Input-to-State Stability of a Bilevel Proximal Gradient Descent Algorithm

Torbjørn Cunis*,** and Ilya Kolmanovsky *

* Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109, USA (e-mail:\{tcunis | illya\}@umich.edu)
** Institute of Flight Mechanics and Control, University of Stuttgart, 70569 Stuttgart, Germany (e-mail: tcunis@ifr.uni-stuttgart.de)

Abstract: This paper studies convergence properties of inexact iterative solution schemes for bilevel optimization problems. Bilevel optimization problems emerge in control-aware design optimization, where the system design parameters are optimized in the outer loop and a discrete-time control trajectory is optimized in the inner loop, but also arise in other domains including machine learning. In the paper an interconnection of proximal gradient algorithms is proposed to solve the inner loop and outer loop optimization problems in the setting of control-aware design optimization and robustness is analyzed from a control-theoretic perspective. By employing input-to-state stability arguments, conditions are derived that ensure convergence of the interconnected scheme to the optimal solution for a class of the bilevel optimization problem.

Keywords: Input-to-state stability, Optimal control theory, Proximal gradient descent, Stability of nonlinear systems, Time-distributed optimization.

1. INTRODUCTION

In bilevel optimization (see Colson et al., 2007, and references therein) the cost function and/or constraints of a high-level optimization problem are informed by the optimal value function of a lower-level optimization problem which, in turn, is parametrized by the decision variables of the higher level optimization problem.

Bilevel optimization has numerous applications. For instance, in control co-design (see, e.g., García-Sanz, 2019), plant and controller parameters need to be simultaneously optimized, while in control-aware design (such as described by Cunis et al., 2022), system design parameters are optimized in the outer loop subject to the existence of an admissible state and control trajectory determined through the inner-loop optimization. In those works, the problem is reformulated in such a way that the gradients of the value function of the inner loop optimization problem with respect to the parameters are employed for the outer loop optimization. Cunis et al. (2022) argued that from the perspective of the integration with existing gradient-based multi-disciplinary design optimization solvers, such an approach is preferable to simultaneous optimization with respect to design and controller parameters. In that work, such an approach is employed for optimizing wing shape parameters subject to glide slope following maneuver constraints at landing of a supersonic aircraft.

Bennett et al. (2008) and Franceschi et al. (2018) have discussed bilevel optimization problems in machine learning, where the outer-loop optimization is performed with respect to the hyperparameters of the machine learning solution. The implementation of robust reference governors (see, e.g., Garone et al., 2017, and references therein) involves outer loop optimization with respect to the reference command and the inner loop optimization with respect to the disturbance sequence aimed at maximizing constraint violation. Finally, leader-follower and Stackelberg games lend themselves naturally to bilevel programming (see, e.g., Basilico et al., 2017).

Typical for these bilevel optimization problems is that evaluating value function and and gradient of the inner-loop optimization is computationally expensive. In this paper, a prototypical problem in bilevel optimization which is typical of control-aware design is considered. Here, an optimal control problem is solved in the inner loop and a parameter optimization problem being solved in the outer loop subject to the solution of the inner loop. The aim is to demonstrate that the exact solution of the inner-loop optimization problem is not necessary for the overall convergence of the bilevel optimization. This has obvious advantages when the time to compute the solution and the available computing power are restricted.

More specifically, we focus on a class of bilevel optimization problems where a discrete-time linear quadratic optimal control problem with input constraints is solved in the inner loop and both the linear model and the quadratic cost function depend on the parameter of the outer-loop optimization problem. The optimal value function of the inner loop optimization problem, which is thus a function of the parameter, is to be minimized subject to additional constraints on the parameter. Within this setting, we propose a framework for the analysis of bilevel optimization based on inexact proximal gradient algorithms, where the number of iterations of the inner-loop optimizer remains fixed. Thus, the gradient of the inner-loop value function computed only approximately is used for a proximal
gradient descent with respect to the outer-loop decision variables. By exploiting the control-theoretic notion of input-to-state stability (ISS) and a small gain theorem in the analysis, we derive sufficient conditions for asymptotic stability and convergence of the overall bilevel optimization algorithm.

Notation $\mathbb{N}$ and $\mathbb{R}$ are the sets of natural and real numbers, respectively. The extended real numbers are $\mathbb{R} = \mathbb{R} \cup \{\infty\}$. A convex function $f : \mathbb{R}^n \to \mathbb{R}$ is proper if and only if its domain, $\{x \in \mathbb{R}^n | f(x) \leq \infty\}$, is nonempty. For some $m \in \mathbb{N}$, the set of positive semi-definite (resp., positive definite) matrices in $\mathbb{R}_m^{n \times m}$ is denoted by $\mathcal{S}_+^m$ (resp., $\mathcal{S}_+^m$). The Euclidean norm and inner product on $\mathbb{R}^n$ are denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$. For a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we write that $\alpha \in C$ if and only if $\alpha$ is continuous, positive definite, and strictly increasing; and that $\alpha \in C_{\infty}$ if and only if $\alpha \in C$ as well as $\alpha(r) \to \infty$ if $r \to \infty$. Moreover, $\mathcal{K}_{\mathcal{L}}$ is the class of continuous functions $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\beta(s, \cdot) \in C_{\infty}$ for all $s \geq 0$ and $\beta(r, \cdot)$ is decreasing with $\lim_{s \to \infty} \beta(r, s) = 0$ for all $r \geq 0$.

2. PRELIMINARIES

Given a proper convex and lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R}$, the proximity operator of $f$ maps $x$ to the unique solution of

$$\text{prox}_f(x) = \arg \min_{y \in \mathbb{R}^n} f(y) + \frac{1}{2}\|x - y\|^2$$

for all $x \in \mathbb{R}^n$. If $f$ is the indicator function of a convex set $C \subseteq \mathbb{R}^n$, the proximity operator of $f$ corresponds to a projection onto $C$.

We make extensive use of the fact that the proximity operator is nonexpansive as detailed in the next statement. Further properties can be found in the appendix.

Property 1. (Rockafellar, 1976). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper convex and lower semicontinuous function and take $\nu > 0$; the proximity operator satisfies

$$\|\text{prox}_{\nu f}(x_1) - \text{prox}_{\nu f}(x_2)\| \leq \|x_1 - x_2\|$$

for all $x_1, x_2 \in \mathbb{R}^n$.

2.1 Problem Statement

Consider a system represented by a parameter-dependent model of the form

$$x_{k+1} = A(p)x_k + B(p)u_k + c(p)$$

with initial condition $x_0 \in \mathbb{R}^n$, control input $u_k \in \mathbb{R}^m$, $k \in \mathbb{N}$, and parameter $p \in \mathbb{R}^l$; the functions $A : \mathbb{R}^l \to \mathbb{R}^{n \times n}$, $B : \mathbb{R}^l \to \mathbb{R}^{n \times m}$, and $c : \mathbb{R}^l \to \mathbb{R}^n$ are smooth.

We define the finite-horizon quadratic cost

$$J = \frac{1}{2}x_N^T S(p)x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T Q(p) x_k + u_k^T R(p) u_k)$$

where $N \in \mathbb{N}$ is the horizon, which is evaluated on the trajectories of (1) for a specified initial condition $x_0$. We assume that $R : \mathbb{R}^l \to \mathcal{S}_{+,+}^m$ and $Q, S : \mathbb{R}^l \to \mathcal{S}^+_n$ are smooth. Collecting $u = (u_0^T, \ldots, u_{N-1}^T)^T$, the state sequence $x = (x_0^T, \ldots, x_N^T)^T$ satisfies

$$x = \hat{A}(p)x_0 + \hat{B}(p)u + \hat{c}(p)$$

for some constant $x_0$, where $\hat{A}(p)$ and $\hat{B}(p)$ are matrix-valued and $\hat{c}(p)$ is vector valued; and the cost $J$ is a quadratic function in $u$ and smooth in $p$, viz.

$$J(u, p) = u^T H(p)u + g(p)^T u + c(p)$$

where $H(p)$ is matrix-valued, $g(p)$ is vector-valued, and $c(p)$ is scalar; all functions are smooth in $p$. Note that, under our assumptions, $H(p)$ is positive-definite.

Suppose $P : \mathbb{R}^l \to \mathbb{R}$ and $U : \mathbb{R}^{Nm} \to \mathbb{R}$ are proper convex and lower semicontinuous penalty functions for $p$ and $u$, respectively, with compact domains $\mathcal{P}$ and $\mathcal{U}$. Given some $p \in \mathbb{R}^l$, the cost function $J(p, u) + U(u)$ is strongly convex in $u$ with optimal value $\hat{J} : p \to \min_u J(u, p) + U(u)$ and thus has a unique optimal solution $\hat{u}(p)$. We consider the following optimization problem.

Problem 2. Minimize $\hat{J}(p) + P(p)$ for $p \in \mathbb{R}^l$.

We adopt the bilevel proximal gradient descent algorithm

$$p_{\ell+1}^{f+1} = \text{prox}_P \left[ p^{f+1} - \nu \nabla_p J(u^{f+1}, p^{f+1})^T \right]$$

$$u_{\ell+1}^{f+1} = \text{Arg min}_u J(\cdot, u^{f+1}) + U(\cdot)$$

with $\kappa, \ell \in \mathbb{N}$ and $\nu > 0$, where $\text{Arg min}_u$ denotes a $\kappa$-step suboptimal solution to the parametrized optimal control problem and $\nabla_p J$ denotes the vector of estimated partial derivatives of $J$ at $p^{f+1}$ based on the suboptimal solution $u^{f+1}$.

Without state constraints, the optimal control solution can be approximated by the inner-loop finite-horizon proximal gradient descent iteration

$$u_{\ell+1} = \text{prox}_U \left[ u_{\ell}^{f} - \mu \nabla_u J(u_{\ell}^{f}, p^{f})^T \right]$$

$$u_{\ell+1} = u_{\ell}^{f}$$

for all $k \in \{0, \ldots, \kappa - 1\}$ and with $\mu > 0$.

Denote by $\mathcal{P} \subseteq \mathcal{P}$ the set of minima of $J + P$. We aim to prove a sufficient condition on $\mu$ and $\nu$ as well as $\kappa$ such that

$$p^f \to \mathcal{P} \quad \text{and} \quad u_{\ell}^f \to \hat{u}(p^f)$$

as $\ell \to \infty$.

The dynamics in (4) can be viewed as an interconnection of discrete-time systems, where the state of one system - here, the parameter $p$ and current solution $u$ - enters as disturbance into the other - through the estimated partial derivative vector and parametrized optimization, respectively. To describe a stable behaviour of the interconnection, we make use of the notion of input-to-state stability (ISS) firstly introduced by Sonntag (1995). As $\mathcal{P}$ may consist of more than one element, we rely on a since developed generalization called $\omega$ISS.

2.2 Size functions

Let $X \subseteq \mathbb{R}^n$ be an open set and $A \subseteq X$ be compact. A measurement function (on $X$) is a continuous function $\omega : X \to \mathbb{R}_{\geq 0}$ that is positive semidefinite.

Definition 3. A measurement function $\omega$ on $X$ is a size function for $A$ if and only if $\omega$ vanishes exactly on $A$ as well as $\omega(\xi_k) \to \infty$ for any sequence $\{\xi_k\}_{k \geq 0} \subseteq X$, if either $\xi_k \to \partial \Omega$ or $|\xi_k| \to \infty$. 
Size functions have also been called proper indicators (Kellett and Dower, 2012; Noroozi et al., 2018). If $X$ is the full space, then the distance mapping

$$\text{dist}_A(x) \overset{\text{def}}{=} \min_{\xi \in A} |x - \xi|$$

is a simple example for a size function. The following result allows us to compare size functions and to establish a notion of equivalence.

**Lemma 4.** (Sontag, 2022, Corollary 2.7). Let $\omega : X \to \mathbb{R}_{\geq 0}$ be a size function for $A$; then any measurement function $\omega_2$ on $X$ is a size function (for $A$) if and only if there exists $\alpha_1, \alpha_2 \in K_{\infty}$ satisfying

$$\alpha_1(\omega_1(x)) \leq \omega_2(x) \leq \alpha_2(\omega_1(x))$$

for all $x \in X$. \hfill <$

### 2.3 $\omega$-Input-to-State Stability

We consider the nonlinear, nonautonomous system on $\omega_{2.3}$.

**Definition 5.** Let $\omega : X \to \mathbb{R}_{\geq 0}$ be a size function for $A$; then any measurement function $\omega_2$ on $X$ is a size function (for $A$) if and only if there exists $\alpha_1, \alpha_2 \in K_{\infty}$ satisfying

$$\alpha_1(\omega_1(x)) \leq \omega_2(x) \leq \alpha_2(\omega_1(x))$$

for all $x \in X$. \hfill <$

If $\omega$ is a size function, then Definition 5 reduces to the classical definition of input-to-state stability (for $A$ on $X$) as introduced by Sontag and Wang (1995). However, it also generalizes notions such as state-independent input-to-output stability (Kellett and Dower, 2012). As for classical input-to-state stability, system (4) is $\omega$-input-to-state stable if and only if there exists $\alpha_1(\omega_1(x)) \leq \omega_2(x) \leq \alpha_2(\omega_1(x))$ (7a)

$$V(f(x, u)) \leq \max\{\alpha_3(V(x)), \gamma(|u|)\}$$

(7b)

for all $x \in X$ and $u \in \mathbb{R}^m$ (Noroozi et al., 2018, Theorem 7).

**Remark 6.** If $\omega$ is a size function, the second condition for an ISS Lyapunov function can equivalently be written as an implication

$$\omega(x) \geq \chi(|u|) \Rightarrow V(f(x, u)) - V(x) \leq -\rho(V(x))$$

where $\rho$ is a positive definite function and $\chi \in K$. In particular, the ISS gain is

$$\gamma(r) = \max\{V(f(x, u)) \mid V(x) \leq \chi(r), |u| \leq r\}$$

for all $r \geq 0$. \hfill <$

We now consider an interconnection of $\phi \in \mathbb{N}$ nonlinear systems

$$x_{i(k+1)} = f_i(x_{1k}, \ldots, x_{Rk}, u_{ik})$$

(Σi)

with $x = (x_1, \ldots, x_p) \in X$ and $u_i \in \mathbb{R}^m$, where $f_i : X \times \mathbb{R}^m \to X_i$ are continuous functions, for all $i \in I = \{1, \ldots, \varphi\}$. Let $\|\cdot\|$ be a monotonic norm on $\mathbb{R}^n$.

**Theorem 7.** (Noroozi et al., 2018, Theorem 10). Suppose, for any $i \in I$, that $\omega_i$ is a measurement function on $X_i$ and the subsystem $(\Sigma_i)$ has an $\omega_i$-ISS Lyapunov function $W_i : X_i \to \mathbb{R}_{\geq 0}$ and gains $\gamma_{1i}, \ldots, \gamma_{pi}, \gamma_{ui} \in K \cup \{0\}$, such that $\gamma_{ui} < 1$ and

$$W_i(f_i(x, u_i)) \leq \max\{\gamma_{1i}(W_i(x_j)), \gamma_{ui}(|u_i|)\}$$

for all $(x, u_i) \in X \times \mathbb{R}^m$; define

$$\omega : (x_1, \ldots, x_p) \to \left\|\omega_1(x_1), \ldots, \omega_p(x_p)\right\|$$

and gains $\gamma_{1i}, \ldots, \gamma_{pi}, \gamma_{ui} < 1$ for any $r \in \mathbb{N}$ and $\{i_1, \ldots, i_r\} \subseteq I$, then the interconnection $(\Sigma_1, \ldots, \Sigma_p)$ is $\omega$-input-to-state stable. \hfill <$

### 2.4 Sensitivity of Optimal Control

Take any $p \in P$. The minimizer $\hat{u}(p)$ is the unique solution of the generalized equation (Dontchev, 2021)

$$L_p(u) \overset{\text{def}}{=} \mathrm{grad}_u J(u, p) + \partial U(u) \ni 0, \quad u \in U,$$

where $\partial U(u)$ is the subdifferential of $U$ at $u$. The inverse of the set-valued mapping $L_p$ is

$$L_p^{-1} : \delta \ni u \in U \ni \delta \in L_p(u)$$

By strong convexity, $u \in L_p^{-1}(\delta)$ if and only if $u_4$ is the (unique) solution of the perturbed optimization problem

$$\min J(u, p) + U(u) - \delta \nabla u$$

and the mapping $\delta \ni u_5$ is Lipschitz continuous (Dontchev, 2021, Theorem 11.1). In other words, $L_p$ is strongly regular$^1$ at $u_4$ for $0$ and hence, the solution map $u^\ast(\cdot)$ is Lipschitz continuous around $p$ (Dontchev, 2021, Theorem 8.5). Moreover, we have that

$$\nabla_p J(p) = \nabla_p J(u, p)_{u = u_4(p)}$$

(9)

by virtue of Toan (2021, Theorem 3.1).

### 3. STABILITY ANALYSIS

We study the dynamics of an interconnection of finite-step proximal gradient descent algorithms and derive sufficient conditions for (4) to be asymptotically stable with respect to $P_r$. To that extend, we show that both the outer and inner optimization are $\omega$-input-to-state stable with respect to appropriate measurement functions and prove that, assuming the sufficient condition of Theorem 7 is satisfied, $\omega$ is a size function for $P_r$. Moreover, we will argue that the small-gain condition can always be met if the number of inner iterations $k$ is sufficiently large.

Recall that, under moderate assumptions for the cost function, the proximal gradient descent algorithm asymptotically converges to a fixed point if the step size is chosen to be smaller than or equal to the inverse of the Lipschitz constant of the gradient.

**Assumption 8.** The gradients $\nabla_p J(\cdot)$ and $\nabla_u J(\cdot, p)$ are Lipschitz continuous (for all $p \in P$) with constants $\nu^{-1}$ and $\mu^{-1}$, respectively, or larger.

Since $\nabla_p J(u, \cdot)$ is smooth, $u^\ast(\cdot)$ is locally Lipschitz, and $P$ is compact, as well as $\nabla_u J(\cdot, p)$ being linear in $u$, Lipschitz continuity of the gradients follows a fortiori. Under Assumption 8, the solution $(u_k)_{k \geq 0}$ of (5) for a fixed $p \in P$ converges to $u^\ast(p)$ if $k \to \infty$.\hfill <$

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$^1$ A set-valued map $L : \xi \ni \zeta$ is said to be strongly regular at $\zeta_0$ for $\zeta_0 \in L(\xi_0)$ if and only if $L^{-1}(\zeta)$ has a single value $\xi$ close to $\zeta_0$ for all $\zeta$ around $\zeta_0$. \hfill <$
We propose to estimate the gradient of $\bar{J}$ by the partial derivative

$$\nabla_p J(u, p) = \nabla_p J(u, p)$$ (10)

The interconnection now deviates from the nominal in two points: The parameter of the inner iteration changes during the optimization and the gradient for the outer iteration is perturbed by the error between $u_\epsilon^*$ and $u(p^\epsilon)$.

### 3.1 Parametrized Gradient Descent

Input-to-state stability of parametrized optimization algorithms has previously been studied, e.g., by Liao-McPherson et al. (2020). That work considered the change of parameter and optimization error for input and state, respectively. We are going to generalize the result to obtain $\omega$-input-to-state stability for the proximal gradient descent. We start by rewriting the $\kappa$-step update by (5) to

$$u^{\kappa+1} = g_\kappa(u^\kappa, \tilde{u}, \Delta p^\kappa)$$ (11a)

$$\tilde{p}^{\kappa+1} = \tilde{p}^\kappa + \Delta p^\kappa$$ (11b)

where $g_\kappa : \mathbb{R}^{Nm} \times \mathbb{R}^l \rightarrow \mathbb{R}^{Nm}$ is obtained by repeating the proximal gradient descent (5) for $\kappa$ times with fixed parameter $p^\kappa = \tilde{p}^\kappa + \Delta p^\kappa$. Here, the additional state $\tilde{p} \in \mathbb{R}^l$ represents the previous parameter, that is, $\tilde{p}_0 = p_{\kappa-1}$ for all $\ell > 0$. We introduce the measurement function

$$\omega_u : (u, \tilde{p}) \mapsto \|u - \bar{u}(\tilde{p})\|_2$$

for the augmented state $(u, \tilde{p})$ of (11). Since the proximity operator is nonexpansive (Property 1), any solution $\{u^\kappa\}_{\kappa \geq 0}$ of (5) satisfies the contraction property

$$\omega_u(u^\kappa, p^\kappa) \leq \eta^\kappa \omega_u(u_0, p^\epsilon)$$ (12)

for any $k \geq 0$, where the convergence rate $\eta \in (0, 1)$ follows from strong convexity of $J(\cdot, p)$ and Assumption 8. We obtain the following result.

**Lemma 9.** Let $u \in \mathbb{R}^{Nm}$ and $\tilde{p}, \Delta p \in \mathbb{R}^l$; the dynamics in (11) satisfy the dissipation-type inequality

$$\omega_u(u_0, \tilde{p} + \Delta p) \leq \eta^\kappa \omega_u(u_0, \tilde{p}) + \lambda_\kappa \eta^\kappa |\Delta p|$$

if $u_0 = g_0(u_0, \tilde{p}, \Delta p)$ and $\tilde{p}_0 = \tilde{p} + \Delta p$, where $\lambda_\kappa \geq 0$ is the Lipschitz constant of the optimal solution $\bar{u}(\cdot)$ on $P$.

**Proof.** From the triangle inequality, we obtain

$$\omega_u(u_0, \tilde{p} + \Delta p) = \|u - \bar{u}(\tilde{p} + \Delta p)\|_2$$

$$\leq \|u - \bar{u}(\tilde{p})\|_2 + \|\bar{u}(\tilde{p} + \Delta p) - \bar{u}(\tilde{p})\|_2$$

$$\leq \omega_u(u_0, \tilde{p}) + \lambda_\kappa |\Delta p|$$ (13)

and combining (12) and (13) yields the desired result. □

It remains to prove that $\omega_u$ is an $\omega_u$ISS Lyapunov function.

**Proposition 10.** The $\kappa$-step update of the proximal gradient descent, as given in (11), is $\omega_u$-input-to-state stable with gain $\gamma_u$; and $\gamma_u \to 0$ uniformly as $\kappa \to \infty$.

**Proof.** Let $u_0 = g_0(u_0, \tilde{p}, \Delta p)$ and $\tilde{p}_0 = \tilde{p} + \Delta p$ for some $u_0 \in \mathbb{R}^{Nm}$ and $\tilde{p}, \Delta p \in \mathbb{R}^l$ and choose $\epsilon \in (0, 1)$; by virtue of Lemma 9, we have that

$$\omega_u(u_0, \tilde{p}_0) \leq (1 - \epsilon)\omega_u(u_0, \tilde{p}) \equiv \alpha(\omega_u(u_0, \tilde{p}))$$

if $(1 - \eta^\kappa - \epsilon)\omega_u(u_0, \tilde{p}) \geq \lambda_\kappa \eta^\kappa |\Delta p|$; and

$$\omega_u(u_0, \tilde{p}_0) < \lambda_\kappa \frac{\eta^\kappa + 1}{1 - \eta^\kappa - \epsilon} \eta^\kappa |\Delta p| \equiv \gamma_u(|\Delta p|)$$

otherwise. If $\epsilon$ is sufficiently small, we have that $\alpha, \gamma_u \in K_\infty$ as well as

$$\omega_u(u_0, \tilde{p}_0) \leq \max\{\alpha(\omega_u(u_0, \tilde{p})), \gamma_u(|\Delta p|)\}$$

proving that $\omega_u$ is an $\omega_u$ISS Lyapunov function. Moreover, $\gamma_u$ is linear with $\gamma_u \to 0$ as $\kappa \to \infty$ by l'Hôpital’s rule. □

In other words, (11) is input-to-output stable, with input $\Delta p$ and output $u$, independently of the state $\tilde{p}$ — that is, the actual parameter $p$.

**Remark 11.** $\omega_u$-input-to-state stability of (11) can also be viewed as parametrized input-to-state stability of the original proximal gradient descent algorithm. In contrast, Liao-McPherson et al. (2020) proved input-to-state stability with both $p$ and $\Delta p$ as inputs (although the gain on $p$ was zero).

### 3.2 Perturbed Gradient Descent

In recent work, Sonntag (2022) established input-to-state stability of a perturbed steepest descent algorithm with respect to the set of minimizers $P_\star$, under the assumptions that the cost function and norm of the gradient are size functions for $P_\star$. Convergence of proximal gradient schemes under perturbed gradients was studied in earlier works (Lemaitre, 1988; Tossings, 1994) and more recently by Atchadé et al. (2017).

Building upon those previous works, we prove input-to-state stability of the proximal gradient descent scheme with perturbed gradient, as given in (4a). We start with the simplified iteration

$$p^{\epsilon+1} = T_\nu(p^\epsilon, d) \overset{\text{def}}{=} \text{prox}_p [p^\epsilon - \nu(\nabla J(p^\epsilon)^T + d)]$$ (14)

where $d \in \mathbb{R}^l$ is a perturbation for the gradient. Adopting from Sonntag (2022), we make the following assumptions.

**Assumption 12.** The cost function $J$ satisfies:

- the function $J_\star : p \mapsto (J + P)(p) - j^\star$ is a size function for $P_\star$, where $j^\star$ is the global minimum of $J + P$;
- the function $\Lambda : p \mapsto |T_\nu(p, 0) - p|$ is a size function for $P_\star$.

Since $P$ has a compact domain, $J_\star$ is a size function if and only if $J + P$ is positive definite with respect to $P_\star$. Moreover, the assumption on $\Lambda$ is equivalent to the necessary conditions for optimality being sufficient, too. If $P \equiv 0$, as for gradient descent, $\Lambda(p)$ corresponds to $|\nabla J(p)|$ for all $p \in \mathbb{R}^l$.

**Theorem 13.** Let $\omega_u : \mathbb{R}^l \rightarrow \mathbb{R}_\geq 0$ be a size function for $P_\star$; the dynamics in (14) are $\omega_u$-input-to-state stable.

**Proof.** Take $p, d \in \mathbb{R}^l$; since $T_\nu(p, d) \in P$, we have that

$$(J + P)(T_\nu(p, d)) - (J + P)(p) \leq -2\nu^{-1}|T_\nu(p, d) - p| - \alpha^{-1}(T_\nu(p, d) - p, -\nu d)$$ (15a)

by Lemma 17. Rewriting $T_\nu(p, d) - p$ and applying the triangle inequality to both sides we obtain

$$|\Lambda(p) - \nu|d| \leq |T_\nu(p, d) - p| \leq \Lambda(p) + |\nu|d|$$

and thus, (15a) yields

$$J_\star(T_\nu(p, d)) - J_\star(p) \leq -(2\nu)^{-1}\Lambda(p)^2 + 2\Lambda(p)|d| + |\nu|d|^2$$ (15b)
where rewriting the left-hand side is trivial. We now argue with Sonntag (2022, Proof of Theorem 4) that, since $J_*$ and $\Lambda$ both are size functions for $P_*$, there exists $k_\infty$ functions $b_1$ and $b_2$ satisfying

$$b_1 J_*(p) \leq \Lambda(p) \leq b_2 J_*(p)$$

for all $p \in \mathbb{R}^l$, where linearity of $b_1$ and $b_2$ can be assumed without loss of generality for $P$ is compact.

Inserting the inequalities into (15b), we have that, for all $p, d \in \mathbb{R}^l$,

$$J_*(T_p(p, d)) - J_*(p) \leq -c J_*(p)^2 + 2b_2 J_*(p) |d| + \nu |d|^2$$

(15c)

for some $c > 0$ satisfying $c < b_2^2(2\nu)^{-1}$. Choose $\chi > 0$ such that, for all $p, d \in \mathbb{R}^l$,

$$J_*(T_p(p, d)) \leq J_*(p) - \nu J_*(p)^2$$

if $J_*(p) \geq \chi |d|$, where $\nu > 0$; and otherwise,

$$J_*(T_p(p, d)) \leq \chi |d| + \chi_0 |d|^2$$

with $\chi_0 > 0$. Hence, as $J_* \circ T_p$ is upper-bounded, there exists $\alpha \in (0, 1)$ and $\gamma > 0$ satisfying

$$J_*(T_p(p, d)) \leq \max\{\alpha J_*(p, \gamma |d|)$$

for all $p, d \in \mathbb{R}^l$, proving that $J_*$ is an $\omega_\infty$ISS-Lyapunov function for (14). This concludes the proof. □

It rests to prove input-to-state stability if the gradient perturbation is caused by the estimation error of (10). From (9), we have that the $i$-th component of the gradient error vector satisfies

$$\left| \left[ \nabla_p J_i(p) \right] \right| (p, u) \leq \left| (u(p) - u)^T \nabla_p J_i(p) \right| (u(p) - u) + \gamma |d| (u(p) - u)$$

(16)

where $\nabla_p J_i$ denotes the element-wise partial derivative with respect to the $i$-th component of $p \in \mathbb{R}^l$. Since $U$ is compact, there exists $F > 0$ such that

$$\left| \left[ \nabla_p J_i(p) \right] \right| (p, u) \leq F |u(p) - u|$$

for all $(p, u) \in P \times U$. We conclude with the following corollary.

Corollary 14. The dynamics in (4a) are $\omega_\infty$-input-to-state stable with respect to the input $u(p) - u$ for any size function $\omega_p$ for $P_*$ on $\mathbb{R}^l$. <

3.3 Interconnected Gradient Descent

We are going to prove asymptotic stability of the bilevel proximal gradient scheme by application of the small-gain theorem.

Suppose $\{(p^l, u^l)\}_{l \geq 0} \subset P \times U$ is a solution to the interconnection of (4) and (5); for any $\ell > 0$, take $\tilde{p}_{\ell+1} = p^\ell$ and $\Delta p = p^\ell - \tilde{p}_{\ell+1}$ and recall that

$$\omega_\infty(u^{\ell} + \tilde{p}_{\ell+1}) \leq \max \{\alpha u, \omega_\infty(u_i, \tilde{p}_i), \gamma |\Delta p_i| \}$$

(17a)

$$J_*(p^{\ell+1}) \leq \max \{\alpha o J_*(p^\ell), \gamma F |\omega_\infty(u^\ell, \tilde{p}_i)\}$$

(17b)

where $\alpha_0, \alpha_\in (0, 1), \gamma, \gamma_i, F > 0$, and $\omega_\infty(p, u) \equiv |u(p) - u|$ have been obtained beforehand, as well as

$$|\Delta p_i| \leq \Lambda(\tilde{p}_i) + \nu F |\omega_\infty(u^\ell, \tilde{p}_i)$$

(17c)

by Lemma 18.

Proposition 15. There exists $K \in \mathbb{N}$ such that, for all $\kappa \geq K$, the interconnection of (4) and (5) is asymptotically stable with respect to the set

$$\Omega = \{(u(p), p) | p \in P_*\}$$

and $|\Delta p_i| \to 0$ as $\ell \to \infty$.

Proof. Take $b > 0$ such that $\Lambda(p) \leq b J_*(p)$ for all $p \in \mathbb{R}^l$; from (17) and Lemma 9 we obtain

$$|\Delta p_{\ell+1}| \leq \Lambda(\tilde{p}_{\ell+1}) + \nu F |\omega_\infty(u^{\ell+1}, \tilde{p}_{\ell+1})$$

\[ \leq \nu F \alpha o |\Delta p_{\ell}| + \nu F \alpha |\omega_\infty(u^\ell, \tilde{p}_i) + F \max \{1 + b \nu |F (b \gamma_i)^{-1}\} \omega_\infty(p, d) \]

\[ \leq \nu F \alpha |\Delta p_{\ell}| + \max \{1 + b \nu F (b \gamma_i)^{-1}\} \omega_\infty(p, d) \]

If $K$ is sufficiently large, then $F \alpha \nu F (b \gamma_i)^{-1} \omega_\infty(p, d)$.

Combining (17) with (18) yields an interconnection of three $\omega_I$ISS-Lyapunov functions for $u_\in P_*$, $\tilde{p}_i$, and $\Delta p$. Clearly, if $K$ is sufficiently large, then

$$\gamma_2 \cdot \gamma_1 < 1$$

\[ \gamma_1 \cdot \gamma_2 \cdot \gamma_1 < 1 \]

and by virtue of Theorem 7, the interconnection of (4) and (5) is $\omega_I$-input-to-state stable, where

$$\hat{\omega}((u, \tilde{p}, \Delta p)) = \left| (\omega_\infty(p), \omega_\infty(u, \tilde{p}), |\Delta p|) \right|$$

for all $\kappa \geq K$. We claim that $\hat{\omega}$ is a size function for $P \times U$:

Take $\tilde{p}(u, \Delta p) \in \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R}^l$, then $\hat{\omega}(\tilde{p}(u, \Delta p)) = 0$ if and only if $\tilde{p} \in P_*$, $u = u(\tilde{p})$, and $|\Delta p| = 0$. Suppose $P_\in \mathbb{R}^l$ is compact; then, for any $(p, u) \in P \times U$,

$$\omega_\infty(u, \tilde{p}) \geq |u| - |u(p)| \geq |u| - |u|$$



\[ \omega_\infty(u, \tilde{p}) \geq |u| - |u(p)| \geq |u| - |u| \]

where $\gamma_1 = \max\{\gamma_2, \gamma_1 \} \cdot |\Delta p| < \infty$ by Lipschitz continuity of $u(\cdot)$ on $P$. Hence, since $\omega_p$ is a size function and $\| \| \cdot \|$ is monotone, $\hat{\omega}(\tilde{p}, u, \Delta p) \to \infty$ as either $|p| \to \infty$, $|u| \to \infty$, or $|\Delta p| \to \infty$.

We conclude that $(\tilde{p}, u) \to \Omega$ and $|\Delta p_{\ell}| \to 0$ as $\ell \to \infty$. Noting that $|\Delta p_{\ell}| = p^\ell - \tilde{p}_{\ell}$, this is the desired result. □

4. SPECIAL CASE

We have shown that the bilevel optimization scheme converges asymptotically to the optimal solution even if the inner-loop problem is solved inexactly. However, specific conditions for the optimization problem of (1) and (2) to satisfy Assumption 12 remain an open research question. In order to illustrate our theoretic results we consider the case that the parameters only enters through an additive linear term, viz.

$$x_{k+1} = Ax_k + Bu_k + Ep$$

(19)

and the quadratic cost is constant in the parameter. In this scenario, the parameter may correspond to the initial condition of the optimal control problem or an additional input that is constant over the control horizon. The inner-loop cost function now simplifies to

$$J(u, p) = (u(p) + g(u, p) + c$$

where the block matrix $H$ is positive semidefinite by positive definiteness of $Q$ and $R$. Hence, the cost $J$ is convex in $(u, p)$ and, since $U$ is convex, $J(p) = \min_{u} J(u, p) + U(u)$ is convex too. Consequently, $(\mathcal{P}, \mathcal{R})$ is convex and the necessary conditions for optimality are sufficient, that is, $\Lambda(p) = 0$ if and only if $p \in P_*$, Assumption 12 thus holds since
5. CONCLUSION

We have shown that convergence conditions for a class of bilevel optimization problems with inexactly implemented proximal gradient optimization can be derived using input-to-state stability arguments. Future work will focus on broadening the class of problems and optimization algorithms for which similar approaches can be followed as well as on applications to real world problems motivating this work.

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APPENDIX

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper convex and lower semicontinuous function; assume that the function $g : \mathbb{R}^n \to \mathbb{R}$ is differentiable and that its gradient $\nabla g$ is Lipschitz continuous on the domain of $f$ with constant $\lambda \geq 0$. Define

$$h(\xi) = f(\xi) + g(\xi)$$

for all $\xi \in \mathbb{R}^n$. The following results are based on Atchadé et al. (2017, Lemmas 8–10).

**Lemma 16.** Take $\nu > 0$; then

$$f(\text{prox}_{\nu f}(\xi)) - f(\xi') \leq -\nu^{-1}(\text{prox}_{\nu f}(\xi) - \xi', \text{prox}_{\nu f}(\xi) - \xi)$$

for all $\xi, \xi' \in \mathbb{R}^n$. <

**Lemma 17.** Let $\nu > 0$ satisfy $\nu \leq \lambda^{-1}$; then

$$h(\text{prox}_{\nu f}(\xi)) - h(\xi) \leq -(2\nu)^{-1}\| \text{prox}_{\nu f}(\xi) - \xi \|^2 - \nu^{-1}(\text{prox}_{\nu f}(\xi) - \xi, \xi' - \nu \nabla f(\xi) - \xi')$$

for all $\xi, \xi' \in \mathbb{R}^n$.

**Proof.** By Lipschitz continuity of $\nabla g$, we have that $g(\zeta) - g(\xi) \leq \| \nabla g(\xi') - \xi' - \nu \nabla f(\xi) - \xi \|^2$ for all $\zeta, \xi' \in \mathbb{R}^n$. Setting $\zeta = \text{prox}_{\nu f}(\xi)$ for any $\xi \in \mathbb{R}^n$ and applying Lemma 16 yields the desired result. □

**Lemma 18.** Take $\nu > 0$ and $\eta \in \mathbb{R}^n$; then

$$| \text{prox}_{\nu f}(\xi + \eta) - \text{prox}_{\nu f}(\xi) | \leq |\eta|$$

for all $\xi \in \mathbb{R}^n$.

**Proof.** Follows immediately from Property 1. □