On stability and instability of standing waves for the nonlinear Schrödinger equation with inverse-square potential
Abdelwahab Bensouilah, Van Duong Dinh, Shihui Zhu

To cite this version:
Abdelwahab Bensouilah, Van Duong Dinh, Shihui Zhu. On stability and instability of standing waves for the nonlinear Schrödinger equation with inverse-square potential. Journal of Mathematical Physics, American Institute of Physics (AIP), 2018, 59 (101505).  hal-01936034
ON STABILITY AND INSTABILITY OF STANDING WAVES FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH INVERSE-SQUARE POTENTIAL

ABDELWAHAB BENSOUILAH, VAN DUONG DINH, AND SHIHUI ZHU

ABSTRACT. We consider the focusing nonlinear Schrödinger equation with inverse square potential
\[ i\partial_t u + \Delta u + c|x|^{-2}u = -|u|^\alpha u, \quad u(0) = u_0 \in H^1, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \]
where \( d \geq 3, c \neq 0, c < \lambda(d) = \left(\frac{d-2}{2}\right)^2 \) and \( 0 < \alpha \leq \frac{4}{d} \). Using the profile decomposition obtained recently by the first author [1], we show that in the \( L^2 \)-subcritical case, i.e. \( 0 < \alpha < \frac{4}{d} \), the sets of ground state standing waves are orbitally stable. In the \( L^2 \)-critical case, i.e. \( \alpha = \frac{4}{d} \), we show that ground state standing waves are strongly unstable by blow-up.

1. Introduction

Consider the focusing nonlinear Schrödinger equation with inverse-square potential
\[
\begin{aligned}
&i\partial_t u + \Delta u + c|x|^{-2}u = -|u|^\alpha u, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
&u(0) = u_0 \in H^1,
\end{aligned}
\tag{1.1}
\]
where \( d \geq 3, u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{C}, u_0 : \mathbb{R}^d \to \mathbb{C}, c \neq 0 \) satisfies \( c < \lambda(d) := \left(\frac{d-2}{2}\right)^2 \) and \( 0 < \alpha \leq \frac{4}{d} \).

The Schrödinger equation (1.1) appears in a variety of physical settings, such as quantum field equations or black hole solutions of the Einstein’s equations [8, 7, 16]. The mathematical interest in the nonlinear Schrödinger equation with inverse-square potential comes from the fact that the potential is homogeneous of degree \(-2\) and thus scales exactly the same as the Laplacian.

Let \( P^0_c \) denote the natural action of \(-\Delta - c|x|^{-2}\) on \( C^\infty_0(\mathbb{R}^d\setminus\{0\})\). When \( c \leq \lambda(d) \), the operator \( P^0_c \) is a positive semi-definite symmetric operator. Indeed, we have the following identity
\[
\langle \varphi, P^0_c \varphi \rangle = \int |\nabla \varphi(x)|^2 - c|x|^{-2}|\varphi(x)|^2 dx = \int |\nabla \varphi(x) + \rho x|x|^{-2}\varphi(x)|^2 dx \geq 0,
\]
for all \( \varphi \in C^\infty_0(\mathbb{R}^d\setminus\{0\}) \), where
\[
\rho := \frac{d-2}{2} - \sqrt{\left(\frac{d-2}{2}\right)^2 - c}.
\]
Denote \( P_c \) the self-adjoint extension of \( P^0_c \). It is known (see [16]) that in the range \( \lambda(d) - 1 < c < \lambda(d) \), the extension is not unique. In this case, we do make a choice among possible extensions such as Friedrichs extension. Note also that the constant \( \lambda(d) \) is the sharp constant appearing in Hardy’s inequality
\[
\lambda(d) \int |x|^{-2}|u(x)|^2 dx \leq \int |\nabla u(x)|^2 dx, \quad \forall u \in H^1.
\tag{1.2}
\]

Key words and phrases. Nonlinear Schrödinger equation, inverse-square potential, Standing waves, Stability, Instability.
Throughout this paper, we denote the Hardy functional
\[ \|u\|_{H^1_c}^2 := \|\sqrt{P_c}u\|_{L^2}^2 = \int |\nabla u(x)|^2 - c|x|^{-2}|u(x)|^2dx, \]
and define the homogeneous Sobolev space \( \dot{H}^1_c \) as the completion of \( C_0^\infty(\mathbb{R}^d\setminus\{0\}) \) under the norm \( \| \cdot \|_{\dot{H}^1_c} \). It follows from (1.2) that for \( c < \lambda(d) \),
\[ \|u\|_{\dot{H}^1_c} \sim \|u\|_{\dot{H}^1}. \] (1.4)

This is an assertion of the isomorphism between the homogeneous space \( \dot{H}^1_c \) defined in terms of \( P_c \) and the usual homogeneous space \( \dot{H}^1 \). We refer the interested reader to [18] for the sharp range of parameters for which such an equivalence holds.

The local well-posedness for (1.1) was established in [22]. More precisely, we have the following result.

**Theorem 1.1** (Local well-posedness [22]). Let \( d \geq 3 \) and \( c \neq 0 \) be such that \( c < \lambda(d) \). Then for any \( u_0 \in H^1 \), there exists \( T \in (0, +\infty) \) and a maximal solution \( u \in C([0,T), \dot{H}^1) \) of (1.1). The maximal time of existence satisfies either \( T = +\infty \) or \( T < +\infty \) and
\[ \lim_{t \uparrow T} \|\nabla u(t)\|_{L^2} = \infty. \]

Moreover, the solution enjoys the conservation of mass and energy, i.e.
\[ M(u(t)) = \int |u(t,x)|^2dx = M(u_0), \]
\[ E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2dx - \frac{c}{2} \int |x|^{-2}|u(t,x)|^2dx - \frac{1}{\alpha + 2} \int |u(t,x)|^{\alpha+2}dx, \]
for any \( t \in [0,T) \). Finally, if \( 0 < \alpha < \frac{4}{d} \), then \( T = +\infty \), i.e. the solution exists globally in time.

We refer the reader to [22, Theorem 5.1] for the proof of this result. Note that the existence of solutions is based on a refined energy method and the uniqueness follows from Strichartz estimates. Note also that Strichartz estimates for the linear NLS with inverse-square potential were first established in [5] except the endpoint case \( (2, 2d/(d-2)) \). Recently, Bouchet-Mizutani [4] proved Strichartz estimates with the full set of admissible pairs for the linear NLS with critical potentials including the inverse-square potential. The local well-posedness for (1.1) can be proved using Strichartz estimates and the equivalence between Sobolev spaces defined by \( P_c \) and the usual ones via the Kato method. However, due to the appearance of inverse-square potential, the local well-posedness proved by Strichartz estimates requires a restriction on the validity of \( c \) and \( d \) (see e.g. [26, 17, 19, 20]).

The main purpose of this paper is to study the stability and instability of standing waves for (1.1). In fact, the stability of standing waves for the nonlinear Schrödinger equations is widely pursued by physicists and mathematicians (see [28], for a review). For the classical nonlinear Schrödinger equation, Cazenave and Lions [3] were the first to prove the orbital stability of standing waves via the concentration-compactness principle. Then, a lot of results on the orbital stability were obtained. For the nonlinear Schrödinger equation with a harmonic potential, Zhang [27] succeed in obtaining the orbital stability by the weighted compactness lemma. Recently, the stability phenomenon was proved for the fractional nonlinear Schrödinger equation by establishing the profile decomposition for bounded sequences in \( H^s \) (see [12, 23, 31]).

The first part of this paper concerns the stability of standing waves in the \( L^2 \)-subcritical case \( 0 < \alpha < \frac{4}{d} \). Before stating our stability result, let us introduce some notations. For \( M > 0 \), we consider the following variational problems
• for $0 < c < \lambda(d)$,
\[ d_M := \inf \{ E(v) : v \in H^1, \| v \|_{L^2}^2 = M \}; \]  
(1.5)

• for $c < 0$,
\[ d_{M, \text{rad}} := \inf \{ E(v) : v \in H^1_{\text{rad}}, \| v \|_{L^2}^2 = M \}, \]  
(1.6)

where $H^1_{\text{rad}}$ is the space of radial $H^1$-functions. Note that in the case $c < 0$, we are only interested in radial data. This is related to the fact that the sharp Gagliardo-Nirenberg inequality for non-radial data (see Section 2) is never attained when $c < 0$. We will see later (Proposition 3.1) that the above variational problems are well-defined. Moreover, the above infimums are attained. Let us denote

• for $0 < c < \lambda(d)$,
\[ S_M := \{ v \in H^1 : v \text{ is a minimizer of (1.5)} \}; \]

• for $c < 0$,
\[ S_{M, \text{rad}} := \{ v \in H^1_{\text{rad}} : v \text{ is a minimizer of (1.6)} \}. \]

By the Euler-Lagrange theorem (see Appendix), we see that if $v \in S_M$, then there exists $\omega > 0$ such that
\[ -\Delta v - c|x|^{-2}v + \omega v = |v|^\alpha v. \]  
(1.7)

Note also that if $v$ is a solution to (1.7), then $u(t, x) := e^{i\omega t}v(x)$ is a solution to (1.1). One usually calls $e^{i\omega t}v$ the orbit of $v$. Moreover, if $v \in S_M$, i.e. $v$ is a minimizer of (1.5), then $e^{i\omega t}v$ is also a minimizer of (1.5) or $e^{i\omega t}v \in S_M$. A similar remark goes for $v \in S_{M, \text{rad}}$.

We next define the following notion of orbital stability which is similar to the one in [24].

**Definition 1.2.** The set $S_M$ is said to be **orbitally stable** if, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any initial data $u_0$ satisfying
\[ \inf_{v \in S_M} \| u_0 - v \|_{H^1} < \delta, \]
the corresponding solution $u$ to (1.1) satisfies
\[ \inf_{v \in S_M} \| u(t) - v \|_{H^1} < \epsilon, \]
for all $t \geq 0$. A similar definition applies for $S_{M, \text{rad}}$.

Our first result is the following orbital stability of standing waves for the $L^2$-subcritical (1.1).

**Theorem 1.3 (Orbital stability).** Let $d \geq 3$, $0 < \alpha < \frac{4}{d}$ and $M > 0$.

1. If $0 < c < \lambda(d)$, then $S_M$ is orbitally stable.
2. If $c < 0$, then $S_{M, \text{rad}}$ is orbitally stable.

Let us mention that the stability of standing waves in the case $0 < c < \lambda(d)$ was studied in [24]. However, they only considered radial standing waves in this case. Here, we remove the radial symmetry assumption and prove the stability for non-radial standing waves in the case $0 < c < \lambda(d)$. Moreover, our approach is based on the profile decomposition which is of particular interest. We also study the stability of radial standing waves in the case $c < 0$, which to our knowledge is new.

The proof of our stability result is based on the profile decomposition related to (1.1). Note that this type of profile decomposition was recently established by the first author in [1]. The main difficulty is the lack of space translation invariance due to the inverse-square potential. A careful analysis is thus needed to overcome the difficulty. We refer the reader to Section 3 for more details.
The second part of this paper is devoted to the strong instability result in the $L^2$-critical case $\alpha = \frac{3}{2}$. There are two main difficulties in studying this problem. The first difficulty is the lack of regularity of solutions to the elliptic equation

$$-\Delta Q - c|x|^{-2}Q + Q = |Q|^\frac{4}{3}Q.$$  

(1.8)

More precisely, we do not know whether $Q \in L^2(|x|^2dx)$ for any solution $Q$ of (1.8). This is a strong contrast with the classical NLS ($c = 0$) where solutions to (1.8) are known to have an exponential decay at infinity. Another difficulty is that the uniqueness (up to symmetries) of positive radial solutions to (1.8) is not yet known. To overcome these difficulties, we need to define properly the notion of ground states. To do this, we follow the idea of Csobo-Genoud [10] and define the set of ground states $G$ and the set of radial ground states $G_{rad}$ (see Section 4 for more details). Using this notion of ground states, we are able to show that any ground state $Q$ satisfies $Q \in L^2(|x|^2dx)$, similarly any radial ground state $Q_{rad}$ satisfies $Q_{rad} \in L^2(|x|^2dx)$. Thanks to this fact, the standard virial identity yields the following instability in the $L^2$-critical case.

**Theorem 1.4.**

1. Let $d \geq 3, 0 < c < \lambda(0)$ and $Q \in G$. Then the standing wave $e^{it}Q(x)$ is unstable in the following sense: there exists $(u_{0,n})_{n \geq 1} \subset H^1$ such that

$$u_{0,n} \to Q \text{ strongly in } H^1,$$

as $n \to \infty$ and the corresponding solution $u_n$ to the $L^2$-critical (1.1) with initial data $u_{0,n}$ blows up in finite time for any $n \geq 1$.

2. Let $d \geq 3, c < 0$ and $Q_{rad} \in G_{rad}$. Then the radial standing wave $e^{it}Q_{rad}(x)$ is unstable in the following sense: there exists $(u_{0,n})_{n \geq 1} \subset H^1$ such that

$$u_{0,n} \to Q_{rad} \text{ strongly in } H^1,$$

as $n \to \infty$ and the corresponding solution $u_n$ to the $L^2$-critical (1.1) with initial data $u_{0,n}$ blows up in finite time for any $n \geq 1$.

If we are interested in radial $H^1$-solutions to (1.8), then we can show another version of instability of standing waves. The interest of this instability is that it allows radial $H^1$-solutions of (1.8) whose $L^2$-norms are greater than the $L^2$-norms of ground states. We refer the reader to Section 4 for more details.

The paper is organized as follows. In Section 2, we recall sharp Gagliardo-Nirenberg inequalities and the profile decomposition related to (1.1). In Section 3, we give the proof of the stability result stated in Theorem 1.3. Finally, we study the strong instability of standing waves in Section 4.

2. Preliminaries

2.1. Sharp Gagliardo-Nirenberg inequalities. In this section, we recall the sharp Gagliardo-Nirenberg associated to (1.1), namely

$$\|u\|_{L^{d+2}_x}^{\frac{d+2}{2}} \leq CGN(c) \|u\|_{L^2}^{\frac{4-(d-2)\alpha}{d}} \|u\|_{H^\frac{\alpha}{2}}^{\frac{\alpha}{d}},$$  

(2.1)

for all $u \in H^1$. The sharp constant $CGN(c)$ is defined by

$$CGN(c) := \sup \left\{ J^0_c(u) : u \in H^1 \setminus \{0\} \right\},$$

where $J^0_c(u)$ is the Weinstein functional

$$J^0_c(u) := \frac{\|u\|_{L^{d+2}_x}^{\frac{d+2}{2}}}{\|u\|_{L^2}^{\frac{4-(d-2)\alpha}{d}} \|u\|_{H^\frac{\alpha}{2}}^{\frac{\alpha}{d}}}.$$  

(2.2)
We also recall the sharp radial Gagliardo-Nirenberg inequality, namely
\[
\|u\|_{L^{\alpha+2}_\infty} \leq C_{GN}(c, \text{rad}) \|u\|_{L^2} \|u\|_{H^1_{\text{rad}}}^{\frac{4-\alpha(d-2)}{2}},
\]  
(2.3)
for all \(u \in H^1_{\text{rad}}\). The sharp constant \(C_{GN}(c, \text{rad})\) is defined by
\[ C_{GN}(c, \text{rad}) := \sup \{ J^\alpha_c(u) : u \in H^1_{\text{rad}} \setminus \{0\} \}. \]
In the case \(c = 0\), it is well known (see [25]) that the sharp constant \(C_{GN}(0)\) is attained by the function \(Q_0\) which is the unique (up to symmetries) positive radial solution of
\[
-\Delta Q_0 + Q_0 = |Q_0|^\alpha Q_0.
\]  
(2.4)
In the case \(c \neq 0\) and \(c < \lambda(d)\), we have the following result (see [17] and also [11]).

**Theorem 2.1** (Sharp Gagliardo-Nirenberg inequality). Let \(d \geq 3\), \(0 < \alpha < \frac{4}{d-2}\) and \(c \neq 0\) be such that \(c < \lambda(d)\). Then \(C_{GN}(c) \in (0, \infty)\) and

- if \(0 < c < \lambda(d)\), then the equality in (2.1) is attained by a function \(Q_c \in H^1\) which is a positive radial solution to the elliptic equation
\[
-\Delta Q_c - c|x|^{-2}Q_c + Q_c = |Q_c|^\alpha Q_c.
\]  
(2.5)
- if \(c < 0\), then \(C_{GN}(c) = C_{GN}(0)\) and the equality in (2.3) is never attained. However, the constant \(C_{GN}(c, \text{rad})\) is attained by a function \(Q_{c, \text{rad}} \in H^1\) which is a positive solution to the elliptic equation
\[
-\Delta Q_{c,\text{rad}} - c|x|^{-2}Q_{c,\text{rad}} + Q_{c,\text{rad}} = |Q_{c,\text{rad}}|^\alpha Q_{c,\text{rad}}.
\]  
(2.6)

We refer the reader to [17, Theorem 3.1] for the proof in the case \(d = 3\) and \(\alpha = 2\) and to [11, Theorem 4.1] for the proof in the general case.

2.2. Profile decomposition. We next recall the profile decomposition related to the nonlinear Schrödinger equation with inverse-square potential.

**Theorem 2.2** (Profile decomposition [1]). Let \(d \geq 3\) and \(c \neq 0\) be such that \(c < \lambda(d)\). Let \((v_n)_{n \geq 1}\) be a bounded sequence in \(H^1\). Then there exist a subsequence still denoted by \((v_n)_{n \geq 1}\), a family \((x_{jn})_{n \geq 1}\) of sequences in \(\mathbb{R}^d\) and a sequence \((V^j)_{j \geq 1}\) of \(H^1\)-functions such that

1. for every \(j \neq k\),
\[
|x_{jn} - x_{kn}| \to \infty,
\]  
(2.7)
as \(n \to \infty\);

2. for every \(l \geq 1\) and every \(x \in \mathbb{R}^d\), we have
\[
v_n(x) = \sum_{j=1}^l V^j(x - x_{jn}) + v^l_n(x),
\]
with
\[
\limsup_{n \to \infty} \|v^l_n\|_{L^p} \to 0,
\]  
(2.8)
as \(l \to \infty\) for any \(2 < p < \frac{2d}{d-2}\).
Moreover, for every \( l \geq 1 \), we have

\[
\|v_n\|_{L^2}^2 = \sum_{j=1}^{l} \|V_j\|_{L^2}^2 + \|v_n\|_{L^2}^2 + o_n(1),
\]

(2.9)

\[
\|
abla v_n\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla V_j\|_{L^2}^2 + \|
abla v_n\|_{L^2}^2 + o_n(1),
\]

(2.10)

\[
\|v_n\|_{H^\frac{1}{2}}^2 = \sum_{j=1}^{l} \|V_j(\cdot - x_j^L_n)\|_{H^\frac{1}{2}}^2 + \|v_n\|_{H^\frac{1}{2}}^2 + o_n(1),
\]

(2.11)

\[
\|v_n\|_{L^{\alpha+2}}^2 = \sum_{j=1}^{l} \|V_j\|_{L^{\alpha+2}}^2 + \|v_n\|_{L^{\alpha+2}}^2 + o_n(1),
\]

(2.12)

for any \( 0 < \alpha < \frac{4}{d-2} \).

We refer the reader to [1, Theorem 4] for the proof of Theorem 2.2. We note here that the profile decomposition argument was first proposed by Gérard in [13]. Later, Hmidi and Keraani [15] gave a refined version and used it to give a simple proof of some dynamical properties of blow-up solutions for the classical nonlinear Schrödinger equation. Using the same idea as in [15], the profile decomposition of bounded sequences in \( H^2 \) and \( H^s \) \((0 < s < 1)\) were then established in [29, 30] to study dynamical aspects of blow-up solutions for the fourth-order nonlinear Schrödinger equation and the fractional nonlinear Schrödinger equation. The profile decomposition was also successfully used to study the stability of standing waves for the fractional nonlinear Schrödinger equation (see e.g. [31, 12, 23]).

3. Orbital stability of standing waves

In this section, we will give the proof of the stability result stated in Theorem 1.3. Let us firstly study the variational problems (1.5) and (1.6) by using the profile decomposition of bounded sequences in \( H^1 \).

**Proposition 3.1.** Let \( d \geq 3, 0 < \alpha < \frac{4}{d} \) and \( M > 0 \).

1. If \( 0 < c < \lambda(d) \), then the variational problem (1.5) is well-defined and there exists \( C_1 > 0 \) such that

\[
d_M \leq -C_1 < 0.
\]

(3.1)

Moreover, there exists \( v \in H^1 \) such that \( E(v) = d_M \).

2. If \( c < 0 \), then the variational problem (1.6) is well-defined and there exists \( C_2 > 0 \) such that

\[
d_{M, \text{rad}} \leq -C_2 < 0.
\]

(3.2)

Moreover, there exists \( v \in H^1_{\text{rad}} \) such that \( E(v) = d_{M, \text{rad}} \).
Proof. (1) Let us firstly consider the case $0 < c < \lambda(d)$. Let $v \in H^1$ be such that $\|v\|_{L^2}^2 = M$. By the sharp Gagliardo-Nirenberg inequality (2.1), we have

$$E(v) = \frac{1}{2} \|v\|^2_{H^1} - \frac{1}{\alpha + 2} \|v\|_{L^{\alpha+2}}^{\alpha+2} \geq \frac{1}{2} \|v\|^2_{H^1} - \frac{C_{GN}(\epsilon)}{\alpha + 2} \|v\|_{L^2}^{4-\frac{4}{d}} \|v\|_{H^1}^\epsilon \geq \frac{1}{2} \|v\|^2_{H^1} - \frac{C_{GN}(\epsilon)}{\alpha + 2} M^{4-\frac{4}{d}} \|v\|_{H^1}^\epsilon.$$

Since $0 < \frac{4\epsilon}{d} < 2$, we apply the Young’s inequality, that is for any $a, b > 0$, any $\epsilon > 0$ and any $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, there exists $C(\epsilon, p, q) > 0$ such that

$$ab \leq \epsilon a^p + C(\epsilon, p, q)b^q,$$

to have

$$\frac{C_{GN}(\epsilon)}{\alpha + 2} M^{4-\frac{4}{d}} \|v\|_{H^1}^\epsilon \leq \epsilon \|v\|_{L^2}^2 + C(\epsilon, d, \alpha, M).$$

We thus get

$$E(v) \geq \left(\frac{1}{2} - \epsilon\right) \|v\|^2_{H^1} - C(\epsilon, d, \alpha, M). \tag{3.3}$$

By choosing $0 < \epsilon < \frac{1}{2}$, we see that $E(v) \geq -C(\epsilon, d, \alpha, M)$. This shows that the variational problem (1.5) is well-defined.

For $\lambda > 0$, we define $v_\lambda(x) := \lambda^{\frac{d}{2}} v(\lambda x)$. It is easy to check that $\|v_\lambda\|_{L^2}^2 = \|v\|_{L^2}^2 = M$ and

$$E(v_\lambda) = \lambda^{\frac{d}{2}} \|v_\lambda\|_{L^2}^2 - \frac{\lambda^{\frac{4\epsilon}{d}}}{\alpha + 2} \|v_\lambda\|_{L^{\alpha+2}}^{\alpha+2}.$$ 

Since $0 < \frac{4\epsilon}{d} < 2$, one can find a value $\lambda_1 > 0$ sufficiently small so that $E(v_{\lambda_1}) < 0$. Taking $C_1 := -E(v_{\lambda_1}) > 0$, one gets (3.1).

Let $(v_n)_{n \geq 1}$ be a minimizing sequence of $d_M$, that is $\|v_n\|_{L^2}^2 = M$ for all $n \geq 1$ and $\lim_{n \to \infty} E(v_n) = d_M$. There exists $C > 0$ such that

$$E(v_n) \leq d_M + C,$$

for all $n \geq 1$. For $0 < \epsilon < \frac{1}{2}$ fixed, we use (3.3) to infer

$$\left(\frac{1}{2} - \epsilon\right) \|v_n\|^2_{H^1} \leq d_M + C + C(\epsilon, d, \alpha, M).$$

This shows that $\|v_n\|_{H^1}$ (hence $\|v_n\|_{H^1}$) is bounded for all $n \geq 1$. In particular $(v_n)_{n \geq 1}$ is a bounded sequence in $H^1$. Therefore, the profile decomposition given in Theorem 2.2 implies that up to a subsequence, we can write for every $l \geq 1$,

$$v_n(x) = \sum_{j=1}^l V^j(x - x_n^j) + v^l_n(x), \tag{3.4}$$

and (2.7) – (2.12) hold. In particular, we have

$$E(v_n) = \sum_{j=1}^l E(V^j(x - x_n^j)) + E(v^l_n) + o_n(1), \tag{3.5}$$

as $n \to \infty$. Denote $V^j_n(x) := \lambda_j V^j(x - x_n^j)$ and $\bar{v}^l_n(x) = \lambda_n v^l_n(x)$, where

$$\lambda_j := \frac{\sqrt{M}}{\|V^j\|_{L^2}} \geq 1, \quad \lambda_n := \frac{\sqrt{M}}{\|v^l_n\|_{L^2}} \geq 1.$$
We readily check that
\[
\|\tilde{V}_n\|_{L^2}^2 = \lambda_j^2 \|V^j(\cdot - x_n^j)\|_{L^2}^2 = M, \quad \|\tilde{v}_n^j\|_{L^2}^2 = (\lambda_n^j)^2 \|v_n^j\|_{L^2}^2 = M.
\]

By definition of \(d_M\), we have
\[
E(\tilde{V}_n^j) \geq d_M, \quad E(\tilde{v}_n^j) \geq d_M. \quad (3.6)
\]

Moreover, a direct computation shows that
\[
E(\tilde{V}_n^j) = \frac{\lambda_j^2}{2} \|V^j(\cdot - x_n^j)\|_{H^1}^2 - \frac{\lambda_j^{\alpha+2}}{\alpha+2} \|V^j\|_{L^{\alpha+2}}^{\alpha+2}.
\]

So,
\[
E(V^j(\cdot - x_n^j)) = \frac{E(\tilde{V}_n^j)}{\lambda_j^2} + \frac{\lambda_j^{\alpha-1}}{\alpha+2} \|V^j\|_{L^{\alpha+2}}^{\alpha+2} \quad (3.7)
\]

Similarly,
\[
E(v_n^j) = \frac{E(\tilde{v}_n^j)}{\lambda_n^j} + \frac{\lambda_n^\alpha - 1}{\alpha+2} \|v_n^j\|_{L^{\alpha+2}}^{\alpha+2} \quad (3.8)
\]

Inserting (3.7), (3.8) to (3.5) and using (3.6), we obtain
\[
E(v_n) \geq \sum_{j=1}^I \left(\frac{E(\tilde{V}_n^j)}{\lambda_j^2} + \frac{\lambda_j^{\alpha-1}}{\alpha+2} \|V^j\|_{L^{\alpha+2}}^{\alpha+2} \right) + \frac{E(\tilde{v}_n^j)}{\lambda_n^j} + o_n(1)
\]
\[
\geq \sum_{j=1}^I \frac{\|V^j\|_{L^2}^2}{M} d_M + \left(\inf_{j \geq 1} \frac{\lambda_j^{\alpha-1}}{\alpha+2}\right) \sum_{j=1}^I \|V^j\|_{L^{\alpha+2}}^{\alpha+2} + \frac{\|v_n^j\|_{L^{\alpha+2}}^2}{M} d_M + o_n(1)
\]
\[
= \frac{d_M}{M} (\|v_n\|_{L^2}^2 + o_n(1)) + \left(\inf_{j \geq 1} \frac{\lambda_j^{\alpha-1}}{\alpha+2}\right) (\|v_n\|_{L^{\alpha+2}}^{\alpha+2} - \|v_n^j\|_{L^{\alpha+2}}^{\alpha+2} + o_n(1)) + o_n(1).
\]

Since \(\sum_{j=1}^\infty \|V^j\|_{L^2}^2\) is convergent, there exists \(j_0 \geq 1\) such that
\[
\|V^{j_0}\|_{L^2}^2 = \sup_{j \geq 1} \|V^j\|_{L^2}^2.
\]

We thus have
\[
\inf_{j \geq 1} \lambda_j^\alpha = \inf_{j \geq 1} \left(\frac{\sqrt{M}}{\|V^j\|_{L^2}}\right)^\alpha = \left(\frac{\sqrt{M}}{\|V^{j_0}\|_{L^2}}\right)^\alpha. \quad (3.9)
\]

On the other hand, by the definition of energy,
\[
\frac{\|v_n\|_{L^{\alpha+2}}^{\alpha+2}}{\alpha+2} \geq -E(v_n) \not\geq -d_M.
\]

Using (3.1), we get
\[
\frac{\|v_n\|_{L^{\alpha+2}}^{\alpha+2}}{\alpha+2} \geq -E(v_n) \geq \frac{C_1}{2}, \quad (3.10)
\]

for \(n\) sufficiently large. By (3.9) and (3.10), we get
\[
E(v_n) \geq d_M + o_n(1) + \left[\left(\frac{\sqrt{M}}{\|V^{j_0}\|_{L^2}}\right)^\alpha - 1\right] \left(\frac{C_1}{2} - \|v_n^j\|_{L^{\alpha+2}}^{\alpha+2} \frac{\alpha}{\alpha+2}\right) + o_n(1).
\]
Taking the limits \( n \to \infty \) and \( l \to \infty \) and using (2.8), we obtain
\[
d_M \geq d_M + \left[ \left( \frac{\sqrt{M}}{\|V^{j_0}\|_{L^2}} \right)^\alpha - 1 \right] C_1 2.
\]
Therefore, \( \|V^{j_0}\|_{L^2}^2 \geq M \). The almost orthogonality (2.9) and the fact \( \|v_n\|_{L^2}^2 = M \) then imply that there is only one term \( V^{j_0} \neq 0 \) in the profile decomposition (3.4) and \( \|V^{j_0}\|_{L^2}^2 = M \).

The identity (3.4) implies for every \( l \geq j_0 \),
\[
v_n(x) = V^{j_0}(x - x_n^{j_0}) + v_n^l(x).
\]
Fix \( l = j_0 \). Using
\[
\|v_n\|_{L^2}^2 = \|V^{j_0}\|_{L^2}^2 + \|v_n^{j_0}\|_{L^2}^2 + O_n(1),
\]
as \( n \to \infty \), and \( \|v_n\|_{L^2}^2 = \|V^{j_0}\|_{L^2}^2 = M \), we get (up to a subsequence)
\[
\lim_{n \to \infty} \|v_n^{j_0}\|_{L^2} = 0.
\]
This shows that the sequence \((v_n^{j_0})_{n \geq 1}\) converges to zero weakly in \( H^1 \) and strongly in \( L^2 \). The boundedness of \((v_n^{j_0})_{n \geq 1}\) in \( H^1 \) along with the strong convergence in \( L^2 \) to zero yield that
\[
\lim_{n \to \infty} \|v_n^{j_0}\|_{L^{\alpha+2}} = 0.
\]
The lower semi-continuity of Hardy’s functional then gives
\[
0 \leq \liminf_{n \to \infty} E(v_n^{j_0}),
\]
thus
\[
\liminf_{n \to \infty} E(V^{j_0}(\cdot - x_n^{j_0})) \leq \liminf_{n \to \infty} E(V^{j_0}(\cdot - x_n^{j_0})) + \liminf_{n \to \infty} E(v_n^{j_0}) \\
\leq \liminf_{n \to \infty} \left( E(V^{j_0}(\cdot - x_n^{j_0})) + E(v_n^{j_0}) \right) \\
= \liminf_{n \to \infty} E(v_n) = d_M.
\]
On the other hand, since \( \|V^{j_0}(\cdot - x_n^{j_0})\|_{L^2}^2 = \|V^{j_0}\|_{L^2}^2 = M \) for all \( n \geq 1 \), we have \( E(V^{j_0}(\cdot - x_n^{j_0})) \geq d_M \) for all \( n \geq 1 \). Therefore,
\[
\liminf_{n \to \infty} E(V^{j_0}(\cdot - x_n^{j_0})) = d_M,
\]
or equivalently,
\[
\frac{1}{2} \|\nabla V^{j_0}\|_{L^2}^2 - \frac{1}{\alpha + 2} \|V^{j_0}\|_{L^{\alpha+2}}^{\alpha+2} - \frac{\alpha}{2} \limsup_{n \to \infty} \int |x|^{-2} |V^{j_0}(x - x_n^{j_0})|^2 dx = d_M. \tag{3.11}
\]
We next prove that the sequence \((x_n^{j_0})_{n \geq 1}\) is bounded. Indeed, if it is not true, then up to a subsequence, we assume that \( |x_n^{j_0}| \to \infty \) as \( n \to \infty \). Without loss of generality, we assume that \( V^{j_0} \) is continuous and compactly supported. We have
\[
\int |x|^{-2} |V^{j_0}(x - x_n^{j_0})|^2 dx = \int_{\text{supp}(V^{j_0})} |x + x_n^{j_0}|^{-2} |V^{j_0}(x)|^2 dx.
\]
Since \( |x_n^{j_0}| \to \infty \) as \( n \to \infty \), we see that \( |x + x_n^{j_0}| \geq |x_n^{j_0}| - |x| \to \infty \) as \( n \to \infty \) for all \( x \in \text{supp}(V^{j_0}) \). This shows that
\[
\int |x|^{-2} |V^{j_0}(x - x_n^{j_0})|^2 dx \to 0,
\]
as \( n \to \infty \). This yields
\[
\frac{1}{2} \|\nabla V^{j_0}\|_{L^2}^2 - \frac{1}{\alpha + 2} \|V^{j_0}\|_{L^{\alpha+2}}^{\alpha+2} = d_M.
\]
By the definition of $E(V^{j_0})$, we obtain

$$E(V^{j_0}) + \frac{c}{2} \int |x|^{-2} |V^{j_0}(x)|^2 dx = d_M.$$ 

Since $0 < c < \lambda(d)$, we get $E(V^{j_0}) < d_M$, which is absurd (note that $E(V^{j_0}) \geq d_M$ due to $\|V^{j_0}\|^2_{L^2} = M$). Therefore, the sequence $(x^{j_n})_{n \geq 1}$ is bounded and up to a subsequence, we assume that $x^{j_n} \to x^{j_0}$ as $n \to \infty$.

We now write

$$v_n(x) = \tilde{V}^{j_n}(x) + \tilde{v}^{j_n}(x),$$

where $\tilde{V}^{j_n}(x) = V^{j_n}(x - x^{j_n})$ and $\tilde{v}^{j_n}(x) := V^{j_n}(x - x^{j_n}) - V^{j_0}(x - x^{j_0}) + v^{j_0}_n(x)$. Using the fact $\|v_n\|^2_{L^2} = \|V^{j_0}\|^2_{L^2} = M$, it is easy to see that

$$\tilde{v}^{j_n} \to 0 \text{ weakly in } H^1 \text{ and } \lim_{n \to \infty} \|\tilde{v}^{j_n}\|_{L^2} = 0.$$ 

The first observation on $\tilde{v}^{j_n}$ allows us to write

$$E(v_n) = E(\tilde{V}^{j_n}) + E(\tilde{v}^{j_n}) + o_n(1).$$

Again, the lower semi-continuity of Hardy’s functional and the fact $\lim_{n \to \infty} \|\tilde{v}^{j_n}\|_{L^{(a+2)}} = 0$, we get that $\lim \inf_{n \to \infty} E(\tilde{v}^{j_n}) \geq 0$. Hence, using the fact that $\|\tilde{V}^{j_0}\|^2_{L^2} = M$, we infer that

$$d_M = \lim \inf_{n \to \infty} E(v_n) \geq \lim \inf_{n \to \infty} (E(\tilde{V}^{j_n}) + E(\tilde{v}^{j_n})) \geq E(\tilde{V}^{j_0}) + \lim \inf_{n \to \infty} E(\tilde{v}^{j_n}) \geq E(\tilde{V}^{j_0}) \geq d_M.$$ 

Therefore, $E(\tilde{V}^{j_0}) = d_M$ which completes the proof of Item (1).

(2) We now consider the case $c < 0$. By the same argument (with $C_{GN}(c, \text{rad})$ in place of $C_{GN}(c)$), the variational problem (1.6) is well-defined and there exists $C_\frac{a}{2} > 0$ such that (3.2) holds. It remains to show that there exists $v \in H^1$ radial such that $E(v) = d_M, \text{rad}$. Let $(v_n)_{n \geq 1}$ be a minimizing sequence of $d_M, \text{rad}$, that is $v_n \in H^{1, \text{rad}}$, $\|v_n\|^2_{L^2} = M$ for all $n \geq 1$ and $\lim_{n \to \infty} E(v_n) = d_M, \text{rad}$. Arguing as in the first case, we see that $(v_n)_{n \geq 1}$ is a bounded radial sequence in $H^1$. Thanks to the fact that

$$H^{1, \text{rad}} \hookrightarrow L^p \text{ compactly,}$$

for any $2 < p < \frac{2d}{d-2}$, there exists $V \in H^{1, \text{rad}}$ (see Appendix) such that

$$v_n \to V \text{ weakly in } H^1 \text{ and } v_n \to V \text{ strongly in } L^{a+2}.$$ 

We write

$$v_n(x) = V(x) + r_n(x),$$

with $r_n \to 0$ weakly in $H^1$ (note that $r_n$ can be taken radially symmetric). We have the following expansions

$$\|v_n\|^2_{L^2} = \|V\|^2_{L^2} + \|r_n\|^2_{L^2} + o_n(1), \quad (3.13)$$

$$E(v_n) = E(V) + E(r_n) + o_n(1), \quad (3.14)$$

as $n \to \infty$. Denote $\tilde{V} = \lambda V$ and $\tilde{r}_n = \lambda_n r_n$, where

$$\lambda := \sqrt{\frac{M}{\|V\|^2_{L^2}}} \geq 1, \quad \lambda_n := \sqrt{\frac{M}{\|r_n\|^2_{L^2}}} \geq 1.$$ 

It is obvious that $\|\tilde{V}\|^2_{L^2} = \|\tilde{r}_n\|^2_{L^2} = M$, hence

$$E(\tilde{V}) \geq d_{M, \text{rad}}, \quad E(\tilde{r}_n) \geq d_{M, \text{rad}}.$$
We also have
\[ E(\tilde{V}) = \frac{\lambda^2}{2} ||V||_{H^1_0}^2 - \frac{\lambda^{\alpha+2}}{\alpha+2} ||V||_{L^{\alpha+2}}^{\alpha+2}. \]
So,
\[ E(V) = \frac{E(\tilde{V})}{\lambda^2} + \frac{\lambda^{\alpha} - 1}{\alpha+2} ||V||_{L^{\alpha+2}}^{\alpha+2}. \]
Similarly,
\[ E(r_n) = \frac{E(\tilde{r}_n)}{\lambda_n^2} + \frac{\lambda_n^{\alpha} - 1}{\alpha+2} ||r_n||_{L^{\alpha+2}}^{\alpha+2} \geq \frac{E(\tilde{r}_n)}{\lambda_n^2}. \]
Plugging above estimates to (3.14), we have
\[ E(v_n) \geq \frac{E(\tilde{V})}{\lambda^2} + \frac{\lambda^{\alpha} - 1}{\alpha+2} ||V||_{L^{\alpha+2}}^{\alpha+2} + \frac{E(\tilde{r}_n)}{\lambda_n^2} + o_n(1) \]
\[ = \frac{||V||_{L^2}^2 d_{\text{M,rad}}}{M} + \frac{\lambda^{\alpha} - 1}{\alpha+2} ||V||_{L^{\alpha+2}}^{\alpha+2} + \frac{||r_n||_{L^2}^2 d_{\text{M,rad}}}{M} + o_n(1) \]
\[ = \frac{d_{\text{M,rad}}}{M} (||V||_{L^2}^2 + ||r_n||_{L^2}^2) + \frac{\lambda^{\alpha} - 1}{\alpha+2} (||v_n||_{L^{\alpha+2}}^{\alpha+2} - ||r_n||_{L^{\alpha+2}}^{\alpha+2}) + o_n(1). \]
Since
\[ \frac{||v_n||_{L^{\alpha+2}}^{\alpha+2}}{\alpha+2} \geq -E(v_n) > -d_{\text{M,rad}}. \]
Using the upper bound (3.2), we see that
\[ \frac{||v_n||_{L^{\alpha+2}}^{\alpha+2}}{\alpha+2} \geq -E(v_n) \geq \frac{C_2}{2}, \]
for \( n \) sufficiently large. Taking \( n \to \infty \), this combined with the fact \( \lim_{n \to \infty} ||r_n||_{L^{\alpha+2}}^{\alpha+2} = 0 \) yield
\[ d_{\text{M,rad}} \geq \frac{d_{\text{M,rad}}}{M} \left( \left( \frac{\sqrt{M}}{||V||_{L^2}} \right)^\alpha - 1 \right) \frac{C_2}{2}. \]
We thus obtain \( ||V||_{L^2}^2 \geq M \). Since \( ||v_n||_{L^2}^2 = M \), we have from (3.13) that \( ||V||_{L^2}^2 = ||v_n||_{L^2}^2 = M \). In particular, we have \( \lim_{n \to \infty} ||r_n||_{L^2}^2 = 0 \) and \( E(V) \geq d_{\text{M,rad}} \). Since \( r_n \to 0 \) weakly in \( H^1 \) and strongly in \( L^2 \). The lower semi-continuity of Hardy’s functional implies
\[ \liminf_{n \to \infty} E(r_n) \geq 0. \]
Therefore,
\[ d_{\text{M,rad}} = \liminf_{n \to \infty} E(v_n) \geq \liminf_{n \to \infty} (E(V) + E(r_n)) \geq E(V) + \liminf_{n \to \infty} E(r_n) \]
\[ \geq E(V) \geq d_{\text{M,rad}}. \]
We thus obtain \( E(V) = d_{\text{M,rad}} \), which implies that the variational problem (1.6) is attained. The proof is complete.

**Remark 3.2.** (1) The proof Proposition 3.1 still holds
- in the case \( 0 < c < \lambda(d) \) if in place of
  \[ ||v_n||_{L^2}^2 = M \text{ for all } n \geq 1 \text{ and } \lim_{n \to \infty} E(v_n) = d_M, \]
we assume that
  \[ \lim_{n \to \infty} ||v_n||_{L^2}^2 = M \text{ and } \lim_{n \to \infty} E(v_n) = d_M. \]
• in the case \( c < 0 \) if in place of
\[
\| v_n \|^2_{L^2} = M \text{ for all } n \geq 1 \text{ and } \lim_{n \to \infty} E(v_n) = d_{M,\text{rad}},
\]
we assume that
\[
\lim_{n \to \infty} \| v_n \|^2_{L^2} = M \text{ and } \lim_{n \to \infty} E(v_n) = d_{M,\text{rad}}.
\]

(2) It follows from the proof of Proposition 3.1 that
• in the case \( 0 < c < \lambda(d) \),
\[
E(v_n) = E(\tilde{V}^j_0) + E(\tilde{r}_n^j) + o_n(1),
\]
as \( n \to \infty \), and
\[
\lim_{n \to \infty} E(v_n) = d_M, \quad \lim_{n \to \infty} \| \tilde{r}_n^j \|^{q+2}_{L^{q+2}} = 0, \quad E(\tilde{V}^j_0) = d_M.
\]
We thus have that up to a subsequence,
\[
\lim_{n \to \infty} \| \tilde{r}_n^j \|^2_{H^1_c} = 0.
\]
Since \( \| \tilde{r}_n^j \|^2_{H^1_c} \sim \| r_n^j \|^2_{H^1} \), we conclude that
\[
\lim_{n \to \infty} \| \nabla \tilde{r}_n^j \|_{L^2} = 0.
\]
So,
\[
\lim_{n \to \infty} \| \nabla v_n \|_{L^2} = \| \nabla \tilde{V}^j_0 \|_{L^2},
\]
which along with \( \lim_{n \to \infty} \| v_n \|_{L^2} = \| \tilde{V}^j_0 \|_{L^2} \) yield
\[
v_n \to \tilde{V}^j_0 \text{ strongly in } H^1,
\]
as \( n \to \infty \).
• in the case \( c < 0 \),
\[
E(v_n) = E(V) + E(r_n) + o_n(1),
\]
as \( n \to \infty \), and
\[
\lim_{n \to \infty} E(v_n) = d_{M,\text{rad}}, \quad \lim_{n \to \infty} \| r_n \|^{q+2}_{L^{q+2}} = 0, \quad E(V) = d_{M,\text{rad}}.
\]
We thus have
\[
\lim_{n \to \infty} \| r_n \|^2_{H^1_c} = 0,
\]
which together with \( \| r_n \|^2_{H^1_c} \sim \| r_n \|^2_{H^1} \) yield
\[
\lim_{n \to \infty} \| \nabla r_n \|_{L^2} = 0.
\]
This implies
\[
\lim_{n \to \infty} \| \nabla v_n \|_{L^2} = \| \nabla V \|_{L^2},
\]
which together with \( \lim_{n \to \infty} \| v_n \|_{L^2} = \| V \|_{L^2} \) imply
\[
v_n \to V \text{ strongly in } H^1,
\]
as \( n \to \infty \).
We are now able to prove the orbital stability given in Theorem 1.3.

Proof of Theorem 1.3. We only consider the case $0 < c < \lambda(d)$, the case $c < 0$ is completely similar. We argue by contradiction. Assume that there exist sequences $(u_{0,n})_{n \geq 1} \subset H^1$, $(t_n)_{n \geq 1} \subset \mathbb{R}^+$ and $\epsilon_0 > 0$ such that for all $n \geq 1$,

$$\inf_{v \in S_M} \|u_{0,n} - v\|_{H^1} < \frac{1}{n},$$

(3.15)

and

$$\inf_{v \in S_M} \|u_n(t_n) - v\|_{H^1} \geq \epsilon_0.$$  

(3.16)

Here $u_n(t)$ is the solution to (1.1) with initial data $u_{0,n}$. We next claim that there exists $v \in S_M$ such that

$$\lim_{n \to \infty} \|u_{0,n} - v\|_{H^1} = 0.$$

Indeed, we have from (3.15) that for each $n \geq 1$, there exists $v_n \in S_M$ such that

$$\|u_{0,n} - v_n\|_{H^1} < \frac{2}{n}.$$  

(3.17)

We thus obtain a sequence $(v_n)_{n \geq 1} \subset S_M$, and we get from the proof of Proposition 3.1 that there exists $v \in S_M$ such that

$$\lim_{n \to \infty} \|v_n - v\|_{H^1} = 0.$$  

(3.18)

The claim follows immediately from (3.17) and (3.18). We thus get

$$\lim_{n \to \infty} \|u_{0,n}\|_{L^2}^2 = \|v\|_{L^2}^2 = M, \quad \lim_{n \to \infty} E(u_{0,n}) = E(v) = d_M.$$  

The conservation of mass and energy then imply

$$\lim_{n \to \infty} \|u_n(t_n)\|_{L^2}^2 = M, \quad \lim_{n \to \infty} E(u_n(t_n)) = d_M.$$  

Again, from the proof of Proposition 3.1 and Remark 3.2, there exists $\bar{v} \in S_M$ such that $(u_n(t_n))_{n \geq 1}$ converges strongly to $\bar{v}$ in $H^1$. This contradicts (3.16) and the proof is complete. \Box

4. Strong instability of standing waves

In this section, we study the stability of standing waves for the nonlinear Schrödinger equation with inverse-square potential in the $L^2$-critical case, i.e. $\alpha = \frac{4}{d}$ in (1.1). Let us start by defining properly the notion of ground states related to the $L^2$-critical (1.1).

Definition 4.1 (Ground states). In the case $0 < c < \lambda(d)$, we call ground states the maximizers of $J_c^{4/d}$ (see (2.2)) which are positive radial solutions of

$$-\Delta Q - c|Q|^{-2}Q + Q = |Q|^{\frac{4}{d}}Q.$$  

(4.1)

The set of ground states is denoted by $\mathcal{G}$.

In the case $c < 0$, we call radial ground states the maximizers of $J_c^{4/d}$ which are positive radial solutions of

$$-\Delta Q_{\text{rad}} - c|x|^{-2}Q_{\text{rad}} + Q_{\text{rad}} = |Q_{\text{rad}}|^{\frac{4}{d}}Q_{\text{rad}}.$$  

(4.2)

The set of radial ground states is denoted by $\mathcal{G}_{\text{rad}}$.

Remark 4.2. This notion of ground states in the case $0 < c < \lambda(d)$ was first introduced in [10] due to the fact that the uniqueness (up to symmetries) of positive radial solutions of (4.1) and (4.2) are not yet known.
• It follows from Theorem 2.1 and its proof that there exists $M_{gs} > 0$ such that $\|Q\|_{L^2} = M_{gs}$ for all $Q \in \mathcal{G}$. The constant $M_{gs}$ is called **minimal mass.** Similarly, there exists $M_{gs,rad} > 0$ such that $\|Q_{rad}\|_{L^2} = M_{gs,rad}$ for all $Q_{rad} \in \mathcal{G}_{rad}$. The constant $M_{gs,rad}$ is called **radial minimal mass.**

• Thanks to the pseudo-conformal invariance, it was shown in [10, Remark 3, p.118] that any solution $Q$ of (4.1) satisfies $\|Q\|_{L^2} \geq M_{gs}$. Similarly, any radial solution $Q_{rad}$ of (4.2) satisfies $\|Q_{rad}\|_{L^2} \geq M_{gs,rad}$.

• If $Q \in \mathcal{G}$, then $Q_\omega := (\sqrt{\omega})^{d/2}Q(\sqrt{\omega}x)$ is also a maximizer of $J_c^{4/d}$ satisfying $\|Q\|_{L^2} = \|Q\|_{L^2} = M_{gs}$ and

$$-\Delta Q_\omega - c|x|^{-2}Q_\omega + \omega Q_\omega = |Q_\omega|^\frac{4}{d}Q_\omega.$$  

Similarly, if $Q_{rad} \in \mathcal{G}_{rad}$, then $Q_{\omega,rad} := (\sqrt{\omega})^{d/2}Q_{rad}(\sqrt{\omega}x)$ is also a maximizer of $J_c^{4/d}$ satisfying $\|Q_{\omega,rad}\|_{L^2} = \|Q_{rad}\|_{L^2} = M_{gs,rad}$ and

$$-\Delta Q_{\omega,rad} - c|x|^{-2}Q_{\omega,rad} + \omega Q_{\omega,rad} = |Q_{\omega,rad}|^\frac{4}{d}Q_{\omega,rad}.$$  

• Let $Q$ be a $H^1$-solution to (4.1). Multiplying both sides of (4.1) with $\overline{Q}$, integrating over $\mathbb{R}^d$ and using the integration by parts, we obtain

$$\|Q\|_{H^1}^2 + \|Q\|_{L^2}^2 = \|Q\|_{L^{\frac{4}{d}+2}}^2.$$  

Similarly, multiplying both sides of (4.1) with $x \cdot \nabla \overline{Q}$, integrating over $\mathbb{R}^d$, the integration by parts gives

$$\frac{d-2}{2} \|Q\|_{H^1}^2 + \frac{d}{2} \|Q\|_{L^2}^2 = \frac{d^2}{2d+4} \|Q\|_{L^{\frac{4}{d}+2}}^2.$$  

Combining (4.3) and (4.4), we obtain the following Pohozaev’s identities:

$$\|Q\|_{H^1}^2 = \frac{d}{d+2} \|Q\|_{L^{\frac{4}{d}+2}}^2 = \frac{d}{2} \|Q\|_{L^2}^2.$$  

In particular, we have

$$E(Q) = \frac{1}{2} \|Q\|_{H^1}^2 - \frac{d}{2d+4} \|Q\|_{L^{\frac{4}{d}+2}}^2 = 0.$$  

Similar identities as (4.3) – (4.6) still hold if $Q$ is replaced by $Q_{rad}$ which is a $H^1$-solution of (4.2).

**Proof of Theorem 1.4.** The proof is based on a standard argument (see e.g. [9, Theorem 8.2.1]).

Let us assume at the moment that the functions $Q$ and $Q_{rad}$ belong to $L^2(|x|^2dx)$. Note that unlike the classical nonlinear Schrödinger equation, we do not know whether solutions of (4.1) and (4.2) enjoy the exponential decay at infinity or not. In the case the exponential decay at infinity holds true, the above assumption is obviously satisfied. The above assumption together with the Pohozaev identity (4.5) imply $Q$ and $Q_{rad}$ are both in $H^1 \cap L^2(|x|^2dx)$. We next recall the standard virial identity related to the $L^2$-critical (1.1) (see [11, Lemma 5.3] or [10, Lemma 3, p.124]).

**Lemma 4.3** (Virial identity). Let $d \geq 3$, $c \neq 0$ and $c < \lambda(d)$. Let $u_0 \in H^1 \cap L^2(|x|^2dx)$ and $u : J \times \mathbb{R}^d \to \mathbb{C}$ the corresponding solution to the $L^2$-critical (1.1). Then $u \in C(J, H^1 \cap L^2(|x|^2dx))$ and

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 16E(u_0),$$  

for any $t \in J$. 
We now denote for $0 < c < \lambda(d)$,

$$u_{0,n}(x) := \mu_n Q(x),$$

and for $c < 0$,

$$u_{0,n}(x) := \mu_n Q_{\text{rad}}(x),$$

for any $n \geq 1$, where $\mu_n := 1 + 1/n$. It is obvious that for $0 < c < \lambda(d)$, $u_{0,n} \to Q$ strongly in $H^1$, and for $c < 0$, $u_{0,n} \to Q_{\text{rad}}$ strongly in $H^1$. Let $u_n$ be the corresponding solution to the $L^2$-critical (1.1) with initial data $u_{0,n}$. We will show that $u_n$ blows up in finite time for any $n \geq 1$. We only consider $Q$, the one for $Q_{\text{rad}}$ is similar. To see this, we fix $n \geq 1$ and compute

$$E(u_{0,n}) = \frac{\mu_n^2}{2} ||Q||^2_{H^1} - \frac{d\mu_n^{\frac{4}{d}+2}}{2d+4} ||Q||^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}}$$

$$= \mu_n^2 E(Q) + \frac{d}{2d+4} \mu_n^2 \left(1 - \mu_n^2\right) ||Q||^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}}.$$

Since $E(Q) = 0$ (see (4.6)) and $\mu_n > 1$, we have that $E(u_{0,n}) < 0$. By Lemma 4.3, we have

$$\frac{d^2}{dt^2} E(u_n(t)) = 16E(u_{0,n}) < 0,$$

for any $t$ as long as the solution exists. The standard convexity argument (see e.g. [14]) shows that $u_n$ must blow up in finite time.

It remains to show that $Q$ and $Q_{\text{rad}}$ belongs to $L^2(|x|^2 dx)$. Let us first consider $Q$. Denote $u(t, x) := e^{it}Q(x)$ the standing waves. It is easy to see that $u$ is a global solution of the $L^2$-critical (1.1). For $0 < T < +\infty$, we denote

$$u_T(t, x) := e^{-i\frac{|x|^2}{4(T-t)^2}} \left(\frac{1}{T-t} \frac{x}{T-t}\right).$$

Since the $L^2$-critical (1.1) is invariant under the pseudo-conformal transformation, we have from [10, Lemma 1, p.117] that $u_T$ is a solution of the $L^2$-critical (1.1) which blows up at $T$ and satisfies $||u_T(t)||_{L^2} = ||u(1/(T-t))||_{L^2}$. We thus construct a solution to the $L^2$-critical (1.1) which blows up in finite time $T$ and its initial data satisfies

$$||u_T(0)||_{L^2} = ||Q||_{L^2} = M_{gs}.$$

We have from [2, Theorem 3.2] that for a time sequence $t_n \nearrow T$ as $n \to \infty$, there exists $\tilde{Q} \in G$, sequences of $\theta_n \in \mathbb{R}$, $\lambda_n > 0$ and $x_n \in \mathbb{R}^d$ such that

$$e^{it\theta_n\lambda_n^\frac{4}{d}} u_T(t_n, \lambda_n \cdot + x_n) \to \tilde{Q} \text{ strongly in } H^1,$$

as $n \to \infty$. We also have from [10, p.127-128] that $u_T(t) \in L^2(|x|^2 dx)$ for any $t \in [0, T)$. In particular, $Q \in L^2(|x|^2 dx)$. This completes the proof for $Q$.

The case for $Q_{\text{rad}}$ is similar. The only different point is that instead of (4.7), we have

$$e^{it\rho_n^\frac{4}{d} \cdot + x_n} u_T(t_n, \rho_n \cdot) \to \tilde{Q}_{\text{rad}} \text{ strongly in } H^1,$$

as $n \to \infty$, for some $\tilde{Q}_{\text{rad}} \in G_{\text{rad}}$. The rest of the proof remains the same as for $Q$. The proof is complete.

**Theorem 4.4** (Strong instability II). Let $d \geq 3$ and $c \neq 0$ be such that $c < \lambda(d)$. Let $\omega > 0$ and $Q$ be a radial $H^1$-solution to the elliptic equation

$$-\Delta Q - c|x|^{-2}Q + \omega Q = |Q|^{\frac{4}{d}}Q.$$

(4.8)
Then the standing wave $e^{i\omega t}Q(x)$ is unstable in the following sense: there exists $(u_{0,n})_{n \geq 1} \subset H^1$ such that $u_{0,n} \to Q$ strongly in $H^1$, as $n \to \infty$ and the corresponding solution $u_n$ to the $L^2$-critical (1.1) with initial data $u_{0,n}$ blows up in finite time for any $n \geq 1$.

**Remark 4.5.**

- The strong instability of Theorem 4.4 allows radial solutions of (4.1) and (4.2) whose the $L^2$-norms may larger than $M_{gs}$ and $M_{gs,rad}$.

- As in Remark 4.2, if $Q$ is a $H^1$-solution to (4.8), then we have the following Pohozaev identities

$$
\|Q\|_{L^4}^2 = \frac{d}{d+2}\|Q\|_{L^{\frac{d+2}{2}}}^\frac{d+2}{2} = \frac{d\omega}{2}\|Q\|_{L^2}^2. \quad (4.9)
$$

In order to show Theorem 4.4, we recall the following virial estimates related to the $L^2$-critical (1.1). Let $\theta : [0, \infty) \to [0, \infty)$ be a function satisfying

$$
\theta(r) = \begin{cases}
r^2 & \text{if } 0 \leq r \leq 1, \\
\text{const.} & \text{if } r \geq 2,
\end{cases} \quad \text{and} \quad \theta''(r) \leq 2 \text{ for all } r \geq 0. \quad (4.10)
$$

Note that the precise constant in (4.10) is not important here. For $R \geq 1$, we define the radial function

$$
\varphi_R(x) = \varphi_R(r) := R^2\theta(r/R), \quad r = |x|. \quad (4.11)
$$

We readily see that

$$
2 - \varphi''_R(r) \geq 0, \quad 2 - \frac{\varphi_R''(r)}{r} \geq 0, \quad 2d - \Delta \varphi_R(x) \geq 0, \quad \forall r \geq 0, \quad \forall x \in \mathbb{R}^d.
$$

Let $u$ be a solution to (1.1). We define the localized virial potential associated to $u$ by

$$
V_{\varphi_R}(u(t)) := \int \varphi_R(x)|u(t,x)|^2 \, dx.
$$

**Lemma 4.6** (Radial virial estimate [11]). Let $d \geq 3, c \neq 0$ be such that $c < \lambda(d)$. Let $R > 1$ and $\varphi_R$ be as in (4.11). Let $u : J \times \mathbb{R}^d \to \mathbb{C}$ be a radial solution to the $L^2$-critical (1.1). Then for any $\epsilon > 0$ and any $t \in J$, it holds that

$$
\frac{d^2}{dt^2}V_{\varphi_R}(u(t)) \leq 16E(u_0) - 4 \int_{|x| > R} \left( \chi_{1,R} - \frac{\epsilon}{d+2}\chi_{2,R} \right) |\nabla u(t)|^2 \, dx
$$

$$
+ O \left( R^{-2} + \epsilon R^{-2} + \epsilon^{-\frac{2}{d+2}}R^{-2} \right), \quad (4.12)
$$

where

$$
\chi_{1,R} = 2 - \varphi''_R, \quad \chi_{2,R} = 2d - \Delta \varphi_R. \quad (4.13)
$$

We refer the reader to [11, Lemma 5.6] for the proof of this result, which is based on the argument of [21].

**Lemma 4.7** (blow-up criteria [11]). Let $d \geq 3$ and $c \neq 0$ be such that $c < \lambda(d)$. If $u_0 \in H^1$ is radial and satisfies

$$
E(u_0) < 0,
$$

then the corresponding solution $u \in C([0,T), H^1)$ to the $L^2$-critical (1.1) blows up in finite time, i.e. $T < +\infty$. 

Proof. The proof of this result is given in [11, Theorem 1.3]. For reader’s convenience, we recall some details. Applying Lemma 4.6, we have
\[
\frac{d^2}{dt^2} V_{\varphi_R}(u(t)) \leq 16E(u_0) - 4 \int_{|x| > R} \left( \chi_{1,R} - \frac{\epsilon}{d+2} \chi_{2,R}^\frac{d}{2} \right) |\nabla u(t)|^2 dx + O \left( R^{-2} + \epsilon R^{-2} + \epsilon^{-\frac{2}{d-2}} R^{-2} \right).
\]
If we choose a suitable function \( \varphi_R \) so that
\[
\chi_{1,R} - \frac{\epsilon}{d+2} \chi_{2,R}^\frac{d}{2} \geq 0, \quad \forall R > R,
\]
for a sufficiently small \( \epsilon > 0 \), then by choosing \( R > 1 \) sufficiently large depending on \( \epsilon \), we obtain
\[
\frac{d^2}{dt^2} V_{\varphi_R}(u(t)) \leq 8E(u_0) < 0,
\]
for any \( t \in [0, T) \). The standard convexity argument then implies \( T < +\infty \) or the solution blows up in finite time. Let us now choose \( \varphi_R \) so that (4.14) is satisfied. To do this, we introduce the smooth function \( \theta : [0, \infty) \to [0, \infty) \) satisfying
\[
\theta(r) := \begin{cases} 
2r & \text{if } 0 \leq r \leq 1, \\
2[r - (r - 1)^3] & \text{if } 1 < r \leq 1 + 1/\sqrt{3}, \\
\theta' < 0 & \text{if } 1 + 1/\sqrt{3} < r < 2, \\
0 & \text{if } r \geq 2,
\end{cases}
\]
and define
\[
\theta(t) := \int_0^t \theta(s) ds.
\]
It is not hard to check that \( \theta \) satisfies (4.10) and the function \( \varphi_R \) defined as in (4.11) satisfies (4.14). We refer the reader to [11] for more details. \( \square \)

We also need the so-called Brezis-Lieb’s lemma (see [6]).

**Lemma 4.8** (Brezis-Lieb’s lemma [6]). Let \( 0 < p < \infty \). Suppose that \( f_n \to f \) almost everywhere and \( (f_n)_{n \geq 1} \) is a bounded sequence in \( L^p \), then
\[
\lim_{n \to \infty} (\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p) = \|f\|_{L^p}^p.
\]

We are now able to prove the instability result given in Theorem 4.4.

**Proof of Theorem 4.4.** Let \( (\mu_n)_{n \geq 1} \) and \( (\lambda_n)_{n \geq 1} \) be sequence of positive real numbers satisfying \( \mu_n > 1 \), \( \lim_{n \to \infty} \mu_n = 1 \) and \( \lim_{n \to \infty} \lambda_n = 1 \). Let \( \omega > 0 \) and \( Q \) be a radial \( H^1 \)-solution to the elliptic equation (4.8). Denote
\[
u_{0,n}(x) := \mu_n \lambda_n^{\frac{d}{2}} Q(\lambda_n x).
\]
It is easy to see that
\[
\|\nu_{0,n}\|_{L^2} = \mu_n \|Q\|_{L^2}, \quad \|\nabla \nu_{0,n}\|_{L^2} = \mu_n \lambda_n \|\nabla Q\|_{L^2}, \quad \|\nu_{0,n}\|_{\dot{H}^1} = \mu_n \lambda_n \|Q\|_{\dot{H}^1},
\]
for all \( n \geq 1 \). Moreover,
\[
\lim_{n \to \infty} \|\nu_{0,n}\|_{L^2} = \lim_{n \to \infty} \mu_n \|Q\|_{L^2} = \|Q\|_{L^2}.
\]
By Lemma 4.8, we see that
\[
u_{0,n} \to Q \text{ strongly in } L^2.
\]
Similarly,
\[
\lim_{n \to \infty} \|\nabla \nu_{0,n}\|_{L^2} = \lim_{n \to \infty} \mu_n \lambda_n \|\nabla Q\|_{L^2} = \|\nabla Q\|_{L^2}.
\]
Lemma 4.8 again implies
\[
u_{0,n} \to Q \text{ strongly in } \dot{H}^1.
\]
as $n \to \infty$. Therefore, $u_{0,n} \to Q$ strongly in $H^1$ as $n \to \infty$. It remains to show that $u_n$ blows up in finite time for $n$ sufficiently large. By Lemma 4.7, it suffices to show that

$$E(u_{0,n}) < 0,$$

for all $n \geq 1$. To see (4.16), we use (4.9) to have

$$E(u_{0,n}) = \frac{1}{2}\|u_{0,n}\|_{H^1}^2 - \frac{d}{2d+4}\|u_{0,n}\|_{L^{\frac{4}{d+2}}}^{\frac{4}{d+2}}$$

$$= \frac{1}{2}\mu_n^2\lambda_n^2\|Q\|_{H^1_0}^2 - \frac{d}{2d+4}\mu_n^2\lambda_n^2\|Q\|_{L^{\frac{4}{d+2}}}^{\frac{4}{d+2}}$$

$$= \frac{1}{2}\left(1 - \mu_n^2\right)\mu_n^2\lambda_n^2\|Q\|_{H^1_0}^2.$$  

By the choice of $\mu_n$ and $\lambda_n$, we conclude that (4.16) holds for any $n \geq 1$. The proof is complete. □

**APPENDIX**

In this short appendix, we will justify (1.7) and the radial symmetry of the limit in the compact embedding (3.12).

Let us first justify (1.7). Let $v \in S_M$, that is, $v \in H^1$, $\|v\|_{L^2}^2 = M$ and $E(v) = d_M$. For $s \in \mathbb{R}$ and $\varphi$ a test function, we set

$$v_s := v + s\varphi, \quad \lambda_s := \frac{\|v\|_{L^2}}{\|v_s\|_{L^2}} \quad \text{and} \quad w_s := \lambda_s v_s.$$  

We see that $\|w_s\|_{L^2}^2 = \|v\|_{L^2}^2 = M$, hence $E(w_s) \geq d_M$ for all $s \in \mathbb{R}$. We will prove that $F : s \mapsto E(w_s)$ is differentiable at $s = 0$. Since the minimum of this function is attained at $s = 0$, we get $F'(0) = 0$. On the other hand, we have from a direct computation that

$$E(w_s) = \frac{\lambda_s^2}{2}\|\nabla v_s\|_{L^2}^2 - \frac{c\lambda_s^2}{2}\|\nabla v_s\|_{L^2}^2 = 2\lambda_s^{\alpha+2} - \frac{\lambda_s^{\alpha+2}}{\alpha+2}\|v_s\|_{L^{\alpha+2}}^{\alpha+2},$$

and

$$\frac{d}{ds}\bigg|_{s=0} \|\nabla v_s\|_{L^2}^2 = 2\text{Re} \langle -\Delta v, \varphi \rangle, \quad \frac{d}{ds}\bigg|_{s=0} \|\nabla v_s\|_{L^2}^2 = 2\text{Re} \langle |x|^{-2}v, \varphi \rangle,$$

$$\frac{d}{ds}\bigg|_{s=0} \|v_s\|_{L^{\alpha+2}}^{\alpha+2} = (\alpha+2)\text{Re} \langle |v|^\alpha v, \varphi \rangle,$$

as well as

$$\frac{d}{ds}\bigg|_{s=0} \lambda_s^2 = -\frac{2}{\|v\|_{L^2}^2}\text{Re} \langle v, \varphi \rangle, \quad \frac{d}{ds}\bigg|_{s=0} \lambda_s^{\alpha+2} = -\frac{\alpha+2}{\|v\|_{L^2}^2}\langle v, \varphi \rangle.$$  

Inserting the above identities to $F'(0) = \frac{d}{ds}\big|_{s=0} E(w_s) = 0$, we obtain

$$\text{Re} \langle -\Delta v - c|x|^{-2}v - \frac{E(v)}{\|v\|_{L^2}^2}v - |v|^\alpha v, \varphi \rangle = 0.$$  

Testing the above equality with $i\varphi$ instead of $\varphi$ and using the fact $\text{Re}(iz) = -\text{Im}(z)$, we get

$$\text{Im} \langle -\Delta v - c|x|^{-2}v - \frac{E(v)}{\|v\|_{L^2}^2}v - |v|^\alpha v, \varphi \rangle = 0.$$  

Therefore, $v$ solves

$$-\Delta v - c|x|^{-2}v - \frac{d_M}{M}v - |v|^\alpha v = 0.$$  

This solves (1.7) with $\omega = -\frac{d_M}{M} > 0$. 
We next justify the radial symmetry of the limit in the compact embedding (3.12). It is well-known that if \((u_n)_{n \geq 1}\) is a bounded sequence of \(H_{1,ad}^1\) functions, then there exist a subsequence still denoted by \((u_n)_{n \geq 1}\) and \(u \in H^1\) such that

\[ u_n \rightharpoonup u \text{ weakly in } H^1 \text{ and } u_n \rightarrow u \text{ strongly in } L^p, \]

for any \(2 < p < \frac{2d}{d-2}\). We will show that \(u\) is radial. Indeed, since \(u_n\) is radial, there exists \(f_n : \mathbb{R}^+ \rightarrow \mathbb{C}\) such that \(u_n(x) = f_n(|x|)\). The sequence \((u_n)_{n \geq 1}\) is strongly convergent in \(L^p\), so it is a Cauchy sequence in \(L^p\). This implies that \((f_n)_{n \geq 1}\) is a Cauchy sequence in \(L^p(\mathbb{R}^+, r^{d-1}dr)\), hence strongly convergent since the latter space is complete. That is, there exists \(f \in L^p(\mathbb{R}^+, r^{d-1}dr)\) such that

\[ f_n \rightarrow f \text{ strongly in } L^p(\mathbb{R}^+, r^{d-1}dr). \]

We infer that there exists a subsequence \((f_{n_k})_{k \geq 1}\) of \((f_n)_{n \geq 1}\) and \(I \subset \mathbb{R}^+\) with \(\int_I r^{d-1}dr = 0\) such that for all \(r \in \mathbb{R}^+ \setminus I\),

\[ f_{n_k}(r) \rightarrow f(r), \]

as \(k \rightarrow \infty\). On the other hand, since \(u_{n_k} \rightarrow u\) strongly in \(L^p\), there exists a subsequence \((u_{n_{k_l}})_{l \geq 1}\) of \((u_{n_k})_{k \geq 1}\) and \(\Omega \subset \mathbb{R}^d\) with \(\int_{\Omega} dx = 0\) such that for all \(x \in \mathbb{R}^d \setminus \Omega\),

\[ u_{n_{k_l}}(x) \rightarrow f(|x|), \]

as \(l \rightarrow \infty\). Set \(X := \Omega \cup \{x \in \mathbb{R}^d : |x| \in I\}\). One easily checks that \(X\) has Lebesgue measure zero. Let \(x \in \mathbb{R}^d \setminus X\). On one hand, \(u_{n_{k_l}}(x) \rightarrow u(x)\) as \(l \rightarrow \infty\) (since \(x \notin \Omega\), and on the other hand, \(u_{n_{k_l}}(x) = f_{n_{k_l}}(|x|) \rightarrow f(|x|)\) as \(l \rightarrow \infty\) (since \(|x| \notin I\)). One thus get \(u(x) = f(|x|)\) or \(u\) is radial.

Acknowledgments

The first author would like to thank his thesis advisor, Pr. Sahbi Keraani, for his suggestions and encouragement. The second author would like to express his deep gratitude to his wife-Uyen Cong for her encouragement and support. The authors would like to thank the reviewers for their helpful comments and suggestions.

References

[1] A. Bensouilah, \(L^2\) concentration of blow-up solutions for the mass-critical NLS with inverse-square potential, preprint arXiv:1803.05944, 2018. 1, 3, 5, 6
[2] A. Bensouilah, V. D. Dinh, Mass concentration and characterization of finite time blow-up solutions for the nonlinear Schrödinger equation with inverse-square potential, preprint arXiv:1804.08752, 2018. 15
[3] T. Cazenave, P.L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. 85 (1982), 549-561. 2
[4] J. M. Bouclet, H. Mizutani, Unifrom resolvent and Strichartz estimates for Schrödinger equations with critical singularities, to appear in Trans. Amer. Math. Soc. 2017. 2
[5] N. Burq, F. Planchon, J. Stalker, A. S. Tahvildar-Zadeh, Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential, J. Funct. Anal. 203 (2003), 519-549. 2
[6] H. Brézis, E. H. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983), 486–490. 17
[7] H. E. Camblong, L. N. Epele, H. Fanchiotti, C. A. Garcia Canal, Quantum anomaly in molecular physics, Phys. Rev. Lett. 87 (2001), No. 22, 220302. 1
[8] K. M. Case, Singular potentials, Physical Rev. 80 (1950), No. 2, 797–806. 1
[9] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics 10, Courant Institute of Mathematical Sciences, AMS, 2003. 14
[10] E. Csoó, F. Genoud, Minimal mass blow-up solutions for the \(L^2\) critical NLS with inverse-square potential, Nonlinear Anal. 168 (2018), 110–129. 4, 13, 14, 15
[11] V. D. Dinh, Global existence and blow-up for a class of the focusing nonlinear Schrödinger equation with inverse-square potential, preprint arXiv:1711.04792, 2017. 5, 14, 16, 17
[12] B. Feng, H. Zhang, Stability of standing waves for the fractional Schrödinger-Hartree equation, J. Math. Anal. Appl. 460 (2018), 352–364. 2, 6
[13] P. Gérard, Description du défaut de compacité de l’injection de Sobolev, ESAIM Control Optim. Calc. Var. 3 (1998), 213–233. 6
[14] R. T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equation, J. Math. Phys. 18 (1977), 1794–1797. 15
[15] T. Hmidi, S. Keraani, Blow-up theory for the critical nonlinear Schrödinger equation revisited, Int. Math. Res. Not. 46 (2005), 2815–2828. 6
[16] R. T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equation, J. Math. Phys. 18 (1977), 1794–1797. 15
[17] T. Hmidi, S. Keraani, blow-up theory for the critical nonlinear Schrödinger equation revisited, Int. Math. Res. Not. 46 (2005), 2815–2828. 6
[18] R. Killip, J. Murphy, M. Visan, J. Zheng, The focusing cubic NLS with inverse-square potential in three space dimensions, Differential Integral Equations 30, No. 3-4 (2017), 759-787. 2, 5
[19] R. Killip, C. Miao, M. Visan, J. Zhang, J. Zheng, Sobolev spaces adapted to the Schrödinger operator with inverse-square potential, To appear in Math. Z. 2017. 2
[20] R. Killip, C. Miao, M. Visan, J. Zhang, J. Zheng, The energy-critical NLS with inverse-square potential, Discrete Contin. Dyn. Syst. 37 (2017), 3831-3866. 2
[21] J. Lu, C. Miao, J. Murphy, Scattering in $H^1$ for the intercritical NLS with an inverse-square potential, J. Differential Equations 264 (2018), No. 5, 3174-3211. 2
[22] N. Okazawa, T. Suzuki, T. Yokota, Energy methods for abstract nonlinear Schrödinger equations, Evol. Equ. Control Theory 1 (2012), 337-354. 2
[23] C. Peng, Q. Shi, Stability of standing waves for the fractional nonlinear Schrödinger equation, J. Math. Phys. 59 (2018), 011508. 2, 6
[24] P. Trachanas, N. B. Zographopoulos, Orbital stability for the Schrödinger operator involving inverse square potential, J. Differential Equations 259 (2015), 4989–5016. 3
[25] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 87 (1983), 567-576. 5
[26] J. Zhang, J. Zheng, Scattering theory for nonlinear Schrödinger with inverse-square potential, J. Funct. Anal. 267 (2014), 2907-2932. 2
[27] J. Zhang, Stability of Attractive Bose-Einstein Condensates, Journal of Statistical Physics, 101 (2000), 731-746. 2
[28] G. Fibich, The nonlinear Schrödinger equation: singular solutions and optical collapse, Springer, 2015. 2
[29] S. H. Zhu, J. Zhang, H. Yang, Limiting profile of the blow-up solutions for the fourth-order nonlinear Schrödinger equation, Dyn. Partial Differ. Equa. 7 (2010), No. 2, 187–205. 6
[30] S. H. Zhu, On the blow-up solutions for the nonlinear fractional Schrödinger equation, J. Differential Equations 261 (2016), No. 2, 1506–1531. 6
[31] S. H. Zhu, Existence of stable standing waves for the fractional Schrödinger equations with combined nonlinearities, J. Evol. Equ. 17 (2017), No. 3, 1003–1021. 2, 6

Laboratoire Paul Painlevé (U.M.R. CNRS 8524), U.F.R. de Mathématiques, Université Lille 1, 59655 Villeneuve d’Ascq Cedex, France
E-mail address: ai.bensouilah@math.univ-lille1.fr

Institut de Mathématiques de Toulouse UMR5219, Université de Toulouse CNRS, 31062 Toulouse Cedex 9, France and Department of Mathematics, HCMC University of Pedagogy, 280 An Duong Vuong, Ho Chi Minh, Vietnam
E-mail address: dinhvan.duong@math.univ-toulouse.fr

College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066, China; School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China
E-mail address: shihuizhumath@163.com; shihuizhumath@uestc.edu.cn