Simple completable contractions of nilpotent Lie algebras

Rutwig Campoamor-Stursberg
Departamento de Geometría y Topología
Fac. CC. Matemáticas Univ. Complutense
28040 Madrid ( Spain )

Abstract
We study a certain class of non-maximal rank contractions of the nilpotent Lie algebra \( g_m \) and show that these contractions are completable Lie algebras. As a consequence a family of solvable complete Lie algebras of non-maximal rank is given in arbitrary dimension.

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1 Generalities
The notion of contraction of a Lie algebra (also called degeneration by some authors) was originally introduced by physicists (Segal) as a tool to relate classical and quantum mechanics. Inoue and Wigner used contractions attending to a particularization, namely, that a subalgebra remains fixed through the contraction. This concept, quite restrictive for some purposes, was later generalized by Saletan [13] and Levi-Nahas. The relation between contractions and deformation theory is an important, but not fully exploited question [7]. Orbit closures of Lie algebras, where contractions play a central role, is not an easy problem, and it constitutes an essential tool in the analysis of the components of the varieties \( \mathfrak{L}^n \) and \( \mathfrak{N}^n \).

Let \( \mathfrak{L}^n \) be the set of complex Lie algebra laws in dimension \( n \). We identify each law with its structure constants \( C^k_{ij} \) on a fixed basis \( \{ X_i \} \) of \( \mathbb{C}^n \). The Jacobi identities

\[
\sum_{l=1}^{n} C^k_{ij} C^s_{kl} + C^l_{jk} C^s_{il} + C^s_{ki} C^l_{jl} = 0
\]

for \( 1 \leq i \leq j < k \leq n \), \( 1 \leq s \leq n \) show that \( \mathfrak{L}^n \) is an algebraic variety. The nilpotent Lie algebra laws \( \mathfrak{N}^n \) are a closed subset in \( \mathfrak{L}^n \). The
linear group $GL(n, \mathbb{C})$ acts on $\Sigma^n$ via changes of basis, i.e., $(g \ast \mu)(x,y) = g(\mu(g^{-1}(x),g^{-1}(y)))$ for $g \in GL(n, \mathbb{C})$. Let $O(\mu)$ denote the orbit of the law $\mu$ by this action, consisting of all structures in a single isomorphism class.

**Definition.** A Lie algebra $\lambda$ is a contraction of a law $\mu$ if $\lambda \in O(\mu)$.

Here the topology on the variety is either the metric topology or the Zariski topology. Both topologies lead to the same contractions. A contraction will be denoted by $\mu \hookrightarrow \lambda$. It follows that the entire orbit $O(\lambda)$ lies in the Zariski-closure of $O(\mu)$. In particular, the following condition implies $\mu \hookrightarrow \lambda$:

$$\exists g_t \in GL_n(\mathbb{C}(t)) \text{ such that } \lim_{t \to 0} g_t \ast \mu = \lambda$$

Contractions are also transitive, i.e., if $\lambda \hookrightarrow \mu$ and $\mu \hookrightarrow \psi$, then $\lambda \hookrightarrow \psi$. Thus not any existing contraction must be shown directly.

For a Lie algebra $\lambda$ to be a contraction of $\mu$, the following conditions are necessary:

**Lemma.** Let $\mu \hookrightarrow \lambda$. Then

1. $\dim \operatorname{Der}(\mu) < \dim \operatorname{Der}(\lambda)$
2. $\dim [\lambda, \lambda] \leq \dim [\mu, \mu]$
3. $\dim \mathcal{Z}(\lambda) \geq \dim \mathcal{Z}(\mu)$
4. $\operatorname{rank}(\lambda) \geq \operatorname{rank}(\mu)$

where $\operatorname{Der}(\mu)$ denotes the algebra of derivations, $\mathcal{Z}(\mu)$ the center of the algebra $\mu$ and $\operatorname{rank}(\mu)$ is the dimension of a maximal toral subalgebra of $\mu$.

Proofs of the assertions can be found in [5], [7] and [14].

For a nilpotent Lie algebra $\mathfrak{g}$, we denote the ideals of the central descending sequence by $C^i(\mathfrak{g})$, i.e., $C^i(\mathfrak{g}) = [\mathfrak{g}, C^{i-1}(\mathfrak{g})]$ for $i \geq 1$ and $C^0(\mathfrak{g}) = \mathfrak{g}$. If the algebra $\mathfrak{g}$ contracts to an algebra $\mathfrak{h}$, it follows from the lemma above that $\dim(C^j(\mathfrak{g})) \geq \dim(C^j(\mathfrak{h}))$. Therefore, if $\mathfrak{g}$ has nilindex $p$, then the nilindex of $\mathfrak{h}$ is $\leq p$.

Although we will not make explicit use of the characteristic sequence, it is convenient to recall this invariant: Consider a complex nilpotent Lie algebra $\mathfrak{g} = (\mathbb{C}^n, \mu)$. For each $X \in \mathbb{C}^n$ we denote $c(X)$ the ordered sequence of dimensions of Jordan blocks of the adjoint operator $ad_{\mu}(X)$.

**Definition.** The characteristic sequence of $\mathfrak{g}$ is an isomorphism invariant $c(\mathfrak{g})$ defined as

$$c(\mathfrak{g}) = \sup_{X \in \mathfrak{g} - C^1 \mathfrak{g}} \{c(X)\}$$

where $C^1 \mathfrak{g}$ denotes the derived subalgebra. A characteristic sequence is called linear if there exists an integer $n$ such that $c(\mathfrak{g}) = (n, 1, \ldots, 1)$. 
2 The algebras $\mathfrak{g}_m(q_1, \ldots, q_k)$

In this section we analyze some families of nilpotent Lie algebras for which certain classes of contractions will be determined.

For $m \geq 4$ let $\mathfrak{g}_m$ be the Lie algebra whose structural equations are

\[
d\omega_1 = d\omega_2 = 0
\]
\[
d\omega_j = \omega_1 \wedge \omega_{j-1}, \quad 3 \leq j \leq 2m
\]
\[
d\omega_{2m+1} = \sum_{j=2}^{m} (-1)^j \omega_j \wedge \omega_{2m+1-j}
\]

where $\{\omega_1, \ldots, \omega_{2m+1}\}$ is a basis of $(\mathbb{C}^{2m+1})^\ast$.

**Proposition.** For any $m \geq 4$ the Lie algebra $\mathfrak{g}_m$ is naturally graded of characteristic sequence $(2m-1, 1, 1)$ satisfying the following property

\[
C_{\mathfrak{g}_m}(C^m(\mathfrak{g}_m)) \supset C^m(\mathfrak{g}_m)
\]
\[
C_{\mathfrak{g}_m}(C^{m-1}(\mathfrak{g}_m)) \not\supset C^{m-1}(\mathfrak{g}_m)
\]

This algebra, which has been analyzed in [3], can be characterized as follows:

**Theorem.** For $m \geq 4$ any naturally graded, central extension $\mathfrak{g}$ of the filiform model Lie algebra $L_{2m}$ whose nilindex is $(2m-1)$ and satisfies

\[
C_{\mathfrak{g}}(C^m(\mathfrak{g})) \supset C^m(\mathfrak{g})
\]
\[
C_{\mathfrak{g}}(C^{m-1}(\mathfrak{g})) \not\supset C^{m-1}(\mathfrak{g})
\]

is isomorphic to $\mathfrak{g}_m$.

Algebras satisfying the preceding "centralizer condition" arise from the analysis of gradations of nilradicals of Borel subalgebras of complex simple Lie algebras, and are defined by a modification of the graded structure of these algebras [3]. In particular, the deformations and extensions of $\mathfrak{g}_m$ have been studied in [4].

Let $m \geq 4$. For any sequence $3 \leq q_1 < q_2 < \ldots < q_k \leq m+1$ let $\mathfrak{g}_m(q_1, \ldots, q_k)$ be the $(2m+1)$-dimensional Lie algebra whose Maurer-Cartan equations are:

\[
d\omega_1 = d\omega_2 = 0
\]
\[
d\omega_{q_i} = d\omega_{2m+2-q_i} = 0, \quad 1 \leq i \leq k
\]
\[
d\omega_j = \omega_1 \wedge \omega_{j-1}, \quad 3 \leq j \leq 2m, \quad j \notin \{q_i, 2m+2-q_i\}_{1 \leq i \leq k}
\]
\[
d\omega_{2m+1} = \sum_{j=2}^{m} (-1)^j \omega_j \wedge \omega_{2m+1-j}
\]

where $\{\omega_1, \ldots, \omega_{2m+1}\}$ is a basis of $(\mathbb{C}^{2m+1})^\ast$. 
Lemma. For $m \geq 4$ and $k \geq 1$ the Lie algebras $g_m(q_1,..,q_k)$ are nonsplit nilpotent of nonlinear characteristic sequence.

Remark. In general, the algebras $g_m(q_1,..,q_k)$ will not be naturally graded. In particular this happens whenever we have $q,q'$ such that $q' = 1 + q$. In fact, the differential form $d\omega_{2m+1}$ determines the gradation in some sense [3]. These algebras are also interesting for the study of solvable rigid Lie algebras whose nilradical has linear characteristic sequence [2].

Theorem. For any $m \geq 4$ and $k \geq 1$ the Lie algebra $g_m(q_1,..,q_k)$ is a contraction of $g$.

Proof. For any $k$-tuple $(q_1,..,q_k)$ let $S(m,q_1,..,q_k)$ be the following linear system

$$a_1 + a_{j-1} = a_j, \quad 3 \leq j \leq 2m, \quad j \notin \{q_i, 2m + 2 - q_i\}_{1 \leq i \leq k} \quad (1)$$
$$a_1 + a_{j-1} - a_{j-1} = -1, \quad j \in \{q_i, 2m + 2 - q_i\}_{1 \leq i \leq k} \quad (2)$$
$$a_j + a_{2m+1-j} = a_{j+1} + a_{2m-j}, \quad 2 \leq j \leq m-1 \quad (3)$$

and let $S'(m,q_1,..,q_k)$ be the system given by (1) and (2). We claim that any solution of $S'(m,q_1,..,q_k)$ is also a solution of $S(m,q_1,..,q_k)$. In fact, if $j \notin \{q_i, 2m + 2 - q_i\}_{1 \leq i \leq k}$ then $2m + 2 - q_i \neq 2m + 1 - j$ for all $i$, and therefore we have

$$a_j + a_{2m+1-j} = a_{j+1} + a_{2m-j} = a_1 + a_j + a_{2m-j}$$

thus

$$a_{2m+1-j} = a_1 + a_{2m-j}$$

If $j \in \{q_i, 2m + 2 - q_i\}_{1 \leq i \leq k}$ then $2m + 2 - q_i = 2m + 1 - j$ for some $i \in \{1,..,k\}$. Then

$$a_j + a_{2m+1-j} = a_{j+1} + a_{2m-j} = 1 + a_1 + a_j + a_{2m-j}$$

and simplifying

$$a_{2m+1-j} = 1 + a_1 + a_{2m-j}$$

This shows that the equations (3) are superfluous. Now, the system $S'(m,q_1,..,q_k)$ clearly has integer solutions, depending on the parameters $a_2 = N_1$ and $a_3 = N_2$. Let $(a_1,..,a_{2m})$ be the solution corresponding to the values $N_1 = N_2 = 1$ and $T_{2m+1}(\mathbb{C}(t))$ denote the Borel subgroup of $GL(2m+1,\mathbb{C})$ consisting of lower triangular matrices. Define $f_{(q_1,..,q_k),t} \in T_{2m+1}(\mathbb{C}(t))$ by

$$\left\{ \begin{array}{l}
 f_{(q_1,..,q_k),t}(X_i) = t^{a_i}X_i, \quad i \neq 2m + 1 \\
 f_{(q_1,..,q_k),t}(X_{2m+1}) = t^{1+a_{2m-1}}
 \end{array} \right.$$
and consider the Lie algebra \( f_{(q_1,...,q_k), t}^{-1} \ast \mu \), where \( \mu \) is the Lie algebra law associated to \( g_m \). Then the structural equations of \( f_{(q_1,...,q_k), t}^{-1} \ast \mu \) are given by

\[
\begin{align*}
\omega_1 &= \omega_2 = 0 \\
\omega_j &= t^{a_1 + a_j - a_{j-1}} \omega_1 \wedge \omega_{j-1}, \quad j \in \{q_i, 2m + 2 - q_i\}_{1 \leq i \leq k} \\
\omega_j &= \omega_1 \wedge \omega_{j-1}, \quad 3 \leq j \leq 2m, \quad j \notin \{q_i, 2m + 2 - q_i\}_{1 \leq i \leq k} \\
d\omega_{2m+1} &= \sum_{j=2}^{m} (-1)^j \omega_j \wedge \omega_{2m+1-j}
\end{align*}
\]

Now, as \( a_1 + a_{j-1} - a_j = -1 \) for \( j \in \{q_i, 2m + 2 - q_i\}_{1 \leq i \leq k} \), it follows easily that

\[
\lim_{t \to \infty} f_{(q_1,...,q_k), t}^{-1} \ast \mu = \mu (q_1, ..., q_k)
\]

where \( \mu (q_1, ..., q_k) \) is the law associated to \( g_m (q_1, ..., q_k) \).

\[\square\]

**Corollary.** For \( m \geq 4 \)

\[\mu (q_1, ..., q_k) \in O (\mu (q'_1, ..., q'_k))\]

if and only if \( \{q_1, ..., q_k\} = \{q'_1, ..., q'_k\} \).

**Corollary.** For \( m \geq 4 \) the algebras \( g_m (q_1, ..., q_k) \) ( included \( g \)) are nontrivial deformations of the algebra \( h_{m-1} \oplus \mathbb{C}^2 \), where \( h_{m-1} \) is the \((2m-1)\)-dimensional Heisenberg Lie algebra.

**Proof.** Let \( f_t \in T_{2m+1} (\mathbb{C} (t)) \) be defined by

\[
f_t (X_i) = t^{a_i} X_i, \quad 1 \leq i \leq 2m+1
\]

where the \( a_i \) satisfy the system \( S' (m, q_1, ..., q_m) \) and the additional equation

\[
a_1 + a_{2m-1} = a_{2m} - 1
\]

Then \( \lim_{t \to \infty} f_t^{-1} \ast \mu \in O (h_{m-1} \oplus \mathbb{C}^2) \).

A similar reasoning shows that \( g_m (q_1, ..., q_k) \to h_{m-1} \oplus \mathbb{C}^2 \). The result follows from the fact that a contraction defines a nontrivial deformation [7].

\[\square\]

### 3 Applications to complete Lie algebras

Recall that a Lie algebra \( g \) is called complete if it is centerless and any derivation is inner [8]. I recent years, a general theory of complete solvable Lie algebras whose nilradical is of maximal rank has been developed ( [9], [10], [12]), and the existence of non-maximal rank algebras has been pointed out. In this section we will see how to obtain families of non-maximal rank completable Lie algebras considering the contractions above.
Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\text{Der}(\mathfrak{g})$ its Lie algebra of derivations. A torus $t$ over $\mathfrak{g}$ is an abelian subalgebra of $\text{Der}(\mathfrak{g})$ consisting of semisimple derivations. The torus $t$ induces a natural representation on $\mathfrak{g}$ such that
\[
\mathfrak{g} = \sum_{\alpha \in t^*} \mathfrak{g}_\alpha
\]
where $t^* = \text{Hom}(t, \mathbb{C})$ and $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} | [t, X] = \alpha(t) X, \forall t \in t \}$. If $t$ is maximal for the inclusion relation, by the conjugation theorems of Morozov, its common dimension is a numerical invariant called the rank of $\mathfrak{g}$, denoted $\text{rank}(\mathfrak{g})$. An algebra is called of maximal rank if $\text{rank}(\mathfrak{g}) = b_1 = \dim H^1(\mathfrak{g}, \mathbb{C})$. Following Favre [6], a weight system of $\mathfrak{g}$ is given by
\[
P_\mathfrak{g}(t) = \{(\alpha, d\alpha) | \alpha \in t^* \text{ such that } \mathfrak{g}_\alpha \neq 0, \text{ da = dim } \mathfrak{g}_\alpha \}
\]
We also recall the following results:

**Proposition.** Let $\mathfrak{g}$ be of maximal rank and $t$ a maximal torus. Then the semidirect product $t \oplus \mathfrak{g}$ is complete solvable.

**Definition.** The algebra $\mathfrak{g}$ is called completable if $t \oplus \mathfrak{g}$ is complete for a maximal torus $t$, and simple completable if it is not the direct sum of nontrivial completable Lie algebras.

The main result we will use is a slight modification of the next

**Theorem.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Assume that following conditions are satisfied:

1. $\mathfrak{h}$ is abelian
2. $\mathfrak{g} = \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ with $\Delta \subset \mathfrak{h}^* - \{0\}$
3. there is a basis $\{\alpha_1, ..., \alpha_l\}$ of $\mathfrak{h}^*$ in $\Delta$ such that $\dim \mathfrak{g}_{\pm \alpha_j} \leq 1$ for $1 \leq j \leq l$ and $[\mathfrak{g}_{\alpha_j}, \mathfrak{g}_{-\alpha_j}] \neq 0$ if $-\alpha_j \in \Delta$
4. $\mathfrak{h}$ and $\{\mathfrak{g}_{\pm \alpha_j}, 1 \leq j \leq l\}$ generate $\mathfrak{g}$

Then $\mathfrak{g}$ is a complete Lie algebra.

The last conditions can be replaced by by more general statements [11]

3’ there is a generating system $\{\alpha_1, ..., \alpha_l\}$ of $\mathfrak{h}^*$ in $\Delta$ such that $\dim \mathfrak{g}_{\alpha_j} = 1$ for all $j$ and $\mathfrak{h}, \mathfrak{g}_{\alpha_1}, ..., \mathfrak{g}_{\alpha_l}$ generate $\mathfrak{g}$.

4’ Let $0 \neq x_j \in \mathfrak{g}_{\alpha_j}$ and a basis $\{\alpha_1, ..., \alpha_r\}$ of $\mathfrak{h}^*$. For $r + 1 \leq s \leq l$,
\[
\alpha_s = \sum_{i=1}^{l} k_{is} \alpha_j - \sum_{i=1+1}^{r} k_{is} \alpha_j
\]
where \( k_{i\alpha} \in \mathbb{N} \cup \{0\}, (j_1, ..., j_r) \) is a permutation of \((1, ..., r)\), and there is a formula

\[
\begin{bmatrix}
  x_{j_1}, ..., x_{j_1}, ..., x_{j_1}, ..., x_{k_m}
\end{bmatrix}_{k_{i\alpha}} = \begin{bmatrix}
  x_{j_1+1}, ..., x_{j_1+1}, ..., x_{j_1+1}, ..., x_{k_m}
\end{bmatrix}_{k_{i\alpha}}
\]

without regard to the order or the way of bracketing, where \( m \neq 0 \) if \( t = r \).

**Theorem [11].** Let \( \mathfrak{g} \) be a Lie algebra satisfying conditions (1), (2), (3'), and (4'). Then \( \mathfrak{g} \) is a complete Lie algebra.

We obtain one more consequence of the theorem in the preceding section

**Corollary.** For \( m \geq 4 \) and \( k \geq 1 \)

\[
\text{rank} (\mathfrak{g}_m (q_1, ..., q_k)) > 2
\]

**Proof.** The fact follows from the linear system \((S(\mathfrak{g}_m (q_1, ..., q_k)))\) associated to the algebras [1] and the fact that \( b_1 (\mathfrak{g}_m (q_1, ..., q_k)) > b_1 (\mathfrak{g}_m) = 2 \), where \( b_1 (\mathfrak{g}) = \dim H^1 (\mathfrak{g}, \mathbb{C}) \). \( \square \)

**Proposition.** Let \( m \geq 4 \) and \( k \geq 1 \) a weight system of \( \mathfrak{g}_m (q_1, ..., q_k) \) is given by

\[
P_{\mathfrak{g}_m} (t_m) = \left\{ (\alpha_i, d\alpha_i)_{1 \leq i \leq 2m+1} \right\}
\]

where \( \dim \mathfrak{g}_{\alpha_i} = 1 \) for all \( i \) and the weights \( \{\alpha_1, ..., \alpha_{2m+1}\} \) satisfy the following linear system

\[
\begin{align*}
\alpha_1 + \alpha_{j-1} &= a_j, & 3 \leq j \leq 2m, & j \notin \{q_i, 2m + 2 - q_i\}_{1 \leq i \leq k} \\
a_{2m-t} - \alpha_{t+1} - \alpha_m &= \alpha_{m+1}, & 1 \leq t \leq m - 2
\end{align*}
\]

\((S_1)\)

In particular,

\[
\text{rank} (\mathfrak{g}_m (q_1, ..., q_k)) \leq m + 1.
\]

**Proof.** The system \((S_1)\) coincides with the linear system \( S(\mathfrak{g}_m (q_1, ..., q_k)) \) associated to the nilpotent Lie algebra \( \mathfrak{g}_m (q_1, ..., q_k) \), thus the \( \alpha_i \) correspond to eigenvalues of semisimple derivations [1]. Therefore we can isolate \( \alpha_{j_1}, ..., \alpha_{j_s} \) \( (1 \leq j_1 < j_2 < ... < j_s < 2 + 2k) \) such that for any \( j \in \{1, ..., 2m+1\} \) \( - \{j_1, ..., j_s\} \) we have

\[
\alpha_j = \sum_{t=1}^{s} a^t_j \alpha_{j_t}, \quad a^t_j \in \mathbb{C}
\]

Now, using the standard techniques [1] it is routine to verify that for \( 1 \leq t \leq s \) the derivations \( f_{j_t} \in Der (\mathfrak{g}_m (q_1, ..., q_k)) \) given by

\[
f_{j_t} (X_i) = a^t_i X_i, \quad 1 \leq i \leq 2m + 1
\]
define a maximal torus $t_m (q_1, ..., q_k)$ of $g_m (q_1, ..., q_k)$. Thus the weight system is given as above.

Clearly any weight space $g_{\alpha_j}$ is at most one dimensional. For the last assertion, observe that for $k = m + 1$ the system $(S_1)$ is

$$\alpha_1 + \alpha_{2m-1} = \alpha_{2m}$$
$$\alpha_t + \alpha_{2m+1-t} = \alpha_{m} + \alpha_{m+1}, \quad 2 \leq t \leq m - 1$$

and that the rank is the maximal possible, namely $m + 1$.

Theorem. For $m \geq 4$ and $k \geq 1$ the semidirect products

$$r_m (q_1, ..., q_k) = t_m (q_1, ..., q_k) \oplus g_m (q_1, ..., q_k)$$

are solvable and complete.

Proof. It is easy to see that for any $f \in \text{Der} (r_m (q_1, ..., q_k))$ we have

$$f (t_m (q_1, ..., q_k)) \subset g_m (q_1, ..., q_k)$$

This is a direct consequence of the particular action of $t_m (q_1, ..., q_k)$ over $g_m (q_1, ..., q_k)$.

If $t_m (q_1, ..., q_k) = \{h_1, ..., h_s\}$, then for any $1 \leq t \leq s$ there exists a permutation $\sigma \in S_s$ such that $f_{j^t} = \text{ad} (h_t)$. The nilradical is clearly generated by $\{X_1, X_2, X_q, X_{2m+2-q}\}_{1 \leq i \leq k}$, where $\{X_1, ..., X_{2m+1}\}$ is a dual basis to $\{\omega_1, ..., \omega_{2m+1}\}$. It follows that the algebra $r_m (q_1, ..., q_k)$ satisfies the conditions of the preceding theorem, so that it is complete.

Lemma. If $k = 1$ and $q_1 \neq m + 1$ or $k \geq 2$, the $g_m (q_1, ..., q_k)$ is not of maximal rank.

Proof. The proof is an immediate consequence of the weight system. For $k = 1$ and $q_1 = m + 1$ we have $\text{rank} (S_1) = b_1 = 3$. Observe that this is the only case where $q_i = 2m + 2 - q_i$.

Corollary. For any odd dimension $n \geq 9$ there exist completable Lie algebras of non-maximal rank.

Thus the contractions of the algebra $g_m$ (it is itself completable since it is of maximal rank) are completable of non-maximal rank, up to an exception. The families above expand the examples obtained in [10],[12] for non-maximal rank. In fact, since the algebras $g_m (q_1, ..., q_k)$ are nonsplit, we obtain even more:

Corollary. For $m \geq 4$ and $k \geq 1$ the algebras $g_m (q_1, ..., q_k)$ are simple completable.
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