ARITHMETIC VERSION OF ANDERSON LOCALIZATION
FOR QUASIPERIODIC SCHröDINGER OPERATORS
WITH EVEN COSINE TYPE POTENTIALS

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ABSTRACT. We propose a new method to prove Anderson localization
for quasiperiodic Schrödinger operators and apply it to the quasiperiodic
model considered by Sinai \[33\] and Fröhlich-Spencer-Wittwer \[16\].
More concretely, we prove Anderson localization for even \(C^2\) cosine type
quasiperiodic Schrödinger operators with large coupling constants, Dio-
phantine frequencies and Diophantine phases.

1. INTRODUCTION

Anderson localization of particles and waves in disordered media is one
of the most intriguing phenomena in solid-state physics found by Anderson
\[5\]. Mathematically, localization has already been extensively studied for
the random cases \[1, 3, 4, 13, 30\]. Anderson localization in quasiperiodic
media even has stronger backgrounds in physics \[9, 32\], which is known to
have strong connection with integer quantum Hall effect \[2, 21, 31\], and also
plays an important role in the emerging subject of optical crystals \[28\].

Although been studied for over sixty years, Anderson localization \(^1\) for
quasiperiodic operators has not been completely understood since it de-
pends sensitively on the frequency, the phase and amplitude of oscillation
of the potential. So far almost sure Anderson localization \(^2\) for fixed Dio-
phantine frequencies was rigorously proved only for the following cosine type
quasiperiodic Schrödinger operators with large coupling constants by Sinai
\[33\], Fröhlich-Spencer-Wittwer \[16\] \(^3\) and Forman-Vandenboom \[15\]

\[
(H_{\lambda v, \alpha, \theta} u)_n = u_{n+1} + u_{n-1} + \lambda v(\theta + n\alpha)u_n, \forall n \in \mathbb{Z},
\]

with \(\alpha \in \mathbb{R}\setminus\mathbb{Q}\) (the frequency), \(\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}\) (the phase), \(\lambda \in \mathbb{R}\) (the
coupling constant) and \(v \in C^2(\mathbb{T}, \mathbb{R})\) (the potential) satisfying

- \(\frac{dv}{d\theta} = 0\) at exactly two points, one is minimal and the other one is
  maximal, which are denoted by \(z_1\) and \(z_2\).
- These two extremals are non-degenerate, that is, \(\frac{d^2 v}{d\theta^2}(z_j) \neq 0\)
  for \(j = 1, 2\).

Anderson localization (AL) with precise arithmetic descriptions on both
the frequencies and full Lebesgue measure set of the phases, is referred as
to arithmetic version of Anderson localization, which is obviously stronger

\(^1\)Pure point spectrum with exponentially decaying eigenfunctions
\(^2\)Anderson localization for almost every phase.
\(^3\)Fröhlich-Spencer-Wittwer [16] requires the potential to be even.
than almost sure Anderson localization. The first arithmetic version of
Anderson localization was given by Jitomirskaya [23] who proved that for any
fixed Diophantine frequency and any fixed Diophantine phase, the almost
Mathieu operator has AL for $|\lambda| > 1$. Such arithmetic description on
the frequency and the phase was explored in a sharp way by Jitomirskaya
and Liu, namely, for Diophantine phase, there is a sharp spectral transition
in frequency [25], and for Diophantine frequency, there is a sharp spectral
transition in phase [26]. Arithmetic Anderson localization for one dimen-
sional long range quasiperiodic operators with cosine potential was proved
in [7, 12]. Recently, a new nonperturbative proof of arithmetic theoretic
Anderson localization was given in [18], which applies to higher dimensional
long range quasiperiodic operators, based on nonperturbative reducibility
method and duality argument.

However, the proofs of all the above arithmetic Anderson localization re-
sults crucially depend on the assumption that the potential is exactly the
cosine function. It is not clear if arithmetic Anderson localization could be
expected for other potentials. The main purpose of this paper is to present a
new method from the point of view of dynamical systems to prove the arith-
etic version of Anderson localization for quasiperiodic Schrödinger operators.
Applying our method to even cosine type quasiperiodic Schrödinger
operator, we give an improvement of Fröhlich-Spencer-Wittwer’s result [16].
Compared to the methods of [15, 16, 33] which are based on certain kind
of multiscale analysis, our method is purely dynamical and gives concrete
description of the localization phases. We are also able to give almost
sharp estimate on the decay rate of the all eigenfunctions.

1.1. Statement of the main results. Before formulating our results, we
first give precise arithmetic description on $\alpha$ and $\theta$. A frequency $\alpha \in \mathbb{R}$ is
called $(\kappa, \tau)$-Diophantine (denoted by $\alpha \in \text{DC}(\kappa, \tau)$) if
\begin{equation}
\text{dist}(k\alpha, \mathbb{Z}) \geq \gamma |k|^{-\tau}, \quad \forall k \in \mathbb{Z}\setminus\{0\}.
\end{equation}
We will use the notation
\begin{equation}
\text{DC} = \bigcup_{\kappa > 0; \tau > 1} \text{DC}(\kappa, \tau).
\end{equation}
For a given irrational number $\alpha$, we say $\theta \in (0, 1)$ is $(\gamma, \tau)$–Diophantine with
respect to $\alpha$ (denoted by $\Theta^\alpha$) if
\begin{equation}
\|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\gamma}{(|k| + 1)\tau},
\end{equation}
for any $k \in \mathbb{Z}$, where $\|x\|_{\mathbb{R}/\mathbb{Z}} = \text{dist}(x, \mathbb{Z})$. Let $\Theta = \bigcup_{\gamma > 0; \tau > 1} \Theta^\alpha$. Clearly, $\Theta$
is a set of full Lebesgue measure for any fixed irrational number $\alpha$.

**Theorem 1.1.** Given $\alpha \in \text{DC}$ and an even $C^2$ cosine type potential $v$, there
exists $\lambda_0(\alpha, v)$ such that $H_{\lambda, \alpha, \theta}$ has Anderson localization for all $\theta \in \Theta$
provided that $\lambda > \lambda_0$.

\footnote{Operator (1.1) with $v(\theta) = 2 \cos 2\pi \theta$.}
\footnote{In [15, 16, 33], Anderson localization was proved for almost every phase without an
arithmetic description.}
Remark 1.1. To give a simple arithmetic description of the localization phases (i.e., the Diophantine phases), the eveness condition seems to be necessary.

Remark 1.2. If $\alpha$ is very Liouvillean or $\theta$ is very $\alpha$-Liouvillean (i.e., for generic $\alpha$ and $\theta$), $H_{\lambda v, \alpha, \theta}$ has purely singular continuous spectrum \cite{8, 26, 27}. Thus to prove localization type results, the arithmetic assumptions on both $\alpha$ and $\theta$ are necessary.

We also have a precise estimate on the decay rate of all eigenfunctions.

**Theorem 1.2.** Given $\alpha \in DC$, $\varepsilon > 0$ and an even $C^2$ cosine type potential $v$, there exists $\lambda_0(\alpha, v, \varepsilon)$ such that all eigenfunctions of the operator $H_{\lambda v, \alpha, \theta}$ satisfy

$$\liminf_{|n| \to \infty} -\frac{\ln (u_E^2(n) + u_E^2(n+1))}{2|n|} \geq (1 - \varepsilon) \ln \lambda$$

provided that $\lambda > \lambda_0$.

Remark 1.3. For the almost Mathieu operator (a typical example), Jitomirskaya \cite{23} proved

$$\liminf_{|n| \to \infty} -\frac{\ln (u_E^2(n) + u_E^2(n+1))}{2|n|} = \ln \lambda.$$ 

Thus, the decay rate in the above theorem is almost sharp.

We point out an interesting phenomenon based on Theorem 1.1: The localization phases do not sensitively depend on the space of even $C^2$ cosine type $v$. This phenomenon can be viewed as the robustness of localization phases introduced in \cite{18}.

**Definition 1.1.** For fixed $\alpha$, $H_{V, \alpha, \theta}$ is said to have $C^r$ robust Anderson localization if there is a $C^r$ neighborhood $B(V)$ of $V$ and a subset $\tilde{\Theta}$, such that

$$\bigcap_{V \in B(V)} \{ \theta | H_{\tilde{V}, \alpha, \theta} \text{ has AL} \} = \tilde{\Theta},$$

moreover $|\tilde{\Theta}| = 1$.

Theorem 1.1 proved that $H_{\lambda v, \alpha, \theta}$ with even cosine like potential has $C^2$ robust Anderson localization in the space of even potentials. It seems that both the symmetry (eveness) and the profile ($C^2$ cosine type) of the potential play key roles in robust Anderson localization.

We also mention some important results related to Anderson localization. Eliasson \cite{14} proved that if $v$ is a Gevrey function satisfying non-degenerate conditions, for any fixed Diophantine $\alpha$, $H_{\lambda v, \alpha, \theta}$ has pure point spectrum for a.e. $\theta$ and large enough $\lambda$ (depending on $\alpha$). Bourgain and Goldstein \cite{10} proved that, in the positive Lyapunov exponent regime, for any fixed $\theta$, $H_{\lambda v, \alpha, \theta}$ has AL for a.e. Diophantine $\alpha$ provided that $v$ is a non-constant real analytic function. See \cite{11, 17, 20, 29} for more results.
1.2. **Strategy of the proof.** As we mentioned above, our method is motivated by the methods introduced in [8, 18, 19, 24, 34]. In [8, 24], Avila-You-Zhou and Jitomirskaya-Kachkovskiy gave criteria to prove almost sure Anderson localization for quasiperiodic operators based on nonperturbative reducibility method and duality argument. More precisely, there are two steps. The first step is to construct a family of eigenvalues and eigenfunctions for almost every phase based on reducibility and duality. The second step is to show the family of eigenfunctions they constructed form a complete basis of $\ell^2(\mathbb{Z})$. However, the arithmetic version of Anderson localization is more difficult to be proved compared with almost sure Anderson localization. In [18], Ge-You found a strategy to recover the phases lost in using the method in [8, 24], by introducing an auxiliary measure defined by reducibility, i.e. the $\mathcal{R}$-measure. By using quantitative reducibility, they proved stratified continuity of the $\mathcal{R}$-measure with respect to the phase on the set of Diophantine phases. Ge-You’s method was further developed and simplified by Ge-You-Zhao in [19] to give a new proof of the arithmetic transition conjecture proposed by Jitomirskaya [22].

In the present paper, instead of using reducibility and duality, we give a new way to construct a family of eigenvalues with exponentially localized eigenfunctions for $C^2$ cosine type quasiperiodic Schrödinger operators by the induction scheme developed in [34]. The intuition is that if the intersection between asymptotic stable and unstable directions of the transfer matrix persist in larger and larger time scale which eventually implies the intersection of stable and unstable directions and the norm of the transfer matrix grows exponentially, then one can construct an eigenfunction. When a family of eigenvalues with exponentially localized eigenfunctions are constructed almost surely, almost sure Anderson localization follows directly by the criteria in [8, 24]. To prove the arithmetic version of AL, i.e., AL for all $\theta \in \Theta$, one possible way is to prove that $d\mu_{\theta}^{pp}$ is continuous in $\Theta$. However, this seems to be a difficult task and we don’t know how to prove it directly since $d\mu_{\theta}^{pp}$ sensitively depends on $\theta$. Our strategy is to introduce a new measure $d\nu_{\theta}$ via the localized eigenfunctions we constructed, which is called $L$-measure, motivated by the $\mathcal{R}$-measure defined in [18]. We will prove that $d\nu_{\theta}$ is absolutely continuous with respect to $d\mu_{\theta}^{pp}$. The advantage of $d\nu_{\theta}$ is its stratified continuity in $\Theta$, more precisely the continuity in $\Theta_{\gamma}$, can be proved by quantitative estimates of the localized eigenfunctions. In this way, we can approximate each lost phase in $\Theta$ by localization phases, and prove $d\mu_{\theta}^{pp}(\mathbb{R}) = d\nu_{\theta}(\mathbb{R}) = 1$ for all phases in $\Theta$.

2. **Wang-Zhang’s induction theorem**

Inspired by [36], Wang and Zhang in [34] developed an induction scheme to study the positivity and continuity of the Lyapunov exponent and Cantor spectrum of quasiperiodic Schrödinger operators with $C^2$ cosine type potentials [34, 35]. We observe that the induction theorem can also be used to construct eigenfunctions, which is one of the corner stones in our proof of

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6the pure point piece of the spectral measure
the arithmetic version of Anderson localization. Now we briefly introduce their induction theorem. The readers are referred to [34] for details.

For $\theta \in \mathbb{R}/\mathbb{Z}$, let

$$R_\theta = \begin{pmatrix} \cos 2\pi \theta & -\sin 2\pi \theta \\ \sin 2\pi \theta & \cos 2\pi \theta \end{pmatrix} \in SO(2, \mathbb{R}).$$

Define

$$s : SL(2, \mathbb{R}) \to \mathbb{R}/(\pi \mathbb{Z})$$
as the most contraction direction of $A \in SL(2, \mathbb{R})$, i.e., $\|A \cdot \hat{s}(A)\| = \|A\|^{-1}$ for unit vector $\hat{s}(A)$ in the direction $s(A)$. Abusing the notation a little, define $u(A) = s(A^{-1})$ and $\hat{u}(A) = \hat{s}(A^{-1})$. Then for $A \in SL(2, \mathbb{R})$, it is clear that

$$A = R_u \cdot \left( \begin{array}{cc} \|A\| & 0 \\ 0 & \|A\|^{-1} \end{array} \right) \cdot R_{\hat{u}}^{-s},$$

where $s, u \in [0, \pi)$ are angles corresponding to the directions $s(A), u(A) \in \mathbb{R}/(\pi \mathbb{Z})$.

The main object of the present paper is the following Schrödinger cocycles $(\alpha, S^\lambda_E)$, where

$$S^\lambda_E(x) := \begin{pmatrix} E - \lambda v(x) & 1 \\ 1 & 0 \end{pmatrix}, \quad E \in \mathbb{R}.$$  

Let $\lambda \geq \lambda_0 = \lambda_0(v) \gg 1$ and $t = \frac{E}{\lambda} \in J = [\inf v - 2, \sup v + 2]$. In this case, there is $B \in C^2(\mathbb{T} \times J, SL(2, \mathbb{R}))$ such that

$$B^{-1}(x + \alpha, t) \begin{pmatrix} E - \lambda v(x) & 1 \\ 1 & 0 \end{pmatrix} B(x, t) = A(x, t),$$

where

$$A(x, t) = \begin{pmatrix} \lambda(x, t) & 0 \\ 0 & \lambda^{-1}(x, t) \end{pmatrix} \cdot R_{\phi(x, t)}, \quad \cot \phi(x, t) = t - v(x).$$

(2.1) in fact gives the polar decomposition of the Schrödinger cocycles.

From now on, let $A(x, t)$ be as above and

$$A_n(x, t) := \begin{cases} A(x + (n-1)\alpha, t) \cdots A(x + \alpha, t)A(x, t), & n \geq 0 \\ A^{-1}(x + n\alpha, t)A^{-1}(x + (n+1)\alpha, t) \cdots A^{-1}(x - \alpha, t), & n < 0 \end{cases}.$$  

Abusing the notation a little bit, for $n \geq 1$, we define

$$s_n(x, t) = s[A_n(x, t)], \quad u_n(x, t) = s[A_{-n}(x, t)].$$

We call $s_n$ (respectively, $u_n$) the $n$-step stable (respectively, unstable) direction.

Set $I_0 = \mathbb{R}/\mathbb{Z}$ and $g_1(x, t) = s_1(x, t) - u_1(x, t) = \tan^{-1}[t - v(x)]$ for all $t \in J$. Let $\left\{ \frac{g_i}{g_j} \right\}_{n \geq 1}$ be the continued fraction approximants of $\alpha$. Fix a large $N = N(v)$. [34] proved the following conclusion by induction. Assume that for $i \geq 1$, the following objects are well defined:

(1) $i$-th step critical points:

$$C_i(t) = \{c_{i,1}(t), c_{i,2}(t)\}$$

with $c_{i,j}(t) \in I_{i-1,j}(t)$ minimizing $\{|g_i(x, t)|, \ x \in I_{i-1,j}(t)\}$. 

(2) \( i \)-th step critical interval:
\[
I_{i,j}(t) = \{ x : |x - c_{i,j}(t)| \leq \frac{1}{2^n q_{N+i-1}^j} \} \text{ and } I_i(t) = I_{i,1}(t) \cup I_{i,2}(t).
\]

(3) \( i \)-th step return times:
\[
q_{N+i-1} \leq r_i^\pm(x,t) : I_i(t) \to \mathbb{Z}^+
\]
are the first return times (back to \( I_i(t) \)) after time \( q_{N+i-1} \). Here \( r_i^+(x,t) \) is the forward return time and \( r_i^-(x,t) \) is backward. Let \( r_i(t) = \min \{ r_i^+(t), r_i^-(t) \} \) with \( r_i^\pm(t) = \min_{x \in I_i(t)} r_i^\pm(x,t) \).

(4) \((i+1)\)-th step angle \( g_{i+1} \):
\[
g_{i+1}(x,t) = s_{r_i(t)}(x,t) - u_{r_i(t)}(x,t) : D_i \to \mathbb{R}^1,
\]
where
\[
D_i := \{ (x,t) : x \in I_i(t), t \in J \}.
\]

The next theorem, which is from [34]'s induction theorem, gives the precise description of the several important quantities mentioned above.

**Theorem 2.1** (Theorem 3 of [34]). Given \( \alpha \in DC(\kappa, \tau) \), \( \varepsilon > 0 \) and a \( C^2 \) cosine type potential \( v \), there exists \( \lambda_0(\alpha, v, \varepsilon) \) such that the following holds for \( \lambda > \lambda_0 \).

1. For each \( i \geq 1 \) and \( t \in J \), it holds that
\[
|c_{i,j}(t) - c_{i+1,j}(t)| < C\lambda^{-\frac{2}{\varepsilon}r_{N-1}}(t), j = 1, 2;
\]

2. For each \( i \geq 1, t \in J \), and all \( x \in I_i(t) \), it holds that
\[
\| A_{x} r_i^\pm(x,t) \| > \lambda^{(1-\varepsilon)r_{N-1}}(x,t) \geq \lambda^{(1-\varepsilon)q_{N+i-1}}.
\]

3. For each \( i \geq 1, t \in J \), and all \( x \in I_{i,j}(t) \), it holds that
\[
|g_i(x,t)| \geq c|x - c_{i,j}(t)|^3 \quad j = 1, 2.
\]

4. For each \( i \geq 1 \) and \( t \in J \). If \( |c_{i,1}(t) - c_{i,2}(t) - k\alpha| \geq \frac{1}{q_{N+i-1}} \) for all \( |k| \leq q_{N+i-1} \), then
\[
\| g_{i+1}(\cdot,t) - g_i(\cdot,t) \|_{C^2} \leq C\lambda^{-\frac{2}{\varepsilon}r_{N-1}}(t).
\]

Theorem 2.1 is a simplified version of Theorem 3 in [34]. See (57)-(59) in Theorem 3 and Lemma 6 of [34].

### 3. Construction of eigenfunctions

In this section, we construct sufficiently many “good” eigenfunctions of \( H_{\lambda\nu,\alpha,\theta} \) by Theorem 2.1. We denote by \( \Sigma_{\lambda\nu,\alpha} \) the spectral set of \( H_{\lambda\nu,\alpha,\theta} \) (It does not depend on \( \theta \) since \( \alpha \) is irrational).
3.1. The critical points and growth of the transfer matrix.

Theorem 3.1. Let $\alpha \in DC(\kappa, \tau)$ and $v$ be an even $C^2$ cosine type potential. For any $\varepsilon > 0$, there exists $\lambda_0(\varepsilon, \alpha, v)$ such that if $\lambda > \lambda_0$, and $t \in \lambda^{-1}\Sigma_{\lambda v, \alpha}$, then there exists a strictly increasing continuous surjection
\[
c_{\infty}(t) : \lambda^{-1}\Sigma_{\lambda v, \alpha} \to [0, 1/2],
\]
and there exist $s_\infty(c_\infty(t), t), u_\infty(c_\infty(t), t) \in \mathbb{R}^1$ if $c_\infty(t) \in \Theta_\gamma$ with
\[
s_\infty(c_\infty(t), t) = u_\infty(c_\infty(t), t),
\]
such that
\[
\|A_n(c_\infty(t), t)s_\infty(c_\infty(t), t)\| \geq c\lambda^{-(1-\varepsilon)|n|}, \quad \forall n \in \mathbb{Z},
\]
where $c = c(\kappa, \gamma, \tau, v, \varepsilon) > 0$ and $s_\infty(c_\infty(t), t)$ is the unit vector in the direction $s_\infty(c_\infty(t), t)$.

Proof. By (2.2) in Theorem 2.1 (See also Theorem 3 [34] and Theorem 2 in [35]), for any $t \in \lambda^{-1}\Sigma_{\lambda v, \alpha}$, for all $j \geq 1$ and $m = 1, 2$, we have
\[
|c_{j+1,m}(t) - c_{j,m}(t)| \leq C\lambda^{-(1-\varepsilon)j-1} \leq C\lambda^{-\frac{\varepsilon}{2}N+j-2},
\]
thus there exists $c_{\infty,m}(t)$ such that
\[
c_{\infty,m}(t) = \lim_{n \to \infty} c_{n,m}(t).
\]
We first prove $c_{\infty,1}(t) = -c_{\infty,2}(t)$, this is because of (2.3) in Theorem 2.1, we have
\[
\|A_{r_j}(c_{j,m}(t), t)\| \geq \lambda^{(1-\varepsilon)r_j(t)},
\]
this implies
\[
\|A_{r_j}(c_{j,m}(t), t) \cdot s_{r_j}(c_{j,m}(t), t)\| \leq \lambda^{-(1-\varepsilon)r_j(t)},
\]
\[
\|A_{-r_j}(c_{j,m}(t), t) \cdot u_{r_j}(c_{j,m}(t), t)\| \leq \lambda^{-(1-\varepsilon)r_j(t)}.
\]
Since $v$ is even, we have
\[
\|A_{r_j}(-c_{j,m}(t), t) \cdot \left(\frac{\pi}{2} - u_{r_j}(c_{j,m}(t), t)\right)\| = \|A_{-r_j}(c_{j,m}(t), t) \cdot u_{r_j}(c_{j,m}(t), t)\| \leq \lambda^{-(1-\varepsilon)r_j(t)},
\]
this implies that
\[
\left|\frac{\pi}{2} - u_{r_j}(c_{j,m}(t), t) - s_{r_j}(-c_{j,m}(t), t)\right| \leq \lambda^{-\frac{\varepsilon}{2}r_j(t)},
\]
similarly
\[
\left|\frac{\pi}{2} - s_{r_j}(c_{j,m}(t), t) - u_{r_j}(-c_{j,m}(t), t)\right| \leq \lambda^{-\frac{\varepsilon}{2}r_j(t)}.
\]
(3.5) and (3.6) imply
\[
|g_{r_j}(c_{j,m}(t), t) - g_{r_j}(-c_{j,m}(t), t)| \leq 2\lambda^{-\frac{\varepsilon}{2}r_j(t)}.
\]
By (2.4) in Theorem 2.1,
\[
|c|c_{j,m}(t) - (-c_{j,m}(t))|^3 \leq |g_{r_j}(c_{j,m}(t), t) - g_{r_j}(-c_{j,m}(t), t)|.
\]
Hence
\begin{equation}
|c_{j,m}(t) - (-c_{j,m}(t))| \leq C\lambda^{-\frac{1}{2}} r_j(t).
\end{equation}
By (3.4), we have
\[ c_{\infty,1}(t) = -c_{\infty,2}(t). \]
We simply denote \( c_{\infty}(t) = c_{\infty,1}(t) \). By the induction theorem (Theorem 2.1) we have
\[ c_{\infty}(t) = \begin{cases} 0 & t = \inf \lambda^{-1}\Sigma_{\lambda v, \alpha}, \\ \frac{1}{2} & t = \sup \lambda^{-1}\Sigma_{\lambda v, \alpha}. \end{cases} \]
Note that \( c_{\infty}(t) \) is continuous on \( \lambda^{-1}\Sigma_{\lambda v, \alpha} \), since \( c_{j,1}(t) \) converges uniformly to \( c_{\infty}(t) \) on \( \lambda^{-1}\Sigma_{\lambda v, \alpha} \) and \( c_{j,1} \) is continuous, combine these together, we get that \( c_{\infty}(t) \) is a continuous surjection from \( \lambda^{-1}\Sigma_{\lambda v, \alpha} \) to \([0, \frac{1}{2}]\).

Now we prove \( c_{\infty}(t) \) is increasing on \( \lambda^{-1}\Sigma_{\lambda v, \alpha} \), we need the following result: for \( t \in \lambda^{-1}\Sigma_{\lambda v, \alpha} \) with \( c_{\infty}(t) \in \Theta_{\lambda v, \alpha} \), i.e.,
\[ |2c_{\infty}(t) - k\alpha| \geq \frac{\gamma}{(|k| + 1)^\tau}, \forall k \in \mathbb{Z}, \]
by (3.4), there exists \( j_0(\gamma) \), such that for all \( j > j_0 \)
\begin{equation}
|c_{j,1}(t) - c_{j,2}(t) - k\alpha| \geq \frac{1}{q_{N+j-1}^2}, \quad |k| \leq q_{N+j-1}
\end{equation}
thus by Corollary 3 in [35],
\begin{equation}
\frac{d (c_{j,1}(t) - c_{j,2}(t))}{dt} \geq c > 0,
\end{equation}
in a small neighborhood of \( t \). In view of (3.7) and (3.9), for any sequence \( t_1 > \cdots > t_n > \cdots \) with \( t_n \to t \), we have \( c_{\infty}(t_n) > c_{\infty}(t) \) for all \( n \) sufficiently large.

We are now ready to prove the monotonicity of \( c_{\infty}(t) \). We prove this by contradiction, otherwise, there exists \( t_1 < t_2 \) such that \( c_{\infty}(t_1) > c_{\infty}(t_2) \), for \( \gamma \ll |c_{\infty}(t_1) - c_{\infty}(t_2)| \), there exists \( y \in \Theta_{\lambda v, \alpha}(c_{\infty}(t_2), c_{\infty}(t_1)) \), let \( t' = \sup\{t \in (t_1, t_2) : c_{\infty}(t) = y\} \) and for \( t \in (t', t_2) \), we have
\[ c_{\infty}(t) < y. \]
Thus there exists sequence \( t_1 > \cdots > t_n > \cdots \) with \( t_n \to t' \), such that \( c_{\infty}(t_n) < c_{\infty}(t') \) for all \( n \) sufficiently large which is a contradiction.

We omit the dependence on \( t \) in the following. For any \( n \geq q_{N+j_0-1}^{100c\tau} \), let \( j_0 < j_1 < \cdots < j_k = n \) be the return times of \( c_{\infty}(t) \) to \( I_{N+j_0} \), by (2.3), (2.4) in Theorem 2.1 and (3.4), we have
\[ \|A_n(c_{\infty})\| \geq \prod_{i=0}^{k-1} \|A_{j_i-j_{i-1}}(c_{\infty} + j_{i-1}\alpha)\| q_{N+j_0-1}^{6k\tau} \lambda^{-n+j_{k-1}} \geq c\lambda^{(1-\varepsilon)|n|}, \]
for some \( c = c(\kappa, \gamma, \tau, v, \varepsilon) > 0 \). Thus we have proved (3.3). Finally, (3.2) follows from (2.5) in Theorem 2.1.

\[ \square \]

\[ \text{In case of (3.8), } \rho_j(t) = c_{j,1}(t) - c_{j,2}(t) \text{ in [35].} \]

\[ \text{Here we use the fact that } |n - j_{k-1}| \leq q_{N+j_0-1}^\varepsilon \text{, it follows from Lemma 3 of [6].} \]
3.2. Construction of good eigenfunctions. Recall that \( t = \lambda^{-1} E \).

**Definition 3.1.** For any \( \gamma > 0 \) and \( C > 0 \), a normalized eigenfunction \(^9 u(n)\) is said to be \((C, \gamma)\)-good, if

\[
|u(n)| \leq C e^{-\gamma|n|}
\]

for any \( n \in \mathbb{Z} \).

**Proposition 3.1.** Assume \( \alpha \in DC(\kappa, \tau) \) and \( v \) is an even \( C^2 \) cosine type potential, for any \( \varepsilon > 0 \), there exists \( \lambda_0(\varepsilon, \alpha, v) \) and \( C(\kappa, \gamma, \tau, v, \varepsilon) \) such that if \( \lambda > \lambda_0 \) and \( c_\infty(t) \in \Theta^\gamma_\alpha \), then \( H_{\lambda_0, c_\infty(t)} \) has a \((C, (1 - \varepsilon) \ln \lambda)\)-good eigenfunction corresponding to eigenvalue \( E = \lambda t \).

**Proof.** We denote by \( A^{E, \lambda}(\theta) = \begin{pmatrix} E - \lambda v(\theta) & -1 \\ 1 & 0 \end{pmatrix} \), for \( n \geq 0 \), by (2.1), we have

\[
\left| A^{E, \lambda}_{n+1}(c_\infty(t)) \cdot (B^{-1}(c_\infty(t), t) \cdot s_{n+1}(c_\infty(t), t)) \right| \leq C \| A_{n+1}(c_\infty(t), t) \|^{-1},
\]

\[
\left| A^{E, \lambda}_{n+1}(c_\infty(t)) \cdot (B^{-1}(c_\infty(t), t) \cdot s_n(c_\infty(t), t)) \right|
\]

\[
= \left| A^{E, \lambda}(c_\infty(t) + na) A^{E, \lambda}_n(c_\infty(t)) \cdot (B^{-1}(c_\infty(t), t) \cdot s_n(c_\infty(t), t)) \right|
\]

\[
\leq C \| A_n(c_\infty(t), t) \|^{-1}.
\]

Hence

\[
\left| (B^{-1}(c_\infty(t), t) \cdot s_{n+1}(c_\infty(t), t)) - (B^{-1}(c_\infty(t), t) \cdot s_n(c_\infty(t), t)) \right|
\]

\[
\leq 2C \| A_n(c_\infty(t), t) \|^{-1} \| A_{n+1}(c_\infty(t), t) \|^{-1}.
\]

This implies that

\[
\left| (B^{-1}(c_\infty(t), t) \cdot s_\infty(c_\infty(t), t)) - (B^{-1}(c_\infty(t), t) \cdot s_n(c_\infty(t), t)) \right|
\]

\[
\leq \sum_{k \geq n} 2C \| A_k(c_\infty(t)) \|^{-1} \| A_{k+1}(c_\infty(t)) \|^{-1}.
\]

We have

\[
\left| A^{E, \lambda}_n(c_\infty(t)) \cdot (B^{-1}(c_\infty(t), t) \cdot u_\infty(c_\infty(t), t)) \right|
\]

\[
\leq C \| A_n(c_\infty(t), t) \|^{-1} + \left| A^{E, \lambda}_n(c_\infty(t)) \cdot (B^{-1}(c_\infty(t), t) \cdot (s_n(c_\infty(t), t) - s_\infty(c_\infty(t), t))) \right|
\]

\[
\leq C \left( \| A_n(c_\infty(t), t) \|^{-1} + \| A_n(c_\infty(t), t) \| \sum_{k \geq n} \| A_k(c_\infty(t), t) \|^{-1} \| A_{k+1}(c_\infty(t), t) \|^{-1} \right).
\]

By (3.3), we have

\[
\sum_{k \geq n} \| A_k(c_\infty(t), t) \|^{-1} \| A_{k+1}(c_\infty(t), t) \|^{-1} \leq C \lambda^2(1 - \varepsilon)|n|,
\]

for some \( C = C(\kappa, \gamma, \tau, v, \varepsilon) \).

---

\(^9\)We say \( u(n) \) is normalized if \( \sum_n |u(n)|^2 = 1 \).
Similarly,
\[
\left| A_n^{E, \lambda}(c_\infty) \cdot (B^{-1}(c_\infty(t), t) \cdot s_\infty(c_\infty(t), t)) \right|
\leq C \left\| A_{-n}(c_\infty(t), t) \right\|^{-1} + \left\| A_{-n}^{E, \lambda}(c_\infty(t)) \cdot (B^{-1}(c_\infty(t), t) \cdot (u_n(c_\infty(t), t) - u_\infty(c_\infty(t), t))) \right\|
\leq C \left( \left\| A_{-n}(c_\infty(t), t) \right\|^{-1} + \left\| A_{-n}(c_\infty(t), t) \right\| \sum_{k \geq n} \left\| A_{-k}(c_\infty(t), t) \right\|^{-1} \left\| A_{-k-11}(c_\infty(t), t) \right\|^{-1} \right).
\]

For \( E = \lambda t \), we denote by
\[
\begin{pmatrix}
  u_E(n + 1) \\
  u_E(n)
\end{pmatrix} = A_n^{E, \lambda}(c_\infty(t))B^{-1}(c_\infty(t), t) \cdot s_\infty(c_\infty(t), t),
\]
then for \( n \in \mathbb{Z} \), by (3.2), we have
\[
\left\| \begin{pmatrix}
  u_E(n + 1) \\
  u_E(n)
\end{pmatrix} \right\| = \left\| A_n^{E, \lambda}(c_\infty(t))B^{-1}(c_\infty(t), t) \cdot s_\infty(c_\infty(t), t) \right\|
\leq C\lambda^{-(1-\varepsilon)|n|}.
\]
Thus \( (u_E)_n \) is a \((C, (1 - \varepsilon) \ln \lambda)\)-good eigenfunction for \( H_{\lambda v, \alpha, c_\infty(t)} \).

\[
\hfill \square
\]

4. Completeness arguments

4.1. \( \mathcal{L} \)-measure. In [18], the authors introduced \( \mathcal{R} \)-measure to prove the arithmetic version of Anderson localization. Inspired by the idea, we introduce similarly a measure by Proposition 3.1 in stead of reducibility in [18]. The measure, we call it \( \mathcal{L} \)-measure, will play an important role in the proof of arithmetic version of Anderson localization. We next define \( E : \mathbb{T} \rightarrow \Sigma_{\lambda v, \alpha} \) as the following:
\[
E(\theta) = \begin{cases} 
\lambda c_\infty^{-1}(\theta), & \theta \in [0, \frac{1}{2}], \\
\lambda c_\infty^{-1}(1 - \theta), & \theta \in (\frac{1}{2}, 1].
\end{cases}
\]
Since \( c_\infty \) is increasing in the spectrum, \( E(\theta) \) takes one value.
For every \( \tau > 1 \) and \( \gamma > 0 \), we define \( \mathcal{E}_\gamma^\tau = E(\Theta_\gamma^\tau) \). For any \( E \in \mathcal{E}_\gamma^\tau \), we define a vector-valued function \( u_E : \mathcal{E}_\gamma^\tau \rightarrow \ell^2(\mathbb{Z}) \) as the following,
\[
(4.1) \quad u_E(n) = \frac{v_E(n)}{\|v_E\|_{L^2}},
\]
where \( v_E \) is the eigenfunction of \( H_{\lambda v, \alpha, c_\infty(t)} \) constructed in Proposition 3.1.

For any fixed \( \theta \in \Theta^\tau = \cup_{\gamma > 0} \Theta_\gamma^\tau \), we denote by \( E_m(\theta) = \lambda c_\infty^{-1}(T^m \theta) \). We can define the following \( \mathcal{L} \)-measure,

**Definition 4.1 (\( \mathcal{L} \)-measure).** \( \nu_\theta : \mathcal{B} \rightarrow \mathbb{R} \) is defined as:
\[
\nu_\theta(B) = \sum_{m \in \mathbb{N}_\theta^B} \frac{|u_{E_m(\theta)}(m)|^2 + |u_{E_m(\theta)}(m + 1)|^2}{2},
\]
for all \( B \) in the Borel \( \sigma \)-algebra \( \mathcal{B} \) of \( \mathbb{R} \), where \( \mathbb{N}_\theta^B = \{ m | E_m(\theta) \in B \} \).

The \( \mathcal{L} \)-measure is well defined since the eigenvalues are simple and has the following property.
Lemma 4.1. For a.e. $\theta$, 
\[ \nu_\theta(\mathcal{E}_\gamma^r) \geq |\Theta_\gamma^r| \]
where $|\cdot|$ is the Lebesgue measure.

Proof. Note that $\nu_\theta(\mathcal{E}_\gamma^r)$ is measurable in $\theta$ and $\nu_\theta(\mathcal{E}_\gamma^r) = \nu_{\theta + \alpha}(\mathcal{E}_\gamma^r)$. Thus $\nu_\theta(\mathcal{E}_\gamma^r) = C$ for a.e. $\theta$.

For any $\theta \in \mathbb{T}$, let $N_\theta = \{m|E_m(\theta) \in \mathcal{E}_\gamma^r\}$. For any $m \in \mathbb{Z}$, if $m \notin N_\theta$, let $P_m(\theta) = 0$. If $m \in N_\theta$, let $P_m(\theta)$ be the spectral projection of $H_{\lambda v, \alpha, \theta}$ onto the eigenspace corresponding to $E_m(\theta)$. By the definition of $E_m(\theta)$, $u_{E_m(\theta)}(n)$ is an normalized eigenfunction of $H_{\lambda v, \alpha, T^m \theta}$, thus $T_m u_{E_m(\theta)}(n)$ \(^{10}\) is an normalized eigenfunction of $H_{\lambda v, \alpha, \theta}$. Now we define a projection operator for any $\theta \in \mathbb{T}$,
\[ P(\theta) = \sum_{T^m \theta \in \Theta_\gamma^r} P_m(\theta). \]

Note that all these $E$’s in $\mathcal{E}_\gamma^r$ are different and all $P_m(\theta)$ are mutually orthogonal. It follows that $P(\theta)$ is a projection. Moreover, we have
\[ \int_{\mathbb{T}} \frac{\langle P(\theta) \delta_0, \delta_0 \rangle + \langle P(\theta) \delta_0, \delta_0 \rangle}{2} d\theta = \int_{\mathbb{T}} \sum_{T^m \theta \in \Theta_\gamma^r} \frac{\langle P_m(\theta) \delta_0, \delta_0 \rangle + \langle P_m(\theta) \delta_0, \delta_0 \rangle}{2} d\theta. \]

By Fubini theorem, we have
\[ \int_{\mathbb{T}} \sum_{T^m \theta \in \Theta_\gamma^r} \frac{\langle P_m(\theta) \delta_0, \delta_0 \rangle + \langle P_m(\theta) \delta_1, \delta_1 \rangle}{2} d\theta = \int_{\Theta_\gamma^r} \sum_{m \in \mathbb{Z}} \frac{\langle P_m(T^{-m} \theta) \delta_0, \delta_0 \rangle + \langle P_m(T^{-m} \theta) \delta_1, \delta_1 \rangle}{2} d\theta. \]

Since $T_m H_{\lambda v, \alpha, T^{-m} \theta} T_m = H_{\lambda v, \alpha, \theta}$, we have
\[ H_{\lambda v, \alpha, T^{-m} \theta} T_m u_{E_m(\theta)} = T_m H_{\lambda v, \alpha, \theta} u_{E_m(\theta)} = E_m(\theta) T_m u_{E_m(\theta)} = E_m(T^{-m} \theta) T_m u_{E_m(\theta)}. \]

It follows that $T_m u_{E_m(\theta)}$ belongs to the range of $P_m(T^{-m} \theta)$, and for each $\delta_n \in \ell^2(\mathbb{Z})$, we have
\[ \langle P_m(T^{-m} \theta) \delta_n, \delta_n \rangle \geq |\langle T_m u_{E_m(\theta)}, \delta_n \rangle|^2. \]

This implies that
\[ \sum_{m \in \mathbb{Z}} \int_{\Theta_\gamma^r} \frac{\langle P_m T^{-m} \theta) \delta_0, \delta_0 \rangle + \langle P_m(T^{-m} \theta) \delta_1, \delta_1 \rangle}{2} d\theta \geq \sum_{m \in \mathbb{Z}} \int_{\Theta_\gamma^r} \frac{|\langle T_m u_{E_m(\theta)}, \delta_0 \rangle|^2 + |\langle T_m u_{E_m(\theta)}, \delta_1 \rangle|^2}{2} d\theta \]
\[ = \sum_{m \in \mathbb{Z}} \int_{\Theta_\gamma^r} \frac{|\langle T_m u_{E_m(\theta)}, \delta_0 \rangle|^2 + |\langle T_m u_{E_m(\theta)}, \delta_1 \rangle|^2}{2} d\theta. \]

\(^{10}\) $T_m$ is a translation defined by $T_m u(n) := u(n + m)$. 
Since \( u_{E_m(\theta)} \) is a normalized eigenfunction, i.e.,
\[
\sum_{m \in \mathbb{Z}} |\langle T_m u_{E_m(\theta)}, \delta_0 \rangle|^2 = \sum_{m \in \mathbb{Z}} |\langle T_m u_{E_m(\theta)}, \delta_1 \rangle|^2 = 1.
\]
Hence we have
\[
\int_T \left( \frac{1}{2} \langle P(\theta) \delta_0, \delta_0 \rangle + \langle P(\theta) \delta_1, \delta_1 \rangle \right) d\theta \geq \sum_{m \in \mathbb{Z}} \int_{\Theta_{\gamma}} \left( \frac{1}{2} |\langle T_m u_{E_m(\theta)}, \delta_0 \rangle|^2 + |\langle T_m u_{E_m(\theta)}, \delta_1 \rangle|^2 \right) d\theta
= |\Theta_{\gamma}^*|.
\]
Thus
\[
\nu_\theta(\mathcal{E}_{\gamma}^*) \geq |\Theta_{\gamma}^*|
\]
for a.e. \( \theta \). This finishes the proof.

4.2. Arithmetic version of Anderson localization.

**Lemma 4.2.** For any \( \epsilon > 0 \), there exists \( N_0(\gamma, \tau, v, \alpha, \epsilon) > 0 \) such that for all \( \theta \in \Theta_{\gamma}^* \), we have
\[
\mathcal{R}_{N_0} \nu_\theta(\mathcal{E}_{\gamma}^*) := \sum_{|m| > N_0; T^m \theta \in \Theta_{\gamma}^*} \left| u_{E_m(\theta)}(m) \right|^2 + \left| u_{E_m(\theta)}(m + 1) \right|^2 \leq \epsilon.
\]

**Proof.** Note that for any \( T^m \theta \in \Theta_{\gamma}^* \), by Proposition 3.1,
\[
\left| u_{E_m(\theta)}(m) \right|^2 + \left| u_{E_m(\theta)}(m + 1) \right|^2 \leq C e^{-\frac{\ln \lambda}{2}|m|}.
\]
Thus for any \( \epsilon > 0 \), there exists \( N_0(\gamma, \tau, v, \alpha, \epsilon) > 0 \) such that for all \( \theta \in \Theta_{\gamma}^* \),
\[
\sum_{|m| > N_0; T^m \theta \in \Theta_{\gamma}^*} \frac{\left| u_{E_m(\theta)}(m) \right|^2 + \left| u_{E_m(\theta)}(m + 1) \right|^2}{2} \leq \epsilon.
\]

We define \( \mathcal{E}^* = \bigcup_{\gamma > 0} \mathcal{E}_{\gamma}^* \). Then

**Lemma 4.3.** For any \( N > 0 \) and \( \epsilon > 0 \), there exists \( \delta(\gamma, \tau, v, \alpha, N, \epsilon) > 0 \) such that
\[
|T_N \nu_\theta(\mathcal{E}^*) - T_N \nu_{\theta'}(\mathcal{E}^*)| \leq \epsilon
\]
for any \( \theta, \theta' \in \Theta_{\gamma}^* \) with \( |\theta - \theta'| \leq \delta \) where \( T_N \nu_\theta(\mathcal{E}^*) := \nu_\theta(\mathcal{E}^*) - \mathcal{R}_N \nu_\theta(\mathcal{E}^*) \).

**Proof.** For any fixed \( N \) and any \( \theta \in \Theta_{\gamma}^* \), we have
\[
\| \theta + k\alpha + n\alpha \|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|k + n| + 1)\tau} \geq \frac{\gamma(1 + N)^{-\tau}}{(n + 1)^{\tau}}, \quad |k| \leq N.
\]
Thus \( T^k \theta \in \Theta^* \) for \( |k| \leq N \). By Proposition 2.1, we have
\[
\| v_{E_m(\theta)} \|_{L^2} \leq C(\gamma, \tau, v, \alpha).
\]
By telescoping and Lemma 4.2, there exists $\delta(\gamma, \tau, v, \alpha, N, \epsilon)$, such that if $|\theta - \theta'| < \delta$, then

\begin{equation}
|v_{E_k}(\theta)(k) - v_{E_k}(\theta')(k)| \leq \frac{\epsilon C^{-4}}{500(2N + 1)}, \quad \forall |k| \leq 2N + 1. \tag{4.3}
\end{equation}

(4.2) and (4.3) imply for any $|k| \leq N$,

\begin{equation}
|u_{E_k}(\theta)(k) - u_{E_k}(\theta')(k)| = \frac{|v_{E_k}(\theta)(k)|}{\|v_{E_k}(\theta')\|_2} - \frac{|v_{E_k}(\theta)(k)|}{\|v_{E_k}(\theta')\|_2} \leq \frac{\epsilon C^{-4}}{500(2N + 1)} \frac{\|v_{E_k}(\theta)\|_2 \|v_{E_k}(\theta')\|_2}{\|v_{E_k}(\theta')\|_2 + C}.
\end{equation}

Thus we have

\begin{equation}
|u_{E_k}(\theta)(k) - u_{E_k}(\theta')(k)| \leq \frac{\epsilon}{100(2N + 1)}.
\end{equation}

By Definition 4.1 and (4.1), one has

\begin{equation}
|\mathcal{T}_N \nu_{\theta}(\mathcal{E}^\tau) - \mathcal{T}_N \nu_{\theta}(\mathcal{E}^\tau)|
\end{equation}

\begin{align*}
&= \sum_{|k| \leq N} |u_{E(T^k \theta)}(k)|^2 - \sum_{|k| \leq N} |u_{E(T^k \theta')}(k)|^2 \\
&\quad + \sum_{|k| \leq N} |u_{E(T^k \theta)}(k + 1)|^2 - \sum_{|k| \leq N} |u_{E(T^k \theta')}(k + 1)|^2 \\
&\leq \frac{\epsilon}{50(2N + 1)} (2N + 1) \leq \epsilon.
\end{align*}

\[\square\]

**Lemma 4.4.** For any $\theta \in \Theta_\gamma^r$, we have $\theta$ is $\Theta_{\gamma/100r}^{100r}$ homogenous, i.e., $|((\theta - \sigma, \theta + \sigma) \cap \Theta_{\gamma/100})| \geq \sigma$ for any $\sigma > 0$.

**Proof.** Let $\Theta_k = \{\theta \in [0, 1) : \|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}} < \frac{\gamma}{100(k+1)^{100r}}\}$, then $\Theta_{\gamma/100}^{100r} = [0, 1) \setminus \cup_{k \in \mathbb{Z}} \Theta_k$. Thus for any $\sigma > 0$ sufficiently small, we have

\[(\theta - \sigma, \theta + \sigma) \cap \Theta_{\gamma/100}^{100r} = (\theta - \sigma, \theta + \sigma) \setminus \cup_{k \in \mathbb{Z}} \Theta_k.\]

Notice that if $\Theta_k \cap (\theta - \sigma, \theta + \sigma) \neq \emptyset$, then

\[\frac{\gamma}{(|k| + 1)^r} \leq \|k\alpha + 2\theta\|_{\mathbb{R}/\mathbb{Z}} \leq 4\sigma.\]

It follows that $|k| \geq \left(\frac{\gamma}{4\sigma}\right)^\frac{1}{r}$, thus $|\Theta_k| \leq C(\gamma, \tau)\sigma^{100}$ which implies that

\[|((\theta - \sigma, \theta + \sigma) \cap \Theta_{\gamma/100}^{100r}| \geq 2\sigma - \sum_{k \geq \left(\frac{\gamma}{4\sigma}\right)^\frac{1}{r}} \Theta_k \geq \sigma.\]
Proof of Theorem 1.1: For any fixed $\theta \in \Theta_\tau^r$, by Lemma 4.4 and Lemma 4.1, for any $\gamma_1 < \gamma/100$, there exists a sequence $\theta_k \in \Theta_{\gamma_1}^{100r}$ such that $\theta_k \to \theta$ and

$$\nu_{\theta_k}(\mathcal{E}_{\gamma_1}^{100r}) \geq |\Theta_{\gamma_1}^{100r}|.$$

By Lemma 4.2, there exists $N_0(\gamma_1, \tau, \alpha) > 0$ such that

$$R_{N_0} \nu_{\theta_k}(\mathcal{E}_{\gamma_1}^{100r}) \leq \gamma_1.$$

By Lemma 4.3,

$$\mathcal{T}_{N_0} \nu_{\theta}(\mathcal{E}^{100r}) = \lim_{k \to \infty} \mathcal{T}_{N_0} \nu_{\theta_k}(\mathcal{E}^{100r}).$$

Thus

$$\nu_{\theta}(\mathcal{E}^{100r}) \geq \mathcal{T}_{N_0} \nu_{\theta}(\mathcal{E}^{100r}) \geq \limsup_{k \to \infty} \mathcal{T}_{N_0} \nu_{\theta_k}(\mathcal{E}_{\gamma_1}^{100r}) \geq |\Theta_{\gamma_1}^{100r}| - \gamma_1 \geq 1 - 2\gamma_1.$$

Let $\gamma_1 \to 0$, we have

$$1 \leq \nu_{\theta}(\mathcal{E}^{100r}) \leq \mu_{\theta}^{pp}(\mathcal{E}^{100r}) \leq \mu_{\theta}(\mathcal{E}^{100r}) \leq 1,$$

where $\mu_{\theta}$ is the spectral measure of $H_{\lambda,\alpha,\theta}$ defined by

$$\frac{1}{2} \left( \langle \delta_0, \chi_B(H_{\lambda,\alpha,\theta})\delta_0 \rangle + \langle \delta_1, \chi_B(H_{\lambda,\alpha,\theta})\delta_1 \rangle \right) = \int_{\mathbb{R}} \chi_B d\mu_{\theta},$$

and $\mu_{\theta}^{pp}$ is pure point piece of $\mu_{\theta}$. It follows that $\mu_{\theta} = \mu_{\theta}^{pp}$ for any $\theta \in \Theta_\tau^r$.

Thus we finish the proof. □

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