Conditions for multiplicity of maximal $\ell_p$-norms of channels for fixed integer $p$

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We introduce a condition for memoryless quantum channels which, when satisfied guarantees the multiplicity of the maximal $\ell_p$-norm with $p$ a fixed integer. By applying the condition to qubit channels, it can be shown that it is not a necessary condition, although some known results for qubits can be recovered. When applied to the Werner-Holevo channel, which is known to violate multiplicity when $p$ is large relative to the dimension $d$, the condition suggests that multiplicity holds when $d \geq 2^{p-1}$. This conjecture is proved explicitly for $p = 2, 3, 4$. Finally, a new class of channels is considered which generalizes the depolarizing channel to maps which are combinations of the identity channel and a noisy one whose image is an arbitrary density matrix. It is shown that these channels are multiplicative for $p = 2$.

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I. INTRODUCTION

A noisy quantum channel can be described by means of a completely-positive, trace-preserving (CPT) map $\mathcal{E}$ which transforms the density matrices $\gamma$ on the Hilbert space $\mathcal{H}$ into the output states $\mathcal{E}(\gamma)$. Such maps can always be represented in the form

$$\mathcal{E}(\gamma) = \sum_k A_k \gamma A_k^\dagger, \quad \sum_k A_k^\dagger A_k = 1,$$

with $\{A_k\}$ called a set of Kraus operators associated with $\mathcal{E}$. When the channel is memoryless, $m$ successive uses of are described by the map $\mathcal{E}^\otimes m$. It is natural to ask if entangled inputs can decrease the effects of noise for memoryless channels in some way.

One measure of the effect of noise is the maximal $\ell_p$-norm of a channel, which is defined as

$$\nu_p(\mathcal{E}) \equiv \sup_{\gamma \in \mathcal{D}(\mathcal{H})} \|\mathcal{E}(\gamma)\|_p, \quad p \geq 1,$$

where $\|A\|_p \equiv (\text{Tr}|A|^p)^{1/p}$ is the $p$-norm of the operator $A$ and where the supremum is taken over all $\mathcal{D}(\mathcal{H})$, the set of density matrices. The quantity $\text{Tr}[\mathcal{E}(\gamma)^p]$ is a measure of the closeness of the output to a pure state, and $\nu_p(\mathcal{E}) = 1$ if and only if some output state $\mathcal{E}(\gamma)$ is pure. Because the Rényi entropy can be written as $S_p(\rho) = -1/\log \|\rho\|_p$, one could define a maximal output Rényi entropy satisfying $(p-1)S_{p,\text{max}}(\mathcal{E}) = -p \log \nu_p(\mathcal{E})$.

Amosov, Holevo and Werner (AHW) conjectured that $\nu_p(\mathcal{E})$ is multiplicative for tensor product channels

$$\nu_p(\mathcal{E}^\otimes m) \equiv \sup_{\Gamma \in \mathcal{D}(\mathcal{H}^\otimes m)} \|\mathcal{E}^\otimes m(\Gamma)\|_p = [\nu_p(\mathcal{E})]^m,$$

where $\mathcal{E}^\otimes m$ is the CPT map which describes $m$ successive memoryless uses of the channel $\mathcal{E}$, and where the maximization in the second term of Eq. (3) is now performed over the density matrices $\Gamma \in \mathcal{D}(\mathcal{H}^\otimes m)$. The AHW conjecture requires that a product state $\Gamma$ saturates the supremum of $\nu_p(\mathcal{E}^\otimes m)$ for the memoryless channel $\mathcal{E}^\otimes m$ so that entangled input states $\Gamma$ do not increase the output norm. One rational for the multiplicity hypothesis is the physical intuition that quantum coherence among successive channel uses should be degraded by the action of a memoryless channel. Since the $\ell_p$-norm “measures” the purity of the states emerging from the channel, one might expect separable inputs to perform better than entangled inputs. The multiplicity of $\nu_p(\mathcal{E}^\otimes m)$ is equivalent to additivity for the minimum Rényi entropy with the same $p$. Moreover, if (3) holds for $p$ arbitrarily close to 1, then it implies the additivity of the minimum output von Neumann entropy, another measure of output purity. This has been shown to be related to a conjectured additivity property of the Holevo information, and to conjectures about additivity and superadditivity of the entanglement of formation.

Subsequently, Werner and Holevo showed that the general multiplicativity conjecture is false by producing a channel that violates (3) for $p > 4.79$. Nevertheless, one might still expect multiplicativity to hold for some range of
II. LINEARIZATION OF \( p \)-NORM FUNCTIONS

A. Basic linearization strategy

In this section we present a method, introduced in [13], that allows one to compute the \( \ell_p \)-norm from the expectation value of an operator defined in an extended Hilbert space. For any integer \( p \), it is possible to find a linear operator \( X(\mathcal{E}, p) \) defined in the extended Hilbert space \( \mathcal{H}^{\otimes p} \) such that, for any density matrix \( \gamma \in \mathcal{H} \), we have

\[
\text{Tr}[\mathcal{E}(\gamma)]^p = \text{Tr}[ (\gamma \otimes \gamma \otimes \cdots \otimes \gamma) \cdot X(\mathcal{E}, p)]
\]

where the trace in the left-hand side is computed with respect to an orthonormal basis of \( \mathcal{H} \), while the trace in the right-hand side is computed with respect to an orthonormal basis of \( \mathcal{H}^{\otimes p} \). In other words, we can represent the \( p \)-purity function \( \text{Tr}[\mathcal{E}(\gamma)]^p \) as the expectation value of \( X(\mathcal{E}, p) \) on \( p \) copies of \( \gamma \). The operator \( X(\mathcal{E}, p) \) is not uniquely defined; in fact, it can be realized by the action of tensor products of the dual map of \( \mathcal{E} \) on any permutation operator acting on \( \mathcal{H}^{\otimes p} \) whose shortest cycle is length \( p \).

To make this explicit, we need some notation, which is explained in more detail in Appendix A, particularly sections A.1 and A.2. We will use a hat to denote the dual, or adjoint, map \( \hat{\mathcal{E}} \) with respect to the Hilbert-Schmidt inner product. Let \( L_p \) and \( R_p \) denote the left and right cyclic shifts which can be defined by their action on an orthonormal product basis as

\[
L_p|\xi_1 \xi_2 \cdots \xi_{p-1} \xi_p\rangle = |\xi_2 \cdots \xi_p \xi_1\rangle \quad \text{(5a)}
\]
\[
R_p|\xi_1 \xi_2 \cdots \xi_{p-1} \xi_p\rangle = |\xi_p \xi_1 \cdots \xi_{p-1}\rangle \quad \text{(5b)}
\]

where \( |\xi_1 \xi_2 \cdots \xi_{p-1} \xi_p\rangle = \otimes_{j=1}^{p} |\xi_j\rangle \) and \( \{|\xi_k\rangle\} \) is an orthonormal basis for \( \mathcal{H} \). Then the operator

\[
\Omega(\mathcal{E}, p) \equiv \hat{\mathcal{E}}^{\otimes p}(L_p)
\]

satisfies (1). This follows from

\[
\text{Tr} \gamma^{\otimes p} \Omega(\mathcal{E}, p) = \text{Tr}(\gamma \otimes \gamma \otimes \cdots \otimes \gamma) \hat{\mathcal{E}}^{\otimes p}(L_p) = \text{Tr}[\mathcal{E}(\gamma) \otimes \mathcal{E}(\gamma) \otimes \cdots \otimes \mathcal{E}(\gamma)] L_p = \text{Tr}[\mathcal{E}(\gamma)]^p
\]

where the last step used (A.2). It follows from (A.4) that \( L_p \) could be replaced by another permutation; however, it is important to make a definite choice for later use.
In previous work [13, 14], a different realization of $X(E, p)$ was used which is valid only for pure states. Let
\[
\Theta(E, p) = \Omega(E, p) R_p = \hat{E} \otimes^p (L_p) R_p \tag{8}
\]
\[
= \sum_{k_1, \ldots, k_p} A_{k_1}^1 A_{k_2}^2 \otimes \cdots \otimes A_{k_p}^p A_{k_1}
\tag{9}
\]
where \{$A_k$} form a set of Kraus operators for $E$ as in [11]. The operator $\Theta(E, p)$ satisfies [4] when $\gamma = |\psi\rangle \langle \psi|$ is a pure state. This relation is proved in Appendix B 1, and implicitly shows that it does not depend on the chosen Kraus representation [11] of $E$. For $p = 2$, [13] and [8] were obtained earlier in Ref. [33].

In general, the operator $\Omega(E, p)$ will not be Hermitian. We have already observed that $X(E, p)$ is not unique and that whenever $P_p$ is a permutation operator whose shortest cycle is length $p$, the operator $\hat{E} \otimes^p (P_p)$ provides another realization. Since $L_p = R_p^\dagger$,
\[
[\hat{E} \otimes^p (L_p)]^\dagger = \hat{E} \otimes^p (L_p^\dagger) = \hat{E} \otimes^p (R_p).
\tag{10}
\]
This implies that $\Omega(E, 2)$ is Hermitian for $p = 2$, and that the operator
\[
\frac{1}{2} [\Omega(E, p) + [\Omega(E, p)]^\dagger] = \frac{1}{2} [\hat{E} \otimes^p (L_p) + \hat{E} \otimes^p (R_p)]
\tag{11}
\]
gives a Hermitian realization of $X(E, p)$ for any $p$. However, we do not expect [11] to have the important multiplicity property [14] for repeated uses of the channel. Further discussion of other realizations $X(E, p)$ is given Appendices B 2 and B 3.

Linear operators satisfying [4] provide a useful tool for studying the $p$-purity functions, which are intrinsically non-linear objects; it reduces some associated problems to the analysis of the linear operator $X(E, p)$ acting on the extended Hilbert space $H \otimes^p$ obtained by adding $p - 1$ “fictitious” copies of the input Hilbert space $H$. In Refs. [13, 14], this approach was used to obtain some additivity properties of Gaussian Bosonic channels. For $p = 2$, Eq. (4) was used in Ref. [15] to study the fidelity obtainable in continuous-variable teleportation with finite two-mode squeezing, and in Ref. [33] to analyze the purity of generic quantum channels.

### B. Tensor product maps

The results derived in the preceding section can also be applied when the basic CPT map is itself a tensor product. Then Eq. (1) becomes
\[
\text{Tr} [\hat{E} \otimes^m (\Gamma)]^p = \text{Tr} [\Gamma \otimes \Gamma \otimes \cdots \Gamma] X(E \otimes^m, p),
\tag{12}
\]
where $\Gamma$ is a generic density matrix in the input Hilbert space $H \otimes^m$ and $X(E \otimes^m, p)$ is a linear operator on $(H \otimes^m)^{\otimes p} = H \otimes^mp$. Following the strategy of Section II A we now choose $X(E \otimes^m, p)$ to be the operator,
\[
\Omega(E \otimes^m, p) \equiv (\hat{E} \otimes^m)^{\otimes p} (L_p) = (\hat{E} \otimes^m)^{\otimes p} (L_p^\dagger)
\tag{13}
\]
The operator $L_p$ is described in more detail in Appendix A 3 where it is proved that $L_p = L_p^\dagger = (L_{mp})^m$. Using $(\hat{E} \otimes^m)^{\otimes p} = (\hat{E})^{\otimes mp}$, we find
\[
\Omega(E \otimes^m, p) = \hat{E} \otimes^m p (L_p^\dagger) = [\hat{E} \otimes^p (L_p)]^\dagger = [\Omega(E, p)]^\dagger.
\tag{14}
\]
Equation (14) is a key result whose simplicity hides a great deal of subtlety. The essential point is that the linear operator $X(E \otimes^m, p)$ which satisfies [11] for the tensor product channel $E \otimes^m$ can be realized by the action of the dual of $E \otimes^m$ on the permutation $L_p^\dagger$. 
III. CONDITIONS FOR MULTIPLICATIVITY

A. Upper bound

We now use the singular value decomposition [16, 17] to observe that one can write

$$\Omega(\mathcal{E}, p) = \sum_j \mu_j |\eta_j\rangle \langle \omega_j|$$

(15)

where \{|$\eta_j\rangle$\} and \{|$\omega_j\rangle$\} denote orthonormal bases for $\mathcal{H}^\otimes p$ and $\mu_j > 0$ are the singular values of $\Omega(\mathcal{E}, p)$, i.e., the non-zero eigenvalues of $|\Omega(\mathcal{E}, p)| \equiv \sqrt{\langle \Omega(\mathcal{E}, p) | \Omega(\mathcal{E}, p) \rangle}$. Before applying this, it is convenient to introduce the convention of using bold uppercase Greek letters to denote tensor product vectors as in $|\Psi\rangle \equiv |\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle \in \mathcal{H}^\otimes p$. Then

$$\text{Tr} \left[ \mathcal{E} \left( |\psi\rangle \langle \psi| \right) \right]^p = \langle \Psi | \Omega(\mathcal{E}, p) | \Psi \rangle = \sum_j \mu_j \langle \Psi | \eta_j \rangle \langle \omega_j | \Psi \rangle \leq \mu_{\max} \sum_j |\langle \Psi | \eta_j \rangle \langle \omega_j | \Psi \rangle| \leq \mu_{\max} \| |\Psi\rangle \|^2 = \| \Omega(\mathcal{E}, p) \|_\infty$$

(16)

where $\mu_{\max} = \sup_j \mu_j = \| \Omega(\mathcal{E}, p) \|_\infty$ is the largest singular value of $\Omega(\mathcal{E}, p)$. Applying this analysis to multiple uses of the channel, one can similarly conclude that

$$\text{Tr} \left[ \mathcal{E}^\otimes m \left( |\psi\rangle \langle \psi| \right) \right]^p \leq \| \Omega(\mathcal{E}^\otimes m, p) \|_\infty,$$

(17)

where $|\Psi\rangle$ is now an arbitrary vector in $\mathcal{H}^\otimes m$. However, it follows from (14) that the singular values of $\Omega(\mathcal{E}^\otimes m, p)$ are products of those of $\Omega(\mathcal{E}, p)$ so that

$$\| \Omega(\mathcal{E}^\otimes m, p) \|_\infty = (\| \Omega(\mathcal{E}, p) \|_\infty)^m = (\mu_{\max})^m.$$ 

(18)

Combining (17) and (18), one finds

$$\text{Tr} \left[ \mathcal{E}^\otimes m \left( |\psi\rangle \langle \psi| \right) \right]^p \leq (\| \Omega(\mathcal{E}, p) \|_\infty)^m = (\mu_{\max})^m.$$ 

(19)

Since, the supremum in (2) is attained using a pure state input and (19) holds for all pure inputs $|\Psi\rangle$, we conclude that the upper bound

$$\nu_p(\mathcal{E}^\otimes m) \leq (\mu_{\max})^{m/p},$$

(20)

holds for all pairs of integers $m$ and $p$.

B. Multiplicativity condition

The bound (20) leads to a sufficient condition for multiplicativity. We state this formally, and give a relate condition as a corollary.

**Theorem 1** The channel $\mathcal{E}$ has the multiplicativity property (3) if the largest singular value of $\Omega(\mathcal{E}, p)$ satisfies

$$\| \Omega(\mathcal{E}, p) \|_\infty = [\nu_p(\mathcal{E})]^p$$

(21)

**Corollary 2** The channel $\mathcal{E}$ has the multiplicativity property (3) if the largest singular value of $\Omega(\mathcal{E}, p)$ is also an eigenvalue of $\Omega(\mathcal{E}, p)$ with a product eigenvector of the form $|\phi\rangle^\otimes p$.

To prove Theorem 1 observe that in the notation of the preceding section (21) can be written as $\mu_{\max} = [\nu_p(\mathcal{E})]^p$ Then (20) implies

$$\nu_p(\mathcal{E}^\otimes m) \leq (\mu_{\max})^{m/p} = [\nu_p(\mathcal{E})]^m.$$ 

(22)
On the other hand, one always has
\[ \nu_p(\mathcal{E}^\otimes m) \geq \|\mathcal{E}(\gamma_{\text{max}}\otimes \mathcal{E})\|_p = \|\mathcal{E}(\gamma_{\text{max}})\|^m_p = [\nu_p(\mathcal{E})]^m. \]
where \( \gamma_{\text{max}} \) denotes the state which achieves the supremum for \( \nu_p(\mathcal{E}) \). Combining these inequalities gives \( \nu_p(\mathcal{E}^\otimes m) = [\nu_p(\mathcal{E})]^m. \)

QED

To prove the corollary, observe that its hypothesis holds if and only if there is a state \( |\phi\rangle \) in \( \mathcal{H} \) such that
\[ \mu_{\text{max}} = \langle \Phi|\mathcal{E}(\mathcal{H}, p)|\Phi \rangle = \text{Tr}[\mathcal{E}(|\phi\rangle\langle\phi|)]^p \]
where the second equality used \( 7 \) and our convention that \( |\mu\rangle \)
so that one finds an eigenvector as well as the largest singular value of an operator, but does not require knowledge of \( \gamma \) which are known to be multiplicative, but do not satisfy (21). Verifying the hypothesis of Corollary 2 requires that
\[ \text{Tr}[\mathcal{E}(|\phi\rangle\langle\phi|)]^p \leq \sup_{\gamma} \text{Tr}[\mathcal{E}(\gamma)]^p \equiv [\nu_p(\mathcal{E})]^p \]
so that \( \mu_{\text{max}} \leq [\nu_p(\mathcal{E})]^p \). Combining this with (20) when \( m = 1 \), implies that \( \mu_{\text{max}} = [\nu_p(\mathcal{E})]^p \) so that the hypothesis of Theorem 1 holds.

QED

In Section 15A we will see that the condition in Theorem 1 is not necessary. There are unital qubit CPT maps, which are known to be multiplicative, but do not satisfy (21). Verifying the hypothesis of Corollary 2 requires that one find an eigenvector as well as the largest singular value of an operator, but does not require knowledge of \( \nu_p(\mathcal{E}) \); condition (21) does require the latter, but does not require computation of any eigenvectors. In general, (21) seems easier to check. However, in the examples we analyzed, both conditions hold and the process of verifying one easily yields the other. It would be interesting to know if (21) implies that the singular value of \( \Omega(\mathcal{E}, p) \) is also an eigenvector with a product eigenvalue as in Corollary 2.

IV. APPLICATIONS

A. Qubit channels

1. Notation

We illustrate our condition by looking at some examples of qubit channels, for which will use notation similar to that introduced in [23, 31]. Any \( 2 \times 2 \) matrix can be represented in the basis consisting the \( 2 \times 2 \) identity matrix \( \mathbb{1} \) and the three Pauli matrices which we often write as a formal vector \( \vec{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3) \). In this basis a density matrix can be written as \( \gamma = \frac{i}{2} [\mathbb{1} + \vec{w} \cdot \vec{\sigma}] \) with \( \vec{w} \in \mathbb{R}^3 \) and \( |\vec{w}| \leq 1 \). The density matrix is pure if and only if \( |\vec{w}| = 1 \). Any linear map \( \Phi \) on a qubit, can be described by two real vectors \( \vec{s}, \vec{t} \in \mathbb{R}^3 \) and by a \( 3 \times 3 \) real matrix \( T \), through the expression
\[ \Phi(z_0 \mathbb{1} + \vec{z} \cdot \vec{\sigma}) = (z_0 + \vec{s} \cdot \vec{z}) \mathbb{1} + (z_0 \vec{t} + T \cdot \vec{z}) \cdot \vec{\sigma}, \]
which holds for all \( z_0 \in \mathbb{C} \) and \( \vec{z} \in \mathbb{C}^3 \). This corresponds to representing \( \Phi \) in the basis \( \{ \mathbb{1}, \vec{\sigma} \} \) by the \( 4 \times 4 \) matrix
\[ \left( \begin{array}{cc} 1 & \vec{s}^T \\ \vec{t} & T \end{array} \right) \]
which we have written in block form (with the convention that \( \vec{t} \) corresponds to a column vector and \( \vec{s}^T \) a row vector, using the superscript \( t \) to denote transpose). It was shown in [24] that it suffices to consider \( T \) diagonal with real elements \( \{\lambda_1, \lambda_2, \lambda_3\} \). In essence, a variant of the SVD (which leads to negative as well as positive \( \lambda \)) can be applied to \( T \) corresponding to rotations on the input and output bases respectively.

In this notation, \( \Phi \) is trace preserving (TP) if and only if \( \vec{s} = 0 \) and it is unital if and only if \( \vec{t} = 0 \). Additional conditions under which the map is positivity preserving or completely positive (CP) are more complex. A complete set of conditions for the map to be CPT was obtained in [31]. When \( t_1 = t_2 = 0 \), these CPT conditions reduce to \( (\lambda_1 \pm \lambda_2)^2 \leq (1 + \lambda_3)^2 - t_3^2 \), as shown in Refs. [11, 31]. Since the dual map of \( \Phi \) is represented by the adjoint matrix, it satisfies,
\[ \hat{\Phi}(z_0 \mathbb{1} + \vec{z} \cdot \vec{\sigma}) = (z_0 + \vec{t} \cdot \vec{z}) \mathbb{1} + (z_0 \vec{s} + T^t \cdot \vec{z}) \cdot \vec{\sigma}. \]

Since \( \mathcal{H} \) is now 2-dimensional, the left shift \( L_2 \) is simply the SWAP operator \( S \) which satisfies
\[ S = \frac{i}{2} [\mathbb{1} \otimes \mathbb{1} + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3] \]
(27)

It is then straightforward to use (60) to show that
\[ \Omega(\Phi, 2) = \hat{\Phi}^\otimes 2(S) = \frac{i}{2} \left[ (1 + |\vec{t}|^2) \mathbb{1} \otimes \mathbb{1} + \sum_{j=1}^3 \lambda_j^2 \sigma_j \otimes \sigma_j + \sum_{j=1}^3 \lambda_j t_j (\mathbb{1} \otimes \sigma_j + \sigma_j \otimes \mathbb{1}) \right]. \]
2. Unitary maps

For qubit channels the conjecture (2) has been extensively studied in [18, 19, 22, 24]. Multiplicativity has been proven for all $p$ for unital qubit channels [18] and for $p = 2$ for all qubit channels (Theorem 2 of [18]). Here we will use the case $p = 2$ to illustrate the multiplicativity-criterion presented in Section 11.

It will be useful to choose the subscript “max” in $\{1, 2, 3\}$ so that $|\lambda_{\max}| = \max_k |\lambda_k|$. For unital qubits maps, the maximum $\ell_2$-norm of $\Phi$ can be achieved with an input state of the form $\frac{1}{2}[I \pm \sigma_{\max}]$ for which the output $\frac{1}{2}[I \pm \lambda_{\max}\sigma_{\max}]$ has eigenvalues $\frac{1}{2}[1 \pm \lambda_{\max}]$ and

$$\nu_2(\Phi) = \frac{1}{\sqrt{2}} \sqrt{1 + \lambda_{\max}^2} \tag{29}$$

When $\Phi$ is unital, $\ell = 0$ and the third term in the expression (28) vanishes. It then follows that in the product basis $\{|00\}, |01\}, |10\}, |11\}$, the operator $\Omega(\Phi, 2)$ is represented by the matrix

$$\frac{1}{2} \begin{pmatrix} 1 + \lambda_3^2 & 0 & 0 & \lambda_1^2 - \lambda_2^2 \\ 0 & 1 - \lambda_2^2 & \lambda_1^2 + \lambda_2^2 & 0 \\ 0 & \lambda_1^2 + \lambda_2^2 & 1 - \lambda_3^2 & 0 \\ \lambda_1^2 - \lambda_3^2 & 0 & 0 & 1 + \lambda_3^2 \end{pmatrix}. \tag{30}$$

This is easily seen to have two non-zero $2 \times 2$ blocks. The “inner” block has eigenvalues $\frac{1}{2} \left[ 1 - \lambda_3^2 \pm (\lambda_1^2 + \lambda_2^2) \right]$ with eigenvectors $\frac{1}{\sqrt{2}}(0, 1, \pm1, 0)^t$ corresponding to the Bell states $\frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. The “outer” block has eigenvalues $\frac{1}{2} \left[ 1 + \lambda_3^2 \pm (\lambda_1^2 - \lambda_2^2) \right]$ with eigenvectors $\frac{1}{\sqrt{2}}(1, 0, 0, \pm1)^t$ corresponding to the Bell states $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$. Since $\Omega(\Phi, 2)$ is Hermitian, its singular values are simply the absolute values of the eigenvalues above.

When the $|\lambda_k|$ are distinct for $k = 1, 2, 3$, the singular values of $\Omega(\Phi, 2)$ are all distinct and correspond to maximally entangled, rather than product, states. Moreover, one of the singular values is always strictly greater than $\nu_2(\Phi)$. For example, when $|\lambda_{\max}| = |\lambda_3|$, one of the “outer” eigenvalues equals $\nu_2(\Phi)^2 + \frac{1}{2}(\lambda_1^2 - \lambda_2^2)$ which is strictly greater than $20$ unless $|\lambda_1| = |\lambda_2|$. Therefore, although $\Phi$ is multiplicative, it does not satisfy $21$. This establishes that $21$ is not a necessary condition for multiplicativity.

Now consider the case $\lambda_3 > \lambda_1 = \lambda_2 \geq 0$; such channels are sometimes called “two-Pauli” channels [3]. The image of the Bloch sphere is an ellipsoid shaped like an American football. For these channels, the “outer” block in (30) is diagonal, its (degenerate) eigenvalue $\frac{1}{2}[1 + \lambda_3^2] = |\nu_2(\Phi)|^2$ is the largest singular value of $\Omega(\Phi, 2)$ and the corresponding eigenvectors $|00\rangle$ and $|11\rangle$ are product states. Thus, Theorem 1 implies that the channel satisfies 3.

3. Non-unitary maps

We now consider channels similar to those above, but with the image ellipsoid shifted along the longest axis. It suffices to consider $|\lambda_3| \geq |\lambda_1 = \lambda_2 \geq 0$ and $t_1 = t_2 = 0$. The same results hold for permutations of $1, 2, 3$ and for $\lambda_1 = \lambda_2 \leq 0$. However, the analysis in the basis we have chosen to represent $\Omega(\Phi, 2)$ is simplest when $|\lambda_{\max}| = |\lambda_3|$. The matrix representing $\Omega(\Phi, 2)$ is

$$\frac{1}{2} \begin{pmatrix} 1 + (t_3 + \lambda_3)^2 & 0 & 0 & 0 \\ 0 & 1 + t_3 - \lambda_3 & \lambda_1^2 - \lambda_2^2 & 0 \\ 0 & 2\lambda_1^2 & 1 + t_3 - \lambda_3 & 0 \\ 0 & 0 & 0 & 1 + (t_3 - \lambda_3)^2 \end{pmatrix}. \tag{31}$$

which has an “inner” block with eigenvalues $\frac{1}{2} \left[ 1 + t_3 - \lambda_3^2 \pm 2\lambda_1^2 \right]$ and a diagonal “outer” block with eigenvalues $\frac{1}{2} \left[ 1 + (t_3 + \lambda_3)^2 \right]$ and product eigenvectors. One can verify that the largest singular value is $\frac{1}{2}[1 + ((t_3 + |\lambda_3|)^2]$. To see that this equals $|\nu_2(\Phi)|^2$, observe that the optimal input state is $\frac{1}{4}[I + (1 + \lambda_3\sigma_3)]$ for which the output state has eigenvalues $\frac{1}{2}\left[ 1 \pm (|t_3| + |\lambda_3|)^2 \right]$. Thus, we can again use Theorem 1 to conclude that 3 holds.

The methods introduced here are able to handle qubit channels for which the image of the Bloch sphere is an elongated ellipsoid with a symmetry axis, i.e., in the shape of an American football, both when the channel is unital and when it is shifted in the direction of the longest axis. However, it can not handle these channels if the shift is orthogonal to the longest axis, i.e., if $t_3 = 0$ but $t_2 \neq 0$ above. When the ellipsoid has a symmetry axis but $|\lambda_1| = |\lambda_2| \geq |\lambda_3|$ so that it is shaped like a flying saucer, the methods used here can not prove multiplicativity. Even for unital channels, for which multiplicativity has been established [19], neither of the conditions in Theorem 1 holds.
B. Shifted depolarizing channels

1. Shifting and generalizing the depolarizing channel

The unital qubit map with \( \lambda_k = \pm |\lambda_{\text{max}}| \) for all \( k \), is a special case of the depolarizing channel which has the form

\[
\mathcal{E}(\gamma) = (1 - x)(\text{Tr}\gamma)\frac{1}{d}\mathbb{1} + x\gamma.
\]

It is CPT for \(-\frac{1}{4} \leq x \leq 1\). The non-unital qubit map which takes

\[
\gamma = \frac{1}{2}\left[\mathbb{1} + \vec{w} \cdot \vec{\sigma}\right] \mapsto \frac{1}{2}\left[\mathbb{1} + (\vec{f} + \lambda\vec{w}) \cdot \vec{\sigma}\right] = (1 - |\vec{f}| - \lambda)\frac{1}{2}\mathbb{1} + |\vec{f}|\frac{1}{2}\left[\mathbb{1} + \vec{f} \cdot \vec{\sigma}\right] + \lambda\gamma
\]

(32)
can be regarded as a shifted depolarizing channel because it shifts the output toward the point \( \vec{f} \) on the Bloch sphere. By rotating coordinates so that \( \vec{f} = (0,0,t_3) \), this is a special case of the qubit maps considered in Section 1. It is then natural to define a shifted depolarizing channel in dimension \( d \) by

\[
\mathcal{E}(\gamma) = a(\text{Tr}\gamma)\frac{1}{d} + b(\text{Tr}\gamma)|\psi\rangle\langle\psi| + c\gamma
\]

(33)
with the state \( |\psi\rangle \) fixed and \( a + b + c = 1 \). When \( a, b, c \) are positive, this channel is a convex combination of the identity map and two completely noisy channels which maps all states to \( \frac{1}{d}\mathbb{1} \) and to \( |\psi\rangle\langle\psi| \), respectively.

We now consider the more general class of channels of the form

\[
\mathcal{E}(\gamma) = (1 - c)(\text{Tr}\gamma)\rho + c\gamma
\]

(34)
where \( \rho \) is a fixed density matrix. For \( \rho = \frac{1}{d}\mathbb{1} \), this is the usual depolarizing channel; for \( \rho = \frac{1}{a+b}\left[\frac{a}{d}\mathbb{1} + b|\psi\rangle\langle\psi|\right] \), it is the shifted depolarizing channel \( 33 \).

When \( c \geq 0 \) additivity was proved for the depolarizing channel in \( d \) dimensions using a majorization argument \( 12 \) from which multiplicativity immediately follows; for \(-\frac{1}{2} \leq c \leq 1\), (which is the range for which the map is CPT) multiplicativity of the depolarizing channel in \( d \) dimensions was proved in \( 21 \). Neither shifted depolarizing channels nor the generalization \( 33 \) seem to have been explicitly considered in the literature before. One could obtain a proof of multiplicativity for \( p = 2 \) when \( c > 0 \) by verifying that the positive element condition in \( 22 \) is satisfied. (In fact, these maps satisfy the stronger condition considered in \( 22 \).) However, neither of these positive element conditions can be verified when \( c < 0 \). By contrast, the method presented here can establish multiplicativity when \( p = 2 \) for all CPT maps of the form \( 33 \), including those with \( c < 0 \).

2. Convex combinations of the identity and completely noisy maps

It will be useful to write the spectral decomposition of \( \rho \) as \( \rho = \sum_j a_j|j\rangle\langle j| \) with the eigenvalues \( a_j \) in decreasing order. Then, for \( c > 0 \), the state \( \mathcal{E}(|1\rangle\langle 1|) \) majorizes all outputs so that \([\nu_p(\mathcal{E})]^p = [ca_1 + (1-c)]^p + c^p\sum_{j=1}^{d} a_j^p \).

Since, \( \mathcal{E}(B) = (1-c)[\text{Tr}\mathcal{B}\rho]\mathbb{1} + c\mathcal{B} \), we have

\[
\hat{\mathcal{E}}(|j\rangle\langle k|) = (1-c)|k\rangle\langle j|\mathbb{1} + c|j\rangle\langle k| = (1-c)\delta_{jk}a_k\mathbb{1} + c|j\rangle\langle k|,
\]

(35)
and

\[
\Omega(\mathcal{E}, 2) = (\hat{\mathcal{E}} \otimes \hat{\mathcal{E}})(S) = \sum_{jk} \hat{\mathcal{E}}(|j\rangle\langle k|) \otimes \hat{\mathcal{E}}(|k\rangle\langle j|)
\]

\[
= \sum_{jk} \left[ (1-c)^2\delta_{jk}k_k^2\mathbb{1} \otimes \mathbb{1} + c(1-c)\delta_{jk}a_k \left( \mathbb{1} \otimes |k\rangle\langle k| + |k\rangle\langle k| \otimes \mathbb{1} \right) + c^2 |j\rangle\langle k| \otimes |k\rangle\langle j| \right]
\]

\[
= (1-c)^2(\text{Tr}\rho^2)\mathbb{1} \otimes \mathbb{1} + c(1-c)[\mathbb{1} \otimes \rho + \rho \otimes \mathbb{1}] + c^2S.
\]

(36)
From this it is easy to see that \( \Omega(\mathcal{E}, 2) \) has \( d \) product eigenvectors of the form \(|kk\rangle\rangle \) with eigenvalues

\[
(1-c)^2(\text{Tr}\rho^2) + 2c(1-c)a_k + c^2 = [(1-c)a_k + c]^2 + (1-c)^2\sum_{j \neq k} a_j^2,
\]

(37)
and \( \binom{d}{2} \) blocks of the form \([(1-c)^2(\text{Tr}\rho^2) + c(1-c)(a_j + a_k)]\mathbb{1}_2 \otimes c^2\sigma_x \), with eigenvalues

\[
(1-c)^2(\text{Tr}\rho^2) + c(1-c)(a_j + a_k) \pm c^2
\]

(38)
and entangled eigenvectors \(2^{-1/2}(|jk⟩ ± |kj⟩)\). When \(c > 0\) all eigenvalues are non-negative and the largest singular value is \([(1-c)a_1 + c]^2 + (1-c)^2 \sum_{j>1} a_j^2 = |\nu_2(\mathcal{E})|^2\) associated with the product eigenvector \(|11⟩\). Therefore, one can use Theorem 11 or Corollary 2 to conclude that the channel \((34)\) is multiplicative for \(p = 2\) when \(c > 0\).

3. CPT Maps with a negative contribution from the identity

To analyze the case \(c < 0\), write \(c = -x\) with \(x = |c| > 0\), and recall that we assumed that the \(|a_j⟩\) are decreasing. It can still happen that all eigenvalues of \(Ω(\mathcal{E}, 2)\) are non-negative, in which case the largest singular value is \([(1+x)a_d - x]^2 + (1+x)^2 \sum_{j<d} a_j^2\) associated with the product eigenvector \(|dd⟩\). It turns out that the requirement that \(\mathcal{E}\) be CPT suffices to ensure that the eigenvalues of \(Ω(\mathcal{E}, 2)\) are non-negative. Therefore, any CPT map of the form \((34)\) is multiplicative for \(p = 2\).

To see the relevance of the CPT condition, observe that the CP requirement that \((\mathcal{E} \otimes 1)|\sum_{jk} |j⟩⟨k| \otimes |j⟩⟨k|\)

(which is the Choi matrix) is positive semi-definite holds if and only if \(B = (1+x)\rho - x \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\) is positive semi-definite. Then \(B\) has non-negative diagonal elements, which gives

\[(1+x)a_j - x \geq 0 \Rightarrow x \leq \frac{a_j}{1+x} \Rightarrow \frac{x}{1+x} \leq a_j.\]  \hspace{1cm} (39)

All \(2 \times 2\) principle minors of \(B\) are non-negative, which implies

\[(1+x)^2a_1a_2 - x(1+x)(a_1 + a_2) \geq 0\]  \hspace{1cm} (40)

Now, all eigenvalues of \(Ω(\mathcal{E}, 2)\) will be positive if \((1-c)^2(\text{Tr} \rho^2) + c(1-c)(a_j + a_k) - c^2 \geq 0\) for all \(j, k\). But the most negative of these is

\[(1+x)^2(\text{Tr} \rho^2) - x(1+x)(a_1 + a_2) - x^2 \geq (1+x)^2(a_1^2 + a_2^2) - (1+x)^2a_1a_2 - x^2 \]
\[= (1+x)^2\left[a_1^2 + a_2^2 - a_1a_2 - \left(\frac{x}{1+x}\right)^2\right]\]  \hspace{1cm} (41)
\[\geq (1+x)^2\left[a_1^2 + a_2^2 - 2a_1a_2\right] = (1+x)^2(a_1 - a_2)^2 \geq 0,

where the second inequality used (39) with \(j = 1, 2\) to conclude that \((\frac{x}{1+x})^2 \leq a_1a_2\).

C. The Werner-Holevo channel

In our final example, we apply our condition for \(p = 3, 4\) as well as \(p = 2\). We study the channels \(\mathcal{W}_d\) introduced in (34) to show that multiplicativity does not hold for sufficiently large \(p\). The channel \(\mathcal{W}_d\) is defined on a \(d\) dimensional Hilbert space as

\[\mathcal{W}_d(\gamma) = \frac{1}{d-1}[(\text{Tr} \gamma) 1_d - \gamma^T] = \frac{1}{d-1} \sum_{j<k} W_{jk}^\dagger \gamma W_{jk}\]  \hspace{1cm} (42)

with \(1_d\) the identity operator on \(\mathcal{H}\), \(\gamma^T\) the matrix transpose with respect to some fixed basis \(|i⟩\), and \(W_{jk}\) the anti-Hermitian operator \(|j⟩⟨k| - |k⟩⟨j|\). (We will often suppress the subscript \(d\) and simply write \(W\) for \(\mathcal{W}_d\).) As observed in (34), any pure input state yields an output state \(\mathcal{W}(|ψ⟩⟨ψ|)\) with eigenvalues \(1/(d-1)\) with multiplicity \(d-1\). This implies

\[\nu_p(\mathcal{W}_d) = (d-1)^{(1-p)/p}\]  \hspace{1cm} (43)

Werner and Holevo showed that for \(d = 3\) and \(p > 4.79\) this map is not \(\ell_p\) multiplicative, by showing that maximally entangled inputs yield output \(\ell_p\) norm greater than \((d-1)^{(1-p)/p}\). For \(d > 3\), they also showed that multiplicativity fails for sufficiently large \(p\). Although their results strongly suggest that multiplicativity does hold for smaller \(p\), they do not preclude the possibility that it fails with inputs that are partially entangled. Our results show that this cannot happen when \(p = 2, 3, 4\) and \(d \geq 2^p - 1\).
The multiplicativity of $\mathcal{W}$ for $p = 2$ was established in [23]; the additivity of minimal output entropy and Holevo capacity was proved in [28] and [11]; and, recently, a short elegant proof of multiplicativity for all $1 \leq p \leq 2$ was given in [2]. Here we use Theorem 4 to give another proof of (3) for $p = 2$, and then consider multiplicativity of $\Omega(\mathcal{W}, p)$ for integer $p > 2$.

For $p = 2$ it is straightforward to show that (or see Appendix C1)

$$\Omega(\mathcal{W}, 2) = (\mathcal{W} \otimes \mathcal{W})(S) = \frac{1}{(d - 1)^2} [(d - 2) \mathbb{1} \otimes \mathbb{1} + S].$$

with $S$ the SWAP on $\mathcal{H} \otimes \mathcal{H}$. The eigenvalues of $\Omega(\mathcal{W}, 2)$ can be computed from those of $S$ which has a diagonal block with $d$ product states $|jj\rangle$ as eigenvectors with eigenvalue $1$, and $(d^2)_2$ blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenvalues $+1$ and $-1$ corresponding to the entangled states $\frac{1}{\sqrt{2}} (|jk\rangle \pm |kj\rangle)$. This yields eigenvalues $\frac{1}{d-1}$ with multiplicity $\frac{d(d+1)}{2}$ and $\frac{d-3}{d-1}$ with multiplicity $\frac{d(d-1)}{2}$. For $d \geq 3$, these are also the singular values of $\Omega(\mathcal{W}, 2)$; for $d = 2$, $\frac{1}{d-1}$ is the only singular value. In both cases $\|\Omega(\mathcal{W}, 2)\|_{\infty} = \frac{1}{d-1} = \nu_2(\mathcal{W})$. Therefore, (21) is satisfied and the result follows from Theorem 1.

To study $p > 2$, we first observe that (C2) implies that $\Omega(\mathcal{W}, p)$ is a linear combination of permutation matrices. This has some important consequences.

a) $\Omega(\mathcal{W}, p)$ has a large number of invariant subspaces, giving it a block diagonal structure. Each block describes the restriction of $\Omega(\mathcal{W}, p)$ to a subspace spanned by all permutations of a vector $|\xi_{k_1}, \xi_{k_2}, \ldots, \xi_{k_p}\rangle$ with indices $k_1 \leq k_2 \leq \ldots \leq k_p$.

b) All row and column sums are equal. Moreover, (C3) implies that every row and column sum of $\Omega(\mathcal{W}, p)$ or, equivalently, of each block, is exactly $(d - 1)^{1-p}$, which is also the value of $\nu_p(\mathcal{W})^p$.

It follows immediately from (b) that $(d - 1)^{1-p}$ is an eigenvalue of each block of $\Omega(\mathcal{W}, p)$ and, hence, an eigenvalue of $\Omega(\mathcal{W}, p)$ with very high degeneracy. Therefore, $\Omega(\mathcal{W}, p)$ can have a singular value greater than $(d - 1)^{1-p}$. The following lemma, which is proved in Appendix C5 shows that it will suffice to consider this question for one of the largest blocks.

**Lemma 3** When $d \geq p$, the largest singular value of $\Omega(\mathcal{W}, p)$ is a singular value of each of the $p! \times p!$ blocks representing the restriction of $\Omega(\mathcal{W}, p)$ to a subspace of $(\mathbb{C}^d)^{\otimes p}$ spanned by all permutations of a vector $|\xi_{k_1}, \xi_{k_2}, \ldots, \xi_{k_p}\rangle$ with distinct $k_p$.

Based on this and the structure of the largest blocks as described in Appendix C4, we make the following

**Conjecture 4** The $\ell_p$ multiplicativity relation (4) holds for the channel $\mathcal{W}_d$ when the dimension $d \geq 2^{p-1}$.

This conjecture is proved for $p = 2, 3, 4$. For larger $p$ we have shown in Appendix C4 that the largest block of $\Omega(\mathcal{W}, p)$ has two eigenvectors which transform as the two one-dimensional representations of $S_p$. The corresponding eigenvalues are $(d - 1)^{1-p}$ and $(d - 1)^{-p}(d - 2^{p-1})$. When $d \geq 2^{p-1}$, $|d - 2^{p-1}| \leq d - 1$. Moreover, no other singular values have the symmetry associated with a one-dimensional representation of $S_p$. Thus, if we knew that the largest singular value of $\Omega(\mathcal{W}, p)$ must be associated with a one-dimensional irreducible representation, we could conclude that the largest singular value of $\Omega(\mathcal{W}, p)$ is $d - 1$, proving the conjecture.

Now we consider $p = 3, 4$. The results in Appendix C1 can be used to write $\Omega(\mathcal{W}, p)$ explicitly as

$$\Omega(\mathcal{W}, 3) = \frac{1}{(d-1)^2} \left[(d - 3) \mathbb{1} + \sum_{a < b} S_{ab} - R_3\right],$$

$$\Omega(\mathcal{W}, 4) = \frac{1}{(d-1)^3} \left[(d - 4) \mathbb{1} + \sum_{a < b} S_{ab} - \sum_{a < b < c} R_3(a, b, c) + R_4\right],$$

where the shift $R_3(a, b, c)$ is defined in Appendix A2. The block structure of $\Omega(\mathcal{W}, p)$ for $p = 3, 4$ is summarized in Table 1. In this table, $i, j, k, \ell$ always denote distinct indices. For readability, $(d - 1)^p \mu_{\text{max}}$ is reported in the last three columns, and should be compared to $(d - 1)^p \nu_p(\mathcal{W})^p = (d - 1)$. For $\Omega(\mathcal{W}, 3)$ and $\Omega(\mathcal{W}, 4)$, all singular values can be found explicitly with the help of Mathematica, with the largest for each block shown in Table 1. The multiplicativity condition (21) holds if the largest singular value is $(d - 1)^{1-p}$. For $p = 3$, this holds for $d \geq 4$; for $p = 4$, it holds for $d \geq 8$. For $p = 3$, an analytic argument, which does not require determining the eigenvalues of $\Omega(\mathcal{W}, 3)$, is presented in Appendix C2.
| number of blocks | size | type of vectors | non-neg | max sing value | $d = 3$ | $d = 4$ | $d \geq 5$ |
|------------------|------|----------------|--------|----------------|--------|--------|--------|
| $d$              | $1 \times 1$ | $|kkk\rangle$ | yes    | 2              | 3      | $d - 1$ |         |
| $d(d - 1)$       | $3 \times 3$ | $|jjk\rangle$ | yes    | 2              | 3      | $d - 1$ |         |
| $\binom{d}{d}$   | $6 \times 6$ | $|ijk\rangle$ | no     | 4              | 3      | max{$d - 1, |7 - d|$} |         |

| $p = 4$ |
|------------------|------|----------------|--------|----------------|--------|--------|--------|
| $d$              | $1 \times 1$ | $|kkkk\rangle$ | yes    | 2              | 3      | $d - 1$ |         |
| $d(d - 1)$       | $4 \times 4$ | $|jjjk\rangle$ | yes    | 2              | 3      | $d - 1$ |         |
| $\binom{d}{d}$   | $6 \times 6$ | $|jjkk\rangle$ | yes    | 2              | 3      | $d - 1$ |         |
| $\frac{1}{2}d(d - 1)(d - 2)$ | $12 \times 12$ | $|ijkk\rangle$ | no     | $\sqrt{18}$ | $\sqrt{13}$ | max{$d - 1, \sqrt{d^2 - 12d + 45}$} |         |
| $\binom{d}{d}$   | $24 \times 24$ | $|ijkl\rangle$ | no     | 11             |         | max{$d - 1, |15 - d|$} |         |

**TABLE I:** Block structure of $\Omega(W, p)$.

V. CONCLUSION

We have extended the method introduced in [13, 14] to study the maximal $\ell_p$-norms of a CPT map when $p$ is a fixed integer. This yields a sufficient condition for multiplicativity which requires only that one find the singular values of a particular matrix, rather than performing a full optimization. Although the matrix will be $d^p \times d^p$, it often has a block structure which makes the problems quite tractable, as shown in several examples. The condition is not necessary, but does allow us to prove new results about multiplicativity in several interesting cases, as well as providing alternative proofs of known results.

APPENDIX A: SOME OPERATOR PROPERTIES

1. Hilbert-Schmidt duality

For a Hilbert space $H$ the subspace of operators satisfying $\text{Tr} A^\dagger A < \infty$ also forms a Hilbert space (the space of Hilbert Schmidt operators) with respect to the inner product

$$\langle A, B \rangle = \text{Tr} A^\dagger B \quad (A1)$$

An operator (sometimes referred to as a “superoperator”) $\mathcal{E}$ acting on this space has an adjoint which we will denote $\hat{\mathcal{E}}$ and which satisfies

$$\text{Tr}[\mathcal{E}(A)]^\dagger B = \text{Tr} A^\dagger \hat{\mathcal{E}}(B) \quad \forall \ A, B. \quad (A2)$$

Because $[\mathcal{E}(A)]^\dagger = \mathcal{E}(A^\dagger)$, by writing $C$ for $A^\dagger$ one easily sees that (A2) is equivalent to the condition

$$\text{Tr}[\mathcal{E}(C)]B = \text{Tr} C \hat{\mathcal{E}}(B) \quad \forall \ B, C. \quad (A3)$$

The map $\hat{\mathcal{E}}$ is often called the dual of $\mathcal{E}$ because it is defined by the duality property of the Riesz representation theorem applied to the inner product (A1). When $\mathcal{E}$ is a CPT map of the form (1), its dual is the unital CP map with the form

$$\hat{\mathcal{E}}(\gamma) = \sum_k A_k^\dagger B A_k. \quad (A4)$$

One can verify, either directly from (A1) or by using (A3), that the dual of the map $\mathcal{E}^{\otimes m}$ is given by the $m$-fold tensor product of the dual map of $\mathcal{E}$, i.e. $\hat{\mathcal{E}}^{\otimes m} = (\hat{\mathcal{E}})^{\otimes m}.$
2. Shift operators

The shift operators defined in (5) are unitary and satisfy $L_p R_p = 1$ so that $L_p^{-1} = R_p$. Moreover, if a vector $|\Psi\rangle$ in $\mathcal{H}^\otimes p$ has the expansion

$$|\Psi\rangle = \sum_{j_1 \cdots j_p} c_{j_1 \cdots j_p} |\xi_{j_1} \xi_{j_2} \cdots \xi_{j_p}\rangle$$

(A5)

then

$$L_p |\Psi\rangle = \sum_{j_1 \cdots j_p} c_{j_1 \cdots j_p} |\xi_{j_2} \xi_{j_3} \cdots \xi_{j_1}\rangle$$

(A6)

$$= \sum_{j_1 \cdots j_p} c_{j_p j_1 \cdots j_{p-1}} |\xi_{j_1} \xi_{j_2} \cdots \xi_{j_p}\rangle$$

(A7)

so that $L_p$ induces a right shift on the expansion coefficients. From this, it follows that, that $L_p$ and $R_p$ induce left and right shifts on all product states, e.g.,

$$L_p |\phi_1, \phi_2, \cdots, \phi_p\rangle = |\phi_2, \phi_3, \cdots, \phi_p, \phi_1\rangle$$

(A8a)

$$R_p |\phi_1, \phi_2, \cdots, \phi_p\rangle = |\phi_p, \phi_1, \cdots, \phi_{p-1}\rangle$$

(A8b)

where $|\phi_1, \phi_2, \cdots, \phi_p\rangle$ denotes $|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_p\rangle$. It also follows from (A8b) that the shift operators are independent of the choice of orthonormal basis in $\mathcal{H}$.

To compute operators associated with the WH-channel, it will be useful to observe that

$$L_p = \sum_{m_1 \cdots m_p} |m_2 \cdots m_p m_1\rangle \langle m_1 m_2 \cdots m_p|$$

(A9a)

$$R_p = \sum_{m_1 \cdots m_p} |m_p m_1 \cdots m_{p-1}\rangle \langle m_1 m_2 \cdots m_p|$$

(A9b)

where $|m_{ij}\rangle$ denotes any orthonormal basis of $\mathcal{H}$. It will also be useful to introduce some notation for shift operators on a subset of $\mathcal{H}^\otimes p$. For example, write $\mathcal{H}^\otimes 4 = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \otimes \mathcal{H}_d$. Then $L_3(a, b, d)$ denotes the operator which acts as a left shift on $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_d$ and the identity on $\mathcal{H}_c$, i.e.,

$$L_3(a, b, d) = \sum_{m_1 \cdots m_4} |m_2 m_4 m_3 m_1\rangle \langle m_1 m_2 m_3 m_4|.$$

(A10)

The SWAP operators $L_2(a, b) = R_2(a, b)$ play such a special role that we denote them as $S_{ab}$. Using the standard method for writing any permutation as a product of cycles, one can see that any shift can be written as a product of SWAP operators, e.g. $L_3(a, b, d) = S_{ab} S_{ad}$ and $L_4(a, b, c, d) = S_{ab} S_{ac} S_{ad}$.

3. Tensor products of shifts

When the underlying Hilbert is itself a tensor product $\mathcal{H}^\otimes m$, we will let $L_p$ denotes the shift operator acting on $p$ copies of $\mathcal{H}^\otimes m$, e.g., $L_3|x, y, z\rangle = |y, z, x\rangle$ with $x, y, z$ denoting vectors in $\mathcal{H}^\otimes m$. Then, $L_p = L_p^\otimes m = (L_mp)^m$. To avoid notation with double subscripts, we prove this in the case $p = 3$. Then

$$L_3|x, y, z\rangle = L_3|x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m, z_1, z_2, \cdots, z_m\rangle$$

$$= |y_1, y_2, \cdots, y_m, z_1, z_2, \cdots, z_m, x_1, x_2, \cdots, x_m\rangle$$

(A11)

$$= L_3^\otimes m|x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m, z_1, z_2, \cdots, z_m\rangle$$

where the last line follows by writing

$$L_3^\otimes m = (L_3 \otimes 1 \otimes \cdots \otimes 1)(1 \otimes L_3 \otimes 1 \otimes \cdots \otimes 1) \cdots (1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes L_3)$$

and observing that

$$(L_3 \otimes 1 \otimes \cdots \otimes 1)|x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m, z_1, z_2, \cdots, z_m\rangle = |y_1, x_2, \cdots, x_m, z_1, y_2, \cdots, y_m, x_1, z_2, \cdots, z_m\rangle.$$

Note that it is also evident from (A11) that $L_p^\otimes m = (L_mp)^m$. 
4. An important trace identity

We now show that for any set of operators \(\{B_1, B_2, \cdots, B_p\}\) acting on \(\mathcal{H}\),

\[
\text{Tr}_{\mathcal{H}}[B_1 B_2 B_3 \cdots B_p] = \text{Tr}_{\mathcal{H}^\otimes p} [B_1 \otimes B_2 \otimes \cdots \otimes B_p] L_p , \tag{A12}
\]

where we have introduced subscripts to emphasize that the trace in the left-hand side of Eq. (A12) is performed on \(\mathcal{H}\), while the trace in the right-hand side is performed on \(\mathcal{H}^\otimes p\). To verify (A12) observe that

\[
\text{Tr}[B_1 B_2 \cdots B_p] = \sum_{\xi_1} \langle \xi_1 | B_1 B_2 \cdots B_{p-1} B_p | \xi_1 \rangle
\]

\[
= \sum_{\xi_1, \xi_2, \cdots, \xi_p} \langle \xi_1 | B_1 | \xi_2 \rangle \langle \xi_2 | B_2 | \xi_3 \rangle \cdots \langle \xi_{p-1} | B_{p-1} | \xi_p \rangle \langle \xi_p | B_p | \xi_1 \rangle
\]

\[
= \sum_{\xi_1, \xi_2, \cdots, \xi_p} \langle \xi_1, \xi_2, \cdots, \xi_p | B_1 \otimes B_2 \otimes \cdots \otimes B_p | \xi_2, \cdots, \xi_p, \xi_1 \rangle
\]

\[
= \text{Tr} [B_1 \otimes B_2 \otimes \cdots \otimes B_p] L_p ,
\]

where a resolution of the identity operator \(I\) of \(\mathcal{H}\) was inserted between the products \(B_jB_{j+1}\).

5. General permutations

Shifts are special cases of permutation operators. Let \(\Pi_p\) denote a permutation of \(\{1, 2, \cdots, p\}\) and \(\mathcal{S}_p\) the set of all such permutations. We will write \(\Pi(j) = k_j\) for the permutation that takes \(j \mapsto k_j\). For example, \(L_p(j) = j + 1\). One can then define a permutation operator on \(\mathcal{H}^\otimes p\) by

\[
\Pi_p | \xi_1, \xi_2, \cdots, \xi_p \rangle = | \Pi(\xi_1), \Pi(\xi_2), \cdots, \Pi(\xi_p) \rangle \tag{A13}
\]

with \(\{|\xi_j\rangle\}\) an orthonormal basis for \(\mathcal{H}\) as in (4). A permutation of the indices \(\{1, 2, \cdots, p\}\) induces a permutation on the \(d^p\) product basis vectors \(\mathcal{H}^\otimes p\) via (A13). Although we abuse notation by using the same letter for both, there should be no confusion. The permutation operator on \(\mathcal{H}^\otimes p\) is represented by a \(d^p \times d^p\) matrix which has precisely one 1 and \(d^p - 1\) 0’s in each row and column.

The permutation which takes \(k_1 \mapsto k_2 \mapsto \cdots \mapsto k_q \mapsto k_1\) is called a cycle and written \(P = (k_1, k_2, \cdots, k_q)\), i.e., \(P(k_q) = (k_{q+1})\) with the understanding that \(P(k_q) = k_1\) and \(\Pi(j) = j\) if \(j\) does not appear as one of the \(k_i\) in the cycle. Any permutation can be written uniquely as a product of disjoint cycles, and the length of the disjoint cycles in \(\Pi\Pi^\dagger\) are the same as those in \(P\). For example \((13)\{13\} = (14532)\). If a permutation of \(\{1, 2, \cdots, p\}\) has a cycle decomposition with cycles whose length is strictly less than \(p\), then some subset of \(\{1, 2, \cdots, p\}\) is invariant. A permutation \(\Pi_p\) whose shortest cycle is of length \(p\) has no invariant subsets. Permutations satisfying this condition, which is equivalent to \((\Pi_p)^s(j) \neq j\) for \(s < p\) and \((\Pi_p)^p(j) = j\) for all \(j\), are of particular interest.

In fact, when all operators \(B_i\) for \(i\) are identical, (A12) can be extended to any permutation \(\Pi_p\) of \(\{1, 2, \cdots, p\}\) whose shortest cycle is length \(p\). One finds

\[
\text{Tr}_{\mathcal{H}} B^p = \sum_{\xi_1, \cdots, \xi_p} \langle \xi_1 | B | \Pi_p(1) \rangle \langle \Pi_p(1) | B | \Pi_p(2) \rangle \cdots \langle \Pi_p(p) | B | \xi_1 \rangle
\]

\[
= \sum_{\xi_1, \cdots, \xi_p} \langle \xi_1 | B | \Pi_p(1) \rangle \langle \xi_2 | B | \Pi_p(2) \rangle \cdots \langle \xi_p | B | \xi_1 \rangle
\]

\[
= \sum_{\xi_1, \cdots, \xi_p} \langle \xi_1, \xi_2, \cdots, \xi_p | B \otimes B \otimes \cdots \otimes B | \xi_2, \cdots, \xi_p, \xi_1 \rangle
\]

\[
= \text{Tr}_{\mathcal{H}^\otimes p} [B \otimes B \otimes \cdots \otimes B] \Pi_p = \text{Tr}_{\mathcal{H}^\otimes p} B^\otimes p \Pi_p , \tag{A14}
\]

To see where the invariance condition is used, consider the permutation (153)(24). Attempting to apply the process above yields

\[
\text{Tr}_{\mathcal{H}} B^5 = \sum_{\xi_1, \xi_2, \xi_3} \langle \xi_1 | B | \xi_5 \rangle \langle \xi_5 | B | \xi_3 \rangle \langle \xi_3 | B^3 | \xi_1 \rangle
\]

\[
= \sum_{\xi_1, \xi_2, \xi_3} \langle \xi_1, \xi_5, \xi_3 | B \otimes B \otimes B^3 | \xi_5, \xi_3, \xi_1 \rangle
\]

\[
= \text{Tr}_{\mathcal{H}^\otimes \mathcal{H}_c} [B \otimes B \otimes B^3] L_3
\]
or $\text{Tr}_H B^5 = \text{Tr}_{H^\otimes 3} [B \otimes B^3 \otimes B] L_3$ or $\text{Tr}_H B^5 = \text{Tr}_{H^\otimes 2} [B \otimes B^4] L_2$.

6. Double stochastic matrices

A double stochastic matrix \[16\] is a matrix with non-negative elements whose row and column sums are all 1, i.e., $B$ is double stochastic if and only if $b_{jk} \geq 0 \ \forall \ j, k$ and $\sum_j b_{jk} = \sum_k b_{jk} = 1$. The vector $(1, 1, \cdots 1)$ is always an eigenvector with eigenvalue 1. Moreover, all other eigenvalues satisfy $|\lambda_j| \leq 1$. A permutation of $\{1, 2 \cdots p\}$ can be represented by a matrix which has precisely one 1 and $p-1$ 0’s in each row and column. This is a special type of double stochastic matrix called a “permutation matrix”. Moreover, a permutation $\Pi_p$ of $\{1, 2 \cdots p\}$ has no non-trivial invariant subspaces if and only if its permutation matrix is indecomposable. Note that the corresponding permutation operator on $H^p$, represented by a $d^p \times d^p$ matrix with precisely one 1 and $d^p - 1$ 0’s in each row and column, can have invariant subspaces. In fact, it will be block diagonal.

APPENDIX B: PROPERTIES OF LINEARIZING OPERATORS $X(\mathcal{E}, p)$

1. Kraus operator form of $\Omega(\mathcal{E}, p)$

We first observe that conjugation of a tensor product of operators by a shift operation induces a shift on the tensor product, e.g.,

$$L_p \left( B_1 \otimes B_2 \otimes \cdots \otimes B_p \right) L_p^{-1} = B_2 \otimes \cdots \otimes B_p \otimes B_1.$$  \hfill (B1)

More generally,

$$\Pi_p \left( B_1 \otimes B_2 \otimes \cdots \otimes B_p \right) \Pi_p^{-1} = B_{\Pi(1)} \otimes B_{\Pi(2)} \cdots \otimes B_{\Pi(p)}.$$  \hfill (B2)

To prove \[11\], one can use \[11\], and \[B1\] to see that

$$\left[ \hat{\mathcal{E}} \otimes^p (L_p) \right] R_p = \left[ \sum_{k_1, \cdots, k_p} (A_{k_1}^\dagger \otimes A_{k_2}^\dagger \otimes \cdots \otimes A_{k_p}^\dagger) L_p (A_{k_1} \otimes A_{k_2} \otimes \cdots \otimes A_{k_p}) \right] L_p^{-1}$$

$$= \sum_{k_1, \cdots, k_p} (A_{k_1}^\dagger \otimes A_{k_2}^\dagger \otimes \cdots \otimes A_{k_p}^\dagger) (A_{k_2} \otimes A_{k_3} \otimes \cdots \otimes A_{k_p} \otimes A_{k_1})$$

$$= \sum_{k_1, \cdots, k_p} A_{k_1}^\dagger A_{k_2} \otimes A_{k_2}^\dagger A_{k_3} \otimes \cdots \otimes A_{k_p}^\dagger A_{k_1}$$

which gives the desired result. Moreover, using a similar argument and \[B2\], one finds

$$R_p \left[ \hat{\mathcal{E}} \otimes^p (L_p) \right] = \left[ \sum_{k_1, \cdots, k_p} (A_{k_1}^\dagger \otimes A_{k_2}^\dagger \otimes \cdots \otimes A_{k_p}^\dagger) L_p^{-1} (A_{k_1} \otimes A_{k_2} \otimes \cdots \otimes A_{k_p}) \right]$$

$$= \sum_{k_1, \cdots, k_p} A_{k_1}^\dagger A_{k_1} \otimes A_{k_2}^\dagger A_{k_2} \otimes \cdots \otimes A_{k_{p-1}}^\dagger A_{k_p}.$$  \hfill (B4)

Then by observing that both \[B3\] and \[B4\] involve tensor products of operators of the form $A_{k_j}^\dagger A_{k_{j+1}}$, one sees that after a change of variable in the summation indices, e.g. $k_j \rightarrow k_{j-1}$ in \[B3\], the two expression are identical. Therefore, $R_p$ commutes with $\hat{\mathcal{E}} \otimes^p (L_p)$ and $\Theta(\mathcal{E}, p) = \Omega(\mathcal{E}, p) R_p = R_p \Omega(\mathcal{E}, p)$.

2. General permutations

Define $X(\mathcal{E}, p)$ the set of operators $X(\mathcal{E}, p)$ of $H^\otimes p$ that satisfy the property \[11\] for all the input states $\gamma$ of $\mathcal{H}$. We have already seen that $\hat{\mathcal{E}} \otimes^p (L_p)$ is in $X(\mathcal{E}, p)$ which implies that it is non-empty. Moreover, the linearity of Eq. \[11\] with respect to $X(\mathcal{E}, p)$ implies that whenever $X(\mathcal{E}, p)$ and $Y(\mathcal{E}, p)$ are in $X(\mathcal{E}, p)$, then $a X(\mathcal{E}, p) + (1-a) Y(\mathcal{E}, p)$ is
in $X(\mathcal{E}, p)$ is also. This true for any real number $a$ including $a < 0$ and $a > 1$, and even for complex $a$. By choosing $0 < a < 1$, we can also conclude that $\mathcal{X}(\mathcal{E}, p)$ is convex; however, $\mathcal{X}(\mathcal{E}, p)$ is not compact. Because $\text{Tr}[\mathcal{E}(\gamma)]^p$ is real, 
\[
\text{Tr}[\gamma \otimes p \mathcal{E}(\mathcal{E}, p)] = \text{Tr}[\mathcal{E}(\mathcal{E}, p)]^p = \text{Tr}[\gamma \otimes p \mathcal{E}(\mathcal{E}, p)] = \text{Tr}[\mathcal{E}(\mathcal{E}, p)]^p \gamma \otimes p = \text{Tr}[\gamma \otimes p \mathcal{E}(\mathcal{E}, p)]^p \tag{B5}
\]
for all density matrices $\gamma$. Therefore, whenever $X(\mathcal{E}, p)$ is in $\mathcal{X}(\mathcal{E}, p)$ so are $[X(\mathcal{E}, p)]^p$ and the self-adjoint operator 
\[
\frac{1}{2} \left( X(\mathcal{E}, p) + [X(\mathcal{E}, p)]^p \right). 
\]
In view of the discussion in Appendix A we can also conclude that the operator $\tilde{\mathcal{E}}^\otimes p(\Pi_p)$ is in $\mathcal{X}$ whenever $\Pi_p$ is a permutation whose shortest cycle is length $p$. Moreover, a modification of the argument in the preceding section shows that, for these permutations,
\[
\tilde{\mathcal{E}}^\otimes p(\Pi_p) = \Pi_p^\dagger \tilde{\mathcal{E}}^\otimes p(\Pi_p) = \sum_{k_1, \ldots, k_p} A_{k_1}^\dagger A_{k_2}^\dagger A_{k_3}^\dagger \cdots A_{k_p}^\dagger A_{k_p} \tag{B6}
\]
Since $\Pi_p^\gamma \Pi_p^\dagger = \Pi_p$ for any permutation,
\[
\text{Tr}[\gamma \otimes p (\Pi_p X(\mathcal{E}, p) \Pi_p^\dagger)] = \text{Tr}[(\Pi_p^\gamma \otimes p \Pi_p) \mathcal{E}(\mathcal{E}, p)] = \text{Tr}[\gamma \otimes p \mathcal{E}(\mathcal{E}, p)] = \text{Tr}[\mathcal{E}(\gamma)]^p \tag{B7}
\]
Note that the map $P_p \mapsto \Pi_p^\gamma \Pi_p^\dagger$ does not change the cycle structure of $P_p$, e.g., if $P_p$ is a product of a 3-cycle and a disjoint 2-cycle, then so is $\Pi_p^\gamma \Pi_p^\dagger$. Thus, $\Pi_p^\gamma \Pi_p^\dagger$ is a permutation whose shortest cycle is length $p$ irrespective of the cycle structure of $\Pi_p$. One can show that $\Pi_p^\gamma \Pi_p^\dagger \mathcal{E}(\gamma)L_p \Pi_p^\dagger = \tilde{\mathcal{E}}^\otimes p(\Pi_p^\gamma \Pi_p^\dagger \Pi_p^\dagger)$, with a similar result when $L_p$ is replaced by any permutation whose shortest cycle is length $p$.

3. Linearizing operators for pure inputs

The set $\mathcal{X}(\mathcal{E}, p)$ is a subset of $\mathcal{X}_{\text{pure}}(\mathcal{E}, p)$, the set of operators, which satisfy the property $\mathbb{1}$ when $\gamma = |\psi\rangle\langle\psi|$ is pure. We have already observed that $\Theta(\mathcal{E}, p) = \Omega(\mathcal{E}, p) R_p$ belongs to $\mathcal{X}_{\text{pure}}(\mathcal{E}, p)$ but need not belong to $\mathcal{X}(\mathcal{E}, p)$. It follows from B6 that the operators $\tilde{\mathcal{E}}^\otimes p(\Pi_p)\Pi_p^\dagger$ are also in $\mathcal{X}_{\text{pure}}(\mathcal{E}, p)$ in addition, for any $X(\mathcal{E}, p) \in \mathcal{X}_{\text{pure}}(\mathcal{E}, p)$ the operators $X(\mathcal{E}, p)\Pi_p$ and $\Pi_p X(\mathcal{E}, p)$ are also in $\mathcal{X}_{\text{pure}}(\mathcal{E}, p)$ for all permutations $\Pi_p$. This follows from
\[
\text{Tr} \left[ (\gamma \otimes \cdots \otimes \gamma) X(\mathcal{E}, p) \Pi_p \right] = \text{Tr} \left[ |\psi\rangle\langle\psi| \otimes \cdots \otimes |\psi\rangle\langle\psi| X(\mathcal{E}, p) \Pi_p \right] = \text{Tr} \left[ \Pi_p^\gamma \otimes \cdots \otimes \Pi_p^\gamma X(\mathcal{E}, p) \right] = \text{Tr} \left[ \gamma \otimes \cdots \otimes \gamma X(\mathcal{E}, p) \right] = \text{Tr} \mathcal{E}(\gamma)^p,
\]
whenever $\gamma = |\psi\rangle\langle\psi|$ is pure.

**APPENDIX C: OPERATORS FOR WERNER-HOLEVO CHANNEL**

1. General form of $\Omega(W, p)$

It follows from B6, A9 and A9 that for the WH channel,
\[
\Omega(W, p) = \sum_{\xi_1 \cdots \xi_p} W(\xi_1 |\xi_1 \rangle \rangle \otimes \cdots \otimes W(\xi_p |\xi_p \rangle \rangle) = \int_{(d-1)^p}^{d} \left( \sum_{\xi_1 \cdots \xi_p} |\tilde{\xi}_1 \rangle \langle \tilde{\xi}_1 | + \sum_{\xi_2} |\tilde{\xi}_2 \rangle \langle \tilde{\xi}_2 | + \cdots + \sum_{\xi_p} |\tilde{\xi}_p \rangle \langle \tilde{\xi}_p | \right) \right]
\]
\[
+ \sum_{a < b} \left( \sum_{\xi_a \xi_b} \xi_a \xi_b \xi_a \xi_b \right) - \cdots + (-1)^p \sum_{\xi_1 \cdots \xi_p} |\tilde{\xi}_1 \xi_2 \cdots \xi_p \rangle \langle \tilde{\xi}_2 \xi_3 \cdots \xi_p | \right) \tag{C1}
\]
\[
= \frac{1}{(d-1)^p} \left[ d - \sum_{\xi_1} \xi_1 \sum_{\xi_2} \xi_2 + \cdots + \sum_{\xi_p} \xi_p \right] \tag{C2}
\]
where we have used the notation introduced at the end of Appendix A. Note that the orthonormal basis \( \{ |\xi_j\rangle\} \) can be chosen real, but even if it is not, \( \{ \tilde{|\xi_j\rangle}\} \) gives another orthonormal basis for \( \mathcal{H} \) for which the representation (A.9) is also valid.

It is useful to compare the structure of (C.2) to that of a binomial expansion. The term in square brackets is a sum of shift operators \( R_k \) of order \( k = 0, 1, 2, \cdots, p \). For \( k \geq 2 \) the number of \( R_k \) is \( {p \choose k} \) with coefficient \( (-1)^k \). In view of (C.1), the \((d - p)\mathbb{I}\) term should be regarded as the sum of a \( k = 0 \) term \( d\mathbb{I} \) and a \( k = 1 \) term \(-p\mathbb{I}\). The coefficient of the \( k = 0 \) term is anomalous, since it has the value \( d \) rather than 1. This implies that the row and column sums of the matrix representing \( \Omega(\mathcal{W}, p) \) in the orthonormal basis \( \{|\xi_j\rangle, |\xi_j\rangle, \cdots, |\xi_j\rangle\} \) of \( \mathcal{H}^{\otimes p} \) are

\[
\frac{1}{(d - 1)^p} \left[ d + \sum_{k=1}^{p} (-1)^k \binom{p}{k} \right] = \frac{d - 1}{(d - 1)^p}.
\]

We similarly find that the sum of the absolute values of elements in any row or column sum is bounded above by

\[
\frac{1}{(d - 1)^p} \left[ d + \sum_{k=1}^{p} \binom{p}{k} \right] = \frac{(d - 1 + 2^p)}{(d - 1)^p},
\]

and we will use the fact that \( \sum_{k=2}^{p} \binom{p}{k} = 2^p - p - 1 \).

2. Singular value analysis for \( p = 3 \)

We first remark that one can reduce the analysis of \( \Omega(\mathcal{W}, 3) \) to that of its \( 6 \times 6 \) blocks without using Lemma 5. When \( p = 3 \), all blocks with basis vectors \( |jjk\rangle \) with \( j \neq k \) have only non-negative elements. To see why, note that the only negative contribution comes from \( R_3 \), for which \( \langle jjk| R_3 |jjk\rangle = -1 \) is the only non-zero element of the row corresponding to \( jjk \). But \( \langle jjk| \Omega |jjk\rangle \geq \langle jjk| (S_{ac} - R_3) |jjk\rangle = 0 \). Therefore, every \( 3 \times 3 \) blocks is represented by a stochastic matrix and, hence, its column sum \((d - 1)^{-p}\) is also its largest singular value. Thus, only the \( 6 \times 6 \) blocks of \( \Omega(\mathcal{W}, 3) \) can have negative elements and, hence, a singular value greater than \((d - 1)^{-p}\).

Using an ordered basis whose first three elements are \( \{ |ijk\rangle, L_3 |ijk\rangle, L_3^* |ijk\rangle \} \) and last three \( S_{ab} \{ |ijk\rangle, S_{ac} |ijk\rangle, S_{ac} |ijk\rangle \} \), one can write each \( 6 \times 6 \) block as \((d - 1)^{-3} F \) with

\[
F = (d - 3)\mathbb{I}_6 + \begin{pmatrix} -L_3 & V \\ V & -L_3 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]  

(C.4)

Then

\[
F^\dagger F = (d - 3)^2 \mathbb{I} + (d - 3)^2 \begin{pmatrix} G + G^\dagger \\ G^\dagger G \end{pmatrix} = (d^2 - 5d + 7) \mathbb{I}_6 + \begin{pmatrix} -d + 6 & 2d - 8 \\ 2d - 8 & -d + 6 \end{pmatrix} \otimes V.
\]

(C.5)

Since the eigenvalues of \( V \) are 3, 0, 0, the non-zero eigenvalues of \( F^\dagger F \) are \( d^2 - 5d + 7 \) (with 4-fold degeneracy) and \( (d^2 - 5d + 7) + 3(6 - d) \pm (2d - 8) \) or \((d - 7)^2\) and \((d - 1)^3\). Now \( (d^2 - 5d + 7) \leq (d - 1)^3\) when \( d \geq 2 \) and \((d - 7)^2 \leq (d - 1)^3\) if and only if \( d \geq 4 \). Therefore, when \( d \geq 4 \) the largest singular value of this block is \( d - 1 \) which implies that the largest singular value of \( \|\Omega(\mathcal{W}, 3)\|_\infty = (d - 1)^{-2}\).

3. Singular value analysis for \( p = 4 \)

For \( p = 4 \), one can show that the \( 4 \times 4 \) and \( 6 \times 6 \) blocks have only non-negative elements. Therefore, their largest singular value is the same as the column sum \((d - 1)^{-3}\). \( \Omega(\mathcal{W}, 4) \) also has \( 12 \times 12 \) blocks corresponding to permutations of \(|ijkk\rangle\), with \( i, j, k \) distinct and \( 24 \times 24 \) blocks corresponding to permutations of \(|ijk\ell\rangle\), with \( i, j, k, \ell \) distinct. By Lemma 5 the largest singular value is associated with the latter. Nevertheless, an analysis of all blocks was performed using Mathematica, yielding the results summarized in Table 5. This confirms that the largest singular value of \( \Omega(\mathcal{W}, 4) \) is \((d - 1)^{-3}\) when \( d \geq 8 \).
TABLE II: Singular value decomposition of $\Omega(W,4)$ on the twelve dimensional subspace generated by the vectors $\{|i j k k\}, |j i k k\}, \cdots, |k j i k\}$ and the twenty four dimensional subspace generated by $\{|i j k \ell\}, |j i k \ell\}, \cdots, |k j i \ell\}$. The singular values of $\Omega(W,4)$ are given in the left column, with the corresponding degeneracies in the central and right columns.

| singular value $d$ | degeneracy $(12 \times 12$ blocks) | degeneracy $(24 \times 24$ blocks) |
|-------------------|-------------------|-------------------|
| $\sqrt{d^2 - 12d + 45}$ | 2 | 6 |
| $|d - 5|$ | 1 | 3 |
| $|d - 3|$ | 3 | 5 |
| $\sqrt{d^2 - 4d + 5}$ | 4 | 6 |
| $|d - 1|$ | 2 | 3 |
| $|d - 15|$ | 0 | 1 |

4. Structure of largest block

a. Preliminaries

Recall that every permutation $P$ in $S_p$ can be classified as even or odd, depending on the number of transpositions (or SWAP) operators needed to write it as a product $P = S_{a_1b_1}S_{a_2b_2} \cdots S_{a_mb_m}$. Although this decomposition is not unique, $m$ is either always even or always odd. Let $|P|$ be the minimal number of swaps needed so that $(-1)^{|P|} = \begin{cases} +1 & \text{if } P \text{ is even} \\ -1 & \text{if } P \text{ is odd} \end{cases}$. Note that $S(a,b)$ and $R_4(a,b,c,d)$ are odd and $R_3(a,b,c)$ is even. More generally, a shift of $j$ elements is even when $j$ is odd and odd when $j$ is even. Thus, one can write

$$\Omega(W,p) = \frac{1}{(d-1)^p} \left[(d-p)I + \Omega_{\text{odd}} - \Omega_{\text{even}}\right]$$

where $\Omega_{\text{odd}}$ is the sum over odd permutations (even shifts) in (C8) and $\Omega_{\text{even}}$ the sum over even permutations (odd shifts) in (C2).

Fix $k_1 < k_2 < \cdots < k_p$ and let $K$ denote the subspace spanned by $\{|P|\xi_{k_1}, \xi_{k_2}, \cdots, \xi_{k_p}\} : P \in S_p\}$ where $|\xi_k|$ is an orthonormal basis for $\mathbb{C}^d$ and the action of $P$ is as defined in (A18). The matrix representing a particular permutation operator $\Pi$ has elements

$$\pi_{st} = \langle \xi_{k_1}, \xi_{k_2}, \cdots, \xi_{k_p} | P_s^\dag \Pi P_t | \xi_{k_1}, \xi_{k_2}, \cdots, \xi_{k_p} \rangle$$

which depends only on the labeling $P_s$, $s = 1, 2, \ldots, p$ of elements of $S_p$ and not on the choice of indices $k_j$ or vectors $\xi_j$. It will be convenient to simply use $|k|$ to denote $|\xi_k|$, and to write $|\Pi(k_1, k_2, \ldots, k_p)|$ for $\Pi|\xi_{k_1}, \xi_{k_2}, \cdots, \xi_{k_p}\rangle$. (The condition $k_j < k_{j+1}$ is only a convenient convention; the essential requirement is that the $k_j$ are distinct.)

b. Irreducible representation structure

The matrix representing the action of a permutation $\Pi$ on the vectors $\{|P|k_1, k_2, \ldots, k_p\} : P \in S_p\}$ is identical to its matrix in the regular representation of $S_p$. Therefore, one can find a unitary transformation to a basis whose components form disjoint subsets which transform as the irreducible representations of $S_p$. This basis change simultaneously converts all permutations to a block diagonal form. Thus, $\Omega(W,p)$, is also block diagonal with each block corresponding to an irreducible representation of $S_p$. The two one-dimensional representations, therefore, yield eigenvectors of $\Omega(W,p)$. In fact

$$\Omega(W,p) |\phi_{\text{sym}}\rangle = \frac{d - 1}{(d-1)^p} |\phi_{\text{sym}}\rangle$$

$$\Omega(W,p) |\phi_{\text{anti}}\rangle = \frac{d - 2p + 1}{(d-1)^p} |\phi_{\text{anti}}\rangle$$
where
\[ |\phi_{\text{sym}}\rangle = \frac{1}{\sqrt{p!}} \sum_{P \in S_p} |P(k_1, k_2, \ldots, k_p)\rangle = \frac{1}{\sqrt{2}} (|u_{\text{even}}\rangle + |u_{\text{odd}}\rangle) \] (C9a)
\[ |\phi_{\text{anti}}\rangle = \frac{1}{\sqrt{p!}} \sum_{P \in S_p} (-)^p |P(k_1, k_2, \ldots, k_p)\rangle = \frac{1}{\sqrt{2}} (|u_{\text{even}}\rangle - |u_{\text{odd}}\rangle) \] (C9b)

with \( |u_{\text{even}}\rangle = \sqrt{\frac{p}{d!}} \sum_{P_{\text{even}}} |P(k_1, k_2, \ldots, k_p)\rangle \), and \( |u_{\text{odd}}\rangle = \sqrt{\frac{p}{d!}} \sum_{P_{\text{odd}}} |P(k_1, k_2, \ldots, k_p)\rangle \). If we could conclude that the largest singular value of \((d-1)p\Omega(W, p)\) is associated with a one-dimensional representation of \(S_p\), then we could conclude that \((d-1)p\|\Omega(W, p)\|_\infty = \max\{d-1, |d-2^p+1|\}\). Note that this maximum is clearly \(d-1\) when \(d \geq 2^p - 1\). For \(d < 2^p\), the maximum is \(d-1\) if and only if \(2^p - d \leq d - 1 \Leftrightarrow 2d \geq 2^p\).

c. Odd/even structure

We now describe the odd/even structure of \(\Omega(W, p)\). We can divide the \(p!\) basis vectors of \(K\) into two equal subsets, those of the form \(P_{\text{even}}[k_1, k_2, \ldots, k_p]\) and those of the form \(P_{\text{odd}}[k_1, k_2, \ldots, k_p]\). We will denote their spans as \(\mathcal{K}_{\text{even}}\) and \(\mathcal{K}_{\text{odd}}\) respectively. Now \(\langle k_1, k_2, \ldots, k_p | \Pi | k_1, k_2, \ldots, k_p \rangle = 0\) unless \(\Pi\) is the identity permutation. Therefore \(\langle P_s(k_1, k_2, \ldots, k_p) | \Pi | P_s(k_1, k_2, \ldots, k_p) \rangle = 0\) unless \(\Pi = P_s P_s^t = I\). Moreover, since the identity is an even permutation

\[ \langle P_{\text{even}}(k_1, k_2, \ldots, k_p) | \Pi_{\text{odd}} | \tilde{P}_{\text{even}}(k_1, k_2, \ldots, k_p) \rangle = \langle P_{\text{odd}}(k_1, k_2, \ldots, k_p) | \Pi_{\text{odd}} | \tilde{P}_{\text{odd}}(k_1, k_2, \ldots, k_p) \rangle = 0 \] (C10)
\[ \langle P_{\text{even}}(k_1, k_2, \ldots, k_p) | \Pi_{\text{even}} | \tilde{P}_{\text{odd}}(k_1, k_2, \ldots, k_p) \rangle = \langle P_{\text{odd}}(k_1, k_2, \ldots, k_p) | \Pi_{\text{even}} | \tilde{P}_{\text{even}}(k_1, k_2, \ldots, k_p) \rangle = 0 \] (C11)

Thus, the largest block of \((d-1)p\Omega(W, p)\) can be written in the form \(B = (d-p)I + \begin{pmatrix} -B_{\text{ee}} & B_{\text{eo}} \\ B_{\text{oe}} & -B_{\text{oo}} \end{pmatrix}\) with \(B_{\text{ee}}\) and \(B_{\text{oo}}\) determined by \(\tilde{\Omega}_{\text{even}}\) and \(B_{\text{eo}}\) and \(B_{\text{oe}}\) determined by \(\tilde{\Omega}_{\text{odd}}\).

It is useful to relate the order of elements within the bases associated with odd and even permutations. Let \(P_1, P_2, \ldots, P_M\) with \(M = p!/2\) denote the even permutations (with \(P_1 = I\)) and \(P_{t+M} = P_s S\) the odd, where \(S\) denotes the swap operator \(S(k_1, k_2, k_3, \ldots, k_p) = k_2, k_1, k_3, \ldots, k_p\). (There is nothing special about applying SWAP to the first two elements. Any fixed choice would do.) Then

\[ b_{s, t+M} = \langle P_s(k_1, k_2, \ldots, k_p) | \Pi | P_t S(k_1, k_2, \ldots, k_p) \rangle = \langle P_s S(k_2, k_1, k_3, \ldots, k_p) | \Pi | P_t(k_2, k_1, \ldots, k_p) \rangle \] (C12)

where we used the fact that the matrix representing a permutation is independent of the initial choice of \(k_1\). Thus, \(B_{\text{eo}} = B_{\text{oe}}\) and, for the same reason, \(B_{\text{ee}} = B_{\text{oo}}\), and we can write

\[ B = (d-p)I + \begin{pmatrix} -W_e & W_o \\ W_o & -W_e \end{pmatrix} = (d-p)I + B_{\text{off}}. \] (C13)

where \(W_e\) and \(W_o\) are determined by \(\tilde{\Omega}_{\text{even}}\) and \(\tilde{\Omega}_{\text{odd}}\) respectively. By conjugating with \(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\), one finds that \(B\) has the same singular values as

\[ G = (d-p)I - \begin{pmatrix} W_e & W_o \\ W_o & W_e \end{pmatrix} = (d-p)I - G_{\text{off}} \] (C14)

Note that we have shown that the restriction of \(\Omega(W, p)\) to \(K\) is similar to \(\frac{1}{(d-1)p} \left[ (d-p)I - \tilde{\Omega}_{\text{odd}} - \tilde{\Omega}_{\text{even}} \right]\) which differs from \(\tilde{\Omega}_{\text{even}}\) by a sign. Although this may seem surprising, it could easily be established directly by observing that any vector \(|v\rangle \in K\) can be written as \(|v\rangle = |v_{\text{even}}\rangle + |v_{\text{odd}}\rangle\) with \(|v_{\text{even}}\rangle \in \mathcal{K}_{\text{even}}\) and \(|v_{\text{odd}}\rangle \in \mathcal{K}_{\text{odd}}\). Using \(\tilde{\Omega}_{\text{even}}\)
and related combinatorics, one finds that the row and column sums of \( B, G, W_e \) and \( W_o \) are, respectively, \( d - 1, d - 2p + 1, 2^{p-1} - p, \) and \( 2^{p-1} - 1. \) It follows that \( d - 1 \) and \( d - 2p + 1 \) are eigenvalues of \( B \) and \( G \).

\[
B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (d-1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (d-2p+1) \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]

\[
G \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (d-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad G \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (d-2p+1) \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

(C15a, b)

where \( 1 \) denotes a vector with all 1's. These are easily seen to be equivalent to (C18). Let \( d \) and related combinatorics, one finds that the row and column sums of basis \( \tilde{B} \) to (C18). One finds \( d \) in (C16) we showed that \( \tilde{B} \) is a multiple of a double stochastic matrix, its column sum \( 2^p - p - 1 \) is both its largest eigenvalue and its largest singular value. Therefore, \( d - 2p + 1 \) is the smallest eigenvalue of \( G \); however, even when it is the most negative eigenvalue, we cannot conclude that it is also the largest singular value because \( G \) could have a positive, or complex, eigenvalue of greater magnitude.

Remark: Conjugating \( B \) with the block Hadamard transform \( H = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \) corresponds to making the change of basis to (C18). One finds

\[
HFH^\dagger = (d - p)\mathbb{1} + \begin{pmatrix} -W_e + W_o & 0 \\ 0 & -W_e - W_o \end{pmatrix}.
\]

(C16)

5. Proof that \( \|\Omega(W,p)\|_\infty \) is attained on the largest blocks

As above, fix \( k_1 < k_2 < \ldots < k_p \) and let \( B \) denote the block of \( \Omega(W,p) \) corresponding to their span \( \mathcal{K} \).

For simplicity, we first compare the singular values of \( B \) to those for a block spanned by vectors of the form

\[
\{\Pi|j,j,k_3,\ldots,k_p\} : \Pi \in S_p \}
\]

(C17)

with \( j < k_3 < \ldots < k_p \). Observe that

\[
\left\{ \frac{1}{\sqrt{2}}\Pi(|k_1,k_2,k_3,\ldots,k_p) \pm |k_2,k_1,k_3,\ldots,k_p) : \Pi \in S_p, \Pi \neq S_{12} \right\}
\]

(C18)

is another orthonormal basis for \( \mathcal{K} \), and let \( V \) be the unitary matrix for the basis change from \( \{P|k_1,k_2,\ldots,k_p\}, P \in S_p \} \) to (C18). Let \( \mathcal{K}_+ \) denote the subspace spanned by vectors with a \( \pm \) sign in (C18), and \( \tilde{B}_{++} \) the submatrix for the restriction of \( VBV^\dagger \) to the subspace \( \mathcal{K}_+ \). The effect of any permutation on vectors of the form (C17) and those with a \( + \) sign in (C18) is the same. Therefore, \( \tilde{B}_{++} \) is identical to the matrix for the restriction of \( \Omega(W,p) \) to the span of (C17), and the largest singular value of the latter is the same as

\[
\|\tilde{B}_{++}\|_\infty = \sup_{\phi \in \mathcal{K}_+} \frac{\langle \phi, \tilde{B}_{++} \phi \rangle}{\|\phi\|^2} = \sup_{\phi \in \mathcal{K}_+} \frac{\langle \phi, VBV^\dagger \phi \rangle}{\|\phi\|^2} \\
\leq \sup_{\phi \in \mathcal{K}} \frac{\langle \phi, VBV^\dagger \phi \rangle}{\|\phi\|^2} = \|B\|_\infty^2.
\]

(C19)

In (C16) we showed that \( \tilde{B}_{++} = -W_e + W_o \) and that \( B \) is block diagonal, which immediately implies that the singular values of \( \tilde{B}_{++} \) are a subset of those for \( B \). This is stronger than (C19), but does not necessarily generalize.

Next, consider a block for a subspace spanned by vectors of the form

\[
\{\Pi|j,j,\ldots,j,k_{m+1},\ldots,k_p\} : \Pi \in S_p \}
\]

(C20)

with \( m \) occurrences of \( j \) and \( j < k_{m+1} \ldots k_p \). We adopt the convention that \( Q \in S_m \) denotes a permutation of \( \{1,2,\ldots,m\} \). Choose \( p! / m! \) permutations \( P_t \in S_p \) such that each \( P_t \) is in a distinct coset of \( S_p/S_m \) or, equivalently \( P_s P_t^{-1} \notin S_m \) \( \forall s \neq t \). Then the vectors

\[
|\phi_t\rangle = \frac{1}{\sqrt{m}} \sum_{Q \in S_m} P_t Q |k_1 \ldots k_m, k_{m+1} \ldots k_p\rangle.
\]

(C21)
transform under permutations exactly as those in (C20). Therefore, the restriction of \(B\) to the span of (C21) is represented by the same matrix as the block of \(\Omega(W, p)\) corresponding to (C20). Then, as in (C19), its largest singular value is bounded above by \(\|B\|_\infty\).

To deal with the general case, note that the restriction \(j < k_{m+1} < k_{m+2} < \ldots < k_p\) does not play an essential role. The same argument works whenever \(j\) is distinct from the remaining \(k_i\) with \(i > m\). Then, for example, the largest singular value of the block for permutations of \(|i, i, i, j, j, k_6 \ldots k_p\rangle\) \(\leq\) largest singular value of the block for permutations of \(|i, i, i, k_4, k_5, \ldots, k_p\rangle\) \(\leq\) largest singular value of the block for permutations of \(|k_1, k_2, \ldots, k_p\rangle = \|B\|_\infty\).

Proceeding in this way, one can complete the argument by induction. Alternatively, one could consider cosets for repeated indices, such as \(S_p/(S_3 \times S_2)\) in this example.

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