On an equation involving fractional powers with prime numbers of a special type

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Abstract

We consider the equation \[ p_1^{c_1} + p_2^{c_2} + p_3^{c_3} = N, \]
where \( N \) is a sufficiently large integer, and prove that if \( 1 < c < \frac{15}{16} \), then it has a solution in prime numbers \( p_1, p_2, p_3 \) such that each of the numbers \( p_1 + 2, p_2 + 2, p_3 + 2 \) has at most \( \left\lfloor \frac{95}{17-16c} \right\rfloor \) prime factors, counted with the multiplicity.

1 Introduction and statement of the result

In 1937 I. M. Vinogradov [15] proved that for every sufficiently large odd integer \( N \) the equation
\[ p_1 + p_2 + p_3 = N \]  \hspace{1cm} (1)
has a solution in prime numbers \( p_1, p_2, p_3 \).

Analogous problem involving diophantine inequality was considered in 1952 by Piatetski-Shapiro [9]. In 1992, Tolev [13] established that if \( 1 < c < \frac{15}{14} \), then the diophantine inequality
\[ |p_1^{c_1} + p_2^{c_2} + p_3^{c_3} - N| < N^{-\kappa} \]
has a solution in prime numbers \( p_1, p_2, p_3 \) for certain \( \kappa = \kappa(c) > 0 \). Several improvements were made and the strongest of them is due to Baker and Weingartner [1], who improved Tolev’s result with \( 1 < c < \frac{10}{9} \).

In 1995, M. B. Laporta and D. I. Tolev [7] considered the equation
\[ [p_1^{c_1}] + [p_2^{c_2}] + [p_3^{c_3}] = N, \]  \hspace{1cm} (2)
where \( c \in \mathbb{R}, c > 1, N \in \mathbb{N} \) and \([t]\) denotes the integer part of \( t \). They showed that if \( 1 < c < \frac{17}{16} \) and \( N \) is a sufficiently large integer, then the equation (2) has a solution in prime numbers \( p_1, p_2, p_3 \).

For any natural number \( r \), let \( \mathcal{P}_r \) denote the set of \( r \)-almost primes, i.e. the set of natural numbers having at most \( r \) prime factors counted with multiplicity. There are

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many papers devoted to the study of problems involving primes and almost primes. For example, in 1973 J. R. Chen \[4\] established that there exist infinitely many primes \( p \) such that \( p + 2 \in \mathcal{P}_2 \). In 2000 Tolev \[12\] proved that for every sufficiently large integer \( N \equiv 3 \mod 6 \) the equation (1) has a solution in prime numbers \( p_1, p_2, p_3 \) such that \( p_1 + 2 \in \mathcal{P}_2, \ p_2 + 2 \in \mathcal{P}_5, \ p_3 + 2 \in \mathcal{P}_7 \). Thereafter this result was improved by Matomäki and Shao \[8\], who showed that for every sufficiently large integer \( N \equiv 3 \mod 6 \) the equation (1) has a solution in prime numbers \( p_1, p_2, p_3 \) such that \( p_1 + 2, p_2 + 2, p_3 + 2 \in \mathcal{P}_2 \).

Recently Tolev \[14\] established that if \( N \) is sufficiently large, \( E > 0 \) is an arbitrarily large constant and \( 1 < c < \frac{17}{16} \) then the inequality
\[
|p_1^2 + p_2^2 + p_3^2 - N| < (\log N)^{-E}
\]
has a solution in prime numbers \( p_1, p_2, p_3 \), such that each of the numbers \( p_1 + 2, p_2 + 2, p_3 + 2 \) has at most \( \left\lfloor \frac{95}{17-16c} \right\rfloor \) prime factors, counted with the multiplicity.

In this paper, we prove the following

**Theorem 1.** Suppose that \( 1 < c < \frac{17}{16} \). Then for every sufficiently large \( N \) the equation (2) has a solution in prime numbers \( p_1, p_2, p_3 \), such that each of the numbers \( p_1 + 2, p_2 + 2, p_3 + 2 \) has at most \( \left\lfloor \frac{95}{17-16c} \right\rfloor \) prime factors, counted with the multiplicity.

We note that the integer \( \left\lfloor \frac{95}{17-16c} \right\rfloor \) is equal to 95 if \( c \) is close to 1 and it is large if \( c \) is close to \( \frac{17}{16} \).

To prove Theorem 1 we combine ideas developed by Laporta and Tolev \[7\] and Tolev \[14\]. First we apply a version of the vector sieve and then the circle method. In section 4 we find an asymptotic formula for the integrals \( \Gamma'_{1} \) and \( \Gamma'_{4} \) (defined by (19) and (22) respectively). In section 5 we estimate \( \Gamma''_{1} \) and \( \Gamma''_{4} \) (defined by (20) and (23) respectively) and we will complete the proof.

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2 Notations and some lemmas

We use the following notations: with \( [t] \) we denote the integer part of \( t \) and \( \{t\} = t - [t] \) is the fractional part of \( t \). With \( ||t|| \) we denote the distance from \( t \) to the nearest integer. As usual with \( \mu(n) \), \( \varphi(n) \) and \( \Lambda(n) \) we denote respectively, Möbius’ function, Euler’s function and von Mangoldt’s function. Also \( e(t) = e^{2\pi i t} \).

We use Vinogradov’s notation \( A \ll B \), which is equivalent to \( A = O(B) \). If we have simultaneously \( A \ll B \) and \( B \ll A \), then we shall write \( A \asymp B \).

We reserve \( p, p_1, p_2, p_3 \) for prime numbers. By \( \epsilon \) we denote an arbitrarily small positive number, which is not necessarily the same in the different formulae.

With \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{R} \) we will denote respectively the set of natural numbers, the set of integer numbers and the set of real numbers.

Now we introduce some lemmas, which shall be used later.
Lemma 2. Suppose that $D \in \mathbb{R}, D > 4$. There exist arithmetical functions $\lambda^\pm(d)$ (called Rosser’s functions of level $D$) with the following properties:

1. For any positive integer $d$ we have

$$|\lambda^\pm(d)| \leq 1, \quad \lambda^\pm(d) = 0 \text{ if } d > D \text{ or } \mu(d) = 0.$$  

2. If $n \in \mathbb{N}$ then

$$\sum_{d|n} \lambda^-(d) \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^+(d).$$

3. If $z \in \mathbb{R}$ is such that $z^2 \leq D \leq z^3$ and if

$$P(z) = \prod_{2 < p < z} p, \quad B = \prod_{2 < p < z} \left(1 - \frac{1}{p-1}\right), \quad N^\pm = \sum_{d|P(z)} \frac{\lambda^\pm(d)}{\varphi(d)}, \quad s_0 = \frac{\log D}{\log z}, \quad (3)$$

then we have

$$B \leq N^+ \leq B \left(F(s_0) + O\left((\log D)^{-\frac{1}{2}}\right)\right),$$  

$$B \geq N^- \geq B \left(f(s_0) + O\left((\log D)^{-\frac{1}{2}}\right)\right),$$  

where $F(s)$ and $f(s)$ satisfy

$$f(s) = 2e^\gamma s^{-1}\log(s - 1), \quad F(s) = 2e^\gamma s^{-1} \text{ for } 2 \leq s \leq 3. \quad (6)$$

Here $\gamma$ is Euler’s constant.

Proof. See Greaves [5, Chapter 4]. \qed

Lemma 3. Suppose that $\Lambda_i, \Lambda_i^\pm$ are real numbers satisfying $\Lambda_i = 0$ or 1, $\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+$, $i = 1, 2, 3$. Then

$$\Lambda_1\Lambda_2\Lambda_3 \geq \Lambda_1^-\Lambda_2^+\Lambda_3^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+ + \Lambda_1^-\Lambda_2^+\Lambda_3^- - 2\Lambda_1^+\Lambda_2^+\Lambda_3^-. \quad (7)$$

Proof. The proof is similar to the proof of Lemma 2 in [2]. \qed

Lemma 4. Suppose that $x, y \in \mathbb{R}$ and $M \in \mathbb{N}, M \geq 3$. Then

$$e(-x\{y\}) = \sum_{|m| \leq M} c_m e(my) + O\left(\min\left(1, \frac{1}{M||y||}\right)\right),$$

where

$$c_m = \frac{1 - e(-x)}{2\pi i(x + m)}. \quad (8)$$

Proof. Proof can be find in Buriev [3, Lemma 12]. \qed
Lemma 5. Consider the integral
\[ I = \int_{a}^{b} e(f(x)) \, dx, \]
where \( f(x) \) is real function with continuous second derivative and monotonic first derivative. If \( |f'(x)| \geq h > 0 \), for all \( x \in [a, b] \) then \( I \ll h^{-1} \).

Proof. See [10, p. 71]. \( \square \)

3 Beginning of the proof

Let \( \eta, \delta, \xi \) and \( \mu \) are positive real numbers depending on \( c \). We shall specify them later. Now we only assume that they satisfy the conditions
\[ \xi + 3\delta < \frac{12}{25}, \quad 2 < \frac{\delta}{\eta} < 3, \quad \mu < 1. \] (9)

We denote
\[ X = N^{\frac{1}{2}}, \quad z = X^\eta, \quad D = X^{\delta}, \quad \Delta = X^{\xi-c} \] (10)
and
\[ P(z) = \prod_{2 < p < z} p. \] (11)

Consider the sum
\[ \Gamma = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3). \] (12)

If we prove the inequality
\[ \Gamma > 0, \] (13)
then the equation (2) would have a solution in primes \( p_1, p_2, p_3 \) satisfying conditions in the sum \( \Gamma \). Suppose that \( p_i + 2 \) has \( l \) prime factors, counted with multiplicity. From (10), (11) and \( (p_i + 2, P(z)) = 1 \) we have
\[ X + 2 \geq p_i + 2 \geq z^l = X^{\eta l} \]
and then \( l < \frac{1}{\eta} \). This means that \( p_i + 2 \) has at most \( \lfloor \eta^{-1} \rfloor \) prime factors counted with multiplicity. Therefore, to prove Theorem 1 we have to establish (13) for an appropriate choice of \( \eta \).

For \( i = 1, 2, 3 \) we define
\[ \Lambda_i = \sum_{d \mid (p_i + 2, P(z))} \mu(d) = \begin{cases} 1 & \text{if } (p_i + 2, P(z)) = 1, \\ 0 & \text{otherwise}. \end{cases} \] (14)
Then we find that

\[ \Gamma = \sum_{\mu X < p_1, p_2, p_3 \leq X} \Lambda_1 \Lambda_2 \Lambda_3 (\log p_1)(\log p_2)(\log p_3). \]

We can write \( \Gamma \) as

\[ \Gamma = \sum_{\mu X < p_1, p_2, p_3 \leq X} \Lambda_1 \Lambda_2 \Lambda_3 (\log p_1)(\log p_2)(\log p_3) \int_{-\frac{1}{2}}^{\frac{1}{2}} e(\alpha([p_1] + [p_2] + [p_3] - N)) d\alpha. \]

Suppose that \( \lambda^{\pm}(d) \) are the Rosser functions of level \( D \). Let also denote

\[ \Lambda^{\pm}_i = \sum_{d \mid (p_i+2, P(z))} \lambda^{\pm}(d), \quad i = 1, 2, 3. \tag{15} \]

Then from Lemma 2, (14) and (15) we find that

\[ \Lambda^{-}_i \leq \Lambda_i \leq \Lambda^{+}_i. \]

We use Lemma 3 and find that

\[ \Gamma \geq \sum_{\mu X < p_1, p_2, p_3 \leq X} \Lambda^{-}_1 \Lambda^{+}_2 \Lambda^{+}_3 (\log p_1)(\log p_2)(\log p_3) \int_{-\frac{1}{2}}^{\frac{1}{2}} e(\alpha([p_1] + [p_2] + [p_3] - N)) d\alpha. \tag{16} \]

Hence, we get

\[ \Gamma \geq 3\Gamma_1 - 2\Gamma_4. \]

Let first consider \( \Gamma_1 \). We have

\[ \Gamma_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} e(-N\alpha)L^{-}(\alpha)L^{+}(\alpha)^2 d\alpha, \tag{17} \]

where

\[ L^{\pm}(\alpha) = \sum_{\mu X < p \leq X} (\log p)e(\alpha[p]) \sum_{d \mid (p+2, P(z))} \lambda^{\pm}(d). \]
Changing the order of summation in $L^\pm(\alpha)$, we get

$$L^\pm(\alpha) = \sum_{d|P(z)} \lambda^\pm(d) \sum_{\mu X < p \leq X \atop p+2 \equiv 0 \pmod{d}} (\log p)e(\alpha [p^2]).$$

We divide the integral from (17) into two parts:

$$\Gamma_1 = \Gamma'_1 + \Gamma''_1,$$  \hspace{1cm} (18)

where

$$\Gamma'_1 = \int_{|\alpha|<\Delta} e(-N\alpha)L^-(\alpha)L^+(\alpha)^2 d\alpha,$$  \hspace{1cm} (19)

$$\Gamma''_1 = \int_{\Delta<|\alpha|<\frac{1}{2}} e(-N\alpha)L^-(\alpha)L^+(\alpha)^2 d\alpha.$$  \hspace{1cm} (20)

Similarly, for $\Gamma_4$ we have

$$\Gamma_4 = \Gamma'_4 + \Gamma''_4,$$  \hspace{1cm} (21)

where

$$\Gamma'_4 = \int_{|\alpha|<\Delta} e(-N\alpha)L^+(\alpha)^3 d\alpha,$$  \hspace{1cm} (22)

$$\Gamma''_4 = \int_{\Delta<|\alpha|<\frac{1}{2}} e(-N\alpha)L^+(\alpha)^3 d\alpha.$$  \hspace{1cm} (23)

### 4 The integrals $\Gamma'_1$ and $\Gamma'_4$

We shall find an asymptotic formula for the integrals $\Gamma'_1$ and $\Gamma'_4$ defined by (19) and (22), respectively. The arithmetic structure of the Rosser weights $\lambda^\pm(d)$ are not important here, so we consider a sum of the form

$$L(\alpha) = \sum_{d \leq D} \lambda(d) \sum_{\mu X < p \leq X \atop p+2 \equiv 0 \pmod{d}} (\log p)e(\alpha [p^2]),$$  \hspace{1cm} (24)

where $\lambda(d)$ are real numbers satisfying

$$|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \text{ if } 2|d \text{ or } \mu(d) = 0.$$  \hspace{1cm} (25)
It is easy to see that
\[
L(\alpha) = \sum_{d \leq D} \lambda(d) \sum_{\mu x < p \leq X \atop p+2 \equiv 0 \pmod{d}} (\log p)e(\alpha \rho \gamma + O(|\alpha|))
\]
\[
= \sum_{d \leq D} \lambda(d) \sum_{\mu x < p \leq X \atop p+2 \equiv 0 \pmod{d}} (\log p)e(\alpha \rho \gamma)(1 + O(|\alpha|))
\]
\[
= \sum_{d \leq D} \lambda(d) \sum_{\mu x < p \leq X \atop p+2 \equiv 0 \pmod{d}} (\log p)e(\alpha \rho \gamma) + O \left( \sum_{d \leq D} \sum_{p \leq X \atop p+2 \equiv 0 \pmod{d}} (\log p)|\alpha| \right).
\]
If \( D \leq X \) then
\[
L(\alpha) = \mathcal{T}(\alpha) + O(\Delta X(\log X)),
\] (26)
where
\[
\mathcal{T} = \sum_{d \leq D} \lambda(d) \sum_{\mu x < p \leq X \atop p+2 \equiv 0 \pmod{d}} (\log p)e(\alpha \rho \gamma).
\]
For \( \mathcal{T}(\alpha) \) we use the asymptotic formula from Lemma 10 in [14]. From (9) and (10) we see that, when \( |\alpha| < \Delta \), then for every constant \( A > 0 \), we have
\[
\mathcal{T}(\alpha) = \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(\alpha) + O(X(\log X)^{-A}),
\] (27)
where
\[
I(x) = \int_{\mu x}^{X} e(\alpha t \gamma)dt.
\] (28)
Hence from (10), (26) and (27) we see that if \( \xi < c \) then
\[
L(\alpha) = \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(\alpha) + O(X(\log X)^{-A}).
\] (29)
If \( |\alpha| < \Delta \), then from (3) and (29) we find
\[
L^\pm(\alpha) = \mathcal{N}^\pm I(\alpha) + O(X(\log X)^{-A}).
\] (30)
Let
\[
\mathcal{M}^\pm = \mathcal{M}^\pm(\alpha) = \sum_{d \leq D} \frac{\lambda^\pm(d)}{\varphi(d)} I(\alpha) = \mathcal{N}^\pm I(\alpha).
\] (31)
It is easy to see that
\[
\mathcal{M}^\pm \ll (\log X)|I(\alpha)|.
\] (32)
We use (30), (31) and the identity
\[
L^-(L^+)^2 = (L^- - \mathcal{M}^-)(L^+)^2 + (L^+ - \mathcal{M}^+ \mathcal{M}^- L^+ + (L^+ - \mathcal{M}^+) \mathcal{M}^+ \mathcal{M}^- + \mathcal{M}^-(\mathcal{M}^+)^2,
\]
to find that
\[ |L^-(L^+)^2 - \mathcal{M}^-(\mathcal{M}^+)^2| \ll X(\log X)^{-A} \left( |L^+|^2 + |\mathcal{M}^-|^2 + |\mathcal{M}^+|^2 \right). \] (33)

Let
\[ B = \int_{|\alpha|<\Delta} e(-N\alpha)\mathcal{M}^-(\mathcal{M}^+)^2 d\alpha. \] (34)

From (19), (32) – (34) we have
\[ \Gamma'_1 - B \ll X(\log X)^{2-A} \left( \int_{|\alpha|<\Delta} |L^+(\alpha)|^2 d\alpha + \int_{|\alpha|<\Delta} |I(\alpha)|^2 d\alpha \right). \]

We need the next lemma, which is an analog of Lemma 11 in [14].

**Lemma 6.** If \( \Delta \leq X^{1-c} \), then for the sum \( L(\alpha) \) defined by (24) and for the integral \( I(\alpha) \) defined by (28) we have
\[ \int_{|\alpha|<\Delta} |L(\alpha)|^2 d\alpha \ll X^{2-c}(\log X)^{6}, \]
\[ \int_{|\alpha|<\Delta} |I(\alpha)|^2 d\alpha \ll X^{2-c}(\log X)^{6}, \]
\[ \int_{|\alpha|<1} |I(\alpha)|^2 d\alpha \ll X(\log X)^5. \]

**Proof.** The proof is similar to the proof of Lemma 11 in [14]. \( \square \)

Hence
\[ \Gamma'_1 - B \ll X^{3-c} (\log X)^{8-A}. \] (35)

Consider now the integral
\[ B_1 = \int_{-\infty}^{\infty} e(-N\alpha)I(\alpha)^3 d\alpha. \] (36)

Using the method in Lemma 5.6.1 in [11] we find
\[ B_1 \gg X^{3-c}. \] (37)

For \( I(\alpha) \) we apply Lemma 5 and see that \( I(\alpha) \ll |\alpha|^{-1}X^{1-c} \). Then from (10), (31), (34) and (36) we find
\[ |\mathcal{N}^-(\mathcal{N}^+)^2B_1 - B| \ll (\log X)^3 \int_{|\alpha|>\Delta} |I(\alpha)|^3 d\alpha \ll (\log x)^3 X^{3-c-2\xi}. \] (38)

If \( A = 12 \), then using (35) and (38) we find
\[ \Gamma'_1 = \mathcal{N}^-(\mathcal{N}^+)^2B_1 + O(X^{3-c}(\log X)^{-4}). \] (39)

We proceed with \( \Gamma'_2 \) in the same way and prove that
\[ \Gamma'_4 = (\mathcal{N}^+)^3B_1 + O(X^{3-c}(\log X)^{-4}). \] (40)
5 The estimation of the integrals $\Gamma''_1$ and $\Gamma''_4$ and the end of the proof

In this section we consider the integrals $\Gamma''_1$ and $\Gamma''_4$ defined by (20) and (23) respectively. We show that the integrals $\Gamma''_1$ and $\Gamma''_4$ are small enough. Now we assume that

$$\xi = \frac{16c - 5}{32}, \quad \delta = \frac{17 - 16c}{32}. \quad (41)$$

It is obvious that for $\Gamma''_1$ defined by (20) we have

$$\Gamma''_1 \ll \max_{\Delta \leq |\alpha| \leq \frac{1}{2}} |L^-(\alpha)| \int_0^1 |L^+(\alpha)|^2 d\alpha. \quad (42)$$

We use Lemma 6 and find that

$$\Gamma''_1 \ll X (\log X)^\delta \max_{\Delta \leq |\alpha| \leq \frac{1}{2}} |L^-(\alpha)|. \quad (43)$$

From (24) we see that

$$L(\alpha) = L_1(\alpha) + O \left( X^{\frac{1}{2} + \epsilon} \right),$$

where

$$L_1(\alpha) = \sum_{d \leq D} \lambda(d) \sum_{\mu X < n \leq X \atop n + 2 \equiv 0 \pmod{d}} \Lambda(n) e(\alpha n^\epsilon).$$

Let $M = X^\kappa$, for some $\kappa$, which will be specified later. Now for $L_1(\alpha)$ applying Lemma 4 with parameters $x = \alpha$, $y = n^\epsilon$ and $M$ (we note that $[t] = t - \{t\}$). Hence

$$L_1(\alpha) = \sum_{|m| \leq M} c_m \sum_{d \leq D} \lambda(d) \sum_{\mu X < n \leq X \atop n + 2 \equiv 0 \pmod{d}} \Lambda(n) e((\alpha + m)n^\epsilon) +$$

$$+ O \left( X^{\epsilon} \sum_{\mu X < n \leq X} \min \left( 1, \frac{1}{M||n^\epsilon||} \right) \right). \quad (44)$$

We need the following

**Lemma 7.** Suppose that $D$, $\Delta$ are define by (10) and $\xi$, $\delta$ are specified by (11). Suppose also that $\lambda(d)$ satisfy (25) and $c_m$ are define by (8). Then

$$\max_{\Delta \leq |\alpha| \leq M + 1} \left| \sum_{|m| \leq M} c_m \sum_{d \leq D} \lambda(d) \sum_{\mu X < n \leq X \atop n + 2 \equiv 0 \pmod{d}} \Lambda(n) e(\alpha n^\epsilon) \right| \ll$$

$$\ll x^{\epsilon} \left( X^{\frac{1}{2} + \frac{\epsilon}{2}} D M^{\frac{1}{2}} + X^{1 - \frac{\epsilon}{2}} \Delta^{-\frac{1}{2}} + X^{\frac{1}{2} + \frac{\epsilon}{2}} D M^{\frac{1}{2}} + X^\frac{1}{2} + X^{1 - \frac{\epsilon}{2}} D \Delta^{-\frac{1}{2}} + X^{1 - \frac{\epsilon}{2}} \Delta^{-\frac{1}{2}} \right).$$
Proof. See Lemma 15 in [14]. □

Also we need the estimate
\[
\sum_{\mu X < n \leq X} \min \left(1, \frac{1}{M||n^c||} \right) \ll X^\varepsilon \left(XM^{-1} + M^{\frac{1}{2}} X^{\frac{1}{2}} \right).
\]

(45)

For the proof we note that the Fourier series of \(\min \left(1, \frac{1}{M||n^c||} \right)\) is given by
\[
\min \left(1, \frac{1}{M||n^c||} \right) = \sum_{k \in \mathbb{N}} b_M(k)e(kn^c),
\]

(46)

where the Fourier coefficients satisfy
\[
|b_M(k)| \leq \begin{cases} 
\frac{4 \log M}{M} & \text{if } k \in \mathbb{Z}, \\
\frac{M}{n^2} & \text{if } k \in \mathbb{Z}, n \neq 0.
\end{cases}
\]

(47)

From (46) we get
\[
\sum_{\mu X < n \leq X} \min \left(1, \frac{1}{M||n^c||} \right) = \sum_{\mu X < n \leq X} \sum_{k \in \mathbb{N}} b_M(k)e(kn^c).
\]

(48)

Changing the order of summation in last formula we get
\[
\sum_{\mu X < n \leq X} \min \left(1, \frac{1}{M||n^c||} \right) = \sum_{k \in \mathbb{N}} b_M(k) H(k),
\]

where
\[
H(k) = \sum_{\mu X < n \leq X} e(kn^c).
\]

Now using (47) and (48) and the identity \(|H(k)| = |H(-k)|\) we find
\[
\sum_{\mu X < n \leq X} \min \left(1, \frac{1}{M||n^c||} \right) \ll X \log M + \frac{\log M}{M} \sum_{1 \leq k \leq M} |H(k)| + M \sum_{k > M} \frac{|H(k)|}{k^2}.
\]

(49)

If \(\theta(x) = kx^c\), then \(\theta''(x) = c(c-1)kx^{c-2} \propto kX^{c-2}\) uniformly for \(x \in [\mu X, X]\). Hence, we can apply Van der Corput’s theorem (see [12], chapter 1, Theorem 5) to get
\[
H(k) \ll k^{\frac{1}{2}} X^{\frac{c}{2}} + k^{-\frac{1}{2}} X^{1-\frac{c}{2}}.
\]

(50)

Hence from (49) and (50) we prove (45).

When combining Lemma 7 (43) – (45) we find that
\[
\max_{\Delta \leq \alpha \leq M + 1} |L(\alpha)| \ll x^\varepsilon \left(X^{\frac{1}{2}+\frac{c}{2}} DM^{\frac{1}{2}} + X^{1-\frac{c}{2}} \Delta^{-\frac{1}{2}} + X^{\frac{1}{2}+\frac{c}{2}} D^{\frac{1}{2}} M^{\frac{1}{2}} + \\
+ X^{\frac{1}{2}} + X^{1-\frac{c}{2}} D M^{-1} \Delta^{-\frac{1}{2}} + X^{1-\frac{c}{2}} \Delta^{-\frac{1}{2}} + XM^{-1}\right).
\]
Then from last formula, (10) and (42) we find
\[ \Gamma'' \ll x^\varepsilon \left( X^{\frac{1}{3}+\frac{\delta}{2}+\frac{\varepsilon}{2}} + X^{\frac{7}{10}+\frac{2\delta}{5}+\frac{\varepsilon}{5}} + X^{\frac{11}{15}} + X^{\frac{2}{3}-\frac{\delta}{5}} + X^{2-\kappa} \right). \] (51)

If we choose \( \kappa = \frac{8c-5}{96} \), then from (41) and (51) we conclude that if \( 1 < c < \frac{17}{16} \) then
\[ \Gamma'' \ll X^{3-c-\varepsilon}. \]

From (15), (18), (21) and (37) – (40) we conclude that
\[ \Gamma \geq |3N^- - 2N^+| |N^+|^3 B_1 + O(X^{3-c}(\log x)^4). \] (52)

Now we shall find a lower bound for the difference \( 3N^- - 2N^+ \). It is easy to see that
\[ B \approx (\log X)^{-1}. \] (53)

From (4) and (5) we see that
\[ 3N^- - 2N^+ \geq B(3f(s_0) - F(s_0)) + O\left(\log X^{-\frac{4}{3}}\right), \]
where \( s_0 \) is defined by (3) and \( F(s) \) and \( f(s) \) are defined by (5). If we choose \( s_0 = 2,95 \) then from (3), (10) and (11) we find
\[ \eta = \frac{\delta}{2,95} = \frac{17 - 16c}{94,4} \]

and also from (6) we find \( 3f(s_0) - F(s_0) > 0. \)

Now from (4), (37), (52) and (53) we obtain
\[ \Gamma \gg X^{3-c}(\log X)^{-3}. \]

Therefore \( \Gamma > 0 \) and this proves Theorem 1. \( \square \)

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