The Isolated Horizons (IH) formalism, together with a simple phenomenological model for colored black holes has been used to predict non-trivial formulae that relate the ADM mass of the solitons and hairy Black Holes of Gravity-Matter system on the one hand, and several horizon properties of the black holes in the other. In this article, the IH formalism is tested numerically for spherically symmetric solutions to an Einstein-Higgs system where hairy black holes were recently found to exist. It is shown that the mass formulae still hold and that, by appropriately extending the current model, one can account for the behavior of the horizon properties of these new solutions. An empirical formula that approximates the ADM mass of hairy solutions is put forward, and some of its properties are analyzed.

PACS numbers: 04.70.Bw, 04.40.Nr, 04.20.Cv

I. INTRODUCTION

In recent years, the introduction of the Isolated Horizon (IH) formalism has proved to be useful to gain insight into the static sector of theories admitting “hair”. Firstly, it has been found that the Horizon Mass of the black hole (BH), a notion constructed out of purely quasi-local quantities, is related in a simple way to the ADM masses of both the colored black hole and the solitons of the theory. Second, a simple model for colored black holes as bound states of regular black holes and solitons has allowed to provide heuristic explanations for the behavior of horizon quantities of those black holes. Third, the formalism is appropriate for the formulation of uniqueness conjectures for the existence of unique stationary solutions in terms of horizon “charges”. Finally, the combination of the Mass formula, together with the fact that in theories such as Einstein-Yang-Mills-Higgs (EYMH) different “branches” of static solutions merge, has allowed to have a formula for the difference of soliton masses in terms of black hole quantities. Many of these predictions have been confirmed in more general situations and for other matter couplings. For a recent review on IH (including hair) see, and for a review of hairy black holes see.

In this article, we explore further the consequences of the IH formalism in the static sector of the theory. In particular, we explore the behavior of a recently found family of hairy static spherically symmetric (SSS) solutions to the Einstein-Higgs system, where the scalar potential is allowed to be negative and therefore, the existing no-hair theorems do not apply. In the standard treatment of stationary black holes with killing horizons, one is always restoring to several concepts that use asymptotic information very strongly. On the other hand, the IH formalism only uses quasi-local information defined on the horizon, allowing it to prove very general results involving only these quasi-local quantities. The IH formalism has proved to be generalizable, in the scalar sector, even to the non-minimal coupling regime, where the energy conditions required for the consistency of the formalism are much weaker. In the present paper we shall restrict our attention to the minimally coupled case, and for a particular form of the scalar potential for which static solutions are known to exist. We will study the one parameter family

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of solutions (that could be labelled by its geometric radius \( r_\Delta \)) and compare its properties with those of hairy black holes in other theories, such as EYM, where the phenomenological predictions of the IH formalism have been shown to work very well. As we will show, we find that the mass formulae relating BH and soliton ADM masses works also well, but the model of a hairy black hole as a bound state of a soliton and a bare black holes exhibits some new unexpected features. As we shall see, one needs to slightly modify the model from its original formulation in Ref.\([2]\). Once this modification is made the model can again explain all the qualitative behavior of the hairy BH solutions.

The structure of the paper is as follows: In Sec. \( \text{I} \) we review the consequences of the IH formalism for hairy solitons and BH solutions. In Sec. \( \text{II} \) we review the SSS found recently in the Einstein-Higgs sector. Section \( \text{III} \) is the main section of the paper, in it, we show the numerical evidence for the mass formulae and the phenomenological predictions of the model. Unlike the EYM case, there are some unexpected features, such as the binding energy becoming positive. We then propose a modification of the formalism to deal with such situations. We show that with these modifications, the model can still account for the geometrical phenomena found in several theories. In Sec. \( \text{IV} \) we explore the situation of the collapse of a hairy black hole and use the model to put bound on the total possible energy to be radiated. These results should be of some relevance to full dynamical numerical evolutions of such black holes. In Sec. \( \text{V} \) we propose an empirical formula for the horizon and ADM masses of scalar hairy black holes that can also be applied to the EYM case. Finally, we end with a discussion in Sec. \( \text{VI} \).

II. CONSEQUENCES OF THE ISOLATED HORIZONS FORMALISM

In recent years, a new framework tailored to consider situations in which the black hole is in equilibrium (“nothing falls in”), but which allows for the exterior region to be dynamical, has been developed. This Isolated Horizons (IH) formalism is now in the position of serving as starting point for several applications. Notably, for the extraction of physical quantities in numerical relativity and also for quantum entropy calculations. The basic idea is to consider space-times with an interior boundary (to represent the horizon), satisfying quasi-local boundary conditions ensuring that the horizon remains “isolated”. Although the boundary conditions are motivated by geometric considerations, they lead to a well defined action principle and Hamiltonian framework. Furthermore, the boundary conditions imply that certain ‘quasi-local charges’, defined at the horizon, remain constant ‘in time’, and can thus be regarded as the analogous of the global charges defined at infinity in the asymptotically flat context. The isolated horizons Hamiltonian framework allows to define the notion of Horizon Mass \( M_\Delta \), as a function of the ‘horizon charges’ (hereafter, the subscript “\( \Delta \)” stands for a quantity at the horizon).

In the Einstein-Maxwell and Einstein-Maxwell-Dilaton systems considered originally, the horizon mass satisfies a Smarr-type formula and a generalized first law in terms of quantities defined exclusively at the horizon (i.e. without any reference to infinity). The introduction of non-linear matter fields like the Yang-Mills field has brought unexpected subtleties to the formalism. However, one still is in the position of defining a Horizon Mass, and furthermore, this Horizon Mass satisfies a first law.

An isolated horizon is a non-expanding null surface generated by a (null) vector field \( l^a \). The IH boundary conditions imply that the acceleration \( \kappa \) of \( l^a \) (\( l^a \nabla_a b^b = \kappa b^b \)) is constant on the horizon \( \Delta \). However, the precise value it takes on each point of phase space (PS) is not determined a-priori. On the other hand, it is known that for each vector field \( t^a \) on time-space, the induced vector field \( X_{t^a} \) on phase space is Hamiltonian if and only if there exists a function \( E_{t^a} \) such that \( \delta E_{t^a} = \Omega(\delta, X_{t^a}) \), for any vector field \( \delta \) on PS. This condition can be re-written as, \( \delta E_{t^a} = \frac{\kappa}{\pi G_0} \delta a_\Delta + \text{work terms} \). Thus, the first law arises as a necessary and sufficient condition for the consistency of the Hamiltonian formulation. Thus, the allowed vector fields \( t^a \) will be those for which the first law holds. Note that there are as many ‘first laws’ as allowed vector fields \( t^a \equiv t^a \) on the horizon. However, one would like to have a Physical First Law, where the Hamiltonian \( E_{t^a} \) be identified with the ‘physical mass’ \( M_\Delta \) of the horizon. This amounts to finding the ‘right \( \kappa \)’. This ‘normalization problem’ can be easily overcome in the EM system. In this case, one chooses the function \( \kappa = \kappa(a_\Delta, Q_\Delta) \) as the corresponding function for the static solution with charges \( (a_\Delta, Q_\Delta) \). However, for the EYM system, this procedure is not as straightforward. A consistent viewpoint is to abandon the notion of a globally defined horizon mass on Phase Space, and to define, for each value of \( n = n_\circ \) (which labels different branches of the solutions), a canonical normalization \( t^a_\circ \) that yields the Horizon Mass \( M_\Delta^{(n_\circ)} \) for the \( n_\circ \) branch. The horizon mass takes the form (from now on we shall omit the \( n_\circ \) label),

\[
M_\Delta(r_\Delta) = \frac{1}{2G_0} \int_0^{r_\Delta} \beta(r) \, dr ,
\]
with $r_\Delta$ the horizon radius. Here $\beta(r_\Delta)$ is related to the surface gravity as follows $\beta(r_\Delta) = 2r_\Delta \kappa(r_\Delta)$.

Furthermore, one can relate the horizon mass $M_\Delta$ to the ADM mass of static black holes. Recall first that general Hamiltonian considerations imply that the total Hamiltonian, consisting of a term at infinity, the ADM mass, and a term at the horizon, the Horizon Mass, is constant on every connected component of static solutions (provided the evolution vector field $t^\mu_0$ agrees with the static Killing field everywhere on this connected component) [2, 3]. In the Einstein-Yang-Mills case, since the Hamiltonian is constant on any branch, we can evaluate it at the solution with zero horizon area. This is just the soliton, for which the horizon area $a_\Delta$, and the horizon mass $M_\Delta$ vanish. Hence we have that $H(t_0) = M_{sol}$. Thus, we conclude [4]:

$$M_{sol} = M_{ADM} - M_\Delta$$

Thus, the ADM mass contains two contributions, one attributed to the black hole horizon and the other to the outside ‘hair’, captured by the ‘solitonic residue’. The formula (2), together with some energetic considerations [5], lead to the model of a colored black hole as a bound state of an ordinary, ‘bare’, black hole and a ‘solitonic residue’, where the ADM mass of the colored black hole of radius $r_\Delta$ is given by the ADM mass of the soliton plus the horizon mass of the ‘bare’ black hole plus the binding energy:

$$M_{ADM} = M_{sol} + M_\Delta = M_{sol}^0 + M_{sol} + E_{bind} \ ,$$

with $E_{bind} = M_\Delta - M_{sol}^0$. Simple considerations about the behavior of the ADM masses of the colored black holes and the solitons, together with some expectations of this model (such as demanding for a non-positive binding energy) give raise to several predictions about the behavior of the horizon parameters [5]. Among the predictions, we have:

i) The absolute value of the binding energy decreases as $r_\Delta$ increases.

ii) $\beta(r_\Delta)$, as a function of $r_\Delta$, is a positive function, bounded above by $\beta(0)(r_\Delta) = 1$.

iii) The curve $\beta(r)$, as functions of $r$ intersect the $r = 0$ axis at distinct points between 0 and 1, and never intersect. Finally,

iv) The curve for $\beta$, for large value of its argument, becomes asymptotically tangential to the curve $\beta(0)(r_\Delta) = 1$.

One of the features of these solutions in, say, EYM is that there is no limit for the size of the black hole. That is, if we plot the ADM mass of the BH as function of the radius $r_\Delta$ we get an infinite number of curves, each of the intersecting the $r_\Delta = 0$ line at the value of the soliton mass, and never intersecting each other.

The purpose of this paper is to test the mass formula (2), for the scalar hairy solutions found in Ref. [12] and also to confront the predictions i)-iv) (obtained for the colored EYM BH model [5]), with the corresponding properties for the scalar hairy BH. In the next section we will review the scalar hairy solutions, and afterwards we shall study their horizon properties.

### III. SCALAR SOLITONS AND BLACK HOLES IN EINSTEIN-HIGGS THEORY

Let us consider the theory of a scalar field minimally coupled to gravity described by the total action:

$$S_{tot}[g_{\mu\nu}, \phi] = \int \sqrt{-g} \left\{ \frac{R}{16\pi} - \left[ \frac{1}{2} (\nabla^\beta \phi)(\nabla_\beta \phi) + V(\phi) \right] \right\} d^4x$$

(units where $G_0 = c = 1$ are employed). The field equations following from the variation of the action [4] are,

$$G_{\mu\nu} = 8\pi \left\{ (\nabla_\mu \phi)\nabla_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} (\nabla^\beta \phi)(\nabla_\beta \phi) + V(\phi) \right] \right\} \ ,$$

and,

$$\Box \phi = \frac{\partial V(\phi)}{\partial \phi} \ .$$

It is well known that asymptotically flat static spherically symmetric solutions representing black holes solutions to the Einstein-Higgs equations do not exist if the scalar matter satisfies the weak energy condition (WEC) due
to the existence of the so called scalar no-hair theorems \[13\]. Recently, numerical evidence for the existence of asymptotically flat and static spherically symmetric solutions representing scalar hairy black holes (SHBH) and scalar solitons (scalarons; hereafter SS) have been found in theories represented by the action \(4\) and with a scalar potential non-positive-semidefinite \[12\] given by the asymmetric potential, \(V(\phi) = \frac{\lambda}{4} \left[ (\phi - a)^2 - \frac{4(\eta_1 + \eta_2)}{3}(\phi - a) + 2\eta_1\eta_2\right](\phi - a)^2\), \(\text{(7)}\)

where \(\lambda, \eta_i\) and \(a\) are constants. For this class of potential one can see that, for \(\eta_1 > 2\eta_2 > 0\), \(\phi = a\) corresponds to the local minimum, \(\phi = a + \eta_1\) is a global minimum and \(\phi = a + \eta_2\) is a local maximum (see Fig. 1). The key point in the shape of the potential, \(V(\phi)\), for the asymptotically flat solutions to exist, is that the local minimum \(V_{\text{local}} = V(a)\) is also a zero of \(V(\phi)\) (to see \[12\] for a detailed analysis for the existence of these solutions). Moreover, \(V(\phi)\) is not positive definite (we assume \(\lambda > 0\)), which leads to a violation of the WEC and therefore the scalar no-hair theorems \[13\] can not be applied to this case.

![Qualitative shape of the scalar-field potential](image)

**FIG. 1:**
Qualitative shape of the scalar-field potential \(V(\phi)\) as given by Eq. \(7\) used to construct the asymptotically flat black hole and soliton solutions.

In order to describe the asymptotically flat SHBH and SS, we use a standard parametrization for the metric and the scalar field describing spherically symmetric and static spacetimes

\[ds^2 = -\left(1 - \frac{2m(r)}{r}\right)e^{2\delta(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1}dr^2 + r^2d\Omega^2,\]

\[\phi = \phi(r),\]

\(\text{(8)}\)
\(\text{(9)}\)

For SHBH we demand regularity on the event horizon \(r_\Delta\) which implies the conditions,

\[m_\Delta = \frac{r_\Delta}{2}, \quad \delta(r_\Delta) = \delta_\Delta, \quad \phi(r_\Delta) = \phi_\Delta,\]

\(\text{(10)}\)

\[(\partial_r \phi)_\Delta = \frac{r_\Delta}{1 - 8\pi r_\Delta^2 V_\Delta}, \quad (\partial_r m)_\Delta = 4\pi r_\Delta^2 V_\Delta.\]

\(\text{(11)}\)

For SS we impose regularity at the origin of coordinates \(r = 0\),

\[m(0) = 0, \quad \delta(0) = \delta_0, \quad \phi(0) = \phi_0, \quad (\partial_r \phi)_0 = 0.\]

\(\text{(12)}\)

where \(\delta_0\) and \(\phi_0\) are to be found such as to obtain the desired asymptotic conditions. In addition to the regularity conditions, we impose asymptotically flat conditions on the spacetime for SHBH and SS:

\[m(\infty) = M_{\text{ADM}}, \quad \delta(\infty) = 0, \quad \phi(\infty) = \phi_\infty.\]

\(\text{(13)}\)
Above, the value \( \phi_\infty \) corresponds to the local minimum of \( V(\phi) \). \( M_{\text{ADM}} \) is the ADM mass associated with a SHBH or SS configuration. For a given theory, the family of SHBH configurations is parametrized by the free parameter \( A_\Delta \) which specifies the area of the black hole horizon. Therefore for SHBH, \( M_{\text{ADM}} = M_{\text{ADM}}(A_\Delta) \), or equivalently \( M_{\text{ADM}} = M_{\text{ADM}}(r_\Delta) \) since in our coordinates the horizon area \( A_\Delta = 4\pi r_\Delta^2 \). The value \( \phi_\Delta = \phi_\Delta(r_\Delta) \) is a shooting parameter rather than an arbitrary boundary value which is determined so that the asymptotic flat conditions are satisfied. On the other hand, for SS the value \( \phi_0 \) is the shooting parameter, and the corresponding configuration is characterized by a unique \( M_{\text{sol}}^{\text{ADM}} \).

Finally, the surface gravity of a spherically symmetric static black hole can be calculated from the general expression of spacetimes admitting a Killing horizon \([14]\):

\[
\kappa = \left[ -\frac{1}{4} \nabla^2 \alpha \right]^{1/2}_{r=r_\Delta},
\]

where \( \nabla^2 \) stands for the Laplacian operator associated with the stationary metric and \( \alpha = (\partial_t, \partial_t) \) is the norm of the time-like (static) Killing field which is null at the horizon. For the present case, \( \alpha = g_{tt} = -\left(1 - \frac{2m(r)}{r}\right)e^{2\delta(r)} \).

From the above formula, one can obtain the following useful expression

\[
\kappa = \lim_{r \to r_\Delta} \left( \frac{1}{2} \frac{\partial_r g_{tt}}{\sqrt{g_{tt}g_{rr}}} \right) .
\]

For the election of the parametrization of the metric \([8]\) we have

\[
\kappa(r_\Delta) = \frac{1}{2r_\Delta} e^{\delta(r_\Delta)} \left[ 1 - 2(\partial_r m) \right] .
\]

Introducing \([11]\) in \([16]\) we obtain the final expression for the surface gravity of the SHBH

\[
\kappa(r_\Delta) = \frac{1}{2r_\Delta} e^{\delta(r_\Delta)} \left[ 1 - 8\pi r_\Delta^2 V_\Delta \right] .
\]

In the next section, we shall analyze these solutions from the perspective of the IH formalism.

**IV. MASS FORMULAE**

Let us now turn to the straightforward application of the IH formalism mentioned in the Sec. II to the case of SHBH and SS in the Einstein-Higgs theory with action given by \([4]\) and \( V(\phi) \) by Eq. \([7]\) \([12]\). As in Ref. \([12]\), we shall take the specific values \( \eta_1 = 0.5, \eta_2 = 0.1 \) and \( a = 0 \); all the quantities (e.g., \( M_{\text{ADM}} \) and \( r_\Delta \)) have been rescaled as appropriate using \( 1/\sqrt{\lambda} \) as a length-unit.

The first consequence coming from the IH formalism is that the horizon mass associated to the SHBH takes the form \(^1\)

\[
M_\Delta(r_\Delta) = \frac{1}{2} \int_0^{r_\Delta} \beta(r) \, dr ,
\]

where \( \beta(r_\Delta) = 2r_\Delta \kappa(r_\Delta) \) [the value of \( \kappa(r_\Delta) \) is given by \([16]\)]. We have dropped the subindex \((u)\) in the expression \([15]\) because in the Einstein-Higgs system considered here there is only one branch of static spherically symmetric SHBH labelled by its horizon radius \( r_\Delta \) (the corresponding scalar configurations do not have nodes). Additionally, there is another branch of static spherically symmetric BH given by the family of Schwarzschild BH’s labelled by its corresponding horizon radius \( r_\Delta \) and with horizon mass

\[
M_\Delta^{\text{Schw}}(r_\Delta) = r_\Delta/2 .
\]

\(^1\) We remind the reader our choice of units \( G_0 = c = 1 \).
The ADM mass (solid lines), the horizon BH mass (dash-dotted lines), and the mass of the Schwarzschild solution (dashed lines) plotted as functions of $r_\Delta$ (first panel). The second panel depicts similar quantities using logarithmic scales to appreciate better their behavior for small $r_\Delta$. The soliton mass $M_{\text{sol}} \approx 3.827$.

FIG. 3:

$\beta(r_\Delta) = 2\kappa r_\Delta$ is plotted as function of $r_\Delta$. Note that it approaches asymptotically the value $\approx 1.24$. Here $\beta(0) \approx 0.324$.

The second consequence coming from the IH formalism is that on the entire branch of SHBH we can expect the following identity to be true

$$M_{\text{ADM}} = M_{\text{sol}} + M_\Delta,$$

where $M_{\text{sol}}$ is the ADM mass of the SS obtained taking the limit $r_\Delta \to 0$ of the branch of SHBH and $M_{\text{ADM}}$ is the ADM mass corresponding to the SHBH with horizon radius $r_\Delta$, and the horizon mass is given by $M_{\text{ADM}}$. Thus, in a similar way to the EYM theory, the total ADM Mass of the solution contains two contributions, one attributed to the horizon of the SHBH and the other to the outside ‘hair’, captured by the SS. We have performed numerical explorations for SHBH for a large range of values of the horizon radius (in normalized units) and have checked the
identity $\Delta$. We have found complete agreement within the numerical uncertainties. This can be seen in Fig. 2, where the identity was checked up to $r_\Delta = 250$.

Figure 3 depicts the behavior of $\beta(r_\Delta)$. Unlike the EYM model, where $\beta \approx 1$ for large $r_\Delta$, in this model $\beta \approx 1.24$ asymptotically.

Figure 4 shows an example of a BH solution with large $r_\Delta$ (for a small BH see fig. 2 of Ref. [12]).

![Graphs showing behavior of $\beta(r_\Delta)$ and mass function.](image)

**FIG. 4:**
Large-black-hole configuration constructed with $V(\phi)$ as given by Eq. 4 with parameters $\eta_1 = 0.5$, $\eta_2 = 0.1$, $a = 0$, and $r_\Delta = 150/\sqrt{\lambda}$, $\phi_\Delta \sim 0.26111$. The upper panels depict the scalar field and the mass function respectively. The latter converges to $M_{ADM} \sim 93.096/\sqrt{\lambda}$. The lower panels depict the metric potentials (the first is a zoom of the second): $\sqrt{-g_{tt}}$ (solid line), $\sqrt{g_{rr}}$ (dashed line), $e^\delta$ (dash-dotted line) and $\delta$ (dotted line).

**A. A physical model of SHBH**

The extrapolation of the model of a hairy black hole as a bound state of an ordinary, 'bare', black hole and a 'solitonic residue' (first applied successfully to the colored BH’s in the EYM theory) [5] does not apply directly to the SHBH because the straightforward generalization of the formula $M_{ADM} = M_{sol} + M_\Delta = M_\Delta^{schwarz} + M_{sol} + E_{bind}$, (21)

where $E_{bind} = M_\Delta - M_\Delta^{schwarz}$, to our case, has the problem that the binding energy changes sign, becoming positive for BH larger that $r_\Delta \approx 30$, and then increasing in absolute value as $r_\Delta$ gets larger. That is,

$$E_{bind} \sim B r_\Delta,$$  (22)
(where $B$ is a constant whose value depends on the specific model) for $r_\Delta \gg 1$, which is contrary to the expected feature of a negative binding energy as in the EYM case (i.e. the prediction (i) mentioned above will not be satisfied).

In order to appreciate the origin of the failure of this feature, let us recall that we can write $M_{\text{ADM}}$ in terms of the Schwarzschild mass and the mass of the ‘hair’ as,

$$M_{\text{ADM}}(r_\Delta) = M_\Delta^{\text{schwartz}}(r_\Delta) + M_{\text{hair}}(r_\Delta),$$  \hspace{1cm} (23)

where

$$M_{\text{hair}}(r_\Delta) = -\int_{r_\Delta}^\infty r^2 T_t^t \, dr,$$  \hspace{1cm} (24)

By rescaling the $r-$coordinate in terms of $r_\Delta$, we can rewrite the mass of the hair. It becomes then,

$$M_{\text{hair}}(r_\Delta) = -r_\Delta \int_1^\infty x^2 \tilde{T}_t^t \, dx,$$  \hspace{1cm} (25)

with

$$x = \frac{r}{r_\Delta},$$  \hspace{1cm} (26)

$$\tilde{T}_t^t = r_\Delta^2 T_t^t,$$  \hspace{1cm} (27)

$$T_t^t = -(1 - 2m(r)) \left[ \frac{\partial_r \phi}{r} \right]^2 + V(\phi).$$  \hspace{1cm} (28)

Now, for $x \gg 1$, the integral in Eq.(25) becomes almost independent of $r_\Delta$, and in fact the numerical analysis provides the following value

$$M_{\text{hair}}(r_\Delta \gg 1) \sim B r_\Delta.$$  \hspace{1cm} (29)

where $B \approx 0.12$. Now, since $M_\Delta^{\text{schwartz}}(r_\Delta) = r_\Delta/2$ we have then

$$M_{\text{ADM}}(r_\Delta \gg 1) \sim C r_\Delta.$$  \hspace{1cm} (30)

where now $C \approx 0.62$. Therefore we conjecture that $C$ is a constant that depends of the matter-theory involved. For the EYM case, one can easily show that the scaling properties of the hair contribution of the energy makes the equivalent of the integral of Eq.(25) to behave like $1/r_\Delta$ rather than $r_\Delta$. Therefore, for the EYM case, $C = 1/2$.

As we now show, this subtle difference in both theories makes that the binding energy expression used in the EYM cannot be used straightforwardly for the Einstein-Higgs theory analyzed here.

From (21) and (23) we find that

$$E_{\text{bind}}(r_\Delta \gg 1) \sim M_{\text{hair}}(r_\Delta \gg 1) - M_{\text{sol}} \sim B r_\Delta - M_{\text{sol}},$$  \hspace{1cm} (31)

then $E_{\text{bind}}$ scales as $r_\Delta$ when $r_\Delta \gg 1$ (remember that $M_{\text{sol}}$ is a constant) \cite{12}. It is clear that the sign of the binding energy as such defined changes sign and grows with the size of the black hole.

To deal with this situation we now proceed to propose a modification of the model of a hairy black hole as a bound state in order to adapt it to the more general case. Our proposal consists in redefining the binding energy as,

$$E_{\text{bind}}^{\text{new}}(r_\Delta) = E_{\text{bind}}(r_\Delta) - B r_\Delta = M_\Delta(r_\Delta) - M_\Delta^{\text{schwartz}}(r_\Delta) - B r_\Delta,$$  \hspace{1cm} (32)

By this procedure we have ‘renormalized’ the binding energy by subtracting the divergent term. That is, the new expression for the binding energy is

$$E_{\text{bind}}^{\text{new}}(r_\Delta) = M_\Delta(r_\Delta) - C r_\Delta.$$  \hspace{1cm} (33)

This definition shares now exactly the same properties as the original expression for the EYM. Thus, it vanishes at $r_\Delta = 0$, and decreases monotonically to the negative value $-M_{\text{sol}}$. It is interesting to note that this new definition reduces to the old one for the EYM case, since as we previously remarked $C_{\text{EYM}} = 1/2$.\[ ]
Both binding energies are plotted. The ‘old’ binding energy \( E_{\text{bind}}(r_{\Delta}) \) is shown to become positive and approaches asymptotically a straight line. The ‘new’ binding energy \( E_{\text{new}}^{\text{bind}}(r_{\Delta}) \) is also plotted (continuous line), showing the expected behavior.

The formula (21) can now be reformulated as,

\[
M_{\text{ADM}}(r_{\Delta}) = M_{\text{sol}} + M_{\Delta}(r_{\Delta}) = M_{\Delta}^{\text{schwarz}}(r_{\Delta}) + M_{\text{sol}} + E_{\text{bind}}^{\text{new}}(r_{\Delta}) + Br_{\Delta} = Cr_{\Delta} + M_{\text{sol}} + E_{\text{bin}}^{\text{new}}(r_{\Delta}),
\]

Thus, we would get the same structure for the ADM mass of the hairy black hole where now the mass of the ‘bare black hole’ would be equal to \( M^{0}_{\Delta} = (1/2 + B)r_{\Delta} = Cr_{\Delta} \). However it is not clear what the origin of this extra term \((Br_{\Delta})\) is, and we could very well have assigned it to the soliton mass to form a new ‘solitonic residue with mass \( M_{\text{sol}} + Br_{\Delta} \) (whose interpretation however seems somewhat obscure). We must explore more in order to decide which interpretation is best suited for our model. As we stressed, the constant \( C \) depends on the theory considered; for the EYM and EYMH theories, \( C = 1/2 \). It would be interesting to explore (in addition to the current analysis, see below) whether other theories admitting hair posses values of \( C \) different from 1/2.

To end this section, let us rewrite the form of the hairy mass in terms of the old binding energy, in order to understand the behavior of the scalar system. First, let us note using Eqs. (21) and (23) that

\[
M_{\text{hair}}(r_{\Delta}) = M_{\text{ADM}}(r_{\Delta}) - M_{\Delta}^{\text{schwarz}}(r_{\Delta}) = M_{\text{sol}} + E_{\text{bind}},
\]

so the binding energy is

\[
E_{\text{bind}} = M_{\text{hair}}(r_{\Delta}) - M_{\text{sol}} = - \left[ \int_{r_{\Delta}}^{\infty} r^2 T_{t}^{t} \, dr - \int_{0}^{\infty} r^2 T_{o}^{t} \, dr \right] ,
\]

where \( T_{o}^{t} \) is the stress-energy tensor of the solitonic regular solution. This equation can be rewritten as,

\[
E_{\text{bind}} = - \left[ \int_{r_{\Delta}}^{\infty} r^2 (T_{t}^{t} - T_{o}^{t} ) \, dr - \int_{0}^{r_{\Delta}} r^2 T_{o}^{t} \, dr \right] .
\]

Here we can identify the first term as the difference between the hair of the BH and the “hair” of the soliton. Of course we are comparing the quantities (the integrands) that live on different manifolds, but the total integral is well defined. What happens in the EYM case is that both stress tensors behave very much alike, for the exterior region \((r > r_{\Delta})\) and for large values of \( r_{\Delta} \), and thus the only term that contributes is the second one, that gives the ADM
mass of the soliton (recall that $r_\Delta \gg 1$, that is for BH’s much larger than the characteristic size of the soliton (of order one in this dimension-less units), so the integral captures most of the soliton mass). In the scalar field case, the fact that the binding energy is proportional to $r_\Delta$, for large black holes, is captured by the fact that the BH contribution to the first term in (37) is dominating. It would be interesting to explore this issue in other gravity-matter systems.

V. INSTABILITY AND FINAL STATE

The next question we want to consider has to do with the following situation. Consider the case where a hairy black hole of geometrical radius $r_\Delta$ is slightly perturbed and therefore it decays. The final state will be, one expects, a black hole that in its near horizon geometry resembles the Schwarzschild solution, with the scalar field taking the value where the potential has a local minima and vanishes. This means that in this process the “scalar charge” at the horizon, namely the value $\phi_\Delta$ must change. One can make an argument similar to the one in Ref. [5] to conclude that, in that situation, the horizon must grow in the process and therefore, the available energy to be radiated can not all be radiated to infinity; part of it must fall into the black hole. Let us now recall the estimate for the upper bound of the total energy to be radiated.

The first step is to assume that the process illustrated in Fig. 6 takes place. Then, we assume that in the initial surface there was an isolated horizon $\Delta_{\text{in}}$ and after the initial unstable configuration has decayed, with part of the energy falling through the horizon and the rest radiating away to infinity, we are left with a horizon $\Delta_{\text{fin}}$ of a hairless black hole (with $r_{\text{fin}} > r_{\text{in}}$). If we denote by $E_{I^+}$ the energy radiated to future null infinity $I^+$, and given that the ADM energy does not change in the process, we have

$$M_{\text{ADM}} = M_{\Delta}(r_{\text{in}}^\Delta) + M_{\text{sol}} = M_{\Delta}^{\text{schwarz}}(r_{\text{fin}}^\Delta) + E_{I^+},$$

which can be rewritten as,

$$M_{\text{sol}} + E_{\text{bind}}^{\text{in}} = M_{\Delta}^{\text{schwarz}}(r_{\text{fin}}^\Delta) - M_{\Delta}^{\text{schwarz}}(r_{\text{in}}^\Delta) + E_{I^+}.$$  

(39)

On the right-hand-side note that the first two terms can be identified with $\Delta M_{\Delta}^0$, namely the change in (bare) horizon mass, while the second term corresponds to the radiated energy. Thus, it is natural to identify the quantity on the left as the available energy $E_{\text{avail}}$ on the system. We can then write,

$$E_{\text{avail}} = M_{\text{ADM}} - M_{\Delta}^{\text{schwarz}}(r_{\text{in}}^\Delta) = M_{\text{sol}} + E_{\text{bind}}(r_{\text{in}}^\Delta) = M_{\text{sol}} + E_{\text{bind}}^{\text{new}}(r_{\text{in}}^\Delta) + B r_{\text{in}}^\Delta$$  

(40)

FIG. 6:
This figure illustrates a physical process where an initial configuration with an isolated horizon $\Delta_{\text{in}}$ is perturbed and the final state contains another isolated horizon $\Delta_{\text{fin}}$. 
There are several comments regarding this quantity. First, we note that there is a qualitative change in the behavior of \( E_{\text{avail}}(r_\Delta) \) as function of the initial horizon radius, as in the EYM case. Its functional dependence is very similar to the binding energy since they differ only by the soliton mass. In the EYM case the available energy was equal to the soliton mass when there was no initial black hole (there is no energy used in binding the BH), and decreases as the radius increases. For very large black holes, the available energy goes to zero. For the scalar case under consideration here, we have a different behavior. The available energy decreases for small black holes but starts to increase and grows linearly with \( r_\Delta \). The fact that in the EYM case the available energy went to zero for large BHs was interpreted as meaning that those black holes were 'less unstable'. This expectation was confirmed by the fact that the frequencies of the linear perturbations was decreasing with the radius of the initial BH \([8, 17]\). It is natural then to ask the same question for the scalar black holes. We have computed the frequencies of the (single) unstable mode \( \psi(t,r) = \chi(r)e^{\sigma t} \) present (where \( \sigma^2 \) turns to be always negative), as a function of the horizon radius and plotted it in Fig. 7 (where \( \psi(t,r) \) represents a linear perturbation of \( \phi(r) \); see Ref. [12] for the details).

![FIG. 7:](image)

The unstable-mode frequency \( \omega = \sqrt{-\sigma^2} \) of the perturbed BH and soliton \( (r_\Delta = 0) \) is plotted as a function of the horizon radius \( r_\Delta \). Note that the frequency decreases as the horizon radius increases.

As can be seen from the figure, the frequencies still decrease as the black holes become larger, which is the same behavior observed in the EYM case. It is convenient then to reconsider the meaning of 'less unstable'. In the scalar field case considered here, numerical investigations of the dynamical evolution of the soliton as initial state show that the system is unstable \([18]\). The dynamical evolution of the system depends on the sign of the initial perturbation on the extrinsic curvature. For one sign of the perturbation, the system collapses and forms a black hole with a final isolated horizon, while for the other sign the system expands as a domain wall and gets therefore radiated to infinity (for details see \([18]\)). One should then expect that the dynamical evolution of slightly perturbed hairy black holes will show a similar qualitative behavior. In that case, for one sign of the perturbation one might expect the situation considered before, namely that the scalar field collapses and the BH grows. For the other sign, one can imagine that there could be, in some situations, an expanding wall that radiates away while leaving a “naked black hole”. The pressing question in whether, in that case, this residue would be a Schwarzschild like or an AdS like black hole. This question arises since, for the soliton collapse in the case of the expanding wall, the region around the origin resembles an AdS spacetime with an effective negative cosmological constant generated by the (non-positive) potential. One might need in that case a new interpretation of the formalism.

It is our belief that one needs to clarify what the criteria should be for regarding the system as slightly unstable or very unstable, other than the frequency of its perturbations. This and a full clarification of the nature of the resulting bare black hole could be achieved whenever full numerical simulations of dynamical evolution staring from scalar hairy black holes become available.

Let us return our discussion to Eq. (40). The first thing to note is that due to the characteristic behavior of
these solutions, for horizons larger than \( r_\Delta \approx 30 \), the horizon mass of the hairy black hole becomes larger than the Schwarzschild horizon mass of the same radius. This is also the point at which the binding energy becomes positive. One can thus speculate that the black holes of this radius and larger will have more violent collapses with a larger fraction of the available energy radiated away. Finally, from Eq. (10) one could interpret that again, the term \((B r_\Delta)\) could be associated to the soliton mass to form a solitonic residue that, together with the new binding energy allows us to have the same qualitative features of the heuristic model of [5]. Again, a more detailed analysis will have to wait for the numerical investigations of the fully dynamical process.

VI. AN EMPIRICAL FORMULA

The non-linear behavior of the Einstein-Matter equations and the non-trivial relations between the masses and the horizon radius posses a challenge to obtain an analytical formula for \( M_{\text{ADM}}(r_\Delta) \). One should expect that there is in general no closed analytical formula for the masses of hairy BH.

We have discovered that the following empirical formula reproduces the qualitative and quantitative features of the numerical analysis

\[
M_{\text{emp}}(r_\Delta) = \sqrt{\left(M_{\text{sol}} + \frac{D}{2C}\right)^2 + C^2 r_\Delta^2 + D r_\Delta - \frac{D}{2C}} = \frac{M_{\text{sol}}}{M_{\text{sol}}} \left\{ \sqrt{1 + \frac{Cr_\Delta}{M_{\text{sol}}}} + F - 1 \right\}
\]

where

\[
D = \frac{2C \beta_0 M_{\text{sol}}}{2C - \beta_0}, \quad M_{\text{sol}} = \frac{\beta_0 M_{\text{sol}}}{2C - \beta_0}, \quad F = \left(\frac{2C}{\beta_0}\right)^2 - 1
\]

\( \beta_0 \) being the value of \( \beta \) at \( r_\Delta = 0 \).

The formula (41) has the following nice properties:

1. \( M_{\text{emp}}(0) = M_{\text{sol}} \).
2. \( M_{\text{emp}}(r_\Delta) \) is a monotonically increasing function of \( r_\Delta \).
3. For large \( r_\Delta \), \( M_{\text{emp}} \to C r_\Delta \).
4. The relative error between \( M_{\text{emp}} \) and the numerical one is less that 10%. These errors become very small for small and large \( r_\Delta \).
5. One can define an empirical binding energy by using \( E_{\text{bind}}(r_\Delta) = M_{\text{emp}} - M_{\text{sol}} - C r_\Delta \) [where the first two terms provide the empirical horizon mass; here we are using Eq. (33)]. This formula reproduces very well the numerical results.
6. One can then obtain a fit for \( \beta \) as follows

\[
\beta_{\text{emp}}(r_\Delta) = 2 \frac{dM_{\text{emp}}}{dr_\Delta} = \frac{2r_\Delta C^2 + D}{\sqrt{(M_{\text{sol}} + \frac{D}{2C})^2 + C^2 r_\Delta^2 + D r_\Delta}} = \frac{2C}{\beta_0} \left(1 + \frac{Cr_\Delta}{M_{\text{sol}}}\right) \sqrt{\left[1 + \frac{Cr_\Delta}{M_{\text{sol}}}\right]^2 + F}.
\]

This formula reproduces the qualitative shape of the numerical \( \beta \), such as its exact value at the origin, its monotonically increasing behavior and its asymptotic value \( 2C \) for large \( r_\Delta \).

7. For the Schwarzschild case \( (C = 1/2, M_{\text{sol}} = 0) \), one obtains the expected results: \( M_{\text{emp}}(r_\Delta) = r_\Delta/2 \), \( \beta_{\text{emp}}(r_\Delta) \equiv 1 \), \( E_{\text{bind}}(r_\Delta) \equiv 0 \).
Clearly by adding terms of the form $r^2_\Delta (1 < \theta < 2)$ inside the square root of Eq. (41) one could improve the fit between the numerical results and the analytical formula. In Figure 8 we compare between the empirical formula and the numerical values of the hairy scalar black holes, for the ADM mass, screened surface gravity $\beta$ and the binding energy.

The empirical formula can be used also for the EYM case with $C = 1/2$ and the corresponding values of $M_{\text{sol}}$ and $D$. Figure 8 compares the numerical values of the EYM $n = 1$-branch with those obtained from the empirical formula.

We conjecture that the empirical formula can work also for different $n$, by using their corresponding values $M_{\text{sol}}^n$, and $\beta^n_0$. Moreover, we also speculate that such a formula can hold for other theories admitting hair, such as in the Einstein-Skyrme and Einstein-sphaleron models. It remains to be investigated what are the values of $C$, $M_{\text{sol}}$ and $\beta_0$ for such cases.

Now, we can further use the first law of thermodynamics $\delta M = \kappa \delta A_\Delta/(8\pi)$ for $M_{\text{ADM}}^{\text{emp}}$, and obtain the following prediction

$$\kappa \left( M_{\text{ADM}}^{\text{emp}} + \frac{D}{2C} \right) = C^2 + \frac{D}{2r_\Delta}.$$ (44)

Since the properties 1)–7) show that the analytical results obtained from Eq. (41) work particularly well for large and small $r_\Delta$, the most reliable consequence of (44) is a remarkable simple relation between the surface gravity and the ADM mass for sufficiently large hairy black holes:

$$\kappa M_{\text{ADM}} \approx C^2.$$ (45)

Note that Eq. (45) is consistent for the Schwarzschild case ($C = 1/2, D = 0, M_{\text{ADM}} = r_\Delta/2$), where the identity $\kappa M_{\text{ADM}} \equiv 1/4$, holds exactly.

![FIG. 8:](image)

The three panels depict the ADM-mass, $\beta$, and binding energy $E_{\text{bind}}$, respectively, as a function of the horizon radius. The solid lines correspond to the values obtained from a numerical analysis and the dashed lines were obtained from the empirical formulae described in the main text. Note the good qualitative behavior of the empirical formulae. Remarkably good fits to the more precise numerical values are obtained for small and large $r_\Delta$. The values for the empirical formula are $C \approx 0.62$, $M_{\text{sol}} \approx 3.827$ and $\beta_0 \approx 0.324$.

We have performed a non-exhaustive analysis of the solutions with respect to variations of some of the parameters of the scalar potential Eq. (7). Notably, we have computed the effect of the variation of $\eta_2$ on the global quantities. It is to note that changing $\eta_2$ modifies the potential barrier between the global and the local minimum. In fact, the closer the value $\eta_2$ to $\eta_1/2$, the less negative is $V(a + \eta_1) = \lambda \eta_1^3 \left[ 2\eta_2 - \eta_1 \right]/12$, and therefore the potential approaches the conditions where the no-hair theorems apply. The details of the solutions then depend in a non-trivial fashion between the interplay of the negative global minimum (in order to avoid the applicability of the non-hair theorems) and the height of the potential barrier.

Figure 10 depicts different global quantities as a function of the horizon radius for five different values of $\eta_2$. The soliton mass ($r_\Delta = 0$) as well as the ADM and horizon masses (for large $r_\Delta$) tend to increase with $\eta_2$. Remarkably,
FIG. 9:
Same as Fig. 8, for the Einstein-Yang-Mills theory ($n = 1$ colored black holes). Here the values for the empirical formulae are $C = 1/2$, $M_{\text{sol}} \approx 0.828$ and $\beta_0 \approx 0.126$.

The empirical formulae continue to provide reasonable good results by changing the corresponding values of their parameters $\bar{P}_n := (M_{\text{sol}}, C, \beta_0)$, $n$. The quality of the fit to the numerical values can be appreciated by the dashed curves of Fig. 10 which were computed with the empirical formulae.

VII. DISCUSSION

Let us first summarize our results. By solving numerically Einstein’s equations for static solutions of a self-gravitating scalar field, we have analyzed the behavior of several spacetime quantities as functions of the black hole horizon radius. We have found that the ADM mass of the spacetimes exhibits two types of behavior: it is similar to other ‘hairy’ theories for small black holes, but its behavior changes dramatically for large black holes. In particular, the ADM mass of large BH scales not as $r_{\Delta}/2$ as in other theories (EYM, EYMH, etc), but the proportionality constant (with respect to horizon radius) takes a different value depending on the form of the potential ($C \approx 0.63$ for $\eta_2 = 0.1$). In this article we have analyzed the consequences of this fact for a model based on the isolated horizons formalism. In such a model, a hairy black hole is viewed as a bound state of a soliton (which we have) and a ‘bare black hole’. The binding energy is found to be negative in EYM and EYMH, but in our case, for large BH, the binding energy becomes positive and grows linearly with $r_{\Delta}$ \(^2\). This fact leads to several possibilities. We have seen that it is possible to modify the original model by ‘renormalizing’ the binding energy in such a way that the newly defined energy has the same qualitative behavior as in the EYM system. The price one has to pay is the need to reinterpret either a new ‘solitonic residue’, or a new bare black hole. As a first attempt towards giving a definite answer to this question, we analyzed the frequency of the unstable mode of the linearized perturbation, and found that the behavior is the same as in EYM. This suggests that the proper physical interpretation is still unclear and that further numerical dynamical investigations are needed to fully settle the question. In particular, the two different regimes of the theory might have some consequences in the dynamical evolution of slightly perturbed BHs, where one could conjecture a different qualitative behavior for small and large black holes, regarding the endpoint of evolution and the nature of the bare black hole to which the solution settles. We have also conjectured that the constant that fixed the proportionality between ADM mass and horizon radius for large BHs is a theory-dependent constant, which would in particular imply that axi-symmetric non-spherical BH solutions to the gravity-scalar field system would have the same asymptotic behavior, for each given potential. It would be worth studying other gravity-matter systems, such as non-minimally coupled scalars, to see whether they possess a different proportionality constant (work is in progress in these directions).

We have shown also that a very simple heuristic analytic formula captures the essential qualitative behavior of the ADM mass of the hairy scalar BH’s, specifically for small and large values of the horizon radius. We have conjectured

\[^2\] Recently, another system in 5 dimensions was shown to possess a positive binding energy as well \[^1\].
Panels 1-4 depict the ADM-mass, horizon mass, $\beta$ and the binding energy respectively as a function of $r_\Delta$. The solid lines were obtained from the numerical analysis while the dashed lines were computed using the empirical formulae. The lines are associated with the five different values used for $\eta_2 = 0.1, 0.11, 0.12, 0.13, 0.14$ with $\eta_1 = 0.5$ fixed. As seen from bottom to top (for large $r_\Delta$) the plots of panels 1-3 correspond to $\eta_2$ in increasing order (in panel 4 the order is reversed). The values of the parameters $\vec{P}_{\eta_2} = (M_{\text{sol}}, C, \beta_0)$ used in the empirical formulae are $\vec{P}_{0.1} \approx (3.82, 0.62, 0.32)$, $\vec{P}_{0.11} \approx (5.22, 0.66, 0.24)$, $\vec{P}_{0.12} \approx (7.47, 0.72, 0.17)$, $\vec{P}_{0.13} \approx (11.78, 0.82, 0.11)$, $\vec{P}_{0.14} \approx (23.51, 1.01, 0.06)$.

that such formula can also be useful for EYM and more general hairy black holes. It remains a theoretical challenge to fully understand the origin of such simple formula.

Perhaps the most important conclusion from the present work is the lesson that hairy black holes for different matter systems exhibit new, and sometimes, unexpected behavior. This also point out to the need of a proper and deeper understanding of the reason why the heuristic hairy black hole model works so well for the system that it does, and whether the phenomenological modifications we have proposed here stand the test of full numerical investigations.

Acknowledgments

We would like to thank A. Ashtekar and D. Sudarsky for discussions. This work was in part supported by grants DGAPA-UNAM IN122002, and IN119005. U.N. acknowledges partial support from SNI, and Grants No. 4.8 CIC-UMSNH, No. PROMEP PTC-61 and No. CONACYT 42949-F.

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The subsection (7.4) from the reference [11]; specifically, we take the equation (7.37):

$dm(r) = N\left[\frac{r^2}{2} + \sin^2 \chi \left(\frac{d\chi}{dr}\right)^2 + \left[r^2 + \frac{\sin^2 \chi}{2}\right] \frac{\sin^2 \chi}{r^2}\right] (46)$

where the metric for static and spherically symmetric configurations is:

$ds^2 = -\sigma^2(r) N(r) dt^2 + \frac{1}{N(r)} dr^2 + r^2 d\Omega^2 (47)$

with

$N(r) = 1 - 2m(r)/r (48)$

and the Skyrme field $\chi(r)$ depending on the dimensionless coordinate $r$; to continuation we define a new coordinate as $x = r/r_\Delta$ and after integration rewrite (46) as

$M_{ADM}(r_\Delta) = \frac{r_\Delta}{2} + M_{hair}(r_\Delta)$

$= \frac{r_\Delta}{2} + r_\Delta \int_1^\infty \left[\frac{N x^2}{2} \left(\frac{d\chi}{dx}\right)^2 + \sin^2 \chi\right] dx$

$+ \frac{1}{r_\Delta} \int_1^\infty \left[N \left(\frac{d\chi}{dx}\right)^2 + \frac{\sin^2 \chi}{2x^2}\right] (\sin^2 \chi) dx (49)$

The behavior for $\chi(x)$ can see from left panel in the fig. 14. of such reference. From equation (7.41) we see the asymptotic behavior for $\chi(x)$ as $x \to \infty$ (with a constant): $\chi \approx ax^{-2}$. Although this analysis shows that $M_{ADM}(r_\Delta)$ grows linearly with $r_\Delta$ for $r_\Delta \gg 1$, we can not apply the limit $r_\Delta \to \infty$ to (19) because the existence of Skyrme’s black holes is limited to configurations with horizon radius $r_\Delta \leq r_{\Delta}^{max}(\kappa)$ (here $\kappa = 4\pi G_0f^2$ is the coupling constant of the theory and $r_{\Delta}^{max}(\kappa)$ is a maximal value depending of $\kappa$) and $\kappa \leq \kappa_{bh}$. The value of $\kappa_{bh}$ is of the order of $0.0315$. At the other hand, the
existence of solitons is permitted for $\kappa \lesssim 0.0437$. In the Einstein-Higgs at hand, $r_\Delta$ seems to be limited by the precision of the shooting method. For large $r_\Delta$, the shooting parameter $\phi_\Delta$ approaches a below limit ($\approx 0.26$ for the model $\tilde{P}_{0,1}$ of fig. [10]). It is unknown if this limit for the SHBH configurations is a fundamental limit or not. It turns then interesting to construct a similar model based in the isolated-horizon formalism for the Einstein-Skyrme system to compare similarities and differences with the model presented here [21].

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