CARTAN CONNECTIONS ASSOCIATED TO A $\beta$-CONFORMAL CHANGE IN FINSLER GEOMETRY

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Abstract. On a Finsler manifold $(M, L)$, we consider the change $L \rightarrow \overline{L}(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y)$, which we call a $\beta$-conformal change. This change generalizes various types of changes in Finsler geometry: conformal, $C$-conformal, $h$-conformal, Randers and generalized Randers changes. Under this change, we obtain an explicit expression relating the Cartan connection associated to $(M, L)$ and the transformed Cartan connection associated to $(M, \overline{L})$. We also express some of the fundamental geometric objects (canonical spray, nonlinear connection, torsion tensors, ...etc.) of $(M, \overline{L})$ in terms of the corresponding objects of $(M, L)$. We characterize the $\beta$-homothetic change and give necessary and sufficient conditions for the vanishing of the difference tensor in certain cases.

It is to be noted that many known results of Shibata, Matsumoto, Hashiguchi and others are retrieved as special cases from this work.

Keywords: $\beta$-conformal change, Conformal change, Randers change, Generalized Randers change, $\beta$-change, Cartan connection, Difference tensor.

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1. Introduction and Notations

Let \((M, L)\) be a Finsler space, where \(M\) is an \(n\)-dimensional \(C^\infty\) manifold and \(L(x, y)\) is a Finsler metric function. If \(\sigma(x)\) is a function in each coordinate neighborhood of \(M\), the change \(L(x, y) \rightarrow e^{\sigma(x)}L(x, y)\) is called a conformal change. This change was introduced by M. S. Kneblman \[10\] and deeply investigated by many authors: \[4\], \[5\], \[6\], ... etc. The change \(L(x, y) \rightarrow L(x, y) + \beta(x, y)\), where \(L(x, y)\) is a Riemannian metric function and \(\beta = b_iy^i\) is a 1-form on \(M\), is called a Randers change after Randers who first introduced it in \[13\]. The geometric properties of such a change have been studied in various works: \[2\], \[9\], \[12\], \[16\],... etc. Matsumoto \[11\] introduced the transformation \(L^*(x, y) = L(x, y) + \beta(x, y)\), where \(L(x, y)\) is a Finsler metric function, and named it a \(\beta\)-change. He obtained the relationship between the Cartan connection coefficients of \((M, L)\) and those of \((M, L^*)\). Since then, this change has been investigated by many authors: Shibata \[14\], Miron \[12\],...etc.

A change generalizing all the above mentioned changes has been introduced by Abed \[1\] in the form:

\[ L(x, y) \rightarrow e^{\sigma(x)}L(x, y) + \beta(x, y) , \tag{1.1} \]

where \(\sigma\) is a function of \(x\) and \(\beta(x, y) = b_i(x)y^i\) is a 1-form on \(M\), and named a \(\beta\)-conformal change. \[1\] In fact, when \(\beta = 0\), the change (1.1) reduces to a conformal change. When \(\sigma = 0\), it reduces to a \(\beta\)-change if \(L\) is a Finsler metric function and to a Randers change if \(L\) is a Riemannian metric function. In \[1\], we have established the relationships between some important tensors associated with \((M, L)\) and the corresponding tensors associated with \((M, L^*)\). We have also investigated some invariant and \(\sigma\)-invariant tensors.

In this paper, we still consider the \(\beta\)-conformal change (1.1). Under this change we obtain an explicit expression relating the Cartan connection \(CT\) associated to the Finsler manifold \((M, L)\) and the transformed Cartan connection \(CT^*\) associated to the Finsler manifold \((M, L^*)\) (cf. Theorem A). This result generalizes various results of Hashiguchi \[3\], Izumi \[5\], \[6\], Matsumoto \[11\], Shibata \[14\] and others \[15\],...etc. (cf. Remark 3.1). Having established this crucial relation, we draw some consequences and conclusions from our fundamental theorem. We relate the two canonical sprays \(S^r, S^s\) and also the two Cartan nonlinear connections \(N^r_j, N^s_j\). We get the relation between the torsion tensors \(C^r_{ij}, P^r_{ij}, R^r_{ij}\) and the corresponding tensors \(\overline{C}^r_{ij}, \overline{P}^r_{ij}, \overline{R}^r_{ij}\). We terminate the paper by two theorems (cf. Theorems B and C) which, roughly speaking, characterize the \(\beta\)-homothetic change and the vanishing of the difference tensor (between the two Cartan connections) in different cases.

It should finally be noted that many known results are retrieved as special cases from the obtained results as indicated in different places of the work.

NOTATIONS. Throughout the present paper, \((M, L)\) denotes an \(n\)-dimensional \(C^\infty\) Finsler manifold, \((x^i)\) denote the coordinates of any arbitrary point of the base manifold \(M\) and \((y^i)\) a supporting element at the same point. We use the following notations:

\[ \text{In } [1], \text{ we called the change (1.1) a “conformal } \beta\text{-change”, but we think that the name “}\beta\text{-conformal change” is rather the appropriate one for such a change. This is the name that we will always employ.} \]
\[ \partial_i: \text{ partial differentiation with respect to } x^i, \]
\[ \hat{\partial}_i: \text{ partial differentiation with respect to } y^i, \]
\[ g_{ij} := \frac{1}{2} \hat{\partial}_i \hat{\partial}_j L^2 = \hat{\partial}_i \hat{\partial}_j E: \text{ the fundamental metric tensor,} \]
\[ L_i := \hat{\partial}_i L = g_{ij} L^j := g_{ij} \frac{\partial}{\partial x^j}: \text{ the normalized supporting element,} \]
\[ h_{ij} := L \hat{\partial}_i L_j = LL_{ij} = g_{ij} - L_i L_j: \text{ the angular metric tensor,} \]
\[ C_{ijk} := \frac{1}{2} \hat{\partial}_k (g_{ij}) = \frac{1}{2} \hat{\partial}_i \hat{\partial}_j \hat{\partial}_k E, \]
\[ C^i_{jk} := g^{ih} C_{hjk}: \text{ the (h)hv-torsion tensor,} \]
\[ \Gamma_{jk}: \text{ the coefficients of the Cartan connection } \Gamma, \]
\[ N^i_j := (y^h \Gamma^i_{jk}): \text{ the coefficients of the canonical nonlinear connection,} \]
\[ C := (\Gamma^i_{jk}, N^i_j, C^i_{jk}): \text{ the Cartan connection associated to } (M, L), \]
\[ \delta_k := \partial_k - N^i_k \hat{\partial}_i, \]
\[ X^i_{jk} := \delta_k X^i_j + X^i_j \Gamma^i_{rk} - X^i_r \Gamma^i_{jk}: \text{ the horizontal or (h)-covariant derivative of } X^i_j. \]

Contraction by \( y^i \) will be denoted by the index 0. For example, we write \( \Gamma^i_{0j} \) for \( \Gamma^i_{k0j}y^k \).

### 2. Basic tensors associated to a \( \beta \)-conformal change

In this section we introduce some basic tensors associated to a \( \beta \)-conformal change. Consider the \( \beta \)-conformal change (2.1). The relation between the associated normalized covariant supporting elements is given by:

\[ T_i(x, y) = e^\sigma(x) L_i(x, y) + b_i(x). \]  
(2.1)

Consequently, if we write \( L_{ij} := \hat{\partial}_j L_i, \ L_{ijk} := \hat{\partial}_k L_{ij}, \ldots \) etc., we get

\[ T_{ij}(x, y) = e^\sigma(x) L_{ij}(x, y). \]  
(2.2)

\[ T_{ijk}(x, y) = e^\sigma(x) L_{ijk}(x, y). \]  
(2.3)

\[ T_{ijkl}(x, y) = e^\sigma(x) L_{ijkl}(x, y). \]  
(2.4)

It is to be noted that Equation (2.2) is equivalent to (2.1); it is then a characterization of \( \beta \)-conformal changes. It is clear that \( L_{ij} = \frac{h_{ij}}{L} \) is \( \sigma \)-invariant (A tensor \( K \) is \( \sigma \)-invariant if \( \overline{K}(x, y) = e^\sigma K(x, y) \) under the \( \beta \)-conformal change (2.1)).

**Lemma 2.1.** Under a \( \beta \)-conformal change \( \overline{T}(x, y) = e^\sigma(x) L(x, y) + \beta(x, y) \), the relation between the fundamental metric tensors \( g_{ij} \) and \( \overline{g}_{ij} \) is given by:

\[ \overline{g}_{ij} = \tau (g_{ij} - L_i L_j) + \overline{T}_i \overline{T}_j \]

and the relation between the corresponding covariant components \( g^{ij} \) and \( \overline{g}^{ij} \) is given by:

\[ \overline{g}^{ij} = \tau^{-1} g^{ij} + \mu l^i l^j - \tau^{-2} (l^i b^j + l^j b^i), \]  
(2.5)

where \( \mu = (e^\sigma L b^2 + \beta) / L \tau^2, \ \tau = e^\sigma L, \ b^2 = b_i b^i \) and \( b^i = g^{ij} b_j \).
Differentiation the angular matrix $h_{ij}$ with respect to $y^k$, we get

$$\dot{h}_{ij} = 2C_{ijk} - L^{-1}(L_i h_{jk} + L_j h_{ik}),$$  \hspace{1cm} (2.6)$$

from which we obtain

$$L_{ijk} = \frac{2}{L} C_{ijk} - \frac{1}{L^2}(h_{ij}L_k + h_{jk}L_i + h_{ki}L_j).$$  \hspace{1cm} (2.7)$$

Taking (2.7) into account, (2.3) can be rewritten in terms of $C_{ijk}$ in the form

$$\overline{C}_{ijk} = \tau [C_{ijk} + \frac{1}{2L} h_{ij}],$$  \hspace{1cm} (2.8)$$

where $h_{ijk} = h_{ij} m_k + h_{jk} m_i + h_{ki} m_j$ and $m_i = b_i - \frac{\beta}{L} L_i$.

(Note that $m_0 = m_i y^i = 0.$)

Using (2.6) and (2.8) we get the following

**Lemma 2.2.** Under a $\beta$-conformal change $\overline{L}(x, y) = e^{\sigma(x)} L(x, y) + \beta(x, y)$, the relation between the (h)hv-torsion tensors $C_{ijk}$ and $\overline{C}_{ijk}$ has the form:

$$\overline{C}_{ijk} = C_{ijk} + A_{ijk},$$

where

$$A_{ijk} = \frac{1}{2L} (h_{ij} m^i + h_{ij} m_k + h_{ij} m_k) - \frac{1}{\tau} C_{jkr} L^i b^s - \frac{1}{2L^2} (2m_j m_k + m^2 h_{jk}) L^i, \hspace{1cm} (2.9)$$

$m^i = g^{ik} m_k$ and $h_{ij} := g^{ik} h_{jk}$.

Differentiation both sides of (2.7) with respect to $y^h$, we get

$$L_{hijk} = \frac{2}{L} \dot{h}_{ij} C_{hij} - \frac{2}{L^2}(L_h C_{ijk} + L_i C_{hjk} + L_j C_{hik} + L_k C_{hij})$$

$$- \frac{1}{L^3} (h_{hj} h_{ik} + h_{kj} h_{ih} + h_{kij})$$

$$+ \frac{2}{L^3} (h_{hi} L_j L_k + h_{hj} L_k L_i + h_{ki} L_j L_h + h_{ij} L_k L_h + h_{jk} L_i L_h + h_{kj} L_j L_h).$$

Taking the above equation into account, (2.4) gives the relation between $\dot{h}_{ij} C_{hij}$ and $\dot{h}_{ij} C_{hij}$:

**Lemma 2.3.** Under a $\beta$-conformal change $\overline{L}(x, y) = e^{\sigma(x)} L(x, y) + \beta(x, y)$, we have

$$\dot{h}_{ij} C_{hij} = \tau \dot{h}_{ij} C_{hij} + \frac{e^{\sigma}}{L} C_{ijk} m_r + \mathcal{G}_{i,j,k} [\frac{e^{\sigma}}{L} C_{ijr} m_k - \frac{e^{\sigma}}{2L^2} (h_{ij} (n_{rk} + \frac{\beta}{L^r} h_{rk}) + h_{ir} n_{jk})],$$

where $n_{rk} = m_r l_k + m_k l_r$ and $\mathcal{G}_{i,j,k}$ denotes cyclic permutation on the indices $i,j,k$. 

4
3. Cartan connections associated to a $\beta$-conformal change

This section is devoted to the determination of the relationship between the Cartan connection $C$ associated to $(M, L)$ and the Cartan connection $C$ associated to $(M, \overline{L})$, under a $\beta$-conformal change $L(x, y) \to \overline{L}(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y)$.

Let $D_{jk}^i$ be the difference tensor between the Cartan connection coefficients $\Gamma_{jk}^i$ and $\overline{\Gamma}_{jk}^i$:

$$D_{jk}^i = \overline{\Gamma}_{jk}^i - \Gamma_{jk}^i \quad \text{(3.1)}$$

Now, we are going to determine an explicit expression of $D_{jk}^i$. We will do so in three steps. Firstly, we determine $D_{00}^i = D_{jk}^i y^j y^k$, then $D_{0j}^i = D_{jk}^i y^j$ and finally $D_{ij}^i$. Here is our fundamental result:

**Theorem A.** Under a $\beta$-conformal change $\overline{L}(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y)$, the relationship between the Cartan connection $\overline{\Gamma}_{jk}^r$ of $(M, \overline{L})$ and the Cartan connection $\Gamma_{jk}^r$ of $(M, L)$ is given by Equations (3.30) below.

**Proof.** As we have said, we proceed as follows: we firstly compute $D_{00}^i$, then $D_{0j}^i$ and finally $D_{ij}^i$.

- **Determination of $D_{00}^i = D_{jk}^i y^j y^k$:**

  From (2.2), we get

  $$\partial_k \overline{L}_{ij} = \partial_k (e^{\sigma}L_{ij}), \quad \text{(3.2)}$$

  since

  $$L_{ij|k} = \partial_k L_{ij} - L_{ijr} N_{k}^{r} - L_{rj} \Gamma_{ik}^{r} - L_{ri} \Gamma_{jk}^{r} \quad \text{(3.3)}$$

  In virtue of $L_{ij|k} = 0$, (3.3) implies

  $$\partial_k L_{ij} = L_{ijr} N_{k}^{r} + L_{rj} \Gamma_{ik}^{r} + L_{ri} \Gamma_{jk}^{r}. \quad \text{(3.4)}$$

  Using (3.4) and (3.1), Equation (3.2) yields

  $$L_{ijr} D_{0k}^{r} + L_{rj} D_{ik}^{r} + L_{ri} D_{jk}^{r} = \sigma_k L_{ij}, \quad \text{(3.5)}$$

  where $\sigma_k = \frac{\partial \sigma}{\partial x^k}$.

  Now, differentiating (2.1) with respect to $x^j$, we get

  $$\partial_j \overline{L}_i = \partial_j (e^{\sigma}L_i + b_i) = e^{\sigma} \sigma_j L_i + e^{\sigma} (\partial_j L_i) + \partial_j b_i. \quad \text{(3.6)}$$

  Taking into account the fact that $L_{ij} = 0$ or, equivalently, that $\partial_j L_i = L_{ri} N_j^r + L_r \Gamma_{ji}^r$, we get

  $$e^{\sigma} L_{ri} D_{0j}^{r} + \overline{L}_r D_{ij}^{r} = e^{\sigma} \sigma_j L_i + b_{ij}. \quad \text{(3.7)}$$

  Equation (3.7) is equivalent to the following two equations:

  $$e^{\sigma} (L_{ir} D_{0j}^{r} + L_{jr} D_{0i}^{r}) + 2 \overline{L}_r D_{ij}^{r} = 2 E_{ij} + e^{\sigma} \mu_{ij}. \quad \text{(3.8)}$$

  $$e^{\sigma} (L_{ir} D_{0j}^{r} - L_{jr} D_{0i}^{r}) = 2 F_{ij} - e^{\sigma} \mu_{ij}, \quad \text{(3.9)}$$
where
\[ E_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad F_{ij} = \frac{1}{2}(b_{ij} - b_{ji}), \]
\[ \sigma_{ij} = \sigma_i L_j + \sigma_j L_i, \quad \mu_{ij} = \sigma_i L_j - \sigma_j L_i. \]  \hspace{1cm} (3.10)

On the other hand, Equation \((3.5)\) is equivalent to
\[ 2L_{jr} D_{r0k} + L_{ijk} D_{r0i} - L_{ikr} D_{r0j} = \sigma_i L_{jk} + \sigma_k L_{ij} - \sigma_j L_{ik}. \]  \hspace{1cm} (3.11)

Contracting \((3.8)\) by \(y_j\), we get
\[ e\sigma_{ij} L_{r} D_{r00} + 2L_{jr} D_{r0i} = 2E_{0i} + e\sigma_{ij} \sigma_{0} L_{ij} - \sigma_{j} L_{ik}. \]  \hspace{1cm} (3.12)

Similarly, from \((3.9)\) and \((3.11)\), we get
\[ e\sigma_{ij} L_{r} D_{r00} = 2F_{i0} + e\sigma_{ij} \sigma_{0} L_{ij} - \sigma_{j} L_{ik}. \]  \hspace{1cm} (3.13)

Again, contracting \((3.12)\) by \(y_i\) gives
\[ L_{r} D_{r00} = E_{00} + e\sigma_{ij} \sigma_{0} L. \]  \hspace{1cm} (3.15)

Equations \((3.13)\) and \((3.15)\) can be written as a system of algebraic equations in \(D_{r00}\):

\[ \begin{align*}
(i) & \quad L_{r} D_{r00} = 2e^{-\sigma} F_{i0} + (\sigma_0 L_{i} - \sigma_{i} L) = B_i, \\
(ii) & \quad L_{r} D_{r00} = E_{00} + e\sigma_{ij} \sigma_{0} L = B.
\end{align*} \]  \hspace{1cm} (3.16)

To determine the tensor \(D_{r00}\), we need the following lemma of Matsumoto \[11\]:

**Lemma 1.** Given \(B\) and \(B_i\) such that \(B_i L^i = 0\), the system of algebraic equations in \(A_r\)

\[ \begin{align*}
(i) & \quad L_{ri} A^r = B_i, \\
(ii) & \quad (L_{r} + b_{r}) A^r = B,
\end{align*} \]

has the unique solution
\[ A^r = LB^r + \frac{L}{L + \beta} (B - LB_{\beta}) L^r, \]

where \(B_{\beta} = B_{\beta} b^i\).

Now, applying Lemma 1 on the system \((3.16)\), noting that \(B_i y^i = 0\), we obtain an explicit expression for the required tensor \(D_{r00}\):
\[ D_{r00} = 2Le^{-\sigma} F_0^r + \frac{L}{L} (E_{00} - 2Le^{-\sigma} F_{\beta 0}) L^r - L^2 \sigma^r + \frac{L}{L} (2Le^\sigma \sigma_0 + L^2 \sigma_{\beta}) L^r, \]  \hspace{1cm} (3.17)

where \(\sigma_{\beta} = \sigma_i b^i, F_0^r = g^{ir} F_{i0}\) and \(F_{\beta 0} = F_{i0} b^i\).

- **Determination of** \(D_{0ij} = D_{ij} y^i\):

Adding Equations \((3.9)\) and \((3.14)\), we get
\[ L_{ir} D_{r0j} = e^{-\sigma} F_{ij} - \frac{1}{2} L_{ijr} D_{r00} + \frac{1}{2} (\sigma_0 L_{ij} - \mu_{ij}) =: G_{ij}. \]  \hspace{1cm} (3.18)
Equation (3.12) can be rewritten as

\[ \mathcal{L}_r D_{i0}^r = E_{0i} - \frac{1}{2} e^\sigma L_{ir} D_{00}^r + \frac{1}{2} e^\sigma (\sigma_0 L_i + \sigma_i L) =: G_i. \]  

(3.19)

Substituting (3.17) in (3.18) and (3.13) in (3.19), we get the following expressions for \( G_{ij} \) and \( G_i \):

\[
(i) \ G_{ij} = e^{-\sigma} F_{ij} - e^{-\sigma} LL_{ijr} F_{r0} + \frac{1}{2L} L_{ij} (E_{00} - 2Le^{-\sigma} F_{r0}) - \\
\quad + \frac{1}{2} [L^2 L_{ijr} \sigma^r + \frac{L^2}{L} L_{ij} \sigma_\beta + (\sigma_0 L_{ij} - \mu_{ij})] + \frac{L}{L} e^\sigma \sigma_0 L_{ij},
\]

\[
(ii) \ G_i = E_{0i} - F_{0i} + e^\sigma \sigma_i L.
\]

(3.20)

From Equations (3.16), (3.18) and (3.19), the tensors \( G_{ij} \) and \( G_j \) have the properties:

\[
G_{ij} y^i = 0, \quad G_{ij} y^j = B_i, \quad G_j y^j = B
\]

(3.21)

To determine \( D_{0j}^r \) we need the following lemma, which is a generalized version of Matsumoto’s lemma:

**Lemma 2.** Given the tensor \( G_{ij} \) and \( G_j \), with the properties (3.21), the system of algebraic equations in \( D_{0j}^r \)

\[
(i) \ L_{ri} D_{0j}^r = G_{ij} \quad \quad (ii) \ L_r D_{0j}^r = G_j.
\]

(3.22)

has the unique solution

\[
D_{0j}^i = LG_{j}^i + \frac{L}{L} (G_j - LG_{\beta j}) L^i,
\]

(3.23)

where \( G_{j}^i = g^{ir} G_{rj} \) and \( G_{\beta j} = b^i G_{ij} \).

Proof of Lemma 2. It follows, from the formula \( g_{ij} = h_{ij} + L_i L_j \), that (i) of (3.22) can be rewritten as

\[
g_{ir} D_{0j}^r = LG_{ij} + L_i L_r D_{0j}^r
\]

(3.24)

Contact of (3.24) by \( b^i \) gives

\[
b_r D_{0j}^r = LG_{\beta j} + \frac{\beta}{L} L_r D_{0j}^r
\]

(3.25)

On the other hand, taking (3.25) into account, (ii) of (3.22) is rewritten as

\[
e^\sigma L_r D_{0j}^r = G_j - b_r D_{0j}^r = G_j - [LG_{\beta j} + \frac{\beta}{L} L_r D_{0j}^r]
\]

which gives

\[
L_r D_{0j}^r = \frac{L}{L} [G_j - LG_{\beta j}]
\]

(3.26)

Substitution of (3.26) into (3.24) ends the proof of the lemma.

The required tensor \( D_{0j}^r \) is determined by Equations (3.23).
• Determination of $D^r_{ij}$:

Equations (3.11) and (3.8) can be rewritten respectively as

\[ L_{ir} D^r_{jk} = 1/2 \left( L_{jkr} D^r_{0i} - L_{ijk} D^r_{0k} - L_{ikr} D^r_{0j} + \sigma_j L_{ik} + \sigma_k L_{ij} - \sigma_i L_{jk} \right) =: H_{ijk}, \]

\[ L_{ir} D^r_{ij} = E_{ij} - 1/2 e^\sigma (L_{ir} D^r_{0j} + L_{jr} D^r_{0i} - \sigma_{ij}) =: H_{ij}, \]  
(3.27)

From Equations (3.22) and (3.27), the tensors $H_{ijk}$ and $H_{jk}$ possess the properties:

\[ H_{ijk} = H_{ikj}, \quad H_{ijk} y^i = 0, \quad H_{ijk} y^j = G_{ik}, \quad \text{and} \quad H_{jk} y^j = G_k. \]  
(3.28)

Finally, to determine $D^r_{jk}$ we need the next lemma, which generalizes Lemma 2 and can be proved similarly:

**Lemma 3.** Given the tensor $H_{ijk}$ and $H_{jk}$, with the properties (3.28), the system of algebraic equations in $D^r_{jk}$

\[ L_{ir} D^r_{jk} = H_{ijk}, \quad L_{ir} D^r_{jk} = H_{jk}, \]  
(3.29)

has the unique solution

\[ D^i_{jk} = L H^i_{jk} + \frac{L}{L} (H_{jk} - L H_{\beta jk}) L^i, \]  
(3.30)

where $H^i_{jk} = g^{ir} H_{rjk}$ and $H_{\beta jk} = b^i H_{ijk}$.

Applying Lemma 3 on the system (3.27) having the properties (3.28), we obtain the expression (3.30) for the required tensor $D^i_{jk}$, which completes the proof of Theorem 3.1. □

It should be noted that the difference tensor $D^r_{ij}$ determined by (3.30) is explicitly expressed in terms of the constituents of the $\beta$-conformal change (1.1) only.

**Remark 3.1.** Consider the $\beta$-conformal change (1.1):

\[ L(x, y) \rightarrow \bar{L}(x, y) = e^{\sigma(x)} L(x, y) + \beta(x, y). \]

- When the $\beta$-conformal change (1.1) is conformal ($\beta = 0$), the difference tensor $D^r_{jk}$ takes the form:

\[ D^r_{jk} = L^2 (C^m_{jm} C^r_{mk} - C^m_{jm} C^r_{km} - C^m_{km} C^m_{jm}) - (C^r_{jk} \sigma_0 - C^m_{jm} y^j - C^r_{jm} y^j + C_{jm} y^r) \]
\[ + (\delta^r_k \sigma_j + \delta^r_j \sigma_k - g_{jk} \sigma^r), \]  
(3.31)

where $C^r_{jk} = C^m_{kj} \sigma^j$ and $C_{rk} = C^m_{rk} \sigma_j$.

This is the case studied by Hashiguchi [4], Izumi ([5], [6]) and others [3], [17], ...etc.

- When the $\beta$-conformal change (1.1) is $C$-conformal ($\beta = 0$ and $C^r_{jk} \sigma^j = 0$), the difference tensor $D^r_{jk}$ takes the form:

\[ D^r_{jk} = \delta^r_k \sigma_j + \delta^r_j \sigma_k - g_{jk} \sigma^r - C^r_{jk} \sigma_0. \]  
(3.32)

This is the case studied by Shibata and Azuma [15], Kim and Park [7] and others.
• When the $\beta$-conformal change (1.1) is $h$-conformal ($\beta = 0$ and $C^r_{jk}\sigma_r = \frac{c^i_{\sigma_i}}{(n-1)}h_{jk}$), the difference tensor $D^r_{jk}$ takes the form:

$$D^r_{jk} = \delta^r_k \rho_j + \delta^r_j \rho_k - g_{jk} \rho^r - C^r_{jk} \rho - \frac{c^i_{\sigma_i}}{(n-1)}LL^r L_j L_k,$$

(3.33)

where $\rho_j := \sigma_j + L \frac{c^i_{\sigma_i}}{(n-1)}L_j$ and $\rho = \rho_j y^j$.

This is the case studied by Izumi [6].

• When the $\beta$-conformal change (1.1) is a Randers change ($\sigma = 0$ and $L$ is Riemannian), the difference tensor $D^i_{jk}$ takes the form:

$$D^i_{jk} = LH^i_{jk} + \frac{L}{L}(H_{jk} - LH_{\beta, jk})L^i,$$

(3.34)

where $H_{ij} = E_{ij} - \frac{1}{2}(G_{ij} + G_{ji})$,

$$H_{ijk} = \frac{1}{2}(L_{jkr} D^r_{0i} - L_{ijr} D^r_{0k} - L_{ikr} D^r_{0j}),$$

$$D^r_{0i} = LG^r_{\xi, i} + \frac{L}{L}(G_i - LG_{\beta i})L^r,$$

with $G_{ij}$ and $G_i$ given by Equation 4.3 below.

This is the case studied by Matsumoto [11], Shibata, Shimada et al [16] and others [8],...etc

• When the $\beta$-conformal change (1.1) is a $\beta$-change ($\sigma = 0$ and $L$ is Finslerian), the difference tensor $D^i_{jk}$ takes the form (3.33), with $G_{ij}$ and $G_i$ given by Equation 4.4 below.

This is the case studied by Shibata [14], Matsumoto [11] and others.

The above discussion shows that our consideration is much more general than various investigations existing in the literature.

4. Consequences of the fundamental theorem

Having obtained an explicit expression for the difference tensor $D^r_{jk}$, we are in a position to draw some consequences and conclusions from our fundamental theorem (Theorem A).

Proposition 4.1. Under a $\beta$-conformal change, the following relations hold:

$$G_{ij} = \frac{1}{2}L_{ir}(\dot{\sigma}_j D^r_{00}) \quad \text{and} \quad G_j = \frac{1}{2}L_{r}(\dot{\sigma}_j D^r_{00})$$

(4.1)

Proof. From equation (3.13), we have

$$\dot{\sigma}_j(L_{ir} D^r_{00}) = \dot{\sigma}_j(2e^{-\sigma}F_{i0} + (\sigma_0 L_i - \sigma_i L))$$

which gives

$$L_{ir}(\dot{\sigma}_j D^r_{00}) = 2e^{-\sigma}F_{ij} - L_{ijr} D^r_{00} + (\sigma_0 L_{ij} - \mu_{ij}) = 2G_{ij}.$$
Similarly, the second relation is obtained from Equation (3.15).

As a consequence of the above Proposition, taking (3.22) into account, we get

$$\dot{\partial}_j D_{00} = 2D_{0j}^r$$

**Proposition 4.2.** The tensor $G_{ij}$ can be written in a form free from the difference tensor $D_{ij}$ as follows:

$$G_{ij} = e^{-\sigma}F_{ii} + \frac{e^{-\sigma}}{L}(L_i F_{j0} + L_j F_{i0}) - 2e^{-\sigma}C_{ij}^k F_{k0} + G_{ij}$$

$$+ \frac{1}{2} [L^2 L_{ij}, \sigma^r + \frac{L^2}{L} L_{ij} \sigma^\beta + (\sigma_0 L_{ij} - \mu_{ij})] + \frac{L}{L} e^\sigma \sigma_0 L_{ij},$$

where we have put $G = \frac{1}{2L^2}(E_{00} - 2Le^{-\sigma}F_{j0})$.

In the case of a Randers metric, we retrieve the Matsumoto’s tensor [11]:

$$G_{ij} = F_{ij} + \frac{1}{L}(L_i F_{j0} + L_j F_{i0}) + Kh_{ij} \text{ and } G_i = E_{0i} - F_{0i},$$

where $K = \frac{1}{2L^2}(E_{00} - 2LF_{j0})$.

In the case of a generalized Randers metric or a $\beta$-metric, we get

$$G_{ij} = F_{ij} + \frac{1}{L}(L_i F_{j0} + L_j F_{i0}) - 2C_{ij}^k F_{k0} + Kh_{ij} \text{ and } G_i = E_{0i} - F_{0i}.$$ (4.4)

**Proposition 4.3.** Under a $\beta$-conformal change $L \to \overline{L} = e^\sigma L + \beta$, the canonical spray $\overline{S}^r$ and the Cartan nonlinear connection $\overline{N}^r_j$ of $(M, \overline{L})$ can be determined in terms of the corresponding objects of $(M, L)$ in the form:

$$\overline{S}^r = S^r + D_{00}^r,$$

$$\overline{N}^r_j = N^r_j + D_{0j}^r,$$

where $D_{00}^r$ and $D_{0j}^r$ are given by (3.17) and (3.23) respectively.

It should be noted that the expression, relating the two nonlinear connections, found by Hashiguchi [11] results as a special case from the above proposition (by setting $\beta = 0$).

**Proposition 4.4.** Under a $\beta$-conformal change, the torsion tensors change as follows:

(a) $\overline{C}^i_{jk} = C^i_{jk} + A^i_{jk},$

(b) $\overline{P}^i_{jk} = P^i_{jk} - D^i_{jk} + B^i_{jk},$

(c) $\overline{R}^i_{jk} = R^i_{jk} + \Omega_{ijk}(D^i_{0j} - (B^i_{jr} + P^i_{jr})D^r_{0k}),$

where $A^i_{jk}$, $D^i_{0j}$ and $D^i_{jk}$ are given by (2.9), (3.23) and (3.30) respectively, $B^i_{jk} := \dot{\partial}_k D^i_{0j}$ and $\Omega_{ijk}(Q_{jk}) := Q_{jk} - Q_{kj}$.
Proposition 4.5. The relation between the \((v)hv\)-torsion tensors \(P_{hjk}\) and \(P_{hjk}\) can be written in the form
\[
P_{hjk} = \tau P_{hjk} - \frac{\tau L}{2} \left[ L_{hjk} D_{00}^r + L_{hjr} D_{0h}^r + L_{hkr} D_{0j}^r + L_{jkr} D_{0h}^r - \sigma_0 L_{hjk} \right].
\]

**Proof.** By Proposition 4.4(b), taking Equation (3.30) and the expressions \(B_{ij} := \dot{\partial}_k D_{0j}^i\) and \(\overline{L} \overline{P}_{jk} \overline{C}_{jk}^0 = 0\) into account, we get
\[
P_{hjk} = g_{ih} P_{ijk} = \left[ \tau (g_{ih} - L_i L_h) + \overline{L}_i \overline{L}_k \right] \overline{P}_{jk}
\]
\[
= \tau (g_{ih} - L_i L_h) [P_{ijk} + \dot{\partial}_k D_{0j}^i - D_{jk}^i]
\]
\[
= \tau P_{hjk} + \tau LL_{ih} (\dot{\partial}_k D_{0j}^i - D_{jk}^i).
\]

Differentiation \(L_{ir} D_{0j}^i\) with respect to \(y^k\) and using equations (3.18), we obtain
\[
\dot{\partial}_k (L_{ir} D_{0j}^i) = L_{ir} D_{0j}^i + L_{ir} \dot{\partial}_k D_{0j}^i \quad \text{and} \quad L_{ih} \dot{\partial}_k D_{0j}^i = \dot{\partial}_k G_{hj} - L_{ihk} D_{0j}^h
\]
Therefore, \(P_{hjk}\) can be rewritten as
\[
P_{hjk} = \tau P_{hjk} + \tau L (\dot{\partial}_k G_{hj} - L_{hkr} D_{0j}^r - H_{hjk}). \quad (4.6)
\]
Now, by Proposition 4.1 and Equation (3.18), one can calculate \(\dot{\partial}_k G_{hj}\) which will be of the form
\[
\dot{\partial}_k G_{hj} = -\frac{1}{2} L_{hjk} D_{00}^r - L_{hjr} D_{0r}^r + \frac{1}{2} [\sigma_0 L_{hjk} + \sigma_k L_{hj} - \sigma_h L_{jk} + \sigma_j L_{hk}].
\]
Substituting the above equation into (4.6), we get the result. □

It should be noted that Proposition 4 of Matsumoto [11] results as a special case from the above proposition (by setting \(\sigma = 0\) and letting \(L\) be Riemannian).

**Theorem B.** Under a \(\beta\)-conformal change \(L \rightarrow \overline{L} = e^\sigma L + \beta\), consider the following two assertions:
(i) The covariant vector \(b_i\) is parallel with respect to the Cartan connection \(CT\).
(ii) The difference tensor \(D_{ij}^0\) vanishes identically.
Then, we have
(a) If (i) and (ii) hold, then \(\sigma\) is homothetic.
(b) If \(\sigma\) is homothetic, then (i) and (ii) are equivalent.

**Proof.**
(a) If \(b_{ij} = 0\), then \(E_{ij} = 0 = F_{ij}\), by (3.10). This, together with (3.16), imply
\[
\sigma_0 L_i - \sigma_i L = L_{ir} D_{00}^r,
\]
\[
e^\sigma \sigma_0 L = \overline{L}_r D_{00}^r.
\]
Now, if, moreover, \(D_{ij}^0 = 0\), then \(D_{00}^r = 0\), and so the above two equations give
\[
\sigma_0 L_i - \sigma_i L = 0,
\]
\[
\sigma_0 L = 0.
\]
These two equations, together, yield $\sigma_i = 0$ and $\sigma$ is homothetic.

(b) Let $\sigma_i = 0$ and $b_{ij} = 0$. Then, it follows from (3.17) that $D^r_{00} = 0$. This, together with (3.18) and (3.19), imply that $G_{ij} = 0$ and $G_i = 0$. Consequently, $D^r_{0j} = 0$, by (3.23). Again, this, together with (3.27), imply that $H_{ijk} = 0$ and $H_{ij} = 0$. Consequently, $D^r_{ij} = 0$, by (3.30).

On the other hand, let $\sigma_i = 0$ and $D^r_{ij} = 0$. Then, (3.18) implies that $F_{ij} = 0$ and (3.27)(ii) implies that $E_{ij} = 0$. Consequently, it follows from (3.10) that $b_{ij} = 0$. □

As a consequence of the above theorem, we have the following interesting special cases which retrieve some results of [3], [14], and [17]:

**Theorem C.**

(i) Let the $\beta$-conformal change $L \rightarrow \overline{L} = e^\sigma L + \beta$ be conformal ($\beta = 0$), then $D^r_{ij}$ vanishes identically if and only if $\sigma$ is homothetic.

(ii) Let the $\beta$-conformal change $L \rightarrow \overline{L} = e^\sigma L + \beta$ be a $\beta$-change ($\sigma = 0$), then $D^r_{ij}$ vanishes identically if and only if $b_i$ is Cartan-parallel.

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