Hamilton Paths in Dominating Graphs of Trees and Cycles

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Abstract
The dominating graph of a graph $H$ has as its vertices all dominating sets of $H$, with an edge between two dominating sets if one can be obtained from the other by the addition or deletion of a single vertex of $H$. In this paper we prove that the dominating graph of any tree has a Hamilton path. We also show how a result about binary strings leads to a proof that the dominating graph of a cycle on $n$ vertices has a Hamilton path if and only if $n$ is not a multiple of 4.

Keywords Reconfiguration · Domination · Hamilton paths

1 Introduction

Let $H$ be a graph with vertex set $V(H)$. A dominating set of $H$ is a set $D \subseteq V(H)$ such that every vertex of $V(H) \setminus D$ is adjacent to a vertex of $D$. The dominating graph of $H$, $\mathcal{D}(H)$, is the graph whose vertices are all the dominating sets of $H$; if $X$ and $Y$ are distinct vertices of $\mathcal{D}(H)$, then there is an edge between $X$ and $Y$ if and only if $Y$ can be obtained from $X$ by adding or deleting a single vertex. Note that we use the same label for a vertex of $\mathcal{D}(H)$ as for the corresponding dominating set of $H$ because it is clear from context whether we are referring to $H$ or $\mathcal{D}(H)$.

The graph $\mathcal{D}(H)$ is the reconfiguration graph of dominating sets of $H$ under the token addition/removal (TAR) model, first considered in [9]. For any graph $H$ and any integer $k$, $1 \leq k \leq |V(H)|$, the $k$-dominating graph of $H$, denoted $\mathcal{D}_k(H)$, is the subgraph of $\mathcal{D}(H)$ induced by the dominating sets of $H$ with cardinality at most $k$. When $k = |V(H)|$, then $\mathcal{D}_k(H) = \mathcal{D}(H)$. There have been several papers about dominating graphs and their subgraphs, the $k$-dominating graphs. Most of these focus on conditions on $k$ that ensure that $\mathcal{D}_k(H)$ is connected. Two recent surveys of reconfiguration of dominating sets are [1] and [11].

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There has been considerable interest in reconfiguration and reconfiguration graphs of other well-known graph structures and operations, including independent sets, cliques, vertex covers of graphs, zero forcing, and graph coloring. Nishimura [12] examines reconfiguration from an algorithmic perspective and considers complexity questions in a wide range of reconfiguration settings. Reconfiguration of graph coloring problems and dominating set problems are surveyed in a recent paper of Mynhardt and Nasserasr [11].

In this paper we investigate Hamilton cycles and Hamilton paths in dominating graphs, properties that have been studied for other types of reconfiguration problems. A Hamilton path or Hamilton cycle in a reconfiguration graph is a combinatorial Gray code, that is, a listing of all the objects in a set so that successive objects differ in some prescribed minimal way [8]. Several recent papers give conditions for the existence of Gray codes for all colorings with $k$ or fewer colors of the following classes of graphs: trees [7], bipartite graphs [6], and 2-trees [5]. A forthcoming survey by Mütze gives a wide variety of combinatorial Gray code results [10].

We consider only finite simple graphs. For a graph $H$, we use the notation $P = x_1, x_2, \ldots, x_j$, where $j \geq 3$, to denote a path $P$ in $H$ where $\{x_1, x_2, \ldots, x_j\}$ is a subset of the vertices of $H$. An edge, i.e., a path with two vertices $x$ and $y$, is written simply as $e = xy$. For basic graph theory notation and terminology not defined here, see [3].

We begin with the question of which dominating graphs have Hamilton cycles. It is clear that if $H$ is a graph, then its dominating graph, $D(H)$, is bipartite, with the bipartition based on the parity of the dominating sets of $H$. It follows that if $D(H)$ has a Hamilton cycle, then $D(H)$ has an even number of vertices (equivalently, the number of dominating sets of $H$ is even). By contrast, we have the following unpublished result of Brouwer [4], an expanded proof of which is included in [1].

**Lemma 1** [4] The number of dominating sets of any graph is odd.

Combining Brouwer’s result with the observation that a bipartite graph with a Hamilton cycle must have an even number of vertices gives a short answer to the question of which dominating graphs have Hamilton cycles.

**Proposition 2** [1] For any graph $H$, the dominating graph $D(H)$ has no Hamilton cycle.

Henceforth, we focus our attention on Hamilton paths in dominating graphs.

In [1] we show that Hamilton paths exist in the dominating graphs of certain classes of graphs. Specifically, we prove the following.

**Theorem 3** [1] Let $m$ and $n$ be positive integers. Then $D(K_n)$ has a Hamilton path, $D(P_n)$ has a Hamilton path, and $D(K_{m,n})$ has a Hamilton path if and only if at least one of $m$ or $n$ is odd.
In this paper we explore the dominating graphs of trees and prove the following.

**Theorem 4** For any tree $T$, $D(T)$ has a Hamilton path.

We also use a result of Baril and Vajnovszki [2] on Lucas strings to characterize cycles whose dominating graphs have Hamilton paths.

## 2 Hamilton Paths in Dominating Graphs of Trees

We begin this section by introducing two operations on a graph $H$. We then prove that if $H'$ is a graph obtained from $H$ by applying either operation, and $D(H')$ has a Hamilton path, then $D(H)$ has a Hamilton path. This is subsequently used to show that the dominating graph of any tree has a Hamilton path.

**Operation I** Let $H$ be a graph with vertices $u$, $v$ and $x$ such that $NH(u) = NH(v) = \{x\}$. We say that $H' := H - v$ is obtained from $H$ by Operation I (Fig. 1a).

**Operation II** Let $H$ be a graph with vertices $u$, $v$ and $w$ such that $NH(v) = \{u, w\}$ and $NH(w) = \{v\}$. We say that $H' := H - w - v$ is obtained from $H$ by Operation II (Fig. 1b).

**Lemma 5** Let $H$ and $H'$ be graphs such that $H'$ is obtained from $H$ by Operation I. If $D(H')$ has a Hamilton path, then $D(H)$ has a Hamilton path.

**Proof** Let $u$, $v$ and $x$ be vertices of $H$ such that $NH(u) = NH(v) = \{x\}$ and $H' := H - v$. To simplify notation, we define $G$ and $G'$ to be the dominating graphs of $H$ and $H'$, respectively, i.e., $G := D(H)$ and $G' := D(H')$.

Let $F_1, F_2, \ldots, F_n$ be a Hamilton path in $G'$. For each $i$, $1 \leq i \leq n$, define $F_i^v := F_i \cup \{v\}$, and for each $i$, $1 \leq i \leq n$, with $u \notin F_i$, define $F_i^u := F_i \cup \{u\}$ and $F_i^{uv} := F_i \cup \{u, v\}$.
By referring to Fig. 2, it is routine to verify that the following subsets of \( V(G) \) form a partition of the dominating sets of \( H \).

\[
\begin{align*}
X' &:= \{ F_i \mid x \in F_i, u \notin F_i, \ 1 \leq i \leq n \}, \\
B' &:= \{ F_i \mid \{ x, u \} \not\subseteq F_i, \ 1 \leq i \leq n \} = \{ F_i^u \mid F_i \in X' \}, \\
X &:= \{ F_i^v \mid x \in F_i, u \notin F_i, \ 1 \leq i \leq n \} = \{ F_i^v \mid F_i \in X' \}, \\
B &:= \{ F_i^v \mid \{ x, u \} \not\subseteq F_i, \ 1 \leq i \leq n \} = \{ F_i^{uv} \mid F_i \in X' \}, \\
U &:= \{ F_i^v \mid u \in F_i, x \notin F_i, \ 1 \leq i \leq n \}.
\end{align*}
\]

Furthermore, the definitions of \( B', X \) and \( B \) in terms of \( X' \) make it clear that

\[
G[B'] \cong G[X] \cong G[B] \cong G[X'],
\]

and that \( G[X \cup B \cup U] \cong G' \). It follows that \( P := F_1^v, F_2^v, \ldots, F_n^v \) is a path in \( G \) and also a Hamilton path of \( G[X \cup B \cup U] \). We now extend \( P \) to a Hamilton path of \( G \).

Let \( Q := F_i^v, F_{i+1}^v, \ldots, F_j^v \) be a maximal subpath of \( P \) in \( G[X] \). There are two cases to consider, depending on the parity of \( j - i + 1 \) (the number of vertices in \( Q \)). First suppose that \( j - i + 1 \) is even. Then for each \( t \in \{ i, i + 2, \ldots, j - 1 \} \), replace the edge \( F_t^v F_{t+1}^v \) of \( P \) by the path

\[
F_t^v, F_t^u, F_{t+1}^u, F_{t+1}^v.
\]

Since \( G[B'] \cong G[X] \), this replacement results in a path in \( G \).

Now assume that \( j - i + 1 \) is odd. In this case, for each \( t \in \{ i, i + 2, \ldots, j - 2 \} \), replace the edge \( F_t^v F_{t+1}^v \) of \( P \) by the path

\[
F_t^v, F_t^u, F_{t+1}^u, F_{t+1}^v.
\]

Again, since \( G[B'] \cong G[X] \), this replacement results in a path in \( G' \). If \( j = n \), replacing vertex \( F_n^v \) in \( P \) by the path \( F_n^v, F_n^u, F_n^u \) results in a path. Otherwise, \( j < n \), so the maximality of \( Q \) implies \( F_j^v \in B \), and hence \( F_{j+1}^v = F_j^{uv} \). Replacing the edge \( F_j^v F_{j+1}^v \) (which equals \( F_j^v F_{j+1}^v \)) of \( P \) with the path
ensures the result is a path in $G$.

Since $G[B'] \cong G[X'] \cong G[X]$, making these replacements for each maximal sub-path $Q$ of $P$ in $G[X]$ incorporates all the vertices of $X'$ and $B'$ into the resulting path and produces a Hamilton path of $G = \mathcal{D}(H)$. \hfill \Box

**Lemma 6**  Let $H$ and $H'$ be graphs such that $H'$ is obtained from $H$ by Operation II. If $\mathcal{D}(H')$ has a Hamilton path, then $\mathcal{D}(H)$ has a Hamilton path.

**Proof**  Let $u$, $v$ and $w$ be vertices of $H$ such that $N_H(v) = \{u, w\}$, $N_H(w) = \{v\}$, and $H' := H - w - v$. As before, we define $G$ and $G'$ to be the dominating graphs of $H$ and $H'$, respectively, i.e., $G := \mathcal{D}(H)$ and $G' := \mathcal{D}(H')$.

Let $Y$ be a dominating set of $H'$. Define 

$$
Y^v := Y \cup \{v\}, Y^w := Y \cup \{w\}, \text{ and } Y^{vw} := Y \cup \{v, w\},
$$

and let 

$$
A := \{Y^v, Y^w, Y^{vw} \mid Y \in V(G')\}.
$$

Then $A$ consists of dominating sets of $H$. The dominating sets of $H$ that are *not* in $A$ can be described as follows. Let 

$$
J := \{S \subseteq V(H') \mid S \text{ is a dominating set of } H' - u \text{ and } S \cap N_{H'}[u] = \emptyset\},
$$

i.e., $J$ consists of the dominating sets of $H' - u$ that are *not* dominating sets of $H'$. It follows that if $S \in J$, then $S \cap \{u, v, w\} = \emptyset$, so we define 

$$
S^u := S \cup \{u\}, S^v := S \cup \{v\}, S^{uw} := S \cup \{u, v\},
$$

$$
S^{vw} := S \cup \{v, w\}, \text{ and } S^{uvw} := S \cup \{u, v, w\}.
$$

We now let 

$$
B := \{S^u, S^{uw} \mid S \in J\}.
$$

It is routine to verify that $\{A, B\}$ is a partition of the dominating sets of $H$.

Let $F_1, F_2, \ldots, F_n$ be a Hamilton path in $G'$. By Lemma 1, $n$ is odd, so replacing vertex $F_i$ with the path $F^v_i, F^{vw}_i, F^w_i$ when $i$ is odd, and with the path $F^w_i, F^{vw}_i, F^v_i$ when $i$ is even produces the path 

$$
P := F^v_1, F^{vw}_1, F^w_1, F^v_2, F^{vw}_2, F^w_2, \ldots, F^v_n, F^{vw}_n, F^w_n
$$

in $G$. Since $P$ consists of all the vertices in $A$, what remains is to incorporate the vertices of $B$ into this path.

First notice that, for each $S \in J$, $S^u$ is a dominating set of $H'$, and hence $S^u = F_i$ for some $i$, $1 \leq i \leq n$. Furthermore, it is clear that if $S_1, S_2 \in J$, then $S_1 \neq S_2$ if and only if $S^u_1 \neq S^u_2$. 

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We now proceed as follows. For $S \in J$, let $t \in \{1, \ldots, n\}$ be the index for which $S^v = F_t$. In the path $P$, we replace the edge $F_i^vF_i^v$ (which is the same as $S^{v^*}S^{v^*}$) with the path

$$F_t^v = S^{v^*}, S^v, S^{v^*}, S^{v^*} = F_t^v.$$

Repeating this for each $S \in J$ results in a path containing all the vertices of $A \cup B = V(G)$, and hence all the dominating sets of $H$. Therefore, $G = D(H)$ has a Hamilton path. 

Together, the two preceding lemmas imply the following.

**Corollary 7** Let $H$ be a graph and let $H'$ be a graph obtained from $H$ by applying any sequence of the Operations I and II. If $D(H')$ has a Hamilton path then $D(H)$ has a Hamilton path.

A particular class of graphs to which we can apply Corollary 7 is trees.

**Theorem 4** For any tree $T$, $D(T)$ has a Hamilton path.

**Proof** Let $P_i$ denote the path on $i \geq 1$ vertices. If $|V(T)| \leq 2$, then $T \cong P_1$ or $T \cong P_2$. Since $D(P_1) \cong P_1$ and $D(P_2) \cong P_3$, $D(T)$ has a Hamilton path.

Now suppose that $|V(T)| \geq 3$, and let $P = v_0, v_1, \ldots, v_k$ be a maximum length path in $T$. Then $k \geq 2$ and $d(v_0) = 1$. If $d(v_1) = 2$, then we let $w = v_0$, $v_1$, $u = v_2$, and Operation II can be applied to $T$. Otherwise, $d(v_1) \geq 3$, so there exists $u \in N_T(v_1)$, $u \notin \{v_0, v_2\}$. By the maximality of $P$, $N_T(u) = v_1$, so we let $v = v_0$ and $x = v_1$, and Operation I can be applied to $T$. Repeatedly applying Operations I and II to $T$ to produces a $T' \cong P_2$. By Corollary 7, $D(T)$ has a Hamilton path.

3 Further Results

3.1 Hamilton Paths in Dominating Graphs of Cycles

Let $C_n$ denote the cycle on $n \geq 3$ vertices with vertex set $V(C_n) = \{0, 1, \ldots, n-1\}$ and edge set $\{ij : i - j \equiv \pm 1 \pmod{n}\}$. We encode $X \subseteq V(C_n)$ as an $n$-digit binary string, $x_0x_1 \cdots x_{n-1}$, by setting $x_i = 1$ if and only if $i \in X$, $0 \leq i \leq n - 1$. It follows that $X \subseteq V(C_n)$, encoded by the binary string $x_0x_1 \cdots x_{n-1}$, is a dominating set of $C_n$ if and only if $x_{i-1}x_ix_{i+1} \neq 000$ for all $i$, $0 \leq i \leq n - 1$, where subscripts are taken modulo $n$. If $X$ and $Y$ are dominating sets of $C_n$, and are represented by binary strings $x_0x_1 \cdots x_{n-1}$ and $y_0y_1 \cdots y_{n-1}$, respectively, then $X$ and $Y$ are adjacent in $D(C_n)$ if and only if $x_0x_1 \cdots x_{n-1}$ and $y_0y_1 \cdots y_{n-1}$ differ in exactly one bit. It follows that $D(C_n)$ has a Hamilton path if and only if the set of binary strings representing the dominating sets of $C_n$ has a Gray code.
It was brought to our attention by T. Mütze (Personal communication) that the set of strings corresponding to the dominating sets of $C_n$ are the bitwise complements of the Lucas strings $L_{n,3}$. The set of Lucas strings of length $n$ and order $p \geq 1$ (see [2]), denoted $L_{n,p}$, is the set of binary strings of length $n$ that have no $p$ consecutive ones when the strings are considered circularly. In particular, the set of Lucas strings of length $n$ and order 3 is so is the set of bitwise complements of elements of $V(D(C_n))$. It turns out that the Gray codes of Lucas strings are already well-understood.

Baril and Vajnovszki [2] construct an ordering of the elements of $L_{n,p}$ called a minimal change list (see [2]), denoted by $\mathcal{L}_{n,p}$. They prove $\mathcal{L}_{n,p}$ is a Gray code if and only if $n$ is not a multiple of $(p + 1)$. Let $\hat{\mathcal{L}}_{n,p}$ denote the sequence obtained by taking bitwise complements of the strings of $\mathcal{L}_{n,p}$, and note that $\mathcal{L}_{n,p}$ is a Gray code if and only if $\hat{\mathcal{L}}_{n,p}$ is a Gray code. Since a Gray code is a Hamilton path, we have the following.

**Theorem 8** For all integers $n \geq 3$, $D(C_n)$ has a Hamilton path if and only if $n$ is not a multiple of 4.

An elegant but computationally inefficient construction of $\mathcal{L}_{n,p}$ is described in [2]. A minor variation of this construction, applied to the standard $n$-bit reflected binary code gives a simple construction of $\hat{\mathcal{L}}_{n,3}$ (a Hamilton path in $D(C_n)$) whenever $n$ is not a multiple of 4. We illustrate this construction in Example 9.

**Example 9** Let $n = 5$. To construct a Hamilton path of $D(C_5)$, begin with the standard 5-bit reflected binary code. The strings are organized in Fig. 3a to be read from top to bottom and left to right. Next, delete any string $x_0x_1x_2x_3x_4$ that has $x_{i-1}x_ix_{i+1} = 000$ for $0 \leq i \leq 4$, subscripts modulo 4.

The reader can easily verify that remaining strings, shown in Fig. 3b, are still a Gray code when read from top to bottom and left to right, and hence describe a Hamilton path in $D(C_5)$.
3.2 Unicyclic Graphs

Corollary 7 applies more generally and can be used to prove the existence of Hamilton paths in classes of dominating graphs that are built up using dominating graphs that are known to have Hamilton paths. These include complete graphs, paths, cycles $C_n$ when $n$ is not a multiple of four, certain complete bipartite graphs (Theorem 3), and trees (Theorem 4). We include one example.

For any graph $H$, we say that $H$ is reducible to subgraph $H'$ if $H'$ can be obtained from $H$ by applying a sequence of Operations I and II as described in Sect. 2. As a consequence of Corollary 7 and Theorem 8, we have the following.

**Corollary 10** Let $G$ be a unicyclic graph whose unique cycle $C_n$ has length $n \geq 3$, where $n$ is not a multiple of 4. Suppose $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, and let $T_i$ be the component (a tree) of $G - E(C_n)$ containing $v_i$ for some $i, 1 \leq i \leq n$. If $T_i$ is reducible to $v_i$ for each $i, 1 \leq i \leq n$, then $D(G)$ has a Hamilton path.

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