HOCHSCHILD AND CYCLIC HOMOLOGY OF THE QUANTUM KUMMER SPACES

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Abstract. We study the quotient space obtained by the flip action on the quantum n-tori. The Hochschild, cyclic and periodic cyclic homology are calculated.

0. Introduction

Spanier [S] studied the Kummer (non-smooth) manifolds obtained by the action of $\mathbb{Z}_2$ on the $2n$ dimensional torus. He concluded that the space is homeomorphic to $\mathbb{R}P^{2n-1}$. It has $2^{2n}$ double points and is simply connected with vanishing odd homology. Alternatively, a link of fixed-points after the quotient is homeomorphic to $\mathbb{R}P^{2n-1}$. The non-commutative geometry currently does not have a well defined notion of “non-commutative knots/links” but we shall see that homologically the $\mathbb{Z}_2$ quotient of the $n$-quantum torus $A_\Theta$ is similar to the Kummer manifold/variety. The dimension of cyclic homology is the same as the Betti numbers for the classical Kummer manifolds.

The Hochschild homology for these orbifolds for the case $n = 2$ was done in [O] [B] and [Q]. We here are inspired by the proof of [Q] and extending the methodology there into higher dimensions. It maybe noted that the periodic cyclic homology of $A_\Theta \times \mathbb{Z}_2$, the noncommutative smooth torus $\mathbb{Z}_2$ computed in a recent work [CTY] matches in dimension to what we have calculated in this article for the quantum/algebraic noncommutative torus with $\mathbb{Z}_2$ action. It is also expected that for “sufficiently” good $\Theta$, the $\mathbb{Z}_2$ quotient of the noncommutative smooth $n$-torus $A_\Theta$ shares similar Hochschild homological property but even for $n = 2$, such a computation was tricky[C]. What is known rather is that the odd periodic homology vanishes which does hint the vanishing of odd Hochschild homology. Other than having a striking similarity with the smooth quotients, the Hochschild homology of the quantum tori themselves have been studied [W1] and [T]. Readers can refer to [BRT, Page 353] for the comparison table therein for various homological and algebraic properties between the smooth non-commutative 2-torus and the quantum/algebraic non-commutative 2-torus.

1. Statement

THEOREM 1.1. Let $\Theta$ be a skew symmetric $n \times n$ matrix such that its entries are unimodular but none are roots of unity.

a) $H_0(\mathcal{A}_\Theta \times \mathbb{Z}_2) \cong \mathbb{C}^{2n+1}$ and

b) $H_\bullet(\mathcal{A}_\Theta \times \mathbb{Z}_2) \cong \mathbb{C}(\bullet)$ for $\bullet = 2k$ for some $k > 0$ and $0$ otherwise.

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COROLLARY 1.2. \( \dim_C(HC_\bullet(\mathcal{A}_\Theta \rtimes \mathbb{Z}_2)) = \sum_{2k \leq \bullet} \binom{n}{2k} + 2^n \) for \( \bullet \) even, 0 otherwise.

COROLLARY 1.3. The periodic homology are as follows:

a) \( HP_{\text{even}}(\mathcal{A}_\Theta \rtimes \mathbb{Z}_2) \cong \mathbb{C}^{3 \cdot 2^{n-1}} \) and

b) \( HP_{\text{odd}}(\mathcal{A}_\Theta \rtimes \mathbb{Z}_2) = 0. \)

2. STRATEGY OF THE PROOF

We shall study the Hochschild homology using the paracyclic spectral decomposition of the homology of the crossed product algebra. [GJ]

\[
H_\bullet(\mathcal{A}_\Theta \rtimes \mathbb{Z}_2) \cong H_\bullet(\mathcal{A}_\Theta)^{\mathbb{Z}_2} \oplus H_\bullet(-\mathcal{A}_\Theta)^{\mathbb{Z}_2},
\]

where \(-\mathcal{A}_\Theta\) is the algebra \(\mathcal{A}_\Theta\) with (left) \(\mathbb{Z}_2\)-twisted \(\mathcal{A}_\Theta^e\) module structure.

Hence our proof will investigate each of the summands in the above decomposition by firstly understanding the associated Hochschild homology and then locating the \(\mathbb{Z}_2\) invariant cycles. We shall use Nest’s resolution (which is similar to Wambst’s resolution for quantum symmetric algebras [W2]) and also Connes’ resolution as and when we find suitable.

3. NEST’S KOZUL RESOLUTION REVISITED

Nest [N] introduced a Koszul resolution for higher dimensional non-commutative tori. We briefly describe his resolution below.

The algebra \(\mathcal{A}_\Theta\) for a skew symmetric complex matrix \(\Theta\) is generated by unitaries \(\nu_1, \nu_2, \ldots, \nu_n\), satisfying the commutation relations

\[
\nu_i \nu_j = \lambda_{ij} \nu_j \nu_i, \quad \text{for } 1 \leq i, j \leq n
\]

such that \(|\lambda_{ij}| = 1\).

The enveloping algebra \(\mathcal{A}_\Theta^e\) is the algebraic tensor of \(\mathcal{A}_\Theta\) and its opposite algebra.

\[
\mathcal{A}_\Theta^e = \mathcal{A}_\Theta \otimes \mathcal{A}_\Theta^{op}.
\]

An element of \(\mathcal{A}_\Theta^e\) is denoted by \(a \otimes b^\circ\) for \(a \in \mathcal{A}_\Theta\) and \(b^\circ \in \mathcal{A}_\Theta^{op}\). We set \(V = \mathbb{C}^n\) with orthonormal basis \(e_1, e_2, \ldots, e_n\). We have the standard bar resolution of \(\mathcal{A}_\Theta\) is given as below:

\[
\cdots \to \Lambda_s(\mathcal{A}_\Theta) \xrightarrow{b} \Lambda_{s-1}(\mathcal{A}_\Theta) \xrightarrow{b} \cdots \xrightarrow{b} \Lambda_2(\mathcal{A}_\Theta) \xrightarrow{b} \Lambda_1(\mathcal{A}_\Theta) \xrightarrow{b} \Lambda_0(\mathcal{A}_\Theta) \xrightarrow{b} \mathcal{A}_\Theta,
\]

where \(\Lambda_s(\mathcal{A}_\Theta) = \mathcal{A}_\Theta^e \otimes \mathcal{A}_\Theta^{s\circ}\), \(\epsilon\) is the augmentation map and \(b : \Lambda_n(\mathcal{A}_\Theta) \to \Lambda_{n-1}(\mathcal{A}_\Theta)\);

\[
a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_ia_{i+1} \cdots a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.
\]

We set \(E_s = \mathcal{A}_\Theta^e \otimes \Lambda^sV\) and consider the following maps:

\[
h_s : E_s \to \Lambda_s(\mathcal{A}_\Theta);
\]

\[
1 \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \mapsto \sum_{\sigma \in S_s} \text{sgn}(\sigma)(\nu_{\sigma(i_1)} \nu_{\sigma(i_2)} \cdots \nu_{\sigma(i_s)})^{-1} \otimes \nu_{\sigma(i_1)} \otimes \nu_{\sigma(i_2)} \otimes \cdots \otimes \nu_{\sigma(i_s)}.
\]

\[
\alpha_s : E_s \to E_{s-1};
\]

where \(E_0 = \mathcal{A}_\Theta^e \otimes \Lambda^0V\).
where

\[ 1 \otimes e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_n} \mapsto \sum_{k=1}^{n} (-1)^k (1 - \nu_{i_k}^{-1} \otimes \nu_{i_k}^0) \otimes e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \wedge \ldots \wedge e_{i_n}. \]

\[ k_s : \Lambda_s(\mathcal{A}_\Theta) \to E_s; \]

\[ k((\nu_{i_1}, \nu_{i_2}, \ldots, \nu_{i_n})^{-1} \otimes \nu_{i_1} \otimes \nu_{i_2} \otimes \cdots \otimes \nu_{i_n}) = \sum_{i_1 > i_2 > \ldots > i_n} \rho_i ((\nu_{i_1})^{-1} \otimes (\nu_{i_1})) \wedge \rho_{i_2} ((\nu_{i_2})^{-1} \otimes (\nu_{i_2})) \wedge \cdots \wedge \rho_{i_n} ((\nu_{i_n})^{-1} \otimes (\nu_{i_n})). \]

Where for \( E_s \cong \mathcal{A}_\Theta \otimes \Lambda^s V \otimes \mathcal{A}_\Theta \) has a graded product structure \([N]\), for \( \pi = (\pi_1, \pi_2, \ldots, \pi_s) \in \mathbb{Z}^s \), \( \nu^\pi := \nu_{1}^{\pi_1} \nu_{2}^{\pi_2} \ldots \nu_{s}^{\pi_s} \) and \( \rho_i : \Lambda_1(\mathcal{A}_\Theta) \to E_1 \) as defined as below.

Note: The formula for \( \rho_i((\nu_{i_1})^{-1} \otimes \nu_{i_1}) \) in \([N][Page \text{1050}]\) has a misprint and the correct formula, which we shall use in our study is as follows:

\[ \rho_i((\nu_{i_1})^{-1} \otimes \nu_{i_1}) = (\nu_{i_1}^{1-i})^{-1} (\sum_{s=0}^{\pi_i-1} \nu_{i_s}^{-k} \otimes \nu_{i_s}^{k})(\nu_{i_1}^{1-i}). \]

where

\[
\begin{aligned}
\sum_{i=0}^{n'} = \begin{cases} 
\sum_{i=0}^{n} & \text{for } n \geq 0, \\
0 & \text{for } n = -1, \\
-1 & \text{for } n < -1.
\end{cases}
\end{aligned}
\]

and \( \nu^{1-i}_{1-p-1} := \nu^{p}_{1} \cdots \nu^{n}_{1}. \)

Though Nest gave this resolution for smooth non-commutative \( n \)-tori \( \mathcal{A}_\Theta \), but it is also a resolution of the quantum tori \( \mathcal{A}_\Theta \), the proof is easy and similar to the proof that Connes’ resolution is a resolution for quantum 2-torus.[Q]

4. Invariant cycles, \( H_*(\mathcal{A}_\Theta)^{\mathbb{Z}^2} \)

Using the Nest resolution we can easily compute \( H_0(\mathcal{A}_\Theta) \) explicitly, they zeroth cocycles are the \( \mathcal{A}_\Theta/im(\alpha_1) \), where

\[ \alpha_1(a^i \otimes 1 \otimes e_i) = a^i \otimes (1 - \nu^{-1}_i \otimes \nu^0_i) = a^i - \nu^{-1}_i a^i \nu_i. \]

Clearly, the zeroth Hochschild homology \( H_0(\mathcal{A}_\Theta) \) is generated by the equivalence class of elements supported at \( a_0 \). These scalars are invariant under \( \nu_i \mapsto \nu_i^{-1} \), hence \( H_0(\mathcal{A}_\Theta)^{\mathbb{Z}^2} = \mathbb{C}. \)

To compute \( H_*(\mathcal{A}_\Theta)^{\mathbb{Z}^2} \) for \( \bullet > 1 \) we shall firstly observe \( H_*(\mathcal{A}_\Theta) \) using the Nest’s Koszul resolution and then locate the invariant cycles. Wambst [W1] computed these \( k \)-Hochschild cycles of the quantum tori \( \mathcal{A}_\Theta \), they were generated by elements \( \{((x^\pi)^{-1} \otimes x^\pi)_{\pi \in (0,1)^n} \} \) with \( |\pi| = k \). But in this article, we shall restrict ourselves with the notation of Nest [N].

Let us consider \( E_s \) in the Nests’ Koszul resolution, using the following map it is straightforward to see that \( H_s(\mathcal{A}_\Theta) \) is generated by elements \( a_0 \otimes e_{i_1} \wedge \ldots \wedge e_{i_s} \) where \( \{i_1, \ldots, i_s\} \subset \{1, 2, \ldots, n\} \). We want to locate the \( E_2 \) Hochschild \( k \)-cycle using the Koszul resolution of \( \mathcal{A}_\Theta. \)

\[ (1 \otimes d_s) : \mathcal{A}_\Theta \otimes E_s \to \mathcal{A}_\Theta \otimes E_{s-1}. \]
Let $a_0 \otimes 1 \otimes e_{i_1} \land \cdots \land e_{i_s} \in ker(1 \otimes d_s)$, to check if it is invariant under $\mathbb{Z}_2$ action we push the cycle into the bar complex using the map $h_s$.

\[
\begin{array}{c}
\cdots \rightarrow E_2 \xrightarrow{\alpha_2} E_1 \xrightarrow{\alpha_1} E_0 \xrightarrow{\alpha_0} \mathcal{A}_\Theta \rightarrow 0 \\
\downarrow h \quad \downarrow k_2 \quad \downarrow h_1 \quad \downarrow k_1 \quad \downarrow h_0 = id \quad \downarrow k_0 = id \quad \downarrow id \\
\cdots \rightarrow \mathcal{A}^{\otimes 4}_\Theta \xrightarrow{b'} \mathcal{A}^{\otimes 3}_\Theta \xrightarrow{b'} \mathcal{A}^{\otimes 2}_\Theta \xrightarrow{b'} \mathcal{A}_\Theta \rightarrow 0
\end{array}
\]

\[(1 \otimes h_s)(a_0 \otimes 1 \otimes e_{i_1} \land \cdots \land e_{i_s}) = a_0 \otimes \sum_{\sigma \in S_s} sgn(\sigma)(\nu_{\sigma(i_1)} \cdots \nu_{\sigma(i_s)})^{-1} \otimes \nu_{\sigma(i_1)} \otimes \cdots \otimes \nu_{\sigma(i_s)}.
\]

Now,

\[
(1 \otimes k_s)(a_0 \otimes \sum_{\sigma \in S_s} sgn(\sigma)(\nu_{\sigma(i_1)}^{-1} \cdots \nu_{\sigma(i_s)}^{-1})^{-1} \otimes \nu_{\sigma(i_1)}^{-1} \otimes \cdots \otimes \nu_{\sigma(i_s)}^{-1})
\]

\[= sgn(\psi)a_0 \otimes k_s((\nu_{\psi(i_1)}^{-1} \cdots \nu_{\psi(i_s)}^{-1})^{-1} \otimes \nu_{\psi(i_1)}^{-1} \otimes \cdots \otimes \nu_{\psi(i_s)}^{-1}) \]

Where $\psi \in S_k$ is the permutation such that $\psi(i_1) > \psi(i_1) > \cdots \psi(i_s)$. We have used the fact that $\rho_i((\nu^\pi)^{-1} \otimes \nu^\pi) = 0$ if $(\pi)_i = 0$.

\[
\rho_{\psi_j}(\nu_{\psi(i_j)}^{-1} \otimes \nu_{\psi(i_j)}^{-1}) = - (\nu_{\psi_j} \otimes e_{\psi_j} \otimes \nu_{\psi_j}^{-1}).
\]

Therefore,

\[
(1 \otimes k_s)(a_0 \otimes \sum_{\sigma \in S_s} sgn(\sigma)(\nu_{\sigma(i_1)}^{-1} \cdots \nu_{\sigma(i_s)}^{-1})^{-1} \otimes \nu_{\sigma(i_1)}^{-1} \otimes \cdots \otimes \nu_{\sigma(i_s)}^{-1})
\]

\[= (-1)^s sgn(\psi)sgn(\psi)^{-1}a_0 \otimes 1 \otimes e_{i_1} \land \cdots \land e_{i_s}.
\]

Hence for $s$ even all the $s$-cycles are $\mathbb{Z}_2$ invariant and for $s$ odd none are. This was exactly the case in [Q, Page 329, 331], for the 1-cycles and the 2-cycle of the quantum 2-torus with $SL_2(\mathbb{Z})$ action. We have the following:

**LEMMA 4.1.** $H_\bullet(\mathcal{A}_\Theta)^{\mathbb{Z}_2} = \mathbb{C}^{(s)}$ if $\bullet = 2k$, 0 otherwise.

5. Twisted invariant cycles, $H_\bullet(\mathcal{A}_\Theta)^{\mathbb{Z}_2}$

To calculate $H_\bullet(\mathcal{A}_\Theta)$ we need to consider the $\mathbb{Z}_2$ twisted Koszul chain complex. For $s = 0$ we can explicitly see that the $\mathbb{Z}_2$ twisted zeroth cycle.

\[
H_0(\mathcal{A}_\Theta) = -\mathcal{A}_\Theta / -\alpha_1.
\]

where

\[
-\alpha_1(a^i \otimes 1 \otimes e_i) = a^i \otimes (1 - \nu_i^{-1} \otimes \nu_i^\pi) = a^i - \nu_i a^i \nu_i.
\]

Hence $H_0(\mathcal{A}_\Theta)$ is generated by the equivalence class of elements of the form $(a_\beta)_{\beta \in \{0,1\}^n}$. Therefore $H_0(\mathcal{A}_\Theta) = \mathbb{C}^{2^n}$. Under the action of $\mathbb{Z}_2$, an element $(a_\beta)$ is mapped to homologous element $(a_{-\beta})$ hence

\[
H_0(\mathcal{A}_\Theta)^{\mathbb{Z}_2} = \mathbb{C}^{2^n}.
\]

Observe that for $a \in H_n(\mathcal{A}_\Theta) = ker(1 \otimes d_n)$, $a = \nu_i a^i \nu_i$ for all $i$. Hence $H_n(\mathcal{A}_\Theta) = 0$.

We shall proceed by induction, we shall induct on the dimension of the torus.

We state that for the $n$-dimensional quantum torus $(n > 1)$, $H_\bullet(\mathcal{A}_\Theta) = 0$ for all $0 < \bullet < n$. As we noted earlier, the above statement holds for the case $n = 2$. Let us assume that for
all torus of dimensions less than \( n \) it holds. We shall prove that \( H_s(-\mathcal{A}_\Theta) = 0 \) for all \( 0 < \bullet < n \). The proof is now divided into two cases:

5.1. **Case I: \( \bullet > 1 \).**

**Lemma 5.1.** In this case we shall prove that \( H_s(-\mathcal{A}_\Theta) = 0 \) for \( n - 1 > \bullet > 0 \) then \( H_s(-\mathcal{A}_\Theta^n) = 0 \) for all \( n > \bullet > 1 \).

**Proof.** We notice that \( \binom{n}{s} = \binom{n-1}{s} + \binom{n-1}{s-1} \). Hence we have the following identification.

\[
(\pi_1, \pi_2) : -\mathcal{A}_\Theta \otimes E_s^n = -\mathcal{A}_\Theta \otimes E_{s-1}^{n-1} \oplus -\mathcal{A}_\Theta \otimes E_{s-1}^{n-1} \wedge e_n
\]

Therefore to show that \( \mathcal{A}_\Theta \otimes E_s^n \) is acyclic at \( s \) it is enough to show that

\[
-\mathcal{A}_\Theta \otimes E_{s+1}^n \bigoplus -\mathcal{A}_\Theta \otimes E_s^n \wedge e_n \xrightarrow{1 \otimes -\alpha_{s+1}} -\mathcal{A}_\Theta \otimes E_s^n \bigoplus -\mathcal{A}_\Theta \otimes E_s^n \wedge e_n \xrightarrow{1 \otimes -\alpha_s} -\mathcal{A}_\Theta \otimes E_{s-1}^{n-1} \bigoplus -\mathcal{A}_\Theta \otimes E_{s-2}^{n-1} \wedge e_n
\]

is middle exact.

We notice that the map \( 1 \otimes -\alpha_s \) does mix the direct summands. Explicitly,

\[
(1 \otimes -\alpha_s)(-\mathcal{A}_\Theta \otimes E_{s-1}^{n-1}) \subset -\mathcal{A}_\Theta \otimes E_{s-1}^{n-1};
\]

\[
(1 \otimes -\alpha_s)(-\mathcal{A}_\Theta \otimes E_{s-1}^{n-1} \wedge e_n) \subset -\mathcal{A}_\Theta \otimes E_{s-1}^{n-1} \bigoplus -\mathcal{A}_\Theta \otimes E_{s-2}^{n-1} \wedge e_n.
\]

Hence for \( \gamma \in \ker (1 \otimes -\alpha_s) \), \( \pi_2(\gamma) \in \ker (1 \otimes -\alpha_{s+1}) \subset -\mathcal{A}_\Theta \otimes E_{s-1}^{n-1} \wedge e_n \). But, since by hypothesis \( H_{s-1}(-\mathcal{A}_\Theta) = 0 \), there exists \( \mu_2 \in -\mathcal{A}_\Theta \otimes E_{s-1}^{n-1} \) such that

\[
(1 \otimes -\alpha_s)(\mu_2) = \pi_2(\gamma).
\]

Therefore, \( \pi_2(1 \otimes -\alpha_{s+1})(\mu_2 \wedge e_n) = \pi_2(\gamma) \).

We are now left to prove that there exists a \( \mu_1 \in -\mathcal{A}_\Theta \otimes E_{s-1}^{n-1} \), such that \( \pi_1(1 \otimes -\alpha_{s+1})(\mu_1) = \pi_2(\gamma) \). A kernel relation over \( -\mathcal{A}_\Theta \otimes E_{s-1}^{n-1} \) having the indices \( e_{r_1} \wedge e_{r_2} \wedge \cdots \wedge e_{r_{s-1}} \) such that \( r_i \neq n \) for any \( i \) looks like

\[
\psi^{p_1} V_{p_1} + \psi^{p_2} V_{p_2} + \cdots + \psi^{p_{n-s}} V_{p_{n-s}} + \psi^{p_n} V_{p_n} = 0.
\]

Where \( V_{e_i} := (1 - \nu_i \otimes \nu_i^{-1}) \), \( p_i \neq e_j \) for any \( i, j \) and \( \psi^k \in -\mathcal{A}_\Theta \). It can be observed that \( \psi^{p_n} = -\epsilon^{p_1} V_{p_1} - \epsilon^{p_2} V_{p_2} - \cdots - \epsilon^{p_{n-1}} V_{p_{n-1}} \), where \( \epsilon^k \) are the coefficients of \( e_{r_1} \wedge e_{r_2} \wedge \cdots \wedge e_{r_{s-1}} \wedge e_{p_k} \wedge e_n \in E_{s-1}^{n} \wedge e_n \). Since for \( a \in -\mathcal{A}_\Theta \), \( aV_i = aV_i V_i \) for all \( i \), the above kernel relation is hence reduced to one of the following form

\[
\psi^{p_1} V_{p_1} + \psi^{p_2} V_{p_2} + \cdots + \psi^{p_{n-s}} V_{p_{n-s}} = 0.
\]

Which has a solution by induction hypothesis, \( H_s(-\mathcal{A}_\Theta) = 0 \) for the \( n - 1 \) dimensional torus. Hence we are done. \( \Box \)
5.2. Case II: \( \bullet = 1 \). We prove for this case by induction over the dimension of torus and using the techniques of \([Q]\). The \( \ker(1 \otimes d_s) \subseteq \mathcal{A}_\Theta \otimes E_s \) are represented as diagrams in the \( (n_s)\)-dimensional lattice space. For \( \gamma \in \ker(1 \otimes d_s) \), consider a its diagram, \( \text{Diag}(\gamma) \).

Without loss of generality we may assume that \( \text{Diag}(\gamma) \) is a connected sub-lattice of \( \mathbb{Z}^{(n_s)} \) assembled by \( (n_s) \) dimensional polytopes. The case \( 2s = n = 2 \) corresponds to the quantum 2-torus. Here \( s = 1 \), hence consider an arbitrary 1-cocycle \( \gamma \) and its \( \text{Diag}(\gamma) \subseteq \mathbb{Z}^n \). We shall prove that \( \gamma \) is homologous to 0 in a similar way as we did in \([Q]\). We firstly change the basis of the Koszul resolution, while the basis of Connes’ Koszul resolution is \( 1 \otimes u_1 \) and \( 1 \otimes u_2 \) for the 2 dimensional case and \( (1 \otimes u_{i_1} \otimes u_{i_2} \otimes \ldots \otimes u_{i_s}) \) in general, the basis for the Nest’s Koszul resolution is anti symmetrised \( \{ (u_{i_1} u_{i_2} \ldots u_{i_s}) \}^{-1} \otimes u_{i_1} \otimes u_{i_2} \otimes \ldots u_{i_s} \).

While Nest’s resolution is computationally convenient, Connes’ basis is more convenient for diagrammatic approach \([Q]\). In this subsection, we shall consider the Generalized Connes’ Koszul resolution. An element of \( \mathcal{A}_\Theta \otimes E_s \) is finitely supported in \( \mathbb{Z}^n \), hence there exists \( l > 0 \) such that \( \text{Diag}(\gamma) \subseteq B_l(\bar{0}) \). The hyperplanes \( x_1 = l \) and \( x_1 = -l \) contain \( \text{Diag}(\gamma) \) between them.

A connected component of \( \text{Diag}(\gamma) \) is an assembly of \( n \)-hypercubes with no edges, i.e. diagram of the following form does not exist.

A typical kernel diagram in \( \mathbb{Z}^3 \) looks like as below. It is cubes connected by the kernel relation, the bullet dots represents a non-zero element of \( \mathcal{A}_\Theta^{\oplus n} \).

**Lemma 5.2.** \( H_1(\mathcal{A}_\Theta) = 0 \).

**Proof.** We prove by induction, let us assume that for all tori of dimension less than \( n \) the first Hochschild homology vanishes. Consider \( \text{Diag}(\gamma) \cap \{ x_1 = l \} \), we choose the \( a_i \otimes e_1 \wedge e_i \) such that the projection of \( (1 \otimes a_i^0)(a_i \otimes e_1 \wedge e_i) \) on the hyperplane \( \{ x_1 = l \} \) kills \( \text{Diag}(\gamma) \cap \{ x = l \} \). This can be done by ordering the non-zero lattice points in the \( i \)th dimension and using \( e_1 \wedge e_i \) with appropriate coefficients to kill them. Thus, what remains is a cycle which is not supported on \( \{ x_1 = l \} \) and is homologous to \( \gamma \). Repeating this for each hyper plane \( \{ x_i = d \} \), \( d < l \) we end up with a diagram which represents a cycle \( \gamma_{d_0} \) homologous to \( \gamma \) and lies entirely in \( \{ x_1 = d_0 \} \) for some \( d_0 \geq -l \). But we observe that \( \gamma_{d_0} \) is also a 1-cycle for \( n - 1 \) torus and hence homologous to 0 by induction hypothesis. \[ \square \]
Proof of Theorem 1.1. It follows from Lemma 4.1, Lemma 5.1 and Lemma 5.2.

\[\square\]

6. Cyclic Homology

Connes introduced an $S, B, I$ long exact sequence relating the Hochschild and cyclic homology of an algebra $A$,

\[ \cdots \xrightarrow{B} HH_n(A) \xrightarrow{L} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{L} \cdots \]

The cyclic homology

Proof of Corollary 1.2 and 1.3. The $\mathbb{Z}_2$ action on $\mathcal{A}_\Theta$ commutes with the map $-1\alpha$, we obtain the following exact sequence

\[ \cdots \xrightarrow{B} (HH_n(\mathcal{A}_\Theta))^{\mathbb{Z}_2} \xrightarrow{L} (HC_n(\mathcal{A}_\Theta))^{\mathbb{Z}_2} \xrightarrow{S} (HC_{n-2}(\mathcal{A}_\Theta))^{\mathbb{Z}_2} \xrightarrow{B} (HH_{n-1}(\mathcal{A}_\Theta))^{\mathbb{Z}_2} \xrightarrow{L} \cdots \]

Using the above long exact sequence we deduce the cyclic homology of $\mathcal{A}_\Theta \times \mathbb{Z}_2$[Corollary 1.2]. The $\mathbb{Z}_2$ invariant periodic cyclic homology for $\mathcal{A}_\Theta$ is $HP^{even}(\mathcal{A}_\Theta)^{\mathbb{Z}_2} = \mathbb{C}^{2^n}$ and $HP^{odd}(\mathcal{A}_\Theta)^{\mathbb{Z}_2} = 0$. Similarly for the untwisted case $HP^{even}(\mathcal{A}_\Theta)^{\mathbb{Z}_2}$ has dimension $\sum_{\bullet=2k}^{n} \left( \begin{array}{c} n \\ \bullet \end{array} \right),$ hence;

\[ HP^{even}(\mathcal{A}_\Theta)^{\mathbb{Z}_2} = \mathbb{C}^{2^{n-1}} \text{ and } HP^{odd}(\mathcal{A}_\Theta)^{\mathbb{Z}_2} = 0. \]

By using the paracyclic spectral decomposition we have:

\[ HP^{even}(\mathcal{A}_\Theta \times \mathbb{Z}_2) \cong HP^{even}(\mathcal{A}_\Theta)^{\mathbb{Z}_2} \oplus HP^{even}(\mathcal{A}_\Theta)^{\mathbb{Z}_2} = \mathbb{C}^{2^n} \oplus \mathbb{C}^{2^{n-1}} = \mathbb{C}^{3 \cdot 2^{n-1}} \]

and

\[ HP^{odd}(\mathcal{A}_\Theta \times \mathbb{Z}_2) = 0. \]

\[\square\]

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