The $4^{\text{th}}$ structure.

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Abstract

In this paper we describe all Lie bialgebra structures on the polynomial Lie algebra $g[u]$, where $g$ is a simple, finite dimensional, complex Lie algebra. The results are based on an unpublished paper Montaner and Zelmanov [8]. Further, we introduce quasi-rational solutions of the CYBE and describe all quasi-rational $r$-matrices for $sl(2)$.

1 Introduction

The aim of the present paper is to describe all Lie bialgebra structures on the polynomial Lie algebra $P = g[u]$, where $g$ is a simple, finite dimensional Lie algebra over $\mathbb{C}$. The starting point of the study became the following simple observation: up to classical twisting (defined in [4]) there are only two Lie bialgebra structures on $g$. The corresponding Lie co-algebra structures on these structures are given by $\delta_1 = 0$ and $\delta_2 = [\Gamma_{DJ}, a \otimes 1 + 1 \otimes a]$, where $\Gamma_{DJ}$ is the classical Drinfeld-Jimbo modified $r$-matrix.

What is more difficult to prove is that of the classical twist does not change the corresponding classical double, and for $g$ we have, $D_1(g) = g \otimes \mathbb{C}[\varepsilon]$, where $\varepsilon^2 = 0$, and $D_2(g) = g \otimes \mathbb{C}[\omega]$, where $\omega^2 = 1$.(see [6])

Similarly, according to an unpublished result of Montaner and Zelmanov, there are four Lie bialgebra structures on $P = g[u]$, again up to twisting. Therefore, we have to consider all the corresponding classical doubles and then to derive Lie bialgebra structures related to one and the same classical
double. So, it is convenient to start with a description of the four possible cases for the double $D(P)$ and the corresponding Lie co-bracket on $P$.

Case 1.

In this case $\delta_1 = 0$ and $D_1(P) = P + \varepsilon P^*$, where $\varepsilon^2 = 0$, and thus, $\varepsilon P^*$ is a commutative ideal. The canonical symmetric nondegenerate invariant form $Q$ on $D_1(P)$ is given by the pairing between $P$ and $P^*$:

$$Q(p, q) = Q(p^*, q^*) = 0, \quad Q(p, \varepsilon q^*) = q^*(p).$$

It is not difficult to show that there is a one-to-one correspondence between Lie bialgebra structures of the first type and finite-dimensional quasi-Frobenius Lie subalgebras of $P$.

Case 2.

Here $\delta_2(p(u)) = [\Gamma_2(u, v), p(u) \otimes 1 + 1 \otimes p(v)]$ for $\Gamma_2(u, v) = \frac{\Omega}{u - v}$, where $\Omega \in g \otimes g$ is the derived Casimir element and $D_2(P) = g((u^{-1}))$. The canonical form $Q$ is given by the formula

$$Q(p(u), q(u)) = Res_{u=0}K(p, q),$$

where $K(p, q)$ is the Killing form of the Lie algebra $g((u^{-1}))$ considered over $\mathbb{C}(u^{-1})$. In this case $P^*$ can be identified with $u^{-1}g[[u^{-1}]]$. The corresponding Lie bialgebra structures are in a one-to-one correspondence with rational solutions of the CYBE (see [5]).

Case 3.

In the third case the $1-cocycle$ $\delta_3$ is defined as $\delta_3(p(u)) = [\Gamma_3(u, v), p(u) \otimes 1 + 1 \otimes p(v)]$ with the corresponding $\Gamma_3(u, v) = \frac{\Omega}{u - v} + r_{DJ}$, where $r_{DJ}$ is the classical Drinfeld-Jimbo modified $r$-matrix. The classical double has the following form $D_3(P) = g((u^{-1})) \oplus g$. The expression for the invariant bilinear $Q$ is:

$$Q((f(u), a), (g(u), b)) = Res_{u=0}u^{-1}K(f, g) - K(a, b),$$

where $a, b \in g$.

The Lie algebra $P^*$ is identified with:

$$u^{-1}g[[u^{-1}]] \oplus \{(l, k) \in b_+ \oplus b_- : l_h + k_h = 0\},$$

where $h$ is a fixed Cartan subalgebra, $b_\pm$ are the corresponding Borel subalgebras of $g$ and $l_h, k_h$ is a Cartan part of $l$ and $k$ respectively. There is a natural one-to-one correspondence between these Lie bialgebra structures with the so-called quasi-trigonometric solutions of the CYBE. [7]

Case 4.

This is a new case. The Lie bialgebra structure on $P$ is given by:

$$\delta_4(p(u)) = [\Gamma_4(u, v), p(u) \otimes 1 + 1 \otimes p(v)],$$

where $\Gamma_4(u, v) = \frac{\Omega}{u - v} = \frac{u \Omega}{v - u}$.

$D_4(P) = g((u^{-1})) \oplus D_1(g)$, where $D_1(g)$ was defined above.

Let us describe the corresponding form $Q$:

$$Q(f(u) + A_0 + A_1 \varepsilon, g(u) + B_0 + B_1 \varepsilon),$$

where $f(u) = \sum_{k=0}^{N} a_k u^k$, $g(u) = \sum_{k=0}^{M} b_k u^k$ and $a_i, b_i, A_i, B_i \in g$.

$$Q(f(u) + A_0 + A_1 \varepsilon, g(u) + B_0 + B_1 \varepsilon) = Res_{u=0}u^{-2}K(f, g) - K(A_0, B_1) - K(A_1, B_0).$$

(1)

P is embedded into $D_4$ as follows: $i(\sum_{k=0}^{N} c_k u^k) = \sum_{k=0}^{N} c_k u^k + c_0 + c_1 \varepsilon$ (It is easy to check that $i(P)$ is a Lagrangian subalgebra of $D_4(P)$ )

$P^*$ can be identified with $g[[u^{-1}]] \oplus g\varepsilon$. In order to verify that $D_4(P)$ corresponds to $\Gamma_4(u, v) = \frac{u \Omega}{v - u}$, we have to find the dual bases in $P$ and $P^*$ with respect to $Q$ and compute the corresponding dual base element.
Let us do this.

Let \( \{I_i\} \) be an orthonormal basis of \( \mathfrak{g} \) with respect to \( K \). Then \( i(\mathfrak{g}[u]) \subset \mathbf{D}_4 \) poses the following basis: \( \{I_iu^k(k \geq 2), (I_iu, I_i\varepsilon), (I_i, I_i)\} \). It is not difficult to find the dual basis in \( P^* \): \( \{I_iu^{-k+1}(k \geq 2), (I_i, 0), (0, -\varepsilon I_i)\} \). The corresponding dual base element is:

\[
\sum_i \sum_{k=2}^\infty I_i u^k \otimes I_i v^{-k+1} + \sum_i (I_iu, I_i\varepsilon) \otimes (I_i, 0) + \sum_i (I_i, I_i) \otimes (0, -\varepsilon I_i). 
\]

If we project this element onto the first component in the decomposition \( \mathbf{D}_4(P) = \mathfrak{g}((u^{-1})) \oplus \mathbf{D}_1(\mathfrak{g}) \), we get:

\[
\sum_i \sum_{k=1}^\infty I_i u^k \otimes I_i v^{-k+1} = u \sum_i \sum_{m=0}^\infty I_i u^m \otimes I_i v^{-m} = \frac{uv\Omega}{v-u}. 
\]

Therefore, we have proved the following result:

**Theorem 1.** The double \( \mathbf{D}_4(P) \) corresponds to the co-bracket on \( P \) given by the formula

\[
\delta_4(p(u)) = [\Gamma_4(u, v), p(u) \otimes 1 + 1 \otimes p(v)], 
\]

with \( \Gamma_4(u, v) = \frac{uv\Omega}{v-u} \)

**Definition 1.** Solutions of the classical Yang-Baxter equation of the form \( q(u, v) = \frac{uv\Omega}{v-u} + p(u, v) \), where \( p(u, v) \) is a skew-symmetric polynomial, are called quasi-rational \( r \)-matrices.

## 2 Quasi-rational solutions of CYBE.

Following methods developed in [3], [4], [6], we can prove that:

**Theorem 2.** (i). All quasi-rational \( r \)-matrices provide one and the same double \( \mathbf{D}_4(P) \).

(ii). Quasi-rational solutions of the CYBE are in a one-to-one correspondence with Lagrangian subalgebras \( \mathbf{W} \subset \mathbf{D}_4(P) \), satisfying the following conditions:

1. \( \mathbf{W} \cap P = \{0\} \)
2. \( \mathbf{W} \oplus P = \mathbf{D}_4(P) \)
3. \( \mathbf{W} \supset u^{-N}\mathfrak{g}[[u^{-1}]] \), for some \( N > 0 \).

Since \( P \subset \mathbf{D}_4(P) \), we can treat the group \( \text{Ad}(P) \), as a subgroup of \( \text{Ad}(\mathbf{D}_4(P)) \). If \( q(u, v) \) is a quasi-rational \( r \)-matrix, then \( \text{Ad}(p(u) \otimes p(v)q(u, v)) \) is also a quasi-rational solution of the CYBE.

We say that two quasi-rational \( r \)-matrices \( q_1(u, v) \) and \( q_2(u, v) \) are gauge equivalent if there exists \( p(u) \in \text{Ad}\mathfrak{g}[u] \) such that:

\[ q_2(u, v) = \text{Ad}(p(u) \otimes p(v))q_1(u, v). \]

It is not difficult to prove that:
**Proposition 1.** Quasi-rational $r$-matrices $q_1(u,v)$ and $q_2(u,v)$ are gauge equivalent iff $W_2 = \text{Ad}(p(u))W_1$, where $W_i$ corresponds to $q_i(u,v)$, $(i = 1, 2)$.

**Theorem 3.** Let $g = \mathfrak{sl}(n)$. For any quasi-rational $r$-matrix $q_1(u,v)$, there exists another quasi-rational $r$-matrix $q_2(u,v)$, which is gauge equivalent to $q_1(u,v)$, and such that the corresponding $W_2 \subset D_4(P)$ is contained in $d_k^{-1}\mathfrak{sl}(n, \mathbb{C}[[u^{-1}]]) d_k \oplus \mathfrak{sl}(n, \mathbb{C}[\varepsilon])$.

Here $d_k = \text{diag}(1, \ldots, 1, u, \ldots, u)$.

Now we are ready to give a more explicit description of the quasi-rational $r$-matrices.

**Proposition 2.** If $W$ corresponds to a quasi-rational $r$-matrix $q(u,v)$ and $W \subset \mathfrak{sl}(n, \mathbb{C}[[u^{-1}]]) \oplus \mathfrak{sl}(n, \mathbb{C}[\varepsilon])$, then $q(u,v) = \frac{u\Omega}{v-u} + q$, where $q$ is a skew-symmetric solution of the CYBE. Conversely, if $q(u,v) = \frac{u\Omega}{v-u} + q$ is a quasi-rational $r$-matrix, then the corresponding $W$ is contained in $\mathfrak{sl}(n, \mathbb{C}[[u^{-1}]]) \oplus \mathfrak{sl}(n, \mathbb{C}[\varepsilon])$.

**Remark 1.** Both statements above are valid for any $g$.

**Remark 2.** Since skew-symmetric $r$-matrices are in a one-to-one correspondence with quasi-Frobenius subalgebras of $g$, we have proved that there is a one-to-one correspondence between quasi-rational $r$-matrices of the form $\frac{u\Omega}{v-u} + q$ and quasi-Frobenius subalgebras of $g$.

We continue with the following:

**Lemma 1.** Let $W_k = d_k^{-1}\mathfrak{sl}(n, \mathbb{C}[[u^{-1}]])d_k \oplus \mathfrak{sl}(n, \mathbb{C}[\varepsilon])$. Then its orthogonal complement $W_k^\perp$ with respect to the form $Q$ on $D_4(\mathfrak{sl}(n))$, is isomorphic to $d_k^{-1}\mathfrak{sl}(n, \mathbb{C}[[u^{-1}]])d_k$ and $\frac{W_k^\perp}{W_k} \cong \mathfrak{sl}(n, \mathbb{C}[\varepsilon])$.

Now we can to describe all the quasi-rational $r$-matrices related to $W_k$.

Let $W$ correspond to $q(u,v)$ and let $W \subset W_k$. Since $W$ is Lagrangian, we get $W_k^\perp \subset W \subset W_k$. Therefore, the image of $W$ in $\frac{W_k^\perp}{W_k}$ is a Lagrangian subalgebra, which is transversal to $P \cap W_k$ (it is easy to see that the canonical map $P \cap W_k \to \frac{W_k^\perp}{W_k}$ is an embedding, since $P \cap W_k = \{0\}$ and $W \supset W_k^\perp$).

The image of $P \cap W_k$ in $\frac{W_k^\perp}{W_k} \cong \mathfrak{sl}(n, \mathbb{C}[\varepsilon])$ can be computed.

Let $P_k$ be the maximal parabolic subalgebra of $\mathfrak{sl}(n)$, which contains $B^+$ and corresponds to the maximal root $\alpha_k$. Then the image of $P \cap W_k$ is isomorphic to $P_k + \varepsilon P_k^\perp$, where $P_k^\perp$ is the orthogonal complement to $P_k$ with respect to the Killing form on $\mathfrak{sl}(n)$. Thus, we proved:

**Theorem 4.** There is a one-to-one correspondence between quasi-rational $r$-matrices related to $W_k$ and Lagrangian subalgebras of $\mathfrak{sl}(n, \mathbb{C}[\varepsilon])$ transversal to $P_k + \varepsilon P_k^\perp$. 
At this point we note that Theorem 4 together with results of \[6\], provide a one-to-one correspondence between rational and quasi-rational $r$-matrices in $\mathfrak{sl}(n)$.

Therefore, the following result is proved.

**Theorem 5.** Quasi-rational $r$-matrices related to $W_k$ are in a one-to-one correspondence with the pairs $(L, B)$, where $L \subset \mathfrak{sl}(n)$ is a subalgebra such that $L + P_k = \mathfrak{sl}(n)$, and $B$ is a 2-cocycle on $L$, such that $B$ is non-degenerate on $L \cap P_k$. (see \[6\] for details).

Finally, we would like to compute all quasi-rational $r$-matrices for $\mathfrak{sl}(2)$. In this case $k = 0, 1$.

If $k = 0$, then the corresponding quasi-rational $r$-matrices are in a one-to-one correspondence with the quasi-Frobenius subalgebras of $\mathfrak{sl}(2)$. We have two cases:

1. $L = 0 \implies q_0(u, v) = \frac{uv\Omega}{v-u}$
2. $L = B^+ \implies q_1(u, v) = \frac{uv\Omega}{v-u} + e \otimes h - h \otimes e$

If $k = 1$, then we have only one pair $(L, B)$ satisfying conditions $L + P_1 = \mathfrak{sl}(2)$, $B$ is non-degenerate on $L \cap P_1$: $L = \mathfrak{sl}(2)$, $B(x, y) = K(f, [x, y])$. Here, \{e, f, h\} is the standard basis of $\mathfrak{sl}(2)$ and $K$ is the Killing form on $\mathfrak{sl}(2)$.

The rational $r$-matrix corresponding to this data is:

$$\frac{\Omega}{u-v} + eu \otimes h - h \otimes ev. \quad (5)$$

However, the quasi-rational $r$-matrix, which corresponds to the same data is rather different:

$$q_2(u, v) = \frac{uv\Omega}{v-u} + \frac{1}{2}h \otimes e - \frac{1}{2}e \otimes h - eu \otimes f + f \otimes ev. \quad (6)$$

One can prove that up to gauge equivalence $q_0(u, v)$, $q_1(u, v)$ and $q_2(u, v)$ exhaust all quasi-rational $r$-matrices for $\mathfrak{sl}(2)$.

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