In the context of the two-body problem in General Relativity, hereditary terms in
the long range gravitational field depend on the history rather than the instantaneous
state of the source at retarded time. We compute the next-to leading effects of such
hereditary terms, that comprise tail and memory, on the two-body dynamics, within
effective field theory methods, including both dissipative and conservative effects.
The former confirm known results at 2.5 post-Newtonian order with respect to the
leading order in the luminosity function; the conservative part is a new result and
is an unavoidable ingredient for a derivation of the conservative two-body dynamics
at fifth post-Newtonian order.

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*stefano.foffa@unige.ch, riccardo@iip.ufrn.br
I. INTRODUCTION

The recent detections of Gravitational Waves (GWs) (see [1] for a summary of all confirmed detections up to the time of writing), beside marking the beginning of the new science named GW Astronomy, have triggered scientific interest over all aspects of GW production and detection.

To maximize the efficiency of the search for signals from compact binary coalescences, the output of GW detectors like LIGO [2] and Virgo [3] is processed via matched-filtering [4], which is particularly sensitive to the phase of the GW signals. This is a mixed blessing having the downside of faithful parameter reconstruction depending on the availability of accurate model of signals, and the advantage of offering a unique probe to the quantitative details of the highly non-linear regime of General Relativity (GR).

One of the pillars to construct waveform templates for the LIGO/Virgo data analysis pipeline has been the post-Newtonian (PN) approximation to GR, see [5] for a review, which is a perturbative method expanding the two-body dynamics around the Newtonian result, with expansion parameter the relative velocity \( v \), where \( v^2 \sim G_N M/r \) for Kepler law (with \( G_N \) the Newton’s constant, \( M \) to total mass of the binary system, \( r \) the binary constituent mutual distance, using natural units for the speed of light \( c = 1 \)), and \( n \)-PN corrections corresponding to terms of the order \( G_N^{n-j+1} v^{2j} \), with \( 0 \leq j \leq n + 1 \). To construct accurate waveform templates describing the entire coalescence, including the merger of the two bodies, the PN-approximation must be completed with non-perturbative results derived from numerical simulation, see e.g. [6] for one of the most complete numerical waveform catalogs, which has brought to the successful implementation of phenomenological models [7–9] merging information from analytic and numerical relativity. In particular the Effective Field Theory (EFT) approach to the PN approximation to GR adopted here, pioneered in [10], see [11–14] for reviews, has, among others, the undeniable advantage of recasting the GR 2-body problem into the powerful language of field theory scattering amplitudes, which has been developed and enriched of deep theoretical insight over decades.

At present the dynamics is known in the conservative sector up to 4th PN order [15–21], i.e. next-next-next-next-to-leading order (N^4LO) for the spin-less terms, and up to 3.5PN and 4PN order [22–31] for terms including spin. In the dissipative sector current knowledge of the luminosity function extends to 3.5PN order for spin independent [32, 33] and to
3.5PN for linear-in-spin \((N^2\text{LO})\) \cite{34} terms (and to 4PN order for tail and linear-in-spin terms \((\text{NLO})\) \cite{35}, see below for tail definition), up 3PN for terms quadratic in spins \((\text{NLO})\) \cite{36} and at leading order \((3.5\text{PN})\) for spin cube effects \cite{37}. The leading PN order for spin interaction to the \(m\)-th power for both dissipative and conservative sector were computed in \cite{38}, corresponding to \((1/2 + m)\)-PN order for \(m\) odd and to \((m)\)-PN order for \(m\) even. In the spin-less case, the fifth PN order in the conservative sector (which is the main focus of this work) is qualitatively different from the lower ones, where finite size effects cannot affect the dynamics as per the effacement principle \cite{39}. In the case of neutron star finite size effects are parametrized by tidal Love numbers \cite{39,40}, whose first preliminary measure has been enabled by the detection of GW170817 \cite{41,42}, whereas for black holes it has actually been shown \cite{43–46} that tidal deformation vanishes in the static case, pushing its effect to higher orders (see \cite{47–49} for slowly spinning black holes).

In the PN approach it is convenient to divide the problem of binary dynamics in a near and a far zone: the former describes the conservative dynamics around the sources at a distance \(\sim r\) at which it is possible to resolve the individual constituents of the binary system, the latter describes the dynamics from a much larger distance \(r/v = \lambda\) with \(\lambda\) the GW wavelength, where the binary system can be described by a single object endowed with multipoles.

An intriguing aspect of the post-Newtonian approximation is that near and far zone are not disconnected: while near zone results determine conservative dynamics only, far zone ones give contribution to both conservative and dissipative dynamics, and their contribution is necessary to obtain consistent results in the conservative sector, in particular the unambiguous and systematic cancellation of spurious infra-red divergences in the near zone \cite{50}.

The contribution of far zone dynamics to conservative physics was first observed in \cite{51}, where the effect of GW emitted by the binary system and scattered off the quasi-static curvature onto the same GW source was (partially) computed and the name of tail terms was coined to indicate the “back-scattering” of GWs. This process was understood as part of phenomena where the near-zone physics depends on the full past history of the source, hence the name hereditary, also coined in \cite{51}, rather than just on the source state at retarded time. Hereditary terms affect the phasing of the gravitational waveform via the tail effect \cite{52,53}, see also \cite{25,54} for an EFT derivation, but also via scattering off the curvature...
induced by GW themselves, that is the memory effect. Memory effect causes a cumulative change in the waveform, that does not vanish after the passage of the radiation, originally derived in [55] and first derived in the binary system context in [52].

The present work is adding another brick to the construction of a complete precision gravity program that maximizes the physics output of GW detection, while at the same time providing further insight into intriguing theoretical aspects of the general relativistic two-body dynamics. More in detail, we present the original result of next-to-leading order hereditary processes with no external radiation, as done in [59, 60] at leading order, from which it is possible to extract contributions to the conservative dynamics and the luminosity function. The leading tail contribution to the luminosity function, determined by the imaginary part of the amplitude, is at 1.5PN order with respect to the leading order quadrupole formula, while the real part contributes at 4PN order to the conservative sector where it has a divergent and a finite piece: the former is regularized by properly adding similarly divergent terms from the near zone dynamics [50], see [61] for a general treatment of divergence cancellation in theories where momentum integrals are separated in regions (like in the near-far zone case), dubbed zero bin subtraction; the latter turns from hereditary to instantaneous when computed over circular orbits, contributing (beside rational terms) with a logarithmic term to the energy of circular orbit, which has been determined at 5PN order in [62] and in [63] from gravitational self-force computation.

The memory term gives no contribution to the luminosity function, however it starts contributing to the conservative dynamics at 5PN order, and its value, which is of the same order of the next-to-leading tail effect, is computed in this work for the first time. Note that differently from the tail effect, which is of hereditary type both in the GW phase and in the conservative dynamics (for generic orbits), the memory terms entering the conservative dynamics are not hereditary, as they affect the gravitational waveform $h$ via a non-local in time term which can be written as the time integral of an instantaneous term, therefore giving an instantaneous contribution to the energy which depends on the time derivative of $h$.

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1 The presence of memory effects was noticed in linearized gravity already in [56], where the passage of GWs sourced by moving massing objects was identified to cause a permanent displacement between test particles, not fading away after the gravitational perturbation as gone quiet. It was later quantified in [57, 58] to leading order $1/r$ in linearized gravity and found to relate the difference in the gravitational radiative field at early and late times to the source velocities at early and late times.

2 See also [64] for related work on 5PN logs.
The plan of the paper is the following: in sec. II we outline the EFT method we use to perform the computation of the 5PN hereditary terms. In sec. III we detail the computation and present the result, including the determination of several new, unpublished terms in the conservative sector, while confirming previous findings in the dissipative one, and we finally conclude in sec. IV.

II. METHOD

On length scales larger than the orbital separation, the multipole moments of the binary system are the relevant degrees of freedom when it comes to describe its interaction with the gravitational field; the effective Lagrangian governing the dynamics of the system is

\[ S_{\text{mult}} = -\frac{1}{\Lambda} \left\{ \int \tau \left[ E + \frac{1}{2} \dot{x}^{\mu} L_{\alpha \beta} \omega_{\mu}^{\alpha \beta} \right. \right.
\]
\[-\frac{1}{2} \sum_{n \geq 0} \left( c_n^I T_{\mu_1 \ldots \mu_n}^{\alpha \beta} \mathcal{E}_{\alpha \beta ; \mu_1 \ldots \mu_n} + c_n^J J_{\mu_1 \ldots \mu_n}^{\alpha \beta} B_{\alpha \beta ; \mu_1 \ldots \mu_n} \right) \right\} \]
\[ \simeq \frac{1}{\Lambda} \int dt \left[ \frac{1}{2} Eh_{00} + \frac{1}{2} \epsilon_{ijk} L^j h_{0j,k} + \frac{1}{2} Q^{ij} \mathcal{E}_{ij} + \frac{1}{6} O^{ijk} \mathcal{E}_{ij,k} - \frac{2}{3} J^{ij} B_{ij} + \ldots \right], \] (1)

where \( \Lambda \) is related to the \( d \) dimensional gravitational constant \( G_d \) by \( \Lambda^2 \equiv 1/(32\pi G_d) \). In the first line of eq. (1) the mass \( E \) and the angular momentum \( L_{\alpha \beta} \) coupling to the spin connection have been singled out (neglecting total momentum since we assume to work in the center-of-mass frame), and electric and magnetic multipoles of generic order have been indicated with \( I \) and \( J \), respectively, and are coupled to the appropriate curvature tensors. In the second line we have expanded the metric around Minkowski and reported only terms needed up to next-to-leading order: the multipole series is an expansion in powers of \( v = r/\lambda \), being \( r \) the size of the source and \( \lambda \) the length curvature scale of the gravitational field coinciding with the GW-length, which is not independent on the size of source and its internal velocity \( v \).

We have also expanded at linear order in the gravitational perturbation \( h_{\mu \nu} \) around Minkowski spacetime, and made explicit space-time decomposition (Latin indices running over spatial dimension only).

The gravitational field is to be evaluated at the center of mass of the system, and the relevant electric and magnetic tensors components read

\[ \mathcal{E}_{ij} \equiv R_{0i0j} \simeq \frac{1}{2} \left( h_{00,ij} + \dot{h}_{ij} - \dot{h}_{0i,j} - \dot{h}_{0,j,i} \right), \]

\[ \epsilon_{ijk} L^j = T^{0j} x^k - T^{0k} x^j, \text{ i.e. with a minus sign with respect to the standard definition.} \]
and $B_{ij} \equiv \frac{1}{2} \epsilon_{ikl} R_{0jkl}$, with $R_{0jkl} \approx \frac{1}{2} \left( \dot{h}_{jk,l} - \dot{h}_{jl,k} + h_{0l,jk} - h_{0k,jl} \right)$, being $R^\mu_{\nu\rho\sigma}$ the standard Riemann tensor.

The multipole moments, $E$, $\vec{L}$, $Q_{ij}$, $O^{ijk}$ are respectively the energy, spin, mass quadrupole and octupole moments of the system, the last two symmetric and traceless, and the current quadrupole moment is defined as $J^{ij} \equiv -\frac{1}{2} \int \left( j^i x^j + j^j x^i \right)$ (with $x^i T_{0j} - x^j T_{0i} \equiv \epsilon_{ijkl} j^k$, $T^{ij}$ denoting the energy momentum tensor, implying that $J^{ij}$ is also traceless).

The explicit expression of the multipole moments in terms of the individual constituent of the binary system will not be needed until next section, where we will derive the logarithmic energy shift for a binary system in circular orbit. Such expressions can be determined by a matching procedure, i.e. computing the coupling of external gravitational field to the binary energy momentum tensor [10, 12], so they include also the contribution of the gravitational interaction at the orbital scale; the last remark is relevant for $E$ and $Q_{ij}$ which need to be considered at next-to-leading order.

Eq. (1) can be explicitly derived from the fundamental coupling $T^{\mu\nu} h_{\mu\nu}$ via multiple derivations by parts and reiterated use of the equations of motion; here we retained only terms which do not vanish on the equations of motion.

The present work reports the computation of a certain class of self-energy diagrams, depicted in fig. 1, representing self-energy corrections due to the source interacting with the GWs produced by itself. The imaginary part of these diagrams is related to the power emission, while the real part gives their direct contribution to the potential ruling the conservative dynamics.

We work in dimensional regularization and the real parts of some of the hereditary diagrams present short-distance (UV) poles, which cancel against long-scale (IR) spurious poles in the effective Lagrangian at the orbital scale (near zone), according to the zero-bin prescription [61, 65, 66] as explicitly shown at 4PN order in [50]. The finite terms remaining after such divergences cancellation depend unambiguously on the finite contributions to the real parts of the hereditary processes we show in the next section, and they are necessary to complete the determination of the near zone dynamics.

The hereditary processes in fig. 1 involve a “bulk” three-field interaction that can be read from the (gauge-fixed) action for gravity, which with our choice of the harmonic gauge reads

$$S_{EH+GF} = \frac{1}{16\pi G_d} \int d^{d+1} x \sqrt{-g} \left[ R(g) - \frac{1}{2} \Gamma^\mu \Gamma_\mu \right], \quad (2)$$

$$\frac{1}{2} \epsilon_{ikl} R_{0jkl}, with R_{0jkl} \approx \frac{1}{2} \left( \dot{h}_{jk,l} - \dot{h}_{jl,k} + h_{0l,jk} - h_{0k,jl} \right), being R^\mu_{\nu\rho\sigma} the standard Riemann tensor.

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Figure 1. Hereditary diagrams contributing from leading order (top one), and from next-to leading order (the remaining four). Wiggled lines represent on-shell gravitons (i.e. gravitational waves), straight dashed and dotted ones stand for instantaneous propagators, i.e. interaction with static fields. A relativistic correction to the instantaneous propagator of the top diagram has not been considered because it gives a physically irrelevant term (see discussion in the text). The horizontal continuous black double line represent the external source given by the binary system.

with $\Gamma^\mu \equiv g^{\nu\rho} \Gamma_{\nu\rho}^\mu$. As in our previous works (see for instance [21] for details), we find convenient to decompose the metric into a scalar $\phi$, a vector $\vec{A}$ and a symmetric tensor $\sigma_{ij}$, which have the virtue of not mixing with each other at quadratic order. Expanding around Minkowski with the metric parametrization [67]

$$g_{\mu\nu} = e^{2\phi/\Lambda} \left( \frac{-1}{\Lambda} \frac{A_j}{\Lambda} e^{-c_4\phi/\Lambda} \left( \delta_{ij} + \frac{\sigma_{ij}}{\Lambda} - \frac{A_i A_j}{\Lambda^2} \right) \right),$$  \hspace{1cm} (3)
with \( c_d \equiv 2(d-1)/(d-2) \), one obtains the following action, truncated to cubic order

\[
S_{EH+GF} \supset \int d^{d+1}x \sqrt{-\gamma} \left\{ \frac{1}{4} \left[ (\nabla \sigma)^2 - 2(\nabla \sigma_{ij})^2 - (\delta^2 - 2(\sigma_{ij})^2) e^{-\frac{c_d^2}{\Lambda}} \right] - c_d \left[ (\nabla \phi)^2 - \phi^2 e^{-\frac{c_d^2}{\Lambda}} \right] \right\} + \frac{F_{ij}^2}{2} + \left( \nabla \cdot \vec{A} \right)^2 - \vec{A}^2 e^{-\frac{c_d^2}{\Lambda}} \right\} e^{-\frac{c_d^2}{\Lambda}} + \frac{2}{\Lambda} \left[ (F_{ij}A^i \dot{A}^j + \vec{A} \cdot \vec{A} (\nabla \cdot \vec{A})) - c_d \dot{\phi} A \nabla \phi \right] \right) + 2c_d \left( \dot{\phi} \nabla \cdot \vec{A} - \vec{A} \nabla \phi \right) \right) + \frac{\delta_{ij}}{\Lambda} \left( -\delta^{ij} A_i \dot{\Gamma}^l_{kk} + 2A_k \dot{\Gamma}^k_{ij} - 2A_i \dot{\Gamma}^i_{jk} \right) \right) - \frac{1}{\Lambda} \left( \sigma_{\delta_{ij}} - \sigma_{\delta_{ij}} \right) \left( \sigma_{\delta_{ik}} \sigma_{\delta_{jl}} - \sigma_{\delta_{ik}} \sigma_{\delta_{jl}} + \sigma_{\delta_{ik}} \sigma_{\delta_{jk}} - \sigma_{\delta_{ik}} \sigma_{\delta_{jk}} \right) \right) ,
\]

where \( F_{ij} \equiv A_{j,i} - A_{i,j} \), and \( \dot{\Gamma}^i_{jk} \) is the connection of the purely spatial \( d \)-dimensional metric \( \gamma_{ij} \equiv \delta_{ij} + \sigma_{ij} / \Lambda \), which is also used above to raise and contract spatial indices. All spatial derivatives are understood as simple (not covariant) derivatives and when ambiguities might raise gradients are always meant to act on contravariant fields, e.g. \( \nabla \vec{A} \equiv \gamma_{ij} A_{i,j} \) and \( F_{ij}^2 \equiv \gamma_{ik} \gamma_{jl} F_{ij} F_{kl} \).

In general, the amplitude for a generic hereditary process \( A_{her} \) has the following structure:

\[
A_{her} = \int_{k,q} \frac{dq_0 dk_0}{2\pi^2} \frac{d_{M^{(1)}_{i_1...i_n}(q_0) M^{(2)}_{j_1...j_m}(k_0) M^{(3)}_{k_1...k_n}(-k_0 - q_0)}}{D} \frac{P_{i_1...i_j...j_m k_1...k_n} (k, q, k_0, q_0)}{D \mid_{q_0=0}}
\]

with \( M^{(i)}_{i_1...i_n} \) being the (Fourier-transformed) generic multipole moment and we introduced the notation \( \int_{p} \equiv \int \frac{dp}{(2\pi)^d} \), while the (inverse of the) factor \( D \equiv (k^2 - k_0^2)(q^2 - q_0^2)[(k + q)^2 - (k_0 + q_0)^2] \) collects the product of the scalar parts of the three propagators involved. Also some more elementary integrals involving only two factors in the denominator are involved in amplitude computations, and they are reported in app. \[B\] in the case of tail integrals one of the sources is actually conserved (all but the bottom right diagram in fig. \[1\]) and substituting \( M^{(1)}_{i_1...i_n}(q_0) \rightarrow 2\pi \delta(q_0) \tilde{M}_{i_1...i_n} \) the amplitude simplifies as one propagator become instantaneous:

\[
A_{tail} = \int_{k,q} \frac{dk_0}{2\pi} \frac{\tilde{M}_{i_1...i_j} M^{(2)}_{j_1...j_m}(k_0) M^{(3)}_{k_1...k_n}(-k_0)}{D \mid_{q_0=0}} \frac{P_{i_1...i_j...j_m k_1...k_n} (k, q, k_0, 0)}{D \mid_{q_0=0}}
\]

In this case we can use Feynman boundary conditions for the propagators, which give the (time-symmetric) real Lagrangian contributing to the near zone conservative dynamics, whereas the imaginary part returns the averaged probability loss (related to the energy loss).

\[4\] Such substitution is identically true for the diagrams of the second line of figure \[1\] for which \( E \approx M \) so \( \tilde{M} = 0 \) identically, while for the upper diagram and for the lower left one this is true only modulo terms which vanish on the equations of motion, which are neglected here because they do not give contribution to gauge-invariant quantities (as they can be removed by an unphysical coordinate shift).
Had we been interested in computing back-reaction force or instantaneous radiation field, we should have used in-in correlators as explained in detail in [68]. On the other hand, when all three sources are dynamical, we will resort to retarded boundary conditions, see [69] for detailed explanations.

Given these premises, all the integrals can be reduced in terms of the following master integral

\[ I_m(k_0, q_0) \equiv \int_{k, q} \frac{1}{D}, \]  

plus other more elementary ones. \( I_m \) is UV-divergent and has been extensively studied in particle physics, see e.g. [70], as it is the master integral of the two-loop vacuum diagram, also relevant for two-loop self-energy diagrams in gauge theories. We note however that only the specific case \( I_m(k_0, 0) \) (still UV-divergent) appears in our final results, namely in the first three diagrams of figure 1; this happens when the diagram is UV-divergent but one of the three sources is conserved, as in eq. (6). On the other hand \( I_m(k_0, q_0) \) appears only in some intermediate steps of the bottom right diagram and cancels in the final answer because such process is UV-finite.

### III. RESULTS

#### A. General properties of tails

We start by reporting the result of the first amplitude (top of fig. 1), which appears at leading order (1.5PN for power emission, and 4PN for the conservative part), and has been already considered within the EFT approach in [59] [60]. At linear order the electric part of the Riemann tensor reads

\[ R_{00ij} = \frac{1}{2} \dddot{\sigma}_{ij} - \frac{1}{2} \dddot{A}_{i,j} - \frac{1}{2} \dddot{A}_{j,i} - \phi_{,ij} - \delta_{ij} \dddot{\phi} + O(h^2). \]  

The leading tail amplitude is represented by the top diagram in fig. 1 and its calculation is detailed here below (omitting the propagator pole displacement in the complex plane, which is understood to follow Feynman prescription) and decomposed for polarizations: the only gravity polarization coupling to the conserved energy is \( \phi \), thus one has six possibilities in terms of different polarizations for the three-point vertex.
After neglecting terms proportional to quadrupole traces the leading tail amplitude reads:

\[
iS_{\text{eff}4\text{PN}}^{EQ} = -i64\pi^2 G_d^2 E \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} Q^{ij}(k_0)Q^{kl}(-k_0) \int_{k,q} \frac{1}{k^2 - k_0^2} \frac{1}{(k + q)^2} \frac{1}{k^2 q^2} x \\
\{ - \frac{1}{8} k_0^6 (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \phi \sigma^2 \\
- k_0^4 (k_ijkl) \phi A \sigma \\
+ \frac{k_0^2}{c_d} q_i k_j k_i k_l \phi^2 \sigma \xi \\
+ \frac{1}{2} k_0^2 [k_i k_j q_i q_l - q_i k_i q_j k_l + \delta_{ik} k_j (k + q) k \cdot (k + q)] \phi A^2 \\
- \frac{k_0^2}{c_d} q_i (k + q) k_i k_j \phi^2 A \\
- \frac{1}{2 c_d} k_0^2 q_i (k + q) k_i (k + q) \phi^3 \} \]

giving the result

\[
S_{\text{eff}4\text{PN}}^{EQ} = -\frac{G_N E}{5} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} k_0^6 \left[ \frac{1}{\epsilon} - \frac{41}{30} - i\pi + \log \left( \frac{k_0^2 \epsilon^2}{\pi \mu^2} \right) + O(\epsilon) \right] Q_{ij}(k_0) Q^{ij}(-k_0) (10)
\]

where \( \epsilon \equiv \mu - 3 \) and \( \mu \) is the dimensional constant introduced in dimensional regularization to relate standard 4-dimensional Newton constant \( G_N \) to the \( d \)-dimensional gravitational coupling \( G_d \equiv \mu^{-1} G_N \). The factor \(-\frac{41}{30}\), first derived in [59], enables to unambiguously determine the regularized near zone Lagrangian at 4PN as predicted in [60] and explicitly done in [18, 50].

From the imaginary part\(^5\) of eq. (10) the power loss \( P_{\text{tail}} \) can be derived by multiplying the integrand by \( k_0 \) and averaging over time. The leading order power loss is the quadrupole formula \( P_{QQ} \) which can be obtained by the imaginary part of the following diagram

\[
S_{\text{eff}2.5\text{PN}}^{EQ} = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_k \left( \begin{array}{c}
\text{Diagram}
\end{array} \right)
\]

\[
= -\pi G_N \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_k \frac{Q_{ij}(k_0)Q_{kl}(-k_0)}{k^2 - k_0^2} \left[ -k_0^4 \delta^{ik}\delta^{jl} + 2k_0^2 \delta^{ik} k^j k^l \right] \\
= \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} |k_0| k_0^4 Q_{ij}(k_0) Q^{ij}(-k_0),
\]

where the three terms in between round brackets, \( A_{\phi}^{ijkl} \equiv -k_0^4 \delta^{ik}\delta^{jl} \), \( A_A^{ijkl} \equiv 2k_0^2 \delta^{ik} k^j k^l \), \( A_{\phi}^{ijkl} \equiv -k^i k^j k^k k^l / 2 \) (apart from a common normalization) are the contributions respectively

\(^5\) Differently from [60], which presents the result in terms of the (+, -) variables if the in-in formalism, the imaginary part of (10) does not present the term \( \text{sgn}(k_0) \).
from the $\sigma, A, \phi$ polarizations. The result of $k$ integration, see app. [B], has vanishing real part and receives a finite imaginary part from the region of integration where $|k| \sim |k_0|$. In the tail diagram, the double integration over space momenta $k, q$ on purely heuristic arguments leads to an amplitude result $\propto G^2_d(-k_0^2)^{d-3}$ from which one can infer that the presence of an imaginary part is invariably linked to a divergence:

$$\frac{G^2_d}{\epsilon} (-k_0^2 - i0^+) = G^2_N \left( \frac{1}{\epsilon} + \log(k_0^2/\mu^2) - i\pi + O(\epsilon) \right),$$

implying that pole residual not only fixes the logarithmic term but also the imaginary one.

Focusing on the divergent part of the tail amplitude, it receives contributions only from processes involving the same graviton polarizations attaching to the two radiative sources, as they are the ones diverging when $q \to 0$ (see app. [B] for explicit computations), and it can be written as

$$S_{\text{pole}}^{\text{tail}} = -64\pi^2 G_N^2 E \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_{k,q} \frac{k_0^2 Q_{ij}(k_0)Q_{kl}(-k_0)}{q^2[(k + q)^2 - k_0^2](k^2 - k_0^2)} \times \left[ -k_0^4 \delta^{ik}\delta^{jl} + 2k^2\delta^{ik}k^j + - \frac{1}{2} k^i k^j k^k k^l \right].$$

Note that the terms in square brackets in (11) and (12) are the same, apart from the substitution $k^2 \to k_0^2$ in $A^{ijkl}_A$, and the $q$ integration of the two propagators involving $q$ factorizes from the rest of the amplitude with result $(-k_0^2)^{d/2-2} f(k_0^2/k_0^2)$ for some function $f$, which is the same for all multipoles as they do not depend on $q$, thus giving:

$$S_{\text{pole}}^{\text{tail}} = -64\pi^2 G_N^2 E \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} (-k_0^2)^{d/2-1} \int_{k} f \left( \frac{k^2}{k_0^2} \right) \frac{Q_{ij}(k_0)Q_{kl}(-k_0)}{k^2 - k_0^2} \times \left[ A_{ijkl} + \frac{k^2}{k_0^2} A^{ijkl}_A + A^{ijkl}_\phi \right].$$

Now observing that the imaginary part must originate from the $k^2 = k_0^2$ region of integration, it follows that $f(k_0^2/k_0^2)$ in (13) reduces to a UV divergent factor common to all multipoles (see appendix [B] for details), hence it can be fixed by its value for the quadrupole case.

In particular assuming that the fundamental source is composed by a binary system with reduced mass $\eta M$ ($\eta$ is the symmetric mass ratio) on a circular orbit of radius $r$ and orbital angular velocity $\omega$, so that the quadrupole component $Q_{xx}(k_0) = \eta M r^2 \pi/2 (\delta(k_0 - 2\omega) + \delta(k_0 + 2\omega))$ and using $\delta(0) = T/(2\pi)$, the leading order Kepler law $G_N M/r = (r\omega)^2$ and the integral in eq. (B2), one can derive the tail corrected quadrupole
formula

\[ P_{QQ+\text{tail}} = \frac{\eta^2 G_N M^2 r^4}{5} \text{Im} \left[ \int_0^\infty dk_0 \left( -k_0^2 \right)^{1/2} k_0^4 \delta(k_0 - 2\omega) \left( 1 + 2\pi G_N E k_0 \right) \right] \]

\[ = \frac{32\eta^2}{5G_N^2} x^5 \left( 1 + 4\pi x^{3/2} \right), \]

where in the final step we have introduced the standard post-Newtonian expansion parameter \( x \equiv (G_N M \omega)^{2/3} \). In the previous formula (14) the factor \( 1 + 2\pi G_N E k_0 \) is the universal leading tail correction for all multipoles\(^6\) universality already noticed in [71] and in [54].

Beside the imaginary term, clearly also the logarithmic term is fixed by the tail divergent piece, which we now understand to be just \( 2G_N E k_0 \) times the leading order imaginary part for all multipoles, hence we have proven how to compute the far zone logarithmic contribution to the conservative binary dynamics at \( n \)-PN order by the result for the flux at \( (n-4) \)-PN order, as suggested in [15]. As a consequence of this universality, one can write down the action for all the non-local simple tails (we are not considering composite effects like tails of tails here), where the coefficient of each term is given by the coefficient of the corresponding non-tail process in the power emission formula \( P = \frac{1}{5} \dot{Q}_{ij}^2 + \frac{1}{189} \dot{O}_{ijk}^2 + \frac{16}{45} J_{ij}^2 \ldots \) [72]:

\[ S_{\text{log}}^{\text{tail}} = -G_N^2 E \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \log \left( \frac{k_0^2}{\mu^2} \right) \sum_{n \geq 2} k_0^{2(n+1)} c_{n(I,J)}^{(\alpha\beta\mu_1\ldots\mu_{n-2})(k_0)(I,J)} \sum_{\alpha\beta\mu_1\ldots\mu_{n-2}} (-k_0) \]

\[ = -2G_N^2 E \int_{-\infty}^{\infty} \text{dt} \sum_{n \geq 2} (-1)^n c_{n(I,J)}^{(\alpha\beta\mu_1\ldots\mu_{n-2})(t)} \int_0^{\infty} d\tau (I,J)^{(2n+3)}_{\alpha\beta\mu_1\ldots\mu_{n-2}} (t - \tau) \log (\mu\tau) \]

with

\[ c_{I}^{n} = \frac{(n + 1)(n + 2)}{n(n - 1)n!(2n + 1)!!}, \]

\[ c_{J}^{n} = \frac{4n(n + 2)}{(n - 1)(n + 1)!(2n + 1)!!}. \]

**B. Next-to-leading order hereditary terms**

In this subsection we compute amplitudes giving hereditary effects at NLO, i.e. which start contributing to the power emission at 2.5PN order (unless they are vanishing) and at 5PN for the conservative part.

\(^6\) This includes also magnetic multipoles, for which a similar calculation can also be performed.
1. Octupole tail

Here we present the computation of the tail-octupole amplitude (upper right diagram of fig. 1)

\[ iS_{eff}^{MO^2} = -i \frac{64}{9} \pi^2 G^2 \int \frac{dk_0}{2\pi} O^{ijw}(k_0) O^{klr}(-k_0) \int_{k,q} \frac{k_w (k + q)_r}{k^2 - k_0^2 (k + q)^2 - k_0^2 q^2} \times \left\{ \begin{array}{l}
- \frac{1}{8} k_0^6 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\
- k_0^4 (k_j q_k \delta_{il}) \\
+ \frac{k_0^2}{c_d} q_i k_i k_j k_j \\
+ \frac{1}{2} k_0^2 [k_i q_k - k_k q_i + \delta_{ik} k \cdot (k + q)] k_j (k + q)_l \\
- \frac{1}{c_d} k_0^2 k_i k_j (k + q)_l \\
- \frac{1}{2c_d} k_0^2 k_i k_j (k + q)_l (k + q)_l \end{array} \right\} \{\phi^2, \phi A, \phi^2 A, \phi^3\}, \]

which adds to

\[ S_{eff}^{MO^2} = -\frac{G^2 N M}{189} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} k_0^8 \left[ \frac{1}{\epsilon} - \frac{82}{35} - i\pi + \log \left( \frac{k_0^2}{\mu^2} \right) + O(\epsilon) \right] O^{ijw}(k_0) O^{ijw}(-k_0). \]

Also in this case, the imaginary part respects the universality of tail terms described in the previous subsection, while the new finite real coefficient \(-\frac{82}{35}\), analogous to the \(-\frac{41}{30}\) of the quadrupole tail, is the finite correction to the near zone conservative binary dynamics originally derived in this paper. Note also that since this diagram is considered here at leading order, we are entitled to trade \(E\) with the total rest mass \(M\) in the result.

2. Magnetic quadrupole tail

The second diagram in the second line in fig. 1 represents the current quadrupole tail, which couples to the magnetic component of the Riemann tensor:

\[ R_{0jkl} = \frac{1}{2} (\hat{\sigma}_{jk,l} - \hat{\sigma}_{jl,k}) + \frac{1}{2} (A_{i,kj} - A_{k,ij}) + \frac{1}{d-2} \left( \hat{\phi}_{,k} \delta_{jl} - \hat{\phi}_{,l} \delta_{jk} \right) + O(h^2). \]
A simplification happens here, as all contributions with a $\phi$ polarization emitted by the magnetic quadrupole vanish, giving the result

$$iS^{JM^2\phi}_{\text{eff}5\text{PN}} = -\frac{1024}{9}\pi G^2 \int \frac{dk_0}{2\pi} J^{ij}(k_0) J^{kl}(-k_0) \int_{k,q} \frac{1}{k^2} \frac{1}{k_0^2} \frac{1}{(k+q)^2} \frac{1}{q^2} \times$$

$$\left\{ -\frac{1}{8} k_0^4 \left[ \delta_{jl} \left( \delta_{ik} (k+q) - k_i (k+q)_k + \epsilon_{jib} \epsilon_{lai} k^b (k+q)^a \right) \right] \{ \phi \sigma^2 \}$$

$$+ \frac{1}{8} k_0^6 \left[ \epsilon_{ijb} \epsilon_{kac} (k+q)^a \right] \left[ \delta_{lai} \delta_{jk} - \delta_{lij} \delta_{ak} \right] \{ \phi A \sigma \}^{(20)}$$

$$+ \frac{1}{8} \epsilon_{i\alpha\beta} \epsilon_{k\gamma\delta} k_j k_\beta (k+q)_l (k+q)_d \left[ \delta_{\alpha\gamma} k \cdot (k+q) + k_\gamma q_\alpha + k_\alpha q_\gamma \right] \{ \phi A^2 \}$$

$$+ 0 \right\} \{ \phi^2 \sigma \} \{ \phi^2 A \} \{ \phi^3 \}.$$ 

The result of this amplitude is

$$S^{JM^2\phi}_{\text{eff}5\text{PN}} = -\frac{16}{45} G^2 N M \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} k_0^6 \left( \frac{1}{\epsilon} - \frac{127}{60} - i\pi + \log \left( \frac{k_0^2 \epsilon \gamma}{\pi \mu^2} \right) + O(\epsilon) \right) J^{ij}(k_0) J^{ij}(-k_0),$$

and we find again the expected coefficients of pole, logarithmic term and imaginary part, as well as a second finite correction to the conservative dynamics, identified by the rational number $-\frac{127}{60}$.

3. Logarithmic contribution to energy of circular orbit

The finite, instantaneous corrections to the conservative dynamics computed above $(\frac{82}{6615} O^{(2)}_{ijk} + \frac{508}{675} \gamma^2_{ij})$ affect the conservative dynamics of the binary system, once expressed the multipoles in terms of individual binary constituent dynamical variables.

The hereditary logarithmic terms from the tail processes also become instantaneous when the generic multipoles are specialized to a binary system in circular orbit and then give finite, logarithmic contributions to the energy.

We expect that such logarithmic terms do not receive contributions from the near zone, as it happens at 4PN order, meaning that tail logarithms embody all of the logarithmic contribution to the energy of circular orbit at 5PN. Using the 1PN corrected expression of the quadrupole moment (see [54]) and the leading PN order of octupole and magnetic moments we can write explicitly the binary system energy of circular orbits

$$E_{\text{circ}} = -\frac{1}{2} \eta M x^2 \left\{ 1 + \ldots + x^4 \eta \log x \left[ \frac{448}{15} + x \left( -\frac{4988}{35} - \frac{656}{5} \eta \right) \right] \right\},$$

where $x = 1 + x^4 \eta$. 

(22)
where we have omitted non-logarithmic terms, completely known up to 4PN order. The 4PN logarithmic correction was first computed in [73] and later confirmed with EFT methods in [59, 60, 74], and the 5PN computed here agrees with the one found in [62], later confirmed in [63], by comparison with extreme mass ratio results, i.e. with a non PN computation.

4. Angular momentum “failed” tail

The results of the two bottom amplitudes of fig. 1 are grouped in literature under the label memory terms, but, as explained in sec. I, we find that both of them actually give finite, local-in-time contributions to the conservative dynamics and no contribution to the dissipative one.

In particular the diagram involving the conserved total angular momentum \( \vec{L} \) can be dubbed as a “failed” tail because, despite having an identical diagrammatic representation to the energy tail, after replacing \( E \rightarrow \vec{L} \), and also being characterized by \( q_0 = 0 \), it gives just an instantaneous contribution to the conservative dynamics.

Indeed, for this diagram at least one of the graviton polarizations must be an \( \vec{A} \) since it is the only polarization directly coupling to the angular momentum of the system, and it presents a gradient coupling proportional to momentum \( q \) that kills any divergence of the amplitude, which in the case \( q_0 = 0 \) occurs for \( q \rightarrow 0 \). We can thus infer that all diagrams involving the conserved quantity \( \vec{L} \) and any higher multipole are qualitatively different from the ones involving \( E \) in that the former are real, finite and local.

Broken in terms of the polarization, the amplitude for the bottom right diagram of fig. 1 is

\[
\begin{align*}
&iS_{\text{eff}5\text{PN}}^{LQ^2} = -i64\pi^2 C_4^2 \rho \epsilon_{pnmn} \int \frac{dk_0}{2\pi} Q_{ij}(k_0) \hat{Q}_{kl}(-k_0) \int_{k,q} \frac{1}{k^2 - k_0^2} \frac{1}{(k + q)^2 - k_0^2} \frac{q^n}{q^2} \times \\
&\left\{ \frac{1}{2} k_0^4 \delta^{kj} (\delta^{im} k^l - \delta^{lm} k^i) \right\} A_{\sigma^2} \\
&- \frac{1}{2} k_0^2 \delta_{im} q^k k^i k^j A_{\sigma \phi} \\
&+ \frac{1}{2c_d} (k + q)^i (k + q)^j k^k k^l k^m A_{\phi^2} \\
&- \frac{1}{2} k^i k^j (q \cdot (k + q) \delta_{ml} + q^m k^l - q^l k^m) (k + q)^k A_{2\phi} \\
&+ \frac{1}{2} k_0^2 k^i (k \cdot q \delta_{mk} \delta_{jl} + q^k k^l \delta_{mj} + \delta^{mk}(k^j q^l - k^l q^j) + \delta^{jl}(q^m k^k - k^m q^k)) A_{2\sigma} \\
&- \frac{1}{2} k_0^2 k^k (k + q)^i (\delta_{ij} k^m - \delta_{mj} k^i + \delta_{ln} k^j) A^3,
\end{align*}
\]
which summed up and written in time domain is

\[ S^{LQ^2}_{\text{eff}_{5PN}} = \frac{8}{15} G_N^2 \int dt \ddot{Q}_d \ddot{Q}_j \epsilon_{ijk} L_k. \]  

(24)

5. GW self interaction

Finally we consider the last diagram, which is qualitatively different from the others studied so far because it involves three mass quadrupoles, which are not conserved quantities, and a triple GW vertex. This is usually considered as a memory term for its effect on the gravitational waveform, see e.g. [75], but it appears as a local-in-time contribution to the energy. The detail of the amplitude is

\[ iS^Q_{\text{eff}_{5PN}} = -i128\pi^2 G_d^2 \int \frac{dq_0 \, dk_0}{2\pi} Q_{ij}(k_0) Q_{mn}(q_0) Q_{kl}(p_0) \times \]

\[ \int_{k,q} \frac{k^2 - k_0^2}{p^2 - p_0^2} \frac{q^2 - q_0^2}{q_0^2} \times \]

\[ \frac{1}{3} \left[ V^{ijmnkl}(k,q,p) + V^{ijmnkl}(k,p,q) + V^{ijmnkl}(p,q,k) \right], \]

(25)

with \( p^{\mu} \equiv -k^{\mu} - q^{\mu} \), retarded (advanced) boundary conditions are used for \( k,q,p \), and

\[ V^{ijmnkl}(k,q,p) = \left\{ \begin{array}{l}
-\frac{1}{8} p_0^2 k_0 q_0^2 \delta_{im} \delta_{jk} \delta_{ln} (-k_0 q_0 + k \cdot q) \quad \left\{ \sigma^3 \right\} \\
-\frac{1}{4} p_0^3 q_0 k_0^2 q_n \delta_{il} (k_k \delta_{im} - k_i \delta_{km}) \quad \left\{ \sigma^2 A \right\} \\
0 \quad \left\{ \sigma^2 \phi \right\} \\
-\frac{1}{4} \delta_{il} p_0^2 k_0 q_i q_m q_n (q_0 k_k - k_0 q_k) \quad \left\{ \sigma A \phi \right\} \\
+\frac{1}{4} p_0^2 k_0 q_0 k_i q_m q_n \left[ (k \cdot q - k_0 q_0) \delta_{nk} \delta_{il} + k_l q_k \delta_{nj} + \delta_{nl} (k_j q_k - k_i q_j) + \delta_{jk} (k_l q_n - k_n q_l) \right] \quad \left\{ \sigma A^2 \right\} \\
-\frac{1}{8} \frac{1}{C_d} p_0^2 k_i k_j k_i q_l q_m q_n \quad \left\{ \sigma \phi^2 \right\} \\
-\frac{1}{8} \frac{1}{C_d} p_0^2 k_0 q_0 k_i k_l q_m q_n (q_j \delta_{ni} - q_i \delta_{nj} + q_n \delta_{ij}) \quad \left\{ A^3 \right\} \\
+\frac{1}{8} k_0 p_0 k_i k_l q_m q_n (k \cdot p \delta_{jl} + k_j p_l - k_l p_j) \quad \left\{ A^2 \phi \right\} \\
+\frac{1}{4} k_0 p_0 k_i k_l q_m q_n q_n \quad \left\{ A \phi^2 \right\} \\
-\frac{1}{8} \frac{1}{C_d} k_0 p_0 k_i k_l k_j p_l q_m q_n \quad \left\{ \phi^3 \right\}.
\]

(26)

Some of the polarizations above give a divergent result due to the presence of the master integral \( I_m \), but poles cancel in the sum, which can be compactly written in time domain as
in the previous case:

$$S_{eff}^{Q_{3}} = -\frac{G_{N}^{2}}{15} \int dt \left[ \dot{Q}_{ij} \dot{Q}_{jl} Q_{ij} + \frac{4}{7} \ddot{Q}_{ij} \dot{Q}_{ij} \right].$$  \hspace{1cm} (27)

This concludes our derivation of the hereditary terms at next-to-leading order for a gravitational source described by the multipolar expansion. With the exception of eq. (22), the validity of all the results of this subsection is not restricted to the compact binary case, but it rather holds for any source which allows a multipole decomposition.

### IV. CONCLUSIONS

Given the advent of Gravitational Astronomy, and the planning of new gravitational wave detectors, like third generation ground based [76] and space detectors [77], able to reach higher signal-to-noise ratios than presently operating LIGO [2] and Virgo [3], the study of high precision gravity is becoming an urgent program.

Within the post-Newtonian approximation to General Relativity, which is the main framework for modeling the signals from coalescing binaries detected so far, it is then of outmost interest to increase our perturbative knowledge of binary dynamics, which at the moment lies at fourth perturbative post-Newtonian order in the conservative sector (see [19, 24, 26, 28, 31] for a complete determination of the 4PN near zone dynamics purely within the effective field theory methods), as well as to gain insight on generic properties of the PN series.

The effective field theory of gravity program, initiated in [10], has been proved very powerful in addressing this problem and within its framework we have derived in the present paper additional bricks concurring to the edification of the complete fifth post-Newtonian order binary dynamics.

In particular at 5PN, like at 4PN, there are contributions from the far, or radiation zone, where the degrees of freedom of gravity couple to source multipoles, to the near zone dynamics, i.e. the region around the source whose size is smaller than the wavelength of gravitational waves. The division in zones leads to several operational simplifications within the post-Newtonian approximation but also introduces spurious divergences in both zones starting from 4PN order, which recompose to a finite physical result once the two computations are consistently combined, as explained in detail in [50].

However computable, finite local-in-time and unambiguous terms remain after near and far zone results are combined, and we have originally derived in the present paper all yet
unknown contributions from the far zone to the near zone conservative dynamics at 5PN order, which we report here for convenience of the reader

\[ S_{\text{eff}5PN}^{(\text{far})} = G_N^2 \int dt \left[ \mathcal{M} \left( \frac{82}{6615} \zeta_{ijk}^2 + \frac{508}{675} \zeta_{ij}^2 \right) \right. \\
+ \left. \frac{8}{15} Q_{ij} Q_{ij} \epsilon_{ijk} L_k - \frac{1}{15} Q_{ij} Q_{ij} Q_{ij} - \frac{4}{105} Q_{ij} Q_{ij} Q_{ij} \right] . \quad (28) \]

In particular, the terms in the second line come from finite, local amplitudes in the far zone, so there are no associated IR-divergent terms in the near zone that could possibly signal the presence of such finite contributions.

Along with the finite local-in-time terms, there comes logarithmic hereditary terms, whose values we obtained in agreement with known previous results \([62, 63]\), which have been extended in \([78]\).

Note that the 5th PN order is qualitative different from previous ones, since it is the lowest one at which finite size effects, for spin-less black holes, are not forbidden by the effacement principle \([39]\), even though they are expected to appear only at higher order because of the vanishing of black hole static Love number \([43–49]\).

To determine the remaining missing terms ruling the 5PN dynamics it is necessary to complete the near zone computations, which has already been solved in the static sector, i.e. at \(O(G_N^6)\), in \([79]\) (independently confirmed in \([80]\)), while it is in principle possible to extract information about the 5PN order at lower power of \(G_N\) from the post-Minkowskian results at \(O(G_N^4)\) \([81, 85]\), \(O(G_N^2)\) \([86, 87]\), \(O(G_N^3)\) \([88]\). Among all the terms needed at 5PN, the ones determined in this paper stand out as the only ones that require knowledge of the far zone dynamics.

As a byproduct of our computation we have also re-derived the universality relations, already observed in \([71]\), between the power flux emitted by any multipole moment via the tail process and the leading order.

Finally the last original finding of the present work has been to relate the flux formula at generic \(n\)-PN order to the logarithm of tail terms affecting the real part of the action at \((n+4)\)-PN order, as summarized by eq. \([15]\). These logarithmic terms embody non-local-in-time (but causal) interactions depending on the past history of the source, which become instantaneous (hence local) for multipoles describing binary systems in circular orbit.
The logarithmic terms come with (unphysical) poles, which then are also constrained by the flux emission formula. Since the far zone poles has to cancel with equally unphysical poles in the near zone, this provides another non-trivial constraints on the results of near zone dynamics that are needed to complete the 5PN order dynamics.

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Appendix A: Amplitude construction

To construct amplitudes the basic building blocks are the multipolar action (1) and the gauge-fixed and bulk gravity action (4). E.g. to derive (11) one has to consider the quadrupole-gravity linear coupling, from the explicit expression eq. (8), which in Fourier domain is written as

\[
\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{Q_{ij}(k_0)}{2\Lambda} \left[ -\frac{1}{2} k_0^2 \sigma_{ij} - \frac{1}{2} k_0 k_i A_j - \frac{1}{2} k_0 k_j A_i + \left( k_i k_j + \frac{k_0^2}{d-2} \delta_{ij} \right) \phi \right],
\]

and the gravity propagators

\[
P[\sigma_{ij}(k)\sigma_{kl}(q)] = -\left( \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d-2} \delta_{ij}\delta_{kl} \right),
\]

\[
P[A_i(k)A_j(q)] = \delta_{ij},
\]

\[
P[\phi(k)\phi(q)] = -\frac{d-2}{2(d-1)}
\]

with vanishing propagators between not-alike fields (i.e. \( P[\sigma_{ij}A_k] = P[\sigma_{ij}\phi] = P[A_i\phi] = 0 \)).

Gluing together two terms of the type of (A1), using the above propagators (A2) and integrating over all possible momenta one finally obtains eq. (11).

To build tail amplitudes one has to consider a tri-linear bulk coupling, which can be read from (4), and pair each of its three fields with a source multipole term. E.g. to build the amplitude corresponding to the first line in the curly bracket of eq. (9) one has to consider bulk vertex

\[
c_d k_0 k'_0 \left( \sigma_{ij}(k)\sigma_{kl}(k')\phi(q) \right) \left( \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} \right),
\]
then Wick-contract each $\phi$ and $\sigma$ field with the the appropriate field in
\[
\left[ \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{(-k_0^2)}{4} Q_{ij}(k_0)\sigma_{ij}(-k) \right]^2 \left( \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \frac{(-E)}{\Lambda} \phi(q) \right),
\]
where in the propagator connected with the source term involving the conserved energy $E$ the denominator has to be expanded for small $q_0$, i.e.
\[
\frac{1}{q^2 - q_0^2} \simeq \frac{1}{q^2} \left( 1 + \frac{q_0^2}{q^2} + \ldots \right), \tag{A4}
\]
since the momentum of the longitudinal gravitational mode $q = (q_0, \mathbf{q})$ has $q_0 \ll |\mathbf{q}|$, according to the integration prescription known as region of momentum (see [89] for a rigorous demonstration), and at leading order all terms involving $q_0$ in eq. (A4) can be neglected.

**Appendix B: Relevant integrals**

In the present work we had to integrate amplitudes like eq. (5), involving numerators with up to six free indices; exploiting spatial rotation invariance and the possibility of relating integrals by means of index contractions, everything can be reduced to the integration of the following scalar factor
\[
\frac{1}{D_{\alpha,\beta,\gamma}} \equiv \left[ (q^2 - q_0^2)^\alpha (\mathbf{k} + \mathbf{q})^2 - (k_0 + q_0)^2 \right]^\beta (k^2 - k_0^2)^\gamma \left[ (q^2 - q_0^2)^\alpha (\mathbf{k} + \mathbf{q})^2 - (k_0 + q_0)^2 \right]^{-1}, \tag{B1}
\]
with $(\alpha, \beta, \gamma)$ integers equal to $-2, -1, 0$ or $1$ and the retarded prescription for propagator poles is understood. The case $\alpha = \beta = \gamma = 1$ corresponds to the master integral $I_m$, while the other relevant cases, using
\[
\int \frac{1}{k^2 - k_0^2} = \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} (-k_0^2)^{d/2-1}, \tag{B2}
\]
give
\[
\int_{k,q} \frac{1}{D_{1,0,1}} = \Theta(-k_0^2)^{d/2-1}(-q_0^2)^{d/2-1}, \quad \Theta \equiv \frac{\Gamma^2 \left( 1 - \frac{d}{2} \right)}{(4\pi)^d},
\]
\[
\int_{k,q} \frac{1}{D_{-n,0,0}} = \int_{k,q} \frac{1}{D_{0,n,-n}} = 0 \quad \text{for} \quad n \geq 0,
\]
\[
\int_{k,q} \frac{1}{D_{1,-1,1}} = \left[ q_0^2 + k_0^2 - (k_0 + q_0)^2 \right] \int_{k,q} \frac{1}{D_{1,0,1}},
\]
\[
\int_{k,q} \frac{1}{D_{1,-2,1}} = \left[ (q_0^2 + k_0^2)^2 - (k_0 + q_0)^4 + \frac{4}{d} k_0^2 q_0^2 \right] \int_{k,q} \frac{1}{D_{1,0,1}} - 2(k_0 + q_0)^2 \int_{k,q} \frac{1}{D_{1,-1,1}}. \tag{B3}
\]
The results for the integrals with two free indices can be written, up to terms $O(d - 3)$, as
\[
\int \frac{k^i k^j}{k^2 - k_0^2} = \frac{\delta_{ij} k_0^2}{d} \Gamma(1 - d/2) \left( -k_0^2 \right)^{-d/2 - 1},
\]
\[
\int \frac{k^i q^j}{D} \approx \frac{\delta_{ij}}{d} \left[ I_m q_0 + \frac{\Theta}{2} \left( k_0^2 + q_0^2 + k_0 q_0 \right) \right],
\]
\[
\int \frac{k^i k^j}{D} \approx \frac{\delta_{ij}}{d} \left[ I_m k_0^2 - \Theta q_0 \left( k_0 + q_0 \right) \right],
\]
\[
\int \frac{q^i q^j}{D} \approx \frac{\delta_{ij}}{d} \left[ I_m q_0^2 - \Theta k_0 \left( k_0 + q_0 \right) \right].
\]

For the four-indices case, in terms of the following parametrization
\[
\int \frac{k^i k^j k^l k^m}{D} = C^{kkkk} \delta_{ijlm}, \quad \int \frac{k^i k^j k^l q^m}{D} = C^{kkkq} \delta_{ijlm}, \quad \int \frac{k^i q^j q^m}{D} = C^{kkqq} \delta_{ijlm} ;
\]
\[
\int \frac{q^i q^j q^m}{D} = C^{qqqq} \delta_{ijlm}, \quad \int \frac{k^i k^j q^m}{D} = C^{kqqq} \delta_{ijlm},
\]
where $\delta_{ijlm} \equiv \delta_{il} \delta_{jm} + \delta_{im} \delta_{jn} + \delta_{ij} \delta_{lm}$ is the completely symmetrized combination of two $\delta$s, one obtains
\[
C^{kkkk} \simeq \frac{1}{d(d+2)} \left[ k_0^4 I_m - 2\Theta q_0 \left( k_0^3 + 2k_0 q_0^2 + 2k_0 q_0^2 + q_0^3 \right) \right],
\]
\[
C^{kkqk} \simeq \frac{1}{d(d+2)} \left[ k_0^3 q_0 I_m + \frac{\Theta}{2} \left( k_0^4 + k_0^3 q_0 + k_0^2 q_0^2 + 2k_0 q_0^3 + q_0^4 \right) \right],
\]
\[
C^{kkqq} \simeq \frac{1}{d(d+2)} \left\{ k_0^2 q_0^2 I_m - \frac{1}{2(d-1)} \Theta \left[ (d-2) k_0^4 - 2k_0^3 q_0 - dk_0^2 q_0^2 - 2k_0 q_0^3 + (d-2) q_0^4 \right] \right\},
\]
\[
C^{kkqq} \simeq -\frac{1}{2d(d-1)} \left[ k_0^4 + 2k_0^3 q_0 + k_0^2 q_0^2 + 2k_0 q_0^3 + q_0^4 \right].
\]
and the results for $C^{qqqq}$ and $C^{kqqq}$ can obviously be obtained from $C^{kkkk}$ and $C^{kkkq}$ by means of $k_0 \leftrightarrow q_0$.

Finally, by using an analogous parametrization for the six-indices integrals
\[
\int \frac{k^i k^j k^l k^m k^r k^s}{D} = C^{kkkkkk} \delta_{ijlmrs}, \quad \int \frac{k^i k^j k^l k^m k^r q^s}{D} = C^{kkkkkq} \delta_{ijlmrs},
\]
\[
\int \frac{k^i q^j q^m q^r q^s}{D} = C^{kkqqqq} \delta_{ijlmrs}, \quad \int \frac{q^i q^j q^m q^r q^s}{D} = C^{qqqqqq} \delta_{ijlmrs},
\]
\[
\int \frac{k^i k^j k^l k^m q^r q^s}{D} = C^{kkkkqq} \delta_{ijlmrs} + C^{kkkkkk} \delta_{ijlm},
\]
\[
\int \frac{k^i k^j q^m q^r q^s}{D} = C^{kkqqqq} \delta_{ijlmrs} + C^{kkkkqq} \delta_{ilmrs} \delta_{ij},
\]
\[
\int \frac{k^i k^j q^m q^r q^s}{D} = C^{kkqqqq} \delta_{ijlmrs} + C^{kkkkqq} \delta_{ijlmrs} \delta_{ij},
\]
\[
\int \frac{k^i k^j q^m q^r q^s}{D} = C^{kkqqqq} \delta_{ijlmrs} + C^{kkkkqq} \delta_{ijlmrs} \delta_{ij},
\]
where $\delta_{ijlmrs}$ is the completely symmetrized combination of three $\delta$s, one gets
\[
C^{kkkkkk} \simeq k_0^6 I_m - \frac{\Theta}{d} q_0 \left[ 3dk_0^5 + 9dk_0^2 q_0 + 4(4d+1)k_0^3 q_0^2 + 6(3d+2)k_0^2 q_0^2 + 12(d+1)k_0 q_0^4 + 4(d+1)q_0^5 \right] \frac{1}{d(d+2)(d+4)}.
\]
\[ C_{\alpha\beta\gamma\delta} \approx \frac{k_0^5 q_0 I_m + \left( \frac{5}{24} \right) [d k_0^6 + k_0^4 q_0 + 4(d+1) k_0^3 q_0^2 + 4(2d+1) k_0^2 q_0^3 + 4(2d+3) k_0 q_0^5 + 4(d+1) q_0^6]}{d(d+2)(d+4)} \]

\[ C_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \approx \frac{k_0^4 q_0 I_m - \frac{\Theta}{2(d-1)} [(d-2) k_0^6 - 2 k_0^5 q_0 - d k_0^3 q_0^2 - 2 k_0^2 q_0^3 + (d-2) k_0^4 q_0 + 2(1-d) k_0 q_0^5 + 2(1-d) q_0^6]}{d(d+2)(d+4)} \]

\[ \bar{C}_{\alpha\beta\gamma\delta} \approx -\frac{\Theta}{d(d+2)(d+4)} \frac{d k_0^6 + 2 d k_0^5 q_0 + d k_0^4 q_0^2 + 2(2d-1) k_0^3 q_0^3 + (7d-6) k_0^2 q_0^4 + 6(1-d) k_0 q_0^5 + 2(1-d) q_0^6}{2d^2(d-1)(d+2)} \]

\[ \bar{C}_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \approx -\frac{\Theta}{d(d+2)(d+4)} \frac{d k_0^6 + 2 d k_0^5 q_0 + d k_0^4 q_0^2 + 2(2d-1) k_0^3 q_0^3 + (7d-6) k_0^2 q_0^4 + 6(1-d) k_0 q_0^5 + 2(1-d) q_0^6}{2d^2(d-1)(d+2)} \]

and the results for \( C_{\alpha\beta\gamma\delta} \), \( C_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \), \( \bar{C}_{\alpha\beta\gamma\delta} \) and \( \bar{C}_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \) can be obtained as above by means of \( k_0 \leftrightarrow q_0 \).

As in most cases one can set \( q_0 = 0 \) because a conserved quantity is involved in the amplitude, one can exploit the following closed formula

\[ \int_{k,q} \left. \frac{1}{D_{\alpha\beta\gamma\delta}} \right|_{q_0=0} = \frac{\Gamma(\alpha + \beta + \gamma - d) \Gamma(\alpha + \gamma - \frac{d}{2}) \Gamma(\alpha + \beta - \frac{d}{2}) \Gamma(\frac{d}{2} - \alpha)}{(4\pi)^d \Gamma(\gamma) \Gamma(2\alpha + \beta + \gamma - d) \Gamma(\frac{d}{2})} (\frac{k_0^2}{-\alpha - \beta - \gamma})^{d-\alpha - \beta - \gamma}. \]

The dimension-less function \( f \) appearing in eq. (13) is explicitly given by \( k_0^2 (-k_0^2)^{1-d/2} \) times

\[ \int_{\mathbf{q}} q^2 (|\mathbf{k} + \mathbf{q}|^2 - k_0^2)^{-2} = \frac{\Gamma(d/2 - 1)}{(4\pi)^{d/2}} \int_0^1 dx (x(1-x)k^2 - x k_0^2)^{d/2 - 2} \]

\[ = \frac{2 \Gamma(d/2 - 1)}{(d-2)(4\pi)^{d/2} k^2 - k_0^2} \text{2F} \left( 1, d/2 - 2; \frac{d}{2}; \frac{k^2}{k_0^2 - k_0^2} \right), \]

where \( \text{2F}(a, b, c; z) \) is the Hypergeometric function which is divergent for \( |z| \to 0 \).

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