Calculation of the $\phi^4$ 6-loop non-zeta transcendental

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Abstract

We present an analytic calculation of the first transcendental in $\phi^4$-Theory that is not of the form $\zeta(2n+1)$. It is encountered at 6 loops and known to be a weight 8 double sum. Here it is obtained by reducing multiple zeta values of depth $\leq 4$. We give a closed expression in terms of a zeta-related sum for a family of diagrams that entails a class of physical graphs. We confirm that this class produces multiple zeta values of weights equal to the crossing numbers of the related knots.

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1 Introduction

The miraculous connection between Feynman graphs, knots, and numbers discovered by D. Kreimer and D.J. Broadhurst reveals a structure underlying generic renormalizable Quantum Field Theories. It is conjectured that via Knot Theory the topology of an amplitude provides its transcendental number contents which for primitively divergent diagrams, is given by the renormalization scheme independent numerical factor in front of the logarithmic singularity.

The calculation of these numbers was achieved in for 56 of the 59 primitively divergent graphs up to 7 loops in massless $\phi^4$-Theory. Diagrams that could not be calculated analytically were evaluated numerically to high precision. A systematic search for the rational coefficients of a conjectured basis of transcendentals led in all but three 7-loop diagrams (related to the hyperbolic knots $10_{139}$ and $10_{152}$) to (with very high probability) exact results.

The first graphs where analytic calculations were not available showed up at 6 loops. These diagrams entailed the 4,3-torus knot (as opposed to the $(2n+1),2$-torus knots leading to $\zeta(2n+1)$) which was determined to be related to the transcendental $\frac{1}{5}(-\zeta(5,3) + 29\zeta(8))$, where $\zeta(n, m) =$

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$\sum_{k \geq 1} k^{-n} \ell^{-m}$. In the terminology of \( \mathbb{P} \) these graphs are the \( M(2, 1, 1, 0) \) and the \( M(1, 1, 1, 1) \) '4-banana' diagrams.

In this paper we focus on the more symmetric \( M(1, 1, 1, 1) \). We keep equations general as far as this does not complicate calculations. In Theorem 10 a closed result is given that easily generalizes to encompass all \( n \)-banana diagrams (defined below). The derivation is organized as a collection of propositions that are all proved in full detail although most of them are elementary.

2 Notation and definitions

We normalize 4-dimensional integrals with \( \frac{1}{4\pi^2} \) and 2-dimensional integrals with \( \frac{1}{2\pi} \) (which is symmetric under Fourier transformation),

\[
\int dx = \frac{1}{4\pi^2} \int_{\mathbb{R}^4} d^4x, \quad \int_C dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} d^2x .
\]  

(1)

In order to handle the general case most efficiently, it is convenient to use a \( \delta \)-symbol (also \( \bar{\delta} \), \( \delta_i \), etc.) with the meaning that \( f(\delta) \) is the constant term in the Laurent expansion of \( f \) at \( \delta = 0 \),

\[
f(\delta) \equiv f_0 = \frac{1}{N!} \left. \frac{\partial^N}{\partial x^N} \right|_{x=0} x^Nf(x) , \quad \text{if } f(x) = \sum_{n=-N}^{\infty} f_nx^n.
\]  

(2)

Example:

\[
\frac{1}{\delta^k} x^\delta = \frac{1}{k!} \ln^k (x) , \quad \text{if } k \geq 0 \text{ (and zero otherwise)}.
\]

Definition 1. For \( x, y \in \mathbb{R}^4 \) let \( f_k(x, y) \) be recursively defined by

\[
f_0(x, y) = \frac{1}{(x - y)^2} \]

(3)

\[
f_{k+1}(x, y) = \int \frac{dz}{(x - z)^2} \frac{1}{z^2} f_k(z, y)
\]  

(4)

Remark 2. The function \( f_k(x, y) \) depends only on \(|x|, |y|\), and the angle between \( x \) and \( y \). We may treat it as a function of two complex variables \( x_C \) and \( y_C \) that coincide with \( x \) and \( y \) if the complex plane is considered as the subspace of \( \mathbb{R}^4 \) spanned by \( x \) and \( y \).

Instead of \( f_k(x_C, 1_C) \) we also write \( f_k(x, \bar{x}) \). We have

\[
f_k(x, y) = f_k(y, x) , \quad f_k(x_C, y_C) = \frac{1}{|y_C|^2} f(x_C/yc, 1_C) = \frac{1}{|x_C|^2} f(y_C/x_C, 1_C).
\]  

(5)

Definition 3 (\( n \)-banana). Let \( \mathbf{k} = (k_1, \ldots, k_n) \) be a vector of non-negative integers with not more than one of the \( k_i = 0 \). Let \(-2 < \alpha < n - 2 \) and \( e_1 \) be a 4-dimensional unit-vector. Then

\[
B_\alpha(\mathbf{k}) = \int dx \left(x^2\right)^\alpha \prod_{i=1}^{n} f_{k_i}(x, e_1).
\]  

(6)

Note, that the integral is independent of \( e_1 \) and converges under the above conditions.

In the notation of \( \mathbb{P} \) we have

\[
G(k_1, k_2, k_3) = B_0(k_1, k_2, k_3) = B_{-1}(k_1, k_2, k_3) \quad \text{and} \quad M(k_1, k_2, k_3, k_4) = B_0(k_1, k_2, k_3, k_4).
\]
3 Calculation of the \( n \)-banana diagram

We restrict ourselves to \(-1 < \alpha < n - 3\) and \(k_i \geq 1 \ \forall i\) which includes \(B_0(1,1,1,1)\).

**Proposition 4.** Let \( f : \mathbb{R}^4 \rightarrow \mathbb{R}, x \mapsto f(x) \) be a function of \(|x|\) and \(x \cdot e_1\) only, then

\[
\int dx f(x) = -\frac{1}{4} \int \mathcal{C} dx (x - \bar{x})^2 f_\mathcal{C}(x, \bar{x}) ,
\]

where \( f_\mathcal{C} : \mathbb{C} \rightarrow \mathbb{R}, x \mapsto f_\mathcal{C}(x, \bar{x}) \) equals \( f \) on the complex plane spanned by \( x \) and \( e_1 = 1_\mathbb{C} \).

**Proof.** Introducing polar coordinates yields

\[
\int dx f(x) = \frac{1}{4\pi^2} \int_0^\infty dx x^4 \int_0^{2\pi} d\theta \sin^2 \theta f_\mathcal{C}(xe^{i\theta}, xe^{-i\theta})
\]

\[
= \frac{1}{2\pi} \int_0^\infty dx x \int_0^{2\pi} d\theta \frac{(xe^{i\theta} - xe^{-i\theta})^2}{(2i)^2} f_\mathcal{C}(xe^{i\theta}, xe^{-i\theta}) = -\frac{1}{4} \int \mathcal{C} dx (x - \bar{x})^2 f_\mathcal{C}(x, \bar{x}) .
\]

Hence,

\[
B_\alpha(k) = -\frac{1}{4} \int \mathcal{C} dx (x\bar{x})^\alpha (x - \bar{x})^2 \prod_{i=1}^n f_{k_i}(x, \bar{x}) .
\]

With Eq. (3) we immediately obtain

\[
B_\alpha(k) = B_{n-4-\alpha}(k) .
\]

Moreover, with \( x = |x|\bar{x}, \bar{x} = |x|/\bar{x} \) we get

\[
B_\alpha(k) = -\frac{1}{4} \int_0^\infty dx x^{2\alpha+1} \int_{\partial B} \frac{d\bar{x}}{2\pi i\bar{x}} (x\bar{x} - x/\bar{x})^2 \prod_{i=1}^n f_{k_i}(x\bar{x}, x/\bar{x}) ,
\]

where we integrate \( \bar{x} \) along the unit circle \( \partial B \) in \( \mathbb{C} \).

**Proposition 5.**

\[
f_k(x, \bar{x}) = -\frac{\delta + \delta}{(4\delta)^k} \int_0^1 dt t^\delta \left( \frac{t}{x\bar{x}} \right) \frac{1}{t-x} \frac{1}{(t-x)} .
\]

With (5) and our definition of the \( \delta \)-symbol we get the following 4-dimensional statement.

**Corollary 6.**

\[
f_k(x, y) = -\frac{1}{4^kk! (k-1)!} \int_0^1 dt \ln^{k-1}(t) \ln^{k-1} \left( \frac{ty^2}{x^2} \right) \ln \left( \frac{t^2y^2}{x^2} \right) \frac{1}{(yt-x)^2} .
\]

**Proof** of Prop. 5. Expansion of the propagator into a sum of Gegenbauer polynomials gives (with \(\cos(\vartheta y) = y \cdot \bar{y}/|xy|\) and \(x < x\) if \(x < 1\), \(x = 1/x\) if \(x > 1\))

\[
\frac{|xy|}{(x-y)^2} = \sum_{n=1}^\infty C_{n-1}(\cos(\vartheta y)) \frac{|x|^n}{|y|^n} = \int \frac{dP}{\pi} \sum_{n=1}^\infty nC_{n-1}(\cos(\vartheta y)) \frac{|x|^{|P|}}{|y|^{|P|}} \frac{1}{(n^2 + P^2)^{k+1}} .
\]

Orthogonality of the Gegenbauer Polynomials yields

\[
f_k(x, e_1) = \int \frac{dP}{\pi} \sum_{n=1}^\infty nC_{n-1}(\cos(\vartheta e_1)) \frac{|x|^{|P|}}{|(n^2 + P^2)^{k+1}}
\]
from which we derive the recursion relation

\[
\frac{\partial}{\partial |x|} |x| f_k (x, e_1) = \int \frac{dP}{\pi} \sum_{n=1}^{\infty} nC_{n-1} \left( \cos (\theta_{xe}) \right) |x|^{|P|-1-i \frac{\partial}{2k \partial P} \left( \frac{1}{n^2 + P^2} \right)}
\]

\[
= \frac{i}{2k} \int \frac{dP}{\pi} \sum_{n=1}^{\infty} nC_{n-1} \left( \cos (\theta_{xe}) \right) \frac{1}{(n^2 + P^2)^{k}} \frac{\partial |x|^{|P|-1}}{\partial P} = - \frac{\ln (|x|)}{2k} f_{k-1} (x, e_1).
\]

If we integrate this relation we encounter an ambiguity of the form \( c(\hat{x})/|x| \) which is removed by the condition \( \lim_{|x| \to \infty} |x| f(x, e_1) = 0 \). Equation [(10)] meets this condition. Moreover with \( g_k(x, \bar{x}) = \int_0^1 dt t^k (t/(x\bar{x}))^\delta (1/(t - x) - 1/(t - \bar{x})) \) we get

\[
- \frac{2k}{\ln (|x|)} \frac{\partial}{\partial |x|} |x|^\delta \frac{1}{x - x} \frac{1}{(4\delta \tilde{k})} \frac{1}{x - \bar{x}} \frac{\partial g_k (x, \bar{x})}{x - x} - \frac{2k}{\ln (|x|)} \frac{\partial}{\partial |x|} |x|^\delta \frac{1}{x - x} \frac{1}{(4\delta \tilde{k})} \frac{1}{x - \bar{x}} \frac{\partial g_k (x, \bar{x})}{x - x}
\]

\[
= - \frac{2k}{4^k \ln(|x|) (x - x)} \int_0^1 dt \left( \frac{\ln^k (t/x\bar{x})}{k!} \frac{\ln^{k-2} (t)}{(k-2)!} \frac{\ln^k (t)}{k!} \right) \left( \frac{1}{t - x} - \frac{1}{t - \bar{x}} \right)
\]

\[
\text{if } k \geq 2 \quad \text{and} \quad \frac{1}{1 - x} \frac{1}{1 - \bar{x}} \text{ if } k = 1.
\]

According to Cor. 6 this equals \( f_{k-1}(x, \bar{x}) \) which completes the proof.

We find the following behaviour of \( f_k(x, \hat{x}, x/\hat{x}) \) as a function of a complex variable \( \hat{x} \).

**Proposition 7.** The function \( f_k(x, \hat{x}, x/\hat{x}) \) is analytic in \( \hat{x} \) save for two cuts on the real axis ranging from 0 to \( x \) and from \( 1/x \) to \( \infty \). Moreover, if \( \hat{x} = |\hat{x}| e^{i\epsilon} \) we have in the limit \( \epsilon \to 0 \)

\[
f_k (x, \hat{x}, x/\hat{x}) - f_k (\bar{x}, \bar{x}/\bar{x}) = 2\pi i \frac{\delta + \tilde{\delta}}{(4\delta \tilde{k})} \frac{(x/\hat{x})^\delta (x/\bar{x})^{-\tilde{\delta}}}{x/\hat{x} - x/\bar{x}} \left( \Theta (x < |\hat{x}|) - \Theta (|\hat{x}| - 1/x) \right).
\]

**(14)**

**Proof.**

\[
f_k (x, \hat{x}, x/\hat{x}) - f_k (\bar{x}, \bar{x}/\bar{x}) = - \frac{\delta + \tilde{\delta}}{(4\delta \tilde{k})} \int_0^1 dt t^\delta \left( \frac{1}{t - x} - \frac{1}{t - x/\bar{x}} \right) \frac{1}{t - x/\hat{x}} \frac{1}{t - x/\bar{x}} \left( \frac{1}{t - x} - \frac{1}{t - x/\bar{x}} \right)
\]

(In the limit \( \epsilon \to 0 \) the poles in the differences move to the points \( t = x/\hat{x} \) and \( t = x/|\hat{x}| \) on the real axis. In the case that one of them falls into the interval \([0,1]\) we deform the \( t \)-integrals to closed curves \( C_{x/\hat{x}} \) and \( C_{x/|\hat{x}|}^{-1} \) encircling the singularities with positive or negative orientation.)

\[
= - \frac{\delta + \tilde{\delta}}{(4\delta \tilde{k})} \left( \Theta (1 - x/|\hat{x}|) \int_{C_{x/|\hat{x}|}} dt + \Theta (1 - x/|\hat{x}|) \int_{C_{x/|\hat{x}|}^{-1}} dt \right) t^\delta \left( \frac{1}{t^2} \right) \left( \frac{1}{t - x/|\hat{x}|} t - x/|\hat{x}| \right)
\]

\[
= 2\pi i \frac{\delta + \tilde{\delta}}{(4\delta \tilde{k})} \frac{1}{x/\hat{x} - x/|\hat{x}|} \left( \Theta (1 - x/|\hat{x}|) (x/\hat{x})^\delta (x/|\hat{x}|)^{-\tilde{\delta}} + \Theta (1 - x/|\hat{x}|) (x/|\hat{x}|)^{-\delta} (x/\hat{x})^{-\tilde{\delta}} \right)
\]

Changing \( \delta \) to \(-\tilde{\delta}\) and \( \tilde{\delta} \) to \(-\delta\) in the second term leads to the claimed result.
Since, for $n \geq 3$ the integrand in Eq. (13) has no pole at $\hat{x} = 0$ we may deform the integration contour to range from $0$ to $e^{-i\varepsilon}$ and back from $e^{i\varepsilon}$ to $0$ (in the limit $\varepsilon \to 0$).

\[
\int_\partial B \frac{d\hat{x}}{2\pi i\hat{x}} \left( x\hat{x} - x/\hat{x} \right)^2 \prod_{i=1}^{n} f_{ki} (x\hat{x}, x/\hat{x}) = \left( \int_0^{e^{-i\varepsilon}} d\hat{x} - \int_0^{e^{i\varepsilon}} d\hat{x} \right) \prod_{i=1}^{n} f_{ki} (x\hat{x}, x/\hat{x})
\]

\[
= \int_0^1 \frac{d\hat{x}}{2\pi i\hat{x}} \left( x\hat{x} - x/\hat{x} \right)^2 \left( \prod_{i=1}^{n} f_{ki} \left( x\hat{x} e^{-i\varepsilon}, (x/\hat{x}) e^{i\varepsilon} \right) - \prod_{i=1}^{n} f_{ki} \left( x\hat{x} e^{i\varepsilon}, (x/\hat{x}) e^{-i\varepsilon} \right) \right)
\]

\[
= -\int_0^{x<} \frac{1}{x} \sum_{\ell=1}^{n} \prod_{i=1}^{\ell} f_{ki} \left( x\hat{x} e^{i\varepsilon}, (x/\hat{x}) e^{-i\varepsilon} \right) \frac{\delta_\ell + \tilde{\delta}_\ell}{(4\delta_\ell \tilde{\delta}_\ell)^{4\ell}} (x/|\hat{x}| - x/|\hat{x}|) (x/|\hat{x}|)^{\delta_\ell} (x/|\hat{x}|)^{-\delta_\ell}
\]

\[
\cdot \prod_{i>\ell} f_{ki} \left( x\hat{x} e^{i\varepsilon}, (x/\hat{x}) e^{-i\varepsilon} \right)
\]

where $\prod_{i<1} \equiv \prod_{i>n} = 1$.

Proposition 8.

\[
\int_0^\infty dx \int_0^{x<} \frac{1}{x} \frac{d\hat{x}}{\hat{x}} F (x\hat{x}, x/\hat{x}) = \frac{1}{2} \int_0^1 dx \int_0^\infty dx \int_1^x \frac{d\hat{x}}{\hat{x}} F (x\hat{x}, x/\hat{x}) \label{eq:prop8}
\]

Proof.

\[
\int_0^1 dx \int_0^x \frac{d\hat{x}}{\hat{x}} F (x\hat{x}, x/\hat{x}) + \int_0^\infty dx \int_0^{1/x} \frac{d\hat{x}}{\hat{x}} F (x\hat{x}, x/\hat{x})
\]

\[
= \int_0^1 dx \int_0^x \frac{d\hat{x}}{\hat{x}} F \left( x^2 \hat{x}, 1/\hat{x} \right) + \int_0^\infty dx \int_0^1 \frac{d\hat{x}}{\hat{x}} F \left( x, x^2/\hat{x} \right)
\]

\[
= \frac{1}{2} \int_0^1 dx \int_0^\infty \frac{d\hat{x}}{\hat{x}} F (x/\hat{x}, \hat{x}) + \frac{1}{2} \int_0^\infty dx \int_0^1 \frac{d\hat{x}}{\hat{x}} F (x, x/\hat{x})
\]

\[
= \frac{1}{2} \left( \int_0^\infty \frac{d\hat{x}}{\hat{x}} \int_0^{1/x} dx + \int_0^1 dx \int_0^\infty \frac{d\hat{x}}{\hat{x}} \right) F (x, \bar{x})
\]

\[
= \frac{1}{2} \int_0^1 dx \int_0^\infty \frac{d\hat{x}}{\hat{x}} (x/|\bar{x}| + x/|\bar{x}|) F (x, \bar{x}) = \frac{1}{2} \int_0^1 dx \int_1^\infty \frac{d\hat{x}}{\hat{x}} F (x, \bar{x}) \]

With the shorthand

\[
g_k^\pm (x, \bar{x}) = \frac{\delta + \tilde{\delta}}{(4\delta \tilde{\delta})^4} \int_0^{e^{i\varepsilon}} dt \frac{t^\delta}{(x\bar{x})^\delta} \left( \frac{1}{t - x} - \frac{1}{t - \bar{x}} \right)
\]

we get from Eqs. (13), (15) (for $-1 < \alpha < n - 3$ all integrals converge)

\[
B_\alpha (k) = \frac{1}{8} \sum_{\ell=1}^{n} \frac{\delta_\ell + \tilde{\delta}_\ell}{(4\delta_\ell \tilde{\delta}_\ell)^4} \int_0^1 dx \int_1^\infty \frac{d\hat{x}}{(x - \bar{x})^{n-2}} \prod_{i<\ell} g_{ki}^\pm (x, \bar{x}) \prod_{i>\ell} g_{ki}^\pm (x, \bar{x}) \label{eq:balpha}
\]
Proposition 9. For $x < 1$ and $\bar{x} > 1$

$$ g_k^\pm (x, \bar{x}) = \frac{\delta + \bar{\delta}}{(4\delta)^k} \left( \sum_{n=1}^{\infty} \frac{x^{-\delta} \bar{x}^{-n-\delta} + x^{n+\delta} \bar{x}^\delta}{n + \delta + \bar{\delta}} + \pi x^{-\delta} \bar{x}^\delta \left( -\cot \left( \pi (\delta + \bar{\delta}) \right) \pm i \right) \right). \quad (19) $$

Proof.

(i) 

$$ \int_0^{e^{\pm i\epsilon}} dt \ t^\delta \left( \frac{t}{x \bar{x}} \right) \frac{1}{t - x} = - \sum_{n=1}^{\infty} \int_0^1 dt \ t^\delta \left( \frac{t}{x \bar{x}} \right) \frac{t^{n-1}}{x^n} = - \sum_{n=1}^{\infty} x^{-\delta} \bar{x}^{-n-\delta} \frac{1}{n + \delta + \bar{\delta}}. $$

(ii) 

$$ \int_0^{e^{\pm i\epsilon}} dt \ t^\delta \left( \frac{t}{x \bar{x}} \right) \frac{1}{t - x} = \left( \int_0^{\infty} dt - \int_1^{\infty} dt \right) t^\delta \left( \frac{t}{x \bar{x}} \right) \frac{1}{t - x} = \bar{x}^\delta \bar{x}^{-\delta} \int_0^{\infty} dt \ t^{\delta+\bar{\delta}} \frac{1}{t - 1} - \sum_{n=0}^{\infty} \int_0^{\infty} dt \ t^\delta \left( \frac{t}{x \bar{x}} \right) \frac{x^n}{t^{n+1}}. $$

Now, $(C_x^{-1}$ is a negatively orientated circle around $x)$

$$ \left( \int_0^{\infty} dt - \int_0^{-\infty} dt \right) t^{\delta+\bar{\delta}} \frac{1}{t - 1} = \int_{C_x^{-1}} dt \ t^{\delta+\bar{\delta}} \frac{1}{t - 1} = -2\pi i, $$

and assuming (without restriction) $\delta + \bar{\delta} < 0$

$$ \left( \int_0^{\infty} dt + \int_0^{-\infty} dt \right) t^{\delta+\bar{\delta}} \frac{1}{t - 1} = 2 \text{Re} \left( \int_0^{e^{i\epsilon}} dt \ t^{\delta+\bar{\delta}} \sum_{n=1}^{\infty} t^{n-1} + \int_{e^{i\epsilon}}^{\infty} dt \ t^{\delta+\bar{\delta}} \sum_{n=-\infty}^{0} t^{n-1} \right) = -2 \sum_{n=-\infty}^{\infty} \frac{1}{n + \delta + \bar{\delta}} = -2\pi \cot \left( \pi (\delta + \bar{\delta}) \right). $$

Hence,

$$ \int_0^{e^{\pm i\epsilon}} dt \ t^\delta \left( \frac{t}{x \bar{x}} \right) \frac{1}{t - x} = - \sum_{n=0}^{\infty} \frac{x^{n-\delta} \bar{x}^{-\delta}}{n - \delta - \bar{\delta}} - \pi x^\delta \bar{x}^{-\delta} \left( \cot \left( \pi (\delta + \bar{\delta}) \right) \pm i \right). $$

The $n = 0$-term drops for $k > 0$ by definition of the $\delta$-symbol. Changing $\delta$ to $-\bar{\delta}$ and $\bar{\delta}$ to $-\delta$ in (ii) gives the result. \hfill \Box

With $(\bar{x} - x)^{-n+2} = \sum_{n_i=1}^{\infty} (n_i+3) x^{n_i-1} \bar{x}^{-n_i-3}$ the $x$, $\bar{x}$-integrals in Eq. (18) are trivially evaluated and we obtain the following result.

**Theorem 10.** Let $-1 < \alpha < n - 3$ and $k_i \geq 1 \forall i$. With $\bar{\alpha} = n - 4 - \alpha$ and

$$ O_j^\pm F (\beta_j, \bar{\beta}_j) = \sum_{n_j=1}^{\infty} \frac{F (-\delta_j, n_j + \delta_j) + F (n_j + \delta_j, -\delta_j)}{n_j + \delta_j + \bar{\delta}_j} + \pi F (-\bar{\beta}_j, -\bar{\beta}_j) \left( -\cot \left( \pi (\delta_j + \bar{\beta}_j) \right) \pm i \right) $$

we obtain (cf. Def. 3)

$$ B_\alpha (k) = \frac{1}{8} \prod_{i=1}^{n} \frac{\delta_i + \bar{\delta}_i}{(4\delta_i \delta_i)} \sum_{\ell=1}^{\infty} \prod_{\ell=1}^{\infty} \left( n_\ell + n - 4 \right) \prod_{j<\ell} O_j^{-1} \prod_{j>\ell} O_j^+ F (\beta, \bar{\beta}) , \quad (21) $$

with

$$ F (\beta, \bar{\beta}) = \frac{1}{n_\ell + \delta_\ell + \sum_{j \neq \ell} \bar{\beta}_j + \alpha} \frac{1}{n_\ell + \delta_\ell + \sum_{j \neq \ell} \bar{\beta}_j + \alpha}. \quad (22) $$

\hfill \Box
In the case $\alpha = \bar{\alpha}$ there exists a symmetry interchanging bared with unbared variables. This allows us to reduce the case $n = 4$, $k_i = k$ \forall i to 6 sums.

With the definitions

\[ A_J = \frac{1}{\sum_{j \in J} n_j + \sum_{j \notin J} \delta_j - \sum_{j \notin J} \delta_j + \alpha}, \quad \bar{A}_J = \frac{1}{\sum_{j \in J} n_j + \sum_{j \notin J} \delta_j - \sum_{j \notin J} \delta_j + \alpha}, \quad \text{for } J \subset \{1, \ldots, n\}, \]

we find

\[ B_0(k, k, k, k) = \prod_{i=1}^{n} \frac{\delta_i + \bar{\delta}_i}{(4\delta_i \bar{\delta}_i)^k} \left( \sum_{n_1, n_2, n_3, n_4 = 1}^{\infty} C_2 C_3 C_4 A_{1234} \bar{A}_1 + 3 \sum_{n_1, n_2, n_3, n_4 = 1}^{\infty} C_2 C_3 C_4 A_{123} \bar{A}_{14} \right) \]

\[ - 3\pi \cot(\pi (\delta_4 + \bar{\delta}_4)) \left( \sum_{n_1, n_2, n_3 = 1}^{\infty} C_2 C_3 A_{123} \bar{A}_1 + 3 \sum_{n_1, n_2, n_3 = 1}^{\infty} C_2 C_3 A_{12} \bar{A}_{13} \right) \]

\[ + \pi^2 (3 \cot(\pi (\delta_3 + \bar{\delta}_3)) \cot(\pi (\delta_4 + \bar{\delta}_4)) - 1) \sum_{n_1, n_2 = 1}^{\infty} C_2 A_{12} \bar{A}_1 \]

\[ - \frac{\pi^3}{2} \cot(\pi (\delta_2 + \bar{\delta}_2)) (\cot(\pi (\delta_3 + \bar{\delta}_3)) \cot(\pi (\delta_4 + \bar{\delta}_4)) - 1) \sum_{n_1 = 1}^{\infty} A_1 \bar{A}_1 \].

Each of the sums is convergent. The last sum is easily evaluated to be a multiple of $\pi^8$.

To convert sums 1, 3, 5 into multiple zeta values it is sufficient to make repeated use of the identity

\[ \frac{1}{n_1 n_2} = \frac{1}{n_1 + n_2} \left( \frac{1}{n_1} + \frac{1}{n_2} \right) \]

either before or after the expansion of the $\delta$'s.

Sums 2, 4 are first to be converted into partial fractions with respect to $k_4$ or $k_3$, respectively. Then one uses the identity ($i = 4, 3$)

\[ \sum_{k_1, k_1 = 1}^{\infty} F(k_i + k_1, k_1) = \sum_{k_1, k_1 = 1}^{\infty} F(k_i, k_1) - \sum_{k_1, k_1 = 1}^{\infty} F(k_i, k_1 + k_1) - \sum_{k_1 = 1}^{\infty} F(k_i, k_1) \]

and (25) to produce multiple zeta values.

Finally we use tables (due to D.J. Broadhurst) to convert the multiple zeta values of weight 8 to a minimum basis consisting of $\zeta(5, 3)$, $\zeta(8)$, $\zeta(5)\zeta(3)$, and $\zeta(3)^2 \zeta(2)$. We obtain

\[ B_0(1, 1, 1, 1) = -\frac{567}{320} \zeta(5, 3) + \frac{11663}{3840} \zeta(8) - \frac{252}{64} \zeta(5) \zeta(3) + 45 \frac{\zeta(3)^2 \zeta(2)}{64} \]

\[ - \frac{351}{320} \zeta(5, 3) - \frac{1261}{3840} \zeta(8) + \frac{9}{64} \zeta(5) \zeta(3) + 45 \frac{\zeta(3)^2 \zeta(2)}{64} \]

\[ - \frac{621}{320} \zeta(5, 3) - \frac{10079}{3840} \zeta(8) + \frac{261}{64} \zeta(5) \zeta(3) - 90 \frac{\zeta(3)^2 \zeta(2)}{64} \]

\[ - \frac{135}{320} \zeta(5, 3) + \frac{14555}{3840} \zeta(8) - \frac{180}{64} \zeta(5) \zeta(3) - 18 \frac{\zeta(3)^2 \zeta(2)}{64} \]

\[ - \frac{270}{320} \zeta(5, 3) - \frac{1710}{3840} \zeta(8) + 54 \frac{\zeta(3)^2 \zeta(2)}{64} \]

\[ - \frac{3840}{3840} \zeta(8) \]
\[ B_{0}(1, 1, 1, 1) = -\frac{27}{20} \zeta(5, 3) + \frac{261}{80} \zeta(8) - \frac{81}{32} \zeta(5) \zeta(3) \] (27)

which reproduces the result in \[1\] up to a factor of 4^5 which is due to a different normalization of the integrals.

4 Conclusions

We presented a closed expression for \( n \)-banana diagrams in terms of sums that easily convert to multiple zeta values.

The derivation is valid for \(-1 < \alpha < n-3\) whereas the \( n \)-banana is defined for \(-2 < \alpha < n-2\). If we approach the limit \( \alpha = -1 \) or \( \alpha = n - 3 \) in Eq. \([21]\) we encounter terms of the form \( 1/0 \) for \( k > 0 \). The limit, however, is finite. We conclude that the coefficients in front of the singular terms add up to zero. We thus expect Eq. \([21]\) to be valid in the region \(-2 < \alpha < n - 2\) if one nullifies singular terms.

The inclusion of the case \( k_i = 0 \) for one \( i \in \{0, \ldots, n\} \) is straightforward and amounts to basically replacing the numerators \( \delta_i + \bar{\delta}_i \) by linear combinations of the \( n_j \)'s. It is hence possible to treat all \( n \)-banana diagrams with our method. Analogous results may be derived in any even dimensions.

Equation \([21]\) can be used to study the cases where level-mixing does not occur. If \( n \geq 5 \) we expect in general level-mixing due to the inhomogeneous factor of \( \binom{n+\alpha-4}{n-3} \).

From Eq. \([21]\) we read off that for \( k_i \geq 1 \) the 4-banana in homogeneous of weight \( 2(k_1 + k_2 + k_3 + k_4) \). We expect the same behaviour if one of the \( k_i \)'s is zero. This confirms the conjecture in \([2]\) that the class \( B_{0}(k, l, 1, 1) \) of graphs related to renormalizable Quantum Field Theories reduces to multiple zeta values of weight equal to the crossing number of the related knots.

The cases where \( \alpha = -1 \), \( \alpha = n - 2 \), or \( n = 3 \) are more subtle because of the occurrence of the singular terms. These cases include a second physical series \( B_{0}(m + n + 2, k, 0) \) which is conjectured to lead to weight \( 2(n + m + k) + 5 \) multiple zeta transcendentals. These together with two more classes of non-banana diagrams are expected to exhaust all independent multiple zeta values up to weight 16 (corresponding to 9 loops).

We would not be surprised if future work reveals a simpler derivation of \( B_{0}(1, 1, 1, 1) \) since \( B_{0}(1, 1, 1, 1) \) seems genuinely to be a two-fold sum: The coefficients in front of the \( \zeta(3)^2 \zeta(2) \)-transcendental add up to zero. Since the reduction of weight 8 depth 2 multiple zeta values (in contrast to depth > 2) never produces \( \zeta(3)^2 \zeta(2) \)-terms the miracle of the vanishing coefficient could be naturally avoided if there was a calculation of \( B_{0}(1, 1, 1, 1) \) that does not make use of depth > 2. Moreover our calculation does not explain the connection to the \((4,3)\)-torus-knot (and the \( T_{5,2}T_{3,2} \)-factor knot).

We close the paper with the conjecture that any amplitude reduces to multiple zeta values if it has an angular graph that reduces to a single line \((\bullet---\bullet)\) after repeated substitutions of double lines \((\bullet=\bullet)\) and iterated lines \((\bullet-\bullet-\bullet\bullet)\) by single lines. Those amplitudes are free of 6- or higher-\(j\)-symbols. They do not entail the full structure of 4 dimensions and are therefore accessible to reducing 4-dimensional integrals to integrals over the complex plane which lies in the heart of our calculations.

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