THE STABLE REGULARITY LEMMA REVISITED

MARYANTHE MALLIARIS AND ANAND PILLAY

(Communicated by Mirna Džamonja)

ABSTRACT. We prove a regularity lemma with respect to arbitrary Keisler measures \( \mu \) on \( V \), \( \nu \) on \( W \) where the bipartite graph \((V,W,R)\) is definable in a saturated structure \( M \) and the formula \( R(x,y) \) is stable. The proof is rather quick, making use of local stability theory. The special case where \((V,W,R)\) is pseudofinite, \( \mu \), \( \nu \) are the counting measures, and \( M \) is suitably chosen (for example a nonstandard model of set theory), yields the stable regularity theorem of a work by the first author and S. Shelah, though without explicit bounds or equitability.

1. INTRODUCTION

We refer to [3] for a discussion of Szemerédi’s Regularity Lemma and various elaborations on it. Our context is a saturated model \( M \) of an arbitrary theory, and a definable (with parameters) bipartite graph \((V,W,R)\) where the edge relation \( R(x,y) \) is stable. Stability of \( R \) means that for some \( k \) there do not exist \( a_i \in V \), \( b_i \in W \) for \( i \leq k \) such that \( R(a_i,b_j) \) holds iff \( i \leq j \). We also have global Keisler measures \( \mu_x \) on \( V \) and \( \nu_y \) on \( W \) (i.e. finitely additive probability measures on the Boolean algebras of definable subsets of \( V,W \) respectively). By a \( \Delta \)-formula we mean a finite Boolean combination of things like \( R(x,b) \) and \( x = a \), and by a \( \Delta^*(x,y) \)-formula we mean a finite Boolean combination of things like \( R(a,y) \) and \( y = b \). We don’t actually need the = formulas but it is convenient to include them. By convention the \( x \) variable is restricted to \( V \) and \( y \) variable to \( W \) so by definition a \( \Delta \)-formula (\( \Delta^* \)-formula) defines a subset of \( V \) (\( W \)).

**Theorem 1.1.** In the context above, namely a definable graph \((V,W,R)\) in a saturated structure \( M \), where \( R(x,y) \) is stable, and Keisler measures \( \mu \) on \( V \), \( \nu \) on \( W \), we have the following: Given \( \epsilon > 0 \), we can partition \( V \) into finitely many definable sets \( V_1, \ldots, V_m \) each defined by a \( \Delta \)-formula, and also partition \( W \) into finitely many \( W_1, \ldots, W_m \) each defined by a \( \Delta^* \)-formula, such that for each \( V_i,W_j \) exactly one of the following holds:

(i) for all \( a \in V_i \) outside a set of \( \mu \) measure \( \leq \epsilon \mu(V_i) \), for all \( b \in W_j \) outside a set of \( \nu \) measure \( \leq \epsilon \nu(W_j) \), we have \( R(a,b) \), AND DUALLY for all \( b \in W_j \) outside a set of \( \nu \) measure \( \leq \epsilon \nu(W_j) \), for all \( a \in V_i \) outside a set of \( \mu \) measure \( \leq \epsilon \mu(V_i) \), we have \( R(a,b) \) holds, or

(ii) for all \( a \in V_i \) outside a set of \( \mu \) measure \( \leq \epsilon \mu(V_i) \), for all \( b \in W_j \) outside a subset of \( \nu \) measure \( \leq \epsilon \nu(W_j) \), we have \( \neg R(a,b) \), AND DUALLY for all \( b \in W_j \) outside a subset of \( \nu \) measure \( \leq \epsilon \nu(W_j) \), we have \( \neg R(a,b) \) holds.

1761
outside a subset of \( \nu \) measure \( \leq \epsilon \nu(W_j) \), for all \( a \in V_i \) outside a set of \( \mu \) measure \( \leq \epsilon \mu(V_i) \), \( \neg R(a,b) \).

Remark 1.2. We obtain \( \delta \)-regularity for suitable \( \delta \) and in a suitable sense, of each of the \((V_i, W_j, R)(V_i \times W_j)\) (as essentially in Claim 5.17 of [3]): Take \( \delta = \sqrt{2\epsilon} \), and suppose we are in case (i) of the conclusion of the theorem. Suppose \( A \subseteq V_i \) is definable (not necessarily by a \( \Delta \)-formula), and \( B \subseteq W_j \) is definable (not necessarily by a \( \Delta^* \)-formula), with \( \mu(A) \geq \delta \mu(V_i) \) and \( \nu(B) \geq \delta \nu(W_j) \). Then for all \( a \in A \) outside a set of \( \mu \)-measure \( \leq \delta \mu(A) \), for all \( b \in B \) outside a set of \( \nu \)-measure \( \leq \delta \nu(B) \) we have \( R(a,b) \).

Remark 1.3. We also recover the regularity lemma for finite stable graphs in the sense of [3] by considering the counting measure on pseudofinite graphs (but without explicit bounds or equitability of the partition).

Explanation. Let \((V_i, W_i, R_i)\) be a family of finite graphs where the relation \( R_i \) does not have the \( k \)-order property for fixed \( k \). Take an infinite ultraproduct, to obtain a definable nonstandard finite graph \((V, W, R)\) in a (saturated) nonstandard model of set theory \( \bar{M} \). Then \( R(x, y) \) is stable. Let \( \mu \) be the counting measure on \( V_i \), namely for definable (in \( \bar{M} \)) \( A \subseteq V \), define \( \mu(A) \) to be the standard part of \(|A|/|V|\), and likewise for \( \nu \) on \( W \). Then \( \mu, \nu \) are global Keisler measures and we can apply the theorem. Note that the definable sets in \( \bar{M} \) are the internal sets in the sense of nonstandard analysis, so the standard compactness arguments give regularity in the finite situation with respect to arbitrary subsets (remembering the parenthesis).

Our model theory notation is standard. Stability theory is due to Shelah and the basic reference is [5]. For the purposes of the current paper it is convenient to refer to Chapter 1 of [4]. We will be in particular making use of “local stability theory” namely the theory of forking for complete \( \Delta \)-type \( p(x) \) where \( \Delta \) is a finite set of stable formulas \( \phi(x,y) \), and where a complete \( \Delta \)-type over a model \( M \) is given (axiomatized) by a maximal consistent set of formulas of the form \( \phi(x,b) \), \( \neg \phi(x,b) \) for \( \phi(x,y) \in \Delta \) and \( b \in M \). The \( \Delta \)-rank refers to \( R_\Delta(\cdot) \) as in Section 1.3 of [4], which is denoted \( R(\cdot, \Delta, \omega) \) in [3]. By a \( \Delta \)-formula we mean a finite Boolean combination of formulas \( \phi(x,b) \) for \( \phi \in \Delta \). Recall that a formula \( \phi(x,b) \) divides over \( A \) if there an \( A \)-indiscernible sequence \( b = b_0, b_1, b_2 \ldots \) such that \( \{ \phi(x,b_i) : i < \omega \} \) is inconsistent. As in this context we will be concerned with stable formulas, we use the expression “forking” to mean dividing.

Keisler measures are of course important in this paper. If \( \bar{M} \) is a saturated model and \( V \) a sort, a Keisler measure \( \mu \) on \( V \) is a finitely additive probability measure on the Boolean algebra of definable (with parameters) subsets of \( V \). If we are only concerned with subsets definable over \( M \) we call \( \mu \) a Keisler measure on \( V \) over \( M \). If \( \Delta \) is a collection of \( L \)-formulas \( \phi(x,y) \) (\( x \) fixed of sort \( V \), \( y \) arbitrary), we can restrict \( \mu \) to \( \Delta \) formulas and call it \( \mu|\Delta \). Likewise \( \mu|\Delta(M) \) is the restriction of \( \mu \) to \( \Delta \)-formulas over \( M \). We will use freely facts (from [2]) such that if \( \mu \) is a global Keisler measure on \( V \) say, then for any type-definable subset \( X \) of \( V \), type-defined over a small set of parameters (equivalently any partial type \( \Sigma \) over a small set of parameters), \( \mu(X) \) is defined and is approximated from above by the \( \mu \)-measure of definable sets (formulas in \( \Sigma \)).

We depend somewhat on Keisler’s seminal paper [2], which is very much concerned with Keisler measures \( \mu|\Delta \) for \( \Delta \) a finite set of stable formulas. A key
observation (Lemma 1.7 of [2]), is that such \( \mu|\Delta \) is a weighted sum of complete \( \Delta \)-types (the proof of which is more or less repeated in Lemma 2.1 in the next section).

We will need the following, which after a suitable translation is Proposition 1.20 of [2]:

**Fact 1.4.** Suppose \( \Delta \) is a finite set of stable formulas and \( \mu_x \) is a global Keisler measure. Then there is a small model \( M \) such that \( \mu|\Delta \) does not fork over \( M \) in the sense that each \( \Delta \)-formula \( \psi(x) \) of positive \( \mu \)-measure does not fork over \( M \).

In fact \( \mu|\Delta \) will be the unique nonforking extension of \( \mu|((\Delta, M)) \), so \( \mu|\Delta \) will be a “generically stable” measure in the sense of [1]. But all we will really need is the

**Fact 1.5.** A complete \( \Delta \)-type over a model \( M \) has a unique nonforking extension to a global complete \( \Delta \)-type.

As in the statement of the theorem we will take \( \Delta \) to be \( \{R(x, y), x = z\} \) and likewise for \( \Delta^* \).

## 2. Proof of Theorem 1.1

First by Fact 1.4, let \( M \) be a small model such that \( (V, W, R) \) is definable over \( M \), and both \( \mu|\Delta, \nu|\Delta^* \) do not fork over \( M \).

**Lemma 2.1.** Given \( \epsilon > 0 \), we can write \( V \) as a disjoint union of \( \Delta \)-formulas (restricted to \( V \) over \( M, V = V_1 \cup \cdots \cup V_m \), such that for each \( i \) there is a complete \( \Delta \)-type \( p_i \) over \( M \) such that \( \mu(p_i) > 0 \), \( V_i \subseteq p_i \) (or \( p_i \subseteq V_i \)) and \( \mu(V_i \setminus p_i) \leq \epsilon \mu(V_i) \).

**Proof.** By induction on \( R_\Delta(V) \) (which is a finite number). If it is 0, then \( V \) is finite and consists of finitely many types, all realized in \( M \), so there is nothing to do. Suppose \( R_\Delta(V) = n > 0 \). Let \( p_1, \ldots, p_k \) be the complete \( \Delta \)-types over \( M \) of \( R_\Delta = n \). Let \( \mu(p_i) = \alpha_i \). We divide into cases.

**Case (i).** \( \alpha_i > 0 \) for \( i = 1, \ldots, k \).

Then we can find \( V_i \subseteq p_i \) for \( i = 1, \ldots, k \) such that \( \mu(V_i) \leq \alpha_i/(1 - \epsilon) \) whereby \( \mu(V_i \setminus p_i) \leq \epsilon \mu(V_i) \). We may assume that the \( V_i \) are disjoint. Let \( U = V_1 \cup \cdots \cup V_k \).

Then \( V \setminus U \) has \( \Delta \)-rank \( < n \) so we can apply the induction hypothesis to it, to obtain the lemma for \( V \). (Noting that if \( \mu(V \setminus U) = 0 \) we can just adjoin it to one of the \( V_i \).)

**Case (ii).** Some but not all of the \( \alpha_i \) are 0.

Without loss \( \alpha_1, \ldots, \alpha_\ell > 0 \) and \( \alpha_j = 0 \) for \( j = \ell + 1, \ldots, k \) where \( 1 \leq \ell < k \). As in Case (i), find pairwise disjoint \( V_i \subseteq p_i \) for \( i = 1, \ldots, \ell \), such that \( \mu(V_i \setminus p_i) \leq \mu(V_i) \). Now as \( \mu(V_j) = 0 \) for \( j = \ell + 1, \ldots, k \) we can find a \( \Delta \)-definable over \( M \) set \( V' \) containing \( p_{\ell+1} \cup \cdots \cup p_k \) and disjoint from \( V_1 \cup \cdots \cup V_\ell \) such that \( \mu(V') \leq (\epsilon \mu(V_1)) - (\mu(V_i \setminus p_i)) \). Put \( V'_1 = V_1 \cup V' \). Then \( \mu(V'_1 \setminus p_i) \leq \epsilon \mu(V'_1) \). Put again \( U = V'_1 \cup V_2 \cup \cdots \cup V_\ell \). Then \( V \setminus U \) has \( \Delta \)-rank \( < n \) and we can apply induction.

**Case (iii).** All \( \alpha_i \) are 0.

By Lemma 1.7 of [2], let \( p \) be a complete \( \Delta \)-type over \( M \) with \( \mu(p) > 0 \). So by our case hypothesis, the \( \Delta \)-rank of \( p \) equals \( m < n \). Let \( W \) be a formula in \( p \) of \( \Delta \)-rank \( m \). So \( \mu(W) > 0 \) and by induction hypothesis we find complete \( \Delta \)-types \( q_1, \ldots, q_r \).
over $M$ of $\mu$-measure $> 0$, and pairwise disjoint $W_i \in q_i$ such that $W = W_1 \cup \cdots \cup W_r$ and $\mu(W_i \setminus q_i) \leq \mu(W_i)$ for $i = 1, \ldots, r$. From our case hypothesis we can find a
$\Delta$-formula over $M$, $V'$ which contains $p_1 \cup \ldots \cup p_k$ and is disjoint from $W$ such that $\mu(V') \leq (\epsilon \mu(W_1) - \mu(W_1 \setminus q_1))/(1 - \epsilon)$. Put $W'_i = W_1 \cup V'$ and as above $\mu(W'_i \setminus q_1) \leq \epsilon \mu(W_i)$. Again let $U = W'_1 \cup W_2 \cup \cdots \cup W_r$. Then the $\Delta$-rank of $V \setminus U$ is $< n$ so we can apply induction to complete the proof in case (iii).

We can do the same thing for $W$ to write $W$ as a disjoint union of $\Delta^*$-definable
(over $M$) sets such that for each $j$ there is a complete $\Delta^*$-type $q_j$ over $M$ such that $\nu(W_j \setminus q_j) \leq \epsilon \nu(W_j)$.

Lemma 2.2. For each $i = 1, \ldots, m$ and $j = 1, \ldots, n$, we have (i) or (ii) of Theorem 1.1

Proof. Fix $i, j$ and we have complete $\Delta$-type $p_i(x)$ over $M$ and complete $\Delta^*$-type $q_j$ over $M$. Now $p_i$ is definable, and its $R$-definition is given by a $\Delta^*$-formula
(1.27 of [4]). Namely there is a $\Delta^*$-formula $\psi(y)$ over $M$ such that for $b \in W(M)$, $R(x, b) \in p_i(x)$ iff $| \psi(b)$. Likewise if $p_i'$ is the unique nonforking extension of $p_i$ to a complete $\Delta$-type, $\psi(y)$ is the $R(x, y)$-definition of $p_i'$. We have two cases:

Case (i). $\psi(y) \in q_j$.

Hence for all $b \in W_j$ other than a set a set of $\nu$-measure $\leq \epsilon \nu(W_j)$ we have $| \psi(b)$. Now suppose that $| \psi(b)$, hence $R(x, b) \in p_i'$, whereby $p \cup \{ \neg R(x, b) \}$
divides over $M$ (by Fact 1.5) so as $\mu$ does not divide over $M$, $\mu(p \cup \{ \neg R(x, b) \}) = 0$, so for all $a \in V_j$ outside a set of $\mu$-measure $\leq \epsilon \mu(V_i)$, we have $| R(a, b)$. We have actually proved the second clause of (i) of the theorem. To obtain the first clause, let $\chi(x)$ be the $\Delta^*$-definition of $q_j$, so by Lemma 2.8 of [4], and our case hypothesis,
$\chi(x)$ (a $\Delta$-formula over $M$) is in $p_i$. Continue as above.

Case (ii). $\neg \psi(y) \in q_j$. In which case we obtain as in Case (i) that for all $a \in V_i$
outside a set of $\mu$-measure $\leq \epsilon \mu(V_i)$, for all $b \in W_j$ outside a set of $\nu$-measure
$\leq \epsilon \nu(W_j)$ we have $\neg R(a, b)$.

This completes the proof. □

Theorem 1.1 follows directly from Lemmas 2.1 and 2.2.

Acknowledgement

The authors thank Sergei Starchenko for comments on a first draft of the paper.

References

[1] Ehud Hrushovski, Anand Pillay, and Pierre Simon, Generically stable and smooth measures
in NIP theories, Trans. Amer. Math. Soc. 365 (2013), no. 5, 2341–2366, DOI 10.1090/S0002-
9947-2012-05626-1. MR3020101

[2] H. Jerome Keisler, Measures and forking, Ann. Pure Appl. Logic 34 (1987), no. 2, 119–169,
DOI 10.1016/0168-0072(87)90069-8. MR890599 (88i:03052)

[3] M. Malliaris and S. Shelah, Regularity lemmas for stable graphs, Trans. Amer. Math. Soc.
366 (2014), no. 3, 1551–1585, DOI 10.1090/S0002-9947-2013-05820-5. MR3145742

[4] Anand Pillay, Geometric stability theory, Oxford Logic Guides, vol. 32, The Clarendon
Press, Oxford University Press, New York, 1996. Oxford Science Publications. MR1429864
(98a:03049)
[5] S. Shelah, *Classification theory and the number of nonisomorphic models*, 2nd ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990. MR1083551 (91k:03085)

Department of Mathematics, University of Chicago, Chicago, Illinois 60637

*E-mail address*: mem@math.uchicago.edu

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556

*E-mail address*: apilla@nd.edu