Statistical convergence of nets on locally solid Riesz spaces

May 28, 2021

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Abstract

The statistical convergence is handled for sequences with the natural density, in general. In a recent paper, the statistical convergence for nets in Riesz spaces has been studied and investigated by developing topology-free techniques in Riesz spaces. In this paper, we introduce the statistically topological convergence for nets on locally solid Riesz spaces with solid topologies. Moreover, we introduce the statistical continuity on locally solid Riesz spaces.

Keywords: Statistical convergence of nets, finitely additive measure, Riesz spaces, statistical continuous operator

2010 AMS Mathematics Subject Classification: 40A35, 46A40, 40A05, 46B42

1 Introduction and preliminaries

Riesz space and statistical convergence are the natural and efficient tools in the theory of functional analysis. Riesz space was introduced
by F. Riesz in [19] and the idea of statistical convergence was firstly introduced by Zygmund [23], after then, Fast [12] and Steinhaus [20] independently improved the idea of statistical convergence. Riesz space is an ordered vector space having many applications in measure theory, operator theory, and applications in economics (cf. [1, 2, 22]). On the other hand, statistical convergence is a generalization of the ordinary convergence of a real sequence. Several applications and generalizations of the statistical convergence of sequences have been investigated by several authors (cf. [5, 6, 12, 13, 16, 20, 21]). In general, the statistical convergence of sequences is considered with the natural density of sets on the natural numbers \( \mathbb{N} \). However, Connor introduced the notion of statistical convergence for sequences with finitely additive set function [9, 10]. After then, some similar works have been done (cf. [8, 11, 18]). The study related to this paper is done by Aydin and Temizsu in [7], where the statistical convergence was introduced for nets. In this work, we introduce the concept of statistical convergence for nets and statistical continuous operators on locally solid Riesz space with solid topologies.

First, let us remember some notations and terminologies used in this paper. A binary relation \( \leq \) on a set \( A \) is called a preorder if it is reflexive and transitive. A non-empty set \( A \) with a preorder binary relation \( \leq \) is said to be a directed upwards (directed set, shortly) if for each pair \( x, y \in A \) there exist \( z \in A \) such that \( x \leq z \) and \( y \leq z \). Unless otherwise stated, we consider all directed sets as infinite in this paper. For given elements \( a \) and \( b \) in a preordered set \( A \) such that \( a \leq b \), the set \( \{ x \in A : a \leq x \leq b \} \) is called an order interval in \( A \). A subset \( I \) of \( A \) is called an order bounded set whenever \( I \) is contained in an order interval. A function whose domain is a directed set is said to be a net. A net is briefly abbreviated as \( (x_\alpha)_{\alpha \in A} \) with its directed domain set \( A \).

We remind that a map from a field \( \mathcal{M} \) to \([0, \infty]\) is called finitely additive measure whenever \( \mu(\emptyset) = 0 \) and \( \mu(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i) \) for all finite disjoint sets \( \{E_i\}_{i=1}^{n} \) in \( \mathcal{M} \) (cf. [14, p.25]). We take the following definitions from [7].

**Definition 1.1.** Let \( A \) be a directed set and \( \mathcal{M} \) be a subfield of \( \mathcal{P}(A) \).

1. An order interval \([a, b]\) of \( A \) is said to be a finite order interval if it is a finite subset of \( A \).

2. \( \mathcal{M} \) is called an interval field on \( A \) whenever it includes all finite order intervals of \( A \).
A finitely additive measure $\mu : \mathcal{M} \rightarrow [0, 1]$ is said to be a directed set measure if $\mathcal{M}$ is an interval field and $\mu$ satisfies the following facts: $\mu(I) = 0$ for every finite order interval $I \in \mathcal{M}$; $\mu(A) = 1$; $\mu(C) = 0$ whenever $C \subseteq B$ and $\mu(B) = 0$ holds for $B, C \in \mathcal{M}$.

Following from [7, Rem.2.5] that the directed set measure is an extension of the natural density. In this paper, we consider all nets with a directed set measure on the interval fields of the power set of the index sets. Moreover, to simplify the presentation, a directed set measure on an interval field $\mathcal{M}$ of directed set $A$ will be expressed briefly as a measure on the directed set $A$.

Recall that a real vector space $E$ with an order relation “$\leq$” is called an ordered vector space if, for each $x, y \in E$ with $x \leq y$, $x + z \leq y + z$ and $\alpha x \leq \alpha y$ hold for all $z \in E$ and $\alpha \in \mathbb{R}_+$. An ordered vector space $E$ is called a Riesz space or a vector lattice if, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf \{x, y\} \quad \text{and} \quad x \vee y = \sup \{x, y\}$$

exist in $E$, respectively. A subset $I$ of a Riesz space $E$ is said to be a solid set if, for each $x \in E$ and $y \in I$ with $|x| \leq |y|$, it follows that $x \in I$. A solid vector subspace is called an order ideal. An order closed ideal is called a band. Also, a band $B$ is called a projection band whenever it satisfies $E = B \oplus B^d$, where $B^d$ is the disjoint complement set of $B$. A Riesz space $E$ has the Archimedean property provided that $\frac{1}{n}x \downarrow 0$ holds in $E$ for each $x \in E_+$. In this paper, unless otherwise stated, all Riesz spaces are assumed to be real and Archimedean. We continue with the crucial notion of Riesz spaces (cf. [1, 2, 15, 22]).

**Definition 1.2.** A net $(x_\alpha)_{\alpha \in A}$ in a Riesz space $E$ is called order convergent to $x \in E$ if there exists another net $(y_\alpha)_{\alpha \in A} \downarrow 0$ (i.e., $\inf y_\alpha = 0$ and $y_\alpha \downarrow$) such that $|x_\alpha - x| \leq y_\alpha$ holds for all $\alpha \in A$.

We refer the reader for some different types of the order convergence and some relations among them to [3].

We remind that a linear topology $\tau$ on a vector space $E$ means that it is a topology on $E$ which makes the addition and the scalar multiplication continuous. For each topological vector space, it is well known that there is a base $\mathcal{N}$ consisting of zero neighborhoods holding the following properties (cf. [11, 14]):

(a) Every $U \in \mathcal{N}$ is a balanced set, i.e., $\lambda U \subseteq U$ for all $|\lambda| \leq 1$;
(b) Each $U \in \mathcal{N}$ is an absorbing set, i.e., for every $u \in U$, there exists $\lambda > 0$ such that $\lambda u \in U$;

(c) For every $U \in \mathcal{N}$, there is another $V \in \mathcal{N}$ with $V + V \subseteq U$;

(d) For any $U_1, U_2 \in \mathcal{N}$, there is $U \in \mathcal{N}$ such that $U \subseteq U_1 \cap U_2$;

(e) For every $U \in \mathcal{N}$ and every scalar $\lambda$, the set $\lambda U$ is in $\mathcal{N}$.

Whenever we mention a basic zero neighborhood, we always assume that it belongs to a base which satisfies the properties (a)-(e).

**Definition 1.3.** Let $\tau$ be a linear topology on a Riesz space $E$. Then $(E, \tau)$ is said to be a locally solid Riesz space or locally solid vector lattice whenever $\tau$ has a base at zero which consists of solid sets.

We denote $\mathcal{N}_{sol}$ as the solid base of a locally solid Riesz space.

## 2 Statistically topological convergence

Let $K$ be a subset of natural numbers and define a new set $K_n = \{k \in K : k \leq n\}$. Then we denote $|K_n|$ for the cardinality of that set $K_n$. If the limit $\delta(K) := \lim_{n \to \infty} |K_n|/n$ exists then $\delta(K)$ is called the natural density or asymptotic density of the set $K$. On the other hand, let $X$ be a topological space and $(x_n)$ be a sequence in $X$. Then $(x_n)$ is said to be statistically convergent to $x \in X$ whenever, for each neighborhood $U$ of $x$, we have $\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0$ (cf. [16, 17]).

Motivated by the above definitions, we give the following notion which is crucial for the present paper.

**Definition 2.1.** Let $(x_\alpha)_{\alpha \in A}$ be a net in a locally solid Riesz space $(E, \tau)$. Then $(x_\alpha)_{\alpha \in A}$ is called $\mu$-statistically topological convergent to $x \in E$ if, for every zero $\tau$-neighborhood $U$, there exists an index $\alpha_u$ related $U$ such that

$$\mu(\{\alpha_u \leq \alpha \in A : (x_\alpha - x) \notin U\}) = 0.$$ 

We abbreviate this convergence as $x_\alpha \xrightarrow{\mu-st\_\tau} x$. Then we shortly say that $(x_\alpha)_{\alpha \in A}$ is $\mu$-statistically $\tau$-convergent to $x$.

Briefly, $x_\alpha \xrightarrow{\mu-st\_\tau} x$ if $\mu(B_{\{\alpha_u,U\}}(x_\alpha, x)) = 0$ for each zero $\tau$-neighborhood $U$ and for some indexes $\alpha_u \in A$, where

$$B_{\{\alpha_u,U\}} = \{\alpha_u \leq \alpha \in A : (x_\alpha - x) \notin U\}.$$
We denote $E_{\mu-st, \tau}$ as the set of all $\mu$-statistically $\tau$-convergent nets in a locally solid Riesz space $(E, \tau)$. Also, we observe the following useful and important fact.

**Remark 2.2.** Consider a net $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \xrightarrow{\mu-st, \tau} x$ in a locally solid Riesz space $(E, \tau)$. Then, for a fixed zero $\tau$-neighborhood $U$, there exists an index $\alpha_u \in A$ such that $\mu(B_{\{\alpha_u, U\}}(\{x_\alpha, x\})) = 0$, where

$$B_{\{\alpha_u, U\}} = \{\alpha \leq \alpha \in A : (x_\alpha - x) \notin U\} = \{\alpha \in A : \alpha_u \leq \alpha\} \cap \{\alpha \in A : (x_\alpha - x) \notin U\}.$$

Thus, the complement set of $B_{\{\alpha_u, U\}}$ is

$$B^c_{\{\alpha_u, U\}} = \{\alpha \in A : \alpha_u \not\leq \alpha\} \cup \{\alpha \in A : (x_\alpha - x) \in U\}.$$

So, it follows that $\mu(B^c) = 1$, and so, one can obtain that

$$\mu(\{\alpha \in A : (x_\alpha - x) \in U\}) = 1$$

does need not hold in general. Therefore, generally, we have $\mu(\{\alpha \in A : \alpha_u \not\leq \alpha\}) \neq 0$.

We can introduce the following notion: a net $x_\alpha \xrightarrow{\mu-st, \tau} x$ is said to be straight $\mu$-statistically topological convergent to $x \in E$ if $\mu(\{\alpha \in A : (x_\alpha - x) \in U\}) = 1$, where $\alpha_u \in A$ is the index of $\mu$-st-$\tau$-convergence. But, we do not dwell on this definition in this study. Hence, unless otherwise stated, we assume that the measure of sets $\{\alpha \in A : \alpha_u \not\leq \alpha\}$ and $\{\alpha \in A : (x_\alpha - x) \in U\}$ in Remark 2.2 are different from zero for all $\mu$-statistically $\tau$-convergent nets $(x_\alpha)_{\alpha \in A}$ and for each zero $\tau$-neighborhood $U$. Following from the solidness property of zero $\tau$-neighborhoods, we give the following observation.

**Lemma 2.3.** It is clear that $x_\alpha \xrightarrow{\mu-st, \tau} x$ if and only if $|x_\alpha - x| \xrightarrow{\mu-st, \tau} 0$ in locally solid Riesz spaces.

The following example which is similar to [5, Exam.1.3] yields two sequences, one is $\mu$-statistically $\tau$-convergent and the other is not on the same space.

**Example 2.4.** Let consider the Riesz space $E := c_0$ the set of all real null sequences. Thus, $E$ is a Banach lattice with the supremum norm $\|\cdot\|_{\infty}$. It follows from [1, Thm.2.28] that $(E, \|\cdot\|_{\infty})$ is also a locally solid
Riesz space. It can be seen that the base of solid topology generated by the supremum norm consists of zero neighborhoods

\[ W_\varepsilon = \{ x \in E : \|x\|_\infty < \varepsilon \}, \]

where \( \varepsilon \) is an arbitrary positive real number. Now, consider the sequence \((e_n)\) of the standard unit vectors in \( E \). Then \( e_n \xrightarrow{\mu-\text{str}} 0 \) in \( E \). Indeed, take an arbitrary zero \( \tau \)-neighborhood \( U \). Then there exists some \( W_\varepsilon \in \mathcal{N}_{\text{sol}} \) for some \( \varepsilon > 0 \) such that \( W_\varepsilon \subseteq U \). Thus, it follows that

\[ B = \{ n \in \mathbb{N} : e_n \notin W_\varepsilon \} = \{ n \in \mathbb{N} : \|e_n\|_\infty \geq \varepsilon \}. \]

It can be seen that \( \mu(B) \neq 0 \) because \( \|e_n\|_\infty = 1 > \varepsilon \) holds for all \( n \in \mathbb{N} \). Therefore, we obtain the desired result.

Now, take another sequence \((x_n)\) in \( E \) which is denoted by

\[ x_n := (0,0,0,\ldots,0,\frac{1}{n},0,\ldots) \]

for each \( n \in \mathbb{N} \). Thus, we have \( e_n \xrightarrow{\mu-\text{str}} 0 \). To see this, consider an arbitrary zero \( \tau \)-neighborhood \( U \). Then there exists a natural number \( n_0 \) such that \( \frac{1}{n_0} \leq \varepsilon \). So, it follows that \( W_{\frac{1}{n_0}} \subseteq W_\varepsilon \). Also, we have

\[ C = \{ n \in \mathbb{N} : x_n \notin W_{\frac{1}{n_0}} \} = \{ n \in \mathbb{N} : \|x_n\|_\infty = \frac{1}{n} > \frac{1}{n_0} \}. \]

Thus, we obtain \( \mu(C) = 0 \). Since \( \{ n \in \mathbb{N} : x_n \notin U \} \subseteq \{ n \in \mathbb{N} : x_n \notin W_\varepsilon \} \subseteq C \), we have \( \mu(\{ n \in \mathbb{N} : x_n \notin U \}) = 0 \). Therefore, we obtain the desired result, \( x_n \xrightarrow{\mu-\text{str}} 0 \).

3 Results of \( \mu \)-statistically topological convergence

Recall that a net \((x_\alpha)_{\alpha \in A}\) topological converges to a point \( x \) in a topological space \( X \) if, for every neighborhood \( U \) of \( x \), there is an index \( \alpha_0 \in A \) such that \( x_\alpha \in U \) for all \( \alpha \geq \alpha_0 \).

**Remark 3.1.** The topological convergence implies the \( \mu \)-statistically topological convergence. Indeed, suppose that \( x_\alpha \xrightarrow{\text{t}} x \) in a locally solid Riesz space \((E, \tau)\). Then, for arbitrary zero \( \tau \)-neighborhood \( U \), we have an index \( \alpha_u \) such that \( (x_\alpha - x) \in U \) for all \( \alpha \geq \alpha_u \). Thus, we get \( \mu(\{ \alpha_u \leq \alpha \in A : (x_\alpha - x) \notin U \}) = 0 \), i.e., \( x_\alpha \xrightarrow{\mu-\text{str}} x \).
The converse of Remark 3.1 does not need to be true. We continue with the following several basic and useful results which are similar to the classical ones for so many kinds of statistical convergences.

**Theorem 3.2.** Let $x_\alpha \xrightarrow{\mu_{\text{st}}} x$ and $y_\alpha \xrightarrow{\mu_{\text{st}}} y$ in a locally solid Riesz space $(E, \tau)$. Then we have the following statements:

(i) if $\tau$ is Hausdorff, $x_\alpha \xrightarrow{\mu_{\text{st}}} x$ and $x_\alpha \xrightarrow{\mu_{\text{st}}} z$ then $x = z$;

(ii) $x_\alpha + y_\alpha \xrightarrow{\mu_{\text{st}}} x + y$;

(iii) $\lambda x_\alpha \xrightarrow{\mu_{\text{st}}} \lambda x$ for any $\lambda \in \mathbb{R}$;

(iv) $x_\alpha \xrightarrow{\mu_{\text{st}}} x$ if and only if $(x_\alpha - x) \xrightarrow{\mu_{\text{st}}} 0$.

**Proof.** The (iv) is straightforward, and so, we show the other statements. Let $U$ be an arbitrary zero $\tau$-neighborhood. Then there exists $W \in \mathcal{N}_{\text{sol}}$ such that $W \subseteq U$. Moreover, there is $V \in \mathcal{N}_{\text{sol}}$ so that $V + V \subseteq W$, and so, $V + V \subseteq U$.

(i) It follows from $x_\alpha \xrightarrow{\mu_{\text{st}}} x$ and $x_\alpha \xrightarrow{\mu_{\text{st}}} z$ that there exist indexes $\alpha_1$ and $\alpha_2$ such that

$$
\mu(B_{\{\alpha_1, \alpha_2\}} \{x_\alpha, x\}) = \mu(B_{\{\alpha_2, \alpha\}} \{x_\alpha, z\}) = 0.
$$

Also, since the index set $A$ of the net $(x_\alpha)_{\alpha \in A}$ is directed, there exists $\alpha_0 \in A$ such that $\alpha_1 \leq \alpha_0$ and $\alpha_2 \leq \alpha_0$. So, we have $\mu(\{\alpha_0 \leq \alpha \in A : (x_\alpha - x) \notin V\}) = 0$ and $\mu(\{\alpha_0 \leq \alpha \in A : (x_\alpha - z) \notin V\}) = 0$. Thus, $x_\alpha - x, x_\alpha - z \in V$ for some $\alpha \in A$. It follows that

$$
x - z = (x - x_\alpha) + (x_\alpha - z) \in V + V \subseteq U
$$

for some $\alpha \in A$. Hence, we get $(x - z) \in U$ for each zero $\tau$-neighborhood $U$. It is well known that the intersection of all zero $\tau$-neighborhood in Hausdorff space is the singleton zero. It means that $x = z$.

(ii) There are some indexes $\alpha_1, \alpha_2 \in A$ such that

$$
\mu(B_{\{\alpha_1, \alpha_2\}} \{x_\alpha, x\}) = \mu(B_{\{\alpha_2, \alpha\}} \{y_\alpha, y\}) = 0
$$

because of $x_\alpha \xrightarrow{\mu_{\text{st}}} x$ and $y_\alpha \xrightarrow{\mu_{\text{st}}} y$. Then there is $\alpha_0 \in A$ such that $\alpha_1 \leq \alpha_0$ and $\alpha_2 \leq \alpha_0$. So, we have $\mu(B_{\{\alpha_0, \alpha\}} \{x_\alpha, x\}) = \mu(B_{\{\alpha_0, \alpha\}} \{y_\alpha, y\}) = 0$. Hence, it follows from the equality

$$
(x_\alpha + y_\alpha) - (x + y) = (x_\alpha - x) + (y_\alpha + y) \in V + V \subseteq U
$$

7
that \((x_\alpha + y_\alpha) - (x+y) \notin U\) implies \((x_\alpha - x) \notin V\) or \((y_\alpha - y) \notin V\) for each \(\alpha \in A\). Without loss of generality, assume that \((x_\alpha + y_\alpha) - (x+y) \notin U\) implies \((x_\alpha - x) \notin V\). Thus, we have

\[
\{\alpha \in A : (x_\alpha + y_\alpha) - (x+y) \notin U\} \subseteq \{\alpha \in A : x_\alpha - x \notin V\}.
\]

Therefore, we obtain \(\mu(\{\alpha \in A : (x_\alpha + y_\alpha) - (x+y) \notin U\}) = 0\), i.e., \(x_\alpha + y_\alpha \xrightarrow{\mu-st} x + y\).

(iii) Take a scalar \(\lambda \in \mathbb{R}\) with \(|\lambda| < 1\). Then \((x_\alpha - x) \in W\) implies \(\lambda(x_\alpha - x) = (\lambda x_\alpha - \lambda x) \in W\) because \(W\) is a balanced set. Hence, one can see that

\[
\{\alpha_w \leq \alpha \in A : \lambda(x_\alpha - \lambda x) \notin W\} \subseteq B_{\{\alpha_w,W\}}.
\]

So, we obtain that \(\mu(\{\alpha_w \leq \alpha \in A : (\lambda x_\alpha - \lambda x) \notin W\}) = 0\). It follows \(\mu(\{\alpha_w \leq \alpha \in A : (\lambda x_\alpha - \lambda x) \notin U\}) = 0\) because of \(W \subseteq U\). Therefore, we obtain that \(\lambda x_\alpha \xrightarrow{\mu-st\tau} \lambda x\) for all \(|\lambda| < 1\).

Next, choose a scalar \(|\lambda| > 1\). For given \(W\), another zero \(\tau\)-neighborhood solid set \(N \in \mathcal{N}_{sol}\) can be found so that

\[
\{N + N + \cdots + N\}_m \subseteq W
\]

holds for \(m \in \mathbb{N}\), the smallest natural number greater or equal \(|\lambda|\). It follows from \(x_\alpha \xrightarrow{\mu-st\tau} x\) that \(\mu(B_{\{\alpha_N,N\}}(x_\alpha, x)) = 0\) for some \(\alpha_N \in A\). We observe that

\[
|\lambda x_\alpha - \lambda x| = |\lambda||x_\alpha - x| \leq m|x_\alpha - x| \in N + \cdots + N \subseteq W \subseteq U
\]

for some \(\alpha \in A\). Then \((\lambda x_\alpha - \lambda x) \notin U\) implies \((x_\alpha - x) \notin N\) for \(\alpha \in A\). By the same argument in the last part of proof (ii), we have \(\mu(\{\alpha_N \leq \alpha \in A : (\lambda x_\alpha - \lambda x) \notin U\}) = 0\), i.e., \(\lambda x_\alpha \xrightarrow{st-\tau} \lambda x\).

**Definition 3.3.** A net \((x_\alpha)_{\alpha \in A}\) in a locally solid Riesz space \((E, \tau)\) is called \(\mu\)-statistically \(\tau\)-bounded whenever there is some \(\lambda > 0\) scalar such that

\[
\mu(\{\alpha_u \leq \alpha \in A : \lambda x_\alpha \notin U\}) = 0
\]

holds for each zero \(\tau\)-neighborhood \(U\) with some index \(\alpha_u \in A\).

One can see that a topological bounded net is \(\mu\)-statistically \(\tau\)-bounded in locally solid Riesz spaces. However, the converse not need to be true. To see this, we consider [7, Exam.2.3].
Example 3.4. Take a field $\mathcal{M}$ consisting of countable or co-countable subsets of the index set of a net $(x_\alpha)_{\alpha \in A}$. Then every subnet of $(x_\alpha)_{\alpha \in A}$ with the countable index is $\mu$-statistically $\tau$-bounded. But $(x_\alpha)_{\alpha \in A}$ might have a subnet with a countable index that is not topologically bounded.

Remark 3.5. It follows from [1, Thm.2.19] every order bounded set is topologically bounded. Thus, every ordered bounded net is $\mu$-statistically $\tau$-bounded in locally solid Riesz spaces.

Theorem 3.6. Every $\mu$-statistically $\tau$-convergent net is $\mu$-statistically $\tau$-bounded.

Proof. Assume that a net $(x_\alpha)_{\alpha \in A}$ is $\mu$-statistically $\tau$-converges to $x$ in a locally solid Riesz space $(E, \tau)$. Fix a zero $\tau$-neighborhood $U$. Then there are $W, V \in \mathcal{N}_{\text{sol}}$ such that $V + V \subseteq W \subseteq U$. So, there is an index $\alpha_0$ such that $\mu(B_{\{\alpha, V\}}\{x_\alpha, x\}) = 0$. By using the absorbing property of $V$, there is a scalar $\beta > 0$ such that $\beta x \in V$. Now, choose a scalar $\sigma_u$ such that $|\sigma_u| \leq 1$ and $|\sigma_u| \leq \beta$. Then it follows from $|\sigma_u| \leq |\beta| x$ that $\sigma_u x \in V$. Moreover, $(x_\alpha - x) \in V$ implies $\sigma_u (x_\alpha - x) \in V$ because $V$ is balanced. Then we observe from

$$
\sigma_u x_\alpha = \sigma_u (x_\alpha - x) + \sigma_u x \in V + V \subseteq U
$$

that $\sigma_u x_\alpha \notin U$ implies $\sigma_u (x_\alpha - x) \notin V$ for $\alpha \in A$. Hence, we obtain

$$
\{\alpha \in A : \sigma_u x_\alpha \notin U\} \subseteq \{\alpha \in A : \sigma (x_\alpha - x) \notin V\}
$$

$$
\subseteq \{\alpha \in A : (x_\alpha - x) \notin V\}.
$$

It follows that $\mu(\{\alpha, \alpha \leq \alpha \in A : \sigma_u x_\alpha \notin U\}) = 0$. If we take the scalar $\lambda$ in Definition 3.7 as the minimum of $\{\sigma_u : U \text{ is zero } \tau\text{-neighborhood}\}$ then we get the desired result.

Definition 3.7. A net $(x_\alpha)_{\alpha \in A}$ in a locally solid Riesz space $(E, \tau)$ is called $\mu$-statistically $\tau$-Cauchy if the net $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha') \in A \times A}$ is $\mu$-statistically $\tau$-convergent to zero.

Question 3.8. Is $\mu$-statistically $\tau$-Cauchy net $\mu$-statistically $\tau$-bounded?

Theorem 3.9. Every $\mu$-statistically $\tau$-convergent net is a $\mu$-statistically $\tau$-Cauchy.
\textbf{Proof.} Suppose that $(x_\alpha)_{\alpha \in A}$ is $\mu$-statistically $\tau$-convergent to $x$ in a locally solid Riesz space $(E, \tau)$. Then, for any zero $\tau$-neighborhood $U$ with zero $\tau$-neighborhood solid sets $V + V \subseteq W \subseteq U$, there exists an index $\alpha_v$ such that $\mu(B_{\{\alpha_v, V\}}(x_\alpha, x)) = 0$. Then we have
\[ x_\alpha - x_{\alpha'} = (x_\alpha - x) + (x - x_{\alpha'}) \in V + V \subseteq U \]
for some $\alpha, \alpha' \in A$. Thus, one can see that $(x_\alpha - x_{\alpha'}) \notin U$ implies $(x_\alpha - x) \notin V$ for some $\alpha$. Thus, we obtain
\[ \mu\{(\alpha_v : (x_\alpha - x_{\alpha'}) \notin U)\} = 0. \]
Therefore, $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha') \in A \times A} \xrightarrow{\mu\text{-st}} 0$. It proves that $(x_\alpha)_{\alpha \in A}$ is $\mu$-statistically $\tau$-Cauchy net. \hfill \square

Recall that a linear topology $\tau$ on a Riesz space $E$ is locally solid if and only if it is generated by a family $\{\rho_j\}_{j \in J}$ of Riesz pseudonorms (cf. [1, Thm.2.28]). It is well known that if a subset $A$ of $E$ is topological bounded then $\rho_j(A)$ is bounded in $\mathbb{R}$ for each $j \in J$.

\textbf{Remark 3.10.} If a net $(x_\alpha)_{\alpha \in A}$ is $\mu$-statistically $\tau$-bounded in a locally solid vector lattice with a family of a Riesz pseudonorms $\{\rho_j\}_{j \in J}$ then $\rho_j(x_\alpha)$ is not need to be bounded in $\mathbb{R}$ for all $j \in J$.

The converse of Remark 3.10 is also not need to be hold. To see this, consider the following example.

\textbf{Example 3.11.} Let $(E, \tau)$ be a locally solid vector lattice with a family of a Riesz pseudonorms $\{\rho_j\}_{j \in J}$. Define $\hat{\rho}_j := \frac{\rho_j}{1 + \rho_j}$ for each $j \in J$. Then it can be seen that $\hat{\rho}_j$ is also a Riesz pseudonorm on $E$ and the topology $\tau$ is generated by the family of $\{\hat{\rho}_j\}_{j \in J}$. Moreover, $\hat{\rho}_j(x_\alpha) \leq 1$ for all $\alpha \in A$. It means that $(x_\alpha)_{\alpha \in A}$ is bounded in $\mathbb{R}$ for every net $(x_\alpha)_{\alpha \in A}$ in $E$. However, we might have a net in $E$ that is not $\mu$-statistically $\tau$-bounded.

\textbf{Proposition 3.12.} Let $(x_\alpha)_{\alpha \in A}$, $(y_\alpha)_{\alpha \in A}$ and $(z_\alpha)_{\alpha \in A}$ nets in a locally solid Riesz space $(E, \tau)$ with $x_\alpha \leq y_\alpha \leq z_\alpha$ for all $\alpha \in A$. Then $x_\alpha \xrightarrow{\mu\text{-st}} w$ and $z_\alpha \xrightarrow{\mu\text{-st}} w$ implies $y_\alpha \xrightarrow{\mu\text{-st}} w$.

\textbf{Proof.} Let $U$ be an arbitrary zero $\tau$-neighborhood in $E$ with $V \in \mathcal{N}_{\text{sol}}$ such that $V + V \subseteq U$. Since $x_\alpha \xrightarrow{\mu\text{-st}} w$ and $z_\alpha \xrightarrow{\mu\text{-st}} w$, there exist $\alpha_v$ such that
\[ \mu(B_{\{\alpha_v, V\}}(x_\alpha, w)) = \mu(B_{\{\alpha_v, V\}}(z_\alpha, w)) = 0. \]
On the other hand, it follows from $x_\alpha \leq y_\alpha \leq z_\alpha$ for all $\alpha \in A$ that

$$|y_\alpha - w| \leq |x_\alpha - w| + |z_\alpha - w| \in V + V \subseteq U$$

holds for some $\alpha \in A$. So, $(y_\alpha - w) \notin U$ implies that $(x_\alpha - w) \notin V$ or $(z_\alpha - w) \notin V$ for $\alpha \in A$. Therefore, one can get $\mu\{\alpha_0 \leq \alpha : (y_\alpha - w) \notin V\} = 0$ for both cases. Hence, $y_\alpha \xrightarrow{\mu-st} w$.

**Theorem 3.13.** Let $C$ be a projection band and $x_\alpha \xrightarrow{\mu-st} x$ in a locally solid Riesz space $(E, \tau)$. Then $P_C(x_n) \xrightarrow{\mu-st} P_C(x)$ for the corresponding order projection $P_C$ of $C$.

**Proof.** Consider a fixed zero $\tau$-neighborhood $U$. Then there is an index $\alpha_u$ such that $\mu(B_{\{\alpha_u, U\}}\{x_\alpha : x\}) = 0$ because of $x_\alpha \xrightarrow{\mu-st} x$. On the other hand, it follows from [15, Thm.24.5 and Thm.24.6] that $P_C$ is an order continuous lattice homomorphism and $0 \leq P_C \leq I$. Also, by considering [2, Thm.2.14], we observe that

$$|P_C(x_\alpha) - P_C(x)| = P_C(|x_\alpha - x|) \leq |x_\alpha - x|,$$

holds for all $\alpha \in A$. Thus, $(P_C(x_\alpha) - P_C(x)) \notin U$ implies $(x_\alpha - x) \notin U$ for all $\alpha \geq \alpha_u$. Thus, it follows that

$$\mu\{\alpha_u \leq \alpha \in A : (P_C(x_\alpha) - P_C(x)) \notin U\} = 0.$$

Therefore, $P_C(x_n) \xrightarrow{st-ur} P_C(x)$ in $E$. \qed

## 4 The $\mu$-statistically continuity

**Definition 4.1.** An operator $T$ between locally solid Riesz spaces $(E, \tau)$ and $(F, \tau')$ is called $\mu$-statistically topological continuous operator whenever $x_\alpha \xrightarrow{\mu-st} x$ in $E$ implies $T(x_\alpha) \xrightarrow{\mu-st} T(x)$ in $F$.

We show that the lattice operators are $\mu$-statistically topological continuous in the following sense.

**Theorem 4.2.** Let $(x_\alpha)_{\alpha \in A}$ and $(y_\alpha)_{\alpha \in A}$ be two nets in a locally solid Riesz space $(E, \tau)$. If $x_\alpha \xrightarrow{\mu-st} x$ and $y_\alpha \xrightarrow{\mu-st} y$ then:

(i) $x_\alpha \land y_\alpha \xrightarrow{\mu-st} x \land y$;

(ii) $x_\alpha \lor y_\alpha \xrightarrow{\mu-st} x \lor y$;

11
(iii) \( x_{a}^{+} \xrightarrow{\mu\text{-st}\tau} x^{+} \);

(iv) \( x_{a}^{-} \xrightarrow{\mu\text{-st}\tau} x^{-} \);

(v) \( |x_{a}| \xrightarrow{\mu\text{-st}\tau} |x| \).

**Proof.** It is enough to show statement (ii) because one can obtain the other statements from [2, Thm.1.7].

Fix an arbitrary zero \( \tau \)-neighborhood \( U \) in \( E \) with \( V \in \mathcal{N}_{\text{sol}} \) such that \( V + V \subseteq U \). Thus, there exists an index \( \alpha_{0} \) such that

\[
\mu(B_{\{\alpha_{0}, V\}}\{x_{\alpha}, x\}) = \mu(B_{\{\alpha_{0}, V\}}\{y_{\alpha}, y\}) = 0
\]

because of \( x_{\alpha} \xrightarrow{\mu\text{-st}\tau} x \) and \( y_{\alpha} \xrightarrow{\mu\text{-st}\tau} y \). On the other hand, we observe from [15, Thm.12.4] that

\[
|x_{\alpha} \lor y_{\alpha} - x \lor y| \leq |x_{\alpha} \lor y_{\alpha} - y_{\alpha} \lor x| + |y_{\alpha} \lor x - x \lor y|
\]

\[
\leq |x_{\alpha} - x| + |y_{\alpha} - y|
\]

holds for each \( \alpha \in A \). So, \( (x_{\alpha} \lor y_{\alpha} - x \lor y) \notin U \) implies that \( (x_{\alpha} - x) \notin V \) or \( (y_{\alpha} - y) \notin V \). Therefore, we get \( \mu(\{0 \leq \alpha \in A : (x_{\alpha} \lor y_{\alpha} - x \lor y) \notin U\}) = 0 \) for two cases, and so, we get the desired result.

The positive cone of a Riesz space is denoted by \( E_{+} := \{x \in E : 0 \leq x\} \). Also, it follows from Theorem 3.2 and Theorem 4.2 that \( E_{+} \) is closed under the \( \mu \)-statistically \( \tau \)-convergence in Hausdorff locally solid Riesz spaces.

**Proposition 4.3.** Every monotone \( \mu \)-statistically \( \tau \)-convergent net in Hausdorff locally solid Riesz spaces is order convergent.

**Proof.** Suppose that a net \( (x_{\alpha})_{\alpha \in A} \) is increasing and \( x_{\alpha} \xrightarrow{\mu\text{-st}\tau} x \) in a Hausdorff locally solid Riesz space \((E, \tau)\). We show that \( x_{\alpha} \uparrow x \). Take an arbitrary zero \( \tau \)-neighborhood \( U \). Then there exists an index \( \alpha_{0} \) such that \( \mu(B_{\{\alpha_{0}, U\}}\{x_{\alpha}, x\}) = 0 \). So, we obtain

\[
\mu(\{\alpha_{\leq} \leq \alpha \in A : (x_{\alpha} - x_{\alpha_{0}}) - (x - x_{\alpha_{0}}) \notin U\}) = 0.
\]

Hence, \( x_{\alpha} - x_{\alpha_{0}} \xrightarrow{\mu\text{-st}\tau} x - x_{\alpha_{0}} \). On the other hand, it follows from the increasing of \((x_{\alpha})_{\alpha \in A}\) that \( x_{\alpha} - x_{\alpha_{0}} \in E_{+} \) for all \( \alpha \geq \alpha_{0} \). Thus, \( x \geq x_{\alpha_{0}} \). It means that \( x \) is an upper bound of \((x_{\alpha})_{\alpha \in A}\). Take another upper bound \( z \) of \((x_{\alpha})_{\alpha \in A}\). Then we have

\[
\mu(\{\alpha_{\leq} \leq \alpha \in A : (z - x_{\alpha}) - (z - x) \notin U\}) = 0.
\]

So, we get \( z \geq x \), i.e., \( x_{\alpha} \uparrow x \) because of \( z - x_{\alpha} \in E_{+} \) for all \( \alpha \in A \).
Proposition 4.4. If \( x_\alpha \overset{\mu\text{-st}}{\longrightarrow} x \) and \( y_\alpha \overset{\mu\text{-st}}{\longrightarrow} y \) in a Hausdorff locally solid Riesz space \( (E, \tau) \) then \( x_\alpha \geq y_\alpha \) for all \( \alpha \) implies \( x \geq y \).

Proof. Assume that \( x_\alpha \overset{\mu\text{-st}}{\longrightarrow} x \), \( y_\alpha \overset{\mu\text{-st}}{\longrightarrow} y \) and \( y_\alpha \leq x_\alpha \) for all \( \alpha \). Then, for any zero \( \tau \)-neighborhood \( U \), there exists an index \( \alpha_0 \) such that \( \mu(B_{\{\alpha_0, V\}} \{x_\alpha, x\}) = \mu(B_{\{\alpha_0, V\}} \{y_\alpha, y\}) = 0 \) holds. Thus, we have
\[
\mu\{\alpha_\mu \leq \alpha \in A : (x_\alpha - y_\alpha) - (x - y) \notin V\} = 0.
\]
That is, \( x_\alpha - y_\alpha \overset{\mu\text{-st}}{\longrightarrow} x - y \). So, we have \( x - y \in E_+ \) because of \( x_\alpha - y_\alpha \in E_+ \). Hence, we obtain the desired result.

Proposition 4.5. The family of all \( \mu \)-statistically topological convergent nets \( E_{\mu\text{-st}} \) is a Riesz space.

Proof. It follows from Theorem 3.2, \( E_{\mu\text{-st}} \) is a vector space. Now, consider an arbitrary element \( x := (x_\alpha)_{\alpha \in A} \) in \( E_{\mu\text{-st}} \) such that \( x \overset{\text{st}}{\longrightarrow} y \) for some \( y \in E \). Thus, by applying Theorem 4.2, we get \( |x| \overset{\text{st}}{\longrightarrow} |y| \). It means that \( |x| \in E_{\mu\text{-st}} \). Therefore, by using [11] Thm.1.3 and Thm.1.7, one can obtain that \( E_{\mu\text{-st}} \) is a Riesz subspace.

Theorem 4.6. Every uniformly continuous operator between locally solid Riesz spaces is a \( \mu \)-statistically continuous operator.

Proof. Suppose that \( T : (E, \tau) \rightarrow (F, \tau') \) is a uniformly continuous operator and \( (x_\alpha)_{\alpha \in A} \) \( \mu \)-statistically \( \tau \)-convergent to \( x \in E \). Let \( U \) be an arbitrary zero \( \tau' \)-neighborhood. Then there exists a zero \( \tau \)-neighborhood \( V \) such that \( T(v) \in U \) for every \( v \in V \). Thus, \( (x_\alpha - x) \in V \) implies \( T(x_\alpha) - T(x) = T(x_\alpha - x) \in U \). It follows from \( x_\alpha \overset{\mu\text{-st}}{\longrightarrow} x \) that there is an index \( \alpha_v \) such that
\[
\mu\{\alpha_v \leq \alpha \in A : (x_\alpha - x) \notin V\} = 0.
\]
Then we observe that \( (x_\alpha - x) \notin V \) whenever \( T(x_\alpha) - T(x) \notin U \). Therefore, we obtain that
\[
\mu\{\alpha_v \leq \alpha \in A : (T(x_\alpha) - T(x)) \notin U\} = 0.
\]
As a result, \( T(x_\alpha) \overset{\mu\text{-st}}{\longrightarrow} T(x) \), i.e., \( T \) is \( \mu \)-statistically continuous.
References

[1] C. D. Aliprantis, O. Burkinshaw, Locally Solid Riesz Spaces with Applications to Economics, Amer. Math. Soc., Providence, RI, 2003.

[2] C. D. Aliprantis, O. Burkinshaw, Positive Operators, Springer, Dordrecht, 2006.

[3] Y. A. Abramovich, G. Sirotkin, On order convergence of nets, Positivity, 9 (3) (2005), 287-292.

[4] C. D. Aliprantis, R. Tourky, Cones and duality, Graduate Studies in Mathematics, vol. 84, American Mathematical Society, Providence, RI, 2007.

[5] A. Aydm, The statistically unbounded $\tau$-convergence on locally solid Riesz spaces, Turk. J. Math. 44(3), 949-956, 2020.

[6] A. Aydm, The statistical multiplicative order convergence in Riesz algebras, to appear Fact. Univ. Ser.: Math. Inform. 2021.

[7] A. Aydm, F Temizsu, Statistical convergence of nets in Riesz spaces, arxiv.org/abs/2105.08420v1.

[8] L. Cheng, G. Lin, Y. Lan, H. Lui, Measure theory of statistical convergence, Sci. China Ser. A-Math. 51(12), 2285-2303, 2008.

[9] J. Connor, Two valued measures and summability, Analysis, 10(4), 373-385, 1990.

[10] J. Connor, $R$-type summability methods, Cauchy criteria, $P$-sets and statistical convergence, Proc Amer. Math Soc. 115(2), 319–327 (1992)

[11] O. Duman, C. Orhan, $\mu$-Statistically Convergent Function Sequences, Czech. Math. 54(2), 413–422, 2004.

[12] H. Fast, Sur la convergence statistique, Colloq. Math. 2, 241-244, 1951.

[13] J. A. Fridy, On statistical convergence, Analysis, 5(4), 301-313, 1985.

[14] G. B. Folland, Real Analysis: Modern Techniques and Their Applications, Wiley, 2013.

[15] W. A. J. Luxemburg, A. C. Zaanen, Riesz Spaces I, North-Holland Pub. Co., Amsterdam, 1971.
[16] I. J. Maddox, Statistical convergence in a locally convex space, Math. Proc. Cambr. Phil. Soc. 104(1), 141-145, 1988.

[17] G. D. Maio, L. D. R. Kocinac, Statistical convergence in topology. Topology and its Applications 2008; 156 (1): 28–45.

[18] F. Moricz, Statistical limits of measurable functions, Analysis, 24(1), 1-18, 2004.

[19] Riesz F. Sur la D´ecomposition des Op´erations Fonctionelles Lin´eaires. Bologna, Atti Del Congresso Internazionale Dei Mathematics Press, 1928.

[20] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2, 73-74, 1951.

[21] C. Şençimen, S. Pehlivan, Statistical order convergence in Riesz spaces, Math. Slov. 62(2), 557-570, 2012.

[22] A. C. Zaanen, Riesz Spaces II, North-Holland Pub. Co. Amsterdam, 1983.

[23] A. Zygmund, Trigonometric Series, Cambridge University Press, Cambridge, UK, 1979.