Intersection and Union Hierarchies of Deterministic Context-Free Languages and Pumping Lemmas

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Abstract

We study the computational complexity of finite intersections and unions of deterministic context-free (dcf) languages. Earlier, Wotschke [J. Comput. System Sci. 16 (1978) 456–461] demonstrated that intersections of \((d+1)\) dcf languages are in general more powerful than intersections of \(d\) dcf languages for any positive integer \(d\) based on the intersection hierarchy separation of Liu and Weiner [Math. Systems Theory 7 (1973) 185–192]. The argument of Liu and Weiner, however, works only on bounded languages of particular forms, and therefore Wotschke’s result is not directly extendable to disprove any given language to be written in the form of \(d\) intersection of dcf languages. To deal with the non-membership of a wide range of languages, we circumvent the specialization of their proof argument and devise a new and practical technical tool: two pumping lemmas for finite unions of dcf languages. Since the family of dcf languages is closed under complementation and also under intersection with regular languages, these pumping lemmas help us establish a non-membership relation of languages formed by finite intersections of non-bounded languages as well. We also refer to a relationship to deterministic limited automata of Hibbard [Inf. Control 11 (1967) 196–238] in this regard.

Key words. deterministic pushdown automata, intersection and union hierarchies, pumping lemma, limited automata, iterative pair, stack-operational, state-stack pair

1 A Historical Account and New Pumping Lemmas

We briefly review a historical account of the subject of this exposition and provide a quick overview of the main contributions.

1.1 Intersection and Union Hierarchies and Historical Background

In formal languages and automata theory, context-free languages constitute a fundamental family, CFL, which is situated in between the family REG of regular languages and that of context-sensitive languages in Chomsky’s hierarchy. Over a half century, a flurry of studies have been conducted toward a better understanding of the nature of the family CFL. It has been well known that, among numerous structural properties, CFL enjoys a closure property under the union operation but not the intersection operation since the non-context-free language \(L_{ab} = \{a^n b^n c^n \mid n \geq 0\}\), for instance, can be expressed as the intersection of the two rather simple context-free languages \(A_1 = \{a^n b^n c^p \mid n, p \geq 0\}\) and \(A_2 = \{a^n b^n c^p \mid n, p \geq 0\}\). This non-closure property can be further generalized to any intersection of \((d \geq 1)\) context-free languages. For later notational convenience, we here write CFL(\(d\)) for the family of such intersection languages, in other words, the \(d\) intersection closure of CFL (see, e.g., [21] for this notation). With the use of this succinct notation, we can say that the above language \(L_{ab}\) belongs to the difference CFL(2) – CFL. In a similar way, the more complicated language \(L_d = \{a_1 a_2 \cdots a_d b_1 b_2 \cdots b_d^n \mid n_1, n_2, \ldots, n_d \geq 0\}\) over the alphabet \(\{a_1, a_2, \ldots, a_d, b_1, b_2, \ldots, b_d\}\) falls into CFL(\(d\)) because \(L_d\) can be expressed as an intersection of \(d\) context-free languages of the form \(\{a_1^{n_1} a_2^{n_2} \cdots a_d^{n_d} b_1^{m_1} b_2^{m_2} \cdots b_d^{m_d} \mid n_k = m_k\}\) for each fixed index \(k\) with \(1 \leq k \leq d\), where \(n_1, n_2, \ldots, n_d, m_1, m_2, \ldots, m_d \geq 0\). In 1973, Liu and Weiner [13] (who actually used a slightly different languages) gave a contrived proof to the following key statement.

\((*)\) For any index \(d \geq 2\), the language \(L_d\) is located outside of CFL(\(d - 1\)).

This result (*) then asserts that the collection \(\{\text{CFL}(d) \mid d \geq 1\}\) truly forms an infinite hierarchy.

Deterministic context-free (dcf) languages, in contrast, have been another focal point of intensive research since an initiation of the systematic study in 1966 by Ginsburg and Greibach [4]. The importance of such languages can be exemplified by the facts that dcf languages are easy to parse and that every context-free language can be expressed simply as the homomorphic image of a dcf language. Unlike CFL, the family DCFL of all dcf languages is closed under neither the union nor the intersection operations. We use the terms of \(d\)-intersection deterministic context-free (dcf) languages and \(d\)-union deterministic

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context-free (dcf) languages to express any intersection of d dcf languages and any union of d dcf languages, respectively. For brevity, we write DCFL(d) and DCFL[d] respectively for the d-intersection closure and the d-union closure of DCFL, namely, the family of all d-intersection dcf languages and that of all d-union dcf languages. As a special case, we obtain DCFL(1) = DCFL[1] = DCFL. Since DCFL is closed under complementation, it follows that the complement of DCFL(d) coincides with DCFL[d]. For our convenience, we call the hierarchies \{DCFL(d) | d \geq 1\} and \{DCFL[d] | d \geq 1\} the intersection hierarchies of dcf languages and the union hierarchies of dcf languages, respectively. The notation DCFL(\omega) is meant to express the entire hierarchy \bigcup_{d \geq 1} DCFL(d), which is the intersection closure of DCFL. Similarly, we write DCFL[\omega] for \bigcup_{d \geq 1} DCFL[d].

In an analogy to the aforementioned union hierarchy of context-free languages, Wotschke [19, 20] discussed the union hierarchy \{DCFL[d] | d \geq 1\} and noted that this hierarchy truly forms an infinite hierarchy. He argued that, since the aforementioned language \(L_d\) is in fact in DCFL(d), the result (*) of Liu and Weiner instantly yields DCFL(d) \(\not\subseteq\) CFL(d − 1), which leads to the conclusion of DCFL(d − 1) \(\not\subseteq\) DCFL(d). Wotschke’s argument, nonetheless, heavily relies on Liu and Weiner’s separation result (*), which uses the property of \textit{stratified semi-linear sets}. The proof technique of Liu and Weiner was particularly developed for \(L_d\), which is a special form of \textit{bounded languages}[^3], and it is therefore not applicable to arbitrary languages. In fact, the key idea of the proof of Liu and Weiner for \(L_d\) is to focus on the number of occurrences of each base symbol of \(\{a_1, \ldots, a_d, b_1, \ldots, b_d\}\) in a given string \(w\) and to translate \(L_d\) into a set \(\Psi(L_d)\) of Parikh images \(\{#_a(w), #_a(w), \ldots, #_a(w), #_b(w), \ldots, #_b(w)\}\) in order to exploit the semi-linearity of \(\Psi(L_d)\), where \(\#_a(w)\) expresses the total number of symbols \(a\) in a string \(w\).

Because of the above-mentioned limitation of Liu and Weiner’s proof technique, the scope of their proof cannot be directly extended to other forms of languages, such as the languages \(L_d^{(S)} = \{a_1^{m_1} \cdots a_d^{m_d}, \ldots, b_1^{n_1} \cdots b_d^{n_d} | \forall i \in \{1, 2, \ldots, d\}, m_i \leq n_i\}\) and \(NPAL^{(S)} = \{w_1^{m_1}#w_2^{m_2} \cdots #w_k^{m_k} | \forall m_i \in \{0, 1\}^*\}\), where \(w_i^{m_i}\) expresses the reverse of \(w_i\). The former is a bounded language expanding \(L_d\) but its Parikh images do not have semi-linearity. The latter is a “non-palindromic” language and it is not even a bounded language. Liu and Weiner’s argument is not directly used to verify that neither \(L_d^{(S)}\) nor \(NPAL^{(S)}\) belongs to CFL(d − 1) (unless we dexterously pick up its core strings that form a certain bounded language). With no such contrived argument, how can we prove \(L_d^{(S)}\) and \(NPAL^{(S)}\) to be excluded from DCFL(d)? Furthermore, for any given language, how can we verify that it is not in DCFL(\omega)? We can ask similar questions for d-union dcf languages and, more generally, the union hierarchy DCFL[\omega] of dcf languages. Ginsburg and Greibach [4] remarked with no proof[^4] that the context-free language \(Pal = \{ww^R | w \in \Sigma^*\}\) (even-length palindromes) for any non-unary alphabet \(\Sigma\) is not in DCFL[\omega]. It is natural to call for a formal proof of the remark of Ginsburg and Greibach. Using a quite different language \(L_{wot} = \{wxc | w, x \in \{a, b\}^*, w \neq x\}\), however, Wotschke [19, 20] proved that \(L_{wot}\) does not belong to DCFL[\omega] (more strongly, the Boolean closure of DCFL) by employing the closure property of DCFL(d) under inverse gsm mappings as well as intersection with regular languages. Wotschke’s proof relies on the following two facts. (i) The language \(L_{d+1}\) can be expressed as the inverse gsm map of the language \(Dup_{\omega} = \{wcw | w \in \{a, b\}^*\}\), restricted to \(a_1^{+}a_2^{+} \cdots a_{d+1}^{+}, (ii) \) \(Dup_{\omega}\) is expressed as the complement of \(L_{wot}\), restricted to a certain regular language. Together with these facts, the final conclusion comes from the aforementioned result (*) of Liu and Weiner because \(Dup_{\omega} \in DCFL(d)\) implies \(L_{d+1} \in DCFL(d)\) by (i) and (ii). To our surprise, all the fundamental results on DCFL(d) and DCFL[d] that we have discussed so far are merely “corollaries” of the main result (*) of Liu and Weiner!

In order to answer more general non-membership questions to DCFL(d), we need to divert from Liu and Weiner’s contrived argument for the statement (*) and to develop a completely different, new, and more practical technical tool. The sole purpose of this exposition is, therefore, set to (i) develop a new proof technique, which can be applicable to many other languages, (ii) present an alternative proof for the fact that the intersection and the union hierarchies of DCFL are infinite hierarchies, and (iii) exhibit other languages in CFL that do not belong to DCFL(\omega) \(\cup\) DCFL[\omega] (in part, verifying Ginsburg and Greibach’s remark for the first time).

In relevance to the union hierarchy of dcf languages, there is another known extension of DCFL within CFL using an access-controlled machine model called \textit{limited automata}[^5] which are originally invented by Hibbard [7] and discussed extensively in, e.g., [14, 15, 22]. A \textit{d-limited deterministic automaton}

[^3]: A \textit{bounded language} satisfies \(L \subseteq w_1^{m_1}w_2^{m_2} \cdots w_k^{m_k}\) for fixed strings \(w_1, w_2, \ldots, w_k\).

[^4]: In page 640 of [1], they wrote “More strongly, \(\{ww^R | w \in \Sigma^*\}\) is not a finite union of deterministic languages if \(\Sigma\) contains at least two elements. We omit the proof.” The proof has never been published as far as we know.

[^5]: Hibbard [7] actually defined a rewriting system, called “scan-limited automata.” Later, Pighizzini and Pisoni [13, 15] re-formulated Hibbard’s system in terms of a restricted form of linear-bounded automata.
(or a d-lda, for short) is a one-tape deterministic Turing machine that can rewrite each tape cell located between two endmarkers only during the first d visits (except that making a turn of a tape head counts as “double visits”). We can raise a question of whether there is any relationship between the union hierarchy and d-lda’s.

### 1.2 New and Useful Pumping Lemmas for DCFL[d]

As noted in Section 1.1, some of the fundamental properties associated with DCFL(d) heavily rely on the single separation result (*) of Liu and Weiner. However, Liu and Weiner’s technical tool that leads to their main result does not seem to withstand a wide variety of direct applications. It is thus desirable to develop a new, simple, and practical technical tool that can find numerous applications for conducting a future study on DCFL(d) as well as DCFL[d]. Our main purpose of this exposition is therefore to present a simple but powerful and practical technical tool, called the pumping lemmas for DCFL[d] for any $d \geq 1$, which also deepen our understanding of both DCFL[d] and DCFL(d). Notice that there have been numerous forms of so-called pumping lemmas (or iteration theorems) for variants of dcf languages in the past literature, e.g., [6] [9] [10] [11] [13] [24]. Our pumping lemmas are a crucial addition to the list of such lemmas.

A pumping lemma generally asserts the existence of a pair of “repeatable” portions in a given string. Such a pair is known as an iterative pair [2]. Given a language L and a string w of the form $uxyz$ (called a factorization), a pair $(x,y)$ is called an iterative pair of w for L if $|yx| \geq 1$ and $ux^iyvy^iz \in L$ for any nonnegative integer i. For instance, Yu’s pumping lemma for DCFL [24, Lemma 1] can be rephrased as follows using the notion of iterative pairs. For a string $x$ of length n and any index $i \in [n]$, the notation $x[i]$ expresses the ith symbol of $x$, where the notation $[n]$ stands for the finite set $\{1, 2, \ldots, n\}$.

**Yu’s Pumping Lemma for DCFL.** Let $L$ be any infinite dcf language. There exists a constant $c > 0$ such that, for any pair $w, w' \in L$, if $w = xy$ and $w' = xz$ with $|x| > c$ and $y[1] = z[1]$, then one of the following two conditions holds. (1) There exists a factorization $x = x_1x_2x_3x_4x_5$ with $|x_2x_4| \geq 1$ and $|x_2x_3x_4| \leq c$ such that $(x_2, x_4)$ is an iterative pair of both $w$ and $w'$ for L. (2) There exist three factorizations $x = x_1x_2x_3, y = y_1y_2y_3$, and $z = z_1z_2z_3$ with $|x_2| \geq 1$ and $|x_2x_3| \leq c$ such that $(x_2, y_2)$ and $(x_2, z_2)$ are respectively iterative pairs of $w$ and $w'$ for L.

We intend to describe our pumping lemmas using the notion of iterative pairs. The first pumping lemma for DCFL[d] is stated as follows.

**Lemma 1.1 [The First Pumping Lemma for DCFL[d]]** Let d be any positive integer and let $L$ denote any infinite d-union dcf language. There exist a constant $c > 0$ such that, for any $d + 1$ strings $w_1, w_2, \ldots, w_{d+1} \in L$, if $w_i$ has the form $xy(i)$ with $|x| > c$ and $|y(i)| \geq 1$ for any index $i \in [d + 1]$, then there exist two distinct indices $j_1, j_2 \in [d + 1]$ for which the following statement holds. For any strings $x'$, $y$, and $z$ with $|x'| > c$, if $x'y = xy(j_1)$ and $x'z = xy(j_2)$, then one of the following conditions (1)–(2) must hold.

1. There is a factorization $x' = x_1x_2x_3x_4x_5$ with $|x_2x_4| \geq 1$ and $|x_2x_3x_4| \leq c$ such that $(x_2, x_4)$ is an iterative pair of both $x'y$ and $x'z$ for L.

2. There are three factorizations $x' = x_1x_2x_3, y = y_1y_2y_3$, and $z = z_1z_2z_3$ with $|x_2| \geq 1$ and $|x_2x_3| \leq c$ such that $(x_2, y_2)$ and $(x_2, z_2)$ are respectively iterative pairs of $x'y$ and $x'z$ for L.

As a special case of $d = 1$ in Lemma 1.1, we instantly obtain Yu’s pumping lemma for DCFL. Since there have been few machine-based analyses known to prove various pumping lemmas in the past literature, one of the important aspects of this exposition is a clear demonstration of the first alternative proof to Yu’s pumping lemma, which is solely founded on an analysis of the behaviors of 1dpda’s instead of derivation trees of LR(k) grammars as in [24]. The proof of Lemma 1.1 in fact, exploits an early result of [23] on an ideal shape form in Section 2.3 together with an approach with $\varepsilon$-enhanced machines by analyzing transitions of state-stack pairs in Section 4.1. These notions will be explained in Sections 2 and 4 and their basic properties will be explored therein.

An iterative pair $(x, y)$ is further said to be nondegenerate if either $\{ux^iyvy^iz \in L \mid j \geq 0\}$ is finite for each $i \geq 0$ or $\{ux^iyvy^iz \in L \mid i \geq 0\}$ is finite for each $j \geq 0$. If either $x$ or $y$ is empty, then the iterative pair $(x, y)$ is called empty; otherwise, it is nonempty. Every nondegenerate iterative pair is obviously...
Figure 1: Containment of language families

nonempty. The nondegenerate iterative pairs for dcf languages have been discussed in the past literature (e.g., [10, 17]).

The second pumping lemma for DCFL[d] is described as follows using nondegenerate iterative pairs.

**Lemma 1.2** [The Second Pumping Lemma for DCFL[d]] Let d be any positive integer and let L denote any infinite d-union dcf language. For arbitrary d + 1 strings \( w_1, w_2, \ldots, w_{d+1} \in L \), if \( w_i \) has the form \( xy^i \) with \( |x|, |y^i| \geq 1 \) for any index \( i \in [d+1] \), then there exist two distinct indices \( j_1, j_2 \in [d+1] \) such that, for any three nonempty strings \( x', y, \) and \( z \), if \( x' = xy^{j_1} \) and \( x'z = xy^{j_2} \), then one of the following conditions (1)–(5) holds.

1. Either \( x'y \) or \( x'z \) has no nondegenerate iterative pair for L.
2. There are two factorizations \( x' = x_1x_2x_3x_4x_5 = \hat{x}_1\hat{x}_2\hat{x}_3\hat{x}_4\hat{x}_5 \) such that \( (x_2, x_4) \) and \( (\hat{x}_2, \hat{x}_4) \) are respectively nondegenerate iterative pairs of \( x'y \) and \( x'z \) for L.
3. There are four factorizations \( x' = x_1x_2x_3 = \hat{x}_1\hat{x}_2\hat{x}_3 \), \( y = y_1y_2y_3 \), and \( z = z_1z_2z_3 \) such that \( (x_2, y_2) \) and \( (\hat{x}_2, z_3) \) are respectively nondegenerate iterative pairs of \( x'y \) and \( x'z \) for L.
4. There exists a string \( u \in \{y, z\} \) such that one of the following cases (a)–(c) occurs.
   a) There are two factorizations \( x' = x_1x_2 \) and \( u = u_1u_2u_3u_4 \) with \( |x_2|, |u_1| \geq 1 \) such that \( (x_2u_1, u_3) \) is a nondegenerate iterative pair of \( x'u \) for L.
   b) There are two factorizations \( x' = x_1x_2x_3x_4 \) and \( u = u_1u_2 \) with \( |x_4|, |u_1| \geq 1 \) such that \( (x_2, x_4u_1) \) is a nondegenerate iterative pair of \( x'u \) for L.
   c) There is a factorization \( u = u_1u_2u_3u_4u_5 \) such that \( (u_2, u_4) \) is a nondegenerate iterative pair of \( x'u \) for L.
5. If \( y \neq z \), then there is a string \( u \in \{y, z\} \) such that one of the following cases (a)–(b) occurs. Let \( u^{(op)} \) denote a unique element in \( \{y, z\} \setminus \{u\} \).
   a) There are two factorizations \( x' = x_1x_2x_3x_4x_5x_6x_7 \) and \( u = u_1u_2u_3 \) such that \( (x_2, u_2) \) and \( (x_4, x_6) \) are respectively nondegenerate iterative pairs of \( x'u \) and \( x'u^{(op)} \) for L.
   b) There are two factorizations \( x' = x_1x_2x_3x_4x_5x_6x_7 \) and \( u = u_1u_2u_3 \) such that \( (x_5x_6, u_2) \) and \( (x_2, x_4x_5) \) (or \( (x_6, u_2) \) and \( (x_2, x_4) \)) are respectively nondegenerate iterative pairs of \( x'u \) and \( x'u^{(op)} \) for L.

The proof of Lemma 1.2 will be given in Section 4.5. For the proof, we need a technical assertion (Proposition 4.4), which links a nondegenerate iterative pair to a specific stack behavior, called “stack-operation”, of an underlying deterministic pushdown automaton explained in Section 4.3.

Using the two pumping lemmas for DCFL[d], we significantly expand the applicable scope of the argument of Liu and Weiner [13], which is limited to specific bounded languages, to other types of languages, including the aforementioned languages \( L^\leq_d \) and \( NPal^d \) for an arbitrary index \( d \geq 2 \).

**Theorem 1.3** For any integer \( d \) at least 2, the languages \( L^\leq_d \) and \( NPal^d \) are not in DCFL(\( d - 1 \)).

Since our pumping lemmas concern with DCFL[d], in the proof of Theorem 1.3 given in Section 3 we first take the complements of the above languages, restricted to suitable regular languages, and we then apply the pumping lemmas appropriately to them. This technique will be formulated in Lemmas 2.1 and 2.2. These lemmas will be given in Section 2.2. From Theorem 1.3 we instantly obtain the
The aforementioned consequence of Wotschke [19, 20], stated as follows. The statement directly comes from Theorem 1.5 because \( \text{DCFL}(d) \) contains both \( L_d^{(\leq)} \) and \( N\text{Pal}_d^{\#} \).

**Corollary 1.4** [19, 20] The intersection hierarchy of dcf languages and the union hierarchy of dcf languages are both infinite hierarchies.

Concerning the limitation of \( \text{DCFL}(\omega) \) and \( \text{DCFL}[\omega] \) in recognition power, since all unary context-free languages are also regular languages and the family \( \text{REG} \) of regular languages is closed under intersection, all unary languages in \( \text{DCFL}(\omega) \) are regular as well. Although it is thus easy to find languages that are not in \( \text{DCFL}[\omega] \), such languages do not serve themselves to separate CFL from \( \text{DCFL}(\omega) \cup \text{DCFL}[\omega] \). As noted in Section 1.1, Ginsburg and Greibach [4] remarked that the context-free language \( \text{Pal} = \{ w^R \mid w \in \{0,1\}^+ \} \) is not in \( \text{DCFL}(\omega) \). As another direct application of our pumping lemmas, we can provide a formal written proof of their remark. Furthermore, we can show that another context-free language \( \text{MPal}^{\#} = \{ w_1#w_2\cdots#w_m#^2v_1#v_2\cdots#v_n \mid m, n \geq 1, \exists i \in [\min\{m, n\}](v_i = w_i^R) \} \), where \( w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_n \in \{0,1\}^+ \), is indeed outside of \( \text{DCFL}(\omega) \cup \text{DCFL}[\omega] \).

**Theorem 1.5**

1. The language \( \text{Pal} \) is in \( \text{CFL} \cap \text{co-CFL} \) but not in \( \text{DCFL}[\omega] \).
2. The language \( \text{MPal}^{\#} \) is in neither \( \text{DCFL}(\omega) \) nor \( \text{DCFL}[\omega] \).

As an immediate consequence of Theorem 1.5, we obtain Wotschke’s separation of \( \text{CFL} \) from \( \text{DCFL}[\omega] \). Here, we stress that, unlike the work of Wotschke [19, 20], our proof does not depend on the main result (\(^*\) of Liu and Weiner. Actually, we can obtain a much stronger consequence from Theorem 1.5, because \( \text{CFL} \cap \text{co-CFL} \subseteq \text{DCFL}(\omega) \) implies \( \text{CFL} \cap \text{co-CFL} = \text{co-}(\text{CFL} \cap \text{co-CFL}) \subseteq \text{co-DCFL}(\omega) = \text{DCFL}[\omega] \).

**Corollary 1.6** \( \text{CFL} \cap \text{co-CFL} \not\subseteq \text{DCFL}(\omega) \cup \text{DCFL}[\omega] \).

We then turn our interest to limited automata. For each index \( d \geq 1 \), we write \( d\text{-LDA} \) for the family of all languages recognized by \( d \)-limited deterministic automata, in which their tape heads are allowed to rewrite tape symbols only during the first \( d \) accesses (except that, in the case where tape heads make a turn, we treat each turn as double visits). Hibbard demonstrated that \( d\text{-LDA} \neq (d-1)\text{-LDA} \) for any integer \( d \geq 3 \) [2]. We expand this separation result to the following in connection to the intersection of dcf languages.

**Proposition 1.7** For any \( d \geq 2 \), \( d\text{-LDA} \cap \text{DCFL}[2d-1] \not\subseteq (d-1)\text{-LDA} \cup \text{DCFL}[2d-1] - 1 \).

The proofs of all the above-mentioned assertions will be given in Section 3 after introducing necessary notions and notation in the subsequent section.

2 Preparations: Notions and Notation

We begin with a detailed explanation of basic notions and notation, which are crucial in the rest of this exposition.

2.1 Fundamental Notions and Notation

The set of all natural numbers (including 0) is denoted by \( \mathbb{N} \). An integer interval \([m, n]_\mathbb{Z}\) for two integers \( m \) and \( n \) with \( m \leq n \) is the set \([m, m+1, m+2, \ldots, n]\). For any integer \( n \geq 1 \), \([1, n]_\mathbb{Z}\) is particularly abbreviated as \([n]\). Given a set \( S \), the notation \( |S| \) indicates the cardinality of \( S \) and the notation \( \mathcal{P}(S) \) refers to the power set of \( S \).

An alphabet is a finite nonempty set of “symbols” or “letters”. A finite sequence of symbols in alphabet \( \Sigma \) is called a string over \( \Sigma \). The length of such a string \( x \), denoted \( |x| \), is the total number of symbols occurring in \( x \). The special symbol \( \varepsilon \) is used to denote the empty string of length 0. We write \( \Sigma^* \) for the set of all strings over \( \Sigma \) and we set \( \Sigma^+ \) to be \( \Sigma^* - \{\varepsilon\} \). Given a number \( n \in \mathbb{N} \), the notation \( \Sigma^n \) (resp., \( \Sigma^{\leq n} \)) further denotes the set of all strings of length exactly \( n \) (resp., at most \( n \)) in \( \Sigma^* \). We say that \( x \) is a substring of \( y \) (denoted by \( x \subseteq y \)) if \( y = xuv \) holds for certain strings \( u \) and \( v \). When a nonempty string \( x \) is expressed as \( x = x_1x_2\cdots x_n \) with strings \( x_1, x_2, \ldots, x_n \), we call such an expression a factorization of \( x \).

Any subset of \( \Sigma^* \) is called a language over \( \Sigma \). For a language \( L \) over \( \Sigma \), \( \Sigma^* - L \) is the complement of \( L \) and expressed as \( \overline{L} \) as long as \( \Sigma \) is clear from the context. Given a family \( \mathcal{F} \) of languages, \( \text{co-}\mathcal{F} \) expresses the complement family, which consists of the complements \( \overline{L} \) of all languages \( L \in \mathcal{F} \).
For two family languages $F_1$ and $F_2$, the notation $F_1 \cap F_2$ (resp., $F_1 \cup F_2$) denotes the family of all languages $L$ over certain alphabets $\Sigma$ such that there are two languages $L_1 \in F_1$ and $L_2 \in F_2$ over $\Sigma$ satisfying $L = L_1 \cap L_2$ (resp., $L = L_1 \cup L_2$). Generally, for a given $k$-ary operation $op$ over $k$ languages, we say that a family $C$ of languages is closed under $op$ if, for any $k$ languages in $C$, $op(L_1, L_2, \ldots, L_k)$ belongs to $C$.

Given two binary strings $w$ and $w'$. $K(w)$ means the Kolmogorov complexity of $w$ and $K(w | w')$ expresses the conditional Kolmogorov complexity of $w$ conditional to $w'$. See, e.g., [12] for basic properties.

### 2.2 Deterministic Pushdown Automata

The machine model of one-way deterministic pushdown automata (abbreviated as 1dpda’s) was introduced in 1966 by Ginsburg and Greibach [4]. Formally, a 1dpda $M$ is a nonuple $(Q, \Sigma, \{\#, \&\}, \Gamma, \delta, q_0, Z_0, Q_{acc}, Q_{rej})$, where $Q$ is a finite set of inner states, $\Sigma$ is an input alphabet, $\Gamma$ is a stack alphabet, $\delta$ is a transition function from $(Q - Q_{halt}) \times (\Sigma \times \epsilon)$ to $\mathcal{P}(Q \times \Gamma^*)$ satisfying $|\delta(p, \sigma, \epsilon)| \leq 1$ for any $(p, \sigma, \epsilon)$, $q_0$ is the initial state in $Q$, $Z_0$ is the bottom marker in $\Gamma$, and $Q_{acc}$ and $Q_{rej}$ are respectively sets of accepting states and of rejecting states, where $Q_{halt} = Q_{acc} \cup Q_{rej}$. $\Sigma = \Sigma \cup \{\#, \&\}$, and $\epsilon \in \Sigma$. Note that $\#$ and $\&$ are respectively the left-endmarker and the right-endmarker. Let $\Gamma^{-} = \Gamma - \{Z_0\}$. The push size of a 1ppda is the maximum length of any string pushed into a stack by any single move. The 1dpda $M$ must satisfy the following deterministic requirement: $|\delta(p, \sigma, a \epsilon) \cup \delta(p, \sigma, a)| = 1$ for any $p \in Q$, any $\sigma \in \Gamma$, and any nonempty symbol $\sigma \in \Sigma$. When $\delta(q, \sigma, a)$ is a singleton, say, $\{(p, w)\}$, we intentionally write $\delta(p, \sigma, a) = (p, w)$. This transition indicates that, when $M$ is in inner state $q$ scanning an input tape and $a$ on the top of a stack, $M$ changes $q$ to $p$, replace $a$ by $w$, and move its tape head to the right whenever $\sigma \neq \epsilon$. Moreover, we require that the bottom marker $Z_0$ is not removable, namely, $\delta(q, \sigma, Z_0) \neq (p, \epsilon)$ for any $p, q \in Q$ and $\sigma \in \Sigma$. A content of a stack is expressed as $a_1a_2\ldots a_k$ so that $a_1$ is the topmost stack symbol, $a_k$ is the bottom marker, and all $a_i$’s are placed in order from the top to the bottom in the stack. The stack height refers to the size of a stack content, namely, the total number of symbols stored in the stack. For instance, a stack content $a_1a_2\ldots a_k$ with $a_k = Z_0$ has stack height $k$. Since $Z_0$ is not removable, the stack height is at least 1 at any step of a computation.

When $M$ enters a halting state, which is either an accepting state or a rejecting state, $M$ must halt. In the rest of this exposition, we always demand that $M$ halts on all inputs. Given a string $w$, we say that $M$ accepts (resp., rejects) $w$ if $M$ is in an accepting (resp., rejecting) state when it halts. We use the notation $L(M)$ to denote the set of all strings accepted by $M$. If a language $L$ satisfies $L = L(M)$, $M$ is said to recognize $L$. Such a language is called a deterministic context-free (dcf) language.

Given any number $d \in \mathbb{N}^+$, a $d$-intersection deterministic context-free (def) language refers to an intersection of $d$ def languages. Let DCFL($d$) denote the family of all $d$-intersection def languages. Similarly, we define $d$-union def languages and DCFL[$d$] by substituting “union” for “intersection” in the above definition. It follows that DCFL[$d$] = co-(DCFL($d$)) because of DCFL = co-DCFL.

The following two lemmas will be quite useful in proving Theorems 1.3 and 1.5 together with Proposition 1.7.

**Lemma 2.1** [19] [20] DCFL($d$) is closed under union, intersection with regular languages. In other words, DCFL($d$) $\cap$ REG $\subseteq$ DCFL($d$) and DCFL($d$) $\cup$ REG $\subseteq$ DCFL($d$). A similar statement holds for DCFL[$d$].

**Proof.** For completeness, we provide the proof. Take any language $L \in$ DCFL[$d$] and a regular language $A$. Take $d$ appropriate languages $L_1, L_2, \ldots, L_d \in$ DCFL satisfying $L = \bigcap_{i \in [d]} L_i$. It then follows that $L \cap A = \bigcap_{i \in [d]} (L_i \cap A)$ and $L \cup A = \bigcap_{i \in [d]} (L_i \cup A)$. Since $L_i \cap A$ and $L_i \cup A$ are both deterministic context-free, we conclude that $L \cap A$ and $L \cup A$ are both in DCFL($d$). The case for DCFL[$d$] is similarly treated. \hfill $\square$

**Lemma 2.2** Let $d \geq 1$ be any integer.

1. DCFL($d$) = DCFL($d+1$) if and only if DCFL[$d$] = DCFL[$d+1$].

2. For any language $L \in$ DCFL($d$), it follows that $A \cap L \in$ DCFL[$d$] for any language $A \in$ REG.

**Proof.** (1) Assume that DCFL($d$) = DCFL($d+1$). By taking complementation, it then follows that co-(DCFL($d$)) = co-(DCFL($d+1$)). This is equivalent to DCFL[$d$] = DCFL[$d+1$]. The other direction is similarly proven. \hfill $\square$
(2) Assume that \( L \in \text{DCFL}(d) \). From this follows \( L \in \text{DCFL}[d] \). Lemma 2.1 then yields \( A \cap L \in \text{DCFL}[d] \) for any regular language \( A \).

From Lemma 2.2(1) follows Corollary 1.4 provided that Theorem 1.3 is true. Theorem 1.3 itself will be proven in Section 3.

2.3 Ideal Shape Lemma

We will give the proofs of the pumping lemmas for \( \text{DCFL}[d] \) (Lemmas 1.1 and 1.2) in Section 4. To make the proofs simpler, we intend to use the fact that any 1dpda can be converted into a specific form. Let us recall from \( \text{22}[23] \) a special “push-pop-controlled” form (called an ideal shape), in which (i) pop operations always take place by first reading an input symbol and then making a series (one or more) of the pop operations without reading any further input symbol and (ii) push operations add single symbols without altering any existing stack content. The original notion was meant for one-way probabilistic pushdown automata (or 1ppda’s); however, in this exposition, we wish to apply this notion to 1dpda’s.

To be more formal, a 1ppda in an ideal shape is restricted to the following moves regarding its stack operations. (1) Scanning \( \sigma \in \Sigma \), preserve the topmost stack symbol (called a stationary operation). (2) Scanning \( \sigma \in \Sigma \), push a new symbol \( \epsilon \in \Gamma \) without changing any other symbol in the stack. (3) Scanning \( \sigma \in \Sigma \), pop the topmost stack symbol. (4) Without scanning an input symbol (i.e., \( \epsilon \)-move), pop the topmost stack symbol. (5) The stack operations (4) comes only after either (3) or (4).

It was shown in 22, 23 that any 1ppda can be converted into its “equivalent” 1ppda in an ideal shape. We say that two 1dpda’s are error-equivalent if, for any input \( x \), their acceptance/rejection coincide.

Lemma 2.3 [Ideal Shape Lemma for 1ppda’s 22[23]] Let \( n \in \mathbb{N}^+ \). Any \( n \)-state 1ppda \( M \) with stack alphabet size \( m \) and push size \( e \) can be converted into another error-equivalent 1ppda \( N \) in an ideal shape with \( O(en^2m^2(2m)^{2emn}) \) states and stack alphabet size \( O(enm(2m)^{2emn}) \).

As noted in 22, 23, by appropriately choosing an error probability of a 1ppda, Lemma 2.3 is applicable to 1dpda’s.

2.4 Deterministic Limited Automata

Hibbard 7 discussed another machine model of scan limited automata in 1967. In this exposition, we follow a reformulation of his model by Pighizzini and Pisoni 14[15] and Yamakami 22 under the name of deterministic d-limited automata (or d-lda’s, for short). Given \( d \geq 1 \), a d-lda \( M \) is formally an octuple \( (Q, \Sigma, \{\{,\}\}, \{\Gamma^{(c)}\}_{c \in [d]}, \delta, q_0, Q_{acc}, Q_{rej}) \) with \( \delta: (Q - Q_{bait}) \times \Sigma \times \Gamma \rightarrow Q \times \Gamma \times \Delta \), where \( \Delta = \{-1,+1\}, \Gamma = \bigcup_{c \in [0,d]} \ \Gamma^{(c)} \), with \( \Gamma^{(0)} = \Sigma \), \( \{\{,\}\} \subseteq \Gamma^{(d)} \), and \( \Gamma^{(i)} \cap \Gamma^{(j)} = \emptyset \) for any distinct pair \( i,j \in [0,d] \). We remark that \( M \) has a single rewritable input/work tape and moves its tape head in both directions. In particular, \( M \) makes no \( \epsilon \)-move. If \( \delta(q,\sigma) = (p,\tau,\ell) \), then \( M \) changes its inner state \( q \) to \( p \), writes \( \tau \) over \( \sigma \), and moves its tape head in direction \( \ell \). Assuming further that \( \sigma \in \Gamma^{(i)} \) and \( \tau \in \Gamma^{(j)} \) for \( i,j \in [0,d] \), we demand that (1) if \( i = d \), then \( \sigma = \tau \) and \( j = d \), (2) if \( i < d \) and \( i \) is even, then \( j = i + 2^{(1-i)/2} \), and (3) if \( i < d \) and \( i \) is odd, then \( j = i + 2^{(1+i)/2} \). These requirements imply that no symbol in \( \Gamma^{(d)} \) is replaced by any other symbol.

Here, we reuse the terminology defined for 1dpda’s in Section 2.2 and we do not intend to repeat the same kinds of definitions for d-lda’s. The notation d-LDA is used for the family of all languages accepted by d-lda’s. It is known that \( \text{DCFL} = 2\text{-LDA} \) 15 and \( d\text{-LDA} \neq (d+1)\text{-LDA} \) for any index \( d \in \mathbb{N}^+ \) 7.

3 Proofs of Three Separation Claims

The proofs of the two pumping lemmas for \( \text{DCFL}[d] \) (namely, Lemmas 1.1 and 1.2) require a lengthy argument and we postpone the proofs of the lemmas until Section 4. Meanwhile, we intend to concentrate on the three separation claims (Theorems 1.3 and 1.5 and Proposition 1.7) announced in Section 2.1 and provide their detailed proofs in this section. To understand these proofs better, we first demonstrate a simple and easy example of how to apply the first pumping lemma for \( \text{DCFL}[d] \) (Lemma 1.1) to obtain the desired separation between \( \text{DCFL}[d] \) and \( \text{DCFL}[d-1] \).

Proposition 3.1 Let \( d \geq 2 \) and let \( L(d) = \{a^{n+k^d}b^n \mid k \in [d], n \geq 0 \} \). It then follows that \( L(d) \in \text{DCFL}[d] - \text{DCFL}[d-1] \).
Proof. Let \( d \geq 2 \). Since \( L(d) \) can be expressed as \( \bigcup_{k \in [d]} L^{(k)} \) using the \( d \) languages \( L^{(k)} = \{ a^n b^kn \mid n \geq 0 \} \) for any number \( k \in [d] \), \( L(d) \) clearly belongs to DCFL\([d]\). Toward the non-membership \( L(d) \notin \text{DCFL}([d]-1) \), we assume otherwise and apply the first pumping lemma for DCFL\([d]\) to \( L(d) \). We take a pumping-lemma constant \( c > 0 \) that satisfies the lemma. Let \( n = c + 1 \) and consider \( x = a^n \) and \( y^{(i)} = b^m \) for each index \( i \in [d] \). Since each string \( w_i = a^n b^m \) belongs to \( L(d) \), there is a special index pair \( j, k \in [d] \) with \( j < k \) such that \( w_j \) and \( w_k \) satisfy the lemma.

Consider the condition (1) of the lemma. Let \( x' = a^n b^{n-1} \), \( y = b \), and \( z = b^{(k-j)n+1} \). Firstly, let us consider a factorization \( x' = x_1 x_2 x_3 x_4 x_5 \) with \( |x_2 x_4| \geq 1 \) and \( |x_2 x_3 x_5| \leq c \). Since \( x_1 x_2 x_3 x_4 x_5 y \in L(d) \) holds for any \( i \in \mathbb{N} \), it follows that \( x_2 \in \{ a \}^* \) and \( x_3 \in \{ b \}^* \). Because of the definition of \( L(d) \), we also conclude that \( x_2 \neq \varepsilon \) and \( x_4 \neq \varepsilon \). Let \( x_2 = a^n \) and \( x_4 = b^m \) for certain numbers \( m, r \in [c] \). Note that \( x_1 x_2 x_3 x_4 x_5 y \in L(d) \) holds. Since \( |x_2 x_3 x_4| = n \), \( (m-r) \) holds for a certain \( q \in [d] \). This implies that \((jg-1)n = (m-qr)(i-1)\), \((m-qr)(i-1)\). Then we obtain \( jg-1 = m-qr = 0 \), which further implies that \( j = g = 1 \) and \( m = r \). Similarly, from \( x_1 x_2 x_3 x_4 x_5 z \in L(d) \), it follows that \( n + (i-1)m = g\left(k + (i-1)r\right) \) for an appropriate number \( g' \in [d] \). Thus, \((kg'-1)m = (m-g'r)(i-1)\). This implies \( k = g' = 1 \) and \( m = r \). Since \( j \neq k \), we obtain a contradiction.

Secondly, let us consider the condition (2) with appropriate factorizations \( x' = x_1 x_2 x_3 \), \( y = g_{12} y_{23} \), and \( z = z_{23} z_{34} \) with \( |x_2| \geq 1 \) and \( |x_2 x_3| \leq c \) such that \( x_1 x_2 x_3 y_{23} y_{34} \in L(d) \) and \( x_1 x_2 x_3 z_{23} z_{34} \in L(d) \) for any number \( i \in \mathbb{N} \). Since \( |x_2 x_3| \leq c \), we obtain \( x_2 \in \{ b \}^* \). Assume that \( x_2 = b^m \) for a certain number \( m \in [c] \). However, this is impossible because \( x_1 x_2 x_3 y_{23} y_{34} \) has the form \( a^n b^{m(n+1)+1} \) and the exponent of \( b \) is not of the form \( m \) for any number \( i \in [d] \). Therefore, \( L(d) \) is not in DCFL\([d]-1] \). \( \square \)

Theorem 1.3 relates to the languages \( L^{(S)}_{d} = \{ b^n a^m b^{n+m} \mid i \in \mathbb{N} \} \) over the alphabet \( \Sigma_d = \{ a, a, \ldots, a, b, b, \ldots, b \} \) and \( N\text{PAL}_d = \{ w_1# \cdots # w_d \mid |\forall i \in [d] \} \) over the alphabet \( \{ 0, 1 \} \). We employ the two pumping lemmas for DCFL\([d]-1] \) to verify the theorem.

**Theorem 1.3 (rephrased) Let \( d \geq 2 \).**

1. The language \( L^{(S)}_{d} \) is not in DCFL\([d]-1] \).
2. The language \( N\text{PAL}_d \) is not in DCFL\([d]-1] \).

**Proof.** Let \( d \geq 2 \) be any positive integer.

1. Our first target is the language \( L^{(S)}_{d} \) over the alphabet \( \Sigma_d = \{ a, a, \ldots, a, b, b, \ldots, b \} \). For each index \( i \in [d] \), we set \( L^{(i)} = \{ a^n b^m b^{n+m} \mid n_i > m_i \geq 0 \} \). It is clear that \( L^{(S)}_{d} \) coincides with \( \bigcup_{i \in [d]} L^{(i)} \), and therefore \( L^{(S)}_{d} \) belongs to DCFL\([d]-1] \). Our next goal is to verify the non-membership of \( L^{(S)}_{d} \) to DCFL\([d]-1] \). To lead to a contradiction, let us assume that \( L^{(S)}_{d} \) is in DCFL\([d]-1] \). Take the special regular language \( A = a^n b^m b^{n+m} \) and consider \( L' = A \cap \Sigma_{d}^{*} \). In other words, \( L' = \{ a^n b^m b^{n+m} \mid \forall i \in [d] \} \). Note by Lemma 2.2 [2] that, since \( L^{(S)}_{d} \) is in DCFL\([d]-1] \), we obtain \( L' \) is not in DCFL\([d]-1] \). This makes it possible to apply to \( L' \) the first pumping lemma for DCFL\([d]-1] \) (Lemma 1.1). We take a pumping-lemma constant \( c > 0 \) that satisfies the first pumping lemma, and then set \( n = c + 1 \). Let us focus on strings \( x y \) with \( |x y| \geq 1 \) and \( |x y| \leq c \) such that \( x_1 x_2 x_3 y \in L(d) \) holds for any \( u \in [y, z] \) and any number \( i \in \mathbb{N} \). We choose an index \( j \) for which \( x_1 x_2 x_3 \in \{ a \}^* \). Let \( x_1 = a^n \) and \( x_2 = a^c \) for two numbers \( c_1, c_2 \) satisfying \( c_1 + c_2 > 0 \). By setting \( i = 0 \), \( x_1 x_2 x_3 y \) contains \( a^n b^{m(n+1)+1} \) and \( b^{m(n+1)+1} \). This is a clear contradiction.

Next, we consider the condition (2), in which there are three factorizations \( x' = x_1 x_2 x_3 \), \( y = g_{12} y_{23} \), and \( z = z_{23} z_{34} \) with \( |x' y| \geq 1 \) and \( |x_2 x_3| \leq c \) such that \( x_1 x_2 x_3 y \in L' \) and \( x_1 x_2 x_3 z \in L' \) for any number \( i \in \mathbb{N} \). Since \( |x_2 x_3| \leq c \), \( x_2 \in \{ b \}^* \) follows. Let \( x_2 = b^m \) with \( m \geq 1 \) and then take \( i = 2 \). Note that \( x_1 x_2 x_3 y \) has factors \( a^n b^{m(n+1)+1} \) and \( b^{m(n+1)+1+2e} \). Thus, we obtain \( j_1 n < j_1 n + 2e - 1 \), a contradiction.

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We next discuss the non-membership of $N\text{Pal}^#_d$ to DCFL$(d-1)$. Let $A$ denote the regular language $\{w_1 \# w_2 \# \cdots \# w_d | \forall i \in [d](w_i \in \{0,1\}^*)\}$ and consider $L' = A \cap \{0,1\}^* - N\text{Pal}^#_d$. Note that $L' = \{w_1 \# \cdots \# w_ix_1 \# \cdots \# w_d | \exists i \in [d](v_i = w_i^R)\}$. It suffices to prove that $L' \notin \text{DCFL}[d-1]$ because, by Lemma 2, this clearly implies that $N\text{Pal}^#_d \notin \text{DCFL}(d-1)$. Assume to the contrary that $L' \in \text{DCFL}[d-1]$. We then apply to $L'$ the second pumping lemma for DCFL$[d-1]$ (Lemma 1.2).

We choose a sufficiently large integer $n$ and take arbitrary strings $w_1, w_2, \ldots, w_d$ of length $n$. For each index $i \in [d]$, we set $v_i$ to be $w_i^R s_i$, where $s_i$ is any string of length $n$. We define $d = w_1 \# w_2 \# \cdots \# w_d \#$ and $y^{(i)} = v_1 \# v_2 \# \cdots \# v_i - 1 \# w_i^R \# v_{i+1} \# \cdots \# v_d$ for any index $i \in [d]$. Clearly, $xy^{(i)}$ is in $L'$ and $(w_i, w_i^R)$ is a nondegenerate iterative pair of $xy^{(i)}$ for $L'$. We then take two distinct indices $j_1, j_2 \in [d]$ satisfying the second pumping lemma. Assuming $j_1 < j_2$, let $x' = w_1 \# \cdots \# w_j \# v_1 \# \cdots \# v_{j_1 - 1} \# w_{j_1}^R, y = \# v_{j_1 + 1} \# \cdots \# v_d$ and $z = s_{j_1} \# v_{j_1 + 1} \# \cdots \# v_{j_2 - 1} \# w_{j_2}^R \# v_{j_2 + 1} \# \cdots \# v_d$. Note that $x'y = xy^{(j_1)}$ and $x'z = xy^{(j_2)}$.

Note that $(w_{j_1}, w_{j_1}^R)$ and $(w_{j_2}, w_{j_2}^R)$ are respectively nondegenerate iterative pairs of $x'$ and $x'$ for $L'$. Thus, the condition (1) fails. Clearly, there is no nondegenerate iterative pair of $x'$ in the $i'$-region. Hence, the condition (2) is false. Since no entry of any nondegenerate iterative pair of $x'y$ crosses over the borderline between $x'$ and $y$ (as well as $x'$ and $z$), the condition (3) as well as the condition (4c) fails. Since $y$ begins with $\#$, the conditions (4a) and (4b) also fail.

We then turn our attention to the condition (5). In the case (a) with setting $u = y$, the pumping lemma provides two factorizations $x = x_1 x_2 x_3 x_4 x_5 x_6 y_1$ and $y = y_1 y_2 y_3$ such that $(x_2, y_2)$ and $(x_4, x_6)$ are nondegenerate iterative pairs of $x'$ and $x'$ for $L'$, respectively. It then follows that $x_1$ and $x_4$ are respectively substrings of $w_{j_1}$ and $v_{j_1} (= w_{j_1}^R)$. Thus, $x_2$ is a substring of $w_1 \# \cdots \# w_{j_1 - 1}$; however, since $y$ does not contain any of the strings $w_1^R, \ldots, w_{j_1 - 1}$, $(x_2, y_2)$ cannot be an iterative pair of $x'y$. If $u = z$, then $(x_2, w_2)$ is a nondegenerate iterative pair of $x'z$, $x_1$ and $x_2$ must be respectively substrings of $w_{j_1}$ and $v_{j_1} (= w_{j_1}^R)$. Thus, both $x_1$ and $x_4$ are substrings of $w_{j_2 + 1} \# \cdots \# v_{j_1 - 1} \# w_{j_1}^R$. Obviously, $(x_4, x_6)$ cannot be any nondegenerate iterative pair of $x'y$.

Next, we consider the case (b). We begin with setting $u = y$. Assuming $x' = x_1 x_2 x_3 x_4 x_5 x_6 y_1$ and $y = y_1 y_2 y_3, (x_5 x_6, y_2)$ and $(x_2, x_4 x_5, (x_2, y_2)$ and $(x_4, x_5)$ are respectively nondegenerate iterative pairs of $x'y$ and $x'z$ for $L'$. This is impossible because any nondegenerate iterative pair of $x'$ exists in $x'$. Similarly, we obtain a contradiction for the case of $u = z$.

Recall that $M\text{Pal}^# = \{w_1 \# w_2 \# \cdots \# w_n \#^2 v_1 \# v_2 \# \cdots \# v_n | m, n \geq 1, \exists i \in [\min\{m, n\}] (v_i = w_i^R)\}$, which extends the language $L'$ defined in the proof of Theorem 1.3(2), where $w_1, w_2, \ldots, w_n, v_1, v_2, \ldots, v_n \in \{0,1\}^*$. It is clear that $M\text{Pal}^#$ is a context-free language. Theorem 1.5 is rephrased in the following way.

**Theorem 1.5** (rephrased)

1. The language $\text{Pal}$ is in $\text{CFL} \cap \text{co-CFL} - \text{DCFL}[\omega]$.
2. The language $M\text{Pal}^#$ is in $\text{CFL} - (\text{DCFL}(\omega) \cup \text{DCFL}[\omega])$.

**Proof.** (1) It is easy to see that $\text{Pal}$ is in CFL. We then claim that $\text{Pal}$ is also in co-CFL. Let $N\text{Pal} = \{xy \mid |x| = |y|, x, y \in \{0,1\}^*, y \neq x^R\}$ and $\text{ODD} = \{w \in \{0,1\}^* \mid |w| \text{ is odd}\}$. Note that $\overline{\text{ODD}} = \text{ODD} \cup \{\text{ODD} \cap N\text{Pal}\}$. Since $\text{ODD} \in \text{REG}$, it suffices to verify that $\overline{\text{ODD}} \cap N\text{Pal} \in \text{CFL}$. Consider the language $1n^d - 1$. Given an input $w$ in $\overline{\text{ODD}}$, split $w$ nondeterministically into $xy$, store $x$ into a stack, and pop $x$ bit by bit while reading $y$. If $|x| \neq |y|$, then we reject $w$. Assume otherwise. If we discover any discrepancy between $y$ and $x^R$, then we accept $w$; otherwise, we reject it. It is obvious that this machine correctly recognizes $\overline{\text{ODD}} \cap N\text{Pal}$, and thus the language belongs to CFL.

Next, we intend to verify that $\text{Pal} \notin \text{DCFL}[d]$ for any $d \geq 2$ because this implies $\text{Pal} \notin \text{DCFL}[\omega]$. Toward a contradiction, we assume that $\text{Pal} \in \text{DCFL}[d]$ for a certain index $d \geq 2$. We apply to $\text{Pal}$ the first pumping lemma for DCFL$[d]$ and take a pumping-lemma constant $c > 0$ guaranteed by the lemma. Let $n = 2c$.

Here, we use “blocks” of $0^n$ and $1^n$. Choose $w_k = 0^n 1^n 0^n 1^n \cdots 0^n (2k + 1 \text{ blocks of } 0^n \text{ and } 1^n)$ for any index $k \in [d + 1]$. For example, $w_1 = 0^n 1^n 0^n$ and $w_2 = 0^n 1^n 0^n 1^n 0^n$. By the first pumping lemma, there is a distinct pair $i, j \in [d + 1]$ with $i < j$ satisfying the lemma. We consider the first case where $i$ and $j$ are even. Let $x^* = 0^n 1^n 0^n \cdots 0^n (i + 1 \text{ blocks of } 0^n), y = 1^n 0^n \cdots 0^n (i \text{ blocks of } 0^n)$, and $z = 1^n 0^n \cdots 0^n (2j - i \text{ blocks of } 0^n)$ such that $w_i = x'y$ and $w_i = x'z$. Consider the condition (1) of the lemma. There is a factorization $x' = x_1 x_2 x_3 x_4 x_5 \in \pi 2x_4 \geq 1$ and $|x_2 x_3| \leq c$ such that $x_1 x_2 x_3 x_4 x_5 \in \pi$ and $x_1 x_2 x_3 x_4 x_5 \in \pi$ for any number $i \geq 0$. It follows that $x_2 x_3 x_4$ must be a substring of $0^n, 1^n 0^n, 1^n 0^n$, or $1^n 0^n$. In any case, nevertheless, we obtain a contradiction.

Let us consider the condition (2) with three factorizations $x = x_1 x_2 x_3, y = y_1 y_2 y_3, z = z_1 z_2 z_3$ with $|x_2| \geq 1$ and $|x_2 x_3| \leq c$ satisfying $x_1 x_2 x_3 y_1 y_2 y_3 \in \pi$ and $x_1 x_2 x_3 z_1 z_2 z_3 \in \pi$ for any number
$i \geq 0$. We then obtain $x_2 \in \{0\}^+$ because of $|x_2 x_3| \leq c$. Thus, in particular, $x_1 x_3 y_3 \notin Pal$ follows, a contradiction.

(2) The claim $MPal^\# \notin DCFL[\omega]$ comes from (1) since $MPal^\# \in DCFL[d]$ implies $Pal \in DCFL[d]$ for any $d \geq 1$. Next, we wish to show that $MPal^\# \notin DCFL[\omega]$. Toward a contradiction, we assume otherwise. For convenience, we define $FORM = \{x_1 \cdots \# x_m \# y_1 \# y_2 \# \cdots \# y_n \mid m, n \geq 1, x_1, \cdots, x_m, y_1, \cdots, y_n \in \{0, 1\}^*\}$ and set $NPal^\# = \{w_1 \# \cdots \# w_m \# v_1 \# \cdots \# v_n \mid m, n \geq 1, v_i \in \{1\}[\{v_i \neq w_i^R\}]\}$, where $x_1 \cdots x_m, y_1 \cdots y_n \in \{0, 1\}^*$. Note that $NPal^\# = FORM \cap MPal^\#$. Since $FORM \in REG$, by Lemma 2.2(2), it follows from our assumption that $NPal^\# \in DCFL[\omega]$. We then take the minimum index $d \in \mathbb{N}^+$ for which $NPal^\#$ is in DCFL[d]. As a special case of $NPal^\#$, we consider $NPal^\#_2 = \{w_1 \# w_2 \# v_1 \# v_2 \mid w_1, w_2, v_1, v_2 \in \{0, 1\}^*, v_1 \neq w_1^R, v_2 \neq w_2^R\}$. Since $NPal^\# \in DCFL[d]$ implies $NPal^\#_2 \in DCFL[d]$, $NPal^\#_2$ is written as the $d$-union $\bigcup_{i \in [d]} L_i$ for appropriate-def languages $L_1, \ldots, L_d$.

We choose a sufficiently large integer $n$ and strings $w_1, w_2, \ldots, w_d \in \{0, 1\}^n$ such that $K(n) \geq \log_2 n$, $K(w_1) \geq n/2$, and $K(w_i \mid w_j) \geq n/2$ for any distinct pair $i, j \in [d]$.

Given a binary string $z$ and a number $i \in [\lfloor z \rfloor]$, let $z(i)$ denote the string obtained from $z$ by flipping the $i$th bit of $z$. For instance, $0110(1) = 1110$, $0110(2) = 0010$, and $0110(3) = 0100$. For convenience, we set $x = w_1 \# w_2$ and $y(i_1, i_2) = w_1^R(i_1)w_2^R(i_2)$ for any pair $i_1, i_2 \in [n]$. We write $A_n$ for the set of all strings of the form $xy(i_1, i_2)$ for any $i_1, i_2 \in [n]$. Since $A_n \subset NPal^\#$, we choose an index $t \in [d]$ for which $A_n \cap L_t$ has the maximum cardinality. For this $L_t$, we consider a 1dpda $M$ in an ideal shape recognizing it. Let $Q$ denote the set of all inner states of $M$. In addition, let $y(i_1, 0)$ denote $w_1^R(i_1)w_2^R(i_2)$ and let $y(i, i_0)$ denote $w_1^R(i_1)w_2^R(i_2)$. Since $n$ is sufficiently large, $M$ accepts at least $n/d$ strings of the form $xy(i_1, i_2)$ in $A_n \cap L_t$. Note that $n/d$ is sufficiently larger than the number of inner states of $M$. Let us consider two computations of $M$ on $xy(i_1, i_2)$ and $xy(i, i_0)$. Since $xy(i_1, i_2) \in L_t$ and $xy(i, i_0) \notin L_t$, for more than $|Q|$ values of $i_1$, $M$ must compare the $(n - i_1)$th bit of $w_1$ and the $i_1$th bit of $w_2^R(i_1)$. We claim that, in order to make such a comparison, $M$ must access the information associated with $w_1$, which is stored inside the stack while reading $w_1$. Assume that $M$ accesses only the information on $w_2$ stored in the stack during the process of $w_2^R(i_1)$. In this case, we can deterministically "generate" $w_1$ from the information on $w_2$, $t_1$, and $Q$. It then follows that $K(w_1 \mid w_2)$ is bounded from above by a certain constant, independent of $n$. This is absurd because of $K(w_1 \mid w_2) \geq n/2$. Since $M$ is in an ideal shape and $M$ cannot remember all the information on $w_1$ in its inner states, to access it, $M$ must pop all (except for finitely many) stack symbols stored on top of the information on $w_2$. Similarly, since $xy(i_1, 0) \notin L_t$ for a large number of $i_2$'s, the same reasoning supports that $M$ must access the stored information that is generated while reading $w_2$. This is impossible because such information has been already removed during the comparison between $w_1$ and $w_2^R(i_1)$. Therefore, $NPal^\#_2$ is not in DCFL[d].

This completes the proof.

Proposition 1.7 sharpens the hierarchy separation of Hibbard [7], which proved $k$-LDA $\neq (k+1)$-LDA for any $k \in \mathbb{N}^+$.

Proposition 1.7 (rephrased) For any $k \geq 2$, $k$-LDA $\cap DCFL[2^{k-1}] \subsetneq (k-1)$-LDA $\cup DCFL[2^{k-1}-1]$.

Proof. In this proof, we intend to use a series of special languages used in [22] arXiv version], each of which is in $k$-LDA but not in DCFL[$k - 1$] for each $k \geq 2$. Let us recall from [22] arXiv version] the language $L_k$, which is a slight modification of Hibbard's [7] Section 4] original example language. As shown in [22] arXiv version], $L_k$ belongs to $k$-LDA but it is not in $(k-1)$-LDA. It suffices to show by induction that $L_k \in DCFL[2^{k-1}]$ and $L_k \notin DCFL[2^{k-1}-1]$.

Let $k \geq 2$ be any integer. For each fixed index $i \in [k]$, we write $w_i$ for an arbitrary string of the form $a^{n_i}b^{m_i}c^{p_i}$ for any numbers $n_i, m_i, p_i \in \mathbb{N}$. We set $L_k$ to be composed of all strings $w$ of the form $w_2#w_3#\cdots#w_{k-1}#w_k#w_{k+1}#\cdots#w_3#w_2#w_1$ if $k$ is even, and $w_2#w_3#\cdots#w_{k-1}#w_k#w_{k+1}#\cdots#w_3#w_2#w_1$ if $k$ is odd, together with the following conditions (a)–(c).

(a) It holds that $n_1 \leq m_1$.

(b) Let $j$ denote any index in $[2, k][Z]$. If $n_j \leq m_j$, then either $n_{j-1} = m_{j-1}$ or $m_{j-1} = p_{j-1}$ holds. If $m_j \leq p_j$, then either $n_{j-1} < m_{j-1}$ or $m_{j-1} < p_{j-1}$ holds.

(c) Either $n_k = m_k$ or $m_k = p_k$ holds.

For instance, $L_2$ equals $\{a^{n_1}b^{m_1}c^{p_1} \mid (n_1 = m_1 \land n_2 = m_2) \lor (n_1 < m_1 \land m_2 = p_2)\}$. For convenience, we also define another language $L'_k$ obtained from $L_k$ by replacing the last term $a^{n_1}b^{m_1}c^{p_1}$ of $L_k$ with $a^{n_1}b^{m_1}c^{p_1}$. 10
(1) As the first step, we prove the membership of both \( L_k \) and \( L'_k \) to DCFL[\( 2^{k-1} \)].

(i) When \( k = 2 \), we define \( A_{(1)} = \{ a^{n_2} b^{m_2} c \# a^{n_1} b^{m_1} c \# | n_1 = m_1, n_2 = m_2 \} \) and \( A_{(2)} = \{ a^{n_2} b^{m_2} c \# a^{n_1} b^{m_1} c \# | n_1 < m_1, n_2 = m_2 \} \), where \( n_1, n_2, m_1, m_2, p_1, p_2 \in \mathbb{N} \). Clearly, \( L_2 \) equals \( A_{(1)} \cup A_{(2)} \), and thus \( L_2 \) belongs to DCFL[2]. Similarly, we can prove that \( L'_k \) is in DCFL[2].

(ii) Consider the case where \( k \) is an even number with \( k \geq 3 \). Given a string \( w \) of the form \( w_3 \# w_2 \# \cdots \# w_4 \# w_k \# w_{k-1} \# \cdots \# w_3 \), we succinctly write \( w^{(-)} \) for the string \( w_3 \# \cdots \# w_4 \# w_k \# w_{k-1} \# \cdots \# w_3 \) so that \( w = w_3 \# w^{(-)} \# w_1 \). Let \( A_{(1)} = \{ w | n_1 = m_1, n_2 = m_2, w^{(-)} \in L_{k-2} \} \), \( A_{(2)} = \{ w | n_1 = m_1, n_2 < m_2, w^{(-)} \in L_{k-2} \} \), \( A_{(3)} = \{ w | n_1 < m_1, m_2 = p_2, w^{(-)} \in L'_{k-2} \} \), \( A_{(4)} = \{ w | n_1 < m_1, m_2 < p_2, w^{(-)} \in L'_{k-2} \} \), where \( w \) has the above-mentioned form with \( n_1, n_2, m_1, m_2, p_1, p_2 \in \mathbb{N} \). The induction hypothesis of \( L_{k-2}, L'_{k-2} \in \text{DCFL}[2^{(k-2)-1}] \) implies that \( A_{(j)} \in \text{DCFL}[2^{(k-2)-1}] \) for every index \( j \in [4] \). Since \( L_k = A_{(1)} \cup A_{(2)} \cup A_{(3)} \cup A_{(4)} \), it instantly follows that \( L_k \in \text{DCFL}[2^{(k-2)-1}] = \text{DCFL}[2^{k-1}] \). In a similar way, we obtain \( L'_k \in \text{DCFL}[2^{k-1}] \) as well.

(iii) The case of odd \( k \geq 3 \) is treated similarly to (ii).

(2) As the second step, we prove that \( L_k \notin \text{DCFL}[2^{k-1} - 1] \). Assume, for the contrary that \( L_k \in \text{DCFL}[2^{k-1} - 1] \). We obtain the first pumping lemma for DCFL[\( 2^{k-1} - 1 \)] from which we obtain \( L_k \notin \text{DCFL}[1] \). Next, consider the case of \( k \geq 3 \). Toward a contradiction, we assume that \( L_k \in \text{DCFL}[2^{k-1} - 1] \). In the case where \( k \) is even, we first define the index set \( I_k = [1, 2, 3, 4]^k - 1 \times [1, 2] \) and choose \( s = s_2 s_3 \cdots s_k \) and \( s' = s_2' s_3' \cdots s_k' \) so that \( s \neq s' \). Let \( k \) be sufficiently large integers \( n_1, n_2, \ldots, n_k \), which are all different from each other. For each index \( j \in [k] \), we set \( w_j^{(1)} = a^s b^{n_j} c^{n_j-1} \), \( w_j^{(2)} = a^s b^{n_j+1} c^{n_j} \), \( w_j^{(3)} = a^s b^{n_j-1} c^{n_j-1} \), and \( w_j^{(4)} = a^s b^{n_j-1} c^n \). We then define \( \alpha_s = w_j^{(3)} \# w_j^{(4)} \# \cdots \# w_{k-2}^{(4)} \# w_k^{(4)} \# w_{k-1}^{(4)} \# \cdots \# w_1^{(4)} \).

For any \( s \in I_k \), let \( x = a^{n_2} \) and \( y \) be the string obtained from \( \alpha_s \) by replacing \( w_2^{(2)} \) with \( b^{n_2} c^{n_2} \). Regarding \( \alpha_s \), for any \( j \in [k] \), we set \( \alpha_s^{(j)} = w_2^{(2)} \# w_3^{(4)} \# \cdots \# w_{k-1}^{(4)} \# w_1^{(4)} \# \). Let us consider \( \alpha_s \) and \( \alpha_s' \). There are 6 cases to consider separately. Our proof strategy is that, in each such case, by choosing \( \hat{x}, \hat{y}, \hat{z} \) and \( z \), we define \( \hat{x}' = \alpha_s(\hat{x}) \), \( \hat{y} = \hat{y}' \), and \( \hat{z} = \hat{z}' \) to form \( \alpha_s = x'y \) and \( \alpha_s' = x'z \). We then apply the conditions (1) – (2) of the pumping lemma to draw a contradiction.

(i) If \((s_j, s'_j) = (1, 2)\), we set \( \hat{x} = a^n b^n, \hat{y} = c^{n_j}, \hat{z} = b c^{n_j} \) and \( \hat{z}' = a^n b^n \). (ii) If \((s_j, s'_j) = (3, 4)\), we set \( \hat{x} = a^{n_j} b^{n_j} c^{n_j-1}, \hat{y} = b^{n_j} c^{n_j}, \hat{z} = b^{n_j} c^{n_j-1} \). (iii) If \((s_j, s'_j) = (1, 3)\), we set \( \hat{x} = a^{n_j} b^{n_j} c^{n_j-1}, \hat{y} = b^{n_j} c^{n_j}, \hat{z} = b^{n_j} c^{n_j-1} \). (iv) If \((s_j, s'_j) = (1, 4)\), we set \( \hat{x} = a^{n_j} b^{n_j} c^{n_j-1}, \hat{y} = b^{n_j} c^{n_j}, \hat{z} = b^{n_j} c^{n_j-1} \). (v) If \((s_j, s'_j) = (2, 3)\), we set \( \hat{x} = a^{n_j} b^{n_j} c^{n_j-1}, \hat{y} = b^{n_j+1} c^{n_j}, \hat{z} = b^{n_j} c^{n_j-1} \). (vi) If \((s_j, s'_j) = (2, 4)\), we set \( \hat{x} = a^{n_j} b^{n_j}, \hat{y} = b^{n_j} c^{n_j}, \hat{z} = b^{n_j} c^{n_j-1} \).

Let us consider the case (i). Since \( \hat{y} \) contains only one \( b \), the matching symbol in \( x' \) is \( b^{n_j-1} \) to form an iterative pair of \( x'y \) for \( L_k \). However, since \( \hat{z} \) has no \( b \), there is no iterative pair of \( x'z \) for \( L_k \), a contradiction. Next, we consider the case (ii). Consider an iterative pair \((a, b)\) of \( x'y \), where \( a \) is in \( \hat{x} \) and \( b \) is in \( \hat{y} \). However, we cannot pump \( a \) in \( x'z \), a contradiction. For the other cases, we can use a similar argument to draw a contradiction. This completes the proof of the proposition.

\( \square \)

4 Proofs of the Pumping Lemmas for DCFL[d]

Throughout this section, we will provide the proofs of the pumping lemmas for DCFL[d] (Lemmas 1.1 and 1.2). In Section 4.1, in Section 4.2, we will explain fundamental notions needed for the proofs. The actual proofs will be given in Sections 4.3, 4.4, 4.5.

4.1 Boundaries, Turns, and \( \varepsilon \)-Enhanced Strings

Let us define two elementary notions of boundaries and boundary blocks, which are necessary to introduce other important notions. For this purpose, we visualize a single move of a 1dpda \( M \) as a series of three successive actions: (i) firstly replacing a topmost stack symbol, (ii) secondly updating an inner
A boundary is a borderline between two adjacent tape cells (except for the case of the leftmost tape cell). We index all such boundaries from 0 to |x| as follows. The boundary 0 is located at the left of cell 0 and boundary i + 1 is located between cell i and cell i + 1 for each i ≥ 0. We often use both “cell indices” and “boundaries” to specify an area of the input tape interchangeably. When a string xy is written in [xy] consecutive cells, the (x, y)-boundary refers to the boundary |x| + 1, which separates between x and y. A boundary block between two boundaries t1 and t2 with t1 < t2 is a consecutive series of boundaries between t1 and t2 (including t1 and t2). These numbers t1 and t2 are called the ends of this boundary block. For brevity, we write [t1, t2] to denote such a boundary block between t1 and t2. For each boundary block [t1, t2], its fringe is either the boundary t1 − 1 or the boundary t2 + 1 if they actually exist. The (t1, t2)-region indicates all the consecutive “cells” located in the boundary block [t1, t2]. When an input string x is written in this (t1, t2)-region, we conveniently call this region the x-region as long as the region is clear from the context. On the contrary, x is called the (t1, t2)-regional string.

The stack height of M at boundary t is the length of the stack content while the tape head is passing through this boundary t. Given t1, t2 ∈ N with t1 < t2, the boundary block [t1, t2] is flat if the stack height does not change in the (t1, t2)-region. A boundary block [t1, t2] is called convex if there is a boundary s between t1 and t2 (namely, s ∈ [t1, t2]) such that there is no pop operation in the (t1, s)-region and there is no push operation in the (s, t2)-region. A boundary block [t1, t2] is pseudo-convex if the stack height at any boundary s ∈ [t1, t2] does not go below h1 + \frac{h2 - h1}{t2 - t1}(s - t1), where h1 is the stack height at the boundary t1 for any index i ∈ {1, 2}. By their definitions, either convex or flat boundary blocks are pseudo-convex.

A peak (resp., a pit) is a boundary t for which the stack heights at the boundaries t − 1 and t + 1 are smaller (resp., greater) than the stack height at boundary t. A plateau is a boundary block [t, t′] such that any stack height at boundary i ∈ [t, t′] is the same. This stack height is called the height of the plateau. A basin (resp., an elevated plateau) [t, t′] is a plateau and the stack heights at all fringes of this plateau are greater (resp., smaller) than the height of the plateau whenever the fringes exist. A hill is a boundary block [t, t′] such that (i) the stack height at the boundary t and the stack height at the boundary t′ coincide, (ii) there exists either a peak or an elevated plateau, but not both, in [t, t′], (iii) there is no pit or basin in [t, t′], and (iv) one of the ends of the hill is either a pit or the left edge of a basin. The lowest stack height of the hill is particularly called the bottom height. The height of the hill is the difference between its bottom height and its highest stack height. We briefly call either a peak or an elevated plateau existing in a hill by a hill top. Those concepts are illustrated in Fig. 2.

An upward slope means a non-flat boundary block in which the stack height never decreases. In contrast, a downward slope is a non-flat boundary block in which no increase of the stack height occurs. A slope refers to either an upward slope or a downward slope.

A turn is intuitively a change of stack height from “nondecreasing” to “decreasing” discussed in 1966 by Ginsburg and Spanier 5 (who actually defined “turn” in a slightly different way). We describe this notion in terms of boundary blocks and use it to partition an entire stack history. A turning point is either a peak or the right edge of an elevated plateau. A turn refers to a boundary block [t1, t2] in which there is exactly one turning point such that the left fringe (if any) is a part of a downward slope and the right fringe is a part of either an upward slope or a basin. The height of a turn is the difference between the maximum stack height and the minimum stack height in the turn. The minimum stack height in particular occurs at either end of the turn and is also called the bottom height.

Given strings over an alphabet Σ, ε-enhanced strings are strings over the extended alphabet Σε.

Figure 2: A history of the changes of stack heights.
Figure 3: Turn Partitioning of a stack history. Each $x^{(i)}$ is the left part of a turn and $y^{(i)}$ is the right part of the same turn.

(= $\Sigma \cup \{\varepsilon\}$). Here, the notation $\varepsilon$ is treated as a special input symbol expressing the “absence” of symbols, not the “empty string.” Even for an $\varepsilon$-enhanced string $x$, the notation $|x|$ is used for the total number of non-$\varepsilon$-symbols in $x$. On the contrary, we use the new notation $|x|_\varepsilon$ to denote the length of $x$ by counting $\varepsilon$ as an independent symbol. For instance, if $x = 001\varepsilon0110\varepsilon10$, then $|x| = 9$ and $|x|_\varepsilon = 11$. An $\varepsilon$-enhanced 1dpda (or an $\varepsilon$-1dpda, for short) is a 1dpda that takes $\varepsilon$-enhanced input strings and works as a standard 1dpda except that a tape head always moves to the right without stopping. This tape head movement is sometimes called “real time.”

**Lemma 4.1** For any 1dpda $M$, there exists an $\varepsilon$-1dpda $N$ such that, for any input string $x$, there is an appropriate $\varepsilon$-enhanced string $\hat{x}$ for which $M$ accepts (resp., rejects) $x$ iff $N$ accepts (resp., rejects) $\hat{x}$. Moreover, $\hat{x}$ is uniquely determined from $x$ and $M$. In this case, $\hat{x}$ is said to be induced from $x$ by $M$. In addition, $N$ can be made to be in an ideal shape.

**Proof.** Let $M$ be any 1dpda. By Lemma 2.3 we can assume that $M$ is in an ideal shape. The desired machine $N$ is defined as follows. On any input, if $M$ makes a non-$\varepsilon$-move, then $N$ simulates this single step. On the contrary, in the case where $M$ makes an $\varepsilon$-move, if $N$ reads a symbol $\varepsilon$ on an input tape, then $N$ simulates $M$’s move and then moves its tape head to the right; otherwise, $N$ enters a rejecting state. This forces an input string to match $M$’s $\varepsilon$-moves and thus ensures the uniqueness of $\hat{x}$. We then define an $\varepsilon$-enhanced string $\hat{x}$ so that $N$ always reads $\varepsilon$ whenever $M$ makes an $\varepsilon$-move. By the above constructions of $N$ and $\hat{x}$, it is obvious that $N$ is also in an ideal shape. \qed

It is thus possible to partition an entire stack history into consecutive turns, as shown in Fig. 3. We call such a partition a turn partition.

### 4.2 State-Stack Pairs and Mutual Correlations

We next introduce another important notion of state-stack pairs. Let $M$ denote either a 1dpda or an $\varepsilon$-1dpda with a set $Q$ of inner states and a stack alphabet $\Gamma$, and assume that $M$ is in an ideal shape. A **state-stack pair** at boundary $i$ is a pair $(q, \gamma)$ of inner state $q$ and stack content $\gamma$ (which is expressed as $a_1a_2 \cdots a_k$ from the top to the bottom of the stack with $a_k = Z_0$), namely, $(q, \gamma) \in Q \times (\Gamma^*)^*Z_0$. In particular, $(q_0, Z_0)$ is the state-stack pair at the boundary 0. In a computation of $M$ on input $x$, a state-stack pair $(q, \gamma)$ at boundary $i$ with $i \geq 1$ refers to the machine’s current status where $M$ is reading a certain input symbol, say, $\sigma$ at cell $i-1$ in a certain state, say, $p$ with a certain stack content $a\gamma'$ for $a \in \Gamma$, and $M$ then changes its inner state to $q$, modifying $a$ by either pushing another symbol $b$ satisfying $\gamma = ba\gamma'$ or popping $a$ to make $\gamma = \gamma'$. It is important to remark that any computation of $M$ on $x$ can be expressed as a series of state-stack pairs at every boundary in the $\varepsilon$-$x$-$\delta$-region. Consider a sequence $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_m)$ in which each $\alpha_i$ is a state-stack pair at step $i$ for any number $i \in [0, m]_\mathbb{Z}$. Such a sequence completely describes a computation of $M$ if $\alpha_0 = (q_0, Z_0)$ and $M$ halts in exactly $m$ steps.

Two boundaries $t_1$ and $t_2$ with $t_1 < t_2$ are **mutually correlated** if (1) there are two state-stack pairs $(q, \gamma)$ and $(p, \gamma)$ at the boundaries $t_1$ and $t_2$, respectively, and (2) the boundary block $[t_1, t_2]$ is pseudo-convex. For four boundaries $t_1, t_2, t_3, t_4$ with $t_1 < t_2 < t_3 < t_4$, the two boundary blocks $[t_1, t_2]$ and $[t_3, t_4]$ are **mutually correlated** if (1’) $[t_1, t_2]$, $[t_2, t_3]$, and $[t_3, t_4]$ are all pseudo-convex, (2’) there are two state-stack pairs $(q, \gamma)$ and $(p, \alpha\gamma)$ at the boundaries $t_1$ and $t_2$, respectively, and (3’) there are two state-stack pairs $(s, \alpha\gamma)$ and $(r, \gamma)$ at the boundaries $t_3$ and $t_4$, respectively, for appropriately chosen $p, q, r, s \in Q, \gamma \in (\Gamma^*)^*Z_0$, and $\alpha \in (\Gamma^*)^*$. Mutually correlated boundaries or boundary blocks are closely related to the iteration of certain parts of strings. Furthermore, a pair $([t_1, t_2], [t_3, t_4])$ of boundary blocks with
t₁ < t₂ < t₃ < t₄ is called good if the two boundary blocks [t₁, t₂] and [t₃, t₄] are mutually correlated, inner states at the boundaries t₁ and t₂ coincide, and inner states at the boundaries t₃ and t₄ coincide.

**Lemma 4.2** Let M denote any ε-1dpda in an ideal shape and let w be any nonempty string.

1. Let t₁, t₂ ∈ N with 1 ≤ t₁ < t₂ ≤ |w| + 1. Let w = x₁x₂x₃ ∈ L(M) with |x₁| ≥ 1 be a factorization of string w such that t₁ is the (x₁, x₂)-boundary and t₂ is the (x₂, x₃)-boundary. If the boundaries t₁ and t₂ are mutually correlated and inner states at the boundaries t₁ and t₂ coincide, then x₁x₂x₃ ∈ L(M) holds for any number i ∈ N.

2. Let t₁, t₂, t₃, t₄ ∈ N with 1 ≤ t₁ < t₂ < t₃ < t₄ ≤ |w| + 1. Let w = x₁x₂x₃x₄x₅ such that each tᵢ is a (xᵢ, xᵢ₊₁)-boundary for each index i ∈ [4]. If the pair ([t₁, t₂], [t₃, t₄]) is good, then (x₂, x₄) is an iterative pair of w for L(M).

**Proof.** (1) Assume that 1 ≤ t₁ < t₂ ≤ |w| + 1. Since the boundaries t₁ and t₂ are mutually correlated, the boundary block [t₁, t₂] is pseudo-convex and there exist two state-stack pairs of the form (q, γ) and (p, γ) at the boundaries t₁ and t₂, respectively. By the premise of the lemma, we obtain p = q. Since [t₁, t₂] is pseudo-convex, any stack height in [t₁, t₂] does not go below |γ|. This makes it possible to repeat this boundary block [t₁, t₂] for an arbitrary number of times without affecting the rest of M’s computation; in particular, the outcomes of the computation does not alter. It therefore follows that x₁x₂x₃ ∈ L(M) iff x₁x₂x₃ ∈ L(M) for any number i ∈ N.

(2) Given any t₁, t₂, t₃, t₄ ∈ N with 1 ≤ t₁ < t₂ < t₃ < t₄ ≤ |w| + 1, since the boundary blocks [t₁, t₂] and [t₃, t₄] are mutually correlated, the boundary blocks [t₁, t₂], [t₂, t₃], and [t₃, t₄] are all pseudo-convex and there are state-stack pairs (q, γ) and (p, αγ) at the boundaries t₁ and t₂ and also (s, αγ) and (r, γ) at the boundaries t₃ and t₄ for appropriate p, q, s, r, α, γ. The premise of the lemma then requires that p = q and r = s. The pseudo-convexity of [t₁, t₂], [t₂, t₃], and [t₃, t₄] makes it possible to repeat [t₁, t₂] and [t₃, t₄] for the same number of times without tampering the outcomes of M. Thus, (x₂, x₄) is an iterative pair of w for L(M).

To support the proof of Lemma 4.2, we present another useful lemma. For readability, we abbreviate a concatenation α₁α₂...αᵢ of i strings as αᵢ<sub>1</sub>...αᵢ<sub>i</sub>.

**Lemma 4.3** Let M be any 1dpda with Q and Γ. We fix an input string w and a computation of M on w. Let p, q, p₁, q₁ ∈ Q for all i ∈ [m], γ ∈ (Γ<sup>-1</sup>)<sup>*</sup>Z₀, and αᵢ ∈ (Γ<sup>-1</sup>)<sup>*</sup> for any index i ∈ [m]. Assume that there are boundaries t₁, t₂, ..., tₘ with 1 ≤ t₁ < t₂ < ... < tₘ for which (q, γ), (q, α₁γ), ..., (q, αₘ₋₁γ), (q, αₘ₋₁γ) are state-stack pairs at the boundaries t₁, t₂, ..., tₘ, respectively, and that there are boundaries r₁, r₂, ..., rₘ with tₘ ≤ r₁ < r₂ < ... < rₘ for which (p₁, αₘ₋₁γ), (p₂, αₘ₋₂γ), ..., (pₘ₋₁, αγ), (pₘ, γ) are state-stack pairs at the boundaries r₁, r₂, ..., rₘ, respectively. Moreover, each pair of boundary blocks [tᵢ, tᵢ₊₁] and [rᵢ, rᵢ₊₁] are pseudo-convex for each index i ∈ [m − 1].

1. If m > |Q|, then there exist two distinct indices i, j ∈ [m] satisfying pᵢ = pⱼ.
2. If m > |Q|^2, then there exists a subset C of [m] of size more than |Q| such that pᵢ = pⱼ holds for any pair i, j ∈ C.

**Proof.** (1) Consider the set P = {p₁, p₂, ..., pₘ} of inner states. Since P ⊆ Q, it follows that |P| ≤ |Q| < m. Thus, there is a distinct pair i, j ∈ [m] satisfying pᵢ = pⱼ by the pigeonhole principle.

(2) Assume that there is no such subset C stated in the lemma. For each inner state q ∈ Q, we introduce Cₕ = {i ∈ [m] | pᵢ = q}. Since |Cₕ| ≤ |Q| for all q ∈ Q, it follows that m = |∪ₕ∈Q Cₕ| = Σₕ∈Q |Cₕ| ≤ |Q|^2. This is a contradiction against m > |Q|^2.

### 4.3 Stack-Operational Pairs

It is known that, for a context-free language L, a structural property given by an iterative pair for L is closely related to a generative property of a “grammar” generating L. This latter notion is described in terms of grammatical pairs in, e.g., [H] Section VIII.2. In connection to nondegenerate iterative pairs for dcf languages, we introduce another notion of *stack-operational pairs* to describe a structural property given by the stack behaviors of 1dpda’s.

Let us consider a dcf language L and a 1dpda M = (Q, Σ, {t, s}, Γ, δ, q₀, Q<sub>acc</sub>, Q<sub>rej</sub>) recognizing L. We further assume that M is in an ideal shape and consider a string w ∈ Σ<sup>*</sup> and its factorization u₁v₁wᵢxᵢyᵢz. We say that (x, y) is a *stack-operational pair w.r.t.* M if the pair ([t₁, t₂], [t₃, t₄]) of boundary blocks is good,
where \( t_1, t_2, t_3, \) and \( t_4 \) represent the \((u,x)\)-boundary, the \((x,v)\)-boundary, the \((u,y)\)-boundary, and the \((y,z)\)-boundary, respectively.

Next, we consider two factorizations \( uxvyz \) and \( u'x'v'y'z' \). We say that \( u'x'v'y'z' \) is deduced from \( uxvyz \) if (1) there exists a number \( n_0 \in \mathbb{N}^+ \) such that \( u'x'v'y'z' = u_{x^n}v_{y^n}z' \) and (2) there are two numbers \( p, q \in \mathbb{N} \) and four strings \( x_1, x_2, y_1, y_2 \) such that \( x = x_1x_2, y = y_1y_2, x' = (x_2x_1)^p, y' = (y_2y_1)^q, u' \in u_{x^n}, v' \in x_2x^n y^n y_1, \) and \( z' \in y_2y^n z \). It instantly follows that \( u'(x')^{k+1}v'(y')^{l+1} = u_{x^n}v_{y^n}z' \) for any pair \( k,l \) in \( \mathbb{N} \). With no proof, we also claim that, if \((x,y)\) is a nondegenerate iterative pair of \( uxvyz \) for \( L(M) \) and \( u'x'v'y'z' \) is deduced from \( uxvyz \), then \((x', y')\) is also a nondegenerate iterative pair of \( u'x'v'y'z' \) for \( L(M) \).

By Lemma 4.3.2, any stack-operational pair is also an iterative pair. The converse in general does not hold. However, the deducibility further establishes a close relationship between iterative pairs and stack-operational pairs. This relation makes it possible to translate nondegenerate iterative pairs to nondegenerate stack-operational pairs.

**Proposition 4.4** Let \( L \) be DCFL and let \( M \) denote any \( 1 \text{dpda} \) in an ideal shape recognizing \( L \). Given a factorization \( w = uxvyz \), assume that \((x,y)\) is a nondegenerate iterative pair of \( w \) for \( L \). There exist strings \( \bar{u}, \bar{x}, \bar{v}, \) and \( z \) such that (1) \( \bar{u}x\bar{v}y\bar{z} \) is deduced from \( uxvyz \) and (2) \((\bar{x},\bar{y})\) is a nondegenerate stack-operational pair w.r.t. \( M \).

In what follows, we try to prove Proposition 4.4 by splitting its proof into three parts, each of which is discussed separately as Lemmas 4.5, 4.6. We will follow in part a basic proof strategy of Rubtsov [10].

As a starting point, let \( M \) denote any def language and a 1 dpda \( M \) in an ideal shape satisfying \( L = L(M) \). Moreover, let us fix an arbitrary input \( w \) of the form \( uxvyz \) and assume that \((x,y)\) is a nondegenerate iterative pair of \( w \) for \( L \). We will analyze the stack history created by \( M \) on an input of the form \( ux^ny^pz \) for any pair \( k,l \) in \( \mathbb{N} \).

As a succinct notation, for any input string \( w \) and for any two state-stack pairs \((q,\gamma)\) and \((q',\gamma')\) in a stack history of \( M \) on \( w \), we briefly write \( (q,\gamma) \rightarrow_w (q',\gamma') \) if \((q',\gamma')\) is located at the left boundary of \( w \) and \((q',\gamma')\) is at the right boundary of \( w \).

**Lemma 4.5** Consider the stack history of \( M \) on an input of the form \( u x^n \) for any sufficiently large number \( n \) in \( \mathbb{N}^+ \). There exist a constant \( d \geq 1 \), a prefix \( x' \) of \( x^n \), and three fixed strings \( \alpha, \beta, \gamma \in (\Gamma^{-1})^* \), \( \gamma \in (\Gamma^{-1})^* Z \) (not depending on \( n \)) such that, for any \( s \in \mathbb{N} \), if \( |x'| |x^n| \leq |x^n| \), then \((q_0, Z_0) \rightarrow_{u x'} (q, \alpha \gamma)\) and \((q, \alpha \beta \gamma) \rightarrow_{x'} (q, \alpha \beta \gamma' + 1)\).

**Proof.** We focus on the computation of \( M \) on an input \( u x^n \) for any sufficiently large number \( n \in \mathbb{N}^+ \). Let \((q_n, \gamma_n)\) denote a state-stack pair of \( M \) at the \((u x^n, x)\)-boundary. We consider two cases below.

**Case 1:** Consider the case where there exists a constant \( c \in \mathbb{N}^+ \) such that \( |\gamma_n| \leq c \) holds for infinitely many numbers \( n \in \mathbb{N}^+ \). Let us consider all state-stack pairs \((q_n, \gamma_n)\). For any \( c \in \mathbb{N} \), we define \( \Gamma_c \) as the set of all stack contents of length at most \( c \), namely, \( \Gamma_c = \{ \gamma \in (\Gamma^{-1})^* Z | |\gamma| \leq c \}. \) Since \( Q \times \Gamma_c \) is a finite set, there exist a pair \( n_1, n_2 \in \mathbb{N}^+ \) with \( n_1 < n_2 \) such that \((q_n, \gamma_n)\) coincides with \((q_{n_2}, \gamma_{n_2})\). Let \( x' = x^n \) and \( d = n_2 - n_1 \). Then we choose three strings \( \alpha, \beta, \gamma \) such that \( \gamma_{n_1} = \alpha \gamma_{n_2} = \alpha \beta \gamma \) so that \( M \) never accesses \( \gamma \) while reading symbols in the boundary block \( ||ux'|, |ux^n|| \). Then obtain \((q_0, Z_0) \rightarrow_{ux'} (q, \alpha \gamma)\) and \((q, \alpha \beta \gamma) \rightarrow_{x'} (q, \alpha \beta \gamma + 1)\) for any number \( i \) in \( \mathbb{N} \) by the choice of \( \gamma \).

**Case 2:** Consider the case where, for any number \( c \in \mathbb{N}^+ \), \( |\gamma_n| > c \) holds for all but finitely many numbers \( n \in \mathbb{N}^+ \). For each value \( c \), we set \( g(c) = \min\{n' \in \mathbb{N}^+ | \forall n \geq n'[|\gamma_n| > c] \}. \) Note that \( g(c) \leq (c + 1) \) holds for all \( c \in \mathbb{N} \) and that \( |\gamma_{g(c)-1}| \leq c < |\gamma_{g(c)}| \). Since \( M \) is in an ideal shape, it follows that \( 0 < |\gamma_{g(c)}| - |\gamma_{g(c)-1}| \leq |x| \). We consider the corresponding state-stack pair \((q, \gamma)\) at \((u x^{g(c)-1}, x)\)-boundary. Note that, for any \( n \geq g(c) \), \( M \) accesses no more than the top \( |x'\) stack symbols of \( \gamma \). Since \( Q \times \{ \alpha' \in (\Gamma^{-1})^* \} \) is finite, there exist a pair \( c_1, c_2 \in \mathbb{N}^+ \) with \( g(c_1) < g(c_2) \), an inner state \( q \in Q \), and three strings \( \alpha, \beta, \gamma \) such that \((q, \alpha \gamma)\) is a state-stack pair at the \((u x^{g(c_1)-1}, x)\)-boundary and \((q, \alpha \beta \gamma)\) is a state-stack pair at the \((u x^{g(c_2)-1}, x)\)-boundary. We set \( x' = x^{g(c_1)-1} \) and \( d = g(c_2) - g(c_1) \). This implies that, for any \( s \in \mathbb{N} \), \((q, \alpha \beta \gamma)\) is the state-stack pair at the \((u x^s, x)\)-boundary. We then obtain \((q_0, Z_0) \rightarrow_{ux'} (q, \alpha \gamma)\) and \((q, \alpha \beta \gamma) \rightarrow_{x'} (q, \alpha \beta \gamma + 1)\) for any number \( i \in \mathbb{N} \).

Next, we consider the string \( v \), which has the form \( v_1v_2 \cdots v_k \) with \( k \) symbols. We concentrate on an input of the form \( ux^n v \) given to \( M \) for any \( n \geq 1 \). For each number \( n \geq 1 \), in what follows, we consider an \( e \)-enhanced string of \( v \) induced by \( M \) on \( ux^n \). We write this \( e \)-enhanced string as \( \tilde{v}_n \) for clarity. This \( \tilde{v}_n \) can be described as \( \tilde{v}_n = v_1 \varepsilon v_2 \varepsilon v_3 \varepsilon \cdots v_k \varepsilon v_{k+1} \) for certain numbers \( n_1, n_2, \ldots, n_k+1 \) in \( \mathbb{N} \).
Lemma 4.6 Assume that $M$'s state-stack pair at the $(ux^n, v)$-boundary has the form $(q, \alpha \beta^s \gamma)$. We assume that $s$ is sufficiently large. If $(q, \alpha \beta^s \gamma) \rightarrow_q \xi_n$, then either (i) there exists a string $\alpha'$ with $|\alpha'| \leq |\alpha| + |v|$ satisfying $\xi_n = \alpha' \beta^s \gamma$ or (ii) there exists a string $\alpha''$ with $|\alpha''| \leq |Q||\beta| + |\alpha|$ and a positive constant $a \leq 2(|v| + 1)|Q||\beta|$ for which $\xi_n = \alpha'' \beta^{s-a} \gamma$.

Proof. From the premise of the lemma, we set $c = |Q||\beta|$ and consider two cases.

Case 1: Consider the case where $M$ does not access any symbol in $\beta^s$ while reading $\overline{v}$. In this case, $M$ produces $\alpha' \beta^s \gamma$ for a certain $\alpha'$ of size at most $|\alpha| + |v|$. The last term $|v|$ is needed because $M$ may push some symbols into the stack, as it reads symbols in $v$.

Case 2: Consider the case where $M$ accesses $\beta^s$ while reading $\overline{v}$. In this case, we want to claim that $n_i \leq c$ holds for all $i \in [k+1]$. To show this claim, we assume otherwise and we take an index $i \in [k+1]$ ensuring $n_i > c$. Consider the set $Q \times [|\beta|] \times \times N$. We say that $M$ admits $(q, m, r)$ in this set if $M$ is in inner state $q$, reading the $r$th symbol of $\beta$ at time $r$. Consider a quintuple $(q, m, r_1, r_2)$ satisfying that $r_1 < r_2$ and $M$ admits $(q, m, r_1)$ and $(q, m, r_2)$ while reading $\epsilon^{|m|}$. This implies that $M$ can repeat the behavior made during the period between $r_1$ and $r_2$. This is a contradiction against the nondegeneracy of $(x, y)$. Therefore, we obtain $n_i \leq c$ for any $i \in [k+1]$. It then follows that $|\overline{v}| \leq (k + 1)c + |v| \leq 2(|v| + 1)c$. Finally, we set $a = 2(|v| + 1)c$. Clearly, after reading $v$, the stack has the form $\alpha'' \beta^{s-a} \gamma$ for a certain string $\alpha''$. Note that $|\alpha''|$ is upper-bounded by $c + |\alpha|$.

Finally, we analyze the stack behavior of $M$ on inputs of the form $ux^kyvy^l$.

Lemma 4.7 Let $k, l \in \mathbb{N}$ be arbitrary numbers. Assume that $M$'s state-stack pair at the $(ux^kyvy^l)$-boundary is of the form $(q, \xi_k \beta^s \gamma)$ for strings $\xi_k, \beta \in (\Gamma^*)^s$ and $\gamma \in (\Gamma^*)^s Z_0$, where $|\xi_k| \leq c$ for a certain constant $c > 0$, which is independent of the choice of $(k, l)$. There exist a constant $c \in \mathbb{N}^+$, a prefix $y'$ of $y$, a string $\alpha \in (\Gamma^*)^s$, and two numbers $t_1, t_2 \geq 1$ such that, for any number $l \in \mathbb{N}$, if $t_1 + t_2 \leq s$, then $(q, \xi_k \beta^s \gamma) \rightarrow_{y'} (q', \alpha' \beta^{s-t_1} \gamma)$ and $(q', \alpha' \beta^{s-t_1} \gamma) \rightarrow_{y''} (q'', \alpha' \beta^{s-t_1-l(t_1 + t_2)} \gamma)$.

Proof. We fix $k, l$ arbitrarily and discuss how the stack content of $M$ changes while reading $y'$ first. Since Cases 1–2 both lead to contradictions, we write $\gamma'$ for the stack content obtained after reading $ux^kyvy^l$. In particular, $\gamma'_0$ is the stack content obtained after reading $ux^kyv$. Take the smallest number $l_0 \in \mathbb{N}$ such that $|\gamma'_0| \geq |\gamma'_0|$ for all numbers $l \in \mathbb{N}$.

Firstly, we claim the existence of two constants $f \in \mathbb{N}$ and $l_f \in \mathbb{N}^+$ satisfying $l_0 \geq l - f$ for any $l \geq l_f$. To lead to a contradiction, we assume that, for any $f \in \mathbb{N}$ and any $l_f \in \mathbb{N}^+$, there exists a number $l$ (and thus $l_0$) satisfying (*) $l \geq l_f$ and $l_0 < l - f$. Let us consider the following two cases.

Case 1: Assume that there exists a constant $c > 0$ such that $|\gamma'_0| \leq |\gamma'_0| \leq |\gamma'_0| + c$ for infinitely many $l \geq l_0$. We take a sufficiently large number $f$ and fix a number $l$ satisfying (*). Notice that $l_0 - l_f$ is sufficiently larger than $c$. For such an $l$, a state-stack pair at the boundary $l$ can be expressed as $(q, \alpha \gamma)$ for appropriate elements $q, \alpha \gamma$ for which $\gamma = \gamma'_0$ and $\gamma'_0 = \alpha \gamma$. Clearly, $|\alpha| \leq c$ follows. Since $\{\alpha \in (\Gamma^*)^s | |\alpha| \leq c\}$ is a finite set, there exist elements $\alpha \in \mathbb{N}^+$, $q' \in Q$, and $\alpha'$ satisfying that $(q', \alpha \gamma') \rightarrow_{y''} (q'', \alpha' \gamma')$. Since this last computation can be repeated an arbitrary number of times, we obtain a contradiction against the nondegeneracy of $(x, y)$.

Case 2: Assume that Case 1 fails. For any number $c > 0$, there exists a number $l' \geq l_0$ for which $|\gamma'_0| > |\gamma'_0| + c$. A similar argument used in the proof of Lemma 4.5 shows the existence of a periodic repetition. This makes $ux^kyvy^lz$ true for infinitely many numbers $n$, a contradiction against the nondegeneracy of $(x, y)$.

Since Cases 1–2 both lead to contradictions, we conclude that, for two fixed constants $f \in \mathbb{N}$ and $l_f \in \mathbb{N}^+$, $l_0 \geq l - f$ holds for any $l \geq l_f$. It then follows that, for any number $c > 0$, there exists a number $l_c \geq l_0$ satisfying $|\gamma'_0| \leq |\gamma'_0| < |\xi_k \beta^s \gamma| - c$ for any $l \geq l_c$. We write $(q_i, \gamma_i)$ to denote the state-stack pair at the $(ux^kyvy^l)$-boundary. Given number $c > 0$, we define $h(c) = \min\{l_c \in \mathbb{N} | l_c \geq l_0, \forall l \geq l_c, |\gamma'_0| \leq |\gamma'_0| < |\xi_k \beta^s \gamma| - c\}$. Notice that $|\gamma'_0| < |\xi_k \beta^s \gamma| - c \leq |\gamma'_0|$. Let $c_0 = |Q||\beta||y|$. We claim that $0 < |\gamma'_0| - |\gamma'_0| \leq c_0$. This claim is shown similarly to the proof of Lemma 4.5 as follows. Remember that $M$ may make a series of consecutive $\epsilon$-moves after reading each symbol of $y'$. Concerning $\beta$, $M$ pops only a prefix of $\beta^m$ in sequence for a certain number $n$. Consider the set $Q \times [|\beta|] \times \times N$, each element $(q, m, r)$ of which indicates that $M$ is in inner state $q$ and reads the $m$th symbol of $\beta$ at time $r$. Recall the notion of “admit” from the proof of Lemma 4.5. If $|\gamma'_0| > |\gamma'_0| + c_0$, then there exist elements $\gamma \\ \gamma \in Q \in [|\beta|]$, and $r_1, r_2 \in \mathbb{N}$ with $r_1 < r_2$ such that $M$ admits both $(q, m, r_1)$ and $(q, m, r_2)$ because of $|Q||\beta| < c_0$. If $M$ repeats the same computation made during the time period between $r_1$ and $r_2$, $M$'s stack goes down below $|\gamma'_0|$. Therefore,
Let $\Gamma_0 = \{ \alpha \in (\Gamma^-)^* \mid |\alpha| \leq c_0 \}$. Since $Q \times \Gamma_0$ is a finite set, there exist a distinct pair $c_1, c_2 \in \mathbb{N}^+$ with $b(c_1) < b(c_2)$ and a string $\alpha' \in (\Gamma^-)^*$ with $|\alpha'| \leq |y'|$ such that $(q, \xi_k \beta^* \gamma) \rightarrow_{y'} (q', \alpha' \beta^* \gamma)$ and $(q', \alpha' \beta^* \gamma) \rightarrow_{y'} (q'', \alpha' \beta^* \gamma)$ for any $i \in \mathbb{N}$, where $e = b(c_2) - b(c_1)$ and $y'' = y^{b(c_1)} - 1$.

The proof follows similarly.}

**Proof of Proposition 4.4.** Assume that $(x, y)$ is a nondegenerate iterative pair of $uxvyz$ for $L$. We fix sufficiently large numbers $k$ and $l$ that satisfy Lemmas 4.5, 4.7, and we then consider the string $u^kxy^l$. Let $(\bar{x}, \bar{y})$ be an $\epsilon$-enhanced string. For readability, we continue using the same notations. Hence, $\bar{x} = x^k$ and $\bar{y} = y^l$. We then obtain $\bar{x} \bar{y} \bar{y} = u^k x^k y^l y^l z = u^k y^l y^l z$ for certain numbers $k', l' \in \mathbb{N}$. Thus, $(x, y)$ is degenerate, a contradiction.

**Case 2:** Consider the case where for any $l \in \mathbb{N}$, there are infinitely many $k \in \mathbb{N}$ satisfying $\bar{x} \bar{y} \bar{y} \in L$. It then follows that $\bar{x} \bar{y} \bar{y} \bar{y} z = u^k (x^k)^k y^l (y^l)^k y^l z = u^k y^l y^l z$ for certain numbers $k', l' \in \mathbb{N}$. Thus, $(x, y)$ is degenerate, a contradiction.

**Case 2:** Consider the case where for any $l \in \mathbb{N}$, there are infinitely many $k \in \mathbb{N}$ for which $\bar{x} \bar{y} \bar{y} \bar{y} \bar{y} \in L$. This case can be handled similarly to Case 1.

**4.4 Proof of the First Pumping Lemma for DCFL[d]**

We are now ready to describe the proof of the first pumping lemma for DCFL[d] (Lemma 4.1). Our proof has two distinguished parts depending on the value of $d$. The first part of the proof targets the basis case of $d = 1$. As noted in Section 1.2, this special case directly corresponds to Yu's pumping lemma [2] Lemma 1. To prove his lemma, Yu utilized a so-called left-part theorem of his own for LR(k) grammars. As a caviary of this exposition, we intend to re-prove the lemma using 1dpda's with no reference to LR(k) grammars.

Let $\Sigma$ be any alphabet and take any infinite dcf language $L$ over $\Sigma$ in DCFL[d]. Our proof argument is easily extendable to one-way nondeterministic pushdown automata (or 1dpda's) and to the "standard" pumping lemma for CFL (cf. [3]). The second part of the proof deals with the general case of $d \geq 2$. Hereafter, we discuss this two parts separately.

**Basis Case of $d = 1$:** We begin our proof for the basis case of $d = 1$. By Lemma 4.1, we take an appropriate $\epsilon$-1dpda $M = (Q, \Sigma, \{ \{ \}, \{ \} \}, \Gamma, \delta, q_0, Z_0, Q_{acc}, Q_{rej})$ in an ideal shape that recognizes $L$. For the desired constant $c$, we set $c = 2^{|Q|^6}$. Firstly, we take two arbitrary strings $w_1 = xy$ and $w_2 = xz$ in $L$ over $\Sigma$ with $|y|, |z| \geq 1$ and $|x| > c$ such that $w_1$ and $w_2$ have iterative pairs for $L$. Following the use of an $\epsilon$-enhanced machine, we naturally expand these given input strings to their corresponding $\epsilon$-enhanced strings. For readability, we continue using the same notations $w_1, w_2, x, y, z$ to denote the corresponding $\epsilon$-enhanced strings.

We assume that the condition $(1)$ fails. Our goal is then to verify that the condition $(2)$ of the lemma indeed holds. There are four specific cases to deal with. Hereafter, we intend to discuss them separately. Remember that, since $M$'s tape head moves in only one direction, every state-stack pair at each boundary in the $x$-region does not depend on the choice of $y$ and $z$.

**Case 1:** Consider the case where there are two boundaries $t_1, t_2$ with $1 \leq t_1 < t_2 \leq |x|$ and $|t_2 - t_1| \leq c$ such that (i) the boundaries $t_1$ and $t_2$ are mutually correlated and (ii) inner states at the boundaries $t_1$ and $t_2$ coincide. In this case, we can factorize $x$ into $x_1 x_2 x_3$ so that $t_1$ is the $(x_1, x_2 x_3)$-boundary and $t_2$ is the $(x_2, x_3)$-boundary. By Lemma 4.1, it then follows that $x_1 x_2 x_3 y \in L$ and $x_1 x_2 x_3 z \in L$ for any number $i \in \mathbb{N}$. Thus, the condition (1) holds. This is a contradiction against our assumption.

**Case 2:** Consider the case where there are four boundaries $t_1, t_2, t_3, t_4$ with $1 \leq t_1 < t_2 < t_3 < t_4 \leq |x|$ and $|t_4 - t_1| \leq c$, and there exist numbers $p, q \in \{ \gamma \}$ such that (i) $(q, \gamma)$ and $(q, \alpha')$ are state-stack pairs at the boundaries $t_1$ and $t_2$, respectively, (ii) $(p, \gamma)$ and $(p, \alpha')$ are state-stack pairs at the boundaries $t_3$ and $t_4$, respectively, and (iii) $t_i, t_{i+1}$ for every index $i \in [3]$ is
pseudo-convex. We then factorize \( x \) into \( x_1x_2x_3x_4x_5 \) so that \( t_i \) is the \((x_i,x_{i+1})\)-boundary for each index \( i \in [4] \). Note that \( |x_2| \geq 2 \) because of \( t_1 \leq t_2 \) and \( t_3 \leq t_4 \). Since \( M \) is in an ideal shape, \( |x_2| \geq 1 \) also follows. By an application of Lemma 12.2(2), it then follows that \((x_2,x_4)\) is an iterative pair of both \( y \) and \( xz \) for \( L \). This also contradicts our assumption.

Case 3: For convenience, we set \( R = ([x] - c, [x]) \). Consider the simple case where there is no pop operation in the \( R \)-region. There exist a number \( m \geq 1 \) and a series of \( m \) boundaries \( s_1, s_2, \ldots, s_m \) in the \( R \)-region with \( 1 \leq s_1 < s_2 < \cdots < s_m \) such that, for appropriately chosen elements \( q \in Q, \gamma \in (\Gamma^-)^*Z_0 \), and \( \alpha_1, \ldots, \alpha_{m-1} \in (\Gamma^+)^* \), there are state-stack pairs of the form \((q,\gamma), (q,\alpha_1), \ldots, (q,\alpha_{m-1})\gamma)\) at the boundaries \( s_1, s_2, \ldots, s_m \) respectively, where \( \alpha_1 \) is the abbreviation of \( \alpha_1 \alpha_2 \cdots \alpha_{m-1} \) introduced in Section 12.

Note that the boundary blocks \([s_1,s_2],[s_2,s_3], \ldots, [s_{m-1},s_m] \) are all convex. We maximize the value of \( m \) and we hereafter assume that \( m \) denotes this maximum value. Notice that \( R \)-region contains more than \(|Q|^3\) boundaries because of \(|Q|^3 < c \). If, for each \( q \in Q \), the corresponding value \( m_q \) is at most \(|Q|^2 \), then the size \( c \) of the \( R \)-region is at most \(|Q|^3 \). This is a contradiction, and thus we conclude that \( m > |Q|^2 \).

Next, we choose \( \{t_i \}_{i \in [m]} \) and \( \{r_i \}_{i \in [m]} \) so that \( t_i \) and \( r_i \) are respectively boundaries in the \( y \)-region and the \( z \)-region satisfying that \((a) s_m \leq t_1 < t_2 < \cdots < t_m \) and \( s_m \leq r_1 < r_2 < \cdots < r_m \) and \((b) [s_1,s_{i+1}] \) is mutually correlated to \([t_{m-i},t_{m-i+1}]\) in the \( y \)-region and also to \([r_{m-i},r_{m-i+1}]\) in the \( z \)-region for any index \( i \in [m-1] \). This is possible because \( M \) is \( \varepsilon \)-enhanced and in an ideal shape. Notice that the boundary blocks \([t_1,t_2],[t_1,t_3],[t_1,t_4],[r_1,r_2],[r_1,r_3],[r_1,r_4] \) are all pseudo-convex. Assume that, at boundaries \( t_1, t_2, \ldots, t_m \), the associated state-stack pairs are respectively of the form \((p_1,\gamma_1),(p_2,\gamma_2),(p_3,\gamma_3), \ldots, (p_m,\gamma)\) for certain inner states \( p_1, p_2, \ldots, p_m \in Q \). Similarly, assume that \((e_1,\gamma_1),(e_2,\gamma_2),(e_3,\gamma_3), \ldots, (e_m,\gamma)\) are state-stack pairs at the boundaries \( r_1, r_2, r_3, r_4 \), respectively.

By Lemma 13.3(2), since \( m > |Q|^2 \), there is a subset \( C \subseteq [m] \) of size more than \(|Q| \) such that, for any index pair \( i,j \in C \), the inner states at the boundaries \( t_i \) and \( t_j \) coincide; namely, \( p_i = p_j \). From this fact, it follows that there is a special pair \( j_1, j_2 \in C \) with \( j_1 < j_2 \) such that the inner states at the boundaries \( t_{j_1} \) and \( t_{j_2} \) coincide; that is, \( e_{j_1} = e_{j_2} \). We fix such a pair \((j_1,j_2)\). We then factorize the strings \( x, y \) and \( z \) as \( x = x_1x_2x_3, \ y = y_1y_2y_3 \), and \( z = z_1z_2z_3 \) so that \( s_{j_1} \) is an \((x_1,x_2)\)-boundary, \( s_{j_2} \) is the \((x_2,x_3)\)-boundary, \( t_{j_1} \) is the \((y_1,y_2)\)-boundary, \( t_{j_2} \) is the \((y_2,y_3)\)-boundary, \( r_{j_1} \) is the \((z_1,z_2)\)-boundary, and \( r_{j_2} \) is the \((z_2,z_3)\)-boundary. It then follows that \((x_1,y_2)\) and \((x_2,y_2)\) are respectively iterative pairs of \( xz \) and \( x \) for \( L \). Therefore, the condition (2) follows.

Case 4: Assume that Case 3 fails. Hence, at least one pop operation must take place in the \( R \)-region. We will proceed our proof by taking the following steps (1)–(5). In what follows, we focus our attention only on the \( R \)-region.

(1) We first claim that any flat boundary block has size at most \(|Q| \). This is because if a flat boundary block has size more than \(|Q| \), then Case 1 occurs, a contradiction against our assumption. Therefore, the claim is true. This implies that there are at least \( c/(|Q|+1) \) operations of “pop” and “push” in the \( R \)-region.

(2) Choose two boundaries \( s, s' \) such that \([x] - c \leq s < s' \leq [x] \), the boundary block \([s,s']\) is pseudo-convex, and \([s,s']\) can be partitioned into a number of turns. Such a turning partition is illustrated in Fig. 3. Hereafter, we consider the specific cases (a)–(b).

(a) We first consider the case where \([s,s']\) is a single turn. We focus on a hill, say, \([t_1,t_2] \) that is contained in the turn and claim that this hill has height at most \(|Q|^2 \). This is shown as follows. If the hill has height more than \(|Q|^2 \), then there exists an inner state \( q \in Q \) that appears at least \(|Q| + 1 \) state-stack pairs in the left part of the hill. Let \((q,\gamma),(q,\alpha_1), \ldots, (q,\alpha_{|Q|+1})\gamma)\) denote those state-stack pairs for appropriate strings \( \gamma, \alpha_1, \ldots, \alpha_{|Q|+1} \). Associated with them, we also choose state-stack pairs \((p_1,\alpha_1,\gamma),(p_2,\alpha_2,\gamma),(p_3,\alpha_3,\gamma),(p_{|Q|+1},\gamma)\) in the right part of the hill for certain inner states \( p_1, \ldots, p_{|Q|+1} \). Lemma 14.2(1) then guarantees the existence of a distinct pair \( i,j \in [|Q|+1] \) satisfying \( p_i = p_j \). This leads to Case 2, a contradiction.

As a consequence, the total size of the boundary blocks, each of which expresses one slope (either upward or downward) of the hill (excluding all plateaus), is at most \( 2|Q|^2 \). Notice that each plateau has size at most \(|Q| \) by (1). No distinct pair of plateaus in the hill has the same height since, otherwise, those two plateaus contain two state-stack pairs with the same inner state, and thus Case 1 occurs, a contradiction. Thus, the hill includes no more than \(|Q|^2 \) plateaus. From this fact, it follows that \( |t_2 - t_1| \leq 2|Q|^2 + |Q|^2 \times |Q| \leq 3|Q|^3 \). Note that the size \( s' - s \) of the turn is the size of the hill plus the size of a certain slope extending one end (either the right or the left) of the hill. Such a slope cannot have size more than \(|Q|^2 \) because, otherwise, we can conclude Condition (1). The size of the turn is therefore at most \(|t_2 - t_1| + |Q|^2 \leq 3|Q|^3 + |Q|^2 \leq 4|Q|^3 \). Similarly, the height of the turn is at most
\[|Q|^2 + |Q|^2 \leq 2|Q|^2.\]

Moreover, the inner states at the boundaries \(t_1\) and \(t_2\) must be different because, otherwise, the corresponding state-stack pairs coincide, and thus Case 2 follows, a contradiction.

(b) Next, we consider the case where \([s, s']\) consists of a series of turns \([s_1, s_2], [s_2, s_3], \ldots, [s_{m-1}, s_m]\) with \(s = s_1, s' = s_m,\) and \(s_1 < s_2 < \cdots < s_m.\) For each index \(i \in [m - 1],\) we take a hill \([t_i, t_i']\) lying in \([s_i, s_{i+1}].\) The height of such a hill is at most \(|Q|^2\) and its boundary-block size is at most \(3|Q|^3\) by (2a). Recall that, for each turn \([s_i, s_{i+1}],\) its bottom height is the stack height at either end of the turn, i.e., \(s_i\) or \(s_{i+1}.) For convenience, we define the \(gain\) of the turn, denoted by \(gain(s_i, s_{i+1}),\) to be the stack height at the boundary \(s_i\) minus the stack height at the boundary \(s_{i+1}.\) It is easy to see that, when the gain of a turn equals 0, the turn is merely a hill. Let \((p_i, \gamma_i)\) denote the state-stack pair at the boundary \(s_i.\) Assume that the gains of all the turns are zero and that their bottom heights are all equal. In this special case, we obtain \(m \leq |Q|\). This is shown as follows. Since the stack height does not go below the bottom height and \(M\) is in an ideal shape, \(\gamma_i = \gamma_j\) follows for any pair \(i, j \in [m - 1].\) By Lemma 4.3(1), if \(m \geq |Q| + 1,\) then there are distinct indices \(i, j \in [m]\) for which \(p_i = p_j,\) a contradiction against Case 1. Hence, \(m \leq |Q|\) follows and \(s' - s\) is upper-bounded by \(m \cdot 4|Q|^3,\) which is at most \(4|Q|^3.\)

(3) We estimate the total number of the turns whose bottom height is a fixed number \(h.\) Given such a number \(h,\) we define \(bh(h)\) to be the set of all indices \(i \in [m - 1]\) such that the bottom height of \([s_i, s_{i+1}]\) is exactly \(h.\) Note that \(bh(0) \leq |Q|\) follows by (2b). We first assume that \(bh(h + 1) > \bigcup_{i \leq h} bh(i)\) holds for a certain number \(h \in \mathbb{N}.\) This implies the existence of more than \(|Q|\) turns in \(bh(h + 1),\) any pair of which contain no turn in \(bh(i).\) Consider \(|Q| + 1\) state-stack pairs at the boundaries whose height matches the bottom heights of those specific turns. By Lemma 4.3(1), at least two of those state-stack pairs coincide, a contradiction. Hence, we conclude that, for any \(h \in \mathbb{N}, bh(h + 1) \leq \bigcup_{i \leq h} bh(i)\) holds. We then assert that \(bh(h) \leq 2^{h-1}|Q|^{h+1}\) for any \(h \in \mathbb{N}+.\) Since \(bh(i) \leq 2^{-i-1}|Q|^{i+1}\) holds for any \(i \in \mathbb{N}+,\) by induction hypothesis, it follows that \(bh(h) \leq |Q| \cdot \bigcup_{i \leq h} bh(i) \leq |Q|(\bigcup_{i=1}^{h} 2^{-(i-1)}|Q|^{i+1}) \leq (1 + \sum_{i=1}^{h-1} 2^{-(i-1)})|Q|^{h+1} = 2^{h-1}|Q|^{h+1}.

(4) We define another notion of “true gain” in the \((s, s')\)-region and intend to estimate its value. Let us consider a series of consecutive turns \([s_1, s_2], [s_2, s_3], \ldots, [s_{m-1}, s_m]\) in the boundary block \([s, s'],\) where \(m\) is the number that maximizes the length of this series.

For each turn, we consider its gain. Given an index \(k \in [m],\) the true gain in \([s_k, s_k],\) denoted by \(tg(s_k, s_k),\) is set to be the sum \(\sum_{i \leq k-1} gain(s_i, s_{i+1}).\) In particular, we set \(tg(s, s') = tg(s_1, s_1).\) We want to evaluate the value of this true gain and prove that \(tg(s, s') > |Q|^3.\) Assuming that \(tg(s, s') \leq |Q|^3,\) we wish to reach a contradiction. We need to examine two separate cases.

(a) Firstly, we consider the case where \(tg(s_1, s_k) \geq 2|Q|^3\) holds for a certain index \(k \in [m - 1].\) We intend to partition all the turns according to the “sign” of their gains. For any pair \(k, l \in [m]\) with \(k < l,\) the notation \(P_{s_k, s_l}\) (resp., \(N_{s_k, s_l}\) and \(Z_{s_k, s_l}\)) denotes the set of all indices \(i \in [k, l - 1]_Z\) such that \(gain(s_i, s_{i+1}) > 0\) (resp., \(gain(s_i, s_{i+1}) < 0\) and \(gain(s_i, s_{i+1}) = 0).\)

Since \(tg(s_k, s_k) \geq 2|Q|^3,\) we can take elements \(q \in Q, \alpha_1, \ldots, \alpha_{|Q|}, \gamma\) satisfying that, at certain \(|Q| + 1\) boundaries in \(\bigcup_{i \leq k} [s_i, s_{i+1}],\) the state-stack pairs have the form \((q, \gamma), (q, \alpha_1, \gamma), \ldots, (q, \alpha_{|Q|}, \gamma).\) Associated with these state-stack pairs, there are also state-stack pairs \((p_1, \alpha_{|Q|}, \gamma), (p_2, \alpha_{|Q|}, \gamma), \ldots, (p_{|Q|+1}, \gamma))\) at certain \(|Q| + 1\) boundaries in \(\bigcup_{i \leq k} [s_i, s_{i+1}]\) for appropriately chosen inner states \(p_1, \ldots, p_{|Q|+1} \) since \(tg(s, s') \leq |Q|^3.\) By Lemma 4.3(1), we can find an appropriate distinct pair \(i, j \in [|Q| + 1]\) for which \(p_i = p_j.\) This implies Case 3, a contradiction.

(b) Consider the case where, for any index \(k \in [|Q| + 1],\) \(tg(s_k, s_k) < 2|Q|^3\) holds. The number \(m\) of all the turns in the \(R\)-region is at most \(\bigcup_{i=0}^{[|Q|^3]} bh(i).\) Thus, we conclude that \(m \leq \sum_{i=0}^{[|Q|^3]} |bh(i)| \leq |Q| + 2^{[|Q|^3]} 2^{-1}|Q|^{[|Q|^3]} \leq |Q| + 2^{[|Q|^3]} |Q|^2 2^{[|Q|^3]} \leq 2^{4|Q|^4}.\) By (2b), there may be series at most \(|Q|\) turns having gain 0 and those turns have size at most \(4|Q|^4.\) Since \(|s' - s|\) is the sum of the sizes of all turns, it follows that \(|s' - s| \leq |Q| \times 4|Q|^4 + m \cdot 4|Q|^4 \leq 4|Q|^2 4|Q|^4 \leq 2^{5/2}|Q|^5.\) Since each turn has size at most \(4|Q|^4\) and \(m\) is the maximum, it follows that \(|s - ([x - c]) \leq 4|Q|^4\) and \(|x - s| \leq 4|Q|^4.\) From these inequalities, we obtain \(|s' - s| \geq c \sim 8|Q|^4.\) We then conclude that \(c \leq 2^{5/2}|Q|^5 + 8|Q|^4.\) This contradicts the definition of \(c.\) From the cases (a)–(b), we conclude that \(tg(s, s') > |Q|^3.\)

(5) Since \(tg(s, s') \geq |Q|^3 + 1,\) there exists a series \((q, \gamma), (q, \alpha_1, \gamma), \ldots, (q, \alpha_{|Q|}, \gamma))\) of \(|Q|^2 + 1\) state-stack pairs respectively at the boundaries \(t_1, t_2, \ldots, t_{|Q|^2+1}\) in the \(R\)-region for appropriately chosen \(q, \alpha_1, \ldots, \alpha_{|Q|},\) Consider the \(y\)-region of \(x'y.\) Let \(r_1, r_2, \ldots, r_{|Q|^2+1}\) denote boundaries in the \(y\)-region such that those boundaries at the form
We have already proven in Section 4.4 the first pumping lemma for DCFL, contradicting our assumption that (\(s\) induces (\(x\) such that (\(s\) holds). Hereafter, we state this argument more formally.

This completes the proof of the case of \(d = 1\).

**General Case of \(d \geq 2\):** We begin our argument by considering \(d\) dcf languages \(L_1, L_2, \ldots, L_d\) satisfying \(L = \bigcup_{i \in [d]} L_i\). Take \(d + 1\) strings \(w_1, w_2, \ldots, w_{d+1}\) of the form \(xy^k\) in \(L\) with \(|x| > c\) for appropriate strings \(x\) and \(y^k\). Since there are only \(d\) languages for the \(d + 1\) strings, there must be an index \(e \in [d]\) and two distinct indices \(j_1, j_2 \in [d + 1]\) for which \(w_{j_1}, w_{j_2} \in L_e\). In what follows, we fix such a triplet \((j_1, j_2, e)\) and set \(w = w_{j_1}, w' = w_{j_2}\), and \(L' = L_e\) for simplicity. Let us take any two factorizations \(w = x'y\) and \(w' = x''z'\) with \(|x'| > c\). We apply the basis case of \(d = 1\) to \(w\) and \(w'\) and obtain one of the conditions (1)–(2) of the lemma. This completes the proof of the general case.

4.5 Proof of the Second Pumping Lemma for DCFL[d]

We have already proven in Section 4.4 the first pumping lemma for DCFL[d] by conducting a detailed analysis on a stack history of a given 1dpda. In this section, we intend to prove the second pumping lemma for DCFL[d] (Lemma 4.3) by taking a slightly different strategy. Instead of applying Lemmas 4.3, we here use Proposition 4.4.

Take any integer \(d \geq 1\) and fix an arbitrary infinite language \(L\) in DCFL[d]. We split the intended proof into two parts, depending on the value of \(d\).

**Basis Case of \(d = 1\):** We first consider the basis case of \(d = 1\). Fix a 1dpda \(M = (Q, \Sigma, \{0, 1\}, \delta, \delta_0, Q_{acc}, Q_{rej})\) in an ideal shape recognizing \(L\). Consider two arbitrary strings \(w = x'y\) and \(w' = x''z'\) in \(L\) with \(|x'|, |y|, |z| \geq 1\).

Let us assume that the conditions (1)–(4) of the lemma all fail. It then suffices to verify the validity of the condition (5). Since the condition (1) in particular fails, both \(w\) and \(w'\) must have nondegenerate iterative pairs for \(L\). We consider such nondegenerate iterative pairs of \(x'y\) and \(x''z'\) for \(L\). Note that, by our assumption, any of them must satisfy none of the conditions (2)–(4). The remaining case is that exactly one of the two nondegenerate iterative pairs of \(x'y\) and \(x''z'\) is in the \(x'\)-region. Fix a string \(u \in \{y, z\}\) such that the \(x'\)-region includes a nondegenerate iterative pair of \(x'u\). For the other string \(x'u'\), where \(u'\) is a unique string in \(y, z\) \(\cap \{u\}\), a half pair of the nondegenerate iterative pair of \(x'u'\) must be in the \(x'\)-region and the other half lies (at least partly) in the \(z\)-region.

We apply Proposition 4.4 to \(w\) and \(w'\) and obtain stack-operational pairs of certain two strings deduced from \(w\) and \(w'\). For readability, we write those deduced strings as \(w\) and \(w'\) as well and focus on their stack-operational pairs. In what follows, we assume that \(x'\) and \(u\) are factorized into \(x' = x_1x_2x_3x_4x_5x_6x_7\) and \(u = u_1u_2u_3\), respectively. For readability, we use the notation \((\alpha_1, \alpha_2, \alpha_3)\) to denote a factorization \(\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5\) of a given string and we say that a quintuple \((t_1, t_2, t_3, t_4)\) of boundaries is induced by \((\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)\) if \(t_i\) is the \((\alpha_i, \alpha_{i+1})\)-boundary for every index \(i \in [4]\). Then we examine the following cases (i)–(iv).

Case (i): Consider the case where \((x_2x_3, x_7u_1)\) and \((x_3x_4, x_6x_7)\) are respectively nondegenerate stack-operational pairs of \(x'u\) and \(x'u'\) w.r.t. \(M\). This case makes the condition (4) true, a contradiction.

Case (ii): Consider the case where \((x_2x_3, u_2)\) and \((x_2x_3, x_6)\) are respectively nondegenerate stack-operational pairs of \(x'u\) and \(x'u'\) w.r.t. \(M\). By stack-operationalism, \(M\) can repeatedly push a series of the stack symbols corresponding to \(x_2x_3\) and then repeatedly pop the same series of symbols while reading a multiple copies of \(x_6\). To do so, however, \(M\) need to pop all the symbols corresponding to \(x_3x_4\). Hereafter, we state this argument more formally.

In relation to \(x'u'\), we take a quintuple \((t_1, t_2, t_3, t_4)\) of boundaries induced by \((x_1x_2x_3|x_5x_6x_7u_1|x_2|x_3)\) and \((x_1x_2|x_3|x_5x_6x_7u_1|x_2)\). Since \((t_1, t_2, t_3, t_4)\) is good, we can choose appropriate elements \(q, p, \alpha, \gamma\) such that \((q, \gamma)\) and \((q, \alpha\gamma)\) are respectively state-stack pairs at the boundaries \(t_1, t_2\) and \((p, \alpha\gamma)\) and \((p, \gamma)\) are respectively state-stack pairs at the boundaries \(t_3, t_4\). In a similar way, we take \((s_1, s_2, s_3, s_4)\) induced by \((x_1x_2|x_3x_4|x_5x_6x_7u_1|x_2|x_3)\) and \((x_1x_2|x_3|x_5x_6x_7u_1|x_2)\) and take state-stack pairs \((q', \gamma')\) and \((q', \beta')\) at the boundaries \(s_1\) and \(s_2\), respectively, and \((p', \beta')\) and \((p', \gamma')\) at the boundaries \(s_3\) and \(s_4\), respectively. If \(|\gamma'| \geq |\gamma|\), then \(|s_2, s_3|\) cannot be pseudo-convex, and thus \(|s_1, s_2|\) and \(|s_3, s_4|\) are not mutually correlated. This contradicts our assumption that \((x_2x_3, x_6)\) is a stack-operational pair of \(x'u'\). Thus, \(|\gamma'| < |\gamma|\) follows.
However, this implies that \( t_2, t_3 \) cannot be pseudo-convex. Hence, \([t_1, t_2] \) and \([t_3, t_4] \) are not mutually correlated. This is also absurd. Therefore, this case (ii) never happens.

Similarly, even if we replace \((x_2x_3, x_6)\) and \((x_3x_4, u_2)\) respectively by \((x_2, x_6)\) and \((x_4, u_2)\), we can obtain another contradiction.

Case (iii): Consider the case where \((x_2, u_2)\) and \((x_4, x_6)\) are respectively nondegenerate stack-operational pairs of \(x'u\) and \(x'u^{(op)}\) w.r.t. \(M\). In this case, we choose two quintuples \((t_1, t_2, t_3, t_4)\) and \((s_1, s_2, s_3, s_4)\) induced respectively by \((x_1|x_2|x_3|x_5|x_6)\) and \((x_1|x_2|x_3|x_4|x_5|x_6)\). Since \([t_2, t_3]\) and \([s_2, s_3]\) are both pseudo-convex, \([s_1, s_2]\) and \([s_3, s_4]\) must appear inside \([t_2, t_3]\). We can process \(x_2\) and \(u_2\) for the same number of times without affecting any other computation of \(M\). Similarly, we can do the same for \(x_4\) and \(x_6\). Thus, the case (a) is true.

Case (iv): Consider the case where \((x_3x_6, u_2)\) and \((x_2, x_4x_5)\) with \(|x_5| \geq 1\) are respectively nondegenerate stack-operational pairs of \(x'u\) and \(x'u^{(op)}\) w.r.t. \(M\). As in the previous cases, let \((t_1, t_2, t_3, t_4)\) and \((s_1, s_2, s_3, s_4)\) denote two quintuples induced respectively by \((x_1|x_2|x_3|x_4|x_5|x_6)\) and \((x_1|x_2|x_3|x_4|x_5|x_6)\). Note that there are a hill top in \([t_2, t_3]\) and another in \([s_2, s_3]\). It is thus possible for \(M\) to repeatedly push the stack symbols associated with \(x_2\) and then pop the same symbols while reading a multiple copies of \(x_4x_5\). In a similar way, \(M\) can repeatedly process the same number of copies \(x_3x_6\) and \(u_2\). These repetitive procedures do not influence the other computation of \(M\). As a consequence, the case (b) is true.

Case (v): Consider the last case where \((x_6, u_2)\) and \((x_2, x_4)\) are respectively nondegenerate stack-operational pairs of \(x'u\) and \(x'u^{(op)}\) w.r.t. \(M\). An argument used for Case (iv) also leads to the case (b).

**General Case of** \(d \geq 2\): We wish to prove the lemma for a general case of \(d \geq 2\). Take \(d+1\) strings \(w_1, w_2, \ldots, w_{d+1}\) in \(L\). Assume that \(L = \bigcup_{e \in [d]} L_e\) to \(d\) dcf languages \(L_1, L_2, \ldots, L_d\). Since the \(d+1\) strings fall into the union of these \(d\) languages, by the pigeonhole principle, there must be an index \(e \in [d]\) and a distinct pair \(j_1, j_2 \in [d+1]\) for which \(w_{j_1}\) and \(w_{j_2}\) both belong to \(L_e\). Consider a 1dpda \(M\) in an ideal shape that recognizes \(L_e\). We then apply the basis case \(d = 1\) to \((w_{j_1}, w_{j_2}, M)\). This completes the proof of the general case.

# 5 A Brief Discussion and Future Challenges

We have presented two new and practical lemmas, called the pumping lemmas for DCFL\([d]\). The first one (Lemma \[1.1\]) is in part viewed as a natural extension of Yu’s pumping lemma for DCFL \([24]\). This lemma has a quite simple form, describing a structural property of iterative pairs. In Section \[5\] we have applied it to separate the intersection and the union hierarchies of deterministic context-free (dcf) languages. Our proof of this pumping lemma is solely founded on an analysis of the behaviors of one-way deterministic pushdown automata (or 1dpda’s) and it therefore provides an alternative proof to Yu’s pumping lemma whose proof actually utilizes LR\((k)\) grammars. In contrast, the second pumping lemma for DCFL\([d]\) (Lemma \[1.2\]) relates to a structural property of nondegenerate iterative pairs. The proof of this pumping lemma relies on the notion and the properties of stack-operational pairs. As a future task, it is desirable to study various aspects of iterative pairs particularly for dcf languages.

Although our pumping lemmas are quite useful, it is not as powerful as to precisely characterize all \(d\)-union dcf languages. For instance, to prove that \(NP{\text{al}}\#\) is not in DCFL\([\omega]\) in the proof of Theorem \[1.5\] 2), we have employed an argument, which relies on none of the pumping lemmas for DCFL\([d]\). One of the most challenging tasks is to expand our pumping lemmas to characterize such languages. See, e.g., \[18\] for the case of context-free languages.

Since finite intersections and unions have been intensively discussed so far, we here wish to discuss their simple extension to “infinite” intersections in a certain controlled way. To a given 1dpda \(M = (Q, \Sigma, \{\#, \}$, \$, \$), \alpha, q_0, Z_0, Q_{acc}, Q_{rej}\), we first assign its description size, which equals the value \(|Q|/\Sigma|/\Sigma^*|\). We express this value as \(\text{des}(M)\). Let us consider an infinite sequence \(\{M_n\}_{n \in \mathbb{N}}\) of 1dpda’s. Such a sequence is said to have polynomial description size if there exists a polynomial (with nonnegative coefficients) \(p\) such that \(\text{des}(M_n) \leq p(n)\) holds for any number \(n \in \mathbb{N}\). Given a function \(\mu : \mathbb{N} \to \mathbb{N}\), we concentrate on \(\mu\)-bounded intersection of the dcf languages \(\{L(M_n)\}_{n \in \mathbb{N}}\), which is defined as \(\{x \mid \forall i \leq n, \mu(|x|)[x \in L(M_i)]\}\). Recall from Section \[1.1\] the language \(\text{Pat}\) of even-length palindromes, which is not in DCFL\([\omega]\) by Theorem \[1.3\] 1). In spite of this fact, we can characterize \(\text{Pat}\) in terms of a certain \(\mu\)-bounded intersection of dcf languages as follows.

**Lemma 5.1** There exists an infinite sequence \(\{M_n\}_{n \in \mathbb{N}}\) of 1dpda’s having polynomial description size.
such that Pal coincides with the $\mu$-bounded intersection of $\{L(M_n)\}_{n \in \mathbb{N}}$, where $\mu(n) = \lceil n/2 \rceil$ for any $n \in \mathbb{N}$.

**Proof.** We wish to construct a family $M = \{M_n\}_{n \in \mathbb{N}}$ of 1dpda’s. Let $M_0$ denote a 1dpda that recognizes $\{x \mid |x| \text{ is even} \}$ without using its stack. For any other $n \geq 1$, we define $M_n$ in the following way. Given an input $x$, if $|x| \leq n$, then we reject $x$. Assume otherwise. We push all symbols of $x$ one by one into a stack and remember the $n$th symbol of $x$, say, $a_n$ in the form of inner states. This is possible because $n$ is treated as a “constant” for $M_n$, not depending on the length of the input $x$. After reaching the endmarker $\$$, we start to pop the last $n$ symbols of $x$ from the stack. We then check if the $n$th symbol matches $a_n$. If so, then we accept $x$; otherwise, we reject $x$. Consider the $\mu$-bounded intersection of $M$, namely, $\{x \mid \forall i \leq \mu(|x|) [x \in L(M_i)]\}$. It is not difficult to show that this intersection coincides with $\text{Pal}$. \qed

It is interesting to determine the precise computational complexity of $\mu$-bounded intersections of dcf languages for various choices of $\mu$.

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