IMPROVED SPECTRAL PROJECTION ESTIMATES

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Abstract. We obtain new improved spectral projection estimates on manifolds of non-positive curvature, including sharp ones for relatively large spectral windows for general tori. Our results are stronger than those in an earlier work of the first and third authors [6], and the arguments have been greatly simplified. We more directly make use of pointwise estimates that are implicit in the work of Bérard [2] and avoid the use of weak-type spaces that were used in the previous works [6] and [22]. We also simplify and strengthen the bilinear arguments by exploiting the use of microlocal $L^2 \rightarrow L^q$ Kakeya-Nikodym estimates and avoiding the use of $L^2 \rightarrow L^2$ ones as in earlier results. This allows us to prove new results for manifolds of negative curvature and some new sharp estimates for tori. We also have new and improved techniques in two dimensions for general manifolds of non-positive curvature.

In memoriam: Steve Zelditch (1953-2022)

1. Introduction.

The purpose of this paper is to improve upon the spectral projection estimates on compact manifolds of two of us [6], while at the same time simplifying the arguments. As in the previous work we shall focus on obtaining improved estimates for the critical exponent

\begin{equation}
q_c = \frac{2(n+1)}{n-1}
\end{equation}

on $n$-dimensional compact boundaryless Riemannian manifolds $(M,g)$ all of whose sectional curvatures are non-positive. If we obtain improved $L^q_c$-estimates compared to the universal bounds of one of us [18], then, by interpolation and dyadic Sobolev estimates, we obtain improvements for all other exponents $q > 2$, including the difficult range of subcritical exponents considered in earlier papers by Bourgain [8], Sogge [19], Sogge and Zelditch [23] and Blair and Sogge [3], [4], [5] and [6].

We are able to obtain our improvements over the earlier results in [6] by simplifying and strengthening the two main steps in the proof of the spectral projection estimates there. As in [6] and its predecessor [22], we shall adapt an argument of Bourgain [7] for Fourier restriction problems to obtain our results. This allows us to split our estimate into two “heights”. The greater “height” (which is easier to handle) corresponds to the size of $L^2$-normalized zonal spherical harmonics on round spheres $S^n$ which saturate the universal $L^q$-estimates of one of us [18] for exponents $q \geq q_c$, while the lesser “height”
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(which is much harder to handle) corresponds to the size of the highest weight spherical harmonics on $S^n$ which saturate the universal bounds for $2 < q \leq q_c$. See also [17].

We are able to obtain the needed estimates for the larger heights in a much more efficient manner than in the earlier works [6] and [22]. In these works we first obtained weak-type $L^{q_c}$-estimates using the argument from [7] along with pointwise estimates for smoothed out spectral projection kernels that were obtained by the techniques of Béard [2]. We then “upgraded” the weak-type $L^{q_c}$-estimates in the end to $L^{q_c}$-estimates using the Lorentz space improvements of Bak and Seeger [1] for “local” spectral projection operators related to the “global” ones we were considering. We now have a much more direct and simple argument based on Béard-type estimates that completely avoids the use of weak-type estimates and the corresponding necessary interpolation argument involving the local estimates in [1] that leads to unnecessary losses. Also, avoiding the use of weak-type norms in the other main step which treats the “lower heights” simplifies the arguments there.

As in [6], the second main step in the proof of our improved spectral projection estimates utilizes bilinear techniques of Lee [16], Tao [24], Tao, Vargas and Vega [25], Wolff [27] and others that were used to study Fourier restriction problems in Euclidean space. We are able to make better use of the bilinear techniques here by another more direct argument, which, like in the other main step in the proof, is based on a single basic estimate and straightforward arguments using “local operators”. The estimate that we use for this is a “microlocal Kakeya-Nikodym” estimate going from $L^2(M)$ to $L^{q_c}(M)$ that involves a microlocalized version of the smoothed out spectral projection operators that we ultimately want to control. These microlocal Kakeya-Nikodym $L^2 \to L^{q_c}$ estimates were also used by two of us in [6]; however, they were also paired with an additional superfluous step that involved $L^2 \to L^2$ Kakeya-Nikodym estimates (such as in [5]) that arose in all of the earlier works and was an artifact of the earliest works in this thread by Bourgain [8] and Sogge [19]. Naturally, removing this unnecessary step involving $L^2$–Kakeya-Nikodym bounds leads to improvements over earlier results. We are able to obtain further improvements in two-dimensions by tightening the bilinear estimates in this case versus the ones in this dimension in [6] and earlier works.

Additionally, in some settings we are able to exploit specific geometric features to obtain improved $L^2 \to L^{q_c}$ Kakeya-Nikodym estimates compared to the ones in [6] for general manifolds of non-positive curvatures. These lead to further improvements in certain settings.

When $M$ is a torus, we are able to exploit favorable commutator properties for operators that arise in the second step of the proof to obtain new spectral projection estimates that we show to be sharp. These involve relatively large spectral projection windows compared to those in the earlier works of Bourgain and Demeter [9], Germain and Myerson [13] and Hickman [15] that allowed very thin windows of width about the spectral parameter $\lambda$ that go all the way to the wavelengths $\lambda^{-1}$. To be able to obtain near sharp estimates for these very narrow spectral windows these authors used decoupling which resulted in modest $\lambda^\epsilon$ losses, while, as we mentioned, our new estimates are best possible for the relatively large spectral scales we are able to handle with bilinear techniques.

Also, for manifolds of strictly negative sectional curvatures, we are able to exploit the rapid dispersive properties of wave kernels to obtain improvements of the microlocal
$L^2 \to L^q$ Kakeya-Nikodym estimates compared to the ones we can obtain for the general case of manifolds of non-positive curvature. By exploiting these, we are able, in the case of manifolds of negative curvature to obtain estimates for spectral windows of width $(\log \lambda)^{-1}$ about $\lambda$ which are stronger in two-dimensions than the (sharp) estimates that we obtain for $T^2$ and, in 3-dimensions, that match up with the (sharp) estimates for $T^3$ for this spectral scale. Unfortunately, unlike in the latter setting, we are unable to show that these natural estimates are optimal and we discuss a problem concerning geodesic concentration of log-quasimodes on manifolds manifolds of negative curvature that seems to be related to recent work of Brooks [10] and Eswarathasan and Nonnenmacher [12].

Let us now state our main results. We shall consider smoothed out spectral projection operators of the form

$$\rho(T(\lambda - P)), \quad T = T(\lambda),$$

with $P = \sqrt{-\Delta_g}$ where $\rho$ is a real-valued function satisfying

$$\rho \in S(\mathbb{R}), \quad \rho(0) = 1, \quad \text{and supp} \hat{\rho} \subset \{|t| \in (\delta/2, \delta)\},$$

where $0 < \delta < 1$ will be specified later. For the most part, we shall take

$$T = c_0 \log \lambda,$$

with $c_0 > 0$ ultimately chosen sufficiently small depending on $(M, g)$. For tori, though, we shall take $T$ to be much larger.

Our main result which improves on estimates in [6] then is the following

**Theorem 1.1.** Assume that $(M, g)$ is an $n$-dimensional compact manifold with non-positive sectional curvatures. Then, for $T$ as above,

$$\|\rho(T(\lambda - P)) f\|_{L^q(M)} \lesssim \lambda^{\frac{2}{(n+1)q_c}} (\log \lambda)^{-\sigma_n} \|f\|_{L^2(M)},$$

where $\sigma_n = \frac{2}{(n+1)q_c}$.

This result improves the earlier results of two of the authors in [6] by obtaining a larger power of $\sigma_n$ in all dimensions. For instance, in [6], this exponent was $\frac{4}{3(n+1)q_c} \ll \frac{n-1}{(n+1)q_c}$. The proof of Theorem 1.1 is also much simpler than earlier proofs, and, as we shall see leads to further improvements in certain geometries.

Let us note that if $\varepsilon(\lambda) = (\log \lambda)^{-1}$ then (1.5) immediately implies bounds for the spectral projection operators

$$\chi_{[\lambda - \varepsilon(\lambda), \lambda + \varepsilon(\lambda)]} = 1_{[\lambda - \varepsilon(\lambda), \lambda + \varepsilon(\lambda)]}(P),$$

with $1_I$ denoting the indicator function of the interval $I$. Indeed since $\rho(0) = 1$, by a simple orthogonality argument, Theorem 1.1 yields the following.

**Corollary 1.2.** If $\chi_{[\lambda - \varepsilon(\lambda), \lambda + \varepsilon(\lambda)]}$ is as in (1.6) and $(M, g)$ as above has non-positive curvatures then for $\lambda \geq 1$ and $\sigma_n = 2/(n+1)q_c$

$$(1.7) \quad \|\chi_{[\lambda - (\log \lambda)^{-1}, \lambda + (\log \lambda)^{-1}]} f\|_{L^q(M)} \leq C\lambda^{\frac{2}{(n+1)q_c}} (\log \lambda)^{-\sigma_n} \|f\|_{L^2(M)}.$$
Recall that the universal bounds of one of us [18] say that
\[ \| \chi_{[\lambda-1, \lambda+1]} \|_{L^2(M)} \lesssim \lambda^\mu(q), \quad 2 < q \leq \infty, \]
with
\[ \mu(q) = \max\left(n\left(\frac{1}{q} - \frac{1}{2}\right) - \frac{1}{2}, \frac{n-1}{2} - \frac{1}{2}\right). \]

Since \( \mu(q_c) = 1/q_c \), (1.7) is a log-power improvement of the universal bounds on manifolds of non-positive curvature. We need to use shrinking spectral windows in order to improve the bounds (1.8) which are saturated on any manifold (see [21]); however, for \( S^n \) there is no improvement of (1.7) even if one uses \((\log \lambda)^{-1}\)-spectral windows as in (1.7). So the assumption of non-positive curvature is needed to obtain the improved spectral projection estimates in (1.7).

We also note that, by interpolating with the trivial \( L^2 \rightarrow L^2 \) estimates for the spectral projection operators, (1.7) yields
\[ \| \chi_{(\lambda-(\log \lambda)^{-1}, \lambda+(\log \lambda)^{-1})} f \|_{L^q(M)} \leq \left((\log \lambda)^{-\frac{2}{n}}\lambda\right)^\mu(q) \| f \|_{L^2(M)}, \quad 2 < q \leq q_c, \]
with \( \mu(q) = \frac{n-1}{2} - \frac{1}{2} \) as in (1.9). By using dyadic Sobolev estimates one can also obtain log-power improvements for \( q > q_c \); however, these would not be as strong as the \((\log \lambda)^{-1/2}\) improvements over the universal bounds for supercritical exponents of Bérard [2] and Hassell and Tacy [14].

After we prove Theorem 1.1 we shall see that stronger estimates hold if one assumes all of the sectional curvatures are negative, and that, in this case the ones obtained for three-dimensions have numerology that is related to recent toral spectral projection bounds of Hickman [15] and Germain and Myerson [13], while in two-dimensions are more favorable. We shall also obtain in all dimensions \( n \geq 2 \) optimal spectral projection estimates on \( n \)-dimensional tori for \( \varepsilon(\lambda)\)-windows with \( \varepsilon(\lambda) \) larger than a fixed negative power of \( \lambda \) which depends on the dimension.

Next, to motivate the two main steps in our proofs, let us recall the eigenfunctions on \( S^n \) that saturate the universal estimates (1.7). If we wish to obtain improved bounds, such as those in (1.8) or (1.7), we need to develop techniques that will be able to show that the types of extreme eigenfunctions that exist on standard spheres cannot exist on manifolds of non-positive curvature.

The eigenfunctions that saturate (1.8) on \( S^n \) with eigenvalue \( \lambda = \sqrt{k(k+n-1)} \) are the zonal spherical harmonics of degree \( k \), \( Z_k \). If they are \( L^2 \)-normalized and if \( B_{\pm} (c\lambda^{-1}) \) are geodesic balls of radius \( c\lambda^{-1} \) about the two poles on \( S^n \) then
\[ |Z_k(x)| \approx \lambda^{\frac{n-1}{2}}, \quad x \in B_{\pm} (c\lambda^{-1}), \]
for some sufficiently small uniform constant \( c > 0 \). (See, e.g., [17]). As a result, an easy calculation shows that for some \( c_0 > 0 \) we have the lower bounds
\[ \| Z_k \|_{L^q(S^n)} \geq c_0 \lambda^{\frac{n(\frac{1}{2} - \frac{1}{q})}{2} - \frac{1}{2}}, \quad \text{if } \lambda = \sqrt{k(k+n-1)}. \]
This implies that (1.8) cannot be improved when \( M = S^n \) for the range of exponents for which \( \mu(q) = n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2} \), which is \( q \geq q_c \), where \( q_c \) is as in (1.1).
The other extreme eigenfunctions for the above values of $\lambda$ are the highest weight spherical harmonics of degree $k$, $Q_k$. If $\gamma \subset S^n$ is the equator, and if $T_{\lambda^{-1/2}}(\gamma)$ is a $\lambda^{-1/2}$-tubular neighborhood of $\gamma$, then

$$\|Q_k\|_{L^2(T_{\lambda^{-1/2}}(\gamma))} \approx 1 \quad \text{if} \quad \lambda = \sqrt{k(k+n-1)} \quad \text{and} \quad \|Q_k\|_{L^2(S^n)} = 1.$$  

(See, e.g., [17].) Therefore, as we shall show in §7, it is a simple exercise using Hölder’s inequality to verify that for some uniform $c_0 > 0$

$$\|Q_k\|_{L^q(S^n)} \geq c_0 \lambda^{\frac{n-1}{4}} \left(\frac{1}{2} - \frac{1}{q}\right), \quad 2 < q \leq \infty. \quad (1.12)$$

As a result, (1.12) cannot be improved for $2 < q \leq q_c$ since $\mu(q) = \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{q}\right)$ for such exponents. To further motivate the splitting that we shall use in order to prove Theorem [17], we recall (see [17]) that the lower bounds in (1.12) are actually upper bounds, and, in particular

$$\|Q_k\|_{L^\infty(S^n)} \approx \lambda^{\frac{n-1}{4}}, \quad \text{if} \quad \lambda = \sqrt{k(k+n-1)}. \quad (1.13)$$

We note that both the zonal spherical harmonics, $Z_k$, and the highest weight spherical harmonics, $Q_k$, saturate the universal bounds (1.8) when $q$ equals the critical exponent $q_c$ in (1.1). So estimates involving this exponent are sensitive to both “point concentration” and “geodesic concentration”. The two main steps in the proof of Theorem [17] will rule out these types of extreme concentration under our curvature assumptions, and, based on the arguments from §7 and (1.13), the “height” at which the transition between these types of concentration can occur should be $\lambda^{\frac{n-1}{4}}$.

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We also are very grateful for the helpful suggestions of the referees which significantly improved the exposition.

2. The height splitting.

As in some of our earlier works, we shall employ bilinear techniques that require us to compose the “global operators”

$$\rho_\lambda = \rho(T(\lambda - P)) \quad (2.1)$$

in Theorem [17] with the “local operators”

$$\sigma_\lambda = \rho(\lambda - P). \quad (2.2)$$

Also, as in earlier works, it will be convenient to localize a bit more. To this end, let us write

$$I = \sum_{j=1}^N B_j(x, D), \quad (2.3)$$
where each $B_j \in S^0(M)$ is a standard zero order pseudo-differential with symbol supported in a small conic neighborhood of some $(x_j, \xi_j) \in S^* M$. The size of the support will be described in §4; however, we note now that these operators will not depend on our spectral parameter $\lambda \gg 1$. Also note that if $\beta \in C^\infty_0((1/2, 2))$ is a Littlewood-Paley bump function which equals one near $t = 1$ then the dyadic operators

$$B = B_{j,\lambda} = B_j \circ \beta(P/\lambda)$$

are uniformly bounded on $L^p$, i.e.,

$$\|B\|_{L^p(M) \to L^p(M)} = O(1) \quad \text{for} \quad 1 \leq p \leq \infty.$$  

See e.g., [26, Chapter II] for definitions of the above symbol classes, and the bounds in (2.5) follow from the results in [21, Theorem 3.1.6] if $1 < p < \infty$ and the proof of this result if $p = 1$ or $p = \infty$.

We then shall further localize $\sigma_\lambda$ by setting

$$\tilde{\sigma}_\lambda = B \circ \sigma_\lambda,$$

where $B$ is one of the $N$ operators in (2.4). We also define the “semi-global” operators

$$\tilde{\rho}_\lambda = \tilde{\sigma}_\lambda \circ \rho_\lambda.$$

We note that the $\sigma_\lambda$ can be thought of as “smoothed out” versions of the operators in (1.8). They enjoy the same operator norms and the two sets of estimates are equivalent. Similarly, it is an easy exercise using orthogonality to see that we have the uniform bounds

$$\|(I - \sigma_\lambda) \circ \rho_\lambda\|_{L^2(M) \to L^{q_c}(M)} \leq CT^{-1} \lambda^{-\frac{1}{4}}, \quad \text{if} \quad T \geq 1,$$

and

$$\|\sigma_\lambda - \beta(P/\lambda) \circ \sigma_\lambda\|_{L^2(M) \to L^{q_c}(M)} = O(\lambda^{-N}), \quad \forall \ N.$$

Therefore by (2.3) and (2.8), in order to prove (1.5), it suffices to show that

$$\|\tilde{\rho}_\lambda f\|_{L^{q_c}(M)} \lesssim \lambda^{\frac{1}{8}} (\log \lambda)^{-\sigma_n} \|f\|_{L^2(M)},$$

if $T$ is as in (1.4).

Due to the bilinear arguments that we shall employ, we need to make the height decomposition using the semi-global operators $\tilde{\rho}_\lambda$. We shall always, as we may, assume that the function $f$ in (2.9) is $L^2$-normalized, i.e.,

$$\|f\|_{L^2(M)} = 1.$$

We shall then split our task (2.9) into estimating the $L^{q_c}$-norm of $\tilde{\rho}_\lambda f$ over the two regions,

$$A_+ = \{x \in M : |\tilde{\rho}_\lambda f(x)| \geq \lambda^{-\frac{1}{4}} + \frac{1}{8}\}$$

and

$$A_- = \{x \in M : |\tilde{\rho}_\lambda f(x)| < \lambda^{-\frac{1}{4}} + \frac{1}{8}\},$$

which basically correspond to the height (1.13) of the highest weight spherical harmonics. There is nothing special about the exponent $1/8$ in (2.11) and (2.12). Just as in [6], it could be replaced by any sufficiently small $\delta > 0$ in what follows.
3. Large height estimates.

Let us describe here how to estimate the \( L^{q_c}(A_+) \) norm of \( \tilde{\rho}_\lambda f \). If \( a \in C_0^\infty((-1, 1)) \) equals one on \((-1/2, 1/2)\), then we can do this using the “global estimate”

\[
G_\lambda(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - a(t)) T^{-1} \tilde{\Psi}(t/T) e^{it\lambda} (e^{-itP})(x,y) \, dt = O\left( \frac{\lambda^{-\frac{1}{2}}}{T} \exp(C_0 T) \right), \quad \Psi = \rho^2 1 \lesssim \log \lambda.
\]

This estimate is valid when \((M, g)\) has nonpositive sectional curvatures (see e.g., [2], [14], [22], [20]). One proves (3.1) by standard arguments after lifting the calculation up to the universal cover. Since \( \tilde{\Psi} \) is compactly supported, the number of terms in the sum that arises grows exponentially in \( T \) if \((M, g)\) has negative curvature, and this accounts for the \( \exp(C_0 T) \) factor in (3.1). (See e.g., [20, §3.6] and the related arguments in §5 below). Later in §6 we shall be able to obtain more favorable estimates for tori than those in Theorem 1.1 in part by using the fact that in this setting, the number of terms grows polynomially. Due to this, for tori we shall use a slightly different splitting into the two heights compared to (2.11) and (2.12).

Let us now see that if (3.1) is valid then we have the following.

**Proposition 3.1.** Let \( f \) and \( A_+ \) be as in (2.10) and (2.11) and \( B \) be as in (2.4). Then for \( \lambda \gg 1 \), if \( T = c_0 \log \lambda \) with \( c_0 > 0 \) sufficiently small,

\[
\| \tilde{\rho}_\lambda f \|_{L^{q_c}(A_+)} \leq C \lambda^{\frac{1}{2}} (\log \lambda)^{-\frac{1}{2}} \| f \|_{L^2(M)}.
\]

**Proof.** We first note that, by (2.5) and (2.8), we have

\[
\| \tilde{\rho}_\lambda f \|_{L^{q_c}(A_+)} \leq \| B \rho_\lambda f \|_{L^{q_c}(A_+)} + C \lambda^{\frac{1}{2}} / \log \lambda,
\]

since we are assuming that \( f \) is \( L^2 \)-normalized. Thus, we would have (3.2) if we could show that

\[
\| B \rho_\lambda f \|_{L^{q_c}(A_+)} \leq C \lambda^{\frac{1}{2}} (\log \lambda)^{-1/2} + \frac{1}{2} \| \tilde{\rho}_\lambda f \|_{L^{q_c}(A_+)},
\]

To prove this we shall adapt an argument of Bourgain [7] and simplify related ones in [22] and [6]. As mentioned before, we shall avoid using arguments that utilize the Lorentz space estimates of Bak and Seeger [1], unlike in [22] and [6].

To this end, choose \( g \) such that

\[
\| g \|_{L^{q_c}(A_+)} = 1, \quad \text{and} \quad \| B \rho_\lambda f \|_{L^{q_c}(A_+)} = \int B \rho_\lambda f \cdot (1_{A_+} \cdot g) \, dx.
\]
Then since \( \rho^*_\lambda = \rho_\lambda \) and \( \Psi(T(\lambda - P)) = \rho_\lambda \circ \rho_\lambda \) for \( \Psi \) as in (3.1), by the Schwarz inequality, since we are assuming that \( \|f\|_2 = 1 \) we have

\[
\|B \rho f\|_{L^2(\mathbb{R}^d)}^2 = \left( \int f \cdot (\rho \circ B^*) (1_{A_+} \cdot g)(x) \, dx \right)^2 
\]

\[
\leq \int \| \rho \circ B^* (1_{A_+} \cdot g) \|_2^2 \, dx 
\]

\[
= \int (B \circ \Psi(T(\lambda - P)) \circ B^*) (1_{A_+} \cdot g)(x) \cdot \overline{1_{A_+}(x) g(x)} \, dx 
\]

\[
= \int (B \circ L_\lambda \circ B^*) (1_{A_+} \cdot g)(x) \cdot \overline{1_{A_+}(x) g(x)} \, dx 
\]

\[
+ \int (B \circ G_\lambda \circ B^*) (1_{A_+} \cdot g)(x) \cdot \overline{1_{A_+}(x) g(x)} \, dx 
\]

\[
= I + II, 
\]

with here, and in what follows \( \|h\|_p = \|h\|_{L^p(M)} \). Here, \( G_\lambda \) is the operator whose kernel is as in (3.1), while \( L_\lambda \) is the “local” operator

\[
L_\lambda = (2\pi T)^{-1} \int a(t) \tilde{\Psi}(t/T) e^{it\lambda} e^{-it\rho^*} \, dt. 
\]

Thus, \( L_\lambda h = T^{-1} \sum_j m(\lambda; \lambda_j) E_j h \), where \( E_j \) denotes the projection onto the \( j \)-th eigenspace of \( P \) and the spectral multiplier \( m(\lambda; \lambda_j) = (\bar{u} \ast \Psi_T)(\lambda - \lambda_j) \), satisfies

\[
m(\lambda; \lambda_j) = O((1 + |\lambda - \lambda_j|)^{-N}), \quad T \geq 1, 
\]

for any \( N \). Consequently, by the universal bounds in [18], we have

\[
\|L_\lambda\|_{L^p(M) \to L^q(M)} \lesssim T^{-1} \lambda^{\frac{1}{p}}. 
\]

Since \( T = c_0 \log \lambda \), if we use Hölder’s inequality and (2.2) we find that

\[
|I| \leq BL_\lambda B^* (1_{A_+} \cdot g) \|_{q'_{c}} \cdot \|1_{A_+} \cdot g\|_{q_{c}} 
\]

\[
\lesssim \|L_\lambda B^* (1_{A_+} \cdot g)\|_{q'_{c}} \cdot \|1_{A_+} \cdot g\|_{q_{c}} 
\]

\[
\lesssim \lambda^{\frac{1}{p'}} (\log \lambda)^{-1} \|B^* (1_{A_+} \cdot g)\|_{q'_{c}} \cdot \|1_{A_+} \cdot g\|_{q_{c}} 
\]

\[
\lesssim \lambda^{\frac{1}{p'}} (\log \lambda)^{-1} \|g\|_{L^q(\mathbb{R}^d)}^2 \cdot \|1_{A_+} \cdot g\|_{q_{c}} 
\]

\[
= \lambda^{\frac{1}{p'}} (\log \lambda)^{-1}. 
\]

To estimate \( II \), we choose \( c_0 > 0 \) small enough so that if \( C_0 \) is the constant in (3.1) then

\[
\exp(C_0 T) \leq \lambda^{\frac{1}{2}} \quad \text{if} \quad T = c_0 \log \lambda \quad \text{and} \quad \lambda \gg 1. 
\]

Consequently, (3.1) yields

\[
\|G_\lambda\|_{L^1(M) \to L^\infty(M)} \leq \lambda^{\frac{1}{p'}} + \frac{1}{p'}. 
\]

As a result, since the dyadic operators \( B \) are uniformly bounded on \( L^1 \) and \( L^\infty \), we can repeat the argument that we used to estimate \( I \) to see that

\[
|II| \leq C \lambda^{\frac{1}{p'}} \lambda^{\frac{1}{2}} \|1_{A_+} \cdot g\|_1^2 \leq C \lambda^{\frac{1}{p'}} \lambda^{\frac{1}{2}} \|g\|_{L^q(\mathbb{R}^d)}^2 \|1_{A_+}\|_2^2 = C \lambda^{\frac{1}{p'}} \lambda^{\frac{1}{2}} \|1_{A_+}\|_{q_{c}}^2. 
\]
If we recall the definition (2.11) of $A_+$, we can estimate the last factor:
\[
\|1_{A_+}\|_{q_c}^2 \leq (\lambda^{\frac{n-1}{2} + \frac{1}{2}}) - 2 \|\hat{\rho}_\lambda f\|_{L^{q_c}(A_+)}^2.
\]
Therefore,
\[
\|II\| \lesssim \lambda^{-1/8} \|\hat{\rho}_\lambda f\|_{L^{q_c}(A_+)}^2 \leq \left( \frac{1}{2}\|\hat{\rho}_\lambda f\|_{L^{q_c}(A_+)} \right)^2,
\]
assuming, as we may, that $\lambda$ is large enough.

If we combine this bound with the earlier one, (3.2), for $I$ we conclude that (3.3) is valid which of course yields (3.2) and completes the proof of Proposition 3.1. $\square$

4. Controlling small heights using microlocal Kakeya-Nikodym estimates.

Note that, under our curvature assumption, Proposition 3.1 rules out the existence of eigenfunctions like the zonal spherical harmonics $Z_k$ on $S^n$ that maximally concentrate near points. On the other hand, given (1.19) it does not rule out the existence of eigenfunctions like the highest weight spherical harmonics since the $Q_k$ vanish on $A_+$. To complete the proof of Theorem 1.1 we also need the following result for the complementary set, $A_-$, which does rule out the existence of these types of eigenfunctions with maximal concentration near geodesics.

**Proposition 4.1.** Let $M$ have nonpositive curvature. Then
\[
\|\hat{\lambda}\|_{L^{q_c}(A_-)} \leq C\lambda^{\frac{n}{4}} (\log \lambda)^{-\sigma_n}\|f\|_{L^2(M)},
\]
where $\sigma_n$ is as in Theorem 1.1

Let us collect the tools from [6] that we shall need to prove (4.1). We first recall that the symbol $B(x, \xi)$ of $B$ in (2.3) is supported in a conic neighborhood of some $(x_0, \xi_0) \in S^*M$. We may assume that its symbol has small enough support so that we may work in a coordinate chart $\Omega$ so that $x_0 = 0$, $\xi_0 = (0, 0, \ldots, 0, 1)$ and $g_{jk}(0) = \delta_k^j$ in the local coordinates. So we shall assume that the $x$-support of $B(x, \xi)$ is contained in $\Omega$. We also may assume that $B(x, \xi)$ vanishes when $\xi$ is outside of a small neighborhood of $(0, \ldots, 0, 1)$. These reductions and those that follow will contribute to the number of summands in (2.3); however, it will be clear that the $N$ there will be independent of $\lambda \gg 1$.

The bilinear arguments we shall need to use will involve microlocal cutoffs corresponding to angular sectors of aperture $\approx \lambda^{-1/8}$. To construct them write $x = (x', x_n)$ so that $x' = (x_1, \ldots, x_{n-1})$ denotes coordinates on the hypersurface where $x_n = 0$ in our coordinates. We shall want these cutoffs to commute with the unit-speed geodesic flow $\chi_t : S^*M \rightarrow S^*M$ on the support of $B(x, \xi)$ if $|t| \leq \delta$, where $\delta$ is as in (1.3). Recall that $\chi_t$ is the flow generated by the Hamilton vector field associated with the principal symbol
\[
p(x, \xi) = \left( \sum_{j,k} g^{jk}(x) \xi_j \xi_k \right)^{1/2}
\]
of $P = \sqrt{-\Delta_g}$. Here $g^{jk}(x) = (g_{jk}(x))^{-1}$ is the cometric and so $S^*M = \{(x, \xi) : p(x, \xi) = 1\}$.

If $|t| \leq 2\delta$ with $\delta > 0$ small enough then the map
\[
(t, x', \eta) \rightarrow \chi_t(x', 0, \eta) \in S^*M, \quad (x', 0, \eta) \in S^*M
\]
is a diffeomorphism from a neighborhood of \( x' = 0, t = 0 \) and \( \eta = (0, \ldots, 0, 1) \) to a neighborhood of \((x_0, \xi_0) = (0, (0, \ldots, 0, 1)) \in S^* M\). Indeed, since we are assuming that \( q_{jk}(0) = \delta^j_k \), the Jacobian of the map is the identity at \((x_0, \xi_0)\). After possibly shrinking the support, we may assume that these properties are valid on a neighborhood of \( \supp B(x, \xi) \).

Write the inverse of \((1.2)\) as
\[
S^* M \ni (x, \omega) \mapsto (\tau(x, \omega), \Phi(x, \omega), \Theta(x, \omega)) \in (-\delta, \delta) \times \{ y' \in \mathbb{R}^{n-1} \} \times S^*_0(\Phi(x, \omega), 0) M.
\]
Thus, the unit speed geodesic passing through \((x, \omega) \in S^* \Omega\) arrives at the plane in our local coordinates where \( y_n = 0 \) at \( \Phi(x, \omega) \), has covector \( \Theta(x, \omega) \in S^*_0(\Phi(x, \omega), 0) M \) there, and \( \tau(x, \omega) = d_y(\Phi(x, \omega), 0) \) is the geodesic distance between \( x \) and the point \((\Phi(x, \omega), 0)\) on this plane.

We can now define the microlocal cutoffs that we shall use. First, we let
\[
\nu^\prime = \left( \frac{8}{\nu}, \frac{8}{\nu} \right)
\]
be \( \nu \)-separated set in \( S^{n-1} \) lying in our conic neighborhood of \( \xi_0 = (0, \ldots, 0, 1) \). We then choose a partition of unity \( \sum \nu^\prime = 1 \) on this set which includes a neighborhood of the \( \xi \)-support of \( B(x, \xi) \) so that each \( \beta^\nu \) is supported in a \( 2\nu^\prime \)-cap about \( \nu \in S^{n-1} \), and so that if \( \beta^\nu(\xi) \) is the homogeneous of degree zero extension of \( \beta^\nu \) to \( \mathbb{R}^n \) we have
\[
|D^a \beta^\nu(\xi)| \leq C_\alpha \lambda^{\alpha/8} \quad \text{if } |\xi| = 1.
\]
Finally, if \( \psi \in C^\infty_0(\Omega) \) equals one in a neighborhood of the \( x \)-support of \( B(x, \xi) \), and if \( \beta \in C^\infty_0((0, \infty)) \) equals one in a neighborhood of the support of the Littlewood-Paley bump function in \((2.4)\), we define
\[
q^\nu(x, \xi) = \psi(x) \beta^\nu(\Theta(x, \xi/p(x, \xi))) \tilde{\beta}(p(x, \xi)/\lambda).
\]
It then follows that the pseudo-differential operators \( Q^\nu \) with these symbols belong to a bounded subset of \( S^0_{7/8, 1/8}(M) \). We have constructed these operators so that
\[
q^\nu(x, \xi) = q^\nu(\chi_n(x, \xi)) \quad \text{on } \supp B(x, \xi) \text{ if } |t| \leq 2\delta.
\]
Since
\[
\sigma_\lambda = \frac{1}{2\pi} \int e^{it\lambda} e^{-itP} \tilde{\rho}(t) \, dt
\]
and \( \tilde{\rho}(t) = 0 \), \( |t| \geq \delta \), as noted in \([5, (3.24)]\) one can use Egorov’s theorem and the universal estimates in \([18]\) to see that
\[
\| \tilde{\sigma}_\lambda Q^\nu - Q^\nu \tilde{\sigma}_\lambda \|_{L^2(M) \to L^\infty(M)} \lesssim \lambda^{-\frac{1}{4}}.
\]
In the proof of Theorem \((1.1)\) we only need weaker \( O(\lambda^{\frac{1}{2} - \frac{1}{4}}) \) bounds, which, by a simple exercise, follow from \([4, 5]\), Sobolev estimates and the bounds in \([18]\). If \( \delta > 0 \) in \((1.3)\) is small enough we also have
\[
\tilde{\sigma}_\lambda - \sum \tilde{\sigma}_\lambda Q^\nu = R_\lambda \quad \text{where } \| R_\lambda \|_{L^p(M) \to L^p(M)} = O(\lambda^{-N}) \quad \forall N \text{ if } 1 \leq p \leq \infty.
\]
Finally, the support properties of the symbols imply that we have the uniform bounds
\[
\sum \nu \| Q^\nu h \|_{L^2(M)}^2 \leq C \| h \|_{L^2(M)}^2.
\]
Next, the global estimates that we shall use to prove the small heights bounds in Proposition 4.1 are the following “global" \textit{“}L^\infty\text{-microlocal Kakeya-Nikodym estimates'}:

\begin{equation}
(4.9) \quad \sup_{\nu} \|Q_{\nu} \rho_{\lambda}\|_{L^{2}(M) \rightarrow L^{2c}(M)} \lesssim \lambda^{\frac{1}{\chi}} \cdot (\log \lambda)^{-\frac{1}{\chi}}.
\end{equation}

This, like (3.1), is valid when \((M, g)\) has non-positive sectional curvatures, although in the next section we shall see that stronger bounds hold if the sectional curvatures all assumed to be negative. The inequality (4.9) is a slight improvement over the corresponding estimate (2.18) in [6].

One obtains (4.9) via a simple interpolation argument and the following pointwise microlocalized estimates

\begin{equation}
(4.10) \quad K_{\lambda, \mu}(x, y) = T^{-1} \int \hat{\Psi}(t/T)\beta(t/\mu) e^{i\mu \lambda} \left( Q_{\nu} \circ \cos(tP) \circ Q_{\nu}^{*}\right)(x, y) \, dt \\
= O\left(T^{-1} \mu^{1-\frac{\nu}{\nu+2}} \lambda^\frac{\nu-1}{2}\right), \quad \Psi = (\rho)^2, \quad 1 \leq \mu \leq T = 4c_0 \log \lambda,
\end{equation}

if \(\beta \in C_{0}^{\infty}(1/2, 2)\) is a Littlewood-Paley bump function satisfying \(1 = \sum_{\nu} \beta(t/2^{\nu})\) for \(t > 0\), and \(c_0 > 0\) is sufficiently small. To prove this one uses the Hadamard parametrix after lifting the calculation up to the universal cover. The argument is almost identical to the proof of (3.8) in [5] or the proof of Theorem 2.1 in [6]. The pointwise bounds in (4.10) are more favorable than those in (3.1) since the result of the microlocalization in (4.10) is that number of significant terms in the calculation only grows linearly in \(T\), unlike in the proof of (3.1) where there is exponential growth. For more details, see the remark at the end of §5.

By (4.10), the integral operator with kernel \(K_{\lambda, \mu}\) maps \(L^{1}(M) \rightarrow L^{\infty}(M)\) with norm \(O(T^{-1} \mu^{1-\frac{\nu}{\nu+2}} \lambda^\frac{\nu-1}{2})\). By orthogonality it also maps \(L^{2} \rightarrow L^{2}\) with norm \(O(T^{-1} \mu)\). So by interpolation, it maps \(L^{q_{c}} \rightarrow L^{q_{c}}\) with norm \(O(T^{-1} \mu^{\frac{1}{\nu}} \lambda^\frac{\nu-1}{2})\). By taking \(\mu = 2^{j} \lesssim T\) and adding up these dyadic estimates we find that if \(a\) is as in (3.1) we have that the operator with kernel

\(T^{-1} \int \hat{\Psi}(t/T)(1-a(t)) e^{i\mu \lambda} \left( Q_{\nu} \circ \cos(tP) \circ Q_{\nu}^{*}\right)(x, y) \, dt\)

maps \(L^{q_{c}} \rightarrow L^{q_{c}}\) with norm \(O((\lambda/T)^{\frac{1}{\chi}})\). Finally, since the local estimates from [18] show that when we replace \((1-a(t))\) above by \(a(t)\) the resulting integral operator maps \(L^{q_{c}} \rightarrow L^{q_{c}}\) with a better norm norm \(O(T^{-1} \lambda^{\frac{1}{\chi}})\), we must have

\begin{equation}
(4.11) \quad \sup_{\nu} \|Q_{\nu} \Psi(\lambda - P)Q_{\nu}^{*}\|_{L^{q_{c}}(M) \rightarrow L^{q_{c}}(M)} \lesssim \lambda^{\frac{1}{\chi}} \cdot (\log \lambda)^{-\frac{1}{\chi}},
\end{equation}

which implies (4.9) by a simple \textit{“}TT\textsuperscript{*}\textit{”} argument.

\textbf{Bilinear decomposition}

We also require the (local) bilinear harmonic arguments from [6] that take advantage of the fact that the norm in (4.1) is taken over \(A_{\cdot}\). These techniques go back to the bilinear Fourier restriction arguments from Tao [22], Tao, Vargas and Vega [25], Lee [14], Wolff [27] and others.
We shall write using (2.10) and (4.7)

\[(\bar{\rho}_\lambda f)^2 = \sum_{\nu, \tilde{\nu}} (\bar{\sigma}_\lambda Q_{\nu} h) \cdot (\tilde{\sigma}_\lambda Q_{\tilde{\nu}} h) + O(\lambda^{-N}), \quad h = \rho_\lambda f.\]

We shall organize the pairs of directions \((\nu, \tilde{\nu}) \in S^{n-1} \times S^{n-1}\) just as in [25]. So, let us write \(\nu = (\nu', \nu_n)\) where \(\nu' \in \mathbb{R}^{n-1}\) is near the origin since \(\nu\) is close to \((0, \ldots, 0, 1)\).

To this end, consider the collection of dyadic cubes \(\{\tau_{j}^\nu\}\) in \(\mathbb{R}^{n-1}\) of side length \(2^j\), where \(\tau_{j}^\nu\) denotes the translation of \((0, 2^j)^{n-1}\) by \(\nu' \in 2^j\mathbb{Z}^{n-1}\). Two such cubes are said to be “close” if they are not adjacent but have adjacent parents of sidelength \(2^{j+1}\). If this is the case, we write \(\tau_{j}^\nu \sim \tau_{j}^{\nu'}\). Close cubes are separated by a distance which is comparable to \(2^j\). As noted in [25], any distinct \(\nu', \tilde{\nu}' \in \mathbb{R}^{n-1}\) lie in a unique pair of close cubes. Thus, there is a unique triple \(j, \mu, \mu'\) such that \((\nu', \tilde{\nu}') \in \tau_{j}^\nu \times \tau_{j}^{\nu'}\) and \(\tau_{j}^{\nu'} \sim \tau_{j}^{\mu'}\). Since \(\nu'\) is close to the origin for our \(\nu \in S^{n-1}\), we need only to consider integers \(j \leq 0\).

If we then let \(J \ll 0\) be the integer satisfying \(2^{J-1} < 8\lambda^{-1/8} \leq 2^{J}\), it follows that the sum in (4.12) can be organized as

\[
(4.13) \quad \left( \sum_{J+1 \leq j \leq 0} \sum_{(\nu', \tilde{\nu}') \in \{\tau_{j}^\nu \times \tau_{j}^{\nu'}: \tau_{j}^{\nu'} \sim \tau_{j}^{\mu'}\}} + \sum_{(\nu', \tilde{\nu}') \in \Xi_J} \right) \left( \bar{\sigma}_\lambda Q_{\nu} h \right) \cdot \left( \tilde{\sigma}_\lambda Q_{\tilde{\nu}} h \right),
\]

where \(\Xi_J\) denotes the remaining pairs not included in the first sum. These include diagonal pairs where \(\nu' = \tilde{\nu}'\), and all \((\nu', \tilde{\nu}') \in \Xi_J\) satisfy \(|\nu' - \tilde{\nu}'| \lesssim \lambda^{-1/8}\). Thus, for each fixed \(\nu'\), we have \#\{(\tilde{\nu}' : (\nu', \tilde{\nu}') \in \Xi_J\} = O(1).

As in [6], write

\[
(4.14) \quad \mathcal{Y}^{\text{diag}}(h) = \sum_{(\nu', \tilde{\nu}') \in \Xi_J} \left( \bar{\sigma}_\lambda Q_{\nu} h \right) \cdot \left( \tilde{\sigma}_\lambda Q_{\tilde{\nu}} h \right),
\]

if \(\lambda^{-1/8} \in (2^{J-4}, 2^{J-3}]\), and \(h = \rho_\lambda f\).

We shall estimate the \(L^{q_c/2}\)-norm of this “diagonal” term using results from [6].

Next, if we consider the first sum in the left of (4.13) over \((\nu', \tilde{\nu}') \not\in \Xi_J\) along with the error term in (4.12),

\[
(4.15) \quad \mathcal{Y}^{\text{far}}(h) = \sum_{J+1 \leq j \leq 0} \sum_{(\nu', \tilde{\nu}') \in \{\tau_{j}^\nu \times \tau_{j}^{\nu'}: \tau_{j}^{\nu'} \sim \tau_{j}^{\mu'}\}} \left( \bar{\sigma}_\lambda Q_{\nu} h \right) \cdot \left( \tilde{\sigma}_\lambda Q_{\tilde{\nu}} h \right) + O(\lambda^{-N}),
\]

we have the favorable bilinear estimates

\[
(4.16) \quad \int |\mathcal{Y}^{\text{far}}(h)|^{q/2} \, dx \leq C \Lambda \left(\lambda^N\right)^{n-1/2} q^{q_c} \|h\|^q_{L^2(M)}, \quad \text{if} \quad q \in \left(\frac{2(n+2)}{n}, q_c\right).
\]

For, in the notation of [6], \(\mathcal{Y}^{\text{far}}(h)\) here equals \(\mathcal{Y}^{\text{off}}(h) + \mathcal{Y}^{\text{smooth}}(h)\) together with the error term in (4.12), and so (4.16) follows from summing over \(j \in [J + 1, 0]\) in inequality (4.7) in [6] and the fact that, like the error term in (4.12), \(\|\mathcal{Y}^{\text{smooth}}(h)\|_{q_c/2} \lesssim \lambda^{-N} \|h\|_2^2\) for all \(N\), due to (4.5) in [6]. Based on this, one obtains (4.16). We note that the estimates in [6] for the non-trivial part, \(\mathcal{Y}^{\text{off}}\) of \(\mathcal{Y}^{\text{far}}\) are based on the bilinear oscillatory integral estimates of Lee [16]. The terms \(\mathcal{Y}^{\text{smooth}}\) in [6] were trivial microlocal error terms to allow
us to use those estimates via parabolic scaling. We have opted to lump together these two terms into (4.15) to simplify the notation and argument a tiny bit in what follows.

The significance of (4.16) is that the second power of \( \lambda \) in the right is negative as \( q < q_c \) there. On the other hand, this makes it somewhat awkward to take advantage of this power-improvement to prove the \( L^{q_c} \) bound in our remaining inequality (4.1). However, as in [6], we shall be able to use (4.16) since the norm in the left hand side of (4.1) is taken over \( A_- \). We also note that, we have rewritten the left side of (4.12) as follows

\[
(\tilde{\sigma}_n h)^2 = (\hat{\sigma}_n h)^2 = Y^{\text{diag}}(h) + Y^\text{far}(h), \quad h = \rho_\lambda f.
\]

Next, let us use the fact that we can use (4.16) and (4.17) to prove a slightly stronger version of estimates in [6] that will allow us to prove (4.1). Specifically, as we shall show in the next subsection, we can use (4.16) to obtain the following

\[
(4.18) \quad \|\tilde{\sigma}_n h\|_{L^{q_c}(A_-)} \leq C \lambda^{\frac{1}{q_c}} \lambda^{-\delta_n} \|h\|_{L^2(M)} + C \left( \sum_\nu \|\tilde{\sigma}_\nu Q_\nu h\|_{L^{q_c}(M)}^q \right)^{1/q_c}, \quad h = \rho_\lambda f,
\]

for certain \( \delta_n > 0 \).

Let us now see how we can use (4.18) along with the microlocal Kakeya-Nikodym estimate (4.9) to prove Proposition 4.1.

**Proof of Proposition 4.1.** In this proof, let us express the microlocal log-power gain in (4.19) as

\[
(4.19) \quad \varepsilon(\lambda) = (\log \lambda)^{-\frac{1}{q_c}}.
\]

With \( h = \rho_\lambda f \), we have from (4.18) that for \( n \geq 2 \)

\[
(4.20) \quad \|\tilde{\sigma}_n f\|_{L^{q_c}(A_-)} \leq C \lambda^{\frac{1}{q_c}} (\log \lambda)^{-1} \|f\|_{L^2(M)} + C \left( \sum_\nu \|\tilde{\sigma}_\nu Q_\nu \rho_\lambda f\|_{L^{q_c}(M)}^q \right)^{1/q_c}.
\]

Note that the universal (local) spectral projection bounds from [18] and (2.5) yield

\[
(4.21) \quad \|\tilde{\sigma}_\lambda\|_{L^2 \to L^{q_c}} = O(\lambda^{\frac{1}{q_c}}).
\]

Using this and the fact that the \( Q_\nu \) are almost orthogonal, we find that

\[
(4.22) \quad \sum_\nu \|\tilde{\sigma}_\nu Q_\nu \rho_\lambda f\|_{L^{q_c}(M)}^2 \lesssim \lambda^{\frac{1}{q_c}} \sum_\nu \|Q_\nu \rho_\lambda f\|_{L^{q_c}(M)}^2 \lesssim \lambda^{\frac{1}{q_c}} \|\rho_\lambda f\|_{L^2(M)}^2 \lesssim \lambda^{\frac{1}{q_c}} \|f\|_{L^2(M)}^2.
\]

Next, note that by (4.6) and the fact that \( \|\rho_\lambda\|_{L^2 \to L^2} = O(1) \), we have

\[
(4.23) \quad \|\tilde{\sigma}_\nu Q_\nu \rho_\lambda f\|_{L^{q_c}(M)} \lesssim \|Q_\nu \sigma_\nu f\|_{L^{q_c}(M)} + \lambda^{\frac{1}{q_c}} \|f\|_{L^2(M)}.
\]

Additionally, \( Q_\nu \tilde{\sigma}_\nu = Q_\nu B \sigma_\lambda = BQ_\nu \sigma_\lambda + [Q_\nu, B] \sigma_\lambda \), and since the commutator \([Q_\nu, B]\) is in \( S_{7/8,1/8}^{-3/4} \) and band limited (see [26, Theorem 4.4, Chapter II]) we have \( \|[Q_\nu, B]\|_{L^{q_c} \to L^{q_c}} = O(\lambda^{-\frac{1}{8}}) \). So, by (4.21) and (2.5)

\[
(4.24) \quad \|Q_\nu \tilde{\sigma}_\nu f\|_{L^{q_c}(M)} \lesssim \|BQ_\nu \sigma_\nu f\|_{L^{q_c}(M)} + \lambda^{\frac{1}{q_c}} \|f\|_{L^2(M)} \lesssim \|Q_\nu \sigma_\nu f\|_{L^{q_c}(M)} + \lambda^{\frac{1}{q_c}} \|f\|_{L^2(M)}.
\]
If we combine (4.23) and (4.24), since the microlocal cutoffs $Q_\nu$ are uniformly bounded on all $L^p$ spaces, we conclude that

$$\|\hat{\sigma}_\lambda \hat{Q}_\nu \hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} \lesssim \|Q_\nu \hat{\sigma}_\lambda \hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} + \lambda^{\frac{1}{q_c}-\frac{1}{2}} \|f\|_{L^2(\mathcal{M})}$$

$$\lesssim \|Q_\nu \hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} + \|Q_\nu \circ (I - \hat{\sigma}_\lambda) \hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} + \lambda^{\frac{1}{q_c}-\frac{1}{2}} \|f\|_{L^2(\mathcal{M})}$$

$$\lesssim \|Q_\nu \hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} + \|(I - \hat{\sigma}_\lambda) \hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} + \lambda^{\frac{1}{q_c}-\frac{1}{2}} \|f\|_{L^2(\mathcal{M})}.$$ 

If we use the first part of (5.3) and our Kakeya-Nikodym bounds (4.14) to estimate the first two terms in the right we conclude that

$$\|\hat{\sigma}_\lambda Q_\nu \hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} \lesssim \lambda^{\frac{1}{q_c}} \varepsilon(\lambda) \|f\|_{L^2(\mathcal{M})},$$

with $\varepsilon(\lambda)$ as in (4.19).

If we combine (4.20), (4.22) and (4.24), we obtain

$$\|\hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} \lesssim \sup_{\nu} \|\hat{\sigma}_\lambda Q_\nu \hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} \cdot \left( \sum_{\nu} \|\hat{\sigma}_\lambda Q_\nu \hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})}^2 \right)^{\frac{1}{2}} + \lambda^{\frac{1}{q_c}} (\log \lambda)^{-1} \|f\|_{L^2(\mathcal{M})}$$

$$\lesssim \left( \lambda^{\frac{1}{q_c}} \varepsilon(\lambda) \right)^{\frac{2}{q_c}} (\log \lambda)^{\frac{1}{2}} \|f\|_{L^2(\mathcal{M})} + \lambda^{\frac{1}{q_c}} (\log \lambda)^{-1} \|f\|_{L^2(\mathcal{M})}$$

$$\lesssim \lambda^{\frac{1}{q_c}} \varepsilon(\lambda)^{\frac{2}{q_c}} \|f\|_{L^2(\mathcal{M})}$$

$$= \lambda^{\frac{1}{q_c}} \varepsilon(\lambda)^{\frac{2}{q_c}} \|f\|_{L^2(\mathcal{M})}.$$ 

This combined with (4.22) yields

$$\|\hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} \lesssim \lambda^{\frac{1}{q_c}} \varepsilon(\lambda)^{\frac{2}{q_c}} \|f\|_{L^2(\mathcal{M})},$$

since $(\varepsilon(\lambda))^{\frac{2}{q_c}} > (\log \lambda)^{-1/2}$. This gives the bounds (4.13) in Theorem 1.1 if one recalls (4.14).

\section*{Proving the bilinear estimates}

Let us prove (4.13) by using estimates from (5.6) along with (4.14).

As in these inequalities $\hat{\rho}_\lambda f = \hat{\sigma}_\lambda h$, with $h = \hat{\rho}_\lambda f$. So, we need to control

$$\|\hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})} = \left( \int_{\{x: |\hat{\sigma}_\lambda h(x)| > \lambda^{\frac{1}{q_c}+\frac{1}{2}} \}} |\hat{\sigma}_\lambda h|^2 \right)^{1/2}.$$ 

If we recall (4.17) then of course we have for $q \in \left( \frac{2(n+2)}{n}, q_c \right)$ as in (4.16)

$$|\hat{\sigma}_\lambda h \hat{\sigma}_\lambda h|^{q_c/2} \leq 2^{q_c/2} |\hat{\sigma}_\lambda h \hat{\sigma}_\lambda h|^{\frac{2-q_c}{q_c}} \left( |U^\text{diag}(h)|^{q/2} + |U^\text{far}(h)|^{q/2} \right).$$

Thus,

$$\|\hat{\rho}_\lambda f\|_{L^{q_c}(\mathcal{M})}^{q_c} = \int_{\mathcal{M}} \left( \int_{\mathcal{M}} |\hat{\sigma}_\lambda h \hat{\sigma}_\lambda h|^{\frac{2-q_c}{q_c}} \right)^{q_c/2} \left( |U^\text{diag}(h)|^{q/2} + |U^\text{far}(h)|^{q/2} \right)^{q_c/2} = \mathcal{I} + \mathcal{II}.$$
To estimate the second term we use (4.16), the ceiling for $A_-$ and the fact that $\|f\|_2 = 1$ to see that
\[
II \lesssim \|\tilde{\sigma}_A h\|^{q_e-q}_{L^{\infty}(A_-)} \lambda^{-(q_e-q)(\frac{n-1}{2}\frac{n+1}{2})}\cdot \lambda \leq \left( \lambda^{\frac{n-1}{2}+\frac{1}{2}} \right)^{(q_e-q)} \cdot \lambda^{-(q_e-q)(\frac{n-1}{2}\frac{n+1}{2})} \cdot \lambda = \lambda^{-(q_e-q)(\frac{3n-1}{16} - \frac{1}{n})} \cdot \lambda = \lambda^{1-\delta_n}, \quad \delta_n > 0,
\]
since $(q_e - q) \cdot \left( \frac{3n-1}{16} - \frac{1}{n} \right) > 0$.

To estimate $I$, we use the fact that by Lemma 4.2 in [6],
\[
\|Y_{\text{diag}}(h)\|_{L^{q_e/2}(M)} \lesssim \left( \sum_{\nu} \|\tilde{\sigma}_A Q_{\nu} h\|_{L^{q_e}(M)}^{2\gamma^*} \right)^{\frac{1}{2\gamma^*}} + \lambda^{-N},
\]
where $q^* = q_e/2$ if $n \geq 3$ and $q^* = 3/2$ if $n = 2$. Thus, by Hölder’s inequality followed by Young’s inequality
\[
I \leq \|\tilde{\sigma}_A h\|^{q_e-q}_{L^{q_e}(A_-)} \cdot \|Y_{\text{diag}}\|^{q_e/2}_{L^{q_e/2}(M)} \\
\leq \frac{2q_e}{q_e-1} \|\tilde{\sigma}_A h\|^{q_e/2}_{L^{q_e}(A_-)} + \frac{4q_e}{q_e} \|Y_{\text{diag}}\|^{q_e/2}_{L^{q_e/2}(M)} \\
\leq \frac{2q_e}{q_e-1} \|\tilde{\sigma}_A h\|^{q_e/2}_{L^{q_e}(A_-)} + \left( \sum_{\nu} \|\tilde{\sigma}_A Q_{\nu} h\|_{L^{q_e}(M)}^{2\gamma^*} \right)^{\frac{1}{2\gamma^*}} + \lambda^{-N}.
\]
Since $\frac{q_e-1}{q_e} < 1$, the first term in the right can be absorbed in the left side of (4.27).

Thus, our bounds imply that for $n \geq 3$
\[
\|\tilde{\sigma}_A f\|_{L^{q_e}(A_-)} \lesssim \left( \sum_{\nu} \|\tilde{\sigma}_A Q_{\nu} h\|_{L^{q_e}(M)}^{2\gamma^*} \right)^{\frac{1}{2\gamma^*}} + \lambda^{\frac{1}{q_e}-\delta_n/q_e} \|f\|_{L^2(M)},
\]
and for $n = 2$
\[
\|\tilde{\sigma}_A f\|_{L^{q_e}(A_-)} \lesssim \left( \sum_{\nu} \|\tilde{\sigma}_A Q_{\nu} h\|_{L^{q_e}(M)}^{2\gamma^*} \right)^{\frac{1}{2}} + \lambda^{\frac{1}{q_e}-\delta_n/q_e} \|f\|_{L^2(M)}.
\]

Note that the first term on the right side of (4.29) is larger than the one appearing in (4.18) due to the inclusion $\ell^3 \subset \ell^3$. In the next subsection we shall see how we can modify the arguments we used for term $I$ to get an improved bilinear type inequality rather than (4.29) when $n = 2$.

**Modified bilinear arguments for $n = 2$**

To be able to have a variant of (4.29) which involves an $\ell^6$ sum instead of $\ell^3$, we need to change the definition of $Q_{\nu}$ a bit. When $n = 2$, instead of $\frac{1}{8}$, we fix a small $\varepsilon_0$, which will be specified later, and let $\nu$ denote a $\lambda^{-\varepsilon_0}$-separated set in $S^1$ lying in our conic neighborhood of $\xi_0 = (0, \ldots, 0, 1)$. We then choose a partition of unity $\sum_{\nu} \beta_{\nu}(\xi) = 1$ on this set which includes a neighborhood of the $\xi$-support of $B(x, \xi)$ so that each $\beta_{\nu}$ is supported in a $2\lambda^{-\varepsilon_0}$ cap about $\nu \in S^1$, and so that if $\beta_{\nu}(\xi)$ is the homogeneous of degree zero extension of $\beta_{\nu}$ to $\mathbb{R}^n$\setminus0 we have
\[
|D^\alpha \beta_{\nu}(\xi)| \leq C_\alpha \lambda^{|\alpha|\varepsilon_0} \quad \text{if } |\xi| = 1.
\]
If we let $g_{\nu}(x, \xi)$ then be defined as in (4.3), it follows that the pseudo-differential operators $Q_{\nu}$ with symbols belong to a bounded subset of $S^0_{1-\varepsilon_0, \xi_0}(M)$. And similar to (4.6),
we can use the arguments in [6], which involve applications of Egorov’s theorem and the universal bound in [18] to see that

\[(4.30) \quad \|\hat{\sigma}_\lambda Q_\nu - Q_\nu \hat{\sigma}_\lambda\|_{L^2(M)} \lesssim \lambda^{\frac{\nu}{2} + 2\varepsilon_0},\]

which is better than the estimate in (4.6) if ε_0 is small since q_c = 6 when n = 2.

It is straightforward to check that (4.7) and (4.8) hold for the Q_\nu operators defined as above, and by repeating the previous arguments, we also have the analog of (4.9), as long as we let T = c_0 \log \lambda with c_0 \ll \varepsilon_0. These estimates ensure that the arguments (4.19)-(4.26), which is the proof of Proposition 4.1, still work in our current setting.

We are choosing the \lambda^{-\varepsilon_0} scale instead of \lambda^{-\frac{\nu}{2}}, since in our later arguments, we shall make explicit use of the fact that the number of choices of \nu is small. See the remark below (4.33) for more details.

Write

\[(\hat{\rho}_\lambda h)^2 = \gamma_{\text{diag}}(h) + \gamma_{\text{far}}(h),\]

where \gamma_{\text{far}}(h) is defined as in (4.15), and

\[(4.31) \quad \gamma_{\text{diag}}(h) = \sum_{(\nu', \nu') \in \Xi_J} (\hat{\sigma}_\lambda Q_\nu h) \cdot (\hat{\sigma}_\lambda Q_\nu h),\]

if \lambda^{-\varepsilon_0} \in (2^{-4}, 2^{-3}], and \ h = \rho_\lambda f.

Here as before, \nu = (\nu', \nu_n), and the definition of \Xi_J is the same as the one in (4.13), which include diagonal pairs where \nu' = \nu', and all (\nu', \nu') \in \Xi_J satisfy |\nu' - \nu'| \leq 32 \lambda^{-\varepsilon_0}.

Thus, for each fixed \nu', we have \#\{\nu' : (\nu', \nu') \in \Xi_J\} = O(1).

By (4.27), to prove improved version of (4.29), it suffices to control the term I involving \gamma_{\text{diag}}(h). The other term II involving \gamma_{\text{far}}(h) can be handled like before and it actually satisfies better bound since we have improved bilinear estimates with the wider angular separation for the off-diagonal terms here. To proceed, let us define

\[T_{\nu, h} = \sum_{\nu' : (\nu', \nu') \in \Xi_J} (\hat{\sigma}_\lambda Q_\nu h)(\hat{\sigma}_\lambda Q_\nu h),\]

and write

\[(4.32) \quad (\gamma_{\text{diag}}(h))^2 = (\sum_{\nu} T_{\nu, h})^2 \quad + \quad \sum_{|\nu_1 - \nu_2| \geq 128 \lambda^{-\varepsilon_0}} T_{\nu_1, h} T_{\nu_2, h} + \sum_{|\nu_1 - \nu_2| \leq 128 \lambda^{-\varepsilon_0}} T_{\nu_1, h} T_{\nu_2, h} = A + B.\]

Then for q \in (4.6), we have

\[(4.33) \quad |\gamma_{\text{diag}}(h)|^{\eta/2} \lesssim \sum_{|\nu_1 - \nu_2| \geq 128 \lambda^{-\varepsilon_0}} T_{\nu_1, h} T_{\nu_2, h}^{\frac{\eta}{4}} \quad + \quad \sum_{|\nu_1 - \nu_2| \leq 128 \lambda^{-\varepsilon_0}} T_{\nu_1, h} T_{\nu_2, h}^{\frac{\eta}{4}} = |A|^{\frac{\eta}{4}} + |B|^{\frac{\eta}{4}}.\]
By splitting \((T^{\text{diag}}(h))^2\) as a sum of \(A\) and \(B\) as above, we essentially want to make use of the bilinear estimates for a second time, where \(A\) corresponds to the off-diagonal terms and \(B\) corresponds to the diagonal ones. However, for the off-diagonal terms, here we are not using the classical Whitney-type decomposition as in \((4.12)\)–\((4.13)\) since the \(T_{\nu}\) operators themselves are defined as product of the \(Q_{\nu}\) operators, which makes it difficult to define the \(\sum_{\nu}T_{\nu}\) operators for certain collection of nearby \(\nu\) indexes. Instead, we assume that the number of choices of \(\nu\) is small, which can be achieved by choosing \(\varepsilon_0\) to be small enough, so that after applying bilinear estimates for each of the single term in \(A\), we can sum them up in a rather crude but still sufficient way. We shall also use the fact that when \(n = 2\), \(q_c = 6\) and thus \(\frac{6}{2} \in (1,2)\), which allows us to follow the strategies in \([6]\) to control the diagonal term \(B\) using an almost orthogonal type inequality (see \((4.37)\) below).

More explicitly, to estimate the term involving \(A\) in \(I\), let us first assume that \(\nu_1, \nu_2\) are fixed, with \(|\nu'_1 - \nu'_2| \geq 128\lambda^{-\varepsilon_0}\). By the Cauchy-Schwarz inequality and the remarks below \((4.31)\), it is easy to see that

\[
|T_{\nu_1}hT_{\nu_2}h| \leq C \left( \sum_{\nu'_1: |\nu'_{1} - \nu'_2| \leq 32\lambda^{-\varepsilon_0}} |\tilde{\sigma}_{\lambda}Q_{\nu_1}h|^2 \right) \left( \sum_{\nu'_2: |\nu'_{2} - \nu'_1| \leq 32\lambda^{-\varepsilon_0}} |\tilde{\sigma}_{\lambda}Q_{\nu_2}h|^2 \right)
\]

Since the number of terms in both summations are bounded independently of \(\lambda\), it suffices to control a single term involving \(|\tilde{\sigma}_{\lambda}Q_{\nu_1}h\tilde{\sigma}_{\lambda}Q_{\nu_2}h|^2\), where \(|\nu'_1 - \nu'_2| \approx 2^j \in (32\lambda^{-\varepsilon_0}, 1)\).

To proceed, let \(\mu_1, \mu_2 \in S^1\) satisfy \(|\mu_1 - \mu_2| \approx 2^j\), and each be associated with an operator \(Q_{\mu_1}, Q_{\mu_2}\), where \(Q_{\mu_1}\) is the pseudo-differential operator with symbol \(q_{\mu_1}(x,\xi)\), and we assume \(q_{\mu_1}(x,\xi)\) is supported in a \(2^j\) conical neighborhood of \(\mu\), with \(q_{\mu_1}(x,\xi) \equiv 1\) in a neighborhood of the support of \(Q_{\nu_1}\). It is not hard to see that

\[
\|(I - Q_{\mu_1}) \circ Q_{\nu_1}\|_{L^2 \to L^2} \lesssim_N \lambda^{-N}, \quad \forall \ N > 0.
\]

Thus, if we define \(Q_{\mu_2}\), similarly, we have

\[
|\tilde{\sigma}_{\lambda}Q_{\nu_1}h\tilde{\sigma}_{\lambda}Q_{\nu_2}h|^2 = |\tilde{\sigma}_{\lambda}Q_{\mu_1}Q_{\nu_1}h\tilde{\sigma}_{\lambda}Q_{\mu_2}Q_{\nu_2}h|^2 + O(\lambda^{-N}).
\]

Now if we apply bilinear estimates at the scale \(2^j\) in \(L^q\) (see e.g., \((4.12)\) in \([6]\)), along with the fact that \(\tilde{\sigma}_{\lambda}h\) is bounded by \(\lambda^{\frac{-q}{2} j + \frac{q}{4}}\) (with \(n = 2\)) on the set \(A_{-}\), we have, similar to before,

\[
(4.34) \quad \int_{A_{-}} |\tilde{\sigma}_{\lambda}h \tilde{\sigma}_{\lambda} h|^\frac{q}{4} \left| (\tilde{\sigma}_{\lambda}Q_{\mu_1}Q_{\nu_1}h)^2(\tilde{\sigma}_{\lambda}Q_{\mu_2}Q_{\nu_2}h)^2 \right|^\frac{q}{4} \leq \lambda^{1 - \delta_2},
\]

for some fixed \(\delta_2 > 0\). Here the \(\lambda^{-\delta_2}\) gain is independent of \(\varepsilon_0\) and \(j\), whenever we have \(2^j \geq \lambda^{-\delta}\).

Since the number of choices of \(\nu'_1, \nu'_2\) in the sum in \(A\) is bounded by \(\lambda^{2\varepsilon_0}\), \((4.34)\) implies

\[
(4.35) \quad \int_{A_{-}} |\tilde{\sigma}_{\lambda}h \tilde{\sigma}_{\lambda} h|^{\frac{q}{4}} |A|^\frac{q}{4} \lesssim \lambda^{1 + 3\varepsilon_0 - \delta_2} + \lambda^{-N}.
\]

By choosing \(\varepsilon_0\) small enough so that \(3\varepsilon_0 \leq \delta_2/2\), this gives us the desired bound.
To control the term involving $B$, let us define

$$S_{\nu_i} h = \sum_{\nu_i' : |\nu_i' - \nu_i| \leq 128 \lambda^{-\varepsilon}} T_{\nu_i'} h T_{\nu_i} h$$

(4.36)

$$= \sum_{\nu_i', \nu_i', \nu_i'} (\tilde{\sigma}_\lambda Q_{\nu_i'} h)(\tilde{\sigma}_\lambda Q_{\nu_i'} h)(\tilde{\sigma}_\lambda Q_{\nu_i'} h)(\tilde{\sigma}_\lambda Q_{\nu_i'} h),$$

where $|\nu_i' - \nu_i| \leq 32 \lambda^{-\varepsilon}$, $|\nu_i' - \nu_i| \leq 32 \lambda^{-\varepsilon}$, $|\nu_i' - \nu_i| \leq 128 \lambda^{-\varepsilon}$, and thus $|\nu_i' - \nu_i| \leq 160 \lambda^{-\varepsilon}$. Overall, the number of terms in the second line is bounded independently of $\lambda$, and we have $B = \sum_{\nu_i} S_{\nu_i} h$. If we use a similar argument as in the proof of Lemma 4.2 in [6], it is not hard to show the following almost orthogonal type inequality

$$\| \sum_{\nu_i} S_{\nu_i} h \|_{L^{3/2}(M)} \lesssim \left( \sum_{\nu_i} \|S_{\nu_i} h\|_{L^{3/2}(M)}^2 \right)^{\frac{2}{3}} + \lambda^{-N},$$

(4.37)

which implies that

$$\| \sum_{\nu_i} S_{\nu_i} h \|_{L^{3/2}(M)} \lesssim \left( \sum_{\nu_i} \|\tilde{\sigma}_\lambda Q_{\nu_i} h\|_{L^6(M)}^6 \right)^{\frac{1}{3}} + \lambda^{-N},$$

(4.38)

where in the second inequality we essentially used the fact that for fixed $\nu_i$, the number of terms in the sum is finite.

Thus, by Hölder’s inequality followed by Young’s inequality

$$\int_{A_-} |\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h|^\frac{6-\varepsilon}{2} |B|^{\varepsilon} \leq \|\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h\|_{L^6(A_-)}^\frac{6-\varepsilon}{2} \|B\|_{L^{3/2}(M)}^\varepsilon \leq \frac{6-\varepsilon}{6} \|\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h\|_{L^6(A_-)}^3 + \frac{\varepsilon}{6} \|B\|_{L^{3/2}(M)}^3 \leq \frac{6-\varepsilon}{6} \|\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h\|_{L^6(A_-)}^3 + \sum_{\nu_i} \|\tilde{\sigma}_\lambda Q_{\nu_i} h\|_{L^6(M)}^6 + \lambda^{-N}.$$

Since $\frac{6-\varepsilon}{6} < 1$, the first term in the right can be absorbed in the left side of (4.27).

Thus, the proof of (4.18) for the case $n = 2$ is complete.

5. **Further improvements for negatively curved manifolds.** As we shall see in the next subsection, if we assume that *all* of the sectional curvatures of $(M, g)$ are negative, then we can improve the second global estimate, (4.19), that we used to

$$\sup_P \|Q_P \rho\|_{L^2(M) \to L^{\varepsilon c}(M)} \lesssim \lambda^{\frac{1}{\varepsilon c}} \cdot \varepsilon(\lambda), \quad \text{with} \quad \varepsilon(\lambda) = (\log \lambda)^{-1/2.}$$

(5.1)

If one repeats the above arguments, one finds that this implies that we can therefore improve Theorem 1.1 under this curvature assumption as follows.
Theorem 5.1. Assume that \((M, g)\) is an \(n\)-dimensional compact manifold with all sectional curvatures being negative. Then, if \(T = c_0 \log \lambda\) with \(c_0 > 0\) sufficiently small, we have
\[
\|\rho(T) f\|_{L^{q_c}(M)} \lesssim \lambda^{1/q_c} (\log \lambda)^{-\alpha_n} \|f\|_2,
\]
where \(\alpha_n = \frac{1}{n+1}\).

Note that when \(n = 2\), \(q_c = 6\), and so (5.2) says that
\[
\|\rho \lambda f\|_{L^6(M)} \lesssim \lambda^{1/6} (\log \lambda)^{-1/3} \|f\|_{L^2(M)}, \quad n = 2,
\]
while when \(n = 3\), \(q_c = 4\), and so (5.2) says that
\[
\|\rho \lambda f\|_{L^4(M)} \lesssim \lambda^{1/4} (\log \lambda)^{-1/4} \|f\|_{L^2(M)}, \quad n = 3.
\]

Estimates (5.3) and (5.4) should be compared to bounds in Hickman [15] and Germain and Myerson [13] who extended the toral eigenfunction estimates of Bourgain and Demeter [9]. As we shall see in §7, the three-dimensional estimates (5.4) would be optimal if \(M\) were a product manifold with \(S^1\) as a factor; however, in this case, \(M\) cannot have all sectional curvatures negative, since some would have to equal zero. Perhaps not unexpectedly, in two-dimensions, the bounds in (5.3) are stronger than the sharp estimates that we have obtained for two-dimensional tori for spectral windows of width \((\log \lambda)^{-1}\). It would be interesting to determine whether or not this phenomena persists in higher dimensions.

Proof of \(L^2 \to L^{q_c}\) Kakeya-Nikodym estimates assuming negative curvature

The main step in the proof of (5.1) will be to establish certain pointwise estimates using an argument that is almost identical one in [5]. Basically, the only difference is that, here, we shall exploit the fact that, if all the sectional curvatures of \((M, g)\) are negative, then the leading term in the Hadamard parametrix for the solution of the wave equation in the universal cover of \((M, g)\) decays exponentially in terms of the time parameter. This fact was not used by two of us in [5] or its predecessor, Sogge and Zelditch [23], since we were interested in results for general manifolds of non-positive curvature and the techniques in these papers or [6] only provided somewhat modest improvements for spectral projection estimates.

Turning to the proof of (5.1), just as in [5], we shall want to make use of the fact that if \(Q_\nu(x, y)\) denotes the kernels of our microlocal cutoffs there, then we have the uniform bounds
\[
\sup_x \int |Q_\nu(x, y)| \, dy \leq C.
\]
This implies that \(Q_\nu : L^\infty(M) \to L^\infty(M)\) uniformly, and, since they are also uniformly bounded on \(L^2(M)\), by interpolation and duality, we conclude that
\[
\|Q_\nu\|_{L^p(M) \to L^p(M)} = O(1), \quad 2 \leq p \leq \infty
\]
\[\text{and} \quad \|Q_\nu^*\|_{L^p(M) \to L^p(M)} = O(1), \quad 1 \leq p \leq 2.
\]

Also, of course if, as in (4.10), \(\Psi = \rho^2\), then (5.1) is equivalent to the estimate
\[
\|Q_\nu \Psi(T(\lambda - P))Q_\nu^*\|_{L^{q_c}(M) \to L^{q_c}(M)} \lesssim \lambda^{2/q_c} (\log \lambda)^{-1},
\]
if, as before,
\begin{equation}
T = c_0 \log \lambda, \tag{5.8}
\end{equation}
with $c_0 > 0$ sufficiently small.

By (5.6), this would be a consequence of
\begin{equation}
\| Q_\nu \Psi(T(\lambda - P)) \|_{L^{q_c}(M) \to L^{q_c}(M)} \lesssim \lambda^{2/q_c} (\log \lambda)^{-1}. \tag{5.9}
\end{equation}
The proof of this will closely follow the related $L^2 \to L^2$ Kakeya-Nikodym estimate (3.6) in [5].

First, as was done in the proof of Proposition 3.1, let us split our smoothed out spectral projection operators $\Psi(T(\lambda - P))$ into the sum of a “local” and “global” part. This time, since we shall want to dyadically decompose the “global” piece, let us assume, as before, that our Littlewood-Paley function $\beta \in C_0^\infty \left( (1/2, 2) \right)$ satisfies
\[ \sum_{j=-\infty}^{\infty} \beta \left( \frac{t}{2^j} \right) = 1, \quad t > 0. \]

We then let
\begin{equation}
\beta_0(t) = 1 - \sum_{j=0}^{\infty} \beta \left( \frac{j}{2^j} \right) \in C_0^\infty (\mathbb{R}_+),
\end{equation}
and note that $\beta_0(t) \equiv 1$ for $t > 0$ near the origin. The “local” operator then is
\begin{equation*}
L_\lambda = \frac{1}{2\pi T} \int e^{i\lambda t} e^{-itP} \beta_0(|t|) \hat{\Psi}(t/T) \, dt,
\end{equation*}
and we have
\begin{equation*}
\| Q_\nu L_\lambda \|_{L^{q_c}(M) \to L^{q_c}(M)} = O(\lambda^{2/q_c} (\log \lambda)^{-1})
\end{equation*}
using (5.6), (5.7) and the universal local spectral projection estimates in [18] just as we did in the proof of Proposition 3.1.

As a result of these local bounds, we would have (5.9) and be done if we could show that
\begin{equation}
\| Q_\nu G_\lambda \|_{L^{q_c}(M) \to L^{q_c}(M)} = O(\lambda^{2/q_c} (\log \lambda)^{-1}), \tag{5.10}
\end{equation}
if
\begin{equation}
G_\lambda = \frac{1}{2\pi T} \int e^{i\lambda t} e^{-itP} \left( 1 - \beta_0(|t|) \right) \hat{\Psi}(t/T) \, dt. \tag{5.11}
\end{equation}
Note that if we replace $e^{-itP}$ by $e^{itP}$ here, then the resulting operator has a kernel which is $O(\lambda^{-N})$ and so trivially enjoys the bounds in (5.10).

Consequently, by Euler’s formula, it suffices to show that if
\begin{equation}
\tilde{G}_\lambda = \frac{1}{\pi T} \int e^{i\lambda t} \cos(t \sqrt{-\Delta_g}) \left( 1 - \beta_0(|t|) \right) \hat{\Psi}(t/T) \, dt, \tag{5.12}
\end{equation}
then
\begin{equation}
\| Q_\nu \tilde{G}_\lambda \|_{L^{q_c}(M) \to L^{q_c}(M)} = O(\lambda^{2/q_c} (\log \lambda)^{-1}). \tag{5.13}
\end{equation}
If we let for $\mu = 2^j \geq 1$
\begin{equation}
\tilde{G}_{\lambda,\mu} = \frac{1}{\pi T} \int e^{i\lambda t} \cos(t \sqrt{-\Delta_g}) \beta(|t|/\mu) \hat{\Psi}(t/T) \, dt, \tag{5.14}
\end{equation}
then since
\[
\tilde{G}_\lambda = \sum_{\{\mu = 2^j, j \geq 0\}} \tilde{G}_{\lambda,\mu},
\]
in order to prove (5.13), it suffices to show that
\[
(5.15) \quad \|Q_{\nu} \tilde{G}_{\lambda,\mu}\|_{L^q_c(M) \to L^q_c(M)} \leq C \lambda^{2/q} (\log \lambda)^{-1} \mu^{-1}.
\]
We note that since \(\hat{\Psi}\) is compactly supported, by Huygens’ principle, \(\tilde{G}_{\lambda,\mu} = 0\) if \(\mu\) is larger than a fixed multiple of \(T\) as in (5.8).

As before, we shall prove (5.15) using interpolation. To this end, we first note that since the Fourier transform of
\[
t \to T^{-1} \beta(|t|/\mu) \hat{\Psi}(t/T)
\]
is \(O(\mu/T)\), by (5.6) and orthogonality, we have
\[
(5.16) \quad \|Q_{\nu} \tilde{G}_\lambda\|_{L^2(M) \to L^2(M)} = O(\mu/T).
\]
The other estimate that we shall use, which requires our curvature assumption, is that
\[
(5.17) \quad \|Q_{\nu} \tilde{G}_\lambda\|_{L^1(M) \to L^\infty(M)} \leq C_{K,N} T^{-1} \lambda^{\frac{1}{2}} \mu^{-N}, \quad N = 1, 2, \ldots,
\]
if all of the sectional curvatures of \((M, g)\) are assumed to be \(\leq -K^2 < 0\). This is analogous to (3.8′) in [5] which did not include the rapid decay \(\mu^{-N}\) since the curvatures there were merely assumed to be non-positive. Also, there, we did not break things up dyadically as we have done so here.

By a simple interpolation argument (as in the proof of (4.9) above), (5.16) and (5.17) imply (5.15). So, we have reduced matters to establishing (5.17).

Just as in [5] and other works, we have switched from the half-wave operator in (5.11) to the use of \(\cos t \sqrt{-\Delta_g}\) so that we can use the Hadamard parametrix and the Cartan-Hadamard theorem to lift the calculations that will be needed for (5.17) up to the universal cover \(\mathbb{R}^n, \tilde{g}\) of \((M, g)\). This approach was used earlier in [5] and [23].

To this end, let \(\{\alpha\} = \Gamma\) denote the group of deck transformation preserving the associated covering map \(\kappa : \mathbb{R}^n \to M\) coming from the exponential map at the point in \(M\) with coordinates \(0\) in \(\Omega\) above. The metric \(\tilde{g}\) on \(\mathbb{R}^n\) that we mentioned above is the pullback of \(g\) via \(\kappa\). Choose a Dirichlet domain \(D \simeq M\) for \(M\) centered at the lift of the point with coordinates \(0\).

Following [2], [23] and others, we recall that if \(\tilde{x}\) denotes the lift of \(x \in M\) to \(D\), then we have the following formula
\[
(\cos t \sqrt{-\Delta_{\tilde{g}}})(x, y) = \sum_{\alpha \in \Gamma} (\cos t \sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y})),
\]
which is a cousin of the classical Poisson summation formula. As a result, for later use, if \(K_{\lambda,\mu}(x, y)\) denotes the kernel of \(\tilde{G}_{\lambda,\mu}\) then we have the formula
\[
(5.18) \quad K_{\lambda,\mu}(x, y) = \sum_{\alpha \in \Gamma} K_{\lambda,\mu}(\tilde{x}, \alpha(\tilde{y})),
\]
if \(K_{\lambda,\mu}(\tilde{x}, \tilde{y}) = \frac{1}{\lambda} \int e^{i\lambda t} (\cos t \sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \tilde{y}) \beta(|t|/\mu) \hat{\Psi}(t/T) dt\).
The key estimate for us, which is an improvement of (3.8) in [3] and requires our curvature assumption is that

\[ |K_{\lambda,\mu}(\tilde{x}, \tilde{y})| \leq C_{K,N}T^{-1} \lambda^{\frac{n+1}{2}} (d_{\tilde{g}}(\tilde{x}, \tilde{y}))^{-\frac{n+1}{2}} \mu^{-N}, \quad N = 1, 2, \ldots. \tag{5.19} \]

To prove this, we note that this kernel vanishes if \( d_{\tilde{g}}(\tilde{x}, \tilde{y}) \) is larger than \( 2\mu \) by the Huygens principle since \( \beta(|t|/\mu) = 0 \) for \(|t| \geq 2\mu \). Also, since this function also vanishes for \(|t| \leq \mu/2 \), it is straightforward to use the Hadamard parametrix to see that \( K_{\lambda,\mu}(\tilde{x}, \tilde{y}) = O(\lambda^{-N}) \) for every \( N \) if \( d_{\tilde{g}}(\tilde{x}, \tilde{y}) \leq \mu/4 \), which is better than the bounds in (5.19) as \( \mu \lesssim \log \lambda \ll \lambda \).

Due to these simple facts, to prove (5.19), we may assume that \( d_{\tilde{g}}(\tilde{x}, \tilde{y}) \approx \mu \). We shall make use of the Hadamard parametrix for \( \cos t\sqrt{-\Delta_{\tilde{g}}} \). As is well known and described, for instance, in [2], [20] and [23], the leading term in this parametrix is of the form

\[ w_0(\tilde{x}, \tilde{y}) \cdot (2\pi)^{-n} \int_{\mathbb{R}^n} e^{id_{\tilde{g}}(\tilde{x}, \tilde{y})\xi_1} \cos(t|\xi|) \, d\xi, \tag{5.20} \]

where in geodesic normal coordinates about \( \tilde{x} \) this leading coefficient is given by

\[ w_0(\tilde{x}, \tilde{y}) = \left( \det \tilde{g}_{ij}(\tilde{y}) \right)^{-1/4}. \tag{5.21} \]

Thus, if in geodesic polar coordinates the volume element is given by

\[ dV_{\tilde{g}}(\tilde{y}) = (A(t, \omega))^{n-1} \, dt \, d\omega, \quad t = d_{\tilde{g}}(\tilde{x}, \tilde{y}), \]

then

\[ w_0(\tilde{x}, \tilde{y}) = \left( t/A(t, \omega) \right)^{n-1}. \]

By the classical Günther comparison theorem from Riemannian geometry (see [11 §III.4])

\[ A(t, \omega) \geq \frac{1}{K} \sinh(Kt), \tag{5.22} \]

if, as we are assuming all the sectional curvatures are \( \leq -K^2 < 0 \). Thus,

\[ w_0(\tilde{x}, \tilde{y}) \leq C_{K,N} \mu^{-N} \quad \text{if} \quad d_{\tilde{g}}(\tilde{x}, \tilde{y}) \approx \mu. \tag{5.23} \]

Also, a routine stationary phase calculation yields

\[ T^{-1} \int e^{i\lambda} e^{id_{\tilde{g}}(\tilde{x}, \tilde{y})\xi_1} \cos(t|\xi|) \beta(|t|/\mu) \tilde{\Psi}(t|T|) \, dt \, d\xi \]

\[ = O(T^{-1} \lambda^{\frac{n-1}{2}} (d_{\tilde{g}}(\tilde{x}, \tilde{y}))^{-\frac{n-1}{2}}), \quad \text{if} \quad d_{\tilde{g}}(\tilde{x}, \tilde{y}) \approx \mu \geq 1. \tag{5.24} \]

Due to (5.23) and (5.24), if we replace \( \cos t\sqrt{-\Delta_{\tilde{g}}}(\tilde{x}, \tilde{y}) \) in (5.18) with the leading term (5.20) in its Hadamard parametrix, the resulting expression will satisfy the bounds in (5.19). Since one can also use stationary phase to see that the lower order terms in the Hadamard parametrix (see, e.g., [23] Theorem 2.4.1) will each make contributions which are \( O(\lambda^{\frac{n-3}{2}}) \) which is better than the bounds posited in (5.19), we obtain this bound.

Unfortunately, although (5.19) is sufficient for obtaining bounds like (5.17), it, by itself, can not be sufficient for proving (5.17) since there are \( \exp(c\mu) \) terms in the sum in (5.18).
with \(d_\beta(x,\tilde{y}) \approx \mu\). To prove (5.17), as in [5], we need to use the fact that (5.17) includes the microlocal cutoff which means that the kernel of \(Q_\nu \hat{G}_{\lambda,\mu}\),

\[
\sum_{\alpha \in \Gamma} \frac{1}{\pi T} \int e^{i\lambda t} \left( Q_\nu \cos t \sqrt{-\Delta_{\beta}} \right)(\tilde{x}, \alpha(\tilde{y})) \beta(|t|/\mu) \hat{\Psi}(t/T) \, dt,
\]

only involves \(O(\mu)\) non-trivial terms. In the above \(Q_\nu\) denotes the pullback of the operator \(Q_\nu\) to the fundamental domain \(D\).

To see this we shall argue exactly as in [5]. First recall that through the point \(x \in \Omega\) there is a unique geodesic \(\gamma_{x,\nu}\) which passes through the plane \(z_n = 0\) with covector \(\xi_{\nu}\) at this point in the plane. After modifying the coordinates in \(\Omega\), we may assume that \(z' = 0\) at the intersection point in the plane. Then, as in [5], we let \(\tilde{\gamma}(t), t \in \mathbb{R}\) denote the lift of the projection of the geodesic \(\gamma\) in \(\Omega\) to the universal cover and

\[
T_R(\tilde{\gamma}) = \{ \tilde{z} : d_\beta(\tilde{\gamma}, \tilde{z}) \leq R \}
\]

denote an \(R\)-tube about \(\tilde{\gamma}\). Then, just as in [5], if \(R\) is fixed sufficiently large and \(\alpha(D) \cap T_R(\tilde{\gamma}) = \emptyset\), the corresponding summand in (5.25) is \(O(\lambda^{-1})\) by Toponogov’s theorem and microlocal arguments. Indeed, this is exactly how (3.9) in [5] was proved and one can simply repeat the arguments there. Since there are \(O(\lambda^{1/2})\) non-zero terms in (5.25) if \(c_0\) in (5.8) is fixed small enough, we obtain in this case

\[
\sum_{\{\alpha : \alpha(D) \cap T_R(\tilde{\gamma}) = \emptyset\}} \frac{1}{\pi T} \int e^{i\lambda t} \left( Q_\nu \cos t \sqrt{-\Delta_{\beta}} \right)(\tilde{x}, \alpha(\tilde{y})) \beta(|t|/\mu) \hat{\Psi}(t/T) \, dt = O(\lambda^{-1/2}).
\]

Therefore, the operator with this kernel maps \(L^1(M)\) to \(L^\infty(M)\) with norm \(O(\lambda^{-1/2})\), which is much better than the bounds in (5.17) as \(\mu \leq \log \lambda\).

Since (5.17) is equivalent to the statement that the kernel of \(Q_\nu \hat{G}_{\lambda,\mu}\) has sup-norm bounded by the right side of (5.17), it suffices to show that the remaining part of the sum in (5.25) where \(\alpha(D) \cap T_R(\tilde{\gamma}) \neq \emptyset\) can be bounded by the right side of (5.17). As above, the terms where \(d_\beta(x, \alpha(\tilde{y}))\) is not comparable to \(\mu\) yield \(O(\lambda^{-N})\) contributions and so can be ignored. Thus, since, by (5.6), \(Q_\nu : L^\infty \to L^\infty\), our task now would be complete if we could show that

\[
\sum_{\{\alpha : \alpha(D) \cap T_R(\tilde{\gamma}) \neq \emptyset, \text{ and } d_\beta(x, \alpha(\tilde{y})) \approx \mu\}} \frac{1}{T} \left| \int e^{i\lambda t} \left( \cos t \sqrt{-\Delta_{\beta}} \right)(\tilde{x}, \alpha(\tilde{y})) \beta(|t|/\mu) \hat{\Psi}(t/T) \, dt \right| \leq C_{K,N} T^{-1} \lambda^{-N} \mu^{-N}, \quad N = 1, 2, \ldots.
\]

It is easy to see directly or by arguments in [5], this sum only involves \(O(\mu)\) terms. By (5.19), each of these is \(O(T^{-1} \lambda^{-N} \mu^{-N})\) for any \(N\), which yields (5.26) and completes the proof.

Remark: This argument also yields the microlocal Kakeya-Nikodym estimate (4.19) in which we were merely assuming that \((M, g)\) has non-positive curvature. In this case, the leading coefficient of the Hadamard parametrix is no longer rapidly decreasing. So, we cannot use (5.23). However, we do have that \(w_0\) is bounded also by the Günther comparison theorem. If we repeat the above arguments, we would see that, up to an \(O(T^{-1} \lambda^{-N} \mu^{-N})\) error, the kernels \(K_{\lambda,\mu}\) in (4.19) would be dominated by the left side.
of (6.20). Since \( w_0 \) is bounded, by (6.24), each summand is \( O(T^{-1} \lambda^{\frac{n-1}{2}} \mu^{-\frac{n-1}{2}}) \) since \( d_\beta(\hat{x}, \alpha(\hat{y})) \approx \mu \) in the sum. Since there are \( O(\mu) \) terms, we conclude that the left side of (6.20) under the assumption of non-positive curvature is \( O(T^{-1} \lambda^{\frac{n-1}{2}} \mu^{1-\frac{n-1}{2}}) \), and so we obtain (4.10).

6. Some sharp estimates on tori.

In this section, we shall see how we can modify the previous arguments to get the following sharp improved spectral projection bounds on tori.

**Theorem 6.1.** Let \( n \geq 2 \), \( \mathbb{T}^n = \mathbb{R}/(\ell_1 \mathbb{Z}) \times \cdots \times \mathbb{R}/(\ell_n \mathbb{Z}) \) denote the rectangular torus with \( \ell_i \geq 1 \). Then,

\[
\|\rho(T(\lambda - P))f\|_{L^{q_c}(\mathbb{T}^n)} \lesssim \lambda^{1/q_c} T^{-1/q_c} \|f\|_2, \quad 1 \leq T \leq \lambda^{\frac{n-1}{2} - \varepsilon}.
\]

Here \( \varepsilon \) is an arbitrary small constant, and for convenience we are assuming \( \ell_i \geq 1 \), which implies that the injectivity radius of the torus is bounded below by \( \frac{1}{2} \). The implicit constants in (6.1) may depend on \( \ell_i \) and \( \varepsilon \), but do not depend on \( T \) and \( \lambda \). As we shall see later in Proposition 6.2, the \( T^{-1/q_c} \) gain in the above estimate is sharp.

To prove (6.1), if we define \( \hat{\rho}_\lambda \) as in (2.4), by arguing as before, it suffices to show that

\[
\|\hat{\rho}_\lambda f\|_{L^{q_c}(\mathbb{T}^n)} \leq \lambda^{1/q_c} T^{-1/q_c} \|f\|_2.
\]

**Proposition 6.2.** Let

\[ A_+ = \{x \in \mathbb{T}^n : |\hat{\rho}_\lambda f(x)| \geq C(T\lambda)^{\frac{n-1}{2}} \} \subset \mathbb{T}^n, \]

where \( C \) is a large enough constant which may depend on \( \ell_i \). Then,

\[
\|\hat{\rho}_\lambda f\|_{L^{q_c}(A_+)} \leq C \lambda^{\frac{n}{2}} T^{-1/2} \|f\|_{L^2(\mathbb{T}^n)}.
\]

**Proof.** We first note that, by (2.5) and (2.8), we have

\[
\|\hat{\rho}_\lambda f\|_{L^{q_c}(A_+)} \leq \|B\rho_\lambda f\|_{L^{q_c}(A_+)} + C\lambda^{\frac{n}{2}}/T,
\]

since we are assuming that \( f \) is \( L^2 \)-normalized. Thus, we would have (6.3) if we could show that

\[
\|B\rho_\lambda f\|_{L^{q_c}(A_+)} \leq C\lambda^{\frac{n}{2}} T^{-1/2} + \frac{1}{2} \|\hat{\rho}_\lambda f\|_{L^{q_c}(A_+)}. \tag{6.4}
\]

If we repeat the argument in (3.3), by duality,

\[
\|B\rho_\lambda f\|_{L^{q_c}(A_+)}^2 \leq \int (B \circ L_\lambda \circ B^*)(1_{A_+} \cdot g)(x) \cdot \overline{1_{A_+}(x)g(x)} \, dx
\]

\[
+ \int (B \circ G_\lambda \circ B^*)(1_{A_+} \cdot g)(x) \cdot \overline{1_{A_+}(x)g(x)} \, dx
\]

\[
= I + II.
\]

Here, \( G_\lambda \) is the operator whose kernel is in (3.1), while \( L_\lambda \) is the “local” operator

\[ L_\lambda = (2\pi T)^{-1} \int a(t) \hat{\Psi}(t/T) e^{it\lambda} e^{-itP} \, dt. \]

As before the local operator \( L_\lambda \) satisfies

\[
\|L_\lambda\|_{L^{q_c}(\mathbb{T}^n) \to L^{q_c}(\mathbb{T}^n)} \lesssim T^{-\frac{1}{2}/q_c}.
\]
By Hölder’s inequality, we have
\[
|I| \leq \|B_L \lambda B^*(1_{A_+} \cdot g)\|_{q_e} \cdot \|1_{A_+} \cdot g\|_{q'_e} \\
\lesssim \|L \lambda B^*(1_{A_+} \cdot g)\|_{q_e} \cdot \|1_{A_+} \cdot g\|_{q'_e} \\
\lesssim \lambda^{2/\eta} T^{-1} \|B^*(1_{A_+} \cdot g)\|_{q'_e} \cdot \|1_{A_+} \cdot g\|_{q'_e} \\
\lesssim \lambda^{2/\eta} T^{-1} \|g\|_{L_{\infty}^q(\Lambda_+)}^2 \\
= \lambda^{2/\eta} T^{-1}.
\]

To handle the second term, we shall use the fact that if \(\beta \in C_0^\infty((1/2, 2))\) and
\[
G^1_\lambda(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - a(t)) \beta(t/2^j) T^{-1} \hat{\psi}(t/T) e^{it\lambda} (e^{-itP})(x, y) dt,
\]
then the integral operator with this kernel maps \(L^1 \to L^\infty\) with norm \(O(T^{-1} 2^{\frac{n-1}{2}} \lambda^{-\frac{n-1}{2}})\).

To see this, note that if we replace \(e^{-itP}\) above by \(e^{itP}\), the kernel is \(O(\lambda^{-N})\) for all \(N > 0\). Thus by Euler’s formula, it suffices to show that for any fixed \(x, y\)
\[
| \int_{-\infty}^{\infty} (1 - a(t)) \beta(t/2^j) T^{-1} \hat{\psi}(t/T) e^{it\lambda} (\cos(tP))(x, y) dt | = O(T^{-1} 2^{\frac{n-1}{2}} \lambda^{-\frac{n-1}{2}}).
\]

We postpone the proof of this to \((6.2)-(6.4)\) below.

As a result, since the dyadic operators \(B\) are uniformly bounded on \(L^1\) and \(L^\infty\), we can repeat the argument that we used to estimate \(I\) to see that
\[
\|G_\lambda\|_{L^1(\mathbb{T}^n) \to L^\infty(\mathbb{T}^n)} \leq \sum_{1 \leq 2^j \leq T} \|G^1_\lambda\|_{L^1(\mathbb{T}^n) \to L^\infty(\mathbb{T}^n)} \lesssim (T \lambda)^{-\frac{n-1}{2}}.
\]

Then \((6.7)\) yields
\[
|II| \leq C_0(T \lambda)^{-\frac{n-1}{2}} \|1_{A_+} \cdot g\|_1^2 \leq C_0(T \lambda)^{-\frac{n-1}{2}} \|g\|_{q_e}^2 \cdot \|1_{A_+}\|_{q_e}^2 = C_0(T \lambda)^{-\frac{n-1}{2}} \|1_{A_+}\|_{q_e}^2.
\]

If we recall the definition of \(A_+\) and use Chebyshev’s inequality we can estimate the last factor:
\[
\|1_{A_+}\|_{q_e}^2 \leq (C(T \lambda)^{-\frac{n-1}{2}})^{-2} \|\tilde{\rho}_\lambda f\|_{L_{q_e}(\Lambda_+)}^2.
\]

By choosing the constant \(C\) large enough such that \(C^{-2} C_0 \leq \frac{1}{4}\), we have,
\[
|II| \leq \frac{1}{4} \|\tilde{\rho}_\lambda f\|_{L_{q_e}(\Lambda_+)}^2.
\]

If we combine this bound with the earlier one, \((6.3)\), for \(I\) we conclude that \((6.3)\) is valid. \(\square\)

**Proposition 6.3.** Let
\[
A_- = \{ x \in \mathbb{T}^n : |\tilde{\rho}_\lambda f(x)| \leq C(T \lambda)^{-\frac{n-1}{2}} \} \subset \mathbb{T}^n.
\]

Then, for any fixed \(\varepsilon > 0\) we have
\[
\|\tilde{\rho}_\lambda f\|_{L_{q_e}(A_-)} \leq C \lambda^{\frac{n-1}{2}} T^{-1/q_e} \|f\|_{L^2(\mathbb{T}^n)}, \quad \forall 1 \leq T \leq \lambda^{\frac{n-1}{2}} \varepsilon.
\]
In order to make use of the bilinear estimates, we shall work in the microlocalized modes. First, let $\nu$ range over a $T^{-1}$-separated set in $S^{n-1}$ lying in a neighborhood of $\xi_0 = (0, \ldots, 0, 1)$. We then choose a partition of unity $\sum \beta_{\nu}(\xi) = 1$ on this set which includes a neighborhood of the $\xi$-support of $B(x, \xi)$ so that each $\beta_{\nu}$ is supported in a $2T^{-1}$ cap about $\nu \in S^{n-1}$, and so that if $\beta_{\nu}(\xi)$ is the homogeneous of degree zero extension of $\beta_{\nu}$ to $\mathbb{R}^n \backslash 0$ we have

$$|D^\alpha \beta_{\nu}(\xi)| \leq C_{\alpha} T^{\alpha} \quad \text{if} \quad |\xi| = 1.$$  

And as in (4.3), we define the symbols

$$q_{\nu}(x, \xi) = \psi(x) \beta_{\nu} \left( \Theta(x, \xi/p(x, \xi)) \right) \hat{\beta}(p(x, \xi)/\lambda).$$

where $\psi \in C^\infty_0(\Omega)$ equals one in a neighborhood of the $x$-support of $B(x, \xi)$, and $\hat{\beta} \in C^\infty_0((0, \infty))$ equals one in a neighborhood of the support of the Littlewood-Paley bump function in (2.4). In particular, in a local coordinate chart of $\mathbb{T}^n$, we always have $p(x, \xi) = |\xi|$, which is independent of $x$. Also recall that as in (4.3), $\Theta(x, \xi/p(x, \xi))$ is the third component function of the inverse of the map

$$(t, x'), (\xi), (t, 0, \eta) \in S^* M, \quad (x', 0, \eta) \in S^* M,$$

in a neighborhood of $x' = 0, t = 0$ and $\eta = (0, \ldots, 0, 1)$. And on the flat torus $\mathbb{T}^n$, we always have $\chi_{(x', 0, \eta)} = (x' + t\eta', \eta)$ for $(x, x', \eta)$ in this small neighborhood, thus $\Theta(x, \xi/p(x, \xi)) = \xi/|\xi|$, which is also independent of $x$. It then follows that, by rescaling, the pseudo-differential operators $Q_{\nu}$ with these symbols have Schwartz kernel

$$Q_{\nu}(x, y) = \frac{\lambda^n}{(2\pi)^n} \int e^{i\lambda(x-y, \xi)} \psi(x) \beta_{\nu}(\xi/|\xi|) \hat{\beta}(|\xi|) d\xi,$$

which satisfies

$$\sup_x \int |Q_{\nu}(x, y)| dy \leq C.$$  

This implies that $Q_{\nu} : L^\infty(M) \to L^\infty(M)$ uniformly, and, since they are also uniformly bounded on $L^2(M)$, by interpolation and duality, we conclude that

$$\|Q_{\nu}\|_{{L^p(M) \to L^p(M)}} = O(1), \quad 2 \leq p \leq \infty$$

and

$$\|Q_{\nu}\|_{{L^p(M) \to L^p(M)}} = O(1), \quad 1 \leq p \leq 2.$$  

For later use, we shall record several other properties the of $Q_{\nu}$ operators. First, as in (4.6) and (4.30), we can use the arguments in [8] to see that

$$\|\tilde{\sigma}_\lambda Q_{\nu} - Q_{\nu} \tilde{\sigma}_\lambda\|_{{L^2(M) \to L^2(M)}} \lesssim \lambda^{\frac{n}{4} - \frac{1}{2}} T^2,$$

Also (4.14) and (4.15) hold for the $Q_{\nu}$ operators defined in (6.12), as long as $\delta > 0$ in (4.1) is small enough. That is

$$\tilde{\sigma}_\lambda - \sum_{\nu} \tilde{\sigma}_\lambda Q_{\nu} = R_\lambda$$

where $\|R_\lambda\|_{{L^p(M) \to L^p(M)}} = O(\lambda^{-N}) \quad \forall \ N$ if $1 \leq p \leq \infty$, and

$$\sum_{\nu} \|Q_{\nu} h\|_{{L^2(M)}}^2 \leq C \|h\|_{{L^2(M)}}^2.$$
Furthermore, as in (4.9), we have the following “global” “$L^\infty$-microlocal Kakeya-Nikodym estimates”:

\begin{equation}
\sup_{\nu} \|Q_{\nu} \rho_{\lambda}\|_{L^2(T^n) \to L^\infty(T^n)} \lesssim_{\delta_0} \lambda^{\frac{1}{2\nu}} \cdot T^{-\frac{1}{\nu}}, \quad \forall 1 \leq T \leq \lambda^{1-\delta_0}.
\end{equation}

Here $\delta_0$ is a fixed constant that can be arbitrarily small, the condition $T \leq \lambda^{1-\delta_0}$ ensures that $\lambda^{-1}T \leq \lambda^{-\delta_0}$, which implies that the $Q_{\nu}$ operator is rapidly decreasing outside a $\lambda^{-\delta_0}$ neighborhood of the diagonal. The estimate (6.18) is analogous to (4.9) where $T = c_0 \log \lambda$ there, while here in the torus case, we can take $T$ to be almost as large as $\lambda$.

If $\Psi = \rho^2$, then (6.18) is equivalent to the estimate

\begin{equation}
\|Q_{\nu} \bar{\Psi}(T(\lambda - P))Q_{\nu}^*\|_{L^\infty(M) \to L^\infty(M)} \lesssim \lambda^{2/\nu} T^{-\frac{2}{\nu}}.
\end{equation}

By (6.14), this would be a consequence of

\begin{equation}
\|Q_{\nu} \bar{\Psi}(T(\lambda - P))\|_{L^\infty(M) \to L^\infty(M)} \lesssim \lambda^{2/\nu} T^{-\frac{2}{\nu}}.
\end{equation}

If we repeat the arguments in (5.9), in order to prove (6.20), it suffices to show that

\begin{equation}
\|Q_{\nu} \tilde{G}_{\lambda,\mu}\|_{L^\infty(M) \to L^\infty(M)} \leq C \lambda^{2/\nu} T^{-1} \mu^{\frac{1}{2\nu}},
\end{equation}

where for $\mu = 2^j \geq 1$

\begin{equation}
\tilde{G}_{\lambda,\mu} = \frac{1}{\pi T} \int e^{i\lambda t} \cos(t\sqrt{-\Delta_y}) \beta(|t|/\mu) \Psi(t/T) dt,
\end{equation}

since, after adding up the dyadic estimates for $\mu = 2^j \leq T$, this gives us the right side of (6.20). And as before, the other operators involving $L_{\lambda}$ and the difference $G_{\lambda} - \tilde{G}_{\lambda}$ satisfy better estimates.

As in (5.16), by using (6.14), it is not hard to see that

\begin{equation}
\|Q_{\nu} G_{\lambda}\|_{L^2(M) \to L^2(M)} = O(\mu/T).
\end{equation}

Thus, by interpolation, (6.21) would be a consequence of

\begin{equation}
\|Q_{\nu} G_{\lambda}\|_{L^2(M) \to L^\infty(M)} \lesssim T^{-1} \lambda^{\frac{1}{2\nu}} \mu^{1-\frac{1}{2\nu}}.
\end{equation}

As before, we prove (6.24) by lifting the calculations up to the universal cover, where in our current case the universal cover is just $\mathbb{R}^n$ with the standard metric. More explicitly, we shall use the fact that

\begin{equation}
(\cos t \sqrt{-\Delta_{\mathbb{R}^n}})(x, y) = \sum_{j \in \mathbb{Z}^n} (\cos t \sqrt{-\Delta_{\mathbb{R}^n}})(x - (y + j_\ell)),
\end{equation}

if $j_\ell = (\ell_1 j_1, \ell_2 j_2, \ldots, \ell_n j_n)$ for $j = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$. This also follows from the classical Poisson summation formula. Here we are abusing notation a bit by viewing $x \in T^n$ as a point in $\mathbb{R}^n$, since the torus can always be identified with the cube $Q = [-\ell_1/2, \ell_1/2] \times \cdots \times [-\ell_n/2, \ell_n/2] \subset \mathbb{R}^n$, which is a fundamental domain for the covering. As a result, if $K_{\lambda,\mu}(x, y)$ denotes the kernel of $G_{\lambda,\mu}$ then we have the formula

\begin{equation}
K_{\lambda,\mu}(x, y) = \sum_{j \in \mathbb{Z}^n} \tilde{K}_{\lambda,\mu}(x, (y + j_\ell)),
\end{equation}
if

$$\tilde{K}_{\lambda, \mu}(x, y) = \frac{1}{\pi T} \int e^{i\lambda \left(\cos t \sqrt{\Delta_{\mathbb{R}^n}}\right)(x, y) \beta(|t|/\mu)} \tilde{\Psi}(t/T) \, dt.$$  

(6.27)

$$= \frac{1}{\pi T} \cdot (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\lambda (x-y, \xi)} (\cos t |\xi|) \beta(|t|/\mu) \tilde{\Psi}(t/T) \, dt \, d\xi.$$  

Note that by finite speed of propagation, the above kernel vanishes if $|x - y| \geq 2\mu$, since $\beta(|t|/\mu) = 0$ for $|t| \geq 2\mu$. Also, since this function also vanishes for $|t| \leq \mu/2$, by integrating by parts, it is straightforward to check that $\tilde{K}_{\lambda, \mu}(x, y) = O(\lambda^{-N})$ for every $N$ if $|x - y| \leq \mu/4$. Thus, we may assume that $|x - y| \approx \mu$. In this case, by using a stationary phase argument, we have

$$\tilde{K}_{\lambda, \mu}(x, y) = \sum_{\pm} T^{-1} \lambda^{-\frac{n+1}{2}} \mu^{-\frac{n-1}{2}} e^{\pm i\lambda|x-y|} a_{\pm}(\lambda, |x-y|) + O(\lambda^{-N})$$

if $|x - y| \approx \mu \geq 1$, where for $j \geq 0$, $|\partial_r^j a_{\pm}(\lambda, r)| \leq C_r r^{-j}$, $r \geq \lambda^{-1}$.

Since for fixed $x, y$ the number of choices of $j_t$ such that $|x - (y + j_t)| \approx \mu$ is $O(\mu^n)$. By summing up all possible choices of $j_t$, (6.28) also implies that

$$|K_{\lambda, \mu}(x, y)| \lesssim T^{-1} \lambda^{-\frac{n-1}{2}} \mu^{-\frac{n+1}{2}},$$

which is equivalent to $\|\tilde{G}_{\lambda, \mu}\|_{L^1 \to L^\infty} = O(T^{-1} \lambda^{-\frac{n-1}{2}} \mu^{-\frac{n+1}{2}})$.

Note that the above estimate is not sufficient for proving (6.24). To prove (6.24), as in the negative curvature case, we need to use the fact that (6.24) includes the microlocal cutoff which means that the kernel of $Q_{\nu, \tilde{G}}_{\lambda, \mu}$

$$\sum_{j \in \mathbb{Z}^n} \frac{1}{\pi T} \int e^{i\lambda (Q_{\nu} \cos t \sqrt{-\Delta_{\mathbb{R}^n}})(x, y + j_t) \beta(|t|/\mu)} \tilde{\Psi}(t/T) \, dt$$

(6.30)

$$= \sum_{j \in \mathbb{Z}^n} K_{\lambda, \mu, \nu}(x, (y + j_t))$$

only involves $O(\mu)$ non-trivial terms for $\mu \gg 1$. In the above $Q_{\nu}$ denotes the pullback of the operator $Q_{\nu}$ to the fundamental domain $Q$, which has the same Schwartz kernel as in (6.12).

More explicitly, for fixed $x, y$ and $\mu$, if we define

$$D_{\text{main}} = \left\{ j \in \mathbb{Z}^n : \pm \frac{x - (y + j_t)}{|x - (y + j_t)|} - \nu \right\} \leq C \mu^{-1}, \quad |x - (y + j_t)| \approx \mu \},$$

and

$$D_{\text{error}} = \left\{ j \in \mathbb{Z}^n : \pm \frac{x - (y + j_t)}{|x - (y + j_t)|} - \nu \right\} \geq C \mu^{-1}, \quad |x - (y + j_t)| \approx \mu \},$$

for some large enough constant $C$. Then the contributions from the terms $j \in D_{\text{error}}$ in (6.30) is negligible. To see this, note that by (6.12) and (6.28), modulo $O(\lambda^{-N})$ errors, the kernel of $K_{\lambda, \mu, \nu}(x, y)$ is

$$C_{T, \lambda, \mu} \int e^{i(\lambda z, \xi) + i\lambda |z-y|} a_{\pm}(\lambda, |z-y|) \psi(x) \tilde{\psi}(z) \beta_{\nu}(\xi/|\xi|) \tilde{\beta}(|\xi|) d\xi dz,$$
where \( C_{T,\lambda,\mu} = (2\pi)^{-n} \lambda^{\frac{n-1}{2}} T^{-1} \mu^{-\frac{n+1}{2}} \) and \( \tilde{\psi} \in C_0^\infty(\Omega) \) equals one in a neighborhood of the \( x \)-support of \( \psi \). Here we introduced the extra the cut-off function \( \tilde{\psi}(z) \), which is allowed since as mentioned before, the \( Q_\nu \) operator is rapidly decreasing outside a \( \lambda^{-\delta_0} \) neighborhood of the diagonal. And since for fixed \( y \), as in (6.28), the integral is taken over the region \( |z-y| \approx u \gg 1 \), we can also assume that \( |x-y| \approx u \).

Note that fixed \( x \) and \( y \) with \( |x-y| \approx \mu \), if

\[
(6.34) \quad \left| \pm \frac{x-y}{|x-y|} - \nu \right| \geq C\mu^{-1},
\]

for some large enough constant \( C \), then it is not hard to check that for all \( z \) in the support of \( \tilde{\psi} \) and \( \xi \) in the support of \( \beta_\nu \)

\[
(6.35) \quad \left| \pm \frac{z-y}{|z-y|} - \xi \right| \geq C(2\mu)^{-1}.
\]

Thus, integration by parts in \( z \) in (6.33) yields

\[
(6.36) \quad |K_{\lambda,\mu,\nu}(x, y)| \lesssim_N (\lambda\mu^{-1})^{-N} \lambda^{\frac{3n-1}{2}} T^{-1} \mu^{-\frac{n+1}{2}},
\]

which implies that

\[
(6.37) \quad \sum_{j \in D_{error}} |K_{\lambda,\mu,\nu}(x, y + j\ell)| \lesssim_N (\lambda\mu^{-1})^{-N} \lambda^{\frac{3n-1}{2}} T^{-1} \mu^{-\frac{n+1}{2}}.
\]

Since we are assuming \( \mu \leq T \leq \lambda^{1-\delta_0} \), (6.37) is better than desired when \( N \) is large enough.

On the other hand, it is straightforward to check that \( \#D_{main} = O(\mu) \), and for each fixed \( j \in D_{main} \),

\[
(6.38) \quad |K_{\lambda,\mu,\nu}(x, y + j\ell)| = \left| \frac{1}{\pi T} \int e^{iM} (Q_\nu \cos t \sqrt{-\Delta_{\mathbb{T}^n}})(x, y + j\ell) \beta(|t|/\mu) \tilde{\Psi}(t/T) \, dt \right|
\]

\[
= \left| \int Q_\nu(x, z) K_{\lambda,\mu}(z, y + j\ell) \, dz \right|
\]

\[
\leq \sup_x \int |Q_\nu(x, z)| \, dz \cdot \sup_{z, \beta} |K_{\lambda,\mu}(z, y)|
\]

\[
= O(T^{-1} \lambda^{\frac{n-1}{2}} \mu^{-\frac{n+1}{2}}),
\]

where used (6.13) and (6.28) in the last line. This implies that

\[
(6.39) \quad \sum_{j \in D_{main}} |K_{\lambda,\mu,\nu}(x, y + j\ell)| \lesssim \lambda^{\frac{n+1}{2}} T^{-1} \mu^{1-\frac{n+1}{2}}.
\]

Thus the proof of (6.24) is complete.

Recall that the standard orthonormal basis for Laplacian on the rectangular torus \( \mathbb{T}^n \) is

\[
\{ e^{2\pi i (\ell_1 x_1 + \cdots + \ell_n x_n)} (\ell_1, \ldots, \ell_n)^{\frac{1}{2}}, \quad j \in \mathbb{Z}^n \}.
\]

For later use, let us also define

\[
(6.40) \quad \hat{Q}_\nu f = \sum_{j \in \mathbb{Z}^n} \eta_\nu (j\ell) a_j e^{2\pi i (j\ell', x)} \quad \text{if} \quad f = \sum_{j \in \mathbb{Z}^n} a_j e^{2\pi i (j\ell', x)},
\]
where \( j_{\nu} = (\frac{j_{1}}{c}, \ldots, \frac{j_{n}}{c}) \) for \( j = (j_{1}, \ldots, j_{n}) \in \mathbb{Z}^{n} \) and \( \eta_{\nu} \in C_{0}^{\infty}(\mathbb{R}^{n}) \) equals one in a neighborhood of the support of \( \beta_{\nu} \) and

\[
\text{supp} \ (\eta_{\nu}) = \{ \xi \in \mathbb{R}^{n} : \frac{\xi}{|\xi|} - \nu \leq 4T^{-1} \},
\]

It is obvious that \( \tilde{Q}_{\nu} \) commute with the operators \( \sigma_{\lambda} \) and \( \rho_{\lambda} \), and

\[
(6.41) \sum_{\nu} \| \tilde{Q}_{\nu} f \|_{2}^{2} \lesssim \| f \|_{2}^{2},
\]

if \( \nu \) is chosen from a \( T^{-1} \) separated set on \( S^{n-1} \).

On the other hand, we also have

\[
(6.42) \| Q_{\nu} f - Q_{\nu} \tilde{Q}_{\nu} f \|_{L^{p}(\mathbb{R}^{n})} \leq \lambda^{-N} \| f \|_{2}, \quad \forall \ p \geq 2, \ 1 \leq T \leq \lambda^{1-\delta_{0}}.
\]

To see this, since \( Q_{\nu} \) is rapidly decreasing outside a \( \lambda^{-\delta_{0}} \) neighborhood of the diagonal, it suffices to show \( (6.42) \) with \( Q_{\nu} \) replaced by

\[
(6.43) \tilde{Q}_{\nu}(x, y) = \frac{\lambda^{n}}{(2\pi)^{n}} \int e^{i\lambda(x-y, \xi)} \psi(x) \tilde{\psi}(y) \beta_{\nu}(\xi/|\xi|) \tilde{\beta}(|\xi|) d\xi,
\]

where \( \tilde{\psi} \in C_{0}^{\infty}(\Omega) \) equals one in a neighborhood of the \( x \)-support of \( \psi \). Now, if \( f = \sum_{j \in \mathbb{Z}^{n}} a_{j} e^{2\pi i (j_{\nu} \cdot x)} \)

\[
(6.44) Q_{\nu} f - Q_{\nu} \tilde{Q}_{\nu} f = \frac{\lambda^{n}}{(2\pi)^{n}} \sum_{j \in \mathbb{Z}^{n}} e^{i\lambda(x-y, \xi)+2\pi i (j_{\nu} \cdot y)} (1 - \eta_{\nu}(j_{\nu})) \psi(x) \tilde{\psi}(y) \beta_{\nu}(\xi/|\xi|) \tilde{\beta}(|\xi|) d\xi dy,
\]

integrating by parts in \( y \) and using Sobolev give us \( (6.42) \).

We have collect all the necessary properties we need for \( Q_{\nu} \). Now write

\[
(\tilde{\sigma}_{\lambda} h)^{2} = \Upsilon_{\text{diag}}(h) + \Upsilon_{\text{far}}(h),
\]

where \( \Upsilon_{\text{far}}(h) \) is defined as in \( (4.15) \), and

\[
(6.45) \Upsilon_{\text{diag}}(h) = \sum_{(\nu', \nu'') \in \Xi_{J}} (\tilde{\sigma}_{\lambda} Q_{\nu'} h) \cdot (\tilde{\sigma}_{\lambda} Q_{\nu''} h),
\]

if \( T^{-1} \in (2^{J-4}, 2^{J-3}) \), and \( h = \rho_{\lambda} f \).

Here as before, \( \nu = (\nu', \nu_{n}) \), and the definition of \( \Xi_{J} \) is the same as the one in \( (4.13) \), which include diagonal pairs where \( \nu' = \nu'' \), and all \( (\nu', \nu'') \in \Xi_{J} \) satisfy \( |\nu' - \nu''| \lesssim T^{-1} \).

Thus, for each fixed \( \nu' \), we have \# \{ \nu'' : (\nu', \nu'') \in \Xi_{J} \} = O(1).

For \( \Upsilon_{\text{far}}(h) \), we have the favorable bilinear estimates

\[
(6.46) \int |\Upsilon_{\text{far}}(h)|^{q/2} dx \leq C_{A} (\lambda T^{-1})^{\frac{n-1}{2} (q - n_{c})} \| h \|_{L^{q}(\mathbb{M})}^{q}, \quad \text{if} \quad q \in \left( \frac{2(n+2)}{n}, q_{c} \right).
\]

This is analogous to \( (4.16) \), where we replaced \( \lambda^{7/8} \) in \( (4.16) \) by \( \lambda T^{-1} \) here. And as is pointed out below \( (4.16) \), \( (6.46) \) essentially follows form \( (4.7) \) in \( [6] \) up to rapidly decaying terms.
If we combine (6.46) with the fact that $|\tilde{p}_\lambda f(x)| \leq C(T,\lambda)^{-\frac{n-1}{q}}$ on $A_-$, we have
\[
\int_{A_-} |\tilde{p}_\lambda h|^\frac{n-1}{q} |Y_{\text{far}}(h)|^{q/2} \lesssim \|\tilde{p}_\lambda h\|_{L^q(A_-)}^{q}\left(\lambda T^{-1}\right)^\frac{n-1}{q} \cdot \lambda 
\]
\[
\leq \left(\lambda T\right)^\frac{n-1}{q} \cdot \left(\lambda T^{-1}\right)^\frac{n-1}{q} \cdot \lambda
\]
\[
= \left(\lambda T\right)^\frac{n-1}{q} \cdot \left(\lambda T^{-1}\right)^\frac{n-1}{q} \cdot \lambda.
\]
Thus, if one repeat the arguments in (4.27)-(4.29) for the $Q_\nu$ operator defined in this section, then for dimensions $n \geq 3$, we have
\[
(6.48) \quad \|\tilde{p}_\lambda h\|_{L^q(A_-)} \leq C\lambda^{\frac{1}{q}} \mu_n(T,\lambda)\|h\|_{L^2(\mathbb{T}^n)} + C\left(\sum_{\nu} \|\tilde{p}_\lambda Q_\nu h\|_{L^q(\mathbb{T}^n)}^q\right)^{1/q_c}, \quad h = \rho_\lambda f.
\]
And for $n = 2$, where $q_c = 6$, this is replaced by
\[
(6.49) \quad \|\tilde{p}_\lambda h\|_{L^6(A_-)} \leq C\lambda^{\frac{1}{6}} \mu_2(T,\lambda)\|h\|_{L^2(\mathbb{T}^2)} + C\left(\sum_{\nu} \|\tilde{p}_\lambda Q_\nu h\|_{L^6(\mathbb{T}^2)}^3\right)^{1/3}, \quad h = \rho_\lambda f.
\]
Here $\mu_n(T,\lambda) = (\lambda^{-\frac{1}{n-1}} T^{\frac{3(n+1)}{n}})^{1/\frac{n}{q}-\frac{1}{q}}$, for any $\frac{2(n+2)}{n} < q < q_c$. If we require $\mu_n(T,\lambda) \leq T^{-\frac{2}{q}}$ and let $\lambda \rightarrow \frac{2(n+2)}{n}$, we get the condition $T \leq \lambda^{\frac{1}{q_c} - \varepsilon}$ for arbitrary small $\varepsilon$.

**Proof of Proposition 2.3.** With $h = \rho_\lambda f$ in (6.48), we have for $n \geq 3$,
\[
(6.50) \quad \|\tilde{p}_\lambda f\|_{L^q(A_-)} \leq C\lambda^{\frac{1}{q}} T^{-\frac{1}{q}} \|f\|_{L^2(\mathbb{T}^n)} + C\left(\sum_{\nu} \|\tilde{p}_\lambda Q_\nu \rho_\lambda f\|_{L^q(\mathbb{T}^n)}^q\right)^{1/q_c},
\]
if $1 \leq T \leq \lambda^{\frac{1}{q_c} - \varepsilon}$. Note that the universal (local) spectral projection bounds from [18] yield
\[
\|\tilde{p}_\lambda\|_{L^2 \rightarrow L^{q_c}} = O(\lambda^{\frac{1}{q_c}}).
\]
Using this and (6.42), we have
\[
\sum_{\nu} \|\tilde{p}_\lambda Q_\nu \rho_\lambda f - \tilde{p}_\lambda Q_\nu \rho_\lambda \hat{Q}_\nu f\|_{L^{q_c}(\mathbb{T}^n)}^2
\]
\[
= \sum_{\nu} \|\tilde{p}_\lambda (Q_\nu - \hat{Q}_\nu) \rho_\lambda f\|_{L^{q_c}(\mathbb{T}^n)}^2
\]
\[
\lesssim \lambda^{\frac{1}{q_c}} \sum_{\nu} \|(Q_\nu - \hat{Q}_\nu) \rho_\lambda f\|_{L^2}^2
\]
\[
\lesssim \lambda^{\frac{1}{q_c} - N} \|\rho_\lambda f\|_{L^2}^2 \lesssim \lambda^{\frac{1}{q_c} - N} \|f\|_{L^2}^2.
\]
Thus, it suffices to show that
\[
(6.52) \quad \left(\sum_{\nu} \|\tilde{p}_\lambda Q_\nu \rho_\lambda \hat{Q}_\nu f\|_{L^{q_c}(\mathbb{T}^n)}^q\right)^{1/q_c} \leq C\lambda^{\frac{1}{q}} T^{-\frac{1}{q}} \|f\|_{L^2}.
\]
Note that by (6.15) and the fact that $\|\rho_\lambda\|_{L^2 \rightarrow L^2} = O(1)$, we have
\[
(6.53) \quad \|\tilde{p}_\lambda Q_\nu \rho_\lambda \hat{Q}_\nu f\|_{L^{q_c}(M)} \leq \|Q_\nu \tilde{p}_\lambda \rho_\lambda \hat{Q}_\nu f\|_{L^{q_c}(M)} + \lambda^{\frac{1}{q_c} - \frac{1}{2}} T^2 \|\hat{Q}_\nu f\|_{L^2(M)}.
\]
As before, write $Q_\nu \tilde{\sigma}_\lambda = Q_\nu B \sigma_\lambda = B Q_\nu \sigma_\lambda + [Q_\nu, B] \sigma_\lambda$, where in the current case the commutator $[Q_\nu, B]$ satisfies $\| [Q_\nu, B] \|_{L^\infty \to L^\infty} = O(\lambda^{-1} T^2)$. So, by (6.21) and (2.5)

\[(6.54) \quad \| Q_\nu \tilde{\sigma}_\lambda \rho_\lambda \tilde{Q}_\nu f \|_{L^\infty(M)} \lesssim \| B Q_\nu \sigma_\lambda \rho_\lambda \tilde{Q}_\nu f \|_{L^\infty(M)} + \lambda^\frac{\delta}{\nu^2} T^{-1} \| \tilde{\tilde{Q}}_\nu f \|_{L^2(M)} \]

\[\lesssim \| Q_\nu \sigma_\lambda \rho_\lambda \tilde{Q}_\nu f \|_{L^\infty(M)} + \lambda^\frac{\delta}{\nu^2} T^{-1} \| \tilde{\tilde{Q}}_\nu f \|_{L^2(M)} .\]

If we combine (6.53) and (6.54), by (2.8), (6.14) along with the Kakeya-Nikodym bounds (6.18), we have

\[\| \tilde{\tilde{Q}}_\nu f \|_{L^2(\mathbb{T}^n)} \]

\[\lesssim \| Q_\nu \sigma_\lambda \rho_\lambda \tilde{Q}_\nu f \|_{L^\infty(\mathbb{T}^n)} \quad \text{with} \quad C \lambda^\frac{\delta}{\nu^2} \frac{1}{T} \quad \text{for} \quad 1 \leq T \leq \lambda^{1/\nu^2} \varepsilon .\]

If we combine (6.41) and (6.55), we have

\[\sum_{\nu} \| \tilde{\tilde{Q}}_\nu f \|_{L^\infty(\mathbb{T}^n)}^{\frac{1}{q}} \leq C \lambda^\frac{\delta}{\nu^2} T^{-\frac{1}{\nu^2}} \left( \sum_{\nu} \| \tilde{\tilde{Q}}_\nu f \|_{L^2(\mathbb{T}^n)} \right)^{\frac{1}{2}} \leq C \lambda^\frac{\delta}{\nu^2} T^{-\frac{1}{\nu^2}} \| f \|_{L^2},\]

which implies (6.52). The proof for the case $n = 2$ is similar. If we combine Proposition (6.2) and Proposition (6.3) we get (6.2).

**Remark:** In the next section, we shall show that the results in Theorem 6.1 are optimal in terms of how the bounds depend on $T$. It would be interesting to see whether the optimal results in Theorem 6.1 are valid for $T \lesssim \lambda^{1/2}\varepsilon$.

### 7. Geodesic concentration of quasimodes and lower bounds of $L^2(M)$-norms.

In this section we shall show that the bounds in Theorem 6.1 are sharp in terms of their dependence on $T$ even though we expect the bounds to hold for a larger range of $T$, at least $T \in [1, \lambda^{1/2}]$. Bourgain and Demeter [9] and Hickman [15] (see also Germain and Myerson [13]) obtained near optimal bounds when $T$ is larger than a fixed power of $\lambda$; however, since their arguments used decoupling they involved $\lambda^2$-losses for arbitrary $\varepsilon > 0$.

We shall consider here relationships between the problem of obtaining lower bounds for the $L^2(M)$ norms of quasimodes $\Psi_\lambda$ of frequency $\lambda \gg 1$ on $M$ and the problem of establishing their nontrivial $L^2$-mass concentration near a fixed closed geodesic $\gamma \in M$. Here, we are assuming that the spectrum associated with $P = \sqrt{-\Delta_\gamma}$ of the quasimodes $\Psi_\lambda$ belongs to intervals of the form $[\lambda - \varepsilon(\lambda), \lambda + \varepsilon(\lambda)]$, where, as $\lambda$ goes to infinity through a sequence $\lambda_\gamma \to +\infty$ we have $\varepsilon(\lambda) \searrow 0$. We shall also assume that these quasimodes are $L^2$-normalized, i.e.,

\[(7.1) \quad \| \Psi_\lambda \|_{L^2(M)} = 1.\]
Typically, we shall have $M_{in} \leq (7.4)$ that is wider than the injectivity radius of $M$ possible concentration about geodesics. It is also not natural to consider tubes which are wider than the injectivity radius of $M$. So, by $(7.2)$, it is natural to assume, as we shall, that

$$1 \leq N(\lambda) = O(\lambda^{1/2-}).$$

Note that if we assume $(7.3)$, then by Hölder’s inequality we have for $q > 2$

$$\delta_0 \leq \|\Psi_L\|_{L^2(\mathcal{T}_{\lambda,N})} \leq |\mathcal{T}_{\lambda,N}|^{\frac{1}{q} - \frac{1}{2}} \|\Psi_L\|_{L^q(M)}.$$  

Since the Riemannian volume of the tubes satisfies $|\mathcal{T}_{\lambda,N}| \approx (N(\lambda) \cdot \lambda^{-1/2})^{(n-1)}$, this gives the lower bounds

$$(n(\lambda))^{-(n-1)}(\frac{1}{q} - \frac{1}{2}) \lambda^{\frac{n+1}{2}(\frac{1}{2} - \frac{1}{q})} \approx \|\Psi_L\|_{L^q(M)}$$

for the lower bounds of our quasimodes satisfying $(7.3)$. Tighter concentration of $L^2$-mass near the geodesic (i.e. smaller $N(\lambda)$) implies stronger lower bounds.

Note that for the standard round sphere, $S^n$, the $L^2$-normalized highest weight spherical harmonics $Q_\lambda$ are eigenfunctions satisfying $(7.3)$ with $N(\lambda) \equiv 1$, and the lower bounds in $(7.3)$ with $\Psi_L = Q_\lambda$ are sharp since $\|Q_\lambda\|_{L^q(S^n)} \approx \lambda^{\frac{n+1}{2}}(\frac{1}{2} - \frac{1}{q})$ if $q > 2$, and this fact shows that the universal bounds in $(18)$ are best possible for $2 < q \leq q_c$. (See $(17)$.)

Let us also see here that we can also obtain natural lower bounds for $L^q$ estimates for this range of $q$ if $M$ is a product manifold with $S^1$ as one of its factors. So let us assume for now that

$$M = S^1 \times X^{n-1}$$

is a Cartesian product of $S^1$ with an $(n-1)$-dimensional compact manifold $X^{n-1}$ equipped with the product metric. If $x_0 \in X^{n-1}$ is fixed, let us consider the geodesic $\gamma = S^1 \times \{x_0\}$ in $M$. Also, for $\lambda = 1, 2, 3, \ldots$ consider the function on $S^1 \times X^{n-1}$ defined by

$$\Psi_\lambda(\theta, x) = (\lambda^{1/2}/N)^{-(n-1)/2} e^{i \lambda \theta} \beta(P_X/(\lambda^{1/2}/N))(x_0, x), \quad N = N(\lambda),$$

where $0 \leq \beta \in C^\infty_0((1/2, 2))$ is a Littlewood-Paley bump function and $P_X = \sqrt{-\Delta_{X^{n-1}}}$ is the first order operator on $X^{n-1}$ coming from its metric Laplacian. Here $\beta(P_X/\mu)(x, y)$ is the kernel of the operator $\beta(P_X/\mu)$ on $X \times X^{n-1}$ defined by the spectral theorem.

Let us now prove the following simple result on such product manifolds.

**Proposition 7.1.** Let $M = S^1 \times X^{n-1}$ be as in $(7.6)$ with $n \geq 2$. Then if $\varepsilon(\lambda) \searrow 0$ and $(7.4)$ is valid with $N(\lambda) = (\varepsilon(\lambda))^{-1/2}$, there is a geodesic $\gamma \in M$ so that we have $(7.3)$. 
Consequently, we have

\[(7.8) \quad \|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}\|_{L^2(S^1 \times X^{n-1})} \leq L \geq c_0 \left( \varepsilon(\lambda) \cdot \lambda \right)^{\frac{2}{n+1}} \left( \frac{1}{\lambda^{\frac{n+1}{2}}} \right),\]

\[q \in (2, q_c), \quad q_c = \frac{2(n+1)}{n-1}.

\textbf{Remark 7.2.} Note that the universal bounds in (7.8) imply that the operators in (7.8) map $L^2 \rightarrow L^q$ with norm $O(\lambda^{\frac{n+1}{2}}(\frac{1}{\lambda^{1/2}}))$, and so (7.8) says that for manifolds as in (7.3) one can, at best, improve these bounds by a factor of $\varepsilon(\lambda)^{\frac{n-1}{n+1}}(\frac{1}{\lambda^{1/2}})$. Consequently, since tori are product manifolds involving $S^1$ (perhaps scaled) as a factor, the bounds in Theorem 6.1 are sharp for the range $\varepsilon(\lambda) \in [\lambda^{\frac{n+1}{2}}, 1]$ there.

As we pointed out before, Hickman [14] showed that for tori if one includes additional arbitrary $\lambda^2$ factors in the right side of (7.8) then this controls the bounds of the projection operators there. Further results were obtained by Germain and Myerson [13] as well as the lower bound (7.8) for tori using a slightly different and more direct method.

\textbf{Proof.} We shall take $\gamma = S^1 \times \{x_0\}$ as mentioned above. We then notice that, if $\{\mu_k\}$ are the eigenvalues of $P_X$ and $\{e_k\}$ is the associated orthonormal basis, then

\[
\|\beta(P_X/(\lambda^{1/2}/N))(x_0, \cdot)\|_{L^2(X^{n-1})}^2 = \sum_k \beta^2(\mu_k/(\lambda^{1/2}/N)) |e_k(x_0)|^2
\approx \# \{\mu_k : \mu_k \approx (\lambda^{1/2}/N) \} \approx (\lambda^{1/2}/N)^{n-1}.
\]

Therefore, the functions in (7.7) have $L^2(S^1 \times X^{n-1})$ norms which are comparable to one.

Consequently, by (7.6) with $N(\lambda) = (\varepsilon(\lambda))^{-1/2}$ we would have the lower-bound (7.8) if we could show that for large $\lambda \in \mathbb{N}$

\[(7.9) \quad \text{Spec } \Psi_{\lambda} \subset [\lambda - C_0 \varepsilon(\lambda), \lambda + C_0 \varepsilon(\lambda)],
\]

for some fixed constant, as well as

\[(7.10) \quad \|\Psi_{\lambda}\|_{L^2((y \in S^1 \times X^{n-1}, d_{\gamma}(y, \gamma) \leq (\varepsilon(\lambda)\lambda)^{-1/2}))} \geq \delta_0, \quad \gamma = S^1 \times \{x_0\}, \ x_0 \in X^{n-1},
\]

for some fixed $\delta_0 > 0$. Here Spec $\Psi_{\lambda}$ of course denotes the $P$-spectrum of $\Psi_{\lambda}$.

To prove (7.9) we note that $\Psi(\theta, x)$ is a linear combination of eigenfunctions on $S^1 \times X^{n-1}$ of the form $e^{i\lambda \theta} e_k(x)$ where $\lambda \in \mathbb{N}$ is fixed and $\mu_k \approx \lambda^{1/2}/N(\lambda) = \sqrt{\varepsilon(\lambda)\lambda}$. Since

\[
\sqrt{-\left(\frac{\partial^2}{\partial \theta^2} + \Delta_X\right)} (e^{i\lambda \theta} e_k(x)) = \sqrt{\lambda^2 + \mu_k^2} (e^{i\lambda \theta} e_k(x))
\]

and

\[
\sqrt{\lambda^2 + \mu_k^2} - \lambda = O(\varepsilon(\lambda)) \quad \text{if} \quad \mu_k \leq C \sqrt{\varepsilon(\lambda)\lambda},
\]

we conclude that (7.9) is valid.

To prove (7.10) it suffices to see that if $B(x_0, r)$ is the geodesic ball of radius $r \ll 1$ about $x_0$ in $X^{n-1}$ then we have the lower bound

\[(7.11) \quad (\varepsilon(\lambda)\lambda)^{-1/2} \|\beta(P_X/(\varepsilon(\lambda)\lambda)^{1/2})(x_0, \cdot)\|_{L^2(B(x_0, (\varepsilon(\lambda)\lambda)^{-1/2}))} \geq \delta_0 > 0.
\]
This is standard calculation. Since \( \varepsilon(\lambda) \lambda \geq \lambda^\sigma \) for some \( \sigma > 0 \) because of our assumption (7.3), one can argue as in [21 §4.3] to see that for large \( \lambda \) we have
\[
\beta(P_N/(\varepsilon(\lambda)\lambda)^{1/2})(x_0,y) \approx (\sqrt{\varepsilon(\lambda)\lambda})^{n-1} \text{ if } y \in B(x_0, c_0(\varepsilon(\lambda)\lambda)^{-1/2})
\]
for some fixed \( c_0 > 0 \), which yields (7.11) and completes the proof. \( \square \)

### Concentration problems for manifolds of negative curvature

As we pointed out before the analog of (7.8) with \( \varepsilon(\lambda) \equiv 1 \) is valid on \( S^n \) since there are \( L^2 \)-normalized eigenfunctions on \( S^n \) for which (7.3) is valid with \( N(\lambda) \equiv 1 \), which means they have a fixed fraction of their \( L^2 \)-mass in \( \lambda^{-1/2} \) tubes about a geodesic. The proof of Proposition 7.1 showed that on tori one can find \( \varepsilon(\lambda) \)-quasimodes with fixed \( L^2 \)-mass in \((\varepsilon(\lambda))^{-1/2}\lambda^{-1/2} \) tubes around a closed geodesic whenever \( \varepsilon(\lambda) \in [\lambda^{-1}, 1] \). So we have natural concentration results for quasimodes near geodesics in manifolds of positive and zero curvature.

We wonder if there are any analogous results for manifolds of negative curvature. Recently there has been somewhat related work by Brooks [10], Eswarathasan and Nonnenmacher [12] and others who showed that there is strong scarring of logarithmic quasi-modes on compact quotients of \( \mathbb{H}^n \). These say that, on such manifolds, if \( \gamma \) is a closed geodesic then there is a sequence of \( L^2 \)-normalized quasimodes \( \Psi_\lambda \) satisfying (7.3) (or a related \( L^2 \)-quasimode condition) so that the quantum probability measures \(|\Psi_\lambda|^2 \text{dVol}_g\) exhibit strong scarring along \( \gamma \). By this we mean that weak limits of these measures must include a positive multiple of the Dirac mass on \( \gamma \), and so the \( \Psi \) are tightly concentrated on this geodesic.

It would be interesting if one could find more quantitative versions of these results that might lead to nontrivial lower bounds for the \( L^2 \to L^{q_c} \) operator norms of the operators in (7.3) with \( \varepsilon(\lambda) = (\log \lambda)^{-1} \) (or perhaps larger). So it would be interesting to see if for certain \( N(\lambda) \not\to \) we could find logarithmic quasimodes \( \Psi_\lambda \) (so \( \varepsilon(\lambda) \approx (\log \lambda)^{-1} \)) on hyperbolic quotients for which we have (7.2). It is routine to see that on any manifold one cannot have for \( \varepsilon(\lambda) \not\to 0 \)
\[
(7.12) \quad \|\chi_{[\lambda^{-\varepsilon(\lambda)},\lambda+\varepsilon(\lambda)]}\|_{L^2 \to L^{q_c}} = o((\sqrt{\varepsilon(\lambda)}\lambda)^{1/4}),
\]
since these bounds would imply that the operators map \( L^2 \to L^\infty \) with norm \( o((\sqrt{\varepsilon(\lambda)}\lambda)^{-1/4}) \), and this would violate the Weyl counting formula.

Even though Hassell and Tacy [14] showed that there is \((\log \lambda)^{-1/2}\) improvements over the universal bounds in [18] on the \( L^q \to L^q \) norms of these operators for \( q > q_c \) when \( M \) has negative curvature and \( \varepsilon(\lambda) = (\log \lambda)^{-1} \), given Proposition 7.1 one might expect such an improvement to not hold when \( q \) is the critical exponent \( q = q_c \). This would follow from showing that there is a closed geodesic \( \gamma \) and \( L^2 \)-normalized logarithmic quasimodes as described above so that one has the non-trivial lower bounds (7.3) with \( N(\lambda) = o((\log \lambda)^{q_c/4}) = o((\log \lambda)^{3/4-\epsilon}) \). So, for instance, when \( n = 2 \), if one could find \( \Psi_\lambda \) as above with fixed lower bounds of \( L^2 \)-mass in \( o((\log \lambda)^{3/2-\epsilon/2}) \) neighborhoods of a closed geodesic \( \gamma \), then the \( \sqrt{\varepsilon(\lambda)} = (\log \lambda)^{-1/2} \) improvement of Hassell and Tacy [14] for \( q > q_c \) could not hold for \( q = q_c \).
Also, it would be interesting to know whether our estimates in \(5.3\) for log-quasimodes on surfaces of negative curvature are sharp. If one could construct a sequence of such quasimodes for which \(\|\Psi\|_{L^2(T)} \geq \delta_0\) for tubes \(T = T_\lambda(\gamma)\) of width \(\log \lambda \cdot \lambda^{-1/2}\) about a periodic geodesic \(\gamma\), then \(\mathbb{R}\) would be saturated. Note that these are much wider tubes than the ones we used for \(S^1 \times X^{n-1}\), which is to be expected due to the much faster divergence of geodesics on manifolds of negative curvature.

As was kindly pointed out to us by Eswarathasan, due to the role of the Ehrenfest time in the constructions, it does not seem likely that the techniques in \([10]\) or \([12]\) could be used to prove this. It would also be interesting to see whether such a result might hold for log-log quasimodes, in which case one would replace the above \(\log \lambda\) terms with \(\log \log \lambda\). So \(\varepsilon(\lambda) = (\log \log \lambda)^{-1}\) and \(N(\lambda) = o((\log \log \lambda)^{\frac{n+1}{n+1}})\). If we then had \(7.3\) this would imply that the \(\sqrt{\varepsilon(\lambda)}\) improvements for the \(L^2 \rightarrow L^q\) norms of the operators in \(7.8\) of Hassell and Tacy for \(q > q_c\) could not hold for \(q = q_c\) with \(\varepsilon(\lambda) = (\log \log \lambda)^{-1}\).

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