Truncated, Censored, and Actuarial Payment-type Moments for Robust Fitting of a Single-parameter Pareto Distribution

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Abstract. With some regularity conditions maximum likelihood estimators (MLEs) always produce asymptotically optimal (in the sense of consistency, efficiency, sufficiency, and unbiasedness) estimators. But in general, the MLEs lead to non-robust statistical inference, for example, pricing models and risk measures. Actuarial claim severity is continuous, right-skewed, and frequently heavy-tailed. The data sets that such models are usually fitted to contain outliers that are difficult to identify and separate from genuine data. Moreover, due to commonly used actuarial “loss control strategies” in financial and insurance industries, the random variables we observe and wish to model are affected by truncation (due to deductibles), censoring (due to policy limits), scaling (due to coinsurance proportions) and other transformations. To alleviate the lack of robustness of MLE-based inference in risk modeling, here in this paper, we propose and develop a new method of estimation – method of truncated moments (MTμM) and generalize it for different scenarios of loss control mechanism. Various asymptotic properties of those estimates are established by using central limit theory. New connections between different estimators are found. A comparative study of newly-designed methods with the corresponding MLEs is performed. Detail investigation has been done for a single parameter Pareto loss model including a simulation study.

Keywords & Phrases. Claim Severity; Deductible; Relative Efficiency; Loss Models; Robust Estimation; Truncated and Censored Moments.
1 Introduction

The research leading to the results of this work is basically motivated to find some trade-offs between robustness and efficiency of parametric estimators for ground-up continuous loss distributions. Parametric statistical loss models for insurance claim severity are continuous, right-skewed, and frequently heavy-tailed Klugman et al. (2019). The data sets that such models are usually fitted to contain outliers that are difficult to identify and separate from genuine data. As a result, there could be a significant difference in statistical inference if there is a small perturbation in the assumed model from the unknown true underlying parametric model. In practice, due to commonly used loss control mechanism in the financial and insurance industries Ergashev et al. (2016), the random variables we observe and wish to model are affected by data truncation (due to deductibles), censoring (due to policy limits), and scaling (due to coinsurance factor). Maximum likelihood estimators (MLEs) typically result in sensitive loss severity models if there is a small perturbation in the underlying assumed model or if the observed sample is coming from a contaminated distribution Tukey (1960). The implementation of MLE procedures even on ground-up loss data is computationally challenging Frees (2017), Lee (2017). This issue is even more evident when one tries to fit complicated multi-parameter models such as mixtures of Erlangs (Reynkens et al., 2017, Verbelen et al., 2015). Thus, beside many ideas from the mainstream robust statistics literature (see, e.g., Hampel, 1974, Huber, 1964, Huber and Ronchetti, 2009), actuaries have to deal with heavy-tailed and skewed distributions, data truncation and censoring, identification and recycling of outliers, and aggregate loss, etc. Based on a general class of $L$–statistics (Chernoff et al., 1967), two board classes of robust estimator – the methods of trimmed moments (MTM) Brazauskas et al. (2009); and winsorized moments (MWM) Zhao et al. (2018) are recently developed with actuarial applications in view. Therefore, it is appealing to search some estimation procedures which directly work with those mentioned loss control mechanism and are insensitive.

If a truncated (both singly and doubly) normal sample data is available then the MLE procedures for such data have been developed by Cohen (1950) and the method of truncated moments estimators can be found in Cohen (1951) and Shah and Jaiswal (1966). But the goal and motivation of this research work is different and is initially purposed by this author in Poudyal (2018). That is,
instead of truncated sample data, we assume a complete ground-up sample loss data is available, i.e., the data set is neither truncated nor censored, and we propose and develop robust estimation procedures for the corresponding ground-up loss severity models. Instead of trimming or winsorizing a fixed lower (say, 2%) and upper proportion (say, 3%) of the observed sample data, in this paper we develop a novel fixed lower and upper thresholds method of truncated moments (say, MTuM) approach where the tail probabilities will be random. Depends on the nature of the loss data mentioned above, some variants of MTuM, called methods of fixed censored moment (MCM) and actuarial payment-type moment (MTCM) will be defined for single parameter Pareto distribution, see Figure 3.3. Asymptotic distributions, such as normality and consistency, along with asymptotic relative efficiency of those estimators with respect to the corresponding MLEs are established. Several theoretical connections between different approaches are also discovered. The newly designed procedures work like the standard method-of-moments but instead of classical moments they are truncated or censored moments for a completely observed sample. Irrespective with the heaviness of the underlying distribution, threshold truncated and censored moments are always finite.

The remainder of the paper is organized as follows. In Section 2, the newly proposed MTuM estimation procedure is defined in general with the establishment of the corresponding asymptotic distributional properties. In Section 3, we develop specific formulas of different estimators (including MTuM) when the underlying loss distribution is Pareto I which is equivalent to an exponential distribution, and compare the asymptotic relative efficiency of all the estimators with respect to the corresponding MLEs for completely observed data. Several connections among different estimators are established. Section 4 summarizes a detail simulation study of different estimators developed in this paper. Concluding remarks are offered in Section 5. Finally, some additional results are provided in Appendix A and B.

2 Method of Truncated Moments

We assume that a complete ground-up loss data is available, i.e., the data set is neither truncated nor censored. Then, instead of trimming or winsorizing fixed proportion from both tails, from a completely observed data, as investigated by Brazauskas et al. (2009), Zhao et al. (2018), in this approach of parametric estimation we truncate the data from below at lower threshold and from
above at upper threshold and then apply the method of moments on the remaining data. We call such an approach \textit{method-of-truncated-moments (MTuM – for short)}.

2.1 Definition

Let $X_1, X_2, ..., X_n$ be i.i.d. random variables with common ground-up cdf $F(\cdot|\theta)$, where $\theta := (\theta_1, \ldots, \theta_k)$, $k \geq 1$ is the parameter vector to be estimated. The truncated moments estimators of $\theta_1, \theta_2, ..., \theta_k$ are computed according to the following procedures.

(i) The sample truncated moments are computed as

$$
\hat{\mu}_j = \frac{\sum_{i=1}^{n} h_j(X_i) 1\{d_j < X_i \leq u_j\}}{\sum_{i=1}^{n} 1\{d_j < X_i \leq u_j\}}, \quad 1 \leq j \leq k, \tag{2.1}
$$

where $1\{\cdot\}$ denotes the indicator function. The $h'_j$s in (2.1) are specially chosen functions as well as the thresholds $d_j$ and $u_j$ are chosen by the researcher. In general, it is reasonable to assume that $X_{1:n} \leq d_j < u_j \leq X_{n:n}$, for all $1 \leq j \leq k$, where $X_{1:n}$ and $X_{n:n}$ are the smallest and the largest order statistics, respectively, from the sample.

(ii) Derive the corresponding population truncated moments as

$$
\mu_j(\theta_1, \theta_2, ..., \theta_k) = \mathbb{E}[h_j(X)|d_j < X \leq u_j] = \frac{\mathbb{E}[h_j(X)1\{d_j < X \leq u_j\}]}{\mathbb{P}(d_j < X \leq u_j)} = \frac{\int_{d_j}^{u_j} h_j(x) f(x|\theta) \, dx}{F(u_j|\theta) - F(d_j|\theta)}, \quad 1 \leq j \leq k. \tag{2.2}
$$

(iii) Now, match the sample and population truncated moments from (2.1) and (2.2) to get the following system of equations for $\theta_1, \theta_2, ..., \theta_k$:

$$
\begin{cases}
\mu_1(\theta_1, \ldots, \theta_k) = \hat{\mu}_1 \\
\vdots \\
\mu_k(\theta_1, \ldots, \theta_k) = \hat{\mu}_k
\end{cases} \tag{2.3}
$$

Definition 2.1. A solution to the system of equations (2.3), say $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k)$, if it exists, is called the method of truncated moments (MTuM) estimator of $\theta$. Thus, $\hat{\theta}_j = g_j(\hat{\mu}_1, \hat{\mu}_2, ..., \hat{\mu}_k), \quad 1 \leq j \leq k$ are the MTuM estimators of $\theta_1, \theta_2, ..., \theta_k$.

Note 2.1. Obviously, it is possible that the system of equations (2.3) does not have a solution, or it is difficult to solve the system even with numerical methods when $k$ is large. To facilitate this issue,
the functions $h_j$ have to be chosen carefully. But most claim severity distributions have a small number $k$ of parameters, usually not exceeding three (see Klugman et al., 2019, Appendix A).

2.2 Asymptotic Properties

For $1 \leq j, j' \leq k$ and for any positive integer $n$, define $1\{d_{jj'} < X \leq u_{jj'}\} := 1\{d_j < X \leq u_j\}1\{d_{jj'} < X \leq u_{jj'}\}$ and consider the following additional notations:

$$Z_j := h_j(X), \quad h_{jj'}(x) := h_j(x)h_{j'}(x), \quad p_j := F(u_j|\theta) - F(d_j|\theta),$$

$$Y_{jj'} := Y_jY_{j'}, \quad Y_j := Z_j1\{d_j < X \leq u_j\}, \quad p_{jj'} := F(u_{jj'}|\theta) - F(d_{jj'}|\theta),$$

$$r_j := h_j(d_j), \quad R_j := h_j(u_j), \quad W_{jj'} := Z_j1\{d_{jj'} < X \leq u_{jj'}\}, \quad p_{j,n} := F_n(u_j) - F_n(d_j),$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n}1\{X_i \leq x\}$ is the empirical distribution function. Note that $Y_{jj'} = Y_{j'j}$ but $W_{jj'} \neq W_{j'j}$ for $j \neq j'$, in general. With those notations, the density of $Y_j$, $(1 \leq j \leq k)$ can be expressed as

$$f_{Y_j}(y) = \begin{cases} 1 - F_{Z_j}(R_j|\theta) + F_{Z_j}(r_j|\theta), & \text{if } y = 0; \\ f_{Z_j}(y|\theta), & \text{if } r_j < y < R_j; \\ 0, & \text{otherwise.} \end{cases}$$

The density of the random variables $Y_{jj'} = Y_{j'j}$ and $W_{jj'}$ can be constructed with the four possible scenarios which are listed in Appendix A. To establish the asymptotic distribution of $\hat{\mu}$, we need the following lemma.

**Lemma 2.1.** For $1 \leq j, j' \leq k$,

$$\text{Cov}(Y_j, Y_{j'}) = \mu_{Y_{jj'}} - \mu_{Y_j}\mu_{Y_{j'}}, \quad \text{Cov}(Y_j; p_{j',1}) = \mu_{W_{jj'}} - \mu_{Y_j}p_{j'}, \quad \text{Cov}(p_{j,1}; p_{j',1}) = p_{jj'} - p_jp_{j'}.$$

Consider a $2k$-dimensional random vector $V := (Y_1, \ldots, Y_k, p_{1,1}, \ldots, p_{k,1})$. Clearly the mean vector of $V$ is $\mu_V = (\mu_{Y_1}, \ldots, \mu_{Y_k}, p_{1,1}, \ldots, p_{k,1})$ and with Lemma 2.1, the variance-covariance matrix is $\Sigma_V = \begin{bmatrix} \sigma_{V,jj'}^2 \end{bmatrix}_{j,j'=1}^{2k}$, where

$$\sigma_{V,jj'}^2 = \begin{cases} \mu_{Y_{jj'}} - \mu_{Y_j}\mu_{Y_{j'}}, & 1 \leq j, j' \leq k; \\ \mu_{W_{jj'(-k)}} - \mu_{Y_j}p_{j'-k}, & 1 \leq j \leq k; k + 1 \leq j' \leq 2k; \\ \mu_{W_{(j-k)j'}} - \mu_{Y_j}p_{j'-k}, & 1 \leq j' \leq k; k + 1 \leq j \leq 2k; \\ p_{(j-k)(j'-k)} - p_{j-k}p_{j'-k}, & k + 1 \leq j, j' \leq 2k. \end{cases}$$
Theorem 2.1. The empirical estimator

\[ \hat{\mu}_V := \frac{1}{n} \left( \sum_{i=1}^{n} Y_{1,i}, \ldots, \sum_{i=1}^{n} Y_{k,i}, \sum_{i=1}^{n} p_{1,i}, \ldots, \sum_{i=1}^{n} p_{k,i} \right) = (\bar{Y}_{1,n}, \ldots, \bar{Y}_{k,n}, \bar{p}_{1,n}, \ldots, \bar{p}_{k,n}) \]

of the mean vector \( \mu_V \) is such that \( \hat{\mu}_V \sim \mathcal{N} (\mu_V, \frac{1}{n} \Sigma_V) \).

Proof. Let \( \{V_n\} \) be a sequence of i.i.d. \( V \) random vectors, then by multivariate Central Limit Theorem (see, e.g., Serfling, 1980, Theorem B, p. 28), we have:

\[ (\bar{Y}_{1,n}, \ldots, \bar{Y}_{k,n}, \bar{p}_{1,n}, \ldots, \bar{p}_{k,n}) = \frac{1}{n} \sum_{i=1}^{n} V_i \sim \mathcal{N} \left( \mu_V, \frac{1}{n} \Sigma_V \right). \]

The system of MTuM equations (2.3) can now be written as:

\[
\begin{cases}
\mu_1(\theta_1, \ldots, \theta_k) = \hat{\mu}_1 = \frac{\bar{Y}_{1,n}}{p_{1,n}}, \\
\vdots \\
\mu_k(\theta_1, \ldots, \theta_k) = \hat{\mu}_k = \frac{\bar{Y}_{k,n}}{p_{k,n}}.
\end{cases}
\]

(2.4)

Lemma 2.2. Consider a function \( g_V : \mathbb{R}^{2k} \rightarrow \mathbb{R}^k \) for \( x = (x_1, x_2, \ldots, x_{2k}) \) defined by

\[ g_V(x) = (g_1(x), \ldots, g_k(x)) := \left( \frac{x_1}{x_{k+1}}, \ldots, \frac{x_k}{x_{2k}} \right), \]

where \( x_i \neq 0, i = k+1, \ldots, 2k \). Then \( g_V \) is totally differentiable at any point \( x_0 \in \mathbb{R}^{2k} \).

Proof. A proof directly follows from (Serfling, 1980, Lemma 1.12.2).

With the help of Theorem 2.1 and Lemma 2.2, we are now ready to state the asymptotic distribution of the truncated sample moment vector \( \hat{\mu} \) whose proof can be found in Appendix B.

Theorem 2.2. The asymptotic joint distribution of the truncated sample moment vector \( (\hat{\mu}_1, \ldots, \hat{\mu}_k) \) is given by \( N (\mu, \frac{1}{n} \Sigma) \) with \( \Sigma = D_V V_V D'_V =: \left[ \sigma_{jj'}^{2} \right]_{k \times k} \), where

\[
\sigma_{jj'}^{2} = \frac{1}{p_{j'}} \left( \frac{\mu_{y_{j'}} - \mu_{y_j} \mu_{y_{j'}}}{p_j} - \frac{\mu_{y_j} \left( \mu_{y_{j'}} - \mu_{y_j} p_j \right)}{p_j^2} \right) - \frac{\mu_{y_{j'}}}{p_{j'}} \left( \frac{\mu_{y_{j'}} - \mu_{y_j} p_{j'}}{p_j} - \frac{\mu_{y_j} \left( p_{j'} - p_j p_{j'} \right)}{p_j^2} \right).
\]

Now, with \( \hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_k) \) and \( g_0(\hat{\mu}) = (g_1(\hat{\mu}), \ldots, g_k(\hat{\mu})) = \hat{\theta} \), then by delta method (see, e.g., Serfling, 1980, Theorem A, p. 122), we have the following main result of this section.
Theorem 2.3. The MTuM estimator of $\theta$, denoted by $\widehat{\theta}$, has the following asymptotic distribution:

$$\widehat{\theta} = (\widehat{\theta}_1, \ldots, \widehat{\theta}_k) \sim \mathcal{N}(\theta, \frac{1}{n}D\Sigma D')$$

where the Jacobian $D$ is given by $D = \left[ \frac{\partial g_{j,\theta}}{\partial \hat{\mu}_{j'}} \right]_{k \times k} =: [d_{jj'}]_{k \times k}$ and the variance-covariance matrix $\Sigma$ has the same form as in Theorem 2.2.

Note 2.2. In view of the above derivations, we notice that data trimming and thus (method of trimmed moments – MTM) investigated by Brazauskas et al. (2009) can be interpreted as special cases of data truncation and thus MTuM, respectively. To see that, let $F$ be the distribution function of $X$. For $1 \leq j \leq k$, consider $F(d_j|\theta) = a_j$ and $F(u_j|\theta) = 1 - b_j$. Then, using integration by substitution with $U = F(X)$, the equation (2.2) becomes

$$\mu_j(\theta_1, \theta_2, ..., \theta_k) = \int_{d_j}^{u_j} h_j(x)f(x|\theta) \, dx \frac{F(u_j|\theta) - F(d_j|\theta)}{F(u_j|\theta) - F(u_j|\theta)} = \int a_j h_j(F^{-1}(u_j|\theta)) \, du \frac{1 - b_j}{1 - a_j - b_j},$$

which is equivalent to the corresponding population trimmed moment.

Note 2.3. For estimation purposes these two approaches (i.e., MTM and MTuM) are very different. With the MTuM approach, the limits of integration as well as the denominator in equation (2.5a) are unknowns, which create technical complications when we want to assess the asymptotic properties of MTuM estimators. On the other hand, with the MTM approach, both the limits of integration and the denominator in equation (2.5b) are constants, which simplify the matters significantly. Indeed, as is evident from complete data examples in Brazauskas et al. (2009) and Zhao et al. (2018), MTM leads to explicit formulas for all location-scale families and their variants, but that is not the case with MTuM. In view of this, we will consider the MTuM approach further only for some data scenarios, but not all.

3 Pareto I Distribution

Let $Y \sim \text{Pareto}(\alpha, x_0)$ with the distribution function $F_Y(y) = 1 - (x_0/y)^\alpha$, $y > x_0$, zero elsewhere, where $\alpha > 0$ is the shape (so called tail) parameter and $x_0 > 0$ is known left threshold. Then
\(X := \log(Y/x_0) \sim \text{Exp}(\theta = 1/\alpha)\) with the distribution function \(F_X(x) = 1 - e^{-x/\theta}\). Therefore, in order to estimate \(\alpha\) it is equivalent to estimate the exponential parameter \(\theta\) that what we will proceed for the rest of this section. MTuM will be derived with asymptotic results for a complete \(i.i.d.\) sample from an exponential distribution. For this particular distribution, we also explore two additional methods: method of censored moments and insurance payment\textendash type moment estimators. Several connections between different approaches are established.

The asymptotic performance of the newly designed estimators will be measured via asymptotic relative efficiency (ARE) with respect to MLE and is defined as (see, e.g., Serfling, 1980, van der Vaart, 1998):

\[
\text{ARE}(\mathcal{C}, \text{MLE}) = \frac{\text{asymptotic variance of MLE estimator}}{\text{asymptotic variance of \(\mathcal{C}\) estimator}}. \tag{3.1}
\]

The main reason why MLE should be used as a benchmark is its optimal asymptotic performance in terms of variability (of course, with the usual caveat of "under certain regularity conditions").

### 3.1 Method of Truncated Moments

In this section, we derive MTuM and related estimators for the parameter of exponential distribution for completely observed data. Since there is a single parameter, \(\theta\), to be estimated, we consider the function \(h(x) = x\). Let \(X_1, \ldots, X_n\) be \(i.i.d.\) random variables given as in Definition 2.1. Consider \(d\) and \(u\) be the left and right truncation points, respectively. Then the sample truncated moment is given by

\[
\hat{\mu}_{\text{MTuM}} := \frac{\left(\sum_{i=1}^{n} X_i \mathbb{1}\{d < X_i \leq u\}\right)/n}{\left(\sum_{i=1}^{n} \mathbb{1}\{d < X_i \leq u\}\right)/n} = \frac{\sum_{i=1}^{n} Y_i / n}{F_n(u) - F_n(d)} = \frac{\sum_{i=1}^{n} Y_i / n}{F_n(u) - F_n(d)} = \frac{\sum_{i=1}^{n} Y_i / n}{p_n}.
\]

where \(Y_1, Y_2, \ldots, Y_n \overset{i.i.d.}{\sim} Y := X \mathbb{1}\{d < X \leq u\}\) and \(p_n := F_n(u) - F_n(d)\) with \(p \equiv p(\theta) = F(u|\theta) - F(d|\theta) = e^{-u/\theta} - e^{-d/\theta}\).

**Theorem 3.1.** The mean and the variance of the random variable \(Y\) are respectively given by

\[
\mu_Y = \theta p + de^{-d/\theta} - ue^{-u/\theta} \quad \text{and} \quad \sigma_Y^2 = 2\theta^2 \left(\Gamma\left(3; \frac{u}{\theta}\right) - \Gamma\left(3; \frac{d}{\theta}\right)\right) = \mu_Y^2,
\]

where \(\Gamma(\alpha; x)\) with \(\alpha > 0, x > 0\) is the incomplete gamma function defined as

\[
\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} dt \quad \text{with} \quad \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.
\]
Proof. See Appendix B.

From Theorem 2.2, $\tilde{\mu}_{\text{MTU}} \sim \mathcal{AN}\left(\frac{\nu_Y}{p} \cdot \frac{1}{n} \left(\frac{\sigma^2}{p^2} - \frac{(1-p)\mu^2}{p^2}\right)\right)$. Note that the asymptotic variance of $\tilde{\mu}_{\text{MTU}}$ is exactly equal to the approximation through the second order Taylor series expansion of the ratio of the asymptotic distribution of $Y_n$ and $p_n$ as mentioned in Hayya et al. (1975). The corresponding population version of $\tilde{\mu}_{\text{MTU}}$ is given by

$$\mu_{\text{MTU}} := \mathbb{E}[X|d < X \leq u] = \frac{\mathbb{E}[Y]}{F(u|\theta) - F(d|\theta)} = \frac{\nu_Y}{p}. \quad (3.2)$$

**Theorem 3.2.** The equation $\mu_{\text{MTU}} = \tilde{\mu}_{\text{MTU}}$ has a unique solution $\tilde{\theta}$ provided that $d < \tilde{\mu}_{\text{MTU}} < \frac{d + u}{2}$. Otherwise, the solution does not exist.

**Proof.** It is clear that $d < \tilde{\mu}_{\text{MTU}} < u$. Also, $\mu_{\text{MTU}}(\theta) = \frac{\nu_Y}{p} = \frac{e^{\frac{d}{\tilde{\theta}}} - e^{-\frac{d}{\tilde{\theta}}}}{e^{\frac{u}{\tilde{\theta}}} - e^{-\frac{u}{\tilde{\theta}}}}$. Then, in order to establish the result, it is enough to prove the following statements:

(a) $\mu_{\text{MTU}}(\theta)$ is strictly increasing, \(b\) $\lim_{\theta \to 0^+} \mu_{\text{MTU}}(\theta) = d$, and \(c\) $\lim_{\theta \to \infty} \mu_{\text{MTU}}(\theta) = \frac{d + u}{2}$.

First of all, let us establish that $\mu_{\text{MTU}}(\theta)$ is strictly increasing.

$$\mu'_{\text{MTU}}(\theta) = \frac{\left(\frac{d}{\tilde{\theta}}\right)^2 e^{-\frac{d}{\tilde{\theta}}} - \left(\frac{u}{\tilde{\theta}}\right)^2 e^{-\frac{u}{\tilde{\theta}}}}{\left(e^{-\frac{d}{\tilde{\theta}}} - e^{-\frac{u}{\tilde{\theta}}}\right)^2} + \left(e^{-\frac{d}{\tilde{\theta}}} - e^{-\frac{u}{\tilde{\theta}}}\right)^2$$

$$= -e^{-\frac{d+u}{\tilde{\theta}}} \left(\frac{d}{\tilde{\theta}}\right)^2 \left(\frac{u}{\tilde{\theta}}\right)^2 + \frac{2d+u}{\tilde{\theta}} e^{-\frac{d+u}{\tilde{\theta}}} + \left(e^{-\frac{d}{\tilde{\theta}}} - e^{-\frac{u}{\tilde{\theta}}}\right)^2$$

$$= 1 - \frac{4(d-u)^2}{4\theta^2 \left(e^{\frac{d-u}{2\theta}} - e^{-\frac{d-u}{2\theta}}\right)^2} = 1 - \left(\frac{d-u}{2\theta}\right)^2 \left(\frac{2}{e^{\frac{d-u}{2\theta}} - e^{-\frac{d-u}{2\theta}}}\right)^2$$

$$= 1 - \left(\frac{d-u}{2\theta}\right)^2 \text{csch}^2 \left(\frac{d-u}{2\theta}\right).$$

Therefore, $\mu'_{\text{MTU}}(\theta) > 0$ if and only if $\left(\frac{d-u}{2\theta}\right)^2 < \sinh^2 \left(\frac{d-u}{2\theta}\right)$, which is true since $x < \sinh x$ for all $x > 0$ and $x > \sinh x$ for all $x < 0$. Further,

$$\lim_{\theta \to 0^+} \mu_{\text{MTU}}(\theta) = \lim_{\theta \to 0^+} \left[\theta + \frac{de^{-\frac{d}{\theta}} - ue^{-\frac{u}{\theta}}}{e^{-\frac{d}{\theta}} - e^{-\frac{u}{\theta}}}\right] = \lim_{\theta \to 0^+} \frac{e^{-\frac{d}{\tilde{\theta}}} \left(\tilde{\theta} - u e^{-\frac{u}{\tilde{\theta}}}\right)}{1 - e^{-\frac{d}{\tilde{\theta}}}} = d,$$

$$\lim_{\theta \to \infty} \mu_{\text{MTU}}(\theta) = \lim_{\theta \to \infty} \left[\theta + \frac{de^{-\frac{d}{\theta}} - ue^{-\frac{u}{\theta}}}{e^{-\frac{d}{\theta}} - e^{-\frac{u}{\theta}}}\right]_{\tilde{\theta} = \frac{1}{y}} = \lim_{y \to 0^+} \left[\frac{1}{y} + \frac{de^{-dy} - ue^{-uy}}{e^{-dy} - e^{-uy}}\right].$$
 \[
\begin{align*}
\text{Proposition 3.1.} & \quad \text{result.} \\
\text{Proof.} & \quad \text{A proof immediately follows from (3.2), if we fix the left, } d, \text{ and right, } u, \text{ truncation thresholds and allow the tail probabilities, i.e., } F(d \mid \theta) \text{ and } 1 - F(u \mid \theta), \text{ be random then the corresponding asymptotic relative efficiency is not stable (see Figure 3.2). However, as in method of trimmed moments (MTM) (see, e.g., Brazauskas et al., 2009, Zhao et al., 2018) if the tail probabilities } F(d \mid \theta) \text{ and } 1 - F(u \mid \theta) \text{ are fixed then we have the following result.}
\end{align*}
\]

\[
\begin{align*}
\text{Proposition 3.1.} & \quad \text{Let } \theta_1 \neq \theta_2 \text{ be two exponential parameters with corresponding left and right truncation thresholds } d_1, d_2 \text{ and } u_1, u_2, \text{ respectively. Assume } F(d_1 \mid \theta_1) = F(d_2 \mid \theta_2) \text{ and } F(u_1 \mid \theta_1) = F(u_2 \mid \theta_2), \text{ then it follows that}
\end{align*}
\]

\[
\begin{align*}
\text{ARE} \left( \hat{\theta}_{1, \text{MTM}}, \hat{\theta}_{1, \text{MLE}} \right) = \text{ARE} \left( \hat{\theta}_{2, \text{MTM}}, \hat{\theta}_{2, \text{MLE}} \right). 
\end{align*}
\]
Numerical values of $\text{ARE} \left( \hat{\theta}_{\text{MTM}}, \hat{\theta}_{\text{MLE}} \right)$ given by (3.6) for some selected values of left and right truncation thresholds $d$ and $u$, respectively are summarized on the first horizontal block of Table 3.1.

As mentioned above, if $Y \sim \text{Pareto I} \left( \alpha, x_0 \right)$ with $x_0$ known then $X := \log \left( \frac{Y}{x_0} \right) \sim \text{Exp} \left( \frac{1}{\alpha} =: \theta \right)$. So, estimators of $\alpha$ of the single-parameter Pareto distribution will share the same AREs with estimators of $\text{Exp} (\theta)$, given that $h(y) = \log \left( \frac{y}{x_0} \right)$. The following result for single-parameter Pareto has been partially derived in Clark (2013), but can easily be extended using the tools of this section.

**Theorem 3.3.** Let $d$ and $u$ be the left and right truncation points, respectively, for $Y \sim \text{Pareto I} \left( \alpha, x_0 \right)$. Also, define $A_{du} := u^\alpha \left( 1 - \alpha \log \left( \frac{x_0}{d} \right) \right) - d^\alpha \left( 1 - \alpha \log \left( \frac{x_0}{u} \right) \right)$ and $g_{du}(\alpha) := \frac{A_{du}}{\alpha u - d u^\alpha}$. Then the equation $\hat{\mu}_{\text{MTM}} = \mu_{\text{MTM}}$ has a unique solution provided that $\lim_{\alpha \to \infty} g_{du}(\alpha) < \hat{\mu}_{\text{MTM}} < \lim_{\alpha \to 0^+} g_{du}(\alpha)$.

**Proof.** See Appendix B.

Note that, given a truncated data, method of truncated moments estimators for a normal population parameters can be found in Cohen (1951) and Shah and Jaiswal (1966).

### 3.2 Method of Fixed Censored Moments

There are several versions of data censoring that occur in statistical modeling: interval censoring (it includes left and right censoring depending on which end point of the interval is infinite), type I censoring, type II censoring, and random censoring. For actuarial work, the most relevant type is **interval censoring**. It occurs when complete sample observations are available within some interval, say $(d, u]$, but data outside the interval is only partially known. That is, counts are available but actual values are not. That is, we observe the *i.i.d.* data

\[ Z_1, Z_2, \ldots, Z_n, \quad (3.8) \]

where each $Z$ is equal to the ground-up variable $X$, if $X$ falls between $d$ and $u$, and is equal to the corresponding end-point of the interval if $X$ is beyond that point. That is, $Z$ is given by

\[
Z := \min \left\{ \max(d, X), u \right\} = d \mathbb{I} \{ X \leq d \} + X \mathbb{I} \{ d < X \leq u \} + u \mathbb{I} \{ X > u \} = \begin{cases} 
  d, & X \leq d; \\
  X, & d < X \leq u; \\
  u, & X > u.
\end{cases}
\]
Therefore, instead of winsorizing fixed proportions of lowest and highest order statistics from an observed sample (Zhao et al., 2018), here we design a method of fixed threshold censored moment for exponential distribution.

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Exp}(\theta)$ random variables. Then, the sample censored mean is given by

$$\hat{\mu}_{\text{MCM}} := \frac{d \sum_{i=1}^{n} \mathbb{1}\{X_i \leq d\} + \sum_{i=1}^{n} X_i \mathbb{1}\{d < X_i \leq u\} + u \sum_{i=1}^{n} \mathbb{1}\{X_i > u\}}{n}.$$  

The corresponding population censored moments are:

$$\mu_{\text{MCM}} := \mathbb{E}[Z] = d \left(1 - e^{-\frac{d}{\theta}}\right) + \mu_Y + u e^{-\frac{u}{\theta}},$$

$$\mu_{\text{MCM},2} := \mathbb{E}[Z^2] = d^2 \left(1 - e^{-\frac{d}{\theta}}\right) + \mathbb{E}[Y^2] + u^2 e^{-\frac{u}{\theta}},$$

where $Y := X \mathbb{1}\{d < X \leq u\}$ as in Section 3.1. Thus, $\sigma^2_{\text{MCM}} = \mu_{\text{MCM},2} - \mu^2_{\text{MCM}}$. Moreover, setting $\mu_{\text{MCM}} = \hat{\mu}_{\text{MCM}}$ implies $d + \theta \left(e^{-\frac{d}{\theta}} - e^{-\frac{u}{\theta}}\right) = \hat{\mu}_{\text{MCM}}$, which needs to be solved to get a method of censored moment (MCM) estimator, $\hat{\theta}_{\text{MCM}}$, of $\theta$.

**Theorem 3.4.** The equation $\hat{\mu}_{\text{MCM}} = \mu_{\text{MCM}}$ has a unique solution $\hat{\theta}_{\text{MCM}}$ provided that $d < \hat{\mu}_{\text{MCM}} < u$. Otherwise, the solution does not exist.

**Proof.** A proof can similarly be established as in Theorem 3.2. \hfill \Box

Moreover, $\theta' := \frac{d\theta}{d\hat{\mu}_{\text{MCM}}} = \frac{\theta}{\rho \theta + d e^{-\frac{d}{\theta}} - u e^{-\frac{u}{\theta}}}$. Then

$$\hat{\theta}_{\text{MCM}} \sim \mathcal{N}\left(\hat{\theta}, \frac{1}{n} \left(\frac{\theta^2 \sigma^2_{\text{MCM}}}{(\rho \theta + d e^{-\frac{d}{\theta}} - u e^{-\frac{u}{\theta}})^2}\right)\right), \tag{3.9}$$

and hence

$$\text{ARE}\left(\hat{\theta}_{\text{MCM}}, \hat{\theta}_{\text{MLE}}\right) = \left(\frac{\rho \theta + d e^{-\frac{d}{\theta}} - u e^{-\frac{u}{\theta}}}{\sigma^2_{\text{MCM}}}\right)^2. \tag{3.10}$$

Again, as in MTuM case, see (3.6), $\text{ARE}\left(\hat{\theta}_{\text{MCM}}, \hat{\theta}_{\text{MLE}}\right)$ given by (3.10) is a function of $\theta$ and hence is not stable with respect to $\theta$. But if we fix the tail probabilities $F(d \mid \theta)$ and $1 - F(u \mid \theta)$, then we have the main result of this section.

**Theorem 3.5.** For method of censored moments (MCM) with $a = F(d \mid \theta)$ and $b = 1 - F(u \mid \theta)$, then the following result holds:

$$\text{ARE}\left(\hat{\theta}_{\text{MCM}}, \hat{\theta}_{\text{MLE}}\right) = \text{ARE}\left(\hat{\theta}_{\text{MTM}}, \hat{\theta}_{\text{MLE}}\right).$$
Proof. Following Brazauskas et al. (2009), we know that

\[ \hat{\theta}_{MTM} \sim AN \left( \theta, \frac{\theta^2}{n} \Delta \right), \quad \text{with} \quad \Delta = \frac{J(a, 1 - b)}{[I(a, 1 - b)]^2}, \]

where

\[ I(a, 1 - b) := \int_a^{1-b} \log (1 - v) dv \quad \text{and} \quad J(a, 1 - b) := \int_a^{1-b} \int_a^{1-b} \min\{v, w\} - vw \frac{1}{(1 - v)(1 - w)} dw dv. \]

Therefore,

\[ \text{ARE} \left( \hat{\theta}_{MTM}, \hat{\theta}_{MLE} \right) = \frac{1}{\Delta} = \frac{[I(a, 1 - b)]^2}{J(a, 1 - b)}. \]

On the other hand,

\[ \text{ARE} \left( \hat{\theta}_{MCM}, \hat{\theta}_{MLE} \right) = \frac{\left( p\theta + de^{-\frac{d}{\theta}} - ue^{-\frac{u}{\theta}} \right)^2}{\sigma^2_{MCM}}. \]

So, we need to show that,

\[ \text{ARE} \left( \hat{\theta}_{MCM}, \hat{\theta}_{MLE} \right) = \text{ARE} \left( \hat{\theta}_{MTM}, \hat{\theta}_{MLE} \right) = \frac{\theta^2[I(a, 1 - b)]^2}{\sigma^2_{MCM}}. \]

That is, \( J(a, 1 - b) = \frac{\sigma^2_{MCM}}{\theta^2} \). For that, we have:

\[ \sigma^2_{MCM} = d^2 \left( 1 - e^{-\frac{d}{\theta}} \right) + 2\theta^2 \left[ \Gamma \left( 3; d \frac{\theta}{\theta} \right) - \Gamma \left( 3; \frac{u}{\theta} \right) \right] + u^2 e^{-\frac{u}{\theta}} - d^2 - 2d\theta p - \theta^2 p^2 \]

\[ = 2\theta^2(1 - a - b) + 2\theta^2(- (1 - a) \log (1 - a) + b \log (b)) \]

\[ - \theta^2(1 - a - b)(-2 \log (1 - a) + (1 - a - b)). \]

\[ \frac{\sigma^2_{MCM}}{\theta^2} = 2(1 - a - b) + 2(b \log (b) - (1 - a) \log (1 - a)) \]

\[ - (1 - a - b)(1 - a - b - 2 \log (1 - a)) \]

\[ = (1 - a - b)[a + \log (1 - a)] - I(a, 1 - b) \]

\[ + (1 - b - 1) \left[ a - 1 + b + \log \left( \frac{1 - a}{b} \right) \right] \]

\[ = (1 - a - b)[a + \log (1 - a)] - I(a, 1 - b) + (1 - b - 1)I_1(a, 1 - b) \]

\[ = J(a, 1 - b). \]

where \( I_1(a, 1 - b) := \int_a^{1-b} \frac{v}{1 - v} dv = (a - 1 + b) + \log \left( \frac{1 - a}{b} \right). \]

Here is an important and a new connection between trimmed and interval censored population means for any \( F \in \mathcal{F} \), where \( \mathcal{F} \) is the family of continuous parametric distributions.
Theorem 3.6. Let $F \in \mathcal{F}$ be an arbitrary continuous ground-up cumulative distribution function (cdf). Consider $d$ and $u$ be the lower and upper thresholds, respectively. Define $a := F(d)$ and $b := 1 - F(u)$. Let

$$\mu_{\text{MCM}} = dF(d) + \int_d^u z f(z) \, dz + u(1 - F(u)) \quad \text{and} \quad \mu_{\text{MTM}} = \frac{1}{1 - a - b} \int_a^{1-b} F^{-1}(v) \, dv,$$

are, respectively, the fixed censored mean and proportion trimmed mean of the same cdf $F$. Then

$$IF(\mu_{\text{MCM}}, x) = (1 - a - b) IF(\mu_{\text{MTM}}, x), \quad -\infty < x < \infty$$

where $IF$ stands for influence function.

Proof. The influence function of the trimmed mean is given by (see, e.g., Hampel, 1974, Huber and Ronchetti, 2009):

$$IF(\mu_{\text{MTM}}, x) = \frac{1}{1 - a - b} \int_a^{1-b} \left( \frac{d}{d\lambda} F^{-1}(v) \right) \bigg|_{\lambda=0} \, dv = \frac{1}{1 - a - b} \int_a^{1-b} v - 1 \{ F(x) \leq v \} \frac{f(F^{-1}(v))}{f(F^{-1}(v))} \, dv,$$

(3.11)

where $F_\lambda := (1 - \lambda)F + \lambda \delta_x$ and $\delta_x$ is the point mass at $x$. Since $d$ and $u$ are left and right censored points, respectively. Then, the censored mean is:

$$\mu_{\text{MCM}}[F] = \int_d^u z dF_Z(z) = dF(d) + \mathbb{E}[X 1\{ d < X < u \}] + u(1 - F(u))$$

$$= dF(d) + \int_d^u z dF(z) + u(1 - F(u)),$$

where $F_Z$ is the distribution function of $Z$ given by:

$$F_Z(z \mid d, u) = \mathbb{P} \left[ \min \left\{ \max(d, X), u \right\} \leq z \right] = \begin{cases} 0, & z < d; \\ F(z), & d \leq z < u; \\ 1, & z \geq u, \end{cases}$$

(3.12)

Further, $\mu_{\text{MCM}}[F_\lambda] = dF_\lambda(d) + \int_d^u z dF_\lambda(z) + u(1 - F_\lambda(u))$. Note that the influence function is just a special case of first order Gâteaux derivative (see, e.g., Hampel et al., 1986, Section 2.3). Thus, a simpler computational formula to get the IF is (see, e.g., Serfling, 1980, Chapter 6):

$$IF(\mu_{\text{MCM}}, x) = \frac{d}{d\lambda} \mu_{\text{MCM}}[F_\lambda] \bigg|_{\lambda=0}.$$

It is clear that $\frac{dF_\lambda(d)}{d\lambda} \bigg|_{\lambda=0} = -F(d) + \delta_x(d)$ and similarly $\frac{d(1-F_\lambda(u))}{d\lambda} \bigg|_{\lambda=0} = F(u) - \delta_x(u)$. Also, by using Leibniz’s rule for differentiation under integral sign, we get

$$\frac{d}{d\lambda} \int_d^u z dF_\lambda(z) = \frac{d}{d\lambda} \int_{F_\lambda(d)}^{F_\lambda(u)} F^{-1}(v) \, dv$$
\[
F^{-1}(F(\lambda(u))) \frac{d}{d\lambda} F(\lambda(u)) - F^{-1}(F(\lambda(d))) \frac{d}{d\lambda} F(\lambda(d)) + \int_{F(\lambda(d))}^{F(\lambda(u))} \frac{d}{d\lambda} F^{-1}(v) dv
\]
\[
= u \frac{d}{d\lambda} F(\lambda(u)) - d \frac{d}{d\lambda} F(\lambda(d)) + \int_{F(\lambda(d))}^{F(\lambda(u))} \frac{d}{d\lambda} F^{-1}(v) dv.
\]

\[
\frac{d}{d\lambda} \int_{d}^{u} z dF(z) \bigg|_{\lambda=0} = u \frac{d F(\lambda(u))}{d\lambda} \bigg|_{\lambda=0} - d \frac{d F(\lambda(d))}{d\lambda} \bigg|_{\lambda=0} + \int_{F(\lambda(d))}^{F(\lambda(u))} \frac{d}{d\lambda} F^{-1}(v) dv \bigg|_{\lambda=0}
\]
\[
= u \frac{d F(\lambda(u))}{d\lambda} \bigg|_{\lambda=0} - d \frac{d F(\lambda(d))}{d\lambda} \bigg|_{\lambda=0} + \int_{a}^{1-b} \left( \frac{d}{d\lambda} F^{-1}(v) \right) \bigg|_{\lambda=0} dv.
\]

Therefore,

\[
IF(\mu_{\text{MCM}}, x) = \frac{d\mu_{\text{MCM}}[F(\lambda)]}{d\lambda} \bigg|_{\lambda=0} = d \frac{d F(\lambda)}{d\lambda} \bigg|_{\lambda=0} + u \frac{d (1 - F(\lambda(u)))}{d\lambda} \bigg|_{\lambda=0} + \frac{d}{d\lambda} \int_{d}^{u} y dF(y) \bigg|_{\lambda=0}
\]
\[
= d \frac{d F(\lambda)}{d\lambda} \bigg|_{\lambda=0} + u \frac{d (1 - F(\lambda(u)))}{d\lambda} \bigg|_{\lambda=0} + \frac{d}{d\lambda} \int_{a}^{1-b} \left( \frac{d}{d\lambda} F^{-1}(v) \right) \bigg|_{\lambda=0} dv
\]
\[
\int_{a}^{1-b} \left( \frac{d}{d\lambda} F^{-1}(v) \right) \bigg|_{\lambda=0} dv. \quad (3.13)
\]

Thus, from Equations (3.13) and (3.11), \( IF(\mu_{\text{MCM}}, x) = (1 - a - b)IF(\mu_{\text{MTM}}, x). \)

\[\square\]

**Figure 3.1:** Influence functions of trimmed mean (left panel) and censored mean (right panel).

The following two points are immediate consequence of Theorem 3.6.

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Censored mean is asymptotically more stable than trimmed mean. This point is more clear from the Figure 3.1 as the graph of $IF(\mu_{MCM}, x)$ is just the vertical contraction of the graph of $IF(\mu_{MTM}, x)$ by the contracting factor $0 < 1 - a - b < 1$ with the assumption $0 < a + b < 1$.

The asymptotic investigation of censored mean could be quite challenging due to thresholds. So, in this situation one can assess the asymptotic distributional properties of censored mean through the corresponding properties of trimmed mean as the asymptotic variance of an estimator is the expectation of the square of the corresponding IF (see, e.g., Hampel, 1974, Maronna et al., 2019).

### 3.3 Method of Actuarial Payment–type Moments

Insurance contracts have coverage modifications that need to be taken into account when modeling the underlying loss variable. Usually the coverage modifications such as deductibles, policy limits, and coinsurance are introduced as loss control strategies so that unfavorable policyholder behavioral effects (e.g., adverse selection) can be minimized. Therefore, the actuarial loss data are left truncated and right censored in nature. Motivated with this nature of the loss data, here we design an estimation approach, called insurance payment–type estimators and is basically left truncated and right censored method of moments.

Let $X_1, \ldots, X_n$ be i.i.d. random variables with common exponential cdf $F(\cdot|\theta)$. Define the left-truncated (at $d$) and right-censored (at $u$) sample moment as:

$$
\hat{\mu}_{MTM} := \frac{\sum_{i=1}^n X_i \mathbb{1}\{d < X_i \leq u\} + u \sum_{i=1}^n \mathbb{1}\{X_i > u\}}{\sum_{i=1}^n \mathbb{1}\{X_i > d\}} \approx \frac{\text{W}_n}{\tau_n},
$$

where $W := X \mathbb{1}\{d < X \leq u\} + u \mathbb{1}\{X > u\}$, $\tau_n = 1 - F_n(d)$, and $\tau = 1 - F(d|\theta)$. The covariance of $W$ and $\tau_1$ is given as $\sigma^2_{W, \tau_1} = \text{Cov}(W, \tau_1) = \mu_W(1 - \tau)$, with

$$
\mu_W = \mathbb{E}[W] = \mu_Y + u(1 - F(u|\theta)) \quad \text{and} \quad \mathbb{E}[W^2] = \mathbb{E}[Y^2] + u^2(1 - F(u|\theta)),
$$

where $Y := X \mathbb{1}\{d < X \leq u\}$ as in Section 3.1. Then, by multivariate Central Limit Theorem, we have

$$
(W_n, \tau_n) \sim \mathcal{N} \left( (\mu_W, \tau), \frac{1}{n} \left[ \begin{array}{cc} \sigma^2_W & \sigma^2_{W, \tau_1} \\ \sigma^2_{W, \tau_1} & \tau(1 - \tau) \end{array} \right] \right).
$$
Then, by delta method with a function $g(x_1, x_2) := \frac{x_1}{x_2}$, $x_2 \neq 0$, we have

$$\hat{\mu}_{\text{MTCM}} = \frac{W_n}{\tau_n} \sim \mathcal{N}\left(\frac{\mu_W}{\tau}, \frac{1}{n} \left(\frac{\sigma^2_W}{\tau^2} - \frac{\mu^2_W (1 - \tau)}{\tau^3}\right)\right).$$

The population version of $\hat{\mu}_{\text{MTCM}}$ is given by

$$\mu_{\text{MTCM}} = \mathbb{E}[W] = \theta \left(\frac{e^{-\frac{\theta}{\tau}} - e^{-\frac{u}{\tau}}}{e^{-\frac{u}{\tau}}}\right) + \frac{de^{-\frac{\theta}{\tau}}}{\tau} = \frac{p\theta + d\tau}{\tau},$$

$$\Rightarrow \theta' := \frac{d\theta}{d\mu_{\text{MTCM}}} = \frac{\tau \theta^2}{p\theta^2 + \theta \left(\frac{de^{-\frac{\theta}{\tau}} - ue^{-\frac{u}{\tau}}}{\tau}\right)} + \frac{d^2\tau - d\tau \mu_{\text{MTCM}}}{p\theta - e^{-\frac{u}{\tau}}(u - d)} = \frac{\tau \theta^2}{p\theta^2 - e^{-\frac{u}{\tau}}(u - d)}.$$

A solution, if exists, of the equation $\hat{\mu}_{\text{MTCM}} = \mu_{\text{MTCM}}$, say $\hat{\theta}_{\text{MTCM}}$, is called the method of truncated and censored moment (MTCM) estimator of $\theta$. Let $b := e^{-\frac{u}{\tau}}$, then by delta method the asymptotic distribution and ARE are, respectively, given by

$$\hat{\theta}_{\text{MTCM}} \sim \mathcal{N}\left(\theta, \frac{(\theta')^2}{n} \left(\frac{\sigma^2_W}{\tau^2} - \frac{(1 - \tau)\mu^2_W}{\tau^3}\right)\right) = \mathcal{N}\left(\theta, \frac{\theta^2}{n} \left(\frac{p\theta^2 (\tau + b) - 2\theta b(u - d)}{\tau \left[p\theta - b \left(\frac{u - d}{\tau}\right)^2\right]^2}\right)\right)$$

(3.14)

**Figure 3.2:** Graphs of $\text{ARE}(\hat{\theta}_C, \hat{\theta}_{\text{MLE}})$ where $C \in \{\text{MTuM}, \text{MCM}, \text{MTCM}\}$ with $(d, u) = (0.50, 23.00)$ and $\theta = 10$. 

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\[
\text{ARE}\left(\hat{\theta}_{\text{MTCM}}, \hat{\theta}_{\text{MLE}}\right) = \frac{\theta^2 \tau^3}{(\theta')^2 (\tau \sigma_W^2 - (1 - \tau) \mu_W^2)} = \frac{[p - b \left(\frac{u - d}{\theta}\right)]^2}{p(1 + b/\tau) - 2b \left(\frac{u - d}{\theta}\right)}.
\] (3.15)

Similar to (3.6) and (3.10), \(\text{ARE}\left(\hat{\theta}_{\text{MTCM}}, \hat{\theta}_{\text{MLE}}\right)\) given by (3.15) is a function of \(\theta\). But if we fix the tail probabilities then we have the following stability result.

**Proposition 3.2.** Let \(\theta_1 \neq \theta_2\) be two exponential parameters with corresponding left and right truncation thresholds \(d_1, d_2\) and \(u_1, u_2\), respectively. Assume \(F(d_1 \mid \theta_1) = F(d_2 \mid \theta_2)\) and \(F(u_1 \mid \theta_1) = F(u_2 \mid \theta_2)\), then it follows that

\[
\text{ARE}\left(\hat{\theta}_{1,\text{MTCM}}, \hat{\theta}_{1,\text{MLE}}\right) = \text{ARE}\left(\hat{\theta}_{2,\text{MTCM}}, \hat{\theta}_{2,\text{MLE}}\right).
\] (3.16)

**Proof.** With the assumptions given, we have

\[
e^{-\frac{d_1}{\theta_1}} = e^{-\frac{d_2}{\theta_2}}, \quad e^{-\frac{u_1}{\theta_1}} = e^{-\frac{u_2}{\theta_2}} \implies \frac{u_1 - d_1}{\theta_1} = \frac{u_2 - d_2}{\theta_2},
\]

and then the conclusion follows directly from (3.15).

---

**Figure 3.3:** Effects of MTuM (left panel), MCM (middle panel), and MTCM (right panel) on the underlying quantile function and thus data. MTuM focuses on the data only between the truncation thresholds, MCM is a threshold censored form that takes into account the upper and lower outside values as well (orange area), and MTCM is a mixed version of both MTuM (left truncated) and MCM (right censored).
Table 3.1: Numerical values of $\text{ARE}\left(\hat{\theta}_C, \hat{\theta}_{\text{MLE}}\right)$ where $C \in \{\text{MTuM}, \text{MCM}, \text{MTCM}\}$, respectively, given by (3.6), (3.10), and (3.15) for various values of left and right truncation thresholds $d$ and $u$ from $\text{Exp}(\theta = 10)$. The truncation thresholds $d$ and $u$ are rounded to two decimal places; for example, $0.51 \approx F^{-1}(0.05)$, $18.97 \approx F^{-1}(0.85)$, etc.

| $d$ | $u$ | ARE ($\hat{\theta}_{\text{MTuM}}, \hat{\theta}_{\text{MLE}}$) | ARE ($\hat{\theta}_{\text{MCM}}, \hat{\theta}_{\text{MLE}}$) | ARE ($\hat{\theta}_{\text{MTCM}}, \hat{\theta}_{\text{MLE}}$) |
|-----|-----|---------------------------------|---------------------------------|---------------------------------|
| 0.00 | 0.00 | 1 | 0.478 | 0.311 | 0.215 | 0.109 | 0.021 | 0.003 | 0.000 |
| 0.51 | 29.96 | 0.950 | .443 | 0.284 | 0.193 | 0.095 | 0.016 | 0.002 | 0.000 |
| 1.05 | 23.03 | 0.900 | .408 | 0.257 | 0.172 | 0.082 | 0.012 | 0.001 | 0.000 |
| 1.63 | 18.97 | 0.850 | .373 | 0.231 | 0.152 | 0.069 | 0.009 | 0.000 | - |
| 2.88 | 13.86 | 0.750 | .307 | 0.182 | 0.114 | 0.047 | 0.004 | 0.000 | - |
| 6.73 | 7.13 | 0.610 | .161 | 0.080 | 0.042 | 0.011 | 0.000 | - |
| 12.04 | 3.57 | 0.300 | .057 | 0.019 | 0.006 | 0.000 | - |
| 18.97 | 1.63 | .150 | .009 | .001 | - |

From Table 3.1, it follows evidently that

$\text{ARE}\left(\hat{\theta}_{\text{MTuM}}, \hat{\theta}_{\text{MLE}}\right) \leq \text{ARE}\left(\hat{\theta}_{\text{MCM}}, \hat{\theta}_{\text{MLE}}\right) \leq \text{ARE}\left(\hat{\theta}_{\text{MTCM}}, \hat{\theta}_{\text{MLE}}\right)$.

This inequality is intuitive because MTuM is more robust than MCM and MTCM. As a result, MTuM estimators lose more efficiency and converge to the asymptotic results slower. For example, if the lower and upper truncation thresholds are, respectively, $d = 0.51$ and $u = 29.96$ then $\text{ARE}\left(\hat{\theta}_{\text{MTuM}}, \hat{\theta}_{\text{MLE}}\right) = 0.443$ and $\text{ARE}\left(\hat{\theta}_{\text{MCM}}, \hat{\theta}_{\text{MLE}}\right) = 0.918$. That is, we lose approximately 52% efficiency by going from MCM to MTuM. The reason that MTuM relative efficiency is much lower than the corresponding MCM is that the censored sample size is always fixed but even if we fix the
truncation thresholds, the truncated sample size is random. Further, MTuM disregards the observations beyond the truncation thresholds in order to control the influence of extremes in statistical inference. MCM controls such influence of extremes differently, i.e., those observations which are beyond the thresholds are adjusted to be equal to the corresponding thresholds and hence increases the efficiency significantly. MTCM controls the influence of extremes by disregarding the observations below lower threshold and adjusting the observations above upper threshold to be equal to the upper threshold which makes the MTCM entries in between the corresponding MTuM and MCM entries. Due to Theorem 3.5, entries for $\text{ARE} \left( \hat{\theta}_{\text{MCM}}, \hat{\theta}_{\text{MLE}} \right)$ are identical to $\text{ARE} \left( \hat{\theta}_{\text{MTM}}, \hat{\theta}_{\text{MLE}} \right)$ entries found in (Brazauskas et al., 2009, Table 1).

**Theorem 3.7.** The equation $\hat{\mu}_{\text{MTCM}} = \mu_{\text{MTCM}}$ has a unique solution $\hat{\theta}_{\text{MTCM}}$ provided that $d < \hat{\mu}_{\text{MTCM}} < u$. Otherwise, the solution does not exist.

*Proof.* A proof can similarly be established as in Theorem 3.2.

4 Simulation Study

This section supplements the theoretical results we developed in Section 3 via simulation. The main goal is to access the size of the sample such that the estimators are free from bias (given that the estimators are asymptotically unbiased), justify the asymptotic normality, and their finite sample relative efficiencies (RE) are approaching to the corresponding AREs. To compute RE of different estimators (MTuM, MCM, and MTCM) we use MLE as a benchmark. Thus, the definition of asymptotic relative efficiency given by equation (3.1) for finite sample performance translates to:

$$RE(C, MLE) = \frac{\text{asymptotic variance of MLE estimator}}{\text{small-sample variance of a competing estimator } C},$$

where the denominator is the empirical mean square error matrix of the competing estimator $C$.

From Exp($\theta = 10$), we first monitor the approximate normality distributional properties of the MTuM, MCM, and MTCM estimators of $\theta$ given, respectively, by (3.5), (3.9), and (3.14) with $(d, u) = (0.50, 23.00)$ and finite sample sizes $n = 30, 50$. We generate 100 samples for each sample size $n = 30, 50$ and estimate the values of $\theta$ from each sample via MTuM, MCM, and MTCM. We plot the histograms of those 100 estimated values of $\theta$ in Figure 4.1. Clearly, the histograms
corresponding to MTuM (for \( n = 30, 50 \)) are positively skewed (but turns out to be symmetric for \( n = 500 \)) and hence the asymptotic normality property of \( \hat{\theta}_{\text{MTuM}} \) given by (3.5) can be achieved slower, i.e., only for bigger sample sizes, than MCM and MTCM. On the other hand the asymptotic normality property of \( \hat{\theta}_{\text{MCM}} \) given by (3.9) and \( \hat{\theta}_{\text{MTCM}} \) given by (3.14) can be justified even for the sample of size \( n = 30 \).

Second, again from exponential distribution \( F(\cdot|\theta = 10) \) we generate 10,000 samples of a specified length \( n \) using Monte Carlo. For each sample we estimate the parameter of \( F \) using various MTuM, MCM, and MTCM estimators and then compute the average mean and RE of those 10,000 estimates. This process is repeated 10 times and the 10 average means and the 10 RE's are again

---

**Figure 4.1:** Histograms of 100 estimated values of the parameter \( \text{Exp}(\theta = 10) \) via MTuM, MCM, and MTCM with \( (d, u) = (0.50, 23.00) \) and sample sizes \( n = 30, 50, 500 \).
averaged and their standard errors are reported. Such repetitions are useful for assessing standard errors of the estimated means and RE’s. Hence, our findings are essentially based on 100,000 samples. The standardized ratio $\hat{\theta}/\theta$ that we report is defined as the average of 100,000 estimates divided by the true value of the parameter that we are estimating. We observe the performance of different methods of estimation for exponential distribution (see Section 3) in the aspects of 

(i) Sample size: $n = 50, 100, 250, 500, 1000$.
(ii) Estimators of $\theta$:

(a) MLE, which is a special case of all others.
(b) MTuM, MCM, MTCM.
(c) For the selected proportions $a = b = 0; a = b = 0.05; a = b = 0.10; a = b = 0.15; a = b = 0.25; a = 0.10$ and $b = 0.70; a = 0.25$ and $b = 0.00$, the left and right truncation (or censored) thresholds $d$ and $u$, respectively, to the nearest two decimal places are chosen specifically as $a = F(d)$ or $d = F^{-1}(a)$ and $1 - b = F(u)$ or $u = F^{-1}(1 - b)$.

Simulation results are recorded in Table 4.1. The entries are mean values (with standard errors in parentheses) based on 100,000 samples. The columns corresponding to $n \rightarrow \infty$, represent analytic $\text{ARE}(\hat{\theta}_C, \hat{\theta}_{\text{MLE}})$ results with $C \in \{\text{MTuM}, \text{MCM}, \text{MTCM}\}$ and are found in Section 3, not from simulations. Among these three columns, the first one is $C = \text{MTuM}$, second $C = \text{MCM}$, and third $C = \text{MTCM}$. As seen from Table 4.1, the ratio $\hat{\theta}/\theta$ of the exponential $\theta$ estimators converges to the true asymptotic value of 1 very fast. Besides MTuM approach, the bias of all other procedures disappears as soon as $n \geq 500$ and the estimators’ RE’s practically reach their ARE levels just for $n \geq 250$. Some of the finite sample entries for MTuM columns in Table 4.1 are not reported, specially for the pair $(d, u) = (1.05, 3.57)$, because the corresponding threshold pair $(d, u)$ does not satisfy the necessary condition of Theorem 3.2 for at least one generated sample. It is evident from the entries that if the difference between the thresholds (i.e., $d$ and $u$) is smaller then the estimators converge slower to the true values. Overall, as expected MCM perform the best in terms of balancing between efficiency and robustness. MTuM performs very poorly specially for small sample sizes in terms of efficiency but this approach produces highly robust estimators.
Table 4.1: Finite-sample performance evaluation of different Estimators for $\text{Exp}(\theta = 10)$.

| $d_{(a)}$ | $u_{(b)}$ | MTuM $n = 50$ | MCM $n = 50$ | MTuM $n = 100$ | MCM $n = 100$ | MTuM $n = 250$ | MCM $n = 250$ |
|-----------|-----------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0.00 $(.00)$ | $\propto (0.00)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ |
| 0.51 $(.05)$ | 29.96 $(.05)$ | 1.03 $(.001)$ | 1.01 $(.000)$ | 1.02 $(.000)$ | 1.02 $(.000)$ | 1.01 $(.000)$ | 1.00 $(.000)$ |
| 1.05 $(.10)$ | 23.03 $(.10)$ | 1.08 $(.002)$ | 1.01 $(.000)$ | 1.01 $(.000)$ | 1.01 $(.000)$ | 1.01 $(.000)$ | 1.00 $(.000)$ |
| $\hat{\theta}/\theta$ | 1.63 $(.15)$ | 18.97 $(.15)$ | - | 1.01 $(.000)$ | 1.02 $(.000)$ | 1.07 $(.002)$ | 1.01 $(.000)$ | 1.00 $(.000)$ |
| 2.88 $(.25)$ | 13.86 $(.25)$ | - | 1.02 $(.001)$ | 1.03 $(.001)$ | - | 1.01 $(.000)$ | 1.02 $(.000)$ | - | 1.00 $(.000)$ | 1.01 $(.000)$ |
| 1.05 $(.10)$ | 03.57 $(.70)$ | - | 1.08 $(.001)$ | 1.16 $(.002)$ | - | 1.04 $(.001)$ | 1.07 $(.001)$ | - | 1.01 $(.000)$ | 1.02 $(.000)$ |
| 2.88 $(.25)$ | $\propto (0.00)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ |

| $d_{(a)}$ | $u_{(b)}$ | MTuM $n = 500$ | MCM $n = 500$ | MTuM $n = 1000$ | MCM $n = 1000$ | MTuM $n \to \infty$ | MCM $n \to \infty$ |
|-----------|-----------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0.00 $(.00)$ | $\propto (0.00)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ |
| 0.51 $(.05)$ | 29.96 $(.05)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ |
| 1.05 $(.10)$ | 23.03 $(.10)$ | 1.01 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ |
| $\hat{\theta}/\theta$ | 1.63 $(.15)$ | 18.97 $(.15)$ | 1.01 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ |
| 2.88 $(.25)$ | 13.86 $(.25)$ | 1.05 $(.001)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ |
| 1.05 $(.10)$ | 03.57 $(.70)$ | - | 1.01 $(.000)$ | 1.01 $(.000)$ | - | 1.00 $(.000)$ | 1.01 $(.000)$ | - | 1.00 $(.000)$ | 1.01 $(.000)$ |
| 2.88 $(.25)$ | $\propto (0.00)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ |

| $d_{(a)}$ | $u_{(b)}$ | MTuM $n = 500$ | MCM $n = 500$ | MTuM $n = 1000$ | MCM $n = 1000$ | MTuM $n \to \infty$ | MCM $n \to \infty$ |
|-----------|-----------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0.00 $(.00)$ | $\propto (0.00)$ | 1.00 $(.001)$ | 1.00 $(.001)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ | 1.00 $(.000)$ |
| 0.51 $(.05)$ | 29.96 $(.05)$ | 0.43 $(.001)$ | 0.91 $(.001)$ | 0.86 $(.000)$ | 0.86 $(.000)$ | 0.44 $(.002)$ | 0.91 $(.004)$ | 0.86 $(.003)$ | .442 | .918 | .868 |
| 1.05 $(.10)$ | 23.03 $(.10)$ | 0.24 $(.001)$ | 0.84 $(.001)$ | 0.74 $(.000)$ | 0.74 $(.000)$ | 0.25 $(.001)$ | 0.84 $(.004)$ | 0.74 $(.003)$ | .257 | .848 | .749 |
| $\hat{\theta}/\theta$ | 1.63 $(.15)$ | 18.97 $(.15)$ | 0.14 $(.001)$ | 0.78 $(.001)$ | 0.63 $(.000)$ | 0.63 $(.000)$ | 0.14 $(.001)$ | 0.78 $(.003)$ | 0.63 $(.002)$ | .152 | .787 | .638 |
| 2.88 $(.25)$ | 13.86 $(.25)$ | 0.03 $(.001)$ | 0.67 $(.001)$ | 0.42 $(.000)$ | 0.43 $(.000)$ | 0.04 $(.000)$ | 0.67 $(.002)$ | 0.43 $(.001)$ | .047 | .679 | .433 |
| 1.05 $(.10)$ | 03.57 $(.70)$ | - | 0.24 $(.001)$ | 0.14 $(.000)$ | - | 0.24 $(.001)$ | 0.15 $(.001)$ | .001 | .250 | .156 |
| 2.88 $(.25)$ | $\propto (0.00)$ | 0.75 $(.001)$ | 0.99 $(.001)$ | 0.75 $(.000)$ | 0.75 $(.000)$ | 0.75 $(.003)$ | 0.99 $(.004)$ | 0.75 $(.003)$ | .750 | .995 | .750 |
5 Concluding Remarks

In this paper, we have developed the methods of truncated (called MTuM), censored (called MCM), and insurance payment-type (called MTCM) moments estimators for completely observed ground-up loss severity data. A series of theoretical results about estimators’ existence and asymptotic normality are established. Our analysis has established new connections between data truncation, trimming, and censoring, which paves the way for more effective modeling of non-linearly transformed loss data. Further, as seen from Table 3.1, there is clear trade-offs between efficiency and robustness between newly designed estimators and the corresponding MLEs when sample size is large. The finite sample performance, for various sample sizes, of all the estimators developed in this paper has been investigated in detail for single parameter Pareto model via simulation study.

The results of this paper motivate open problems and generate several ideas for further research. First, most of the results of Section 3 (beside Theorem 3.6) are limited to complete exponentially (equivalently single parameter Pareto) distributed data but they could be extended to more general situations and models. For example, similar estimation approaches could be designed for (log) location-scale and exponential dispersion families which could lead to more challenging non-linear equations to be solved (see Theorems 3.2, 3.4, and 3.7). Second, several contaminated loss severity models are proposed in the literature (see, e.g., Brazauskas and Kleefeld, 2016, Chan et al., 2018, Scollnik and Sun, 2012), so it could even produce a better model still maintaining a reasonable balance between efficiency and robustness if one implements the procedures developed in this paper on the body part of the data and some heavier distributions (say, for e.g., Pareto) on the right tail. Further, it is yet to measure how the newly designed estimation procedures act with different risk analysis in practice.

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**Appendix A: All four possible scenarios for Section 2.2**

**Scenario 1**: \( d_j \leq d_j' < u_j \leq u_j' \)

\[
Y_{jj'} = h_{jj'}(X) \mathbb{1}\{d_j < X \leq u_j\} = h_{jj'}(X) \mathbb{1}\{d_j < X \leq u_j\},
\]

\[
W_{jj'} = Z_j \mathbb{1}\{d_j < X \leq u_j\}, \quad \text{and} \quad W_{j'j} = Z_j' \mathbb{1}\{d_j < X \leq u_j\}.
\]

**Scenario 2**: \( d_j \leq d_j' < u_j' \leq u_j \)

\[
Y_{jj'} = h_{jj'}(X) \mathbb{1}\{d_j < X \leq u_j\} = h_{jj'}(X) \mathbb{1}\{d_j < X \leq u_j\},
\]

\[
W_{jj'} = Z_j \mathbb{1}\{d_j < X \leq u_j\}, \quad \text{and} \quad W_{j'j} = Z_j' \mathbb{1}\{d_j < X \leq u_j\}.
\]

**Scenario 3**: \( d_j' \leq d_j < u_j \leq u_j' \)

\[
Y_{jj'} = h_{jj'}(X) \mathbb{1}\{d_j < X \leq u_j\} = h_{jj'}(X) \mathbb{1}\{d_j < X \leq u_j\},
\]

\[
W_{jj'} = Z_j \mathbb{1}\{d_j < X \leq u_j\}, \quad \text{and} \quad W_{j'j} = Z_j' \mathbb{1}\{d_j < X \leq u_j\}.
\]

**Scenario 4**: \( d_j' \leq d_j < u_j' \leq u_j \)

\[
Y_{jj'} = h_{jj'}(X) \mathbb{1}\{d_j < X \leq u_j\} = h_{jj'}(X) \mathbb{1}\{d_j < X \leq u_j\},
\]
\[ W_{jj'} = Z_j \mathbb{1}\{d_j < X \leq u_{j'}\}, \quad \text{and} \quad W_{jj'} = Z_{j'} \mathbb{1}\{d_j < X \leq u_{j'}\} \].

Therefore, depending on the scenario the expected values are given by:

\[
\begin{align*}
\mu_{Y_{jj'}} &= \mathbb{E}[Y_{jj'}] = \int_{F(d_{jj'}\theta)}^{F(u_{jj'}\theta)} h_{jj'}(F^{-1}(v|\theta)) \, dv, \\
\mu_{W_{jj'}} &= \mathbb{E}[W_{jj'}] = \int_{F(d_{jj'}\theta)}^{F(u_{jj'}\theta)} h_j(F^{-1}(v|\theta)) \, dv.
\end{align*}
\]

Appendix B: Proofs

**Proof of Theorem 2.2:**

Clearly, \( g_V(\mu_V) = \left( \frac{\mu_Y}{p_1}, \ldots, \frac{\mu_Y}{p_k} \right) =: (\mu_1, \ldots, \mu_k) =: \mu \). From Lemma 2.2, it follows that

\[ D_V := \left[ \frac{\partial g_j}{\partial x_{jj'}} \bigg|_{x=\mu_V} \right]_{k \times 2k} = [d_{V, jj'}]_{k \times 2k}, \]

where

\[ d_{V, jj'} := \begin{cases} 
\frac{1}{p_j}, & \text{if } 1 \leq j = j' \leq k; \\
-\frac{\mu_Y}{p_j}, & \text{if } j' - j = k; \\
0, & \text{otherwise.}
\end{cases} \]

Now, with an application of delta method corresponding with the function \( g_V \) above, (see Serfling, 1980, §3.3 Theorem A), we have

\[ (\hat{\mu}_1, \ldots, \hat{\mu}_k) \sim \mathcal{AN}(g_V(\mu_V) = \mu, \frac{1}{n} D_V \Sigma_V D_V'). \]

**Proof of Theorem 3.1:**

The r.v. \( Y \) can be expressed in the form of

\[ Y = X \wedge u - u \mathbb{1}\{u < X < \infty\} - X \wedge d + d \mathbb{1}\{d < X < \infty\}. \]

Define, \( I_{a,b} := \mathbb{1}\{a < X < b\} \). Therefore,

\[
\begin{align*}
\mu_Y &= \mathbb{E}[Y] = \mathbb{E}[X \wedge u] - \mathbb{E}[u I_{u,\infty}] - \mathbb{E}[X \wedge d] + \mathbb{E}[d I_{d,\infty}] \\
&= \theta(1 - e^{-\frac{d}{\theta}}) - ue^{-\frac{d}{\theta}} - \theta(1 - e^{-\frac{d}{\theta}}) + de^{-\frac{d}{\theta}} = \theta \left( e^{-\frac{d}{\theta}} - e^{-\frac{u}{\theta}} \right) + de^{-\frac{d}{\theta}} - ue^{-\frac{u}{\theta}}.
\end{align*}
\]
Since $Y = X \wedge u - u \mathbb{1}\{u < X < \infty\} - X \wedge d + d \mathbb{1}\{d < X < \infty\}$, then

$$Y^2 = (X \wedge u)^2 - (X \wedge d)^2 - 2d [X \wedge u - X \wedge d] - u^2 I_{u, \infty} - d^2 I_{d, \infty} + 2d (X I_{d,u} + u I_{u,\infty}).$$

Therefore, for $X \sim \text{Exp}(\theta)$ then $\mu_{Y^2} := \mathbb{E}[Y^2]$ is computed as below:

$$\mu_{Y^2} = \mathbb{E}[Y^2] = \mathbb{E}[(X \wedge u)^2] - \mathbb{E}[(X \wedge d)^2] - 2d \mathbb{E}[X \wedge u] - \mathbb{E}[X \wedge d] - u^2 \mathbb{E}[I_{u, \infty}] - d^2 \mathbb{E}[I_{d, \infty}] + 2d (\mathbb{E}[X I_{d,u}] + u \mathbb{E}[I_{u,\infty}])$$

$$= 2\theta^2 \Gamma \left( 3; \frac{u}{\theta} \right) + u^2 e^{-\frac{u}{\theta}} - 2\theta^2 \Gamma \left( 3; \frac{d}{\theta} \right) - d^2 e^{-\frac{d}{\theta}}$$

$$- 2d \left[ \theta \left( 1 - e^{-\frac{d}{\theta}} \right) - \theta \left( 1 - e^{-\frac{d}{\theta}} \right) \right] - u^2 e^{-\frac{u}{\theta}} - d^2 e^{-\frac{d}{\theta}}$$

$$+ 2d \left[ \theta e^{-\frac{d}{\theta}} + de^{-\frac{d}{\theta}} - \theta e^{-\frac{d}{\theta}} - \theta e^{-\frac{d}{\theta}} + ue^{-\frac{u}{\theta}} \right]$$

$$= 2\theta^2 \left( \Gamma \left( 3; \frac{u}{\theta} \right) - \Gamma \left( 3; \frac{d}{\theta} \right) \right).$$

Therefore,

$$\sigma_Y^2 = \mu_{Y^2} - \mu_Y^2 = 2\theta^2 \left( \Gamma \left( 3; \frac{u}{\theta} \right) - \Gamma \left( 3; \frac{d}{\theta} \right) \right) - \mu_Y^2.$$

**Proof of Theorem 3.3:**

Note that the parameter vector is given by $\theta = (\alpha, x_0)$ with $x_0$ known in advance. The population version of $\hat{\mu}_{MTM}$ is given by

$$\mu_{MTM} = \mathbb{E}[h(Y)|d < Y \leq u] = \mathbb{E}[h(Y)\mathbb{1}\{d < Y \leq u\}] = \int_{F(d|\theta)} F(u|\theta) f(y|\theta) dy$$

$$= \int_{F(d|\theta)} h(F^{-1}(v|\theta))) dv$$

$$= \frac{1}{\alpha \left( \frac{x_0}{d} \right)^\alpha - \left( \frac{x_0}{u} \right)^\alpha} \left[ F(d|\theta) - F(u|\theta) + \alpha \left( 1 - F(d|\theta) \right) \log \left( \frac{x_0}{d} \right) - \alpha \left( 1 - F(u|\theta) \right) \log \left( \frac{x_0}{u} \right) \right]$$

$$= \frac{x_0^\alpha(u)^\alpha}{\alpha x_0^\alpha(u)^\alpha(a - d^\alpha)} \left[ u^\alpha \left( 1 - \alpha \log \left( \frac{x_0}{d} \right) \right) - d^\alpha \left( 1 - \alpha \log \left( \frac{x_0}{u} \right) \right) \right]$$

$$= \frac{1}{\alpha \left( u^\alpha - d^\alpha \right)} \left[ u^\alpha \left( 1 - \alpha \log \left( \frac{x_0}{d} \right) \right) - d^\alpha \left( 1 - \alpha \log \left( \frac{x_0}{u} \right) \right) \right]$$

$$= \frac{A_d}{\alpha \left( u^\alpha - d^\alpha \right)}.$$
Now, to establish the proof of the statement, it is enough to prove that the function $g_{du}$ is strictly decreasing with respect to $\alpha$. For that,

$$g'_{du}(\alpha) = \frac{dg_{du}(\alpha)}{d\alpha} = \frac{(du)^{\alpha^2} (\log \left( \frac{u}{d} \right))^2 - (u^\alpha - d^\alpha)^2}{\alpha^{2}(u^\alpha - d^\alpha)^2}. $$

Now, in order to show that $g'_{du}(\alpha) < 0$, it is enough to establish $(du)^{\frac{\alpha}{2}} \alpha \log \left( \frac{u}{d} \right)^2 - (u^\alpha - d^\alpha)^2 < 0$ which is equivalent to establish that $(du)^{\frac{\alpha}{2}} \alpha \log \left( \frac{u}{d} \right) < u^\alpha - d^\alpha$. Now,

$$(du)^{\frac{\alpha}{2}} \alpha \log \left( \frac{u}{d} \right) < u^\alpha - d^\alpha$$

$$\iff \quad \alpha \log \left( \frac{u}{d} \right) < \left( \frac{u}{d} \right)^{\frac{\alpha}{2}} - \left( \frac{u}{d} \right)^{-\frac{\alpha}{2}}$$

$$\iff \quad \alpha \log \left( \frac{u}{d} \right) < 2 \sinh \left( \frac{\alpha}{2} \log \left( \frac{u}{d} \right) \right)$$

$$\iff \quad \frac{\alpha}{2} \log \left( \frac{u}{d} \right) < \sinh \left( \frac{\alpha}{2} \log \left( \frac{u}{d} \right) \right).$$

But we know that $x < \sinh x$ for all $x > 0$, therefore, $g'_{du}(\alpha) < 0$ for all $\alpha > 0$ which implies that $g_{du}$ is strictly decreasing. Finally, note that

$$\lim_{\alpha \to 0^+} g_{du}(\alpha) = \frac{(\log(u))^2 - (\log(d))^2 - 2 \log(u) \log \left( \frac{x_0}{d} \right) + 2 \log(d) \log \left( \frac{x_0}{u} \right)}{2 \log \left( \frac{u}{d} \right)}, \quad \text{and}$$

$$\lim_{\alpha \to \infty} g_{du}(\alpha) = - \log \left( \frac{x_0}{d} \right). \quad \square$$