We give an explicit affine algebraic variety whose coordinate ring is isomorphic (as a $W$-algebra) with the equivariant cohomology of some Springer fibers.

1. Introduction

Let $G$ be a connected simply-connected semisimple complex algebraic group with a Borel subgroup $B$ and a maximal torus $T \subset B$. Let $P \supseteq B$ be a (standard) parabolic subgroup of $G$. Let $L \supset T$ be the Levi subgroup of $P$ and let $S$ be the connected center of $L$ (i.e., $S$ is the identity component of the center of $L$). Then, $S \subset T$. We denote the Lie algebras of $G, T, B, P, L, S$ by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}, \mathfrak{p}, \mathfrak{l}, \mathfrak{s}$ respectively. Let $W$ be the Weyl group of $G$ and $W_L \subset W$ the Weyl group of $L$. Let $\sigma = \sigma_l$ be a principal nilpotent element of $\mathfrak{l}$. Let $X = G/B$ be the full flag variety of $G$ and let $X_\sigma \subset X$ the Springer fiber corresponding to the nilpotent element $\sigma$ (i.e., $X_\sigma$ is the subvariety of $X$ fixed under the left multiplication by $\text{Exp} \sigma$ endowed with the reduced subscheme structure). Observe that $S$ keeps the variety $X_\sigma$ stable under the left multiplication of $S$ on $X$.

Definition 1.1. Let $Z_l$ be the reduced closed subvariety of $\mathfrak{t} \times \mathfrak{t}$ defined by:

$$Z_l := \{ (x, wx) : w \in W, x \in \mathfrak{s} \}.$$  

Since $Z_l$ is a cone inside $\mathfrak{t} \times \mathfrak{t}$, the affine coordinate ring $\mathbb{C}[Z_l]$ is a non-negatively graded algebra. Moreover, the projection $\pi_1 : Z_l \to \mathfrak{s}$ on the first factor gives rise to a $S(\mathfrak{s}^*)$-algebra structure on $\mathbb{C}[Z_l]$. Also, define an action of $W$ on $Z_l$ by:

$$v \cdot (x, wx) = (x, vwx), \text{ for } x \in \mathfrak{s}, v, w \in W.$$  

This action gives rise to a $W$-action on $\mathbb{C}[Z_l]$, commuting with the $S(\mathfrak{s}^*)$ action on $\mathbb{C}[Z_l]$.

In fact, even though we do not need it, $W$ is precisely the automorphism group of $\mathbb{C}[Z_l]$ as $S(\mathfrak{s}^*)$-algebra.

For $\mathfrak{p} = \mathfrak{b}$, the Levi subalgebra $\mathfrak{l}$ coincides with $\mathfrak{t}$, $\sigma_l = 0$ and $X_\sigma = X$. In this case, $\mathfrak{s} = \mathfrak{t}$ and we abbreviate $Z_l$ by $Z$. Clearly, $Z_l$ (for any Levi subalgebra $\mathfrak{l}$) is a closed subvariety of $Z$.

The following theorem is our main result.

Theorem 1.2. With the notation as above, assume that the canonical restriction map $H^*(X) \to H^*(X_\sigma)$ is surjective, where $H^*$ denotes the singular cohomology with complex coefficients. Then, there is a graded $S(\mathfrak{s}^*)$-algebra isomorphism

$$\phi_l : \mathbb{C}[Z_l] \to H_S^*(X_\sigma),$$  

where $H_S^*$ denotes the $S$-equivariant cohomology with complex coefficients.

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Moreover, the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbb{C}[Z] & \xrightarrow{\phi_l} & H^*_T(X) \\
\downarrow & & \downarrow \\
\mathbb{C}[Z_l] & \xrightarrow{\phi_l} & H^*_S(X_\sigma),
\end{array}
$$

where the vertical maps are the canonical restriction maps.

In particular, we get an isomorphism of graded algebras

$$\phi^o_l : \mathbb{C} \otimes_{S(s^*)} \mathbb{C}[Z_l] \to H^*(X_\sigma),$$

making the following diagram commutative:

$$
\begin{array}{ccc}
\mathbb{C} \otimes_{S(t^*)} \mathbb{C}[Z] & \xrightarrow{\phi^o_l} & H^*(X) \\
\downarrow & & \downarrow \\
\mathbb{C} \otimes_{S(s^*)} \mathbb{C}[Z_l] & \xrightarrow{\phi^o_l} & H^*(X_\sigma),
\end{array}
$$

where the vertical maps are the canonical restriction maps and $\mathbb{C}$ is considered as a $S(s^*)$-module under the evaluation at 0.

Moreover, the isomorphism $\phi^o_l$ is $W$-equivariant under the Springer’s $W$-action on $H^*(X_\sigma)$ and the $W$-action on $\mathbb{C} \otimes_{S(s^*)} \mathbb{C}[Z_l]$ induced from the $W$-action on $\mathbb{C}[Z_l]$ defined above.

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2. Proof of the Theorem

Before we come to the proof of the theorem, we need the following lemma. (See, e.g., [C, Theorem 2].)

Lemma 2.1. For any $w \in W$, there exists a unique $w' \in W_L$ such that

$$w'wB \in X_\sigma^S \subset X.$$

Moreover, this induces a bijection

$$W_L \backslash W \leftrightarrow X_\sigma^S.$$

We also need the following simple (and well known) result.

Lemma 2.2. Let $S = S(V^*)$ be the symmetric algebra for a finite dimensional vector space $V$ and let $M, N, R$ be three $S$-modules. Assume that $N$ and $R$ are $S$-free of the same finite rank and $M$ is a $S$-submodule of $R$. Then, any surjective $S$-module morphism $\phi : M \to N$ is an isomorphism.

We now come to the proof of the theorem.

Proof of the theorem. Consider the equivariant Borel homomorphism

$$\beta : S(t^*) \to H_T(X)$$
obtained by $\lambda \mapsto c_1(\mathcal{L}_\lambda)$, where $\lambda \in \mathfrak{t}^*$ and $c_1(\mathcal{L}_\lambda)$ is the $T$-equivariant first Chern class of the line bundle $\mathcal{L}(\lambda)$ on $X$ corresponding to the character $e^\lambda$, and extended as a graded algebra homomorphism. This gives rise to an algebra homomorphism

$$\chi: \mathbb{C}[t \oplus t] \simeq S(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*) \to H_T(X), \quad p \otimes q \mapsto p \cdot \beta(q),$$

where $\cdot$ denotes the multiplication in the $T$-equivariant cohomology by $p \in S(\mathfrak{t}^*) \simeq H_T(pt)$. It is well known that $\chi$ is surjective. Moreover, both the restriction maps

$$H_T(X) \to H_S(X) \to H_S(X_\sigma)$$

are surjective; this follows since both the spaces $X$ and $X_\sigma$ have cohomologies concentrated in even degrees (cf. [DLP]). (Use the degenerate Leray-Serre spectral sequence and the assumption that the restriction map $H^*(X) \to H^*(X_\sigma)$ is surjective.)

Consider the canonical surjective map $\theta: \mathbb{C}[t \oplus t] \twoheadrightarrow \mathbb{C}[Z_\pi]$. Then, of course,

$$(3) \quad \text{Ker } \theta = \left\{ \sum_i p_i \otimes q_i : p_i, q_i \in S(\mathfrak{t}^*) \text{ and } \sum_i p_i(x) q_i(wx) = 0, \text{ for all } x \in \mathfrak{s} \text{ and } w \in W \right\}.$$

We claim that

$$(4) \quad \text{Ker } \theta \subset \text{Ker } \gamma,$$

where $\gamma$ is the composite map

$$\mathbb{C}[t \oplus t] \xrightarrow{\chi} H_T(X) \to H_S(X_\sigma).$$

Since $X_\sigma$ has cohomologies only in even degrees, by the degenerate Leray-Serre spectral sequence, $H_S(X_\sigma)$ is a free $S(\mathfrak{s}^*)$-module. In particular, by the Borel-Atiyah-Segal Localization Theorem (cf. [AP, Theorem 3.2.6]),

$$H_S(X_\sigma) \hookrightarrow H_S(X_\sigma^S).$$

Thus, to prove the claim (4), it suffices to prove that for any $\sum_i p_i \otimes q_i \in \text{Ker } \theta$,

$$\gamma \left( \sum_i p_i \otimes q_i \right) \big|_{X_\sigma^S} \equiv 0.$$

It is easy to see that the Borel homomorphism $\beta$ restricted to the $T$-fixed points $X^T$ satisfies:

$$\beta(q)(wB) = wq, \quad \text{for any } q \in S(\mathfrak{t}^*) \text{ and } w \in W.$$

Thus, for any $w \in W$,

$$\gamma \left( \sum_i p_i \otimes q_i \right)(w'wB) = \left( \sum_i (p_i)(w'wq_i) \right)|_w,$$

where $w'$ is as in Lemma 2.1. From the description of $\text{Ker } \theta$ given in (3), we thus get that the claim (4) is true. Hence, the map $\theta$ descends to a surjective $S(\mathfrak{s}^*)$-module map

$$\phi_\theta: \mathbb{C}[Z_\pi] \twoheadrightarrow H_S(X_\sigma).$$

Again using the Localization Theorem, the free $S(\mathfrak{s}^*)$-module $H_S(X_\sigma)$ is of rank $= \#W_L \setminus W$, since $\#X_\sigma^S = \#W_L \setminus W$ by Lemma 2.1. Also, the projection on the first factor $\pi_1: Z_\pi \to \mathfrak{s}$ is a finite morphism with all its fibers of cardinality $\leq \#W_L \setminus W$. To see this, consider the surjective morphism $\alpha: \mathfrak{s} \times W/W_L \to Z_\pi$, $(x, wW_L) \mapsto (x, wx)$. Then, $\pi_1 \circ \alpha: \mathfrak{s} \times W/W_L \to \mathfrak{s}$ is again the projection on the first factor, which is clearly a finite morphism and hence so is $\pi_1$. 
Now, taking $M = \mathbb{C}[Z]$, $N = H^*_S(X_\sigma)$, $R = \mathbb{C}[s \times W/W_L]$ and $V = s$ in Lemma 2.2, we get that $\phi$ is an isomorphism, where the inclusion $M \subset R$ is induced from the surjective morphism $\alpha : s \times W/W_L \rightarrow Z$.

The commutativity of the diagram (1) clearly follows from the above proof.

Since $H^*(X_\sigma)$ is concentrated in even degrees, by the degenerate Leray-Serre spectral sequence, we get that

$$H^*(X_\sigma) \simeq \mathbb{C} \otimes_{S(l^*)} H^*_S(X_\sigma).$$

From this the ‘In particular’ part of the theorem follows.

From the definition of the map $\phi$, it is clear that $\phi^\sigma$ is $W$-equivariant with respect to the action of $W$ on $\mathbb{C} \otimes_{S(l^*)} \mathbb{C}[Z]$ induced from the action of $W$ on $\mathbb{C}[Z]$ as defined in Definition 1.1 and the standard action of $W$ on $H^*(X)$. Moreover, the restriction map $H^*(X) \rightarrow H^*(X_\sigma)$ is $W$-equivariant with respect to the Springer’s $W$ action on $H^*(X_\sigma)$ (cf. [HS, §2]). Thus, the $W$-equivariance of $\phi^\sigma$ follows from the commutativity of the diagram (2). This completes the proof of the theorem.

\[\square\]

**Remark 2.3.** (1) By the Jordan block decomposition, any nilpotent element $\sigma \in sl(N)$ (up to conjugacy) is a regular nilpotent element in a standard Levi subalgebra $l$ of $sl(N)$. Moreover, the canonical restriction map $H^*(X) \rightarrow H^*(X_\sigma)$ is surjective in this case. In fact, as proved by Spaltenstein [S], in this case there is a paving of $X$ by affine spaces as cells such that $X_\sigma$ is a closed union of cells (cf. also [DLP]). Thus, the above theorem, in particular, applies to any nilpotent element $\sigma$ in any special linear Lie algebra $sl(N)$.

(2) A certain variant (though a less precise version) of our Theorem 1.2 for $\mathfrak{g} = sl(N)$ is obtained by Goresky-MacPherson [GM, Theorem 7.2].

(3) For a general semisimple Lie algebra $\mathfrak{g}$, it is not true that the restriction map $H^*(X) \rightarrow H^*(X_\sigma)$ is surjective for any regular nilpotent element in a Levi subalgebra $l$. Take, e.g., $\mathfrak{g}$ of type $C_3$ and $\sigma$ corresponding to the Jordan blocks of size $(3, 3)$. In this case, the centralizer of $\sigma$ in the symplectic group $Sp(6)$ is connected and $X_\sigma$ is two dimensional. The cohomology of $X_\sigma$ as a $W$-module is given as follows:

Of course, $H^0(X_\sigma)$ is the one dimensional trivial $W$-module; $H^2(X_\sigma)$ is the sum of the three dimensional reflection representation with a one dimensional representation; and $H^4(X_\sigma)$ is a three dimensional irreducible representation.

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