SOLUTIONS FOR QUASILINEAR FOURTH ORDER ELLIPTIC EQUATIONS ON $\mathbb{R}^N$ WITH SIGN-CHANGING POTENTIAL

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ABSTRACT. We obtain existence and multiplicity results for quasilinear fourth order elliptic equations on $\mathbb{R}^N$ with sign-changing potential. Our results generalize some recent results on this problem.

1. INTRODUCTION

In this paper we consider the following fourth order quasilinear elliptic equations on $\mathbb{R}^N$,

$$
\begin{aligned}
\Delta^2 u - \Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) &= f(x,u) \quad \text{in } \mathbb{R}^N, \\
u &\in H^2(\mathbb{R}^N).
\end{aligned}
$$

(1.1)

Here $N \leq 6$, $V \in C(\mathbb{R}^N)$ is the potential and $f \in C(\mathbb{R}^N \times \mathbb{R})$ is the nonlinearity, $\Delta^2 = \Delta(\Delta)$ is the biharmonic operator.

Fourth order elliptic problems involving the biharmonic operator have been studied by many authors. Lazer-McKenna [15] pointed out that this type of problems furnish a model to study traveling waves in suspension bridges. For recent results about semilinear biharmonic equations on bounded domain $\Omega \subset \mathbb{R}^N$, we refer the readers to [25, 26, 29, 32].

To the best of our knowledge, the first result about semilinear biharmonic equations on $\mathbb{R}^N$ seems to be Chabrowski-do Ō [6]. In Yin-Wu [28], for the following fourth order equation

$$
\begin{aligned}
\Delta^2 u - \Delta u + V(x)u &= f(x,u) \quad \text{in } \mathbb{R}^N, \\
u &\in H^2(\mathbb{R}^N)
\end{aligned}
$$

(1.2)

with Laplacian and linear potential terms, the authors obtained a sequence of high energy solutions of (1.2) assuming that the potential $V$ has positive infimum and satisfies condition $(V_1)$ given below. The nonlinearity $f(x,u)$ there is odd and superlinear at infinity. Then, their results were improved by Ye-Tang [27], where the case that $f(x,u)$ is sublinear was also studied with the aid of a critical point theorem of Kajikiya [13]. See also Zhang et al. [30] for related results for sublinear problems and [31] for the asymptotically linear case.

As far as we know, the first work on fourth order quasilinear elliptic equations (1.1) on $\mathbb{R}^N$ is due to Chen et al. [10], where the positive potential $V$ satisfies condition $(V_1)$ below and the nonlinearity $f(x,u)$ is 4-superlinear in the sense of (1.8). The results in [10] were then extended by Cheng-Tang [11]. Another recent paper about the problem (1.1) is Che-Chen [8], where the nonlinearity $f(x,u)$ is sublinear so that the variational functional is coercive. By applying the three critical points theorem from Liu-Su [17] and the classical Clark theorem, multiple solutions are obtained.

It is interesting to note that unlike many works in this field, in [11] the potential $V$ is allowed to be sign-changing. To deal with this situation, the authors chose a constant $W_0 > 0$ such that

$$W(x) = V(x) + W_0 \geq 1$$

for all $x \in \mathbb{R}^N$. They then considered the equivalent problem with positive potential $W$:

$$
\begin{aligned}
\Delta^2 u - \Delta u + W(x)u - \frac{1}{2}u\Delta(u^2) &= g(x,u) \quad \text{in } \mathbb{R}^N, \\
u &\in H^2(\mathbb{R}^N).
\end{aligned}
$$

(1.3)

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Here among other requirements, the new nonlinearity \( g(x,t) = f(x,t) + W_0 t \) satisfies
\[
|g(x,t)| \leq C_1 |t|^3 + C_2 |t|^{p-1}, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}
\] (1.4)
for some \( C_1, C_2 > 0 \) and \( p \in (4,2_*) \); where \( 2_* \) is the critical Sobolev exponent for \( H^2(\mathbb{R}^N) \), namely \( 2_* = +\infty \) for \( N \leq 4 \) and \( 2_* = 2N/(N-4) \) for \( N > 4 \). Back to the original problem (1.1), we have
\[
\lim_{|t| \to 0} \frac{f(x,t)}{t} = \lim_{|t| \to 0} \frac{g(x,t) - W_0 t}{t} = -W_0 < 0
\]
uniformly in \( x \in \mathbb{R}^N \). Therefore, the results in Cheng-Tang [11] have not resolved the case that \( V \) is sign-changing and at the same time
\[
\lim_{|t| \to 0} \frac{f(x,t)}{t} = 0, \quad \text{uniformly in } x \in \mathbb{R}^N. \quad (1.5)
\]
Motivated by the above observation and our early works [9, 21] on Schrödinger-Poisson systems, the purpose of the present paper is to study this issue.

Let \( H^2 = H^2(\mathbb{R}^N) \) be the standard Sobolev space. If \( V \in C(\mathbb{R}^N) \) is bounded from below, we can choose a constant \( m > 0 \) such that \( \tilde{V}(x) = V(x) + m \geq 1 \) for \( x \in \mathbb{R}^N \). On the linear subspace
\[
X = \left\{ u \in H^2 \left| \int V(x) u^2 < \infty \right. \right\},
\]
where from now on all integrals are taken over \( \mathbb{R}^N \) except stated explicitly, we equip with the inner product
\[
(u,v) = \int (\Delta u \Delta v + \nabla u \cdot \nabla v + \tilde{V}(x) uv)
\]
and corresponding norm \( \| \cdot \| \). Then \((X, \| \cdot \|)\) is a Hilbert space that will be denoted by \( X \) for simplicity. Note that if \( V \in C(\mathbb{R}^N) \) is bounded, then \( X \) is precisely the standard Sobolev space \( H^2 \).

Now we are ready to state our assumptions on \( V \) and \( f \).

\( (V_1) \) \( V \in C(\mathbb{R}^N) \) is bounded from below, \( \mu(V^{-1}(-\infty,M)) < \infty \) for all \( M > 0 \), where \( \mu \) is the Lebesgue measure on \( \mathbb{R}^N \).

\( (V_2) \) \( V \in C(\mathbb{R}^N) \) is a bounded function such that the quadratic form \( \mathcal{B} : X \to \mathbb{R} \),
\[
\mathcal{B}(u) = \frac{1}{2} \int \left( |\Delta u|^2 + |\nabla u|^2 + V(x) u^2 \right)
\]
(1.6)
is non-degenerate and the negative space of \( \mathcal{B} \) is finite-dimensional.

\( (f_0) \) \( f \in C(\mathbb{R}^N \times \mathbb{R}) \) and there exist \( C > 0 \) and \( p \in (4,2_*) \) such that
\[
|f(x,t)| \leq C \left( |t| + |t|^{p-1} \right).
\]
(1.7)
\( (f_1) \) for \( (x,t) \in \mathbb{R}^N \times \mathbb{R} \) we have \( 0 \leq 4F(x,t) \leq tf(x,t) \), moreover, for almost all \( x \in \mathbb{R}^N \)
\[
\lim_{|t| \to \infty} \frac{F(x,t)}{t^4} = +\infty, \quad \text{where } F(x,t) = \int_0^t f(x,s) ds.
\]
(1.8)
\( (f_2) \) for any \( r > 0 \), we have
\[
\lim_{|t| \to \infty} \sup_{0 < |t| \leq r} \frac{|f(x,t)|}{t} = 0.
\]

If \( (V_1) \) holds, by a well-known compact embedding established by Bartsch-Wang [4], it can been shown as in Chen et.al. [10, Lemma 2.1] that for \( s \in [2,2_*) \), the embedding \( X \hookrightarrow L^s(\mathbb{R}^N) \) is compact.

The spectral theory of self-adjoint compact operators implies that the eigenvalue problem
\[
\Delta^2 u - \Delta u + V(x) u = \lambda u, \quad u \in X
\]
(1.9)
possesses a complete sequence of eigenvalues
\[
-\infty < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_k \to +\infty.
\]
Each \( \lambda_k \) has been repeated in the sequence according to its finite multiplicity. We denote by \( \phi_k \) the eigenfunction of \( \lambda_k \) with \( |\phi_k|_2 = 1 \), where \( |\cdot|_q \) is the \( L^q \)-norm.

Now we are ready to state the main results of this paper.
Theorem 1.1. Suppose \((V_1), (f_0)\) and \((f_1)\) are satisfied. If \((1.5)\) holds and 0 is not an eigenvalue of \((1.9)\), then \((1.1)\) has a nontrivial solution \(u \in X\).

Theorem 1.2. Suppose \((V_1), (f_0)\) and \((f_1)\) are satisfied. If \(f(x, \cdot)\) is odd for all \(x \in \mathbb{R}^N\), then \((1.1)\) has a sequence of solutions \(\{u_n\}\) such that \(\Phi(u_n) \to +\infty\).

Since 0 is not an eigenvalue of \((1.9)\), we may assume that 0 ∈ \((\lambda_\ell, \lambda_{\ell+1})\) for some \(\ell \geq 1\). Of course it is possible that 0 < \(\lambda_1\). In this case from the argument we presented below it is easy to see that the zero function \(u = 0\) is a local minimizer of \(\Phi\), then as in \([10, 11]\) the mountain pass theorem of Ambrosetti-Rabinowitz \([1]\) can be used to get a nonzero critical point of \(\Phi\), which is a nontrivial solution of \((1.1)\). Therefore we will omit this easy situation. Note that if 0 ∈ \((\lambda_\ell, \lambda_{\ell+1})\) for some \(\ell \geq 1\), the zero function is not local minimizer of \(\Phi\) anymore, this is the main difference between our Theorems 1.1, 1.2 and the results of \([10, 11]\).

For the case that \(V\) satisfies \((V_2)\), \(X\) is exactly the standard Sobolev space \(H^2\), we do not have the compact embedding \(X \hookrightarrow L^s(\mathbb{R}^N)\) for \(s \in [2, 2_s]\) any more. But we still have the following result.

Theorem 1.3. Suppose \((V_2), (f_0), (f_1)\) and \((f_2)\) are satisfied. If \((1.5)\) holds, then \((1.1)\) has a nontrivial solution \(u \in X\).

Remark 1.4. Note that in \((f_1)\), the limit \((1.8)\) is point-wise. Thus, if \(a : \mathbb{R}^N \to (0, \infty)\) is continuous and decay to zero at infinity, \(p \in (4, 2_s)\), then
\[f(x, t) = a(x)|t|^{p-2}t\]
satisfies all our assumptions on \(f\) in Theorem 1.3.

Under the assumptions \((V_1)\) or \((V_2)\), and \((f_0)\), similar to Chen et.al. \([10, \text{Lemma 2.2}]\) we can show that the functional \(\Phi : X \to \mathbb{R}\)
\[\Phi(u) = \mathfrak{B}(u) + \frac{1}{2} \int u^2 |\nabla u|^2 - \int F(x, u)\]
is well defined and is of class \(C^1\). The derivative of \(\Phi\) is given by
\[\langle \Phi'(u), v \rangle = \int (\Delta u \nabla v + \nabla u \cdot \nabla v + V(x) uv) + \int (u |\nabla u|^2 + u^2 \nabla u \cdot \nabla v) - \int f(x, u)v\]
for \(u, v \in X\). Consequently, critical points of \(\Phi\) are weak solutions of problem \((1.1)\).

To study the functional \(\Phi\), it will be convenient to rewrite the quadratic part \(\mathfrak{B}\) in a simpler form. It is well known that, if \((V_1)\) holds and 0 is not an eigenvalue of \((1.9)\), or if \((V_2)\) holds, then there exists an equivalent norm \(\| \cdot \|_V\) on \(X\) such that
\[\mathfrak{B}(u) = \frac{1}{2} \left( \| u^+ \|_V^2 - \| u^- \|_V^2 \right),\]
where \(u^\pm\) is the orthogonal projection of \(u\) on \(X^\pm\) being \(X^\pm\) the positive/negative space of \(\mathfrak{B}\). Using this new norm, \(\Phi\) can be rewritten as
\[\Phi(u) = \frac{1}{2} \left( \| u^+ \|_V^2 - \| u^- \|_V^2 \right) + \frac{1}{2} \int u^2 |\nabla u|^2 - \int F(x, u)\] (1.10)

The paper is organized as follows. In Section 2 we explain why the usual linking theorem is not applicable to our problem \((1.1)\). We will prove Theorem 1.1 by applying Morse theory. Therefore we will introduce some concepts and results of the theory, verify the assumptions required and then give the proof of Theorem 1.1. In Section 3 we will prove Theorem 1.2 via the symmetric mountain pass theorem. Finally, after investigating the weak continuity of the functional \(u \mapsto \int u^2 |\nabla u|^2\) on \(H^2\) and its derivative (see Lemma 4.1), we use Morse theory again to prove Theorem 1.3 in Section 4.

2. PROOF OF THEOREM 1.1

In this section and the next section, we always assume that \((V_1)\) holds. Consider the quadratic form \(\mathfrak{B}\) defined in (1.6). The negative space of \(\mathfrak{B}\) is given by
\[X^- = \text{span} \{\phi_1, \ldots, \phi_\ell\}.\]
Let $X^+$ be the orthogonal complement of $X^-$ in $X$, then $X = X^- \oplus X^+$. It is well known that there exists a constant $\eta > 0$ such that
\begin{equation}
\pm \mathfrak{B}(u) \geq \eta \|u\|^2, \quad \text{for } u \in X^\pm.
\end{equation}

To find critical points of the functionals with indefinite quadratic part, a natural idea is to apply the linking theorem. More precisely, set
\begin{align*}
N &= \{ u \in X^+ \mid \|u\| = \rho \}, \quad M = \{ u \in X^- \oplus \mathbb{R}^\phi \|u\| \leq R \},
\end{align*}
where $\phi \in X^+ \setminus 0$. If $\Phi$ satisfies the Palais-Smale $(PS)$ condition and for some $R > \rho > 0$,
\begin{equation}
b = \inf_{\mathcal{N}} \Phi > \max_{\partial M} \Phi,
\end{equation}
then the linking theorem [24, Theorem 5.3] gives rise to a nonzero critical point of $\Phi$. In applications, to verify (2.2) we usually need to show that $\Phi \leq 0$ on $X^-$. However, because the second integral $\int u^2 |\nabla u|^2$ in our functional $\Phi$ is positive for $u \neq 0$, it seems impossible to obtain $\Phi|_{X^-} \leq 0$ even if we assume $F(x, t) \geq 0$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. Therefore unlike many other indefinite problems (see e.g. [12, 14, 20]) the linking theorem is not applicable for our problem (1.1).

Fortunately, as in our previous works [9, 21] on Schrodinger-Poisson systems, we observe that our functional $\Phi$ has a local linking at the origin. Therefore, we can apply local linking theory (see [16, 19]) and infinite dimensional Morse theory (see, e.g., Chang [7] and Mawhin-Willem [22, Chapter 8]) to prove Theorem 1.1. We start by recalling some concepts and results about Morse theory and local linking.

Let $X$ be a Banach space, $\varphi : X \to \mathbb{R}$ be a $C^1$-functional, $u$ is an isolated critical point of $\varphi$ and $\varphi(u) = c$. Then
\begin{equation}
C_i(\varphi, u) := H_i(\varphi, \varphi_\varepsilon \setminus \{0\}), \quad i \in \mathbb{N} = \{0, 1, 2, \ldots\},
\end{equation}
is called the $i$-th critical group of $\varphi$ at $u$, where $\varphi_\varepsilon := \varphi^{-1}(-\infty, c]$ and $H_\varepsilon$ stands for the singular homology with coefficients in $\mathbb{Z}$.

If $\varphi$ satisfies the $(PS)$ condition and the critical values of $\varphi$ are bounded from below by $\alpha$, then following Bartsch-Lu [2], we define the $i$-th critical group of $\varphi$ at infinity by
\begin{equation}
C_i(\varphi, \infty) := H_i(X, \varphi_\alpha), \quad i \in \mathbb{N}.
\end{equation}
It is well known that the homology on the right hand side does not depend on the choice of $\alpha$.

**Proposition 2.1** ([2, Proposition 3.6]). If $\varphi \in C^1(X, \mathbb{R})$ satisfies $(PS)$ and $C_i(\varphi, 0) \neq C_i(\varphi, \infty)$ for some $\ell \in \mathbb{N}$, then $\varphi$ has a nonzero critical point.

**Proposition 2.2** ([18, Theorem 2.1]). Suppose $\varphi \in C^1(X, \mathbb{R})$ has a local linking at 0 with respect to the decomposition $X = Y \oplus Z$, i.e., for some $\varepsilon > 0$,
\begin{align*}
\varphi(u) &\leq 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\
\varphi(u) &> 0 \quad \text{for } u \in (Z \setminus \{0\}) \cap B_\varepsilon,
\end{align*}
where $B_\varepsilon = \{ u \in X \mid \|u\| \leq \varepsilon \}$. If $\ell = \dim Y < \infty$, then $C_i(\varphi, 0) \neq 0$.

Now, we can begin the investigation of our functional $\Phi$.

**Lemma 2.3.** If $(V_1)$ $(f_0)$ and (1.5) hold, 0 is not an eigenvalue of (1.9), then $\Phi$ has a local linking at 0 with respect to the decomposition $X = X^- \oplus X^+$.

**Proof.** As in the proof of [10, Lemma 2.2], we have for all $u \in X$,
\begin{equation}
\int u^2 |\nabla u|^2 \leq \left( \int |u|^6 \right)^{1/3} \left( \int |\nabla u|^3 \right)^{2/3} \leq |u|_6^2 \|u\|^2_{W^{1,3}} \leq S \|u\|^4,
\end{equation}
where we have used the continuity of the embeddings $X \hookrightarrow H^2 \hookrightarrow W^{1,3}$ and $X \hookrightarrow L^6$.

By $(f_0)$ and (1.5), for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
\begin{equation*}
|F(x, t)| \leq \varepsilon t^2 + C_\varepsilon |t|^p.
\end{equation*}
Therefore, since \( p > 4 \), using (2.3) we see that as \( ||u|| \to 0 \),

\[
\int u^2 |\nabla u|^2 = o(||u||^2), \quad \int F(x,u) = o(||u||^2).
\]

Hence as \( ||u|| \to 0 \),

\[
\Phi(u) = \mathfrak{B}(u) + \frac{1}{2} \int u^2 |\nabla u|^2 - \int F(x,u) = \mathfrak{B}(u) + o(||u||^2).
\]

From this estimate and (2.1), the conclusion of the lemma follows easily.

Set \( g(x,t) = f(x,t) + mt \), then by (f1),

\[
G(x,t) := \int_0^t g(x,s)ds = F(x,t) + \frac{m}{2} t^2 \leq \frac{t}{4} g(x,t) + \frac{\tilde{b}}{4} t^2,
\]

(2.4)

where \( \tilde{b} = b + m > 0 \). Note that from (f1) we have \( G(x,t) \geq 0 \) for \( (x,t) \in \mathbb{R}^N \times \mathbb{R} \) and

\[
\lim_{|t| \to \infty} \frac{G(x,t)}{t^4} = +\infty, \quad \text{for a.e.} \quad x \in \mathbb{R}^N.
\]

(2.5)

The functional \( \Phi \) can be rewritten as follows:

\[
\Phi(u) = \frac{1}{2} ||u||^2 + \frac{1}{2} \int u^2 |\nabla u|^2 - \int G(x,u),
\]

with derivative given by

\[
\langle \Phi'(u), v \rangle = (u,v) + \int (uv|\nabla u|^2 + u^2 \nabla u \cdot \nabla v) - \int g(x,u)v.
\]

To apply variational methods, it is important to study the convergence of asymptotically critical sequences. Based on the compact embedding \( X \to L^s \) for \( s \in [2,2_*) \), it has been shown in Cheng-Tang [11, Lemma 2.2] that if \( g \) satisfies (1.4) then any bounded (PS) sequence \( \{u_n\} \) of \( \Phi \) has a convergent subsequence. Under our assumptions, the (new) nonlinearity \( g \) only satisfies the weaker condition

\[
|g(x,t)| \leq C_1 |t| + C_2 |t|^{p-1}, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}.
\]

However, this is sufficient to get Eq. (2.9) in [11]. In fact, because up to a subsequence \( u_n \to u \) in \( L^2 \) and in \( L^p \),

\[
\left| \int (g(x,u_n) - g(x,u)) (u_n - u) \right| \leq (C_1 + C_2) \int \left( |u_n| + |u| + |u_n|^{p-1} + |u|^{p-1} \right) |u_n - u|
\]

\[
\leq C \left( |u_n|_2 + |u|_2 |u_n - u|_2 + \left( |u_n|^{p-1} + |u|^{p-1} \right) |u_n - u|_p \right)
\]

\[
\to 0, \quad \text{as} \quad n \to \infty.
\]

Therefore, under the assumptions of Theorem 1.1 (or Theorem 1.2), any (PS) sequence of our functional \( \Phi \) also has a convergent subsequence.

**Lemma 2.4.** If \( (V_1) \), \( (f_0) \) and \( (f_1) \) hold, then \( \Phi \) satisfies the (PS) condition.

**Proof.** Let \( \{u_n\} \subset X \) be a (PS) sequence of \( \Phi \), that is

\[
c := \sup_n |\Phi(u_n)| < \infty, \quad \Phi'(u_n) \to 0.
\]

(2.6)

By the above remark, it suffices to show that \( \{u_n\} \) is bounded. Suppose \( \{u_n\} \) is unbounded, we may assume \( ||u_n|| \to \infty \). Then using (2.6) and (2.4) we have

\[
4c + ||u_n|| \geq 4\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle
\]

\[
= ||u_n||^2 - \int (4G(x,u_n) - g(x,u_n)u_n)
\]

\[
\geq ||u_n||^2 - \tilde{b} \int u_n^2.
\]

(2.7)
Let \( v_n = \|u_n\|^{-1}u_n \). Up to a subsequence, by the compact embedding \( X \hookrightarrow L^2(\mathbb{R}^N) \) we see that

\[
v_n \rightharpoonup v \quad \text{in} \quad X, \quad v_n \rightarrow v \quad \text{in} \quad L^2(\mathbb{R}^N), \quad v_n(x) \rightharpoonup v(x) \quad \text{a.e.} \quad \mathbb{R}^N
\]

for some \( v \in X \). Multiplying by \( \|u_n\|^{-2} \) to both sides of (2.7) and letting \( n \rightarrow \infty \) yield

\[
\tilde{b} \int v^2 \geq 1.
\]

Therefore, \( v \neq 0 \).

For \( x \in \{ v \neq 0 \} \) we have \( |u_n(x)| \rightarrow +\infty \). Hence by (2.5),

\[
\frac{G(x, u_n(x))}{\|u_n\|^4} = \frac{G(x, u_n(x))}{u_n^4(x)} v_n^4(x) \rightarrow +\infty.
\]

Since \( \mu(\{ v \neq 0 \}) > 0 \), by Fatou’s lemma we deduce from (2.8) that

\[
\int \frac{G(x, u_n)}{\|u_n\|^4} \geq \int_{v \neq 0} \frac{G(x, u_n)}{\|u_n\|^4} \rightarrow +\infty.
\]

It follows from (2.3) and (2.6) that

\[
o(1) = \frac{\Phi(u_n)}{\|u_n\|^4} = \frac{1}{\|u_n\|^4} \left( \frac{1}{2} \frac{\|u_n\|^2}{v_n^2} + \frac{1}{2} \int u_n^2 \|
abla u_n\|^2 - \int G(x, u_n) \right)
\]

\[
\leq \frac{1}{2\|u_n\|^2} + \frac{S}{2} - \int \frac{G(x, u_n)}{\|u_n\|^4} \rightarrow -\infty,
\]

this is a contradiction. Therefore \{ \{u_n\} \} is bounded in \( X \).

To investigate \( C_\mathbf{s}(\Phi, \infty) \), using the idea of [21, Lemma 3.3] we will prove the following lemma. For this purpose we will use the equivalent norm \( \| \cdot \|_V \) on \( X \) and rewrite \( \Phi \) in the form given in (1.10).

Note that by (2.3), there exists a constant \( S_1 > 0 \) such that

\[
\int u^2 |\nabla u|^2 \leq S_1 \|u\|^4_V.
\]

**Lemma 2.5.** If \((V_1), (f_0)\) and \((f_1)\) hold, 0 is not an eigenvalue of (1.9), then there exists \( A > 0 \) such that if \( \Phi(u) \leq -A \), then

\[
\frac{d}{dt}_{|t=1} \Phi(tu) < 0.
\]

**Proof.** Otherwise, there exists a sequence \( \{ u_n \} \subset X \) such that \( \Phi(u_n) \leq -n \) but

\[
\langle \Phi'(u_n), u_n \rangle = \frac{d}{dt}_{|t=1} \Phi(tu_n) \geq 0.
\]

Consequently, \( \|u_n\|_V \rightarrow \infty \) and

\[
\|u_n^+\|^2_V - \|u_n^-\|^2_V \leq (\|u_n^+\|^2_V - \|u_n^-\|^2_V) + \int [f(x, u_n) - 4F(x, u_n)]
\]

\[
= 4\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \leq -4n.
\]

Let \( v_n = \|u_n\|^{-1}u_n \) and \( v_n^\pm \) be the orthogonal projection of \( v_n \) on \( X^\pm \). Then up to a subsequence \( v_n \rightharpoonup v^- \) for some \( v^- \in X^- \), because \( \dim X^- < \infty \).

If \( v^- \neq 0 \), then \( v_n \rightharpoonup v \) in \( X \) for some \( v \in X \setminus \{0\} \). By assumption \((f_1)\) we have

\[
\frac{f(x, t)t}{t^4} \geq \frac{4F(x, t)}{t^4} \rightarrow +\infty, \quad \text{as} \quad t \rightarrow \infty.
\]

Thus, similar to the proof of (2.9), we obtain

\[
\frac{1}{\|u_n\|^4_V} \int f(x, u_n)u_n \rightarrow +\infty.
\]

Now, using (2.10) we have a contradiction

\[
0 \leq \frac{\langle \Phi'(u_n), u_n \rangle}{\|u_n\|^4_V} = \frac{1}{\|u_n\|^4_V} \left( (\|u_n^+\|^2_V - \|u_n^-\|^2_V) + 2 \int u_n^2 |\nabla u_n|^2 - \int f(x, u_n)u_n \right)
\]
we know that we can does not depend on the compactness of the embedding \( X \hookrightarrow L^2 \). Therefore the conclusion of Lemma 2.5 remains valid if instead of \((V_1)\), \(V\) satisfies \((V_2)\).

**Remark 2.6.** We emphasize that the proof of Lemma 2.5 does not depend on the compactness of the embedding \( X \hookrightarrow L^2 \). Therefore the conclusion of Lemma 2.5 remains valid if instead of \((V_1)\), \(V\) satisfies \((V_2)\).

**Remark 2.7.** Let \( B \) be the unit ball in \( X \). Using (1.8) and (2.3), it is easy to see that for all \( u \in \partial B \),

\[
\Phi(tu) \to -\infty, \quad \text{as } t \to +\infty.
\]

Therefore, as in the proof of [21, Lemma 3.4], for \( A > 0 \) large enough, using Lemma 2.5 we can construct a deformation from \( X \setminus B \) to the level set \( \Phi^{-1}(-\infty,-A) \), and deduce

\[
C_i(\Phi,\infty) = H_i(X,\Phi^{-1}(-\infty,-A)) \equiv H_i(X,X \setminus B) = 0, \quad \text{for all } i \in \mathbb{N}.
\]

**Proof (Proof of Theorem 1.1).** We have shown that \( \Phi \) satisfies \((PS)\) and has a local linking at \( 0 \) with respect to the decomposition \( X = X^- \oplus X^+ \). Since \( \dim X^- = \ell \), Proposition 2.2 yields \( C_\ell(\Phi,0) \neq 0 \). From (2.13) we see that

\[
C_\ell(\Phi,0) \neq C_\ell(\Phi,\infty).
\]

Therefore by Proposition 2.1 we know that \( \Phi \) has a nonzero critical point \( u \), which is a nontrivial solution of the problem (1.1).

### 3. Proof of Theorem 1.2

To prove Theorem 1.2, we will apply the following symmetric mountain pass theorem due to Ambrosetti-Rabinowitz [1].

**Proposition 3.1 ([23, Theorem 9.12]).** Let \( X \) be an infinite dimensional Banach space, \( \varphi \in C^1(X,\mathbb{R}) \) be even, satisfies \((PS)\) condition and \( \varphi(0) = 0 \). If \( X = Y \oplus Z \) with \( \dim Y < \infty \), and \( \varphi \) satisfies

- \((I_1)\) there are constants \( \rho, \alpha > 0 \) such that \( \varphi|_{\partial B_{\rho} \cap Z} \geq \alpha \),
- \((I_2)\) for any finite dimensional subspace \( W \subset X \), there is an \( R = R(W) \) such that \( \varphi \leq 0 \) on \( W \setminus B_{R(W)} \),

then \( \varphi \) has a sequence of critical values \( c_j \to +\infty \).

**Lemma 3.2.** For \( i \in \mathbb{N} \), let \( Z_i = \text{span} \{ \phi_i, \phi_{i+1}, \ldots \} \) and set

\[
\beta_i = \sup_{u \in Z_i, \|u\| = 1} |u|_2.
\]

Then \( \beta_i \to 0 \) as \( i \to \infty \).

**Proof.** For \( u \in Z_i \) with \( \|u\| = 1 \), we have

\[
\int \left( |\Delta u|^2 + |\nabla u|^2 + V(x)u^2 \right) \geq \lambda_i \int u^2
\]

equivalently (note that \( \bar{V} = V + m \)),

\[
1 = |u|^2 = \int \left( |\Delta u|^2 + |\nabla u|^2 + \bar{V}(x)u^2 \right) \geq (\lambda_i + m) \int u^2 = (\lambda_i + m) |u|_2^2.
\]

Therefore, because \( \lambda_i \to +\infty \), we have

\[
|\beta_i| \leq \frac{1}{\sqrt{\lambda_i + m}} \to 0.
\]
Proof (Proof of Theorem 1.2). Under the assumptions of Theorem 1.2, the functional $\Phi$ is even and satisfies $(PS)$ condition. It suffices to show that $\Phi$ verifies the assumptions $(I_1)$ and $(I_2)$ of Proposition 3.1.

Verification of $(I_1)$. By $(f_0)$, there exist positive constants $C_1$ and $C_2$ such that
\begin{equation}
|G(x,t)| \leq C_1 |t|^2 + C_2 |t|^p . \tag{3.1}
\end{equation}
For $i \in \mathbb{N}$, set $Z_i$ and $\beta_i$ as in Lemma 3.2. Then we have $\beta_i \to 0$. Choose $k \in \mathbb{N}$ such that
\[ \lambda = \frac{1}{2} - C_1 \beta_k^2 > 0 , \]
then set
\[ Y = \text{span} \{ \phi_1, \ldots, \phi_{k-1} \} , \quad Z = \text{span} \{ \phi_k, \phi_{k+1}, \ldots \} . \]
We have $X = Y \oplus Z$.

For $u \in Z = Z_k$, using (3.1) and note that $p > 4$, we have
\begin{align*}
\Phi(u) &= \frac{1}{2} \| u \|^2 + \frac{1}{2} \int u^2 |\nabla u|^2 - \int G(x,u) \\
&\geq \frac{1}{2} \| u \|^2 - \int G(x,u) \geq \frac{1}{2} \| u \|^2 - C_1 \| u \|^2 - C_2 |u|^p \\
&\geq \left( \frac{1}{2} - C_1 \beta_k^2 \right) \| u \|^2 - C_2 \| u \|^p \\
&= \lambda \| u \|^2 + o(\| u \|^2) ,
\end{align*}
as $\| u \| \to 0$. From this estimate it is easy to see that $(I_1)$ is valid.

Verification of $(I_2)$. We only need to show that $\Phi$ is anti-coercive on any finite dimensional subspace $W$. If this is not true, there exists $\{ u_n \} \subset W$ and $A > 0$ such that $\| u_n \| \to \infty$ but $\Phi(u_n) \geq -A$. Let $v_n = \| u_n \|^{-1} u_n$, then up to a subsequence $v_n \rightharpoonup v$ for some $v \in W \setminus \{ 0 \}$, because $\text{dim} W < \infty$. Similar to (2.9) we deduce that
\[ \frac{1}{\| u_n \|^4} \int G(x,u_n) \to +\infty . \]
Thus using (2.3) we have
\begin{align*}
\Phi(u_n) &= \frac{1}{2} \| u_n \|^2 + \frac{1}{2} \int u_n^2 |\nabla u_n|^2 - \int G(x,u_n) \\
&\leq \| u_n \|^4 \left( \frac{1}{2 \| u_n \|^2} + S - \frac{1}{\| u_n \|^4} \int G(x,u_n) \right) \to -\infty ,
\end{align*}
a contradiction to $\Phi(u_n) \geq -A$.

Remark 3.3. In Cheng-Tang [11, Theorem 1.3], the authors essentially studied the problem (1.3) with positive potential $W$ and nonlinearity $g(x,u)$ satisfying (1.4). While in our Theorem 1.2, the potential $V$ can be indefinite and the nonlinearity $f(x,u)$ only need to satisfy the much weaker growth condition (1.7).

4. PROOF OF THEOREM 1.3

We now assume that $V$ satisfies $(V_2)$, then $X$ is exactly the standard Sobolev space $H^2$, the embedding $X \hookrightarrow L^2$ is not compact anymore. Therefore we need to recover the $(PS)$ condition with the help of condition $(f_3)$. Firstly, we need to investigate the $C^1$-functional $\mathcal{N} : H^2 \to \mathbb{R}$,
\[ \mathcal{N}(u) = \frac{1}{2} \int u^2 |\nabla u|^2 . \]
It is known that the derivative of $\mathcal{N}$ is given by
\[ \langle \mathcal{N}'(u), v \rangle = \int \left( uv |\nabla u|^2 + u^2 \nabla u \cdot \nabla v \right) , \quad u,v \in H^2 . \]

Lemma 4.1. The functional $\mathcal{N}$ is weakly lower semi-continuous, its derivative $\mathcal{N}' : H^2 \to H^{-2}$ is weakly sequentially continuous.
Proof. Let \( \{u_n\} \) be a sequence in \( H^2(\mathbb{R}^N) \) such that \( u_n \rightharpoonup u \) in \( H^2 \), we need to show

\[
\mathcal{N}(u) \leq \limsup \mathcal{N}(u_n), \quad \langle \mathcal{N}'(u_n), \phi \rangle \to \langle \mathcal{N}'(u), \phi \rangle.
\]

for all \( \phi \in H^2 \).

Since \( u_n \rightharpoonup u \) in \( H^2 \), by the compactness of the embedding \( H^2 \hookrightarrow H^1_{\text{loc}} \), we have \( u_n \to u \) in \( H^1_{\text{loc}} \).

Therefore up to a subsequence,

\[
\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N, \quad u_n \to u \quad \text{a.e. in } \mathbb{R}^N. \tag{4.1}
\]

By Fatou lemma,

\[
\mathcal{N}(u) = \int |\nabla u|^2 u^2 \leq \liminf \int |\nabla u_n|^2 u_n^2 = \liminf \mathcal{N}(u).
\]

Hence \( \mathcal{N} \) is weakly lower semi-continuous.

The Hölder conjugate number of \( 2_s \) is

\[
(2_s)' = \frac{2_s}{2_s - 1} = \frac{2N}{N + 4}.
\]

Since \( N \leq 6 \), we have

\[
r = \frac{N + 4}{2N - 4} > 1, \quad r' = \frac{r}{r - 1} = \frac{N + 4}{8 - N}, \quad \frac{2N}{N + 4}r' = \frac{2N}{8 - N} \leq 2_s.
\]

For \( u \in H^2 \), since \( H^2 \hookrightarrow W^{1,2_s^*} \) and \( H^2 \hookrightarrow L^{2N/(8-N)} \) continuously, we have

\[
\int |\nabla u|^2 u^{(2_s)'2} = \int |\nabla u|^2 u^{2N/(N+4)} = \int |\nabla u|^{4N/(N+4)} |u|^{2N/(N+4)}
\]

\[
\leq \left( \int |\nabla u|^{4N/(N+4)} \right)^r \left( \int |u|^{2N/(N+4)} \right)^{r'} \left( \int |u|^{2N/(8-N)} \right)^{8-N/(N+4)}
\]

\[
= \left( \int |\nabla u|^{2N/(N+4)} \right)^{2N/(8-N)} \left( \int |u|^{2N/(8-N)} \right).
\]

Since \( \{u_n\} \) is bounded in \( H^2 \), from (4.2) we see that \( \{ |\nabla u_n|^2 u_n \} \) is bounded in \( L^{2_s'} \). From (4.1) we have \( |\nabla u_n|^2 u_n \rightharpoonup |\nabla u|^2 u \) a.e. in \( \mathbb{R}^N \). The Brézis-Lieb lemma [5] implies that \( |\nabla u_n|^2 u_n \to |\nabla u|^2 u \) in \( L^{(2_s)'} \). Consequently, for \( \phi \in H^2 \), we have \( \phi \in L^{2_s} \) and

\[
\int |\nabla u_n|^2 u_n \phi \to \int |\nabla u|^2 u \phi. \tag{4.3}
\]

Similarly, because \( N \leq 6 \) implies \( N \leq 2_s \), for \( u \in H^2 \) we have

\[
\int |u^2 \nabla u|^{(2_s)'} = \int |u^2 \nabla u|^{2N/(N+2)} = \int |\nabla u|^{2N/(N+2)} |u|^{4N/(N+2)}
\]

\[
\leq \left( \int \left( |\nabla u|^{2N/(N+2)} \right)^s \right)^{1/s} \left( \int \left( |u|^{4N/(N+2)} \right)^{s'} \right)^{1/s'}
\]

\[
= \left( \int |\nabla u|^{2N/(N+2)} \right)^{(N-2)/(N+2)} \left( \int |u|^{4/(N+2)} \right)^{4/(N+2)}
\]

\[
= |\nabla u|^{2N/(N+2)} |u|^{4N/(N+2)}
\]

\[
\leq C \|u\|^{2N/(N+2)} \|u\|^{4N/(N+2)},
\]

where

\[
s = \frac{N + 2}{N - 2}, \quad s' = \frac{s}{s - 1} = \frac{N + 2}{4}.
\]
Therefore, for all $i$, the sequence $\{u_n^2\partial_i u_n\}$ is bounded in $L^{(2^*)'}$ and converges point-wise to $u^2 \partial_i u$.

Applying the Brézis-Lieb lemma [5] again we deduce

$$u_n^2 \nabla u_n \rightharpoonup u^2 \nabla u,$$

in $[L^{(2^*)'}]^N$.

For $\phi \in H^2$ (which implies $\phi \in W^{1,2^*}$ and $\partial_i \phi \in L^{2^*}$) we have

$$\int u_n^2 \nabla u_n \cdot \nabla \phi \to \int u^2 \nabla u \cdot \nabla \phi.$$  \hfill (4.4)

From (4.3) and (4.4), for $\phi \in H^2$ we have

$$\langle \mathcal{N}'(u_n), \phi \rangle = \int |\nabla u_n|^2 u_n \phi + \int u_n^2 \nabla u_n \cdot \nabla \phi$$

$$\to \int |\nabla u|^2 u \phi + \int u^2 \nabla u \cdot \nabla \phi = \langle \mathcal{N}'(u), \phi \rangle.$$  \hfill (4.5)

Therefore, we have proved that $\mathcal{N}'$ is weakly sequentially continuous.

As a consequence of Lemma 4.1, if $u_n \rightharpoonup u$ in $X = H^2(\mathbb{R}^N)$, then

$$\lim \int \left( |\nabla u_n|^2 u_n (u_n - u) + u_n^2 \nabla u_n \cdot \nabla (u_n - u) \right) = \lim \left( 4 \mathcal{N}(u_n) - \langle \mathcal{N}'(u_n), u \rangle \right)$$

$$\geq 4 \mathcal{N}(u) - \langle \mathcal{N}'(u), u \rangle = 0.$$  \hfill (4.5)

Having established (4.5), we can follow the idea of [21] to prove that $\Phi$ satisfies the (PS) condition.

**Lemma 4.2.** Under the assumptions of Theorem 1.3, the functional $\Phi$ satisfies the (PS) condition.

**Proof.** Let $\{u_n\} \subset X$ be a (PS) sequence of $\Phi$. We claim that $\{u_n\}$ is bounded. Otherwise, up to a subsequence we may assume $\|u_n\|_V \to \infty$. Let $v_n = \|u_n\|_V^{-1} u_n$, then

$$v_n = v_n^- + v_n^+ \rightharpoonup v = v^- + v^+ \in X, \quad v_n^\pm, v^\pm \in X^\pm.$$  

If $v = 0$, then $v_n^- \to v^- = 0$ because $\dim X^- < \infty$. Since

$$\|v_n^+\|_V^2 + \|v_n^-\|_V^2 = 1,$$  

for $n$ large enough we have

$$\|v_n^+\|_V^2 - \|v_n^-\|_V^2 \geq \frac{1}{2}.$$  

Therefore, by assumption (f2) we deduce

$$1 + \sup_n |\Phi(u_n)| + \|u_n\|_V \geq \Phi(u_n) - \frac{1}{4} \langle \Phi'(u_n), u_n \rangle$$

$$= \frac{1}{4} \|u_n\|_V \left( \|v_n^+\|_V^2 - \|v_n^-\|_V^2 \right) + \int \left( \frac{1}{4} f(x, u_n) u_n - F(x, u_n) \right)$$

$$\geq \frac{1}{8} \|u_n\|_V^2,$$

contradicting $\|u_n\|_V \to \infty$.

If $v \neq 0$, using $F(x, t) \geq 0$ and (1.8), similar to (2.9), we have

$$\frac{1}{\|u_n\|_V^2} \int F(x, u_n) \to +\infty.$$  

This will lead to a contradiction as well because by (2.10), for $n$ large

$$\frac{1}{\|u_n\|_V^2} \int F(x, u_n) = \frac{\|u_n^+\|_V^2 - \|u_n^-\|_V^2}{2 \|u_n\|_V^4} + \frac{1}{2 \|u_n\|_V^2} \int u_n^2 |\nabla u_n|^2 - \frac{\Phi(u_n)}{\|u_n\|_V^4}$$

$$\leq \frac{S_1}{2} + 1.$$  

Therefore, $\{u_n\}$ is bounded in $X$. 

Now, up to a subsequence we have \( u_n \to u \) in \( X \). Hence
\[
\int (\Delta u_n \Delta u + \nabla u_n \cdot \nabla u + V(x)u_n u) \to \int \left( (\Delta u)^2 + |\nabla u|^2 + V(x)u^2 \right) = \|u^+\|_V^2 - \|u^-\|_V^2.
\]
Because \( \text{dim} X^- < \infty \), we have \( u_n^- \to u^- \) and \( \|u_n^-\|_V \to \|u^-\|_V \). Consequently
\[
o(1) = (\Phi'(u_n), u_n - u)
= \int (\Delta u_n \Delta (u_n - u) + \nabla u_n \cdot \nabla (u_n - u) + V(x)u_n(u_n - u))
+ \int \left( |\nabla u_n|^2 u_n(u_n - u) + u_n^2 \nabla u_n \cdot \nabla (u_n - u) \right) - \int f(x, u_n)(u_n - u)
= (\|u_n^+\|_V^2 - \|u_n^-\|_V^2) - (\|u^+\|_V^2 - \|u^-\|_V^2) - \int f(x, u_n)(u_n - u)
+ \int \left( |\nabla u_n|^2 u_n(u_n - u) + u_n^2 \nabla u_n \cdot \nabla (u_n - u) \right)
= \|u_n^+\|_V^2 - \|u^+\|_V^2 - \int f(x, u_n)(u_n - u)
+ \int \left( |\nabla u_n|^2 u_n(u_n - u) + u_n^2 \nabla u_n \cdot \nabla (u_n - u) \right) + o(1).
\]
(4.6)

Using the condition \((f_2)\), similar to [3, p. 29] we can prove
\[
\lim \int f(x, u_n)(u_n - u) \leq 0.
\]
We deduce from (4.5) and (4.6) that
\[
\lim (\|u_n^+\|_V^2 - \|u^+\|_V^2)
= \lim \left( \int f(x, u_n)(u_n - u) - \int \left( |\nabla u_n|^2 u_n(u_n - u) + u_n^2 \nabla u_n \cdot \nabla (u_n - u) \right) \right)
= \lim \int f(x, u_n)(u_n - u) - \lim \left( \int |\nabla u_n|^2 u_n(u_n - u) + u_n^2 \nabla u_n \cdot \nabla (u_n - u) \right) \leq 0.
\]
Therefore \( \|u_n^+\|_V \to \|u^+\|_V \) because
\[
\|u^+\|_V \leq \lim \|u_n^+\|_V \leq \lim \|u_n^+\|_V \leq \|u^+\|_V.
\]
We conclude that \( \|u_n\|_V \to \|u\|_V \). Hence from \( u_n \to u \) in \( X \) we deduce \( u_n \to u \) in \( X \).

As we have pointed out in Remark 2.6, the validity of Lemma 2.5 does not rely on the compactness of the embedding \( X \hookrightarrow L^2 \). Therefore, using the same proof we deduce that under the assumptions of Theorem 1.3, there exists \( A > 0 \) such that if \( \Phi(u) \leq -A \), then
\[
\frac{d}{dt}_{t=1} \Phi(tu) < 0.
\]
Applying Remark 2.7, we deduce \( C_i(\Phi, \infty) = 0 \) for all \( i \in \mathbb{N} \). On the other hand, similar to the proof of Lemma 2.3 we can show that \( \Phi \) has a local linking at 0 with respect to the decomposition \( X = X^- \oplus X^+ \), thus for \( l = \text{dim} X^- \) we have \( C_i(\Phi, 0) \neq 0 \). That is
\[
C_i(\Phi, 0) \neq C_i(\Phi, \infty).
\]
By Proposition 2.1, \( \Phi \) has a nonzero critical point. This completes the proof of Theorem 1.3.

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