A Linear Time Algorithm for $L(2, 1)$-Labeling of Trees

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Abstract An $L(2, 1)$-labeling of a graph $G$ is an assignment $f$ from the vertex set $V(G)$ to the set of nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $x$ and $y$ are adjacent and $|f(x) - f(y)| \geq 1$ if $x$ and $y$ are at distance 2, for all $x$ and $y$ in $V(G)$. A $k$-$L(2, 1)$-labeling is an $L(2, 1)$-labeling $f : V(G) \rightarrow \{0, \ldots, k\}$, and the $L(2, 1)$-labeling problem asks the minimum $k$, which we denote by $\lambda(G)$, among all possible assignments. It is known that this problem is NP-hard even for graphs of treewidth 2, and tree is one of very few classes for which the problem is polynomially solvable. The running time of the best known algorithm for trees had been $O(\Delta^{4.5}n)$ for more than a decade, and an $O(\min\{n^{1.75}, \Delta^{1.5}n\})$-time algorithm has appeared recently,

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where \( \Delta \) and \( n \) are the maximum degree and the number of vertices of an input tree, however, it has been open if it is solvable in linear time. In this paper, we finally settle this problem by establishing a linear time algorithm for \( L(2, 1) \)-labeling of trees. Furthermore, we show that it can be extended to a linear time algorithm for \( L(p, 1) \)-labeling with a constant \( p \).

**Keywords** Frequency/channel assignment · Graph algorithm · \( L(2, 1) \)-Labeling · Vertex coloring

1 **Introduction**

Let \( G \) be an undirected graph. An \( L(p, q) \)-labeling of a graph \( G \) is an assignment \( f \) from the vertex set \( V(G) \) to the set of nonnegative integers such that \( |f(x) - f(y)| \geq p \) if \( x \) and \( y \) are adjacent and \( |f(x) - f(y)| \geq q \) if \( x \) and \( y \) are at distance 2, for all \( x \) and \( y \) in \( V(G) \). A \( k \)-\( L(p, q) \)-labeling is an \( L(p, q) \)-labeling \( f : V(G) \rightarrow \{0, \ldots, k\} \), and the \( L(p, q) \)-labeling problem asks the minimum \( k \) among all possible assignments. We call this invariant, the minimum value \( k \), the \( L(p, q) \)-labeling number and denote it by \( \lambda_{p,q}(G) \). Notice that we can use \( k + 1 \) different labels when \( \lambda_{p,q}(G) = k \) since we can use 0 as a label for conventional reasons.

The original notion of \( L(p, q) \)-labeling can be seen in the context of frequency assignment, where ‘close’ transmitters must receive frequencies that are at least \( q \) frequencies apart and ‘very close’ transmitters must receive frequencies that are at least \( p (\geq q) \) frequencies apart so that they can avoid interference.

Among several possible settings of \( p \) and \( q \), \( L(2, 1) \)-labeling problem has been intensively and extensively studied due to its practical importance. From the graph theoretical point of view, since this is a kind of vertex coloring problems, it has attracted a lot of interest [3, 10, 13, 16]. We can find various related results on \( L(2, 1) \)-labelings and \( L(p, q) \)-labelings for other parameters in comprehensive surveys by Calamoneri [2] and by Yeh [17].

**Related Work** There are also a number of studies on the \( L(2, 1) \)-labeling problem, from the algorithmic point of view [1, 5, 15]. It is known to be NP-hard for general graphs [10], and it still remains NP-hard for some restricted classes of graphs, such as planar graphs, bipartite graphs, chordal graphs [1], and it turned out to be NP-hard even for graphs of treewidth 2 [6]. In contrast, only a few graph classes are known to have polynomial time algorithms for this problem, e.g., we can determine the \( L(2, 1) \)-labeling number of paths, cycles, wheels within polynomial time [10].

As for trees, Griggs and Yeh [10] showed that \( \lambda(T) \) is either \( \Delta + 1 \) or \( \Delta + 2 \) for any tree \( T \), and also conjectured that determining \( \lambda(T) \) is NP-hard, however, Chang and Kuo [3] disproved this by presenting a polynomial time algorithm for computing \( \lambda(T) \). Their algorithm exploits the fact that \( \lambda(T) \) is either \( \Delta + 1 \) or \( \Delta + 2 \) for any tree \( T \). Its running time is \( O(\Delta^{4.5}n) \), where \( \Delta \) is the maximum degree of a tree \( T \) and \( n = |V(T)| \). This result has a great importance because it initiates to cultivate polynomially solvable classes of graphs for the \( L(2, 1) \)-labeling problem and related problems. For example, Fiala et al. showed that it can be determined
in $O(\lambda^{2r+4.5}n)$ time whether a $t$-almost tree has $\lambda$-$L(2, 1)$-labeling for $\lambda$ given as an input, where a $t$-almost tree is a graph that can be a tree by eliminating $t$ edges [5]. Also, it was shown that the $L(p, 1)$-labeling problem for trees can be solved in $O((p + \Delta)^{2.5}n) = O(\lambda^{2.5}n)$ time whether a $t$-almost tree has $\lambda$-$L(2, 1)$-labeling for $\lambda$ given as an input, where a $t$-almost tree is a graph that can be a tree by eliminating $t$ edges [5]. Both results are based on Chang and Kuo’s algorithm, which is called as a subroutine in the algorithms. Moreover, the polynomially solvable result for trees holds for more general settings. The notion of $L(p, 1)$-labeling is generalized as $H(p, 1)$-labeling, in which graph $H$ defines the metric space of distances between two labels, whereas labels in $L(p, 1)$-labeling (that is, in $L(p, q)$-labeling) take nonnegative integers; i.e., it is a special case that $H$ is a path graph. In [7], it has been shown that the $H(p, 1)$-labeling problem of trees for arbitrary graph $H$ can be solved in polynomial time, which is also based on Chang and Kuo’s idea. In passing, these results are unfortunately not applicable for $L(p, q)$-labeling problems for general $p$ and $q$. Recently, Fiala et al. [8] showed that the $L(p, q)$-labeling problem of trees for arbitrary graph $H$ can be solved in polynomial time, which is also based on Chang and Kuo’s idea. In passing, these results are unfortunately not applicable for $L(p, q)$-labeling problems for general $p$ and $q$. Recently, Fiala et al. [8] showed that the $L(p, q)$-labeling problem of trees is NP-hard if $q$ is not a divisor of $p$, which is contrasting to the positive results mentioned above.

As for $L(2, 1)$-labeling of trees again, Chang and Kuo’s $O(\Delta^{4.5}n)$ algorithm is the first polynomial time one. It is based on dynamic programming (DP) approach, and it checks whether $(\Delta + 1)$-$L(2, 1)$-labeling is possible or not from leaf vertices to a root vertex in the original tree structure. The principle of optimality requires to solve at each vertex of the tree the assignments of labels to subtrees, and the assignments are formulated as the maximum matching in a certain bipartite graph. Recently, an $O(\min(n^{1.75}, \Delta^{1.5}n))$ time algorithm has been proposed [11]. It is based on the similar DP framework to Chang and Kuo’s algorithm, but achieves its efficiency by reducing heavy computation of bipartite matching in Chang and Kuo’s and by using an amortized analysis. We give a concise review of these two algorithms in Sect. 2.2.

**Our Contributions** Although there have been a few polynomial time algorithms for $L(2, 1)$-labeling of trees, it has been open if it can be improved to linear time [2]. In this paper, we present a linear time algorithm for $L(2, 1)$-labeling of trees, which finally settles this problem. The linear time algorithm is based on DP approach, which is adopted in the preceding two polynomial time algorithms [3, 11]. Besides using their ideas, we newly introduce the notion of “label compatibility”, which indicates how we flexibly change labels with preserving its $(\Delta + 1)$-$L(2, 1)$-labeling. The label compatibility is a quite handy notion and in fact, it deduces several useful facts on $L(2, 1)$-labeling of trees in fact. For example, it is used to show that only $O(\log \Delta n)$ labels are essential for $L(2, 1)$-labeling in any input tree, which enables to replace the bipartite matching of graphs with the maximum flow of much smaller networks as an engine for filling out DP tables. Roughly speaking, by utilizing the label compatibility in several ways, we can obtain a linear time algorithm for $L(2, 1)$-labeling of trees. It should be noted that the notion of label compatibility can be generalized for $L(p, 1)$-labeling of trees, and therefore, we can obtain a linear time algorithm for $L(p, 1)$-labeling of trees with a constant $p$ by extending the one for $L(2, 1)$-labeling together with the generalized label compatibility.

**Organization of this Paper** Sections from 2 to 4 deal with $L(2, 1)$-labeling of trees, and Sect. 5 deals with its extension to $L(p, 1)$-labeling of trees. Section 2 gives basic
definitions and introduces as a warm-up the ideas of Chang and Kuo’s $O(\Delta^{4.5}n)$ time algorithm and its improvement into $O(n^{1.75})$ time. Section 3 introduces the crucial notion of label compatibility that can bundle a set of compatible vertices and reduce the size of the graph constructed for computing bipartite matchings. Moreover, this allows to use maximum-flow based computation for them. In Sect. 4, we give precise analyses to achieve linear running time for $L(2, 1)$-labeling of trees. In Sect. 5, after generalizing the techniques introduced in Sect. 3 to $L(p, 1)$-labeling, we show that analyses similar to those in Sect. 4 are possible for a constant $p$; $L(p, 1)$-labeling of trees with a constant $p$ is linearly solvable also. Section 6 provides a proof of the key lemma used in Sects. 3–5, named Level Lemma, and Sect. 7 concludes the paper.

## 2 Preliminaries

### 2.1 Definitions and Notations

A graph $G$ is an ordered pair of its vertex set $V(G)$ and edge set $E(G)$ and is denoted by $G = (V(G), E(G))$. We assume throughout this paper that all graphs are undirected, simple and connected, unless otherwise stated. Therefore, an edge $e \in E(G)$ is an unordered pair of vertices $u$ and $v$, which are end vertices of $e$, and we often denote it by $e = (u, v)$. Two vertices $u$ and $v$ are adjacent if $(u, v) \in E(G)$. For a graph $G$, we may denote $|V(G)|$, the number of vertices of $G$, simply by $|G|$. A graph $G = (V(G), E(G))$ is called bipartite if the vertex set $V(G)$ can be divided into two disjoint sets $V_1$ and $V_2$ such that every edge in $E(G)$ connects a vertex in $V_1$ and one in $V_2$; such $G$ is denoted by $(V_1, V_2, E)$. Throughout the paper, we also denote by $n$ the number of vertices of input tree $T$.

For a graph $G$, the (open) neighborhood of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) | (u, v) \in E(G)\}$, and the closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v$ is $|N_G(v)|$, and is denoted by $d_G(v)$. We use $\Delta(G)$ to denote the maximum degree of a graph $G$. A vertex whose degree is $\Delta(G)$ is called major. We often drop $G$ in these notations if there are no confusions. A vertex whose degree is 1 is called a leaf vertex, or simply a leaf.

When we describe algorithms, it is convenient to regard the input tree to be rooted at a leaf vertex $r$. Then we can define the parent-child relationship on vertices in the usual way. For a rooted tree, its height is the length of the longest path from the root to a leaf. For any vertex $v$, the set of its children is denoted by $C(v)$. For a vertex $v$, define $d'(v) = |C(v)|$.

### 2.2 Chang and Kuo’s Algorithm and Its Improvement

Before explaining algorithms, we give some significant properties on $L(2, 1)$-labeling of graphs or trees that have been used so far for designing $L(2, 1)$-labeling algorithms. We can see that $\lambda(G) \geq \Delta + 1$ holds for any graph $G$. Griggs and Yeh [10] observed that any major vertex in $G$ must be labeled 0 or $\Delta + 1$ when $\lambda(G) = \Delta + 1$, and that if $\lambda(G) = \Delta + 1$, then $N_G[v]$ contains at most two major vertices for any $v \in V(G)$. Furthermore, they showed that $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$ for any tree $T$. By using this fact, Chang and Kuo [3] presented an $O(\Delta^{4.5}n)$ time algorithm for computing $\lambda(T)$.
Chang and Kuo’s Algorithm  Now, we first review the idea of Chang and Kuo’s
dynamic programming algorithm (CK algorithm) for the $L(2, 1)$-labeling problem
of trees, since our linear time algorithm also depends on the same formula of the
principle of optimality. The algorithm determines if $\lambda(T) = \Delta + 1$, and if so, we can
easily construct the labeling with $\lambda(T) = \Delta + 1$.

To describe the idea, we introduce some notations. We assume for explanation
that $T$ is rooted at some leaf vertex $r$. Given a vertex $v$, we denote the subree of $T$
rooted at $v$ by $T(v)$. Let $T(u, v)$ be a tree rooted at $u$ that forms $T(u, v) = (\{u\} \cup
V(T(v)), \{(u, v)\} \cup E(T(v)))$. Note that this $u$ is just a virtual vertex for explanation
and $T(u, v)$ is uniquely determined by $T(v)$. For $T(u, v)$, we define

$$
\delta((u, v), (a, b)) = \begin{cases} 
1, & \text{if } \lambda(T(u, v)) | f(u) = a, f(v) = b \leq \Delta + 1, \\
0, & \text{otherwise},
\end{cases}
$$

where $\lambda(T(u, v)) | f(u) = a, f(v) = b$ denotes the minimum $k$ of a $k$-$L(2, 1)$-
labeling $f$ on $T(u, v)$ satisfying $f(u) = a$ and $f(v) = b$. This $\delta$ function satisfies
the following formula:

$$
\delta((u, v), (a, b)) = \begin{cases} 
1, & \text{if there is an injective assignment } g: C(v) \rightarrow \\
[0, 1, \ldots, \Delta + 1] - \{a, b - 1, b, b + 1\} \text{ such that } \\
\delta((v, w), (b, g(w))) = 1 \text{ for each } w \in C(v), \\
0, & \text{otherwise}.
\end{cases}
$$

The existence of such an injective assignment $g$ is formalized as the maximum
matching problem: For a bipartite graph $G(u, v, a, b) = (C(v), X, E(u, v, a, b))$, where
$X = \{0, 1, \ldots, \Delta, \Delta + 1\}$ and $E(u, v, a, b) = \{(w, c) \mid \delta((v, w), (b, c)) = \\
1, c \in X - \{a\}, w \in C(v)\}$, we can see that there is an injective assignment $g:
C(v) \rightarrow \{0, 1, \ldots, \Delta + 1\} - \{a, b - 1, b, b + 1\}$ if there exists a matching of size
d$'(v)$ in $G(u, v, a, b)$. Namely, for $T(u, v)$ and two labels $a$ and $b$, we can easily
(i.e., in polynomial time) determine the value of $\delta((u, v), (a, b))$ if the values of $\delta$
function for $T(v, w), w \in C(v)$ and any two pairs of labels are given. Now let $t(v)$ be
the time for calculating $\delta((u, v), (\ast, \ast))$ for vertex $v$, where $\delta((u, v), (\ast, \ast))$ denotes
$\delta((u, v), (a, b))$ for all possible pairs of $a$ and $b$ under consideration. (We use $\ast$ also in
some other places in a similar manner.) CK algorithm solves this bipartite matching
problems of $O(\Delta)$ vertices and $O(\Delta^2)$ edges $O(\Delta^2)$ times for each $v$, in order to ob-
tain $\delta$-values for all pairs of labels $a$ and $b$. This amounts $t(v) = O(\Delta^{2.5}) \times O(\Delta^2) = 
O(\Delta^{4.5})$, where the first $O(\Delta^{2.5})$ is the time complexity of the bipartite matching
problem [14]. Thus the total running time is $\sum_{v \in V} t(v) = O(\Delta^{4.5} n)$.

An $O(n^{1.75})$-time Algorithm  Next, we review the $O(n^{1.75})$-time algorithm proposed
in [11]. The running time $O(n^{1.75})$ is roughly achieved by two strategies. One is that
the problem can be solved by a simple linear time algorithm if $\Delta = \Omega(\sqrt{n})$, and the
other is that it can be solved in $O(\Delta^{1.5} n)$ time for any input tree.

The first idea of the speedup is that for computing $\delta((u, v), (\ast, b))$, the algorithm
does not solve the bipartite matching problems every time from scratch, but reuse the
obtained matching structure. More precisely, the bipartite matching problem is solved
for $G(u, v, -, b) = (C(v), X, E(u, v, -, b))$ instead of $G(u, v, a, b)$ for a specific $a$, where $E(u, v, -, b) = \{(w, c) | \delta((v, w), (b, c)) = 1, c \in X, w \in C(v)\}$. A maximum matching of $G(u, v, -, b)$ is observed to satisfy the following properties:

**Property 1** If $G(u, v, -, b)$ has no matching of size $d'(v)$, then $\delta((u, v), (i, b)) = 0$ for any label $i$.

**Property 2** $\delta((u, v), (i, b)) = 1$ if and only if vertex $i$ can be reached by an $M$-alternating path from some vertex in $X$ unmatched by $M$ in $G(u, v, -, b)$, where $M$ denotes a maximum matching of $G(u, v, -, b)$ (of size $d'(v)$).

From these properties, $\delta((u, v), (\ast, b))$ can be computed by a single bipartite matching and a single graph search, and its total running time is $O(\Delta^{1.5}d'(v)) + O(\Delta d'(v)) = O(\Delta^{2.5}d'(v))$ for solving the bipartite matching of $G(u, v, -, b)$, which has $O(\Delta)$ vertices and $O(\Delta d'(v))$ edges, and for a single graph search). Since this calculation is done for all $b$, we have $t(v) = O(\Delta^{2.5}d'(v))$.

The other technique of the speedup introduced in [11] is based on preprocessing operations for amortized analysis. By some preprocessing operations, the shape of input trees can be restricted while preserving $L(2, 1)$-labeling number, and the input trees can be assumed to satisfy the following two properties.

**Property 3** All vertices adjacent to a leaf vertex are major vertices.

**Property 4** The size of any path component of $T$ is at most 3.

Here, a sequence of vertices $v_1, v_2, \ldots, v_\ell$ is called a path component if $(v_i, v_{i+1}) \in E$ for all $i = 1, 2, \ldots, \ell - 1$ and $d(v_i) = 2$ for all $i = 1, 2, \ldots, \ell$, and $\ell$ is called the size of the path component.

Furthermore, this preprocessing operations enable the following amortized analysis. Let $V_L$ and $V_Q$ be the set of leaf vertices and the set of major vertices whose children are all leaf vertices, respectively. Also, let $d''(v) = |C(v) - V_L|$ for $v \in V$. (Note that $d''(v) = 0$ for $v \in V_L \cup V_Q$.)

By Property 3, if we go down the resulting tree from a root, then we reach a major vertex in $V_Q$. Then, the following facts are observed: (i) for $v \in V_Q$ $\delta((u, v), (a, b)) = 1$ if and only if $b = 0$ or $\Delta + 1$ and $|a - b| \geq 2$, (ii) $|V_Q| \leq n/\Delta$. Note that (i) implies that it is not required to solve the bipartite matching to obtain $\delta$-values. Also (ii) and Property 4 imply that $|V - V_Q - V_L| = O(n/\Delta)$ (this can be obtained by pruning leaf vertices and regarding $V_Q$ vertices as new leaves). Since $\sum_{v \in V - V_L - V_Q} d''(v) = |V - V_L - V_Q| + |V_Q| - 1$ holds, we have the following lemma:

**Lemma 1** For a tree satisfying Properties 3 and 4,

$$\sum_{v \in V - V_L - V_Q} d''(v) = O(n/\Delta),$$

holds.
Algorithm 1 LABEL-TREE
1: Do PREPROCESSING (Algorithm 3). Let the output trees be $T_1, T_2, \ldots, T_k$.
2: for $i = 1$ to $k$ do
3: Let $T := T_i$, $V := V(T)$, and $V_Q$ be the set of major vertices whose children are all leaf vertices in $V$.
4: If $N[v]$ contains at least three major vertices for some vertex $v \in V$, output “No”. Halt.
5: If $|V_Q|$ is at most $\Delta - 6$, goto Step 2.
6: For $T( u, v )$ with $v \in V_Q$ (its height is 2), let $\delta((u, v), (a, 0)) := 1$ for each label $a \neq 0, 1, \delta((u, v), (a, \Delta + 1)) := 1$ for each label $a \neq \Delta, \Delta + 1$, and $\delta((u, v), (*, *)) := 0$ for any other pair of labels. Let $h := 3$.
7: For all $T( u, v )$ of height $h$, compute $\delta((u, v), (*, *))$ (Compute $\delta((u, v), (i, b))$ for each $b$ by MAINTAIN-MATCHING $(G(u, v, -, b))$ (Algorithm 2).
8: If $h = h^*$ where $h^*$ is the height of root $r$ of $T$, then goto Step 9. Otherwise let $h := h + 1$ and goto Step 6.
9: If $\delta((r, v), (a, b)) = 1$ for some $(a, b)$, then goto Step 2; otherwise output “No”. Halt.
10: end for
11: Output “Yes”. Halt.

Algorithm 2 MAINTAIN-MATCHING$(G(u, v, -, b))$
1: Find a maximum bipartite matching $M$ of $G(u, v, -, b)$. If $G(u, v, -, b)$ has no matching of size $d'(v)$, output $\delta((u, v), (*, *))$ as $\delta((u, v), (i, b)) = 0$ for every label $i$.
2: Let $X'$ be the set of unmatched vertices under $M$. For each label vertex $i$ that is reachable from a vertex in $X'$ via $M$-alternating path, let $\delta((u, v), (i, b)) = 1$. For the other vertices $j$, let $\delta((u, v), (j, b)) = 0$. Output $\delta((u, v), (*, *))$.

Since it is not necessary to compute bipartite matchings for $v \in V_L \cup V_Q$, and this implies that the total time to obtain $\delta$-values for all $v$’s is $\sum_{v \in V} t(v) = O(\sum_{v \in V-V_L-V_Q} t(v))$, which turned out to be $O(\Delta^{2.5} \sum_{v \in V-V_L-V_Q} d'(v)) = O(\Delta^{1.5} n)$ by Lemma 1. Since we have a linear time algorithm if $\Delta = \Omega(\sqrt{n})$ as mentioned above, we can solve the problem in $O(n^{1.75})$ time in total.

2.3 Concrete Form of $O(n^{1.75})$-Time Algorithm

We describe the main routine of $O(n^{1.75})$-time algorithm [11] as Algorithm 1. This algorithm calls two algorithms as subroutines. One is Algorithm 2, which is used to quickly compute $\delta((u, v), (*, b))$, as explained in Sect. 2.2. The other is Algorithm 3, which does the preprocessing operations for an input tree.

They are carried out (i) to remove the vertices that are ‘irrelevant’ to the $L(2, 1)$-labeling number, and (ii) to divide $T$ into several subtrees that preserve the $L(2, 1)$-labeling number. It is easy to show that neither of the operations affects the $L(2, 1)$-labeling number. Note that, these operations may not reduce the size of the input tree, but more importantly, they restrict the shape of input trees, which enables an amortized analysis.

We can verify that after the preprocessing operations, the input trees satisfy Properties 3 and 4.
Algorithm 3 PREPROCESSING

1: Check if there is a leaf \( v \) whose unique neighbor \( u \) has degree less than \( \Delta \). If so, remove \( v \) and edge \((u, v)\) from \( T \) until such a leaf does not exist.

2: while \( T \) contains a path component whose size is at least 4 do

3: Let the path component be \( v_1, v_2, \ldots, v_\ell \), and let \( v_0 \) and \( v_{\ell+1} \) be the unique adjacent vertices of \( v_1 \) and \( v_\ell \) other than \( v_2 \) and \( v_{\ell-1} \), respectively. Assume \( d(v_0) \geq d(v_{\ell+1}) \).

4: if \( d(v_0) = d(v_{\ell+1}) = \Delta \) then

5: Remove \( v_2, \ldots, v_{\ell-1} \) from \( T \).

6: else if \( d(v_0) = \Delta > d(v_{\ell+1}) \) then

7: Remove \( v_2, \ldots, v_{\ell} \) from \( T \).

8: else if \( \Delta > d(v_0), d(v_{\ell+1}) \) then

9: Remove \( v_1, \ldots, v_{\ell} \) from \( T \).

10: end if

11: end while

12: Output subtrees \( T_1, T_2, \ldots, T_k \), connected components of the obtained forest.

3 Label Compatibility and Flow-Based Computation of \( \delta \)

As reviewed in Sect. 2.2, one of keys of an efficient computation of \( \delta \)-values was reusing the matching structures. In this section, for a further speedup of the computation of \( \delta \)-values, we introduce a new notion, which we call ‘label compatibility’, that enables to treat several labels equivalently under the computation of \( \delta \)-values. Then, the faster computation of \( \delta \)-values is achieved on a maximum flow algorithm instead of a maximum matching algorithm. Seemingly, this sounds a bit strange, because the time complexity of the maximum flow problem is greater than the one of the bipartite matching problem. The trick is that by this notion the new flow-based computation can use a smaller network (graph) than the graph \( G(u, v, -, b) \) used in the bipartite matching.

3.1 Label Compatibility and Neck/Head Levels

Let \( L_h = \{h, h + 1, \ldots, \Delta - h, \Delta - h + 1\} \). Let \( T \) be a tree rooted at \( v \), and \( u \notin V(T) \).

We say that \( T \) is head-L\(_h\)-compatible if \( \delta((u, v), (a, b)) = \delta((u, v), (a', b)) \) for all \( a, a' \in L_h \) and \( b \in L_0 \) with \( |a - b| \geq 2 \) and \( |a' - b| \geq 2 \). Analogously, we say that \( T \) is neck-L\(_h\)-compatible if \( \delta((u, v), (a, b)) = \delta((u, v), (a, b')) \) for all \( a \in L_0 \) and \( b, b' \in L_h \) with \( |a - b| \geq 2 \) and \( |a - b'| \geq 2 \). The neck and head levels of \( T \) are defined as follows:

Definition 1 Let \( T \) be a tree rooted at \( v \), and consider the tree obtained from \( T \) by adding a new vertex \( u \) adjacent to \( v \).

(i) The neck level (resp., head level) of \( T \) is 0 if \( T \) is neck-L\(_0\)-compatible (resp., head-L\(_0\)-compatible).

(ii) The neck level (resp., head level) of \( T \) is \( h \geq 1 \) if \( T \) is not neck-L\(_{h-1}\)-compatible (resp., head-L\(_{h-1}\)-compatible) but neck-L\(_h\)-compatible (resp., head-L\(_h\)-compatible).
An intuitive explanation of neck-$L_h$-compatibility (resp., head-$L_h$-compatibility) of $T$ is that if for $T(u, v)$, a label in $L_h$ is assigned to $v$ (resp., $u$) under $(\Delta + 1)$-$L(2, 1)$-labeling of $T(u, v)$, the label can be replaced with another label in $L_h$ without violating a proper $(\Delta + 1)$-$L(2, 1)$-labeling; labels in $L_h$ are compatible. The neck and head levels of $T$ represent the bounds of $L_h$-compatibility of $T$. Thus, a trivial bound on neck and head levels is $(\Delta + 1)/2$.

For the relationship between the neck/head levels and the tree size, we can show the following lemma and theorem that are crucial throughout analyses for the linear time algorithm.

**Lemma 2** (Level Lemma) Let $T'$ be a subtree of $T$. If $|T'| < (\Delta - 3 - 2h)^{h/2}$ and $\Delta - 2h \geq 2$, then the head level and neck level of $T'$ are both at most $h$.

**Theorem 1** For a tree $T$, both the head and neck levels of $T$ are bounded by $O(\log |T|/\log \Delta)$.

We give proofs of Lemma 2 (Level Lemma) and Theorem 1 in Sect. 6 as those in more generalized forms (see Lemma 8 and Theorem 4).

### 3.2 Flow-Based Computation of $\delta$

We are ready to explain the faster computation of $\delta$-values. Recall that $\delta((u, v), (a, b)) = 1$ holds if there exists a matching of $G(u, v, a, b)$ in which all $C(v)$ vertices are just matched; which vertex is matched to a vertex in $C(v)$, we can obtain the following properties:

For a maximum flow $\psi$, $\psi((c, t)) \geq 0$. By the flow integrality and arguments similarly to Properties 1 and 2, we can obtain the following properties:

**Lemma 3** If $\mathcal{N}(u, v, - , b)$ has no flow of size $d'(v)$, then $\delta((u, v), (i, b)) = 0$ for any label $i$. 
Lemma 4 For $i \in L_0 - L_{h(v)}$, $\delta((u,v),(i,b)) = 1$ if and only if vertex $i$ can be reached by a $\psi$-alternating path from some vertex in $X'$ in $N(u,v,-,b)$. For $i \in L_{h(v)}$, $\delta((u,v),(i,b)) = 1$ if and only if vertex $h(v)$ can be reached by a $\psi$-alternating path from some vertex in $X'$ in $N(u,v,-,b)$.

Here, a $\psi$-alternating path is defined as follows: Given a flow $\psi$, a path in $E_\delta$ is called $\psi$-alternating if its edges alternately satisfy $\text{cap}(e) - \psi(e) \geq 1$ and $\psi(e) \geq 1$.

By these lemmas, we can obtain $\delta((u,v),(a,b))$-values for $b$ by solving the maximum flow of $N(u,v,-,b)$ once and then applying a single graph search.

The current fastest algorithm for the maximum flow problem runs in $O\left(\min\{m^{1/2}, n^{2/3}\}m \log(n^2/m) \log U\right) = O(n^{2/3}m \log n \log U)$ time, where $U$, $n$ and $m$ are the maximum capacity of edges, the number of vertices and edges, respectively [9]. Thus the running time of calculating $\delta((u,v),(a,b))$ for a pair $(a,b)$ is

$$O((h(v) + d''(v))^{2/3}(h(v)d''(v)) \log(h(v) + d''(v)) \log \Delta)$$

$$= O\left(\Delta^{2/3}(h(v)d''(v)) \log^2 \Delta\right),$$

since $h(v) \leq \Delta$ and $d''(v) \leq \Delta$ (recall that $d''(v) = |C(v) - V_L|$). Note that this includes the time to construct the network $N(u,v,-,b)$. By using a similar technique of updating matching structures (see [11]), we can obtain $\delta((u,v),(\ast,\ast))$ in $O(\Delta^{2/3}(h(v)d''(v)) \log^2 \Delta) + O(h(v)d''(v)) = O(\Delta^{2/3}(h(v)d''(v)) \log^2 \Delta)$ time. Since the number of candidates for $b$ is also bounded by $h(v)$ from the neck/head level property, we have the following lemma.

Lemma 5 $\delta((u,v),(\ast,\ast))$ can be computed in $O(\Delta^{2/3}(h(v))^{2}d''(v) \log^2 \Delta)$ time, that is, $t(v) = O(\Delta^{2/3}(h(v))^{2}d''(v) \log^2 \Delta)$. 

Fig. 1 An example of $N(u,v,-,b)$ where $h = h(v)$
Combining this lemma with Lemma 1, we can show the total running time for the $L(2, 1)$-labeling is $O(n(\max\{h(v)\})^2 \cdot (\Delta^{-1/3} \log^2 \Delta))$. By applying Theorem 1, we have the following theorem:

**Theorem 2** For trees, the $L(2, 1)$-labeling problem can be solved in $O(\min(n \log^2 n, \Delta^{1.5} n))$ time. Furthermore, if $n = O(\Delta^{poly(\log \Delta)})$, it can be solved in $O(n)$ time.

**Corollary 1** For a vertex $v$ in a tree $T$, we have $\sum w \in V(T(v)) t(w) = O(|T(v)|)$ if $|T(v)| = O(\Delta^{poly(\log \Delta)})$.

Only by directly applying Theorem 1 (actually Lemma 2), we obtain much faster running time than the previous one. In the following section, we present a linear time algorithm, in which Lemma 2 is used in a different way.

### 4 A Linear Time Algorithm and Its Proofs

As mentioned in Sect. 2.2, one of keys for achieving the running time $O(\Delta^{1.5} n) = O(n^{1.75})$ is equation $\sum v \in V_{\delta} d''(v) = O(n/\Delta)$, where $V_{\delta}$ is the set of vertices in which $\delta$-values should be computed via the matching-based algorithm; since the computation of $\delta$-values for each $v$ is done in $O(\Delta^{2.5} d''(v))$ time, it takes $\sum v \in V_{\delta} O(\Delta^{2.5} d''(v)) = O(\Delta^{1.5} n)$ time in total. This equation is derived from the fact that in leaf vertices we do not need to solve the matching to compute $\delta$-values, and any vertex with height 1 has $\Delta - 1$ leaves as its children after the preprocessing operation.

In our linear time algorithm, we generalize this idea: In the above argument, the vertices of $V - V_{\delta}$, whose $\delta$-values can be obtained without heavy computation (matching), are leaf vertices. We enlarge $V - V_{\delta}$ to the vertices in subtrees with size between $\Delta^c$ and $\Delta^{c+1}$ for some constant $c \geq 2.5$. In fact, $\delta$-values of such vertices can be obtained in linear time by Corollary 1. Also, $\sum v \in V_{\delta} d''(v)$ could be $O(n/\Delta^c)$. Then, in total, the running time $\sum v \in V_{\delta} O(\Delta^{2.5} d''(v)) = O(n)$ could be roughly achieved. Actually, this argument contains a cheating, because a subtree with size between $\Delta^c$ and $\Delta^{c+1}$ is not always connected to a major vertex, whereas a leaf is, which plays a key role to obtain $\sum v \in V_{\delta} d''(v) = O(n/\Delta)$. We resolve this problem by best utilizing the properties of neck/head levels and the maximum flow techniques introduced in Sect. 3.

#### 4.1 Efficient Assignment of Labels for Computing $\delta$

In this section, by compiling observations and techniques for assigning labels in the computation of $\delta((u, v), (\ast, \ast))$ for $v \in V$, given in Sects. 2 and 3, we will design an algorithm to run in linear time within the DP framework.

Throughout this section, we assume that an input tree $T$ satisfies Properties 3 and 4. Also we assume $\Delta \geq 18$, because otherwise the problem can be solved in linear time by our $O(\Delta^{1.5} n)$-time algorithm or even by Chang and Kuo’s algorithm. Furthermore, we assume $n = |T| = o(\Delta^{poly(\log \Delta)})$ by Theorem 2. Below, we first
partition the vertex set $V$ into five types of subsets defined later, and give a linear time algorithm for computing the value of $\delta$ functions, specified for each type.

We here start with defining such five types of subsets $V_i$ ($i = 1, \ldots, 5$). Let $V_M$ be the set of vertices $v \in V$ such that $T(v)$ is a “maximal” subtree of $T$ with $|T(v)| \leq \Delta^5$; i.e., for the parent $u$ of $v$, $|T(u)| > \Delta^5$. Divide $V_M$ into two sets $V_M^{(1)} := \{ v \in V_M | |T(v)| \geq (\Delta - 19)^4 \}$ and $V_M^{(2)} := \{ v \in V_M | |T(v)| < (\Delta - 19)^4 \}$ (notice that $V_L \subseteq \cup_{v \in V_M} V(T(v))$). Define $\tilde{d}(v) := |C(v) - V_M^{(2)}| (= d'(v) - |C(v) \cap V_M^{(2)}|)$. Let

\[
V_1 := \cup_{v \in V_M} V(T(v)), \\
V_2 := \{ v \in V - V_1 | \tilde{d}(v) \geq 2 \}, \\
V_3 := \{ v \in V - V_1 | \tilde{d}(v) = 1, C(v) \cap (V_M^{(2)} - V_L) = \emptyset \}, \\
V_4 := \{ v \in V - (V_1 \cup V_3) | \tilde{d}(v) = 1, \sum_{w \in C(v) \cap (V_M^{(2)} - V_L)} |T(w)| \leq \Delta(\Delta - 19) \}, \\
V_5 := \{ v \in V - (V_1 \cup V_3) | \tilde{d}(v) = 1, \sum_{w \in C(v) \cap (V_M^{(2)} - V_L)} |T(w)| > \Delta(\Delta - 19) \}.
\]

Notice that $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$, and $V_i \cap V_j = \emptyset$ for each $i, j$ with $i \neq j$ (see Fig. 2). In this classification, we call a vertex in $V_M$ a generalized leaf, because it plays a similar role of a leaf vertex in $O(n^{1.75})$-time algorithm.

Here we describe an outline of the algorithm for computing $\delta((u, v), (\ast, \ast)), v \in V$, named COMPUTE-$\delta(v)$ (Algorithm 4), which can be regarded as a subroutine of the DP framework.

Below, we show that for each $V_i$, $\delta((u, v), (\ast, \ast)), v \in V_i$ can be computed in linear time in total; i.e., $O(\sum_{v \in V_i} t(v)) = O(n)$. Namely, we have the following theorem.

**Theorem 3** For trees, the $L(2, 1)$-labeling problem can be solved in linear time.
Algorithm 4 COMPUTE-δ(v)

1: /** Assume that the head and neck levels of \(T(v)\) are at most \(h\). **/
2: If \(v \in V_1 \cup V_2\), then for each \(b \in (L_0 - L_h) \cup \{h\}\), compute \(\delta((u, v), (b, b))\) by the max-flow computation in the network \(\mathcal{N}(u, v, -, b)\) defined in Sect. 3.2.
3: If \(v \in V_3\), execute the following procedure for each \(b \in L_0\) in the case of \(C(v) \cap V_L = \emptyset\), and for each \(b \in \{0, \Delta + 1\}\) in the case of \(C(v) \cap V_L \neq \emptyset\).
   /** Let \(w^*\) denote the unique child of \(v\) not in \(V_M^{(2)}\). **/
3-1: If \(|\{c | \delta((v, w^*), (b, c)) = 1\}| \geq 2\), then let \(\delta((u, v), (a, b)) := 1\) for all labels \(a \notin \{b - 1, b, b + 1\}\), and let \(\delta((u, v), (a, b)) := 0\) for \(a \in \{b - 1, b, b + 1\}\).
3-2: If \(|\{c | \delta((v, w^*), (b, c)) = 1\}| = 1\), then let \(\delta((u, v), (a, b)) := 0\) for \(a \in \{c^*, b - 1, b, b + 1\}\) and \(\delta((u, v), (a, b)) := 1\) for all the other labels \(a \notin \{c^*, b - 1, b, b + 1\}\).
3-3: If \(|\{c | \delta((v, w^*), (b, c)) = 1\}| = 0\), then let \(\delta((u, v), (b, c)) := 0\).
4: If \(v \in V_4 \cup V_5\), then similarly to the case of \(v \in V_1 \cup V_2\), compute \(\delta((u, v), (\ast, \ast))\) by the max-flow computation in a network such as \(\mathcal{N}(u, v, -, \ast)\) specified for this case (details will be described in Sects. 4.3 and 4.4).

We first show \(O(\sum_{v \in V_1} t(v)) = O(|V_1|)\). For each \(v \in V_M\), \(O(\sum_{w \in V(T(v))} t(w)) = O(|T(v)|)\) holds, by Corollary 1 and \(|T(v)| = O(\Delta^5)\). Hence, we have \(O(\sum_{v \in V_1} t(v)) = O(\sum_{v \in V_M} \sum_{w \in V(T(v))} t(w)) = O(\sum_{v \in V_M} |T(v)|) = O(|V_1|)\).

The proofs for \(V_2, V_3, V_4\) and \(V_5\) are given in the subsequent subsections.

4.2 Computation of \(\delta\)-Value for \(V_2\)

By Lemma 5, it can be seen that \(\sum_{v \in V_2} t(v) = O(\sum_{v \in V_2} \Delta^{2/3} d'(v) h^2 \log^2 \Delta) = O(\Delta^{8/3} \log^2 \Delta \sum_{v \in V_2} d'(v))\) (note that \(h \leq \Delta\) and \(d''(v) \leq d'(v)\)). Since \(d'(v) \leq \Delta\), we have \(\sum_{v \in V_2} t(v) = O(\Delta^{11/3} \log^2 \Delta |V_2|)\). Below, in order to show that \(\sum_{v \in V_2} t(v) = O(n)\), we prove that \(|V_2| = O(n/\Delta^4)\).

By definition, there is no vertex whose all children are vertices in \(V_M^{(2)}\), since if there is such a vertex \(v\), then for each \(w \in C(v)\), we have \(|T(w)| < (\Delta - 19)^4\) and hence \(|T(v)| < \Delta^5\), which contradicts the maximality of \(T(w)\). It follows that in the tree \(T'\) obtained from \(T\) by deleting all vertices in \(V_1 - V_M^{(1)}\), each leaf vertex belongs to \(V_M^{(1)}\) (note that \(V(T') = V_M^{(1)} \cup V_2 \cup V_3 \cup V_4 \cup V_5\)). Hence,

\[
|V(T')| - 1 = |E(T')| = \frac{1}{2} \sum_{v \in V(T')} d_T(v) = \frac{1}{2} \left( |V_M^{(1)}| + \sum_{v \in V_2 \cup V_3 \cup V_4 \cup V_5} (\tilde{d}(v) + 1) - 1 \right) \\
= \frac{1}{2} \left( |V_M^{(1)}| + \sum_{v \in V_2} (\tilde{d}(v) + 1) + 2|V_3| + 2|V_4| + 2|V_5| - 1 \right) \\
\geq \frac{1}{2} |V_M^{(1)}| + \frac{3}{2} |V_2| + |V_3| + |V_4| + |V_5| - \frac{1}{2}
\]
(the last inequality follows from \( \tilde{d}(v) \geq 2 \) for all \( v \in V_2 \)). Thus, \( |V_M^{(1)}| - 1 \geq |V_2| \). It follows by \( |V_M^{(1)}| = O(n/\Delta^4) \) that \( |V_2| = O(n/\Delta^4) \).

Besides, we can observe that \( \sum_{v \in V_2} \tilde{d}(v) = |E(T')| - |V_3| - |V_4| - |V_5| = |V_M^{(1)}| + |V_2| - 1 \leq 2|V_M^{(1)}| - 2 \) (the first equality follows from \( |E(T')| = \sum_{v \in V_2 \cup V_3 \cup V_4 \cup V_5} \tilde{d}(v) = \sum_{v \in V_2} \tilde{d}(v) + |V_3| + |V_4| + |V_5| \) and the second equality follows from \( |E(T')| - |V(T')| - 1 = |V_M^{(1)}| + |V_2| + |V_3| + |V_4| + |V_5| - 1 \). Thus, it holds that \( \sum_{v \in V_2} \tilde{d}(v) = O(n/\Delta^4) \). This fact is used in Sect. 4.5.

4.3 Computation of \( \delta \)-Value for \( V_4 \)

We first claim that \( |V_4| = O(n/\Delta) \). Since each leaf is incident to a major vertex by Property 3, note that we have

\[
|T(w)| \geq \Delta, \quad \text{for each } w \in V_M - V_L. \tag{1}
\]

Now, we have \( C(v) \cap (V_M^{(2)} - V_L) \neq \emptyset \) for \( v \in V_4 \) by definition. This implies that the total number of descendants of a child in \( V_M^{(2)} - V_L \) of each vertex in \( V_4 \) is at least \( \Delta|V_4| \), and hence \( |V_4| = O(n/\Delta) \).

Here, we observe several properties of vertices in \( V_4 \) before describing an algorithm for computing \( \delta \)-values. Let \( v \in V_4 \). By (1) and \( \sum_{w \in C(v) \cap (V_M^{(2)} - V_L)} |T(w)| \leq \Delta(\Delta - 19) \), we have

\[
|C(v) \cap (V_M^{(2)} - V_L)| \leq \Delta - 19. \tag{2}
\]

Intuitively, (2) indicates that the number of labels to be assigned for vertices in \( C(v) \cap (V_M^{(2)} - V_L) \) is relatively small. Now, the head and neck levels of \( T(w) \) are at most 8 for each \( w \in V_M^{(2)} \) (3) by Lemma 2 and \( |T(w)| < (\Delta - 19)^4 \) (note that we assume that \( \Delta \geq 18 \)). Hence, we can observe that there are many possible feasible assignments for \( C(v) \cap (V_M^{(2)} - V_L) \); i.e., we can see that if we can assign labels to some restricted children of \( v \) properly, then there exists a proper assignment for the whole children of \( v \), as observed in the following Lemma 6. For a label \( b \), we divide \( C(v) \cap (V_M^{(2)} - V_L) \) into two subsets \( C_1(b) := \{ w \in C(v) \cap (V_M^{(2)} - V_L) \mid \delta((v, w), (b, c)) = 1 \text{ for all } c \in L_8 - \{b - 1, b, b + 1\} \} \) and \( C_2(b) := \{ w \in C(v) \cap (V_M^{(2)} - V_L) \mid \delta((v, w), (b, c)) = 0 \text{ for all } c \in L_8 - \{b - 1, b, b + 1\} \} \) (notice that by (3), the neck level of \( T(w) \), \( w \in C(v) \cap (V_M^{(2)} - V_L) = C_1(b) \cup C_2(b) \)). Let \( w^* \) be the unique child of \( v \) in \( C(v) - V_M^{(2)} \). By the following property, we have only to consider the assignments for \( \{w^*\} \cup C_2(b) \).

**Lemma 6** For \( v \in V_4 \), let \( a \) and \( b \) be labels with \( |b - a| \geq 2 \) such that \( b \in L_0 \) if \( C(v) \cap V_L = \emptyset \) and \( b \in \{0, \Delta + 1\} \) otherwise. Then, \( \delta((a, v), (a, b)) = 1 \) if and only if there exists an injective assignment \( g : \{w^*\} \cup C_2(b) \to L_0 - \{a, b - 1, b, b + 1\} \) such that \( \delta((v, w), (b, g(w))) = 1 \) for each \( w \in \{w^*\} \cup C_2(b) \).
Proof The only if part is clear. We show the if part. Assume that there exists an injective assignment \( g_1 : \{w^*\} \cup C_2(b) \rightarrow L_0 - \{a, b - 1, b, b + 1\} \) such that \( \delta((v, w), (b, g_1(w))) = 1 \) for each \( w \in \{w^*\} \cup C_2(b) \). Notice that by definition of \( C_2(b) \), all \( w \in C_2(b) \) satisfies \( g_1(w) \in L_0 - L_8 \). Hence, there exists at least \( |L_8 - \{a, b - 1, b, b + 1, g_1(w^*)\}| = \Delta - 19 \) labels which are not assigned by \( g_1 \). By (2), we can assign such remaining labels to all vertices in \( C_1(b) \) injectively; let \( g_2 \) be the resulting labeling on \( C_1(b) \). Notice that for all \( w \in C_1(b) \), we have \( \delta((v, w), (b, g_2(w))) = 1 \) by definition of \( C_1(b) \) and \( g_2(w) \in L_8 \). It follows that the function \( g_3 : C_1(b) \cup C_2(b) \cup \{w^*\} \rightarrow L_0 - \{a, b - 1, b, b + 1\} \) such that \( g_3(w) = g_1(w) \) for all \( w \in C_2(b) \cup \{w^*\} \) and \( g_3(w) = g_2(w) \) for all \( w \in C_1(b) \) is injective and satisfies \( \delta((v, w), (b, g_3(w))) = 1 \) for all \( w \in C_1(b) \cup C_2(b) \). Thus, if \( C(v) \cap V_L = \emptyset \), then we have \( \delta((u, v), (a, b)) = 1 \).

Consider the case where \( C(v) \cap V_L \neq \emptyset \). Let \( b = 0 \) without loss of generality. Then by Property 3, \( v \) is major. Hence, \( |C(v) \cap V_L| = \Delta - 1 - |C_1(0)| - |C_2(0)| - |\{w^*\}| \).

Notice that the number of the remaining labels (i.e., labels not assigned by \( g_3 \)) is \( |L_0 - \{0, 1, a\} - C_1(0) - C_2(0) - \{w^*\}| = \Delta - 2 - |C_1(0)| - |C_2(0)| \). Hence, we can see that by assigning the remaining labels to vertices in \( C(v) \cap V_L \) injectively, we can obtain a proper labeling; \( \delta((u, v), (a, b)) = 1 \) holds also in this case.

Below, we will show how to compute \( \delta((u, v), (\ast, b)) \) in \( O(1) \) time for a fixed \( b \), where \( b \in L_0 \) if \( C(v) \cap V_L = \emptyset \) and \( b \in \{0, \Delta + 1\} \) otherwise. If \( |C_2(b)| \geq 17 \), then \( \delta((u, v), (\ast, b)) = 0 \) because in this case, there exists some \( w \in C_2(b) \) to which no label in \( L_0 - L_8 \) can be assigned since \( |L_0 - L_8| = 16 \). Assume that \( |C_2(b)| \leq 16 \). There are the following three possible cases: (Case-1) \( \delta((v, w^*), (b, c_i)) = 1 \) for at least two labels \( c_1, c_2 \in L_8 \), (Case-2) \( \delta((v, w^*), (b, c_1)) = 1 \) for exactly one label \( c_1 \in L_8 \), and (Case-3) otherwise.

(Case-1) By assumption, for any \( a \), \( \delta((v, w^*), (b, c)) = 1 \) for some \( c \in L_8 - \{a\} \). By Lemma 6, we only have to check whether there exists an injective assignment \( g : C_2(b) \rightarrow L_0 - L_8 - \{a, b - 1, b, b + 1\} \) such that \( \delta((v, w), (b, g(w))) = 1 \) for each \( w \in C_2(b) \). According to Sect. 3.2, this can be done by utilizing the maximum flow computation on the subgraph \( N' \) of \( \tilde{N}(u, v, \ast, b) \) induced by \( \{s, t\} \cup C_2(b) \cup X' \) where \( X' = \{0, 1, \ldots, 7, \Delta - 6, \Delta - 5, \ldots, \Delta + 1\} \). Obviously, the size of \( N' \) is \( O(1) \) and it follows that its time complexity is \( O(1) \).

(Case-2) For all \( a \neq c_1 \), the value of \( \delta((u, v), (a, b)) \) can be computed similarly to Case-1. Consider the case where \( a = c_1 \). In this case, if \( \delta((v, w^*), (b, c)) = 1 \) holds, then it turns out that \( c \in L_0 - L_8 \). Hence, by Lemma 6, it suffices to check whether there exists an injective assignment \( g : \{w^*\} \cup C_2(b) \rightarrow L_0 - L_8 - \{b - 1, b, b + 1\} \) such that \( \delta((v, w), (b, g(w))) = 1 \) for each \( w \in \{w^*\} \cup C_2(b) \). Similarly to Case-1, this can be done in \( O(1) \) time, by utilizing the subgraph \( N'' \) of \( \tilde{N}(u, v, \ast, b) \) induced by \( \{s, t\} \cup (C_2(b) \cup \{w^*\}) \cup X' \).

(Case-3) By assumption, if \( \delta((v, w^*), (b, c)) = 1 \) holds, then it turns out that \( c \in L_0 - L_8 \). Similarly to the case of \( a = c_1 \) in Case-2, by using \( N'' \), we can compute the values of \( \delta((u, v), (\ast, \ast)) \) in \( O(1) \) time.

We analyze the time complexity for computing \( \delta((u, v), (\ast, \ast)) \). It is dominated by that for computing \( C_1(b) \), \( C_2(b) \), and \( \delta((u, v), (\ast, \ast)) \) for each \( b \in L_0 \). By (3), we have \( C_i(b) = C_i(b') \) for all \( b, b' \in L_8 \) and \( i = 1, 2 \). It follows that the computation of
$C_1(b)$ and $C_2(b)$, $b \in L_0$ can be done in $O(|C(v) \cap (V_M^{(2)} - V_L)|)$ time. On the other hand, the values of $\delta((u, v), (\ast, b))$ can be computed in constant time in each case of Cases-1, 2 and 3 for a fixed $b$. Thus, $\delta((u, v), (\ast, \ast))$ can be computed in $O(\Delta)$ time.

4.4 Computation of $\delta$-Value for $V_5$

In the case of $V_4$, since the number of labels to be assigned is relatively small, we can properly assign labels to $C_1(b)$ even after determining labels for $\{w^\ast\} \cup C_2(b)$; we just need to solve the maximum flow of a small network in which vertices corresponding to $C_1(b)$ are omitted. In the case of $V_5$, however, the number of labels to be assigned is not small enough, and actually it is sometimes tight. Thus we need to assign labels carefully, that is, we need to solve the maximum flow problem in which the network contains vertices corresponding to $C_1(b)$. To deal with this, we utilize the inequality $|V_5| \leq \frac{n}{\Delta(\Delta - 19)} = O(n/\Delta^2)$, which is derived from $\sum_{w \in C(v) \cap (V_M^{(2)} - V_L)} |T(w)| > \Delta(\Delta - 19)$.

Below, in order to show that $O(\sum_{v \in V_5} T(v)) = O(n)$, we prove that the values of $\delta((u, v), (\ast, b))$ can be computed in $O(\Delta)$ time for a fixed $b$. A key is that the children $w \in C(v) \cap V_M^{(2)}$ of $v$ can be classified into $2^{17}$ ($= O(1)$) types, depending on its $\delta$-values ($\delta((v, w), (b, i))$) $i \in (L_0 - L_8) \cup \{\tilde{c}_8\}$, where $\tilde{c}_8$ is some label in $L_8 - \{b - 1, b, b + 1\}$, since for each such $w$, the head and neck levels of $T(w)$ are at most 8, as observed in (3) (i.e., $\delta((v, w), (b, c)) = \delta((v, w), (b, \tilde{c}_8))$ for any $c \in L_8 - \{b - 1, b, b + 1\}$). We denote the characteristic vector $(\delta((v, w), (b, i)))$ $i \in (L_0 - L_8) \cup \{\tilde{c}_8\}$ by $(x_w)$. Furthermore, by the following lemma, we can see that $\delta((u, v), (\ast, b))$ can be obtained by checking $\delta((u, v), (a, b))$ for $O(1)$ candidates of $a$, where we let $w^\ast$ be the unique child of $v$ not in $V_M^{(2)}$.

**Lemma 7** Let $v \in V_5$ and $b \in L_0$. If $\delta((u, v), (a_1, b)) \neq \delta((u, v), (a_2, b))$ for some $a_1, a_2 \in L_8 - \{b - 1, b, b + 1\}$ (say, $\delta((u, v), (a_1, b)) = 1$), then we have $\delta((v, w^\ast), (a_2, b)) = 1$ and $\delta((v, w^\ast), (b, a)) = 0$ for all $a \in L_8 - \{a_2, b - 1, b, b + 1\}$, and moreover, $\delta((u, v), (a, b)) = 1$ for all $a \in L_8 - \{a_2, b - 1, b, b + 1\}$.

**Proof** Let $f$ be a labeling on $T(u, v)$ with $f(u) = a_1$ and $f(v) = b$, achieving $\delta((u, v), (a_1, b)) = 1$. By $\delta((u, v), (a_2, b)) = 0$, there exists a child $w_1$ of $v$ with $f(w_1) = a_2$, since otherwise the labeling from $f$ by changing the label for $u$ from $a_1$ to $a_2$ would be feasible.

Assume for contradiction that $w_1 \neq w^\ast$ (i.e., $w_1 \in V_M^{(2)}$). Then the neck level of $T(w_1)$ is at most 8 and we have $\delta((v, w_1), (b, a_1)) = \delta((v, w_1), (b, a_2)) = 1$ by $a_1, a_2 \in L_8$. This indicates that $\delta((u, v), (a_2, b)) = 1$ would hold. Indeed, the function $g: C(v) \to L_0 - \{a_2, b - 1, b, b + 1\}$ such that $g(w_1) = a_1$ and $g(w^\ast) = f(w^\ast)$ for all other children $w'$ of $v$, is injective and satisfies $\delta((v, w), (b, g(w))) = 1$ for all $w \in C(v)$.

Hence, we have $w_1 = w^\ast$. Note that $\delta((v, w^\ast), (b, a_2)) = 1$ since $f$ is feasible. Then, assume for contradiction that some $a_3 \in L_8 - \{a_2, b - 1, b, b + 1\}$ satisfies $\delta((v, w^\ast), (b, a_3)) = 1$ ($a_3 = a_1$ may hold). Suppose that there exists a child $w_2$ of $v$ with $f(w_2) = a_3$ (other cases can be treated similarly). Notice that the neck level of $T(w_2)$ is at most 8 and $\delta((v, w_2), (b, a_1)) = \delta((v, w_2), (b, a_3)) = 1$. Then we can
see that \( \delta((u, v), (a_2, b)) = 1 \) would hold. Indeed, the function \( g : C(v) \to L_0 - \{a_2, b - 1, b, b + 1\} \) such that \( g(w^*) = a_3, g(w_2) = a_1 \) \( g(w') = f(w') \) for all other children \( w' \) of \( v \), is injective and satisfies \( \delta((v, w), (b, g(w))) = 1 \) for all \( w \in C(v) \). Furthermore, similarly to these observations, we can see that \( \delta((u, v), (a, b)) = 1 \) for all \( a \in L_8 - \{a_2, b - 1, b, b + 1\} \).

This lemma implies the following two facts.

**Fact 1** *In the case of \(|\{c \in L_8 - \{b - 1, b, b + 1\} \mid \delta((v, w^*), (b, c)) = 1\}| \neq 1, we have \( \delta((u, v), (a, b)) = \delta((u, v), (a', b)) \) for all \( a, a' \in L_8 - \{b - 1, b, b + 1\} \).*

**Fact 2** *In the case of \(|\{c \in L_8 - \{b - 1, b, b + 1\} \mid \delta((v, w^*), (b, c)) = 1\}| = |\{c^*\}, we have \( \delta((u, v), (a, b)) = \delta((u, v), (a', b')) \) for all \( a, a' \in L_8 - \{b - 1, b, b + 1, c^*\} \) (note that otherwise Lemma 7 implies that there would exist some label \( a'' \in L_8 - \{b - 1, b, b + 1\} \) other than \( c^* \) with \( \delta((v, w^*), (b, a'')) = 1 \)).*

From these facts, we can observe that in order to obtain \( \delta((u, v), (\ast, b)) \), we only have to check \( a \in (L_0 - L_8) \cup \{c_8\} \) in the former case, and \( a \in (L_0 - L_8) \cup \{c^*, c'\} \) where \( c' \in L_8 - \{b - 1, b, b + 1, c^*\} \) in the latter case.

From these observations, it suffices to show that \( \delta((u, v), (a, b)) \) can be computed in \( O(\Delta) \) time for a fixed pair \( a, b \). We will prove this by applying the maximum flow techniques observed in Sect. 3.2 to the network \( N''(u, v, a, b) \) with \( O(1) \) vertices, \( O(1) \) edges, and \( O(\Delta) \) units of capacity, defined as follows:

\[
N''(u, v, a, b) = \{(s, t) \cup U_8 \cup \{w^*\} \cup X_8, E_8 \cup E'_X \cup E''_X, cap\},
\]

where \( U_8 = \{(x_w) \in [0, 1]^{X_8} \mid w \in C(v)\}, X_8 = \{0, 1, \ldots, 7, \Delta - 6, \Delta - 5, \ldots, \Delta + 1\} \cup \{8\}, E_8 = \{s\} \cup (U_8 \cup \{w^*\}), E''_X = (U_8 \cup \{w^*\}) \times X_8, E'_X = X_8 \times \{t\} \). We define \( (x_w) = \delta((v, w), (b, c)) \). For example, for \( w \), vector \( (x_w) = (101\ldots1|0) \) means that \( w \) satisfies \( (x_w) = \delta((v, w), (b, 0)) = 1 \), \( (x_w) = \delta((v, w), (b, 1)) = 0 \), \( (x_w) = \delta((v, w), (b, 2)) = 1 \), and \( (x_w) = \delta((v, w), (b, 8)) = 0 \), where “\(|\)” is put to indicate the position of \( (x_w) \).

Notice that \( U_8 \) is a set of 0-1 vectors with length 17, and its size is bounded by a constant. When we refer to a vertex \( i \in X_8 \), we sometimes use \( c_i \) instead of \( i \) to avoid a confusion. For the edge sets, its \( cap(e) \) function is defined as follows: For \( e = (s, (x)) \in \{s\} \times U_8 \subset E_8, cap(e) = |\{w \mid (x) = (x_w)\}| \). For \((s, w^*) \in E_8, cap(e) = 1 \). If \( a \in X_8 - \{c_8\} \), then for \( e = (a', a) \in E'_X \) with \( a' \in U_8 \cup \{w^*\}, cap(e) = 0 \). For \( e = ((x_w), c) \in U_8 \times (X_8 - \{a, c_8\}) \subset E'_X, cap(e) = (x_w)c \), and for \( e = (w^*, c) \in E'_X \) with \( c \in X_8 - \{a, c_8\}, cap(e) = \delta((v, w^*), (b, c)) \). For \( e = ((x_w), c) \in U_8 \times \{c_8\} \subset E'_X, cap(e) = |L_8 - \{a, b - 1, b, b + 1\}| if (x_w) = 1, and cap(e) = 0 otherwise. For \( (w^*, c_8) \in E'_X, cap(e) = 1 if \delta((v, w^*), (b, c)) = 1 for some \( c \in L_8 - \{a, b - 1, b + 1\} \), and cap(e) = 0 otherwise. For \( e = (c, t) \in (X_8 - \{c_8\}) \times \{t\} \subset E_X \), cap(e) = 1. For \( e = (c_8, t) \in E'_X, cap(e) = |L_8 - \{a, b - 1, b, b + 1\}| \). Figure 3 shows an example of relationship between \( N(u, v, a, b) \) and \( N''(u, v, a, b) \). where edges with capacity 0 are not drawn. In this example, since \( \delta((v, w_1), (b, \ast)) = \delta((v, w_2), (b, \ast)) \), \( w_1 \) and \( w_2 \) in the left figure are treated as a single vertex \( x = (110\ldots10|1) \) in the right figure.
This network is constructed differently from $\mathcal{N}(u,v,a,b)$ in two points. One is that in the new $\mathcal{N}'(u,v,a,b)$, not only label vertices but also $C(v)$ vertices are bundled to $U_8$. For each arc $e=((x_w),c_8) \in E'_8$, $\operatorname{cap}(e)$ is defined by $|L_8 - \{a,b-1,b,b+1\}|$ if $(x_w)_8=1$ and 0 otherwise. This follows from the neck level of $T(w)$ for $w \in C(v)-\{*\}$ is at most 8; i.e., we have $\delta((v,w),(b,c)) = \delta((v,w),(b,c'))$ for all $c,c' \in L_8 - \{b-1,b,b+1\}$. The other is that the arc $(w',c_8)$ is set in a different way from the ones in $\mathcal{N}(u,v,a,b)$. Notice that neck level of $T(w')$ may not be at most 8; $\operatorname{cap}((w',c_8))=1$ does not imply that $\delta((v,w'),(b,c))$ is equal for any $c \in L_8$. Despite the difference of the definition of $\operatorname{cap}$ functions, we can see that $\delta((u,v),(a,b))=1$ if and only if there exists a maximum flow from $s$ to $t$ with size $d'(v)$ in $\mathcal{N}'(u,v,a,b)$. Indeed, even for a maximum flow $\psi$ with size $d'(v)$ such that $\psi(w^*,c_8)=1$ (say, $\delta((v,w^*),(b,e^*))=1$ for $c^* \in L_8 - \{a,b-1,b,b+1\}$), there exists an injective assignment $g: C(v) \to L_0 - \{a,b-1,b,b+1\}$ such that $\delta((v,w),(b,g(w)))=1$ for each $w \in C(v)$, since we can assign injectively the remaining labels in $L_8$ (i.e., $L_8 - \{a,b-1,b,b+1,c^*\}$) to all vertices corresponding to $x_w$ with $\psi((x_w),c_8)>0$.

To construct $\mathcal{N}'(u,v,a,b)$, it takes $O(d'(v))$ time. Since $\mathcal{N}'(u,v,a,b)$ has $O(1)$ vertices, $O(1)$ edges and at most $\Delta$ units of capacity, the maximum flow itself can be solved in $O(\log \Delta)$ time. Thus, the values of $\delta((u,v),(a,b))$ can be computed in $O(d'(v) + \log \Delta) = O(\Delta)$ time.
4.5 Computation of $\delta$-Value for $V_3$

First, we show the correctness of the procedure in the case of $v \in V_3$ in algorithm Compute-$\delta(v)$. Let $v \in V_3$, $u$ be the parent of $v$, $w^*$ be the unique child of $v$ not in $V_M^{(2)}$, and $b$ be a label such that $b \in L_0$ if $v \in V_3^{(1)} := \{ v \in V_3 \mid C(v) \cap V_L = \emptyset \}$, and $b \in \{0, \Delta + 1\}$ if $v \in V_3^{(2)} := V_3 - V_3^{(1)}$. Notice that if $v \in V_3^{(2)}$ (i.e., $C(v) \cap V_L \neq \emptyset$), then by Property 3, $v$ is major and hence $\delta((u, v), (a, b)) = 1$, $a \in L_0$ indicates that $b = 0$ or $b = \Delta + 1$; namely we have only to check the case of $b \in \{0, \Delta + 1\}$. Then, if there is a label $c \in L_0 - \{b - 1, b, b + 1\}$ such that $\delta((v, w^*), (b, c)) = 1$, then for all $a \in L_0 - \{b - 1, b, b + 1\}$, we have $\delta((u, v), (a, b)) = 1$. It is not difficult to see that this observation shows the correctness of the procedure in this case.

Next, we analyze the time complexity of the procedure. Obviously, for each $v \in V_3$, we can check which case of 3-1, 3-2, and 3-3 in algorithm Compute-$\delta(v)$ holds, and determine the values of $\delta((u, v), (\ast, b))$, in $O(1)$ time. Therefore, the values of $\delta((u, v), (\ast, \ast))$ can be determined in $O(\Delta)$ time. Below, in order to show that $\sum_{v \in V_3} t(v) = O(n)$, we prove that $|V_3| = O(n/\Delta)$.

As observed above, each vertex in $v \in V_3^{(2)}$ is major and we have $d(v) = \Delta$. Thus, it holds that $|V_3^{(2)}| = O(n/\Delta)$.

Finally, we show that $|V_3^{(1)}| = O(n/\Delta)$ also holds. By definition, we can observe that for any $v \in V_3^{(1)}$, $d'(v) = \tilde{d}(v) = 1$ (i.e., $d(v) = 2$). By Property 4, the size of any path component of $T$ is at most 3. This means that at least $|V_3^{(1)}|/3$ vertices in $V_3^{(1)}$ are children of vertices in $V_2 \cup V_3^{(2)} \cup V_4 \cup V_5$. Thus, $|V_3^{(1)}|/3 \leq \sum_{v \in V_2 \cup V_3^{(2)} \cup V_4 \cup V_5} \tilde{d}(v)$.

From the discussions in the previous subsections (and this subsection), $\sum_{v \in V_2} \tilde{d}(v) = O(n/\Delta^2)$, $\sum_{v \in V_3^{(2)}} \tilde{d}(v) = \sum_{v \in V_3^{(2)}} 1 = |V_3^{(2)}| = O(n/\Delta)$, $\sum_{v \in V_4} \tilde{d}(v) = \sum_{v \in V_4} 1 = |V_4| = O(n/\Delta)$, and $\sum_{v \in V_5} \tilde{d}(v) = \sum_{v \in V_5} 1 = |V_5| = O(n/\Delta^2)$. Therefore, $|V_3^{(1)}| = O(n/\Delta)$.

5 Extension for $L(p, 1)$-Labeling of Trees

In the previous sections, we presented an algorithm that determines whether a given tree $T$ has $(\Delta + 1)$-$L(2, 1)$-labeling or not and showed its linear running time. In this section, we extend the result to an algorithm that determines, for a given tree $T$ and $\lambda$, whether $T$ has $\lambda$-$L(p, 1)$-labeling or not. For this purpose, we generalize several ideas that contributed to the efficient running time of $L(2, 1)$-labeling algorithms, especially including the notion of label compatibility, to be applicable for $L(p, 1)$-labeling of trees.

As seen in the previous sections, the linear time $L(2, 1)$-labeling algorithm are realized by several ideas and their combination. They are summarized as follows: (1) the equality $\sum_{v \in V_5} d''(v) = O(n/\Delta)$ by the preprocessing computation, (2) label compatibility and the flow-based computation of $\delta$, which enables a linear time computation for $L(2, 1)$-labeling of small $T$, and (3) the notion of generalized leaf and classification by the size of subtrees and the number of generalized leaves, which is
also supported by (1) and (2). Namely, if these ideas (especially, (1) and (2)) can be arranged into \(\lambda\)-\(L(p, 1)\)-labeling, the linear time \(L(2, 1)\)-labeling algorithm could be extended to a (linear time) algorithm for \(\lambda\)-\(L(p, 1)\)-labeling.

In the succeeding subsections, we present two properties of \(L(p, 1)\)-labelings corresponding to (1) and (2). Since it is known that \(\Delta + p - 1 \leq \lambda_{p,1}(T) \leq \min\{\Delta + 2p - 2, 2\Delta + p - 2\}\), we assume \(\lambda \in [\Delta + p - 1, \min\{\Delta + 2p - 2, 2\Delta + p - 2\}]\). The extended algorithm is based on the same scheme as the linear time \(L(2, 1)\)-labeling algorithm, that is, it is a dynamic programming algorithm that computes \(\delta\)-values from leaves to root. In the following, we assume \(p\) is a constant, unless otherwise stated.

5.1 Generalized Major Vertex

The extended algorithm also does a similar preprocessing operation to Algorithm 3. In the \(L(2, 1)\)-labeling algorithms, Algorithm 3 is applied to an input tree so that it satisfies Properties 3 and 4. Property 4 is not affected by the change from \(L(2, 1)\) to \(L(p, 1)\), whereas Property 3 is so. The reason why we can assume Property 3 for an \(L(2, 1)\)-labeling is that \(T\) has \((\Delta + 1)\)-\(L(2, 1)\)-labeling if and only if \(T - \{v\}\) has \((\Delta + 1)\)-\(L(2, 1)\)-labeling, where a leaf vertex \(v\) of \(T\) is adjacent to a non-major vertex. In order to fix this property for \(\lambda\)-\(L(p, 1)\)-labeling, we generalize the notion of major vertex: A vertex whose degree is \(\lambda - p - i + 1\) (resp., at least \(\lambda - p - i + 1\)) is called \(i\)-major (resp., \(i^\ge\)-major) with respect to \(\lambda\). For example, for \(\lambda = \Delta + p - 1\), a vertex whose degree is \(\Delta - 1\) is not 0-major but 1-major, \(1^\ge\)-major, \(2^\ge\)-major, and so on, with respect to \(\Delta + p - 1\). We often omit “with respect to \(\lambda\)”, if no confusion arises.

We can show that \(T\) has \(\lambda\)-\(L(p, 1)\)-labeling if and only if \(T - \{v\}\) has \(\lambda\)-\(L(p, 1)\)-labeling where leaf \(v\) is adjacent to a vertex \(u\) whose degree is at most \(\lambda - 2p + 2\). In fact, we can construct \(\lambda\)-\(L(p, 1)\)-labeling of \(T\) from \(\lambda\)-\(L(p, 1)\)-labeling of \(T - \{v\}\), because in the labeling of \(T - \{v\}\), \(u\) freezes at most \(2p - 1\) labels and adjacent vertices of \(u\) freeze at most \(\lambda - 2p + 1\); \((\lambda + 1) - (2p - 1) - (\lambda - 2p + 1) = 1\) label is always available and we can use it for labeling \(v\) in \(T\). Thus we obtain the corresponding version of Property 3:

**Property 5** All vertices adjacent to a leaf vertex are \((p - 2)^\ge\)-major vertices with respect to \(\lambda\).

Consequently, we modify Step 1 of Algorithm 3 (i.e., replace “less than \(\Delta\)” with “less than \(\lambda - 2p + 3\)”)) so that the resulting \(T\) satisfies Property 5 instead of Property 3.

We can see that this change in Algorithm 3 and the generalized notion of \((p - 2)^\ge\)-major vertex absorb several gaps caused by the difference between \(L(2, 1)\) and \(L(p, 1)\). In the analyses of \(O(n^{1.75})\)-time algorithm for \(L(2, 1)\)-labeling in Sect. 2.2, for example, Lemma 1 holds. By a similar argument, the corresponding term in the analysis for \(L(p, 1)\)-labeling is estimated as \(\sum_{v \in V - V_L - V_Q} d''(v) = O(n/\lambda - 2p + 3)) = O(n/(\Delta - p + 2)) = O(n/\Delta),\) which implies that an \(O(\Delta^{1.5}n)\)-time algorithm for \(\lambda\)-\(L(p, 1)\)-labeling of trees is obtained only by this minor modification.
5.2 Generalized Label Compatibility and Its Application

As mentioned several times, the most important idea in the analyses of the linear time algorithm for \(L(2,1)\)-labeling of trees is the label compatibility. In this subsection, we generalize it to \(\lambda-L(p,1)\)-labeling of trees. In the following, we use \(\delta_\lambda((*,*),(*,*))\) to denote the DP-tables in the \(\lambda-L(p,1)\)-labeling algorithm.

We say that \(T\) is \(\lambda\)-head-\(L_h\)-compatible if \(\delta_\lambda((u,v),(a,b)) = \delta_\lambda((u,v),(a',b))\) for all \(a,a' \in L_h\) and \(b \in L_0\) with \(|a-b| \geq p\) and \(|a'-b| \geq p\). Analogously, we say that \(T\) is \(\lambda\)-neck-\(L_h\)-compatible if \(\delta_\lambda((u,v),(a,b)) = \delta_\lambda((u,v),(a,b'))\) for all \(a \in L_0\) and \(b,b' \in L_h\) with \(|a-b| \geq p\) and \(|a-b'| \geq p\). We often omit “\(\lambda\)’ if it is clear. The neck and head levels of \(T\) with respect to \(\lambda\) are defined similarly to Definition 1:

**Definition 2** Let \(T\) be a tree rooted at \(v\), and consider the tree obtained from \(T\) by adding a new vertex \(u\) adjacent to \(v\).

(i) The neck level (resp., head level) of \(T\) with respect to \(\lambda\) is 0 if \(T\) is \(\lambda\)-neck-\(L_0\)-compatible (resp., \(\lambda\)-head-\(L_0\)-compatible).

(ii) The neck level (resp., head level) of \(T\) with respect to \(\lambda\) is \(h(\geq 1)\) if \(T\) is not \(\lambda\)-neck-\(L_{h-1}\)-compatible (resp., \(\lambda\)-head-\(L_{h-1}\)-compatible) but \(\lambda\)-neck-\(L_h\)-compatible (resp., \(\lambda\)-head-\(L_h\)-compatible).

For \(L(2,1)\)-labeling, Lemma 2 gives a relationship between the neck/head levels and the tree size. This lemma is generalized as follows:

**Lemma 8** (Level Lemma (Generalized Form)) Let \(T'\) be a subtree of \(T\). (i) If \(|T'| < (\lambda - 2h - 4p + 4)^{h/(2p-2)}\), \(\lambda - 2h \geq 3p - 3\), and \(p > 1\), then the head level and neck level of \(T'\) are both at most \(h\). (ii) If \(p = 1\), then the head level and neck level of \(T'\) are 0.

Note that \(p\) is a general positive integer in Lemma 8. By letting \(\lambda = \Delta + 1\) and \(p = 2\) in Lemma 8, we immediately obtain Lemma 2. Similarly to Theorem 1, we obtain the following theorem:

**Theorem 4** For a tree \(T\) and a positive integer \(p\), both the head and neck levels of \(T\) with respect to \(\lambda\) are \(O(\min\{\Delta, p \log |T|/\log \lambda\})\).

**Corollary 2** If \(p\) is bounded by a constant and \(\lambda \in [\Delta + p - 1, \Delta + 2p - 2]\), the head and neck levels of \(T\) with respect to \(\lambda\) are \(O(\min\{\Delta, \log |T|/\log \Delta\})\).

The proofs of Lemma 8 and Theorem 4 are shown in Sect. 6.

Similarly to Sect. 3.2, this theorem enables to speed up flow-based computation of \(\delta\). The running time of calculating \(\delta_\lambda((u,v),(*,*))\) for a pair \((a,b)\) is

\[
O\left((h(v) + d''(v))^{2/3} (h(v)d''(v)) \log(h(v) + d''(v)) \log \Delta\right)
\]

\[= O\left(\Delta^{2/3} (h(v)d''(v)) \log^2 \Delta\right).\]
where $h$ and $d''$ are defined similarly as in Sect. 3.2. We thus have the following lemma.

**Lemma 9** $\delta_{h}(u, v, (**))$ can be computed in $O(\Delta^{2/3}(h(v))^{2}d''(v)\log^{2}\Delta)$ time for a positive integer $p$.

Since we assume $p$ is bounded by a constant, we have $\sum_{v\in V-V_{L}-V_{Q}}d''(v) = O(n/\Delta)$ (as we saw in the previous subsection) and $h(v) = O(\min(\Delta, \log |T_{i}/\log \Delta))$ (Corollary 2). By these, the following theorem is obtained:

**Theorem 5** Given a tree $T$ and positive integers $\lambda$ and $p$, it can be determined in $O(\min(\Delta^{2/3}p^{2}n\log^{2}n, \Delta^{2.5}n))$ time if $T$ has $\lambda$-L($p$, 1)-labeling. If $p$ is a positive constant integer, it can be done in $O(\min(n\log^{2}n, \Delta^{1.5}n))$ time. Furthermore, if $n = O(\Delta^{\text{poly}(\log \Delta)})$, it can be determined in $O(n)$ time.

As seen in these subsections, the crucial properties in the analyses for that $L(2, 1)$-labeling of trees is solvable in linear time are generalized to $\lambda$-L($p$, 1)-labeling algorithm for a constant $p$, though we omit the details to avoid tedious repetitions. Since the $L(p, 1)$-labeling number of a tree $T$ is in $[\Delta + p - 1, \min(\Delta + 2p - 2, 2\Delta + p - 2)]$, an optimal $L(p, 1)$-labeling of $T$ can be obtained by applying the $\lambda$-L($p$, 1)-labeling algorithm for $T$ at most $p$ times; the total running time for solving $L(p, 1)$-labeling problem for $T$ is $O(n)$ if $p$ is a constant.

**Theorem 6** For trees, the $L(p, 1)$-labeling problem can be solved in linear time, if $p$ is bounded by a positive constant.

### 6 Proof of Level Lemma

In this section, we provide proofs of Lemma 8 and Theorem 4. For a tree $T'$ rooted at $v$, denote by $T' + (u, v)$ the tree obtained from $T'$ by adding a vertex $u \notin V(T')$ and an edge $(u, v)$. This is similar to $T(u, v)$ defined in Sect. 2.2, however, for $T(u, v)$, $u$ is regarded as a virtual vertex, while for $T' + (u, v)$, $u$ may be an existing vertex.

**Proof of Lemma 8** (i) Let $p > 1$, $\delta((u, v), (**)) := \delta_{h}((u, v), (**))$, and $h > 0$ (note that the case of $h = 0$ means $|T'| = 0$). When $h \leq 2p - 3$, we have $|T'| \leq \lambda - 4p + 1$ and hence $\Delta(T' + (u, v)) \leq \lambda - 4p + 1$, where $v$ denotes the root of $T'$. It follows that $T' + (u, v)$ can be labeled by using at most $\Delta(T' + (u, v)) + 2p - 1 \leq \lambda - 2p$ labels. Thus, in these cases, it is not difficult to see that the head and neck levels are both 0.

Now, we assume for contradiction that this lemma does not hold. Let $T_{1}$ be such a counterexample with the minimum size, i.e., $T_{1}$ satisfies the following properties (4)–(7):

$$|T_{1}| < (\lambda - 2h - 4p + 4)^{h/(2p-2)},$$

(4)
the neck or head level of $T_1$ is at least $h + 1,$ \hspace{1cm} (5)

$h \geq 2p - 2$ (from the arguments of the previous paragraph), \hspace{1cm} (6)

and

for each tree $T'$ with $|T'| < |T_1|$, the lemma holds. \hspace{1cm} (7)

By (5), there are two possible cases (Case-I) the head level of $T_1$ is at least $h + 1$ and (Case-II) the neck level of $T_1$ is at least $h + 1$. Let $v_1$ denote the root of $T_1$.

(Case-I) In this case, in $T_1 + (u, v_1)$ with $u \not\in V(T_1)$, for some label $b$, there exist two labels $a, a' \in L_h$ with $|b - a| \geq p$ and $|b - a'| \geq p$ such that $\delta((u, v_1), (a, b)) = 1$ and $\delta((u, v_1), (a', b)) = 0$. Let $f$ be a $\lambda$-$L(p, 1)$-labeling with $f(u) = a$ and $f(v_1) = b$. If any child of $v_1$ does not have label $a'$ in the labeling $f$, then the labeling obtained from $f$ by changing the label for $u$ from $a$ to $a'$ is also feasible, which contradicts $\delta((u, v_1), (a', b)) = 0$.

Consider the case where some child $w$ of $v_1$ satisfies $f(w) = a'$. Then by $|T(w)| < |T_1|$ and (7), the neck level of $T(w)$ is at most $h$. Hence, we have $\delta((v_1, w), (b, a')) = \delta((v_1, w), (b, a)) = 1$ by $a, a' \in L_h$. Let $f_1$ be a $\lambda$-$L(p, 1)$-labeling on $T(w) + (v_1, w)$ achieving $\delta((v_1, w), (b, a)) = 1$. Now, note that any vertex $v \in C(v_1) - \{w\}$ satisfies $f(v) \notin \{a, a'\}$, since $f$ is feasible. Thus, we can observe that the labeling $f_2$ satisfying the following (i)–(iii) is a $\lambda$-$L(p, 1)$-labeling on $T_1 + (u, v_1)$: (i) $f_2(u) := a'$, (ii) $f_2(v) := f_1(v)$ for all vertices $v \in V(T(w))$, and (iii) $f_2(v) := f(v)$ for all other vertices. This also contradicts $\delta((u, v_1), (a', b)) = 0$.

(Case-II) By the above arguments, we can assume that the head level of $T_1$ is at most $h$. In $T_1 + (u, v_1)$ with $u \not\in V(T_1)$, for some label $a$, there exist two labels $b, b' \in L_h$ with $|b - a| \geq p$ and $|b' - a| \geq p$ such that $\delta((u, v_1), (a, b)) = 1$ and $\delta((u, v_1), (a, b')) = 0$. Similarly to Case-I, we will derive a contradiction by showing that $\delta((u, v_1), (a, b')) = 1$. Now there are the following three cases: (II-1) there exists such a pair $b, b'$ with $b' = b - 1$, (II-2) the case (II-1) does not hold and there exists such a pair $b, b'$ with $b' = b + 1$, and (II-3) otherwise.

First we show that we have only to consider the case of (II-1); namely, we can see that if

(II-1) does not occur for any $a, b$ with $|b - a| \geq p$, $|b - 1 - a| \geq p$,

and $\{b - 1, b\} \subseteq L_h$, \hspace{1cm} (8)

then neither (II-2) nor (II-3) occurs. Assume that (8) holds. Consider the case (II-2). Then, since we have $\delta((u, v_1), (\lambda - a, \lambda - b)) = \delta((u, v_1), (a, b)) = 1$ and $\delta((u, v_1), (\lambda - a, \lambda - b - 1)) = \delta((u, v_1), (a, b + 1)) = 0$, it contradicts (8). Consider the case (II-3), that is, there is no pair $b, b'$ such that $|b - b'| = 1$. Namely, in this case, for some $a \in L_{h+p}$, we have $\delta((u, v_1), (a, h)) \neq \delta((u, v_1), (a, \lambda - h))$ and $\delta((u, v_1), (a, b_i)) = \delta((u, v_1), (a, b_2))$ only if (i) $b_i \leq a - p$, $i = 1, 2$ or (ii) $b_i \geq a + p$, $i = 1, 2$. Then, since the head level of $T_1$ is at most $h$, it follows by $\delta((u, v_1), (a, h)) = \delta((u, v_1), (\lambda - a, h))$. By $\delta((u, v_1), (\lambda - a, h)) = \delta((u, v_1), (a, \lambda - h))$, we have $\delta((u, v_1), (a, h)) = \delta((u, v_1), (a, \lambda - h))$, a contradiction.
Below, in order to show (8), we consider the case of \( b' = b - 1 \). Let \( f \) be a \( \lambda - L(p, 1) \)-labeling with \( f(u) = a \) and \( f(v_1) = b \). We first start with the labeling \( f \), and change the label for \( v_1 \) from \( b \) to \( b - 1 \). Let \( f_1 \) denote the resulting labeling. If \( f_1 \) is feasible, then it contradicts \( \delta((u, v_1), (a, b - 1)) = 0 \). Here, we assume that \( f_1 \) is infeasible, and will show how to construct another \( \lambda - L(p, 1) \)-labeling by changing the assignments for vertices in \( V(T_1) - \{v_1\} \). Notice that since \( f_1 \) is infeasible, there are

some child \( w \) of \( v_1 \) with \( f_1(w) = b - p \), \( \text{(9)} \)
or

some grandchild \( x \) of \( v_1 \) with \( f_1(x) = b - 1 \). \( \text{(10)} \)

Now we have the following claim.

**Claim 1** Let \( f' \) be a \( \lambda - L(p, 1) \)-labeling on \( T_1 \) and \( T(v) \) be a subtree of \( T_1 \). There are at most \( \lambda - 2h - 4p + 3 \) children \( w \) of \( v \) with \( f'(w) \in L_h \) and \( |T(w)| \geq (\lambda - 2h)^{(h-2p+2)/(2p-2)} \).

*Proof* Let \( C'(v) \) be the set of children \( w \) of \( v \) with \( f'(w) \in L_h \) and \( |T(w)| \geq (\lambda - 2h)^{(h-2p+2)/(2p-2)} \). If this claim does not hold, then we would have \( |T_1| \geq |T(v)| \geq 1 + \sum_{w \in C'(v)} |T(w)| > (\lambda - 2h)^{(h-2p+2)/(2p-2)} > 1 + (\lambda - 2h - 4p + 4)^{(h-2p+2)/(2p-2)} > |T_1|, \) a contradiction. \( \square \)

This claim indicates that given a feasible labeling \( f' \) of \( T_1 \), for each vertex \( v \in V(T_1) \), there exist at least \( 2p - 2 \) labels \( \ell_1, \ell_2, \ldots, \ell_{2p-2} \in L_h \) such that \( \ell_i \) is not assigned to any vertex in \( \{v, p(v)\} \cup C(v) \) (i.e., \( \ell_i \notin \{f'(v') \mid v' \in \{v, p(v)\} \cup C(v)\}\)) or assigned to a child \( c_1 \in C(v) \) with \( |T(c)| < (\lambda - 2h)^{(h-2p+2)/(2p-2)} \), since \( |L_h - \{f'(p(v))\}, f'(v) - p + 1, f'(v) - p, \ldots, f'(v) + p - 1| \geq \lambda - 2h - 2p + 1 \), where \( p(v) \) denotes the parent of \( v \). For each vertex \( v \in V(T_1) \), denote such labels by \( \ell_i(v; f') \) and such children by \( c_i(v; f') \) (if exists) for \( i = 1, 2, \ldots, 2p - 2 \). We note that by (7), if \( c_i(v; f') \) exists, then the head and neck levels of \( T(c_i(v; f')) \) are at most \( h - 2p + 2 \).

First consider the case where the vertex of (9) exists; denote such a vertex by \( w_1 \). We consider this case by dividing into two cases (II-1-1) \( b \geq h + p \) and (II-1-2) \( b \leq h + p - 1 \).

(II-1-1) Suppose that we have \( \ell_1(v_1; f) \neq b - p \), and \( c_1(v_1; f) \) exists (other cases can be treated similarly). By (7), the head and neck levels of \( T(w) \) are at most \( h \) for each \( w \in C(v_1) \), and especially, the head and neck levels of \( T(c_1(v_1; f)) \) are at most \( h - 2p + 2 \). Hence, we have

\[
\delta((v_1, w_1), (b, b - p)) = \delta((v_1, w_1), (b, \ell_1(v_1; f))) = \delta((v_1, w_1), (b - 1, \ell_1(v_1; f))), \quad \text{(11)}
\]

\[
\delta((v_1, c_1(v_1; f)), (b, \ell_1(v_1; f))) = \delta((v_1, c_1(v_1; f)), (b - 1, \ell_1(v_1; f))) = \delta((v_1, c_1(v_1; f)), (b - 1, b + p - 1)),
\]

\[
\delta((v_1, c_1(v_1; f)), (b, \ell_1(v_1; f))) = \delta((v_1, c_1(v_1; f)), (b - 1, \ell_1(v_1; f))) = \delta((v_1, c_1(v_1; f)), (b - 1, b + p - 1)),
\]
since \( \ell_1(v_1; f) \not\in \{b - p, b - p + 1, \ldots, b + p - 1 \} \) and \( \{b - p, b - 1, b, \ell_1(v_1; f)\} \subseteq L_h \) (note that in the case where \( c_1(v_1; f) \) does not exist, (12) is not necessary). Notice that \( b + p - 1 \not\in L_h \) may hold, however we have \( b + p - 1 \in L_{h-p+1} \) by \( b \in L_h \). By these observations, there exist labelings \( f'_1 \) and \( f'_2 \) of \( T(w_1) + (v_1, w_1) \) and \( T(c_1(v_1; f)) + (v_1, c_1(v_1; f)) \), achieving \( \delta((v_1, w_1), (b - 1, \ell_1(v_1; f))) = 1 \) and \( \delta((v_1, c_1(v_1; f)), (b - 1, b - p - 1)) = 1 \), respectively. Let \( f^* \) be the labeling such that \( f^*(v) = b - 1 \), \( f^*(v) = f'_1(v) \) for all \( v \in V(T(w_1)) \), \( f^*(v) = f'_2(v) \) for all \( v \in V(T(c_1(v_1; f))) \), and \( f^*(v) = f(v) \) for all other vertices.

(II-1-2) In this case, we have \( b + 2p - 2 \in L_h \) by \( \lambda - 2h \geq 3p - 3 \). Now we have the following claim.

**Claim 2** For \( T(w_1) + (v_1, w_1) \), we have \( \delta((v_1, w_1), (b - 1, b + p - 1)) = 1 \).

**Proof** Let \( f_1 \) be the labeling such that \( f_1(v_1) := b - 1 \), \( f_1(w_1) := b + p - 1 \), and \( f_1(v) := f(v) \) for all other vertices \( v \). Assume that \( f_1 \) is infeasible to \( T(w_1) + (v_1, w_1) \) since otherwise the claim is proved. Hence, (A) there exists some child \( x \) of \( w_1 \) with \( f_1(x) \in \{b + 1, b + 2, \ldots, b + 2p - 2\} \) or (B) some grandchild \( y \) of \( w_1 \) with \( f_1(y) = b + p - 1 \) (note that any child \( x' \) of \( w_1 \) has neither label \( b - 1 \) nor \( b \) by \( f(w_1) = b - p \) and \( f(v_1) = b \)).

First, we consider the case where vertices of (A) exist. Suppose that there are \( 2p - 2 \) children \( x_1, \ldots, x_{2p-2} \in C(w_1) \) with \( f_1(x_i) = b + i \), we have \( \{\ell_i(w_1; f) | i = 1, 2, \ldots, 2p - 2\} \cap \{f(x_i) | i = 1, 2, \ldots, 2p - 2\} = \emptyset \), and all of \( c_i(w_1; f) \) exist (other cases can be treated similarly).

Now by (7), the head and neck levels of \( T(x_i) \) (resp., \( T(c_i(w_1; f)) \)) is at most \( h \) (resp., \( h - 2p + 2 \)) for \( i = 1, 2, \ldots, 2p - 2 \). Hence, for each \( i = 1, 2, \ldots, 2p - 2 \), we have

\[
\delta((w_1, x_i), (b - p, b + i)) = \delta((w_1, x_i), (b - p, \ell_i(w_1; f)))
\]

\[
\delta((w_1, c_i(w_1; f)), (b - p, \ell_i(w_1; f)))
\]

\[
= \delta((w_1, c_i(w_1; f)), (\lambda - h + p, \ell_i(w_1; f)))
\]

\[
= \delta((w_1, c_i(w_1; f)), (\lambda - h + p, b - 2p + i))
\]

\[
= \delta((w_1, c_i(w_1; f)), (b + p - 1, b - 2p + i)).
\]

(13) and (14) hold. Hence, for each \( i = 1, 2, \ldots, 2p - 2 \), we have

\[
\delta((w_1, x_i), (b - p, b + i)) = \delta((w_1, x_i), (b - p, \ell_i(w_1; f)))
\]

\[
= \delta((w_1, c_i(w_1; f)), (b - p, \ell_i(w_1; f)))
\]

\[
= \delta((w_1, c_i(w_1; f)), (\lambda - h + p, \ell_i(w_1; f)))
\]

\[
= \delta((w_1, c_i(w_1; f)), (\lambda - h + p, b - 2p + i))
\]

\[
= \delta((w_1, c_i(w_1; f)), (b + p - 1, b - 2p + i)).
\]

By (13) and (14), there exist \( \lambda-L(p, 1) \)-labelings \( f''_i \) and \( f''_i \), \( i = 1, 2, \ldots, 2p - 2 \), of \( T(x_i) + (v_1, x_i) \), \( T(c_i(w_1; f)) + (w_1, c_i(w_1; f)) \), achieving \( \delta((w_1, x_i), (b - p, \ell_i(w_1; f))) = 1 \) and \( \delta((w_1, c_i(w_1; f)), (b + p - 1, b - 2p + i)) = 1 \), respectively. Let \( f_2 \) be the labeling of \( T(w_1) + (v_1, w_1) \) such that \( f_2(v_1) = b - 1 \), \( f_2(w_1) = b + p - 1 \), \( f_2(v) = f'_1(v) \) for all \( v \in V(T(x_i)) \), \( f_2(v) = f''_1(v) \) for all \( v \in V(T(c_i(w_1; f))) \), and \( f_2(v) = f(v) \) for all other vertices. Observe that we have
\( f_2(x_i) = \ell_i(w_1; f) \), \( f_2(c_1(w_1; f)) = b - 2p + i \), and \( f_2(x) \notin \{b - 1, b, \ldots, b + 2p - 2\} \) for all \( x \in C(w_1) \), every two labels in \( C(w_1) \) are pairwise disjoint, and \( f_2 \) is a \( \lambda \)-\( L(p, 1) \)-labeling of each subtree \( T(x) \) with \( x \in C(w_1) \).

Assume that \( f_2 \) is still infeasible. Then, there exists some grandchild \( y \) of \( w_1 \) with \( f_2(y) = b + p - 1 \). Observe that from \( f(w_1) = b - p \), no sibling of such a grandchild \( y \) has label \( b \) in the labeling \( f_2 \), while such \( y \) may exist in the subtree \( T(x) \) with \( x \in C(w_1) - \{c_i(w_1; f) \mid i = 1, 2, \ldots, 2p - 2\} \). Also note that for the parent \( x_p = p(y) \) of such \( y \), we have \( f(x_p) \notin \{b - 2p + 1, \ldots, b - 1\} \). Suppose that \( \ell_1(x_p; f_2) \neq b + p - 1 \) holds and \( c_1(x_p; f_2) \) exists (other cases can be treated similarly). Now by (7), the neck level of \( T(y) \) (resp., \( T(c_1(x_p; f_2)) \)) is at most \( h \) (resp., \( h - 2p + 2 \)). Hence, we have \( \delta((x_p, y), (f_2(x_p), b + p - 1)) = \delta((x_p, y), (f_2(x_p), \ell_1(x_p; f_2))) \) and \( \delta((x_p, c_1(x_p; f_2)), (f_2(x_p), \ell_1(x_p; f_2))) = \delta((x_p, c_1(x_p; f_2)), (f_2(x_p), b - p)) \).

It follows that there exist \( \lambda \)-\( L(p, 1) \)-labelings \( f'' \) and \( f'' \) on \( T(y) + (x_p, y) \) and \( T(c_1(x_p; f_2)) + (x_p, c_1(x_p; f_2)) \) which achieve \( \delta((x_p, y), (f_2(x_p), \ell_1(x_p; f_2))) = 1 \) and \( \delta((x_p, c_1(x_p; f_2)), (f_2(x_p), b - p)) = 1 \), respectively. It is not difficult to see that the labeling \( f'' \) such that \( f''(v) = f''(v) \) for all \( v \in V(T(y)) \), \( f''(v) = f''(v) \) for all \( v \in V(T(c_1(x_p; f_2))) \), and \( f''(v) = f_2(v) \) for all other vertices is a \( \lambda \)-\( L(p, 1) \)-labeling of \( T(x_p) + (w_1, x_p) \).

Thus, by repeating these observations for each grandchild \( y \) of \( w_1 \) with \( f_2(y) = b + p - 1 \), we can obtain a \( \lambda \)-\( L(p, 1) \)-labeling \( f_3 \) of \( T(w_1) + (v_1, w_1) \) with \( f_3(v_1) = b - 1 \) and \( f_3(w_1) = b + p - 1 \). □

Let \( f^* \) be the labeling such that \( f^*(v) = f_2(v) \) for all \( v \in \{v_1\} \cup V(T(w_1)) \) and \( f^*(v) = f(v) \) for all other vertices. Thus, in both cases (II-1-1) and (II-1-2), we have constructed a labeling \( f^* \) such that \( f^*(u) = a, f^*(v_1) = b - 1, \) and \( f^*(w) \notin \{a, b - p, b - p + 1, \ldots, b + p - 2\} \) for all \( w \in C(v_1) \), every two labels in \( C(v_1) \) are pairwise disjoint, and \( f^* \) is a \( \lambda \)-\( L(p, 1) \)-labeling on each subtree \( T(w) \) with \( w \in C(v_1) \).

Assume that \( f^* \) is still infeasible. Then, there exists some grandchild \( x \) of \( v_1 \) of (10). Notice that for each vertex \( v \in \{p(x)\} \cup V(T(x)) \), we have \( f^*(v) = f(v) \) from the construction; \( f^*(p(x)) \notin \{b - p + 1, \ldots, b + p - 1\} \) and \( f^*(x') \neq b \) for any sibling \( x' \) of \( x \). Moreover, by (7), the neck level of \( T(x) \) is at most \( h \); \( \delta((p(x), x), (f^*(p(x)), b - 1)) = \delta((p(x), x), (f^*(p(x)), b)) = 1 \). Hence, there exists a \( \lambda \)-\( L(p, 1) \)-labeling \( f' \) of \( T(x) + (p(x), x) \) which achieves \( \delta((p(x), x), (f^*(p(x)), b)) = 1 \). It follows that the labeling \( f'' \) such that \( f''(v) = f''(v) \) for all \( v \in V(T(x)) \) and \( f''(v) = f^*(v) \) for all other vertices, is a \( \lambda \)-\( L(p, 1) \)-labeling of \( T(p(x)) + (v_1, p(x)) \). Thus, by repeating these observations for each grandchild of \( v_1 \) of (10), we can obtain a \( \lambda \)-\( L(p, 1) \)-labeling \( f^\** \) for \( T_1 + (u, v_1) \) with \( f^\**(u) = a \) and \( f^\**(v_1) = b - 1 \). This contradicts \( \delta((u, v_1), (a, b - 1)) = 0 \).

(ii) When \( p = 1 \), the constraint \( |a - b| \geq 1 \) for any labels \( a \) and \( b \) is equivalent to \( a \neq b \); in any \( L(1, 1) \)-labeling, we can exchange any two different labels. This implies (ii). □

As mentioned before, this lemma holds for general \( p \), so does Theorem 4, and to show this, notice that the following easy lemma holds.

**Lemma 10** If \( |T| \geq 2 \) and \( p \geq 2\Delta \), no label in \( \{2\Delta - 1, \ldots, p - 1\} \) is used in any \( \lambda \)-\( L(p, 1) \)-labeling of a tree \( T \) for any \( \lambda \in \{\Delta + p - 1, \ldots, 2\Delta + p - 2\} \).
Proof of Theorem 4  We first show that the head level and neck level of $T$ is $O(\Delta)$. The case of $p = O(\Delta)$ is clear because $\lambda(T) \leq \Delta + p - 2$. If $p \geq 2\Delta$, then Lemma 10 indicates that the head level and neck level of $T$ is at most $2\Delta - 2$ (note that if $|T| = 1$, both levels are 0).

We next show that the head level and neck level of $T$ is $O(p \log n / \log \lambda)$. The case of $\lambda = O(p \log n / \log \lambda)$ is clear. Consider the case where $\lambda > \frac{8p \log n}{\log (\lambda/2)} + 8p - 8$. Then, for $h = \frac{2p \log n}{\log (\lambda/2)}$, we have

$$(\lambda - 2h - 4p + 4) \left( \frac{h}{2} \right) > \left( \frac{4p \log n}{\log \frac{\lambda}{2}} + 4p - 4 \right) - \frac{4p \log n}{\log \frac{\lambda}{2}} - 4p + 4 \right) \frac{\log n}{\log \frac{\lambda}{2}} = n.$$ 

Now note that $\lambda - 2h > \frac{4p \log n}{\log (\lambda/2)} + 8p - 8 > 3p - 3$. Hence, by Lemma 8, it follows that the head and neck levels of $T$ are both at most $\frac{2p \log n}{\log (\lambda/2)}$. □

7 Concluding Remarks

This paper presents a linear time algorithm for $L(p, 1)$-labeling of trees, when $p$ is bounded by a constant, especially by describing the one in case of $p = 2$, which is the $L(2, 1)$-labeling problem. Although the main contribution of the paper is the linear time algorithm itself, the introduction of the notion of label-compatibility might have more impact, because it could be generalized for other distance constrained labelings. Also, it is a quite interesting open problem to obtain a combinatorial characterization of trees with $\lambda(T) = \Delta(T) + 1$ and $\lambda(T) = \Delta(T) + 2$.

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