A fourth-order 9-point finite difference method for the Helmholtz equation

Jianjun Chen\textsuperscript{1,2}, D S Cheng\textsuperscript{2*}, Rong Jie\textsuperscript{1} and Xiaoling Zhu\textsuperscript{1}

\textsuperscript{1} School of Mathematics and Statistics, Nanning Normal University, Nanning, Guangxi, 530299, PR China
\textsuperscript{2} School of Software Engineering, Shenzhen Institute of Information Technology, Shenzhen, Guangdong, 518172, PR China

*Corresponding author’s e-mail: chengds@sziit.edu.cn

Abstract. In this paper, we develop a new fourth-order 9-point finite difference scheme for solving the Helmholtz equation. The central fourth-order difference scheme is used to discretize the second-order derivative, and the matched interface and boundary (MIB) method is employed to deal with the resulting boundary problem. For the discretization of the zero-order term, a weighted average method is designed by utilizing all of the 9 points, and the weight parameters are determined by minimizing the numerical dispersion. The new method is simple and efficient in suppressing the numerical dispersion. Finally, numerical examples are presented to illustrate the numerical convergence and effectiveness of the new scheme.

1. Introduction

The Helmholtz equation is widely used to characterize the physical phenomena such as wave propagation and inverse scattering in the scientific fields of acoustics, optics, and electromagnetics. Many problems in the technical fields of aerospace, marine technology and oil and gas exploration are attributed to numerical modeling of the Helmholtz equation. To solve the Helmholtz equation, the finite difference method (FDM) and the finite element method (FEM) are commonly used. Since it is simple, flexible and possesses less calculation, the finite difference method is usually preferred in many areas of technology.

Due to the “pollution effect” [1], the traditional second-order central difference scheme generates so large numerical dispersion that it can hardly be used in practical application. To reduce the numerical dispersion, mixed-grid and staggered-grid finite difference methods were developed [2-6]. These methods showed improvement over the traditional central difference scheme. However, most of them are of second-order in accuracy, hence the numerical solutions converge slowly to the exact ones. In literatures [7-9], some high-order compact finite difference schemes were constructed to improve the accuracy as well as the numerical dispersion. However, these compact high-order methods require high smoothness on the right side term, which is unpractical in applications.

In this paper, we propose a fourth-order 9-point finite difference for solving the two-dimensional Helmholtz equation based on the traditional fourth-order central difference scheme. The new method is not a compact one, hence it is not demanding for the smoothness of the right side term. However, it brings the boundary problem that is rather trouble. To overcome this problem, we utilize the matched interface and boundary (MIB) method which is proposed in [10]. Thus, we use the traditional central fourth-order difference scheme to discretize the second-order derivative, and use a weighted average
of all the 9 points to discretize the zero-order term, which aims to reduce the numerical dispersion. That is, the weight parameters are obtained by minimizing the numerical dispersion via the dispersion relation formula. Numerical experiments show that the new scheme is exactly a fourth-order one, and it also performs efficiently in suppressing the numerical dispersion error.

2. Fourth-order 9-point finite difference scheme

Consider the two-dimensional Helmholtz equation:

\[-\Delta u - k^2 u = g\]  \hspace{1cm} (1)

where \(\Delta\) is the Laplacian, \(u\) usually represents the displacement, \(k\) is the wavenumber, and \(g\) denotes the source term.

To discretize equation (1), we first present two numbered 9-point finite difference stencils, as shown in Figure 1, where \((0,0)\) represents the central point, while others denote the neighboring points of \((0,0)\). Let \(u(m,n) := u(x_0 + mh, y_0 + nh)\) denote the discretization of \(u\) at location \((x_m := x_0 + mh, y_m := y_0 + nh)\), where \(h\) is the step size, and \((x_0, y_0)\) is a given initial point.

![9-point difference stencils]  \hspace{1cm} Figure 1. 9-point difference stencils. (a) is for \(\Delta u\), and (b) is for \(k^2 u\).

Define \(\phi_x, \phi_y\) as follows

\[
\phi_x(m,n) := \frac{1}{h^2} \left[ \frac{4}{3} (u_{(m+1,n)} - 2u_{(m,n)} + u_{(m-1,n)}) - \frac{1}{12} (u_{(m+2,n)} - 2u_{(m,n)} + u_{(m-2,n)}) \right].
\]  \hspace{1cm} (2)

\[
\phi_y(m,n) := \frac{1}{h^2} \left[ \frac{4}{3} (u_{(m,n+1)} - 2u_{(m,n)} + u_{(m,n-1)}) - \frac{1}{12} (u_{(m,n+2)} - 2u_{(m,n)} + u_{(m,n-2)}) \right].
\]  \hspace{1cm} (3)

Then, the approximation of the second-order derivatives \(\partial^2 u/\partial x^2\) and \(\partial^2 u/\partial y^2\) at \((x_m, y_n)\) are performed by the following formulas:

\[
\frac{\partial^2 u}{\partial x^2}(m,n) = \phi_x(m,n), \quad \frac{\partial^2 u}{\partial y^2}(m,n) = \phi_y(m,n)
\]  \hspace{1cm} (4)

Thus, we obtain the fourth-order central difference scheme for discretizing \(\Delta u\). We next approximate the zeroth term \(k^2 u\). Define \(\varphi_j(k^2 u)_{(m,n)}, j = 1, 2, 3\) as follows:

\[
\varphi_1(k^2 u)_{(m,n)} := (k^2 u)_{(m,n)}.
\]  \hspace{1cm} (5)

\[
\varphi_2(k^2 u)_{(m,n)} := \frac{2}{3} \left[ (k^2 u)_{(m+1,n)} + (k^2 u)_{(m-1,n)} \right] - \frac{1}{6} \left[ (k^2 u)_{(m+2,n)} + (k^2 u)_{(m-2,n)} \right].
\]  \hspace{1cm} (6)

\[
\varphi_3(k^2 u)_{(m,n)} := \frac{2}{3} \left[ (k^2 u)_{(m,n+1)} + (k^2 u)_{(m,n-1)} \right] - \frac{1}{6} \left[ (k^2 u)_{(m,n+2)} + (k^2 u)_{(m,n-2)} \right].
\]  \hspace{1cm} (7)
Then, the approximation of $k^2u$ is given by:

$$ \phi(k^2u) = b_1\phi_1(k^2u)_{(m,n)} + b_2\phi_2(k^2u)_{(m,n)} + b_3\phi_3(k^2u)_{(m,n)}, $$

where parameters $b_1, b_2$ satisfy $b_1 + 2b_2 = 1$.

It follows from the Taylor expansion that the approximation scheme (8) is also fourth-order in accuracy. Finally, we obtain the fourth-order 9-points difference scheme for equation (1) as following:

$$ -\phi_k(m,n) - \phi(m,n) - \phi(k^2u) = g(m,n), $$

where $g(m,n)$ is the discretization of $g$ at $(x_m, y_n)$. For the parameters $b_1, b_2$, they will be determined later by minimizing the numerical dispersion.

3. Determination of the weight parameters

Substituting equations (2)-(3) and (5)-(7) into equation (9), the new difference scheme is given by

$$ T_1U_{m,n-2} + T_2U_{m,n-1} + T_2U_{m-2,n} + T_2U_{m-1,n} + T_2U_{m,n} + T_2U_{m+1,n} + T_2U_{m+2,n} + T_2U_{m,n+1} + T_2U_{m,n+2} = g_{m,n} $$

where $U_{m,n+j}$ denote the unknowns, and

$$ T_1 = \frac{1}{12h^2} + \frac{b_2}{2}k^2, \quad T_2 = -\frac{4}{3h^2} - \frac{2b_2}{3}k^2, \quad T_3 = \frac{5}{h^2} + b_1k^2. $$

In order to determine the parameters $b_1, b_2$, we now perform the classical dispersion analysis. It is known that the classical plane-wave solution is $U(x, y) = \exp[-ik(x\cos\theta + y\sin\theta)]$, where $\theta$ is the propagation angle from the $y$-axis. Let $f$ and $v$ represent the frequency and velocity respectively, then the wavenumber is $k = 2\pi f / \lambda$. Let $\lambda$ be the wavelength and $G$ be the number of grid points per wavelength, then $G = \lambda h^{-1}$, $\lambda = vf^{-1}$ and $kh = 2\pi G^{-1}$. Substituting the discrete plane-wave solution $U_{m+n+j} = \exp[-ik(x_{m+n}\cos\theta + y_{n+j}\sin\theta)]$ into equation (10) leads to

$$ k^2h^2 \left\{ b_1 + \frac{b_2}{3} \left[ 4(P_1 + Q_1) - (P_2 + Q_2) \right] \right\} = M $$

where

$$ P_1 = \cos \left( \frac{2\pi}{G} \sin\theta \right), \quad P_2 = \cos \left( \frac{4\pi}{G} \sin\theta \right), \quad Q_1 = \cos \left( \frac{2\pi}{G} \cos\theta \right), \quad Q_2 = \cos \left( \frac{4\pi}{G} \cos\theta \right), $$

$$ M = 5 - \frac{8Q_1}{3} + \frac{Q_2}{6} - \frac{8P_1}{3} + \frac{P_2}{6}. $$

By replacing $k$ in equation (11) with variable $k_N$ (numerical wavenumber), we have

$$ k_N^2h^2 \left\{ b_1 + \frac{b_2}{3} \left[ 4(P_1 + Q_1) - (P_2 + Q_2) \right] \right\} = M $$

Let

$$ L := b_1 + \frac{b_2}{3} \left[ 4(P_1 + Q_1) - (P_2 + Q_2) \right], \quad R := M. $$

Then, with $kh = 2\pi/G$, we have the following dispersion relation formula
The relationship between the exact wave number and the numerical wave number is revealed by equation (14). In particular, when the difference scheme is an exact scheme, equation (14) is equal to one. Therefore, we determine the parameters in such a way that the making the error between them as small as possible. To do this, we define the function \( J \) as follow
\[
J(b_1, b_2) := \frac{k_n}{k} - 1 = \frac{G}{2\pi} \left( \frac{R}{L} \right)^{1/2} - 1.
\] (15)

Then, the determination of the parameters \( b_1, b_2 \) is equivalent to solving the optimization problem
\[
(b_1, b_2) = \arg \min_{b_1, b_2} \left\{ J(b_1, b_2) \right\},
\] (16)
with \( b_1 + 2b_2 = 1 \), \( \theta \in \left[0, \pi/2\right] \) and \( G \in \left[ G_{\min}, G_{\max} \right] \). Generally, we choose \( G_{\max} = 400 \) and \( G_{\min} \geq 2 \) according to the Nyquist sampling limit. Finally, we solve the optimization problem (16) by the least squares method as in [5-6].

4. Numerical simulations
In this Section, numerical experiments are presented to demonstrate the efficiency of the new difference scheme.

4.1. Normalized phase velocity curves
For \( G_{\min} = 2 \) and \( G_{\min} = 4 \), using the least-squares method to solve (16), we obtain two groups of parameters \( (b_1, b_2) = (0.81072, 0.09464) \) and \( (b_1, b_2) = (0.86778, 0.06611) \).

We now plot the normalized phase velocity curve for the new difference scheme. In figure 2, we present the normalized phase velocity curves for the new difference scheme and the traditional central difference scheme. For the normalized phase velocity curves, the x-axis represents \( 1/G \), and the y-axis represents \( k_n/k \). The propagation angles are chosen to be \( 0^0, 15^0, 30^0, 45^0 \). As is observed from figure 2 that the new difference scheme performs much better in reducing the numerical dispersion, since it is designed based on minimizing the numerical dispersion error. As the \( G \) decreases, both the normalized phase velocity curves deviate from 1, however, the amplitude of the curve for the new scheme is much smaller than that of the central difference scheme. It means that the newly proposed 9-point difference scheme performs better in improving the numerical error.
4.2. Convergence Order

Example 1:
\[ -\Delta u - k^2 u = \left(2\pi^2 - k^2 \right)\sin(\pi x)\sin(\pi y), \quad (x, y) \in \Omega := [0,1] \times [0,1]. \]  
(17)

Given the boundary condition \( u(x_0, y_0) = 0, \quad (x_0, y_0) \in \partial\Omega \), then the exact solution is \( u(x, y) = \sin(\pi x)\sin(\pi y) \).

Example 2:
\[ -\Delta u - k^2 u = \left(2n^2 - 1 \right)\sin(nkx)\sin(nky), \quad (x, y) \in \Omega := [0,1] \times [0,1]. \]  
(18)

Given the boundary condition \( u(x_0, y_0) = 0, \quad (x_0, y_0) \in \partial\Omega \), then the exact solution is \( u(x, y) = k^2 \sin(nkx)\sin(nky) \).

We use the proposed difference scheme to solve the Helmholtz equations (19) with \( k = 10, 20, 30 \) and solve (20) with \( n = 1, k = 3\pi, n = 3, k = 5\pi \) and \( n = 3, k = 12\pi \), respectively. The results are presented in Table 1 and Table 2, where C.O. represents the numerical convergence order. As is seen that the new difference scheme is exactly fourth order in accuracy.

In computation, the number of points used in the x-axis (or y-axis) exceeds that in the computation area, which brings the boundary problem of spillover. To tackle this problem, we employ the matched interface and boundary (MIB) method in [10], which performs effectively.

| \( k = 10 \) | \( k = 20 \) | \( k = 30 \) |
|---|---|---|
| \( k \) | \( \| u-u_n \|_\infty \) | C.O. | \( \| u-u_n \|_\infty \) | C.O. | \( \| u-u_n \|_\infty \) | C.O. |
| 1/8 | 8.47584e-05 | 7.56143e-05 | 7.42249e-05 | |
| 1/16 | 5.40004e-06 | 3.97231 | 4.81539e-06 | 3.97293 | 4.72654e-06 | 3.97304 |
| 1/32 | 3.39174e-07 | 3.99287 | 3.02415e-07 | 3.99305 | 2.96829e-07 | 3.99308 |
| 1/64 | 2.12247e-08 | 3.99921 | 1.89238e-08 | 3.99825 | 1.85742e-08 | 3.99826 |
| 1/128 | 1.32694e-09 | 3.99957 | 1.183099-09 | 3.99956 | 1.16124e-09 | 3.99961 |
Table 2. Numerical results for Example 2

| $h$  | $\|u-u_{h}\|_\infty$ C.O. | $\|u-u_{h}\|_\infty$ C.O. | $\|u-u_{h}\|_\infty$ C.O. |
|------|-----------------|-----------------|-----------------|
| 1/8 | 2.57995e-05 | 7.90003e-03 | 1.87269e-03 |
| 1/16 | 2.40867e-06 | 3.42103 | 4.95141 | 1.62817e-04 | 3.52379 |
| 1/32 | 2.01140e-07 | 3.58196 | 3.93382 | 1.05813e-05 | 3.94367 |
| 1/64 | 1.34232e-08 | 3.90540 | 3.87765 | 5.81891e-06 | 4.18462 |
| 1/128 | 8.52478e-10 | 3.97693 | 3.95893 | 4.03100e-07 | 3.85154 |

5. Conclusion
We propose a fourth-order 9-point finite difference for solving the two-dimensional Helmholtz equation. The central difference scheme is used to discretize the second derivative term, and a weighted average method is used to discretize the zero-order term, with the weight parameters determined by minimizing the numerical dispersion. To overcome the boundary problem caused by the spillover of grid points, the matched interface and boundary (MIB) method is employed. Simulation experiments illustrate the efficiency of the new scheme, which possesses a high convergence rate as well as a small numerical dispersion error.

Acknowledgments
Authors wishing to acknowledge the financial support from the Innovation Project of Guangxi Graduate Education under grants YCSW2019183, the Natural Science Foundation of China under grant 11701389, and the Characteristic Innovation Project from the Educational Department of Guangdong Province under grant 2018GKTSCX043 and 2017GKQNCX069.

References
[1] Babuška I., Sauter S. (2000) Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers? SIAM Rev., 42: 451-484.
[2] Hustedt B., Operto S., Virieux J. (2004) Mixed-grid and staggered-grid finite-difference methods for frequency-domain acoustic wave modelling. Geophys. J. Int., 157: 1269-1296.
[3] Chen Z.Y., Cheng D.S., Feng W., Wu T.T. (2013) An optimal 9-point finite difference scheme for the Helmholtz equation with PML. Int. J. Numer. Anal. Model., 10: 389-410.
[4] Stolk C.C.. (2016) A dispersion minimizing scheme for the 3-D Helmholtz equation based on ray theory. J. Comput. Phys., 314: 618-646.
[5] Cheng D.S., Tan X., Zeng T.S. (2017) A dispersion minimizing finite difference scheme for the Helmholtz equation based on point-weighting. Comput. Math. Appl., 73: 2345-2359.
[6] Cheng D.S., Chen B.W., Chen X.L. (2019) A robust optimal finite difference scheme for the three-dimensional Helmholtz equation. Math. Prob. Eng., 2019: 1-13.
[7] Turkel E., Gordon G., Gordon R., Tsynkov S. (2013) Compact 2D and 3D sixth order schemes for the Helmholtz equation with variable wave number. J. Comput. Phys., 232: 272-287.
[8] Wu T.T. (2017) A dispersion minimizing compact finite difference scheme for the 2d helmholtz equation. J.Comp. Appl. Math., 311: 497-512.
[9] Wu T.T., Xu R.M. (2018) An optimal compact sixth-order finite difference scheme for the Helmholtz equation. Comput. Math. Appl., 75: 2520-2537.
[10] Zhao S. (2010) High order matched interface and boundary methods for the Helmholtz equation in media with arbitrarily curved interfaces. J. Comput. Phys., 229: 3155-3170.