Spinor Residue Family Operators and Spectral Theory of Dirac Operator for Poincaré-Einstein Metrics

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Abstract

We study conformal $Spin$-subgeometry of submanifolds in a semi-Riemannian $Spin$-manifold, focusing on conformal $Spin$-manifolds $(M, [h])$ and their Poincaré-Einstein metrics $(X, g_+)$. Our approach is based on the spectral theory of Dirac operator in the ambient $Spin$-manifold, and associated spinor valued meromorphic family of distributions with residues given by the residue family operators $\mathcal{B}_N^{m}(k; \lambda)$ on spinors. We develop basic aspects and properties of $\mathcal{B}_N^{m}(h; \lambda)$ including conformal covariance, factorization properties by conformally covariant operators for both flat and curved semi-Riemannian $Spin$-manifolds, and Poisson transformation.

Keywords: Conformal semi-Riemannian $Spin$-geometry and subgeometry, Spectral theory of Dirac operator, Invariant distributions, Poisson transform, Conformal powers of the Dirac operator.

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1 Introduction

The fact that the orthogonal Lie group may be regarded as the isometry group of a space and at the same time as the conformal group of another space has far-reaching consequences in geometrical analysis, representation theory and topology of manifolds.

The representative example of this phenomenon is given by conformally equivariant operators for principal series representations on the conformal sphere $S^n$, which can be interpreted as scattering operators on the hyperbolic space; the Poisson transform allows to realize tensor-spinors on compactified boundary (the conformal sphere) of the hyperbolic space as asymptotics of eigenspaces for the algebra of invariant differential operators in the interior of conformal compactification, cf. [He70].

A curved generalization of the previous example is called ambient metric, see [FG11]. It associates to a conformal manifold $(M, [h])$ of dimension $n \geq 3$ a pseudo-Riemannian Ricci flat ambient manifold of dimension $n + 2$, so that conformal invariants of $(M, [h])$ are induced by pseudo-Riemannian invariants of the ambient metric. In particular, the ambient metric allows to construct conformally covariant operators including GJMS operators as conformal modifications.
of powers of the Laplace operator, \([GJMS92, GZ03, GP03]\), and conformal powers of the Dirac operator, \([HS01, GMP12, Fis13]\). In the ambient space there is a Lorentzian hypersurface \(X = M \times (0, \varepsilon)\), and the ambient metric induces on \(X\) an Einstein metric \(g_+\) termed Poincaré-Einstein metric. Thus \((M, [h])\) is realized as the conformal infinity of the Poincaré-Einstein metric \((X, g_+)\) and the geometric scattering theory for the Poincaré-Einstein metric produces the scattering operator \(S(h, \lambda), [GZ03, GMP10]\), fulfilling covariant transformation property in the conformal class \([h]\).

Beside the class of conformally covariant operators acting on sheaves over the same base manifold \(M\) like the scattering or GJMS operators mentioned above, there is a sequence of 1-parameter families of differential operators depending on \(\lambda \in \mathbb{C}\) and called residue families, \(D_{\lambda}^{res}(h, \lambda) : \mathcal{C}^\infty (M \times [0, \varepsilon)) \rightarrow \mathcal{C}^\infty (M),\) \(N \in \mathbb{N}_0 (0 \leq N \leq n \text{ for even } n)\), for \(n\)-dimensional conformal manifold \((M, [h])\), cf. [Juh09]. The residue families exhibit a covariant transformation property for conformal change of the metric \(h\), specialize to GJMS operators at specific values of \(\lambda\) and obstruct the existence of a continuation of the function \(r^{\mu} u, u \in \ker (\Delta g_+ + \lambda (n - \lambda)), \mu \in \mathbb{C},\) to a distribution on \(M \times [0, \varepsilon)\). The residue families encode neat invariants of conformal structure \([h]\) on \(M\), e.g., the Branson’s \(Q\)-curvature is produced as a derivative in the variable \(\lambda\) of the critical residue families \(D_{\lambda}^{res}(h; \lambda)\), for even \(n\), at the value \(\lambda = 0\), evaluated on the constant function 1. Furthermore, the residue families enjoy a system of factorization identities, given by pre-compositions resp. post-compositions with GJMS operators on \((M, h)\) resp. on the conformal compactification \((M \times [0, \varepsilon), \bar{g})\) of \((X, g_+)\). In the flat case, the factorization identities are the consequences of factorization properties in the representation theory, describing compositions of homomorphisms of generalized Verma modules, see [KØSS13]. To summarize, the residue families are differential invariants of conformal submanifolds and can be analytically realized in residues of a distribution constructed out of the defining function of the submanifold and an eigenfunction of the Laplace operator in the ambient manifold.

The main results of our article are the construction and basic properties of spinor analogues of residue families, thereby producing differential invariants of conformal \(Spin\)-submanifolds. In this case are the basic building blocks of residue family operators on spinors the conformally covariant operators with leading terms given by odd powers of the Dirac operator \(D\) on \((M, h)\), cf. [HS01, GMP12, Fis13]. We focus again on the case of Poincaré-Einstein metric and its conformal compactification.

This theme conceptually fits into the framework of boundary valued problems and Poisson transform for the systems of partial differential equations with regular singularities along a submanifold, cf. [KKM+78, Hel70]. A closely parallel topics include analytic aspects of geometric scattering theory for spinors on Poincaré-Einstein metrics \((X, g_+)\), cf. [GMP10, GMP12], or algebraic aspects of the residue family operators on spinors based on the classification of homomorphisms of generalized Verma modules for conformal parabolic subalgebras and inducing spinor representations, cf. [KØSS13].

Let us briefly review the content of the present article. We start with the formulation of boundary valued problem for spinors, focusing on the eigen-
value equation for the Dirac operator on Poincaré-Einstein metric associated to \((M, [\mathfrak{h}])\), cf. Section 2. In Section 3, we construct the residue family operators on spinors, cf. Definition 3.1, through the analysis of obstructions for extending formal asymptotic eigenspinors as spinor-valued distributions, supported on \(M \times [0, \varepsilon)\) and acting on the functional space of compactly supported smooth spinors. We discuss the general case of curved semi-Riemannian Spin-manifold in a way that for the flat case the residue family operators correspond to conformally covariant differential operators induced by homomorphisms of Verma modules for codimension one orthogonal Lie algebras and their conformal parabolic subalgebras, cf. Theorem 3.7. Section 4 contains a short digression on the conformal covariance of residue family operators, cf. Theorem 4.2. In addition, we define a family of first order differential operators associated to arbitrary hypersurface in a conformal manifold and prove its conformal covariance, cf. Proposition 4.4. In the flat case, we determine the full system of factorization identities for residue family operators on spinors, cf. Theorems 5.2 and 5.6 in Section 5. The abstract conclusion is the consequence of the classification of conformally covariant differential operators on spinors, and the proof is based on combinatorial identities for the hypergeometric functions. We also prove factorization identities for a few residue family operators in the general curved case, exploiting the metric construction of generalized cylinders over conformal boundary \((M, [\mathfrak{h}])\). Sections 6 and 7 explain the relationship between the residue family operators on spinors and \(F\)-method used to produce them in the flat case, resp. the relative Poisson kernel allowing to introduce the integral intertwining operator on spinors. In Section 8 we comment on several interesting questions unresolved in the article. The Appendices A, B and C briefly review the concepts of Spin-geometry, Poincaré-Einstein metrics and Gegenbauer polynomials.
2 Boundary value problem for the Dirac operator associated to a Poincaré-Einstein metric

In this section we discuss the construction of formal powers series asymptotic solutions of the eigenvalue equation for the Dirac operator associated to Poincaré-Einstein metrics, cf. Appendix B, which we exploit later on in the construction of the residue family operators on spinors. For certain eigenvalues these formal asymptotic solutions are obstructed and the obstructions yield conformal powers of the Dirac operator, cf. [GMP12, Fis13].

Let \((M, h)\) be a \(n\)-dimensional Riemannian Spin-manifold, see Remark 2.6 for the discussion of pseudo-Riemannian Spin-manifolds. Let \(g_+\) be the associated Poincaré-Einstein metric on \(X = M \times (0, \varepsilon), \varepsilon > 0\) and \(r\) the coordinate on \([0, \varepsilon)\), cf. Appendix B. The conformal compactification of \((X, g_+)\) is

\[
(X, \bar{g} := r^2 g_+ = dr^2 + h_r),
\]

where \(\bar{g}\) extends smoothly to \(r = 0\). Note that we denote by the same letter \(X\) the extension of \(M \times (0, \varepsilon)\) to \(r = 0\). The embedding \(\iota_r : M \hookrightarrow X, M \ni x \mapsto \iota_r(x) := (x, r)\), pulls back \(\bar{g}\) to \(h_r\), hence \((M, \iota_r^* \bar{g})\) is a hypersurface in \((X, \bar{g})\) with spacelike normal vector field \(\partial_r\). Let us denote by

\[
S(M, h), \quad S(X, g_+), \quad S(X, \bar{g}),
\]
corresponding spinor bundles, respectively. The Gauß equation (with respect to \(\iota_r\))

\[
\nabla_Y^2 Z = \nabla_Y^{h_r} Z + \bar{g}(W_r(Y), Z)\partial_r, \quad Y, Z \in \Gamma(TM)
\]
lifts to the spinor bundle
\[
\nabla^g_S \theta = \bar{\nabla}^h_r \theta - \frac{1}{2} \partial_r \cdot W_r(Y) \cdot \theta, \quad \theta \in \Gamma(S(X, \bar{g})|_M, Y) \in \Gamma(TM), \quad (2.1)
\]
where
\[
W_r(Y) = -\nabla^g_r \partial_r \text{ denotes the Weingarten map associated to } \tau_r, \quad \text{The Dirac operator } D^g \text{ on } S(X, \bar{g}) \text{ and the leaf-wise (or, hypersurface) Dirac operator}
\]
\[
\tilde{D}^h_r := \partial_r \cdot \sum_{i=1}^n s_i \cdot \bar{\nabla}^h_{s_i} : \Gamma(S(X, \bar{g})) \to \Gamma(S(X, \bar{g})) \quad (2.2)
\]
are related by
\[
\iota^*_r \partial_r \cdot D^g = \tilde{D}^h_r + \frac{n}{2} \iota^*_r H_r - \iota^*_r \nabla^g_S, \quad (2.3)
\]
where \( H_r = \frac{1}{n} \text{tr}_{h_r} (W_r) \) is the \( h_r \)-trace of the Weingarten map associated to \( \tau_r \) and \( \iota^*_r \) when acting on spinors, denotes the restriction to \( M \times \{ r \} \), cf. [BGM05].

Since \( g_r \) and \( \bar{g} \) are conformally equivalent metrics, there exists a vector bundle
\[
F_r : S(X, g_r) \to S(X, \bar{g}).
\]
It follows from the conformal covariance of Dirac operator, equation (2.3) and the isomorphism \( F_r \) that \( D^g \varphi = i \lambda \varphi, \lambda \in \mathbb{C} \) and \( \varphi \in \Gamma(S(X, g_r)) \), is equivalent to
\[
D(\bar{g}) \theta = i \lambda \theta, \quad \theta = F_r \varphi \in \Gamma(S(X, \bar{g})), \quad (2.4)
\]
and
\[
D(\bar{g}) := -r \partial_r \cdot \tilde{D}^h_r + \frac{n}{2} r H_r \partial_r + r \partial_r \cdot \nabla^g_S - \frac{n}{2} \partial_r, \quad (2.5)
\]
Let us consider the following formal power series expansion around \( r = 0 \):
\[
\tilde{D}^h_r = \sum_{k \geq 0} \frac{\tilde{D}^{(h,k)} r^k}{k!}, \quad H_r = \sum_{k \geq 0} \frac{H^{(k)} r^k}{k!}
\]
We shall use the notation \( \tilde{D} := \tilde{D}^0_h \), see Remark 2.1 for its identification with the Dirac operator \( D \) on \( (M, h) \). The first few terms in the previous expansion depend on \( \tilde{D} \), Schouten tensor \( P := \frac{1}{n-2} (P - Jh) \) and normalized scalar curvature \( J := \frac{1}{2(n-1)} \) only, cf. [BGM05]:
\[
\tilde{D}^h_r = \tau_r^0 \circ \tilde{D} \circ \tau_r^0 + \frac{1}{2} \tau_r^2 \circ \partial_r \circ h(P, \bar{\nabla}^h) \circ \tau_r^0 + \cdots,
\]
\[
H_r = \frac{1}{n} r J + \frac{1}{3!} r^3 \left( -\frac{12}{n} \text{tr}_h(P^2) \right) + \cdots,
\]
where
\[
h(P, \bar{\nabla}^h) := \sum_{i=1}^n P(s_i) \cdot \bar{\nabla}^h_{s_i} \text{ and } \tau_r^x : \Gamma(S(X, \bar{g})|_{M \times \{ s \}}) \to \Gamma(S(X, \bar{g})|_{M \times \{ r \}})
\]
denotes the parallel transport along the geodesic coordinate \( r \) with respect to \( \nabla^g_S \).
Remark 2.1  The restriction of the spin representation \(\kappa_{n+1} : \text{Spin}(n+1) \to \text{Gl}(\Delta_{n+1})\), cf. Appendix A, to \(\text{Spin}(n) \subset \text{Spin}(n+1)\) yields a representation of \(\text{Spin}(n)\). For even \(n\) this is irreducible, whereas for odd \(n\) it decomposes into two equivalent irreducible representations. In all cases they are equivalent to \(\kappa_n\). Thus there are vector bundle isomorphisms of induced \(\text{Spin}(n)\)-modules: For even \(n\), we have

\[
\Xi : S(X,\tilde{g})|_{r=0} \to S(M,h),
\]

and in case \(n\) is odd, we have

\[
\Xi^\pm : S^\pm(X,\tilde{g})|_{r=0} \to S(M,h).
\]

We denoted by \(S^\pm(X,\tilde{g})\) the splitting of \(S(X,\tilde{g})\) with respect to the volume element. For any vector field \(Y\) on \(M\) and spinors \(\psi \in S(X,\tilde{g})|_{r=0}\) (even \(n\)) or \(\psi^\pm \in S^\pm(X,\tilde{g})|_{r=0}\) (odd \(n\)), we have

\[
\Xi(\partial_r \cdot Y \cdot \psi) = Y \cdot \Xi(\psi), \quad \Xi^\pm(\pm \partial_r \cdot Y \cdot \psi^\pm) = Y \cdot \Xi^\pm(\psi^\pm),
\]

respectively. In this identification we have \(\tilde{\mathcal{D}} = \mathcal{D}\) for even \(n\), and \(\tilde{\mathcal{D}} = \begin{pmatrix} \mathcal{D} & 0 \\ 0 & -\mathcal{D} \end{pmatrix}\) for odd \(n\). Here \(\mathcal{D}\) is the Dirac operator on \((M,h)\) and by an abuse of notation, we set \(\Xi := \Xi^+\) for even \(n\) odd.

The linear map \(\partial_r : S(X,\tilde{g}) \to S(X,\tilde{g})\) decomposes spinor bundle on \(X\) into

\[
S^\pm(\partial_r)(X,\tilde{g}) := \{ \theta \in S(X,\tilde{g})|_{\partial_r \cdot \theta = \pm i\theta} \},
\]

since it squares to \(-1\), and the corresponding projection operators are

\[
\theta^\pm := P_\theta \theta := \frac{1}{2}(1 - (\pm i)\partial_r)\theta. \tag{2.6}
\]

This leads to the definition of convenient function space

\[
A^\pm := \{ \theta = \sum_{j \geq 0} \theta_j r^j | \theta_{j+1} \in \Gamma(S^{+\partial_r}(X,\tilde{g})), \theta_{j+1} \in \Gamma(S^{-\partial_r}(X,\tilde{g})), \nabla_{\partial_r}^{\theta_j} S \theta_j = 0 \},
\]

such that formal asymptotic solutions of equation (2.4) will be constructed inside \(A^\pm\).

Proposition 2.2  Let \(\varphi_1 \in \Gamma(S^{+\partial_r}(X,\tilde{g})|_{r=0})\), \(\varphi_2 \in \Gamma(S^{-\partial_r}(X,\tilde{g})|_{r=0})\) and \(\lambda \notin -\mathbb{N} + \frac{1}{2}\). Then there exists a unique, up to \(O(r^{n+1})\) if \(n\) is even, \(\theta \in A^+\) and \(\phi \in A^-\) such that \(r^\frac{1}{2} + \lambda \theta\) and \(r^\frac{1}{2} - \lambda \phi\) are solutions of equation (2.4) with \((\theta_j^+)|_{r=0} = \varphi_1\) and \((\phi_j^-)|_{r=0} = \varphi_2\).

Proof.  Let us consider a formal power series

\[
\theta = \sum_{j \geq 0} (\theta_j^+ + \theta_j^-) r^j,
\]

in \(A^\pm\).
such that \( \theta_j^\pm \in \Gamma(S^{\pm \theta_j}(X, \bar{g})), \) \( j \in \mathbb{N}_0, \) are parallel along the \( r \)-coordinate with respect to \( \nabla^{g,S}. \) The equation \( D(\bar{g}) \phi = i\lambda \nabla^\pm \phi \) is equivalent to

\[
i\lambda \sum_{j \geq 0} r^\pm \lambda+j_\theta(\phi_\theta^+ + \phi_\theta^-) = - \partial_r \sum_{j,k \geq 0} r^\pm \lambda+j+k+1 \left( D^{(h,k)} + \frac{\lambda}{2} H^{(\theta)} \right)(\phi_\theta^+ + \phi_\theta^-) + \partial_r \sum_{j \geq 0} r^\pm \lambda+j_\theta(\phi_\theta^+ + \phi_\theta^-).
\]

It follows that \( \theta_0^+ \) can be chosen arbitrarily, hence we set it to \( \varphi_1 \) at \( r = 0 \) and extend it over \( X \) by parallel transport. Furthermore, we have \( \theta_0^- = 0. \) Now the initial data \( \varphi_1 \) uniquely, up to \( O(r^{n+1}) \) for every \( n, \) determine \( \theta \) inductively.

Similarly, let us consider the formal power series

\[
\tau^\pm r \lambda = \sum_{j \geq 0} \tau^\pm \lambda+j_\theta(\phi_j^+ + \phi_j^-)
\]

for \( \phi_j^\pm \in \Gamma(S(X, \bar{g})), \) \( j \in \mathbb{N}_0, \) which are parallel along the \( r \)-coordinate with respect to \( \nabla^{g,S}. \) The initial data \( \phi_0^+ = 0 \) and \( \phi_0^- \) equal at \( r = 0 \) to \( \varphi_2, \) determine \( \phi \) uniquely up to \( O(r^{n+1}) \) for even \( n \) provided \( r^\pm \lambda \phi \) solves equation (2.4). The proof is complete. \( \square \)

Let us recall an important application of Proposition 2.2.

**Theorem 2.3 [GMP12, Theorem 2]**

Let \( (M, h) \) be a Riemannian Spin-manifold of dimension \( n. \) For every \( N \in \mathbb{N}_0 \) \( (N < \frac{1}{2} \text{ for even } n) \) there exists a conformally covariant linear differential operator

\[
D_{2N+1} : \Gamma(S(M, h)) \rightarrow \Gamma(S(M, h))
\]

such that \( D_{2N+1} = \mathcal{D}_{2N+1}^2 + LOT, \) where \( LOT \) denotes lower order terms.

These operators, termed conformal powers of the Dirac operator, are the obstruction to extend a boundary spinor into interior, solving equation (2.4) for \( \lambda \in -\mathbb{N} + \frac{1}{2} \) recursively.

**Remark 2.4** Let us consider the solutions \( \tau^\pm r \lambda \theta \) and \( \tau^\pm r \lambda \phi \) of equation (2.4) in Proposition 2.2. By construction, all their coefficients at \( r = 0 \) are given by \( h \)-natural linear differential operators \( T_l(h; \lambda) \), termed solution operators, of order \( l, l \in \mathbb{N}. \) They act on boundary data as follows:

\[
\begin{align*}
\theta_{2l}^+|_{r=0} &= T_{2l}(h; \lambda)\varphi_1, & \theta_{2l+1}^-|_{r=0} &= T_{2l+1}(h; \lambda)\varphi_1, & \forall l \in \mathbb{N}_0, \\
\phi_{2l}^+|_{r=0} &= T_{2l}(h; -\lambda)\varphi_2, & \phi_{2l+1}^-|_{r=0} &= T_{2l+1}(h; -\lambda)\varphi_2, & \forall l \in \mathbb{N}_0.
\end{align*}
\]

All other coefficients are zero. Note that \( T_l(h; \lambda) \) depends rationally on \( \lambda. \)

**Example 2.5** The first order solution operator is

\[
T_1(h; \lambda) = \frac{1}{2\lambda + 1} \tilde{\mathcal{D}}.
\]
the second order solution operator is
\[ T_2(h; \lambda) = \frac{1}{2(2\lambda + 1)} \tilde{\Phi}^2 + \frac{1}{4} J, \]
and the third order solution operator is
\[ T_3(h; \lambda) = \frac{1}{2(2\lambda + 1)(2\lambda + 3)} \tilde{\Phi}^3 + \frac{1}{2(2\lambda + 3)} \partial_\tau \cdot h \left( P, \nabla^{h \tau} \right) \\
+ \frac{1}{4(2\lambda + 1)} J \tilde{\Phi} + \frac{1}{4(2\lambda + 3)} \partial_\tau \cdot \text{grad}^{M}(J). \]
Note that residues of solution operators at \( \lambda = -\frac{1}{2}, -\frac{3}{2} \) yield conformal first and third power of the Dirac operator.

Now we apply the results obtained so far to the case of flat euclidean manifold \((\mathbb{R}^{n-1}, h = \langle \cdot, \cdot \rangle_{n-1})\). The associated Poincaré-Einstein metric is given by the hyperbolic metric \( g_+ = x_+^{-2}(dx_+^2 + h) \) on \( \mathbb{R}^n_{x_+ > 0} \). Its conformal compactification is given by the flat structure \((\mathbb{R}^n_{x_+ \geq 0}, \tilde{g} = \langle \cdot, \cdot \rangle_n)\), and the operator (2.2) simplifies to
\[ \tilde{\Phi} = e_n \cdot \sum_{i=1}^{n-1} e_i \cdot \partial_i. \]
(2.7)
The solution operators \( T_l(h; \lambda), l \in \mathbb{N}_0 \), obey an explicit formula
\[ T_{2l}(h; \lambda) = \frac{1}{2^{2l}l!(\frac{1}{2} + \lambda)} \tilde{\Phi}^{2l}, \quad T_{2l+1}(h; \lambda) = \frac{1}{2^{2l+1}l!(\frac{1}{2} + \lambda)_{l+1}} \tilde{\Phi}^{2l+1}, \]
where \((a)_l := a(a+1) \ldots (a+l-1), a \in \mathbb{C} \) and \( l \in \mathbb{N} \), denotes the Pochhammer symbol. We set conventionally \((a)_0 := 1\).

**Remark 2.6** (Pseudo-Riemannian structures)
Let \((M^n, h)\) be a pseudo-Riemannian Spin\(-\)manifold of signature \((p, q)\), \( p + q = n \). In this case we can decompose \( S(X, \tilde{g}) \) into \((\pm i)^{p+1}\)-eigenspaces with respect to \( \partial_\tau \). The projector operators are defined by
\[ P_\pm := \frac{1}{2}(1 - (\pm i)^{p+1} \partial_\tau), \]
hence formally self-adjoint with respect to the spinor scalar product \( < \cdot, \cdot > \) on \( S(X, \tilde{g}) \). This is compatible with the eigenvalue equation analogous to (2.4), replacing \( i\lambda \) by \( i^{p+1}\lambda \), which does not change the structure of formal solutions obtained in Proposition 2.2.

### 3 Residue family operators on spinors
The present section introduces a definition of residue family operators on spinors. Firstly, we define the residue family operators on spinors for general semi-Riemannian \( \text{Spin} \)-manifolds. Secondly, we prove that in the flat case the resulting residue family operators agree with a family of intertwining operators introduced in [KOSS13].
3.1 Residue family operators on spinors - curved case

The present subsection is devoted to the general definition of the residue family operators on spinors for semi-Riemannian Spin-manifolds. Again, we restrict ourselves to Riemannian Spin-manifolds and discuss the pseudo-Riemannian case in Remark 3.3.

Let us introduce the volume function

$$v(r) := \sqrt{\frac{\det(h_r)}{\det(h)}} = 1 + r^2v_2 + r^4v_4 + \cdots,$$

where \(v_{2j}, j > 0\), are called renormalized volume coefficients, see [Gra99]. We have the following relation among volume forms,

$$Vol(\hat{g}) = v(r)drVol(h).$$

Now, we consider the formal asymptotic solution

$$\hat{\theta} := r^{\frac{2}{h}+\lambda}\theta = \sum_{j \geq 0} r^{\frac{2}{h}+\lambda+j}\theta_j$$

of the eigenvalue equation [2.4] for the eigenvalue \(i\lambda\), cf. Proposition 2.2. For \(\mu \in \mathbb{C}\) such that \(Re(\mu) \gg 0\), we define a family of spinor valued distributions by

$$M_\theta(\mu; r)(\varphi) := \int_{(0,\epsilon)} \int_M <r^\mu \hat{\theta}, \varphi> Vol(\hat{g}),$$

where \(\varphi \in \Gamma_c(S(X, \hat{g}))\) is compactly supported near \(r = 0\) and \(<\cdot, \cdot>\) denotes the scalar product on the spinor bundle \(S(X, \hat{g})\).

After fixing the eigenvalue \(\lambda\), we can meromorphically extend \(M_\theta(\mu; r)\) to \(\mu \in \mathbb{C}\) with simple poles at \(\mu = -\frac{2}{h} - \lambda - 1 - N_0\). By partial integration with respect to \(\nabla^{\partial_\theta}_{\partial_\theta}\), the residue of \(M_\theta(\mu; r)\) at \(\mu = -\frac{2}{h} - \lambda - 1 - N, N \in N_0\) (\(N \leq n\) for even \(n\)), is

$$\text{Res}_{\mu= -\frac{2}{h} - \lambda - 1 - N} (M_\theta(\mu; r)(\varphi)) =$$

$$\int_M \sum_{j=0}^N \frac{1}{(N-j)!} \left< \sum_{k=0}^{j} v_k \mathcal{T}_{j-k}(h; \lambda)\theta_0^+, (\nabla^{\partial_\theta}_{\partial_\theta})^{N-j}(\varphi)(0, \cdot) > Vol(h).$$

Taking adjoints (with respect to the induced \(L^2\)-scalar product), indicated by \(\ast\), of the operators on the left hand side of the scalar product \(<\cdot, \cdot>\), we arrive at a family of linear \(h\)-natural differential operators

$$\delta^+_N(h; \lambda) : \Gamma_c(S(X, \hat{g})) \rightarrow \Gamma(S^{\partial_\theta}_{\partial_\theta}(X, \hat{g})_{|r=0})$$

defined by

$$\text{Res}_{\mu= -\frac{2}{h} - \lambda - 1 - N} (M_\theta(\mu; r)(\varphi)) =$$

$$\int_M <\theta_0^+, \delta^+_N(h; \lambda)\varphi> Vol(h).$$

They are given by explicit formulas

$$\delta^+_N(h; \lambda) = \sum_{j=0}^{N-1} \frac{1}{(2N - 2j - 1)!} \sum_{k=0}^{j} \left(\mathcal{T}_{j+1-2k}(h; \lambda)\right)^* v_{2k} \left(\nabla^{\partial_\theta}_{\partial_\theta}\right)^{2N-2j-1}$$
where \( \bar{\lambda} \) are different. We define, by continuous extension sections, 
\[
\lambda \in (0, \infty)
\]
It follows from equations (3.1) and (3.2) that 
\[
\phi = 0. \tag{3.1}
\]
Similarly, one defines for \( N \in \mathbb{N}_0 \)
\[
\delta_N(h; \lambda) := \int_M \phi_0^\pm \rho \delta_N(h; \lambda) > Vol(h),
\]
where \( \phi_0 := r^{2+\lambda} \phi \) is a solution of the eigenvalue equation (3.1) for the eigenvalue \(-i\lambda\). Again, \( \delta_N(h; \lambda) \) are structurally the same except that their target spaces are different. We define, by continuous extension \( \delta_N^\pm(h; \lambda) \) to the space of smooth sections,
\[
\delta_N(h; \lambda) := \delta_N^+(h; \lambda) + \delta_N^-(h; \lambda) : \Gamma(S(X, g)) \rightarrow \Gamma(S(X, g)|_{r=0}). \tag{3.3}
\]
It follows from equations (3.1) and (3.2) that \( \delta_{2N}(h; \lambda) \) has \( N \) simple poles at \( \lambda \in \{-N + \frac{1}{2}, \ldots, -\frac{1}{2}\} \), whereas \( \delta_{2N+1}(h; \lambda) \) has \( (N + 1) \) simple poles at \( \lambda \in \{-N + \frac{1}{2}, \ldots, -\frac{3}{2}\} \).

Now we define the residue family operators for spinors.

**Definition 3.1** Let \( \lambda \in \mathbb{C} \) and \( N \in \mathbb{N}_0 \) \( (N \leq n \text{ for even } n) \). Then the differential operators
\[
\mathcal{B}_N^{\text{res}}(h; \lambda) : \Gamma(S(X, g)) \rightarrow \Gamma(S(X, g)|_{r=0})
\]
given by
\[
\mathcal{B}_{2N}^{\text{res}}(h; \lambda) := 2^{2N}N! \left[ (\lambda + \frac{n}{2} - 2N + \frac{1}{2}) \cdot \ldots \cdot (\lambda + \frac{n}{2} - N - \frac{1}{2}) \right] \times \delta_{2N}(h; \lambda + \frac{n}{2} - 2N),
\]
\[
\mathcal{B}_{2N+1}^{\text{res}}(h; \lambda) := 2^{2N+1}N! \left[ (\lambda + \frac{n}{2} - 2N + \frac{1}{2}) \cdot \ldots \cdot (\lambda + \frac{n}{2} - N - \frac{1}{2}) \right] \times \delta_{2N+1}(h; \lambda + \frac{n}{2} - 2N - 1),
\]
are called the residue family operators on spinors.

All statements about residue family operators \( \mathcal{B}_N^{\text{res}}(h; \lambda) \) for even \( n \) are in general valid only for \( N \leq \frac{n}{2} \). In what follows this restriction will be omitted.
**Remark 3.2** The last definition is analogous to the definition of residue families in the scalar case, cf. [Juh09]. However, contrary to the scalar case, the target space of residue family operators is $S(X, \bar{g})|_{\tau=0}$ rather than $S(M, h)$. An explanation for this choice will be given in Remark 5.8.

**Remark 3.3** It follows from the definition of residue family operators:

1. $\mathcal{D}_N^{res}(h; \lambda)$ is a family of linear $h$-natural differential operators of order $N$.
2. $\mathcal{D}_N^{res}(h; \lambda)$ is a polynomial in $\lambda$ of degree $\lfloor \frac{N}{2} \rfloor + 1$, where $\lfloor k \rfloor$ denotes the integer part of $k \in \mathbb{R}$.

We present a few low order examples.

**Example 3.4** The first order residue family operator is

$$\mathcal{D}_1^{res}(h; \lambda) = 2(\lambda + \frac{n}{2} - 1)\frac{1}{2}\frac{1}{J},$$

Because the adjoint of solution operator is $(T_1(h; \lambda))^* = \frac{1}{2\lambda + 1} \tilde{\mathcal{D}}$, we get

$$\mathcal{D}_1^{res}(h; \lambda) = (2\lambda + n - 1)e^*\nabla^S_{\partial_\nu} + \tilde{\mathcal{D}}e^*.$$ 

The second order residue family operator is

$$\mathcal{D}_2^{res}(h; \lambda) = 2^2(\lambda + \frac{n}{2} - 2)\frac{3}{2}\frac{1}{J},$$

$$= 2(2\lambda + n - 3)(\lambda + \frac{n}{2} - 2)^*e^* + (T_2(h; \lambda + \frac{n}{2} - 2))^*e^* + v_2e^* + (T_1(h; \lambda + \frac{n}{2} - 2))^*e^*\nabla^S_{\partial_\nu}.$$ 

Using $v_2 = -\frac{1}{2}J$, cf. [Juh09] Theorem 6.9.2, and

$$(T_2(h; \lambda))^* = \frac{1}{2(2\lambda + 1)} \tilde{\mathcal{D}}^2 + \frac{1}{4}J,$$

we have

$$\mathcal{D}_2^{res}(h; \lambda) = (2\lambda + n - 3)e^*(\nabla^S_{\partial_\nu})^2 + \tilde{\mathcal{D}}^2e^* - \frac{1}{2}(2\lambda + n - 3)J^*e^* + 2\mathcal{D}e^*\nabla^S_{\partial_\nu}.$$ 

The third order residue family operator is

$$\mathcal{D}_3^{res}(h; \lambda) = 2^3(\lambda + \frac{n}{2} - 3)\frac{5}{2}(\lambda + \frac{n}{2} - 2)^*e^* + \frac{3}{2}\frac{1}{J},$$

$$= 2(2\lambda + n - 5)(2\lambda + n - 3)^*e^*(\nabla^S_{\partial_\nu})^3 + (T_2(h; \lambda + \frac{n}{2} - 3))^*e^*\nabla^S_{\partial_\nu}.$$
\[ + v_2 \ast \nabla_{\partial_\nu} + \frac{1}{2}(T_1(h; \lambda + \frac{n}{2} - 3))^\ast (\nabla g, S) \partial_r + (T_3(h; \lambda + \frac{n}{2} - 3))^\ast + (T_1(h; \lambda + \frac{n}{2} - 3))^\ast v_2 \ast ] \]

Using
\[
(T_3(h; \lambda))^\ast = \frac{1}{2(2\lambda + 1)(2\lambda + n - 3)} \tilde{\partial}^3 + \frac{1}{2(2\lambda + n - 3)} \partial_r \cdot h(P, \tilde{\nabla}) + \frac{1}{2(2\lambda + n - 3)} \partial_r \cdot \text{grad}^M(J) \cdot + \frac{1}{4(2\lambda + 1)} J \tilde{\partial}
\]
\[
+ \frac{1}{2(2\lambda + 1)(2\lambda + n - 3)} \partial_r \cdot \text{grad}^M(J),
\]

where we note that \( \partial_r \cdot h(P, \tilde{\nabla}) \) is not self-adjoint because its adjoint contributes by the gradient of \( J \), we get
\[
B_{res}^N(h; \lambda) = \frac{1}{3}(2\lambda + n - 5)(2\lambda + n - 3)e_i^\ast (\nabla g, S) \partial_i + (2\lambda + n - 3) \tilde{\partial}^3 \ast \nabla g, S
\]
\[
- \frac{1}{2}(2\lambda + n - 5)(2\lambda + n - 3)J \ast \nabla g, S + (2\lambda + n - 3) \tilde{\partial} \ast (\nabla g, S) ^2
\]
\[
+ \tilde{\partial} \ast (2\lambda + n - 5) \partial_r \cdot h(P, \tilde{\nabla}) \ast
\]
\[
- \frac{1}{2}(2\lambda + n - 3)J \tilde{\partial} \ast - \partial_r \cdot \text{grad}^M(J) \ast.
\]

Later on these formulas will be used to prove the factorization identities in the curved case.

**Remark 3.5** Let \((M, h)\) be a pseudo-Riemannian Spin-manifold. In Remark 2.6 we suggested how to solve the eigenvalue equation in the presence of signature. The adjustments made there can be used to define residue family operators in a similar way as for the Riemannian Spin-manifolds.

### 3.2 Residue family operators on spinors - flat case

We now specialize the definition of residue family operators \( B_{res}^N(h; \lambda) \), cf. Definition 3.1, to the flat case \((\mathbb{R}^{n-1}, h)\). Notice that in this case there is no restriction on \( N \in \mathbb{N}_0 \) for even \((n - 1)\). Due to explicit knowledge of solution operators (2.8), we are able to derive an explicit formula for residue family operators in terms of Gegenbauer polynomials.

First note that formal self-adjointness (with respect to the \( L^2 \)-scalar product \( \langle \cdot, \cdot \rangle_{L^2} \) on \( S(\mathbb{R}^n, \tilde{g}) \)) of the operator
\[
\tilde{\partial} = e_n \cdot \sum_{i=1}^{n-1} e_i \cdot \partial_i,
\]
i.e., \( \langle \tilde{\partial} \theta_1, \theta_2 \rangle_{L^2} = \langle \theta_1, \tilde{\partial} \theta_2 \rangle_{L^2} \) for compactly supported spinors \( \theta_1, \theta_2 \), implies that solution operators, cf. equation (2.8), are formally self-adjoint,
The even and odd order family (3.1) and (3.2) specialize, using $v(\tau) = 1$ and $\nabla_{\partial_r}^{\beta,S} = \partial_{\tau}$ in the flat case, to

$$\delta_{2N}^+(h; \lambda) = \sum_{j=0}^{N-1} \frac{1}{(2N-2j)!} \frac{1}{4j! \Gamma(\frac{1}{2} + \lambda)} \tilde{D}^{2j} \right) t^* \tilde{D}_{2N-2j}^2 D_N \right)$$

$$- \sum_{j=0}^{N-1} \frac{1}{(2N-2j)!} \frac{1}{4j! \Gamma(\frac{1}{2} + \lambda)} \tilde{D}^{2j} \right) t^* \tilde{D}_{2N-2j}^2 D_N \right)$$

$$\delta_{2N+1}^+(h; \lambda) = e_n \cdot \sum_{j=0}^{N-1} \frac{1}{(2N-2j)!} \frac{1}{4j! \Gamma(\frac{1}{2} + \lambda)} \tilde{D}^{2j} \right) t^* \tilde{D}_{2N-2j}^2 D_N \right)$$

where $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is canonical embedding on first $n - 1$ coordinates. Note that similar statement holds for $\delta_{N}^-(h; \lambda)$.

For $N \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ and $a^{(N)}_N(\lambda), b^{(N)}_N(\lambda) \in \mathbb{R}$, we define

$$a^{(N)}_j(\lambda) := \frac{N!(-2)^{N-j}}{j!(2N-2j)!} \prod_{k=j}^{N-1} (2\lambda - 4N + 2k + n + 1) a^{(N)}_N(\lambda), \quad (3.6)$$

$$b^{(N)}_j(\lambda) := \frac{N!(-2)^{N-j}}{j!(2N-2j+1)!} \prod_{k=j}^{N-1} (2\lambda - 4N + 2k + n + 1) b^{(N)}_N(\lambda), \quad (3.7)$$

$0 \leq j \leq N - 1$. They satisfy the following recurrence relations:

$$(N - j + 1)(2N - 2j + 1)a^{(N)}_{j+1}(\lambda) + j(2\lambda + n - 4N + 2j - 1)a^{(N)}_j(\lambda) = 0, \quad (3.8)$$

$$(N - j + 1)(2N - 2j + 3)b^{(N)}_{j+1}(\lambda) + j(2\lambda + n - 4N + 2j - 3)b^{(N)}_j(\lambda) = 0, \quad (3.9)$$

for all $1 \leq j \leq N$ and $\lambda \in \mathbb{C}$ and are known as the coefficients of Gegenbauer polynomials of even and odd degree, respectively; cf. Appendix C for the origin and wider framework of this notion. We use the conventions $a^{(N)}_{N+1} := 0$ and $b^{(N)}_{N+1} := 0$. Note that empty products are set to be 1, and we shall consider the normalizations $a^{(N)}_N(\lambda) = b^{(N)}_N(\lambda) = (-1)^N$.

**Remark 3.6** The normalization among odd and even Gegenbauer polynomials is a consequence of equations (6.3), (6.4) and (6.6), (6.7).
The next Theorem identifies $D_N^{res}(h; \lambda)$ as a family of intertwining differential operators $D_N(\lambda)$ introduced in [KÖSS13], though we use the opposite convention in the sign of the density given by $\lambda$.

**Theorem 3.7** Let $\lambda \in \mathbb{C}$ and $N \in \mathbb{N}_0$. Then it holds

\[
D_{2N}^{res}(h; \lambda) = (-1)^N \left[ \sum_{j=0}^{N} a_j^{(N)}(\lambda - \frac{1}{2}) D_T^{2j} t^* D_N^{2N-2j} + 2N \sum_{j=0}^{N-1} b_j^{(N-1)}(\lambda - \frac{1}{2}) D_T^{2j+1} t^* D_N^{2N-2j-1} \right],
\]

(3.10)

\[
D_{2N+1}^{res}(h; \lambda) = (-1)^{N+1} e_n \cdot \left[ \sum_{j=0}^{N} b_j^{(N)}(\lambda - \frac{1}{2}) c^{(N)}(\lambda) D_T^{2j} t^* D_N^{2N-2j+1} - \sum_{j=0}^{N} a_j^{(N)}(\lambda - \frac{1}{2}) D_T^{2j+1} t^* D_N^{2N-2j} \right],
\]

(3.11)

where $c^{(N)}(\lambda) := (2\lambda + n - 2N - 2)$.

**Proof.** The proof is based on direct computations. Let us start with the even order family. From equation (3.3) we obtain

\[
D_{2N}^{res}(h; \lambda) = 2^{2N} N! \left[ (\lambda + \frac{n}{2} - 2N) \cdots (\lambda + \frac{n}{2} - N - 1) \right] \times
\]

\[
\times \delta_{2N}(h; \lambda + \frac{n}{2} - 2N - \frac{1}{2})
\]

\[
= 2^{2N} N! \left[ (\lambda + \frac{n}{2} - 2N) \cdots (\lambda + \frac{n}{2} - N - 1) \right] \times
\]

\[
\times \sum_{j=0}^{N} \frac{1}{(2N-2j)!} 2^{2j} j! (\lambda + \frac{n}{2} - 2N)_{j} D_T^{2j} t^* \partial_n^{2N-2j}
\]

\[
- \sum_{j=0}^{N-1} \frac{1}{(2N-2j-1)!} 2^{2j+1} j! (\lambda + \frac{n}{2} - 2N)_{j+1} D_T^{2j+1} t^* \partial_n^{2N-2j-2} D_T D_N
\]

\[
= \sum_{j=0}^{N} \frac{2^{2N-2j} N!}{(2N-2j)! j!} \prod_{k=j}^{N-1} (\lambda + \frac{n}{2} - 2N + k) D_T^{2j} t^* \partial_n^{2N-2j}
\]

\[
- \sum_{j=0}^{N-1} \frac{2^{2N-2j-1} N!}{(2N-2j-1)! j!} \prod_{k=j}^{N-2} (\lambda + \frac{n}{2} - 2N + k + 1) D_T^{2j+1} t^* \partial_n^{2N-2j-2} D_T D_N.
\]

Introducing $1 = (-1)^j e_n^j$ for $l = N - j \in \mathbb{N}$ and using definitions (3.6) and (3.7), we get

\[
D_{2N}^{res}(h; \lambda) = (-1)^N \sum_{j=0}^{N} a_j^{(N)}(\lambda - \frac{1}{2}) D_T^{2j} t^* D_N^{2N-2j}
\]

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and it is compatible with Clifford multiplication
\[ \tilde{\ast} \]
the pull-back and by \( \Phi \) \( \Gamma \) diffeomorphism \( \Phi \) lifts to a vector bundle isomorphism
pulls-back sections of associated vector bundles on \( X \) and \( \psi \) structure preserving diffeomorphism such that \( \Phi \ast \) semi-Riemannian
\[ \square \]
variance property of the residue family operators. Let \( (X, g) \) on spinors in the flat case uniquely.

The odd order family is treated in an analogous way and we obtain
\[
\mathcal{B}_{2N+1}^{res}(h; \lambda) = (-1)^{N+1}e_n \cdot \left[ \sum_{j=0}^{N} b_j^{(N)} (\lambda - \frac{1}{2}) (2\lambda + n - 2N - 2j^2) + 2N \sum_{j=0}^{N-1} b_j^{(N-1)} (\lambda - \frac{1}{2}) D_j^{2j+1} \ast D_N^{2N-2j-1} \right].
\]

which completes the proof. \( \square \)

Example 3.8 Let us give several low order examples:
\[
\begin{align*}
\mathcal{B}_1^{res}(h; \lambda) &= e_n \cdot \left[ D_T \ast - (2\lambda + n - 2) \ast D_N \right], \\
\mathcal{B}_2^{res}(h; \lambda) &= D_T^2 \ast - 2D_T \ast D_N - (2\lambda + n - 4) \ast D_N^2, \\
\mathcal{B}_3^{res}(h; \lambda) &= e_n \cdot \left[ D_T^3 \ast - (2\lambda + n - 4)(D_T^2 \ast D_N + D_T \ast D_N^2) + \frac{1}{3}(2\lambda + n - 6)(2\lambda + n - 4) \ast D_N^3 \right], \\
\mathcal{B}_4^{res}(h; \lambda) &= D_T^4 \ast - 4D_T^3 \ast D_N + \frac{4}{3}(2\lambda + n - 6)D_T^3 \ast D_N^4 - 2(2\lambda + n - 6)D_T^2 \ast D_N^4 + \frac{1}{3}(2\lambda + n - 8)(2\lambda + n - 6) \ast D_N^4.
\end{align*}
\]

4 Conformal transformation properties

An important property of the residue family operators on spinors \( \mathcal{B}_N^{res}(h; \lambda) \), implicit in the invariance properties of solution operators in the original eigenvalue problem for the Poincaré-Einstein Dirac operator, is its conformal covariance with respect to a conformal change of the boundary metric. The conformal covariance property determines the residue family of natural differential operators on spinors in the flat case uniquely.

We shall start with several remarks used to understand the conformal covariance property of the residue family operators. Let \( (X_1, g_1), (X_2, g_2) \) be semi-Riemannian Spin-structure preserving diffeomorphism such that \( \Phi^* (g_2) = g_1 \). We denote by \( \Phi^* \) the pull-back and by \( \Phi_* \) the push-forward map induced by \( \Phi \). For example, \( \Phi^* \) pulls-back sections of associated vector bundles on \( X_2 \) to \( X_1 \). Note that the diffeomorphism \( \Phi \) lifts to a vector bundle isomorphism
\[
\Phi^* : S(X_1, g_1) \rightarrow S(X_2, g_2),
\]
and it is compatible with Clifford multiplication \( \Phi^* (X \cdot \psi) = \Phi^* X \cdot \Phi^* \psi, X \in TX_1 \) and \( \psi \in S(X_1, g_1) \). For \( \varphi \in \Gamma(S(X_2, g_2)) \) we set \( \Phi^* \varphi := \Phi^{-1} \circ \varphi \circ \Phi \in \Gamma(S(X_1, g_1)) \), which is the pull-back of \( \varphi \) with respect to \( \Phi \).
Lemma 4.1  In the setting described above, the Levi-Civita connections are related by
\[ \nabla_{g_1} = (\Phi^{-1})_* \nabla_{g_2} \Phi_*, \]  
and induced spinor covariant derivatives are related by
\[ (\tilde{\Phi}^{-1})_* \nabla_{g_1}^S \tilde{\Phi}^* \varphi = \nabla_{g_2}^S \Phi_* \varphi, \]  
for \( \varphi \in \Gamma(S(X_2, g_2)) \), \( X \in TX_1 \).

As for the Dirac operators on \((X_1, g_1)\) and \((X_2, g_2)\), we have
\[ (\tilde{\Phi}^{-1})_* \slashed{D}_{g_1} \tilde{\Phi}^* \varphi = \slashed{D}_{g_2} \varphi, \]  
for \( \varphi \in \Gamma(S(X_2, g_2)) \).

Proof.  The first statement is the isometry invariance of Levi-Civita connections. The second claim follows from the local description of spinor covariant derivatives, equation (4.1) and \((\tilde{\Phi}^{-1})_* (X \cdot \tilde{\Phi}^* \varphi) = \Phi_* X \cdot \varphi\) for \( X \in \Gamma(TM) \) and \( \varphi \in \Gamma(S(X_2, g_2)) \). Finally, using equation (4.2) we get for Dirac operators
\[ (\tilde{\Phi}^{-1})_* \slashed{D}_{g_1} \tilde{\Phi}^* \varphi = (\tilde{\Phi}^{-1})_* \sum_{i=1}^n \varepsilon_i \Phi_* s_i \cdot \nabla_{g_2}^S \Phi_* s_i \varphi = \sum_{i=1}^n \varepsilon_i \Phi_* s_i \cdot \nabla_{g_2}^S \Phi_* s_i \varphi = \slashed{D}_{g_2} \varphi, \]
which completes the proof. \( \square \)

These results will be applied to the diffeomorphism of the Poincaré-Einstein metric induced by a conformal change of the boundary metric \( h \), cf. Appendix B.

Let \( g_+ = r^{-2}(dr^2 + h_r) \) be the Poincaré-Einstein metric on \( X \) for a representative \( h \in \mathcal{H} \) in the conformal class on the conformal infinity \((M, [h])\) of \( X \). For a smooth function \( \sigma \in C^\infty(M) \), the Poincaré-Einstein metrics of \( h \) and \( \tilde{h} = e^{2\sigma} h \) are related by
\[ \Phi^* \left( r^{-2}(dr^2 + h_r) \right) = r^{-2}(dr^2 + \tilde{h}_r), \]
where \( \Phi \) is the induced diffeomorphism on \( X \) which restricts to the identity on the hypersurface \( r = 0 \), cf. Appendix B. Then we have
\[ \Phi^*(dr^2 + h_r) = \left( \frac{\Phi^*(r)}{r} \right)^2 (dr^2 + \tilde{h}_r) \]  
with
\[ \lim_{r \to 0} \frac{\Phi^*(r)}{r} = e^{-\sigma}, \quad \varphi \left( \frac{\Phi^*(r)}{r} \right)^\mu = e^{-\mu \sigma}, \]
where \( \varphi \) is the usual embedding of \( M \) into \( X \) at \( r = 0 \).
Theorem 4.2  The residue family operators $\mathcal{D}^{\text{res}}_{N}(h; \lambda)$ are conformally covariant differential operators in the sense that

$$\mathcal{D}^{\text{res}}_{N}(\tilde{h}; \lambda) = e^{(\lambda-N)\sigma} \circ \mathcal{D}^{\text{res}}_{N}(h; \lambda) \circ (\tilde{\Phi}^{-1})^* \circ \left( \frac{\Phi^*(r)}{r} \right)^\lambda$$

for all $\sigma \in C^\infty(M)$, $N \in \mathbb{N}_0$.

Remark 4.3  Notice that the behavior of $\mathcal{D}^{\text{res}}_{N}(h; \lambda)$ under the conformal transformation is actually not conformally covariant in the usual sense, cf. [Kos75].

Proof.  Let us consider the conformal compactification $(M \times [0, \varepsilon), \tilde{g})$ of the Poincaré-Einstein metric $g_+$ associated to $h \in [\tilde{h}]$ on $M$, $\tilde{\theta} := \frac{\tilde{\theta}^+}{\tilde{\theta}^+ + \lambda} \tilde{\theta} \in \Gamma(S(X, \tilde{g}))$ a formal asymptotic solution of the eigenvalue equation (2.10), cf. Proposition (2.2) and $\phi \in \Gamma_c(S(X, \tilde{g}))$ a smooth compactly supported spinor field.

It follows from (4.3) that once $\tilde{\theta}$ is an eigenspinor of $D(\tilde{g})$, see (2.25), $\tilde{\Phi}^* \tilde{\theta}$ is an eigenspinor of $D(r^2\Phi^* g_+)$ with the same eigenvalue. Thus we can calculate the residues of meromorphic spinor valued distribution

$$\text{Res}_{\mu} = -\frac{2}{2} - \lambda - N - 1 \left( M_{\Phi}^*(\mu; r)(\phi) \right)$$

$$= \text{Res}_{\mu} = -\frac{2}{2} - \lambda - N - 1 \int_X r^\mu < \tilde{\Phi}^*(\tilde{\theta}), \phi > V ol(\tilde{h})$$

in two independent ways. On the one hand, integrating $r$ over $[0, \varepsilon)$, using partial integrations and extracting residues, (4.3) equals to

$$\int_M e^{-(\lambda+\frac{2}{2} - \lambda)\sigma} < \theta_0, \Phi^*(h; \lambda) \phi > V ol(h)$$

because $\frac{V ol(\tilde{h})}{V ol(h)} = e^{n\sigma}$ for the conformal transformation $h \mapsto \tilde{h} = e^{2\sigma} h$. On the other hand, equation (4.3) and compactness of the support of $\phi$ implies

$$\text{Res}_{\mu} = -\frac{2}{2} - \lambda - N - 1 M_{\Phi}^*(\lambda; r)(\phi)$$

$$= \text{Res}_{\mu} = -\frac{2}{2} - \lambda - N - 1 \int_X r^{\mu+n+1} < \Phi^*(\tilde{\theta}), \phi > (\Phi^*(r))^{-n-1} V ol(\Phi^*(dr^2 + h_r))$$

$$= \text{Res}_{\mu} = -\frac{2}{2} - \lambda - N - 1 \int_X ((\Phi^{-1})^*(r))^{\mu+n+1} < \tilde{\theta}, (\Phi^{-1})^*(\phi) > r^{-n-1} V ol(dr^2 + h_r)$$

$$= \text{Res}_{\mu} = -\frac{2}{2} - \lambda - N - 1 \int_X \left( \frac{(\Phi^{-1})^*(r)}{r} \right)^{\mu+n+1} r^\mu < \tilde{\theta}, (\Phi^{-1})^*(\phi) > V ol(h)$$

Since the boundary value $\theta_0$ of $\theta$ was chosen arbitrarily, we get

$$\Phi^*(h; \lambda) = e^{(\lambda+\frac{2}{2} - \lambda)\sigma} \circ \Phi^*(h; \lambda) \circ \left( \frac{(\Phi^{-1})^*(r)}{r} \right)^{-\lambda+\frac{2}{2} - \lambda} \circ (\Phi^{-1})^*.$$

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For $\lambda \mapsto \lambda + \frac{n}{2} - N$ this formula amounts to

$$
\delta_{N}^{\alpha}(h, \lambda + \frac{n}{2} - N) = e^{\lambda - N} \cdot \delta_{N}^{\alpha}(h, \lambda + \frac{n}{2} - N) \circ (\Phi^{-1})^{*} \circ \left(\frac{\Phi^{*}(r)}{r}\right)^{\lambda}.
$$

A similar statement holds for $\delta_{N}^{\alpha}(h; \lambda + \frac{n}{2} - N)$, which completes the proof. □

Inspired by [Juh09] Theorem 6.2.1, we construct the first order residue family operators on spinors for general hypersurfaces in conformal manifolds. Let $(X, g)$ be a $(n+1)$-dimensional semi-Riemannian Spin-manifold. Consider a hypersurface $\iota : M \to X$ equipped with induced metric $h := \iota^{*}g$, induced Spin-structure and the unit normal vector field $N(g)$. Let $H(g)$ (note the different sign convention for second fundamental form in [Juh09]) be the corresponding mean curvature. Then we define, for $\lambda \in \mathbb{C}$, a family of first order operators

$$
D_{1}(X, M; g, \lambda) : \Gamma(S(X, g)) \to \Gamma(S(X, g)|_{M})
$$

then holds for $\lambda \in S(X, g)$

$$
\theta \mapsto N(g) \cdot \iota^{*}D\Phi^{\lambda} \theta + (2\lambda + n)\iota^{*}\nabla_{\lambda}^{S} \theta
$$

$$
+ (2\lambda + n)(\lambda - \frac{1}{2})\iota^{*}H(g)\theta.
$$

(4.6)

Note that $D_{1}(X, M; g, \lambda)$ is a polynomial in $\lambda$ of degree two, in contrast to the first order residue family operators. Nevertheless, it satisfies the following conformal transformation law:

**Proposition 4.4** Let $\widehat{g} = e^{2\sigma} g$ be conformally equivalent to $g$, for $\sigma \in C^\infty(X)$. Then it holds

$$
e^{-(\lambda-1)\iota^{*}\sigma} D_{1}(X, M; \widehat{g}, \lambda)(e^{\lambda\sigma} \widehat{\theta}) = D_{1}(X, M; g, \lambda)\theta,
$$

for $\theta \in \Gamma(S(X, g))$. Note that $\widehat{\cdot}$ denotes evaluation with respect to $\widehat{g}$.

**Proof.** The proof based on direct computations. Using

$$
\widehat{Y} = e^{-\sigma} Y, \quad \widehat{\theta} = \widehat{\theta}\cdot \widehat{\theta}, \quad N(\widehat{g}) = e^{-\sigma} N(g),
$$

$$
H(\widehat{g}) = e^{-\sigma} (H(g) - N(g)(\sigma)),
$$

$$
\nabla_{\lambda}^{\widehat{g} S} \theta = \nabla_{\lambda}^{\sigma S} \theta - \frac{1}{2}(\lambda - \frac{1}{2})H(\widehat{g})\theta,
$$

$$
\Phi^{\lambda}(e^{-\sigma} \theta) = e^{\lambda\sigma} \Phi^{\lambda} \theta,
$$

for $Y \in \Gamma(TM)$, $\theta \in \Gamma(S(X, g))$, we compute

$$
N(\widehat{g}) \cdot \nabla_{\lambda}^{\sigma} (e^{\lambda\sigma} \widehat{\theta}) = e^{(\lambda-1)\iota^{*}\sigma} \left[N(\widehat{g}) \cdot \nabla_{\lambda}^{\sigma} \theta + (\frac{n}{2} + \lambda)N(g) \cdot \nabla(\sigma) \cdot \theta\right],
$$

$$
(2\lambda + n) \nabla_{\lambda}^{\sigma} \theta = e^{(\lambda-1)\iota^{*}\sigma} \left[(2\lambda + n) \nabla_{\lambda}^{\sigma} \theta - (\frac{n}{2} + \lambda)N(g) \cdot \nabla(\sigma) \cdot \theta + (2\lambda + n)(\lambda - \frac{1}{2})H(\widehat{g})\theta\right],
$$

$$
(2\lambda + n)(\lambda - \frac{1}{2})H(\widehat{g})e^{\lambda\sigma} \widehat{\theta} = e^{(\lambda-1)\iota^{*}\sigma}(2\lambda + n)(\lambda - \frac{1}{2})[H(\widehat{g})\theta - N(g)(\sigma)\widehat{\theta}],
$$
Remark 4.5  Consider the conformal compactification \((X, \tilde{g})\) of the Poincaré-Einstein metric \(g_+\) associated to \((M, h)\). In this case we have that \(\iota^* H = 0\) and \(N(g) = 0\). Thus, using equation (2.3) and Example 3.4, we obtain

\[
D_1(X, M; \tilde{g}, \lambda) = \tilde{\nabla}^S_{\partial_r} = \nabla^S_{\partial_r}
\]

hence \(D_1(X, M; g, \lambda)\) generalizes the first order residue family operator on spinors.

The last Proposition indicates the existence of a collection of natural family operators acting on spinors,

\[
\mathcal{H}_N(X, M; g, \lambda) : \Gamma(S(X, g)) \to \Gamma(S(X, g)|_{M})
\]

conformally covariant in the sense that

\[
e^{-(\lambda - N)(\iota^* - \sigma)} \circ \mathcal{H}_N(X, M; \tilde{g}, \lambda) \circ e^{\lambda \sigma} = \mathcal{H}_N(X, M; g, \lambda)
\]

for all \(\tilde{g} = e^{2\sigma} g\) and \(\lambda \in \mathbb{C}\). Notice that in the flat case was the existence of such operators abstractly concluded and their explicit construction was given in [KØSS13], based on the techniques of the classification of homomorphisms of generalized Verma modules.

5 Factorization identities

This section presents a complete set of factorization identities for the residue family operators on spinors in the flat euclidean case \((\mathbb{R}^{n-1}, h)\), while for a semi-Riemannian \(Spin\)-manifold \((M^n, h)\) the factorizations will be demonstrated for residue family operators up to order three.

5.1 Factorization identities - flat case

Let us start with some low order examples of factorizations:

**Example 5.1**  It directly follows from Example 3.8 that low order residue family operators in the flat case obey the following factorizations:

\[
\begin{align*}
\mathcal{H}_1^{res}(h; -\frac{n - 2}{2}) &= \iota^* \cdot D_T \iota^*, \\
\mathcal{H}_1^{res}(h; -\frac{n - 1}{2}) &= e_n \cdot \iota^*(D_T + D_N), \\
\mathcal{H}_2^{res}(h; -\frac{n - 4}{2}) &= e_n \cdot \iota^* \mathcal{H}_1^{res}(h; -\frac{n - 4}{2}), \\
\mathcal{H}_2^{res}(h; -\frac{n - 1}{2}) &= -e_n \cdot \mathcal{H}_1^{res}(h; -\frac{n + 1}{2}) (D_T + D_N), \\
\mathcal{H}_3^{res}(h; -\frac{n - 4}{2}) &= e_n \cdot D_T^3 \iota^*, \\
\mathcal{H}_3^{res}(h; -\frac{n - 3}{2}) &= e_n \cdot \iota^*(D_T + D_N)^3, \\
\mathcal{H}_3^{res}(h; -\frac{n - 6}{2}) &= e_n \cdot D_T \mathcal{H}_2^{res}(h; -\frac{n - 6}{2}), \\
\mathcal{H}_3^{res}(h; -\frac{n - 1}{2}) &= e_n \cdot \mathcal{H}_2^{res}(h; -\frac{n + 1}{2}) (D_T + D_N).
\end{align*}
\]
Now we prove the general form of factorization identities.

**Theorem 5.2** For $N \geq 1$ and $0 \leq M \leq N - 1$, the even spinor residue family operators have factorization properties of the form

$$\mathcal{B}_{2N}^{\text{res}}(h; 2N - \frac{n}{2} - M) = e_n \cdot D_T^{2M+1} \mathcal{B}_{2N-2M-1}^{\text{res}}(h; 2N - \frac{n}{2} - M), \quad (5.1)$$

and for $N \geq 1$ and $0 \leq M \leq N$, the odd ones fulfill

$$\mathcal{B}_{2N+1}^{\text{res}}(h; 2N + 1 - \frac{n}{2} - M) = e_n \cdot D_T^{2M+1} \mathcal{B}_{2N-2M}^{\text{res}}(h; 2N + 1 - \frac{n}{2} - M). \quad (5.2)$$

**Proof.** In what follows we omit the restriction map $\iota^*$. We start to prove equation $(5.1)$, using explicit formulas for the spinor residue family operators, cf. Theorem 5.7. First of all, we have

$$\mathcal{B}_{2N}^{\text{res}}(h; 2N - \frac{n}{2} - M) = (-1)^N N \prod_{j=0}^{N-1} a_j^{(N)}(2N - \frac{n}{2} - M) D_T^{2j} D_N^{2N-2j-1}$$

$$+ 2N \sum_{j=0}^{N-1} b_j^{(N-1)}(2N - \frac{n}{2} - M) D_T^{2j+1} D_N^{2N-2j-1}.$$  

On the other hand, we have

$$\mathcal{B}_{2N-2M-1}^{\text{res}}(h; 2N - \frac{n}{2} - M) = \mathcal{B}_{2(N-M-1)+1}^{\text{res}}(h; 2N - \frac{n}{2} - M) =$$

$$= (-1)^{N-M} e_n \cdot \left[ \frac{N-M-1}{N-M} \prod_{j=0}^{N-M-1} a_j^{(N-M-1)}(2N - \frac{n}{2} - M) D_T^{2j} D_N^{2N-2M-2j-1}$$

$$- \sum_{j=0}^{N-M-1} b_j^{(N-M-1)}(2N - \frac{n}{2} - M) D_T^{2j+1} D_N^{2N-2M-2j-2} \right].$$

Multiplying the last formula by $e_n \cdot D_T^{2M+1} = -D_T^{2M+1} e_n$, and shifting the summation index gives

$$e_n \cdot D_T^{2M+1} \mathcal{B}_{2(N-M)-1}^{\text{res}}(h; 2N - \frac{n}{2} - M) =$$

$$= (-1)^{N-M} \left[ 2N \sum_{j=M}^{N-M-1} b_j^{(N-M-1)}(2N - \frac{n}{2} - M) D_T^{2j+1} D_N^{2N-2j-1-2}$$

$$- \sum_{j=M+1}^{N} a_j^{(N-M-1)}(2N - \frac{n}{2} - M) D_T^{2j} D_N^{2N-2j-2} \right].$$

This equals to $\mathcal{B}_{2N}^{\text{res}}(h; 2N - \frac{n}{2} - M)$ provided we have for all $j = M + 1, \ldots, N$

$$(-1)^{N-M} a_j^{(N)}(2N - \frac{n}{2} - M) = (-1)^{N-M-1} a_{j-M-1}^{(N-M-1)}(2N - \frac{n}{2} - M), \quad (5.3)$$

$$\text{for } j = M + 1, \ldots, N.$$
and for all $j = 0, \ldots, M$ we have $a_j^{(N)}(2N - \frac{n+1}{2} - M) = 0$, as well as for all $j = M, \ldots, N - 1$ we have
\[
(-1)^{M-N-j}(2N - \frac{n}{2} - M - \frac{1}{2}) = (-1)^{N-M-j}(N-M-1)(2N - \frac{n}{2} - M - \frac{1}{2}),
\]

and for all $j = 0, \ldots, M-1$ holds $b_j^{(N-1)}(2N - \frac{n+1}{2} - M) = 0$. Using definition (3.6), it directly follows that equation (5.3) is equivalent to
\[
\sum_{j=1}^{N-M} b_j^{(N)}(2N + 1 - \frac{n}{2} - M) D_T^2 D_N^{2j+1} = 0 \quad \text{for all } j = 0, \ldots, N.
\]

Using definition (3.7), similar statement for $b_j^{(N-1)}$. Now we discuss equation (5.2). From Theorem 3.7 we have
\[
(2N - D_T^2 D_N^{2j+1}) = (2N + 1 - \frac{n}{2} - M) D_T^2 D_N^{2j+1} D_N^{2j+1} D_N^{2j+1}.
\]

We multiply the last formula by $c_n \cdot D_T^{2j+1}$ and shift the summation index, resulting in
\[
e_n D_T^{2j+1} D_N^{2j+1} D_N^{2j+1} = (2N + 1 - \frac{n}{2} - M) D_T^2 D_N^{2j+1} D_N^{2j+1} D_N^{2j+1}.
\]

This equals to $b_j^{(N-1)}(2N + 1 - \frac{n}{2} - M)$ provided we have for all $j = M, \ldots, N$
\[
(-1)^{N-j}(2N + 1 - \frac{n}{2} - M) = (-1)^{N-M-j}(N-M-j)(2N + 1 - \frac{n}{2} - M),
\]

for all $j = 0, \ldots, M$. One can also prove, using definition (3.7), similar statement for $b_j^{(N-1)}$. From Theorem 3.7 we have
\[
\prod_{j=0}^{N-1} (2k - 2M) = \prod_{k=0}^{N-M-2} (2k + N + 4).
\]
and for all $j = 0, \ldots, M - 1$ we have $a_j^{(N)}(2N + \frac{1}{2} - M) = 0$, as well as for all $j = M + 1, \ldots, N$ we have 

$$(-1)^M b_j^{(N)}(2N + \frac{1}{2} - M) = (-1)^{N-M-1}b_j^{(N-M-1)}(2N + \frac{1}{2} - M),$$

and for all $j = 0, \ldots, M$ we have $b_j^{(N)}(2N + \frac{1}{2} - M) = 0$. These properties are easily checked from definitions (3.6) and (3.7), and this completes the proof. □

The next theorem states that the residue family operators afford factorization properties from the right by powers of the Dirac operator $D_T + D_N$ on $\mathbb{R}^n$. Since this proof is more difficult than the left factorization, we start with two preparatory lemmas.

**Lemma 5.3** For $M \in \mathbb{N}_0$, we have

$$(D_T + D_N)^{2M+1} = \sum_{l=0}^{2M+1} \binom{M}{\frac{l}{2}} D_T^l D_N^{2M+1-l}$$

$$= \sum_{l=0}^{M} \binom{M}{l} D_T^{2l} D_N^{2M+1-2l} + \sum_{l=0}^{M} \binom{M}{l} D_T^{2l+1} D_N^{2M-2l},$$

where $[k]$ denotes the integer part of $k$.

**Proof.** The proof goes by induction. The case $M = 0$ holds for trivial reasons, and we assume the statement holds for $M$. Then we have for $M + 1$:

$$(D_T + D_N)^{2(M+1)+1} = (D_T + D_N)^{2M+1} (D_T + D_N)^2$$

$$= \sum_{l=0}^{2M+1} \binom{M}{\frac{l}{2}} D_T^l D_N^{2M+1-l} (D_T + D_N)^2$$

$$= \sum_{l=2}^{2M+3} \binom{M}{\frac{l-2}{2}} D_T^l D_N^{2M+3-l} + \sum_{l=0}^{2M+1} \binom{M}{\frac{l}{2}} D_T^l D_N^{2M+3-l}$$

$$= \sum_{l=0}^{2(M+1)+1} \binom{M+1}{\frac{l}{2}} D_T^l D_N^{2(M+1)+1-l},$$

which completes the induction step. Decomposing the sum into even and odd values of the summation index $l$ gives the desired last statement. □

The next lemma treats several combinatorial identities involving binomial coefficients and Pochhammer symbols. We recall the definition of Pochhammer symbol $(a)_l = a(a + 1) \ldots (a + l - 1)$ and $(a)_0 := 1$, for $a \in \mathbb{R}$ and $l \in \mathbb{N}_0$.

**Lemma 5.4**

1. For all $j, M, N \in \mathbb{N}$ such that $1 \leq M \leq N - M - 1$ and $M \leq j \leq N - M - 1$, the identity

$$(N - M)_M (N - M + \frac{1}{2})_M =$$

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\[ M \sum_{l=0}^{M}(M \atop l)(j-l+1)_{l}(N-M-j+l)_{M-l} \times \\
\times (N-M-j+l+\frac{1}{2})_{M-l}(j-l-2N+M+\frac{3}{2})_{l}, \quad (5.5) \]

holds true.

2. For all \( j, M, N \in \mathbb{N} \) such that \( 1 \leq M \leq N-M-1 \) and \( M \leq j \leq N-M-1 \), the identity

\[
(N-M+\frac{1}{2})_{M}(N-M+1)_{M} = \\
\sum_{l=0}^{M}(-1)^{l}(M \atop l)(j-l+1)_{l}(N-M-j+l+\frac{1}{2})_{M-l} \times \\
\times (N-M-j+l+1)_{M-l}(j-l-2N+M+\frac{1}{2})_{l},
\]

holds true.

3. For all \( j, M, N \in \mathbb{N} \) such that \( 1 \leq M \leq N-M \) and \( M+1 \leq j \leq N-M \), the identity

\[
(N-M+1)_{M}(N-M+\frac{3}{2})_{M} = \\
\sum_{l=0}^{M}(-1)^{l}(M \atop l)(j-l+1)_{l}(N-M-j+l+1)_{M-l} \times \\
\times (N-M-j+l+\frac{3}{2})_{M-l}(j-l-2N+M+\frac{1}{2})_{l},
\]

holds true.

**Proof.** We shall prove only the first statement, since the second and third are proved analogously. Recall the definition of a generalized hypergeometric function \( pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) \) defined by

\[
pF_q = \begin{pmatrix} a_1, a_2, \ldots, a_p \cr b_1, b_2, \ldots, b_q \end{pmatrix} z = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_p)_n}{(b_1)_n(b_2)_n \ldots (b_q)_n} \frac{z^n}{n!}. \quad (5.6)
\]

A direct comparison converts our binomial expression into balanced \( 3F_2 \) series,

\[
3F_2\left( \begin{array}{c} -j, 2N-M-j+\frac{1}{2}, -M \\
N-M-j, N-M-j+\frac{1}{2} \end{array} ; 1 \right) \times \\
\times 4^{-M}(-2j-2M+2N)_{2M} = \sum_{l=0}^{\infty}(-1)^{l}(M \atop l)(j-l+1)_{l} \times \\
\times 4^{l-M} \times (2N-2M-2j+2l)_{2(M-l)}(j-l-2N+M+\frac{3}{2})_{l},
\]

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which terminates the summation index by \( l = M \). Now we use the Pfaff-Saalschütz summation formula, cf. [Sla66, Appendix III.2],

\[
\binom{a, b, -n}{c, 1 + a + b - c - n; 1} = \frac{(-a + c) \cdot (-b + c)_{-n}}{(c)_{-a - b + c} n}.
\]  

(5.7)

for \( n \in \mathbb{N} \). This converts the hypergeometric terminating expansion for \( \binom{a, b, -n}{c, 1 + a + b - c - n; 1} \) into required binomial identity and the proof is complete. □

Remark 5.5 The proof of Lemma 5.4 was communicated to the authors by Ch. Krattenthaler. He also informed us that there is a Gosper-Zeilberger algorithm, cf. [PS95] and references therein for its implementation, allowing to find a recurrence for binomial sums. In particular, denoting the right hand side of equation (5.5) by \( S[N] \), one finds the functional recurrence relation

\[
N(1 + 2N)S[N] = (-1 + 2M - 2N)(M - N)S[N + 1] = 0.
\]  

(5.8)

Now we can prove the second set of factorizations.

Theorem 5.6 For \( N \geq 1 \) and \( 0 \leq M \leq N - 1 \), the even spinor residue family operators have the right factorization property

\[
\mathcal{D}_{2N}^{res}(h; \frac{1 - n}{2} + M) = -e_n \cdot \mathcal{D}_{2(N-M-1)+1}^{res}(h; -\frac{1 + n}{2} - M)(D_T + D_N)^{2M+1},
\]  

(5.9)

while for \( N \geq 1 \) and \( 0 \leq M \leq N \), the odd ones fulfill

\[
\mathcal{D}_{2N+1}^{res}(h; \frac{1 - n}{2} + M) = e_n \cdot \mathcal{D}_{2N-2M}^{res}(h; -\frac{n + 1}{2} - M)(D_T + D_N)^{2M+1}.
\]  

(5.10)

Proof. It follows from Theorem 4.7 that

\[
\mathcal{D}_{2N}^{res}(h; \frac{1 - n}{2} + M) = (-1)^N \sum_{j=0}^{N} a_j^{(N)}(-\frac{n}{2} + M)D_T^{2j}D_N^{2N-2j}
\]

\[
+ 2N \sum_{j=0}^{N-1} b_j^{(N-1)}(-\frac{n}{2} + M)D_T^{2j+1}D_N^{2N-2j-1},
\]

and

\[
\mathcal{D}_{2(N-M-1)+1}^{res}(h; -\frac{1 - n}{2} - M) =
\]

\[
= (-1)^{N-M} e_n \cdot \left[ - \sum_{j=0}^{N-M-1} a_j^{(N-M-1)}(-\frac{n}{2} - M - 1)D_T^{2j+1}D_N^{2(N-M-1)-2j} \right].
\]
\[-(2N+1)^{N-M-1} \sum_{j=0}^{N-M-1} b_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) D_T^{2j} D_N^{2(N-M-1)-2j+1}\].

Multiplying the last formula from the left by \(-e_n\) and from the right by \((D_T + D_N)^{2M+1}\) gives

\[-e_n \cdot b_0^{(N-M-1)+1}(h; -\frac{1}{2} - M) (D_T + D_N)^{2M+1} =
\]

\[= -(N-M)^{-1} \left[ -(2N+1) \sum_{j=0}^{N-M-1} b_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) D_T^{2j} D_N^{2(N-M-1)-2j+1}\right]
\]

\[- \sum_{j=0}^{N-M-1} a_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) D_T^{2j+1} D_N^{2(N-M-1)-2j+2j+1}\]

\[= (2N+1) \sum_{j=0}^{N-M-1} \sum_{l=0}^{M} b_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) \left(\frac{M}{l}\right) D_T^{2j+2l} D_N^{2(N-2j-2l-1)}\]

\[- \sum_{j=0}^{N-M-1} \sum_{l=0}^{M} a_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) \left(\frac{M}{l}\right) D_T^{2j+2l+1} D_N^{2(N-2j-2l-1)}\]

\[= -(N-M)^{-1} \left[ -(2N+1) \sum_{l=0}^{M} b_0^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) \left(\frac{M}{l}\right) D_T^{2l} D_N^{2(N-2l)}\right]
\]

\[- \sum_{l=0}^{N-M-1} a_{l+1}^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) \left(\frac{M}{l}\right) D_T^{2l+1} D_N^{2(N-2l)}\]

\[- \sum_{j=0}^{N-M-1} \sum_{l=0}^{M} \left[(2N+1)b_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right)\right]\]

\[- \left(a_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) \left(\frac{M}{l}\right) D_T^{2j+2l} D_N^{2(N-2j-2l)}\right] (5.11)
\]

\[+ \sum_{l=0}^{N-M-1} \sum_{j=0}^{M} \left[(2N+1)b_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right)\right]\]

\[- \left(-a_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) \left(\frac{M}{l}\right) D_T^{2j+2l+1} D_N^{2(N-2j-2l-1)}\right].\]

Let us further assume \(M \leq N-M-1\), the case \(N-M-1 < M\) is analogous and follows from the first by transformation \(M \rightarrow N-M-1\). We introduce the abbreviations

\[c_j := (2N+1)b_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right) + a_j^{(N-M-1)} \left(-\frac{n}{2} - M - 1\right), \quad (5.13)\]
respectively,

\[ d_j := (2N + 1) b_j^{(N-M-1)} \left( -\frac{n}{2} - M - 1 \right) - c_j^{(N-M-1)} \left( -\frac{n}{2} - M - 1 \right), \quad (5.14) \]

for which two sums in equation (5.11) and (5.12) can be reformulated as

\[
\sum_{j=1}^{N-M-1} \sum_{l=0}^{M} c_j \left( \begin{array}{c} M \\ l \end{array} \right) D_T^{2j+2l} D_N^{2N-2j-2l} = \\
= M \sum_{i=1}^{i-1} c_{i-1} \left( \begin{array}{c} M \\ l \end{array} \right) D_T^{2i} D_N^{2N-2i} + \sum_{i=M+1}^{N-M-1} \sum_{l=0}^{M} c_{i-1} \left( \begin{array}{c} M \\ l \end{array} \right) D_T^{2i} D_N^{2N-2i} + \\
+ \sum_{i=N-M}^{N-1} \sum_{l=i-N+M+1}^{M} c_{i-1} \left( \begin{array}{c} M \\ l \end{array} \right) D_T^{2i} D_N^{2N-2i},
\]

respectively,

\[
\sum_{j=0}^{N-M-1} \sum_{l=0}^{M} d_j \left( \begin{array}{c} M \\ l \end{array} \right) D_T^{2j+2l+1} D_N^{2N-2j-2l-1} = \\
= \sum_{i=0}^{i-1} d_{i-1} \left( \begin{array}{c} M \\ l \end{array} \right) D_T^{2i+1} D_N^{2N-2i-1} + \sum_{i=M}^{N-M-1} \sum_{l=0}^{M} d_{i-1} \left( \begin{array}{c} M \\ l \end{array} \right) D_T^{2i+1} D_N^{2N-2i-1} + \\
+ \sum_{i=N-M}^{N-1} \sum_{l=i-N+M+1}^{M} d_{i-1} \left( \begin{array}{c} M \\ l \end{array} \right) D_T^{2i+1} D_N^{2N-2i-1}.
\]

Then equation (5.19) is equivalent, when comparing the coefficients by \( D_T^k D_N^{2N-k} \), to the following system of equations: for even \( k \) we get

\[ (-1)^N c_0^{(N)} \left( -\frac{n}{2} + M \right) = (-1)^{N-M-1} (2N + 1) b_0^{(N-M-1)} \left( -\frac{n}{2} - M - 1 \right), \quad (5.15) \]

\[ (-1)^N c_j^{(N)} \left( -\frac{n}{2} + M \right) = (-1)^{N-M-1} \sum_{l=0}^{j} \left( \begin{array}{c} M \\ l \end{array} \right) c_{j-l}, \quad \forall 1 \leq j \leq M, \quad (5.16) \]

\[ (-1)^N c_j^{(N)} \left( -\frac{n}{2} + M \right) = (-1)^{N-M-1} \sum_{l=0}^{M} \left( \begin{array}{c} M \\ l \end{array} \right) c_{j-l}, \quad \forall M + 1 \leq j \leq N - M - 1, \quad (5.17) \]

\[ (-1)^N c_j^{(N)} \left( -\frac{n}{2} + M \right) = (-1)^{N-M-1} \sum_{l=j-N+M}^{M} \left( \begin{array}{c} M \\ l \end{array} \right) c_{j-l}, \quad \forall N - M \leq j \leq N - 1, \quad (5.18) \]

whereas for odd \( k \) we have

\[ (-1)^N 2Nb_j^{(N-1)} \left( -\frac{n}{2} + M \right) = (-1)^{N-M} \sum_{l=0}^{j} \left( \begin{array}{c} M \\ l \end{array} \right) d_{j-l}, \quad \forall 0 \leq j \leq M - 1, \quad (5.19) \]

\[ (-1)^N 2Nb_j^{(N-1)} \left( -\frac{n}{2} + M \right) = (-1)^{N-M} \sum_{l=0}^{M} \left( \begin{array}{c} M \\ l \end{array} \right) d_{j-l}, \quad \forall M \leq j \leq N - M - 1, \quad (5.20) \]
\[
(-1)^N 2N b_j^{(N-1)} \left( -\frac{n}{2} + M \right) = (-1)^N \sum_{l=j-N+M+1}^{M} \binom{M}{l} \ d_{j-l}, \quad \forall \ N - M \leq j \leq N - 1.
\]

(5.21)

Based on definitions \((5.6)\) and \((5.7)\), an easy computation proves equation \((5.15)\).

Equations \((5.10)\) up to \((5.21)\) can be checked in a quite uniform way. First of all, definitions \((5.6)\) and \((5.7)\) allow us to compute

\[
(-1)^{N-M-1} c_{j-l} = (2N+1) b_j^{(N-M-1)} \left( -\frac{n}{2} - M - 1 \right) + a_j^{(N-M-1)} \left( -\frac{n}{2} - M - 1 \right)
\]

\[
= (2N+1) \left( \frac{N-M-1}{j-l} \right) ! (N-M-1-j+l) \left( \frac{N-M-j+l}{j-l} + \frac{1}{2} \right) (N-M-j+l)
\]

\[
\times (j-l) \left( \frac{N-M}{j-l} \right) ! (N-M-j+l)
\]

\[
\times (j-l) \left( \frac{N-M-2j+2l-1}{j-l} \right) ! (N-M-2j+2l)
\]

\[
= (N-M-1) ! (N-M-j+l)
\]

and similarly one gets

\[
(-1)^{N-M-1} d_{j-l} = (2N+1) b_j^{(N-M-1)} \left( -\frac{n}{2} - M - 1 \right) - a_j^{(N-M-1)} \left( -\frac{n}{2} - M - 1 \right)
\]

\[
= 2N \left( \frac{N-M-1}{j-l} \right) ! (N-M-1-j+l-1) \left( \frac{N-M-j+l-1}{j-l} + \frac{3}{2} \right) (N-M-j+l-1)
\]

Thus, for example, equation \((5.17)\) is equivalent to

\[
\frac{N!}{j!} \left( \frac{N-M}{j-l} \right) ! (j-l) ! (N-M-j+l+1) (j-l) ! (N-M-2j+2l) ! (j-l) ! (N-M-j+l)
\]

\[
= \sum_{l=0}^{M} \left( \frac{M}{l} \right) \left( \frac{N-M}{j-l} \right) ! (N-M-j+l+1) (j-l) ! (N-M-2j+2l) ! (j-l) ! (N-M-j+l)
\]

which can be reduced, using Pochhammer identities, to

\[
(N-M+1) M (N-M+1) \left( \frac{1}{2} \right) M = \sum_{l=0}^{M} (-1)^l \binom{M}{l} \left( \frac{1}{2} \right) M \times
\]

\[
\times (j-l) ! (2N-2j+2l+1) \left( \frac{1}{2} \right) M (j-l+1)
\]

\[
= \sum_{l=0}^{M} (-1)^l \binom{M}{l} (j-l) ! (2N-2j+2l+1) \left( \frac{1}{2} \right) M (j-l+1)
\]

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\[ \times (N - M - j + l + \frac{1}{2})_{M-1}(N - M - j + l + 1)_{M-1}(j - l + 1)t. \]

Lemma 5.3 allows us to conclude that the last equality is true, and therefore equation (5.17) holds. Similarly, one observes that equations (5.16) and (5.18) follow from Lemma 5.4 due to the range of the index \( j \) making trivial some of the Pochhammer symbols and allowing to rearrange the summation array accordingly. Let us proceed to the proof of equation (5.20). This amounts to check

\[ 2N \frac{(N - 1)!(-4)^{N-j-1}}{j!(2N - 2j - 1)!}(j - 2N + M + \frac{3}{2})_{N-j} = \]

\[ = 2N \sum_{l=0}^{M} \binom{M}{l} \frac{(N - M - 1)!(-4)^{N-M-j+l-1}}{(j - l)!(2N - 2M - 2j + 2l - 1)!} \times \]

\[ \times (j - l - 2N + M + \frac{3}{2})_{N-M-j+l-1}, \quad \text{(5.22)} \]

which is equivalent, using Pochhammer identities, to

\[ (N - M)_M(N - M + \frac{1}{2})_M = \sum_{l=0}^{M} (-1)^j \binom{M}{l} (j - 2N + M + \frac{3}{2})_l \times \]

\[ \times (N - M - j + l)_M(N - M - j + l + \frac{1}{2})_{M-1}(j - l + 1)t, \quad \text{(5.23)} \]

and this holds true due to Lemma 5.4. Again, equations (5.19) and (5.21) can be checked analogously using Lemma 5.4 and taking into account the range of the index \( j \).

Now we pass to the second set of factorizations. Based on abbreviations

\[ c_j := -2(N - M) b_{j-1}^{(N-M-1)}(-\frac{n}{2} - M - 1) + a_j^{(N-M)}(-\frac{n}{2} - M - 1), \quad \text{(5.24)} \]

\[ d_j := 2(N - M) b_j^{(N-M-1)}(-\frac{n}{2} - M - 1) - a_j^{(N-M)}(-\frac{n}{2} - M - 1), \quad \text{(5.25)} \]

equation (5.10) is equivalent to the system of equalities

\[ (-1)^N(2N - 2M - 1)b_0^{(N)}(-\frac{n}{2} + M) = (-1)^{N-M}a_0^{(N-M)}(-\frac{n}{2} - M - 1), \quad \text{(5.26)} \]

\[ (-1)^N(2N - 2M - 1)b_j^{(N)}(-\frac{n}{2} + M) \]

\[ = (-1)^{N-M} \sum_{l=0}^{j} \binom{M}{l} c_{j-l}, \forall 1 \leq j \leq M, \quad \text{(5.27)} \]

\[ (-1)^N(2N - 2M - 1)b_0^{(N)}(-\frac{n}{2} + M) \]

\[ = (-1)^{N-M} \sum_{l=0}^{j} \binom{M}{l} c_{j-l}, \forall M + 1 \leq j \leq N - M, \quad \text{(5.28)} \]

\[ (-1)^N(2N - 2M - 1)b_0^{(N)}(-\frac{n}{2} + M) \]

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\[(N - M) \sum_{l=0}^{j} \binom{M}{l} c_{j-l} \forall N - M + 1 \leq j \leq N, \]  
(5.29)

and

\[(-1)^N a_j^{(N)}(-\frac{n}{2} + M) = (-1)^{N-M} \sum_{l=0}^{j} \binom{M}{l} d_{j-l} \forall 0 \leq j \leq M - 1, \]  
(5.30)

\[(-1)^N a_j^{(N)}(-\frac{n}{2} + M) = (-1)^{N-M} \sum_{l=0}^{M} \binom{M}{l} d_{j-l} \forall M \leq j \leq N - M - 1, \]  
(5.31)

\[(-1)^N a_j^{(N)}(-\frac{n}{2} + M) = (-1)^{N-M} \sum_{l=j-N+M}^{M} \binom{M}{l} d_{j-l} \forall N - M \leq j \leq N - 1, \]  
(5.32)

By similar arguments as for the factorizations of \(\mathcal{D}_{\text{res}}^{2N}(h; \lambda)\) it follows that equations (5.26) - (5.32) holds. This completes the proof of the theorem. \(\square\)

A direct consequence of the last theorem (the case \(M = N\)) together with the conformal covariance of residue family operators, cf. Theorem 4.2, is a well known observation:

**Corollary 5.7** Let \(N \in \mathbb{N}_0\). The \((2N + 1)\)-th power of the Dirac operator on the flat euclidean space is conformally covariant.

**Remark 5.8** We finish by several remarks on the target space of residue family operators

\[\mathcal{D}^\text{res}_N(h; \lambda) : \Gamma(S(\mathbb{R}^n, \bar{g})) \to \Gamma(S(\mathbb{R}^n, \bar{g})|_{x_n=0}).\]

In the case when the identification map \(\Xi\), see Remark 2.1, is included into the definition of residue family operators, the first and second order family operators can be written as

\[\mathcal{D}^\text{res}_1(h; \lambda) = \mathcal{D}_T \circ \Xi \circ \iota^* - (2\lambda + n - 2)\Xi \circ \iota^* e_n \cdot \mathcal{D}_N,\]

\[\mathcal{D}^\text{res}_2(h; \lambda) = \mathcal{D}_T^2 \circ \Xi \circ \iota^* - 2\mathcal{D}_T \circ \Xi \circ \iota^* e_n \cdot \mathcal{D}_N + (2\lambda + n - 4)\Xi \circ \iota^* \mathcal{D}_N^2,\]

hence they are maps from \(\Gamma(S(\mathbb{R}^n, \bar{g}))\) to \(\Gamma(S(\mathbb{R}^{n-1}, \bar{h}))\). The simultaneous change of \(\Xi := \Xi^\rightarrow\), cf. Remark 2.1 then leads to a partial change of signs. Consequently, one can not prove the factorization properties in general: some factorizations still survive, but for example for \(\mathcal{D}^\text{res}_2\) we have

\[\mathcal{D}^\text{res}_2(h; -\frac{n-1}{2}) = \mathcal{D}_T^2 \circ \Xi \circ \iota^* - 2\mathcal{D}_T \circ \Xi \circ \iota^* e_n \cdot \mathcal{D}_N + 3\Xi \circ \iota^* \mathcal{D}_N^2,\]

whereas

\[\mathcal{D}^\text{res}_1(h; -\frac{n+1}{2})(\mathcal{D}_T + \mathcal{D}_N) = \mathcal{D}_T^2 \circ \Xi \circ \iota^* e_n \cdot -2\mathcal{D}_T \circ \Xi \circ \iota^* \mathcal{D}_N\]
Factorization identities hold true if one considers $S(\mathbb{R}^n, \tilde{g})|_{z_n=0}$ as a target space for residue family operators, i.e., omitting the identification map $\Xi$.

### 5.2 Factorization identities of low order - curved case

We start with a general description of the geometry of embedded hypersurfaces and related techniques associated to the geometry of Spin-structures, cf. [Bur93, BGM05], used to materialize the factorization identities of low order residue families operators on spinors.

Let $\iota : M^n \hookrightarrow Z^{n+1}$ be an embedded hypersurface or boundary of a semi-Riemannian manifold $(Z, g)$ with induced metric $h := \iota^*(g)$ on $M$. The restriction and projection of the Levi-Civita connection $\nabla^g$ on $Z$ to $TM$ is the Levi-Civita connection $\nabla^h$ on $(M, h)$. Assuming the existence of a unit normal vector field $\nu$ to $M$ in $Z$, the second fundamental form of $\iota : M \hookrightarrow Z$ is a scalar valued symmetric 2-tensor $II : \Gamma(TM) \times \Gamma(TM) \rightarrow C^\infty(M)$, $X, Y \mapsto g(\nabla^g_X Y, \nu)$.

The mean curvature of $\iota : M \hookrightarrow Z$ is the normalized $h$-trace of the second fundamental form $II$, i.e., in a local $h$-orthonormal frame $(s_1, \ldots, s_n)$ we have

$$H = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i II(s_i, s_i).$$

For our purposes it is sufficient to consider the concept of generalized cylinders, i.e., $(Z^{n+1} = M^n \times I, g = dr^2 + h_r)$, where $h_r$ is a 1-parameter family of metrics on $M$ and $\partial_r$ denotes the coordinate vector field, cf. [BGM05]. For every $r \in I$ let $M_r := (M \times \{r\}, h_r)$ be the $r$-leaf of $M$ inside $Z$. Recall the Gauß equation

$$\nabla^g_X Y = \nabla^h_X + g(W_r(X), Y)\partial_r, \quad X, Y \in \Gamma(TM)$$

where $W_r$ denotes the Weingarten map associated to the embedding $\iota_r : M \rightarrow M \times \{r\} \subset Z$. For $X \in \Gamma(TM)$ a local coordinate vector field on $M$, we have

$$g(X, \partial_r) = 0, \quad [X, \partial_r] = 0, \quad \nabla^g_{\partial_r} \partial_r = 0,$$

hence for $m \in M$ are the curves $t \mapsto (t, m)$ geodesics parametrized by the arclength.

We define the 1-parameter families of smooth symmetric 2-tensors $\frac{d}{dr} h_r$, $\frac{d^2}{dr^2} h_r$ on $M_r$ by

$$\frac{d}{dr} h_r(X, Y) := \frac{d}{dr}(h_r(X, Y)), \quad \frac{d^2}{dr^2} h_r(X, Y) := \frac{d^2}{dr^2}(h_r(X, Y))$$

for $X, Y \in T_m M$, which appear in the following set of curvature identities, cf. [BGM05]:

$$g(W_r(X), Y) = -\frac{1}{2} \frac{d}{dr} h_r(X, Y),$$

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The formal power series expansions, in the normal coordinate terms on the right hand side of the last equality are
\[
\text{Ric}^Z(\partial_r, \partial_r) = \text{tr}_{h_0}(W_r^2) - \frac{1}{2}\text{tr}_{h_0}(\frac{d^2}{dr^2} h_r),
\]
\[
\text{Ric}^Z(X, \partial_r) = X(\text{tr}_{h_0}(W_r)) - g(\text{div}^M(W_r), X),
\]
for all \(X, Y \in T_{mM}\).

In fact, our local considerations of a neighborhood of conformal compactification \((X, \tilde{g} = dr^2 + h_r)\) of the Poincaré-Einstein metric, cf. Appendix B, fit into the concept of the generalized cylinder. A tubular neighborhood of the conformal boundary \(M\) is isomorphic to \(M \times [0, \varepsilon)\). We can assume that the last coordinate \(r \in I\) generates geodesic curve, i.e., the Killing vector frame \(\partial_r\) is parallel with respect to the Levi-Civita connection of \(\tilde{g}\).

For later purposes we collect a few useful facts.

**Lemma 5.9** [Jahl09, Lemma 6.11.1, Lemma 6.11.2]

For \((M, h)\) and the associated Poincaré-Einstein metric \(g_+\), the two functions \(r \mapsto J(dr^2 + h_r)\) and \(r \mapsto P(dr^2 + h_r)\) satisfy
\[
t^*J = t^*J(h_0) = J, \quad \frac{d}{dr}|_{r=0}(J) = 0,
\]
\[
t^*P = t^*P(h_0) = P,
\]
where \(\tilde{\cdot}\) denotes the evaluation with respect to \(\tilde{g} = r^2 g_+\).

Let us remind some notation and formulas from Section 2.

\[
\widetilde{\nabla}^S = \partial_r - \sum_{i=1}^n \varepsilon_i s_i \cdot \nabla^h_{s_i}, \quad \nabla = \sum_{i=1}^{n+1} \varepsilon_i s_i \cdot \nabla^g_{s_i},
\]
\[
\partial_r \cdot t^* \nabla = \nabla^h - \frac{n}{2} \partial_r H_r - t^* \nabla^g, \quad \nabla^g = \partial_r + \frac{n}{2} H_r,
\]
\[
H_r = \frac{J}{n} - \frac{2}{n} \text{tr}_h(P^2) r^3 + \ldots .
\]

It is then elementary to verify, using the commutator formula [BCM05 Proposition 3.1], the fundamental identity
\[
[\nabla, \nabla^g_{s_i}] = \frac{1}{2} \text{Ric}^X(\partial_r) \cdot \sum_{i=1}^n \varepsilon_i W_r(s_i) \cdot \nabla^h_{s_i} - \frac{1}{2} \partial_r \cdot \sum_{i=1}^n \varepsilon_i W_r(s_i) \cdot W_r(s_i) \cdot \frac{1}{2} \partial_r.
\]

The formal power series expansions, in the normal coordinate \(r\), of the three terms on the right hand side of the last equality are
\[
\frac{1}{2} \text{Ric}^X(\partial_r) = \frac{1}{2} \sum_{i=1}^n \varepsilon_i \text{Ric}^X(\partial_r, s_i) s_i + \frac{1}{2} \text{Ric}^X(\partial_r, \partial_r) \partial_r = \frac{1}{2} J \partial_r + O(r^2),
\]
\[
\sum_{i=1}^n \varepsilon_i W_r(s_i) \cdot \nabla^h_{s_i} = r \sum_{i=1}^n \varepsilon_i P(s_i) \cdot \nabla^h_{s_i} + O(r^2),
\]
\[
\frac{1}{2} \partial_r \cdot \sum_{i=1}^n \varepsilon_i W_r(s_i) \cdot W_r(s_i) = O(r^2),
\]

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and collecting all terms together gives
\[
\frac{1}{2} \text{Ric}^X(\partial_r) = \sum_{i=1}^n \varepsilon_i W_r(s_i) \cdot \nabla_{s_i}^h r - \frac{1}{2} \partial_r \cdot \sum_{i=1}^n \varepsilon_i W_r(s_i) \cdot W_r(s_i).
\]

\[
= \frac{1}{2} J \partial_r \cdot -r h(P, \tilde{\nabla}^h) + O(v^2).
\]  
(5.37)

Based on the review of the fundamental curvature identities for the embedding of the conformal manifold \((M, h)\) into the conformal compactification \((X, \bar{g})\) of the Poincaré-Einstein metric \(g_+\), we are ready to discuss the curved version of factorization identities for residue family operators on spinors of low order. We shall first observe the factorization properties for the first order residue family operator. From the explicit formula in Example 3.4 and equation (5.35) we get the following lemma.

**Lemma 5.10** The residue family operator \(\tilde{\mathcal{D}}_{\text{res}}^1(h; \lambda)\) has the following factorization properties:

1. \(\tilde{\mathcal{D}}_{\text{res}}^1(h; -\frac{n+1}{2}) = \tilde{D} \iota^*\),
2. \(\tilde{\mathcal{D}}_{\text{res}}^1(h; -\frac{n-1}{2}) = \partial_r \cdot \iota^* \tilde{D}\).

It then follows

\[
\tilde{\mathcal{D}}_{\text{res}}^1(h; \lambda) = (2\lambda + n) \tilde{D} \iota^* - (2\lambda + n - 1) \iota^* \partial_r \cdot \tilde{D}.
\]  
(5.38)

Now we pass to factorization properties for second order residue family operator.

**Lemma 5.11** The residue family operator \(\tilde{\mathcal{D}}_{\text{res}}^2(h; \lambda)\) has the following factorization properties:

1. \(\tilde{\mathcal{D}}_{\text{res}}^2(h; -\frac{n+3}{2}) = \tilde{D} \tilde{\mathcal{D}}_{\text{res}}^1(h; -\frac{n+3}{2})\),
2. \(\tilde{\mathcal{D}}_{\text{res}}^2(h; -\frac{n}{2}) = -\partial_r \cdot \tilde{\mathcal{D}}_{\text{res}}^1(h; -\frac{n+3}{2})\).

The residue family operator \(\tilde{\mathcal{D}}_{\text{res}}^2(h; \lambda)\) can be written as

\[
\tilde{\mathcal{D}}_{\text{res}}^2(h; \lambda) = (2\lambda + n) \tilde{D}^2 \iota^* - (2\lambda + n - 3) \iota^* \tilde{D}^2 - 2\partial_r \cdot \tilde{D} \iota^* \tilde{D}.
\]  
(5.39)

**Proof.** The first factorization follows from Example 3.4:

\[
\tilde{\mathcal{D}}_{\text{res}}^2(h; -\frac{n+3}{2}) = \tilde{D} \tilde{\mathcal{D}}_{\text{res}}^1(h; -\frac{n+3}{2}).
\]

The second one follows from Example 3.4 and equation (5.38):

\[
\tilde{\mathcal{D}}_{\text{res}}^2(h; -\frac{n}{2}) = -3 \iota^* \nabla_{\partial_r} \nabla_{\partial_r}^S \nabla_{\partial_r}^S + \tilde{D}^2 \iota^* + \frac{3}{2} \partial_r \cdot \tilde{D} \iota^* \tilde{D} + 2 \tilde{D} \iota^* \nabla_{\partial_r}^S.
\]
\[ \mathcal{B}_1^{res}(h; -\frac{n+2}{2}) = -2\hat{\theta} v^* \hat{\theta} + \partial_r \cdot \iota^* \hat{\theta}^2 \]
\[ = 2\partial_r \cdot \hat{\theta} v^* \nabla_{\partial_r} = 2\partial_r \cdot \hat{\theta}^2 + 3\partial_r \cdot \hat{\theta}^2 v^* \]
\[ - 3\partial_r \cdot \hat{\theta} v^* \nabla_{\partial_r} = 3\partial_r \cdot \hat{\theta}^2 \nabla_{\partial_r} + 3\partial_r \cdot \hat{\theta}^2 \nabla_{\partial_r}. \]

The combination of equations (5.35), (5.36) and (5.37) yields
\[ \mathcal{B}_1^{res}(h; -\frac{n+2}{2}) \hat{\theta} = \partial_r \cdot \mathcal{B}_2^{res}(h; -\frac{n}{2}). \]

As for the explicit formula for \( \mathcal{B}_2^{res}(h; \lambda) \), it follows from its factorization properties and equation (5.38) that
\[ \mathcal{B}_2^{res}(h; \lambda) = \frac{1}{3}(2\lambda + n)\tilde{\hat{\theta}} \mathcal{G}_1^{res}(h; -\frac{n+2}{2}) + \frac{1}{3}(2\lambda + n - 3)\partial_r \cdot \mathcal{B}_1^{res}(h; -\frac{n+2}{2}) \hat{\theta} \]
\[ = \frac{1}{3}(2\lambda + n)\tilde{\hat{\theta}}((-n + 3 + n)\hat{\theta}^\ast - (-n + 3 + n - 1)\partial_r \cdot \iota^* \hat{\theta}) \]
\[ + \frac{1}{3}(2\lambda + n - 3)\partial_r \cdot ((-n + 2 + n)\hat{\theta}^\ast - (-n + 2 + n - 1)\partial_r \cdot \iota^* \hat{\theta}) \hat{\theta} \]
\[ = (2\lambda + n)\tilde{\hat{\theta}} \hat{\theta}^\ast - (2\lambda + n + 3)\partial_r \cdot \hat{\theta}^\ast + 2\partial_r \cdot \hat{\theta} \iota^* \hat{\theta}. \]

The proof is complete. \( \square \)

The factorization properties for the third order residue family operator demonstrate the complexity of the growth of the number of curvature contributions. Before stating the result, we recall the conformal third power of the Dirac operator \( \mathcal{D}_3 = \mathcal{B}^3 - 2h(P, \nabla h, S) - \text{grad}^M(J) \cdot \) on \( (M, h) \), cf. references in Appendix A. We denote by \( \mathcal{D}_3 \) the conformal third power of the Dirac operator on \( (X, \tilde{g}) \). Hence, due to the relation of the Clifford multiplication on \( S(M, h) \) realized inside \( S(X, \tilde{g}) \), we identify \( \mathcal{D}_3 \) inside \( S(X, \tilde{g}) \) with \( \tilde{\mathcal{D}}_3 \), where all Clifford multiplications are replaced by additional multiplication with \( \pm \partial_r \), cf. Remark 2.1.

Then we have

**Lemma 5.12** The residue family operator \( \mathcal{B}_3^{res}(h; \lambda) \) on spinors has the following factorization properties:

1. \( \mathcal{B}_3^{res}(h; -\frac{n-3}{2}) = \tilde{\mathcal{D}}_3^\ast \),
2. \( \mathcal{B}_3^{res}(h; -\frac{n-5}{2}) = \tilde{\mathcal{D}}_3 \mathcal{B}_2^{res}(h; -\frac{n-5}{2}) \),
3. \( \mathcal{B}_3^{res}(h; -\frac{n+1}{2}) = \partial_r \cdot \mathcal{B}_2^{res}(h; -\frac{n+1}{2}) \hat{\theta} \),
4. \( \mathcal{B}_3^{res}(h; -\frac{n+2}{2}) = \partial_r \cdot \iota^* \tilde{\mathcal{D}}_3 \).

**Proof.** Let us start with the first identity. From Example 3.4 we have
\[ \mathcal{B}_3^{res}(h; -\frac{n-3}{2}) = \hat{\theta}^3 \hat{\theta}^\ast - 2\partial_r \cdot h(P, \tilde{\nabla} h) \hat{\theta}^\ast + \partial_r \cdot \text{grad}^M(J) \cdot \hat{\theta}^\ast. \]
Finally we come to the last identity. Again, Example 3.4 implies
\[ \tilde{\mathcal{B}}^3 = (\tilde{\mathcal{B}} - 2\partial_r \cdot h(\mathcal{P}, \tilde{\nabla}^h) - \partial_r \cdot \text{grad}^M(J) \cdot \mathcal{I}^* \]
\[ = \tilde{\mathcal{B}}_2 \mathcal{I}^*. \]

As for the second identity, Example 3.4 gives
\[ \tilde{\mathcal{B}}_2^\text{res} (h; -\frac{n-5}{2}) = \tilde{\mathcal{B}} (2\mathcal{I}^* (\nabla^S_{\partial_r})^2 + \tilde{\mathcal{B}}^2 \mathcal{I}^* - J\mathcal{I}^* + 2\tilde{\mathcal{B}} \mathcal{I}^* (\nabla^S_{\partial_r}) \]
\[ = 2\tilde{\mathcal{B}} \mathcal{I}^* (\nabla^S_{\partial_r})^2 + \tilde{\mathcal{B}}^3 \mathcal{I}^* - \partial_r \cdot \text{grad}^M(J) \cdot \mathcal{I}^* \]
\[ + J \tilde{\mathcal{B}} \mathcal{I}^* + 2\tilde{\mathcal{B}} \mathcal{I}^* \nabla^S_{\partial_r} \]
\[ = \tilde{\mathcal{B}}_3^\text{res} (h; -\frac{n-5}{2}). \]

To prove the third identity, we need Example 3.4 again and equations (5.39), (5.36), (5.37). On the one hand,
\[ \tilde{\mathcal{B}}_3^\text{res} (h; -\frac{n}{2}) = 5\mathcal{I}^* (\nabla^S_{\partial_r})^3 - 3\tilde{\mathcal{B}}^2 \mathcal{I}^* \nabla^S_{\partial_r} - \frac{15}{2} \mathcal{I}^* (\nabla^S_{\partial_r})^2 \]
\[ + \tilde{\mathcal{B}} \mathcal{I}^* - 5\partial_r \cdot h(\mathcal{P}, \tilde{\nabla}^h) \mathcal{I}^* + \frac{3}{2} J \tilde{\mathcal{B}} \mathcal{I}^* - \partial_r \cdot \text{grad}^M(J) \cdot \mathcal{I}^*, \]

while
\[ \tilde{\mathcal{B}}_2^\text{res} (h; -\frac{n+2}{2}) \tilde{\mathcal{B}} = (-5\mathcal{I}^* (\nabla^S_{\partial_r})^2 + \tilde{\mathcal{B}}^2 \mathcal{I}^* + \frac{5}{2} \mathcal{I}^* + 2\tilde{\mathcal{B}} \mathcal{I}^* (\nabla^S_{\partial_r}) \]
\[ = 5\partial_r \cdot \tilde{\mathcal{B}} \mathcal{I}^* (\nabla^S_{\partial_r})^2 - 5\partial_r \cdot \mathcal{I}^* (\nabla^S_{\partial_r})^3 + 5\mathcal{I}^* [\tilde{\mathcal{B}}, \nabla^S_{\partial_r}] \nabla^S_{\partial_r} \]
\[ + 5\mathcal{I}^* \nabla^S_{\partial_r} [\tilde{\mathcal{B}}, \nabla^S_{\partial_r}] - \partial_r \cdot \tilde{\mathcal{B}}^3 \mathcal{I}^* + \partial_r \cdot \tilde{\mathcal{B}}^2 \mathcal{I}^* \nabla^S_{\partial_r} - \frac{5}{2} J\partial_r \cdot \tilde{\mathcal{B}} \mathcal{I}^* \]
\[ + \frac{5}{2} J\partial_r \cdot \mathcal{I}^* \nabla^S_{\partial_r} + 2\partial_r \cdot \tilde{\mathcal{B}} \mathcal{I}^* \nabla^S_{\partial_r} - 2\partial_r \cdot \tilde{\mathcal{B}} \mathcal{I}^* (\nabla^S_{\partial_r})^2 - 2\tilde{\mathcal{B}} \mathcal{I}^* [\tilde{\mathcal{B}}, \nabla^S_{\partial_r}] \]
\[ = 3\partial_r \cdot \tilde{\mathcal{B}} \mathcal{I}^* (\nabla^S_{\partial_r})^2 - 5\partial_r \cdot \mathcal{I}^* (\nabla^S_{\partial_r})^3 + 3\partial_r \cdot \tilde{\mathcal{B}} \mathcal{I}^* \nabla^S_{\partial_r} - \partial_r \cdot \tilde{\mathcal{B}}^3 \mathcal{I}^* \]
\[ - \frac{3}{2} J\partial_r \cdot \tilde{\mathcal{B}} \mathcal{I}^* - \text{grad}^M(J) \cdot \mathcal{I}^* + 5J\partial_r \cdot \mathcal{I}^* \nabla^S_{\partial_r} + 5\mathcal{I}^* \nabla^S_{\partial_r} [\tilde{\mathcal{B}}, \nabla^S_{\partial_r}]. \]

After multiplication by \( \partial_r \cdot \), we arrive at
\[ \tilde{\mathcal{B}}_3^\text{res} (h; -\frac{n}{2}) = \partial_r \cdot \tilde{\mathcal{B}}_2^\text{res} (h; -\frac{n+2}{2}) \tilde{\mathcal{B}}. \]

Finally we come to the last identity. Again, Example 3.4 implies
\[ \tilde{\mathcal{B}}_3^\text{res} (h; -\frac{n-2}{2}) = \mathcal{I}^* (\nabla^S_{\partial_r})^3 - \tilde{\mathcal{B}}^2 \mathcal{I}^* \nabla^S_{\partial_r} - \frac{3}{2} J\mathcal{I}^* \nabla^S_{\partial_r} - \tilde{\mathcal{B}} \mathcal{I}^* (\nabla^S_{\partial_r})^2 + \tilde{\mathcal{B}}^3 \mathcal{I}^* \]
\[ - 3\partial_r \cdot h(\mathcal{P}, \tilde{\nabla}^h) \mathcal{I}^* + \frac{1}{2} J\tilde{\mathcal{B}} \mathcal{I}^* - \partial_r \cdot \text{grad}^M(J) \cdot \mathcal{I}^*. \]
Proof. From equations (5.33) and (5.34) we conclude its inverse, cf. Remark 2.1. Note that one could alternatively use $\Xi := \Xi^\sharp$. Because we have

$$\iota^*D_3 = \iota^*D^3 - 2\iota^*\bar{g}(P, \nabla^{g.S}) - \iota^* \text{grad}^X(J),$$

we have

$$\iota^*D_3 = \iota^*D^3 - 2\iota^*\bar{g}(P, \nabla^{g.S}) - \iota^* \text{grad}^X(J).$$

From equations (5.33) and (5.34) we conclude

$$-2\iota^*\bar{g}(P, \nabla^{g.S}) = -2\iota^*\sum_{i=1}^{n+1} P(s_i) \cdot \nabla^{g.S}_{s_i} = -2h(P, \nabla^{h.S})\iota^*,$$

$$-\iota^* \text{grad}^X(J)_{\cdot} = -\iota^*\left(\sum_{i=1}^{n} \iota(s_i) s_i + \partial_{\iota}(J)\partial_{\iota}\right) = -\text{grad}^M(J)\iota^* \cdot .$$

Furthermore, using equations (5.33), (5.36) and (5.37) we compute

$$\iota^*D_3 = -\partial_{\iota} \cdot \bar{D}(\iota^* - \iota^* \nabla_{\partial_{\iota}}S) + \partial_{\iota} \cdot \iota^* \nabla_{\partial_{\iota}}S + \partial_{\iota} \cdot \iota^* \nabla_{\partial_{\iota}}S \bar{D}^2$$

Taking into account all contributions we get

$$\iota^*D_3 = -\partial_{\iota} \cdot \bar{D}^3 \iota^* + \partial_{\iota} \cdot \bar{D}^2 \iota^* \nabla_{\partial_{\iota}}S + \partial_{\iota} \cdot \bar{D} \iota^* (\nabla_{\partial_{\iota}}S)^2 - \partial_{\iota} \cdot \iota^* (\nabla_{\partial_{\iota}}S)^3$$

and the proof is complete. \(\square\)

From the Lemma above and Theorem 5.2 we conclude:

**Corollary 5.13** Let \((M, h)\) be a semi-Riemannian Spin-manifold. Then the conformal first and third power of the Dirac operator are given by

$$D_1 = D^1,$$

$$D_3 = D^3 - 2h(P, \nabla^{h.S}) - \text{grad}^M(J) \cdot .$$

**Proof.** The results follow from the factorizations of \(D_1^{et} = (h; -\frac{n-1}{2})\) and \(D_3^{et} = (h; -\frac{n-3}{2})\) by post- and pre-composition with the identification map \(\Xi\) and its inverse, cf. Remark 2.1. Note that one could alternatively use \(\Xi := \Xi^{-1}\) in the case of odd \(n\), with the effect of an additional sign. \(\square\)
Remark 5.14 The third order residue family operator is a polynomial in \( \lambda \) of degree two, cf. Example 3.4. Lemma 5.12 shows that it satisfies four factorization identities. Thus making an ansatz for \( D_3^{res}(h; \lambda) \) as a polynomial of degree three in \( \lambda \) shows that due to four factorizations the operator valued coefficient by the third power of \( \lambda \) is trivial. However, this operator is a multiple of

\[ M_3 \iota^* - \iota^* \overline{M}_3, \]

where \( M_3 := -2h(P, \nabla^h, S) - \text{grad}^M(J) \) and \( \overline{M}_3 \) is given by analogous formula evaluated with respect to \( \bar{g} \). Notice that this operator is one of the three operators \( \{M_1, M_3, M_5\} \) described in [Fis13, Chapter 6], which determine the conformal powers of the Dirac operator, up to order five, as a non-commutative free algebra.

Now we discuss the factorization identities for \( D_1(X, M; g, \lambda) \), see (4.6) for its definition. Based on equation (5.35) and direct computations we obtain

Lemma 5.15 The family of first order operators \( D_1(X, M; g, \lambda) \) has the following factorization properties:

1. \( D_1(X, M; g, -\frac{n-1}{2}) = N(g) \cdot \iota^* B^g \)
2. \( D_1(X, M; g, -\frac{n}{2}) = \tilde{B}^h \iota^* \).

The family \( D_1(X, M; g, \lambda) \) can be written as

\[
D_1(X, M; g, \lambda) = - (2\lambda + n - 1)N(g) \cdot \iota^* B^g + (2\lambda + n)\tilde{B}^h \iota^* + 2(\lambda + \frac{n}{2})(\lambda + \frac{n-1}{2})\iota^* H(g).
\]

6 Representation theory and residue family operators

In the present section we review the interpretation of our results, concerning factorization identities, in the framework of the classification of homomorphisms of generalized Verma modules for couples of Lie algebras \( g, g' \) and their conformal parabolic subalgebras \( p, p' \). As for the technique and general background we refer to [KÖSS13]. The singular vectors describing such homomorphisms correspond to conformally covariant differential operators in the pairing of generalized Verma modules with induced representations.

Let \( g \) resp. \( g' \) be the conformal Lie algebras in dimension \( n \) resp. \( n - 1 \). Let \( \lambda \in \mathbb{C}, \ N \in \mathbb{N}_0 \), and let \( s_\lambda \in S_\lambda \) for the spinor representation \( S_\lambda = \Delta_{n-1} \otimes \mathbb{C}_{\lambda} \) of \( \mathbb{R}^{n-1} \) tensored by 1-dimensional representation of the Levi factor of \( p \) on \( \mathbb{C}_{\lambda} \), \( a \mapsto a^\lambda, \ a \in \mathbb{R} \). It is a result of [KÖSS13] that \( g' \)-singular vectors of odd and even homogeneity are given by two Gegenbauer polynomials, cf. Appendix C:

\[
\tilde{P}_N(t) := (-t)^N C_{2N}^{\lambda-\frac{n}{2}} \left( \frac{i}{\sqrt{t}} \right)
\]
\[
\tilde{P}_{2N} \cdot s_\lambda = \left( \eta^{2N} \tilde{P}_N(t) + \tilde{Q}_{N-1}(t)\xi \cdot \eta \right) \cdot s_\lambda,
\]
determines a \(\mathfrak{g}'\)-singular vector of homogeneity \(2N\), \(N \in \mathbb{N}_0\), in the Fourier dual of the generalized Verma \(\mathfrak{g}\)-module induced from the twisted spinor representation of the conformal parabolic subalgebra \(\mathfrak{p}\). Note that \(\tilde{Q}_{-1}(t) := 0\) by convention. Furthermore, it follows from the system of differential equations satisfied by \(\tilde{P}_N(t)\) and \(\tilde{Q}_{N-1}(t)\), cf. [KOSS13], that the coefficients of \(\tilde{P}_N(t), \tilde{Q}_{N-1}(t)\) are related by
\[
N(2j - 2N + 1)b_j^{(N-1)}(\lambda + \frac{1}{2}) - (j + 1)a_{j+1}^{(N)}(\lambda + \frac{1}{2}) = 0 \quad (6.3)
\]
for all \(j = 0, \ldots, N - 1\), and
\[
N(2\lambda + n - 4N + 2j + 2)b_j^{(N-1)}(\lambda + \frac{1}{2}) + (j - N)a_j^{(N)}(\lambda + \frac{1}{2}) = 0 \quad (6.4)
\]
for all \(j = 0, \ldots, N\).

If we did not fix the normalizations of Gegenbauer polynomials, cf. Subsection 6.2, the relation between the two sets of coefficients would be \(a_j^{(N)}(\lambda) = -b_j^{(N-1)}(\lambda)\).

In the case of odd homogeneity \(2N + 1\), \(N \in \mathbb{N}_0\),
\[
\tilde{P}_{2N+1} \cdot s_\lambda = \eta^{2N}(\tilde{P}_N(t)\xi + \tilde{Q}_N(t)\eta) \cdot s_\lambda
\]
determine \(\mathfrak{g}'\)-singular vectors provided the coefficients of \(\tilde{P}_N(t), \tilde{Q}_N(t)\) satisfy
\[
-(2\lambda + n - 2N)(2N + 1 - 2j)b_j^{(N)}(\lambda + \frac{1}{2}) + (2\lambda + n - 4N + 2j)a_j^{(N)}(\lambda + \frac{1}{2}) = 0
\]
for all \(j = 0, \ldots, N\), and
\[
(N - j)a_j^{(N)}(\lambda + \frac{1}{2}) + (j + 1)(2\lambda + n - 2N)b_{j+1}^{(N)}(\lambda + \frac{1}{2}) = 0
\]
for all \(j = 0, \ldots, N - 1\).

If we did not fix the normalizations of Gegenbauer polynomials, cf. Subsection 6.2, the relation between the two sets of coefficients would be \(a_j^{(N)}(\lambda) = b_j^{(N)}(\lambda)\).
Remark 6.1 The relations among the coefficients of Gegenbauer polynomials can be used to prove the factorization identities of residue family operators stated in Theorem 5.6. For example, the left hand side of both equations (5.16) and (5.20) satisfies a recurrence relation, cf. equation (3.8) and (3.9). The right hand side of both equations (5.16) and (5.20) satisfies the same recurrence relation due to equations (6.3), (6.4), (6.6) and (6.7).

7 Poisson transformation and residue family operators

In the present section we indicate the origin of the family of distributions \( \delta_N(h; \lambda) : C^\infty(\mathbb{R}^n_{\geq 0}) \to C^\infty(\mathbb{R}^{n-1}) \) on the real hyperbolic space (upper half-space), whose residues produce the residue families on functions, cf. [Juh09]. We also comment on the relation between \( \delta_N(h; \lambda) \) and Poisson transformation acting on induced representations on the boundary of the rank one symmetric space \( H_n := (\mathbb{R}^n_{x_n > 0}, g_{hyp} := x_n^{-2}(dx_1^2 + \cdots + dx_{n-1}^2)) \).

All representation spaces are considered in the non-compact picture. In the second part of the section we pass to the spinor valued case and suggest analogous distribution valued in the algebra of endomorphisms of spinor representation.

We shall start with the scalar case, so we denote by \( x \) a point in the vector space \( \mathbb{R}^n \) and write it as \( x = (x', x_n) \) with respect to the splitting \( \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \). The Poisson kernel on \( \mathbb{H}^n \) is given by locally integrable function

\[
P(y, x') := \frac{y_n}{|\langle y', y_n \rangle - \langle x', 0 \rangle|^2}, \quad y \in \mathbb{R}^n, x' \in \mathbb{R}^{n-1},
\]

where the absolute value is considered with respect to the euclidean scalar product. The Poisson transform \( \mathcal{P}_\mu : C^\infty(\mathbb{R}^{n-1}) \to C^\infty(\mathbb{R}^n) \) with Poisson kernel \( P(y, x') \), given by

\[
\mathcal{P}_\mu(f)(y) := \int_{\mathbb{R}^{n-1}} P(y, x')^\mu f(x') dx', \quad \mu \in \mathbb{C},
\]

is an eigenfunction of the hyperbolic Laplace operator, see [Hel70, Theorem 1.7],

\[
-\Delta_{g_{hyp}} u = \mu(n - 1 - \mu) u,
\]

for \( u \in C^\infty(\mathbb{R}^n) \).

For \( f \in C^\infty_c(\mathbb{R}^{n-1}) \), we denote by \( u := \mathcal{P}_\mu(f) \) its Poisson transform. Let us consider the family of locally integrable functions \( x_n^{\lambda-n} u \) for \( Re(\lambda) \gg 0 \), and define the family of distributions \( M_u(\lambda; x_n) \) supported on \( \mathbb{R}^n_{x_n \geq 0} \) by

\[
(M_u(\lambda; x_n))(\varphi) = \int_{\mathbb{R}_{x_n \geq 0}} y_n^{\lambda-n} u(y', y_n) \varphi(y', y_n) dy' dy_n.
\]
for compactly supported \( \varphi \in C_c^\infty(\mathbb{R}^n_{\geq 0}) \), \( \Re(\lambda) \gg 0 \). Because \( u \) is the Poisson transform of \( f \in C_c^\infty(\mathbb{R}^{n-1}) \), we get

\[
(M_u(\lambda; y_0))(\varphi) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n_{\geq 0}} \varphi(y', y_0) \frac{y_0^{\lambda-n+\mu}}{|y' - x|^2 + y_0^2} f(x') dy' dy_n dx'. \tag{7.1}
\]

As noticed in [Kob13], for \( \lambda, \mu \in \mathbb{C} \) such that \( \Re(\lambda-\mu) > 0 \) and \( \Re(\lambda+\mu) > n-1 \), the locally integrable function

\[
K_{\lambda,\mu}(x', x_n) := \frac{|x_n|^{\lambda+\mu-n}}{(|x'|^2 + x_n^2)^{\mu}} \tag{7.2}
\]

allows to introduce an integral operator

\[
K_{\lambda,\mu} : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^{n-1})
\]

\[
f \mapsto (K_{\lambda,\mu} f)(x') := \int_{\mathbb{R}^n} f(y', y_0) K_{\lambda,\mu}(x' - y', -y_0) dy' dy_n
\]

intertwining the action of the conformal group associated to the boundary \( \mathbb{R}^{n-1} \). Regarding the image of that integral operator as a distribution supported on \( \mathbb{R}^{n-1} \), we can evaluate it on \( g \in C_c^\infty(\mathbb{R}^{n-1}) \),

\[
(K_{\lambda,\mu} f)(g) = \int_{\mathbb{R}^{n-1}} (K_{\lambda,\mu} f)(x') g(x') dx' = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} f(y', y_0) K_{\lambda,\mu}(x' - y', -y_0) g(x') dy' dy_n dx'. \tag{7.3}
\]

A direct comparison of equations (7.1) and (7.3) gives

**Proposition 7.1** Let \( f \in C_c^\infty(\mathbb{R}^{n-1}) \), \( \varphi \in C_c^\infty(\mathbb{R}^n_{\geq 0}) \) be compactly supported functions, \( u = \mathcal{P}_\mu(f) \). Then

\[
(M_u(\lambda, x_n))(\varphi) = (K_{\lambda,\mu} \varphi)(f), \tag{7.4}
\]

and the equality extends continuously to the space of smooth functions.

The meromorphic continuation of the distribution \( K_{\lambda,\mu} \) to \( \lambda, \mu \in \mathbb{C} \), [Kob13], yields the residue families on densities, cf. [Juh09].

Now we turn our attention to an analogous question on Poisson transformation for spinor valued functions, i.e., we suggest a family of distributions whose meromorphic continuation presumably yields the residue family operators on spinors discussed in the present article.

The Poisson kernel for spinors on the real hyperbolic space \( \mathbb{H}^n \) is a locally integrable function valued in the Clifford algebra \( Cl(\mathbb{R}^n) \),

\[
P^S_\mu(y, x') := \frac{y_n^\mu}{|(y', y_n) - (x', 0)|^n} \sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n^* e_n, \quad \mu \in \mathbb{C}, \tag{7.5}
\]

cf. [RC01]. Here \( y \in \mathbb{R}^n, x' \in \mathbb{R}^{n-1} \), where the absolute value is given by the euclidean scalar product.
Proposition 7.2  The Poisson transformation

\( P^S_\mu : \Gamma_c(S(\mathbb{R}^n, h)|_{\partial \mathbb{R}^n}) \rightarrow \Gamma(S(\mathbb{R}^n, \bar{g})) \)

with Poisson kernel \( P^S_\mu(y, x') \), given by

\[
P^S_\mu(s)(y) := \int_{\mathbb{R}^n} P^S_\mu(y, x') s(x') dx',
\]

is an eigenvector of the hyperbolic Dirac operator,

\[
P^{hyp} P^S_\mu(s) = (\lambda - \frac{n-1}{2}) P^S_\mu(s), \ s \in \Gamma_c(S(\mathbb{R}^n, h)|_{\partial \mathbb{R}^n}),
\]

with the eigenvalue \( \lambda - \frac{n-1}{2} \).

Proof.  The proof is a direct consequence of the explicit formula

\[
P^{hyp} = x_n e_n \cdot \partial_n - \frac{1}{2}(n-1)e_n \cdot + x_n \sum_{i=1}^{n-1} e_i \cdot \partial_i,
\]

the hyperbolic Dirac operator, and the definition of \( P^S_\mu(s) \). \( \square \)

Let us denote by \( D(S(\mathbb{H}^n)) \) the algebra of invariant differential operators acting on spinors for rank one symmetric space \( \mathbb{H}^n \). It is well-known, cf. [RC01], that for \( n \) even, \( D(S(\mathbb{H}^n)) \simeq \mathbb{C}[\{\mathcal{D}^{hyp}\}][S^1(\mathbb{R}^n)] \), and for \( n \) odd, \( D(S(\mathbb{H}^n)) \simeq \mathbb{C}[\mathcal{D}^{hyp}] \). Consequently, the Poisson transform maps smooth spinors on the boundary of \( \mathbb{H}^n \) into eigenspaces of the Dirac operator on the hyperbolic space \( \mathbb{H}^n \).

The distribution on spinors \( \{7.5\} \) allows a generalization along the lines of the scalar valued distribution, cf. equation \( \{7.2\} \). We define, for \( \lambda, \mu \in \mathbb{C} \) such that \( Re(\lambda) \gg 0 \) and \( Re(\mu) \gg 0 \), a locally integrable \( Cl(\mathbb{R}^n) \)-valued function on \( \mathbb{R}^n \times \mathbb{R}^{n-1} \)

\[
K^{S \lambda, \mu}(x', x_n) := \frac{|x_n|^\lambda}{(|x'|^2 + x_n^2)^{\frac{\lambda}{2}}} \left( \sum_{i=1}^{n-1} (y_i - x_i)e_i + y_ne_n \right) e_n \cdot e'_i,
\]

which allows to introduce an integral operator on spinors

\[
K^{S \lambda, \mu} : \Gamma_c(S(\mathbb{R}^n, \bar{g})) \rightarrow \Gamma(S(\mathbb{R}^n, h)|_{\partial \mathbb{R}^n}),
\]

\[
s \rightarrow (K^{S \lambda, \mu} \cdot s)(x') := \int_{\mathbb{R}^n} K^{\lambda, \mu}(x' - y', -y_n) \cdot s(y', y_n) dy'dy_n.
\]

Regarding the image of the previous integral operator as a distribution supported on \( \mathbb{R}^{n-1} \), the integration of a smooth compactly supported spinor valued function \( \varphi \in \Gamma_c(S(\mathbb{R}^n, h)|_{\partial \mathbb{R}^n}) \) yields

\[
(K^{S \lambda, \mu} \cdot s)(\varphi) = \int_{\mathbb{R}^{n-1}} < K^{S \lambda, \mu} \cdot s(x'), \varphi(x') > dx'
\]

\[
= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} < K^{S \lambda, \mu}(x' - y', -y_n) \cdot s(y', y_n), \varphi(x') > dy'dy_n dx'.
\]

(7.7)
It remains to explain the definition of the family of distributions
\[ ((\sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n)\cdot)^\mu, \quad \mu \in \mathbb{C}. \]

Let us define two \( Cl(\mathbb{R}^n) \)-valued functions \( \Pi_{\pm} := \frac{1}{2}(Id \pm i\omega) \), \( \omega \in S^{n-1} \subset \mathbb{R}^n \subset Cl(\mathbb{R}^n) \), fulfilling \(-i\omega \Pi_{\pm} = \pm \Pi_{\pm} \), or
\[
\left(\sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n\right)e_n = r^2 \left(\sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n\right)\left|\sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n\right|
\]
for \( \omega := \frac{\sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n}{\left|\sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n\right|} \in S^{n-1} \) in the spherical coordinates on \( \mathbb{R}^n \) with the radial coordinate \( r \). Then
\[
(-i\sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n)\mu \Pi_{\pm} = (-i\omega)^\mu r^\mu \Pi_{\pm} = (\pm 1)^\mu r^\mu \Pi_{\pm},
\]
and because \( \Pi_{+} + \Pi_{-} = Id \), we get
\[
(-i\sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n)\mu = 1^\mu r^\mu \Pi_{+} + (-1)^\mu r^\mu \Pi_{-}
\]
\[
= \frac{1}{2}(1^\mu + (-1)^\mu) r^\mu + \frac{1}{2}(1^\mu - (-1)^\mu) (-i\sum_{i=1}^{n-1} (y_i - x_i)e_i + y_n e_n) r^{\mu-1}.
\]

As for the distribution \( r^\mu \), we have the standard definition as in the scalar valued case,
\[
(r^\mu, \varphi) = \int_0^\infty r^\mu \varphi(r)dr, \quad \varphi \in C_0^\infty(\mathbb{R}_+),
\]
locally integrable for \( \text{Re}(\mu) > -1 \), and
\[
(1)^\mu = e^{2\pi il\mu}, \quad (-1)^\mu = e^{(2l+1)\pi i\mu}, \quad l \in \mathbb{Z}.
\]
In this way, \( (7.6) \) reduces to a linear combination of two scalar valued distributions with coefficients in \( Cl(\mathbb{R}^n) \).

We expect that this distribution will play a decisive role in the spinor version of the scalar distribution kernel, cf. Proposition \( 7.4 \).

8 Discussion and outlook

In this last short section we comment on several unresolved questions related to our work.

For a manifold with conformal structure \((M, [h])\), there is an invariant theory of tractors based on the existence of conformally invariant connection and compatible invariant metric on indecomposable bundles called tractor bundles. This structure allows, for a hypersurface \( M \) inside \((X, g)\), to produce natural intrinsically conformally covariant operators for \( N \in \mathbb{N}_0 \) and \( \lambda \in \mathbb{C} \),
\[
D_N^T(X, M; g, \lambda) : C^\infty(X) \to C^\infty(M)
\]
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can clearly produce the spectral derivatives of spinor residue family operators. Whether there is a reasonable construct for a spinor $Q$-curvature, but one can clearly produce the spectral derivatives of spinor residue family operators. In [Kob13], the Riesz potential and corresponding Knapp-Stein integral operator acting between induced representations for two consecutive orthogonal Lie algebra on the flat space (or the sphere) was constructed. Quite remarkably, the residue analysis of analytic continuation of Knapp-Stein intertwining map produces residue families on densities, and the factorization identities correspond to Kummer’s relation for Gauss hypergeometric function. We expect that an analogous, though vector valued version of such identities, will “explain” the origin of spinor residue family operators and their factorization properties.

Yet another issue is closely related to the spinor version of the holographic deformation property. It is not clear to the authors, whether there is a reasonable construct for a spinor $Q$-curvature, but one can clearly produce the spectral derivatives of spinor residue family operators $D^{\kappa}_{2N}(h;\lambda)$. In other words, it is desirable to construct a holographic deformation of the Dirac operator.

Let $(M^n,h)$ be a semi-Riemannian $Spin$-manifold of signature $(p,q)$, $n = p + q$. Then any orthonormal frame $\{s_i\}_i$ fulfills $h(s_i,s_j) = \varepsilon_i\delta_{ij}$, where $\varepsilon_i = -1$ for $1 \leq i \leq p$ and $\varepsilon_i = 1$ for $p + 1 \leq i \leq n$.

The Clifford algebra of $(\mathbb{R}^n,\langle \cdot,\cdot \rangle_{p,q})$, denoted by $Cl(\mathbb{R}^{p,q})$, is the quotient of tensor algebra of $\mathbb{R}^n$ by two sided non-homogeneous ideal generated by relations $x \otimes y + y \otimes x = -2\langle x,y \rangle_{p,q}$ for all $x,y \in \mathbb{R}^n$.

In the even case $n = 2m$ the complexified Clifford algebra $Cl_{\mathbb{C}}(\mathbb{R}^{p,q})$ has up to an isomorphism a unique irreducible representation, whereas in the odd case $n = 2m + 1$ it has up to an isomorphism two non-equivalent irreducible representations on $\Delta_{p,q} := \mathbb{C}^{2^m}$. The restriction of this representation to the spin group $Spin(p,q)$, regarded as a subgroup of the group of units $Cl^*(\mathbb{R}^{p,q})$, is denoted by $\kappa_{p,q}$.

The choice of a $Spin$-structure $(Q,f)$ on $(M^n,h)$ gives an associated spinor bundle $S(M^n,h) := Q \times (Spin_{p,q}(\kappa_{p,q}) \Delta_{p,q}$, where $Spin_{p,q}$ denotes the connected component of the spin group containing the identity element. Then the Levi-Civita connection $\nabla^h$ on $(M^n,h)$ lifts to a covariant derivative $\nabla^{h,S}$ on spinors, and the operator $D^{\kappa}_{2N}(h;\lambda)$ will produce the spectral derivatives of spinor residue family operators. In [Kob13], the Riesz potential and corresponding Knapp-Stein integral operator acting between induced representations for two consecutive orthogonal Lie algebra on the flat space (or the sphere) was constructed. Quite remarkably, the residue analysis of analytic continuation of Knapp-Stein intertwining map produces residue families on densities, and the factorization identities correspond to Kummer’s relation for Gauss hypergeometric function. We expect that an analogous, though vector valued version of such identities, will “explain” the origin of spinor residue family operators and their factorization properties.

Yet another issue is closely related to the spinor version of the holographic deformation property. It is not clear to the authors, whether there is a reasonable construct for a spinor $Q$-curvature, but one can clearly produce the spectral derivatives of spinor residue family operators $D^{\kappa}_{2N}(h;\lambda)$, $N \in \mathbb{N}_0$.

**Appendix A: Spin geometry**

Let $(M^n,h)$ be a semi-Riemannian $Spin$-manifold of signature $(p,q)$, $n = p + q$. Then any orthonormal frame $\{s_i\}_i$ fulfills $h(s_i,s_j) = \varepsilon_i\delta_{ij}$, where $\varepsilon_i = -1$ for $1 \leq i \leq p$ and $\varepsilon_i = 1$ for $p + 1 \leq i \leq n$.

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The choice of a $Spin$-structure $(Q,f)$ on $(M^n,h)$ gives an associated spinor bundle $S(M^n,h) := Q \times (Spin_{p,q}(\kappa_{p,q}) \Delta_{p,q}$, where $Spin_{p,q}$ denotes the connected component of the spin group containing the identity element. Then the Levi-Civita connection $\nabla^h$ on $(M^n,h)$ lifts to a covariant derivative $\nabla^{h,S}$ on
the spinor bundle. Furthermore, there is a scalar product \(< \cdot, \cdot >\) on the spinor bundle, which is parallel with respect to \(\nabla^{h,S}\) and compatible with the Clifford multiplication:

\[ < X \cdot \psi, \phi > = (-1)^{p+1} < \psi, X \cdot \phi > \]

holds for all \(X \in \Gamma(TM)\) and \(\psi, \phi \in S(M, h)\). The Dirac operator \(\slashed{D} = \sum_i e_i s_i \cdot \nabla^{h,S}_{s_i}\) acting on \(S(M, h)\) is formally anti-selfadjoint,

\[ < \slashed{D} \psi, \phi >_{L^2} = (-1)^p < \psi, \slashed{D} \phi >_{L^2} ,\]

where \(<,\cdot,\cdot >_{L^2}\) denotes the induced \(L^2\)-scalar product.

Let \(\hat{h} := e^{2\sigma} h\) be a metric conformally related to \(h\), \(\sigma \in C^\infty(M)\). The spinor bundles for \(\hat{h}\) and \(h\) can be identified through a vector bundle isomorphism \(F_\sigma : S(M, h) \to S(M, \hat{h})\), and the Dirac operator satisfies the following conformal covariance:

\[ \hat{\slashed{D}}(e^{\frac{2\sigma}{n-2}} \hat{\psi}) = e^{-\frac{2\sigma}{n-2}} \slashed{D}\hat{\psi}, \]

for any \(\psi \in \Gamma(S(M, h))\) and \(\hat{\cdot}\) denotes evaluation with respect to \(\hat{h}\). Conformal odd powers of the Dirac operator were constructed in [HS01, GMP 12, Fis13], and are denoted by \(D^{2N+1} = \slashed{D}^{2N+1} + LOT\), for \(N \in \mathbb{N}_0\) \((N \leq \frac{n}{2}\) for even \(n\)). They satisfy

\[ \hat{D}^{2N+1}(e^{\frac{2N+1-n}{2}} \hat{\psi}) = e^{-\frac{2N+1+n}{2}} \slashed{D}^{2N+1}\hat{\psi}, \]

for any \(\psi \in \Gamma(S(M, h))\).

**Appendix B: Poincaré-Einstein metric construction**

Here we briefly review the content of Poincaré-Einstein metric construction, [FG11]. Let \((M^n, h)\) be an \(n\)-dimensional semi-Riemannian manifold, \(n \geq 3\). On \(X := M \times (0, \varepsilon)\), for \(\varepsilon > 0\), we consider the metric

\[ g_+ = r^{-2}(dr^2 + h_\varepsilon), \]

for a 1-parameter family of metrics \(h_\varepsilon\) on \(M\) such that \(h_0 = h\). The requirement of Einstein condition on \(g_+\) for \(n\) odd,

\[ Ric(g_+) + ng_+ = O(r^\infty), \]

uniquely determines the family \(h_\varepsilon\), while for \(n\) even the conditions

\[ Ric(g_+) + ng_+ = O(r^{n-2}), \]
\[ tr(Ric(g_+) + ng_+) = O(r^{n-1}), \]

uniquely determine the coefficients \(h_{(2)}, \ldots, h_{(n-2)}, \hat{h}_{(n)}\) and the trace of \(h_{(n)}\) in the formal power series

\[ h_\varepsilon = h + r^2 h_{(2)} + \cdots + r^{n-2} h_{(n-2)} + r^n (h_{(n)} + \hat{h}_{(n)} \log r) + \cdots. \]
For example, we have

\[ h_{(2)} = -P, \quad h_{(4)} = \frac{1}{4}(P^2 - \frac{B}{n - 4}), \]

where \( P \) is the Schouten tensor and \( B \) is the Bach tensor associated to \( h \).

The metric \( g_+ \) on \( X \) is called Poincaré-Einstein metric associated to a semi-Riemannian manifold \((M, h)\).

All constructions in the present paper, based on the Poincaré-Einstein metric, depend for even \( n \) on the coefficients \( h_{(2)}, \ldots, h_{(n-2)} \) and \( tr(h_{(n)}) \) only.

Choosing different representatives \( h, \tilde{h} \in [h] \) in the conformal class leads to Poincaré-Einstein metrics \( g_+^1 \) and \( g_+^2 \) related by a diffeomorphism \( \Phi : U_1 \subset X \to U_2 \subset X \), where both \( U_i, \ i = 1, 2 \), contain \( M \times \{0\} \), \( \Phi|_M = \text{id}_M \), and \( g_+^1 = \Phi^*g_+^2 \) (up to a finite order in \( r \), for even \( n \)).

Appendix C: Gegenbauer polynomials

We summarize several basic conventions and properties related to Gegenbauer polynomials used throughout the article.

The Gegenbauer polynomials are defined in terms of their generating function

\[ \frac{1}{(1 - 2xt + t^2)\alpha} = \sum_{n=0}^{\infty} C_n^\alpha(x)t^n, \]

and satisfy the recurrence relation

\[ C_n^\alpha(x) = \frac{1}{n} \left( 2x(n + \alpha - 1)C_{n-1}^\alpha(x) - (n + 2\alpha - 2)C_{n-2}^\alpha(x) \right) \]

with \( C_0^\alpha(x) = 1, C_1^\alpha(x) = 2\alpha x \). Gegenbauer polynomials are solutions of the Gegenbauer differential equation

\[ \left( (1 - x^2)\frac{d^2}{dx^2} - (2\alpha + 1)x\frac{d}{dx} + n(n + 2\alpha) \right) y(x) = 0, \]

and thus can be written as terminating hypergeometric series

\[ C_n^\alpha(x) = \frac{(2\alpha)_n}{n!} \, _2F_1 \left( \frac{-n}{\alpha + \frac{1}{2}}, 2\alpha + n; \frac{1 - x}{2} \right). \]

The explicit form of Gegenbauer polynomials is

\[ C_n^\alpha(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{\Gamma(n - k + \alpha)\Gamma(k)!} \left( 2\alpha + n \right)_{n-2k}, \]

whose consequence is the basic formula for derivative of Gegenbauer polynomials

\[ \frac{d}{dx}C_{2N}^\alpha(x) = 2\alpha C_{2N+1}^\alpha(x). \]
The even and odd Gegenbauer polynomials are given, in terms of (3.6) and (3.7), by

\[
\begin{align*}
\frac{(-1)^N N!}{(-\lambda - \frac{n+1}{2})_N} C_{2N}^{-\lambda} - \frac{n+1}{2} (x) &= \sum_{j=0}^{N} (-1)^j a_j^{(N)}(\lambda) x^{2N-2j}, \\
\frac{(-1)^N N!}{2(-\lambda - \frac{n+1}{2})_{N+1}} C_{2N+1}^{-\lambda} - \frac{n+1}{2} (x) &= \sum_{j=0}^{N} (-1)^j b_j^{(N)}(\lambda) x^{2N+1-2j}.
\end{align*}
\]

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