We develop a quantum effective action for scalar-tensor theories of gravity which is both space-time diffeomorphism invariant and field reparameterisation (frame) invariant beyond the classical approximation. We achieve this by extending the Vilkovisky–DeWitt formalism, treating both the scalar fields and the components of the gravitational tensor field as coordinates describing a manifold, which we name the grand field space. By using tensors covariant under diffeomorphisms of this manifold, we show that scalar-tensor theories can be written in a form that is manifestly frame invariant at both classical and quantum levels. We term this formalism grand covariance.

In the same context, we show that in order to maintain manifest frame invariance, we must modify the Feynman rules of theories with a non-trivial grand field space. We show that one such theory is General Relativity by demonstrating explicitly that it has a non-zero grand field-space Riemann tensor. Thus, when constructing theories of quantum gravity, we must deal not only with curved spacetime, but also with a curved field space. Finally, we address the cosmological frame problem by tracing its origin to the existence of a new model function that appears in the path integral measure. Once this function is fixed, we find that frame transformations have no effect on the quantisation of the theory. Therefore, the grand covariant formalism ensures that our improved quantum effective action is truly unique.

* kieran.finn@manchester.ac.uk
† s.karamitsos@lancaster.ac.uk
‡ apostolos.pilaftsis@manchester.ac.uk
## Contents

I. Introduction 3

II. Covariance in Scalar Field Theories 8

III. Non-Covariance of the Ordinary Effective Action 10

IV. Vilkovisky and DeWitt’s Solution: The Covariant Effective Action 12

V. Covariant Feynman Rules 14

VI. The Geometric Structure of Gravity 18

VII. The Cosmological Frame Problem in Scalar-Tensor Theories 20

VIII. The Grand Field Space 22

IX. The Grand Configuration Space 23

X. Summary of the Grand Covariant Formalism 26

XI. Conclusions 27

Appendices 30

A. Theory with a Complex Scalar Field 30
   a. Standard Approach: Linear Parametrisation 30
      (i). Effective Potential 31
      (ii). Feynman Rules and Renormalisation 31
   b. Standard Approach: Non-Linear Parametrisation 33
      (i). Effective Potential 34
      (ii). Feynman Rules and Renormalisation 35
   c. Covariant Approach 37
      (i). Vilkovisky DeWitt Effective Potential 38
      (ii). Covariant Feynman Rules and Renormalisation 38

B. Curved Field-Space Example 39
   a. Standard Approach 40
   b. Covariant Approach 42

C. Example with Linear Potential 43

D. Field-Space Riemann Tensor for General Relativity 47

References 48
I. Introduction

The laws of nature should not depend on the way we choose to describe them. This seemingly obvious fact has historically had far-reaching consequences. For example, imposing that the laws of physics not care about the way we label space and time leads inevitably to Einstein’s celebrated theory of relativity [1]. In this paper, we shall take this principle (which we shall refer to throughout as the invariance principle) as a guiding light, and develop a formalism to construct theories that obey it manifestly.

When we measure a quantity, what we are really doing is comparing it to a particular unit, which is an entirely arbitrary quantity that we use as a standard. It can come from a physical artefact (such as the platinum-iridium cylinder at the International Bureau of Weights and Measures that was used to define the kilogram up until May this year) or an experimental measurement (the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom, used to define the second). Expressing the measurement of a quantity mathematically, requires exactly two ingredients: its numerical value with respect to some unit, and the definition of the unit itself. For instance, the length $\ell$ of a stick is expressed as

$$\ell = l \times \ell_0,$$  \hspace{1cm} (I.1)

where $\ell_0$ is the unit for length (such as the length of the King’s foot) and $l$ is a dimensionless number describing the relative magnitude of the measured quantity with respect to that unit.

Of course, we could use a different unit, such as the distance travelled by light in $1/299,792,458$ of a second. In this case, the length of the stick can be expressed as

$$\ell = \tilde{l} \times \tilde{\ell}_0.$$  \hspace{1cm} (I.2)

The invariance principle tells us that our choice of units will not affect the physical length $\ell$ and thus we can write

$$\ell_0 = (\tilde{l}/l) \times \ell'_0.$$  \hspace{1cm} (I.3)

This is simply a conversion between units, with $C \equiv \tilde{l}/l$ acting as the conversion factor. It might appear, for instance, as $1 \text{ ft} = 0.3048 \text{ m}$.

For spacetime independent changes of units, the invariance principle is made manifest by use of the Buckingham-$\pi$ theorem [2]. This theorem states that any physically meaningful equation involving dimensionful quantities can always be rewritten in terms of the dimensionless ratios $\pi_i$ of those quantities. Since such ratios are independent of the system of units, it follows that laws written in such a non-dimensionallised manner will automatically be invariant under spacetime independent changes of units.

The situation becomes more subtle, however, when we consider changes of units that
depend on spacetime. In this case, the invariance principle becomes less transparent. Any
derivatives in the theory will now act not only on the dimensionless ratios, but also on
the conversion factors and therefore extreme care must be taken to transform all quantities
appropriately. Invariance under this more general change of units was first advocated by
Weyl [3] and later by Dicke [4].

In the usual formulation of Quantum Field Theories (QFTs), invariance under such space-
time dependent changes of units is far from manifest. Making such a change of units corre-
sponds to performing a Weyl transformation

\[ X \rightarrow \tilde{X} = C^D(x)X \]

(I.4)
to all dimensionful quantities. Here \( C \) is the (spacetime dependent) conversion factor, \( X \)
is a dimensionful quantity in the theory and \( D \) is its dimension. Unless the theory is a
specifically Weyl invariant theory, it will look very different after the transformation (I.4).

There have been several previous attempts to make invariance under local changes of
units manifest. One method [5, 6] involves adding a new scalar field that transforms in a
way that compensates the effects of (I.4) in a process similar to the well-known Stueckelberg
trick [7]. Alternatively [8, 9], one can rewrite the action in terms of a conformally covariant
derivative that cancels out all terms proportional to \( dC/dx \).

However, invariance under a change of units is just one form of reparameterisation. When
writing down a QFT, we must define a set of quantum fields in which to express it. We are
always free to re-express the same theory in terms of a different set of fields. This is known
as a change of frame.

There has been much debate in the literature [10–27] as to whether such a change of frame
represents an observable change to the theory or merely a change of description. From the
above discussion, it appears that it is just a reparameterisation and therefore should not
affect any physical observables according to the invariance principle. However, it has been
shown that in the ordinary formulation of QFTs, one can get different predictions at the
quantum level depending on whether one quantises before or after changing frame. This
has become known as the cosmological frame problem. For a historical overview of the issue
see [28].

Our aim is therefore to develop a new formalism that does not suffer from the cosmo-
logical frame problem. In such a formalism, invariance under any reparameterisation of our
theory, including local changes of units, spacetime diffeomorphisms, and frame transforma-
tions, will be manifest at both the classical and the quantum levels. This will allow for a
clear distinction to be drawn between the content of the theory (the physical phenomena it
predicts) and its representation (how we have chosen to write it down).

In this paper we shall focus on scalar-tensor theories of quantum gravity [29–35] with
a field content that consists of a spin-2 graviton field \( g_{\mu\nu} \) and a set of scalar fields \( \phi^A \)
(collectively denoted as $\phi$) and with an action of the form

$$S \equiv \int d^4x \sqrt{-g} \mathcal{L} = \int d^4x \sqrt{-g} \left[ -\frac{f(\phi)}{2}R + \frac{1}{2}g^{\mu\nu}k_{AB}(\phi)\partial_\mu \phi^A \partial_\nu \phi^B - V(\phi) \right],$$

(I.5)

where $g \equiv \det(g_{\mu\nu})$. Here $f(\phi)$, $k_{AB}(\phi)$ and $V(\phi)$ are the effective Planck mass, the scalar field-space metric and the potential, respectively. We shall refer to these three functions as model functions and together they fully define our theory at the classical level.

In the context of such theories, there are two types of transformations that amount to nothing more than a change of description – spacetime diffeomorphisms and field reparameterisations. We must ensure that our theory is invariant under both of these in order to obey the invariance principle.

Spacetime diffeomorphisms consist of a changing the coordinates of spacetime:

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x^\mu).$$

(I.6)

This is just a relabelling of the points on the spacetime manifold and thus should not affect any physical observables. Diffeomorphism invariance is the backbone of General Relativity and, as such, has been much studied in the literature. We will therefore not focus on it here.

Field reparameterisations involve changing the definition of the fields of the theory by making the transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}(g_{\rho\sigma}, \phi),$$

$$\phi^A \to \tilde{\phi}^A = \tilde{\phi}^A(g_{\rho\sigma}, \phi).$$

(I.7)

Again, this is just a relabelling of the degrees of freedom in the theory and should not have a physical effect.

Spacetime diffeomorphism invariance restricts the class of field redefinitions that we have to consider. When performing the transformation (I.7), we must maintain the spacetime covariant structure of the fields and should not introduce any new spacetime tensors. This restricts the admissible set of transformations to those of the form

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu},$$

$$\phi^A \to \tilde{\phi}^A = \tilde{\phi}^A(\phi).$$

(I.8)

We will refer to these transformations as a conformal transformation and a scalar field reparameterisation, respectively. Together, they constitute a frame transformation. Under such a frame transformation, the model functions in (I.5) transform as

$$f \to \tilde{f} = \Omega^{-2} f,$$

$$k_{AB} \to \tilde{k}_{AB} = \left(k_{CD} - 6f \Omega^{-2} \Omega_{,C} \Omega_{,D} + 3\Omega^{-1}f_{,C} \Omega_{,D} + 3\Omega^{-1} \Omega_{,C} f_{,D} \right) K^C_A K^D_B,$$

$$V \to \tilde{V} = \Omega^{-4} V,$$

(I.10)

(I.11)

(I.12)
where a comma \( \partial_A \equiv \partial/\partial \phi^A \) denotes differentiation with respect to the field \( \phi^A \) and 
\( K^A_B \equiv \partial \phi^A / \partial \tilde{\phi}^B \) is the Jacobian of the scalar field reparameterisation.

We must also consider unit redefinitions of the form (1.4). However, the only units that are physically meaningful are those that are set by some physical observable. For such units, the conversion factor can depend on spacetime only through the fields in our theory so that 
\( \mathcal{C}(x) = \mathcal{C}[^\phi(x)] \). In this case, (1.4) is merely a subclass of (1.8) and (1.9). Thus, building a theory that is frame invariant and spacetime diffeomorphism invariant is already sufficient to fully obey the invariance principle.

In this paper, we will show an explicit construction of such a theory using the well-known technique of field-space covariance [36–39], whose relevance to resolving the cosmological frame problem was first pointed out in [40]. We treat both the scalar fields and the components of the graviton field as coordinates describing a manifold, which we call the grand field space. Frame transformations of the form (I.7) are then simply diffeomorphisms of this manifold. Provided we construct our theory out of objects that are both spacetime and grand field-space tensors, and then fully contract any indices, the theory will manifestly obey the invariance principle.

With this technique, the theory of General Relativity (which is just a scalar-tensor theory without the scalars) can also be expressed in terms of a grand field-space manifold. This manifold is separate from the spacetime manifold and comes with its own Riemann tensor, Ricci tensor and Ricci scalar. As we will see in Section VI, all these curvature invariants are non-zero. Thus, when studying quantum theories of gravity, we must necessarily deal not only with curved spacetime, but with a curved field space as well.

We shall express our frame invariant theory using the quantum effective action [41–44]. All predictions of the theory can be obtained from this effective action and thus defining it is sufficient to fully define the theory. However, as we shall see in Section III, the ordinary construction of the effective action depends on our choice of parametrisation and thus disobeys the invariance principle.

If we can treat gravity as a classical background, the Vilkovisky–DeWitt (VDW) formalism [45, 46], reviewed in Section IV, is enough to solve this problem. However, if we wish to treat gravity as a field and place it on the same footing as the other fields in our theory, we encounter ambiguities, which inevitably lead to the cosmological frame problem.

As we shall show, these ambiguities arise from a frame-dependent choice that must be made in the standard approach to scalar-tensor theories of gravity. The graviton field \( g_{\mu\nu} \) is normally identified as the metric of spacetime. However, \( g_{\mu\nu} \) transforms under a frame transformation (I.7) whereas the metric of spacetime does not. This identification is therefore only valid in a particular frame [25] and thus the frame invariance of the VDW formalism is ruined.

In this paper we overcome the cosmological frame problem by defining the metric of spacetime in a frame invariant manner. We achieve this through the introduction of a new model function, \( \ell = \ell[^\phi(x)] \), so that the metric of spacetime is given by \( \tilde{g}_{\mu\nu} = g_{\mu\nu} / \ell^2 \). We are therefore able to construct, for the first time, a fully frame and spacetime diffeomorphism
invariant quantum effective action for scalar-tensor theories of gravity.

In practice, the quantum effects of a theory are usually calculated using Feynman diagrams. However, as we shall see in Section V, the usual way in which these diagrams are calculated crucially depends on the frame in which they are evaluated. Feynman rules, when calculated in the usual way, are not covariant field-space tensors and thus different parameterisations of the fields will yield different sets of rules. We will show how the Feynman rules must be modified in the presence of a non-trivial field space in order to preserve reparameterisation invariance.

We adopt the following conventions throughout this paper. Lowercase Greek letters ($\mu$, $\nu$ etc.) will be used for spacetime indices and repeated indices will imply summation in accordance with the Einstein summation convention. Upper case Latin letters ($A$, $B$ etc.) will be used for field-space indices with repeated indices again implying summation. Lowercase Latin letters ($a$, $b$ etc.) will be used for configuration-space indices and will thus simultaneously represent both a discrete field-space index and a point in spacetime. For such indices we shall use the Einstein–DeWitt notation [47] in which repeated configuration-space indices imply summation over the discrete index and integration over spacetime, e.g.

$$J_a \phi^a \equiv \int d^D x_A \sqrt{-g} \sum_A J_A(x_A) \phi_A(x_A), \quad (I.13)$$

where $D$ is the number of spacetime dimensions and $g_{\mu\nu}$, with determinant $g$, is the metric of spacetime.

This paper is laid out as follows. We begin in Section II by reviewing the construction of the field and configuration spaces for scalar field theories. We then review the effective action formalism in Section III explicitly demonstrating that it is dependent on the parametrisation of the fields, therefore violating the invariance principle. We show in Section IV how Vilkovisky and DeWitt’s reformulated effective action resolves these issues when gravity can be treated as a background. In Section V we show the effect of reparameterisations on ordinary quantum calculations using Feynman diagrams and develop a method for calculating Feynman rules in a reparameterisation-invariant manner.

We show how the same geometric approach of Vilkovisky and DeWitt can be applied to gravity in Section VI, explicitly constructing the field space for General Relativity. We add scalar fields to the theory in Section VII showing that when we do, there is an ambiguity in the definition of the spacetime metric, which is responsible for the cosmological frame problem. In Section VII we construct a grand field space for the scalar and tensor fields and use it to enforce manifest invariance under a frame transformation (I.7). We then incorporate the spacetime dependence of the fields in order to construct a grand configuration space in Section VIII. This allows us to construct a fully frame and spacetime diffeomorphism invariant path integral measure, which we can then use to quantise the theory in a frame invariant way. We provide a concise description of our grand covariant formalism in Section IX before discussing our findings in Section X.
II. Covariance in Scalar Field Theories

Let us begin by reviewing the construction of the field space for scalar field theories without gravity. Such theories have actions of the form

$$S \equiv \int d^{D}x \sqrt{-g} \mathcal{L} = \int d^{D}x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} k_{AB}(\phi) \partial_{\mu} \phi^{A} \partial_{\nu} \phi^{B} - V(\phi) \right], \quad (\text{II.1})$$

where $D$ is the dimension of spacetime. In this section we will take the metric of spacetime $g_{\mu\nu}$ to be fixed and will not consider any redefinitions of the form (I.8). We will relax this assumption in Section VII.

As discussed in the Introduction, we could just as easily describe this theory in terms of a different set of fields $\tilde{\phi}$ and, according to the invariance principle, the transformation

$$\phi^{A} \rightarrow \tilde{\phi}^{A} = \tilde{\phi}^{A}(\phi) \quad (\text{II.2})$$

is just a change of description should therefore not affect any calculations. In order to make this fact explicit, we will construct a manifold known as the field space \cite{8, 9, 36–39} and treat the fields $\phi$ as coordinates describing that manifold. With such a construction, the transformation (II.2) is simply a diffeomorphism of the field space. We can then construct a theory that is explicitly reparameterisation invariant by simply building it out of field-space covariant objects.

The field space is a Riemannian manifold, and so it is equipped with a metric. Such a metric needs to satisfy the following three properties \cite{45}:

1. It should transform as a symmetric rank 2 tensor under (II.2).
2. It should be determined from the classical action (II.1).
3. It should be Euclidean for a canonically normalised theory.

The only quantity that satisfies these conditions is the model function $k_{AB}(\phi)$ and so that is what is used in the literature.

In this paper we want to introduce a new expression for the field-space metric; one that is constructive, rather than relying on the identification of a particular term in Lagrangian. We will thus define the field-space metric to be

$$G_{AB} \equiv \frac{g_{\mu\nu}}{D} \frac{\partial^{2} \mathcal{L}}{\partial (\partial_{\mu} \phi^{A}) \partial (\partial_{\nu} \phi^{B})}. \quad (\text{II.3})$$

Notice that for the theory described by (II.1), this new prescription still gives $G_{AB} = k_{AB}$. However, this new prescription is now constructive and can thus be applied to any field theory – even, for example, those with higher derivative terms.\footnote{In the case of a higher derivative theory, (II.3) would lead to a Finslerian metric \cite{48} – one that depends on both the fields and their derivatives. We will not discuss such theories here, but will save them for future work.}
ensures that the field space metric is unique for a given theory.

With the field-space metric thus defined, we can straightforwardly define a connection on the field-space manifold:

\[ \Gamma^A_{BC} \equiv \frac{1}{2} G^{AD} \left[ \frac{\partial G_{BD}}{\partial \phi^C} + \frac{\partial G_{DC}}{\partial \phi^B} - \frac{\partial G_{BC}}{\partial \phi^D} \right], \]

where \( G^{AB} \) is the inverse of \( G_{AB} \). We can also define a field-space covariant derivative

\[ \nabla_C X^A = \frac{\partial X^A}{\partial \phi^C} + \Gamma^A_{CD} X^D, \quad \nabla_C X_A = \frac{\partial X_A}{\partial \phi^C} - \Gamma^D_{CA} X_D, \quad \text{etc.} \]

When quantising the theory, the field-space manifold alone is not sufficient. In the path integral formalism, we must integrate not just over the fields, but over all configurations of the fields. In order to construct this integral in a covariant manner, we define an infinite dimensional configuration-space manifold. Each direction on this manifold represents a different configuration of the fields and thus we can describe it using coordinates

\[ \phi^a \equiv \phi^A(x_A). \]

The lowercase Latin index \( a = \{ A, x_A \} \) is a continuous index that runs over all points in spacetime in addition to all the scalar fields in the theory, as described in the Introduction.

In order to define a metric for the configuration space, we need to add one more property to the list above. The configuration-space metric should be ultra-local, i.e. it should be proportional to a Dirac delta function only and contain no derivatives of the fields. We therefore define the configuration-space metric as

\[ G_{ab} \equiv \frac{g_{\mu\nu}}{D} \frac{\delta^2 S}{\delta (\partial_\mu \phi^a) \delta (\partial_\nu \phi^b)} = G_{AB} \delta^{(D)}(x_A - x_B), \]

where \( D \) is the number of spacetime dimensions. Here we have normalised the Dirac delta function so that

\[ \int d^D x \sqrt{-g} \delta^{(D)}(x) = 1. \]

Such a definition allows \( \delta^{(D)}(x) \) to be diffeomorphism invariant.

The connection on the configuration-space manifold is as follows:

\[ \Gamma^a_{bc} \equiv \frac{1}{2} G^{ad} \left[ \delta \frac{G_{bd}}{\delta \phi^c} + \delta \frac{G_{dc}}{\delta \phi^b} - \delta \frac{G_{bc}}{\delta \phi^d} \right] = \Gamma^A_{BC} \delta^{(D)}(x_A - x_B) \delta^{(D)}(x_A - x_C), \]

and thus the configuration-space covariant functional derivative is

\[ \nabla_c X^a = \frac{\delta X^a}{\delta \phi^c} + \Gamma^a_{cd} X^d, \quad \nabla_c X_a = \frac{\delta X_a}{\delta \phi^c} - \Gamma^d_{ca} X_d, \quad \text{etc.} \]
With the configuration-space manifold defined, it is straightforward to construct theories that are reparameterisation invariant. We simply need to build our theory out of configuration-space tensors and ensure that all indices are fully contracted.

It is also easy to identify quantities that are not reparameterisation invariant. Two examples of non-invariant objects are the quantum effective action and Feynman diagrams, as we shall show in the following sections.

### III. Non-Covariance of the Ordinary Effective Action

The ordinary effective action formalism \[41–44\] fundamentally stems from the one-particle irreducible (1PI) approach in QFT. Through its application, it is possible to define an action that inherently incorporates all quantum effects beyond tree level, in principle allowing us to study radiative corrections non-perturbatively.

The starting point for the derivation of the effective action is the generating functional

\[
\mathcal{Z}[\mathcal{J}] \equiv \exp \left( \frac{i}{\hbar} \mathcal{W}[\mathcal{J}] \right) = \int [\mathcal{D}\phi] \, \mathcal{M}[\phi] \exp \left[ \frac{i}{\hbar} \mathcal{S}[\phi] + \mathcal{J}_a \phi^a \right],
\]

(III.1)
defined in the presence of an external source field \( \mathcal{J}_a \equiv \mathcal{J}_A(x_A) \) (also collectively denoted as \( \mathcal{J} \)). Here the functional integral element is \([\mathcal{D}\phi] \equiv \mathcal{D}\phi^1(x_1) \ldots \mathcal{D}\phi^n(x_n)\) and the \(\mathcal{M}[\phi]\) is the measure of the configuration space for the quantum fields \(\phi^a\). We have also introduced the reduced Planck constant \(\hbar\) as a means of keeping track different orders of quantum loops. The generating functional is reminiscent of the partition function in statistical mechanics, which is a weighted sum of Boltzmann factors over the different microstates of the system. In a similar vein, the generating functional is defined as a weighted integral over all possible configurations of the quantum fields \(\phi^a\) of the system.

From the generating functional, it is possible to arrive at the effective action via the Legendre transformation

\[
\Gamma[\phi] = \mathcal{W}[\mathcal{J}] + i\hbar \, \mathcal{J}_a \phi^a,
\]

(III.2)
where the \(\phi^a\) (collectively denoted as \(\phi\)) are the mean fields and \(\mathcal{J}_a = \mathcal{J}_a[\phi]\) is considered to be a functional of \(\phi\). In the presence of the source terms \(\mathcal{J}_a\), the mean fields and the sources are related by

\[
\phi^a = -i\hbar \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}_a}, \quad \mathcal{J}_a = -\frac{i}{\hbar} \frac{\delta \Gamma[\phi]}{\delta \phi^a}.
\]

(III.3)
The usefulness of the effective action is thus that extremising it generates the quantum-corrected equations of motion.

Already at this point, it is possible to observe that this construction lacks covariance. Since \(\phi^a\) is not a configuration-space vector, \(\mathcal{J}_a \phi^a\) is a frame-dependent expression and
thus all three equations (III.1)-(III.3) are sensitive to the way in which we parametrise the fields in our theory. This is a major drawback for this approach and one we shall try to overcome in the rest of the paper. Nonetheless, let us proceed in order to illustrate how the ordinary effective action is usually derived and pave the way for the derivation of the covariant expression.

The effective action $\Gamma[\phi]$ satisfies the following implicit functional integro-differential equation:

$$\exp \left( \frac{i}{\hbar} \Gamma[\phi] \right) = \int [D\phi] \mathcal{M}[\phi] \exp \left\{ \frac{i}{\hbar} \left[ S[\phi] + \frac{\delta \Gamma[\phi]}{\delta \phi^a} (\phi^a - \phi^a) \right] \right\}. \quad \text{(III.4)}$$

Equation (III.4) may be derived by substituting (III.2) and (III.3) in (III.1). Evidently, solving (III.4) exactly is prohibitively hard. Fortunately, it is possible to solve for $\Gamma[\phi]$ in a perturbative loop-wise expansion with the help of the background field method [49], where we split the quantum field $\phi^a$ into a background component, which we treat classically, and a quantum perturbation. Similarly, we expand $\Gamma[\phi] = S_0[\phi] + \hbar \Gamma^{(1)}[\phi] + \hbar^2 \Gamma^{(2)}[\phi] + \cdots$. At each loop order, the path integral can be evaluated explicitly. In detail, at one and two-loop order, we have

$$\Gamma^{(1)}[\phi] = i \ln \mathcal{M}[\phi] - \frac{i}{2} \ln \det S_{ab}[\phi], \quad \text{(III.5)}$$

$$\Gamma^{(2)}[\phi] = \frac{1}{8} \Delta^{ab} \Delta^{cd} S_{abcd} - \frac{1}{12} \Delta^{ab} \Delta^{cd} \Delta^{ef} S_{ace} S_{bdf}, \quad \text{(III.6)}$$

where a comma $a \equiv \delta/\delta \phi^a$ indicates a functional derivative with respect to the field $\phi^a$ and

$$\Delta^{ab} \equiv \left( \frac{\delta^2 S}{\delta \phi^a \delta \phi^b} \right)^{-1} \quad \text{(III.7)}$$

is the propagator.

As we shall explore in detail in Section V (III.6) can be represented graphically by the Feynman diagrams

$$\Gamma^{(2)}[\phi] = \begin{array}{c}
\text{Feynman Diagram}
\end{array} \quad \text{(III.8)}$$

Note that $\Gamma^{(2)}[\phi]$ contains only 1PI graphs. Other possible one-particle reducible diagrams, such as

$$\text{Diagram}
\begin{array}{c}
\text{(III.9)}
\end{array}$$

evaluate to zero and so do not contribute to the final expression (III.8).

For our theory to be fully frame invariant, we require that the effective action be a scalar under reparameterisations of the mean fields

$$\phi^a \rightarrow \tilde{\phi}^a = \tilde{\phi}^a(\phi). \quad \text{(III.10)}$$
We saw above that the explicit dependence of the generating functional on the fields $\varphi^a$ spoils the covariance, and as a result, such a transformation will not leave (III.5) and (III.6) invariant. This occurs because the difference $\varphi^a - \phi^a$ does not transform as a vector in configuration space, spoiling the covariance of the term $\frac{\delta \Gamma[\varphi]}{\delta \varphi^a}(\varphi^a - \phi^a)$ in (III.4). Similarly, the presence of ordinary functional derivatives in (III.5) and (III.6) induces extra terms in the expression for $\Gamma^{(1)}$ and $\Gamma^{(2)}$, which means that the expression for the effective action is not a configuration-space scalar.

IV. Vilkovisky and DeWitt’s Solution: The Covariant Effective Action

The so-called Vilkovisky–DeWitt (VDW) effective action formalism [36, 45, 46, 50–52] was originally developed in order to address the problems of non-covariance of the ordinary effective action that were outlined in the previous section. Unlike the conventional approach, this formalism does not unduly privilege a particular frame. In this section, we review the key results of the VDW formalism.

As noted in Section III, the non-invariance of the ordinary effective action stems from the term $\frac{\delta \Gamma[\varphi]}{\delta \varphi^a}(\varphi^a - \phi^a)$ in (III.4), which is not a configuration-space scalar. Vilkovisky’s proposal [45] was therefore to replace the difference $\varphi^a - \phi^a$ with a two-point quantity $\Sigma^a[\varphi, \phi]$ that transforms as a vector with respect to the mean field $\varphi$, a scalar with respect to the quantum field $\phi$ and satisfies $\Sigma^a[\phi, \phi] = 0$. Making this replacement in (III.4) gives

$$
\exp \left( \frac{i}{\hbar} \Gamma[\varphi] \right) = \int [D\phi] \mathcal{M}[\phi] \exp \left\{ \frac{i}{\hbar} \left[ S[\phi] + \frac{\delta \Gamma[\varphi]}{\delta \varphi^a} \Sigma^a[\varphi, \phi] \right] \right\}.
$$

(IV.1)

There are no frame-dependent terms in (IV.1) and therefore this newly defined action is fully frame invariant.

Vilkovisky’s original proposal was to use $\Sigma^a[\varphi, \phi] = \sigma^a[\varphi, \phi]$, where $\sigma^a[\varphi, \phi]$ is the tangent vector to the geodesic connecting $\varphi$ and $\phi$ evaluated at $\varphi$. The affinely normalised tangent vector can be found by solving

$$
\sigma^b[\varphi, \phi] \nabla_b \sigma^a[\varphi, \phi] = \sigma^a[\varphi, \phi],
$$

(IV.2)

along with the boundary conditions

$$
\sigma^a[\varphi, \phi] \big|_{\varphi=\phi} = 0, \quad \nabla_b \sigma^a[\varphi, \phi] \big|_{\varphi=\phi} = \delta^A_B \delta^{(D)}(x_A - x_B) \equiv \delta^a_b, \quad (IV.3)
$$

where $\nabla_a$ is the covariant derivative as defined in (II.10) and is taken to act on the first argument $\varphi$. It is possible to expand $\sigma^a[\varphi, \phi]$ in terms of the configuration-space connection $\Gamma^a_{bc}[\varphi]$ as

$$
-\sigma^a[\varphi, \phi] = -(\varphi^a - \phi^a) + \frac{1}{2} \Gamma^a_{bc}[\varphi](\varphi^b - \phi^b)(\varphi^c - \phi^c) + \cdots.
$$

(IV.4)
However, $\sigma^a$ is not the only possible choice of two-point quantity that satisfies the required properties to make the action frame invariant. In fact, any superposition of tangent vectors $\Sigma^a[\varphi, \phi] = (C^{-1}[\varphi])^a_b \sigma^b[\varphi, \phi]$ (IV.5) will do. We therefore need to introduce another requirement to fix the matrix $C^a_b$. For theories with a flat configuration space we can always go to a frame in which the metric is Euclidean and all the connections vanish. In such a frame there should be no non-trivial field-space effects and thus the VDW effective action should agree with the ordinary effective action calculated in the previous section. It can be shown [53] that this requirement forces us to choose $C^a_b = \delta^a_b$ for such theories. However, for theories with non-zero configuration-space curvature, no such frame exists and so a different condition is required to fix $C^a_b$.

The choice made by DeWitt [46] is the condition of vanishing tadpoles

$$\langle \Sigma^a[\varphi, \phi] \rangle = 0,$$

(IV.6)

where the expectation value is defined as

$$\langle F[\varphi, \phi] \rangle = \exp \left(-\frac{i}{\hbar} \Gamma[\varphi] \right) \int [D\phi] M[\phi] F[\varphi, \phi] \exp \left\{ \frac{i}{\hbar} \left[ S[\phi] + \frac{\delta \Gamma[\varphi]}{\delta \phi^a} \Sigma^a[\varphi, \phi] \right] \right\}.$$

(IV.7)

This choice was made for two main reasons. First, it allows the effective action to be calculated perturbatively as a sum of 1PI Feynman diagrams [54]. Second, when the formalism is extended to gauge theories, (IV.6) is vital in ensuring that the resulting effective action is independent of the choice of gauge-fixing conditions [50, 53].

In order to satisfy (IV.6), we find that we require

$$C^a_b[\varphi] = \langle \nabla_b \sigma^a[\varphi, \phi] \rangle = \langle \delta^a_b - \frac{1}{3} R^a_{cbd}[\varphi] \sigma^c[\varphi, \phi] \sigma^d[\varphi, \phi] + \ldots \rangle.$$

(IV.8)

Here $R^a_{cbd}$ is the Riemann tensor of the configuration-space manifold. Notice that the Riemann tensor for a flat manifold is $R^a_{cbd} = 0$ and thus we recover Vilkovisky’s original proposal in this case.

We can use the background field method to expand (IV.1) perturbatively, exactly as we did for the ordinary effective action (III.4). This gives us the following equations for the one and two-loop corrections to the VDW effective action [53]

$$\Gamma^{(1)}[\varphi] = -\frac{i}{2} \ln \det \nabla^a \nabla_b S,$$

(IV.9)

$$\Gamma^{(2)}[\varphi] = \frac{1}{8} \Delta^{ab} \Delta^{cd} \nabla_a \nabla_b \nabla_c \nabla_d S - \frac{1}{12} \Delta^{ab} \Delta^{cd} \Delta^{ef} \left( \nabla_a \nabla_c \nabla_b \nabla_d S \right) \left( \nabla_b \nabla_d \nabla_f S \right),$$

(IV.10)

where $\Delta^{ab} = (\nabla_a \nabla_b S)^{-1}$ is the covariant propagator and the parentheses $(\ldots)$ denote symmetrisation with respect to the indices enclosed. Notice that $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are both now
invariant under a frame transformation (III.10) as expected.

It is important to note that, on shell, we have \[ \frac{\delta \Gamma}{\delta \phi^a} = 0, \] in which case the expressions for the ordinary effective action (III.4) and the VDW effective action (IV.1) are identical. Thus, we are guaranteed to get the same results for on-shell observables regardless of whether we use the ordinary effective action (III.4) or the VDW effective action (IV.1). This also means that any parametrisation dependence that arises when using the ordinary effective action must vanish when the calculations are performed on shell. We show some examples of this in Appendix A.

The fact that the VDW formalism remains covariant off-shell is important for a few reasons, even if off-shell quantities will never appear in observables. First, from a geometric point of view, we expect covariance to be satisfied for the entirety of the configuration space, not just the geodesics. The ordinary approach is parametrisation-independent only for a severely restricted subspace (the on-shell region), and so the VDW approach is required to restore covariance for the whole configuration space. Second, off-shell formulations of QFTs have many important applications, such as in supersymmetry [55] and the analysis of quantum anomalies [56]. Finally, inflationary observables are often computed in the slow-roll approximation [57]. Such an approximation forces us to perform calculations in the off-shell regime.

V. Covariant Feynman Rules

In the previous section, we showed how the quantum effective action can be constructed in a fully covariant way. However, in practice, radiative corrections are often calculated perturbatively with the help of Feynman diagrams. As we will show in this section, usual Feynman diagrams are also inherently non-covariant. As such, their form depends on the parametrisation used to calculate them. Since this violates the invariance principle, we will develop an alternative method of calculating Feynman rules that is fully covariant.

We first review how ordinary Feynman diagrams may be employed to calculate correlation functions (as well as S-matrix elements through the LSZ reduction formula [58]). The derivation can be found in most textbooks on QFT (see, e.g., [59]), but our treatment here most closely follows [60] and [61].

In the path integral formulation of QFT, a correlation function in the presence of a source \( J \), is given by

\[
\langle \phi^a \phi^b \ldots, J \rangle = \frac{\int [D\phi] \mathcal{M}[\phi](\phi^a \phi^b \ldots) e^{\frac{i}{\hbar} S[\phi_0 + \phi] + J_a \phi^a}}{\int [D\phi] \mathcal{M}[\phi] e^{\frac{i}{\hbar} S[\phi_0 + \phi] + J_a \phi^a}}, \tag{V.1}
\]

where \( \phi_0 \) is an arbitrary point around which we quantise – usually taken to be the classical vacuum. This can be calculated using the generating functional \( Z[J] \) defined in (III.1).
terms of this generating functional, the correlation function becomes

\[
\langle \phi^a \phi^b \ldots, J \rangle = \frac{1}{Z[J]} \left( \frac{\delta}{\delta J_a} \frac{\delta}{\delta J_b} \ldots \right) Z[J],
\]

where \( E \) is the number of external particles.

In order to perform perturbative calculations, we use a Taylor series expansion of the action

\[
S[\phi_0 + \phi] = \sum_N S^{(N)}_{a_1 \ldots a_N} \phi^{a_1} \ldots \phi^{a_N}
\]

where

\[
S^{(N)}_{a_1 \ldots a_N} = \frac{1}{N!} \frac{\delta^N S}{\delta \phi^{a_1} \ldots \delta \phi^{a_N}} \bigg|_{\phi_0}.
\]

The constant term \( S^{(0)} \) gives factors which cancel out in (V.1) and therefore we will therefore it. We will also take \( \phi_0 \) to be the classical vacuum so that we have \( S^{(1)}_{a} = 0 \). The lowest order non-trivial term in our expansion is therefore

\[
S[\phi_0 + \phi] \approx S^{(2)}_{ab} \phi^a \phi^b,
\]

about which we shall expand the generating functional \( Z[J] \).

We must also Taylor expand the path integral measure. Using Vilkovisky’s suggestion of \( \mathcal{M}[\phi] = \sqrt{\text{det} G_{ab}} \) we find

\[
\mathcal{M}[\phi_0 + \phi] = 1 + \delta^{(D)}(0) \int d^D x \sqrt{-g} \text{Tr} \ln G_{AB}[\phi(x)] + \delta^{(D)}(0)(\ldots).
\]

From this expansion, we see that all non-trivial effects of the measure are proportional to \( \delta^{(D)}(0) \). This is a simple divergence equal to the total volume of the spacetime manifold and, as such will be removed by our regularisation procedure. Therefore, the functional form of the measure will have no impact on perturbative results and so we can set \( \mathcal{M} = 1 \).

With this knowledge, and the expansion given in (V.4), we can write

\[
Z[J] = \exp \left( \frac{i}{\hbar} \sum_{N>2} S^{(N)}_{a_1 \ldots a_N} \frac{\delta}{\delta J_{a_1}} \ldots \frac{\delta}{\delta J_{a_N}} \right) \int [D\phi] e^{\frac{i}{\hbar} S^{(2)}_{ab} \phi^a \phi^b + J_a \phi^a},
\]

The functional integral is now Gaussian and so can be calculated explicitly. The result is

\[
Z[J] = \mathcal{N} \exp \left( \frac{i}{\hbar} \sum_{N>2} S^{(N)}_{a_1 \ldots a_N} \frac{\delta}{\delta J_{a_1}} \ldots \frac{\delta}{\delta J_{a_N}} \right) \exp \left( -i\hbar J_a \Delta^{ab} J_b \right),
\]

where \( \Delta^{ab} \) is the inverse of \( S^{(2)}_{ab} \), often known as the propagator, and \( \mathcal{N} \) is an irrelevant normalisation factor.
Expanding out the two exponentials, we see that the correlation function (V.2) is

$$\langle \phi^a \phi^b \ldots, J \rangle = \frac{N}{Z[J]} \left( \frac{\delta}{\delta J_a} \frac{\delta}{\delta J_b} \ldots \prod_{N>2} \sum_{V_N=0}^{\infty} \frac{1}{V_N!} \left( \frac{i}{\hbar} S_{a_1\ldots a_N}^{(N)} \frac{\delta}{\delta J_{a_1}} \ldots \frac{\delta}{\delta J_{a_N}} \right)^{V_N} \right) \times \sum_{P=0}^{\infty} \frac{1}{P!} (-i\hbar J_c \Delta_{cd} J_d)^P.$$  

(V.9)

Feynman diagrams [62] are a beautiful graphical way to keep track of the non-zero terms in (V.9). If we represent each propagator by a line,

$$a \leftrightarrow b = \Delta^{ab},$$  

(V.10)

and each term of the expansion (V.3) with a vertex,

$$a_2 \bigcirc a_3 = N! \frac{S_{a_1\ldots a_N}^{(N)}}{\delta \phi^{a_1} \ldots \delta \phi^{a_N}} \bigg|_{\phi_0},$$  

(V.11)

then each term in (V.9) can be expressed as a diagram with $P$ propagators and $V_N$ vertices of order $N$. Calculating the correlation function then simply amounts to summing up all possible diagrams with the correct of external legs. Finally, it can easily be shown that the prefactor $N\frac{Z[J]}{Z}$ on the RHS of (V.9) has the effect of removing all diagrams that are not fully connected.

The above derivation is very elegant and has been used extensively in QFT calculations. However, it is not frame invariant. This is because, as we have seen, the quantity $\phi^a$ is not a configuration-space vector. Therefore, it will not transform in a covariant manner and cannot be contracted to form reparameterisation invariant quantities.

This means that the individual terms on the RHS of (V.3) will change under a field redefinition. Although the full sum will remain invariant (since the LHS is a configuration-space scalar), the individual terms will mix into each other and hence any finite truncation of the sum will not be invariant. Moreover, the term $J_a \phi^a$ in (III.1), as well as the definition of the correlation function (V.1) are not field covariant. As such, their form is dependent on our choice of parametrisation. Some examples of the parametrisation dependence of ordinary Feynman calculations are shown in Appendix A.

It is therefore clear that a new, covariant approach to Feynman diagrams is required if we are to construct a fully covariant QFT. The simplest way to achieve such invariance is to replace the coordinate $\phi^a$ with a configuration-space vector, much like we did in Section IV. However, in contrast to the previous section, we will employ Vilkovisky’s original choice and choose it to be the tangent vector in configuration space $\sigma^a[\phi_0, \phi_0 + \phi]$. We shall therefore
calculate the covariant correlation functions
\[ \langle \sigma^a \sigma^b \ldots, J \rangle_\sigma = \frac{\int [\mathcal{D} \sigma] (\sigma^a \sigma^b \ldots) e^{\frac{i}{\hbar} S[\phi_0 + \phi] + J_\sigma \sigma^a}}{\int [\mathcal{D} \sigma] e^{\frac{i}{\hbar} S[\phi_0 + \phi] + J_\sigma \sigma^a}}. \quad (V.12) \]

Notice that \([\mathcal{D} \phi] \mathcal{M}[\phi] = [\mathcal{D} \sigma]\) and thus the measure is trivial in this case.

We note that \(\sigma^a[\phi_0, \phi_0 + \phi] = \delta^a + O(\phi^2)\) and therefore the correlation functions \((V.1)\) and \((V.12)\) have the same pole structure. This means that the renormalised on-shell S matrix elements
\[ \prod_{I=1}^{E} \lim_{k_I^2 \to m_I^2} \frac{k_I^2 - m_I^2}{Z_I^2} \langle \sigma^a(k_1) \sigma^b(k_2) \ldots, 0 \rangle_\sigma = \prod_{I=1}^{E} \lim_{k_I^2 \to m_I^2} \frac{k_I^2 - m_I^2}{Z_I^2} \langle \phi^a(k_1) \phi^b(k_2) \ldots, 0 \rangle \quad (V.13) \]
are identical \([63]\). Here \(E\) is the number of external fields in the correlation function and \(m_I\) and \(Z_I\) are the (renormalised) mass and wavefunction renormalisation of particle \(I\), respectively. Off shell, however, the correlation functions \((V.1)\) and \((V.12)\) will not be equal in general. Note that we should continue to use the correlation functions \((V.12)\) to calculate S-matrix elements even in the presence of field-space curvature. The correlation functions of DeWitt’s modified two-point quantity \(\Sigma\) give only linear combinations of \((V.13)\), as can be seen from \((IV.5)\), and therefore should not be used.

Let us modify the definition of the generating function to make it frame invariant:
\[ \tilde{Z}[J] = \int [\mathcal{D} \sigma] e^{\frac{i}{\hbar} S[\phi_0 + \phi] + J_\sigma \sigma^a}. \quad (V.14) \]

We then find that the correlation functions are given by
\[ \langle \sigma^a \sigma^b \ldots, J \rangle_\sigma = \frac{1}{\tilde{Z}[J]} \left( \frac{\delta}{\delta J_a} \frac{\delta}{\delta J_b} \ldots \right) \tilde{Z}[J]. \quad (V.15) \]

Finally, we consider an alternative, but equivalent, covariant expansion of the action \([45]\), given by
\[ S[\phi_0 + \phi] = \sum_\mathcal{N} \tilde{S}^{(N)}_{a_1 \ldots a_n} \sigma^{a_1}[\phi_0, \phi_0 + \phi] \ldots \sigma^{a_n}[\phi_0, \phi_0 + \phi], \quad (V.16) \]
\[ \text{where} \]
\[ \tilde{S}^{(N)}_{a_1 \ldots a_n} = \frac{1}{N!} \nabla_{a_1} \ldots \nabla_{a_n} S|_{\phi_0}. \quad (V.17) \]

Now, since \(\sigma^a\) is a genuine field-space vector, all \(\tilde{S}^{(N)}\) are fully covariant field-space tensors and every term in \((V.16)\) is independently reparameterisation invariant.

We can repeat the same derivation as above to calculate the correlation functions graphically by using Feynman diagrams. Now, however, the Feynman rules must be calculated covariantly with the propagator being given by
\[ a \leftrightarrow b = (\nabla_a \nabla_b S)^{-1}, \quad (V.18) \]
and the vertex factor given by

\[ \nabla_{(a_1} \cdots \nabla_{a_n)} S \big|_{\phi_0}. \]  \hfill (V.19)

Notice that the Feynman rule is symmetrised over its indices. This is because only the symmetrised version of \((V.17)\) appears in \((V.16)\). For \((V.11)\), this symmetrisation had no effect since the ordinary functional derivative is already symmetric. However, for theories with curved field space, covariant functional derivatives do not commute and as a result, this symmetrisation is vital in fixing the order of differentiation.

In Appendices A, B, and C we perform some explicit calculations using the covariant Feynman approach, demonstrating its relation to results obtained in the ordinary approach.

### VI. The Geometric Structure of Gravity

So far we have treated gravity as a background and have not considered the metric \(g_{\mu\nu}\) to be a field. However, the Vilkovisky–DeWitt covariant approach explored in the previous sections can be readily applied to tensor fields. Doing so will lead us to the construction of the field space for gravitational theories. This space is a Riemannian manifold and is distinct from the manifold of spacetime. The goal of this section is to illustrate the geometrical features of gravity as described by General Relativity.

We begin by examining the action for General Relativity, described by the Einstein–Hilbert action,

\[ S = -\frac{1}{2} \int d^D x \sqrt{-g} R. \]  \hfill (VI.1)

We use the standard definitions

\[ \Gamma^\alpha_{\mu\nu} = \frac{g^{\alpha\beta}}{2} \left( g_{\beta\nu,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta} \right), \]  \hfill (VI.2)

\[ R^\alpha_{\mu\beta\nu} = \Gamma^\alpha_{\nu\mu,\beta} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\gamma_{\beta\gamma,\nu} \Gamma^\gamma_{\gamma,\mu} - \Gamma^\gamma_{\beta\gamma,\mu} \Gamma^\gamma_{\gamma,\nu}, \quad R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}, \]  \hfill (VI.3)

for the spacetime Christoffel symbols, Riemann tensor, Ricci tensor and Ricci scalar, respectively.

For \(D\)-dimensional gravity, the \(D(D + 1)/2\) degrees of freedom of the field space will be represented by an unordered pair of spacetime indices \((\mu\nu)\). In order to maintain consistency with the position of the indices, we take the fundamental field to be \(g^{\mu\nu}\). This means that \(\delta g^{\mu\nu}\) is a contravariant vector in field space and \(\delta g_{\mu\nu}\) is a covariant vector. \(^2\)

\(^2\) Diffeomorphism invariance reduces the physical number of degrees of freedom by \(D\). Nonetheless, even when the gauge is taken into account, the gravitational field \(g_{\mu\nu}\) is commonly expressed in terms of both the physical and gauge degrees of freedom. The effect of gauge freedom on the field space was studied in more detail in [45, 50, 52] and we shall return to it in future work.
As discussed in Section II, the field-space metric can be explicitly calculated from the classical action (VI.1) by using (II.3). However, there is a subtlety with gravity, stemming from its gauge freedom. This freedom requires us to add to the action a gauge fixing term of the form

\[
S_{GF} = -\frac{1}{2} \int d^D x \gamma \sqrt{-g} \chi^\mu g_{\mu\nu} \chi^\nu. \quad (VI.4)
\]

Here, \( \chi^\nu = 0 \) is the gauge fixing condition and \( \gamma \) is a non-negative constant. When we apply (II.3) to the sum of (VI.1) and (VI.4), we get the metric

\[
G_{(\mu\nu)(\rho\sigma)} = \frac{1}{2} \left( g_{\mu\rho} g_{\sigma\nu} + g_{\mu\sigma} g_{\rho\nu} - \alpha g_{\mu\nu} g_{\rho\sigma} \right), \quad (VI.5)
\]

where \( \alpha = \alpha(\chi^\mu, \gamma) \) is a constant that depends on the gauge fixing condition \( \chi^\mu \) and the constant \( \gamma \). For example, in de Donder gauge \( g^{\sigma\rho} \Gamma^\mu_{\rho\sigma} = 0 \), we have \( \alpha = 2 - \gamma \).

Fortunately, we have another condition that we can impose to further restrict the form of the metric. Given that \( G_{(\mu\nu)(\rho\sigma)} \) must also transform as a spacetime tensor, the inverse metric should be simply given by

\[
G^{(\mu\nu)}_{(\rho\sigma)} = g^{\alpha\mu} g^{\beta\nu} g^{\kappa\rho} g^{\lambda\sigma} G_{(\alpha\beta)(\kappa\lambda)}, \quad (VI.6)
\]
i.e. by simply raising the indices of (VI.5). However, the inverse metric can also be calculated by requiring

\[
G_{(\mu\nu)(\rho\sigma)} G^{(\rho\sigma)}_{(\kappa\lambda)} = \frac{1}{2} (\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho). \quad (VI.7)
\]
The definition (VI.7) leads to the following expression for the inverse metric:

\[
G^{(\mu\nu)}_{(\rho\sigma)} = \frac{1}{2} \left( g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\rho\nu} - \frac{2\alpha}{D\alpha - 2} g^{\mu\nu} g^{\rho\sigma} \right). \quad (VI.8)
\]
Consistency between (VI.6) and (VI.8) implies:

\[
\alpha = \frac{4}{D}. \quad (VI.9)
\]
Thus, in four dimensions, (VI.5) reduces to

\[
G_{(\mu\nu)(\rho\sigma)} = P_{\mu\nu\rho\sigma} = \frac{1}{2} (g_{\mu\rho} g_{\sigma\nu} + g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma}), \quad (VI.10)
\]
where \( P_{\mu\nu\rho\sigma} \) is Vilkovisky’s metric for gravity, derived in [45] by different considerations.

Note that this differs from the DeWitt metric [64], which imposes a time slicing condition.

3 Note that we can also arrive at (VI.5) up to an irrelevant normalisation simply by enforcing that \( G_{(\mu\nu)(\rho\sigma)} \) transforms as a spacetime tensor and is symmetric under \( \mu \leftrightarrow \nu \) and \( \rho \leftrightarrow \sigma \).

4 The solution \( \alpha = 0 \) also allows for consistency between the two equations. However, we choose to use the solution in (VI.9), since it agrees with Vilkovisky’s original calculation [45] in \( D = 4 \), as well as other results in the literature [65, 66].
and focuses only on the spatial part of the spacetime metric. In contrast, our calculation considers all components of the spacetime metric equally. This allows the metric (VI.10) to transform as a tensor under diffeomorphisms of the full spacetime.

We are now equipped to determine the curvature of the field space for gravity. This is a cumbersome but straightforward task. The expressions for the curvature tensors are exactly the same as those for the usual spacetime curvature tensors, except the indices are replaced with field-space indices. For example, the field-space Christoffel symbols and Riemann tensor are given as

\[ \Gamma^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{2} P^{\alpha\beta\gamma\delta} \left( \partial_{(\mu\nu)} P_{\gamma\delta\rho\sigma} + \partial_{(\rho\sigma)} P_{\mu\nu\gamma\delta} - \partial_{(\gamma\delta)} P_{\mu\nu\rho\sigma} \right), \]  

(VI.11)

\[ R^{\alpha\beta\gamma\delta}_{(\mu\nu)(\rho\sigma)} = \partial_{(\rho\sigma)} \Gamma^{(\mu\nu)}_{(\gamma\delta)(\alpha\beta)} - \partial_{(\gamma\delta)} \Gamma^{(\mu\nu)}_{(\rho\sigma)(\alpha\beta)} + \Gamma^{(\mu\nu)}_{(\rho\sigma)(\kappa\lambda)} \Gamma^{(\kappa\lambda)}_{(\gamma\delta)(\alpha\beta)} - \Gamma^{(\mu\nu)}_{(\gamma\delta)(\kappa\lambda)} \Gamma^{(\kappa\lambda)}_{(\rho\sigma)(\alpha\beta)}, \]  

(VI.12)

respectively, where \( \partial_{(\mu\nu)} \equiv \partial / \partial g^{\mu\nu} \). Correspondingly, the field-space Ricci tensor and Ricci scalar for gravity are given by

\[ R^{\alpha\beta}_{(\gamma\delta)} = R^{(\mu\nu)}_{(\alpha\beta)(\gamma\delta)}, \quad R = P^{\alpha\beta\gamma\delta} R^{(\alpha\beta)(\gamma\delta)}. \]  

(VI.13)

To cope with the complexity of this calculation, we employed the symbolic computer algebra system Cadabra2 \[67, 68\]. In this way we find the following explicit forms for the Riemann tensor \( R^{(\mu\nu)}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)} \) (shown in Appendix D), the Ricci tensor

\[ R_{(\mu\nu)(\rho\sigma)} = \frac{1}{4} g_{\mu\nu} g_{\rho\sigma} - \frac{D}{8} g_{\mu\rho} g_{\nu\sigma} - \frac{D}{8} g_{\mu\sigma} g_{\nu\rho}, \]  

(VI.14)

and the Ricci scalar

\[ R = \frac{D}{4} - \frac{D^2}{8} - \frac{D^3}{8}. \]  

(VI.15)

These tensors are all non-zero (except, as expected, when \( D = 1 \)). Therefore this shows that gravity has a genuinely curved field space. Indeed, we can see from (VI.15) that the field space is always negatively curved. It would be interesting to explore whether this negative curvature is the origin for the non-convergence of the path integral for pure gravity.

VII. The Cosmological Frame Problem in Scalar-Tensor Theories

After studying scalar field theories and gravitational theories separately, we now wish to combine the methods of the previous sections and look at theories with both scalar fields and gravity. In the following two sections, we will therefore construct a covariant formalism for scalar-tensor theories with an action of the form (I.5).

However, before we do so, we must address the cosmological frame problem. As discussed in the Introduction, in the standard formulation of scalar-tensor theories, one must choose a “preferred frame” in which to quantise the theory. In this case, the theoretical predictions
will depend, at the quantum level, on which frame is chosen \[25\]. This appears to violate the invariance principle and thus we must understand its origin if we are to construct a fully frame invariant formalism.

The problem stems from a subtlety regarding spacetime diffeomorphism invariance in scalar-tensor theories. Diffeomorphism invariance is normally achieved by identifying the graviton field \( g_{\mu\nu} \) as the metric of spacetime and then constructing the theory out of co-variant tensors of the spacetime manifold. However, this identification is frame dependent. Specifically, making a frame transformation (1.7) will change the definition of \( g_{\mu\nu} \), but should not change the metric of spacetime. One must therefore choose a particular frame in which to make this identification and this choice leads to the cosmological frame problem.

In order to avoid the cosmological frame problem, we shall use a different, frame invariant, definition of the metric of spacetime. The most general such definition that does not require the introduction of any new spacetime tensors is

\[
\bar{g}_{\mu\nu} \equiv \frac{g_{\mu\nu}}{\ell^2(\phi(x))} \tag{VII.1}
\]

where \( \bar{g}_{\mu\nu} \) is the metric of spacetime and \( \ell \) is a (generally spacetime dependent) length scale. In this paper, we will restrict ourselves to the case where \( \ell \) depends on \( x \) only through the scalar fields \( \phi \) in which case \( \ell(\phi) \) represents another non-singular model function in our theory.

Provided that \( \ell \) transforms as

\[
\ell \rightarrow \tilde{\ell} = \Omega \ell \tag{VII.2}
\]

under conformal transformations (1.8) and does not transform under scalar field redefinitions (1.9), then \( \bar{g}_{\mu\nu} \) is frame invariant. Thus, we may define a spacetime line element

\[
ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu \tag{VII.3}
\]

which is both frame and diffeomorphism invariant. This line element is also dimensionless, in contrast to the standard definition, and therefore qualifies as an observable according to the Buckingham-\( \pi \) theorem \[2\]. Previous authors \[17, 69–71\] have defined similar frame invariant line elements, but have assumed a particular form of \( \ell \). To the best of our knowledge, we are the first to identify \( \ell(\phi) \) as a freely selectable model function that must be specified when defining a scalar-tensor theory.

At first glance, it may appear that \( \ell \) has no physical meaning. After all, it does not appear anywhere in the classical action (1.5), and so will have no effect on any classical observable. However, as we will show below, \( \ell \) does appear in the functional measure of the path integral and therefore the choice of \( \ell \) will have an observable impact at the quantum level.

Specifying a particular form of \( \ell(\phi) \) is equivalent to specifying a “preferred frame” in

---

Note that if we do not make this assumption, then \( \ell(x) \) would act as a new field in the theory and we would have to quantise it accordingly.
the ordinary approach. The frame in which the metric of spacetime is $\bar{g}_{\mu\nu} = g_{\mu\nu}$ is the one in which $\ell(\phi) = 1$, which we shall refer to as the metric frame. However, this frame is no more “preferred” than any other in our new formalism and thus we are able to construct a scalar-tensor theory of gravity without ever singling out a particular frame. Note that in general the Einstein frame and the metric frame are different, and it is not always possible to choose a frame both with minimal coupling $f(\phi) = 1$, and with $\ell(\phi) = 1$.

VIII. The Grand Field Space

In this section, we will construct an augmented field-space manifold (the grand field space) that incorporates both the scalar fields $\phi^A$ and the gravitational tensor field $g^{\mu\nu}$ [25, 72]. To this end, we shall define the following coordinate chart:

$$\Phi^I = \begin{pmatrix} g^{\mu\nu} \\ \phi^A \end{pmatrix},$$

(VIII.1)

where $I = \{\mu\nu, A\}$.

As mentioned in the Introduction, the invariance principle tells us that any theory should be invariant under reparameterisations of the fields. Such reparameterisations are nothing but diffeomorphisms of the grand field space. In fact, the transformations (I.7) can be re-expressed in this notation as

$$\Phi^I \to \tilde{\Phi}^I(\Phi).$$

(VIII.2)

Again, as discussed in the Introduction, not all grand field-space diffeomorphisms are allowed. Invariance under spacetime diffeomorphisms restricts us to conformal transformations (I.8) and scalar field redefinitions (I.9).

We now equip our grand field space with a metric. We wish to define the metric in such a way that both spacetime and grand field-space diffeomorphism invariance remains manifest. To do so, we first define the invariant Lagrangian

$$\bar{L} = \ell^D \mathcal{L}$$

(VIII.3)

such that

$$S = \int d^D x \sqrt{-\bar{g}} \mathcal{L} = \int d^D x \sqrt{-\bar{g}} \bar{\mathcal{L}}.$$  

(VIII.4)

This definition allows $\bar{\mathcal{L}}$ to be invariant under both spacetime diffeomorphisms and grand field redefinitions. This is in contrast to the standard Lagrangian $\mathcal{L}$, which picks up a conformal factor under the conformal transformation (I.8).

We can now define the grand field-space metric in a way analogous to (II.3). Explicitly, we have

$$G_{IJ} = \frac{\bar{g}_{\mu\nu}}{D} \frac{\partial^2 \bar{\mathcal{L}}}{\partial(\partial^\mu \Phi^I)\partial(\partial^\nu \Phi^J)}.$$  

(VIII.5)
It is important to note that the effective Planck length $\ell$ is now part of the definition of the field-space metric, since it appears in the definition of $\bar{g}_{\mu\nu}$.

For the scalar-tensor theory described by (I.5) in four dimensions the grand field space metric is

$$G_{I,J} = \left( \frac{1}{2} \ell^2 f P_{\mu\rho\sigma} - \frac{1}{2} \ell^2 f B g_{\mu\nu} \right),$$

where $P_{\mu\rho\sigma}$ is defined in (VI.10).

This metric can be used to define a frame-invariant field-space line element, given by

$$d\sigma^2 = G_{I,J} d\Phi^I d\Phi^J.$$  

By construction, this line element is both spacetime-diffeomorphism invariant and frame invariant. We can also define the grand field-space connection as

$$\Gamma^I_{JK} = \frac{1}{2} G^{IL} \left[ \partial_J G_{LK} + \partial_K G_{JL} - \partial_L G_{JK} \right],$$

where $\partial_I \equiv \partial/\partial \Phi^I$. The form of the connection can then be used to construct the grand field-space covariant derivative

$$\nabla_J X^I = \frac{\partial X^I}{\partial \Phi^J} + \Gamma^I_{JK} X^K, \quad \nabla_J X_I = \frac{\partial X_I}{\partial \Phi^J} - \Gamma^K_{JI} X_K, \quad \text{etc.}$$

Anything constructed out of grand field-space tensors and the grand field-space covariant derivative will be invariant under (VIII.2) provided all indices are properly contracted.

**IX. The Grand Configuration Space**

We now wish to extend the geometric construction of the grand field space in order to take into account the spacetime dependence of the fields. This means that each coordinate now comes with a spacetime argument,

$$\Phi^i \equiv \Phi^I(x_I).$$

As in section II, the lowercase Latin index $i = \{x_I, I\}$ is a continuous index and runs over all points in spacetime as well as all the fields in our theory.

In order to maintain both manifest diffeomorphism and frame invariance, we will make use of the invariant spacetime metric (VII.11) and define the spacetime line element as in (VII.3). We will also use the corresponding invariant volume element when performing spacetime integrals and from now on, integrations of repeated configuration space indices
will be performed as

\[ X_a Y^a \equiv \int d^D x_A \sqrt{-\bar{g}} \sum_A X_A(x_A) Y^A(x_A). \tag{IX.2} \]

This choice of spacetime metric directly affects the definition of both the functional derivative and the functional determinant, and we will be explicit in defining them such that their dependence on the metric is made clear.

With the help of the spacetime metric \( \bar{g}_{\mu\nu} \), we can define functional differentiation as follows:

\[ \frac{\delta F[\Phi(x)]}{\delta \Phi(y)} \equiv \lim_{\epsilon \to 0} \frac{F[\Phi(x) + \epsilon \delta^{(D)}(x - y)] - F[\Phi(x)]}{\epsilon}, \tag{IX.3} \]

where we have defined

\[ \delta^{(D)}(x) \equiv \ell^{\mu} \delta^{(D)}(x) \tag{IX.4} \]

such that

\[ \int d^D x \sqrt{-\bar{g}} \delta^{(D)}(x) = 1. \tag{IX.5} \]

With the definition \( \text{(IX.4)} \), \( \delta^{(D)} \) is both diffeomorphism and frame invariant. As a result, functional derivatives defined as in \( \text{(IX.3)} \) will inherit their transformation properties from the functional \( F \) and field \( \Phi \).

The choice of metric \( \bar{g}_{\mu\nu} \) also affects how we take the functional determinant, since for an infinite dimensional matrix, the determinant involves an integral over the continuous degrees of freedom. We must therefore explicitly choose which volume measure we will use to count them. Using the invariant volume element derived from \( \text{(VII.3)} \), the functional determinant is given by

\[ \overline{\text{det}}(M(x, y)) \equiv \exp \left[ i \int d^D x \sqrt{-\bar{g}} \ln(M(x, x)) \right], \tag{IX.6} \]

which by construction, is both diffeomorphism and frame invariant.

Notice that we have written both the functional derivative and the functional determinant with an overbar to emphasise that these are defined with respect to the metric \( \bar{g}_{\mu\nu} \). Using any other metric (e.g. \( g_{\mu\nu} \)) would lead to a non-equivalent definition and, in general, would not maintain diffeomorphism and frame invariance.

These definitions allow us to define the metric of the grand configuration space as follows:

\[ \mathcal{G}_{ij} \equiv \frac{\bar{g}_{\mu\nu}}{\delta^2 S} \overline{\delta(\partial_\mu \Phi)} \overline{\delta(\partial_\nu \Phi)} = G_{IJ}(x_I) \delta^{(D)}(x_I - x_J). \tag{IX.7} \]

The uniqueness of the configuration-space metric was questioned by DeWitt (see discussion in Section 14 of [46]). Indeed, without the introduction of the model function \( \ell \), there would be an ambiguity as to which spacetime metric should be used in the definition \( \text{(IX.7)} \). In our prescription, however, \( \ell \) is a fundamental part of the theory, no less important than \( f \), \( k_{AB} \) or \( V \). Therefore, for a given theory, \( \ell \) must have a fixed functional form and hence
the definition (IX.7) is unique.

With the help of the grand configuration space metric, we may write down the grand configuration-space line element as

\[ D\Sigma^2[\Phi] = G_{ij}D\Phi^iD\Phi^j \]
\[ = \int d^Dx\sqrt{-\bar{g}} G_{IJ}(x)D\Phi^I(x)D\Phi^J(x) . \]  

(IX.8)

We can also construct the configuration-space connection, given by the Christoffel symbols

\[ \Gamma^i_{jk} \equiv \frac{1}{2}G^{il} \left[ \frac{\delta G_{jl}}{\delta \Phi^k} + \frac{\delta G_{lk}}{\delta \Phi^j} - \frac{\delta G_{jk}}{\delta \Phi^l} \right] = \Gamma^I_{JK}[\delta(D)(x_I - x_J)\delta(D)(x_I - x_K)]. \]  

(IX.9)

Finally, since we have an invariant configuration-space line element, we can use it to accordingly construct an invariant path-integral volume element, i.e.

\[ \sqrt{\det (G_{ij})}[D\Phi]. \]  

(IX.10)

Note that the volume element (IX.10) is, by construction, fully diffeomorphism and frame invariant. Thus, it may be used it to define a fully grand covariant effective action:

\[ \exp \left( \frac{i}{\hbar} \Gamma[\phi] \right) = \int \sqrt{\det (G_{ij})}[D\Phi] \exp \left[ \frac{i}{\hbar} \left( S[\Phi] + \frac{\delta \Gamma}{\delta \Phi^i} \Sigma^I[\phi, \Phi] \right) \right] . \]  

(IX.11)

Given that \( \det(G_{ij}) \) depends crucially on the definition of \( \bar{g}_{\mu\nu} \) through the model function \( \ell(\phi) \), there will be a non-trivial effect on quantum corrections arising from the measure (IX.10). This effect leads to the cosmological frame problem if the transformation of \( \ell(\phi) \) is not properly accounted for.

Note that in our approach, the choice of measure does not introduce an ambiguity in the definition of the effective action. Instead, it is part of the theory itself. Therefore, theories with different measures will have different but unique effective actions.

We will now comment on the meaning of the effective Planck length term \( \ell \) and its significance in the unambiguous specification of a theory. We know that \( \ell \) is irrelevant classically, as it does not appear in the classical action (I.5). However, we know that \( \ell \) will have an effect beyond the tree level. Let us therefore explicitly calculate the effect of the model function \( \ell \) on the radiative corrections by comparing (IX.11) to a theory with a different set of model functions \( \hat{\ell}, \hat{f}, \hat{k}_{AB} \) and \( \hat{V} \). The effective action for such a theory is

\[ \exp \left( \frac{i}{\hbar} \hat{\Gamma}[\phi] \right) = \int \sqrt{\det (\hat{G}_{ij})}[D\Phi] \exp \left[ \frac{i}{\hbar} \left( \hat{S}[\Phi] + \frac{\delta \hat{\Gamma}}{\delta \Phi^i} \hat{\Sigma}^I[\phi, \Phi] \right) \right] . \]  

(IX.12)

Here we have defined \( \hat{\det} \) and \( \hat{\delta}/\hat{\delta} \Phi^i \) by replacing \( \bar{g}_{\mu\nu} \) with \( \hat{g}_{\mu\nu} = g_{\mu\nu}/\hat{\ell}^2 \) in the definitions (IX.3) and (IX.6).
From (IX.7), we can see that the two grand configuration space metrics are related by

\[ \hat{G}_{ij} = \left( \frac{\ell}{\hat{\ell}} \right)^{-D-2} G_{ij}. \]  

(IX.13)

By carefully following through all the relations between the hatted and barred quantities, we can rewrite (IX.12) as

\[ \exp \left( \frac{i}{\hbar} \hat{\Gamma}[\varphi] \right) = \int \sqrt{\det (G_{ij})} \left[ D\Phi \right] \exp \left[ \frac{i}{\hbar} \int d^D x \sqrt{-g} \left\{ \left( \frac{\ell}{\hat{\ell}} \right)^D \hat{\Delta}[\Phi] + \frac{\delta \hat{\Gamma}}{\delta \Phi^I} \Sigma^I[\varphi, \Phi] 
+ \frac{\hbar}{2} N(D + 2) \ln \left( \frac{\ell}{\hat{\ell}} \right) + \frac{\hbar}{2} \left[ \left( \frac{\ell}{\hat{\ell}} \right)^D - 1 \right] \text{Tr} \ln(\hat{G}_{IJ}) \right\} \right] \]

(IX.14)

where \( N \) is the number of degrees of freedom in the theory and \( \text{Tr} \) denotes the trace over the field space.

It is now important to observe that if we consider

\[ f = \hat{f}, \quad k_{AB} = \hat{k}_{AB}, \]

\[ V = \hat{V} - \frac{\hbar}{2} \frac{N(D + 2)}{\ell^D} \ln \left( \frac{\ell}{\hat{\ell}} \right) + \frac{\hbar}{2} \left( \frac{1}{\ell^D} - \frac{1}{\hat{\ell}^D} \right) \text{Tr} \ln(\hat{G}_{IJ}), \]

(IX.15)

then the effective action (IX.14) will become identical to (IX.12). Thus, the effect of \( \ell \) can be entirely captured by the potential given in (IX.15). In fact, we could even consider a theory for which \( \hat{V} = 0 \), in which case the form of the potential would be determined entirely by \( \ell \).

We emphasise here that this is not a perturbative result, and so holds exactly to all orders.

X. Summary of the Grand Covariant Formalism

In this section, we summarise our grand covariant formalism for scalar-tensor theories. To fully specify a scalar-tensor theory, we require four model functions:

1. the effective Planck mass \( f \),
2. the scalar field-space metric \( k_{AB} \),
3. the scalar potential \( V \),
4. the effective Planck length \( \ell \).

In detail, with these model functions, the classical action is given by

\[ S = \int d^D x \sqrt{-g} \left[ -\frac{f}{2} R + \frac{1}{2} g^{\mu\nu} k_{AB} \partial_\mu \phi^A \partial_\nu \phi^B - V \right]. \]

(X.1)
We can then extract the metric of the grand configuration space, $G_{ij}$, from the classical action using

$$G_{ij} = \frac{\bar{g}_{\mu\nu}}{D} \frac{\delta^2 S}{\delta (\partial_\mu \Phi_I(x_I)) \delta (\partial_\nu \Phi_J(x_J))},$$

where $\bar{g}_{\mu\nu} = g_{\mu\nu}/\ell^2$ is the metric of spacetime as given in (VII.1). Therefore, the configuration space metric (and therefore the action) is fully determined by the content of the theory.

We can calculate the quantum effects of this theory in two equivalent ways. One way is to use the VDW action $\Gamma[\Phi]$. This can be calculated from the implicit equation

$$\exp\left(\frac{i}{\hbar} \Gamma[\varphi]\right) = \int \sqrt{\det (G_{ij})} \left[ D\Phi \right] \exp\left[ \frac{i}{\hbar} \left( S + \frac{\delta\Gamma}{\delta \Phi^i} \Sigma^i[\varphi, \Phi] \right) \right],$$

where $\Sigma^i[\Phi, \varphi]$ is defined in Section [V] and $\det$ and $\delta/\delta \Phi^i$ are defined in Section [IX].

An alternative way in which quantum corrections can be calculated is through the use of covariant Feynman diagrams as described in Section [V]. In this approach, Feynman rules are calculated in a covariant manner with the propagators given by

$$i \longleftrightarrow j = (\nabla_i \nabla_j S)^{-1}$$

and the vertices given by

$$= \nabla_{(i_1 \ldots \n)} S.$$  

Feynman diagrams can then be calculated in the usual way.

Both of the above approaches agree with the standard calculation for on-shell observables, but they additionally preserve frame invariance off shell. Moreover, the calculation of cosmological observables, which is based on the slow-roll approximation, requires us to calculate the effective potential, which is an off-shell quantity. Therefore, we should use the grand covariant effective action (X.3) instead of (III.4) for calculations such as the tensor-to-scalar ratio, spectral indices etc.

XI. Conclusions

We have developed a grand covariant formalism for scalar-tensor theories of quantum gravity. By extending the Vilkovisky–DeWitt effective action and the geometric structure of the grand configuration space, we have constructed a Quantum Field Theory that is manifestly frame and spacetime diffeomorphism invariant and thus explicitly satisfies the invariance principle.

This is in contrast to the usual approach, which requires us to identify a “preferred frame” in which the expression for the ordinary effective action (III.4) holds. The non-covariance
of (III.4) leads to an inequivalence in the standard (ordinary) approach between theories with different choices of preferred frame. This is the root of the cosmological frame problem.

Our formalism resolves this issue by identifying a new model function $\ell(\phi)$ that transforms covariantly under a change of frame. This model function relates the spacetime metric $\bar{g}_{\mu\nu}$ and the gravitational tensor field $g_{\mu\nu}$, which are equal only in the particular frame where $\ell = 1$, which we call the metric frame. Choosing the form of $\ell(\phi)$ in our formalism is equivalent to choosing a preferred frame in the conventional approach but does not unduly privilege a particular frame.

Our formalism draws a clear dividing line between two often confused concepts: the content of a theory and its representation. Once we have picked a particular form for the model functions $f, k_{AB}, V,$ and $\ell$, we have uniquely specified our QFT, and therefore all of its physical predictions. However, we may still change the representation of the theory by performing a frame transformation (VIII.2). The model functions will be different after this change of frame, but the QFT as defined by $\Gamma[\phi]$ will still have the same functional form and will make the same predictions.

The freedom of choosing the frame in which the theory is quantised still exists in our formalism, as it does in the conventional approach. However, it is now explicitly part of the content of the theory, captured by the model function $\ell$, as opposed to being expressed in terms of a particular parametrisation (the preferred frame). After all, the relation between the tensor field $g_{\mu\nu}$ and the metric of spacetime $\bar{g}_{\mu\nu}$ is a physical one and not just a convention. Two theories between which this relation differs (with all else being equal) cannot be related by a frame transformation (VIII.2), and they will give rise to different quantum predictions.

Since the covariant quantum effective action (X.3) is frame invariant and the ordinary effective action (III.4) is not, it is clear that they can only agree in at most one frame. This frame is the one in which $\ell = 1$ and additionally all the scalar fields are canonically normalised. However, such a canonical frame does not exist for theories with intrinsic field-space curvature. Thus, for such theories, the usual approach is not suitable in any frame and we must adopt the formalism developed in this paper in order to obey the invariance principle.

The above observation may be important for the development of a UV-complete quantum theory of gravity. As we have shown in Section VII, the field space of gravity is necessarily curved. This means that a quantum theory of gravity should be built in a way that respects this field-space curvature.

The transformation rules given in (IX.15) show that it is possible to capture the degree of freedom encoded in $\ell$ by simply redefining the potential. Although the scalar-tensor Lagrangian is specified by four model functions, the observables of the theory will depend only on three combinations. This means that we could, for example, consider only theories with $\ell = 1$ and, provided we consider all possibilities for $f, k_{AB}$ and $V$, we would still be able to capture the entire space of physically distinct scalar-tensor theories. However, for some theories, it may be more convenient to choose a non-trivial form of $\ell$, since this may make the potential (or other model functions) simpler.
Even if we set $\ell = 1$, there is still a redundancy in our description of scalar-tensor theories. Since our formulation of scalar-tensor theories is fully frame invariant, we can always transform to the Einstein frame where $f = 1$ and specify the rest of the content of our theory there. Thus, we can actually fully specify our theory using only the field-space metric in the Einstein frame $\hat{k}_{AB}$ and the potential $\hat{V}$.

By identifying the model function $\ell$, we have identified the source of the cosmological frame problem. Any frame transformation that does not take into account the transformation of $\ell$ will lead to a different theory with different quantum predictions. This implies that the classical action is not sufficient to fully define a QFT. One must, in addition, specify the form of $\ell$. As a consequence, the classical action corresponding to a particular QFT is not unique. In fact, one can obtain an infinite set of classical actions from a given QFT by performing redefinitions of the form (IX.15).

In this paper, we have taken the invariance principle as a fundamental guiding principle. We argue that a theory should not depend on the way it is parametrised, and therefore Lagrangians related by a frame transformation are different expressions of the same underlying theory. Based on this principle, we have developed a formalism that gives the same predictions for a given theory regardless of its parametrisation. Our formalism can be used to derive a quantum effective action that is manifestly invariant under frame transformations that include $\ell$. Therefore, once this additional degree of freedom is taken to be a model function, the quantum effective action becomes truly unique.

Acknowledgements

The authors would like to thank Jack Holguin and Chris Shepherd for useful comments and discussion. We would also like to thank Daniel Martin for his help in writing the Mathematica code used to calculate covariant Feynman rules. KF is supported by the University of Manchester through the President’s Doctoral Scholar Award. The work of AP and SK is supported by the Lancaster–Manchester–Sheffield Consortium for Fundamental Physics under STFC research grant ST/L000520/1.

---

6 It may even be possible to reduce the number of model functions required to fully specify our theory even further. The Eisenhart lift [73, 74] allows the effects of the potential to be incorporated into an extended field-space metric. However, this has only been shown at the classical level so far.
Appendices

A. Theory with a Complex Scalar Field

As an example to highlight the parametrisation dependence of the standard formulation of QFTs, we consider the example of a single complex scalar field $\phi$ with action

$$S = \int d^4x \left[ \partial_\mu \phi \partial^\mu \phi - m^2 |\phi|^2 - \lambda |\phi|^4 \right].$$

(A.1)

We choose our parameters with $m^2 < 0$ so that the vacuum is

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \rho_0 \equiv \sqrt{\frac{-m^2}{2\lambda}}.$$  

(A.2)

This theory has $U(1)$ symmetry $\phi \rightarrow e^{i\theta} \phi$, which is spontaneously broken by the vacuum (A.2). Therefore, the perturbations will have two modes, a massive Higgs mode and a massless Goldstone mode.

For simplicity we will assume a flat, static, background spacetime with Minkowski metric $\eta_{\mu\nu} = \text{diag} (1, -1, -1, -1)$.

a. Standard Approach: Linear Parametrisation

The complex field $\phi$ contains two real degrees of freedom, which we can parametrise in terms of its real and imaginary parts as

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}.$$ 

(A.3)

In this parametrisation, the action (A.1) is

$$S = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2 \right],$$

(A.4)

and the vacuum (A.2) is

$$\langle \phi_1 \rangle = \rho_0, \quad \langle \phi_2 \rangle = 0.$$ 

(A.5)

As stated before, the perturbations consist of a massive Higgs mode, corresponding to perturbations of $\phi_1$, and a massless Goldstone mode corresponding to perturbations of $\phi_2$. 

30
(i). Effective Potential

Let us start by calculating the one-loop correction to the effective action via (III.5). The inverse propagator for this theory is

\[
\frac{\delta^2 S}{\delta \phi^A(x) \delta \phi^B(y)} = \begin{pmatrix}
-\partial^2 - m^2 - 3\lambda \phi_1^2 & -2\lambda \phi_1 \phi_2 \\
-2\lambda \phi_1 \phi_2 & -\partial^2 - m^2 - 3\lambda \phi_2^2 - \lambda \phi_1^2
\end{pmatrix} \delta^{(4)}(x-y). \tag{A.6}
\]

Without loss of generality, we can use the $U(1)$ symmetry to set $\phi_2 = 0$. Thus, the one-loop effective action evaluated for a static configuration (the effective potential) in the $\overline{\text{MS}}$ renormalisation scheme is

\[
V_{\text{eff}}(\varphi) \equiv -\frac{1}{V_4} \Gamma[\phi_1 = \varphi, \phi_2 = 0] = V(\varphi) - \frac{i}{2} \ln \det G_{AB} + \frac{i}{2} \ln \det [\partial^2 + m^2 + 3\lambda \varphi^2] + \frac{i}{2} \ln \det [\partial^2 + m^2 + \lambda \varphi^2] \\
= \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4 + \frac{1}{64\pi^2} \left\{ (m^2 + 3\lambda \varphi^2)^2 \left[ \ln \left( \frac{m^2 + 3\lambda \varphi^2}{\mu^2} \right) - \frac{3}{2} \right] \right. \\
\left. + (m^2 + \lambda \varphi^2)^2 \left[ \ln \left( \frac{m^2 + \lambda \varphi^2}{\mu^2} \right) - \frac{3}{2} \right] \right\}, \tag{A.7}
\]

where $V_4$ is the total four-volume of spacetime. Notice that, for a static configuration, $\ln \det G_{AB} = 0$ in dimensional regularisation.

(ii). Feynman Rules and Renormalisation

The standard calculation (V.11) leads to the following Feynman rules:

\[
\begin{align*}
\begin{array}{c}
\text{solid line} \\
\text{dashed line}
\end{array} & = \frac{i}{p^2 - m_1^2}, & & \frac{i}{p^2}, \\
\begin{array}{c}
\text{solid line} \\
\text{double line}
\end{array} & = -6i\lambda \rho_0, & -2i\lambda \rho_0, \\
\begin{array}{c}
\text{double line} \\
\text{double line}
\end{array} & = -6i\lambda, & -6i\lambda, & -2i\lambda,
\end{align*}
\tag{A.8}
\]

where

\[
m_1^2 \equiv m^2 + 3\lambda \rho_0^2 = -2m^2 \tag{A.9}
\]

is the mass of the Higgs mode. Here we represent the Higgs mode $\phi_1$ by a solid line and the Goldstone mode $\phi_2$ by a dashed line.

Let us use these Feynman rules to calculate the renormalisation of the Higgs mass. At
one loop order we have

\[ i\Gamma_{\phi_1\phi_1}(p) = \quad + \quad + \quad + \quad = i(p^2 - m_1^2) + \frac{3i\lambda}{(4\pi)^2} A(m_1^2) + \frac{i\lambda}{(4\pi)^2} A(0) + 18i \frac{\lambda^2\rho_0^2}{(4\pi)^2} B_0(p^2, m_1, m_1) \]

\[ + 2i \frac{\lambda^2\rho_0^2}{(4\pi)^2} B_0(p^2, 0, 0) - 18i \frac{\lambda^2\rho_0^2}{(4\pi)^2 m_1^2} A(m_1^2) - 6i \frac{\lambda^2\rho_0^2}{(4\pi)^2 m_1^2} A(0). \] (A.10)

Here we have defined the following two integrals

\[ A(m^2) \equiv \int \frac{d^4k}{i\pi^2} \frac{1}{k^2 - m^2}, \] (A.11)

\[ B_0(p^2, m_1, m_2) \equiv \int \frac{d^4k}{i\pi^2} \frac{1}{k^2 - m_1^2 (p + k)^2 - m_2^2}, \] (A.12)

which we can perform using dimensional regularisation scheme to give

\[ A(m^2) = m^2 \left[ C_{UV} + 1 - \ln \left( \frac{m^2}{\mu^2} \right) \right], \] (A.13)

\[ B_0(p^2, m_1, m_2) = C_{UV} - \int_0^1 dx \ln \left( \frac{m_1^2(1-x) + m_2^2x - x(1-x)p^2}{\mu^2} \right), \] (A.14)

where \( \mu \) is the renormalisation scale and

\[ C_{UV} = \frac{2}{4 - D} - \gamma_E + \ln(4\pi) \] (A.15)

is the UV divergence that is cancelled by counterterms in the \( \overline{\text{MS}} \) renormalisation scheme. Here \( D = 4 - 2\epsilon \), and \( \gamma_E = 0.577 \ldots \) is the Euler–Mascheroni constant. We therefore have

\[ \Gamma_{\phi_2\phi_2}(p) = (p^2 - m_1^2) + \frac{\lambda m_1^2}{4\pi^2} \ln \left( \frac{p^2}{\mu^2} \right) \]

\[ - \frac{\lambda m_1^2}{(4\pi)^2} \left[ -4C_{UV} + 4 + 9 \int_0^1 dx \ln \left( \frac{x(x-1)p^2 + m_1^2}{\mu^2} \right) \right]. \] (A.16)

Note that \( A(0) = 0 \) and thus the third and final diagrams in (A.10) give no contribution.

From (A.16) we see that there is no wavefunction renormalisation, as expected, and the beta function of the Higgs mass is

\[ \beta_{m_1^2} = -\mu \frac{\partial \tilde{\Gamma}_2}{\partial \mu} = \frac{\lambda m_1^2}{2\pi^2}. \] (A.17)
We can also calculate the Goldstone self energy. At one loop we have

\[ i \Gamma_{\phi_2 \phi_2}(p) = \quad + \quad + \quad + \quad + \quad + \quad + \]

\[ = ip^2 + \frac{i \lambda}{(4\pi)^2} A(m_1^2) + \frac{3i}{(4\pi)^2} \lambda A(0) \]

\[ + 4i \frac{\lambda^2 \rho_0^2}{(4\pi)^2} B_0(p^2, m_1, 0) - 6i \frac{\lambda^2 \rho_0^2}{(4\pi)^2 m_1^2} A(m_1^2) - 2i \frac{\lambda^2 \rho_0^2}{(4\pi)^2 m_1^2} A(0) \]

\[ = ip^2 - \frac{2i \lambda m_1^2}{16\pi^2} \int_0^1 dx \ln \left(1 - \frac{x p^2}{m_1^2}\right). \quad \text{(A.18)} \]

Since (A.18) has no dependence on \( \mu \), the Goldstone mass is not renormalised and remains zero in accordance with Goldstone’s theorem.

Finally, let us compute the coupling renormalisation using the Callan Symanzic equation \[75, 76\]

\[ \left[ \mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_{m^2} \frac{\partial}{\partial m^2} \right] \tilde{V}_{\text{eff}} = 0. \quad \text{(A.19)} \]

where

\[ \tilde{V}_{\text{eff}}(\varphi) = V_{\text{eff}}(\varphi) - V_{\text{eff}}(0) \quad \text{(A.20)} \]

is the modified effective potential. From the expression for \( V_{\text{eff}} \) in (A.7) we have, at leading order,

\[ - \frac{(m^2 + 3 \lambda \varphi^2)^2 + (m^2 + \lambda \varphi^2)^2 - 2m^4}{32\pi^2} + \frac{1}{4} \beta_\lambda \varphi^4 - \frac{1}{4} \beta_{m^2} \varphi^2 = 0, \quad \text{(A.21)} \]

where we have used the identity \( \beta_{m^2} = -2 \beta_{m^2} \), which derives from (A.9).

Rearranging, and using the expression for \( \beta_{m^2} \) from (A.17) we see that the beta function for the coupling renormalisation, evaluated at the vacuum \( \varphi = \rho_0 \), is

\[ \beta_\lambda = \frac{5}{4\pi^2} \lambda^2. \quad \text{(A.22)} \]

b. Standard Approach: Non-Linear Parametrisation

Alternatively we could have used a non-linear parametrisation of the complex field

\[ \phi = \frac{1}{\sqrt{2}} \rho e^{i \sigma}. \quad \text{(A.23)} \]
In this parametrisation the action (A.1) is

\[ S = \int d^4x \left[ \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \left( \frac{\rho}{\rho_0} \right)^2 \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m^2 \rho^2 - \frac{\lambda}{4} \rho^4 \right] \]  

(A.24)

and the vacuum (A.2) is

\[ \langle \rho \rangle = \rho_0, \quad \langle \sigma \rangle = 0. \]  

(A.25)

In this parametrisation the Higgs mode is in the direction of \( \rho \) and the Goldstone mode is in the direction of \( \sigma \).

(i). Effective Potential

Let us calculate the one-loop effective action in this parametrisation using (III.5). The inverse propagator in this parametrisation is

\[ \frac{\delta^2 S}{\delta \phi^a(x) \delta \phi^b(y)} = \begin{pmatrix} -\partial^2 + \partial_\mu \sigma \partial^\mu \sigma - m^2 - 3\lambda \rho^2 & -2\partial_\mu \rho \partial^\mu \sigma - 2\rho \partial^2 \sigma - 2\rho \partial_\mu \sigma \partial^\mu & -2\rho \partial_\mu \rho \partial^\mu - \rho^2 \partial^2 \\ -2\partial_\mu \rho \partial^\mu \sigma - 2\rho \partial^2 \sigma - 2\rho \partial_\mu \sigma \partial^\mu & -2\partial_\mu \rho \partial^\mu \sigma - 2\rho \partial^2 \sigma - 2\rho \partial_\mu \sigma \partial^\mu & -2\rho \partial_\mu \rho \partial^\mu - \rho^2 \partial^2 \\ -2\partial_\mu \rho \partial^\mu \sigma - 2\rho \partial^2 \sigma - 2\rho \partial_\mu \sigma \partial^\mu & -2\partial_\mu \rho \partial^\mu \sigma - 2\rho \partial^2 \sigma - 2\rho \partial_\mu \sigma \partial^\mu & -2\rho \partial_\mu \rho \partial^\mu - \rho^2 \partial^2 \end{pmatrix} \delta^{(4)}(x-y). \]  

(A.26)

As before we can, without loss of generality, use the U(1) symmetry to set \( \sigma = 0 \). Again, we will consider a static configuration in order to calculate the effective action. In the \( \overline{\text{MS}} \) scheme, this is given by

\[ V_{\text{eff}}(\varphi) \equiv -\frac{1}{V_4} \Gamma[\rho = \varphi, \sigma = 0] \]

\[ = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 - \ln \det [\partial^2 + m^2 + 3\lambda \varphi^2] - \ln \det [\varphi^2 \partial^2] \]

\[ = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 + \frac{1}{64\pi^2} \left( m^2 + 3\lambda \varphi^2 \right)^2 \ln \left( \frac{m^2 + 3\lambda \varphi^2}{\mu^2} \right) - \frac{3}{2}. \]  

(A.27)

Notice that (A.7) and (A.27) differ off shell, highlighting the parametrisation dependence of the standard effective action. However, on shell when \( \varphi = \rho_0 = \sqrt{-m^2/\lambda} \), the two expressions agree.

Note that if we had instead taken \( m^2 > 0 \), the vacuum would lie at \( \varphi = 0 \). Surprisingly, in this case (A.7) and (A.27) do not agree even on shell. This is due to a peculiarity with the particular coordinate chart (A.23). The point \( \phi = 0 \) is multiply covered by this chart and therefore represents a coordinate singularity – a point where the chart cannot be trusted.

To rectify this problem we can define an offset parametrisation \( \phi = \frac{1}{\sqrt{2}} \left( \rho e^{i\delta} - \delta \right) \) so that the vacuum \( \phi = 0 \) is no longer at the singular point. In the offset parametrisation, the
effective potential is
\[
V_{\text{eff}}(\tilde{\rho} = \tilde{\varphi}, \tilde{\sigma} = 0) = \frac{1}{2}m^2(\tilde{\varphi} - \delta)^2 + \frac{\lambda}{4}(\tilde{\varphi} - \delta)^4
+ \frac{(m^2 + 3\lambda(\tilde{\varphi} - \delta)^2)^2}{64\pi^2} \left[ \ln \left( \frac{m^2 + 3\lambda(\tilde{\varphi} - \delta)^2}{\mu^2} \right) - \frac{3}{2} \right]
+ \frac{1}{64\pi^2} \frac{\delta^2}{\tilde{\varphi}^2} (m^2 + \lambda(\tilde{\varphi} - \delta)^2)^2 \left[ \ln \left( \frac{\delta m^2 + \lambda(\tilde{\varphi} - \delta)^2}{\varphi^2} \right) - \frac{3}{2} \right], \tag{A.28}
\]
which we can see does agree with (A.7) at \(\tilde{\varphi} = \delta\). Thus, in order to calculate the effective action (A.27) at \(\varphi = 0\), we should take the limit \(\tilde{\varphi} \to \delta \to 0\) in (A.28), which gives us
\[
V_{\text{eff}}(0) = \frac{2m^2}{64\pi^2} \left[ \ln \left( \frac{m^2}{\mu^2} \right) - \frac{3}{2} \right] \tag{A.29}
\]
in agreement with (A.7). This expression will be needed for the Callan–Symanzic equation.

(ii). Feynman Rules and Renormalisation

The standard Feynman rules from (V.11) in this parametrisation are:
\[
\parbox{1cm}{\begin{picture}(1,1)
\put(0,0){\line(1,0){1}}
\put(0,0){\line(0,1){1}}
\put(0,0){\line(1,1){0.5}}
\end{picture}} = \frac{i}{p^2 - m_1^2}, \quad \parbox{1cm}{\begin{picture}(1,1)
\put(0,0){\line(-1,0){1}}
\put(0,0){\line(0,1){1}}
\put(0,0){\line(-1,1){0.5}}
\end{picture}} = \frac{i}{p^2},
\]
\[
\parbox{1cm}{\begin{picture}(1,1)
\put(0,0){\line(1,0){1}}
\put(0,0){\line(0,-1){1}}
\put(0,0){\line(1,-1){0.5}}
\end{picture}} = -6i\lambda \rho_0, \quad \parbox{1cm}{\begin{picture}(1,1)
\put(0,0){\line(1,0){1}}
\put(0,0){\line(0,-1){1}}
\put(0,0){\line(1,-1){0.5}}
\put(-0.1,0.1){\makebox[0cm][c]{\(k_1\)}}
\put(0.1,-0.1){\makebox[0cm][l]{\(k_2\)}}
\end{picture}} = -\frac{2i}{\rho_0} k_1 \cdot k_2,
\]
\[
\parbox{1cm}{\begin{picture}(1,1)
\put(0,0){\line(-1,0){1}}
\put(0,0){\line(0,-1){1}}
\put(0,0){\line(-1,-1){0.5}}
\end{picture}} = -6i\lambda, \quad \parbox{1cm}{\begin{picture}(1,1)
\put(0,0){\line(-1,0){1}}
\put(0,0){\line(0,-1){1}}
\put(0,0){\line(-1,-1){0.5}}
\put(-0.1,0.1){\makebox[0cm][c]{\(k_1\)}}
\put(0.1,-0.1){\makebox[0cm][l]{\(k_2\)}}
\end{picture}} = -\frac{2i}{\rho_0} k_1 \cdot k_2. \tag{A.30}
\]
As before, the solid line represents the Higgs mode and the dashed line represents the Goldstone mode. As expected these Feynman rules are different from (A.8), showing explicitly the parametrisation non-invariance of this approach.
If we calculate the Higgs mass renormalisation with this parametrisation, we will find

\[
\begin{align*}
  i\Gamma_\rho(p) &= \frac{3i\lambda}{(4\pi)^2} A(m_1^2) + \frac{2i}{\rho_0} \int \frac{d^4k}{(2\pi)^4} + 18i \frac{\lambda^2 \rho_0^2}{(4\pi)^2} B_0(p^2, m_1, m_1) \\
  &+ \frac{i}{2(4\pi)^2} p^4 B_0(p^2, 0, 0) - 18i \frac{\lambda^2 \rho_0^2}{(4\pi)^2 m_1^2} A(m_1^2) + 6i \frac{\lambda^2 \rho_0^2}{m_1^2} \int \frac{d^4k}{(2\pi)^4} \\
  &= i(p^2 - m_1^2) + \frac{3i\lambda m_1^2 + \lambda p^4}{(4\pi)^2} \ln \left( \frac{p^2}{\mu^2} \right) \\
  &+ \frac{i\lambda m_1^2}{(4\pi)^2} \left[ \left( 3 + \frac{p^4}{m_1^4} \right) C_{UV} - 6 + 2 \frac{p^4}{m_1^4} - 9 \int_0^1 dx \ln \left( \frac{(x-1)p^2 + m_1^2}{\mu^2} \right) \right].
\end{align*}
\]

We see that, as expected, this differs from (A.16) off shell. In fact, due to the presence of the \( p^4 \) divergence, this theory is naively non-renormalisable. However, if we only consider on-shell momentum, so that \( p^2 = m_1^2 \) the two expressions, (A.16) and (A.31) are equal. As a result, the beta function will be given by (A.17).

We can also calculate the Goldstone mass renormalisation

\[
\begin{align*}
  i\Gamma_\sigma(p) &= \quad + \quad + \quad + \\
  &= i(p^2 - m_1^2) + \frac{ip^2}{(4\pi)^2 \rho_0^2} A(m_1^2) + 6i \frac{\lambda p^2}{(4\pi)^2 m_1^2} A(m_1^2) \\
  &- 2i \frac{p^2}{\rho_0^2} \int \frac{d^4k}{(2\pi)^4} + \left[ \frac{i3p^2 - m_1^2}{(4\pi)^2 \rho_0^2} A(m_1^2) + i \frac{(p^2 - m_1^2)^2}{(4\pi)^2 \rho_0^2} B_0(p^2, m_1, 0) \right] \\
  &= ip^2 + \frac{5p^2 - m_1^2}{16\pi^2 \rho_0^2} m_1^2 \left[ C_{UV} + 1 - \ln \left( \frac{m_1^2}{\mu^2} \right) \right] \\
  &+ \frac{i(p^2 - m_1^2)^2}{16\pi^2 \rho_0^2} m_1^2 \left[ C_{UV} - \int_0^1 dx \ln \left( \frac{(1-x)m_1^2 - x(1-x)p^2}{\mu^2} \right) \right].
\end{align*}
\]

As before, this expression differs from the expression obtained using the linear parametrisation (A.18) and also contains non-renormalisable terms. However, on-shell, when \( p^2 = 0 \), we have

\[
\Gamma_\sigma(p^2 = 0) = 0
\]

in agreement with (A.18) and the Goldstone mass is not renormalised as expected by Goldstone’s theorem.
Finally we look at the coupling renormalisation, which we shall calculate through the Callan-Symanzic equation (A.19) as before. Using the expression (A.27) for the effective action gives us

\[- \frac{(m^2 + 3\lambda\varphi^2)^2 - 2m^4}{32\pi^2} + \frac{1}{4}\beta_\lambda \varphi^4 - \frac{1}{4}\beta_{m^4} \varphi^2 = 0. \tag{A.34}\]

On shell when \(\varphi = \rho_0\), this becomes identical to (A.21), which means that the beta function for the coupling renormalisation is

\[\beta_\lambda = \frac{5}{4\pi^2}\lambda^2 \tag{A.35}\]

as before.

Although the two approaches led to several differences in the intermediate, off-shell results, as the above calculation demonstrates all physical observables are the same regardless of the parametrisation.

c. Covariant Approach

We have shown in the main text how to alleviate the parametrisation dependence of quantum calculations by using an explicitly covariant formalism. Let us now repeat the above calculations using this formalism to show how parametrisation invariance is maintained.

For the linear parametrisation (A.3), the field-space is trivial, and so there is no difference between the covariant approach and the standard (ordinary) approach. Thus, the VDW effective potential will be (A.7), the covariant Feynman rules will give (A.8), and the renormalisation group calculations will be identical to those in Section A.a.

We will therefore focus on the non-linear parametrisation (A.23). In this parametrisation, the configuration-space metric (II.7) is

\[G_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & (\rho/\rho_0)^2 \end{pmatrix} \delta^{(4)}(x_a - x_b) \tag{A.36}\]

and the non-zero configuration-space Christoffel symbols can be calculated as

\[\Gamma^{\rho(z)}_{\sigma(x)\sigma(y)} = -\frac{\rho(z)}{\rho_0^2} \delta^{(4)}(z - x)\delta^{(4)}(z - y), \tag{A.37}\]

\[\Gamma^{\sigma(z)}_{\rho(x)\sigma(y)} = \frac{1}{\rho(z)} \delta^{(4)}(z - x)\delta^{(4)}(z - y). \tag{A.38}\]

\footnote{Note that we had to use (A.28) to calculate \(V_{\text{eff}}(\varphi = 0)\). Had we used (A.27) instead, the two results would not have agreed. As stated earlier, this is because the parametrisation (A.23) features a coordinate singularity at \(\rho = 0\) and so cannot be trusted there.}
(i). Vilkovisky DeWitt Effective Potential

Let us first calculate the Vilkovisky DeWitt effective action for this theory using (IV.9). The covariant $2 \times 2$ inverse propagator is

$$\nabla_a \nabla_b S = \begin{pmatrix} -\partial^2 + \partial_\mu \sigma \partial^\mu \sigma - m^2 - 3\lambda \rho^2 & \rho \partial_\mu \sigma \partial^\mu \\ \rho \partial_\mu \sigma \partial^\mu & -\rho \partial_\mu \rho \partial^\mu - \rho^2 \partial^2 + \rho^2 \partial_\mu \sigma \partial^\mu \sigma - m^2 \rho^2 - \lambda \rho^4 \end{pmatrix} \delta(x_I - x_J),$$

where $\phi^a = (\rho, \sigma)$. As before, we can use the $U(1)$ symmetry to set $\sigma = 0$ without loss of generality. We will also consider a static configuration as before. Therefore, the one-loop VDW effective potential in the $\overline{\text{MS}}$ scheme reads

$$V_{\text{eff}}(\phi) \equiv -\frac{1}{V_4} \Gamma[\rho = \phi, \sigma = 0] = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 - \ln \det \left[ \partial^2 + m^2 + 3\lambda \varphi^2 \right] - \ln \det \left[ \partial^2 + m^2 + \lambda \varphi^2 \right] - \ln(\varphi^2) + \ln \det[G_{ab}]$$

$$= \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 + \frac{1}{64\pi^2} \left\{ (m^2 + 3\lambda \varphi^2)^2 \left[ \ln \left( \frac{m^2 + 3\lambda \varphi^2}{\mu^2} \right) - \frac{3}{2} \right] \right\},$$

where $V_4$ is the four-volume. Observe that the expression (A.40) is identical to (A.7). As expected, the Vilkovisky DeWitt effective action is independent of parametrisation.

(ii). Covariant Feynman Rules and Renormalisation

With the help of (V.19), we can calculate the covariant Feynman rules for this theory. We find them to be

$$\begin{array}{c}
\text{\tiny \includegraphics[width=0.2\textwidth]{C1.png}} = \frac{i}{p^2 - m_1^2}, & \text{\tiny \includegraphics[width=0.2\textwidth]{C2.png}} = \frac{i}{p^2}, \\
\text{\tiny \includegraphics[width=0.2\textwidth]{C3.png}} = -6i\lambda \rho_0, & \text{\tiny \includegraphics[width=0.2\textwidth]{C4.png}} = -2i\lambda \rho_0, \\
\text{\tiny \includegraphics[width=0.2\textwidth]{C5.png}} = -6i\lambda, & \text{\tiny \includegraphics[width=0.2\textwidth]{C6.png}} = -6i\lambda, & \text{\tiny \includegraphics[width=0.2\textwidth]{C7.png}} = -2i\lambda.
\end{array}$$

By construction, these Feynman rules are identical to (A.8) and thus all RG calculation are identical both on and off shell.
We wish to consider a simple toy model with genuine field-space curvature in order to study the effect this has on the quantum observables. Since it is impossible to have curvature in one dimension, we consider a theory with two fields $\rho$ and $\sigma$, and take $\sigma$ to be an angular variable with a shift symmetry. In order to avoid ghosts, the metric of the field space must be positive-definite. Consequently, we take our field-space metric to be

$$G_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & (\rho/\rho_0)^{2n} \end{pmatrix}. \tag{B.1}$$

The non-zero Christoffel symbols of this metric are easily calculated

$$\Gamma^\rho_{\sigma\sigma} = -n\frac{\rho^{2n-1}}{\rho_0^{2n}}, \quad \Gamma^\sigma_{\rho\sigma} = \frac{n}{\rho}. \tag{B.2}$$

From these, we obtain the non-zero components of the field-space Riemann tensor

$$R^\rho_{\sigma\rho\sigma} = \frac{n(n-1)}{\rho^2} \frac{\rho^{2n}}{\rho_0^{2n}}, \quad R^\rho_{\sigma\sigma\rho} = \frac{n(n-1)}{\rho^2} \frac{\rho^{2n}}{\rho_0^{2n}}, \quad R^\sigma_{\rho\rho\sigma} = \frac{n(n-1)}{\rho^2}, \quad R^\sigma_{\rho\sigma\rho} = -\frac{n(n-1)}{\rho^2}. \tag{B.3}$$

We see that provided $n \neq 0, 1$, the Riemann tensor is non-zero and thus the field space is curved. Notice that $n = 0$ and $n = 1$ correspond to the two flat field space examples we have looked at already in Appendix A.a and A.b respectively.

The simplest model with curvature is therefore the case $n = 2$, which has the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \left( \frac{\rho}{\rho_0} \right)^4 \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m^2 \rho^2 - \frac{\lambda}{4} \rho^4. \tag{B.4}$$

We will consider the symmetry-broken vacuum

$$\langle \rho \rangle = \rho_0 \equiv \sqrt{-\frac{m^2}{\lambda}}, \tag{B.5}$$

as we have done before.
a. Standard Approach

We start by calculating the renormalisation group flow in the standard way by looking at the standard Feynman rules \([V.11]\). For the Lagrangian \([B.4]\), these are given by

\[
\frac{i}{p^2 - m_1^2}, \quad \frac{i}{p^2}, \\
-6i\lambda_0, \quad -4i\frac{k_1 \cdot k_2}{\rho_0}, \\
-6i\lambda, \quad -12i\frac{k_3 \cdot k_4}{\rho_0^2},
\]

(B.6)

where \(m_1^2 \equiv m^2 + 3\lambda_0^2\) as before. Higher order interactions also exist, however they will not be necessary for the one-loop calculations we will perform.

First, we will calculate the self-energy of the \(\rho\) field. This is given by

\[
\Gamma_{\rho\rho}(p) = \frac{i}{p^2 - m_1^2} + \frac{3i\lambda}{(4\pi)^2} A(m_1^2) + 18i\frac{\lambda^2\rho_0^2}{(4\pi)^2} B_0(p^2, m_1, m_1) - 18i\frac{\lambda^2\rho_0^2}{(4\pi)^2 m_1^2} A(m_1^2)
\]

\[
+ 12\frac{\lambda}{m_1^2} \int \frac{d^4k}{(2\pi)^4} - 6\frac{1}{\rho_0^2} \int \frac{d^4k}{(2\pi)^4} + \frac{2ip^4}{(4\pi)^2 \rho_0^2} B_0(p^2, 0, 0)
\]

\[
=i(p^2 - m_1^2) + \frac{3i\lambda}{(4\pi)^2} A(m_1^2) + 18i\frac{\lambda^2\rho_0^2}{(4\pi)^2} B_0(p^2, m_1, m_1) - 18i\frac{\lambda^2\rho_0^2}{(4\pi)^2 m_1^2} A(m_1^2)
\]

\[
+ 12\frac{\lambda}{m_1^2} \int \frac{d^4k}{(2\pi)^4} - 6\frac{1}{\rho_0^2} \int \frac{d^4k}{(2\pi)^4} + \frac{2ip^4}{(4\pi)^2 \rho_0^2} B_0(p^2, 0, 0)
\]

\[
= i(p^2 - m_1^2) + \frac{3i\lambda m_1^2}{(4\pi)^2} \left[ 3 + 4\frac{p^4}{m_1^4} \right] C_{\text{UV}} + 6 \ln \left( \frac{m_1^2}{\mu^2} \right) - 4\frac{p^4}{m_1^4} \ln \left( \frac{p^2}{\mu^2} \right)
\]

\[
- 9 \int_0^1 dx \ln \left( \frac{m_1^2 - x(1-x)p^2}{\mu^2} \right) - 6 + 8\frac{p^4}{m_1^4} \right]. \quad \text{(B.7)}
\]

As expected, the non-renormalisability of the theory leads to a UV-divergent term proportional to \(p^4\), which cannot be absorbed into a counterterm. In order to compare to the covariant approach, we calculate the on-shell self energy as follows:

\[
\Gamma_{\rho\rho}(p^2 = m_1^2) = \frac{\lambda m_1^2}{(4\pi)^2} \left[ -7C_{\text{UV}} - 7 \ln \left( \frac{m_1^2}{\mu^2} \right) + 20 - 3\sqrt{3}\pi \right]. \quad \text{(B.8)}
\]
We now compare to the Goldstone self-energy. This is given by

\[ i \Gamma_{\sigma\sigma}(p) = \quad \quad + \quad + \]

\[ = i p^2 + \frac{6i p^2}{(4\pi)^2 \rho_0^2} A(m_1^2) - 12i \frac{p^2 \lambda}{(4\pi)^2 m_1^2} A(m_1^2) \]

\[ - 16 \frac{p^2}{\rho_0^2 m_1^2} \int \frac{d^4 k}{(2\pi)^4} + 4i \left\{ \frac{3p^2 - m_1^2}{(4\pi)^2 \rho_0^2} A(m_1^2) + \frac{(p^2 - m_1^2)^2}{(4\pi)^2 \rho_0^2} B_0(p^4, m_1, 0) \right\} \]

\[ = i p^2 + 4i \frac{3p^2 - m_1^2}{(4\pi)^2 \rho_0^2} m_1^2 \left[ C_{UV} + 1 - \ln \left( \frac{m_1^2}{\mu^2} \right) \right] \]

\[ + 4i \frac{(p^2 - m_1^2)^2}{(4\pi)^2 \rho_0^2} \left[ C_{UV} + 1 - \int_0^1 dx \ln \left( \frac{x p^2 - m_1^2}{\mu^2} \right) \right] . \quad (B.9) \]

As before, due to the non-renormalisability of the theory, this expression contains divergences that cannot be absorbed by a counterterm. However, on-shell we have \( p^2 = 0 \) and the expression reduces to

\[ \Gamma_{\sigma\sigma}(p^2 = 0) = 0, \quad (B.10) \]

implying that the Goldstone boson receives no correction to its mass as expected.

Finally, it is instructive to calculate the tree-level S-matrix element for \( \rho \rho \rightarrow \sigma \sigma \). Taking into account the contributing diagrams, find

\[ i M(\rho \rho \rightarrow \sigma \sigma) = \]

\[ = -6i \frac{s}{\rho_0^2} - 6i \frac{s m_1^2}{\rho_0^2 (s - m_1^2)} - 4i \frac{(m_1^2 - t)^2}{\rho_0^2 t} - 4i \frac{(m_1^2 - u)^2}{\rho_0^2 u} \]

\[ = -2i \frac{\rho_0^2}{\rho_0^2} \left[ 3 \frac{s^2}{s - m_1^2} + 2 \frac{(m_1^2 - t)^2}{t} + 2 \frac{(m_1^2 - u)^2}{u} \right] , \quad (B.11) \]

where \( s = (k_1 + k_2)^2, t = (k_1 - k_3)^2, u = (k_1 - k_4)^2 \) are the standard Mandelstam variables. Note that in the high energy limit, \( M(\rho \rho \rightarrow \sigma \sigma) \propto -2s/\rho_0^2 \).
b. Covariant Approach

We now perform analogous calculations in the covariant formalism using (V.19). Here we show a limited set of the covariant Feynman rules for this theory

\[
\begin{align*}
\hline & = \frac{i}{p^2 - m_1^2}, & \hline & = \frac{i}{p^2}, \\
\hline & = -6i\lambda \rho_0, & \hline & = -4i\lambda \rho_0, \\
\hline & = -6i\lambda, & \hline & = -24i\lambda.
\end{align*}
\]

(B.12)

There are also an infinite set of higher order vertices, which we do not calculate since they do not affect the one-loop calculations we make in this section.

Finally, there is the \(\rho\rho\sigma\sigma\) vertex. Due to the curvature of the field space there is an ambiguity in the order in which the covariant derivatives are taken when calculating this vertex. In Section V we argued that the correct approach was to symmetrise over all possible orderings. Nevertheless, we calculate each ordering explicitly. We have

\[
\begin{align*}
\begin{tikzpicture}
\begin{feynman}
\vertex (a) at (1,0);
\vertex (b) at (0,0);
\vertex (c) at (0,1);
\vertex (d) at (1,1);
\diagram [edges={->}, every loop/.style={min distance=7cm}]
{ (a) -- (b) -- (c) -- (d) -- (a),
  (a) -- (c) -- (b) -- (d) -- (a),
  (a) -- (d) -- (c) -- (b) -- (a),
  (b) -- (d) -- (c) -- (a) -- (b),
  (b) -- (c) -- (d) -- (a) -- (b),
  (c) -- (d) -- (a) -- (b) -- (c),
  (d) -- (a) -- (b) -- (c) -- (d)
};
\end{feynman}
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
\begin{feynman}
\vertex (a) at (1,0);
\vertex (b) at (0,0);
\vertex (c) at (0,1);
\vertex (d) at (1,1);
\diagram [edges={->}, every loop/.style={min distance=7cm}]
{ (a) -- (b) -- (c) -- (d) -- (a),
  (a) -- (c) -- (b) -- (d) -- (a),
  (a) -- (d) -- (c) -- (b) -- (a),
  (b) -- (d) -- (c) -- (a) -- (b),
  (b) -- (c) -- (d) -- (a) -- (b),
  (c) -- (d) -- (a) -- (b) -- (c),
  (d) -- (a) -- (b) -- (c) -- (d)
};
\end{feynman}
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
\begin{feynman}
\vertex (a) at (1,0);
\vertex (b) at (0,0);
\vertex (c) at (0,1);
\vertex (d) at (1,1);
\diagram [edges={->}, every loop/.style={min distance=7cm}]
{ (a) -- (b) -- (c) -- (d) -- (a),
  (a) -- (c) -- (b) -- (d) -- (a),
  (a) -- (d) -- (c) -- (b) -- (a),
  (b) -- (d) -- (c) -- (a) -- (b),
  (b) -- (c) -- (d) -- (a) -- (b),
  (c) -- (d) -- (a) -- (b) -- (c),
  (d) -- (a) -- (b) -- (c) -- (d)
};
\end{feynman}
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
\begin{feynman}
\vertex (a) at (1,0);
\vertex (b) at (0,0);
\vertex (c) at (0,1);
\vertex (d) at (1,1);
\diagram [edges={->}, every loop/.style={min distance=7cm}]
{ (a) -- (b) -- (c) -- (d) -- (a),
  (a) -- (c) -- (b) -- (d) -- (a),
  (a) -- (d) -- (c) -- (b) -- (a),
  (b) -- (d) -- (c) -- (a) -- (b),
  (b) -- (c) -- (d) -- (a) -- (b),
  (c) -- (d) -- (a) -- (b) -- (c),
  (d) -- (a) -- (b) -- (c) -- (d)
};
\end{feynman}
\end{tikzpicture}
\end{align*}
\]

where the ordering is denoted by the indices after the diagram. In the above, we did not display the other orderings of the two individual \(\rho\) and \(\sigma\) particles when calculating these rules, which may be obtained by exchanging \(k_1 \leftrightarrow k_2\) and/or \(k_3 \leftrightarrow k_4\).

It is interesting to calculate the form of the vertices when taking all external particles to be on-shell. We set

\[
\begin{align*}
k_1^2 = k_2^2 = m_1^2 = 2\lambda \rho_0^2,
\quad k_3^2 = k_4^2 = 0.
\end{align*}
\]

(B.14)
Conservation of momentum then implies that
\[ 0 = k_1 + k_2 + k_3 + k_4, \quad k_1 \cdot k_2 = k_3 \cdot k_4 - m_1^2, \quad k_1 \cdot k_3 = k_2 \cdot k_4, \quad k_1 \cdot k_4 = k_2 \cdot k_3. \] (B.15)

Employing these relations, we see that on-shell, all six orderings are equal
\[
\begin{align*}
\kappa_1 \kappa_2 \kappa_3 \kappa_4 &\quad \rho \rho \sigma \sigma = \kappa_1 \kappa_2 \kappa_3 \kappa_4 = \kappa_2 \kappa_3 \kappa_4 \rho \sigma \sigma = \kappa_2 \kappa_3 \kappa_4 \sigma \rho \rho = \kappa_2 \kappa_3 \kappa_4 \sigma \rho \sigma = \kappa_3 \kappa_4 \kappa_1 \kappa_2 \rho \rho \sigma \\
&= 4i\lambda - 4i\frac{k_3 \cdot k_4}{\rho_0^2}. \quad \text{(B.16)}
\end{align*}
\]

Notice that the expression on (B.16) is invariant under \( k_1 \leftrightarrow k_2 \) and \( k_3 \leftrightarrow k_4 \).

We have found that ordering does not matter when the particles are on shell. However, any quantum calculation will involve off-shell particles and for these, the ordering will make a difference. It is therefore important to use the fully symmetrised rule
\[
\begin{align*}
\kappa_1 \kappa_2 \kappa_3 \kappa_4 &\quad \rho \rho \sigma \sigma = \frac{2i\lambda}{3} - \frac{2i}{3}\left(\frac{(k_1 - k_3) \cdot (k_2 - k_4) + (k_1 - k_4) \cdot (k_2 - k_3)}{\rho_0^2}\right), \quad \text{(B.17)}
\end{align*}
\]
as discussed in Section V.

Let us now calculate the \( \rho \rho \) and \( \sigma \sigma \) self-energy as we did above. For the \( \sigma \sigma \) self-energy, we obtain
\[
\begin{align*}
i\Gamma_{\rho \rho}(p) &\quad = i(p^2 - m_1^2) + \frac{3i\lambda}{(4\pi)^2} A(m_1^2) + 18i\frac{\lambda^2 \rho_0^2}{(4\pi)^2} B_0(p^2, m_1, m_1) - 18i\frac{\lambda^2 \rho_0^2}{(4\pi)^2 m_1^2} A(m_1^2) \\
&\quad - 12i\frac{\lambda^2 \rho_0^2}{(4\pi)^2 m_1^2} A(0) + 8i\frac{\lambda^2 \rho_0^2}{(4\pi)^2} B_0(p^2, 0, 0) \\
&\quad + i\frac{1}{2}\left[\frac{3\lambda + p^2}{(4\pi)^2} \frac{1}{\rho_0^2} A(0) + \frac{1}{\rho_0^2} \int d^4k(2\pi)^4\right] \\
&\quad = i(p^2 - m_1^2) + \frac{i\lambda m_1^2}{(4\pi)^2} \left[ 7C_{\text{UV}} + 6 \ln \left(\frac{m_1^2}{\mu^2}\right) + 4 \ln \left(\frac{p^2}{\mu^2}\right) - 2 + 9 \int_0^1 dx \ln \left(\frac{m_1^2 - x(1-x)p^2}{\mu^2}\right) \right]. \quad \text{(B.18)}
\end{align*}
\]

Notice that in the covariant approach, there is no non-renormalisable divergence. If we set
the particle on-shell, we get

\[
\Gamma_{\rho\rho}(p^2 = m_1^2) = \frac{\lambda m_1^2}{(4\pi)^2} \left[ 7C_{\text{UV}} - 7 \ln \left( \frac{m_1^2}{\mu^2} \right) + 20 - 3\sqrt{3}\pi \right]. \tag{B.19}
\]

This is in agreement with (B.8).

In the previous calculation, only the final diagram of (B.18) depends on the ordering of the Feynman rule and it vanishes regardless of the ordering. As such, the ordering was not really tested in this calculation. Let us instead calculate the self-energy of the \(\sigma\) field, which will test the ordering. This is given by

\[
i \Gamma_{\sigma\sigma}(p) = \begin{array}{c}
\bullet \\
\end{array}
= \begin{array}{c}
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\end{array}. \tag{B.20}
\]

Before completing this calculation, let us focus on the last diagram, which is the only one in which the ordering makes a difference. Although we will eventually symmetrise over all possible orderings, let us first consider them all individually:

\[
\begin{align*}
\rho\rho\sigma & = -\frac{2i\lambda}{(4\pi)^2} A(m_1^2), \\
\rho\sigma\rho & = -\frac{2i\lambda}{(4\pi)^2} A(m_1^2), \\
\rho\sigma\rho & = -\frac{2i\lambda}{(4\pi)^2} A(m_1^2), \\
\sigma\rho\rho & = -\frac{2i\lambda}{(4\pi)^2} A(m_1^2), \\
\sigma\rho\rho & = -\frac{2i\lambda}{(4\pi)^2} A(m_1^2).
\end{align*} \tag{B.21}
\]

Note that the different orderings of the above diagrams lead to different results. However, these results converge to a single expression when the external particles are taken to be on-shell, i.e \(p^2 = 0\), despite the fact that the particle in the loop is off-shell. For the diagram with off-shell external particles, we will use the symmetrised Feynman rule, which gives

\[
= -\frac{2i\lambda}{(4\pi)^2} A(m_1^2). \tag{B.22}
\]
In this way, we find
\[
\Gamma_{\sigma\sigma} = p^2 + \frac{12\lambda}{(4\pi)^2} A(0) + 16 \frac{\lambda^2 \rho_0^2}{(4\pi)^2} B_0(p^2, m_1^2, 0)
\]
\[
- 12 \frac{\lambda^2 \rho_0^2}{(4\pi)^2} A(m_1^2) - 8 \frac{\lambda^2 \rho_0^2}{(4\pi)^2} m_1^2 A(0) - \frac{2\lambda}{(4\pi)^2} \left( 1 + \frac{1}{6} \frac{p^2}{m_1^2} \right) A(m_1^2)
\]
\[
= p^2 \left( 1 - \frac{\lambda}{3} C_{UV} \right) + 8 \lambda m_1^2 \left[ \ln \left( \frac{m_1^2}{\mu^2} \right) - \int_0^1 dx \ln \left( \frac{m_1^2 - x p^2}{\mu^2} \right) \right] + \frac{1}{24} \frac{p^2}{m_1^2} \left( \ln \left( \frac{m_1^2}{\mu^2} \right) - 1 \right).
\]

For on-shell Goldstone particles, we have
\[
\Gamma_{\sigma\sigma}(p^2 = 0) = 0 \quad \text{(B.24)}
\]
in agreement with (B.10).

As in Appendix B.a, we calculate the tree-level S matrix element for \(\rho \rightarrow \sigma\sigma\) in the covariant approach. The contributing diagrams are

\[
i M(\rho \rightarrow \sigma\sigma) = \frac{k_1}{k_3} \frac{k_2}{k_4} = \frac{k_1}{k_2} \frac{k_3}{k_4} + \frac{k_1}{k_3} \frac{k_2}{k_4} + \frac{k_1}{k_2} \frac{k_3}{k_4} + \frac{k_1}{k_3} \frac{k_2}{k_4}.
\]

As discussed earlier, the ordering in the first diagram does not matter when all particles are on shell. Therefore, we have
\[
M(\rho \rightarrow \sigma\sigma) = 2 \left( 2\lambda - \frac{s}{\rho_0^2} \right) - 24\lambda^2 \rho_0^2 \frac{1}{s - m_1^2} - 16\lambda^2 \rho_0^2 \frac{1}{t} - 16\lambda^2 \rho_0^2 \frac{1}{u}
\]
\[
= -2 \frac{s^2}{\rho_0^2} \left[ \frac{3}{s - m_1^2} + 2 \frac{(m_1^2 - t)^2}{t} + 2 \frac{(m_1^2 - u)^2}{u} \right]. \quad \text{(B.26)}
\]
This result coincides with (B.11).

\section*{C. Example with Linear Potential}

We now consider an example with Lagrangian
\[
\mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \left( \frac{\rho}{\rho_0} \right)^2 \partial_\mu \sigma \partial^\mu \sigma - t_{\rho \rho} - \frac{1}{2} m_1^2 \rho^2.
\]
This has a flat field space – the kinetic part on its own is just a reparameterisation of two canonical kinetic terms. Additionally, the potential has no interaction terms between the \(\rho\) and \(\sigma\) fields. However, as we shall see, the theory described by (C.1) is nonetheless interacting.
The theory has a symmetry-broken vacuum, which we parametrise as

\[ \langle \rho \rangle = \rho_0 \equiv -t_\rho/m^2, \quad \langle \sigma \rangle = 0. \] (C.2)

We can then calculate the covariant Feynman rules for this theory using (V.19).

The propagators are

\[ = \frac{i}{p^2 - m^2}, \quad = \frac{i}{p^2}, \] (C.3)

where a solid line represents the Higgs mode \( \rho \) and a dashed line represents the Goldstone mode \( \sigma \).

The three-, four-, five-, and six-point interactions are:

\[ = \frac{it_\rho}{\rho_0^2}, \quad = -2i \frac{t_\rho}{\rho_0^2}, \quad = 3i \frac{t_\rho}{\rho_0^2}, \quad = 6i \frac{t_\rho}{\rho_0^2}, \quad = -9i \frac{t_\rho}{\rho_0^2}, \] (C.4)

\[ = -24i \frac{t_\rho}{\rho_0^2}, \quad = 36i \frac{t_\rho}{\rho_0^2}, \quad = 45i \frac{t_\rho}{\rho_0^2}. \]

Notice that there is an infinite series of higher-point vertices, which are proportional to \( t_\rho \). Moreover, these infinite series include interactions that are absent in the standard approach.

To better understand why this theory has an infinite tower of interactions, we switch to a canonical parametrisation and define

\[ \phi_1 = \rho \cos \left( \frac{\sigma}{\rho_0} \right), \quad \phi_2 = \rho \sin \left( \frac{\sigma}{\rho_0} \right). \] (C.5)

Then, (C.1) takes the form

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) - t_\rho \sqrt{\phi_1^2 + \phi_2^2}. \] (C.6)

With this parametrisation, the field space metric becomes manifestly Euclidean and thus ordinary and covariant Feynman rules will be identical. The final term in (C.6) is non-polynomial and thus has an infinite Taylor series expansion. This term therefore leads to an infinite tower of Feynman rules as presented.
D. Field-Space Riemann Tensor for General Relativity

In Section VI we have only presented the field-space Ricci tensor and Ricci scalar, but not the full expression for the Riemann tensor $\mathcal{R}^{(\mu\nu)}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)}$ due to its length. In this appendix, we explicitly display $\mathcal{R}^{(\mu\nu)}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)}$. With the aid of the symbolic computer algebra system Cadabra2 \cite{77,78}, we find that the field-space Riemann tensor for General Relativity reads

\[
\mathcal{R}^{(\mu\nu)}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)} = -\frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\gamma\delta} g_{\alpha\beta} - \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\alpha\delta} - \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\alpha\delta} - \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\sigma\gamma} g_{\alpha\delta} \\
- \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\sigma\gamma} g_{\alpha\delta} - \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} - \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} - \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\sigma\gamma} g_{\beta\delta} \\
- \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\sigma\gamma} g_{\beta\delta} - \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} - \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} - \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\sigma\gamma} g_{\beta\delta} \\
+ \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\alpha\delta} + \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} + \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} + \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} \\
+ \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\alpha\delta} + \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} + \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} + \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} \\
+ \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\alpha\delta} + \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} + \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} + \frac{1}{32} \delta^\mu_\rho \delta^\nu_\sigma g_{\rho\gamma} g_{\beta\delta} \\
+ \frac{1}{4D} g_{\rho\gamma} g^{\mu\nu} g_{\sigma\beta} g_{\alpha\delta} + \frac{1}{4D} g_{\rho\delta} g^{\mu\nu} g_{\sigma\beta} g_{\alpha\gamma} + \frac{1}{4D} g_{\rho\gamma} g^{\mu\nu} g_{\sigma\beta} g_{\gamma\delta} + \frac{1}{4D} g_{\rho\sigma} g^{\mu\nu} g_{\delta\beta} g_{\gamma\delta} \\
+ \frac{1}{4D} g_{\rho\gamma} g^{\mu\nu} g_{\sigma\alpha} g_{\beta\delta} + \frac{1}{4D} g_{\rho\delta} g^{\mu\nu} g_{\sigma\alpha} g_{\gamma\delta} + \frac{1}{4D} g_{\rho\gamma} g^{\mu\nu} g_{\sigma\delta} g_{\alpha\gamma} + \frac{1}{4D} g_{\rho\delta} g^{\mu\nu} g_{\sigma\gamma} g_{\alpha\delta} \\
- \frac{1}{4D} g^{\mu\nu} g_{\rho\delta} g_{\sigma\gamma} g_{\beta\delta} - \frac{1}{4D} g^{\mu\nu} g_{\rho\gamma} g_{\sigma\delta} g_{\beta\gamma} - \frac{1}{4D} g^{\mu\nu} g_{\rho\gamma} g_{\sigma\beta} g_{\alpha\delta} - \frac{1}{4D} g^{\mu\nu} g_{\rho\beta} g_{\sigma\gamma} g_{\alpha\delta}. \tag{D.1}
\]

We note that this tensor vanishes for $D = 1$. This is to be expected since the field space of gravity in one dimension cannot be anything other than trivial.

Note that these results differ from those reported in \cite{77}, where the DeWitt metric was used instead.
[1] A. Einstein, “The Field Equations of Gravitation,” Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) (1915) 844.
[2] E. Buckingham, “On Physically Similar Systems; Illustrations of the Use of Dimensional Equations”, Phys. Rev. 4, 4 (1914) 345-376.
[3] H. Weyl, “Gravitation and electricity,” Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) (1918) 465.
[4] R. H. Dicke, “Mach’s principle and invariance under transformation of units,” Phys. Rev. 125 (1962) 2163.
[5] A. Codello, G. D’Odorico, C. Pagani and R. Percacci, “The Renormalization Group and Weyl-invariance,” Class. Quant. Grav. 30 (2013) 115015.
[6] I. Bars, P. Steinhardt and N. Turok, “Local Conformal Symmetry in Physics and Cosmology,” Phys. Rev. D 89 (2014) no.4, 043515.
[7] E. C. G. Stueckelberg, “Interaction energy in electrodynamics and in the field theory of nuclear forces,” Helv. Phys. Acta 11 (1938) 225.
[8] D. Burns, S. Karamitsos and A. Pilaftsis, “Frame-Covariant Formulation of Inflation in Scalar-Curvature Theories,” Nucl. Phys. B 907 (2016) 785.
[9] S. Karamitsos and A. Pilaftsis, “Frame Covariant Nonminimal Multifield Inflation,” Nucl. Phys. B 927 (2018) 219.
[10] M. Fierz, “On the physical interpretation of P.Jordan’s extended theory of gravitation,” Helv. Phys. Acta 29 (1956) 128.
[11] A. O. Barvinsky and G. A. Vilkovisky, “The Generalized Schwinger-Dewitt Technique in Gauge Theories and Quantum Gravity,” Phys. Rept. 119 (1985) 1. doi:10.1016/0370-1573(85)90148-6
[12] S. Capozziello, R. de Ritis and A. A. Marino, “Some aspects of the cosmological conformal equivalence between 'Jordan frame' and 'Einstein frame','” Class. Quant. Grav. 14 (1997) 3243.
[13] V. Faraoni, E. Gunzig and P. Nardone, “Conformal transformations in classical gravitational theories and in cosmology,” Fund. Cosmic Phys. 20 (1999) 121.
[14] E. Alvarez and J. Conde, “Are the string and Einstein frames equivalent,” Mod. Phys. Lett. A 17 (2002) 413.
[15] S. Capozziello, P. Martin-Moruno and C. Rubano, “Physical non-equivalence of the Jordan and Einstein frames,” Phys. Lett. B 689 (2010) 117.
[16] C. F. Steinwachs and A. Y. Kamenshchik, “Non-minimal Higgs Inflation and Frame Dependence in Cosmology,” AIP Conf. Proc. 1514 (2013) no.1, 161.
[17] L. Järv, P. Kuusk, M. Saal and O. Vilson, “Invariant quantities in the scalar-tensor theories of gravitation,” Phys. Rev. D 91 (2015) no.2, 024041.
[18] A. Y. Kamenshchik and C. F. Steinwachs, “Question of quantum equivalence between Jordan frame and Einstein frame,” Phys. Rev. D 91 (2015) no.8, 084033.
[19] M. Postma and M. Volponi, “Equivalence of the Einstein and Jordan frames,” Phys. Rev. D 90 (2014) no.10, 103516.
[20] G. Doménech and M. Sasaki, “Conformal Frame Dependence of Inflation,” JCAP 1504 (2015) no.04, 022.
[21] L. Järv, K. Kannike, L. Marzola, A. Racioppi, M. Raidal, M. Rünkla, M. Saal and H. Veermäe, “Frame-Independent Classification of Single-Field Inflationary Models,” Phys. Rev. Lett. 118 (2017) no.15, 151302.
[22] M. Herrero-Valea, “Anomalies, equivalence and renormalization of cosmological frames,” Phys. Rev. D 93 (2016) no.10, 105038.
[23] S. Pandey and N. Banerjee, “Equivalence of Jordan and Einstein frames at the quantum level,” Eur. Phys. J. Plus 132 (2017) no.3, 107.
[24] A. Karam, A. Lykkas and K. Tamvakis, “Frame-invariant approach to higher-dimensional scalar-tensor gravity,” Phys. Rev. D 97 (2018) no.12, 124036.
[25] K. Falls and M. Herrero-Valea, “Frame (In)equivalence in Quantum Field Theory and Cosmology,” Eur. Phys. J. C 79 (2019) no.7, 595.
[26] D. Nandi and P. Saha, “Einstein or Jordan: seeking answers from the reheating constraints,” arXiv:1907.10295 [gr-qc].
[27] J. Francfort, B. Ghosh and R. Durrer, “Cosmological Number Counts in Einstein and Jordan frames,” arXiv:1907.03606 [gr-qc].
[28] S. Karamitsos and A. Pilaftsis, “On the Cosmological Frame Problem,” PoS CORFU 2017 (2018) 036.
[29] P. Jordan, “Zur empirischen Kosmologie”, Naturwissenschaften 26 (1938) 417-421.
[30] P. Jordan. Schwerkraft und Weltall. Die Wissenschaft, Bd. 107, Braunschweig, 1952.
[31] C. Brans and R. H. Dicke, “Mach’s principle and a relativistic theory of gravitation,” Phys. Rev. 124 (1961) 925.
[32] P. G. Bergmann, “Comments on the scalar tensor theory,” Int. J. Theor. Phys. 1 (1968) 25.
[33] R. V. Wagoner, “Scalar tensor theory and gravitational waves,” Phys. Rev. D 1 (1970) 3209.
[34] Y. Fujii, K. Maeda. The scalar-tensor theory of gravitation, Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2007.
[35] V. Faraoni, “Cosmology in scalar tensor gravity”, Fundam. Theor. Phys 139 (2004).
[36] G. A. Vilkovisky, “The Unique Effective Action in Quantum Field Theory,” Nucl. Phys. B 234 (1984) 125.
[37] S. Groot Nibbelink and B. J. W. van Tent, “Density perturbations arising from multiple field slow roll inflation,” hep-ph/0011325.
[38] S. Groot Nibbelink and B. J. W. van Tent, “Scalar perturbations during multiple field slow-roll inflation,” Class. Quant. Grav. 19 (2002) 613.
[39] B. van Tent, “Multiple-field inflation and the cmb,” Class. Quant. Grav. 21 (2004) 349.
[40] C. F. Steinwachs, “Non-minimal Higgs inflation and frame dependence in cosmology,” doi:10.1007/978-3-319-01842-3
[41] F. Sauter, “Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs,” Z. Phys. 69 (1931) 742.
[42] W. Heisenberg and H. Euler, “Consequences of Dirac’s theory of positrons,” Z. Phys. 98 (1936) no.11-12, 714.
[43] V. Weisskopf, “The electrodynamics of the vacuum based on the quantum theory of the electron,” Kong. Dan. Vid. Sel. Mat. Fys. Med. 14N6 (1936) 1.
[44] J. S. Schwinger, “On gauge invariance and vacuum polarization,” Phys. Rev. 82 (1951) 664.
[45] G. A. Vilkovisky, “The Gospel According To Dewitt” (in Christensen, S.M. (Ed.): Quantum Theory Of Gravity, 169-209).
[46] “The Effective Action”, Architecture of Fundamental Interactions at Short Distances: Proceedings, Les Houches 44th Summer School of Theoretical Physics: Les Houches, France, July 1-August 8, 1985, pt2 (1987) 1023-1058.
[47] B. S. DeWitt, “Quantum Theory of Gravity. 2. The Manifestly Covariant Theory,” Phys. Rev. 162 (1967) 1195.
[48] P. Finsler. Über Kurven und Flächen in allgemeinen Räumen. Göttingen, Zürich: O. Füssli, 120 S. 8° (1918).
[49] L. F. Abbott, “Introduction to the Background Field Method,” Acta Phys. Polon. B 13 (1982) 33.
[50] A. Rebhan, “The Vilkovisky-de Witt Effective Action and Its Application to Yang-Mills Theories,” Nucl. Phys. B 288 (1987) 832.
[51] S. R. Huggins, G. Kunstatter, H. P. Leivo and D. J. Toms, “On the Unique Effective Action in Five-dimensional Kaluza-Klein Theory,” Phys. Rev. Lett. 58 (1987) 296.
[52] G. Kunstatter, “Vilkovisky’s Unique Effective Action: An Introduction And Explicit Calculation” (in Vancouver 1986, Proceedings, Super Field Theories, 503-517 and preprint – Kunstatter, G. (86,Rec.Nov.) 28p.
[53] P. Ellicott and D. J. Toms, “On the New Effective Action in Quantum Field Theory,” Nucl. Phys. B 312 (1989) 700.
[54] C. P. Burgess and G. Kunstatter, “On the Physical Interpretation of the Vilkovisky-de Witt Effective Action,” Mod. Phys. Lett. A 2 (1987) 875 Erratum: [Mod. Phys. Lett. A 2 (1987) 1003].
[55] M. F. Sohnius, “Introducing Supersymmetry,” Phys. Rept. 128 (1985) 39.
[56] D. R. Grigore, “Off-Shell Fields and Quantum Anomalies,” Ann. U. Craiova Phys. 21 S117.
[57] A. R. Liddle, P. Parsons and J. D. Barrow, “Formalizing the slow roll approximation in inflation,” Phys. Rev. D 50 (1994) 7222 doi:10.1103/PhysRevD.50.7222 [astro-ph/9408015]
[58] H. Lehmann, K. Symanzik and W. Zimmermann, “On the formulation of quantized field theories,” Nuovo Cim. 1 (1955) 205.
[59] S. Pokorski, Gauge Field Theories, Second Edition, Cambridge University Press, Cambridge, 2000.
[60] M. Srednicki. Quantum Field Theory. Cambridge University Press, Cambridge, 2007.
[61] L.H. Ryder. Quantum Field Theory. Cambridge University Press, Cambridge, 1996.
[62] R. P. Feynman, “The Theory of positrons,” Phys. Rev. 76 (1949) 749.
[63] B. W. Lee, “Gauge Theories,” Conf. Proc. C 7507281 (1975) 79.
[64] B. S. DeWitt, “Quantum Theory of Gravity. 1. The Canonical Theory,” Phys. Rev. 160 (1967) 1113.
[65] S. D. Odintsov, “Does the Vilkovisky-De Witt effective action in quantum gravity depend on the configuration space metric?,” Phys. Lett. B 262 (1991) 394. doi:10.1016/0370-2693(91)90611-S
[66] D. Giulini and C. Kiefer, “Wheeler-DeWitt metric and the attractivity of gravity,” Phys. Lett. A 193 (1994) 21.
[67] K. Peeters, “Introducing Cadabra: A Symbolic computer algebra system for field theory problems,” hep-th/0701238
[68] K. Peeters, “Cadabra2: computer algebra for field theory revisited,” Journal of Open Source Software 3 (2018) 1118.
[69] P. W. Higgs, “Quadratic lagrangians and General Relativity,” Nuovo Cim. 11 (1959) no.6, 816.
[70] E. E. Flanagan, “The Conformal frame freedom in theories of gravitation,” Class. Quant. Grav. 21 (2004) 3817.
[71] R. Catena, M. Pietroni and L. Scarabello, “Einstein and Jordan reconciled: a frame-invariant approach to scalar-tensor cosmology,” Phys. Rev. D 76 (2007) 084039.
[72] K. Fujikawa, “Path Integral Measure for Gravitational Interactions,” Nucl. Phys. B 226 (1983) 437.
[73] L. P. Eisenhart, “Dynamical Trajectories and Geodesics,” Annals of Mathematics 1/4 30 (1928) 591-606.
[74] K. Finn, S. Karamitsos and A. Pilaftsis, “Eisenhart lift for field theories,” Phys. Rev. D 98 (2018) no.1, 016015.
[75] C. G. Callan, Jr., “Broken scale invariance in scalar field theory,” Phys. Rev. D 2 (1970) 1541.
[76] K. Symanzik, “Small distance behavior in field theory and power counting,” Commun. Math. Phys. 18 (1970) 227.
[77] C. F. Steinwachs and M. L. van der Wild, “Quantum gravitational corrections from the Wheeler-DeWitt equation for scalar-tensor theories,” Class. Quant. Grav. 35 (2018) no.13, 135010 doi:10.1088/1361-6382/aaec587 arXiv:1712.08543 [gr-qc].