Approximately low-rank recovery from noisy and local measurements by convex program

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Abstract

Low-rank matrix models have been universally useful for numerous applications starting from classical system identification to more modern matrix completion in signal processing and statistics. The nuclear norm has been employed as a convex surrogate of the low-rankness since it induces a low-rank solution to inverse problems. While the nuclear norm for low-rankness has a nice analogy with the \( \ell_1 \) norm for sparsity through the singular value decomposition, other matrix norms also induce low-rankness. Particularly as one interprets a matrix as a linear operator between Banach spaces, various tensor product norms generalize the role of the nuclear norm. We provide a tensor-norm-constrained estimator for recovery of approximately low-rank matrices from local measurements corrupted with noise. A tensor-norm regulizer is designed adapting to the local structure. We derive statistical analysis of the estimator over matrix completion and decentralized sketching through applying Maurey’s empirical method to tensor products of Banach spaces. The estimator provides a near optimal error bound in a minimax sense and admits a polynomial-time algorithm for these applications.

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1 Introduction

We consider the estimation of an “approximately” low-rank matrix $M_0 \in \mathbb{R}^{d_1 \times d_2}$ from its noisy and “local” linear measurements given by

$$y_k = \text{tr}(A_k^\top M_0) + \eta_k, \quad k = 1, \ldots, n, \quad (1)$$

where $A_1, \ldots, A_n$ denote given measurement matrices and $\eta_1, \ldots, \eta_n$ correspond to additive random noise. The locality implies that each measurement determined by a subset of the entries of $M_0$. To simplify notation, let $y = [y_1; \ldots; y_n] \in \mathbb{R}^n$ and $A : \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^n$ generate the measurements in (1) without noise.

In practice, it is often the case that the unknown matrix $M_0$ is not exactly low-rank but well approximated by a low-rank matrix. Thus we consider a set of approximately rank-$r$ matrices in the form of

$$\mathcal{C} = \{ M \in \mathbb{R}^{d_1 \times d_2} : ||| M ||| \leq \sqrt{\nu} \| M \|_F \},$$

where $||| \cdot |||$ is a norm determined adapting to the measurement operator $A$. The parameter $\nu$ is set so that $\mathcal{C}$ includes all rank-$r$ matrices. This model is a special case of the generalized sparsity model studied in [17]. One example of $\mathcal{C}$ is the set of matrices so that the magnitudes and nuclear norm are bounded simultaneously, which has been studied for matrix completion [28].

We consider the estimator given by

$$\min_{M} \| y - A(M) \|_2^2$$

subject to $||| M ||| \leq \alpha, \quad (2)$

which generalizes the original LASSO estimator [39], and demonstrate that this estimator provides a near optimal error bound in a minimax sense in selected applications.

1.1 Regularizer for local measurement operator

We first show a design principle to obtain a regularizer $||| \cdot |||$ in (2) adapting to the structure in the measurement operator $A$. This will be illustrated over two selected applications.

The first application considers decentralized sketching of an unknown subspace over a network. We assume that vectors sampled from the subspace are scattered over the network.
and the fusion center collects random linear sketches of these vectors. The acquisition model is described by (1) with \( A_k \)’s in the form of the outer product of a random vector and a standard basis vector, i.e., \( A_k = \xi_k e^\top_{j_k} \), where \( j_k \in [d_2] \) and \( e_j \) denotes the \( j \)th column of the identity matrix of size \( d_2 \). In view of the generative model \( \xi_k^\top M_0 e_{j_k} \), the matrix \( M_0 \) is thought of as a linear operator acting only on \( e_1, \ldots, e_{d_2} \). It is natural to interpret \( M_0 \) as a linear operator from \( \ell^d_1 \) to \( \ell^d_2 \). (The choice of the domain space is with respect to the convex hull of all possible inputs. The range space was chosen to ease the computation of the regularizer.) Then we consider the operator norm and the mixed-norm on \( M_0 \). The former corresponds to the maximum column \( \ell^2_2 \) norm, denoted by \( \| \cdot \|_{1 \to 2} \), and the latter is defined by

\[
\|M\|_{\text{mixed}} := \inf_{U, V : UV^\top = M} \|U\|_F \|V^\top\|_{1 \to 2}.
\]

In fact, the mixed-norm is a special case of the 2-summing norm when the domain and range are \( \ell^d_1 \) and \( \ell^d_2 \), respectively. As will be shown later, these norms satisfy that \( \|M\|_{1 \to 2} \leq \|M\|_{\text{mixed}} \leq \sqrt{r} \|M\|_{1 \to 2} \) for all rank-\( r \) matrices. Finally, the regularizer is given by

\[
\|\| M \| \| = \max \left( \|M\|_{1 \to 2}, \frac{\|M\|_{\text{mixed}}}{\sqrt{r}} \right).
\]

Regularization with respect to this norm together with an appropriate choice of the threshold \( \alpha \) enforces the following two properties: i) The estimator is approximately rank-\( r \); and ii) All column norms are similar in the sense that the maximum is within a factor of the root-mean-square. To elaborate further, consider the extreme case when the unknown matrix has only few nonzero columns. In this case, since the measurements are equally distributed over all columns, most measurements contain no information on the nonzero columns. On the contrary, if the energy spreads evenly across the correlated columns in a low-rank matrix, then all the measurements aid in the joint recovery of all of the columns. While the former case would require more measurements per column irrespective of the recovery algorithm, the latter case can be better handled by a disciplined optimization approach. Our choice of \( \|\| M \| \| \) induces a bias towards the latter set of matrices, thus aiding recovery from a near-optimal number of measurements. Unlike the nuclear norm which favors all low-rank matrices equally, our choice of matrix norms favors solutions that are low-rank and additionally have a more uniform spread of energy across their columns.

We apply a similar design principle to the approximately low-rank matrix completion
problem. The linear operator here is given with measurement matrices so that \( A_k \) is proportional to \( \hat{e}_{i_k} e_{j_k}^\top \), where \( i_k \in [d_1] \), \( j_k \in [d_2] \), and \( \hat{e}_i \) (resp. \( e_i \)) denotes the \( i \)th column of the identity matrix of size \( d_1 \) (resp. \( d_2 \)). Since we only consider \( \hat{e}_{i_k}^\top M_0 e_{j_k} \), we may deduce that \( M_0 \) is a linear map from \( \mathbb{R}^{d_2} \) to \( \mathbb{R}^{d_1} \) that acts only on \( e_1, \ldots, e_{d_2} \) and its images interact only with \( \hat{e}_1, \ldots, \hat{e}_{d_1} \). In other words, \( M_0 \) acts on \( \ell_1^{d_1} \) or equivalently \( M_0 e_{j_k} \in \ell_1^{d_1} \). So it is natural to consider \( M_0 \) as a linear operator from \( \ell_1^{d_2} \) to \( \ell_1^{d_1} \). Then the corresponding operator norm is the largest entry magnitude, denoted by \( \| \cdot \|_\infty \) and the \( \gamma_2 \) norm reduces to the max-norm \([23]\) defined by

\[
\|M\|_{\max} := \inf_{U,V: UV^\top = M} \|U\|_{1\rightarrow 2} \|V\|_{1\rightarrow 2}.
\]

It has been shown that \( \|M\|_{\infty} \leq \|M\|_{\max} \leq \sqrt{\theta} \|M\|_{\infty} \) for all rank-\( r \) matrices. The regularizer is given by

\[
\|\|M\|\| = \max \left( \|M\|_{\infty}, \frac{\|M\|_{\max}}{\sqrt{r}} \right).
\]

enforces the solution approximately rank-\( r \) and of similar magnitudes simultaneously.

1.2 A geometric characterization of estimation error bound

We present a generalized analysis framework to derive recovery guarantees for the optimization program in (2). Our main goal is to bound the error between the estimate by (2) and the ground truth \( M_0 \). In the following proposition, we provide a deterministic upper bound on the estimation error conditioned on the event when the measurement operator \( A \) satisfies certain properties. The proof of the proposition follows the technique used by Cai and Zhou [4] to derive their error bound. We provide the proof in Appendix A for the sake of completeness.

**Proposition 1.1.** Let \( \eta_1, \ldots, \eta_n \) in (1) be i.i.d. drawn from \( \mathcal{N}(0, \sigma^2) \). Suppose that there exist parameters \( \theta, \Gamma > 0 \) determined by \( d_1, d_2, n, \alpha, r \) so that

\[
\sup_{\|\|M\|\| \leq 1} \left| \sum_{k=1}^{n} \text{tr}(A_k^\top M)^2 - \|M\|_F^2 \right| \leq \theta \tag{5}
\]

and

\[
\mathbb{E} \left\| \sum_{k=1}^{n} g_k A_k \right\|_* \leq \Gamma, \tag{6}
\]

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where \( \| \cdot \|_* \) denotes the dual norm of \( \| \cdot \| \) and \( (g_k)_{k=1}^n \) is a sequence of i.i.d. standard Gaussian random variables. Let \( R = \sup_{\|M\| \leq 1} \|M\|_F \) be the radius of the \( \| \cdot \| \) unit ball in the Frobenius norm. Then the following statement holds with probability at least \( 1 - \zeta \): for every \( M_0 \in \{ M : \|M\| \leq \alpha \} \), the estimate by (2) from the noisy measurements in (1) satisfies

\[
\| \hat{M} - M_0 \|_F^2 \leq 4\alpha^2 \theta + 4\alpha \sigma \Gamma + 2\pi \alpha \sigma \sqrt{2 \log(2\zeta^{-1})(\theta + R^2)}.
\]

(7)

The quantities \( \theta \) and \( \Gamma \) help characterize the geometry of the constraint set in (2) for a given observation model. In particular, (5) describes how well the \( \ell_2 \)-norm of the measurements are concentrated around that of the ground truth, and (6) describes the Gaussian complexity of the measurement operator over the convex set of candidate matrices. In order to obtain tight estimates of \( \theta \) and \( \Gamma \) for specific applications, we first interpret the matrix as an operator between suitably chosen Banach spaces. We then obtain tight upper bounds on \( \theta \) and \( \Gamma \) by computing the entropy numbers of operators between particular tensor products of the chosen Banach spaces. In the following sections, we provide upper bounds on the estimation error for the two applications of decentralized sketching and low-rank matrix completion. These error bounds are particular realizations of 1.1.

### 1.3 Statistical analysis of decentralized sketching

Our main result provides an error bound for the estimator by (2) when it is applied to the decentralized sketching problem.

**Theorem 1.2.** Let \( A_k = L^{-1/2} b_k e_{j_k}^\top \) with \( j_k \in [d_2] \) satisfying \( j_k \equiv k \) modulo \( [d_2] \) for \( k = 1, \ldots, Ld_2 \) and \( \| \cdot \| \) be defined as in (3). Suppose that \( b_1, \ldots, b_{Ld_2} \) are i.i.d. \( \mathcal{N}(0, I_{d_1}) \) and \( \eta_1, \ldots, \eta_{Ld_2} \) are i.i.d. \( \mathcal{N}(0, \sigma^2) \). Then the following statement holds with probability \( 1 - (d_1 + d_2)^{-1} \): For any matrix \( M_0 \in K_{\text{mixed}} := \{ M \in \mathbb{R}^{d_1 \times d_2} : \|M\|_{1\to2} \leq \alpha, \|M\|_{\text{mixed}} \leq \sqrt{r\alpha} \} \), the estimate \( \hat{M} \) by (2) satisfies

\[
\frac{1}{d_1 d_2} \| \hat{M} - M_0 \|_F^2 \leq \frac{\alpha^2}{d_1} \cdot \max \left( 1, \frac{\sigma \sqrt{L \log(Ld_2)}}{\alpha} \right) \cdot \sqrt{\frac{r(d_1 + d_2) \log^4(d_1 + d_2)}{Ld_2}}.
\]

(8)

Let \( \alpha = \mu/\sqrt{d_2} \) for some \( \mu \geq 1 \). Then the intersection of \( K_{\text{mixed}} \) with the unit Frobenius-norm sphere \( S_F \) coincides with the intersection of \( C_{\text{mixed}} := \{ M : \|M\| \leq (\mu/\sqrt{d_2}) \|M\|_F \} \)
with $S_F$. The latter set consists of unit-Frobenius norm matrices which are approximately rank-$r$ and the maximum of column norms is not too larger compared to the root-mean-square. The error bound in (8) applies to $C_{\text{mixed}} \cap S_F$. It is straightforward to verify that
\[ \mu = \sup_{M \in C_{\text{mixed}} \setminus \{0\}} \sqrt{d_2} \|M\|_{1-2}. \]
Then $\mu$ denotes the “spikiness” of the column norms. With
\[ \text{SNR} = \frac{\sum_{k=1}^{Ld_1} \mathbb{E}[\text{tr}(A_k^T M_0)^2]}{\sum_{k=1}^{Ld_2} \mathbb{E}[\eta_k^2]} = \frac{\|M_0\|_F^2}{Ld_2\sigma^2}, \]
the error bound in (8) is rewritten as
\[ \frac{\|\hat{M} - M_0\|_F^2}{\|M_0\|_F^2} \lesssim \mu^2 \cdot \max \left(1, \frac{\mu^{-1} \log(Ld_2)}{\text{SNR}^{1/2}} \right) \sqrt{\frac{r(d_1 + d_2) \log^4(d_1 + d_2)}{Ld_2}}. \quad (9) \]
Particularly, in a high noise regime where $\text{SNR} = O(\mu^{-2} \log(Ld_2))$, we note that (9) further reduces to
\[ \frac{\|\hat{M} - M_0\|_F^2}{\|M_0\|_F^2} \lesssim \sqrt{\frac{\text{SNR}^{-1} \cdot \mu^2 r(d_1 + d_2) \log^4(d_1 + d_2)}{Ld_2}}. \quad (10) \]
Since the error bound in (10) is invariant under scaling of $M_0$, it indeed applies to $C_{\text{mixed}}$, which consists of all non-spiky and approximately rank-$r$ matrices. That is, in order to achieve an $\epsilon$-accurate estimation in the normalized error, it suffices to obtain $\tilde{O}(\epsilon^{-2} \cdot \text{SNR}^{-1} \cdot \mu^2 r(d_1 + d_2))$ random local measurements. To show the tightness of the error bound in Theorem 1.2, we compare it to a matching lower bound given in the following theorem.

**Theorem 1.3.** For the measurement model in (1), suppose that the parameters satisfy
\[ \frac{48\alpha^2}{d_1 \vee d_2} \leq \alpha^2 r \leq \frac{\sigma^2 d_1 d_2}{128}. \]
Then the minimax $\|\cdot\|_F$-risk is lower-bounded as
\[ \inf_{\hat{M}} \sup_{M_0 \in K_{\text{mixed}}} \mathbb{E}\|\hat{M} - M_0\|_F^2 \geq \frac{\alpha^2}{16d_1} \left(1 \wedge \frac{\sigma \sqrt{L}}{\alpha} \sqrt{\frac{r(d_1 + d_2)}{Ld_2}} \right). \quad (11) \]
Further, when $Ld_2 > r(d_1 + d_2)$, we have that
\[ \inf_{\hat{M}} \sup_{M_0 \in K_{\text{mixed}}} \mathbb{E}\|\hat{M} - M_0\|_F^2 \geq \frac{\alpha^2}{16d_1} \left(\sqrt{\frac{r(d_1 + d_2)}{Ld_2}} \left(1 \wedge \frac{\sigma \sqrt{L}}{\alpha} \right) \right). \quad (12) \]
Note that, in a low SNR regime, the error bound in (8) matches the minimax lower bound in (11) up to a logarithmic factor. However, in a high SNR regime, where the $\text{SNR}^{1/2} \mu^{-1}$ term is $o(1)$, the error bound in (8) is suboptimal. In particular, the estimator does not provide exact parameter recovery in the noiseless case ($\sigma = 0$).

The only results in the literature that are directly comparable to Theorem 1.3 are [25, 26]. These works show how $M_0$ can be recovered using an alternating minimization or gradient descent algorithm. When the unknown matrix is exactly rank-$r$ and there is no noise in the measurements, they showed that alternating minimization (resp. gradient descent) provides an $\epsilon$-accurate estimate from $\tilde{O}(\tilde{\mu}^2 \max(d_1 r^3, d_2 r) \log(1/\epsilon))$ (resp. $\tilde{O}(\tilde{\mu}^2 r^2(d_1 + d_2) \log(1/\epsilon))$) measurements, where the incoherence parameter $\tilde{\mu} = \|V^\top\|_{1-2\sqrt{d_2/r}}$ corresponds to the “spikiness” of the right singular vectors. We compare these results to Theorem 1.2 from the following perspectives.

**Low-spikiness vs incoherence:** The incoherence parameter $\tilde{\mu}$ on singular vectors is upper-bounded by the product of the condition number $\kappa$ and the spikiness $\mu$ of $M_0$. This implies that their sample complexity depends on conditioning of the unknown matrix $M_0$. This dependence has been pointed out as a weakness of nonconvex algorithms for low-rank recovery compared to the convex counterpart. It might be possible to alleviate this dependence by adopting a scaled gradient descent algorithm as in [40]. However, in the presence of noise, the spectral initialization, on which the performance of these algorithms depends critically, cannot be made free from the dependence on $\kappa$. Furthermore, it is easy to construct matrices such that $\tilde{\mu}$ is much larger than $\mu$. To see this, let $M$ be of rank $r - 1$. Suppose that all left (resp. right) singular vectors are orthogonal to $u \in S^{d_1 - 1}$ (resp. $e_1 \in S^{d_2 - 1}$). Then $M_0 = M + tue_1^\top$ is of rank-$r$. It follows that the right singular vectors of $M_0$ are given as those of $M$ and $e_1$. For small $t$, the spikiness $\mu$ of $M_0$ is similar to that of $M$. However, the incoherence of $\mu$ is close to the maximal value $\sqrt{d_2/r}$. Therefore, one may argue that the convex estimator in (2) has advantages in terms of its stable operation regardless of conditioning of $M_0$.

**Flexibility of model:** In an ideal scenario where $M_0$ is exactly rank-$r$ and measurements are noise-free, if one ignores the difference between $\mu$ and $\tilde{\mu}$, then the sample complexity for alternating minimization and gradient descent [25, 26] is significantly better than that in Theorem 1.2 in terms of their dependence on $\epsilon$. To get within accuracy $\epsilon$, [25, 26] require
\(O(\log(1/\epsilon))\) measurements whereas Theorem 1.2 requires \(O(1/\epsilon^2)\). However, Theorem 1.2 provides a uniform error bound over all matrices that are approximately low-rank. It is unclear how the results in [25, 26] could be extended to this more flexible model.

Furthermore, our result shows that the convex estimator in (2) is consistent and provides a near optimal error bound in the presence of (not too weak) noise. This result matches the minimax lower bound presented in Theorem 1.3.

1.4 Low-rank matrix completion without incoherence conditions

As shown earlier, a local measurement operator also arises in matrix completion. Even in the absence of noise, the unique identification of \(M_0\) in matrix completion has been shown only under certain incoherence conditions imposed on its singular vectors. However, in practice, it is often not possible to verify that such incoherence conditions are satisfied by the ground-truth matrix \(M_0\). On the contrary, in many applications like collaborative filtering, the entries of the unknown matrix are bounded by a certain threshold. Motivated by this observation, Negahban and Wainwright [28] considered an alternative estimation problem, in which the unknown matrix satisfies the above milder condition. Specifically, they demonstrated that a LASSO estimator with the nuclear norm provides the minimax-optimal error bound when the noise level is higher than some threshold. Later the max-norm has been proposed as an alternative regularizer to the nuclear norm [23, 33, 34]. Particularly for completing low-rank matrices with bounded entries, it has been shown that the max-norm regularized estimator empirically outperform that with the nuclear norm [34]. Foygel and Srebro [11] presented a near optimal error bound. This result was extended to sampling with a non-uniform distribution and sharpened to an optimal error bound by Cai and Zhou [4].

We provides an alternative derivation of the statistical analysis of the max-norm LASSO estimator for matrix completion. Cai and Zhou [4] employed Bousquet’s version of Talagrand’s concentration inequality for empirical processes indexed by bounded functions. Indeed, this enabled to drop the logarithmic factor in the previous result [11]. Our approach is based on a unifying characterization of the \(\gamma_2\) norm via the projective tensor norm and Maurey’s empirical method by Carl [6]. Although our error bound resulted in an extra poly-log factor, we believe that this alternative analysis can be useful to understand
other regression problems with a local measurement operator sharing similar structure. The result is presented in Section 5.

1.5 Summary of contributions and related work

In this paper, we present a unifying design principle and statistical analysis for a convex estimator for approximately low-rank matrices from local noisy measurements. The results are given as a near optimal error bound specifically for two illustrating examples of decentralized sketching and matrix completion. The convex regularizer we design adapting to the structure in local measurement operators generalizes the max-norm and mixed-norm to a set of tensor norms. It has been empirically shown that tensor norms beyond the two examples also provide successful regularization for denoising [3].

The relations among various tensor norms of rank-constrained linear operators are derived for a selected pairs of Banach spaces via fundamental properties studied in classical functional analysis [15]. In particular, we show that the mixed-norm and regularized inference with respect to it can be rewritten as a standard semidefinite program.

Furthermore we present upper bounds on the entropy integral with respect to covering number between tensor products of Banach spaces via Maurey’s empirical method [6]. This result leads to a version of restricted isometry property for the local linear operators arising in decentralized sketching and matrix completion. Finally, we show that these concentration results provide a near optimal error bound for the corresponding applications.

Besides the aforementioned related work on decentralized sketching and matrix completion, there has been discussions on related problems in the literature. Multilinear regression [27] and sketching low-rank covariance [1, 2] share similar structure to decentralized sketching. However, the problem settings and objectives are different from decentralized sketching in various perspectives and the results are not directly comparable. Kliesch et al. [19] considered a set of novel ideas on related regularization problem. They extended the nuclear norm to square and diamond norms. The former improved the stability of inverse problem by optimizing the descent cone of the regularizer and the latter enabled regularization applies to a more general objects in operator spaces, which was inspired from key applications in quantum tomography.

The rest of this paper is organized as follows: Preliminaries on tensor product and tensor
norms are provided in Section 2, followed by the analysis of tensor norms on selected pairs of Banach spaces in Section 3. We present the covering number analysis of tensor products in Section 4. Then we apply these results to get an error bound in Section 5. A matching information-theoretic lower bound is derived in Section 6. We conclude the paper with discussions and future directions.

2 Notations and preliminaries

In this section, we introduce nations used throughout the paper, and recall the definition of a tensor product of two Banach spaces and various norms defined on it.

For a positive integer $d$, let $[d]$ denote the set $\{1, 2, \ldots, d\}$. For real numbers $a$ and $b$, let $a \lor b$ and $a \land b$ denote the maximum and minimum of $\{a, b\}$. For a matrix $M$, $M^\top$ denotes its transposition. For a linear operator $A$, we use $A^*$ to denote the adjoint operator. For a Banach space $X$, its norm dual is denoted by $X^\ast$. We use $\otimes$ for both tensor product and Kronecker product, the distinction will be clear in the context. By $a \lesssim b$, we mean that there is an absolute constant $C$ such that $a \leq Cb$. Throughout the paper, $c, C, c_k, C_k$ will denote absolute constants which might vary line to line.

2.1 Tensor product of Banach spaces

A matrix $M \in \mathbb{R}^{m \times n}$ can be interpreted as the matrix representation of a linear operator from a vector space of dimension $n$ to another vector space of dimension $m$. We will also interpret as an element in the tensor product. The algebraic tensor product of two Banach spaces $X$ and $Y$, denoted by $X \otimes Y$, is a set of all finite sum of outer products of two vectors respectively from $X$ and $Y$. It is regarded as a vector space $L(X^\ast, Y)$, which consists of all linear operators from $X^\ast$ to another Banach space $Y$, where $X^\ast$ denotes the norm dual of $X$. There are various ways to define a norm on $X \otimes Y$.

2.2 Tensor norms

A norm $\| \cdot \|$ on $X \otimes Y$ is called a tensor norm $[15]$ if it satisfies

$$\|x \otimes y\| \leq \|x\|_X \|y\|_Y, \quad \forall x \in X, y \in Y$$
and its dual norm denoted by $\| \cdot \|_\pi$ satisfies

$$\| x^* \otimes y^* \|_\pi \leq \| x^* \|_{X^*} \| y^* \|_{Y^*}, \quad \forall x^* \in X^*, y^* \in Y^*.$$ 

We consider a selected tensor norms defined below.

The injective tensor norm is the smallest tensor norm on $X \otimes Y$ and is defined by

$$\| T \|_\vee := \sup_{x^* \in B_{X^*}, y^* \in B_{Y^*}} |\langle x^* \otimes y^*, T \rangle|,$$

where $B_{X^*}$ and $B_{Y^*}$ are the unit balls in $X^*$ and $Y^*$ respectively. Then $X \otimes Y$ will denote the corresponding Banach space. The injective tensor norm of $T$ coincides with the operator norm of $T \in L(X^*, Y)$ denoted by $\| T \|$.

The projective tensor norm is the largest tensor norm on $X \otimes Y$ and is defined by

$$\| T \|_\wedge := \inf \left\{ \sum_n \| x_n \|_X \| y_n \|_Y : T = \sum_n x_n \otimes y_n \right\}.$$ 

The corresponding Banach space is denoted as $X \bar{\otimes} Y$. Particularly when $X$ and $Y$ are finite-dimensional, the projective tensor norm of $T$ coincides with the 1-nuclear norm denoted by $\nu_1(T)$ [15].

The 2-summing norm of $T \in L(X^*, Y)$, denoted by $\pi_2(T)$, is defined as the smallest constant $c$ that satisfies

$$\left( \sum_i \| T x_i^* \|^2 \right)^{1/2} \leq c \sup \left\{ \left( \sum_i |\langle x, x_i^* \rangle|^2 \right)^{1/2} : x \in B_X \right\}$$

for all sequence $(x_i^*) \subset X^*.$

There exists a tensor norm defined though the optimal factorization through a finite dimensional Hilbert space. The $\gamma_2$ norm of $T \in X \otimes Y$ is defined by

$$\gamma_2(T) := \inf \{ \| T_1 \| \| T_2 \| : d \in \mathbb{N}, T_1 \in B(X^*, \ell_2^d), T_2 \in B(\ell_2^d, Y), T = T_2 T_1 \}.$$ 

### 3 Relation between tensor norms of rank-$r$ operators

In this section, we demonstrate the relation between the injective and projective tensor norms of rank-$r$ operators in a tensor product of Banach spaces. We first recall the known results on $\ell_\infty^n \otimes \ell_\infty^m$ in the literature.
Linial et al. [23] showed that the $\gamma_2$ norm on $\ell^n_\infty \otimes \ell^m_\infty$ is upper-bounded by the operator norm multiplied by the square root of the rank. In fact, their result is derived from the fact that the Banach-Mazur distance (see e.g. [30]) between a finite-dimensional Banach space and a Hilbert space is no larger than the square root of the dimension. Therefore their result [23, Lemma 4.2] applies to any pair of Banach spaces. We paraphrase it as the following lemma.

**Lemma 3.1** (A paraphrased version of [23, Lemma 4.2]). *Suppose that $T \in X \otimes Y$ with Banach spaces $X, Y$ satisfies $\text{rank}(T) \leq r$. Then*

$$\|T\| \leq \gamma_2(T) \leq \sqrt{r}\|T\|.$$  

Furthermore, for a linear operator from $\ell^n_\infty$ to $\ell^m_1$, it is well known that its $\gamma_2$ norm and the 1-nuclear norm are equivalent up to the Grothendieck constant (e.g., [15]).

**Lemma 3.2** (little Grothendieck (e.g., [15])). *Let $T \in \ell^n_\infty \otimes \ell^m_\infty$. Then*

$$\gamma_2(T) \leq \nu_1(T) \leq K_G \gamma_2(T),$$

*where $K_G$ denotes the Grothendieck constant that satisfies $1.67 \leq K_G \leq 1.79$.*

Then we obtain the following corollary as a direct consequence of Lemmas 3.1 and 3.2.

**Corollary 3.3.** *Let $T \in \ell^n_\infty \otimes \ell^m_\infty$ satisfy $\text{rank}(T) \leq r$. Then*

$$\|T\| \leq \nu_1(T) \leq K_G \sqrt{r}\|T\|.$$  

Corollary 3.3 shows the equivalence of the injective and projective tensor norms of a rank-$r$ linear operator in $\ell^n_\infty \otimes \ell^m_\infty$ up to $K_G\sqrt{r}$.

One might deduce from Lemma 3.1 that $\gamma_2$ norm might be a good regularizer to induce low-rankness for any choice of Banach spaces. Unfortunately this is not the case. As we show in the next section, the covering number from the projective tensor product to the injective tensor product admits a meaningful upper bound that leads to a near optimal error bound. Therefore we pursue a tensor-norm regularizer, which is equivalent to the projective tensor norm. In fact, Lemma 3.2 shows that the $\gamma_2$ norm is equivalent up to a constant to the 1-nuclear norm or the projective tensor norm. This is the reason why the $\gamma_2$ norm on $\ell^m_\infty \otimes \ell^m_\infty$, which is the max-norm, was an effective regularizer for low-rankness.
3.1 Equivalence of tensor norms for linear operators to a Hilbert space

Next we show that analogous equivalence results are available for rank-$r$ linear operators in $X \otimes \ell_2^m$, where $X$ belongs to a set of Banach spaces. The first lemma shows that the 2-summing norm of the adjoint does not exceed the operator norm multiplied by the square root of the rank.

Lemma 3.4. Let $T \in X \otimes \ell_2^m$, where $X$ is a Banach space. Suppose that $T^*$ is 2-summing and $\text{rank}(T) \leq r$. Then we have

$$\|T\| \leq \pi_2(T^*) \leq \sqrt{r}\|T\|.$$

Proof. Let $E = \ker(T^*)^\perp$. Then it follows that $T^* = T^*P_E$, where $P_E$ denotes the orthogonal projection onto $E$. Furthermore, since $\text{rank}(T^*) \leq r$, we have $\dim(E) \leq r$. By [15, Proposition 3.1], the 2-summing norm of $T^*$ is upper-bounded by

$$\pi_2(T^*) \leq \|T^*\|_{\pi_2}(P_E).$$

Since the $\ell_2$ norm is unitarily-invariant, there exists a diagonal operator $\Lambda$ such that $\pi_2(P_E) = \pi_2(\Lambda)$. Then, since $\dim(E) \leq r$, it follows by [15, Proposition 3.5] that

$$\pi_2(P_E) \leq \sqrt{r}.$$

Finally it trivially holds that $\|T^*\| = \|T\|$. Combining the results provides the upper bound. Next, the lower bound is obtained from the fact that the 2-summing norm is always lower-bounded by the operator norm (see [15, Proposition 3.1]). This completes the proof. \qed

The result in Lemma 3.4 applies beyond the case $X = \ell_2^m$. For example, when $X = \ell_2^n$, the 2-summing and operator norms become the Frobenius and spectral norms, respectively.

Next we present the relation between the 2-summing norm and the 1-nuclear norm of a linear operator in $\ell_2^n \otimes Y$ where $Y$ is a Banach space of type-2 defined as follows.

Definition 3.5. A Banach space $X$ has type $p$ if there exists a constant $C$ such that for all finite sequence $(x_j)$ in $X$

$$\left( \mathbb{E} \left\| \sum_j \epsilon_j x_j \right\|_X^p \right)^{1/p} \leq C \left( \sum_j \|x_j\|_X^p \right)^{1/p},$$

where $(\epsilon_j)$ is a Rademacher sequence [30]. The type-$p$ constant of $X$, denoted by $\tau_p(X)$, is the smallest constant $C$ that satisfies (13).
The triangle inequality implies that every normed space is of type 1 with constant \( \tau_1(X) = 1 \). In this paper, we are interested in Banach spaces of type 2. For example, \( L_p(\Omega, \Sigma, \mu) \) satisfies
\[
\tau_{\min(p,2)}(L_p) \leq \sqrt{p}
\]
for \( 1 \leq p < \infty \) (e.g. [6, Lemma 3]). This implies that \( \ell_p^n \) is of type 2 for all \( 2 \leq p < \infty \).

The following lemma shows that the 1-nuclear norm of \( T \in \ell_\infty^n \otimes Y \) is equivalent to the 2-summing norm of the adjoint \( T^* \) up to the type-2 constant of \( Y \).

**Lemma 3.6.** Let \( T \in \ell_\infty^n \otimes Y \), where \( Y \) is a finite-dimensional Banach space of type-2. Suppose that \( T^* \) is 2-summing. Then we have
\[
\pi_2(T^*) \leq \nu_1(T) \leq \sqrt{2} \tau_2(Y) \pi_2(T^*). 
\]

**Proof of Lemma 3.6.** We first compute the nuclear-1 norm of the adjoint \( T^* \in Y \otimes \ell_\infty^n \). By the trace duality, the projective tensor norm of \( T^* \) is written as
\[
\nu_1(T^*) = \sup \{ \text{tr}(S^*T^*) : S^* \in \ell_1^n \otimes Y^*, \|S^*\| \leq 1 \}. \tag{14}
\]
Recall that the 2-summing norm \( \pi_2 \) is self-dual with respect to the trace duality. Therefore, we have
\[
\text{tr}(S^*T^*) \leq \pi_2(S^*) \pi_2(T^*). \tag{15}
\]
Since \( S^* \in \ell_1^n \otimes Y^* \) and \( Y \) is of type-2, by [15, Propositions 9.3 and 9.8], it follows that
\[
\pi_2(S^*) \leq \sqrt{2} \tau_2(Y) \|S^*\|. \tag{16}
\]
By plugging in (15) and (16) into (14), we obtain
\[
\nu_1(T^*) \leq \sqrt{2} \tau_2(Y) \pi_2(T^*). 
\]
Moreover, since all Banach spaces here are finite-dimensional, it follows from [15, Proposition 1.13] that
\[
\nu_1(T) = \nu_1(T^*). 
\]
The lower bound is obtained from the fact that the nuclear-1 norm is the largest operator ideal norm and the 2-summing norm is an operator ideal norm. This completes the proof. \(\square\)
By combining Lemmas 3.4 and 3.6 with the fact that $\tau_2(\ell_p^m) \leq \sqrt{p}$ for $2 \leq p < \infty$ (e.g., see [6, Lemma 3]), we obtain the following corollary.

**Corollary 3.7.** Let $T \in \ell^m_2 \otimes \ell^m_2$ be of rank-$r$. Then, we have

$$\|T\| \leq \pi_2(T^*) \leq \nu_1(T) \leq \sqrt{2r}\|T\|.$$  

Corollary 3.7 implies that both the 2-summing and 1-nuclear norms of a rank-$r$ operator are equivalent to the operator norm up to $\sqrt{2r}$. Moreover, the 1-nuclear and operator norms correspond to the projective and injective tensor norms in this setting. In the next section, we will use the interlacing property of these norms to compute entropy integrals, which arise in the analysis of relevant low-rank recovery problems.

### 3.2 Semidefinite program characterization of tensor norms

While the relation between the injective and projective tensor norms of rank-$r$ operators is useful for the entropy estimate, it is often easier to compute the 2-summing norm of the adjoint than the projective norm. Below we show that $\pi_2(T^*)$ is written through an optimal factorization similarly to the $\gamma_2$ norm on the tensor product $\ell^m_\infty \otimes \ell^m_\infty$.

**Lemma 3.8.** Let $T \in X \otimes Y$ with $X$ complete. Then

$$\pi_2(T^*) := \inf\{\pi_2(T^*_1)\|T^*_2\| : d \in \mathbb{N}, T^*_1 \in L(Y^*, \ell^d_2), T^*_2 \in L(\ell^d_2, X), T^* = T^*_2T^*_1\}. \quad (17)$$

**Proof of Lemma 3.8.** We use the following result by Pietsch to prove Lemma 3.8.

**Lemma 3.9 ([15, Theorem 5.8]).** Let $X,Y$ be normed linear spaces ($Y$ complete) and $T \in L(X, Y)$ be a 2-summing operator. Then there exists a Hilbert space $H$ and operators $T_1 \in L(X, H)$ and $T_2 \in L(H, Y)$ such that $T = T_2T_1$, $\pi_2(T_1) = \pi_2(T)$, and $\nu_1(T_2) = 1$.

By Lemma 3.9, there exist $d \in \mathbb{N}, T^*_1 \in L(Y^*, \ell^d_2)$, and $T^*_2 \in L(\ell^d_2, X)$ such that $T^* = T^*_2T^*_1$, $\pi_2(T^*_1) = \pi_2(T^*)$ and $\|T^*_2\| = 1$. Furthermore, by [15, Proposition 3.1], we have

$$\pi_2(T^*_2T^*_1) \leq \pi_2(T^*_1)\|T^*_2\|$$

for any $T^*_1 \in L(Y^*, \ell^d_2)$ and $T^*_2 \in L(\ell^d_2, X)$. Therefore, the 2-summing norm of $T^*$ is written as in (17). \qed
In a special case when $T \in \ell_2^m \otimes \ell_2^n$, the 2-summing norm of $T^*$ is computed by the following optimization problem:

$$\pi_2(T^*) = \inf \{ \|T_1\|_F \|T_2\| : d \in \mathbb{N}, T_1 \in L(\ell_2^d, \ell_2^m), T_2 \in L(\ell_1^n, \ell_2^n), T = T_1 T_2 \}. \quad (18)$$

Indeed, it trivially holds that $\|T^*_1\| = \|T_1\|$. Furthermore, for every $T^*_1 \in L(\ell_2^m, \ell_2^n)$, we have

$$\pi_2(T^*_1) = \|T_1^*\|_F = \|T_1\|_F = \pi_2(T_1).$$

Therefore, (18) follows from (17). Below we show that (18) can also be expressed as a semidefinite program. Note that if $T_1 T_2 = T$, then $T'_1 = aT_1$ and $T'_2 = a^{-1}T_2$ for any $a \neq 0$ also satisfy $T'_1 T'_2 = T$. Therefore, from the arithmetic-geometric-harmonic-means inequality, it follows that (18) is equivalently rewritten as

$$\pi_2(T^*) = \frac{1}{2} \inf \{ \|T_1\|^2_F + \|T_2\|^2 : d \in \mathbb{N}, T_1 \in L(\ell_2^d, \ell_2^m), T_2 \in L(\ell_1^n, \ell_2^n), T = T_1 T_2 \}.$$

We also have:

$$\|T_1\|^2_F = \text{trace}(T_1 T_1^*) = \langle T_1, T_1 \rangle \quad \text{and} \quad \|T_2\|^2 = \|\text{diag}(T^*_2 T_2)\|_F,$$

where $\text{diag}(\cdot)$ constructs a column vector consisting of the diagonal entries of the input matrix. Furthermore, by the proof of [34, Lemma 3], there exists $T_1$ and $T_2$ such that $T = T_1 T_2$, $W_1 = T_1 T_1^*$, and $W_2 = T_2 T_2^*$ if and only if

$$\begin{bmatrix} W_1 & T \\ T^* & W_2 \end{bmatrix} \succeq 0.$$

Combining these results implies that $\pi_2(T^*)$ is written as a semidefinite program given by

$$\pi_2(T^*) = \inf_{W_1, W_2} \max \{ \text{trace}(W_1), \|\text{diag}(W_2)\|_F \}$$

s.t. $$\begin{bmatrix} W_1 & T \\ T^* & W_2 \end{bmatrix} \succeq 0.$$

### 4 Entropy estimates of tensor products

This section is devoted to derive estimates on covering number and the resulting entropy integral for tensor products of Banach spaces. Let us first recall relevant definitions to state the main results.
For symmetric convex bodies $D$ and $E$, the covering number $N(D, E)$ and the packing number $M(D, E)$ are respectively defined by

\[
N(D, E) := \min \left\{ l : \exists y_1, \ldots, y_l \in D, D \subseteq \bigcup_{1 \leq j \leq l} (y_j + E) \right\},
\]

\[
M(D, E) := \max \left\{ l : \exists y_1, \ldots, y_l \in D, y_j - y_k \notin E, \forall j \neq k \right\}.
\]

Indeed, they are related to each other by

\[N(D, E) \leq M(D, E) \leq N(D, E)/2.\]

For $T \in L(X, Y)$, the dyadic entropy number \cite{dyadic} is defined by

\[e_k(T) := \inf \{ \epsilon > 0 : M(T(B_X), \epsilon B_Y) \leq 2^{k-1} \}.
\]

We will use the following shorthand notation for the weighted summation of the dyadic entropy numbers:

\[
\mathcal{E}_{2,1}(T) := \sum_{k=0}^{\infty} \frac{e_k(T)}{\sqrt{k}},
\]

which is up to a constant equivalent to the entropy integral

\[\int_0^{\infty} \sqrt{\ln N(T(B_X), \epsilon B_Y)} d\epsilon \leq 30,
\]

which plays a key role in analyzing properties on random linear operators on low-rank matrices.

We derive the $\mathcal{E}_{2,1}$ of the identity operator from the injective tensor product to the projective tensor product of a set of Banach space pairs. The main machinery in deriving these estimates is Maurey’s empirical method \cite{maurey}, summarized in the following lemma.

**Lemma 4.1.** Let $T \in L(\ell_1^n, \ell_\infty^m)$. Then

\[\mathcal{E}_{2,1}(T) \leq C \sqrt{1 + \ln(m \lor n) (1 + \ln(m \land n))^{3/2}} \|T\|.
\]

**Proof of Lemma 4.1.** The case when $n \geq m$ has been shown in \cite[Lemma 3.7]{maurey}. Therefore, we only show the proof for the other case with $n < m$. We use Maurey’s empirical method given in the following lemma.

**Lemma 4.2** \cite[Proposition 3]{maurey}. Let $T : \ell_1^n \to \ell_\infty^m$. Then there exists a numerical constant $C$ such that

\[e_k(T) \leq Ch(k, n, m) \|T\|,
\]

where
\[ h(k, n, m) := 2^{-(k/n \vee k/m \vee 1)} \]
\[ \cdot \left\{ 1 \wedge \left[ 1 \vee \log_2 \left( \frac{m}{k} + 1 \right) \right] \cdot \left[ 1 \wedge \left( \frac{\log_2 \left( \frac{m}{k} + 1 \right)}{k} \vee \frac{1}{n} \right) \right] \right\}^{1/2}. \]

Lemma 4.2 implies
\[ e_k(T) \leq C_5 \sqrt{1 + \ln m} \|T\| \sqrt{1 + \ln N} k^{-1/2}, \quad \forall k \leq n, \]
and
\[ e_k(T) \leq C_5 \sqrt{1 + \ln m} \|T\| 2^{-k/(2m)} N^{-1/2}, \quad \forall k > n. \]

Therefore,
\[ \sum_{k=1}^{\infty} \frac{e_k(T)}{\sqrt{k}} \leq C_5 \sqrt{1 + \ln m} \|T\| \left( \sqrt{1 + \ln N} \sum_{k=1}^{2m} k^{-1} + N^{-1/2} \sum_{k=2m+1}^{\infty} k^{-1/2} 2^{-k/(2m)} \right), \quad (19) \]

where (§) is upper-bounded by
\[ (§) \leq 1 + \int_1^{2m} t^{-1} dt \leq 1 + \ln 2 + \ln m \quad (20) \]
and (§§) is upper-bounded by
\[ (§§) \leq \int_{2m}^{\infty} t^{-1/2} e^{-t/(2m)} dt \leq 2 \int_{\sqrt{2m}}^{\infty} \exp \left( -\frac{\zeta^2}{2m/\ln 2} \right) d\zeta \leq \sqrt{\frac{\pi 2m}{\ln 2}}. \quad (21) \]

Then the assertion follows by plugging in (20) and (21) to (19).

When \( m > N \), instead we have
\[ e_k(T) \leq C_5 \sqrt{1 + \ln m} \|T\| \sqrt{1 + \ln N} k^{-1/2}, \quad \forall k \leq 2N, \]
and
\[ e_k(T) \leq C_5 \sqrt{1 + \ln m} \|T\| 2^{-k/(2N)} N^{-1/2}, \quad \forall k > 2N. \]

The logarithmic factor in the resulting is \( \sqrt{1 + \ln m (1 + \ln N)^3/2} \), which is less than \( \sqrt{1 + \ln N (1 + \ln m)^3/2} \). This completes the proof. \( \square \)
In order to apply Lemma 4.1 to $\ell^m_x \otimes \ell^m_x$, we use the fact that $\ell^m_x \otimes \ell^m_x$ is isometrically isomorphic to $\ell^{mn}_x$. In fact,
\[
\|M\|_{\ell^m_x \otimes \ell^m_x} = \max_{1 \leq j \leq n} \|Me_j\|_\infty = \|\text{vec}(M)\|_\infty,
\]
where vec($M$) rearranges $M \in \mathbb{R}^{m \times n}$ into $\mathbb{R}^{mn}$ by stacking its columns vertically. Furthermore, the trace dual and the Banach space dual of $\ell^m_x \otimes \ell^m_x$ are $\ell^m_x \hat{\otimes} \ell^m_x$ and $\ell^m_x \hat{\otimes} \ell^m_x$, respectively. Therefore, it follows that $\ell^m_x \hat{\otimes} \ell^m_x$ is isometrically isomorphic to $\ell^{mn}_x$. With these isometric isomorphisms, Maurey’s empirical method in Lemma 4.1 provides the following estimate.

**Proposition 4.3.** There exists a numerical constant $C$ such that
\[
\mathcal{E}_{2,1}(\text{id}: \ell^m_x \otimes \ell^m_x \to \ell^m_x \otimes \ell^m_x) \leq C\sqrt{1 + m + n (1 + \ln m + \ln n)^3/2}.
\]

**Proof of Proposition 4.3.** Let $\iota_m : \ell^m_1 \to \ell^m_x$ be defined by
\[
\iota_m ([a_j]_{j=1}^m) = \left( \sum_{j=1}^m \epsilon_j a_j \right)_{(\epsilon_j)_{j=1}^m \in \{\pm 1\}^m}.
\]
Then we have
\[
\|\iota_m([a_j]_{j=1}^m)\|_\infty = \max \left\{ \left\| \sum_{j=1}^m \epsilon_j a_j \right\| : (\epsilon_j)_{j=1}^m \in \{\pm 1\}^m \right\} = \|([a_j]_{j=1}^m)\|_1.
\]
This implies that $\ell^m_1$ is isometrically isomorphic to $E_m = \iota_m(\ell^m_1) \subset \ell^m_x$. Since $\iota_m \in L(\ell^m_1, E_m)$ is an isomorphism, there exists $J_m \in L(\ell^m_x, \ell^m_1)$ such that $\iota_m J_m|_{E_m}$ (resp. on $J_m \iota_m$) is the identity on $E_m$ (resp. $\ell^m_1$). Then $J_m^*(\ell^m_x) = E_m^*$ and $\ell^m_x$ is isometrically isomorphic to $E_m^*$. Furthermore, by the Hahn-Banach theorem, $\iota_m^* \in L(\ell^m_x, \ell^m_1)$ is surjective and isometric. It also follows that $J_m$ is also surjective and isometric.

Then there exists an isometry $\iota$ that embeds $\ell^m_1 \hat{\otimes} \ell^m_1$ into $\ell^m_x \hat{\otimes} \ell^m_x$. Indeed, $\ell^m_1 \hat{\otimes} \ell^m_1$ is identified to $L(\ell^m_x, \ell^m_1)$. By $\iota_m \in L(\ell^m_1, \ell^m_x)$ and $J_n^* \in L(\ell^m_x, \ell^m_1)$, $L(\ell^m_x, \ell^m_1)$ is isometrically isomorphic to $L(E_n^*, E_m)$. Since $E_n^*$ and $E_m$ are subspaces of $\ell^m_1$ and $\ell^m_x$ respectively, $L(E_n^*, E_m)$ embeds into $L(\ell^m_x, \ell^m_x)$, which is identified to $\ell^m_x \hat{\otimes} \ell^m_x$. Furthermore, similarly to the above argument, by the Hahn-Banach theorem, $\iota^* : \ell^m_1 \hat{\otimes} \ell^m_1 \to \ell^m_x \hat{\otimes} \ell^m_x$ is surjective and isometric.

Then we obtain the following commutative diagram:
Since the map \( \iota^* \) is surjective and isometric, we have

\[
e_k(\text{id} : \ell_2^n \otimes \ell_1^m \to \ell_\infty^n \otimes \ell_\infty^m) \leq e_k(T : \ell_1^{2n} \otimes \ell_1^{2m} \to \ell_\infty^n \otimes \ell_\infty^m) = e_k(\text{id} : \ell_2^{2n+m} \to \ell_\infty^{nm}).
\]

Finally, the assertion follows by Maurey’s empirical method in Lemma 4.1. \( \square \)

Next, in order to apply Lemma 4.1 to \( \ell_\infty^n \otimes \ell_p^m \) with \( 2 \leq p < \infty \), we use the following result that shows embedding of finite-dimensional \( \ell_p \) space to \( \ell_1 \) up to a small Banach-Mazur distance.

**Lemma 4.4.** [6, Lemma 5] Let \( 1 < p < 2 \). For each \( \epsilon > 0 \), there exists a constant \( c(p, \epsilon) > 0 \) such that for each \( m \), \( \ell_1^m \) contains a subspace \( (1 + \epsilon) \)-isomorphic to \( \ell_1^k \) with \( k \geq c(p, \epsilon) m \).

The following corollary is a direct consequence of Lemma 4.4 via the Hahn-Banach theorem.

**Corollary 4.5.** Let \( 2 \leq p < \infty \) and \( p' \) satisfy \( 1/p + 1/p' = 1 \). Then there exists a subspace \( E' \) of \( \ell_\infty^{[m/c(p')]} \) such that \( \ell_1^m \) is isomorphic to \( E' \) with the Banach-Mazur distance at most \( 2 \), where \( c(p') \) is a constant that only depends on \( p' \).

**Proof.** By Lemma 4.4, there exists an isomorphism \( A \in L(\ell_\infty^{m/c(p')}, E) \) with \( E \subset \ell_1^{[m/c(p')]} \) such that \( \|A\| A^{-1} \| \leq 2 \), where \( A^{-1} \) is defined on \( E \) and \( c(p') \) only depends on \( p' \). By the Hahn-Banach theorem, there also exists an isomorphism \( \Upsilon \) that maps \( \ell_p^m \) to \( E^* \subset \ell_\infty^{[m/c(p')]} \). Let \( \Upsilon^{-1} \) be a map on \( E^* \) such that \( \Upsilon^{-1} \Upsilon = \text{id}_{\ell_p^m} \) and \( \Upsilon \Upsilon^{-1} = \text{id}_{E^*} \). Then \( \|\Upsilon\| \|\Upsilon^{-1}\| \leq 2 \). \( \square \)

Then we can obtain the following entropy estimate for \( \ell_\infty^n \otimes \ell_p^m \) with \( 2 \leq p < \infty \) by combining Lemmas 4.1 and 4.4.

**Proposition 4.6.** Let \( 2 \leq p < \infty \). Then

\[
\mathcal{E}_{2,1}(\text{id} : \ell_\infty^n \otimes \ell_p^m \to \ell_\infty^n \otimes \ell_p^m) \leq C \sqrt{1 + n + m(1 + \ln n + \ln m)^{3/2}}.
\]
Proof of Proposition 4.6. Let $p'$ satisfy $1/p + 1/p' = 1$. By Lemma 4.5, there exist $\Upsilon: \ell_p^m \to E^* \subset \ell_1^{[m/c(p')]}, \Upsilon^{-1} : E^* \to \ell_p^m$ such that $\Upsilon^{-1} \Upsilon = \text{id}_{\ell_p^m}$, $\Upsilon \Upsilon^{-1} = \text{id}_{E^*}$ and $\|\Upsilon\| \|\Upsilon^{-1}\| \leq 2$. Without loss of generality, we may assume that $\|\Upsilon\| = 1$ and $\|\Upsilon^{-1}\| \leq 2$.

Next similarly to the proof of Proposition 4.3, one can embed $\ell_2^{[m/c(p')]},$ through an isometric isomorphism. By taking composition of the two embedding maps, we obtain that $\ell_p^m$ is embedded into a subspace of $\ell_1^{[m/c(p')]},$ with the Banach-Mazur distance up to 2.

Then there exists a map $\phi$ that embeds the tensor product $\ell_1^n \otimes \ell_p^m$ into $\ell_1^2 \otimes \ell_1^{[m/c(p')]},$ with the Banach-Mazur distance between $\ell_1^n \otimes \ell_p^m$ and $\phi(\ell_1^n \otimes \ell_p^m)$ is upper-bounded by 2.

Therefore we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\ell_1^n \otimes \ell_1^{[m/c(p')]}, & \xrightarrow{T} & \ell_1^n \otimes \ell_1^{[m/c(p')]}, \\
\phi & & \|\| \| \| \| \\
\ell_1^n \otimes \ell_p^m & \xrightarrow{\text{id} \otimes \Upsilon} & \ell_1^n \otimes E^* & \xrightarrow{\text{id} \otimes \Upsilon^{-1}} & \ell_1^n \otimes \ell_p^m \\
\end{array}
$$

Since $\|\text{id} \otimes \Upsilon^{-1} : \ell_1^n \otimes E^* \to \ell_1^n \otimes \ell_p^m \| \leq 2$, the assertion follows by the consequence of Maurey’s empirical method in Lemma 4.1.

\section{Estimation error upper bounds}

In this section, we derive statistical error bounds for decentralized subspace sketching and matrix completion by using Proposition 1.1. We obtain tail bounds of the geometric quantities $\theta$ and $\Gamma$.

\subsection{Decentralized subspace sketching}

We first prove Theorem 1.2 for decentralized subspace sketching. The following lemma, as a corollary of Proposition 4.6, provides a tail bound on $\theta$.

**Lemma 5.1.** Let $(b_k)_k^{Ld_2}, (\eta_k)_k^{Ld_2},$ and $\|\cdot\|$ be as in Theorem 1.2. Then there exists an
absolute constant $C$ such that

$$
\sup_{\|M\| \leq 1} \left\| M^2 - \frac{1}{L} \sum_{k=1}^{Ld_2} \langle b_k e_{jk}^T, M \rangle^2 \right\|
\leq C d_2 \left( \frac{r(d_1 + d_2) \log^3(d_1 + d_2) \lor \log(2\zeta^{-1})}{Ld_2^2} \lor \sqrt{\frac{r(d_1 + d_2) \log^3(d_1 + d_2) \lor \log(2\zeta^{-1})}{Ld_2}} \right)
$$

holds with probability $1 - \zeta$.

**Proof.** Let $Q_M \in \mathbb{R}^{Ld_1 d_2 \times Ld_1 d_2}$ be a block diagonal matrix defined by

$$
Q_M := \frac{1}{\sqrt{L}} \begin{bmatrix}
I_L \otimes (Me_1)^\top & 0 & \ldots & 0 \\
0 & I_L \otimes (Me_2)^\top & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & I_L \otimes (Me_{d_2})^\top
\end{bmatrix}.
$$

Then we have

$$
\frac{1}{L} \sum_{k=1}^{Ld_2} \langle b_k e_{jk}, M \rangle^2 = \|Q_M \xi\|^2
$$

for $\xi \sim \mathcal{N}(0, I_{Ld_1 d_2})$. It follows that $\|Q_M \xi\|^2$ is a Gaussian chaos indexed by $M$. Furthermore, it satisfies

$$
\mathbb{E} \|Q_M \xi\|^2 = \|M\|^2_F.
$$

Therefore, it suffices to obtain a tail estimate on the supremum of $\|Q_M \xi\|^2 - \mathbb{E} \|Q_M \xi\|^2$ over $\{M : \|M\| \leq 1\}$. To this end, we use the concentration inequality by Krahmer et al. [20], which is summarized in the following theorem.

**Theorem 5.2** (Theorem 3.1 in [20]). Let $\xi \in \mathbb{R}^n$ be a Gaussian vector with $\mathbb{E}[\xi] = 0$ and $\mathbb{E}[\xi^\top] = I_n$. Let $\Delta \subset \mathbb{R}^{m \times n}$. There exists an absolute constant $C$ such that

$$
\sup_{Q \in \Delta} \left| \|Q \xi\|^2_2 - \mathbb{E} \|Q \xi\|^2_2 \right| \leq C \left( E + V \sqrt{\log(2\zeta^{-1})} + U \log(2\zeta^{-1}) \right)
$$

holds with probability $1 - \zeta$, where

$$
E := \gamma_2(\Delta, \|\cdot\|) \left[ \gamma_2(\Delta, \|\cdot\|) + d_F(\Delta) \right],
V := d_S(\Delta) \left[ \gamma_2(\Delta, \|\cdot\|) + d_F(\Delta) \right],
U := d_S^2(\Delta).
$$
Here \( \tilde{\gamma}_2(\Delta, \|\cdot\|) \) denotes the Talagrand \( \gamma_2 \) functional of the metric space \( \Delta, \|\cdot\| \) given by the spectral norm, and \( d_S(\Delta) \) and \( d_F(\Delta) \) denotes the radii of \( \Delta \) with respect to the spectral norm and the Frobenius norm, respectively.\[^1\]

We apply Theorem 5.2 to the set \( \Delta := \{ Q_M : M \in B_{\|\cdot\|} \} \), where the unit ball \( B_{\|\cdot\|} := \{ M : \| M \| \leq 1 \} \) is written as \( B_{\|\cdot\|} = \{ M : \| M \|_{1\to 2} \leq 1, \| M \|_{\text{mixed}} \leq \sqrt{r} \} \). Note that \( d_F(\Delta) \) and \( d_S(\Delta) \) are respectively upper-bounded by

\[
d_F(\Delta) \leq \sqrt{d_2} \quad \text{and} \quad d_S(\Delta) \leq \frac{1}{\sqrt{L}}.
\]

Let \( B_S \) denote the unit ball with respect to the spectral norm. Then, by Dudley’s inequality, \( \tilde{\gamma}_2(\Delta) \) is upper-bounded through by

\[
\tilde{\gamma}_2(\Delta, \|\cdot\|) \lesssim \int_0^\infty \sqrt{\log N(\Delta, \eta B_S)} \, d\eta \\
\lesssim \frac{1}{\sqrt{L}} \int_0^\infty \sqrt{\log N(B_{\|\cdot\|}, \eta B_{1\to 2})} \, d\eta \\
\lesssim \frac{\sqrt{r(d_1 + d_2) \log^3(d_1 + d_2)}}{\sqrt{L}},
\]

where the second inequality holds since \( \| Q_M - Q_{M'} \| = L^{-1/2} \| M - M' \|_{1\to 2} \) and the last inequality follows from Proposition 4.6. Then \( E, U, \) and \( V \) in Theorem 5.2 are upper-bounded respectively by

\[
E \lesssim d_2 \left( \sqrt{\frac{r(d_1 + d_2) \log^3(d_1 + d_2)}{Ld_2}} + \frac{r(d_1 + d_2) \log^3(d_1 + d_2)}{Ld_2} \right), \quad U \lesssim \frac{d_2}{Ld_2},
\]

and

\[
V \lesssim \frac{d_3}{\sqrt{Ld_2}} \left( \sqrt{\frac{r(d_1 + d_2) \log^3(d_1 + d_2)}{Ld_2}} + 1 \right).
\]

Then the assertion follows by plugging in these upper estimates to Theorem 5.2. \( \square \)

Next we obtain a tail bound on \( \Gamma \) by the following lemma.

\[^1\]We used a non-standard notation \( \tilde{\gamma}_2 \) to distinguish it from the \( \gamma_2 \) factorization norm in Section 2.
Lemma 5.3. Let \((b_k)_{k=1}^{Ld_2}, (\eta_k)_{k=1}^{Ld_2}\), and \(\|\cdot\|\) be as in Theorem 1.2. Then there exists an absolute constant \(C\) such that
\[
E_{(g_k)}\left\| \frac{1}{\sqrt{L}} \sum_{k=1}^{Ld_2} g_k b_k e_{j_k}^\top \right\|_\ast \leq C \sqrt{\log(\zeta^{-1})} \sqrt{d_2 \sqrt{r(d_1 + d_2)} \log^{3/2}(d_1 + d_2) \sqrt{\log(Ld_2 + 1)}}
\]
holds with probability \(1 - \zeta\).

Proof. The left-hand side is equivalent (up to a logarithmic factor of the size of the summation) to the corresponding Rademacher complexity (e.g. [21], Equation (4.9)), i.e.
\[
E_{(g_k)}\left\| \frac{1}{\sqrt{L}} \sum_{k=1}^{Ld_2} g_k b_k e_{j_k}^\top \right\|_\ast \leq \sqrt{\log(Ld_2 + 1)} E_{(r_k)}\left\| \frac{1}{\sqrt{L}} \sum_{k=1}^{Ld_2} r_k b_k e_{j_k}^\top \right\|_\ast,
\]
where \((r_k)_{k=1}^{Ld_2}\) is a Rademacher sequence and the expectation is conditioned on \((b_k)_{k=1}^{Ld_2}\). Then by the symmetry of the standard Gaussian distribution, we obtain
\[
E_{(r_k)}\left\| \sum_{k=1}^{Ld_2} r_k b_k e_{j_k}^\top \right\|_\ast = \sup_{\|M\| \leq 1} \left| \sum_{k=1}^{Ld_2} \langle r_k b_k, M e_{j_k} \rangle \right| = \sup_{\|M\| \leq 1} \left| \sum_{k=1}^{Ld_2} \langle b_k, M e_{j_k} \rangle \right|,
\]
where the second equation holds in the sense of distribution.

A tail bound on \((\xi)\) is derived by the following lemma, which is a direct consequence of the moment version of Dudley’s inequality (e.g., p. 263 in [10]) and a version of Markov’s inequality (e.g., Proposition 7.11 in [10]).

Lemma 5.4. Let \(\xi \sim \mathcal{N}(0, I_n), \Delta \subset \mathbb{R}^n\), and \(0 < \zeta < e^{1/2}\). Then there exists constant \(c\) such that
\[
\sup_{f \in \Delta} |f^* \xi| \leq c \sqrt{\log(\zeta^{-1})} \int_0^\infty \sqrt{\log N(\Delta, \eta B_2)} d\eta
\]
with probability \(1 - \zeta\).

Since the set \(\{M : \|M\| \leq \alpha\}\) is symmetric, we can omit the absolute value in the objective function in \((\xi)\). Let \(f_M := [1_{L,1} \otimes (M e_1); \ldots; 1_{L,1} \otimes (M e_{d_2})] \in \mathbb{R}^{Ld_2}\), where \(1_{L,1}\) denotes the column vector of length \(L\) with all entries set to 1. Then \((\xi)\) is written as the maximum of \(f^* \xi\) with \(\xi \sim \mathcal{N}(0, I_n)\) over the set \(\Delta = \{f_M : \|M\| \leq 1\}\). Since
\[
\|f_M - f_M'\|_2 = \sqrt{L} \left\| M - M' \right\|_F \leq \left\| M - M' \right\|_{1\rightarrow 2} \sqrt{Ld_2},
\]

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it follows that
\[ N(\Delta, \eta B_2) \leq N \left( B_{|| \cdot ||}, \frac{\eta}{\sqrt{Ld_2}} B_{1-2} \right) \leq N \left( \sqrt{\tau} B_{\text{mixed}}, \frac{\eta}{\sqrt{Ld_2}} B_{1-2} \right). \]
Recall that the mixed-norm is equivalent to the projective tensor norm up to a constant. Therefore Proposition 4.6 provides an upper bound on \( N(\Delta, \eta B_2) \). The assertion is obtained by applying this to Lemma 5.4.

Finally, plugging in the tail estimates of \( \theta \) and \( \Gamma \) respectively by Lemmas 5.1 and 5.3 to Proposition 1.1 provides Theorem 1.2. We choose \( \zeta = (d_1 + d_2)^{-1} \) in applying those lemmas.

### 5.2 Completion of bounded and approximately low-rank matrices

Similarly, Proposition 1.1 can also be used to obtain the error bound for max-norm-constrained matrix completion. The error bound matches the best known result [4] (up to a logarithmic factor). In fact Proposition 1.1 is an abstraction of the proof strategy by Cai and Zhou [4] to general tensor-norm-based LASSO. Cai and Zhou [4] have already provided tail estimates on \( \theta \) and \( \Gamma \). Here we provide an alternative tail bound on \( \theta \) through Maurey’em empirical method.

By applying the entropy estimate in Proposition 4.3 to a version of the Rudelson-Vershynin lemma [17, Proposition 2.6], which generalizes upon previous works [8, 31, 32], we obtain the following concentration inequality on the quadratic form with random entrywise sampling operator.

**Lemma 5.5.** Let \( (i_k, j_k) \) for \( k = 1, \ldots, n \) be independent copies of a uniform random variable on \( \{1, \ldots, d_1\} \times \{1, \ldots, d_2\} \). Let \( || \cdot || \) be defined as in (4). Then there exists an absolute constant \( C \) such that

\[
\sup_{||M|| \leq 1} \left| \frac{d_1 d_2}{n} \sum_{k=1}^{n} \langle e_{i_k} \otimes e_{j_k}, M \rangle^2 - ||M||_F^2 \right| \\
\leq Cd_1 d_2 \left( \frac{r(d_1 + d_2) \ln^3(d_1 + d_2) \vee \ln^2(\zeta^{-1})}{n} \vee \sqrt{\frac{r(d_1 + d_2) \ln^3(d_1 + d_2) \vee \ln^2(\zeta^{-1})}{n}} \right)
\]

holds with probability at least \( 1 - \zeta \).
Proof. The assertion is obtained as a consequence of [17, Proposition 2.6]. Let $X$ be a Banach space of $d_1$-by-$d_2$ matrices equipped with the norm defined by $\|M\|_X = \sqrt{d_1 d_2} \|M\|$. Then it immediately follows that $\|M\|_F \leq \|M\|_X$. We consider a generalized sparsity model given by $\{M : \|M\|_X \leq \sqrt{s} \|M\|_F\}$. Let $v_k : X \to \mathbb{R}$ denote the linear operator defined by $v_k(M) = \langle \sqrt{d_1 d_2} e_{ik} \otimes e_{jk}, M \rangle$ for $k = 1, \ldots, n$. Since the random indices are uniformly distributed, we have the isotropy, i.e. $E v_k^* v_k = \text{Id}$. Let $v : X \to \ell_n^\infty$ denote the composite operator such that $v(M) = [v_1(M), \ldots, v_n(M)]^\top$. Then [17, Proposition 2.6] implies that there exists an absolute constant $C_1$ with which

$$\sup_{\|M\|_X \leq \sqrt{s}} \left| \frac{1}{n} \sum_{k=1}^n |v_k(M)|^2 - \|M\|_F^2 \right| \leq C_1 \left( \sqrt{\frac{s \varrho}{n}} + \frac{s \varrho}{n} \right)$$

holds with probability at least $1 - \zeta$, where

$$\varrho := \sup_{k \in \mathbb{N}} \left[ (E \mathcal{E}_{2,1}(v))^{2k/1} + \sqrt{\ln(1/\zeta)} (E \|v\|^{2k/1}) \right].$$

Therefore, to get an upper bound in (22), it suffices to calculate the moments of $\mathcal{E}_{2,1}(v)$ and $\|v\|$.

First we obtain an upper bound on $\mathcal{E}_{2,1}(v)$ by Proposition 4.3 as follows.

$$X \xrightarrow{\text{Id}} \ell_n^\infty \xrightarrow{v} \ell_n^\infty$$

$$\xrightarrow{\ell_{d_2}^\infty \otimes \ell_{d_1}^\infty}$$

By the above commutative diagram, we have

$$\mathcal{E}_{2,1}(v : X \to \ell_n^\infty) \leq \|v : \ell_{d_2}^\infty \otimes \ell_{d_1}^\infty \to \ell_n^\infty\| \cdot \mathcal{E}_{2,1}(\text{Id} : X \to \ell_{d_2}^\infty \otimes \ell_{d_1}^\infty)$$

$$\leq \sqrt{d_1 d_2} \cdot \sqrt{\frac{r}{d_1 d_2}} \cdot \mathcal{E}_{2,1}(\text{Id} : \ell_{d_2}^\infty \otimes \ell_{d_1}^\infty \to \ell_{d_2}^\infty \otimes \ell_{d_1}^\infty)$$

$$\leq C_2 \sqrt{r (d_1 + d_2) \ln^3(d_1 + d_2)},$$

$^2$[17, Proposition 2.6] has an extra constraint $\|M\|_F = 1$ in the supremum in the left-hand side. This leads to a multiplicative deviation, i.e. the upper bound is proportional to $\|M\|_F$. However, the proof remains valid when this unit-norm constraint is dropped.
where the last inequality follows from Proposition 4.3.

Moreover, the operator norm of $v$ satisfies

$$\|v : X \to \ell^\infty_n\| \leq \frac{1}{\sqrt{d_1 d_2}} \|v : \ell^d_\infty \otimes \ell^d_\infty \to \ell^\infty_n\| = 1.$$  

Note that the above upper bounds on $E_{2,1}(v)$ and $\|v\|$ hold with probability 1. Therefore the corresponding moment terms are upper-bounded in the same way. Then the assertion follows by plugging in these upper estimates to (23) with $s = d_1 d_2$. \hfill \Box

This concentration inequality provides an alternative tail bound on $\theta$. Lemma 5.5 provides

$$\sup_{\|M\| \leq 1} \left| \frac{1}{n} \sum_{k=1}^n |\langle e_{i_k} \otimes e_{j_k}, M \rangle|^2 - \|M\|_F^2 \right| \leq C \sqrt{\frac{d_1 d_2 r(m+n) \ln^3(m+n)}{n}}$$  

with high probability. Compared to the analogous result by Cai and Zhou [4], which is summarized as follows:

$$\left| \frac{1}{n} \sum_{k=1}^n |\langle e_{i_k} \otimes e_{j_k}, M \rangle|^2 - \|M\|_F^2 \right| \leq \frac{1}{2} \|M\|_F^2 + C \sqrt{\frac{r(d_1 + d_2)}{n}}$$  

holds for all $M$ with $\|M\| \leq 1$ with high probability. Our upper bound in (24) has an extra logarithmic factor compared to (25). Therefore, unlike the result in [4], we do not achieve an optimal rate only up to an absolute constant. However, our proof by Maurey’s empirical method [6] might apply to a broader class of tensor norms beyond the max-norm by leveraging sophisticated embedding theorems (e.g. [9, 12, 14, 16, 18, 24, 29, 36–38]).

6 Information-theoretic lower bound

Cai and Zhou [4] has established a minimax lower bound for matrix completion where the unknown matrix is approximately low-rank and bounded. We present the proof of Theorem 1.3 by adapting their strategy to our setting of decentralized sketching. Let $\kappa(\alpha, R) = \{M : \|M\|_{1-2} \leq \alpha, \|M\|_{\text{mixed}} \leq R\}$ with $R = \sqrt{\alpha}$. The first step is to show that there exists a packing set of $\kappa(\alpha, R)$ of a desirable size and packing density, which is followed by a multiway hypothesis testing argument and Fano’s inequality. These steps will establish the minimax lower bound.

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The following lemma, obtained by adapting the statement and proof of [4, Lemma 3.1] to our setting, provides a random construction of a packing set of $\kappa(\alpha, R)$.

**Lemma 6.1.** Let $r = R^2/\alpha^2$ and $\gamma \leq 1$ satisfy $r \leq \gamma^2(d_1 \land d_2)$ is an integer. Then there exists a subset $\mathcal{M} \subset \kappa(\alpha, R)$ with cardinality

$$|\mathcal{M}| = \exp\left(\frac{r(d_1 \lor d_2)}{16\gamma^2}\right)$$

with the following properties:

1. Every $M \in \mathcal{M}$ satisfies that $\text{rank}(M) \leq r/\gamma^2$ and $M_{kl} \in \{\pm \gamma\alpha/\sqrt{d_1}\}$ for all $k \in [d_1]$ and $l \in [d_2]$, thereby

$$\|M\|_1 = \gamma \alpha, \quad \text{and} \quad \|M\|_2^2 = \gamma^2 \alpha^2 d_2.$$

2. Any two distinct $M^i, M^j \in \mathcal{M}$ satisfy

$$\|M^i - M^j\|_F^2 \geq \frac{\gamma^2 \alpha^2 d_2}{2}.$$

**Proof.** The idea is to show the existence of $\mathcal{M}$ by leveraging the empirical method. Without loss of generality, we may assume that $d_2 \geq d_1$. (Otherwise we only need to flip $d_1$ and $d_2$ in the first assumption.) Let $N = \exp(rd_2/16\gamma^2)$, $B = r/\gamma^2$ and for each $i = 1, \cdots, N$, we draw a random matrix $M^i$ as follows: The matrix $M^i$ consists of i.i.d. blocks of dimensions $B \times d_2$, stacked up from top to bottom, with entries of the first block being i.i.d. symmetric random variables, taking values in $\pm \alpha\gamma/\sqrt{d_1}$ such that

$$M^i_{kl} = M^i_{k'l'}, \quad \forall l, \forall k, k' : k' \equiv k \pmod{B}.$$

It can be verified that all the matrices $M^1, \ldots, M^N$ drawn in such a manner satisfy the first property above. We can then define the packing set $\mathcal{M}$ as the set $\{M_1, M_2, \cdots, M_N\}$. It remains to show that the second property is also satisfied.

For any $M^i \neq M^j$, we have

$$\|M^i - M^j\|_F^2 = \sum_{k,l} (M^i_{kl} - M^j_{kl})^2 \geq \left[\frac{d_1}{B}\right] \sum_{k=1}^B \sum_{l=1}^{d_2} (M^i_{kl} - M^j_{kl})^2 = \frac{4\alpha^2 \gamma^2}{d_1} \left[\frac{d_1}{B}\right] \sum_{k=1}^B \sum_{l=1}^{d_2} \delta_{kl},$$

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where \((\delta_{kl})\) denotes an array of i.i.d. Bernoulli random variables with mean \(1/2\). Then Hoeffding’s inequality implies
\[
\Pr \left( \sum_{k=1}^{B} \sum_{l=1}^{d_2} \delta_{kl} \geq \frac{Bd_2}{4} \right) \leq e^{-Bd_2/8}.
\]
By using the union bound argument over all \(\binom{N}{2}\) possible pairs, we obtain that
\[
\min_{i \neq j} \| M^i - M^j \|_F^2 > \alpha^2 \gamma^2 \left( \frac{d_1}{B} \right)^{2} \frac{Bd_2}{d_1} \geq \frac{\alpha^2 \gamma^2 d_2}{2}
\]
holds with probability at least \(1 - \binom{N}{2} \exp(-Bd_2/8) \geq 1/2\). In other words, the second property is satisfied with nonzero probability, thereby, there exists such an instance satisfying (26). This concludes the proof. 

**Lemma 6.2.** Let \(M \subset \kappa(\alpha, R)\) be \(\delta\)-separated, i.e., every distinct pair in \(M\) is at least separated by \(\delta\) in the Frobenius norm. Let \(\widetilde{M} = \arg\min_{M \in \mathcal{M}} \| M - \widetilde{M} \|_F\) and \(M^*\) be uniformly distributed over \(\mathcal{M}\). Then
\[
\inf_{M} \sup_{M \in \kappa(\alpha, R)} \mathbb{E} \| M - \widetilde{M} \|_F^2 \geq \frac{\delta^2}{4} \min_{\tilde{M} \in \mathcal{M}} \Pr(\widetilde{M} \neq M^*).
\]

**Proof.** Suppose that there exists \(M^j \in \mathcal{M}\) such that \(M^j \neq \widetilde{M}\). By the optimality of \(\widetilde{M}\) and the triangle inequality, we have
\[
\| \widetilde{M} - M^j \|_F \geq \| M^j - \widetilde{M} \|_F - \| \widetilde{M} - M^j \|_F \geq \| M^j - \widetilde{M} \|_F - \| \widetilde{M} - M^j \|_F,
\]
which implies
\[
\| \widetilde{M} - M^j \|_F \geq \frac{\| M^j - \widetilde{M} \|_F}{2}.
\]
Thus, since \(M^j, \widetilde{M} \in \mathcal{M}\) satisfy \(\| M^j - \widetilde{M} \|_F \geq \delta\), we obtain
\[
\| \widetilde{M} - M^j \|_F^2 \geq \frac{\delta^2}{4}.
\]
Hence we deduce that
\[
M^j \neq \widetilde{M} \implies \| \widetilde{M} - M^j \|_F^2 \geq \frac{\delta^2}{4}.
\]
(27)
Finally, we have
\[
\inf_{\hat{M}} \sup_{M \in \mathcal{M}(\alpha, R)} \mathbb{E}\|\hat{M} - M\|^2_F \geq \inf_{\hat{M}} \max_{M' \in \mathcal{M}} \mathbb{E}\|\hat{M} - M'\|^2_F
\]
\[
\geq \inf_{\hat{M}} \max_{M' \in \mathcal{M}} \frac{\delta^2}{4} \mathbb{P}\left(\|\hat{M} - M'\|^2_F \geq \frac{\delta^2}{4}\right)
\]
\[
\geq \frac{\delta^2}{4} \min_{\hat{M} \in \mathcal{M}} \max_{M' \in \mathcal{M}} \mathbb{P}(\hat{M} \neq M')
\]
\[
\geq \frac{\delta^2}{4} \min_{\hat{M} \in \mathcal{M}} \mathbb{P}(\hat{M} \neq M^*),
\]
where (a) holds by Markov’s inequality; (b) follows from (27); and (c) holds since the worst-case error probability is larger than the error probability with respect to the uniformly distributed random matrix \(M^*\). This completes the proof.

We now proceed to provide a lower bound on \(\min_{\hat{M} \in \mathcal{M}} \mathbb{P}(\hat{M} \neq M^*)\) by using the following lemma, which is a consequence of Fano’s inequality (Theorem 2.10.1, [7]).

**Lemma 6.3** ([4, Eq. (6.17)]). Let \(P(y|M, (A_k)_{k=1}^n)\) denote the conditional probability density of \(y\) in (1) given \(M\) and \((A_k)_{k=1}^n\). Let \(KL(M||M')\) denote the Kullback-Leibler divergence between \(P(y|M, (A_k)_{k=1}^n)\) and \(P(y|M', (A_k)_{k=1}^n)\). Then
\[
\mathbb{P}(\hat{M} \neq M^*) \geq 1 - \frac{1}{2} \text{KL}(M || M^*) + \frac{\log 2}{\log |\mathcal{M}|},
\]
where \(\mathcal{M}\) is the packing set derived in Lemma 6.1.

Moreover, with Gaussian noise, \(KL(M || M^*)\) is simplified as follows.

**Lemma 6.4.** Suppose that \(\eta_1, \ldots, \eta_n\) are i.i.d. \(\mathcal{N}(0, \sigma^2)\). Then
\[
KL(M^j || M^{j'}) = \frac{1}{2\sigma^2} \sum_{k=1}^n |\langle A_k, M^j - M^{j'} \rangle|^2.
\]
Furthermore, if \(\mathbb{E}A_k\langle A_k, M \rangle = M\) for all \(M\), then
\[
\mathbb{E} KL(M^j || M^{j'}) = \frac{1}{2\sigma^2} \|M^j - M^{j'}\|^2_F.
\]
Proof. Since the conditional distribution of \( y \) is written as

\[
P(y|M, (A_k)_{k=1}^n) = \prod_{k=1}^n \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( - \frac{1}{2\sigma^2} (y_k - \langle A_k, M \rangle)^2 \right),
\]

it follows that

\[
\text{KL}(P(y|M^j||M^{j'}) = \int P(y|M^j, (A_k)_{k=1}^n) \log \left( \frac{P(y|M^j, (A_k)_{k=1}^n)}{P(y|M^{j'}, (A_k)_{k=1}^n)} \right) dy
\]

\[
= \mathbb{E} \left( \sum_{k=1}^n \frac{(y_k - \langle A_k, M^{j'} \rangle)^2 - (\sqrt{\lambda} y_k - \langle A_k, M^j \rangle)^2}{2\sigma^2} \right) | M^j, (A_k)_{k=1}^n)
\]

\[
= \frac{1}{2\sigma^2} \sum_{k=1}^n |A_k, M^{j'} - M^j|^2.
\]

The second part follows immediately from the isotropy assumption. \( \square \)

Lemma 6.1 implies that \( |M^j - M^{j'}|^2 \leq 4\alpha^2 \gamma^2 d_2 \) for all distinct \( M^j \) and \( M^{j'} \) in \( \mathcal{M} \). Furthermore, we also have \( \log |\mathcal{M}| \geq r(d_1 \lor d_2)/(16\gamma^2) \). With this, we want to identify the conditions under which the above probability is non-zero. Let us suppose that \( \gamma^4 \leq \sigma^2 r(d_1 \lor d_2)/(128\alpha^2 d_2) \) and \( r(d_1 \lor d_2) \geq 48 \). Then, using the above estimate of the KL divergence in from (28) provides

\[
\mathbb{P}(\hat{M} \neq M^*) \geq 1 - \frac{16\gamma^2}{r(d_1 \lor d_2)} \left( \frac{2\alpha^2 \gamma^2 d_2}{\sigma^2} + \log 2 \right) \geq \frac{1}{2}.
\]

We already have that \( r(d_1 \lor d_2) \geq 48 \) from the assumption stated in Theorem 1.3. We now show how the condition on \( \gamma^4 \) can be satisfied. If \( \sigma^2 r(d_1 \lor d_2) \geq 128\alpha^2 d_2 \), then we can choose \( \gamma^2 = 1 \). In this case, we obtain

\[
\inf_{\tilde{M}} \sup_{M \in \mathcal{B}(\alpha, R)} \mathbb{E} \| \tilde{M} - M \|^2 \geq \frac{\delta^2}{4} \cdot \frac{1}{2} \geq \frac{\alpha^2 d_2}{16},
\]

since \( \delta = \alpha \gamma \sqrt{d_2/2} \). Hence

\[
\inf_{\tilde{M}} \sup_{M \in \mathcal{B}(\alpha, R)} \mathbb{E} \frac{1}{d_2} \| \tilde{M} - M \|^2 \geq \frac{\delta^2}{4} \cdot \frac{1}{2} \geq \frac{\alpha^2}{16}.
\]

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Otherwise, we choose \( \gamma^2 = \sqrt{\sigma^2 r(d_1 \lor d_2)/(128\alpha^2 d_2)} \) and obtain
\[
\inf_M \sup_{M \in \mathcal{K}(\alpha, R)} \mathbb{E}\|\hat{M} - M\|_F^2 \geq \frac{\delta^2}{4} \cdot \frac{1}{2}
\]
\[
\geq \alpha^2 \sigma \sqrt{\frac{r(d_1 \lor d_2)}{128}} \cdot \frac{1}{\alpha\sqrt{d_2}} \cdot \frac{1}{2} \cdot \frac{d_1}{4}
\]
\[
\geq \alpha \sigma \sqrt{\frac{r(d_1 \lor d_2)}{128}} \cdot \frac{\sqrt{d_2}}{2}
\]

since \( \delta = \alpha \gamma \sqrt{d_2/2} \). Note that this choice of \( \gamma^2 \) obeys the conditions in Lemma 6.1. Hence
\[
\inf_M \sup_{M \in \mathcal{K}(\alpha, R)} \mathbb{E}\left[\frac{1}{d_2} \|\hat{M} - M\|_F^2 \right] \geq \frac{\alpha^2 \sigma \sqrt{L}}{16} \cdot \frac{1}{\alpha} \sqrt{\frac{r(d_1 \lor d_2)}{Ld_2}}
\]

Finally, combining the above results, we obtain
\[
\inf_M \sup_{M \in \mathcal{K}(\alpha, R)} \frac{1}{d_2} \mathbb{E}\|\hat{M} - M\|_F^2 \geq \min \left( \frac{\alpha^2 \cdot \alpha^2 \sigma \sqrt{L}}{16}, \frac{1}{16}, \frac{16}{\alpha} \right) \cdot \frac{\sqrt{r(d_1 + d_2)}}{Ld_2}
\]

To simply further, we can consider the case where the number of measurements \( Ld_2 \) is greater than the degrees of freedom:
\[
Ld_2 \geq r(d_1 + d_2) = \frac{R^2(d_1 + d_2)}{\alpha^2}
\]

Then, we have that \( \alpha^2 \geq \frac{R^2(d_1 + d_2)}{Ld_2} \). Under this condition, the above minimax lower bound reduces to
\[
\inf_M \sup_{M \in \mathcal{K}(\alpha, R)} \frac{1}{d_2} \mathbb{E}\|\hat{M} - M\|_F^2 \geq \frac{\alpha^2 \sigma \sqrt{L}}{16} \cdot \frac{1}{\alpha} \sqrt{\frac{r(d_1 + d_2)}{Ld_2}}
\]
\[
\geq \frac{\alpha}{16} \sqrt{\frac{d_1 + d_2}{Ld_2}} \cdot \frac{\sigma \sqrt{L}}{\alpha} \cdot \frac{1}{16} \cdot R \cdot \sqrt{\frac{d_1 + d_2}{Ld_2}}
\]
\[
\geq \frac{R}{16} \sqrt{\frac{d_1 + d_2}{Ld_2}} \left( \alpha \cdot \sigma \sqrt{L} \right)
\]

Therefore, we have the following minimax lower bound:
\[
\inf_M \sup_{M \in \mathcal{K}_{mixed}} \frac{\mathbb{E}\|\hat{M} - M\|_F^2}{d_1 d_2} \geq \frac{\alpha^2}{16d_1} \sqrt{\frac{r(d_1 + d_2)}{Ld_2}} \left( 1 \cdot \frac{\sigma \sqrt{L}}{\alpha} \right)
\]
7 Discussion

We presented a tensor-norm-constrained LASSO estimator for low-rank recovery and its statistical analysis. When the observations are obtained by a local measurement operator, we proposed a principled design of a tensor-norm regularizer adapting to the local structure of measurements. Such a tensor norms that generalizes the max-norm and mixed-norm has provided a near optimal error bound over decentralized subspace sketching and matrix completion. Importantly, the relaxed low-rank model allows inexact modeling of data. Furthermore, the error bound applies uniformly to all approximately low-rank matrices satisfying a less stringent condition than the conventional incoherence condition. We will provide more applications of the presented framework in companion papers including subspace blind deconvolution and recovery of jointly low-rank matrices.

There are several interesting directions to which the presented results may extend. Much less is known about tensor products of three or more Banach spaces compared to the case of two Banach spaces. Recently, the geometry of selected triple tensor products has been studied [13]. The extension along this direction will advance understanding on the regression with multilinear data. With applications in quantum tomography, it will be also fruitful to study the extension to tensor products of operator spaces similarly to the diamond norm approach in [19].

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A Proof of Proposition 1.1

First, note that since both $M_0$ and $\hat{M}$ are feasible, we have

$$\|\hat{M} - M_0\| \leq 2\alpha,$$
and since $\hat{M}$ is optimal,
\[
\sum_{k=1}^{n} \left( y_k - \text{tr}(A_k^T \hat{M}) \right)^2 \leq \sum_{k=1}^{n} \left( y_k - \text{tr}(A_k^T M_0) \right)^2.
\]

Then under the assumption (5), and using the fact that $y_k = \text{tr}(A_k^T M_0) + \eta_k$, we have
\[
\|\hat{M} - M_0\|_F^2 \leq 4\alpha^2 \theta + \sum_{k=1}^{n} \text{tr}(A_k^T (\hat{M} - M_0))^2
\]
\[
= 4\alpha^2 \theta + \sum_{k=1}^{n} (y_k - \text{tr}(A_k^T \hat{M}))^2 + 2\eta_k (\text{tr}(A_k^T \hat{M}) - y_k) + \eta_k^2
\]
\[
\leq 4\alpha^2 \theta + \sum_{k=1}^{n} (y_k - \text{tr}(A_k^T M_0))^2 + 2\eta_k (\text{tr}(A_k^T M_0) - y_k) + \eta_k^2
\]
\[
= 4\alpha^2 \theta + 2 \sum_{k=1}^{n} \eta_k \text{tr}(A_k^T (\hat{M} - M_0)).
\]

To upper bound the right hand side, we take the supremum over all feasible $\hat{M}$,
\[
\sup_{\|\hat{M}\| \leq \alpha} \sum_{k=1}^{n} \eta_k \text{tr} \left( A_k^T (\hat{M} - M_0) \right) = \sup_{\|\hat{M}\| \leq 2\alpha} \sum_{k=1}^{n} \langle \eta_k A_k, M \rangle = 2\alpha \left\| \sum_{k=1}^{n} \eta_k A_k \right\|_*.
\]

The quantity on the right is a Gaussian empirical process for which [30, Theorem 4.7] provides the concentration bound
\[
\left\| \sum_{k=1}^{n} \eta_k A_k \right\|_* \leq \sigma \mathbb{E} \left\| \sum_{k=1}^{n} g_k A_k \right\|_* + \sigma \sqrt{\frac{\log(2\zeta^{-1})}{2\alpha}} \sup_{\|M\| \leq 1} \sum_{k=1}^{n} \text{tr}(A_k^T M)^2,
\]
which holds with probability $1 - \zeta$, where the $(g_k)_{k=1}^{n}$ are i.i.d. Gaussian with zero mean and unit variance. Thus
\[
\|\hat{M} - M_0\|_F^2 \leq 4\alpha^2 \theta + 4\alpha \sigma \Gamma + 4\alpha \sigma \pi \sqrt{\frac{\log(2\zeta^{-1})}{2}} \sup_{\|M\| \leq 1} \sum_{k=1}^{n} \text{tr}(A_k^T M)^2
\]
\[
\leq 4\alpha^2 \theta + 4\alpha \sigma \alpha \Gamma + 4\alpha \sigma \alpha \pi \sqrt{\frac{\log(2\zeta^{-1})}{2}} \left( \theta + \sup_{\|M\| \leq 1} \|M\|_F^2 \right)
\]
\[
\leq 4\alpha^2 \theta + 4\alpha \sigma \Gamma + 2\pi \alpha \sigma \sqrt{2\log(2\zeta^{-1})} (\theta + R^2).
\]
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