YONEDA STRUCTURES AND KZ DOCTRINES

CHARLES WALKER

Abstract. In this paper we strengthen the relationship between Yoneda structures and KZ doctrines by showing that for any locally fully faithful KZ doctrine, with the notion of admissibility as defined by Bunge and Funk, all of the Yoneda structure axioms apart from the right ideal property are automatic.

1. Introduction

The majority of this paper concerns Kock-Zöberlein doctrines, which were introduced by Kock [3] and Zöberlein [8]. These KZ doctrines capture the free cocompletion under a suitable class of colimits Φ, with a canonical example being the free small cocompletion KZ doctrine on locally small categories. On the other hand, Yoneda structures as introduced by Street and Walters [6] capture the presheaf construction, with the canonical example being the Yoneda structure on (not necessarily locally small) categories, whose basic data is the Yoneda embedding \( \mathcal{A} \to [\mathcal{A}^{op}, \text{Set}] \) for each locally small category \( \mathcal{A} \). When \( \mathcal{A} \) is small this coincides with the usual free small cocompletion, but not in general. In this paper we prove a theorem tightening the relationship between these two notions, not just in the context of this example, but in general.

A key feature of a Yoneda structure (which is not present in the definition of a KZ doctrine) is a class of 1-cells called admissible 1-cells. In the setting of the usual Yoneda structure on \( \text{CAT} \), a 1-cell (that is a functor) \( L: \mathcal{A} \to \mathcal{B} \) is called admissible when the corresponding functor \( \mathcal{B}(L -, -): \mathcal{B} \to [\mathcal{A}^{op}, \text{SET}] \) factors through the inclusion of \( [\mathcal{A}^{op}, \text{Set}] \) into \( [\mathcal{A}^{op}, \text{SET}] \).

In order to compare Yoneda structures with KZ doctrines, we will also need a notion of admissibility in the setting of a KZ doctrine. Fortunately, such a notion of admissibility has already been introduced by Bunge and Funk [1]. In the case of the free small cocompletion KZ doctrine \( \mathcal{P} \) on locally small categories, these admissible 1-cells, which we refer to as \( \mathcal{P} \)-admissible, are those functors \( L: \mathcal{A} \to \mathcal{B} \) for which the corresponding functor \( \mathcal{B}(L -, -): \mathcal{B} \to [\mathcal{A}^{op}, \text{SET}] \) factors through the inclusion of \( \mathcal{P} \mathcal{A} \) into \( [\mathcal{A}^{op}, \text{SET}] \).

The main result of this paper; Theorem 18, shows that given a locally fully faithful KZ doctrine \( \mathcal{P} \) on a 2-category \( \mathcal{C} \), on defining the admissible maps to be those of Bunge and Funk, one defines all the data and axioms for a Yoneda structure except for the “right ideal property” which asks that the class of admissible 1-cells \( \mathcal{I} \) satisfies the property that for each \( L \in \mathcal{I} \) we have \( L \cdot F \in \mathcal{I} \) for all \( F \) such that the composite \( L \cdot F \) is defined.

2. Background

In this section we will recall the notion of a KZ doctrine \( \mathcal{P} \) as well as the notions of left extensions and left liftings, as these will be needed to describe Yoneda structures, and to discuss their relationship with KZ doctrines.

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Definition 1. Suppose we are given a 2-cell \( \eta : I \to R \cdot L \) as in the left diagram

in a 2-category \( \mathcal{C} \). We say that \( R \) is exhibited as a left extension of \( I \) along \( L \) by the 2-cell \( \eta \) when pasting 2-cells \( \sigma : R \to M \) with the 2-cell \( \eta : I \to R \cdot L \) as in the right diagram defines a bijection between 2-cells \( R \to M \) and 2-cells \( I \to M \cdot L \). Moreover, we say such a left extension is respected by a 1-cell \( E : C \to D \) when the whiskering of \( \eta \) by \( E \) given by the following pasting diagram

exhibits \( E \cdot R \) as a left extension of \( E \cdot I \) along \( L \).

Dually, we have the notion of a left lifting. We say a 2-cell \( \eta : I \to R \cdot L \) exhibits \( L \) as a left lifting of \( I \) through \( R \) when pasting 2-cells \( \delta : L \to K \) with the 2-cell \( \eta : I \to R \cdot L \) defines a bijection between 2-cells \( L \to K \) and 2-cells \( I \to R \cdot K \). We call such a lifting absolute if for any 1-cell \( F : X \to A \) the whiskering of \( \eta \) by \( F \) given by the following pasting diagram

exhibits \( L \cdot F \) as a left lifting of \( I \cdot F \) through \( R \).

There are quite a few different characterizations of KZ doctrines, for example those due to Kelly-Lack or Kock \([2, 3]\). For the purposes of relating KZ doctrines to Yoneda structures, it will be easiest to work with the following characterization given by Marmolejo and Wood \([5]\) in terms of left Kan extensions.

Definition 2. \([5, \text{Definition 3.1}]\) A KZ doctrine \((P, y)\) on a 2-category \( \mathcal{C} \) consists of

(i) An assignation on objects \( P : \text{ob} \mathcal{C} \to \text{ob} \mathcal{C} \);
(ii) For every object \( A \in \mathcal{C} \), a 1-cell \( y_A : A \to PA \);
(iii) For every pair of objects \( A \) and \( B \) and 1-cell \( F : A \to PB \), a left extension

of \( F \) along \( y_A \) exhibited by an isomorphism \( c_F \) as above.

Moreover, we require that:
(a) For every object \(A \in \mathcal{C}\), the left extension of \(y_A\) as in \(2.1\) is given by

\[
P_A \xrightarrow{\text{id}_{P_A}} P_A \xrightarrow{\text{id}} \xRightarrow{\mu} P_A \xrightarrow{\text{id}} A \xleftarrow{\text{id}} y_A
\]

Note that this means \(c_{y_A}\) is equal to the identity 2-cell on \(y_A\).

(b) For any 1-cell \(G : B \to PC\), the corresponding left extension \(\overline{G} : PB \to PC\) respects the left extension \(\overline{F}\) in \(2.1\).

Remark 3. This definition is equivalent (in the sense that each gives rise to the other) to the well known algebraic definition, which we refer to as a KZ pseudomonad \([5, 4]\). A KZ pseudomonad \((P, y, \mu)\) on a 2-category \(\mathcal{C}\) is taken to be a pseudomonad \((P, y, \mu)\) on \(\mathcal{C}\) equipped with a modification \(\theta : Py \to yP\) satisfying two coherence axioms \([3]\).

Just as KZ doctrines may be defined algebraically or in terms of left extensions, one may also define pseudo algebras for these KZ doctrines algebraically or in terms of left extensions.

The following definitions in terms of left extensions are equivalent to the usual notions of pseudo \(P\)-algebra and \(P\)-homomorphism, in the sense that we have an equivalence between the two resulting 2-categories of pseudo \(P\)-algebras arising from the two different definitions \([5, \text{Theorems 5.1, 5.2}]\).

Definition 4 \([5]\). Given a KZ doctrine \((P, y)\) on a 2-category \(\mathcal{C}\), we say an object \(X \in \mathcal{C}\) is \(P\)-cocomplete if for every \(G : B \to X\)

\[
P_B \xrightarrow{\text{id}_{P_B}} P_B \xrightarrow{\text{id}} \xRightarrow{\mu} P_B \xrightarrow{\text{id}} X \xleftarrow{\text{id}} y_B
\]

there exists a left extension \(\overline{G}\) as on the left exhibited by an isomorphism \(c_{y_B}\), and moreover this left extension respects the left extensions \(\overline{F}\) as in the diagram on the right. We say a 1-cell \(E : X \to Y\) between \(P\)-cocomplete objects \(X\) and \(Y\) is a \(P\)-homomorphism when it respects all left extensions along \(y_B\) into \(X\) for every object \(B\).

Remark 5. It is clear that \(P_A\) is \(P\)-cocomplete for every \(A \in \mathcal{C}\).

The relationship between \(P\)-completeness and admitting a pseudo \(P\)-algebra structure is as below.

Proposition 6. Given a KZ doctrine \((P, y)\) on a 2-category \(\mathcal{C}\) and an object \(X \in \mathcal{C}\), the following are equivalent:

1. \(X\) is \(P\)-cocomplete;
2. \(y_X : X \to PX\) has a left adjoint with invertible counit;
3. \(X\) is the underlying object of a pseudo \(P\)-algebra.

Proof. For (1) \(\iff\) (2) see the proof of \([5, \text{Theorem 5.1}]\), and for (2) \(\iff\) (3) see \([2]\). \(\square\)

We now recall the notion of Yoneda structure as introduced by Street and Walters \([6]\).

Definition 7. A Yoneda structure \(\mathcal{Y}\) on a 2-category \(\mathcal{C}\) consists of:
(1) A class of 1-cells $\mathbf{I}$ with the property that for any $L \in \mathbf{I}$ we have $L \cdot F \in \mathbf{I}$ for
all $F$ such that the composite $L \cdot F$ is defined; we call this the class of admissible
1-cells. We say an object $A \in \mathcal{C}$ is admissible when $id_A$ is an admissible 1-cell.

(2) For each admissible object $A \in \mathcal{C}$, an admissible map $y_A : A \to P.A$.

(3) For each $L : A \to B$ such that $L$ and $A$ are both admissible, a 1-cell $R_L$ and
2-cell $\varphi_L$ as in the diagram

Such that:

(a) The diagram above exhibits $L$ as a absolute left lifting and $R_L$ as a left
extension via $\varphi_L$.

(b) For each admissible $A$, the diagram

exhibits $id_{PA}$ as a left extension.

(c) For admissible $A, B$ and $L, K$ as below, the diagram

exhibits $R_{y_B \cdot L} : R_K$ as a left extension.

Remark 8. We note that when the admissible maps form a right ideal, the admis-
sibility of $L$ in condition (c) is redundant. However, in the following sections we
will consider a setting in which the admissible maps are closed under composition,
but do not necessarily form a right ideal.

Remark 9. There is an additional axiom (d) discussed in “Yoneda structures” [6]
which when satisfied defines a so called good Yoneda structure [7]. This axiom asks
for every admissible $L$ and every diagram

that if $\phi$ exhibits $L$ as an absolute left lifting, then $\phi$ exhibits $M$ as a left extension.
This condition implies axioms (b) and (c) in the presence of (a) [6, Prop. 11].

However, this condition is often too strong. For example one may consider the
free $\text{Cat}$-cocompletion, and take $\mathbb{N}$ to be the monoid of natural numbers seen as a
one object category, yielding the absolute left lifting diagram
It is then trivial, as we would be extending along an identity, that the left extension property is not satisfied.

3. Admissible Maps in KZ Doctrines

Yoneda structures as defined above require us to give a suitable class of admissible maps, and so in order to compare Yoneda structures with KZ doctrines we will need a suitable notion of admissible map in the setting of a KZ doctrine. Bunge and Funk defined a map \( L : A \to B \) in the setting of a KZ pseudomonad \( P \) to be \( P \)-admissible when \( P L \) has a right adjoint, and showed this notion of admissibility may also be described in terms of left extensions [1]. Our definition in terms of left extensions and KZ doctrines is as follows.

**Definition 10.** Given a KZ doctrine \((P, y)\) on a 2-category \( C \), we say a 1-cell \( L : A \to B \) is \( P \)-admissible when there exists a left extension \((R_L, \varphi_L)\) of \( y_A \) along \( L \) as in the left diagram, and moreover the left extension is respected by any \( H \) as in the right diagram where \( X \) is \( P \)-cocomplete.

**Remark 11.** Note that such a \( H \) is a \( P \)-homomorphism, and conversely that a \( P \)-homomorphism \( H : PA \to X \) is a left extension of \( H := PH \cdot y_A \) along \( y_A \) as above. Thus this is saying the left extension \( R_L \) is respected by \( P \)-homomorphisms.

**Lemma 12.** Suppose we are given a KZ doctrine \((P, y)\) and a \( P \)-admissible 1-cell \( L : A \to B \) where \( B \) is \( P \)-cocomplete, then the 1-cell \( R_L \) in

![Diagram](image)

has a left adjoint \( \overline{L} : PA \to B \).

**Proof.** Taking \( \overline{L} \) to be the left extension

![Diagram](image)

we then have \( \overline{L} \dashv R_L \) since we may define \( n : id_{PA} \to R_L \cdot \overline{L} \) and \( e : L \cdot R_L \to id_B \) respectively as (since \( L \) is \( P \)-admissible) the unique solutions to

![Diagram](image)

Verifying the triangle identities is then a simple exercise.

\( \square \)
The following is an easy consequence of this Lemma.

**Lemma 13.** Suppose we are given a KZ doctrine \((P, y)\) on a 2-category \(\mathcal{C}\) and a \(P\)-admissible 1-cell \(L: A \to B\). Then the 1-cell \(\text{res}_L\) defined here as the left extension in the top triangle

\[
\begin{aligned}
& PA \xrightarrow{\text{res}_L} PB \\
& \downarrow_{y_A} \quad \quad \downarrow_{y_B}
& A \xleftarrow{\varphi_{RL}} B
\end{aligned}
\]

has a left adjoint \(\text{lan}_L\), and when \(RL\) is \(P\)-admissible, a right adjoint \(\text{ran}_L\).

**Proof.** First note that it is an easy consequence of the left extension pasting lemma (the dual of [6, Prop. 1]) that \(y_B \cdot L\) is \(P\)-admissible, which is to say the left extension \(\text{res}_L\) above is respected by any \(P\)-homomorphism \(\overline{\mathcal{F}}: PA \to X\). This is since such a \(\overline{\mathcal{F}}\) will respect the left extension \(RL\) of \(y_A\) along \(L\) as well as the left extension \(\text{res}_L\) of \(RL\) along \(y_B\). Hence by Lemma 12 \(\text{res}_L\) has a left adjoint \(\text{lan}_L\) given as the left extension as on the left (which is how \(PL\) is defined given the data of Definition 2),

\[
\begin{aligned}
& PA \xrightarrow{\text{lan}_L} PB \\
& \downarrow_{y_A} \quad \quad \downarrow_{y_B}
& A \xleftarrow{\varphi_{RL}} B
\end{aligned}
\]

and if \(RL\) is \(P\)-admissible then we may define \(\text{ran}_L := RL\) (which is the left extension as on the right) and since \(PA\) is \(P\)-cocomplete \(\text{ran}_L\) has a left adjoint given by \(\text{res}_L = RL\) again by Lemma 12.

**Remark 14.** We have shown that when both \(L\) and \(RL\) are \(P\)-admissible we have the adjoint triple \(PL \dashv RL \dashv RL\). Of particular interest is the case where \(L = y_A\) for some \(A \in \mathcal{C}\). Clearly in this case both \(L\) and \(RL\) are \(P\)-admissible and so we may define \(\mu_A := \overline{y_A} = y_{PA}\) and observe \(R_{RL} = R_{y_A} = y_{PA}\) to recover the well known sequence of adjunctions \(P_{y_A} \dashv y_{PA}\) as in [4].

The following result is mostly due to Bunge and Funk [1], though we state it in our notation and from the viewpoint of KZ doctrines in terms of left extensions. Also, we will prove the following proposition in full detail in order to clarify some parts of the argument given by Bunge and Funk [1]. For example, in order to check that certain left extensions are respected we will need to know their exhibiting 2-cells. These exhibiting 2-cells will also be needed later to prove our main result.

**Proposition 15.** Given a KZ doctrine \((P, y)\) on a 2-category \(\mathcal{C}\) and a 1-cell \(L: A \to B\), the following are equivalent:

1. \(L\) is \(P\)-admissible;
2. every \(P\)-cocomplete object \(X \in \mathcal{C}\) admits, and \(P\)-homomorphism respects, left extensions along \(L\). This says that for any given 1-cell \(K: A \to X\), where \(X\) is \(P\)-cocomplete, there exists a 1-cell \(J\) and 2-cell \(\delta\) as on the left

\[
\begin{aligned}
& B \xrightarrow{J} X \\
& \downarrow_{L} \quad \downarrow_{K}
& A
\end{aligned}
\]

\[
\begin{aligned}
& B \xrightarrow{J} X \\
& \downarrow_{L} \quad \downarrow_{K}
& A
\end{aligned}
\]

exhibiting \(J\) as a left extension, and moreover this left extension is respected by any \(P\)-homomorphism \(E: X \to Y\) for \(P\)-cocomplete \(Y\) as in the right diagram.
(3) $PL := \text{lan}_L$ given as the left extension

\[
\begin{array}{ccc}
PA & \xrightarrow{PL} & PB \\
\downarrow{y_A} & & \downarrow{y_B} \\
A & \xrightarrow{L} & B
\end{array}
\]

has a right adjoint. We denote the inverse of the above 2-cell as $y_L := c^{-1}_{\text{res}_L}$ for every 1-cell $L$.

**Proof.** The following implications prove the logical equivalence.

(2) $\Rightarrow$ (1): This is trivial as $PA$ is $P$-cocomplete.

(1) $\Rightarrow$ (2): Given a $K : A \to X$ as in (2). We take the pasting

\[
\begin{array}{ccc}
B & \xrightarrow{R_L} & PA & \xrightarrow{\bar{\pi}} & X \\
\downarrow{s_L} & & \downarrow{s_A} & & \downarrow{K} \\
L & \xrightarrow{\phi_L} & A
\end{array}
\]

as our left extension using that $L$ is $P$-admissible. This is respected by any $P$-homomorphism $E : X \to Y$ where $Y$ is $P$-cocomplete as a consequence of the second part of the definition of $P$-admissibility.

(1) $\Rightarrow$ (3): This was shown in Lemma 13.

(3) $\Rightarrow$ (1): This implication is where the majority of the work lies in proving this proposition. We suppose that we are given an adjunction $\text{lan}_L \dashv \text{res}_L$ with unit $\eta$ where $\text{lan}_L$ is defined as in (3). We split the proof into two parts.

**Part 1:** The given right adjoint, $\text{res}_L$, is a left extension of $\text{res}_L \cdot y_B$ along $y_B$ as in the diagram

\[
\begin{array}{ccc}
P_B & \xrightarrow{\text{res}_L} & PA \\
\downarrow{y_B} & & \downarrow{\phi_L} \\
B & \xrightarrow{\eta_B} & PA
\end{array}
\]

exhibited by the identity 2-cell.\(^1\)

To see this, we consider the isomorphism in the square on the left

\[
\begin{array}{ccc}
PA & \xrightarrow{PL} & PB \\
\downarrow{y_A} & & \downarrow{y_B} \\
A & \xrightarrow{L} & B
\end{array}, \quad \begin{array}{ccc}
P^2A & \xrightarrow{PL} & P^2B \\
\downarrow{P\phi_A} & & \downarrow{P\phi_B} \\
P^2A & \xrightarrow{\text{res}_L} & P^2B
\end{array}, \quad \begin{array}{ccc}
P^2A & \xrightarrow{\text{res}_L} & P^2B \\
\downarrow{\mu_A} & & \downarrow{\mu_B} \\
P^2A & \xrightarrow{\text{res}_L} & P^2B
\end{array}
\]

and then apply $P$ to get the isomorphism of left adjoints in the middle square (suppressing pseudofunctoriality constraints\(^2\)), which corresponds to an isomorphism of right adjoints in the right square (which we leave unnamed). Now by [5, Theorem 4.2] (and since $\mu_A \cdot \text{res}_L$ respects the left extension $P\phi_B$) we have the left extension

\(^1\)This may be seen as an analogue of [1, Prop. 1.3]. However, we emphasize here that considering right adjoints tells us $\text{res}_L$ is a $P$-homomorphism since the adjunctions may be used to construct an isomorphism between $\text{res}_L$ and a known $P$-homomorphism.

\(^2\)These pseudofunctoriality constraints are those arising from the uniqueness of left extensions up to coherent isomorphism.
\(\mu_A \cdot \text{Res}_L \cdot P y_B\) of \(\text{Res}_L \cdot y_B\) along \(y_B\) as below

\[
\begin{align*}
\text{Res}_L \\
y_B \\
\end{align*}
\]

and so pasting with the isomorphism \(\mu_A \cdot \text{Res}_L \cdot P y_B \cong \text{Res}_L\) constructed as above tells us \(\text{Res}_L\) is also an extension of \(\text{Res}_L \cdot y_B\) along \(y_B\). It follows that \(\text{Res}_L\) respects the left extension

\[
\begin{align*}
\text{Res}_L \\
y_B \\
\end{align*}
\]

and this gives the result.

**Part 2:** The following pasting exhibits

\[
\begin{align*}
\text{Res}_L \\
y_B \\
\end{align*}
\]

the composite \(\overline{\text{H}} \cdot \text{Res}_L \cdot y_B\) as a left extension of \(H\) along \(L\).

Suppose we are given a 1-cell \(K : B \to X\). We then see that our left extension is exhibited by the sequence of natural bijections

\[
\begin{align*}
H & \to K \cdot L \\
\text{Res}_L & \to \text{Res}_L \cdot y_B \\
L & \to \text{Res}_L \cdot \text{lan}_L \cdot y_A \\
\text{Res}_L \cdot y_B & \to K \cdot y_B \\
\end{align*}
\]

It is easily seen this left extension is exhibited by the above 2-cell since when taking \(K = \overline{\text{H}} \cdot \text{Res}_L \cdot y_B\) we may take \(\overline{K} = \overline{\text{H}} \cdot \text{Res}_L\) as a consequence of Part 1 (with the left extension \(\overline{K}\) exhibited by the identity 2-cell). Tracing through the bijection to find the exhibiting 2-cell is then trivial.

**Remark 16.** Considering Part 2 in the above proposition with \(H = y_A\) and \(\overline{\text{H}}\) and \(c_H\) being an identity 1-cell and 2-cell respectively, we see that for any \(P\)-admissible 1-cell \(L : A \to B\) and corresponding adjunction \(P L \dashv \text{Res}_L\) with unit \(\eta\), we may define our 1-cell \(R_L\) and 2-cell \(\varphi_L\) as in Definition 10 by

\[
\begin{align*}
B & \xrightarrow{R_L} P A \\
A & \xrightarrow{\varphi_L} P A \\
\end{align*}
\]

We will make regular use of this definition in the next section.
Remark 17. It is clear from the above proposition that $P$-admissible 1-cells are closed under composition as noted by Bunge and Funk [1]. We may also note, as in [1], that every left adjoint is $P$-admissible, as taking $PL := \text{lan}_L$ defines a pseudofunctor [5, Theorem 4.1] and so preserves the adjunction.

4. Relating KZ doctrines and Yoneda Structures

We are now ready to prove our main result. In the following statement we call a KZ doctrine locally fully faithful if the unit components are fully faithful; indeed Bunge and Funk [1] noted that a KZ pseudomonad is locally fully faithful precisely when its unit components are fully faithful. Here the admissible maps of Bunge and Funk refer to those maps $L$ for which $PL := \text{lan}_L$ has a right adjoint (which we denote by $\text{res}_L$).

Theorem 18. Suppose we are given a locally fully faithful KZ doctrine $(P, y)$ on a 2-category $C$. Then on defining the class of admissible maps $L$ to be those of Bunge and Funk, with chosen left extensions $(RL, \varphi_L)$ those of Remark 16, we obtain all of the definition and axioms of a Yoneda structure with the exception of the right ideal property (though the admissible maps remain closed under composition).

Proof. We need only check that:

1. $\varphi_L$ exhibits $L$ as an absolute left lifting. Thus, we must exhibit a natural bijection between 2-cells $L \cdot W \to H$ and 2-cells $y_A \cdot W \to RL \cdot H$ for 1-cells $W: D \to A$ and $H: D \to B$ as in the diagram

   \[
   \begin{array}{ccc}
   D & \xrightarrow{\varphi} & PA \\
   ^L \downarrow & & \downarrow \psi_L \\
   B & \xrightarrow{\psi_{RL}} & PB \\
   _{\varphi_{RL}} \downarrow & & \downarrow \varphi_L \\
   A & \xrightarrow{\psi_{A}} & PA \\
   \\
   \end{array}
   \]

   Such a natural bijection is given by the correspondence

   \[
   \begin{array}{c}
   y_B \cdot L \cdot W \to y_B \cdot H \\
   \text{lan}_L \cdot y_A \cdot W \to \text{lan}_L \cdot y_B \cdot H \\
   y_A \cdot W \to \text{res}_L \cdot y_B \cdot H \\
   y_A \cdot W \to RL \cdot H
   \end{array}
   \]

   and the 2-cell exhibiting this absolute left lifting is easily seen to be the 2-cell as given in Remark 16 by following the above bijection.

2. $\text{res}_L \cdot RK$ is a left extension. Considering the diagram

   \[
   \begin{array}{ccc}
   PA & \xrightarrow{\text{res}_L} & PB \\
   ^{\text{res}_L} \downarrow & & \downarrow ^{\text{res}_L} \\
   A & \xrightarrow{\psi_{RL}} & B \\
   _{\varphi_{RL}} \downarrow & & \downarrow \varphi_{RL} \\
   C & \xrightarrow{\psi_{RL}} & C \\
   ^{\psi_{RL}} \downarrow & & \downarrow ^{\psi_{RL}} \\
   \end{array}
   \]

   we first note that $\text{res}_L \cdot RK$ is a left extension of $RL$ along $K$ since $K$ is $P$-admissible. We then apply the pasting lemma for left extensions to see the outside diagram also exhibits $\text{res}_L \cdot RK$ as a left extension. □

Remark 19. We observe that to ask that $\text{res}_L \cdot RK$ be a left extension in the diagram above for every $P$-admissible $L$ and $K$, is to ask by the pasting lemma that the
pasting of $\varphi_K$ and $c_{RL}$ exhibit $\text{res}_L \cdot R_K$ as a left extension. As $c_{RL}$ is invertible, this is to say that $\text{res}_L$ respects every left extension arising from admissibility. This is equivalent to asking $\text{res}_L$ be a $P$-homomorphism.

Remark 20. We note here that we do not necessarily have the right ideal property. Indeed given a KZ doctrine on a 2-category every identity arrow is admissible, and so the right ideal property would require all arrows into all objects being admissible (that is all arrows being admissible). This fails for example with the identity KZ doctrine on any 2-category $\mathcal{C}$ which contains an arrow $L$ with no right adjoint.

Remark 21. Given an object $A \in \mathcal{C}$ with a $P$-admissible generalized element $a : S \to A$ we have a version of the Yoneda lemma in the sense that we have bijections

$$\begin{align*}
y_A \cdot a &\to K \\
\text{lan}_a \cdot y_S &\to K \\
y_S &\to \text{res}_a \cdot K
\end{align*}$$

for generalized elements $K : S \to PA$. In the case where $P$ is the usual free small cocompletion KZ doctrine on locally small categories and $S = 1$ is the terminal category, maps $y_S \to \text{res}_a \cdot K$ are elements of $\text{res}_a \cdot K$ (which may be viewed as $K$ evaluated at $a$).

The purpose of the following is to give an example in which absolute left liftings (also known as relative adjunctions or partial adjunctions) are preserved\(^3\). Also, the following proposition does not require locally fully faithfulness, whereas Theorem 18 does.

Proposition 22. Suppose we are given a KZ doctrine $(P, y)$ on a 2-category $\mathcal{C}$. Then for every $P$-admissible 1-cell $L : A \to B$ as on the left,

\[
\begin{array}{c}
\phantom{\sum} \\
\sum_{RL} \phantom{\sum} \\
\phantom{\sum} \\
PA \\
\downarrow \\
A
\end{array}
\begin{array}{c}
\phantom{\sum} \\
\sum_{PRL} \phantom{\sum} \\
\phantom{\sum} \\
PB \phantom{\sum} \ phantom{\sum} \\
\downarrow PRL \\
P A
\end{array}
\]

the 2-cell $P\varphi_L$ as on the right (in which we have suppressed the pseudofunctoriality constraints) exhibits $PL$ as an absolute left lifting of $Py_A$ through $PRL$.

Proof. Without loss of generality, we define $\varphi_L$ as in Remark 16. We then have the sequence of natural bijections

\[
\begin{array}{c}
PL \cdot W \to H \\
P_{y_S} \cdot PL \cdot W \to P_{y_S} \cdot H \\
P L \cdot P y_A \cdot W \to P y_B \cdot H \\
P L \cdot P y_A \cdot W \to P L \cdot res_L \cdot P y_B \cdot H \\
P y_A \cdot W \to P L \cdot res_L \cdot P y_B \cdot H \\
P y_A \cdot W \to P L \cdot res_L \cdot P y_B \cdot H \\
P y_A \cdot W \to P L \cdot res_L \cdot P y_B \cdot H \\
P y_A \cdot W \to P L \cdot res_L \cdot P y_B \cdot H
\end{array}
\]

for 1-cells $W$ into $PA$. Following the bijection we see that the absolute left lifting is exhibited by $P\varphi_L$, suppressing the pseudofunctoriality constraints. \(\square\)

Some observations made in “Yoneda structures” [6] may be seen more directly in this setting of a KZ doctrine. For example Street and Walters defined an admissible morphism $L$ (in the setting of a Yoneda structure) to be fully faithful when the 2-cell $\varphi_L$ is invertible (which agrees with a representable notion of fully faithfulness, that is fully faithfulness defined via the absolute left lifting property, when axiom (d) is satisfied). Here we see this in the context of a (locally fully faithful) KZ doctrine.

\(\text{In this case respected by the KZ pseudomonad resulting from the KZ doctrine as in [5].}\)
Proposition 23. Suppose we are given a KZ doctrine \((P, y)\) on a 2-category \(\mathcal{C}\), and a \(P\)-admissible 1-cell \(L : A \to B\) with a left extension \(R_L\) as in the above diagram. Then the exhibiting 2-cell \(\varphi_L\) is invertible if and only if \(P_L := \text{lan}_L\) is fully faithful.

Proof. We use the well known fact that the left adjoint of an adjunction is fully faithful precisely when the unit is invertible. Now, given that \(\varphi_L\) is invertible we may define our 2-cell \(\eta^*\) as the unique solution to

That \(\eta\) is the inverse of \(\eta^*\) follows from an easy calculation using Remark 16. Conversely, if the unit \(\eta\) is invertible then so is \(\varphi_L\) by Remark 16. \(\Box\)

Remark 24. If we define a map \(L\) to be \(P\)-fully faithful when \(P_L\) is fully faithful, then as a consequence of Proposition 15 (Part 2) and Proposition 23 we see that for any \(P\)-admissible map \(L\), this \(L\) is \(P\)-fully faithful if and only if every left extension along \(L\) into a \(P\)-cocomplete object is exhibited by an invertible 2-cell.

In the following remark we compare \(P_L\) being fully faithful with \(L\) being fully faithful, and point out sufficient conditions for these notions to agree.

Remark 25. Note that if \(P_L\) is fully faithful then \(L\) is fully faithful assuming \(P\) is locally fully faithful, as \(y\) is pseudonatural. Conversely if \(L\) is fully faithful, then (supposing our corresponding left extension \(R_L\) is pointwise) the exhibiting 2-cell is invertible [7, Prop. 2.22], equivalent to \(P_L\) being fully faithful by the above. This converse may also be seen when the KZ doctrine is locally fully faithful and good (meaning axiom (d) is satisfied for \(P\)-admissible maps) as we can use the argument of [6, Prop. 9]. However, as we now see, this converse need not hold in general.

An example in which \(L\) is fully faithful but \(P_L\) is not is given as follows. Take \(\mathcal{A}\) to be the 2-category containing the two objects \(0, 1\) and two non-trivial 1-cells \(x, y: 0 \to 1\), and take \(\mathcal{B}\) to be the same but with an additional 2-cell \(\alpha: x \to y\). Define \(L\) as the inclusion of \(\mathcal{A}\) into \(\mathcal{B}\). Then for the free \(\text{Cat}\)-cocompletion of \(\mathcal{A}\) given by \(y_A: A \to [\mathcal{A}^{\text{op}}, \text{Cat}]\) we note that \(y_A\) and \(R_L \cdot L\) are not isomorphic, and so the 2-cell \(\varphi_L\) is not invertible meaning \(P_L\) is not fully faithful (despite \(L\) being fully faithful).

5. Future Work

We have seen that the notions of pseudo algebras and admissibility for a given KZ doctrine, and KZ doctrines themselves, may be expressed in terms of left extensions. In a soon forthcoming paper we show that pseudodistributive laws over a KZ doctrine may be simply expressed entirely in terms of left extensions and admissibility, allowing us to generalize some results of Marmolejo and Wood [5].
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Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: charles.walker1@mq.edu.au