Annihilating polynomials for quadratic forms and Stirling numbers of the second kind

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We present a set of generators of the full annihilator ideal for the Witt ring of an arbitrary field of characteristic unequal to two satisfying a non-vanishing condition on the powers of the fundamental ideal in the torsion part of the Witt ring. This settles a conjecture of Ongenae and Van Geel. This result could only be proved by first obtaining a new lower bound on the $2$-adic valuation of Stirling numbers of the second kind.

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1 Introduction

In 1937, Witt already observed that the Witt ring was integral in the sense that each element was annihilated by a monic integer polynomial. Fifty years later, in 1987, Lewis was the first to give explicit examples of such polynomials [5]. He showed that the monic polynomial, $p_n(X)$, defined as

\[ p_n(X) = (X - n)(X - (n - 2)) \ldots (X + (n - 2))(X + n) \]

annihilates every nonsingular quadratic form of dimension $n$ over every field $F$ of characteristic unequal to two.

Since then, many other polynomials in $\mathbb{Z}[X]$ were found annihilating all or a family of classes of nonsingular quadratic forms in the Witt ring and we refer the reader to [4] for a nice survey of the main results on this topic.

Let $F$ be a field of characteristic not 2. The object we want to consider here is the torsion annihilator ideal

\[ A_t(F) = \{ f(X) \in \mathbb{Z}[X] \mid f(\phi) = 0, \text{ for all } \phi \in I_t(F) \} \]

where $I_t(F) = W_t(F) \cap I(F)$, $W_t(F)$ is the torsion part of the Witt ring and $I(F)$ is the ideal of all even-dimensional forms in the Witt ring. Since $A_t(F)$ is an ideal in the noetherian ring $\mathbb{Z}[X]$, it is finitely generated. The main problem is to find a set of generators for this ideal.

We will prove the following result.

For fields $F$ satisfying the conditions that $2^rW_t(F) = 0$ and $2^r(I_t(F))^{2k-1} \neq 0$ with $k$ uniquely determined by $r$, the torsion annihilator ideal $A_t(F)$ is the ideal generated by the monomials

- $2^r \cup \{2^{r-\nu_2(2j!)}X^{2j} \}_{1 \leq j \leq k-1} \cup \{X^{2k} \}$ for non-real fields and
- $2^rX \cup \{2^{r-\nu_2(2j!)}X^{2j} \}_{1 \leq j \leq k-1} \cup \{X^{2k} \}$ for real fields,

where $\nu_2$ denotes the 2-adic valuation.

In the case of a non-real field $F$ this theorem was conjectured (see Corollary 3.4) by Ongenae and Van Geel [6]. They gave a proof for fields with level $s(F) \leq 16$ and using the same technique, one can check that the theorem holds for all non-real fields $F$ with level $s(F) \leq 64$, but a general method was lacking.

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The general method, used to prove the theorem, consists in evaluating a polynomial \( f(X) \in \mathbb{Z}[X] \) in the even-dimensional forms \( \perp_{n=1}^k \langle a_i \rangle \) with \( 1 \leq n \leq \text{deg}(f) \). This evaluation can be rewritten as a linear combination of sums of Pfister forms and the coefficients that appear turn out to be related to the Stirling numbers of the second kind.

The result about the (torsion) annihilator ideal could only be proved by first obtaining a new lower bound for the 2-adic valuation of all Stirling numbers \( S(n, k) \) of the second kind, namely
\[
\nu_2(S(n, k)) \geq d(k) - d(n) \quad \text{for } 0 < k \leq n
\]
where \( d(k) \) is the sum of the binary digits in the binary representation of \( k \).

## 2 Stirling numbers of the second kind

### 2.1 Preliminaries

Let \( n \in \mathbb{N} \). The Stirling numbers \( S(n, k) \) (\( k \in \mathbb{N} \)) of the second kind are given by
\[
x^n = \sum_{k=0}^{\infty} S(n, k)(x)_k,
\]
where \( (x)_k = x(x-1)(x-2) \ldots (x-k+1) \) for \( k \in \mathbb{N} \setminus \{0\} \) and \( (x)_0 = 1 \). Actually \( S(n, k) \) is the number of ways in which it is possible to partition a set with \( n \) elements in \( k \) classes.

The Stirling numbers of the second kind can be computed in several ways.

**Proposition 2.1**

\[
S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{k-i}(k-i)^n,
\]
\[
S(n, k) = S(n-1, k-1) + kS(n-1, k) \quad \text{with } S(n, 0) = S(0, k) = 0 \text{ and } S(0, 0) = 1,
\]
\[
S(n, k) = \frac{1}{k!} \sum_{n_1, n_2, \ldots, n_k} \binom{n}{n_1, n_2, \ldots, n_k},
\]
where \( n_1, n_2, \ldots, n_k \) are non-zero and their sum equals \( n \).

**Proof.** See [1] and [7].

### 2.2 2-adic valuation of Stirling numbers of the second kind

The 2-adic valuation of Stirling numbers of the second kind and the 2-adic valuations of other combinatorial numbers have been widely studied, but many problems in this area are still unsolved. We will give a new lower bound for the 2-adic valuation of all Stirling numbers of the second kind.

Denote by \( d(n) \) the sum of the digits in the binary representation of \( n \) and define the 2-adic valuation function \( \nu_2(n) \) for all non-zero integers \( n \) by \( \nu_2(n) = p \), where \( 2^p | n \) and \( 2^{p+1} \nmid n \).

Recall the following properties (see [1]).
\[
\nu_2(n!) = n - d(n) \quad \text{(Legendre)},
\]
\[
\nu_2 \left( \binom{n}{k} \right) = d(k) + d(n-k) - d(n) \quad \text{(Kummer)},
\]
for all \( k, n \in \mathbb{N} \) with \( 0 \leq k \leq n \).

A new lower bound on the 2-adic valuation of Stirling numbers of the second kind can be obtained as follows.

**Corollary 2.2** Let \( n, k \in \mathbb{N} \) and \( 0 < k \leq n \). Then
\[
\nu_2(S(n, k)) \geq d(k) - d(n).
\]
We have the following

\[ \nu_2 \left( \frac{1}{k!} \binom{n}{n_1, \ldots, n_k} \right) = \nu_2(n!) - \nu_2(k!) - \sum_{i=1}^{k} \nu_2(n_i!) \]

\[ = n - d(n) - (k - d(k)) - \left( \sum_{i=1}^{k} (n_i - d(n_i)) \right) \]

(by the Legendre identity)

\[ \geq d(k) - d(n) \] (since \( d(n_i) \geq 1 \)).

\[ \square \]

### 2.3 Relationship between Stirling numbers of the second kind and quadratic forms

Consider the \( \mathbb{Z} \)-algebra \( B = \mathbb{Z}[X_1, X_2, \ldots, X_n]/K \) where \( K = (X_1^2 - 2X_1, X_2^2 - 2X_2, \ldots, X_n^2 - 2X_n) \). This algebra has a \( \mathbb{Z} \)-basis given by the elements \( \{1, X_1, X_2, \ldots, X_n\} \mod K \).

Stirling numbers of the second kind turn up in a natural way by making calculations in this \( \mathbb{Z} \)-algebra. The calculations frequently use the following relation

\[ Y_{(i)}^p = 2^{p-1} Y_{(i)}. \]

We have the following

**Proposition 2.3** Let \( f(X) = c_d X^d + \ldots + c_1 X \in \mathbb{Z}[X] \). Then

\[ f \left( \sum_{i=1}^{n} Y_{(i)} \right) = \sum_{q=1}^{d} 2^{q-1} S(q, 1)c_q \left( \sum_{i=1}^{n} Y_{(i)} \right) \]

\[ + \left( \sum_{q=2}^{d} 2^{q-2}2! S(q, 2)c_q \left( \sum_{i<j}^{n} Y_{(i,j)} \right) \right) \]

\[ + \ldots \]

\[ + \left( \sum_{q=n}^{d} 2^{q-n}n! S(q, n)c_q \right) Y_{\{1,2,\ldots,n\}}. \]

**Proof.** The evaluation of the polynomial \( f(X) = c_d X^d + \ldots + c_1 X \) in sums of the basis elements \( \sum_{i=1}^{n} Y_{(i)} \) can be written in the following unique way.

\[ f \left( \sum_{i=1}^{n} Y_{(i)} \right) = A_1(c_1, \ldots, c_d) \left( \sum_{i=1}^{n} Y_{(i)} \right) + A_2(c_1, \ldots, c_d) \left( \sum_{i<j}^{n} Y_{(i,j)} \right) \]

\[ + \ldots + A_n(c_1, \ldots, c_d) Y_{\{1,2,\ldots,n\}}, \]

where

\[ A_p(c_1, \ldots, c_d) = \sum_{q=1}^{d} \gamma_{p,q} c_q \]

with \( \gamma_{p,q} \in \mathbb{N} \) being the coefficient of \( Y_{\{1,2,\ldots,p\}} \) in \( \left( \sum_{i=1}^{n} Y_{(i)} \right)^q \).

For \( p \leq q \) we can write

\[ \gamma_{p,q} = \sum_{q_1+q_2+\ldots+q_p=q} \left( \begin{array}{c} q \nonumber \end{array} \right) 2^{q_1-1}2^{q_2-1} \ldots 2^{q_p-1} = 2^{q-p} \sum_{q_1+q_2+\ldots+q_p=q} \left( \begin{array}{c} q \nonumber \end{array} \right) S(q, p). \]

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If \( p > q \) then clearly no basis-element \( Y_{\{1,2,\ldots,p\}} \) can occur in \( (\sum_{i=1}^{p} Y_{\{i\}})^q \).

So,

\[
\gamma_{p,q} = \begin{cases} 
2^{q-p} p! S(q,p) & \text{if } p \leq q, \\
0 & \text{otherwise}.
\end{cases}
\]

Applying this to the coefficients \( A_{p}(c_1, \ldots, c_d) \) we obtain

\[
f \left( \sum_{i=1}^{n} Y_{\{i\}} \right) = \left( \sum_{q=1}^{d} 2^{q-1} \cdot 1! S(q,1)c_q \right) \left( \sum_{i=1}^{n} Y_{\{i\}} \right) \\
+ \left( \sum_{q=2}^{d} 2^{q-2} \cdot 2! S(q,2)c_q \right) \left( \sum_{i<j}^{n} Y_{\{i,j\}} \right) \\
+ \ldots \\
+ \left( \sum_{q=n}^{d} 2^{q-n} n! S(q,n)c_q \right) Y_{\{1,2,\ldots,n\}}.
\]

We can evaluate

\[
f(X) = c_d X^d + \ldots + c_1 X \in \mathbb{Z}[X]
\]

in classes of quadratic forms \( \phi \in W(F) \) by defining

\[
f(\phi) = c_d \phi^d \perp \ldots \perp c_1 \phi \in W(F)
\]

where \( c_i \phi = \text{sign}(c_i) \left( \phi \perp \ldots \perp \phi \right) \) and \( \phi^k = \phi \otimes \ldots \otimes \phi \).

For arbitrary \( k > 0 \) and \( a_1, a_2, \ldots, a_k \in F^* = F \setminus \{0\} \), let \( \langle a_1, a_2, \ldots, a_k \rangle \) denote the \( 2^k \)-dimensional \( k \)-fold Pfister form

\[
\langle a_1, a_2, \ldots, a_k \rangle := \langle 1, a_1 \rangle \otimes \langle 1, a_2 \rangle \otimes \ldots \otimes \langle 1, a_k \rangle.
\]

Observe that the following relation holds.

\[
\langle a \rangle^p = 2^{p-1} \langle a \rangle.
\]

**Corollary 2.4** Let \( f(X) = c_d X^d + \ldots + c_1 X \in \mathbb{Z}[X] \).

Then

\[
f(\bot_{i=1}^{n} \langle a_i \rangle) = \left( \sum_{q=1}^{d} 2^{q-1} \cdot 1! S(q,1)c_q \right) (\bot_{i=1}^{n} \langle a_i \rangle) \\
\bot \left( \sum_{q=2}^{d} 2^{q-2} \cdot 2! S(q,2)c_q \right) (\bot_{i,j}^{n} \langle a_i, a_j \rangle) \\
\bot \ldots \\
\bot \left( \sum_{q=n}^{d} 2^{q-n} n! S(q,n)c_q \right) \langle a_1, \ldots, a_n \rangle.
\]

**Proof.** Since we have a \( \mathbb{Z} \)-algebra homomorphism \( B \rightarrow W(F) \) with \( Y_{\{i\}} \rightarrow \langle a_i \rangle \) and \( Y_{\{J\}} \rightarrow \langle a_J \rangle \) for all \( J \subseteq \{1,2,\ldots,n\} \) this result follows from the previous proposition by substituting \( \sum_{i=1}^{n} Y_{\{i\}} \) by \( \bot_{i=1}^{n} \langle a_i \rangle \).
Corollary 2.5 Let \( f(X) = c_dX^d + \ldots + c_1X \in \mathbb{Z}[X] \) and let \( \phi = (a_1, a_2, \ldots, a_n) \) be a quadratic form of dimension \( n \). Then
\[
\begin{align*}
  f(\phi) &= \left( \sum_{q=1}^{d} \sum_{t=q}^{d} 2^{q-1} S(q, 1) \left( \frac{t}{q} \right) n^{t-q} c_t \right) \langle \langle 1_{i=1}^n \rangle \rangle \\
  &\quad \vdots \\
  &\quad \vdots \\
  &\quad \vdots \\
  \end{align*}
\]

Proof. Note that for all \( \phi = (a_1, a_2, \ldots, a_n) \)
we have
\[
\phi = \perp_{i=1}^n \langle \langle a_i \rangle \rangle - n(1).
\]
We can also rewrite \( f(X) \) as
\[
\begin{align*}
  f(X) &= \sum_{q=0}^{\infty} \frac{1}{q!} f^{(q)}(n)(X - n)^q \\
  &= \sum_{q=0}^{d} \sum_{t=q}^{d} \left( \frac{t}{q} \right) n^{t-q} c_t (X - n)^q \quad \text{(by the Taylor series of } f) \\
  &=: g(X - n).
\end{align*}
\]
The result follows from the previous proposition applied to \( g(Y) \) and the \( \mathbb{Z} \)-algebra homomorphism \( B \to W(F) \) described in the proof of the previous lemma.

Lemma 2.6 Let \( f(X) = c_dX^d + \ldots + c_1X \in \mathbb{Z}[X] \). Let \( n \in \mathbb{N}, n \leq d, a_1, a_2, \ldots, a_d \in F^* \). If \( f(\perp_{i=1}^k \langle \langle a_{\tau(i)} \rangle \rangle) = 0 \), for all \( 1 \leq k \leq n, \text{ and } \sigma \in S_d \), then
\[
\left( \sum_{q=n}^{d} 2^{q-n} n! S(q, n)c_q \right) \langle \langle a_{\tau(1)}\ldots a_{\tau(n)} \rangle \rangle = 0 \quad \text{for all } \tau \in S_d.
\]
Proof. By induction on \( n \).
For \( n = 1 \),
\[
\begin{align*}
  0 &= f(\langle \langle a_i \rangle \rangle) \\
  &= c_d\langle \langle a_i \rangle \rangle^d + c_{d-1}\langle \langle a_i \rangle \rangle^{d-1} + \ldots + c_1\langle \langle a_i \rangle \rangle \\
  &= \left( 2^{d-1}c_d + 2^{d-2}c_{d-1} + \ldots + c_1 \right) \langle \langle a_i \rangle \rangle \\
  &= \left( \sum_{q=1}^{d} 2^{q-1} S(q, 1)c_q \right) \langle \langle a_i \rangle \rangle
\end{align*}
\]
since \( S(q, 1) = 1 \) for all \( q \geq 1 \).
Assume now that the lemma is true for all \( i < n \). We will prove the lemma for \( n \).
Let \( \tau \in S_d \).
\[ 0 = f \left( \sum_{q=1}^{d} 2^{q-1} S(q, 1) c_q \right) \left( \sum_{i=1}^{n} \langle a_{\tau(i)} \rangle \right) \]
\[ = \left( \sum_{q=2}^{d} 2^{q-2} S(q, 2) c_q \right) \left( \sum_{i=1}^{n} \langle a_{\tau(i)}, a_{\tau(j)} \rangle \right) \]
\[ \perp \left( \sum_{q=n}^{d} 2^{q-n} n! S(q, n) c_q \right) \langle a_{\tau(1)}, \ldots, a_{\tau(n)} \rangle \]
by Corollary 2.4.

Let \( 1 \leq k \leq n - 1 \). Note that for every subset \( U \subset \{1, \ldots, d\} \) of \( k \) elements, there exists a permutation \( \sigma \in S_k \) such that \( U = \{ \sigma(1), \ldots, \sigma(k) \} \).

So,
\[ \left( \sum_{q=k}^{d} 2^{q-k} k! S(q, k) c_q \right) \langle a_{\tau(1)}, \ldots, a_{\tau(k)} \rangle = 0 \]
by the induction hypothesis. So, (2.1) becomes
\[ 0 = \left( \sum_{q=n}^{d} 2^{q-n} n! S(q, n) c_q \right) \langle a_{\tau(1)}, \ldots, a_{\tau(n)} \rangle. \]

### 3 Polynomials annihilating the Witt ring

#### 3.1 Preliminaries

The *fundamental ideal* \( I(F) \) of \( W(F) \) is the ideal consisting of all even-dimensional forms of \( W(F) \). Put \( I_t(F) := I(F) \cap W_t(F) \).

A field \( F \) is called *non-real* if \(-1\) is a sum of squares in \( F \), in which case we define the *level* of \( F \) to be
\[ s(F) := \min \{ n \in \mathbb{N} \mid a_1^2 + a_2^2 + \ldots + a_n^2 = -1, a_i \in F \}. \]

Otherwise, \( F \) is called *(formally)* real and the level is defined to be \( s(F) = \infty \).

It is a well-known fact that \( s(F) \) is a power of two, if \( F \) is non-real ([3] or [8]). There is no odd torsion in the Witt ring. The *height* of \( F \) is defined to be \( h(F) := \infty \) if there is no 2-power 2\(^d\) with \( 2^d W_t(F) = 0 \). Otherwise, \( h(F) \) is the smallest such 2-power with \( 2^d W_t(F) = 0 \).

We define the *torsion annihilator ideal* \( A_t(F) \) in \( \mathbb{Z}[X] \) by
\[ A_t(F) = \{ f(X) \in \mathbb{Z}[X] \mid f(\varphi) = 0 \text{ for all } \varphi \in I_t(F) \}. \]

For \( F \) non-real, define the *full annihilator ideal* \( A(F) \) in \( \mathbb{Z}[X] \) by
\[ A(F) = \{ f(X) \in \mathbb{Z}[X] \mid f(\varphi) = 0 \text{ for all } \varphi \in W(F) \}, \]
the *even annihilator ideal* by
\[ A_e(F) = \{ f(X) \in \mathbb{Z}[X] \mid f(\varphi) = 0 \text{ for all even-dimensional } \varphi \in W(F) \} \]
and the odd annihilator ideal by

$$A_o(F) = \{ f(X) \in \mathbb{Z}[X] \mid f(\varphi) = 0 \text{ for all odd-dimensional } \varphi \in W(F) \}. $$

In what follows, let $k = k(r)$ be the natural number uniquely determined by $\nu_2((2k - 2)!) < r \leq \nu_2((2k)!)$ (see [6]).

We define the ideals

$$J_{e,r} = (2^r X, 2^r X - 1, 2^r - 3, X^2, \ldots, 2^r - \nu_2((2k)!) - 2, X^2k - 2, X^{2k}) \cap F \subseteq F,$$

$$J_{e,r} = (2^r X, 2^r - 1, X^2, 2^r - 3, X^4, \ldots, 2^r - \nu_2((2k)!) - 2, X^2k - 2, X^{2k}),$$

$$J_{e,r} = (2^r X, 2^r - 1, X^2, 2^r - 3, X^4, \ldots, 2^r - \nu_2((2k)!) - 2, X^2k - 2, X^{2k}),$$

and

$$J_{r} = (2^r X, 2^r - 1, X^2, 2^r - 3, X^4, \ldots, 2^r - \nu_2((2k)!) - 2, X^2k - 2, X^{2k}).$$

**Example 3.1**

$$J_{e,1} = (2X, X^2),$$

$$J_{e,2} = (4X, 2X^2, X^4),$$

$$J_{e,3} = (8X, 4X^2, X^4),$$

$$J_{e,4} = (16X, 8X^2, 2X^4, X^6),$$

$$J_{e,5} = (32X, 16X^2, 4X^4, 2X^6, X^8),$$

$$J_{e,6} = (64X, 32X^2, 8X^4, 4X^6, X^8).$$

**3.2 Generators for the full annihilator ideal**

Since $\mathbb{Z}[X]$ is noetherian, the ideals $A_e(F), A_o(F), A(F)$ and $A_t(F)$ are finitely generated. Under certain conditions we give a set of generators.

All $f(X) \in A_t(F)$ have to vanish at $X = 0 \in W_t(F)$. This implies that the constant term of $f(X)$ is zero in the case of a real field $F$ and that the constant term of $f(X)$ is a multiple of $2^r$ in the case of a non-real field $F$ with level $s(F) = 2^{r-1}$. So, $A_t(F) = A_t(F) \cap \mathbb{Z}[X]$ in the real case and $A_t(F) = (A_t(F) \cap X \mathbb{Z}[X]) + 2^r \mathbb{Z}[X]$ in the non-real case where the level is $s(F) = 2^{r-1}$. From now on we will study $A_t'(F) := A_t(F) \cap X \mathbb{Z}[X]$.

The following lemma is, in the non-real case, proved in [6]. We will give a more general proof, using Stirling numbers of the second kind.

**Lemma 3.2** Let $F$ be a field for which $2^r W_t(F) = 0$. Then

$$J_{e,r} \subseteq A_t'(F).$$

**Proof.** For the generator $f(X) = 2^r X \in J_{e,r}$ we have $f(\varphi) = 2^r \varphi = 0$ for all $\varphi \in W_t(F)$.

Let $f(X) = 2^r - \nu_2((2k)!) X^2k, 1 \leq i < k$, be one of the other generators of $J_{e,r}$ and $\phi \simeq \langle a_1, a_2, \ldots, a_n \rangle$ an arbitrary even-dimensional element of $W_t(F)$.
By Corollary 2.5

\[ f(\phi) = \left( \sum_{q=1}^{2i} \binom{2q-1}{1} S(q,1) \binom{2i}{q} n^{2i-q} q^{2r-\nu_2((2i)!)} \right) \left( \prod_{i=1}^{n} \langle a_i \rangle \right) \]

\[ \perp \left( \sum_{q=2}^{2i} \binom{2q-2}{2} S(q,2) \binom{2i}{q} n^{2i-q} q^{2r-\nu_2((2i)!)} \right) \left( \prod_{1<i<j}^{n} \langle a_i, a_j \rangle \right) \]

\[ \perp \ldots \]

\[ \perp \left( \sum_{q=n}^{2i} \binom{2q-n}{n} S(q,n) \binom{2i}{q} n^{2i-q} q^{2r-\nu_2((2i)!)} \right) \langle a_1, \ldots, a_n \rangle. \]

For all \( j \leq q \) we have

\[ \nu_2 \left( \binom{2q-j}{j} S(q,j) \binom{2i}{q} n^{2i-q} q^{2r-\nu_2((2i)!)} \right) \]

\[ = q-j+j-d(j)+\nu_2(S(q,j)) + d(q) + d(2i-q) - d(2i) + (2i-q)\nu_2(n) + r - 2i + d(2i) \]

(by Kummer and Legendre)

\[ \geq q-d(j) + d(j) + d(q) + d(2i-q) + (2i-q)\nu_2(n) + r - 2i \]

(by Corollary 2.2)

\[ \geq q + d(2i-q) + (2i-q) + r - 2i \]

(since \( n \) is even)

\[ \geq r \]

Since \( 2^r W_1(F) = 0 \) it follows that

\[ f(\phi) = 0 \]

or equivalently that

\[ f(X) \in A'_1(F). \]

A similar argument holds for the generator \( f(X) = X^{2k} \), using the fact that \( r \leq \nu_2((2k)!). \)

This brings us to the main result of this paper:

**Theorem 3.3** Let \( F \) be a field such that \( 2^r W_1(F) = 0 \) and \( 2^{r-1}(I_1(F))^{2k-1} \neq 0 \) with \( k \) uniquely determined by \( \nu_2((2k-2)!)) < r \leq \nu_2((2k)!). \) Then

\[ J'_{c,r} = A'_1(F). \]

**Proof.** Let \( F \) be a field such that \( 2^r W_1(F) = 0 \) and \( 2^{r-1}(I_1(F))^{2k-1} \neq 0 \). Let \( k \) be the unique natural number such that \( \nu_2((2k-2)!) < r \leq \nu_2((2k)!). \)

Since \( X^{2k} \in J'_{c,r} \) annihilates every even-dimensional torsion quadratic form, we will have to prove that every polynomial of degree \( 2k-1 \),

\[ f(X) = c_{2k-1} X^{2k-1} + \ldots + c_1 X, \quad \text{with} \quad c_i \in \mathbb{Z}, \]

annihilating every even-dimensional torsion quadratic form, lies in \( J'_{c,r} \).

\( I_1(F) \) is generated by the elements \( \langle a \rangle \in I_{1}(F) \). The condition on the power of the fundamental ideal implies the existence of elements \( a_1, \ldots, a_{2k-1} \in F^* \) such that the form \( 2^{r-1} \langle a_1, \ldots, a_{2k-1} \rangle \) is not zero.

Fix such elements.

We will evaluate the polynomial \( f(X) \) in the even-dimensional torsion quadratic forms of shape

\[ \perp_{i=1}^{n} \langle a_{\sigma(i)} \rangle \quad \text{where} \quad 1 \leq n \leq 2k-1, \quad \sigma \in S_{2k-1}. \]
By Lemma 2.6 we get the following set of equations in the Witt ring:

\[
\left( \sum_{q=n}^{2k-1} 2^{r-n} n! S(q,n) c_q \right) \langle a_1, \ldots, a_n \rangle = 0, \quad 1 \leq n \leq 2k - 1.
\]

(3.1)

For \( n = 2k - 1 \), this becomes

\[
0 = (2k - 1)! S(2k - 1, 2k - 1)c_{2k-1} \langle a_1, \ldots, a_{2k-1} \rangle = (2k - 1)! c_{2k-1} \langle a_1, \ldots, a_{2k-1} \rangle.
\]

Since

\[
2^{r-1} \langle a_1, \ldots, a_{2k-1} \rangle \neq 0,
\]

and \( 2^r W_t(F) = 0 \) it follows that

\[
(2k - 1)! c_{2k-1} = b_{2k-1} 2^r \quad \text{for some} \quad b_{2k-1} \in \mathbb{Z},
\]

and since \( \nu_2((2k - 1)!) = \nu_2((2k - 2)!) < r \) that

\[
c_{2k-1} = 2^{r-\nu_2((2k-1)!)} b_{2k-1} \quad \text{for some} \quad b_{2k-1} \in \mathbb{Z}.
\]

For \( n = 2k - 2 \), (3.1) becomes

\[
((2k - 2)! c_{2k-2} + 2(2k - 2)! S(2k - 1, 2k - 2)c_{2k-1}) \langle a_1, \ldots, a_{2k-1} \rangle = 0.
\]

(3.2)

The second term \( 2(2k - 2)! S(2k - 1, 2k - 2)c_{2k-1} \langle a_1, \ldots, a_{2k-1} \rangle \) vanishes since

\[
\nu_2(2(2k - 2)! S(2k - 1, 2k - 2)c_{2k-1})
\]

\[
= 1 + \nu_2((2k - 2)! + \nu_2(S(2k - 1, 2k - 2)) + \nu_2(c_{2k-1})
\]

\[
= 1 + 2k - 2 - d(2k - 2) + \nu_2(S(2k - 1, 2k - 2)) + \nu_2(c_{2k-1})
\]

(by Legendre)

\[
\geq 2k - 1 - d(2k - 2) + d(2k - 2) - d(2k - 1) + \nu_2(c_{2k-1})
\]

(by Corollary 2.2)

\[
\geq 2k - 1 - d(2k - 1) + r - (2k - 1) + d(2k - 1)
\]

\[
= r
\]

and \( 2^r W_t(F) = 0 \).

Equation (3.2) is thus equivalent to

\[
(2k - 2)! c_{2k-2} \langle a_1, \ldots, a_{2k-2} \rangle = 0
\]

and it follows, since \( \nu_2((2k - 2)!) = \nu_2((2k - 1)!) < r \) that

\[
c_{2k-2} = 2^{r-\nu_2((2k-2)!)} b_{2k-2} \quad \text{for some} \quad b_{2k-2} \in \mathbb{Z}.
\]

Using the same technique and observing that

\[
\nu_2(2^{q-n} n! S(q,n) c_q) = q - n + \nu_2(n!) + \nu_2(S(q,n)) + \nu_2(c_q)
\]

\[
= q - d(n) + \nu_2(S(q,n)) + \nu_2(c_q)
\]

(by Legendre)

\[
\geq q - d(n) + d(q) + r - q + d(q)
\]

(by Corollary 2.2)

\[
= r
\]

for all \( n < q \) and that \( \nu_2(n!) < r \) for all \( 1 \leq n \leq 2k - 1 \), the set of equations (3.1) is equivalent to the set of
The non-vanishing condition on the power of the fundamental ideal implies that
\[ n! c_n(a_1, \ldots, a_n) = 0, \quad \text{where} \quad n = 1, \ldots, 2k - 1. \]

The solutions are
\[ c_n = 2^{r - \nu_2(n)} b_n \quad \text{for some} \quad b_n \in \mathbb{Z}. \]

We can thus rewrite,
\[ f(X) = c_{2k - 1} X^{2k - 1} + \ldots + c_1 X \]
as
\[
\begin{align*}
f(X) &= 2^{r - \nu_2((2k - 1)!) b_{2k - 1}} X^{2k - 1} + 2^{r - \nu_2((2k - 2)!) b_{2k - 2}} X^{2k - 2} + \ldots + 2^{r - \nu_2((2 + 1)!) b_{2 + 1}} X^{2 + 1} \\
&\quad + b_2 2^{r - \nu_2((2 + 1)!) X^{2 + 1}} + \ldots + 2^{r - \nu_2(1!) b_1} X \\
&= 2^{r - \nu_2((2k - 1)!) X^{2k - 1}}(b_{2k - 1} X + b_{2k - 2}) + \ldots + 2^{r - \nu_2((2 + 1)!) X^{2 + 1}}(b_{2 + 1} X + b_2) + \ldots + 2^r b_1 X
\end{align*}
\]
or equivalently,
\[ f(X) \in J_{e,r}', \]
i.e.,
\[ A'_e(F) \subset J_{e,r}' \]
and using the other inclusion of the previous lemma
\[ A'_e(F) = J_{e,r}'. \]
\[ \square \]

**Corollary 3.4** Let \( F \) be a non-real field of finite level \( s(F) = 2^{r - 1} \) such that \( s(F)(I(F))^{2k - 1} \neq 0 \), where \( k = k(r) \) is uniquely determined by \( \nu_2((2k - 2)! < r \leq \nu_2((2k)!). \) Then
\[ A(F) = J_{r}. \]

**Proof.** Since \( s(F) = 2^{r - 1} \), we have that \( 2^r W(F) = 0. \) Moreover, \( F \) is non-real and hence \( I_{e}(F) = I(F). \)
The non-vanishing condition on the power of the fundamental ideal implies that
\[ A_e(F) = A'_e(F) + (2^r) = J_{e,r}' + (2^r) = J_{e,r}. \]
If \( \phi \) is an odd-dimensional form in \( W(F) \), then \( \phi \perp -1 \) is an even-dimensional form in \( W(F) \). This implies that
\[ A_o(F) = J_{o,r}. \]
\( J_{e,r} \) and \( J_{o,r} \) are comaximal ideals, since they contain the comaximal ideals \((X^{2k})\) and \((X - 1)^{2k})\) respectively. So, we get
\[ A(F) = A_e(F) \cap A_o(F) = J_{e,r} \cap J_{o,r} = J_{e,r} \cdot J_{o,r} = J_{r}. \]
\[ \square \]

**Corollary 3.5** Let \( F \) be a real field of finite height \( h(F) = 2^r \) such that \( \frac{1}{2} h(F)(W_r(F))^{2k - 1} \neq 0, \) where \( k = k(r) \) is uniquely determined by \( \nu_2((2k - 2)! < r \leq \nu_2((2k)!). \) Then
\[ A_t(F) = J_{e,r}. \]
Proof. By the definition of the height $h(F) = 2^r$, we have that $2^r W(F) = 0$. Since $F$ is a real field, we have $I_t(F) = W_t(F)$. The non-vanishing condition on the power of the torsion ideal implies that
\[ A_r(F) = A'_r(F) + (2^r) = J_{e,r} + (2^r) = J_{e,r}. \]

Remark 3.6 In [6], examples are given of non-real fields $F$ satisfying the conditions of Corollary 3.4. Starting with a field $F$ of level $2^{r-1}$, the purely transcendental extension of transcendence degree $2k - 1$, $K = F(X_1, \ldots, X_{2k-1})$ satisfies the conditions, since the form $2^{r-1}\langle 1, X_1 \rangle \otimes \ldots \otimes \langle 1, X_{2k-1} \rangle \in s(F)\langle I(F) \rangle^{2k-1}$ is anisotropic over $K$.

Real fields $F$ of arbitrary height $h(F) = 2^r$ satisfying the conditions of Corollary 3.5 can be found using Merkurjev’s method of iterated function fields (for a description we refer the reader to [2]). We will sketch the construction of such a real field, the details can be found in the author’s thesis (UCD - 2006). Let $k$ be a real field and consider the purely transcendental extension of degree $(2k - 1)2^r$,
\[ K = k(X_{1,1}, \ldots, X_{1,2^r}, X_{2,1}, \ldots, X_{2k-1,2^r}). \]
Put $\pi_i := X_{i,1}^2 + \ldots + X_{i,2^r}^2$ and consider the form $\varphi = 2^{r-1}\langle 1, -\pi_1 \rangle \otimes \ldots \otimes \langle 1, -\pi_{2k-1} \rangle$. One can show that $\varphi$ is anisotropic over $K(2^r)$, the function field of the quadric $X_1^2 + X_2^2 + \ldots + X_{2^r}^2 = 0$ over $K$. For a real extension $L$ of $K$ such that $\varphi$ is anisotropic over $L(2^r)$ and for any $a \in \Sigma L^2$ one shows that $\varphi$ is anisotropic over $L(\psi)(2^r)$, with $\psi = 2^r(1, -a)$. We construct a tower of fields $K_0 = K \subset K_1 \subset K_2 \subset \ldots$ by defining $K_{i+1}$ as the free compositum of all function fields $K_i(\psi)$ over $K_i$ with $\psi \in \{2^{r-1}-b \mid b \in \Sigma K_i^2\}$. Let $F = \bigcup_{i=0}^{\infty} K_i$, then $F$ is a real field with Pythagoras number $p(F) = 2^r$ such that $\varphi_F \in 2^{r-1}(W_t(F))^{2k-1}$. Since the height $h(F)$ of a field $F$ is the smallest $2$-power greater than or equal to the Pythagoras number $p(F)$ we have constructed a field satisfying the conditions of Corollary 3.5.

Remark 3.7 One can show that, for a field $F$, satisfying $2^r W(F) = 0$, but not satisfying the non-vanishing condition $2^{r-1}I_t(F))^{2k-1} \neq 0$, the torsion annihilator ideal $A_r(F)$ always differs from $J_{e,r}$. To show this, one proceeds as follows. Using Corollary 2.5 and the lower bound on the $2$-adic order of Stirling numbers of the second kind, one shows that $q_l(\varphi) \in 2^{r-1}I_l$ for all $\varphi \in I$, with $q_l(X) = X(X-2)(X-4)\ldots(X-2(l-1)) \in \mathbb{Z}[X]$. Choosing the appropriate index $l = l(r)$, $q_l(X)$ annihilates all even torsion forms and doesn’t belong to $J_{e,r}$.

A set of generators for the ideal $A_r(F)$ in the other cases is, in general, not known.

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