A New Spin on the Dirac Electron

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Abstract

We re-examine the non Hermitian position coordinate of Dirac’s equation, in the light of his own insights and conclude that this, and the Dirac equation itself is symptomatic of an underlying Noncommutative Geometry.

1 Introduction

Albert Einstein had once observed ”you know, it would be sufficient to really understand the electron” [1]. At the turn of the Twentieth Century several valiant attempts were made to model the electron in terms of Classical Electrodynamics, but all these attempts ultimately failed[2]. As Wheeler pointed out[3], the real challenge was to introduce into the theory the purely Quantum Mechanical spin half of the electron. The first successful theory of the electron emerged with Dirac’s equation, which also at the same time brought about the unification of Quantum Theory with the Special Theory of Relativity.

However the Dirac theory also encountered several problems. One of these was that the position coordinate turned out to be complex or Non Hermitian, and as we will see, a related problem, namely that the velocity of the electron turned out to be the velocity of light. Dirac himself recognized the reason for all this. He remarked[4], ”... since the theoretical velocity in the above conclusion is the velocity at one instant of time while observed velocities are always average velocities through appreciable time intervals...”, and again,
"To measure the velocity we must measure the position at two slightly different times and then divide the change of position by the time interval. (It will not do to measure the momentum and apply a formula, as the ordinary connection between velocity and momentum is not valid.) In order that our measured velocity may approximate to the instantaneous velocity, the time interval between the two measurements of position must be very short and hence these measurements must be very accurate. The great accuracy with which the position of the electron is known during the time-interval must give rise, according to the principle of uncertainty, to an almost complete indeterminacy in its momentum. This means that almost all values of the momentum are equally probable, so that the momentum is almost certain to be infinite. An infinite value for a component of momentum corresponds to the value $\pm c$ for the corresponding component of velocity.”

This realization, which highlights the limitation of space time points in Quantum Theory highlights the fact that we have to deal instead, with minimum space time intervals, within which there are negative energy solutions and the zitterbewegung type of unphysical effects. On the other hand negative energy components of the Dirac bi-spinor are negligible outside the Compton scale. Thus the averaging prescribed by Dirac eliminates these components and gives us back a physical theory in terms of Hermitian operators (Cf.[5]). This realization is the seed of what in recent years has been termed a Non Commutative Geometry.

2 Positive and Negative Energy Solutions

Let us consider in a little more detail[5] the implications of Dirac’s averaging over the Compton scale.

We consider for simplicity, the free particle Dirac equation. The solutions are of the type,

$$\psi = \psi_A + \psi_S$$

(1)

where

$$\psi_A = e^{i \gamma^\mu E \gamma^\mu} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } e^{i \gamma^\mu E \gamma^\mu} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and}$$
\[ \psi_S = e^{-\frac{i}{\hbar}Et} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } e^{-\frac{i}{\hbar}Et} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \]

denote respectively the negative energy and positive energy solutions. From (1) the probability of finding the particle in a small volume about a given point is given by

\[ |\psi_A + \psi_S|^2 = |\psi_A|^2 + |\psi_S|^2 + (\psi_A\psi_S^* + \psi_S\psi_A^*) \]

Equations (2) and (3) show that the negative energy and positive energy solutions form a coherent Hilbert space and so the possibility of transition to negative energy states exists. This difficulty however can be overcome by the well known Hole theory which uses the Pauli exclusion principle, and is described in many standard books on Quantum Mechanics.

However the last or interference term on the right side of (3) is like the zitterbewegung term. When we remember that we really have to consider averages over spacetime intervals of the order of \( \hbar/mc \) and \( \hbar/mc^2 \), this term disappears and effectively the negative energy solutions and positive energy solutions stand decoupled in what is now the physical universe. In other words, the Hole theory and other artifices of point space time theory are circumvented if, self consistently we use space time intervals instead of points.

The spirit of Dirac’s average spacetime intervals rather than spacetime points has received attention over the years in the form of minimum spacetime intervals– from the work of Snyder and Schild to Quantum Superstring theory \[ \text{[6, 7, 8, 9, 10, 11, 12]} \]. In modern language, it is symptomatic of a Non commutative spacetime geometry which again brings out the nature of the mysterious Quantum Mechanical spin \[ \text{[13, 14]} \]. This is what we will briefly examine.
The Non commutative Structure

Very early on, Newton and Wigner \[15\] showed that the correct physical coordinate operator is given by

\[
x^k = (1 + \gamma^0) \frac{p_0^{3/2}}{(p_0 + \mu)^{1/2}} \left( -\frac{i\partial}{\partial p_k} \right) \frac{p_0^{-1/2}}{(p_0 + \mu)^{1/2}} P \tag{4}
\]

where \(P\) is a projection operator eliminating negative energy components and the gammas denote the usual Dirac matrices.

To appreciate the significance of (4), let us consider the case of spin zero. Then (4) becomes

\[
x^k = \frac{i}{8\pi} \int \frac{\exp(-\mu|x - y|)}{|x - y|} \frac{\partial}{\partial y} dy
\]

The first term on the right side of (5) denotes the usual position operator, but the second term represents an imaginary part, which has an extension \(\sim 1/\mu\), the Compton wavelength, exactly as in the case of the Dirac electron.

Returning to Dirac’s treatment \[4\], the position coordinate is given by

\[
\vec{x} = c^2 pt + \frac{1}{2} \bar{\alpha}(\vec{\sigma} - cpH^{-1})H^{-1} \equiv \frac{c^2 pt}{H} t + \hat{x}
\]

\(H\) being the Hamiltonian operator and \(\alpha’s\) the non-commuting Dirac matrices, given by

\[
\bar{\alpha} = \begin{bmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{bmatrix}
\]

The first term on the right hand side of (3) is the usual (Hermitian) position. The second term of \(\vec{x}\) is the small oscillatory term of the order of the Compton wavelength, arising out of zitterbewegung effects which averages out to zero.

On the other hand, if we were to work with the (non Hermitian) position operator in (3), then we can easily verify that the following Non-commutative geometry holds,

\[
[x_i, x_j] = \alpha_{ij}l^2 \tag{7}
\]

where \(\alpha_{ij} \sim 0(1)\).

The relation (3) shows on comparison with the position-momentum commutator that the coordinate \(\vec{x}\) also behaves like a "momentum". This can be
seen directly from the Dirac theory itself where we have

$$c\vec{\alpha} = \frac{c^2 \vec{p}}{H} - \frac{2i}{\hbar} \hat{x}H$$

(8)

In (8), the first term gives the usual momentum. The second term is the extra "momentum" $\vec{p}$ due to the relations (7).

In fact we can easily verify from (8) that

$$\vec{p} = \frac{H^2}{\hbar c^2} \hat{x}$$

(9)

where $\hat{x}$ has been defined in (8).

Let us now see what the angular momentum $\sim \vec{x} \times \vec{p}$ gives at the Compton scale. Using (8), we can easily show that

$$(\vec{x} \times \vec{p})_z = \frac{c}{E}(\vec{\alpha} \times \vec{p})_z = \frac{c}{E}(p_2 \alpha_1 - p_1 \alpha_2)$$

(10)

where $E$ is the eigen value of the Hamiltonian operator $H$. The right side of (10) is a super position of the $\alpha$’s which again contain the Pauli $\sigma$ matrices. This shows that at the Compton scale, the angular momentum leads to the "mysterious" Quantum Mechanical spin.

It may be mentioned that Zakruzewski [16] deduced from a different point of view that spin implies non commutativity. On the other hand, the zitterbewegung contribution to spin has been shielded by Barut and coworkers and Hestenes (Cf.ref.[5] and several references therein).

In the above considerations, we started with the Dirac equation and deduced the underlying Noncommutative geometry of spacetime. Interestingly, starting with Snyder’s Non commutative geometry, based solely on Lorentz invariance and a minimum spacetime length, at the Compton scale,

$$[x, y] = \frac{i l^2}{\hbar} L_z etc.$$ 

that is, in effect starting with (7), it is possible to deduce the relations (10), (1) and the Dirac equation itself as has been shown in detail elsewhere [11, 16, 17, 18].

We have thus established the correspondence between considerations starting from the Dirac theory of the electron and Snyder’s (and subsequent) approaches based on a minimum spacetime interval and Lorentz covariance. We will now show using Nelson’s analysis that that the above non commutativity is also symptomatic of an underlying stochastic behaviour.
4 The Stochastic Underpinning

In Nelson’s approach [19, 20], there is a double Weiner process arising from the fact that the forward and backward time derivatives,

\[ \frac{d}{dt^+}, \quad \frac{d}{dt^-} \]  

are unequal. Let us consider first the problem in one dimension (Cf. [20]) we have

\[ \frac{d_+}{dt} x(t) = b_+, \quad \frac{d_-}{dt} x(t) = b_- , \]  \(12\)

From (12) we define two new velocities

\[ V = \frac{b_+ + b_-}{2}; \quad U = \frac{b_+ - b_-}{2} \]  \(13\)

It may be pointed out that in general \( U \), given in (13) vanishes while \( V \) gives the usual velocity. It is now possible to introduce a complex velocity

\[ \nabla = V - iU \]  \(14\)

From (14) it appears that the coordinate \( x \) goes over to a complex coordinate

\[ x \rightarrow x + ix' \]  \(15\)

This is also true for the Dirac equation (8). In fact it can be shown that this leads to the special relativistic metric in \(1 + 1\) dimensions [4].

Following this line of reasoning, in the usual theory, as is well known we work with (14) to deduce the Schrodinger equation. This could be done in three dimensions also.

But let us now look upon (15) from a different angle and ask, "Can we generalise (15) itself to the three dimensional case?" It is known [21] that in this case, surprisingly,

\[ (1, i) \rightarrow (I, \vec{\sigma}) \]  \(16\)

where \( I \) is the unit \( 2 \times 2 \) matrix and \( \vec{\sigma} \) are the Pauli matrices. We get the Lorentz invariant metric at the same time.

Equation (14) gives a quarternionic description (Cf. [21]). It would lead to a two component neutrino type equation. However this \( 2 \times 2 \) description does
not preserve spatial and time reflections, which are necessary in physical theories. If we incorporate these reflections also then it is known that \[ [x_i, x_j] = i^2 \Theta_{ij} \] 

where \( l \) represents a length scale. Equation (17) is, of course, the same as (4)!

All this need not be surprising - equation (6) represents the zitterbewegung effects due to the interference of the negative and positive energy solutions. The positive and negative time derivatives (11) of the double Weiner process described above represent exactly the positive and negative energy interference effects, contained in the term \( U \) of (13). (It is these interference terms which, even in the non relativistic Nelsonian theory lead to the Quantum Mechanical Schrodinger equation).

The important point to note is that whenever we have a complex space coordinate as in (15), then the generalization to three dimensions, in fact leads to (16) and non commutativity.

There is another very interesting and apparently different situation where complex coordinates are used - this is in the derivation of the Kerr-Newman metric [23, 24, 25]. In this case we consider the Maxwell equations,

\[ \nabla \times \vec{W} = \imath \vec{W}, \nabla \cdot \vec{W} = 0 \]  

where

\[ \vec{W} = \vec{E} + \imath \vec{B} \]

\( \vec{E} \) and \( \vec{B} \) being the usual electric and magnetic components of the electromagnetic field. The interesting point is that if we effect an imaginary shift of the coordinates, then we obtain the Quantum Mechanical electron anomalous gyro magnetic ratio \( g = 2 \) of the Kerr-Newman metric (Cf. [23] for details). Newman himself found this association of the imaginary spatial shift with Quantum Mechanical spin inexplicable [26]. However in the light of the above comments the connection between the two becomes clear (Cf. also [27]). Infact this leads to a model of the electron as a Kerr-Newman black hole, with the naked singularity shielded by a fuzzyness induced by the non commutativity (17) as was independantly confirmed by Nottale [28].

For completeness we mention that interestingly, as is well known, the hydrodynamical formulation of Quantum Mechanics also leads to equations similar
to the Nelsonian formulation (Cf.[5] for a brief review). Here also if we consider a one dimensional laminar flow we get a velocity that is both solenoidal and irrotational and satisfies
\[ \vec{\nabla} \cdot \vec{V} = 0, \quad \vec{\nabla} \times \vec{V} = 0 \] (19)
From (19) we can define a complex velocity potential by standard methods which again leads to complex coordinates (15) and ultimately we end up with (17).
We finally observe that a phase space approach based on relations like (13) has been worked out in detail by Kaiser [29, 30], though he does not follow the route through (16), and therefore does not arrive at the above conclusions. One can see that equation (13) also suggests this approach. Nevertheless it is interesting to note that in this reverse approach in which we introduce complex coordinates the zitterbewegung as manifested in the complex coordinate \( \hat{x} \) of (8) disappears.

5 Conclusion

Thus the purely Quantum Mechanical "mysterious" spin half is symptomatic of non commutativity and vice versa. Interestingly, this is also symptomatic of an underlying double Weiner or stochastic process.

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