A necessary and sufficient condition for the existence of \( \{p, p + 1, q - 1, q\} \)-orientations in simple graphs

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Abstract

Let \( G \) be a simple graph and let \( p \) and \( q \) be two integer-valued functions on \( V(G) \) with \( p < q \) in which for each \( v \in V(G) \), \( q(v) \geq \frac{1}{2}d_G(v) \) and \( p(v) \geq \frac{1}{2}q(v) - 2 \). In this note, we show that \( G \) has an orientation such that for each vertex \( v \), \( d^+_G(v) \in \{p(v), p(v) + 1, q(v) - 1, q(v)\} \) if and only if it has an orientation such that for each vertex \( v \), \( p(v) \leq d^+_G(v) \leq q(v) \) where \( d^+_G(v) \) denotes the out-degree of \( v \) in \( G \). From this result, we refine a result due to Addario-Berry, Dalal, and Reed (2008) in bipartite simple graphs on the existence of degree constrained factors.

Keywords: Orientation; out-degree; factor; degree; bipartite graph.

1 Introduction

In this note, graphs have no loops, but multiple edges are allowed, and a simple graph have neither multiple edges nor loops. Let \( G \) be a graph. The vertex set and the edge set of \( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. We denote by \( d_G(v) \) the degree of a vertex \( v \) in the graph \( G \), whether \( G \) is directed or not. If \( G \) has an orientation \( D \), the out-degree and in-degree of \( v \) are denoted by \( d^+_D(v) \) and \( d^-_D(v) \); when \( D \) is clear from the context, we only write \( d^+_G(v) \) and \( d^-_G(v) \). We denote by \( G[A] \) the induced subdigraph of \( D \) with the vertex set \( A \) containing precisely those edges of \( G \) whose ends lie in \( A \). Likewise, we denote by \( D[A] \) the induced subdigraph of \( D \) with the vertex set \( A \) containing precisely those edges of \( D \) whose ends lie in \( A \). Let \( L : V(G) \rightarrow 2^\mathbb{Z} \) be a function. An orientation of \( D \) of \( G \) is said to be (i) \( L \)-orientation, if for each vertex \( v \), \( d^+_D(v) \in L(v) \), (ii) \( (p, q) \)-orientation, if for each vertex \( v \), \( p(v) \leq d^+_D(v) \leq q(v) \), where \( p \) and \( q \) are two integer-valued functions on \( V(G) \). Likewise, a factor \( F \) of the graph \( G \) is said to be \( L \)-factor, if for each vertex \( v \), \( d^+_F(v) \in L(v) \), (ii) \( (g, f) \)-factor, if for each vertex \( v \), \( g(v) \leq d^+_F(v) \leq f(v) \), where \( g \) and \( f \) are two integer-valued functions on \( V(G) \).

In 1976 Frank and Gyárfás formulated the following a criterion for the existence of \( (p, q) \)-orientations
which generalizes a result of Hakimi [6] who gave a criterion for the existence of orientations with given upper bound on out-degrees.

**Theorem 1.1.** ([4]) Let $G$ be a graph and let $p$ and $q$ be two integer-valued function on $V(G)$ with $p \leq q$. Then $G$ has a $(p, q)$-orientation if and only if for all $S \subseteq V(G)$,

$$e_G(S) \leq \min \{ \sum_{v \in S} q(v), \sum_{v \in S} (d_G(v) - p(v)) \}.$$

Recently, the present author introduced the following criterion for the existence of $(p, q)$-orientations in highly edge-connected graphs. In this note, we prove that under some conditions, as mentioned in the abstract, a simple graph has a $(p, p + 1, q - 1, q)$-orientation if and only if it has a $(p, q)$-orientation.

**Theorem 1.2.** ([7]) Let $G$ be a $8k^2$-edge-connected graph and let $p$ and $q$ be two integer-valued functions on $V(G)$ in which for each vertex $v$, $p(v) \leq d_G(v)/2 \leq q(v)$ and $|q(v) - p(v)| \leq k$. Then $G$ has a $(p, q)$-orientation if and only if there is an integer-valued function $t$ on $V(G)$ such that $|E(G)| = \sum_{v \in V(G)} t(v)$ and $t(v) \in \{p(v), q(v)\}$ for each $v \in V(G)$.

As an application, we refine the following result in bipartite simple graphs which is due to Addario-Berry, Dalal, and Reed (2008). More precisely, we conclude that under simpler conditions, a bipartite simple graph has a $(g, g + 1, f - 1, f)$-factor if and only if it has a $(g, f)$-factor.

**Theorem 1.3.** ([2]) Let $G$ be a simple graph and let $g$ and $f$ be two integer-valued functions on $V(G)$ satisfying $f \leq d_G$. If for each $v \in V(G)$, $\frac{1}{2}f(v) - 2 \leq g(v) \leq \lfloor \frac{1}{4}d_G(v) \rfloor < f(v) \leq \frac{1}{2}(d_G(v) + g(v)) + 2$, then $G$ has a factor $F$ such that for each $v \in V(G)$,

$$d_F(v) \in \{g(v), g(v) + 1, f(v) - 1, f(v)\}.$$

## 2 \ $(p, p + 1, q - 1, q)$-orientations of simple graphs

The following theorem gives a criterion for existence of $(p, p + 1, q - 1, q)$-orientations in simple graphs.

**Theorem 2.1.** Let $G$ be a simple graph and let $p$ and $q$ be two integer-valued functions on $V(G)$ with $p < q$ in which for each $v \in V(G)$, $q(v) \geq \frac{1}{4}d_G(v)$ and $p(v) \geq \frac{1}{4}q(v) - 2$. Then $G$ admits a $(p, q)$-orientation if and only if it has an orientation such that for each $v \in V(G)$,

$$d_C(v) \in \{p(v), p(v) + 1, q(v) - 1, q(v)\}.$$

**Proof.** Consider an orientation for $G$ such that for each vertex $v$, $p(v) \leq d_C(v) \leq q(v)$. Now, among such orientations, consider $D$ with the minimum $\sum_{v \in W_D} |d_D(v) - p(v)|$, where $X(D) = \{v \in V(G) : p(v) + 1 <
\(d_D^+(v) < q(v) - 1\). If \(X(D) = \emptyset\), then the proof is completed. Suppose, to the contrary, that there is a vertex \(x \in X(D)\). Define \(S\) to be the set of all vertices \(v\) such that there is a directed path from \(x\) to \(v\). Note that we must have \(d_D^+(v) \notin \{p(v), q(v) - 1\}\); otherwise, we can reverse the orientation of that path to obtain a better orientation, which derives a contradiction. Obviously, \(x \in S\). By the definition of \(S\), there is no directed edge from \(S\) to \(V(G) \setminus S\). Therefore,

\[
\sum_{v \in S} d_D^{-|S|}(v) = \sum_{v \in S} d_D^+(v).
\]

This implies that \(d_D^{-|S|}(x) \geq d_D^+(x)\) or there is a vertex \(y \in S \setminus \{x\}\) such that \(d_D^{-|S|}(y) \geq d_D^+(y) + 1\). In the first case, since \(d_D^+(x) \geq p(x) + 2 \geq \frac{1}{2}q(x)\), we must have \(d_D^{-|S|}(x) \geq d_D^+(x) \geq q(x) - 1 - d_D^+(x) \geq 1\). Thus, we can reverse the orientation of \(q(x) - 1 - d_D^+(x)\) edges of \(D[S]\) incident to \(x\) which is directed toward it. In the second case, since \(d_D^-(y) \geq p(y) + 1 \geq \frac{1}{2}q(y) - 1\), similarly we must have \(d_D^{-|S|}(y) \geq d_D^-(y) + 1 \geq q(y) - 1 - d_D^-(y)\).

In addition, the inequality \(q(y) \geq d_G(y)/2\) implies that \(d_D^+(y) \neq q(y)\) and hence \(q(y) - 1 - d_D^+(y) \geq 1\).

Therefore, we can first reverse the orientation of a directed path from \(x\) to \(y\), and next reverse the orientation of \(q(y) - 2 - d_D^+(y)\) edges incident to \(y\) which is directed toward it.

Let \(D_0\) be the new orientation of \(G\) and let \(X(D_0) = \{v \in V(G) : p(v) + 1 < d_{D_0}^+(v) < q(v) - 1\}\). Since \(G\) has no multiple edges, each \(v \in S \setminus \{x, y\}\) is incident to at most one modified edge of the last step. This implies that \(d_{D_0}^+(v) - d_D^+(v) \in \{-1, 0\}\) and \(p(v) \leq d_{D_0}^+(v) \leq q(v)\). Recall that \(d_D^+(v) \notin \{p(v), q(v) - 1\}\) for all \(v \in S\). For the first case, we have \(d_{D_0}^+(x) = q(x) - 1\) and hence \(X(D_0) \subseteq X(D) \setminus \{x\}\). For the second case, we have \(d_{D_0}^+(x) - d_D^+(x) \in \{-2, -1\}\) and \(d_{D_0}^+(y) = q(y) - 1\), and hence \(X(D_0) \subseteq X(D) \setminus \{y\}\). Therefore, \(D_0\) is a \((p, q)\)-orientation of \(G\) while \(\sum_{v \in X(D_0)} |d_{D_0}^+(v) - p(v)| < \sum_{v \in X(D)} |d_D^+(v) - p(v)|\). This is a contradiction and consequently the theorem is proved.

\[\square\]

**Corollary 2.2.** Let \(G\) be a simple graph and let \(p\) and \(q\) be two integer-valued functions on \(V(G)\) with \(p < q\). If for each \(v \in V(G)\), \(\frac{1}{2}d_G(v) - \frac{4}{3} \leq p(v) \leq \frac{1}{2}d_G(v) \leq q(v) \leq \frac{2}{3}d_G(v) + \frac{4}{3}\), then \(G\) admits an orientation such that for each \(v \in V(G)\),

\[
d_G^+(v) \in \{p(v), p(v) + 1, q(v) - 1, q(v)\}.
\]

**Proof.** Obviously, the graph \(G\) has an orientation such that for each vertex \(v\), \(|d_G^+(v) - d_G^-(v)| \leq 1\) which implies that \(q(v) \leq \lfloor \frac{1}{2}d_G(v) \rfloor \leq d_G^+(v) \leq \lceil \frac{1}{2}d_G(v) \rceil \leq p(v)\). Since \(p(v) \geq \frac{1}{3}d_G(v) - \frac{4}{3} \geq \frac{2}{3}q(v) - 2\), the proof can be completed by Theorem 2.1.

\[\square\]

**Remark 2.3.** Theorem 2.1 can be reformulated by replacing the conditions \(q(v) \leq (d_G(v) + q(v))/2 + 2\) and \(p(v) \leq d_G(v)/2\). To see this, it is enough to work with restricted in-degrees in the proof. By the same arguments in the proof, one can also develop Theorem 2.1 to multigraphs \(G\) provided that for each vertex \(v\), \(p(v) - q(v)/2 + 2 \geq d_G(v) - |N_G(v)|\), where \(N_G(v)\) denotes the set of all neighbours of \(v\) in \(G\).
3 Applications to degree constrained factors

Addario-Berry, Dalal, McDiarmid, Reed, and Thomason (2007) established the following theorem on the existence of degree constrained factors in simple graphs. This result was a prototype of Theorem 1.3. In this section, we are going to introduce a new stronger version for both of Theorems 1.3 and 3.1 in bipartite simple graphs based on Theorem 2.1.

**Theorem 3.1.** ([1]) Let $G$ be a simple graph and let $g$ and $f$ be two integer-valued functions on $V(G)$. If for each $v \in V(G)$, $\frac{1}{3}d_G(v) - \frac{1}{3} \leq g(v) \leq \frac{1}{3}d_G(v) \leq f(v) \leq \frac{2}{3}d_G(v) + \frac{2}{3}$, then $G$ has a factor $F$ such that for each $v \in V(G)$,

$$d_F(v) \in \{g(v), g(v) + 1, f(v) - 1, f(v)\}.$$ 

For this purpose, we need the following lemma that provides a useful relation between orientation and factors of bipartite graphs. A special case of this lemma was also used by Thomassen (2014) [9] to form a result on modulo factors of edge-connected graphs.

**Lemma 3.2.** Let $G$ be a bipartite graph with bipartition $(X, Y)$ and $L : V(G) \to 2^\mathbb{Z}$ be a function. Then $G$ admits an $L$-orientation if and only if $G$ admits an $L_0$-factor, where for each vertex $v$,

$$L_0(v) = \begin{cases} L(v), & \text{when } v \in X; \\ \{d_G(v) - i : i \in L(v)\}, & \text{when } v \in Y. \end{cases}$$

**Proof.** If $D$ is an orientation of $G$, then the factor $F$ consisting of all edges of $G$ directed from $X$ to $Y$ satisfies $d_F(v) = d_D^+(v)$ for each $v \in X$, and $d_F(v) = d_G(v) - d_D^-(v)$ for each $v \in Y$. Conversely, from every factor $F$, we can make an orientation $D$ whose edges directed from $X$ to $Y$ are exactly the same edges of $F$. \hfill \square

The following theorem is an equivalent version of Theorem 2.1 in bipartite graphs in terms of factors. Note that every bipartite graph $G$ has a factor $F$ such that for each vertex $v$, $[d_G(v)/2] \leq d_F(v) \leq [d_G(v)/2]$. To see this, it is enough to apply Lemma 3.2 along with an orientation of $G$ such that for each vertex $v$,

$$|d_G^+(v) - d_G^-(v)| \leq 1.$$ 

**Theorem 3.3.** Let $G$ be a bipartite simple graph with bipartition $(X, Y)$ and let $g$ and $f$ be two integer-valued functions on $V(G)$ with $g < f$. Assume that for each $v \in X$, $f(v) \geq d_G(v)/2$ and $g(v) \geq f(v)/2 - 2$, and for each $v \in Y$, $g(v) \leq d_G(v)/2$ and $f(v) \leq (d_G(v) + g(v))/2 + 2$. Then $G$ has a $(g, f)$-factor if and only if it has a factor $F$ such that for each $v \in V(G)$,

$$d_F(v) \in \{g(v), g(v) + 1, f(v) - 1, f(v)\}.$$ 

**Proof.** Apply Lemma 3.2 and Theorem 2.1 by setting $p(v) = g(v)$ and $q(v) = f(v)$ for each $v \in X$, and setting $p(v) = d_G(v) - f(v)$ and $q(v) = d_G(v) - g(v)$ for each $v \in Y$. \hfill \square
Remark 3.4. Note that one can use Lemma 3.2 to rediscover Theorem 4 and Lemma 6 in [3], which gave sufficient conditions for the existence of list orientations, from Theorem 1 in [5] and Theorem 2 in [5], which gave sufficient conditions for the existence of list factors.

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