Natural frequency analysis of curved pipe conveying non-uniform axial flow

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Abstract. In this article, the natural frequencies of curved pipe conveying non-uniform axial flow are calculated. The non-uniformity of velocity distribution on the section is considered and the parabola fluid is adopted in the section. The Flügge curved beam is used to build the pipe model. The dynamic equation of pipe conveying non-uniform fluid is derived based on Hamilton principle. The wave propagation method is used to solve the equation. With different flow velocity, the natural frequency is computed in the assumption of non-uniform fluid and uniform fluid. Compared with the uniform fluid, the non-uniform fluid will have a higher frequency value.

Keywords: non-uniform axial flow, curved pipe conveying fluid, wave motion method, natural frequency,

1. Introduction
Curved pipe conveying fluid is widely used in aerospace, nuclear engineering, ocean engineering and petrochemical fields. The resonance failure will occur when the external exciting force frequency is close to the natural frequencies of the curved pipe, which can cause serious engineering problems. Therefore, it is necessary to study the resonance reliability of the curved pipe conveying fluid.

Pipelines are usually composed of straight and curved pipes. There have been a lot of studies on straight pipes, but relatively few studies on curved pipes [1]. Chen [2] established the differential vibration equation of the curved pipe based on Hamiltonian theory, calculated the critical velocity of curved pipe under four boundary conditions and discussed the influence of velocity on the natural frequency of curved pipe. Li et al. [3] presented the solution method for calculating the natural frequency under the scalable and non-scalable axis of the curved pipe. Hu et al. [4] established the dynamic model of the axially extensible curved pipe in the curvilinear coordinate system, and obtained the natural frequency and critical velocity of the curved pipe by using the partition matrix method. Kuroda et al. [5] used the transfer matrix method to calculate the vibration of the curved pipe. In the research of pipes conveying fluid, most of the internal fluid is assumed to be uniform flow, but the existence of boundary effect and the fluid viscous resistance will lead the non-uniform velocity distribution in pipes [6]. Although some progress has been made in the study of the curved pipe conveying fluid, few studies have considered the uniform fluid. In this paper, the velocity of flow is assumed to follow a parabolic distribution on the cross section of the pipe. The vibration control equation of the curved pipe with non-uniform axial flow is derived and the natural frequency of the curved pipe is obtained with wave method.
2. Free vibration analysis of curved pipe conveying fluid

![Curved Pipe Diagram]

Figure 1. Schematic diagram of fluid-conveying curved pipe

The curved pipe is shown in figure 1, where $R_o$ is the curvature radius of the center line of the curved pipe, $v$ is the velocity of the fluid, $u$ is the radial displacement perpendicular to the axis of the pipe, and $w$ is the axial displacement along the axis of the pipe.

The viscous force between the fluid and the pipe wall is neglected, and the non-uniformity of velocity distribution on the section which caused by fluid viscosity is considered. It is assumed that the Reynolds number $R_e < 2300$ and the fluid in the pipe is in laminar flow state. The pressure loss caused by friction on the pipe wall is ignored. The flow velocity profile of the fluid on the interface is a quadratic parabola, and the flow velocity can be shown as [7]:

$$v = v_0 \left(1 - \frac{r^2}{R^2}\right)$$  \hspace{1cm} (1)

Where, $v_0$ is the fluid velocity at the center of the pipe, which is the maximum velocity in the pipe, and $r$ is the distance between fluid particles in the pipe and the center of the pipe.

![Non-Uniform Flow Diagram]

Figure 2. Pipe conveying fluid with non-uniform axial flow

Hamilton principle is adopted to deduce the vibration equation of the curved pipe [1]:

$$\delta \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \delta W dt - \int_{t_1}^{t_2} M_f \left(\frac{\partial R_e}{\partial t} + v \tau_e\right) \cdot \delta R_e dt = 0$$  \hspace{1cm} (2)

where, $L$ ($L = T_p + T_f - V_p - V_f$) is the Lagrange function, $T_p$ and $T_f$ are the fluid kinetic energy of the pipe, $V_p$ and $V_f$ are fluid potential energy of the pipe, $\delta W$ is the virtual work of the system, $M_f$ is the fluid mass per unit length, $v$ is the speed of the fluid in pipe, $R_e$ is the he bend position vector of the center line which is locate in the outlet of the pipe. $\tau_e$ is the unit vector at the exit which is tangent to the center line of the pipe.

The vibration in the plane is considered only, the movement of the curved pipe includes displacement $u$ along the $N$ axis and displacement $w$ along the $T$ axis. The kinetic energy of the curved pipe can be expressed as:

$$T_p = \frac{\dot{u}^2 + \dot{w}^2}{2} M_p R_e d\theta$$  \hspace{1cm} (3)

Where, $\dot{\#}$ and $\ddot{\#}$ respectively represent the derivative of the spatial and time coordinates.

The potential energy of the curved pipe can be expressed as:
$$V_p = \int_0^a \frac{EI}{2R_0^2} \left( u' + w' \right)^2 R_0 d\theta \quad (4)$$

Where, $EI$ is the bending stiffness of the pipe.

Assuming that the fluid is incompressible and the influence of gravity is ignored, then the potential energy of the fluid is zero \([8]\):

$$V_f = 0 \quad (5)$$

Bend deformation of any point on the $P_0$ after moving to point $P$, in which $\tau_0, n_0$ are the tangent and normal vector of point $P_0$, $\tau, n$ are the tangent and the normal vector of point $P$, $r_{p0}, r_p$ are the position vector of point $P_0$ and point $P$, using $s_0, s$ as the natural coordinates of the curved pipe deformation before and after, the direction displacement along $\tau_0, n_0$ can be shown as $w, u$.

![Diagram](image)

**Fig. 3** The coordinate relationship before and after deformation of the curved pipe

In this paper, the pipe that cannot be extended in axial direction is considered and simulated based on Flügge curved beam model \([3]\), so there is $u = \partial w/\partial \theta$. The relation between the two curvilinear coordinates before and after deformation is \([3]\):

$$\begin{bmatrix} \tau \\ n \end{bmatrix} = \begin{bmatrix} 1 & u/R_0 + \partial w/\partial s \\ -u/R_0 - \partial w/\partial s & 1 \end{bmatrix} \begin{bmatrix} \tau_0 \\ n_0 \end{bmatrix} \quad (6)$$

The fluid in the paper is axial and incompressible, and the velocity direction of the fluid in the pipe is always along the tangential direction of the pipe axis. The absolute velocity of the fluid $v_f$ can be expressed as the sum of the pipe velocity $v_p$ and the fluid velocity $v$:

$$v_f = v_p + v\tau \quad (7)$$

The pipe velocity $v_p = \partial u/\partial t \tau_0 + \partial w/\partial t n_0$, combining the relationship between the local coordinate system and the global coordinate system before and after the pipe deformation, the following equation can be obtained:

$$v_f = \left( u + v_0 \left( 1 - \frac{r^2}{R^2} \right) \right) \tau_0 + \left( w + v_0 \left( 1 - \frac{r^2}{R^2} \right) \left( \frac{u}{R_0} + w' \right) \right) n_0 \quad (8)$$

Due to the non-uniform flow velocity, the calculation of fluid kinetic energy in the pipe is different from that of the uniform flow, and the fluid flow velocity is the same on the concentric circle centered at the interface center of the pipe. Let the mass of the fluid per unit length be $M_f = \rho \pi R^2$, the kinetic energy of the fluid in the whole pipeline is:

$$T_f = \int_0^a \int_0^{2\pi} \rho \pi v_r^2 R_0 dr d\theta = \frac{M_f}{6} \int_0^a \left[ 3u^2 + 3w^2 + v_0^2 + 3u v_0 \left( \frac{u}{R_0} + w' \right) \right] R_0 d\theta \quad (9)$$
When the pipe is fixed at both ends, the boundary condition is:

\[ \delta u_0 = \delta u_\alpha = 0, \quad \delta w_0 = \delta w_\alpha = 0 \]  
\( t_1 \) and \( t_2 \) are the two moments which given as:

\[ \delta u_{t_1} = \delta u_{t_2} = 0, \quad \delta w_{t_1} = \delta w_{t_2} = 0 \]

Thus, the governing equation in the plane of the curved pipe with non-uniform axial and non-scalable flow can be obtained as follows:

\[
\frac{EI}{R_0} \left( \frac{\partial^6 w}{\partial \theta^6} + 2 \frac{\partial^4 w}{\partial \theta^4} + \frac{\partial^2 w}{\partial \theta^2} \right) - \frac{M_0}{\rho_0} \left( \frac{1}{R_0} + 1 \right)^2 \frac{\partial^2 w}{\partial \theta^2} - M_0 \frac{1}{R_0} + 1 \left( M_r + M_f \right) \left( \frac{\partial^4 w}{\partial \theta^4} - \frac{\partial^2 w}{\partial \theta^2} \right) = 0
\]

Compared with the dynamic control equation of the curved pipe with uniform flow [2]:

\[
\frac{EI}{R_0} \left( \frac{\partial^6 w}{\partial \theta^6} + 2 \frac{\partial^4 w}{\partial \theta^4} + \frac{\partial^2 w}{\partial \theta^2} \right) + \frac{M_0}{\rho_0} \left( \frac{1}{R_0} + 1 \right)^2 \frac{\partial^2 w}{\partial \theta^2} + 2 M_0 \frac{1}{R_0} \left( M_r + M_f \right) \left( \frac{\partial^4 w}{\partial \theta^4} - \frac{\partial^2 w}{\partial \theta^2} \right) = 0
\]

Assuming that the system is in simple harmonic vibration, the radial displacement can be written as follows [3]:

\[ u(\theta, t) = C_u \exp(i(k\theta + \omega t)), \quad w(\theta, t) = C_w \exp(i(k\theta + \omega t)) \]

Where, \( i = \sqrt{-1} \) is the imaginary unit, \( k, \omega \) respectively represents the wave number and circular frequency, and \( C_u, C_w \) are the undetermined coefficients.

The equation is substituted into the vibration control equation to obtain a cubic polynomial equation about the wave number \( k \), which is called as dispersion equation:

\[
\frac{EI}{R_0} k^6 + 2 \frac{EI}{R_0} k^4 + \left( \frac{EI}{R_0} + M_r \frac{1}{R_0} + 1 \right)^2 \left( M_r + M_f \right) \omega^2 k^2 + M_0 \frac{1}{R_0} \left( M_r + M_f \right) \omega^2 = 0
\]

The equation has six roots, which respectively represent the wave propagating and attenuating along the positive and negative directions of the pipeline. In order to facilitate the subsequent analysis, it is stipulated that \( k_1, k_2, k_3 \) is the wave propagating in the negative direction along the pipeline \( k_4, k_5, k_6 \) is the wave propagating in the positive direction along the pipeline

\[ u(\theta, t) = \sum_{j=1}^{6} C_{uj} \exp(i(k_j\theta + \omega t)), \quad w(\theta, t) = \sum_{j=1}^{6} C_{wj} \exp(i(k_j\theta + \omega t)) \]

The propagation matrix of left and right traveling waves is as follows:

\[
T_l = \begin{bmatrix} \exp(-ik_0 R_0 \pi) & 0 & 0 \\ 0 & \exp(-ik_0 R_0 \pi) & 0 \\ 0 & 0 & \exp(-ik_0 R_0 \pi) \end{bmatrix}, \quad T_r = \begin{bmatrix} \exp(ik_0 R_0 \pi) & 0 & 0 \\ 0 & \exp(ik_0 R_0 \pi) & 0 \\ 0 & 0 & \exp(ik_0 R_0 \pi) \end{bmatrix}
\]

The reflection matrix of the wave at the left and right end of the pipe can be obtained as:

\[
R_l = \begin{bmatrix} k_1 & k_2 & k_3 \\ 1 & 1 & 1 \\ k_4 & k_5 & k_6 \end{bmatrix}, \quad R_r = \begin{bmatrix} k_1 & k_2 & k_3 \\ 1 & 1 & 1 \\ k_4 & k_5 & k_6 \end{bmatrix}
\]

\[ I \text{ is the third-order identity matrix. The non-zero solution of this equation is provided that the following determinant is zero:} \]

\[ \det[I - R_l T_l R_r T_r] = 0 \]

Since the determinant contains frequency \( \omega \), the natural frequency of the curved pipe can be solved by the formula.
3. Example
A semicircular flow pipeline with a constant radius of curvature is taken as the calculation model. Both ends of the pipeline are supported by fixed ends, and the material parameters of the pipeline are shown in table 1.

| Material parameters | 
|---------------------|
| Radius of curvature | Young modulus | Outside radius | Inside radius | Pipeline density | Density of liquid |
| 0.5m                | 70GPa         | 0.0116m        | 0.0075m        | 2700kg/m³        | 1000kg/m³        |

First, the natural frequency of the non-uniform flow pipeline is calculated. The uniform flow hypothesis holds that the flow velocity is the same everywhere on the cross section of the pipe, and the parabolic non-uniform flow believes that due to the existence of fluid viscosity, the flow velocity at each point on the cross section of the pipe is related to the distance from the point to the center of the cross section, and presents a parabolic relationship. When the flow rate is zero, the fluids under the assumption of uniform flow and non-uniform flow are both static fluids, and the infusion pipeline becomes a liquid-filled pipeline. At this time, the natural frequencies of the pipelines with uniform flow and non-uniform flow should be consistent. Calculated in this paper, under the assumption of the non-uniform flow velocity is zero in the first order natural frequency for 43 rad/s, and USES the uniform flow hypothesis of first-order natural frequency for 43 rad/s, the results are the same, the other 2-4 order pipeline inherent frequency is consistent, as shown in figure 7, to verify the effectiveness of the proposed in this paper the rationality of the non-uniform axial flow hypothesis.

**Figure 4.** Natural frequency of the curved pipe at different flow velocity
(a) first-order natural frequency, (b) second-order natural frequency, (c) third-order natural frequency, (d) fourth-order natural frequency, the red line with circle is Non-uniform fluid and the blue line with asterisk is uniform fluid.

The curve of the natural frequency of the first-four orders of the pipeline changing with the flow velocity is shown in FIG. 4. In the figure, the red circle is the natural frequency under the assumption of non-uniform flow, and the blue solid dot is the natural frequency under the assumption of uniform flow. The first-order natural frequency of the curved pipe decreased with the increase of the flow rate, but compared with that of the uniform flow, the decrease rate is smaller under the assumption of non-uniform flow, and the change of the third-order and fourth-order natural frequency with the flow rate also showed such trend. This is because the maximum velocity at the center of the pipeline under the
assumption of non-uniform flow is greater than that under the assumption of uniform flow when the interfacial flow of the pipeline is the same at the same time. Different from the uniform flow hypothesis, the second order natural frequency of the curved-pipe in the case of non-uniform flow and uniform flow was first increased and then decreased, and the increase amplitude under the non-uniform flow hypothesis was greater than that under the uniform flow hypothesis. This is also related to the fact that the flow velocity at the center of the curved pipe is greater than the average flow velocity under the assumption of uniform flow under the condition that the cross-section flow is the same. The change of pipe natural frequency caused by the increase of flow velocity is not consistent. For example, the first order, third order and fourth order natural frequencies decrease with the increasing flow velocity while the second order natural frequency first increases and then decreases. The change in natural frequency of the non-uniform flow situation is smaller than that of the uniform flow conditions.

4. Conclusion
In this paper, the natural frequency of curved pipe with conveying non-uniform fluid is computed. Applying the typical parabolic non-uniform axial flow assumption and the Flügge curved beam is used for building the pipe model. With the Hamilton principle, the dynamic equation of the curved fluid-conveying pipe is derived. Only the on-plane motion is considered. The natural frequency of the semi-circular curved pipe with fixed support at both ends is obtained with the wave method. Compared with the uniform flow model, the non-uniformity of fluid is considered in this paper. Because of the non-uniform of fluid, the natural frequency of non-uniform fluid is high than that of uniform fluid.

5. References
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