BKMs's Criterion and Global Weak Solutions for Magnetohydrodynamics with Zero Viscosity

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Abstract

In this paper we derive a criterion for the breakdown of classical solutions to the incompressible magnetohydrodynamic equations with zero viscosity and positive resistivity in $\mathbb{R}^3$. This result is analogous to the celebrated Beale-Kato-Majda’s breakdown criterion for the inviscid Euler equations of incompressible fluids. In $\mathbb{R}^2$ we establish global weak solutions to the magnetohydrodynamic equations with zero viscosity and positive resistivity for initial data in Sobolev space $H^1(\mathbb{R}^2)$.

Keyword: Beale-Kato-Majda’s criterion, weak solutions, magnetohydrodynamics, zero viscosity.

1 Introduction

The incompressible magnetohydrodynamic equations in $\mathbb{R}^n$, $n = 2, 3$, take the form

$$
\begin{align*}
    u_t + u \cdot \nabla u + \nabla (p + \frac{1}{2}|h|^2) &= \mu \Delta u + (h \cdot \nabla)h, \\
    h_t - \nabla \times (u \times h) &= \nu \Delta h, \\
    \nabla \cdot u &= 0, \quad \nabla \cdot h = 0,
\end{align*}
$$

(1.1)

where $u = (u_1, \cdots, u_n)^T$ is the velocity of the flows, $h = (h_1, \cdots, h_n)^T$ is the magnetic field, $p$ is the scalar pressure, $\mu$ is the viscosity of the fluid which is the inverse of the Reynolds number and $\nu$ is the resistivity constant which is inversely proportional to the electrical conductivity constant. The system (1.1) describes the macroscopic behavior of electrically conducting incompressible fluids (see [17]).

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In the extremely high electrical conductivity cases which occur frequently in the cosmical and geophysical problems, we ignore the resistivity to have the following partially viscous magnetohydrodynamic system (see [5]):

\[
\begin{align*}
&u_t + u \cdot \nabla u + \nabla (p + \frac{1}{2}|h|^2) = \mu \Delta u + (h \cdot \nabla) h, \\
&h_t - \nabla \times (u \times h) = 0, \\
&\nabla \cdot u = 0, \quad \nabla \cdot h = 0.
\end{align*}
\]

(1.2)

In the turbulent flow regime which occurs when the Reynolds numbers is very big, we ignore the viscosity of fluids to have the following partially viscous magnetohydrodynamic system:

\[
\begin{align*}
&u_t + u \cdot \nabla u + \nabla (p + \frac{1}{2}|h|^2) = (h \cdot \nabla) h, \\
&h_t - \nabla \times (u \times h) = \nu \Delta h, \\
&\nabla \cdot u = 0, \quad \nabla \cdot h = 0.
\end{align*}
\]

(1.3)

The local well-posedness of the Cauchy problem of the partially viscous magnetohydrodynamic systems (1.2) and (1.3) is rather standard and similar to the case of fully viscous magnetohydrodynamic system which is done in [21]. At present, there is no global-in-time existence theory for strong solutions to systems (1.2) and (1.3). The notable difference between (1.2) or (1.3) and its Newtonian counterpart, the incompressible Euler equations, is that for (1.2) or (1.3), global-in-time existence has not been established even in two dimensions (global-in-time existence is only established in the fully viscous case \(\mu > 0, \nu > 0\) in two dimensions, see [21]), even with small initial data. The question of spontaneous apparition of singularity from a local classical solution is a challenging open problem in the mathematical fluid mechanics, which is similar as the cases of ideal magnetohydrodynamics and fully viscous magnetohydrodynamics. We just refer some of the studies on the finite time blow-up problem in the ideal magnetohydrodynamics (see [4, 10, 8, 7] and references therein).

In the absence of a well-posedness theory, the development of blowup/non-blowup theory is of major importance for both theoretical and practical purposes. For incompressible Euler and Navier-Stokes equations, the well-known Beale-Kato-Majda’s criterion [1] says that any solution \(u\) is smooth up to time \(T\) under the assumption that \(\int_0^T \|\nabla \times u(t, \cdot)\|_{L^\infty} dt < \infty\). Beale-Kato-Majda’s criterion is slightly improved by Kozono-Taniuchi [15] under the assumption (1.4). Recently, a logarithmically improved Beale-Kato-Majda’s criterion is proven by Zhou and Lei [23]. Caflisch-Klapper-Steele [4] extended the Beale-Kato-Majda’s criterion to the 3D ideal magnetohydrodynamic equations, under the assumption on both velocity field and magnetic field: \(\int_0^T \left(\|\nabla \times u(t, \cdot)\|_{L^\infty} + \|\nabla \times h(t, \cdot)\|_{L^\infty}\right) dt < \infty\). Motivated by numerical experiments [11, 19] which seem to indicate that the velocity field plays the more important role than the magnetic field in the regularity theory of solutions to the magnetohydrodynamic equations, a lot of work are focused on the regularity problem of magnetohydrodynamic equations under assumptions only on velocity field, but not on magnetic...
field [12, 13, 6]. Especially, for fully viscous magnetohydrodynamic equations, the analogy of Beale-Kato-Majda’s criterion is studied by Chen-Miao-Zhang [6] under the assumption only on the vorticity of velocity field, where the authors made clever use of Littlewood-Paley theory to avoid using the logarithmic Sobolev inequality, which successfully avoids assuming any control of the magnetic field.

In this paper, we establish the analogous Beale-Kato-Majda’s criterion to magnetohydrodynamic equations (1.3) even in the case of zero viscosity. The partially viscous magnetohydrodynamic system (1.2) will be studied with an incompressible viscoelastic fluid system of the Oldroyd type in our forthcoming paper [16]. The first goal of this paper is to prove that

\[ \text{Theorem 1.1.} \]

Let \( T > 0 \) and \( u_0, h_0 \in H^s(\mathbb{R}^n) \) for \( s \geq 3 \) and \( n = 2 \) or 3. Suppose that \((u, h)\) is a smooth solution to the partially viscous magnetohydrodynamic system (1.3) with initial data \( u(0, x) = u_0, h(0, x) = h_0 \). Then \((u, h)\) is smooth up to time \( T \) provided that

\[ \int_0^T \| \nabla \times u(t, \cdot) \|_{\text{BMO}} dt < \infty. \]  

Our key observation is that under the assumption (1.4), if we start from the time \( T_\ast < T \) which is as close to \( T \) as possible, we find that

\[ \|(u, h)(t, \cdot)\|_{H^1} \leq C_\ast \sup_{T_\ast \leq s \leq t} \|(u, h)\|_{H^3}^\delta, \quad \delta \text{ is as small as we want, } T_\ast \leq t < T. \]

From this point of view, the nonlinearity of velocity field \( u \) and magnetic field \( h \) can be as weak as what we want in the sense of \( H^1 \) norm. By using this observation, we find that logarithmic Sobolev inequalities can still be used to prove Theorem 1.1.

Our second result is the global existence and regularity of weak solutions to partially viscous magnetohydrodynamic system (1.3) in \( \mathbb{R}^2 \):

\[ \text{Theorem 1.2.} \]

Let \( u_0, h_0 \in H^1(\mathbb{R}^2) \). Then there exists a global weak solution \((u, h)\) with \( u \in L^\infty(0, \infty; H^1) \) and \( h \in L^\infty(0, \infty; H^1) \cap L^2(0, \infty; H^2) \) to the partially viscous magnetohydrodynamic system (1.3) with initial data \( u(0, x) = u_0, h(0, x) = h_0 \).

The observation of the above result lies in that we can obtain the \( H^1 \) estimate of the velocity field and magnetic field. Then the proof of Theorem 1.2 is elementary and standard. Thus, we will just present the \( H^1 \) estimate of the velocity field and magnetic field in this paper, leaving the establishment of global weak solutions to interesting readers (for example, see the books [22, 18] as references). See section 4 for more details.

The rest of this paper is organized as follows: In section 2 we recall some inequalities which will be used in this paper. In section 3 we derive the Beale-Kato-Majda’s criterion for the partially viscous magnetohydrodynamic equation (1.3) and present the proof of Theorem 1.1. Then in section 4 we prove the \( H^1 \) a priori estimate of the velocity field \( u \) and the magnetic field \( h \) in \( \mathbb{R}^2 \) which implies the result in Theorem 1.2.
2 Preliminaries

First of all, let us recall the following multiplicative inequalities.

**Lemma 2.1.** The following interpolation inequalities holds.

\[
\begin{cases}
\|f\|_{L^\infty} \leq C_0 \|f\|_{L^2}^{\frac{3}{2}} \|\nabla^2 f\|_{L^2}^{\frac{1}{2}}, \\
\|f\|_{L^2} \leq C_0 \|f\|_{L^2}^{\frac{1}{2}} \|\nabla f\|_{L^2}^{\frac{1}{2}}, \\
\|f\|_{L^p} \leq C_0 \|f\|_{L^2}^{\frac{3}{2}} \|\nabla^2 f\|_{L^2}^{\frac{3}{2}}, \\
\|\nabla f\|_{L^2} \leq C_0 \|f\|_{L^2}^{\frac{1}{2}} \|\nabla^2 f\|_{L^2}^{\frac{1}{2}},
\end{cases}
\]

(2.1)

where \(C_0\) is an absolute positive constant.

**Proof.** The above inequalities are of course well-known. In fact, they can just proved by Sobolev embedding theorems and the scaling techniques. As an example, let us present the proof of the first inequalities. By Sobolev embedding theorem \(H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)\), one has

\[
\|f\|_{L^\infty} \leq C_0 (\|f\|_{L^2} + \|\nabla^2 f\|_{L^2}).
\]

The constant \(C_0\) in above inequality is independent of \(f \in H^2(\mathbb{R}^2)\). Thus, for any given \(0 \neq f \in H^2(\mathbb{R}^2)\) and any \(\lambda > 0\), let us define \(f^\lambda(x) = f(\lambda x)\). Then one has

\[
\|f^\lambda\|_{L^\infty} \leq C_0 (\|f^\lambda\|_{L^2} + \|\nabla^2 f^\lambda\|_{L^2}),
\]

which is equivalent to

\[
\|f\|_{L^\infty} \leq C_0 (\lambda^{-\frac{1}{2}} \|f\|_{L^2} + \lambda \|\nabla^2 f\|_{L^2}).
\]

Taking \(\lambda = (\|f\|_{L^2} \|\nabla^2 f\|_{L^2})^{\frac{1}{4}}\), one gets the first inequality in (2.1).

Next, let us recall the following well-known inequalities. In fact, the first one is Gagliardo-Nirenberg inequality and the second one is a direct consequence of the chain rules and Gagliardo-Nirenberg inequality.

**Lemma 2.2.** The following inequalities holds:

\[
\begin{cases}
\|\nabla^i u\|_{L^\infty} \leq C_0 \|u\|_{L^s} \|\nabla^s u\|_{L^2}^{\frac{s}{s - i}}, & 0 \leq i \leq s, \\
\|\nabla^s (u \cdot \nabla u) - u \cdot \nabla^s u\| \leq C_0 \|\nabla u\|_{L^\infty} \|\nabla^s u\|_{L^2}, & s \geq 1.
\end{cases}
\]

(2.2)

At last, let us recall the following logarithmic Sobolev inequality which is proved in [15] and is an improved version of that in \([1]\) (see also \([2, 3]\)).

**Lemma 2.3.** Let \(n = 2\) or \(3\) and \(p > n\). The following logarithmic Sobolev embedding theorem holds for all divergence free vector fields:

\[
\|\nabla f\|_{L^\infty(\mathbb{R}^n)} \leq C_0 [1 + \|f\|_{L^2(\mathbb{R}^n)} + \|\nabla \times f\|_{\text{BMO}(\mathbb{R}^n)}] \ln \left(1 + \|f\|_{W^{2,p}(\mathbb{R}^n)}\right).
\]

(2.3)
3 Beale-Kato-Majda’s Criterion for Magnetohydrodynamic Equations with Zero Viscosity

In this section we prove the analogous Beale-Kato-Majda’s criterion for the partially viscous magnetohydrodynamic equations (1.3) in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). First of all, for classical solutions to (1.3), one has the following basic energy law

\[
\frac{1}{2} \frac{d}{dt}(\|u\|_{L^2}^2 + \|h\|_{L^2}^2) + \nu \|\nabla h\|_{L^2}^2 = 0. 
\tag{3.1}
\]

For any integer \( s \geq 1 \), applying \( \nabla^s \) to the velocity field equation and magnetic field equation, and taking the \( L^2 \) inner product of the resulting equations with \( \nabla^s u \) and \( \nabla^s h \) respectively, one has

\[
\frac{1}{2} \frac{d}{dt}(\|\nabla^s u\|_{L^2}^2 + \|\nabla^s h\|_{L^2}^2) + \nu \|\nabla^2 h\|_{L^2}^2 = \nonumber
\]

\[
= -\int_{\mathbb{R}^n} \nabla^s u \left[ \nabla^s (u \cdot \nabla u) - u \cdot \nabla \nabla^s u \right] dx 
\]

\[
- \int_{\mathbb{R}^n} \nabla^s h \left[ \nabla^s (u \cdot \nabla h) - u \cdot \nabla \nabla^s h \right] dx 
\]

\[
+ \int_{\mathbb{R}^n} \nabla^s u \left[ \nabla^s (h \cdot \nabla h) - h \cdot \nabla \nabla^s h \right] dx 
\]

\[
+ \int_{\mathbb{R}^n} \nabla^s h \left[ \nabla^s (h \cdot \nabla u) - h \cdot \nabla \nabla^s u \right] dx, 
\tag{3.2}
\]

where we have used the divergence free condition \( \nabla \cdot u = \nabla \cdot h = 0 \).

For simplicity, we just set \( \nu = 1 \) below. Let \( s = 1 \) in (3.2). One can easily derive the following estimate:

\[
\frac{1}{2} \frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla h\|_{L^2}^2) + \|\nabla^2 h\|_{L^2}^2 \leq C \|\nabla u\|_{L^\infty} \left( \|\nabla u\|_{L^2}^2 + \|\nabla h\|_{L^2}^2 \right) 
\]

which gives that

\[
\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla h(t, \cdot)\|_{L^2}^2 + \int_{t_0}^{t} \|\nabla^2 h(s, \cdot)\|_{L^2}^2 ds 
\]

\[
\leq C \left( \|\nabla u(t_0, \cdot)\|_{L^2}^2 + \|\nabla h(t_0, \cdot)\|_{L^2}^2 \right) \exp \left\{ C \int_{t_0}^{t} \|\nabla u\|_{L^\infty} ds \right\}. \tag{3.3}
\]

Noting (1.4), one concludes that for any small constant \( \epsilon > 0 \), there exists \( T_* < T \) such that

\[
\int_{T_*}^{T} \|\nabla \times u\|_{BMO} ds \leq \epsilon. \tag{3.4}
\]
Let us denote

$$M(t) = \sup_{T_\ast \leq t} (\|\nabla^3 u(s, \cdot)\|_{L^2}^2 + \|\nabla^3 h(s, \cdot)\|_{L^2}^2), \quad T_\ast \leq t < T. \quad (3.5)$$

By (3.1), (3.3), (3.4), (3.5) and Lemma 2.3, one has

$$\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla h(t, \cdot)\|_{L^2}^2 + \int_{T_\ast}^t \|\nabla^2 h(s, \cdot)\|_{L^2}^2 ds \leq C_\ast \exp \left\{C_0 \int_{T_\ast}^t \|\nabla \times u\|_{MBO} \ln (1 + \|u\|_{H^1}^2 + \|h\|_{H^1}^2) ds \right\} \leq C_\ast \exp \left\{C_0 \epsilon \ln (1 + M(t)) \right\} = C_\ast (1 + M(t))^{C_0 \epsilon}, \quad T_\ast \leq t < T,$$

where $C_\ast$ depends on $\|\nabla u(T_\ast, \cdot)\|_{L^2}^2 + \|\nabla h(T_\ast, \cdot)\|_{L^2}^2$, while $C_0$ is an absolute positive constant given in Lemma 2.3. We remark here that in the case of $n = 2$, one can just set $\epsilon = 0$. See section 4 for more details.

For simplicity below, we will set $s = 3$ and present the estimate of the right hand side of (3.2). First of all, by Lemma 2.2, it is easy to see that

$$\left| \int_{\mathbb{R}^n} \nabla^3 u \left[ \nabla^3 (u \cdot \nabla u) - u \cdot \nabla \nabla^3 u \right] dx \right| \leq C \|\nabla u\|_{L^\infty} \|\nabla^3 u\|_{L^2}^2. \quad (3.7)$$

Next, by integrating by parts, one has

$$\left| \int_{\mathbb{R}^n} \nabla^3 h \left[ \nabla^3 (u \cdot \nabla h) - u \cdot \nabla \nabla^3 h \right] dx \right| + \left| \int_{\mathbb{R}^n} \nabla^3 h \left[ \nabla^3 (h \cdot \nabla u) - h \cdot \nabla \nabla^3 u \right] dx \right| \leq 4 \|\nabla u\|_{L^\infty} \|\nabla^3 h\|_{L^2}^2 + 14 \|\nabla u\|_{L^\infty} \|\nabla^3 h\|_{L^2}^2 + 10 \|\nabla u\|_{L^\infty} \|\nabla^2 h\|_{L^2} \|\nabla^4 h\|_{L^2} + 4 \|\nabla^2 u\|_{L^4} \|\nabla h\|_{L^4} \|\nabla^4 h\|_{L^2}.$$

By Lemma 2.1 and Lemma 2.2 we estimate

$$10 \|\nabla u\|_{L^\infty} \|\nabla^2 h\|_{L^2} \|\nabla^4 h\|_{L^2} \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} \|\nabla^2 h\|_{L^2}^2 + \frac{3}{16} C_{0 \epsilon} \|\nabla u\|_{L^\infty} \|\nabla^3 h\|_{L^2} \|\nabla^2 h\|_{L^2} \|\nabla^3 h\|_{L^2}.$$

By Lemma 2.3 we have

$$\|\nabla u\|_{L^\infty} \|\nabla^2 h\|_{L^2} \|\nabla^4 h\|_{L^2} \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla^3 u\|_{L^2} \|\nabla h\|_{L^2} \|\nabla^3 h\|_{L^2} \|\nabla^2 h\|_{L^2} \|\nabla^3 h\|_{L^2} \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla^3 u\|_{L^2} \|\nabla h\|_{L^2} \|\nabla^3 h\|_{L^2} \|\nabla^2 h\|_{L^2} \|\nabla^3 h\|_{L^2} \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} M(t)^{\frac{3}{2}} \left(1 + M(t) \right)^{\frac{5 C_{0 \epsilon}}{8}}.$$
in 3D case and
\[ 10\|\nabla u\|_{L^\infty} \|\nabla^2 h\|_{L^2} \|\nabla^4 h\|_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C\|\nabla u\|_{L^\infty} \|\nabla^3 u\|_{L^2} \|\nabla^2 h\|_{L^2} \|\nabla^3 h\|_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C_*\|\nabla u\|_{L^\infty} \|\nabla u\|_{L^\infty} M(t)^{\frac{2}{3}} \left(1 + M(t)\right) \frac{3C_0}{4} \]

in 2D case, where we used (3.6). On the other hand, one can similarly estimate
\[ 4\|\nabla^2 u\|_{L^4} \|\nabla h\|_{L^4} \|\nabla^4 h\|_{L^2} \]
\[ \leq 4\|\nabla u\|_{L^\infty} \frac{1}{2}\|\nabla^3 u\|_{L^2} \|\nabla^2 h\|_{L^2} \|\nabla^3 h\|_{L^2} \|\nabla^4 h\|_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C\|\nabla u\|_{L^\infty} \|\nabla h\|_{L^2} \|\nabla^3 u\|_{L^2} \|\nabla^3 h\|_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C_*\|\nabla u\|_{L^\infty} M(t)^{\frac{2}{3}} \left(1 + M(t)\right) \frac{5C_0}{8} \]

in 3D case and
\[ 4\|\nabla^2 u\|_{L^4} \|\nabla h\|_{L^4} \|\nabla^4 h\|_{L^2} \]
\[ \leq 4\|\nabla u\|_{L^\infty} \frac{1}{2}\|\nabla^3 u\|_{L^2} \|\nabla^2 h\|_{L^2} \|\nabla^3 h\|_{L^2} \|\nabla^4 h\|_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C\|\nabla u\|_{L^\infty} \|\nabla h\|_{L^2} \|\nabla^3 u\|_{L^2} \|\nabla^3 h\|_{L^2} \|\nabla^4 h\|_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla^4 h\|_{L^2}^2 + C_*\|\nabla u\|_{L^\infty} M(t)^{\frac{2}{3}} \left(1 + M(t)\right) \frac{3C_0}{4} \]

in 2D case. Consequently, one has
\[ \left| \int_{\mathbb{R}^n} \nabla^s h \left[ \nabla^s (u \cdot \nabla h) - u \cdot \nabla \nabla^s h \right] dx \right| \quad (3.8) \]
\[ + \left| \int_{\mathbb{R}^n} \nabla^s h \left[ \nabla^s (h \cdot \nabla u) - h \cdot \nabla \nabla^s u \right] dx \right| \]
\[ \leq \frac{1}{4} \|\nabla^4 h\|_{L^2}^2 + C_*\|\nabla u\|_{L^\infty} \left[1 + M(t)\right] \]

provided that
\[ \epsilon \leq \frac{1}{5C_0}. \quad (3.9) \]

It remains to estimate the last term on the right hand side of (3.2). By integrating by parts:
\[ \left| \int_{\mathbb{R}^n} \nabla^3 u \nabla^2 h \cdot \nabla \nabla h dx \right| \leq \left| \int_{\mathbb{R}^n} \nabla^2 u \nabla^3 h \cdot \nabla \nabla h dx \right| + \left| \int_{\mathbb{R}^n} \nabla^2 u \nabla^2 h \cdot \nabla \nabla^2 h dx \right| , \]
one similarly has the following estimate

\[
\left| \int_{\mathbb{R}^n} \nabla^3 u \left[ \nabla^3 (h \cdot \nabla h) - h \cdot \nabla \nabla^3 h \right] dx \right|
\]

(3.10)

\[
\leq \left| \int_{\mathbb{R}^n} \nabla^3 u \nabla^3 h \cdot \nabla dx \right| + 3 \left| \int_{\mathbb{R}^n} \nabla^3 u \nabla^2 h \cdot \nabla \nabla^3 h dx \right|
\]

\[
\leq \frac{1}{4} \| \nabla^4 h \|_{L^2}^2 + C \| \nabla u \|_{L^\infty} \left[ 1 + M(t) \right]
\]

Combining (3.2) with (3.7), (3.8) and (3.10) and using Lemma 2.3, we arrive at

\[
\frac{d}{dt} \left( \| \nabla^3 u \|_{L^2}^2 + \| \nabla^3 h \|_{L^2}^2 \right) + \| \nabla^3 \nabla h \|_{L^2}^2
\]

\[
\leq C_s \| \nabla u \|_{L^\infty} \left[ 1 + M(t) \right] + C_s \| \nabla \times u \|_{BMO} \ln \left( 1 + M(t) \right) \left[ 1 + M(t) \right]
\]

for all \( T_* \leq t < T \). Integrating the above inequality with respect to time from \( T_* \) to \( t \in [s, T) \) and using (3.4), we have

\[
1 + \| \nabla^3 u(s, \cdot) \|_{L^2}^2 + \| \nabla^3 h(s, \cdot) \|_{L^2}^2
\]

\[
\leq 1 + \| \nabla^3 u(T, \cdot) \|_{L^2}^2 + \| \nabla^3 h(T, \cdot) \|_{L^2}^2
\]

\[
+ C_s \int_{T_*}^s \| \nabla \times u(\tau, \cdot) \|_{BMO} \left[ 1 + M(\tau) \right] \ln \left( 1 + M(\tau) \right) d\tau,
\]

which implies

\[
1 + M(t) \leq 1 + \| \nabla^3 u(T, \cdot) \|_{L^2}^2 + \| \nabla^3 h(T, \cdot) \|_{L^2}^2
\]

\[
+ C_s \int_{T_*}^t \| \nabla \times u(\tau, \cdot) \|_{BMO} \left[ 1 + M(\tau) \right] \ln \left( 1 + M(\tau) \right) d\tau.
\]

Then Gronwall’s inequality gives

\[
1 + \| \nabla^3 u(t, \cdot) \|_{L^2}^2 + \| \nabla^3 h(t, \cdot) \|_{L^2}^2
\]

(3.11)

\[
\leq \left( 1 + \| \nabla^3 u(T, \cdot) \|_{L^2}^2 + \| \nabla^3 h(T, \cdot) \|_{L^2}^2 \right) \exp \exp \{ C_s \epsilon \}
\]

for all \( T_* \leq t < T \). Noting that the right hand side of (3.11) is independent of \( t \) for \( T_* \leq t < T \), one concludes that (3.11) is also valid for \( t = T \) which means that \( (u(T, \cdot), h(T, \cdot)) \in H^3(\mathbb{R}^n) \).

4  Global Weak Solutions to the Magnetohydrodynamic Equations with Zero Viscosity in 2D

In this section we prove Theorem 1.2. To this end, following a standard procedure, we establish an approximate system of (1.3) with smoothed initial data which admits a
unique local classical solution. We will establish the global a priori $H^1$-estimate for the approximate system in terms of the $H^1$-norm of the original initial data $u_0$ and $h_0$, which implies that the magnetohydrodynamic equations (1.3) in $\mathbb{R}^2$ possess a global weak solution. Below we will just present the global a priori estimate for classical solutions to the magnetohydrodynamic equations (1.3) with $n = 2$, while we refer the reader to as [22, 18] for the standard procedure to establish the global weak solutions and to construct approximate systems.

Now let us denote curl($v$) = $\nabla \times v = \partial_{x_1} v_2 - \partial_{x_2} v_1$ for two dimensional vector $v$. Applying curl to (1.3), one has

\[
\begin{cases}
(u \cdot \nabla)(\nabla \times u) + (\nabla \times u)_t = (h \cdot \nabla)(\nabla \times h), \\
(h \cdot \nabla)(\nabla \times h) = \nu \Delta (\nabla \times h) + 2 \text{tr}(\nabla u \nabla \perp h).
\end{cases}
\]

Taking the $L^2$ inner product of the above equations with $\nabla \times u$ and $\nabla \times h$ respectively, and noting the interpolation inequality in Lemma 2.1, one has

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \times u\|_{L^2}^2 + \|\nabla \times h\|_{L^2}^2 \right) + \nu \|\nabla (\nabla \times h)\|_{L^2}^2 
\]

\[
= \int_{\mathbb{R}^2} \left[ (\nabla \times u)(h \cdot \nabla)(\nabla \times h) + (\nabla \times h)(h \cdot \nabla)(\nabla \times u) \right] dx 
\]

\[
+ 2 \int_{\mathbb{R}^2} (\nabla \times h) \text{tr}(\nabla u \nabla \perp h) dx 
\]

\[
\leq 0 + 2\|h\|_{L^4}^2 \|\nabla u\|_{L^2} \leq C\|\nabla h\|_{L^2} \|\nabla u\|_{L^2} \|\Delta h\|_{L^2} 
\]

\[
\leq \frac{\nu}{2} \|\nabla \perp (\nabla \times h)\|_{L^2}^2 + \frac{C}{\nu} \|\nabla h\|_{L^2}^2 \|\nabla \times u\|_{L^2}^2,
\]

where we used Calderon-Zygmund theory and the fact that $\Delta h = \nabla \perp (\nabla \times h)$. Using the basic energy law (3.1), we derive from (4.2) that

\[
\|\nabla \times u\|_{L^2}^2 + \|\nabla \times h\|_{L^2}^2 + \nu \int_0^t \|\nabla (\nabla \times h)\|_{L^2}^2 ds 
\]

\[
\leq \left( \|\nabla \times u_0\|_{L^2}^2 + \|\nabla \times h_0\|_{L^2}^2 \right) \exp \left\{ \frac{2C}{\nu} \int_0^t \|\nabla h\|_{L^2}^2 ds \right\} 
\]

\[
\leq \left( \|\nabla \times u_0\|_{L^2}^2 + \|\nabla \times h_0\|_{L^2}^2 \right) \exp \left\{ \frac{C\|u_0\|_{L^2}^2 + \|h_0\|_{L^2}^2}{\nu^2} \right\}.
\]

This combining with the basic energy law (3.1) gives a global uniform bound for the velocity field $u$ and the magnetic field $h$ in $u \in L^\infty(0, \infty; H^1)$ and $h \in L^\infty(0, \infty; H^1) \cap L^2(0, \infty; H^2)$.

**Final Remark:** In fact, one may improve the regularity of the weak solutions if the initial data is more regular. For example, for some $p, q \in (2, \infty)$, using the estimates
for linear Stokes system (see [2]) and transport equations, one can get a uniform bound for $\|\nabla^2 h\|_{L^q_t(L^p_x)} + \|\nabla \times u\|_{L^\infty_t(L^p_x)}$. However, at the moment we are not able to get the global classical solutions of the partially viscous magnetohydrodynamic equations (1.3) for $n = 2$. We will investigate this issue further in our future work.

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