FLATNESS IMPLIES SMOOTHNESS FOR SOLUTIONS
OF THE POROUS MEDIUM EQUATION

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Abstract. One of the major problems in the theory of the porous medium equation
\[ \partial_t \rho = \Delta_x \rho^m, \quad m > 1, \]
is the regularity of the solutions \( \rho(t, x) \geq 0 \) and the free boundaries
\[ \Gamma = \partial \{(t, x) : \rho > 0 \}. \] Here we assume flatness of the solution and derive \( C^\infty \) regularity of the interface after a small time, as well as \( C^\infty \) regularity of the solution in the positivity set and up to the free boundary for some time interval.

We use these facts to prove the following eventual regularity result: solutions with compactly supported initial data are smooth after a finite time \( T_r \) that depends on \( \rho_0 \). More precisely, \( \rho^{m-1} \) is \( C^\infty \) in the positivity set and up to the free boundary, which is a \( C^\infty \) hypersurface for \( t \geq T_r \). Moreover, \( T_r \) can be estimated in terms of only the initial mass and the initial support radius. This result eliminates the condition of non-degeneracy on the initial data that has been carried on for decades in the literature. Let us recall that regularization for small times is false, and that as \( t \to \infty \) the solution increasingly resembles a Barenblatt function and the support looks like a ball.

1. Introduction

We consider the porous medium equation (PME), that we will write as
\[ \partial_t \rho = k \Delta_x \rho^m, \quad m > 1. \]
We assume that the solution is defined in a time-space cylinder \( Q := I \times \Omega \) with an open time interval \( I := (t_1, t_2) \) and an open set \( \Omega \subset \mathbb{R}^n \). This is convenient for the local regularity theory. In the last part we consider global solutions defined in \( Q := (0, \infty) \times \mathbb{R}^n \).

We introduce a constant \( k > 0 \) for convenience in the calculations though it is irrelevant in the results since it can be absorbed for instance into the \( \rho \) variable, without changing time or space. To be precise, the change \( \bar{\rho} = k^{1/(m-1)} \rho \) allows to pass from \( k > 0 \) to the value \( \bar{k} = 1 \) of the usual PME. We will fix the value \( k = \frac{m-1}{m} \) throughout the paper, since in this way the explicit formulas that enter our computations in the local regularity theory will be easier to read. In particular, the equation can then be written as
\[ \partial_t \rho = \nabla \cdot (\rho \nabla \rho^{m-1}), \]
and we use the formula \( v = \rho^{m-1} \) to define the ‘pressure’ and then \( v(t, x) \) satisfies
\[ \partial_t v = (m-1)v \Delta_x v + |\nabla_x v|^2. \]
It is well-known that the equation can be solved in a unique way in the sense for instance of continuous weak solutions, after giving Dirichlet data in the parabolic boundary of \( Q \), or just giving initial data if the space domain is \( \mathbb{R}^n \). It is also well-known that initial data that vanish say in the closure of an nontrivial open set at the initial time \( t_1 \) will vanish for a certain time interval \( J = (t_1, t_0), \) \( t_0 \leq t_2 \), in a possibly smaller set. This

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property is called finite speed of propagation. If we denote the positivity set of the solution by $\mathcal{P}(\rho) = \{(t, x) : \rho(t, x) > 0\}$ and write the section at time $t$ as

$$\mathcal{P}(t) = \{x \in \Omega \mid \rho(t, x) > 0\},$$

then finite propagation means that $\mathcal{P}(t)$ is smaller than $\Omega$ for all $t \in J$. The consequence of finite propagation is the existence of a free boundary or interface $\Gamma = \partial \mathcal{P} \cap Q$, where the transition from the ‘gas region’ $\mathcal{P}$ to the ‘empty region’ $\{\rho = 0\}$ takes place.

The study of the free boundary $\Gamma$ plays a key role in the regularity theory for the PME (1.1), since the equation is not parabolic at those points. The positivity set $\mathcal{P}(t)$ is open and increasing with respect to $t$ for $t \geq t_1$, and strictly increasing outside the original support. Thus, the free boundary is given as a graph of the form $t = h(x)$, where $h$ is a function defined in the closure of the complement of the support of $\rho(t_1, \cdot)$. Let us consider global solutions for simplicity. It was shown in [10] that the function $h$ is at least Hölder continuous. As a consequence of [11], $h$ is Lipschitz continuous for $t$ sufficiently large when the initial datum is a compactly supported and non-degenerate function (roughly that the velocity $|\nabla v|_{\partial \mathcal{P}(t)}$ at the free boundary $\partial \mathcal{P}(t)$ is bounded from below, see [11] for a nondegeneracy condition on the initial data ensuring non-degeneracy of the solution for large time, which was relaxed in [18]), and hence the solution is monotone in a cone of directions including the time direction locally near the free boundary. A decade later, the second author [18] proved that solutions satisfying this monotonicity and non-degeneracy conditions are actually smooth up to the free boundary.

We recall that regularity has an important consequence for the dynamics of the free boundary $\Gamma$. Indeed, once you prove that $\Gamma$ is (at least locally) a smooth hypersurface in $\mathbb{R}^{n+1}$ and the pressure is also smooth and non-degenerate on the occupied side $v > 0$ and up to the free boundary, then standard theory shows that Darcy’s law holds on that part of $\Gamma$, in both forms: $v_t = |\nabla v|^2$ or $V_n = |\nabla v|$, cf. [26] (where $V_n$ denotes the normal advancing speed of the free boundary, and both $v_t$ and $\nabla v$ are calculated as lateral limits from the region $\{v > 0\}$).

This paper was motivated by the wish to obtain a higher regularity theory for local solutions, and to eliminate the extra conditions of monotonicity and non-degeneracy on the initial data that were needed to establish the higher regularity of solutions and free boundaries in dimensions $n \geq 2$. Such result was known for $n = 1$, cf. [6]. For years the efforts have been unsuccessful in several dimensions. In this paper we contribute a key step by considering locally flat solutions, in the sense that they are sandwiched on a cylinder between two traveling wave solutions lying very close to each other, and then we show that they are smooth up to the free boundary in half of the cylinder (thus, a bit later in time). See Definition 1 below for the concept of $\delta$-flatness, and Theorem 1 for the precise formulation of our main result.

Important steps in the direction of this result were taken in the Dissertation of the first author, [16], which deals with the special case of flat solutions on a global scale, and such results will be used in the course of our project. On the other hand, our main result has as a consequence the result we were originally looking for, concerning the large time behavior of global solutions (i.e., defined on the whole $\mathbb{R}^n$) with compactly supported initial data. As a consequence of the well-known asymptotic convergence towards a Barenblatt profile [24], such solutions satisfy our flatness condition for large enough times. This allows to dispense completely with the regularity and non-degeneracy assumptions used in [11] and [15] on the initial data.
In particular, we will prove that the free boundary function $h$ above is $C^\infty$ smooth outside a large ball, and it converges to the free boundary of the Barenblatt solution with same mass, see Theorem 2. Moreover, the pressure of the solution is $C^\infty$ in the positivity set and up to the free boundary for times $t \geq T(u_0)$, a lower bound that can be estimated in terms of the initial mass and the initial radius. Much is known about the precise behavior of solutions when $t \to \infty$. Recently, [21, 22] has obtained very precise asymptotics as $t \to \infty$ for solutions with compact support.

1.1. Preliminaries, definitions and main result. Our work uses many different tools of the PME theory, most of them can be found in [26]. Two basic facts will have a special relevance, the use of the scaling group and the existence of a family of traveling wave solutions.

- The first one consists of the observation that, due to the symmetries of the PME, whenever $\rho(t, x)$ is a solution in a given cylinder, so is the expression
  \[ \tilde{\rho}(t, x) = A \rho(C(t-t_0), L(x-x_0)) \]
  for any constants $A, C, L > 0$ such that $A^{n-1} = CL^{-2}$, and all displacement constants $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$. Of course, the domain of definition varies accordingly. We may add also a rotation or symmetry of the space variables without affecting the validity of the solution.

- The traveling waves are a family of solutions that are perfectly flat in terms of the pressure variable $v = \rho^{n-1}$. They are given by the formula
  \[ \rho_{tw}^{n-1}(t, x; a, c, d) := c (c t + (a, x) + d)_+, \]
  defined for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. There are a number of parameters that can be fixed at will: the wave speed $c > 0$; $a \in \mathbb{R}^n$ is a unit vector indicating the direction ($-a$ would be more appropriate) in which the wave front propagates; the displacement $d \in \mathbb{R}^n$ is also arbitrary and incorporates possible displacements of the $x$ or $t$ axis or both. If $a$ points into the $n$-th coordinate direction and the wave travels at unit speed $c = 1$, the formula for the traveling wave at $(t_0, x_0) = (0, 0)$ simplifies to
  \[ \rho_{tw}(t, x) := (t + x_n + d)_+^{\frac{1}{n-1}}, \]
  We will drop the subscript when the displacement $d = 0$.

- Next, we need a rough concept of flatness of a solution. For easier reference in more general applications we state it for any $a$, $c$, and some $d > 0$.

**Definition 1.** Consider $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and $\delta, R > 0$. We say that $\rho$ is a $\delta$-flat solution of the PME at $(t_0, x_0)$ on scale $R$ if there exist $V > 0$ and a unit vector $a \in \mathbb{R}^n$ such that

1. $\rho$ is a nonnegative and continuous weak solution of the PME on the cylinder
   \[ Q_{R,V}(t_0, x_0) := (t_0 - \frac{R}{V}, t_0] \times B_R(x_0) \]
2. and
   \[ \rho_{tw}(t - t_0, x - x_0; a, V, -\delta R) \leq \rho(t, x) \leq \rho_{tw}(t - t_0, x - x_0; a, V, \delta R) \]
   on $Q_{R,V}(t_0, x_0)$.
   We call $V$ the speed and $a$ the direction of $\rho$. 

In what follows we use the normalized values $V = 1$ and $a = e_n$ unless mention to the contrary. Moreover, we should say a $\delta$-approximate speed and a $\delta$-approximate direction to be precise, since they are not unique for any given solution, they may admit small variations.

Let us perform the scaling reduction. Given a $\delta$-flat solution $\rho$ at $(t_0, x_0)$ on scale $R$ with $\delta$-approximate speed $V$ and $\delta$-approximate direction $a$, we may use the two-parameter scaling group with parameters $R, V > 0$ and introduce any orthogonal matrix $O$ with $O e_n = a$ to obtain a function

$$\tilde{\rho}(t, x) = (V R)^{-\frac{1}{m-1}} \rho \left( V^{-1} R t + t_0, R O x + x_0 \right)$$

which is another solution of the PME (with the same $k$) and this function is $\delta$-flat at $(0, 0)$ on scale 1 with $\delta$-approximate unit speed and $\delta$-approximate direction $e_n$; in other words, we have

$$\rho_{tw,-\delta} \leq \tilde{\rho} \leq \rho_{tw,\delta} \text{ in } Q_{1,1}(0, 0),$$

meaning that $\tilde{\rho}$ is trapped between two traveling wave solutions with velocity 1 that lie at a distance $2\delta$ in the unit cylinder centered at zero. Note that exactly one of the traveling wave solutions is positive in $(0, 0)$. We are now ready to state our main contribution.

**Theorem 1.** There exists $\delta_0 > 0$ such that the following holds:

If $\rho$ is a nonnegative $\delta$-flat solution of the PME at $(0, 0)$ on scale 1 with $\delta$-approximate direction $e_n$ and $\delta$-approximate speed 1, and $\delta \leq \delta_0$, then for all derivatives we have uniform estimates

$$|\partial_t^k \partial_x^m \nabla_x (\rho^{m-1} - (x_n + t))| \leq C \delta$$

at all points $(t, x) \in \left( [-1/2, 0] \times B(0, \frac{1}{2}) \right) \cap \mathcal{P}(\rho)$ with $C = C(n, m, k, \alpha) > 0$. In particular, $\rho^{m-1}$ is smooth up to the boundary of the support in $(-\frac{1}{2}, 0] \times B_{1/2}$, and

$$|\nabla_x \rho^{m-1} - e_n|, \quad |\partial_t \rho^{m-1} - 1| \leq C \delta.$$

Moreover, the level sets for positive values of $\rho$ and the free boundary are uniformly smooth hypersurfaces inside $(-\frac{1}{2}, 0] \times B_{1/2}(0)$.

Theorem 1 is formulated in a normalized setting. It can be combined with the symmetries of the porous medium equation in the obvious fashion to get a result in a not normalized setting. We point out that though our strategy follows the general ideas of the proofs developed by Caffarelli for similar free boundary problems, we considerably deviate from his arguments in the proof of Proposition 2 below, mainly because the proper linearization of geometry and density at the free boundary is a nonstandard degenerate equation, the solutions of which must replace the use of harmonic and caloric functions used in the standard theory of free boundary problems.

Theorem 1 implies the eventual $C^\infty$-regularity result for global solutions that we have already mentioned as our second contribution. We use the notation $R_B(t) = c_1(n, m) M^{(m-1)\lambda} t^{\lambda}$ with

$$\lambda = 1/(n(m - 1) + 2)$$

for the Barenblatt radius for the solution with mass $M$ located at the origin.

**Theorem 2.** Let $\rho \geq 0$ be a solution of the PME posed for all $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and let the initial data $\rho_0$ be nonnegative, bounded and compactly supported with mass $M = \int \rho_0 dx > 0$. Then, there exists a time $T_r$ depending on $\rho_0$ such that for all $t > T_r$ we have:
The pressure of the solution \( \rho^{m-1} \) is a \( C^\infty \) function inside the support and is also smooth up to the free boundary, with \( \nabla \rho^{m-1} \neq 0 \) at the free boundary. Moreover, the free boundary function \( t = h(x) \) is \( C^\infty \) in the complement of the ball of radius \( R(T_r) \) and, there exists \( c > 0 \) such that

\[
t^{-n\lambda}(a^2 M^{2(m-1)\lambda} - ct^{-2\lambda} - \frac{\lambda |x - x_0|^2}{2t^{2\lambda}})^{\frac{m-1}{m}} \leq \rho(t, x)
\]

where \( x_0 = M^{-1} \int x \rho(x) dx \) is the conserved center of mass, and \( a \) is the constant defined in \( \text{(8.3)} \). Moreover,

\[
B_{R(t) - ct^{-\lambda}}(x_0) \subseteq \text{supp } \rho \subseteq B_{R(t) + ct^{-\lambda}}(x_0)
\]

(ii) Moreover, if the initial function is supported in the ball \( B_R(0) \), then we can write the upper estimate of the regularization time as

\[
T_r = T(n, m) M^{1-m} R^\frac{2}{\lambda}.
\]

By scaling and space displacement we can reduce the proof to the case \( M = 1 \) and \( x_0 = 0 \). We remind the reader that a minimum delay \( T_r \) is needed for general initial data, even under the assumptions of compact support and smoothness, for the regularity result to hold. Indeed, it is known that the initial regularity of a solution can be lost for some intermediate times because of the phenomenon called focusing, whereby typically a hole in the support of the initial data gets filled in finite time by the evolving solution, and then \( \rho^{m-1} \) is not Lipschitz continuous near the focusing point at the focusing moment, cf. \( \text{(2.5)} \).

But this phenomenon disappears in finite time for compactly supported solutions as shown in \( \text{(11)} \), and eventually solutions and interfaces are \( C^\infty \) smooth as the theorem claims. The asymptotics in part 1 are sharp, see Seis \( \text{(22)} \), who also discusses finer asymptotics.

We conclude this introduction with a conjecture.

**Conjecture 1.** Suppose that \( \rho \) is a solution to the porous medium equation on \([-1, 0] \times B_2(0) \).

We assume that there is a nonempty open cone \( C \) so that \( \rho(t, x) \leq \rho(t, y) \) for \( y - x \in C \). Then \( \rho^{m-1} \) is smooth up to the boundary of the support outside the initial support and \( \nabla \rho^{m-1} \neq 0 \) at the free boundary.

1.2. Outline. We gather in Section 2 the precise statements of the main propositions that form the basis of the proof of Theorem 1 we state and prove the main body of the proof of the theorem in Sections 3 and 4 and then we prove the list of auxiliary propositions in the later sections. We derive Theorem 2 in Section 8 after the detailed quantitative analysis of the evolution of global solutions and their interfaces is done.

We will use the following rather standard notations for the space-time cylinders that appear, \( Q_r(t_0, x_0) = (t_0 - r, t_0) \times B_r(x_0) \) and \( Q_r = Q_r(0, 0) \). The open space ball of center \( x_0 \in \mathbb{R}^n \) and radius \( r > 0 \) will be denoted by \( B_r(x_0) \).

2. Idea of the proof. Basic Propositions

The key step in the proof of Theorem 1 is the following self-improvement result.

**Proposition 1** (Improvement of flatness). There exist \( \delta_0 \) and \( r \),

\[
0 < 2\delta_0 < r < \frac{1}{2}
\]
such that if $\rho$ is a solution of the PME on $Q_1$ with $\rho(0, 0) = 0$, $(0, 0)$ lies in the boundary of the support, and $\rho$ is $\delta$-flat with velocity 1 and scale 1 at $(0, 0)$ with $\delta < \delta_0$, then there exist $\Lambda > 0$ and a unit vector $a$ which satisfy

$|1 - \Lambda| \leq \frac{\delta}{r}$, $|a - e_n| \leq c\sqrt{\delta}$,

such that the rescaled solution

$$\tilde{\rho}(t, x) = r^{-\frac{1}{m-1}} \rho\left(\Lambda^2 rt, \Lambda r x\right)$$

is $\delta/2$-flat with velocity 1 and scale 1 at $(0, 0)$, in direction $a$.

To establish these facts we prove and then use a number of regularity results:

- Proposition 2 for global solutions to the PME with initial pressure which is Lipschitz close to a traveling front,
- the decay estimate from Proposition 3 - a consequence of a Gaussian kernel estimate,
- the local regularity results Proposition 4 for non-degenerate parabolic equations and non-degenerate local solutions to the porous medium equations. The estimates there are more precise than the ones of the theorems.

Together these propositions allow to prove the $\delta$-improvement of Proposition 1. From there, standard and easier arguments yield $C^{1,\alpha}$ regularity of the boundary of the support, and the pressure at the boundary, and a lower bound on the velocity $\nabla \rho^{m-1}$ at the free boundary. After that, full regularity follows from Proposition 4.

The first auxiliary result we have mentioned deals with solutions of the PME, equation (1.1), that are global in space on a time interval $(t_1, t_2) \subset \mathbb{R}$ with finite $t_1 < t_2$.

**Proposition 2.** There exists $\mu > 0$ such that we have:

If $\rho_0 : \mathbb{R}^n \to [0, \infty)$ satisfies

$$\sup_{P(\rho_0)} \left| \nabla_x \left( \rho_0^{m-1} - (x_n + t_1) \right) \right| \leq \mu$$

and if $\rho$ is the solution of (PME) on $(t_1, t_2) \times \mathbb{R}^n$ with initial data $\rho_0$ at $t = t_1$, then for $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$

$$\sup_{P(\rho)} (t - t_1)^{k+|\alpha|} \left| \partial_t^k \partial_x^{\alpha} \nabla_x \left( \rho^{m-1} - (x_n + t) \right) \right| \leq C_1 \sup_{P(\rho_0)} \left| \nabla_x \left( \rho_0^{m-1} - (x_n + t_1) \right) \right|$$

with $C_1 = C_1(n, m, k, \alpha) > 0$.

Since the equation is invariant under time translations, this is equivalent to restricting to $t_1 = 0$. The statement is the main result of Kienzler [17]. We announce that we will take $\mu$ equal to the constant of Proposition 2 throughout the whole paper.

The next statement gives a pointwise control of differences of the graphs of the pressures of two solutions assuming uniform control on the gradients of the pressure. After a change of coordinates in dependent and independent variables, this is a consequence of Gaussian estimates of the second author [18].

**Proposition 3.** Let $\rho$ and $\tilde{\rho}$ be global solutions to the PME on $(0, \infty) \times \mathbb{R}^n$ with initial data $\rho_0$ and $\tilde{\rho}_0$ which satisfy (2.2).

(i) Suppose that

$$|\rho_0^{m-1} - \tilde{\rho}_0^{m-1}| \leq \delta$$

for all $x \in \mathbb{R}^n$,
and let $R \geq \delta$, $0 < t \leq R$ and $x \in \mathbb{R}^n$. Then there exists $c = c(n, m)$ such that for

$$a := c \left( R^2 e^{-\frac{R^2}{1 - \rho_0(x)}} \right)^t + R^{-n}(R + \rho_0^{-1}(x))^{\frac{m-1}{m}} \int_{B_R(x)} |\rho_0(y) - \tilde{\rho}_0(y)| dy + \delta \left( \frac{\delta}{t} \right)^{\frac{1}{m-1}}$$

we have the pointwise comparison

$$\rho(t, x) \leq \rho(t, x + ae_n).$$

(ii) If moreover

$$\rho_0(y) = \tilde{\rho}_0(y) \text{ for } y \in B_R(x),$$

then the conclusion (2.4) holds with

$$a = c\delta \left( \frac{\delta}{t} \right)^{\frac{1}{m-1}} e^{-\frac{R^2}{1 - \rho_0(x)}}.$$

Note that the first part of Proposition 3 is only useful for $t \geq \delta$. We also need a local regularity statement under non-degeneracy conditions.

**Proposition 4.** There exist $\delta_5 > 0$ and $\kappa_5 > 0$ such that the following holds if $\rho$ is a $\delta$-flat solution of (PME) on $(0, 1) \times B_2(0)$ for a $\delta < \delta_5$:

(i) We have

$$\left| \partial_t^k \partial_x^\alpha (\rho^{m-1}(t, x) - (x_n + t)) \right| \leq C_5 \delta \left( t^{-(k+|\alpha|)}(t + x_n)^{-\frac{|\alpha|}{2}} + (t + x_n)^{-k-\alpha} \right).$$

if $x_n + t \geq \kappa_5 \delta$ and $(t, x) \in (0, 1) \times B_1(0)$.

(ii) If $(0, 0)$ is contained in the free boundary, if

$$\sup_{P(\rho) \cap (0, 1) \times B_2(0)} \left| \nabla_x (\rho^{m-1} - (x_n + t)) \right| \leq \mu,$$

then for $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$,

$$\sup_{(0, 1) \times B_1(0) \cap P(\rho)} \left| \partial_t^k \partial_x^\alpha (\rho^{m-1} - (x_n + t)) \right| \leq C_5 \delta t^{\frac{k+|\alpha|}{2}}(t + \rho(m-1))^{\frac{1}{2} - \frac{|\alpha|}{2}}.$$

The first part is a purely parabolic estimate up to rescaling. The second part gives localized estimates of the type of Proposition 2.

3. Proof of Proposition 4

**Proof.** The proof is organized in several steps.

3.1. Initial data for comparison solutions. We construct global solutions much closer to $\rho$ than the traveling wave solutions. Let $\rho$ be $\delta$-flat in $Q_1$ with $\delta \leq \delta_0$ where $\delta_0$ will be chosen later on. In the first step we will construct comparison functions $\rho_{\pm}$ as global solutions of the PME on the strip $(t_1, 0) \times \mathbb{R}^n$ for some $t_1 \in \left[ -\frac{1}{2}, 0 \right)$ by specifying initial data $\rho_{\pm, 0}$ at time $t_1$ that satisfy the following properties:

$$\rho_{\pm, -2\delta}(t_1, x) \leq \rho_{-, 0}(x) \leq \rho_{+, 0}(x) \leq \rho_{\pm, 2\delta}(t_1, x) \text{ for all } x \in \mathbb{R}^n,$$

$$\rho_{\pm, 0}(x) = \rho_{\pm, \pm 2\delta}(t_1, x) \text{ for all } x \text{ with } |x| \geq \frac{3}{4},$$

$$\rho_{-, 0}(x) \leq \rho(t_1, x) \leq \rho_{+, 0}(x) \text{ for all } x \text{ with } |x| \leq 1,$$
(3.4) \( \rho_{-0}(x) = \rho(t_1, x) = \rho_{+0}(x) \) for all \( x \) with \( |x| \leq \frac{1}{2} \) with \( \rho^{m-1}(t_1, x) \geq \frac{4\delta}{\mu} \), and

\[
(3.5) \quad |\nabla \rho_{-0}^{m-1}(x) - e_n| \leq \mu \quad \text{for all } x \in \mathcal{P}(\rho_{\pm0})
\]

where \( \mu \) is the constant from Proposition 2 and can be chosen as small as needed without loss of generality.

We postpone the construction of \( \rho_{\pm0} \) and the verification of its stated properties and explore first the consequences for the proof of the proposition. In this process we will choose \( t_1 \) and \( \delta_0 \).

3.2. The comparison solutions. The gradient estimate (3.3) and standard theory of the PME ensure that there exist unique solutions \( \rho_{\pm} \) of the PME on \( (t_1, 0) \times \mathbb{R}^n \) with initial data \( \rho_{\pm}(t_1, x) = \rho_{\pm0}(x) \). By the maximum principle, the ordering (3.1) is preserved for these solutions for all time. By Proposition 2 the solutions \( \rho_{\pm} \) are smooth in the sense of the proposition and satisfy the estimate (2.3) and the estimates of Proposition 4.

Due to the identity condition (3.2) we can apply the second part of Proposition 3 to \( \rho_{tw,-2\delta}(\tilde{t} + t_1, \cdot) \) and \( \rho_{-}(\tilde{t} + t_1, \cdot) \) resp. to \( \rho_{+}(\tilde{t} + t_1, \cdot) \) and \( \rho_{tw,2\delta}(\tilde{t} + t_1, \cdot) \) with \( R = \frac{1}{4} \) and \( 0 \leq \tilde{t} \leq |t_1| \) to get

\[
\rho_{-}(t, x) \leq \rho_{tw,-2\delta}(t, x + a e_n) \leq \rho_{tw,-\delta}(t, x)
\]

for \( t_1 \leq t \leq 0 \) and \( |x| = 1 \) provided (see Proposition 3 (2.5))

\[
(3.6) \quad \frac{a}{\delta} = c \left( \frac{\delta}{t - t_1} \right)^{\frac{1}{m-1}} e^{-\frac{1}{m-1} \frac{\delta}{t - t_1}} \leq c e^{\frac{1}{m-1} \frac{\delta}{t - t_1}} \leq 1
\]

for \( c_2 = c_2(n, m) \) which is the first restriction on \( t_1 \). It holds provided |t_1| is bounded by a constant depending only on \( m \) and \( n \). Likewise, an application of Proposition 3 with \( R = \frac{1}{4} \) yields \( \rho_{+}(t, x) \geq \rho_{tw,+\delta}(t, x) \) on the same boundary set. Under this restriction we then have

\[
(3.7) \quad \rho_{-}(t, x) \leq \rho_{tw,-\delta}(t, x) \leq \rho_{\pm}(t, x) \leq \rho_{+}(t, x)
\]

for \( |x| = 1 \), and \( t_1 \leq t \leq 0 \) by the flatness of \( \rho \). Once we have this information at the boundary we can apply the comparison principle and deduce from (3.7) and (3.3) that

\[
(3.8) \quad \rho_{-}(t, x) \leq \rho(t, x) \leq \rho_{+}(t, x)
\]

for \( t_1 \leq t \leq 0 \) and \( |x| \leq 1 \).

3.3. The distance of the comparison solutions. Next, we apply Proposition 3 with \( |x| \leq \frac{1}{4} \), \( R = \frac{1}{4} \) to \( \rho_{-}(\tilde{t} + t_1, \cdot) \) and \( \rho_{+}(\tilde{t} + t_1, \cdot) \) and \( 0 < \tilde{t} \leq |t_1| \). By construction (more precisely condition (3.4))

\[
\int_{B_{\frac{1}{4}}(x)} |\rho_{-0}(y) - \rho_{+0}(y)| dy = \int_{B_{\frac{1}{4}}(x) \cap \{ \rho^{m-1}(t_1, y) \leq \frac{4\delta}{\mu}\}} |\rho_{-0}(y) - \rho_{+0}(y)| dy
\]

\[
\leq c_n \left( \frac{\delta}{\mu} \right)^{\frac{m}{m-1}}
\]

and hence \( a \) in the first part of Proposition 3 is bounded by

\[
a \leq c \left[ e^{rac{1}{\alpha-11}} + \left( \frac{\delta}{t - t_1} \right)^{\frac{1}{m-1}} + \frac{1}{\mu} \left( \frac{\delta}{\mu} \right)^{\frac{1}{m-1}} \right].
\]

The exponential term is bounded by

\[
e^{-\frac{1}{\alpha-11}} \leq \delta(t - t_1)^4 \leq \delta t_1^4
\]
We define 

\[ \rho(t, x) = \rho_{n, m} \left( t, x + c_{n, m} \left[ \left( \frac{\delta}{|t - t_1|} \right)^{-1} + e^{-c_\delta |t - t_1|} \right] \delta e_n \right), \]

with a constant \( c = c(n, m) \). If \( |x| \leq \frac{1}{4}, |t_1| \ll \frac{1}{2} \) (which we used above) and \( \frac{1}{4} \leq t \leq 0 \) (to replace \( t - t_1 \) by \( t_1 \)) we obtain

\[ \rho_-(t, x) \leq \rho(t, x) \leq \rho_+(t, x) \leq \rho_-(t, x + \tau \delta e_n), \quad \tau := c_{n, m} \left( \frac{\delta}{|t_1|} \right)^{-1} + |t_1|^4. \]

We will complete the proof by studying the distance of the two solutions to the first order Taylor expansion of \( \rho^{m-1} \) at a boundary point.

### 3.4. Improved flatness.

The pressure formulation of the porous medium equation shows that the affine part of the Taylor expansion of solutions at the free boundary defines traveling wave solutions at the level of linear approximations: If \( \rho \) is a solution with smooth pressure \( v = \rho^{m-1} \) and \( \nabla_x v(t_0, x_0) \neq 0 \) at the point \((t_0, x_0)\) of the boundary of the support then

\[ \partial_t v(t_0, x_0) = |\nabla v(t_0, x_0)|^2 \]

and hence

\[ [(t - t_0)\partial_t v + (x - x_0)\nabla v(t_0, x_0)]^+ = \rho^{m-1}_t (t - t_0, x - x_0; V, 1, a) \]

with \( V = |\nabla v(t_0, x_0)| \) and \( a = V^{-1}\nabla v(t_0, x_0) \). We use this on \( \rho_- \), which has smooth pressure by Proposition 4 at the point \( (t_0, x_0) = (0, he_n) \), where \( h \) is the unique non-negative number for which \( he_n \) is in the boundary of the support of \( \rho_-(0, \cdot) \), hence

\[ V = |\nabla \rho^{m-1}_(0, he_n)|, \quad a = V^{-1}\nabla \rho^{m-1}_(0, he_n). \]

Higher derivatives of \( \rho_\pm \) in \((t_1, 0) \times B_{\frac{1}{2}}(0)\) are controlled by Part (ii) of Proposition 4 which is applicable because of (3.5), hence in the positivity set

\[ |t - t_1| \frac{\delta}{2} (t + \rho^{m-1}_\pm (t, x))^\frac{|n|}{2} \left| \partial_t^k \partial_x^j (\rho^{m-1}_\pm - (x_n + t)) \right| \leq C_2 \delta. \]

In particular the remainder term of Taylor’s formula applied in a ball with radius \( r \) centered at \((0, he_n)\) can be bounded in the positivity set of \( \rho_- \) by

\[ |\rho^{m-1}_-(t, x) - \rho^{m-1}_t(t, x; V, 1, a)| \leq c(r/|t_1|)^2 \delta \]

with \( \tau \) from (3.9) provided

\[ |t| \leq 2r \leq |t_1|, \quad |x - he_n| \leq 4r. \]

Using this estimate and (3.9), by Taylor’s formula the graph of \( \rho^{m-1} \) is sandwiched as

\[ \rho_t (t, x - c(r/|t_1|)^2 \delta e_n; V, 1, a) \leq \rho(t, x) \leq \rho_t (t, x + [c(r/|t_1|)^2 + \tau |\delta e_n]; V, 1, a). \]

We define

\[ \tilde{\rho}(t, x) = r^{-\frac{1}{m-1}} \rho(rt, rx) \]

which is sandwiched as

\[ \rho_t (t, x - cr|t_1|^{-2} \delta e_n; V, 1, a) \leq \tilde{\rho}(t, x) \leq \rho_t (t, x + cr|t_1|^{-2} + \tau/t |\delta e_n]; V, 1, a). \]

We choose \( t_1 \) so small that the previous conditions are satisfied, and in addition \( (c + 1)|t_1| < \frac{1}{40} \), next \( r = |t_1|^{\frac{1}{2}} \) and finally \( \delta_0 \leq \mu r^{m+2}/40 \) which ensures \( \tilde{\rho} \) is sandwiched as

\[ \rho_t (t, x - \delta e_n/10; V, 1, a) \leq \tilde{\rho}(t, x) \leq \rho_t (t, x + \delta e_n/10; V, 1, a). \]

for \( |x| \leq 2 \) and \(-2 \leq t \leq 0\).
3.5. Rescaling. We now compare the two approximations we constructed for \( \rho \) near \( (0, 0) \) after the rescaling with parameter \( r \) to estimate \( \lambda \) and \( \nu \) of the statement of the Proposition.

On the one hand, \( \delta \)-flatness in the direction \( e_n \) at \( t = 0 \) implies that

\[
(x_n - \delta/r)_+ \leq \tilde{\rho}^{m-1}(0, x) \leq (x_n + \delta/r)_+
\]

for \( |x| \leq 1 \). On the other hand, we have

\[
\left( V^{-1}a \cdot (x - \frac{1}{10} \delta e_n) \right)_+ \leq \tilde{\rho}^{m-1}(0, x) \leq \left( V^{-1}a \cdot (x - \delta e_n) \right)_+
\]

for \( |x| \leq 2 \). We evaluate the inequalities at \( e_n \) and at \( a \):

\[
1 - \frac{\delta}{r} \leq c(a_n + \delta/10), \quad c(a_n - \delta/10) \leq 1 + \frac{\delta}{r},
\]

\[
a_n - \frac{\delta}{r} \leq c(1 + a_n \delta/10), \quad \frac{1}{\lambda}(1 - \delta a_n \delta/10) \leq a_n + \frac{\delta}{r}.
\]

Thus, if \( \delta_0 \leq r/10 \) - which we can satisfy - then

\[
V^{-1} \leq \frac{1 + \delta}{1 - \frac{\delta}{r}} \leq 1 + 2\frac{\delta}{r}, \quad V^{-1} \geq \frac{1 - \delta}{1 + \frac{\delta}{r}} \geq 1 - 2\frac{\delta}{r},
\]

\[
a_n \geq V^{-1}(1 - \delta/r) - \frac{\delta}{1 - \frac{\delta}{r}} \geq 1 - 4\frac{\delta}{r}, \quad |V|^2 = 1 - V_n^2 \leq 8\frac{\delta}{r}.
\]

Thus \( r/V < 1 \) and the proof is complete, up to a construction of the initial data \( \rho_{\pm,0}(x) \).

3.6. Construction of the comparison functions \( \rho_{\pm,0} \). We define

\[
\rho_{\pm,R}^{m-1}(x) := \rho_{t_0, \pm, 2\delta}(t_1, x) \mp 12\delta(\frac{3}{4} - |x|)_+
\]

and

\[
\rho_{\pm,L}^{m-1}(x) := [\rho_{t_0, \pm, 2\delta}(t_1, x) + \mu(x + t_1 \pm 2\delta)]_+,
\]

and with these notations

\[
\rho_{-,0}(x) := \begin{cases} \min\{\rho_{-,R}(x), \rho_{-,L}(x), \rho(t_1, x)\} & \text{if } |x| \leq 1 \\ \rho_{t_0, -, 2\delta}(t_1, x) & \text{if } |x| > 1 \end{cases}
\]

and

\[
\rho_{+,0}(x) := \begin{cases} \max\{\rho_{+,R}(x), \rho_{+,L}(x), \rho(t_1, x)\} & \text{if } |x| \leq 1 \\ \rho_{t_0, +, 2\delta}(t_1, x) & \text{if } |x| > 1 \end{cases}
\]

With this, (3.11), (3.2), and (3.3) are obvious. But the definition also implies that

\[
(3.12) \quad \rho_{-,0}(x) = \rho(t_1, x) = \rho_{+,0}(x) \quad \text{if } |x| \leq \frac{1}{2}, \quad \text{and } t_1 + x_n \geq 3 \max\{2\delta, \frac{\mu}{C_5 \delta}\},
\]

hence (3.4).

We set \( t_1 \geq -\frac{1}{2} \). For \( \Lambda > \max\{\kappa_3 \delta_0, 4\delta\} \) (which we will ensure later on) we use the first part of Proposition 4 to obtain

\[
|\nabla_x \rho_{\pm,0}^{m-1}(x) - e_n| \leq C_5 \delta(x_n + t_1)^{-1}
\]

for \( |x| \leq \frac{1}{2} \) and \( -\frac{1}{2} \leq t_1 \) and \( x_n + t_1 \geq \Lambda \). We will choose

\[
\Lambda = \frac{4\delta}{\mu}
\]

assuming without loss of generality that \( \mu \leq \min\{\kappa_3^{-1}, \frac{1}{16}\} \) and \( \delta \leq \delta_0 \leq \mu \).
For $|x| \geq \frac{1}{2}$ or $t_1 + x_n < \Lambda$ the same bound follows directly from the definition of $\rho_{\pm,0}$.

Altogether, we then have

$$\left| \nabla_x (\rho_{\pm,0}^{m-1}(x) - (x_n + t_1)) \right| \leq \mu$$

for $x \in \mathcal{P}(\rho_{\pm,0})$ and thus (3.3) holds.

\[ \square \]

4. Proof of Theorem 4

In this section we deduce $C^{1,\alpha}$ regularity claimed in Theorem 4 from the Proposition 1 of the previous section. Full regularity follows from Proposition 4. The iteration argument using the improvement of flatness of Proposition 1 goes back to Caffarelli’s work, see [12] for example.

**Proof.** We first prove the claim:

**Claim 4.1.** There exists $\alpha > 0$ and $\delta > 0$ such that the following holds:

If the solution $\rho$ is $\delta_1$-flat on $Q_1 = (-1,0) \times B_1(0)$ for a $\delta_1 < \delta$ with speed $1$ and $a = e_n$, then

$$\| \nabla_x \rho^{m-1} - e_n \|_{L^\infty((-1/2,0) \times B_{1/2}(0) \cap \mathcal{P}(\rho))} < c\delta.$$

Moreover, $\rho^{m-1} \in C^{1,\alpha}((-1/2,0) \times B_{1/2}(0) \cap \mathcal{P}(\rho))$ in the sense that the derivatives are Hölder continuous up to the boundary.

The claim implies that the free boundary is the graph of a function $h \in C^{1,\alpha}$.

**4.1. Proof of the claim: Setup.** By assumption, $\rho$ is a $\delta$-flat solution with $\delta \leq \delta_1 := \delta_0 t_0 / 2$, with $\delta_0$ and $t_0$ as in Proposition 1. Let us change the origin to a point $(s, y)$ of the free boundary with $-\frac{1}{2} \leq s \leq 0$ and $|y| \leq \frac{1}{2}$. If $\delta \leq \delta_1$ and if $\rho$ is $\delta$-flat then the function

$$\rho_0(t, x) = \frac{1}{2} \rho(t + s, \frac{1}{2} x + y)$$

is $2\delta$-flat, and we can apply Proposition 1 because of our assumption on the smallness of $\delta$.

Thus, there exists $V$ and a unit normal vector $a$ so that the newly rescaled function

$$\rho_1(t, x) = (2/t_0)^{\frac{1}{m-1}} \rho\left( \frac{t_0}{2V^2} t + s, \frac{t_0}{2V} x + y \right)$$

satisfies

$$(t + \langle a, x \rangle - \delta)_+ \leq \rho_1(t, x)^{m-1} \leq (t + \langle a, x \rangle + \delta)_+$$

i.e., $\rho_1(t, x)$ is $\delta$-flat in direction $a$ with velocity $V$, and $2V \delta / t_0$ flat in direction $e_n$ with velocity $1$, hence

$$|a - e_n| + |V - 1| \leq 2\delta / t_0.$$

**4.2. Proof of the claim: Iteration.** We repeat this construction recursively, but now we keep the point $(0,0)$ fixed since it is in the free boundary. Then there exist a sequence of unit vectors $a_j$ and numbers $W_j$ so that

$$\rho_j(t, x) = r^{\frac{1}{m-1}} \rho\left( W_j^{-2} t + s, W_j^{-1} x + y \right)$$

is $2^{1-j}\delta$-flat in direction $a_j$ at every step. More precisely

$$(t + \langle a_j, x \rangle - 2^{1-j} \delta)_+^{\frac{1}{m-1}} \leq \rho_j(t, x) \leq (t + \langle \nu_j, x \rangle + 2^{1-j} \delta)_+^{\frac{1}{m-1}}.$$


Moreover,
\[
\left| \frac{W_{j+1}}{W_j} - 1 \right| = |V_j - 1| \leq 2^{1-j} \delta
\]
\[
|a_{j+1} - a_j| \leq 2^{1-j} \delta.
\]
since \( \rho_j \) is \((\frac{1}{2})^{j-1} \delta\) flat. Summing a geometric series,
\[
|a_j - e_n| + |A_j - 1| \leq c\delta,
\]
and both quantities are Cauchy sequences with limit \( a(s, y) \) and \( W(s, y) \) which are functions of the point considered initially and which satisfy
\[
|W(s, y) - 1| + |a(s, y) - 1| \leq c\delta.
\]
Without loss of generality we may assume that \( W(s, y) = 1 \) at the expense of increasing \( \delta \) by a fixed factor, and that \( a(s, y) = e_n \) for a fixed point \((s, y)\) in the free boundary.

4.3. **Proof of Theorem 1** \( C^{1,\alpha} \) regularity at the free boundary. We claim that this analysis shows that there exists \( C > 0 \) so that
\[
(4.1) \quad (t + \langle x, a(s, y) \rangle) - CR^{\alpha_1})^{\frac{1}{\alpha_1}} \leq \rho(s + \Lambda^2(s, y)t, y + \Lambda x) \leq (t + \langle x, a \rangle + CR^{\alpha_1})^{\frac{1}{\alpha_1}}
\]
if \( |x - y| + |t - s| \leq R/C, \) \( \nu \) depends on the free boundary point \((s, y)\) and
\[
\alpha_1 = \frac{\ln 1/2}{\ln r} > 1.
\]
This is a rather standard counting argument, but we will give the details for the reader’s convenience. Place yourself at one such point \((t, x)\) and count the maximal number of iterations \( N \) so that the initial unit cylinder of the definition of \( \delta\)-flatness for \( \rho \) is shrunk but still contains \((t, x)\). Looking at the scalings of Proposition 1, this means for \( R \) small (so that \( N \) is large) we have roughly (but it will be precise enough)
\[
r^N \sim R.
\]
But the final flatness after \( N \) steps is \( \delta_N = \delta 2^{1-N} \), hence
\[
\delta_N \sim \delta 2^{-\log(R\Lambda^{-2})/\log r} \leq C \delta R^{-\log 2/\log r} = C \delta R^{\log(1/2)/\log r}
\]
and this immediately implies (4.1).

4.4. **Proof of the Theorem 1** \( C^{1,\alpha} \) regularity of the pressure. Equation (4.1) implies one sided differentiability of \( \rho^{m-1} \) at the free boundary, and Hölder continuity of the derivative at the free boundary.

For \((t, x)\) and \((s, y)\) in the positivity set we want to prove
\[
|\nabla \rho^{m-1}(t, x) - \nabla \rho^{m-1}(s, y)| \leq c(t-s) + |x-y|^\alpha.
\]
By the triangle inequality it suffices to consider two cases. If \((t_0, x_0)\) is at the free boundary, and \((t_1, x_1)\) is not we denote by \( d \) be the distance to the free boundary. By (4.1) the rescaled function
\[
\tilde{\rho}(t, x) = (d/2)^{-\frac{m-1}{m-\alpha}} \rho(dt/2, dx/2)
\]
is \(d^\alpha \delta\) flat. By Proposition 3 the higher order derivative derivatives of \( \tilde{\rho}^{m-1} \) are uniformly bounded in a ball of radius 1/4 around \((dt_1/2, dx_1/2)\). This is only compatible with \(d^\alpha \delta\) flatness if the second order derivatives in the ball of radius 1/4 are bounded by a constant times \(d^\alpha \delta\). Then the same is true for first order derivatives, and the derivatives are \(d^\alpha \delta\) close to the corresponding derivatives of the powers of the traveling wave solutions, and hence also to \(D_{t,x} \rho^{m-1}(t_0, x_0)\).
In the second case both \((t, x)\) and \((s, y)\) have the same distance to the free boundary and to another. After scaling this reduces to estimating second derivatives at points of distance 1 to the free boundary in terms of \(\delta\). This is the contents of Proposition 4, which also implies full regularity. □

5. **Von Mises Transform and Intrinsic Geometry**

For the proofs of Proposition 3 and 4 we want to linearize simultaneously the geometry and the porous medium equation. For that purpose we change coordinates with respect to dependent and independent variables simultaneously. After this change we obtain an intrinsic subelliptic degenerate parabolic equation. This section is devoted to this change of coordinates and a discussion of the sub-Riemannian geometry associated to linearization of the traveling wave.

Consider the von-Mises-transform of \(\rho^{m-1}\) denoted by \(w\), assuming that the pressure \(\rho^{m-1}\) is Lipschitz continuous and \(\partial_{x_n}\rho^{m-1}\) is bounded from below by a positive constant on the positivity set. We decompose \(\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}\) and write \(x = (x', x_n)\). We define the bi-Lipschitz map
\[
(t, x) \rightarrow (t, x', \rho^{m-1}(t, x)) = (t, y)
\]
from the closure of the positivity set to \(\{(t, y) : y_n \geq 0\}\). We now introduce the inverse map: \(w(t, y) = x_n\), and we will use \(w(t, y)\) instead of \(\rho(t, x)\) as the basic unknown in our computations. A tedious but standard calculation gives
\[
\partial_t w - y_n \Delta_y' w + y_n^{-\sigma} \partial_{y_n} \left[ y_n^{1+\sigma} \frac{1 + |\nabla_y' w|^2}{1 + \partial_{y_n} w} \right] = 0
\]
with
\[
(5.1) \quad \sigma := \frac{2 - m}{m - 1} > -1.
\]
Here \(\Delta'\) and \(\nabla'\) denote the operators with respect to the first \((n - 1)\) space variables. In this setting, the traveling wave solution described in the introduction becomes
\[
w(t, x) = x_n - (1 + \sigma)t,
\]
and the deviation from the traveling wave solution
\[
u = w - (x_n - (1 + \sigma)t)
\]
satisfies the equation
\[
\partial_s u - y_n \Delta_y' u - y_n^{-\sigma} \partial_{y_n} \left[ y_n^{1+\sigma} \frac{\partial_{y_n} u - |\nabla_y' u|^2}{1 + \partial_{y_n} u} \right] = 0 \text{ on } (s_1, s_2) \times \mathbb{T},
\]
where \((s_1, s_2)\) is a suitable rescaling of the original time interval \((t_1, t_2)\) (see [17]). This equation can be rewritten as a quasilinear equation
\[
\partial_s u - y_n a^{ij}(Du)\partial_{ij} u - (1 + \sigma) b^i \partial_j u = 0
\]
with symmetric coefficients \(a^{ij}\) given by
\[
a^{ij} = a^{ij}(\nabla_y u) = \begin{cases} 
\delta^{ij} & \text{for } i, j < n \\
-\frac{\partial_{y_n} u}{1 + \partial_{y_n} u} & \text{for } i < n, j = n \text{ or } i = n, j < n \\
\frac{1 + |\nabla_y' u|^2}{(1 + \partial_{y_n} u)} & \text{for } i = j = n,
\end{cases}
\]
and

\[ b^j = b^j(\nabla_y u) = \begin{cases} a^{nj} & \text{for } j < n \\ \frac{1}{1 + \partial_n u} & \text{for } j = n. \end{cases} \]

We may also write the equation as a perturbation of the linear equation

(5.4)

\[ \partial_s u - L_\sigma u = f[u] \]

with inhomogeneity

\[ f[u] := -y_n^{-\sigma} \partial_n \left( y_n^{1+\sigma} \frac{|\nabla_y u|^2}{1 + \partial_n u} \right) \]

and spatial linear operator

\[ L_\sigma u := y_n \Delta_y u + (1 + \sigma) \partial_n u. \]

The second order part of \( L_\sigma \) defines a Riemannian metric \( g \) on \( y_n > 0 \),

\[ g(x)(v, w) = x_n^{-1} v \cdot w \]

which in turn defines a nonstandard metric on the closed upper half space which is the Carnot-Carathéodory metric \( d \) defined by the vector fields

\[ x_n^{1/2} e_j \]

(see [17]).

We denote intrinsic balls by \( B^*_R(x) \). They are related to Euclidean balls as follows. Given arbitrary \( r > 0 \), \( y_0 \in \mathbb{H} \) and using the abbreviation

\[ R := r (r + \sqrt{y_0, n}), \]

\[ B_{R/C}(x) \subset B^*_R(x) \subset B_{cR}(x) \]

for some \( c > 1 \).

6. Proof of Proposition [3]

For the proof of this proposition we need a general Gaussian estimate in terms of the intrinsic metric and weighted measure, as already contained in [18]. We provide a new and simpler proof with a standard strategy as follows.

**Lemma 6.1.** Let \( \sigma > -1 \) and \( a^{ij}(t, x) \) measurable, uniformly bounded and coercive,

(6.1)

\[ a^{ij}(t, x) \xi_i \xi_j \geq \nu |\xi|^2 \]

for almost all \( t \) and \( x \). We consider the equation

(6.2)

\[ u_t - x_n^{-\sigma} \partial_t \left( x_n^{1+\sigma} a^{ij} \partial_{ij} u \right) = 0. \]

Then there is a unique Green’s function \( g(t, x, s, y) \) such that the unique solution to the initial value problem for (6.2) is given by

\[ u(t, x) = \int g(t, s, x, y) u_0(y) dy. \]

Moreover, there exist \( c \) and \( \delta > 0 \) so that we have the estimate

\[ |g(t, x, s, y)| \leq c y_n^\sigma (x_n + y_n + |t - s|^{\frac{1}{2}} |t - s|^{\frac{1}{2}})^{-n - \sigma} e^{-\delta \frac{|x - y|^2}{(x_n + y_n + |t - s|)(t - s)}}. \]
Proof. The formal energy identity
\[ \frac{1}{2} \int y_n^s |u(t, x)|^2 dx - \frac{1}{2} \int y_n^s |u(s, x)|^2 dx + \int_s^t \int y_n^{1+\sigma} a^{ij} \partial_i u \partial_j u dx d\tau = 0 \]
can be used with a Galerkin approximation to construct a weak solution for given initial data which satisfies this energy identity. Let \( \phi \) be a bounded Lipschitz function (6.5)
\[ x_n|\nabla \phi|^2 \leq 1. \]
Note that this condition is equivalent to the requirement that \( \eta \) is a Lipschitz function with Lipschitz constant 1 with respect to the special Riemannian metric adapted to the problem. Then, again formally, but with standard justification, we have the weighted energy estimate (6.4)
\[ \int e^{L \phi(x, y)} y_n^s |u(t, x)|^2 dx \leq e^{cL^2(t-s)} \int y_n^s e^{L \phi(x, y)} |u(s, x)|^2 dx. \]
Such estimates are called Davies-Gaffney estimates, see [14]. The next ingredient is the Moser iteration.

Lemma 6.2. Let \( B_{2R}^2(x) \) be the intrinsic ball. We assume that \( u \) is a weak solution on \( Q_{2R} = (-2R)^2, 0 \times B_{2R}(x) \). Then (6.5)
\[ \|u\|_{L^\infty(Q_{2R})} \leq c|Q_{2R}|^{1-\sigma}\|u\|_{L^2(Q_{2R}, x, dy, dx)}. \]
Proof. Let \( k > 1 \). We define the cylinders \( Q_j = Q_{(1+2^{-j} R)} \) and Lipschitz functions \( \eta_j \) with \( \eta_j = 1 \) on \( Q_j \) and \( \eta_j = 0 \) on \( Q \setminus Q_j \),
\[ |\partial_t \eta_j|, x_n^j |D_x \eta_j| \leq 2^j. \]
The starting points for Nash’s inequality (6.7) are the Sobolev inequality
\[ \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq 2\|\nabla u\|_{L^1(\mathbb{R}^n)} \]
and Hardy’s inequality for \( s > -1/p \),
\[ \|x_n^s u\|_{L^p(H)} \leq \frac{1}{s + 1/p}\|x_n^{s+1} \nabla u\|_{L^p(H)}. \]
By an even reflection the Sobolev inequality holds in \( H \),
\[ \|u\|_{L^{\frac{n}{n-1}}(H)} \leq 2\|\nabla u\|_{L^1(H)} \]
and hence, for \( s \geq 0 \), by an application of Hölder’s inequality and Sobolev’s inequality
\[ \|x_n^s u\|_{L^{\frac{n}{n-1}}(H)} = \int_0^\infty (x_n^s \|u(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})})^{\frac{n-1}{n}} x_n^s \|u(\cdot, x_n)\|_{L^{\frac{n}{n-1}}} dx_n \leq \sup_{x_n} (x_n^s \|u(\cdot, x_n)\|_{L^1})^{\frac{n-1}{n}} \|x_n^s \nabla u\|_{L^1} \]
and
\[ t^s \|u(\cdot, t)\|_{L^1} \leq t^s \|\partial_t u\|_{L^1(\{x_n > t\})} \leq \|x_n^s \partial_t u\|_{L^1}, \]
thus for \( s \geq 0 \)
\[ \|x_n^s u\|_{L^{\frac{n}{n-1}}(H)} \leq 2\|x_n \nabla u\|_{L^1(H)} \]
We combine this inequality with the Hardy inequality for \( p = 1 \) and we get for \( 1 \leq p \leq \frac{n}{n-1} \)
\[ \|x_n^{s-\frac{n-1}{p}} u\|_{L^p} \leq c_{n,p} \|x_n^s \nabla u\|_{L^1}. \]
We apply this inequality to \(|u|^p\) and obtain for \( \frac{1}{q} - \frac{1}{n} \leq \frac{1}{p} \leq \frac{1}{q} \) the weighted Hardy-Sobolev inequality.
which in turn implies Nash’s inequality for \( s > -1 \),

\[
\|u\|_{L^{2+2s/(n-2)}(x_n)} \leq c_n\|u\|_{L^2(x_n)}\|\nabla u\|_{L^2(x_n^{1+s})}^2
\]

We formally calculate with \( \eta = \eta_j, \ p = p^j \) and \( R = 1 \)

\[
\frac{d}{dt} \int x_n^\sigma \eta^2 u^p dx = p \int x_n^\sigma \eta^2 u^{p-1} u_t dx + 2 \int x_n^\sigma \eta \eta v^2 dx
\]

\[
= -p \int x_n^{1+\sigma} \partial_t u a^{ij} \partial_j (\eta^2 u^{p-1}) dx + 2 \int x_n^\sigma \eta \eta u^2 dx
\]

\[
= - \int_H \frac{4p}{(p-1)2} x_n^{1+\sigma} \eta^2 a^{ij} \partial_i |u|^{p/2} \partial_j |u|^{p/2} dx
\]

\[
- 4 \int_H x_n^{1+\sigma} (\partial_t u^{p/2}) |a^{ij} \eta (\partial_j \eta)| u^{p/2} dx + 2 \int x_n^\sigma \eta \eta u^p dx
\]

and hence, with \( v = u^{p^j/2} \),

\[
\sup_t \int x_n^\sigma \eta^2 |v(t, x)|^2 dx + \frac{1}{p} \int x_n^{1+\sigma} \eta^2 |\nabla v|^2 dx \leq p^{2j} \|u\|_{L^2(Q_{j-1})}^{2j}.
\]

We combine this with Nash’s inequality to

\[
\|u\|_{L^{p^j}(Q_j)} \leq c p^{2j} 2^{2j} \|u\|_{L^{p^{j-1}}}
\]

and hence

\[
\|u\|_{L^{p^j}(Q_j)} \leq (c p^{2j} 2^{2j})^{1/p^j} |Q|^{\frac{1}{p^j-1} - \frac{1}{p}} \|u\|_{L^{p^{j-1}}(Q_{j-1})}
\]

which becomes

\[
\|u\|_{L^{p^k}(Q_k)} \leq \prod_{j=1}^k \left( c (2p)^{2j} \right)^{1/p^j} |Q|^{\frac{1}{p^j-1} - \frac{1}{p}} \|u\|_{L^2}.
\]

Then

\[
\sum_{j=1}^k \frac{1}{p^j} \leq \Lambda
\]

for some \( \Lambda \) and

\[
P \sum_{j=1}^k \frac{2j}{p^j} \leq P
\]

. Now we let \( j \) tend to \( \infty \) to obtain Moser’s inequality (6.2). \qed

We continue with the proof of the Gaussian estimates of Lemma 6.1. In the next step we combine the consequences of the Moser iteration and the Davies-Gaffney estimate, and we obtain the Gaussian kernel bounds. This argument is general and standard, and we only indicate the steps. First, let \( x_1 \) and \( x_2 \) be two points with distance \( R \). If \( u_0 \in L^2 \) is supported in an intrinsic ball \( B^c_{R/4}(x_1) \) then by the Davies Gaffney estimate

\[
\|u(t)\|_{L^2(B^c_{R/4})} \leq c e^{tL^2 - LR/2} \|u_0\|_{L^2}
\]

for which we optimize with \( L = \frac{R}{4} \) which yields

\[
\|u(t)\|_{L^2(B^c_{R/4})} \leq c e^{-\frac{R^2}{16t}} \|u_0\|_{L^2}.
\]
Now we use the Moser estimate (6.2) to conclude that for \( x \in B^i_{R/4}(x_2) \)
\[
(6.8) \quad |u(t, x)| \leq c|B^i_{R/4}(x_2)|^{-\frac{1}{2}}e^{-\frac{t^2}{16|\sigma|}}\|u_0\|_{L^2(B^i_{R/4}(x_1))}.
\]
By the Riesz representation theorem the map
\[
L^2 \ni u_0 \mapsto u(t, x)
\]
has a kernel
\[
g(t, s, x, y)
\]
which by duality satisfies
\[
\partial_s g + \partial_t a^{ij} \partial_j g = 0.
\]
Let \( v(s, y) = g(t, s, x, y) \). Then estimate (6.8) combined with the Riesz representation theorem reads as
\[
\|v(s)\|_{L^2(B_R(x_1))} \leq |B^i_{R/2}|^{-\frac{1}{2}}e^{-\frac{t^2}{16|\sigma|}}
\]
and a second application of Moser’s estimate implies the Gaussian bounds. \( \square \)

Proof of Proposition 3. The proof consists in tracing the assumptions through the von Mises transform. The conclusion is a consequence of the pointwise Gaussian bound. Let \( w_1 \) and \( w_2 \) be two solutions of the transformed equation. For the difference \( w_2 - w_1 = w \) we have
\[
\partial_t w - x_n \Delta_x w + x_n^{-\sigma} \partial_{x_n} \left[ x_n^{1+\sigma} \left( \frac{1 + |\nabla' w_2|^2}{\partial_{x_n} w_2} - \frac{1 + |\nabla' w_1|^2}{\partial_{x_n} w_1} \right) \right] = 0.
\]
We consider this as a linear equation
\[
\partial_t \tilde{w} - x_n \Delta_x \tilde{w} - x_n^{-\sigma} \partial_{x_n} \left[ x_n^{1+\sigma} \left( \frac{1 + |\nabla' w_2|^2}{\partial_{x_n} w_2} - \frac{1 + |\nabla' w_1|^2}{\partial_{x_n} w_1} \right) \right] = 0
\]
with coefficients
\[
a^{nj} = -\frac{\partial_j (w_2 + w_1)}{\partial_n w_2}, \quad a^{nn} = \frac{1 + |\nabla' w_2|^2}{\partial_n w_2 \partial_n w_1}, \quad a^{ij} = \delta_{ij} \quad \text{if } i < n
\]
satisfying
\[
|a^{ij} - \delta^{ij}| \leq c \delta
\]. For such an equation we have the Gaussian estimate from Lemma 6.1. The representation of the solution yields
\[
|w(t, x)| \leq c|B_{\sqrt{t}}(x)|^{-1} \int_H y_\nu^\gamma e^{-C\frac{|\nabla^2(y, \nu)|}{\nu}}|w_2(0, y) - w_1(0, y)|dy.
\]
The claim of the proposition follows if we prove
\[
|B_{\sqrt{t}}(x)|^{-1} \int_H y_\nu^\gamma e^{-C\frac{|\nabla^2(y, \nu)|}{\nu}}|\tilde{w}(0, y) - w(0, y)|dy
\]
\[
\leq c \left\{ \delta e^{-C\frac{t^2}{(\nu + \rho_0^{-1}(\nu))\nu}} + (\delta/t)^{m-1} + r^{-n}(r^{m-1} + \rho_0(x))^{m-2} \int_{B_{\nu}(x)} |\rho_0(y) - \tilde{\rho}_0(y)|dy \right\}
\]
The assumptions imply \( |\tilde{w} - w| \leq 2\delta \), and we claim that with the intrinsic balls \( B^i_{R/4} \subset H \)
\[
|B_{\sqrt{t}}^i(x)|^{-1} \int_{H \setminus B_{R/4}^i(x)} y_\nu^\gamma e^{-C\frac{|\nabla^2(y, \nu)|}{\nu}}dy \leq ce^{-C\frac{R^2}{\nu}} \leq ce^{-C\frac{(r + \rho_0^{m-1}(\nu))}{\nu}}
\]
which follows by a straight-forward calculation and the observation that $B_r(x) \subset B_R^k(x)$ provided $R \geq r(r + x_n)$. To complete the proof we show that

$$|B^i_{\sqrt{\sigma} \gamma}(x_0)|_\sigma^1 \int_{H \cap B^i_{\sqrt{\sigma} \gamma}(x_0)} y_n^\sigma |w_2 - w_1|dx$$

$$\leq C \left[ r^{-n}(r^{m-1} + \rho_0(y_0))^{m-2} \int_{B_r(y_0)} |\tilde{\rho}(y) - \rho_0(y)|dy + \delta(\delta/t)^{m-1} \right]$$

where $y_0$ and $x_0$ are related by the coordinate change, i.e. $y_0 = (x'_0, \rho^{m-1}(t_1, x_0))$ and $r(r + \rho^{m-1}(x_0)) \sim \sqrt{t}$.

Then

$$\int_{H \cap B^i_{\sqrt{\sigma} \gamma}(x_0) \cap \{x_n > 8r\}} y_n^\sigma |\bar{w} - w|dx \sim \int_{B_r(x_0) \cap (\rho^{m-1} > 4\delta)} \rho^{2-m}(x)|\tilde{\rho}^{m-1}(x) - \rho^{m-1}(x)|dx$$

$$\sim \int_{B_r(y_0) \cap (\rho^{m-1} > 2\delta)} |\tilde{\rho}(x) - \rho(x)|dx.$$ 

This is the whole integral if $x_{0, n} > ct$. In that case $r \sim \sqrt{t}/x_{0, n} \sim \sqrt{t}\rho^{m-1}(y_0)$ and

$$|B^i_{\sqrt{\sigma} \gamma}(x_0)|_\sigma \sim \rho^{(m-1)(2-m+n/2)}(y_0)t^{n/2} \sim r^n \rho^2 \rho^{m-1}(y_0)$$

completes the estimate. If $x_{0, n} \leq 2t$ we may as well enlarge the ball and assume that $x_{0, n} = 0$. Then

$$\int_{H \cap B^i_{\sqrt{\sigma} \gamma}(x_0) \cap \{x_n < 2\delta\}} |w_2(x) - w_1(x)|dx \leq ct^{n-1}\delta^{2+\sigma} = ct^{n-1}\delta^{\frac{1}{m-1}}.$$ 

In this case $|B^i_{\sqrt{\sigma} \gamma}(x_0)|_\sigma \sim t^{n+\sigma}$ which again completes the estimate.

\[ \square \]

7. Proof of Proposition 4

7.1. The interior estimate. We notice that the traveling wave is given by $(x_n + t)^{\frac{1}{m-1}}$, so that the free boundary is the hyperplane with equation $x_n = -t$. By the estimates on $\rho$ we know that the positivity set of $\rho$ inside $Q$ is contained in the set $x + t \geq -\delta$ and contains the set $x + t \geq \delta$. We take a point $Y_0 = (x_0, t_0)$ in this last set, and denote by $L = x_{0, n} + t_0$ which is close to the distance from $Y_0$ to the free boundary $\Gamma$. We assume that $\delta \ll L < 1$.

Take now a cylinder $C_0$ centered on $Y_0$ with measures $2L/100$ in the $t$ direction and $2mL/(m - 1)100$ in the $x$-direction. Since $p_{tw}(T_0) = cL > 0$, by our estimates we conclude that the pressure $p$ is bounded uniformly in $C_0$ from below by .9L and from above by 1.1L. We now introduce the rescaling

$$\hat{\rho}(t', x') = \frac{1}{T} \rho(Lt', x_0 + Lx')$$

and then $\hat{\rho}$ is uniformly bounded by constants 0.9 and 1.1 in $C_0'$ which is a certain cylinder centered at $(0, 0)$ with size independent of $\delta, L$. By the theory of uniformly parabolic equations in divergence form applied to the corresponding density $\hat{\rho}$ (see [20]) we conclude that

$$|\partial_k \partial_\alpha \hat{\rho}| \lesssim \max\{1, t^{-k-|\alpha|/2}\}$$

holds in the interior of the cylinder, for every $k$ and $\alpha$.

Consider now the difference $v = p - p_{tw}$. It satisfies the equation

$$v_t = (m - 1)p \Delta v + \vec{b} \cdot \nabla v, \quad \vec{b} = \nabla p + (m/(m - 1))e_n$$
or in divergence form
\[(7.1) \quad v_t = (m - 1)\nabla \cdot (p \nabla v) + \vec{b}_1 \cdot \nabla v, \quad \vec{b}_1 = \vec{b} + (1 - m)\nabla p.\]

Applying again the theory from [20] and using the estimates obtained in step (i) on the rescaled version of \(\hat{v}\) we get uniform estimates for \(\hat{v}\) in \(1/2C_0\) of the form
\[(7.2) \quad \|\partial^k \partial^\alpha \hat{v}\|_{L^\infty(\mathbb{R}^n)} \leq t^{-k - |\alpha|/2} \|\hat{v}\|_{L^\infty(\mathbb{R}^n)} = t^{-k - |\alpha|/2} C_0.\]

Undoing the rescaling we obtain the interior estimate of Proposition 4.

7.2. Change of coordinates. We use the notation introduced in the Section 5. Since we restrict our attention to balls centered at the boundary we do not need to work with intrinsic balls and define \(B_R(y)^+ = B_R(y) \cap H\). We consider the problem written as in (5.2). The condition
\[(7.3) \quad |\nabla \rho^{m-1} - e_n| \leq \mu/4\]
for \(\mu \leq \frac{1}{2}\) translates into
\[(7.4) \quad \sup_H |\nabla_g u| \leq \mu,\]
and
\[(7.5) \quad \|u\|_{sup} \leq 2(\|\rho^{m-1} - (t + x_n)\|_{sup}\]
and the second part of Proposition 4 follows from the following local regularity result:

**Lemma 7.1.** There exists \(\mu > 0\) so that for any any bounded function
\[u : (0, 1] \times B_2^+ (x) \cap H\]
which satisfies equation (5.2) and
\[|D_x u| \leq \mu\]
satisfies also
\[|\partial_t^k \partial_y^\alpha u(t, y)| \leq c_k,\alpha, m, n t^{-k - |\alpha|} \|u\|_{L^\infty((0, 1] \times B_2^+ (0))}\]
for \(0 < t \leq 1\) and \(y_n \in B_1^+ (0)\).

This immediately implies half of the desired estimate in Part ii). The second half, when \(y_n > t\), follows from the interior estimate of the first part:
\[|\partial_t^k \partial_y^\alpha u(t, y)| \leq c_k,\alpha, m, n t^{-k - |\alpha|/2} (t + y_n)^{-|\alpha|/2} \|u\|_{L^\infty((0, 1] \times B_2^+ (0))}\]
In particular
\[(7.6) \quad \|D_{t,x} u\|_{L^\infty} + \|y_n D_y^2 u\|_{L^\infty} \leq c\mu.\]

7.3. Real analysis lemmata. We collect three different real analysis results. We will encounter equations of the type
\[(7.7) \quad \frac{d}{dx} (xF(x, v)) + sG(x, v) = 0\]
satisfying
\[\left| \frac{\partial F}{\partial v} - 1 \right| + \left| \frac{\partial G}{\partial v} - 1 \right| < \delta.\]
Lemma 7.2. Suppose that $\frac{1}{p+1} < p \leq \infty$ and that $\delta$ is sufficiently small. Then there exists a unique solution $v \in L^p$ which satisfies

$$\|v\|_{L^p} \lesssim \|F(\cdot, 0)\|_{L^p} + \|G(\cdot, 0)\|_{L^p}.$$ 

If

$$xf(x)v' + (s + 1)G(x, v) = 0$$

where

$$|f - 1| < \delta, \quad |\partial_v G(x, v) - 1| < \delta$$

then

$$\|x_n v'\|_{L^p} + \|v\|_{L^p} \lesssim \|G(\cdot, 0)\|_{L^p}.$$ 

Proof. The linear equation

$$(xv)' + sv = (xg)' + f$$

has the general solution - at least for regular $f$ and $g$ -

$$x^{-1-s} \int_0^x y^s(f(y) - sg(y))dy + cx^{-1-s}.$$ 

We claim that

$$\left\|x^{-1-s} \int_0^x y^s f(y)dy\right\|_{L^p} \leq \frac{1}{1 + s + \frac{1}{p}} \|f\|_{L^p},$$

which implies that $c = 0$ for every solution in $L^p$. An easy fixed point argument implies the first estimate and uniqueness, and existence in $L^p$ and uniqueness in the second part. The equation shows that in the second part

$$\|x_n v'\|_{L^p} \leq \|G(\cdot, 0)\|_{L^p}.$$ 

It remains to verify claim (7.9). We prove the more general inequality for $s > 0$

$$\left\|x^{-1-\frac{1}{p}-s} \int_0^x y^{s+\frac{1}{p}}dx\right\|_{L^p} \leq \frac{1}{s + 1} \|f\|_{L^p}.$$ 

It follows by interpolation from the trivial estimates

$$\left|x^{-1-s} \int_0^x y^s f(y)\right| \leq \frac{1}{s + 1} \|f\|_{L^\infty}$$

and

$$y^{1+s} \int_y^\infty x^{-s-2}dx = \frac{1}{s + 1}.$$ 

□

Lemma 7.3. Let $K \subset J \subset \mathbb{R}$ be an open intervals with $|K| \geq |J|/2$. Further let

$$\eta : J \to [0, 1]$$

have compact support, $\eta|_K = 1$ with $\eta^{-1}(t, \infty))$ connected for $0 \leq t \leq 1$. Suppose $\vartheta \geq 0$ and $u \in C^2(J) \cap L^p(J)$. Then there exists a constant $c = c(p) > 0$ such that

$$\|y^\vartheta \eta u'\|^2_{L^p(J)} \leq c\|y^\vartheta u\|_{L^p(J)} \left(|J|^{-2} \|y^\vartheta \eta^2 u\|_{L^p(J)} + \|y^\vartheta (\eta^2 u')'\|_{L^p(J)}\right).$$
Proof. We claim
\begin{equation}
\|u\|_{L^p(\mathbb{R})}^2 \lesssim_p \|u\|_{L^p(\mathbb{R})} \|u''\|_{L^p(\mathbb{R})}
\end{equation}
for \(u \in W^{2,p}\) which is contained in [15]. It can be proven by an integration by parts argument if \(p \geq 2\):
\[
\int (u')^p dx = \int |u'|^{p-2}u' u' dx = - \int (p-1)|u'|^{p-2}u'' u dx \leq (p-1)\|u\|_{L^p(\mathbb{R})}^{p-2} \|u''\|_{L^p(\mathbb{R})} \|u\|_{L^p(\mathbb{R})}
\]
by dividing by \(\|u''\|_{L^p(\mathbb{R})}^{p-2}\). A simple extension argument shows that
\begin{equation}
\|u\|_{L^p(\mathbb{R})}^2 \leq 3(p-1)\|u\|_{L^p(\mathbb{R})} \|u''\|_{L^p(\mathbb{R})}
\end{equation}
for \(p \geq 2\) - if \(1 \leq p < 2\) the constant will be different. For \(\vartheta > 0\), by an application of Fubini, for \(a > 0\) and \(x_0 \geq 0\)
\[
\|x^\vartheta u\|_{L^p([x_0, \infty))}^p - x^\vartheta_0 \|u\|_{L^p([x_0, \infty))}^p = \int_{x_0}^\infty x^{p\vartheta-1} |u'|^p_{L^p([x, \infty))} dx
\leq 3(p-1) \int_{x_0}^\infty x^{p\vartheta-1} |u''|^p_{L^p([x, \infty))} \|u\|_{L^p([x, \infty))}^p dx
\leq 3(p-1) \int_{x_0}^\infty x^{p\vartheta-1} \left( \frac{a}{2} |u''|_{L^p([x, \infty))}^p + \frac{1}{2a} \|u\|_{L^p([x, \infty))}^p \right) dx
= (3(p-1))^{p/2} \left( \frac{a}{2} |x^\vartheta u''|_{L^p([x_0, \infty))} - |x^\vartheta_0 u''|_{L^p([x_0, \infty))} \right)
+ \frac{1}{2a} \left( |x^\vartheta u|_{L^p}^p - |x^\vartheta_0 u|_{L^p([x_0, \infty))}^p \right).
\]
We optimize \(a\) and combine the estimate with (7.11) to arrive at
\[
\|x^\vartheta u\|_{L^p([x_0, \infty))}^2 \leq 3(p-1)\|x^\vartheta u\|_{L^p([x_0, \infty))} \|x^\vartheta u''\|_{L^p([x_0, \infty))}.
\]
By extension to the right we obtain
\begin{equation}
\|x^\vartheta u\|_{L^p(I)}^2 \lesssim_p \|x^\vartheta u\|_{L^p(I)} \left( |I|^{-2} \|x^\vartheta u\|_{L^p(I)} + \|x^\vartheta u''\|_{L^p(I)} \right)
\end{equation}
for any bounded interval \(I\). Again using Fubini
\[
\|y^\vartheta \eta u\|_{L^p} = \int_0^1 \|y^\vartheta u\|_{L^p}\eta > t^{1/p}) dt
\leq c \int_0^1 \|y^\vartheta u\|_{L^p(\eta > t^{1/p})} \left( |J|^{-2} \|y^\vartheta u\|_{L^p(\eta > t^{1/p})} + \|y^\vartheta u''\|_{L^p(\eta > t^{1/p})} \right) dt
\leq \|y^\vartheta u\|_{L^p(J)} \left( |J|^{-2} \|y^\vartheta \eta^ju\|_{L^p(J)} + \|y^\vartheta (\eta^j u)\|_{L^p(J)} \right),
\]
we complete the proof of the lemma. \qed

The third tool is a Calderón-Zygmund type estimate, [17] Proposition 3.23:

Lemma 7.4. Suppose that \(p > (1 + \sigma)^{-1}\) and \(f \in L^p(\mathbb{R} \times H)\).

Then there exists \(v\) satisfying
\[
\|D_{x,y}v\|_{L^p(\mathbb{R} \times H)} + \|x_n D_x^2 v\|_{L^p(\mathbb{R} \times H)} \leq c \|f\|_{L^p(\mathbb{R} \times H)}
\]
and
\[
v_t - x_n \Delta v - \sigma \partial_n v = f.
\]
Moreover $v$ is unique up to the addition of a constant. Similarly, if $F^i \in L^p$ then there is a unique $v$ with $\nabla v \in L^p$ and

$$v_t - x_n \Delta v - \sigma \partial_n v = x_n^{-s} \partial_n x_n^{1+s} F^i.$$  

7.4. Improved estimates. After this preparation we prove Lemma 7.3 for solutions satisfying (6.2) and (6.3). Since we can choose $\mu$ smaller if need be, we can assume that $\mu_1$ and hence $|\nabla_y u|$ are as small as we like. Then also

$$|a^{ij} - \delta^{ij}| + |b^i - \delta^{nj}| \leq C|\nabla_y u| \leq C\mu_1,$$

with a constant depending only on the space dimension, $m$ and

$$|\nabla_{s,y} a^{ij}| + |\nabla_{s,y} b^i| \leq C(1 + |\nabla_y u|) |\nabla_y \nabla_{s,y} u| \leq C(1 + \mu_1) |\nabla_y \nabla_{s,y} u|$$
on $(s_1, s_2) \times \overline{H}$.

We define a cutoff function

$$\eta(s, y) = s \prod_{j=1}^n \eta_0(y_j)$$

where

$$\eta_0 \in C_0^\infty(-2, 2), \quad \eta_0 \leq 1, \quad \eta_0 = 1 \text{ on } (-1, 1) \quad (\eta_0)^{-1}(\{(\sigma, \infty)\}) \text{ is connected for } \sigma \leq 1.$$  

A direct calculation shows that

$$w := \eta^2 u$$

satisfies the equation

$$(7.13) \quad \partial_s w - y_n \Delta_y w - (1 + \sigma) \partial_{y_n} w = F[u, w, \eta]$$
on $L^p \times \overline{H}$ with zero initial value, where

$$F[u, w, \eta] := y_n (a^{ij} - \delta^{ij}) \partial_{ij} w + (1 + \sigma) (b^i - \delta^{nj}) \partial_j w$$

$$+ u (\partial_s(\eta^2) - y_n a^{ij} \partial_{ij}(\eta^2) - (1 + \sigma) b^i \partial_j(\eta^2))$$

$$- y_n (a^{ij} + a^{ji}) \partial_i(\eta^2) \partial_j u.$$  

We fix $p = 2 \max\{(1 + \sigma)^{-1}, n + 1\}$, apply Lemma 7.4 to the function $w$ in (7.13) and obtain

$$\|\nabla_{s,y} w\|_{L^p(\mathbb{R} \times \overline{H})} + \|y_n D_y^2 w\|_{L^p(\mathbb{R} \times \overline{H})} \lesssim_{n, \sigma} \|F[u, w, \eta]\|_{L^p(\mathbb{R} \times \overline{H})}$$

$$\lesssim_{n, \sigma, \mu} \left(\|\nabla_{s,y} w\|_{L^p(\mathbb{R} \times \overline{H})} + \|y_n D_y^2 w\|_{L^p(\mathbb{R} \times \overline{H})}\right)$$

$$+ \|\partial_s \eta^2\| + \|y_n D_y^2 \eta^2\| + \|D_y \eta^2\|_{L^\infty(\mathbb{R} \times \overline{H})}\|u\|_{L^\infty}$$

$$+ \|y_n D_y(\eta^2) D_y u\|_{L^p(\mathbb{R} \times \overline{H})}.$$  

By Lemma 7.3

$$(7.14) \quad \|y_n D_y(\eta^2) D_y u\|_{L^p(\mathbb{R} \times \overline{H})} \lesssim \left(\|u\|_{L^\infty} + \|y_n D_y^2(\eta^2 u)\|_{L^p(\mathbb{R} \times \overline{H})}\right)\|u\|_{L^\infty}$$

hence

$$(7.15) \quad \|y_n D_y(\eta^2) D_y u\|_{L^p(\mathbb{R} \times \overline{H})} \lesssim \varepsilon \|y_n D_y^2 u\|_{L^p(\mathbb{R} \times \overline{H})} + C(\varepsilon)\|u\|_{L^\infty}. $$

The first term and the third term on the right hand side can be controlled by the left hand side if we choose $\varepsilon$ and $\mu$ sufficiently small, $\mu \varepsilon \leq \mu_0$ with $\mu_0$ depending only on $n$ and $\sigma$, which is a function of $m$. The first factor of the second term is bounded and gives the desired supremum norm of $u$. Altogether

$$(7.16) \quad \|\nabla_{s,y} u\|_{L^p((1,2] \times B_1(0) \cap H)} + \|y_n D_y^2 u\|_{L^p((1,2] \times B_1(0) \cap H)} \leq c\|u\|_{L^\infty((0,2) \times B_2(0) \cap H)}.$$
7.5. Higher order derivatives. We deal inductively with higher order derivatives. We assume that \( k \geq 0 \) and
\begin{equation}
\|D_{t,y}^{k+2}u\|_{L^p((0,2) \times B_2(0))} + \|y_nD_{x}^2D_{t,y}^{k+1}u\|_{L^p((0,2) \times B_2(0))} \leq \delta
\end{equation}
and we claim that then
\begin{equation}
\|D_{t,y}^{k+2}u\|_{L^p((0,2) \times B_2(0))} + \|y_nD_{x}^2D_{t,y}^{k+1}u\|_{L^p((0,2) \times B_2(0))} \lesssim \delta.
\end{equation}
Let \( \alpha \) be a multi-index of length \( k \) and \( v = \eta^2 \partial^\alpha u \). It satisfies
\begin{equation}
\partial_s w - y_n \Delta_y w - (1 + \sigma) \partial_{y_n} w = F_\alpha[u, w, \eta]
\end{equation}
where
\begin{align*}
F_\alpha[u, w, \eta] &:= y_n(a^{ij} - \delta^{ij}) \partial_{ij} w + (1 + \sigma)(b^j - \delta^{ij}) \partial_j w \\
&+ \partial^\alpha u \left( \partial_s (\eta^2) - y_n a^{ij} \partial_{ij}(\eta^2) - (1 + \sigma) b^j \partial_j(\eta^2) \right) \\
&- y_n(a^{ij} + a^{ji}) \partial_i(\eta^2) \partial_j \partial^\alpha u \\
&+ G_\alpha
\end{align*}
where \( G_\alpha \) contains all the terms with at least two factors with at least two derivatives.
We begin with the considerations for \( |\alpha| = 1, 1 \leq i \leq n - 1 \). Then
\begin{align*}
\|G_\alpha\|_{L^p} &\lesssim \|y_n\eta^2 D^2 u D\partial_i u\|_{L^p} + \|y_n\eta D\eta Du D\partial_i u\|_{L^p} \\
&\lesssim C \|D^2 \eta^2 u\|_{L^p} \|x_n D^2 u\|_{L^\infty} \\
&\lesssim C \|x_n D^2 u\|_{L^p}.
\end{align*}
The very same argument works for \( w = \eta^2 \partial_i u \). It is easy to make the argument rigorous by using finite differences.
Let \( D' = D'_{t,x} \) denote all derivatives besides the one in direction \( e_n \). Then we have proven
\begin{equation}
\|D_{t,x} D' u\|_{L^p([0,1] \times B_1(0))} + \|x_n D^2 D' u\|_{L^p([0,1] \times B_1(0))} \leq C \mu.
\end{equation}
To control the vertical derivative we recall that
\begin{equation}
\partial_n \left[ x_n \frac{\partial_n u + |\nabla u|^2}{1 + \partial_n u} \right] + s \left[ x_n \frac{\partial_n u + |\nabla u|^2}{1 + \partial_n u} \right] = u_t - x_n \Delta' u
\end{equation}
hence, with \( v = \partial^2_n u \)
\begin{equation}
\partial_n \left[ x_n a^{nn} v \right] + (s + 1)a^{nn} v = \partial^2_n u - \Delta' u - x_n \Delta' u - \partial_n \left[ x_n a^{nn} \partial^2_n u \right] - sa^{nn} \partial^2_n u.
\end{equation}
By Lemma 7.2
\begin{equation}
\|Dv\|_{L^p} \lesssim \|DD' u\|_{L^p} + \|x_n D^2 u\|_{L^p}.
\end{equation}
Now we rewrite the derivative term on the right hand side
\begin{equation}
\partial_n \left[ x_n a^{nn} \partial^2_n u \right] = a^{nn} \partial^2_n u + x_n D^2 u DD' u + x_n a^{nn} \partial^2_n v,
\end{equation}
and use the second estimate to conclude
\begin{equation}
\|x_n D^2 u\|_{L^p} \leq C \|DD' u\|_{L^p} + \|x_n D^2 u\|_{L^p}.
\end{equation}
We may choose \( p \) as large and hence
\begin{equation}
||u||_{C^{1,\alpha}} + \|x_n D^2 u\|_{\sup} \leq \mu.
\end{equation}
We iterate the argument. For second tangential derivatives we have to bound terms like
\begin{equation}
\|x_n D^2 u D^2 u\|_{L^p} \leq \|x_n D^2 u\|_{L^\infty} \|D^2 u\|_{L^p} \leq C \mu \|x_n D^3 u\|_{L^p}
\end{equation}
with at least one tangential derivative (or difference quotient for the rigorous proof) on the RHS. Next we differentiate (7.19) twice in the vertical direction. The details are similar but simpler than before.

8. Estimates for global compactly supported solutions

The proof of Theorem 2 follows from the preceding analysis but it is not immediate because we want to establish $C^\infty$ regularity after a time $T(\rho_0)$ that can be estimated in terms of simple information on the initial data. This is a rather precise result that needs a careful qualitative analysis. Such an analysis is interesting in itself. It occupies this section as follows. First, we review the Barenblatt solutions, and the qualitative result on convergence of any nonnegative, integrable solution $\rho(t,x)$ towards the Barenblatt profile with the same mass. Then we start the quantitative analysis of sizes of solutions and location of the free boundaries. These solutions do resemble the Barenblatt solution but for constant factors that must be controlled. The theory of entropy and entropy dissipation allows us to transform this resemblance into convergence with rate as time goes to infinity in a very precise way. We use these results and the flatness criterion of Theorem 1 to obtain $C^\infty$ regularity for large times. We recall the definition of $\lambda$ in (1.13).

8.1. The Barenblatt solution and plain asymptotic convergence. The Barenblatt solution is given by the formula

$$B(t,x;M) = t^{-n\lambda}F(\sqrt{\lambda}xt^{-\lambda}), \quad F(\xi) = \left(A^2 - \frac{|\xi|^2}{2}\right)^{\frac{1}{m-1}}$$

where $\lambda$ is the above-mentioned constant and $A > 0$ is a free constant that can be easily calculated in terms of the mass $M$ of the solution,

$$A = aM^{-\lambda},$$

and $a = a(m,n)$ is given by

$$a = \left(\int_{B_1(0)}(1 - \frac{1}{2}|\xi|^2)^{\frac{1}{m-1}}d\xi\right)^{\frac{1}{n\lambda}}$$

see the details in [24], Section 2.1. Recall that writing the equation as $\rho_t = k\Delta \rho^m$ with a constant $k = (m-1)/m$ changes the coefficient in the expression of $F$ written in that reference (in fact, it simplifies it).

The Barenblatt solutions play an important role in describing general global nonnegative solutions for large times. This is reflected in the result on asymptotic convergence of any global solution $\rho(t,x)$ of the PME with nonnegative, integrable initial data towards the Barenblatt solution with same mass $M$, cf. [24], Theorem 18.1. The uniform convergence result, formula (18.7) of that reference, says that

$$\lim_{t \to \infty} t^{n\lambda} \sup_x |\rho(t,x) - B(t,x;M)| = 0,$$

with $\lambda$ as defined above and $M = \int_{\mathbb{R}^n} \rho_0(x) dx > 0$. It is known that this initial mass is conserved in time.

It is convenient to reformulate the result as uniform convergence in space-time around the time $t = 1$ for some rescaled solutions. Indeed, if we define for $k > 1$ the family of rescalings

$$\rho_k(t,x) = k^{n\lambda}\rho(kt,k^\lambda x),$$

see the details in [24], Section 2.1. Recall that writing the equation as $\rho_t = k\Delta \rho^m$ with a constant $k = (m-1)/m$ changes the coefficient in the expression of $F$ written in that reference (in fact, it simplifies it).
then the \( \rho_k \) are again solutions of the PME with the same mass \( M \). The previous convergence result can be equivalently stated as the uniform convergence

\[
\lim_{k \to \infty} |\rho_k(t,x) - B(t,x,M)| = 0
\]

in every cylinder of the form \( \hat{Q} = (t_1,t_2) \times \mathbb{R}^n \) with \( 0 < t_1 < t_2 \). It is also proved that for compactly supported data the free boundary \( \Gamma_k \) of \( \rho_k \) converges uniformly to the free boundary \( \Gamma(B) \) of \( B \), which is an expanding ball with radius

\[
R(t) = R_B(M^{(m-1)}t)^{\lambda}
\]

with constant \( R_B = a(2/\lambda)^{1/2} \), a function of \( m \) and \( n \).

8.2. Quantitative question. Reduction and first bounds. We need to prove a quantitative version of the error that is committed in such approximation for a class of initial data and for \( t \) large enough. This is better done after some reduction of the problem based on the scale invariance of the equation. Let \( M = \int_{\mathbb{R}^n} \rho_0(x) \, dx > 0 \) be the mass of the initial data, that is conserved in time, and let \( \rho_0 \) be supported in the ball of radius \( R \). Then, by defining

\[
\tilde{\rho}(t,x) = M^{-1} \rho(M^{1-m} R^2 t, Rx)
\]

we get yet a solution of the same PME, but now it has mass 1 and \( \tilde{\rho}(0,x) \) is supported in \( B_1(0) \). Thanks to this transformation we may assume without loss of generality that \( \rho_0 \) has mass 1, and is supported in a ball of radius 1. Also, by space translation we may also assume that \( \rho_0 \) has center of mass \( x = 0 \). Let us call this class of solutions \( C \).

Let us start by obtaining uniform estimates for the whole class \( C \), and let us see how these estimates look like the values for \( B(t,x;1) \), at least when \( t \) is large enough.

8.2.1. Sup estimates for the solutions. The so-called \( L^1-L^\infty \) smoothing effect with best constant says that

\[
\sup_x \rho(t,x) \leq B(t,0;1) = A_{\frac{2}{m-1}} t^{-n\lambda},
\]

cf. [23]. This holds for every \( t > 0 \).

8.2.2. Bounds from below. The next step consists in deriving lower bounds on the solution \( \rho(t,x) \). We use the results by Aronson and Caffarelli in [4], according to which there exists a constant \( C > 0 \) depending only on \( n \) and \( m \) such that every nonnegative global solution satisfies the inequality (rather, family of inequalities)

\[
\int_{B_r(x_0)} \rho_0(x) \, dx \leq C \left( r^{\frac{1}{m-1}} t^{-\frac{m}{m-1}} + t^{\frac{n}{m-1}} \rho_{\frac{\partial}{\partial t}}(x_0,t) \right)
\]

for every \( x_0 \in \mathbb{R}^n \), \( r > 0 \), and \( t > 0 \). Therefore, if \( \rho_0 \) belongs to the class \( C \) and if \( r > |x_0|+1 \), the left-hand side of the formula is just 1, so that for

\[
t \geq (C/2)^{m-1} r^{1/\lambda},
\]

it follows from (8.8) that \( 1/2 \leq C t^{n/2} \rho^{1/2}(t,x_0) \), or

\[
\rho(t,x_0) \geq C t^{-n\lambda}
\]

a size to be compared with (8.7) and with the Barenblatt solution. Since the time condition can be written as

\[
|x_0| + 1 \leq C_2 t^{\lambda},
\]
these estimates show that \( \rho(t, \cdot) \) is larger than a Barenblatt solution of small but comparable mass at the same time, and this holds for all \( t \geq t_1 \) with fixed \( t_1 > 0 \). Such estimate is not precise enough in the constants, but it gives the correct dependence in time, and it is uniform for all the class of data we consider.

8.2.3. Support estimates from below. This time they will be sharp. We define \( R(t) \) as the smallest real number such that \( \rho(t, \cdot) \) is supported in \( B_{R(t)} \). By known theory this radius is monotonically increasing in time. Again, by a result of the last author based on symmetrization the support is at least as large in measure as the one of the Barenblatt solution and hence

\[
R(t) \geq R_B t^\lambda,
\]

where \( R_B = R_B(m, n) \) is the radius of the unit Barenblatt solution at \( t = 1 \).

8.2.4. Upper bounds on \( R(t) \). They are more difficult and not so accurate. Bénilan, Crandall and Pierre have introduced in [7] the weighted norm

\[
\| \mu \|_{r,m} = \sup_{R \geq r} R^{-\frac{2}{m-1} - n} \mu(B_R(0))
\]

for given \( r \geq 1 \), and the corresponding end time \( T(\rho_0) = c_1(n, m)/\| \rho_0 \|_{r,m}^{m-1} \). Then their estimate (1.7) asserts that for all times \( 0 < t < T(\rho_0) \) we have the very explicit upper estimate:

\[
t^{n\lambda} \rho(x, t) \leq c_2(n, m) \rho_0^{2\lambda} \| \rho_0 \|_{r,m}^{2\lambda}
\]

if \( |x| \leq r \). As explained in [11], Lemma 1.2, the restriction \( r \geq 1 \) is unimportant since it can be eliminated by rescaling, but in letting \( r \to 0 \) we have to be careful with the possible divergence of the quotient in the right-hand side of formula (8.12). The way to make this term finite is to shift the origin of coordinates to a point \( x_0 \) away from the support of \( \rho_0 \), i.e., \( |x_0| \geq 1 \). Then we use the shifted norm \( \| \mu \|_{x_0,r,m} \) and get the estimate

\[
t^{n\lambda} \rho(x, t) \leq c_3(n, m) \rho_0^{2\lambda} \| \rho_0 \|_{x_0,r,m}^{2\lambda}
\]

if \( 0 < t < T(\rho_0) = c_1/\| \rho_0 \|_{x_0, r,m}^{m-1} \) and \( |x - x_0| \leq r, r > 0 \). In particular, letting \( r \to 0 \) for fixed \( t \) we get the result: if \( |x_0| > 1 \) then \( \rho(t, x_0) = 0 \) if

\[
t \leq c_3(n, m)(|x_0| - 1)^{n(m-1)+2}.
\]

In this way we get the upper bound

\[
R(t) \leq 1 + (t/c_3)^\lambda.
\]

It is an important feature of this bound that the right-hand side amounts for large \( t \) (at least \( t > 2 \)) to at most a fixed constant \( L \) times the lower bound, which uses the Barenblatt radius.

8.2.5. Roundness of level sets via moving planes. Using comparison and moving plane techniques, Caffarelli et al. [11] have shown that \( B_{R(t)-2} \) is contained in the support and the following monotonicity holds; if

\[
1 \leq \frac{|y|^2 - |x|^2}{2|y - x|} \quad \text{then} \quad \rho(y, t) \leq \rho(x, t).
\]

In particular, if \( r(t) > 1 \) and \( |x_0| > r \) then the level sets

\[
S = \{ x : \rho(t, x) = \rho(t, x_0) \}
\]
are graphs over the sphere, i.e., if \( \theta(x) \) are spherical angle coordinates in \( \mathbb{R}^n \), then there exists a unique function \( h \) such that
\[
S = \{ x : |x| = h(\theta(x)) \}
\]
and we also have: \( \max h - \min h \leq 2 \). Write \( h(x) = h(\theta(x)) \). Then for \( x, y \in S \)
\[
|h(y)|^2 - |h(x)|^2 \leq 2|x - y|
\]
and the ratio between the maximum and minimum is bounded uniformly. Interior estimates
\[
Q
\]
that
\[
\max \left| \frac{\nabla v}{v} \right| \leq \frac{h(x)}{h(x) - 2}.
\]
In particular, the level sets become rounder with time, both in the supremum norm, and in
the Lipschitz norm, to finally look like balls as \( t \to \infty \).

8.2.6. The gradient bound for the pressure. The upper and lower bounds for the pressure,
plus the monotonicity imply a uniform gradient bound for the pressure. We follow the proof
by Caffarelli et al. [11], Theorem 1. Let \( t_1 \) be a time at which the solutions of \( C \) have
expanded to cover twice the unit ball, a time that can be uniformly estimated according to
the preceding paragraphs. Then we have

**Lemma 8.1.** If \( \rho \in C \) and \( t \geq kt_1 \), there is a uniform constant \( C \) such that
\[
|\nabla \rho^{m-1}(t, x)| \leq C t^{-(1-\lambda)}.
\]
The constant \( k \) is also uniform, i.e. it depends only on \( m \) and \( n \).

**Proof.** (i) To see this we take pressure \( v := \rho^{m-1} \) and define
\[
v_\varepsilon(t, x) = (1 + \varepsilon)^{-\varepsilon} v((1 + \varepsilon)t + t_0, (1 + \varepsilon)x)
\]
with \( t_0 > t_1 \) conveniently chosen. We show that this new variable satisfies
\[
v_\varepsilon(0, x) \leq v_{\varepsilon=0}(0, x) = v(t_0, x)
\]
if we put \( t_0 = kt_1 \) with a constant \( k \geq 2 \) that is estimated below. This comparison can
be done as a variant of the proof of [11], Theorem 1. We add a sketch of the comparison
argument with the novelties that allow to obtain uniform constants. We write
\[
v_\varepsilon(0, x) - v(t_0, x) = - \frac{\varepsilon}{1+\varepsilon} v(t_0, (1 + \varepsilon)x) + v(t_0, (1 + \varepsilon)x) - v(t_0, x).
\]
The first term in the r.h.s. is negative so we only have to consider the last one. Now for
\( |x| > 1 \) (i.e., outside of the initial support), the monotonicity condition proved in [11] implies
that \( v(t_0, (1 + \varepsilon)x) \leq v(t_0, x) \).

We have to examine carefully what happens for \( |x| \leq 1 \). By the estimates of previous
paragraphs we know that in the cylinder \( Q_1 = (kt_1/2, kt_1) \times B_{R_1}(0) \), \( R_1 = C(kt_1)^\lambda \) we have
\[
0 < c_1(k) < v(t, x) \leq c_2(k)
\]
and the ratio between the maximum and minimum is bounded uniformly. Interior estimates
for uniformly parabolic equations with smooth coefficients imply then that \( |\nabla v| \leq K_1c_2/R_1 \)
with uniform \( K_1 \). In this way we get for \( |x| \leq 1 \)
\[
v(t_0, (1 + \varepsilon)x) - v(t_0, x) \leq |\varepsilon x| |\nabla v| \leq K_1 \varepsilon \frac{||v||_{L^\infty}}{R_1}.
\]
This means that for \( k \) not so small \( v(t_0, (1 + \varepsilon)x) - v(t_0, x) \leq \varepsilon c ||v(t_0, \cdot)||_{L^\infty} \) with a very small \( c \). We conclude that \( v_\varepsilon(0, x) - v(t_0, x) \leq 0 \) in \( \mathbb{R}^n \).
(ii) Once \( v_\varepsilon(0,x) \leq v_\varepsilon=0(0,x) \), by the Maximum Principle this is true for all \( t > 0 \), \( v_\varepsilon(t,x) \leq v_\varepsilon=0(t,x) \). Differentiation with respect to \( \varepsilon \) at \( \varepsilon = 0 \) gives

\[
v(t + t_0, x) - tv_t(t + t_0, x) - x \cdot \nabla v(t + t_0, x) \geq 0
\]

at least in the distribution sense, and in fact almost everywhere by known regularity theory. Since we also have [26]

\[
v_t = |\nabla v|^2 + (m - 1)\Delta v \geq |\nabla v|^2 - \frac{n}{n + 2(m - 1)} \frac{v}{t},
\]

we obtain the a.e. inequality

\[
|\nabla v|^2 \leq \left( \frac{n\lambda}{t} + \frac{1}{t - t_0} \right) v + \frac{|x|}{t - t_0} |\nabla v|.
\]

Compare with formula (2.5) of [11] (page 381) which is not so precise since uniformity was not under study. The gradient bound of the Lemma is an immediate consequence of this formula and the uniform bounds \( v = O(t^{-n(m-1)\lambda}) \) and \( R(t) = O(t^\lambda) \). \( \square \)

8.3. Continuous rescaling and entropy. Pursuing again the idea of making asymptotic calculations in renormalized settings where the solution does not tend to zero but evolves into some nontrivial profile, and copying from the sizes of the Barenblatt solution we introduce the continuous scaling [26]

\[
\tilde{\rho}(s, y) = e^{sn} \rho(e^{s/\lambda}, \lambda^{-1/2}e^s y)
\]

with inverse

\[
\rho(t, x) = t^{-n\lambda} \tilde{\rho}(\lambda \log(t), \lambda^{1/2} x t^{\lambda}), \quad s = \lambda \log(t).
\]

It is easy to see that \( \tilde{\rho} \) satisfies the equation

\[
\partial_s \tilde{\rho} = \nabla_y \cdot \left( \tilde{\rho} \nabla_y (\tilde{\rho}^{m-1} + y \tilde{\rho}) \right) = \nabla_y \cdot \left( \tilde{\rho} \nabla_y \left[ \tilde{\rho}^{m-1} + \frac{1}{2} |y|^2 \right] \right).
\]

Notice that the Barenblatt solutions transform into the following family of stationary solutions of the right-side

\[
\tilde{F}^{m-1}(y) = (A^2 - \frac{1}{2} |y|^2)_+.
\]

As a consequence of our previous analysis we know that \( \tilde{\rho} \) satisfies a number of estimates:

1. It is bounded from above \( \tilde{\rho} \leq A^{2m-1}; \)
2. For \( s \geq s_1 \) the support of \( \tilde{\rho}(s, \cdot) \) contains a ball of radius \( C_2 \) and is contained in the ball of radius \( LC_2 \), both centered at \( 0; \)
3. The solution is bounded below in the form
   \[
   \tilde{\rho}(s, y) \geq C_3 \quad \text{for} \quad |y| \leq C_2/2, \ s \geq s_1;
   \]
4. Gradient bound:
   \[
   |\nabla_y \tilde{\rho}^{m-1}| \leq C_4, \quad s \geq s_1.
   \]

All the constants are uniform for the class \( \mathcal{C} \), they depend on \( n \) and \( m \).
8.3.1. Properties of the entropy. The entropy is defined by

\begin{equation}
H(\tilde{\rho}) = \int_{\mathbb{R}^n} \left( \frac{\tilde{\rho}^m}{m} + \frac{1}{2} |y|^2 \tilde{\rho} \right) dy
\end{equation}

and the entropy dissipation by

\begin{equation}
I(\tilde{\rho}) = \int |y + \nabla (\tilde{\rho}^{m-1})|^2 \tilde{\rho} dy.
\end{equation}

They are functions of the new time \( s = \lambda \log(t) \). They satisfy

\[
\frac{d}{ds} H(\tilde{\rho}) = -I(\tilde{\rho}), \quad \frac{d}{ds} I(\tilde{\rho}) \leq -2I(\tilde{\rho}),
\]

see Carrillo and Toscani [13] where the coefficients are a bit different without affecting the result. Both \( H \) and \( I \) are decreasing functions, and we also know that \( H(0) \leq c(n, m) \). It is also shown that \( H(\tilde{\rho}(s)) \) is bounded below by the entropy of the stationary state \( H(\bar{F}) \) that has the same mass, here set to 1. More detailed asymptotic information is obtained by integration

\[
H(\tilde{\rho}(s)) + \int_{s_0}^s I(\tilde{\rho}(\tau)) d\tau = H(\tilde{\rho}(s_0)), \quad I(\tilde{\rho}(s)) \leq I(\tilde{\rho}(s_0)) e^{-2(s-s_0)}.
\]

It follows that

\[
I(\tilde{\rho}(s)) \leq c(n, m) e^{-2s}, \quad H(\tilde{\rho}(s)) - H(F) \leq c(n, m) e^{-2s},
\]

since we already know that \( \tilde{\rho}(s) \) converges to the Barenblatt profile \( F = \bar{B} \) of mass 1.

8.3.2. Application to our problem. We take \( 0 < \varepsilon \leq \varepsilon_{n,m} \) and work for any large time in the convex round set

\[
U_\varepsilon(s) = \{ x : \tilde{\rho}^{n-1}(s, y) > \varepsilon \},
\]

which is uniformly bounded. By the definition of the entropy dissipation and our decay inequality for it we have

\[
\int |\nabla \tilde{\rho}^{n-1} + y|^2 dx \leq c \varepsilon^{-1} e^{-2s}.
\]

By the Poincaré’s inequality there exists a constant \( c_0(s, \varepsilon) \) such that

\begin{equation}
\| \tilde{\rho}^{n-1} + \frac{1}{2} |y|^2 - c_0(s, \varepsilon) \|_{H^1(U_\varepsilon)} \leq c \varepsilon^{-1/2} e^{-s}.
\end{equation}

We interpolate with the Lipschitz bound \( |\nabla \tilde{\rho}^{n-1}| \leq C_3 \) to get

\[
\| \tilde{\rho}^{n-1} - (c_0^2 - \frac{1}{2} |y|^2) \|_{L^\infty(U_\varepsilon)} \leq c \varepsilon^{-\frac{1}{2}} e^{-\frac{s}{2}},
\]

if \( n > 2 \). Thus, on the set \( \{ u > \varepsilon \} \) we have a very precise estimate

\begin{equation}
c_0^2 - c \varepsilon^{-\frac{1}{2}} e^{-\frac{s}{4}} - \frac{1}{2} |y|^2 \leq \tilde{\rho}^{n-1}(s, y) \leq c_0 + c \varepsilon^{-\frac{1}{2}} e^{-\frac{s}{4}} - \frac{1}{2} |y|^2.
\end{equation}

This means that for \( t \) large \( \tilde{\rho} \) looks like a Barenblatt profile on the set \( U_\varepsilon = \{ \tilde{\rho} > \varepsilon \} \), that in turn must look like a ball \( B_{R_0(t)} \). The calculations are a bit different in lower dimensions, we get

\[
\| \tilde{\rho}^{n-1} - (c_0^2 - \frac{1}{2} |y|^2) \|_{L^\infty(U_\varepsilon)} \leq c \varepsilon^{-\frac{1}{2}} e^{-s}
\]

if \( n = 1 \), while for \( n = 2 \) the expression is

\[
\| \tilde{\rho}^{n-1} - (c_0 - \frac{1}{2} |y|^2) \|_{L^\infty(U_\varepsilon)} \leq c \varepsilon^{-\frac{1}{2}} e^{-s} (s + |\ln \varepsilon|).
\]
8.3.3. Improved upper bounds on \( c_0 \). We already know that at the maximum value \( \tilde{\rho} \) must be bounded above by the Barenblatt profile \( \tilde{F} \), hence from (8.23) we have the upper bound (for \( n \geq 3 \))
\[
 c_0^2 - c e^{-\frac{1}{2} t} e^{-\frac{2}{n} s} \leq A^2.
\]
This allows to improve the upper bound for the solution to the form
\[
 \tilde{\rho}^{m-1}(s, y) \leq \max \left\{ A^2 + 2 c e^{-\frac{1}{2} t} e^{-\frac{2}{n} s} - \frac{1}{2} |y|^2, \varepsilon^{m-1} \right\},
\]
for \( n \geq 3 \).
The upper bound immediately implies that for \( s \) large \( U_{\varepsilon} \subset B_r(0) \) with
\[
 r = \sqrt{2 c_0} + c e^{-\frac{1}{2} t} e^{-\frac{2}{n} s}
\]
if \( n > 2 \), and
\[
 r = \sqrt{2 c_0} + c e^{-\frac{1}{2} t} e^{-s}
\]
if \( n = 1 \) with a logarithmic correction if \( n = 2 \). Recall now the upper approximation for \( c_0 \) and we get an approximation of \( U_{\varepsilon} \) to the Barenblatt radius.

We point out a difficulty in taking the limit in the upper bounds of the sets \( R \) as \( \varepsilon \to 0 \), since the support may have a thin tail where \( \rho \) is smaller than the error that we have calculated. We already have a uniform upper bound for the support in a possibly larger ball \( B_{\tilde{R}_1}(0) \). We still have to prove below that such an external region \( B_{\tilde{R}_1} \setminus U_{\varepsilon}(s) \) (the ‘tail’) is small, but in any case we know that \( \tilde{\rho} \leq \varepsilon \) there.

8.3.4. An upper bound of the radius. We turn to an upper estimate of the support, for which we have to bound the possible ‘tail’ where \( \tilde{\rho} \) is small. We do this by a comparison argument with a Barenblatt solution outside the Barenblatt radius for the Barenblatt solution with same center and mass. We recall that (see (8.23))
\[
 \text{supp } \rho(t, \cdot) \subset B_{c t^{\lambda}}(0)
\]
for \( t \geq t_0, t_0 \) and \( c \) depending only on \( n \) and \( m \). Then
\[
 \rho(t, x) \leq \varepsilon t^{-n \lambda}
\]
provided
\[
 |x| \geq r(t)t^{\lambda} := R_B t^{\lambda}(1 + e^{-\frac{1}{2} - \left(\frac{m-1}{m}\right) t^{-\lambda}} + e^{-\frac{1}{2} t^{-\lambda/n} + \varepsilon^{m-1}})
\]
if \( n \leq 2 \) (again with logarithmic correction if \( n = 2 \) and \( m \leq 2 \)) resp.
\[
 |x| \geq r(t)t^{\lambda} := R_B t^{\lambda}(1 + C e^{-\frac{1}{2} - \left(\frac{m-1}{m}\right) t^{-\lambda}} + e^{-\frac{1}{2} t^{-\lambda/n} + \varepsilon^{m-1}})
\]
for \( n > 2 \). This information suffices to construct a comparison solution by rescaling and time translating a Barenblatt solution. We fix \( \tilde{t} \geq t_0 \) and define
\[
 \rho_B(t, x) = (t + t_1)^{-n \lambda} \left( \frac{A^2}{4} - \frac{|x|^2}{2(t + t_1)^{2\lambda}} \right)^{\frac{1}{m-1}}
\]
which satisfies (with \( c \) from (8.23))
\[
 \rho_B(t, x) \geq \varepsilon \tilde{t}^{-n \lambda} \quad \text{for } |x| \leq \tilde{c} \tilde{t}^{\lambda}
\]
if
\[
 \left( \frac{\tilde{t}}{t + t_1} \right)^{n(m-1)\lambda} \left( \frac{A^2}{4} - c^2 \left( \frac{\tilde{t}}{t + t_1} \right)^{2\lambda} \right) \geq \varepsilon^{m-1}
\]
which holds if
\[
 \tilde{t} \ll t_1 \ll \varepsilon^{-\frac{1}{\lambda}}.
\]
Moreover
\[
 \rho_B(t, x) \geq \varepsilon \quad \text{for } \tilde{t} \leq t \leq T, |x| = r(t)
\]
if
\[ \varepsilon^{m-1} \leq \frac{1}{(T + t_1)^{(m-1)n\lambda}} \left( \frac{A^2}{4} - \frac{1}{2} \int^T_{t_1} T^{2\lambda} \frac{t}{(t_1 + T)^{2\lambda}} \right) \]
which holds if
\[ T = \left[ \frac{(A^2)^{\frac{m-1}{2m}}}{1 - (A^2)^{\frac{m-1}{2m}}} - \varepsilon^{m-1} + \epsilon - \frac{1}{2} \frac{(m-2)}{m-1} t^{-\lambda} + \varepsilon - \frac{1}{n} T^{\frac{2\lambda}{n}} \right] t_1 \]
with obvious modifications if \( n = 1 \) or \( n = 2 \).

But then \( \rho_B(T, \cdot) \) is supported in \( B_R(0) \) with
\[ R = D(T + \delta t \varepsilon^{-\frac{2}{n}})^{\lambda} \leq A(T^\lambda + \epsilon^{m-1} + \epsilon - \frac{1}{2} \frac{(m-2)}{m-1} t^{-\lambda} + \varepsilon - \frac{1}{n} T^{\frac{2\lambda}{n}}) \]
Now we optimize \( \varepsilon \) and arrive at
\[ \text{supp} \; \rho(T, \cdot) \subset \text{supp} \; \rho_B(T, \cdot) \subset B_{R(T) + cT^{-\beta}}. \]
for some positive constant \( \beta \) depending on \( n \) and \( m \) resp.

(8.29) \[ \text{supp} \; \tilde{\rho}(s) \subset B_{1 + e^{-\beta s/\lambda}}(0). \]

8.3.5. Lower bounds. With this information we derive a lower bound on \( c_0 \) by
\[ \int (A^2 - y^2)^{\frac{m-1}{2m}} dy = \int \tilde{\rho}(s, y) dy \leq \int (c_0 - |y|^2)^{\frac{m-1}{2m}} dy + \epsilon c - \frac{1}{2} \frac{(m-2)}{m-1} t^{-\lambda} + \epsilon c e^{-\beta s/\lambda} \]
and hence
\[ (A^2 - c e^{-\beta s/\lambda} - \frac{1}{2} |y|^2)^{\frac{m-1}{2m}} \leq \rho^{m-1} \leq (A^2 + c e^{-\beta s/\lambda} - \frac{1}{2} |y|^2)^{\frac{m-1}{2m}} \]
for some positive \( c \) and \( \beta \). We obtain

**Proposition 5.** Let \( \rho_0 \) belong to the class \( C \). There exists \( t_0(n, m) > 0 \) such that for \( t \geq t_0 M^{-1} R^\frac{1}{L} \) the solution \( \rho(t, x) \) is sandwiched between

(8.30) \[ t^{-n\lambda} \left( A^2 - c_1 t^{-\lambda\beta} - \frac{\lambda |x|^2}{2t^{2\lambda}} \right)^{\frac{1}{m-1}} \leq \rho(t, x) \leq t^{-n\lambda} \left( A^2 + c_1 t^{-\lambda\beta} - \frac{\lambda |x|^2}{2t^{2\lambda}} \right)^{\frac{1}{m-1}} \]
where \( A \) is given in (8.22). Moreover, the free boundary of \( \rho(t, x) \) is contained in an annulus with radii \( R_\pm(t) \) such that
\[ R_B^{n\lambda} \leq R_- \leq R_+ \leq R_B^{n\lambda}(1 + c_3 t^{-\beta}). \]
The constants \( t_0, A, c_1 \) and \( c_3 \) are uniform for the class \( C \).

The proof that we have done assumes \( M = 1 \) and \( R = 1 \), and uses rescaled variables, the usual variables are restored via formula (8.10). For general \( u_0 \in C \) with \( M, R > 0 \) use the scaling transformation (1.5) with \( L = 1/R, A = 1/(MR^n) \), \( x_0 = 0 \), \( t_0 = 0 \), hence \( 1/C = M^{-1} R^{2+n(m-1)} \), which is the time scale factor.

8.4. Flatness for large time. We proceed with the proof of Theorem 2. We start from a time large enough so that (8.30) applies near the time \( t = t_1 \) after rescaling. An elementary geometric estimate shows that, as a consequence of (8.30), there exists \( c \) depending only on \( n \) and \( m \) so that if \( t_1 \geq T \geq 2t_0, (t_1, x_1) \in \partial \mathcal{P}(\rho) \) and \( a = x_1/|x_1| \) then
\[ \rho_{tu}(t, x; a, 1, -cT^{-\beta/2}) \leq T^{\frac{\alpha}{(m-1)}} t_1^{-\frac{1}{m-1}} \rho(t_1 + \frac{2\lambda-1}{2} T^{-\frac{\beta}{2}} t, x_1 + t_1^\lambda T^{-\frac{\beta}{2}} x) \]
\[ \leq \rho_{tu}(t, x; a, 1, cT^{-\beta/2}) \]
for $|x| \leq 1$ and $-1 \leq T \leq 0$. Hence, if $\delta := cT^{-\frac{\beta}{2}} \leq \delta_0$ with the constants of Theorem 1 we apply that Theorem to
\[
\mathfrak{p}(t, x) = T^{\frac{\beta}{2(m-1)}} t_1^{\frac{1}{m-1}} \rho(t_1 + t_1^{2\lambda - 1} T^{-\frac{\beta}{2}} t, x_1 + t_1^{2} T^{-\frac{\beta}{2}} x)
\]
and conclude that at near the new time $t = 0$ the solution is $C^\infty$ regular in $x$ and $t$ up to the free boundary which is $C^\infty$ hypersurface very close to the rescaled Barenblatt free boundary. This gives the proof of the regularity part of Theorem 2.

8.5. Changing dependent and independent variables for large time. To complete the proof we have to obtain the rate of convergence. We rewrite the equation with self-similar coordinates by parametrizing the graph of the pressure $\tilde{\rho}^{m-1}(s, y)$ as
\[
\Phi : B\sqrt{2}(0) \times (0, \infty) \to (xu, u^2(1 - |x|^2/2)).
\]
Then $u = \text{const}$ is the Barenblatt solution and $y$ and $v = \tilde{\rho}^{m-1}$ are related to $x$ and $u$ by
\[
y = xu \quad v = u^2(1 - |x|^2/2)
\]
and a tedious calculation gives
\begin{equation}
(8.31) \quad u_s - (m - 1) \sum_i \partial_i \left[ \left( 1 - \frac{|y|^2}{2} \right) F^i \right] - (m - 2) y_i F^i = 0 \quad \text{on } [T, \infty) \times B\sqrt{2}(0),
\end{equation}
where
\[
F^i = \partial_{x_i} u - \frac{x_i |\nabla u|^2}{u + x_k \partial_k u}.
\]
It is not hard to see that (8.10) is a consequence of (8.14) which in these coordinates becomes
\begin{equation}
(8.32) \quad |u - \sqrt{A_{m,n}}| \leq ce^{-2s}.
\end{equation}
It remains to prove (8.32).

The linearization at $u = 1$ is
\begin{equation}
(8.33) \quad \hat{u}_s - (m - 1) \sum_i \partial_i \left[ \left( 1 - \frac{1}{2} |x|^2 \right) \partial_i \hat{u} \right] - (m - 2) x_i \partial_i \hat{u} = 0 \quad \text{on } [0, \infty) \times B\sqrt{2}(0).
\end{equation}

The mass is given by
\[
M = \int \tilde{\rho} dx
\]
\[
= \int u^{n-\frac{2}{m-1}} (1 - \frac{1}{2} |x|^2) \frac{1}{n-\frac{2}{m-1}} \det(u \delta_{ij} + \partial_i u x_j)_{1 \leq i, j \leq n}
\]
\[
= \int u^{n+1-\frac{2}{m-1}} (u + x_k \partial_k u) \left( 1 - \frac{1}{2} |x|^2 \right) \frac{1}{n-\frac{2}{m-1}} dx
\]
\[
= \int \left[ u^{n+1-\frac{2}{m-1}} + \frac{1}{n+\frac{2}{m-1}} x_k \partial_k u^{n+\frac{2}{m-1}} \right] \left( 1 - \frac{1}{2} |x|^2 \right) \frac{1}{n-\frac{2}{m-1}}
\]
\[
= 2\lambda \int_{B\sqrt{2}(0)} u^{n+\frac{2}{m-1}} \left( 1 - \frac{1}{2} |x|^2 \right) \frac{2-n}{n-\frac{2}{m-1}} dx
\]
and the entropy by
\[ H = \int \tilde{\rho} \left( \frac{\tilde{\rho}^{m-1}}{m} + \frac{1}{2} |y|^2 \right) dy \]
\[ = \frac{1}{m} \int_{B_{\sqrt{2}}(0)} u^{n+1} \left( u + x_k \partial_k u \right) (1 + (m - 1)|x|^2/2) \left( 1 - \frac{1}{2} |x|^2 \right)^{\frac{1}{m-1}} dx \]
\[ = \frac{2}{(n+2)(m-1)+2} \int_{B_{\sqrt{2}}(0)} u^{n+2} \left( 1 - \frac{1}{2} |x|^2 \right)^{\frac{2}{m-1}} dx. \]

8.6. Relating entropy dissipation and an energy for \( u \).

**Lemma 8.2.** Suppose that \( \lambda s \geq \ln T(n, m) \). Then \( u \) is well defined and the following estimates hold

\[ |\sqrt{A_{n,m}} - u| \leq C e^{-\beta \lambda s}, \]
\[ |\nabla u| \leq C e^{-\beta \lambda s}. \]

Moreover,
\[ \int \left( 1 - \frac{|y|^2}{2} \right)^{\frac{1}{m-1}} |\nabla u(s, .)|^2 \leq c \int \tilde{\rho} + \nabla \tilde{\rho}^{m-1} |y|^2 dy \leq c e^{-2s}. \]

**Proof.** The pointwise bound is equivalent to (8.30). Elementary geometric considerations show that

\[ |\nabla \tilde{\rho}^{m-1} - y| \sim |\nabla u| \]

if \( |\nabla v - y| \leq \frac{1}{2} \) or \( |\nabla u| \leq \frac{1}{2} \). In an \( \sqrt{\delta_0} \) neighborhood of the free boundary this estimate is a consequence of the regularity estimates of Theorem 1. In the interior it follows from the bound on entropy dissipation rate (8.21) and Moser’s \( L^\infty \) bounds. We obtain (8.36) from the decay of the entropy dissipation. \( \square \)

Since \( u \) can be written as a solution to
\[ u_t - (m-1) \left( 1 - \frac{|x|^2}{2} \right)^{\frac{2-m}{m-1}} \sum_i \partial_i \left[ \left( 1 - \frac{|y|^2}{2} \right)^{\frac{1}{m-1}} a^{ij} \partial_j u \right] = 0 \quad \text{on } [0, \infty) \times B_{\sqrt{2}}(0) \]
we obtain for \( s \geq s_0 + 1 \) the existence of a constant \( \kappa \) so that
\[ \| u(s) - \kappa \|^2_{\text{sup}} \leq c \| u(s-1) - \kappa \|^2_{L^2} \leq c \int (1 - |x|^2/2)|\nabla u(s-1)|^2 dx \leq c e^{-2s} \]
where we used Moser’s estimate for the first and Poincaré’s inequality for the second inequality. Clearly \( \kappa = \sqrt{A_{nm}} \). Arguing similarly in the interior and at the boundary we arrive at
\[ \| \partial^2_x \partial^k_x u(s, x) \|_{\text{sup}} \leq c e^{-s}. \]

8.7. The spectrum of the linearization [8.33]. The spectrum of the elliptic operator in [8.33] determines the convergence rate. We summarize the relevant results of Seis, Proposition 6 in [22]. The operator can be written as
\[ u \to Lu = -(m-1) \left( 1 - \frac{|x|^2}{2} \right)^{\frac{2-m}{m-1}} \nabla \left[ \left( 1 - \frac{|x|^2}{2} \right)^{\frac{1}{m-1}} \nabla u \right] = 0. \]
Let $H$ be the Hilbert space $L^2(B_{\sqrt{2}(0)}, (1 - |x|^2/2)^{\frac{2-m}{2}})$. Then $L$ is the positive semi-definite operator on $H$ defined by the quadratic form

$$E(u) = (m - 1) \int \left( 1 - \frac{|x|^2}{2} \right)^{\frac{m-1}{2}} |\nabla u|^2 dx.$$  

The Hilbert space $H^1$ defined by

$$\|u\|_{H^1}^2 = E(u) + \|u\|_H^2$$

embeds compactly into $H$ and hence the spectrum of $L$ consists of a sequence of eigenvalues tending to $\infty$. Both norms are invariant under rotations and we can diagonalize it into spherical harmonics of degree $l$. For each degree we obtain a sequence of simple eigenvalues tending to infinity:

$$\lambda_{lk} = (l + 2k) + (m - 1)k(2k + 2l + n - 2)$$

where $l$ and $k$ are nonnegative integers. The first eigenvalues and eigenfunctions are:

1. $\lambda_{00} = 0$, the eigenspace is spanned by $u = 1$.
2. $\lambda_{10} = 1$, the eigenspace is spanned by $u = x_i$.
3. $\lambda_{20} = 2$, $u = x' Ax$ for a traceless matrix $A$.
4. $\lambda_{l0} = l$, $u$ is a harmonic polynomial of degree $l$.
5. $\lambda_{01} = \lambda^{-1}$, $u = |x|^2 - 2n(m - 1)\lambda$ with a crossover at $m = 1 + \frac{1}{n}$ and a second at $m = 1$. This mode corresponds to a time shift in the original variables.
6. $\lambda_{02} = 2\lambda^{-1} + 4(m - 1)$ with radial eigenfunction

$$u(x) = |x|^4 + \frac{4(m - 1)(n + 2)}{2 - \lambda_{02}} \left( |x|^2 + \frac{4(m - 1)n}{\lambda_{02}} \right).$$

This is the first relevant eigenvalue in the radial case (resp. the case with vanishing harmonic moments).

### 8.8. Consequences for stability.

We now discuss their relevance for the nonlinear dynamics.

1. $\dot{u} = 0$ with eigenvalue $0$ is the least stable mode. It corresponds to changes of mass. We eliminate this mode by fixing the mass of the solution.
2. A mismatch in the center of mass corresponds to the modes

$$\dot{u} = y_i$$

with eigenvalue $-1$. This implies the convergence rate $t^{-\lambda}$ in self-similar coordinates. Note that the integral $\int \rho(x)x_i dx$ is conserved by the PME flow. In particular, if the center of mass is $0$ initially then this remains so for all time. This can be done by a mere displacement of the axes. Equation (8.31) can now be written as a perturbation of the linear equation (8.37).

3. The next eigenvalue is $-2$ and hence

$$u = A_{n,m} + \sum_{i=1}^{n} a_i y_i e^{-s} + O(e^{-(2-\varepsilon)s})$$

in $L^2$. Since

$$0 = \int y_i \tilde{\rho}(s, y) dy = \int a_i y_i^2 e^{-t} + O(e^{-(2-\varepsilon)s}),$$
we conclude that $a_i = 0$ and, as above
$$
\sup_{|\alpha| \leq M} |\partial^\alpha u| \leq ce^{-(2-\varepsilon)s}.
$$
Thus,
$$
u_s - (m-1)(1-|y|^2/2) \sum_{i} \partial_i [(1-|y|^2/2)^{m-1}] \partial_i u = O(e^{-(4-\varepsilon)s})$$
on $[0, \infty) \times B_{\sqrt{2}}(0),$
and we expand into the next modes (in $L^2$)
$$u = (x^t A x e^{-2s} + h_3(x)e^{-3s} + \sqrt{\lambda} e^{-s/\lambda} + O(e^{-(4-\varepsilon)s}),$$
where $A$ is a traceless matrix, $h_3$ is a homogeneous harmonic polynomial of degree 3. It is not hard to show that this expansion holds in a smooth sense, and to determine the next term.

(4) One can determine $A$ and the coefficients $h_3$ from the initial data as follows: If $f$ is a smooth function then
$$\frac{df}{dt} \int \rho f(x) dx = \int \rho^m \Delta f dx.$$
In particular, if $h$ is a harmonic polynomial then $\int \rho h dx = 0$. Then, $A$ is given by the harmonic second order moments.

(5) In the radial case there exists $t_0$ so that
$$\left| e^{n\lambda s} \rho(e^{s\lambda} + t_0, \sqrt{\lambda} e^{s\lambda} y)^{m-1} - (A_{n,m} - |x|^2/2) \right| \leq ce^{-(2+4\lambda(m-1))s}.$$

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