Higher order superintegrability, Painlevé transcendents and representations of polynomial algebras

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Résumé. In recent years, progress toward the classification of superintegrable systems with higher order integrals of motion has been made. In particular, a complete classification of all exotic potentials with a third or a fourth order integrals, and allowing separation of variables in Cartesian coordinates. All doubly exotic potentials with a fifth order integral have also been completely classified. It has been demonstrated how the Chazy class of third order differential equations plays an important role in solving determining equations. Moreover, taking advantage of various operator algebras defined as Abelian, Heisenberg, Conformal and Ladder case of operator algebras, we re-derived these models. These new techniques also provided further examples of superintegrable Hamiltonian with integrals of arbitrary order. It has been conjectured that all quantum superintegrable potentials that do not satisfy any linear equation satisfy nonlinear equations having the Painlevé property. In addition, it has been discovered that their integrals naturally generate finitely generated polynomial algebras and the representations can be exploited to calculate the energy spectrum. For certain very interesting cases associated with exceptional orthogonal polynomials, these algebraic structures do not allow to calculate the full spectrum and degeneracies. It has been demonstrated that alternative sets of integrals which can be build and used to provide a complete solution. This this allow to make another conjecture i.e. that higher order superintegrable systems can be solved algebraically, they require alternative set of integrals than the one provided by a direct approach.

1. Introduction

The connection between the n-dimensional Kepler-Coulomb and harmonic oscillator and Lie algebras $su(n)$ and $so(n+1)$ have been central to the development of algebraic techniques in physics and study of exactly solvable models [34, 6, 59, 92, 5, 64, 91, 4, 83, 65, 10, 7, 35, 56]. They also motivated application of finite dimensional unitary representations and Casimir operators and development of algebraic methods to study Hamiltonians. These two models are exactly solvable but as well superintegrable. A systematic search for two-dimensional superintegrable Hamiltonians with two second order integrals was started in the 60ties by Winternitz [36]. Many of their properties have been discovered for these models such as multiseparability, exact solvability and degenerate spectrum. Over the years, a complete classification of quadratically superintegrable systems on two-dimensional conformally flat space was performed [82]. Progress has been reported for the three dimensional case [32].

A program to study superintegrable systems with a second order integrals and another higher order integrals of Nth order is more recent and has been started fifteen years ago. A first results
consisted in obtaining the determining equations \((N=3)\) from a third order integral \([48]\) and classifying all Hamiltonians allowing separation of variable in Cartesian coordinates with a third order integrals \([49]\). A connection with Painlevé transcendent was also established for the first time and demonstrated the significance of studying higher order superintegrability.

Much recently, it has been recognized how the search can be narrowed to exotic potentials i.e. the potential do not satisfy any linear differential equation. All exotic potentials for integrals of fourth order \((N=4)\) \([67]\) and doubly exotic systems with integral of fifth order \((N=5)\) \([2]\) were obtained. These potentials can be rewritten in term of the first, second, third, fourth and fifth Painlevé transcendent. Superintegrable systems allowing separation of variables in parabolic coordinates for \(N=3\) \([88]\), as well as systems allowing separation of variables in polar coordinates for \(N=3\) \([94]\) and \(N=4\) \([26, 30]\). Models were obtained in terms of the sixth Painlevé transcendent. Some work have been done in regard of the case of \(N\)th order integrals related to superintegrable systems with separation of variables in Cartesian coordinates \([60]\) and \([89]\). In particular various alternative form for the integrals of motion have been obtained. Similar work for \(N\)th order integrals have also been done in regard of polar coordinates \([31]\).

Recently alternative approach based on one-dimensional operator algebras have been used to obtained new superintegrable systems allowing separation of variable in Cartesian coordinates with integrals of arbitrary order \([68]\). Systems involving Painlevé transcendents with integrals of arbitrary order have been generated.

These results had lead to formulate a conjecture that for all maximally superintegrable systems with exotic potentials in quantum mechanics the corresponding nonlinear differential equations possess the Painlevé property \([69]\).

Another program has been started in regard of superintegrable systems which is to construct their symmetry algebra and calculate their energy spectrum algebraically via representations. It has been discovered more than 25 years ago \([46, 26]\) that the integrals of motion generate naturally quadratic algebra and in particular the case of the Racah algebra \([40]\). Since they have been applied widely to obtain energy spectrum of superintegrable systems. It has been observed that higher rank quadratic algebra can be exploited \([55, 27, 63, 93]\) as well. Example of polynomial algebras find applications \([9, 60, 71, 58]\) and in particular the cubic algebras \([72, 73]\) in regard of fourth Painlevé transcendent models. A puzzling phenomena has been noticed \([73]\) consisting in incomplete algebraic description of the spectrum and the degeneracies for certain values of the parameters of the fourth Painlevé transcendent.

The story of that problem took an interesting turn as it has been shown that it was connected \([78, 79, 80, 81]\) with exceptional orthogonal polynomials \([90, 41, 42, 43, 85, 86, 44, 45]\). New integrals can be created via Krein-Adler and Darboux-Crum for k-step extention models and allowed as well a complete algebraic description of the levels and degeneracies of the fourth Painlevé transcendent Hamiltonian.

The purpose of this paper is two fold. First, this paper intend to review recent results on the classification of superintegrable systems \([67, 68, 69]\), Painlevé property of quantum superintegrable and connection with Chazy class of equations. Secondly, we will describe algebraic solution for the fourth Painleve transcendent via results on Darboux-Crum and Krein-Adler chains. The paper is organized as follow in section 2, we recall results on the quadratic and more generally polynomial algebras. In section 3, we review in particular classification using a direct approach for \(N=3\), \(N=4\) and cubic algebra in one example. In Section 4, we review one dimensional operator algebra method introduced recently. In Section 5, we present discussion of how ladder operator can be factorized in term of supercharge which allow to obtain the wavefunction for case in term of fourth and fifth Painleve and apply to an example together with cubic algebra of section 4 to point out the incomplete solution. In section 6, we discuss how ladder operator are in fact not unique and one can find for a given Hamiltonian alternative ladder which allow to obtain alternative set of integrals.
2. Polynomial algebras and superintegrable systems

An Hamiltonian system on a n-dimensional space (generally on Riemannian manifold) with Hamiltonian $H$

\[ H = \frac{1}{2} g^{ik} p_i p_k + V(\vec{x}) \]

is integrable if it allows n integrals of motion that are well defined, in involution \( \{ H, X_a \}_p = 0 \), \( \{ X_a, X_b \}_p = 0 \), a,b=1,...,n-1 and functionally independent. A system is superintegrable if it admits \( n + k \) (with \( k = 1, ..., n - 1 \)) functionally independent constants of the motion (well defined). Maximally superintegrable if \( k = n - 1 \). In quantum mechanics \( \{ H, X_a, Y_b \} \) are well defined quantum mechanical operators and form an algebraically independent set.

\[ H = \frac{1}{2} p^2 + V(x, y) \]

was considered and two integrals admit the general form

\[ A = \sum_{i,k=1}^{2} \{ f^{ik}(x,y), p_i p_k \} + \sum_{i=1}^{2} g^i(x,y)p_i + \phi(x,y), j = 1, 2. \]

\[ B = \sum_{i,k=1}^{2} \{ v^{ik}(x,y), p_i p_k \} + \sum_{i=1}^{2} w^i(x,y)p_i + \psi(x,y), j = 1, 2. \]

The quadratic algebra take the form, it includes the Racah algebra as a special case

\[ [A, B] = C \]  
\[ [A, C] = \alpha A^2 + \gamma \{ A, B \} + \delta A + \epsilon B + \zeta \]  
\[ [B, C] = a A^2 - \gamma B^2 - \alpha \{ A, B \} + d A - \delta B + z. \]

It admit one Casimir operator which is cubic in the generator

\[ K = C^2 - \alpha \{ A^2, B \} - \gamma \{ A, B^2 \} + (\alpha \gamma - \delta) \{ A, B \} + (\gamma^2 - \epsilon) B^2 \]

\[ + (\gamma \delta - 2 \zeta) B + \frac{2a}{3} A^3 + (d + \frac{\alpha \gamma}{3} \alpha^2) A^2 + (\frac{ae}{3} + a \delta + 2\zeta) A. \]

It has been shown it can be exploited to obtain finite dimensional unitary representations of the quadratic algebras. Recent results have shown the importance of that algebra in the classification of two dimensional quadratically superintegrable system on conformally flat spaces. The complex spaces admitting at least three 2nd order symmetries, flat space, complex 2-sphere, the four Darboux spaces, eleven 4 parameter Koenigs spaces. There are 59 2nd order superintegrable systems in 2D, under the Stackel transform, the systems divide into 12 equivalence classes. 6 with nondegenerate 3-parameter potentials (S9,E1,E2,E30,E8,E10), 6 with degenerate 1-parameter potentials (S3,E3,E4,E5,E6,E14), all these systems are related via process of contraction of quadratic algebra related to $e(2, C)$ and $o(3, C)$ along the idea of Wigner-Inonu. These contraction allow to related to the full askey scheme of orthogonal polynomials. More generally Bocher contraction have been identified \[62, 52, 33\].

Algebraic derivation of spectrum using finite dimensional unitary representation has been extended to classes of superintegrable systems that decomposes into direct sums of higher rank
Lie algebras and quadratic algebras. Their structure constants also depend on central element or even on elements in the centre of the universal enveloping algebra of some Lie algebra. Recently, an embedded type of structure has been discovered for a n-dimensional superintegrable system and even an extension of that higher rank quadratic algebra with structure constant that depend on Casimir operator of a Lie algebra \[18\]. These quadratic algebras of superintegrable systems, are still poorly understood, as their classification remain an open problem as well as systematic study of their representations and properties of their universal enveloping algebra and its centre remains to be obtained. In recent years, the construction of Casimir operator as well as realization as deformed oscillator algebra have been pursued for polynomial algebra generated from underlying second and Nth order integrals of motion \[58\].

Further constraints are imposed from the Jacobi identity on the structure constants. Various commutator and anti-commutator identities were established and as well as recurrence relations. Another algebraic structure which play a role in superintegrability is the deformed oscillator algebras \[26\] \{\(b^\dagger, b, N\)\}

\[
\begin{align*}
[A, B] &= C \\
[A, C] &= \sum_{i=1}^{N} \alpha_i A^3 + \delta B + \epsilon + \beta \{A, B\} \\
[B, C] &= \sum_{i=1}^{N} \lambda_i A^2 + \rho B^2 + \eta B + \sum_{i} \omega_i \{A^i, B\} + \zeta
\end{align*}
\]

(3a) (3b) (3c)

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\[
\begin{align*}
[N, b^\dagger] &= b^\dagger \\
[N, b] &= -b \\
b^\dagger b &= \Phi(N) \\
b b^\dagger &= \Phi(N + 1)
\end{align*}
\]

(4a) (4b) (4c) (4d)

It has been demonstrated that realization as deformed oscillator algebra exist for polynomial algebra of arbitrary order generated by integrals of order 2 and N of the form \(A = A(N)\) and \(B = b(N) + b^\dagger \rho(N) + \rho(N)b\). Explicit expression have been obtained for the quadratic \[26\], cubic \[72, 73\] and quartic case \[74\]. Let us consider the cubic case, which take the form

\[
\begin{align*}
[A, B] &= C \\
[A, C] &= \alpha A^3 + \beta \{A, B\} + \gamma A + \delta B + \epsilon \\
[B, C] &= \mu A^3 + \nu A^2 - \beta B^2 - \alpha \{A, B\} + \xi A - \gamma B + \zeta
\end{align*}
\]

(5a) (5b) (5c)

The Casimir operator
\[ K = C^2 - \alpha \{ A^2, B \} - \beta \{ A, B^2 \} + (\alpha \beta - \gamma) \{ A, B \} + (\beta^2 - \delta) B^2 \]
\[ + (\beta \gamma - 2\epsilon) B + \frac{\mu}{2} A^4 + \frac{2}{3}(\nu + \mu \beta) A^3 + (-\frac{1}{6} \mu \beta^2 + \frac{\beta \nu}{3} + \frac{\delta \mu}{2} + \alpha^2 + \xi) A^2 \]
\[ + (-\frac{1}{6} \mu \beta \delta + \frac{\delta \nu}{3} + \alpha \gamma + 2\zeta) A \]

Two types of realization have been constructed (i.e. for \( \beta = 0 \) and \( \beta \neq 0 \)) and explicit formulae for \( A(N), b(N), \rho(N) \) and \( \Phi(N) \) have been provided. The structure function \( \Phi(N) \) take the form of polynomials of order 4 and 10 with coefficients involving the structure constants of the cubic algebra. These structure constants in general may depend on parameters of the quantum model such as a coupling constant, but can involve the Hamiltonian itself. The algebraic method requires to write the Casimir operator in term of the central elements only which can be found factorized form to the integrals of motion or using advantage of presence inner structures to the integrals in term of supercharges, ladder or shift operators for which the action on wave function take simpler form. In the case of polynomial algebras, contrary to the quadratic case, it has been observed that there are cases for which these algebraic approaches provide the spectrum, but an incomplete description of the deneneracies or even missing levels. These cases are exactly the one related to fourth Painlevé transcendent and specific values of the parameter for which the solution in term of generalized Hermite and Okamoto in fact take the form of exceptional orthogonal polynomials of Hermite of type III.

3. N=3 and N=4 : Exotic potential and Painlevé transcendent

In this section, we will review superintegrable systems on two dimensional Euclidean space separable in Cartesian coordinates with a second and additional integrals which is a third or a fourth order integral of motion. We will discuss how they are connected with Painlevé transcendent [57, 87, 37, 39] and more generally the Chazy class of nonlinear differential equations and their reduction to Painlevé transcendent [17, 22, 23, 24, 25, 11, 12, 13, 14]. The Painlevé transcendent have long history and they have been obtained by Painleve, Gambier and Fush. These works have been can be found in particular where 50 type of equations whose only movable singularities are poles are presented. The most interesting of the fifty types are those which are irreducible and serve to define new transcendent Painlevé transcendent. The other 44 can be integrated in terms of classical functions and transcendent or transformed into the remaining six equations. Several of their properties have been studied over the years in particular of particular solution [11, 19, 20, 21]. They take the following form
\[ P_0''(z) = 6P_1^2(z) + z \quad (9a) \]
\[ P_2''(z) = 2P_2(z)^3 + zP_2(z) + \alpha \quad (9b) \]
\[ P_3''(z) = \frac{P_3''(z)}{P_3(z)} - \frac{P_3'(z)}{z} + \frac{\alpha P_2^2(z) + \beta}{z} + \gamma P_3^3(z) + \frac{\delta}{P_3(z)} \quad (9c) \]
\[ P_4''(z) = \frac{P_4'(z)}{2P_3(z)} + \frac{3}{2} \frac{P_3^2(z)}{P_3(z)} + 4zP_4^2(z) + 2(z^2 - \alpha)P_4(z) + \frac{\beta}{P_4(z)} \quad (9d) \]
\[ P_5''(z) = \left( \frac{1}{2P_3(z)} + \frac{1}{P_5(z) - 1} \right) P_5'(z)^2 - \frac{1}{z} P_5'(z) + \frac{(P_3(z) - 1)^2}{z^2} \left( \frac{P_5'(z) + \delta}{P_5(z)} \right) \quad (9e) \]
\[ P_6''(z) = \frac{1}{2} \left( \frac{1}{P_b(z)} + \frac{1}{P_b(z) - 1} + \frac{1}{P_b(z) - z} \right) P_6'(z)^2 - \frac{1}{z} P_6'(z) + \frac{1}{P_b(z) - z} \left( \frac{P_5'(z) + \delta}{P_5(z)} \right) \quad (9f) \]

Many of their properties have been studied in particular particular solutions of \( P_2 \) to \( P_6 \). They find many applications in various domain of mathematical physics Statistical mechanics, quantum field theory, relativity, Symmetry reduction of various equations (Kdv, Boussineq, Sine-Gordon, Kadomstev-Petviashvile, nonlinear Schrodinger) [20]. The connection with non relativistic quantum mechanics and superintegrable systems is much more recent.

### 3.1. Superintegrable systems with \( N=3 \)

Let us consider the case of superintegrable systems with second order integral \( A \) and third order integrals \( B \)

\[ B = \sum_{i+j+k=3} A_{ijk} \{ L_3^i, P_1^j P_2^k \} + \{ g_1(x,y), p_1 \} + \{ g_2(x,y), p_2 \} \]

The constants \( A_{ijk} \) and functions \( V, g_1 \) and \( g_2 \) are subject to determining equation from the commutator \( [H, B] = 0 \)

\[ g_{1,x} = 3f_1 V_x + f_2 V_y \quad (10a) \]
\[ g_{2,y} = f_3 V_x + 3f_4 V_y \quad (10b) \]
\[ g_{1,y} + g_{2,x} = 2(f_2 V_x + f_3 V_y) \quad (10c) \]
\[ g_1 V_x + g_2 V_y = \frac{\hbar^2}{4} (f_1 V_{xxx} + f_2 V_{xxy} + f_3 V_{xyy} + f_4 V_{yxy}) + 8A_{300}(x_1 V_y - x_2 V_x) + 2(A_{210} V_x + A_{201} V_y) \quad (10d) \]

The functions \( f_i \) are polynomial involving the constants \( A_{ijk} \). The problem consists in 10 constants and 3 functions to be determined from the overdetermined systems of 4 equations. 20 quantum potential were obtained and among them 5 exotic potentials written in terms of Painlevé transcendent. One models in term of the fourth Painlevé transcendent can be written as
Q18

\[ V(x, y) = \frac{\omega^2}{2}(x^2 + y^2) + \frac{\hbar^2}{2} P_4^2 \left( \frac{\omega}{\hbar} x \right) + 2\omega \sqrt{\omega \hbar} P_4 \left( \frac{\omega}{\hbar} x \right) + \frac{\epsilon \hbar}{2} P_4' \left( \frac{\omega}{\hbar} x \right) + \frac{\hbar \omega}{3} (\epsilon - \alpha). \]

3.2. Cartesian and N=4

In the case the integrals B is of the fourth order, this integrals can be written in the following form

\[ B = \sum_{j+k+l=4} A_{jkl} \left( L_j^2, p_1^k, p_2^l \right) + \frac{1}{2} \left\{ \{ g_1(x, y), p_1^2 \} + \{ g_2(x, y), p_1 p_2 \} + \{ g_3(x, y), p_2^2 \} \right\} + l(x, y) \]

where the quantities \( f_i, i = 1, 2, \ldots, 5 \) are polynomials in the variables \( x \) and \( y \). The commutator \([H, B] = 0\) lead to a set of 6 linear PDEs for the functions \( g_1, g_2, g_3, \) and \( l \). We \( V \) is not known, this is a system of 6 nonlinear PDEs for \( g_i, l \) and \( V \). They take the form

\[ g_{1,x} = 4f_1 V_x + f_2 V_y \]
\[ g_{2,x} + g_{1,y} = 3f_2 V_x + 2f_3 V_y \]
\[ g_{3,x} + g_{2,y} = 2f_3 V_x + 3f_4 V_y \]
\[ g_{3,y} = f_4 V_x + 4f_5 V_y, \]

and

\[ \ell_x = 2g_1 V_x + g_2 V_y + \frac{\hbar^2}{4} \left( (f_2 + f_4) V_{xxx} - 4(f_1 - f_5) V_{xxy} - (f_2 + f_4) V_{yy} \right) + (3f_2 - f_5) V_{xx} - (13f_1 + f_4) V_{xy} - 4(f_2 - f_5) V_{yy} - 2(6A_{400} x^2 + 62A_{400} y^2 + 3A_{301} x - 29A_{310} y + 9A_{220} + 3A_{202}) V_x + 2(56A_{400} x y - 13A_{310} x + 13A_{301} y - 3A_{211}) V_y, \]
\[ \ell_y = g_2 V_x + 2g_3 V_y + \frac{\hbar^2}{4} \left( -(f_2 + f_4) V_{xxx} + 4(f_1 - f_5) V_{xxy} + (f_2 + f_4) V_{yy} \right) + 4(f_1 - f_4) V_{xx} - (f_2 + 13f_5) V_{xy} - (f_1 - 3f_4) V_{yy} + 2(56A_{400} x y - 13A_{310} x + 13A_{301} y - 3A_{211}) V_x - 2(62A_{400} x^2 + 64A_{400} y^2 + 29A_{301} x - 3A_{310} y + 9A_{202} + 3A_{220}) V_y. \]

The quantities \( f_i, i = 1, 2, \ldots, 5 \) are polynomials. The cases can be simplified via transformation such translation and removing trivial integrals. The leading terms split in 3 types and further analysis will provide 12 exotic cases. For these 12 cases nonlinear fourth order ordinary differential equation are obtained. They can be further integrated and transformed into an equation of type Chazy-I. These equations consist in third order differential equations in the polynomial class of the form

\[ W''' = aW W'' + bW'^2 + cW^2 W' + dW^4 + A(y)W'' \]

(13)
where \(a, b, c,\) and \(d\) are certain rational or algebraic numbers, and the remaining coefficients are locally analytic functions of \(y\). Among work on higher order analog\[17, 22, 23, 24, 25, 11, 12, 13, 14\], the Chazy class\[17, 12, 13\] has revealed useful in classifying the exotic potential. We relied on the canonical form of Chazy-I equation and its first integral. In addition, the relation with Painlevé transcendent has been facilitated by relying on the following class in which belong the integrals of Chazy-I

\[
A(W', W, y)W'' + B(W', W, y)W'' + C(W', W, y) = 0
\]

where \(A, B\) and \(C\) are polynomials in \(W\), and \(W'\) with coefficients analytic in \(y\). Cosgrove and Scoufis\[22\] gave a complete classification of Painlevé type equations of second order and second degree \(W'' = F(W', W, y)\) where \(F\) is rational in \(W'\), and \(W\) and analytic in \(y\) which divide into six classes of them (denoted by SD-I, SD-II, ..., SD-VI). The reduction to Painlevé transcendent and transformation were presented. These results allowed to connect for the 12 classes the exotic potential to Painlevé transcendent. One of them is \(Q_5^3\) and given by

\[
Q_5^3 : V(x, y) = c_1 x + \frac{\hbar^2}{2} \sqrt{\alpha} P_3'(y) + \frac{3}{4} \alpha (P_3(y))^2 + \frac{\delta}{4P_3^2(y)} + \frac{\beta P_3(y)}{2y} + \frac{\gamma}{2yP_3(y)} - \frac{P_3'(y)}{2yP_3(y)} + \frac{P_3^2(y)}{4P_3^2(y)}.
\]

In addition, a complete classification of doubly exotic potential has been performed for \(N = 5\)[2]. Another part of that program on the classification of higher order superintegrable systems is the construction of the algebra generated by the conserved quantities. Among these potentials cases with fourth Painlevé transcendent as revealed to lead to interesting algebraic structure formed by the integrals of motion. For more detail in the case \(N=3\) we refer the reader to\[72, 73\]. The explicit form of the integrals have been obtained and involving calculation lead to cubic algebra. The key step consist in using the general formula for the Casimir operator which is a quartic polynomial of the generators and by relying on the explicit differential operator realization to rewrite the Casimir invariant as a polynomial of the Hamiltonian only. Algebraic expression of the structure function were calculated in regard of the structure constants and finite dimensional unitary representations were obtained.

4. Constructive approaches

The works described in previous sections rely on a direct approach to classify the superintegrable systems as well as a direct approach to construct the symmetry algebra. There are advantages to use the direct approach as one exhaust all possibilities and allow to obtain the lowest possible order integrals of motion. On the other hand such approach become harder as the order increase. In recent years different constructive approaches have been proposed where integrals of motion are generate via building block and the corresponding factorized form facilitate the construction of the symmetry algebra. Among the recent approach are the recurrence and ladder operators method and their classical analog\[15, 61, 75, 76, 77\]. More recently constructive method based on four type of one dimensional operator algebra has been discussed\[68\].

\[
H_1 = \frac{P_x^2}{2} + V(x), \quad K_1 = \sum_{j=0}^{M} f_j(x)p_x^j
\]
where \( f_M(x) \neq 0 \), and \( f_j(x) \) are locally smooth functions.

Consider the 4 following forms for \( \alpha = \beta = \gamma = 0; \alpha = \beta = 0, \gamma \neq 0; \alpha = \gamma = 0, \beta \neq 0; \) and \( \alpha \neq 0 \), respectively

\[
[H_1, K_1] = 0,
\]
\[
[H_1, K_1] = \alpha_1,
\]
\[
[H_1, K_1] = \alpha_1H_1,
\]
\[
[H_1, K_1] = -\alpha_1K_1, \quad \alpha_1 \in \mathbb{R}\backslash 0
\]

where \( \alpha_1 \neq 0 \) is a constant. We shall refer to these relations as Abelian type (a), Heisenberg type (b), conformal type (c), and ladder type (d), respectively. We shall call the systems \( \{H_1, K_1\} \) in one dimension “algebraic Hamiltonian systems”. Some of these cases have been studied for particular cases (a) [53, 54], (b) [38, 84, 50], (c) [28] and (d) [95, 16, 77, 76].

4.1. Case (d)

We consider the one dimensional Hamiltonian and the \( M \)th order operator \( K_1 \) and their commutator \( [H_1, K_1] \). Once \( [H_1, K_1] \) is chosen to be equal to \( 0, \alpha_1, \alpha_1H_1 \) or \( -\alpha_1K_1 \), this will provide us with determining equations for the potential \( V(x) \) and the coefficients \( f_j(x) \), \( 0 \leq j \leq M \) in the operator \( K_1 \). We obtain the following operator of order \( M + 1 \)

\[
[H_1, K_1] = \sum_{l=0}^{M+1} Z_l D^l
\]

with

\[
Z_{M+1} = (-ih)^{M+2} f'_M,
\]
\[
Z_M = -\frac{\hbar^2}{2}(-ih)^{M-1}(2f'_{M-1} - ihf''_M),
\]
\[
Z_l = -\frac{\hbar^2}{2}(-ih)^{l-1}(2f'_{l-1} - ihf''_l) - \sum_{j=l+1}^{M} (-ih)^j f_j C^j_l V^{(j-l)}, \quad 1 \leq l \leq M - 1,
\]
\[
Z_0 = -\frac{\hbar^2}{2} f''_0 - \sum_{j=1}^{M} (-ih)^j f_j V^{(j)}
\]

\( C^j_l \) are the Newton binomial coefficients. The determining equations are

\[
Z_{M+1} = 0, \quad Z_l = \alpha_1 f_l, \quad 0 \leq l \leq M.
\]

The solution to these determining equation gives the following solution for the potentials in terms of Painlevé transcendent.
\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
Case & Type & Integral type & K & Order of K \\
\hline
(b,b) & polynomial & $\alpha_2 K_1 - \alpha_1 K_2$ & $\alpha_2 K_1 - \alpha_1 K_2$ & max($k_1, k_2$) \\
(c,b) & polynomial & $\alpha_2 K_1 - \alpha_1 H_1 K_2$ & $\alpha_2 K_1 - \alpha_1 H_1 K_2$ & max($k_1, k_2 + 2$) \\
(d,d) & polynomial & $(K_1^1)^m (K_2^-)^n - (K_1^-)^m (K_2^1)^n$ & $(m k_1 + n k_2 - 1)$ \\
(c,c) & polynomial & $\alpha_2 H_2 K_1 - \alpha_1 H_1 K_2$ & $\alpha_2 H_2 K_1 - \alpha_1 H_1 K_2$ & max($k_1 + 2, k_2 + 2$) \\
\hline
\end{tabular}
\caption{Integrals of motion in $E_2$}
\end{center}
\end{table}

\begin{equation}
V_{d_1} = \frac{\alpha_1^2}{2\hbar^2} x^2, \\
V_{d_2} = \frac{\alpha_1^2}{8\hbar^2} x^2 + \frac{\beta}{x^2}, \\
V_{d_3} = \epsilon\alpha_1 P'_4 + \frac{2\alpha_1^2}{\hbar^2} (P_4^2 + x P_4) + \frac{\alpha_1^2}{2\hbar^2} x^2 + (\epsilon - 1) \frac{\alpha_1}{3} - \frac{\hbar^2}{6} k_1, \\
V_{d_4} = \frac{\alpha_1^2}{8\hbar^2} x^2 + \hbar^2 \left( \frac{\gamma}{P_3 - 1} + \frac{1}{x^2} (P_3 - 1)(\sqrt{2\lambda} + \lambda(2P_3 - 1) + \frac{\beta}{P_3}) \\
+ x^2 \left( \frac{P_2^2}{2P_3} - \frac{\alpha_1^2}{8\hbar^2} P_3 \right)(2P_3 - 1) - \frac{P_2^2}{P_3 - 1} - 2\sqrt{2\lambda} P_3 \right) + \frac{3\hbar^2}{8x^2}.}
\end{equation}

For $M = 5$ setting $u(x) = \int V_{d_i} dx$, we get the function $f_i$ in term of the function $u$ and its derivatives, $u$ satisfies a sixth order differential equation satisfy the Painlevé property. These models are interesting from a one dimensional perspective and as illustrated by the Table they allow to generate superintegrable systems. In fact, the corresponding superintegrable systems belonging to $(d, d)$ also lead to polynomial algebras as these ladder operator generate in fact polynomial Heisenberg algebras which can be further use to build the polynomial algebras of the integrals of motion.

4.2. Case (d,d)

The operator algebras Case (a), (b), (c) and (d) generate integrals of motion for

A 2D system with separation of variables in Cartesian :

\[ H = H_1 + H_2 = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + V_1(x) + V_2(y) \]

Recently the role played by these operators in context of superintegrability for the classical and quantum models has been discussed. Let us present part of the construction related to the quantum case.

In the case (d,d) there is existence of polynomial Heisenberg algebras in both axis

\[ [H_i, K_i^1] = \alpha_i K_i^1, \quad [H_i, K_i^-] = -\alpha_i K_i^- \]

\[ K_i^- K_i^1 = Q(H_i + \alpha_i), \quad L_i^1 L_i = Q(H_i) \]

$\alpha_1$ and $\alpha_2$ are constants while $Q(x)$ and $S(y)$ are polynomials. This is similar to bosonic realization of Lie algebras and in particular boson realizations of $su(2)$ and $su(3)$. Explicit expression for the polynomial algebras $A = H_1 - H_2$ and $B_1 = (K_1^1)^m (K_2^-)^n$, $B_2 = (K_1^-)^m (K_2^1)^n$ have been obtained and been used to provide spectrum of the quantum models.
5. Fourth Painlevé transcendent model $Q_{18}$: ladder and SUSYQM

The part of the potential $Q_{18}$ in the $x$ variable and involving the fourth Painlevé transcendent has been obtained and rediscovered by different approaches [3, 10, 68]. In particular, one can generate the model using two types of supercharges in supersymmetric quantum mechanics, of first and second order [3, 16]. It has been demonstrated that they imply that the ladder operators can be factorized and this form allow to obtain wavefunctions by determining the zero modes of the annihilation and creation operators. At most three of the six possible states annihilated by $a^- (\psi_i)$ and $a^+ (\phi_i)$ in total can be square integrable [3, 16, 73].

\[
\psi_i = f_1(P_4, P'_4)e^{i g_1(P_4, P'_4)}, \quad i = 1, 2, 3
\]

\[
\phi_i = f_2(P_4, P'_4)e^{i g_2(P_4, P'_4)}, \quad i = 1, 2, 3
\]

For some ranges of the parameters $\alpha$ and $\beta$, $H_1$ may admit one, two, or three infinite sequences of equidistant levels, or one infinite sequence of equidistant levels with either one additional singlet or one additional doublet. It has been discussed [73] that these results can be used to compare the spectrum obtained from the cubic algebras for the two dimensional systems which is superintegrable.

5.1. Example, part 1

We considered a case ($\epsilon = 1$) with the two parameter of the fourth Painlevé transcendent taking the value $\alpha = 5$, $\beta = -8$. The fourth Painlevé admit a rational solution of the form

\[
f(z) = \frac{4z(2z^2-1)(2z^2+3)}{(2z^2+1)(4z^2+3)}.
\]

\[
V(x, y) = \frac{\omega^2}{2}(x^2 + y^2) - \frac{8\hbar^2 \omega}{(2\omega x^2 + \hbar)^2} + \frac{4\hbar^2 \omega}{(2\omega x^2 + \hbar)} + \frac{2\hbar \omega}{3}.
\]

From the cubic algebra we get unitary representations. Due to constraints for unitary irreducible representations we have one infinite sequence. The spectrum decomposes into an infinite unirreps, a singlet and a doublet.

\[
\phi(x) = 4\hbar^2 \omega^2 x(p + 1 - x)(x + 3)(x + 2), \quad E = \hbar \omega(p + \frac{8}{3}), p = 0, 1, ...
\]

\[
\phi(x) = 4\hbar^2 \omega^2 x(p + 1 - x)(x - 3)(x - 1), \quad E = \hbar \omega(p - \frac{1}{3}), p = 0
\]

\[
\phi(x) = 4\hbar^2 \omega^2 x(p + 1 - x)(x + 1)(x - 2), \quad E = \hbar \omega(p + \frac{2}{3}), p = 0, 1
\]

\[
\psi_0(x) = N_0(a^\dagger)^n \psi_0
\]
\[ \psi_0 = e^{-\omega x^2/2} \frac{(3\hbar + 2\omega x^2)}{(\hbar + 2\omega x^2)}, \chi(x) = \frac{e^{-\omega x^2}}{\hbar + 2\omega x^2}. \] 

(23)

The corresponding spectrum for the two-dimensional models decomposes into the following two infinite sequences of states

\[ \psi_{n,k} = \psi_n(x)e^{-\omega x^2/2} H_k(\sqrt{\frac{\omega}{\hbar}} y), \quad E = \hbar \omega(n + k + \frac{8}{3}) \] 

(24)

\[ \phi_m = \chi(x)e^{-\omega y^2/2} H_m(\sqrt{\frac{\omega}{\hbar}} y), \quad E_m = \hbar \omega(m - \frac{1}{3}) \] 

(25)

which lead to the following degeneracies

\[ \text{deg}(E_N) = \begin{cases} 
1 & \text{if } N = 0, \\
1 & \text{for } N = 1, \\
1 & \text{if } N = 2, \\
N - 1 & \text{if } N = 3, \ldots 
\end{cases} \] 

(27)

The total number of degeneracies coming from counting all the states via ladder operators and acting on physical states and given by eq.(27) is different than the one generated via the cubic algebra formed by integrals of motion and given by eq.(22). The algebraic approaches allow to get the level, but not the correct degeneracies. This is due to these singlet states, and more precisely to the fact the third order integral obtained is connected to certain type of ladder operators, which is hidden when one classify higher order superintegrable systems. This problem can be observed for other rational solution of the fourth Painlevé transcendent and point out how one need to look for additional inner structures and operators for exotic potentials for certain value of their parameters. Here this particular solutions is connected to Hermite exceptional orthogonal polynomials.

6. Ladder type (d), factorization and EOP

In this section we will provide further details on how the Case (d) of ladder operators is in fact much more richer. It was demonstrated that not only these ladder operators can be factorized, but ladder operators are not unique and even for a same order (i.e. as a differential operators) distinct lowering and raising operators exist with different patterns of finite and infinite dimensional unitary representations. It means the spectrum can be decomposed into direct sums of distinct finite and infinite unitary representations which belong to different polynomial algebras. In recent years, the case of k-step extension of harmonic oscillator has been related to type III Hermite exceptional orthogonal polynomials. Such orthogonal polynomial has been discovered in 2008, related to quantum models in supersymmetric quantum mechanics and since large body of literature has been devoted to their properties. Let us present some results of SUSYQM. The intertwining of Hamiltonian by two nth-order differential operators \( A \) and \( A^\dagger \)

\[ AH^{(1)} = H^{(2)} A, \quad A = A^{(n)} \cdots A^{(2)} A^{(1)} \]

\[ A^{(i)} = \frac{d}{dx} + W^{(i)}(x), \quad W^{(i)}(x) = -\frac{d}{dx} \log \varphi^{(i)}(x), i = 1, 2, \ldots, n, \]

From seed solution \( \varphi_i(x) \) of the Schrödinger equation associated with \( H^{(1)} \)
They have the form \( \varphi H \) of \( m_1 \) deleting approach (at least) in the state deleting approach which generate the same final partner Hamiltonian. In the state equivalence i.e. there exist another chain (not necessarily of the same length) of supercharges is indeed a (equivalence means

7. Families of superintegrable models related to k-step extension

Many superintegrable families from these k-step rational extension have been constructed \([78, 79, 80, 70, 81]\) and in fact these results can be straightforwardly extended in n-dimensional systems in the same way the isotropic and anisotropic harmonic oscillator extend in n dimensional. However, a complete algebraic description of their spectrum would be non trivial. Among these families there are the quantum system

\[
H = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2 + y^2 - 2k - 2\frac{d^2}{dx^2}\log W(H_{m_1}, H_{m_2}, \ldots, H_{m_k}),
\]

\( E_{i,N} = 2N, \quad N = \nu_x + \nu_y + 1 \)

\( \nu_x = -m_k - 1, \ldots, -m_1 - 1, 0, 1, 2, \ldots \)

which display the following spectrum \( E_{i,N} = 2N, N = \nu_x + \nu_y + 1 \) with \( \nu_x = -m_k - 1, \ldots, -m_1 - 1, 0, 1, 2, \ldots \). The degeneracies take the form

\[
\psi_1, \psi_2, \ldots, \psi_{m_k - m_k-1}, \ldots, \psi_{m_k - m_2}, \ldots, \psi_{m_k - m_1}, \ldots, \psi_{m_k}, \psi_{m_k}.
\]
\( \deg(E_N) = \begin{cases} 
\kappa - \lambda + 1 & \text{if } N = -m_j, -m_j + 1, \ldots, -m_{j-1} - 1, \\
\kappa & \text{for } j = 2, 3, \ldots, k, \\
N + \kappa & \text{if } N = 1, 2, 3, \ldots
\end{cases} \)

This example coincide with the particular solution of the fourth Painlevé transcendent we presented in previous section. In the case \( m_1 = 2 \) the degeneracies well coincide

\[
\deg(E_N) = \begin{cases} 
1 & \text{if } N = -2, -1, 0, \\
N + 1 & \text{if } N = 1, 2, 3, \ldots
\end{cases}
\]

In general these k-step extensions exhibit very unusual patterns of degeneracies and bands of levels. One application of the ladder operators obtained which only admit infinite dimensional unitary representations is that they allow to build integrals and polynomial algebras whose finite-dimensional unitirreps can be combined to provide the correct total number of degeneracies and level. Two approaches have been taken to provide algebraic derivation \cite{78, 79, 80, 70}, one using the deformed oscillator realizations approach and the other by constructing explicitly in combinatorial ways the multiplets corresponding to representations. We classified all multiplets for a given energy level. Let us summarize briefly this approach. The unirreps may be characterized by \( (N, s) \) and their basis states by \( |N, \tau, s, \sigma \rangle \) \( \sigma = -s, -s + 1, \ldots, s \) and \( \tau \) distinguishes between repeated representations specified by the same \( s \) (integer or half-integer).

\[
b^\dagger |N, \tau, s, s \rangle = b |N, \tau, s, -s \rangle = 0.
\]

The \( \sigma \) is associated with each state forming this sequence. Using notation \( N = \alpha n_1 n_2 + \mu \) with appropriate values of \( \alpha \) and \( \mu, |N, \nu_x \rangle = |\nu_x \rangle_1 |N - \nu_x - 1 \rangle_2 \). The Tables for the general k-step extended systems, the seven families and generalization to singular oscillator can be find in previous references. Let us only present the case \( k = 1 \) in the Table 2.

| \( \lambda \) | \( \mu \) | \( s \) | \( N \) | \( \deg(E_N) \) |
|---|---|---|---|---|
| \(-1\) | \( 0 \) | \( 0 \) | \( 1 \) | \( 1 \) |
| \( 0 \) | \( 1, \ldots, m_1 \) | \( \mu \) | \( N + 1 \) | \( 0^{\mu-1} \) |
| \( 1, 2, \ldots \) | \( \frac{1}{2} \) | \( m_1 + 1 \) | \( N + 1 \) | \( (\frac{\lambda - 1}{2})^{m_1} \) |
| \( 1, 2, \ldots \) | \( \frac{\lambda + 1}{2} \) | \( m_1 + 1 \) | \( N + 1 \) | \( (\frac{\lambda}{2})^{\mu-1} \) |
| \( 1, 2, \ldots \) | \( \frac{\lambda + 1}{2} \) | \( m_1 + 1 \) | \( N + 1 \) | \( (\frac{\lambda - 1}{2})^{m_1 - \mu + 1} \) |
When the parameter \( m_1 = 2 \), we well recover the total number of degeneracies for the particular case related to the fourth Painlevé transcendent given by eq(27) and thus this provide an algebraic derivation of the spectrum via another set of integrals.

8. Conclusion

We presented an overview of the classification of superintegrable quantum systems with a second and an additional third or fourth order integrals, and allowing separation of variable in Cartesian coordinates. In addition to the direct approach to the classification of superintegrable systems, we reviewed results obtained via operator algebras of one-dimensional Hamiltonians. It has been demonstrated that all superintegrable quantum systems are connected to the Painlevé property and it is conjectured that it is always the case. The role of the Chazy class of equation has also been highlighted. The conserved quantities of these superintegrable models lead to finitely generated polynomial algebras. For particular values of parameters for models involving the fourth Painlevé transcendent an algebraic solution based on integrals generated from a direct approach does not provide an accurate counting of the degeneracies and levels. This was shown by eq(22) and (27). An algebraic solution was determined using new integrals build from ladder constructed from Darboux-Crum and Krein-Adler supercharges and this is given by Table 2 (for the specific value \( m_1 = 2 \)). This show how algebraic techniques can be extended for higher order superintegrable systems in the case the integrals obtained via the direct approach do not provide a complete solution.

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