Numerical Quantum Gravity by Dynamical Triangulation

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Abstract. Recently an alternate technique for numerical quantum gravity, dynamical triangulation, has been developed. In this method, the geometry is varied by adding and subtracting equilateral simplices from the simplicial complex. This method overcomes certain difficulties associated with the traditional approach in Regge calculus of varying geometry by varying edge lengths. However additional complications are introduced: three of the four moves in dynamical triangulation can violate the simplicial nature of the complex. Simulations indicate that the rate of these violations is significant. Thus additional conditions must be placed on the dynamical triangulation moves to ensure that the simplicial complex and its topology are preserved.

1 Introduction

Numerical evaluations of path integral expressions for quantum amplitudes can address many issues not tractable to analytic or perturbative methods. It is thus natural to formulate such methods for the study of quantum gravity and quantum cosmology. A standard approach to numerically computing quantum amplitudes for gravity is Regge calculus. In this method, the topology of a history in the path integral is given in terms of a simplicial complex. Its geometry is specified by fixing the edge lengths of all simplices in the complex. The geometry is varied by varying the edge lengths. A sum over all geometries is carried out by evaluating the contribution of these varied geometries to the path integral. This approach has been used by several workers with some interesting results (See review by Williams and Tuckey [1992], sect. 5). However, the requirement that simplices remain nondegenerate under variation of edge length and the necessity of recomputing the curvature after each variation are factors that make this technique somewhat involved.

An alternate approach to numerical computation of quantum gravity, dynamical triangulation, has been used recently by several workers. This work has concentrated on searching for phase transitions and characterizing their behavior; recent
results have found some evidence for such a phase transition in four dimensions (Agishtein and Migdal [1992], Ambjorn and Jurkiewicz [1992], Varsted [1992], Brugmann and Marinari [1993]). In the dynamical triangulation approach, the topology is again given in terms of a simplicial complex. However, now all simplices in the complex are fixed to be equilateral. The geometry is then varied by changing the number of simplices in the complex. Simplices are thus always nondegenerate and additionally always produce the same contribution to the curvature which allegedly allows for more rapid numerical evaluation.

It is key to the use of a simplicial complex in dynamical triangulation that any changes in geometry preserve both the simplicial nature of the complex and the topology.\footnote{Indeed it is this requirement that restricts edge lengths in the Regge approach.} Surprisingly, as described in detail in this paper, this is not the case for the standard moves for three dimensional dynamical triangulation: three of the four moves used to change the number of simplices in the complex can violate its simplicial nature. Therefore restrictions on these moves must be enforced if the simplicial nature of the complex is to be preserved by dynamical triangulation. Fortunately, as detailed below, such restrictions can be enforced in 3 dimensions.

\section*{2 Problems with Dynamical Triangulation}

A discretization of histories appropriate for quantum gravity is given by a simplicial complex.\footnote{For a more detailed presentation of this introductory material, see Schleich and Witt [1993].} First, a \textit{n-simplex} is the convex hull of \((n+1)\) affinely independent points in \(\mathbb{R}^{n+1}\). A 0-simplex is a point, a 1-simplex is an edge, a 2-simplex is a triangle, a 3-simplex is a tetrahedra and so on. A simplex that is the convex hull of a subset of the vertices of an \(n\)-simplex is a \textit{face} of the \(n\)-simplex. Then

\begin{definition}
A simplicial complex \(K\) is a topological space \(|K|\) and a collection of simplices \(K\) such that
\begin{enumerate}
\item \(|K|\) is a closed subset of some finite dimensional euclidean space,
\item If \(F\) is a face of a simplex in \(K\), then \(F\) is also contained in \(K\),
\item If \(B,C\) are simplices in \(K\), then \(B \cup C\) is a face of both \(B\) and \(C\).
\end{enumerate}
The topological space \(|K|\) is the union of all simplices in \(K\).
\end{definition}

Simplicial complexes can describe very general topological spaces; however those relevant for the current discussion are manifolds. First, the \textit{star} of vertex \(v\) is the complex consisting of all simplices containing \(v\). The \textit{link} of \(v\) is the complex consisting of the subset of simplices in the star of \(v\) that do not contain \(v\) itself. Then a closed combinatorial \(n\)-manifold is a simplicial complex for which the link of every vertex is a combinatorial \((n-1)\)-sphere. The key element in the definition of simplicial complex is that all elements are uniquely determined by the vertices they contain; no two elements in the complex are specified by the same vertices. Were elements of the complex not uniquely specified by their vertices, additional information would be needed to differentiate the elements. Moreover, it is precisely this feature that allows the geometry of a given space to be modeled by filling each simplicial complex with flat space and fixing all edge lengths. As the geometry is flat on the interior of the simplices, the curvature is no longer distributed over the space but is carried on \((n-2)\)-simplices. In dynamical triangulation, one chooses all \(n\)-simplices to have equal edge lengths. An advantage of this choice is that the
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curvature at each \((n-2)\)-simplex is determined very simply by counting the number \(k\) of \(n\)-simplices containing it: \(R = 2\pi - k\delta\) where \(\delta\) is the deficit angle between adjacent faces of an equilateral \(n\)-simplex. For three dimensions \(\delta = \cos^{-1}(\frac{1}{3})\). The Euclidean action for a closed 3-manifold then reduces to especially simple form, \(I = \kappa_3 N_3 - \kappa_1 N_1\) where \(N_i\) is the number of \(i\)-simplices in the manifold and \(\kappa_i\) are constants depending on \(\delta, G, \Lambda\) and the edge length. Thus changes in the action can be computed directly from changes in the number of simplices in the complex.

Clearly changing the number of \(3\)-simplices adjacent to a given edge will change the curvature of the space and correspondingly the geometry of the combinatorial manifold. Such changes are carried out by a special set of moves. In three dimensions, there are four moves as shown in Figure 1; a subdivision of one tetrahedra into four tetrahedra, a flip move in which two tetrahedra sharing a triangle are taken into three tetrahedra sharing an edge, and their inverses (Godfrey and Gross [1991], Ambjorn and Varsted [1992]). Note that a necessary condition for performing the inverse flip move is that the edge is in precisely three tetrahedra. Similarly a necessary condition for the inverse subdivision move is that the vertex is in precisely four tetrahedra. The subdivision changes the number of edges and tetrahedra in the complex by \(\Delta N_1 = 4, \Delta N_3 = 3\), and the flip move by \(\Delta N_1 = 1, \Delta N_3 = 1\). Clearly the inverse moves change the counting by the opposite amount.

At first glance, it would appear that as these moves preserve the boundary of the region, they preserve the simplicial nature of the complex. This is not true; three of the four moves can violate the simplicial properties of the 3-manifold. This fact is not noted in the literature on three dimensional dynamical triangulation. It is useful to demonstrate how such moves can break the complex in two dimensions, where we can clearly illustrate this. The flip move in two dimensions is illustrated at the top of figure 2. Now consider performing the flip move on the 2-manifold at the bottom of figure 2; by the nature of the curvature at the flip location, the edge to be added into the complex is already present. Thus the simplicial nature of the complex is broken at this location and the standard assignment of curvature in Regge calculus cannot be used on the resulting space.

In three dimensions analogous violations occur: The flip move can attempt to add an edge that is already in the simplicial 3-manifold. The inverse flip move can similarly attempt to add a triangle that is already present. Finally the inverse subdivision can attempt to add a tetrahedra that is already in the complex. Although it is hard to illustrate pictorially, examples of all of these violations occur for the smallest combinatorial 3-sphere consisting of the surface of a 4-simplex, \(K = \{v_0, v_1, \ldots, v_4, e_0, e_1, \ldots, e_9, f_0, f_1, \ldots, f_9, q_0, q_1, \ldots, q_9\}\) with the higher dimensional simplices containing the following vertices:

\[
\begin{align*}
e_0 & \supset v_0, v_1, & e_1 & \supset v_0, v_2, & e_2 & \supset v_0, v_3, & e_3 & \supset v_0, v_4, & e_4 & \supset v_1, v_2, & e_5 & \supset v_1, v_3, & e_6 & \supset v_1, v_4, & e_7 & \supset v_2, v_3, & e_8 & \supset v_2, v_4, & e_9 & \supset v_3, v_4, & t_0 & \supset v_2, v_3, v_4, & t_1 & \supset v_1, v_3, v_4, & t_2 & \supset v_1, v_2, v_4, & t_3 & \supset v_1, v_2, v_3, & t_4 & \supset v_0, v_3, v_4, & t_5 & \supset v_0, v_2, v_4, & t_6 & \supset v_0, v_2, v_3, & t_7 & \supset v_0, v_1, v_4, & t_8 & \supset v_0, v_1, v_3, & t_9 & \supset v_0, v_1, v_2, & q_0 & \supset v_0, v_1, v_2, v_3, & q_1 & \supset v_0, v_1, v_2, v_4, & q_2 & \supset v_0, v_1, v_3, v_4, & q_3 & \supset v_0, v_2, v_3, v_4, & q_4 & \supset v_1, v_2, v_3, v_4
\end{align*}
\]

All triangles are in two tetrahedra (this is true for all closed 3-manifolds) and are therefore candidates for a flip. Consider \(t_0\); it is in \(q_3\) and \(q_4\) and the vertices that appear in the new edge will be \(v_0, v_1\). However, this edge is already in the complex; it is \(e_0\). Therefore this move cannot be performed. Similarly, all edges are
contained in precisely 3-tetrahedra (this is not generic to 3-manifolds, but true for this special case) and are thus candidates for an inverse flip move. Consider $e_5$; it is in $q_0, q_2, q_4$ and thus would add a triangle with vertices $v_0, v_2, v_4$. But again this triangle is already there; it is $t_5$. Finally, the inverse subdivision move at vertex $v_0$ attempts to replace the four tetrahedra $q_0, q_1, q_2, q_3$ with one containing vertices $v_1, v_2, v_3, v_4$. Again this is already there; it is $q_5$. Therefore, all of these moves fail. Note that these violations can occur for more general 3-manifolds; it is clear that this 3-sphere can be joined to any more general space to produce a region with edges, triangles and vertices that have similar properties.

These violations of the simplicial nature of the complex are serious. Fortunately, they are local. Therefore, the simplicial nature of the complex can be preserved so long as one tests for these violations before performing a move. Such a test is easy to design: Check to see that all new elements to be added to the complex are not currently present. Finally, after performing the move, verify that the manifold topology is preserved by an algorithm. Such a violation testing code has been developed for three dimensional dynamical triangulation by the authors. This code is based on a data structure that encodes the topology of the simplicial complex in hierarchical form. Moves are only implemented after testing for violations of the simplicial nature of the complex. Data is kept on the frequency of attempts to perform a move that violates the simplicial nature of the complex. This code has been run on small scale simulations. Results of these simulations indicate that the violation rate of the flip move, defined as the percentage of successful flips on a random sample of all triangles is about 60 percent. The violation rate of the inverse flip move defined as the percentage of successful inverse flips on a random sample of all edges that are contained in precisely three tetrahedra is around 40 percent. These high violation frequencies can be intuitively understood in terms of the subdivision move: it adds 4 edges to the simplex that contain precisely 3 tetrahedra, but these edges also contain a vertex that contains only 4 tetrahedra. Therefore, these edges cannot be used in an inverse flip move. It also adds 6 triangles that cannot be used in a flip move as the new vertex contains only 4 tetrahedra. As the subdivision move can be performed at any tetrahedra in the complex, one expects a continued violation rate for arbitrarily sized complexes. Finally, the inverse subdivision move has a very low violation rate, 0.2 percent. Clearly the violation rates for the flip move and inverse flip move are significant.

3 Conclusions

Any application of dynamical triangulation should preserve the topology and simplicial nature of the complex. It has been demonstrated here that additional conditions on the neighborhoods of the elements subject to moves must be enforced to do so. Preliminary simulations indicate that failing to enforce these conditions leads to a high violation rate. Thus enforcement is important if dynamical triangulation simulations are to implement a discretized version of three dimensional gravity. Furthermore, these results have implications for the four dimensional case as the problems discussed in this paper will appear in four dimensions. These difficulties will be magnified by the lack of an easily implementable algorithm for testing for a 4-manifold. Thus it may be difficult to verify the additional conditions on the moves needed to preserve the simplicial nature of the complex are sufficient.
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Figure Captions

Figure 1. The dynamical triangulation moves in three dimensions. The first picture illustrates the subdivision move: the original tetrahedra is replaced with 4 new tetrahedra sharing a common vertex. Observe that the boundary of the subdivision is the same as that of the original tetrahedra. The second illustrates the flip move; two tetrahedra sharing a common triangle with vertices $v_0, v_1, v_2$ are replaced with three tetrahedra sharing an edge $v_3, v_4$. Again the boundary of the region is preserved.

Figure 2. Two dimensional example of a failed move. The top picture illustrates the flip move in two dimensions. The bottom picture illustrates the consequences of this move for a vertex $v_3$ of high curvature. Note edges specified by $v_0, v_1, v_2$ are not the boundary of a triangle in the complex. Before the move, all edges are uniquely specified by their vertices. After the move there are two edges with vertices $v_0, v_2$. To distinguish them, the new edge has been curved. Curvature is thus no longer carried only at vertices and the Regge formulas for it are no longer valid.