ON $C^0$-CONTINUITY OF THE SPECTRAL NORM FOR SYMPLECTICALLY NON-ASPHERICAL MANIFOLDS

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Abstract. The purpose of this paper is to study the relation between the $C^0$-topology and the topology induced by the spectral norm on the group of Hamiltonian diffeomorphisms of a closed symplectic manifold. Following the approach of Buhovsky-Humilière-Seyfaddini, we prove the $C^0$-continuity of the spectral norm for complex projective spaces and negative monotone symplectic manifolds. The case of complex projective spaces provides an alternative approach to the $C^0$-continuity of the spectral norm proven by Shelukhin. We also prove a partial $C^0$-continuity of the spectral norm for rational symplectic manifolds. Some applications such as the Arnold conjecture in the context of $C^0$-symplectic topology are also discussed.

Contents

1. Introduction 2
   1.1. Set-up 2
   1.2. $C^0$-topology 3
   1.3. Spectral norms 3
   1.4. Main results 4
   1.5. Application 1: $C^0$-continuity of barcodes 4
   1.6. Application 2: The $C^0$-Arnold conjecture 6
   1.7. Application 3: The displaced disks problem 8
   1.8. Acknowledgments 8
2. Preliminaries 8
   2.1. Hamiltonian Floer theory 11
   2.2. Quantum homology and Seidel elements 12
   2.3. Hamiltonian spectral invariants 14
   2.4. Barcodes 16
3. Proofs 17
   3.1. Proofs of Theorem 4 and 6 17
   3.2. Proof of Theorem 3 23
   3.3. Proof of Theorem 2 27
4. Proofs of applications 29
   4.1. The displaced disks problem 29
   4.2. The $C^0$-Arnold conjecture 30
1. Introduction

The study of topological properties of the group of Hamiltonian diffeomorphisms of a symplectic manifold has been one of the central topics in symplectic topology. The group of Hamiltonian diffeomorphisms is known to carry different metrics such as the Hofer metric, the spectral metric and the $C^0$-metric and their relations have been studied extensively. This paper studies the relation between the $C^0$-topology and the topology induced by the spectral metric. More precisely, we study the $C^0$-continuity of the spectral norm which has been already verified for certain cases: for $\mathbb{R}^{2n}$ by Viterbo [Vit92], for closed surfaces by Seyfaddini [Sey13-1], for symplectically aspherical manifolds by Buhovsky-Humilière-Seyfaddini [BHS18b] and for complex projective spaces by Shelukhin [Sh18]. In this paper, we push the method developed by Buhovsky-Humilière-Seyfaddini [BHS18b] forward to the symplectically non-aspherical setting and confirm the $C^0$-continuity of the spectral norm for negative monotone symplectic manifolds. We also obtain a partial $C^0$-continuity of the spectral norm for rational symplectic manifolds and an alternative proof of the $C^0$-continuity of the spectral norm for complex projective spaces.

1.1. Set-up. Throughout this paper, $(M, \omega)$ will denote a closed symplectic manifold. A symplectic manifold $(M, \omega)$ is called

- rational if $\langle \omega, \pi_2(M) \rangle = \lambda_0 \mathbb{Z}$ for some constant $\lambda_0 > 0$. We refer to the constant $\lambda_0$ as the rationality constant.
- monotone (resp. negative monotone) if $\omega|_{\pi_2(M)} = \lambda \cdot c_1|_{\pi_2(M)}$ for some positive (resp. negative) constant $\lambda$ where $c_1 := c_1(TM)$ denotes the first Chern class of $TM$. We refer to the constant $\lambda$ as the monotonicity constant.
- symplectically aspherical when $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$.

The positive generator of $\langle c_1, \pi_2(M) \rangle \subset \mathbb{Z}$ is called the minimal Chern number $N_M$ i.e.

$$\langle c_1, \pi_2(M) \rangle = N_M \mathbb{Z}, \quad N_M > 0.$$ 

Example.

- The complex projective space equipped with the standard Fubini-Study form $(\mathbb{C}P^n, \omega_{FS})$ is monotone and its minimal Chern number $N_{\mathbb{C}P^n}$ is $n + 1$.
- The degree $k$ Fermat hypersurfaces of $\mathbb{C}P^{n+1}$

$$M := \{ (z_0 : z_1 : \cdots : z_n) \in \mathbb{C}P^{n+1} : z_0^k + z_1^k + z_2^k + \cdots + z_n^k = 0 \}$$

is negative monotone for $k > n + 1$. The minimal Chern number $N_M$ is $|k - (n + 2)|$ if $k \neq n + 2$ and $+\infty$ otherwise.
A Hamiltonian $H$ on $M$ is a smooth time dependent function $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$. We define its Hamiltonian vector field $X_H$ by $-dH_t = \omega(X_H, \cdot)$. The Hamiltonian flow of $H$, denoted by $(\phi^t_H)_{t \in \mathbb{R}}$, is by definition the flow of $X_H$. Its time-one map $\phi^1_H$ is called the Hamiltonian diffeomorphism of $H$ and will be denoted by $\phi_H$. The set of Hamiltonian diffeomorphisms and its universal cover will be denoted respectively by $\Ham(M, \omega)$ and $\tilde{\Ham}(M, \omega)$.

1.2. $C^0$-topology. We define the $C^0$-metric by $d_{C^0}(\phi, \psi) := \max_{x \in M} d(\phi(x), \psi(x))$, where $d$ denotes the distance on $M$ induced by the Riemannian metric on $M$. Note that the topology induced by the $C^0$-distance is independent of the choice of a Riemannian metric. We denote the $C^0$-closure of $\Ham(M, \omega)$ in the group of homeomorphisms of $M$ by $\Ham(M, \omega)$. Their elements are called the Hamiltonian homeomorphisms. Hamiltonian homeomorphisms are central objects in $C^0$-symplectic topology.

1.3. Spectral norms. We roughly outline the notion of the spectral norm. For precise definitions, we refer to Section 2.3. First of all, a Hamiltonian $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})$ is called non-degenerate if for each $x \in \Fix(\phi_H)$, the set of eigenvalues of $d\phi_H(x) : T_xM \to T_xM$ does not include 1. For a non-degenerate Hamiltonian $H$ and a fixed ground field $\mathbb{K}$ (see Remark 2.3 for more information about the choice of a ground field), one can define the Floer homology group $HF(H) = HF(H; \mathbb{K})$ as well as their filtration with respect to the action functional which will be denoted by $\{HF^\tau(H)\}_{\tau \in \mathbb{R}}$. For each $\tau \in \mathbb{R}$, we denote the natural map induced by the inclusion map of the chain complex by $i^\tau_*$:

$$i^\tau_* : HF^\tau(H) \to HF(H).$$

The quantum homology ring is defined by

$$QH_\ast(M; \mathbb{K}) := H_\ast(M; \mathbb{K}) \otimes_{\mathbb{K}} \Lambda$$

where

$$\Gamma := \pi_2(M)/\ker(\omega) \cap \ker(c_1),$$

$$\Lambda := \left\{ \sum_{A \in \Gamma} a_A \otimes e^A : a_A \in \mathbb{K}, (\forall \tau \in \mathbb{R}, \#\{a_A \neq 0 : \omega(A) < \tau\} < +\infty) \right\}.$$ 

The ring structure of $QH_\ast(M; \mathbb{K})$ is given by the quantum product $*$: for its definition, see Section 2.2. Floer homology group is ring isomorphic to the quantum homology ring $QH_\ast(M; \mathbb{K})$ by the PSS-isomorphism

$$\Phi_{PSS,H; \mathbb{K}} : QH_\ast(M; \mathbb{K}) \to HF(H).$$

For a Hamiltonian $H$ and $a \in QH_\ast(M; \mathbb{K}) \setminus \{0\}$, the spectral invariant of $H$ and $a$ is defined by

$$c(H,a) := \inf\{\tau : \Phi_{PSS,H; \mathbb{K}}(a) \in \text{Im}(i^\tau_*)\}.$$
The spectral norm of a Hamiltonian \( H \) is defined by
\[
\gamma(H) := c(H, [M]) + c(\overline{H}, [M])
\]
where \( \overline{H}(t, x) := -H(t, \phi_H^t(x)) \) which is a Hamiltonian that generates the Hamiltonian flow
\[
t \mapsto (\phi_H^t)^{-1}.
\]
Since \( \gamma \) is invariant under homotopy i.e. if \( \phi_H^t \sim \phi_G^t \) rel. endpoints, then \( \gamma(H) = \gamma(G) \), it can be seen as a map defined on the universal cover of \( \text{Ham}(M, \omega) \), namely
\[
\gamma : \widetilde{\text{Ham}}(M, \omega) \to \mathbb{R}.
\]
We define spectral norms for Hamiltonian diffeomorphisms by
\[
\gamma : \text{Ham}(M, \omega) \to \mathbb{R},
\]
\[
\gamma(\phi) := \inf_{\phi = \phi_H} \gamma(H).
\]

1.4. Main results. Throughout the paper \( \lambda_0 > 0 \) denotes the rationality constant i.e. \( \langle \omega, \pi_2(M) \rangle = \lambda_0 \mathbb{Z} \). We first state our result for rational symplectic manifolds.

**Theorem 1.** Let \((M, \omega)\) be a rational symplectic manifold. For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( d_{C^0}(\text{id}, \phi_H) < \delta \), then
\[
|\gamma(H) - l \cdot \lambda_0| < \varepsilon
\]
for some integer \( l \in \mathbb{Z} \) depending on \( H \).

Theorem 1 gives us the candidates of the value of the spectral norm of a \( C^0 \)-small \( \phi_H \). When the values of spectral norms are bounded by a number strictly smaller than \( \lambda_0 \), then Theorem 1 implies the \( C^0 \)-continuity. The complex projective space \( \mathbb{C}P^n \) meets this condition.

**Theorem 2.** Let \((\mathbb{C}P^n, \omega_{FS})\) be the complex projective space equipped with the Fubini-Study form.

(1) For any \( \phi \in \text{Ham}(\mathbb{C}P^n, \omega_{FS}) \),
\[
\gamma(\phi) \leq \frac{n}{n + 1} \cdot \lambda_0
\]
where \( \lambda_0 \) denotes the rationality constant.

(2) The spectral norm is \( C^0 \)-continuous i.e.
\[
\gamma : (\text{Ham}(\mathbb{C}P^n, \omega_{FS}), d_{C^0}) \to \mathbb{R}
\]
is continuous. Moreover, \( \gamma \) extends continuously to \( \widetilde{\text{Ham}}(\mathbb{C}P^n, \omega_{FS}) \).

**Remark 3.**
(1) Theorem 2 is already proven in other papers: (1) appears as Theorem G in [KS18] and (2) appears as Theorem C in [Sh18]. Shelukhin’s argument in [Sh18], which is different from ours, is based on barcode techniques and is specific for \( \mathbb{C}P^n \).

(2) We will prove an a priori more general result in Section 5.3.

For general monotone symplectic manifolds, instead of the \( C^0 \)-continuity, we only obtain the following \( C^0 \)-control of the spectral norm.

**Theorem 4.** Let \((M, \omega)\) be a monotone symplectic manifold.

1. For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( d_{C^0}(\text{id}, \phi_H) < \delta \), then
   \[
   \gamma(H) < \frac{\dim(M)}{N_M} \cdot \lambda_0 + \varepsilon.
   \]

2. If \( N_M > \dim(M) \), then the spectral norm is \( C^0 \)-continuous i.e.
   \[
   \gamma : (\text{Ham}(M, \omega), d_{C^0}) \to \mathbb{R}
   \]
   is continuous. Moreover, \( \gamma \) extends continuously to \( \widetilde{\text{Ham}}(M, \omega) \).

**Remark 5.**

1. The author does not know any example satisfying the assumptions in Theorem 4 (2). Note that Theorem 4 (2) follows immediately from Theorem 1 and Theorem 4 (1).
2. Theorem 4 applies to spectral norms of any ground ring \( R \) (i.e. a commutative ring with unit). See Remark 25 for a comment on the choice of ground rings/fields.

We now consider the case of negative monotone symplectic manifolds.

**Theorem 6.** Let \((M, \omega)\) be a negative monotone symplectic manifold.

1. For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( d_{C^0}(\text{id}, \phi_H) < \delta \), then
   \[
   \gamma(H) < \varepsilon.
   \]
   In particular, if \( \phi_H = \phi_G \) for \( H, G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R}) \), then
   \[
   \gamma(H) = \gamma(G)
   \]
   i.e. \( \gamma : \widetilde{\text{Ham}}(M, \omega) \to \mathbb{R} \) descends to
   \[
   \gamma : \text{Ham}(M, \omega) \to \mathbb{R}.
   \]

2. The spectral norm is \( C^0 \)-continuous i.e.
   \[
   \gamma : (\text{Ham}(M, \omega), d_{C^0}) \to \mathbb{R}
   \]
   is continuous. Moreover, \( \gamma \) extends continuously to \( \overline{\text{Ham}}(M, \omega) \).

**Remark 7.** The independence of the spectral norm of the choice of Hamiltonian follows also from Lemma 3.2.(iv) in [McD10].
1.5. Application 1: $C^0$-continuity of barcodes. Barcodes are roughly speaking finite sets of intervals which are bounded from below but can be unbounded from above. The set of barcodes carries a metric called the bottleneck distance denoted by $d_{\text{bot}}$. Barcodes have been a common tool in topological data analysis. Polterovich-Shelukhin brought barcodes into symplectic topology in [PS16] where they defined barcodes of (non-degenerate) Hamiltonian diffeomorphisms on symplectically aspherical manifolds and found applications to Hofer geometry. Later, as we will explain in Section 2.4 the definition of barcodes was extended to Hamiltonian diffeomorphisms on (negative) monotone symplectic manifolds [LSV18], [PSS17] after considering a completion of the set of barcodes with respect to the bottleneck distance which we will denote by $\hat{\text{Barcodes}}$. An estimate of the bottleneck distance due to Kislev–Shelukhin [KS18] (see the inequality 1 in Section 2.4) combined with the $C^0$-continuity of the spectral norm implies the $C^0$-continuity of barcodes for negative monotone symplectic manifolds.

Corollary 8. Let $(M,\omega)$ be a negative monotone symplectic manifold. The barcode map is $C^0$-continuous i.e.

$$B : (\text{Ham}(M,\omega), d_{C^0}) \to (\hat{\text{Barcodes}}, d_{\text{bot}})$$

is continuous. Moreover, $B$ extends continuously to $\text{Ham}(M,\omega)$.

Remark 9. Of course, Theorem 2 (2) directly implies the $C^0$-continuity of barcodes in the case of $(\mathbb{C}P^n, \omega_{FS})$. This is proven by Shelukhin in Corollary 6 in [Sh18].

1.6. Application 2: The $C^0$-Arnold conjecture. The Arnold conjecture has been historically one of the central topics in symplectic geometry.

Conjecture 10. (The Arnold conjecture)

Let $(M^{2n}, \omega)$ be a closed symplectic manifold.

1) For a non-degenerate $\phi \in \text{Ham}(M,\omega)$,

$$\#\text{Fix}(\phi) \geq \sum_j \dim C H_j(M; \mathbb{C}).$$

2) For $\phi \in \text{Ham}(M,\omega)$,

$$\#\text{Fix}(\phi) \geq \text{cl}(M)$$

where

$$\text{cl}(M) := \# \max \{ k + 1 : \exists a_1, a_2, \ldots, a_k \in H_{*<2n}(M) \text{ s.t. } a_1 \cap a_2 \cap \cdots \cap a_k \neq 0 \}$$

and $\cap$ denotes the intersection product.

Since the advent of Floer homology, there has been a huge progress in the two versions of the Arnold conjecture: (1) is now completely settled [FO99], [LT98] and (2) has been confirmed for symplectically aspherical manifolds [Fl89], $\mathbb{C}P^n$ [For85], [ForW85] and negative monotone symplectic manifolds with $N_M \geq n$ [LO94].
It caught attention whether or not the Arnold conjecture is $C^0$-robust i.e. if Hamiltonian homeomorphisms satisfy similar properties. For closed surfaces, this question was answered in the positive by Matsumoto [Ma00]. However, Buhovsky-Humilière-Seyfaddini [BHS18a] discovered that in higher dimension, this turns out not to be the case.

**Theorem 11.** ([BHS18a])

Let $(M,\omega)$ be any closed symplectic manifold of dimension $\geq 4$. There exists a Hamiltonian homeomorphism $\phi \in \overline{\text{Ham}}(M,\omega)$ such that

$$\#\text{Fix}(\phi) = 1.$$ 

In their subsequent paper [BHS18b], Buhovsky-Humilière-Seyfaddini have reformulated the Arnold conjecture in a way that is more suited to study the rigidity of Hamiltonian homeomorphisms when the ambient manifold is symplectically aspherical. We will follow their idea to obtain similar results for symplectic manifolds that are not symplectically aspherical by using the quantum product $\ast$ of $QH_\ast(M;\mathbb{K})$ (for its definition, see Section 2.2).

**Definition 12.** Let $(M^{2n},\omega)$ be a symplectic manifold. Let $a,b \in H_\ast(M;\mathbb{K})\setminus\{0\}$. For a Hamiltonian $H$, define

$$\sigma_{a,a\ast b}(H) := c(H,a) - c(H,a \ast b)$$

and for a Hamiltonian diffeomorphism $\phi$, define

$$\sigma_{a,a\ast b}(\phi) := \inf_{\phi \in \phi_0} \sigma_{a,a\ast b}(H).$$

For $(M,\omega)$ for which $\gamma$ is $C^0$-continuous (e.g. negative monotone symplectic manifolds and $(\mathbb{C}P^n,\omega_{FS})$), $\sigma_{a,a\ast b}$ turns out to be $C^0$-continuous and it extends continuously to $\overline{\text{Ham}}(M,\omega)$: see Section 2.2 for details.

**Theorem 13.** Let $(M^{2n},\omega)$ be either a negative monotone symplectic manifold or $(\mathbb{C}P^n,\omega_{FS})$. For $\phi \in \overline{\text{Ham}}(M,\omega)$, if there exist homology classes $a,b \in H_\ast(M;\mathbb{K})\setminus\{0\}, b \neq [M]$ such that

$$\sigma_{a,a\ast b}(\phi) = 0,$$

then $\text{Fix}(\phi)$ is homologically non-trivial, hence it is an infinite set.

**Remark 14.**

1. Recall that, a subset $A \subset M$ is homologically non-trivial if for every open neighborhood $U$ of $A$ the map $i_* : H_j(U;\mathbb{K}) \to H_j(M;\mathbb{K})$, induced by the inclusion $i : U \to M$, is non-trivial for some $j > 0$. Homologically non-trivial sets are infinite sets.

2. In [How12], Howard considers the smooth version of this statement.
1.7. **Application 3: The displaced disks problem.** A topological group $G$ is a Rokhlin group if it possesses a dense conjugacy class i.e. for some $\phi \in G$, $\text{ Conj}(\phi) := \{ \psi^{-1} \phi \psi : \psi \in G \}$ is dense. Béguin-Crovisier-Le Roux formulated the following question so-called the "displaced disks problem", in order to study whether or not the group of area-preserving homeomorphisms on a sphere is a Rokhlin group.

**Question.** (Béguin-Crovisier-Le Roux)

Let 
\[ G_r := \{ \phi \in \overline{\text{Ham}}(M,\omega) : \phi(f(B_r)) \cap f(B_r) = \emptyset \} \]
for any $r > 0$ where $f : B_r \to (M,\omega)$ is a symplectic embedding. Does the $C^0$-closure of $G_r$ contain the identity element for some $r > 0$?

This original question which was for $(M,\omega) = (S^2,\omega_{\text{area}})$ was solved by Seyfaddini in [Sey13-2] as a consequence of his earlier result on $C^0$-continuity of spectral norms for closed surfaces [Sey13-1]. Other cases, also deduced by $C^0$-continuity of spectral norms, has also been considered: [BHS18b] deals with symplectically aspherical manifolds and [Sh18] treats $\mathbb{C}P^n$. Here we add the case of negative monotone symplectic manifolds.

**Theorem 15.** Let $(M,\omega)$ be a negative monotone symplectic manifold. For any $r > 0$, there exists $\delta > 0$ such that if $\phi \in \overline{\text{Ham}}(M,\omega)$ displaces a symplectically embedded ball of radius $r$, then $d_{C^0}(\text{id},\phi) > \delta$.

We obtain the following as a direct consequence.

**Corollary 16.** Let $(M,\omega)$ be a negative monotone symplectic manifold. The group $\overline{\text{Ham}}(M,\omega)$ seen as a topological group with respect to the $C^0$-topology is not a Rokhlin group.

**Remark 17.** The case of $(\mathbb{C}P^n,\omega_{FS})$ was considered by Shelukhin in [Sh18].

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2. **Preliminaries**

In this section, we briefly review the notions and basic propositions needed later in the proof. For further details, we refer to [MS04].
Let \((M, \omega)\) be a symplectic manifold. A Hamiltonian \(H\) on \(M\) is a smooth time dependent function \(H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}\). We define its Hamiltonian vector field \(X_H\) by \(-dH_t = \omega(X_H, \cdot)\). The Hamiltonian flow of \(H\), denoted by \(\phi_H^t\), is by definition the flow of \(X_H\). A Hamiltonian diffeomorphism is a diffeomorphism which arises as the time-one map of a Hamiltonian flow. The set of all Hamiltonian diffeomorphisms is denoted by \(\text{Ham}(M, \omega)\).

Denote the set of smooth contractible loops in \(M\) by \(\mathcal{L}M\) and consider its universal cover. Two elements in the universal cover, say \([z_1, w_1]\) and \([z_2, w_2]\), are called equivalent if their boundary sum \(w_1 \# w_2\) i.e. the sphere obtained by gluing \(w_1\) and \(w_2\) along their common boundary with the orientation on \(w_2\) reversed, satisfies \(\omega(w_1 \# w_2) = 0\), \(c_1(w_1 \# w_2) = 0\).

We denote by \(\tilde{\mathcal{L}}_0M\) the space of equivalence classes. For a Hamiltonian \(H\), define the action functional \(A_H : \tilde{\mathcal{L}}M \to \mathbb{R}\) by

\[
A_H([z, w]) := \int_0^1 H(t, z(t)) dt - \int_{D^2} w^* \omega
\]

where \(w : D^2 \to M\) is a capping of \(z : \mathbb{R}/\mathbb{Z} \to M\). Note that in general, the action functional depends on the capping and not only on the loop. Critical points of this functional are precisely the capped 1-periodic Hamiltonian orbits of \(H\) which will be denoted by \(\mathcal{P}(H)\). The set of critical values of \(A_H\) is called the action spectrum and is denoted by \(\text{Spec}(H)\):

\[
\text{Spec}(H) := \{A_H(z) : z \in \mathcal{P}(H)\}.
\]

We briefly explain some notions of indices used later to construct Floer homology. The Maslov index \(\mu : \pi_1(\text{Sp}(2n)) \to \mathbb{Z}\) maps a loop of symplectic matrices to an integer. Given a loop of Hamiltonian diffeomorphisms \((\psi^t)_{t \in [0,1]} \in \pi_1(\text{Ham}(M, \omega))\) and a point \(x \in M\), denote the capped orbit of \(t \mapsto \psi^t(x)\) with a capping \(w\) by \([\psi^t(x), w]\). We define its Maslov index \(\mu([\psi^t(x), w])\) via the trivialization of \(w^*TM\) and the loop of symplectic matrices \(d\psi^t(x) : T_xM \to T_{\psi^t(x)}M\). The definition of Maslov indices cannot be directly applied to periodic orbits of a Hamiltonian \(H\) since given a periodic orbit \([\phi_H^t(x), w]\), \(d\phi_H^t(x) : T_xM \to T_{\phi_H^t(x)}M\) might not define a loop. To overcome this difficulty, Conley-Zehnder modified the definition of the Maslov index and introduced the Conley-Zehnder index

\[
\mu_{CZ} : \{A : [0,1] \to \text{Sp}(2n) | \det(A(1) - \text{id}) \neq 0\} \to \mathbb{Z}
\]

which maps paths of symplectic matrices to integers. Thus, as in the case of Maslov indices, we define the Conley-Zehnder index of a non-degenerate periodic orbit of a Hamiltonian \(H\) \([\phi_H^t(x), w]\) (i.e. \(d\phi_H^t(x) : T_xM \to T_xM\) has no eigenvalue which equals
to 1), denoted by \( \mu_{CZ}(\phi^t_H(x), w) \), via the trivialization of \( w^*TM \) and the path of symplectic matrices \( d\phi^t_H(x) : T_x M \to T_{\phi^t_H(x)} M \).

The following elementary properties are often used to calculate the action.

**Proposition 18.** Let \((M, \omega)\) be a symplectic manifold. Assume the Hamiltonian paths generated by \( H \) and \( G \) are homotopic rel. end points i.e. there exists \( W : [0,1] \times [0,1] \to \text{Ham}(M, \omega) \) such that

1. \( W(0,t) = \phi^t_H, \ W(1,t) = \phi^t_G. \)
2. \( W(s,0) = \text{id}, \ W(s,1) = \phi_H = \phi_G. \)

Let \( x \in \text{Fix}(\phi_H) = \text{Fix}(\phi_G) \) and \( w \) be a capping of the orbit \( \phi^t_H(x) \). Then the action of the capped orbit \([\phi^t_G(x), w']\) where \( w' := w \# W (W \text{ glued to } x \text{ along } \phi^t_H(x)) \) coincides with the action of \([\phi^t_H(x), w]\):

\[
A_H([\phi^t_H(x), w]) = A_G([\phi^t_G(x), w'])
\]

**Proposition 19.** Let \((M, \omega)\) be a symplectic manifold.

1. For any Hamiltonian \( H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R}), \)
   \[
   \overline{H}(t,x) := -H(t, \phi^t_H(x))
   \]
   generates the Hamiltonian flow
   \[
   t \mapsto (\phi^t_H)^{-1}
   \]
   and has the time-1 map \( \phi_H^{-1} \).
2. For any Hamiltonian \( H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R}), \)
   \[
   \tilde{H}(t,x) := -H(-t, x)
   \]
   generates the Hamiltonian flow
   \[
   t \mapsto \phi_H^{-t}
   \]
   and has the time-1 map \( \phi_H^{-1} \).
3. Hamiltonian paths generated by \( \overline{H} \) and \( \tilde{H} \) are homotopic rel. end points.

**Proposition 20.** Let \((M, \omega)\) be a symplectic manifold.

1. For any Hamiltonians \( H, G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R}), \)
   \[
   H \# G(t,x) := H(t,x) + G(t, (\phi^t_H)^{-1}(x))
   \]
   generates the Hamiltonian flow
   \[
   t \mapsto \phi^t_H \circ \phi^t_G
   \]
   and has the time-1 map \( \phi_H \circ \phi_G \).
(2) For any Hamiltonians $H, G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})$,

$$H \wedge G(t, x) := \begin{cases} 2\chi'(2t)G(\chi(2t), x) & (0 \leq t \leq 1/2) \\ 2\chi'(2t-1)H(\chi(2t-1), \phi_G(x)) & (1/2 \leq t \leq 1) \end{cases}$$

generates the Hamiltonian flow

$$t \mapsto \begin{cases} \phi_G^{(2t)} & (0 \leq t \leq 1/2) \\ \phi_H^{(2t-1)} \circ \phi_G & (1/2 \leq t \leq 1) \end{cases}$$

and has the time-1 map $\phi_H \circ \phi_G$. Here,

$$\chi : [0, 1] \to [0, 1], \ t \mapsto \chi(t)$$

is a smooth map that is identically 0 around $t = 0$ and 1 around $t = 1$ and plays the role of gluing two Hamiltonians smoothly.

(3) Hamiltonian paths generated by $H \# G$ and $H \wedge G$ are homotopic rel. end points (for any choice of $\chi : [0, 1] \to [0, 1], \ t \mapsto \chi(t)$).

The following two propositions will be used in Section 3.1. Proofs will be omitted as they follow from elementary arguments.

**Proposition 21.** Let $(M, \omega)$ be a symplectic manifold, $U$ a simply connected non-empty open set and $H$ a Hamiltonian such that $\phi_H(p) = p$ for all $p \in U$. Take any $x_0 \in U$ and a capping $w_0 : D^2 \to M$ of the orbit $\phi_H^t(x_0)$ and fix them.

For any $x \in U$, define a capping $w_x : D^2 \to M$ of the orbit $\phi_H^t(x)$ by

$$w_x(se^{2\pi it}) := \phi_H^t(c(s))\#w_0$$

where $c : [0, 1] \to M$ is a smooth path from $x_0$ to $x$ and $\phi_H^t(c(s))\#w_0$ denotes the gluing of $\phi_H^t(c(s))$ and $w_0$ along $\phi_H^t(x_0)$. Then we have the following:

1. $A_H([\phi_H^t(x), w_x]) = A_H([\phi_H^t(x_0), w_0])$.
2. $\mu([\phi_H^t(x), w_x]) = \mu([\phi_H^t(x_0), w_0])$.

**Proposition 22.** Let $(M, \omega)$ be a symplectic manifold, $H$ a Hamiltonian and $[\phi_H^t(x), w]$ any capped 1-periodic orbit of $H$. Then

1. $\overline{w} : D^2 \to M, \ \overline{w}(se^{2\pi it}) := w(se^{2\pi i(-t)})$ is a capping of the orbit $\phi_H^t(x)$
2. $\mu([\phi_H^t(x), w]) = -\mu([\phi_H^t(x), \overline{w}])$
3. $A_H([\phi_H^t(x), w]) = -A_H([\phi_H^t(x), \overline{w}])$ where $\overline{H}(t, x) = -H(-t, x)$.

### 2.1. Hamiltonian Floer theory.

We fix a ground field $\mathbb{K}$ of zero characteristic (see Remark 23). We say that a Hamiltonian $H$ is non-degenerate if the diagonal set $\Delta := \{(x, x) \in M \times M\}$ intersects transversally the graph of $\phi$, $\Gamma_\phi := \{(x, \phi(x)) \in M \times M\}$. For a non-degenerate $H$, we define the Floer chain complex $CF_*(H)$ as follows:

$$CF_*(H) := \{ \sum_{z \in \mathcal{P}(H)} a_z \cdot z : a_z \in \mathbb{K}, (\forall \tau \in \mathbb{R}, \ #\{z : a_z \neq 0, A_H(z) \leq \tau\} < +\infty) \}. $$
The Floer chain complex has a \( \mathbb{Z} \)-grading by the Conley-Zehnder index \( \mu_{CZ} \). The boundary map counts certain solutions of a perturbed Cauchy-Riemann equation for a chosen \( \omega \)-compatible almost complex structure \( J \) on \( TM \), which can be viewed as isolated negative gradient flow lines of \( \mathcal{A}_H \). This gives us a chain complex \( (CF_\ast(H), \partial) \) called the Floer chain complex. Its homology is called the Floer homology of \( (H, J) \) and is denoted by \( HF_\ast(H, J) \). Often it is abbreviated to \( HF_\ast(H) \) as Floer homology does not depend on the choice of an almost complex structure.

Recapping of a capped orbit by \( A \in \pi_2(M) \) changes the action and the Conley-Zehnder index as follows:

- \( \mathcal{A}_H([z, w # A]) = \mathcal{A}_H([z, w]) - \omega(A) \).
- \( \mu_{CZ}([z, w # A]) = \mu_{CZ}([z, w]) - 2c_1(A) \).

We define the filtered Floer complex of \( H \) by

\[
CF_\tau^+(H) := \{ \sum a_z z \in CF_\ast(H) : \mathcal{A}_H(z) < \tau \}.
\]

Since the Floer boundary map decreases the action, \( (CF_\tau^+(H), \partial) \) forms a chain complex. The filtered Floer homology of \( H \) which is denoted by \( HF_\tau^+(H) \) is the homology defined by the chain complex \( (CF_\tau^+(H), \partial) \).

It is useful to clarify our convention of the Conley-Zehnder index since conventions change according to literature. We fix our convention by requiring that for a \( C^2 \)-small Morse function \( f : M \to \mathbb{R} \), each critical point \( x \) of \( f \) satisfy

\[
\mu_{CZ}([x, w_x]) = i(x)
\]

where \( i \) denotes the Morse index and \( w_x \) is the trivial capping.

### 2.2. Quantum homology and Seidel elements

We sketch some basic definitions and properties concerning the quantum homology. Once again, we fix a ground field \( \mathbb{K} \). For further details of the concepts sketched in this section, we refer to [MS04].

Let \( (M, \omega) \) be a closed symplectic manifold. Define

\[
\Gamma := \pi_2(M)/(\ker(\omega) \cap \ker(c_1)).
\]

The Novikov ring \( \Lambda \) is defined by

\[
\Lambda := \{ \sum a_A \otimes e^A : a_A \in \mathbb{K}, (\forall \tau \in \mathbb{R}, \# \{ a_A \neq 0, \omega(A) < \tau \} < \infty) \}.
\]

The quantum homology of \( (M, \omega) \) is defined by

\[
QH_\ast(M; \mathbb{K}) := H_\ast(M; \mathbb{K}) \otimes_\mathbb{K} \Lambda.
\]

The quantum homology has a ring structure with respect to the quantum product denoted by \( * \). It is defined as follows:

\[
\forall a, b, c \in H_\ast(M; \mathbb{K}), \ (a \ast b) \circ c := \sum_{A \in \Gamma} GW_{3, A}(a, b, c) \otimes e^A
\]
where $\circ$ denotes the intersection index and $GW_{3,A}$ denotes the 3-pointed Gromov-Witten invariant in the class $A$. See [MS04] for details.

When $(M, \omega)$ is either monotone or negative monotone, then the quantum homology ring $QH_*(M; \mathbb{K})$ can be expressed in a simple way using the field of Laurent series. We first explain the case of monotone symplectic manifolds. In this case, $\Gamma \simeq \mathbb{Z}$ with a generator $A$ such that

$$\omega(A) = \lambda_0, \quad c_1(A) = N_M.$$ 

Thus the Novikov ring $\Lambda$ is the ring of formal Laurent series

$$\mathbb{K}[[s^{-1}, s]] := \left\{ \sum_{k \geq k_0} a_k s^k : a_j \in \mathbb{K}, k_0 \in \mathbb{Z} \right\}$$

where $s := e^A$ and the quantum homology ring $QH_*(M; \mathbb{K})$ is

$$QH_*(M; \mathbb{K}) = H_*(M; \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[[s^{-1}, s]].$$

The quantum product is expressed by

$$\forall a, b \in H_*(M; \mathbb{K}), \quad a \ast b = a \cap b + \sum_{k>0} (a \ast b)_k \cdot s^k.$$ 

The series on the right hand side runs over only non-positive powers since the elements of $\Gamma$ appearing in the sum represents pseudo-holomorphic spheres and pseudo-holomorphic spheres has non-negative $\omega$-area (remember that $s$ represents a sphere $A$ such that $\omega(A) = \lambda_0$).

When $(M, \omega)$ is negative monotone, by denoting the generator $A$ of $\Gamma$ which satisfies $\omega(A) = +\lambda_0, \quad c_1(A) = -N_M$

and by denoting $s := e^A$, we have

$$QH_*(M; \mathbb{K}) = H_*(M; \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[[s^{-1}, s]]$$

just as in the monotone case.

**Example.** The quantum homology ring of $(\mathbb{C}P^n, \omega_{FS})$ is expressed as follows:

$$QH_*(\mathbb{C}P^n; \mathbb{K}) = \frac{\mathbb{K}[[s^{-1}, s] [h]]}{\langle h^{*(n+1)} = [\mathbb{C}P^n] \cdot s^{-1} \rangle}$$

where $h \in H_{2n-2}(\mathbb{C}P^n; \mathbb{K})$ denotes the projective hyperplane class, $s$ denotes the generator of the Novikov ring and $h^{*(n+1)} := \underbrace{h \ast h \ast \cdots \ast h}_{n+1\text{-times}}$.

There is a canonical isomorphism called the PSS-isomorphism between Floer homology and quantum homology which will be denoted by $\Phi$:

$$\Phi_{PSS,H;\mathbb{K}} : QH_*(M; \mathbb{K}) \xrightarrow{\sim} HF_*(H).$$

PSS-isomorphism preserves the ring structure: for $a, b \in QH_*(M; \mathbb{K})$,

$$\Phi_{PSS,H;\mathbb{K}}(a) \ast_{pp} \Phi_{PSS,H;\mathbb{K}}(b) = \Phi_{PSS,H;\mathbb{K}}(a \ast b)$$

where $\ast_{pp}$ denotes the product of projection type.
where $*_pp$ denotes the pair-of-pants product.

Next, we briefly recall the definition of the Seidel element. The idea goes back to Seidel [Sei97]. For a Hamiltonian loop $\psi \in \pi_1(\text{Ham}(M, \omega))$, we can define a Hamiltonian fiber bundle

$$(M, \omega) \hookrightarrow (M_\psi, \Omega_\psi) \rightarrow (S^2, \omega_{\text{area}})$$

where, unit disks $D_j^2$, $j = 1, 2$,

$$M_\psi := (D_1^2 \times M) \coprod (D_2^2 \times M) / \sim,$$

$$(z_1, x) \sim (z_2, y) \iff z_1 = z_2 = e^{2\pi it}, y = \psi^i(x).$$

The form $\Omega_\psi$ is a family of symplectic form on $TM_\psi^{vert} = \text{Ker}(d\pi)$ parametrized by points of $S^2$. We fix almost complex structures $j$ on $S^2$ and $J$ on $M_\psi$ such that $d\pi$ is pseudo-holomorphic i.e. $j \circ d\pi = d\pi \circ J$ and for every $z \in S^2$, $J_{|\pi^{-1}(z)}$ defines a $\Omega_\psi$-compatible almost complex structure on $M_\psi$. For a section class $\sigma \in \pi_2(M_\psi)$, we denote the set of $(j, J)$-pseudo-holomorphic spheres in the class $\sigma$ by $\text{SecCl}(j, J, \sigma)$. The image of $\text{SecCl}(j, J, \sigma)$ by the evaluation map $ev : S^2 \to M$ at $z_0 \in S^2$ defines a homology class $[ev_{z_0}(\text{SecCl}(j, J, \sigma))]$ of $M$. We thus define the Seidel element $S_{\psi, \sigma} \in QH_*(M; \mathbb{K})$ by

$$S_{\psi, \sigma} := \sum_{A \in \Gamma} [ev_{z_0}(\text{SecCl}(j, J, \sigma + A))] \otimes e^A.$$

### 2.3. Hamiltonian spectral invariants.

Let $i^* : CF^+(H) \to CF_*(H)$ be the natural inclusion map and denote by $i^*_r : HF_*(H) \to HF_*(H)$ the induced map on homology. For a quantum homology class $a \in QH_*(M; \mathbb{K})$, define the spectral invariant by

$$c(H, a) = c(H, a; \mathbb{K}) := \inf \{ \tau \in \mathbb{R} : \Phi_{PSS, H, \mathbb{K}}(a) \in \text{Im}(i^*_r) \}.$$

Spectral invariants were introduced by Viterbo [Vit92] in terms of generating functions and later their counterparts in Floer theory were studied by Schwarz for aspherical symplectic manifolds [Sch00] and Oh for closed symplectic manifolds [Oh05]. We list some basic properties of spectral invariants.

**Proposition 23.** Spectral invariants satisfy the following properties where $H, G$ are Hamiltonians:

1. For any $a \in QH_*(M; \mathbb{K}) \setminus \{0\}$,
   $$\mathcal{E}^-(H - G) \leq c(H, a) - c(G, a) \leq \mathcal{E}^+(H - G)$$

   where
   $$\mathcal{E}^-(H) := \int_{t=0}^1 \inf_x H_t(x) dt, \quad \mathcal{E}^+(H) := \int_{t=0}^1 \sup_x H_t(x) dt,$$
   $$\mathcal{E}(H) := \mathcal{E}^+(H) - \mathcal{E}^- (H).$$

2. For any $a \in QH_*(M; \mathbb{K}) \setminus \{0\}$,
   $$c(H, a) \in \text{Spec}(H)$$
• for any Hamiltonian $H$ when $(M, \omega)$ is rational.

• for any non-degenerate Hamiltonian $H$ when $(M, \omega)$ is a general closed symplectic manifold.

(3) For any $a, b \in QH_*(M; \mathbb{K}) \setminus \{0\}$,
\[
c(H \# G, a \ast b) \leq c(H, a) + c(G, b)
\]

(4) Let $U$ be a non-empty subset of $M$.
\[
c(H, [M]) \leq e(\text{Supp}(H)) := \inf \{E(G) : \phi_G(\text{Supp}(H)) \cap \text{Supp}(H) = \emptyset\}.
\]

(5) Let $f : M \to \mathbb{R}$ be an autonomous Hamiltonian and $a \in H_*(M; \mathbb{K})$. For a sufficiently small $\varepsilon > 0$, we have
\[
c(\varepsilon f, a) = c_{LS}(\varepsilon f, a) = \varepsilon \cdot c_{LS}(f, a)
\]
where $c_{LS}(f, a)$ is the Lusternik–Schnirelmann spectral invariant defined by
\[
c_{LS}(f, a) := \inf \{\tau : a \in \text{Im}(H_*(\{f \leq \tau\}) \to H_*(M))\}.
\]

(6) For $\psi \in \pi_1(\text{Ham}(M, \omega))$, a section class $\sigma$ of the Hamiltonian fiber bundle $M_\psi \to S^2$, and $a \in QH_*(M; \mathbb{K}) \setminus \{0\}$ we have
\[
c(\psi^* H, a) = c(H, S_\psi, \sigma \ast a) + \text{const}(\psi, \sigma)
\]
where
\[
(\psi^* H)_t := (H_t - K_t) \circ \psi^t, \quad \psi^t := H_t^t, \quad \phi_{H \circ \psi}^{-1} \circ \phi_H^t
\]
and $\text{const}(\psi, \sigma)$ denotes a constant depending on $\sigma$ and $K$.

(7) For any $\psi \in \text{Symp}_0(M, \omega)$ and $a \in QH_*(M; \mathbb{K}) \setminus \{0\}$,
\[
c(H \circ \psi, a) = c(H, a).
\]

Remark 24.

(1) For a set $A$, $e(A) := \inf \{E(G) : \phi_G(A) \cap A = \emptyset\}$ is called the displacement energy of $A$.

(2) Strictly speaking, spectral invariants $c(H, \cdot)$ can be defined only if $H$ is non-degenerate since they are defined via Floer homology of $H$. However, by Proposition 23 (1), one can define $c(H, \cdot)$ for a degenerate Hamiltonian $H$ by considering an approximation of $H$ by non-degenerate Hamiltonians.

The spectral norm of $H$ is defined by
\[
\gamma(H) := c(H, [M]) + c(H^\perp, [M])
\]
where $[M]$ denotes the fundamental class. We also define a spectral norm for Hamiltonian diffeomorphisms by
\[
\gamma : \text{Ham}(M, \omega) \to \mathbb{R}
\]
\[
\gamma(\phi) := \inf_{\phi^\ast H = \phi} \gamma(H).
\]
Remark 25. In this paper, we work with a fixed ground field \( \mathbb{K} \) of zero characteristic but some results hold for other ground rings. More precisely, Theorem 2 and 4 hold for spectral norms respectively with any ground field \( \mathbb{K} \) and with any ground ring \( R \) that is commutative and unital e.g. \( \mathbb{Z} \). In fact, Usher proved in [Ush08] that whenever one can define a Floer chain complex with a ground ring that is Noetherian, spectral invariants can be defined as above and satisfy properties listed in Proposition 23. For weakly-monotone symplectic manifolds, one can define Floer chain complexes with any ground field \( \mathbb{K} \) [MS04]. Moreover, for monotone symplectic manifolds, one can define Floer chain complexes with any ground ring \( R \) that is commutative and unital e.g. \( \mathbb{Z} \). For general closed symplectic manifolds where one needs to use virtual cycle techniques in order to build Floer chain complexes [FO99], [FOOO09], [LT98], the ground field \( \mathbb{K} \) should have zero characteristic.

2.4. Barcodes. In this subsection, we roughly explain how to define barcodes for Hamiltonian diffeomorphisms on (negative) monotone symplectic manifolds following [LSV18]. We also refer to [PSS17] and [UZ16] for constructions of barcodes in symplectic topology.

A finite barcode is a finite set of intervals

\[
B = \{I_j = (a_j, b_j) : a_j \in \mathbb{R}, b_j \in \mathbb{R} \cup \{+\infty\}\}_{1 \leq j \leq N}.
\]

Two finite barcodes \( B, B' \) are said to be \( \delta \)-matched if, after deleting some intervals of length less than \( 2\delta \), there exists a bijective matching between the intervals of \( B \) and \( B' \) such that the endpoints of the matched intervals are less than \( \delta \) of each other. The bottleneck distance of \( B, B' \) is defined as follows:

\[
d_{\text{bot}}(B, B') := \inf \{\delta > 0 : B \text{ and } B' \text{ are } \delta - \text{matched}\}.
\]

Barcodes of non-degenerate Hamiltonian diffeomorphisms were first defined in [PS16] for symplectic manifolds that are symplectically aspherical via filtered Floer homology. For symplectically non-aspherical manifolds, (filtered) Floer homology groups do not satisfy the "finiteness" condition and in order to overcome this issue, [PSS17] defines barcodes for non-degenerate Hamiltonian diffeomorphisms on monotone symplectic manifolds by fixing a degree. Later, in order to define barcodes of degenerate Hamiltonian diffeomorphisms on spheres, Le Roux–Seyfaddini–Viterbo [LSV18] considered a completion of the set of finite barcodes with respect to the bottleneck distance. Their method applies more generally to (negative) monotone symplectic manifolds. We will briefly review their idea.

Let \( \text{Barcodes} \) denote the set of a collection of intervals \( B = \{I_j\}_{j \in \mathbb{N}} \) such that for any \( \delta > 0 \) only finitely many of the intervals \( I_j \) have lengths greater than \( \delta \). The bottleneck distance \( d_{\text{bot}} \) extends to \( \text{Barcodes} \). The space \( (\text{Barcodes}, d_{\text{bot}}) \) is indeed the completion of the space of finite barcodes. Given a barcode \( B = \{I_j\}_{j \in \mathbb{N}} \) and \( c \in \mathbb{R} \), define \( B + c = \{I_j + c\}_{j \in \mathbb{N}} \), where \( I_j + c \) is the interval obtained by adding \( c \) to the endpoints of \( I_j \). Define an equivalence relation \( \sim \) by \( B \sim B' \) if \( B' = B + c \) for
some $c \in \mathbb{R}$. We will denote the quotient space of $\text{Barcodes}$ with the relation $\sim$ by $\hat{\text{Barcodes}}$.

We explain briefly how to map a (possibly degenerate) Hamiltonian diffeomorphism on a (negative) monotone symplectic manifold to a barcode following [LSV18]. Given a non-degenerate Hamiltonian $H$ and an integer $k \in \mathbb{Z}$, the filtered $k$-th Floer homology group $\{HF^k_\tau(H)\}_{\tau \in \mathbb{R}}$ forms a persistence module. For this filtered vector spaces, one can define a barcode in the same way as in [PS16] and we denote the barcode by $B_k(H)$. We define the barcode of $H$ by

$$B(H) := \sqcup_k B_k(H) \in \hat{\text{Barcodes}}.$$  

For two Hamiltonians $H, G$ such that $\phi_H = \phi_G$, their Floer homology groups coincide up to shifts of index and action filtration i.e. $HF^\tau_\kappa(H) \simeq HF^\tau_{\kappa+k_0}(G)$ for some $k_0 \in \mathbb{Z}, \tau_0 \in \mathbb{R}$ (see, for example, [MS04] Section 12.5). Thus $B(H) = B(G)$ and therefore, we define the barcode map $B$ as follows:

$$B(H) := \sqcup_k B_k(H) \in \hat{\text{Barcodes}}.$$  

for any $H$ such that $\phi_H = \phi$.

Kislev-Shelukhin [KS18] proved the following inequality to estimate the bottleneck distance between barcodes of $\phi, \psi \in \text{Ham}(M, \omega)$:

$$d_{\text{bot}}(B(\phi), B(\psi)) \leq \frac{1}{2}\gamma(\phi^{-1}\psi). \quad (1)$$

This implies that once we obtain the $C^0$-continuity of $\gamma$, the map

$$B : (\text{Ham}(M, \omega), d_{C^0}) \to (\hat{\text{Barcodes}}, d_{\text{bot}})$$

is continuous. Thus, Corollary 8 is a direct consequence of Theorem 6.

3. Proofs

In this section, we prove the results claimed in the introduction. We start from the case of negative monotone symplectic manifolds since the proof is based on a similar idea to the case of rational symplectic manifolds but it is simpler.

3.1. Proofs of Theorem 4 and 6. We prove Theorem 4 (1) and Theorem 6. It is achieved by combining the following Propositions 26 and 27.

**Proposition 26.** Let $(M, \omega)$ be a monotone or negative monotone symplectic manifold, $U$ be a simply connected open subset of $M$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})$ satisfies $d_{C^0}(\text{id}, \phi_H) < \delta$ and $\phi_H(x) = x$ for all $x \in U$, then
when \((M, \omega)\) is monotone,
\[\gamma(H) < \frac{\dim(M)}{N_M} \lambda_0 + \varepsilon.\]

- when \((M, \omega)\) is negative monotone,
\[\gamma(H) < \varepsilon.\]

**Proposition 27.** ([BHS18b] Lemma 4.2)

Let \((M, \omega)\) be any closed symplectic manifold. For any \(\varepsilon > 0\), there exists a non-empty open ball \(B \subset M\) satisfying the following properties: its displacement energy is estimated by \(e(B) < \varepsilon\) and for any \(\varepsilon' > 0\), there exists \(\delta' > 0\) such that if \(\phi_H \in \text{Ham}(M, \omega), d_{C^0}(\text{id}_M, \phi_H) < \delta'\), then there exist \(G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M \times M, \mathbb{R})\) such that

\[
\begin{align*}
1) & \quad \gamma(G) < \varepsilon \\
2) & \quad d_{C^0}(\text{id}_{M \times M}, \phi_G) < \varepsilon' \\
3) & \quad (\phi_H \times \phi^{-1}_H) \circ \phi_G|_{B \times B} = \text{id}_{B \times B}
\end{align*}
\]

Proposition 27, proven by Buhovsky-Humilière-Seyfaddini [BHS18b], claims that given a Hamiltonian diffeomorphism \(\phi\) on \(M\), one can always deform the Hamiltonian diffeomorphism \(\phi \times \phi^{-1}\) to a Hamiltonian diffeomorphism on \(M \times M\) that does not move any point on a certain open set by composing with a both \(C^0\)- and \(\gamma\)-small Hamiltonian diffeomorphism on \(M \times M\).

We postpone the proof of Proposition 26 and first briefly review the proof of Proposition 27 due to [BHS18b] as we will need some parts of the proof in the proof of Claim 34.

**Proof.** (of Proposition 27 by [BHS18b])

Let \(\varepsilon > 0\) and fix any non-empty open ball \(B'\) whose displacement energy satisfies \(e(B') < \varepsilon/4\).

**Claim 28.** (Claim 4.3 [BHS18b])

There exists a Hamiltonian \(Q\) on \(M \times M\) and an open ball \(B''\) in \(M\) such that

- \(\text{Supp}(Q) \subset B' \times B''\).
- \(\forall (x, y) \in B'' \times B'', \phi_Q(x, y) = (y, x)\).
- Denote the origin point of the ball \(B'\) by \(x_0\). The point \((x_0, x_0)\) is fixed by the flow of \(Q\): \(\forall t, \phi_t^Q((x_0, x_0)) = (x_0, x_0)\).

Now, define
\[G := (0 \oplus H)\#Q\#(0 \oplus H)\#Q.\]  \hspace{1cm} (2)

This Hamiltonian \(G\) satisfies the following:
\[
c(G, [M \times M]) = c((0 \oplus H)\#Q\#(0 \oplus H)\#Q, [M \times M])
\leq c((0 \oplus H)\#Q\#(0 \oplus H), [M \times M]) + c(Q, [M \times M])
= c(Q, [M \times M]) + c(Q, [M \times M])
\]
\[ \leq e(\text{Supp}(Q)) + e(\text{Supp}(Q)) \leq 2e(B' \times B') \]

and as the ball \( B' \) was chosen so that \( e(B') \leq \varepsilon/4 \), we get

\[ c(G, [M \times M]) < \varepsilon/2. \tag{3} \]

We can estimate \( c(\overline{G}, [M \times M]) \) in the same way and we get

\[ \gamma(G) < \varepsilon. \]

Now, let \( B \) be a ball whose closure is included in \( B'' \) and make sure that the origin of \( B \) is the same as the origin of \( B' \), namely \( x_0 \). If we require \( \phi_H \) to be \( C^0 \)-close enough to \( \text{id} \) so that \( \phi_H(B) \subset B'' \), then for all \((x, y) \in B \times B\)

\[ (\phi_H \times \phi_H^{-1}) \circ \phi_G(x, y) = (x, y). \]

This finishes the proof of Proposition \( 27 \)

Now, before proving Proposition \( 26 \), we prove Theorem \( 4 \) (1) and Theorem \( 6 \).

**Proof.** (of Theorem \( 4 \) (1) and Theorem \( 6 \))

Note that if \((M, \omega)\) is (negative) monotone, then so is \((M \times M, \omega \oplus \omega)\). Given any \( \varepsilon > 0 \), we can take a ball \( B \) in \( M \) as in Proposition \( 27 \). By Proposition \( 27 \), for any \( \varepsilon' > 0 \), there exists \( \delta' > 0 \) such that if \( d_{C^0}(\text{id}_M, \phi) < \delta' \), then there exist \( G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M \times M, \mathbb{R}) \) such that

1. \( \gamma(G) < \varepsilon \)
2. \( d_{C^0}(\text{id}_{M \times M}, \phi_G) < \varepsilon' \)
3. \( \|(\phi \times \phi^{-1}) \circ \phi_G\|_{B \times B} = \text{id}_{B \times B} \)

We take \( \varepsilon' > 0 \) small enough so that

\[ d_{C^0}(\text{id}_{M \times M}, (\phi \times \phi^{-1}) \circ \phi_G) < \delta \]

is satisfied where \( \delta > 0 \) is a positive number as in Proposition \( 26 \) which is determined by \( B \times B \) and \( \varepsilon > 0 \). This is achievable as

\[ d_{C^0}(\text{id}_{M \times M}, (\phi \times \phi^{-1}) \circ \phi_G) \leq d_{C^0}(\text{id}_{M \times M}, \phi_G) + d_{C^0}(\phi_G, (\phi \times \phi^{-1}) \circ \phi_G) \]

\[ = d_{C^0}(\text{id}_{M \times M}, \phi_G) + d_{C^0}(\text{id}_{M \times M}, (\phi \times \phi^{-1}) \circ \phi_G) \leq \varepsilon' + 2\delta'. \]

Now, take any Hamiltonian \( H \) generating \( \phi: \phi_H = \phi \). Then \((H \oplus H) \# G\) generates \((\phi \times \phi^{-1}) \circ \phi_G\) so if by Proposition \( 26 \) we have

- if \((M, \omega)\) is monotone, then
  \[ \gamma((H \oplus H) \# G) < \frac{\dim(M \times M)}{N_{M \times M}} \lambda_0 + \varepsilon = 2 \cdot \frac{\dim(M)}{N_M} \lambda_0 + \varepsilon. \]
- if \((M, \omega)\) is negative monotone,
  \[ \gamma((H \oplus H) \# G) < \varepsilon. \]
As $\gamma(H \oplus \overline{H}) = 2\gamma(H)$ (by Theorem 5.1. in [EP09]), we have
\begin{align*}
2\gamma(H) &= \gamma(H \oplus \overline{H}) \leq \gamma((H \oplus \overline{H})\#G) + \gamma(G) \\
&= \gamma((H \oplus \overline{H})\#G) + \gamma(G) < \gamma((H \oplus \overline{H})\#G) + \varepsilon.
\end{align*}
Therefore,
\begin{itemize}
  \item if $(M, \omega)$ is monotone, then
    \begin{equation}
    2\gamma(H) < 2 \cdot \dim(M) \frac{\lambda_0 + \varepsilon}{N_M} + \varepsilon,
    \end{equation}
    thus
    \begin{equation}
    \gamma(H) < \frac{\dim(M)}{N_M} \lambda_0 + \varepsilon.
    \end{equation}
    This proves Theorem 4 (1).
  \item if $(M, \omega)$ is negative monotone, then $2\gamma(H) < 2\varepsilon$, thus
    \begin{equation}
    \gamma(H) < \varepsilon.
    \end{equation}
    This proves Theorem 6.
\end{itemize}
\end{proof}

\begin{proof} (of Theorem 6 (2))
Once we know that spectral norms are well-defined on $\text{Ham}(M, \omega)$, the $C^0$-continuity at id follows directly from Theorem 6 (1). The $C^0$-continuity at $\phi \in \text{Ham}(M, \omega)$ is a consequence of the triangle inequality: for any $\varepsilon > 0$, if we take $d_{C^0}(\phi, \psi)$ small enough so that $d_{C^0}(\text{id}, \phi^{-1} \circ \psi) < \delta$ where $\delta$ is taken as in Theorem 6 (1). Then,
\begin{equation}
|\gamma(\psi) - \gamma(\phi)| \leq \gamma(\phi^{-1} \circ \psi) < \varepsilon.
\end{equation}
By using the $C^0$-continuity, we can define the spectral norm for Hamiltonian homeomorphisms in the following way: for $\phi \in \text{Ham}(M, \omega)$, take a sequence $\phi_k \in \text{Ham}(M, \omega)$ that $C^0$-converges to $\phi$. Define $\gamma(\phi) := \lim_{k \to +\infty} \gamma(\phi_k)$. Note that any approximating sequence will give the same limit. This completes the proof of Theorem 6 (2).
\end{proof}

We now prove Proposition 26.

\begin{proof} (of Proposition 26)
Take a Morse function $f : M \to \mathbb{R}$ whose critical points are located in $U$. We assume that $f$ is $C^2$-small enough so that its Hamiltonian flow does not admit any non-constant periodic points and that $\text{osc}(f) := \max f - \min f < \varepsilon$. Since $\phi_f$ has no fixed points in $M \setminus U$, there exists $\delta > 0$ such that
\begin{equation}
\forall x \in M \setminus U, \ d(x, \phi_f(x)) > \delta.
\end{equation}
We will now see that if $\phi_H$ is $C^0$-close enough to id, then
\begin{equation}
\text{Crit}(f) = \text{Fix}(\phi_H \circ \phi_f).
\end{equation}
\end{proof}
First, \( \text{Crit}(f) \subset \text{Fix}(\phi_H \circ \phi_f) \) follows from \( \forall x \in U, \phi_H(x) = x \). Next, we will see \( \text{Fix}(\phi_H \circ \phi_f) \subset \text{Crit}(f) \) if \( \phi_H \) is \( C^0 \)-close enough to \( \text{id} \). Let \( x \in \text{Fix}(\phi_H \circ \phi_f) \).

1. Assume \( x \in U \). Then, \( \phi_f(x) = \phi_H \circ \phi_f(x) = x \) and since \( \text{Crit}(f) = \text{Fix}(\phi_f) \), we have \( x \in \text{Crit}(f) \).
2. Assume \( x \notin U \). Then, \( \phi_H(x) \notin U \) and

\[
\begin{align*}
d_{C^0}(x, \phi_H \circ \phi_f(x)) & \geq d_{C^0}(\phi_f(x), x) - d_{C^0}(\phi_f(x), \phi_H \circ \phi_f(x)) \\
& \geq \delta - d_{C^0}(\text{id}, \phi_H).
\end{align*}
\]

If we take \( \phi_H \) to be \( C^0 \)-close enough to \( \text{id} \) so that the last equation become positive, then \( x \notin \text{Fix}(\phi_H \circ \phi_f) \). Thus \( x \in \text{Fix}(\phi_H \circ \phi_f) \) implies \( x \in U \) and \( x = \phi_H \circ \phi_f(x) = \phi_f(x) \). Thus \( x \in \text{Crit}(f) \).

We have proven that if \( \phi_H \) is \( C^0 \)-close enough to \( \text{id} \), then

\[
\text{Crit}(f) = \text{Fix}(\phi_H \circ \phi_f).
\]

Thus, for such \( \phi_H \) and for any \( x \in \text{Crit}(f) = \text{Fix}(\phi_H \circ \phi_f) \), its orbit is \( \phi_H^{f}(x) = \phi^f_H(x) \) and thus,

\[
\text{Spec}(H \# f) = \{ f(x) + \mathcal{A}_H([\phi^f_H(x), w]) : x \in \text{Crit}(f), [\phi^f_H(x), w] \in \text{Crit}(\mathcal{A}_H) \}.
\]

Take any \( x_0 \in \text{Crit}(f) \) and a capping \( w_0 : D^2 \to M \) of the orbit \( \phi^f_H(x_0) \) i.e. \( w_0(e^{2\pi i t}) = \phi^f_H(x_0) \). We fix this capped orbit \([\phi^f_H(x_0), w_0]\) in the sequel.

For any \( x \in \text{Crit}(f) \), define a capping \( w_x : D^2 \to M \) of the orbit \( \phi^f_H(x) \) by

\[
w_x(se^{2\pi i t}) := \phi^f_H(c(s)) \# w_0
\]

where \( c : [0, 1] \to U \) is a smooth path from \( x_0 \) to \( x \) and \( \phi^f_H(c(s)) \# w_0 \) denotes the gluing of \( \phi^f_H(c(s)) \) and \( w_0 \) along \( \phi^f_H(x_0) \).

Recall that \( \gamma(H) = c(H, [M]) + c(\overline{H}, [M]) \) and we will estimate \( c(H, [M]) \) and \( c(\overline{H}, [M]) \) separately.

By the triangle inequality,

\[
c(H, [M]) \leq c(H \# f, [M]) + c(\overline{f}, [M]).
\]

For the second term we know that

\[
c(\overline{f}, [M]) = c(-f, [M]) \leq \varepsilon
\]

as \( f \) is \( C^2 \)-small and \( osc(f) < \varepsilon \).

For the first term, \( c(H \# f, \cdot) \in \text{Spec}(H \# f) \) so there exists a point \( x \in \text{Crit}(f) \) and a sphere \( A : S^2 \to M \) such that

- \( \mathcal{A}_{H \# f}([\phi^f_H(x), w_x \# A]) = c(H \# f, [M]) \).
- \( \mu_{CZ}([\phi^f_H(x), w_x \# A]) = \text{deg}([M]) = 2n \).
The sphere $A$ plays the role of correcting the capping of the capped orbit $[\phi_H^t(x), w_x]$ to achieve the appropriate capped orbit which realizes the spectral invariant $c(H\#f, [M])$.

The action and the index can be rewritten in the following way where $i$ denotes the Morse index:

- $A_{H\#f}([\phi_H^t(x), w_x\#A]) = f(x) + A_H([\phi_H^t(x), w_x]) - \omega(A).
- \mu_{CZ}([\phi_H^t(x), w_x\#A]) = i(x) + 2\mu([\phi_H^t(x), w_x]) - 2c_1(A).

Thus we get the following two equations.

\[
c(H\#f, [M]) = f(x) + A_H([\phi_H^t(x), w_x]) - \omega(A). \tag{4a}
2n = i(x) + 2\mu([\phi_H^t(x), w_x]) - 2c_1(A). \tag{4b}
\]

In the same way, there exist a point $y \in \text{Crit}(f)$ and a sphere $B : S^2 \to M$ such that

\[
c(\overline{H}\#f, [M]) = f(y) + A_{\overline{H}}([\phi_\overline{H}(y), \overline{w}_y]) - \omega(B). \tag{5a}
2n = i(y) + 2\mu([\phi_\overline{H}(y), \overline{w}_y]) - 2c_1(B). \tag{5b}
\]

Here, the capping $\overline{w}_y$ is

\[
\overline{w}_y(se^{2\pi it}) := w_y(se^{2\pi i(-t)}).
\]

Thus, by adding the equations \ref{4a} and \ref{5a} we obtain

\[
\gamma(H) \leq 2c(-f, [M]) + c(H\#f, [M]) + c(\overline{H}\#f, [M])
= 2c(-f, [M]) + f(x) + f(y) + A_H([\phi_H^t(x), w_x]) + A_{\overline{H}}([\phi_\overline{H}(y), \overline{w}_y]) - \omega(A + B)
\leq 4\varepsilon - \omega(A + B)
\]

where Proposition \ref{21} and \ref{22} were used in the last line. In the same way, by adding the equalities \ref{4b} and \ref{5b} we obtain

\[
4n = i(x) + i(y) + 2\mu([\phi_H^t(x), w_x]) + 2\mu([\phi_\overline{H}(y), \overline{w}_y]) - 2c_1(A + B)
= i(x) + i(y) - 2c_1(A + B).
\]

Now, since $i(x), i(y)$ are Morse indices, we have

\[
0 \leq i(x), i(y) \leq 2n = \dim(M)
\]

and thus,

\[
0 \leq 4n + 2c_1(A + B) \leq 4n.
\]

Thus,

\[
-2n \leq c_1(A + B) \leq 0.
\]

Note that up to now, we have not used the (negative) monotonicity of $(M, \omega)$. Now,

- if $(M, \omega)$ is negative monotone, then

\[
-\omega(A + B) = -\lambda \cdot c_1(A + B) \leq 0.
\]
• if \((M, \omega)\) is monotone, then

\[-\omega(A + B) = -\lambda \cdot c_1(A + B) \leq 2n\lambda = \frac{2n}{N_M} \lambda_0.\]

Therefore,

• if \((M, \omega)\) is negative monotone, then

\[\gamma(H) \leq 4\varepsilon.\]

• if \((M, \omega)\) is monotone, then

\[\gamma(H) \leq \frac{2n}{N_M} \lambda_0 + 4\varepsilon.\]

This completes the proof of Proposition 26. □

3.2. **Proof of Theorem 31.** The goal of this subsection is to prove Theorem 31 which includes Theorem 1 as a special case. The argument is similar to the negative monotone case. We start by some additional definitions.

**Definition 29.** Let \((M, \omega)\) be any closed symplectic manifold and \(a, b \in H_*(M; \mathbb{K})\setminus\{0\} \). We define the following:

\[\gamma_{a,b} : C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R}) \to \mathbb{R},\]

\[\gamma_{a,b}(H) := c(H,a) + c(H,b).\]

**Remark 30.** Of course, \(\gamma_{[M],[M]} = \gamma\) where \(\gamma\) is the usual spectral norm.

**Theorem 31.** Let \((M, \omega)\) be a rational symplectic manifold and \(a, b \in H_*(M; \mathbb{K})\setminus\{0\}\). For any \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \(d_{C^0}(\text{id}, \phi_H) < \delta\), then

\[|\gamma_{a,b}(H) - l \cdot \lambda_0| < \varepsilon\]

for some integer \(l \in \mathbb{Z}\) depending on \(a, b \in H_*(M; \mathbb{K})\setminus\{0\}\) and \(H\).

Before proving Theorem 31, we will see the following consequence on the \(C^0\)-continuity of the spectral norm.

**Corollary 32.** Let \((M, \omega)\) be a rational symplectic manifold. Assume that there exist constants \(0 < \kappa < 1\) and \(\delta' > 0\) such that if \(\phi \in \text{Ham}(M, \omega)\), \(d_{C^0}(\text{id}, \phi) \leq \delta'\), then \(\gamma(\phi) \leq \kappa \cdot \lambda_0\). Then, \(\gamma : \text{Ham}(M, \omega) \to \mathbb{R}\) is \(C^0\)-continuous.

Corollary 32 will be used to obtain the \(C^0\)-continuity of the spectral norm for \(\mathbb{C}P^n\) in Theorem 2.

**Proof.** (of Corollary 32)

It is enough to prove the continuity at \(\text{id}\) since \(|\gamma(\phi) - \gamma(\psi)| \leq \gamma(\psi^{-1}\phi)\). For a given \(\varepsilon \in (0, \frac{1}{2}(1 - \kappa)\lambda_0)\), take \(\delta > 0\) as in Theorem 1. Let \(\phi \in \text{Ham}(M, \omega)\), \(d_{C^0}(\text{id}, \phi) < \min\{\delta, \delta'\}\).
There exists a Hamiltonian $H$ such that $\phi_H = \phi$ and

$$
\gamma(H) < \gamma(\phi) + \epsilon < \kappa \cdot \lambda_0 + \frac{1}{2}(1 - \kappa)\lambda_0
$$

$$
= \frac{1}{2}(1 + \kappa)\lambda_0 < \lambda_0 - \epsilon.
$$

Thus, by Theorem 1

$$
\gamma(H) < \epsilon.
$$

Thus,

$$
\gamma(\phi) \leq \gamma(H) < \epsilon.
$$

This implies the continuity of $\gamma$ at $id$ and hence completes the proof of Corollary 32. □

Now, we move to the proof of Theorem 31. The following Proposition will be needed.

**Proposition 33.** Let $(M, \omega)$ be a closed symplectic manifold. Fix an arbitrary point $x_0 \in M$. There exists a constant $C > 0$ satisfying the following property: For any point $x \in M$, there exists $\psi \in \text{Ham}(M, \omega)$ such that

1. $\psi(x) = x_0$
2. $\|d\psi^{-1}\| \leq C$

The proof is elementary and thus will be omitted.

**Proof.** (of Theorem 31)

The proof is similar to the proof of Theorem 6. For a given $\epsilon > 0$, we take a ball $B$ as in Proposition 27. We will denote the origin of the ball $B$ by $x_0$. For the open set $B \times B$, consider a Morse function $F : M \times M \rightarrow \mathbb{R}$ such that

- $\text{Crit}(F) \subset B \times B$.
- $F$ is $C^2$-small enough so that $\text{Fix}(\phi_F) = \text{Crit}(F)$ and that $\text{osc}(F) := \max F - \min F < \epsilon$.

As $\phi_F$ has no fixed points in $M \setminus (B \times B)$, there exists $\delta > 0$ such that for any $x \in M \times M \setminus (B \times B)$, $d(x, \phi_F(x)) > \delta$.

For any $\epsilon' > 0$, we can take $\delta' > 0$ as in Proposition 27. By Proposition 33, for $x_0$, there exists a constant $C > 0$ such that for any $x \in M$, there exists $\psi \in \text{Ham}(M, \omega)$ such that

- $\psi(x) = x_0$
- $\|d\psi^{-1}\| \leq C$

We consider $\phi_H$ so that $d_{C^0}(\text{id}, \phi_H) < \delta'/C$. For any $x_\ast \in \text{Fix}(\phi_H)$, we can take $\psi \in \text{Ham}(M, \omega)$ such that $\psi(x_\ast) = x_0$ and $\|d\psi^{-1}\| \leq C$.

Let $H' := H \circ \psi^{-1}$. We have

$$
d_{C^0}(\text{id}, \phi_{H'}) = d_{C^0}(\text{id}, \psi^{-1}\phi_H\psi) = d_{C^0}(\psi^{-1}, \psi^{-1}\phi_H)
$$
\[ \leq \|d\psi^{-1}\| d_{C^0}(\text{id}, \phi_H) \leq C \cdot \delta' / C = \delta'. \]

By Proposition 27 there exists \( G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M \times M) \) such that

- \( \gamma(G) < \varepsilon. \)
- \( d_{C^0}(\text{id}_{M \times M}, \phi_G) < \varepsilon'. \)
- \( (\phi_H^{-1} \times \phi_H) \circ \phi_G|_{B \times B} = \text{id}_{B \times B}. \)

In addition, we have seen in the proof of Proposition 27 that \( G \) is defined by
\[
G = (0 \oplus H')#Q#(0 \oplus H')#Q
\]
where \( Q \) is an autonomous Hamiltonian on \( M \times M \) whose flow fixes the point \((x_0, x_0)\) for all time \( t: \phi^t_Q((x_0, x_0)) = (x_0, x_0)\). The spectral invariant of \( G \) was estimated as
\[
c(G, [M \times M]) < \frac{1}{2} \varepsilon.
\]

All these properties of \( G \) and \( Q \) will be used in the following.

We will now split the proof into four steps.

- **Step 1:** The aim of this step is to prove the following:

  **Claim 34.**
  \[
  |c(\overline{H} \oplus H', a \otimes b) - c((\overline{H} \oplus H')#G#F, a \otimes b)| < \frac{3}{2} \varepsilon.
  \]

  **Proof.** By the triangle inequality, we have
  \[
  c((\overline{H} \oplus H')#G#F, a \otimes b) - c(\overline{H} \oplus H', a \otimes b)
  \leq c(G#F, [M \times M]) \leq c(G, [M \times M]) + c(F, [M \times M]) < \frac{3}{2} \varepsilon.
  \]

  Note that the final inequality uses,
  \[
  c(F, [M \times M]) \leq \max(F) < \varepsilon
  \]
  and the estimate
  \[
  c(G, [M \times M]) < \frac{1}{2} \varepsilon.
  \]

  The other side of the inequality follows from a similar estimate. \( \square \)

- **Step 2:** The aim of this step is to prove the following:

  **Claim 35.**
  \[
  c((\overline{H} \oplus H')#G#F, a \otimes b)
  = F(x, y) + A((\overline{H} \oplus H')#G([\phi^t_{(\overline{H} \oplus H')#G}((x, y)), w_{x,y}]) + (\omega \oplus \omega)(A_1)
  \]
  for some critical point \((x, y)\) of \( F \), some capping \( w_{x,y} \), and some \( A_1 \in \pi_2(M \times M) \).

  **Proof.** As
  \[
  d_{C^0}(\text{id}, (\phi_H^{-1} \times \phi_H) \circ \phi_G) \leq d_{C^0}(\text{id}, \phi_G) + d_{C^0}(\phi_G, \phi_H^{-1} \times \phi_H) \circ \phi_G)
  = d_{C^0}(\text{id}, \phi_G) + d_{C^0}(\phi_G, \phi_H^{-1} \times \phi_H) \leq \varepsilon' + \delta',
  \]
we can take $\varepsilon' > 0$ small enough so that
\[ d_{C^0}(\text{id}, (\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G) \leq \delta. \]

Therefore, as
- for all $x \notin B \times B$, $d_{C^0}(x, \phi_F(x)) > \delta$,
- $d_{C^0}(\text{id}, (\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G) \leq \delta$,
- $(\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G|_{B \times B} = \text{id}_{B \times B}$,

we have $\text{Fix}((\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G \circ \phi_F) = \text{Crit}(F)$. Thus the spectral invariant $c((H' \oplus H') \# G \# F, a \otimes b)$ can be expressed as follows:
\[ c((H' \oplus H') \# G \# F, a \otimes b) = F(x, y) + A_{(H' \oplus H') \# G}([\phi^t_{(H' \oplus H') \# G}((x, y)), w_{x,y}]) + (\omega \oplus \omega)(A_1) \]
where
- $(x, y)$ is a certain critical point of $F$ which is located in $B \times B$.
- $w_{x,y}$ denotes an arbitrary chosen capping of the orbit $\phi^t_{(H' \oplus H') \# G}((x, y))$. We fix this capping in the sequel.
- $A_1$ denotes the sphere which plays the role of correcting the capping $w_{x,y}$.

\[ \Box \]

Step 3: The aim of this step is to prove the following:

Claim 36.
\[ A_{(H' \oplus H') \# G}([\phi^t_{(H' \oplus H') \# G}((x, y)), w_{x,y}]) = (\omega \oplus \omega)(A_2) \]
for some $A_2 \in \pi_2(M \times M)$.

Proof. By Proposition 21 (2), we obtain
\[ A_{(H' \oplus H') \# G}([\phi^t_{(H' \oplus H') \# G}((x, y)), w_{x,y}]) = A_{(H' \oplus H') \# G}([\phi^t_{(H' \oplus H') \# G}((x_0, x_0)), w_{x_0,x_0}]) \]
where $w_{x_0,x_0}$ is the capping of the orbit $\phi^t_{(H' \oplus H') \# G}((x_0, x_0))$ corresponding to the capping $w_{x,y}$ in the sense of Proposition 21 (2). As $Q$ is a Hamitonian which generates a time-1 map that switches the coordinate i.e. $(p, q) \mapsto (q, p)$ in $B \times B$ and satisfies $\forall t, \phi^t_Q((x_0, x_0)) = (x_0, x_0)$, we have
\[ A_{(H' \oplus H') \# G}([\phi^t_{(H' \oplus H') \# G}((x_0, x_0)), w_{x_0,x_0}]) = \int Q(\phi^t_Q(x_0, x_0)) dt + \int (0 \oplus H')(t, x_0, \phi^t_H(x_0)) dt - \omega(\phi^t_H(x_0)) + \int Q(\phi^t_Q(x_0, x_0)) dt + \int (0 \oplus H')(t, x_0, \phi^t_H(x_0)) dt - \omega(\phi^t_H(x_0)) + (\omega \oplus \omega)(A_2) \]
where
• $\phi_{H}(x_0)$ denotes the capped orbit of $\phi_{H}(x_0)$ whose capping is chosen arbitrarily.
• $\phi_{H}(x_0)$ denotes the capped orbit of $\phi_{H}(x_0)$ whose capping is the same as the capping of $\phi_{H}(x_0)$ chosen above.
• $A_2$ denotes the sphere to which corrects the capping of the RHS so that it will meet the capping on the LHS.
Thus, by employing Proposition 22 (3) for $\int H'_t(\phi_{H}(x_0))dt$ and $\int H'_t(\phi_{H}(x_0))dt$, we obtain,
\[ A_{(H'\oplus H')#}(\phi_{H}(x_0),w_{x_0,x_0}) = (\omega \oplus \omega)(A_2). \]

• Step 4: The aim of this step is to complete the proof.
By Step 2 and 3, we have
\[ c((H'\oplus H')#G#F,a \otimes b) = F(x,y) + (\omega \oplus \omega)(A_2) + (\omega \oplus \omega)(A_1) \]
\[ = F(x,y) + l \cdot \lambda_0 \]
for some integer $l \in \mathbb{Z}$ such that $(\omega \oplus \omega)(A_1 + A_2) = l \cdot \lambda_0$ and
\[ c((H'\oplus H',a \otimes b) = \gamma_{a,b}(H') = \gamma_{a,b}(H \circ \psi) = \gamma_{a,b}(H) \]
where the last equality uses Proposition 23 (7).
By Step 1, we conclude that
\[ |\gamma_{a,b}(H) - l \cdot \lambda_0| \leq \frac{5}{2} \varepsilon. \]
Hence we complete the proof.

3.3. Proof of Theorem 2. The aim of this section is to prove Theorem 2. We prove the following a priori more general result.

Theorem 37. Let $(M^{2n},\omega)$ be a monotone symplectic manifold with a minimal Chern number $N_{M} > n$. Assume that there exist $\psi \in \pi_1(\text{Ham}(M,\omega))$ and a section class $\sigma$ of the Hamiltonian fibration $M_{\psi} \to S^2$, such that its Seidel element $S_{\psi,\sigma} \in QH_{*}(M;\mathbb{K})$ satisfies the following:
• $(S_{\psi,\sigma})^{k} = a_1 \cdot [pt]$ for some $a_1 \in \mathbb{K}\{0\}$ and $k \in \mathbb{N}$ where $[pt]$ denotes the point class in $H_0(M;\mathbb{K})$.
• $(S_{\psi,\sigma})^{k} = a_2 \cdot [M] \cdot s^{-l'}$ for some $a_2 \in \mathbb{K}\{0\}$ and $k',l' \in \mathbb{N}$ where $[M]$ denotes the fundamental class and $s$ denotes the generator of the Novikov ring of $(M,\omega)$.

Then the spectral norm satisfies the following.
(1) For any $\phi \in \text{Ham}(M,\omega)$,
\[ \gamma(\phi) \leq \frac{n}{N_{M}} \cdot \lambda_0. \]
(2) The spectral norm is $C^0$-continuous i.e.

$$\gamma : (\text{Ham}(M, \omega), d_{C^0}) \to \mathbb{R}$$

is continuous. Moreover, $\gamma$ extends continuously to $\text{Ham}(M, \omega)$.

**Remark 38.**

(1) Theorem 37 (1) is essentially contained in Proposition 15 in [KS18] where Kislev-Shelukhin considers Lagrangian spectral invariants instead of Hamiltonian ones.

(2) So far, $(\mathbb{C}P^n, \omega_{FS})$ seems to be the only example that satisfies the assumptions in Theorem 37.

**Proof.** (of Theorem 37)

Let $\phi \in \text{Ham}(M, \omega)$ and take any Hamiltonian $H$ such that $\phi_H = \phi$. Let $\psi \in \pi_1(\text{Ham}(M, \omega))$ and $\sigma$ be as in the statement. Denote

$$a := S_{\psi, \sigma} \in QH_*(M; \mathbb{K}), \quad a^k := a \ast a \ast \cdots \ast a, \text{ } k \text{-times.}$$

By looking at the degree, we have

- $\deg(a^k) = \deg([pt]) = 0$,
- $\deg(a^{sk}) = \deg([M] \cdot s^{-l}) = 2n - 2Nl'$,
- For any $m \in \mathbb{N}$, $\deg(a^{sm}) = m \cdot \deg(a) - (m - 1) \cdot 2n$.

These equations will give us the following:

$$\frac{k'}{k} = \frac{Nl'}{n} \quad \text{(6)}$$

and our assumption $N > n$ implies $k' > k$. As $N_M > n$ and $\mathbb{K}$ is a field, the formula in [EP03] Section 2.7 gives us

$$c(H, [M]) = -c(H, [pt]),$$

and by Proposition 23 we get the following.

- $\gamma(H) = c(H, [M]) - c(H, [pt]) = c(H, [M]) - c(H, a^k)$,
- $\gamma(\psi^*H) = c(H, S_{\psi, \sigma} \ast [M]) - c(H, S_{\psi, \sigma} \ast a^k)) = c(H, a) - c(H, a^{*(k+1)})$.
- $\gamma((\psi^2)^*H) = c(H, a^2) - c(H, a^{*(k+2)})$.
- $\gamma((\psi^k H) = c(H, a^{*(k-k)}) - c(H, a^{k'})$
  $$= c(H, a^{*(k-k)}) - c(H, [M]) + l' \lambda_0.$$
\[
\gamma((\psi^{k'-k+1})^*H) = c(H, a^{(k'-k+1)}) - c(H, a) + l'\lambda_0.
\]

\[\vdots\]

\[
\gamma((\psi^{k'-1})^*H) = c(H, a^{(k'-1)}) - c(H, a^{(k-1)}) + l'\lambda_0.
\]

We used that for \(j \in \mathbb{Z}\),
\[
c(H, a^{*j+k'}) = c(H, a^j) - l'\lambda_0.
\]

Adding up these \(k'\)-equations will give us the following.
\[
\sum_{0 < j < k' - 1} \gamma((\psi^j)^*H) = kl' \cdot \lambda_0.
\]

As \(\gamma(\phi) \leq \gamma((\psi^j)^*H)\) for all \(0 \leq j \leq k' - 1\),
\[
k' \cdot \gamma(\phi) \leq kl' \cdot \lambda_0.
\]

By equation 6 we conclude
\[
\gamma(\phi) \leq \frac{kl'}{k'} \cdot \lambda_0 = \frac{n}{N} \cdot \lambda_0.
\]

The continuity of \(\gamma\) is a direct consequence of Corollary 32. \(\square\)

Theorem 2 is a direct consequence of Theorem 37.

\textbf{Proof.} (of Theorem 2)

We explain briefly that \(\mathbb{C}P^n\) meets the assumptions in Theorem 37. Consider a loop of Hamiltonian diffeomorphism of \(\mathbb{C}P^n\) defined by
\[
\psi^t ([z_0 : z_1 : \cdots : z_{n-1} : z_n]) := [z_0 : e^{2\pi it} z_1 : e^{2\pi it} z_2 : \cdots : e^{2\pi it} z_{n-1} : e^{2\pi it} z_n].
\]

It is known that there exists a section class \(\sigma\) such that \(S_{\psi,\sigma} = [\mathbb{C}P^{n-1}]\) where \([\mathbb{C}P^{n-1}]\) denotes the generator of \(H_{2n-2}(\mathbb{C}P^n; \mathbb{R})\). See Example 9.6.1 and Proposition 9.6.4 in [MS04]. This shows that \(\mathbb{C}P^n\) satisfies the assumptions in Theorem 37. \(\square\)

\section{Proofs of applications}

\textbf{4.1. The displaced disks problem.} We prove Theorem 15. We use the following energy-capacity inequality proven by Usher in [Ush10].

\textbf{Proposition 39.} ([Ush10])

Let \(B := B(r)\) be an open ball in \((\mathbb{R}^{2n}, \omega_{std})\). If \(B(r)\) is symplectically embedded to \((M, \omega)\)
\[
f : B(r) \hookrightarrow (M, \omega)
\]
and \(\phi(f(B)) \cap f(B) = \emptyset\) for \(\phi \in \text{Ham}(M, \omega)\), then
\[
\pi r^2 \leq \gamma(\phi).
\]
Notice that for \((M, \omega)\) for which the spectral norm is \(C^0\)-continuous, Proposition 39 holds for Hamiltonian homeomorphisms as well.

**Proof.** (of Theorem 15) By Theorem 6, we can apply Proposition 39 for Hamiltonian homeomorphisms. Let \(r > 0\) and take \(\delta > 0\) so that if \(\phi \in \text{Ham}(M, \omega)\), \(\gamma(\phi) \geq \pi r^2\), then \(d_{C^0}(\text{id}, \phi) > \delta\). Now, we will prove that if \(\phi \in \text{Ham}(M, \omega)\) displaces an embedded ball of radius \(r\), then \(d_{C^0}(\text{id}, \phi) > \delta\). By Proposition 39 we have \(\gamma(\phi) \geq \pi r^2\) and from our choice of \(\delta\), this implies \(d_{C^0}(\text{id}, \phi) > \delta\). □

### 4.2. The \(C^0\)-Arnold conjecture

We start by looking at properties of \(\sigma_{a,a^*b}\) defined earlier in Section 1.6.

**Proposition 40.** Let \((M, \omega)\) be a symplectic manifold and \(a, b \in H_*(M; K)\setminus\{0\}\). For Hamiltonians \(H, G\), we have the following triangle inequality:

\[
|\sigma_{a,a^*b}(H) - \sigma_{a,a^*b}(G)| \leq \gamma(H \# G).
\]

**Proof.**

\[
\sigma_{a,a^*b}(H) - \sigma_{a,a^*b}(G) = c(H, a) - c(H, a * b) - (c(G, a) - c(G, a * b)) 
\leq c(G \# H, [M]) + c(\overline{H} \# G, [M]) = \gamma(\overline{H} \# G).
\]

By changing the role of \(H\) and \(G\), we get \(\sigma_{a,a^*b}(G) - \sigma_{a,a^*b}(H) \leq \gamma(\overline{H} \# G)\) too. This completes the proof. □

Proposition 40 allows us to define the following: Let \((M, \omega)\) be a negative monotone symplectic manifold and \(a, b \in H_*(M; K)\).

\[
\sigma_{a,a^*b} : \text{Ham}(M, \omega) \rightarrow \mathbb{R}
\]

\[
\sigma_{a,a^*b}(\phi) := \sigma_{a,a^*b}(H)
\]

for any \(H\) such that \(\phi_H = \phi\). Note that the well-definedness is due to Theorem 6. Similarly, we define the following for \(\mathbb{C}P^n\): Let \(h := [\mathbb{C}P^{n-1}]\) and \(l_1, l_2 \in \mathbb{N}, l_1 < l_2\).

\[
\sigma_{h_1, h_2} : \text{Ham}(\mathbb{C}P^n, \omega) \rightarrow \mathbb{R}
\]

\[
\sigma_{h_1, h_2}(\phi) := \inf_{\phi_H = \phi} \sigma_{h_1, h_2}(H).
\]

**Corollary 41.** Let \((M, \omega)\) be either a negative monotone symplectic manifold or \((\mathbb{C}P^n, \omega_{FS})\). For \(a, b \in H_*(M; K)\), we have the following triangle inequality: For \(\phi, \psi \in \text{Ham}(M, \omega)\),

\[
|\sigma_{a,a^*b}(\phi) - \sigma_{a,a^*b}(\psi)| \leq \gamma(\phi^{-1}\psi).
\]
Proof. We only explain the case of \((\mathbb{C}P^n, \omega_{FS})\) since the other is simpler. By Proposition 40,

\[ \sigma_{h^1, h^2}(H \# G) \leq \sigma_{h^1, h^2}(H) + \gamma(G). \]

Take an infimum on both sides as in the definition.

\[ \sigma_{h^1, h^2}(\phi \psi) \leq \inf_{\phi_H = \phi, \phi_G = \psi} \sigma_{h^1, h^2}(H \# G) \leq \sigma_{h^1, h^2}(\phi) + \gamma(\psi). \]

Since \(\sigma_{h^1, h^2}\) are finite,

\[ \sigma_{h^1, h^2}(\phi \psi) - \sigma_{h^1, h^2}(\phi) \leq \gamma(\psi). \]

This implies the triangle inequality

\[ |\sigma_{h^1, h^2}(\phi) - \sigma_{h^1, h^2}(\psi)| \leq \gamma(\phi^{-1}\psi) \]

where \(\phi, \psi \in \text{Ham}(\mathbb{C}P^n, \omega_{FS}). \)

This corollary and the \(C^0\)-continuity of \(\gamma\) implies the \(C^0\)-continuity of \(\sigma_{a, ab}\). This allows us to define \(\sigma_{a, a^* b}\) for Hamiltonian homeomorphisms i.e. for a Hamiltonian homeomorphism \(\phi\), define \(\sigma_{a, a^* b}(\phi) := \lim_{n \to \infty} \sigma_{a, a^* b}(\phi_n)\) where \(\phi_n \in \text{Ham}(M, \omega), \phi_n \xrightarrow{C^0} \phi. \)

We are now ready to prove Theorem 13.

Proof. (of Theorem 13)

Since the negative monotone case is simpler than the case of \((\mathbb{C}P^n, \omega_{FS})\), we only prove the latter. We assume that for \(\phi \in \text{Ham}(\mathbb{C}P^n, \omega_{FS})\) and \(l_1 < l_2\), we have

\[ \sigma_{h^1, h^2}(\phi) = 0. \]

It is enough to prove that an arbitrary open neighborhood \(U\) of \(\text{Fix}(\phi)\) is homologically non-trivial. Let \(f : M \to \mathbb{R}\) be a sufficiently \(C^2\)-small smooth function such that \(f < 0\) on \(M \setminus U, f|_U = 0\) and \(c_{LS}(f, \cdot) = c(f, \cdot). \) (See Proposition 23 (5) for the definition of \(c_{LS}. \))

First of all, take a sequence \(\phi_j \in \text{Ham}(M, \omega), j \in \mathbb{N}\) such that

\[ d_{C^0}(\phi, \phi_j) \leq 1/j. \]

The \(C^0\)-continuity of \(\gamma\) allows us to take a subsequence \(\{f_k\}_{k \in \mathbb{N}}\) so that for each \(k, \)

\[ \gamma(\phi^{-1}\phi_j) < 1/k. \]

Next, for each \(k\), take a Hamiltonian \(H_k\) which generates \(\phi_j\) and

\[ \sigma_{h^1, h^2}(H_k) \leq \sigma_{h^1, h^2}(\phi_j) + 1/k. \]

We borrow the following claim proved in [BHS18a].

Claim 42. (Claim 5.3 in [BHS18b]) Assume \(\phi_{H_k} \xrightarrow{C^0} \phi. \) For any \(a \in H_\ast(M; \mathbb{K}) \setminus \{0\}, \)

there exists \(0 < \varepsilon_0 < 1\) and an integer \(k_0\) such that for any \(k \geq k_0, \) we have

\[ c(H_k \# \varepsilon_0 f, a) = c(H_k, a). \]
From this Claim, there exist $\varepsilon_0 > 0$ and $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, then
\[ c(H_k, h^{l_2}) = c(H_k, h^{l_2}) = c(H_k, h^{l_1}) + c(\varepsilon_0 f, h^{l_2-l_1}) \]
for all $a \in H_4(\mathbb{C}P^n; \mathbb{K})$. For $k \geq k_0$,
\[ c(H_k, h^{l_2}) = c(H_k, h^{l_2}) \leq c(\varepsilon_0 f, h^{l_2-l_1}) \]
and thus,
\[ -\sigma_{h_1, h_2}(H_k) \leq c(\varepsilon_0 f, h^{l_2-l_1}) \leq c(f, h^{l_2-l_1}). \]
By our choices of $\phi_{j_k}$ and $H_k$, we have the following.
\[ \sigma_{h_1, h_2}(H_k) \leq \sigma_{h_1, h_2}(\phi_{j_k}) + 1/k \leq \sigma_{h_1, h_2}(\phi) + \gamma(\phi^{-1} \phi_{j_k}) + 1/k \]
\[ \leq \sigma_{h_1, h_2}(\phi) + 2/k = 2/k. \]
Thus,
\[ 2/k \leq -\sigma_{h_1, h_2}(H_k) \leq c(f, h^{l_2-l_1}). \]
By taking a limit $k \to +\infty$, we obtain
\[ 0 \leq c(f, h^{l_2-l_1}). \]
Thus,
\[ 0 \leq c(f, h^{l_2-l_1}) \leq c(f, [M]) \leq 0. \]
The last inequality follows from $f \leq 0$. Since $f$ was taken to satisfy $c_{LS}(f, \cdot) = c(f, \cdot)$, we have
\[ c_{LS}(f, h^{l_2-l_1}) = c_{LS}(f, [M]) = 0. \]
This implies that $\overline{U}$ is homologically non-trivial. \hfill \Box

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