Z) | 0          | 1          |
|----|------------|------------|
| 0  | (0) 0.0082 | (0) 0.0219 |
|    | (1) 0.0006 | (1) 0.0202 |
| 1  | (0) 0.0729 | (0) 0.3928 |
|    | (1) 0.0905 | (1) 0.3889204545 |

Because of

$$(0.0082+0.0006) \cdot (0.03928+0.3889204545) = (0.0219+0.0202) \cdot (0.0729+0.0905)$$

we have $I(X; Y) = 0$, thus $I(X; Y \downarrow Z) = 0$. On the other hand, the corresponding quantum state, for which the above distribution results by measuring in the standard bases, can be shown to be entangled.
Appendix C: A Protocol for Advantage Distillation

The following protocol for classical advantage distillation is called repeat-code protocol and was first proposed in [20]. Note that there exist more efficient protocols in terms of the amount of extractable secret key. However, since we only want to prove qualitative possibility results, it is sufficient to look at this simpler protocol. Assume the scenario of Example 1.

Let \( N > 0 \) be an even integer, and let Alice choose a random bit \( C \) and send the block

\[
X^N \oplus C^N := [X_1 \oplus C, X_2 \oplus C, \ldots, X_N \oplus C]
\]

over the classical channel. Here, \( X^N \) stands for the block \( [X_1, X_2, \ldots, X_N] \) of \( N \) consecutive realizations of the random variable \( X \), whereas \( C^N \) stands for the \( N \)-bit block \( [C, C, \ldots, C] \). Bob then computes \( [(C \oplus X_1) \oplus Y_1, \ldots, (C \oplus X_N) \oplus Y_N] \) and (publicly) accepts exactly if this block is equal to either \([0,0,\ldots,0]\) or \([1,1,\ldots,1]\). In other words, Alice and Bob make use of a repeat code of length \( N \) with the only two codewords \([0,0,\ldots,0]\) and \([1,1,\ldots,1]\).

Bob’s conditional error probability \( \beta_N \) when guessing the bit sent by Alice, given that he accepts, is

\[
\beta_N = \frac{1}{p_{a,N}} \cdot D^N \leq \left( \frac{D}{1 - D} \right)^N,
\]

where \( p_{a,N} = D^N + (1 - D)^N \) is the probability that Bob accepts the received block. It is obvious that Eve’s optimal strategy for guessing \( C \) is to compute the block \( [(C \oplus X_1) \oplus Z_1, \ldots, (C \oplus X_N) \oplus Z_N] \) and guess \( C \) as 0 if at least half of the bits in this block are 0, and as 1 otherwise. Given that Bob correctly accepts, Eve’s error probability when guessing the bit \( C \) with the optimal strategy is lower bounded by \( 1/2 \) times the probability that she decodes to a block with \( N/2 \) bits 0 and the same number of 1’s. Hence we get that

\[
\gamma_N \geq \frac{1}{2} \left( \frac{N}{N/2} \right)^{(1 - \delta)^N/2 \delta^N/2} \geq \frac{K}{\sqrt{N}} \cdot \left( \frac{2 \sqrt{(1 - \delta)}}{1 - \delta} \right)^N
\]

holds for some constant \( K \) and for sufficiently large \( N \) by using Stirling’s formula. Note that Eve’s error probability given that Bob accepts is asymptotically equal to her error probability given that Bob correctly accepts because Bob accepts erroneously only with asymptotically vanishing probability, given that he accepts.
Although it is not the adversary’s ultimate goal to guess the bits $C$ sent by Alice, it has been shown that the fact that $\beta_N$ decreases exponentially faster than $\gamma_N$ implies that for sufficiently large $N$, Bob has more (Shannon) information about the bit $C$ than Eve (see for example [22]). Hence Alice and Bob have managed to generate new random variables with the property that Bob obtains more information about Alice’s random bit than Eve has. Thus $S(X; Y||Z) > 0$ holds.