The three-nucleon (3N) force is important for understanding the structure and dynamics of atomic nuclei and nuclear matter from first principles and it is routinely used in many ab-initio structure calculations [1-4]. It is also included in calculations describing reactions involving nuclei that can be modelled as few-body systems [2, 5, 6]. However, this force is not considered in analyses of experimental data involving direct reactions with complex nuclei. One particular class of such reactions, deuteron stripping \((d, p)\) and pick-up \((p, d)\), is an important experimental tool for testing the shell-model picture of atomic nuclei, which is often used for indirect determination of nucleon capture reaction rates at astrophysical energies [7].

The \((d, p)\) and \((p, d)\) reactions are usually described either within the Distorted Wave Born Approximation (DWBA) [8] or the Adiabatic Distorted Wave Approximation (ADWA) [9]. Both use phenomenological optical potentials where the contribution from the 3N force is present implicitly. However, there are two other ways for the 3N force to manifest itself in \((d, p)\) and \((p, d)\) reactions. One of them is through an additional term in the DWBA or ADWA \(T\)-matrix element containing the \(n-p\) interaction \(W_{npi}\), where \(i\) belongs to the target, in the transition operator [10]. Its importance has been assessed in [11] using a simplified hypercentral model, similar to that used in [10], showing the importance of the consistency between the NN and 3N models. We then proceed with a local chiral EFT at N2LO model with parameters taken from [2] that reproduce well the properties of light nuclei and nuclear matter. We evaluate the contributions from the contact, one-pion-exchange plus contact and two-pion-exchange parts of these interaction in Sec. IV, V, and VI, respectively. We demonstrate influence of the choice of the 3N force format and reveal the role of the deuteron \(d\)-state. The ADWA calculations will then be presented in Sec. VII for \(^{40}\)Ca and \(^{27}\)Al targets using a range of nucleon-target optical potentials. In Sec. VIII we calculate the 3N force contribution in the Watanabe folding \(d-A\) model where the coupling to deuteron breakup states is missing, but 3N effects could be expected to be more similar to those one might expect beyond the ADWA. We will show that the 3N contribution has a different effect on the Watanabe \(d-A\) folding potential and the corresponding \((d, p)\) cross sections. The obtained results are discussed in Sec. IX where conclusions will also be given and the need for specific future developments described. The Appendix provides expressions for a few functions needed to calculate the \(2\pi\)-exchange contribution.
II. ADIABATIC MODEL WITH 3N FOLDING POTENTIAL

The ADWA assumes that the wave function of the d–A system is described by a n + p + A three-body model and that it can be represented by the first term in the Weinberg expansion [9], which has been shown to dominate the (d, p) cross sections due to the short-range n-p potential in the (d, p) T-matrix [13]. The ADWA approximates the first Weinberg component by a product of the adiabatic distorted-wave function $\chi_{ad}(R)$ and a deuteron wave function $\phi_0$. The distorted wave $\chi_{ad}(R)$ is found from the solution of the two-body Schrödinger equation

$$\langle T + \langle \phi_1 | U_{nA} + U_{pA} | \phi_0 \rangle - E_d \rangle \chi_{ad}(R) = 0,$$

where $T$ is the kinetic energy operator associated with the coordinate $R$ of the relative d–A motion, $E_d$ is the centre-of-mass deuteron energy and $U_{nA}$ and $U_{pA}$ are the nucleon-target optical potentials that depend on coordinates $R + r/2$ and $R - r/2$, respectively, where $r$ is the $n$-$p$ separation, shown in Fig. 1. Also in Eq. (1), $\phi_0$ is the deuteron wave function and

$$\phi_1 = \frac{V_{np} \phi_0}{\langle \phi_0 | V_{np} | \phi_0 \rangle}.$$

The ADWA, as well as other direct reaction theories, was derived on the assumption that the nuclear Hamiltonian contains NN interactions only. In the presence the 3N force one can expect that the adiabatic equation (1) should include an additional term

$$U_{3N}(R) = \langle \phi_1 \phi_A | \sum_{i \in A} W_{npi} | \phi_A \phi_0 \rangle,$$

where $W_{npi}$ is the 3N interaction between $n$, $p$ and a nucleon $i$ belonging to the target $A$, $\phi_A$ is the many-body wave function describing $A$ and the integration in the matrix element (3) is carried out over all internal coordinates of $A$ and over $r$. We will start with considering the matrix element

$$W_{di}^{eff}(r_{di}) = \langle \phi_1 | W_{npi} | \phi_0 \rangle,$$

where integration is done over $r$. It has a meaning of an effective two-body potential between nucleon $i$ and the centre-of-mass of deuteron. The 3N contribution $U_{3N}$ to the adiabatic potential is then described by the well-known folding potential

$$U_{3N}(R) = \int ds \rho_A(s) W_{di}^{eff}(R - s),$$

where $\rho_A$ is the target density for which we will use some available phenomenological representations. The adiabatic distorted wave $\chi_{ad}(R)$ is now found from

$$(T + \langle \phi_1 | U_{nA} + U_{pA} | \phi_0 \rangle + U_{3N}(R) - E_d) \chi_{ad}(R) = 0.$$

III. ADIABATIC d – A POTENTIAL WITH A HYPERCENTRAL 3N INTERACTION

To have a general idea about the sensitivity of the d–A potential $U_{3N}(R)$ to the 3N force we first consider a hypercentral 3N interaction

$$W(\rho_{ijk}) = |W_0|^2 \rho_{ijk}^{2/3},$$

where $\rho_{ijk} = (r_{ij}^2 + r_{ik}^2 + r_{jk}^2)/3$, with $r_{ij}$ being the distance between nucleons $i$ and $j$, and $\rho_0$ is the range of the 3N force. We can also consider another format of this force, such as used in [10],

$$W^{(\tau)}(\rho_{ijk}) = \frac{1}{3} (\tau_i \cdot \tau_j + \tau_i \cdot \tau_k + \tau_k \cdot \tau_j) W^{(\tau)} e^{-\frac{r_{ijk}^2}{\rho_0^2}},$$

where $\tau_i$ is the isospin Pauli matrix. The two different representations of the hypercentral force are reminiscent of different possible formats of the 3N contact interactions in $\chi$EFT at N2LO that we will use in the sections below. We notice that the matrix element in Eq. (3) that determines $W_{di}^{eff}$ uses $W_{npi}$ together with the functions $\phi_0$ and $\phi_1$, which correspond to zero deuteron isospin. In this case the isospin $T_{npi}$ of the $i$ – d system is equal to $\frac{1}{2}$ so that

$$(\tau_n \cdot \tau_p + \tau_n \cdot \tau_i + \tau_p \cdot \tau_i) \chi d \chi_i = 2 (T_{npi}^2 - t_n^2 - t_p^2 - t_i^2) \chi d \chi_i = -3 \chi d \chi_i,$$

where $\chi_d$ and $\chi_i$ are the isospin functions of the deuteron and nucleon $i$, respectively, and $T_{npi}$ and $t_i$ are the isospin operators of the three-body $n$-$p$-$i$ system and nucleon $i$, respectively. Therefore, using the formats (7) and (8) of the 3N force with $W_{0}^{(\tau)} = -W_{0}$ will give identical results.
The hyperradius $\rho_{np\pi}$ can be expressed via normalized Jacobi coordinates $x_1 = r/\sqrt{2}$ and $x_2 = \sqrt{3/4} r_{di}$ as $\rho_{np\pi}^2 = x_1^2 + x_2^2$ (see Fig. 1 for coordinate definitions). Then using (8) or (9) in Eq. (4) we obtain

$$W_{di}^{\text{eff}}(r_{di}) = W_0 \phi \left( \sqrt{\frac{2}{3}} r_{di} \right) e^{-\frac{2}{3} r_{di}^2 c_0} \int dr \phi_1^*(r) e^{-\frac{2}{3} r_{di}^2 c_0} \phi_0(r). \quad (10)$$

Let is consider first a zero-range 3N force with a strength fixed by the volume integral $I_3 = 3 W_0 \pi^2 \rho_0^2$, consistent with the definition given by Eq. (45) of Ref. 10. Such a force is analogous to the unregularized contact interaction of the $\chi$EFT at N2LO [2]. In the $\rho_0 \to 0$ limit we obtain

$$W_{di}^{\text{eff}}(r_{di}) = I_3 \left( \frac{2}{3} \right)^3 \delta \left( \sqrt{\frac{2}{3}} r_{di} \right) \phi_1(0) \phi_0(0). \quad (11)$$

This leads to the folding 3N force

$$U_{3N}(R) = I_3 \phi_1(0) \phi_0(0) \rho_A(R), \quad (12)$$

which has exactly the same shape as the target density profile with the absolute values determined by the volume integral $I_3$ of the 3N interaction and the values of the deuteron wave function $\phi_0$ and the vertex function $\phi_1$ at zero $n$-$p$ separations. The deuteron wave function $\phi_0(0)$ is very sensitive to high $n$-$p$ momentum content in the deuteron and is strongly dependent on the NN model used, which is illustrated in Table I of Ref. 10. The same is valid for $\phi_1(0)$. This value can be obtained from $\phi_0(0)$ by using the Schrödinger equation $(T - \epsilon_d)\phi_0 = -\phi_1(0) V_{np}(\phi_0)$, where $\epsilon_d$ is the deuteron binding energy. Taking $\phi_0$ from the N2LO $\chi$EFT [2] we deduce $\phi_1(0)$ and obtain $\phi_1(0) = -1.17 \text{ fm}^{-3/2}$. We also use $\phi_0(0) = 0.0796 \text{ fm}^{-3/2}$ from the same NN model and assume $I_3 = 2186 \text{ MeV fm}^6$, which is consistent with the set II of the N2LO $\chi$EFT considered below. Then taking an average density $\rho_A(0) \approx 0.16 \text{ fm}^{-3}$ we obtain $U_{3N}(0) \approx -33 \text{ MeV}$. The origin of this attraction is the negative value of $\phi_1$ at $r = 0$. However, beyond zero-range approximation, with increasing range $\rho_0$, the integral in (10) would span a larger fraction of both $\phi_0$ and $\phi_1$ which can affect the depth of the folding potential $U_{3N}(R)$.

In Fig. 2 we show the $U_{3N}$ potential calculated for the $d+^{26}\text{Al}$ system. The $^{26}\text{Al}$ matter density was assumed to be the same as the two-parameter Fermi charge density of $^{26}\text{Al}$ obtained in [14] from the analysis of electron scattering. The same strength $I_3 = 2186 \text{ MeV fm}^6$ was used while the range $\rho_0$ was varied. One can see that the choice of $\rho_0$ has a dramatic effect on $U_{3N}(R)$. First, its depth decreases from 33 to 0 MeV when $\rho_0$ increases from 0 to $\sim 0.5 \text{ fm}$. Then $U_{3N}$ changes sign, becoming repulsive and increasing to $\sim 5.7 \text{ MeV}$ at $0.5 \lesssim \rho_0 \lesssim 0.8 \text{ fm}$ after which it decreases again.

This simple exercise demonstrates the important role of the range of the 3N interaction and shows the necessity of using a 3N force model compatible with the NN interaction model. Below, we will proceed with using both of them from the N2LO $\chi$EFT of [2], where they have been consistently fitted to describe some properties of light nuclei, neutron-$\alpha$ scattering and neutron matter. This 3N force has contact, one-pion-contact ($\pi\pi\pi$-c) and double-pion ($2\pi\pi$) exchange parts. The strength of the first two of these is governed by low-energy constants $c_E$ and $c_D$, respectively. Several best sets of $(c_E, c_D)$ are available in Ref. [2], corresponding to different relative contributions from these two components, and they are shown in Table I. The low-energy constants defining the $2\pi$-exchange contribution are fixed from pion-nucleon or NN scattering [15].

**TABLE I.** The regulator $R_{3N}$ of the two-nucleon formfactor (in fm) and the low-energy constants $c_E$ and $c_D$ for a chosen format of the contact 3N forces taken from [2].

| set | format | $R_{3N}$ | $c_E$ | $c_D$ |
|-----|--------|---------|-------|-------|
| I   | $E_1$  | 1.0     | 0.62  | 0.5   |
| II  | $E_\tau$ | 1.0     | -0.63 | 0.0   |
| III | $E_\tau$ | 1.2     | 0.09  | 3.5   |
| IV  | $E_\pi$ | 1.0     | 0.59  | 0.0   |

**IV. ADIABATIC $d$ – A POTENTIAL WITH CONTACT 3N POTENTIAL**

We will use for $W_{np\pi}$ the contact $\chi$EFT interaction at N2LO from [2] where it was represented by three different formats,

$$V_{E1} = \frac{c_E}{\Lambda_n F_2^2} \sum_{cyc} \delta_{R_{3N}}(r_{ij}) \delta_{R_{3N}}(r_{kj}), \quad (13)$$

$$V_{E\tau} = \frac{c_E}{\Lambda_n F_2^2} \sum_{cyc} \left( \mathbf{r}_i \cdot \mathbf{T}_k \right) \delta_{R_{3N}}(r_{ij}) \delta_{R_{3N}}(r_{kj}), \quad (14)$$

FIG. 2. The adiabatic potential $U_{3N}(R)$ calculated for $d+^{26}\text{Al}$ system using hypercentral 3N potential of a fixed volume integral and several values of the hyperradius $\rho_0$. 

- **TABLE I.** The regulator $R_{3N}$ of the two-nucleon formfactor (in fm) and the low-energy constants $c_E$ and $c_D$ for a chosen format of the contact 3N forces taken from [2].
\[ V_{EP} = \frac{c_E}{\Lambda^2 F^4} \sum_{\text{cyc}} \mathcal{P} \delta_{R_3N}(\mathbf{r}_{ij}) \delta_{R_3N}(\mathbf{r}_{kj}), \]

where cyc denotes all possible cyclic permutations in the n-p-i system, \( \Lambda = 700 \text{ MeV}, \) \( F_\pi = 92.4 \text{ MeV} \) and the operator \( \mathcal{P} \) will be discussed later. The different formats are a consequence of the need for regularization of the \( \delta \)-functions present in the \( \chi \)EFT [2, 3]. The regularized \( \delta \)-functions of range \( R_{3N} \) used here are given by

\[
\delta_{R_{3N}}(r) = \frac{1}{\pi \Gamma(3/4)} R_{3N}^{3/4} e^{-r/R_{3N}},
\]

Let us first start with the \( V_{E1} \) force. It does not contain any isospin operators. All one needs is to use in Eqs. (3) and (4) the following quantities:

\[
D(r_{di}, r) \equiv \delta_{R_{3N}}(\mathbf{r}_{ni}) \delta_{R_{3N}}(\mathbf{r}_{pi}) = e^{-\frac{16r^4 + 28r^2 + r^4}{8R_{3N}^4}} \left( \frac{3}{2} \right) R_{3N}^2
\]

and

\[
[\delta_{R_{3N}}(\mathbf{r}_{ni}) + \delta_{R_{3N}}(\mathbf{r}_{pi})] = 8\pi \sum_{\lambda = \text{even}, \mu} \delta_\lambda \left( \frac{r_{di}}{r} \right) \delta_\lambda \left( \frac{r_{pi}}{r} \right) \times Y_\lambda^*(\hat{r}_{di}) Y_\lambda(\hat{r}_{pi}),
\]

where

\[
\delta_\lambda \left( \frac{r_{di}}{r} \right) = \frac{1}{2} \int_{-1}^{1} d\mu P_\lambda(\mu) \delta_{R_{3N}} \left( \left| \frac{r_{di}}{r} - \frac{1}{2} \right| \right).
\]

Here, the integration is performed over the cosine of the angle between vectors \( r \) and \( r_{di} \). Then, the corresponding effective \( d-i \) force is

\[
W_{E1}^{\text{eff}}(r_{di}) \equiv \langle \phi_1 | V_{E1} | \phi_0 \rangle
\]

\[
= \sqrt{3\pi} \sum_{\lambda \mu} (\lambda \mu J_d M_d J'_d M'_d) W_{E1, \lambda}^{\text{eff}}(r_{di}) Y_\lambda^*(\hat{r}_{di}),
\]

where \( \lambda \) takes only even values, \( J_d(J'_d) \) and \( M_d(M'_d) \) are the ket (bra) deuteron angular momentum and its projection, respectively, coupled by the Clebsch-Gordan coefficient (we have to note that ADWA only deals with \( J'_d = J_d \)), while the radial part of \( W_{E1}^{\text{eff}}(r_{di}) \) is

\[
W_{E1, \lambda}^{\text{eff}}(r_{di}) = \frac{c_E}{\Lambda^2 F^4} \hat{\lambda} J_d \sum_{l'} i(l0l0l') W(\lambda l J_d ; l' J_d') \int_0^\infty dr \sqrt{v_0(r)} u_0(r)
\]

\[
\times W(\lambda J_d l' J_d l) \int_0^\infty dr v_1(r) u_1(r)
\]

\[
\times \left[ D(r_{di}, r) \delta_{\lambda_0} + 2 \delta_{R_{3N}}(r) \delta_\lambda \left( \frac{r_{di}}{r} \right) \right],
\]

where \( W \) is the Racah coefficient. Also, the \( u_l(r) \) and \( v_l(r) \) are the radial parts of the deuteron wave function \( r \phi_0(r) \) and vertex function \( r \phi_1(r) \), respectively, in the partial wave \( l \). They are shown in Fig. 3 for two regulators \( R_{3N} = 1.0 \text{ fm} \) and \( 1.2 \text{ fm} \). For realistic deuteron wave function, with non-zero contribution from the \( d \)-state, the effective \( d-i \) interaction also has a quadrupole component.

The second force, \( V_{E2} \), includes isospin operators. Let us consider the first of them: \( \tau_1 \cdot \tau_n \). To evaluate \( W_{E1}^{\text{eff}} \), we need the following matrix element:

\[
\langle T_d M_d(n, p) , \frac{1}{2} \tau^*(i) | \tau_1 \cdot \tau_n | T_d M_d(n, p) , \frac{1}{2} \tau(i) \rangle,
\]

where \( | T_d M_d(n, p) \rangle \) represents the deuteron isospin state vector with isospin \( T_d \) and its projection \( M_d \), \( \frac{1}{2} \tau(i) \) represents the isospin state of nucleon \( i \) with projection \( \tau \). The ADWA involves the deuteron isospin \( T_d = 0 \) both in the entrance and exit channels and \( \tau^*(i) = \tau(i) \). It is easy to show that in this case the contribution from \( \tau_1 \cdot \tau_n \) vanishes. For the same reason there will be no contribution from \( \tau_1 \cdot \tau_p \), while the contribution from \( \tau_n \cdot \tau_p \) is calculated in a way similar to Eq. (9),

\[
(\tau_n \cdot \tau_p) \chi_B \chi_i = 2(T_{np}^2 - t_n^2 - t_p^2) \chi_B \chi_i = -3 \chi_B \chi_i.
\]

Therefore, the \( W_{E2}^{\text{eff}} \) interaction has a simpler structure than \( W_{E1}^{\text{eff}} \) being

\[
W_{E2}^{\text{eff}}(r_{di}) = -\frac{3c_E}{\Lambda^2 F^4} \sum_l \int_0^\infty dr u_l(r) u_l(r) D(r_{di}, r).
\]

The third force, \( V_{EP} \), contains the operator
\[ P = \frac{1}{36} \left( 3 - \sum_{i<j} \sigma_i \cdot \sigma_j \right) \left( 3 - \sum_{i<j} \tau_i \cdot \tau_j \right) \] acting on (\( S = \frac{1}{2}, T = \frac{3}{2} \)) state of the \( n-p+i \) system only. If we take Eq. \( \text{[7]} \) into account, then for \( T_d = 0 \) this operator reduces to \[ P = \frac{1}{3} \left( 3 - \sum_{i<j} \sigma_i \cdot \sigma_j \right) \] which is needed in the context of the matrix element

\[ \langle S_d M_d'(n,p), \frac{1}{2} \sigma'(i) | P | S_d M_d(n,p), \frac{1}{2} \sigma(i) \rangle = \sum_{M_s} \langle S_d M_d' | \frac{1}{2} \sigma'(i) | S_d M_d | \frac{1}{2} \sigma(i) \rangle, \]

(25)

where \( | S_d M_d(n,p) \rangle \) represents the deuteron spin state with spin \( S_d \) and projection \( M_d \) and \( | \frac{1}{2} \sigma(i) \rangle \) is the spin state of nucleon \( i \) with projection \( \sigma \). In general, such a matrix element leads to an effective \( d-i \) force that depend on spin projections \( \sigma_i \) of nucleon \( i \), which would require the corresponding densities of the target \( A \). Nondiagonal on spin projection parts of such densities can be nonzero and, moreover, they can explicitly depend on \( \sigma_i \). However, in a specific case when the expectation value of the operator \( S_A^2 = \left( \sum_{i=1}^{N} s_i \right)^2 \) is zero, which is a good approximation for double-closed shell nuclei, we have \( \sigma'_i = \sigma_i \) and

\[ \sum_{\sigma_i} \langle S_d M_d', \frac{1}{2} \sigma | P | S_d M_d, \frac{1}{2} \sigma \rangle = \frac{2}{3} \delta_{S_{d'}S_d} \delta_{M_{d'}M_d}, \]

(26)

assuming that density distribution of spin projections \( \frac{1}{2} \) and \(-\frac{1}{2} \) are exactly the same. Therefore, we can conclude that the effective \( d-i \) interaction \( W_{\text{E}1}^{\text{eff}} \) associated with the force \( V_{\text{E}1} \) is given by Eqs. [21] and [20], the same as in the case of \( V_{\text{E}2} \) but multiplied by \( 2/3 \). We stress that this conclusion is valid only for a target that has spin-zero component.

Figure 4 compares \( W_{\text{E}1}^{\text{eff}} \) and \( W_{\text{E}2}^{\text{eff}} \), corresponding to set I and II from Table I, respectively. Both sets have the same regulator \( R_{\text{E}N} = 1.0 \text{ fm} \) and very similar strengths given by low-energy constants \( c_E \). However, \( W_{\text{E}1}^{\text{eff}} \) is significantly larger than \( W_{\text{E}2}^{\text{eff}} \) at \( \tau_{d_i} \approx 0 \). Although both of them have a common part containing \( D(r_{d_i}, r) \delta_{\lambda,0} \), this part is three times stronger for \( W_{\text{E}2}^{\text{eff}} \) due to the contribution from \( \tau_{n_1} \tau_{p} \) that benefits from a factor of \(-3 \). However, \( W_{\text{E}1}^{\text{eff}} \) receives the contribution from the \( n \) and \( p \) interacting via the target nucleon \( i \) only, while \( W_{\text{E}2}^{\text{eff}} \) also includes interaction of \( n_p \) with \( i \) via \( p(n) \). The latter are of a significantly longer range, shown on Fig. 4 by a thin line. As a result, the volume integrals of the \( \lambda = 0 \) components in \( W_{\text{E}2}^{\text{eff}} \) and \( W_{\text{E}1}^{\text{eff}} \) are similar, being 31.1 and 37.4 MeV·fm\(^3\), respectively. Figure 4 also presents results obtained with the deuteron \( s \)-state only. One can see that the \( d \)-state contribution is very important, which is a consequence of a large magnitude of \( v_2(r) \). The \( d \)-state gives rise to a quadrupole \( \lambda = 2 \) component in \( W_{\text{E}1}^{\text{eff}} \) but it is very small. No quadrupole component is present in \( W_{\text{E}2}^{\text{eff}} \).

For any \( W_{\text{eff}}^{\text{E}1} \) interaction the folding \( d-A \) potential can be represented by a partial wave expansion

\[ U_{\lambda N}^{(E)}(R) = \sqrt{4\pi} \sum_{\lambda \mu} \lambda \mu J_d M_d | J_d' M_d' \rangle U_{\lambda N}^{(\lambda)}(R) Y_{\lambda \mu}^*(\hat{R}), \]

(27)

where the radial part is

\[ U_{\lambda N}^{(\lambda)}(R) = 4\pi \int_0^\infty ds s^2 W_{\text{d}i,\lambda}(s) \rho_\lambda(R, s), \]

(28)

with \( \rho_\lambda(R, s) \) given by Eq. [19] in which \( \delta \) is replaced by density \( \rho \). Fig. 5 shows \( U_{\lambda N}^{(E)}(R) \) for the \( d-^{40}\text{Ca} \) system, calculated using sets I and II of the contact 3N

![Figure 4](image4.png)

![Figure 5](image5.png)
force, given by the $E1$ and $E\tau$ formats, respectively. The parameterisation of the $^{40}$Ca density has been taken from electron scattering studies using the two-parameter Fermi model of [14]. One can see that the $d-^{40}$Ca potentials originating from the contact 3N force differ by about 20%, which is explained by a similar difference in the corresponding volume integrals of $W_{d}^{\text{eff}}$. However, for $E1$ the $U_{3N}$ potential is wider. Once again, we see an important contribution (about 50%) comes from the deuteron $d$-state, which is demonstrated in Fig. [5] by plotting the calculations retaining the deuteron s-wave state only. We did not show the contribution from the 3N force in the $E\mathcal{P}$ format (set IV) but it should be the same as the $E1$ contribution scaled down by a factor of 2/3 and corrected for the corresponding value of the low-energy constant $c_{E}$. The $E1$ format also give rise to the quadrupole part of the $d-^{40}$Ca potential but this component is small.

V. CONTRIBUTION FROM ONE-PION-CONTACT EXCHANGE 3N FORCE

Following [2] we will consider the $1\pi-c$ force in the $n-p-i$ system defined as

$$V_{D} = \frac{g_{ACD}m_{\pi}^{2}}{96\pi\Lambda_{\pi}F_{\pi}^{2}} \sum_{\text{cyc}} \tau_{i} \cdot \tau_{k} \left[ \delta_{R_{3N}}(r_{ij}) + \delta_{R_{3N}}(r_{kj}) \right] \times \left[ X_{ik}(r_{ik}) - \frac{4\pi}{m_{\pi}^{2}} \sigma_{i} \cdot \sigma_{k} \delta_{R_{3N}}(r_{ik}) \right],$$

(29)

where $g_{A} = 1.267$, the pion mass $m_{\pi} = 138.03$ MeV/c$^{2}$,

$$X_{ik}(r) = [S_{ik}(r)T(r) + \sigma_{i} \cdot \sigma_{k}]Y(r)$$

(30)

is the coordinate-space pion propagator that contains the tensor operator

$$S_{ik}(r) = 3(\sigma_{i} \cdot \hat{r})(\sigma_{k} \cdot \hat{r}) - \sigma_{i} \cdot \sigma_{k},$$

(31)

and the tensor Yukawa functions, $T$ and $Y$, are defined as $T(r) = 1 + 3/(m_{\pi}r) + 3/(m_{\pi}r)^{2}$ and $Y(r) = 1 - e^{-(r/\Lambda_{\pi})^{2}}(r/\Lambda_{\pi})^{2},$ where the latter contains the long-range regulator. As in the case of the contact interaction, the cyclic sum runs over all cyclic permutations of the $n-p-i$ system. We should note that one more format of the $1\pi-c$ force has been considered in [2] but the best fit of the data returned a value of $c_{D} = 0$ for that force.

We will need to act with the operator $V_{D}$ on the product of isospin function $\chi_{i}$ of a nucleon $i$ in $A$ and the deuteron isospin function $\chi_{d}$ and, as in the case of the contact interaction, to evaluate the matrix element [22].

The contributions from $\tau_{n} \cdot \tau_{i}$ and $\tau_{p} \cdot \tau_{i}$ to this matrix element in the ADWA are zero. Therefore, this selects in Eq. (29) the $\tau_{n} \cdot \tau_{p}$ and $\sigma_{n} \cdot \sigma_{p}$ operators only. The latter, acting on the deuteron spin-function $\chi_{1}$ with the deuteron spin $S_{d} = 1$, gives

$$(\sigma_{n} \cdot \sigma_{p})\chi_{1} = 2(\vec{S}_{d}^{2} - s_{1}^{2} - s_{2}^{2})\chi_{1} = \chi_{1}. $$

(32)

The corresponding effective interaction between nucleon $i$ and the deuteron is then

$$W_{1\pi}^{\text{eff}}(r_{di}) = -\frac{g_{ACD}m_{\pi}^{2}}{32\pi\Lambda_{\pi}F_{\pi}^{2}} \langle \phi_{1}(r) | [X_{np}(r) - \frac{4\pi}{m_{\pi}^{2}} \delta_{R_{3N}}(r)] \times [\delta_{R_{3N}}(r_{ni}) + \delta_{R_{3N}}(r_{pi})] | \psi_{0}(r) \rangle. $$

(33)

To evaluate the matrix element we will need

$$\langle [Y_{1}(\vec{r}) \otimes \chi_{1}]_{M_{d}}^{J_{d}'} | (\sigma_{n} \cdot \vec{r}) Y_{A_{n}}(\vec{r}) | [Y_{1}(\vec{r}) \otimes \chi_{1}]_{M_{d}}^{J_{d}''} \rangle = (4\pi)^{-1/2}(\lambda_{\mu}J_{d}M_{d}J_{d}'M_{d}')A_{\lambda\mu}',$$

(34)

where

$$A_{\lambda\mu'} = 18\hat{M}_{d} \sum_{\lambda'\lambda''}^{\lambda'\lambda''} (1010|\lambda'0\lambda'0\lambda'0\lambda'0|\lambda'00\lambda'00),$$

(35)

We will also need

$$\langle [Y_{1}(\vec{r}) \otimes \chi_{1}]_{M_{d}}^{J_{d}'} | (\sigma_{n} \cdot \vec{r}) Y_{A_{n}}(\vec{r}) | [Y_{1}(\vec{r}) \otimes \chi_{1}]_{M_{d}}^{J_{d}''} \rangle = (4\pi)^{-1/2}(\lambda_{\mu}J_{d}M_{d}J_{d}'M_{d}')B_{\lambda\mu'},$$

(36)

with

$$B_{\lambda\mu'} = \hat{M}_{d} (0000) W(\lambda J_{d}'; 1; l') J_{d}. $$

(37)

Assuming the same partial wave decomposition of $W_{d}^{\text{eff}}$ as given by Eq. (20) for $W_{1\pi}^{\text{eff}}$ and using (34) and (36) in Eq. (33) we obtain

$$W_{1\pi c, \lambda}(r_{di}) = -\frac{g_{ACD}m_{\pi}^{2}}{16\pi\Lambda_{\pi}F_{\pi}^{2}} \sum_{ll'} \int_{0}^{\infty} dr v_{\lambda}(r) u_{\lambda}(r) \times \delta_{\lambda}(r_{di} - \frac{r}{2}) f_{\lambda\mu'}(r),$$

(38)

FIG. 6. The $\lambda = 0$ component of the adiabatic effective $d-i$ potential $W_{1\pi c}^{\text{eff}}$ arising from the $1\pi-c$ force, obtained with sets I and III. Calculations with and without the deuteron $d$-state are shown by solid and dashed lines, respectively. The inset represents the quadrupole part of this force multiplied by $\sqrt{4\pi}$.  

\[ \frac{1}{2} \] \[ \frac{1}{2} \] \[ \frac{1}{2} \] \[ \frac{1}{2} \] \[ \frac{1}{2} \] \[ \frac{1}{2} \]
where

\[ f_{\text{eff}}(r) = 3 A_{\text{eff}} T(r) Y(r) - B_{\text{eff}} \left( (T(r) - 1) Y(r) + \frac{4\pi}{m^2} \delta_{R_3N}(r) \right). \]

(39)

Figure 6 shows the \( W_{\text{eff}}^{1\pi c} \) potential for two 3N interactions, given by set I and set III from Table I. The contribution from the deuteron \( s \)-state only is also shown. With the \( s \)-state only, the matrix element associated with the tensor operator \( S_{np} \) is zero so that \( W_{\text{eff}}^{1\pi c} \) depends only on \( (\sigma_n \cdot \sigma_p) \left( Y(r) - \frac{4\pi}{m^2} \delta_{R_3N}(r) \right) \). This contribution is positive and it does not exceed 5 MeV at zero \( d \)-\( i \) separation. Thus, the main contribution to \( W_{\text{eff}}^{1\pi c} \) comes from the \( d \)-state and is determined by the 1\( r \)-\( c \) part associated with the tensor operator \( S_{np} \). The \( \lambda = 0 \) monopole part of \( W_{\text{eff}}^{1\pi c} \) dominates. Contrary to the case of the contact interaction, the \( W_{\text{eff}}^{1\pi c} \) is attractive.

VI. CONTRIBUTION FROM TWO-PION-EXCHANGE 3N FORCE.

The 2\( \pi \)-exchange force \( V_C \) has contributions from three terms, \( V_{C,1} \), \( V_{C,3} \) and \( V_{C,4} \) but the latter will not contribute to the \( d \)-\( A \) potential because of isospin arguments similar to those explained in the sections above. The potentials \( V_{C,1} \) and \( V_{C,3} \) are given in Ref. \( 3 \) by

\[ V_{C,1} = \frac{c_1 m^2 g_\Lambda^2}{2 f_\pi^2 (4\pi)^2} \sum_{ijc} (\tau_i \cdot \tau_k)(\sigma_i \cdot \hat{r}_{ij})(\sigma_k \cdot \hat{r}_{kj}) \times U(r_{ij})Y(r_{ij})U(r_{kj})Y(r_{kj}) \]

(40)

and

\[ V_{C,3} = \frac{c_3 g_\Lambda^2}{36 f_\pi^2} \sum_{ijk} (\tau_i \cdot \tau_k) \left[ \frac{m^4}{(4\pi)^2} X_{ij}(r_{ij})X_{kj}(r_{kj}) - \frac{m^2}{4\pi} X_{ik}(r_{ij})\delta_{R_3N}(r_{kj}) - \frac{m^2}{4\pi} X_{ik}(r_{kj})\delta_{R_3N}(r_{ij}) + \sigma_i \cdot \sigma_j \delta_{R_3N}(r_{ij}) \right], \]

(41)

where \( c_1 = -0.81 \text{ MeV}^{-4} \) and \( c_3 = -3.4 \text{ MeV}^{-4} \) are the low-energy constants \( 15 \), \( U(r) = 1 + 1/(m \pi r) \) and \( \delta_{R_3N} \) and \( X \) are defined in the sections above. As in the previous sections, only \( (\tau_n \cdot \tau_p) \) will contribute to \( V_{C,1} \) and \( V_{C,3} \) for isospin reasons. To obtain expressions of the corresponding contributions to \( W_{\text{eff}}^{1\pi} \) we use

\[ (\sigma_n \cdot \hat{r}_{ni})(\sigma_p \cdot \hat{r}_{pi}) = \frac{1}{r_{ni} r_{pi}} \sum_{\lambda=0;2;\mu} (-)^{\lambda+\mu} |\sigma_n \times \sigma_p|^\lambda \mu \times \left[ r_{di} \times r_{di} \right]^\lambda \mu + \frac{1}{2}(-)^{\lambda} \left[ r_{di} \times r \right]^\lambda \mu - \frac{1}{4} |r \times r|^\lambda \mu \]

and the fact that an arbitrary function \( H(i) \) that depends on \( r_{ni} \) and \( r_{pi} \) only can be expanded as

\[ H(i)(r_{ni}, r_{pi}) = 4\pi \sum_{\lambda'\mu'} H^{(i)}_{\lambda\lambda'} \left( \frac{r_{di}}{2} \right) Y_{\lambda'\mu'}(\hat{r}_{di}) Y_{\lambda\mu}(\hat{r}), \]

(43)

where

\[ H^{(i)}_{\lambda\lambda'} \left( \frac{r_{di}}{2} \right) = \frac{1}{2} \int_{-1}^{1} d\mu P_{\lambda}(\mu) H^{(i)}(x_-, x_+), \]

(44)

with \( x_k = \sqrt{r_{di}^2 \pm r_{di}r_{pi} + \frac{4}{9}} \). Then, performing standard Racah algebra we obtain:

\[ W_{2\pi c}(r_{di}) = \sum_{\lambda'} \int_{0}^{\infty} dr u_{\lambda'}(r)u_{\lambda'}(r) \times \left[ \mathcal{F}^{(c_1)}_{\lambda\lambda'} \left( \frac{r_{di}}{2} \right) + \mathcal{F}^{(c_3)}_{\lambda\lambda'} \left( \frac{r_{di}}{2} \right) \right], \]

(45)

where

\[ \mathcal{F}^{(c_1)}_{\lambda\lambda'} = \sum_{\lambda'} C^{(c_1)}_{\lambda'\lambda'} \left[ r_{di}^2 H^{(0)}_{\lambda'} - \left( \frac{r_{di}}{2} \right)^2 H^{(0)}_{\lambda} \right], \]

(46)

\[ \mathcal{F}^{(c_3)}_{\lambda\lambda'} = \sum_{\lambda'} \left[ C^{(c_1)}_{\lambda'\lambda'} \left( r_{di}^2 H^{(1)}_{\lambda'} + r_{di}^2 H^{(2)}_{\lambda'} \right) - r_{di} C^{(2)}_{\lambda'\lambda'} H^{(3)}_{\lambda'} \right] \]

(47)

\[ + \hat{\lambda} \hat{J}_d (10\lambda 0 l' 0) W(\lambda J_d' \lambda; l' J_d) \mathcal{H}_{\lambda}^{(4)}. \]

Here \( \mathcal{F}^{(c_1,c_3)}_{\lambda\lambda'} \) as well as all \( \mathcal{H}^{(i)}_{\lambda} \) are functions of \( (r_{di}, \frac{r_{di}}{2}) \). The \( \mathcal{H}^{(i)}_{\lambda} \) are obtained from Eq. \( 44 \) using functions \( H^{(i)}(x_-, x_+ \right) \) whose expressions can be found in the Appendix. The other quantities in Eqs. \( 46 \) and \( 47 \) are the coefficients \( C^{(1,2)}_{\lambda'\lambda'} \) given by

\[ C^{(1)}_{\lambda'\lambda'} = 18\hat{J}_d \hat{\lambda}'^2 (10\lambda 0 l' 0) \sum_{\lambda'=0;2} \hat{\lambda}'^2 (10\lambda 0 l' 0) \times (\lambda' 0 \lambda 0) \left\{ \begin{array}{ccc} 1 & 1 & \lambda' \\ 1 & 1 & \lambda \\ 1 & 1 & \lambda' \\ 1 & 1 & \lambda \end{array} \right\} \]

(48)

\[ C^{(2)}_{\lambda'\lambda'} = 18\hat{J}_d \hat{\lambda}'^2 (10\lambda 0 l' 0) \times (10\lambda 0 \lambda 0) \left\{ \begin{array}{ccc} 1 & 1 & \lambda' \\ 1 & 1 & \lambda \end{array} \right\} \]

and

Numerical calculations have shown that the contribution from the \( V_{C,1} \) force is negligible for both regulators used (see Fig. 7). The contribution from \( V_{C,3} \) strongly depends on \( R_{3N} \) and it is dominated by contribution from
the deuteron $d$-state. Without this state the $W_{2\pi}$ is repulsive, while adding it makes the $d$-$i$ potential attractive. It should be noted that this potential has a large quadrupole part which, unlike in the case of the potential $1\pi-c$, is plotted in Fig. 7 without multiplication by a factor of $\sqrt{4\pi}$. However, for the spherical targets considered here its contribution to the folding $d$-$A$ potential is very small.

The sum of the $d$-$i$ interactions $W_{d_i}^{\text{eff}}$ obtained with contact, $1\pi-c$ and $2\pi$ contributions is shown in Fig. 8 for two sets (I and III) with $c_D \neq 0$. The contribution from the contact interaction is repulsive for set I, but attractive for set III due to the different signs of the low-energy constant $c_E$ in these two cases. This has a profound effect on the total $d$-$i$ potential, $W_{d_i}^{\text{eff}}$, which has a repulsive core for set I, but is purely attractive for set III. While in the first case all three 3N force components contribute to the $W_{d_i}^{\text{eff}}$ shape, in the second case this shape is dominated by the $1\pi-c$ part.

The total $W_{d_i}^{\text{eff}}$ interactions calculated with all four sets of the 3N potentials are compared to each other in Fig. 9. They depend strongly on the choice of 3N force. Their volume integrals and r.m.s. radii are shown in Table II. Since these radii are smaller than the typical sizes of nuclei the shape of the $d$-$A$ potentials $U_{3N}$ will be mainly determined by the nuclear density $\rho$. Since the volume integrals of $W_{d_i}^{\text{eff}}$ and $U_{3N}$ are related by a simple factor equal to the number of nucleons, $A$, in the target, the depths of the $U_{3N}$ potentials obtained for different 3N sets are in the same proportions as the corresponding volume integrals of $W_{d_i}^{\text{eff}}$. This can be seen in Fig. 9b where $U_{3N}$ are shown for the case of $d-^{40}\text{Ca}$ for all four 3N sets.

VII. THE ADWA CALCULATIONS WITH THE $d$-$A$ POTENTIALS ARISING FROM THE 3N FORCE

The example of the $d$-$A$ potential shown in Fig. 9 shows that additional contribution to the adiabatic potential due to the 3N force can be sufficiently strong to give a noticeable contribution to the $(d,p)$ cross section. Intuitively, one could expect that the largest influence on $(d,p)$ cross sections can be expected from the 3N force

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**FIG. 7.** The monopole and quadrupole adiabatic effective $d$-$i$ potentials $W_{2\pi,i}^{\text{eff}}$ calculated with $V_{C,c_3}$ force only (dashed lines) and with full $V_C$ potential (solid lines) for two different regulators: (a) 1.0 fm and (b) 1.2 fm.

**FIG. 8.** The individual contributions from the contact, $1\pi-c$ and $2\pi$ parts of the 3N potential to the monopole part of the adiabatic effective $d$-$i$ potential $W_{d_i}^{\text{eff}}$ obtained for sets I (a) and III (b). The sums of these contributions are shown by solid lines.
set III. However, we discovered that importance of the 3N contribution varies with the choice of the nucleon-target optical potentials used to construct the adiabatic potential \( U_{3N} \). By changing \( N_R \) monotonously within the range of 0.9 to 1.6 we learned that the corresponding changes in the \( (d, p) \) cross sections are not linear, moreover, they can behave differently for different \( U_{ADWA} \) choices. This observation explains the cross sections results shown in Figs. 10(a) and 11.

In Fig. 10 we have plotted the ADWA zero-range differential cross sections for \( ^{40}\text{Ca}(d, p)^{41}\text{Ca} \) reaction calculated at two incident deuteron energies, 11.8 MeV and 56 MeV, using the code TWOFNR [16]. Three different nucleon-target optical potentials were used both in the deuteron and proton channels: (i) local potentials from the Koenig-Delaroche (KDO3) nucleon optical model systematics [17], (ii) nonlocal energy-independent optical potentials from the \( N = Z \) Gianinni-Ricco (GR) systematics [18] and (iii) energy-dependent nonlocal dispersive optical model (NLDOM) potential [19] with modified parameters from [20] that reproduce the neutron separation from \( ^{41}\text{Ca} \). When employing nonlocal potentials we used their leading order local-equivalent representations both in the entrance deuteron and exit proton channels as explained in [21] and [20] [22], respectively. The overlap between \( ^{40}\text{Ca} \) and \( ^{41}\text{Ca} \) was taken from [20].

The KDO3 Johnson-Tandy potential for \( E_d = 11.8 \) MeV has a depth of about 107 MeV and one could expect that adding attractive 3N contributions with depths between 13 and 37 MeV will introduce proportionate changes to the cross sections. However, adding set I with the depth of 19 MeV does not change the cross sections much, while slightly shallower sets II and IV affect the cross sections decreasing it by \( \sim 8\% \) (see Fig. 10b). At the same time adding deep attractive Set III (\( \sim 37 \) MeV) increases the cross sections by 15\%, which is the result of non-linear nature of the optical potential behaviour with the renormalization factor \( N_R \). The situation changes with the deuteron incident energy, when for \( E_d = 56 \) MeV the \( U_{ADWA} \) depth is about 10\% smaller than that at 11.8 MeV. Figure 10 shows that the slope of all four \( ^{40}\text{Ca}(d, p)^{41}\text{Ca} \) cross sections in the angular range of \( 20^\circ < \theta < 30^\circ \) is similar their behaviour at \( \theta < 20^\circ \) is very different and it does seem to scale with the depth of the added 3N contribution.

For the case of nonlocal energy-independent optical \( p-A \) and \( n-A \) potentials the local-equivalent \( d-A \) potential is shallower than the Johnson-Tandy potential constructed from local-equivalents of nonlocal \( p-A \) and \( n-A \) potentials.

### Table II. Absolute values of the volume integrals (V.I., in MeV) and r.m.s. radii (in fm) of the effective \( d-i \) interactions \( W_{di}^{\text{eff}} \) and the \( d-^{40}\text{Ca} \) folding potentials \( U_{3N} \) calculated in the ADWA and Watanabe models.

|       | \( W_{di}^{\text{eff}} \) | \( U_{3N} \) |       | \( W_{di}^{\text{eff}} \) | \( U_{3N} \) |
|-------|----------------|----------------|-------|----------------|----------------|
|       | V.I. radius | V.I. radius |       | V.I. radius | V.I. radius |
| I     | 110.8 | 0.838 | 4430 | 3.531 | 0.343 | 1.228 | 13.72 |
| II    | 76.2  | 0.875 | 3047 | 3.540 | 2.137 | 1.259 | 1378 |
| III   | 215.7 | 1.183 | 8626 | 3.628 | 34.46 | 1.259 | 3.653 |
| IV    | 83.5  | 0.711 | 3341 | 3.503 | 2.957 | 0.888 | 118.3 |

**FIG. 9.** The monopole part of the total effective \( d-i \) potential \( W_{di}^{\text{eff}} \) (a) and the corresponding adiabatic potential \( U_{3N}(R) \) (b) calculated for the \( d+^{40}\text{Ca} \) system using four sets of the \( \chi \)EFT 3N interaction.
FIG. 10. The $^{40}$Ca($d,p)^{41}$Ca reaction at $E_d = 11.8$ MeV (left column) and 56 MeV (right column) calculated with nucleon optical potentials KD03 (a, b), GR (c, d), NLDOM (e, f) and NLDOM with I3B effects described in the text (g, h). For each case, four 3N force sets are used and calculations without 3N force (3NF) are also shown.

Thus, for the case of $d-^{40}$Ca at $E_d = 11.8$ MeV considered above the depth of the GR local-equivalent potential is 86 MeV as compared to 107 MeV associated with the KD03 potential. For $E_d = 56$ MeV these depths are 73 and 91 MeV, respectively. Unlike in the case of KD03, the spread between the 11.8 MeV cross sections calculated with four different 3N sets is smaller and they all differ more noticeably from those calculated without
the 3N force (see Fig. 10c). Stronger 3N effects are seen in Fig. 10d at 56 MeV for sets II, III and IV while set I does not affect the cross section despite the $U_{3N}$ depth is about 26% of $U_{ADWA}$.

It has been shown in [23] that in the adiabatic limit the energy-dependence of nonlocal optical potentials requires using the nucleon optical potentials evaluated at an energy shifted with respect to the traditionally used $E_d/2$ value by a large number given by the $n$-$p$ kinetic energy averaged over the short range of their interaction. This number is 57 MeV for a deuteron model without high $n$-$p$ momenta, but it can be much larger [24]. The depth of an optical potential usually decreases with the nucleon energy, and for $d-^{40}$Ca ADWA potential with NLDOM it is 82 and 69 MeV for $E_d=11.8$ and 56 MeV, respectively. Adding 3N force at $E_d=11.8$ Mev is more noticeable than in the case of the GR nonlocal potential and the spread between $(d,p)$ cross sections obtained with four 3N sets is larger (see Fig. 10e). Even bigger 3N effect is seen in Fig. 10f at $E_d=56$ MeV but the spread between the four cross sections is smaller.

The $(d,p)$ cross sections calculated with nonlocal optical potentials evaluated at shifted energies are overestimated [20, 23]. They lack the contribution from the induced $n+p+A$ three-body force (13B) arising due to interaction between $n$ and $p$ via excited states of the tar-
FIG. 12. The effective \( d-i \) potentials \( W_{d_i}^{\text{eff}} \) calculated in the Watanabe model. Decomposition into contact, \( 1\pi-c \) and \( 2\pi \) exchange is shown in (a) and (b) for set I and set III, respectively. Comparison between \( W_{d_i}^{\text{eff}} \) obtained with all four 3N sets is shown in (c). Thin lines represent calculations with the deuteron \( s \)-state only. Figure (d) shows \( U_{3N}(R) \) calculated for the \( d+^{40}\text{Ca} \) system using the same four sets and with \( s \)-state only shown by thin lines.

It was shown in [11] that accounting for this force in the ADWA increases the imaginary part of the adiabatic deuteron potential by a factor of two, thus reducing the \( (d,p) \) cross sections. It was also shown in [11] that with increased absorption the \( (d,p) \) cross sections are less sensitive to small changes in the real potential. Here, we increase the dynamical part of NLDOM by a factor of two to account for I3B and observe that in this case adding \( U_{3N} \) produces a smaller effect on the \( (d,p) \) cross sections for both deuteron incident energies (see Fig. 10 g and h).

The calculations with the \(^{40}\text{Ca} \) target involve neutron transfer to the \( f_{7/2} \) orbit in \(^{41}\text{Ca} \). To get an idea of how the 3N force manifests itself in \( (d,p) \) reactions for other angular momenta we have considered the \( d_5/2 \) and \( s_{1/2} \) transfers in the \(^{26}\text{Al}(d,p)^{27}\text{Al} \) reaction at \( E_d = 12 \) MeV for two \(^{27}\text{Al} \) final states: the ground state and the astrophysically important state at \( E_x = 7806 \) keV. The calculations are performed with the local KD03 and non-local GR optical potentials only. The standard geometry of the potential well for the transferred neutron (radius \( r_0 = 1.25 \) fm and diffuseness \( a = 0.65 \) fm) was used with the spectroscopic factor set to one. The results, shown in Fig. 11 suggest that the \( (d,p) \) sensitivity to the 3N force is smaller for the \( s_{1/2} \) transfers, while \( d_{5/2} \) transfer can be more strongly influenced by it, with the ground state of \(^{27}\text{Al} \) being particularly affected. Again, as in the \(^{40}\text{Ca}(d,p)^{41}\text{Ca} \) case, quantitative conclusions about importance of the 3N contribution depend on the choice of the optical potential.

**VIII. THE WATANABE MODEL WITH 3N FORCE**

The adiabatic calculations of the \( d-A \) folding potential, \( U_{3N} \), revealed its strong sensitivity to the choice of the 3N model and that for all models a large contribution to \( U_{3N} \) comes from the deuteron \( d \)-state. This sensitivity arises because the ADWA uses the short-range function \( v_l(r) \) representing the \( n-p \) interaction, which is large when \( l = 2 \).

The ADWA accounts for deuteron breakup only in an approximate way. An exact three-body treatment of deuteron breakup in Faddeev and continuum-discretized coupled channel (CDCC) approaches using local nucleon
FIG. 13. The $^{40}$Ca$(d,p)^{41}$Ca cross sections at $E_d = 11.8$ MeV (a) and 56 MeV (b) and the $^{26}$Al$(d,p)^{27}$Al cross sections at $E_d = 12$ MeV (c, d, e) calculated in the Watanabe model calculated without 3N force (3NF) and with set III. The final states in $^{27}$Al are the ground state (c) and the $E_x = 7806$ keV state for which the $s_{1/2}$ and $d_{5/2}$ transfers are presented separately in (d) and (e), respectively.

optical potentials suggests that non-adiabatic corrections can be important (see, for example, [25, 26]). Non-adiabatic effects are even stronger when $N-A$ optical potentials, employed in $(d,p)$ calculations, are nonlocal [27]. To understand if 3N-model sensitivity persists beyond the ADWA we have performed a few Watanabe folding calculations of $W_{\text{eff}}^{di}$ and $U_{3N}$. Although the Watanabe folding model does not connect the deuteron ground state wave function with its breakup states, it represents the leading order term in the CDCC expansion. One might expect that dependence on the 3N force would be similar for all CDCC matrix elements involving the lowest continuum bins since for a fixed NN interaction the $n$-$p$ continuum wave functions are similar at low $n$-$p$ energies and at small $n$-$p$ separations that determine $W_{\text{eff}}^{di}$.

The expressions for $W_{\text{eff}}^{di}$ in the Watanabe folding model are obtained by replacing all $v_l$ by $u_l$ in the corresponding equations of Sections IV, V and VI. The $W_{\text{eff}}^{di}$ are shown in Fig. 13 for all four sets of the 3N force and in the case of sets I and III, where $c_D \neq 0$, separate contributions from the contact, $1\pi$-c and $2\pi$-exchange are shown. For the total $W_{\text{eff}}^{di}$ force we also show the calculations with deuteron $s$-state only.

The Watanabe potentials $W_{\text{eff}}^{di}$ are wider than those obtained in ADWA because they contain a product of two long-range functions $u_l(r)$ and $u_l'(r)$ (see Table II for r.m.s. radii of $W_{\text{eff}}^{di}$). The overlap of such a product with the short-range 3N force is weaker than the one
for \( u_j(r) v_l(r) \) so that the resulting Watanabe \( W^{\text{eff}} \) potentials are weaker as well (see Table II for their volume integrals). The probability of the \( d \)-state in \( u_l \) is smaller than in \( v_l \) but the \( d \)-state still gives noticeable contribution to \( W^{\text{eff}} \) mainly through the tensor operator \( S_{np} \) in the \( 1\pi-c \) and \( 2\pi \) cases.

The \( d \)-state influence propagates to the \( d \)-\( A \) potentials \( U_{3N} \), shown in Fig. 12. Without it \( U_{3N} \) is repulsive with the height showing a small spread between 3 and 5 MeV. With \( d \)-state included, the contribution from the \( 2\pi \)-exchange almost cancels out the contribution from the contact interaction for sets I, II and IV, leading to negligible contribution from the 3N force. A different story occurs for set III where all three contributions, the contact, the \( 1\pi-c \) and \( 2\pi \), are attractive, resulting in a noticeable total 3N force. It is still significantly smaller than any \( U_{3N} \) of the adiabatic case but does affect the \( (d,p) \) cross sections, which is shown in Fig. 13 for the same targets, \( {}_4^{10}\text{Ca} \) and \( {}_2^{26}\text{Al} \), and at the same deuteron incident energies that are considered in the previous section. In these calculations we used the Watanabe model to calculate the folding potentials \( \langle \phi_0 | U_{AA} + U_{pA} | \phi_0 \rangle \), with KD03 chosen for \( U_{AA} \) and \( U_{pA} \). For \( {}_4^{10}\text{Ca} \) target the effect of including the 3N force (set III) has an opposite effect to the one seen in the ADWA for both deuteron energies. For \( {}_2^{26}\text{Al} \), this force has a weaker effect on populating the \( 2\pi \) state than the ADWA does but it shows a slightly stronger influence for the final \( 2\pi \) state at \( E_x = 7806 \) keV. In the latter case the 3N effect is still small, no more than 10% in the maximum.

IX. SUMMARY, CONCLUSIONS AND FUTURE CHALLENGES

We have presented the first calculations of a three-body optical potential due to bare 3N interactions of the neutron and proton in the incoming deuteron with target nucleons. With the aim of using this potential in modelling \( (d,p) \) reactions, we have evaluated its expectation value in ADWA motivated by the widespread use of the latter in the analysis of deuteron stripping experiments. Unlike standard \( N-A \) folding potentials, which are mainly sensitive to the volume integral of the NN force along with the target density, the three-body optical potential arising due to bare 3N interactions probes the short-range part of the deuteron wave function and both the strength and the range of the 3N interactions. A simple hypercentral model considered in Sec. III shows the necessity of a consistent approach to the choice of both the deuteron model and the 3N force.

In the present work the deuteron wave function has been taken from the \( \chi \text{EFT} \) at N2LO where 3N force arises naturally. We used consistently four N2LO sets of local 3N interactions from [2] that describe equally well some light nuclei and nuclear matter. These sets involve three different formats of the contact interactions, with two sets having a \( 1\pi-c \) component as well, and \( 2\pi \)-exchange.

The last two are heavily dominated by the contribution from the deuteron \( d \)-wave. Within ADWA, the deuteron \( d \)-wave state also gives an important contribution to the contact interaction.

We have used all four 3N force sets in the ADWA calculations of \( (d,p) \) cross sections and have shown that, potentially, they could distinguish between the choices made for the 3N force. However, the relative importance of these choices also depends on the nucleon-target optical potential employed in calculations. Future development of \( ab\text{-initio} \) theories of optical potentials, such as those initiated in [28–30], may on the one hand fix this optical potential dependence, but on the other they might introduce an additional new 3N-dependence in the optical potentials.

The ADWA overemphasizes the role of short \( n-p \) distances, which may lead to overestimation of 3N effects. We have compared the ADWA predictions to those of the Watanabe model, which also treats \( d-A \) as a three-body system but does not allow for deuteron excitation to the \( n-p \) continuum. The 3N Watanabe \( d-A \) potential is close to zero for three sets of the 3N force (I, II and IV), where near-cancellation occurs between the contributions from the contact and \( 2\pi \) forces, but for set III it is sufficiently large to affect the \( (d,p) \) cross sections. This particular set has a large contribution from the \( 1\pi-c \) force and attractive contact interaction so that amplification rather than cancellation takes place when adding all three 3N contributions. A noticeable effect on \( (d,p) \) cross sections in the Watanabe model suggests that it can also be manifest beyond the ADWA.

This work is the first step in the investigation of the bare 3N force effects in the deuteron channel of \( (d,p) \) reactions, but it does not yet give the full picture of this contribution. Other effects, such as non-adiabatic deuteron breakup into \( n-p \) states, including those with different orbital momenta and spins (for example breakup into singlet channels), and additional 3N term in the \( T \)-matrix, as introduced in [10], could modify the conclusions of this work. Both of these effects require an extension of the folding formalism considered in this paper.

We should also point out that a fixed phenomenological density of the target \( A \) was used in our calculations. Ideally, the target density should be calculated in a microscopic model that uses the same NN and 3N interactions. This could induce an additional 3N-dependence sensitivity of the \( U_{3N} \) potential and the corresponding \( (d,p) \) cross sections. The target density for non-zero spin nuclei can also have significant non-spherical components, which may give rise to stronger quadrupole contributions to the incident channel deuteron scattering. These are intriguing problems awaiting further exploration.

Finally, we should point out that choosing a different NN model and the 3N interactions associated with it can result in a different \( (d,p) \) sensitivity to the 3N force. Changing the NN model, for example by going beyond the N2LO within the \( \chi \text{EFT} \), requires calculation of new matrix elements involving 3N forces. They contain
high-momentum cut-off regulators and uncertainty in the choice of the latter can significantly affect the short-range form of the deuteron wave function important for \( W_{\text{eff}} \) and \( U_{3N} \) calculations. It would be fascinating to see if \((d,p)\) reactions could offer any help in restricting the model parameters of the NN and 3N force.

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**APPENDIX**

Here expressions for \( H^{(i)}(x_-, x_+) \) are given for \( x_\pm = \sqrt{r_{di}^2 \pm rr_{di} + \frac{r_d^2}{4}} \). The notation \( F_{x_\pm} \) everywhere stands for \( F(x_\pm) \) and \( F(x) \) can be any of the functions \( U(x) \), \( T(x) \), \( Y(x) \) or \( \delta_{R_{3N}}(x) \). So,

\[
H^{(0)}(x_-, x_+) = -\frac{3c_3 m_A^4 g_A^2 U_+ Y U_+ Y_+}{2 f^8_\pi (4\pi)^2} \frac{x_- x_+}{x_- x_+},
\]

(50)

\[
H^{(1)}(x_-, x_+) = -\frac{3c_3 g_A^2}{36 f^4_\pi} \left\{ \frac{m_4^4}{16\pi^2} \left[ 9 \left( r_{di}^2 - \frac{r^2}{4} \right) \frac{Y_+ T_+ T_+}{x_-^2 x_+^2} + 3Y_+ \left( \frac{T_-(1-T_+)}{x_-^2} + \frac{T_+(1-T_-)}{x_+^2} \right) \right] \right. \\
\left. - \frac{3m_2^2}{4\pi} \left( \frac{Y_- T_+ \delta_+}{x_+^2} + \frac{Y_+ T_- \delta_-}{x_-^2} \right) \right\},
\]

(51)

\[
H^{(2)}(x_-, x_+) = -\frac{3c_3 g_A^2}{36 f^4_\pi} \left\{ \frac{m_4^4}{16\pi^2} \left[ -\frac{9}{4} \left( r_{di}^2 - \frac{r^2}{4} \right) \frac{Y_+ T_+ T_+}{x_-^2 x_+^2} + \frac{3}{4} Y_+ \left( \frac{T_-(1-T_+)}{x_-^2} + \frac{T_+(1-T_-)}{x_+^2} \right) \right] \right. \\
\left. - \frac{3m_2^2}{4\pi} \left( \frac{Y_- T_+ \delta_+}{x_+^2} + \frac{Y_+ T_- \delta_-}{x_-^2} \right) \right\},
\]

(52)

\[
H^{(3)}(x_-, x_+) = -\frac{3c_3 g_A^2}{36 f^4_\pi} \left\{ \frac{3m_4^4}{16\pi^2} \frac{Y_+}{x_-^2} \left( \frac{T_-(1-T_+)}{x_-^2} - \frac{T_+(1-T_-)}{x_+^2} \right) + \frac{3m_2^2}{4\pi} \left( \frac{Y_- T_+ \delta_+}{x_+^2} - \frac{Y_+ T_- \delta_-}{x_-^2} \right) \right\},
\]

(53)

\[
H^{(4)}(x_-, x_+) = -\frac{3c_3 g_A^2}{36 f^4_\pi} \left\{ \frac{m_4^4}{16\pi^2} \frac{Y_+}{x_-^2} (1-T_-(1-T_+)) - \frac{m_2^2}{4\pi} \left[ Y_-(1-T_-) \delta_+ + Y_+(1-T_+) \delta_- \right] + \delta_- \delta_+ \right\}.
\]

(54)

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