PALEV STATISTICS AND THE CHRONON

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Abstract A finite relativistic quantum space-time is constructed. Its unit cell has Palev statistics defined by a spin representation of an orthogonal group. When the Standard Model and general relativity are physically regularized by such space-time quantization, their gauges are fixed by nature; the cell groups remain.

1 Novus ordo seclorum

The goal is still a finite physical theory that fits our finite physical experiments. The classical space-time continuum led to singular (divergent) quantum field theories: Infinity in, infinity out. In ancient times, a continuum was the only way to understand the translational and rotational invariance of Euclid's geometry. Today there are quantum spaces with a finite number of quantum points that still have the continuous symmetries of gravity and the Standard Model, at least within experimental error. If electron spin were any simpler it would not exist. The strategy is build the cosmos from such atoms.

Quantum spaces are represented by probability spaces that define the lowest-order logic of their points. Modular architecture requires the higher-order logic, classically dealt with by set theory. Classical space-time and field theory are formulated within classical set theory; perhaps quantum space-time and field theory need a quantum set theory. Classical set theory was invented by Cantor to represent the mind of the Eternal. Quantum set theory is intended to represent the system under study as quantum computer. Like any quantum theory it statistically represents input/outtake (IO) beams of systems by what Heisenberg called probability vectors.

Quantum logic is a square root of classical logic: Transition probabilities in the classical sense are squares of components of the probability vector; transition probability amplitudes. Call the quadratic space of probability vectors a probability space.

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In general, $\mathcal{P}$, like the quantum space of Saller [11], includes both input (ket) and outtake (bra) vectors. Distinguish these by the signs of their norms. $\mathcal{P}$ is not a Hilbert space.

A quantum theory should describe populations as well as individuals. Enrich the quadratic probability space $\mathcal{P}$ to a probability algebra $\mathcal{P}$ whose product $ab$ represents successive application of input/outtake (IO) operators. The one-system probability vectors form a generating subspace $\mathcal{P}_1 \subset \mathcal{P}$. $\mathcal{P}$ consists of polynomials over $\mathcal{P}_1$, subject to constraints and identifications said to define the statistics.

A regular quantum theory is one with a finite-dimensional probability algebra [1].

The probability algebra $\mathcal{P}^-$ for fermions is a Clifford algebra, defined by anticommutation relations among the one-fermion vectors:

$$\forall x \in \mathcal{P}_1^- : \quad x^2 = \|x\| = x \cdot x.$$  \hspace{1cm} (1)

Its dimension is $\dim \mathcal{P}^- = 2^{\dim \mathcal{P}_1}$, so Fermi statistics is regular if the one-fermion probability space is.

The probability algebra $\mathcal{P}^+$ for even quanta is commonly assumed to be a Bose (Heisenberg, canonical) algebra whose generators obey

$$\forall x \in \mathcal{P}_1 : \quad xy - yx = \varepsilon(x, y)$$  \hspace{1cm} (2)

with a given skew-symmetric bilinear form $\varepsilon$ on $\mathcal{P}_1 \otimes \mathcal{P}_1$. This is not exactly right: Bose statistics is always singular. Pairs of fermions, however, obey a regular statistics whose probability algebra envelops a Lie algebra, with Bose statistics and the Heisenberg algebra as a singular limit. This is a special case of Palev statistics [8, 9], which is reviewed next.

2 Palev statistics

For any semisimple Lie algebra $\mathfrak{p}$, a Palev statistics of the $\mathfrak{p}$ class is one whose probability algebra $\mathcal{P}$ is a finite-dimensional enveloping algebra of $\mathfrak{p}$ (has $\mathfrak{p}$ as commutator Lie algebra).

Fermi and Bose statistics have graded Lie algebras $\mathfrak{f}, \mathfrak{b}$ that specify their commutation relations. $\mathfrak{f}$ and $\mathfrak{b}$ as vector spaces are also one-quantum probability spaces. They have essentially unique irreducible unitary representations; these serve as many-quantum probability spaces. There are, however, an infinity of irreducible representations of a Palev algebra $\mathfrak{p}$ that might serve as many-quantum probability space. Empirical choices must be made for Palev statistics that are already decided for Fermi and Bose statistics.

Palev gives a representation of $\mathfrak{sl}(n+1)$ on a Fock space $W_\mathfrak{p}$ of symmetric tensors of degree $\mathfrak{p}$. $W_\mathfrak{p}$ is a Hilbert space, appropriate for his applications and not for these.

The probability space of a hypothetical quantum event must have enough dimensions to allow for the observed quantum systems. It is not clear that events in
space-time can be experimentally located to within much less than a fermi, corresponding to a localization in time of about $10^{-25}$ s. The Planck limit at $10^{-43}$ s was initially a conjecture based on pure quantum gravity. Our instruments, to be sure, do not seem to be made of gravitons alone, but take part in all the interactions. The Planck time seems at best a poor lower bound to the quantum of time.

However the energy at which all the running coupling constants seem to converge is not very much greater than the Planck energy, so it may indeed have universal significance. Yet crystals have many scales besides cell size, such as Debye shielding length, skin depth, coherence length, and mean free paths. The Planck time and the unification energy might correspond more closely to one of these than to the cell size $X$. To avoid a premature commitment, call the natural time $X$ the *chronon*.

How many dimensions must the event probability space have? Suppose the lifetime of the four-dimensional universe is $10^{21}$ s; an error by a factor of 100 will not matter much. If $X \sim 10^{-43}$ s then the dimensionality of the history probability space of the cosmos—which we cannot observe maximally—is about $10^{1024}$. The largest system that can be maximally observed by a co-system within such a cosmos—here we renounce the perspective of the Eternal—is much simpler. Its probability space might have no more than $\log_2 10^{256} \sim 3000$ dimensions.

Here are two examples of Palev statistics:

### 2.1 Rotatons

The so(3) Lie algebra with commutation relations $L \times L = L$ defines an aggregate of palevons of the so(3) kind, whose quanta must be called rotatons, since Landau preempted the term “roton”. Relative to an arbitrary component $L_3$ as generator of a Cartan subalgebra, the root vectors $L_{\pm} = L_1 \pm iL_2$ represent the input and outtake of spin-1 rotatons. An irreducible representation with extreme eigenvalues $\pm il$ for $L_3$ represents a Palev statistics with no more than $2l + 1$ rotatons present in a single aggregation.

### 2.2 Di-fermions

Di-fermions are palevons. If the fermion probability space is $2^N \mathbb{R}$ then the di-fermion is a palevon of type $\text{so}(N,N)$.

This refers to the elementary fact that the Fermi commutation relations for a fermion with $N$ independent probability vectors define a Clifford algebra $\text{Cliff}(N,N)$, and the second grade of $\text{Cliff}(N,N)$ is both the probability space for a fermion pair, and the Lie algebra $\text{spin}(N,N)$ defining a Palev statistics of class $D_N$. 
2.3 Regular space-times

Call a spacetime regular if its coordinate algebra is regular. All its coordinates then have finite spectra. There are not many regular spacetimes in the literature.

Singular (non-semisimple) Lie algebras can be regularized by slightly changing some vanishing commutators, undoing the flattening contraction that led, presumably, from the regular to the singular.

The prototype of such regularization by decontraction is (special) relativization \([13, 6]\). This regularizes the Galilean Lie algebra \(g = g(L, K)\) of Euclidean rotations \(L\) and Galilean boosts \(K\), to the Lorentz Lie algebra so\((3, 1)\). Write such relations as

\[
\text{so}(3, 1) \rightarrow g \text{ or } g \leftarrow \text{so}(3, 1),
\]

directed from the regular algebra to the singular.

The sole remaining culprit today is Bose statistics, the canonical Lie algebra. Its de-contraction requires adding new variables. This is the general case; special relativity and quantum theory were exceptional in this respect.

Some physical self-organization must then freeze these extra variables out near the singular limit. This can be tested experimentally in principle by disrupting this organization. Hopefully, a suitable regularization of the remaining singular theories will once again improve the fit with experiment.

The Killing form of classical observables is as singular as can be: identically 0. It is nearly regularized by canonical quantization:

\[
a_{\text{comm}}(x, p, i) \leftrightarrow h(N),
\]

where the canonical (Heisenberg) Lie algebra is

\[
h(N) : [x^{\nu}, p_{\nu'}] = i\hbar \delta^{\nu'}_{\nu}, \quad \nu, \nu' = 1, \ldots, N,
\]

other commutators vanishing. The solvable radical \(C\) generated by \(i\) survives canonical quantization. Recall that \(hp\) does not fit into any \(\text{sl}(N\mathbb{R})\). (The left-hand sides of the canonical commutation relations would have well-defined trace 0, and the right-hand side \(i\) would have non-zero trace.) Canonical quantization is a quantization interrupted by premature canonization.

2.4 Feynman space-time

Feynman [3] seems to have constructed the first regular relativistic quantum space-time \(\mathcal{F}\). Its positional coordinates are finite spin sums:

\[
x^\mu \leftrightarrow \bar{x}^\mu = X[\gamma^\mu (1) + \ldots + \gamma^\mu (N)], \quad \mu = 1, 2, 3, 4.
\]
The $\gamma^\mu$ have unit magnitudes, and $X$ is the natural quantum unit of time, or *chrono*. If the commutators $[\gamma^\mu(n'), \gamma^\mu(n)]$ vanish for $n \neq n'$ then the probability vector space for $\mathcal{F}$ is a $16^N$-dimensional Clifford algebra. Quantification theory abbreviates (6) to
\[ \mathcal{F} = X\psi\gamma^\mu\psi. \]

Each term in the sum represents a hypothetical quantum element of the space-time event; call it a chronon. The Feynman chronon has spin 0 or 1, because $\gamma^\mu$ has both a scalar part $\gamma^0$ and a vector part $(\gamma^1, \gamma^2, \gamma^3)$.

Nature seems to have a unit of space-time size like $X$ at every event, fixing the gauge in the original sense of Weyl. If each event has a space-time measure $X^4$, then the dimension of the one-event probability space is proportional to the space-time volume; as if the event statistics is extensive in the sense of Haldane [5, 9].

### 2.5 Yang space-time

Yang [15] proposed the first regular relativistic space-time-momentum-energy Lie algebra, leaving the signature somewhat open:
\[ \eta = (\text{so}(5,1) \text{ or } \text{so}(3,3)) \rightarrow \mathfrak{sp}(4). \]

$\eta$ can represent the orbital variables of a spinless relativistic quantum. Yang restricted consideration to representations in Hilbert space, however, blocking regularity. Regular version of the Yang theory uses a finite-dimensional representation of the Yang Lie algebra $\text{so}(3,3)$. Its probability algebra must then have an indefinite norm (§2.6).

The Yang Lie algebra $\eta$ is not to be confused with the conformal $\text{so}(3,3)$ Lie algebra. They are isomorphic but act on different physical variables and have different physical effects. We deal with groups of physical operations, not abstract groups.

### 2.6 Interpretation of the indefinite norm

In special relativity the sign of the Minkowski metric form $g$ distinguishes allowed (timelike) directions from forbidden (space-like) ones.

In a relativistic quantum theory of the Dirac kind, the probability-amplitude form $\beta$ is neutral (of signature 0).
\[ \beta \Psi \Psi = \Psi^\beta \circ \Psi \]
gives the mean flux of systems from the experiment (not the absolute flux). The sign distinguishes input probability vectors from outtake probability vectors [2].
Let source kets bras have positive norm and sink bras negative. This changes no physics in the usual quantum theory, which does not add bras and kets. Here it enlarges the group and must be tested by experiment.

A regular theory might associate an elementary particle with an irreducible finite-dimensional isometric representation of a simple Lie algebra \( \eta \) that approximates the Poincaré Lie algebra:

\[
\eta \mapsto \hbar p.
\]  

Then all one-particle observables have finite spectra.

The constant \( \hbar \) of the usual quantum physics is then another non-zero vacuum expectation value, a frozen variable like the Minkowski metric \( g_{\mu\nu} \) and the Higgs field. Centralizing a hypercomplex number by a condensation gives mass to any gauge boson that transports that number; this was shown for the quaternion case, for example. Such a frozen \( \bar{i} \) must be assumed in the Yang and Segal space-time quantizations [15, 13] based on the Lie algebra

\[
\eta = \text{so}(3,3) \cong \text{spin}(3,3) \cong \text{sl}(4\mathbb{R}).
\]

Infinitesimal generators of \( \text{so}(3,3) \cong \text{spin}(3,3) \cong \text{sl}(4\mathbb{R}) \) make up a tensor \( [L_{\gamma'\gamma}] \) \((y,y' = 1, \ldots ,6)\) with 15 independent components, representing orbital variables of the Yang scalar quantum. This \( \eta \) is also a candidate for the scalar particle Lie algebra \( \eta \) of (10). The Feynman quantum space-time and the Penrose quantum space [10] can be regarded as spin representations of the Yang Lie algebra.

\( \hbar \), the quantized \( \bar{i} \), is a normalized \( 2 \times 2 \) sector \( [L_{z'z}] \) \((z,z' = 1,2)\) of \( [L_{\gamma'\gamma}] \) in an adapted frame. The tensor \( [L_{\gamma'\gamma}] \) then breaks up according to

\[
[L_{\gamma'\gamma}] = \begin{bmatrix}
L_{\mu'\mu} & \frac{i p_{\gamma'}}{i p_{\mu}} \\
\frac{i x_{\mu}}{i p_{\mu}} & L_{56}
\end{bmatrix}
\sim \begin{bmatrix}
4 \times 4 & 4 \times 2 \\
2 \times 4 & 2 \times 2
\end{bmatrix}, \quad y,y' = 1, \ldots ,6,
\]

which includes the Lorentz generator \( L_{\mu'\mu} \) as a \( 4 \times 4 \) block, position \( x^\mu \) and momentum \( p^\mu \) \((\mu,\mu' = 1,2,3,4)\) as \( 4 \times 1 \) blocks, and \( L_{56} \) \((z,z' = 1,2)\) as a \( 2 \times 2 \) block.

Posit a self-organization, akin to ferromagnetism, that causes the extra component \( L_{56} \) to assume its maximum magnitude in the vacuum. Small first-order departures from perfect organization of \( \bar{i} \) make second-order errors in \( |\bar{i}| \).

### 2.7 Locality

One more limit stands between the regular Yang Lie algebra \( \eta \) and singular canonical field theories. The special-relativistic kinematics and \( \eta \) have a canonical symmetry between the \( x^\mu \) and \( p^\mu \). Yet there are great physical differences between these variables. Under the composition of systems, \( p_\mu \) is extensive and \( x^\mu \) is intensive. The fundamental gauge interactions of the Standard Model and gravity are local in
x^\mu$ and not in $p_\mu$; unless asymptotic freedom can be regarded as a weak form of locality in $p_\mu$.

This suggests that there is a richer class of regular quantum structures that have classical differential geometry and gauge field theories as organized singular limits, with at least three quantification levels: the chronon, the event, and the field.

Wigner proposed that an elementary particle corresponds to an irreducible unitary representation of the Poincaré group. Up-dates in this concept are called for by his later work. The Wigner concept of elementary particle gives no information about interactions between particles. Gauge theory requires an elementary particle to have a location in space-time where it interacts with a gaugeon. It would then seem useful to associate an elementary particle with an irreducible representation of the Heisenberg-Poincaré Lie algebra $\mathfrak{hp}(x^\mu, p_\mu, L_\mu^\nu, i)$ instead, which fuses the Poincaré and the Heisenberg (canonical) Lie algebras. This is the Lie algebra that Yang regularized.

### 2.8 Quantization and quantification

Quantification and quantization are related like archeology and architecture. They concern similar structures, but quantification synthesizes them from the bottom up, which likely the order of formation, while quantization analyzes them from the top down, the order of discovery.

Quantization re-introduces a quantum constant that the classical limit eliminates. Quantification does not introduce one because the quantum individual already provides it.

In particular, space-time quantization introduces a new quantum entity, the chronon, carrying a time unit, the chrone $X$, and an energy unit, the erge $E$. The chronon is no particle in the usual sense but a least part of the history of a particle.

Canonical quantization can also be interpreted as a quantification with a singular statistics. What is sometimes called “second quantization” is more accurately a second quantification.

### 2.9 Gauge

A gauge is an arbitrarily fixed movable standard used in measurements. It is part of the co-system, the complement of the system in the cosmos. As part of the co-system, a gauge is normally studied under low resolution and treated classically. A field theory may postulate a replica of the gauge at every event in space-time, forming an infinite field of infinitesimal gauges. Weyl’s original gauge was an infinitesimal analogue of a carpenter’s gauge or a machinist’s gauge block, a movable standard of length; hence the name.
A gauge transformation changes the gauges but fix the system. They form a Lie group, the gauge group, which indicates the arbitrariness of the gauges. It is customary to assume that the relevant dimensions of the gauge field are fixed during an experimental run, so that the experimental results can be compared meaningfully with each other. This means that the gauge must be stiff. For example, machinist’s gauge blocks are often made of tungsten carbide. Such rigid constraints become a source of infinities.

In a simple quantum theory, however, all variables have discrete spectra. All eigenvalues can be defined by counting instead of by measuring. There is no need for arbitrary units, external gauges, or gauge group; Nature provides the gauge within the system. Thus a gauge group is another sign of interrupted quantization.

One well-known way to break a gauge group is by self-organization. It is often supposed that the Higgs field, which breaks an isospin group, is such a condensate.

A gauge group is also broken, however, when further quantization discovers a natural quantum unit, fixing a gauge. The quantized $i$ that breaks $su(2)$ and imparts mass in quaternion quantum gauge theory is of that kind. So is the quantized imaginary $\hat{i}$ that breaks Yang $so(3,3)$. The Higgs $\hat{\eta}$ that breaks isospin $so(3)$, and the gravitonic $\hat{g}_{\mu'\mu}$ that breaks $sl(4\mathbb{R}) \cong so(3,3)$ may also be such natural quantum gauges, to be recovered by regularizing the kinematical Lie algebra of the Standard Model through further quantization.

Notation: The one-quantum total momentum-energy vector is, up to a constant, the differentiator $[\partial_{\mu}]$, canonically conjugate to the space-time position vector $[x^\mu]$. $[\partial_{\mu}]$ reduces to a gauge-invariant differentiator $[D_{\mu}]$, also canonically conjugate to $x^\mu$, related to kinetic energy, and a vector $\Gamma_{\mu}(x)$ that commutes with position, related to potential energy:

$$\partial_{\mu} = D_{\mu} + \Gamma_{\mu}. \quad \text{Total} = \text{Kinetic} + \text{Potential} \quad (13)$$

The gauge commutator algebra $a(x^\mu, D_{\mu}(x), F_{\mu'\mu}, \ldots)$ is generated by the space-time coordinates $x^\mu$ and the kinetic differentiator $D_{\mu}(x)$, and includes the field variables $F_{\mu'\mu} = [D_{\mu'}, D_{\mu}]$ and their higher covariant derivatives. Its radical includes all functions of the $x^\mu$. This makes it singular too.

Gauging semi-quantizes. It converts the commutative operators $\partial_{\mu}$ into the noncommutative ones $D_{\mu}$, and for individual quanta these are observables. Its contraction parameter is the coupling constant. Landau quantization in a magnetic field is of that kind.

Gauging also quantifies: It converts one finite-dimensional global gauge group $G$ into many isomorphs of $G$, one at each space-time point. Gauging introduces infinities because the number of gauges is assumed to be infinite. Thus quantum gauge physics can be regularized by regularizing its quasi-Lie algebras. This eliminates gauge groups as well as theory singularities. This will be taken up elsewhere.
3 Higher-order quantum set theory

Classical set theory iterates the power-set functor to form the space of all “regular” (ancestrally finite, hereditarily finite) sets. A regular set theory might therefore iterate the Fermi quantification functor \[4\], as follows.

The Peano \( \iota \), with
\[
\{a, b, c, \ldots\} := \{a\}\{b\}\{c\} \ldots = ta\,tb\,tc\,\ldots,
\]
defines membership \( a \in b \):
\[
a \in b \;:=
\;ta \subset b
\]
(15)

Let \( S \) designate the classical algebra of finite sets finitely generated from the empty set 1 by bracing \( \iota x = \{x\} \) and the disjoint union \( x \lor y \) (a group product with identity 1, the empty set). Sets of \( S \) are here called \textit{perfinite} (elsewhere, ancestrally or hereditarily finite). They are finite, and so are their elements, and their elements, and so forth, all the way down to the empty set. Let \( \sqcup s \) be the set of finite subsets of \( s \). Then
\[
\bigvee : S \to S = \bigvee S.
\]
(16)

An element of \( S \) is a set or simplex whose vertices may be sets or simplices. \( S \) is supposedly complex enough to represent any finite classical structure.

A quantum analogue \( \hat{S} \) is a kind of linearization of \( S \):
For any quadratic space \( S \), let \( \sqcup S \) designate the Clifford algebra of finite-degree polynomials over \( S \), modulo the exclusion principle
\[
\forall s \in S : s \lor s = 0.
\]
(17)

\( \sqcup S \) and \( \sqcup \) correspond to the classical power set and the symmetric union (XOR). If \( \mathcal{P}_1 \) is a one-fermion probability space then \( \mathcal{P} = \text{Cliff} \mathcal{P}_1 \) is the many-fermion probability algebra.

Each quantum subclass of a system is associated with a subspace \( \mathcal{C} \subset \mathcal{P} \) in the probability space of the system, and so with a Clifford probability vector \( e_\mathcal{C} \), a top vector of the Clifford algebra \( \text{Cliff} \mathcal{C} \subset \text{Cliff} \mathcal{P} \).

Then define \( \iota : \mathcal{P} \to \text{Cliff} \mathcal{P} \) as a Cantor brace, modulo linearity:
\[
\forall p \in \mathcal{P} : \iota p := \{p\}, \quad \mod\quad \iota(ax + by) \equiv atx + bty.
\]
(18)

Take \( \hat{S} \) (as a first trial) to be the least Clifford algebra that is its own Clifford algebra:
\[
\bigcup : \hat{S} \to \bigcup \hat{S} = \hat{S}.
\]
(19)

Call the quantum structures with probability vectors in \( \hat{S} \) \textit{quantum sets}. The quantum set is supposedly complex enough to represent any finite quantum structure.

Table 1 arranges basic probability vectors \( 1_n \) of \( \hat{S} \) by rank \( r \) and serial number \( n \).
Table 1 Quantum and classical sets $1_n$ by rank $r$ and serial number $n$

| $r$ | $1_n$ |
|-----|-------|
| $n$ |       |

3.1 Spin structure of $\mathcal{S}$

For each rank $r$, $\mathcal{S}[r]$ is naturally a spinor space:

- $D[r] = \text{hexpr}$ is its dimension.
- $\mathcal{S}[r-1]$ is its Cartan semivector space.
- $\mathcal{W}[r-1] := \mathcal{S}[r-1] \oplus \text{Dual } \mathcal{S}[r-1]$ is its underlying quadratic space.
- $\text{SO}(D[r-1], D[r-1])$ is its orthogonal group.
- There is a neutral symmetric Pauli form $\beta[r] : \mathcal{S}[r] \rightarrow \text{Dual } \mathcal{S}[r]$ for which the first grade $\gamma^w \in \text{Cliff}[r]$ are hermitian symmetric.
The Pauli form can be chosen to be a Berezin integral with respect to the top Grassmann element (or volume element) \( \gamma^\top \in \mathcal{W}[r] \):

\[
\beta[r-1] : \mathcal{W}[r-1] \otimes \mathcal{W}[r-1] \to \mathbb{R},
\]

\[
\forall \psi = w \oplus w' \in \mathcal{W}[r] : \quad \|\psi\|_{r-1} = \beta[r] \psi \psi := \int d\gamma^\top \psi^2 = \partial(\gamma^\top)\psi^2. \tag{20}
\]

Cliff(\( \mathcal{W}[r-1] \)) \( \cong \) EndoVec\( \mathcal{S}[r] \): the algebra of linear operators on the spinor space is isomorphic as algebra to the associated Clifford algebra Cliff\([r]\).

This \( \beta \) is just the \( \beta \) of Pauli and Chevalley expressed in the more powerful notation used by physicists. Since \( L^2(\mathcal{H}) \) designates a quadratic space defined by a quadratic Lebesgue integral over \( \mathcal{M} \), write the quantum space defined by a quadratic Berezin integral over \( \mathcal{W} \) as \( B^2(\mathcal{W}) \).

Every real Grassmann algebra \( \mathcal{G} = \text{Grass}_N \mathbb{R} \) is a spinor space for the orthogonal group whose quadratic space \( \mathcal{W} \) is the direct sum of the polar and axial vectors of \( \mathcal{G} \), grades 1 and \( N-1 \) of \( \mathcal{G} \):

\[
\mathcal{W} = \text{Grade}_1 \mathcal{G} \oplus \text{Grade}_{N-1} \mathcal{G}. \tag{21}
\]

The norm on \( \mathcal{W} \) is the quadratic Berezin form

\[
\beta : \mathcal{W} \otimes \mathcal{W} \to \mathbb{R},
\]

\[
\forall \psi = w \oplus w' \in \mathcal{W} : \quad \|\psi\| = \beta \psi \psi := \int d\gamma^\top \psi^2 = \partial(\gamma^\top)\psi^2. \tag{22}
\]

This imbedding of the quadratic space \( \mathcal{W} \) in its spinor space is isometric but not invariant under spin(\( \mathcal{W} \)), which mixes \( \text{Grade}_1 \mathcal{G} \) and \( \text{Grade}_{N-1} \mathcal{G} \).

### 4 Revised quantum set theory

Here are some adaptations of \( \mathcal{S} \) to current physics.

#### 4.1 Bosons

The number of times one set belongs to another (\( a \in b \)) is either 0 or 1. In this respect classical sets have Fermi (odd) statistics. Classical thought did not allow for Bose (even) statistics, which grossly violates the Leibniz doctrine that indistinguishable objects are one. Nor does \( \mathcal{S} \) describe elementary bosons. The Standard Model, however, requires them. Moreover, \( a \) and \( b \) are monads (first-grade elements) of the space \( \mathcal{S} \), hence fermionic, then \( \{a, b\} \) is a fermionic monad too, although \( a \lor b \) is an approximate boson and an exact palevon. Two odds make an odd in set theory, and an even in nature. \( \mathcal{S} \) violates conservation of statistics.
This is resolved by modeling the even quanta as pairs of odd quanta. These pairs can then be associated by unition. To be sure the result is odd, a unit set. But every rank \( r \) in \( \hat{S} \) has its own product \( \lor \), and this can be applied to rank \( r + 1 \) as well as \( r \): One disunites, multiplies, and reunites. The spin-statistics relation refers to one rank in the theory.

4.2 Reducibility

The power set contains subsets of every cardinality up to the maximum. In the quantum theory the corresponding construction leads to a representation of the group of the previous rank that is reducible by degree. The known quanta provide irreducible representations, and interact by irreducible couplings.

Nevertheless the Standard Model uses the unreduced classical brace \{ \ldots \} to assemble (say) a fermion probability vector from orbital, isospin, spin, and other probability vectors. The same practice works in the regular theory \( \hat{S} \).

4.3 Irreversibility

\( \hat{S} \) is founded on the highly asymmetric operation \( \iota \) of degree \( \binom{n}{1} \). All the interactions in the Standard Model are symmetric with respect to the exchange of input and output. If the input has degree 2, then so does the output, and the tensor is of degree \( \binom{2}{2} \). This degree does not occur among the basic operations in \( \hat{S} \). But it is a term in \( \iota \beta \iota \), where \( \beta \) is the Pauli adjoint operation.

Proposition: \( \delta_r \) is asymptotically neutral:

\[
\lim_{n \to \infty} \frac{p_r}{n_r} \to 1. \tag{23}
\]

This approaches the canonical signature of the probability space. The proof is straightforward. For finite ranks \( r > 2 \), however, the space is never exactly neutral. One must wonder whether this slight departure from symmetry between input and output can be used to represent a slight asymmetry between matter and anti-matter, or a non-zero vacuum energy density.

Naturally Cartan based his spinor theory on a classical space-time. There is none in nature, so the generators of the spin group should be interpreted in the earlier manner of Schur [12], as binary flips, not rotations, and then quantized. The Clifford algebra \( \text{Cliff}(n) \) now represents a finite quantum group corresponding to the classical finite group \( 2^n \).

The probability vector space of the Standard Model is a tensor product of fermionic Clifford algebras and Bose symmetric tensor algebras. Regularization replaces every Bose Lie algebra by a Palev algebra. But the fermion algebra already
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contains many palevon ones. This leads one to suspect that the bosons of the regularized Standard Model can be economically represented as fermion pairs held together by binding rather than by unition.

Such a possibility was already raised by the de Broglie two-neutrino photon, and the four-neutrino graviton considered and rejected by Feynman. The main obstacle to such constituent theories is that according to the Heisenberg indeterminacy principle, fermions near each other in position must be far apart in momentum. Then they require a correspondingly high interaction-energy for their binding. But experiment finds no such intense interaction but only the asymptotic freedom implied by the Standard Model.

In a Feynman or Yang quantum space-time, however, the operator $i\hbar$ that replaces $i\hbar$ has a finite spectrum of magnitudes with extreme values $\pm \hbar$. Presumably a self-organization akin to ferromagnetization freezes $i\hbar$ to its maximum value $i\hbar$. The Heisenberg uncertainty principle is then weakened wherever a local disorganization reduces the magnitude of $i\hbar$. Such a local defect might allow two leptons to bind into a photon, say. This re-opens the question of a di-fermion theory of the gauge bosons.

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References

1. E. Bopp and R. Haag. Über die Möglichkeit von Spinmodellen. Zeitschrift für Naturforschung 5a:644 (1950).
2. P. A. M. Dirac. Spinors in Hilbert Space. Plenum, New York (1974).
3. R. P. Feynman. Personal communication ca. 1961. Feynman did this in about 1941, before his work on the Lamb shift, and probably published this formula in a footnote, but we did not find the reference.
4. Finkelstein, D. Quantum Relativity. Heidelberg: Springer, 1996.
5. F. D. M. Haldane. “Fractional Statistics” in Arbitrary Dimensions: A Generalization of the Pauli Principle. , Physical Review Letters 67: 937 1991.
6. E. Inönü and E. P. Wigner. On the contraction of groups and their representations. Proceedings of the National Academy of Sciences 39:510-525 (1952).
7. OPERA Collaboration. Measurement of the neutrino velocity with the OPERA detector in the CNGS beam. arXiv:1109.4897 (2011)
8. T. D. Palev. Lie algebraical aspects of the quantum statistics. Unitary quantization (A-quantization). Joint Institute for Nuclear Research Preprint JINR E17-10550. Dubna (1977). hep-th/9705032.
9. T. D. Palev and J. Van der Jeugt. Jacobson generators, Fock representations and statistics of $sl(n+1)$. Journal of Mathematical Physics 43:3850-3873 (2002).
10. R. Penrose. Angular momentum: an approach to combinatorial space-time. In T. Bastin (ed.), Quantum Theory and Beyond, 151–180, Cambridge 1971. Penrose kindly shared much of this seminal work with me ca. 1960.
11. H. Saller. Operational Quantum Theory I. Nonrelativistic Structures. Springer, New York (2006).
12. I. Schur. Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen. Journal für die reine und angewandte Mathematik 139:155–250 (1911)
13. I. E. Segal. A class of operator algebras which are determined by groups. Duke Mathematical Journal 18:221–265 (1951). Especially §6A.
14. R. M. Unger. The Self Awakened: Pragmatism Unbound. Harvard University Press (2007).
15. C. N. Yang. On Quantized Space-Time. Physical Review 72:874 (1947).