The Hyperbolic Geometry of the Sinh-Gordon Equation

Preprint

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Abstract

This preliminary report studies immersed surfaces of constant mean curvature in $H^3$ through their adjusted Gauss maps (as harmonic maps in $S^2$) and their adjusted frames in $SU(2)$. Lawson’s correspondence between Euclidean CMC surfaces and their hyperbolic cousins is interpreted here under a different perspective: the equivalence of their Weierstrass representations (normalized potentials). This work also presents a construction algorithm for the moving frame, the adjusted frame, their Maurer-Cartan forms, and ultimately the CMC immersion.

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1 Introduction

The classical Weierstrass representation formula for minimal surfaces in $E^3$ consists of a meromorphic function (the classical Gauss map) and a holomorphic 1-form. Several years ago, a method now referred to as DPW ([8]) was introduced for nonzero constant mean curvature surfaces (abbreviated as CMC) in $E^3$. The method gives a characterization of these surfaces in terms of a pair of functions, called normalized (or meromorphic) potential, and also a method to construct all associate immersions based on loop group factorization. Among the classes of surfaces for which such a potential was found explicitly, and used to construct surfaces, we mention: constant mean curvature surfaces in Euclidean space $E^3$ ([5]); minimal surfaces in $E^3$ ([7]); weakly regular pseudospherical surfaces in $E^3$ ([16]); timelike surfaces in the Minkowski 3-space ([6]); Willmore surfaces in $E^3([11])$; timelike minimal surfaces ([15]).

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In this paper, we consider only cousins of the Euclidean CMC surfaces, that is, surfaces with constant mean curvature $|H| > 1$. The case $|H| < 1$, consisting of solutions to the cosh-Gordon equation, has only one distinguished representative, the minimal surfaces $H = 0$. The author has devoted an entirely different approach (and separate paper) to this case. More precisely, for minimal surfaces, the adjusted Maurer-Cartan and the adjusted (harmonic) Gauss map are different, and they require different loop groups.

As a byproduct, we here show that for any associated family of CMC surfaces with $|H| > 1$, there is a (strongly conformal) corresponding family, obtained by spectral deformation, whose frame is unitary (SU(2)-valued, as opposed to SL(2, C)). This is not the case for $|H| < 1$.

A first step of this study was communicated in 2002 as a preliminary report, but a DPW method for constant mean surfaces in $\mathbb{H}^3$ was yet to be found. The author then introduced the normalized potentials and conjectured that they represent the input for a construction algorithm. A proposed DPW algorithm was publicized in 2004, through an outstanding work by four authors, [12], for surfaces with constant mean curvature $|H| > 1$.

Our spectral deformations, potentials and DPW method are different from [12]. We study the normalized potential and show that it basically reduces to a ‘Weierstrass pair’: the Hopf differential, together with the holomorphic part of the metric conformal factor. We here analyze how the normalized potential is used in order to generate the adjusted SU(2) frame, the regular SL(2, C) frame, their two Maurer-Cartan forms, and the CMC immersion.

Among the advantages of this particular approach are the simple form of the normalized potential compared to other representations one may use, as well as the fact that one does not have to keep track of the monodromy representation. Also, although the factorizations are not explicit, the resulting frames and immersions are.

We prompt the reader to check and note the following:

Even for the spectral deformation from [12], the usual frame $F$ is not r-unitary. Due to the off-diagonal entries of the Lax matrices, containing $H - 1$ and $H + 1$, the usual Maurer-Cartan form $F^{-1} \cdot dF$ is not su(2)-valued (even when the spectral parameter takes values on $S^1$).

The r-unitarization that is aimed in [12] (see (1.10), (2.4) and (3.1)) actually takes place
in our context, for a different reason (see formula (17)).

Let \( \mathbb{H}^3(-1) \) denote the hyperbolic 3-space of constant sectional curvature \(-1\). Surfaces of constant mean curvature \( |H| = 1 \) represented the topic of many papers over the past fifteen years. An important result of R. Bryant ([4]) gave a representation formula for these surfaces. In [1], R. Aiyama and K. Akutagawa gave a Kenmotsu-Bryant type representation formula for (branched) surfaces in \( \mathbb{H}^3(-c^2) \) of constant mean curvature \( |H| \geq c \). In [2], the same two authors showed even further that there exists a Kenmotsu-Bryant type representation formula for surfaces in \( \mathbb{H}^3(-c^2) \) of constant mean curvature \( |H| < c \).

A general result known as Lawson’s correspondence has the following theorem as a corollary:

**Theorem 1.** There is a bijective correspondence between the space of isometric immersions of constant mean curvature \( H > 0 \) in \( \mathbb{E}^3 \) and the space of isometric immersions of constant mean curvature \( \sqrt{H^2 + 1} \) in \( \mathbb{H}^3(-1) \).

From now on, we will assume \( \mathbb{H}^3 \) as being of sectional curvature \(-1\), unless otherwise stated.

Through the Lawson correspondence, CMC surfaces in \( \mathbb{E}^3 \) are corresponded to CMC \( |H| > 1 \) surfaces in \( \mathbb{H}^3 \). On the other hand, minimal surfaces in \( \mathbb{E}^3 \) are corresponded to CMC \( |H| = 1 \) surfaces in \( \mathbb{H}^3 \). In the past decade, significant progress has been made in the area of surfaces in \( \mathbb{H}^3 \), especially surfaces of constant mean curvature greater than one. Recently, there has been some progress in visualizing some surfaces. For example, N. Schmitt used a loop group splitting in order to construct surfaces of constant mean curvature \( |H| > 1 \) in \( \mathbb{H}^3 \). Based on these methods, he wrote a program that produces hyperbolic analogues of some CMC surfaces in \( \mathbb{E}^3 \), such as CMC bubbletons, CMC cylinders, Smyth surfaces, and N-noids. The pictures of these CMC surfaces in \( \mathbb{H}^3 \) can be viewed at the GANG’s gallery of CMC surfaces ([http://www.gang.umass.edu/](http://www.gang.umass.edu/)).

2 Integrable Systems of Constant Mean Curvature Surfaces in Hyperbolic 3-Space \( \mathbb{H}^3 \)

Let us consider the 4-dimensional Lorentzian space

\[
\mathbb{R}^{3,1} = \{(x^0, x^1, x^2, x^3)|ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2\}.
\]

The hyperbolic 3-space is the spacelike 3-manifold

\[
\mathbb{H}^3 = \mathbb{H}^3(-1) = \{x \in \mathbb{R}^{3,1} | <x, x> = -1, x^0 > 0 \}.
\]
of constant sectional curvature $-1$.

Note that the following correspondence
\[
x = (x^0, x^1, x^2, x^3) \mapsto x = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix}
\]
provides an identification between $\mathbb{H}^3$ and the space of $2 \times 2$ Hermitian matrices. The complex Lie group $\text{SL}(2, \mathbb{C})$ acts isometrically and transitively on $\mathbb{H}^3(-1)$ by
\[
\text{SL}(2, \mathbb{C}) \times \mathbb{H}^3(-1) \rightarrow \mathbb{H}^3(-1)
\]
\[
(g, h) \mapsto g \cdot h = ghg^*,
\]
where $g^* = \bar{g}'$. Thus, $\mathbb{H}^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2)$.

Let $M$ be a simply connected Riemann surface and $f : M \rightarrow \mathbb{H}^3$ an immersion. Consider $(e^0 = f, e^1, e^2, e^3)$ the local orthonormal frame of the immersion $f$. Then we have
\[
de e^0 = df = \omega_i e^i, \ i = 1, 2,
\]
\[
de e^j = \omega_j e^0 + \omega_i^j e^i, \ i = 1, 2, 3,
\]
where $\omega_j^i = -\omega_i^j$ and $\omega_i^i = 0$.

For the adapted frame of the immersion $f$, Cartan’s structure equations can be written on short as
\[
d\omega_i = \omega_j^i \wedge \omega_j
\]
\[
d\omega_j^i + \omega_k^i \wedge \omega_j^k + \omega_i \wedge \omega_j = 0.
\]
Let $\sigma_i, \ i = 0, 1, 2, 3$, be the following matrices
\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
These matrices are called Pauli spin matrices.

By the action of $\text{SL}(2, \mathbb{C})$ on $\mathbb{H}^3$, there exists a map $F$ from an open set in $M$ to $\text{SL}(2, \mathbb{C})$ such that
\[
F(\sigma_i) = F\sigma_i F^* = e^i, \ i = 0, 1, 2, 3.
\]
This map represents the local moving frame associated to the immersion $f$. Let $\Omega := F^{-1}dF \in \text{sl}(2, \mathbb{C})$. The Gauss and Codazzi equations are equivalent to
\[
d\Omega + \frac{1}{2}[\Omega \wedge \Omega] = 0,
\]
which is the null curvature condition of the Maurer-Cartan (connection) form Ω.

It is well known [3, for example] that every surface with constant mean curvature (in $E^3$, $S^3$, $H^3$) admits conformal (isothermal) coordinates, $z = x + iy$, so that

$$I = ds^2 = df \otimes df = e^{2u}dz \otimes d\bar{z}.$$ 

So, we may rewrite $f : D \rightarrow \mathbb{H}^3$ (by abuse of notation), with $D \subset \mathbb{C}$ open, simply connected, and containing the origin.

Thus, $<f_z, f_{\bar{z}}>$, $<f_z, f_{\bar{z}}>$, and $<f, f_{\bar{z}}>$ are related by $e^{2u}$, where $f_z = \frac{1}{2}(f_x - if_y)$, $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$. We also have

$$<f_z, N> = <f_{\bar{z}}, N> = 0,$$

$$<N, N> = 1.$$

The form $Qdz^2 := <f_{zz}, N> dz^2$ is called Hopf differential.

It is also well known [3] that an immersion $f$ has constant mean curvature if and only if the Hopf differential is holomorphic. The second fundamental form is defined as

$$II = -<df, dN> = ldx^2 + 2mdxdy + ndy^2.$$  

Then

$$<f_{zz}, N> = \frac{1}{4}(l - n - 2im) = Q,$$

$$<f_{\bar{z}}, N> = \frac{1}{4}(l + n) = \frac{1}{2}He^{2u},$$

where $N \equiv e_3$ represents the usual Gauss map (unit normal vector field on $M$). The Maurer-Cartan form $\Omega$ can be written as

$$\Omega = Adz + Bd\bar{z},$$

where

$$A = \begin{pmatrix} \frac{1}{2}u_z & \frac{1}{2}e^u(1 + H) \\ -e^{-u}Q & -\frac{1}{2}u_z \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2}u_{\bar{z}} & e^{-u}Q \\ \frac{1}{2}e^u(1 - H) & \frac{1}{2}u_{\bar{z}} \end{pmatrix}.$$  

The moving frame $F$ satisfies the following Lax equations

$$\begin{cases} F_z = FA \\ F_{\bar{z}} = FB \end{cases}. \quad (1)$$

The compatibility condition $F_{z\bar{z}} = F_{\bar{z}z}$ gives

$$A_{\bar{z}} - B_z - [A, B] = 0, \quad (2)$$

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which can be written as

\[
\begin{aligned}
  \left\{ 
  u_{\bar{z}z} - \frac{e^{2u}}{4}(1 - H^2) - e^{-2u}Q\bar{Q} = 0, \\
  Q_{\bar{z}} = 0.
\right.
\end{aligned}
\]  

(3)

3 Spectral Deformations

Let us consider an arbitrary immersion \( f \), of metric conformal factor \( e^{u(z, \bar{z})} \), constant mean curvature \( H \) and Hopf differential \( Qdz^2 \). We start with the desire to find isometric or quasi-isometric spectral deformations of this immersion. We are mainly interested in new surfaces characterized by the triple \((\bar{u}, \bar{H}, \bar{Q})\), such that they satisfy the same Gauss and Codazzi equations as the initial \((u, H, Q)\). Here, it should be remarked that our approach is different from the one in [3] and [14]. More precisely, these references consider a spectral transformation given by the complex non-zero parameter \( \lambda \) such that

\[
(1 + H) \rightarrow \lambda(1 + H), \quad (1 - H) \rightarrow \lambda^{-1}(1 - H)
\]

(see, for example, [14, formulas 23-25]); this accordingly changes the matrices \( A \) and \( B \) of the Lax system.

\( H \) and \( u \) are both real-valued, while \( Q \) is complex-valued. In our opinion, the following two spectral deformations have more geometric meaning:

A). A positive real parameter \( s \) is introduced in the second term of the Gauss-Codazzi equation via \((1 + H) \rightarrow s(1 + H)\) and \((1 - H) \rightarrow s^{-1}(1 - H)\), so that the new \((\bar{u}, \bar{H}, \bar{Q})\) satisfy the same Gauss-Codazzi equation.

B). A complex parameter of modulus one, \( \theta = e^{it} \), is introduced in the third term of the Gauss-Codazzi equation, via \( Q \rightarrow \theta^{-2}Q \), so that the Gauss-Codazzi equation does not change.

A). \( s \)-Spectral Deformations. Proper Deformations

The name of spectral parameter comes from mathematical physics, where it was interpreted as a spectral parameter in a corresponding linear problem.

Definition 1. We call \( s \)-spectral deformation of the constant mean curvature immersion \( f \) the effect (on the surface) of introducing the positive parameter \( s \) via \((1 + H) \rightarrow s(1 + H)\) and \((1 - H) \rightarrow s^{-1}(1 - H)\), respectively.
This effect depends on the geometric interpretation we give this transformation, that is:

\[ k(1 + H^s) := s(1 + H) \]  

and

\[ k(1 - H^s) := s^{-1}(1 - H), \]

where \( k \) is a nonzero real number.

As a direct consequence of equations above, we obtain:

\[ k = \frac{s(1 + H) + s^{-1}(1 - H)}{2} \]  

\[ H^s = \frac{s(1 + H) - s^{-1}(1 - H)}{s(1 + H) + s^{-1}(1 - H)} \]  

**Theorem 2.** For any fixed positive parameter \( s \), the \( s \)-spectral transformation

\[ (1 + H) \rightarrow s(1 + H)(= k(1 + H^s)) \]  

and

\[ (1 - H) \rightarrow s^{-1}(1 - H)(= k(1 - H^s)), \]

deforms an immersion \( f \) of metric \( e^{2u}dzd\bar{z} \), Hopf differential \( Qdz^2 \) and mean curvature \( H \), into a conformal immersion, \( f^s \), of metric \( e^{2u^s}dzd\bar{z} \), Hopf differential \( Q^s dz^2 := k \cdot Qdz^2 \) and mean curvature \( H^s \), as defined by the formulas above.

**Proof.** Note that the Gauss-Codazzi equation satisfied by \((u, H, Q)\) is equivalent to the following Gauss-Codazzi equation satisfied by \((u^s, H^s, Q^s)\):

\[
\begin{cases}
  u_{zz}^s - \frac{e^{2u^s}}{4}[1 - (H^s)^2] - e^{-2u^s}Q^s \overline{Q} = 0, \\
  Q^s_{\bar{z}} = 0,
\end{cases}
\]

where \( u^s := u + \ln |k| \), \( H^s = \frac{s(1 + H) - s^{-1}(1 - H)}{s(1 + H) + s^{-1}(1 - H)} \), and \( Q^s = kQ \).

This spectral deformation of \( f \) to \( f^s \) may be interpreted as a substitute for similarity transformations, since similarity does not exist in hyperbolic 3-space \( \mathbb{H}^3(-1) \).

**Remark 1.** Any such \( s \)-spectral deformation is interesting in itself; it *rescales* both the metric \( I \) and the 2-form \(|Q|^2dz \cdot d\bar{z}\) (by multiplication with the same positive constant) and so the new surface looks similar to the first one, although the mean curvature changes.
We say that the immersion $f^s$ obtained from $f$ via the spectral deformations (8) and (9) is **strongly conformal** to $f$.

**Definition 2.** The $s$-spectral deformation is called proper if $s \neq 1$ and it leaves the metric unmodified, that is, $k = 1$ or $k = -1$ (see Theorem 3).

Note that whenever $s$ is not equal to 1, the deformation is proper iff $s = \frac{|1 - H|}{|1 + H|}$, $H \neq -1, 1$.

In this work, we will use general $s$-deformations (strongly conformal deformations), and will specify those particular instances when deformations are proper (isometric).

**B). $\theta$-Spectral Deformation:**

**Definition 3.** We call $\theta$-spectral deformation of the constant mean curvature immersion $f$ the effect of introducing the $S^1$-parameter $\theta = e^{it}$ such that the Hopf differential changes according to $Q \rightarrow \theta^{-2}Q$.

The $\theta$-deformation does not change the metric or the mean curvature, only the Hopf differential. It gives the well-known family of **associate surfaces**.

**4 The $\lambda$-Spectral Deformation**

Let us consider a simply connected Riemann surface, immersed in $\mathbb{H}^3$. Let the immersion be $f$, of constant mean curvature $H$ and Hopf differential $Qdz^2$.

**Remark 2.** It is easy to see that the two types of spectral deformations have different geometric effects on the surface. We will combine the two deformations, and introduce a parameter that covers both spectral deformations mentioned above.

**Definition 4.** We define $\lambda = s \cdot \theta$, where $s > 0$, and $\theta = e^{it}$. We call $\lambda$ **generalized spectral parameter**.

For the case of an isometric deformation (that is a $\theta$-deformation while $s = 1$, or a theta-deformation combined with a proper $s$-deformation), the mean curvature $H$ and Hopf differential $Q$ remain the same - up to an eventual change in sign.

**Definition 5.** By $\lambda(= s \cdot \theta)$-spectral deformation we mean the effect of performing both of the following deformations on the initial immersion $f$ or mean curvature $H$:

A). an $s$-deformation ($s > 0$),
B). a $\theta = e^{it}$-deformation.
Note that order does not matter: since \( s \)-deformations are independent from \( \theta \)-deformations, they commute.

Case A). gives a genuine (and strongly conformal) surface deformation in general, as described in Theorem 3. The surface stays the same in just two cases: the trivial case \( s = 1 \) (identity) and the case of proper deformation \( (s = \frac{|1-H|}{|1+H|}) \), both being isometries.

Case B). describes the associate family.

While performing a general \( \lambda \)-deformation, that is a \( \theta \)-deformation and an \( s \)-deformation, keep in mind the changes described in Theorem 2. In terms of Lax matrices, we obtain:

\[
A(s, \theta) = \begin{pmatrix}
\frac{1}{2}u_z & \frac{s}{2}e^u(1 + H) \\
-e^{-u}\theta^{-2}Q & -\frac{1}{2}u_z
\end{pmatrix},
\]
\[
B(s, \theta) = \begin{pmatrix}
-\frac{1}{2}u_z & e^{-u}\theta^2Q \\
\frac{s^{-1}}{2}e^u(1 - H) & \frac{1}{2}u_z
\end{pmatrix}.
\]

Remark 3. For loop group reasons, we conjugate these matrices with the \( z \)-independent matrix

\[
G = i \begin{pmatrix}
0 & \theta^{1/2} \\
\theta^{-1/2} & 0
\end{pmatrix},
\]

and obtain the matrices

\[
A^\lambda = \begin{pmatrix}
\frac{1}{2}u_z & \theta^{-1}e^{-u}Q \\
\theta^{-1} \cdot \frac{s}{2}e^u(1 + H) & \frac{1}{2}u_z
\end{pmatrix},
\]
\[
B^\lambda = \begin{pmatrix}
\frac{1}{2}u_z & \theta \cdot \frac{s^{-1}}{2}e^u(1 - H) \\
\theta \cdot e^{-u}Q & -\frac{1}{2}u_z
\end{pmatrix}.
\]

Note that these conjugated matrices will satisfy the Lax system and the compatibility condition associated to it.

Remark 4. While looking for the right type of spectral transformation, eventually an isometric one, instead of our deformation, one may have been tempted to perform the traditional one: \( Q \rightarrow \lambda^{-1}Q \), with \( \lambda \in \mathbb{C}^* \), hoping to obtain a frame \( F \) with the property \( F \cdot \overline{F(\lambda^{-1})} = I \), which in particular would be unitary for \( \lambda \) in \( S^1 \). Note that this type a deformation does not lead to such a frame.

Also, if we made such a choice, the off-diagonal terms that contain \( H + 1 \) and \( H - 1 \) would destroy the hope for a \( \text{su}(2) \)-valued Maurer-Cartan form.
The \( \lambda \) deformation we just introduced is convenient, in the sense that the Maurer-Cartan form becomes a \( \text{su}(2) \)-valued form for a specific real value \( s_0 \) of the parameter \( s \), and all values of \( \theta \) in \( S^1 \).

Via our \( \lambda \) deformation, the frame \( F \) changes to \( F^\lambda \) (which can be considered fixed at a point \( p \in M \)). The Lax system

\[
\begin{align*}
F^\lambda_z &= F^\lambda A^\lambda \\
F^\lambda_{\bar{z}} &= F^\lambda B^\lambda
\end{align*}
\]

can be also written as

\[
(F^\lambda)^{-1} dF^\lambda = \Omega^\lambda,
\]

so that the Maurer-Cartan form \( \Omega^\lambda \) writes

\[
\Omega^\lambda = A^\lambda dz + B^\lambda d\bar{z}
\]

A solution \( F^\lambda \) of the above equation, together with the initial condition \( F^\lambda(0,0,\lambda) = I \), in a simply connected domain \( D \), \( F^\lambda : D \rightarrow \Lambda^s\text{SL}(2,\mathbb{C}) \), is called extended frame corresponding to the spectral deformations \( f \mapsto f^s \), and \( Q \mapsto \theta^{-2}Q \).

Here, \( \Lambda^s\text{SL}(2,\mathbb{C}) \) represents the “twisted” loop group over \( \text{SL}(2,\mathbb{C}) \) given by the automorphism

\[
\sigma : g \mapsto (\text{Ad}_{\sigma_3})(g),
\]

\[
\sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\]

\( \Lambda^s\text{SL}(2,\mathbb{C}) := \{ g : C_s \rightarrow \text{SL}(2,\mathbb{C}) | g(-\lambda) = \sigma(g(\lambda)) \} \), where \( s \) is the absolute value of the parameter \( \lambda \) and \( C_s \) is the circle of center \( O \) and radius \( s \) in the complex plane.

Note these maps could be also written as \( g_s(\theta) : S_1 \rightarrow \text{SL}(2,\mathbb{C}) \), with \( s \) fixed, real and positive, and the property \( g_s(-\theta) = \sigma(g_s(\theta)) \), and that there is no significant difference between these loop groups and the usual loop group considered in \[5\].

We will denote \( \Lambda\text{SL}(2,\mathbb{C}) := \{ g : S^1 \rightarrow \text{SL}(2,\mathbb{C}) | g(-\theta) = \sigma(g(\theta)) \} \),

It is customary to denote by \( \Lambda^s_+\text{SL}(2,\mathbb{C}) \) the set of all maps of \( \Lambda\text{SL}(2,\mathbb{C}) \) that can be holomorphically extended outside the disk enclosed by the circle, and equal to identity at infinity. Also, \( \Lambda_\pm\text{SL}(2,\mathbb{C}) \) stands for those maps that can be holomorphically extended inside the same disk.
Similar notations are used for $\Lambda SU(2)$.

In order to make such a loop group into a complete Banach Lie group, we consider the same $H^p$-norm for $p > \frac{1}{2}$ as used in [5]. Elements of this loop group are matrices with off-diagonal entries that are odd in $\theta$ and diagonal entries that are even in $\theta$. We view the elements as formal series in $\theta$.

Whenever we use loop group factorizations, we will always split inside the loop group $\Lambda SL(2, \mathbb{C})$. The reason why we use loop group factorizations is related to the methods of constructing surfaces starting from the generalized Weierstrass representation formula. Such a method was first presented in [5].

**Theorem 3.** For any associated family of CMC surfaces with given frame $F = F(\theta)$, $\theta \in S^1$, and mean curvature $|H| > 1$, there exists a certain $s$-deformation, for some $s = s_0$, that generates a unitary frame $\tilde{F} = \tilde{F}(\theta) \in \Lambda SU(2)$. The unitary frame $\tilde{F}$ represents the lift of a harmonic map $\tilde{N}$ in $S^2$.

**Proof.** It is easy to see that choosing $s = s_0 := \sqrt{\frac{H-1}{H+1}}$ gives the only deformation that makes (changes) the Maurer-Cartan $\Omega$ into an $\text{su}(2)$-valued form $\tilde{\Omega}$.

Remark that as $\lambda$ we approaches $\lambda_0 = s_0 \cdot \theta$, the mean curvature will go to infinity, and this particular deformation degenerates. From the Gauss-Codazzi equations, it follows that there exists a map $\tilde{F}$ from $D$ to $SU(2)$ such that $\tilde{F}^{-1}d\tilde{F} = \tilde{\Omega}$. The harmonic map $\tilde{N}$ represents the natural projection of the frame $\tilde{F}$ to $S^2$. $\square$

**Definition 6.** We call $\tilde{F}$ the adjusted frame of $F$ and the form $\tilde{F}^{-1}d\tilde{F}$ the adjusted Maurer-Cartan form.

Hence, the explicit form of the adjusted Maurer-Cartan is

$$\tilde{\Omega} = \left( \begin{array}{cc} -\frac{1}{2}u_z & \theta^{-1} \cdot e^{-uQ} \\ \theta^{-1} \cdot e^{u} \sqrt{H^2 - 1} & \theta \cdot e^{-uQ} \end{array} \right) dz + \left( \begin{array}{cc} \frac{1}{2}u_z & -\theta \cdot \frac{1}{2} e^u \sqrt{H^2 - 1} \\ \theta \cdot e^{-uQ} & -\frac{1}{2} u_z \end{array} \right) d\bar{z}. \quad (17)$$

5 Weierstrass Type Representation Formula for CMC Surfaces in $\mathbb{H}^3$

Let $M$ be any simply connected Riemann surface immersed in $\mathbb{H}^3$, via immersion $f$, corresponding to the moving frame $F$. 11
It is well-known that for every local framing $F$ and connection form $\Omega := F^{-1}dF$, we have the identity (Maurer-Cartan equation):

$$d\Omega + \frac{1}{2}[\Omega \wedge \Omega] = 0.$$ 

An arbitrary $\lambda \in \mathbb{C}^*$ deformation transforms $\Omega$ into $\Omega^\lambda := (F^\lambda)^{-1}dF^\lambda = A^\lambda dz + B^\lambda d\bar{z}$, which can be also written as

$$\Omega^\lambda = \Omega'_1 dz + \Omega''_0 + \Omega'_1 d\bar{z},$$

where $\Omega_0 = \Omega'_0 dz + \Omega''_0 d\bar{z}$. Here $\Omega_0$ is, as usual, a one form with values on the diagonal elements of $\mathfrak{sl}(2, \mathbb{C})$, and the rest of the terms are off-diagonal. Hence,

$$\Omega_0' = \begin{pmatrix} -\frac{1}{2}u_z & 0 \\ 0 & \frac{1}{2}u_z \end{pmatrix},$$

$$\Omega_0'' = \begin{pmatrix} \frac{1}{2}u_{\bar{z}} & 0 \\ 0 & -\frac{1}{2}u_{\bar{z}} \end{pmatrix},$$

$$\Omega_1' = \begin{pmatrix} 0 & -\theta^{-1} \cdot e^{-u} \theta \\ \theta^{-1} \cdot \frac{\bar{u}}{2} e^{u}(1 + H) & 0 \end{pmatrix},$$

$$\Omega_1'' = \begin{pmatrix} 0 & \theta \cdot e^{-u} \theta \\ \theta \cdot e^{-u} \theta & 0 \end{pmatrix}.$$

Here $\lambda = s \cdot \theta$ and $\theta = e^{it}$, as usual.

Let us consider the associate form $\tilde{\Omega}$ which is a $\mathfrak{su}(2)$-valued 1-form and hence decomposed via Cartan decomposition, as $\mathfrak{su}(2) = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the diagonal part, and $\mathfrak{p}$ is the off-diagonal one. Thus, the form $\tilde{\Omega}$ writes $\tilde{\Omega} = \tilde{\Omega}_0 + \tilde{\Omega}_1$. Let $\tilde{\Omega}_1 = \tilde{\Omega}_1' + \tilde{\Omega}_1''$ be the splitting into $(1, 0)$ and, respectively, $(0, 1)$-forms. We compare matrices $\tilde{\Omega}_1'$ and $\tilde{\Omega}_1''$ to their correspondents from $\Omega^\lambda$, namely $\Omega_1'$ and $\Omega_1''$. A straight-forward computation leads us to the following result:

**Theorem 4.** Let $\tilde{\Omega} : D \to S^2$ be a non-conformal harmonic map with lift $\tilde{F} = \tilde{F}(\theta) : D \to \Lambda \mathbb{SU}(2)$, where $D$ is a simply connected domain as before.

Let $\tilde{\Omega}(\theta) = \tilde{F}^{-1}d\tilde{F} = \tilde{\Omega}_1' + \tilde{\Omega}_0 + \tilde{\Omega}_1''$. Let $a > 0$ be an arbitrary real constant, and let

$$\beta_1'(a) = \frac{1}{4} \cdot (a - 1) \cdot (\sigma_0 - \sigma_3)\tilde{\Omega}_1'(\sigma_0 + \sigma_3)dz,$$

respectively.
\[ \beta''_1(a) = \frac{1}{4} \cdot (a^{-1} - 1) \cdot (\sigma_0 + \sigma_3) \bar{\Omega}''_1(\sigma_0 - \sigma_3) d\bar{z} \]

Let

\[ \Omega = \Omega(a, \theta) := \bar{\Omega}(\theta) + \beta'(a) + \beta''_1(a). \]

Then we have the following:

i). \[ d\Omega(a, \theta) + \frac{1}{2} [\Omega(a, \theta) \wedge \Omega(a, \theta)] = 0. \]

ii). If \( F \) is a \( \text{SL}(2, \mathbb{C}) \)-valued solution of \( \Omega = F^{-1} dF \), then \( f = F \cdot F^{*} \) is a conformal immersion with isolated singularities and constant mean curvature \( H = \frac{a^2 + 1}{a^2 + 1} \).

**Proof.** One may also see Theorem 4.4, [14], for the construction of a form that is similar to \( \Omega \).

Let us now assume that \( \bar{\Omega}(\theta) = F^{-1} dF \) is of the form given by equation (17). We are looking for a parameter \( \lambda \in \mathbb{C}^* \) such that \( \Omega^\lambda \) coincides with \( \Omega(a, \theta) \). By direct computation, we obtain that \( s = |\lambda| \) must satisfy the relation \( s = a \cdot \sqrt{\frac{H^2 - 1}{H^2 + 1}} \). Therefore, \( \lambda = s \cdot \theta \in C_s \), where \( s \) is uniquely determined by \( a \), from the above formula. The frame \( F^\lambda \) is a solution to \( \Omega = F^{-1} dF \), and its corresponding immersion is \( f^\lambda = F^\lambda \cdot F^{\lambda *}, \) with mean curvature \( H^s \).

We substitute \( s \) in the simplified formula (7) and obtain \( H^s = \frac{a^2 - 1}{a^2 + 1} \). This proves ii). \( \square \)

**Remark 5.** Assuming that the initial Maurer-Cartan form \( \bar{\Omega} \) corresponds to the adjusted frame \( F \) of a certain family of CMC surfaces with initial mean curvature \( H \), let us compute the matrices explicitly. We obtain

\[ \beta'_1 = (a - 1) \begin{pmatrix} 0 & 0 \\ \theta^{-1} \cdot \frac{1}{2} e^{\theta \sqrt{H^2 - 1}} & 0 \end{pmatrix} dz, \]

respectively

\[ \beta''_1 = (a^{-1} - 1) \begin{pmatrix} 0 & -\theta \cdot \frac{1}{2} e^{\theta \sqrt{H^2 - 1}} \\ 0 & 0 \end{pmatrix} d\bar{z} \]

The sum of these matrices is \( \text{Asl}(2, \mathbb{C}) \)-valued. Note that the first defines a \((1, 0)\) form in \( \theta^{-1} \), while the other one is a \((0, 1)\) form in \( \theta \). These expressions will be of use in the next section.

Note that while \( \Omega \) is not \( \text{su}(2) \)-valued, \( \beta'_1 + \beta''_1 \) measures its ‘defect’ from \( \text{su}(2) \). In a sense, this measures the ‘defect’ of the Gauss map \( N \) from being a harmonic map in the symmetric space \( S^2 \).
Let us now denote

\[ G(\theta) = F^\lambda \cdot \tilde{F}^{-1}(\theta) \]  

A very important remark is that \( G \cdot G^* = F \cdot F^* = f^\lambda \). We view the matrix \( G \) exclusively as a function of \( \theta \). It will play a significant role in Section 7.

6 Normalized potentials

The notion of normalized potential was introduced in the most general case - for harmonic maps in symmetric spaces, and their extended frames - [5]. Next, [5] and [19] gave the expression and computation of this potential in particular for the case of constant mean curvature surfaces in Euclidean space. We recall the following adaptation of (see [19]):

**Theorem 5.** Let \( \tilde{N} : D \to S^2 \) be a harmonic map based at identity, and \( \tilde{F}(\theta) : D \to \Lambda SU(2, \mathbb{C}) \) an extended frame corresponding to it. Then there exists a discrete subset \( S \) of \( D - 0 \) such that for any \( z \in D - S \) we have \( \tilde{F}(z, \theta) = \tilde{F}_-(z, \theta) \cdot \tilde{F}_+(z, \theta) \), with \( \tilde{F}_-(\theta) \in \Lambda^+ \text{SL}(2, \mathbb{C}) \) and \( \tilde{F}_+(\theta) \in \Lambda_+ \text{SL}(2, \mathbb{C}) \). Remark that here minus and plus refer to the power series in \( \theta \). The form \( P(z) = \tilde{F}_-^{-1} d\tilde{F}_- \theta \) is a meromorphic \((1,0)\) - form on \( D \), with poles in \( S \). This form is called meromorphic potential or normalized potential.

Conversely, any such harmonic map \( \tilde{N} \) can be constructed from a meromorphic potential by integration, obtaining first \( \tilde{F}_- : D - S \in \Lambda^+ \text{SL}(2, \mathbb{C}) \) where the discrete subset \( S \) consists of poles of \( P \) and then obtaining an extended frame \( \tilde{F} \) of \( f \) via the Iwasawa factorization \( \Lambda \text{SL}(2, \mathbb{C}) = \Lambda \text{SU}(2) \cdot \Lambda^B \text{SL}(2, \mathbb{C}) \), \( \tilde{F} = \tilde{F}_- \cdot \tilde{F}_+^{-1} \).

For details on the Iwasawa factorization, one may consult [13] and [5]. Note that this type of decomposition may be done in minus-plus form or in plus-minus form (with different, unique factors).

The above stated theorem has the following important consequence:

**Theorem 6.** The normalized potential corresponding to constant mean surfaces in the hyperbolic space is identical to the one corresponding to their Euclidean correspondents.

**Proof.** The Lawson correspondence is performed via the same harmonic maps. More precisely, the harmonic maps that represent Gauss maps for the Euclidean CMC surfaces correspond to the adjusted Gauss maps of their hyperbolic counterparts. Via the above theorem, it becomes natural that the normalized potential in the two cases is the same.

\[ \square \]
In view of the above theorem and using formula (3.24) of [19], we deduce the normalized potential corresponding to a CMC surface with $|H| > 1$ in hyperbolic space, of Hopf differential $Qdz^2$ and metric factor $u(z, \bar{z})$, as

$$P = \begin{pmatrix} 0 & -e^{-2h(z)+h(0)}Q \\ \frac{1}{2}e^{2h(z)-h(0)}\sqrt{H^2-1} & 0 \end{pmatrix}$$

where $h(z) := u(z, 0)$ is the holomorphic part of $u(z, \bar{z})$.

Remark that $\theta^{-1}P$ can be deduced directly from the form $\tilde{\Omega}_1'$, as we had expected.

Note that we did not use Lawson’s correspondence in order to obtain this result. In some other words, we did not ‘cheat’, by replacing some Euclidean mean curvature $c$ with its hyperbolic correspondent $\sqrt{c^2-1}$. The normalized potential $P$ that we arrived at simply came as a byproduct of our loop group techniques!

In the spirit of Wu, the holomorphic part $e^{2h(z)}$ of the conformal factor $e^{2u(z,\bar{z})}$ in the induced metric $e^{2u}dzd\bar{z}$ on a CMC immersion $f$ in the hyperbolic space is meromorphic on the entire domain $D$. This meromorphic function and the Hopf differential uniquely determine the induced metric and the surface, up to spectral deformations.

The Weierstrass type data (potential) is the “genetic material” (like the classical Weierstrass representation formula for minimal surfaces in $E^3$) for surface construction.

7 Constructing CMC surfaces in $\mathbb{H}^3$

This represents a DPW type of algorithm to be used in constructing CMC surfaces in $\mathbb{H}^3$:

i). Start from a normalized potential $P$.

Solve the initial value problem

$$\theta^{-1}P = \tilde{F}_-^{-1}d\tilde{F}_-$$

with $\tilde{F}_-(z = 0, \theta) = I$.

ii). Given the solution $\tilde{F}_-$ from i), perform the Iwasawa decomposition in $\Lambda SL(2, \mathbb{C}) = \Lambda SU(2) \cdot \Lambda B SL(2, \mathbb{C})$,

$$\tilde{F}_- = \tilde{F} \cdot \tilde{F}_+^{-1}$$

in order to obtain the $\Lambda SU(2)$-valued extended frame $\tilde{F}$.

Denote by $\tilde{F}_0$ be the coefficient of $\theta^0$ in the $\theta$-series expansion of $\tilde{F}$. 

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iii). Write $\Omega = \tilde{F}^{-1}d\tilde{F}$ and name it adjusted Maurer-Cartan form. Separate its components $\Omega'_1$ and $\Omega''_1$. Pick a value $a > 0$. Apply formulas of Theorem 4 in order to obtain $\beta'_1$ and $\beta''_1$.

iv). Using $\beta'_1$ from iii), solve the initial value problem

$$G^{-1}_-dG_- = \tilde{F}_0 \cdot \beta'_1 \cdot \tilde{F}_0^{-1},$$

with $G_-(z = 0, \theta) = I$.

v). Using $\beta''_1$ from iii), solve the initial value problem

$$G^{-1}_+dG_+ = \tilde{F}_0 \cdot \beta''_1 \cdot \tilde{F}_0^{-1},$$

with $G_+(z = 0, \theta) = I$.

vi). Compute

$$L := G^{-1}_- \cdot G_+,$$

where $G_-$ is the solution of iv), and $G_+$ is the solution of v).

Then split again,

$$L = p_+ \cdot p_-^{-1}.$$

Let

$$G := G_- \cdot p_+.$$

vii). $f := GG^*$ will be an associate family of immersions of mean curvature $H = \frac{a^2 + 1}{a^2 - 1}$, corresponding to the normalized potential (Weierstrass representation) $P$. (Note that there is such a family for each value of $a > 0$).

The extended frame of this immersion is the $\Delta SL(2, \mathbb{C})$-valued frame $F := G \cdot \tilde{F}$, and $f = FF^*$. The associate frame is $\tilde{F}$, which is $SU(2)$-valued. The Gauss map $N$ of this immersion is not harmonic, but the adjusted Gauss map $\tilde{N}$ is harmonic, and it represents the natural projection on $S^2$ of the associate frame $\tilde{F}$.

**Proof.** Steps i) and ii) represent a parallel to the standard DPW procedure for CMC surfaces in Euclidean space. They come as a direct consequence of Theorems 5 and 6.

Step iii) is justified by Theorem 4.

For the rest of the steps, let us note the consequences of defining $F := G \cdot \tilde{F}$. A direct consequence is that
\[ G^{-1}dG = \tilde{F}[\beta'_1 + \beta''_1]\tilde{F}^{-1}. \]

Iwasawa factorization for \( G \) (both ways) give \( G_- \) and \( G_+ \), uniquely, such that
\[ G = G_- \cdot p_+ \quad \text{and} \quad G = G_+ \cdot p_- . \]
After replacing these expressions of \( G \) into the formula for \( G^{-1}dG \) and differentiating, we compare the terms corresponding to negative respectively positive powers of \( \theta \). We obtain
\[ G_-^{-1}dG_- = \tilde{F}_0 \cdot \beta'_1 \cdot \tilde{F}_0^{-1}, \]
\[ G_+^{-1}dG_+ = \tilde{F}_0 \cdot \beta''_1 \cdot \tilde{F}_0^{-1}, \]
where \( \tilde{F}_0 \) represents the constant matrix in the \( \theta \)-expansion of \( \tilde{F} \).

The rest of the steps are clear, and represent a standard technique of regaining a matrix from the first factor of its Iwasawa factorization. Since \( F := G \cdot \tilde{F} \), and \( \tilde{F} \) is unitary, the CMC immersion is obtained as \( f = FF^* = GG^* \). The mean curvature of the associated family of \( f \) is \( \frac{a^2+1}{a^2-1} \), as a consequence of Theorem 4.

8 Open Problems

Besides its direct computational applications, the above representation formula will hopefully lead to a better understanding of global period problems for CMC immersions in hyperbolic space, as well as symmetries and singularities.

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References

[1] R. Aiyama, K. Akutagawa, Kenmotsu-Bryant type representation formulas for constant mean curvature surfaces in \( H^3(-c^2) \) and \( S^3_1(c^2) \), Annals of Global Analysis and Geometry 17, 1998, 49-75.

[2] R. Aiyama, K. Akutagawa, Representation formulas for surfaces in \( H^3(-c^2) \) and harmonic maps arising from CMC surfaces, Harmonic Morphisms, Harmonic Maps, and
Related Topics, Chapman Hall/CRC Research Notes in Mathematics, \textbf{413}, 2000, 275-285.

[3] A. I. Bobenko, \textit{Constant mean curvature surfaces and integrable systems}, Russian Math. Surveys, \textbf{46}:4, 1991, 1-45.

[4] R. Bryant, \textit{Surfaces of constant mean curvature one in hyperbolic 3-space}, Astérisque, \textbf{154-155}, 1987, 321-347.

[5] J. Dorfmeister, G. Haak, \textit{Meromorphic potentials and smooth surfaces of constant mean curvature}, Math. Z., \textbf{224}, 1997, 603-640.

[6] J. Dorfmeister, J. Inoguchi, M. Toda, \textit{Weierstrass-type representation of timelike surfaces with constant mean curvature in Minkowski 3-Space}, Differential Geometry and Integrable Systems, Contemporary Mathematics, \textbf{308}, AMS, 2002, 77-100.

[7] J. Dorfmeister, F. Pedit, M. Toda, \textit{Minimal surfaces vis loop groups}, Balkan J. Geom. Appl. \textbf{2}, 1997, 25-40.

[8] J. Dorfmeister, F. Pedit, H. Wu, \textit{Weierstrass-type represetation of harmonic maps into symmetric spaces}, Comm. in Analysis and Geometry \textbf{6}, 1998, 633-668.

[9] A. Fujioka, \textit{Harmonic maps and associated maps from simply connected Riemann surfaces into 3-dimensional space forms}, Tôhoku Math. J., \textbf{47}, 1995, 431-439.

[10] M. A. Guest, Y. Ohnita, \textit{Loop group actions on harmonic maps and their applications}, Harmonic Maps and Integrable Systems, A. P. Fordy and J. C. Wood, editors, Aspects of Math., E23, Viewig, Braunschweig/Viesbaden, 1994, 273-292.

[11] F. Helein, \textit{Willmore immersions and loop groups}, Journal of Differential Geometry, \textbf{50}, no. 2, 1998, 331-388.

[12] M. Kilian, S-P. Kobayashi, W. Rossman, N. Schmitt, \textit{Constant mean curvature surfaces in 3-dimensional space forms}, \texttt{arXiv:math.DG/0403366v1}, Mar 2004.

[13] A. N. Pressley, G.B. Segal, \textit{Loop groups}, Oxford University Press, 1986.

[14] C. Qing, C. Yi, \textit{Spectral transformation of constant mean curvature surfaces in $H^3$ and Weierstrass representation}, Science in China, A, \textbf{45}:8, 2002, 1066-1075.

[15] J. Inoguchi, M.Toda \textit{Timelike minimal surfaces via loop groups}, Acta Applicandae Mathematicae, \textbf{83}, no. 3, II, 2004, 313-355.

[16] M. Toda, \textit{Weierstrass-type representation of weakly regular pseudospherical surfaces in Euclidean space}, Balkan J. Geom. Appl., \textbf{7}, 2002, no.2, 87-136.

[17] M. Umehara, K. Yamada, \textit{Complete surfaces of constant mean curvature-1 in the hyperbolic 3-space}, Ann. of Math., \textbf{137}, 1993, 611-638.
[18] M. Umehara, K. Yamada, *Surfaces of constant mean curvature \( c \) in the space \( H^3(-c^2) \) with prescribed hyperbolic Gauss map*, Math. Ann., **304**, 1996, 203-224.

[19] H. Wu, *A simple way for determining the normalized potentials for harmonic maps*, Ann. Global Anal. Geom., **17**, 1999, 189-199.