Spectral fluctuations in $^{24}$Mg nucleus using the framework of the nuclear shell model

Thoraia A Abdul Hussian$^1$ and Adel K Hamoudi$^2$
$^1$Radiography techniques department, Al-Turath University, Baghdad, Iraq
$^2$Department of Physics, College of Science, University of Baghdad, Baghdad, Iraq
E-mail: thoraiamsc@yahoo.com

Abstract

Random matrix theory is used to study the chaotic properties in nuclear energy spectrum of the $^{24}$Mg nucleus. The excitation energies (which are the main object of this study) are obtained via performing shell model calculations using the OXBASH computer code together with an effective interaction of Wildenthal (W) in the isospin formalism. The $^{24}$Mg nucleus is assumed to have an inert $^{16}$O core with 8 nucleons (4 protons and 4 neutrons) move in the $1d_{5/2}$, $2s_{1/2}$ and $1d_{3/2}$ orbitals. The spectral fluctuations are studied by two statistical measures: the nearest neighbor level spacing distribution $P(s)$ and the Dyson-Mehta statistics ($\Delta_8$ statistics).

For calculations with the full diagonalization of the Hamiltonian, the spectral fluctuations are found to be in agreement with the Gaussian orthogonal ensemble of random matrices.

Keywords: Quantum chaos; Random matrix theory; Spectral fluctuations; Shell model calculations

1. Introduction:

Quantum chaos was investigated intensely through the last three decades [1]. Bohigas et al. [2] suggested a linking between chaos in a classical system and the spectral fluctuations of the analogous quantum system, where an analytical proof of the Bohigas et al. conjecture has been found in [3]. It is now known that quantum analogs of most classically chaotic systems reveal spectral fluctuations that agree with the random matrix theory (RMT) [4,5] while quantum analogs of classically regular systems reveal spectral fluctuations that agree with a Poisson distribution. For time-reversal-invariant systems, the suitable form of RMT is the Gaussian orthogonal ensemble (GOE). RMT was firstly utilized to characterize the statistical fluctuations of neutron resonances in compound nuclei [6]. RMT has become a standard tool for analyzing the universal statistical fluctuations in chaotic systems [7-10].

The mean field approximation can be used to study the chaotic nature of the single particle dynamics in the nucleus. However, the nuclear residual interaction mixes different mean field configurations and affects the statistical fluctuations of the many particle spectrum and wave functions. These fluctuations may be investigated with different nuclear structure models. The statistics of the low-lying collective part of the nuclear spectrum were studied in the framework of the interacting boson model [11, 12], in which the nuclear fermionic space is mapped onto a much smaller space of bosonic degrees of freedom. Because of the relatively small number of degrees of freedom in this model, it was also possible to relate the statistics to the underlying mean field collective dynamic. At higher excitations, additional degrees of freedom (such as broken pair) become important [13], and the effects of interactions on the statistics must be studied in larger model spaces. The interacting shell model offers an attractive framework for such studies. In this model, an effective interactions are available and the basis states are labeled by exact quantum numbers of angular momentum ($J$), isospin ($T$) and parity ($\pi$) [14].

In the studies [15-19], the distribution of eigenvector components was examined using the framework of the shell model. Brown and Bertsch [17] found that the basis vector amplitudes are consistent with Gaussian distribution (which is the GOE prediction) in regions of high level
density but deviated from Gaussian behavior in other regions unless the calculation employs degenerate single particle energies. Later, the study of Ref. [19] also suggested that calculations with degenerate single particle energies are chaotic at lower energies than more realistic calculations.

The electromagnetic transition intensities in a nucleus are observables that are sensitive to the wave functions, and the study of their statistical distributions should complement [11, 12] the more common spectral analysis and serve as another signature of chaos in quantum systems. Hamoudi et al carried out [20] the fp-shell model calculations to study the statistical fluctuations of energy spectrum and electromagnetic transition intensities in A=60 nuclei using the $F5P$ [21] interaction. The calculated results were in agreement with RMT and with the previous finding of a Gaussian distribution for the eigenvector components [15-19]. Hamoudi studied [22] the effect of the one-body Hamiltonian on the fluctuation properties of energy spectrum and electromagnetic transition intensities in $^{136}Xe$ using an effective interaction for the $N82$—model space defined by $2d_{5/2}, 1g_{7/2}, 1h_{11/2}, 3s_{1/2}$ and $2d_{1/2}$ orbitals. A clear quantum signature of breaking the chaoticity was observed as the values of the single particle energies are increased. Later, Hamoudi et al carried out [23] full $fp$—shell model calculations to investigate the regular to chaos transition of the energy spectrum and electromagnetic transition intensities in $^{44}V$ using the interaction of FPD6 as an effective interaction in the isospin formalism. The calculated spectral fluctuations and the distribution of electromagnetic transition intensities were found to have a regular dynamic at $\beta = 0$, ($\beta$ is the strength of the off-diagonal residual interaction), a chaotic dynamic at $\beta \geq 0.3$ and intermediate situations at $0 < \beta < 3$. Recently, Hamoudi and Murad have carried out [24] statistical calculations of nuclear excitation energies in $^{138}Ba$ nucleus, where this nucleus is described as a core of $^{132}Sn$ with 6 active protons move in the space of N82. The interaction of N82K has been used to compute the excitation energies required in the study. The obtained outcome has been in agreement with the study [23], Zelevinsky et al. [14] carried out sd-shell model calculations to examine the spectral fluctuations in $^{24}Mg$ nucleus using the proton-neutron formalism interaction of Wildenthal (WPN-interaction) [27]. Here the nuclear states are not described by the isospin $T$ but only by the angular momentum ($J$) and parity ($\pi$). The computed spectral fluctuations [14] were intermediate between the Wigner distribution (which characterizes the chaotic dynamic of systems) and Poisson distribution (which characterizes the regular dynamic of systems) due to the absence of mixing and repulsion between levels with different isospin. To explore the effect of hidden integral of motion (i.e., the hidden isospin $T$) on the spectral fluctuations in $^{24}Mg$ nucleus, we have (in this study) repeated the calculations of $^{24}Mg$ as in [14] but this time we have used the isospin formalism interaction of Wildenthal (W-interaction) [27]. Here the nuclear states are described by the angular momentum ($J$), isospin ($T$) and parity ($\pi$). The computed results of the present study is totally different than that of Zelevinsky et al. [14], where our computed spectral fluctuations in $^{24}Mg$ nucleus are found to be in consistent with the GOE (Wigner distribution) of RMT. This is attributed to the presence of mixing and repulsion between levels with different isospin.

2. Theory
The effective shell-model Hamiltonian, which describes a many-body system, is represented by

$$H = H_0 + H', \tag{1}$$

where $H_0$ and $H'$ represent the independent particle (one body) part and the residual two-body interaction of $H$. The unperturbed Hamiltonian

$$H_0 = \sum_{\lambda} e_{\lambda} a_{\lambda}^\dagger a_{\lambda} \tag{2}$$
describes non-interacting fermions in the mean field of the appropriate spherical core. The single-particle orbitals \( |\lambda \rangle \) have quantum numbers \( \lambda = (ljm) \) of orbital \( l \) and total angular momentum \( j \), projection \( j_z = m \) and isospin projection \( \tau \). The antisymmetrized two-body interaction \( H' \) of the valence particles is written as

\[
H' = \frac{1}{4} \sum \langle \lambda \mu ; \nu \sigma | a^\dagger_\mu a^\dagger_\sigma a_\nu a_\sigma \rangle.
\]

(3)

To construct the many-body wave functions with good spin \( J \) and isospin \( T \) quantum numbers, we use the \( m \)– scheme determinants which have, for given \( J \) and \( T \), the maximum spin and isospin projection,

\[ |M = J, T = T_3; m\rangle, \]

(4)

where \( m \) span the \( m \)– scheme subspace of states with \( M = J \) and \( T_3 = T \).

The matrix of the many-body Hamiltonian

\[
H_{\text{MT}} = \sum_k \langle JT; k | H | JT; k' \rangle
\]

(5)

is eventually diagonalized to obtain the eigenvalues \( E_\alpha \) and the eigenvectors

\[ |JT; \alpha \rangle = \sum_k C_\alpha^k |JT; k\rangle \]

(6)

Here, the eigenvalues \( E_\alpha \) are considered as the main object of the present investigation.

The fluctuation properties of nuclear energy spectrum are obtained via two statistical measures: the nearest-neighbors level spacing distribution \( P(s) \) and the Dyson-Mehta or \( \Delta_3 \) statistics [4, 25]. The staircase function of the nuclear shell model spectrum \( N(E) \) is firstly build. Here, \( N(E) \) is defined as the number of levels with excitation energies less than or equal to \( E \). In this study, a smooth fit to the staircase function is performed with polynomial fit. The unfolded spectrum is then defined by the mapping [12]

\[
\widetilde{E}_i = \widetilde{N}(E_i).
\]

(7)

The real spacings reveal strong fluctuations whereas the unfolded levels \( \widetilde{E}_i \) have a constant average spacing.

The level spacing distribution (which exemplifies the fluctuations of the short-range correlations between energy levels) is defined as the probability of two neighboring levels to be a distance \( s \) apart. The spacings \( s_i \) are determined from the unfolded levels by \( s_i = \widetilde{E}_{i+1} - \widetilde{E}_i \). A regular system is forecasted by the Poisson statistics

\[
P(s) = \exp(-s).
\]

(8)

If the system is classically chaotic, we foresee to get the Wigner distribution

\[
P(s) = (\pi / 2)s \exp(-s^2 / 4),
\]

(9)

which is consistent with the GOE statistics.

The \( \Delta_3 \) statistic (which characterizes the fluctuations of the long-range correlations between energy levels) is utilized to measure the rigidity of the nuclear spectrum and defined by [4]
\[ \Delta_3(\alpha, L) = \min_{A,B} \frac{1}{L} \int_{0}^{L} \left[ N(E) - (AE + B) \right] dE. \] (10)

It measures the deviation of the staircase function (of the unfolded spectrum) from a straight line. A rigid spectrum corresponds to smaller values of \( \Delta_3 \) whereas a soft spectrum has a larger \( \Delta_3 \).

To get a smoother function \( \overline{\Delta_3}(L) \), we average \( \Delta_3(\alpha, L) \) over several \( n_{\alpha} \) intervals \( (\alpha, \alpha + L) \)

\[ \overline{\Delta_3}(L) = \frac{1}{n_{\alpha}} \sum_{\alpha} \Delta_3(\alpha, L). \] (11)

The successive intervals are taken to overlap by \( L/2 \).

In the Poisson limit, \( \Delta_3(L) = L/15 \). In the GOE limit, \( \Delta_3 \approx L/15 \) for small \( L \), while \( \Delta_3 \approx \pi^{-2} \ln L \) for large \( L \).

3. Results and discussion

Shell model calculations with the computer program OXBASH [26] are performed for \( ^{24}\text{Mg} \) nucleus (with \( T = 0 \)). This nucleus is assumed to have an inert \( ^{16}\text{O} \) core with 8 valence nucleons (4 protons and 4 neutrons) move in the sd-shell (1d \( 5/2 \), 2s \( 1/2 \) and 1d \( 3/2 \) orbitals) model space. The W-interaction [27] is chosen as an effective interaction in the isospin formalism together with realistic spe’s. Many-body basis states \( |k \rangle \) were constructed with good total angular momentum \( J \) (its projection \( M \) ) parity \( \pi \) and isospin \( T \) (its projection \( T_3 \)). It is significant to indicate that all computed results are gained with diagonalization of the full Hamiltonian. In the following, we have considered the levels \( J^\pi T = O^+O, 2^+O, \) and \( 4^+O \) as test cases in the present analysis.

Table 1 shows the dimensions for the considered \( J^\pi T \) states formed with isospin formalism interaction of Wildenthal [27] for the \( ^{24}\text{Mg} \) nucleus, which is assumed to have 8 valence nucleons (4 protons and 4 neutrons) move in the sd-model space.

| \( J^\pi \) | Dimensions |
|------|-------------|
| 0^+  | 325         |
| 2^+  | 1206        |
| 4^+  | 1311        |

Figure 1 demonstrates the level density \( \rho(E) \) (histograms) of \( ^{24}\text{Mg} \) nucleus, computed with Wildenthal interaction, for \( J^\pi T = 0^+O, 2^+O, \) and \( 4^+O \) levels. It is clear from this figure that the distributions of \( \rho(E) \) (histograms) have a bell-shaped curve that is symmetric about the mean energy \( E_0 \), where these histograms are fitted by the Gaussian shape (the dashed line). The corresponding values of \( E_0 \) (the mean energy) and \( \sigma \) (the standard deviation) utilized
in the fitting with the Gaussian shape [28] for the considered \( J^\pi T \) levels in \(^{24}\text{Mg} \) nucleus are displayed in Table 2.

**Table 2.** Values of \( E_0 \) (the mean energy) and \( \sigma \) (the standard deviation) for considered \( J^\pi \) levels in \(^{24}\text{Mg} \) generated with isospin formalism interaction of Wildenthal [27].

| \( J^\pi \) | 24Mg \( E_0 \) (MeV) | \( \sigma \) |
|---|---|---|
| 0\(^+\) | 41.8153 | 12.928 |
| 2\(^+\) | 42.3150 | 12.591 |
| 4\(^+\) | 41.646 | 11.940 |

It is obvious from Figure 1 that the level density sharply grows along with excitation energy, arrives its maximum at the middle of the spectrum and then reduces again for the highest energy. This performance of the high energy, and the rough symmetry with respect to the middle of the spectrum, are artificial features of models with limited Hilbert space which is in contrast to real many-body systems. It is noticed that the computed \( \rho(E) \) (histograms) of all considered \( J^\pi T \) levels has a Gaussian shape, which is in agreement with the predicted study [7] for a many body system with two-body residual interaction. It is also noticed that the computed histograms have no dependency on the angular momentum \( J \).

**Figure 1.** The level density of \(^{24}\text{Mg} \) nucleus obtained with full Hamiltonian (histogram) is compared with the Gaussian fit (solid line) for \( J^\pi T = 0^+\), 2\(^+\), and 4\(^+\) states.
Figure 2 shows the nearest-neighbors level spacing distributions \( P(s) \) for the unfolded \( J^zT = 0^+0, 2^+0, \) and \( 4^+0 \) levels in \( ^{24}\text{Mg} \) nucleus. The distribution of computed \( P(s) \) is displayed by histograms. The GOE distribution (which describes chaotic systems) is displayed by the solid line. The Poisson distribution (which corresponds to a random sequence of levels and describes regular systems) is displayed by the dashed line. The level repulsion at the region of small spacing \( 0 \leq s \leq 0.2 \), which is considered as a distinctive feature of chaotic level statistics, is very well reproduced (not reproduced) by the present study (by the study [14]) due to the presence (absence) of the mixing and repulsion between levels with different isospin. In addition, the occurrence of the level repulsion at the region \( 0 \leq s \leq 0.2 \) is also attributed to the mixing of nuclear energy levels by the off-diagonal Hamiltonian of W-interaction. Moreover, the computed histograms obtained by the present study (by the study [14]) for the unfolded \( J^zT \) levels are in very well agreement with the GOE limit (clearly deviated from the GOE limit, i.e., intermediate between the GOE and Poisson limits). It is so obvious that the computed histograms, displayed in Figure 2, show no dependency on the angular momentum \( J \).

![Figure 2. The nearest neighbor level spacing \( P(s) \) distributions in \( ^{24}\text{Mg} \) nucleus for \( J^zT = 0^+0, 2^+0, \) and \( 4^+0 \) states. The histograms are the calculated \( P(s) \) with full Hamiltonian. The solid and dashed lines are the GOE and Poisson distributions, respectively.](image)

Figure 3 illustrates the \( \Delta_3(L) \) statistics for the unfolded \( J^zT = 0^+0, 2^+0, \) and \( 4^+0 \) levels in \( ^{24}\text{Mg} \) nucleus. The computed distribution of \( \Delta_3(L) \) statistics is denoted by the symbols of the open circle, the Poisson distribution is denoted by the dashed line while the GOE
distribution is denoted by the solid line. Again, the computed $\Delta_3$ statistics of the present study (the study [14]) for the unfolded $J^zT$ levels are in very good accordance with the GOE distribution (noticeably diverged from the GOE distribution, i.e., in-between the GOE and Poisson distributions). However, the computed distribution of $\Delta_3$ (the symbols of open circle) for $J^zT = 0^00$ levels demonstrates some slight oscillations around the GOE distribution. These oscillations are attributed to the number of intervals $n_\alpha$ (which is related to the dimension of $J^zT$ states) used in averaging the statistic $\Delta_3(\alpha, L)$, see Eq. (11). A smooth statistic corresponds to a large $n_\alpha$ and non-smooth statistic corresponds to a small $n_\alpha$. However, the symbols of the open circle of the $J^zT = 0^00$ levels are still very close to the GOE limit (the solid line). Once more, the distributions of computed $\Delta_3$ statistics, presented in Figure 3, exhibit also no dependency on the angular momentum $J$.

![Figure 3](image-url).

**Figure 3.** The average $\Delta_3$ statistics in $^{24}$Mg nucleus for $J^zT = 0^00$, $2^+0$ and $4^+0$ states. The calculated $\Delta_3$ statistics (open circles) correspond to the results obtained with full Hamiltonian. The solid and dashed lines are the GOE and Poisson distributions, respectively.

Additionally, the outcome of $\Delta_3$ statistic, displayed in Figure 3, confirms the outcome that we have obtained from the analysis of the $P(s)$ distributions, displayed in Figs. 2, i.e., the spectral fluctuations of the considered energy spectrum are in accordance with the GOE of RMT.
4. Conclusions

Chaotic properties of the nuclear energy spectrum in the $^{24}$Mg nucleus have been analyzed using the framework of RMT. The nuclear energy levels for the states $J^T = 0^+, 2^+, 0^-$, and $4^-$ are obtained by carrying out shell model calculations utilizing the OXBASH computer code together with an isospin formalism interaction of Wildenthal. It is noticed that the computed level density $\rho(E)$ of all considered $J^T$ levels have a Gaussian shape, which is in accordance with the predicted study of Brody et al. [7] for a many body system with two-body residual interaction.

It is seen that the level repulsion, presented in the computed distributions of $P(s)$, at the region of small spacing $0 \leq s \leq 0.2$ is very well imitated (not imitated) by the present study (by the study [14]) due to the existence (nonexistence) of mixing and repulsion between levels with different isospin. However, the computed distributions of $P(s)$ [Figure 2] and $\Delta_3$ statistics [Fig. 3] for different unfolded $J^T$ levels exhibit a very good accordance with the GOE limit (a noticeable discrepancy from the GOE limit, i.e., intermediate between the GOE and Poisson limits). Besides, both $P(s)$ and $\Delta_3$ statistics reveal no dependency on the angular momentum $J$.

Acknowledgment

The authors would like to express their thanks to Professor B. A. Brown of the National Superconducting Cyclotron Laboratory, Michigan State University, for providing the computer code OXBASH.

References

[1] Haak F 2001, Quantum signature of chaos, 2nd enlarged ed. Springer-Verlag, Berlin
[2] Bohigas O, Giannoni M, and Schmit C 1986, Phys. Rev. Lett. 52, 1
[3] Heusler S, Muller S, Altland A, Braun P, and Haak F 2007, Phys. Rev. Lett. 98, 044103
[4] Mehta M L 2004, Random Matrices, 3rd ed. Academic, New York
[5] Papenbrock T, and Weidenmuller H A 2007, Rev. Mod. Phys., Vol. 79, No. 3, 997
[6] Porter C E 1965, Statistical Theories of Spectra: Fluctuations Academic, New York
[7] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S.M 1981, Rev. Mod. Phys. 53, 385
[8] Guhr T, Müller-Groeling A and Weidenmüller H A 1998, Phys. Rep. 299, 189
[9] Alhassid Y 2000, Rev. Mod. Phys. 72, 895
[10] Gutzwiller M C 1990, Chaos in Classical and Quantum Mechanics Springer-Verlag, New York
[11] Alhassid Y, Novoselsky A and Whelan N 1990, Phys. Rev. Lett. 65, 2971; Alhassid Y and Whelan N1991, Phys. Rev. C 43, 2637
[12] Alhassid Y and Novoselsky A 1992, Phys. Rev. C 45, 1677
[13] Alhassid Y and Vretenar D 1992, Phys. Rev. C 46, 1334
[14] Zelevinsky V, Brown B A, Frazier N and Horoi M 1996, Phys. Rev. 276, 87
[15] Whitehead R R et al. 1978, Phys. Lett. B 76, 149
[16] Verbaarschot J J M and Brussaard P J 1979, Phys. Lett. B 87, 155
[17] Brown B A and Bertsch G 1984, Phys. Lett. B 148, 5
[18] Dias Het al. 1989, J. Phys. G 15, L79
[19] Zelevinsky V, Horoi M and Brown B A 1995, Phys. Lett. B 350, 141
[20] Hamoudi A, Nazmitdinov R G, Shahaliev E and Alhassid Y 2002, Phys. Rev. C 65, 064311
[21] Glaudemans P W M, Brussaard P J and Wildenthal R H 1976, Nucl. Phys. A 102, 593
[22] Hamoudi A 2011, Nucl. Phys. A 849, 27
[23] Hamoudi A and Abdul Majeed Al-Rahmani A 2012, Nucl. Phys. A 892, 21
[24] Hamoudi A and Murad S 2018, Iraqi journal of Science Vol. 59, No. 4C, Page 2225-2233
[25] Stephens F S, Deleplanque M A, Lee I Y, Macchiavelli A O, Ward D, Fallon P, Cromaz M, Clark R M, Descovich M, Diamond R M, and Rodriguez-Vieitez E 2005, Phys. Rev. Lett. 94, 042501
[26] Brown A B, Etchegoyen A, Godin N S, Rae W D M, Richter W A, Ormand W E, Warburton E K, Winfield J S, Zhao L and Zimmermam C H, MSU-NSCL Report Number 1289
[27] Wildenthal B H, Prog 1984. Part. Nucl. Phys. 11, 5
[28] Ribeiro M I 2004, Instituto Superior Tenico, Av. Rovisco Pais, 1; 1049-001 Lisboa Portugal, February