Real Tunneling and Black Hole Creation

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Abstract

We discuss the Hawking theory of quantum cosmology with regard to approximation at the lowest order of the Planck constant. At this level, the quantum scenario will be reduced to its classical evolutions in real and imaginary times. We restrict our attention to the so-called real tunneling case. It can be shown that, even at this level, there still exist some quantum effects, the classical field equation may not hold at the transition surface. One can introduce the concept of constrained gravitational instanton. It may play some important role in the scenario of black hole creation in the inflationary background at the Planckian era of the universe. From the constrained gravitational instanton, the real tunneling can occur through different ways. Consequently, it will lead to the creation of different parts of the black hole spacetime in the de Sitter background. The global aspects of the black hole creation are discussed.

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I. Introduction

The Hawking theory of the No-Boundary Universe is the first success in obtaining a self-contained cosmology. Now, in principle, one can predict everything in the universe solely from the theory.

According to the no-boundary proposal of Hartle and Hawking, the quantum state of the universe is in the ground state, which is defined by a path integral over compact Euclidean metrics \[1\]. Conceptually, this proposal is very appealing. Technically, like any other theory of quantum cosmology, it encounters enormous difficulty in calculations due to the lack of a complete theory of quantum gravity. The task of this article is to deal with the situation at the modest level, i.e., the lowest order in the Planck constant \(\hbar\). It turns out that the problem is not really as trivial as it looks at the first glance.

In quantum field theory, it is well known that quantum tunneling can be studied by using the instanton theory. Instanton is a sort of Euclidean solution of the field equation. Quantum penetrating can be described by an analytic continuation from the Euclidean solution to its Lorentzian counterpart. In flat spacetime background, one can readily realize this by simply changing the time value from imaginary to real. However, except for some very special cases, a complex solution of the Einstein field equation does not typically have both purely Euclidean and Lorentzian sectors. Therefore, the instanton theory cannot be used here without modification.

In some simple models, the creation of the universe can be considered as a quantum tunneling from an Euclidean spacetime to a Lorentzian one. If the field equation is not only satisfied in the Euclidean and Lorentzian spacetimes, respectively, but also at the location of the transition, then the instanton theory can be used as it is. Unfortunately, very few models in quantum cosmology share such luck \[2\]. The de Sitter model is an exception.

The de Sitter model is the first nontrivial model in quantum cosmology. Since the 4-sphere and the de Sitter spacetime are the Euclidean and Lorentzian sectors of a complex solution to the vacuum Einstein equation with a cosmological constant, the instanton theory does apply here. The main reason that the creation of the de Sitter universe can be considered as tunneling from the 4-sphere is that it is a model with only one degree of freedom, the scale of the 3-metric. Indeed, the instanton theory can be used to all models with only one degree of freedom.

However, if one discusses a model with more than one degrees of freedom, then the situation be-
comes more complicated. For more realistic models such as the Hawking massive scalar or primordial black hole creation, the instanton theory has to be modified and generalized.

In the Hawking model a massive scalar $\phi$ with mass $m$ is coupled to an isotropic and homogeneous universe [3]. It can be shown that there does not exist any instanton solution in this model. In fact, there does not exist any compact sector of a complex solution, as the 4-sphere solution to the de Sitter model.

Section II will review the Wheeler-DeWitt equation at the lowest order. It is known that, in general, the wave packet represents two ensembles of classical evolutions, one is in real time and other in imaginary. They satisfy classical equations with modification due to their mutual interactions. These two ensembles of trajectories are mutually orthogonal with respect to the supermetric of the configuration space.

Section III will deal with the modification of the instanton theory for the so-called real tunneling case. That is, there is no interaction between Euclidean and Lorentzian evolutions, or the two ensembles decouple. To classically interpret the quantum tunneling, one has to begin with the action from first principles. Then we can introduce the concept of a constrained gravitational instanton [4], which is an Euclidean stationary action solution under some constraints. It satisfies the Einstein equation with the possible exception at the transition surface where the constraints are imposed.

A very interesting application of the theory will be the primordial black hole creation in quantum cosmology [5]. Although some attempts were made one decade ago on the Schwarzschild-de Sitter black hole creation in quantum cosmology [6][7], its conclusive solution was obtained only very recently. This will be the content of Section IV. The problem of a black hole of the whole Kerr-Newman family in the de Sitter background has been completely resolved. Its probability, at the \textit{WKB} level, is the exponential of a quarter of the sum of the black hole and cosmological horizons.

It turns out that the de Sitter evolution is the most probable trajectory in the Planckian era of the universe.

Section V shows that there are alternative ways of real tunneling in the black hole creation with different probabilities. They will lead to the Lorentzian evolutions of parts of interior of the black hole horizon or exterior of the cosmological horizon. Section VI is devoted to the global aspects of
Section VII is a summary.

II. The wave function at the lowest approximation

In the No-Boundary Universe the wave function of the universe is given by [1]

$$\Psi(h_{ij}, \phi) = \int_C d[g_{\mu\nu}] d[\phi] \exp(-\bar{I}(g_{\mu\nu}, \phi)), \quad (1)$$

where the path integral is over the class $C$ of compact Euclidean 4-metrics and matter field configurations, which agree with the given 3-metrics $h_{ij}$ of the only boundary and matter configuration $\phi$ on it. Here $\bar{I}$ means the Euclidean action.

The Euclidean action for the gravitational part for a smooth spacetime manifold $M$ with boundary $\partial M$ is

$$\bar{I} = -\frac{1}{16\pi} \int_M (R - 2\Lambda) - \frac{1}{8\pi} \int_{\partial M} K, \quad (2)$$

where $\Lambda$ is the cosmological constant, $R$ is the scalar curvature and $K$ is the trace of the second fundamental form of the boundary.

The dominant contribution to the path integral comes from some stationary action manifolds with matter fields on them, which are the saddle points of the path integral. In general, the wave function takes a superposition form of wave packets

$$\Psi \approx C \exp(-S/\bar{h}), \quad (3)$$

where we have written $\bar{h}$ explicitly; $C$ is a slowly varying prefactor; and $S = S_r + iS_i$ is a complex phase.

Since the wave packets of form (3) are not independent in the decomposition of the wave function, one more restriction should be imposed. That is, the wave packets itself should obey the Wheeler-DeWitt equation. Classically, it means that the evolutions represented by the wave packet should satisfy the Einstein equation with some quantum corrections, as it will be shown below.

The Wheeler-DeWitt equation takes the following form modulo to some operator ordering ambiguities:

$$\left(-\frac{1}{2} \Delta + V\right) \Psi = 0, \quad (4)$$

black hole creation.
where $\triangle$ is the Laplacian in the supermetric of the configuration space and $V$ is considered as the potential term. Classically, it means that the evolutions represented by the wave packet should satisfy the Einstein equation with some quantum corrections. We have implicitly assumed the universe to be closed.

Inserting the wave packet form into the Wheeler-DeWitt equation, one obtains:

$$
\left( -\frac{1}{2}(\nabla S)^2 + V \right) C + \left( \frac{1}{2} \triangle S + \nabla S \cdot \nabla \right) Ch - \frac{1}{2} \triangle Ch^2 = 0. 
$$

(5)

We can separate it into a real part

$$
\left( -\frac{1}{2}(\nabla S_r)^2 + \frac{1}{2}(\nabla S_i)^2 + V \right) C + \left( \frac{1}{2} \triangle S_r + \nabla S_r \cdot \nabla \right) Ch - \frac{1}{2} \triangle Ch^2 = 0, 
$$

(6)

and an imaginary part

$$
-C \nabla S_r \cdot \nabla S_i + \left( \frac{1}{2} C \triangle S_i + \nabla S_i \cdot \nabla C \right) h = 0. 
$$

(7)

If we ignore the quantum effects represented by the terms associated with powers of $\hbar$ in these equations, then Eqs. (6) and (7) become

$$
-\frac{1}{2}(\nabla S_r)^2 + \frac{1}{2}(\nabla S_i)^2 + V = 0, 
$$

(8)

and

$$
\nabla S_r \cdot \nabla S_i = 0. 
$$

(9)

Eq. (8) is the Lorentzian (or Euclidean) Hamilton-Jacobi equation, with $S_i$ (or $S_r$) and $\nabla S_i$ (or $\nabla S_r$) identified as the classical action and the canonical momenta, respectively. One can define Lorentzian (or Euclidean) orbits along integral curves with $\frac{d}{dt} \equiv \nabla S_i \cdot \nabla$ (or $\frac{d}{\tau} \equiv \nabla S_r \cdot \nabla$). The wave function represents an ensemble of classical trajectories. The Lorentzian (or Euclidean) trajectories will trace out orbits in the presence of the potential $V - \frac{1}{2}(\nabla S_r)^2$ (or $-V - \frac{1}{2}(\nabla S_i)^2$).

Here, two kinds of quantum corrections are involved: one due to the higher $\hbar$ terms of Eq. (6) and the other due to the nonzero value of $\nabla S_r$ (or $\nabla S_i$). For the Lorentzian case, if the modification is not negligible, then the evolutions should deviate quite dramatically from classical dynamics. This deviation, which has little effect on the short range behavior, may be crucial to the global properties of the universe. At the lowest approximation the first kind of corrections are neglected.
The quantity $\Psi^*\Psi$ or the factor $\exp(-2S_r)$ in the wave packet can be interpreted as the relative probability of the Lorentzian trajectories in the region of the configuration space with varying $S_i$. From Eq. (9) we know that the Lorentzian and Euclidean trajectories are mutually perpendicular, or $S_r$ remains constant along the orbits at the lowest order in $\hbar$. The $\hbar$ term of Eq. (7) represents the probability creation rates during the Lorentzian evolutions, and the probabilities are conserved if and only if the first term vanishes. The $\hbar$ term of Eq. (6) represents the dynamic effects of probability creation due to the Euclidean evolution. Briefly speaking, the evolution in real time is causal, while the evolution in imaginary time is stochastic.

While imaginary time becomes a commonly accepted notion in quantum cosmology, it has seemed not accepted yet by workers in other fields. Many people on quantum optics are talking about light propagation at a speed higher than $c$ when transpassing a classically prohibited region. They notice that the light takes zero real time lapse for the tunneling. If one agrees that the time lapse is imaginary within this region and the only observational effect of imaginary time is an exponential decay of the signal, then there would not be any puzzle left [8].

Eqs. (8) (9) represents the Wheeler-DeWitt equation for quantum cosmology at the lowest order approximation. The second order partial functional differential equation has been degraded into the first order one. One of the consequences of this approximation is that the propagation property associated with the wavelike equation of the exact equation has been ignored at this level. We shall restrict our following arguments to the lowest order approximation unless otherwise stated.

In some models there exists a so-called Euclidean regime in configuration space with a purely real phase in the wave packet. At the boundary of this region a transition from Euclidean evolution to Lorentzian evolution occurs through a 3-geometry $\Sigma$. This transition is called real tunneling by Gibbons and Hartle [2].

During real tunneling, the Euclidean spacetime is connected to the Lorentzian spacetime with common boundary $\Sigma$. Gibbons and Hartle argue that if the Einstein equation holds at the both sides of $\Sigma$, then the second fundamental form $K_{ij}$ has to vanish from the both sides [2],

$$K_{ij} = 0.$$  \(10\)

There is no such restriction on the normal derivative of the matter field there. Since $\Sigma$ has a vanishing second fundamental form, one can construct the vacuum instanton manifold by joining...
the Euclidean manifold with its orientation reversal across $\Sigma$.

In the next section we shall show that the presumption made by Gibbons and Hartle at the transition surface is too restrictive, it would exclude many interesting phenomena in quantum cosmology, in particular the scenario of black hole creation at the birth of the universe [5].

From the above argument, we learn that at the lowest level, for the real tunneling, the Lorentzian or Euclidean classical equations are satisfied along the trajectories in the interiors of the Euclidean or Lorentzian regimes.

One may wonder whether at the lowest approximation the quantum theory can be fully reduced into the classical theory. We shall argue that this is not always the case. So we have to look closely at the transition surface of the manifold, or the boundary of the two regimes in the configuration space. At some cases the classical equation will not be recovered there, and consequently condition (10) may not always hold there.

We shall deal with real tunneling cases in the rest of the paper unless stated otherwise.

**III. The constrained gravitational instanton**

The probability of the Lorentzian trajectory emanating from the 3-surface $\Sigma$ with the matter field $\phi$ on it can be written as

$$P = \Psi^* \Psi = \int_C d[g_{\mu\nu}] d[\phi] \exp(-\bar{I}([g_{\mu\nu}, \phi])), \quad (11)$$

where the class $C$ is composed by all no-boundary compact Euclidean 4-metrics and matter field configurations which agree with the given 3-metric $h_{ij}$ and the matter field $\phi$ on $\Sigma$.

Here, we do not restrict the class $C$ to contain regular metrics only, since the derivation from Eq. (1) to Eq. (11) has already led to some jump discontinuities in the extrinsic curvature at $\Sigma$. This point is crucial for the wave function of the universe, otherwise it would be impossible to factorize expression (11) into the ground state definition (1).

The main contribution to the path integral in Eq. (11) is due to the stationary action 4-metric, which meets all requirements on the 3-surface $\Sigma$. At the $WKB$ level, the exponential of the negative of the stationary action is the probability of the corresponding Lorentzian trajectory.
From the above viewpoint, an extension of the class $C$ to include metrics with some mild singularities is essential. Indeed, it is recognized that, in some sense, the set of all regular metrics is not complete. For many cases, under the usual regularity conditions and the requirements at the equator $\Sigma$, there may not exist any stationary action metric, i.e., a gravitational instanton. It is not clear, how large the class $C$ should be. A necessary condition for a metric to be a member is that its scalar curvature should be well-defined mathematically. It is reasonable to include metrics with jump discontinuities of extrinsic curvature and their degenerate cases, that is, the conical or pancake singularities. For this kind of singularity, the quantity $g^{1/2}R$ can be interpreted as a distribution-valued density [9].

If we lift the requirement on the 3-metric of the equator, then the stationary action solution becomes the regular gravitational instanton, as it satisfies the Einstein equation everywhere. Then the Gibbons-Hartle condition (10) should hold at the equator. The probability of the corresponding trajectory takes stationary value; it may be maximum, minimum or neither [10].

The wave packet represents an ensemble of Lorentzian trajectories. If the regular gravitational instanton has minimum action, then the 3-metric from which the Lorentzian evolution supposes to emanate is determined, and the most probable trajectory is therefore singled out. As a result, quantum cosmology fully realizes its prediction power: there is no degree of freedom left, with the exception of physical time [10]. If one does not use the instanton theory, the degree of freedom is reduced to half by the ground state proposal, roughly speaking, due to the regularity condition at the south pole of the Euclidean manifolds in the path integral.

In general, the regularity conditions on the 4-metrics and the requirements from the equator $\Sigma$ sometimes are so strong that no gravitational instanton exists. The reason is that one cannot require the regularity condition and the given 3-metric at the equator simultaneously in the variational calculation. Therefore, hopefully, one can only find a nonregular gravitational instanton with some mild singularities within the class $C$. If this is the case, then Eq. (10) will no longer hold at the singularities. The probability of the real tunneling will not be stationary when lifting the 3-metric requirements.

It has been proven [9] that a stationary action regular solution keeps its status under the extension of the class $C$. However, if a stationary action regular solution cannot be found, then it can probably
be expected with some singularities at its equator among the class $C$.

One can rephrase this by saying that the solution obeys the generalized Einstein equation in the whole manifold. Since this result is derived from first principles, and if one believes that Nature is quantum, then one should not feel upset about this situation.

IV. The creation of a black hole

In this section, we shall apply the constrained gravitational instanton theory to the problem of primordial black hole. The creation of a black hole in the whole Kerr-Newman family has been resolved [5].

In the Hawking model, the universe at the Euclidean and inflation stage can be approximated by a $S^4$ space and the de Sitter space with an effective cosmological constant $\Lambda \equiv 3m^2\phi_0^2$, where $\phi_0$ is the initial value of the scalar field. This is the motivation to discuss the black hole creation in the de Sitter background. A chargeless and nonrotating black hole sitting in the de Sitter background can be described by the Schwarzschild-de Sitter spacetime. It is the unique spherically symmetric vacuum solution to the Einstein equation with a cosmological constant $\Lambda$. The $S^2 \times S^2$ Nariai spacetime is its degenerate case.

Its Euclidean metric can be written as [11]

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)dr^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2.$$  \hspace{1cm} (12)

For convenience one can factorize the potential [11]

$$\Delta = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} = -\frac{\Lambda}{3r}(r - r_0)(r - r_2)(r - r_3).$$  \hspace{1cm} (13)

$$r_2 = 2\sqrt{\frac{1}{\Lambda}} \cos\left(\frac{\alpha + \pi}{3}\right), \quad r_3 = 2\sqrt{\frac{1}{\Lambda}} \cos\left(\frac{\alpha - \pi}{3}\right), \quad r_0 = -2\sqrt{\frac{1}{\Lambda}} \cos\alpha,$$  \hspace{1cm} (14)

and

$$\alpha = \frac{1}{3} \arccos(3m\Lambda^{1/2}),$$  \hspace{1cm} (15)

where $r_2, r_3$ are the black hole and cosmological horizons, and $r_0$ is the horizon for the negative $r$. We are interested in the Euclidean sector $r_2 \leq r \leq r_3$ for $0 \leq m \leq m_c = \Lambda^{-1/2}/3$. For the extreme case $m = m_c$ the sector degenerates into the $S^2 \times S^2$ space.
The black hole and cosmological surface gravities $\kappa_2$ and $\kappa_3$ are [11]

\begin{align}
\kappa_2 &= \frac{\Lambda}{6r_2^2}(r_3^2 - r_2^2)(r_2 - r_0), \tag{16}
\kappa_3 &= \frac{\Lambda}{6r_3^2}(r_3^2 - r_2^2)(r_3 - r_0). \tag{17}
\end{align}

Now we are making a constrained gravitational instanton. In the $(\tau - r)$ plane $r = r_2$ is an axis of symmetry, the imaginary time coordinate $\tau$ is identified with period $\beta_2 = 2\pi\kappa_2^{-1}$, and $\beta_2^{-1}$ is the Hawking temperature. This makes the Euclidean manifold regular at the black hole horizon.

One can also apply this procedure to the cosmological horizon with period $\beta_3 = 2\pi\kappa_3^{-1}$, and $\beta_3^{-1}$ is the Gibbons-Hawking temperature. For the $S^2 \times S^2$ case these two horizons are identical, thus one obtains a regular instanton. Except for the $S^2 \times S^2$ spacetime, one cannot simultaneously regularize at both horizons. In fact, there is no way to avoid singularity in compacting the Euclidean spacetime because of the inequality $\beta_2^{-1} > \beta_3^{-1}$.

To form a constrained gravitational instanton [5], one can have two cuts at $\tau = consts.$ between $r = r_2$ and $r = r_3$ and then glue them. Then the $f_2$-fold cover turns the $(\tau - r)$ plane into a cone with a deficit angle $2\pi(1 - f_2)$ at the black hole horizon. In a similar way one can have an $f_3$-fold cover at the cosmological horizon. Both $f_2$ and $f_3$ can take any pair of real numbers with the relation

$$f_2\beta_2 = f_3\beta_3 \tag{18}$$

for a fairly symmetric Euclidean manifold.

If $f_2$ or $f_3$ is different from 1 (at least one should be), then the cone at the black hole or cosmological horizon will have an extra contribution to the action of the manifold. We shall see that after the transition to Lorentzian spacetime, the conical singularities will only affect the real part of the phase of the wave function, i.e. the probability of the black hole creation. The black hole creation can be described by an analytic continuation from imaginary time to real time of the constrained gravitational instanton at the equator which is two joint $\tau$ sectors, say $\tau = \pm f_2\beta_2/4$ through the two horizons.

Since the integral of $K$ with respect to the 3-area in the boundary term of the action (2) is the area increase rate along its normal, then the extra contribution due to the conical singularities can
be considered as the degenerate form shown below

\[ \tilde{I}_{2,\text{deficit}} = -\frac{1}{8\pi} \cdot 4\pi r_2^2 \cdot 2\pi (1 - f_2), \tag{19} \]

\[ \tilde{I}_{3,\text{deficit}} = -\frac{1}{8\pi} \cdot 4\pi r_3^2 \cdot 2\pi (1 - f_3). \tag{20} \]

The volume term of the action for the manifold can be calculated

\[ \tilde{I}_{\text{vol}} = -\frac{\Lambda}{6} (r_3^3 - r_2^3) f_2 \beta_2. \tag{21} \]

Using Eqs. (18) - (21), one can get the total action

\[ \tilde{I}_{\text{total}} = -\pi (r_2^2 + r_3^2). \tag{22} \]

This is one quarter of the negative of the sum of the two horizon areas. One quarter of the sum is the total entropy of the universe.

It is remarkable to note that the action is independent of the choice of \( f_2 \) or \( f_3 \). Our manifold satisfied the Einstein equation everywhere except for the two horizons at the equator. Consequently, the parameter \( f_2 \) or \( f_3 \) is the only degree of freedom left. In order to check whether or not we get a stationary action solution or a constrained instanton, one only needs to see whether the above action is stationary with respect to this parameter. Our result (22) shows that our gravitational action has a stationary action, the manifold is qualified as a constrained instanton, and can be used for the \( WKB \) approximation to the wave function. It also means no matter which flat fragment of the constrained gravitational instanton is chosen, the same black hole should be created with the same probability. Of course, the most dramatic case is that of no volume, i.e. \( f_2 = f_3 = 0 \).

Therefore, the probability of the black hole creation is

\[ P_m \approx \exp(\pi (r_2^2 + r_3^2)). \tag{23} \]

Our result interposes two special cases [12]. The first is the de Sitter model with \( m = 0 \),

\[ P_0 \approx \exp \left( \frac{3\pi}{\Lambda} \right) \tag{24} \]

and the second is the Nariai model, or pair black hole creation, with \( m = m_c \),

\[ P_c \approx \exp \left( \frac{2\pi}{\Lambda} \right). \tag{25} \]
For the case $m \ll m_c$, we have
\[ \bar{I}_{\text{instanton}} \approx -\pi \left[ \frac{3}{\Lambda} - 2m \sqrt{\frac{3}{\Lambda}} + 2m^2 \right] \]
and
\[ P_m \approx P_0 \exp(-\pi r_3 r_2). \tag{27} \]

For the case that $m$ is close to $m_c$, i.e. $\alpha \approx 0$, one then has
\[ \bar{I}_{\text{instanton}} \approx \frac{2\pi}{\Lambda} (1 + 2\alpha^2) \]
and
\[ P_m \approx \exp \left( \frac{2\pi}{\Lambda} (1 + 2\alpha^2) \right) \approx P_c \exp \left( \frac{4\pi\alpha^2}{\Lambda} \right). \tag{29} \]

The probability is an exponentially decreasing function in terms of the mass parameter. The de Sitter case has the maximum probability and the Nariai case has the minimum probability.

The topology of the 3-metric of the equator is $S^2 \times S^1$, the configuration space has two degrees of freedom, one being the size of the universe (or the scale of $S^2$), the other being the mass parameter. This situation is very fortunate in that the quantum creation of a black hole can be realized by a real tunneling.

If one includes an electromagnetic field into the model, one would be able to carry out a similar calculation. One simply replaces the potential by
\[ \Delta = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} = -\frac{\Lambda}{3r^2} (r - r_0)(r - r_1)(r - r_2)(r - r_3), \tag{30} \]
where $Q$ is the charge parameter of the black hole.

For the magnetically charged black hole case, the configuration of the wave function is the 3-metric and magnetic charge. However, the configuration for the wave function of an electrically charged black hole is not well defined [5][13][14][15], if one naively uses the folding and gluing techniques described above. For the electric case the configuration of the wave function is the 3-metric and the canonical momentum conjugate to the charge. In order to get the wave function for
the charge, one has to appeal to a Fourier transformation, by which the duality between electric and magnetic black holes is recovered.

The quantum creation scenario of a Schwarzschild-de Sitter black hole and the Reissner-Nordström-de Sitter black hole can be clearly depicted by using the so-called synchronous coordinates [15]:

\[ ds^2 = -d\eta^2 + \frac{1}{\cos^2 \xi} \left( \frac{dr}{d\xi} \right)^2 d\xi^2 + r^2(\eta, \xi)(d\theta^2 + \sin^2 \theta d\phi^2) \]  
\[ (31) \]

The Einstein constraint implies that

\[ \dot{r}^2 + 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} = \cos^2 \xi, \]  
\[ (32) \]

where dot represents the derivative with respect to time, which is \( \eta \) here, and \( m \) is the integral constant, which is identified as the mass parameter.

The relations between coordinates \((t, r)\) and \((\eta, \xi)\) are:

\[ \eta = \int \frac{dr}{(E^2 - \Delta)^{1/2}}, \]  
\[ (33) \]

\[ t = \int \frac{E dr}{(E^2 - \Delta)^{1/2} \Delta}, \]  
\[ (34) \]

where

\[ E = |\cos \xi|, \]  
\[ (35) \]

and

\[ \Delta = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}. \]  
\[ (36) \]

Its classical evolution is equivalent to a coherent motion of particles along a congruence of timelike geodesics, labeled by \((\xi, \theta, \phi)\), in a potential hill \( \Delta \). One can release a particle from the potential hill between \( r_2 \) and \( r_3 \). Except for the case with the initial position at the top of the potential, the particle will approach infinity or hit the singularity \( r = 0 \) for the chargeless black hole case. Therefore, the synchronous coordinates cover the whole spacetime manifold. For the charged case, the potential will blow up near \( r = 0 \), and the particle will transpass the inner horizon \( r = r_1 \) and is bounced back by the singularity at \( r = 0 \).
By using this coordinate system one can obtain the wave function for the Schwarzschild-de Sitter and the Reissner-Nordström-de Sitter spacetimes [15] for the spacelike 3-geometry covered by the coordinates.

The whole scenario of the chargeless black hole creation is shown in Fig. 1. The $S^2$ space $(\theta - \phi)$ is represented by a $S^1$ space around the vertical axis. The radius of $S^2$ is $r$. The bottom part shows the instanton, the upper part shows the black hole created. The inner edge of the donut collapses from the black horizon to the singularity $r = 0$. The outer edge expands from the cosmological horizon to $r = \infty$. The conical singularities are not shown in this very sketchy picture. The scenario of the charged black hole creation is similar except that the motion of the infalling trajectories are bounced by the singularity.

For the rotating and charged black hole case, the spacetime metric takes the Kerr-Newman form

$$ds^2 = \rho^2(\Delta_r^{-1}dr^2 + \Delta_\theta^{-1}d\theta^2) + \rho^{-2}\Xi^{-2}\Delta_\theta \sin^2 \theta(adt - (r^2 + a^2)d\phi)^2 - \rho^{-2}\Xi^{-2}\Delta_r(dt - a\sin^2 \theta d\phi)^2,$$

(37)

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$

(38)

$$\Delta_r = (r^2 + a^2)(1 - \Lambda r^3^{-1}) - 2mr + Q^2 + P^2,$$

(39)

$$\Delta_\theta = 1 + \Lambda a^2 r^3 \cos^2 \theta,$$

(40)

$$\Xi = 1 + \Lambda a^2 r^3$$

(41)

and $m, a, Q$ and $P$ are constants, $m$ and $ma$ represent mass and angular momentum. $Q$ and $P$ are electric and magnetic charges.

One can factorize $\Delta_r$ as follows:

$$\Delta_r = -\frac{\Lambda}{3}(r - r_0)(r - r_1)(r - r_2)(r - r_3),$$

(42)

where the roots $r_0, r_1, r_2$ and $r_3$ are in ascending order, $r_2$ and $r_3$ are the black hole and cosmological horizons.

To form the constrained gravitational instanton, one can imaginarily set the time coordinate by setting $t = -i\tau$, and then do folding and gluing as in the nonrotating cases.
If one were to naively factorize the probability from Eq. (11), he would get the wave function for the 3-metric and the differential rotation of two horizons only. So one has to use another Fourier transformation to obtain the wave function for angular momentum [5]. If the hole is electrically charged, one has to appeal again to the Fourier transformation, as we did for the nonrotating case. Here, two Fourier transformations are involved.

At any case, the probability of a black creation in the de Sitter background, at the WKB level, is the exponential of a quarter of the sum of the black hole and cosmological horizons, a quarter of the sum is the total entropy of the universe. By the no-hair theorem, the problem of a single black hole creation in quantum cosmology has been resolved completely.

The probability is an exponentially decreasing function of mass, charge magnitude and angular momentum. The de Sitter evolution is the most probable one at the Planckian era [5].

V. The alternative tunnelings

Now we are going to discuss a black hole creation from an alternative route. We begin with the vacuum Kantowski-Sachs model [16] with the positive cosmological constant. The 3-surface is homogeneous and has topology $S^1 \times S^2$. The treatment in this section is suitable for the Kaluza-Klein $S^1 \times S^n$ case [10], but we shall study the $n = 2$ case only below for simplicity. In Ref. [16] a black hole will be formed when the massive scalar field rolls down the potential hill and starts to oscillate. The effect of the massive scalar field is approximated by a cosmological constant during the imaginary time stage and the inflationary stage. Here, we investigate the case of a black hole creation at the exact moment of the birth of the universe.

The Euclidean metric of the Kantowski-Sachs model takes the form:

$$ds^2 = d\tau^2 + a^2(\tau)d\omega^2 + b^2(\tau)d\Omega^2,$$

where $\omega$ is identified with a period $2\pi$.

The Euclidean field equation is

$$b\ddot{b} + \frac{\dot{b}^2}{2} + \frac{1}{2} + \frac{\Lambda b^2}{2} = 0,$$

$$\ddot{ab} + b\dot{a} + \dot{b} + \Lambda ab = 0$$

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and
\[ bb\dot{a} + \frac{\dot{b}^2}{2} - \frac{a}{2} + \frac{\Lambda ab^2}{2} = 0, \]  
where the dot means the derivative with respect to \( \tau \).

There are two kinds of gravitational instantons. They correspond to the regularity condition at the south pole. (I): \( a = a_0, \dot{a} = 0, b = 0 \) and \( \dot{b} = 1 \); (ii): \( b = b_0, \dot{b} = 0, a = 0 \) and \( \dot{a} = 1 \).

The gravitational instanton with the boundary condition (i) is
\[ ds^2 = d\tau^2 + a_0^2 \cos^2 H\tau d\omega^2 + H^{-2} \sin^2 H\tau d\Omega_2^2, \]  
where \( H = \sqrt{\Lambda/3} \) is the Hubble constant.

After the scale of \( S^2 \) reaches maximum, then one can make an analytical continuation along real time direction to get its Lorentzian counterpart
\[ ds^2 = -dt^2 + a_0^2 \sinh^2 Htd\omega^2 + H^{-2} \cosh^2 Htd\Omega_2^2. \]  

The Lorentzian metric is a part of the de Sitter space with a cone singularity at the birth of the universe due to the identification of the circle \( S^1 \) with the finite period. The Einstein equation does not hold at the transition as \( \tau = \pi/2H \) and \( t = 0 \) unless at the Euclidean side as \( a_0 = H^{-1} \). The solution has a scale invariance associated with the circle. Thus there essentially exists only one trajectory. This mild singularity is acceptable. The fact that there are so many beautiful representations for the de Sitter spacetime is a manifestation of its versatility.

The gravitational instanton with the boundary condition (ii) is the Schwarzschild-de Sitter manifold of the last section, in which \( r \) is identified as \( b \), the size of \( S^2 \), \( \Delta \) is identified as \( a^2 \), and \( \Delta^{-1}dr^2 \) becomes \( d\tau^2 \) here.

If one regularizes the black hole horizon by setting \( f_2 = 1 \) as in the last section, then one can consider the horizon as the south pole. It is noted that the expansion rate of \( S^2 \) space vanishes at both horizons. The spacetime becomes Lorentzian as one enters the exterior of the cosmological horizon \( r > r_3 \). Here, the \( r \) coordinate becomes timelike. At the transition there is a conical singularity, while the scale of \( S^1 \) shrinks to zero. The Lorentzian evolution represents a part of the exterior of cosmological horizon with \( S^1 \) of extension \( \Delta t = \beta_2 \). The constrained gravitational instanton is formed by joining the above manifold with its orientation reversal across the cosmological horizon.
Therefore, the action should be twice as much as the action of the constrained instanton discussed in the last section. The probability of the creation, at the $WKB$ level, is

$$P_3 \approx \exp(2\pi(r_2^2 + r_3^2)).$$  (49)

On the other hand, if one lets $f_3 = 1$, then the cosmological horizon can be considered as the south pole and quantum transition will occur at the black horizon. The Lorentzian evolution describes an interior part of the black hole. One just exchanges the two horizons in the statement of the preceding paragraph. One will obtain the same creation probability at the $WKB$ level

$$P_2 \approx \exp(2\pi(r_2^2 + r_3^2)).$$  (50)

The manifolds created by the alternative tunnelings are shown as regions $OGH$ and $EMN$ in Fig. 2, respectively. The curves $OG$ and $OH$ (curves $EM$ and $EN$) are identified due to the periodicity condition required by Euclidean compactification at the cosmological (black hole) horizon.

To apply the above arguments on the alternative tunnelings to the black holes of the whole Kerr-Newman family is straightforward.

**VI. Global aspects of the black hole creation**

Now we discuss the global property of the black hole creation.

The synchronous coordinates cover the whole manifold of the Schwarzschild-de Sitter spacetime. This can be shown clearly by the Penrose-Carter diagrams [11] in Fig. 2. There is an infinite sequence of diamond shape regions, singularities $r = 0$ and spacelike infinities $r = \infty$. Therefore, the 3-geometry is not closed and not suitable for No-Boundary Universe. However, in the scenario of a black hole creation the whole manifold created can be obtained by an identification, for instance, with lines $ABC$ and $DEF$. The equator in the instanton is identified as the closed line $BE$ here.

It is worth mentioning that $t$ coordinate is future (past) directed in the right (left) triangle above line $BE$. This is consistent with the analytic continuation of the imaginary time at the two cuts $\tau = \text{consts}$ for the south hemisphere of the constrained gravitational instanton, or the two ends of
the imaginary time lapse. The same argument applies to the other members of the Kerr-Newman family. In the Schwarzschild space, \( r_3 \) becomes \( r = +\infty \), and no identification is necessary.

Fig. 3 shows the Penrose-Carter diagram for Reissner-Nordström-de Sitter black hole creation. The lines \( ABC \) and \( DEF \) are identified. The line \( BE \) is the quantum transition surface, from which the region of \( HABOEDLG \) is created. The synchronous coordinates cover the whole manifold minus neighborhoods of the singularity. One may travel to other universes by passing through the “wormholes” made by the charge. In particular, if one follows an infalling trajectory in the synchronous coordinates, he will be bounced by the singularity and enter the another universe. But this does not bother us right now, since the relevant 3-metric will no longer be spacelike, and the wave function is not well defined under this circumstance. In the Reissner-Nordstrom space, \( r_3 \) becomes \( r = +\infty \), and no identification is necessary.

Fig. 4 shows the Penrose-Carter diagram of the symmetry axis of the Kerr-Newman-de Sitter black hole creation. The infinities \( r = +\infty \) and \( r = -\infty \) are not joined together. The open circles mark where the ring singularity occurs, although it is not on the symmetry axis. The lines \( ABC \) and \( DEF \) are identified. The line \( BE \) is the quantum transition surface, from which the region of \( HABOEDLG \) is created. In the Kerr-Newman space, \( r_3 \) becomes \( r = +\infty \), no identification is necessary.

The alternative tunnelings are shown by the shaded bands in Figs. 2 and 3 with the two boundaries identified. It should be similar for the Kerr-Newman case.

VII. Summary

The complex tunnelings are common phenomena in Nature. We investigate the real tunneling problem in this paper. However the conventional concept of real tunneling is too narrow. If one insists on this, then many interesting phenomena, such as creation of a single black hole in quantum cosmology, would be excluded from studying. On the other hand, if one begins with the first principles in quantum framework, it is naturally leads to the constrained gravitational instanton. The field equation holds in the instanton except for the location where the constraint is imposed; here is where one will find the equator of the instanton. If one believes that the nature is quantum,
then the field equation becomes secondary, and one should welcome this kind of generalization.

As an example, we have shown that the real tunnelings occur at the quantum creation of a black hole in the de Sitter background. This is a quite rare case in nature, taking into consideration the fact that the model has more than one degree of freedom.

By the analysis of this article, one learns that real tunnelings of quantum transition originates from the same Schwarzschild-de Sitter instanton in several ways. No matter what kind analytic continuation is made, one always obtain the whole or parts of the same Lorentzian spacetime, but with different probabilities. If we are working with the de Sitter model, the situation is not so transparent, since it is of the maximum symmetry. All equators of the 4-sphere are identical.

It is also interesting to examine how the Euclidean manifold is joined to the Lorentzian counterpart in the black hole creation. The Lorentzian spacetimes have to be compactified by a periodic identification, then the global aspects of the whole scenario is clarified.

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Figure captions:

Fig. 1: The whole scenario of the black hole creation. The $S^2$ space ($\theta - \phi$) is represented by a $S^1$ space around the vertical axis. The radius of $S^2$ is $r$. The bottom part shows the instanton; the upper part shows the black hole created.

Fig. 2: The Penrose-Carter diagram for a Schwarzschild-de Sitter black hole creation.

Fig. 3: The Penrose-Carter diagram for a Reissner-Nordström-de Sitter black hole creation.

Fig. 4: The Penrose-Carter diagram of the symmetry axis of the Kerr-Newman-de Sitter black hole creation.