Entangled light from Bose–Einstein condensates

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New Journal of Physics 11 (2009) 043009 (9pp)
Received 16 November 2008
Published 3 April 2009
Online at http://www.njp.org/
doi:10.1088/1367-2630/11/4/043009

Abstract. We propose a method to generate entangled light with a Bose–Einstein condensate trapped in a cavity, a system realized in recent experiments. The atoms of the condensate are trapped in a periodic potential generated by a cavity mode. The condensate is continuously pumped by a laser and spontaneously emits a pair of photons of different frequencies in two distinct cavity modes. In this way, the condensate mediates entanglement between two cavity modes, which leak out and can be separated and exhibit continuous variable entanglement. The scheme exploits the experimentally demonstrated strong, steady and collective coupling of condensate atoms to a cavity field.

Quantum communications can outperform their classical counterparts, for example, quantum cryptography enables secure distribution of quantum information. ‘Quantum correlations’ or entanglement, when shared between distant parties, is a key resource for quantum communication tasks such as quantum cryptography [1], teleportation [2] and dense coding [3]. These applications provide a very strong motivation for entangled light, which is widely regarded as the most ideal entity for the sharing of entanglement between genuinely distant parties [4]. Additionally, as the interface between light and matter matures as a technology [5, 6], entangled light can also link up distinct matter registers of a quantum computer and thereby aid in scaling up quantum computers.

One important form of entangled light [7, 8] is continuous variable (CV) entanglement between phase quadratures of two distinct modes of the light field of the type discussed in the famous Einstein–Podolsky–Rosen (EPR) paper [9]. Such entanglement has been used for quantum teleportation and has applications in quantum dense coding [10] and quantum cryptography [11]. Additionally, if the entanglement is sufficiently ‘narrow band’ in frequencies then the quantum states of the light will efficiently interface with those of atomic ensembles [6] for applications in quantum repeaters and linking quantum registers. Thus the motivation for having entangled sources of EPR light is very strong.
The prevalent sources of EPR-entangled light are nonlinear crystals. It was noticed long ago that light fields produced from nonlinear crystals seem to be non-classically correlated [12]. For non-degenerate optical parametric oscillators, Reid demonstrated that the quadrature phases of the output fields have EPR-type entanglement [7] and this is indeed one of the sources in recent experiments [8]. Alternatively, the outputs of two degenerate optical parametric oscillators are interfered to obtain EPR-entangled light [13]. For such crystals, the Hamiltonian is actually phenomenologically constructed to describe the observed nonlinear processes, and can be expressed in terms of the nonlinear susceptibility of the macroscopic crystal. This is why there has been a recent interest in deriving EPR-entangled light from a more fundamental Hamiltonian, such as from the ‘quantized motion’ of a single atom trapped in a cavity [14, 15]. This provides a coherent control of the entanglement generator in a microscopic system, as opposed to a bulk crystal. In addition to this fundamental interest, such alternative sources may also have a practical interest if they can surpass the squeezing parameter (a parameter that controls the amount of entanglement in the EPR-entangled light) possible from crystal sources, as many of the restrictions such as the lower finesse of cavities around crystals, or the unbalanced absorption of the entangled modes while traversing the crystal, do not directly apply.

Recently, the strong coupling of an atomic Bose–Einstein condensate (BEC) to a single-mode photon field of an optical cavity has been experimentally achieved [16, 17]. The ultracold atoms are trapped in a periodic potential generated by a quantized field mode [18]. Since the $N$ two-level atoms are identically coupled to the single-mode photon field, this gives a collective enhancement of a factor $\sqrt{N}$ [16, 17]. In fact, such strong atom–photon couplings are very useful in performing quantum information processing (QIP) before decoherence sets in. The potential applications include long-lived quantum memory [8] and quantum networks for light–matter interfaces [19].

In this paper, we consider an atomic BEC trapped inside an optical cavity [16]. Each atom is located at an anti-node of the quantized cavity field so that one atom per site can identically couple to the cavity field. This means that it is equivalent to the medium composed of two-level particles localized at the antinodes of the cavity field. Compared to a thermal cloud of atoms with the inhomogeneous atom–photon couplings in the cavity, we can truely apply the Tavis–Cummings model [20] to study our system. In addition, the BECs with reduced Doppler broadening lead to much longer coherence times than those of the thermal clouds [21]. These are advantages relative to having a similar scheme with a thermal cloud of atoms where all the atoms do not interact with the light equally and the interaction Hamiltonian can only be an approximation obtained by integrating the atom–light coupling strength along the light propagation (for instance, an atom–light entangling scheme through scattering in the atomic ensembles has been studied [22] with this approximation).

Even if there is more than one atom per site, all these atoms will experience exactly the same interaction with the light field. In addition, we consider the light–atom coupling strength to be much stronger than the tunnelling strength and the interactions between the atoms at each site. Therefore, it is still valid to use the Tavis–Cummings model to describe the BEC in the superfluid regime. If all the lattice sites are not commensurate with an antinode, the atom–light interaction can be averaged out by integrating the atom–light coupling strength along the lattices. The average interaction is expected to be equal for all the atoms in a BEC as they all have exactly the same wavefunction across the lattice sites.
We consider that the BEC is continuously driven by an external laser and spontaneously emits photons with the two different frequencies in pairs. Hence, the BEC acts as a medium to mediate the entanglement between the two cavity modes. The two quantum-correlated light modes are emitted through a one-sided mirror as shown in figure 1. Since the ultracold atoms have long coherence times [23], this can provide a robust way to generate the entangled light. We will show that the degree of entanglement depends on the ratio of the decay rate of the cavity and the effective Rabi frequency. This means that the degree of entanglement can be controlled by adjusting the atom–photon coupling strength.

We consider a two-component condensate trapped inside a cavity in which the atoms are trapped in a one-dimensional optical lattice as shown in figure 2. The classical lasers, with frequencies $\omega_{L_1}$ and $\omega_{L_2}$, are used to pump the two internal states $|1\rangle$ and $|2\rangle$ to a higher
level \( |3 \rangle \). Then, the two different quantized light fields with frequencies \( \omega_1 \) and \( \omega_2 \) are spontaneously emitted due to large detuning [24]. The Hamiltonian is written as \((\hbar = 1)\)

\[
H = \sum_{j=1}^{2} \omega_j a_j^\dagger a_j + \sum_{k=1}^{N} \left[ \omega_{3k} |3\rangle_k \langle 3| + \omega_{2k} |2\rangle_k \langle 2| + \Omega (|1\rangle_k \langle 2| + |2\rangle_k \langle 1|) e^{i\omega_{\nu} t} + \text{h.c.} \right] \\
+ \sum_{j=1}^{2} (\Omega_j |j\rangle_k \langle 3| e^{i\omega_{\nu} t} + \lambda_j a_j^\dagger |j\rangle_k \langle 3| + \text{h.c.}) ,
\]

(1)

where \( a_j \) is the annihilation operator of the cavity modes with frequency \( \omega_j \), \( \omega_{j1} \) are the energy splitting between the states \( |j\rangle_k \) and \( |1\rangle_k \), and \( \omega_{Lj} \) is the frequency of the laser field on the states \( |j\rangle_k \) and \( |3\rangle_k \) \((|1\rangle_k \) and \( |2\rangle_k \) for \( j = 1, 2 \). The parameters \( \Omega \) \((\Omega_j)\) and \( \lambda_j \) are the Rabi frequencies associated with the atomic transitions \( |1\rangle_k \) and \( |2\rangle_k \) \((|j\rangle_k \) and \( |3\rangle_k \)), and the coupling strength between the cavity field \( a_j \) and the atomic states \( |j\rangle_k \) and \( |3\rangle_k \), respectively. The frequencies of the two modes satisfy the two-photon Raman resonance condition \( \omega_{L1} + \omega_{L2} = \omega_1 + \omega_2 \) [24] so that we can write \( \omega_{1,2} = (\omega_{L1} + \omega_{L2}) \pm \nu \). These two modes must satisfy the boundary condition of the cavity.

It is instructive to work in the rotating frame by using the unitary transformation \( U(t) = \exp [i(\omega_{L1} a_1^\dagger a_1 + \omega_{L2} a_2^\dagger a_2 + \sum_{k=1}^{N} \omega_{Lk} |3\rangle_k \langle 3| + \omega_{L2} |2\rangle_k \langle 2|) t] \). The transformed Hamiltonian reads as

\[
\tilde{H} = \sum_{k=1}^{N} (\Delta_3 |3\rangle_k \langle 3| + \Delta_2 |2\rangle_k \langle 2|) \\
+ \sum_{k=1}^{N} \sum_{j=1}^{2} \left[ \delta_j a_j^\dagger a_j + \Omega (|1\rangle_k \langle 2| + \text{h.c.}) + \Omega_j (|j\rangle_k \langle 3| + \text{h.c.}) + \lambda_j (a_j^\dagger |j\rangle_k \langle 3| + \text{h.c.}) \right],
\]

(2)

where \( \Delta_3 = \omega_{31} - \omega_{L1}, \Delta_2 = \omega_{21} - \omega_{L1}, \delta_j = \omega_j - \omega_{Lj} \) and we set \( \omega_L = \omega_{L1} - \omega_{L2} \). For \( \Delta_3 \gg \Omega_j, \lambda_j \), this enables us to adiabatically eliminate the upper level \( |3\rangle \).

The effective Hamiltonian is given by

\[
H' = \sum_{j=1}^{2} \left[ \delta_j a_j^\dagger a_j + \tilde{\omega} J_z - \left[ (g_1 a_1 + g_2 a_2^\dagger) J_z + \text{h.c.} \right] \right],
\]

(3)

where \( J_z = \sum_{k=1}^{N} |2\rangle_k \langle 1| \) and \( J_z = \sum_{k=1}^{N} |2\rangle_k \langle 2| - |1\rangle_k \langle 1| \). The parameters \( \tilde{\omega} \) and \( g_j \) are \( \Delta_2 + (\Omega_1^2 - \Omega_2^2) / \Delta_3 \) and \( \lambda_j \Omega_j / \Delta_3 \), respectively and \( C = J_z + a_1^\dagger a_1 - a_2 a_2^\dagger \) is a constant of motion. Here we have omitted the terms with the coefficients \( \lambda_i \) are much smaller than \( \Omega_j \). We also require \( \Omega = \Omega_1 \Omega_2 / \Delta_3 \) to eliminate the transition between the states \( |1\rangle \) and \( |2\rangle \) with the classical laser.

We consider that only the low-lying collective spin excitations in the condensates are involved throughout the dynamics. Then, we can approximate the angular momentum operator as a harmonic oscillator as [25]: \( J_+ \approx \sqrt{N} b^\dagger, \quad J_- \approx \sqrt{N} b \) and \( J_z \approx b^\dagger b - N / 2 \). The Hamiltonian can be rewritten as

\[
H' \approx \omega' b^\dagger b - \left[ \sqrt{N} (g_1 a_1 + g_2 a_2^\dagger) b^\dagger + \text{h.c.} \right].
\]

(4)
We assume $\omega' = (\tilde{\omega} - \nu) \gg g_j \sqrt{N}$ such that the low excitations approximation is valid. In the large detuning limit, the mode of the collective spin excitations $b$ can be adiabatically eliminated. We thus can write

$$H_{\text{eff}} = \chi_i a_i^\dagger a_i + \chi_j a_j a_j + \chi (a_i^\dagger a_i + a_j a_j),$$

(5)

where $\chi_j = g_j^2 N/\omega'$ and $\chi = g_1 g_2 N/\omega'$.

For convenience, we represent the Hamiltonian in terms of the operators

$$K_3 = \frac{1}{2} (a_i^\dagger a_i + a_j a_j),$$

(6)

$$K_\pm = a_i^\dagger a_j = K_\mp^-.$$

(7)

These operators satisfy the commutation relations $[K_3, K_\pm] = \pm K_\pm$ and $[K_+, K_-] = -2K_3$. The Hamiltonian is represented in the form of (we have ignored the constant term)

$$H_{\text{eff}} = \chi_i + \chi_j K_3 + \chi (K_+ + K_-),$$

(8)

We consider the initial state is the vacuum state and the time evolution of the two-mode state is $|\Psi(\tau)\rangle = e^{-iH_{\text{eff}}\tau}|0, 0\rangle$. According to the operator-ordering theorem [26], we have

$$e^{-iH_{\text{eff}}\tau} = e^{\Gamma K_+} e^{\tilde{\Gamma} K_3} e^{\Gamma K_-},$$

(9)

where

$$\tilde{\Gamma} = \left( \cosh \beta - \frac{\tilde{\gamma}}{2\beta} \sinh \beta \right)^{-2},$$

(10)

$$\Gamma = \frac{2\gamma \sinh \beta}{2\beta \cosh \beta - \tilde{\gamma} \sinh \beta}.$$  

(11)

The parameters $\gamma$, $\tilde{\gamma}$, $\beta^2$ are $-i\chi \tau$, $-i(\chi_1 + \chi_2)\tau/2$ and $\tilde{\gamma}^2/4 - \gamma^2$, respectively. Therefore, the state can be readily obtained as

$$|\Psi(\tau)\rangle = \tilde{\Gamma} \sum_{n=0}^{\infty} \Gamma^n |n, n\rangle.$$  

(12)

Clearly, this state is an entangled state in which the two cavity modes are entangled in pairs.

We have shown that the entanglement between the two cavity modes can be produced inside the cavity. However, this intra-cavity entanglement is neither easy to detect nor useful (useful entanglement needs to have the entangled modes well separated in space). Therefore, it is necessary to detect entanglement in the light leaking out of the cavity. The entangled light is emitted through the mirror and splits into the two different frequency components via a prism as shown in figure 1. They are measured through a homodyne detection. The difference in the photon current is then recorded. The resulting squeezing spectrum can be obtained by the spectrum analyser [27].

To evaluate the entangled light outside the cavity, we thus have to take account of the input–output theory [27]. Here we consider the cavity modes coupled through the number of harmonic oscillators from the one-sided cavity. It can be described by the Langvein equations of motion which are given by [28]

$$\dot{a}_j = ig_j \sqrt{N} b - \kappa_j a_j - \sqrt{2\kappa_j} a_{j\text{in}},$$

(13)

$$\dot{b} = -i\omega' b + i\sqrt{N} (g_1 a_1 + g_2 a_2^\dagger),$$

(14)
Figure 3. The squeezing spectra $S^0(\omega)$ and $S^{\pi/2}(\omega)$ are plotted as functions of the frequency $\omega/g$ in (a) and (b), where $g_1 = g_2 = g$, $\omega = 10^4 g$ and $N = 10^4$. The solid, dash and dotted lines represent the different values of $\kappa = 10$, 5 and 2.5 (in units of $g$), respectively.

where $a_{in}$ is the input field for the one-sided cavity and $\kappa_j$ is the damping constant. The output fields are $a_{o} = a_{in} + \sqrt{2\kappa_j}a_j$ which relate the input fields and the internal cavity fields [27]. We assume that the radiative noise from the cavity is much larger than the noise coming from the BEC and also that the input noise source is in vacuum.

Now we study the squeezing spectrum by transforming to the Fourier space as [15]

$$I_{\pm}^\theta(\omega) = \frac{1}{\sqrt{2\pi}} \int dt e^{-i\omega t} I_{\pm}^\theta(t),$$

where $I_{\pm}^\theta = a_{10}e^{-i\theta} + a_{10}^\dagger e^{i\theta} - a_{20}e^{-i\theta} - a_{20}^\dagger e^{i\theta}$ and $I_{\mp}^\theta = -i[a_{10}e^{-i\theta} - a_{10}^\dagger e^{i\theta} + a_{20}e^{-i\theta} - a_{20}^\dagger e^{i\theta}]$. The squeezing spectrum can be defined as $(I_+^\theta(\omega)I_+^\theta(\omega') + I_-^\theta(\omega')I_-^\theta(\omega)) = 2S_{\pm}^\theta(\omega + \omega')$. We can take the Fourier transform for the output fields $a_{o}(\omega)$ which can be expressed in terms of the input fields $a_{in}(\omega)$ and $a_{in}^\dagger(\omega)$. After some algebra, we found that $S_{\pm}^\theta(\omega) = S_{\mp}^\theta(\omega)$. It can be shown that the two output modes are entangled if $S_{\pm}^\theta(\omega) < 1$ [15, 29].

We investigate the squeezing spectrum $S^\theta(\omega)$ for the regime of $\omega' \gg g_{1,2}\sqrt{N}$. For simplicity, we take $g_{1,2} = g$ and $\kappa_{1,2} = \kappa$. In figure 3(a), we plot the squeezing spectrum $S^0(\omega)$ as a function of $\omega$ (in units of $g$) for the different cavity decay rates. It shows that the squeezing occurs in the negative frequency domain, whereas unsqueezing occurs in the positive frequency domain. We also plot the out-of-phase squeezing spectrum $S^{\pi/2}(\omega)$ in figure 3(b). In contrast, the squeezing (unsqueezing) occurs in the positive (negative) frequency domain. Apart from that, we can see that a larger squeezing compensates a narrower range of frequencies as shown in figures 3(a) and (b).

We also study the maximal degree of squeezing attainable in the system for different values of cavity decay and different numbers of atoms. The minimal values of squeezing for the...
Figure 4. The minimal values of squeezing for the squeezing spectrum $S^0_{\text{min}}$ are plotted as a function of $\kappa/g$, where, $\omega' = 10^4 g$ and $N = 10^4$. The inset shows the minimal squeezing $S^0_{\text{min}}$ as a function of $\log_{10} N$ for $\kappa = g$ and $\omega' = 10^4 g$.

The degree of squeezing increases significantly as the cavity decay parameter $\kappa$ decreases. Nearly perfect squeezing can be obtained for the low values of $\kappa/g$. Of course, there is no squeezing for the case of the closed cavity ($\kappa = 0$). In the inset of figure 4, the minimal value of squeezing is plotted as a function of $\log_{10} N$. This means that stronger squeezing can be attained as the number of atoms increases.

For $\omega' \gg g_{1,2}^2 N$, the approximated analytical expression of the squeezing spectrum $S^0(\omega)$ can be found as

$$S^0(\omega) \approx 1 + \frac{4\kappa^2 g_1 g_2 N}{(k^2 + \omega^2)^2(\omega^2 - \omega'^2)} \left[ \frac{4\omega' \omega^2}{k^2 + \omega^2} + \frac{g_1 g_2 N (\omega'^2 + \omega^2)}{\omega^2 - \omega'^2} \right]. \quad (16)$$

The approximate solution equation (16) is compared with the numerical solution in figure 5. This shows that it is indeed a very good approximation in the limit of this large detuning.

We now briefly examine the range of parameters available for our scheme if the setups of some recent experiments are directly used. If one literally uses the parameters from the experiment by Brennecke et al [16] then one can have $\lambda_j = 2\pi \times 10.6$ MHz $\approx 67$ MHz (what we say below holds for both $j = 1, 2$). Typically, the parameter $\Omega_j$ can be made even larger as it is proportional to the strength of the external laser field, so we assume it to be $\eta \lambda_j$, where $\eta$ is a numerical factor which can be varied between 1 and 4. The detuning $\Delta_3 \gg \lambda_j^2, \Omega_j^2, \lambda_j \Omega_j$ is required for the adiabatic elimination of the level [3]. So we choose $\Delta_3 \sim 10 \lambda_j$. Thus the effective Rabi frequency $g_j \sim \lambda_j \Omega_j / \Delta$ is roughly equal to $6.7 \eta$ MHz and the cavity decay $\kappa$ is about $2\pi \times 1.3$ MHz $\approx 8.1$ MHz. Thus the ratio $\kappa/g_j$ can be made to vary between 1 and 0.3 by varying $\eta$ between 1 and 4. The energy splitting $\omega'$ is about 10 GHz (which means $\omega'$, when expressed in terms of $g_j$ is $10^3$ to $10^4 g_j$). Though the number of atoms $N$ can be up to $2 \times 10^5$ [16], we restrict the number to be about $\sim 10^4$ which is also a typical number in experiments, so that $\omega' \gg g_j \sqrt{N}$ is fulfilled and the BEC is not excited. We noted that the cavity...
Figure 5. The squeezing spectrum $S^0(\omega)$ plotted as a function of the frequency $\omega/g$, where $\omega' = 10^5 g$ and $N = 10^4$. The black and red lines represent the exact and approximated solutions for $\kappa = g$.

The decay rate $\kappa$ can be adjusted because $\kappa = \pi c/(2LF)$ depends on the length $L$ and the finesse $F$ of the cavity [30]. Hence, different amounts of squeezing can be observed in experiment by varying the parameters $\kappa$ and $g$.

We have studied the generation of entangled light with a cavity–BEC system in which the atoms are continuously driven by the external laser field. The entangled light can be emitted from the cavity through the one-side mirror and then the entanglement can be verified through a homodyne detection. We have shown that the degree of entanglement can be controlled by adjusting the strength of atom–photon couplings. Our scheme to generate entangled light can be realized with the current experimental technology [16, 17].

Acknowledgments

We thank Ling Zhou for his useful comment. The work of HTN is supported by the quantum information processing IRC (QIPIRC) (GR/S82176/01). SB also thanks the Engineering and Physical Sciences Research Council (EPSRC) UK and the Royal Society and the Wolfson Foundation.

References

[1] Ekert A K et al 1991 Phys. Rev. Lett. 67 661
[2] Bennett C H et al 1993 Phys. Rev. Lett. 70 1895
[3] Bennett C H and Wiesner S 1992 Phys. Rev. Lett. 69 2881
[4] Bouwmeester D et al (ed) 2001 The Physics of Quantum Information (Berlin: Springer)
[5] Blinov B B, Moehring D L, Duan L M and Monroe C 2004 Nature 428 153
[6] Sherson J, Julsgaard B and Polzik E S 2006 arXiv:quant-ph/0601186
[7] Reid M D 1989 Phys. Rev. A 40 913
[8] Schori C et al 2002 Phys. Rev. A 66 033802
[9] Einstein A, Podolsky B and Rosen N 1935 Phys. Rev. 47 777

New Journal of Physics 11 (2009) 043009 (http://www.njp.org/)
[10] Li X 2002 Phys. Rev. Lett. 88 047904
[11] Ralph T C 2000 Phys. Rev. A 62 062306
[12] Burnham D C and Weinberg D L 1970 Phys. Rev. Lett. 25 84
[13] Furusawa A et al 1998 Science 282 706
[14] Morigi G et al 2006 Phys. Rev. Lett. 96 023601
  Morigi G et al 2006 Phys. Rev. A 73 033822
[15] Vitali D et al 2006 Phys. Rev. A 74 053814
[16] Brennecke F et al 2007 Nature 450 268
[17] Colombe Y et al 2007 Nature 450 272
[18] Maschler C and Ritsch H 2005 Phys. Rev. Lett. 95 260401
[19] Duan L M et al 2001 Nature 414 413
[20] Tavis M and Cummings F W 1969 Phys. Rev. 188 692
[21] Inouye S et al 1999 Science 285 571
[22] Muschik et al 2006 Phys. Rev. A 73 062329
[23] Harber D M, Lewandowski H J, McGuirk J M and Cornell E A 2002 Phys. Rev. A 66 053616
[24] Law C K, Wang L and Eberly J H 1992 Phys. Rev. A 45 5089
  Law C K and Eberly J H 1993 Phys. Rev. A 47 3195
[25] Ng H T and Leung P T 2005 Phys. Rev. A 71 013601
  Ng H T and Burnett K 2007 Phys. Rev. A 75 023601
[26] Barnett S M and Radmore P M 2002 Methods in Theoretical Quantum Optics (New York: Oxford University Press)
[27] Walls D F and Milburn G J 1994 Quantum Optics (Berlin: Springer)
[28] Gardiner C W and Zoller P 2004 Quantum Noise (Berlin: Springer)
[29] Duan L M et al 2000 Phys. Rev. Lett. 84 2722
  Simon R 2000 Phys. Rev. Lett. 84 2726
[30] Hinds E A et al 2007 Proc. ICOLS07 Conf. arXiv:0802.0987