Research Article

Remarks on the Initial and Terminal Value Problem for Time and Space Fractional Diffusion Equation

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The fractional problem for partial differential equation has many applications in science and technology. The main objective of the paper is to investigate the convergence of the mild solution of the diffusion equation with time and space fractional. We consider the problem in two cases which are forward problem and inverse problem. We use new techniques to overcome some of the complex assessments.

1. Introduction

Fractional calculation has been shown to provide many important applications in natural sciences, such as in biological systems, signal processing, fluid mechanics, electrical networks, optical, and viscosity [1–8]. With the development of mathematics, there are now many different definitions of fractional derivatives, for example, Riemann-Liouville, Caputo, Hadamard, and Riesz. Let us refer many various papers on fractional differential equation, for example, Manimaran et al., Tuan et al., Long et al., Long L.D. et al., and Ngoc et al. [9–14]; Adiguzel et al., Li et al., Afsahari et al., Alqahtani et al., Karapinar et al., Salim et al., Karapinar et al., and Abdeljawad et al. [15–22]; and Bachir et al., Salim et al., and Baitichea et al. [23–25]. Although most of them have been extensively studied, most mathematicians are interested and studied two derivatives which are Caputo and Riemann-Liouville derivatives.

In this paper, for \( \alpha, \beta \in (0, 1) \), we are interested to study the following problem:

\[
\begin{aligned}
\partial_t^\alpha u(x, t) + (-\Delta)^\beta u(x, t) &= H(x, t), \quad x, t \in (0, \pi) \times (0, T), \\
(u(0, t) = u(\pi, t) &= 0, \quad t \in (0, T), \\
u(x, 0) &= u_0(x), 0 < x < \pi,
\end{aligned}
\] (1)

with the initial condition

\[
u(x, 0) = u_0(x), 0 < x < \pi,
\] (2)

or the terminal condition

\[
u(x, T) = f(x), 0 < x < \pi.
\] (3)

There are many results related to the Problem (1) in both aspects: theoretical analysis and numerical analysis. The existence and well-posedness of Problems (1)–(2) and (1)–(3) has been studied in [26]. Jin et al. [27] applied two semidiscrete schemes of Galerkin FEM method in order to approximate the solution of Problems (1) and (2). In [28], the authors investigated a reaction-diffusion equation with a Caputo fractional derivative in time. In [29], the authors established the existence and uniqueness of the weak solution and the regularity of the solution for coupled fractional diffusion system. Mu et al. [30] investigated some initial-boundary value problems for time-fractional diffusion equations. Let us now mention some previous works on terminal value problem Problems (1)–(3). The main current applications of the terminal value problem are hydrodynamic inversion and spoil the image. In [31], the authors used variable total variation to approximate the backward problem for a time-space fractional diffusion equation. Under the
interesting paper [32], Ngoc et al. considered the terminal value problem for nonlinear model.

\[
\mathbf{D}_t^u u - u_{xx} = F(u). \tag{4}
\]

Our main purpose of this paper is to study the convergence of Problem (1) when \( \beta \to 1^- \). This result gives us the relationship between the solutions of the two Problem (1) with the case \( 0 < \beta < 1 \) and \( \beta = 1 \). To the best of our knowledge, the research direction on this convergence topic is still limited. The main techniques to solve our problem is to use Mittag-Leffler evaluations with the combination of the Wright function.

This paper is organized as follows. In Section 2, we focus preliminaries with some background on the definition and evaluations of Mittag-Leffler functions.

2. Preliminaries

Let us consider the Mittag-Leffler function, which is defined by

\[
E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}. \tag{5}
\]

For \( z \in \mathbb{C} \), for \( \alpha > 0 \) and \( \beta \in \mathbb{R} \). When \( \beta = 1 \), it is abbreviated as \( E_{\alpha}(z) = E_{\alpha,1}(z) \).

**Lemma 2.1.** The following equality holds (See [33]):

\[
E_{\alpha,1}(-z) = \int_0^\infty \Phi_\alpha(\theta)e^{-\theta z}d\theta, \quad \text{for } z \in \mathbb{C}, \tag{6}
\]

where the Wright function \( \Phi_\alpha(\theta) \) is defined by

\[
\Phi_\alpha(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{j! \Gamma(-\alpha j + 1 - \alpha)}, \quad 0 < \alpha < 1. \tag{7}
\]

In addition, \( \Phi_\alpha(\theta) \) is a probability density function, that is,

\[
\Phi_\alpha(\theta) \geq 0, \quad \text{for } \theta > 0 \text{ and } \int_0^\infty \Phi_\alpha(\theta) = 1. \tag{8}
\]

**Lemma 2.2.** For \( \alpha \in (0, 1) \) and \( b > -1 \), the following properties hold (See [33]):

\[
\int_0^\infty \theta^b \Phi_\alpha(\theta)d\theta = \frac{\Gamma(b+1)}{\Gamma(\alpha b + 1)}. \tag{9}
\]

Let a given positive number \( \sigma \geq 0 \). Let us also define the Hilbert space scale as follows:

\[
H^\sigma(\Omega) = \left\{ \psi \in L^2(\Omega); \sum_{j=1}^{\infty} \beta^2 \left\langle \psi, \varphi_j \right\rangle^2 < \infty \right\}, \tag{10}
\]

with the following norm \( \|\psi\|_{H^\sigma(\Omega)} = \left( \sum_{j=1}^{\infty} \beta^2 \left( \left\langle \psi, \varphi_j \right\rangle \right)^2 \right)^{1/2} \).

Here we give the following lemma, which will help our proofs later:

**Lemma 2.3.** Let \( \epsilon, \epsilon' > 0 \). Then we get the following:

\[
E_{\alpha,1}\left(-j^{2\beta}t^\alpha\right) - E_{\alpha,1}\left(-j^2t^\alpha\right) \leq C_1(\alpha, \epsilon) t^{\epsilon} \left( 1 - \beta \right)^{\epsilon'} j^{2\epsilon+\epsilon'} \tag{11}
\]

\[
E_{\alpha,\alpha}\left(-j^{2\beta}t^\alpha\right) - E_{\alpha,\alpha}\left(-j^2t^\alpha\right) \leq C_2(\alpha, \epsilon) t^{\epsilon} \left( 1 - \beta \right)^{\epsilon'} j^{2\epsilon+\epsilon'} \tag{12}
\]

**Proof.** Let us now study the difference \( |E_{\alpha,1}(-j^{2\beta}t^\alpha) - E_{\alpha,1}(-j^2t^\alpha)| \) for \( 0 < \beta < 1 \). Since the definition of Wright function as in Lemma 2.1, we get that

\[
E_{\alpha,1}\left(-j^{2\beta}t^\alpha\right) - E_{\alpha,1}\left(-j^2t^\alpha\right) = \int_0^\infty \Phi_\alpha(\theta) \exp\left(-j^{2\beta}t^\alpha \theta\right)d\theta \tag{13}
\]

\[
- \int_0^\infty \Phi_\alpha(\theta) \exp\left(-j^2t^\alpha \theta\right)d\theta.
\]

Since \( j \geq 1 \) and \( 0 < \beta \leq 1 \), we know easily that \( \exp(-j^{2\beta}t^\alpha \theta) \geq \exp(-j^2t^\alpha \theta) \). Hence, we find that

\[
\exp\left(-j^{2\beta}t^\alpha \theta\right) - \exp\left(-j^2t^\alpha \theta\right) = \exp\left(-j^{2\beta}t^\alpha \theta\right) \left( 1 - \exp\left(-j^2t^\alpha \theta\right) \right). \tag{14}
\]

Using the inequality \( 1 - e^{-\tau} \leq C_\epsilon \epsilon \) for any \( \epsilon > 0 \), we find that

\[
\exp\left(-j^{2\beta}t^\alpha \theta\right) - \exp\left(-j^2t^\alpha \theta\right) \leq C_\epsilon \left(j^{2 - j^{2\beta}t^\alpha} \theta \right). \tag{15}
\]

Combining Problems (13) and (15), we derive that

\[
E_{\alpha,\alpha}\left(-j^{2\beta}t^\alpha\right) - E_{\alpha,\alpha}\left(-j^2t^\alpha\right) \leq C_\epsilon \left(j^{2 - j^{2\beta}t^\alpha} \theta \right) \left( \int_0^\infty \theta^\alpha \Phi_\alpha(\theta)d\theta \right)
\]

\[
= C_\epsilon \frac{\Gamma(\epsilon + 1)}{\alpha \epsilon + 1} \left(j^{2 - j^{2\beta}t^\alpha} \theta \right)^\epsilon. \tag{16}
\]

For any \( \epsilon' > 0 \) and noting that \( \log(j) \leq j \) for any \( j \geq 1 \), it is obvious to see that

\[
\left(j^{2 - j^{2\beta}t^\alpha} \theta \right)^\epsilon \leq \left(j - 2\beta \theta \right)^\epsilon \left(j - j^{2\beta}t^\alpha \theta \right)^\epsilon' \leq \left(j - 2\beta \theta \right)^\epsilon \theta^{\epsilon'}. \tag{17}
\]

This implies that

\[
\left(j^{2 - j^{2\beta}t^\alpha} \theta \right)^\epsilon \leq (1 - \beta)^{\epsilon'} j^{2\epsilon + \epsilon'}. \tag{18}
\]
From some above observations, we get that
\[
E_{a,1} \left( -j^{2} \beta \, t^{a} \right) - E_{a,1} \left( -j^{2} \, t^{a} \right) \leq C_{1} (\alpha, \varepsilon) t^{\alpha \varepsilon} (1 - \beta)^{2 + \pi \varepsilon}.
\] (19)

By a similar argument as above, we also obtain the desired result, Problem (12).

3. Initial Value Problem

In this section, we focus the following initial value problem under the linear case:

\[
\begin{aligned}
\varphi'' + \sqrt{(-\Delta)^{\beta}} \varphi &= H(x, t), & (x, t) \in (0, \pi) \times (0, T), \\
\varphi(0, t) &= \varphi(\pi, t) = 0, & t \in (0, T), \\
\varphi(x, 0) &= \varphi_{0}(x), & x \in (0, \pi),
\end{aligned}
\] (20)

where \( \varphi_{0} \) and source function \( H \) are defined later.

**Theorem 3.1.** Let \( \varphi_{0} \in H^{p} (\Omega) \) and \( H \in L^{\infty} (0, T; H^{p} (\Omega)) \) for any \( p > 0 \). Then we get

\[
\begin{aligned}
\| \varphi_{p} (., t) - v^{*} (., t) \|_{H^{p} (\Omega)} &
\leq (1 - \beta)^{\frac{p - 2}{2}} \left[ \| \varphi_{0} \|_{H^{p} (\Omega)} + \| H \|_{L^{\infty} (0, T; H^{p} (\Omega))} \right]
\end{aligned}
\] (21)

for any \( 0 < s < p \).

**Proof.** The mild solution to Problem (20) with \( 0 < \beta < 1 \) is defined by

\[
\begin{aligned}
\varphi_{p} (x, t) &= \sum_{j=1}^{\infty} E_{a,1} \left( -j^{2} \, \beta \, t^{a} \right) \left( \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right) \varphi_{j}(x) \\
&\quad + \sum_{j=1}^{\infty} \left[ \int_{0}^{t} \left( t - r \right)^{a-1} E_{a,a} \left( -j^{2} \, (t - r)^{a} \right) H_{j}(r) dr \right] \varphi_{j}(x),
\end{aligned}
\] (22)

and the mild solution to Problem (20) with \( \beta = 1 \) is defined by

\[
\begin{aligned}
v^{*} (x, t) &= \sum_{j=1}^{\infty} E_{a,1} \left( -j^{2} \, t^{a} \right) \left( \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right) \varphi_{j}(x) \\
&\quad + \sum_{j=1}^{\infty} \left[ \int_{0}^{t} \left( t - r \right)^{a-1} E_{a,a} \left( -j^{2} \, (t - r)^{a} \right) H_{j}(r) dr \right] \varphi_{j}(x),
\end{aligned}
\] (23)

By subtracting both sides of the two expressions above, we get the following difference:

\[
\begin{aligned}
\varphi_{p} (x, t) - v^{*} (x, t) &= \sum_{j=1}^{\infty} E_{a,1} \left( -j^{2} \, \beta \, t^{a} \right) \left( \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right) \varphi_{j}(x) \\
&\quad + \sum_{j=1}^{\infty} \left[ \int_{0}^{t} \left( t - r \right)^{a-1} E_{a,a} \left( -j^{2} \, (t - r)^{a} \right) H_{j}(r) dr \right] \varphi_{j}(x) \\
&\quad - E_{a,a} \left( -j^{2} \, (t - r)^{a} \right) \left( \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right) \varphi_{j}(x) \\
&\quad - E_{a,a} \left( -j^{2} \, (t - r)^{a} \right) \left( \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right) \varphi_{j}(x) \\
&= \mathcal{M}_{1} (x, t) + \mathcal{M}_{2} (x, t).
\end{aligned}
\] (24)

Let us first consider the term \( \mathcal{M}_{1} \). By applying Parseval’s equality and Lemma 2.3, we find that

\[
\begin{aligned}
\| \mathcal{M}_{1} (., t) \|_{H^{p} (\Omega)}^{2} &= \sum_{j=1}^{\infty} E_{a,1} \left( -j^{2} \, \beta \, t^{a} \right) \left( \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right)^{2} \\
&\quad + \sum_{j=1}^{\infty} \left[ \int_{0}^{t} \left( t - r \right)^{a-1} E_{a,a} \left( -j^{2} \, (t - r)^{a} \right) H_{j}(r) dr \right] \varphi_{j}(x) \\
&\quad \leq |C_{1} (\alpha, \varepsilon, \delta)| t^{2a} (1 - \beta)^{2a} \sum_{j=1}^{\infty} \left( j^{2} + 4 \pi^{2} \right)^{2} \left( \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right) \\
&\quad \times \left[ \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right]^{2},
\end{aligned}
\] (25)

where any \( \delta > 0 \). Hence, we know that the upper bound

\[
\| \mathcal{M}_{1} (., t) \|_{H^{p} (\Omega)} \leq (1 - \beta)^{\delta} \| \varphi_{0} \|_{H^{2+4\pi^{2}} (\Omega)}.
\] (26)

Let us now treat the second term \( \mathcal{M}_{2} \). By using Parseval’s equality, we get that

\[
\begin{aligned}
\| \mathcal{M}_{2} (., t) \|_{H^{p} (\Omega)}^{2} &= \sum_{j=1}^{\infty} E_{a,1} \left( -j^{2} \, \beta \, t^{a} \right) \left( \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right)^{2} \\
&\quad + \sum_{j=1}^{\infty} \left[ \int_{0}^{t} \left( t - r \right)^{a-1} E_{a,a} \left( -j^{2} \, (t - r)^{a} \right) H_{j}(r) dr \right] \varphi_{j}(x) \\
&\quad \leq |C_{2} (\alpha, \varepsilon, \varepsilon')| t^{2a} (1 - \beta)^{2a} \left( j^{2} + 2 \pi^{2} \right)^{2} \left( \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right) \\
&\quad \times \left[ \int_{0}^{t} \varphi_{0}(x) \varphi_{j}(x) dx \right]^{2},
\end{aligned}
\] (27)

In view of the second estimate of Lemma 2.3, we derive that

\[
\begin{aligned}
\left( E_{a,a} \left( -j^{2} \, (t - r)^{a} \right) - E_{a,a} \left( -j^{2} \, (t - r)^{a} \right) \right)^{2} &\leq |C_{2} (\alpha, \varepsilon, \varepsilon')| t^{2a} (1 - \beta)^{2a} \left( j^{2} + 2 \pi^{2} \right)^{2}.
\end{aligned}
\] (28)
Combining Problems (27) and (28), we derive that
\[
\|\mathcal{M}_2(\cdot,t)\|_{\mathbb{H}^s(\Omega)}^2 \leq C_2 \left( (1 - \beta)^{2\epsilon'} \left( \int_0^t (t - r)^{s-1+2\epsilon} \right) dr \right)^2 + \left( \int_0^t (t - r)^{s-1+2\epsilon} |H(r)|^2 dr \right) + C_2 \left( (1 - \beta)^{2\epsilon'} \left( \int_0^t (t - r)^{s-1+2\epsilon} \right) dr \right)^2 \left( \int_0^t (t - r)^{s-1+2\epsilon} |H(r)|^2 dr \right).
\]

It is obvious to see that the integral term \( \int_0^t (t - r)^{s-1+2\epsilon} dr \) is convergent. Hence, we obtain that the following estimate:
\[
\|\mathcal{M}_2(\cdot,t)\|_{\mathbb{H}^s(\Omega)}^2 \leq C_2 \left( (1 - \beta)^{2\epsilon'} \left( \int_0^t (t - r)^{s-1+2\epsilon} \right) dr \right)^2 \left( \int_0^t (t - r)^{s-1+2\epsilon} |H(r)|^2 dr \right).
\]

(30)

Combining Problems (24), (25), and (30), we find that
\[
\left\| v_\beta(t) - v^*(t) \right\|_{\mathbb{H}^s(\Omega)} \leq \left( 1 - \beta \right)^{2\epsilon} \left[ \left\| v_0 \right\|_{\mathbb{H}^{2s-1}(\Omega)} + \left\| H \right\|_{L^2(0,T;\mathbb{H}^{2s-1}(\Omega))} \right].
\]

(31)

Since \( p > s \), we can choose
\[
\epsilon = \frac{p - s}{4}, \quad \delta = \epsilon' = 2.
\]

(32)

This implies that
\[
\left\| v_\beta(t) - v^*(t) \right\|_{\mathbb{H}^s(\Omega)} \leq \left( 1 - \beta \right)^{p-s} \left[ \left\| v_0 \right\|_{\mathbb{H}^p(\Omega)} + \left\| H \right\|_{L^2(0,T;\mathbb{H}^p(\Omega))} \right].
\]

(33)

\[
\frac{\mathbb{H}^s(\Omega)}{\mathbb{H}^s(\Omega)} \leq \frac{\mathbb{H}^s(\Omega)}{\mathbb{H}^s(\Omega)} + \mathbb{H}^s(\Omega)
\]

4. Terminal Value Problem

\[
\text{Theorem 4.1}. \text{ Let } f \in \mathbb{H}^p(\Omega) \text{ and } H \in L^{\infty}(0,T;\mathbb{H}^p(\Omega)). \quad \text{Then we get}
\]
\[
\left\| u_\beta(x,t) - u^*(x,t) \right\|_{L^2(0,T;\mathbb{H}^s(\Omega))} \leq \left( 1 - \beta \right)^{p-s} \left[ \left\| f \right\|_{\mathbb{H}^p(\Omega)} + \left\| H \right\|_{L^2(0,T;\mathbb{H}^p(\Omega))} \right] + \left( 1 - \beta \right)^{p-s} \left[ \left\| H \right\|_{L^2(0,T;\mathbb{H}^p(\Omega))} \right],
\]

(34)

for \( 1 < m < 1/\alpha \) and \( b > s + 2\beta + 2 \).

\[
\text{Proof.} \text{ The mild solution to terminal value Problem (1) for } 0 < \beta < 1 \text{ is given by}
\]
\[
u_\beta(x,t) = \sum_{j=1}^{\infty} E_{a_n} \left( -\beta^2 T^a \right) \left( \int_0^T f(x) \phi_j(x) dx \right) \phi_j(x)
\]

(35)

where
\[
H_j(r) = \int_0^T H(x,r) \phi_j(x) dx.
\]

(36)

The mild solution to terminal value Problem (1) for \( \beta = 1 \) is given by
\[
u_1(x,t) = \sum_{j=1}^{\infty} E_{a_n} \left( -T^a \right) \left( \int_0^T f(x) \phi_j(x) dx \right) \phi_j(x)
\]

(37)

Taking the difference of Problems (35) and (37) on both sides, we get the following bound:
\[
u_\beta(x,t) - u_\beta(x,t) = \sum_{j=1}^{\infty} E_{a_n} \left( -\beta^2 T^a \right) - \frac{E_{a_n} \left( -T^a \right)}{E_{a_n} \left( -T^a \right)} \left( \int_0^T f(x) \phi_j(x) dx \right) \phi_j(x)
\]

(38)

\[
= \sum_{j=1}^{\infty} E_{a_n} \left( -\beta^2 T^a \right) \left( \int_0^T (T - r)^{a-1} E_{a_n} \left( -\beta^2 (T - r)^a \right) \right) \phi_j(x)
\]

(39)

\[
+ \sum_{j=1}^{\infty} \left( \int_0^T (T - r)^{a-1} E_{a_n} \left( -\beta^2 (T - r)^a \right) H_j(r) dr \right) \phi_j(x).
\]

(40)
\[ + \sum_{j=1}^{\infty} \int_{0}^{T} (t-r)^{\alpha-1} \left( E_{\alpha,a} \left( -j^{2} (t-r)^{\alpha} \right) \right) dr \]

\[ = j_{1}(x,t) + j_{2}(x,t) + j_{3}(x,t) + j_{4}(x,t). \]

**Step 1. Estimation of the Term \( J_{1} \).**

In order to evaluate \( J_{1} \), we need to control the component

\[ M_{1} = \frac{E_{\alpha,1} \left( -j^{2} T^{\alpha} \right)}{E_{\alpha,1} \left( -j^{2} T^{\alpha} \right)} - \frac{E_{\alpha,1} \left( -j^{2} T^{\alpha} \right)}{E_{\alpha,1} \left( -j^{2} T^{\alpha} \right)}. \]

It is obvious to compute the above term as follows:

\[ M_{1} = \frac{E_{\alpha,1} \left( -j^{2} T^{\alpha} \right) - E_{\alpha,1} \left( -j^{2} T^{\alpha} \right)}{E_{\alpha,1} \left( -j^{2} T^{\alpha} \right)} \]

\[ \leq \frac{C_{\alpha}}{1 + j^{2} T^{\alpha}} \leq \frac{C_{\alpha}}{j^{2} (T^{\alpha} + 1)}, \]

we know that

\[ E_{\alpha,1} \left( -j^{2} T^{\alpha} \right) \geq \frac{C_{\alpha}}{1 + j^{2} T^{\alpha}} \leq \frac{C_{\alpha}}{j^{2} (T^{\alpha} + 1)}. \]

By a similar explanation as above, we find that

\[ E_{\alpha,1} \left( -j^{2} T^{\alpha} \right) - E_{\alpha,1} \left( -j^{2} T^{\alpha} \right) \leq T^{\alpha} \left( 1 - \beta \right)^{\epsilon} \beta j^{2} e^{\epsilon} \beta j^{2} (T^{\alpha} + 1). \]

where the hidden constant depends on \( \epsilon, T, \epsilon' \). From two above observation, we find that

\[ M_{1} = \frac{E_{\alpha,1} \left( -j^{2} T^{\alpha} \right)}{E_{\alpha,1} \left( -j^{2} T^{\alpha} \right)} - E_{\alpha,1} \left( -j^{2} T^{\alpha} \right) \leq \frac{C_{\alpha}}{1 + j^{2} T^{\alpha}} \leq \frac{C_{\alpha}}{j^{2} (T^{\alpha} + 1)}. \]

where the hidden constant depends on \( \alpha, T, \epsilon \). Hence, we obtain that

\[ \|J_{1}\|_{\overline{H}^{1}(\Omega)}^{2} = \sum_{j=1}^{\infty} \int_{0}^{T} \left( j_{1}(x,t) \right)^{2} \right) \]

\[ \leq (1 - \beta)^{2} \left( \int_{0}^{T} \left( j_{1}(x,t) \right)^{2} \right). \]

It implies that the following bound

\[ \|J_{1}\|_{\overline{H}^{1}(\Omega)} \leq (1 - \beta)^{2} \|f\|_{H^{1,2\alpha,2\beta}^{1,2}(\Omega)}. \]

**Step 2. Estimation of the Term \( J_{3} \).**

By using Parseval’s equality and noting that Problem (44), we find that

\[ \|J_{3}\|_{\overline{H}^{1}(\Omega)}^{2} = \sum_{j=1}^{\infty} \int_{0}^{T} \left( j_{3}(x,t) \right)^{2} \right) \]

\[ \leq (1 - \beta)^{2} \left( \int_{0}^{T} \left( j_{3}(x,t) \right)^{2} \right). \]

where we have used the fact that \( E_{\alpha,a} \left( -j^{2} (T-r)^{\alpha} \right) \leq C_{\alpha} \)

Hence, we find that

\[ \|J_{3}\|_{\overline{H}^{1}(\Omega)} \leq \frac{C_{\alpha}}{1 + j^{2} T^{\alpha}} \leq \frac{C_{\alpha}}{j^{2} (T^{\alpha} + 1)}. \]

**Step 3. Estimation of the Term \( J_{5} \).**

By using Parseval’s equality, we derive that

\[ \|J_{5}\|_{\overline{H}^{1}(\Omega)}^{2} = \sum_{j=1}^{\infty} \int_{0}^{T} \left( j_{5}(x,t) \right)^{2} \right) \]

\[ \leq (1 - \beta)^{2} \left( \int_{0}^{T} \left( j_{5}(x,t) \right)^{2} \right). \]

where the hidden constant depends on \( \alpha, T, \epsilon \). Hence, we obtain that

\[ \|J_{5}\|_{\overline{H}^{1}(\Omega)} \leq (1 - \beta)^{2} \|f\|_{H^{1,2\alpha,2\beta}^{1,2}(\Omega)}. \]
It is easy to verify that
\[
\frac{E_{\alpha,a}(-j^\beta r^a)}{E_{\alpha,a}(-j^{2\beta} r^a)} \leq 1 + T^{\beta} j^{\beta} \leq T^{\alpha} r^a. \tag{50}
\]

Using Hölder’s inequality, we derive that
\[
\left( \int_0^T (T-r)^{a-1} \left( E_{\alpha,a}(-j^\beta (T-r)^a) - E_{\alpha,a}(-j^{2\beta} (T-r)^a) \right) H_j(r) dr \right)^2 \\
\leq \left( \int_0^T (T-r)^{a-1} \right)^2 \left( \int_0^T (T-r)^{a-1} \left( E_{\alpha,a}(-j^\beta (T-r)^a) \right)^2 |H_j(r)|^2 dr \right) \\
\leq \int_0^T (T-r)^{a-1} \left( E_{\alpha,a}(-j^\beta (T-r)^a) \right)^2 |H_j(r)|^2 dr.
\tag{51}
\]

By a similar explanation, we can get that the following bound:
\[
E_{\alpha,a}(-j^\beta r^a) - E_{\alpha,a}(-j^{2\beta} r^a) \leq C_2(\alpha, \epsilon, \gamma) r^{1+2\epsilon+2\gamma},
\tag{52}
\]
for any \( \gamma > 0 \). This implies that
\[
\left( E_{\alpha,a}(-j^\beta (T-r)^a) - E_{\alpha,a}(-j^{2\beta} (T-r)^a) \right)^2 \leq (T-r)^{2\alpha} (1-\beta)^{2\alpha} j^{2\epsilon+2\gamma}. \tag{53}
\]

Hence, we get that the following bound:
\[
\int_0^T (T-r)^{a-1} \left( E_{\alpha,a}(-j^\beta (T-r)^a) - E_{\alpha,a}(-j^{2\beta} (T-r)^a) \right)^2 |H_j(r)|^2 dr \\
\leq (1-\beta)^{2\alpha} \int_0^T (T-r)^{a+2\alpha-1} j^{2\epsilon+2\gamma} |H_j(r)|^2 dr. \tag{54}
\]

Combining Problems (49), (50), and (54), we derive that
\[
\| J_2 \|_{l_{\beta}^{\epsilon,\gamma}(\Omega)} \leq \int_0^T (T-r)^{a+2\alpha-1} dr \\
\cdot \left( \sum_{j=1}^{\infty} \left( \int_0^T (T-r)^{a+2\alpha-1} |H_j(r)|^2 \right)^2 \right)^{1/2} \\
= (1-\beta)^{2\alpha} \int_0^T (T-r)^{a+2\alpha-1} |H_j(r)|^2 dr \\
\leq (1-\beta)^{2\alpha} \int_0^T (T-r)^{a+2\alpha-1} dr \\
\cdot \| H \|_{l_{\beta}^{\epsilon,\gamma}(\Omega)}^2. \tag{55}
\]

It is obvious to see that
\[
\int_0^T (T-r)^{a+2\alpha-1} dr = \frac{T^{a+2\alpha}}{\alpha + 2\alpha \epsilon}. \tag{56}
\]

So, we obtain that the following confirmation
\[
\| J_2 \|_{l_{\beta}^{\epsilon,\gamma}(\Omega)} \leq (1-\beta)^{2\alpha} T^{\alpha+2\alpha \epsilon} \| H \|_{L^{\infty}(0,T;H^{2-\alpha\epsilon}(\Omega))}. \tag{57}
\]

**Step 4. Estimation of the Term J_4.**

By using Parseval’s equality and Hölder’s inequality, we get that
\[
\| J_4 \|_{l_{\beta}^{\epsilon,\gamma}(\Omega)}^2 = \sum_{j=1}^{\infty} \int_0^T (t-r)^{a-1} \left( E_{\alpha,a}(-j^\beta (t-r)^a) \right)^2 |H_j(r)|^2 dr \\
\leq \left( \int_0^T (t-r)^{a-1} dr \right)^2 \sum_{j=1}^{\infty} \int_0^T (t-r)^{a-1} \left( E_{\alpha,a}(-j^\beta (t-r)^a) \right)^2 |H_j(r)|^2 dr.
\tag{58}
\]

By a similar techniques as in Problem (54), we derive that
\[
\int_0^T (t-r)^{a-1} \left( E_{\alpha,a}(-j^\beta (t-r)^a) \right)^2 |H_j(r)|^2 dr \\
\leq (1-\beta)^{2\alpha} \int_0^T (t-r)^{a+2\alpha-1} j^{2\epsilon+2\gamma} |H_j(r)|^2 dr. \tag{59}
\]

By review two latter observations, we can deduce that
\[
\| J_4 \|_{l_{\beta}^{\epsilon,\gamma}(\Omega)} \leq (1-\beta)^{2\alpha} \int_0^T (t-r)^{a+2\alpha-1} \left( \sum_{j=1}^{\infty} \left( \int_0^T (t-r)^{a+2\alpha-1} |H_j(r)|^2 \right)^2 \right) dr \\
= (1-\beta)^{2\alpha} \int_0^T (t-r)^{a+2\alpha-1} \| H \|_{l_{\beta}^{\epsilon,\gamma}(\Omega)}^2 dr \\
\leq (1-\beta)^{2\alpha} \int_0^T (t-r)^{a+2\alpha-1} \| H \|_{L^{\infty}(0,T;H^{2-\alpha\epsilon}(\Omega))}^2 dr. \tag{60}
\]

The above inequality implies that the following estimate:
\[
\| J_4 \|_{l_{\beta}^{\epsilon,\gamma}(\Omega)} \leq (1-\beta)^{2\alpha} \int_0^T (t-r)^{a+2\alpha-1} dr \| H \|_{l_{\beta}^{\epsilon,\gamma}(\Omega)}^2 \tag{61}
\]

By similar computation as above, we deduce that
\[
\| J_4 \|_{l_{\beta}^{\epsilon,\gamma}(\Omega)} \leq (1-\beta)^{2\alpha} \int_0^T (t-r)^{a+2\alpha-1} dr \| H \|_{L^{\infty}(0,T;H^{2-\alpha\epsilon}(\Omega))}^2. \tag{62}
\]
Combining four steps as above, we deduce that
\[
\|u_{\beta}(.t) - u_{s}(.t)\|_{H^{1}(\Omega)} \\
\leq \|J_{1}\|_{H^{1}(\Omega)} + \|J_{2}\|_{H^{1}(\Omega)} + \|J_{3}\|_{H^{1}(\Omega)} + \|J_{4}\|_{H^{1}(\Omega)}
\]
\[
\leq (1 - \beta)^{\epsilon_{2}} \left(\|f\|_{L^{1}}^{2} + \|H\|_{L^2}^{2} \right) + (1 - \beta)^{\gamma} (t^{-\alpha} + 1) \|H\|_{L^2}^{2}.
\]  
(63)

Let us choose
\[
\epsilon = \frac{b - s - 2\beta - 2}{4}, \quad \epsilon' = 2, \quad \gamma = \frac{b - s + 2\beta + 2}{b - s - 2\beta - 2}.
\]  
(64)

Then from some above observations, we deduce that the following estimate:
\[
\|u_{\beta}(.t) - u_{s}(.t)\|_{H^{1}(\Omega)} \\
\leq (1 - \beta)^{\epsilon_{2}} \left(\|f\|_{H^{1}(\Omega)} + \|H\|_{L^2}^{2} \right) + (1 - \beta)^{\gamma} (t^{-\alpha} + 1) \|H\|_{L^2}^{2}.
\]  
(65)

This estimate implies that the desired result, Problem (34).

\[\square\]

5. Conclusion

In this work, we consider the fractional problem for partial differential equation. We investigate the convergence of the mild solution of the diffusion equation with time and space fractional. Moreover, we consider the problem in two cases which are forward problem and inverse problem by using new techniques to overcome some of the complex assessments.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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