FILTRATIONS ON HOMOTOPY INVARIANT SHEAVES WITH TRANSFERS

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Abstract. We construct filtrations on homotopy invariant sheaves with transfers and show that under Ayoub’s conjectures on $n$-motives, our filtration agrees with the one conjectured by Ayoub and Barbieri-Viale if the latter exists. Our construction is directly motivated by the work of Pelaez.

1. Introduction

Let $DM^\text{eff}(k, \mathbb{Q})$ be Voevodsky’s triangulated category of unbounded effective motives over a perfect field $k$ with rational coefficients. The triangulated subcategory $DM^{\leq n}(k, \mathbb{Q})$ of $n$-motives is defined as the smallest localizing subcategory of $DM^\text{eff}(k, \mathbb{Q})$ that contains the motives of smooth schemes of dimension $\leq n$. While the inclusion $DM^{\leq n}(k, \mathbb{Q}) \hookrightarrow DM^\text{eff}(k, \mathbb{Q})$ has a right adjoint for an arbitrary $n$ for abstract reasons, left adjoints exist only for $n = 0$ and 1, but they are closely related to the theory of the algebraic equivalence relation and Albanese varieties. The existence of left adjoints for $n \geq 2$ leads to a contradiction as observed in [ABV09, 2.5].

On the other hand, one may focus on the heart of $DM^\text{eff}(k, \mathbb{Q})$ with respect to the homotopy $t$-structure. The heart is known to be equivalent to the category $HI^{\text{et}}(k, \mathbb{Q})$ of homotopy invariant étale sheaves of $\mathbb{Q}$-modules with transfers. In this category, Ayoub and Barbieri-Viale ([ABV09]) defined cococomplete abelian subcategories $HI^{\leq n}_{\text{et}}(k, \mathbb{Q})$ of $n$-motivic sheaves for non-negative integers $n$ (see Subsection 2.2 for a quick review). It is conjectured that $HI^{\leq n}_{\text{et}}(k, \mathbb{Q})$ is the heart of $DM^{\leq n}(k, \mathbb{Q})$ with respect to (the restriction of) the homotopy $t$-structure ([Ayo17, Conjecture 4.27], [ABV09, Conjecture 2.5.3]). This conjecture is affirmative for $n = 0$ and 1 ([Voe09, Org01, BYK16, ABV09]).

Now, let $h^0_{\text{et}}(X)\mathbb{Q}$ be the 0-th cohomology of the motive $M(X)$ of a smooth $k$-scheme $X$ with respect to the homotopy $t$-structure. More explicitly, it is the étale sheafification of the $0$-th homology of the Suslin complex $C_*\mathbb{Q}_{\text{et}}(X)$. It is shown in [ABV09] that if the sheaves $h^0_{\text{et}}(X)\mathbb{Q}$ carry filtrations satisfying a certain set of axioms (see Conjecture 2.7), then the inclusions $HI^{\leq n}_{\text{et}}(k, \mathbb{Q}) \hookrightarrow HI^{\text{et}}(k, \mathbb{Q})$ admit left adjoints for all $n$. In [ibid.], it was remarked that when $X$ is smooth and projective, the filtration induced on $h^0_{\text{et}}(X)\mathbb{Q}$ of $HI^{\text{et}}(k, \mathbb{Q})$ admits a left adjoint for all $n$.

The purpose of this article is to construct a filtration on $h^0_{\text{et}}(X)\mathbb{Q}$ for $\tau \in \{\text{Nis,} \text{ ét}\}$ with $R$ a commutative ring in which the characteristic of $k$ is invertible, and show that this filtration is a candidate for the Ayoub-Barbieri-Viale filtration in the sense of Theorem 3.14. By construction, our filtration agrees with that of Pelaez ([Pel17]) when evaluated at finitely generated fields over $k$ for $\tau = \text{Nis}$ or for a $\mathbb{Q}$-algebra $R$. The relation with the Bloch-Beilinson filtration is provided by the following: Using the work of Voisin ([Vo10, Proposition 6]) under the Lefschetz standard conjecture, Pelaez ([Pel17, 6.1.9]) showed that his filtration for $0$-cycle Chow groups is contained in the conjectural Bloch-Beilinson filtration.

Notation and conventions.

We assume that the base field $k$ is perfect and of exponential characteristic $p$, and schemes are separated and of finite type over $k$. The category of schemes (resp., smooth schemes) over $k$ with $k$-morphisms is denoted by $\text{Sch}/k$ (resp., $\text{Sm}/k$).

The symbol $\text{Hom}^\text{eff}$ stands for the internal hom in $DM^\text{eff}(R)$ for arbitrary $R$ or that in $DM^\text{eff}(k, R)$ for a $\mathbb{Q}$-algebra $R$. This shall not cause confusions as the two categories are equivalent as tensor triangulated categories when $R$ is a $\mathbb{Q}$-algebra. The definitions of triangulated category of motives are recalled in Section 3.

Assumptions stated at the beginning of a section (resp., subsection) run through the section (resp., subsection).
We summarize necessary facts on $n$-motivic sheaves and recall the conjecture of Ayoub and Barbieri-Viale [ABV09 Conjecture 1.4.1]. We assume that the base field $k$ is perfect and of exponential characteristic $p$, and $R$ denotes the ring of coefficients. In this section, $\tau \in \{\text{triv}, \text{Nis}, \text{ét}\}$, where triv stands for the trivial topology.

2.1. Homotopy invariant sheaves with transfers. Let $\text{Cor}(k)$ be the category of finite correspondences over $k$ as in [MVW06 Lecture 1]. The category $\text{PST}(k, R)$ of presheaves of $R$-modules with transfers on $\text{Sm}/k$ is the category of contravariant additive functors from $\text{Cor}(k)$ to the category $R\text{Mod}$ of $R$-modules. $R_{\text{tr}}(-) : \text{Cor}(k) \to \text{PST}(k, R)$ denotes the embedding given by $R_{\text{tr}}(X) = \text{Hom}_{\text{Cor}(k)}(-, X) \otimes R$. A presheaf with transfers is called a $\tau$-sheaf with transfers on $\text{Sm}/k$ if it is a $\tau$-sheaf when restricted to $\text{Sm}/k$. The full subcategory of sheaves with transfers in $\text{PST}(k, R)$ is denoted by $\text{Sh}^\tau(k, R)$. A presheaf with transfers of the form $R_{\text{tr}}(X)$ is in fact a $\tau$-sheaf with transfers ([MVW06 Lemma 6.2]).

A sheaf with transfers $F$ is said to be homotopy invariant if, for any $X \in \text{Sm}/k$, the morphism $F(X) \to F(X \times_k A_n)$ induced by the projection $X \times_k A_n \to X$ is an isomorphism. The full subcategory of homotopy invariant $\tau$-sheaf with transfers in $\text{Sh}^\tau(k, R)$ is denoted by $\text{HI}^\tau(k, R)$. It is classical ([Swa72 Lemma 4.2]) that the inclusion $\text{HI}^\tau_{\text{tr}}(k, R) \to \text{Sh}^\tau_{\text{tr}}(k, R) = \text{PST}(k, R)$ has a left adjoint $h^\tau_{\text{tr}}$ given by $h^\tau_{\text{tr}}(F) = H_0(C, F)$, where $C$ is the singular simplicial complex of $F$ as in [Voe00 3.2]. For details, we refer the reader to loc. cit. or [MVW06 Lecture 2]. More generally, we have the following.

Proposition 2.1 (Suslin, Voevodsky). Let $\tau \in \{\text{triv}, \text{ét}, \text{Nis}\}$ and $R$ be a ring. If $\tau = \text{ét}$, assume that $p$ is invertible in $R$. Then, the inclusions $\text{HI}^\tau_{\text{tr}}(k, R) \to \text{Sh}^\tau_{\text{tr}}(k, R) \to \text{PST}(k, R)$ admit left adjoints $\text{PST}(k, R) \overset{a_{\tau}}\to \text{Sh}^\tau_{\text{tr}}(k, R) \overset{h^\tau_{\text{tr}}}{\to} \text{HI}^\tau(k, R)$. Here, $a_{\tau}$ is given by the $\tau$-sheafification and $h^\tau_{\text{tr}}$ by the composition $a_{\tau} \circ h_0 \circ i$.

Proof. This is part of [ABV09 Lemma 1.1.1 and Proposition 1.1.2]. (When $\tau = \text{ét}$, $p$ needs to be invertible because the proof depends on Suslin’s rigidity theorem.)

Let us introduce the sheaf of our main interest.

Definition 2.2. For $X \in \text{Sm}/k$ and a ring $R$, a $\tau$-sheaf $h^\tau_0(X)_R$ is defined as $h^\tau_0(X)_R = h^\tau_0(R_{\text{tr}}(X))$.

This sheaf is closely related to Suslin homology, which, for proper schemes, is nothing but Chow groups modulo rational equivalence.

Proposition 2.3. Let $X \in \text{Sm}/k$. If $\tau \in \{\text{triv}, \text{Nis}\}$, then $h^\tau_0(X)_Q(k)$ is canonically isomorphic to the 0-th Suslin homology $H^S_0(X, \mathbb{Z})$. If $\tau = \text{ét}$, this is still true rationally: $h^\text{ét}_0(X)_Q(k) \cong H^S_0(X, \mathbb{Q})$.

Proof. Let $R$ be a ring. The Suslin homology of $X$ is defined as $H^S_n(X, R) = H_n(C, (\mathbb{Z}/p^n \mathbb{Z} \otimes \mathbb{Q})_R(k))$. Thus, if $\tau = \text{triv}$, there is nothing to prove. If $\tau = \text{Nis}$, the statement follows because a field does not have nontrivial Nisnevich coverings. The case $\tau = \text{ét}$ is also true because any Nisnevich sheaf of $\mathbb{Q}$-modules with transfers is a sheaf in the étale topology as well ([MVW06 Corollary 14.22]).

2.2. $n$-motivic sheaves. Let $(\text{Sm}/k)_{\leq n}$ (resp., $\text{Cor}(k)_{\leq n}$) be the full subcategory in $\text{Sm}/k$ (resp., $\text{Cor}(k)$) that consists of schemes of dimension at most $n$. Endow $(\text{Sm}/k)_{\leq n}$ with the $\tau$-topology. Note that any $\tau$-covering of a scheme $X$ has the same dimension as $X$. We define the category $\text{PST}(k_{\leq n}, R)$ of presheaves with transfers on $(\text{Sm}/k)_{\leq n}$ as the category of contravariant additive functors from $(\text{Sm}/k)_{\leq n}$ to $R\text{Mod}$. A presheaf with transfers on $(\text{Sm}/k)_{\leq n}$ is called a $\tau$-sheaf with transfers on $(\text{Sm}/k)_{\leq n}$ if it is a $\tau$-sheaf when restricted to $(\text{Sm}/k)_{\leq n}$. We write $\text{Sh}^\tau_n(k_{\leq n}, R)$ for the full subcategory in $\text{PST}(k_{\leq n}, R)$ of $\tau$-sheaves with transfers on $(\text{Sm}/k)_{\leq n}$.

The exact functor $\sigma_{\leq n} : \text{Sh}^\tau_n(k, R) \to \text{Sh}^\tau_n(k_{\leq n}, R)$ induced by the inclusion $\sigma_n : (\text{Sm}/k)_{\leq n} \to \text{Sm}/k$ has a left adjoint $\sigma^*_n : \text{Sh}^\tau_n(k_{\leq n}, R) \to \text{Sh}^\tau_n(k, R)$, which is given by $\sigma^*_n(F) = \text{colim}_X \to_R R_{\text{tr}}(X)$. Here, the colimit is computed in $\text{Sh}^\tau_n(k, R)$ and the index category is the category $\text{Cor}(k_{\leq n})/R$ whose objects are
arrows $R_{\tau}(X) \rightarrow F$ in $\text{Sh}^\tau_{\text{et}}(k_{\leq n}, R)$ with $X \in \text{Cor}(k_{\leq n})$ and morphisms are given by commutative diagrams

$$
\begin{array}{ccc}
R_{\tau}(X) & \xrightarrow{F} & R_{\tau}(Y) \\
& & \\
\end{array}
$$

of $\tau$-sheaves with transfers ([ABV09] Lemma 1.1.12]).

**Definition 2.4 ([ABV09] Definition 1.1.20).** A homotopy invariant sheaf $F \in \text{HI}^\tau_{\text{et}}(k, R)$ is $n$-motivic if the counit of the adjunction $\sigma_n^* \dashv \sigma_n*$ induces an isomorphism $h_0^\tau(\sigma_n^* \sigma_n*(F)) \xrightarrow{\sim} h_0^\tau(F)$.

The full subcategory of $n$-motivic $\tau$-sheaves in $\text{HI}^\tau_{\text{et}}(k, R)$ is denoted by $\text{HI}^\tau_{\tau, \leq n}(k, R)$.

**Remark 2.5.** A sheaf $F \in \text{Sh}^\tau_{\text{et}}(k, R)$ is called $n$-generated (resp., strongly $n$-generated) if the counit $\sigma_n^* \sigma_n* F \rightarrow F$ is a surjection (resp., isomorphism). Any $n$-motivic $\tau$-sheaf is the $h_0^\tau$ of a strongly $n$-generated $\tau$-sheaf, and conversely, $h_0^\tau$ of any strongly $n$-generated $\tau$-sheaf is $n$-motivic. In particular, if $X \in (\text{Sm}/k)_{\leq n}$, then $h_0^\tau(X)$ is an $n$-motivic $\tau$-sheaf. See [ABV09], Remark 1.1.21 for the proof of these.

For $n$-motivic sheaves, we generally know the following. (See Remark 2.8 for what is conjecturally expected.)

**Proposition 2.6 ([ABV09] Lemma 1.1.22, Corollary 1.1.24).** Let $\tau \in \{\text{triv, ét, Nis}\}$. Assume that $p$ is invertible in $R$ when $\tau = \text{ét}$. Then,

(i) the property of being $n$-motivic is stable under taking cokernels and extensions in $\text{HI}^\tau_{\text{et}}(k, R)$.

(ii) The category $\text{HI}^\tau_{\tau, \leq n}(k, R)$ is abelian and cocomplete, and the inclusion $\text{HI}^\tau_{\tau, \leq n}(k, R) \rightarrow \text{HI}^\tau_{\text{et}}(k, R)$ is right exact.

2.3. The conjectures. In this subsection, $R$ denotes a ring in which $p$ is invertible. The conjecture is concerned with étale sheaves $h_0^{\text{ét}}(X)_R$ of $R$-modules.

**Conjecture 2.7 ([ABV09] Conjecture 1.4.1).** For any $X \in \text{Sm}/k$, there exists a decreasing filtration $F^n h_0^{\text{ét}}(X)_R \supseteq F^{n-1} h_0^{\text{ét}}(X)_R$ such that

(A) $F^n h_0^{\text{ét}}(X)_R = h_0^{\text{ét}}(X)_R$ and $F^n h_0^{\text{ét}}(X)_R = 0$ for $n \geq \text{dim } X + 1$.

(B) The filtration is compatible with the action of correspondences, i.e. for $\gamma \in \text{Cor}(X, Y)$, the induced morphism of sheaves $h_0^{\text{ét}}(X)_R \rightarrow h_0^{\text{ét}}(Y)_R$ is compatible with the filtration.

(C) If $U$ is a dense open subscheme of $X$, then $h_0^{\text{ét}}(U)_R \rightarrow h_0^{\text{ét}}(X)_R$ is strict for the filtration.

(D) For $n \geq 0$, the quotient $h_0^{\text{ét}}(X)_R/F^{n+1} h_0^{\text{ét}}(X)_R$ is $n$-motivic.

(Axiom (D) actually follows from (A), (B) and a Weaker Version of (D): For $n \geq 0$, the quotient $h_0^{\text{ét}}(X)_R/F^{n+1} h_0^{\text{ét}}(X)_R$ is $n$-generated; see [ABV09] Lemma 1.4.3.)

**Remark 2.8 (cf. Proposition 2.9).** Under (A), (B) and (D), [ABV09] Corollary 1.4.5] states that $\text{HI}^\tau_{\text{ét}, \leq n}(k, R)$ is a Serre subcategory of $\text{HI}^\tau_{\text{et}}(k, R)$ (i.e. closed under subobjects, quotients and extensions) and the inclusion $\text{HI}^\tau_{\text{ét}, \leq n}(k, R) \rightarrow \text{HI}^\tau_{\text{et}}(k, R)$ is exact. For $n = 0$ and $1$, this result is unconditionally proved in [ibid., Proposition 1.2.7, Corollary 1.3.5].

Another beautiful consequence of Conjecture 2.7 is the following ([ABV09] Proposition 1.4.6): If the conjecture is true for $R = \mathbb{Q}$, then the inclusion $\text{HI}^\tau_{\text{ét}, \leq n}(k, \mathbb{Q}) \rightarrow \text{HI}^\tau_{\text{et}}(k, \mathbb{Q})$ admits a left adjoint $(-)^{\leq n} : \text{HI}^\tau_{\text{et}}(k, \mathbb{Q}) \rightarrow \text{HI}^\tau_{\text{ét}, \leq n}(k, \mathbb{Q})$ for an arbitrary $n$. As we have explained in the Introduction, this is in contrast to the derived $\tau$-category where the existence of left adjoints of the inclusion $\text{DM}^{\text{eff}}_{\text{ét}, \leq n}(k, \mathbb{Q}) \rightarrow \text{DM}^{\text{eff}}_{\text{et}}(k, \mathbb{Q})$ for $n \geq 2$ leads to a contradiction at least when $k$ is algebraically closed and has infinite transcendence degree over $\mathbb{Q}$ ([ABV09] 2.5). It is also known that, conversely, if the left adjoints $(-)^{\leq n}$ exist and $\text{HI}^\tau_{\text{ét}, \leq n}(k, \mathbb{Q})$ are Serre subcategories of $\text{HI}^\tau_{\text{et}}(k, \mathbb{Q})$, then Conjecture 2.7 holds. The filtration and the left adjoints are related by the equation $F^n h_0^{\text{ét}}(X)_Q = \ker\{h_0^{\text{ét}}(X)_Q \rightarrow h_0^{\text{ét}}(X)_Q^{n-1}\};$ hence the filtration as in Conjecture 2.7 is unique if it exists.
3. The filtration on $h^n_0(X)_R$

Our construction is motivated by the work of [Pel17], especially Corollary 5.3.3 thereof. In this section, $\tau \in \{\text{Nis}, \text{ét}\}$ and we assume that $p$ is invertible in the coefficient ring $R$ in order to avoid the use of resolution of singularities and to ensure the existence of the homotopy $t$-structures on $\mathbf{DM}^{\text{eff}}_\text{Nis}(k, R)$ for $\tau = \text{ét}$. We write $\mathbf{DM}_\tau(k, R)$ for Voevodsky’s triangulated category of unbounded $\tau$-motives with coefficients in $R$. By this, we mean the homotopy category (with respect to the stable model structure defined in [Ayo07 Définition 4.3.29]) of the model category $\text{Spt}_T(\text{Ch}(\mathbf{Sh}^{\text{eff}}_\text{Nis}(k, R)))$ of symmetric $T$-spectra, where $\text{Ch}(\mathbf{Sh}^{\text{eff}}_\text{Nis}(k, R))$ is endowed with the $\mathbb{A}^1$-local model structure, i.e. the model structure obtained by the Bousfield localization of the injective model structure with respect to the class of morphisms $C_{\mathbb{A}^1} = \{R_{\text{tr}}(X \times_k \mathbb{A}^1_n)[n] \to R_{\text{tr}}(X)[n] | X \in \text{Sm}/k, n \in \mathbb{Z}\}$. We write $M : \text{Sm}/k \to \mathbf{DM}_\tau(k, R)$ for the canonical functor that associates smooth schemes with their motives.

3.1. Construction of the filtration. A triangulated category $T$ with arbitrary coproducts is compactly generated if there is a set $S$ of generators consisting of compact objects ([Nec01 Definition 1.7]). A subcategory $T'$ of $T$ is called localizing if it is closed under coproducts in $T$. A triangulated category $T$ with arbitrary coproducts is compactly generated with a set of compact generators $S$ if and only if $T$ itself is the only localizing subcategory that contains $S$ ([SS03 Lemma 2.21]).

The category $\mathbf{DM}_{\text{Nis}}(k, R)$ is a compactly generated triangulated category with a set of compact generators $G = \{M(X)(n) | X \in \text{Sm}/k, n \in \mathbb{Z}\}$ ([Ayo07 Théorème 4.5.67]). For $n \in \mathbb{Z}$, let $G^{\text{eff}}(n) := \{M(X)(p) | X \in \text{Sm}/k, p \geq n\}$. We define $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)(n)$ to be the smallest localizing subcategory that contains $G^{\text{eff}}(n)$ ([Pel17 3.1.5]). Note that $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)(0)$ is equivalent to the category of effective motives $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$ under the infinite suspension functor $\Sigma^\infty : \mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R) \to \mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$. Here, $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$ is the homotopy category of $\text{Ch}(\mathbf{Sh}^{\text{Nis}}_{\text{Nis}}(k, R))$ with respect to the $\mathbb{A}^1$-local model structure, or equivalently, the Verdier localization of the derived category of the abelian category $\mathbf{Sh}^{\text{Nis}}_{\text{Nis}}(k, R)$ with respect to the class of morphisms $C_{\mathbb{A}^1}$. From now on, we identify $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)(0)$ and $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$. The canonical functor $M : \text{Sm}/k \to \mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$ factors through $\Sigma^\infty$ by the constructions. We write the image of $X$ in $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$ by the same symbol $M(X)$. This is by definition the image of $R_{\text{tr}}(X)$ in $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$.

Consider the inclusion $i_n : \mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)(n) \to \mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$ $(n \geq 0)$. Since both target and source are compactly generated, by Neeman’s Brown representability theorem (applied in the form of [Pel17 Theorem 2.1.3]), the functor $i_n$ has a right adjoint $r_n : \mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R) \to \mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)(n)$, and $r_n$ is a triangulated functor. We write $f_n := i_n \circ r_n$ and the counit of the adjunction is denoted by $\epsilon_n^M = \epsilon_n : f_n M \to M$. The functor $f_n$ is called the $(n - 1)$-th effective cover and discussed further in [Pel17 Subsection 3.3].

The above constructions work for the étale topology as long as $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$ is compactly generated (for example, if $k$ has finite cohomological dimension or $R$ is a $\mathbb{Q}$-algebra). However, we do not use this fact because the following lemma does not hold for the étale topology. From now on, we write $M(n) := M(n)[2n]$ for short.

**Lemma 3.1.** Let $X$ be a smooth $k$-scheme of dimension $d$ and let $M^c(X) \in \mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$ be the motive of $X$ with compact supports ([MVW06 Definition 16.13]). Then, there is an isomorphism $M(X) \cong \text{Hom}^{\text{eff}}(M^c(X), R(d))$ in $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$.

**Proof.** By [Voe00 Theorem 4.3.7 (3)], there is an isomorphism $M^c(X)^\vee \cong M(X)(-d)$ in $\mathbf{DM}^{\text{Nis,gm}}_{\text{Nis}}(k, R)$, where $M^c(X)^\vee$ is the dual of $M^c(X)$. Since $\text{Hom}^{\text{eff}}(M^c(X), R(d))$ is geometric by [MVW06 Corollary 20.4] and a localization triangle ([MVW06 Theorem 16.5] under resolution of singularities; [Kel17 Proposition 5.3.5] unconditionally for $\mathbb{Z}[1/p]$-coefficients), we have $M^c(X)^\vee = \text{Hom}^{\text{eff}}(M^c(X), R(d))(-d)$ in $\mathbf{DM}^{\text{Nis,gm}}_{\text{Nis}}(k, R)$ by the definition of dual objects. Therefore, there is an isomorphism $M(X) \cong \text{Hom}^{\text{eff}}(M^c(X), R(d))$ in $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$.

The following definition is directly motivated by [Pel17 Corollary 5.3.3].

**Definition 3.2.** For a scheme $X \in \text{Sm}/k$ of dimension $d$, set

$$F^n h^0_{\text{Nis}}(X)_R := \ker\{h^0_{\text{Nis}}(X)_R \cong H^0(M(X)) \cong H^0(\text{Hom}^{\text{eff}}(M^c(X), R(d))) \xrightarrow{\epsilon_{r_n}^M} H^0(\text{Hom}^{\text{eff}}(f_{d+1-n}M^c(X), R(d)))\},$$

for $n \in \{0, 1, \cdots, d+1\}$, where $H^0$ is the cohomology with respect to the homotopy $t$-structure on $\mathbf{DM}^{\text{eff}}_{\text{Nis}}(k, R)$.

The second isomorphism is induced by the one in Lemma 3.1.
A filtration $F^n h^{	ext{eff}}_0(X)_R$ on $h^{	ext{eff}}_0(X)_R$ is defined by étale sheafification $a_{d_1}$:

$$F^n h^{	ext{eff}}_0(X)_R := \text{im}\{a_{d_1} F^n h^\text{Nis}_0(X)_R \to a_{d_1} h^\text{Nis}_0(X)_R = h^{	ext{eff}}_0(X)_R\}.$$ 

If $R$ is a $\mathbb{Q}$-algebra, the sheafification $a_{d_2}$ induces an equivalence of categories $\alpha: \text{DM}_{\text{Nis}}^\text{eff}(k, R) \to \text{DM}_{\text{et}}^\text{eff}(k, R)$ ([MVW06, Theorem 14.30]). Hence, in this situation, the filtration on $h^0_0(X)_R$ can be described as

$$F^n h^0_0(X)_R := \ker\{h^0_0(X)_R \cong H^0(M(X)) \cong H^0(\text{Hom}(M^c(X), R[d])) \xrightarrow{\epsilon_{n-2}^{0 \to n}} H^0(\text{Hom}(f_{d_1+n} M^c(X), R[d]))\},$$

for $n \in \{0, 1, \ldots, d+1\}$, where $H^0$ is the cohomology with respect to the homotopy $t$-structure on $\text{DM}_{\text{et}}^\text{eff}(k, R)$ and $\text{Hom}^\text{eff}$ is the internal hom in $\text{DM}_{\text{et}}^\text{eff}(k, R)$.

Let us show that our filtration satisfies (A) and (B) of Conjecture [??] and their analogues in the Nisnevich topology.

**Proposition 3.3** (cf. Conjecture [??] (A)). Let $X \in \text{Sm}_k$ be a smooth scheme of dimension $d$. Then, $F^n h^0_0(X)_R$ is a decreasing filtration on $h^0_0(X)_R$ such that $F^0 h^0_0(X)_R = h^0_0(X)_R$ and $F^{d+1} h^0_0(X)_R = 0$.

**Proof.** It is enough to prove the claim for $\tau = \text{Nis}$. The case $\tau = \text{et}$ is immediate from this. Let $M \in \text{DM}_{\text{Nis}}(k, R)$. By definition, $r_n+1 M$ belongs to $\text{DM}_{\text{Nis}}(k, R)(n+1)$, so a fortiori, to $\text{DM}_{\text{Nis}}(k, R)(n)$. Therefore, by the universal property of $\epsilon_n^M: f_n M \to M$, there is a unique morphism $h: f_{n+1} M \to f_n M$ that satisfies $\epsilon_{n+1}^M = \epsilon_n^M \circ h$. Apply this to $M = M^c(X)$. It follows that $F^{n+1} h^0_0(X)_R \subset F^n h^0_0(X)_R$. Hence, $F^n$ is a decreasing filtration.

For the triviality of the $(d+1)$-th filter, simply observe that $r_0$ is the identity as it is right adjoint to the identity $i_0$ on $\text{DM}_{\text{Nis}}^\text{eff}(k, R)(0) = \text{DM}_{\text{Nis}}^\text{eff}(k, R)$.

To show that $F^0 h^0_0(X)_R = h^0_0(X)_R$, we need to show that $\epsilon_{d+1}: f_{d+1} M^c(X) \to M^c(X)$ is trivial. We claim more generally that $M^c(X) \in \text{DM}_{\text{Nis}}^\text{eff}(k, R)(d+1)$, where $\text{DM}_{\text{Nis}}^\text{eff}(k, R)(d+1)$ is the full subcategory of $\text{DM}_{\text{Nis}}^\text{eff}(k, R)$ consisting of objects $M$ such that $\text{Hom}_{\text{DM}_{\text{Nis}}^\text{eff}(k, R)}(N, M) = 0$ for every $N \in \text{DM}_{\text{Nis}}(k, R)(d+1)$. For this, it suffices to show that $\text{Hom}_{\text{DM}_{\text{Nis}}^\text{eff}(k, R)}(M(Y)(d+r)[s], M^c(X)) = 0$ for each $Y \in \text{Sm}_k, r \geq 1$ and $s \in \mathbb{Z}$ (see [Pel17, Remark 2.1.2]). By the Suslin-Friedlander duality ([Voe00, Theorem 8.2]), there is an isomorphism

$$\text{Hom}_{\text{DM}_{\text{Nis}}^\text{eff}(k, R)}(M(Y)(d+r)[s], M^c(X)) \cong \text{Hom}_{\text{DM}_{\text{Nis}}^\text{eff}(k, R)}(M(X \times Y)(r)[s-2d], R),$$

but the right hand side vanishes as shown in the proof of [Pel17, Lemma 5.1.1].

**Proposition 3.4** (cf. Conjecture [??] (B)). The filtration $F^n h^0_0(X)_R$ is compatible with the action of correspondences, i.e. for any $\gamma \in \text{Cor}(X, Y)$, the induced morphism of sheaves $h^0_0(X)_R \to h^0_0(Y)_R$ is compatible with the filtration.

**Proof.** We only need to deal with the case $\tau = \text{Nis}$. Let $d_X = \dim X$ and $d_Y = \dim Y$. For any smooth scheme $X$ and $r \geq 0$, there is a commutative diagram in $\text{DM}_{\text{Nis}}^\text{eff}(k, R)$

$$\begin{array}{ccc}
\text{Hom}^\text{eff}(M^c(X), R[d_X]) & \xrightarrow{\cong} & \text{Hom}^\text{eff}(f_{d_X+1-n} M^c(X), R[d_X]) \\
\downarrow & & \downarrow \\
\text{Hom}^\text{eff}(M^c(X)(r), R[d_X + r]) & \xrightarrow{\cong} & \text{Hom}^\text{eff}(f_{d_X+1-n} M^c(X) \otimes R(r), R[d_X + r]) \\
& & \cong f \\
& & \text{Hom}^\text{eff}(f_{d_X+1-n+r} M^c(X)(r), R[d_X + r]),
\end{array}$$

where the vertical arrows in the square are isomorphisms by Voevodsky’s cancellation theorem ([Voe10]) and the isomorphism $f$ is induced by the isomorphism

$$t^\text{eff}_r(M^c(X)): (f_{d_X+1-n} M^c(X)) \otimes R(r) \to f_{d_X+1-n+r} M^c(X)(r)$$

in [Pel17, Proposition 3.3.3 (2)] (and shifting it by $2r$).

In particular, setting $r = d_Y$, we see that

$$F^n h^\text{Nis}_0(X)_R = \ker\{h^\text{Nis}_0(X)_R \cong H^0(M(X)) \cong H^0(\text{Hom}(M^c(X)(d_Y), R[d_X + d_Y])$$

on $\text{DM}_{\text{Nis}}^\text{eff}(k, R)$.
Similarly, we have
\[ F^n h^0_Nis(Y)_R = \ker \left(h^0_Nis(Y)_R \cong H^0(M(Y)) \cong H^0(\text{Hom}^\text{eff}(M^c(Y)(d_X), R(d_X + d_Y))) \right) \]
\[ \longrightarrow H^0(\text{Hom}^\text{eff}(f_{d_X + d_Y + 1 - n}(M^c(Y)(d_X)), R(d_X + d_Y))). \]

Now, a finite correspondence \( \gamma \in \text{Cor}(X, Y) \) induces a morphism \( M(X) \to M(Y) \) in \( \text{DM}^\text{eff}_Nis(k, R) \), and this induces the morphism in question \( h^0_Nis(X)_R \to h^0_Nis(Y)_R \). Thus, the commutativity of the following diagram implies the proposition:
\[
\begin{array}{cc}
M(X) & \xrightarrow{\gamma} & \text{Hom}^\text{eff}(M^c(Y)(d_X), R(d_X + d_Y)) \\
\text{Hom}^\text{eff}(M^c(Y)(d_X), R(d_X + d_Y)) & \xrightarrow{g^*} & \text{Hom}^\text{eff}(f_{d_X + d_Y + 1 - n}(M^c(Y)(d_X)), R(d_X + d_Y)) \\
M(Y) & \xrightarrow{\gamma} & \text{Hom}^\text{eff}(M^c(Y)(d_X), R(d_X + d_Y)) \\
\end{array}
\]
where \( g \) is the composition
\[
M^c(Y)(d_X) \cong \text{Hom}^\text{eff}(M(Y), R(d_X + d_Y)) \overset{\gamma*}{\longrightarrow} \text{Hom}^\text{eff}(M(X), R(d_X + d_Y)) \cong M^c(X)(d_Y)
\]
of the canonical isomorphisms and the composition with \( \gamma \). (In other words, \( g \) is the \((d_X + d_Y)-\)twist and \(2(d_X + d_Y)\)-shift of the morphism in \((\text{non-effective}) \text{DM}^\text{eff}_{Nis,gm}(k, R)\))
\[
M^c(Y)(-d_Y) \cong M(Y)^\vee \overset{\gamma*}{\longrightarrow} M(X)^\vee \cong M^c(X)(-d_X),
\]
where \( \vee \) stands for the dual in \( \text{DM}^\text{eff}_{Nis,gm}(k, R) \). As the left square is in the category of geometric motives \( \text{DM}^\text{eff}_{Nis,gm}(k, R) \) \([\text{MVW06}] \text{ Theorem 16.15, Corollary 20.4}; [\text{Kel17}] \text{ Proposition 5.3.5}\) to remove the hypothesis of resolution of singularities, the commutativity of the square is immediate from this description of \( g \).

\[ \square \]

3.2. Under Ayoub’s conjectures on \( n \)-motives. We derive properties of the filtration \( F^n h^0_Nis(X) \) from Ayoub’s conjectures on \( n \)-motives. In the end, under these conjectures, we conclude that if the filtration in Conjecture 2.7 exists, it agrees with the one in Definition 3.2. In this subsection, \( \tau \) is either \( Nis \) or \( \acute{e}l \).

We write \( \text{DM}^\text{eff}_{\tau \leq n}(k, R) \) for the smallest localizing subcategory of \( \text{DM}^\text{eff}_{\acute{e}l}(k, R) \) that contains the set of objects \( \{R_\tau(X) | X \in (\text{Sm}/k)_{\leq n}\} \). The category \( \text{DM}^\text{eff}_{\tau \leq n}(k, R) \) is called the triangulated category of \( n \)-motives. The conjectures of Ayoub are concerned with motives with coefficients in a \( \mathbb{Q} \)-algebra \( R \).

Conjecture 3.5 ([\text{Ayo17}] Conjecture 4.22]). Let \( R \) be a \( \mathbb{Q} \)-algebra. Then, the functor
\[
\text{Hom}^\text{eff}(R(1), -) : \text{DM}^\text{eff}_{\acute{e}l}(k, R) \to \text{DM}^\text{eff}_{\acute{e}l}(k, R)
\]
takes \( \text{DM}^\text{eff}_{\tau \leq n}(k, R) \) to \( \text{DM}^\text{eff}_{\tau \leq n-1}(k, R) \), where \( \text{Hom}^\text{eff} \) stands for the internal hom in \( \text{DM}^\text{eff}_{\acute{e}l}(k, R) \) and \( R(1) \) is the Tate motive.

Conjecture 3.6 ([\text{Ayo17}] Conjecture 4.27). [\text{ABV09}] Conjecture 2.5.3]). Let \( R \) be a \( \mathbb{Q} \)-algebra. Then, the homotopy t-structure on \( \text{DM}^\text{eff}_{\acute{e}l}(k, R) \) restricts to a t-structure on \( \text{DM}^\text{eff}_{\tau \leq n}(k, R) \), and the heart of this is the category \( \text{H}^\text{tr}_{\acute{e}l, \leq n}(k, R) \) of \( n \)-motivic sheaves.

Remark 3.7. In [\text{Ayo17}] Definition 4.19], an \( n \)-motivic sheaf of \( R \)-modules is called an \( n \)-presented homotopy invariant sheaf with transfers. The latter is defined as a sheaf \( F \in \text{H}^\text{tr}_{\acute{e}l}(k, R) \) such that there is an exact sequence of sheaves in \( \text{H}^\text{tr}_{\acute{e}l}(k, R) \)
\[
\bigoplus_{j \in J} h^0_{\acute{e}l}(Y)_R \longrightarrow \bigoplus_{i \in I} h^0_{\acute{e}l}(Y)_R \longrightarrow F \longrightarrow 0,
\]
where \( X_i \) and \( Y_j \) are objects in \( (\text{Sm}/k)_{\leq n} \). As remarked in [ibid., Remark 4.20], the two notions agree, so Conjecture 2.7 stands in the present context. Indeed, by Proposition 2.8 any \( n \)-presented homotopy invariant sheaf with transfers is an \( n \)-motivic sheaf. Conversely, by Remark 2.9 any \( n \)-motivic étale sheaf \( F \) is the \( h^0_{\acute{e}l} \) of some strongly \( n \)-generated sheaf \( G \). Now, by the description of \( \sigma_n^* \) recalled in Subsection 2.3, we have \( G \cong \text{colim}_{X \to \sigma_n,G} R_\tau(X) \), where the colimit is taken over the category \( \text{Cor}(k_{\leq n})/\sigma_n,G \). Since \( h^0_{\acute{e}l} \) commutes with colimits, we obtain
\[
F \cong h^0_{\acute{e}l}(G) \cong h^0_{\acute{e}l}(\text{colim}_{X \to \sigma_n,G} R_\tau(X)) \cong \text{colim}_{X \to \sigma_n,G} h^0_{\acute{e}l}(R_\tau(X)) = \text{colim}_{X \to \sigma_n,G} h^0_{\acute{e}l}(X)_R.
\]
The functor \( r_n \) in Subsection 3.14 can be expressed in terms of internal hom in \( \text{DM}^\text{eff}_{\text{et}, k} \). This description of \( r_n \) enables us to use the power of Conjectures 3.5 and 3.6. The author learned this method from [Pel17 6.1.9].

**Proposition 3.8** ([HK06 Proposition 1.1]). Suppose \( n \geq 0 \). The functor \( \nu_n : \text{DM}^\text{eff}_{\text{et}, k} \to \text{DM}^\text{eff}_{\text{et}, k}(n) \) defined by \( \nu_n(M) = \text{Hom}^\text{eff}(R(n), M)(n) \) is right adjoint to the inclusion \( \iota_n \), i.e. \( \nu_n \cong r_n \).

**Proof.** The proof in [HK06 Proposition 1.1] works verbatim. (Let us remark that the proof uses Voevodsky’s cancellation theorem [Voe10].)

Here is a (very) weak version of (D).

**Proposition 3.9.** Let \( R \) be a \( \mathbb{Q} \)-algebra. Assume Conjectures 3.3 and 3.7. Then, for any \( X \in \text{Sm}/k \), the sheaf \( h^0_\text{et}(X)_R/F^n h^0_\text{et}(X)_R \) can be embedded into an \( (n-1) \)-motivic sheaf.

**Proof.** Let \( d := \dim X \). By Definition 3.2 and Conjecture 3.6, it is enough to prove that \( \text{Hom}^\text{eff}(f_{d+1-n}M^\text{c}(X), R(d)) \) is \((n-1)\)-motivic. By Proposition 3.8 we have

\[
\text{Hom}^\text{eff}(f_{d+1-n}M^\text{c}(X), R(d)) \cong \text{Hom}^\text{eff}(\text{Hom}^\text{eff}(R(d+1-n), M^\text{c}(X))(d+1-n), R(d)).
\]

Since the right hand side is isomorphic to \( \text{Hom}^\text{eff}(R(d+1-n), \text{Hom}^\text{eff}(\text{Hom}^\text{eff}(R(d+1-n), M^\text{c}(X)), R(d))) \) and since we are under Conjecture 3.3, it remains to show that \( \text{Hom}^\text{eff}(\text{Hom}^\text{eff}(R(d+1-n), M^\text{c}(X)), R(d)) \) is \( d \)-motivic. Now, \( R(d) \) is \( d \)-motivic because there is a decomposition \( M(\mathbb{P}^d_k) \cong \bigoplus_{i=0}^d R(i) \) ([MVW06 Exercise 15.11]). Therefore, it suffices to show the following lemma. 

**Lemma 3.10.** Let \( R \) be a \( \mathbb{Q} \)-algebra. Assume Conjecture 3.7. For any motive \( M \in \text{DM}^\text{eff}_{\text{et}, k} \) and any \( N \in \text{DM}^\text{eff}_{\text{et}, \leq n}(k, R) \), the internal hom \( \text{Hom}^\text{eff}(M, N) \) belongs to \( \text{DM}^\text{eff}_{\text{et}, \leq n}(k, R) \).

**Proof.** This is [Ayo17 Proposition 4.26] if \( M \) belongs to \( \text{DM}^\text{eff}_{\text{et}, \text{gm}}(k, R) \). By [Pel17 Theorem 2.1.3], the inclusion \( j_n : \text{DM}^\text{eff}_{\text{et}, \leq n}(k, R) \to \text{DM}^\text{eff}_{\text{et}}(k, R) \) has a right adjoint \( d_n : \text{DM}^\text{eff}_{\text{et}}(k, R) \to \text{DM}^\text{eff}_{\text{et}, \leq n}(k, R) \), which is a triangulated functor, and there is another triangulated functor \( l_n : \text{DM}^\text{eff}_{\text{et}}(k, R) \to \text{DM}^\text{eff}_{\text{et}}(k, R) \) together with a natural triangle in \( \text{DM}^\text{eff}_{\text{et}}(k, R) \) for any \( E \in \text{DM}^\text{eff}_{\text{et}}(k, R) \):

\[
j_n d_n E \to E \to l_n E \to [1]
\]

with \( l_n E \in \text{DM}^\text{eff}_{\text{et}, \leq n}(k, R)^\perp \).

Applying this to \( E = \text{Hom}^\text{eff}(M, N) \), we see that it suffices to show that \( l_n \text{Hom}^\text{eff}(M, N) = 0 \) for all \( d \)-motivic effective motives \( M \). This is known under Conjecture 3.5 for effective geometric motives by [Ayo17 Proposition 4.26]. Therefore, it suffices to show that the full triangulated subcategory \( T \) of \( \text{DM}^\text{eff}_{\text{et}}(k, R) \) consisting of objects \( X \) such that \( l_n \text{Hom}^\text{eff}(X, N) = 0 \) is localizing.

Let \( S \) be the full triangulated subcategory of \( \text{DM}^\text{eff}_{\text{et}, \leq n}(k, R)^\perp \) consisting of objects \( E \) such that \( d_n E = 0 \). It follows from the above distinguished triangle that \( l_n \) has a image in \( S \). Let us write \( l'_n : \text{DM}^\text{eff}_{\text{et}}(k, R) \to S \) for the functor induced by \( l_n \). We claim that \( l'_n \) is left adjoint to the inclusion \( S \to \text{DM}^\text{eff}_{\text{et}}(k, R) \). (This claim is a variant of [HK06 Corollary 1.4 (ii)].) Indeed, for any \( E \in \text{DM}^\text{eff}_{\text{et}}(k, R) \) and any \( F \in S \), the distinguished triangle for \( E \) gives rise to an exact sequence of hom groups in \( \text{DM}^\text{eff}_{\text{et}}(k, R) \)

\[
\cdots \to \text{Hom}(j_n d_n E[1], F) \to \text{Hom}(l'_n E, F) \to \text{Hom}(E, F) \to \text{Hom}(j_n d_n E, F) \to \cdots
\]

Since \( F \) belongs to \( \text{DM}^\text{eff}_{\text{et}, \leq n}(k, R)^\perp \), we have \( \text{Hom}(j_n d_n E[1], F) = \text{Hom}(j_n d_n E, F) = 0 \). Thus, the middle map \( \text{Hom}(l'_n E, F) = \text{Hom}_{\text{DM}^\text{eff}_{\text{et}}(k, R)}(l'_n E, F) \to \text{Hom}_{\text{DM}^\text{eff}_{\text{et}}(k, R)}(E, F) \) is an isomorphism. Therefore, \( l'_n : \text{DM}^\text{eff}_{\text{et}}(k, R) \to S \) is a left adjoint functor. In particular, it commutes with coproducts. This immediately implies that \( T \) is a localizing subcategory.

It is shown that if there are filtrations \( F^n h^0_\text{et}(X) \) for all \( X \in \text{Sm}/k \) that satisfies (A), (B) and the Weaker Version of (D) in Conjecture 2.7 then any homotopy invariant subsheaf of an \( n \)-motivic sheaf is again \( n \)-motivic ([ABY09 Corollary 1.4.5]). Therefore, we have the following.

**Corollary 3.11.** Suppose \( R \) is a \( \mathbb{Q} \)-algebra and assume Conjectures 3.3 and 3.7. If the filtration as in Conjecture 2.7 exists, then the filtration in Definition 3.5 satisfies Conjecture 2.7(D).
Let us define a filtration on an arbitrary homotopy invariant $\tau$-sheaf $S$ of $R$-modules with transfers. $S$ can be written as a colimit of sheaves $h^0_{\tau}(X)_R$ with $X \in \mathrm{Sm}/k$: $S \cong \colim_{\mathrm{Cor}(k)/S} h^0_{\tau}(X)_R$ ([ABV09 Corollary 1.1.8]). We define a filtration on $S$ as follows. First, set

$$S^{<i} := \colim_{\mathrm{Cor}(k)/S} h^0_{\tau}(X)_R/F^{i+1}h^0_{\tau}(X)_R.$$ 

Then, define

$$F^iS := \ker\{S \to S^{<i-1}\}.$$ 

This is clearly compatible with the filtration on $h^0_{\tau}(X)_R$ in Definition 3.2. Note also that $F^0S = S$ holds for any $S \in \text{HI}^r_{\tau}(k, R)$.

**Lemma 3.12.** Any morphism $S \to S'$ in $\text{HI}^r_{\tau}(k, R)$ respects the filtration. If $S$ is an $n$-motivic $\tau$-sheaf, then $F^iS = 0$ for $i \geq n + 1$.

**Proof.** Any morphism respects the filtration because morphisms between sheaves of the form $h^0_{\tau}(X)_R$ ($X \in \text{Sm}/k$) have this property by Proposition 3.3. For the triviality of the filtration on $n$-motivic sheaves for $i \geq n + 1$, note that we have $S \cong \colim_{\mathrm{Cor}(k)_{\tau < n}/S} h^0_{\tau}(X)_R$ by Remark 3.7. With Propositions 3.3 and 3.4, this means that for $j \geq n$ we have

$$S^{<j} = \colim_{\mathrm{Cor}(k)/S} h^0_{\tau}(X)_R/F^{j+1}h^0_{\tau}(X)_R = \colim_{\mathrm{Cor}(k)_{\tau < n}/S} h^0_{\tau}(X)_R/F^{j+1}h^0_{\tau}(X)_R = S.$$ 

Hence, $F^iS = \ker\{S \to S^{<i-1}\} = 0$ for $i \geq n + 1$. \hfill $\square$

**Proposition 3.13.** Let $X$ be an arbitrary smooth $k$-scheme. Then, any morphism $f: h^0_{\tau}(X)_R \to F$ of homotopy invariant sheaves of $R$-modules with transfers with $F \in \text{HI}_{\tau \leq n-1}(k, R)$ factors through $h^0_{\tau}(X)_R \to h^0_{\tau}(X)_R/F^n h^0_{\tau}(X)_R$.

**Proof.** By Lemma 3.12, $f$ respects the filtration $F^i$, so the image of $F^n h^0_{\tau}(X)_R$ under $f$ belongs to $F^n F$, but the latter is trivial again by the lemma. \hfill $\square$

**Remark 3.14.** Proposition 3.13 is unconditional. When $\tau = \et$ and $R = \mathbb{Q}$, it may be regarded as a very weak form of the conjectured existence of the left adjoint to the inclusion $\text{HI}^r_{\et \leq n}(k, \mathbb{Q}) \to \text{HI}^r_{\et}(k, \mathbb{Q})$.

**Theorem 3.15.** Suppose $R$ is a $\mathbb{Q}$-algebra and assume Conjectures 3.3 and 3.4. If the filtration in Conjecture 2.7 exists, then it agrees with the one in Definition 3.2.

**Proof.** With Proposition 3.3, Proposition 3.4 and Corollary 3.11, it remains to show the property (C) in Conjecture 2.7. Slightly more strongly, we show that if $U$ is a dense open subscheme of $X \in \text{Sm}/k$, then the morphism $F^n h^0_{\tau}(U)_R \to F^n h^0_{\tau}(X)_R$ is surjective. We follow the proof of [ABV09 Proposition 1.4.6], especially the last part.

Consider the commutative diagram

$$
\begin{array}{ccc}
0 & \to & F^n h^0_{\tau}(U)_R \\
\downarrow a & & \downarrow b \\
0 & \to & F^n h^0_{\tau}(X)_R \\
\end{array}
$$

We shall show that coker $a$ is trivial. But for this, it is enough to prove that coker $a$ is $(n-1)$-motivic. Indeed, the kernel-cokernel sequence associated with the composition $F^n h^0_{\tau}(U)_R \to F^n h^0_{\tau}(X)_R \to h^0_{\tau}(X)_R$ gives rise to an exact sequence

$$0 \to \text{coker } a \to F^n h^0_{\tau}(X)_R \to F^n h^0_{\tau}(U)_R \xrightarrow{f} h^0_{\tau}(X)_R \to 0.$$ 

By Corollary 3.11, $h^0_{\tau}(X)_R/F^n h^0_{\tau}(X)_R$ is $(n-1)$-motivic. If coker $a$ is $(n-1)$-motivic, $h^0_{\tau}(X)_R/F^n h^0_{\tau}(U)_R$ is also $(n-1)$-motivic by Proposition 2.6(i). Thus, Proposition 3.13 implies that $f$ is an isomorphism; hence coker $a$ is trivial.

Let us show that coker $a$ is $(n-1)$-motivic. The map $b$ in the diagram is an epimorphism by [MVW06 Corollary 22.8] and Yoneda’s lemma. Therefore, coker $a$ is a subquotient of $h^0_{\tau}(U)_R/F^n h^0_{\tau}(U)_R$. Because, if the filtration in Conjecture 2.7 exists, $\text{HI}^r_{\et \leq n-1}(k, R)$ is a Serre subcategory of $\text{HI}^r_{\et}(k, R)$ ([ABV09 Corollary 1.4.5]), it is enough to show that $h^0_{\tau}(U)_R/F^n h^0_{\tau}(U)_R$ is $(n-1)$-motivic. But this is Corollary 3.11. \hfill $\square$
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