STANDARD EXAMPLES AS SUBPOSETS OF POSETS

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Abstract. We prove that a poset with no induced subposet \( S_k \) (for fixed \( k \geq 3 \)) must have dimension that is sublinear in terms of the number of elements.

1. Introduction

Dushnik and Miller (1941) first studied order dimension. Let \( P \) be a poset. A set of its linear extensions \( \{L_1, \ldots, L_d\} \) forms a realizer, if \( L_1 \cap \cdots \cap L_d = P \). The minimum cardinality of a realizer is called the dimension of the poset \( P \). This concept is also sometimes called the order dimension or the Dushnik-Miller dimension of the partial order.

The standard example \( S_n \) on \( 2^n \) elements is the poset formed by considering the 1-element subsets, and the \((n-1)\)-element subsets of a set of \( n \) elements, ordered by inclusion. It is known that the dimension of \( S_n \) is half of the number of elements of \( S_n \), that is, \( \dim(S_n) = \frac{n}{2} \). It is also known that there are posets of arbitrarily large dimension without an \( S_3 \) subposet. For further study of the dimension of posets we refer the readers to the monograph [7], and to the survey article [8].

We use the notation \(|P|\) for the number of elements (of the ground set) of the poset \( P \), and we use the standard notation \([n]\) for the set \( \{1,2,\ldots,n\} \). The upset (sometimes called ideal) of the element \( x \) is the set \( U(x) = \{y \in P : y > x\} \). Similarly, the downset of the element \( x \) is \( D(x) = \{y \in P : y < x\} \). The symbol \( x \parallel y \) is used to denote that \( x \) is incomparable with \( y \) in the poset.

Let \( P \) and \( Q \) be two posets. \( Q \) is a subposet of \( P \), if there is an induced copy of \( Q \) in \( P \). More precisely, \( Q \) is a subposet of \( P \) if there is a bijection \( f \) from a subset of the ground set of \( P \) to the ground set of \( Q \) with the property that \( f(x) < f(y) \) if and only if \( x < y \). If \( P \) does not contain \( Q \) as a subposet, \( P \) is called \( Q \)-free. Note that some authors use the term “subposet” in the “non-induced” sense. We conform to the more standard usage, and we say “an extension of \( Q \) is a subposet of \( P \)” if we ever need the non-induced meaning.

Hiraguchi [5] proved that for any poset with \( |P| \geq 4 \), \( \dim(P) \leq |P|/2 \). Bogart and Trotter [3] showed that if \( |P| \geq 8 \), then the only extremal examples of Hiraguchi’s theorem are the standard examples \( S_n \) with \( \dim(S_n) = |S_n|/2 \). A natural question arises: if the dimension is just a bit less than the maximum, can we expect largely the same structure as in the extremal case?

Biró, Füredi, and Jahanbekam [2] conjectured the following.

Conjecture 1.1. For every \( t < 1 \), but sufficiently close to 1 there is a \( c > 0 \) and \( N \) positive integer, so that if \( |P| \geq 2n \geq N \), and \( \dim(P) \geq tn \), then \( P \) contains \( S_{\lfloor cn \rfloor} \).
Conjecture 1.1 was mentioned at the “Problems in Combinatorics and Posets” workshop [1] in 2012 and it is listed as Problem #1 of the workshop. Even the existence of an $S_3$ was unclear.

In this paper we show that a poset with no induced subposet $S_k$ (for fixed $k \geq 3$) must have dimension that is sublinear in terms of the number of elements.

A few more definitions are needed. The height of a poset is the size of a maximum chain. Height two posets are called bipartite. A bipartition of a bipartite poset $P$ is a partition $(A, B)$ of its ground set together with a fixed linear order on the parts, such that $x < y$ in $P$ implies $x \in A$ and $y \in B$. Note that the bipartition includes a linear ordering on both $A$ and $B$, respectively. They are not denoted to avoid clutter, but they are important in the discussion.

A critical pair $(x, y)$ is an ordered pair of elements of $P$ with the properties $x \parallel y$, $D(x) \subseteq D(y)$, and $U(y) \subseteq U(x)$. A linear extension $L$ reverses the critical pair $(x, y)$, if $y < x$ in $L$. A set of linear extensions $L$ reverses $(x, y)$ if there is a linear extension in $L$ that reverses $(x, y)$. It is known that a set of linear extensions is a realizer if and only if it reverses every critical pair of $P$. This makes the investigation simpler in bipartite posets, where all incomparable minimum–maximum pairs are critical, and they are the only (interesting) critical pairs; the remaining critical pairs have a very special structure and can be usually handled in a simple way.

Let $P$ be a poset. A poset $Q$ is the dual of $P$, if $Q$ is defined on the same ground set, but every order is reversed, that is, $x < y$ in $P$ iff $x > y$ in $Q$. Obviously, $\dim(P) = \dim(Q)$.

2. Preliminaries

Definition 2.1. Let $k \geq 3$ integer. Let $F_k$ denote the set of finite $S_k$-free posets.

\[ D(n, k) = \max \{ \dim(P) : P \in F_k, |P| = n \} \]
\[ \Delta(n, k) = \max \{ \dim(P) : P \in F_k, P \text{ is bipartite}, |P| = n \} \]

The main goal of the paper is to prove the following theorem.

Theorem 2.2.

\[ D(n, k) = o(n). \]

The proof is in Section 4.

Let $H$ be a hypergraph. A coloring of $H$ is an assignment of positive integers (colors) to its vertices (with no restrictions). A $\ell$-coloring is a coloring with $\ell$ colors. A subset $H$ of $V(H)$ is monochromatic, if every edge $E \subseteq H$ receives the same color in the coloring.

$K^k_n$ is used to denote the complete $k$-uniform hypergraph on $n$ vertices. We need the following version of Ramsey’s Theorem.

Theorem 2.3. [9] For all $k, q, \ell$ positive integers there exists an $N$ such that if $n \geq N$, then every $\ell$-coloring of $K^k_n$ contains a monochromatic set of size $q$.

The (hypergraph) Ramsey number $R(k, q, \ell)$ is the least $N$ in Theorem 2.3. For a fixed $k$, the function $R(k, q, k)$ is a function of one variable; the following function may be regarded as its inverse.
Definition 2.4. Let \( k \geq 3 \) fixed. Let \( r(n) \) denote the minimum size of a largest monochromatic subset in a \( k \)-coloring of \( K_n^k \).

We use a simple corollary of Theorem 2.3, that is, we use that \( r(n) \to \infty \) as \( n \to \infty \).

Throughout this paper a positive integer \( k \geq 3 \) is fixed. The purpose of this is that most of the time we assume that our posets are \( S_k \)-free.

Let \( P \) be an \( S_k \)-free bipartite poset with bipartition \((A,B)\). We fix an ordering of the vertices of \( A \). Let \( S = \{a_1, \ldots, a_k\} \subseteq A \), and assume that the indexing preserves the ordering on \( A \). Call an element \( b \in B \) a mate of \( a_i \), if \( a_i \parallel b \), but \( a_j < b \) for all \( j \neq i \). Clearly any \( b \in B \) can not be a mate of more than one \( a_i \), so the set of mates of \( a_1, \ldots, a_k \) form disjoint subsets of \( B \). The condition that \( P \) is \( S_k \)-free means that there exists a \( a_{i_0} \) that has no mate. In this case, we say that \( i_0 \) is a valid color for \( S \).

An upset based coloring (or UB-coloring for short) of the \( k \)-element subsets of \( A \) is such that assigns a valid color to each subset. Note that UB-coloring is only defined in the context of \( S_k \)-free bipartite posets with a fixed ordering on the minimal elements.

3. Bipartite posets

We begin with a simple technical statement of probability that will contain the key computation.

Lemma 3.1. Let \( t,q \) be positive integers, \( 2 \leq q \leq t,q \), and let \( r \geq \ell 2^t \ln q \). Let \( X = [x_{i,j}] \) be a random \( r \times q \) binary matrix, in which each entry is 1 independently with probability \( \frac{1}{2} \). Let \( E \) be the event that for all sequences \( 1 \leq j_1 < j_2 < \cdots < j_t \leq q \) and all integers \( 1 \leq \ell \leq t \) there is a row \( i \) in \( X \) with the property \( x_{i,j_1} = \cdots = x_{i,j_{\ell-1}} = x_{i,j_{\ell+1}} = \cdots = x_{i,j_t} = 0 \) and \( x_{i,j_\ell} = 1 \). Then \( \Pr(E) > 0 \).

Proof. For a fixed sequence \( s = (j_1, \ldots, j_t) \) and integer \( \ell \), let \( E_{s,\ell} \) be the event that at least one row has the property. Any given row has the property with probability \( 2^{-t} \), so \( \Pr(E_{s,\ell}) = (1 - 2^{-t})^{r} \). Hence

\[
\Pr(E) = \Pr(\cap E_{s,\ell}) = 1 - \Pr(\cup E_{s,\ell}) \geq 1 - \sum \Pr(E_{s,\ell}) = 1 - t \binom{q}{t} (1 - 2^{-t})^{r} > 1 - q^r e^{-r2^{-t}} = 1 - e^{t \ln q - r2^{-t}} \geq 0,
\]

where the last inequality follows from the condition on \( r \). \( \square \)

Recall that \( k \geq 3 \) is a fixed integer.

Lemma 3.2. Let \( P \) be a bipartite poset with bipartition \((A,B)\). Let \( q \geq 2 \), and \( Q = \{a_1, \ldots, a_q\} \subseteq A \) be a monochromatic set in a UB-coloring, and assume that the indexing preserves the ordering on \( A \). Then there exists a set of linear extensions \( L = \{L_1, \ldots, L_{q'}\} \) with \( q' = 2\lceil k2^k \ln q \rceil \) that reverses every critical pair \((a_i, b)\) with \( b \in B \).

Proof. \( Q \) is a monochromatic set, denote its color by \( \ell \). Let \( t = \max\{\ell - 1, k - \ell\} \). Since \( t < k \), we have that \( q'/2 > t 2^t \ln q \), so we can apply Lemma 3.1 with \( t, q \), and \( r = q'/2 \). We conclude that there exists a matrix \( X \) with the property described in the lemma.
We construct the set \( L \) using \( X \). For each row of \( X \) we construct two linear extensions. For a given binary row \( \mathbf{x} = (x_1, \ldots, x_q) \), first construct two permutations \( \sigma_1 \) and \( \sigma_2 \) of \([q]\) as follows. For \( \sigma_1 \), first list elements of \([q]\) for which the corresponding bit of \( \mathbf{x} \) is 1 in order from left to right, then list the elements corresponding to 0 bits in order from left to right. For \( \sigma_2 \), do the same, except list the elements of \([q]\) from right to left.

In the next step, we construct a linear extension for each permutation. We refer to the permutation as \( \sigma \), which will be first \( \sigma_1 \), then \( \sigma_2 \), thereby resulting two linear extensions. Let \( U_i = \{ b \in U(a_{\sigma(i)}): b \notin U(a_{\sigma(j)}) \text{ for any } j < i \} \) for every \( i \in [q] \). In other words, \( \{U_i\} \) forms a partition of the union of the upsets of the elements of \( Q \) such that any element that belongs to multiple upsets will be placed into the first one, where the order is determined by \( \sigma \). Let \( R = P - Q - \bigcup_{i=1}^{q} U_i \). Then define the linear extension with

\[
U_1 > a_{\sigma(1)} > U_2 > a_{\sigma(2)} > \cdots > U_q > a_{\sigma(q)} > R
\]

where the order within each \( U_i \) and \( R \) are arbitrary (e.g. we can respect some predetermined order of the elements).

Repeating the process for every row we construct the set \( L \).

It remains to be shown that \( L \) reverses every critical pair of the form \((a_i, b)\) with \( b \in B \). Consider such a pair \((a_i, b)\). Let

\[
M_1 = \{ a_m \in D(b) : m < i \} \quad \text{and} \quad M_2 = \{ a_m \in D(b) : m > i \}.
\]

Since any \( k \)-subset \( K = \{a_{\ell_1}, \ldots, a_{\ell_k}\} \) with \((\ell_1 < \cdots < \ell_k)\) in which \( a_{\ell_i} = a_i \) is colored \( \ell \), we know that \( a_i \) cannot have a mate with respect to \( K \). Since \( a_i \parallel b \), we have that either

i) \( |M_1| \leq \ell - 2 \), or

ii) \( |M_2| \leq k - \ell - 1 \).

In case \( \square \), find a row \( \mathbf{z} \) of \( X \) such that \( x_j = 0 \) for all \( j \) for which \( a_j \in M_1 \), and \( x_i = 1 \). Such a row exists, because \( |M_1| + 1 \leq t \) (we need to force a few more zeros in the row to directly use the lemma, if \( |M_1| < t - 2 \)). The first linear extension corresponding to this row places \( a_i \) over \( b \). In case \( \square \), find a row \( \mathbf{x} \) of \( X \) such that \( x_j = 0 \) for all \( j \) for which \( a_j \in M_2 \), and \( x_i = 1 \). The second linear extension corresponding to this row places \( a_i \) over \( b \). \( \square \)

**Lemma 3.3.** Let \( P \) be a bipartite \( S_k \)-free poset with a bipartition \((A, B)\), and \(|A| \geq n\), where \( n \) is such that \( r(n) \geq 2 \), (see \( r(n) \) in Definition 2.4). Then there exist \( c = c(k) \) such that for all \( q = 2, 3, \ldots, r(n) \) there exists \( Q \leq A \) with \(|Q| = q\) such that \( \dim(P) \leq \dim(P - Q) + c \ln q \).

**Proof.** We show that \( c = 3k2^k \) works. Let \( 2 \leq q \leq r(n) \) be an arbitrary integer. Consider a UB-coloring of the \( k \)-subsets of \( A \). By Ramsey’s Theorem there is a monochromatic subset \( Q \) with \(|Q| = q \). Let \( P' = P - Q \), and let \( \mathcal{L}' \) be a realizer of \( P' \). By Lemma 2.2, there exists a set of linear extensions \( \mathcal{L}'' \) of size \( 2k2^k \ln q \) that reverses every critical pair in \( P \) of the form \((a, b)\), where \( a \in Q \), and \( b \in B \). Now consider the set \( M \), the set of all minimal elements of \( P \). Add a linear extension \( L \) in which we reverse the order of elements of \( M \) (compared to a fixed element of, say, \( \mathcal{L}'' \)) but the rest of the elements are inserted arbitrarily.
Hence \( \Delta(n, k) \geq o(n) \)

**Theorem 3.4.** Let \( k \geq 3 \) integer.

\[
\Delta(n, k) = o(n)
\]

**Proof.** Fix \( \epsilon > 0 \). Let \( c \) be as in Lemma 3.3 and let \( q \geq 2 \) be such that \( c \ln q / q \leq \epsilon / 2 \). Furthermore, let \( N \) be such that \( r([N/2]) \geq q \). We show that for all \( \ell \geq 0 \) integer,

\[
\Delta(N + \ell q, k) \leq \Delta(N, k) + \ell \frac{\epsilon}{2} q.
\]

To show this we use induction on \( \ell \). It is obvious for the case \( \ell = 0 \). Assume \( \ell \geq 1 \). Consider an \( S_k \)-free bipartite poset \( P \) on \( N + \ell q \) elements with \( \dim(P) = \Delta(N + \ell q, k) \). Without loss of generality we may assume that \( P \) has a bipartition \((A, B)\) with \( |A| \geq \lfloor N/2 \rfloor \), for otherwise we may consider the dual of \( P \) instead.

By Lemma 3.3, \( P \) has a subset \( Q \) of size \( q \), such that \( \dim(P) \leq \dim(P - Q) + c \ln q \). Hence

\[
\Delta(N + \ell q, k) \leq \dim(P - Q) + c \ln q \leq \Delta(N + (\ell - 1)q, k) + c \ln q
\]

\[
\leq \Delta(N, k) + (\ell - 1) \frac{c}{2} q + c \ln q \leq \Delta(N, k) + (\ell - 1) \frac{c}{2} q + \frac{\epsilon}{2} q = \Delta(N, k) + \ell \frac{\epsilon}{2} q.
\]

This, with the fact that \( \Delta(n, k) \) is a monotone increasing sequence of \( n \), finishes the proof. \( \square \)

4. KIMBLE SPLITS AND GENERAL POSETS

We need the notion of a “split” of a poset. Kimble \([6]\) introduced this notion. We only need a special version of his definition, and a special case of his theorem; we only mention those.

**Definition 4.1.** Let \( P \) be a poset with ground set \( \{x_1, \ldots, x_n\} \). The Kimble split of \( P \) is the poset on the ground set \( \{x_1', x_1'', \ldots, x_n', x_n''\} \) with \( x_i' \leq x_i'' \) if and only if \( x_i \leq x_j \) in \( P \).

**Theorem 4.2.** \([6]\) Let \( P \) be a poset and \( Q \) be its Kimble split. Then

\[
\dim(P) \leq \dim(Q) \leq \dim(P) + 1
\]

We also need the following simple lemma.

**Lemma 4.3.** Let \( P \) be an \( S_k \)-free poset and \( Q \) be its Kimble split. Then \( Q \) is \( S_k \)-free.

**Proof.** Suppose \( Q \) has an \( S_k \) subposet, call the set of its vertices \( S \). If all \( 2k \) elements of \( S \) come from distinct elements of \( P \), then they formed an \( S_k \) in \( P \). Otherwise there is a pair \( a', a'' \in S \) such that \( a' < a'' \) in \( Q \), and they are the split versions of the original vertex \( a \) of \( P \). Since \( k \geq 3 \), there is \( b', c'' \in S \) with \( b' > a' \), and \( c'' < a'' \), and \( b' \parallel c'' \in Q \). Clearly \( b' \) and \( c'' \) came from distinct elements of \( P \), because they are incomparable, and these elements are also distinct from \( a \); call them \( b \) and \( c \) respectively. Due to the definition of the Kimble split, \( b < a < c \) in \( P \), but then \( b' < c'' \) in \( Q \), a contradiction. \( \square \)
4.1. **Proof of Theorem 2.2.** Lemma 4.3 and the first inequality of Theorem 4.2 imply that

\[ D(n, k) \leq \Delta(2n, k), \]

so the theorem is a consequence of Theorem 3.4.

We end the discussion with a corollary in the style of (the still open) Conjecture 1.1.

**Corollary 4.4.** For all \( k \geq 3 \) integer, and \( t < 1 \) there is an \( N \) integer, so that if \( |P| \geq 2n \geq N \), and \( \dim(P) \geq tn \), then \( P \) contains \( S_k \).

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