Torsional Oscillations of Magnetized Relativistic Stars

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Abstract. We present the formalism and numerical results for torsional oscillations of relativistic stars endowed with a strong dipole magnetic field, assumed to be confined to the crust. In our approach, we focus on axisymmetric modes and neglect higher-order couplings induced by the magnetic field. Using extended numerical results we derive empirical relations for the effect of the magnetic field on torsional modes as well as for the crust thickness. We compare our numerical results to observed frequencies in SGRs and find that certain high-density EoS and mass values are favored over others in the non-magnetized limit.

1. Introduction
There is growing evidence that Soft Gamma Repeaters (SGRs) could be magnetars experiencing starquakes that are connected (through the intense magnetic field) to gamma ray flare activity. Magnetars are thought to be neutron stars with very strong magnetic fields, greatly exceeding $\sim 5 \times 10^{13}$ G [8]. The flare activity consist of an initial short peak in the hard part of the spectrum, followed by a decaying softer part (tail) which last for hundreds of seconds. Recent analysis of detailed observations from several SGRs [11, 28, 31] revealed that the decaying part of the spectrum exhibits a number of oscillations with frequencies in the range of a few tenths of Hz to a few hundred Hz. There are three events detected up to now which can be associated with crust oscillations of magnetars. The first event was detected already in 1979 from the source SGR 0526-66 [15, 2], the second in 1998 from SGR 1900+14 [10], while the third and most energetic one was observed in December 2004 from the source SGR 1806-20 [30, 19]. Analysis of the tail oscillations of SGR 1806-20 revealed the presence of oscillations at approximately 18, 26, 29, 92.5, 150, 626.5, and 1837 Hz [11, 31, 29], for SGR 1900+14 the detected frequencies were 28, 54, 84, and 155 Hz [28], while for the case of SGR 0526-66 there was only one frequency identified at 43.5 Hz [2].

Magnetar starquakes may be driven by the evolving intense magnetic field, which accumulates stress and eventually leads to crust fracturing. The excited seismic oscillations are known to be of two types: shear modes, which are polar-type oscillations and torsional modes, which are axial-type oscillations. The latter are thought to be more easily excited during a fracturing of the crust, since they only involve oscillations of the velocity, with the shear
modulus of the crust acting as the necessary restoring force. Slowly rotating stars still maintain a nearly spherical shape and the velocity field of torsional oscillations is then divergence-free, with no radial component. Torsional modes (\( t \)-modes) are labeled as \( \ell t_n \), where \( \ell \) is the angular index, while the index \( n \) corresponds to the number of radial nodes in the eigenfunctions of the overtones for a specific \( \ell \). Shear and torsional modes have been studied mainly in Newtonian theory, see e.g. [9, 16, 5, 24, 13], while there are only a few studies of torsional modes in general relativity [23, 14, 17, 25, 22, 26].

In the Newtonian, non-magnetized limit [9] found that the period \( \ell P_0 \) of the fundamental torsional modes depends mostly on the radius of the star, \( R \), the speed of shear waves, \( u_s \), and the angular index \( \ell \), via

\[
\ell P_0 \approx 2\pi[\ell(\ell + 1)]^{-1/2}R/u_s \approx 26\text{ms} \tag{1}
\]

where \( u_s = (\mu/\rho)^{1/2} \), with \( \mu \) and \( \rho \) being the shear modulus and the density, respectively. Furthermore, it was found that the period of overtones of index \( n \) is essentially independent of the angular index \( \ell \) and is basically determined by the crust thickness \( \Delta r \) i.e. \( \ell P_n \approx 2n^{-1}\Delta r/u_s \). From the above two relations it follows that the relative crust thickness, \( \Delta r/R \), is given by

\[
\frac{\Delta r}{R} \approx \frac{\pi n}{\sqrt{\ell(\ell + 1)} \ell f_0} \ell f_n, \tag{2}
\]

which is independent of the details of the equation of state in the crust and basically only depends on the ratio of the frequency of the overtones to the fundamental frequency, see also [22]. It thus becomes obvious that the successful identification of both the fundamental torsional frequencies and their overtones could allow for the determination of the crust thickness, which in turn could lead to information on the high-density part of the equation of state.

Relativistic effects have been found to significantly increase the fundamental \( \ell = 2 \) torsional mode period by roughly 30\% for a typical \( M = 1.4M_\odot, R = 10\text{km} \) model [7]. On the other hand, the effect of a strong magnetic field, \( B \), on torsional modes was considered by [7] and [17]. Assuming that the shear modulus \( \mu \) is augmented by the magnetic field tension, \( B^2/4\pi \), one can derive the following estimate for the period of a torsional mode

\[
P \approx P^{(0)} [1 + (B/B_\mu)]^{-1/2}, \tag{3}
\]

where \( P^{(0)} \) is the period in the limit of vanishing magnetic field and \( B_\mu = \sqrt{4\pi\mu} \).

An essential ingredient in computing torsional mode frequencies is the assumption one makes about the shear modulus \( \mu \). Here, we adopt the zero-temperature limit of the approximate formula derived by [27]

\[
\mu = 0.1194\frac{n_i(Ze)^2}{a}, \tag{4}
\]

where \( n_i \) is the ion number density, \( a \) is the average ion spacing, and \( +Ze \) is the ion charge.

In the article we present a short overview of an extended study published mainly in [25] (Paper I) and a few results from [26] (Paper II). We specialize the magnetic field to a dipole structure, which allows us to reduce the system of equations to only one dimension. A large number of torsional modes is obtained for selected magnetar models differing in the high-density EOS, the crust EoS and mass. Our study is valid for moderate magnetic field strengths, up to roughly \( 10^{16} \text{ G} \), assuming that the magnetic field is confined to the crust and neglecting the possible presence of a thin fluid ocean. The reason for this is twofold: on one hand, for significantly stronger magnetic fields the distortion of the equilibrium shape of the star as well the magnetic coupling to higher harmonics should also be taken into account; on the other
hand, the Alfvén velocity \( u_A \equiv B/(4\pi \rho)^{1/2} \approx 3 \times 10^7 \text{ cm/s} \), becomes comparable to the speed of shear waves in the crust when the magnetic field reaches values of \( 10^{15}-10^{16} \text{ G} \), which implies that global magnetoionic waves will be strongly coupled to shear waves and the torsional oscillations may no longer be confined to the crust. The effect of including global magnetoionic waves even at smaller magnetic field strengths has also been studied in more detail [26].

2. General-Relativistic Ideal MHD

The energy momentum tensor \( T^{\mu\nu} \) for a magnetized relativistic star in equilibrium, is the sum of the stress-energy tensors of a perfect fluid \( T^{\mu\nu}(pf) \) and the magnetic field \( T^{\mu\nu}(M) \)

\[
T^{\mu\nu} = T^{\mu\nu}(pf) + T^{\mu\nu}(M),
\]

where

\[
T^{\mu\nu}(pf) = (\epsilon + p)u^{\mu}u^{\nu} + pg^{\mu\nu} \quad \text{and} \quad T^{\mu\nu}(M) = H^2 u^{\mu}u^{\nu} + \frac{1}{2} H^2 g^{\mu\nu} - H^{\mu}H^{\nu}.
\]

Above, \( \epsilon, p, \) and \( H^\mu \) are the energy density, the pressure, and a normalized magnetic field, which we define by absorbing a factor of \( \sqrt{4\pi} \), i.e., \( H^\mu \equiv B^\mu/\sqrt{4\pi} \), and \( H^2 = H_{\alpha}H^{\alpha} \). Here and throughout the paper we will assume the ideal MHD approximation. In addition, we assume the shear modulus to be isotropic, so that there is no contribution of the shear stress in equilibrium. Using the stress-energy tensor (5), one obtains the equations of motion of the fluid by projecting the conservation of the energy-momentum tensor on to the hypersurface normal to \( u^\mu \), i.e.,

\[
h^{\mu\alpha}T^{\alpha\nu} = 0,
\]

where \( h^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu} \) is the projection tensor. The equations of motion are

\[
(\epsilon + p + H^2)u^{\mu}u^{\nu} = -h^{\mu\nu}\left(p + \frac{1}{2} H^2\right) + h^{\mu}_{\alpha}(H^\alpha H^{\nu})_{,\nu}.
\]

Due to the ideal MHD approximation the electric field 4-vector vanishes, \( F_{\mu\nu} = F_{\mu\nu}u^{\nu} = 0 \) so that the electric field is zero for a comoving observer. In this approximation, the Maxwell’s equations \( F_{[\mu\nu]\gamma] = 0 \), where \( F_{\mu\nu} \) is the Faraday tensor, can be written in the simple form:

\[
(u^\mu H^{\nu} - u^{\nu}H^\mu)_{,\mu} = 0.
\]

In deriving the above equation the following definitions are used

\[
F_{\mu\nu} = u_{\mu}E_{\nu} - u_{\nu}E_{\mu} - \epsilon_{\mu\nu\alpha\beta}u^{\alpha}B^{\beta} \quad \text{and} \quad B_{\mu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}u^{\nu}F^{\rho\sigma},
\]

From Equation (9) one derives the magnetic induction equation

\[
H^{\mu}_{,\nu}u^{\nu} = -u^{\alpha}_{,\mu}H^{\mu} + u^{\mu}_{,\nu}H^{\nu} + H^{\alpha}u_{\alpha;\beta}u^{\beta}u^{\mu},
\]

\[
= \left(\sigma^{\mu}_{\nu} + \omega^{\mu}_{\nu} - \frac{2}{3}\delta^{\mu}_{\nu}\Theta\right)H^{\nu} + H^{\alpha}u_{\alpha;\beta}u^{\beta}u^{\mu},
\]

where \( \sigma_{\mu\nu} \) is the rate of shear tensor, \( \omega_{\mu\nu} \) is the twist tensor i.e.

\[
\sigma_{\mu\nu} \equiv \frac{1}{2} (u_{\mu;\alpha}h^{\alpha}_{,\nu} + u_{\nu;\alpha}h^{\alpha}_{,\mu}) - \frac{1}{3}\Theta h_{\mu\nu}, \quad \omega_{\mu\nu} \equiv \frac{1}{2} (u_{\mu;\alpha}h^{\alpha}_{,\nu} - u_{\nu;\alpha}h^{\alpha}_{,\mu}).
\]

and \( \Theta \) is the expansion, defined as \( \Theta \equiv u^{\mu}_{,\mu} \). In a following section, the above equations will be linearized, in order to obtain the perturbed equations of motion describing small-amplitude oscillations. But first, we describe the construction of the background equilibrium models.
3. Equilibrium Configuration

A strongly magnetized relativistic star has a non-spherical shape, due to the tension of the magnetic field, which is expressed via the off-diagonal components in the stress-energy tensor $T^{\mu\nu}$. However, the deformations from spherical symmetry induced by the magnetic field are small for neutron stars with magnetic field usually assumed for magnetars, because the energy of the magnetic field is considerably smaller than the gravitational energy. For this reason, we neglect the deformations due to magnetic fields in the construction of equilibrium models. Since magnetars are also extremely slowly rotating (the intense magnetic field spins down the star on a short timescale) we can also neglect any rotational deformations, so that the equilibrium models can be considered as spherically symmetric solutions of the well-known TOV equations described by a metric of the form

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

(14)

where $\Phi$ and $\Lambda$ are the function of the Schwarzschild radial coordinate $r$. The four-velocity of the equilibrium model is thus $u^\mu = (e^{-\Phi}, 0, 0, 0)$. We supplement the equilibrium model by a dipole magnetic field (but see [4] for other assumptions), following the approach of [12] i.e. we consider an axisymmetric, poloidal magnetic field, which is created by a 4-current $J_\mu = (0, 0, 0, J_\phi)$. In ideal MHD the electromagnetic 4-potential $A_\mu$ is very simple and has only one component in spherical polar coordinates $A_\phi = (0, 0, 0, A_\phi)$, and $F^{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, [3, 12]. For the metric chosen above, Maxwell’s equations $F^{\mu\nu} = 4\pi J^\mu$ leads to an elliptic equation for $A_\phi$ which can be numerically calculated estimate and the components of the vector $H_\mu$ describing the normalized magnetic field are given by the following set of equations

$$H_\rho = \frac{-e^\Lambda}{\sqrt{4\pi r^2 \sin \theta}} \partial_\theta A_\phi = \frac{e^\Lambda \cos \theta}{\sqrt{4\pi r^2}} a_1 \quad \text{and} \quad H_\theta = \frac{-e^{-\Lambda}}{\sqrt{4\pi \sin \theta}} \partial_\rho A_\phi = -\frac{e^{-\Lambda} \sin \theta}{\sqrt{4\pi}} a_{1,\rho}.$$ (15)

We emphasize that, even though the magnetic field configuration has been computed throughout the star, the boundary conditions we will assume for the torsional oscillations will neglect the presence of the magnetic field outside the crust region, so that, effectively, that magnetic field will be assumed to be confined to the crust only.

4. Perturbation Equations

The perturbation equations are derived by linearizing the equations of motion (8) and the magnetic induction equation (11). Since we only consider torsional oscillations, which are of axial type and do not induce density variations in spherical stars, there is no significant variation in the radiative part of the metric describing the gravitational field. In addition, since torsional modes are essentially material velocity oscillations, the imaginary part of their frequency due to the emission of current-multipole gravitational radiation is much smaller than the real part. For these reasons the frequency of torsional oscillations is determined with satisfactory accuracy even when neglecting entirely the metric perturbations by setting $\delta g_{\mu\nu} = 0$ (relativistic Cowling approximation). The linearized form of the equation of motion (8) is

$$(\epsilon + p + H^2)\delta u^{\mu}_{\nu}u^{\nu} = -(\delta \epsilon + \delta p + 2H_{\alpha}\delta H^\alpha)u^{\mu}_{\nu}u^{\nu} - (\epsilon + p + H^2)u^{\mu}_{\nu}\delta u^{\nu}$$

$$+ (u^\mu\delta u_\alpha + \delta u^\mu u_\alpha)\left[H^{\alpha\beta}H^{\nu} - g^{\alpha\nu}\left(p + \frac{1}{2}H^2\right)\right]^{\mu}_{\nu}$$

$$+ h^{\mu}_\alpha \left[H^{\alpha\delta}H^{\nu} - \delta H^{\alpha\beta}H^{\nu} - g^{\alpha\nu}\delta p + H_\beta \delta H^\beta\right]^{\mu}_{\nu} - h^{\mu}_\alpha \delta T^{\alpha\nu(\delta)}.$$ (16)
where $\delta T^{\mu\nu(s)}$ is the linearized shear stress tensor. The latter is assumed to be related to the linearized shear tensor $\delta S^{\mu\nu}$, through $\delta T^{(s)}_{\mu\nu} = -2\mu\delta S^{\mu\nu}$, see [23]. In addition, linearization of the magnetic induction equation (11) yields

$$
\delta H^\mu_{\nu} u^\nu = -H^\mu_{\nu} \delta u^\nu + h^\mu_{\nu} \delta u_{\alpha;\beta} H^{\beta}_{\gamma} + h^{\mu\alpha} u_{\alpha;\beta} \delta H^{\beta}_{\gamma} + u^\mu \delta u^\alpha H^\beta u_{\alpha;\beta} - \delta \Theta H^\mu - \Theta \delta H^\mu
$$

where we used $u^\alpha \delta u_{\alpha;\beta} = -\delta u^\alpha u_{\alpha;\beta}$.

The usual method for studying matter perturbations of spherically symmetric backgrounds is to decompose the perturbed quantities into scalar and vector spherical harmonics of definite indices $\ell$ and $m$. In this way, the different oscillation modes characterized by $\ell$ and $m$ decouple and one can study them independently. Here we restrict attention to axial-type perturbations, for which the perturbed matter quantities can be written as $\delta \epsilon = \delta p = 0$, $\delta u^r = \delta u^\theta = 0$ and

$$
\delta u^\phi = e^{-\Phi} \partial_t \mathcal{Y}(t, r) b(\theta), \quad \text{where} \quad b(\theta) \equiv \frac{1}{\sin \theta} \partial_\theta P_\ell(\cos \theta),
$$

where $\partial_\theta$ denotes the partial derivative with respect to $\theta$. Above, $\mathcal{Y}(t, r)$ describes the radial dependence of the angular displacement of the stellar material, $P_\ell(\cos \theta)$ is the Legendre polynomial of order $\ell$ and we have set $m = 0$, due to the degeneracy in $m$ for spherically symmetric backgrounds. The above assumptions for a dipole magnetic field lead to the following form of the perturbation equation

$$
0 = [\epsilon + p + H^r H_r + H^\theta H_\theta] \omega^2 e^{-2\Phi} \mathcal{Y} + e^{-2\Lambda} \mu \mathcal{Y}'' + \left[\frac{4}{\ell - 2}(\Phi + \Lambda') \mu + \mu'\right] e^{-2\Lambda} \mathcal{Y}'
$$

$$
- (\ell + 2)(\ell - 1) \left[\frac{\mu}{\ell^2} + (H^\theta)^2\right] \mathcal{Y} + (H^r)^2 \mathcal{Y}''
$$

$$
+ \left[2 \cot \theta H^r H^\theta + \left(\Phi' + \frac{2}{r}\right) (H^r)^2 + H^r H^r + H^\theta H^\theta, r - \cot \theta (H^\theta)^2 + H^\theta H^\theta, \theta\right] \mathcal{Y}'
$$

$$
+ \left\{\left[\left(\Phi' + \frac{2}{r}\right) H^r H^\theta + H^r H^r, r - \cot \theta (H^\theta)^2 + H^\theta H^\theta, \theta\right] \mathcal{Y} + 2H^r H^\theta \mathcal{Y}'\right\} \frac{b_\theta}{b},
$$

where we used the following equation satisfied by the angular function $b(\theta)$

$$
\frac{b_{,\theta}}{b} + 3 \cot \theta b_{,\theta} + (\ell + 2)(\ell - 1)b = 0.
$$

The above eigenvalue equation can be simplified further by assuming a dipole magnetic field of the form (15). The eigenvalue equations takes the 1-dimensional form

$$
\ell(\ell + 1) (A_\ell + B_\ell) + \mathcal{L}^{\pm 2}_1 (C_\ell - B_\ell) + \mathcal{L}^{\pm 2}_2 D_\ell = 0.
$$

In deriving this equation we used the specific properties of spherical harmonic functions described in [25]. An operator of the form $\mathcal{L}^{\pm 2}$ actually couples perturbations with angular index $\ell$ with those of parity $\ell \pm 2$.

As mentioned in the introduction, we limit our current study to moderate magnetic field strengths of a few times $10^{15}$G, since for larger values one would also have to take into account global magnetoacoustic oscillations. We further simplify the numerical problem, by neglecting the couplings to $\ell \pm 2$ terms in Equation (21). This is motivated by the expectation that higher-order contributions will be less significant than the lowest-order piece of an eigenfunction, since
the higher-order pieces are sampling the background magnetic field at smaller scales. Our numerical results will thus be valid only to the extend that magnetic-field-induced couplings to \( \ell \pm 2 \) terms can be neglected this leads to the following eigenvalue equation

\[ 0 = \left[ \mu + (1 + 2\lambda_1) \frac{a_1^2}{\pi r^4} \right] Y'' + \left\{ \left( \frac{4}{r} + \Phi' - \lambda \right) \mu + \mu' + (1 + 2\lambda_1) \frac{a_1^2}{\pi r^4} \left( (\Phi' - \lambda') a_1 + 2a_1' \right) \right\} Y' \]

\[ + \left\{ \left( \varepsilon + p + (1 + 2\lambda_1) \frac{a_1^2}{\pi r^4} \right) e^{-2\lambda} - \frac{\lambda_1 a_1^2 r}{2\pi r^4} \right\} \omega^2 e^{-2\Phi} \]

\[- (\lambda - 2) \left( \frac{\mu e^{2\lambda}}{r^2} - \frac{\lambda_1 a_1^2}{2\pi r^4} \right) + (2 + 5\lambda_1) \frac{a_1^2}{2\pi r^4} \left\{ (\Phi' - \lambda') a_1' + a_1'' \right\} Y, \]  

(22)

where \( \lambda = \ell(\ell + 1) \) and \( \lambda_1 = -\lambda/(2\ell - 1)(2\ell + 3) \). Here, we only consider torsional modes confined to the crust and impose a zero traction condition at the base of the crust. At the stellar surface, the “zero-torque-at-surface” condition is imposed, i.e. \( \delta T^{(e)}_{\phi} = 0 \) while we neglect the possible presence of a thin fluid ocean.

5. Numerical Results

We have studied torsional modes for a variety of neutron star models using four different equations of state for the core, ranging from a very soft EoS (EoS A) [20] to a very stiff (EoS L) [21], with two intermediate ones, EoS WFF3 [32] and APR [1]. For each EoS we constructed a number of models, starting from a gravitational mass of 1.4\( M_\odot \) and reaching close to the maximum mass limit in increments of 0.2\( M_\odot \). In order to separately investigate the effect of the composition of the crust, we matched the various high-density EoS to two different proposed equations of state for the crust, one recent derived by [6] (DH) and, for reference, and older EoS by [18] (NV). The individual models for which we compute torsional modes are listed in Table 1.

From Table 1 it is evident that for the soft equations of state (A and WFF3) the choice of the crust EoS does not affect significantly the bulk properties of the star, while for the stiffest EoS L models with the same mass have considerably different radii. The reason for this is that the models constructed with the stiff EoS have central densities that are much closer to the density of the base of the crust than for soft EoS. The two crust EoS differ significantly both in the detailed composition, as well as in the density at the base of the crust, which is at \( \rho \approx 2.4 \times 10^{14} \text{gr/cm}^3 \) for [18] and at \( \rho \approx 1.28 \times 10^{14} \text{gr/cm}^3 \) for [6]. For the stiff EoS these different properties of the crust have a considerable effect on the bulk properties. For the two crust EoS the shear modulus \( \mu \) can be computed using Equation (4).

Even though the excitation of these modes and the SGR activity are likely related to the strong magnetic field, the torsional mode frequencies are not affected by the magnetic field for \( B < 10^{15} \text{G} \) (at least when the latter is confined to the crust). It is not clear whether magnetars have magnetic fields that only approach this value or that are even stronger. One should therefore first consider torsional mode frequencies in the non-magnetized limit.

In Table 2 we list the frequencies of the fundamental torsional modes \( \ell_0 \) for \( \ell = 2 \) to \( \ell = 10 \), for the equilibrium models discussed earlier and shown in Table 1. Among these models, the frequency of the fundamental \( \ell = 2 \) mode varies from 17 to 29 Hz i.e. depending on the stellar parameters it can vary by up to 30-50%. Limiting attention to models with \( M = 1.4M_\odot \) only, the variation is from 21 to 29 Hz, which still significant. Therefore, strong constraints of the high-density EoS could be placed by observations of the fundamental \( \ell = 2 \) torsional mode in
magnetars, especially if the mass is deduced by other means. On the other hand, the choice of the crust EoS does not seem to affect significantly the frequency of the fundamental torsional mode. The typical variation observed for equal mass models with different crust EoS is on the order of 1-5%. The same comments apply to the behavior of higher $\ell$ modes, which are nearly equally spaced for $\ell \geq 3$, such that a scaling law can be derived for each model.

The above picture is drastically altered when one considers the first overtone, $\ell_1$, for which the obtained frequencies are shown in Table 3. We observe that our numerical data are in agreement with the conclusion in [9] that the frequencies are practically independent of the harmonic index $\ell$. A second observation is that the variations in the frequencies due to different choices of both the high-density and crust EoS are significant.

The frequencies of the first overtones vary from 500-1200 Hz, while even for models with the same high-density EoS and same mass the frequency varies by up to 30% when changing the crust EoS. This behavior is quite useful given that there are suggestions for an observation in the spectrum of SGR 1806-20 of a torsional mode with $n = 1$ with a frequency of 626.5 Hz [31] and at least one more at 1837 Hz. Comparing this to our results in Table 3 one could immediately exclude the soft and intermediate EoS A and WFF3, if the magnetic field is such that it does not affect the torsional mode frequencies. The 1.4$M_\odot$ model of EoS APR with an NV description of the crust agrees with the observation, as does the 2.0$M_\odot$ model with EoS L and NV for the crust. EoS L also seems to be compatible with the observation when combined with the DH crust EoS for a mass somewhat larger than 1.6$M_\odot$. Finally, one can observe also that as the stellar mass increases the fundamental mode frequency decreases while the frequency of the 1st overtone increases. Note that this last observation also applies for the frequencies of the higher overtones as well.

According to our estimates, the observed frequencies of 18, 29, 92.5, 150, 626.5, and 1837 Hz would correspond to $2f_0$, $3f_0$, $9f_0$, $15f_0$, $f_1$, and $f_4$, respectively, although it is difficult to explain the observational data of 26 Hz because this data is very close to the other data of 29 Hz and it may be difficult to explain both frequencies by crustal torsional modes only. In this non-magnetized limit, the currently available observational data appear to exclude softer EoS, such as EoS A and WFF3, if crustal torsional modes are invoked in interpreting the data.

As mentioned in the Introduction, one expects a simple relation between the thickness of the crust, $\Delta r/R$ and the ratio of the frequencies of the fundamental mode and higher harmonics, Equation (2). Using our numerical results, we can test the validity of the expected relation and attempt to infer the thickness of the crust from the observed frequencies in several SGRs. We construct the following empirical formula for the crust thickness

$$\frac{\Delta r}{R} \approx \ell \beta_n \frac{f_0}{fn}.$$  (23)
Table 2. Frequencies (in Hz) of the fundamental torsional modes in the non-magnetized limit (i.e. \( n = 0 \) and \( B = 0 \)).

| Model    | \( \ell = 2 \) | \( \ell = 3 \) | \( \ell = 4 \) | \( \ell = 5 \) | \( \ell = 6 \) | \( \ell = 8 \) | \( \ell = 10 \) |
|----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| A+DH\(_{14}\) | 28.5          | 45.1           | 60.5           | 75.4           | 90.1           | 119.2          | 148.1          |
| WFF3+DH\(_{14}\) | 26.3          | 41.6           | 55.9           | 69.7           | 83.3           | 110.2          | 136.8          |
| APR+DH\(_{14}\) | 24.6          | 38.9           | 52.2           | 65.1           | 77.8           | 102.9          | 127.8          |
| L+DH\(_{14}\) | 21.6          | 34.1           | 45.7           | 60.0           | 68.1           | 90.1           | 111.9          |
| A+NV\(_{14}\) | 28.7          | 45.4           | 60.9           | 76.0           | 90.8           | 120.2          | 149.2          |
| WFF3+NV\(_{14}\) | 26.7          | 42.2           | 56.6           | 70.6           | 84.4           | 111.6          | 138.6          |
| APR+NV\(_{14}\) | 25.2          | 39.8           | 53.4           | 66.6           | 79.5           | 105.2          | 130.7          |
| L+NV\(_{14}\) | 23.2          | 36.6           | 49.2           | 61.3           | 73.3           | 96.9           | 120.3          |

Table 3. Frequencies (in Hz) of the first overtone of torsional modes in the non-magnetized limit (i.e. \( n = 1 \) and \( B = 0 \)).

| EoS       | \( \ell = 2 \) | \( \ell = 3 \) | \( \ell = 4 \) | \( \ell = 5 \) | \( \ell = 6 \) | \( \ell = 8 \) | \( \ell = 10 \) |
|-----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| A+DH\(_{14}\) | 1206.2        | 1206.8         | 1207.6         | 1208.6         | 1209.7         | 1212.7         | 1216.4         |
| WFF3+DH\(_{14}\) | 942.4        | 943.0           | 943.9           | 945.0           | 946.3           | 949.6           | 953.7           |
| APR+DH\(_{14}\) | 761.3        | 762.0           | 762.9           | 764.1           | 765.5           | 769.0           | 773.4           |
| L+DH\(_{14}\) | 530.2        | 531.0           | 532.0           | 533.3           | 534.8           | 538.7           | 543.4           |
| A+NV\(_{14}\) | 951.0        | 951.7           | 952.8           | 954.0           | 955.6           | 959.4           | 964.2           |
| WFF3+NV\(_{14}\) | 740.8        | 741.6           | 742.7           | 744.1           | 745.8           | 750.0           | 755.3           |
| APR+NV\(_{14}\) | 616.5        | 617.3           | 618.5           | 621.8           | 623.9           | 626.3           | 631.8           |
| L+NV\(_{14}\) | 483.6        | 484.6           | 485.9           | 487.5           | 489.4           | 494.1           | 500.1           |

where \( \ell \beta_n \) are fitting coefficients. Using our numerical results presented in previous sections we find for the most interesting case of \( \ell = 2 \) and for \( n = 0 \) and \( n = 1 \) the fitted values \( 2\beta_1 = 2.23 \pm 0.29 \) for all models with a crust described by the NV EoS and \( 2\beta_1 = 2.19 \pm 0.28 \) for the corresponding models with a crust described by the DH EoS. These values are somewhat larger than the ones shown in the approximate relation (2), but this should be expected, since the Equation (2) was based on Newtonian dynamics and heuristic arguments.

Using the above empirical relation, important conclusions can be drawn for the stellar models listed in Table 1. The observed frequencies of SGR 1806-20 SGR, \( \ell f_0 = 18 \text{Hz} \) and \( \ell f_1 = 626.5 \text{Hz} \) suggest a crust ratio \( \Delta r/R \approx 0.06 \) which combined with the observed frequencies favors the model L+NV\(_{22}\). Although, this conclusion is based on a single observation, it is still very important because it favors neutron star models with a very stiff equation of state, implying considerably larger masses than the typical ones. One should still keep in mind that there is an uncertainty regarding the observed frequency at 18Hz. If the actual frequency for the \( \ell f_0 \) mode is at 29Hz then the empirical formula (23) implies \( \Delta r/R \approx 0.10 \), which suggest stellar models with considerable smaller mass such as APR+NV\(_{14}\) again.

We have already discussed earlier the possible effect of the magnetic field on the frequencies of the various torsional modes, when the magnetic field is confined to the crust. An approximate formula for the effect of the magnetic field based on Newtonian physics has been suggested by [7], see also a detailed discussion by [17]. The formula suggests that the corrections in the
frequency coming from the magnetic field scale as $B^2$ i.e.

$$\frac{\ell f_n}{\ell f_n^{(0)}} = \left[1 + \left(\frac{B}{B_\mu}\right)^2\right]^{1/2}$$

(24)

where $\ell f_n^{(0)}$ is the frequency in the absence of any magnetic field and $B_\mu$ is a typical magnetic field strength at which magnetic field effects on torsional modes have become important, which we take here to be $B_\mu = 4 \times 10^{15}$ G. In the limit of $B << B_\mu$, this result agrees with earlier studies of the effect of the magnetic field on spheroidal modes derived by [5], where the effect of the magnetic field has been shown to have the behavior

$$f \approx f^{(0)} + \ell \tilde{\alpha}_n B^2$$

(25)

where $\ell \tilde{\alpha}_n$ is a coefficient depending on the parameters of the star ($M$, $R$ and EoS) and it can be derived easily if the eigenfunctions of a specific mode are available. When one uses general relativity to calculate the frequencies of the torsional modes the relativistic effects such as the relativistic form of the sound speed, the redshift corrections and the EoS affect the weighting factor $\ell \alpha_n$ of equation (25) and have to be taken properly into account.

In our numerical calculations of the torsional modes of magnetized neutron stars for the various EoS mentioned earlier we derived lists of frequencies for every model, for magnetic field strengths up to $B = 10^{17}$ G. By numerical fitting we have thus found the coefficients $\ell \alpha_n$ of the following empirical formula

$$\frac{\ell f_n}{\ell f_n^{(0)}} \approx \left[1 + \ell \alpha_n \left(\frac{B}{B_\mu}\right)^2\right]^{1/2}.$$  

(26)

The coefficients $\ell \alpha_n$ are listed in Table 4 and the way that the various EoS (both for the core and the crust) affect the torsional frequencies becomes apparent. In general, it seems that models constructed with the DH equation of state are affected significantly more by the magnetic field than models following the NV EoS, for all harmonics. As a result, the large differences in the torsional mode frequencies between models constructed with the stiffest EoS L but with different crust EoS are diminished by magnetic field effects around $B = 4 \times 10^{15}$ G and reversed for larger values of $B$.

When the magnetic field strength is equal to $B = B_\mu = 4 \times 10^{15}$ G the frequencies have increased by up to 35% for the fundamental torsional mode (independently of the value of $\ell$) and up to 100% for the first overtone. The results suggest that for magnetic field strengths exceeding roughly $10^{15}$ G the shift in the frequencies is significant and should be taken into account in any attempt to fit the observational data with specific stellar models.

In Figure 1 we show some examples of the effect of the magnetic field on the torsional mode frequencies. Notice that for $B < B_\mu$ the effect of the magnetic field on the frequencies follows a quadratic increase, in agreement with the approximate relation (25). On the other hand, when $B > B_\mu$ the modes change character and become dominated by the magnetic field, while the frequencies tend to become less sensitive to the stellar parameters. In this regime, additional effects, such as the coupling to global magnetosonic waves and to higher-order harmonics should be taken into account.
Table 4. The values for the fitting factors $\ell\alpha_n$ of equation (26). The fitting factors have been calculated for magnetic field strength up to $10^{17}$ G.

| Model       | $2\alpha_0$ | $2\alpha_1$ | $5\alpha_0$ | $5\alpha_1$ | $8\alpha_0$ | $8\alpha_1$ |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| A+DH$_{14}$ | 0.38        | 1.44        | 0.67        | 1.63        | 0.70        | 1.65        |
| WFF3+DH$_{14}$ | 0.45      | 1.55        | 0.70        | 1.75        | 0.72        | 1.77        |
| APR+DH$_{14}$ | 0.54      | 1.63        | 0.71        | 1.84        | 0.72        | 1.87        |
| L+DH$_{14}$ | 0.80        | 1.83        | 0.77        | 2.06        | 0.75        | 2.09        |
| A+NV$_{14}$ | 0.26        | 0.56        | 0.32        | 0.63        | 0.33        | 0.64        |
| WFF3+NV$_{14}$ | 0.34      | 0.60        | 0.33        | 0.68        | 0.33        | 0.69        |
| APR+NV$_{14}$ | 0.42      | 0.63        | 0.35        | 0.72        | 0.33        | 0.73        |
| L+NV$_{14}$ | 0.55        | 0.69        | 0.37        | 0.78        | 0.34        | 0.79        |

Figure 1. The frequencies of the fundamental $n=0$ and the first overtone $n=1$ for $\ell=2,3$ and 4 torsional modes as functions of the normalized magnetic field ($B/B_\mu$) [left panel]. The frequencies of the fundamental torsional mode $\ell_0$ for four EoS. The lines correspond to fits according to the empirical formula, Eq. (26), with coefficient values from Table 4.

6. Summary and Discussion

We have derived the formalism for computing torsional oscillations of relativistic stars endowed with a strong dipole magnetic field, confined to the crust. Our equations are valid in the Cowling approximation (no spacetime perturbations) and we have neglected the effect of the magnetic field on the equilibrium configuration and on the coupling of torsional modes to $\ell\pm 2$ terms and to global magnetosonic modes. Under these approximations, our formalism allows us to obtain an estimate of the magnetic field effects on torsional modes, up to moderate values of the magnetic field strength (up to a few times $B_\mu = 4 \times 10^{15}$ G). These moderate values of the magnetic field strength are appropriate for models of magnetars, for which there is strong evidence that they are at the heart of the SGR phenomenon.

Our numerical results have shown that torsional mode frequencies are sensitive to the crust model if the high-density equation of state is very stiff (such as EoS L). In addition, torsional mode frequencies are drastically affected by a dipole magnetic field, if the latter has a strength exceeding roughly $10^{15}$ G. The effect of the magnetic field is surprisingly sensitive to the adopted
crust model. Using our extended numerical results we have derived empirical relations for the
effect of the magnetic field on torsional modes as well as for the crust thickness. We compare
our numerical results to observed frequencies in SGRs and find that certain high-density EoS
and mass values are favored over others in the non-magnetized limit. On the other hand,
if the magnetic field is strong, then its effect has to be taken into account in attempts to
formulate a theory of asteroseismology for magnetars. This topic, as well as the inclusion
of global magnetosonic modes are discussed in [26].

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