Dark energy cosmology with generalized linear equation of state

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Abstract
Dark energy with the usually used equation of state \( p = w \rho \), where \( w = \text{const} < 0 \), is hydrodynamically unstable. To overcome this drawback, we consider the cosmology of a perfect fluid with a linear equation of state of a more general form \( p = \alpha (\rho - \rho_0) \), where the constants \( \alpha \) and \( \rho_0 \) are free parameters. This non-homogeneous linear equation of state provides the description of both hydrodynamically stable (\( \alpha > 0 \)) and unstable (\( \alpha < 0 \)) fluids. In particular, the considered cosmological model describes the hydrodynamically stable dark (and phantom) energy. The possible types of cosmological scenarios in this model are determined and classified in terms of attractors and unstable points by using phase trajectories analysis. For the dark energy case, some distinctive types of cosmological scenarios are possible: (i) the universe with the de Sitter attractor at late times, (ii) the bouncing universe, (iii) the universe with the big rip and with the anti-big rip. In the framework of a linear equation of state the universe filled with a phantom energy, \( w < -1 \), may have either the de Sitter attractor or the big rip.

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1. Introduction
The astronomical observations indicate that the expansion of our universe accelerates [1]. In the framework of general relativity, this means that about two-thirds of the total energy density of the universe consists of dark energy: the still unknown component with a relativistic negative pressure \( p < -\rho/3 \). The simplest candidate for dark energy is the cosmological \( \Lambda \)-term or vacuum energy. During the cosmological evolution, the \( \Lambda \)-term component has the constant (Lorentz invariant) energy density \( \rho \) and pressure \( p = -\rho \). However, the \( \Lambda \)-term requires that the vacuum energy density be fine tuned to have the observed very low value.
For this reason, the different forms of dynamically changing dark energy with an effective equation of state \( w \equiv p/\rho < -1/3 \) were proposed instead of the constant vacuum energy density. As a particular example of dark energy, the scalar field with some slow rolling potential (quintessence) [2] is often considered. The possible generalization of quintessence is a k-essence [3], the scalar field with a non-canonical Lagrangian.

Present observation data constrain the range of the equation of state of dark energy as \(-1.38 < w < -0.82\) [4]. Recently, the Chandra observations [5] gave similar restrictions on the value of \( w \). These data do not exclude the possibility of our universe being filled with a phantom energy [6]: energy with a super-negative equation of state \( w < -1 \) (note, however, that dark energy with the equation of state \( w \) evolving from the quintessence-like \( w > -1 \) in the past, to phantom-like \( w < -1 \) at present, provides the best fit for supernova data [7]). The different aspects of phantom cosmology were considered in [8]. The phantom energy is usually associated with phantom or ghost fields—scalar fields with wrong-sign kinetic term. It is known that such fields are unstable due to quantum instability of vacuum, unless the kinetic term is stabilized by the ultra-violet cut-off [9]. The intriguing possibility of constructing effective \( w < -1 \) from the scalar field with correct-sign kinetic term was proposed in [10]. In [11], the authors constructed the theory with the effective \( w < -1 \) in the braneworld models.

The presence of phantom energy in the universe leads to interesting physical phenomena: the possibility of the big rip scenario [6], the black-hole mass decreasing by phantom energy accretion [13] and a new type of wormhole evolution [14]. The alternatives to a scalar field model are the perfect fluid models such as a Chaplygin gas [15]. Hao and Li [16] demonstrated that \( w = -1 \) state is an attractor for the Chaplygin gas and the equation of state of this gas could approach this attractor from either \( w < -1 \) or \( w > -1 \) sides.

In this paper, we analyse the perfect fluid model with a general linear equation of state \( p = \alpha (\rho - \rho_0) \) where \( \alpha \) and \( \rho_0 \) are constants. This is a generalization of a homogeneous linear equation of state \( (\rho_0 = 0) \) and is suitable for modelling either the linear gas with \( p > 0 \) or the dark energy with \( p < 0 \). The advantage of using the general linear equation of state is the possibility of describing the dark energy with a positive squared sound speed (for the usually considered equation of state \( p = w \rho \) the dark energy is hydrodynamically unstable, because \( \partial p/\partial \rho = w < 0 \)). The considered linear model is reduced to the perfect fluid model with \( w = \) const for the particular case of \( \rho_0 = 0 \).

In the framework of this model, it is possible to find analytical cosmological solutions for the arbitrary values of parameters \( \alpha \) and \( \rho_0 \). We show below that this linear model (unlike the Chaplygin gas model) describes distinctively different types of cosmological scenarios: the big bang, big crunch, big rip, anti-big rip, solutions with the de Sitter attractor, bouncing solutions and their various combinations.

The paper is organized as follows. The principal part of the paper is section 2 in which the basic properties and particular analytical solutions for a perfect fluid cosmology with a linear equation of state are derived. We show that dark energy with the equation of state \( p = \alpha (\rho - \rho_0) \) may be effectively reduced to the \( \Lambda \)-term and to the simplified linear equation of state \( p_{\text{eff}} = w_{\text{eff}} \rho_{\text{eff}} \). In this case, either the effective density \( \rho_{\text{eff}} \) or the density of an effective \( \Lambda \)-term may in general have negative values. Depending on the signs of parameters \( \alpha \) and \( \rho_0 \), the four different cosmological scenarios are considered. To study the cosmological dynamics of the universe for different scenarios, phase plane analysis is used. In section 3, we discuss the restrictions which can be set on this cosmological model from the astronomical observations. The discussion of the results and their further possible generalization are presented in section 4.
2. Linear equation of state

We consider the flat Friedman–Robertson–Walker universe filled with a perfect fluid. For the sake of simplicity we will call this fluid dark energy. Using the unit conventions \(8\pi G/3 = c = 1\), the corresponding Einstein equations for this cosmology can be written as follows:

\[
H^2 = \rho_{\text{DE}},
\]
\[
\dot{\rho}_{\text{DE}} = -3H(\rho_{\text{DE}} + p_{\text{DE}}),
\]

where \(H = \dot{a}/a\) is the Hubble parameter and \(\rho_{\text{DE}}\) and \(p_{\text{DE}}\) are the energy density and pressure of dark energy, respectively. In this paper, we consider a perfect fluid with the linear equation of state of the following general form:

\[
p_{\text{DE}} = \alpha(\rho_{\text{DE}} - \rho_0),
\]

where \(\alpha\) and \(\rho_0\) are constants. Certain features of cosmology with \(\rho_0 \neq 0\) and \(0 < \alpha < 1\) were considered in [12]. The equation of state (3) was used in [13] to describe the dark energy accretion onto the black hole. Note the difference of this equation of state (3) from the commonly used one \(p = w\rho\) with \(w = \text{const}\). Our simple generalization allows us to include dark (and also phantom) energy with a positive squared sound speed of linear perturbations \(c_s^2 \equiv \partial p/\partial \rho = \alpha \geq 0\).

One can use the observational restrictions [4] on the allowable range of equation of state, \(-1.38 < w < -0.82\), recalculated and presented for parameters \(\alpha\) and \(\rho_0\) of the linear model of dark energy. Here \(\alpha\) and \(\rho_0\) are the parameters in (3) and \(\rho_{\text{DE}}\) is the present density of dark energy.

Substituting (3) into the equation of state \(w = p_{\text{DE}}/\rho_{\text{DE}}\), we obtain the range of allowable parameters \(\alpha\) and \(\rho_0\), shown in figure 1.

The equation of state (3) can be reduced to the ‘effective’ cosmological constant and the dynamically evolving dark energy by redefining the fluid density and pressure in the following way:

\[
\rho_{\text{DE}} = \rho_{\Lambda} + \rho_a, \quad p_{\text{DE}} = p_{\Lambda} + p_a,
\]

where

\[
p_{\Lambda} = -\rho_{\Lambda}, \quad p_a = \alpha \rho_a,
\]

and

\[
\rho_{\Lambda} = \frac{\alpha \rho_0}{1 + \alpha}, \quad \rho_a = \rho_{\text{DE}} - \frac{\alpha \rho_0}{1 + \alpha}.
\]
In equations (4)–(6), we denote $\rho/\Lambda_1$ and $p/\Lambda_1$ as the density and the pressure of the ‘effective’ cosmological constant. Correspondingly, $\rho_\alpha$ and $p_\alpha$ are the density and pressure of the dynamically evolving part of dark energy. The values $\rho/\Lambda_1$ and $p/\Lambda_1$ obviously obey the relation $\dot{\rho}/\Lambda_1 = -3H(\rho + p)$ and therefore $\dot{\rho}_\alpha$ and $p_\alpha$ obey the equation:

$$\dot{\rho}_\alpha = -3H(\rho_\alpha + p_\alpha).$$

(7)

The cosmology with $\Lambda$-term and a linear equation of state (for $p/\rho > -1$) was considered previously in [17]. Note, however, that in our consideration either $\rho/\Lambda_1$ or $\rho_\alpha$ may be negative (nevertheless the sum $\rho/\Lambda_1 + \rho_\alpha$ must be positive) and also the equation of state for the dynamically changing part of dark energy may be super-negative. This leads to the new distinctive types of cosmological scenarios considered below. The signs of $\rho_\alpha$ and $\rho_\Lambda$ are conserved during the universe evolution as can be seen from (7) and (2). Below, we obtain the full analytical solutions of evolution of the universe. In both cases $\rho + p > 0$ and $\rho + p < 0$ we find from (7) and (2):

$$\rho_\alpha = \frac{\Lambda}{a^{3(1+\alpha)}},$$

(8)

where constant $\Lambda$ may be either positive or negative. Two different asymptotic regimes are possible: for $|\rho_\alpha| \ll |\rho_\Lambda|$ the universe behaves like the de Sitter universe with

$$\rho \approx \rho_\Lambda, \quad p \approx -\rho_\Lambda,$$

(9)

and for $|\rho_\alpha| \gg |\rho_\Lambda|$ the universe is filled with the dynamical part of dark energy:

$$\rho \approx \rho_\alpha.$$

(10)

Using (1) and (8) we obtain the relation between the differentials $da$ and $dt$:

$$\frac{da}{a\sqrt{\rho_\Lambda + \rho_\alpha}} = \pm dt.$$

(11)

Depending on the signs of $\rho_\Lambda$ and $\Lambda$ there can be three different results for integration of relation (11). For $\rho_\Lambda > 0$ and $\Lambda > 0$ we find

$$a(t) = \left\{ \sqrt{\beta} \sinh \left[ \pm \frac{\kappa}{2} \sqrt{\rho_\Lambda t} \right] \right\}^{2/\kappa}.$$

(12)

Here we denote $\kappa = 3(1 + \alpha)$ and $\beta = \Lambda/\rho_\Lambda$. The choice of the upper or lower sign in (12) depends on the sign ‘+’ or ‘−’ in (11). Expression (12) was first obtained in [17]. The asymptotic behaviour of (12) for $t \to 0$ is given by

$$a = \left( \pm \frac{\kappa}{2} \sqrt{\Lambda t} \right)^{2/\kappa}.$$

(13)

Equation (12) for $t \to \pm \infty$ can be rewritten as follows:

$$a = \left( \frac{|\beta|}{4} \right)^{1/\kappa} \exp(\pm \sqrt{\rho_\Lambda t}).$$

(14)

Expression (14) is valid only if $\kappa > 0$. For $\kappa < 0$, the signs in the exponent in this expression should be changed to opposite ones.

In the case $\rho_\Lambda > 0$ and $\Lambda < 0$ the integration of (11) gives

$$a(t) = \left\{ \sqrt{-\beta} \cosh \left[ \pm \frac{\kappa}{2} \sqrt{\rho_\Lambda t} \right] \right\}^{2/\kappa}.$$

(15)
Solution (15) was obtained in slightly different form in [18] for the restricted case \(-1 < \alpha < 0\). The asymptotic behaviour of a scale factor for \(t \to 0\) is given by

\[
a(t) = (-\beta)^{1/\kappa} \left( 1 + \frac{\kappa}{4} \rho_\Lambda t^2 \right). \tag{16}
\]

For \(t \to \pm \infty\) the scale factor evolution is described by (14). Finally, for \(\rho_\Lambda < 0\) and \(A > 0\) one can find the evolution of the scale factor of the universe:

\[
a(t) = \left\{ -\beta \sin \left[ \frac{\kappa}{2} \sqrt{-\rho_\Lambda t} \right] \right\}^{2/\kappa}. \tag{17}
\]

In the limitation \(-1 < \alpha < 1\) a similar solution was found in [19] for the cosmology with negative anti-de Sitter \(\Lambda\)-term plus a scalar quintessential field with special form of a potential. For \(t \to 0\) the asymptotic behaviour of the scale factor is given by (13). The behaviour of a scale factor near \(t \simeq \pi/(\kappa \rho_\Lambda)\) can be described as follows:

\[
a(t) = \left\{ -\beta \left[ 1 - \frac{1}{2} \left( \frac{\kappa}{2} \sqrt{-\rho_\Lambda t - \frac{\pi}{2}} \right)^2 \right] \right\}^{2/\kappa}. \tag{18}
\]

Correspondingly, the asymptotic behaviour of the scale factor at \(t \to \pm 2\pi/(\kappa \rho_\Lambda)\) is given by

\[
a = \left( \pi \sqrt{-\beta} - \frac{\kappa}{2} \sqrt{\Lambda t} \right)^{2/\kappa}. \tag{19}
\]

To find the attractors and unstable points of the solution of the Friedman equations, we use the method of the phase trajectories. Denoting \(y = \rho_a + p_a\) we obtain the following system of equations:

\[
\begin{align*}
\dot{H} &= -3y/2, \\
\dot{y} &= -3(\alpha + 1)H y.
\end{align*} \tag{20}
\]

All phase trajectories of the system lie on the curve \(H^2 = \rho_{DE}\) which can be rewritten as \(y = (\alpha + 1)H^2 - \alpha \rho_0\). The particular form of this curve and direction of the phase trajectory depend on the sings of \(1 + \alpha\) and \(\alpha \rho_0\). Four cases are possible:

(i) \(1 + \alpha > 0\) and \(\alpha \rho_0 > 0\). There are two singular points at the \(y = 0\) axis. At the point \((H = \rho_\Lambda^{1/2}, y = 0)\) the linearized system of evolution equations is

\[
\begin{align*}
\dot{H} &= -3y/2, \\
\dot{y} &= -3(1 + \alpha)\rho^{1/2}_\Lambda y.
\end{align*} \tag{21}
\]

The first eigenvalue and the first eigenvector of the system are equal to zero. The second eigenvalue is equal to \(\lambda_2 = 3(1 + \alpha)\rho^{1/2}_\Lambda < 0\) and the second eigenvector is \(\{3/2, 3(1 + \alpha)\rho^{1/2}_\Lambda \}\). We can see that the considered point is the attractor. The same linearization method is used for the universe evolution analysis near other singular points. The second singular point \((H = -\rho^{1/2}_\Lambda, y = 0)\) corresponds to the unstable equilibrium state. Another interesting singular point is \((H = 0, y = -\alpha \rho_0)\), at which the universe reaches zero density \(\rho_{DE} = 0\) and its contraction is changed to expansion. The phase trajectories near the singular points are plotted in the left panel of figure 2 with the directions of evolution marked by arrows. In the auxiliary right panel of figure 2 the evolution of pressure \(p\) as a function of the dark energy density \(\rho\) is shown.

The right branch of the parabola in the left panel of figure 2 \((y > 0, H > 0)\) corresponds to solution (12) with an upper sign at \(t > 0\). In this case, the universe is filled with non-phantom energy. The universe starts from the big bang, corresponding to the initial values of density \(\rho_{DE,i} = +\infty\) and scale factor \(a_i = 0\). The asymptotic behaviour of a scale factor at \(a \to 0\) is described by (13) with an upper sign. Relation (13) with an upper sign also describes the evolution of a scale factor for \(\rho_\Lambda = 0\) for all \(t\). At late times the universe approaches the de Sitter regime (9). The sign of \(\rho + p\) is conserved during the evolution of the universe.
Figure 2. Phase trajectory of the universe and diagram of evolution in coordinates $(p, \rho)$ are shown in the case of dark energy parameters $1 + \alpha > 0$ and $\alpha \rho_0 > 0$. In the left panel, the phase-space diagram of evolution in coordinates $H$ and $y = \rho_u + p_u$ is presented. The direction of evolution is shown by arrows. There are two unstable singular points, $(H = -\rho_1^2/2, y = 0)$ and $(H = 0, y = -\alpha \rho_0)$, and one attractor $(H = \rho_1^2/2, y = 0)$. In the right panel, the evolution is shown in coordinates $\rho$ and $p$. The evolution from $\rho = \infty$ to $\rho = \rho_0$ corresponds to the right branch of the parabola $(y > 0, H > 0)$ of the phase-space diagram. The reverse process (from $\rho = \rho_0$ to $\rho = \infty$) corresponds to the left branch of the parabola $(y > 0, H < 0)$ of the phase-space diagram. The evolution from $\rho = \rho_0$ to bounce $\rho = 0$ and back to $\rho = \rho_0$ corresponds to the middle branch of the parabola $(y < 0)$ of the phase-space diagram.

The asymptotic behaviour of $a$ for $t \to \infty$ is given by (14) with an upper sign. The left branch of the parabola in the left panel of figure 2 $(y > 0, H < 0)$ corresponds to the reverse process with respect to that examined above and is described by (12) with a lower sign. Note that $t$ is taken to be negative in this case. The asymptotic behaviour for $t \to 0$ and $(t \to \infty)$ is given by (13) and (14) respectively with lower signs in both cases. A middle part of the parabola in the left panel of figure 2 $(y < 0)$ corresponds to solution (15). This particular case was considered in [18] as the simplest example of phantom cosmology without a big rip. The universe in this case is filled with a phantom energy. Because of the specifics of a linear equation of state, the considered universe with a phantom energy is not born in the big bang. Instead of this, the universe starts in this case from the initial scale factor $a_i = +\infty$. For $t \to -\infty$ it behaves like the de Sitter universe (however, reversed in time) according to (14) with an upper sign and bounces at the minimal value of a scale factor $a_{\text{min}} = (-\beta)^{1/\kappa}$. At this moment the state of dark energy is very special: the pressure is finite but the total energy density is zero, $\rho_u + \rho_\Lambda = 0$. Near the bounce this universe can be described by (16). After the bouncing the universe expands and at the late times it approaches the de Sitter state (14) with an upper sign.

(ii) $1 + \alpha > 0$ and $\alpha \rho_0 < 0$. The density of the $\Lambda$-term is negative, $\rho_\Lambda < 0$. The linearized system of the evolution equations at the point $(H = 0, y = -\alpha \rho_0)$ is

$$H = 3\alpha \rho_0/2 - 3(y + \alpha \rho_0)/2, \quad \dot{y} = 3(1 + \alpha)\alpha \rho_0 H. \quad (22)$$

The universe expands starting from the big bang, reaches zero density at the point $(H = 0, y = -\alpha \rho_0)$ and then its expansion changes to contraction. The phase trajectory and diagram of evolution $p - \rho$ are shown in figure 3 (similar cosmological behaviour was obtained in [19] for the universe with negative $\Lambda$-term and quintessence with special potential). The universe is filled with non-phantom energy and starts from the big bang. The evolution of a scale factor is described by (17) and at early times by the asymptotic (13) with an upper sign. This asymptotic is also valid for $\rho_\Lambda = 0$ at all $t$. During the finite time $\Delta t = \pi/(\kappa \rho_\Lambda)$, a scale
Figure 3. Phase trajectory of the universe and diagram of evolution in coordinates \((p, \rho)\) are shown in the case of dark energy parameters \(1 + \alpha > 0\) and \(\alpha \rho_0 < 0\). In the left panel, the phase-space diagram of evolution in coordinates \(H\) and \(y = \rho_u + p_u\) is presented. The direction of evolution is shown by arrows. There is only one singular point, \((H = 0, y = -\alpha \rho_0)\), which is unstable. In the right panel, the evolution is shown in coordinates \(\rho\) and \(p\). The universe evolves from \(\rho = \infty\) to \(\rho = 0\), which corresponds to the right branch of the parabola \((H > 0)\) of the phase-space diagram. At the point \(\rho = 0\) (corresponding to the point \((H = 0, y = -\alpha \rho_0)\) of the phase-space diagram) there is a bounce. Then the universe evolves back to \(\rho = \infty\) which corresponds to the left branch of the parabola \((H < 0)\) of the phase-space diagram.

Figure 4. Phase trajectory of the universe and diagram of evolution in coordinates \((p, \rho)\) are shown in the case of dark energy parameters \(1 + \alpha < 0\) and \(\alpha \rho_0 < 0\). In the left panel, the phase-space diagram of evolution in coordinates \(H\) and \(y = \rho_u + p_u\) is presented. The direction of evolution is shown by arrows. There are two unstable singular points, \((H = \rho_\Lambda^{1/2}, y = 0)\) and \((H = 0, y = -\alpha \rho_0)\), and one attractor \((H = -\rho_\Lambda^{1/2}, y = 0)\). In the right panel evolution of the universe is shown in coordinates \(\rho\) and \(p\). The evolution from \(\rho = \rho_\Lambda\) to \(\rho = \rho_\infty\) corresponds to the right branch of the parabola \((y < 0, H > 0)\) of the phase-space diagram. The reverse process from \(\rho = \infty\) to \(\rho = \rho_\Lambda\) corresponds to the left branch of the parabola \((y < 0, H < 0)\) of the phase-space diagram. The evolution from \(\rho = \rho_\Lambda\) to the bounce at \(\rho = 0\) and back to \(\rho = \rho_\Lambda\) corresponds to the middle branch of the parabola \((y > 0)\) of the phase-space diagram.

factor \(a\) reaches the maximum value \(a_{\text{max}} = (-\beta)^{1/\kappa}\) at which the universe bounces. Near the bounce the behaviour of a scale factor can be described by (18). After the bounce the universe contracts and in a finite time \(\Delta t\) it collapses to the big crunch. The scale factor of the universe near the collapse is given by (19).

(iii) \(1 + \alpha < 0\) and \(\alpha \rho_0 < 0\). At zero density at the point \((H = 0, y = -\alpha \rho_0)\) the universe expansion is changed to contraction. The singular point \((H = -\rho_\Lambda^{1/2}, y = 0)\) is an attractor and respectively \((H = \rho_\Lambda^{1/2}, y = 0)\) is an unstable equilibrium point. The phase trajectory and diagram of evolution \(p - \rho\) are shown in figure 4.
Figure 5. Phase trajectory of the universe and diagram of evolution in coordinates \((p, \rho)\) are shown in the case of dark energy parameters \(1 + \alpha < 0\) and \(\alpha \rho_0 > 0\). In the left panel, the phase-space diagram of evolution in coordinates \(H\) and \(y = \rho_0 + p_0\) is presented. The direction of evolution is shown by arrows. There is only one singular point, \((H = 0, y = -\alpha \rho_0)\), which is unstable. In the right panel the evolution of the universe is shown in coordinates \(\rho\) and \(p\). The universe evolves from ‘anti-big rip’ state \((\rho = \infty)\) to the bouncing point \((\rho = 0)\), which corresponds to the left branch of parabola \((H < 0)\) of the phase-space diagram. Then the universe evolves to the big rip \((\rho = \infty)\) which corresponds to the right branch of parabola \((H > 0)\) of the phase-space diagram.

First we consider the right branch of the parabola \((y < 0, H > 0)\). In this case the universe is filled with phantom energy, \(\rho + p < 0\). The evolution of a scale factor is given by (12) with an upper sign. Note that \(t\) is taken to be negative in this case. Starting from the scale factor \(a_i = 0\), the universe expands during an infinite time to the final big rip. At \(t \to -\infty\), the scale factor of the universe is given by (14) with an upper sign. Near the big rip, the scale factor \(a\) is described by (13) with an upper sign (note that \(\kappa\) and \(t\) are both negative). The left branch of the parabola \((y < 0, H < 0)\) corresponds to the reverse process with respect to that examined above and is described by (12) with a lower sign. Time \(t\) is positive during the whole evolution. In this case, the universe is filled with phantom energy and it starts from the ‘anti-big rip’ solution (with infinite value of the initial scale factor), contracts during an infinite time to the final \(a_f = 0\). The asymptotic behaviour for \(t \to 0\) and \(t \to +\infty\) is given by (13) with the lower sign and (14) with the lower sign, respectively. The middle part of the parabola \((y > 0)\) corresponds to solution (15). The universe is filled with a non-phantom energy and starts from \(a_i = 0\). At \(t \to -\infty\), the scale factor of the universe is given by (14) with the upper sign. After the bouncing at the maximum value of the scale factor \(a_{\text{max}} = (-\beta)^{1/\kappa}\), the universe begins to contract. Near the bouncing point, the scale factor is given by (16). At \(t \to +\infty\) the universe behaves like (14) with the lower sign.

(iv) \(1 + \alpha < 0\) and \(\alpha \rho_0 > 0\). In this case the density of the \(\Lambda\)-term is negative: \(\rho_\Lambda < 0\). The universe reaches the zero density at the point \((H = 0, y = -\alpha \rho_0)\), where the contraction is changed to expansion. The phase trajectory and diagram of evolution \(p - \rho\) are shown in figure 5. The universe is filled with a phantom energy and the solution for the scale factor is given by (17). The universe is born in the ‘anti-big rip’ state and the scale factor is given by (13) with the lower sign near \(t \approx 0\). Then the universe contracts to the minimum state \(a_{\text{min}} = (-\beta)^{1/\kappa}\) in a finite time \(\Delta t = \pi/(\kappa \rho_\Lambda)\), and after bouncing it begins to expand. In time \(\Delta t\), the universe comes to the big rip where the scale factor of the universe is given by (19).

The different cosmological scenarios discussed above in this section are summarized in table 1.
limit only the behaviour of the universe at small redshifts

If we add the usual matter (dark matter, baryons and radiation) to the considered universe, then the differential equation for scale factor evolution would have the following form:

\[
\left( \frac{\dot{a}}{a} \right)^2 = H_0^2 \left[ \Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0} + \Omega_{q,0}(1+z)^{3(\alpha+1)} \right],
\]

(23)

where \( H_0 \) is the current value of the Hubble constant, \( z = \frac{a_0}{a} - 1 \) is a redshift, \( \Omega_{r,0} \) and \( \Omega_{m,0} \simeq 0.3 \) are the cosmological density parameters of radiation and non-relativistic matter, respectively. The value of the cosmological density parameter of ‘\( \Lambda \)-term’ is \( \Omega_{\Lambda,0} = [\alpha/(\alpha+1)]\rho_0/\rho_{c,0} \), where \( \rho_{c,0} \) is the current critical density. The contribution of the dynamically changing part of dark energy is as follows: \( \Omega_{q,0} = \Omega_{q,0} - \Omega_{\Lambda,0} \), where \( \Omega_{q,0} \simeq 0.7 \) is the quintessence density parameter. We suppose that the universe is flat with \( \Omega_{r,0} + \Omega_{m,0} + \Omega_{q,0} = 1 \). In the case of pure \( \Lambda \)-term (\( \alpha = -1, \rho_0 = 0 \)), one should omit the last term in the brackets in (23) and take \( \Omega_{\Lambda,0} = \Omega_{q,0} \).

The observational restrictions on the \( w (-1.38 < w < -0.82) \), derived from the SN data, limit only the behaviour of the universe at small redshifts \( z \simeq 0 - 1 \). In the case of \( \rho_0/\rho_{DE} < 1 \) (see figure 1) the dark energy does not drastically change the evolution of the universe at high redshift in comparison with a pure \( \Lambda \)-term case. The only restriction is the age of the universe which is limited (e.g., by the oldest globular clusters) to \( t_0 > 12 \times 10^9 \) yr. We find the age of the universe in our model by integration of (23) from \( z = 0 \) to \( z = \infty \) for different values of \( \alpha \) and \( \rho_0 \) and for present-day values of \( H_0 = 65 \text{ km s}^{-1} \text{ Mpc}^{-1} \), \( \Omega_m = 0.3 \) and \( \Omega_r = 4.54 \times 10^{-5}/h^2 \), where \( h \) is the \( H_0 \) in the units 100 km s\(^{-1}\) Mpc\(^{-1}\). The connection of commonly used parameter \( w \) with our parameters \( \alpha \) and \( \rho_0 \) is given by the relation \( w = \alpha(1 - \rho_0/\rho_{DE}) \). For example, if \( w = -0.82 \) then the age restriction is important only at \( \rho_0/\rho_{DE} \leq -100 \). In the case of \( w < -1 \) the universe is older than in a pure \( \Lambda \)-term case (\( w = -1 \)) and the age restriction is unimportant.

In contrast, at \( \rho_0/\rho_{DE} > 1 \) only a small range of parameters may correspond to the real universe. In addition to restriction on \( w \) and the universe age restriction, one must require the universe to start from the big bang and not from the bounce at a finite scale factor. The bounce \( \dot{a} = 0 \) appears in the case of \( \alpha > 1/3 \) and \( \Omega_{q,0} < 0 \). Therefore, to avoid the bounce we must additionally require that \( \alpha \leq 1/3 \). One more condition \( \Omega_{q,0} + \Omega_{r,0} > 0 \) should be satisfied in the boundary case of \( \alpha = 1/3 \). Strictly speaking, the bounce at redshift

### Table 1. Possible cosmological scenarios in the case of a generalized linear equation of state

| Values of $\Omega_1$ | Expansion | Expansion, Attractor | Contraction | Contraction |
|---------------------|-----------|----------------------|-------------|-------------|
| $H < -\sqrt{\Omega_{\Lambda}}$ | Contraction | Expansion, bounce and contraction | Attractor | Contraction |
| $H = -\sqrt{\Omega_{\Lambda}}$ | Non-steady equilibrium point | Expansion, bounce and contraction | Non-steady equilibrium point | Non-steady contraction |
| $-\sqrt{\Omega_{\Lambda}} < H < \sqrt{\Omega_{\Lambda}}$ | Attractor | Expansion | Attractor | Contraction |
| $H > \sqrt{\Omega_{\Lambda}}$ | Expansion | Expansion | Expansion | Expansion |

3. Universe with dark energy and matter

If we add the usual matter (dark matter, baryons and radiation) to the considered universe, then the differential equation for scale factor evolution would have the following form:

\[
\left( \frac{\dot{a}}{a} \right)^2 = H_0^2 \left[ \Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0} + \Omega_{q,0}(1+z)^{3(\alpha+1)} \right],
\]
$1 + z_b = \left(\Omega_{r,0}/|\Omega_{\nu,0}|\right)^{1/(3\alpha - 1)}$ could not occur later than the end of inflation or reheating moment at the temperature $T_{RH} \sim 10^{13}$ GeV corresponding to $z_{RH} \sim 10^{26}$. Therefore, it is necessary to satisfy the condition $z_b \gg z_{RH}$. The last condition puts a restriction on the allowable values of $\alpha$ and $\Omega_{r,0}$. This restriction can be considered in the particular inflation model taking into account the additional dynamical components (say inflaton). This consideration is out of the scope of this paper.

4. Discussion and conclusion

In this paper, we examined the dynamical evolution of the universe filled with a dark energy obeying the linear equation of state (3). It turns out that this simple linear model for the different choices of parameters $\alpha$ and $\rho_0$ has a rich variety of cosmological dynamics. Depending on the signs of $1 + \alpha$ and $\alpha \rho_0$ and the initial conditions for $\rho$ and $p$ there can be a set of distinctive types of cosmological scenarios: big bang, big crunch, big rip, anti-big rip, solutions with the de Sitter attractor, bouncing solutions and their various combinations. In the framework of the linear model (3), the analytical solutions of the dynamics of the universe were obtained. Using the phase plane analysis, we gave the full classification of the solutions based on the parameters $\alpha$ and $\rho_0$.

We distinguish four main types of the evolution of the universe filled with dark energy with a linear equation of state (3):

(i) For parameters $1 + \alpha > 0$ and $\alpha \rho_0 > 0$, the universe may contain either non-phantom or phantom energy. For non-phantom energy there are two types of evolution: (a) starting from the big bang the universe approaches the de Sitter phase during an infinite time; (b) reversed process with respect to that described above when the universe starts from a reversed in time de Sitter phase and evolves to the big crunch. In the phantom case the universe starts from the reversed in time de Sitter phase then it reaches the bouncing point $(\rho = 0, p \neq 0)$ and after the bounce the universe approaches de Sitter phase.

(ii) For parameters $1 + \alpha > 0$ and $\alpha \rho_0 < 0$, the universe may contain only non-phantom energy. In this case, only one type of evolution is possible: the universe expands starting from the big bang, reaches the bounce point $(\rho = 0, p \neq 0)$ in finite time and then its expansion changes to the contraction, resulting in the big crunch.

(iii) For parameters $1 + \alpha < 0$ and $\alpha \rho_0 < 0$ the universe may contain either non-phantom or phantom energy. In the case of a phantom universe there are two possible scenarios: (a) starting from $a_i = 0$ the universe expands in infinite time to the final big rip; (b) the reversed in time process with respect to that described above when the universe starts from the anti-big rip and evolves to a final state with $a_f = 0$. In the non-phantom case, the universe starts from the reversed in time de Sitter phase. Then it reaches the bouncing point $(\rho = 0, p \neq 0)$ and after the bounce the universe approaches the de Sitter phase.

(iv) For parameters $1 + \alpha < 0$ and $\alpha \rho_0 > 0$ the universe may contain only phantom energy. The universe is born in the anti-big rip state and contracts reaching the bouncing point in a finite time. After the bounce, the universe begins to expand. In a finite time the universe comes to the big rip.

Table 1 summarizes the above results for the evolution of the universe filled with dark energy with a generalized equation of state (3).

It should be stressed that the linearity of a considered dark energy equation of state is not crucial for the general properties of cosmological evolution. Instead of (3), one may consider any rather smooth curves $p = p(\rho)$ as shown in figure 6. It is clear that the general behaviour of the evolution and properties of attractors and bounce points do not change in this case.
because any sufficiently smooth function $p = p(\rho)$ can be linearized in the local vicinity of any point. Thus, we can reduce the general problem for $p = p(\rho)$ to the analysis of a linear cosmology considered in this paper. From the above, it follows in particular that the universe filled with dark energy with an equation of state $p_{DE} = p_{DE}(\rho_{DE})$ always approaches the de Sitter attractor if additionally the physically reasonable conditions $0 < d_{pDE}/d_{\rho_{DE}} < 1$ are satisfied. The first inequality in the last expression means that the considered dark energy is hydrodynamically stable. While the second inequality restricts the sound speed to the speed of the light, a more detailed analysis of the cosmology with an arbitrary continuous equation of state $p = p(\rho)$ will be presented elsewhere [20].

Our analysis is limited by the consideration of the cosmological model of the universe evolution filled only with dark energy. Taking into account the presence of ordinary matter and radiation makes the evolution of the universe much more complicated and requires special consideration [20].

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