RESONANCE TYPE INSTABILITIES
IN THE GASEOUS DISKS
OF THE FLAT GALAXIES

III. The gyroscopical resonance type instability
C.M. Bezborodov, J.V. Mustsevaya and V.V. Mustsevoy

The stability of thin homogeneous gaseous disk with step discontinuity of velocity and density profile has been investigated analytically in the limitary case of large compressibility and numerically at spatially separated velocity and density kinks. The expressions for reflection and transmission coefficients at the axisymmetrical vortex sheet are derived. The possibility of gyroscopical resonance type instability development in addition to Kelvin–Helmholtz and centrifugal instabilities is shown and its physical mechanism is clarified. It has been discussed the possibility of application of obtained results to the stability analysis of the gaseous disks of the real flat galaxies.

Introduction.

The applied problems of astrophysics about stability of supersonic currents (i.e. disks of accreting matter, galactic disks, jets from young stars etc.) from the academic point of view are interesting first of all by allowing to allocate many specific effects and mechanisms of amplification of disturbances in the pure state. This circumstances are related to absence of such complicating factors as heat exchange with firm borders, non-absolute rigidity of these borders, and also extremely small viscosity. Therefore it is possible to combine successfully relative simplicity of models with practical application of results obtained at their analytis. In the present part of work with reference to gaseous disks of flat (spiral) galaxies amplification of unstable disturbances in a waveguide layer between jumps of rotation velocity and density caused by over-reflection is discussed.

The effect of approaching infinity of reflection and transmission coefficients of a flat monochromatic sound wave falling at certain angles on a flat surface of supersonic vortex sheet (so called the effect of over-reflection) was discovered by Miles (1957) and Ribner (1957). Subsequently the similar effect was investigated for magnetohydrodynamic jump (Acheson 1967). It was also shown that the presence of jump is not obligatory for over-reflection and that it also takes place on a critical layer in a flow with smooth velocity distribution (Kolyhalov 1985) (that is on a layer where flow speed and wave phase speed along flow speed coincide). The mechanism of over-reflection is caused by that the energy flux in a transmitted wave is directed to jump (Miles 1957, Ribner 1957). Thus the transmitted wave outgoing from jump (or critical layer) picks up kinetic energy from the basic current and transfers it to a reflected wave through jump (see e.g. Landau & Lifshits 1988).

Although Betchov & Criminale (1967) marked that if to arrange a reflecting surface (acoustic screen) in parallel to a critical layer, an acoustical resonance type instability may develop. However, their speculations did not go further than statement of such opportunity. The first rather serious researches of acoustic resonance type instability are probably works of Ferrari et al. 1982, Payne & Cohn 1985, Hardee & Norman 1988 (see also Norman et al. 1982), in which stability of astrophysical jets in models of a flat layer and cylindrical jet with discontinuous borders was investigated. The large number of works on the specified theme have appeared subsequently but all of them concerned to plainparallel gas flows.
The unstable modes caused by over-reflection from a vicinity of corotation in gaseous disks with monotonous dependence of rotation velocity on radius were found for the first time by Papaloisou and Pringle (1985, 1987). Then such modes (and also other resonant modes, for which amplification a continuity of rotation velocity is essential) were investigated by Glatzel (1987), Savonije & Heemskerk (1990), Papaloizou & Savonije (1991) etc. Nevertheless as for disks with continuous radial parameters distribution the linearized equations of gas dynamics can be investigated only numerically, there was the number of unsolved questions connected to dependence of characteristic time of development of unstable disturbances and angular phase rotation velocity of a pattern created by them upon unperturbed disk parameters. Besides, in quoted works the consideration was performed with reference to accretion disks.

In the present part of the article the effect of over-reflection and instability caused by it in the elementary discontinuous model of a thin gaseous disk is considered. Its applicability is caused by observable peculiarities of density and rotation velocity distributions in disks of flat galaxies.

The basis for consideration is such fact that the wave can be many times reflected from rather sharp density jump and from corotation radius, and from the latter with strengthening (Savonije & Heemskerk 1990). It is clear that even when “centres” of smoothed jumps of velocity and density distributions coincide, specified points of turn generally speaking are separated on radius. Thus in a disk the waveguide layer presents, in which wave collects energy in due course.

Is the most simple model of such waveguide in a gaseous disk is model of two jumps, namely vortex sheet\(^1\) and internal rather it contact jump. The choice of discontinuous model for the analysis requires some explanations, as to the present there is the number of works where stability of gaseous disks with continuous velocity and density profiles (see, for example, Papaloizou & Pringle 1985, 1987, Glatzel 1987, Savonije & Heemskerk 1990, Papaloizou & Savonije 1991, Torgashin 1986, Morozov 1989, Morozov et al. 1992) was analyzed.

However in a disk with continuous parameters distributions the instability as a rule is developed simultaneously by several mechanisms (for example, over-reflection and over-transmission of waves in a corotation vicinity (Savonije & Heemskerk 1990), resonant radiation of waves from corotation radius (ibidem), centrifugal (Torgashin 1986, Morozov 1989, Morozov et al. 1992), centrifugal-resonant (Morozov et al. 1992), Kelvin–Helmholtz instability (KHI) mechanism (Torgashin 1986) etc.). So it is almost impossible to tell them from each other and to reveal their relative influence. In the contrary the discontinuous model will allow us in section 3 to show that instability developing in it is of a really new type. Though the physical mechanism of this instability has similar features with the mechanism of acoustic resonance type instability in plainparallel current (Payne & Cohn 1985), there is conceptual difference consisting that a resonance occurs rather at sound frequency than at gyroscopic one. Accordingly frequency areas of unstable modes localization are determined not by “sound” frequency \(|\mathbf{k}|c_s\) but gyroscopic one \(\Omega_{\text{in};\text{ex}}(m \pm 2)\), where \(m\) is azimuthal mode number (number of an arms of spiral pattern). Besides such model will allow to allocate in the pure state unstable superficial modes: centrifugal

\(^1\) In discontinous model corotation radius with necessity coincides with vortex sheet, because phase speed of a unstable wave should lay between the minimum and maximum speeds of the basic current, that it is possible to prove with the help of a theorem (Kolyhalov 1985).
connected with velocity jump and mode connected to density drop, and to investigate
dependence of their properties on parameters of a disk. Thus an independent problem
is to determine the expressions for wave falling on axisymmetric vortex sheet reflection
and transmission coefficients (section 2), as though they have large importance for our
consideration but were not received former.

Finally, it seems useful to discuss in section 4 the probable astrophysical applications
of obtained results.

In the present part of the work we shall use the references to the formulae from Parts
I and II, adding in this case a symbol “1.” or “2.” accordingly before number of the
formula.

1. **Model and dispersion equation.**

The basic characteristic features of model used in the present part of our work are the
same as in model described in a Part II. We will therefore stop only at differences.

The model contains not one but two jumps: contact jump at radius $R_{\rho}$ and vortex
sheet at $R_{\Omega}$. Let’s assume for definiteness $R_{\rho} < R_{\Omega}$ as it is seems such situation is more
characteristic for real objects (at least it is valid for the Galaxy). The radial dependences
of unperturbed parameters are set by relations:

\[
\rho_0(r, z) = \rho_{in}(z) + (\rho_{ex}(z) - \rho_{in}(z)) \theta(r - R_{\rho}),
\]

\[
c_s(r, z) = c_{sin}(z) \left[ 1 + \left( \sqrt{\frac{\rho_{in}(z)}{\rho_{ex}(z)}} - 1 \right) \theta(r - R_{\rho}) \right],
\]

\[
\Omega(r, z) = \Omega_{in}(z) + (\Omega_{ex}(z) - \Omega_{in}(z)) \theta(r - R_{\Omega}),
\]

\[
P_0(r, z) = \frac{c_s^2(r, z)\rho_0(r, z)}{\gamma} = const(z).
\]

As the magnitudes of disturbances should be limited in zero and at infinity, the solution
of the appropriate modified Bessel equation in the given model takes a form:

\[
p(r) = \begin{cases} 
  AI_m(k_{in}r), & r < R_{\rho}, \\
  BI_m(k_{mer}r) + CK_m(k_{mer}r), & R_{\rho} < r < R_{\Omega}, \\
  DK_m(k_{ex}r), & r > R_{\Omega}.
\end{cases}
\]

Here $k_{in}$ and $k_{ex}$ are defined by (2.6) and in area between jumps

\[
k_{me}^2 = \frac{4\Omega_{in}^2 - (\omega - m\Omega_{in})^2}{c_{s ex}^2}.
\]

It is obvious the conditions at jumps do not differ from (2.8), (2.9):

\[
p(R_{\rho} + 0) - p(R_{\rho} - 0) = 0,
\]

\[
p(R_{\Omega} + 0) - p(R_{\Omega} - 0) = \rho_{ex} \xi(R_{\Omega}) \Omega_{\Omega}(\Omega_{in} - \Omega_{ex}^2),
\]

\[
\xi(R_{\rho} + 0) - \xi(R_{\rho} - 0) = \xi(R_{\Omega} + 0) - \xi(R_{\Omega} - 0) = 0.
\]

Substituting the solution as (2) in conditions (4) we come to the system of linear algebraic
equations on coefficients $A, B, C, D$. A condition of simultaneity of this system will be
equality to zero of its determinant. Excepting neutral gyroscopic modes of fluctuations
\( \omega = (m \pm 2)\Omega_{in} \), \( \omega = (m \pm 2)\Omega_{ex} \) we finally write:

\[
\begin{vmatrix}
1 & 1 & 1 & 0 \\
\beta_{in}^{(p)} & Q_{\alpha_{me}}^{(p)} & Q_{\beta_{me}}^{(p)} & 0 \\
0 & \delta_K & \delta_I & 4q^2 - (x - mq)^2 - \alpha_{ex}^{(\Omega)} (1 - q^2) \\
0 & \delta_I \alpha_{me}^{(\Omega)} & \delta_I \beta_{me}^{(\Omega)} & \alpha_{ex}^{(\Omega)} [4 - (x - m)^2]
\end{vmatrix} = 0. \tag{5}
\]

The equality (5) represents the dispersion equation for disturbances in considered model.
Designations of Parts I and II in combinations \( \alpha_i^{(j)} \) and \( \beta_i^{(j)} \) setting by (1.8) and (1.9) are here kept; top index specifies that they are written on radius of velocity or density jump, the bottom index takes values “\( \text{in} \)”, “\( \text{ex} \)” or “\( \text{me} \)” (with the account \( \Omega_{me} \equiv \Omega_{in} \)). Besides following designations are introduced:

\[
\delta_I = \frac{I_m(k_{me}R_{\Omega})}{I_m(k_{me}R_{\rho})}; \quad \delta_K = \frac{K_m(k_{me}R_{\Omega})}{K_m(k_{me}R_{\rho})}. \tag{6}
\]

Before solving directly the dispersion equation (5) in the following section we consider a problem on reflection of a neutral wave from axisymmetric vortex sheet to have an opportunity to apply its results at interpretation of the instability mechanism.

2. Reflection and transmission coefficients.

From theoretical works (Miles 1957, Ribner 1957) it is known that at incidence on plainparallel vortex sheet the neutral wave can reflect and refract with amplification on amplitude (over-reflection effect). At presence of the second reflecting surface (for example, density jump or kink) spatial resonator will be formed, in which the wave can collect energy in due course. Therefore the over-reflection is capable to result in development of instabilities of a waveguide-resonant type, in particular gyroscopic resonance type instability considered in the given section.

For plainparallel shift currents the problem of over-reflection up to now is investigated well even for systems with magnetic fields (Acheson 1967). However, coefficients of reflection and transmission for a neutral wave falling on axisymmetric jump were not obtained till now.

Note at once that in cylindrical geometry there is the allocated direction on centre. It leads to distinction of coefficients for waves falling on axisymmetric jump from within and from outside. Nevertheless, in a limit of large jumps radii or azimuthal short-wave disturbances the expressions for coefficients received in our model should lead to similar expressions from a plainparallel problem.

Following Miles (1957) and Ribner (1957), we shall consider incidence of a neutral monochromatic wave on vortex sheet being not interested in this section by a question of its stability (on formal correctness of such approach see Landau & Lifshits 1988).

Further in this section the index “\( i \)” concerns to a incident wave, “\( r \)” — to reflected, index “\( t \)” — to transmitted ones. The magnitudes in internal area from jump are designated by index 1, outside — 2.

a) Incidence from within on density and velocity jump
By applying results of a Part II and condition of limitary absorption (i.e. the disturbances should fade as they propagate) at once write out peak functions of pressure of all three waves:

\[ p_i = AK_m(k_1 r); \quad p_r = BI_m(k_1 r); \quad p_t = CK_m(k_2 r). \tag{7} \]

Thus displacement magnitudes on dividing border are

\[ \xi_1 = \xi_i + \xi_r = \frac{AK_m(k_1 R)\alpha_1 + BI_m(k_1 R)\beta_1}{\rho_1 c_{s1}^2 k_1 R}, \tag{8} \]

\[ \xi_2 = \xi_t = \frac{CK_m(k_2 R)\alpha_2}{\rho_2 c_{s2}^2 k_2 R}. \tag{9} \]

The parameters \( \alpha_i, \beta_i \) are governed by the formulae (1.8), (1.9).

Now introduce complex reflection and transmission coefficients with respect to pressure by a usual way:

\[ \mathcal{R}(-) \equiv \frac{p_r}{p_i} = \frac{BI_m(k_1 R)}{AK_m(k_1 R)}; \quad \mathcal{T}(-) \equiv \frac{p_t}{p_i} = \frac{CK_m(k_2 R)}{AK_m(k_1 R)}. \tag{10} \]

Taking into account general conditions (2.8), (2.9) come to the equations on \( \mathcal{R}(-), \mathcal{T}(-) \):

\[ \frac{k_2^2 R^2}{k_1^2 R^2} \frac{\alpha_2}{\alpha_1} \mathcal{T}(-) - \frac{\beta_1}{\alpha_1} \mathcal{R}(-) = 1, \tag{11} \]

\[ \frac{\mu^2}{Q} \mathcal{T}(-) \left\{ 1 - \frac{M^2 (1 + Q)}{2 \mu^2} \frac{\alpha_2}{k_2^2 R^2} (1 - q^2) \right\} = 1 + \mathcal{R}(-). \tag{12} \]

Solving this system let us write coefficients in an obvious kind:

\[ \mathcal{R}(-) = \frac{Q k_1^2 R^2 \alpha_2}{\mu^2 k_2^2 R^2 \beta_1 - Q k_1^2 R^2 \alpha_2} + \frac{1}{2} (1 + Q) (1 - q^2) M^2 \alpha_1 \alpha_2, \tag{13} \]

\[ \mathcal{T}(-) = \frac{Q k_2^2 R^2 (\beta_1 - \alpha_1)}{\mu^2 k_2^2 R^2 \beta_1 - Q k_1^2 R^2 \alpha_2} - \frac{1}{2} (1 + Q) (1 - q^2) M^2 \beta_1 \alpha_2. \tag{14} \]

\textit{b) Incidence from outside on density and velocity jump}

In a case when the neutral wave falls on vortex sheet from the large radii the solution of modified Bessel equation takes a form:

\[ p_i = AI_m(k_2 r); \quad p_r = BK_m(k_2 r); \quad p_t = CI_m(k_1 r). \tag{15} \]

With the help (15) write down displacement on jump:

\[ \xi_1 = \xi_t = \frac{CI_m(k_1 R)\beta_1}{\rho_1 c_{s1}^2 k_1^2 R}, \tag{16} \]

\[ \xi_2 = \xi_i + \xi_r = \frac{AI_m(k_2 R)\beta_2 + BK_m(k_2 R)\alpha_2}{\rho_2 c_{s2}^2 k_2^2 R}. \tag{17} \]
Applying again the condition on jump and making simple transformations we obtain coefficients $\mathcal{R}^{(+)}$ and $\mathcal{T}^{(+)}$:

\[
\mathcal{R}^{(+)} = \frac{\mu^2 k_2^2 R^2 \beta_1 - Q k_1^2 R^2 \beta_2 - \frac{1}{2} (1 + Q)(1 - q^2) M^2 \beta_1 \beta_2}{Q k_1^2 R^2 \alpha_1 - \mu^2 k_2^2 R^2 \beta_1 + \frac{1}{2} (1 + Q)(1 - q^2) M^2 \beta_1 \alpha_1},
\]

\[
\mathcal{T}^{(+)} = \frac{\mu^2 k_1^2 R^2 (\alpha_1 - \beta_2)}{Q k_1^2 R^2 \alpha_1 - \mu^2 k_2^2 R^2 \beta_1 + \frac{1}{2} (1 + Q)(1 - q^2) M^2 \beta_1 \alpha_1}.
\]

Note that classical connection between $\mathcal{R}$ and $\mathcal{T}$ ($\mathcal{T} = 1 + \mathcal{R}$) known for a plainparallel problem (Miles 1957, Ribner 1957, Landau & Lifshits 1988) in our case is valid only on density jump as just on it the boundary conditions coincide identically with those of a flat problem, namely both $\xi$ and $p$ are continuous.

It is interesting to note taking the determinant (5) it is possible to write dispersion equation through the reflection coefficients production:

\[
\mathcal{R}^{(+)} \mathcal{R}^{(-)} = \frac{I_m(k_{me} R_\Omega) K_m(k_{me} R_\rho)}{I_m(k_{me} R_\rho) K_m(k_{me} R_\Omega)}.
\]

Here the index “$\rho$” marks reflection coefficient on density jump ($R = R_\rho$, $q = 1$) and index “$\Omega$” — on velocity jump ($R = R_\Omega$, $Q = \mu = 1$). The latter equation allows to demonstrate evidently that reflection coefficients are associated with the instability mechanism that we consider in the following section. Really, in the limit of short radial wavelength ($|k_{me} R_\rho| \gg m$) from (20) we find:

\[
\mathcal{R}^{(+)} \mathcal{R}^{(-)} \simeq \exp \left\{ 2 k_{me}(R_\Omega - R_\rho) \right\} \simeq \exp \left\{ -2iD \frac{M}{\mu} (x - m) \right\}.
\]

Dimensionless width of a backlash between jumps is here introduced:

\[
D = \frac{R_\Omega - R_\rho}{R_\Omega}.
\]

Taking a square of the (21) module we get:

\[
\left| \mathcal{R}^{(+)} \mathcal{R}^{(-)} \right|^2 \simeq \exp \left\{ 4 D \frac{M}{\mu} I m x \right\} = \exp \left\{ 4 \frac{(R_\Omega - R_\rho)}{c_{s e x}} I m \omega \right\}.
\]

Thus for unstable ($I m \omega > 0$) disturbances $\left| \mathcal{R}^{(+)} \mathcal{R}^{(-)} \right| > 1$ is always valid. From (22) follows

\[
I m \omega \simeq \frac{1}{2\tau_c} \ln \left| \mathcal{R}^{(+)} \mathcal{R}^{(-)} \right|,
\]

where $\tau_c$ is characteristic run-time of a sound wave between jumps. Note that the result (23) is typical for models admitting development of acoustic resonance type instabilities (Payne & Cohn 1985).

3. The stability analysis.
Before receiving the approximated solutions of the equation (5) it is necessary to make one remark. For existence of over-reflection and unstable reflective harmonics caused by it a nonzero wave energy flux in a radial direction is necessary at least in two areas (at \( r > R_\Omega \) and at \( R_\rho < r < R_\Omega \)) the same as and in a flat case (see Morozov et al. 1991). It is possible if the eigenfunctions \( p \) and \( \xi \) in these areas are oscillating on radial coordinate (as for value of group velocity a value of radial component of wave vector \( k_r \) is crucial). The latter may occur only when arguments of modified Bessel functions are imaginary (the proceeding concerns to a limit \( Im \omega \rightarrow 0 \), as for such disturbances only an energy flux is defined). Writing \( k^2 = -k^2_r \) in each area we come to restriction on a real part of frequency, namely, the following inequalities should not occur:

\[
(m - 2)\Omega_{in} \leq Re \omega \leq (m + 2)\Omega_{in},
\]

\[
(m - 2)\Omega_{ex} \leq Re \omega \leq (m + 2)\Omega_{ex}.
\]

As for increasing disturbances

\[
m\Omega_{ex} \leq Re \omega \leq m\Omega_{in}
\]

is valid (at \( \Omega_{ex} < \Omega_{in} \)), the reflective harmonics can take place only in following frequency range:

\[
(m + 2)\Omega_{ex} < Re \omega < (m - 2)\Omega_{in},
\]

that it is possible at \( m > 2 \). Note that here is the strict inequality, as neutral gyroscopic modes have already been excluded from consideration.

Will jump-orthogonal component of a wave vector be real or imaginary (and accordingly presence or absence of energy flux in this direction) is determined in a flat case by a relation of squares of Doppler \( \omega - k_{\parallel}V \) and sound \( k_{\parallel}c_s \) frequencies, and in considered case — by Doppler \( \omega - m\Omega_i \) and gyroscopic \( 2\Omega_i \) ones. And in the latter case changing \( c_s \) simply scales \( k \) (\(|k| \rightarrow \infty \) at \( c_s \rightarrow 0 \) and \(|k| \rightarrow 0 \) at \( c_s \rightarrow \infty \)), not breaking proportions between \( Im k \) and \( Re k \). Accordingly the resonance will take place on harmonics of gyroscopic frequency \( (m - 2)\Omega_{in} \) in a layer of gas between velocity and density jumps. This allows on the one hand to limit ourselves in searching of reflective harmonics by permitted area of frequencies (27) (this area is limited by continuous lines on Fig. 3.1) and on the other hand to set the value \( (m - 2)\Omega_{in} \) as initial approximation of frequency of the basic unstable mode in a waveguide layer between jumps.

Let us obtain the solution of the dispersion equation (5) in a limit of strong compressibility of media in all three areas divided by jumps \((M = R_\Omega, \Omega_{in}/c_{sin} \gg 1, M/\mu \gg 1)\). Remember, magnitude of arguments of modified Bessel functions is determined in the basic by Mach number. It allows to put \(|k_iR_\rho| \gg m\); in this case the dispersion equation is simple to bring to a kind:

\[
\frac{Q \th a + \mu}{\mu \th a + Q} \simeq \frac{x - m}{-(x - mq) + i \frac{M}{\mu}(1 - q^2)},
\]

where \( a = k_{me}R_\Omega D = \frac{MD}{\mu} \sqrt{4 - (x - m)^2} \). For the case of uniform on density at \( z = const \) disk \((Q = \mu = 1)\) from (28) result (2.12) describing centrifugal instability mode directly
follows. In general case ($Q \neq 1$) if only $D \to 0$ has not place with accuracy to the terms of order $|e^{-2a}(Q - \mu)/(Q + \mu)| \ll 1$ we find:

$$x \simeq \frac{1}{2} m(1 + q) + \frac{iM}{2\mu}(1 - q^2).$$

(29)

Expression (29) means that if the jumps are not too close the centrifugal mode of velocity jump in the main order is not “sensitive” to the density jump.

For the further calculations transform (28) as follows:

$$\text{th}a \simeq - \frac{Q(x - m) + \mu(x - mq) - iM(1 - q^2)}{\mu(x - m) + Q(x - mq) - i\frac{M}{\mu}(1 - q^2)}. $$

(30)

In a case of very close incoincident jumps ($D \ll 1$) the left-hand part of (30) can be presented as: $\text{th}a \simeq -iMD(x - m)/\mu$. Substituting this result in (30) we get the square equation on $x$, but despite its simplicity the solution appears extremely inconvenient for the analysis because of bulkness. Therefore let us limit by a case $D \to 0$ at which we find:

$$x \simeq \frac{1}{\mu + Q} \left\{ m(Q + q\mu) + iM(1 - q^2) \right\}. $$

(31)

It is easy to note, that (31) does not coincide with the asymptotic solution (2.11) for concurrent jumps. Thus the limiting transition to a case $D \equiv 0$ is absent. It is a direct consequence that we work with model of “jump” with finite thickness ($\xi \ll \Lambda_{\Omega}, \Lambda_\rho$, where $\Lambda_{\Omega}, \Lambda_\rho$ are characteristic radial scales of jumps of angular velocity and density), in which as it was specified by Fridman & Khoruzhii (1993) the large role is played with a specific kind of equilibrium parameters inside “jump” profile. It is obvious, the result (31) could be obtained also in model of concurrent jumps if to set “asymmetrical” concerning middle of “jump” $\rho(r)$ and $\Omega(r)$ distributions that would change boundary conditions (Fridman & Khoruzhii 1993). The physical sense of (31) becomes perfectly transparent if to write growth rate (31) in the dimensional form:

$$\text{Im} \omega = \frac{\rho_{ex}R_{\Omega}(\Omega_{in}^2 - \Omega_{ex}^2)}{c_{sin}\rho_{in} + c_{sex}\rho_{ex}}. $$

(32)

So the converse characteristic time of instability development is determined by the ratio of centrifugal force density drop on angular velocity jump to the sum of media wave resistances on both sides from a layer containing both jumps.

To find out reflective harmonics we transform the left-hand part of (30) by presenting dimensionless frequency as $x = x_0 + \delta x$ and $|\delta x| \ll x_0$:

$$\text{th} \left( \frac{MD}{\mu} \sqrt{4 - (x - m)^2} \right) \simeq i \tan \left( \frac{MD}{\mu} \sqrt{(x_0 - m)^2 - 4 + C \delta x} \right),$$

(33)

where

$$C = \frac{MD}{\mu} \frac{(x_0 - m)}{\sqrt{(x_0 - m)^2 - 4}}.$$
Transforming (33) further, we assume \( \frac{MD}{\mu} \sqrt{(x_0 - m)^2 - 4} = \pi n \).

Then (33) degenerates to \( i \tan(\pi n + C \delta x) = i \tan(C \delta x) \). Making the additional assumption \( |C \delta x| \ll 1 \) we get \( \tan(C \delta x) \simeq C \delta x \), with respect of told let us write approximated expression for frequency of reflective harmonics (number of harmonic \( n = 1, 2, 3, \ldots \) determines number of zeros of eigenfunctions between jumps):

\[
x_0 = m - \sqrt{4 + \left(\frac{\mu \pi n}{MD}\right)^2};
\]

(34)

\[
\delta x \simeq \frac{m(Q + \mu q) - x_0(Q + \mu) + iM(1 - q^2)}{(Q + \mu) + \frac{CMQ}{\mu}(1 - q^2) + iC [x_0(Q + \mu) - m(\mu + qQ)]}.
\]

(35)

Remind, by virtue of told in the section beginning, \( x_0 \) lays in limits:

\[
(m + 2)q < x_0 < (m - 2).
\]

(36)

At last let us find approximated expression for basic (with \( n = 0 \)) unstable waveguide mode. Note, by simple substitution \( n = 0 \) in (34), (35) the frequency of this mode cannot be described as then \( C \) turns to infinity.

We believe \( x \sim m - 2 \). Then in approximation of \( |k_{in} R_\rho|, |k_{me} R_\Omega| \ll 1, \ |k_{ex} R_\Omega| \gg 1 \) substituting frequency as \( x = m - 2 + \delta x \) in the equation (5) we determine:

\[
\delta x \simeq \frac{4 \left[ 1 - q^2 + \frac{i\mu}{M}(m - 2 - mq) \right]}{8 \frac{1}{m} - 4 \frac{i\mu}{M} + \left[ (1 + Q) + \Xi \right] \left( 1 - q^2 + \frac{i\mu}{M}(m - 2 - mq) \right)},
\]

(37)

where \( \Xi \equiv (1 - Q)(1 - D)^{m} \).

Remarkable peculiarity of expressions (34), (35), (37) is the fact that they describe unstable roots both at \( \Omega_{in} > \Omega_{ex} \) and at \( \Omega_{ex} > \Omega_{in} \) and in an essentially supersonic case \( (R_\Omega |\Omega_{in} - \Omega_{ex}| / c_{sex} \gg 1) \), and are similar to analogous asymptotics of reflective harmonics and basic mode developing between density and velocity jumps in a plainparallel flow (see Morozov et al. 1991). According to told in the beginning of section instability described by expressions (34), (35), (37) we shall name the gyroscopic resonance type instability (GRTI). The absence of centrifugal suppression of instability at \( \Omega_{ex} > \Omega_{in} \) is caused by overreflective character of its development mechanism that is by specifically wave effect not connected directly to mass forces density distribution in a disk.

The case of small compressibility \( (M \ll 1) \) cannot be investigated analytically because of bulkness of the approximated algebraic equation obtained in this limit.

The results of the numerical solution of (5) (see Fig. 3.2–3.7) as a whole confirm conclusions made on the basis of asymptotic analytical research and at essential compressibility \( (R_\Omega |\Omega_{in} - \Omega_{ex}| / c_{sex} \gg 1) \) and small relative distance between jumps \( D \ll 1 \) these results show presence besides mode of centrifugal instability a discrete set of poorly unstable GRTI modes. The phase angular velocity of rotation of a pattern of these modes \( \Omega_p = Re \omega / m \) extremely weakly depends on \( Q \) and \( q \) (Fig. 3.2, 3.4) and essentially changes with \( D \) and \( M \) change (Fig. 3.5, 3.6), remaining thus in permitted frequency area (27) (see also Fig. 3.1). Exception makes basic (not having zeros of eigenfunctions between jumps) GRTI mode for which in all parameters ranges \( Re \omega \) insignificantly exceeds \( (m - 2) \Omega_{in} \).
The number of reflective \((n \geq 1)\) harmonics is increased with growth of Mach number, azimuthal mode number \(m\) and distance between jumps reduction.

As it is seen from Fig. 3.8 the asymptotic solution of (34), (35) gives only qualitative conformity to numerical results. It is connected that at approximated taking square roots in \(k_iR_\Omega\) we have neglected a square of gyroscopic frequency in comparison with a squared Doppler one, whereas the effect of over-reflection resulting in GRTI occurrence is caused by a resonance just between these frequencies. Nevertheless, (34), (35) begin to work well at large numbers \(m\) when the calculation of modified Bessel functions on the computer is complicated.

Let us specify at last, that as well as in a Part II the considered disturbances with \(\tilde{v}_z \equiv 0\) are allocated by a natural way by the instability mechanism itself. Really, in this sense all told in Part II is valid for centrifugal instability mode, and resonant harmonics with a wave vector component \(k_z\) comparable with \(k_r\) and \(k_\phi\) during the time of \(h/c_s\) order (here \(h\) is the characteristic disk half-thickness) leave a waveguide layer. As this time is near to inverted growth rate of resonant modes, such disturbances will have no time to amplify up to appreciable amplitudes. If to take into account an opportunity of returning of these disturbances to a waveguide layer because of refraction, their growth rate anyway will be less than at disturbances not having \(k_z\).

4. Conclusions.

Let us generalize numerical and analytical results by formulating the conclusions about stability of a uniform on pressure at \(z = \text{const}\) gas disk containing rotation angular velocity jump and internal rather it non-homentopic density jump (in models of “jumps” of finite thickness, that is \(\xi \ll \Lambda_\Omega, \Lambda_\rho\), where \(\Lambda_\Omega, \Lambda_\rho\) are characteristic radial scales of angular velocity and density jumps).

1. In a limit of weak compressibility in considered system only one unstable mode resulting Kelvin–Helmholtz instability development on vortex sheet can develop.
2. With growth of compressibility the development mechanism of the vortex sheet superficial mode essentially changes from Bernoulli effect resulting in Kelvin–Helmholtz instabilities up to centrifugal one in a supersonic case.
3. The presence of density jump in the system appreciably impacts to centrifugal instability development only when the jumps are very close, i.e. \(D \ll 1\).
4. At essentially supersonic velocity difference on jump and \(m > 2\) a discrete set of weakly unstable modes appears. These modes differ by number of zeros of eigenfunctions between jumps. The growth rate of these modes appreciably differs from zero only when \(D \ll 1\) and their amount grows with growth of compressibility on jump, azimuthal mode number, with distance between jumps reduction and practically does not depend on other parameters. These modes are caused by over-reflection on velocity jump and collect energy in a waveguide layer between jumps exponentially in time, picking it up from kinetic energy of the main movement (rotation) of outer gas layer.
5. The effect of over-reflection is caused by a resonance between Doppler and gyroscopic frequencies\(^2\) and the waveguide unstable modes described above are displayed near to the basic gyroscopic frequency and its higher harmonics. So the given instability is named as gyroscopic resonance type instability (GRTI).

\(^2\) The last in considered model with discontinuous velocity profile represents a degenerate case of epicyclic frequency.
6. As against centrifugal instability the GRTI develops and at “negative” velocity drop on jump (when the periphery of a disk rotates faster than the central region).

Finishing the article, we make some final remarks on probable applied meaning of obtained results.

It is necessary to note, GRTI by virtue of small scale and small growth rates is hardly capable to compete to centrifugal and Kelvin–Helmholtz instabilities as the generator of large-scale spiral structure, its role rather can be reduced unless to creation of multi-arm secondary (imposed on basic) pattern (see Morozov et al. 1992). At the same time it is known, the similar resonant Papaloisou–Pringle instabilities and acoustic resonance type ones are much more weakly subject to stabilizing influence of velocity jump smoothing and besides are multimode (Savonije & Heemskerk 1990, Papaloizou & Savonije 1991, Mustsevoy & Hoperskov 1991). Add that the resonant modes by virtue of its development slowness do not result to rough rearrangement of system parameters distributions and to system exiting on stability margin. Therefore the simultaneous development of large number of various harmonics of different azimuthal GRTI modes probably is capable to lead to effective removal of an angular moment from a vicinity of a waveguide layer on disk periphery and to smooth durable change of radial profiles of density and rotation velocity.

Make at last one more remark on possible GRTI and other hydrodynamic resonant instabilities influence on magnitude of stars velocities dispersion in a galactic disk.

Besides gyroscopic resonance type instability considered in the previous sections the development of other resonant instabilities (e.g. Papaloisou-Pringle instability (Papaloizou 1985, Savonije & Heemskerk 1990), resonant-centrifugal one (Morozov et al. 1992)) is possible in a differentially rotating gaseous disk. All of them are multimode and are characterized by discrete sets of radial, azimuthal and transverse to a disk wave numbers and frequency eigenvalues. Thus in result of development of resonant type instabilities in a gaseous galactic disk there is the complex system of disturbances of velocity and thermodynamic parameters with hierarchy of spatial and temporal scales. It is rather essential that the energy of these disturbances exceeds sound energy or tends to it from above. The basis for the last statement give laboratory (Norman et al. 1982) and nonlinear numerical (Norman & Hardee 1988) modelling of supersonic jets. In these experiments it is shown that the growth of disturbances caused by resonant instabilities stops at a nonlinear stage at the formation of system of weak slanted shock waves (disturbance with wavelength comparable to characteristic jet size) on a background of advanced acoustic turbulence (the most short wavelength disturbances). On the other hand it is known that in galactic disks rich by gas (∼ 10%), the gravitational stability of a disk is determined by more “cold” gas component (Jog & Solomon 1984). Therefore in numerical experiments on modeling binary (gas + stars) disk the gravitational instability of a gas subsystem develops and results in formation of giant molecular complexes (GMC). In turn relaxational processes (stars scattering on massive GMC) result to fast “warm up” of a star disk. As a result stars velocity dispersion becomes so high that the star disk appears gravitationally stable even at an unstable initial condition (described scenario was observed, for example, in experiments of Shlosman & Noguchi 1993). So in numerical experiments the specified process could result in a conclusion of impossibility of spiral structure or bar generation by development of gravitational instability of a star disk.

As the margin of gravitational stability is determined by balance of energy of an ordered movement under action of gravitational forces and stochastic thermal movements
and as the energy of small-scale system of weak slanted shock waves caused and supported by multimode resonant instabilities, dissipates in thermal because of viscosity (though small), it is necessary to make a conclusion that the really gas disk in an extended vicinity of a waveguide layer is warmed up and consequently appears to be more gravitationally stable. Besides velocity and density fluctuations caused by resonant instabilities should slow down GMC formation, partially to destroy already formed complexes and accordingly make scattering of stars on them less effective.

So the account of the really working resonant type instabilities ought cause reduction of stars velocities dispersion in numerical modeling “gas + stars”. We have used a subjunctive inclination as the resonant modes conceptually are short-wave and cannot be observed in really feasible experiments as their characteristic scales are less or comparable to the characteristic size of a cell and they are inevitably absorbed by numerical viscosity.

Acknowledgement. One of us (VVM) is grateful to INTAS for support of this work by grant project N 95-0988.

References

Acheson, D.J., 1967, J. Fluid Mech., 77, 433
Betchov, R., & Criminale, W.O., 1967, Stability of Parallel Flows (New York: Academic Press)
Ferrari, A., Massaglia, S., & Trussoni, E., 1982, MNRAS, 198, 1065
Fridman, A.M., & Khoruzhii, O.V., 1993, Sov. Phys. Uspekhi, 163, 79
Glatzel, W., 1987, MNRAS, 225, 227
Hardee, P.E., & Norman, M.L., 1988, Ap. J., 334, 70
Jog, C.J., & Solomon, P.M., 1984, Ap. J., 276, 114
Kolyhalov, P.I., 1985, Sov. Phys. Dokl., 180, 95 (in Russian)
Landau L.D., Lifshits E.M., 1986, Gidrodinamika (Hydrodynamics), 3rd ed. (Moscow: Nauka) (There exist an English edition, 1987).
Miles, J.W., 1957, J. Acoust. Soc. Amer., 29, 226
Morozov, A.G., 1989, KFNT, 5, 75 (in Russian)
Morozov, A.G., Mustsevaya, J.V., & Mustsevoy, V.V., 1991, Preprint of Volgograd State Univ., 2-91
Morozov, A.G., Mustsevoy, V.V., & Prosvirov, A.E., 1992, SvA. Lett., 18, 46
Mustsevoy, V.V., & Khoperskov, A.V., 1991, SvA. Lett., 17, 281
Norman, M.L., Smarr, L., & Vinkler, K.-H., 1982, in “Numerical Astrophysics. Proceedings of a symposium in honor of James R. Wilson...” (Boston: Jones and Bartlett Publishers, Inc.)
Norman, M.L., & Hardee, P.E., 1988, Ap. J., 334, 80
Papaloizou, J.C.B., & Pringle, J.E., 1985, MNRAS, 213, 799
Papaloizou, J.C.B., & Pringle, J.E., 1987, MNRAS, 225, 267
Papaloizou, J.C.B., & Savonije, G.J., 1991, MNRAS, 248, 353
Payne, D.G., & Cohn, H., 1985, Ap. J., 334, 80
Ribner, H.S., 1957, J. Acoust. Soc. Amer., 29, 435
Savonije, G.J., & Heemskerk, M.H.M., 1990, A&A, 240, 191
Shlosman, I., & Noguchi, M., 1993, Ap. J., 414, 474
Torgashin, Yu.M., 1986, PhD diss., Tartu (in Russian)
Fig. 3.1. Areas of frequencies permitted for harmonics caused by over-reflection (limited by continuous lines).
Fig. 3.2. Dependence of dimensionless frequency (a) and dimensionless growth rate (b) from relative density drop on internal jump. The letter designations have dispersion curves of centrifugal mode, the digits correspond to number of harmonics of gyroscopic resonance type instability. Centrifugal instability growth rate is not shown since it significantly exceeds GRTI growth rate.
Fig. 3.3. The same as on Fig. 3.2 (b) but in scale allowing to show centrifugal instability growth rate.
Fig. 3.4. Dependence of dimensionless frequency (a) and dimensionless growth rate (b) from angular velocity relative difference on external jump.
Fig. 3.5. Dependence of dimensionless frequency (a) and dimensionless growth rate (b) from Mach number.
Fig. 3.6. Dependence of dimensionless frequency (a) and dimensionless growth rate (b) from the ratio of density jump radius to velocity jump one $\Delta = R_\rho/R_\Omega = 1 - D$. 

$m = 8 \quad M = 10 \quad q = 0.2 \quad Q = 20$
Fig. 3.7. The same as on Fig. 3.6 but the most interesting fragments are shown in larger scale.
Fig. 3.8. Comparative behaviour of dispersion curves by results of the numerical solution of (5) — continuous lines and on asymptotics (34), (35), (37) — dashed lines. It is visible concurrence only qualitative.