FURTHER RESULTS
ON REGULAR FREDHOLM PAIRS AND CHAINS

ENRICO BOASSO

The main objective of the present article is to characterize regular Fredholm pairs and chains in terms of Fredholm operators.

AMS 2000 Subject Classification: Primary 47A13; Secondary 47A53.

Key words: Fredholm pairs and chains, regular operators.

1. INTRODUCTION

In multiparameter operator theory there exist objects that in the context of Hilbert spaces can be characterized using linear and bounded maps, to mention only some of the works related to this subject, see for example [9, 10, 11, 12]. Since the adjoint plays a key role in this relationship between one and several variable operator theory, it is not possible to directly translate these results to the frame of Banach spaces. However Harte and Lee [7] showed that generalized inverses could be used to reformulate the Hilbert space situation for regular Fredholm Banach space chains and complexes.

On the other hand, Fredholm pairs and chains have been recently introduced and their main properties have been studied, see [1, 2, 4, 5, 8] and the monograph [3]. These objects consist in generalizations of the notions of Fredholm operators and Fredholm Banach space complexes respectively. In particular, in [4] Fredholm pairs and chains in Hilbert spaces were characterized using the idea mentioned in the previous paragraph. Furthermore, in [5] regular Fredholm pairs, that is Fredholm pairs whose operators admit generalized inverses, were characterized and classified.

The main objective of the present article is to characterize regular Fredholm pairs and chains in Banach spaces using Fredholm operator extending to these objects the characterizations of [4] using the approach of Harte and Lee [7]. Naturally, as in [7], since the adjoint cannot be considered any more, suitable generalized inverses must be defined to prove the main results of this work.

REV. ROUMAINE MATH. PURES APPL., 55 (2010), 3, 149–157
The article is organized as follows. In the next section the definitions of the objects under consideration will be recalled. In Section 3 two characterizations of regular Fredholm pairs will be given. Finally, in Section 4 the results of Section 3 will be applied to regular Fredholm chains to obtain two characterizations of them.

2. PRELIMINARY DEFINITIONS

From now on \( X \) and \( Y \) will denote two Banach spaces and \( L(X, Y) \) will stand for the algebra of all linear and continuous operators defined on \( X \) with values in \( Y \). As usual, when \( X = Y \), \( L(X, X) \) is denoted by \( L(X) \). For every \( S \in L(X, Y) \), the null space of \( S \) is denoted by \( \text{N}(S) = \{ x \in X : S(x) = 0 \} \), and the range of \( S \) by \( \text{R}(S) = \{ y \in Y : \exists x \in X \text{ such that } y = S(x) \} \).

Next follow the definitions of Fredholm pairs and chains, see for instance [1, 3, 8].

**Definition 2.1.** Let \( X \) and \( Y \) be two Banach spaces and let \( S \in L(X, Y) \) and \( T \in L(Y, X) \) be such that the following dimensions are finite:

(i) \( a := \dim \frac{\text{N}(S)}{(\text{N}(S) \cap \text{R}(T))} \), \( b := \dim \frac{\text{R}(T)}{(\text{N}(S) \cap \text{R}(T))} \);

(ii) \( c := \dim \frac{\text{N}(T)}{(\text{N}(T) \cap \text{R}(S))} \), \( d := \dim \frac{\text{R}(S)}{(\text{N}(T) \cap \text{R}(S))} \).

A pair \((S, T)\) with the above properties is said to be a Fredholm pair.

Let \( P(X, Y) \) denote the set of all Fredholm pairs. If \((S, T) \in P(X, Y)\), then the index of \((S, T)\) is defined by the equality

\[
\text{ind}(S, T) := a - b - c + d.
\]

In particular, if \((S, T) \in P(X, Y)\) is such that \( ST = 0 \) and \( TS = 0 \), that is if \( b = d = 0 \), then \((S, T)\) and \((T, S)\) are Fredholm chains in the sense of [6, Section 10.6] and [7].

**Definition 2.2.** A Fredholm chain \((X_p, \delta_p)_{p \in \mathbb{Z}}\) is a sequence of Banach spaces \(X_p\) and bounded operators \(\delta_p \in L(X_p, X_{p-1})\) such that there is a natural number \(n\) with the property \(X_p = 0, p < 0\) and \(p \geq n + 1, \delta_p = 0, p \leq 0\) and \(p > n\), and

\[
\frac{\text{N}(\delta_p)}{(\text{N}(\delta_p) \cap \text{R}(\delta_{p+1}))} \quad \text{and} \quad \frac{\text{R}(\delta_{p+1})}{(\text{N}(\delta_p) \cap \text{R}(\delta_{p+1}))}
\]

are finite dimensional subspaces of \(X_p, p \in \mathbb{Z}\).

Given a Fredholm chain, it is possible to associate an index to it. In fact, if \((X_p, \delta_p)_{p \in \mathbb{Z}}\) is such an object, then define

\[
\text{ind}(X_p, \delta_p)_{p \in \mathbb{Z}} = \sum_{p=0}^{n} (-1)^p d_p,
\]
where \( d_p = \dim N(\delta_p)/(N(\delta_p) \cap R(\delta_{p+1})) - \dim R(\delta_{p+1})/(N(\delta_p) \cap R(\delta_{p+1})) \), see [8].

Recall that in [8] the more general concept of semi-Fredholm chain was introduced. However, since the main concern of this article consists in Fredholm objects, only Fredholm chains will be considered.

Remark 2.3. There is a natural relationship between Fredholm pairs and chains. In fact, given as in Definition 2.2 a sequence of spaces and maps \((X_p, \delta_p)_{p \in \mathbb{Z}}\), consider the Banach spaces

\[
X = \bigoplus_{p=2k} X_p, \quad Y = \bigoplus_{p=2k+1} X_p,
\]

and the Banach space operators \(S \in L(X, Y)\) and \(T \in L(Y, X)\) defined by

\[
S = \bigoplus_{p=2k} \delta_p, \quad T = \bigoplus_{p=2k+1} \delta_p,
\]

where \(X_p = 0, \ p < 0\) and \(p \geq n + 1, \ \delta_p = 0, \ p \leq 0\) and \(p > n\), and \(n\) is a natural number.

Now, it is not difficult to prove that \(\dim R(ST)\) and \(\dim R(TS)\) are finite dimensional if and only if \(\dim R(\delta_{p+1})/(N(\delta_p) \cap R(\delta_{p+1}))\) are finite dimensional, \(p = 0, \ldots, n\). Furthermore, a straightforward calculation shows that necessary and sufficient for \(\dim N(T)/(N(T) \cap R(S))\) to be finite dimensional is the fact that \(\dim N(\delta_p)/(N(\delta_p) \cap R(\delta_{p+1}))\) are finite dimensional, \(p = 0, \ldots, n\). Consequently, \((X_p, \delta_p)_{p \in \mathbb{Z}}\) is a Fredholm chain if and only if \((S, T)\) is a Fredholm pair. Finally, in this case,

\[
\text{ind}(X_p, \delta_p)_{p \in \mathbb{Z}} = \text{ind}(S, T),
\]

see [4, Remark 2.4].

On the other hand, recall that an operator \(T \in L(X, Y)\) is called regular, if there is \(S \in L(Y, X)\) such that \(T = TST\). The map \(S\) is said to be a generalized inverse of \(T\). In addition, if \(T\) is also a generalized inverse of \(S\), that is if \(S = STS\), then \(S\) is said to be a normalized generalized inverse of \(T\). Note that if \(T\) is regular, then \(T\) always has a normalized generalized inverse. In fact, if \(S\) is a generalized inverse of \(T\), then \(S' = STS\) is a normalized generalized inverse of \(T\). In the following definition, the notions of regular Fredholm pairs and chains will be recalled, see [5].

Definition 2.4. Let \(X\) and \(Y\) be two Banach spaces and consider \(S \in L(X, Y)\) and \(T \in L(Y, X)\) such that \((S, T) \in P(X, Y)\). The pair \((S, T)\) will be said to be a regular Fredholm pair, if \(S\) and \(T\) are regular operators. Similarly, given a Fredholm chain \((X_p, \delta_p)_{p \in \mathbb{Z}}\), then \((X_p, \delta_p)_{p \in \mathbb{Z}}\) will be said to be a regular Fredholm chain, if the Fredholm pair defined by \((X_p, \delta_p)_{p \in \mathbb{Z}}\) is regular.
Note that given \((S, T) \in P(X, Y)\), several statements equivalent to the fact that \((S, T)\) is a regular Fredholm pair were considered in [5, Proposition 2.4]. Concerning regular Fredholm chains, see [5, Remark 2.7].

3. CHARACTERIZATIONS OF REGULAR FREDHOLM PAIRS

First of all, a preliminary remark is presented.

Remark 3.1. Let \(X\) and \(Y\) be two Banach spaces, and let \(S \in L(X, Y)\) and \(T \in L(Y, X)\) be two operators such that \(R(ST)\) and \(R(TS)\) are finite dimensional subspaces of \(Y\) and \(X\) respectively. Then, define the Banach spaces \(X = X/R(TS)\) and \(Y = Y/R(ST)\), and the linear and bounded maps \(\tilde{S} \in L(\tilde{X}, Y)\) and \(\tilde{T} \in L(Y, \tilde{X})\) as the factorization of \(S\) and \(T\) through the respective invariant subspaces. Clearly, \(\tilde{S}\tilde{T} = 0\) and \(\tilde{T}\tilde{S} = 0\), that is \((\tilde{S}, \tilde{T})\) and \((\tilde{T}, \tilde{S})\) are chains in the sense of [7]. Moreover, the regularity of \(S\) and \(T\) is equivalent to the one of \(\tilde{S}\) and \(\tilde{T}\).

Theorem 3.2. Under the conditions of Remark 3.1, the following statements are equivalent.

(i) \(S\) and \(T\) are regular operators.
(ii) \(\tilde{S}\) and \(\tilde{T}\) are regular operators.

Proof. Since \(S \in L(X, Y)\) is a regular operator, according to [6, Theorem 3.8.2], there is \(M\) a linear subspace of \(Y\) such that \(R(S) \oplus M = Y\). Consequently, \(R(\tilde{S}) + \pi_Y(M) = Y\), where \(\pi_Y : Y \to Y\) is the quotient map.

Next, suppose that there is \(\tilde{x} \in \tilde{X}\) such that \(\tilde{S}(\tilde{x}) = \tilde{m} \in \pi_Y(M)\). In particular, there exist \(x \in X\), \(m \in M\), and \(y \in Y\) such that \(\pi_X(x) = \tilde{x}\), \(\pi_Y(m) = \tilde{m}\), and \(S(x) - m = ST(y)\), where \(\pi_X : X \to \tilde{X}\) is the quotient map. However, in this case \(S(x - T(y)) = m\). Then, \(m \in M \cap R(S) = 0\), and \(\tilde{S}(\tilde{x}) = \tilde{m} = 0\). Therefore, \(R(\tilde{S}) \oplus \pi_Y(M) = Y\).

On the other hand, according to [6, Theorem 3.8.2], there exists \(N\), a vector subspace of \(X\) such that \(N(S) \oplus N = X\). However, since

\[
(N(S) + R(T))/N(S) \cong R(T))/((N(S) \cap R(T)) \cong R(ST),
\]

there is a finite dimensional subspace \(X_1\) such that \(N(S) \oplus X_1 = N(S) + R(T)\). Then, a straightforward calculation proves that there exists a closed vector subspace \(R\) of \(X\) such that \((N(S) + R(T)) \oplus R = X\).

Now, according to [1, Remark 2.1], \(N(\tilde{S}) = \pi_X(N(S) + R(T))\). In particular, \(N(\tilde{S}) + \pi_X(R) = \tilde{X}\).

Next, suppose that there are \(n \in N(S)\), \(y \in Y\), and \(r \in R\) such that \(\pi_X(n + T(y)) = \pi_X(r)\). Then, there is \(x_0 \in X\) such that \(n + T(y) - r = TS(x_0)\). In particular, \(n + T(y - S(x_0)) = r\). Thus \(r = 0\), which implies that
\[ \pi_X(r) = \pi_X(n + T(x)) = 0. \] Therefore, \( N(\tilde{S}) \oplus \pi_X(R) = \mathcal{X}. \) As a result, according to what has been proved and to [6, Theorem 3.8.2], the operator \( \tilde{S} \) is regular.

Interchanging \( S \) with \( T \) and \( \tilde{S} \) with \( \tilde{T} \), it can be proved that \( \tilde{T} \) is regular.

In order to prove the converse implication, suppose that \( \tilde{S} \) and \( \tilde{T} \) are regular operators.

Consider \( \mathcal{V} \), a closed vector subspace of \( \mathcal{Y} \), such that \( R(\tilde{S}) \oplus \mathcal{V} = \mathcal{Y} \). Let \( V_1 = \pi^{-1}_\mathcal{V}(\mathcal{V}) \cap R(ST) \). Since \( V_1 \) has finite dimension, there is a vector subspace \( W_1 \) such that \( V_1 \oplus W_1 = \pi^{-1}_\mathcal{V}(\mathcal{V}) \). Moreover, since \( V_1 \subseteq R(ST) \) and \( \pi_\mathcal{V} \) is surjective, \( \mathcal{V} = \pi_\mathcal{V}(W_1) \). Consequently, \( \pi_\mathcal{V}(R(S) + W_1) = R(\tilde{S}) + \mathcal{V} = \mathcal{Y} \), which implies that \( R(S) + W_1 + R(ST) = \mathcal{Y} \). However, since \( R(ST) \subseteq R(S) \), \( R(S) + W_1 = \mathcal{Y} \).

Next, define \( L = R(S) \cap W_1 \). Then, \( \pi_\mathcal{V}(L) \subseteq R(\tilde{S}) \cap \mathcal{V} = 0 \). Thus, \( L \subseteq R(ST) \). However, since \( W_1 \cap R(ST) = 0 \), \( L = 0 \) and \( R(S) + W_1 = X \).

Similarly, suppose that \( \mathcal{U} \) is a closed subspace of \( \mathcal{X} \) such that \( N(\tilde{S}) \oplus \mathcal{U} = \mathcal{X} \). Let \( U_1 = \pi^{-1}_\mathcal{X}(\mathcal{U}) \cap R(TS) \). Since \( U_1 \) is a finite dimensional subspace, there exists a subspace \( Z_1 \) such that \( U_1 \oplus Z_1 = \pi^{-1}_\mathcal{X}(\mathcal{U}) \). Furthermore, since \( \pi_\mathcal{X} : X \to \mathcal{X} \) is a surjective map, \( \pi_\mathcal{X}(Z_1) = \mathcal{U} \).

Now, according to [1, Remark 2.1], \( \pi_X(N(S) + R(T) + Z_1) = \pi_X(N(S) + R(T)) + \pi_X(Z_1) = N(\tilde{S}) + \mathcal{U} = \mathcal{X} \). Therefore, since \( R(TS) \subseteq R(T) \), \( N(S) + R(T) + Z_1 = X \).

Let \( P = (N(S) + R(T)) \cap Z_1 \). According to [1, Remark 2.1], \( \pi_X(P) \subseteq N(\tilde{S}) \cap \mathcal{U} = 0 \). Thus, \( P \subseteq R(TS) \). Consequently, since \( Z_1 \cap R(TS) = 0 \), \( P = 0 \), and \( (N(S) + R(T)) \oplus Z_1 = X \). However, since \( N(S) \oplus X_1 = N(S) + R(T) \), where \( X_1 \) is a finite dimensional vector subspace of \( X \), \( N(S) \oplus (X_1 + Z_1) = X \), which, according to what has been proved and to [6, Theorem 3.8.2], implies that \( S \) is a regular operator.

Interchanging \( \tilde{S} \) with \( \tilde{T} \) and \( S \) with \( T \), it can be proved that \( T \) is a regular operator. \( \square \)

In order to prove the first characterization concerning regular Fredholm pairs, some preparation is needed.

**Remark 3.3.** Let \( X, Y, S, T, \mathcal{X}, \mathcal{Y}, \tilde{S}, \mathcal{X}, \mathcal{Y}, \tilde{T} \) be as in Remark 3.1. Recall that \( \tilde{S} \tilde{T} = 0 \) and \( \tilde{T} \tilde{S} = 0 \). Furthermore, according to Theorem 3.2, necessary and sufficient for \( S \) and \( T \) to be regular bounded and linear maps is the fact that \( (\tilde{S}, \tilde{T}, \tilde{S}) \) is a regular chain in the sense of [7, p. 283–284]. In addition, according again to [7, p. 284], there exist \( \tilde{S}' \in L(\mathcal{Y}, \mathcal{X}) \) and \( \tilde{T}' \in L(\mathcal{X}, \mathcal{Y}) \) such that \( (\tilde{T}', \tilde{S}', \tilde{T}') \) is a regular chain in the sense of [7] with the property that \( \tilde{S}' \) (respectively \( \tilde{T}' \)) is a normalized generalized inverse of \( \tilde{S} \) (respectively \( \tilde{T} \)).
Now, since $R(ST)$ and $R(TS)$ are finite dimensional subspaces of $Y$ and $X$ respectively, $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic to finite codimensional closed subspaces of $X$ and $Y$ respectively. Consequently, using Banach space isomorphisms, $\tilde{S}'$ and $\tilde{T}'$ can be thought of operators defined on and to finite codimensional closed subspaces of $X$ and $Y$. Denote by $S' \in L(Y, X)$ and $T' \in L(X, Y)$ the extension of $\tilde{S}'$ and $\tilde{T}'$ respectively, such that $S' = 0$ on $R(TS)$ and $T' = 0$ on $R(ST)$. Note that $S'$ and $T'$ are regular maps.

Next follows the first characterization of regular Fredholm pairs.

**Theorem 3.4.** Under the conditions of Remark 3.3, the following statements are equivalent.

(i) $(S, T)$ is a regular Fredholm pair.

(ii) $S + T' \in L(X, Y)$ is a Fredholm operator.

(iii) $T + S' \in L(Y, X)$ is a Fredholm operator.

Furthermore, in this case

$$\text{ind}(S, T) = \text{ind}(S + T') = -\text{ind}(T + S').$$

**Proof.** According to [1, Remark 2.1] and to Theorem 3.2, $(S, T)$ is a regular Fredholm pair if and only if $(\tilde{S}, \tilde{T}, \tilde{S})$ is a Fredholm regular chain in the sense of [7]. Then, according to [7, Theorem 5], the first statement of the Theorem is equivalent to the fact that $\tilde{S} + \tilde{T}' \in L(\mathcal{X}, \mathcal{Y})$ is a Fredholm operator.

Now, since as in Remark 3.3 $\mathcal{X}$ and $\mathcal{Y}$ can be thought of finite codimensional closed subspaces of $X$ and $Y$ respectively, it is possible to define $S_1 \in L(Y, X)$ as the extension of $\tilde{S}$ with the property $S_1 = 0$ on $R(TS)$. Consequently, since $R(ST)$ and $R(TS)$ are finite dimensional vector subspaces, according to what has been proved, the first statement of the theorem is equivalent to the fact that $S_1 + T' \in L(Y, X)$ is a Fredholm operator. However, since $S - S_1$ is a compact operator, the first and the second statements of the theorem are equivalent.

As regard the index, according to [1, Remark 2.1] and to [7, Theorem 5]

$$\text{ind}(S, T) - \dim R(TS) + \dim R(ST) = \text{ind}(\tilde{S}, \tilde{T}) = \text{ind}(\tilde{S} + \tilde{T}').$$

On the other hand, note that

$$\text{ind}(S + T') = \text{ind}(S_1 + T') = \text{ind}(\tilde{S} + \tilde{T}') + \dim R(TS) - \dim R(ST).$$

Therefore, $\text{ind}(S, T) = \text{ind}(S + T').$

A similar argument proves that the first and the third statements are equivalent as well as the relationship between the indexes. □

The second characterization of regular Fredholm pairs is more general in the sense that the generalized inverses can be chosen more freely, however, the formula regarding the index cannot be considered. On the other hand, to state this characterization, a Banach space operator is introduced.
Remark 3.5. Let \( X, Y, S, T, \mathcal{X}, \mathcal{Y}, \tilde{S}, \) and \( \tilde{T} \) be as in Remark 3.3. Since \( \tilde{S} \) and \( \tilde{T} \) are regular operators, there exist \( \tilde{S}' \in L(\mathcal{Y}, \mathcal{X}) \) and \( \tilde{T}' \in L(\mathcal{X}, \mathcal{Y}) \) generalized inverses of \( \tilde{S} \) and \( \tilde{T} \) respectively. In addition, proceeding as in Remark 3.3, denote by \( \tilde{S}' \in L(\mathcal{Y}, \mathcal{X}) \) and \( \tilde{T}' \in L(\mathcal{X}, \mathcal{Y}) \) any extensions of \( \tilde{S}' \) and \( \tilde{T}' \) respectively.

On the other hand, define the map \( V \in L(X \oplus Y) \) as follows: \( V |_X \in L(X, Y), V |_X = S + T', \) and \( V |_Y \in L(Y, X), V |_Y = T + S'. \)

Theorem 3.6. Under the conditions of Remark 3.5, the following statements are equivalent.

(i) \((S, T)\) is a regular Fredholm pair.
(ii) \(S'S + TT'\) and \(T'T + SS'\) are Fredholm operators.
(iii) \(V\) is a Fredholm operator.

Proof. As in Theorem 3.4, according to [1, Remark 2.1] and to Theorem 3.2, the first statement is satisfied if and only if \((\tilde{S}, \tilde{T}, \tilde{S})\) is a Fredholm regular chain in the sense of [7], which, according to [7, Theorem 3], is equivalent to the fact that \(\tilde{S}'\tilde{S} + \tilde{T}'\tilde{T} \in L(\mathcal{X})\) and \(\tilde{T}'\tilde{T} + \tilde{S}'\tilde{S} \in L(\mathcal{Y})\) are Fredholm operators. Now, since \(R(TS)\) and \(R(ST)\) are finite dimensional subspaces of \(X\) and \(Y\) respectively, it is not difficult to prove that what has been proved is equivalent to the second statement of the theorem.

On the other hand, \(V\) is a Fredholm operator if and only if \(V^2\) also is Fredholm. Now, a straightforward calculation proves that there exists a finite range operator \(F \in L(X \oplus Y)\) such that \(V^2 - F\) is a diagonal operator with entries \(S'S + TT' \in L(X)\) and \(SS' + TT' \in L(Y)\), which clearly implies that the second and the third statements are equivalent. \(\square\)

4. CHARACTERIZATIONS OF REGULAR FREDHOLM CHAINS

In order to prove the main results of this section, some preliminary facts will be considered.

Remark 4.1. Let \((X_p, \delta_p)_{p \in \mathbb{Z}}\) be a sequence of spaces and maps such that \(X_p = 0, p < 0\) and \(p \geq n + 1, \) and \(\delta_p = 0, p \leq 0\) and \(p > n, \) where \(n\) is a natural number. In addition, suppose that \(R(\delta_p \delta_{p-1})\) is finite dimensional and \(\delta_p \in L(X_p, X_{p-1})\) is a regular operator, \(p \in \mathbb{Z}\). Then, if \(X, Y, S \in L(X, Y),\) and \(T \in L(Y, X)\) are as in Remark 2.3, the properties of \((X_p, \delta_p)_{p \in \mathbb{Z}}\) are equivalent to the fact that \(R(ST)\) and \(R(TS)\) are finite dimensional and \(S\) and \(T\) are regular operators.
Now, if $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{S}$ and $\mathcal{T}$ are as in Remark 3.3, note that

$$\mathcal{X} = \bigoplus_{p=2k} \mathcal{X}_p, \quad \mathcal{Y} = \bigoplus_{p=2k+1} \mathcal{X}_p,$$

$$\mathcal{S} = \bigoplus_{p=2k} \tilde{\delta}_p, \quad \mathcal{T} = \bigoplus_{p=2k+1} \tilde{\delta}_p,$$

where $\mathcal{X}_p = X_p/R(\delta_{p+1}\delta_{p+2})$ and $\tilde{\delta}_p : \mathcal{X}_p \to \mathcal{X}_{p-1}$ is the quotient map induced by $\delta_p : X_p \to X_{p-1}$, $p \in Z$. Since $\mathcal{S}\mathcal{T} = 0$ and $\mathcal{T}\mathcal{S} = 0$, the sequence of spaces and maps $(\mathcal{X}_p, \tilde{\delta}_p)_{p \in Z}$ is a finite complex of Banach spaces. In addition, according to Theorem 3.2, $\mathcal{S}$ and $\mathcal{T}$ are regular operators, which is equivalent to the fact that $\tilde{\delta}_p : \mathcal{X}_p \to \mathcal{X}_{p-1}$ is regular, $p \in Z$. A straightforward calculation proves that there exists a sequence of linear and bounded maps $(\tilde{\delta}_p'_{p \in Z}, \tilde{\delta}_p'_{p \in Z})$ is a finite complex of Banach spaces. Consequently, if

$$\mathcal{S}' = \bigoplus_{p=2k} \tilde{\delta}'_p, \quad \mathcal{T}' = \bigoplus_{p=2k+1} \tilde{\delta}'_p,$$

then $(\tilde{T}', \mathcal{S}', \mathcal{T}')$ is regular chain in the sense of [7] such that $\mathcal{S}'$ and $\mathcal{T}'$ are normalized generalized inverses of $\mathcal{S}$ and $\mathcal{T}$. Finally, as in Remark 3.3, consider $S' \in L(Y, X)$ and $T' \in L(X, Y)$, the extensions of $\mathcal{S}'$ and $\mathcal{T}'$ such that $S' = 0$ on $R(TS)$ and $T' = 0$ on $R(ST)$, and denote by $\delta'_p : X_{p-1} \to X_p$ the extension of $\tilde{\delta}'_p$ such that $\delta'_p = 0$ on $R(\delta_p\delta_{p-1})$, $p \in Z$.

**Theorem 4.2.** Under the conditions of Remark 4.1, the following statements are equivalent.

(i) $(X_p, \delta_p)_{p \in Z}$ is a regular Fredholm chain.
(ii) $\bigoplus_{p=2k} (\delta_p + \delta'_{p+1})$ is a Fredholm operator.
(ii) $\bigoplus_{p=2k+1} (\delta_p + \delta'_{p+1})$ is a Fredholm operator.

Furthermore, in this case,

$$\text{ind}(X_p, \delta_p)_{p \in Z} = \text{ind} \bigoplus_{p=2k} (\delta_p + \delta'_{p+1}) = -\text{ind} \bigoplus_{p=2k+1} (\delta_p + \delta'_{p+1}).$$

**Proof.** Apply Remark 2.3, Theorem 3.4, and Remark 4.1. □

**Remark 4.3.** Under the conditions of Remark 4.1, consider the same spaces and maps but, as in Remark 3.5, $\mathcal{S}'$ and $\mathcal{T}'$ denote any generalized inverses of $\mathcal{S}$ and $\mathcal{T}$. In addition, $\mathcal{S}'$ and $\mathcal{T}'$ can be any extensions of $\mathcal{S}'$ and $\mathcal{T}'$. Similarly, $\delta'_p$ is any extension of $\tilde{\delta}'_p$, $p \in Z$. 

Theorem 4.4. Under the conditions of Remark 4.3, the following statements are equivalent.

(i) \((X_p, \delta_p)_{p \in \mathbb{Z}}\) is a regular Fredholm chain.

(ii) \(\delta_{p+1}\delta_{p+1}^* + \delta_p\delta_p \in L(X_p), \ p \in \mathbb{Z}\).

Proof. Apply Remark 2.3, Theorem 3.6, and Remarks 4.1 and 4.3. \(\square\)

REFERENCES

[1] C.-G. Ambrozie, On Fredholm index in Banach spaces. Integral Equations Operator Theory 25 (1996), 1–34.
[2] C.-G. Ambrozie, The Euler characteristic is stable under compact perturbations. Proc. Amer. Math. Soc. 124 (1996), 2041–2050.
[3] C.-G. Ambrozie and F.-H. Vasilescu, Banach Space Complexes. Kluwer Academic Publishers, Dordrecht–Boston–London, 1995.
[4] E. Boasso, Characterizations of Fredholm pairs and chains in Hilbert spaces. Rev. Roumaine Math. Pures Appl. 51 (2006), 151–165.
[5] E. Boasso, Regular Fredholm pairs. J. Operator Theory 55 (2006), 311–337.
[6] R. Harte, Invertibility and Singularity for Bounded Linear Operators. Marcel Dekker, Inc., New York–Basel, 1988.
[7] R. Harte and W.Y. Lee, An index formula for chains. Studia Math. 116 (1995), 283–294.
[8] V. Müller, Stability of index for semi-Fredholm chains. J. Operator Theory 37 (1997), 247–261.
[9] F.-H. Vasilescu, A characterization of the joint spectrum in Hilbert spaces. Rev. Roumaine Math. Pures Appl. 22 (1977), 1003–1009.
[10] F.-H. Vasilescu, On pairs of commuting operators. Studia Math. 62 (1978), 203–207.
[11] F.-H. Vasilescu, Analytic perturbations of the \(\delta\)-operator and integral representation formulas in Hilbert spaces. J. Operator Theory 1 (1979), 187–205.
[12] F.-H. Vasilescu, The stability of the Euler characteristic for Hilbert complexes. Math. Ann. 248 (1980), 109–116.

Received 15 June 2009
