Connecting local structure to interface formation: a molecular scale van der Waals theory of nonuniform liquids

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ABSTRACT: This article reviews a new and general theory of nonuniform fluids that naturally incorporates molecular scale information into the classical van der Waals theory of slowly varying interfaces. The method optimally combines two standard approximations, molecular (mean) field theory to describe interface formation and linear response (or Gaussian fluctuation) theory to describe local structure. Accurate results have been found in many different applications in nonuniform simple fluids and these ideas may have important implications for the theory of hydrophobic interactions in water.
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1 Introduction

This article reviews recent progress we and our coworkers have made in developing a new and general theory of nonuniform fluids (1-7), based on a reexamination of the ideas that lead to the classic van der Waals (VDW) theory (8, 9) of the liquid-vapor interface. The VDW interface theory, developed twenty years after the VDW equation of state for the uniform fluid, is equally far-reaching and has a great many virtues that merit further consideration in the light of modern developments in statistical mechanics. It is physically motivated, treating separately the excluded volume effects associated with the short-ranged and harshly repulsive intermolecular forces and the averaged effects of the longer ranged attractive interactions. Both thermodynamic and structural features are connected together naturally in an elegant and self-consistent approach.

In principle the VDW theory can be applied to a very general problem: the structure and thermodynamics of a fluid in the presence of a general external field. However, the usual theory makes a crucial assumption that the fluid density in the presence of the field is in some sense slowly varying. While this assumption seems appropriate for the liquid-vapor interface in zero field where the VDW theory had its original and spectacularly successful application (9), it fails badly when applied to the more general and often rapidly varying fields associated with a number of problems of current interest. By trying to understand precisely where and why the classical theory failed, we have been able to develop a new perspective that allows us to address these more general problems.

For example, the field can describe the interaction of fixed solutes with a solvent liquid. The solvation free energy is related to the changes in free energy as the interaction field is “turned on” from zero to full strength (10). A significant part of the solvation free energy arises from the required expulsion of solvent molecules from the region occupied by the solute. This involves very strong “excluded volume” interactions that can significantly perturb the density around the solutes. These considerations also play a major role in the theory of hydrophobic interactions (4, 11-29), and we will later review and clarify work (4) by Lum, Chandler, and Weeks (LCW) based on this perspective. (As emphasized by Lawrence Pratt in his review of hydrophobic interactions in this volume (30), the general theory of hydrophobic interactions is incredibly complicated, and we will only deal with certain important but limited aspects here.) Similar issues arise in trying to understand effective interactions between solutes arising from depletion forces (31).

An external field representing one or more fixed solvent particles leads to theories for multi-particle correlations functions describing the molecular scale structure of the bulk solvent liquid (32). Nonuniform fluids confined by walls, slits, pores, etc., can be described using appropriate external fields. Particular fields can enhance and alter features seen in the local structure of the uniform fluid. Confining fields can also produce shifts and rounding of bulk liquid-vapor and critical point phase transitions and can change the effective dimensionality of bulk fluid correlations; e.g., from three to two dimensional behavior in sufficiently narrow slits. In addition the field can induce new density correlations associated with interface formation, leading to wetting or drying transitions and related phenomena like capillary condensation (33-42). Thus, in an example we will consider in some detail later, a vapor-like drying region of lower density can form near a hard wall in a Lennard-Jones (LJ) fluid over a range of thermodynamic states near coexistence. We refer to the recent review by Gelb et al (33) for a detailed discussion of many of these possibilities.
This possible mixing of local structural components along with interfacial components in the density response to an external field leads to the great variety of behavior seen in nonuniform fluids. The classical VDW theory gives a qualitatively accurate description of the smooth interfacial components but fails for the local structure features that also must be taken into account for even a qualitatively accurate description of problems like those discussed above.

Of course, it has long been recognized that the VDW assumption of a slowly varying interface is at fault. But that realization alone does not give us much insight into how to improve things. Most modern attempts (43-60) to address these problems have used density functional theory (DFT). Here one tries to express the free energy as a (generally nonlocal) functional of the density, and the VDW theory emerges when a local density approximation, appropriate for a slowly varying interface, is made. But what is that functional when the local density is rapidly varying and even gradient corrections are inadequate? Most workers have used a weighted density approximation in which one averages in some way over the rapidly varying local structure, and this approach has had great success in some applications. However, there is no “theory of theories” (61) for how to choose suitable weighting functions, and a host of different and often highly formal schemes have been proposed. These complications stand in strong contrast to the simplicity and physical appeal of the original VDW theory. We will discuss DFT further in Section 4.

We show here that there is another and very simple way to derive the VDW interface equation directly without first approximating the free energy. This allows one to think about the “local density” or “slowly varying profile” approximation in a new way where the theory itself suggests what is needed to correct it. Our work can be viewed as simply implementing these corrections to the classical VDW interface equation and it seems appropriate to refer to it for the purposes of this review as a molecular scale van der Waals (MVDW) theory.

More precisely, we show that there are two key approximations in this interpretation of the VDW theory: i) the introduction of an effective single particle potential or “molecular field” to describe the locally averaged effects of the attractive interactions in the nonuniform fluid and ii) the use of a hydrostatic approximation to determine the density response to the effective field that takes into account only the local value of the field. The latter approximation is accurate only for very slowly varying fields, where it reduces to the local density approximation, and this is what limits the utility of the VDW theory in most of the applications discussed above. But from this perspective, it is easy to see how the theory can be corrected by considering nonlocal effects from the effective field. Our new MVDW theory does this using what is probably the simplest possible method, linear response theory (10, 62), which we argue (5, 6) is especially accurate when used to correct the hydrostatic approximation.

In the following, we will present this interpretation of the VDW theory, first for the nonuniform LJ fluid. Then we will describe the corrections and generalizations that lead to the MVDW theory, and review results for several applications to nonuniform LJ and hard sphere fluids, and to water. Not all the simplifications that arise from this viewpoint were realized originally, so the present review can serve as a simpler and more concise guide to the theory. In particular, we have tried to clarify certain aspects of the LCW theory (4) for hydrophobic interactions. A different view of the LCW theory that may go beyond the MVDW picture considered originally and discussed here is given in (63).
We first consider the LJ fluid, where the physics is particularly clear. One reason why the LJ fluid is aptly called a simple liquid is that its local structure at typical liquid densities can be well approximated by an even simpler model, the hard sphere fluid. As pointed out by Widom (64, 65), in most typical configurations of the uniform LJ fluid the vector sum of the longer-ranged attractive forces on a given molecule from pairs of oppositely situated neighbors tends to cancel. This leaves only the excluded volume correlations induced by the harshly repulsive molecular cores, well described by hard spheres of an appropriate size (10, 66). The hard sphere model is thus of fundamental importance in the theory of liquids as the simplest model that can give a realistic description of structural correlations arising from excluded volume effects.

As emphasized by VDW himself (9), this cancellation argument must fail for nonuniform fluids, and there will exist “unbalanced” attractive forces, whose effect on the structure can also be quite significant, leading in some cases to interface formation. In general there is a competition between these two sources of structural correlations that a proper theory of nonuniform fluids should account for (1-7). This physical picture will help us understand what is missing in the classical VDW theory and how it might be improved. The insights gained from a careful study of this simple system may suggest physically well-grounded approximations that could be applied more generally. We believe this is the case for the MVDW theory.

We start by defining the interactions in the nonuniform LJ fluid. Particles interact with a known external field \( \phi(r) \), which we will initially assume is nonzero only in a local region, e.g., the potential arising when a LJ or hard core solute is fixed at the origin. We describe the system using a grand ensemble with fixed chemical potential \( \mu_B \), which determines \( \rho_B \), the uniform bulk fluid density far from the perturbation where \( \phi(r) = 0 \). The LJ pair potential \( w(r) \equiv u_0(r) + u_1(r) \) is separated into rapidly and slowly varying parts associated with the intermolecular forces (67-69) so that all the harshly repulsive forces arise from \( u_0 \) and all the attractive forces from \( u_1 \). Thus \( u_0(r) = w(r) + \epsilon \) for \( r \leq r_0 \), where \( r_0 \equiv 2^{1/6}\sigma \) is the distance to the potential minimum where the LJ force changes sign, and is zero otherwise, while \( u_1(r) = -\epsilon \) for \( r \leq r_0 \) and equals \( w(r) \) otherwise. Here \( \sigma \) and \( \epsilon \) are the usual length and energy parameters in the LJ potential. With this separation \( u_1(r) \) is relatively slowly varying and smooth, with a continuous derivative even at \( r_0 \). We will make use of these features in the theory described below.

## 2 Molecular field approximation

### 2.1 Structure of the nonuniform reference fluid

We now begin our derivation of the VDW interface equation. The fundamental approximation in this interpretation of the VDW theory is the introduction of an effective single particle potential or “molecular field” that describes the locally averaged effects of the attractive interactions (8) in the nonuniform LJ fluid. Since the attractive interactions are relatively slowly varying, such an averaged treatment seems physically reasonable. The structure of the resulting “reference” or “mimic” system, where the attractive intermolecular interactions have been replaced by the effective single particle potential, is supposed to accurately approximate that of the original system. Thus the structure of the nonuniform LJ fluid is assumed to be given by that of the simpler nonuniform reference fluid (1-3), with only repulsive intermolecular pair interactions \( u_0(r) \) (equal
to the LJ repulsions) and a chemical potential $\mu_0^B$ corresponding to the same bulk density $\rho^B$ but in a different renormalized or effective reference field (ERF) $\phi_R(r)$. Because $u_0$ is harshly repulsive, many properties of the reference fluid can be accurately approximated by those of a fluid of hard spheres with a diameter chosen by the usual “blip function” expansion (10, 66), as described in detail in (5, 6). While this approximation is not essential, it is numerically very convenient, and for most purposes we will treat the reference system as a hard sphere fluid in the presence of the ERF.

Before we discuss the specific molecular field equation usually used to determine the ERF, let us consider some general physical consequences arising from the use of any ERF to describe the effects of attractive interactions that will be important in what follows. Since the goal is to produce structure in the reference fluid approximating that of the full fluid to the extent possible, it is natural to choose $\phi_R(r)$ in principle so that the local (singlet) densities (70) at every point $r$ in the two fluids are equal:

$$\rho_0(r; [\phi_R], \mu_0^B) = \rho(r; [\phi], \mu^B).$$

(1)

Of course this density is not known in advance, so in practice we will make approximate choices for $\phi_R$ motivated by molecular field ideas. Here the subscript 0 denotes the reference fluid, the absence of a subscript the LJ fluid, and the notation $[\phi]$ indicates that the correlation functions are functionals of the external field. Unless we want to emphasize this point, we will suppress this functional dependence, e.g., writing Equation 1 as $\rho_0(r) = \rho(r)$.

But the reference fluid can also describe certain features of higher order correlation functions. We expect that when $\phi_R$ is chosen so that Equation 1 holds, this will produce similar local environments for the (identical) repulsive cores in the two fluids, which at high density will mainly determine higher order density correlations through excluded volume effects. Thus, when Equation 1 is satisfied, it seems plausible that pair correlations are also approximately equal (1, 2), so that

$$\rho^{(2)}(r_1, r_2) \simeq \rho_0^{(2)}(r_1, r_2).$$

(2)

In the dense uniform LJ fluid it is well known that correlations are dominated by the repulsive forces, and this near equality lies behind the success of perturbation theories of liquids (10, 68, 69). (However, this structural assumption is rigorously true only in the artificial “Kac limit” where the attractive interactions are infinitely weak and long-ranged (64, 71).) The most general use of this idea asserts that all structural effects of attractive forces in the nonuniform LJ fluid can be approximately described in terms of the structure of the reference fluid in the appropriately chosen ERF.

There are many advantages in using the simpler reference system to define structure. In particular, the uniform reference fluid is well defined for all densities from dilute vapor to dense fluid and there are no conceptual problems that arise from densities in the two phase region, as would be the case in the traditional theories requiring such densities in the original LJ fluid. See, e.g., Section 4 below. Moreover, as we will discuss in detail later, there exist simple and accurate theories for the reference fluid structure based on linear response theory.

However there are inherent limitations in such an approximation, most notably in describing long wavelength correlations such as those found near the critical point. Some other shortcomings of the molecular field approximation when applied to realistic fluid models are easily understood.
For example, the effects of long wavelength capillary wave fluctuations (72) of the free liquid-vapor interface clearly cannot be described using such reference system correlation functions however $\phi_R$ is chosen.

2.2 Simple molecular field equation

We now turn to the choice of $\phi_R$. In our interpretation, the classical VDW theory uses the simple molecular field (MF) approximation for the ERF, given in Equation 3 below. This is just a transcription of the usual molecular field equation for the Ising model to a continuum fluid with attractive interactions $u_1(r)$ and can be arrived at in a number of different ways (8, 44). Particularly relevant for our purposes here is the derivation discussed in detail in (1-3, 7) that starts from the balance of forces in the reference and LJ fluids as described by the exact Yvon-Born-Green (YBG) hierarchy (10) and uses Equations 1 and 2 to arrive at the MF equation by an approximate integration. This can be looked on as a modern version of the closely related calculation VDW originally carried out (9). The final result is well known:

$$\phi_R(r_1) = \phi(r_1) + \int d\mathbf{r}_2 \rho_0(\mathbf{r}_2; [\phi_R], \mu_0^B) u_1(r_{12}) + 2a\rho^B, \quad (3)$$

where

$$a \equiv -\frac{1}{2} \int d\mathbf{r}_2 u_1(r_{12}) \quad (4)$$

corresponds to the attractive interaction parameter $a$ in the uniform fluid VDW equation, as discussed below. The last term in Equation 3 represents a constant of integration in the derivation in (1-3, 7) and is chosen so that $\phi_R$ vanishes far from a localized perturbation where the density becomes equal to $\rho^B$.

Because of the integration over the slowly varying attractive component of the intermolecular potential $u_1(r_{12})$, the second term on the right in Equation 3 is smooth and relatively slowly varying even when $\rho_0$ itself has discontinuities and oscillations, as could arise from a $\phi$ with a hard core. Thus the ERF $\phi_R(r_1)$ is quite generally comprised of the original field $\phi(r_1)$ and an additional slowly varying term

$$\phi_s(r_1) \equiv \int d\mathbf{r}_2 \rho_0(\mathbf{r}_2) u_1(r_{12}) + 2a\rho^B \quad (5)$$

that takes account of spatial variations of the attractive interactions, i.e., the unbalanced attractive forces in the nonuniform LJ fluid.

Some thermodynamic implications of the MF approximation can be seen when we consider the MF equation 3 in the limit of a constant field. In the grand ensemble, this is equivalent to a shift of the chemical potential, producing a shift in the uniform density, as discussed in more detail in the next Section. Thus we can use Equation 3 to relate chemical potentials in the reference and LJ systems. Let $\mu(\rho)$ and $\mu_0(\rho)$ denote the chemical potential as a function of density in the uniform LJ system and reference system respectively. (These also depend on the temperature, but we are usually interested in density variations along particular isotherms, so we will not indicate the temperature dependence explicitly.) Then $\phi_0 = \mu^B - \mu(\rho)$ is the exact value of that constant field such that the density in the LJ fluid changes from $\rho^B$ to $\rho$, and $\phi_R^0 = \mu_0^B - \mu_0(\rho)$ is the analogous field in the reference fluid producing the same density change. Using Equation 3, the MF equation
gives the MF approximation relating $\phi^0$ and $\phi_R^0$, or equivalently, the MF approximation relating $\mu(\rho)$ and $\mu_0(\rho)$. This can be written in the familiar VDW form (64):

$$
\mu(\rho) = \mu_0(\rho) - 2a\rho.
$$

In the limit of a uniform system, Equation 3 describes all effects of attractive interactions in terms of the constant parameter $a$. Indeed the theory then reduces to the generalized uniform fluid VDW theory of Longuet-Higgins and Widom (64, 65), where one combines an accurate description of the uniform (hard sphere) reference system with the simple treatment of the attractive interactions in terms of a constant background potential $a$. In the MF approximation $\mu(\rho)$ is defined by the right side of Equation 3 and has meaning even for values of $\rho$ in the two phase region.

In general, to determine $\phi_R$ a self-consistent solution of Equation 3 is required, since $\phi_R$ appears explicitly on the left side and implicitly on the right side through $\rho_0(\mathbf{r}; [\phi_R], \mu_R^B)$. In principle, since a unique density $\rho_0(\mathbf{r}; [\phi_R], \mu_R^B)$ is associated with a given external field $\phi_R$ through the partition function, Equation 3 is self-contained and hence self-consistent values for both $\phi_R$ and $\rho_0$ can be found, by iteration, for example. Such solutions were found in (1, 2), using computer simulations to accurately determine the associated density $\rho_0(\mathbf{r}; [\phi_R], \mu_R^B)$ for a variety of external fields.

In practice, one must make additional approximations beyond the molecular field assumption to obtain $\rho_0(\mathbf{r}; [\phi_R], \mu_R^B)$ in an accurate and computationally practical way. It is here that the main limitation of the classical VDW theory arises. In our derivation the classical VDW theory results when a second approximation, appropriate for a slowly varying density field (8), is used to determine the density $\rho_0(\mathbf{r}; [\phi_R], \mu_R^B)$ induced by $\phi_R$. This approximation takes account only of the local value of the field through a shifted chemical potential and can be used for a general $\phi_R(\mathbf{r})$.

When $\phi_R(\mathbf{r})$ is very slowly varying it is exact. But in more general cases it gives inaccurate results, spoiling the predictions of the VDW theory. To see how this comes about, we first define this local hydrostatic density response for a general field, and then show how its use transforms Equation 3 into the VDW interface equation as it is usually presented.

### 3 Hydrostatic density

#### 3.1 Local response to general field

Consider first a given general external field $\phi_R$. Since the chemical potential acts like a constant field in the grand partition function, the associated density $\rho_0(\mathbf{r}; [\phi_R], \mu_R^B) \equiv \rho_0(\mathbf{r})$ is a functional of $\phi_R$ and a function of $\mu_0^B$ and depends only on the difference between these quantities. Thus for any fixed position $\mathbf{r}_1$ we can define a shifted chemical potential $\mu_R^{*1} \equiv \mu_0^B - \phi_R(\mathbf{r}_1)$ and shifted field $\phi_R^{*1}(\mathbf{r}) \equiv \phi_R(\mathbf{r}) - \phi_R(\mathbf{r}_1)$, whose parametric dependence on $\mathbf{r}_1$ is denoted by a superscript, and we have for all $\mathbf{r}$ the exact relation $\rho_0(\mathbf{r}; [\phi_R], \mu_R^{*1}) = \rho_0(\mathbf{r}; [\phi_R^{*1}], \mu_R^{*1})$.

By construction the shifted field $\phi_R^{*1}(\mathbf{r})$ vanishes at $\mathbf{r} = \mathbf{r}_1$. If $\phi_R$ is very slowly varying, then it remains very small for $\mathbf{r}$ near $\mathbf{r}_1$. In that case, to determine the density at $\mathbf{r}_1$ we can approximate $\rho_0(\mathbf{r}_1; [\phi_R^{*1}], \mu_R^{*1})$ by $\rho_0(\mathbf{r}_1; [0], \mu_R^{*1}) = \rho_R^{*1}$, the density of the uniform fluid (in zero field) at the shifted chemical potential $\mu_R^{*1}$. This defines $\rho_R^{*1}$, the hydrostatic density response to the field $\phi_R$. Note that $\rho_R^{*1}$ depends only on the local value of the field $\phi_R$ at $\mathbf{r}_1$ through its dependence on $\mu_R^{*1}$. When the field varies slowly enough the hydrostatic approximation, where $\rho_0(\mathbf{r}_1)$ at each $\mathbf{r}_1$ is replaced by the
corresponding uniform fluid density \( \rho_0^{r_1} \), is very accurate \(^{(5)}\).

One can equivalently define the hydrostatic density \( \rho_0^{r_1} \) using \( \mu_0(\rho) \), the chemical potential of the uniform reference fluid as a function of density; it is defined by

\[
\mu_0(\rho_0^{r_1}) = \mu_0^B - \phi_R(r_1). \tag{7}
\]

This equation plays a central role in all that follows. The hydrostatic density is easy to determine for a general \( \phi_R \). In particular if \( \phi_R \) has a hard core with \( \phi_R = \infty \) in a certain region of space, then the hydrostatic density \( \rho_0^{r_1} \) correctly vanishes in that same region, corresponding to the vanishing density of the uniform fluid at the chemical potential \( \mu_0^{r_1} = -\infty \). However, because of the strictly local response to the field, any nonlocal excluded volume correlations induced by the hard core potential outside the hard core region are not properly described by the hydrostatic approximation. Still, it remains well defined even in this limit whereas approximations based on using properties of a uniform fluid evaluated at the local density (which can easily exceed close packing) break down completely \(^{(43, 44)}\).

### 3.2 Local response to ERF

To obtain the VDW and MVDW theories we now determine the local hydrostatic response to the ERF in Equation \(3\). This arises after Equation \(3\) for \( \phi_R(r_1) \) is substituted into Equation \(\ref{eq:hydrostatic_density}\):

\[
\mu_0(\rho_0^{r_1}) = \mu_0^B - \phi(r_1) - \int dr_2 \rho_0(r_2) u_1(r_{12}) - 2a\rho_0^B. \tag{8}
\]

As written, Equation \(8\) just defines the hydrostatic density \( \rho_0^{r_1} \) in terms of the local value of the ERF at \( r_1 \), which itself involves an integral over the full density \( \rho_0(r_2; [\phi_R], \mu_0^B) \) at all other points \( r_2 \). To explicitly determine \( \rho_0^{r_1} \) we need to specify \( \Delta \rho_0(r_2) \equiv \rho_0(r_2) - \rho_0^{r_2} \), as is clear when Equation \(8\) is exactly rewritten in the form:

\[
\mu_0(\rho_0^{r_1}) - 2a\rho_0^{r_1} = \mu_0^B - 2a\rho_0^B - \phi(r_1) - \int dr_2 [\rho_0^{r_2} - \rho_0^{r_1}] u_1(r_{12}) - \int dr_2 \Delta \rho_0(r_2) u_1(r_{12}). \tag{9}
\]

As we now show, different approximations for \( \Delta \rho_0(r_2) \) immediately give the classical VDW and the MVDW interface equations.

### 4 Hydrostatic approximation and the classical VDW theory

The classical VDW interface equation arises when one assumes that the full density response to the ERF \( \rho_0(r_2) \) for all \( r_2 \) is accurately approximated by \( \rho_0^{r_2} \), the local hydrostatic response. This hydrostatic approximation is consistent when the ERF and the induced density are slowly enough varying. Thus in this derivation of the classical VDW theory the last term in Equation \(8\) is ignored and Equation \(8\) or \(9\) is approximated by an integral equation involving only the hydrostatic density:

\[
\mu(\rho_0^{r_1}) = \mu^B - \phi(r_1) - \int dr_2 [\rho_0^{r_2} - \rho_0^{r_1}] u_1(r_{12}), \tag{10}
\]
along with the assumption that $\rho_0(\mathbf{r}) = \rho_0^s$ for all $\mathbf{r}$. Here we have also used Equation 6 to replace $\mu_0(\rho)$ by $\mu(\rho)$, with the understanding that the latter is really defined by the right side of Equation 6. Consistent with the assumption of a slowly varying profile and the fact that $u_1$ is reasonably short-ranged, $\rho_0^{r_2}$ in the last term on the right is often expanded to second order in a Taylor series about $\rho_0^{r_1}$, yielding in the simple case of a liquid-vapor interface with planar symmetry and $\phi = 0$ a differential equation for the interface profile $\rho_0^z$:

$$\mu(\rho_0^z) = \mu^B + m d^2 \rho_0^z / dz^2$$ (11)

where

$$m \equiv -\frac{1}{6} \int d\mathbf{r} r^2 u_1(\mathbf{r}).$$ (12)

While there is a strictly local response to the value of the ERF at $\mathbf{r}_1$ in Equation 10 yielding $\rho_0^{r_1}$, the ERF itself at $\mathbf{r}_1$ depends nonlocally on $\rho_0^{r_2}$ through the integration over the attractive interactions. In the classical VDW theory this provides the only source of nonlocality.

Equations 10 and 11 are completely equivalent to the VDW theory for the density profile as it is usually presented (8). In our derivation the theory describes hydrostatic densities in the reference system, and $\mu(\rho)$ is also defined in terms of reference system quantities given on the right side of Equation 6. This provides a simple and consistent interpretation that avoids all problems associated with densities in the two phase region of the LJ fluid.

However, in view of Equation 6, one can replace $\rho_0^{r_1}$ by $\rho^{r_1}$ in Equations 10 and 11 and formally eliminate all explicit mention of the reference system. This is the way the standard theory is usually presented. This formal rewriting of Equations 10 and 11 seems to require only properties of the original LJ system and could be useful when applying the theory to other fluids where the appropriate reference system is not so apparent. Work for water (4) along these lines will be reviewed later. However, this obscures some of the physical underpinnings of the theory in terms of the basic MF approximation and it is not clear from the form of these equations how they should be corrected in cases where the density profile is more rapidly varying.

Of course, one might hope that there could exist a more general formulation of the theory where one does not need the MF approximation at all. Despite some formal results from density functional theory, which we will discuss later, we believe in practice this is not likely to be the case. Long-standing conceptual problems (8) arise in interpreting what is meant by $\mu(\rho^{r_1})$ and $\rho^{r_1}$ itself in Equations 10 and 11 for density values in the two phase region of the uniform LJ fluid unless they are defined by using a MF approximation either explicitly as in the right hand side of Equation 6, or implicitly by assuming some kind of analytic continuation of values from the stable phases. See Section 3 below for further discussion.

Thus to improve the classical VDW theory, we return to Equations 8 and 1. Here the essential physics involving the interplay between the MF approximation for the ERF and the hydrostatic density response is clear. If we can understand in detail how to improve the VDW theory for the LJ system, we may gain insights that could apply more generally.

5 Correcting the hydrostatic approximation
As emphasized in Section 3.2, the VDW theory determines the local density response $\rho_0^{r_1}$ to the ERF $\phi_R(r_1)$ in Equations 3 and 9. Problems arise in the classical VDW theory from the second approximation of ignoring the density difference $\Delta \rho_0(r_2) \equiv \rho_0(r_2) - \rho_0^{r_2}$, produced by definition from nonzero values of the shifted field $\phi_R^{r_2}(r)$ away from $r_2$.

But one can calculate $\Delta \rho_0$ in a very simple way by using linear response theory, which exactly relates small changes in the density and field. This approach is clearly correct when $\Delta \rho_0$ and $\phi_R^{r_2}$ are uniformly small, and in this sense is analogous to a gradient correction to the local density approximation in DFT. However, in contrast to the gradient correction, we will see that there are good physical reasons to believe that our linear response treatment could remain accurate even for large perturbations (3, 5, 6). This will allow us to develop a new and generally accurate theory for the nonuniform reference fluid in an arbitrary external field and also will help us correct one of the major shortcomings of the classical VDW theory for fluids with attractive interactions.

We start from the general linear response equation (10) for a nonuniform system in a general field $\phi_R$, with chemical potential $\mu_0^B$, inverse temperature $\beta = (k_B T)^{-1}$ and associated density $\rho_0(r)$:

$$-\beta \delta \phi_R(r_1) = \int d r_2 \chi_0^{-1}(r_1, r_2; [\rho_0]) \delta \rho_0(r_2),$$

which relates small perturbations in the density and potential through the (inverse) linear response function

$$\chi_0^{-1}(r_1, r_2; [\rho_0]) \equiv \delta(r_1 - r_2)/\rho_0(r_1) - c_0(r_1, r_2; [\rho_0]).$$

Here $c_0$ is the direct correlation function of the system with density $\rho_0(r)$. In most cases we will consider perturbations about a uniform system, so $c_0$ will take the simple form $c_0(r_{12}, \rho)$. Since we want to focus on effects of the perturbing field, we have used the inverse form of linear response theory, where the field appears explicitly only on the left hand side of Equation 13, evaluated at $r_1$.

Could Equation 13 also be used to determine the finite density response $\Delta \rho_0$ to a large field perturbation $\Delta \phi_R$? Such a linear relation between a (possibly infinite) external field perturbation on the left hand side and the finite induced density change on the right must certainly fail for values of $r_1$ where the field is very large. Conversely, Equation 13 should be most accurate for those values of $r_1$ where the field is small — in particular where the field vanishes — and then through the integration over all $r_2$ it relates density changes in the region where the field vanishes to density changes in the regions where the field is nonzero (3, 5).

This optimal condition for the validity of linear response theory holds true automatically when we use Equation 13 to determine the change at each $r_1$ from the uniform fluid with density $\rho_0^{r_1}$ induced by the shifted field. Thus we set $\chi_0^{-1} = \chi_0^{-1}(r_{12}; \rho_0^{r_1})$ in Equation 13 and take $\delta \phi_R = \phi_R^{r_1}$ and $\delta \rho_0(r_2) = \rho_0(r_2) - \rho_0^{r_1}$. Since $\phi_R^{r_1}(r)$ is zero at $r_1$ by construction, the left side of Equation 13 vanishes, and we have

$$0 = \int d r_2 \chi_0^{-1}(r_{12}; \rho_0^{r_1})[\rho_0(r_2) - \rho_0^{r_1}],$$

which can be rewritten using Equation 14 as

$$\rho_0(r_1) = \rho_0^{r_1} + \rho_0^{r_1} \int d r_2 c_0(r_{12}; \rho_0^{r_1})[\rho_0(r_2) - \rho_0^{r_1}].$$

5.1 Optimized linear response and the HLR equation

This optimal condition for the validity of linear response theory holds true automatically when we use Equation 13 to determine the change at each $r_1$ from the uniform fluid with density $\rho_0^{r_1}$ induced by the shifted field. Thus we set $\chi_0^{-1} = \chi_0^{-1}(r_{12}; \rho_0^{r_1})$ in Equation 13 and take $\delta \phi_R = \phi_R^{r_1}$ and $\delta \rho_0(r_2) = \rho_0(r_2) - \rho_0^{r_1}$. Since $\phi_R^{r_1}(r)$ is zero at $r_1$ by construction, the left side of Equation 13 vanishes, and we have

$$0 = \int d r_2 \chi_0^{-1}(r_{12}; \rho_0^{r_1})[\rho_0(r_2) - \rho_0^{r_1}],$$

which can be rewritten using Equation 14 as

$$\rho_0(r_1) = \rho_0^{r_1} + \rho_0^{r_1} \int d r_2 c_0(r_{12}; \rho_0^{r_1})[\rho_0(r_2) - \rho_0^{r_1}].$$

This optimal condition for the validity of linear response theory holds true automatically when we use Equation 13 to determine the change at each $r_1$ from the uniform fluid with density $\rho_0^{r_1}$ induced by the shifted field. Thus we set $\chi_0^{-1} = \chi_0^{-1}(r_{12}; \rho_0^{r_1})$ in Equation 13 and take $\delta \phi_R = \phi_R^{r_1}$ and $\delta \rho_0(r_2) = \rho_0(r_2) - \rho_0^{r_1}$. Since $\phi_R^{r_1}(r)$ is zero at $r_1$ by construction, the left side of Equation 13 vanishes, and we have

$$0 = \int d r_2 \chi_0^{-1}(r_{12}; \rho_0^{r_1})[\rho_0(r_2) - \rho_0^{r_1}],$$

which can be rewritten using Equation 14 as

$$\rho_0(r_1) = \rho_0^{r_1} + \rho_0^{r_1} \int d r_2 c_0(r_{12}; \rho_0^{r_1})[\rho_0(r_2) - \rho_0^{r_1}].$$

This optimal condition for the validity of linear response theory holds true automatically when we use Equation 13 to determine the change at each $r_1$ from the uniform fluid with density $\rho_0^{r_1}$ induced by the shifted field. Thus we set $\chi_0^{-1} = \chi_0^{-1}(r_{12}; \rho_0^{r_1})$ in Equation 13 and take $\delta \phi_R = \phi_R^{r_1}$ and $\delta \rho_0(r_2) = \rho_0(r_2) - \rho_0^{r_1}$. Since $\phi_R^{r_1}(r)$ is zero at $r_1$ by construction, the left side of Equation 13 vanishes, and we have

$$0 = \int d r_2 \chi_0^{-1}(r_{12}; \rho_0^{r_1})[\rho_0(r_2) - \rho_0^{r_1}],$$

which can be rewritten using Equation 14 as

$$\rho_0(r_1) = \rho_0^{r_1} + \rho_0^{r_1} \int d r_2 c_0(r_{12}; \rho_0^{r_1})[\rho_0(r_2) - \rho_0^{r_1}].$$
This is our final result, which we refer to as the *hydrostatic linear response* (HLR) equation. A more formal derivation involving a coupling parameter integration, and yielding another related equation for \( \rho_0(r_1) \) is given in (5). Note that the field appears only implicitly through its local effect on \( \rho_1^{r_1} \). The HLR equation is useful by itself as a way to determine the density change induced by a known external field \( \phi_R(r) \). It builds on properties of the uniform reference system, requiring in particular \( c_0(r_{12}; \rho) \) and \( \mu_0(\rho) \). Quantitatively accurate and computationally convenient approximations for these functions are known from the GMSA theory of Waisman (73, 74); in many cases the simpler Percus-Yevick (PY) approximation (10, 62) discussed below gives sufficient accuracy. Given these functions and a known external field \( \phi_R(r) \), Equation 16 is a linear integral equation that can be solved self-consistently to determine the induced density \( \rho_0(r_1) \) for all \( r_1 \).

### 5.2 Properties of the HLR equation for fixed \( \phi_R \)

Equation 16 relates the density \( \rho_0(r_1) \) at each \( r_1 \) to an integral involving the density \( \rho_0(r_2) \) at all other points and a locally optimal uniform fluid kernel \( c_0(r_{12}; \rho_0^{r_1}) \) that depends implicitly on \( r_1 \) through \( \rho_0^{r_1} \). This \( r_1 \) dependence is the most important new feature of the HLR equation and it represents the main reason for improved results over conventional methods, which generally use only \( c_0(r_{12}; \rho_0^B) \) along with various nonlinear closures that try to directly relate the field and the density in regions where the field is nonzero (10). See, e.g., Equation 17 below.

#### 5.2.1 Hard sphere solute and the PY equation

Consider in particular the important test case where \( \phi_R \) represents the potential of a hard sphere solute fixed at the origin. The local hydrostatic response \( \rho_0^{r_1} \) is simply a step function in this case, equal to zero for \( r_1 \) inside the core region and to \( \rho_0^B \) outside. In reality at high density there are very large nonlocal oscillatory excluded volume correlations in the true \( \rho_0(r_1) \). As we will see, the HLR equation gives results in this special case equivalent to the generally accurate Percus-Yevick (PY) approximation. In addition, we can use results from recent computer simulations (75) and related work on the Gaussian field model (76) to provide us with further insights into why this simple linear response treatment can remain surprisingly accurate even for hard cores.

We can connect the HLR equation with conventional integral equation theory leading to the PY equation by noting that the usual solute-solvent direct correlation function \( C_S(r_1) \) can be defined by

\[
C_S(r_1) = \int dr_2 \chi_0^{-1}(r_{12}; \rho_0^B)[\rho_0(r_2) - \rho_0^B],
\]

where exact values for all functions are used on the right side. Thus \(-C_S(r_1)/\beta\) is that function that must replace the perturbing solute-solvent potential to give exact results when the full induced density change from \( \rho_0^B \) is used in the linear response equation 13. (Using Equation 14, we see that Equation 17 is equivalent to the standard solute-solvent Ornstein-Zernike equation, which is the usual way (10) of defining \( C_S(r_1) \).)
Similarly, in the derivation of the HLR equation we can replace the zero on the left side of Equation 13 by a new function $\tilde{C}_S(r_1)$ that exactly satisfies:

$$
\tilde{C}_S(r_1) = \int dr_2 \chi_0^{-1}(r_{12}; \rho^*_0)[\rho_0(r_2) - \rho^*_0].
$$

(18)

Note that the local value of the perturbing field $\phi^*_R$ associated with $\tilde{C}_S(r_1)$ always vanishes at $r_1$, unlike the case for $C_S(r_1)$ above. The HLR equation 16 follows from Equation 18 by setting $\rho^*_0 \tilde{C}_S(r_1)$ equal to zero everywhere, and to the extent that the HLR equation is accurate, we expect $\rho^*_0 \tilde{C}_S(r_1)$ to be generally very small. In particular the HLR equation automatically satisfies the core condition $\rho_0 = 0$ inside the hard core region where both $\rho^*_0$ and $\rho_0(r_1)$ vanish. Corrections to the HLR equation arise from nonzero values of $\rho^*_0 \tilde{C}_S(r_1)$ outside the core.

If we compare Equations 17, with the core condition imposed, and 18 for $r_1$ outside the core region where $\rho^*_0 = \rho^*_B$, we see they are identical. Thus $\tilde{C}_S(r_1)$ outside the core exactly equals $C_S(r_1)$, the “tail” of the hard sphere solute-solvent direct correlation. This is generally believed to be small and in the very accurate GMSA equation it is approximated by a rapidly decaying Yukawa function (73, 74). In the HLR approximation this tail is set equal to zero. Thus the HLR equation predicts that the hard sphere solute-solvent direct correlation vanishes identically outside the core. As is well known, this is equivalent to the PY closure for hard core systems (10). A GMSA treatment could be used in Equation 18 to correct the HLR results if more accuracy is needed. A self-consistent application of these ideas when the solute is the same size as the solvent, requiring that $C_S = c_0 = 0$ outside the core, yields the standard PY equation for hard spheres as interpreted by Stell (77), with $c_0(r)$ equal to zero outside the core and $\rho_0(r) = \rho^*_0 g_0(r)$ equal to zero inside. Similarly, the HLR equation reduces to the PY wall-particle equation when the radius of the solute tends to infinity and the PY $c_0$ is used.

5.2.2 Computer simulations and Gaussian fluctuations

Further insight into the surprising accuracy of linear response theory for uniform hard sphere fluids comes from recent computer simulations by Crooks and Chandler (75), following related work on water (17). They have shown that even large spontaneous density fluctuations in a uniform hard sphere fluid can be accurately described using the same Gaussian probability distribution that describes small fluctuations. Small density changes induced by a small perturbing field are described by the linear response equation 13. By the fluctuation-dissipation theorem, the same linear response function $\chi_0^{-1}(r_{12}; \rho)$ controls the spontaneous Gaussian density fluctuations in the uniform fluid (10, 76). In particular, they considered the probability distribution for finding $N$ hard spheres in a spherical volume of the fluid and found that even cavity or void formation with $N = 0$ was reasonably well described by the Gaussian theory. Since a cavity influences the rest of the fluid in exactly the same way as a hard sphere solute of the same size, the density response of the hard sphere fluid to such a solute (or imposed region of zero density) should indeed be well described using linear response theory with the uniform fluid response function, as in Equation 13. Thus a key feature exploited in Equation 16 is that density fluctuations in the reference fluid are to a remarkable extent Gaussian. The simulation results pertain to fluctuations in a uniform fluid. Similarly, to describe effects of a general external field, the HLR equation considers perturbations at
each \( r_1 \) to the uniform hydrostatic fluid density \( \rho_{01}^r \) induced by the shifted field \( \phi_R^r \), which vanishes at least locally at \( r_1 \).

5.2.3 Other rapidly varying model potentials

Thus we see that the HLR equation \( [14] \) satisfies the following limits. i) It is exact when \( \phi_R(r) \) is very slowly varying and exactly describes to linear order small corrections to the hydrostatic approximation ii) It is exact for any \( \phi(r) \) at low enough density, where there is a local relation between the potential and induced density. iii) For a field \( \phi(r) \) from a general hard core solute, Equation \( [16] \) reduces to the PY approximation, as discussed above. To give more indications of the accuracy of Equation \( [16] \), we review solutions \( (5) \) for some model potentials designed to show both the strengths and the weaknesses of the present methods and compare with computer simulations.

Figure 1 shows the correlation functions \( g_0(r) \equiv \rho_0(r)/\rho_B^0 \) for a hard sphere system at a moderately high bulk density \( \rho_B^0 = 0.49 \) induced by the deep attractive spherical parabolic model potentials shown in the inset. (Reduced units, with distances measured in units of the hard sphere diameter are used.) The HLR equation reproduces the increased density inside the well, and the nonlocal oscillatory excluded volume correlations, which show a local density minimum at the center of the well due to packing effects.

Since the external field enters Equation \( [16] \) only locally through its effect on \( \rho_{01}^r \), it is also easy to use it for the inverse problem of determining the field associated with a given density profile. As an example, the crosses in the inset gives the potentials predicted by Equation \( [16] \) given the simulation data for \( g(r) \).

While these results are qualitatively very satisfactory for the most part, there are some problems. Note in particular the slightly negative density at the center of the well; a positive density is not guaranteed in this linear theory. (The related HM equation derived in \( (5) \) always yields a nonnegative density and does slightly better here but it performs less well in the hard wall limit.) One would expect methods based on expanding about the uniform hydrostatic fluid to be least accurate for potentials with very steep gradients, and poor results at high density were found for a repulsive planar triangular barrier potential that rose from 0 to 10\( k_B T \) over a distance of one hard sphere diameter. Results actually improve as the barrier height increases and the potential approaches a hard wall potential: the theory does better for hard cores because there is no region of space where there is a large gradient in the external field while at the same time the local density is nonzero. A GMSA like treatment based on Equation \( [18] \) may improve matters here. But large repulsive potentials with very steep gradients are better treated by “blip function” methods \((66)\) or other expansions about a hard core system.

6 Correcting the molecular field approximation

We now return to fluids with attractive interactions as described using a self-consistently determined ERF. Thinking about the hydrostatic limit also can help us correct some quantitatively inadequate features of the simple MF approximation \( [3] \) for the ERF \( (7) \). In the limit of a uniform system this describes all effects of attractive interactions in terms of the constant VDW parameter \( a \). While this very simple approximation captures much essential physics and gives a qualitative description of the
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uniform fluid thermodynamic properties it certainly is not quantitatively accurate. For example, when used to describe a slowly varying liquid-vapor interface, it will predict shifted (molecular field) values for the densities of the coexisting bulk liquid and vapor phases. The main problem with the theory is not so much its description of the local density gradients, but its predictions for the thermodynamic properties of the coexisting bulk phases (60).

To achieve quantitative agreement with bulk thermodynamic properties one can replace the constant $a$ by a function $\alpha$ that depends (hopefully weakly, to the extent the classical VDW theory is reasonably accurate) on temperature and density. This has been suggested many times in the literature, usually in the context of perturbation theories for uniform fluids (10), where approximate expressions giving such functions have been derived (78). We review here a simpler and more empirical approach (7) that can be used to determine $\alpha$ if an accurate analytic equation of state for the bulk fluid is known, and we will use this in a simple modification of the MF expression for the ERF to insure that exact thermodynamic results (consistent with the given uniform fluid equation of state) are found in the hydrostatic limit of a very slowly varying ERF.

To that end, let us consider again the MF equation in the limit of a constant field. Instead of using the MF approximation for $\mu(\rho)$ as in Equation 3, we assume that $\mu(\rho)$ is known from an accurate bulk equation of state. In particular, we determine $\mu(\rho)$ from the 33-parameter equation of state for the LJ fluid (79) given by Johnson, et al. This provides a very good global description of the stable liquid and vapor phases in the LJ fluid and provides a smooth interpolation in between by using analytic fitting functions. Thus it naturally produces a modified “van der Waals loop” in the two phase region and seems quite appropriate for our use here in improving the simplest MF description of the uniform fluid.

Using known properties of the hard sphere fluid, we also have an essentially exact expression for $\mu_0(\rho)$. Using this we can relate $\mu(\rho)$ and $\mu_0(\rho)$ in analogy to Equation 3:

$$\mu(\rho) = \mu_0(\rho) - 2\alpha(\rho) \rho$$

(19)

but with the constant $a$ replaced by a function $\alpha(\rho)$ of density and temperature chosen so that Equation 19 holds. Since even the simplest molecular field approximation is qualitatively accurate we expect that the ratio $\alpha(\rho)/a$ will be of order unity and rather weakly dependent on density and temperature.

We indeed found that the constant $a$ was a good overall compromise (7). However the true $\alpha(\rho)$ fit to the equation of state exhibits variations of up to about fifteen per cent as a function of density and temperature, illustrating the need for an accurate equation of state for quantitative accuracy.

Because of the strictly local response, these results for a constant field can be used to determine exact results in the hydrostatic limit of very slowly varying fields. We want to modify Equation 3 so that in the hydrostatic limit it will reproduce these exact values, while still giving reasonable results for more rapidly varying fields.

There is no unique way to do this, but the following simple prescription seems very natural, and gives our final result, which we will call the modified molecular field (MMF) approximation for the ERF:

$$\phi_R(\mathbf{r}_1) - \phi(\mathbf{r}_1) = \frac{\alpha(\rho_0^{\mathbf{r}_1})}{a} \int d\mathbf{r}_2 \rho_0(\mathbf{r}_2; [\phi_R, \mu_B^0]) u_1(\mathbf{r}_{12}) + 2\alpha(\rho_B^0) \rho_B.$$  

(20)

Thus the molecular field integral in Equation 3 is multiplied by a factor $\alpha(\rho_0^{\mathbf{r}_1})/a$ of order unity.
that depends on \( r_1 \) through the dependence of the hydrostatic density \( \rho_{0r} \) on the local value of the field \( \phi_R(r_1) \), and the constant of integration \( 2ap^2 \) is replaced by the appropriate limiting value of the modified integral. Note that the hydrostatic density \( \rho_{0r} \) remains smooth and relatively slowly varying outside the core even when \( \phi_R(r_1) \) contains a hard core. The nonlocal oscillatory excluded volume correlations that can exist in the full density \( \rho_0(r_1) \) do not appear in \( \rho_{0r} \) because of the strictly local response to \( \phi_R \). Results using Equation 20 will be discussed below.

7 MVDW theory of nonuniform fluids with attractive interactions

7.1 Two step method

The ERF must be calculated self-consistently for fluids with attractive interactions. Our new MVDW theory corrects the classical VDW theory by using Equation 16 to accurately determine \( \rho_0(r_2) \) in Equation 8. Thus the MVDW theory requires the simultaneous solution of two equations: Equation 8 (or the closely related equation that arises if the Equation 20 is used for the ERF), and Equation 16. Equation 8 determines \( \rho_{0r} \), the local hydrostatic response to the ERF as in the VDW theory described above, and Equation 16 determines the full nonlocal response \( \rho_0(r_1) \). (Similarly, the VDW theory can be viewed as replacing Equation 16 by the hydrostatic approximation \( \rho_0(r_1) = \rho_{0r} \).)

One can think about solving these equations by a two step iterative method (3). For a given approximation to the ERF, determine in a first step the associated smooth hydrostatic density \( \rho_{0r} \) from Equation 8. Then in a second step take account of nonlocal and usually oscillatory corrections to this profile, generally induced by rapidly varying features in the external field \( \phi \), using a locally optimal application of linear response theory, Equation 16. This new density \( \rho_0(r_1) \) is then used to compute a new approximation for the ERF, and the two steps are iterated to self consistency. This process is easy to implement numerically and rapid convergence has been found in most cases we have examined (3, 6, 7).

We noted in Section 2.2 that the simple MF ERF can quite generally be written as \( \phi_R = \phi + \phi_s \), where \( \phi_s \) in Equation 5 takes account of the unbalanced attractive interactions and is smooth and relatively slowly varying. We are often interested in cases where \( \phi \) is a hard core potential. (If \( \phi \) also has a slowly varying part, say from weak attractive interactions between a solute and solvent, the latter should be added (6) to \( \phi_s \) in the discussion that follows.) In this case one can implement the two step process in a slightly different way, which is physically suggestive and was in fact how we first thought about the problem (3, 4, 6).

Let us consider in the first step the hydrostatic response \( \rho_{sr} \) to the slowly varying part \( \phi_s \) of the ERF alone. Since there is a strictly local response to the ERF, \( \rho_{sr} \) differs from \( \rho_{0r} \) for the full ERF only inside the core region, where \( \rho_{0r} \) vanishes while \( \rho_{sr} \) remains continuous and smooth. This smoothness allows us to use the gradient approximation in the next to last term of Equation 9 in determining \( \rho_{sr} \) if we wish. Since \( \phi_s \) describes the unbalanced attractive interactions associated with interface formation, we can interpret the smooth \( \rho_{sr} \) as an interfacial component of the full density response \( \rho_0(r_1) \).

Then in the second step we take into account the response to the remaining hard core part of \( \phi_R \) and all nonlocal effects. This will cause \( \rho_0(r_1) \) to vanish inside the core and, depending on the extent
to which the density $\rho^{r_1}_s$ near the core has been reduced, can induce nonlocal oscillatory excluded volume corrections to $\rho^{r_1}_s$ outside the core, which we again calculate using linear response theory. Thus speaking pictorially, the second step takes into account the nonlocal Gaussian fluctuations induced by the hard core potential about the smoothly varying interfacial component of the density profile. The strongly non-Gaussian component associated with interface formation and arising from the unbalanced attractive interactions is taken into account in the first step. The final profile results from the self-consistent interplay between the components described in each step.

Since we are using linear response theory in all cases to correct a hydrostatic response, these alternate ways of implementing the two step procedure are completely equivalent. In particular in the first interpretation described above we calculate the hydrostatic response $\rho^{r_1}_0$ to the full ERF and correct it for all $r_1$ using Equation 15. The hard core condition $\rho_0(r) = 0$ for all $r$ inside the core region follows automatically from perturbing about the local hydrostatic density. In the second interpretation we calculate in the first step the hydrostatic response $\rho^{r_1}_s$ to the interfacial component $\phi_s$ alone and correct it by using linear response only for $r_1$ outside the core, while imposing $\rho_0(r_2) = 0$ in the core region:

$$0 = \int dr_2 \chi^{-1}_0(r_{12}; \rho^{r_1}_s)[\rho_0(r_2) - \rho^{r_1}_s].$$

(21)

Since $\rho^{r_1}_s$ equals $\rho^{r_1}_0$ outside the core, the solution $\rho_0(r)$ to Equation 21 with the core condition imposed is identical to that given by Equation 15.

### 7.2 Hard sphere solute in LJ fluid near liquid-vapor coexistence

These ideas have been applied to the nonuniform LJ fluid in a variety of different situations, including fluids near a hard wall, fluids confined in slits and tubes, and the liquid-vapor interface, with generally very good results (1-3, 5-7, 80-82). We will review here results (7) for a system studied by Huang and Chandler (HC) that combines many of these limits (83), and is physically very relevant for our later discussion about hydrophobic interactions.

HC carried out extensive computer simulations to determine properties of the LJ liquid at a state very near the triple point with $\rho^B = 0.70$ and $T = 0.85$ as the radius of a hard sphere solute is varied, and compared the results to the LCW theory reviewed below. (We use the standard LJ reduced units.) By definition the solute centered at the origin interacts with the LJ particles through the hard core potential:

$$\phi(r; S) = \begin{cases} 
\infty, & r \leq S, \\
0, & r > S.
\end{cases}$$

(22)

The MMF theory discussed above allows us to reduce this problem to that of the reference fluid in the presence of the effective field $\phi_R(r; S)$ satisfying Equation 21. We have calculated self-consistently the ERF $\phi_R(r; S)$ and the associated density response $\rho_0(r; S)$ of the reference fluid, solving Equations 21 and 16 by iteration. In Figure 2 we compare these results for the density profiles in the presence of the hard sphere solutes with $S$ equal to 1.0, 2.0, 3.0 and 4.0 in reduced units with the simulation results of the same LJ system by HC. There is very good agreement between theory and simulation.
Figure 3 shows the corresponding ERF’s obtained in these calculations. For small solutes with $S$ less than about 0.7, attractive interactions do not give rise to any substantial modification of the bare external field, as can be seen from the plot of $\phi_R(r; S)$ for $S = 0.5$. (Clearly, for $S = 0$ there are no solute induced interactions of any kind and the profile reduces to the constant $\rho^B$). However, the effects due to unbalanced attractions become important even for $S = 1.0$, which is about the same size as the LJ core, and all larger sizes give rise to a very strong and relatively soft repulsion in $\phi_R(r; S)$. The corresponding density profiles show pronounced depletion near the surface of the solute, characteristic of surface induced drying.

### 7.3 Solvation free energy

Another quantity of great interest is the free energy of the nonuniform system. This is the main focus of attention in density functional theory, briefly discussed in Section 9 below. In contrast, the MVDW approach focuses first on the liquid structure. We believe this permits physical insight to play a more direct role. However, since we can determine the density response to an arbitrary external field, the free energy can be easily calculated from a coupling parameter type integration that connects some initial state (e.g., the bulk fluid) whose free energy is known to the final state as the field is varied. In the present case there is a very simple route to the free energy of the nonuniform LJ system that uses structural features that we know from simulations are accurately determined.

In particular let us consider the change in free energy of the LJ fluid as the range of the external field representing the hard core solute is varied from zero to its full extent $S$. This construction is the basis of scaled particle theory (84), and it is well known that the free energy change takes a particularly simple form:

$$\beta \Delta \Omega_S = 4\pi S^3 \int_0^1 d\lambda \lambda^2 \rho_\lambda(\lambda S^+),$$

which requires only the contact value $\rho_\lambda(\lambda S^+)$ of the density profile. This is very accurately given by the theory described above. To use this “virial route”, we can replace the $\lambda$-integration by a sum and calculate the density profile for several values of $\lambda$ at the fixed bulk chemical potential $\mu^B$.

Using Equation 23 we obtain the dependence of solvation free energy on the size of the hard sphere solute. The free energy per unit surface area of the solute $\Delta \Omega_S/4\pi S^2$ we obtain is shown in Figure 4. For small solutes unbalanced attractive interactions do not play an important role and the solvation free energy agrees well with a pure hard sphere model, which completely neglects attractions by using the bare hard core solute potential, as shown by the dotted line. At the solute size of about 0.7 the behavior changes drastically and the reduced free energy rapidly crosses over to the practically constant plateau in agreement with the simulation results. The small slope of the curves in Figure 4 for large $S$ can be understood by separating the free energy into volume ($V_S = 4\pi S^3/3$) and surface ($A_S = 4\pi S^2$) contributions as discussed by HC (83):

$$\Delta \Omega_S \approx V_S p^B + A_S \gamma_S.$$  

The first term in this expression corresponds to the work required to remove liquid particles from
the volume occupied the solute, where $p^B$ is the bulk liquid pressure, and is very small for the values of $S$ considered here. The second term determines the cost of forming the liquid-solute interface and is proportional to the interface tension $\gamma_S$, which is essentially independent of the solute size for large solutes.

8 Hydrophobic interactions in water

We now discuss an important extension of these ideas to hydrophobic interactions in water (4), as described by Lum, Chandler and Weeks (LCW). We first consider water at ambient conditions as the radius of a hard sphere solute at the origin is varied. Because this state is very close to the liquid-vapor phase boundary, phenomena involving interfaces can be very important. This system serves as a simple model of a hydrophobic object in water — the solute does not participate in hydrogen bonding and it creates an excluded volume region where the density of water molecules vanishes. Weak van der Waals attractions between the solute and solvent do not change the qualitative nature of the phenomena we will discuss and can easily be taken into account (6, 85).

But how can one sensibly apply the theory to water? The local structure of water is certainly very different from that of the LJ fluid and anything relying on the detailed properties of a hard sphere reference system cannot be trusted. However the two systems do have certain essential features in common that can be exploited in a properly generalized MVDW theory.

First, small fluctuations in liquid water are Gaussian, and computer simulations had earlier shown that even relatively large fluctuations leading to the formation of molecular sized cavities can be well described using the same Gaussian distribution (17). Indeed Pratt and Chandler (13) developed a quantitative theory for the solvation of small apolar molecules using the experimental linear response function for water that takes into account the structural and free energy changes induced by the excluded volume of the solute. These ideas have been significantly extended and placed on a firmer conceptual basis in recent work by Pratt, Hummer and coworkers (17, 19, 20, 22, 30, 86). See also (76). This suggests that the Gaussian/linear response ideas used in the second step of the two step method in the MVDW theory, using response functions appropriate for water, could be modified to apply to water.

Second, just as for the LJ fluid above, the main non-Gaussian feature to be expected in this application is associated with interface formation. Long ago, Stillinger (11) gave a qualitative and physically very suggestive description of what would be expected as the radius of the hard sphere solute is increased. While small solutes should not significantly disturb the hydrogen bond network, which can simply go around the solute, in the vicinity of a sufficiently large solute or wall the network must be completely disrupted. In the latter case, Stillinger argued, the arrangement of molecules near the solute and the interface free energy should resemble that of the liquid-vapor interface, which optimally solves the similar problem of going from a complete hydrogen-bond network in the liquid to no hydrogen bonds in the dilute vapor.

Thus Stillinger envisioned a drying transition very like that studied in the last Section for the LJ fluid as the solute size is increased. This phenomena is very general. The differences in local structure should have little effect on the generic and qualitative physics leading to interface formation. Thus it seems plausible that the interfacial component determined in the first step of the MVDW
theory could be described at least qualitatively by a MF treatment similar to that used for the LJ fluid provided appropriate thermodynamic parameters for water are used.

We see that key features of both steps of the MVDW theory have some analogues for water. The hard sphere reference fluid picture for the LJ fluid was incisive in developing the general ideas leading to the MVDW theory. Given that understanding, we may be able to develop an analogous approach for water and other fluids that does not rely on the details of the reference fluid, or indeed explicitly introduce a reference fluid at all.

Thus, following LCW, let us examine the essential features of the MVDW theory and see how they can be modified to apply to water. As in the last part of Section 4, we will try to describe everything formally in terms of the properties of water itself, and not a reference system, though molecular field ideas will be introduced to define what is meant in two-phase regions, etc. A crucial part of the physics of interface formation in the MVDW theory is the description of the unbalanced attractive interactions in Equation 5. In the LJ fluid this can be rewritten and reinterpreted in terms of an averaged or coarse grained density $\rho_1(r_1)$ with a normalized “weighting function” proportional to $u_1$:

$$-2a\rho_1(r_1) \equiv \int dr_2 \rho(r_2) u_1(r_{12}).$$  \hspace{1cm} (25)

LCW argue that the unbalanced attractive interactions in water can be described by a similar coarse graining of the water density, with the coarse graining carried out over the appropriate range $\lambda$ of the attractive interactions in water.

Now consider the first step of the two step method, as described in the last part of Section 7.1 where the smooth interfacial component $\rho_{s1}^r$ associated with the field $\phi_s$ in Equation 5 is determined from Equation 8. Using the notation of Equation 25 and expanding the next to last term, Equation 9 becomes:

$$\mu(\rho_{s1}^r) = \mu^B + m\nabla^2 \rho_{s1}^r + 2a[\rho_1(r_1) - \rho_{s1}^r].$$  \hspace{1cm} (26)

With appropriate change of notation this is exactly Equation 5 of LCW. However LCW did not consistently interpret $\rho_{s1}^r (= n_s(r_1)$ in LCW) as the hydrostatic density and some later simplifications that arise from this were not exploited. Since the slowly varying and generic interfacial component $\rho_{s1}^r$ in Equation 26 should be essentially independent of local structure, LCW used a simple VDW form for $\mu(\rho)$, which automatically interpolates in the two phase region, but with VDW parameters $a$ and $b$ chosen to reproduce the density and compressibility of liquid water at phase coexistence and $T = 298K$. The parameter $m$ was fit to the surface tension of water and the VDW relationship between $a$ and $m$ determined the coarse-graining scale $\lambda$ for $\rho(r)$, which LCW carried out using a simple Gaussian weight. While the details of this fitting procedure are rather arbitrary, and the VDW equation is probably inadequate to describe the quantitative relation between the energy density and surface tension in water (63), LCW showed that small variations in these parameter values did not change the qualitative picture that emerged.

Now turn to the second linear response step, described for the LJ fluid by Equation 21. What is the analogue of the LJ reference $\chi^{-1}_{12}(r_{12};\rho)$ for water? In view of Equation 2, one can formally consider fluctuations in the full fluid instead. For $\rho$ in the stable liquid phase, small fluctuations are Gaussian and the uniform fluid $\chi^{-1}_{12}(r_{12};\rho)$ can be used directly. However when interfaces form, we need an approximation for $\chi^{-1}$ in our MF treatment that remains well defined for all values of $\rho$ in
the two-phase region as well as in the vapor phase, as is the case for the reference $\chi_0^{-1}$ for the LJ fluid. In effect LCW devised an interpolation scheme for a $\chi^{-1}$ for water that has these properties.

LCW essentially considered an alternate but equivalent version of Equation 21:

$$\rho(r_1) - \rho_s^{r_1} = \int d\mathbf{r}_2 \chi(r_{12}; \rho_s^{r_1}) C_S(\mathbf{r}_2),$$

which involves the standard linear response function $\chi(r_{12}; \rho) \equiv \rho \delta(r_1 - r_2) + \rho^2 h(r_{12}; \rho)$, the inverse to $\chi^{-1}(r_{12}; \rho)$. Here $h+1$ is the radial distribution function for water and $C_S$ is a generalized solute-solvent direct correlation function, similar to that in Equation 17. This is nonzero only inside the core (PY closure), and is completely determined by the imposed requirement that $\rho(\mathbf{r})$ equals zero for all $\mathbf{r}$ in the core region. They replaced $h(r_{12}; \rho)$ by $h(r_{12}; \rho^B)$ in the definition of $\chi$, and used experimental values for the latter. This approximation is exact in the bulk liquid phase, reduces to the correct ideal gas value at very low density, and smoothly interpolates in between. Equation 6 in LCW reduces to our Equation 27 when it is realized that $\rho_s^{r_1}$ is the hydrostatic density. Since $\rho_s^{r_1}$ is slowly varying, no essential differences should result from using either equation.

Thus LCW implement in an approximate, but plausible way both steps of the MVDW theory, making use of experimental thermodynamic and structural data. They obtained results for the solute-solvent density distribution function qualitatively very similar to that of Figure 2 for the analogous LJ-hard sphere system. As the radius of the solute increased they found a crossover on the biologically relevant length scale of nanometers from “wetting” with peak densities greater than the bulk to “drying” with peak densities less than the bulk. (The unbalanced attractive interactions cause significant density perturbations in both regimes, so this terminology is somewhat misleading.) Weak attractive van der Waals attractions between the solute and water can shift inward the position of the interface when partial drying occurs and suppress complete drying, but should not have an important effect on the basic interface structure and free energy changes or the length scale for the crossover (85).

LCW also calculated solvation free energies, using a Gaussian approximation that is quantitatively somewhat less accurate than the coupling parameter method discussed above for the LJ system, but quite sufficient for qualitative purposes. Again they found behavior very similar to Figure 4, with free energies scaling with surface area only on large length scales of order nanometers. LCW also studied assemblies of extended idealized hydrophobic objects (plates and rods) and found that the drying can lead to strong attractions between sufficiently close pairs of such surfaces as the intervening water is expelled. Thus there is a length scale dependence of hydrophobic interactions. LCW suggest that such phase transitions could play an important role in aspects of protein folding where extended mostly hydrophobic regions approach one another.

It is beyond the scope of this article (and the expertise of this author!) to assess the validity of that last conjecture, given the many complications occurring in nature. Clearly much more theoretical and experimental effort is called for on all aspects of the theory. Some interesting recent work along these lines can be found in (21-29, 87-93). Our purpose is merely to argue that interface formation is a fundamental piece of physics that almost certainly occurs for the idealized models discussed herein and that LCW have developed a qualitatively reasonable MF treatment of that process based on sound statistical mechanical principles. From that perspective, the scaling of hydrophobic solvation energies with exposed surface area with a value close to the surface tension
of water can be justified only on large length scales (4, 93).

9 Density functional theory

As discussed in the Introduction, the most commonly used theory for the structure and thermodynamics of nonuniform fluids is density functional theory (DFT). Space limitations permit only a few general remarks here focusing on possible relations to the MVDW theory and the advantages and disadvantages of each approach. Two standard reviews of DFT are found in (43, 44).

A main focus of DFT is the intrinsic free energy density functional \( F(\rho, w) \), which arises from the usual grand canonical free energy \( \Omega \) by a Legendre transform (10), where the functional dependence on the chemical potential and external field \( \phi(r) \) is replaced by a functional dependence on the (uniquely associated) singlet density \( \rho(r) \). Our notation emphasizes that \( F \), just like \( \Omega \), remains a functional of the intermolecular pair (and any higher order) potentials \( w \). In principle this formalism is exact and \( F \) contains all the information in \( \Omega \). An exact hierarchy of (direct) correlation functions can be derived by successive functional derivatives of \( F \) with respect to \( \rho \).

While it must be as hard to determine \( F \) exactly as it is \( \Omega \), the hope is that practically useful approximations may suggest themselves more naturally in this representation, and the sound theoretical basis of DFT may provide a means for systematic corrections. By starting from the free energy, certain exact sum rules relating integrals of those correlation functions to the thermodynamic properties are automatically and consistently satisfied in any DFT (45, 59, 60); this is not the case for structurally based methods such as the MVDW theory. Particularly suggestive is the fact that the classical VDW theory can be viewed as a DFT for \( F \) where a MF approximation is made for the functional dependence on \( w \) and a local density approximation is made for the functional dependence on \( \rho \). While this certainly seems promising, it turns out to be rather difficult to do significantly better from this starting point.

Thus, can the DFT formalism help us improve on the MF approximation? Consider the first functional derivative of \( F \), which from basic properties of the Legendre transform is easily seen to satisfy

\[
\delta F(\rho, w) / \delta \rho(r) = \mu - \phi(r).
\]

In principle if we knew the “exact” \( F \) we could solve this equation, determining the density \( \rho(r) \) associated with a given potential \( \phi(r) \). While most discussions have focused on the density dependence, the main problem in determining an accurate \( F \) from a fundamental point of view is its dependence on \( w \). When there are attractive interactions, the exact \( F \) must describe critical phenomena, capillary waves, and a host of other properties for which we have essentially no idea what the true functional dependence on \( w \) should be. A treatment incorporating even the simplest capillary wave correlations for the liquid-vapor interface produced a density functional very different from conventional ideas (44, 72, 94), and this does not begin to address the range of phenomena arising from the general functional dependence on \( w \). So as a practical matter we are essentially forced to accept the MF treatment of the attractive interactions, perhaps modified slightly as in Section 6 to give a better description of the uniform fluid (60).

The true \( F \) for a system with repulsive forces only must be significantly simpler. Even here the dependence on a general repulsive \( w_0 \) can cause problems; any density functional must also explicitly
or implicitly approximate the dependence on $w_0$. It is not obvious from a fundamental point of view how to modify a functional that gives a good description for say hard spheres, so that it can describe much softer repulsions. See (53, 54) for interesting work on this question. Unlike most quantum-mechanical applications of DFT, where there is a single coulomb interaction potential, we must deal with a range of interactions and hence potentially many different functionals.

Most workers have quite reasonably focused on the basic hard sphere system, and here some significant progress has been made in determining accurate approximations for $F_0$, particularly in the development of fundamental measure theory (50-52, 58). However, the latter is very complicated and one might hope for a simpler and perhaps more physically motivated approach.

Unfortunately, the DFT formalism itself gives few indications of how to proceed in a practical manner. The basic problem can be seen when we consider Equation 28 for a system of hard spheres and a general $\phi_R$, i.e.,

$$\delta F_0([\rho_0], [u_0])/\delta \rho_0(\mathbf{r}) = \mu_0^R - \phi_R(\mathbf{r}).$$

This equation can be viewed as a generalization of our hydrostatic equation 7, which in principle could be used to determine the exact density if a proper $F_0$ could be found. Indeed Equation 7 follows from Equation 28 for a very slowly varying profile when a local density approximation is made. But how should this be corrected for rapidly varying densities? To go beyond the local density approximation, various weighted density approximations have been proposed. However, from a fundamental point of view except in special low dimensional cases (58) it is not even clear that $F_0$ can be written as a simple functional of weighted densities; certainly the original $\Omega_0$ is not a functional of a weighted $\phi_R(\mathbf{r})$! This contrasts with the development of the MVDW theory from the VDW theory. There the simplest correction (linear response) is clearly correct for small perturbing potentials, and it remains reasonably accurate even in the hard core limit, reducing naturally to the PY approximation. Usually in DFT this limit is imposed by hand.

Since many treatments of DFT have discussed the formal advantages of the method, we have mostly focused here on some of the difficulties as we see them. Despite these (perhaps pedantic!) objections, there have been impressive successes arising from DFT, and there clearly are close connections between some of the ideas of DFT and the MVDW theory. In particular, while we do think about molecular fields, the HLR equation 16 explicitly involves only densities. Indeed the reason for its success is the removal of any explicit dependence on the field through the expansion about the hydrostatic density. However, a deeper connection to DFT has so far escaped us; this is an active topic of our current research.

10 Final Remarks

We conclude with a few general remarks. The MVDW theory combines in a self-consistent way two standard and widely used ingredients: molecular (or mean) field theory and linear response (or Gaussian field) theory. The molecular field equation for the ERF takes account of the unbalanced attractive forces discussed in the original VDW theory (9); all that is required is to determine the molecular field and the induced structure accurately. While the hydrostatic approximation used in our interpretation of the VDW theory is not generally accurate, it serves as an optimal starting point for corrections based on linear response theory. The resulting theory can handle
problems involving fluctuations on a variety of length scales where each ingredient alone would fail. Because of its sound physical basis, we are hopeful that the MVDW theory will prove useful in many different applications. In particular, assessing and further developing both the physical and biophysical implications of the LCW theory in realistic environments seems an important topic for future research. One new direction we are thinking about involves fluids with long-ranged coulomb forces. Here there are a wide range of phenomena such as charge ordering and double layer formation arising from very strong and competing interactions on many length scales. This will certainly put our current ideas to a most severe test.

There are also many basic theoretical questions left open. Can the MVDW theory be understood as some kind of weighted DFT? Is it possible to extend these ideas to solid-fluid interfaces? Can one go significantly beyond the molecular field picture in describing the effects of attractive interactions while still maintaining a tractable theory? The LCW theory, despite its use of properties only of water, is definitely a molecular field theory. But the idea of short wavelength Gaussian fluctuations related to local structure and long wavelength slowly varying fluctuations related to interface formation seems more general. A reinterpretation of LCW theory from this perspective is found in (63). We close with what is probably the most basic question: Is it is generally correct to imagine that fluctuations are essentially Gaussian in most non-critical liquids except when interfaces form? A deeper understanding of whether and why this is true is called for. If this physically suggestive picture remains valid, then the MVDW theory has captured in a surprisingly simple way much of the non-critical physics of the liquid state.

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Figure 1: Correlation functions for hard spheres in the presence of spherical parabolic potentials shown in the inset (solid lines) as given by theory and simulation. The upper curve corresponds to the smaller potential and has been displaced upward by one unit. Also shown in the inset (crosses) are the potentials predicted by Equation 16 given the simulation data.
Figure 2: Density profiles of the LJ fluid ($T = 0.85, \rho^B = 0.70$) in the presence of the hard sphere solute with $S = 1.0, 2.0, 3.0$ and $4.0$. Circles denote simulation results (83). Lines are results of the self-consistent approach based on the modified molecular field determined from Equation 20. For ease of viewing, the density profiles for $S = 1.0, 2.0$ and $3.0$ have been shifted vertically by $0.6$, $0.4$ and $0.2$ units respectively.
Figure 3: Self-consistent molecular field of the LJ fluid for the solute with $S = 0.5, 1.0, 2.0, 3.0$ and 4.0, obtained from Equation 20.
Figure 4: Dependence of the solvation free energy on the cavity size $S$. Circles denote results of simulations (83). Lines are obtained from Equation 23 by using the results of the molecular field equation 20 (solid), and by neglecting the molecular field (dotted).