Black hole quasinormal modes using the asymptotic iteration method

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Abstract
In this paper we show that the asymptotic iteration method (AIM) allows one to numerically find the quasinormal modes of Schwarzschild and Schwarzschild de Sitter black holes. An added benefit of the method is that it can also be used to calculate the Schwarzschild anti-de Sitter quasinormal modes for the case of spin-zero perturbations. We also discuss an improved version of the AIM, more suitable for numerical implementation.

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1. Introduction

The study of quasinormal modes (QNMs) of the Schwarzschild black hole is an old and well-established subject (for a recent review see [1] and references therein), where the various frequencies have been well determined, typically by applying a Frobenius series solution approach, leading to continued fractions for QNM boundary conditions, in the manner of Leaver [2]. Recently a new method for obtaining analytic/numerical solutions of second-order ordinary differential equations with bound potentials has been developed called the asymptotic iteration method (AIM) [3], which was found to be closely connected to continued fractions developed from exact WKB solutions, see [4] and references therein\(^6\).

In this paper we will demonstrate that the AIM can also be applied to the case of black hole QNMs, which have unbounded (scattering)-like potentials. On a related note, the AIM

\(^6\) The phrase ‘continued fraction’ used for WKB solutions should not be confused with the type of continued fractions developed from Frobenius series for QNMs by Leaver [2]. In the rest of the paper ‘continued fraction’ will mean those generated from Frobenius series.
was used to find QNMs for Scarf II (upside-down Poschl–Teller-like) potentials [5], based on observations made by one of the current authors [6] relating QNMs from quasi-exactly solvable models. Indeed, bound state Poschl–Teller potentials have been used for QNM approximations previously by inverting black hole potentials [7]. However, the AIM does not require any inversion of the black hole potential as we shall show.

In this paper we shall focus on spherically symmetric backgrounds in four dimensions with field equations of the form

$$\frac{d^2 \psi(x)}{dx^2} + [\omega^2 - V(x)]\psi(x) = 0,$$  \hspace{1cm} (1)

where $V(x)$ is a master potential of the form [1]

$$V(r) = f(r) \left[ \frac{\ell(\ell + 1)}{r^2} + (1 - s^2) \left( \frac{2M}{r^3} - \frac{(4 - s^2)\Lambda}{6} \right) \right],$$  \hspace{1cm} (2)

and $dx = dr/f(r)$ where

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2.$$  \hspace{1cm} (3)

with the cosmological constant $\Lambda$. Here $s = 0, 1, 2$ denotes the spin of the perturbation: scalar, electromagnetic and gravitational (for half-integer spin see [8–10]). In the following we shall use the definition that QNMs are defined as solutions of the above equations with boundary conditions

$$\psi(x) \rightarrow \begin{cases} e^{i\omega x} & x \rightarrow \infty \\ e^{-i\omega x} & x \rightarrow -\infty \end{cases},$$  \hspace{1cm} (4)

for $e^{-i\omega t}$ time dependence (which corresponds to ingoing waves at the horizon and outgoing waves at infinity). Note that the boundary condition such as $x \rightarrow \infty$ does not apply to asymptotically anti-de Sitter spacetimes, where instead something like a Dirichlet boundary condition is imposed, e.g see [11]. We shall see how the AIM can be applied not only to Schwarzschild and Schwarzschild de Sitter (SdS) QNMs for general spin, $s$, but also to the Schwarzschild anti-de Sitter case (SAdS), at least for spin-zero ($s = 0$) fields.

The structure of the paper is as follows. In the next section we describe how the AIM can be applied to exterior eigenvalue problems for the Schwarzschild background. Then in section 3 we show how the AIM can be applied to backgrounds with more than one horizon, such as SdS. In section 4 we show how the AIM can be applied to scalar fields on SAdS, while in section 5 we discuss the results and make conclusions. In the appendix we present a brief derivation of the continued fraction method for SdS for spin $s = 1, 2$ fields.

2. Schwarzschild black holes

To explain the AIM we shall start with the simplest case of the radial component of a perturbation of the Schwarzschild metric outside the event horizon [8]. For an asymptotically flat Schwarzschild solution ($\Lambda = 0$)

$$f(r) = 1 - \frac{2M}{r},$$ \hspace{1cm} (5)

where from $dx = dr/f(r)$ we have

$$x(r) = r + 2M \ln \left( \frac{r}{2M} - 1 \right).$$  \hspace{1cm} (6)
for the tortoise coordinate, $x$. Note, for the Schwarzschild background the maximum for this potential, in terms of $r$, is given by
\[
\frac{3 M}{2} \frac{1}{\ell(\ell+1)} \left[ \frac{\ell(\ell+1)}{r^2} - \sigma + \left( \frac{\ell(\ell+1)}{r^2} + \frac{14}{9} \frac{\ell(\ell+1)}{r^2} \sigma^2 \right)^{1/2} \right],
\]
where $\sigma = 1 - \frac{\ell^2}{9} [12]$.

The choice of coordinates is somewhat arbitrary and in the next section (for SdS) we will see how an alternative choice leads to a simpler solution. First, consider the change of variable
\[
\xi = 1 - \frac{2M}{r},
\]
with $0 \leq \xi < 1$. In terms of $\xi$, equation (1) then becomes
\[
\frac{d^2 \psi}{d\xi^2} + \frac{1 - 3\xi - 3\xi(1-\xi)}{\xi(1-\xi)} \frac{d\psi}{d\xi} + \left[ \frac{4M^2\omega^2}{\xi^2(1-\xi)^2} - \frac{\ell(\ell+1)}{\xi(1-\xi)} - \frac{1 - s^2}{\xi(1-\xi)} \right] \psi = 0.
\]

To accommodate the outgoing wave boundary condition $\psi \to e^{i\omega x}$ as $(x, r) \to \infty$ in terms of $\xi$ (which is the limit $\xi \to 1$) and the regular singularity at the event horizon ($\xi \to 0$), we define
\[
\psi(\xi) = \xi - 2iM\omega (1-\xi) - 2iM\omega e^{2iM\omega} [1-\xi] \chi(\xi),
\]
where the Coulomb power law is included in the asymptotic behaviour (cf [2] equation (5)).

The radial equation then takes the form
\[
\chi'' = \lambda_0(\xi) \chi' + s_0(\xi) \chi,
\]
where
\[
\lambda_0(\xi) = \frac{4M\omega(2\xi^2 - 4\xi + 1) - (1 - 3\xi)(1-\xi)}{\xi(1-\xi)^2},
\]
\[
s_0(\xi) = \frac{16M^2\omega^2(\xi - 2) - 8M\omega(1 - \xi) + \ell(\ell + 1) + (1 - s^2)(1 - \xi)}{\xi(1-\xi)^2}.
\]

Note that primes of $\chi$ denote derivatives with respect to $\xi$.

The crucial observation in the AIM is that differentiating the above equation $n$ times with respect to $\xi$ leaves a symmetric form for the right-hand side:
\[
\chi^{(n+2)} = \lambda_n(\xi) \chi' + s_n(\xi) \chi,
\]
where
\[
\lambda_n(\xi) = \lambda_{n-1}(\xi) + s_{n-1}(\xi) \lambda_{n-1}(\xi) \quad \text{and} \quad s_n(\xi) = s_{n-1}(\xi) + s_0(\xi) \lambda_{n-1}(\xi).
\]

For sufficiently large $n$ the asymptotic aspect of the ‘method’ is introduced, that is
\[
\frac{s_n(\xi)}{\lambda_n(\xi)} \equiv \beta(\xi),
\]
where the QNMs are obtained from the ‘quantization condition’
\[
\delta_n = s_n\lambda_{n-1} - s_{n-1}\lambda_n = 0,
\]
which is equivalent to imposing a termination to the number of iterations [13]. This leads to the general solution
\[
\chi(\xi) = \exp \left[ -\int^\xi \beta(\xi') d\xi' \right] \left( C_2 + C_1 \int^\xi \exp \left\{ \int^\xi [\lambda_0(\xi'') + 2\beta(\xi'')] d\xi'' \right\} d\xi' \right).
\]
2.1. Improved AIM

One unappealing feature of the recursion relations in equations (15) is that at each iteration one must take the derivative of $s$ and $\lambda$ of the previous iteration. This can slow down the numerical implementation of the AIM and also lead to problems with numerical precision. To circumvent these issues we have developed an improved version of the AIM which bypasses the need to take derivatives at each step. This greatly improves both the accuracy and speed of the method. We expand $\lambda_n$ and $s_n$ in a Taylor series around the point where the AIM is performed, $\xi$:

$$\lambda_n(\xi) = \sum_{i=0}^{\infty} c_i^n (x - \xi)^i,$$

$$s_n(\xi) = \sum_{i=0}^{\infty} d_i^n (x - \xi)^i,$$

where $c_i^n$ and $d_i^n$ are the $i$th Taylor coefficients of $\lambda_n(\xi)$ and $s_n(\xi)$, respectively. Substituting these expressions into equations (15) leads a set of recursion relations for the coefficients:

$$c_i^n = (i + 1) c_{i+1}^{n-1} + d_{i+1}^{n-1} + \sum_{k=0}^{i} c_k^0 c_{i-k}^n,$$

$$d_i^n = (i + 1) d_{i+1}^{n-1} + \sum_{k=0}^{i} d_k^0 c_{i-k}^n.$$

In terms of these coefficients the ‘quantization condition’ (17) can be reexpressed:

$$d_0^n c_0^n - d_0^{n-1} c_0^n = 0$$

and thus we have reduced the AIM into a set of recursion relations which no longer require derivative operators.

Observing that the right-hand sides of equations (21) and (22) involve terms of order at most $n - 1$, one can recurse these equations until only $c_0^n$ and $d_0^n$ terms remain (that is the coefficients of $\lambda_0$ and $s_0$ only). However for large numbers of iterations, due to the large number of terms, such expressions become impractical to compute. We avert this combinatorial problem by beginning at the $n = 0$ stage and calculating the $n + 1$ coefficients sequentially until the desired number of recursions is reached. Since the quantization condition only requires the $i = 0$ term, at each iteration $n$ we only need to determine coefficients with $i < N - n$, where $N$ is the maximum number of iterations to be performed.

The QNMs that we calculate in this paper will be determined using this improved AIM. Indeed in a previous work of ours for spheroidal harmonics [14], we found that the AIM was somewhat slower than the continued fraction method. However, the Mathematica code implementing this new version of the AIM is found to be on a par with the continued fraction approach; these can be downloaded from [http://www-het.phys.sci.osaka-u.ac.jp/~naylor/AIM.html](http://www-het.phys.sci.osaka-u.ac.jp/~naylor/AIM.html).

3. Schwarzschild dS

The QNMs for Schwarzschild gravitational perturbations are presented in table 1; however, to further justify the use of this method, it is instructive to consider some more general cases. As
such, we shall now consider the SdS case, where we have the same WKB-like wave equation and potential as in equation (1), but now
\[ f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \] (24)
where \( \Lambda > 0 \) is the cosmological constant. Interestingly, the choice of coordinates we use here leads to a simpler AIM solution, because there is no Coulomb power law tail; however, in the limit \( \Lambda = 0 \) we recover the Schwarzschild results. Note that although it is possible to find an expression for the maximum of the potential in equation (1), for the SdS case, it is the solution of a cubic equation, which for brevity we refrain from presenting here. In our AIM code\(^7\) we use a numerical routine to find the root to make the code more general.

In the SdS case it is more convenient to change coordinates to \( \xi = \frac{1}{r} \)\(^11\), which leads to the following master equation (cf equation (2)):
\[ \frac{d^2 \psi}{d\xi^2} + \frac{p'}{p} \frac{d\psi}{d\xi} + \left[ \frac{\ell(\ell + 1) + (1 - s^2)\left(2M\xi - (4 - s^2)\frac{\Lambda}{6\xi^2}\right)}{p^2} \right] \psi = 0, \] (25)
where we have defined
\[ p = \xi^2 - 2M\xi^3 - \Lambda/3 \Rightarrow p' = 2\xi(1 - 3M\xi). \] (26)
It may be worth mentioning that for SdS we can express \(^11\)
\[ e^{i\omega_\xi} = (\xi - \xi_1)\frac{\omega_\xi}{\omega_\xi}(\xi - \xi_2)\frac{\omega_\xi}{\omega_\xi}(\xi - \xi_3)\frac{\omega_\xi}{\omega_\xi}, \] (27)
in terms of the roots of \( f(r) \), where \( \xi_1 \) is the event horizon and \( \xi_2 \) is the cosmological horizon (and \( \kappa_n \) is the surface gravity at each \( \xi_n \)). This is useful for choosing the appropriate scaling behaviour for QNM boundary conditions.

Based on the above equation an appropriate choice for QNMs is to scale out the divergent behaviour at the cosmological horizon\(^8\):
\[ \psi(\xi) = e^{i\omega_\xi}u(\xi), \] (28)
which implies
\[ pu'' + (p' - 2i\omega)u' - \left[ \ell(\ell + 1) + (1 - s^2)\left(2M\xi - (4 - s^2)\frac{\Lambda}{6\xi^2}\right) \right] u = 0, \] (29)
in terms of \( \xi \). Furthermore, based on the scaling in equation (28), the correct QNM condition at the horizon \( \xi_1 \) implies
\[ u(x) = (\xi - \xi_1)^{\frac{\kappa_1}{2}} \chi(x), \] (30)
where \( \kappa_1 \) is the surface gravity at the event horizon \( \xi_1 \):
\[ \kappa_1 = \left. \frac{1}{2} \frac{df}{dr} \right|_{r=r_1} = M\xi_1^2 - \frac{1}{3\xi_1} \] (31)
with \( \xi_1 = 1/r_1 \), where \( r_1 \) is the smallest real solution of \( f(r) = 0 \), implying \( p = 0 \). The differential equation then takes the standard AIM form
\[ \chi'' = \lambda_0(\xi)\chi' + s_0(\xi)\chi, \] (32)
where
\(^7\) Webpage with *Mathematica* notebooks for generating SdS QNMs: [http://www.het.phys.osaka-u.ac.jp/~maylor/AIM.html](http://www.het.phys.osaka-u.ac.jp/~maylor/AIM.html) or [http://www.j.yukawa.kyoto-u.ac.jp/~jasonad/AIM.html](http://www.j.yukawa.kyoto-u.ac.jp/~jasonad/AIM.html)
\(^8\) Note that this is opposite to the case presented in [11], where they define the QNMs as solutions with boundary conditions: \( \psi(x) \propto e^{\pm i\omega_\xi} \) as \( x \to \pm \infty \), cf equation (4), for the \( e^{i\omega_\xi} \) time dependence.
Table 1. QNMs to four decimal places for gravitational perturbations ($\sigma = -3$) where the fifth column is taken from [12]. Note that the imaginary part of the $n = 0, \ell = 2$ result in [12] has been corrected to agree with [2] (see [*]). Note, if the number of iteration in the AIM is increased to say about 50 then we find agreement with [2] accurate to six significant figures.

| $\ell$ | $n$ | $\omega_{\text{Leaver}}$ | $\omega_{\text{AIM (after 15 iterations)}}$ | $\omega_{\text{WKB}}$ |
|-------|-----|--------------------------|---------------------------------------------|-----------------|
| 2     | 0   | 0.3737 - 0.0896i[*]      | 0.3737 - 0.0896i                           | 0.3732 - 0.0892i|
|       |     | (<0.01%) (<0.01%)        | (<0.13%) (0.44%)[*]                       |                 |
| 1     | 0.3467 - 0.2739i              | 0.3467 - 0.2739i                         | 0.3460 - 0.2749i                  |
|       | (<0.01%) (<0.01%)        | (<0.20%) (<0.36%)                       |                 |
| 2     | 0.3011 - 0.4783i              | 0.3012 - 0.4785i                         | 0.3029 - 0.4711i                  |
|       | (0.03%) (<0.04%)       | (0.60%) (1.5%)                          |                 |
| 3     | 0.2515 - 0.7051i              | 0.2523 - 0.7023i                         | 0.2475 - 0.6703i                  |
|       | (0.32%) (0.40%)          | (<1.6%) (4.6%)                          |                 |
| 3     | 0.5994 - 0.0927i              | 0.5994 - 0.0927i                         | 0.5993 - 0.0927i                  |
|       | (<0.01%) (<0.01%)        | (<0.02%) (0.0%)                         |                 |
| 1     | 0.5826 - 0.2813i              | 0.5826 - 0.2813i                         | 0.5824 - 0.2814i                  |
|       | (<0.01%) (<0.01%)        | (<0.03%) (<0.04%)                       |                 |
| 2     | 0.5517 - 0.4791i              | 0.5517 - 0.4791i                         | 0.5532 - 0.4767i                  |
|       | (<0.01%) (<0.01%)        | (0.27%) (0.50%)                         |                 |
| 3     | 0.5120 - 0.6903i              | 0.5120 - 0.6905i                         | 0.5157 - 0.6774i                  |
|       | (<0.01%) (<0.03%)        | (0.72%) (1.9%)                          |                 |
| 4     | 0.4702 - 0.9156i              | 0.4715 - 0.9156i                         | 0.4711 - 0.8815i                  |
|       | (0.28%) (<0.01%)        | (0.19%) (3.7%)                          |                 |
| 5     | 0.4314 - 1.152i               | 0.4360 - 1.147i                          | 0.4189 - 1.088i                   |
|       | (1.07%) (0.43%)          | (<2.9%) (5.6%)                          |                 |
| 4     | 0.8092 - 0.0942i              | 0.8092 - 0.0942i                         | 0.8091 - 0.0942i                  |
|       | (<0.01%) (<0.01%)        | (<0.01%) (0.0%)                         |                 |
| 1     | 0.7966 - 0.2843i              | 0.7966 - 0.2843i                         | 0.7965 - 0.2844i                  |
|       | (<0.01%) (<0.01%)        | (<0.01%) (<0.04%)                       |                 |
| 2     | 0.7727 - 0.4799i              | 0.7727 - 0.4799i                         | 0.7736 - 0.4790i                  |
|       | (<0.01%) (<0.01%)        | (0.12%) (0.19%)                         |                 |
| 3     | 0.7398 - 0.6839i              | 0.7398 - 0.6839i                         | 0.7433 - 0.6783i                  |
|       | (<0.01%) (<0.01%)        | (0.47%) (0.82%)                         |                 |
| 4     | 0.7015 - 0.8982i              | 0.7014 - 0.8985i                         | 0.7072 - 0.8813i                  |
|       | (<0.01%) (<0.03%)        | (0.81%) (1.9%)                          |                 |

\[
\begin{align*}
\lambda_0(\xi) & = -\frac{1}{p} \left[ p' - \frac{2i\omega}{\kappa_1(\xi - \xi_1)} - 2i\omega \right], \quad (33) \\
\rho_0(\xi) & = \frac{1}{p} \left[ \ell(\ell + 1) + (1 - s^2) \left( 2M\xi - (4 - s^2) \frac{\Lambda}{6\xi^2} \right) + \frac{i\omega}{\kappa_1(\xi - \xi_1)^2} \left( \frac{i\omega}{\kappa_1} + 1 \right) \right] \\
& \quad + \left( p' - 2i\omega \right) \frac{i\omega}{\kappa_1(\xi - \xi_1)}. \quad (34)
\end{align*}
\]

where in table 2 results are presented for SdS with $s = 2$.

For completeness in the appendix we derive a set of three-term recurrence relations for the continued fraction method, valid for electromagnetic and gravitational perturbations ($s = 1, 2$). It may be worth mentioning that via the AIM we can treat the $s = 0, 1, 2$
Table 2. QNMs to six significant figures for SdS gravitational perturbations ($s = 2$) for $\ell = 2$ and $\ell = 3$ modes. We only present results for the AIM method, because the results are identical to those of the continued fraction method after a given number of iterations (in this case 50 iterations for both methods). The $n = 1, 2$ modes can be compared with the results in [21] for $s = 2$.

| $\Lambda(\ell = 2)$ | $n = 1$ | $n = 2$ | $n = 3$ |
|---------------------|--------|--------|--------|
| 0       | 0.373 672 $-$ 0.088 9623i | 0.346 711 $-$ 0.273 915i | 0.301 050 $-$ 0.478 281i |
| 0.02    | 0.338 391 $-$ 0.081 7564i | 0.318 759 $-$ 0.249 197i | 0.282 732 $-$ 0.429 484i |
| 0.04    | 0.298 895 $-$ 0.073 2967i | 0.285 841 $-$ 0.221 724i | 0.259 992 $-$ 0.377 092i |
| 0.06    | 0.253 289 $-$ 0.063 0425i | 0.245 742 $-$ 0.189 791i | 0.230 076 $-$ 0.319 157i |
| 0.08    | 0.197 482 $-$ 0.049 8773i | 0.194 115 $-$ 0.149 787i | 0.187 120 $-$ 0.250 257i |
| 0.09    | 0.162 610 $-$ 0.041 3665i | 0.160 789 $-$ 0.124 152i | 0.157 042 $-$ 0.207 117i |
| 0.10    | 0.117 916 $-$ 0.030 2105i | 0.117 243 $-$ 0.090 6409i | 0.115 876 $-$ 0.151 102i |
| 0.11    | 0.037 2699 $-$ 0.009 615 65i | 0.037 2493 $-$ 0.028 8470i | 0.037 2081 $-$ 0.048 0784i |

$
\Lambda(\ell = 3) \quad n = 1 \quad n = 2 \quad n = 3$

| 0       | 0.599 443 $-$ 0.092 7030i | 0.582 644 $-$ 0.281 298i | 0.551 685 $-$ 0.479 093i |
| 0.02    | 0.543 115 $-$ 0.084 4957i | 0.530 744 $-$ 0.255 363i | 0.507 015 $-$ 0.432 059i |
| 0.04    | 0.480 058 $-$ 0.075 1464i | 0.471 658 $-$ 0.226 395i | 0.455 011 $-$ 0.380 773i |
| 0.06    | 0.407 175 $-$ 0.064 1396i | 0.402 171 $-$ 0.192 807i | 0.392 053 $-$ 0.322 769i |
| 0.08    | 0.317 805 $-$ 0.050 3821i | 0.315 495 $-$ 0.151 249i | 0.310 803 $-$ 0.252 450i |
| 0.09    | 0.261 843 $-$ 0.041 6439i | 0.260 572 $-$ 0.124 969i | 0.257 998 $-$ 0.208 412i |
| 0.10    | 0.189 994 $-$ 0.030 3145i | 0.189 517 $-$ 0.090 9507i | 0.188 555 $-$ 0.151 609i |
| 0.11    | 0.060 0915 $-$ 0.009 61888i | 0.060 0766 $-$ 0.028 8567i | 0.060 0469 $-$ 0.048 0945i |

perturbations on an equal footing (for the scalar case the continued fraction method reduces to a five-term recurrence relation, see the appendix).

4. Schwarzschild AdS

There are various approaches to finding QNMs for the SAdS case (an eloquent discussion is given in the appendix of [15]). One approach is that of Horowitz and Hubeny [16], which uses a series solution chosen to satisfy the SAdS QNM boundary conditions. This method can easily be applied to all perturbations ($s = 0, 1, 2$). The other approach is to use the Frobenius method of Leaver [2], but instead of developing a continued fraction, the series must satisfy a boundary condition, such as Dirichlet, at infinity [11].

The AIM does not seem easy to apply to metrics where there is an asymptotically anti-de Sitter background, because for general spin, $s$, the potential at infinity is a constant and hence would include a combination of ingoing and outgoing waves, leading to a sinusoidal dependence [17]. However, for the scalar spin-zero ($s = 0$) case, the potential actually blows up at infinity and is effectively a bound state problem. In this case the AIM can easily be applied as we show below.

Let us consider the scalar wave equation in SAdS spacetime, where $\Lambda = -3/R^2$, and $R$ is the AdS radius. The master equation takes the same form as for the gravitational case, equation (1), except that the potential becomes

$$V = \left(1 - \frac{2}{r} + r^2\right) \left(\frac{2}{r^2} + 2\right) = \frac{2(r - 1)(r^2 + r + 2)(r^3 + 1)}{r^4}.$$ (35)
Table 3. Comparison of the first few QNMs to six significant figures for SAdS scalar perturbations \((s = 0)\) for \(\ell = 0\) modes with \(r_+ = 1\). The second column corresponds to data [18] using the Horowitz and Hubeny (HH) method [16], while the third column is for the AIM using 70 iterations. Note the mismatch for real part of the \(n = 3\) mode in [18], see [*] below; we have confirmed this using the Mathematica notebook provided in [1].

| \(n\) | HH method | AIM |
|-----|------------|-----|
| 0   | 2.7982 − 2.6712i | 2.798 23 − 2.671 21i |
| 1   | 4.758 49 − 5.037 57i | 4.758 50 − 5.037 57i |
| 2   | 6.719 27 − 7.394 49i | 6.719 31 − 7.394 50i |
| 3   | 8.682 23[*] − 9.748 52i | 8.682 33 − 9.748 54i |
| 4   | 10.646 7 − 12.101 2i | 10.646 9 − 12.101 3i |
| 5   | 12.612 1 − 14.453 3i | 12.612 5 − 14.453 3i |
| 6   | 14.578 2 − 16.805 9i | 14.578 8 − 16.805 0i |
| 7   | 16.544 9 − 19.156 3i | 16.545 7 − 19.156 3i |

Here for simplicity we have taken the AdS radius \(R = 1\), the mass of the black hole \(M = 1\) and the angular momentum number \(l = 0\). Hence, the horizon radius \(r_+ = 1\). Thus, with this choice we can compare with the data in table 3.2 on p 37 of [18] (see table 3).

To implement the AIM we first look at the asymptotic behaviour of \(\psi\). One could obtain this by using the WKB approximation:

\[
\psi \sim \frac{1}{|\omega^2 - V|^{1/4}} e^{\pm i \sqrt{\frac{|\omega^2 - V|}{4 \omega^2}}} x ,
\]

(36)

As \(r \to r_+ = 1\), the potential \(V\) goes to zero. In addition,

\[
x = \int \frac{r \, dr}{(r - 1)(r^2 + r + 2)} \sim \frac{1}{4} \ln(r - 1) + \cdots
\]

(37)

\[
(\omega^2 - V)^{-1/4} \sim \frac{1}{\sqrt{\omega}} + \frac{4}{(\sqrt{\omega})^3} (r - 1) + \cdots
\]

(38)

\[
\int dr \sqrt{\omega^2 - V} \sim \int dr \left[ \frac{\omega}{4(r - 1)} + \cdots \right] \sim \frac{\omega}{4} \ln(r - 1) + \cdots
\]

(39)

\[
\psi \sim e^{i[\pm \frac{1}{2} \ln(r - 1) - i \omega x/4]} \sim (r - 1)^{\pm i \omega/4} \sim \left(1 - \frac{1}{r}\right)^{\pm i \omega/4} .
\]

(40)

For QNMs we choose the ‘outgoing’ (into the black hole) boundary condition, that is

\[
\psi \sim e^{-i \omega x} \sim \left(1 - \frac{1}{r}\right)^{-i \omega/4} .
\]

(41)

On the other extreme of our space, \(r \to \infty\), the potential goes to infinity. This is a crucial difference from the case of gravitational perturbations. In that case, the potential goes to a constant. However, in the scalar case, as \(r \to \infty\),

\[
x \sim -\frac{1}{r} + \cdots
\]

(42)

\[
(\omega^2 - V)^{-1/4} \sim \left(\frac{1}{r}\right)^{1/2} + \cdots
\]

(43)
\[
\int dr \sqrt{\frac{\omega^2 - V}{1 - \frac{2}{r} + r^2}} \sim -i\sqrt{2} \ln \left( \frac{1}{r} \right) + \ldots
\]

and to implement the Dirichlet boundary condition, we take

\[
\psi \sim \left( \frac{1}{r} \right)^{\frac{1}{2} + i\sqrt{2}}
\]

For the AIM one possible choice of variables is

\[
\xi = 1 - \frac{1}{r}
\]

and we see that to accommodate the asymptotic behaviour of the wavefunction we should take

\[
\psi = \xi^{-i\omega/4} \left( 1 - \xi \right)^{2^{1/4} + i \chi}.
\]

Finally, after some calculation we find the scalar perturbation equation to be

\[
\chi'' = \lambda_0(\xi) \chi' + s_0(\xi) \chi,
\]

where

\[
\lambda_0 = -\frac{1}{4 \xi^2} \left[ -4 + 2(9 + 4\sqrt{2})\xi - (21 + 10\sqrt{2})\xi^2 + 4(2 + \sqrt{2})\xi^3 \right],
\]

\[
s_0 = \frac{1}{16\xi^2} \left[ 4i\omega[9 + 8\sqrt{2} - 2(7 + 5\sqrt{2})\xi + (6 + 4\sqrt{2})\xi^2] + \omega^2(-1 + \xi)^2(-40 + 41\xi - 20\xi^2 + 4\xi^3) - 4(4 - 5\xi + 2\xi^2) \right. \]
\[
\left. + 8(10 + 7\sqrt{2})\xi - (91 + 64\sqrt{2})\xi^2 + (34 + 24\sqrt{2})\xi^3 \right],
\]

and \( q = (-4 + 9\xi - 7\xi^2 + 2\xi^3) \). Using the AIM we find the results presented in table 3 , which are discussed in the next section.

5. Results and discussion

We first applied the AIM to the Schwarzschild background defined in section 2, setting \( M = 1 \) and choosing \( \xi = \xi_0 \) (the black hole maximum), which typically leads to the fastest converging results. In table 1 the QNMs for graviton perturbations using the AIM are compared to some other methods. These results show that after 15 iterations the AIM is in good agreement with Leaver’s method to four significant figures, with the disagreement becoming most pronounced for the lowest \( \ell \) modes with larger values of \( n \). We also found that by increasing the number of iterations the AIM can be made to agree exactly with the results of Leaver’s continued fraction method [2] to any number of decimal places.

For completeness we have also included results from an approximate semi-analytic third-order WKB method [12] (more accurate semi-analytic results with better agreement to Leaver’s method can be obtained by extending the WKB method to sixth order [19]). In this regard, it may be worth mentioning that a different semi-analytic approach has recently been discussed by Dolan and Ottewill [20], which has the added benefit of easily being extended to any order in their perturbative scheme.

We also discussed spin \( s = 0, 1, 2 \) in SdS (see table 2) where results to six significant figures are presented for the graviton \( (s = 2) \) case. Identical results were generated by the
AIM and continued fraction method, both after 50 iterations. Though results are presented for \( n = 1, 2 \) and 3, \( \ell = 2 \) and 3 modes only, the AIM is robust enough to be applied to any other case, where like for the \( \Lambda = 0 \) case, agreement with other methods in more extreme parameter choices would only require further iterations. For comparison with the AIM we have also used the continued fraction method, and for completeness we presented a derivation in the appendix (for spin \( s = 1, 2 \)).

As far as we are aware, only [21] (who used a semi-analytic WKB approach) has presented tables for general spin fields for the SdS case. We have also compared our results to those in [21] for the \( s = 0, 1 \) cases and find identical results (to a given accuracy in the WKB method). As briefly discussed in the appendix another benefit of the AIM is that no Gaussian elimination procedure is required, unlike for continued fractions on Reissner–Nördstrom [22] or higher dimensional Schwarzschild backgrounds [23], for example. All that is necessary is to factor out the correct asymptotic behaviour at the horizon(s) and infinity (we showed this for higher dimensional angular spheroids in [14]). Of course, there is no need to find the maximum of the black hole potential for continued fractions, unlike with the AIM.

The reader might wonder about approximate results for cases where Poschl–Teller approximations can be used, such as for Schwarzschild and SdS backgrounds, e.g. see [7, 11]. In fact when the black hole potential can be modelled by a Scarf-like potential the AIM can be used to find the eigenvalues exactly [5] and hence the approximate QNMs. Indeed the authors of [5] also independently suggested the idea of using the AIM to find black hole QNMs from exact potentials numerically (based on previous comments by one of the present authors [6]). On a related note the large \( \ell \) eikonial limit (e.g. for \( \Lambda = 0 \) see [12]) is also easily confirmed numerically as we have verified using the continued fraction method.

The SAdS case is a little more subtle, and we briefly discussed some preliminary results for a spin-zero field (the \( l = 0, r_+ = 1 \) case) in table 3. In the scalar case the WKB form of the potential is essentially a bound state potential (it blows up at infinity) and allowed us to apply the AIM numerically. For other spins (\( s = 1, 2 \)) the potential goes to a constant at infinity and thus would be a combination of ingoing and outgoing waves. This implies a sinusoidal boundary condition at infinity (e.g. see [17]) which is somewhat out of the scope of this paper.

In summary, we have demonstrated how the AIM can also be applied to radial QNMs and not just to spheroidal eigenvalue problems [13, 14]. Given the fact that the AIM can be used in both the radial and angular wave equations [14] we expect no problems in obtaining QNMs for Kerr or Kerr–dS black holes for example. Furthermore, a perusal of the work on the AIM (e.g. see [3, 5, 13] and references therein) indicates that there are still some mathematical problems/questions of numerical refinement (such as the improved AIM we discussed), which was another reason for bringing this method to light.

As for future work, it remains to be seen if the AIM can be tailored to handle asymptotic QNMs (see [24] for an adapted version of the continued fraction method). However, given the close relationship between the AIM and the exact WKB approach [4] it might be possible to adapt the AIM to find asymptotic QNMs [25–28] numerically or even semi-analytically.

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\[^9\text{see footnote 7.}\]
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Appendix. Continued fraction method for SdS with \( s = 1, 2 \)

For completeness and for the purpose of making an accurate comparison with the AIM method, for the SdS case we present some the steps that lead to the continued fraction method. The actual recursion coefficients themselves do not commonly appear in the literature; however, see [29].

The correct scaling for QNM boundary conditions in SdS implies

\[
\psi(x) = e^{i\omega x} (\xi - \xi_1) - \frac{i}{\kappa} \chi(x),
\]

where we can expand \( \chi \) as a power series:

\[
\chi = \sum_{n=1}^{\infty} a_n y^n, \quad y = \frac{\xi - \xi_1}{\xi_2 - \xi_1} \Rightarrow \xi = (\xi_2 - \xi_1) y + \xi_1, \tag{A.1}
\]

where \( a_0 = 1 \). Note that the form of \( y \), above, depends on \((\xi_2 - \xi_1)\) in order to give a converging solution in the Frobenius solution. It then follows that if we define

\[
\varphi = \sum_{n=1}^{\infty} a_n y^{n-\frac{\omega}{\kappa}} \tag{A.2}
\]

then

\[
\psi(x) = e^{i\omega x} (\xi_2 - \xi_1) - \frac{i}{\kappa} \varphi(x), \tag{A.3}
\]

and hence the \((\xi_2 - \xi_1)\) term factors out of the master equation (25). The rescaled equation for \( \varphi \) then takes an identical form to equation (29):

\[
\xi^2 p \varphi'' + \xi^2 (p' - 2i\omega) \varphi' - \left[ \ell(\ell + 1)\xi^2 + (1 - s^2) \left( 2M\xi^3 - (4 - s^2)\frac{A}{\ell} \right) \right] \varphi = 0, \tag{A.4}
\]

where for general integer spin we multiplied both sides of the equation by \( \xi^2 \). We could then expand \( \varphi \) as a power series where

\[
\varphi' = \sum_{n=1}^{\infty} a_n \left( n - \frac{i\omega}{\kappa} \right) y^{n-\frac{\omega}{\kappa}} (\xi_2 - \xi_1)^{-1}, \tag{A.5}
\]

and

\[
\varphi' = \sum_{n=1}^{\infty} a_n \left( n - \frac{i\omega}{\kappa} \right) \left( n - \frac{i\omega}{\kappa} - 1 \right) y^{n-\frac{\omega}{\kappa} - 2}(\xi_2 - \xi_1)^{-2}, \tag{A.6}
\]

where we should also power expand \( \xi^2 p \) and \( \xi^2 p' \) in a power series about \( \xi_1 \) (for a detailed discussion of this point see [30]).

It is straightforward to see that this will lead to a five-term recurrence relation, because \( \xi^2 p \sim O(y^3) \), with \( p(\xi_1) = 0 \), for the \( u'' \) term. This would then require the use of two Gaussian eliminations [22] to reduce it to a three-term recurrence relation. Note, in the Reissner–Nordström–(A)dS case, even the axial graviton perturbations reduce to a four-term recurrence relation [15].

Thus, in order to obtain a set of three-term recurrence relations (which are simpler to deal with as no Gaussian eliminations are required) we consider only spin \( s = 1, 2 \). This drops the last constant term in equation (A.5) and allows us to divide out by a factor of \( \xi^2 \). Then, because \( p \sim O(y^3) \) for the \( u'' \) term, this leads to a three-term recurrence relation. After
substituting these expansions into equation (A.5) and equating coefficients we obtain a set of recurrence coefficients:
\[
\alpha_n = 2(\xi_2 - \xi_1)^{-1} \left( n - \frac{i\omega}{\kappa} + 1 \right) \left( n - \frac{i\omega}{\kappa} + 1 \right) \xi_1 (1 - 3M\xi_1) - i\omega \right],
\]
\[
\beta_n = -\ell(\ell + 1) - 2M(1 - s^2)\xi_1 + (1 - 6M\xi_1) \left( n - \frac{i\omega}{\kappa} \right) \left( n - \frac{i\omega}{\kappa} + 1 \right), \quad (A.7)
\]
\[
\gamma_n = -2M(\xi_2 - \xi_1) \left( n - \frac{i\omega}{\kappa} + s \right) \left( n - \frac{i\omega}{\kappa} - s \right)
\]
which satisfy the following recurrence relations:
\[
\alpha_0\alpha_1 + \beta_0\alpha_0 = 0 \quad (A.8)
\]
\[
\alpha_n\alpha_{n+1} + \beta_n\alpha_n + \gamma_n\alpha_{n-1} = 0. \quad (A.9)
\]

The exterior eigenvalue problem (QNMs) then leads to a continued fraction [2]:
\[
0 = \frac{\beta_0 - \alpha_0\gamma_1\alpha_1\gamma_2 \cdots}{\beta_1 - \beta_2 - \cdots} \quad (A.10)
\]
where this equation can be solved for numerically and allows us to check the results with the AIM, see footnote 7. In all cases we find perfect agreement between the two results (to a given precision) for roughly the same number of iterations in both the AIM and continued fraction. Note that in the recurrence coefficients above, the overall factors of \((\xi_1 - \xi_2)\) cancel out in the continued fraction.

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