ON INTRINSIC NEGATIVE CURVES

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ABSTRACT. Let \( K \) be an algebraically closed field of characteristic 0. A curve of \((K^\ast)^2\) arising from a Laurent polynomial in two variables is intrinsic negative if its tropical compactification has negative self-intersection. The aim of this note is to start a systematic study of these curves and to relate them with the problem of computing Seshadri constants of toric surfaces.

INTRODUCTION

Following the work of González Anaya, González, Karu [16], Kurano [21] and Kurano Matsnoka [22], we define a class of curves on the blowing-up of toric surfaces at a general point. Let \( f \) be a Laurent polynomial in two variables and let \( \Gamma \subseteq (K^\ast)^2 \) be its zero locus. The normal fan to the Newton polygon \( \Delta \) of \( f \) defines a toric variety \( \mathbb{P} \) such that the compactification of \( \Gamma \) is contained in the smooth locus of \( \mathbb{P} \). Such a compactification is called tropical, see [28]. Denote by \( X := \text{Bl}_e\mathbb{P} \) the blowing-up of \( \mathbb{P} \) at the image \( e \) of \((1,1)\) and let \( C \) be the strict transform of the compactified curve. We say that \( C \) is an intrinsic negative curve (resp. non positive) if \( C^2 < 0 \) (resp. \( C^2 \leq 0 \)), cfr. [21, Definition 3.1]. Our first result is the construction of infinite families of intrinsic non-positive curves. In the following table \( \text{lw}(\Delta) \) is the lattice width of \( \Delta \), defined in Section 1, while \( g(C) \) is the genus of the curve \( C \).

**Theorem 1.** There exist infinite families of non-positive intrinsic curves, whose Newton polygons are listed in the following table

| vertices of \( \Delta \) | \( \text{lw}(\Delta) \) | \( C^2 \) | \( g(C) \) |
|------------------------|-------------------|-------|--------|
| (i) \([0, \frac{m}{1}, \frac{1}{m}]\) | \( m \geq 2 \) | \(-1\) | \(0\) |
| (ii) \([0, \frac{m}{1}, \frac{1}{m}, \frac{1}{m}]\) | \( m \geq 4 \) | \(-1\) | \(0\) |
| (iii) \([0, \frac{m}{1}, \frac{2}{m}, \frac{m}{1}, \frac{1}{m}, \frac{1}{m}, \frac{1}{m}]\) | \( m = 2k \geq 8 \) | \(-2\) | \(0\) |
| (iv) \([0, \frac{m}{1}, \frac{2}{m}, \frac{m}{1}, \frac{1}{m}, \frac{1}{m}, \frac{1}{m}]\) | \( m \geq 4 \) | \(0\) | \(0\) |

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Before stating the next result we recall that given a projective variety \(P\), an ample class \(H\) and a point \(x \in P\), the Seshadri constant of \(H\) at \(x\) can be defined as

\[
\varepsilon(H, x) := \inf_{x \in C} \frac{H \cdot C}{\text{mult}_x(C)}
\]

where the infimum is taken over all irreducible curves through \(x\). The problem of finding Seshadri constants of algebraic surfaces have been widely studied (see for instance [4,14,15,26] and the references therein). When \(P\) is a toric surface there are three possibilities for \(x \in P\): either the point is torus-invariant, or it lies on a torus-invariant curve, or it is general. In the first two cases, since the blowing-up \(\text{Bl}_x P\) admits the action of a torus of dimension two and one respectively, it is possible to describe the effective cone (see [11,27] and [2,§5.4] for a description of the Cox ring), and hence to compute the Seshadri constant (see [5,§4] and [20,§3.2]). Concerning a general point, in [20, Thm. 1.3]) a lower bound for the Seshadri constant is given.

In this note we focus on the case of a general point. In particular we prove some relations between the geometry of a lattice polygon \(\Delta\) and the Seshadri constant \(\varepsilon(H_\Delta, e)\), where \((\mathbb{P}_\Delta, H_\Delta)\) is the toric pair defined by \(\Delta\) (see § 1 for the definition) and \(e \in \mathbb{P}_\Delta\) is a general point. The recent interest in these Seshadri constants and more generally in the Cox ring of blow-ups of toric varieties at a general point has been motivated by the work of Castravet and Tevelev [9] where the authors prove that the finite generation of the the Cox ring of \(\text{M}_{0,n}\) implies that of certain blow-ups of toric varieties at a general point.

In order to state our result, given a non-negative integer \(m\) denote by \(L_\Delta(m)\) the linear system of Laurent polynomials whose exponents are integer points of \(\Delta\) and such that all the partial derivatives up to order \(m - 1\) vanish at \((1,1)\). If we denote by \(\text{vol}(\Delta)\) the normalized volume of \(\Delta\) (that is twice its euclidean area), we have the following (the first inequality is well known [1, Thm. 0.1] and [20], but we state it anyway for the sake of completeness).

**Theorem 2.** Let \(\Delta \subseteq \mathbb{Q}^2\) be a lattice polygon, let \((\mathbb{P}_\Delta, H_\Delta)\) be the corresponding toric pair and let \(\varepsilon := \varepsilon(H_\Delta, e)\) be the Seshadri constant at \(e \in \mathbb{P}_\Delta\). Then the following hold.

(i) \(\varepsilon \leq \text{lw}(\Delta)\).

(ii) If \(\text{vol}(\Delta) > \text{lw}(\Delta)^2\) then \(\varepsilon \in \mathbb{Q}\).

(iii) If there exists \(m \in \mathbb{N}\) such that \(\text{vol}(\Delta) \leq m^2\) and \(L_\Delta(m) \neq \emptyset\), then \(\varepsilon \in \mathbb{Q}\) and \(\varepsilon \leq \text{vol}(\Delta)/m\).

(iv) If moreover \(L_\Delta(m)\) contains an irreducible curve, then \(\varepsilon = \text{vol}(\Delta)/m\).

We remark that Theorem 2 provides in some cases (like e.g. [18, Example 5.7]) an alternative proof for the rationality of the Seshadri constant of a toric surface at a general point. Moreover it allows to compute the exact value of the Seshadri constant \(\varepsilon(H_\Delta, e)\) when \((X_\Delta, H_\Delta)\) is the toric pair associated to the Newton polygon of an intrinsic non-positive curve.
Corollary 3. Let $f \in \mathbb{K}[u^{\pm 1}, v^{\pm 1}]$ be a Laurent polynomial with Newton polygon $\Delta$ and multiplicity $m$ at $(1, 1)$, such that the corresponding intrinsic curve $C \subseteq X_\Delta$ is non-positive, i.e. $C^2 \leq 0$. Then the Seshadri constant of the ample divisor $H_\Delta$ of the toric surface $\mathbb{P}_\Delta$ at a general point $e \in \mathbb{P}_\Delta$ is

$$
\varepsilon = \frac{\text{vol}(\Delta)}{m}.
$$

In particular the polygons of the infinite families appearing in Theorem 1 have Seshadri constant $\varepsilon = \text{vol}(\Delta)/\text{lw}(\Delta) \in \mathbb{Q}$.

In order to prove the last statement we are going to apply Theorem 2(iv), showing that in each case there exists an irreducible curve in $\mathcal{L}(m)$, where $m = \text{lw}(\Delta)$ and $\text{vol}(\Delta) \leq m^2$. We remark that for the triangles of type (i) in Theorem 1, the upper bound of Theorem 2(iii) coincides with the lower bound given by [20, §3.2], so that it is also possible to deduce the exact value of the Seshadri constant without producing the irreducible curve, but for all the other families of polygons appearing in Theorem 1, the two bounds are different (see also Remark 3.2).

The paper is structured as follows. In § 1, after recalling some definitions and results about toric varieties and lattice polytopes we introduce intrinsic curves and we prove some preliminaries result. In § 2 we consider infinite families of intrinsic curves: we first prove Theorem 1, and then we construct an infinite family of intrinsic negative curves on a given toric surface (Example 2.4). Finally, § 3 is devoted to Seshadri constants on toric surfaces. We first prove Theorem 2 and Corollary 3, and then we discuss some possible applications to the study of the blowing up of weighted projective planes at a general point.

1. INTRINSIC CURVES

Let us first recall some definitions and set some notations we are going to use throughout this note.

Let $\Delta \subseteq \mathbb{Q}^n$ be a lattice polytope i.e. a polytope whose vertices have integer coordinates. We recall that given a non zero primitive vector $v \in \mathbb{Z}^n$ the lattice width of $\Delta$ in the direction $v$ is $\text{lw}_v(\Delta) := \max(\Delta, v) - \min(\Delta, v)$ and the lattice width of $\Delta$ is $\text{lw}(\Delta) := \min\{\text{lw}_v(\Delta) : v \in \mathbb{Z}^n\}$, see [24, Def. 1.8].

Given a lattice polytope $\Delta \subseteq \mathbb{Q}^n$ we can define a pair $(\mathbb{P}_\Delta, H_\Delta)$ consisting of a toric variety $\mathbb{P}_\Delta$ together with a very ample divisor $H_\Delta$. The toric variety is the normalization of the closure of the image of the following monomial morphism

$$
g_\Delta: (\mathbb{K}^*)^n \to \mathbb{P}[\Delta \cap \mathbb{Z}^n]^{-1}, \quad u \mapsto [u^w : w \in \Delta \cap \mathbb{Z}^n],
$$

where $u = (u_1, \ldots, u_n) \in (\mathbb{K}^*)^n$. It is possible to show that the action of the torus $(\mathbb{K}^*)^n$ on itself extends to an action on $\mathbb{P}_\Delta$ and that the subset of prime torus-invariant divisors is finite and in bijection with the set of facets of $\Delta$. Let $D_1, \ldots, D_r$ be such divisors and let $v_1, \ldots, v_r$ be the inward normal vectors to the facets of $\Delta$. Each $v_i$ defines a linear form $Q_i^* \to \mathbb{Q}$ by $w \mapsto w \cdot v_i$ and the very ample divisor is [18, Prop. 3.1]:

$$
H_\Delta := -\sum_{i=1}^r \min_{w \in \Delta} \{w \cdot v_i\} D_i.
$$

On the other hand, any divisor $D$ on $\mathbb{P}_\Delta$ is equivalent to a combination $\sum_i a_i D_i$, so that we can associate to it the Riemann-Roch polytope

$$
\Delta_D := \{w \in \mathbb{Q}^n : w \cdot v_i \geq -a_i, \forall i = 1, \ldots, r\}.$$

We remark that if $D$ is very ample then the toric variety associated to $\Delta_D$ coincides with $\mathbb{P}_\Delta$. We recall that if $\Delta$ is a very ample polytope [11, Def. 2.2.17] then the closure of the image of $g_\Delta$ is a normal variety by [11, Thm. 2.3.1] and thus it coincides with $\mathbb{P}_\Delta$. Moreover, by [11, Cor. 2.2.19], $\Delta$ is very ample if $n = 2$.

From now on we restrict to the case $n = 2$, i.e. $\Delta \subseteq \mathbb{Q}^2$ is a lattice polygon, so that $\mathbb{P}_\Delta$ is a normal toric surface. We will denote by $e \in \mathbb{P}_\Delta$ the image via $g_\Delta$ of the neutral element of the torus, by $\pi: X_\Delta \to \mathbb{P}_\Delta$ the blowing up of $\mathbb{P}_\Delta$ at $e$ and by $E$ the exceptional divisor. Given an $m \in \mathbb{N}$ we will denote by $\mathcal{L}_\Delta(m)$ the sublinear system of $|H_\Delta|$ consisting of sections having multiplicity at least $m$ at $e$.

**Definition 1.1.** Let $f \in \mathbb{K}[u^1, v^1]$ be an irreducible Laurent polynomial, let $\Delta \subseteq \mathbb{Q}^2$ be the Newton polygon of $f$, i.e. the convex hull of its exponents, and let $\Gamma \subseteq \mathbb{P}_\Delta$ be the closure of $V(f) \subseteq (\mathbb{K}^*)^2$. We say that the strict transform $C \subseteq X_\Delta$ of $\Gamma$ is the *intrinsic curve* defined by $f$ and that $C$ is:

- an *intrinsic negative* (resp. *non-positive*) curve if $C^2 < 0$ (resp. $C^2 \leq 0$);
- an *intrinsic* $(-n)$-curve if $C^2 = -n < 0$ and $p_\Delta(C) = 0$;
- *expected* in $X_\Delta$, with $\Delta \subseteq \Delta'$ if $|\Delta' \cap \mathbb{Z}^2| > \binom{m+1}{2}$.

In what follows we will often use the notation $\mathbb{P}$, $X$ and $H$, omitting the subscript when it is clear from the context.

We remark that, with the notation above, $\Gamma \subseteq \mathbb{P}$ is an element of the very ample linear series $|H|$ and $\Gamma \in \mathcal{L}_\Delta(m)$ if $f$ has multiplicity at least $m$ at $(1, 1)$, that is all the partial derivatives of $f$ up to order $m - 1$ vanish at $(1, 1)$. Moreover if the multiplicity is $m$ then the strict transform $C \subseteq X$ of $\Gamma$ is a Cartier divisor such that

$$C^2 = \text{vol}(\Delta) - m^2 \quad p_\Delta(C) = \frac{1}{2} \left(\text{vol}(\Delta) - |\partial \Delta \cap \mathbb{Z}^2| + m - m^2\right) + 1,$$

see for instance [8, §4]. By abuse of notation we will sometimes refer to the Newton polygon $\Delta$ as the *Newton polygon of $C$*, and we will simply say that $C$ is *expected* if it is expected in $X_\Delta$, that is the linear system $\mathcal{L}_\Delta(m)$ has a non-negative expected dimension.

Our first result is about the characterization of Newton polygons of expected non-positive curves.

**Proposition 1.2.** Let $C$ be an intrinsic non-positive expected curve with Newton polygon $\Delta$ and multiplicity $m$ at $e$. Then one of the following holds:

- $\text{vol}(\Delta) = m^2$ and $|\partial \Delta \cap \mathbb{Z}^2| = m$;
- $\text{vol}(\Delta) = m^2$ and $|\partial \Delta \cap \mathbb{Z}^2| = m + 2$;
- $\text{vol}(\Delta) = m^2 - 1$ and $|\partial \Delta \cap \mathbb{Z}^2| = m + 1$.

In particular $C^2 \in \{-1, 0\}$ and $p_\Delta(C) \in \{0, 1\}$.

**Proof.** Let us denote by $b := |\partial \Delta \cap \mathbb{Z}^2|$ the number of boundary lattice points of $\Delta$ and by $i := |\Delta \cap \mathbb{Z}^2| - b$ the number of interior lattice points. Recall that by Pick’s formula $\text{vol}(\Delta) = 2i + b - 2$. Since $C$ is expected and non-positive, we have $|\Delta \cap \mathbb{Z}^2| \geq \binom{m+1}{2} + 1$ and $\text{vol}(\Delta) \leq m^2$. By (1.2), the non-negativity of the arithmetic genus of $C$ gives $\frac{1}{2}(\text{vol}(\Delta) - b + m - m^2) + 1 \geq 0$. The three inequalities in terms of $i$ and $b$ are $2i + 2b \geq m(m+1)$, $2i + b - 2 \leq m^2$, $2i - m^2 + m \geq 0$. From these one deduces that one of the following holds:

\[
\begin{align*}
\begin{cases}
  b = m & \\
  i = \frac{m^2 - m}{2} + 1
\end{cases},
\begin{cases}
  b = m + 2 & \\
  i = \frac{m^2 - m}{2}
\end{cases},
\begin{cases}
  b = m + 1 & \\
  i = \frac{m^2 - m}{2}
\end{cases}
\end{align*}
\]
Since $C^2 = \text{vol}(\Delta) - m^2$ and $C \cdot K = m - b$, in the first two cases we have $C^2 = 0$ and $p_a(C) = 1$ and 0 respectively, while in the last one $C^2 = -1$ and $p_a(C) = 0$. □

**Proposition 1.3.** All the non-equivalent polygons for intrinsic non-positive curves of size $=2$

| $m$ | $\Delta$ |
|-----|--------|
| 2   | ![Polygon 2] |
| 3   | ![Polygon 3] |
| 4   | ![Polygon 4] |

Proof. We use the database of polygons with small volume [3] to analyze all the polygons with volume at most 16 and such that $\text{vol}(\Delta) - m^2 \leq 0$. For each such polygon $\Delta$ we compute $L_\Delta(m)$, where $m \leq 4$, with the aid of the function `FindCurves` of the Magma library:

https://github.com/alaface/non-polyhedral/blob/master/lib.m

We take only the pairs $(\Delta, m)$ such that $L_\Delta(m)$ contains exactly one element, which is irreducible. Finally we check that in each of these cases the Newton polygon coincides with $\Delta$. □

Remark 1.4. The above intrinsic curves are all expected. In all but the last case they are intrinsic $(-1)$-curves, in the last case the curve has self-intersection 0 and genus 1. The smallest value of $m$ for an unexpected intrinsic negative curve is 5.

The lattice polygon $\Delta$ is the following

One has $|\partial \Delta \cap \mathbb{Z}^2| = m - 1$ and $|\Delta \cap \mathbb{Z}^2| = \binom{m+1}{2}$ which imply that the corresponding curve has arithmetic genus 1. The curve is defined by the Laurent polynomial

$$
1 - 8uv + 3u^2v^2 + 6u^2v^4 - u^2v^6 + 3u^2v + 20u^2v^2 \\
-18u^2v^3 - 18u^3v^2 + 8u^3v^3 + 6u^4v^2 - u^4v^4 - u^5v^2,
$$

which is the unique one whose Newton polygon is contained in $\Delta$ and has multiplicity 5 at $(1, 1)$. Its strict transform in $X_\Delta$ is smooth of genus 1 and self-intersection $-1$.

The proof of Proposition 1.3 suggests the following definitions for a pair $(\Delta, m)$.

**Definition 1.5.** Let $\Delta$ be a lattice polygon, $m$ a positive integer, and set $p_a := \frac{1}{2}(\text{vol}(\Delta) - |\partial \Delta \cap \mathbb{Z}^2| + m - m^2) + 1$. We say that $(\Delta, m)$ is:

- numerically negative (resp. non positive) if $\text{vol}(\Delta) - m^2 < 0$ (resp. $\leq$);
- a $(-n)$-pair if $\text{vol}(\Delta) - m^2 = -n < 0$ and $p_a = 0$;
- expected if $|\Delta \cap \mathbb{Z}^2| > \binom{m+1}{2}$. 


Remark 1.6. Clearly, if $C$ is an intrinsic negative curve, then the pair $(\Delta, m)$, consisting of the Newton polygon of $C$ and the multiplicity of $\Gamma = \pi(C)$ at $e$, is numerically negative. On the other hand, if a pair $(\Delta, m)$ is numerically negative, in general there does not exist an intrinsic negative curve associated to it. Indeed, first of all it can happen that $L_{\Delta}(m)$ is empty (see Example 1.7). Furthermore, even if $(\Delta, m)$ is expected (so that $L_{\Delta}(m)$ is not empty), in some cases it contains only reducible curves (see Example 1.8).

Example 1.7. Consider the following polygon $\Delta$

We have that $\text{vol}(\Delta) = 14$ and $|\partial \Delta \cap \mathbb{Z}|^2 = 4$, so that the pair $(\Delta, 4)$ is numerically a $(-2)$-pair. A direct computation shows that $\mathcal{L}_\Delta(4) = \emptyset$, so that there does not exists an intrinsic $(-2)$-curve with Newton polygon $\Delta$ and multiplicity 4.

Example 1.8. The polygon $\Delta$, whose Minkowski decomposition $\Delta = \Delta_1 + \Delta_2$ is given in the below picture has $\text{vol}(\Delta) = 48$, $|\partial \Delta \cap \mathbb{Z}|^2 = 8$ and $|\Delta \cap \mathbb{Z}|^2 = \binom{7+1}{2} + 1$, so that $(\Delta, 7)$ is an expected $(-1)$-pair. The only element in $\mathcal{L}_\Delta(7)$ is defined by

$$(u^2 v + uv^2 - 3uv + 1) \cdot (u^5 v^3 - 2u^5 v^2 - 6u^4 v^3 + 11u^4 v^2 - 2u^3 v^4 + 17u^3 v^3 - 24u^3 v^2 - u^2 v^2 + 7u^2 v^2 - 22u^2 v^3 + 21u^2 v^2 + 5u^2 v + 4uv^2 - 9uv + 1).$$

The factorization implies the Minkowski decomposition $\Delta = \Delta_1 + \Delta_2$. The polygon $\Delta_1$ corresponds to an intrinsic $(-1)$-curve $C_1$, while $\Delta_2$ corresponds to an intrinsic curve $C_2$ of self-intersection 0 and genus 1. By Proposition 1.12 it follows that $C_1 \cdot C_2 = 0$.

Remark 1.9. Finally we observe that if the pair $(\Delta, m)$ is numerically non positive and $\mathcal{L}_\Delta(m)$ contains an irreducible curve $\Gamma$, then a Laurent polynomial $f$ of $\Gamma \cap (\mathbb{K}^*)^2$ defines an intrinsic non positive curve. Indeed, by definition the strict transform $C \subseteq X_{\Delta}$ of $\Gamma$ satisfies $C^2 = \text{vol}(\Delta) - m^2 \leq 0$. Moreover, since the Newton polygon $\Delta'$ of $f$ is contained in $\Delta$, we also have that $\text{vol}(\Delta') - m^2 \leq \text{vol}(\Delta) - m^2 \leq 0$. We remark that if $\Delta'$ is strictly contained in $\Delta$, then the self intersection of the intrinsic curve is strictly smaller than $C^2$ (see Example 1.10).

Example 1.10. Let $\Delta$ and $\Delta'$ be the following polygons, respectively from left to right
One has \( \text{vol}(\Delta) = 35 \), \( |\partial \Delta \cap \mathbb{Z}^2| = 7 \) and \( |\Delta \cap \mathbb{Z}^2| = \binom{6+1}{2} + 1 \), so that \( (\Delta, 6) \) is an expected \((-1)\)-pair. The linear system \( L_\Delta(6) \) contains a unique irreducible curve defined by the following polynomial
\[
f := -4u^6v^3 + 3u^6v^2 - 6u^5v^4 + 30u^5v^3 - 18u^5v^2 - u^4v^6 + 2u^4v^5 + 17u^4v^4 - 62u^4v^3 + 25u^4v^2
+ 4u^4v + 4u^4v^5 - 26u^3v^4 + 50u^3v^3 + 2u^3v^2 - 10u^3v + 6u^2v^3 - 27u^2v^2 + 6u^2v + 6uv - 1.
\]

The Newton polygon of \( f \) is \( \Delta' \), since \( u \) is the only monomial (corresponding to a lattice point of \( \Delta \)) that does not appear in \( f \). In particular, \( \text{vol}(\Delta') = 34 \) and \( |\partial \Delta \cap \mathbb{Z}^2| = 6 \), so that \( f \) defines an intrinsic (unexpected) \((-2)\)-curve.

\textbf{Definition 1.11.} Given two polygons \( \Delta_1, \Delta_2 \) their \textit{mixed volume} is:
\[
\text{vol}(\Delta_1, \Delta_2) := \frac{1}{2}(\text{vol}(\Delta_1 + \Delta_2) - \text{vol}(\Delta_1) - \text{vol}(\Delta_2)).
\]

We conclude the section by showing how the mixed volume of two lattice polygons relates with the intersection product of curves on a toric variety whose fan refines the normal fans of the two polygons.

\textbf{Proposition 1.12.} Let \( (\Delta_1, m_1), (\Delta_2, m_2) \), be two pairs, each of which consists of a lattice polygon together with a positive integer. Assume that \( L_{\Delta_1}(m_1) \) is non-empty and let \( C_i \subseteq X_{\Delta_i} \) be the strict transform of a curve in the linear system. Let \( X \) be a surface which admits birational morphisms \( \phi_i : X \to X_{\Delta_i} \) for \( i = 1, 2 \). Then
\[
\phi_1^*(C_1) \cdot \phi_2^*(C_2) = \text{vol}(\Delta_1, \Delta_2) - m_1m_2.
\]

\textbf{Proof.} Let \( \pi_i : X_{\Delta_i} \to \mathbb{P}_{\Delta_i} \), be the blowing-up at \( e \in \mathbb{P}_{\Delta_i} \) with exceptional divisor \( E_i \). Since \( C_i \sim \pi_i^*H_i - m_iE_i \), where \( H_i \) is a very ample divisor on \( \mathbb{P}_{\Delta_i} \), we can assume that the support of the divisor \( \pi_i^*H_i - m_iE_i \) does not contain any singular point of \( X_{\Delta_i} \). Thus \( C_i \) is a Cartier divisor of \( X_{\Delta_i} \) and the pullback \( \phi_i^* \) is defined on \( C_i \). The intersection product \( \phi_1^*(C_1) \cdot \phi_2^*(C_2) \) does not depend on the surface \( X \) because all such surfaces differ by exceptional divisors, which have zero intersection product with the pullbacks of \( C_1 \) and \( C_2 \). We can then choose \( X := X_{\Delta} \) to be the blowing-up of \( \mathbb{P}_{\Delta} \) at the general point \( e \), where \( \Delta := \Delta_1 + \Delta_2 \). Since \( H_i \) is very ample on \( \mathbb{P}_{\Delta_i} \), its pullback is base point free in \( \mathbb{P}_{\Delta} \). By Bertini’s theorem the general elements of these two linear systems intersect transversely at distinct points which, without loss of generality, we can assume to be contained in \((\mathbb{K}^* )^2\). By Bernstein-Kushnirenko theorem the number of these intersections is \( \text{vol}(\Delta_1, \Delta_2) \), so that the statement follows after taking into account the intersections of the two curves at \( e \in \mathbb{P}_{\Delta} \). \( \square \)

2. Infinite families

In this section we construct infinite families of intrinsic negative curves. First of all we produce infinite families of toric surfaces, each of which corresponds to an intrinsic negative curve. Then, in Example 2.4, we construct an infinite family of intrinsic negative curves on a given toric surface.

\textbf{Lemma 2.1.} Let \( f_1, f_2, f_3, f_4 \in \mathbb{K}[t] \) and let \( m \) be the maximal degree of the four polynomials. Assume that \( f_1 - f_2 = f_4 - f_3 \), \( \text{Gcd}(f_1, f_2) = \text{Gcd}(f_3, f_4) = 1 \), and \( \text{deg}(f_1 - f_2) = m \). Then the image of the following rational map
\[
\mathbb{P}^1 \longrightarrow (\mathbb{K}^* )^2, \quad t \mapsto (f_1/f_2, f_3/f_4)
\]
has multiplicity \( m \) at \((1, 1)\).
Proof. Since \( \deg(f_1 - f_2) = m \) and \( \mathbb{K} \) algebraically closed, it follows that \( f_1 - f_2 \) has \( m \) roots. Any root of \( f_1 - f_2 \) is also a root of \( f_3 - f_4 \), so that we conclude that there are \( m \) values of \( t \) (counting multiplicities) whose image is the point \((1,1)\). \( \square \)

Proof of Theorem 1. First of all, each polygon \( \Delta \) of type (v) satisfies \( \text{lw}(\Delta) = m \), \( \text{vol}(\Delta) = m^2 \) and \( |\partial \Delta \cap \mathbb{Z}^2| = m \), so that the pair \((\Delta, m)\) is numerically 0 and expected (in particular it has arithmetic genus 1). Moreover, in [8, Section 6] it has been shown that if we set \( m = 2k + 4 \), for each \( k \geq 1 \) there exists an irreducible curve of genus 1 whose Newton polygon is \( \Delta \). Therefore we are left with cases (i) to (iv), for which the arithmetic genus of the pair is 0. In these cases, consider the polynomial functions \( f_1, f_2, f_3, f_4 := f_1 - f_2 + f_3 \) given in the following table.

|   | \( f_1 \)  | \( f_2 \)   | \( f_3 \)   |
|---|----------------|----------------|----------------|
| (i) | \(-1\)         | \( \sum_{i=1}^{m} t^i \) | \( t^m \)         |
| (ii) | \((m-1)t - (m-2)\) | \(-t(t-1)^{m-1} \) | \(-t(t-1)^{m-1} \) |
|     | \( a^{2k-2}(t-1) \) | \( \frac{t^{2k-3}(t-a^2)(t^2-a^2)}{a^2} \) | \( \frac{t^{2k-1}(t-a^2)}{a^2} \) |
| (iv) | \( 2t - 1 \)    | \((1-t)t^{m-1} \)   | \( -(t-1)^2 \) |

These polynomials satisfy the hypotheses of Lemma 2.1, so that the image of the map \( \varphi(t) = (f_1/f_2, f_3/f_4) \) has a point of multiplicity \( m \) at \( e \). In order to conclude we have to show that in each case the Newton polygon is the one given in the first column of the table within the proposition. To this aim we will use [13, Thm. 1.1] which, given a parametric curve \( \Gamma \subseteq (\mathbb{K}^*)^2 \), provides a description of the normal fan of the Newton polygon of \( \Gamma \) together with the length of the edges, in terms of the zeroes of the four polynomials \( f_1, \ldots, f_4 \). For the sake of completeness we explain in detail case (i). In this case the map \( \varphi \) is defined by

\[
\varphi(t) = \left( -\frac{1}{t \sum_{i=0}^{m-1} t^i}, -\frac{t^m}{\sum_{i=0}^{m-1} t^i} \right).
\]

Since \( \varphi \) satisfies \( \text{ord}_0(\varphi) = (-1, m) \), \( \text{ord}_\infty(\varphi) = (m, -1) \) and \( \text{ord}_{q_1}(\varphi) = (-1, -1) \), for all the \( m - 1 \) roots \( q_1, \ldots, q_{m-1} \) of \( \sum_{i=0}^{m-1} t^i \), these are the only values of \( t \) for which \( \text{ord}(\varphi) \) does not vanish. By [13, Thm. 1.1], the rays of the normal fan of the Newton polygon of \( \varphi(\mathbb{P}^1) \) are \((-1, m), (m, -1) \) and \((-1, -1) \). Moreover, the first two rays correspond to two edges of lattice length 1 while the third one has length \( m - 1 \). We conclude that the Newton polygon has vertices \((0,0), (m,1), (1,m)\). \( \square \)
Remark 2.2. The triangles of type (i) in Theorem 1 are indeed equivalent to the ones with vertices \((0, 0), (m - 1, 0), (m, m + 1)\), i.e. \(IT(m - 1, 1)\) in the notation of [16, Thm. 1.1.A]. Therefore, as a byproduct of Theorem 1 we obtain an alternative (short) proof of [17, Thm. 1.1].

Observe that for each infinite family of Theorem 1, the slopes of (some of) the edges change with \(m\), so that also the toric surfaces change. We are now going to give an example of an infinite family of negative curves lying on the blowing-up of a fixed toric surface. First of all we recall a construction from [8]. Given an expected lattice polygon \(\Delta \subseteq \mathbb{Q}^2\) of width \(m := \text{lw}(\Delta)\), with \(\text{vol}(\Delta) = m^2\) and \(|\partial \Delta \cap \mathbb{Z}^2| = m\), if \(\mathcal{L}_\Delta(m)\) contains an unique irreducible element, then its strict transform \(C \subseteq X := X_\Delta\) is a curve of arithmetic genus one with \(C^2 = 0\). Whenever \(C\) is smooth we denote by

\[
\text{res}: \text{Pic}(X) \to \text{Pic}(C)
\]

the pullback induced by the inclusion. It is not difficult to show that the image of the above map is contained in \(\text{Pic}(C)(\mathbb{Q})\). If \(\text{res}(C) \in \text{Pic}^0(C)\) is non-torsion then, by [8, Sec. 3], the divisor \(K_X + C\) is linearly equivalent to an effective divisor whose support can be contracted by a birational morphism \(\phi: X \to Y\). The surface \(Y\) has at most Du Val singularities and nef anticanonical divisor \(-K_Y \sim C\) (here with abuse of notation we denote by the same letter \(C\) a curve which lives in different birational surfaces and is disjoint from the exceptional locus).

Lemma 2.3. If \(\text{Pic}(Y)\) has rank three then \(K^+_Y \cap \text{Eff}(Y) = \mathbb{Q}_{\geq 0} \cdot [C]\).

Proof. The class \([C]\) spans an extremal ray of \(\text{Eff}(Y)\) because the curves contracted by \(\phi\) are disjoint from \(C\) and thus \(\text{res}(C)\) is non-torsion also on \(Y\). As a consequence \(\text{Eff}(Y)\) is non-polyhedral by [8, Lem. 3.3]. By [8, Lem. 3.14] the minimal resolution of singularities \(\pi: Z \to Y\) is a smooth rational surface \(Z\) of Picard rank 10, nef anticanonical class \(-K_Z\) and non-polyhedral effective cone \(\text{Eff}(Z)\). Observe that the root sublattice of \(\text{Pic}(Z)\) spanned by classes of \((-2)\)-curves over singularities of \(Y\) has rank \(R = 7\). Assume now that \(D\) is an effective divisor such that \(D \cdot K_Y = 0\). Then \(D\) is push-forward of an effective divisor \(D'\) of \(Z\) with \(D' \cdot K_Z = 0\). Since \(-K_Z\) is nef, by adjunction \(D' = \sum a_i C_i + n C\), where each \(C_i\) is a \((-2)\)-curve and \(a_i \geq 0\).

By [8, Cor. 3.17], the fact that \(\text{res}(C_i) = 0\) for any \(i\) and the fact that \(\text{Eff}(Y)\) is non-polyhedral we conclude that all the \(C_i\) are contracted by \(\pi\). Thus \(D\) is linearly equivalent to a positive multiple of \(C\). \(\square\)

We are now going to consider a particular lattice polygon \(\Delta\) satisfying the above conditions (it is number 24 in [8, Table 3]) and we are going to show that the blowing-up of the corresponding toric surface contains infinitely many negative curves (see Remark 2.5).

Proposition 2.4. Let \(X\) be the blowing-up at a general point of the toric surface \(\mathbb{P}\), defined by the following lattice polygon \(\Delta \subseteq \mathbb{Q}^2\)
Then there is a birational morphism $\phi: X \to Y$ onto a rational surface $Y$ of Picard rank three with only Du Val singularities. Moreover if $D_1, \ldots, D_6$ are the pullbacks of the prime invariant divisors of $\mathbb{P}$, ordered according to the picture, the pushforward on $Y$ of the divisor

$$E_k := (7k^2 - 1)D_4 + (161k^2 - 49k - 9)D_4 + (70k^2 - 53k + 9)D_5$$

$$(14k^2 - 12k + 2)D_6 - (42k^2 - 19k)E$$

is linearly equivalent to a $(−1)$-curve for any integer $k \neq 0$.

Proof. Let $C$ be the curve of $X$ defined by the unique element in $\mathcal{L}_\Delta(6)$, which has equation

$$f := -1 + 2v + 7uv - 3u^2v - 23uw^2 + 6u^2v^2 + 2u^3v + 18uv^3 + 20u^2v^3 - 26u^3v^3 + 10u^4v^3 - 2$$

$$v^5v^3 - 12uv^4 - 11u^2v^4 + 6u^3v^4 + 5u^4v^4 - 4u^5v^4 + u^6v^4 + 5uv^5 + 3u^2v^5 - 2u^3v^5 - uv^6.$$ 

Let us denote by $v_1, \ldots, v_6$ the primitive generators of the rays of the normal fan to $\Delta$, which are the columns of the following matrix

$$P := \begin{pmatrix}
-1 & -2 & -1 & -2 & 5 & 1 \\
2 & 3 & 1 & -5 & -1 & 0
\end{pmatrix}. $$

By (1.1) the divisor $\pi(C)$ is linearly equivalent to $-\sum_{i=1}^6 \min_{w \in \Delta} \{ w \cdot v_i \} \pi(D_i)$ so that

$$C \sim D_2 + 2D_3 + 32D_4 + D_5 - 6E.$$ 

By (1.2) the curve $C$ has self-intersection $C^2 = 0$ and it is smooth of genus 1. It is isomorphic, over the rational numbers, to the elliptic curve of equation $y^2 + y = x^3 + x^2$, labelled 43.a1 in the LMFDB database. Its Mordell-Weil group $\text{Pic}^0(C)(\mathbb{Q})$ is free of rank one so that $\text{res}(C)$ is either trivial or non-torsion. The first possibility is ruled out by the fact that $\dim |C| = \dim \mathcal{L}_\Delta(6) = 0$ and the exact sequence of the ideal sheaf of $C$ in $X$ (see [8, Lem. 3.2] for details). Thus $\text{res}(C)$ is non-torsion, which implies that $h^0(X, nC) = 1$, for any positive integer $n$, so that $[C]$ spans an extremal ray of $\text{Eff}(X)$. By [8, Cor. 3.12] the divisor $K_X + C$ is linearly equivalent to an effective divisor whose support can be contracted. This contraction is the morphism $\phi$ in the statement. We claim that

$$K_X + C \sim 3C_1 + 2C_2,$$

where $C_1 \sim 5D_4 + D_5 - E$ and $C_2 \sim 5D_5 + D_6 - E$ are the strict transforms of the two one-parameter subgroups corresponding to the width directions $(1, 0)$ and $(0, 1)$ of $\Delta$. To prove the claim it suffices to observe that the divisor $K_X + C \sim 3C_1 - 2C_2 \sim -D_1 + D_3 + 16D_4 - 13D_5 - 3D_6$ is principal, being a linear combination of the rows of the above matrix $P$. Using the intersection matrix of $D_1, \ldots, D_6, E$

$$
\begin{pmatrix}
-3/2 & 1 & 0 & 0 & 0 & 1/2 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -16/7 & 1/7 & 0 & 0 & 0 \\
0 & 0 & 1/7 & 4/189 & 1/27 & 0 & 0 \\
0 & 0 & 0 & 1/27 & -5/27 & 1 & 0 \\
1/2 & 0 & 0 & 0 & 1 & -9/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$$
we can see that $C_i \cdot C = C_1 \cdot C_2 = 0$ for $i = 1, 2$ and $E_k$ has integer intersection product with all the $D_i$, in particular it is a Cartier divisor. Moreover, 

$$E_k \cdot C_i = E_k \cdot C_2 = 0 \quad E_k^2 = E_k \cdot K = -1,$$

so that by Riemann-Roch $E_k$ is effective. To prove that $\phi_* E_k$ is irreducible we proceed as follows. Since $X$ has Picard rank 5 and $\phi_* X \to Y$ contracts $C_1 \cup C_2$, the surface $Y$ has Picard rank 3. The push-forward $\phi_* E_k$ is an effective and Cartier divisor, because we are contracting curves which have intersection product zero with $E_k$. Moreover, since $-K_Y$ is nef and

$$-K_Y \cdot \phi_* E_k = 1,$$

we deduce that $\phi_* E_k$ is either irreducible or it can be written as $D + D'$, with $D$ irreducible and reduced and $D'$ orthogonal to $K_Y$. By Lemma 2.3, $D'$ must be equivalent to a multiple of $C$, so that we can write $\phi_* E_k = D + nC$, for some $n > 0$. Since $D$ is Cartier, both $D^2$ and $D \cdot K_Y$ are integers, and moreover, being $-K_Y$ nef, by the genus formula $D^2 \geq -2$. The case $D^2 = -2$ can be ruled out since otherwise $D$ would be in $K_Y$. Thus $D^2 = D \cdot K_Y = -1$ and $-1 = \phi_* E_k^2 = D^2 + 2n D \cdot C = -1 + 2n > 0$ gives again a contradiction. It follows that $\phi_* E_k$ is linearly equivalent to a $(-1)$-curve. \hfill \square

**Remark 2.5.** The way we determined the divisors $E_k$ has been by solving the diophantine equations 2.1. We also remark that a priori the curve $E_k$ could be reducible of the form $E_k = \Gamma_k + a_1(k)C_1 + a_2(k)C_2$, with $\Gamma_k$ irreducible and $a_1(k), a_2(k) \geq 0$. So we are only showing the existence of infinitely many negative curves $\Gamma_k$ on $X$, which are not necessarily $(-1)$-curves. Moreover even if $E_k = \Gamma_k$, the Newton polygon of $E_k$ does not necessarily coincide with the Riemann-Roch polygon $\Delta_k$ of the curve $E_k$. So that $E_k$ could be a $(-1)$-curve but not an intrinsic one. The Riemann-Roch polygon has vertices corresponding to the columns of the following matrix

$$
\begin{pmatrix}
0 & 0 & 7k^2 - 4k & 14k^2 - 12k + 2 & 35k^2 - 12k - 1 & 42k^2 - 19k \\
0 & 7k^2 + k & 42k^2 - 19k & 7k^2 - 6k + 1 & 21k^2 - 6k - 1 & 28k^2 - 13k
\end{pmatrix}.
$$

In particular, if we set $m = 42k^2 - 19k$, we have that $\text{vol}(\Delta_k) = m^2 - 1$, $|\partial \Delta_k \cap \mathbb{Z}^2| = m + 1$ and $\text{lw}(\Delta_k) = m$, so that $(\Delta_k, m)$ is numerically a $(-1)$-pair. For small values of $k$ it is possible to check, with the help of the software Magma [6], that the Newton polygon of the $(-1)$-curve is indeed $\Delta_k$, but for general $k$ we are not able to prove it. Therefore $\Gamma_k$ is not necessarily an intrinsic $(-1)$-curve, but it is anyway an intrinsic negative curve (see Remark 1.6).

3. Seshadri constants

In this section we first prove Theorem 2 and Corollary 3, and then we discuss some consequences on the study of the effective cone of the blowing up of weighted projective planes.

We will need the following preliminary result about Seshadri constants on projective surfaces.

**Lemma 3.1.** Let $Y$ be a projective surface, $H$ an ample divisor of $Y$, and let $\pi: X \to Y$ be the blowing-up of $Y$ at a smooth point $p \in Y$ with exceptional divisor $E$. 


(i) If there is a positive integer \( m \) such that \( \pi^* H - mE \) is the class of an effective curve \( C = \sum_{i=1}^r a_i C_i \) with \( C^2 \leq 0 \), then
\[
\varepsilon(H,p) = \min_i \left\{ \pi^* H \cdot C_i \right\} \leq m + \frac{C^2}{m}.
\]

(ii) If furthermore \( C \) is irreducible, then \( \varepsilon(H,p) = m + \frac{C^2}{m} \).

**Proof.** We prove (i). Let \( \varepsilon \coloneqq \varepsilon(H,p) \). Observe that we can write \( C^2 + (m-\varepsilon)E \cdot C = (\pi^* H - mE + (m-\varepsilon)E) \cdot C = (\pi^* H - \varepsilon E) \cdot C \geq 0 \), and, since \( E \cdot C = m \), we get
\[
\varepsilon \leq m + \frac{C^2}{m} \leq m.
\]
If \( C \) is nef then \( \varepsilon \geq m \), so that \( \varepsilon = m \) and \( C^2 = 0 \). This implies \( C \cdot C_i = 0 \), and hence \( \varepsilon = \pi^* H \cdot C_i / C_i \) for any \( i \), proving the statement in this case. If \( C \) is not nef then \( \varepsilon < m \). If \( \alpha \) is such that \( \varepsilon < \alpha < m \) then \( \pi^* H - \alpha E \) is effective and non-nef. Let \( C' \) be an irreducible curve such that \( (\pi^* H - \alpha E) \cdot C' < 0 \), then \( (\pi^* H - mE - C') < 0 \) as well. Thus \( C' = C_i \) for some \( i \). Since \( \alpha \) can be chosen arbitrarily close to \( \varepsilon \), we conclude that \( (\pi^* H - \varepsilon E) \cdot C_j = 0 \) for some \( j \), and the statement follows. Statement (ii) is an immediate consequence of (i) and the equality \( \pi^* H \cdot C = C^2 + m^2 \). \( \square \)

**Proof of Theorem 2.** We prove (i). Let \( v \in N \) be a width direction, that is \( \text{lw}_v(\Delta) = \text{lw}(\Delta) \) and let \( C_v \in X_\Delta \) be the strict transform of the one-parameter subgroup of the torus defined by \( v \). If \( \mu > \text{lw}(\Delta) \), then \( (\pi^* H - \mu E) \cdot C_v < 0 \), so that \( \pi^* H - \mu E \) is not nef. This proves the statement.

We prove (ii). Observe that \( (\pi^* H - \varepsilon E)^2 \geq (\pi^* H - \text{lw}(\Delta) E)^2 = \text{vol}(\Delta) - \text{lw}(\Delta)^2 > 0 \), where the first inequality is by (i). By the Riemann-Roch theorem, the class of \( \pi^* H - \varepsilon E \) is in the interior of the effective cone \( \text{Eff}(X_\Delta) \). It follows that the Seshadri constant is computed by a curve \( C \in X_\Delta \). From \( (\pi^* H - \varepsilon E) \cdot C = 0 \) one concludes that \( \varepsilon \) is a rational number (see also [23, Rem. 2.3]).

Statements (iii) and (iv) are consequence of Lemma 3.1 and the fact that \( C^2 = \text{vol}(\Delta) - m^2 \). \( \square \)

**Proof of Corollary 3.** Observe that by hypothesis \( \text{vol}(\Delta) = m^2 = C^2 \leq 0 \), so that the hypothesis (iii) of Theorem 2 is satisfied. Moreover, being \( C \) irreducible we have \( \text{lw}(\Delta) - m = C \cdot C_v \geq 0 \), where \( C_v \) is the strict transform of the one-parameter subgroup of the torus defined by the width direction \( v \). Thus also hypothesis (iv) of Theorem 2 is satisfied and the statement follows. \( \square \)

**Remark 3.2.** Observe that the best lower bound for \( \varepsilon \) we can get from [20, Thm. 1.3] in the case of a toric surface is either the width \( \text{lw}(\Delta) \), or the biggest length of a segment inside \( \Delta \). For instance, if \( \Delta \) is a triangle of type (i) in Theorem 1, consider the projection \( \pi: \mathbb{Q}^2 \to \mathbb{Q} \) onto the second factor. If we take the fiber \( \Delta \cap \pi^{-1}(1) \), by [20, Thm. 1.3] we have the inequality
\[
\varepsilon(H,e) \geq \min\{m, m - 1/m\} = m - 1/m.
\]
Since \( \text{vol}(\Delta) = m^2 - 1 \), by Theorem 2 (iii), we also have the inequality \( \varepsilon(H,e) \leq (m^2 - 1)/m \), so that we can conclude that \( \varepsilon(H,e) \) is indeed equal to \( m - 1/m \), no need of showing that there exists an irreducible curve \( C \in \mathcal{L}_\Delta(m) \).
We also remark that in the remaining cases of Theorem 1 the bound given by [20, Thm. 1.3] is not sharp.

In the same vein, if the lattice polygon $\Delta$ contains a segment of lattice length $lw(\Delta)$, then [20, Thm. 1.3] gives the bound $\varepsilon(H,e) \geq lw(\Delta)$. But since by Theorem 2 (i) we also have the opposite inequality, we can immediately conclude that the Seshadri constant $\varepsilon(H,e)$ is indeed equal to $lw(\Delta)$.

For instance, this shows that for any $m \geq 4$, the polygon with vertices $(0,0)$, $(0,1)$, $(m,1)$, $(1,m)$ corresponds to a toric surface with Seshadri constant $\varepsilon(H,e) = m$ (even if it is not hard to find a parametrisation as we did with the families of Theorem 1).

3.1. **Weighted projective planes.** We briefly recall an open problem about the existence of certain irreducible curves in weighted projective planes, and its relation with intrinsic curves. For a comprehensive reference on known facts and open problems on blow-ups of weighted projective planes see [7, Sec. 6].

Let $a, b, c$ be three positive pairwise coprime integers, let $\mathbb{P}(a,b,c)$ be the corresponding weighted projective plane and let $\pi: X(a,b,c) \to \mathbb{P}(a,b,c)$ be the blowing-up at the general point $e : = (1,1,1)$ with exceptional divisor $E$. The divisor class group of $X := X(a,b,c)$ is free of rank 2 and the effective cone $\text{Eff}(X)$ is in general unknown. By the Riemann-Roch theorem, $\text{Eff}(X)$ contains the positive light cone $Q$ (shaded region) with extremal rays generated by $R_\pm = \pi^*H \pm \frac{1}{\sqrt{abc}}E$.

The question is whether $\text{Eff}(X)$ is bounded by the $\mathbb{R}$-divisor $R_-$, so that $\varepsilon(H,e) = 1/\sqrt{abc}$, or by the class of a negative curve (lying below the ray $R_-$). In many examples (see for instance [16] and [19]) the existence of the negative curve has been proved, but in general the question is still open, and in fact it is conjectured that for some triples $a, b, c$ (such as 9, 10, 13) that negative curve does not exist. We remark that proving this conjecture would not only give an example of a surface having non rational Seshadri constant, but it would also imply the following well known conjecture (see [25]), in some particular cases, as explained in Proposition 3.4.

**Conjecture 3.3** (Nagata’s Conjecture). Let $\pi:X_r \to \mathbb{P}^2$ be the blowing-up at $r \geq 10$ points in very general position and let $E_1, \ldots, E_r$ be the exceptional divisors. Then the class $\pi^*\mathcal{O}(1) - \sum E_i$ is nef on $X_r$.

So far, Nagata’s conjecture have been proved only when $r$ is a perfect square ([25]), but the following result shows that finding a triple $a,b,c$, such that the
Seshadri constant at the general point \( e \in X(a, b, c) \) is \( 1/\sqrt{abc} \), would imply Nagata’s conjecture for \( r = abc \) (it can be found for instance in [12, Prop. 5.2], but we give anyway a brief proof for the sake of completeness).

**Proposition 3.4.** If \( \varepsilon(H, e) = 1/\sqrt{abc} \) then the Nagata’s conjecture holds for \( abc \) points in the plane.

**Proof.** Let \( f: \mathbb{P}^2 \to \mathbb{P}(a, b, c) \) be the morphism defined by \((x, y, z) \mapsto (x^a, y^b, z^c)\) and let \( Y_r \) be the blowing-up of \( \mathbb{P}^2 \) at the \( r := abc \) points of \( f^{-1}(e) \). Since \( R_- \) is nef, also

\[
f^* R_- = L - \frac{1}{\sqrt{abc}} \sum_{i=1}^{abc} E_i
\]

is nef on \( Y_r \). If we denote by \( X_r \) the blowing-up of \( \mathbb{P}^2 \) at \( r \) points in very general position then \( \text{Eff}(X_r) \subseteq \text{Eff}(Y_r) \) by semicontinuity of the dimension of cohomology. Thus \( \text{Nef}(X_r) \supseteq \text{Nef}(Y_r) \) so that \( f^* R_- \) is nef also on \( X_r \). \( \square \)

**Remark 3.5.** If \( C \) is a very general smooth irreducible curve of positive genus \( g \), then the Néron-Severi group, over the rational numbers, of the symmetric product \( \text{Sym}^2(C) \) has rank two. In [10, Prop. 3.1] the authors show that if the Nagata’s conjecture is true and \( g \geq 9 \), then the effective cone of \( \text{Sym}^2(C) \) is open on one side.

The equality \( \varepsilon(H, e) = 1/\sqrt{abc} \) holds if and only if there does not exist a negative curve in \( X(a, b, c) \) having class \( d\pi^* H - mE \), with \( d/m < \sqrt{abc} \). A partial result in this direction is given by [22, Thm. 5.4], where the authors show that if the negative curve is expected, then \( d \) is bounded from above and this result allows them to conclude that there are no such curves on certain \( X(a, b, c) \), like e.g. \( X(9, 10, 13) \).

On the other hand we can also say that the equality \( \varepsilon(H, e) = 1/\sqrt{abc} \) holds if and only if there exists a sequence \( \pi^* d_n H - m_n E \) of classes of positive irreducible curves in \( X(a, b, c) \) such that \( d_n/m_n \to \sqrt{abc} \), that is these classes approach the ray \( R_- \) from the inside of the light cone. In this direction observe that an intrinsic negative curve can appear as a positive curve in \( X(a, b, c) \). We discuss this approach for \( X(9, 10, 13) \) by producing many intrinsic \((-1)\)-curves which are positive curves on the surface. We proceed using the fact that the Cox ring of \( X(a, b, c) \) is isomorphic to the extended saturated Rees algebra (see [19] and [2, Prop. 4.1.3.8]):

\[
R[I]^{\text{sat}} := \bigoplus_{m \in \mathbb{Z}} \left(I^m : J^\infty\right) t^{-m} \subseteq R[t^\pm],
\]

where \( R = \mathbb{K}[x, y, z] \) is the Cox ring of the weighted projective plane, \( I \) is the ideal of \((1, 1, 1)\) in Cox coordinates, \( J = \langle x, y, z \rangle \) is the irrelevant ideal and \( I^m = R \) for any \( m \geq 0 \). Using this we can compute a minimal generating set consisting of homogeneous elements of given bounded multiplicity at \((1, 1, 1)\). In the case of \( X(9, 10, 13) \), fixing the maximum of the multiplicity to be 30, we found 52 generators. In the following table we display the degrees of these generators together with the self-intersection of the corresponding intrinsic curve and its genus (while the self-intersection of the curve on \( X(9, 10, 13) \) is \( d^2/abc - m^2 \)).
The slope \( d/m \) which best approximates \( \sqrt{9 \cdot 10 \cdot 13} \sim 34.20526 \) is 959/28 = 34.25, realized by the last curve. The best approximation given by an intrinsic \((-1)\)-curve of the above list is 891/26 \sim 34.26923. The following question naturally arises.

**Question 3.6.** Is it possible to construct an infinite family of intrinsic \((-1)\)-curves appearing as positive curves in \(X(9, 10, 13)\), and whose slopes approach \( \sqrt{9 \cdot 10 \cdot 13} \)?

**References**

[1] Florin Ambro and Atsushi Ito. Successive minima of line bundles. *Adv. Math.* 365:107045, 38, 2020.

[2] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface. *Cox rings*. Cambridge Studies in Advanced Mathematics, vol. 144. Cambridge University Press, Cambridge, 2013.

[3] Gabriele Balletti. Enumeration of Lattice Polytopes by Their Volume. *Discrete Comput. Geom.*, 2020.

[4] Thomas Bauer. Seshadri constants on algebraic surfaces. *Math. Ann.* 313 (3):547–583, 1999.

[5] Thomas Bauer, Sandra Di Rocco, Brian Harbourne, Michael Kapustka, Andreas Knutsen, Wioletta Syzdek, and Tomasz Szemberg. A primer on Seshadri constants. Interactions of classical and numerical algebraic geometry. *Contemp. Math.*, vol. 496. Amer. Math. Soc., Providence, RI., pages 33–70, 2009.

[6] Weib Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *Journal of Symbolic Computation* 24 (3–4):235–265, 1997. Computational algebra and number theory (London, 1993).

[7] Ana-Maria Castravet. Mori dream spaces and blowing-ups. *Algebraic geometry: Salt Lake City 2015. Proc. Sympos. Pure Math.*, vol. 97. Amer. Math. Soc., Providence, RI., pages 143–167, 2018.

[8] Ana-Maria Castravet, Antonio Laface, Jenia Tevelev, and Luca Ugaglia. Blown-up toric surfaces with non-polyhedral effective cone. *arXiv:2009.14298*, 2020.

[9] Ana-Maria Castravet and Jenia Tevelev. \( \overline{M}_{0,n} \) is not a Mori dream space. *Duke Math. J.* 164 (8):1641–1667, 2015.

[10] Ciro Ciliberto and Alexis Kouvidakis. On the symmetric product of a curve with general moduli. *Geom. Dedicata* 78 (3):327–343, 1999.

[11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*. Graduate Studies in Mathematics, vol. 124. American Mathematical Society, Providence, RI, 2011.

[12] Steven Dale Cutkosky and Kazuhiko Kurano. Asymptotic regularity of powers of ideals of points in a weighted projective plane. *Kyoto J. Math.* 51 (1):25–45, 2011.

[13] Carlos D’Andrea and Martín Sombra. The Newton polygon of a rational plane curve. *Math. Comput. Sci.* 4 (1):3–24, 2010.

[14] Lawrence Ein and Robert Lazarsfeld. Seshadri constants on smooth surfaces. *Astérisque* 218:177–186, 1993. Journées de Géométrie Algébrique d’Orsay (Orsay, 1992).

[15] Lucja Farnik, Tomasz Szemberg, Justyna Szpond, and Halszka Tutaj-Gasińska. Restrictions on Seshadri constants on surfaces. *Taiwanese J. Math.* 21 (1):27–41, 2017.

[16] Javier González Anaya, José Luis González, and Kalle Karu. Constructing non-Mori dream spaces from negative curves. *J. Algebra* 539:118–137, 2019.

[17] On a family of negative curves. *J. Pure Appl. Algebra* 223 (11):4871–4887, 2019.
[18] Christian Haase, Alex Küronya, and Lena Walter. Toric Newton-Okounkov functions with an application to the rationality of certain Seshadri constants on surfaces. arXiv:2008.04018, 2020.

[19] Jürgen Hausen, Simon Keicher, and Antonio Laface. On blowing up the weighted projective plane. Math. Z. 290 (3-4):1339–1358, 2018.

[20] Atsushi Ito. Seshadri constants via toric degenerations. J. Reine Angew. Math. 695:151–174, 2014.

[21] Kazuhiko Kurano. Equations of negative curves of blow-ups of Ehrhart rings of rational convex polygons. arXiv:2101.02448, 2021.

[22] Kazuhiko Kurano and Naoyuki Matsubara. On finite generation of symbolic Rees rings of space monomial curves and existence of negative curves. J. Algebra 322 (9):3268–3290, 2009.

[23] Alex Küronya, Victor Lozovanu, and Catriona Maclean. Convex bodies appearing as Okounkov bodies of divisors. Adv. Math. 229 (5):2622–2639, 2012.

[24] Antonio Laface and Luca Ugaglia. On base loci of higher fundamental forms of toric varieties. J. Pure Appl. Algebra 224 (12):106447, 18, 2020.

[25] Masayoshi Nagata. On the 14-th problem of Hilbert. Amer. J. Math. 81:766–772, 1959.

[26] Michael Nakamaye. Seshadri constants and the geometry of surfaces. J. Reine Angew. Math. 564:205–214, 2003.

[27] P. Orlik and P. Wagreich. Algebraic surfaces with k*-action. Acta Math. 138 (1-2):43–81, 1977.

[28] Jenia Tevelev. Compactifications of subvarieties of tori. Amer. J. Math. 129 (4):1087–1104, 2007.

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