On necessary and sufficient conditions for finite element convergence

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December 26, 2017

Abstract

In this paper we derive a necessary condition for finite element method (FEM) convergence in $H^1(\Omega)$ as well as generalize known sufficient conditions. We deal with the piecewise linear conforming FEM on triangular meshes for Poisson’s problem in 2D. In the first part, we prove a necessary condition on the mesh geometry for $O(h^\alpha)$ convergence in the $H^1(\Omega)$-seminorm with $\alpha \in [0,1]$. We prove that certain structures, bands consisting of neighboring degenerating elements forming an alternating pattern cannot be too long with respect to $h$. This is a generalization of the Babuška-Aziz counterexample and represents the first nontrivial necessary condition for finite element convergence. Using this condition we construct several counterexamples to various convergence rates in the FEM. In the second part, we generalize the maximum angle and circumradius conditions for $O(h^\alpha)$ convergence. We prove that the triangulations can contain many elements violating these conditions as long as their maximum angle vertexes are sufficiently far from other degenerating elements or they form clusters of sufficiently small size. While a necessary and sufficient condition for $O(h^\alpha)$ convergence in $H^1(\Omega)$ remains unknown, the gap between the derived conditions is small in special cases.

1 Introduction

The finite element method (FEM) is perhaps the most popular and important general numerical method for the solution of partial differential equations. In its classical, simplest form, the space of piecewise linear, globally continuous functions on a given partition (triangulation) $\mathcal{T}_h$ of the spatial domain $\Omega$ is used along with a weak formulation of the equation. In our case, we will be concerned with Poisson’s problem in 2D.

*This work is a part of the research project P201/13/00522S of the Czech Science Foundation. V. Kučera is currently a Fulbright visiting scholar at Brown University, Providence, RI, USA, supported by the J. William Fulbright Commission in the Czech Republic.
Much work has been devoted to the a priori error analysis of the FEM. Namely, the question arises, what is the necessary and sufficient condition for the convergence of the method when the meshes $T_h$ are refined, i.e., we have a system of triangulation $\{T_h\}_{h \in (0, h_0)}$. In the simplest case, we are interested in the energy norm, i.e., $H^1(\Omega)$-estimates of the error
\[ |u - U|_1 \leq C(u)h, \] (1)
where $u$ and $U$ are the exact and approximate solutions, respectively, $h$ is the maximal diameter of elements from $T_h$, and $C(u)$ is a constant independent of $h$. Typically, one is interested in deriving (1) for some larger class of functions, e.g., for all $u \in H^2(\Omega)$.

Historically, the first sufficient condition for (1) to hold is the so-called minimum angle condition derived independently in [13], [14]. This condition states that there should exist a constant $\gamma_0$ such that for any triangulation $T_h, h \in (0, h_0)$, and any triangle $K \in T_h$ we have
\[ 0 < \gamma_0 \leq \gamma_K, \] (2)
where $\gamma_K$ is the minimum angle of $K$. This condition was later weakened independently by [1], [2] and [7] to the maximum angle condition: there exists a constant $\alpha_0$ such that for any triangulation $T_h, h \in (0, h_0)$, and any triangle $K \in T_h$ we have
\[ \alpha_K \leq \alpha_0 < \pi, \] (3)
where $\alpha_K$ is the maximum angle of $K$. Finally, a condition for convergence was recently derived in [8] and [12], the circumradius condition: Let
\[ \max_{K \in T_h} R_K \to 0, \] (4)
where $R_K$ is the circumradius of $K$. Then the FEM converges in $H^1(\Omega)$. We note that all these conditions are derived by taking the piecewise linear Lagrange interpolation $\Pi_h u$ of $u$ in Céa’s lemma.

Conditions (2), (3) are sufficient conditions for $O(h)$ convergence. It was shown in [6] that the maximum angle condition is not necessary for (1) to hold. The argument is simple and can be directly extended to the circumradius condition. We take a system of triangulations satisfying (3) – thus exhibiting $O(h)$ convergence – and refine each $K \in T_h$ arbitrarily to obtain $\tilde{T}_h$. Thus $\tilde{T}_h$ can contain an arbitrary amount of arbitrarily bad ‘degenerating’ elements, however since $\tilde{T}_h$ is a refinement of $T_h$, it also exhibits $O(h)$ convergence.

Since (2)–(3) are only sufficient and not necessary, the question is what is a necessary and sufficient condition for (1) to hold. The only step in this direction is the Babuška-Aziz counterexample of [1] consisting of a regular triangulation of the square consisting only of ‘degenerating’ elements violating the maximum angle condition (except for several cut-off elements adjoining to the boundary $\partial \Omega$), cf. Figure 8. Recently a more detailed, optimal analysis of this counterexample was performed in [11]. By controlling the speed of degeneration of

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the aspect ratio of the elements, one can produce a counterexample to \(O(h)\) convergence and even an example of nonconvergence of the FEM. From this one counterexample, Babuška and Aziz conclude that the maximum angle condition is essential for \(O(h)\) convergence, carefully avoiding the word necessary.

In this paper, we will be concerned with a more general version of (1), namely \(O(h^\alpha)\) estimates of the form

\[
|u - U|_1 \leq C(u) h^\alpha,
\]

for \(\alpha \in [0, 1]\). We will derive a necessary condition on the mesh geometry and also a new sufficient condition for \(O(h^\alpha)\) to hold. To the author’s best knowledge, the derived necessary condition is the first nontrivial necessary condition for \(O(h)\) or any other convergence of the finite element method, apart from the trivial necessary condition \(h \to 0\). We note that the results of this paper are in fact about the approximation properties of the piecewise linear finite element space with respect to the \(H^1(\Omega)\)-seminorm, rather than about the FEM itself.

The structure of the paper is as follows. In Section 2, we derive a necessary condition for FEM convergence, i.e. (5) to hold. The condition roughly states that certain structures \(B \subset T_h\), bands of neighboring elements forming an alternating pattern such as in Figure 3 cannot be too long if the maximum angles of elements in \(B\) go to \(\pi\) sufficiently fast w.r.t. \(h \to 0\). The necessary condition is based on an estimate from below of the error \(|u - \Pi_h u|_{H^1(B)}\) on the band \(B\) – Theorem 1 for a single band and Theorem 2 for \(T_h\) containing multiple bands. Corollaries 9 and 13 then state (very roughly, cf. Remark 12) that if (5) holds and if \(B\) has length \(L \geq C h^{2\alpha/5}\) then \(\pi - \alpha_K \geq C h^{3-2\alpha} L\) for all \(K \in B\). This is the case of one band \(B \subset T_h\), for multiple bands this can be improved to \(\pi - \alpha_K \geq C h^{1-\alpha} L\). These results enable us to construct many simple counterexamples to \(O(h^\alpha)\) convergence, which is done in Sections 2.2 and 2.4. We note that the Babuška-Aziz counterexample (where \(T_h\) consists only of the considered bands, cf. Figure 8) can be obtained by this technique. In fact, we recover the optimal results of [11]. However unlike the Babuška-Aziz counterexample, the regular periodic structure of \(T_h\) is not necessary in the presented analysis.

In Section 3, we deal with sufficient conditions for (5) to hold. For simplicity we deal with the \(O(h)\) case, i.e. \(\alpha = 1\) and the general case is then a simple extension (Remark 18). We split \(T_h\) into two parts: \(T_h^1\) consisting of elements \(K\) satisfying (3) for a chosen \(\alpha_0\) and \(T_h^2\) consisting of \(K\) violating this condition (‘degenerating’ elements). While on \(T_h^1\) it is safe to use Lagrange interpolation, on \(T_h^2\) we use a modified Lagrange interpolation operator to construct a linear function \(v^K_h\) on \(K\) which is \(O(h_K)\)-close to \(u\) in the \(H^1(K)\)-seminorm independently of the shape of \(K\). The price paid is that \(v^K_h\) no longer interpolates \(u\) in \(A_K\), the maximal-angle vertex of \(K\), exactly but with a small perturbation of the order \(O(h_K^2)\), cf. Lemma 19. The question then arises how to connect these piecewise linear functions continuously, if they no longer interpolate the continuous function \(u\) exactly in some vertices. Instead of changing the interpolation procedure, we change the interpolated function \(u\) itself, so that the Lagrange interpolation of the new function \(\tilde{u}\) corresponds to the modified Lagrange interpolation of \(u\) on \(T_h^2\). Then it is possible to prove that \(|u - \Pi_h \tilde{u}|_1 \leq C h\), hence
\[ |u - U|_1 \leq Ch \] by Céa’s lemma. For this purpose, we introduce the concept of correction functions and Sections 3.3, 3.4 are devoted to constructing the correction functions corresponding to different situations. Roughly speaking, \( T_h \) can contain degenerating elements which can form arbitrarily large structures, chains, cf. Figure 12, as long as their maximal-angle vertices are not too close to other vertices from \( T_h^2 \). In general, if the degenerating elements form nontrivial clusters, such as the bands of Section 2, then these can have diameter up to \( O(h^{\alpha/2}) \). These results are contained in Theorems 5 and 6.

We note that while a necessary and sufficient condition for any kind of FEM convergence remains an open question, the gap between the derived conditions is small in some special cases, cf. Remark 20. For example, in the case of \( T_h \) containing one band of length \( L \) consisting of sufficiently degenerating elements, the necessary condition for \( O(h^\alpha) \) convergence is

\[ L \leq Ch^{2\alpha/5} = Ch^{0.4\alpha} \]

while the sufficient condition is

\[ L \leq Ch^{\alpha/2} = Ch^{0.5\alpha} \]

1.1 Problem formulation and notation

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded polygonal domain with Lipschitz continuous boundary. We treat the following problem: Find \( u : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \) such that

\[ -\Delta u = f, \quad u|_{\partial\Omega} = u_D. \quad (6) \]

There are several possibilities how to treat nonhomogeneous Dirichlet boundary conditions in the finite element method, here we consider the standard lifting technique, cf. [3]. We note that this choice is not necessary nor important in this paper, as we are essentially interested in the best \( H^1 \)-approximation of \( u \) in the discrete space, independent of the specific form of the weak formulation.

Let \( g \in H^1(\Omega) \) be the Dirichlet lift, i.e. an arbitrary function such that \( g|_{\partial\Omega} = u_D \), and write \( u = u_0 + g \), \( u_0 \in H^1_0(\Omega) \). Defining \( V = H^1_0(\Omega) \) and the bilinear form \( a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \), the corresponding weak form of (6) reads: Find \( u_0 \in V \) such that

\[ a(u_0 + g, v) = (f, v), \quad \forall v \in V. \]

The finite element method constructs a sequence of spaces \( \{X_h\}_{h \in (0,h_0)} \) on conforming triangulations \( \{T_h\}_{h \in (0,h_0)} \) of \( \Omega \), where \( X_h \subset H^1(\Omega) \) consists of globally continuous piecewise linear functions on \( T_h \). Furthermore, let \( V_h = X_h \cap V \). We do not assume any properties of \( T_h \), since it is the goal of this paper to derive necessary and also sufficient geometric properties of \( T_h \) for finite element convergence.

We choose some approximation \( g_h \in X_h \) of \( g \), e.g. a piecewise linear Lagrange interpolation of \( g \) on \( \partial\Omega \) such that \( g(x) = 0 \) in all interior vertices. We seek the FEM solution \( U \in X_h \) in the form \( U = U_0 + g_h \), where \( U_0 \in V_h \). The FEM formulation then reads: Find \( U_0 \in V_h \) such that

\[ a(U_0 + g_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \]
Formally, we should use the notation $U_h$ instead of $U$ to indicate that $U_h \in X_h$, however we drop the subscript $h$ for simplicity of notation. One must have this in mind e.g. when dealing with estimates of the form $|u - U|_{H^1(\Omega)} \leq Ch$.

For simplicity of notation, Sobolev norms and seminorms on $\Omega$ will be denoted using simplified notation, e.g. $|\cdot|_1 := |\cdot|_{H^1(\Omega)}$, $|\cdot|_{2,\infty} := |\cdot|_{W^{2,\infty}(\Omega)}$, etc. Throughout the paper, $C$ will denote a generic constant independent of $h$ and other geometric quantities of $T_h$. The constant $C$ has in general different values in different parts of the paper, even within the same chain of inequalities.

### Remark 1.
Especially of interest are the cases $\alpha = 1$ and $\alpha = 0$ corresponding to $O(h)$ convergence and (possibly) nonconvergence of the FEM. We note that we can just as well consider the more general case of $|u - U|_{H^1(\Omega)} \leq C(u)f(h)$ for some (nonincreasing) function $f$. For simplicity we consider only (7) since this coincides well with the concept of orders of convergence.

### Remark 2.
Assuming (7) holds for all $u \in H^2(\Omega)$ is not necessary, however it is typical for FEM analysis. In Section 2 we will only use (7) for one specific quadratic function (19). Hence $H^2(\Omega)$ can be replaced by any space that contains quadratic functions.

### Remark 3.
We note that $U$ in (7) need not be the FEM solution, since we do not use the FEM formulation in any way in the following. We are essentially dealing only with approximation properties of the space $V_h$. If there exists $U \in V_h$ such that (7) holds, what are the necessary properties of $T_h$.

The basic result upon which we will build the estimates of Section 2 is a simple geometric identity concerning gradients of a continuous piecewise linear function $U$ (not necessarily the FEM solution) on a triplet of neighboring elements $K_0, \tilde{K}_1, K_1$ in the configuration from Figure 1. We denote $U_0 := U|_{K_0}$, $U_1 := U|_{K_1}$ and $\tilde{U}_1 := U|_{\tilde{K}_1}$. The key observation is that if $U_0, U_1$ are given,
Figure 2: Proof of Lemma 1.

then $\tilde{U}_1$ is uniquely determined due to inter-element continuity. We note that the following result is not an estimate, it is an equality, hence optimal.

**Lemma 1.** Let $\xi$ be the angle between the vectors $\nabla(U_0 - U_1)$ and $v$, where $v$ corresponds to the common edge of $K_0, \tilde{K}_1$, cf. Figure 2. Then

$$|\nabla(\tilde{U}_1 - U_1)| = \frac{\cos(\xi)}{\sin(\pi - \tilde{\alpha})}|\nabla(U_0 - U_1)|,$$  

(8)

where $\tilde{\alpha}$ is the maximum angle of $\tilde{K}_1$.

**Proof:** By globally subtracting the function $U_1$ (i.e. its extension to the whole $\mathbb{R}^2$) from all functions on $K_1, \tilde{K}_1, K_0$, we can assume that $U_1 \equiv 0$ without loss of generality. From the continuity of $U$, we have $U_0(A) = \tilde{U}_1(A)$, where $A$ is the common vertex of $K_0$ and $\tilde{K}_1$ denoted in Figure 2. Furthermore, by continuity and since $U_1 \equiv 0$, then $\tilde{U}_1 = 0$ on the edge $e$. Therefore

$$|\nabla(\tilde{U}_1 - U_1)| = |\nabla\tilde{U}_1| = \frac{|\tilde{U}_1(A)|}{|v|\sin(\pi - \tilde{\alpha})}. \quad (9)$$

On the other hand, by continuity and since $U_1 \equiv 0$, we have $U_0(B) = 0$ and $U_0 = 0$ on the line $l$ passing through $B$ in the direction $\nabla U_0^\perp = \nabla(U_0 - U_1)^\perp$. Therefore,

$$|\nabla(U_0 - U_1)| = |\nabla U_0| = \frac{|\tilde{U}_1(A)|}{|v|\cos(\xi)}. \quad (10)$$

Expressing $|\tilde{U}_1(A)|$ from (10) and substituting into (9), we obtain the desired result.

□

**Remark 4.** Since $(a, b) = |a||b|\cos \alpha$, where $\alpha$ is the angle between vectors $a, b$, we can reformulate (9) as

$$|\nabla(\tilde{U}_1 - U_1)| = \frac{1}{\sin(\pi - \tilde{\alpha})}|(\frac{1}{|v|}v, \nabla(U_0 - U_1))|. \quad (11)$$

In other words, on the right-hand side of (11), we have the magnitude of the projection of $\nabla(U_0 - U_1)$ into the direction given by the unit vector $\frac{1}{|v|}v$.  

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6
Lemma 1 states that if $|\nabla (U_0 - U_1)|$ is nonzero, then $|\nabla (\tilde{U}_1 - U_1)|$ is huge, since it is magnified by the factor $\frac{1}{\sin(\pi - \tilde{\alpha})}$ which tends to $\infty$ as $\tilde{\alpha} \to \pi$. If $u$ is e.g. a reasonable quadratic function and if $\nabla U_1$ is a good approximation of $\nabla u|_{\tilde{K}_1}$, then $|\nabla U_1|$ cannot be small, hence $|\nabla u|_{\tilde{K}_1}$, which is reasonably bounded.

We will proceed as follows: We will choose an exact solution $u$ with nonzero second derivatives (a quadratic function). If $|u - U|_1$ is small, then we can expect $|\nabla (U_0 - U_1)| \neq 0$, since this is essentially an approximation of second order derivatives. Therefore $|\nabla (\tilde{U}_1 - U_1)|$ will be enormous on degenerating triangles due to the factor $1/\sin(\pi - \tilde{\alpha})$. Hence $\nabla U_1$ cannot be a good approximation of $u|_{\tilde{K}_1}$, therefore $|u - U|_1$ cannot be small. From this we can obtain restrictions e.g. on $\sin(\pi - \tilde{\alpha})$ for (7) to hold. The main difficulty is that the assumption of $|u - U|_1$ being small does not imply $|\nabla (U_0 - U_1)| \neq 0$ for any given triplet of elements such as in Figure 1, we know this only “on average”, in the $L^2(\Omega)$-sense due to the $H^1(\Omega)$ error estimate. However, if we connect the considered triplets of elements into larger structures (bands $B$, cf. Figure 3), then if these bands are large enough, for some triplet of elements in $B$ necessarily $|\nabla (U_0 - U_1)| \neq 0$ and we can apply Lemma 1.

Our goal will be to prove that if (7) holds and $B$ is long enough, then the differences $|\nabla (U_{i+1} - U_i)|$ must be nonzero on average, hence $\nabla U_i$ is a bad approximation of $\nabla u|_{\tilde{K}_i}$ and therefore the global error cannot be $O(h^\alpha)$ on the elements $\tilde{K}_i$. Such contradictions allow us to formulate a necessary condition for (7) and to construct counterexamples to $O(h)$ and other convergences.

**Definition 2.** We define a band $B \subset T_h$ as the union $B = (\cup_{i=0}^N K_i) \cup (\cup_{i=1}^N \tilde{K}_i)$ such that $K_{i-1}, K_i, \tilde{K}_i$ for $i = 1, \ldots, N$ are neighboring elements forming a triplet as in Lemma 1 such that their maximum angles form an alternating pattern as in Figure 3. We denote the union of longest edges of all $K_i$ by $\Gamma$.

**Remark 5.** In the following, we assume that the edge $\Gamma$ of band $B$ lies on a line, i.e. $B$ is ‘straight’. This makes the notation and proofs less technical, however it is not a necessary assumption. We will indicate in the relevant places how the proofs should be modified to allow for $B$ ‘curved’.

**Lemma 3.** Let $B$ be a band as in definition 2 and let the error estimate (7) hold. Then for sufficiently small $h$

\[ \|u - U\|_{L^2(\Gamma)} \leq C h^\alpha, \tag{12} \]

where $C$ depends only on $\Omega$ and the constant in (7).
Figure 4: Proof of Lemma 3 – splitting of $\Omega$ by $\Gamma$.

Proof: The proof is essentially a straightforward application of the trace inequality. However, we must be careful, since in the standard trace inequality the constant depends heavily on the geometry of the domain. Therefore, we give a detailed proof of (12). We proceed similarly as in [5].

For simplicity, we denote $e := u - U$. We define the constant vector function $\mu = -n_\Gamma$, where $n_\Gamma$ is the unit outer normal to $\mathcal{B}$ on $\Gamma$, cf. Figure 4. Then $\mu$ is the unit outer normal to $\tilde{\Omega}$ on $\Gamma$, where $\tilde{\Omega} \subset \Omega$ is the subdomain containing $\mathcal{B}$ defined by the line on which $\Gamma$ lies. On one hand, we have

$$\int_{\tilde{\Omega}} \nabla (e^2) \cdot \mu \, dx = 2 \int_{\tilde{\Omega}} e \nabla e \cdot \mu \, dx.$$  

On the other hand, by Green’s theorem,

$$\int_{\tilde{\Omega}} \nabla (e^2) \cdot \mu \, dx = \int_{\partial \tilde{\Omega}} e^2 \mu \cdot n \, dS,$$

since $\text{div} \mu = 0$. As $\mu \cdot n = 1$ on $\Gamma$, by combining these two equalities we get

$$\int_{\Gamma} e^2 \, dS \leq 2 \int_{\tilde{\Omega}} e \nabla e \cdot \mu \, dx - \int_{\partial \tilde{\Omega} \cap \partial \Omega} e^2 \mu \cdot n \, dS,$$  

where we have omitted the boundary integral over $\partial \tilde{\Omega} \setminus (\partial \Omega \cup \Gamma)$, which is nonnegative since $\mu \cdot n = 1$ on this part of $\partial \tilde{\Omega}$. The second right-hand side term in (13) can be estimated as

$$\int_{\partial \tilde{\Omega} \cap \partial \Omega} e^2 \mu \cdot n \, dS \leq \int_{\partial \Omega} e^2 \, dS \leq C h^4,$$  

since on $\partial \Omega$, $U$ is the piecewise linear Lagrange interpolation of $u$, hence $|e| = |u - U| \leq C h^2$ on $\partial \Omega$. The first right-hand side term of (13) can be estimated using Young’s inequality by

$$2 \int_{\tilde{\Omega}} e \nabla e \cdot \mu \, dx \leq \int_{\tilde{\Omega}} e^2 + |\nabla e \cdot \mu|^2 \, dx \leq \|e\|_{L^2(\Omega)}^2 + |e|_{H^1(\Omega)}^2.$$  

Combining (13)–(15) with (7), we get

$$\|e\|_{L^2(\Gamma)}^2 \leq \|e\|_{L^2(\Omega)}^2 + C h^4 + C h^{2\alpha}.$$  

(16)
In (16), it remains to estimate \( \| e \|_{L^2(\Omega)}^2 \) by \( | e |_{H^1(\Omega)}^2 \). Ordinarily, one could simply use Poincaré’s inequality, however \( e \notin H^1_0(\Omega) \). Nonetheless, as mentioned above, \( e |_{\partial \Omega} \) is of the order \( O(h^2) \), which allows us to obtain a similar estimate to Poincaré’s inequality. Using Green’s theorem and the trivial identity \( \partial x_1 \partial x_1 = 1 \), we estimate

\[
\| e \|_{L^2(\Omega)}^2 = \int_{\Omega} 1 e^2 \, dx = \int_{\partial \Omega} x_1 n_1 e^2 \, dS - \int_{\Omega} x_1 2e \frac{\partial e}{\partial x_1} \, dx. \tag{17}
\]

Without loss of generality, let \( 0 \in \Omega \). Therefore we can estimate (17) using Young’s inequality as

\[
\| e \|_{L^2(\Omega)}^2 \leq \text{diam} \Omega \| e \|_{L^2(\Gamma)}^2 + \frac{1}{2} \| e \|_{L^2(\Omega)}^2 + (\text{diam} \Omega)^2 | e |_{H^1(\Omega)}^2. \]

Therefore,

\[
\| e \|_{L^2(\Omega)}^2 \leq 2 \text{diam} \Omega \| e \|_{L^2(\Gamma)}^2 + 2(\text{diam} \Omega)^2 | e |_{H^1(\Omega)}^2 \leq C h^4 + C h^{2\alpha}. \tag{18}
\]

Combining (16) and (18) gives us estimate (12) for sufficiently small \( h \). □

**Remark 6.** If we do not assume that \( \Gamma \) lies on a straight line, cf. Remark 5, we need to find a constant vector function \( \mu \) such that \( \mu \cdot n \geq \mu_0 > 0 \) for some \( \mu_0 > 0 \). This can be done e.g. by taking \( \mu = n_{K_i}^T \) for some \( i \in \{0, \ldots, N\} \), i.e. the normal to \( K_i \) on \( \Gamma \), and assuming that \( n_{K_i}^T \cdot n_{K_i}^j \geq \mu_0 > 0 \) for all \( j \neq i \), i.e. \( \mathcal{B} \) does not ‘bend too much’. The constant \( \mu_0 \) then figures in the left-hand sides of (13) and (16). Consequently, we get the constant \( 1/\mu_0 \) in the right-hand side of (12).

In the following, we shall assume that the exact solution of (6) is

\[
u(x_1, x_2) = x_1^2 + x_2^2, \tag{19}\]

i.e. \( f = -4 \) with corresponding Dirichlet boundary conditions. If \( L \) is the length of \( \Gamma \), then \( \Gamma \) can be parametrized by \( t \in (0, L) \) as

\[
\Gamma = \{ x = a + tg, t \in (0, L) \},
\]

where \( a \) is an endpoint of \( \Gamma \) and \( |g| = 1 \) is a vector in the direction of \( \Gamma \). The restriction \( u|_{\Gamma} \) can then also be parametrized by \( t \in (0, L) \). We denote the corresponding function \( u_{\Gamma} \) and observe that

\[
u_{\Gamma}(t) = u(a + tg) = t^2 + p_1(t), \tag{20}\]

where \( p_1 \in P^1(0, L) \), i.e. a linear function. We have the following approximation result:

**Lemma 4.** Let \( u_{\Gamma} \in P^2(0, L) \) be given by (20). Let \( \Pi^1_{(0, L)} \) be the \( L^2(0, L) \)-orthogonal projection onto \( P^1(0, L) \). Then

\[
u_{\Gamma} - \Pi^1_{(0, L)} u_{\Gamma} \|_{L^2(0, L)} = \frac{1}{6 \sqrt{3}} L^{5/2}. \tag{21}\]
Proof: For convenience, $u_T$ can be rewritten in the form $u_T(t) = (t - L/2)^2 + \tilde{p}_1(t)$, where $\tilde{p}_1 \in P^1(0,L)$. Then $u_T - \Pi^1_{(0,L)}u_T = (t - L/2)^2 - \Pi^1_{(0,L)}(t - L/2)^2$, since $\Pi^1_{(0,L)}\tilde{p}_1 = \tilde{p}_1$.

On $(0,L)$, the quadratic function $(t - L/2)^2$ is symmetric with respect to $L/2$, hence its projection must be a constant function. It can be therefore easily computed that $\Pi^1_{(0,L)}(t - L/2)^2 = \frac{1}{12}L^2$. Therefore the norm in (21) can be straightforwardly computed, giving the desired result. To save space, we omit the elementary, yet lengthy calculations. □

**Corollary 5.** Let $\alpha \in [0,1]$ and $L \geq C h^{2\alpha/5}$. Then
\begin{equation}
\|u_T - \Pi^1_{(0,L)}u_T\|_{L^2(0,L)} \geq Ch^\alpha.
\end{equation}

**Remark 7.** Specifically of interest are the cases $\alpha = 1$, corresponding to $L \geq Ch^{2/5}$ and $\alpha = 1$, corresponding to $L \geq C$. These two cases will lead to necessary conditions and counterexamples to $O(h)$ convergence, and convergence of the FEM, respectively.

**Remark 8.** If $\Gamma$ does not lie on a straight line, cf. Remark 5, the length-parametrized restriction $u_T : (0,L) \to \mathbb{R}$ of $u|_\Gamma$ will be continuous, piecewise quadratic. However, if $\Gamma$ is close to a straight line, then $u_T$ will be close to the globally quadratic function $t^2 + p_1(t)$ of (20). We can then estimate
\begin{equation}
\|u_T - \Pi^1_{(0,L)}u_T\|_{L^2(0,L)} \geq \|t^2 + p_1(t) - \Pi^1_{(0,L)}u_T\|_{L^2(0,L)} - \|u_T - t^2 + p_1(t)\|_{L^2(0,L)}.
\end{equation}

For the first right-hand side term we have
\begin{equation}
\|t^2 + p_1(t) - \Pi^1_{(0,L)}u_T\|_{L^2(0,L)} \geq \|t^2 + p_1(t) - \Pi^1_{(0,L)}(t^2 + p_1(t))\|_{L^2(0,L)} = \frac{1}{6\sqrt{3}}L^{5/2}
\end{equation}
by Lemma 4. If we assume e.g. that
\begin{equation}
\|u_T - t^2 + p_1(t)\|_{L^2(0,L)} \leq \frac{1}{12\sqrt{3}}L^{5/2}
\end{equation}
then by (23) we get the statement of Lemma 4 as an estimate from below with the lower bound $\frac{1}{12\sqrt{3}}L^{5/2}$. Condition (24) is effectively a condition on how $\Gamma$ deviates from a straight line.

Now we focus on the discrete solution on $\Gamma$. Again, if $\Gamma$ lies on a straight line, then the length-parametrized restriction $U_T := U|_\Gamma \in L^2(0,L)$ is a continuous piecewise linear function on the partition of $(0,L)$, i.e. $\Gamma$, induced by $T_h$. Hence $U_T \in H^1(0,L)$.

For convenience, to deal with functions from $H^1_0(0,L)$ instead of $H^1(0,L)$, we define $\tilde{U} \in P^1(0,L)$ such that $\tilde{U}(0) = U_T(0)$ and $\tilde{U}(L) = U_T(L)$. Let
\begin{equation}
w := U_T - \tilde{U},
\end{equation}
then $w(0) = w(L) = 0$. We have the following estimate.
Lemma 6. Let \( L \geq C_L h^{2\alpha/5} \), where \( C_L \) is sufficiently large with respect to the constant \( C(u) \) from (7). Then
\[
|w|_{H^1(0,L)} \geq C_L^{3/2}.
\]

Proof: By the reverse triangle inequality, Lemma 4 and 3,
\[
\|w\|_{L^2(0,L)} \geq \|u_\Gamma - \bar{U}\|_{L^2(0,L)} - \|u_\Gamma - u_\Gamma\|_{L^2(0,L)} \geq \frac{1}{6\sqrt{5}} L^{5/2} - C h^{\alpha},
\]

since \( L^{5/2} \geq C_L^{5/2} h^{\alpha} \) and this term dominates the error term \( u - U_\Gamma \) for \( C_L \) sufficiently large w.r.t. the constant in (12), i.e. (7).

To obtain (26), we use Poincaré’s inequality on \((0,L)\). This is possible, since \( w \) is continuous, piecewise linear and \( w(0) = w(L) = 0 \), hence \( w \in H^1(0,L) \).

Since \( L \) is not fixed, one must use the appropriate scaling from the Poincaré inequality on the fixed interval \((0,1)\). From (27) we obtain
\[
\frac{C_P}{2} L^{5/2} \leq \|w\|_{L^2(0,L)} \leq C_P L |w|_{H^1(0,L)},
\]
where \( C_P \) is the constant from Poincaré’s inequality on \((0,1)\). Cancelling \( L \) from both sides gives us (26).

Since \( w \) is piecewise linear on \((0,L)\), \( w' \) is piecewise constant on the corresponding partition \((0,L) = \bigcup_{i=0}^N I_i \) induced on \( \Gamma \) by \( T_h \). We denote \( h_i = |I_i| \) and \( w_i = (w|_{I_i})' \). Then Lemma 6 can be rewritten as
\[
\sum_{i=0}^N h_i w_i^2 \geq C L^3.
\]

Furthermore, since \( w(0) = w(L) = 0 \), we have \( \int_0^L w'(x) \, dx = 0 \), i.e.
\[
\sum_{i=0}^N h_i w_i = 0.
\]

From these two inequalities, we derive an estimate of \( w_i' - w_{i-1}' \) i.e. how much \( w \) differs from a globally linear function. This is already related to the right-hand side of (8).

Lemma 7. Let \( \{h_i\}_{i=0}^N \), \( \{a_i\}_{i=0}^N \) be such that \( h_i > 0, \ i = 0, \ldots, N \) and
\[
\sum_{i=0}^N h_i a_i^2 = A, \quad \sum_{i=0}^N h_i a_i = 0, \quad \sum_{i=0}^N h_i = L,
\]

where \( A \geq 0 \). Then
\[
\sum_{i=1}^N (a_i - a_{i-1})^2 \geq \frac{A}{LN}.
\]
Proof: Assume on the contrary, that there exists \( \{a_i\}_{i=0}^{N} \) such that (30) holds and
\[
\sum_{i=1}^{N} (a_i - a_{i-1})^2 < \frac{A}{LN}.
\] (32)
Let \( i_0 \) be such that \( |a_{i_0}| = \max_{i=0,\ldots,N} |a_i| \). Without loss of generality assume e.g. \( a_{i_0} > 0 \). Since all \( h_i > 0 \) and \( \sum_{i=0}^{n} h_i a_i = 0 \), there exists \( i_1 \) such that \( a_{i_1} < 0 \). Without loss of generality assume e.g. \( i_0 < i_1 \). Then for all \( i \)
\[
|a_i| \leq a_{i_0} < a_{i_0} - a_{i_1} = (a_{i_0} - a_{i_0+1}) + \ldots + (a_{i_1-1} - a_{i_1})
\leq \sum_{k=1}^{N} |a_k - a_{k-1}| \leq N^{1/2} \left( \sum_{k=1}^{N} |a_k - a_{k-1}|^2 \right)^{1/2} < A^{1/2} L^{-1/2}
\] (33)
due to the Cauchy-Schwartz inequality and (32). From (30), we have
\[
\sum_{i=0}^{N} h_i a_i^2 = A.
\] On the other hand, using (33), we have \( \sum_{i=0}^{N} h_i a_i^2 = A \), i.e. \( A < A \) which is a contradiction. \( \square \)

Lemma 8. Let \( L \geq C_L h^{2\alpha/5} \), where \( C_L \) is sufficiently large. Then
\[
\sum_{i=1}^{N} (U'_i - U'_{i-1})^2 \geq C L^2 N^{-1},
\] (34)
where \( U'_i = (U_i|_{I_i})' \).

Proof: We apply Lemma 7 to inequalities (28), (29) with \( a_i := w'_i \). Then \( a_i - a_{i-1} = w'_i - w'_{i-1} = U'_i - U'_{i-1} \), since \( \bar{U}' \) is constant. The assumption \( L \geq C_L h^{2\alpha/5} \) is needed because (28) follows from Lemma 6. \( \square \)

Remark 9. The phrase ”\( C_L \) sufficiently large” in Lemmas (6), (8) should read in full ”\( C_L \) sufficiently large w.r.t. the constant \( C(u) \) from (7) with \( u \) given by (12)”. This can be seen from the proof of Lemma 6 from which it carries on to further results, where we will use the shortened version ”\( C_L \) sufficiently large”. We note that the dependence of the necessary \( C_L \) on \( C(u) \) is such that when \( C(u) \to 0 \) the minimal necessary \( C_L \) also goes to zero. We use this fact in Counterexample 3.

2.1 Main estimate – one band
Let \( B = \bigcup_{i=0}^{N} K_i \cup (\bigcup_{i=1}^{N} \tilde{K}_i) \) be the band with edge \( \Gamma \), cf. Figure 3. We denote \( U_i = U|_{K_i} \) and \( \bar{U}_i = U|_{\tilde{K}_i} \). By \( g \) we denote the unit vector in the direction of \( \Gamma \), i.e. \( |g| = 1 \) and by \( \alpha_i, \tilde{\alpha}_i \) we denote the maximum angles of \( K_i \) and \( \tilde{K}_i \), respectively.
For $U'_i$ from Lemma 8 we have $U'_i = \nabla U_i \cdot g$, since by definition $U'_i$ is the derivative of $U_i$ in the direction $g$. Hence we can reformulate estimate (34) as

$$\sum_{i=1}^{N} |(\nabla U_i - \nabla U_{i-1}) \cdot g|^2 \geq CL^2 N^{-1}.$$ (35)

We denote $\nabla u_i = \nabla u(C_i)$, where $C_i$ is the centroid of $K_i$ and similarly $\nabla \tilde{u}_i = \nabla u(\tilde{C}_i)$, where $\tilde{C}_i$ is the centroid of $\tilde{K}_i$. Trivially,

$$|\nabla u(x) - \nabla u_i| \leq Ch, \quad \forall x \in K_i,$$

$$|\nabla u(x) - \nabla \tilde{u}_i| \leq Ch, \quad \forall x \in \tilde{K}_i.$$ (36)

Now we will prove the main theorem of this section. We show that if $|u - U|_1 \leq Ch^\alpha$, then the error on $B$ can be split into two parts. The first “dominant” part $A_1$ can be bounded from below, while the remainder $A_2$ is of the order $O(h^\alpha)$. The estimate for $A_1$ depends on the geometry of $B$ and can be made arbitrarily large e.g. for the maximal angles in $B$ going to $\pi$ sufficiently fast. Thus contradictions with the $O(h^\alpha)$ bound of the error can be created, producing counterexamples to $O(h^\alpha)$ convergence as in Section 2.2.

**Theorem 1.** Let $u$ be given by (19) and let $U \in X_h$ satisfy the error estimate (7). Let the band $B = (\cup_{i=0}^{N} K_i) \cup (\cup_{i=1}^{N} \tilde{K}_i)$ be such that there exist constants $C_K, C_\alpha$ independent of $h$ such that for all $i = 1, \ldots, N$

$$|\tilde{K}_i| \leq C_K |K_i|,$$

$$|\tilde{K}_i| \leq C_K |K_{i-1}|,$$

$$\sin(\pi - \alpha_{i-1}) \leq C_\alpha \sin(\pi - \tilde{\alpha}_i).$$ (37)

Finally, let $L \geq C_L h^{2\alpha/5}$, where $C_L$ is sufficiently large. Then

$$|u - U|_{H^1(B)} \geq A_1 - A_2,$$ (38)

where

$$A_1 \geq C_1 \min_{i=1, \ldots, N} \frac{1}{\sin(\pi - \alpha_i)} \min_{i=1, \ldots, N} |\tilde{K}_i|^{1/2}LN^{-1/2},$$

$$|A_2| \leq C_2 h^\alpha.$$ (39)

**Proof:** We consider the error only on the elements $\tilde{K}_i$:

$$|u - U|_{H^1(B)} \geq \left( \sum_{i=1}^{N} \int_{\tilde{K}_i} |\nabla u - \nabla U|^2 \, dx \right)^{1/2} = A_1 - A_2,$$ (40)

where the splitting of the error is given by

$$A_1 := \left( \sum_{i=1}^{N} \frac{1}{\sin^2(\pi - \alpha_i)} |(\nabla U_i - \nabla U_{i-1}) \cdot g|^2 |\tilde{K}_i| \right)^{1/2},$$

$$A_2 := \left( \sum_{i=1}^{N} \int_{\tilde{K}_i} |\nabla u - \nabla U|^2 \, dx \right)^{1/2} - A_1.$$ (41)
Due to (35) we immediately have

\[ A_1 \geq \min_{i=1, \ldots, N} \frac{1}{\sin(\pi - \alpha_i)} \min_{i=1, \ldots, N} |\tilde{K}_i|^{1/2} (CL^2N^{-1})^{1/2}, \]  

(42)

which is the first inequality in (39). It remains to estimate \( A_2 \).

Since \( \nabla u|_{\tilde{K}_i} = \nabla \tilde{U}_i \), a series of triangle inequalities gives us

\[ A_2 \leq \left( \sum_{i=1}^{N} \int_{\tilde{K}_i} |\nabla u - \nabla \tilde{u}_i|^2 \, dx \right)^{1/2} + \left( \sum_{i=1}^{N} \int_{\tilde{K}_i} |\nabla \tilde{u}_i - \nabla \tilde{U}_i|^2 \, dx \right)^{1/2} - A_1 \]

\[ = \left( \sum_{i=1}^{N} \int_{\tilde{K}_i} |\nabla u - \nabla \tilde{u}_i|^2 \, dx \right)^{1/2} + \left( \sum_{i=1}^{N} |\nabla \tilde{u}_i - \nabla \tilde{U}_i|^2 |\tilde{K}_i| \right)^{1/2} \]

\[ \leq \left( \sum_{i=1}^{N} \int_{\tilde{K}_i} |\nabla u - \nabla \tilde{u}_i|^2 \, dx \right)^{1/2} + \left( \sum_{i=1}^{N} |\nabla \tilde{u}_i - \nabla \tilde{U}_i|^2 |\tilde{K}_i| \right)^{1/2} \]

\[ + \left( \sum_{i=1}^{N} |\nabla u_i - \nabla U_i|^2 |\tilde{K}_i| \right)^{1/2} - A_1 \]

\[ =: (A) + (B) + (C) + (D) - A_1. \]

We estimate the individual terms \((A) - (D)\) of (42). Due to (36),

\[ (A) \leq Ch|B|^{1/2} \leq Ch|\Omega|^{1/2} \leq Ch \leq Ch^\alpha, \]

(44)

since \( \alpha \in [0, 1] \). Similarly,

\[ (B) = \left( \sum_{i=1}^{N} |\nabla u(\tilde{C}_i) - \nabla u(C_i)|^2 |\tilde{K}_i| \right)^{1/2} \leq Ch|B|^{1/2} \leq Ch \leq Ch^\alpha, \]

(45)

since \( \nabla u \) is a linear function due to (19) and the centroids of the neighboring elements \( K_i, \tilde{K}_i \) satisfy \( |C_i - \tilde{C}_i| \leq 2h \).

Under the assumption \( |\tilde{K}_i| \leq C_K |K_i| \) for all \( i \), we have

\[ (C) \leq \left( C_K \sum_{i=1}^{N} |\nabla u_i - \nabla U_i|^2 |K_i| \right)^{1/2} \leq \left( C_K \sum_{i=1}^{N} \int_{K_i} |\nabla u_i - \nabla U|^2 \, dx \right)^{1/2} \]

\[ \leq \left( C_K \sum_{i=1}^{N} \int_{K_i} |\nabla u_i - \nabla U|^2 \, dx \right)^{1/2} + \left( C_K \sum_{i=1}^{N} \int_{K_i} |\nabla u - \nabla U|^2 \, dx \right)^{1/2} \]

\[ \leq Ch|B|^{1/2} + C_K^{1/2} |u - U| \leq Ch^\alpha \]

(46)

due to (36) and (7).

Using Lemma 1 (or Remark 4) and denoting by \( v_{i-1} \) the unit vector corre-
sponding to the common edge of $K_{i-1}, \tilde{K}_i$, we can rewrite $(D)$ as

\[
(D) = \left( \sum_{i=1}^{N} \frac{1}{\sin^2(\pi - \tilde{\alpha}_i)} \left| v_{i-1} \cdot \nabla(U_i - U_{i-1}) \right|^2 |\tilde{K}_i| \right)^{1/2}
\]

\[
\leq \left( \sum_{i=1}^{N} \frac{1}{\sin^2(\pi - \tilde{\alpha}_i)} \left| (\nabla U_i - \nabla U_{i-1}) \cdot g \right|^2 |\tilde{K}_i| \right)^{1/2}
\]

\[
+ \left( \sum_{i=1}^{N} \frac{1}{\sin^2(\pi - \tilde{\alpha}_i)} \left| (\nabla U_i - \nabla U_{i-1}) \cdot (v_{i-1} - g) \right|^2 |\tilde{K}_i| \right)^{1/2}
=: A_1 + (E). \tag{47}
\]

We notice that we have obtained the term $A_1$ in the estimate of $(D)$, cf. (41). This will cancel with the final term $-A_1$ in (43), while all the remaining terms are estimated by $Ch^\alpha$.

It remains to estimate the term $(E)$ from (47):

\[
(E) \leq \left( \sum_{i=1}^{N} \frac{|\tilde{K}_i|}{\sin^2(\pi - \tilde{\alpha}_i)} \left| (\nabla U_i - \nabla U_{i-1}) \cdot (v_{i-1} - g) \right|^2 \right)^{1/2}
\]

\[
+ \left( \sum_{i=1}^{N} \frac{|\tilde{K}_i|}{\sin^2(\pi - \tilde{\alpha}_i)} \left| (\nabla U_i - \nabla U_{i-1}) \cdot (v_{i-1} - g) - (\nabla U_{i-1} - \nabla u_{i-1}) \cdot (v_{i-1} - g) \right|^2 \right)^{1/2}. \tag{48}
\]

A simple geometric argument, cf. Figure 5, gives us, due to assumption (37),

\[
|v_{i-1} - g| = 2 \sin(\beta_{i-1}/2) |g| \leq 2 \sin(\pi - \alpha_{i-1}) \leq 2C_\alpha \sin(\pi - \tilde{\alpha}_i), \tag{49}
\]

since $\beta_{i-1}/2 < \beta_{i-1} < \pi - \alpha_{i-1}$ and $\sin(\cdot)$ is a monotone function on $[0, \pi/2]$. Furthermore, by the definition of $u$, we have $|\nabla u_i - \nabla u_{i-1}| \leq Ch$. Therefore, we can estimate the first right-hand side term in (48) as

\[
\left( \sum_{i=1}^{N} \frac{1}{\sin^2(\pi - \tilde{\alpha}_i)} \left| (\nabla U_i - \nabla U_{i-1}) \cdot (v_{i-1} - g) \right|^2 |\tilde{K}_i| \right)^{1/2}
\]

\[
\leq \left( \sum_{i=1}^{N} \frac{1}{\sin^2(\pi - \tilde{\alpha}_i)} Ch^2 \sin^2(\pi - \tilde{\alpha}_i) |\tilde{K}_i| \right)^{1/2} \leq Ch|B|^{1/2} \leq Ch^\alpha. \tag{50}
\]
fying the assumptions of Theorem 1 must satisfy Corollary 9. A necessary condition for (7) to hold is that every band (up to a constant) of $A$ for some sufficiently large $\tilde{\alpha}$.

Proof: Assume on the contrary that $C$ for

\[
\sum_{i=1}^{N} \frac{|\tilde{K}_i|}{\sin^2(\pi - \tilde{\alpha}_i)} \left| (\nabla U_i - \nabla u_i) \cdot (v_{i-1} - g) - (\nabla U_{i-1} - \nabla u_{i-1}) \cdot (v_{i-1} - g) \right|^2 \right)^{1/2}
\]

is chosen large enough so as to dominate the term $C\tilde{\alpha}$ in (54). From (54) and (7) we have

\[
C_3 \alpha \leq |u - U|_{H^1(B)} \leq |u - U|_{H^1(\Omega)} \leq C(u) \alpha.
\]

For $C$ sufficiently large, $C_3$ can be made larger than $C(u)$, leading to a contradiction. □
Remark 10. Here we shall comment on the case when $\Gamma$ does not lie on a straight line, cf. Remark 5. By $g_i$ we denote the unit vector given by the edge $K_i \cap \Gamma$. Then in (34), we have $U'_i = \nabla U_i \cdot g_i$ and (35) now reads
\[
\sum_{i=2}^{N} |\nabla U_i \cdot g_i - \nabla U_{i-1} \cdot g_{i-1}|^2 \geq CL^2 N^{-1},
\] which leads to the definition of $A_1$ in (11) as
\[
A_1 := \left( \sum_{i=1}^{N} \frac{1}{\sin^2(\pi - \alpha_i)} |\nabla U_i \cdot g_i - \nabla U_{i-1} \cdot g_{i-1}|^2 |\tilde{K}_i| \right)^{1/2},
\] for which we immediately have the estimate (12) due to (55). The remaining terms in the proof of Theorem (1) are the same, except for (E), which is now
\[
\left( \sum_{i=2}^{N} \frac{1}{\sin^2(\pi - \alpha_i)} |\nabla U_i \cdot (v_{i-1} - g_i) - \nabla u_{i-1} \cdot (v_{i-1} - g_{i-1})|^2 |\tilde{K}_i| \right)^{1/2}.
\] Similarly as in (48),
\[
(E) \leq \left( \sum_{i=2}^{N} \frac{|\tilde{K}_i|}{\sin^2(\pi - \alpha_i)} |\nabla u_i \cdot (v_{i-1} - g_i) - \nabla u_{i-1} \cdot (v_{i-1} - g_{i-1})|^2 \right)^{1/2}
\]
\[
+ \left( \sum_{i=2}^{N} \frac{|\tilde{K}_i|}{\sin^2(\pi - \alpha_i)} |(\nabla U_i - \nabla u_i) \cdot (v_{i-1} - g_i) - (\nabla U_{i-1} - \nabla u_{i-1}) \cdot (v_{i-1} - g_{i-1})|^2 \right)^{1/2}
\] (58)
where the second term can be estimated as (51), since $|v_i - g_i| \leq 2C \sin(\pi - \alpha_i)$ and similarly for $v_{i-1} - g_{i-1}$ by Figure 5. The first term in (58) can be rewritten using the triangle inequality
\[
|\nabla u_i \cdot (v_{i-1} - g_i) - \nabla u_{i-1} \cdot (v_{i-1} - g_{i-1})| \\
\leq |(\nabla u_i - \nabla u_{i-1}) \cdot (v_{i-1} - g_{i-1})| + |\nabla u_{i-1} \cdot (v_{i-1} - g_{i-1})| \\
\leq C h \sin(\pi - \alpha_i) + C |g_{i-1} - g_i|.
\] (59)

In order to eliminate the denominator $\sin(\pi - \alpha_i)$ in (58) and to obtain an $O(h^\alpha)$ estimate for (E), one must e.g. assume that $|g_{i-1} - g_i| \leq C \sin(\pi - \alpha_i) h^{\alpha - 1/2}$ along with $|B| \leq Ch$. Another possibility is to estimate $|B| \leq CLh \max_{i=1,\ldots,N} \sin(\pi - \alpha_i)$, which (under additional assumptions similar to (37)) eliminates the unpleasant denominator and assume $|g_{i-1} - g_i| \leq Ch^{\alpha - 1/2}$. This is possible for one band. For multiple bands as in Section 2.3, one must assume e.g. $|g_{i-1} - g_i| \leq C \sin(\pi - \alpha_i) h^{\alpha}$. We note that without any assumptions it naturally holds $|g_{i-1} - g_i| \leq \sin(\pi - \alpha_i) + \sin(\pi - \alpha_{i-1}) + \sin(\pi - \alpha_i)$, hence we assume an additional factor of e.g. $O(h^{\alpha - 1/2})$. Overall, these variants correspond to $B$ not lying on a straight line but with a restriction on how much it ‘bends’.
2.2 Examples and counterexamples – one band

Here we present several specific applications of Corollary \ref{cor:main}. For simplicity, we assume that the band $B \subset T_h$ consists of identical isosceles triangles with base length $h$ and height $\bar{h}$, cf. Figure \ref{fig:isosceles}. Such a regular structure is by no means required by Theorem \ref{thm:main}, however the evaluation of all quantities in estimate \ref{eq:main} is very simple in this case and also the specific examples can be directly compared to the results of \cite{1, 11}.

We assume that $B = (\bigcup_{i=0}^{N} K_i) \cup (\bigcup_{i=1}^{N} \tilde{K}_i)$ has length $L = (N + 1)h$ for some $N \in \mathbb{N}$. Then

$$N^{-1/2} \geq (N + 1)^{-1/2} = h^{1/2}L^{-1/2}. \quad (60)$$

For all $i$, we have $|\tilde{K}_i| = \frac{1}{2}h\bar{h}$. Finally, we need to estimate $\sin(\pi - \tilde{\alpha}_i)$. Since all $\tilde{K}_i$ are identical, we denote simply $\tilde{\alpha}_i = \tilde{\alpha}$.

Lemma 10. We have

$$\frac{1}{\sin(\pi - \tilde{\alpha})} > \frac{h}{4\bar{h}}. \quad (61)$$

Proof: Let $\tilde{K}$ be the triangle $ABC$ in Figure \ref{fig:proof}. Triangles $ABD$ and $ACE$ are similar, therefore

$$\frac{|CB|\sin(\pi - \tilde{\alpha})}{h} = \frac{|DB|}{h} = \frac{h}{|AC|} < 2,$$

since $\frac{1}{2}h < |AC|$. Hence $|CB|\sin(\pi - \tilde{\alpha}) < 2\bar{h}$. Since $\frac{1}{2}h < |CB|$, we get $\frac{1}{2}h\sin(\pi - \tilde{\alpha}) < 2\bar{h}$, which is \ref{eq:proof}.

Using \ref{eq:proof} and \ref{eq:proof2}, the left-hand side of the necessary condition \ref{eq:main} from Corollary \ref{cor:main} can be estimated as

$$\min_{i=1, \ldots, N} \frac{1}{\sin(\pi - \tilde{\alpha}_i)} \min_{i=1, \ldots, N} |\tilde{K}_i|^{1/2}LN^{-1/2} \geq \frac{h}{4\bar{h}} \left(\frac{1}{2}h\bar{h}\right)^{1/2}L(h^{1/2}L^{-1/2}) = \frac{1}{4\sqrt{2}}h^{2}h^{-1/2}L^{1/2}. \quad (62)$$
Corollary 9 along with (62) then gives us the necessary condition for $O(h^\alpha)$ convergence in the form

$$\frac{1}{4\sqrt{2}} h^2 \tilde{h}^{-1/2} L^{1/2} \leq C h^\alpha$$

(63)

for $C$ sufficiently small. Expressing $\tilde{h}$, we get the necessary condition

$$\tilde{h} \geq C h^{4-2\alpha} L$$

(64)

for $C$ sufficiently small.

### 2.2.1 Bands of minimal length

Theorem 1 requires that $L \geq C_L h^{2\alpha/5}$ for $C_L$ sufficiently large. If we take $L = C_L h^{2\alpha/5}$ in (64), i.e. the minimal possible length of $B$, we get the following.

**Lemma 11.** Let $B \subset T_h$ be a band of length $L = C_L h^{2\alpha/5}$ with the regular geometry considered above, then a necessary condition for (7) to hold, i.e. for $O(h^\alpha)$ convergence is

$$\tilde{h} \geq C h^{4-8\alpha/5}$$

(65)

for $C$ sufficiently large.

Of course, $L = C_L h^{2\alpha/5}$ need not be satisfied exactly for all $h$, we can have $L \sim C_L h^{2\alpha/5}$. As a straightforward consequence of Lemma 11, we get the following counterexamples:

**Counterexample 1:** Let $\{T_h\}_{h \in (0,h_0)}$ be such that a sequence of bands $B_k \subset T_{h_k}, k \in \mathbb{N}$, exists for $h_k \to 0$ as $k \to \infty$, with the regular geometry considered above. Let the length and shape parameters of each $B \in \{B_k\}_{k \in \mathbb{N}}$ satisfy

$$L = C_L h^{2\alpha/5}, \quad \tilde{h} = o(h^{4-8\alpha/5}),$$

(66)

for $C_L$ large enough, then (7) does not hold for $u$ given by (19) for any $C(u) \geq 0$, i.e. the FEM cannot have $O(h^\alpha)$ convergence on $\{T_h\}_{h \in (0,h_0)}$.

We note that since we essentially consider only the error on $B$, the counterexample is independent of the properties of the rest of the triangulation, i.e. $T_h \setminus B$ can be as 'nice' as possible, structured, uniform, etc.

As special cases of Counterexample 1, we take the most interesting values $\alpha = 1$ and $\alpha = 0$.

**Counterexample 2:** As in Counterexample 1, let there be an infinite sequence of bands $B \subset T_h$ with $h \to 0$ (we omit the subscript $k$ for simplicity) such that

$$L = C_L h^{2/5}, \quad \tilde{h} = o(h^{12/5}) = o(h^{2.4}),$$

(67)

then the FEM cannot have $O(h)$ convergence on $\{T_h\}_{h \in (0,h_0)}$ for general $u$. 
Counterexample 3: As in Counterexample 1, let there be an infinite sequence of bands $B \subset T_h$ with $h \to 0$ such that
\[ L \geq C_L, \quad \bar{h} = o(h^4), \tag{68} \]
for any fixed $C_L > 0$. Then the FEM does not converge in $H^1(\Omega)$ on $\{T_h\}_{h \in (0,h_0)}$ for general $u$.

Proof: Assume on the contrary that $|u - U|_1 \to 0$ as $h \to 0$. Choose $\varepsilon > 0$ arbitrary but fixed. Then for all $h$ sufficiently small $|u - U|_1 \leq \varepsilon$, i.e. estimate (7) holds with $\alpha = 0$, $C(u) = \varepsilon$. From Theorem 1, if $L \geq C_L$ sufficiently large w.r.t. $C(u) = \varepsilon$, we have
\[ |u - U|_1 \geq A_1 - A_2 \to \infty \quad \text{as} \quad h \to 0, \tag{69} \]
since $|A_2| \leq C_2$, a constant, and due to (62) and the assumption $\bar{h} = o(h^4)$, $A_1 \to \infty$ as $h \to 0$. Estimate (69) is in contradiction with the assumption $|u - U|_1 \leq \varepsilon$ for all $h$ sufficiently small. Hence $|u - U|_1 \not\to 0$.

We note that to be able to use Theorem 1, $C_L$ must be sufficiently large with respect to the fixed $\varepsilon$. However, for the sake of the proof by contradiction, $\varepsilon$ can be taken arbitrarily small. By Remark 9 for a given lower bound $C_L > 0$ for $L$ in (68), we can always choose $\varepsilon$ small enough so that this particular $C_L$ satisfies the assumptions of Theorem 1. The ability to take $C_L$ arbitrarily small but fixed is important, since for e.g. $\varepsilon = 1$ a band of length $C_L$ might not even fit into $\Omega$. □

2.2.2 Bands of constant length

Taking $L \geq C_L$ in Counterexample 3 was necessary for $\alpha = 0$ due to the condition $L \geq C_L h^{2\alpha/5}$ in Theorem 1. On the other hand, one can take $L \sim C_L$ for all $\alpha \in [0,1]$ instead of the minimal $L$ from Section 2.2.1. Taking $L = C_L$ in (64), we get:

Lemma 12. Let $B \subset T_h$ be a band of length $L = C_L$ with the regular geometry considered above, then a necessary condition for $O(h^{\alpha})$ convergence is
\[ \bar{h} \geq C h^{4-2\alpha} \tag{70} \]
for $C$ sufficiently large.

Counterexample 4: Let there be an infinite sequence of bands $B \subset T_h$ with $h \to 0$ such that
\[ L = C_L, \quad \bar{h} = o(h^{4-2\alpha}), \tag{71} \]
then the FEM cannot have $O(h^{\alpha})$ convergence on $\{T_h\}_{h \in (0,h_0)}$ for general $u$.

Again, as a special case, we take the most interesting value $\alpha = 1$. The case $\alpha = 0$ is covered by Counterexample 3.

Counterexample 5: Let there be an infinite sequence of bands $B \subset T_h$ with $h \to 0$ such that
\[ L = C_L, \quad \bar{h} = o(h^2), \tag{72} \]
then the FEM cannot have $O(h)$ convergence on $\{T_h\}_{h \in (0,h_0)}$ for general $u$. 20
2.3 Main estimate – multiple bands

Up to now, we dealt with estimates on one band only. However, more can be gained by considering the case when there are multiple bands \( \mathcal{B} \) in \( \mathcal{T}_h \). Specifically, we shall consider \( N_B \) disjoint bands \( \{ \mathcal{B}_b \}_{b=1}^{N_B} \subset \mathcal{T}_h \). We use the notation \( \mathcal{B} = \bigcup_{b=1}^{N_B} \mathcal{B}_b \). Of course, Lemmas 1–8 remain valid for each \( \mathcal{B}_b \). It is the analysis of Section 2.1 that needs to be modified to combine these individual estimates together. For example, simply summing the necessary condition (53) over all bands would not yield anything new. However, estimating the error on the whole union \( \bigcup_{b=1}^{N_B} \mathcal{B}_b \) gives the following stronger result of Theorem 2.

Since we now consider multiple bands, we need to take this into account in the notation, which is analogous to that of Section 2.1. Let \( \bigcup_{b=1}^{N_B} \mathcal{B}_b \subset \mathcal{T}_h \). For \( b = 1, \ldots, N_B \), let \( \mathcal{B}_b = (\bigcup_{i=0}^{N_i} K^b_i) \cup (\bigcup_{i=1}^{N_i} \tilde{K}^b_i) \) be the band with edge \( \Gamma_b \). We denote \( U_b^b = U|_{K^b_i} \) and \( \tilde{U}_b^b = U|_{\tilde{K}^b_i} \). By \( \mathbf{g}_b \) denote the unit vector in the direction of \( \Gamma_b \) and by \( L_b \) the length of \( \mathcal{B}_b \).

Similarly as in (35), we can reformulate Lemma 8 for \( \mathcal{B}_b \) as

\[
\sum_{i=1}^{N_i} \left( |\nabla U_b^b - \nabla U_{i-1}^b| \cdot \mathbf{g}_b \right)^2 \geq CL_b^2 N_b^{-1}. \tag{73}
\]

Again, we denote \( \nabla u_b = \nabla u(C_i^b) \), where \( C_i^b \) is the centroid of \( K^b_i \) and similarly for \( \tilde{K}^b_i \). We have approximation properties similar to (36). By \( \alpha_i^b, \tilde{\alpha}_i^b \) we denote the maximum angles of \( K^b_i \) and \( \tilde{K}^b_i \), respectively.

**Theorem 2.** Let \( u \) be given by (19) and let \( U \in X_h \) satisfy the error estimate (7). Let the set of bands \( \mathcal{B} = \bigcup_{b=1}^{N_B} \mathcal{B}_b \subset \mathcal{T}_h \) with \( \mathcal{B}_b = (\bigcup_{i=0}^{N_i} K^b_i) \cup (\bigcup_{i=1}^{N_i} \tilde{K}^b_i) \) be such that there exist constants \( C_K, C_\alpha \) independent of \( h \) such that for all \( b = 1, \ldots, N_B \) and \( i = 1, \ldots, N_i \)

\[
|\tilde{K}^b_i| \leq C_K |K^b_i|, \\
|\tilde{K}^b_i| \leq C_K |K^b_{i-1}|, \\
\sin(\pi - \alpha_i^b) \leq C_\alpha \sin(\pi - \tilde{\alpha}_i^b).
\tag{74}
\]

Finally, let \( L_b \geq C_L h^{2\alpha/5} \), for all \( b = 1, \ldots, N_B \), where \( C_L \) is sufficiently large. Then

\[
|u - U|_{H^1(\mathcal{B})} \geq A_1 - A_2, \tag{75}
\]

where

\[
A_1 \geq C_1 \left( \sum_{b=1}^{N_B} \min_{i=1, \ldots, N_i} \frac{1}{\sin^2(\pi - \alpha_i^b)} \min_{i=1, \ldots, N_b} |\tilde{K}^b_i| L_b^2 N_b^{-1} \right)^{1/2}, \tag{76}
\]

\[
|A_2| \leq C_2 h^\alpha.
\]

**Proof:** The proof is completely analogous to that of Theorem 1 therefore we only point out the main differences. Similarly as in (40), we consider the error...
only on the elements $\tilde{K}_i^b$:

$$|u - U|_{H^1(B)} \geq \left( \sum_{b=1}^{N_B} \sum_{i=1}^{N_h} \int_{\tilde{K}_i^b} |\nabla u - \nabla U|^2 \, dx \right)^{1/2} = A_1 - A_2,$$  \hspace{1cm} (77)

where the splitting of the error is given by

$$A_1 := \left( \sum_{b=1}^{N_B} \sum_{i=1}^{N_h} \frac{1}{\sin^2(\pi - \alpha_i^b)} \left| (\nabla U_i^b - \nabla U_{i-1}^b) \cdot g_b \right|^2 |\tilde{K}_i^b| \right)^{1/2},$$

$$A_2 := \left( \sum_{b=1}^{N_B} \sum_{i=1}^{N_h} \int_{\tilde{K}_i^b} |\nabla u - \nabla U|^2 \, dx \right)^{1/2} - A_1.$$  \hspace{1cm} (78)

From (35) we immediately get the first estimate of (76). Similarly as in (43), we estimate

$$A_2 \leq \left( \sum_{b=1}^{N_B} \sum_{i=1}^{N_h} \int_{\tilde{K}_i^b} |\nabla u - \nabla U_i^b|^2 \, dx \right)^{1/2} + \left( \sum_{b=1}^{N_B} \sum_{i=1}^{N_h} |\nabla U_i^b - \nabla U_{i-1}^b| |\tilde{K}_i^b| \right)^{1/2}$$

$$+ \left( \sum_{b=1}^{N_B} \sum_{i=1}^{N_h} |\nabla U_i^b - \nabla U_{i-1}^b| \right)^{1/2} - A_1$$

$$=: (A) + (B) + (C) + (D) - A_1.$$  \hspace{1cm} (79)

To demonstrate the differences between the proof of Theorems 1 and 2, we estimate only e.g. term (C). Similarly as in (46), we have

$$(C) \leq \left( C_K \sum_{b=1}^{N_B} \sum_{i=1}^{N_h} \int_{\tilde{K}_i^b} |\nabla U_i^b - \nabla u|^2 \, dx \right)^{1/2} + \left( C_K \sum_{b=1}^{N_B} \sum_{i=1}^{N_h} \int_{\tilde{K}_i^b} |\nabla u - \nabla U|^2 \, dx \right)^{1/2}$$

$$\leq C_h |B|^{1/2} + C_K |u - U|_{H^1(B)} \leq C h^\alpha$$  \hspace{1cm} (80)

due to (39) and (7). This is the main difference from the proof of Theorem 1, the estimation $|B|^{1/2} \leq |\Omega|^{1/2}$ instead of $|B|^{1/2} \leq |\Omega|^{1/2}$ and using the error estimate (7) on the whole union $B$ instead of individual bands $B_b$ and then summing over $b$, which would yield the undesired estimate $N_B Ch^\alpha$. All the other terms can be estimated as in Theorem 1 with this in mind. \hfill \Box

From Theorem 2 we get a necessary condition for $O(h^\alpha)$ convergence similar to Corollary 9.

**Corollary 13.** A necessary condition for (7) to hold is that every set of bands $B \subset \mathcal{T}_h$ satisfying the assumptions of Theorem 7 must satisfy

$$\left( \sum_{b=1}^{N_B} \min_{i=1,\ldots,N_h} \frac{1}{\sin^2(\pi - \alpha_i^b)} \min_{i=1,\ldots,N_b} \left| \tilde{K}_i^b \right| L_b^2 N_b^{-1} \right)^{1/2} \leq C h^\alpha$$  \hspace{1cm} (81)

for $C$ sufficiently small.
2.4 Examples and counterexamples – multiple bands

Here we present some applications of Corollary 13. As in Section 2.2, we assume for simplicity that the bands $B_b \subset T_h$ consists of identical isosceles triangles with base length $h$ and height $\bar{h}$. Moreover, we assume that the union of clusters $B$ has the structure of the Babuška-Aziz counterexample, cf. Figure 8. Again we note that such a regular structure of the bands is not required by Theorem (1) and the bands do not need to lie next to each other as in Figure 8. We consider such a simple geometry to make the estimates as clear as possible and directly comparable to the optimal results of [11] on the Babuška-Aziz counterexample.

Similarly as in Section 2.2, we have the estimates

$$N_b^{-1} \geq hL^{-1}, \quad |\tilde{K}_i^b| = \frac{1}{2} h\bar{h}, \quad \frac{1}{\sin^2(\pi - \tilde{\alpha}_b)} > \frac{h^2}{16\bar{h}^2}. \quad (82)$$

Finally, due to the structure of $B$ considered, we have $N_B = L\bar{h}^{-1}$.

Then the left-hand side of the necessary condition (81) from Corollary 13 can be estimated as

$$\left( \sum_{b=1}^{N_B} \min_{i=1,\ldots,N_b} \frac{1}{\sin^2(\pi - \tilde{\alpha}_b)} \min_{i=1,\ldots,N_b} |\tilde{K}_i^b| L_b^2 N_b^{-1} \right)^{1/2} \geq \left( N_B \frac{h^2}{16\bar{h}^2} \left( \frac{1}{2} h\bar{h} \right) L^2 (hL^{-1}) \right)^{1/2} = \left( (L\bar{h}^{-1}) \frac{1}{32} h^4 \bar{h}^{-1} L \right)^{1/2} = \frac{1}{4\sqrt{2}} h^2 \bar{h}^{-1} L. \quad (83)$$

Corollary 13 along with (83) then gives us the necessary condition for $O(h^\alpha)$ convergence in the form

$$\frac{1}{4\sqrt{2}} h^2 \bar{h}^{-1} L \leq Ch^\alpha \quad (84)$$

for $C$ sufficiently small. Expressing $\bar{h}$ as in (64), we get the necessary condition

$$\bar{h} \geq Ch^{2-\alpha} L \quad (85)$$
for $C$ sufficiently small.

### 2.4.1 Multiple bands of minimal length

As in Lemma 11, we can take the minimal length $L = C_L h^{2\alpha/5}$ in (85) to obtain:

**Lemma 14.** Let $B \subset T_h$ be a set of bands of length $L = C_L h^{2\alpha/5}$ with the regular geometry considered above, then a necessary condition for $O(h^\alpha)$ convergence is

$$\bar{h} \geq C h^{2-3\alpha/5} \quad (86)$$

for $C$ sufficiently large.

**Counterexample 6:** Let there be an infinite sequence of sets of bands $B \subset T_h$ with $h \to 0$ (again omitting the subscript $k$ for simplicity) such that

$$L = C_L h^{2/5}, \quad \bar{h} = o(h^{7/5}) = o(h^{1.4}), \quad (87)$$

then the FEM cannot have $O(h)$ convergence on $\{T_h\}_{h \in (0,h_0)}$ for general $u$.

**Counterexample 7:** Let there be an infinite sequence of sets of bands $B \subset T_h$ with $h \to 0$ such that

$$L \geq C_L, \quad \bar{h} = o(h^2), \quad (88)$$

for any fixed $C_L > 0$. Then the FEM does not converge in $H^1(\Omega)$ on $\{T_h\}_{h \in (0,h_0)}$ for general $u$.

### 2.4.2 Multiple bands of constant length

By taking $L \geq C_L$ in (85), we get:

**Lemma 15.** Let $B \subset T_h$ be a set of bands of length $L = C_L$ with the regular geometry considered above, then a necessary condition for $O(h^\alpha)$ convergence is

$$\bar{h} \geq C h^{2-\alpha} \quad (89)$$

for $C$ sufficiently large.

**Counterexample 8:** Let there be an infinite sequence of sets of bands $B \subset T_h$ with $h \to 0$ such that

$$L = C_L, \quad \bar{h} = o(h), \quad (90)$$

then the FEM cannot have $O(h)$ convergence on $\{T_h\}_{h \in (0,h_0)}$ for general $u$.

**Remark 11.** We note that the result of Lemma 15 is optimal when applied to the Babuška-Aziz counterexample, since in [11] it is proven that in this case (using our notation)

$$|u - U|_1 \sim \min(1, h^2 / \bar{h}). \quad (91)$$

If we want to obtain a condition on $\bar{h}$ for $O(h^\alpha)$-convergence from (91), we get exactly (89).
In the Babuška-Aziz counterexample, \cite{11}, the unit square is divided into a triangulation consisting of bands of the considered type, as in Figure 8. If we consider only one band in $T_h$, as in Section 2.2, we need stronger assumptions to produce the corresponding counterexamples. For example, comparing Counterexamples 5 and 8, if there is only one band we need $\bar{h} = o(h^2)$ for counterexamples to $O(h)$ convergence with $L = C_L$, however only $\bar{h} = o(h)$ is needed if we have multiple bands.

Remark 12. Another possibility how to view the necessary conditions \cite{63} and \cite{84} is by expressing these conditions not for $\bar{h}$, but for $\pi - \tilde{\alpha}_i$. Trivially, $\pi - \tilde{\alpha}_i > \sin(\pi - \tilde{\alpha}_i)$. Moreover, similarly as in Lemma 10 it is possible to prove that $\sin(\pi - \tilde{\alpha}_i) > \bar{h}/h$. Combining these results with \cite{63} gives us the necessary condition for $O(h^\alpha)$-convergence

$$\pi - \tilde{\alpha}_i \geq Ch^{3-2\alpha}L$$

for all $i = 1, \ldots, N_b$ and for $C$ sufficiently small. Similarly, from \cite{84}, we get the necessary condition

$$\pi - \tilde{\alpha}_b \geq Ch^{1-\alpha}L$$

for all $i = 1, \ldots, N_b$, $b = 1, \ldots, N_B$ and for $C$ sufficiently small. These angle conditions are mentioned in the Introduction.

3 A sufficient condition for FEM convergence

A basic tool in FEM error analysis is Céa’s lemma, cf. \cite{3}. In fact, this seems to be the only general tool in FEM analysis that is completely independent of the geometry of $T_h$.

Lemma 16 (Céa).

$$|u - U|_1 = \inf_{v_h \in V_h} |u - v_h|_1.$$ \hfill (92)

In a priori error analysis, one typically proceeds by taking $v_h$ in \eqref{92} to be the Lagrange interpolation of $u$. This choice leads to the well known maximum angle condition for $O(h)$ convergence and the newer, lesser known circumradius condition \cite{8}, \cite{12} for convergence of the FEM. We note that by using Céa’s lemma, we are again dealing only with the approximation properties of the space $V_h$ (or $X_h$), the specific form of the FEM scheme is not taken into account, cf. Remark 3.

We proceed as follows. Since Lagrange interpolation gives suitable estimates only on elements satisfying the maximum angle condition, we will use a modified version of Lagrange interpolation on the remaining elements. This is done by dropping the interpolation condition at the maximum angle vertex of $K$. The question is then how to connect such interpolants together continuously. For this purpose we introduce the concept of a correction function which is used to construct a globally continuous function from $V_h$. Basically, we modify the interpolated function $u$ locally so that this new function $\tilde{u}$ is $O(h)$-close to $u$ and can be interpolated by the standard Lagrange procedure with an $O(h)$ error on $T_h$. 

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3.1 Standard Lagrange interpolation

Let $K \in \mathcal{T}_h$ have vertices $A_K, B_K, C_K$, where $A_K$ will denote the maximum angle vertex of $K$ throughout this section, cf. Figure 9. For each element $K \in \mathcal{T}_h$, we seek $v^K_h \in P^1(K)$ such that the following conditions hold:

\begin{align}
    v^K_h(A_K) &= u(A_K), \\
    v^K_h(B_K) &= u(B_K), \\
    v^K_h(C_K) &= u(C_K),
\end{align}

(93)

For the $H^1$-error of such an interpolation, we have the following result, [1], [9]:

**Lemma 17.** Let $\alpha_0 < \pi$ be fixed. Let $K \in \mathcal{T}_h$ with maximum angle $\alpha_K \leq \alpha_0$. Then for $v^K_h$ defined by (93), we have

\[ |u - v^K_h|_{H^1(K)} \leq C(\alpha_0) h_K |u|_{H^2(K)}, \]

(94)

where $C(\alpha_0)$ is independent of $u, K$.

We now define the (global) Lagrange interpolation of $u$ as the function $\Pi_h u \in V_h$ such that $(\Pi_h u)|_K = v^K_h$ for all $K \in \mathcal{T}_h$. By taking $\Pi_h u$ in Céa’s lemma, we get the maximum angle condition.

**Theorem 3** (Maximum angle condition, [1], [9]). Let $\alpha_K \leq \alpha_0 < \pi$ for all $K \in \mathcal{T}_h, h \in (0, h_0)$. Then

\[ |u - U|_1 \leq C(\alpha_0) h |u|_2. \]

The maximum angle condition is a condition for $O(h)$ convergence, which is the subject of Section 3. The general case of $O(h^\alpha)$ convergence (as in Section 2) is a simple extension treated in Remark 18. In this case, the maximum angle condition is too strong an assumption. Recently, a generalization of Lemma 17 and the maximum angle condition was derived, cf. [8], [12].

**Lemma 18.** Let $R_K \leq 1$ be the circumradius of the triangle $K \in \mathcal{T}_h$. Then

\[ |u - v^K_h|_{H^1(K)} \leq CR_K |u|_{H^2(K)}. \]

(95)

**Theorem 4** (Circumradius estimate, [8], [12]).

\[ |u - U|_1 \leq C \max_{K \in \mathcal{T}_h} R_K |u|_2. \]

The circumradius condition of [8] then reads

\[ \max_{K \in \mathcal{T}_h} R_K \to 0 \]

and is a condition on convergence (not $O(h)$ convergence) of the FEM by Theorem 3. We note that the law of sines states

\[ \frac{h_K}{\sin \alpha_K} = 2R_K. \]
If we substitute this expression into (95), we get $O(h)$ convergence if and only if the denominator $\sin \alpha_K$ is uniformly bounded away from zero for all $K$, which is exactly the maximum angle condition. Therefore, as far as $O(h)$ convergence is concerned, Lemmas 17 and 18 are equivalent.

In [6], it is shown that the maximum angle condition is not necessary for $O(h)$ convergence of the finite element method, i.e. $T_h$ can contain elements whose maximum angle degenerates to $\pi$ with respect to $h \to 0$. The argument in [6] can also be used to show that the circumradius condition is not necessary for convergence, i.e $T_h$ can contain arbitrarily ‘bad’ elements. In the following section, we modify the Lagrange interpolation on such elements to obtain a generalization of the maximum angle and circumradius conditions.

3.2 Modified Lagrange interpolation

We split $T_h$ into two subsets of elements: those satisfying the maximum angle condition and those not satisfying this condition. Formally, we choose $\alpha_0 < \pi$, and construct $T_h^1, T_h^2$:

$$T_h^1 = \{ K \in T_h : \alpha_K \leq \alpha_0 \},$$
$$T_h^2 = T_h \setminus T_h^1. \quad (96)$$

Since elements from $T_h^1$ satisfy the maximum angle condition, we can use Lagrange interpolation on them to obtain the element-wise $O(h)$ estimate implied by Lemma 17. However, on $T_h^2$, Lemma 17 does not hold, therefore we modify the Lagrange interpolation conditions in this way: We seek $v^K_h \in P^1(K)$ such that

$$\nabla v^K_h(x_K), v_2 = (\nabla u(x_K), v_2),$$
$$v^K_h(B_K) = u(B_K),$$
$$v^K_h(C_K) = u(C_K). \quad (97)$$

Here $v_1 = \frac{C_K - B_K}{|C_K - B_K|}$ is the unit vector in the direction of the edge $(C_K, B_K)$, vector $v_2$ is the unit vector perpendicular to $v_1$ and $x_K$ is the foot of the altitude from vertex $A_K$ (cf. Figure 9). The first condition from (97) says that $v^K_h$ and $u$ have the same derivative in the direction $v_2$ at point $x_K$. The following estimates hold.

Lemma 19. For $K \in T_h^2$ and $v^K_h$ defined by (97), we have

$$|u - v^K_h|_{H^1(K)} \leq \sqrt{13h_K|u|_{2, \infty}|K|^{1/2}}, \quad (98)$$
$$|u(A_K) - v^K_h(A_K)| \leq (|x_K - B_K||x_K - C_K| + |x_K - A_K|^2)|u|_{2, \infty}. \quad (99)$$

Proof: $H^1(K)$-estimate: Let $x \in K$. Since $v_1$ and $v_2$ are orthogonal vectors of unit length, we can write

$$\nabla u(x) = (\nabla u(x), v_1)v_1 + (\nabla u(x), v_2)v_2,$$
$$\nabla v^K_h(x) = \frac{u(C_K) - u(B_K)}{|C_K - B_K|}v_1 + (\nabla u(x), v_2)v_2.$$
Figure 9: Lagrange interpolation conditions – left. Modified Lagrange interpolation – right.

The expression for $\nabla v_h^K$ is obtained from the fact that $(\nabla v_h^K(x), v_1)$, the coefficient at $v_1$, is the directional derivative of $v_h^K$ in the direction $v_1$, which is $u(C_K)-u(B_K)$, due to (97). Since $v_1 \perp v_2$,

$$|\nabla(u-v_h^K)(x)|^2 = \left( (\nabla u(x), v_1) - \frac{u(C_K)-u(B_K)}{|C_K-B_K|} \right)^2 + \left( (\nabla u(x), v_2) - (\nabla u(x_K), v_2) \right)^2 =: (A) + (B).$$

Estimate of $(B)$: By the multivariate Taylor expansion, we have

$$\frac{\partial u}{\partial x_j}(x) = \frac{\partial u}{\partial x_j}(x_K) + (x-x_K) \cdot \nabla \frac{\partial u}{\partial x_j}(\xi) \Rightarrow |\frac{\partial u}{\partial x_j}(x) - \frac{\partial u}{\partial x_j}(x_K)| \leq h_K \sqrt{2} |u|_{2,\infty},$$

where $\xi$ lies on the line between $x$ and $x_K$. Therefore

$$|\nabla u(x) - \nabla u(x_K)| \leq \sqrt{2} h_K \sqrt{2} |u|_{2,\infty} = 2h_K |u|_{2,\infty}.$$

We can conclude that

$$(B) = (\nabla u(x) - \nabla u(x_K), v_2)^2 \leq |\nabla u(x) - \nabla u(x_K)|^2 |v_2|^2 \leq 4h_K^2 |u|_{2,\infty}^2.$$

Estimate of $(A)$: We have

$$(A) = \left( \frac{\nabla u(x) - \nabla u(B_K), v_1}{(A_1)} \right) + \left( \frac{\nabla u(B_K), v_1}{(A_2)} - \frac{u(C_K)-u(B_K)}{|C_K-B_K|} \right)^2.$$

The term $(A_1)$ can be estimated similarly to $(B)$, hence $|(A_1)| \leq 2h_K |u|_{2,\infty}$. As for $(A_2)$, by the multivariate Taylor expansion,

$$u(C_K) = u(B_K) + (C_K-B_K) \cdot \nabla u(B_K) + \frac{1}{2} (C_K-B_K)^T \nabla^2 u(\xi)(C_K-B_K),$$

where $\nabla^2 u$ is the Hessian matrix of $u$. Therefore, by the definition of $v_1$,

$$(A_2) = \frac{\nabla u(B_K) - (C_K-B_K)}{|C_K-B_K|} - \frac{u(C_K)-u(B_K)}{|C_K-B_K|} = -\frac{1}{2|C_K-B_K|} (C_K-B_K)^T \nabla^2 u(\xi)(C_K-B_K).$$

Thus $|(A_2)| \leq h_K |u|_{2,\infty}$. Finally

$$|\nabla(u-v_h^K)(x)|^2 = ((A_1) + (A_2))^2 + (B) \leq 13h_K^2 |u|_{2,\infty}^2.$$
Therefore, we can estimate the $H^1(K)$-error of $v_h^K$:

$$ |u - v_h^K|_{H^1(K)}^2 = \int_K |\nabla(u - v_h^K)(x)|^2 \, dx \leq 13h_K^2 |u|_{2,\infty}|K|. $$

**Vertex estimate:** On the edge $(B_K, C_K)$, the function $v_h^K$ is simply the two-point linear Lagrange interpolation of $u$. For this we have the well known error expression $e(x) = \frac{1}{2}(x - x_0)(x - x_1)f''(\xi)$, where $f$ is the interpolated function and $\xi$ is between $x_0$ and $x_1$, cf. [4]. In our case, $f$ is the restriction of $v_h^K - u$ to the edge $(B_K, C_K)$ and $f''(\xi) = v_1^2\nabla^2(v_h^K - u)(\xi)v_1 = -v_1^T\nabla^2u(\xi)v_1$ is the second derivative in the direction $v_1$. Therefore

$$ |v_h^K(x_K) - u(x_K)| \leq \frac{1}{2}|x_K - B_K||x_K - C_K||v_1^T\nabla^2u(\xi)v_1| \leq |x_K - B_K||x_K - C_K||u|_{2,\infty}, \quad (100) $$

since $|v_1| = 1$. By the multidimensional Taylor expansion, we have

$$ |u(A_K) - v_h^K(A_K)| \leq |u(x_K) - v_h^K(x_K)| + |(\nabla u(x_K) - \nabla v_h^K(x_K)) \cdot (A_K - x_K)| + \frac{1}{2}|(A_K - x_K)^T\nabla^2u(\xi)(A_K - x_K)| = (C_1) + (C_2) + (C_3). $$

Term $(C_1)$ can be estimated by (100). For $(C_2)$, we have

$$ (C_2) = |(\nabla u(x_K) - \nabla v_h^K(x_K)) \cdot \frac{A_K - x_K}{|A_K - x_K|}|A_K - x_K|| $$

$$ = |(\nabla u(x_K) - \nabla v_h^K(x_K)) \cdot v_2||A_K - x_K|| = 0, $$

due to the first condition in (97). Finally, we have $(C_3) \leq |A_K - x_K|^2|u|_{2,\infty}$. Collecting these estimates gives us (99).

**Remark 13.** In (98), instead of the expected $|u|_{H^2(K)}$, we have $|u|_{2,\infty}|K|^{1/2}$. This term is an upper bound for $|u|_{H^2(K)}$ and mimics its behavior in that summing its squares over $K \in T_h$ gives $|u|_{2,\infty}^2|\Omega|$, a constant. In this sense, (98) is an $O(h_K)$ estimate.

**Remark 14.** A crude estimate of (99) gives us $|u(A_K) - v_h^K(A_K)| \leq 2h_K^2|u|_{2,\infty}$. In other words, we can fix the $O(h_K)$ interpolation property of the Lagrange interpolation in $H^1(K)$ if we make an $O(h_K^2)$ perturbation to the interpolated value of $u$ at $A_K$. At $B_K$ and $C_K$ we interpolate $u$ exactly.

If $v_h^K$ is constructed by (97), then $v_h^K(A_K) \neq u(A_K)$, but by Lemma 19 the difference of these values is $O(h_K^2)$. If $A_K$ also belongs to some element $K' \in T_h$, then $v_h^{K'}(A_K) = u(A_K)$, by the conditions (93). Therefore, the global $V_h$-interpolation would not be continuous at $A_K$. We could fix the value $v_h^K(A_K)$ and modify $v_h^{K'}$ so that $v_h^{K'}(A_K) = v_h^K(A_K)$. That would mean imposing more restrictive conditions on $K'$ so that $v_h^{K'}$ still satisfies estimate (94), even though we changed one of its vertex values by $O(h_K^2)$. For example, if the neighboring element $K'$ was very small, e.g. diam $K' = O(h_K^2)$, then a change of vertex value of $O(h_K^2)$ would lead to an $O(1)$ change in $\nabla v_h^{K'}$. 29
Another possibility is the following. We want to “distribute” the $O(h^2)$ perturbation of the vertex value to a neighborhood of $A_K$, not only to the immediately neighboring element. If we can do this smoothly and locally, we can manage to preserve the $O(h)$ interpolation estimates on elements in that neighborhood. For this purpose we construct a so-called correction function for the modified Lagrange interpolation.

**Definition 20.** Let $\Omega_1 = \cup_{K \in T_h^2} K$ be the subset of $\Omega$ containing $T_h^2$. We call $w : \Omega \to \mathbb{R}$ a correction function corresponding to $u$ and $T_h$, if $w \in C(\bar{\Omega})$, $w \in H^2(\Omega_1)$ and

\[
\begin{align*}
(i) & \ w(x) = v_h^K(x) - u(x) \text{ for all } K \in T_h^2 \text{ and all } x \in \{A_K, B_K, C_K\}, \\
(ii) & \ |w|_{H^1(\Omega_1)} \leq C(u)h, \\
(iii) & \ |w|_{H^2(\Omega_1)} \leq C(u).
\end{align*}
\]

**Remark 15.** In our context, the constants $C(u)$ in (102), (103) will take on the form $C(u) = C(|u|_2 + |w|_{2,\infty})$, where $C$ is independent of $h, u$.

**Remark 16.** Since $w \in C(\bar{\Omega})$ and (i) holds, for every $K \in T_h^2$ the maximum-angle vertex $A_K$ cannot be the vertex of any other $K' \in T_h^2$, since (i) would prescribe two different values of $w$ at this point. On the other hand, $K$ and $K'$ can share the $B, C$ vertices, so we can have e.g. $B_K = C_{K'}$. This is because (i) prescribes $w(B_K) = w(C_{K'}) = 0$ due to (97). This is used in Lemma 24.

Now we shall proceed as follows. Instead of taking $v_h := \Pi_h u$ in (92), we shall take $v_h := \Pi_h \tilde{u}$, where $\tilde{u} = u + w$. Since $w \in C(\bar{\Omega})$, $\Pi_h \tilde{u}$ is well defined. Moreover, due to condition (101) for $K \in T_h^2$ and $x \in \{A_K, B_K, C_K\}$

\[
\tilde{u}(x) = u(x) + v_h^K(x) - u(x) = v_h^K(x).
\]

Thus $(\Pi_h \tilde{u})|_K = v_h^K$ for all $K \in T_h^2$, hence $u - \Pi_h \tilde{u}$ satisfies the estimates of Lemma 19 on $T_h^2$. Finally, on $T_h^2$, where $\Pi_h u$ is a good approximation of $u$, $|w|_{H^1(\Omega_1)} = O(h)$, hence $\tilde{u} - u = O(h)$ in $H^1(\Omega)$, and something similar can be expected to hold for $\Pi_h \tilde{u} - \Pi_h u$.

**Lemma 21.** Let $u \in W^{2,\infty}(\Omega)$ and $w$ be as in Definition 20. Let $\tilde{u} = u + w$. Then

\[
|u - \Pi_h \tilde{u}|_1 \leq C(u)h,
\]

where $C(u)$ is independent of $h$.

**Proof:** We have

\[
|u - \Pi_h \tilde{u}|_1^2 = \sum_{K \in T_h^2} |u - \Pi_h \tilde{u}|_{H^1(K)}^2 + \sum_{K \in T_h^2} |u - \Pi_h \tilde{u}|_{H^1(K)}^2.
\]

Since $(\Pi_h \tilde{u})|_K = v_h^K$ for all $K \in T_h^2$, due to Lemma 19 we have

\[
\sum_{K \in T_h^2} |u - \Pi_h \tilde{u}|_{H^1(K)}^2 \leq \sum_{K \in T_h^2} 13h_K^2 |u|_{2,\infty}^2 |K| \leq 13h^2 |\Omega| |u|_{2,\infty}^2.
\]
On the other hand, due to (102), (103) and Lemma 17
\[
\sum_{K \in T_h^1} |u - \Pi_h \tilde{u}|^2_{H^1(K)} \leq 2 \sum_{K \in T_h^1} \left( |u - \tilde{u}|^2_{H^1(K)} + |\tilde{u} - \Pi_h \tilde{u}|^2_{H^1(K)} \right)
\leq 2 \sum_{K \in T_h^1} \left( |w|^2_{H^1(K)} + C h^2 \| \tilde{u} \|^2_{H^1(K)} \right) \leq 2 |w|^2_{H^1(\Omega_1)} + C h^2 (|u|^2 + |w|^2_{H^2(\Omega_1)})
\leq (C(u) + |u|^2) h^2.
\]
We note that Lemma 17 can be applied since $T_h^1$ consists of elements satisfying the maximum angle condition, cf. (96). Combining the last three estimates gives us the desired result. □

3.3 Construction of the correction function $w$

We shall construct $w$ as a linear combination of disjoint local ‘bumps’. We define the cubic spline function $\tilde{\varphi} : [0, \infty) \to \mathbb{R}$ as
\[
\tilde{\varphi} = \begin{cases} 
2(x-1)^3 + 3(x-1)^2, & x \in [0,1], \\
0, & x > 1.
\end{cases}
\]
This function satisfies $\tilde{\varphi} \in C^1(0, \infty) \cap H^2(0, \infty)$ and $\tilde{\varphi}(0) = 1$, $\tilde{\varphi}'(0) = 0$, $\tilde{\varphi}(1) = 0$, $\tilde{\varphi}'(1) = 0$. Its derivatives are bounded by $|\tilde{\varphi}'| \leq \frac{3}{2}$, $|\tilde{\varphi}''| \leq 6$. Using this function, we construct a local 2D bump with radius $r$:
\[
\varphi_r(x) = \tilde{\varphi}(\frac{|x|}{r}),
\]
cf. Figure 10. We have $\varphi_r \in C^1(\mathbb{R}^2) \cap H^2(\mathbb{R}^2)$ and $\text{supp} \varphi_r = B_r(0)$, the disk with diameter $r$ centered at 0.

Lemma 22. For all $x \in \mathbb{R}^2$
\[
|\nabla \varphi_r(x)| \leq \frac{3}{2r},
\]
\[
\|\nabla^2 \varphi_r(x)\|_F \leq \frac{36}{r^2},
\]
where $\| \cdot \|_F$ is the matrix Frobenius norm.
Figure 11: Support of correction function $w_K$ for three different elements $K$ with the same maximum angle $\alpha_K$.

**Proof:** The presence of $1/r$ and $1/r^2$ can be seen from simple scaling arguments. The specific constants in the estimates can be obtained by a straightforward, yet somewhat tedious calculation. □

First, we consider the case when $T_h^2$ consists of only one element $K$. Then $w$, corresponding to the single element $K$, can be constructed e.g. by

$$w_K(x) = (v_h^K(A_K) - u(A_K)) \varphi_{r_K}(x - A_K),$$

$$r_K = \frac{1}{2} \min\{|A_K - B_K|, |A_K - C_K|\}.$$  \hspace{1cm} (105)

The support of $w_K$ is shown in Figure 11. We note that the constant $\frac{1}{2}$ in (106) can be replaced by any $c \in (0, 1]$, however we must keep in mind that this constant will appear in the denominator in estimates of Lemma 22. Finally, it is important to note that $B_K$ nor $C_K$ never lie in $\text{supp} w_K$, since $r_K$ is half the distance from $A_K$ to the nearer of $B_K, C_K$.

**Lemma 23.** The function $w_K$ defined by (105) is a correction function for $T_h^2$ consisting of one element $K$.

**Proof:** Since $\varphi_{r_K}(0) = 1$, we have $w_K(A_K) = v_h^K(A_K) - u(A_K)$. Furthermore, since $B_K, C_K \notin \text{supp} \varphi_{r_K}(-A_K)$, we have $w_K(x) = 0 = v_h^K(x) - u(x)$ for $x = B_K, C_K$, since in these two points $v_h^K$ is an exact interpolation of $u$, due to (97). Hence (101) is valid.

Since $w_K \in H^2(\Omega)$, the regularity condition is satisfied. It remains to prove (102), (103). Due to (106), either $|A_K - B_K| = 2r_K$ and $|A_K - C_K| \leq h_K$ or vice versa. Moreover $|x_K - A_K| \leq 2r_K \leq h_K$. Hence, by (99)

$$|v_h^K(A_K) - u(A_K)| \leq (|x_K - B_K||x_K - C_K| + |x_K - A_K|^2)|u|_{2,\infty}$$

$$\leq (|A_K - B_K||A_K - C_K| + |x_K - A_K|^2)|u|_{2,\infty} \leq 4r_K h_K |u|_{2,\infty}.$$  \hspace{1cm} (106)

By Lemma 22

$$\left|w_K\right|_1^2 = \int_{B_{r_K}(A_K)} |\nabla w_K(x)|^2 \, dx \leq \int_{B_{r_K}(A_K)} \left|v_h^K(A_K) - u(A_K)\right|^2 \left(\frac{3}{2r_K}\right)^2 \, dx$$

$$\leq \left|B_{r_K}(A_K)\right|(4r_K h_K |u|_{2,\infty})^2 \left(\frac{3}{2r_K}\right)^2 \leq |B_{r_K}(A_K)|36h_K^2 |u|_{2,\infty}^2.$$  \hspace{1cm} (107)
Therefore $|w_K|_1 \leq |\Omega|^{1/2} 6h |u|_{2,\infty} = C(u)h$. Similarly,

$$|w_K|_2^2 = \int_{B_{r_K}(A_K)} \|\nabla w_K(x)\|^2_F \, dx \leq \int_{B_{r_K}(A_K)} |v_K^h(A_K) - u(A_K)|^2 \left( \frac{36}{r_K^2} \right)^2 \, dx$$

$$\leq |B_{r_K}(A_K)| (4r_Kh_K |u|_{2,\infty})^2 \left( \frac{36}{r_K^2} \right)^2 \leq \pi r_K^2 144^2 \left( \frac{h_K}{r_K} \right)^2 |u|^2_{2,\infty}$$

$$= \pi 144^2 h_K^2 |u|_{2,\infty}^2.$$

Therefore $|w_K|_2 \leq \pi^{1/2} 144 h_K |u|_{2,\infty} = C(u)h \leq C(u)$. □

In the proof of Lemma 23, we have used the very crude estimates $|B_{r_K}(A_K)| \leq |\Omega|$ in the estimation of $|w_K|_1$ and $h_K \leq 1$ in the estimation of $|w_K|_2$. However, these quantities can be used in an additive way in the case that $T_h^2$ consists of multiple elements, where the correction function $w$ is simply the sum of all $w_K$. One must however take care that the resulting $w$ satisfies Definition 20.

**Lemma 24.** Let $T_h^2$ be such that

(i) $\sum_{K \in T_h^2} h_K^2 \leq C,$

(ii) $|A_K - A_{K'}| \geq r_K + r_{K'}$ for $K, K' \in T_h^2,$

(iii) $|A_K - B_{K'}| \leq r_K, |A_K - C_{K'}| \geq r_K$ for $K, K' \in T_h^2,$

where $r_K, r_{K'}$ are defined by (106). Then

$$w(x) := \sum_{K \in T_h^2} w_K(x)$$

(108)

is a correction function for $T_h^2$, where $w_K$ is defined by (105).

**Proof:** Condition (ii) merely states that $\text{supp } w_K \cap \text{supp } w_{K'} = \emptyset$ and condition (iii) states that no vertex $B_{K'}$ or $C_{K'}$ of another element $K'$ is contained in $\text{supp } w_K$. Together, (ii) and (iii) ensure that condition (101) is satisfied, since for any $A_K, B_K, C_K$ of any element $K \in T_h^2$, the sum (108) contains at most one nonzero term giving the correct value of $w$ in that particular point: $w(B_K) = w(C_K) = 0$ and $w(A_K) = v_K^h(A_K) - u(A_K)$ for all $K \in T_h^2$.

As for (102) and (103),

$$|w|_1^2 = \sum_{K \in T_h^2} \int_{B_{r_K}(A_K)} |\nabla w_K(x)|^2 \, dx \leq \sum_{K \in T_h^2} |B_{r_K}(A_K)|(4r_Kh_K |u|_{2,\infty})^2 \left( \frac{3}{2r_K} \right)^2$$

$$\leq |\Omega| 36h^2 |u|_{2,\infty}^2,$$

similarly as in (107). Therefore $|w|_1 \leq |\Omega|^{1/2} 6h |u|_{2,\infty} = C(u)h$. Finally,

$$|w|_2^2 = \sum_{K \in T_h^2} \int_{B_{r_K}(A_K)} \|\nabla w_K(x)\|^2_F \, dx \leq \sum_{K \in T_h^2} |B_{r_K}(A_K)|(4r_Kh_K |u|_{2,\infty})^2 \left( \frac{36}{r_K^2} \right)^2$$

$$= \sum_{K \in T_h^2} \pi r_K^2 144^2 \left( \frac{h_K^2}{r_K^2} \right) |u|^2_{2,\infty} \leq \pi 144^2 C |u|_{2,\infty}^2,$$
Figure 12: Chain of degenerating elements with supp $w_K$ depicted for each element.

Figure 13: Band with subdivided even elements satisfying Theorem 5.

due to condition $(i)$. Therefore $|w|_2 \leq \pi^{1/2}144C^{1/2}|u|_{2,\infty} = C(u)$.

\begin{proof}

Theorem 5. Let $T_h = T^1_h \cup T^2_h$, where $T^1_h$ satisfies the maximum angle condition and $T^2_h$ satisfies the conditions of Lemma 24. Then $|u - U|_1 \leq C(u)h$ for all $u \in W^{2,\infty}(\Omega)$.

Proof: The proof is a trivial consequence of Lemmas 24 and 21 and Céa’s lemma.
\end{proof}

Example 1: In Theorem 5, the maximum angle vertex $A_K$ of $K \in T_h^2$ must be sufficiently far from all other vertices of $T_h^2$, as in conditions $(ii), (iii)$. Since in $B_K, C_K$ the function $u$ is interpolated exactly, there is no condition at these vertices and elements from $T_h^2$ can be connected by these vertices. Still, one must be careful that supp $w_K \cap$ supp $w_{\tilde{K}} = \emptyset$, and that supp $w_K$ does not contain any other vertices in $T_h^2$. Once this is satisfied, elements from $T_h^2$ can be linked together to form arbitrarily long chains such as in Figure 12. We note that the rest of the triangulation not indicated in Figure 12, i.e. $T^1_h$, can have an arbitrary structure, as long as it satisfies the maximum angle condition. Such triangulations then satisfy the assumptions of Theorem 5.

Example 2: Theorem 5 does not allow $T_h^2$ to contain the structures called bands in Section 2, since they violate condition $(iii)$ of Lemma 24, cf. Figure 3. However, by subdividing the ‘even’ elements of the band into two elements satisfying the maximum angle condition, we obtain the structure from Figure 13, which satisfies the conditions of Theorem 5.

Example 3: Since linking of elements in $T_h^2$ is allowed as long as $A_K$ is not near another vertex from $T_h^2$ (i.e. supp $w_K$ does not contain any other vertex from $T_h^2$), one can construct more complicated structures allowed by Theorem 5, such as in Figure 14. Again only the elements of $T_h^2$ are depicted, $T^1_h$ can fill in the remaining white spaces arbitrarily. Such structures can cover the whole domain $\Omega$, as long as condition $(i)$ of Lemma 24 is satisfied. As in the previous examples, by Theorem 5 the FEM has $O(h)$ convergence on such meshes.
3.4 Clustering of elements

In Section 3.2, we have modified the linear Lagrange interpolation on individual "degenerating" elements $K \in T_h^2$. As we have seen, such elements can be connected to form "chains", however their maximum angle vertices cannot be too close to other vertices from $T_h^2$, cf. Theorem 5. The question arises whether the presented construction can be extended also to situations such as Figure 15, where the degenerating elements form nontrivial structures, which we will call clusters, similar e.g. to the "bands" of Section 2.

Definition 25. Let $T_h^1, T_h^2$ be as in (96). A cluster of elements $\mathcal{C}$ is a simply connected union of elements $K \in T_h^2$ such that there is at least one element $K \in T_h^2$ that lies in $\mathcal{C}$.

Remark 17. We note that clusters, as from the previous definition, are not uniquely defined. Their choice for the purpose of proving $O(h)$ convergence is 'user dependent' based on the specific geometry of $T_h$.

In general, neither standard Lagrange interpolation nor the modified version (97) can be used on elements from $\mathcal{C}$. The first is not suitable for $K \in T_h^2$, the latter for $K, \tilde{K} \in T_h^2$ connected as in Figure 15. The idea is to construct one linear function $v_h^\mathcal{C}$ globally on $\mathcal{C}$ as the interpolation. Then we will use the idea of correction functions as in Section 3.3 to connect $v_h^\mathcal{C}$ continuously to the elements neighboring $\mathcal{C}$. From $v_h^\mathcal{C}$ we require only that it is $O(h)$-close
to $u|_{\mathcal{G}}$ in the $H^1(\mathcal{G})$-seminorm with a constant independent of the geometry of $\mathcal{G}$. The simplest possibility is to take $v^e_h$ corresponding to the tangent plane of $u$ at some point $x_{e} \in \mathcal{G}$.

Let $x_{e} \in \mathcal{G}$ be an arbitrary but fixed point, we define $v^e_h \in P^1(\mathcal{G})$ by

$$v^e_h(x) = u(x_{e}) + \nabla u(x_{e})(x - x_{e}). \quad (109)$$

The following estimates are a straightforward consequence of Taylor’s theorem.

**Lemma 26.** Let $v^e_h$ be defined by (109), then for $x \in \Omega$

$$|u(x) - v^e_h(x)| \leq |x - x_{e}|^2|u|_{2,\infty},$$
$$|\nabla u(x) - \nabla v^e_h(x)| \leq 2|x - x_{e}| |u|_{2,\infty}.$$  

Similarly as in Definition 20, we define the correction function, which is used to continuously “connect” the function $v^e_h$ to the rest of $T_h$ in the interpolation procedure.

**Definition 27.** Let $\{\mathcal{G}_i\}_{i=1}^{N_\mathcal{G}}$ be a set of clusters of elements from $T_h$. Let $\Omega_1 = \Omega \setminus \cup_{i=1}^{N_\mathcal{G}} \mathcal{G}_i$. We call $w : \Omega \to \mathbb{R}$ a correction function corresponding to $u$ and $T_h$, if $w \in C(\overline{\Omega})$, $w \in H^2(\Omega_1)$ and

(i) $w(x) = v^e_h(x) - u(x)$, $\forall i = 1,\ldots,N_\mathcal{G}$, $\forall x \in \{A_K, B_K, C_K\}$, $\forall K \in \mathcal{G}_i$,

(ii) $|w|_{H^1(\Omega_1)} \leq C(u)h$,  

(iii) $|w|_{H^2(\Omega_1)} \leq C(u).$  

Similarly as in Section 3.2, we shall use the correction function $w$ to construct a special interpolation of $u$ to use in Céa’s lemma. We get the following theorem, the proof of which is essentially identical to that of Lemma 21.

**Lemma 28.** Let $u \in W^{2,\infty}(\Omega)$ and $w$ be as in Definition 27. Let $\tilde{u} = u + w$. Then

$$|u - \Pi_h \tilde{u}| \leq C(u)h,$$

where $C(u)$ is independent of $h$.

As in Section 3.3, we shall construct $w$ as a linear combination of disjoint local ‘bumps’ around the individual clusters. We define the spline function $\tilde{\psi} : [0, \infty) \to \mathbb{R}$ as

$$\tilde{\psi} = \begin{cases} 1, & x \in [0, 1], \\ 2(x - 2)^3 + 3(x - 2)^2, & x \in [1, 2], \\ 0, & x > 2. \end{cases}$$

This function satisfies $\tilde{\psi} \in C^1(0, \infty) \cap H^2(0, \infty)$. Its derivatives are bounded by $|\tilde{\psi}^{(1)}| \leq \frac{5}{2}$, $|\tilde{\psi}^{(2)}| \leq 6$. Using this function, we construct a local 2D ‘table mountain’ bump with radius $2r$:

$$\psi_r(x) = \tilde{\psi}(\frac{|x|}{r}), \quad (113)$$
We have $\psi_r \in C^1(\mathbb{R}^2) \cap H^2(\mathbb{R}^2)$, $\text{supp} \psi_r = B_{2r}(0)$ and $\psi_r \equiv 1$ on $B_{r}(0)$. Finally, we can estimate its derivatives similarly as in Lemma 22.

**Lemma 29.** For all $x \in \mathbb{R}^2$

$$|\nabla \psi_r(x)| \leq \frac{3}{2r},$$

$$\|\nabla^2 \psi_r(x)\|_F \leq \frac{36}{r^2}.$$

First, we consider the case when only one cluster $C$ is present in $T_h$. We define the corresponding correction function by

$$w_C(x) = (v^e_h(x) - u(x))\psi_{r_e}(x - x_C),$$

$$r_e = \text{diam } C.$$  \hspace{1cm} (114)

**Lemma 30.** Let $\text{diam } C \leq Ch^{1/2}$. The function $w_e$ defined by (114) is a correction function for $T_h$ containing one cluster $C$.

**Proof:** Since $\psi_r \equiv 1$ on $B_r(0)$, we have $\psi_r(x - x_C) = 1$ for all $x \in C$, therefore $w_C(x) = v^e_h(x) - u(x)$ for all $x \in C$. Thus (110) is valid.

Trivially, $w_e \in H^2(\Omega)$. It remains to prove (111), (112). By Lemmas 26 and 29

$$|w_e|_1 = \int_{\text{supp } w_e} |\nabla w_e|^2 \, dx$$

$$= \int_{\text{supp } w_e} |\nabla(v^e_h(x) - u(x))\psi_{r_e}(x - x_C) + (v^e_h(x) - u(x))\nabla \psi_{r_e}(x - x_C)|^2 \, dx$$

$$\leq \int_{\text{supp } w_e} \left(4r_e|u|_{2,\infty} + 4r_e^2|u|_{2,\infty} \frac{3}{2r_e} \right)^2 \, dx \leq 400r^2_e|u|_{2,\infty}^2|\text{supp } w_e| \leq Ch^2,$$

since $r^2_e = (\text{diam } C)^2 \leq Ch$ and $|\text{supp } w_e| = \pi(2r_e)^2 \leq Ch$. Hence $|w_e|_1 = O(h)$. Similarly (for brevity we omit the arguments of functions),

$$|w_e|_2 = \int_{\text{supp } w_e} |\nabla^2 w_e|^2 \, dx$$

$$= \int_{\text{supp } w_e} \|\nabla^2(v^e_h - u)\psi_{r_e} + 2\nabla(v^e_h - u) \otimes \nabla \psi_{r_e} + (v^e_h - u)\nabla^2 \psi_{r_e}\|_F^2 \, dx$$

$$\leq \int_{\text{supp } w_e} \left(2|u|_{2,\infty}^2 + 2r_e|u|_{2,\infty}^2 + \frac{3}{2r_e} + 4r_e^2|u|_{2,\infty} \frac{36}{r_e^2} \right)^2 \, dx$$

$$\leq C|\text{supp } w_e||u|_{2,\infty}^2 \leq C(u),$$  \hspace{1cm} (116)

hence (112) is satisfied. \hfill \square
3.4.1 Multiple clusters

In the previous section, we have constructed a correction function for a single cluster $\mathcal{C}$. If there are multiple clusters $\{\mathcal{C}_i\}_{i=1}^{NC} \subset \mathcal{T}_h$, we can simply sum the individual correction functions for each $\mathcal{C}_i$. In order to preserve the properties of the individual correction functions, we need to suppose their supports are disjoint. We get an analogy of Lemma 24.

**Lemma 31.** Let $\{\mathcal{C}_i\}_{i=1}^{NC} \subset \mathcal{T}_h$ be such that

\[
(i) \sum_{i=1}^{NC} r_{\mathcal{C}_i}^2 \leq Ch,
(ii) \text{dist}(\mathcal{C}_i, \mathcal{C}_j) \geq 2(r_{\mathcal{C}_i} + r_{\mathcal{C}_j}) \quad \text{for } i, j = 1, \ldots, NC, \ i \neq j,
\]

where $r_{\mathcal{C}_i} = \text{diam} \mathcal{C}_i$. Then

\[
w(x) = \sum_{i=1}^{NC} w_{\mathcal{C}_i}(x) \quad (117)
\]

is a correction function for $\mathcal{T}_h$, where $w_{\mathcal{C}_i}$ is defined by (114).

**Proof:** As in Lemma 24, condition (ii) means that $\text{supp } w_{\mathcal{C}_i} \cap \text{supp } w_{\mathcal{C}_j} = \emptyset$ for $i \neq j$, i.e. property (110) is satisfied. As for (102) and (103), due to (115)

\[
|w|_1^2 = \sum_{i=1}^{NC} \int_{\text{supp } w_{\mathcal{C}_i}} |\nabla w_{\mathcal{C}_i}(x)|^2 \, dx \leq \sum_{i=1}^{NC} 400r_{\mathcal{C}_i}^2|u|_{2,\infty}^2 |\text{supp } w_{\mathcal{C}_i}| \leq \sum_{i=1}^{NC} Cr_{\mathcal{C}_i}^4|u|_{2,\infty}^2
\]

\[\leq C|u|_{2,\infty}^2 \left(\sum_{i=1}^{NC} r_{\mathcal{C}_i}^2\right)^2 \leq Ch^2|u|_{2,\infty}^2,
\]
due to assumption (i). Therefore $|w|_1 \leq C(u)h$. Similarly, we have due to (116)

\[
|w|_2^2 = \sum_{i=1}^{NC} \int_{\text{supp } w_{\mathcal{C}_i}} \|\nabla^2 w_{\mathcal{C}_i}(x)\|_F^2 \, dx \leq C|u|_{2,\infty}^2 \sum_{i=1}^{NC} |\text{supp } w_{\mathcal{C}_i}| \leq C|u|_{2,\infty}^2 |\Omega|,
\]

due to assumption (i).

We note that the constant 2 in condition (ii) of Lemma 31 can be replaced by any $c > 1$ by constructing the function $\psi$ such that $\psi \equiv 1$ on $[0, 1]$ and $\text{supp } \psi = [0, c]$. However, we will get the factor $1/c$ in the corresponding version of Lemma 29.

**Theorem 6.** Let $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$, where $\mathcal{T}_h^1$ satisfies the maximum angle condition and $\mathcal{T}_h^2 = \{\mathcal{C}_i\}_{i=1}^{NC}$ satisfies the conditions of Lemma 31. Then $|u - U|_1 \leq C(u)h$ for all $u \in W^{2,\infty}(\Omega)$.
Proof: The proof is a trivial consequence of Lemmas 31 and 28 and Céa’s lemma. □

Example 4: Let \( \{ \mathcal{C}_i \}_{i=1}^{N_e} \subset T_h \) be such that the cluster \( \mathcal{C}_i \) have diameter at most \( Ch^{1/2} \) and are at least \( Ch^{1/2} \) apart. Let \( N_e \leq N \) for all \( h \in (0, h_0) \), where \( N \) is independent of \( h \). Then the assumptions of Theorem 6 are satisfied, hence the finite element method has \( O(h) \) convergence.

Example 5: Let \( \{ \mathcal{C}_i \}_{i=1}^{N_e} \subset T_h \) be such that the cluster \( \mathcal{C}_i \) have diameter at most \( Ch^\alpha \) and are at least \( Ch^{1-\alpha} \) apart. Let \( N_e \leq Ch^{-\alpha} \) for all \( h \in (0, h_0) \). Then the finite element method has \( O(h^{\alpha}) \) convergence.

Example 6: Theorems 5 and 6 can be combined together by considering \( T_{h_0} \) where clusters of elements of diameter at most \( Ch^{1/2} \) exist together with the chains and other structures of Theorem 5. The corresponding correction function is simply the sum of all the particular correction functions. One only needs to ensure that the individual supports stay disjoint.

We conclude with several remarks.

Remark 18. Throughout Section 3 we have been interested in sufficient conditions for \( O(h) \) convergence. If we want to generalise Theorems 5 and 6 to the case of \( O(h^\alpha) \) convergence as in Section 2, we can make the following changes:

- Everywhere, the maximum angle condition valid on \( T_h^1 \) can be replaced by the weaker condition: There exists \( C_R > 0 \) such that \( T_h^1 = \{ K \in T_h : R_K \leq C_R h^{\alpha} \} \). By Lemma 18, this gives an \( O(h^\alpha) \) estimate on \( T_h^1 \).

- In condition (106), we can take \( r_K = \frac{1}{2} h_1^{\alpha-\alpha} \min \{|A_K - B_K|, |A_K - C_K|\} \), therefore, in condition (ii) of Lemma 24, the vertices \( A_K, A_K \) can be much closer together for \( \alpha < 1 \). On the other hand, condition (i) becomes \( \sum_{K \in T_h^2} h_2^{\alpha} K \leq C \), i.e. there must be fewer elements in \( T_h^2 \).

- On the other hand, we can keep the original choice of \( r_K \) from (106). Thus we get an \( O(h^\alpha) \) estimate on \( T_h^1 \) and \( O(h) \) on \( T_h^2 \) with the original conditions (i), (ii) of Lemma 24.

- In Lemma 30 we can have \( \text{diam} \mathcal{C} \leq Ch^{\alpha/2} \), i.e the clusters can be much larger for \( \alpha < 1 \). Consequently, in Lemma 31 condition (i) must be replaced by \( \sum_{i=1}^{N_e} r_{\mathcal{C}_i}^2 \leq Ch^\alpha \). Then Theorem 6 gives us \( O(h^\alpha) \) convergence.

Remark 19. The question arises whether the technique of Section 3 is more general than the construction of 9, where a ‘nice’ triangulation satisfying the maximum angle condition is subdivided arbitrarily to still obtain \( O(h) \) convergence on the subdivision. In other words, if \( T_h \) satisfies the assumptions of Theorems 5 and 6 does there exist a coarser triangulation \( \tilde{T}_h \) with elements of diameter at most \( Ch \) such that \( T_h \) is a refinement of \( \tilde{T}_h \)? However in Theorems 5 and 6 nothing is assumed about \( T_h^1 \) except that it satisfies the maximum angle condition, it can have arbitrary structure. Then it is easy to construct \( T_h \).
containing e.g. one element in $T^2_h$, such that it is not a refinement of any triangulation. For example, if no two edges in $T_h$ sharing a vertex lie on a common line, then $T_h$ cannot be a refinement.

Another deeper reason why $T_h$ in general is not a subdivision of a ‘nice’ triangulation is the following lemma proved in [10]:

Lemma 32. Let $\alpha \in \left(\frac{\pi}{4}, \pi\right)$. Let $K$ be a triangle with all angles less than $\alpha$. Then there does not exist a finite conforming partition of $K$ into triangles which all contain an angle greater than or equal to $\alpha$.

In other words, if a triangle $K \in \tilde{T}_h$ is subdivided, the resulting subdivision must contain a triangle at least ‘as nice as’ $K$. However Theorem 6 allows for clusters of arbitrarily ‘bad’ elements of size up to $Ch^{1/2}$, much larger than $Ch$ – the size $K \in \tilde{T}_h$ would have to have in order to still have $O(h)$ convergence on $\tilde{T}_h$ and the subdivision $T_h$.

Remark 20. The question of finding a necessary and sufficient condition for $O(h^{\alpha})$-convergence of the FEM remains open. However in some special cases the derived necessary (Section 2) and sufficient (Section 3) conditions are not far apart. Take for example the situation in Counterexample 1 on page 19. Let $T^2_h$ consist of the band $B$ considered in this counterexample, i.e. with the shape parameter $\bar{h} = o(h^{4-8\alpha/5})$. Then for the length $L$ of $B$ the necessary condition for $O(h^{\alpha})$ convergence is $L \leq C_L h^{2\alpha/5} = C_L h^{0.4\alpha}$, while the sufficient condition is $L \leq C_L h^{\alpha/2} = C_L h^{0.5\alpha}$, since we can consider $B$ as a cluster and apply Theorem 6. We note the difference in exponents is not large.

Similarly, if we consider $T^2_h$ consisting of multiple bands as in Counterexamples 6 and 7 (page 24) forming a cluster of dimensions $L \times L$, then if $\bar{h} = o(h^{2-3\alpha/5})$ again the necessary condition for $O(h^{\alpha})$ convergence is $L \leq C_L h^{2\alpha/5}$, while the sufficient condition is $L \leq C_L h^{\alpha/2}$.

4 Conclusion

We have presented necessary and sufficient conditions on the triangulations $T_h$ for $O(h^{\alpha})$ convergence of the piecewise linear conforming finite element method in $H^1(\Omega)$. The derived necessary condition limits the size of certain structures consisting of degenerating elements and is the first such condition for FEM convergence. The sufficient condition generalizes the maximum angle and circumradius conditions and shows that degenerating elements are allowed in $T_h$ as long as they form structures obeying certain rules. The analysis concerns the approximation properties of the discrete space $V_h$, since the finite element formulation itself is never used and only appears via Céa’s lemma. Specifically:

- We have introduced the notion of bands, sets of neighboring elements connected by edges, such that their maximum angles form an alternating ‘zig-zag’ pattern. Due to the geometry of the bands, the gradients of a continuous piecewise linear function $U$ on odd elements of the band determine the gradients on the intermediate even elements.
• Provided \(|u - U|_1 = O(h^\alpha)|\), this allowed us to produce a lower bound on the approximation error \(|u - U|_{H^1(B)}|\) on the band \(B\), provided \(B\) is long enough \((L \geq C_L h^{2\alpha/5})\), cf. Theorem 1.

• The lower bound on \(|u - U|_{H^1(B)}|\) can be made arbitrarily large e.g. by letting the maximal angles of \(K \in B\) go sufficiently fast to \(\pi\) as \(h \to 0\). Thus the basic assumption \(|u - U|_1 = O(h^\alpha)|\) can be brought to a contradiction, giving necessary conditions on the geometry of \(B\) for the \(O(h^\alpha)|\) estimate to hold, cf. Corollary 9.

• Theorem 2 then gives the lower bound on \(|u - U|_{H^1(B)}|\) for a system of bands \(B\), which gives the necessary condition for \(O(h^\alpha)|\) convergence of Corollary 13.

• Based on the necessary conditions, various counterexamples to \(O(h^\alpha)|\) convergence were constructed, even examples of nonconvergence of the FEM. As a special case, we recovered the Babuška-Aziz counterexample, [1], specifically the optimal results of [11] thereon.

• As for the sufficient conditions for \(O(h^\alpha)|\) convergence, we treated \(\alpha = 1\), the general case follows by simple arguments. We split \(T_h\) into two parts – \(T_h^1\) satisfying and \(T_h^2\) violating the maximum angle condition. To construct a suitable interpolant from \(V_h\), on \(T_h^1\) we use Lagrange interpolation and a modified Lagrange procedure on \(T_h^2\). The modified variant has \(O(h_K)|\) approximation properties in \(H^1(K)|\) independent of the geometry of \(K\), however \(u\) is not interpolated exactly in the maximum angle vertex of \(K\) but with an \(O(h_K^{2\alpha})|\) perturbation, cf. Lemma 19.

• Due to the \(O(h_K^{2\alpha})|\) perturbations at certain vertices of \(T_h^2\), the interpolants on individual elements cannot be continuously connected. To construct a globally continuous interpolant, we introduced the concept of correction functions, which modify \(u\) locally so that the Lagrange interpolation of the new function \(\tilde{u}\) corresponds to the modified Lagrange interpolation of \(u\) on \(T_h^2\). We gave constructions of the correction functions in several cases, leading to Theorems 5 and 6. Elements violating the maximum angle condition can form clusters of diameter up to \(O(h^{1/2})|\) or arbitrarily large ‘chains’ as long as their maximal-angle vertices are not too close to other vertices in \(T_h^2\). Examples of such triangulations were provided.

Although a condition for FEM convergence that would be both necessary and sufficient remains unknown, future work will be devoted to narrowing and perhaps closing the gap between the derived conditions at least in special cases, if not in general. The unifying idea behind the presented analysis is considering the size of sets of elements on which the interpolant can or cannot be one globally defined linear function. Such considerations led to conditions on the diameter – \(O(h^{0.5\alpha})|\) being sufficient and \(O(h^{0.4\alpha})|\) being necessary, as in Remark 20. Another subject for future work is to strengthen the results of Section 3 to hold for \(u \in H^2(\Omega)|\), since currently \(u \in W^{2,\infty}(\Omega)|\) is needed.
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