I. INTRODUCTION

Topologically ordered phases have been long studied not just because of their exotic behaviours [1] that are beyond Landau’s paradigm, but also for their potential application as error correcting codes [2, 3]. One of the most salient features of topological phases in 2D is the existence of anyons: quasiparticle excitations that follow neither bosonic nor fermionic statistics [4] and recently, direct evidence of their existence has been found [5].

A paradigmatic example of topological order is the fractional quantum Hall effect [6, 7] whose anyons host just a fraction of the electron’s charge, i.e. they fractionalize the charge conservation symmetry. Spin liquids form another prominent example where excitations often carry fractional quantum numbers [8]. The quantum phases of matter that possess such a non-trivial interplay between symmetries and topological order have been dubbed symmetry enriched topological (SET) phases.

Bosonic SET phases are well understood: they have been classified [9–13], exactly solvable Hamiltonians for each phase have been constructed [14–17], their ground states and anyons have been given in terms of tensor networks [18–20] and different methods to detect SET phases have been proposed [21–23], to name a few. Notably, most of the aforementioned works focused on renormalization-group (RG) fixed points.

In this manuscript, we study the effect of symmetry-preserving perturbations of a simple SET Hamiltonian (see [24–27] for similar perturbed SET studies). Our starting SET Hamiltonian, proposed in [28, 29], is a decorated toric code model whose ground states are equal weight superpositions of loops decorated with 1D cluster states. This Hamiltonian has an on-site $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry along with time-reversal and lattice inversion symmetry, all of which are fractionalized by the charge excitation in some way. We study this model in the presence of external magnetic fields, a nearest-neighbour Ising interaction, and also a direct interpolation to the usual non-decorated toric code. We use variational infinite projected entangled pair states (PEPS) [30–32] to obtain the ground states and compute observables in the thermodynamic limit, although we stress that the usage of PEPS is not necessary, and other numerical methods could be used here as well.

Our goals for this study are two-fold. First, we extend the validity of the string order parameters that were proposed in [23] to detect symmetry fractionalization in RG fixed-point PEPS. We define similar order parameters that do not rely on a PEPS representation of ground states, and show that they correctly characterize the symmetry fractionalization of the perturbed SET Hamiltonians in all cases, up until the phase transition point. This shows that they are a reliable tool to detect symmetry fractionalization.

Secondly, we study the phenomenon wherein the condensation of an anyon that fractionalizes the symmetry must result in that symmetry being spontaneously broken. This was proven in Ref. [33] using the framework of G-graded tensor categories and it has been identified also in other phase diagrams [27, 34]. We also notice that this phenomenon has been studied previously in high energy physics, in particular, it has been proposed to explain the electroweak symmetry breaking [35]. We provide an alternative proof of this fact in the framework of PEPS. The phenomenon leads to an interesting modification of...
Figure 1. Schematic phase diagram of the toric code (left) and symmetry-enriched toric code (right) in parallel X and Z magnetic fields. Solid (dashed) lines indicate second (first) order phase transitions. The arrows indicate a path connecting the two condensed phases. In the symmetry-enriched case, this path must encounter a phase transition since the two phases have distinct symmetry breaking patterns.

The well-known 2D phase diagram of the toric code in parallel magnetic fields [36–40], see Fig. 1. We also find that the symmetry is spontaneously broken across phase transitions that cannot be described as condensation transitions, such as that driven by a transverse magnetic field [41].

The structure of the manuscript is as follows. In section II we first introduce the target SET Hamiltonian and the string order parameters that we use to capture symmetry fractionalization. Then, we also explain the spontaneous symmetry breaking (SSB) from anyon condensation. In section III we show the numerical results of the Hamiltonian interpolation and conclusions.

II. TC HAMILTONIAN ENRICHED WITH CLUSTER STATE LOOPS

In this section, we first define a Hamiltonian which realizes an SET phase and will form the basis of our analysis throughout the paper. Second, we derive the order parameter we use to characterize the SET phases obtained by perturbing the Hamiltonian. Finally, we discuss with the interesting phenomenon of SSB induced by anyon condensation.

A. SET Hamiltonian

Our SET model can be described as a toric code, whose ground states can be viewed as equal-weight superpositions over closed loop configurations, where these loops are further decorated with 1D symmetry protected topological (SPT) orders, namely cluster states [42]. Such a model was first proposed in [28] and also studied in Refs. [13, 29]. This decoration enriches the toric code with a global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, time-reversal symmetry (TRS), and inversion symmetry which are fractionalized by the charge excitations.

To define the model precisely, let us begin with the toric code with spins on the edges of a honeycomb lattice. The Hamiltonian is

$$H_{TC} = - \sum_{v \in V} A_v - \sum_{f \in F} B_f,$$

where $V$ ($F$) denotes the set of all vertices (faces) in the lattice and the terms in the Hamiltonian are

$$A_v = \prod_{e \ni v} Z_e \quad \text{and} \quad B_f = \prod_{e \in f} X_e,$$

where $X$ and $Z$ are the spin-1/2 Pauli operators, $e \ni v$ denotes all edges terminating on $v$ and $e \in f$ denotes all edges surrounding $f$. Let $C$ be a subset of edges that form closed loops on the lattice. We denote by $|C\rangle$ the state where all edges in $C$ are in the state $|1\rangle$ and the rest are in $|0\rangle$. Then, the ground states of $H_{TC}$ can be written as an equal-weight superposition of loop configurations:

$$|TC\rangle = \frac{1}{N} \sum_C |C\rangle,$$

for some normalization factor $N$. Here, we are assuming trivial topology such that $|TC\rangle$ is the unique ground state of $H_{TC}$.

Now, we introduce new spins on the vertices of the lattice and initialize them in the state $|+\rangle^{|V|} = \bigotimes_{v \in V} \frac{1}{\sqrt{2}}(|0\rangle_v + |1\rangle_v)$. We couple these to the edge spins using a unitary circuit $U_{CCZ}$ which is constructed using CCZ operators acting on every triplet consisting of an edge $e$ and its two vertices $v_e^+ - v_e^-$, where $CCZ = |0\rangle_v^+ \otimes |0\rangle_{v_e^-} \otimes |1\rangle_v^+ \otimes |Z_{v_e^-} v_e^+ \rangle$ and $CZ_{v_e^+ v_e^-} = |0\rangle_v^+ \otimes |I_{v_e^+} \otimes |I_{v_e^-} \rangle + |1\rangle_{v_e^-} \otimes |Z_{v_e^+} \rangle$. We denote the resulting state as

$$|SET\rangle = U_{CCZ} \left(|TC\rangle \otimes |+\rangle^{|V|}\right).$$

This circuit acts as $CZ$ along the vertices of a loop and it acts trivially away from them. Acting on initial $|+\rangle$ states with $CZ$’s between nearest neighbours creates the 1D cluster state. Thus, the vertices along loops form cluster states, which are an example of 1D SPT orders [43].

To obtain the Hamiltonian for which $|SET\rangle$ is the ground state, we can simply conjugate the initial uncoupled Hamiltonian

$$\tilde{H}_{TC} = H_{TC} - \sum_{v \in V} X_v,$$

by $U_{CCZ}$ to obtain

$$H_{SET} = - \sum_{v \in V} A_v - \sum_{f \in F} B_f - \sum_{v \in V} C_v \frac{1 + A_v}{2}$$
This gives a four-fold degenerate subspace of states associated to each $\Gamma$ spanned by the states $|\alpha, \beta\rangle := S_{\Gamma}(\alpha, \beta)|\text{SET}\rangle$. Within this subspace, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry acts projectively on each charge as

$$X_A|\alpha, \beta\rangle = |\alpha \oplus 1, \beta \oplus 1\rangle,$$
$$X_B|\alpha, \beta\rangle = (-1)^\alpha(-1)^\beta|\alpha, \beta\rangle,$$

where we have assumed that $\alpha$ and $\beta$ are both odd vertices, a similar result holds in the general case (see [28]). If we let $V_{(1,0)}(V_{(0,1)})$ denote the local action of $X_A$ ($X_B$) on a single charge, we have $V_{(1,0)} = X$, $V_{(0,1)} = Z$ and $V_{(1,1)} = XZ$, where we have labelled $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(a, b) : a, b = 0, 1\}$. Since $X$ and $Z$ anticommute, the charge carries a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which demonstrates the fractionalization. Concretely, if we write $V_qV_k = \omega(q, k)V_{qk}$, where $q, k \in \mathbb{Z}_2 \times \mathbb{Z}_2$, the symmetry fractionalization (SF) pattern is given by

$$\omega(q, k) = \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 \end{pmatrix}_{q, k}. \tag{10}$$

The local action of $KX_AX_B$ is $T = XZ$ so that $T^2 = -1$. Then, the charge also fractionalizes time-reversal symmetry. Finally, since $X_A$ and $X_B$ anticommute near a charge, and inversion swaps $A \leftrightarrow B$, it follows that inversion anticommutes with $X_AX_B$ near a charge. Conversely, it is easy to see that the SF pattern of the charge for $\tilde{H}_{TC}$ of (5) is $\omega(q, k) = +1$ for all $q, k \in \mathbb{Z}_2 \times \mathbb{Z}_2$, so that it is trivial. In Appendix A we construct a PEPS representation of $|\text{SET}\rangle$ which provides another viewpoint on the SF pattern.

We note that the flux excitations are created by a string of $Z$ operators corresponding to a path on the dual lattice, and they have no symmetry fractionalization. The dyon, which is a composite of flux and charge, fractionalizes in the same way as the charge.

Since we have used decoration by 1D SPT phases to construct our 2D SET model, some comments on their classifications are in order. For a global on-site symmetry group, $Q$, 1D SPT phases are classified by the group $H^2(Q, U(1))$. When $Q = \mathbb{Z}_2 \times \mathbb{Z}_2$, $H^2(Q, U(1)) = \mathbb{Z}_2$ which means that there are only two phases: the trivial one and the non-trivial SPT phase (the one of the cluster state, i.e. the Haldane phase). In the case of SET phases, the classification of SF patterns is given by $H^2(Q, G)$, where $G$ is an abelian group that depends on the topological order. In the present case, a toric code enriched with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry that fractionalizes the charge and not the flux (without anyon permutation) could host four inequivalent SF patterns.

By simple counting, this implies that not all SET phases can be constructed by decorating topological orders with 1D SPT phases as we have done here. This fact affects how to detect SET phases since it implies that 1D

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**Figure 2.** Hamiltonian terms in $H_{\text{SET}}$ (Eq. (6)) and an example of a string operator $S_{\Gamma}$ that creates pairs of charge excitations at the endpoints (Eq. 7). A wavy line connecting two sites marked with circles represents a CZ operation on these sites.

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where $A_v$ is unchanged from before, and $\tilde{B}_f$ is a modified version of $B_f$ decorated by additional CZ operators, see Fig. 2. $C_v = U_{CCZ}X_vU_{CCZ}^\dagger$ is also defined pictorially in Fig. 2. We have additionally modified $C_v$ by adding a projector $1 + A_v^\dagger$ onto the closed-loop subspace. This only affects the energy of some excitations, and serves the purpose of making $H_{\text{SET}}$ commute with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group generated by $X_A$ and $X_B$, which are defined to flip every spin on all vertices of the $A$ and $B$ sublattice, respectively. Since the Hamiltonian is real, it also has time-reversal symmetry generated by $KX_AX_B$, where $K$ denotes complex conjugation. Moreover, the Hamiltonian terms are invariant by bond-centered inversion of the lattice.

**B. Symmetry fractionalization in $H_{\text{SET}}$.**

$H_{\text{SET}}$ represents a non-trivial SET order in the presence of either the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry or the time-reversal symmetry. To see why, consider the following string operator,

$$S_{\Gamma} = \prod_{e \in \Gamma} X_e CZ_{v_e^+v_e^-} \tag{7}$$

where $\Gamma$ is a path of edges and $v_e^+v_e^-$ are the two vertices attached to $e$, see Fig. 2. This operator creates charge excitations at the endpoint vertices $v_i$ and $v_j$ of $\Gamma$ corresponding to violations of $A_v$. Because $A_v = -1$ at these vertices, the Hamiltonian term $C_v$ is disabled by the projector $\frac{1 + A_v^\dagger}{2}$. Therefore, we can dress the endpoints of $S_{\Gamma}$ with $\bar{Z}$ operators without changing the energy of the excitation,

$$S_{\Gamma}(\alpha, \beta) = Z_{v_e^+}Z_{v_e^-}S_{\Gamma}. \tag{8}$$

This gives a four-fold degenerate subspace of states associated to each $\Gamma$ spanned by the states $|\alpha, \beta\rangle := S_{\Gamma}(\alpha, \beta)|\text{SET}\rangle$. Within this subspace, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry acts projectively on each charge as

$$X_A|\alpha, \beta\rangle = |\alpha \oplus 1, \beta \oplus 1\rangle,$$
$$X_B|\alpha, \beta\rangle = (-1)^\alpha(-1)^\beta|\alpha, \beta\rangle,$$
SWAP

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ize the SOPs to length

symmetry.

applied to any anyon that fractionalizes a global on-site

motivated by the SF pattern of

H

mark that the use of the charge in this explanation is

metry group to detect different SF patterns [23]. We re-

SWAP

in the case of 1D SOPs [45, 46]. In Appendix C, we

SWAP

The SOP evaluates the overlap of two charges placed at

anyon affected by the braiding with a non-affected one.

To begin, let us summarize the SOPs of Ref. [23]. Con-

cretely, the goal of the order parameter is to capture the

projective action on the charge. Since in our case

q ∈ Z_2 × Z_2, i.e. the group is order two, a value

ω(q, q) = 1 implies a projective action on the charge.

Here, we modify the SOPs in order to extend their range

to a fixed-point tensor network states.

To test this by applying symmetry-preserving deformations

to a certain fixed-point tensor which drive across a phase

transition to a topologically trivial phase. We find that

as the length ℓ is increased, the value of the SOP conver-

vexes to the value corresponding to the fixed-point ten-

sor. In contrast, when the deformation explicitly breaks

the symmetry, the order parameters show no clear signa-

ture of convergence.

One limitation of the SOPs described above is that

they rely on a tensor network representation of the

ground state in order to create an excited state with

charges. Even when such a representation is available, the

order parameters of Ref. [23] require imposing an

additional topological structure on the tensor network,

which makes variational optimization of the tensors more

costly. As such, it is desirable to modify the SOPs in such

a way that they are not reliant on the structure provided

by topological tensor networks. For this purpose, we now

introduce a new set of SOPs that are designed to detect

the SF pattern of H_{SET}, and have no reliance on ten-

sor networks. To insert charges into the ground state,

we use the string operator S_T (Eq. 7). When H_{SET}

is perturbed, the state created by acting with S_T on

the ground state will no longer be an eigenstate in general,

but we nonetheless expect it to have finite overlap with

the corresponding eigenstate having charges at the end-

points of Γ. Since we will be interested in ratios of the

SOPs, we do not expect this to be an issue.

The SOPs we use are given in terms of the expectation

value of some string operators Λ[q], that depend on the

elements q ∈ Z_2 × Z_2 of the on-site symmetry group.

To define the string operator let us consider a line of 5

vertices with their 4 inside edges on the hexagonal lattice

define

\[ Λ^{[a,b]} = Sγ_1...γ_6[XX X X X] × \]

\[ \text{SWAP}_{e_1,e_3}\text{SWAP}_{e_2,e_4}\text{SWAP}_{1,3}\text{SWAP}_{2,4} S_{Γ=5,6} \]

\[ = (\prod_{i=1}^{4} CZ_{e_i+1,e_i}) (XX X X X) × \]

\[ \text{SWAP}_{e_1,e_3}\text{SWAP}_{e_2,e_4}\text{SWAP}_{1,3}\text{SWAP}_{2,4}. \]

This latter equality can be written pictorially as:

\[ \gamma_{1,2,3,4,5}, \]

where circles labelled a and b denote symmetry operators

X^a and X^b. Using Λ^{[a,b]} we define the following SOPs:

\[ O^{[a,b]} = \frac{⟨Λ^{[a,b]}⟩}{⟨Λ^{[0,0]}⟩}, \]

C. Order parameters for SET phases

To measure the effect of a perturbation on the SET

phase, we need to use an order parameter which detects

the SF pattern of the anyons. Such an order parameter

was proposed in Ref. [23]. Therein, the authors proposed

a set of string order parameters (SOPs) O^{[q]} index by

elements q of the on-site symmetry group Q (= Z_2 × Z_2

here), that, for topological PEPS with zero correlation

length, reveal the lower diagonal \{ω(q,q): 0 ≠ q ∈ Q\} of

the SF pattern, which is sufficient to completely charac-

terize the phase. The value of ω(q, q) characterizes the

action of q applied twice on the charge. Since in our

case q ∈ Z_2 × Z_2, i.e. the group is order two, a value

ω(q, q) ≠ 1 implies a projective action on the charge.

Here, we modify the SOPs in order to extend their range

of applicability beyond fixed-point tensor network states.

In general, the sites permuted by the SWAP can be

changed depending on the topological order and the sym-

metry group to detect different SF patterns [23]. We re-

mark that the use of the charge in this explanation is

motivated by the SF pattern of H_{SET} but it could be

applied to any anyon that fractionalizes a global on-site

symmetry.

These SOPs were only shown to work for fixed-point

tensor network states with zero correlation length. For

states with non-zero correlation length, we can general-

ize the SOPs to length ℓ, with the expectation that they

will better capture the SF pattern with increasing ℓ, as

in the case of 1D SOPs [45, 46]. In Appendix C, we

test this by applying symmetry-preserving deformations

to a certain fixed-point tensor which drive across a phase

transition to a topologically trivial phase. We find that

as the length ℓ is increased, the value of the SOP con-

verges to the value corresponding to the fixed-point ten-

sor. In contrast, when the deformation explicitly breaks

the symmetry, the order parameters show no clear signa-

ture of convergence.

One limitation of the SOPs described above is that

they rely on a tensor network representation of the

ground state in order to create an excited state with

charges. Even when such a representation is available, the

order parameters of Ref. [23] require imposing an

additional topological structure on the tensor network,

which makes variational optimization of the tensors more

costly. As such, it is desirable to modify the SOPs in such

a way that they are not reliant on the structure provided

by topological tensor networks. For this purpose, we now

introduce a new set of SOPs that are designed to detect

the SF pattern of H_{SET}, and have no reliance on ten-

sor networks. To insert charges into the ground state,

we use the string operator S_T (Eq. 7). When H_{SET}

is perturbed, the state created by acting with S_T on

the ground state will no longer be an eigenstate in general,

but we nonetheless expect it to have finite overlap with

the corresponding eigenstate having charges at the end-

points of Γ. Since we will be interested in ratios of the

SOPs, we do not expect this to be an issue.

The SOPs we use are given in terms of the expectation

value of some string operators Λ[q], that depend on the

elements q ∈ Z_2 × Z_2 of the on-site symmetry group.

To define the string operator let us consider a line of 5

vertices with their 4 inside edges on the hexagonal lattice

define

\[ Λ^{[a,b]} = Sγ_1...γ_6[XX X X X] × \]

\[ \text{SWAP}_{e_1,e_3}\text{SWAP}_{e_2,e_4}\text{SWAP}_{1,3}\text{SWAP}_{2,4} S_{Γ=5,6} \]

\[ = (\prod_{i=1}^{4} CZ_{e_i+1,e_i}) (XX X X X) × \]

\[ \text{SWAP}_{e_1,e_3}\text{SWAP}_{e_2,e_4}\text{SWAP}_{1,3}\text{SWAP}_{2,4}. \]

This latter equality can be written pictorially as:

\[ \gamma_{1,2,3,4,5}, \]

where circles labelled a and b denote symmetry operators

X^a and X^b. Using Λ^{[a,b]} we define the following SOPs:
where the expectation value $\langle \cdot \rangle$ is taken with respect to the ground state of the perturbed Hamiltonian.

For systems with a non-zero correlation length, we can define a family of SOPs with increasing length $\ell$. The corresponding operator acts on $4\ell + 1$ vertex spins,

$$
\Lambda_{\ell}^{[a,b]} = \prod_{j=1}^{4\ell} CZ_{j,j+1} X_{c_j} \prod_{i=1}^{\ell} \left[ X_{2i-1}^a X_{2i}^b X_{2i+1}^a X_{2i+2}^b \times \text{SWAP}_{2i-1,2i+2} \times \text{SWAP}_{2i,2i+1} \right]
$$

so that the SOP of length $\ell$ is defined as follows:

$$
\mathcal{O}_{\ell}^{[a,b]} = \frac{\langle \Lambda_{\ell}^{[a,b]} \rangle}{\langle \Lambda_{\ell}^{[0,0]} \rangle}.
$$

This SOP is expected to better capture the SF pattern with increasing $\ell$, namely,

$$
\lim_{\ell \to \infty} \mathcal{O}_{\ell}^{[a,b]} = \omega((a,b), (a,b)).
$$

We will focus mainly on $\mathcal{O}_{\ell}^{[1,1]}$ since it is the one that distinguishes the SF patterns of $H_{TC}$ and $H_{SET}$.

We remark that these SOPs are defined for SET phases, that is, topological phases with an unbroken symmetry. Because of this, the meaning of the value of (15) after the phase transition point is not clear if the perturbation drives the model to a non-SET phase, such as one where the symmetry is spontaneously broken or the topological order is trivial. For example, when inversion symmetry is broken, the order parameter can depend on whether the endpoints lie on the $A$ or $B$ sublattice. Furthermore, we will find that the value becomes undefined in trivial phases where the charge excitation is confined since in that case, the expectation value of $S_\ell$ goes to zero (and then $\langle \Lambda_{\ell}^{[0,0]} \rangle$ goes to zero as well).

## D. Spontaneous symmetry breaking from anyon condensation

In this section, we explain one interesting phenomenon that we will observe in the phase diagram given by perturbing $H_{SET}$. That is, we find that the ground state subspace exhibits spontaneous symmetry breaking (SSB) when an anyon that fractionalizes the symmetry (the charge) is condensed. We note that this has been proven in full generality, using the language of $G$-graded tensor categories, to be a necessary outcome of condensing a fractionalized anyon [33]. In appendix B3 we give an alternative proof of this based on PEPS. Our proof aims to reach a broader audience using a simpler mathematical formalism, at the cost of some generality.

To understand the physics behind this phenomenon, let us make the following cartoon picture. The starting point is some initial topological phase, invariant under some global symmetry, which is undergoing an anyon condensation process. We denote by $b$ the anyon that is condensing (we assume there is only one for the sake of simplicity) and by 1 the vacuum of the initial topological phase. The vacuum of the final phase, $\varphi$, is an anyon condensate of $b$ together with 1, which we can write as $\varphi = 1 + b$. If $b$ transforms non-trivially under the symmetry, this affects how the composition $1 + b$, and therefore $\varphi$, behaves under the action of the symmetry.

In the case where $b$ fractionalizes the symmetry and the symmetry is preserved, this would imply that either $\varphi$ transforms projectively, or in an undefined way since it is a superposition of $b$ and 1. However, a vacuum that transforms non-trivially is not a valid theory, so the symmetry has to break spontaneously in order to act trivially on the vacuum.

For the case where $b$ is permuted to another anyon $c$ of the theory, $\varphi$ would be also identified with $1 + c$. But $c$, in general, will be of another nature of $b$ in the anyon condensation process, that is, it can be a confined anyon. Therefore, since by definition no confined anyon can condense, the symmetry has to break spontaneously to consistently act on the vacuum. This leads to the conclusion that the set of condensed anyons must be closed under the permutation action of the symmetry.

## III. PHASE DIAGRAM OF $H_{SET}$

In this section, we first study a Hamiltonian interpolation between $H_{SET}$ and $H_{TC}$ to understand the behaviour of the SOPs of Eq. (15) in the different SET phases. Afterward, we introduce perturbations to $H_{SET}$ that commute with the symmetry (or part of it) and we study the resulting phase diagrams numerically. We focus on the behaviour of the SOPs as well as the possibility of SSB driven by anyon condensation. For the latter, we employ local order parameters $(Z_{v_A})$, $(Z_{v_B})$, and $(X_{v_A} - X_{v_B})$ which anticommute (only) with $X_A$, $X_B$, and inversion, respectively, where $v_{A/B}$ denotes a vertex on the $A/B$ sublattice. We also measure local magnetizations on the edge degrees of freedom to help understand the structure of the perturbed ground states.

The perturbations that entirely commute with the symmetry are magnetic fields $\sigma_z = (X_v, Y_v, Z_v)$ on edges that are meant to break the topological order in different ways. We also apply a ZZ Ising interaction on the vertices of $H_{SET}$ that changes the symmetry fractionalization pattern, since it breaks explicitly the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry down to a $\mathbb{Z}_2$ subgroup generated by $X_A X_B$, but preserves the topological order. In the next subsections, we examine the result of applying each perturbation separately. We also consider the combination of $X$ and $Z$ fields and examine their 2D phase diagram.

We use the PEPS formalism to approximate ground states. We use a five site unit tensor for the PEPS description that consists of 5 (i.e., 3 edge and 2 vertex) spins
on the hexagonal lattice, as depicted in Appendix A. It is important to note that we choose a local tensor that contains vertices $v_A$ and $v_B$ to allow for the spontaneous breaking of inversion symmetry. The variational optimization has been performed on an infinite cylinder for computing the ground states. Moreover, the expectation values of observables (e.g., SOPs) have been calculated on an infinite plane.

A. Hamiltonian interpolation

In this section we study the phase diagram of the following Hamiltonian interpolation:

$$H(\lambda) = \lambda \tilde{H}_{TC} + (1 - \lambda)H_{SET},$$

(17)

such that $H(0) = H_{SET}$ and $H(1) = \tilde{H}_{TC}$. This path therefore interpolates between a SET phase with a non-trivial SF pattern and one with trivial SF. In Fig. 3 we show the numerical results for the evaluation of the order parameters and the different local magnetizations. We see in Fig. 3(a) how the order parameter characterizes correctly the SF patterns of the two ends of the interpolation.

In particular, $O^{[1,1]}_{\ell}$ is $+1$ or $-1$ in the two ends of the interpolation and surprisingly it goes to zero in the middle part, pointing towards an intermediate phase between the two SET phases. Moreover, we can see that for increasing $\ell$, $O^{[1,1]}_{\ell}$ gets sharper and approaches to $-1$ in the whole range of the non-trivial SET phase. That affirms (15) as a reliable order parameter to probe an SET phase. For $\lambda = 1$, in the ground state of $\tilde{H}_{TC}$, it is straightforward to see that $O^{[a,b]}_{\ell} = 1$ for all $a, b$, as we confirm numerically even for the entire trivial SET phase. Therefore, Eq. 16 is perfectly satisfied with $\omega = +1$ for all $a, b$, this indicates the triviality of the SF pattern.

The nature of the intermediate phase can be seen clearer by the behaviour of the magnetizations in Fig. 3(b). This intermediate phase spontaneously breaks all of the symmetries, as indicated by non-zero values of the SSB order parameters. We note that, by nature of our numerical technique (that involves initialization of tensors for the variational optimization with an optimal ground state tensor of a nearby point), we see only one of the SSB “branches” corresponding to one of SSB ground states. For other ground states, $\langle Z_{v_A} \rangle$ is zero while $\langle Z_{v_B} \rangle$ is non-zero. We have also checked that it is possible to construct ground states in all the branches if we randomly initialize the variational optimization at every point in the intermediate phase. We defer a closer examination of the SSB pattern to Sec. III B, since it is the same in both cases.

B. $X_e$ – $Z_e$ fields: transitions to trivial phases

We now present our findings regarding the phase diagram of $H_{SET}$ in the presence of parallel $X$- and $Z$-fields on the edges:

$$H_{SET} = h_X^e \sum_e Z_e - h_Z^e \sum_e X_e.$$  

(18)

While the phase diagram exhibits some similarities to the well-known phase diagram of regular toric code [36–40], it also has some novel features. In analogy to the regular toric code, the $X$- and $Z$-fields’ action drives the condensation of charge and flux excitations. However, as discussed in Sec. IID, the presence of an anyon that fractionalizes the symmetry leads to SSB after condensation, which leads to a different phase diagram.

Fig. 4 shows the 2D phase diagram in terms of the order parameter $O^{[1]}$ and the vertex $Z$-magnetization $\langle Z_{v_A} \rangle$ ($\langle Z_{v_B} \rangle$ is 0 through the phase diagram for the particular symmetry-breaking ground state we obtain). We see that the former is able to detect the entire SET phase in a broad 2D range of the phase diagram (the cross-out region indicates where the charge is confined, such that $O^{[1]}$ becomes ill-defined). While the latter detects the transition to the charge-condensed trivial phase. The non-zero value of $\langle Z_e \rangle$ in the charge-condensed phase indicates that the symmetry is spontaneously broken. This supports the general claim given in section IID that the condensation of a fractionalized excitation (the charge) must be accompanied by SSB. As in the usual
The toric code phase diagram (with no symmetry fractionalization), there is a line of phase transitions between the two trivial condensed phases. In the toric code case, this line ends at a finite point, meaning that the two condensed phases are in fact the same phase. Here due to the non-trivial SF, we expect that the line cannot end at a finite point since the two topologically trivial phases differ in their symmetry-breaking pattern.

In the following, we further study the lines labeled as (I), (II), and (III) in Fig. 4(b) to access the universal features of the three transition lines in the phase diagram.

**Line(I): flux condensation via $Z_e$-field**

Here we study the one-parameter Hamiltonian:

$$H_{SET} - h_e^Z \sum_e Z_e. \quad (19)$$

The $Z_e$-field commutes with the circuit $U_{CCZ}$ that was used to construct $H_{SET}$. Therefore, the same physics as the regular toric code primarily governs the resulting phase transition [36–40]. It can be seen that the $Z_e$-field penalizes configurations with longer loops so that it condenses the flux. As a consequence of the breakdown of the topological order due to flux condensation, we observe an increase in the expectation value of the $Z_e$-field on the edges (Fig. 5). All SSB order parameters are 0 and we find that there is one symmetric ground state, i.e. no symmetry breaking.

From Fig.5(a) we see that $O_{1}^{1,-1} = -1$ for any value of $h_e^Z$, so the SOP works as expected in the SET phase. However, the value after the phase transition is ill-defined since, dual to the condensation of the flux, the phase transition also confines the charge. This effect can be measured by the confinement fraction of the charge [47, 48] which is given by $\langle \Lambda^{[0,0]} \rangle$. As shown in Fig.5(a), $\langle \Lambda^{[0,0]} \rangle$ goes to zero after the phase transition, indicating the charge confinement and invalidating the use of $O^{[a,b]}$ after this point.

**Line(II): charge condensation and SSB via $X_e$-field**

We now examine the following Hamiltonian:

$$H_{SET} - h_e^X \sum_e X_e. \quad (20)$$

Analogous to the case of regular toric code [36–40], this perturbation drives the SET phase into a trivial topological phase by condensing the charge. In Fig. 5(c) we can see that $O_{1}^{[1,-1]}$ gets sharper with increasing $\ell$ and that the value correctly approaches −1 in the whole SET phase. The condensation of the charge is also indicated by the saturation of $\langle X_e \rangle$ in Fig. 5(d).

The condensation of the charge is accompanied by the SSB of the symmetries that fractionalize this excitation. As in Sec. III A, we find that all symmetries are spontaneously broken, as indicated by the non-zero SSB order parameters. To better understand the SSB pattern, we can obtain the other SSB ground states by starting with one ground state and applying all possible symmetry actions generated by $X_{A/B}$ and inversion. Doing this, we obtain a total of four orthogonal ground states, with each ground state being invariant under either $X_A$ or $X_B$, but never both. It is interesting to compare this SSB pattern of the ground states with another model with a similar SSB pattern. We consider two 2D Ising models placed on each sublattice of the hexagonal lattice (i.e. next-nearest neighbour ZZ Ising interactions on the hexagonal lattice), and we denote this model as Ising$_A$ ⊗ Ising$_B$. This model is invariant under the same symmetries on the vertex spins of $H_{SET} - h_e^X \sum_e X_e$ and moreover, it also breaks all of them spontaneously, resulting in 4 degenerate ground states. However, the four ground states have a different SSB pattern as seen by the SSB order parameters in Table I. The unusual SSB pattern of the $H_{SET}$ under a $X_e$ field, comparing to the one of Ising$_A$ ⊗ Ising$_B$.  

![Figure 4](image-url)
could be due to the fact that the SSB is induced by the condensation of an excitation which fractionalizes on-site and inversion symmetries in a joint manner.

By studying the behavior of the local order parameters across the transition, we can see that it is second order, and we can extract an estimate of the corresponding critical exponents, as shown in Fig. 5(d-Inset). We see that \langle Z_B \rangle, which indicates the breaking of \langle X_B \rangle, has a different critical exponent than \langle X_{vA} - X_{vB} \rangle, the order parameter for inversion symmetry breaking.

**Line(III): combination of \langle X_e \rangle and \langle Z_e \rangle fields**

The condensation of the charge and flux drives the SET in two different topologically trivial phases. In the former, the trivial phase manifests SSB while the latter does not. This shows a clear difference from the regular toric code in parallel magnetic fields where both condensations end up in the same trivial phase and there is a finite first order line between them [37, 40]. Here we study the one-parameter Hamiltonian

\[ H_{SET} = (2 - h_e^Z) \sum_i X_i - h_e^Z \sum_i Z_i, \]  

(21)

that describes line(III). Along that line, the SOP shows a jump that is indicative of a first order phase transition (Fig. 5(e)). Furthermore, expectation values of on-site magnetizations on the edges and vertices also exhibit a behavior that is consistent with first order phase transition (Fig. 5(f)).

**C. \langle Z_0 \rangle-field: transition to a SSB toric code**

In this section, we analyze the effect of the nearest-neighbour Ising interaction between vertices, i.e.

\[ H_{SET} - J_{eZ} \sum_{\langle e,v \rangle} Z_e Z_{v^r}. \]

(22)

The \langle Z_0 \rangle interaction on the vertices of the honeycomb lattice does not preserve the whole global symmetry, it explicitly breaks part of the symmetry. It only commutes with TRS, inversion, and a \langle Z_2 \rangle subgroup of the global \langle Z_2 \times Z_2 \rangle (the one generated by acting with X on all vertices). These symmetries still fractionalize the charge, so we expect the symmetry fractionalization to persist as the interaction strength is increased, up to the phase transition.

The physics of the resulting phase transition are easy to describe, since the \langle Z_0 \rangle interaction again commutes with \langle U_{CCZ} \rangle, as in the case of Eq. 19. If we conjugate Eq. 22 by \langle U_{CCZ} \rangle, we are left with \langle H_{TC} \rangle on the edge spins, and a 2D transverse-field Ising model on the vertices. Therefore, the physics of this transition should be as in this Ising model. Indeed, the local order parameter \langle Z_e \rangle indicates the SSB, and the corresponding critical exponent is in agreement with the 3D classical Ising universality class [see inset in Fig. 6(b)]. The large \langle J_{eZ} \rangle limit of Eq. 22 will be equivalent to a toric code on the edges with one of the SSB ground states of the Ising model on the vertices, which is consistent with the small values of \langle X_e \rangle, \langle Y_e \rangle, and \langle Z_e \rangle.
Figure 6. Second order phase transition from the SET phase to a toric code with SSB driven by the Ising field. (a) Values of \( O^{[1,1]}_i \), see Eq.\((15)\), for different block lengths \( \ell \). (b) Magnetization per site along the different field directions on the edges and vertices of the honeycomb lattice. The inset shows the critical exponent \( \beta \) of the transition.

Despite the equivalence of Eq.\((22)\) to the Ising model, the SOPs still show non-trivial behaviour, as seen in Fig.\(6(a)\). We see that, with increasing \( \ell \), \( O^{[1,1]}_i \) approaches \(-1\) in the whole SET phase, indicating that the symmetry fractionalization does indeed remain non-trivial despite partially breaking the on-site symmetry. While the order parameter is not designed to be used when the symmetry is spontaneously broken, it is easy to show that \( O^{[a,b]} = 0 \) for all \([a, b] \neq [0, 0]\) in the limit \( J^{ZZ}_v \to \infty \), and this is consistent with our results.

D. \( Y_e \)-field

We now study the effects of \( Y_e \)-field,

\[
H_{SET} - h_e^{Y} \sum_e Y_e. \tag{23}
\]

In [41] the authors studied the regular toric code in transverse \( Y_e \)-field and found a first order phase transition to a fully polarized phase. Here, with SET order, we also find a first order phase transition to a phase where the edge spins are polarized in the \( Y_e \)-direction, as indicated by the discontinuities in the SOP and the local magnetizations, see Fig.\(7\).

Interestingly, this phase transition is also accompanied by SSB, as indicated by the non-zero SSB order parameters. The SSB pattern is the same as observed in Fig.\(7(a)\). We see that, with increasing \( \ell \), \( O^{[1,1]}_i \) approaches \(-1\) in the whole SET phase, indicating that the symmetry was spontaneously broken, as indicated by the non-zero SSB order.

Figure 7. (a) Behaviour of the \( O^{[1,1]}_i \) for different \( \ell \) under the action of \( Y_e \)-field (b) Different magnetizations for the \( Y_e \)-field perturbation.

Sec. III B. Unlike the phase transitions induced by parallel \( X_e \) or \( Z_e \) fields, this transition cannot be interpreted as a condensation transition, so we cannot use the general arguments of Section II D to understand the presence of SSB. Nevertheless, it remains possible that the charge excitation is condensed (in the sense of condition (II) in Section B 3), so our proof of SSB presented in Appendix B may still apply. We leave a more detailed analysis of the nature of this transition to future work.

IV. CONCLUSIONS AND OUTLOOK

We have studied the phase diagram of the Hamiltonian \( H_{SET} \) of Eq.\((6)\), which hosts a symmetry enriched topological phase, under different field perturbations—see table II for a summary. We introduced a modified version of the string order parameter defined in [23] which does not rely on tensor network states, and used it to characterize the SET phases until the different phase transitions. In all cases, we have found that these order parameters can reliably extract the pattern of symmetry fractionalization in SET phases.

We have also observed the presence of spontaneous symmetry breaking when an anyon that fractionalizes the symmetry condenses, and provided an alternate proof of this in the framework of PEPS. This results in a qualitative change in the phase diagram under parallel magnetic fields as compared to that of the regular toric code.

Curiously, we also found that the symmetry was spontaneously broken when a topological phase transition was driven by a transverse \( Y_e \)-field, even though this transition is not a simple condensation transition. We note that the general proof of Ref.[33] is specific to continuous phase transitions, whereas we observe a first order phase transition in this case. Therefore, we believe that there is yet more to uncover about the connection between topological phase transitions and spontaneous symmetry breaking.

We have also studied the Hamiltonian interpolation between \( H_{SET} \) and a trivial-enriched toric code Hamiltonian. The string order parameter characterizes correctly both phases. The interpolation does not go directly from one phase to the other: we have found an SSB intermedi-
Table II. Summary of the effect and characterization of the different perturbations on $H_{\text{SET}}$. All symmetries stand for $\mathbb{Z}_2 \times \mathbb{Z}_2$, inversion (i.e., reflection across bonds), and time-reversal symmetry. All broken stands for the breaking of all the symmetries that were present before the phase transition (this does not include the ones that are explicitly broken by the perturbation).

| Perturbation | Transition(s) | Symmetries of GS(s) |
|--------------|--------------|---------------------|
| $X_v$        | SSB from charge condensation | All $\rightarrow$ All broken |
| $Z_e$        | Flux condensation, no SSB     | All $\rightarrow$ All, (No SSB, GS is unique) |
| $Y_e$        | SSB                | $(\mathbb{Z}_2 \times \mathbb{Z}_2)$, Inversion $\rightarrow$ All broken |
| $Z_eZ_{e'}$  | Ising SSB           | $\mathbb{Z}_2$, Inversion, and TRS $\rightarrow$ All broken |


date phase. We point out that the interpolation of $(\mathbb{Z}_2)^3$ SPT phases studied in [49] can be connected to ours by gauging a $\mathbb{Z}_2$ subgroup of the $(\mathbb{Z}_2)^3$ symmetry [29]. We left for future work the details and implications of this connection.

On top of the perturbation analyzed in the main text we have also studied the effect of other perturbations applied to $H_{\text{SET}}$. In particular, a $X_v$-field preserves all symmetries but it drives the SET phase to a trivial topological phase via flux condensation (analogous to the $Z_e$-field). A $Y_v$-field spontaneously breaks the symmetry and drives the SET to a trivial topological phase. We left for future work the combination of $Z_eZ_{e'}$ and $Z_e$-fields. That interplay could be interesting since one field drives the SET to an SSB toric code and the other to a symmetric flux condensed phase, so the two differ in both topological order and symmetry-breaking pattern. The nature of the triple point of our 2D phase diagram, where SSB induced by charge condensation, flux condensation, and SET co-exist, should also be interesting to study. We wonder how the symmetry enrichment modifies the physics behind the analysis of the triple point in the regular toric code in Ref. [38].

Finally, we emphasize the importance of using SOPs designed for SET phases, as in the present study. This is because there are SF patterns that cannot be characterized just by 1D SPT order parameters. Two examples related to toric code topological order are worth to mention. First, two of the non-trivial SET phases of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ global symmetry cannot be distinguished by 1D SPT order parameters. Second, the non-trivial SF pattern given by a global $\mathbb{Z}_2$ symmetry has no analogous 1D SPT phase; this is the case of the model studied in [14] and the ones in appendix C (which we are able to characterize).

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Appendix A: PEPS for the GS of $H_{\text{SET}}$

The ground state of $H_{\text{SET}}$ has a tensor network description. We divide the lattice into the two types of vertices, $A$ and $B$. The PEPS is constructed by assigning different tensors to each type of vertex and to the edges. The resulting PEPS has bond dimension 3. The non-zero components of the tensor of the edges, $T_e$, are:

$$1 = 2 \quad 0 \quad 2 \quad 1 = 0 \quad 1 \quad 0 \quad 1 = 1 \quad 1 \quad 1 \quad 1 \quad .$$

This tensor projects the virtual legs on each edge onto the state labelled 2 if there is no loop running along that edge, and otherwise onto the span of the states labelled by 0 and 1.

The non-zero components of the tensors of vertices $A$ and $B$, $T_A$ and $T_B$, are respectively

$$1 = 2 \quad 0 \quad 1 \quad 2 , \quad 1 = 1 \quad T_A \quad - = 0 \quad T_A \quad + ,$$

$$1 = 2 \quad \frac{0,1}{T_B} \quad 2 = 0 \quad 0 \quad T_B \quad 1 = 1 \quad \frac{1}{T_B} \quad 2 ,$$

plus all their rotations where $\pm = |0\rangle \pm |1\rangle$. The first role of the vertex tensors is to enforce an even number of virtual $|2\rangle$ states at each vertex which gives the closed-loop condition. Then, the rest of the structure serves to create the decoration by cluster states. This can be seen by imagining removing the legs labelled 2 in the above equations. Then, the resulting tensors resemble those defining the 1D cluster state [50].
The symmetries of these tensors are the following:
\[ XT_A = T_A(Z \oplus 1)^{\otimes 3}, \quad T_A(X \oplus 1)^{\otimes 3} = T_A, \]
\[ XT_B = T_B(X \oplus 1)^{\otimes 3}, \quad T_B(Z \oplus 1)^{\otimes 3} = T_B, \]
\[ T_A(\bar{Z})^{\otimes 3} = T_A, \quad T_B(\bar{Z})^{\otimes 3} = T_B, \] (A4)
where \( \bar{Z} = -1 \oplus -1 \oplus 1 \).

We block the hexagonal lattice to a square one by contracting \( T_A, T_e \) and \( T_B \) (independent of the direction) to result in the tensor \( T \):

\[ T = \begin{array}{c}
\otimes \otimes \otimes \\
v_A & v_B \\
\cdots & \cdots \\
e_1 & e_2 & e_3 & e_4
\end{array}. \] (A5)

This tensor has the following symmetries:
\[ T = T\bar{Z}^{\otimes 4}, \]
\[ X_{vb} T = T(X \oplus 1)^{\otimes 4}, \]
\[ X_{va} T = T(Z \oplus 1)^{\otimes 4}, \]
\[ X_{vb} \otimes X_{vb} T = T(iY \oplus 1)^{\otimes 2} \otimes (-iY \oplus 1)^{\otimes 2}, \] (A6)
where the first equation accounts for the \( \mathbb{Z}_2 \)-injectivity. Then, the state has a global symmetry corresponding to the \( \mathbb{Z}_2 \)-injection. Finally, we prove the emergence of SSB when an anyon that transforms non-trivially under the symmetry is condensing.

1. Background of topological order and global symmetries on PEPS

PEPS are pure states completely characterized by a tensor \( A \). We focus on bosonic and translational invariant systems on the square lattice for the sake of simplicity. The tensor \( A_{\alpha, \beta, \gamma, \delta} \) see Fig.8(a)–then has five indices; one \( i = 1, \ldots, d \) for the physical Hilbert space \( \mathbb{C}^d \) of each particle and four \( \alpha, \beta, \gamma, \delta = 1, \ldots, D \) which correspond to the virtual degrees of freedom (d.o.f.). The PEPS is constructed by placing \( A \) on each vertex and contracting the neighbour virtual d.o.f. (identifying and summing the indices) as depicted in Fig.8(b). When the chosen boundary conditions are applied, the resulting tensor contraction \( c_{i_1, \ldots, i_N} = \langle A^{i_1}, \ldots, A^{i_N} \rangle \) describes a quantum many-body state \( |\psi_A\rangle = \sum c_{i_1, \ldots, i_N} |i_1 \cdots i_N\rangle \).

![Figure 8](image-url)

(a) \( A_{\alpha, \beta, \gamma, \delta} \) on the square lattice. (b) The PEPS constructed via the contraction of the virtual d.o.f. of the tensors. The contraction of indices is represented by joining the corresponding legs associated to the indices. We place the PEPS on a torus however, we will leave the boundaries open in the drawings for the sake of clarity. (c) Invariance of the \( G \)-injective tensor under the action of the group \( G \) on the virtual d.o.f. (d) Pair of charge operators placed on the virtual d.o.f.

a. \( G \)-injective PEPS and global symmetries

We focus on the family of \( G \)-injective PEPS [51] whose tensors have the following virtual symmetry (\( G \)-invariance) illustrated in Fig.8(c):
\[ A = A(u_g \otimes u_g \otimes u_g^{1} \otimes u_g^{1}), \] (B1)

where here \( A \) is represented as a map from the virtual to physical indices, and \( u_g \) is some unitary representation of the finite group \( G \). Given a \( G \)-injective PEPS, the

Appendix B: SET on PEPS, order parameters and SSB

In this section, we first introduce PEPS and show how topological order, global symmetries, and their interplay are characterized. Then, we elaborate on the order parameter that we have used in the main text to study the SET phases. Finally, we prove the emergence of SSB when an anyon that transforms non-trivially under the symmetry is condensing.
associated parent Hamiltonian [51], defined on a torus, has the ground state degeneracy $D(G)$ of the quantum double model of $G$. In this work, we focus on abelian $G$ for the sake of simplicity.

The anyons of $D(G)$ are charges, fluxes, and dyons (a combined excitation of the previous two). Charges are labelled by the irreducible representations (irreps) of $G$ and fluxes by elements of $G$. We represent the virtual operator of a pair of charges, $\Pi = C_{\sigma} \otimes C_{\tau}$, as two orange rectangles placed on two different edges of the lattice, see Fig.8(d).

We consider $G$-injective PEPS, $|\psi_A\rangle$, with a global on-site symmetry

$$U_q^{\otimes n}|\psi_A\rangle = |\psi_A\rangle \forall q \in Q,$$

(B2)

where $U_q$ is a linear unitary representation of some finite group $Q$ and $n$ is the number of lattice sites. For all $q \in Q$, there is an invertible matrix $v_q$ which translates $U_q$ through the local tensor $A$ on the virtual d.o.f. as follows [52]:

$$U_q = v_q^{-1} v_q^{\dagger} \forall q \in Q. \quad (B3)$$

The operators $v_q$ do not have to form a linear representation. It actually turns out that

$$v_k v_q = u_{\omega(k,q)} v_{kq}, \quad (B4)$$

where $\omega(k,q) \in G$. This means that $\{v_q\}$ form a homomorphism up to the matrix $u_{\omega(k,q)}$, where $u$ is the representation of $G$ introduced earlier for the $G$-invariance.

The action of the global symmetry on a charge sitting on a virtual bond is given by $C_{\sigma} \rightarrow \Phi_q(C_{\sigma})$, where $\Phi_q(X) = v_q X v_q^{-1}$. Diagrammatically:

$$U_q = v_q^{-1} v_q^{\dagger} \equiv \Phi_q(C_{\sigma}). \quad (B5)$$

If the symmetry is applied for two elements $q, k \in Q$, we see that

$$(\Phi_k \circ \Phi_q)(C_{\sigma}) = (\tau_{\omega(k,q)} \circ \Phi_{kq})(C_{\sigma}), \quad (B6)$$

where $\tau_{\omega(k,q)}$ denotes the conjugation by $u_{\omega(k,q)}$. This implies that the symmetry action over the charge sector can be projective, i.e. the symmetry fractionalizes. Let us assume that $u_q v_q = v_q u_q$ for all $q \in G$ and $q \in Q$ which means that the global symmetry does not permute between anyons. Formally we define $\omega : Q \times Q \rightarrow G : (k, q) \rightarrow \omega(k,q) = v_k v_q v_{kq}^{-1}$ that satisfies a 2-cocycle condition. The possible 2-cocycles are classified by the second cohomology group $H^2(Q, G)$ which characterizes the different SF patterns of $G$-injective PEPS.

2. Order parameters for symmetry fractionalization

In Ref.[23] the authors constructed order parameters that identify the SF pattern of the charges without relying on the knowledge of the virtual symmetry operators $v_q$ for the RGFP of $G$-injective PEPS. Here, we extend their definition to $G$-injective PEPS where a perturbation is applied.

We consider on-site perturbations of the tensor $A$ of the form $A(\theta) = T(\theta) A$, where $T(\theta)$ is some invertible matrix that depends on $\theta$. We notice that this kind of perturbation keeps the $G$-injectivity of the tensor. Then, the virtual charge operator can be the same as in the RGFP.

For $G = Z_2$, $Q = Z_2 = \{e, a\}$, the order parameter that we propose to capture the SF pattern is the following:

$$\mathcal{O}_{\ell}^{[a]} = \frac{\mathcal{L}_{\ell}^{[a]}(\theta)}{\mathcal{L}_{\ell}^{[e]}(\theta)}. \quad (B7)$$

The value of $\mathcal{L}_{\ell}^{[a]}(\theta)$ is given by

$$\mathcal{L}_{\ell}^{[a]}(\theta) = \frac{\bar{A}^{[a]}(\theta)}{\bar{A}^{[e]}(\theta)}, \quad (B8)$$

where the depicted tensor is $A(\theta)$ and the blue lines correspond to the permuted sites (also where the symmetry operators act before doing the scalar product). It is important to note that $\mathcal{L}_{\ell}^{[a]}(\theta)$ is not given in terms of an expectation value of an operator by $|\psi_A(\theta)\rangle$. This is because of the presence of virtual charge operators.

For $Q = Z_2 \times Z_2 = \{e, a, b, ab\}$ the order parameter is the triple $\{\mathcal{O}_{\ell}^{[a]}, \mathcal{O}_{\ell}^{[b]}, \mathcal{O}_{\ell}^{[ab]}\}$. The key point is that this order parameter behaves as the fractionalization class of the charge: $\mathcal{O}_{\ell}^{[a]} \approx \omega(q,q)$, that is, it reveals the sign of this action.

The order parameter of Eq.(B7) is meant for zero correlation length states. However, a perturbation will generally increase the correlation length until the phase transition point is achieved. To account for the growth of the correlation length the following order parameters are defined by blocking sites:

$$\mathcal{O}_{\ell}^{[a]} = \frac{\mathcal{L}_{\ell}^{[a]}(\theta)}{\mathcal{L}_{\ell}^{[e]}(\theta)}. \quad (B9)$$

where $\mathcal{L}_{\ell}^{[a]}(\theta)$ is equal to

$$\mathcal{L}_{\ell}^{[a]}(\theta) = \frac{\bar{A}^{[a]}(\theta)}{\bar{A}^{[e]}(\theta)}, \quad (B10)$$

and $\ell$ should be taken greater than the correlation length.
3. SSB from anyon condensation in PEPS

In this section, we use the framework of PEPS to prove that there is SSB in the ground subspace if a condensed anyon transforms non-trivially under the symmetry, i.e. it either fractionalizes or it is permuted. We remark that this has been proven in [33] using the language of G-graded tensor categories.

The proof is done by analyzing the fixed point structure of the transfer operator of the PEPS. We show that there is a contradiction if we suppose that the following conditions hold at the same time: (i) there is no symmetry breaking on the fixed point structure, i.e. expectation values are independent of the fixed point with whom they are evaluated, (ii) the global symmetry fractionalizes on an anyon $b$ or it permutes $b$ to $c$, (iii) the anyon $b$ condenses or the anyon $b$ condenses and not $c$. To do so we use the framework developed in [47, 48] to describe condensation of anyons in $G$-injective PEPS together with the characterization of symmetries in [52] that we revisit now.

In $G$-injective PEPS the transfer operator $\mathbb{T}$ has a degenerate fixed point structure: $\{|\rho_c\rangle\}$ such that $\mathbb{T}|\rho_c\rangle = |\rho_c\rangle$, see fig. 9(a). This comes from the symmetries of the transfer operator: $|u_q^{\mathbb{T}}\rangle = 0$, where $u_q \equiv u_q \otimes u_{q'}$ defined for all $q \in G \times \bar{G}$. As in Ref.[47] we assume that $G$ is abelian, that the fixed point subspace is spanned by injective MPS with tensors $\{|\rho_c\rangle\}$, and that for each $c, c'$ there is a $g$ such that $v_g^{\mathbb{T}}|\rho_c\rangle = |\rho_{c'}\rangle$.

![Diagram](image)

Figure 9. (a) Transfer operator $\mathbb{T}$, it fixed points vectors $|\rho_c\rangle$ and the equation that relates $\mathbb{T}$ and $|\rho_c\rangle$. (b) Any fixed point $|\rho_c\rangle$ is an injective MPS constructed with the tensor $M_c$ This implies the uniqueness of the left/right fixed points, $\sigma^{c}_{L/R}$, of the transfer matrix constructed with $M_c$. (c) Transformation rule of $M_c$ and $\sigma^{c}_{L}$ under $v_q$. (d) Transformation rule of $M_c$ under a operator $u_g$ corresponding to an unconfined flux, see [47].

A global symmetry on the PEPS is reflected in the transfer operator as follows $[v_q^{\mathbb{T}}, \mathbb{T}] = 0$, where $v_q = v_q \otimes v_q^{-1}$. The global symmetry acts on the fixed point subspace as follows:

$$v_q^{\mathbb{T}}|\rho_c\rangle = \lambda_q |\rho_{c'}\rangle, \quad \forall q \in Q,$$

where $|\lambda_q| = 1$, $\lambda_q$ does not depend on $c$, and we assume that the symmetry operators can only permute between the fixed points. Since we assumed that $|\rho_c\rangle$ are injective MPS, the transfer operator constructed with any of these states has a unique right/left fixed point, $\sigma^{c}_{R/L}$ depicted in fig. 9(b). Eq.(0), is translated into how the virtual operators transform those fixed points, see fig. 9(c).

The main assumption here is that the expectation values do not depend on the fixed point where they are evaluated:

$$\langle \rho, \mathbb{T}|\rho_c\rangle = \langle \rho |\rho_{c'}\rangle, \quad \forall c, c' \in \mathcal{O},$$

where $\mathcal{O}$ is an observable and $\mathbb{T}|\rho_c\rangle$ is it virtual representation using the transfer operator of the tensor. This is a non-symmetry breaking condition since the fixed points are related to the different ground states so that Eq.(I) is equivalent to $\langle \mathbb{T}|\rho_c\rangle = \langle \mathbb{T}|\mathbb{T}|\rho_c\rangle \in \mathcal{O}$. For example, a SB pattern like the magnetization reads $\langle m_\mathcal{O} \rangle = -\langle m_\mathcal{O} \rangle$. In particular, this holds for the evaluation on the fixed points of an anyon operator $[g, \alpha]$ defined as:

$$u_g^{\mathbb{T}}|\rho_c\rangle = |\rho_{c'}\rangle, \quad \forall c, c' \in \mathcal{O},$$

where $C_\alpha$ (orange square) corresponds to the charge part of the anyon and the operator $y_g$ (grey circle), defined in fig. 9(d), corresponds to the flux part. Following Ref. [47], the condition for an anyon $[g, \alpha]$ to be condensed is

$$y_g|\rho_c\rangle = 0 \Rightarrow |y_g, x_q\rangle = 0 \quad \forall q \in Q, g \in \mathbb{G}$$

and that there is a non-trivial SF pattern

$$\omega(k, q) 

a. SF case

In this case, we focus on global symmetries that do not permute the anyons, that is

$$[v_q, u_g] = 0 \Rightarrow [y_q, x_q] = 0 \quad \forall q \in Q, g \in \mathbb{G}$$

and that there is a non-trivial SF pattern

$$v_k^\omega u_q = u_{\omega(k, q)} \omega \forall q \in Q, g \in \mathbb{G}$$
We notice that whenever (III) holds, there must exist a charge $\alpha$ that fractionalizes the symmetry:

$$u_{\omega(k,q)} C_{\alpha} u_{\omega(k,q)}^{-1} = \chi_{\alpha} (\omega(k,q)) C_{\alpha}, \text{ s.t. } \chi_{\alpha} (\omega(k,q)) \neq 1$$

(III')

this is because there must exist a charge $\alpha$ that braids non-trivially the flux labelled by $\omega(k,q) \in G$.

Let us now suppose that (III), (I), and (II) hold and that the anyon that fractionalizes the symmetry in (III'), concretely its charge part, is the one that condenses, satisfying (II). Then,

$$\chi_{\alpha} (\omega(k,q)) C_{\alpha}$$

We use now (I'), arriving to the following equation:

where we find a contradiction between (III') and (II). This means that if a perturbation that preserves the global symmetry also induces an anyon condensation of $[g, \alpha]$ whose charge part $\alpha$ also fractionalizes the symmetry, then the symmetry has to be spontaneously broken after the phase transition in the final phase.

\[ \sigma_L \sigma_R = \chi_{\alpha} (\omega(k,q)) \sigma_L' \sigma_R' \]

where $\chi_{\alpha} (\omega(k,q)) \neq 1$.

\[ \Phi_q (u_g) = v_q u_g v_q^{-1} = u_{\varphi_q (g)} \]

where this corresponds to a permutation of the fluxes. Similarly, the permutation of charges is: $\Phi_q (C_\sigma) = C_{\varphi_q (\sigma)}$ where $\varphi_q (\sigma)$ is another irrep of $G$. We remark that for $G$ abelian, the irreps form a group isomorphic to $G$ so $\varphi_q (g) \in \text{Aut}(G)$ also characterizes the permutation of the irreps. Generally, for dyons, the transformation is $[g, \alpha] \mapsto [\varphi_q (g), \varphi_q (\alpha)]$.

\[ \sigma_L \sigma_R = \chi_{\alpha} (\omega(k,q)) \sigma_L' \sigma_R' \]

\[ \Phi_q (u_g) = v_q u_g v_q^{-1} = u_{\varphi_q (g)} \]

We use the following PEPS tensor to model the ground state:

If we assume (I) that there is no symmetry breaking, (II) that $[g, \alpha]$ is condensing and that the symmetry permute $[g, \alpha]$ as explained before we obtain that

$$0 \neq \sigma_L \sigma_R$$

This implies that the set of condensed anyons are closed under the symmetry. Therefore, if a perturbation that preserves the global symmetry induces an anyon condensation of $[g, \alpha]$ and this is permuted to a non-condensed anyon $[\varphi_q (g), \varphi_q (\alpha)]$, the symmetry has to be spontaneously broken after the phase transition in the final phase.

**Appendix C: A toric code state with a $\mathbb{Z}_2$ global symmetry on the square lattice**

In this appendix, we consider deformations of a fixed-point tensor representing a toric code enriched with $\mathbb{Z}_2$ symmetry in order to test that the string order parameters of Eq. B9 function as intended away from the fixed-point.

We use the following PEPS tensor to model the ground...
state of the symmetry-enriched toric code,

\[ A = \frac{1}{\sqrt{2}} \sum_{b=\{0,1\}} (X^2)^b \otimes 4 \]  

(C1)

where here, and throughout this section, \( X = \sum_{i=0}^{3} |i+1 \mod 4 \rangle \langle i | \) is the generator of the left regular representation of \( \mathbb{Z}_4 \) such that \( X^4 = \mathbb{I} \) and \( X^2 = \sigma_x \otimes \mathbb{I}_2 = g \).

The tensor \( A \) is \( \mathbb{Z}_2 \)-invariant, i.e. \( A \) has the following virtual symmetry \( A = A(g \otimes g \otimes g \otimes g) \). The PEPS constructed with \( A, |\psi_A\rangle \), is left invariant by the action of the operator \( U = X \otimes X \otimes X^{-1} \otimes X^{-1} \) on each lattice site: \( U \otimes U \psi_A = \psi_A \) which would correspond to a global \( \mathbb{Z}_4 \) symmetry. However, the state \( |\psi_A \rangle \) has also a local symmetry generated by \( U^2 \) such that \( U_i^2 |\psi_A\rangle = |\psi_A\rangle \), where \( i \) is any site. Therefore, the \textit{a priori} global \( \mathbb{Z}_4 \) symmetry is reduced to a global \( \mathbb{Z}_2 \) symmetry when quotiented by the local symmetry: this is the one we consider here.

The interesting feature of this model, when considering the parent Hamiltonian of \( |\psi_A\rangle \), is that the anyonic excitation corresponding to the charge fractionalizes that global \( \mathbb{Z}_2 \) symmetry. More explicitly, the virtual operator of the charge is \( C_\sigma = \sigma_x \otimes \mathbb{I}_2 \) and the virtual action of the global \( \mathbb{Z}_2 \) symmetry is given by the conjugation with \( X \) such that \( C_\sigma \to XC_\sigma X^{-1} \). Then, if we apply twice the global symmetry an isolated charge exchanges as \( C_\sigma \to -C_\sigma \) which corresponds to a projective representation of \( \mathbb{Z}_2 \).

In the following, we study the effects of different perturbations on the wavefunction generated by \( (C1) \)

1. Diagonal string perturbation

We apply the perturbation \( \exp[\theta X^2/2] \otimes 4 \) to the tensor \( A \) where \( \exp[\theta X^2] = \cosh(\theta) \mathbb{I} + \sinh(\theta) X^2 \). The perturbation commutes with the symmetry and it drives the system from the TC phase to a product state. This is because \( \lim_{\theta \to \infty} \frac{\sinh(\theta)}{\cosh(\theta)} = 1 \) so that for large values of \( \theta \) we have \( \exp[\theta X^2/2] \propto \mathbb{I} + X^2 \). This implies that

\[ \lim_{\theta \to \infty} - \exp[\theta X^2/2] \otimes 4 A \propto (\mathbb{I} + X^2) \otimes 4 A = A_{\infty}. \]

Since all the operators involved are real the on-site transfer operator can be written as

\[ E = A^*(\theta) A(\theta) = \exp[\theta X^2] \otimes 4 [\mathbb{I} \otimes 4 + (X^2) \otimes 4]. \]

The norm of the PEPS with the perturbation is calculated with the contraction of this tensor \( E \) on each vertex \( v \) of the square lattice \( \mathcal{V} \). When two sites, \( i \) and \( j \), coincide the resulting factor is

\[ L_0(b_i, b_j) = \text{Tr} [\cosh(2\theta)(X^2)^{b_i-b_j} + \sinh(2\theta)(X^2)^{b_i-b_j+1}] = \begin{cases} 4 \cosh(2\theta) & \text{if } b_i - b_j = 0 \mod 2 \\ 4 \sinh(2\theta) & \text{if } b_i - b_j = 1 \mod 2 \end{cases}, \]

instead of the value \( 4\delta_{b_i-b_j,0} \) in the non-perturbed case \( \theta = 0 \). Then, the norm can be expressed as the following sum

\[ \langle \psi_A(\theta) | \psi_A(\theta) \rangle = C_{\epsilon \in \mathcal{V}} \{ \mathbb{E}_\epsilon \} = \sum_{(i,j)} L_0(b_i, b_j), \]

(C2)

so that \( \langle \psi_A(0) | \psi_A(0) \rangle = 2 \cdot 4^{2N_s} \) where \( N_s \) is the number of sites of \( \mathcal{V} \). The wavefunction \( |\psi_A(\theta)\rangle \) can be normalized locally by modifying the weight of the tensor \( A \). We find that the string order parameters are \( L^{[a]}_1(0) = (-1)^g \cdot 2 \cdot 4^{2N_s-4} \) at the fixed-point, and

\[ L^{[a]}_1(\theta) = \sum_b A(b_0, b_7) B(b_3, b_4, b_5, b_6) C_{b_2, b_6}(b_1, b_2) C_{b_4, b_8}(b_8, b_9), \]

and similarly for \( L^{[c]}_1(\theta) \). The values of the functions \( A, B, C \) are

\[ A(b_0, b_7) = \begin{cases} 0 & \text{if } b_7 = 1, \ b_6 = 0 \\ -4 & \text{if } b_7 = 1, \ b_6 = 1 \\ 4 & \text{if } b_7 = 0, \ b_6 = 0 = 2(s_6 + s_7), \\ 0 & \text{if } b_7 = 0, \ b_6 = 1 \end{cases}, \]

\[ B(b_3, b_4, b_5, b_6) = \begin{cases} 4 & \text{if } b_3 + b_6 = 0, \ b_4 + b_5 + b_6 = 0 \\ 0 & \text{if } b_3 + b_6 = 1, \ b_4 + b_5 + b_6 = 1 \\ 0 & \text{if } b_3 + b_6 = 1, \ b_4 + b_5 + b_6 = 0 \\ -4 & \text{if } b_3 + b_6 = 0, \ b_4 + b_5 + b_6 = 1 = 2s_4 s_5 (s_3 + s_6), \end{cases} \]

and

\[ C_{b_2, b_6}(b_1, b_2) = \begin{cases} 2(e^{4\theta} - e^{-4\theta}) & \text{if } b_1 + b_2 + b_4 + b_5 = 0 \\ 2(e^{4\theta} + e^{-4\theta}) & \text{if } b_1 + b_2 + b_4 + b_5 = 1 \end{cases} = 2(e^{4\theta} - s_5 s_4 s_5 e^{-4\theta}), \]

where all the sums are modulo 2 and the transformation to spin variables is \( s_i = (-1)^{b_i} \). The sublattice \( \Omega \) corresponds to the following set of edges that are involved in the SOP:

\[ \Omega = \{ \langle 1,4 \rangle, \langle 2,5 \rangle, \langle 3,4 \rangle, \langle 4,5 \rangle, \langle 5,6 \rangle, \langle 6,7 \rangle, \langle 4,8 \rangle, \langle 5,9 \rangle \}, \]

which are placed as follows:

\[ \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ & s_5 & s_6 & s_7 \\ & & s_8 & s_9 \end{array} \]

It can be checked that for \( \theta = 0 \), \( b_i = b_j \) except for \( b_4, b_5 \) in \( L^{[a]}_1(0) \), such that we obtain \( O_1(0) = -1 \).

1. Mapping to the classical 2D Ising model

The norm of Eq. (C2) can be written as the partition function of the classical Ising model

\[ \sum_{\{s_i\}} \prod_{(i,j)} e^{2s_i s_j}, \]

(C3)
where \( s_i = (-1)^{b_i} \) are the Ising variables and the different weights correspond to

\[
\begin{align*}
\{ e^\beta &= 4 \cosh(2\theta) = 2(e^{2\theta} + e^{-2\theta}), \\
\exp(-\beta) &= 4 \sinh(2\theta) = 2(e^{2\theta} - e^{-2\theta}) \}
\end{align*}
\] (C4)

If we compare the ratio of the above expressions we obtain \( e^{-2\beta} = \tanh(2\theta) \). So, the critical temperature \( \beta_c = \ln(1 + \sqrt{2})/2 \) corresponds to \( \theta_c = \beta_c/2 \) since \( \theta = \tanh^{-1}(e^{-2\beta})/2 \).

In order to compute Eq. (B9), we need to express its numerator in the spins variables.

The calculation results in the following

\[
\mathcal{L}_1^a(\theta) \propto \sum (s_6 + s_7)s_4s_5(s_3 + s_6)(1 - s_1s_2s_4s_5e^{-8\theta})
\times (1 - s_8s_9s_4s_5e^{-8\theta})e^{8\theta} \prod_{(i,j) \in V} e^{\beta s_i s_j}
\] (C5)

where \( \beta \) depends on \( \theta \) via Eq.(C4) as \( e^{-2\beta} = \tanh(2\theta) \).

We can also map to the Ising model the order parameter for a blocking of \( \ell \) sites:

\[
\mathcal{L}_\ell^a(\theta) \propto \sum (s_6 + s_7)s_4s_5(s_3 + s_6)(1 - s_1s_2s_4s_5e^{-8\theta})
\times \prod_{i=1,\ell} (1 - s_2s_1 + s_3s_1 + s_1s_2 + s_1s_2s_4s_5e^{-8\theta})
\times \prod_{(i,j) \in V} e^{\beta s_i s_j},
\]

where \( \Omega_\ell \) is now the sublattice composed by the following red edges:

\[
\begin{align*}
&\{ s_1 \} \\
&\{ s_2 \} \\
&\{ s_3 \} \\
&\{ s_4 \} \\
&\{ s_5 \} \\
&\{ s_6 \} \\
&\{ s_7 \}
\end{align*}
\]

In Fig.10(a) we show the values of \( \mathcal{L}_\ell^a(\theta) \) and \( \mathcal{L}_\ell^c(\theta) \). It easy to see that \( \mathcal{L}_\ell^c(\theta) \) corresponds to the confinement fraction of the charge [47, 48] and it goes to zero after the phase transition. Therefore, (B7) is no longer valid before the phase transition, we can see that \( \mathcal{O}_\ell^{[\sigma]} = -1 \).

We use the infinite matrix product state (iMPS) algorithm for these computations [32]. Furthermore, since, in this case, the deformation keeps the model exactly solvable, we use also use the Metropolis-Hastings algorithm [53], which allows us to evaluate \( \mathcal{O}_\ell(\theta) \) for even larger values of \( \ell \) (data not shown).

2. Dual string perturbations

Now, we analyze the behavior of the string order parameter by considering two further perturbations.

1. We start by defining \( Z = \text{diag}(1, -1, 1, -1) = 1_2 \otimes \sigma_x \). Since \( Z \) commutes with \( X^2 \), by applying the perturbation \( P(\theta) = (\exp[\frac{\theta}{2}Z])^{\otimes 4} + (\exp[-\frac{\theta}{2}Z])^{\otimes 4} \), the resulting tensor in the limiting case can be given as,

\[
\lim_{\theta \to \infty} P(\theta) A \propto (\sigma_x^{\otimes 4} + \sigma_z^{\otimes 4}) \otimes [1_2^{\otimes 4} + X^{\otimes 4}] .
\]

The perturbed state corresponds to two copies of regular toric code times product states. Moreover the perturbation commutes with the symmetry, the results of the numerics are shown in Fig. 10(b) where it can be seen that \( \mathcal{O}_\ell(\theta) \) gets sharper with increasing \( \ell \).

2. On the other hand, if we restrict the perturbation to \( P(\theta) = (\exp[\frac{\theta}{2}Z])^{\otimes 4} \) the resulting tensor in the limiting case is

\[
\lim_{\theta \to \infty} (\exp[\theta Z])^{\otimes 4} A \propto [0|0|0^4 + |0|1|0^4] \otimes 1_2^{\otimes 4},
\]
which corresponds to a product state. It is important to note that in this example the perturbation doesn’t commute with the symmetry so that the SOP is no longer well defined. This is what Fig. 10(c) shows, a slow decay of $O(\ell, \theta)$ in contrast with the other sharp behaviours when the perturbations commute with the symmetry.

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