SHAPE-INVARIANCE AND MANY-BODY PHYSICS

A. B. BALANTEKIN
University of Wisconsin, Department of Physics
Madison, WI 53706, USA
E-mail: baha@nucth.physics.wisc.edu

Recent developments in the study of shape-invariant Hamiltonians are briefly summarized. Relations between certain exactly solvable problems in many-body physics and shape-invariance are explored. Connection between Gaudin algebras and supersymmetric quantum mechanics is pointed out.

1 Introduction

Supersymmetric Quantum Mechanics (SSM) is the name given to the study of particular pairs of Hamiltonians. SSM can be motivated by considering the ground state wavefunction, \( \psi_0(x) \), for a one-dimensional bound system. Since \( \psi_0(x) \) has no nodes it can be written as

\[
\psi_0(x) = \exp \left( -\frac{\sqrt{2m}}{\hbar} \int W(x) dx \right),
\]

where the function \( W(x) \) is related to the potential energy of the system. Introducing the operators

\[
\hat{A} = W(x) + \frac{i}{\sqrt{2m}} \hat{p},
\]

\[
\hat{A}^\dagger = W(x) - \frac{i}{\sqrt{2m}} \hat{p},
\]

one can write the Hamiltonian of the system as

\[
\hat{H} - E_0 = \hat{A}^\dagger \hat{A},
\]

where \( E_0 \) is the ground state energy. The ground state wavefunction satisfies the condition

\[
\hat{A} |\psi_0\rangle = 0.
\]

It is straightforward to show that the supersymmetric partner potentials

\[
\hat{H}_1 = \hat{A}^\dagger \hat{A}
\]

\[
\hat{H}_2 = \hat{A} \hat{A}^\dagger
\]

have the same energy spectra except the ground state of \( \hat{H}_1 \), the energy of which is zero. Potentials corresponding to these Hamiltonians are

\[
V_1(x) = |W(x)|^2 - \frac{\hbar}{\sqrt{2m}} \frac{dW}{dx}
\]

\[
V_2(x) = |W(x)|^2 + \frac{\hbar}{\sqrt{2m}} \frac{dW}{dx}.
\]
The partner potentials in Eq. (6) are called shape-invariant if they can be obtained from one another by changing their parameters:

\[ V_2(x; a_1) = V_1(x; a_2) + R(a_1), \]  

where \( a_2 \) is a function of \( a_1 \), and the remainder \( R(a_1) \) is independent of \( x \). Eq. (7) is equivalent to the operator relation

\[ \hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1). \]  

1.1 Algebraic Approach

Shape-invariance problem was formulated in algebraic terms in Ref. [4]. In this formulation one introduces an operator which transforms the parameters of the potential:

\[ \hat{T}(a_1)O(a_1)\hat{T}^{-1}(a_1) = O(a_2). \]  

Defining the operators

\[ \hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \]
\[ \hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1) \]  

one can show that the Hamiltonian can be written as

\[ \hat{H} - E_0 = \hat{A}^\dagger \hat{A} = \hat{B}_+ \hat{B}_-. \]  

Using the definitions given in Eq. (10), the shape-invariance condition of Eq. (8) takes the form

\[ [\hat{B}_-, \hat{B}_+] = R(a_0), \]  

where \( R(a_0) \) is defined via

\[ R(a_n) = \hat{T}(a_1)R(a_{n-1})\hat{T}^\dagger(a_1). \]  

In terms of these new operators Eq. (4) takes the form

\[ \hat{B}_-|\psi_0\rangle = 0, \]  

i.e. the ground state is annihilated by the lowering operator \( \hat{B}_- \). One can easily establish the commutation relations

\[ [\hat{H}, \hat{B}_+^n] = (R(a_1) + R(a_2) + \cdots + R(a_n))\hat{B}_+^n \]  

\[ [\hat{H}, \hat{B}_-^n] = -\hat{B}_+^n(R(a_1) + R(a_2) + \cdots + R(a_n)). \]  

i.e., \( \hat{B}_+^n|\psi_0\rangle \) is an eigenstate of the Hamiltonian with the eigenvalue \( R(a_1) + R(a_2) + \cdots + R(a_n) \). The normalized eigenstate is

\[ |\psi_n\rangle = \frac{1}{\sqrt{R(a_1) + \cdots + R(a_n)}} \hat{B}_+ \cdots \frac{1}{\sqrt{R(a_1) + R(a_2)}} \hat{B}_+ \frac{1}{\sqrt{R(a_1)}} \hat{B}_+|\psi_0\rangle. \]
To identify the algebra we consider the commutation relations

\[ [\hat{B}_-, \hat{B}_+] = R(a_0) \] (18)

\[ [\hat{B}_+, R(a_0)] = (R(a_1) - R(a_0))\hat{B}_+, \] (19)

\[ [\hat{B}_+, (R(a_1) - R(a_0))\hat{B}_+] = \{(R(a_2) - R(a_1)) - (R(a_1) - R(a_0))\}\hat{B}_+, \] (20)

and so on. In general there are an infinite number of such commutation relations. If the quantities \( R(a_n) \) satisfy certain relations one of the commutators in this series may vanish. For such a situation the commutation relations obtained up to that point plus their complex conjugates form a Lie algebra with a finite number of elements. For example if the condition

\[ (R(a_2) - R(a_1)) - (R(a_1) - R(a_0)) = 0 \] (21)

is satisfied then the algebra is either \( SU(2) \) or \( SU(1,1) \). Most of the exactly solvable one-dimensional problems in quantum mechanics can be described by this algebra. It can be shown that this algebra also describes for example both the bound and scattering states of the Pöschl-Teller potential as well as associated transfer matrices.

1.2 Outlook on future applications

Almost all exactly solvable one-dimensional potential problems encountered in quantum mechanics textbooks are shape invariant where the parameters are related by a translation

\[ a_2 = a_1 + \eta. \] (22)

It should be emphasized that shape-invariance is not the most general integrability condition one can write for such potentials as there are exactly solvable problems which are not shape invariant. There is a second class of shape invariant potentials where the parameters of the partner potentials are related by a scaling

\[ a_2 = qa_1. \] (23)

In this latter class, corresponding one-dimensional potentials are defined implicitly, but explicit expressions are not given.

In searching for integrable models in two-dimensional statistical mechanics a relationship was uncovered between those models, three-dimensional Chern-Simons gauge theory and quantum groups\(^{10}\). These models, being completely integrable, can be written in a shape-invariant way\(^{11}\), corresponding to a shift in the parameters

\[ a_2 = qa_1 + \eta. \] (24)

The associated algebras are called up-down algebras\(^{12}\). These developments suggest that there may be shape-invariant potentials where the parameters are related by linear-fractional transformations:

\[ a_2 = (qa_1 + \eta)/(a_1 + \eta'). \] (25)
This is a completely unexplored direction of research as nothing is known about such integrable systems. Recall that the notation $a_1, a_2$, etc. may represent not only single parameters, but also a set of them. In general one may suggest to simply relate these parameters by the transformation

$$\hat{T}(a_1)\mathcal{O}(a_1)\hat{T}^{-1}(a_1) = \mathcal{O}(a_2).$$  \hfill(26)

where $\hat{T}$ is an element of any group, not just of SL(2,R) as suggested by the linear-fractional transformation and its limits that were so far employed. What kind of exactly solvable problems do we obtain? At the moment this is an open question.

The basic philosophy of this approach is to consider the parameters of the Hamiltonians as auxiliary dynamical variables. This is reminiscent of the path leading to the Interacting Boson Model \cite{13}. To describe the quadrupole collectivity in nuclei one needs to consider a five-dimensional space. It is possible to formulate this problem in terms of boson variables \cite{14}, however the problem is nonlinear written in terms of quadrupole bosons. By considering a parameter of the problem (boson number) as an additional degree of freedom, Interacting Boson Model introduced a scalar boson as a dynamical variable. This has led to the subsequent realization \cite{15} of $s$ and $d$ bosons as pairs of nucleons coupled to the angular momentum $L = 0$ and $L = 2$

So far we talked about considering parameters of the shape-invariant problem as auxiliary dynamical variables. One can imagine an alternative approach of classifying some of the dynamical variables as “parameters”. An example of this is provided by the supersymmetric approach to the spherical Nilsson model of single particle states \cite{16}. The Nilsson Hamiltonian is given by

$$H = \sum_i a_i^\dagger a_i - 2kL \cdot S + k\nu L^2.$$  \hfill(27)

Introducing the variable

$$F^\dagger = l \sum_i \sigma_i a_i^\dagger$$  \hfill(28)

one can show that the “Hamiltonians”

$$H_1 = F^\dagger F = \sum_i a_i^\dagger a_i - \sigma \cdot L$$  \hfill(29)

and

$$H_2 = FF^\dagger = \sum_i a_i a_i^\dagger + \sigma \cdot L$$  \hfill(30)

can be considered as supersymmetric partners of each other \cite{16}. The shape-invariance condition of Eq. (8) can be written as

$$FF^\dagger = F^\dagger F + R,$$  \hfill(31)

where the remainder is

$$R = \sigma \cdot L - 3/4,$$  \hfill(32)
i.e. in this example the radial variables are considered as the main dynamical variables and the angular variables are considered as the “parameters”.

A number of applications of shape-invariance are available in the literature. These include i) Quantum tunneling through supersymmetric shape-invariant potentials\(^{17}\); ii) Study of neutrino propagation through shape-invariant electron densities\(^{18}\); iii) Investigation of coherent states for shape-invariant potentials\(^{19,20}\); and iv) As attempts to devise exactly solvable coupled-channel problems, generalization of Jaynes-Cummings type Hamiltonians to shape-invariant systems\(^{21,22}\). In this article we focus on the applications to many-body systems.

### 2 Many-Body Hamiltonians

One can ask if these methods can be used to search for exactly-solvable many-body systems. It has been shown that the concept of supersymmetric shape-invariance can be utilized to derive the energy spectrum of Calogero-Sutherland model\(^{23}\). Here we discuss an alternative approach and first write down multiple commutators for a shape-invariant Hamiltonian

\[
[\hat{H}, \hat{B}_+] = R(a_1)\hat{B}_+
\]

(33)

\[
[[\hat{H}, \hat{B}_+], \hat{B}_+] = (R(a_1) - R(a_2))\hat{B}_+^2
\]

(34)

\[
[[[\hat{H}, \hat{B}_+], \hat{B}_+], \hat{B}_+] = (R(a_1) - 2R(a_2) + R(a_3))\hat{B}_+^3
\]

(35)

\[
[[[[\hat{H}, \hat{B}_+], \hat{B}_+], \hat{B}_+], \hat{B}_+] = (R(a_1) - 3R(a_2) + 3R(a_3) - R(a_4))\hat{B}_+^4
\]

(36)

and so on. We wish to address the possibility of defining an exactly solvable problem through these commutation relations. We will consider \(\hat{B}_+\) as a raising operator. We assume that the Hamiltonian \(\hat{H}\) may or may not be in the form given by Eq. (11). We consider a generalized pairing problem with

\[
\hat{B}_+ = \sum_j c_j S_j^+.
\]

(37)

In Eq. (37) the pair creation operator in a single-\(j\) shell is defined as

\[
S_j^+ = \sum_m \frac{1}{2} (-)^{j-m} a_{j,m}^\dagger a_{j,-m}^\dagger,
\]

(38)

where \(a_{j,m}^\dagger\) is the particle creation operator. If we assume that the shape-invariant Hamiltonian has only one- and two-body terms the commutator \([[\hat{H}, \hat{B}_+], \hat{B}_+]\) will only involve products of four creation operators. Consequently the next nested commutator will vanish:

\[
[[[\hat{H}, \hat{B}_+], \hat{B}_+], \hat{B}_+] = 0
\]

(39)

Higher nested commutators will also vanish. This will place strong constraints on \(R(a_n)\), i.e.

\[
R(a_3) = -R(a_1) + 2R(a_2),
\]

(40)
\[ R(a_4) = R(a_1) - 3R(a_2) + 3R(a_3) \]  
and so on. Consequently we can immediately write the energy eigenvalues and eigenstates of the Hamiltonian

\[ \hat{H}\hat{B}_n^\dagger|\psi_0\rangle = \left(nR(a_1) + \frac{1}{2}W(n(n-1))\right)\hat{B}_n^\dagger|\psi_0\rangle, \]  
where

\[ W = R(a_2) - R(a_1). \]  
A similar approach was first given by Talmi.24

### 3 Connection to Gaudin Algebras

The pairing model with a constant two-body interaction was solved exactly by Richardson.25 In a parallel development Gaudin developed an algebraic approach to solve many-body spin Hamiltonians.26,27 Here we will explore the relationship between Gaudin’s methods, algebraic methods developed to search for quasi-exactly solvable models28 and supersymmetric quantum mechanics.

Following the notation of Ref. [29] we consider the function defined as

\[ \Psi(\lambda) = \prod_{i=1}^{N} (\lambda - \xi_i) e^{-\int W d\lambda}, \]  
where \( W(\lambda) \) is an arbitrary function of \( \lambda \) and \( \xi_i \) are numbers to be determined. Introducing the operators

\[ A = W + ip, \quad A^\dagger = W - ip, \]  
it can be shown that the function defined in Eq. (44) satisfies the equation

\[ A^\dagger A \Psi = \left[ 2 \sum_{i \neq j} \frac{1}{(\lambda - \xi_i)(\lambda - \xi_j)} - 2 \sum_{i} \frac{W(\lambda)}{(\lambda - \xi_i)} \right] \Psi. \]  
Requiring the residue at \( \xi_i \) to vanish yields the Bethe-ansatz conditions:

\[ W(\xi_i) = \sum_{i \neq j} \frac{1}{\xi_i - \xi_j}. \]  
Inserting Eq. (47) into Eq. (46) we obtain

\[ A^\dagger A \Psi = 2 \sum_{i} \left( \frac{W(\lambda) - W(\xi_i)}{\lambda - \xi_i} \right) \Psi. \]  
Provided that their superpotentials satisfy the condition given in Eq. (47), factorized supersymmetric Hamiltonians satisfy Eq. (48). Note that the right side of Eq. (48) in general depends on \( \lambda \), hence we cannot interpret the term that multiplies
the function $\Psi$ as an energy eigenvalue. However, for a number of limited cases (certain functions $W(\lambda)$ such as those that correspond to a harmonic oscillator) this $\lambda$ dependence drops out and one can recover the standard expressions for the energy eigenvalues.

The three generators of Gaudin’s algebra ($J_0(\lambda)$, $J_\pm(\lambda)$) can be defined through the commutation relations

\[
[J_0(\lambda), J_+(\mu)] = -\frac{J_+(\lambda) - J_+(\mu)}{\lambda - \mu},
\]

\[
[J_-(\lambda), J_+(\mu)] = -2\frac{J_0(\lambda) - J_0(\mu)}{\lambda - \mu},
\]

and

\[
[J_{\pm,0}(\lambda), J_{\pm,0}(\mu)] = 0,
\]

where $\lambda$ is, in general, a continuous parameter. Gaudin studied the eigenstates of the “Hamiltonian” $H(\lambda)$

\[
H(\lambda) = J_0(\lambda)J_0(\lambda) - \frac{1}{2}J_-(\lambda)J_+(\lambda) - \frac{1}{2}J_+(\lambda)J_-(\lambda)
\]

If a state $|0\rangle$ which is annihilated by all $J_-(\lambda)$ can be identified

\[
J_-(\lambda) |0\rangle = 0,
\]

then $W(\lambda)$ is introduced as the eigenvalue of $J_0(\lambda)$ on that state:

\[
J_0(\lambda) |0\rangle = W(\lambda) |0\rangle.
\]

One can then find the eigenvalues and eigenstates of the “Hamiltonian” of Eq. (52):

\[
H(\lambda) |\Phi\rangle = E(\lambda) |\Phi\rangle,
\]

where the eigenstates are

\[
|\Phi\rangle = J_+(\xi_N)J_+(\xi_{N-1})\cdots J_+(\xi_1) |0\rangle,
\]

and the eigenvalues are

\[
E(\lambda) = W^2(\lambda) + W'(\lambda) + 2 \sum_i \left( \frac{W(\lambda) - W(\xi_i)}{\lambda - \xi_i} \right).
\]

In deriving the above equations the conditions

\[
W(\xi_i) = -\sum_{i \neq j} \frac{1}{\xi_i - \xi_j}, \quad i, j = 1, \cdots, N
\]

were assumed to be fulfilled.

The strategy of using Richardson-Gaudin methods to deal with many-body problems were employed by a number of authors. Clearly there is a mapping between the solutions of the Gaudin problem (Eq. (55)) and those of the
factorized supersymmetric Hamiltonians (Eq. (48)). One may ask if this correspondence can be exploited to study pairing and related problems.

The pair creation operator\(^{16}\) in a single-\(j\) shell is defined in Eq. (38),

\[
S_j^+ = \sum_m \frac{1}{2} (-)^{j-m} a_{j,m}^\dagger a_{j,-m}^\dagger,
\]

its Hermitian conjugate, and the number operator span an \(SU(2)\) algebra (the so-called quasi-spin algebra). One can obtain a Gaudin algebra from the quasi-spin algebra by defining

\[
S_+(\lambda) = \sum_j \frac{S_j^+}{\lambda - \epsilon_i},
\]

(and similar formulas for the other elements). This realization of the Gaudin algebra can be very useful in many-body systems. As a simple example we consider a system with \(s\) and \(p\) bosons and define three operators that satisfy Gaudin’s commutation relations

\[
B_+(\lambda) = \frac{1}{2} \left[ \frac{s^\dagger s^\dagger}{\lambda - \alpha_s} + \frac{(p^\dagger \cdot p^\dagger)}{\lambda - \alpha_p} \right],
\]

\[
B_-(\lambda) = [B_+(\lambda)]^\dagger,
\]

and

\[
B_0(\lambda) = \frac{1}{2} \left[ \frac{\hat{n}_s}{\lambda - \alpha_s} + \frac{\hat{n}_p + 3/2}{\lambda - \alpha_p} \right].
\]

It is easy to show that as \(\alpha_s \rightarrow \alpha_p\) the quantity \(B_+(\lambda)B_-(\lambda)\) reduces to

\[
\frac{1}{\lambda - \alpha_p} \hat{P}_4
\]

where \(\hat{P}_4\) is the O(4) pairing operator. One can then study a Gaudin-type Hamiltonian which generalizes this operator

\[
H(\lambda) = B_0(\lambda)B_0(\lambda) - \frac{1}{2} B_-(\lambda)B_+(\lambda)B_+(\lambda)B_-(\lambda).
\]

Following steps above one can show that this Hamiltonian is associated with the one-dimensional potential

\[
V(x) = \frac{1}{2} \left( \frac{1}{x - \alpha_s} + \frac{1}{x - \alpha_p} \right)^2.
\]

Similar ideas could conceivably be useful in dealing with other many-body systems.

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References

1. E. Witten, *Nucl. Phys.* B 188, 513 (1981).
2. F. Cooper, A. Khare and U. Sukhatme, *Phys. Rept.* 251, 267 (1995).
   [arXiv:hep-th/9405029].
3. L. E. Gendenshtein, *JETP Lett.* 38, 356 (1983).
4. A. B. Balantekin, *Phys. Rev.* A 57, 4188 (1998) [arXiv:quant-ph/9712018].
5. F. Iachello and R.D. Levine, *Algebraic theory of Molecules* (Oxford University Press, New York, 1995).
6. Y. Alhassid, F. Gursey and F. Iachello, *Phys. Rev. Lett.* 50, 873 (1983).
7. F. Cooper, J. N. Ginocchio and A. Khare, *Phys. Rev.* D 36, 2458 (1987).
8. A. Khare and U. P. Sukhatme, *J. Phys.* A 26, L901 (1993) [arXiv:hep-th/9212147].
9. D. T. Barclay, R. Dutt, A. Gangopadhyaya, A. Khare, A. Pagnamenta and U. Sukhatme, *Phys. Rev.* A 48, 2786 (1993) [arXiv:hep-ph/9304313].
10. E. Witten, *Nucl. Phys.* B 330, 285 (1990).
11. A. B. Balantekin, in preparation.
12. G. Benkart, *Contemp. Math.* 224, 29 (1999).
13. A. Arima and F. Iachello, *Annals Phys.* 99, 253 (1976).
14. D. Janssen, R.V. Jolos, and F. Donau, *Nucl. Phys.* A 224, 93 (1974).
15. A. Arima, T. Otsuka, F. Iachello, and I. Talmi, *Phys. Lett.* B 66, 205 (1977).
16. A. B. Balantekin, O. Castanos and M. Moshinsky, *Phys. Lett.* B 284, 1 (1992).
17. A. N. Aleixo, A. B. Balantekin and M. A. Candido Ribeiro, *J. Phys.* A 33, 1503 (2000) [arXiv:quant-ph/9910051].
18. A. B. Balantekin, *Phys. Rev.* D 58, 013001 (1998) [arXiv:hep-ph/9712304].
19. A. B. Balantekin, M. A. Candido Ribeiro and A. N. Aleixo, *J. Phys.* A 32, 2785 (1999) [arXiv:quant-ph/9811061].
20. A. N. Aleixo, A. B. Balantekin and M. A. Candido Ribeiro, *J. Phys.* A 35, 9063 (2002) [arXiv:math-ph/0209033].
21. A. N. Aleixo, A. B. Balantekin and M. A. Candido Ribeiro, *J. Phys.* A 33, 3173 (2000) [arXiv:quant-ph/0001049].
22. A. N. Aleixo, A. B. Balantekin and M. A. Candido Ribeiro, *J. Phys.* A 34, 1109 (2001) [arXiv:quant-ph/0101024].
23. P. K. Ghosh, A. Khare and M. Sivakumar, *Phys. Rev.* A 58, 821 (1998). [arXiv:cond-mat/9710206].
24. I. Talmi, *Simple Models of Complex Nuclei: the Shell Model and Interacting Boson Model* (Harwood Academic Publishers, Chur, 1993).
25. R.W. Richardson, *Phys. Lett.* 3, 277 (1963).
26. M. Gaudin, *J. Phys. (Paris)* 37, 1087 (1976).
27. M. Gaudin, *La fonction d’onde de Bethe* (Masson, Paris, 1983).
28. A.G. Ushveridze, *Quasi-Exactly Solvable Problems in Quantum Mechanics* (IOP Publishing, Bristol, 1994).
29. A. G. Ushveridze, *Annals Phys.* 266, 81 (1998) [arXiv:hep-th/9707151].
30. G. Akemann and A.B. Balantekin, in preparation.
31. F. Pan, J. P. Draayer and W. E. Ormand, *Phys. Lett.* B 422, 1 (1998). [arXiv:nucl-th/9709036].
32. J. Dukelsky and S. Pittel, *Phys. Rev. Lett.* 86, 4791 (2001). [arXiv:nucl-th/0102034].
33. J. Dukelsky, C. Esebbag and P. Schuck, *Phys. Rev. Lett.* 87, 066403 (2001). [arXiv:cond-mat/0107477].
34. F. Pan and J. P. Draayer, *Phys. Rev. C* 66, 044314 (2002).