Unraveling non-Hermitian pumping: emergent spectral singularities and anomalous responses

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Within the expanding field of non-Hermitian physics, non-Hermitian pumping has emerged as a key phenomenon, epitomized through the skin effect via extensive boundary mode accumulation modifying the conventional Bloch picture. Beyond redefining bulk-boundary correspondences, we show that non-Hermitian pumping induces an unprecedented type of spectral topology: it admits a graph-theoretic classification, which is distinct from conventional topological classifications of the eigenstate or energy Riemann surface. Each topological class is characterized by a conformally deforms the complex band structure into a graph-like structure with characteristic branching singularities, with gap-preserving transitions giving rise to emergent band geometry and Berry curvature discontinuities physically manifested as anomalous response kinks. To observe such behavior as cusps in wavepacket trajectories, we propose a cold atom setup that builds upon established protocols. By placing all Hermitian and non-Hermitian lattice Hamiltonians on equal footing, our comprehensive framework also enables the first analytic constructions of topological phase diagrams in the presence of multiple non-reciprocal coupling scales, as prototypically demonstrated for the extended non-Hermitian Chern and Kitaev models. Based on general algebraic geometry properties of the energy dispersion, our framework can be directly generalized to multiple bands, dimensions, and even interacting systems. Overall, it reveals the conspiracy of band representations, spectral topology, and complex geometry as it unfolds in directly measurable quantities.

I. INTRODUCTION

The realm of non-Hermiticity harbors a variety of spectacular yet non-intuitive phenomena. Novel Fermi surface properties such as enhanced amplification emerge when energy bands intersect along highly degenerate exceptional manifolds [1,19], and likewise new topological classes appear when the constraints of Hermiticity are relaxed [17,20]. Capturing the attention of much recent theoretical and experimental effort is the phenomenon of non-Hermitian pumping (skin effect) [27,37], where eigenmodes are relentlessly “pumped” towards the boundaries due to effectively asymmetric gain/loss [38]. As intuitively expected, this pumping results in extreme sensitivity to finite-size and boundary effects, as well as the intensely studied modifications of topological bulk-boundary correspondences (BBCs).

The physical and formal implications of non-Hermitian pumping, however, extend far beyond modified band structures or topological descriptions. As we shall reveal, non-Hermitian pumping also leads to a new type of graph-theoretic topological classification marked by physical signatures such as response kinks. It generically deforms the complex band structure into a graph-like structure with characteristic branching singularities, with topological transitions corresponding to novel anomalous linear responses. Characterized by a graph-theoretic spectral classification, this new type of topology is distinct from conventional topological characterizations [7,17] i.e. winding properties of either the eigenstates or the energy Riemann surface, which in the simplest instances give rise to Z windings and vorticies. Transitions between our topological singularities manifest as discontinuities in the Fubini Study metric of the eigenbands, whose imaginary part corresponds to the Berry curvature. Most importantly, such topological transitions are physically detectable as signature anomalies in transport, noise, and scattering behavior, such as through kinks in semi-classical wavepacket trajectories that have no analog in local Hermitian systems. Adding to their enigmatic allure is that such complex singularity transitions do not necessitate band gap closures, unlike conventional topological transitions which rely on gap closures for any discontinuous evolution of the bands.

With the rise of experimental platforms such as topological lasers, photonic crystals, mechanical frameworks and circuits for non-Hermitian phenomena [39–51], and non-Hermitian pumping in particular [48–51], a comprehensive understanding of these exotic singularities and responses is of practical and theoretical exigence. As such, we devise a new universal framework that puts generic non-Hermitian lattice Hamiltonians on equal footing as their Hermitian counterparts, which are immune to the skin effect. Specifically, we formulate a unitary restoration procedure that maps any non-Hermitian model to a quasi-reciprocal surrogate Hamiltonian at the operator level, such that the effectively restored bulk Hilbert space allows topological invariants and metrics to faithfully predict topological phase boundaries just like in genuinely

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reciprocal or Hermitian systems. Literally, this procedure “unravels” non-Hermitian pumping through a redefined non-local basis where the accumulated eigenmodes appear equilibrated (Fig. 1a). Figuratively, it illuminates the deeper implications of non-Hermitian pumping beyond what can be predicted from simply defining a generalized Brillouin zone (BZ) \[27, 33, 37\]. Most salient are the non-perturbative effects implied by additional couplings: While we ordinarily expect weak couplings across distant sites to at most trivially modify the band structure, they, no matter how weak, can generically produce more complex topological singularities when non-Hermitian pumping is present. Such enigmatic behavior is a consequence of the emergent non-locality that also underscores Berry curvature discontinuities in the absence of band touchings.

For concreteness, we shall illustrate our findings with two quintessential non-Hermitian models that have so far eluded rigorous characterization: the extended non-Hermitian Kitaev chain and the extended non-Hermitian Chern insulator. To explore the interesting singularities, we introduce in both models asymmetric couplings beyond nearest neighbors (NNs), which are also physically relevant in realistic photonics and plasmonics setups governed by non-compact orbitals or long-ranged Coulomb forces. In the minimal description of the extended non-Hermitian Kitaev chain, which is of D1-class topology \[52\] (\(\mathbb{Z}_2\) with conjugated particle-hole symmetry (cPHS) and a real line-gap) \[18, 21\], both NN and next-nearest-neighbor (NNN) couplings are in fact necessary and hence essential aspects of this non-Hermitian topological class. Our extended non-Hermitian Chern insulator, which is pedagogically designed to reduce to a minimal 1D description with two effective non-reciprocal couplings, describes the only other known singularity class (besides the well-known NN non-Hermitian Chern insulator \[29\]) where the surrogate Berry curvature, which reliably predicts Chern edge modes, can be analytically computed.

As complex non-analyticities in momentum space, emergent singularity transitions from non-Hermitian pumping should in principle be detectable via any observable that depends on the derivatives of eigenstates. Practically, one of the most feasible avenues is the detection of discontinuous Berry curvature in ultracold atomic systems, where wavepackets experience an anomalous force due to Berry curvature \[53, 54\], and are expected to execute sudden changes in acceleration (cusp in their trajectories) when they cross a discontinuity. With recent advances that allowed for precise optical lattice parameter tuning and atomic-resolution imaging \[55, 57\], pioneering observations of various topological states have already been made with cold atoms. Lately, proposals have further elaborated on inducing non-Hermitian pumping with depopulation losses \[58\]. In this work, we provide an explicit cold atom setup exhibiting a non-Hermitian Chern phase, and illustrate how kinks can be introduced into its semi-classical wavepacket trajectories due to non-Hermitian pumping. Alternatively, classical setups such as electronic circuits \[59, 62\] provide a much simpler route to measuring discontinuous Berry curvature textures, although they involve indirect reconstructions that cannot capture all interesting anomalous response behaviors.

This paper is organized as follows. In Section II, we review known properties of non-Hermitian pumping, and subsequently introduce the concept of a quasi-reciprocal surrogate Hamiltonian that implements the generalized BZ at the operator level. The consequent emergent non-locality of the surrogate basis will be a recurring theme of this work. In Section III, we continue with a detailed treatment of a few common spectral singularities, followed by a discussion of their classification and topological transitions. In particular, we illustrate our formalism with two detailed examples: the 1D non-Hermitian extended Kitaev model and the 2D extended non-Hermitian Chern model, where the construction of the surrogate basis with nontrivial generalized BZ proves crucial for topological characterization and extraction of Berry curvature discontinuities. In Section IV, we elaborate on such discontinuities as anomalous physical responses, and provide a cold-atom proposal for their experimental demonstration. Finally, we conclude in Section V that we have developed the appropriate classification scheme of non-Hermitian physics from the viewpoint of non-Hermitian pumping, which elucidates and encodes new principal phenomena expected to emanate from non-Hermitian systems.

II. UNRAVELING NON-HERMITIAN PUMPING

A. Preliminaries

We first briefly review the rudiments of non-Hermitian pumping in non-Hermitian lattices. Consider a 1D effective Hamiltonian described by

\[
H = \sum_{ij, \alpha \beta} h^{\alpha \beta}_{ij} c_{i \alpha}^\dagger c_{j \beta} = \sum_{k, \alpha \beta} H^{\alpha \beta}(k) c_{k \alpha}^\dagger c_{k \beta},
\]

where \(h^{\alpha \beta}_{ij}\) and \(H^{\alpha \beta}(k)\) are its real space and momentum space representations, and \(ij, k, \alpha \beta\) indexes unit cells, momentum and intra-cell orbitals, respectively. Non-Hermitian pumping, also known as the non-Hermitian skin effect, is an extensive accumulation of eigenmodes that occur when all eigenmodes are “pumped” towards the boundaries under open boundary conditions (OBCs). Intuitively, it occurs when the 1D effective description contains gain/loss terms that couple asymmetrically in real space. Indeed, it can be shown that \[33\] the necessary condition for non-Hermitian pumping is that the effective 1D description \(H\) is simultaneously non-Hermitian and non-reciprocal \[85\], which can be respectively expressed as the first and second of the following inequalities: \(h^{\alpha \beta}_{ij} \neq h^{\alpha \beta}_{ji} \neq h^{\alpha \beta}_{ji}\). In momentum space, these
conditions take the form \( H^T(-k) \neq H(k) \neq H^\dagger(k) \). In other words, there must either exist coupling terms whose magnitudes are direction-dependent, or there must be the simultaneous presence of magnetic flux and on-site gain/loss [64]. In 2D or higher, \( H(k) \) also implicitly contain momentum parameters in other directions perpendicular to the boundary, and it is possible that a fully reciprocal (but still non-Hermitian) lattice can still exhibit non-reciprocity in the effective 1D description [49] \( (H^T(-k) \neq H(k)) \). Thereafter, we shall refer to this mode accumulation only as non-Hermitian pumping, with the implicit understanding that it occurs only when the 1D description is also non-reciprocal.

The conditions of non-reciprocity and non-Hermiticity, which leads to non-Hermitian pumping, also has intuitive interpretations in terms of the energy spectrum. In Hermitian cases, the periodic boundary condition (PBC) spectrum \( \epsilon(k), k \in [0,2\pi] \) is confined to the real line, and in reciprocal cases, \( \epsilon(-k) = \epsilon(k) \) mandates a degenerate loop. But the simultaneous presence of non-reciprocity and non-Hermiticity relaxes both of these constraints, allowing \( \epsilon(k) \) to generically trace closed loops with non-vanishing areas in the complex energy plane (Fig. 1b,c). Since \( k \) is a periodic parameter label, these loops are necessarily closed even if the eigenenergy bands switch partners after every period [33, 66, 65].

Non-Hermitian pumping under OBCs causes extensive boundary accumulation which is not in line with the conventional Bloch picture. As such, we expect the OBC spectrum to deviate considerably from the PBC spectrum, which corresponds to Bloch-type eigensolutions at real momenta (Fig. 1). This ostensible breakdown of BBC is the hallmark of non-Hermitian pumping [33]. Mathematically, it can be expressed as the extensive spectra flow \( \epsilon(k) \rightarrow \bar{\epsilon}(k) \) into the interior of PBC loops/s \{\( \epsilon(k) \)\} as we interpolate between PBCs and OBCs (or, more generally, by adding spatial non-uniformity). In the thermodynamic limit, \( \bar{\epsilon}(k), k \in [0,2\pi] \) converges to the OBC spectrum, excluding its topological modes which are isolated protected eigensolutions [33]. Although we have explicitly referred to \( H(k) \) and \( \epsilon(k), \bar{\epsilon}(k) \) as the Hamiltonian and eigenenergies, the above conceptual review and most of the rest of this paper applies equally well to generic operators, i.e., also the circuit Laplacian and their eigenspectra.

B. Quasi-reciprocal surrogate Hamiltonian

To study the effects of non-Hermitian pumping, we introduce the surrogate Hamiltonian

\[
\tilde{H}(k) = H(k + i\kappa(k)),
\]

where \( \kappa(k) \) is defined such that the \( \tilde{H}(k) \) eigenstate of interest experiences no non-Hermitian pumping when it is put under OBCs - our so-called property of quasi-reciprocity [66] (Fig. 1). To be specific, \( \kappa(k) \) is given by the smallest (magnitude-wise) complex deformation of the momentum \( k \) such that the eigenvalues \( \bar{\epsilon}(k) \) of \( \tilde{H}(k) \) lie at the endpoint of the PBC-OBC spectral flow. This formalism fully encodes the effects of non-Hermitian pumping at the operator level, beyond existing works [27, 33–37] that introduce a generalized BZ (complex analytic continuation of the momentum) for finding the skin eigenmodes. In other words, given any physical Hamiltonian \( H(k) \), we define a surrogate Hamiltonian \( \tilde{H}(k) \) possessing almost identical OBC spectra \( \bar{\epsilon}(k) \) but avoids the complications of non-Hermitian pumping. This is further elaborated on in Sect. II C. Physically, \( \tilde{H}(k) \) provides an effective description of the OBC system after the non-reciprocally pumped modes have “equilibrated” at the boundaries. Most importantly, it experiences no further pumping, and is hence characterized by all approaches valid for reciprocal or Hermitian systems which obey the BBC. By representing the effects of non-Hermitian pumping as a generically non-analytic momentum deformation \( \kappa(k) \), we shall soon uncover manifold exotic non-analytic behavior not present in the simplest case of constant \( \kappa \) as in commonly studied models [27, 29]. We emphasize that the OBC and PBC systems possess their own distinct eigenspaces, and it has to be the OBC \( \tilde{H}(k) \), not the PBC \( H(k) \), that determines all physical responses of a bounded system (which is under OBCs by definition), even those concerning “bulk” properties such as the Berry curvature.

We now describe how the quintessential complex deformation \( \kappa(k) \) can be computed. Intuitively, it is the \( k \)-dependent deformation of the PBC spectral loops \( \text{Det}[H(z) - E I] = 0 \), which generically can be written as a bivariate polynomial \( P(E, z) = 0 \). We shall use \( E \) to refer to the energy as an algebraic variable, and use \( \epsilon(k), \bar{\epsilon}(k) \) when it is also an eigenenergy of the original/surrogate Hamiltonian. For particle-hole symmetric 2-component Hamiltonians with only off-diagonal entries for instance, the eigenenergy equation assumes the form [69]

\[
E^2 = \sum_{n=-l_L}^{l_R} t_n z^n
\]

with \( z = e^{i(k + i\kappa(k))} \) and \( l_L, l_R \) the sum of the maximal ranges of the left/right couplings over both components/sublattices. While the \( t_n \)‘s coincide with the physical couplings in single-component systems, they are sums of product of the couplings. In more general cases, \( t_n \) also depends on \( E \), and thus do not directly correspond to any particular group of couplings.

For each \( k \), \( \kappa(k) \) is thus the smallest complex deformation for which there exists another momentum \( k' \) such that both \( z = e^{i(k + i\kappa(k))} \) and \( z' = e^{i(k' + i\kappa(k'))} \) are roots satisfying the same eigenenergy \( E \). In other words, \( \kappa(k) = \kappa(k') \) are the symmetric deformations necessary
nonconstant $\kappa$ a non-local basis redefinition (Bottom Center) dictated by mode accumulation can be nullified with a simple spatial basis resulting in a non-local $\bar{\epsilon}$ the equilibration of the combination of two or more pumps $\epsilon$ corresponding to NNs in Hamiltonians $H_1(k), H_2(k), H_{12}(k)$ (Left Column) which are affected by non-Hermitian pumping. With non-reciprocal couplings across only one distance scale (Top or Middle Row, corresponding to NNs in $H_1$ or NNNs in $H_2$), spatial eigenmode accumulation can be nullified with the simple spatial basis rescaling corresponding to a constant $\kappa = \kappa_1$ or $\kappa_2$. But the equilibration of the combination of two or more pumping length scales in $H_{12} = H_1 + H_2$ (Bottom Row) requires a non-local basis redefinition (Bottom Center) dictated by nonconstant $\kappa(k)$ which is not equivalent to $\kappa_1$ or $\kappa_2$, resulting in a non-local $\bar{H}_{12}(k)$. (b-c) PBC $\epsilon(k)$ (red), OBC $\epsilon(k) = \epsilon(k + i\kappa(k))$ (black) spectra and their interpolations $\epsilon(k + i\kappa(k))$ (various shades of brown) where $r = 0.4, 0.8, 1.1$. (b) and (c) depict $E = 2z + \frac{1}{2}$ and $E = z^2 + \frac{1}{4}$ where $z = e^{ik}$, representing cases with one and two pumping scales respectively. Background light green curves are contours of constant $\kappa$. (d-e) $\kappa(k)$ for cases (b) and (c), contrasting the constant $\kappa = \log 2 \approx 0.69$ cases (d) with a non-constant $\kappa(k)$ with cusps that indicate emergent non-locality (e).

FIG. 1. (a) Construction of quasi-reciprocal Hamiltonians $\bar{H}_1(k), \bar{H}_2(k), \bar{H}_{12}(k)$ (Right Column) from various physical Hamiltonians $H_1(k), H_2(k), H_{12}(k)$ (Left Column) which are affected by non-Hermitian pumping. With non-reciprocal couplings across only one distance scale (Top or Middle Row, corresponding to NNs in $H_1$ or NNNs in $H_2$), spatial eigenmode accumulation can be nullified with a simple spatial basis rescaling corresponding to a constant $\kappa = \kappa_1$ or $\kappa_2$. But the equilibration of the combination of two or more pumping length scales in $H_{12} = H_1 + H_2$ (Bottom Row) requires a non-local basis redefinition (Bottom Center) dictated by nonconstant $\kappa(k)$ which is not equivalent to $\kappa_1$ or $\kappa_2$, resulting in a non-local $\bar{H}_{12}(k)$. (b-c) PBC $\epsilon(k)$ (red), OBC $\epsilon(k) = \epsilon(k + i\kappa(k))$ (black) spectra and their interpolations $\epsilon(k + i\kappa(k))$ (various shades of brown) where $r = 0.4, 0.8, 1.1$. (b) and (c) depict $E = 2z + \frac{1}{2}$ and $E = z^2 + \frac{1}{4}$ where $z = e^{ik}$, representing cases with one and two pumping scales respectively. Background light green curves are contours of constant $\kappa$. (d-e) $\kappa(k)$ for cases (b) and (c), contrasting the constant $\kappa = \log 2 \approx 0.69$ cases (d) with a non-constant $\kappa(k)$ with cusps that indicate emergent non-locality (e).

Note that while $\bar{H}(k)$ is quasi-reciprocal, i.e., immune to non-Hermitian pumping, it is not necessarily reciprocal. Reciprocity requires symmetric physical couplings ($h_{ij}^{\alpha\beta} = h_{ij}^{\beta\alpha}$) for all pairs $ij$ and $\alpha\beta$, and is a stronger condition than quasi-reciprocity, which requires $\kappa(k) = 0 \forall k$, a constraint [12] on the relatively small number of $t_n$ coefficients from Eq. 3 formed by products and sums of the physical couplings. An important point that will be shortly elaborated on is that $\kappa(k)$ has nontrivial $k$-dependence whenever couplings exist across a range of distances such that $t_n \neq t_{-n}$ for more than one $n$. This is actually the case in most realistic systems, since non-Hermiticity from just one type of coupling is enough to cause $t_n \neq t_{-n}$ for many different $n$. Such $\kappa(k)$ dependences are nonanalytic in general, and lead to various forms of spectral singularities depending on the algebraic form of the characteristic polynomial.

C. Surrogate non-local basis

We next discuss the physical interpretation of the $k \rightarrow k + i\kappa(k)$ deformation in terms of the Hilbert space basis, going beyond existing generalized BZ descriptions. A central motivation of our framework is that this deformation can be regarded not just as an esoteric BZ redefinition, but as a physical change of basis orbitals. This is because OBCs allow for much greater freedom in basis transforms than PBCs, which require the Bloch nature (translation invariance) to be preserved. More precisely, the surrogate Hamiltonian $\bar{H}$ (its PBC version was defined in Eq. 2) admits a similarity transform $S$ such that

$$H^{\text{OBC}} = S^{-1} \bar{H}^{\text{OBC}} S \approx S^{-1} \bar{H}^{\text{PBC}}(k) S,$$

where $S$ undoes, or “unravels”, the complex deformation by implementing the complex gauge transform associated with $k \rightarrow k - i\kappa(k)$. In terms of eigenenergies, we have

$$\{\epsilon^{\text{OBC}}\} = \{\epsilon^{\text{OBC}}\} \approx \{\epsilon^{\text{PBC}}(k)\},$$

where the $\approx$ sign denotes an approximate equivalence that projects out, i.e., excludes isolated topological eigenvalues. In other words, spectrum of $H^{\text{OBC}}$, which is an OBC spectrum of a system that has been subject to non-Hermitian pumping, is exactly equivalent to the spectrum of $\bar{H}^{\text{OBC}}$ without the pumping, which is further equivalent up to a set of measure zero (in the thermodynamic limit) to that of $\bar{H}^{\text{PBC}}$. The upshot of this discussion is that, due of the existence of $S$, the surrogate Hamiltonian $\bar{H}(k)$ describes a bona-fide physical lattice system whose bulk properties are not susceptible to non-Hermitian pumping, and can be used to predict the topology and responses of the original Hamiltonian $\bar{H}(k)$. Note that this procedure is not applicable to disordered systems which also experience non-Hermitian pumping due to spatial non-homogeneity, since the pumping can no longer be “gauged” away by a unique $\kappa(k)$.

While $S$ can be numerically computed by taking the quotient of the matrices that diagonalize $H^{\text{OBC}}$ and $\bar{H}^{\text{OBC}}$, insight into its physical ramifications can be gleaned from Fourier expanding the rescaling factor $e^{-\kappa(k)}$. A nonconstant $\kappa(k)$ renders the eigenequation nonanalytic in $e^{ik}$, leading to emergent non-locality in real-space that is difficult to realize in models with few
hoppings \[71\]. To see that, note that at each \(x\) site, \(k \to k + ik(k)\) replaces the Bloch prefactor \(e^{ikx}\) by
\[
\left(e^{ik} e^{-\kappa(k)}\right)^z \approx \left(e^{ik} \sum_{l=-m}^{l_m} \Gamma_l e^{ikl}\right)^z = \sum_{l'=-l_m}^{l_m} \Gamma_{l'}(x)e^{i(l'+x)k},
\]
with \(\Gamma_l\) denoting the Fourier coefficients of \(e^{-\kappa(k)}\) and \(\Gamma_{l'}(x)\) their multinomial sum. \(\Gamma_l\) are generically power-law decaying \[72\] due to the non-analyticity, which we can truncate at large orders \(\pm l_m\) for convenient numerical treatment. Hence we can alternatively interpret the complex deformation as a non-local basis redefinition, where \(\tilde{H}(k)\) is re-interpreted as the non-deformed \(H(k)\) acting in a non-local basis with each site replaced by a linear combination of sites according to Eq. \[7\] each rescaled by \(\Gamma_{l'}(x)\) (Fig. 1b).

### III. Emergent Classification of OBC Spectral Singularities

Having discussed the formal aspects of our framework, we next provide a few canonical illustrations on how specific singularities emerge in the OBC \(\tilde{\epsilon}(k)\), and connect their branching patterns with the number of coexisting non-reciprocal length scales.

#### A. Classification of spectral singularities

1. One non-reciprocal length scale: As a warm-up, we consider the simplest case where the characteristic polynomial can be represented as
\[
F(E) = t_+ z + \frac{t}{z} + t_0, \tag{7}
\]
with \(t_+ \neq t_-\), and \(t_0\) denoting an unimportant energy offset. We refer to this as the case with one non-reciprocal length scale because asymmetry only occurs in the \(z, z^{-1}\) terms, and no other higher powers. \(F(E)\) is an arbitrary function of \(E\) which will turn out to have no non-trivial bearing on the singularity. For single-component models, \(F(E) = E\), and \(t_+, t_-\) are the asymmetric (non-reciprocal) right and left couplings. Note, however, that for multi-band models, \(t_+, t_-\) may not directly correspond to the bare couplings. More complicated models that possess one non-reciprocal length scale can be described by Eq. \[7\] with modified \(F(E)\) i.e. the non-Hermitian SSH model \[83\] \(H_{SSH}(z) = \sum_{+} \sigma \sum_{+} (t_+ + z^{-1})\sigma\), where \(\sigma = (\sigma_x \pm i \sigma_y) / 2\) are linear combinations of the Pauli matrices. For \(H_{SSH}(z)\), the characteristic polynomial reads \(E^2 = t_+ z + \frac{t}{z} + t_0 + 1\), such that \(F(E) = E^2 - t_+ t_- - 1\).

To determine \(\kappa(k)\), we transform Eq. \[7\] to a more convenient form by substituting \(z = e^{-\kappa(k)} w\), \(|w| = 1\), such that
\[
F(E) = w e^{-\kappa(k)} \sqrt{\frac{t_+}{t_-} + w^* e^{\kappa(k)} \sqrt{\frac{t_-}{t_+}}}, \tag{8}
\]
where \(F(E) = \frac{z^2 - b}{z^2 + \frac{b}{z}}\). Evidently, \(w = e^{ik}\) and \(w^* = e^{-ik}\) will take symmetrical roles if \(\kappa(k)\) is equal to a constant \(\kappa_0\) defined by \(e^{\kappa_0} = \sqrt{\frac{t_0}{t_+}}\). In this case, both \(z = w e^{-\kappa_0}\) and \(z' = w^* e^{-\kappa_0}\) are simultaneously roots of Eq. \[7\] for the same \(E\). As such, we obtain a constant complex deformation \(k \to k + i\kappa_0\) with
\[
\kappa(k) = \kappa_0 = \log \sqrt{\frac{t_0}{t_+}}. \tag{9}
\]
From Eq. \[6\] we find that there exists only one nonzero basis deformation coefficient \(\Gamma_0 = e^{-\kappa_0}\) resulting from one non-reciprocal length scale. This simple result can be visualized as a spatial exponential rescaling \(\tilde{\epsilon}(k) = e^{-\kappa_0 x}\) that counteracts the non-Hermitian pumping (Fig. 1a), which geometrically takes the form of a nonconformal contraction of the spectral loop in the complex energy plane (Fig. 1b).

2. Two non-reciprocal length scales: Going beyond analytic characterizations in the existing literature \[27-31\], we consider the next simplest characteristic polynomial with broken reciprocity at two scales (two different powers of \(z, z^{-1}\)):
\[
F(E) = z^2 + \frac{b}{z}, \tag{10}
\]
which is also schematically illustrated in the bottom row of Fig. 1b. Eq. \[10\] represents the simplest classes of non-Hermitian Hamiltonians with couplings beyond NNs, as in our two illustrative models presented earlier. As before, we have collected all dependence on \(E\) in the function \(F(E)\), whose detailed form will be irrelevant for the branching topology of \(\tilde{\epsilon}(k)\). To find the \(\tilde{\epsilon}(k)\) spectral loci and hence \(\kappa(k)\), we search for \(E = \tilde{\epsilon}(k)\) where there exist roots \(z, z'\) of Eq. \[10\] with the same \(\kappa(k)\), i.e. satisfying \(|z| = |z'|\). By analytically solving the cubic polynomial as detailed in Appendix A3, one obtains the \(\tilde{\epsilon}(k)\) loci as values of \(E\) satisfying
\[
F(E) \propto \left(\frac{b}{2}\right)^{2/3} \omega_j, \tag{11}
\]
i.e., 3 straight lines radiating from the origin of the complex plane \(F(E)\) plane aligned along the cube roots of unity \(\omega_j\), \(j = 0, 1, 2\) (Fig. 1b and 3a). To find the deformation \(\kappa(k)\) needed to bring \(\epsilon(k)\) to \(\tilde{\epsilon}(k)\), we particularize the eigenenergy equation Eq. \[10\] to the \(F(\tilde{\epsilon}(k))\) loci we have just derived (Eq. \[11\]). The resultant expression can be elegantly expressed as
\[
\text{Im} \left[ \left( e^{2\kappa(k)} e^{2ik} + e^{-\kappa(k)} e^{-ik} \right) \omega^{-j} \right] = 0,
\]
which yields
\[
\kappa(k) = -\frac{1}{3} \log \left| \frac{b}{2 \cos (k - 2 \pi j/3)} \right|, \tag{12}
\]
where \(j\) is chosen to give the branch the smallest complex deformation \(|\kappa(k)|\) (Fig. 1b). Notably, the form of \(F(E)\) explicitly appears neither in the OBC \(\tilde{\epsilon}(k)\) loci nor even
3. Generic non-reciprocal couplings: By generalizing the above arguments, it can be shown that for characteristic polynomials of the form

$$E^N = az^p + \frac{b}{z^q}$$

where $p, q > 0$, the OBC $\tilde{\epsilon}(k)$ spectrum takes the shape of a $N(p+q)$-pointed star, generalizing the above-mentioned $p=2, q=1$ case which gives a 3-pointed OBC star (Fig. 2b(ii)). This result can also be intuitively obtained by regarding $p+q$ as the number of times the BZ is folded.

Most generally, the characteristic polynomial is a bivariate polynomial

$$P(E, z) = \sum_{m,n} p_{m,n} E^m z^n = 0.$$  

that contains multiple coefficients of $m$ and $n$, and may not be separable into parts that depend separately on $E$ and $z$, as in Eqs. [7] and [10]. The exact correspondence between the graph topology of the OBC singularity and the algebrao-geometric properties of its associated $P(E, z)$ is an open problem. However, from a single well-understood case, one can already understand that all other cases relate to each other via a conformal transformation of $E$. As illustrated in Fig. 2b(ii) to (iv), $E \rightarrow E^2 + 1 = (E + i)(E - i)$ splits the 5-pointed OBC star into two stars centered at $E = \pm i$, while $E^2 \rightarrow E^3$ produces three images of the star from two. These mappings can be easily implemented by increasing the number of components. For instance, to map an arbitrary single-component (band) $E = E_0(z)$ into $E^2 + 1 = E_0(z)$, one turns to the Hamiltonian $H(z) = \begin{pmatrix} 0 & E_0(z) - 1 \\ 1 & 0 \end{pmatrix}$. Likewise, to map it to $E^3 + 1 = E_0(z)$, one can enlarge the Hamiltonian to $H(z) = \begin{pmatrix} 0 & 0 & E_0(z) - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. More generally, given a Hamiltonian with complicated $P(E, z)$, the trick will be to attempt to bring it into a simpler known form through a conformal transformation of $E$, with branch cuts introducing multiple Riemann sheets corresponding to multiple images of the original (both OBC and PBC) spectrum.

Still, there of course exist many exotic possibilities not transformable to simple star patterns. Consider going from the model in Fig. 2b(i) to (v) via a mapping $E^2 \rightarrow E^2 - \frac{a}{b} E$ which involves $z = e^{ik}$ as well. Since that modifies $\kappa(k)$, the OBC spectrum of (v) cannot be un-
understood in terms of that in (i), and in fact forms a different pattern, containing even a closed graph cycle [69]. The resultant characteristic polynomial \( P(E, z) = E^2 - \frac{0.7}{z^2}E - z^2 = 0 \) can be obtained, for instance, from a Hamiltonian of the form \( H(z) = \begin{pmatrix} 0.7/z^2 & z^2 + 1/(2z) \\ 1 & 0 \end{pmatrix} \), which also includes a term on the diagonal. Note, however, that graph cycles in the OBC spectrum \( \tilde{\varepsilon} \) do not necessary require complete \( E \) dependence, and can in fact arise in single-component models with multiple powers of \( z \), for instance \( E = (z^3 + 2z^2 + z + z^{-1} + 4z^{-2})/2 \) from Ref. [36]. We conclude this discussion by reiterating that the graph topology of the OBC spectrum is, in its essence, a property of the characteristic polynomial \( P(E, z) \), not the Hamiltonian per se, with the exact nature of this graph topology being an open topic for future studies.

B. Singularity transitions

We have just seen how the OBC spectral graph can be drastically modified as the characteristic polynomial \( P(E, z) \) varies. When the spectral graph topology changes discontinuously, at least part of the OBC spectrum shrinks to a point i.e. assumes a complex “flat-band”. The tuning of physical parameters that affect such transitions will generate a phase diagram containing regions of different spectral graph topologies, as in Fig. 2, where \( E = a/z^2 + b/z \). Singularity transitions occur when \( a = 0 \) or \( b = 0 \), since the OBC spectrum shrinks to a point and, optionally, flip across these transitions. More sophisticated transitions are possible in other models, like in those appearing in Fig. 2.

Notably, these topological transitions of the OBC spectral graph generically do not coincide with OBC bandgap closures, which occur when two or more components of the graphs (i.e. stars in Fig. 2) intersect. Yet, because of the emergent non-locality, the eigenstates get to converge non-analytically and mix at the transition degeneracy, eigenstate properties like the Berry curvature can still change discontinuously. In Sect. IV we shall elaborate on the physical consequences of these singularity transitions, and how to measure them.

C. Two illustrative examples

We next elaborate on two models where our surrogate Hamiltonian formalism is essential for deriving the topological phase diagram. While the presence of topological boundary modes is conceptually unrelated to the OBC spectral graph topology, topological phase boundaries are determined by gap closures (intersections) of these OBC spectral graphs. As such, we emphasize that it is the OBC quasi-reciprocal surrogate Hamiltonian \( \tilde{H}(k) \), not the PBC \( H(k) \), which should be used to compute topological invariants that correctly predict the presence of boundary modes.

1. 1D: Non-Hermitian extended Kitaev chain

Our first example is a non-Hermitian version of the extended Kitaev model which, as suggested by the trajectories in Fig. 3, minimally contains both NN and NNN coupling terms. These couplings contain unequal phase factors to break its chiral symmetry into particle-hole symmetry (PHS) described by \( \mathcal{C}H^T(k)\mathcal{C}^{-1} = -H(-k) \) with \( \mathcal{C} \) a unitary matrix, such that \( Z \) topology is broken into \( Z_2 \) (D-class) topology [75]. As complex conjugation does not coincide with transposition for non-Hermitian systems, one can alternatively define a conjugated PHS (cPHS) as \( \mathcal{C}H^*(k)\mathcal{C}^{-1} = -H(-k) \) [18]. Here we shall consider an example with cPHS belonging to the \( D \)-class, as PHS does not allow non-Hermitian pumping with simple constant non-Hermitian terms [21]. As explained below, the necessary presence of both NN and NNN couplings results in a complicated characteristic polynomial with more than one non-reciprocal length scale, whose identification of topological phase diagram requires our surrogate Hamiltonian formalism. A minimal representation of the non-Hermitian extended Kitaev model is given by \( H_D = \hat{h}(k) \cdot \sigma \), where

\[
\begin{align*}
\hat{h}_x &= \Delta_2 \sin \phi \sin 2k + ig_x \\
\hat{h}_y &= \Delta_2 \cos \phi \sin 2k + \Delta_1 \sin k + ig_y \\
\hat{h}_z &= m - t_1 \cos k - t_2 \cos 2k.
\end{align*}
\]

(15)
cPHS, which enforces \( \sigma_x H^*(k)\sigma_x = -H(-k) \), allows only two types of constant [76] non-Hermitian terms: \( ig_x\sigma_x \) and \( ig_y\sigma_y \), and will be broken if any of the parameters \( m, t_1, t_2, g_x, g_y \) become complex. Unlike in the SSH model, both the \( ig_x\sigma_x \) and \( ig_y\sigma_y \) terms can separately lead to the skin effect, since \( \sin 2k \) from the NNN couplings appear in both \( h_x \) and \( h_y \).

In principle, there is no further restriction on the parameters of \( H_D \) (Eq. 15), whose characteristic polynomial generically takes the form \( E^2 = \tilde{P}_h(z)/z^4 \), with \( \tilde{P}_h(z) \)
kined via the constant complex deformation with φ \text{tra} (gray) of H. FIG. 4. (a-c) OBC spectra (blue) enclosed by the PBC spectra (gray) of H. These three cases respectively possess a real line gap, imaginary line gap and real line gap with topological modes, as classified by their distinct topological invariants \( \nu = 1, 0, -1 \) defined by Eq. 16 and its preceding discussion.

To further predict the presence of topological boundary modes, we turn to the biorthogonal pseudospin \( S^{LR}(k) \). In the cases with imaginary line gap, \( E(0) \) and \( E(\pi) \) are imaginary and so are \( S^{LR}(0) \) and \( S^{LR}(\pi) \). The case when \( E(0) \) and \( E(\pi) \) are both real i.e. with real line gap, is more interesting, containing the possibility of hosting topological modes. It can be shown that \( \text{Sign}[S^{LR}(0)] = \epsilon \text{Sign}[S^{LR}(\pi)] \), with \( \epsilon = 1(-1) \) corresponding to the scenario without (with) topological edge states [Figs. 3(b) and (c)]. All in all, there are three distinct phases \( \nu = 1, 0, -1 \) characterized by the topological invariant [21]

\[
\nu = \text{Sign}\{(\text{Re}[S^{LR}_z(0)]\text{Re}[S^{LR}_z(\pi)]\},
\]

as mapped out by the phase diagram of Fig. 4(d):

- Case (i), the \( \nu=1 \) phase with real line gap and no topological boundary mode. \( S^{RR}(k) \) of both bands are mirror-symmetric with each other about the equator.
- Case (ii), the \( \nu=0 \) phase with imaginary line gap and no topological boundary mode. \( S^{RR}(k) \) of each band is mirror-symmetric with itself.
- Case (iii), the \( \nu=-1 \) phase with real line gap and isolated topological boundary modes. \( S^{RR}(k) \) of both bands are mirror-symmetric with each other, each with nontrivial winding about the Bloch sphere axis.

These three phases \( \nu = -1, 0, 1 \) are all the possible gapped phases of this PH-symmetric system, since \( E(0), E(\pi) \) must be both real or both imaginary. Note that in computing \( \nu \), we have made crucial use of the \( \kappa(k) \) deformation introduced by our formalism, without which it is difficult to obtain the surrogate \( S^{RR}(k) \) and \( S^{LR}(k) \) that correctly predict the OBC behavior. In particular, as evident from the nontrivial shapes of the trajectories of \( S^{RR}(k) \) in Fig. 3, NNN couplings and hence two or more non-reciprocal length scales are necessary for realizing the variety of topological configurations afforded by this symmetry class.

2. 2D: Extended non-Hermitian Chern insulator

We now illustrate the use of quasi-reciprocal surrogate quantities in a 2D setting, where quantities like
band geometry and Berry curvature can be modified by non-Hermitian pumping through the non-analytic complex deformation. In the case of cylindrical boundary conditions, we have WLOG OBCs in the $x$-direction and PBCs in the $y$-direction, such that $k_y$ is still a well-defined parameter. For each $k_y$-slice, the surrogate Hamiltonian is defined by $\hat{H}(k_x; k_y) = \hat{H}(k_x + i\kappa_x(k), k_y)$. In the case of OBCs in both directions, which we shall not consider in-depth here, quasi-reciprocity also has to be restored in the $y$-direction, giving rise to \[ \hat{H}(k) = \hat{H}(k_x + i\kappa_x(k), k_y) = \hat{H}(k_x + i\kappa_x(k), k_y + i\kappa_y(k)) \] where $\kappa_x$ is taken as a spectator parameter in the second iteration, and $\kappa_y$ determined by how $H$ depends on $k_y$ both explicitly and through $\kappa_y(k)$.

Focusing on cylindrical boundary conditions from now on, we see that different $k_y$ slices of the same system can possess different OBC spectral graph topologies, with discontinuous gapped transitions between them as further discussed in the next section. Below, we introduce a minimal and analytically tractable example of such a non-Hermitian model with both Chern topology and signature $Y$-shaped spectral topology:

\[ H_{Ch}(k) = (v + z^{-1})\sigma_+ + (v + z - v z^2)\sigma_- + \sin k_y \sigma_z, \]

where $z = e^{ik_y}$ and $u = M + \cos k_y - \mu$ and $v = v_0(M + \cos k_y + \mu)$. As contrasted with the NN non-Hermitian Chern model commonly studied in the literature \cite{29}, our model Eq. \[19\] contains fundamentally asymmetric couplings (detailed in Appendix C), and is not adiabatically connected to any Hermitian Chern model. In other words, the role of its NNN couplings in the $x$-direction (coefficient of $z^2$) is not just to perturb away from the known phase of the NN Chern model, but to define a new Chern phase existing on a $Y$-shaped spectral graph.

By design, the characteristic polynomial of $H_{Ch}$ with $x$-OBCs assumes the classic form of Eq. \[10\]

\[ F_{Ch}(E) = z^2 + b/z, \]

with $F_{Ch}(E) = (1 + \sin^2 k_y + u^2 - E^2)/v^2$ and $b = -u/v^2$. Hence, as in Fig. \[1\], each $k_y$-slice of the OBC skin spectrum $\tilde{\epsilon}(k_x; k_y)$ consists of two cubic singularities, with the size and origin of each $Y$-shaped star controlled by the $k_y$-dependent $b$ and $F_{Ch}(E)$ respectively (Figs. \[5\] and \[6\]). Additionally, isolated in-gap topological modes can also exist. Their existence in each 1D $k_y$-slice can be predicted by the generalized topological criterion of Ref. \[33\] or chiral-symmetric winding number of $H_{Ch}(k_x; k_y)$ over $k_x$. However, most important for our context is whether they traverse the gap over a full cycle of $k_y$, and that is determined by the Chern number $\tilde{c} = \frac{1}{2\pi} \int \Omega_{xy} d^2k$, where $\Omega_{xy} = \text{Im} \Gamma_{xy}$ is the biorthogonal Berry curvature corresponding to the imaginary part of the gauge-invariant quantity

\[ \Gamma_{\mu\nu} = (\partial_\mu \tilde{\psi}_L) \bar{Q} (\partial_\nu \tilde{\psi}_R). \]

Here $\bar{Q} = I - \bar{P}$ where $\bar{P} = |\tilde{\psi}_R \rangle \langle \tilde{\psi}_L|$ the biorthogonal projector onto the band of lower Re $E$ that is biorthogonally spanned \cite{28, 79} by left/right eigenvectors $\tilde{\psi}_L, \tilde{\psi}_R$. Note that

Since our $\bar{H}_{Ch}$ is quasi-reciprocal, the Chern number of its occupied eigenstate must always be a quantized integer corresponding to the number of gap-traversing topological edge modes, at least when the gap is well-defined. Fig. \[5a\] shows a gapped case with $\tilde{c} = 1$ topological edge mode (yellow curve), while Fig. \[5b\] shows a gapless case with no well-defined edge mode. Note that gapless cases can occur as typically as gapped cases in non-Hermitian systems, as illustrated by their extended black regions in the phase diagram of Fig. \[5\].

![Fig. 5. x-OBC spectra of $H_{Ch}$](image) for (a) a topologically nontrivial case $M = 0.5, \mu = 0.3, v_0 = 2$ with $\tilde{c} = 1$ edge mode and (b) a gapless case $M = 1, \mu = 0.5, v_0 = 1.5$ without well-defined separate bands. (c) Phase diagram for $v_0 = 1.3$, with black gapless regions separating phases with topologically nontrivial edge modes (green), trivial edge modes (blue) and no edge modes (pink). The stars denote representative cases presented in the following figure.
FIG. 6. $x$-OBC spectra (Left), surrogate Berry curvature (Center) and FS metric trace $Tr g$ (Right) of $H_{ch}$ for parameters in three different regimes: $M=0.7, \mu=0.4, v_0=1.3$ (Top), $M=0.2, \mu=1.8, v_0=1.3$ (Middle) and $M=0, \mu=0.4, v_0=1.3$ (Right). While all three scenarios contain cubic singularities, only the Top scenario is topologically nontrivial. The Middle scenario also contains edge modes (yellow), but is deformable to the trivial case. While all three scenarios exhibit Berry curvature of FS metric discontinuities at $k_x=0, \pm 2\pi/3$, only the Top and Bottom cases possess discontinuities at $k_y=\pm \cos^{-1}(\mu-M)$ (The Middle case admits no such $k_y$ solutions). Despite their discontinuities, the Berry curvatures integrate to quantized Chern numbers 1, 0 and 0 respectively.
Very interestingly, we observe discontinuities in both the Berry curvature \( \Omega_{xy} \) and the trace of the Fubini-Study (FS) metric \( Tr \tilde{g} = Re \tilde{\Gamma} \), as shown for three contrasting cases in the Center and Right columns of Fig. 6 respectively. Both of these quantities are derived from \( \tilde{\Gamma} \), which contain momentum-space derivatives that pick up qualitative transitions in the behavior of the eigenstate. Since the branching behavior of the OBC spectrum \( \tilde{\epsilon}(k) \) is controlled by \( \kappa(k) \) (i.e. in Eq. 12) which in turn enters the eigenstate, a singularity transition will qualitatively modify the form of \( \kappa(k) \) and lead to non-analytic discontinuities. Physically, these non-analyticities arise from the emergent non-locality induced by the non-Hermitian pumping. Note that because \( \kappa(k) \) is continuous at its kinks, \( \tilde{\Gamma}_{\mu\nu} \) do not diverge at its discontinuities, as further elaborated in Appendix C.

For our model Eq. 19, singularity transitions occur at \( b = -u/v^2 = 0 \) (Eq. 20), i.e. at \( k_y = \cos^{-1}(\mu - M) \). Furthermore, \( \kappa(k) \) also exhibits kinks at \( k_x = 0, \pm 2\pi/3 \), where the OBC spectrum jumps from one branch to the next \[69\]. Indeed, these two sets of lines exactly correspond to the discontinuities in \( \Omega_{xy} \) and \( Tr \tilde{g} \) in Fig. 6. In the \( \tilde{\epsilon} = 1 \) case of the Top row (\( M = 0.7, m = 0.4, v_0 = 1.3 \)), the discontinuities along \( k_y \) exist also seen to correspond to flips (reflections) in the Y-shaped spectra, corroborating with results depicted in Fig. 2b. No true flips and hence discontinuities along \( k_y \) exist for the Center row case (\( M = 0.2, m = 1.8, v_0 = 1.3 \)), for which \( k_y = \pm \cos^{-1}(\mu - M) \) admits no real solution. Despite the discontinuities, it is remarkable that the Berry curvature in all gapped cases integrate to integer multiples of 2\( \pi \).

IV. ANOMALOUS RESPONSES FROM SPECTRAL SINGULARITY TRANSITIONS

Discontinuous transitions between different spectral singularity classes do not close the gap, but can still be physically detect through observables that depend on the momentum-space gradients of eigenstates. A simplest example is the gauge-invariant quantity \( \Gamma_{\mu\nu} = (\partial_\mu \psi_L | Q | \partial_\nu \psi_R), \quad Q = 1 - | \psi_R \rangle \langle \psi_L | \) introduced earlier, whose real and imaginary parts correspond to the FS metric trace \( Tr \tilde{g} = Re Tr Q \) and Berry curvature respectively. In PT-symmetric quantum mechanics, the symmetrized form \( \frac{1}{2}(\Gamma_{\mu\nu} + \Gamma_{\nu\mu}) \) has also been proposed as a quantum geometric tensor \[80\]. Quantities containing higher order gradients correspond to higher-order cumulants in the noise spectrum, and are expected to exhibit discontinuities too. Below, we shall just focus on one salient example, where Berry curvature discontinuities can be detected via kinks in wavepacket trajectories implementable in cold atom lattices.

A. Detection of discontinuous Berry curvature

We now show that the abovementioned discontinuities in the Berry curvature are not just mathematical artifacts, but are actually experimentally detectable via the dynamics of wavepackets. In Hermitian systems, a generic wavepacket can always be decomposed into real momentum eigenstates obeying well-known semi-classical equations of motion \[81\]–\[84\]. Even when boundaries are present, this momentum eigenstate decomposition remains valid throughout the bulk, since bulk eigenmodes are unaffected by the boundaries. But in non-Hermitian lattices, two ostensible hurdles exist. Extensively many OBC eigenstates (considering OBCs to be along \( \delta \) WLOG) are generically indexed by complex momenta \( \bar{k}_x = k_x + i\kappa(k) \) due to non-Hermitian pumping, and a translation-invariant semi-classical picture no longer exists. Moreover, the wavepacket diverges or decays to zero if the spectrum is not real, although interesting Bloch oscillations may also emerge \[65\]–\[88\].

To meaningfully observe non-Hermitian OBC wavepacket dynamics, then, one has to consider (i) only systems with real OBC spectra, or at least those with sufficiently small imaginary energies, and (ii) switch to the modified real space basis conjugate to \( \bar{k}_x \), as described by Eq. 9 and illustrated in Fig. 1. In this spatially inhomogeneous basis, quasi-reciprocity is restored and the semi-classical wavepackets behave like ordinary wavepackets with real momenta \( k = (Re(k_x), k_y) \). Subject to a force \( F = \hbar \dot{\bar{k}} \), a wavepacket in the bounded 2D \( x-y \) plane with peak complex momentum \( \bar{k} = (\bar{k}_x, k_y) \) and position \( \bar{r} \) (in the modified basis) undergoes dynamics described by \[53\]–\[80\],

\[
\frac{d\bar{r}}{dt} = i\frac{1}{\hbar} \frac{\partial E(\bar{k})}{\partial \bar{k}} - \frac{1}{\hbar} (F \times \dot{\bar{r}}) Re \left[ \Omega_{xy}(\bar{k}) \right] \tag{22}\]

While the OBC spectrum \( E(\bar{k}) \) has to be evaluated at \( \bar{k} = k + i\kappa(k) \) due to non-Hermitian pumping, its gradient is taken with respect to \( \bar{k} \) to give the physical velocity in the modified basis. Since we only consider systems with real OBC spectra, the energy gradient is always real, even if the PBC spectra \( E(k) \) is complex. The imaginary part of the Berry curvature \( \Omega_{xy} \) contributes additional growth or decay to the wavepacket, but is often small enough to be neglected. Note that in the modified basis, the wavepacket still has the same peak as in the unmodified basis, although its center-of-mass will be different due to its modified amplitude profile. To obtain the dynamics back in the physical real-space basis, one just needs to spatially rescale the wavepackets according to the results in Sect. II C.

Inspired by the protocol by Price and Cooper originally devised for Hermitian setups \[53\], we can isolate the effect of discontinuous Berry curvature by examining the difference in the time-reversed semi-classical trajectories \( \Gamma_{\pm} \) due to constant forces \( F \) and \(-F\). For odd dispersions \( E(\bar{k}) \), we can take the difference between the \( \pm F \)
FIG. 7. Trajectories $\mathbf{r}_+(t)$ (Top), $\mathbf{r}_-(t)$ (Middle) and $\mathbf{r}_{\text{diff}}(t)$ (Bottom) at various Berry curvature discontinuity strengths $\Sigma = 0, 2, 3$ of $\Omega_{xy}^{\text{Toy}}$ in our Toy model (Eq. 24) with $\mathbf{F} = \hat{y}$ and $\mathbf{k}(0) = 0$, as shown in Panels (a-c) respectively. Trajectories kinks appear for nonzero $\Sigma$ due to anomalous impulses at fixed intervals as the wavepacket orbits pass the discontinuities. Very strong discontinuities can even give rise to loops, as in $\mathbf{r}_-(t)$ when $\Sigma = 3$. The effects of Berry curvature can be isolated from dispersion effects by looking at the trajectory difference $\mathbf{r}_{\text{diff}}(t)$, which is in the direction of $\hat{x} \propto \mathbf{F} \times \hat{z}$. We have set the discontinuities at $\pm k_0 = \pm \pi/3$; other values of $k_0$ give rise to qualitatively similar kinks in $\mathbf{r}_{\text{diff}}(t)$, although the $\mathbf{r}_\pm(t)$ trajectories may assume different shapes.

trajectories from Eq. 22 and obtain

$$\mathbf{r}_{\text{diff}}(t) = -\frac{1}{\hbar} (\mathbf{F} \times \hat{z}) \int_0^t \text{Re} \left[ \Omega_{xy}(\mathbf{k}(t')) \right] dt', \quad (23)$$

where $\mathbf{r}_{\text{diff}} = \mathbf{r}_+ - \mathbf{r}_-$ and $\mathbf{k}(t) = \mathbf{k}(t) + i\kappa_x(t)$, $\mathbf{k}(t) = \mathbf{k}(0) + \mathbf{F} t$. Due to the oddness of the dispersion in momentum, the gradient terms are even and consequently cancel after taking the difference. The Berry curvature contribution for more general dispersions can still be isolated via more sophisticated engineering of the trajectories [54, 89], but doing so will not be necessary for our demonstrative purposes.

When a discontinuity is present in $\Omega_{xy}(\mathbf{k})$, the trajectories $\mathbf{r}_\pm(t)$ will experience a sudden change in velocity as $\mathbf{k}(t)$ sweeps over it. This will be manifested as a kinks in the trajectories, as illustrated in Fig. 7 for the toy model introduced below. Line discontinuities in the Berry curvature cannot be avoided by the wavepacket as its momentum $\mathbf{k}(t)$ orbits around the BZ torus, and will always produce kinks unless $\mathbf{F}$ is parallel to the discontinuity. Even rarer point discontinuities can be probed if the wavepacket is arranged to pass over them via specially prepared paths. In fact, they will be eventually encountered with certainty if $\mathbf{F}$ is incommensurate with the reciprocal lattice vectors of the real BZ, since the wavepacket will ergodically explore the entire BZ.

1. Toy model

To acquaint the reader with Berry curvature discontinuity measurements, we shall first apply Eq. 23 on an illustrative toy model whose dispersion is simply given by a real periodic potential $E(\mathbf{k}) = \sin k_x + \sin k_y$. We specialize to a boundary normal to $\hat{x}$, such that we have $\mathbf{k} = (k_x, k_y)$, with $\mathbf{r}$ accordingly inhomogeneous only in the $x$-direction.

In this toy model, the discontinuous Berry curvature profile is specified by hand as

$$\Omega_{xy}^{\text{Toy}}(\mathbf{k}) = 1 + \Sigma \tanh[\lambda \cos(k_y - k_0)]. \quad (24)$$

Taking $\lambda \to \infty$, it assumes a step function profile, taking uniform values of $1 \pm \Sigma$ depending on whether $k_y$ is within $\pi/2$ of $k_0$. The line discontinuities at $|k_y - k_0| = \pi/2$ mimic those of actual models, for instance the previous Chern model with discontinuities at $k_y = \pm \cos^{-1}(\mu - M)$ or the cold atom setup below with discontinuities at $k_y = \pm \pi/2$. Shown in Fig. 7 are the $\mathbf{r}_\pm(t)$ trajectories (Upper and Middle rows) and their difference
$f_{\text{diff}}(t)$ (Bottom row) for various values of discontinuity strengths $\Sigma = 0, 2, 3$. Evidently, the effects of the discontinuities manifest as trajectory kinks which become more obvious with stronger Berry curvature discontinuity $\Sigma$. These kinks already exist in the $f_{\perp}(t)$ trajectories despite the dependence on the energy dispersion. With effects of the dispersion eliminated in $f_{\text{diff}}(t)$ (which is along the $x \times F \times z$), the kinks can also be quantitatively compared. In Fig. 7a (Bottom) with uniform $\Omega_{xy}(k) = 1$, $|f_{\text{diff}}(t)|$ increases linearly due to the uniform anomalous velocity contribution of the Berry curvature. But with increasing discontinuity $\Sigma$, we observe kinks due to abrupt impulses as the wavepacket goes over the line discontinuities. Since our toy model possess equal and opposite impulses as the wavepacket goes over the line discontinuities, successive impulses alternate in direction, giving rise to an uniformly “jagged” displacement graph.

B. Proposal for a cold-atom demonstration

We now propose how the above-described response kink can be detected in a cold-atom setup. As popular quantum simulators, ultracold atoms in optical lattices can be made to perform wavepacket (atomic cloud) dynamics as described by Eqs. 22 and 23 in a very tunable setting. In our case, the key ingredient - non-Hermitian pumping - can also be implemented via gain/loss mechanisms that are mathematically analogous to non-reciprocal couplings 63 70. For this purpose, we consider a 2D lattice consisting of a series of two-leg ladders in the $x$-direction, which are also coupled perpendicularly in the $y$-direction, as illustrated in Fig. 8a. The two-leg ladders can be realized with two-level ultracold atoms loaded onto a resonantly driven 1D optical lattice 71, where the sublattices A and B are simulated by the atomic levels. Intra-leg couplings amplitudes $\pm t_{y}$ result from orbital differences. To realize the crucial non-Hermitian pumping, we introduce level-dependent depopulation loss of $i\sigma_{x}$ by exciting particles to a third atomic level 73 75. In all, our setup is described by the effective Hamiltonian

$$H_{\text{eff}}(k) = (t_{\perp} \sin \theta + 2t_{x}^{2} \sin k_{x}) \sigma_{y} + (t_{\perp} \cos \theta + 2t_{y}^{2} \cos k_{y}) \sigma_{x} + (i g - 2t_{x} \cos k_{x} - 2t_{y} \cos k_{y}) \sigma_{z}$$

(25)

with the other parameters generically dependent on phenomenological physical quantities 92 93. Time-reversal symmetry and hence reciprocity is broken by the term $t_{\perp} \sin \theta \sigma_{y}$, which is however Hermitian.

To understand how non-Hermitian pumping can emerge, it is instructive to first consider the limit of $t_{d}^{2} = t_{y} = 0$, where the 1D two-leg ladders are decoupled. If furthermore $\theta = \pm \pi/2$ and $t_{x} = t_{d}^{2}$, the Hamiltonian restricted to each 1D ladder maps exactly onto the well-known non-Hermitian SSH model via a unitary pseudospin rotation and a $\pi/2$ shift of $k_{x}$ that repackages Hermitian time-reversal breaking and non-Hermitian dissipation as asymmetric couplings. Reintroducing the couplings, a simple expression for $\kappa_{x}(k)$ still exists if $\theta = \pi/2$, $t_{d}^{2} = t_{x}$ and $t_{\perp} = g$ (Appendix D):

$$\kappa_{x}(k) = \frac{1}{4} \log \left( 1 + \frac{g^{2}}{t_{d}^{2} \cos^{2} k_{y}} \right),$$

(26)

which is independent of $k_{x}$, but diverges at $k_{y} = \pm \pi/2$ where the OBC spectrum shrinks into two points $E_{\pm} = \pm 2t_{x}$. This gives rise to distinct Berry curvature discontinuities at $k_{y} = \pm \pi/2$, as plotted in Fig. 8b. With more generic parameters, the physics remain qualitatively similar, at most hosting slightly more complicated spectral topologies. Note that since the PBC Hamiltonian satisfies $H_{\text{eff}}(k_{x}, k_{y}) = H_{\text{eff}}(k_{x}, -k_{y})$, it is already its own surrogate Hamiltonian for $y$-OBCs, for which no Berry curvature singularity can thus appear.

Shown in Fig. 8c are some illustrative semiclassical wavepacket trajectory differences with kinks due to the Berry curvature discontinuities. We have chosen a force $F \| y$ instead of $x$ since the discontinuities are parallel to the $x$ axis. While the initial momentum $k(0)$ of the wavepacket has significant impact on the shape of the trajectory, the themselves kinks remain robust.

V. DISCUSSION

In our quest for a quasi-reciprocal picture of non-Hermitian systems where non-Hermitian pumping is eliminated, non-locality and its concomitant non-analyticity emerges as unavoidable consequences. These effects lead to enigmatic properties such as discontinuous Berry curvature and band geometry, which universally result in anomalous transport and noise responses in generic systems with non-Hermitian descriptions. For concreteness, we have illustrated in this work how they can, for instance, be detected as kinks in atomic cloud trajectories in an ultracold atomic setup.

At the level of formalism, we developed a restoration procedure to map any non-Hermitian model to its quasi-reciprocal surrogate model with reinstalled bulk-boundary correspondence, from where its topological nature unfolds unambiguously. By encoding the equilibration behavior of accumulated pumped states as non-holomorphic complex momentum deformations, the effective non-locality leads to not only gap-preserving topological transitions, but also ultimately a completely new graph-theoretical topological classification of OBC spectra related to the classification of algebraic varieties.

Our approach applies universally to any system whose characteristic polynomial (energy dispersion) admits a surrogate non-local basis construction. Tailored for realistic setups with multiple effective components and long-ranged couplings, it uncovers possibly non-perturbative topological contributions, unique to non-Hermitian systems, that would not be revealed by oversimplified short-ranged representations. Through repeated application, it can be extended to higher dimensional lattices which support exceptional nodal structures and generalized skin-topological modes 10 24. Being based on unitary trans-
formations, our formalism remains valid in the realm of interacting systems, and can shed light on the interesting interplay between non-Hermitian pumping and many-body phenomena such as emergent Fermi surfaces [95].

VI. ACKNOWLEDGEMENTS

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Appendix for “Unraveling non-Hermitian pumping: emergent spectral singularities and anomalous responses”

Appendix A: Various OBC singularity types from the number of non-reciprocal length scales

Here, we supplement the main text discussions of the various OBC singularity classes with more details.

1. Reciprocal \((t_n = t_{-n})\) case

Consider a characteristic polynomial (eigenenergy equation) of the form

\[
F(E) = \sum_n t_n z^n + t_{-n} z^{-n},
\]

(A1)

\(z = e^{ik}\). It is instructive to first prove the absence of non-Hermitian pumping i.e. the skin effect when \(t_n = t_{-n}\) for all \(n\). Recall that non-Hermitian pumping occurs when \(|z_1| = |z_2| \neq 1\), where \(z_1, z_2\) are the pair of roots of Eq. A1 closest to the unit circle. When \(t_n = t_{-n}\) for all \(n\), one can always turn Eq. A1 into a simpler polynomial through the substitution

\[
u = z + \frac{1}{z}, \tag{A2}
\]

Assuming non-vanishing terms from only a single \(n\), doing so gives \(z_1, z_2 = (-u \pm \sqrt{u^2 - 4})/2\), which are of equal magnitude iff \(u\) and \(\sqrt{u^2 - 4}\) differ by a phase of \(\pi/2\), i.e.

\[
\sqrt{u^2 - 4} = i ru \tag{A3}
\]

where \(r\) is a real multiplier. This implies that \(u = \frac{2}{\sqrt{1+\gamma}}\) or \(0 < u < 2\). Evidently then, \(z + \frac{1}{z} = 2 \cos k = u\) will always have a real \(k\) solution, thereby obviating any skin effect. In the case of multiple \(n\), a similar analysis applies for the more complicated resultant polynomial in \(u\).

2. One length scale

For completeness, we review and elaborate on the simplest case where non-Hermitian pumping affects only NN couplings. Due to the presence of only one length scale, non-Hermitian pumping can be completely “gauged away” with a \(k\)-independent \(\kappa\).

With only one length scale, the RHS of Eq. A1 contains only two dissimilar terms \(t_{\pm}\), in addition to a constant term containing \(t_0\). To be concrete, consider the non-reciprocal SSH model with \(H_{SSH}(z) = (t_- + z)\sigma_+ + (t_+ + z^{-1})\sigma_-\) where \(t_{\pm} = t \pm \gamma\). Eq. A1 takes the form

\[
E^2 = t_+ z + \frac{t_-}{z} + t_+ t_- 1
\]

(A4)

which can be expressed as

\[
\frac{E^2}{\sqrt{t_+ t_-}} - 2 \cosh \log \sqrt{t_+ t_-} = u' = z' + \frac{1}{z'}\tag{A5}
\]

where \(z' = \sqrt{\frac{t}{t_-}}, \) i.e. \(k' = k + i \log \sqrt{\frac{t}{t_+}}\). This is manifestly of the form Eq. A2 with \(u'\) defined as \(\frac{t^2}{\sqrt{t_+ t_-}} - 2 \cosh \log \sqrt{t_+ t_-}\), except that \(k\) is now deformed into \(k'\) by a constant imaginary displacement \(i \log \sqrt{\frac{t}{t_-}}\).

Note that this result applies to any system obeying Eq. A4 and not just the non-reciprocal SSH model. Physically, the complex deformation of \(k\) (or rescaling of \(z\)) corresponds to a spatial exponential rescaling that counteracts the mode accumulation from the pumping. Evidently, it will no longer work when more than one non-reciprocal length scale is at play, as studied below.
3. Two length scales

A minimal characteristic polynomial with more than one length scale is given by

$$F(E) = z^2 + \frac{b}{z},$$  \hspace{1cm} (A6)

Here $F(E)$ is a function of $E$ that absorbs the constant term, if any, and rescales the coefficient of $z^2$ to unity. It’s exact form is immaterial for the branching topology of the skin spectrum - it is the algebraic form of the polynomial in $z$ that matters.

Using Cardano’s formula, the three roots of Eq. A6 are

$$z_{1,2} = \frac{2\sqrt{3}F(E) + \sqrt[3]{12D^2} \pm i \left(\sqrt[3]{18D^2} - 6F(E)\right)}{-2\sqrt[3]{36\sqrt{3}D}}$$

$$z_3 = -\frac{\sqrt[3]{12F(E)} + \sqrt[3]{D^2}}{\sqrt[3]{18D}},$$ \hspace{1cm} (A7)

where $D = \sqrt{81b^2 - 12F(E)^3} - 9b$. Again, to obtain the OBC modes $\bar{c}(k)$, we need $|z_1| = |z_2|$ (and permutations), which occur when

$$\sqrt[3]{18D^2} - 6F(E) = r \left(2\sqrt{3}F(E) + \sqrt[3]{12D^2}\right),$$ \hspace{1cm} (A8)

$r$ being a real multiplier. This can be inverted to yield

$$F(E) = \frac{r^2 - 3}{\sqrt{(1 + r^2)^2}} \left(\frac{b}{2}\right)^{2/3} \omega_j$$ \hspace{1cm} (A9)

where $\omega^3 = 1$. The three branches with $j = 0, 1, 2$ are selected such that they are indeed the solution to Eq. A6 closest to the unit circle. Note that the precise functional dependence on $r$ is not important, $r$ being an auxiliary multiplier. Rather, what is important is the maximal and minimal range of $F(E)$; in this case, the main observation is that $F(E)$ fans out as three straight lines from the origin, as illustrated in Fig. A1 for $F(E) = E^2$ (the power of 2 gives $2 \times 3 = 6$ straight skin mode branches.) Given the generality of Eq. A6 we have established that the cubic (Y-shaped) junction of OBC skin modes as a hallmark of non-Hermitian pumping with 2 length scales.

a. Deformation of the momentum

Now that we have established the skin spectrum which describes a bona-fide quasi-reciprocal lattice by virtue of its imperviousness to non-Hermitian pumping, we shall obtain the complex momentum deformation that transforms the PBC spectrum to the OBC skin spectrum. In general, this can be done by substituting $z \to |z|e^{i\theta}$ for fixed real momentum $\theta \in \mathbb{R}$ in the characteristic polynomial. This will yield $\kappa = -\log |z|$. From Eq. A6 we solve for

$$\text{Im} \left[\left(|z|^2e^{2i\theta} + \frac{b}{|z|}e^{-i\theta}\right)\omega^{-j}\right] = 0$$

to obtain

$$z \to z \sqrt[2]{\frac{b}{3}} \sec (k - 2\pi j/3)$$ \hspace{1cm} (A10)

or

$$k \to k - \frac{i}{3} \log \left|\frac{b}{2 \cos (k - 2\pi j/3)}\right|. \hspace{1cm} (A11)$$

This is illustrated in Fig. 1e of the main text. Note that the form of $F(E)$ does not have to explicitly appear in this deformation.
b. Coupling approximation to deformed basis

To relate the above \( k \)-dependent deformation to an effective physical coupling lattice, one can perform a Fourier decomposition of the characteristic polynomial in terms of the deformed \( z \rightarrow z e^{-\kappa(k)} = z e^{-\kappa(-i \log z)} \). For instance, with \( b = 0.5 \), an approximation of Eq. (A6) up to 5 Fourier harmonics via Eq. (A11) gives

\[
F(E) = -0.051 z^5 + 0.492 z^2 + \frac{0.700}{z} - \frac{0.047}{z^4}.
\]  

Further physical understanding can be obtained by studying the modification to a single NN coupling \( z \): In the above example,

\[
z \rightarrow z \left[ 0.677 - 0.030 \left( z^3 + \frac{1}{z^3} \right) + 0.010 \left( z^6 + \frac{1}{z^6} \right) \right]
\]  

up to six additional harmonics. This is a real-space rescaling consisting of multiple scales: for instance, a coupling over \( X \) sites is not just suppressed by a factor of 0.677\(^X\), but is also approximately equivalent to the superposition of many other terms with different rates of exponential suppression with \( X \). Shown in Fig. A1 is an illustration of the Fourier approximation, where one sees that the spectrum converges to the quasi-reciprocal OBC spectrum as more harmonics are added. The convergence is power-law due to the non-analyticity of \( \kappa(k) \).

4. Generic cases

The precise prediction of their graph structure of the OBC spectrum from \( P(E, z) \) is an open problem. Generically, the OBC spectrum lie along intersections of the various solutions of \( \kappa(k) \) in energy space, as shown in Fig. A2 for two illustrative cases mentioned in the main text. As we can see, the OBC spectrum accumulate along branch-like trenches, but their global topology depends also on the PBC spectral loops. The reader is encouraged to refer to Refs. [10, 33, 36] for further discussions.

Appendix B: Analytic treatment of the extended non-Hermitian Kitaev model

The non-Hermitian extension of the extended Kitaev model [74, 75] realizes a prime example of a non-Hermitian topological phase (D-class) that minimally require NN and NNN couplings and hence more than one reciprocal length scale. It is given by \( H = \sigma \cdot h \), where

\[
\begin{align*}
h_x &= d_x + ig_x \\
h_y &= d_y + ig_y \\
h_z &= d_z,
\end{align*}
\]  

with \( d_x = \Delta_2 \sin \phi \sin 2k \), \( d_y = \Delta_2 \cos \phi \sin 2k + \Delta_1 \sin k \), \( d_z = m - t_1 \cos k - t_2 \cos 2k \). The two terms \( ig_x \sigma_x \) and \( ig_y \sigma_y \) are the only constant non-Hermitian terms that do not violate particle-hole symmetry (PHS)

\[
\sigma_x H^*(k) \sigma_x = -H(-k),
\]
That occurs when the parameters satisfy the constraints $u$ such that

$$\Delta_1 \cos \phi - \Delta_2 \cos \phi = 0$$

which can always be satisfied via a global rescaling. Doing so, the above expression simplifies to

$$E^2 = \Sigma u^2 + u + u_0$$

where

$$u = u_+ z + \frac{u_-}{z}, \quad u_\pm = \Delta_1 [(\Delta_2 - 1) \cos \phi \pm g_y]$$

such that $u^2 = u_+^2 z^2 + u_-^2 z^{-2} + 2u_+ u_-$ reproduces the rest of $E^2$ up to a linear transformation i.e.

$$E^2 = \Sigma u^2 + u + u_0$$

which protects the $Z_2$ topology inherited from the Hermitian version of this model. To ensure no residual $Z$-topology [75], we also break chiral symmetry via the parameter $\phi$, which represents a phase difference between the NN and NNN parings $\Delta_1$ and $\Delta_2$.

To analytically solve for its OBC skin modes, one considers the complex continuation of $z = e^{ik}$, and finds values of $E$ for which the two roots $z_\mu$, $z_\nu$ of the eigenenergy equation

$$E^2 = h_x^2 + h_y^2 + h_z^2$$

nearest to the real circle satisfies $|z_\mu| = |z_\nu|$. The spectrum will converge to these energy loci during the PBC-OBC interpolation. Here, the eigenenergy equation explicitly takes the form

$$E^2 = h_x^2 + h_y^2 + h_z^2$$

In order to explore analytic solutions, we shall consider special cases in which the cubic and quartic terms vanish. That occurs when the parameters satisfy the constraints $t_2 = \Delta_2$ and $t_1 = \Delta_1 \cos \phi$. Also, we shall WLOG set $m = 1$, which can always be satisfied via a global rescaling. Doing so, the above expression simplifies to

$$E^2 = \left( \frac{\Delta_1^2 (1 + \cos \phi)}{2} - (g_y^2 - g_y - \Delta_2^2 - 1) \right) + \Delta_1 (g_y - \cos \phi + \Delta_2 \cos \phi) z - \frac{\Delta_1 (g_y + \cos \phi - \Delta_2 \cos \phi)}{z}$$

Although we have reduced the above from an 8-th order to a 4-th polynomial in $z$, an additional constraint is still needed for a simple analytic solution. We will like to substitute the terms linear in $z$ and $z^{-1}$ by a single variable $u$, i.e.

$$u = u_+ z + \frac{u_-}{z}, \quad u_\pm = \Delta_1 [(\Delta_2 - 1) \cos \phi \pm g_y]$$

such that $u^2 = u_+^2 z^2 + u_-^2 z^{-2} + 2u_+ u_-$ reproduces the rest of $E^2$ up to a linear transformation i.e.

$$E^2 = \Sigma u^2 + u + u_0,$$
where $\Sigma$ and $u_0$ will be computed shortly. This is only possible if $\Delta_1$ is chosen to satisfy
\begin{equation}
\left(\frac{u_-}{u_+}\right)^2 = -\frac{1}{2} \Delta_1^2 \sin^2 \phi - \Delta_2 - \Delta_2 (g_x \sin \phi + g_y \cos \phi) - \frac{1}{2} \Delta_2^2 \sin^2 \phi - \Delta_2 + \Delta_2 (g_x \sin \phi + g_y \cos \phi)
\end{equation}

or, more explicitly, (defining $A=(\Delta_2-1)\cos \phi$)
\begin{equation}
\Delta_1^2 = -\frac{2\Delta_2 \left[ 2Ag_y + (g_y^2 + A^2)(g_x \cos \phi + g_y \sin \phi) \right]}{g_y A \sin^2 \phi}
\end{equation}

In order to avoid complications, we require that $\Delta_1, \Delta_2$ are real. This possible along some interval within $0<\Delta_2<1$ giving a positive value to the RHS of Eq. B8. With some labor, we can also show that
\begin{equation}
\Sigma = \frac{\sin^2 \phi (g_y \cos \phi + g_x \sin \phi)}{4(\Delta_2-1)^2 \cos^2 \phi (g_x + g_y \cos \phi) + 4g_y (g_x g_y + (g_y^2 + 2(\Delta_2-1)) \cos \phi)},
\end{equation}

\begin{equation}
u_0 = -\frac{2g_x (g_y^2 + (\Delta_2-1)^2) \Delta_2 \cos \phi \sin \phi + g_y \sin^2 \phi (-1 + g_y^2 + g_x^2 + \Delta_2 (1 + g_y^2 - g_x^2 + 3(-1 + \Delta_2) \Delta_2)) - 2\Delta_2 (-1 + g_y^2 + \Delta_2^2)}{g_y (\Delta_2-1)}.
\end{equation}

To summarize, if we enforce Eq. B8 as well as $t_2=\Delta_2$ and $t_1=\Delta_1 \cos \phi$, the eigenvalue equation Eq. B3 will reduce to Eq. B6 with $\nu$ given in terms of $z$ via Eq. B5 and $\Sigma$ and $u_0$ given by Eqs. B9 and B10 respectively. After adhering to these constraints we still have 4 free parameters left: $\Delta_2, \phi, g_x, g_y$.

2. OBC skin mode solutions of surrogate system

The roots of Eq. B3 are given via Eq. B5 as
\begin{equation}z_\pm = \frac{u \pm \sqrt{u^2 - 4uvu_-}}{2u_+} \end{equation}

for each $u$ satisfying Eq. B6. To find the skin mode loci satisfying $|z_+|=|z_-|$, the key observation is that $u$ has to be purely imaginary if $u_+, u_-$ are real. This is very similar to the case of Eq. A4 considered in the previous section. More concretely, note that, in the large regime where $u_\pm$ are real and of opposite signs (indication of large non-reciprocity), $|z_+|=|z_-|$ requires $0 > u^2 > 4u_+ u_-$, which is in the regime of imaginary $u$. Then, enforcing Re $u = \left(u_+ |z| + \frac{u_-}{|z|} \right) \cos(\text{Re} k) = 0$ yields the condition
\begin{equation}k \to k - i \log |z| = k - \frac{i}{2} \log \left| \frac{u_-}{u_+} \right|
\end{equation}

where the PBC modes shall be continued into skin modes. After solving for $u$, $E$ can be obtained via the addition step Eq. B6 ($E = \sqrt{u_0 + iv - \Sigma v^2}$ where $v \in \mathcal{R}$), as computed in Fig. A3.

3. Topological phase transition

Topological phase transitions occur when the OBC skin modes intersect. Setting $E=0$ in Eq. B6 and noting that $u$ has to be purely imaginary, it is not hard to see that $u_0=0$ is the condition for skin gap closure i.e.
\begin{equation}
-2g_x (g_y^2 + (\Delta_2-1)^2) \Delta_2 \cos \phi \sin \phi + g_y \sin^2 \phi (-1 + g_y^2 + g_x^2 + \Delta_2 (1 + g_y^2 - g_x^2 + 3(-1 + \Delta_2) \Delta_2)) - 2\Delta_2 (-1 + g_y^2 + \Delta_2^2) = 0
\end{equation}

For the parameters of Fig. A3 it occurs around $\Delta_2 = t_2 \approx 0.548$. For generic parameter cases that are not analytical tractable, we will need to numerically solve for the skin modes to find when they intersect.
where $\Delta$ extremely close to the phase transition, where the OBC eigenenergies almost touch the line gap without topological modes, (c) real line gap with isolated in-gap topological modes and (b) their intermediate case extremely close to the phase transition, where the OBC eigenenergies almost touch the $E=0$ origin. Parameters used are: (a) $\Delta_1=t_2=0.5, \phi=\pi/3, \Delta_1=0.64144, t_1=0.3207, g_x=g_y=0.6,$ (b) $\Delta_2=t_2=0.55\approx 0.548, \phi=\pi/3, \Delta_1=0.8503, t_1=0.4252, g_x=g_y=0.6,$ (c) $\Delta_2=t_2=0.7, \phi=\pi/3, \Delta_1=1.664, t_1=0.832, g_x=g_y=0.6.$ Faint light green curves represent contours of constant $\kappa(k)$.

4. Topological properties from pseudospin vectors

We next characterize the abovementioned topological transition in terms of the topological properties of the pseudospin expectation vector. In a non-Hermitian system, one can construct four possible pseudospin expectations out of the left (L) and right (R) eigenvectors

$$S_{i,\pm}^{\alpha\beta} = \langle \psi_{\pm}^{\alpha}\mid \bar{\sigma}_{i}^{\beta}\rangle, \quad (B14)$$

where $i=x, y, z, \pm$ label the two eigenvectors and $\alpha, \beta$ indicate whether the L or R eigenvector was used. Of the four expectation vectors, only two types are qualitatively distinct: the expectation $S_{x}^{RR}$ or $S_{z}^{LL}$ or a single R or L eigenvector, and the biorthogonal expectation $S_{x}^{LR}$ (or its conjugate). Since we are interested in the OBC topological boundary modes, we shall only work within the eigenspace of the surrogate Hamiltonian $\hat{H}(k)=\hat{H}(k+\text{i}k)$ obtained via complex momentum deformation.

For a geometric interpretation of the topology \cite{77}, we shall first consider the pseudospin expectation of the left or right eigenvectors, which IS always real. Writing $\bar{E} = \sqrt{\bar{h}_{x}^{2} + \bar{h}_{y}^{2} + \bar{h}_{z}^{2}}$, the pseudospin components are given by

$$\bar{S}_{x,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{x}^{R}\rangle,$$

$$\bar{S}_{y,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{y}^{R}\rangle,$$

$$\bar{S}_{z,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{z}^{R}\rangle,$$

$$\bar{S}_{y,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{y}^{R}\rangle,$$

$$\bar{S}_{z,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{z}^{R}\rangle,$$

$$\bar{S}_{x,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{x}^{R}\rangle,$$

$$\bar{S}_{y,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{y}^{R}\rangle,$$

$$\bar{S}_{z,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{z}^{R}\rangle,$$

$$\bar{S}_{x,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{x}^{R}\rangle,$$

$$\bar{S}_{y,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{y}^{R}\rangle,$$

$$\bar{S}_{z,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{z}^{R}\rangle,$$

$$\bar{S}_{x,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{x}^{R}\rangle,$$

$$\bar{S}_{y,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{y}^{R}\rangle,$$

$$\bar{S}_{z,\pm}^{RR} = \langle \psi_{\pm}^{R}\mid \bar{\sigma}_{z}^{R}\rangle,$$
At these two points \( k \) and \( \alpha \), with eigenenergies \( E \), the surrogate Hamiltonian, simultaneoulsy be both imaginary or real, since otherwise the four eigenenergies at these two points shall lie on each these two cases are also characterized by imaginary and real line gaps respectively. Note that \( \bar{\alpha} \) yields two distinct scenarios:

\[
\bar{\alpha} = \pm \bar{\bar{\bar{x}}} = \pm \bar{\bar{\bar{y}}} = \pm \bar{\bar{\bar{z}}},
\]

\[
\bar{\alpha}' = \pm \bar{\bar{\bar{x}}}' = \pm \bar{\bar{\bar{y}}}' = \pm \bar{\bar{\bar{z}}}',
\]

We next specialize to \( \bar{\alpha} \) WLOG, and consider its behavior at high symmetric points. Since PHS also holds for the surrogate Hamiltonian, \( \sigma_x \bar{H}_D(k) \sigma_x = -\bar{H}_D^*(-k) \), the right eigenvectors satisfy

\[
|\tilde{\psi}_\alpha(k)\rangle = \sigma_x |\tilde{\psi}_{\alpha'}(-k)\rangle^*,
\]

with eigenenergies

\[
\tilde{E}_\alpha(k) = -\tilde{E}_{\alpha'}(-k),
\]

and \( \alpha, \alpha' \in \{+, -\} \) labelling possibly different band indices. In terms of the pseudospin vector components,

\[
\tilde{S}_{x,\alpha}(k) = \tilde{S}_{x,\alpha'}(-k), \quad \tilde{S}_{y,\alpha}(k) = \tilde{S}_{y,\alpha'}(-k), \quad \tilde{S}_{z,\alpha}(k) = -\tilde{S}_{z,\alpha'}(-k),
\]

i.e. the \( \alpha \)-band at \( k \) is symmetric to the \( \alpha' \)-band at \( -k \) about the equator of their Bloch sphere. Qualitatively different possibilities arise depending on whether \( \alpha \) is the same as \( \alpha' \) at the high symmetric points \( k=0 \) and \( \pi \). At these two points \( k=0 \), PHS allows only purely imaginary \( \bar{\bar{\bar{h}}}_x(0) \) and real \( \bar{\bar{\bar{h}}}_z(0) \), such that the eigenenergy \( \bar{E}(0) = \sqrt{\bar{\bar{\bar{h}}}_x^2 + \bar{\bar{\bar{h}}}_y^2 + \bar{\bar{\bar{h}}}_z^2} \) is also either real or purely imaginary. As PHS always ensures that \( \bar{E}_+(0) = -\bar{E}_-(0) \), Eq. \[B18\] yield two distinct scenarios:

- \( \alpha = \alpha' \) when \( \bar{E}_\alpha(0) \) is imaginary, suggesting that each \( \tilde{S}_{RR,\alpha} \) trajectory is symmetric to itself. In particular, from Eq. \[B15\] we have

\[
\tilde{S}_{x,\alpha}(0) = \pm \frac{2\bar{h}_x}{E(0)} \frac{2\bar{h}_y \bar{h}_z}{E(0)^2},
\]

\[
\tilde{S}_{y,\alpha}(0) = \pm \frac{2\bar{h}_y}{E(0)} + \frac{2\bar{h}_y \bar{h}_z}{E(0)^2},
\]

\[
\tilde{S}_{z,\alpha}(0) = 0,
\]

indicating that at the high symmetric points \( k=0 \) or \( \pi \), the pseudospin vectors for both bands lie on the equator, but are not symmetric to each other.

- \( \alpha \neq \alpha' \) when \( \bar{E}_\alpha(0) \) is real, suggesting that the two trajectories of \( \tilde{S}_{RR,\alpha} \) are symmetric to each other. Eq. \[B15\] yields normalized pseudospin vectors

\[
\tilde{S}_{\pm,\alpha}(0) = \frac{1}{\sqrt{1-4\bar{h}_x^2 - 4\bar{h}_y^2}} \left( -2\bar{h}_y \frac{2\bar{h}_x}{E(0)^2}, 1 \right),
\]

with the two bands corresponding to two pseudospin vectors symmetric about the equator.

These two types of behaviors, distinguished simply by \( \text{Sign}[\bar{E}^2(k_0)] \) at \( k_0 = 0 \) and \( \pi \), are respectively illustrated in Fig. 4b and 4a,c of the main text. As depicted in their corresponding spectral plots in Fig. 3 of the main text, these two cases are also characterized by imaginary and real line gaps respectively. Note that \( \bar{E}(0) \) and \( \bar{E}(\pi) \) must simultaneously be both imaginary or real, since otherwise the four eigenenergies at these two points shall lie on each
of the positive and negative branches of the real and imaginary axis respectively, giving rise to a full OBC spectrum assuming a loop enclosing the origin, in contradiction to the quasi-reciprocity of the surrogate Hamiltonian.

To further determine the existence of topological boundary modes, i.e. to distinguish the cases in Figs. 3a and 3c of the main text, we will need to turn to the biorthogonal pseudospin expectation vector \( \vec{S}_{LR} \) defined by

\[
\begin{align*}
\bar{S}_{x,\pm} &= \langle \psi^L_{x} | \sigma_x | \psi^R_{x} \rangle = \pm \tilde{h}_x / E, \\
\bar{S}_{y,\pm} &= \langle \psi^L_{y} | \sigma_y | \psi^R_{y} \rangle = \pm \tilde{h}_y / E, \\
\bar{S}_{z,\pm} &= \langle \psi^L_{z} | \sigma_z | \psi^R_{z} \rangle = \pm \tilde{h}_z / E,
\end{align*}
\]

(B22)

which is however generally complex for non-Hermitian systems and not directly visualizable on a Bloch sphere. As discussed in the main text and Ref. [21], it allows the computation of the topological invariant \( \nu \)

is however generally complex for non-Hermitian systems and not directly visualizable on a Bloch sphere. As discussed in the main text and Ref. [21], it allows the computation of the topological invariant \( \nu \)

\( \nu \) is equivalent to the non-Hermitian SSH model in

\( \nu = 0 \) case is the scenario with imaginary line gap corresponding to Fig. A3(a) or Fig. 3b of the main text. The \( \nu = 1 \) cases are scenarios with real line gaps with/without topological boundary modes, corresponding to Fig. 3c (or Fig. A3c) and 3a of the main text.

\[ \begin{align*}
\langle \psi^L_{x} | \sigma_x | \psi^R_{x} \rangle &= \pm \tilde{h}_x / E, \\
\langle \psi^L_{y} | \sigma_y | \psi^R_{y} \rangle &= \pm \tilde{h}_y / E, \\
\langle \psi^L_{z} | \sigma_z | \psi^R_{z} \rangle &= \pm \tilde{h}_z / E,
\end{align*} \]

Appendix C: Further details of the extended non-Hermitian Chern model

1. Extended non-Hermitian Chern model

The Chern model considered in our work is

\[ H_{\text{Ch}}(k) = (v + z^{-1})\sigma_x (u + z - vz^2)\sigma_+ + \sin k_y \sigma_z, \]  

(C1)

where \( u = M + \cos k_y - \mu \) and \( v = v_0(M + \cos k_y + \mu) \). Containing both linear and quadratic terms in \( z, z^{-1} \), it has both NN and NNN couplings. It is specially designed such that \( E^2 = (v + z^{-1})(u + z - vz^2) + \sin^2 k_y \) only has \( z^2, z^{-1} \) and constant terms, thereby reducing to Eq. A6 with \( k_y \)-dependent coefficients and amenable to analytic treatment. Being fundamentally unlike the NN SSH model (in fact \( v\sigma_+ + v\sigma_- \) is equivalent to the non-Hermitian SSH model in \( k_y \)), its phase diagram has never been studied. In real space, its various highly non-reciprocal couplings across \(-2\) sites to \(2\) sites in both \( x \)

\[ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v_0/2 & 0 & 0 \\
0 & 1 & Mv_0 + \mu v_0 & 0 & 0 \\
0 & 0 & v_0/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \]  

(C2)

from pseudospin A to B and

\[ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & -v_0/2 & 0 \\
0 & 0 & M - \mu & -Mv_0 - \mu v_0 & 0 \\
0 & 0 & 1/2 & -v_0/2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \]  

(C3)

from pseudospin B to A. Between the same pseudospin sublattice, we simply have Hermitian NN \( \pm i/2 \) couplings from \( \sin k_y \sigma_z \).

The OBC spectrum takes the form of

\[ F_{\text{Ch}}(E) = \frac{1 + \sin^2 k_y + uv - E^2}{v^2} = z^2 + \frac{b}{z} \]  

(C4)

where \( b = -u/v^2 \), with \( z \) deformed according to \( z \rightarrow z^2 \sqrt{\frac{b}{2\cos(\pi j/3)}} \) where \( j = 0, 1, 2 \) depends on the branch. Simplifying, we can show that for the OBC gap to close,

\[ 4(2 - \cos^2 k_y + v_0(M + \cos k_y)^2 - \mu^2))^3 - 27 v_0^2 ((M + \cos k_y)^2 - \mu^2)^2 = 0. \]  

(C5)

Gapless regions correspond to parameter sets where a real \( k_y \) exists. If not, the system is gapped, characterized by a nonzero \( \kappa_y = \text{Im} k_y \), as plotted in Fig. A4.
2. Finiteness of response discontinuities

Here, we discuss the origin of the discontinuities of $\bar{\Gamma}_{\mu\nu}$ from $\kappa(k)$ discontinuities. The starting point is the functional dependence of the momenta through $\kappa(k)$:

$$|\bar{\psi}_R\rangle = |\bar{\psi}_R(k + i\kappa(k))\rangle$$

and likewise for the left eigenvector. As such, writing $p = k + i\kappa(k)$,

$$|\partial_\nu \bar{\psi}_R\rangle = \frac{d}{dk_\nu} |\bar{\psi}_R(p)\rangle = \frac{d|\bar{\psi}_R(p)\rangle}{dp} \frac{d}{dk_\nu} (k + i\kappa(k)) = \frac{d|\bar{\psi}_R(p)\rangle}{dp} + i \frac{d|\bar{\psi}_R(p)\rangle}{dp} \frac{d\kappa(k)}{dk_\nu}. $$

(C7)

The notation above is somewhat subtle: In line one, the partial derivative $\partial_\nu$ on the LHS refers to a derivative with respect to $k_\nu$ that treats other momenta as independent. However, on the RHS, we have rewritten it as a total derivative in $k_\nu$ to emphasize its total derivative nature with respect to $p = k + i\kappa(k)$.

The key takeaway from Eq. (C7) is that $|\partial_\nu \bar{\psi}_R\rangle$ contains discontinuities from $\frac{d\kappa(k)}{dk_\nu}$, which is discontinuous whenever there is a kink $\kappa(k)$ i.e. as depicted in Fig. 1 of the main text. Yet, the gradient $\frac{d\kappa(k)}{dk_\nu}$ never diverges as long as there are no essential singularities in the complex band structure. This finiteness is inherited in the OBC response quantities from $\bar{\Gamma}_{\mu\nu} = \langle \partial_\mu \bar{\psi}_L | \bar{Q} | \partial_\nu \bar{\psi}_R \rangle$.

Appendix D: Cold-atom system for detecting discontinuous Berry curvature

Here, we provide more details on our model atom model described by the Hamiltonian

$$H_{\text{eff}}(k) = (t_\perp \sin \theta + 2t_x^2 \sin k_x) \sigma_y + (t_\perp \cos \theta + 2t_y^2 \cos k_y) \sigma_x + (ig - 2t_x \cos k_x - 2t_y \cos k_y) \sigma_z. $$

(D1)

To obtain its surrogate quasi-reciprocal counterpart, we specialize to an analytically tractable regime given by $\theta = \pi/2$, $t_d^2 = t_x$ and $t_\perp = g$. The eigenenergies take the form

$$E_{\pm}^2(k_x, k_y) = [t_\perp - it_d(z - 1/z)]^2 + (2t_y^2 \cos k_y)^2 + (ig - t_x(z + 1/z) - 2t_y \cos k_y)^2$$

$$= [g - it_d(z - 1/z)]^2 + (2t_d^2 \cos k_y)^2 + (ig - t_x(z + 1/z) - 2t_y \cos k_y)^2$$

$$= g^2 - t_d^2(z^2 + 1/z^2 - 2) - 2igt_x(z - 1/z) + (2t_d^2 \cos k_y)^2 - g^2 + t_d^2(z^2 + 1/z^2 + 2) + 4t_d^2 \cos^2 k_y$$

$$- 2igt_x(z + 1/z) - 4igt_y \cos k_y + 4t_x t_y \cos k_y(z + 1/z)$$

$$= 4t_x t_y \cos k_y(z + 1/z) - 4igt_x z + (2t_y^2 \cos k_y)^2 + 4t_x^2 \cos^2 k_y - 4igt_y \cos k_y + 4t_d^2.$$  

(D2)
FIG. A5. x-OBC spectra and the Berry curvatures. (Top) $t_\parallel^y = t_x = 0.2$ and $t_\perp = g = 2$. (Bottom) $t_\parallel^y = t_x = 2$ and $t_\perp = g = 0.5$. Other parameters are $t_y = 1$, $t_y^d = 0$, and $\theta = \pi/2$.

with $z = e^{ik_x}$. After the complex deformation $k_x \to k_x + i\kappa_x(k)$, we obtain

$$\tilde{E}_\pm^x(k_x, k_y) = f_0 + f_1 \cos k_x + f_2 \sin k_x + i(f_3 \cos k_x + f_4 \sin k_x),$$

with

$$f_0 = (2t_x^y \cos k_y)^2 + 4t_x^2 \cos^2 k_y - 4igt_y \cos k_y + 4t_x^2$$

$$f_1 = 8t_x k_y \cos k_y \frac{e^{-\kappa} + e^{\kappa}}{2}, \quad f_2 = 4gt_x e^{-\kappa}$$

$$f_3 = -4gt_x e^{-\kappa}, \quad f_4 = 8t_x t_y \cos k_y \frac{e^{-\kappa} - e^{\kappa}}{2}. \quad (D4)$$

$\kappa_x(k)$ is found either through the solution of $f_1 f_4 = f_2 f_3$, or simply Eq. 9 of the main text:

$$e^{4\kappa_x} = 1 + \frac{g^2}{t_y^2 \cos^2 k_y}, \quad (D5)$$

where $\tilde{E}_\pm(\alpha + k_x, k_y) = \tilde{E}_\pm(\alpha - k_x, k_y)$, with $\tan \alpha = f_2/f_1$. In particular, at $k_y = \pm \pi/2$, $\kappa_x$ tends to infinity and the spectrum shrinks into two points at $\tilde{E}_\pm = \pm 2t_x$.

In Fig. A5, we illustrate the spectra and the Berry curvature for this regime with parameters different from that in the main text. In both cases, the discontinuities remain at $k_y = \pm \pi/2$.

In the second case, the Berry curvature jumps from negative nonzero to positive nonzero across $k_y = \pi/2$.

The discontinuity of Berry curvature at $k_y = \pm \pi/2$ corresponds to the point where the shape of the x-OBC spectrum undergoes a non-smooth transition i.e. where the spectrum shrinks into points. Away from the critical $k_y$, the shape of x-OBC spectrum may be homotopy equivalent but not homeomorphic, e.g. a Y-shaped spectrum deforming into a line. In Fig. A6, we illustrate such a scenario with more general parameters, where the x-OBC spectrum deforms from a Y-shape into a line. At the transition point, the spectrum does not shrink into a point, instead, one branch of the Y-shape merges into the curve formed by the other two branches, as shown in Fig. A6 (bottom). In this case, the
FIG. A6. $x$-OBC spectra and the Berry curvatures. Parameters are $t_{xx} = 1$, $t_{xx}^d = 0.5$, $t_y = 1$, $t_y^d = 0.5$, $t_\perp = 2$, and $\theta = \pi/2$. The transition of the $x$-OBC spectrum from a Y-shape to a line is seen to occur at $k_y \approx \pm(0.5 - 0.11)\pi$ and $\pm(0.5 + 0.11)\pi$, where a branch of the Y-shape merges into the curve at $k_x \approx 0.76\pi$ and $0.24\pi$ respectively. The second row shows the $x$-OBC spectra with $k_y = 0.5\pi$, $0.61\pi$, and $\pi$ respectively.

discontinuous Berry curvature occurs only at the merging point, resulting in a singularity point in the 2D BZ. For the parameters we choose, the $x$-OBC spectrum forms a Y-shape when $0.39\pi \lesssim k_y \lesssim 0.61\pi$ or $-0.61\pi \lesssim k_y \lesssim -0.39\pi$, forming a curve otherwise. Note that here, $\kappa_x(k)$ depends on both $k_x$ and $k_y$, and has to be obtained numerically.