Scalar Field Theories On The World Sheet: A Non-Trivial Ground State

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Abstract

The present article completes an earlier publication, which was the culmination of a series of papers dedicated to the study of the planar graphs of the scalar $\phi^3$ theory on a light cone world sheet. In the earlier work, a field theory on a continuous world sheet that reproduces these planar graphs was constructed, and the mean field approximation was applied to it. This led to the formation of a soliton, and the fluctuations around the soliton were identified with stringy excitations. We point out, however, that in this earlier work, a complete treatment of the ground state of the model was missing. This was due to an unnecessary decompactification of the world sheet; by keeping it compactified, we show that, in addition to a trivial ground state, there

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is also a non-trivial one. We investigate fluctuations around the non-trivial ground state in the limit of a densely populated world sheet, and show string formation in this limit. We also show that this limit can be systematically studied by means of an expansion in terms of a conveniently defined coupling constant.
1 Introduction

The present work can be thought of as a supplement to a previous article [1]: it completes that article by providing a crucial final step that was missing. We could have written a short note on just this final step, but instead we decided on a longer article that aims to be self contained for the convenience of the reader. Sections 1 through 6 are essentially a rewrite of [1] with only a few minor modifications and a section deleted. The reader who is only interested in the new material could skip directly to section 7.

Reference [1] was the culmination of a long development starting with [2]. The idea behind this program was to sum the planar graphs of a field theory on a world sheet parametrized by the light cone variables, based on 't Hooft’s pioneering paper [3]. The original field theory studied in this approach was a scalar with $\phi^3$ interaction, and this was later generalized to more complicated and more interesting models [4, 5]. The model under consideration here is again scalar $\phi^3$ in transverse dimensions $D = 1, 2, 4$. For the sake of brevity, section 7 in [1], where an additional $\phi^4$ interaction was introduced, has been omitted.

The starting point is the world sheet field theory, which reproduces the planar graphs of $\phi^3$ [1]. This theory is based on a complex scalar field and a two component fermion field that live on the world sheet. Using the mean field approximation, solitonic classical solutions on the world sheet were constructed, and a certain set of quantum fluctuations about the solitonic solutions were shown to have a string like spectrum. The solitonic solutions are of interest because they describe a non-perturbative feature of field theory. Also, as we shall see later, the soliton emerges from the summation of a dense set of graphs on the world sheet, which can be thought of as the condensation of these graphs. The existence of such a condensate on the world sheet is naturally expected to lead to a string description, an old idea that motivated some of the early work on this subject [6, 7].

These computations suffer from two kinds of divergences: One of them is the field theoretic ultraviolet divergences, which are eliminated by the standard renormalization procedure. The second one is a spurious infrared divergence due to the choice of the light cone coordinates. In the previous work, this infrared problem was temporarily avoided by the discretizing the $\sigma$ coordinate on the world sheet in steps of $a$, but then, several quantities of physical interest were singular in the limit $a \to 0$. The main result of [1] is that
this singularity is indeed spurious, and it can be eliminated by a mass counter
term. It is surprising and highly satisfying that the same counter terms that
are needed to cancel the ultraviolet mass divergences also automatically cancel
the infrared singularity at \( a = 0 \). The mean field approximation can then
be applied to the continuum limit on the world sheet, without encountering
any problems, except for a log singularity in the coupling constant at \( D = 4 \),
which can be circumvented by coupling constant renormalization. The re-
results about soliton formation and and stringy excitations remain unchaned,
except now they are on a firmer basis.

The continuum limit comes with an additional bonus: The model is now
invariant under the subgroup of Lorentz transformations that preserve the
light cone, including the boost \( K_1 \) along the special direction 1. Invariance
under this boost, broken when the sigma coordinate is discretized, is restored
in the continuum limit. We will always make sure that the approximations
employed in this work preserve this important symmetry.

After these preliminaries, we are ready to discuss the new results of this
paper, stating with section 7. In this section, the ground state of the model
is investigated in the mean field approximation. In this approximation, the
classical Hamiltonian, \( H_c \), depends on two parameters: \( \lambda \) and \( \rho \), or two
convenient combinations of these, \( \tilde{\lambda} \) and \( \tilde{\rho} \) (see eq.(6.1)). \( \lambda \) is a Lagrange
multiplier and \( \rho \) measures the average density of the graphs on the world
sheet. The ground state energy is determined by setting the variation of \( H_c \)
with respect to \( \tilde{\lambda} \) equal to zero, and then minimizing the result with respect
to \( \tilde{\rho} \). We find two different solutions: One of them is a trivial solution, with
\( \rho = 0 \) and an empty world sheet. The other one has \( \rho \neq 0 \) and therefore a
non-trivial world sheet populated with graphs. In this approximation, both
solutions are degenerate with vanishing ground state energy. Of course, both
the solitonic configurations and the resulting string picture exist only in the
non-trivial ground state.

The purpose of the present article is to establish the existence of the non-
trivial ground state and investigate some of its features. Unfortunately, this
possibility was missed in reference [1]. In retrospect, the reason for this is
simple. In the light cone set up, the \( \sigma \) coordinate on the world sheet is com-
 pactified on a circle of circumference \( p^+ \). In [1], the model was decompactified
right from the start by letting \( p^+ \to \infty \), and, as explained in section 7, the
non-trivial ground state is then lost. To avoid \( \rho = 0 \) and the resulting empty
world sheet, one can fix \( \rho \) at a non-zero value by, for example, coupling it
to fixed external source. But this is both artificial and unnecessary; keeping
the world sheet compactified avoids the loss of the non-trivial ground state.

Having constructed the solitonic solutions, in section 8, we study the quantum fluctuations in the solitonic background. Here, we focus exclusively on a particular set of fluctuations that come about because the soliton, having a definite location, breaks the translation symmetry of the model (eq.(5.5)). It is then the standard procedure to introduce collective coordinates corresponding to translations. Upon quantization, these collective modes restore the spontaneously broken translation symmetry. They can therefore be identified as the Goldstone modes, and are expected to dominate the low energy regime.

In [1], it was shown that the spectrum of the fluctuations differ from those of the conventional string theory; the Regge trajectories are no longer linear. Only in the asymptotic limit when the density of graphs tends to infinity, the standard string model with linear trajectories is recovered. In section 9, we investigate a weak coupling expansion around the high density limit systematically by introducing a redefined coupling constant $\beta_D$ as an expansion parameter (eq.(9.5)). We find that in addition to the power series dependence on $\beta_D$ expected from a perturbation expansion, there also exponential factors (eq.(9.6)) that are usually associated with tunneling. The leading term is the usual string action in the light cone picture; the non-leading terms introduce corrections that tend to curve the originally straight string trajectories. We end the section with a conjecture: The exponentially suppressed terms could come from the tunneling between the two ground states. Finally, the last section summarizes our conclusions.

2 The World Sheet Picture

The planar graphs of $\phi^3$ can be represented [3] on a world sheet parameterized by the light cone coordinates $\tau = x^+$ and $\sigma = p^+$ as a collection of horizontal solid lines (Fig.1), where the $n$'th line carries a D dimensional transverse momentum $q_n$. Two adjacent solid lines labeled by $n$ and $n+1$ correspond to the light cone propagator

$$\Delta(p_n) = \frac{\theta(\tau)}{2p^+} \exp \left( -i\tau \frac{p_n^2 + m^2}{2p^+} \right),$$

where $p_n = q_n - q_{n+1}$ is the momentum flowing through the propagator. A factor of the coupling constant $g$ is inserted at the beginning and at the
end of each line, where the interaction takes place. Ultimately, one has to integrate over all possible locations and lengths of the solid lines, as well as over the momenta they carry.

The propagator (2.1) is singular at $p^+ = 0$. It is well known that this is a spurious singularity peculiar to the light cone picture. To avoid this singularity temporarily, it is convenient to discretize the $\sigma$ coordinate in steps of length $a$. A useful way of visualizing the discretized world sheet is pictured in Fig.2. The boundaries of the propagators are marked by solid lines as before, and the bulk is filled by dotted lines spaced at a distance $a$. For convenience, the $\sigma$ coordinate is compactified by imposing periodic boundary conditions at $\sigma = 0$ and $\sigma = p^+$. In contrast, the boundary conditions at $\tau = \pm \infty$ are left arbitrary. In sections 4 and 5, it was shown how to go from a discrete to a continuous world sheet after eliminating the singularity at $p^+ = 0$.

3 The World Sheet Field Theory

It was shown in [8] that the light cone graphs described above are reproduced by a world sheet field theory, which we now briefly review. We introduce the complex scalar field $\phi(\sigma, \tau, q)$ and its conjugate $\phi^\dagger$, which at time $\tau$ annihilate (create) a solid line with coordinate $\sigma$ carrying momentum $q$. They satisfy
the usual commutation relations

$$[\phi(\sigma, \tau, q), \phi^\dagger(\sigma', \tau, q')] = \delta_{\sigma,\sigma'} \delta(q - q').$$  \hfill (3.1)

The vacuum, annihilated by the $\phi$'s, represents the empty world sheet.

In addition, we introduce a two component fermion field $\psi_i(\sigma, \tau), i = 1, 2,$
and its adjoint $\bar{\psi}_i$, which satisfy the standard anticommutation relations. The
fermion with $i = 1$ is associated with the dotted lines and $i = 2$ with the
solid lines. The fermions are needed to avoid unwanted configurations on
the world sheet. For example, multiple solid lines generated by the repeated
application of $\phi^\dagger$ at the same $\sigma$ would lead to overcounting of the graphs.
These redundant states can be eliminated by imposing the constraint

$$\int dq \phi^\dagger(\sigma, \tau, q)\phi(\sigma, \tau, q) = \rho(\sigma, \tau),$$  \hfill (3.2)

where

$$\rho = \bar{\psi}_2\psi_2,$$  \hfill (3.3)

which is equal to one on solid lines and zero on dotted lines. This constraint
ensures that there is at most one solid line at each site.

Fermions are also needed to avoid another set of unwanted configurations.
Propagators are assigned only to adjacent solid lines and not to non-adjacent
ones. To enforce this condition, it is convenient to define,

$$\mathcal{E}(\sigma_i, \sigma_j) = \prod_{k=i+1}^{k=j-1} (1 - \rho(\sigma_k)), $$  \hfill (3.4)
for $\sigma_j > \sigma_i$, and zero for $\sigma_j < \sigma_i$. The crucial property of this function is that it acts as a projection: It is equal to one when the two lines at $\sigma_i$ and $\sigma_j$ are separated only by the dotted lines; otherwise, it is zero. With the help of $\mathcal{E}$, the free Hamiltonian can be written as

$$H_0 = \frac{1}{2} \sum_{\sigma, \sigma'} \int dq \int dq' \frac{\mathcal{E}(\sigma, \sigma')}{\sigma' - \sigma} ((q - q')^2 + m^2) \times \phi^\dagger(\sigma, q)\phi(\sigma, q)\phi^\dagger(\sigma', q')\phi(\sigma', q') + \sum_{\sigma} \lambda(\sigma) \left( \int dq \phi^\dagger(\sigma, q)\phi(\sigma, q) - \rho(\sigma) \right), \quad (3.5)$$

where $\lambda$ is a Lagrange multiplier enforcing the constraint (3.2). The evolution operator $\exp(-i\tau H_0)$, applied to states, generates a collection of free propagators, without, however, the prefactor $1/(2p^+)$.

One can also think of the Lagrange multiplier $\lambda(\sigma, \tau)$ as an Abelian gauge field on the world sheet. The corresponding gauge transformations are [14]

$$\psi \rightarrow \exp \left( -\frac{i}{2} \alpha \sigma_3 \right) \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp \left( \frac{i}{2} \alpha \sigma_3 \right),$$

$$\phi \rightarrow \exp(-i\alpha) \phi, \quad \phi^\dagger \rightarrow \exp(i\alpha) \phi^\dagger,$$

$$\lambda \rightarrow \lambda - \partial_\tau \alpha. \quad (3.6)$$

This gauge invariance comes about because constraint (3.2) is time independent. Using the equations of motion,

$$\partial_\tau \left( \int dq (\phi^\dagger \phi) - \rho \right) = 0,$$

and therefore the constraint is really needed only at a fixed $\tau$, say, as an initial condition. This can be implemented by gauge fixing by requiring $\lambda$ to be independent of the time $\tau$,

$$\lambda(\sigma, \tau) \rightarrow \lambda(\sigma),$$

by a suitable choice of gauge parameter $\alpha$. In this time independent form, which we will assume from now on, $\lambda$ is not a dynamical variable but a convenient tool for implementing the constraint (3.2) on the initial states.
Using (3.2), the free Hamiltonian can be written in a form more convenient for later application:

\[
H_0 = \frac{1}{2} \sum_{\sigma,\sigma'} G(\sigma,\sigma') \left( \frac{1}{2} m_0^2 \rho(\sigma) \rho(\sigma') + \rho(\sigma') \int dq \left( q^2 + \mu^2 \right) \phi^\dagger(\sigma,q)\phi(\sigma,q) \right) \\
- \int dq \int dq' (q \cdot q') \phi^\dagger(\sigma,q)\phi(\sigma,q)\phi^\dagger(\sigma',q')\phi(\sigma',q') \\
+ \sum_\sigma \lambda(\sigma) \left( \int dq \phi^\dagger(\sigma,q)\phi(\sigma,q) - \rho(\sigma) \right),
\]

(3.7)

where we have defined

\[
G(\sigma,\sigma') = \frac{\mathcal{E}(\sigma,\sigma') + \mathcal{E}(\sigma',\sigma)}{|\sigma - \sigma'|}.
\]

(3.8)

There is a redundancy in the above equation: the mass is split into two pieces according to

\[
m^2 = m_0^2 + \mu^2.
\]

This redundancy will prove useful later on.

Next, we introduce the interaction term. Two kinds of interaction vertices, corresponding to \(\phi^\dagger\) creating a solid line or \(\phi\) destroying a solid line, are pictured in Fig.3.

The interaction term in the Hamiltonian, including the prefactors of the
form $1/(p^+)$ in (2.1), can now be written as

$$H_I = g \sqrt{a} \sum_{\sigma} \int dq \left( \mathcal{V}(\sigma) \rho_+^{(\sigma)} \phi(\sigma, q) + \rho_-^{(\sigma)} \mathcal{V}(\sigma) \phi^\dagger(\sigma, q) \right), \quad (3.9)$$

where $g$ is the coupling constant, and

$$\mathcal{V}(\sigma) = \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 > \sigma} \frac{W(\sigma_1, \sigma_2)}{\sqrt{(\sigma - \sigma_1)(\sigma_2 - \sigma_1)(\sigma_2 - \sigma)}}, \quad (3.10)$$

where,

$$W(\sigma_1, \sigma_2) = \rho(\sigma_1) E(\sigma_1, \sigma_2) \rho(\sigma_2). \quad (3.11)$$

and

$$\rho_+ = \bar{\psi}_1 \psi_2, \quad \rho_- = \bar{\psi}_2 \psi_1. \quad (3.12)$$

A detailed explanation of the origin of various terms in $H_I$ was given in [8].

The total Hamiltonian is given by

$$H = H_0 + H_I \quad (3.13)$$

and the corresponding action by

$$S = \int d\tau \left( \sum_{\sigma} \left( i \bar{\psi} \partial_\tau \psi + i \int dq \phi^\dagger \partial_\tau \phi \right) - H(\tau) \right). \quad (3.14)$$

4 Classical Solutions And Mass Renormalization

In this section, we look for classical solutions to the equations motion resulting from the above action. However, it was pointed out in [9] that treating the $\rho$’s as classical fields is problematic. It implies factorization of the expectation values of the products of the $\rho$’s, which violates the spin algebras they satisfy:

$$\rho^2(\sigma) = \rho(\sigma), \quad \rho_+(\sigma)\rho_-(\sigma) = 1 - \rho(\sigma), \quad \rho_-(\sigma)\rho_+(\sigma) = \rho(\sigma). \quad (4.1)$$
From the spin algebra, one can derive the overlap relations
\[
G(\sigma, \sigma') \rho(\sigma') \rho_- (\sigma) W(\sigma_1, \sigma_2) = \\
\left( \delta_{\sigma', \sigma_2} \frac{1}{\sigma_2 - \sigma} + \delta_{\sigma', \sigma_1} \frac{1}{\sigma - \sigma_1} \right) \rho_- (\sigma) \ W(\sigma_1, \sigma_2), \quad (4.2)
\]
and
\[
W(\sigma_1, \sigma_2) \rho_+ (\sigma) \rho_- (\sigma) W(\sigma'_1, \sigma'_2) = \delta_{\sigma_1, \sigma'_1} \delta_{\sigma_2, \sigma'_2} \ W(\sigma_1, \sigma_2). \quad (4.3)
\]

These overlap relations turn out to be crucial for the elimination of both ultraviolet divergences and the singularity at \(a = 0\), which is the reflection of the original \(p^+ = 0\) singularity in the propagator (2.1). If present, this singularity would prevent us from taking the continuum limit of the model.

In the classical approximation, operators are replaced by their expectation values. However, this violates the overlap relations. To overcome this problem, we treat the \(\phi\)'s as classical fields, but keep the \(\rho\)'s as operators satisfying eqs.(4.2, 4.3) in the intermediate stages of the computation. The strategy is first to simplify the expressions as much as possible using the overlap relations before making any approximations.

We will now search for solutions \(\phi_0(\sigma, q)\) that are time independent (solitonic) and whose dependence on \(q\) is rotationally invariant. The equation motion for \(\phi_0\) then simplifies to
\[
\left( 2\lambda(\sigma) + \sum_{\sigma'} G(\sigma, \sigma') \rho(\sigma') (q^2 + \mu^2) \right) \phi_0(\sigma, q) = 2g\sqrt{a} \rho_- (\sigma) \mathcal{V}(\sigma). \quad (4.4)
\]

To solve this equation, we make the following ansatz for \(\phi_0\):
\[
\phi_0(\sigma, q) = \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 > \sigma} \rho_- (\sigma) W(\sigma_1, \sigma_2) \tilde{\phi}_0(\sigma, \sigma_1, \sigma_2, q). \quad (4.5)
\]
where \(\tilde{\phi}\) is a c-number and all the operator dependence is in \(\rho_- W\). Using the overlap relations, the solution for \(\tilde{\phi}_0\) is given by
\[
\tilde{\phi}_0(\sigma, q) = - \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 > \sigma} \frac{2g\sqrt{a}}{2\lambda(\sigma) + (q^2 + \mu^2) \left( \frac{\sigma_2 - \sigma}{\sigma - \sigma_1} \right)} \\
\times \frac{1}{\sqrt{(\sigma - \sigma_1)(\sigma_2 - \sigma_1)(\sigma_2 - \sigma)}}. \quad (4.6)
\]
and the solution for $\phi_0^\dagger$ is the Hermitian conjugate expression.

Next, we define $H_c$ by replacing $\phi$ by the above $\phi_0$ in the Hamiltonian,

$$H_c = H(\phi = \phi_0),$$

and simplify again using the overlap relations until we have a linear result in $W$:

$$H_c = -2g^2 a \sum_\sigma \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 < \sigma} \int dq W(\sigma_1, \sigma_2) \times \left( (\sigma - \sigma_1)(\sigma_2 - \sigma_1)(\sigma_2 - \sigma) \left( 2\lambda(\sigma) + (q^2 + \mu^2)\frac{\sigma_2 - \sigma_1}{(\sigma_2 - \sigma)(\sigma - \sigma_1)} \right) \right)^{-1} - \sum_\sigma \lambda(\sigma) \rho(\sigma) + \frac{m_0^2}{2} \sum_{\sigma' > \sigma} W(\sigma, \sigma') \sigma' - \sigma. \quad (4.7)$$

In the above expression, the integral over $q$ is ultraviolet divergent at $D = 2$ and $D = 4$. This divergence can be eliminated by the mass renormalization and at $D = 4$ by also coupling constant renormalization. We observe that as $|q| \to \infty$, the first term on the right, after doing the sum over $\sigma$, reaches a limit identical in form to the mass term. It can therefore be cancelled by setting

$$m_0^2 = 4g^2 a \int dq \frac{1}{q^2 + \mu^2}. \quad (4.8)$$

We note that at $D = 2$, there is no divergence, and at $D = 4$, a quadratic divergence is reduced to a logarithmic divergence in the coupling constant. Although there is no divergence at $D = 1$, we will still use the same expression for $m_0$ in this case.

At the beginning, we started with two independent masses in the problem. But now that $m_0$ is fixed, only $\mu$ remains. We could have given a treatment based on a single mass from the start, however, having an extra mass temporarily is more convenient. For example, it enables us to give a uniform treatment for all dimensions.

Up to this point, the world sheet is still discrete, and the continuum limit $a \to 0$ is problematic. This problem will be addressed in the next section.
The Continuum Limit

The continuum limit is taken by letting \(a \to 0\), after suitably scaling the field variables by
\[
\phi \to \sqrt{a} \phi, \quad \psi \to \sqrt{a} \psi.
\]

From its definition, \(\rho\) scales as
\[
\rho \to a \rho.
\]

In this limit, all the sigma sums become integrals, and all the factors of \(a\) are used up in this process. Also, the product in the definition of \(E\) (3.4) becomes
\[
E(\sigma_1, \sigma_2) = \prod_{\sigma_1} (1 - a \rho(\sigma)) \to \exp \left( - \int_{\sigma_1}^{\sigma_2} d\sigma \rho(\sigma) \right).
\]

After a change of variables by
\[
\sigma = \sigma_1 + x (\sigma_2 - \sigma_1),
\]

\(H_c\) can be written as
\[
H_c = -2g^2 \int d\sigma_2 \int_{\sigma_1}^{\sigma_2} d\sigma_1 \int_0^1 dx \int dq \rho(\sigma_1) E(\sigma_1, \sigma_2) \rho(\sigma_2) \\
\times \left( (\sigma_2 - \sigma_1) (2\lambda(\sigma) x (1-x) (\sigma_2 - \sigma_1) + (q^2 + \mu^2)) \right)^{-1} \\
+ 2g^2 \int d\sigma_2 \int_{\sigma_1}^{\sigma_2} d\sigma_1 \int dq \frac{1}{q^2 + \mu^2} \frac{\rho(\sigma_1) E(\sigma_1, \sigma_2) \rho(\sigma_2)}{\sigma_2 - \sigma_1} \int d\sigma \lambda(\sigma) \rho(\sigma).
\]

The first and the second terms on the right are divergent as \(|q| \to \infty\) at \(D = 2, 4\), and also they are also logarithmically divergent as \(\sigma_2 - \sigma_1 \to 0\). The first is the ultraviolet mass divergence and we have already fixed \(m_0\) by eq.(4.8) so that it cancels between the two terms. The second singularity is a logarithmic singularity at \(\sigma_2 - \sigma_1 = 0\). Since \(\sigma_2 - \sigma_1\) is the \(p^+\) flowing through the propagator, this is the \(p^+ = 0\) singularity in disguise. Surprisingly, this divergence also cancels between the first and second terms in all dimensions. It is highly satisfying that the mass counter term introduced to eliminate an ultraviolet divergence also automatically cancels the infrared divergence at
\(p^+ = 0\). This cancellation is quite non-trivial and absolutely essential, since otherwise, having only one adjustable constant \(m_0\) at our disposal, we would be stuck with one divergence or other at \(D = 2, 4\). We also note that we cannot add an arbitrary ultraviolet finite term to \(m_0^2\) without spoiling the infrared cancellation. Although we started with two masses, in the end only \(\mu\) remains as an arbitrary parameter.

Another important feature of \(H_c\) is its symmetries. In addition to translation invariance in \(q\)
\[
q \rightarrow q + r,
\]
the light cone dynamics is manifestly invariant under a subgroup of Lorentz transformations. The original action (3.14) is trivially invariant under all the generators of this subgroup except for the generator \(K_1\) of boosts along the special direction 1. The discretization of the \(\sigma\) coordinate breaks this symmetry even at the classical level. We expect this symmetry will be at least classically restored in the continuum limit. To see this, we note that under \(K_1\), various fields transform as
\[
\phi(\sigma, \tau, q) \rightarrow \sqrt{u} \phi(u\sigma, u\tau, q), \quad \psi(\sigma, \tau) \rightarrow \sqrt{u} \psi(u\sigma, u\tau),
\]
\[
\rho(\sigma, \tau) \rightarrow u \rho(u\sigma, u\tau), \quad \lambda(\sigma, \tau) \rightarrow u \lambda(u\sigma, u\tau), \quad p^+ \rightarrow \frac{1}{u} p^+,
\]
where \(u\) parametrizes the \(K_1\) transformations. In the expression for \(H_c\), this amounts to letting
\[
\sigma \rightarrow u \sigma, \quad \tau \rightarrow u \tau,
\]
and transforming \(\rho\) according to eq.(31). The classical Hamiltonian then transforms as
\[
H_c \rightarrow u H_c,
\]
and as expected, the corresponding action is therefore invariant. As we shall see, this invariance will be respected by the mean field approximation, and it will play an important role in what follows.

Eq.(5.4), which is free of divergences and independent of \(a\), will be the starting point of the mean field approximation in the next section.

6 The Meanfield Approximation

The mean field approximation consists of replacing \(\rho\) and \(\lambda\) in \(H_c\) by their ground state expectation values, which we assume to be independent of \(\sigma\) and
\( \tau \). (translation invariance of the ground state). Afterwards, the equation of motion with respect to the gauge fixed \( \lambda \) should be imposed as a constraint, and the resulting \( H_c \) should be minimized with respect to \( \rho \) to find the ground state. We remind the reader that this is the standard procedure in fixing an axial gauge: The equations of motion with respect to gauge fixed variable are imposed as constraints.

In eq.(5.4), the \( q \) integration can be done, and the result can be simplified by the following change of variables:

\[
\tilde{\lambda} = \lambda / (\rho \mu^2), \quad \sigma = \sigma' / \rho, \quad \tilde{\rho} = \rho p^+.
\]

(6.1)

These variables have advantage of being both invariant under \( K_1 \) and scale independent. Also \( \tilde{\rho} \) is a physically significant variable; it counts the number of solid lines and hence the number of propagators on the world sheet. \( p^+ \) and \( \rho \) separately are not physically meaningful: They depend on the choice of the Lorentz frame since they are not \( K_1 \) invariant.

In terms of these new variables, the classical Hamiltonian for various transverse dimensions \( D \) can then be written as

\[
p^+ H_c = \tilde{\rho}^2 F_D(\tilde{\lambda}, \tilde{\rho}),
\]

(6.2)

where,

\[
F_D = \mu^2 \left( -\tilde{\lambda} + \alpha_D \int_0^{\tilde{\rho}} d\sigma' \int_0^1 dx \frac{\exp(-\sigma')}{\sigma'} L_D(x, \sigma', \tilde{\lambda}) \right),
\]

(6.3)

with

\[
\alpha_1 = 2\pi g^2 / \mu^3, \quad \alpha_2 = 2\pi g^2 / \mu^2, \quad \alpha_4 = 2\pi^2 g^2,
\]

(6.4)

and,

\[
\begin{align*}
L_1 &= 1 - \frac{1}{\left(1 + 2\tilde{\lambda} x(1 - x) \sigma' \right)^{1/2}}, \\
L_2 &= \ln \left(1 + 2\tilde{\lambda} x(1 - x) \sigma' \right), \\
L_4 &= 2\tilde{\lambda} x(1 - x) \sigma' \ln \left(\Lambda^2 / \mu^2 \right) \\
&\quad - \left(1 + 2\tilde{\lambda} x(1 - x) \sigma' \right) \ln \left(1 + 2\tilde{\lambda} x(1 - x) \sigma' \right).
\end{align*}
\]

(6.5)

In the last equation, \( \Lambda \) is an ultraviolet cutoff. These equations fix \( H_c \) in terms of dimensionless coupling constants \( \alpha_{1,2} \) at \( D = 1, 2 \). At \( D = 4 \),
the expression for $L_4$ has a logarithmic dependence on the cutoff $\Lambda$. This is related to coupling constant renormalization. We recall that $\phi^3$ is asymptotically free in 6 space-time dimensions ($D = 4$), and the above relation is the well known lowest order renormalization group result obtained by summing the leading logarithmic divergences in the perturbation series. To get a finite result, one should first renormalize the coupling constant before summing the logs. This amounts to replacing the cutoff $\Lambda$ by a large but finite value. The coupling constant on the left should then be identified with the running coupling constant $g(\Lambda)$, defined at the energy scale $\Lambda$. For this leading log. approximation to be reliable, $g(\Lambda)$ should be small, which means that $\Lambda^2/\mu^2$ should be large. All the additional terms on the right hand side only make a small change in the scale of the running coupling constant. From now on, we will only keep the leading first term for $L_4$.

7 The Ground State

We will now investigate the ground state of the model in various dimensions, using the meanfield approximation developed in the last section. We remind the reader that $\tilde{\lambda}$ and $\tilde{\rho}$ are taken to be constants independent of $\sigma$ and $\tau$, and the equation

$$\frac{\partial H_c}{\partial \tilde{\lambda}} = 0$$

(7.1)

is imposed as a constraint. Since $H_c$ is proportional to $\tilde{\rho}^2$, this equation always has the trivial solution

$$\tilde{\rho} = 0, \; H_c = 0.$$  

(7.2)

This corresponds to an uninteresting empty world sheet.

We will now show that there is another more interesting solution with

$$\tilde{\rho} \neq 0.$$

and with again

$$H_c = 0.$$  

This non-trivial ground state, degenerate in energy with the trivial one, corresponds to a world sheet populated with Feynman graphs. This solution is obtained by setting

$$\frac{\partial F_D}{\partial \tilde{\lambda}} = 0.$$  

(7.3)
We will now study this equation for various $D$. Starting with $D = 1$, it reduces to
\[
\alpha_1 \int_0^\tilde{\rho} d\sigma' \int_0^1 dx \frac{x(1-x) \exp(-\sigma')}{\left(1 + 2\tilde{\lambda} x(1-x) \sigma'\right)^{3/2}} = 1. \tag{7.4}
\]

Now a few comments:
a) Because of this constraint, we are left with only one independent variable, which we take to be $\tilde{\rho}$. $\tilde{\lambda}$ is treated as a function of $\tilde{\rho}$.
b) Both $\tilde{\rho}$ and $\tilde{\lambda}$ are positive semi-definite; the first by definition and the other by virtue of the above equation.
c) The left hand side is an increasing function of $\tilde{\rho}$ and a decreasing function of $\tilde{\lambda}$. It is then easy to see that the minimum value of $\tilde{\lambda}$,
\[
\tilde{\lambda} = 0, \tag{7.5}
\]
corresponds also to the minimum value of $\tilde{\rho}$, which we label $\rho_1^c$ (1 refers to D). Solving (7.4) for $\tilde{\rho}$ at $\tilde{\lambda} = 0$, we have,
\[
\rho_1^c = -\ln \left(1 - \frac{6}{\alpha_1}\right). \tag{7.6}
\]
For this solution to exist, $\alpha_1$ must satisfy
\[
\alpha_1 > 6. \tag{7.7}
\]
Clearly, this corresponds to the strong coupling regime.
d) $\tilde{\lambda}$ is a monotonically increasing function of $\tilde{\rho}$. As $\tilde{\rho}$ ranges from $\rho_1^c$ to $\infty$, $\tilde{\lambda}$ ranges from 0 to $\infty$.

Next, we show that $F_1$ is also a monotonically increasing function $\tilde{\rho}$, and therefore, its minimum is at $\tilde{\rho} = \rho_1^c$, the minimum value of $\tilde{\rho}$. Differentiating $F_1$ (eq.(6.3)) with respect to $\tilde{\rho}$ and remembering that $\tilde{\lambda}$ is a function of $\tilde{\rho}$ through eq.(7.4), we have
\[
\frac{dF_1}{d\tilde{\rho}} = \mu^2 \alpha_1 \frac{\exp(-\tilde{\rho})}{\tilde{\rho}} \int_0^1 dx \left(1 - \left(1 + 2\tilde{\lambda} x(1-x) \tilde{\rho}\right)^{-1/2}\right). \tag{7.8}
\]
Since the right hand side is positive for
\[
\tilde{\lambda} > 0,
\]
it follows that

$$\frac{dF_1}{d\tilde{\rho}} > 0$$

for

$$\tilde{\rho} > \rho_c^1.$$

Finally, it is easy to show that since $\tilde{\lambda}$ vanishes at $\tilde{\rho} = \rho_c^1$, both $F_1$ and its derivative with respect to $\tilde{\rho}$ vanish at the same point.

Having shown that $F_1(\tilde{\rho})$ has a global minimum at

$$\tilde{\rho} = \rho_c^1,$$

with

$$F_1(\rho_c^1) = 0,$$  \hspace{1cm} (7.9)

we will now show that $H_c$ also has a vanishing minimum at the same point.

From eq.(6.2), $p^+ H_c$ is the product of $F_1(\tilde{\rho})$ and $\tilde{\rho}^2$. Since both factors reach their minimum at $\tilde{\rho} = \rho_c^1$, $H_c$ also reaches its minimum value zero at the same point. Being a global minimum, this corresponds to a stable ground state within the parameter space we have been considering. Of course, this is only a classical result; quantum fluctuations could destabilize it.

Next, we consider $D = 2$, which can be treated in exactly same fashion as $D = 1$, with only some obvious minor changes. Eq.(7.4) is now replaced by

$$\alpha_2 \int_0^{\tilde{\rho}} d\sigma' \int_0^1 dx \frac{x(1-x) \exp(-\sigma')} {1 + 2\tilde{\lambda} x (1-x) \sigma'} = 1. \hspace{1cm} (7.10)$$

We can repeat the argument following eq.(7.4), with the only change that the minimum value of $\tilde{\rho}$ is now

$$\rho_c^2 = - \ln \left(1 - \frac{6}{\alpha_2}\right),$$ \hspace{1cm} (7.11)

and for a solution to exist, $\alpha_2$ must be greater than 6.

The results following (7.4) are still valid, but eq.(7.8) is now replaced by

$$\frac{dF_2}{d\tilde{\rho}} = \mu^2 \alpha_2 \frac{\exp(-\tilde{\rho})} {\tilde{\rho}} \int_0^1 dx \ln \left(1 + 2\tilde{\lambda} x (1-x) \tilde{\rho}\right). \hspace{1cm} (7.12)$$

From this equation, one can easily show that, replacing $F_1$ by $F_2$, the argument following (7.9) is still valid, and therefore

$$\tilde{\rho} = \rho_c^2$$
corresponds to a stable classical ground state.

Finally, we will briefly discuss the $D = 4$ case. Because of the running coupling constant, there are additional complications compared to $D = 1, 2$, and our treatment will be less complete. Eq.(7.3) at $D=4$ gives

$$\tilde{\rho} = \rho_4^c = -\ln \left( 1 - \frac{3}{\bar{\alpha}_4} \right),$$

(7.13)

where

$$\bar{\alpha}_4 = \alpha_4 \ln \left( \frac{\Lambda^2}{\mu^2} \right),$$

and we have kept only the leading log term. In this case, since $H_c$ is linear in $\lambda$, this variable acts as a Lagrange multiplier, $\tilde{\rho}$ is fixed at $\rho_4^c$, and no fluctuations are allowed. $\lambda$ remains arbitrary, and the classical energy is again zero. Although we will not pursue it further here, higher order corrections could easily change this picture.

We now return to the question of why the non-trivial ground state corresponding to $\tilde{\rho} \neq 0$ was missed in reference [1]. As explained in the Introduction, this was because, in [1], only the decompactified model, with

$$p^+ \to \infty,$$

and consequently,

$$\tilde{\rho} \to \infty$$

was studied. Actually, $\tilde{\rho} \to \infty$ is not a solution for the ground state, but the asymptotic limit of the ground states described by eqs.(7.6), (7.11) and (7.13) as $\alpha_D$ tends to its limiting values

$$\alpha_{1,2} \to 6$$

(7.14)

for $D = 1, 2$ and,

$$\bar{\alpha}_4 \to 3,$$

(7.15)

for $D = 4$. By setting $\tilde{\rho} = \infty$ from the very beginning, this subtle point was missed in [1]. We will study this interesting asymptotic limit, which we call the high density limit, in the following sections.
8 Fluctuations Of The Transverse Momentum Around The Classical Background

Given the classical solutions developed in the previous sections, it is natural to study quantum fluctuations about these backgrounds. This can be done explicitly to quadratic order for all the fluctuations. We will, instead, focus on a particular set of fluctuations; namely, the fluctuations of the transverse momentum $q$, which can be studied by quantizing the collective coordinates corresponding to the breaking of the translation invariance of $q$ (eq.(5.5)).

The classical solution, placed at a definite location in the $q$ space, breaks this symmetry, and it is restored by quantizing the so-called collective modes. These modes are very important not only for their role in restoring translation invariance, but also, because, they are the low lying Goldstone modes connected with the spontaneously broken translation symmetry. Also, they were crucial to the formation of a string on the world sheet.

The collective coordinate corresponding to translations is introduced by letting

$$\phi = \phi_0 + \phi_1,$$  \hspace{1cm} (8.1)

where $\phi_1$ is the fluctuating part of the field, and setting,

$$\phi_1(\sigma, \tau, q) = \phi_0(\sigma, q + v(\sigma, \tau)) - \phi_0(\sigma, q),$$  \hspace{1cm} (8.2)

where $\phi_0$ is the classical solution and $v$ is the collective coordinate. The contribution of $\phi_1$ to the action can be written as the sum of kinetic and potential terms:

$$S^{(1)} = S_{k.e} - \int d\tau H_0(\phi_1) = S_{k.e} + S_{p.e},$$  \hspace{1cm} (8.3)

where the kinetic term depends on $\partial_\tau v$ and the potential has no $\tau$ derivatives. We note that only $H_0$ contributes to $S_{p.e}$; so substituting the ansatz (8.2) directly into $H_0$ (eq.(3.7)) and simplifying, we have the following result for all $D$:

$$S_{p.e} = -\frac{1}{4} \int d\tau \int d\sigma \int d\sigma' \frac{W(\sigma, \sigma')}{|\sigma - \sigma'|} (v(\sigma, \tau) - v(\sigma', \tau))^2.$$  \hspace{1cm} (8.4)

We note that so far no approximation was made, and therefore, this result is exact so long as only the contribution of the collective coordinate $v$ is
concerned. Also, there is no singularity at \( \sigma = \sigma' \) and so there is no obstacle to taking the continuum limit immediately. At this point, we introduce the mean field approximation by setting \( \rho p^+ = \rho_D \), its ground state value, and change variables by \( \sigma' = \sigma + z p^+ \).

\[
S_{p.e} \rightarrow -\frac{(\rho_D^c)^2}{2(p^+)^2} \int d\tau \int_0^{p^+} d\sigma \int_0^1 dz \frac{\exp(-\rho_D^c z)}{z} \left( v(\sigma + z p^+, \tau) - v(\sigma, \tau) \right)^2.
\]

(8.5)

We will study this action in detail later on, but before that, we turn our attention to the kinetic energy term. To compute this term to quadratic order in \( \partial_\tau v \), one has to split \( \phi_1 \) into its real and imaginary (Hermitian and anti-Hermitian) parts:

\[
\phi_1 = \phi_{1,r} + \phi_{1,i},
\]

(8.6)

and eliminate one of them by integrating over it. In this case, since the classical solution \( \phi_0 \) is real, \( \phi_{1,i} \) will be integrated out. The kinetic energy term in the action (3.14) can then be rewritten as

\[
i \sum_{\sigma} \int d\tau \int d\mathbf{q} \phi_1^+ \partial_\tau \phi = 2 \sum_{\sigma} \int d\tau \int d\mathbf{q} \phi_{1,i} \partial_\tau \phi_{1,r}
\]

\[
\rightarrow 2 \sum_{\sigma} \int d\tau \int d\mathbf{q} \phi_{1,i} \partial_\tau \phi_0(\sigma, \mathbf{q} + v(\sigma, \tau)).
\]

(8.7)

Integrating over \( \phi_{1,i} \) then amounts to solving the equations of motion for \( \phi_{1,i} \) and substituting in the action. The left hand side of the equation of motion is the same as in (4.4), but the right hand side comes from the variation of the above kinetic term with respect to \( \phi_{1,i} \):

\[
\left( 2\lambda(\sigma) + \sum_{\sigma'} G(\sigma, \sigma') \rho(\sigma') (\mathbf{q}^2 + \mu^2) \right) \phi_{1,i}(\sigma, \tau, \mathbf{q}) = 2\partial_\tau \phi_0(\sigma, \mathbf{q} + v(\sigma, \tau)).
\]

(8.8)

This equation can be solved by letting

\[
\phi_{1,i}(\sigma, \tau, \mathbf{q}) = \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 < \sigma} \rho_-(\sigma) W(\sigma_1, \sigma_2) \bar{\phi}_{1,i}(\sigma, \sigma_1, \sigma_2, \tau, \mathbf{q}),
\]

(8.9)

as in (4.5). Following the same steps as before, this can then be simplified
using the overlap relations, and after some algebra, we have the solution
\[\tilde{\phi}_{1,i}(\sigma, \sigma_1, \sigma_2, \tau) = \frac{2 \partial_{\tau} v(\sigma, \tau) \cdot \nabla_{\sigma} \tilde{\phi}_0(\sigma, \sigma_1, \sigma_2, \tau, q)}{2 \lambda(\sigma) + (q^2 + \mu^2) \left(\frac{\sigma_2 - \sigma_1}{(\sigma_2 - \sigma)(\sigma_2 - \sigma_1)}\right)} , \tag{8.10}\]
where \(\tilde{\phi}_0\) is given by (4.6). It is now easy to take the continuum limit, and apply the mean field approximation by replacing \(\lambda\) and \(\rho\) by their ground state values
\[\lambda \rightarrow 0, \quad \rho \rightarrow \rho_D^c/p^+ .\]
We skip the intermediate steps give the final result for only \(D = 1, 2:\)
\[S_{k.e} = \int d\tau \int_0^{p^+} d\sigma \frac{1}{2} E(\rho_D^c) (\partial_{\tau} v(\sigma, \tau))^2 , \tag{8.11}\]
where,
\[E = \frac{128}{D} g^2 \int_0^{p^+} dy \int_0^y dx \int dq \frac{x^2 (y - x)^2 (\rho_D^c)^2 q^2}{y^4 (q^2 + \mu^2)^5} \exp(-\rho_D^c y) = \frac{\alpha_D C_D}{\mu^4} (1 - (1 + \rho_D^c) \exp(-\rho_D^c)), \tag{8.12}\]
and,
\[C_1 = \frac{1}{12}, \quad C_2 = \frac{4}{45} .\]

9 String Formation In The High Density Limit

In this section, we will study the spectrum of the collective coordinate \(v\), with the action given by the sum of \(S_{p.e}\) (eq.(8.4)) and \(S_{k.e}\) (eqs.(8.11, 8.12)). This is a free field theory and therefore it is exactly solvable. In fact, without any further approximations, the spectrum of \(S_{p.e}\) was determined in [1]. Here, we will only consider the high density (large \(\rho_D^c\)) limit, which corresponds to the coupling constants approaching the bound given by eq.(7.15). In this limit, we expand the term involving \(v\) in eq.(8.5) in powers of \(z\) (derivative expansion):
\[v(\sigma + z p^+, \tau) - v(\sigma, \tau) = z p^+ \partial_{\sigma} v(\sigma, \tau) + \cdots . \tag{9.1}\]
Keeping only the leading term in the expansion and adding the kinetic energy term results in the action

\[
S^{(1)} \to \frac{1}{2} \left( 1 - (1 + \rho_D^c) \exp(-\rho_D^c) \right) \\
\times \int d\tau \int_0^{p^+} \left( \frac{\alpha_D C_D}{\mu^4} \left( \partial_{\sigma} \mathbf{v}(\sigma, \tau) \right)^2 - \left( \partial_{\tau} \mathbf{v}(\sigma, \tau) \right)^2 \right). \tag{9.2}
\]

This is the string action in the lightcone picture with the slope

\[\alpha' = \left( \frac{\alpha_D C_D}{2\pi^2 \mu^4} \right)^{1/2}.\]

We would like to emphasize that only the leading term was kept in the derivative expansion; the inclusion of the higher order terms produces deviations from the string picture by introducing higher derivatives in \(\sigma\) which tend to curve the string trajectories. To see this, we exhibit the next order term:

\[
S^{(2)}_{p.e} = -\frac{(p^+)^2}{8(\rho_D^c)^2} \left( 6 - (6 + 6\rho_D^c + 3(\rho_D^c)^2 + (\rho_D^c)^3) \exp(-\rho_D^c) \right) \\
\times \int d\tau \int_0^{p^+} d\sigma \left( \partial_{\sigma}^2 \mathbf{v}(\sigma, \tau) \right)^2. \tag{9.3}
\]

Let us compare this to the leading term (eq.(9.2)), neglecting terms exponentially suppressed in \(\rho_D^c\). We have

\[
S^{(2)}_{p.e} \approx \frac{3(p^+)^2}{2(\rho_D^c)^2} S^{(1)}_{p.e}. \tag{9.4}
\]

We note the two additional derivatives in \(S^{(2)}_{p.e}\) compared to \(S^{(1)}_{p.e}\) and the extra factor of

\[
\frac{(p^+)^2}{(\rho_D^c)^2}
\]
on the right. In fact, it is easy to show that, apart from numerical factors, each extra derivative with respect to \(\sigma\) goes with a factor of \(p^+/\rho_D^c\). The factor of \(p^+\) is needed for invariance under \(K_1\), and \(1/\rho_D^c\) can be associated with a perturbative expansion in a new coupling constant \(\beta_D\) defined by

\[
\rho_D^c = 1/(\beta_D)^2. \tag{9.5}
\]
An expansion in powers of \((\beta_D)^2\) coincides with the derivative expansion around the high density limit \(\rho_D = 0\).

Such a perturbative treatment of the model is an attractive possibility; however, we have now consider the so far neglected exponential factor

\[
\exp(-\rho_D) = \exp\left(-\frac{1}{(\beta_D)^2}\right).
\]

This clearly not perturbative but looks very much like the tunneling factors familiar from instanton calculations. What is missing is the physical picture of tunneling: Between which two or more states does the tunneling take place? An natural conjecture is to identify these states with the two (trivial and non-trivial) ground states. However, so far we have not been able to construct an instanton configuration that connects them.

\section{Conclusions}

As emphasized in the introduction, the present paper supplements reference [1] by providing an important missing step. The main contribution of that reference was a singularity free treatment of scalar \(\phi^3\) on the world sheet. In particular, by eliminating the singularity at \(p^+ = 0\), the original discretized world sheet could be replaced by a continuous one. What was missing was a complete treatment of the ground state of the model. As explained in the text, this was due to an unnecessary decompactification of the \(\sigma\) coordinate on the world sheet. By keeping the model compactified, we show here that there are two ground states: One of them corresponds to a trivial empty world sheet, and the other to a non-trivial populated world sheet. They are degenerate at zero energy.

In [9, 10, 1], it was shown that a densely populated world sheet leads to string formation. To investigate this high density limit more systematically, we consider an expansion in terms of a redefined coupling constant, and show that the leading term in this expansion reproduces the light cone string action. This expansion has the interesting feature that, in addition to the usual perturbative terms, it has also exponentially suppressed terms. We speculate that these terms may arise from tunneling between the two ground states.

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