Sums of products of power sums

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Abstract

For any two arithmetic functions $f, g$ let • be the commutative and associative arithmetic convolution $(f \bullet g)(k) := \sum_{m=0}^{k} \binom{k}{m} f(m) g(k-m)$ and for any $n \in \mathbb{N}$, $f^n = f \bullet \cdots \bullet f$ be $n$-fold product of $f \in S$. For any $x \in \mathbb{C}$, let $S_0 = e$ be the multiplicative identity of the ring $(S, \bullet, +)$ and $S_x(k) := \frac{B_{x+1}(k+1) - B_1(k+1)}{k+1}$, $x \neq 0$ denote the power sum defined by Bernoulli polynomials $B_x(k) = B_k(x)$. We consider the sums of products $S^N_x(k)$, $N \in \mathbb{N}_0$. A closed form expression for $S^N_x(k)(x)$ generalizing the classical Faulhaber formula, is derived. Furthermore, some properties of $\alpha$–Euler numbers [7] (a variant of Apostol Bernoulli numbers) and their sums of products, are considered using which a closed form expression for the sums of products of infinite series of the form $\eta_{\alpha}(k) := \sum_{n=0}^{\infty} \alpha^n k^n$, $0 < |\alpha| < 1$, $k \in \mathbb{N}_0$ and the related Abel sums, is obtained which in particular, gives a closed form expression for well known Bernoulli numbers. A generalization of the sums of products of power sums to the sums of products of alternating power sums is also obtained. These considerations generalize in a unified way to define sums of products of power sums for all $k \in \mathbb{Z}$ hence connecting them with zeta functions.

Keywords: Power sums, $\alpha$–Euler numbers, Bernoulli numbers, Riemann zeta function, Sums of products, Higher order Bernoulli numbers.

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1 Introduction

The discrete power sum of a positive integer $n$ is defined as $S_k(n) := 1^k + 2^k + \cdots + n^k$, $k \in \mathbb{N}_0$. These sums are not new and have been studied since Pascal (1654) who obtained closed form formula for them by expressing $S_k(n)$ inductively in terms of $S_{k-1}(n), \ldots, S_0(n)$ via the following nice formula

$$(n + 1)^{k+1} - 1 = \sum_{m=0}^{k} \binom{k+1}{m} S_m(n).$$

It immediately follows from this formula that $S_k(n)$ is a polynomial of degree $k + 1$ in $n$. Properties of these polynomials were observed by Faulhaber (1631). The study of the power sums is so amusing that over the years, it has attracted the attention of researchers in one or other way since the work of Faulhaber, Bernoulli, and Euler. As a result, literature is vast in context of different approaches and generalizations of the power sums. In fact the notion of the power sums is standard nowadays and a reasonable material regarding their properties, can be found in any number theory text. However, for a quick review of power sums, we recommend the reader a relatively new work in the Refs. [1, 2, 3, 4, 5] where Beardon [4] studies the general algebraic properties of the polynomial functions like the ones defined by $S_k(n)$. More recently, Shirali [5] also explains quite beautifully, the various properties of the power sums. Kim and Park [6] have obtained many identities concerning symmetries of power sums and Bernoulli polynomials.

As an extension of the discrete power sums, a power sum $S_k(x)$ for a nonnegative integer $k$ and a real or complex variable $x$, is defined via the generating function [7]

$$G(x, t) := \frac{e^{(1+x)t} - e^t}{e^t - 1} := \sum_{k=0}^{\infty} S_x(k) \frac{t^k}{k!}, \ |t| < 2\pi$$  

(1.1)

where we define $S_x \in \mathcal{S}$ such that $S_x(k) = S_k(x), \ x \neq 0$ and $S_0(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$ For a fixed $k$, the power sum $S_k(x)$ is a polynomial of degree $k + 1$ in $x$ over $\mathbb{Q}$ and is expressed in terms of the Bernoulli numbers by the
The classical Faulhaber formula

$$S_k(x) := \frac{1}{k+1} \sum_{m=0}^{k} \binom{k+1}{m} (-1)^m B_m x^{k+1-m}. \quad (1.2)$$

The rational sequence $B_k, k \in \mathbb{N}_0$ of Bernoulli numbers, is defined via the generating function \[3, 8\]

$$\frac{t}{e^t - 1} := \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \quad |t| < 2\pi$$

and was known to Faulhaber and Bernoulli. Many explicit formulas for the Bernoulli numbers are well known in the literature. One such formula is given by

$$B_k := \sum_{m=1}^{k} \frac{1}{m+1} \sum_{n=1}^{m} (-1)^n \binom{m}{n} n^k, \quad k \in \mathbb{N}_0. \quad (1.3)$$

For the other explicit formulas on Bernoulli numbers, the reader is referred to see \[9, 10, 11\] and the exposition by Gould \[12\], in which a survey is done on the explicit formulas for the Bernoulli numbers. Another recent work of Agoh and Dilcher \[13\] gives an infinite class of closed form expressions for Bernoulli numbers via Stirling numbers of second kind. For a representation of Bernoulli numbers and some other related rational sequences in terms of determinants, one may see the work of Malenfant \[14\]. Sums of products of the Bernoulli numbers are defined by $n-$fold convolution $B^N$ of $B \in S$ where $B(k) = B_k$ and their study has attracted the attention of researchers for several number theoretical interests such as, in evaluating sums of products of Riemann zeta functions. To this connection, Srivastava and Todorov \[15\] have obtained the following closed form expression for the sums of products of the Bernoulli numbers

$$B^N(k) = \sum_{m=0}^{k} \binom{k + N - 1}{k - m} \binom{k + N - 1}{m} \frac{k!}{(k+m)!} \sum_{n=0}^{m} (-1)^n \binom{m}{n} n^{k+m}. \quad (1.4)$$

Dilcher \[16\] has done excellent work in this direction and obtained a closed form expressions for the sums of products of Bernoulli numbers; we mention here the one which we will use in the present study

$$B^N(k) = N \binom{k}{N} \sum_{i=0}^{N} (-1)^{N-i} s(N, N-j) \frac{B_{k-i}}{k-i}, \quad k > N \geq 1. \quad (1.5)$$
Dilcher [16] has obtained the sums of products of Bernoulli polynomials using which he has studied sums of products involving Riemann zeta function at even integers.

In a short paper, Petojević and Srivastava [17] have computed the Dilcher sums of products of the Bernoulli numbers in an elegant way. More generally, Kamano [18] has studied sums of products of hypergeometric Bernoulli numbers and used them to study the multiple hypergeometric zeta functions. On the other hand, Kim [19] has obtained sums of products of Bernoulli numbers, using an analytic continuation of the multiple hypergeometric zeta functions using which Kim and Hu [20] have described sums of products of Apostol Bernoulli polynomials.

In this review of sums of products, the author did not find any work on the sums of products of the power sums and their connection with the Riemann zeta. One may argue that the power sums are defined via Bernoulli polynomials up to a constant, so all properties of power sums and their sums of products are contained in the study of Bernoulli polynomials and their sums of products. Well, power sums are fundamental and we feel that the sums of products of power sums are important in their own right and their separate study deserve an equal attention. The present work is an attempt to fill this gap by considering sums of product of power sums and, obtaining closed form expressions for them with promising generalizations in a very natural way via a considerably simple theory.

Rest of the paper is organized as follows. Sums of products of the power sums are discussed in Sec. 2. The Sec. 3 is central in the present study and is devoted to continue the work of Singh [7] in exploring properties of the higher order $\alpha$–Euler numbers (a variant to Apostol Bernoulli numbers), their connection with the higher order Bernoulli numbers, and the sums of products of a generalization of the power sums to alternating power sums. In the same section, the related sums of products of the Abel sums of alternating infinite series, are obtained in the closed form.

2 Sums of products

We now describe the sums of products of the power sums $S_k(x)$ which are related to the sums of products of Bernoulli numbers.
**Definition:** A sum of products of the power sums is defined as the function $S^N_x \in S$ such that $S^N_x(k) = T^N_k(x)$ where the latter is described by generating function

$$H(x, t, N) := \left( \frac{e^{(1+x)t} - e^t}{e^t - 1} \right)^N = \sum_{k=0}^\infty T^N_k(x) \frac{t^k}{k!} , \quad |t| < 2\pi, \; N \in \mathbb{N}_0. \quad (2.1)$$

Observe from (2.1) above that $T^0_0(x) = 1$; $T^0_k(x) = 0$ for all $k \in \mathbb{Z}^+$, $T^N_1(x) = N^k$, $T^N_1(x) = S_k(x)$. Since for each nonnegative integer $k$, $S_k(x)$ is a polynomial in $x$ of degree $k+1$ over $\mathbb{Q}$ with coefficient of the leading term $(k+1)^{-1}$ (see (1.2)), it follows that for each fixed positive integer $N$ and partition $\{k_1, \ldots, k_N\}$ of $k$, the product $S_{k_1}(x) \cdots S_{k_N}(x)$ is a polynomial in $x$ over $\mathbb{Q}$ of degree $\sum_{i=1}^N (k_i + 1) = k + N$ with leading coefficient $\prod_{i=1}^N (k_i + 1)^{-1} > 0$.

Also as $S_{k_i}(0) = 0$ for all $i = 1, \ldots, k_N$, $x^N$ is always a factor of the polynomial $S_{k_1}(x) \cdots S_{k_N}(x)$. Thus $T^N_k(x)$ is a polynomial in $x$ of degree $k+N$ over $\mathbb{Q}$ since coefficient of $x^{k+N}$ in $T^N_k(x)$ is equal to the $N$-fold product $\xi^N_1(k)$ where $\xi_1(k) = (k+1)^{-1}$. Thus we see that $T^N_k(x)$ is a linear combination of $\{x^N, x^{N+1}, \ldots, x^{N+k}\}$ over $\mathbb{Q}$. In particular, we note that the coefficient of $x^i$, $i = 0, 1, \ldots, N-1$, $N > 0$, in expansion of $T^N_k(x)$ in powers of $x$, is always zero. It also follows that

$$T^N_k(x) = (-1)^{k_1+1+\cdots+k_{N+1}} T^N_k(-1 - x) = (-1)^{k+N} T^N_k(-1 - x),$$

from which we observe that $T^N_k(-1) = 0$ for all $k \in \mathbb{Z}^+$. Alternatively, the identity $T^N_1(-1) = 0$ also follows from the generating function in Eq. (2.1).

With this, we conclude that $x^N(x + 1)$ is always a factor of $T^N_k(x)$ for all positive integers $k$ and $N$. These properties of $T^N_k(x)$ can be used to establish that

$$\int_{-1}^0 T^N_k(x) dx = 0 \text{ if } k + N \equiv 1 \mod 2 \text{ and } k, N \geq 1.$$

Further, it would be interesting to obtain properties of the polynomials $T^N_k(x)$ in the context of Faulhaber’s considerations. We do not wish to pursue that here; however, we have the following obvious result.

**Proposition 2.1.** For all $N \in \mathbb{N}_0,$
(1) $T_0^N(x) = x^N$,

$$T_k^N(x) := Nk \int_0^x T_{k-1}^N(t) \, dt + N \sum_{m=0}^{k} \binom{k}{m} C_{k-m} \int_0^x T_{m-1}^N(x), \; k \in \mathbb{N}.$$  

(2) $T_k^N(x) = (-1)^{k+1} \sum_{n=0}^{N} \left( \frac{N}{n} \right) T_n^k(-1-x)$ for all $k \in \mathbb{N}_0$.

Proof. (1) Observe that $T_0^N(x) = \lim_{t \to 0} H(x, t, N) = x^N$. The recurrence follows from $\frac{dH}{dx} := NtH(x, t, N) + NH(x, t, N-1) \frac{e^t-1}{e^t-1}$.

(2) Follows from symmetry of $H$

$$(-1)^N H(x, t, N) = \left( \frac{e^{(1-x)(t-1)}-e^{-t}}{e^{t-1}} + 1 \right)^N = \sum_{n=0}^{N} \left( \frac{N}{n} \right) H(-1-x, t, n) \quad \square$$

Example: Using proposition 2.1 $T_k^N(x)$ for some $k$ and $N$ is given by

$$T_1^N(x) = \frac{N}{2} x^N(x+1); \; T_2^N(X) = \frac{N}{12} x^N(x+1)\{x - 1 + 3N(x+1)\},$$

$$T_3^N(x) = \frac{N^2}{8} x^N(x+1)^2\{x - 1 + N(x+1)\},$$

$$T_4^N(x) = \frac{N}{240} x^N(x+1)\{2(1-x)(1+x^2)+5N(1-x)^2(1+x)-30N^2(1-x)(1+x)^2$$

$$+15N^3(1+x)^3\}etc.$$  

In the foregoing discussion, we connect the sums of products of power sums with the known sums of products and obtain closed form expressions for them. For this we need the following.

Definition: Higher order $k$-th Bernoulli polynomial $B_k^N(x) := B_k^N(x)$ is defined by $\frac{t^N e^t}{(e^t-1)^{k+1}} := \sum_{k=0}^{\infty} B_k^N(x) \frac{t^k}{k!}$, for all $N \in \mathbb{N}_0, |t| < 2\pi$. Higher order Bernoulli numbers are defined by $B^N := B_0^N$.

Note that $B_k^1(1) = B_k(x)$ is the $k$-th Bernoulli polynomial and $B_0^1(k) = B_k$ for all $k \in \mathbb{N}_0$. Following recurrences are immediate $B_k^0(0) = 1; B_k^0(k) = x^k$ for all $k \in \mathbb{N}$; $B_k^N(x) = (B \circ B_{x}^{N-1})(x)$ which for $N = 1$ reduce to the usual
identity for the Bernoulli polynomials $B_x(0) = 1$; $B_x(k) = (B \cdot \epsilon_x)(k)$ where $\epsilon_x = B^0_x$, $x \neq 0$ and $\epsilon_0 = e$. It is also easy to observe the recurrence

$$B^{N+1}(0) = 1; \quad B^{N+1}(k) = \left(1 - \frac{k}{N}\right)B^N(k) - kB^N(k - 1), \quad k \in \mathbb{N}.$$  

This recurrence can be used to calculate $B^N(k)$ explicitly for $N \geq 2$. It is also not hard to see that the sum of products of the power sums $T^N_k(x)$ can be expressed in terms of the higher order Bernoulli polynomials by the formula

$$T^N_k(x) := \frac{k!}{(N + k)!} \sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i}B^N_{N+k}(N + (1 + x)i), \quad (2.2)$$

for all $N, k \in \mathbb{N}_0$, from which for $N = 1$, we recover the Faulhaber formula in terms of Bernoulli polynomials in the form $T^1_k(x) := \frac{B^1_{k+1}(1+x)-B^1_{k+1}(1)}{k+1}$. Note that (2.2) connects the sums of products of power sums with the sums of products of Bernoulli polynomials, however this connection is not trivial as it was for $N = 1$. This is the reason we treat sums of products of power sums in their own right!

**Proposition 2.2.** Let $N, k \in \mathbb{N}_0$. Then

$$T^N_k(x) := \frac{k!}{(k+N)!} \sum_{m=0}^{k} \binom{k+N}{m} B^N_m \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} (N + nx)^{k+N-m}. \quad (2.3)$$

where $B^N_k = B^N(k)$. The case $N = 1$ recovers the classical Faulhaber formula.
Proof. Follows from the following easy computations

\[
\sum_{k=0}^{\infty} T_N^k(x) \frac{t^k}{k!} = \left( \frac{e^{(1+x)t} - e^t}{e^t - 1} \right)^N
\]

\[
= \frac{1}{t^N} \left( \frac{t}{e^t - 1} \right)^N (e^{(1+x)t} - e^t)^N
\]

\[
= \sum_{m=0}^{\infty} B_m^N \frac{t^{m-N}}{m!} \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} e^{(N+nx)t}
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_m^N \frac{t^{m-N}}{m!} \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{s=0}^{\infty} (N+nx)^s \frac{t^s}{s!}
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} (N+nx)^k \frac{k^{k-N}}{k!}
\]

which on comparing the like powers of \(t\) gives

\[
\sum_{m=0}^{k} \binom{k}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} (N+nx)^{k-m} = 0 \text{ for } k = 0, 1, \ldots, N-1,
\]

and

\[
T_N^k(x) = \frac{k!}{(k+N)!} \sum_{m=0}^{k+N} \binom{k+N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} (N+nx)^{k+N-m}. \quad (2.5)
\]

As remarked earlier, \(T_N^k(x)\) is a linear combination of \(x^N, \ldots, x^{N+k}\), and for \(N \in \mathbb{N}\), coefficient of \(x^i\) for \(i = 0, 1, \ldots, N-1, k+N+1, \ldots\) vanishes in (2.5). So we have

\[
\sum_{m=k+1}^{k+N} \binom{k+N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} (N+nx)^{k+N-m} = 0. \quad (2.6)
\]

Therefore (2.5) reduces through (2.6) to (2.3). This completes the proof. \(\Box\)

**Corollary 2.3.** For any nonnegative integers \(k\) and \(N\),

\[
T_N^k(x) = \sum_{m=0}^{k} \binom{k}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{s=0}^{k-m} \binom{k-m-s}{s} \frac{s!(nx)^{N+s} N^{k-m-s}}{(N+s)!}
\]
Proof. For \( N = 0 \), we get from (2.3), \( T_0^0(x) = \sum_{m=0}^{k} \binom{k}{m} B_m^0 (1 + x)^{k-m} \) which takes value 1 at \( k = 0 \) and vanishes for all \( k \in \mathbb{N} \). This establishes the case \( N = 0 \). Now let \( N \geq 1 \) and observe for each \( i = 0, 1, \ldots, N - 1 \),

\[
\sum_{m=0}^{k} \binom{k + N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \binom{k + N - m}{i} (nx)^i N^{k+N-m-i} = 0
\]

since coefficient of \( x^i \), \( i = 0, \ldots, N - 1 \) vanishes in \( T_k^N(x) \). Using this in (2.3) we have

\[
T_k^N(x) = \sum_{m=0}^{k} \binom{k + N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{i=N}^{k+N-m} \binom{k+N-m}{i} \binom{k + N - m}{i} \binom{k + N - m}{i}
\]

\[
\times \frac{k!(nx)^i N^{k+N-m-i}}{(k+N)!}
\]

\[
= \sum_{m=0}^{k} \binom{k + N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{s=0}^{k-m} \binom{k + N - m}{N + s} \binom{k + N - m}{N + s}
\]

\[
\times \frac{k!(nx)^{N+s} N^{k-m-s}}{(k+N)!}
\]

\[
= \sum_{m=0}^{k} \binom{k + N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{s=0}^{k-m} \binom{k + N - m}{N + s} \binom{k + N - m}{N + s}
\]

\[
\times \frac{(k-m)!(nx)^{N+s} N^{k-m-s}}{(k+N-m)!}
\]

\[
= \sum_{m=0}^{k} \binom{k + N}{m} B_m^N \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \sum_{s=0}^{k-m} \binom{k + N - m}{N + s} \binom{k + N - m}{N + s}
\]

\[
\times \frac{s!(nx)^{N+s} N^{k-m-s}}{(N+s)!}
\]

as required. \( \square \)

3 \( \alpha \)-Euler numbers

Consider the vector space \( V(\mathbb{R}) \) of all \( f : \mathbb{R} \to \mathbb{R} \) with the following vector addition and scalar multiplication rules \((f+g)(x) := f(x) + g(x); (\alpha f)(x) := \alpha f(x)\) for all \( f, g \in V, \alpha, x \in \mathbb{R} \). For a fixed \( h \in \mathbb{R} \), we consider the linear operators \( E_h : V \to V, \Delta_h : V \to V \) defined by \( E_h(f)(x) := f(x+h), \Delta_h(f)(x) := (E_h-I)(f)(x) := f(x+h) - f(x) \) where \( I = E_0 \) is the
identity operator. The map $E_h$ is called the shift operator and $\Delta_h$ is called the difference operator. For a nonnegative integer $n$, the $n$–th difference for $f$ at $x$, is defined as $\Delta^n_h(f(x)) := (\Delta_h \circ \cdots \circ \Delta_h(n - \text{times})(f(x)) = (E_h - I)^n(f(x)) = \sum_{m=0}^n \binom{n}{m} (-1)^{n-m}E^m_h(f(x)).$ Note that $E^m_h(f(x)) = f(x + mh)$. In particular, if we take $f(x) := x^k$ then

$$\Delta^n_h(x^k) = \sum_{m=0}^n \binom{n}{m} (-1)^{n-m} (x + mh)^k. \tag{3.1}$$

Singh [7] introduced $\alpha$–Euler numbers denoted $E_k(\alpha)$ which are defined by

$$\frac{\alpha e^t - 1}{\alpha e^t - 1} := \sum_{k=0}^{\infty} E_k(\alpha) \frac{t^k}{k!}, \alpha \neq 1; \quad E_k(1) := \frac{E_k(-1)}{1-2^k}, \quad k \in \mathbb{N}_0 \quad \text{which are variant of Apostol Bernoulli numbers. The } \alpha\text–Euler numbers are known to satisfy the following beautiful identities}

$$\alpha^{-1} E_{k+1}(\alpha) = \frac{d}{d\alpha}(\alpha^{-1} E_k(\alpha)), \quad \alpha^{-1} E_k(\alpha) = (-1)^{k+1} \alpha E_k(\alpha^{-1}).$$

for all $\alpha \neq 0, 1$ and nonnegative integers $k$. $\alpha$–Euler numbers were defined to express the power sums of the type

$$S_k(\alpha, x) := \alpha + \alpha^2 2^k + \cdots + \alpha^x x^k, \quad x \in \mathbb{N}_0$$

and their generalizations. Infact for $0 < |\alpha| < 1 \lim_{x \to \infty} S_k(\alpha) = -\alpha^{-1} E_k(\alpha)$. We now derive the closed form expression for the $\alpha$–Euler numbers.

**Lemma 3.1.** The $k$–th $\alpha$–Euler number is

$$E_k(\alpha) = \sum_{n=1}^{k} \binom{n}{\alpha - 1} \frac{\alpha}{\alpha - 1} \sum_{m=1}^{n \alpha} \binom{n}{m} (-1)^m m^k \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

**Proof.** Consider the geometric power series

$$\sum_{k=0}^{\infty} E_k(\alpha) \frac{t^k}{k!} = G(\alpha, t) := \frac{\alpha}{\alpha e^t - 1} = -\sum_{n=0}^{\infty} \alpha^{n+1} e^{nt}$$

which is valid for all $|\alpha e^t| < 1$. Then for each positive integer
We have the following

\( E_k(\alpha) := \lim_{t \to 0^-} \frac{d^k}{dt^k}(G(\alpha, t)) = -\alpha \lim_{t \to 0^-} \sum_{n=0}^{\infty} \alpha^n n^k e^{nt} \)

\( = -\alpha \lim_{t \to 0^-} \left( \sum_{n=0}^{\infty} \alpha^n E_n^1 \right) t^k \)

\( = \alpha \lim_{t \to 0^-} (\alpha E_1 - I)^{-1} t^k \)

\( = \alpha \lim_{t \to 0^-} (\alpha(I + \Delta_1) - I)^{-1} t^k \)

\( = \frac{\alpha}{\alpha - 1} \lim_{t \to 0^-} \left( 1 + \frac{\alpha}{\alpha - 1} \Delta_1 \right)^{-1} t^k \)

\( = \frac{\alpha}{\alpha - 1} \sum_{i=1}^{k} \left( \frac{\alpha}{1 - \alpha} \right)^i \lim_{t \to 0^-} \Delta^i_1(t^k) \)

\( = \frac{\alpha}{\alpha - 1} \sum_{i=1}^{k} \left( \frac{\alpha}{1 - \alpha} \right)^i \sum_{j=1}^{i} \binom{i}{j} (-1)^j j^k \)

where we have used the fact that \( \Delta^i_1(t^k) = 0 \) for all \( i > k \) and (3.1).

\[ \square \]

### 3.1 A formula for Bernoulli numbers

We note that

\( \frac{-\alpha}{2} \frac{-\alpha}{\alpha e^t - 1} = \frac{\alpha^2}{\alpha^2 e^{2t} - 1} - \frac{\alpha}{2} \frac{\alpha}{\alpha e^t - 1} \)

from which we obtain

\( \frac{-\alpha}{2} \sum_{k=0}^{\infty} E_k(-\alpha) \frac{t^k}{k!} = \sum_{k=0}^{\infty} E_k(\alpha^2) \frac{t^k}{k!} - \frac{\alpha}{2} \sum_{k=0}^{\infty} E_k(\alpha) \frac{t^k}{k!} \)

and subsequently arrive at the following

\( E_k(\alpha^2) = \frac{\alpha}{2e^t} (E_k(\alpha) - E_k(-\alpha)) \).

Using this relation, Singh [7] defined the \( \alpha \)-Euler number for \( \alpha = 1 \) by

\( E_k(1) = \frac{1}{1 - 2k+1} E_k(-1) = \frac{1}{1 - 2k+1} \sum_{i=1}^{k} \frac{1}{2^i+1} \sum_{j=1}^{i} \binom{i}{j} (-1)^j j^k \) (3.3)

where the last expression in (3.3) utilizes lemma 3.1 for \( \alpha = -1 \). In fact, this determines a closed form expression for the Bernoulli numbers since for
\[ \alpha = 1, \] we have \[ \frac{-1}{2} \frac{1}{e^t + 1} = \frac{1}{e^{2t} - 1} \frac{-1}{2} \frac{1}{e^t - 1}, \] using which we obtain
\[ E_k(-1) = (1 - 2^{k+1}) \frac{B_{k+1}}{k+1}, \text{ for all } k = 0, 1, 2, \ldots \] (3.4)

From (3.3)-(3.4) we get
\[ B_{k+1} = \frac{(k + 1)}{1 - 2^{k+1}} E_k(-1) = (k + 1)E_k(1) \] which gives
\[ B_{k+1} = \frac{(k + 1)}{1 - 2^{k+1}} \sum_{i=1}^{k} \frac{1}{2^{i+1}} \sum_{j=1}^{i} \binom{i}{j} (-1)^j j^k. \] (3.5)

### 3.2 Sums of products

**Definition:** For \( \alpha \in \mathbb{C} \) define higher order \( \alpha \)-Euler number \( E_N^\alpha(k) := E_N^\alpha(k) \) where \( E_N^\alpha \in \mathcal{S} \) such that \( \left( \frac{\alpha}{\alpha e^t - 1} \right)^N = \sum_{k=1}^{\infty} E_N^\alpha(x) \frac{t^k}{k!}, |\alpha e^t| < 1. \)

Following important recurrence relation holds for \( E_N^\alpha(k). \)

**Lemma 3.2.** For each \( k \in \mathbb{N}_0 \)
\[ E_{N+1}^\alpha(k) = -\frac{\alpha^2}{N} \frac{d}{d\alpha} E_N^\alpha(k), \ N \in \mathbb{N}, \ \alpha \neq 0, 1. \] (3.6)

**Proof.** For \( \alpha \) as before and \( N \in \mathbb{Z}^+ \), we find that
\[ \frac{d}{d\alpha} \left( \frac{\alpha}{\alpha e^t - 1} \right)^N = -\frac{N}{\alpha^2} \left( \frac{\alpha}{\alpha e^t - 1} \right)^{N+1} \] from which the proof follows at once. \( \square \)

It is worthwhile to note that (3.6) connects \( E_N^\alpha(k) \) with \( E_k(\alpha) \) in a very simple way and makes it easy to compute them. In fact it is possible to obtain a closed form expression for \( E_N^\alpha(k) \) as we see next.

**Theorem 3.3.** For \( N \in \mathbb{N}_0, k \in \mathbb{N}, \) and \( \alpha \neq 0, 1, \)
\[ E_{N+1}^\alpha(k) = \sum_{n=1}^{k} \binom{N + n}{N} \left( \frac{\alpha}{\alpha - 1} \right)^{n+N+1} \sum_{m=1}^{n} \binom{n}{m} (-1)^m m^k \] (3.7)
Proof. We will prove by induction on \( N \). Clearly the result holds for \( N = 0 \). For \( N = 1 \) using lemma 3.1 and lemma 3.2, we get for all \( k \in \mathbb{N} \) and \( \alpha \neq 0, 1 \),

\[
E_k^2(\alpha) = -\frac{\alpha^2}{1}\frac{d}{d\alpha} \sum_{n=1}^{k} \left( \frac{\alpha}{\alpha - 1} \right)^{n+1} \sum_{m=1}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) (-1)^m m^k 
= \sum_{n=1}^{k} \left( \begin{array}{c} n + 1 \\ m \end{array} \right) \frac{\alpha}{\alpha - 1} \sum_{m=1}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) (-1)^m m^k.
\]

This establishes the result for \( N = 1 \). Let the formula 3.7 holds for all positive integer less than or equal to \( N \). Then using lemma 3.2 we have

\[
E_N^{N+1}(\alpha) = -\frac{\alpha^2}{N}\frac{d}{d\alpha} E_N^N(\alpha) 
= -\frac{\alpha^2}{N} \sum_{n=1}^{k} \left( \frac{N - 1 + n}{N - 1} \right) (n + N) \left( \frac{\alpha}{\alpha - 1} \right)^{n+N-1} \frac{d}{d\alpha} \left( \frac{\alpha}{\alpha - 1} \right) \sum_{m=1}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) (-1)^m m^k 
= -\alpha^2 \sum_{n=1}^{k} \left( \frac{N + n}{N} \right) \left( \frac{\alpha}{\alpha - 1} \right)^{n+N-1} \frac{1}{(\alpha - 1)^2} \sum_{m=1}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) (-1)^m m^k 
= \sum_{n=1}^{k} \left( \frac{\alpha}{\alpha - 1} \right)^{n+N+1} \sum_{m=1}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) (-1)^m m^k
\]

which proves the final step of induction. \( \square \)

There is another way of representing the higher order \( \alpha \)–Euler numbers; namely, the following.

**Theorem 3.4.** For each \( N \in \mathbb{N} \), \( k \in \mathbb{N}_0 \), and \( \alpha \neq 1 \),

\[
E_0^N(\alpha) = \frac{\alpha^N}{(\alpha - 1)^N}; \quad E_k^N(\alpha) := \frac{(-\alpha)^{N-1}}{(N-1)!} (\Delta_1 + 2I)_{N-3} E_k(\alpha), k \in \mathbb{N} \quad (3.8)
\]

where \( \Delta_1(E_k(\alpha)) = E_{k+1}(\alpha) - E_k(\alpha) \) and \( (\Delta_1 + 2I)_{N-3} := (\Delta_1 + 2I)(\Delta_1 + 3I) \cdots (\Delta_1 + NI) \).

**Proof.** It follows that \( E_0^N(\alpha) = \lim_{t \to 0} \left( \frac{\alpha}{\alpha e^t - 1} \right)^N = \frac{\alpha^N}{(\alpha - 1)^N} \). Now observe that \( \frac{d}{dt} \left( \frac{\alpha}{ae^t - 1} \right)^N = N \left( \frac{\alpha}{ae^t - 1} \right)^{N-1} \frac{\alpha}{(ae^t - 1)} \) from which we obtain the
recurrence given by
\[ E_{k+1}^{N+1}(\alpha) := -\frac{\alpha}{N}(E_{k+1}^{N}(\alpha) + NE_k^{N}(\alpha)) \]
\[ = -\frac{\alpha}{N}(E_{k+1}^{N}(\alpha) - E_k^{N}(\alpha) + (N + 1)E_k^{N}(\alpha)) \]
\[ = -\frac{\alpha}{N}((\Delta_1 + (N + 1)I)E_k^{N}(\alpha)) \]
\[ = \frac{(-\alpha)^2}{N(N-1)}((\Delta_1 + (N + 1)I)(\Delta_1 + NI)E_k^{N-1}(\alpha)) \]
\[ : \]
\[ = \frac{(-\alpha)^N}{N!}((\Delta_1 + (N + 1)I)\cdots(\Delta_1 + 2I)E_k^{N}(\alpha)) \]
which is valid for \( N \to 0 \) in addition to \( N = 1, 2, \ldots \). The result follows on changing \( N \) to \( N - 1 \) in (3.9).

Recall that for \( n, m \in \mathbb{N}_0 \) Stirling number of first kind \( s(n, m) \) is defined by
\[ x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n}s(n, m)x^m \]
which on changing \( x \) to \( -x \) gives
\[ x(x+1)\cdots(x+n-1) = \sum_{m=0}^{n}(-1)^{n-m}s(n, m)x^m. \]
A closed form expression for \( s(n, m) \) is also available due to Qi [22] which is given by
\[ s(n+1, m+1) := (-1)^{n+m}n!\sum_{\ell_1=1}^{n}\sum_{\ell_2=1}^{\ell_1-1}\cdots\sum_{\ell_m=1}^{\ell_{m-1}-1}\frac{1}{\ell_1\ell_2\cdots\ell_m}. \]

Replacing the symbol \( x \) in the generating function for \( s(n, m) \) above by the shift operator \( E_1 = \Delta_1 + I \), we have
\[ (\Delta_1 + 2I)\cdots(\Delta_1 + (N-1)I)E_k^{N}(\alpha) = E_1(E_1 + I)\cdots(E_1 + (N-1)I)E_{k-1}^{N}(\alpha) \]
\[ = \sum_{i=0}^{N}(-1)^{N-i}s(N, i)E_1^{i}(E_{k-1}^{N}(\alpha)) \]
\[ = \sum_{i=0}^{N}(-1)^{N-i}s(N, i)E_{k+i-1}(\alpha). \]

Substituting for \( (\Delta_1 + 2I)\cdots(\Delta_1 + (N-1)I)E_k^{N}(\alpha) \) from (3.11) in (3.8) we get another equivalent expression for \( E_k^{N}(\alpha) \)
\[ E_k^{N}(\alpha) = \frac{\alpha^{N-1}}{(N-1)!}\sum_{i=0}^{N}(-1)^{i-1}s(N, i)E_{k+i-1}(\alpha) \]
The expression (3.12) is special in a sense that it connects \( B_{k+N}^N \) to \( E_k^N(1) \). More precisely, we have the next result.

**Theorem 3.5.** For all positive integers \( k \) and \( N \),

\[
B_{k+N}^N := \frac{(k+N)!}{k!} E_k^N(1) \tag{3.13}
\]

**Proof.** From (3.4) \( E_k(1) = \frac{B_{k+1}}{k+1} \). Using this for \( \alpha = 1 \) in (3.12), we get

\[
\frac{(k+N)}{k!} E_k^N(1) = \frac{(k+N)}{k!} \frac{1}{(N-1)!} \sum_{i=0}^{N} (-1)^{i-1} s(N, i) \frac{B_{k+i}}{k+i} \]

\[
= N \left( \frac{k+N}{N} \right) \sum_{i=0}^{N} (-1)^{N-i-1} s(N, N-i) \frac{B_{k+N-i}}{k+N-i} \]

\[
= B_{k+N}^N.
\]

where the last step has been obtained using (1.5).

**3.3 \( \alpha \)-power sums**

Singh [7] defined for any \( \alpha \in \mathbb{C} \), \( k \)-th \( \alpha \)-power sum \( S_k(\alpha, x) \) by generating function \( \frac{e^{\alpha x(1+x)}-e^{\alpha x}}{e^x-1} = \sum_{k=0}^{\infty} S_k(\alpha, x) \frac{x^k}{k!} \), \( \alpha \neq 1 \), and obtained closed form expression for \( S_k(\alpha, x) \) in terms of \( E_k(\alpha) \)

\[
S_k(\alpha, x) = \begin{cases} 
\sum_{m=0}^{k} \binom{k}{m} E_m(\alpha) \{ \alpha^x(1+x)^{k-m} - 1 \} & \text{if } \alpha \neq 1 \\
S_k(x) & \text{if } \alpha = 1
\end{cases} \tag{3.14}
\]

Note that \( \lim_{\alpha \to 1} S_k(1, x) \neq S_k(x) \) even when \( x \) is a positive integer. However, for \( x \) a positive integer, it is easy to observe that \( S_k(\alpha, x) := \alpha + \alpha^2 2^k + \cdots \alpha^x x^k \). Further if \( 0 < |\alpha| < 1 \), \( \lim_{x \to \infty} S_k(\alpha, x) \) exists and is equal to \(-\alpha^{-1} E_k(\alpha)\) for all \( k \in \mathbb{N} \) where for \( k = 0 \) this limit is \(-E_0(\alpha)\). In particular the Abel sum for the divergent series \( \eta(-k) := \sum_{n=0}^{\infty} (-1)^n n^k \) can be summed up in closed form as the one sided limit at \( \alpha = -1 \), i.e.

\[
\eta(-k) := \lim_{\alpha \to -1^+} (\lim_{x \to \infty} S_k(\alpha, x)) = -E_k(-1) = -\sum_{i=1}^{k} \frac{1}{2i+1} \sum_{j=1}^{i} \binom{i}{j} (-1)^{i-j} j^k.
\]

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We now consider sums of products of $\alpha$—power sums and the related abel sums.

**Definition:** For a nonnegative integer $N$, we define sum of products of $\alpha$—power sums $T^N_k(\alpha, x)$ by the generating function
\[
G(\alpha, x, N, t) := \left(\frac{\alpha x^1 e^{(x+1)t} - \alpha e^t}{\alpha e^t - 1}\right)^N = \sum_{k=0}^{\infty} T^N_k(\alpha, x) \frac{t^k}{k!}, \quad \alpha \neq 1. \tag{3.15}
\]

It is immediate to see that $T^N_k(\alpha, 0) = 0$ and $T^1_k(\alpha, x) = S_k(\alpha, x)$. Also for $0 < |\alpha| < 1$, $\alpha^x \to 0$ as $x \to \infty$. If this is the case then it follows from (3.14) that $\lim_{x \to \infty} T^N_k(\alpha, x) = (-1)^N \sum_{m=0}^{k} \binom{k}{m} E^N_m(\alpha) N^{k-m}$, $0 < |\alpha| < 1$ which in particular gives the Abel sum of products
\[
\eta^N(-k) = (-1)^N \sum_{m=0}^{k} \binom{k}{m} E^N_m(-1) N^{k-m}. \tag{3.16}
\]

Next result deals with obtaining a closed form expression for the $\alpha$—power sums $T^N_k(\alpha, x)$.

**Theorem 3.6.** For $N \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $\alpha \neq 0, 1$,
\[
T^N_k(\alpha, x) := \sum_{n=0}^{k} \binom{k}{m} E^N_m(\alpha) \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \alpha^{nx} (N+nx)^{k-m}. \tag{3.17}
\]

**Proof.** Follows from following computational steps.
\[
\sum_{k=0}^{\infty} T^N_k(\alpha, x) \frac{t^k}{k!} = \left(\frac{\alpha x^1 e^{(x+1)t} - \alpha e^t}{\alpha e^t - 1}\right)^N (\alpha^x e^x - e^t)^N
\]
\[
= \sum_{m=0}^{\infty} E_m(\alpha) \frac{t^m}{m!} \sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i} \alpha^{ix} e^{(N+ix)t}
\]
\[
= \sum_{m=0}^{\infty} E_m(\alpha) \frac{t^m}{m!} \sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i} \alpha^{ix} \sum_{n=0}^{\infty} (N + ix)^n \frac{t^n}{n!}
\]
\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} E_m(\alpha) \frac{t^m}{m!} \sum_{i=0}^{N} \binom{N}{i} \alpha^{ix} (N + ix)^n \frac{t^{m+n}}{m!n!}
\]
\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} E_m(\alpha) \sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i} \alpha^{ix} (N + ix)^{k-m} \frac{t^k}{k!}
\]
from which the result can be obtained on comparing the like powers of \(t\).

\[ \square \]

4 Further extensions

The sums of products \(T^N_k(\alpha, x)\) satisfy simple recurrence relation from which these can be obtained successively. Here is the special one which we state as the following.

**Proposition 4.1.** Let \(k, N \in \mathbb{N}_0 \alpha \neq 1\). Then

\[
T^N_0(\alpha, x) := \alpha^N \left(\frac{x^\alpha - 1}{\alpha - 1}\right)^N, \quad T^N_k(\alpha, x) := \alpha \frac{\partial}{\partial \alpha}T^N_{k-1}(\alpha, x), \quad k \in \mathbb{N}. \tag{4.1}
\]

**Proof.** Expression for \(T^N_0\) is trivially true. It is also easy to see that (4.1) holds for \(N = 0\). For \(N > 0\) and \(k \geq 1\), the proof follows from \(\frac{\partial}{\partial t}G(\alpha, x, N, t) = \alpha \frac{\partial}{\partial \alpha}G(\alpha, x, N, t)\), where \(G(\alpha, x, N, t)\) is as in (3.15). \(\square\)

The recurrence (4.1) can be used to extend the definition of the sums of products of the power sums to negative integer values of \(k\).

**Definition:** The sum of products \(T^N_k(\alpha, x)\) for a complex \(N\) and real nonzero \(\alpha\), is defined successively by

\[
T^N_0(\alpha, x) = \alpha^N \left(\frac{x^\alpha - 1}{\alpha - 1}\right)^N; \quad T^N_k(\alpha, x) = \int_0^\alpha \frac{T_{-k+1}(\theta, x)}{\theta} d\theta, \quad k \in \mathbb{N}. \tag{4.2}
\]

As an example, from (4.2), we have \(T^N_1(\alpha, x) = \int_0^\alpha \frac{1}{\theta} \left(\frac{\theta^{\alpha+1} - \theta}{\theta - 1}\right)^N d\theta\). Further if we also assume \(0 < |\alpha| < 1\) and define \(\lim_{x \to 0} T^N_k(\alpha, x) := T^N_k(\alpha, \infty)\), we have

\[
T^N_{-1}(\alpha, \infty) = \int_0^\alpha \theta^{N-1}(1 - \theta)^{-N} d\theta
= \beta_\alpha(N, 1 - N), \quad Re(N) > 0, \quad 0 < \alpha \leq 1. \tag{4.3}
\]

where \(\beta_\alpha(N, 1 - N)\) is the standard incomplete beta function. For \(N \in \mathbb{N}\) we obtain from (4.3)

\[
T^N_{-1}(\alpha, \infty) = \sum_{i=1}^{N-1} (-1)^{N-1-i} \frac{T_i(\alpha, \infty)}{i} + (-1)^N \log |1 - \alpha|, \quad 0 < |\alpha| < 1. \tag{4.4}
\]
\[ T^{-N}_2(\alpha, \infty) = (-1)^N \sum_{i=1}^{N-1} \sum_{j=1}^{i-1} (-1)^{ij} \frac{T^j_0(\alpha, \infty)}{ij} - (-1)^N \log |1 - \alpha| \sum_{i=1}^{N-1} \frac{1}{i} \] (4.5)

For the case \( N = 1 \), Singh [7] proved the following result for \( T^{-1}_1(\alpha, \infty) \).

\[ T^{-1}_1(\alpha, \infty) := \frac{(-1)^{i-1}}{(i - 1)!} \int_0^{\alpha} \frac{(\log(t))^{i-1}}{1 - t} dt - \sum_{\beta=1}^{i-1} \frac{(-\log |\alpha|)^\beta}{\beta!} T_{-i+\beta}(\alpha, \infty), \] (4.6)

for all \( i \in \mathbb{N} \).

Also note from (4.6) that \( \lim_{\alpha \to 1^-} T^{-1}_k(\alpha, \infty) = \zeta(k) \) while \( \lim_{\alpha \to 1^+} T^{-1}_k(\alpha, \infty) = -\eta(k) \) for all \( k \in \mathbb{N} \). In the next result, we generalize the formula (4.6) for the underlying sums of products i.e. for all positive integers \( N \geq 2 \). To this end, we have following main result.

**Theorem 4.2.** For all \( k, N \in \mathbb{N}_0 \) and \( 0 < |\alpha| < 1 \),

\[ T^{-N}_k(\alpha, \infty) = \sum_{i_1=1}^{N-1} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \left( \frac{\alpha}{1 - \alpha} \right)^{i_k} \frac{(-1)^{N-k-i_k}}{i_1 \cdots i_k} \]

\[ + \sum_{i=1}^{k-1} s(N, k + 1 - i) T^{-1}_i(\alpha, \infty) + (-1)^N T^{-1}_k(\alpha, \infty) \]

**Proof.** Using (4.2) and (4.6) we obtain inductively

\[ T^{-N}_k(\alpha, \infty) = \sum_{i_1=1}^{N-1} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \left( \frac{\alpha}{1 - \alpha} \right)^{i_k} \frac{(-1)^{N-k-i_k}}{i_1 \cdots i_k} \]

\[ + \sum_{i_1=1}^{N-1} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_{k-1}=1}^{i_{k-2}-1} \frac{(-1)^{N-k}}{i_1 \cdots i_{k-1} i_k} T^{-1}_1(\alpha, \infty) \]

\[ + \cdots \]

\[ + \sum_{i_1=1}^N \frac{(-1)^{N-1}}{i_1} T^{-1}_{k+1}(\alpha, \infty) + (-1)^N T^{-1}_k(\alpha, \infty), \]

which, after substituting for the expression of the Stirling numbers of first kind from (3.10) gives the desired result. \( \Box \)
Corollary 4.3. For \( k, N \in \mathbb{N} \)

\[
\lim_{\alpha \to 1^-} T_N^k(\alpha, \infty) = \frac{1}{(N - 1)!} \sum_{i=2}^{k-1} s(N, k+1-i)\zeta(i)+(-1)^{N-1}\zeta(k) \text{ for } N < k, \quad \text{and} \tag{4.7}
\]

\[
\lim_{\alpha \to -1^+} T_N^k(\alpha, \infty) = \sum_{i_1=1}^{N-1} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \left( \frac{1}{2} \right)^{i_k} (-1)^{N-k-i_k} \frac{i_k}{i_1 \cdots i_k} + \sum_{i=1}^{k-1} s(N, k+1-i)\eta(i) + (-1)^{N-1}\eta(k). \tag{4.8}
\]

For a positive integer \( N \) and \( 0 < |\alpha| < 1 \), we have

\[
\frac{(-\alpha)^N e^{Nt}}{(\alpha e^t - 1)^N} = \sum_{k=0}^{\infty} T_N^k(\alpha, \infty) \frac{t^k}{k!}.
\]

It follows that

\[
\lim_{\alpha \to 1^-} T_N^k(\alpha, \infty) = (-1)^N \lim_{t_1 \to 0} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_k} \frac{e^{Nt_{k+1}}}{(e^{t_{k+1}} - 1)^N} dt_{k+1} \cdots dt_2
\]

\[
= \frac{1}{(N - 1)!} \sum_{i=k+1-N}^{k} s(N, k+1-i)\zeta(i) \text{ if } N < k. \tag{4.9}
\]

As an example, the table below shows the values of \( \lim_{\alpha \to 1^-} T_N^k(\alpha, \infty) \) for few values of \( k \) and \( N \). All computations in table below have been done in Mathematica with reference to (4.7).

| \( k \) \ \( \backslash \ \ N \) | 2     | 3     | 4     |
|-----------------|------|------|------|
| 3               | \( \pi^2 \) - \( \zeta(3) \) | \( \frac{\pi^2}{6} \) - \( \zeta(3) \) | \( \frac{\pi^2}{6} \) - \( \zeta(3) \) |
| 4               | \( -\frac{\pi^4}{90} + \z(3) \) | \( \pi^4 \) + \( \z(3) \) + \( \frac{\pi^4}{90} \) | \( \frac{\pi^4}{90} \) + \( 3\z(3) \) |
| 5               | \( \frac{\pi^6}{945} - \z(5) \) | \( \pi^6 \) + \( \z(5) \) + \( \z(3) \) | \( \frac{\pi^6}{945} \) + \( \z(5) \) |
| 6               | \( -\frac{\pi^6}{945} + \z(5) \) | \( \pi^6 \) + \( \z(5) \) - \( \frac{\pi^6}{945} \) | \( \frac{\pi^6}{945} \) + \( \z(3) \) + \( \z(3) \) |
**Theorem 4.4.** For all \( N \in \mathbb{N} \), the generalized Euler formula is given by

\[
T_{-2}^N \left( \frac{1}{2}, \infty \right) = (-1)^N \left( \sum_{i=1}^{N-1} \sum_{j=1}^{i-1} \frac{(-1)^j}{ij} + \log(2) \sum_{i=1}^{N-1} \frac{1}{i} \right) \\
+ (-1)^{N-1} \left\{ -\frac{(\log(2))^2}{2} + \frac{\zeta(2)}{2} \right\}.
\]  
(4.10)

If we take \( N = 1 \) in theorem 4.4 we arrive at the classical Euler’s formula

\[
- \int_0^{\frac{1}{2}} \frac{\log(1-t)}{t} \, dt := T_{-2}^1 \left( \frac{1}{2}, \infty \right) = -\frac{(\log(2))^2}{2} + \frac{\zeta(2)}{2}.
\]

**Proof.** First observe that for \( \alpha = \frac{1}{2} \), \( T_{-2}^N(\alpha, \infty) = T_{-2}^N(1-\alpha, \infty) \). Along with this if we use the following identity due to Singh \([7]\)

\[
- \int_0^{1-\alpha} \frac{\log(1-t)}{t} \, dt = -\log(\alpha) \log(1-\alpha) + \int_0^{1} \frac{\log(t)}{1-t} \, dt + \int_0^{\alpha} \frac{\log(1-t)}{t} \, dt,
\]

and that \( T_0^N \left( \frac{1}{2} \right) = 1, \forall N \), we obtain Eq.(4.10) by taking \( \alpha = \frac{1}{2} \) in Eq.(4.3) where we note that \( \int_0^{1} \frac{\log(t)}{1-t} \, dt = \zeta(2) \).

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