Here we examine classical $O(n)$ spin systems with arbitrary two spin interactions (of unspecified range) within a general framework. We shall focus on translationally invariant interactions. In the this case, we determine the ground states of the $O(n \geq 2)$ systems. We further illustrate how one may establish Peierls bounds for many Ising systems with long range interactions. We study the effect of thermal fluctuations on the ground states and derive the corresponding fluctuation integrals. The study of the thermal fluctuation spectra will lead us to discover a very interesting odd-even $n$ (coupling-decoupling) effect. We will prove a generalized Mermin-Wagner-Coleman theorem for all two dimensional systems (of arbitrary range) with analytic kernels in $k$ space. We will show that many three dimensional systems have smectic like thermodynamics. We will examine the topology of the ground state manifolds for both translationally invariant and spin glass systems. We conclude with a discussion of $O(n)$ spin dynamics in the general case.

I. INTRODUCTION

In this article we aim to unveil some of general properties of $O(n)$ spins systems having two-spin interactions. We shall mostly concern ourselves with translationally invariant systems.

The outline is as follows: In section (III) we introduce frustrated toy models rich enough to illustrate certain of the general features that we aim to highlight. These models will be employed for illustrative purposes only. The bored reader is encouraged to skip the somewhat verbose exposition and merely study the two Hamiltonians.

In section (IV) we discuss the ground states of Ising spin models and show what patterns one should expect in general. Once the ground states will be touched on, we will head on to show how Peierls bounds may be established for many systems having infinite range interactions if the ground states are simple. In section (V) we shortly review mean field solutions of the general two spin Ising models.

In section (VI) we prove that, sans special commensurability effects, the ground states of all $O(n \geq 2)$ will typically have a spiral like structure.

In section (VII) we will make an exceedingly simple spin wave stiffness analysis to gauge the effect of thermal fluctuations on the various $O(n \geq 2)$ ground states.

In section (VIII) we will discuss thermal fluctuations within the framework of “soft-spin” XY model. We will see that the normalization constraint gives a Dirac like equation. In the aftermath, the fluctuation spectrum will be seen to match with that derived in section (VII). We will show possible links to smectic like behavior in three dimensions.

Next, we go one step further to study the fully constrained “hard-spin” $O(2)$ and $O(3)$ models and show (in section IX) that all translationally invariant systems in two dimensions with an analytic interaction kernel never develop spontaneous magnetization. At the end of the section our analysis will match that of sections (VIII) and (X).

We extend the Mermin-Wagner-Coleman theorem to all two dimensional interactions with an analytic interaction kernel in momentum space.

In section (XI) we will examine the “soft-spin” version of Heisenberg spins. We will see that it might be naively expected that the spin fluctuations in odd $n$ spin systems are larger than in those with an even number of spin components. One of the fluctuating spin components will remain unpaired.

Next, in section (XII), we carry out the spin fluctuation analysis for four component soft spins to see that their spectra coincides with that predicted in the earlier spin stiffness analysis.

In section (XIII) we compute the critical temperature of all translationally invariant $O(n \geq 2)$ spin models with mean field theory.

In section (XIV) we show that in the limit of large $n$ both odd and even component spin systems behave in the same manner. Essentially, they all tend towards an “odd” behavior.

In section (XV) we briefly remark that much of analysis is not changed for arbitrary two spin interactions (including spin glass models).

We conclude with a discussion of $O(n)$ spin dynamics. A central theme which will be repeatedly touched on throughout the paper is the possibilities of non-trivial ground state manifolds. If the system is degenerate the effective topology of the low temperature phase of the system may be classified in momentum (or other basis). In such instances the low temperature behavior of the systems will be exceedingly rich.
II. DEFINITIONS

We will consider simple classical spin models of the type

\[ H = \frac{1}{2} \sum_{\bar{x}, \bar{g}} \hat{V}(\bar{x}, \bar{g})[\hat{S}(\bar{x}) \cdot \hat{S}(\bar{g})] \]  

(1)

Here, the sites \( \bar{x} \) and \( \bar{g} \) lie on a (generally hypercubic)

lattice of size \( N \). The spins \( \{S(\bar{x})\} \) are normalized and

have \( n \) components:

\[ \sum_{i=1}^{n} S_i^2(\bar{x}) = 1 \]  

(2)

at all lattice sites \( \bar{x} \).

We shall, for the most part, consider translationally

invariant interactions \( V(\bar{x}, \bar{g}) = V(\bar{x} - \bar{g}) \).

We employ the non-symmetrical Fourier basis convention

\[ (f(\bar{k}) = \sum_{\bar{x}} F(\bar{x}) e^{-i\bar{k} \cdot \bar{x}}; \quad F(\bar{x}) = \frac{1}{N} \sum_{\bar{k}} f(\bar{k}) e^{i\bar{k} \cdot \bar{x}} \]  

wherein the Hamiltonian is diagonal and reads

\[ H = \frac{1}{2N} \sum_{\bar{k}} |v(\bar{k})|\tilde{S}(\bar{k})|^2 \]  

(3)

where \( v(\bar{k}) \) and \( \tilde{S}(\bar{k}) \) are the Fourier transforms of \( V(\bar{x}) \)

and \( \tilde{S}(\bar{x}) \).

More generally, for some of the properties that we will

illustrate, one could consider any arbitrary real two spin

interactions \( \langle \bar{x}|V|\bar{g} \rangle \) which would be diagonalized in an-\n
other basis \( \{\bar{\mu}\} \) instead of the Fourier basis.

For simplicity, we will set the lattice constant to unity-
i.e. on a hypercubic lattice (of side \( L \)) with periodic

boundary conditions the wave-vector components \( k_i = \frac{2\pi r_i}{L} \)

where \( r_i \) is an integer (and the real space coordinates

\( x_i \) are integers).

Throughout this work we will use \( \bar{K} \) to denote reciprocal

lattice vectors and \( \Delta(\bar{k}) \) as a shorthand for the lattice

lattice Laplacian:

\[ \Delta(\bar{k}) = \sum_{i=1}^{d} (1 - \cos k_i). \]  

(4)

In some of the frustrated systems that we will soon

consider, \( v(\bar{k}) \) may be written explicitly as the sum of

several terms: those favoring homogeneous states \( \bar{k} \to 0 \),

and those favoring zero wavelength \( \bar{k} \to \infty \) (or \( \bar{k} \to (\pi, \pi, ..., \pi) \)
on a lattice.) As a result of this competition, modulated structures

arise on an intermediate scale.

III. TOY MODELS- FOR ILLUSTRATIVE

PURPOSES ONLY

Although we will keep the discussion very general, it might be useful to have a few explicit applications in

mind. There is a lot of physical intuition which underlies

the upcoming models. Unfortunately, insofar as we are

concerned, they will merely serve as nontrivial toy mod-
els on which we will able to exercise our newly gained

intuition.

The systems to be presented are frustrated: not all two

spin interactions can be simultaneously satisfied.

We choose these rather nontrivial toy examples as they

highlight some possible richness which is typically absent

in the more standard spin models. As such, they will

point to typically ignored subtleties.

The Coulomb Frustrated Ferromagnet

Let us introduce our first toy model “The Coulomb

Frustrated Ferromagnet”.

This is a toy model of a doped Mott insulator, where the
tendency, of holes, to phase separate at low doping

is frustrated, in part, by electrostatic repulsion \( [2] \). In

dimensions, a simple spin Hamiltonian \( [6] \) which

represents a small region of space in which

\( \langle \bar{x}|V|\bar{y} \rangle \)

represents the local density of mobile holes. Each site \( \bar{x} \)

represents a small region of space in which \( S(\bar{x}) > 0 \), and

\( S(\bar{x}) < 0 \) correspond to hole-rich and hole-poor phases

respectively. In this Hamiltonian, the first “ferromag-
netic” term represents the short-range (nearest-neighbor)\n
tendency of the holes to phase-separate and form a hole-
rich “metallic” phase, whereas the frustrating effect of

the electrostatic repulsion between holes is present in the

second term. Non-linear terms in the full Hamiltonian

typically fix the locally preferred values of \( S(\bar{x}) \). One

may consider \( d \neq 3 \) dimensional variants wherein the

spins lie on a hypercubic lattice, and the Coulomb kernel

in \( H_0 \) is replaced by \( \frac{Q}{2d} \ln |\bar{x} - \bar{y}|^{2-d} \) (or by \( \frac{Q}{2\pi} \ln |\bar{x} - \bar{y}| \)
in two dimensions) where \( \Omega_d = 2\pi^{d/2}/\Gamma(d/2) \). Here the

competition between both terms, when \( Q \ll 1 \) favors

states with wave-numbers \( \approx Q^{1/d} \). The introduction of

the Coulomb interaction is brute force non-perturbative:

it is long range. Moreover, the previous ferromagnetic

ground state becomes, tout a’ coup, infinite in energy.

We will, for the large part, focus on the continuum limit

of this Hamiltonian where the kernel becomes

\[ H_{\text{Mott}} = -\sum_{\langle \bar{x}, \bar{g} \rangle} S(\bar{x})S(\bar{g}) + \frac{Q}{8\pi} \sum_{\bar{x} \neq \bar{y}} \frac{S(\bar{x})S(\bar{y})}{|\bar{x} - \bar{y}|} \]

\[ = \frac{1}{2N} \sum_{\bar{k}} |\Delta(\bar{k})| + \sum_{K} |\bar{k} - \bar{K}|^{-2} |S(\bar{k})|^2. \]  

(5)
\[ v_{Mott}(\vec{k}) = Qk^{-2} + k^2[1 + \sum_{\vec{k} \neq 0} \left( \frac{4K^2}{K^6} - \frac{1}{K^4} \right)] \]  

(6)

After rescaling, this may also be regarded as the small \( \vec{k} \) limit of the more general

\[ v_Q(\vec{k}) = \Delta(\vec{k}) + Q[\Delta(\vec{k})]^{-1} + A[\Delta(\vec{k})]^2 + \lambda \sum_{i \neq j} (1 - \cos k_i)(1 - \cos k_j) + O(k^6). \]  

(7)

The constants \( A \) and \( \lambda \) are pinned down if we identify \( v_Q(\vec{k}) = v_{Mott}(\vec{k}) \). Here, we will modify them in order to streamline the quintessential physics of this system. First, we set \( A = 0 \): In the continuum limit this term is not large nor does it lift the “cubic rotational symmetry” of the lattice (i.e. those transformations which leave \( \Delta(\vec{k}) \) invariant) present to lower order. Next, we allow \( \lambda \) to vary in order to turn on and off “cubic rotational symmetry” breaking effects.

Note that \( v_Q(\vec{k}) \) with may be regarded as \( v_{Mott} \) augmented by all possible next to nearest neighbor interactions. As our Hamiltonian respects the hypercubic \( d = 2 \) point symmetry group, by surveying all possible values of \( \lambda \) we should be able to make general statements regarding the possible phases (within the planes) of real doped Mott insulators. When \( \lambda > 0 \) the minimizing wave-vectors will lie along the cubic axis and “horizontal” order will be expected. When \( \lambda < 0 \) the minimizing wave-vectors lie along the principal diagonals and diagonal order is expected. At large values of \( Q \), when the continuum limit no longer applies, trivial extensions of these minimizing modes are encountered where one or more of the wave vector components is set to \( \pi \).

Note that if the ferromagnetic system were frustrated by a general long range kernel of the form \( V(|\vec{x} - \vec{y}|) \sim |\vec{x} - \vec{y}|^{-p} \) we could replace the \( \Delta(\vec{k})^{-1} \) in \( v_Q(\vec{k}) \) by the more general \( \Delta(\vec{k})^{(p-d)/2} \). Here, in the continuum limit, the minimizing modes are \( \sim Q^{1/(2+d-p)} \) and as the reader will later be able easily verify all our upcoming analysis can be reproduced for any generic long range frustrating interactions with identical conclusions.

Shown above is the manifold \( (M) \) of the minimizing modes in \( \vec{k} \) space. When no symmetry breaking terms \( (\lambda = 0) \) are present, in the continuum limit it is \( M \) is the surface of sphere of radius \( Q^{1/4} \). If \( \lambda \neq 0 \) this degeneracy will be lifted: only a finite number of modes will minimize the energy. When \( \lambda > 0 \) there will be \( 2d \) minimizing modes (denoted by the big \( X \) in the figure) along the coordinate axes. In the up and coming we will focus mainly on \( \lambda \geq 0 \). When \( \lambda < 0 \), a moment’s reflection reveals that there will be \( 2d \) minimizing modes along the diagonals, i.e. parallel to \( (\pm 1, \pm 1, \pm 1) \) (and in this case, they will have a modulus which differs from \( Q^{1/4} \)).

Unless explicitly stated otherwise, we will set \( \lambda = 0 \) for calculational convenience and when a finite \( \lambda \) is invoked it will be made positive (to avoid the \( \lambda \) dependence of \( |\vec{q}| \) incurred when the former is negative). At times, we will present results for \( v_Q(\vec{k}) \) at sizable \( Q \), even though the model was motivated as a good caricature of \( v_{Mott} \) only in the continuum limit (at small wave-vectors \( \sim Q^{1/4} \)).

**Membranes**

In several fluctuating membrane systems, the affinity of the molecular constituents (say A and B) for regions of different local curvature frustrates phase separation.

Let us define \( S(\vec{x}) \) to be the difference between the A and B densities at \( \vec{x} \).

In the continuum, the energy of the system contains a contribution,

\[ H_{mix} = \frac{b}{2} \int d^2x \, |\nabla S|^2 \]  

(8)

reflecting the demixing of A and B species. Instead of considering long-range interactions, we now allow for out-of-plane (bending) distortions of the sheet. Specifically, we assume that the two molecular constituents display an affinity for regions of different local curvature of the sheet. This tendency can be modeled by introducing a coupling term between the local composition \( S(\vec{x}) \) and the curvature of the sheet.

Provided that the distortions remain small, we may write

\[ H_c = \int d^2x \left[ \frac{1}{2} \sigma |\nabla h(\vec{x})|^2 + \frac{\kappa}{2} |\nabla^2 h(\vec{x})|^2 + \Lambda S(\vec{x}) |\nabla^2 h(\vec{x})|^2 \right] \]
where \( h(\vec{x}) \) represents the height profile of the sheet (relative to a flat reference state), \( \sigma \) is its surface tension, and \( \kappa \) is its bending modulus; \( \Lambda \), the coefficient of the last term in the expression measures the strength of the coupling of the local curvature \( \nabla^2 h \) and the local composition \( \phi \), which we have included here to lowest (bilinear) order. This coupling term reflects the different affinities of the molecular constituents A (\( S = 1 \) corresponds to pure A composition) and B (\( S = 0 \) corresponds to pure B composition) for, respectively, convex (\( \nabla^2 h > 0 \)) and concave (\( \nabla^2 h < 0 \)) regions of the interface. We now minimize the total energy \( H = H_c + H_e \), w.r.t. the membrane shape \( \{ h(\vec{x}) \} \).

\[
0 = \delta H_c = \int d^2 x \left[ \frac{\partial H_c}{\partial h} \delta h + \frac{\partial H_c}{\partial (\partial_i h)} \delta (\partial_i h) + \frac{\partial H_c}{\partial (\partial_i^2 h)} \delta (\partial_i^2 h) \right] = \int d^2 x \left[ \frac{\partial H_c}{\partial h} - \frac{\partial H_c}{\partial (\partial_i h)} + \frac{\partial^2 H_c}{\partial (\partial_i^2 h)} \right] \delta h(\vec{x}),
\]

the variational eqns, where in obtaining to the last line we have employed \( \delta (\partial_i^2 h(\vec{x})) = \partial_i \delta h(\vec{x}) \), \( \delta (\partial_i h(\vec{x})) = \partial_i \delta h(\vec{x}) \), and integrated by parts twice.

Thus

\[
-\sigma \nabla^2 h + \kappa \nabla^2 (\nabla^2 h) + \Lambda \nabla^2 S = 0.
\]

If \( |\kappa \nabla^2 (\nabla^2 h)| \ll \min\{|\sigma \nabla^2 h|,|\Lambda \nabla^2 S|\} \), then an approximate solution to the last eqn is

\[
\Lambda S \simeq \sigma h,
\]

and in \( H_c \), after an integration by parts,

\[
\Lambda S \nabla^2 h \approx \Lambda S \frac{\Lambda}{\sigma} \nabla^2 S \to -\frac{\Lambda^2}{\sigma} (\nabla S)^2.
\]

This effective energy reads

\[
H = \int d^2 k \; v_{\text{membrane}}(\vec{k}) |\phi(\vec{k})|^2,
\]

where \( v_{\text{membrane}}(\vec{k}) = \frac{b'}{2} k^2 + \frac{\Lambda^2 \kappa}{2\sigma^2} k^4 \) is the 2D Fourier transform. A negative \( b' \to b' = -\frac{\Lambda^2}{\sigma} \) obtained when \( b < \Lambda^2 / \sigma \), signals the onset of a curvature instability of the sheet. This instability generates a pattern of domains that differ in composition as well as in local curvature and thus assume convex or concave shapes. The characteristic domain size corresponds to the existence of the minimum of the free energy at a non-zero wave number. The modulation length \( d \simeq \sqrt{(\Lambda^2 \kappa / \sigma^2) / |b'|} \).

After scaling, this model may be regarded as the continuum version of the frustrated short range kernel

\[
v_{z}(\vec{k}) = z \Delta^2 (\vec{k}) - \Delta(\vec{k})
\]

(14)

(where \( z = -\Lambda^2 \kappa / (\sigma^2 b') \)) on the lattice.

The real lattice Laplacian

\[
(\vec{x} | \Delta^2 | \vec{y}) = \begin{cases} 2d & \text{for } \vec{x} = \vec{y} \\ -4d & \text{for } ||\vec{x} - \vec{y}||_\infty = 1 \\ 2 & \text{for } (\vec{x} - \vec{y}) = (\pm \delta \ell, \pm \delta \ell') \text{ where } \ell \neq \ell' \\ 1 & \text{for } \pm 2 \delta \ell \text{ separation.}
\end{cases}
\]

We shall extend the investigation of this model over a broader range of parameters than suggested by its initial physical motivation.

Note that, in the continuum limit, theories with high order derivative terms will generally give rise to

\[
v_{\ell}(\vec{k}) = P(k^2)
\]

where \( P \) is some polynomial. Although \( v_{z}(\vec{k}) \) and its likes are artificial on the lattice, their continuum limit is quite generic. Later on we will show that if \( P(k^2) \) attains its global minima at finite \( |\vec{k}| \), then thermal instabilities can incur an extremely low value of \( T_c \).

**IV. ISING GROUND STATES**

In an “Ising” system \( S(\vec{x}) = \pm 1 \) everywhere on the lattice. Stated alternatively, the scalar \( (n = 1) \) spins satisfy a normalization constraints

\[
\{ S^2(\vec{x}) = 1 \}
\]

(18)

at all \( N \) lattice sites \( \vec{x} \). Henceforth, we will adopt the latter point of view.

Let us define the manifold \( M \) spanned by the set of minimizing wave-vectors \( \vec{q} \)

\[
v_{\ell}(\vec{q} \in M) \equiv \min_{\vec{k}} \{ v_{\ell}(\vec{k}) \}.
\]

(19)

If the local normalization constraints are swept aside then it is clear that the ground states are superpositions of sinusoidal waves with wave-vectors \( \vec{q} \in M \). One would expect this to be true, in spirit, also in the highly constrained Ising case, if \( v_{\ell}(\vec{k}) \) is sharply dipped at its global minima. “Digitizing” a particular plane wave
\[ S(\vec{x}) = \text{sign}(\cos(\vec{q}_1 \cdot \vec{x})) \]  

(20)

and comparing it with the exact (numerical) ground state, one finds encouraging agreement in certain cases.

For instance, this gives reasonable accord when \( H = H^{\text{Mott}} \).

This Hamiltonian (with some twists) was investigated in [2] on a square \((d = 2)\) lattice.

Note that in the continuum limit (i.e. if the lattice is thrown away) we might naively anticipate a huge ground state degeneracy - a “digitized plane wave” for each wave vector \( \vec{q} \) lying on the \((d - 1)\) dimensional manifold \( \{ M_Q : q^4 = Q \} \). This large degeneracy might give rise to a loss of stability against thermal fluctuations.

It is found that striped phases (i.e. “digitized plane waves”) were found in virtually all of the parameter range. Only for a very small range of parameters were more complicated periodic structures found.

An intuitive feeling can be gained by considering a one dimensional pattern such as

\[ + - + - + - + - ... \]

(21)

This pattern is a pure mode

\[ S_{\text{period}=4}(x) = \sqrt{2} \cos[\pi x - \pi/4]. \]

(22)

A double checkerboard pattern such as

\[ + - + - + - + - \]

\[ + - + - + - + - \]

\[ - - + - + - + - \]

\[ - - + - + - + - \]

(23)

extending in all directions in the plane is thus trivially given by

\[ S(\vec{x}) = 2 \cos[\pi x_1 - \pi/4] \cos[\pi x_2 - \pi/4] = \cos[\pi(x_1 + x_2) - \pi/2] + \cos[\pi(x_1 - x_2)]. \]

(24)

Such a \( 4 \times 4 \times 4 \) periodic pattern in three dimensions would include the eight modes \( 1/2(\pm \pi, \pm \pi, \pm \pi) \). This trivial example serves to illustrate an simple point. If one has a periodic building block of dimensions \( p_1 \times p_2 \times p_3 \), then

\[ S(\vec{x}) = S_{p_1}(x_1)S_{p_2}(x_2)S_{p_3}(x_3). \]

(25)

If a configuration \( S_{p}(x) \) contains the modes \( \{k^m_p\} \) with amplitudes \( \{S_p(k_m)\} \), then Fourier transforming the periodic configuration \( S(\vec{x}) \) one will find the modes \( \{\pm k^m_{p_1}, \pm k^m_{p_2}, \pm k^m_{p_3}\} \) appearing with a weight \( \sim |S_{p_1}(k_1) \times S_{p_2}(k_2) \times S_{p_3}(k_3)|^2 \). For high values of the periods \( p \), the weight gets scattered over a large set of wavevectors. If \( v(\vec{k}) \) has sharp minima, such states will not be favored. The system will prefer to generate patterns s.t. in all directions \( i \) albeit one \( p_i = 1 \) (or perhaps 2). For a \( p_1 \times p_2 \times p_3 \) repetitive pattern, the discrete Fourier Transform will be nonzero for only \( \Pi_{i=1}^3 p_i \) values of \( \vec{k} \).

This trivial observation suggests the phase diagram obtained by U. Low et al. [4] in the two dimensional case.

The intuition is obvious. We have derived [6], rigorously, the ground states in only several regions of its parameter space (those corresponding to ordering with half a reciprocal lattice vector), and on a few special surfaces (corresponding to ordering with a quarter of a reciprocal lattice vector). In all of these cases the Ising states may be expressed as superpositions of the lowest energy modes \( \exp[i\vec{q}_1 \cdot \vec{x}] \). Lately, a beautiful extension was carried out by [5].

We now ask whether commensurate lock-in is to be expected. The energy of the Ising “digitized plane wave” on an \( L \times L \times L \) lattice where \( \vec{q} = (q_1, 0, 0) \) with \( q_1 = 2\pi/m \), with even \( m \), reads

\[ E = \frac{1}{2N} \sum_{\vec{k}} v(\vec{k})|S(\vec{k})|^2 = \frac{8}{m} \sum_{j=1,3,\ldots,m-1} \frac{v(\vec{k} = (2\pi j, 0, 0))}{|\exp[2\pi ij/m] - 1|^2} = \frac{2}{m} \sum_{j=1,3,\ldots,m-1} \frac{v(\vec{k} = (2\pi j, 0, 0))}{\sin^2(\pi j/m)}. \]

(26)

The lowest energy state amongst all states of the form considered is a possible candidate for the ground state.

For the particular model long-range introduced above, it seems that for small values of \( Q \), it might be worthwhile to have an incommensurate phase. This is, in a sense, obvious - all low energy modes are of very small wave-number and hence not of low commensurability. The energy

\[ E = \frac{1}{N} \sum_{n=0}^{\infty} 16v(\vec{k} = (2n + 1)\vec{q})/(2n + 1)^2\pi^2. \]

(27)

For \( q \sim Q^{1/(d+1)} \ll 1 \), the higher harmonics \( \vec{k} = (2n + 1)\vec{q} \) do not entail high energies. For large values of \( Q \), \( q \approx O(1) \), and \( v(\vec{k} = (2n + 1)\vec{q}) \) can be very large if \( (2n + 1)\vec{q} \) approaches a reciprocal lattice vector \( \vec{K} \). Under these circumstances it will pay off to have a commensurate structure; for a \( u_1 \times u_2 \times \ldots \times u_d \) repetitive block only the modes \( \vec{k} = 2\pi(u_1/m_1, u_2/m_2, \ldots, u_d/m_d) \) will be populated (i.e. have a non-vanishing \( |S(\vec{k})|^2 \)) - the ferromagnetic point [a reciprocal lattice point] will not be approached arbitrarily close if that is not true weight will be smeared over energetic modes. Generically, we will not expect commensurate lock-in in \( \lim q \rightarrow 0 \) for any theory with a frustrating long range interaction.

Although we have considered only striped phases (which have previously argued are the only ones generically expected), it is clear that this argument may be reproduced for more exotic configurations.
For finite range interactions it is easy to prove, by covering the system with large maximally overlapping blocks, that there will be a sliver about $\vec{q} = 0$, for which we will find the ferromagnetic ground state.

A polynomial in $\Delta(\vec{k})$ will have its minima at $\Delta(\vec{k}) = \text{const}$, i.e. on a (d-1) dimensional hypersurface(s) in $\vec{k}$-space or at the (anti)ferromagnetic point.

The kernel $v_z(\vec{k}) = z\Delta^2 - \Delta$ has its minima $(z > 0)$ at

$$\vec{q} \in M_z : \Delta(\vec{q}) = \min \left\{ \frac{1}{2z}, 4d \right\}$$

(28)

For $z > \frac{1}{5d}$: $M_z$ is $(d - 1)$ dimensional.

We may divide the lattice into all maximally overlapping $5 \times 5 \times \ldots \times 5$ hyper-cubes centered about each site of the lattice.

$$\text{Energy} = \frac{1}{5 \times 6^{d-1}} \sum_{\text{hypercubes}} \epsilon(\text{hypercube})$$

(29)

and evaluate the energies $\epsilon$ of all $5 \times 5 \times \ldots \times 5$ Ising configurations. Of all $2^d$ configurations the Neel state will have the lowest energy for a sliver about $z = \frac{1}{5d}$. Analogously for $z > z_{\text{top}} >> 1$, by explicit evaluation, the ground state will be ferromagnetic. Contour arguments can be employed and a finite lower bound on $T_c$ generated.

In this system on the square lattice with periodic boundary conditions along all diagonals $\tilde{e}_\pm$ : defined by $x_1 \mp x_2 = \text{const}$ Ising ground states for $z = \frac{1}{8}$ can be synthesized. All minimizing modes lie on

$$\vec{q} \in M_{\pm} : |q_1 \pm q_2| = \pi.$$  

(30)

By prescribing an arbitrary spin configuration along $x_-$ and fixing $S(x_-, x_+) = S(x_-, 0)(-1)^{x_+}$:

$$S(\vec{k}) = \sum_{x_-} S(x_-, 0) \exp(ik_- x_-) \sum_{x_+} (-1)^{x_+} \exp(ik_+ x_+)$$

(31)

vanishes for $|k_+| = |k_1 + k_2| \neq \pi$. Similarly by taking the transpose of these configurations we can generate patterns having $S(\vec{k}) = 0$ unless $|k_-| = |k_1 - k_2| = \pi$.

The ground state degeneracy is bounded from below by, the number of independent spin configurations that can be fashioned along $x_+$ or $x_-$, $(2^L + 1 - 2)$ where $L$ is the length of the system along the $x_+$ axis. The number of $\vec{q}$ values, commensurate with the diagonal periodic boundary conditions, lying on $M_{\pm} = \pm (4L - 2)$.

[Similarly for $d > 2$, one can set $(d - 2)$ of the $\vec{q}$ components to zero. There are $d(d - 1)/2 \times L$-dimensional $M_{\pm} = \pm (4L - 2)$, all looking like the two-dimensional $M$ just discussed (i.e. $|q_1 \pm q_2| = \pi$). The real-space ground state degeneracy is bounded from below by $d(d - 1)/[2^L - 1]$ (along the $(d - 2)$ zero-mode]

A) FLIP SPINS DIAGONALLY

\[ \begin{pmatrix} + & - & \vdots & \pm & \ldots \end{pmatrix} \]

B)

\[ \begin{pmatrix} + & + & \vdots & + & \ldots \end{pmatrix} \]

FIG. 2. Simple ground states for $v_z(\vec{k})$ with $z = 1/8$ (and for $v_Q(\vec{k})$ when $Q = 16$): A) Their construction—along the arrowed line we arbitrarily prescribe spins. For each move along the dotted diagonal lines we flip the spins. B) In this chain of diagonal super-spins there is no stiffness against flipping.

directions the ground state spin configurations display no flip).

If we regard each diagonal row of spins as a “super-spin” then we will see that flipping any “super-spin” entails no energy cost. This is reminiscent to a nearest neighbor Ising chain where the energy cost for flipping a spin is dwarfed by comparison to the (logarithmically) extensive entropy. We might expect that here, too, ordering might be somewhat inhibited.

In two dimensions

$$\lim_{z \to \infty} M_z : \vec{q}^2 = \frac{1}{2z}$$

(32)

the “average” number of allowed $\vec{q} \in M_z$ values $\ll O(L)$ (and similarly for the onset $\lim_{z \to \infty} M_z : (\vec{q} - (\pm \pi, \pm \pi))^2 = (1/8d - 1/2z)^2$). For a hypercubic lattice of size $L_1 \times L_2 \times \ldots \times L_d$ in $d > 2$ many discrete reciprocal points will give rise to the same value of $\Delta(\vec{k}) \sim \vec{k}^2$.

The proof is trivial: if all $L_i = L$, then the number of possible $\vec{k}^2$ values is bounded by $dL^2$, whereas there are $L^d$ $\vec{k}$-values.

Therefore, on “average”, the number of $\vec{k}$ points lying on $M_z$, or more precisely lying the closest to $M_z$, s.t. $|\Delta(\vec{k}) - \frac{1}{2z}|$ is min, is, at least, $O(L^{d-2})$.

[Of these, $\frac{1}{4\pi} \frac{d^d}{n(2\pi)^d} d\vec{k}$ wave-vectors, with $n_i$ (and $z$) denoting the number of identical components (and the number of zero components) of a certain $\vec{k}$, nearest to $M_z$, are related to $\vec{k}_1$ by symmetry.]
As we have stated previously, in the continuum limit ($q \to 0$) any short range kernel (including this one) will have a uniform (ferromagnetic) ground state. However the impossibility of constructing ground states that contain only “good” Fourier modes $S(k \in M)$ when $M$ shrinks to a curved surface enclosing the origin is more general and will proved in the next section.

For this short range model, even for $z \neq \frac{1}{2}$ a huge ground state degeneracy is expected. A “plane wave” might correspond to each wave-vector $\vec{q}$ (or commensurate wave-vectors nearby) lying on the $(d-1)$ dimensional manifold $M$.

As we shall prove later on, even in high dimensions, and even if the interactions are long ranged, in the continuum limit it will not be possible to construct Ising states in which $S(k \notin M) = 0$ unless the minimizing manifold $M$ contains flat non-curved segments (or more generally intersects a plane at many points).

V. A UNIVERSAL PEIERLS BOUND

If a real hermitian kernel $v(k)$ attains its minima in only a finite number of commensurate reciprocal lattice points $\{q_i\}$, then a Peierls bound can, in some instances, be proven for an infinite range model: When possible this is suggestive of a finite $T_c$.

For instance, the bound for a (lattice) Coulomb gas (with the kernel solving the discrete Laplace equation on the lattice) is trivially generated.

$$v_e(k) - v_e(q) = e/\Delta(k) - e/\Delta(q) = \pi, \pi, \pi)$$

$$\geq -A(\Delta(k) - \Delta(q) = \pi, \pi, \pi)), (33)$$

with $A = 16d^2$. The right hand side is the kernel of an antiferromagnet. Both system share the same ground states. For a given configuration the energy penalty for the Coulomb gas

$$\Delta E_c = 1/(2N) \sum_k |v_e(k) - v_e(q)| S(k)|^2 (34)$$

is bounded from below by the corresponding penalty in an antiferromagnet of strength $A$. In $d = 2$ the contour penalty of the antiferromagnet is $2A|\Gamma|$, $|\Gamma| = \text{length of the contour } \Gamma)$. A similar trick may frequently be employed when the minimizing wave-vectors attain other commensurate values. It relies on comparison to a short range kernel for which a Peierls bound is trivial.

This can be extended quite generally. All translationally invariant system with commensurate minimizing wave-vectors ($q = (0,0,\ldots,0), (\pi,0,\ldots,0), \ldots$) at which the minimum of $v(k)$ at $k = q$ is quadratic may be bounded by a kernel of a system having nearest neighbor ferromagnetic and antiferromagnetic bonds for which the Peierls bound is trivial (linear in the perimeter of the domain wall $\Gamma$).

VI. ISING WEISS MEAN-FIELD THEORY

Assuming that for $T < T_c$: $\langle S(\vec{x}) \rangle = s \text{ sign}[\cos(\vec{q} \cdot \vec{x})]$, then

$$\sum_{\vec{y}} \langle S(\vec{y}) \rangle V(\vec{x} = 0, \vec{y}) = \frac{1}{N} \sum_k \langle S(k) \rangle v(-k)$$

$$= s \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n + 1)\pi} [v(\vec{k} = (2n + 1)\vec{q}) + v(-k = -(2n + 1)\vec{q})] (35)$$

where we have assumed $q_i = \frac{\pi}{u_i}$ with $u_i \gg 1$ for all $d$ components in replacing a discrete Fourier transform sum by an integral.

$$s = \langle S(\vec{x} = 0) \rangle = -\tanh[\beta \sum_{\vec{y}} V(\vec{x} = 0, \vec{y}) \langle S(\vec{y}) \rangle] (36)$$

yields

$$\beta_c^{-1} = \sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n + 1)\pi} v(\vec{k} = (2n + 1)\vec{q}), (37)$$

The latter is an upper bound on $T_c$ as the non-trivial solution [with a non-zero magnetization $s$] should be self-consistent at all sites (not only at $\vec{x} = 0$).

Summary and Outlook

We have argued that for translation invariant two-spin kernels, the ground states are generically uniform, Neel ordered, or thin stripes (all periodic configurations in which the net weight $\sum_{\vec{y}} |S(\vec{k})|^2$ is not smeared over many non-identical modes.) If the Fourier transformed kernel $v(k)$ is sharply peaked about its global minima (at $\{q_i\}$ and if all other local minima have a much higher “energy” $v(k)$ then the modulation length is $O(|q|^1).$ All this might seem trivial/naive and incorrect. (It is also possible to get a feel for this via the examination of the local magnetization $\langle S(\vec{x}) \rangle$ at low temperatures as computed within the dual unconstrained Hubbard Stratonovich Hamiltonian.)

To emphasize that this is only generic but non-universal, we have constructed non-periodic ground states of a certain finite ranged Hamiltonian.

As we have seen, frustrating interactions can give rise to massive degeneracy. These could, potentially, give rise to a small value of $T_c$.

In the continuum limit ($|q| \to 0$), the ground states in the presence of a frustrating long range interaction are expected to be non-commensurate; if short range frustration is present then the continuum limit ground state will be homogeneous (ferromagnetic).

In this context, it might be worthwhile to point out that the triangular antiferromagnet was solved exactly by Wannier. This system has finite entropy at $T = 0$. There is no transition at finite temperature- this, again, is presumably related to this high ground state degeneracy.
VII. $O(N \geq 2)$ GROUND STATES

By an $O(n)$ system we mean that the spins $\{\vec{S}(\vec{x})\}$ have $n$ components and are all normalized to unity: $\vec{S}^2(\vec{x}) = 1$.

Our previous ansatz $S(\vec{x}) = \text{sign}(\cos(\vec{q}_1 \cdot \vec{x}))$ is readily fortified in the $O(n \geq 2)$ scenario: here there is no need to “digitize” - in the spiral

$$S_1(\vec{x}) = \cos(\vec{q} \cdot \vec{x})$$
$$S_2(\vec{x}) = \sin(\vec{q} \cdot \vec{x})$$
$$S_{1>2}(\vec{x}) = 0$$

(38)

the only non-zero Fourier components are $\vec{S}(\vec{q}), \vec{S}(\vec{-q})$. In plain terms, this state can be constructed with the minimizing wave-vectors only.

It follows that any ground state $g$ must be of the form

$$S_1(\vec{x}) = \sum_{m} a_i^m \cos(\vec{q}_m \cdot \vec{x} + \phi_i^m).$$

(39)

Let us turn to the normalization of the spins at all sites.

$$1(\vec{x}) \equiv \sum_{i=1}^{n} S_i^2(\vec{x}) = \frac{1}{2} \sum_{m,m'} i=1^n a_i^m a_i^{m'}$$

$$\left(\cos(\phi_i^m + \phi_i^{m'}) \cos((\vec{q}_m + \vec{q}_m') \cdot \vec{x}) \right)$$
$$\left(- \sin(\phi_i^m + \phi_i^{m'}) \sin((\vec{q}_m + \vec{q}_m') \cdot \vec{x}) \right)$$
$$\left(+ \cos(\phi_i^m - \phi_i^{m'}) \sin((\vec{q}_m - \vec{q}_m') \cdot \vec{x}) \right)$$
$$\left(- \sin(\phi_i^m - \phi_i^{m'}) \cos((\vec{q}_m - \vec{q}_m') \cdot \vec{x}) \right)$$

(40)

If $1(\vec{x}) = 1$ is to hold identically for all sites $\vec{x}$, then all non-zero Fourier components must vanish.

For the \{cos($\vec{A} \cdot \vec{x}$))\} Fourier components:

$$0 = \sum_{\vec{q}_m + \vec{q}_m' = \vec{A}} \sum_{i=1}^{n} a_i^m a_i^{m'} \left( \cos(\phi_i^m + \phi_i^{m'}) \right)$$
$$+ \sum_{\vec{q}_m - \vec{q}_m' = \vec{A}} \sum_{i=1}^{n} a_i^m a_i^{m'} \left( \cos(\phi_i^m - \phi_i^{m'}) \right) \right)$$

(41)

and a similar relation is to be satisfied by the \{sin($\vec{A} \cdot \vec{x}$))\} components:

$$0 = \sum_{\vec{q}_m + \vec{q}_m' = \vec{A}} \sum_{i=1}^{n} a_i^m a_i^{m'} \left( \sin(\phi_i^m + \phi_i^{m'}) \right)$$
$$+ \sum_{\vec{q}_m - \vec{q}_m' = \vec{A}} \sum_{i=1}^{n} a_i^m a_i^{m'} \left( \sin(\phi_i^m - \phi_i^{m'}) \right) \right)$$

(42)

$$a_i^m \cos(\phi_i^m) \equiv v_i^m, \quad a_i^m \sin(\phi_i^m) \equiv u_i^m.$$

(43)
The \{cos($\vec{A} \cdot \vec{x}$))\} and \{sin($\vec{A} \cdot \vec{x}$))\} conditions read

$$0 = \sum_{\vec{q}_m + \vec{q}_m' = \vec{A}} \sum_{i=1}^{n} \left[ v_i^m v_i^{m'} - u_i^m u_i^{m'} \right]$$
$$+ \sum_{\vec{q}_m - \vec{q}_m' = \vec{A}} \sum_{i=1}^{n} \left[ v_i^m u_i^{m'} + u_i^m v_i^{m'} \right]$$

(44)

In the case of two pairs of wave-vectors $\pm \vec{q}_1$ and $\pm \vec{q}_2$, both not equal half a reciprocal lattice vector: $(0,0,0,...),(\pi,0,...,0), (0,\pi,0,...,0), ..., (\pi,\pi,0,...,0), ..., (\pi,\pi,...,\pi)$, the vector $\vec{A}$ (up to an irrelevant sign) may attain four non-zero values: $\vec{A} = 2\vec{q}_1, 2\vec{q}_2, \vec{q}_1 \pm \vec{q}_2$.

When $\vec{A} = \vec{q}_1 + \vec{q}_2$, the conditions are

$$0 = \sum_{i=1}^{n} \left[ v_i^1 v_i^2 - u_i^1 u_i^2 \right],$$
$$0 = \sum_{i=1}^{n} \left[ v_i^1 u_i^2 + u_i^1 v_i^2 \right].$$

(45)

When $\vec{A} = \vec{q}_1 - \vec{q}_2$, these conditions read

$$0 = \sum_{i=1}^{n} \left[ v_i^1 v_i^2 + u_i^1 u_i^2 \right]$$
$$0 = \sum_{i=1}^{n} \left[ v_i^1 u_i^2 - u_i^1 v_i^2 \right]$$

(46)

For $\vec{A} = 2\vec{q}_\alpha$ ($\alpha = 1, 2$):

$$0 = \sum_{i=1}^{n} \left[ u_i^\alpha v_i^\alpha - u_i^\alpha u_i^\alpha \right]$$
$$0 = 2 \sum_{i=1}^{n} u_i^\alpha v_i^\alpha.$$

(47)

Define

$$\vec{U}^\alpha = (u_1^\alpha, u_2^\alpha, ..., u_n^\alpha)$$
$$\vec{V}^\alpha = (v_1^\alpha, ..., v_n^\alpha)$$

The previous conditions imply that

$$\vec{V}_1 \cdot \vec{V}_2 = \vec{U}_1 \cdot \vec{U}_2 = 0$$
$$\vec{V}_1 \cdot \vec{V}_2 = \vec{U}_1 \cdot \vec{U}_2 = 0$$
$$\vec{U}_1 \cdot \vec{V}_1 = \vec{V}_2 = 0.$$

(48)

The four vectors $\{\vec{U}_1, \vec{U}_2, \vec{V}_1, \vec{V}_2\}$ are all mutually orthogonal. The number of spin components $n \geq 4$. 

8
Two additional demands that follow are
\[ \tilde{V}^\alpha \cdot \tilde{V}^\alpha = \tilde{U}^\alpha \cdot \tilde{U}^\alpha \]
\[ \sum_{\alpha=1}^{2} [\tilde{V}^\alpha \cdot \tilde{V}^\alpha + \tilde{U}^\alpha \cdot \tilde{U}^\alpha] = 2 \sum_{\alpha} \tilde{V}^\alpha \cdot \tilde{V}^\alpha = 2. \]

The last equation is the normalization condition: the statement that the coefficient of \( \cos(\vec{A}\cdot\vec{x}) \), when \( \vec{A} = 0 \), is equal to 1.

For the case of a single pair of wave-vectors, \( \vec{q}_1, \vec{A} = 2\vec{q}_1, 0 \) and the sole conditions are encapsulated in the last of equations (48) and in equation (43).

A moment’s reflection reveals that this only allows for a spiral in the plane defined by \( \tilde{U}^1 \) and \( \tilde{V}^1 \).

When \( n < 4 \) there are no configurations which satisfy \( S^2(\vec{x}) = 1 \) identically for all sites \( \vec{x} \) (excusing those having \( 2(\vec{q}_1' + \vec{q}_2') = \vec{A} \) is equal to a reciprocal lattice vector) that are a superposition of exactly two modes.

For instance, a (double checkerboard state along the \( i = 1 \) axis) \( \otimes \) (a spiral in the 23 plane) has pairs \( i, j \) for which \( \vec{A} = 2(\vec{q}_1' + \vec{q}_2') \) is a reciprocal lattice vector.

As the number of minimizing modes \( \{\vec{q}_m\} \) increases, some of the conditions may degenerate into one, e.g. if \( \vec{q}_1' + \vec{q}_2' = \vec{q}_3' - \vec{q}_2' \) (i.e. the modes are collinear). This degeneracy is a second route that might allow for Ising configurations which are superpositions of several “good” minimum energy modes \( \exp(iq \cdot \vec{x}) \).

The highly degenerate Ising ground states that we have constructed previously can fall under either one of these categories.

If neither one of these situations occurs, Ising states cannot be superpositions of several minimum energy modes: we will be left with too many equations of constraints with too few degrees of freedom.

For three pairs of minimizing modes, none of which is half a reciprocal lattice vector, \( \{\pm \vec{q}_m\}_{m=1}^{3} \) with
\[ \vec{q}_w \mp \vec{q}_t \neq \vec{q}_r \pm \vec{q}_s \neq 2\vec{q}_w \]
for all \( w \neq t \), and \( r \neq s \), conditions similar to those that previously written for \( \vec{A} = \vec{q}_1 \pm \vec{q}_2 \), now hold for all \( \vec{q}_w \pm \vec{q}_t \).

\[ \tilde{U}_\alpha \cdot \tilde{U}_\beta = \tilde{U}_\alpha^2 \delta_{\alpha,\beta} \]
\[ \tilde{V}_\alpha \cdot \tilde{V}_\beta = \tilde{V}_\alpha^2 \delta_{\alpha,\beta} \]
\[ \tilde{U}_\alpha \cdot \tilde{V}_\beta = 0 \]

The relation \( \tilde{U}_\alpha \cdot \tilde{V}_\alpha = 0 \) (\( \alpha = \beta \) in the last eqn above) is enforced by setting \( \vec{A} = 2\vec{q}_t \). Thus, when exactly three pairs of minimizing wave-vectors satisfying the equation are present, the vectors \( \{\tilde{U}_\alpha, \tilde{V}_\alpha\} \) define a 6-dimensional space, and hence \( n \geq 6 \). For \( p \) pairs of minimizing wave-vectors, \( n \) must be at least \( 2p - 1 \) -dimensional. This bound is saturated when \( \tilde{S} \) is a (spiral state in the 12 - plane) \( \otimes \) (a spiral in the 34 - plane) \( \otimes \ldots \otimes \) (a spiral in the 2p - 1, 2p plane), i.e.

\[ (a_1 \cos(q_1 \cdot \vec{x} + \phi_1), a_2 \cos(q_2 \cdot \vec{x} + \phi_2), \ldots, a_p \cos(q_p \cdot \vec{x} + \phi_p)) \]

with \( \sum_{\alpha=1}^{p} a_\alpha^2 = 1 \). When wave-vectors with \( \vec{q}_w \pm \vec{q}_t \) is not a reciprocal lattice vector, s.t. \( \sin(\vec{A} \cdot \vec{x}) \) is not identically zero at all \( \vec{x} \in \mathbb{Z}^d \).

We term the such a \( p = 2 \) configuration a bi-spiral. It is simple to see by counting the number of degrees of freedom for \( n = 4 \), that the bi-spirals overwhelm states having only one mode \( \pm \vec{q}_1 \). This is a simple instance of a general trend: High \( p \) states are statistically preferred. Moreover, as we shall see later, they are more stable against thermal fluctuations.

**Summary and Outlook**

We have outlined a way to determine all \( O(n \geq 2) \) ground states for a given kernel \( V(\vec{x}, \vec{y}) = V(\vec{x} - \vec{y}) \).

Whenever \( n \geq 2 \), any ground state configuration can be decomposed into Fourier components, \( S^0(\vec{x}) = \sum_{|M|} \{ \cos(q \cdot \vec{x}) + b_i \sin(q \cdot \vec{x}) \} \)

\( \vec{q} \) are chosen from the set of wave vectors which minimize \( v(\vec{k}) \).

\( |M| \) is the number of minimizing modes (the “measure” of the modes on the minimizing surface \( M \).

So long as these wave-vectors \( \vec{q} \) which minimize \( v(\vec{k}) \) are “non-degenerate”, in the sense that the sum of any pair of wave vectors, \( \vec{q}_t \mp \vec{q}_s \) is not equal to the sum of any other pair of wave vectors, and “incommensurate” in the sense that for all \( i \) and \( j \), \( 2(\vec{q}_t + \vec{q}_s) \) is not equal to a reciprocal lattice vector, then the that the condition \( S^2(\vec{x}) = 1 \) can be satisfied only if \( |M| \leq n/2 \). (In our toy model of the doped Mott insulator, these conditions are always satisfied for \( Q < 4 \).) Thus, for \( n \leq 3 \) only simple spiral (\( |M| = 1 \)) ground-states are permitted, while for \( n = 4 \), a double spiral saturates the bound. Thus, generally, for \( 2 < n < 4 \) all ground states will be spirals containing only one mode.

The reader should bear in mind that in the usual short range ferromagnetic case, the ground states are globally \( SO(n) \) symmetric and are labeled by only \( (n - 1) \) continuous parameters.

Here, for each minimizing mode there are \( (2n - 3) \) continuous internal degrees of freedom labeling all possible spiral ground states. For \( n > 2 \) this guarantees a much higher degeneracy than that of the usual ferromagnetic ground state.

If there are many minimizing modes (e.g. if the minimizing manifold \( M \) were endowed with \( SO(d - 1) \) symmetry) then the ground state degeneracy is even larger.

When \( n \geq 4 \), there are (generically) even many more ground states (poly-spirals). These poly-spiral states have a degeneracies larger than those of simple spiral. Their degeneracy
\[ g = p(2n - 2p - 1)|M|^p, \]  
where \(|M|\) is the number of minimizing modes.

To capitate: we have just proved that if frustrating interactions cause the ground states to be modulated then the associated ground state degeneracy (for \( n > 2 \)) is much larger by comparison to the usual ferromagnetic ground states.

VIII. SPIN STIFFNESS

When \( Q > 0 \) the minimizing modes lie of \( v_{\text{continuum}}(\vec{k}) \) lie on the surface of a sphere \( \{ M_Q : q^2 = \sqrt{Q} \} \).

As \( Q \to 0 \), this surface \( M_Q \) shrinks and shrinks yet is still a \((d-1)\) dimensional surface of a sphere. When \( Q = 0 \), the minimizing manifold evaporates into a single point \( q = 0 \). This sudden change in the dimensionality has profound consequences. As we shall see shortly, it lends itself to suggest (quite strongly) that order is inhibited for a Heisenberg \((n = 3)\) realization of our model.

Before doing so, let us indeed convince ourselves, on an intuitive level, that the large degeneracy in \( k - \) space brought about by the frustration gives rise to a reduced spin stiffness.

A. Longitudinal

Let us assume twisted boundary conditions

\[ \hat{S}(\vec{x}) = \cos(\frac{2\pi}{L} x + qx)\hat{e}_1 + \sin(\frac{2\pi}{L} x + qx)\hat{e}_2 + \delta \vec{S} \]

\[ = \hat{e}_1 \cos(\vec{k} \cdot \vec{x}) + \hat{e}_2 \sin(\vec{k} \cdot \vec{x}) + \delta \vec{S} \]

with \( \vec{k} = (\frac{2\pi}{L} + q)\hat{e}_1 \). The energy cost of this state relative to the ground state is

\[ \Delta H[\{ \hat{S}(\vec{x}) \}] = \frac{1}{2N} \sum_{\vec{k}'} [v(\vec{k}) - v(\vec{q})]|\hat{S}(\vec{k}')|^2 \]  

Ignoring \( \delta \vec{S} \) contributions:

\[ \Delta H = \frac{N}{2} [v(\vec{k}) - v(\vec{q})] = \frac{N}{2\sqrt{Q}}[(\frac{2\pi}{L} + q)^2 - q^2] \approx \frac{8\pi^2 N}{L^2}. \]

For the usual nearest neighbor ferromagnetic \( XY \) model:

\[ \hat{S}(\vec{x}) = \cos(\frac{2\pi}{L} x)\hat{e}_1 + \sin(\frac{2\pi}{L} x)\hat{e}_2. \]

Here

\[ v(\vec{k}) = 2 \sum_{l=1}^{3} (1 - \cos k_l) \]

\[ \Delta H_{\text{XY}} = [1 - \cos(\frac{2\pi}{L})]N \to \frac{2\pi^2 N}{L^2} \]  

(exactly the same).

B. Transverse

\[ \hat{S}(\vec{x}) = \cos(\frac{2\pi}{L} x + qy)\hat{e}_1 + \sin(\frac{2\pi}{L} x + qy)\hat{e}_2 + \delta \vec{S} \]  

\[ \vec{k} = \frac{2\pi}{L} \hat{e}_1 + q\hat{e}_2. \]

\[ v(\vec{k}) - v(\vec{q}) = \frac{16\pi^4}{L^4 \sqrt{Q}} \]

Ignoring \( \delta \vec{S} \) contributions:

\[ \Delta H = \frac{8\pi^4 N}{L^4 \sqrt{Q}} \]

\[ N \sim L^d. \]

\[ \Delta H \to 0 \]

as \( L \to \infty \) in \( d = 3 \) [a complete loss of stiffness against transverse fluctuations]. For a one dimensional chain of nearest neighbor \( XY \) spins:

\[ \Delta H \sim \frac{2\pi^2}{L^2} N = \frac{2\pi^2}{L} = O(\frac{1}{L}). \]

For our frustrated three-dimensional system, the transverse fluctuations obey

\[ \Delta H \sim \frac{8\pi^4}{\sqrt{Q} L} = O(\frac{1}{L}). \]

C.

In principle, one may envision other impositions of twisted boundary conditions [say in the 3-4 plane]:

\[ \hat{S}(\vec{x}) = \cos(\frac{2\pi}{L} x + \sin(\frac{2\pi}{L} x)\hat{e}_2) \sqrt{1 - \epsilon^2} \]

\[ + [\cos(\frac{2\pi y}{L})\hat{e}_3 + \sin(\frac{2\pi y}{L})\hat{e}_4] \epsilon + \delta \vec{S} \]

\[ \Delta H \sim 1 \frac{2N}{2N} [v(\vec{k}) = \frac{2\pi}{L} \hat{e}_y] - v(\vec{q})] N^2 \epsilon^2 \]

\[ = \frac{N}{2} [v(\vec{k} = 0) - v(\vec{q})] = \infty \]

where in the last line we have taken the limit \( L \to \infty \).

The system exhibits infinite spin stiffness to these sorts of fluctuations.
D.

To summarize, the system responds to fluctuations with an effective kernel

\[ E_{\text{low}}(\delta) \sim A_\perp \delta_\perp^4 + A_{||} \delta_{||}^2 \]  

(68)

where \( \delta_\perp \) and \( \delta_{||} \) denote the the transverse and longitudinal fluctuations.

For a nearest neighbor one dimensional system embedded in \( \delta \)-dimensions, \( v(\vec{k}) - v(\vec{q}) \sim A_\perp \delta_\perp^2 \) \( (A_\perp = 0) \), and thus the fluctuations are even larger than in our case.

IX. THERMAL FLUCTUATIONS OF AN XY MODEL

Let me begin by treating the “soft-spin” version of the XY model, in which we include the non-linear interaction

\[ H_{\text{soft}} = H_0 + u \sum_x [\vec{S}^2(\vec{x}) - 1]^2 \]

\[ \equiv H_0 + H_1 \]  

(69)

with \( u > 0 \) and forget about the normalization conditions \( |\vec{S}(\vec{x})| = 1 \) at all lattice sites \( \vec{x} \).

(The normalized “hard-spin” version can be viewed as the \( u \rightarrow \infty \) limit of the soft-spin model.)

As we have seen previously, the only generic ground states (for both hard- and soft-spin models) when the spins \( \vec{S}(\vec{x}) \) have two (and also three) components are spirals

\[ S_1^{\text{ground-state}}(\vec{x}) = \cos(\vec{q} \cdot \vec{x}); \quad S_2^{\text{ground-state}}(\vec{x}) = \sin(\vec{q} \cdot \vec{x}). \]  

(70)

We will expand \( H_{\text{soft}} \) about these ground states, keeping only the lowest order (quadratic) terms in the fluctuations \( \delta S \). The quadratic term in \( \{\delta S(\vec{k})\} \) stemming from \( H_{\text{soft}} \) is the bilinear \( \frac{u}{2}(\delta S)^+ M(\delta S) \) where

\[ (\delta S)^+ = (\delta S_1(-\vec{k}_1), \delta S_2(-\vec{k}_1), \delta S_1(-\vec{k}_2), \delta S_2(-\vec{k}_2), \delta S_1(-\vec{k}_3), \ldots, \delta S_N(-\vec{k}_N)). \]  

(71)

and the matrix \( M \) reads

\[
\begin{pmatrix}
4 & 0 & 1 & i & \ldots & \\
0 & 4 & i & -1 & \ldots & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & -i & 4 & 0 & 1 & i \\
-i & -1 & 0 & 4 & i & -1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & 1 & -i & 4 & 0 \\
\ldots & \ldots & -i & -1 & 0 & 4 \\
\end{pmatrix}
\]

A few off-diagonal sub-matrices have been omitted. [There are two along each horizontal row (by periodic b.c.).] The sub-matrices are \((2 \times 2)\) matrices in the internal spin indices. The off diagonal blocks are separated from the diagonal ones by wave-vectors \((\pm 2\vec{q})\).

Note that \( \langle \vec{k}|M|\vec{k}' \rangle = M(\vec{k} - \vec{k}') \). Making a unitary (symmetric Fourier) transformation to the real space basis:

\[ |\vec{x}\rangle \equiv N^{-1/2} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{x}} |\vec{k}\rangle \]  

(72)

the matrix \( M \) becomes block diagonal

\[ \langle \vec{x}|M|\vec{x}' \rangle = M(\vec{x} - \vec{x}'). \]  

(73)

Diagonalizing in the internal spin basis:

\[ \lambda = 6, 2. \]  

(74)

These eigenvalues may be regarded, in the usual ferromagnetic case \( (\vec{q} = 0) \) as a two step (state) potential barrier separating the two polarizations. I.e., the normalization constraint of the XY spins (embodied in \( M \)) gives rise to an effective binding interaction. As we shall later see, when the number of spin components \( n \) is odd, one spin component will remain unpaired. \( H_1 \) literally “couple”s the spin polarizations.

Employing Eqn.(74), we note that the corrected fluctuation spectrum \( \{\psi_m\}_{m=1}^N \) (to quadratic order) satisfies a Dirac like equation

\[ [U^+ v(-i\partial_x) U + 2u \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}] U^+ |\psi_m(\vec{x})\rangle = E_m U^+ |\psi_m(\vec{x})\rangle, \]  

(75)

Here

\[ U = \begin{pmatrix} \sin(2\vec{q} \cdot \vec{x}) & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \cos(2\vec{q} \cdot \vec{x}) \end{pmatrix} \]  

(76)

Alternatively, expanding in the fluctuations \( \delta S(\vec{k}) \), leads to bilinear \( (\delta S)^+ \mathcal{H}(\delta S) \)

\[ \mathcal{H}_{\vec{k},\vec{k}'} = \begin{pmatrix} v(\vec{k}) + 8u & 0 \\ 0 & v(\vec{k'}) + 8u \end{pmatrix} \]  

(77)

along the diagonal, and

\[ \mathcal{H}_{\vec{k},\vec{k}'+2q} = \begin{pmatrix} 2u & 2iu \\ 2iu & -2u \end{pmatrix}; \quad \mathcal{H}_{\vec{k},\vec{k}-2q} = \begin{pmatrix} 2u & -2iu \\ -2iu & 2u \end{pmatrix} \]  

(78)

off the diagonal.

\[ \mathcal{H}_{\vec{k},\vec{k}+2q} = 2u (\sigma_3 \pm i\sigma_1). \]  

(79)

\[ \exp(i\frac{\pi}{4} \sigma_1) \mathcal{H}_{\vec{k},\vec{k}+2q} \exp(-i\frac{\pi}{4} \sigma_1) = 2u\sigma^\pm = 4u [ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ] \]

\[ \mathcal{H}_{\vec{k},\vec{k}} = [v(\vec{k}) + 8u] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]  

(80)
If $Q = 16$, $\mathbf{q} = (\pi, 0, 0) \in \mathcal{M}_Q$, and $2\mathbf{q} \equiv 0 \pmod{2\pi}$. The fluctuation matrix is diagonal in the $\mathbf{k}$ basis, and the fluctuations diverge at finite temperatures. An identical situation occurs for $\mathbf{q} = (\pi, \pi, 0) \in \mathcal{M}_Q$. When $Q = 4$, $\mathbf{q} = (\frac{\pi}{2}, 0, 0) \in \mathcal{M}_Q$, and it is easy to show that determining the eigenvalue spectrum degenerates into a problem in two parameters ($\Delta_2, K_1$) where $\Delta_2 \equiv 2 \sum_{-2}^{3} \Delta_2(\mathbf{k} + \mathbf{q})$ (s.t. $\Delta(\mathbf{k} + \mathbf{q}) = \Delta_2 + 2 \cos k_1$, $\Delta = \Delta_2 - 2 \cos k_1$). The fluctuation integral about the chosen ground state exhibits a $(d-2)$ dimensional minimizing manifold (parameterized, in our case, by $(\Delta_2, K_1)$). Note that, at higher order commensurabilities, the dimensionality of the minimizing manifold is low. In fact, the interaction will no longer be diagonalized in $\mathbf{k}$-space.

Till now, all that was stated, held for arbitrarily large $u$-our only error was neglecting $O((\delta S)^3)$ terms by comparison to $O((\delta S)^2)$. Note that the main difficulty with the approach taken till now was the coupling between $\mathbf{k}$ and $\mathbf{k} + 2\mathbf{q}$: i.e. $\mathbf{k}$ is coupled to $\mathbf{k} + 2\mathbf{q}$, while $\mathbf{k} + 2\mathbf{q}$ is coupled to $\mathbf{k} + 4\mathbf{q}$ and $\mathbf{k}$, and so on. Unless $\mathbf{q}$ is of low commensurability an exact solution to this problem is impossible.

To make progress let us assume that $u$ is very small. In this case the lowest eigenstates of the fluctuation matrix will contain only a superposition of the low lying $\mathbf{k}$-states [i.e. those close to the $(d-1)$ dimensional $M$ ($Q > 0$)]. If $\mathbf{k}_1 = \mathbf{q} + \delta$ is close to $M$, then the only important modes in the sequence, $\mathcal{S}_i(\mathbf{k} = \mathbf{k}_1 + 2\mathbf{q})$ are $\mathbf{k}_1$, and $\mathbf{k}_2 = \mathbf{k}_1 - 2\mathbf{q} = -\mathbf{q} + \delta$. The sub-matrix in the relevant sector reads

$$
\begin{pmatrix}
 v(\mathbf{k}_1) + 8u & 0 & 0 & 4u \\
 0 & v(\mathbf{k}_1) + 8u & 0 & 0 \\
 0 & 0 & v(\mathbf{k}_2) + 8u & 0 \\
 4u & 0 & 0 & v(\mathbf{k}_2) + 8u
\end{pmatrix}
$$

The lowest eigenvalue reads

$$E_{low} = \frac{1}{2} [v(\mathbf{k}_1) + v(\mathbf{k}_2)] + 4u - \frac{1}{2} \sqrt{[v(\mathbf{k}_1) - v(\mathbf{k}_2)]^2 + 64u^2}$$

Equivalently, this can be determined from the direct computation of the determinant to $O(u^2)$ to obtain $O(u^2)$ contributions we need to swerve off the diagonal twice.

$$\det \mathcal{H} = \sum_{i_1, i_2, \ldots, i_{2N}} \epsilon_{i_1i_2i_3 \ldots i_{2N}} \mathcal{H}_{i_1i_1} \mathcal{H}_{i_2i_2} \ldots \mathcal{H}_{i_{2N}i_{2N}}.$$  \hspace{1cm} (82)

$$\det \mathcal{H} = \Pi_{i=1}^{N} [v(\mathbf{k}_i) + 8u]^2 - (4u)^2 \sum_j [v(\mathbf{k}_j) + 8u] [v(\mathbf{k}_j + 2\mathbf{q}) + 8u] \Pi_{i \neq j} [v(\mathbf{k}_i + 2\mathbf{q}) + 8u]^2$$

The fluctuation spectrum is trivially determined by replacing $\mathbf{k}$ by $[v(\mathbf{k}) - E]$ in det $\mathcal{H}$ and setting it to zero.

To this order we re-derive $E_{low}$. To higher order

$$\ldots + (4u)^4 \sum_{\mathbf{k}_1 \neq \mathbf{k}_2, \mathbf{k}_3 \neq 2\mathbf{q}} [v(\mathbf{k}_{j_1}) + 8u][v(\mathbf{k}_{j_2} + 2\mathbf{q}) + 8u]$$

$$\times \Pi_{\mathbf{k}_i \neq \mathbf{k}_j, \mathbf{k}_1 + 2\mathbf{q}, \mathbf{k}_2 + 2\mathbf{q}} [v(\mathbf{k}_i) + 8u]^2$$

$$+ (4u)^4 \sum_{\mathbf{k}_1} [v(\mathbf{k}_{j_1}) + 8u][v(\mathbf{k}_{j_1} + 4\mathbf{q}) + 8u]$$

$$\times \Pi_{\mathbf{k}_i \neq \mathbf{k}_j, \mathbf{k}_1 + 2\mathbf{q}, \mathbf{k}_2 + 4\mathbf{q}} [v(\mathbf{k}_i) + 8u] - \ldots$$

(The partition function is trivially $Z = \text{const}\det\mathcal{H}^{-1/2}$.)

For any $u$, no matter how small, there exists a neighborhood of wave-vectors $\mathbf{k}$ near $\mathbf{q}$ such that $[v(\mathbf{k}) - v(\mathbf{k} + 2\mathbf{q})] \ll u$ and as before we may re-expand the characteristic equation for these low lying modes, solve a simple quadratic equation, expand in the components of $(\mathbf{k} - \mathbf{q})$ and obtain a simple dispersion relation.

If $\delta = \delta_{\bot} + \delta_|| \mathbf{e}_||$, with $\bot$ and $||$ denoting directions orthogonal, and parallel to $\hat{n} \bot M$, then

$$E_{low} = A_{\bot} \delta_{\bot}^2 + A_{||} \delta_{||}^2 \hspace{1cm} (83)$$

where

$$A_{||} = \frac{1}{2} \frac{d^2}{dk^2} v(\mathbf{k}) || \mathbf{k} = \mathbf{q}; \hspace{0.5cm} A_{\bot} = \frac{A_{||}}{4q^2} \hspace{1cm} (84)$$

We have already found this form when examining spin stiffness of the “hard” XY model. (In both cases there is also an identical $\delta_{||}^2 \delta_{\bot}^2$ term which may be omitted.) Having reassuringly re-derived this dispersion from another vista, let us note this dispersion is akin to the fluctuation spectrum of the smectic liquid crystals which is well known to give rise to algebraic decay of correlations at low temperatures. In our case, one notes that $\mathbf{q} \neq 0$ and thus the correlations should have an oscillatory prefactor. To be more precise, the correlator, in cylindrical coordinates,

$$G(\mathbf{x}) \sim \frac{4d^2}{x_\bot^2} \exp[-2\eta r - \eta E_1(\frac{x_\bot^2 Q^{1/4}}{2x_\bot})] \times \cos[qx_||] \hspace{1cm} (85)$$

where $\eta = \frac{k_B T}{\hbar \pi} Q^{1/4}$, $d = \frac{2}{\Lambda}$ where $\Lambda$ is the ultra violet momentum cutoff, $\gamma$ is Euler’s const., and

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n(n!)} \hspace{1cm} (86)$$

is the exponential integral. For the uninitiated reader, we present this standard derivation in the appendix.

The thermal fluctuations $\int d^d k E_{low}(k)$ diverge as $[-\ln |\mathbf{e}|]$ with a $\epsilon$ a lower cutoff on $|\mathbf{k} - \mathbf{q}|$. Such a logarithmic divergence is also encountered in ferromagnetic
two dimensional $O(2)$ system if it were exposed to the same analysis. Indeed, both models share similar characteristics albeit having different physical dimensionality.

**Summary and outlook**

We have found that if $H_{soft}$ is indeed soft ($u \ll 1$) then, within our continuum XY systems, only quasi long range (algebraic) order will be observed at low temperatures in the presence of generic long range (e.g. $vq(k)$) frustration. These systems essentially display smectic phases.

**Some smectic trivia**

Smectic A ordering involves a displacement $u$ of the smectic along the $z$-direction (the direction of the molecular axis). It is customary to denote this displacement by $u(x)$. As we shortly see, the kernel for the smectic is identical to ours.

When the wave-vector $\vec{k}$ is along the $z$-direction the displacement is longitudinal, and the energy of the elastic form is $\frac{1}{2} B K^2 |u(\vec{k})|^2$, where $B$ is the compressibility for the smectic layers. When $\vec{k}$ is normal to $z$, the displacement is transverse and the layer separation. No second order in $\vec{k}$ the displacement costs no energy. In this case the restoring force is associated with a director splay distortion. In this case, the elastic energy density is $\frac{K}{2} (\nabla \cdot \vec{n})^2$ with $K$ the splay constant. As $\vec{n}$ is normal to the layers, one has $\delta \vec{n} = -\nabla_{\perp} u$ and an elastic energy $\frac{1}{2} K \nabla_{\perp} u(\vec{k})^2$. Thus the kernel

$$v(\vec{k}) = BK^2 + K \nabla_{\perp}^2.$$

As seen in the form that we obtained for $G(x)$, the penetration depth $\lambda = \sqrt{K/B}$ determines the decay of an undulation distortion (splay director distortion) imposed at the surface of the smectic.

**X. A GENERALIZED MERMIN-WAGNER-COLEMAN THEOREM**

We will now generalize the Mermin-Wagner-Coleman theorem [9, 10]:

All systems with translationally invariant two-spin interactions in two dimensions with a real twice differentiable Fourier transformed kernel $v(k)$ show no spontaneous symmetry breaking at finite temperatures $T > 0$.

Our approach is the standard one. We will merely keep it more general instead of specializing to ferromagnetic order or interactions of one special sort or another.

Following the previous section, the magnetic field

$$\vec{h}(\vec{x}) = h \cos(\vec{q} \cdot \vec{x}) \hat{e}_\alpha$$

applied by itself would cause the spins to take on their ground state values.

If $n = \alpha = 2$ the unique spiral ground state $(S^g(\vec{x}))$, to which a low temperature system would collapse to under the influence of such a perturbation is

$$S^g(\vec{x}) = \sin(\vec{q} \cdot \vec{x}).$$

When $n = 3$ the ground state is not unique:

$$S^g_{i < n}(\vec{x}) = r_i \sin(\vec{q} \cdot \vec{x})$$

and a magnetic field may be applied along two directions, with all the ensuing steps trivially modified.

With the magnetic field applied

$$H = \frac{1}{2} \sum_{\vec{x}, \vec{y}} \sum_{i=1}^{n} V(\vec{x} - \vec{y}) S_i(\vec{x}) S_i(\vec{y}) - \sum_{\vec{x}} h_n(\vec{x}) S_n(\vec{x}).$$

Note that the knowledge of the ground state is not imperative in providing the forthcoming proof [11].

The standard idea [12] that we are about to exploit is the rotational invariance of the measure.

$$\int d\mu \cdot = Z^{-1} \int \Pi_{\vec{x}} d^n S(\vec{x}) \delta(S^2(\vec{x}) - 1) e^{-\beta H}$$

The generators of rotation in the $[\alpha, \beta]$ plane are

$$L_{\alpha \beta} = S_{\alpha} \frac{\partial}{\partial S_{\beta}} - S_{\beta} \frac{\partial}{\partial S_{\alpha}}.$$  

$$0 = \frac{d}{d\theta} \int d^n S \delta(S^2 - 1)$$

$$f(S_1, ..., S_n) \cos \theta + S_{\beta} \sin \theta, ... , S_{\beta} \cos \theta - S_{\alpha} \sin \alpha, ..., S_n).$$

$$0 = \int d^n S \delta(S^2 - 1) L_{\alpha \beta} f(S).$$

Now let us consider in particular the generators of rotation from the axis of the applied field to another internal spin axis

$$(E_{\vec{x}})_\alpha = S_{\alpha}(\vec{x}) \frac{\partial}{\partial S_{\beta}(\vec{x})} - S_{\beta}(\vec{x}) \frac{\partial}{\partial S_{\alpha}(\vec{x})}.$$  

In the up and coming $\perp$ will denote the the projection along the $\beta$ direction.

Let us define the following operators

$$\vec{A}(\vec{k}) = \sum_\vec{x} \exp[i\vec{k} \cdot \vec{x}] \vec{S}_\perp(\vec{x})$$

$$\vec{B}(\vec{k}) = \sum_\vec{x} \exp[i(\vec{k} + \vec{q}) \cdot \vec{x}] \vec{L}_x(\beta H).$$

By the Schwarz inequality.
\[ |(\sum_{i=\alpha,\beta} A_i^* B_i)|^2 \leq \left( \sum_{i=\alpha,\beta} A_i^* A_i \right) \sum_{i=\alpha,\beta} B_i^* B_i \] (98)

We will let \( i = \alpha, \beta \) in the sum span a two element subset of the \( n \) spin components. For any functional \( C \):

\[ \tilde{L}_x(e^{-\beta H} C) = e^{-\beta H} \{ \tilde{L}_x(C) + C \tilde{L}_x(-\beta H) \} \] (99)

\[ 0 = \int \Pi_x d^a S(\vec{x}) \delta(S^2(\vec{x}) - 1) \tilde{L}_x[e^{-\beta H} C] \] (100)

\[ \langle CB(\vec{p}) \rangle = \sum_{\vec{x}} \exp[i\vec{p} \cdot \vec{x}] \tilde{L}_x(C) \] (101)

\[ \tilde{L}_x^2[\beta H] = 2 \times 1/2 \times \beta \sum_{\vec{x}} \{ |S_{\alpha}(\vec{y})S_{\beta}(\vec{x})| - S_{\beta}(\vec{y})S_{\alpha}(\vec{x})|V(\vec{y} - \vec{x}) - h_{\alpha}(\vec{x})S_{\beta}(\vec{x}) \} \] (102)

\[ \sum_{i=\alpha,\beta} \langle L_i^x(L_i^y(\beta H)) \rangle = \beta \{ \sum_{i=\alpha,\beta} S_{\alpha}(\vec{x})S_{\beta}(\vec{y}) V(\vec{x} - \vec{y}) - h(\vec{x})S_{\alpha}(\vec{x}) \} \] (103)

\[ \langle \hat{B}(\vec{k})^* \cdot \hat{B}(\vec{k}) \rangle = \beta \sum_{\vec{x},\vec{y}} \{ \cos(\vec{k} + \vec{q}) \cdot (\vec{x} - \vec{y}) - 1 \} \] (104)

Henceforth, for simplicity, we specialize to \( n = 2 \).

Fourier expanding the interaction kernel

\[ V(\vec{x} - \vec{y}) = \frac{1}{N} \sum_{\vec{u}} v(\vec{u}) e^{i\vec{u} \cdot (\vec{x} - \vec{y})} \] (105)

and substituting

\[ \langle \vec{S}(\vec{x}) \cdot \vec{S}(\vec{y}) \rangle = \frac{1}{N^2} \sum_{\vec{u}} \langle |\vec{S}(\vec{u})|^2 \rangle e^{i\vec{u} \cdot (\vec{x} - \vec{y})} \] (106)

we obtain that

\[ 0 \leq \langle \hat{B}(\vec{k})^* \cdot \hat{B}(\vec{k}) \rangle = \beta \Delta_k(2) E = \frac{\beta}{2N} \sum_{\vec{u}} [v(\vec{u} + \vec{k}) + v(\vec{u} - \vec{k}) - 2v(\vec{u}) \langle |\vec{S}(\vec{u})|^2 \rangle - \beta h_{\alpha}(\vec{S}_{\alpha}(\vec{u})) \] (107)

where \( \Delta_k(2) E \) measures the finite difference of the internal energy with respect to a boost of momentum \( \vec{k} \).

\[ \langle \hat{A}(\vec{k})^* \cdot \hat{A}(\vec{k}) \rangle = \sum_{\vec{x},\vec{y}} \langle \vec{S}_1(\vec{x}) \cdot \vec{S}_1(\vec{y}) \rangle \times \exp[i\vec{k} \cdot (\vec{x} - \vec{y})] \] (108)

\[ \langle \hat{A}(\vec{k})^* \cdot \hat{B}(\vec{k}) \rangle = \sum_{\vec{x},\vec{y}} L_{\vec{x}}(\vec{x}) \exp[i(\vec{k} + \vec{q}) \cdot \vec{x}] \] (109)

where \( m_q \equiv \langle S_{\alpha}(\vec{q}) \rangle \). Note that with our convention for the Fourier transformations, a macroscopically modulated state of wave-vector \( \vec{q} \), \( m_q = \mathcal{O}(N) \) as is the energy difference in Eqn. 107.

The Schwarz inequality reads

\[ 2N|m_q|^2 \left( \beta \sum_{\vec{k}} \langle |\vec{S}(\vec{u})|^2 \rangle (v(\vec{p} + \vec{u}) + v(\vec{p} - \vec{u})) - 2v(\vec{u}) \right) \leq N \sum_{\vec{x},\vec{y}} \langle \vec{S}_1(\vec{x}) \cdot \vec{S}_1(\vec{y}) \rangle e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \] (110)

Explicitly, as the integral \( \int |k| > \delta \frac{d^d k}{(2\pi)^d} \) is non-negative (as \( \langle \hat{B}(\vec{k})^* \cdot \hat{B}(\vec{k}) \rangle \geq 0 \) the denominator in Eqn. 108 is always positive for each individual value of \( \vec{k} \), and as \( \langle \vec{S}_1^2(\vec{x}) \rangle \leq 1 \), we obtain in the thermodynamic limit

\[ \frac{2\beta}{m_q^2} \int |k| < \delta \frac{d^d k}{(2\pi)^d} \int \frac{d^d u}{(2\pi)^d} \langle |\vec{S}(\vec{u})|^2 \rangle (v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{u})) + 2|h||m_q| \leq 1. \] (111)

Taking \( \delta \) to be small we may bound from above (for each value of \( \vec{k} \)) the positive denominator in the square brackets and consequently

\[ \int |k| < \delta \frac{d^d k}{(2\pi)^d} \int \frac{d^d u}{(2\pi)^d} \langle |\vec{S}(\vec{u})|^2 \rangle (v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{u})) + 2|h||m_q| \leq \frac{A_1 k^2 \lambda_{\alpha}(\langle |\vec{S}(\vec{u})|^2 \rangle + 2|h||m_q|)}{B_1 k^2} \] (112)

with \( \lambda_{\alpha} \) chosen to be the largest principal eigenvalue of the \( d \times d \) matrix \( \partial_\alpha \partial_\beta |v(\vec{u})| \), and \( A_1 \) a constant.

For a twice differentiable \( v(\vec{u}) \), and for \( |\vec{k}| \leq \delta \) where \( \delta \) is finite,

\[ (v(\vec{k} + \vec{u}) + v(\vec{k} - \vec{u}) - 2v(\vec{u})) \leq A_1 \lambda_{\alpha} k^2 \leq B_1 k^2 \] (113)

for all \( \vec{u} \) within the Brillouin Zone with the additional positive constant \( B_1 \) introduced.

In \( d \leq 2 \), the integral

\[ \int |k| < \delta \frac{d^d k}{B_1 k^2} \] (114)
diverges making it possible to satisfy eqn. [111] when the external magnetic field \( h \to 0 \) only if the magnetization \( m_{
abla} = 0 \). If finite size effects are restored, in a system of size \( N = L \times L \) where the infrared cutoff in the integral is \( O(2^L, 2^L) \) the latter integral diverges as \( O(\ln N) \). This implies that the upper bound on \( |m_{\nabla}| \) scales as \( O(N/\sqrt{\ln N}) \), much lower than the \( O(N) \) requisite for finite on-site magnetization. For further details see [11].

If there are \( M \geq 2 \) pairs of minimizing modes and \( 2p + 1 \geq n \geq 2p \) (with an integer \( p \)) then we may apply an infinitesimal symmetry breaking magnetic field \( \alpha \beta \) (peratures we cannot assume the zero temperature ground state with the natural dispersion relation \( \Delta E \) for fluctuations about it.

A case in point is the dispersion relation derived in Eqn. (8) for the Coulomb Frustrated Ferromagnet. The reader might recognize the denominator in Eqn. (111) as nothing but a finite temperature extension of the zero temperature dispersion relations \( (\Delta^{(2)} \epsilon) \) derived in Eqn. (8) and Eqn. (3) with the boost parameter \( \vec{k} \) portrayed by \( \vec{d} \). In general, at zero temperature,

\[
\langle |\vec{S}(\vec{k})|^2 \rangle = \frac{N^2}{2} [\delta k_x - \delta k_y - \delta k_z] \tag{118}
\]

and the integral in Eqn. (116) is nothing but

\[
\int \frac{d^d k}{v(\vec{k} + \vec{q}) + v(\vec{k} - \vec{q}) - 2v(\vec{q})}. \tag{119}
\]

Whenever this integral diverges, a bold assumption of an almost ordered ground state at arbitrarily low temperatures \( (T = 0^+) \) with the ensuing zero temperature dispersion relation \( (\Delta^{(2)} \epsilon) = 2(E_{k} - E_0) \) about that ordered state is flawed: Eqn. (116) is strongly violated.

This poses no problem for all canonical \( d \geq 2 \) dimensional models where the integral in Eqn. (119) is finite. For the three dimensional Coulomb Frustrated Ferromagnet and other high dimensional models the divergence of this integral hints possible non-trivialities.

XI. MERMIN-WAGNER-COLEMAN BOUNDS IN HIGH DIMENSIONS

In any dimension, Eqn. (111) reads in the limit \( h \to 0 \)

\[
2|m_{\nabla}|^2 T \int \frac{d^d k}{(2\pi)^d} \frac{1}{\Delta^{(2)} \epsilon} \leq 1 \tag{116}
\]

with the shorthand \( \Delta^{(2)} \epsilon \) defined by Eqn. (107).

By parity invariance and the “realness” of the spins \( \{\vec{S}(\vec{x})\} \):

\[
\sum_{\vec{u}} v(\vec{k} + \vec{u})(|\vec{S}(\vec{u})|^2) = \sum_{\vec{u}} v(\vec{k} - \vec{u})(|\vec{S}(\vec{u})|^2). \tag{117}
\]

Thus the denominator of Eqn. (116) reads \( \Delta^{(2)} \epsilon = [2(E_{\vec{k}} - E_0)] \) where \( E_{\vec{k}} \) is the internal energy of system after undergoing a boost of momentum \( \vec{k} \).

In some instances when the dispersion relation about an assumed zero temperature ground state is inserted into Eqn. (116), we will find that the integral in Eqn. (116) diverges. This merely signals that at arbitrarily low temperatures we cannot assume the zero temperature ground

\[
|\psi_m| \otimes |\delta S(\vec{k})\rangle. \tag{121}
\]

XII. O(\(N = 3\)) FLUCTUATIONS

Let us now return to the more naive “soft-spin” \( O(3) \) models in order to witness an intriguing even-odd binding-unbinding effect that could have otherwise been missed.

As we have proved, for an \( n = 3 \) system the generic ground states are simple spirals. If we rotate the helical ground-state to the \( 1 \leq 2 \) plane; the single quadratic term in \( \delta S_{i=3}(\vec{x}) \) is \( \sum_{\vec{q}} (\delta S_{i=3}(\vec{x})^2. \)

\[
\lambda_{\geq 3} = 2 = \lambda_+ = \lambda_{\min}. \tag{120}
\]

Here, all that follows holds for arbitrarily large \( u \)- the only approximation that we are making is neglecting \( O(|\delta S|^3) \) terms by comparison to quadratic terms: i.e. assuming that \( |\delta S(\vec{x})| \ll 1 \). Unlike the above treatment of the XY spins, no small \( u \) is necessary in order to make headway on the Heisenberg problem.

For \( n = 3 \) we find that the fluctuation eigenstates of the are the products of an eigenstate of \( H \) within the plane of the spiral ground state and a fluctuation eigenstate of \( v(\vec{k}) \) along the direction orthogonal to the spiral plane. Written formally, to quadratic order, the fluctuation eigenstates are
Fluctuations along the $i = 3$ axis are orthogonal (in a geometrical and formal sense) to the ground-state plane. This is expected as fluctuations in any hyper-plane perpendicular to the $[12]$ plane do not change, to lowest order, the norm of the spin. $|\delta S_{(\vec{k})}\rangle$ is literally the “odd” man out. As foretold, this is a general occurrence. Whenever the number of spin components is odd, one unpaired spin component is unaffected by the interaction enforcing the spin normalization constraint.

Within the $i = 3$ subspace $|\langle \vec{k}|M|\vec{k}'\rangle\rangle = 2\delta_{\vec{k},\vec{k}'_3}$ and our previous analysis follows. The dispersion $E_{k=3} = [v(\vec{k}) + 2 u \lambda_{i=3}]$ does not have a higher minimum than with $E_m$ [3].

Both have the same $\lambda$ value and the $\delta S_{i=3}(\vec{x})$ fluctuation is minimized at wave-vectors $\vec{\ell} \in M_Q$ s.t. $v(\vec{\ell}) = v(\vec{q}) = \min_\vec{k} v(\vec{k})$. As $|\psi_m\rangle$ is a normalized superposition of $\langle \delta S(\vec{k})\rangle$ modes (the latter spin vectors being in the $1-2$ plane) and $v(\vec{\ell})$ is diagonal, and attains its minimum at $\vec{\ell} \in M_Q$:

$$\min_m \{E_m\} \geq E_{\vec{\ell} \in M_Q}.$$  (122)

Here, we invoked the trivial inequality

$$\min_\psi \langle \psi | [H_0 + H_1] | \psi \rangle \geq \min_\phi \langle \phi | H_0 | \phi \rangle + \min_\xi \langle \xi | H_1 | \xi \rangle.$$  (123)

Thus, there exist Goldstone modes corresponding to $\delta S_{i=3}$ fluctuations, and one must adjust additive constants s.t. $\min_\vec{k} \{E_{k=3}\} = 0$.

The resulting fluctuation integral reads

$$\langle (\delta S_{i=3}(\vec{x} = 0))^2 \rangle = k_B T \int \frac{d^4k}{(2\pi)^3} \frac{1}{v(\vec{k}) - v(\vec{q})}.$$  (124)

(Where we noted translational invariance $\langle \delta S_i(\vec{k}) \delta S_i(\vec{k'}) \rangle = \delta_{\vec{k},\vec{k'}}\delta_{\vec{q},\vec{q'}}|\langle \delta S_i(\vec{k}) \rangle|^2$ and employed equipartition). As pointed out earlier, when $Q > 0$ in $v_{\text{continuum}}(\vec{k})$ the minimizing manifold is $(d - 1)$ dimensional. The fluctuation integral receives divergent contributions from the low energy modes nearby. By quadratic expansion about the minimum along $\tilde{\vec{n}} \perp M_Q$, a divergent one-dimensional integral for the bounded $\langle (\Delta \vec{S}(\vec{x} = 0))^2 \rangle$ signals that are quadratic fluctuation analysis calculation is inconsistent. We are led to the conclusion that higher order constraining terms are imperative: We cannot throw away cubic and quartic spin fluctuation terms $\langle \delta S_{i=3}(\vec{x}) \rangle$ relative to the quadratic $\langle \delta S_{i=3}(\vec{x}) \rangle$ terms (notwithstanding the fact that all of these terms appear with $O(u)$ prefactors irrespective of how large $u$ is). The spin fluctuations $\delta S_{i=3}(\vec{x})$ are of order unity at all finite temperatures and $T_c(Q > 0) \approx 0$.

Note that this is “almost a theorem”. Here we do not demand that $u$ be small (only $\delta S$). This is an important point. $|\psi_m\rangle$ is an eigenstate for arbitrarily large $u$. To quadratic order in the fluctuations, the minimum belongs to $|\delta S_{(\vec{k})}\rangle$ (or is degenerate with it).

The divergent fluctuations here signal that $O(\delta S^4) = O(\delta S^2)$. Thus assuming that $\delta S \lesssim \sqrt{J/u}$ (with $J = 1$ the exchange constant) we reach a paradox. Thus, if the integral in Eqn. (124) diverges for an $O(3)$ model then at all finite $T$: $\delta S \geq \sqrt{J/u}$.

When $Q = 0$ the minimizing manifold shrinks to a point- the number of nearby low energy modes is small and our fluctuation integral converges in $d > 2$. This is in accord with the well known finite temperature phase transition of the nearest neighbor Heisenberg ferromagnet: $T_c(Q = 0) = O(1)$ (or when dimensions are fully restored- it is of the order of the exchange constant).

Notice that a discontinuity in $T_c$ occurs as $M_Q \rightarrow M_{Q=0} = \{\vec{q} = 0\}$.

We anticipate that small lattice corrections ($\lambda \neq 0$) in $v_Q(\vec{k})$ to yield insignificant modifications to $T_c(Q)$: one way to intuit this is to estimate $T_c$ by the temperature at which the fluctuations, as computed within the quadratic Hamiltonian $\langle \delta \vec{S}^2(\vec{x} = 0) \rangle = O(1)$.

**Summary and Outlook**

We have argued that in (essentially hard spin) Heisenberg realizations of our models no long range order is possible in the continuum limit: $T_c(Q > 0) = 0$. More generally we claim that if the integral

$$\int \frac{d^4k}{v(\vec{k}) - v(\vec{q})}$$  (125)

diverges then no long range order is possible at finite temperature. If lattice effects are mild then $T_c(Q > 0)$ is expected to be small. In the unfrustrated ($Q = 0$) Heisenberg ferromagnet in $d = 3$: $T_c = O(1)$.

Fusing these facts, a discontinuity in $T_c$:

$$\delta T_c = \lim_{Q \rightarrow 0} [T_c(0) - T_c(Q)]$$  (126)

is seen to exist. $T_c(Q = 0)$ is an avoided critical point (see fig. 1 in the introduction).

**XIII. $O(N \geq 4)$ Fluctuations**

The fluctuation analysis of any $O(n > 2)$ system about a spiral ground state is qualitatively similar to that of the Heisenberg system.

For a vanishing lower cutoff $\epsilon$ on $|\vec{k} - q|$, the fluctuations of an $n$-component spin ($d = 3$) about a helical ground state are given by

$$\frac{\langle (\Delta \vec{S})^2 \rangle}{k_B T} = \frac{(n - 2)\sqrt{Q}}{4\pi^2\epsilon} - \frac{1}{16\pi} Q^{1/4} \ln |\epsilon|.$$  (127)

More generally, in $d > 2$ dimensions, the leading order infrared contribution reads
As noted previously, poly-spiral states will tend to dominate at large $n$.

The reader can convince him/herself that for even $n$ with $p < n/2$ poly-spiral ground state and for all odd $n$ the fluctuations will give rise to a leading order $e^{-\epsilon}$ divergence. The reasoning is simple: the poly-spiral states extend along an even number of axis. If $n$ is odd then there will be at least one internal spin direction $i$ along which $S_i^0 = 0$ and our analysis of the Heisenberg model can be reproduced.

The lowest eigen-energy associated with the fluctuations $|\psi_m\rangle$ in the $(2p)$ dimensional space spanned by the ground state is no less than the lowest eigen-energy for fluctuations along an orthogonal direction.

$$\langle \psi | H_{\text{soft}} | \psi \rangle \geq 0.$$ If $|\delta S| \ll 1$, this implies that the quadratic term in $\delta S(\vec{x})$ stemming from $H_1$ is non-negative definite.

For $S_i^0(\vec{x}) = 0$, this quadratic term in $\delta S_i(\vec{x})$ is zero.

The eigenvalue $\lambda_{\min} = \lambda_\epsilon = 2$ corresponds to the zero contribution in $O(\delta S^2(\vec{x}))$ from $H_1$.

And once again

$$\min_m \{ E_m \} \geq E^{i}_{\epsilon \in M_{\Omega}}$$

from the trivial inequality

$$\min_{\vec{q}} \{ \langle \psi | H_0 + H_{\text{soft}} | \psi \rangle \} \geq \min_{\phi} \{ \langle \psi | H_0 | \phi \rangle \} + \min_{\xi} \{ \langle \xi | H_{\text{soft}} | \xi \rangle \}.$$  

The fluctuations of even component spin about a $p = n/2$ poly-spiral ground state are more complicated.

Once again coupling between different modes occurs. In this case they are more numerous.

For $n = 4$, the fluctuation energy, to quadratic order, about a bi-spiral reads

$$\delta H = \frac{1}{N} \sum_{k_1, k_2} \left[ \langle \psi | H_0 + H_{\text{soft}} | \psi \rangle \right] | \delta S_1(\vec{k}_1) \delta S_2(\vec{k}_2) + \delta S_1(\vec{k}_1) \delta S_2(\vec{k}_2) |^2 +$$

$$+ \sum_{k_1, k_2} \left[ \langle \xi | H_0 + H_{\text{soft}} | \xi \rangle \right] | \delta S_1(\vec{k}_1) \delta S_2(\vec{k}_2) + \delta S_1(\vec{k}_1) \delta S_2(\vec{k}_2) |^2$$

and by expansion of $\Delta H$ for different sorts of twists, the dispersion relations of the two spiral simply lumped together. When $d = 3$, as in the $O(2)$ case ($p = 1$) this dispersion gives rise (in the Gaussian approximation) to diverging logarithmic fluctuations: $O(\ln \epsilon)$. Applying equipartition, the Gaussian spin fluctuations in the $[2i - 1, 2i]$ plane:
\[ \Delta S_{[2i-1,2j]}^2(\vec{x} = 0) > \Delta S_{\text{low}}^2|_{[2i-1,2j]}(\vec{x} = 0) = k_B T \int \frac{d^3k}{(2\pi)^3} \frac{1}{a_i^2|A_i|^2 + A_\perp \delta_{i,1}}. \] (134)

For all odd \( n \) and for all even \( n \) with \( p < n/2 \) there will be divergent fluctuations similar to those encountered for the \( O(3) \) model.

We can now update and summarize our conclusions: We have just proved that if frustrating interactions cause the ground states to be modulated then the associated ground state degeneracy (for \( n > 2 \)) is much larger by comparison to the usual ferromagnetic ground states. For even \( n \) we have found that, generically, the a three dimensional system will not have long range order when \( M \) is two dimensional. When \( n \) is odd the system will never show long range order if \( M \) is \((d-1)\) or \((d-2) \) dimensional.

If, in the continuum limit, the uniform (ferromagnetic) state is higher in energy than any other state then, by rotational symmetry, the manifold of minimizing modes in Fourier space is \((d-1) \) dimensional.

In reality, small symmetry breaking terms (e.g. \( \lambda \neq 0 \) in \( v_Q(\vec{k}) \)) will always be present; these will favor ordering at a discrete set of \( \{ \pm \vec{q}_m \}_{m=1}^{|M|} \). If \( n \), then, irrespective of the even/odd parity of \( n \), then will be a
\[
\langle \Delta S^2(\vec{x} = 0) \rangle \geq (n-2|M|) \int \frac{d^d k}{(2\pi)^d} \frac{k_B T}{v(k) - v(\vec{q})}. \] (135)

This contribution is monotonically increasing in \( n \); Within our scheme, \( T_c \) is finite and may be estimated by the temperature at which the fluctuations are of order unity. By tweaking the symmetry breaking terms to smaller and smaller values, the fluctuation integral becomes larger and larger. For instance, if take \( \lambda \ll 1 \) in \( v_Q(\vec{k}) \) then the integral is very large and \( T_c \) extremely low (in can be made arbitrarily low). Thus as the system will be cooled from high temperatures, it might first undergo a Kosterlitz-Thouless like transition at \( T_{KT} \) to an algebraically ordered state and develop true long range order at critical temperatures \( T_c < T_{KT} \).

**XIV. LARGE \( N \) LIMIT**

So far we have seen that the even \( n \) systems are more “gapped” than its odd counterparts. There is never a paradox in the large \( n \) (or spherical model) limit. In this limit wherein a single normalization constraint is imposed
\[
\sum_{\vec{x}} S^2(\vec{x}) = N, \] (136)

the effective number of spin components \( n \) is of the order of the number of sites in the system \( N \). The span of the system \( N \) (the number of Fourier modes allowed within the Brillouin zone) is always larger than the number of minimizing modes \( \{ \vec{q}_i \} \).

In such a case we will be left with a divergence as in equation (135) due to the many unpaired spin components.

In fact, within the spherical model, which is easily solvable the fluctuation integral exactly marks the value of the inverse critical temperature
\[
\frac{1}{k_B T_c} = \int_{B.Z.} \frac{d^d k}{(2\pi)^d} \frac{1}{v(\vec{k}) - v(\vec{q})}. \] (137)

Thus, \( T_c = 0 \) if the latter integral diverges and our circle of ideas nicely closes on itself.

**XV. \( O(N \geq 2) \) WEISS MEAN FIELD THEORY OF ANY TRANSLATIONAL INVARIANT THEORY.**

Let us begin by examining the situation simple spiral states. In this case for \( O(n \geq 2) \) when \( T < T_c \):
\[
\langle \vec{S}(\vec{x}) \rangle = s \vec{S}_{\text{ground-state}}(\vec{x}). \] (138)

For the particular case
\[
\begin{align*}
S_1^\text{ground-state}(\vec{x}) &= \cos(\vec{q} \cdot \vec{x}) \\
S_2^\text{ground-state}(\vec{x}) &= \sin(\vec{q} \cdot \vec{x}) \\
S_{\geq 3}^\text{ground-state}(\vec{x}) &= 0
\end{align*} \] (139)

Now only the \( \pm \vec{q} \) modes have finite weight. Repeating the previous steps
\[
\sum_{\vec{y}} V(\vec{x} = 0, \vec{y}) S_2^\text{ground-state}(\vec{y}) = 0 \] (140)

\[ |\langle \vec{S}(\vec{x} = 0) \rangle| = |\langle S_1(\vec{x} = 0) \rangle| \] (141)

Define
\[
M[z] = -\frac{d}{dz} \ln[(2/z)^{n/2 - 1}I_{n/2 - 1}(z)], \] (142)

with \([I_{n/2 - 1}(z)]\) a Bessel function. The mean-field equation reads
\[
|\langle S_1(\vec{x} = 0) \rangle| = s = M[| \sum_{\vec{y}} V(\vec{x}, \vec{y}) S_1(\vec{y})|]. \]

The onset of the non-zero solutions is at
\[
|\vec{q}_c v(\vec{q})| = n. \] (143)

If \( V(\vec{x} = 0) = 0 \) (no on-site interaction), then
\[
\int d^d k \ v(\vec{k}) = 0, \] (144)
implying that \( v(\bar{q}) < 0 \) and \( T_c > 0 \). Note that within the mean field approximation, \( T_c \) is a continuous function of the parameters.

Here the ground state is symmetric with respect to all sites. The above is the exact value of \( T_c \) within Weiss mean field theory for the helical ground-states.

For poly-spirals we will get \( p \) identical equations: both sides of the self consistency equations are multiplied by \( \alpha_i^2 \) where \( \alpha_i \) is the amplitude of the \( l-th \) spiral in the \([2l-1],2l\) plane. As \( v(\bar{q}_m) = v(\bar{q}) \), we will arrive at the same value of \( T_c \) as for the case of simple spirals.

**XVI. EXTENSIONS TO ARBITRARY TWO SPIN INTERACTIONS: \( \langle \vec{U} \rangle \) SPACE \( \otimes O(N) \) TOPOLOGY ETC.**

Any real kernel \( V(\vec{x}, \vec{y}) \) may be symmetrized \([V(\vec{x}, \vec{y}) + V(\vec{y}, \vec{x})]/2 \rightarrow V(\vec{x}, \vec{y})\) to a hermitian form.

Consequently, by a unitary transformation, it will become diagonal. The Fourier modes are the eigen-modes of \( V \) when it is translationally invariant. We may similarly envisage extensions to other, arbitrary, \( V(\vec{x}, \vec{y}) \) which will become diagonal in some other complete orthogonal basis \( |\vec{u}\rangle \):

\[
\langle \vec{u}_i|V|\vec{u}_j\rangle = \delta_{i,j}\langle \vec{u}_i|V|\vec{u}_i\rangle \quad (145)
\]

Many of the statements that we have made hitherto have a similar flavor in this more general case.

For instance, the large \( n \) fluctuation integrals are of the same form

\[
\int \frac{d^d u}{(2\pi)^d} \frac{1}{v(\vec{u}) - v_{\text{min}}} \quad (146)
\]

with the wave-vector \( \vec{k} \) traded in for \( \vec{u} \).

Once again, one may examine the topology of the minimizing manifold in \( \vec{u} \) space. If the surface if \((d-1)\) dimensional and \( v(\vec{u}) \) is analytic in its enivrons then, for large \( n, T_c = 0 \).

The topology of the ground state sector of \( O(n) \) models will once again be governed by a direct product of the topology of the minimizing manifold in \( \vec{u} \) space with the spherical manifold of the \( O(n) \) group. In the general case in will be dramatically rich.

We may similarly extend the Peierls bounds to some infinite range interactions also in this case by contrasting the energy penalties in the now diagonalizing \( \vec{u} \) basis with those that occur for short range systems in Fourier space

\[
\Delta E_{\text{disordered}} = \frac{1}{2N} \sum_{\vec{u}} |\vec{S}(\vec{u})|^2 \langle |\vec{u}|V_{\text{dis}}|\vec{u}\rangle - v_{\text{min}}(\vec{u}) \geq \frac{1}{2N} \sum_{\vec{k}} |\vec{S}(\vec{k})|^2 [v_{\text{short}}(\vec{k}) - v_{\text{short min}}] \quad (147)
\]

if they share the same lowest energy eigenstate of \( V \).

**XVII. O(N) SPIN DYNAMICS AND SIMULATIONS**

For our repeated general Hamiltonian

\[
H = \frac{1}{2} \sum_{\vec{x}, \vec{y}} V(\vec{x}, \vec{y}) \vec{S}(\vec{x}) \cdot \vec{S}(\vec{y}) \quad (148)
\]

the force on given spin

\[
\vec{F}(\vec{z}) = -\frac{\partial H}{\partial \vec{S}(\vec{z})} = -\frac{1}{2} \sum_{\vec{y}} [V(\vec{x}, \vec{y}) + V(\vec{y}, \vec{z})] \vec{S}(\vec{y}) \quad (149)
\]

For \( V(\vec{x}, \vec{y}) = V(\vec{x} - \vec{y}) = V(\vec{y} - \vec{x}) \) or otherwise we may symmetrize \([V(\vec{x}, \vec{y}) + V(\vec{y}, \vec{x})]/2 \rightarrow V(\vec{x}, \vec{y})\), as we have done repeatedly, without changing \( H \). For a more general two spin kernel \( V(\vec{x}, \vec{y}) \) which is not translationally invariant the Fourier space index \( \vec{k} \) should be replaced by the more general \( \vec{u} \).

\[
\begin{align*}
\frac{d^2 \vec{S}(\vec{x})}{dt^2} &= -\sum_{\vec{y}} V(\vec{x} - \vec{y}) \vec{S}(\vec{y}) \frac{d^2 \vec{S}(\vec{k})}{dt^2} \\
&= -\frac{1}{N} V(\vec{\vec{z}}) \vec{S}(\vec{\vec{z}}). \quad (150)
\end{align*}
\]

Alternatively, the last equation can be derived by starting with the Hamiltonian expressed directly in Fourier space

\[
H = \frac{1}{2N} \sum_{\vec{k}} v(\vec{k}) \vec{S}(\vec{k}) \cdot \vec{S}(\vec{-k}) \]

\[
\frac{d\Pi(\vec{p}, t)}{dt} = \frac{d^2 \vec{S}(\vec{p})}{dt^2} = -\frac{\partial H}{\partial \vec{S}(\vec{p})} = -\frac{1}{2N} [v(\vec{p}) + v(-\vec{p})] \vec{S}(\vec{p}) \]

\[
= -\frac{1}{N} v(\vec{p}) \vec{S}(\vec{p}), \quad (151)
\]

where in the last equation the momentum \( \Pi(\vec{p}) \) conjugate to \( \vec{S}(\vec{x}) \) is trivially

\[
\frac{\partial L}{\partial (d\vec{S}(\vec{x})/dt)} = \frac{\partial (\frac{1}{2} \sum_{\vec{x}} [d\vec{S}(\vec{x})/dt]^2 - \frac{1}{2} \sum_{\vec{x}, \vec{y}} V(\vec{x}, \vec{y}) \vec{S}(\vec{x}) \cdot \vec{S}(\vec{y}))}{\partial (d\vec{S}(\vec{x})/dt)} = \frac{d\vec{S}(\vec{x})}{dt}, \quad (152)
\]

and upon Fourier transforming \( \Pi(\vec{p}) = (d\vec{S}(\vec{x})/dt) \). Let an arbitrary \( A \) satisfy
\[ A > - \min_{k} \{ v(\vec{k}) \}. \] (153)

The equations of motion
\[ \frac{d^2 \vec{S}(\vec{k})}{dt^2} = -[A + v(\vec{k})] \vec{S}(\vec{k}) \]
\[ = -\omega_0^2 \vec{S}(\vec{k}) \] (154)
may be trivially integrated. [Adding the constant \( A \)
merely shifts \( H \to H + A/2. \)]

\[ \vec{S}_{un}(\vec{k}, t) = \vec{S}(\vec{k}, 0) \cos \omega_k t + \frac{d \vec{S}(\vec{k}, t)}{dt}|_{t=0} \times \omega_k^{-1} \sin \omega_k t, \]
\[ \vec{S}_{un}(\vec{k}, t + \delta t) = \vec{S}(\vec{k}, t) \cos \omega_k \delta t \]
\[ + \frac{\delta \vec{S}(\vec{k}, t)}{\delta t} \omega_k^{-1} \sin \omega_k \delta t. \]

This suggests the following simple algorithm:

(i) At time \( t \), start off with initial values \( \{ \vec{S}(\vec{x}, t) \} \).

(ii) Fourier transform to find \( \{ \vec{S}(\vec{k}, t) \} \).

(iii) Integrate to find the un-normalized \( \{ \vec{S}_{un}(\vec{k}, t + \delta t) \} \).

(iv) Fourier transform back to find the un-normalized real-space spins \( \{ \vec{S}_{un}(\vec{x}, t + \delta t) \} \).

(v) Normalize the spins:
\[ \vec{S}(\vec{x}, t + \delta t) = \frac{\vec{S}_{un}(\vec{x}, t + \delta t)}{|\vec{S}_{un}(\vec{x}, t + \delta t)|}. \] (155)

(vi) Compute \( \{ \delta \vec{S}(\vec{x}, t + \delta t) \} = \{ [\vec{S}(\vec{x}, t + \delta t) - \vec{S}(\vec{x}, t)] \} \).

(vii) Fourier transform to find \( \{ \delta \vec{S}(\vec{k}, t + \delta t) \} \).

(viii) Go back to (ii).

Thus far we have neglected thermal effects. To take these into account, one could integrate these equations with a thermal noise term augmented to the restoring force
\[ \frac{d^2 \vec{S}(\vec{k})}{dt^2} = -\frac{1}{N} v(\vec{k}) \vec{S}(\vec{k}) + \vec{F}_k^{\text{noise}}(T, t). \] (156)

Expressed in this format, the execution of this algorithm for continuous \( O(n \geq 2) \) spins seems easier than that for a discrete Ising system. Here the equations of motion may be integrated to produce arbitrarily small updates at all sites.

This could, perhaps, be better than a brute force approach whereby the torque equations in angular variables (the spins \( \vec{S}(\vec{x}) \) are automatically normalized) are integrated whereby the eqs of motion would explicitly read
\[ \frac{d^2 \phi(\vec{x})}{dt^2} = \sum_{\vec{y}} V(\vec{x} - \vec{y}) \sin[\phi(\vec{x}) - \phi(\vec{y})]. \] (157)
for an \( O(2) \) system.

For the three-component spin system:
\[ \sin^2 \theta(\vec{x}) \frac{d^2 \phi(\vec{x})}{dt^2} + \sin 2\theta(\vec{x}) \frac{d\phi(\vec{x})}{dt} \frac{d\theta(\vec{x})}{dt} \]
\[ = \sum_{\vec{y}} V(\vec{x} - \vec{y}) \sin \theta(\vec{x}) \times \sin \theta(\vec{y}) \sin[\phi(\vec{x}) - \phi(\vec{y})]. \]
\[ \cos \theta(\vec{y}) - \cos \theta(\vec{x}) \sin \theta(\vec{y}) \cos[\phi(\vec{x}) - \phi(\vec{y})]. \] (158)

VIII. APPENDIX

Here we follow the beautiful treatment of Als-Nielsen et al. [16].

Within the (hard spin) fully constrained XY model:
\[ G(\vec{x} - \vec{y}) = \langle \vec{S}(\vec{x}) \cdot \vec{S}(\vec{y}) \rangle = \langle \cos[\theta(\vec{x}) - \theta(\vec{y})] \rangle. \] (159)

Here \( \theta(\vec{x}) = \vec{q} \cdot \vec{x} + \Delta \theta(\vec{x}) \), i.e. \( \Delta \theta \) denotes the phase fluctuations about our spiral ground state and
\[ G(\vec{x} - \vec{y}) = \cos(\vec{q} \cdot (\vec{x} - \vec{y})) e^{i(\Delta \theta(\vec{x}) - \Delta \theta(\vec{y}))}. \] (160)

In our harmonic approximation \( \{ \delta \theta(\vec{x}) \} \) are random Gaussian variables and only the first term in the cumulant expansion is non-vanishing.

The correlator
\[ G(\vec{x}) = \exp[-\frac{1}{2} \sum_{\vec{x}} (\Delta \theta(\vec{x}) - \Delta \theta(0))^2] \]
\[ = \exp \left[ k_B T \int \frac{d^4 k}{(2\pi)^d} \frac{1 - \cos \vec{q} \cdot \vec{x}}{A_{||} k_{||}^2 + A_\perp k_{\perp}^2} \right]. \] (161)

Now let us shift variables \( \vec{k} \to \vec{k} - \vec{q} \equiv \delta \), and for purposes of convergence explicitly introduce an upper bound on \( k_{\perp} \): \( 0 < k_{\perp} < \Lambda \)
\[ I(\vec{x}, x_{||}) \equiv \int \frac{1 - \cos \vec{q} \cdot \vec{x}}{A_{||} k_{||}^2 + A_\perp k_{\perp}^2}. \] (162)

This may be computed by first integrating over \( k_{||} \) employing
\[ \int_{-\infty}^{\infty} \frac{1 - \cos(\phi(x))}{x^2 + c^2} \, dx = \frac{\pi}{c} \left[ 1 - e^{-ac} \cos(ab) \right] \] (163)

to obtain

\[ \frac{1}{A||} \int_{0}^{\infty} \frac{1 - \cos\left(\frac{k_\parallel}{k_\perp} \cdot \vec{x}_{\perp}\right)}{k_\perp^2 + (A\parallel k_\parallel^2/A||)} \, dk_\parallel \]

\[ = \frac{1}{2} \pi^2 k_\perp^2 \left( \frac{1}{A||} - 1 - \exp\left(-\frac{A\parallel k_\parallel^2}{A||} \sin^2 \left(\vec{k}_{\perp} \cdot \vec{x}_{\perp}\right)\right) \right) \] (164)

If \( \phi \) denotes the angle between \( \vec{k}_{\perp} \) and \( \vec{x}_{\perp} \) then

\[ \int_{0}^{2\pi} \left[ 1 - \exp\left(-\sqrt{\frac{A\parallel}{A||} k_\perp^2} \sin^2 \left(\vec{k}_{\perp} \cdot \vec{x}_{\perp}\right)\right) \right] \, d\phi \]

\[ = 2\pi - \exp\left(-\sqrt{\frac{A\parallel}{A||} k_\perp^2} \right) \int_{0}^{2\pi} \cos(k_\perp x_{\perp} \cos \phi) \, d\phi. \] (165)

As

\[ J_0(x) = \frac{1}{\pi} \int_{0}^{\pi} \cos(x \cos \phi) \, d\phi, \] (166)

\[ I(\vec{x}) = \frac{1}{2A||} \pi^2 2\pi \int_{A}^{1} 1 - \exp\left(-\sqrt{\frac{A\parallel}{A||} k_\perp^2} \right) J_0(k_\perp x_{\perp}) \, dk_\perp \] (167)

We may now insert the series expansion of \( J_0(x) \) and integrate term by term.

\[ J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2^n n!^2}. \] (168)

Comparing the result to the series form for the exponential

\[ E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n(n!)} \] (169)

\((\gamma \) is Euler’s constant\) we find that

\[ G(\vec{x}) \sim \frac{4d^2}{x_{\perp}} \exp[-2\eta \gamma - \eta E_1(\frac{x_{\perp}^2 q}{4x_{\perp}^2 \sqrt{\frac{A\parallel}{A||}}} \) \times \cos[q x_{\perp}]] \] (170)

where \( \eta = \frac{k_B T}{\hbar \omega} \) which in our case is \( \frac{k_B T}{\hbar \omega} Q^{1/4} \) and \( d = \frac{2 \pi}{\Lambda} \) where \( \Lambda \) is the aforementioned ultra violet momentum cutoff.

**XIX. ACKNOWLEDGMENTS**

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[8] In the usual case \( q = 0 \) and the quadratic fluctuations converge as \( \ln L \) as \( L \) (the size of the system) sets a lower bound: \( k_i \geq 2\pi/L \) in the integral.
[9] N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966)
[10] S. Coleman, Comm. Math. Phys. 31, 259 (1973)
[11] Note that even if we had not known the ground states we could still prove that there is no magnetization. All one would need to do is to apply an infinitesimal magnetic field

\[ \vec{h}(\vec{x}) = h S_{\text{ground state}}(\vec{x}) \] (171)

with as yet an unknown ground state \( S_{\text{ground state}}(\vec{x}) \). Replacing any appearance of the minimizing wave-vector \( \vec{q} \) in the Schwarz inequality and the definition of \( B(\vec{k}) \) by the more general wave-vector \( \vec{\ell} \) and setting \( h \rightarrow 0^+ \) one would arrive at the conclusion that \( m_\ell = 0 \) for each mode \( \vec{\ell} \). Consequently

\[ \langle S_{\ell}(\vec{x}) \rangle = \frac{1}{N} \sum_{\ell} e^{i\ell \cdot \vec{x}} m_\ell = 0 \] (172)

Stated alternatively, by Parseval’s theorem,

\[ \sum_{\vec{x}} \langle \vec{S}(\vec{x}) \rangle^2 = \frac{1}{N^2} \sum_{\ell} \langle \vec{S}(\vec{\ell}) \rangle \langle \vec{S}(\vec{-\ell}) \rangle. \] (173)

Taking careful note of the system size \( N \) dependence in Eqs.\([14]\) and Eqs.\([111]\),

\[ \sum_{\vec{x}} \langle \vec{S}(\vec{x}) \rangle^2 = O\left( \frac{N}{\ln N} \right) \] (174)

and thus the average value of \( \langle \langle \vec{S}(\vec{x}) \rangle \rangle \) at any given site \( \vec{x} \) diminishes as \( O(1/\ln N) \) which vanishes in the thermodynamic \((N \rightarrow \infty)\) limit.

[12] C. Itzykson and J-M. Drouffe “Statistical field theory”, Cambridge University Press (1989)
[13] P. Bruno, cond-mat/0105128
[14] S. Chakravarty and C. Dasgupta, Phys. Rev. B. 22, 369 (1980)
[15] Looking at the real space form of the constraining term it is readily seen that \( \langle \psi | H_1 | \psi \rangle \geq 0 \). If \( |\delta S| < 1 \) this implies
that the quadratic term in $\delta S(\vec{x})$ originating from $H_1$ is non-negative definite. For $S^g_{i-\lambda}(\vec{x}) = 0$, this quadratic term in $\delta S_{i\lambda}(\vec{x})$ vanishes. The lowest eigenvalue $\lambda_{\min} = 2$ corresponds to this vanishing $O(\delta S^2)$ contribution from $H_1$. An analogous occurs in the fluctuation analysis about a poly-spiral state.

[16] J. Als-Nielsen et al., Phys. Rev. B 22, 312 (1980)