Meta-universality classes at criticality

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Abstract

Inferring the presence of critical dynamics from continuous measurements is a challenging problem. We solve this problem by showing that continuous narrowband dynamics from a critical system exhibit qualitatively differing behaviors which depend on the universality class; we term each region of critical avalanche parameters which generates qualitatively constant behavior a meta-universality class. This theoretical observation allows us to infer membership of a given meta-universality class and thus yields a robust test for criticality. We validate these theoretical predictions in simulations and provide unequivocal evidence for criticality in the human brain on the basis electrophysiological recordings.

In their seminal paper [1], Bak et al. showed how a power-law spectrum results from critical dynamics. However, the converse does not hold: the presence of a power-law spectrum does not necessarily imply critical dynamics, which has been a source of recent debate over the origin of power-law spectra [2, 3]. A more rigorous and standard check for criticality is to perform an avalanche analysis, testing for power-law size and lifetime distributions. Such an analysis requires the assumption that clearly separated individual avalanches are observable [4, 5]. We show in this letter that it is not necessary to consider discrete avalanches, clearly separated from one another, to infer criticality; we show it is possible to analyse the continuous data from a potentially critical system whose activity is composed of many simultaneous avalanches. As a consequence we confirm the hypothesis of criticality for the human brain from continuous non-invasive neural data, where several explanations for the power-spectrum have been proposed [2].

Suppose we measure activity $X(t)$, which we call the avalanche process, from a system which consists, at each time $t$, of a superposition of $q$ critical avalanches:

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\[ X(t) = \sum_{s=1}^{T} \sum_{i=1}^{q} L_{s,i}^{1/\sigma \nu z - 1} a_{s,i} \left( \frac{t - s}{L_{s,i}} \right) \]  

\( L_{s,i} \) denotes the lifetime of the \( i \)-th avalanche starting at time \( s \) which is distributed as \( L^{-\alpha} \). The exponent \( 1/\sigma \nu z - 1 \) indicates the power-law relation between the avalanche size \( S \sim L^{1/\sigma \nu z} \) (number of activations in avalanche) and \( L \). Equivalently \( S \) is characterized by a power law distribution \( S^{-\tau} \). The exponent \( \frac{1}{\sigma \nu z} - 1 \) indicates the power-law relation between the avalanche size \( S \sim L^{1/\sigma \nu z} \) (number of activations in avalanche) and \( L \). Equivalently \( S \) is characterized by a power law distribution \( S^{-\tau} \).

\( a(t) \) denotes the within avalanche process on \([0,1]\) and each \( a_{s,i}(t) \) is an independent draw from \( a(t) \). Thus the exponents \( \alpha \) and \( \beta := 1/\sigma \nu z - 1 \) characterize the universality class (equivalently \( \alpha \) and \( \tau \)). Important to note is that we assume that multiple avalanches occur simultaneously.

The autocorrelation of \( X(t) \) (equiv. power spectrum) takes the form of a power-law. For large \( s \):

\[ \mathbb{E}(X(t)X(t + s)) \sim s^{2 - 2H_{\text{raw}}} \]  

\( H_{\text{raw}} \) denotes the Hurst exponent of \( X(t) \). A power-law form of the power spectrum at low frequencies implies \( H_{\text{raw}} > 1/2 \) and \( X(t) \) is said to be long-range temporally correlated (LRTC) (sometimes dependent - LRTD) \[6\].

Given \( X(t) \) and the impulse response of a narrow-band filter, \( \phi_\omega(t) \), centered on a frequency \( \omega \), we may then consider the filtered time-series \( X_\omega(t) \):

\[ X_\omega(t) = \int_{0}^{\infty} \phi_\omega(s)X(t - s)ds \]  

\( X_\omega(t) \) is thus a narrowband signal; however, the amplitude envelope of this signal, as computed by the absolute value of the analytic signal which we denote, \( \mathcal{E}(X_\omega(t)) = |X_\omega(t) + i\mathcal{H}(X_\omega(t))| \), where \( \mathcal{H}(X_\omega(t)) \) is the Hilbert transform of \( X_\omega(t) \), is a broadband signal. Since \( \mathcal{E}(X_\omega(t)) \) is a broadband signal, we may also consider its Hurst exponent, which we denote \( H_{\text{amp}} \) and which is not a priori identical to \( H_{\text{raw}} \).

Having defined these quantities in relation to a frequency, \( \omega \), we can ask the questions (1) is \( \mathcal{E}(X_{\omega_1}(t)) \) LRTC (i.e. \( H_{\text{amp}} > 1/2 \)) and what is the relation between \( H_{\text{amp}} \) and \( \alpha, \beta \)? (2) are \( \mathcal{E}(X_{\omega_1}(t)) \) and \( \mathcal{E}(X_{\omega_2}(t)) \) correlated? The answers to (1) and (2) depend on the universality class \( \alpha \) and \( \beta \). We are able to derive theoretically and confirm in simulation that there are 4 qualitatively differing subsets of universality classes, which we term meta-universality classes, giving differing answers to (1) and (2); see Figure 2 for an illustration of these subsets (labelled MU1-4). The analysis places many constraints upon data stemming from a critical system, since the results hold for arbitrary frequency ranges: thus our results allow for far more stringent testing for criticality than inspection of the power-spectrum alone. Note that our analysis uses a recently introduced refined notion of correlation, viz. the detrended cross correlation coefficient \[7, 8\], denoted \( \rho_{\text{DCCA}}(n) \), which may be thought of as correlation at a time-scale \( n \). See the supplement for the theoretical derivations and details of \( \rho_{\text{DCCA}}(n) \).
\[
\alpha = 2.5; \beta = 1
\]

\[
\alpha = 1.5; \beta = 1
\]

**Figure 1:** Sample processes from the model. The figure displays sample paths, in blue, of the avalanche process for fixed \( \beta = 1 \) but varying \( \alpha = 1.5, 2.5 \) resp. In the first case, individual avalanches are clearly visible. In the second case the process displays random walk behaviour and individual avalanches are indiscernible. Sample paths of narrow band filtered data (from the same process) are displayed for comparison for 2 frequencies (narrow pass filtered in resp. 0.48 and 0.52, and 0.78 and 0.82 of the Nyquist frequency—in red and green).

A key step in our analysis is the observation that the number of avalanches of lifetime greater than \( L' \) active, for fixed \( q \), is asymptotically \( L_c^{\alpha-2} \), where \( L_c \) is the limit set on lifetimes by the size of the system. This implies that for \( \alpha < 2 \) the number of avalanches active at a given time grows without bound with the size of the system: this behavior allows application of the central limit theorem. On the other hand for \( \alpha > 2 \) large avalanches overlap with vanishing probability: for these universality classes, correlations between frequencies \( (\rho_{DCCA}(n)) \) and LRTC of narrowband envelopes \( (H_{amp}) \) requires that \( \beta \) is sufficiently large. See Figure 1 for sample processes for distinct \( \alpha \).

The first meta-universality class (MU1) is defined by the region \( \alpha < 2 \); here our analysis shows that \( \mathcal{X}(t) \) converges to a Gaussian process, for fixed \( q \), when one first takes the limit \( t \to \infty \) and then \( L_c \to \infty \). We show then in simulation (Section 3.2 of the Supplementary Material) that this implies that \( \rho_{DCCA}(n) \) approaches zero at large time-scales \( n \), that \( H_{amp} = 1/2 \), but \( H_{raw} > 1/2 \). Thus this class is defined by LRTC of the avalanche process but uncorrelated narrowband amplitudes (in time and across frequencies).

The second meta-universality class (MU2) we compute is the region such that \( \alpha > 2, \tau < 2 \). Here we have that \( H_{amp} > 1/2 \) and for fixed frequencies \( \omega_1, \omega_2 \) we find that allowing \( n \to \infty \) followed by \( L_c \to \infty \) then we have \( \rho_{DCCA}(\mathcal{E}(X_{\omega_1}(t)), \mathcal{E}(X_{\omega_2}(t)), n) \to 1 \). Thus this class is defined by LRTC of the avalanche process and narrowband amplitudes and cross-correlation over frequencies between the narrowband amplitudes.
The third meta-universality class (MU3) is defined by $\alpha > 2$, $2 < \tau < 3$. Here we can show that $\rho_{\text{DCCA}} \to 1$ for large time-scales and low frequencies. More formally, we are able to derive that if one allows $\omega_1, \omega_2 \to 0$, then $n \to \infty$, followed by $L_c \to \infty$, then we have that $\rho_{\text{DCCA}}(\mathcal{E}(X_{\omega_1}(t)), \mathcal{E}(X_{\omega_2}(t)), n) \to 1$. We find in simulation that this convergence is faster for smaller $\tau$ and that, for small sample sizes, values of $\rho_{\text{DCCA}}(n)$ approach 1 close to the interior of the boundaries at $\tau = 3$ and $\alpha = 2$. Thus this class is defined by LRTC of the avalanche process, but not its narrowband amplitudes, and cross correlation between low frequencies of the narrowband amplitudes.

In the second and third meta-universality classes, we are also able to derive formulae relating the critical exponents $\alpha$ and $\beta$ to $H_{\text{amp}}$ and $H_{\text{raw}}$, giving:

$$H_{\text{amp}} = \begin{cases} \frac{\beta}{2} - \frac{\alpha}{2} + 2 & \text{if } \tau < 2 \\ 1/2 & \text{if } 2 < \tau < 3 \end{cases}$$

(4)

$$H_{\text{raw}} = \beta - \frac{\alpha}{2} + 2$$

(5)

(The second formula is implied by an existing paper [9]). Thus for $\tau < 2$ both $X(t)$ and $\mathcal{E}(X_{\omega}(t))$ are LRTC.

The fourth and final meta-universality class (MU4), $\alpha > 2, \tau > 3$ covers the regimes with the fastest decaying autocorrelations and shortest tails. In this regime we find that $\rho_{\text{DCCA}}(n) \to 0$ and $H_{\text{raw}} = H_{\text{amp}} = 1/2$. Thus in this final class, all measures display no LRTC behaviour and no correlation.

We checked these subdivisions in simulation (Figure 3) and find good agreement in the finite sample (further simulations are presented in the supplement). Since we wish to manually adjust the universality class, we model:

$$a(t) = b(t) + c(t)\epsilon(t)$$

(6)

$b(t)$ is the average avalanche shape, $c(t)$ is the level of fluctuation at time $t$ and $\epsilon(t)$ is a colored noise with the spectrum required at criticality $P(\omega) \sim \omega^{-1/\sigma\nu}$. We set $b(t) = c(t) = -4(t - 1/2)^2 + 1$ as predicted by mean field theory [10]. For estimation of $H_{\text{raw}}$ and $H_{\text{amp}}$, we use Detrended Fluctuation Analysis (DFA) [11].

Numerous theoretical and empirical papers have argued that criticality plays a role in brain function [4, 12, 13]. However, counterarguments have been put forward: in some experiments avalanche dynamics were not observed [2] and alternative explanations for the power-law spectrum have been proposed [3]. Other authors hypothesize that critical states are indeed present in the brain but that the failure to detect avalanche dynamics is instead related to not obtaining a fine enough sampling of the critical neural networks [14]. We contribute to this debate by confirming the hypothesis of criticality for the human brain. This
Figure 2: Division of critical exponents $\alpha$ and $\beta$ into meta-universality classes. The figure displays the range of qualitative behaviours we predict with our theory. Areas marked in green display no LRTC behaviour in sub-bands or DCCA correlations between sub-bands. Areas in red display LRTC and/or cross correlations between amplitudes of sub-bands ($H_{amp} = 1/2$, $\rho_{DCCA}(n) = 0$ for large $n$). The labels (MU1-MU4) designate the meta-universality classes as described in the text.

Figure 3: The figure displays the results of the simulations for varying exponent values. The left hand panel displays amplitude LRTC (Hurst) exponents ($H_{amp}$), the middle panel displays cross-correlations between frequencies (DCCA correlation coefficient at the highest scale—$\rho_{DCCA}(n)$) and the right hand panel displays the Hurst exponents of the raw avalanche process $H_{raw}$ over a range of critical exponents. The theoretically derived transitions in values which occur between meta-universality classes are displayed in black.
is achieved by verifying the predictions of our model on electroencephalography (EEG) recordings. The preprocessing steps are described in the supplement. Important to note, however, is that we analyse 3 frequency ranges which do not exhibit oscillations (no local maximum in power-spectrum). The results of our analysis were pooled across subjects and are displayed in Figure 6. For 244 of 261 signals filtered in 3 narrow frequency bands, the DFA estimate of $H_{amp}$ was higher than 0.5 thus clearly indicating the presence of LRTC ($\text{median } H = 0.61, 5^{th} \text{ and } 95^{th} \text{ percentiles } 0.49 \text{ and } 0.85, p \ll 0.0001$). Likewise, 238 of 261 DCCA correlations between frequency ranges were positive ($p \ll 0.0001$). We found, moreover that the $\rho_{DCCA}$ values and $H_{amp}$ measured from the same neural data were highly correlated with one another ($p \ll 0.0001$) and that measurements of $H_{amp}$ in distinct frequency ranges were highly correlated ($p \ll 0.0001$) (likewise for $\rho_{DCCA}(n)$). Finally we found that the $\rho_{DCCA}$ and $H_{amp}$ values were not significantly correlated with $H_{raw}$ ($p > 0.05$).

We now discuss how these results relate to our theory. The presence of $H_{raw} > 0.5$ and $\rho_{DCCA} > 0$ confirms the predictions of the theory for criticality in MU2. Moreover the correlation between $H_{amp}$ and $\rho_{DCCA}$ confirms our theory, since we predict that high and low values of both measures coincide, especially for small to intermediate $\beta$ (see Figures 2 and 3). The absence of correlation of $\rho_{DCCA}$ and $H_{amp}$ with $H_{raw}$ can be explained if there are brain regions generating the EEG data in all 4 meta-universality classes; this is because, in such a scenario, low values of $\rho_{DCCA}$ and $H_{amp}$ may coincide with both large and small values of $H_{raw}$. This hypothesis is corroborated by a wide range of $H_{raw}$ values, which suggest that a range of critical exponents in distinct meta-universality classes govern neuronal dynamics.

Thus these findings provide conclusive evidence for the presence of criticality in the human brain. These results cannot be explained away in terms of passive filtering properties of the extracellular media [3]: passive filtering of neural signals may generate a power-law spectrum but will not induce narrow-band amplitude LRTC or DCCA correlations between narrowband amplitudes. Moreover, the failure to detect avalanches in the experiments of [2, 14] may be explained by criticality in the first meta-universality class where we have shown that even the largest avalanches are not discernible.

In conclusion the proposed framework allows testing for criticality from continuous data by providing empirical criteria for distinguishing between meta-universality classes.

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Figure 4: Results of data analysis of human EEG. The figure displays for the EEG data the pairwise relationships (over frequencies) between the amplitude noise Hurst exponents, $H_{\text{amp}}$, the large scale DCCA correlations $\rho_{\text{DCCA}}$ and the low frequency DC Hurst exponent, $H_{\text{raw}}$. The frequency ranges analysed $i = 1, 2, 3$ are 35-40Hz, 60-65Hz and 72-77 Hz respectively, which are displayed as superscripts in the plots. The correlations within and between $H_{\text{amp}}$ and $\rho_{\text{DCCA}}$ are highly significant in each case ($p \ll 0.0001$).

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A Estimators

A.1 Detrended Fluctuation Analysis

Detrended Fluctuation Analysis (DFA) \[11\] is a methodology for the estimation of the Hurst exponent of a (possibly non-stationary) time-series. Its advantage over covariance analysis or analysis of the power-spectrum, are its robustness to trends contaminating the empirical time-series and its desirable convergence properties \[15\].

The steps involved in DFA are as follows. First one forms the aggregate sum of the empirical time-series \(X(t)\):

\[
x(t) = \sum_{i=1}^{t} X(i)
\]  

(From now on whenever we refer to \(x(t)\) we mean the time-series obtained from \(X(t)\) by way of this operation.) Analysis of the fluctuations in \(X(t)\) may then be performed by measuring the variance of \(x(t)\) in windows of varying size \(n\) after detrending, i.e., \(x(t)\) is split into windows of length \(n\), \(x_n^{(1)}, \ldots, x_n^{(j)}, \ldots, x_n^{(\lfloor N/n \rfloor)}\) and the average variance after detrending the data in these windows is formed; i.e. let \(P_d\) be the operator which generates the mean-squares estimate of the polynomial fit of degree \(d\), then the DFA coefficients or detrended variances of degree \(d\) are:

\[
F_{DFA}^{2}(n) = \frac{1}{\lfloor N/n \rfloor} \cdot n \sum_j \left( x_n^{(j)} - P_d \left( x_n^{(j)} \right) \right)^\top \left( x_n^{(j)} - P_d \left( x_n^{(j)} \right) \right)
\]  

Crucially, in the limit of data the slope of \(\log(F_{DFA}^{2}(n))\) against \(\log(n)\) converges to \(H\). Thus \(X(t)\) is power-law correlated LRTC if and only if the estimate of \(H\), \(\hat{H}\), converges to a number greater than 0.5 in the limit of data. We note here that there are numerous methods for the Hurst exponent; these include wavelet estimators \[16\], log-periodogram based methods \[17\], among others \[18\]. We chose DFA since it is standard in the physics and neuroimaging literature, and yields competitive estimates \[19\].

A.2 Detrended Cross-Correlation Analysis

In precise analogy, Podobnik and Stanley propose Detrended Cross-Correlation Analysis (DCCA) \[7\], an extension of DFA to two time-series, by considering the detrended covariance:

\[
F_{DCCA}^{2}(n) = \frac{1}{\lfloor N/n \rfloor} \cdot n \sum_j \left( (x_1)^{(j)} - P_d \left( (x_1)^{(j)} \right) \right)^\top \left( (x_2)^{(j)} - P_d \left( (x_2)^{(j)} \right) \right)
\]  

Thus DCCA generalizes DFA in the sense that if \(X_1 = X_2\) then \(F_{DCCA}^{2}(n) = F_{DFA}^{2}(n)\). To simplify the fact that we will need to consider the DFA coefficients of \(X_1\) and \(X_2\) simultaneously, we will also refer to the DCCA coefficients as:

\[
F_{X_1, X_2}^{2}(n) = F_{DCCA}^{2}(n)
\]  

Thus, DCCA quantifies the behaviour of the covariance between $X_1$ and $X_2$ over a range of time-scales given by $n$. In analogy to the Pearson correlation coefficient, we may consider the detrended cross correlation coefficient [8]:

$$\rho_{DCCA}(n, X_1, X_2) = \frac{F_{X_1,X_2}^2(n)}{\sqrt{F_{X_1,X_1}^2(n)} \sqrt{F_{X_2,X_2}^2(n)}} \quad (11)$$

$\rho_{DCCA}$ quantifies the correlation between $X_1$ and $X_2$ over a range of time-scales. Note that while the machinery involved in the estimation of $\rho_{DCCA}$ are more complex than for Pearson correlation, both coefficients estimate the same quantity for stationary time-series, not contaminated by trends. Thus $\rho_{DCCA}$ generalizes the Pearson correlation coefficient. Applicability to non-stationary time-series is particularly important for the neural data analysis. Whenever the context allows for no ambiguity we abbreviate $\rho_{DCCA}(n, X_1, X_2)$ to $\rho_{DCCA}(n)$.

**B Theory**

**B.1 Overlap Probability**

The division of the critical parameter space at $\alpha = 2$ is due to a transition in the number of large avalanches ($L > L'$) active. For $\alpha > 2$, as $L_c \to \infty$ and $L' \to \infty$, the measure of the time spent in avalanches of length greater $L'$, tends to zero; for $\alpha < 2$, the number of large avalanches active is unbounded. The time spent in large avalanches divided by the total recorded time is:

$$\frac{1}{T} \sum_{L_i > L'} L_i \sim qE(1_{(L', \infty)}(L))$$

$$\sim \int_{L'}^{L_c} L^{1-\alpha} dL$$

$$\sim L'^{2-\alpha} - L^{2-\alpha}$$

These terms decay to zero for $\alpha > 2$, whereas, we have for $\alpha < 2$ the number of avalanches active is asymptotically unbounded.

**B.2 Gaussianity for Small $\alpha$**

In the preceding section (Section B.1), we showed that the number of avalanches active at any particular time $k \sim L_c^{2-\alpha} \overset{\alpha \leq 2}{\longrightarrow} \infty$. We prove a Lyapunov central limit condition [20] on:

$$\sum_{j=1}^{\beta} L_i^j \alpha \left( \frac{t - s_i}{L_i} \right),$$
after an appropriate centering and normalization. This implies that the avalanche time-series is Gaussian. Let the mean of \( a(t) \) on \([0, 1] \) be \( \mu \). Then the mean of \( L^\beta a \left( \frac{t-s_L}{L} \right) \) on \([0, L] \), where \( s \) is treated as a uniformly distributed random variable is:

\[
L^\beta \int_0^L \frac{E(a(t/L))}{L} dt = \mu L^\beta
\]

Thus the centering required is:

\[
\sum_{j=1}^{k} L_i^\beta \left( a \left( \frac{t-s_L}{L} \right) - \mu \right),
\]

The normalization is computed by calculating the variance. If we call the second non-central moment of \( a(t) \), \( \kappa^2 \) then we have:

\[
\text{var} \left( L^\beta a \left( \frac{t-s_L}{L} \right) \right) = L^{2\beta} \int_0^L \frac{E(a(t/L)^2)}{L} dt - L^{2\beta} (\mu)^2
\]

\[
= L^{2\beta} \int_0^1 E(a(u)^2) du - L^{2\beta} \mu^2
\]

\[
= L^{2\beta} \kappa^2 - L^{2\beta} \mu^2
\]

\sim L^{2\beta}

(12)

The Lyapunov condition requires bounding a higher order moment of order \( 2 + \delta \), which is given by:

\[
E \left( \left| L^\beta a \left( \frac{t-s_L}{L} \right) - L^\beta \mu \right|^{2+\delta} \right) = L^{(2+\delta)\beta} \int_0^L \frac{E(|a(t/L) - \mu/L|^{2+\delta})}{L} dt
\]

\[
= L^{(2+\delta)\beta} \int_0^1 E(|a(u) - \mu|^{2+\delta}) du
\]

\sim L^{(2+\delta)\beta}

(13)

Thus if \( \sigma_i^2 = \text{var} \left( a \left( \frac{t-s_L}{L} \right) \right) \), then:

\[
s_k^2 = \sum_{i=1}^{k} \sigma_i^2 \sim k \int_0^{L_c} L^{2\beta+\alpha} dL \sim kL_c^{2\beta+\alpha+1}
\]

The normalized and centered sum we consider is:
\[
\frac{1}{\sqrt{kL^{\beta+1/2}}} \sum_{j=1}^{k} L_{i}^{\beta} \left( a \left( \frac{t-s_{i}}{L_{i}} \right) - \mu \right),
\]

In order for Gaussianity to hold via the Lyapunov condition we need to show that:

\[
\frac{1}{s_{k}^{2+\delta}} \sum_{j=1}^{k} \mathbb{E} \left[ L_{i}^{\beta} \left( a \left( \frac{t-s_{i}}{L_{i}} \right) - \mu \right) \right]^{2+\delta} \to 0
\]

Thus:

\[
\frac{1}{s_{k}^{2+\delta}} \sum_{j=1}^{r} \mathbb{E} \left[ L_{i}^{\beta} \left( a \left( \frac{t-s_{i}}{L_{i}} \right) - \mu \right) \right]^{2+\delta}
\]

\[
\sim \frac{1}{k^{1+\delta/2}L_{c}^{(\beta+\alpha/2+1/2)(2+\delta)}} \sum_{i=1}^{k} L_{i}^{(2+\delta)\beta}
\]

\[
\sim \frac{1}{k^{1+\delta/2}L_{c}^{(\beta+\alpha/2+1/2)(2+\delta)}} kL_{c}^{\beta(2+\delta)+\alpha+1}
\]

\[
\sim \frac{1}{k^{\delta/2}L_{c}^{(\alpha/2+1/2)(2+\delta)}}
\]

\[
\leq \frac{1}{k^{\delta/2}L_{c}^{\delta/2}} \to 0
\]

This completes the proof for univariate Gaussianity. The proof that convergence is to a multivariate Gaussian process is a simple extension via the Cramer-Wold device.

We check in the simulations of Section C.4 that LRTC Gaussian processes have \( H_{amp} = 0.5 \) and \( \rho_{DCCA} = 0 \).

**B.3 Filtered avalanche exponent**

By filtering an avalanche in a frequency band we get that:

\[
a_{\omega}(t/L) \sim \int_{-\infty}^{\infty} \phi_\omega(s-t)a(s/L)ds
\]

\[
= \int_{-\infty}^{\infty} \phi_{L\omega}(s-t/L)a(s)ds
\]

\[
\sim L^{-\beta'} a_{\omega}(s/L)
\]
We are able to calculate this exponent and find that:

\[ \beta' = \frac{\beta}{2} \]  

(14)

This is because, for a filtered avalanche of length \( L \), the size is proportional to:

\[ s' \sim \sqrt{L^{1-1/\sigma_\nu \sigma_z} L^{1/\sigma_\nu \sigma_z}} \]  

(15)

\[ = L^{1/2 \sigma_\nu + 1/2} \]  

(16)

The pre-factor on the first line comes from downweighting by the contribution to the standard deviation in the \( \omega^\text{th} \) frequency band. Thus \( h' \sim L^{1/2 \sigma_\nu - 1/2} = L^{\beta/2} \) and \( \beta' = \beta/2 \). A simulation checking this formula for a range of \( \beta \) is presented in Section C.3.

B.4 Cross correlations

In this section consider the value of \( \rho_{\text{DCCA}}(n) \) in the two regimes when \( \tau < 3, \alpha > 2 \) and \( \tau < 2, \alpha > 2 \).

When \( \tau < 3, \alpha > 2 \) we consider the limit \( \omega_1, \omega_2 \to 0 \) followed by \( L_c \to \infty \) and finally \( n \to \infty \). In this regime we have that there exists an \( L' \) such that each avalanches longer than \( L' \) do not overlap with large probability. Moreover, since the scales \( \log(1/\omega_1), \log(1/\omega_2) \) may assumed larger than \( L_c \), then \( \mathcal{E}(L^\beta a_{\omega_1}(t/L)), \mathcal{E}(L^\beta a_{\omega_2}(t/L)) \sim L^\beta \).

Since \( \tau < 3 \) this implies that:

\[ \text{var}(X(t)) \approx \text{var} \left( \sum_{L_{s,i} > L'} L_1^{1/\sigma_\nu \sigma_z - 1} a_{s,i} \frac{t-s}{L_{s,i}} \right) \]  

(17)

Moreover we may assume that with large probability each DFA epoch contains either 1 or 0 avalanches larger than \( L' \). The effect of smaller avalanches may be neglected by virtue of Equation 17.

Consider an epoch containing only one such avalanche, which extends from time point \( s \) to \( s + L \). Then the integrated process \( Y(t) \) (see description of DCCA, DFA) takes the value 0 up to \( s \) and \( \int_s^t (L^\beta \mathcal{E}(a_{\omega_1}(t/L)))dt \) after \( s + L \), for \( i = 1, 2 \). This implies that for large epoch sizes \( n \), \( \rho_{\text{DCCA}}(n) \to 1 \).

The convergence is stronger when \( \tau < 2 \). In this regime we consider fixed \( \omega_1, \omega_2 \). Then we have that \( \mathcal{E}(L^\beta a_{\omega_1}(t/L)), \mathcal{E}(L^\beta a_{\omega_2}(t/L)) \sim L^{\beta/2} \) (Equation 14). Then the relation given by Equation 17 holds only for \( \tau < 2 \) (not for \( \tau > 2 \)). In this case we again have that \( \rho_{\text{DCCA}}(n) \to 1 \) for large \( n \).

B.5 Hurst exponent expressions

Since for \( \alpha < 2 \) and \( \tau < 3 \) the large avalanches are separated in time and we see negligible variance in small avalanches, we may assume that the Hilbert transforms of the filtered avalanches superimpose linearly to yield \( H(X_\omega(t)) \).
This linearization allows us to apply Equation 5 of [9], which computes
the correlation function $\gamma(t)$ of an avalanche process, by approximating the
avalanches as boxcars:

\[
\gamma(t) \propto \int_{|t|}^{\infty} (L - |t|) \int_{0}^{\infty} P(S, L) \left( \frac{S}{L} \right)^2 dSdL
\]

(18)

\[
\propto \int_{|t|}^{\infty} (L - |t|) L^{2\beta' + \alpha} dL
\]

(19)

\[
\sim |t|^{2\beta' - \alpha + 2}
\]

(20)

This implies that:

\[
H_{\text{amp}} = \beta' - \frac{\alpha}{2} + 2
\]

(21)

\[
= \frac{\beta}{2} - \frac{\alpha}{2} + 2
\]

(22)

Although the integral in Equation (19) converges only for $\alpha - 2 > 2\beta$, we
find in simulation that the formula calculated generalizes to $\alpha - 2 \leq 2\beta$. For
fixed $\alpha$, we get that $\beta_1 > \beta_2 \implies H_{\text{amp}}(\beta_1) > H_{\text{amp}}(\beta_2)$.

The same argumentation gives:

\[
H_{\text{raw}} = \beta - \frac{\alpha}{2} + 2
\]

(23)

(24)

Thus we get LRTC behaviour for $2\beta - \alpha + 3 > 0$. This is equivalent to $\tau < 3$
by the relation [21]:

\[
\frac{\alpha - 1}{\tau - 1} = \frac{1}{\sigma \nu z}
\]

(25)

C Simulations

In all simulations we model the average avalanche shape as a quadratic func-
tion: $b(t) = -4(t - 1/2)^2 + 1$. Moreover we set the variance swell identically
so that $b(t) = c(t)$. For the noise component we take the implementation of
[22]. For the power-law cutoff sampling, we perform a density transformation
of the uniform distribution. (In MATLAB $x = \text{rand}(1,T) \cdot (1-L_c) + L_c$; $x =
1./x.^{1/(\alpha-1)}$)
C.1 Simulation 1 (from main text)

In the first simulation, for each pair of exponents in the ranges $\alpha = 1.5, \ldots, 6.5$ and $\beta = 0.25, 0.5, \ldots, 3$, we generate a sample path $X(t)$ of length $T = 300,000$, with a cutoff at $L_c = 100,000$, and number of superpositions $q = 5$. The first 100,000 time points are discarded, to ensure stationarity. We then design Butterworth filters of order 2 between 0.29 and 0.31 and between 0.39 and 0.41 of the sampling frequency. The data from $X(t)$ are then filtered forwards and backwards (yielding effective filter order of 4) and the amplitude envelopes are calculated to yield $Y_1(t)$ and $Y_2(t)$. We measure the Hurst exponent of $E(X_{\omega_1}(t))$, using DFA, and the DCCA correlation coefficients between $E(X_{\omega_1}(t))$ and $E(X_{\omega_2}(t))$, setting $n$ to log spaced values between 100 and 200000. This setup is repeated 100 times, and the results of the simulations are averaged.

C.2 Simulation 2

In the second simulation we check the prediction of $H_{\text{amp}}$ and $H_{\text{raw}}$, setting set $\alpha = 2.5$, $q = 1$, $L_c = 10^6$, with a burn in time of $10^5$ time points (less time is required for convergence for this value of $\alpha$), setting $n$ to log spaced values between 7000 and 60000 for the estimation of $H_{\text{amp}}$ and between 100 and 7000 for the estimation of $H_{\text{raw}}$ (according to where scaling regions were observed). The results are displayed in Figure 5.

The quality of the $H_{\text{raw}}$ estimate is greater for small $\beta$, whereas the quality of the $H_{\text{amp}}$ estimate is greater for larger $\beta$. This discrepancy may be explained as follows: since $X(t)$ has longer tails than $E(X_{\omega}(t))$, the convergence of its moments is slow for large $\beta$, thus the quality of the estimate decreases for large $\beta$. On the other hand, the estimate of $H_{\text{raw}}$ requires a linear approximation to the non-linear transform given by the amplitude of the analytic signal: this approximation increases in quality for larger $\beta$.

C.3 Simulation 3

The aim of this simulation is to verify the theory of Equation (16).

We check the heights of the filtered avalanches $a_{\omega}(t)$. We set $\beta = 0.25$, $L = 2^{10}, \ldots, 2^{13}$ and for each $L$ considered, we simulate $10^4$ avalanches of this length, and calculate the mean avalanche profile. We then log-regress the height of these profiles against $\log(L)$. The results are displayed in Figure 6. Close agreement is observed between the theoretical estimate $\beta' = \beta/2$ and the simulated results; the prediction improves for higher $\beta$.

C.4 Simulation 4

The aim of this simulation is to verify that long-range dependent Gaussian processes satisfy $H_{\text{amp}} = 0.5$ and $\rho_{DCCA} = 0$, where the time series are generated as filtered Gaussian white noise processes.

To this end we simulate 100 long-range dependent Gaussian processes ($H > 0.5$) (implementation) of length 15000 time points. In each case the data are
Figure 5: Scaling of $H_{amp}$ and $H_{raw}$ (second simulation) compared to theory. The theoretical estimate is $H_{amp} \sim 2 - \alpha/2 + \beta'$ for $\beta' = 1$. The middle plot displays the individual log-log plots on the basis of which $H_{amp}$ exponents are calculated. The right hand plot displays the individual log-log plots on the basis of which $H_{raw}$ exponents are calculated.

Figure 6: Comparison of the estimate $\beta/2$ for the scaling of heights of filtered avalanches vs. simulation.
then filtered forward and backwards in two separate frequency bands (between 0.39 and 0.41 of the sampling frequency and 0.29 and 0.31 of the sampling frequency) with butterworth filters of order $n$. We then measure DCCA correlations between the Hilbert transforms of these signals and measure their Hurst exponents with DFA, in both cases using window lengths between $10^3$ and $10^4$.

We repeat this setup for $H = 0.9, 1.4$ and $n = 2, 4$.

The results are displayed in Figure 7 and show that, up to small sample effects, we expect $H_{amp} = 0.5$ and zero cross correlations $\rho_{DCCA} = 0$ between frequency bands.

## D Data Analysis

EEG recordings were obtained at rest with subjects seated comfortably in a chair with their eyes open. Recordings were made of three sessions, each 5 minutes long so that each data set comprises roughly 15 minutes of data. EEG data were recorded with 96 Ag/AgCl electrodes, using BrainAmp amplifiers and BrainVision Recorder software (Brain Products GmbH, Munich, Germany). The signals were recorded in the 0.016–250 Hz frequency range at a 1000Hz sampling frequency and subsequently subsampled to 200Hz. The experimental protocol was approved by the Institutional Review Board of the Charité, Berlin.

The data analytic steps taken on the EEG data were as follows. Outlier channels were rejected after visual inspection for abrupt shifts in voltage and poor signal quality. The data were then re-referenced according to the common average. Spatial filters were computed on the data using Spatio-Spectral Decomposition [23], in order to extract components with pronounced alpha os-
cillations. Spatial filters with poor signal quality or topographies were rejected. We then restricted our analysis to those filters displaying a peak in the alpha range; this step ensured a high signal quality with low levels of artifactual activity. The fact that the spatial filters yield clear oscillatory signals ensured that the neuronal processes in the adjacent frequency ranges similarly originated from cortical areas relating to neuronal rather than artifactual activity. For DFA and DCCA estimation we set \( n \) to log-spaced values between 1000 and 25000.