On Einstein - Weyl unified model of dark energy and dark matter

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Abstract

Here I give a more detailed account of the part of the conference report that was devoted to reinterpreting the Einstein ‘unified models of gravity and electromagnetism’ (1923) as the unified theory of dark energy (cosmological constant) and dark matter (neutral massive vector particle having only gravitational interactions). After summarizing Einstein’s work and related earlier work of Weyl and Eddington, I present an approach to finding spherically symmetric solutions of the simplest variant of the Einstein models that was earlier mentioned in Weyl’s work as an example of his generalization of general relativity. The spherically symmetric static solutions and homogeneous cosmological models are considered in some detail. As the theory is not integrable we study approximate solutions. In the static case, we show that there may exist two horizons and derive solutions near horizons. In cosmology, we propose to study the corresponding expansions of possible solutions near the origin and derive these expansions in a simplified model neglecting anisotropy. The structure of the solutions seems to hint at a possibility of an inflation mechanism that does not require adding scalar fields.

1 Introduction

In this report I give a new interpretation of the ‘unified theory of gravity and electromagnetism’ proposed by A.Einstein in 1923 in [1] and briefly summarized in [2]. Einstein gave no details of his derivations, presented no exact or approximate solutions, and did not explain why he completely abandoned his theory (I failed to find any reference to his papers [1] - [2] in his later work). Apparently these papers were soon forgotten by the scientific community and I could not find any reference to these papers in the second half of the 20-th century except for interesting remarks by Schrödinger [3] and a critical discussion by Pauli in addenda to the English translation of his famous book [4]. For these reasons, I first give a brief historical introduction summarizing Einstein’s ideas and results as well as earlier related work of Weyl and Eddington.

Immediately after the general relativity was formulated in its final form (1915 -1916) some attempts to modify it started. Einstein himself added the cosmological constant term \( \Lambda \) to save (unsuccessfully) his static cosmology. After Friedmann’s work (1922-1924) this modification was becoming more and more dubious. Weyl, after 1918, developed a much more serious modification aimed at unifying gravity and electromagnetism (most clearly summarized in [5]). Starting from Levy-Civita’s ideas on a general (non-Riemannian) connection (1917) he developed the theory of a special space in which the connection depends both on metric tensor and on a vector field which he tried to identify with the electro-magnetic potential. To get a consistent theory he introduced a general idea of gauge invariance which survived although the theory itself failed as he admitted...
later. In paper [6] Einstein discussed Weyl’s theory and expressed (like Pauli in [4]) the opinion that the theory is mathematically very interesting but probably not physical, at least, in its original formulation.

In 1919 Eddington proposed a more radical modification of general relativity [7], [8]. His idea was to start with the pure affine formulation of the gravitation, i.e. using first the general symmetric affine connection and only at some later stage introducing a metric tensor. Indeed, the curvature tensor can be defined without metric (here we use Einstein’s notation [1] but denote differentiations by commas):

\[ r_{klm}^i = -\Gamma_{kl,m}^i + \Gamma_{il}^m \Gamma_{km}^l + \Gamma_{km,l}^i - \Gamma_{nm}^i \Gamma_{kl}^n. \]  

(1)

Then the Ricci-like (but non-symmetric) curvature tensor can be defined by contracting the indices \( i, m \) (or, \( i, l \)):

\[ r_{kl} = -\Gamma_{kl,m}^m + \Gamma_{nl}^m \Gamma_{km}^l + \Gamma_{km,l}^m - \Gamma_{lm}^m \Gamma_{kl}^l \]  

(2)

(let us stress once more that \( \Gamma_{nl}^m = \Gamma_{ln}^m \) but \( r_{kl} \neq r_{lk} \)). Using only these tensors and the anti-symmetric tensor density one can build up a rather rich geometric structure. In particular, Eddington discussed different sorts of tensor densities [8]. A notable scalar density is

\[ \hat{L} \equiv \sqrt{-\det(r_{kl})} \equiv \sqrt{-r} \]  

(3)

which resembles the fundamental scalar density of the Riemannian geometry, \( \sqrt{-\det(g_{kl})} \equiv \sqrt{-g} \). For this and some other reasons Eddington suggested to identify the symmetric part of \( r_{kl} \) with the metric tensor. The anti-symmetric part,

\[ \phi_{kl} \equiv \frac{1}{2}(\Gamma_{km,l}^m - \Gamma_{lm,k}^m), \quad \phi_{kl,m} + \phi_{lm,k} + \phi_{mk,l} \equiv 0, \]  

(4)

strongly resembles the electro-magnetic field tensor and it seems natural to identify it with this tensor. Eddington tried to write consistent equations of the generalized theory but this problem was solved only by Einstein.

The starting point of Einstein in his first paper of the series [1] was to write the action principle and to suppose [3] to be the Lagrangian density depending on 40 connection functions \( \Gamma_{kl}^m \). Varying the action w.r.t. these functions he derived 40 equations that allowed him to find the general expression for \( \Gamma_{kl}^m \) (the derivation is similar to that of the standard general relativity):

\[ \Gamma_{kl}^m = \frac{1}{2}[s_{mn}(s_{kn,l} + s_{ln,k} - s_{kl,n}) - s_{kl} i^n + \frac{1}{3}(\delta_{m}^n i_l + \delta_{l}^n i_k)]. \]  

(5)

Here \( s_{kl} \) is a symmetric tensor \( (s_{mn} \) is the inverse matrix to \( s_{kl} \), which Einstein interpreted as the metric tensor (then the first term is the Christoffel symbol for this metric), and \( i_n \) is a vector which he tried to connect with the electro-magnetic field. This identification apparently follows from the equations

\[ r_{kl} = R_{kl} + \frac{1}{6}[(i_{k,l} - i_{l,k}) + i_k i_l], \]  

(6)

\[ \phi_{kl} = \frac{1}{6}(i_{k,l} - i_{l,k}), \]  

(7)

which can be obtained by inserting the expression \( [5] \) into \( [2] \), [1]; \( R_{kl} \) is the standard Ricci curvature tensor for the metric \( s_{kl} \). Einstein’s interpretation of \( \phi_{kl} \) as the Maxwell field is not so natural because of the term \( i_k i_l \) in the r.h.s. of Eq.(6) which in fact makes this interpretation impossible. First, this term is not gauge invariant (but the gauge invariance was not yet discovered, the first clear formulation of the gauge principle was given by V.Fock in 1926). For Einstein, the main problem was that the electro-magnetic field in this theory could not exist without charges (i.e. there is no free field). To solve this problem he suggested to make this term ‘infinitesimally small’
by choosing the corresponding dimensional constant (above, we omit all dimensional constants that can easily be restored). But we, today, cannot be satisfied with this solution because this term violates gauge invariance and makes the photon effectively massive (while it is known that there exist no continuous transition from the massless to massive photon theory). We return to discussing these facts, on which our interpretation of the Einstein theory is based, after considering the final proposals of Einstein.

In his first paper (‘Zur allgemeinen Relativitätstheorie’), he considered two limiting cases. He showed that, when the $i_m$-terms in the connection vanish, the theory is equivalent to the standard general relativity with the cosmological term that emerges naturally and cannot be removed. In the flat space limit he demonstrated that weak fields $\phi_{kl}$ (linear approximation) satisfy the free Maxwell equations provided that the $i_m i_n$-terms can be neglected. In the second paper of the series \[1\] he gave the following expression for the effective Lagrangian density:

$$\hat{\mathcal{L}} = -2\sqrt{-\det(r_{mn})} + \hat{R} - \frac{1}{6} s^{mn} i_m i_n.$$  

(8)

This should be varied w.r.t. $\hat{s}_{kl}$ and $\hat{f}_{kl}$, which are the tensor densities defined with the aid of the scalar density $\sqrt{-\det(s_{kl})}$ and corresponding to the tensors in the decomposition,

$$r_{kl} = s_{kl} + \phi_{kl};$$  

(9)

$\hat{R}$ is the scalar curvature density for the metric $s_{kl}$. The Lagrangian \[s\] contains a very complex term $\sqrt{-\det(r_{mn})}$ which is more general than the so called Born-Infeld Lagrangian proposed ten years later \[10\] (the first attempts to construct nonlinear electro-dynamics were undertaken in \[11\]). Apparently, Einstein did not try to find any particular solution of this theory and, instead, in the beautiful third paper ‘Zur affinen Feldtheorie’ (\[1\]) he proposed a significantly simpler effective Lagrangian that is the main subject of this paper. As he mentioned in the first two papers the actual form of the Lagrangian is unimportant for getting the connection \[5\], the only important thing is on which variables it depends.

The main idea of the third paper is to take for the Lagrangian $\hat{\mathcal{L}}$ an arbitrary function of $s_{kl}$ and $\phi_{kl}$. Then he introduces the Legendre transformation and the transformed (effective) Lagrangian density $\hat{\mathcal{L}}^*$:

$$\frac{\partial \hat{\mathcal{L}}}{\partial s_{kl}} \equiv \hat{g}^{kl}, \quad \frac{\partial \hat{\mathcal{L}}}{\partial \phi_{kl}} \equiv \hat{f}^{kl}; \quad s_{kl} = \frac{\partial \hat{\mathcal{L}}^*}{\partial \hat{g}^{kl}}, \quad \phi_{kl} = \frac{\partial \hat{\mathcal{L}}^*}{\partial \hat{f}^{kl}}.$$  

(10)

Introducing the Riemann metric tensor $g_{kl}$ and the $i^k$-vector,

$$\hat{g}^{kl} \sqrt{-g} = \hat{g}^{kl}, \quad g_{kl} \hat{g}^{lm} = \delta^{lm}_{k}; \quad \hat{i}^k = \partial_i \hat{f}^{kl},$$  

(11)

he claims (without proof) that Eq.\[5\] is valid with $s_{kl}$ replaced by $g_{kl}$ and thus the affine geometry is the same for any $\hat{\mathcal{L}}(s_{kl}, \phi_{kl})$. Finally, he uses the freedom in choosing $\hat{\mathcal{L}}^*(\hat{g}^{kl}, \hat{f}^{kl})$ and proposes the following effective Lagrangian density:

$$\hat{\mathcal{L}}^* = 2\alpha \sqrt{-g} - \frac{1}{2} \beta f_{kl} \hat{f}^{kl};$$  

(12)

where $\alpha$ and $\beta$ are some constants not defined by the theory. This Lagrangian incorporates main properties of the theory discussed in two previous papers but is easier to deal with.

\[2\] The present best experimental upper bound on the photon mass is $m_\gamma < 10^{-51}$ g \[9\]. Theoretical wisdom says that it must be zero.

\[3\] In his previous work Einstein implied that $\hat{\mathcal{L}}$ depends on $r_{kl}$. At this point he quoted an unpublished work of ‘Droste (from Leiden)’ who ‘two years ago expressed similar views’. The meaning of this rather cryptic remark is clarified in the second brief paper of ref. \[2\], where he confirms that (Johannes) Droste proposed to use a similar effective Lagrangian and, possibly a similar model (maybe, without cosmological constant).
To further clarify the relation of the new theory to general relativity Einstein rewrites the Lagrangian so that the equations of motion can be obtained by varying it in the metric and the vector field tensors, $g_{kl}$ and $f_{kl}$. Neglecting dimensions (for example, taking $\hbar = c = \kappa = 1$) and changing Einstein’s notation we write it as follows:

$$\hat{\mathcal{L}} = \sqrt{-g} \left[ R - 2\Lambda - F_{kl}F^{kl} - \mu^2 A_k A^k \right], \quad F_{kl} \equiv A_{k,l} - A_{k,l}.$$  \hfill (13)

Now it is absolutely evident that the vector field $A_k$ is not the Maxwell field. Obviously, $A_k$ is a neutral massive vector field with coupling to gravity only. We will call it vecton, that is an old fashioned but proper term for this ‘geometric’ particle. This particle has not been directly observed but it can be considered as one of the possible candidates for dark matter. In view of the fact that the affine theory also predicted the cosmological constant term which is one of the best candidates for explaining dark energy, Einstein’s theory may be considered as the first unified model of dark energy and dark matter.

Before we turn to further study of this model let us finish our presentation of its history. If you compare the Einstein model with the concrete models proposed in Weyl’s book [5], you will find that Lagrangian similar to Eq.\((13)\) is one of Weyl’s examples. Einstein’s and Weyl motivations and approaches were quite different, and the Weyl connection does not coincide with Eq.\((5)\) (see Addendum). Weyl’s approach was mostly geometrical and he wrote the Lagrangian as a simplest illustration of possible physical applications, responding to criticism by Einstein, Pauli and other physicists. Einstein was most interested in physics and, especially, in cosmology. Weyl criticized Einstein for his departure from geometric foundations of physics, in particular, for his derivation of geometry from the variational (action) principle which, probably, was his main achievement in the third paper. Note also that Weyl included the cosmological term only to avoid contradiction to Einstein cosmology of that time (‘before Friedmann’) while in the original Einstein model \((3), (7)\) it was unavoidable. I think that, conceptually, the model \((13)\) is a step backwards, in comparison with the original theory, \((3), (8)\). There were, probably, two reasons for this step. First, Einstein’s deep belief in simplicity of fundamental laws (‘...aber boshaft ist Er nicht’). Second, his disappointment in static cosmology after accepting Friedmann’s results, \([12]\). Anyway, in his last papers on affine theory [2] he set the cosmological term to zero what is impossible in the original theory and quite unnatural in the framework of the affine approach.

Above, we also mentioned work and ideas of Eddington. The intensive exchange of ideas between Einstein, Weyl and Eddington resulted in interrelations in their work (published in 1918-1923) that are difficult (and, possibly, unnecessary) to disentangle. As the constructive ideas of the affine theory were mostly created by Weyl, Eddington, and Einstein, the resulting model should probably be called Einstein-Weyl-Eddington unified model of dark energy and dark matter. However, as far as I am here discussing the concrete Lagrangian \((13)\), I call it Einstein-Weyl model.

Before turning to new results let us briefly summarize the results and thoughts of Weyl, Eddington, and Einstein. 1. Weyl had a very clear and original geometric ideas, but: a) his physics was rightly criticized by Einstein, Pauli, and other physicists, b) he considered the theory as a unified theory of gravity and electromagnetism but his vector field was also not electro-magnetic, c) his discussion of dynamics was incomplete and he himself regarded it as preliminary. Nevertheless, it is possible that not all the potential of the Weyl ideas is understood and used. 2. Eddington proposed to use, instead of the Weyl’s non-Riemannian ‘metrical spaces’, the most general
spaces with symmetric affine connection (without torsion). He discussed possible invariants that can be used in physics, in particular, the square root of the determinant of $R_{kl}$\footnote{In addendum to\cite{8}, where he gave a clear and detailed account of the Einstein work\cite{1}, he also discussed another scalar density that was proposed by R. Weitzenböck}. He proposed to consider the symmetric part of the curvature matrix as the metric in the general space and the anti-symmetric one as the electro-magnetic field tensor. In later works he discussed a possibility to use this as a Lagrangian (long before the proposal of Born and Infeld). However, he did not find a consistent approach to dynamics. 3. **Einstein** started with formulating dynamics by use of the Hamilton principle similar to one proposed by Palatini in general relativity. The new (and crucial) idea was not to introduce any metric at the beginning and not to fix any special form of the affine connection (apart of the symmetry condition). He soon realized (in the second apper), that he does not need to use a concrete form of the Lagrangian that can be just any function (tensor density) of $s_{kl}$ and $\phi_{kl}$-matrices (see (9-11)). For any such Lagrangian he proved that the affine connection allows one to introduce a symmetric metric and found the expression for connection. Both Einstein’s and Weyl’s expressions are special cases of the general formula for the symmetric connection (see Appendix).

The most important thing is the following: supposing that the equations of motion follow from an action principle with the general Lagrangian fixes the geometry (connection) and, eventually, allows one to fix some metric compatible with this non-Riemannian connection. Another important thing is that the action can be (and should be) written without metric. Using the Legendre transformation Einstein bypassed difficulties that were met on this way and wrote more tractable effective Lagrangian, but some conceptually beautiful and important features of his new theory were thus hidden (or even lost).

Apparently, Einstein was disappointed in the cosmological constant and also gradually realized that his interpretation of the anti-symmetric field as the electro-magnetic field was not quite satisfactory. Anyway, he completely abandoned this model and left no detailed account of his work. He did not mention any static or cosmological solutions even in the simplified version of the theory, (13). In this paper we try to fill this gap and establish grounds for comparing this model to the present day cosmology.

2 Spherical reduction - static and cosmological solutions

2.1 Vecton-dilaton gravity

At first sight, the theory (13) is very close to the well-understood Einstein-Maxwell theory which can be obtained when $\mu = 0$. However, we will show that the two theories are qualitatively different and it is hardly possible to construct a reasonable perturbation theory in the parameter $\mu^2$. We start our qualitative analysis without assuming that this parameter is small. The natural object for this analysis is the spherically reduced theory. When $\mu = 0$, the theory automatically reduces to rather simple one-dimensional equations that can be explicitly solved. The solution is the Reissner-Nordström black hole (when the electric charge vanishes it reduces to the Schwarzschild black hole). In general, when gravity couples to other (not electro-magnetic) fields the spherically reduced theory is described by two-dimensional differential equations which are not integrable except very special cases (for many examples and references see, e.g., [13]-[17]).

Following the approach to dimensional reduction and to resulting 1+1 dimensional dilaton gravity (DG) developed in papers [15]-[19] it is not difficult to derive these equations. The general spherically symmetric metric is $(i,j = 0, 1; x^0 = t, x^1 = r)$:

$$ds^2 = g_{ij}(t,r)\, dx^i dx^j + \varphi(t,r)(\sin^2 \theta\, d\theta + d\phi^2).$$

(14)
Supposing that all other functions also depend on \( t, r \), inserting the metric (13) into the action with the Lagrangian (13), and integrating out the angle variables \( \theta, \phi \) one can derive the following effective two-dimensional Lagrangian\[^7\]:

\[
\mathcal{L}^{(2)} = \sqrt{-g} [\varphi R^{(2)} + 2 - 2\Lambda \varphi + (\partial \varphi)^2/2\varphi - \varphi F_{ij} F^{ij} - \varphi \mu^2 A_i A^i],
\]

where \( R^{(2)} \) is the two-dimensional Ricci curvature depending on the \( g_{ij} \). It is convenient to remove the fourth term by the Weyl rescaling of the metric, \( g_{ij} = \varphi^{-\frac{1}{2}} g_{ij}^W \). Below we use the transformed Lagrangian,

\[
\mathcal{L}_W^{(2)} = \sqrt{-g} [\varphi R^{(2)} + 2\varphi^{-1/2} - 2\Lambda \varphi^{1/2} - \varphi^{-3/2} F^2 - \varphi \mu^2 A^2].
\]

It is easy to derive the equations of motion which in a generic metric \( g_{ij} \) are equivalent to the Einstein equations for the spherically symmetric solutions of the four-dimensional theory (13). By varying w.r.t. the diagonal metric functions \( g_{ii} \) we fist write the energy and momentum constraints. In the light cone (LC) metric, \( ds^2 = -4f(u,v) du dv \), these constraints are simple:

\[
f \partial_i \left( \partial_i \varphi/f \right) + \varphi \mu^2 A_i^2 = 0, \quad i = u, v.
\]

The constraints (17) should be derived using the general metric \( g_{ij} \). The other equations of motion may be obtained directly in the LC-metric:

\[
\partial_u \partial_v \varphi + f(2\varphi^{-1/2} - 2\Lambda \varphi^{1/2} - \frac{1}{2} \varphi^{3/2} f^{-2} F_{uv}^2) = 0, \quad F_{uv} \equiv A_{u,v} - A_{v,u},
\]

\[
\partial_j \left( \varphi^{3/2} f^{-1} F_{ij} \right) = \varphi \mu^2 A_j \quad i, j = u, v.
\]

From the last equation immediately follows that \( \partial_u (\varphi A_u) + \partial_v (\varphi A_v) = 0 \). In the original four-dimensional theory this is the \( \partial_{\mu} (\sqrt{-g} A^\mu) = 0 \) condition eliminating spin 0. Weyl, Eddington and Einstein called it the Lorentz condition although we know that its origin and meaning are quite different from the gauge fixing condition in the Maxwell theory first introduced by L.Lorenz and later popularized by H.A.Lorentz.

This dilaton gravity coupled to massive vector field (I suggest to call it **vector-dilaton gravity, VDG**) is more complex than the well studied models of dilaton gravity coupled to scalar fields and thus it requires a separate study. The natural first question is: are there exact analytical solutions like Schwarzschild or Reissner-Nordström black holes? If the vector field is constant, we return to exactly soluble DG having explicit solutions with horizons. Otherwise, when the vector field is nontrivial, the answer is more difficult to find but it is worked out in some detail below. The second question is: what are the simplest cosmological solutions in this theory? Thus, the first thing to do is to further reduce the theory to static or cosmological configurations. Consider first the static reduction.

### 2.2 Static states and horizons

The simplest way to derive the corresponding equations is to suppose that all the functions in the equations depend on \( r = u + v \). But this is not the most general dimensional reduction of the two-dimensional theory. There exist more general ones that allow us to simultaneously treat black holes, cosmologies and some waves. These generalized reductions were proposed in papers [20], [21], [17] devoted to dilaton gravity coupled to scalar fields and Abelian gauge fields; here we only discuss in some detail the static and cosmological reductions. In both cases it can be seen that the perturbed \[^7\] Very similar effective Lagrangians can be obtained from the higher-dimensional analogs of the Lagrangian (13). On the (1+1)-dimensional level it is not difficult to include other sorts of matter that appear in reductions of higher-dimensional supergravity theories (for references see, e.g., [13] – [17]). I hope to discuss some of these generalizations in future publications.
theory (with a nonvanishing mass term) is qualitatively different from the non-perturbed one. Indeed, the non-perturbed theory is just dilaton gravity coupled to electromagnetism. This model is equivalent to pure dilaton gravity, which is a topological theory. In particular, it automatically reduces to one-dimensional static or cosmological models that can be analytically solved. Static states are the Reissner-Nordström black holes perturbed by the cosmological constant and having two horizons, while the space between horizons may be considered as an unrealistic cosmology. This object is known from 1916 times; certainly it was familiar to Einstein in 1923 but he did not discuss the static configuration and apparently did not consider black holes or horizons as having any relation to physics.

Let us now write the static equations corresponding to the naive reduction to one spatial dimension. To obtain them one can reduce either the equations or the Lagrangian. Following [19], [22], we write the equations of motion in a somewhat unusual form. Let us define two additional functions, \( \chi \) and \( B \), by the equations (the prime denotes differentiations w.r.t. \( r \))

\[
\varphi'(r) = \chi(r), \quad \tilde{A}'(r) = f(r)\varphi^{-3/2}(r)B(r),
\]

where, as follows from Eq.(19), \( A_v(r) = -A_u(r) \equiv -A(r) \). Then the other equations are

\[
\chi' = -fU, \quad B' = -\frac{1}{2}\varphi \mu^2 A, \quad f' = (f/\chi)[-fU + \varphi \mu^2 A^2],
\]

where we defined the potential

\[
U \equiv 2(\varphi^{-1/2} - \Lambda \varphi^{1/2} - \varphi^{-3/2}B^2),
\]

These equations are not integrable and cannot be solved analytically. To get numerical solutions we first have to study the analytic and asymptotic properties of their solutions.

Here we only consider solutions near possible horizons that are defined as zeroes of the metric, \( f \to 0 \) for finite values of \( \varphi \to \varphi_0 \). It is not difficult to understand that we also should require that \( A \) is finite near the horizon. To study the behaviour of the solutions for small values of \( \varphi \equiv \varphi - \varphi_0 \) it is most convenient to consider the solutions as functions of \( \varphi \). Further analysis shows that the solutions can be expanded in power series of \( \varphi \) and that the functions \( \tilde{F} \equiv f/\chi \) and \( \tilde{A} = A/\chi \) should be finite. Thus we have:

\[
\tilde{F}'(\varphi) = \tilde{F}(\varphi) \mu^2 \tilde{A}^2(\varphi), \quad \chi'(\varphi) = -\tilde{F}(\varphi)U(\varphi),
\]

\[
B'(\varphi) = -\frac{1}{2}\varphi \mu^2 \tilde{A}(\varphi), \quad \tilde{A}'(\varphi)\chi(\varphi) = \tilde{F}(\varphi)[\varphi^{-3/2}B(\varphi) + U(\varphi)\tilde{A}(\varphi)],
\]

where now the prime denotes differentiation in the new variable \( \varphi \). It is not very difficult to show that \( \varphi_0, \tilde{A}_0, B_0, \tilde{F}_0 \) can be taken arbitrary up to one relation that should be satisfied due to the second equation [24]:

\[
\tilde{A}_0 U_0 + \varphi_0^{-3/2}B_0 = 0, \quad U_0 \equiv U(\varphi_0, B_0).
\]

This equation can be solved w.r.t. any parameter. It is interesting to see that it has two solutions for \( \varphi_0 \) which means that there may exist two horizons\(^8\) as distinct from the Schwarzschild black hole. Note that the solutions with different \( \tilde{F}_0 \) are equivalent because the equations are invariant under the scale transformation \( \tilde{F} \Rightarrow CF, \chi \Rightarrow C\chi \).

Now, following the method of [22], one can find several terms in the expansion of the solution. Unfortunately, it is not clear how to construct the complete expansion and therefore our derivations do not allow us to study global properties of the solutions. They say nothing about asymptotic

\(^8\) This is similar to the charged Reissner-Nordström black hole or to black holes in higher-dimensional theories, [23], although in the present model there is no conserved electric charge.
properties and singularities which should be the subject of separate investigations.

When the qualitative properties of the black hole type solutions will be understood, the static solutions and their formation can be studied by numerical simulations. As far as I know, the coupling of massive neutral vector particles to gravity did not attract much attention (see, however, numerical simulations of the critical collapse of a massive vector field in [24]).

2.3 Cosmology

2.3.1 General formulation

Let us turn to cosmological reductions. The simplest cosmology can be obtained by the same naive reductions as was used for static states. However, this is not the most general dimensional reduction giving all possible spherically symmetric cosmological solutions (similarly, the above naive reduction does not give all possible static spherically symmetric solutions). A more general procedure is described in [21]. Following this procedure we return to the two-dimensional Lagrangian [15] but add to it a scalar field $\psi$ that should represent ‘matter’ in low dimensions. To simplify comparison to the standard cosmological solutions let us now use the $(t, r)$-coordinates and write the general spherically symmetric metric as:

$$ds^2_4 = e^{2\alpha} dr^2 + e^{2\beta} d\Omega^2(\theta, \phi) - e^{2\gamma} dt^2 + 2e^{2\delta} dr dt,$$

where $\alpha, \beta, \gamma, \delta$ depend on $t, r$ and $d\Omega^2(\theta, \phi)$ is the metric on the 2-dimensional sphere $S^2$. Then the two-dimensional reduction of the four-dimensional EW theory coupled to a scalar field $\psi$ can easily be found (here the prime denotes differentiations in $r$ and the dot - in $t$):

$$\mathcal{L}^{(2)} = e^{2\beta}[e^{-\alpha - \gamma}(\dot{A}_1 - A'_0)^2 - e^{-\alpha + \gamma}(\dot{\psi}^2 + \mu^2 A_1^2) + e^{\alpha - \gamma}(\ddot{\psi}^2 + \mu^2 A_0^2) - e^{\alpha + \gamma}(V + 2\Lambda)] + \mathcal{L}_{gr},$$

where $\psi = \psi(t, r), V = V(\psi), A_i = A_i(t, r), \dot{A}_1 - A'_0 \equiv F_{10}$ and

$$\mathcal{L}_{gr} \equiv e^{-\alpha + 2\beta + \gamma}(2\beta^2 + 4\beta'\gamma') - e^{\alpha + 2\beta - \gamma}(2\beta^2 + 4\beta\gamma) + 2ke^{\alpha + \gamma}$$

is the gravitational Lagrangian, up to the omitted total derivatives that do not affect the equations of motion. Variations of this Lagrangian give all the equations of motion except one constraint,

$$-\ddot{\beta}' - \dot{\beta}\dot{\beta}' + \dot{\alpha}\beta' + \dot{\beta}\gamma' = \frac{1}{2}[\dot{\psi}\psi' + A_0 A_1],$$

which should be derived before we omit the $\delta$-term in the metric (taking the limit $\delta \to -\infty$). All other equations of motion can be obtained from the effective Lagrangian [27][10].

Now, the distinction between static and cosmological solutions is in the dependence of their ‘matter’ fields $A_i$ and $\psi$ on the space-time coordinates. We call static the solution for which $A_i = A_i(t)$ and $\psi = \psi(r)$. If $A = A_i(t)$ and $\psi = \psi(t)$ we call the solution cosmological. There may exist also the wave-like solutions for which $A$ and $\psi$ depend on linear combinations of $t$ and $r$ but here we do not discuss this possibility (see, e.g., [23] and references therein). For both static and cosmological solutions the gravitational variables in general depend on $t$ and $r$. To solve the equations of motion we may reduce them by separating $t$ and $r$. It is clear that to separate the variables $r$ and $t$ in the metric we should require that

$$\alpha = \alpha_0(t) + \alpha_1(r), \quad \beta = \beta_0(t) + \beta_1(r), \quad \gamma = \gamma_0(t) + \gamma_1(r),$$

Inserting this into the equations of motion one can find the restrictions on the gravitational (and, possibly on the matter) variables that must be fulfilled. The details can be found in [21], where

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9 The asymptotic expansions for $r \to 0$ and $r \to \infty$ can be obtained by following the approach proposed below for the cosmological solutions.

10 The equations are equivalent to Eqs. [17] - [19] but are written in the Einstein frame and in the $(t, r)$ coordinates.
one can find the complete list of the static and cosmological spherically symmetric solutions when
the vector field identically vanishes (we call this case the ‘scalar cosmology’). Here we only give a
very brief summary and a simple generalization to nonvanishing vector field.

The naive cosmological reduction (that supposes all the fields to be independent of \( r \)) does not
give the standard FRW scalar cosmology. As was shown in [20] and [21] (see also the earlier paper
[15]), all homogeneous isotropic cosmologies should satisfy the following conditions:

\[
\dot{\alpha} = \dot{\beta}, \quad \gamma' = 0, \quad \dot{\beta}_1'' + ke^{-2\beta_1} = 0, \quad ke^{-2\beta_1} - 3\beta_1'^2 - 2\beta_1'' = C, \tag{31}
\]

where \( C \) is a constant proportional to the 3-curvature (its time dependence is given by the factor
\( e^{-2\alpha} \)) and the third equation is the isotropy condition. Neglecting inessential constant
factors, we also have chosen \( \alpha_1 = \gamma_1 = 0 \). We see that for the naive reduction the isotropy conditions in (31)
can be satisfied only if \( k = 0 \) and that the first condition is not dictated by (29). Thus, the naive
reduction gives, in general, a homogeneous non-isotropic cosmology.

For the FRW scalar cosmology \( \beta' \neq 0 \) and all the conditions (31) must be satisfied. Then
the effective one-dimensional Lagrangian describing both the naive and FRW cosmology is

\[
\mathcal{L}^{(1)} = 6ke^{\alpha+\gamma} - e^{2\beta}[e^{\alpha+\gamma}(V + 2\Lambda) - e^{\alpha-\gamma}(2\beta^2 + 4\beta\dot{\alpha} - \dot{\psi}^2)], \tag{32}
\]

where \( \bar{k} \) is a new real constant related to \( k \) and \( C; \alpha, \beta, \gamma \) and \( \psi \) depend only on \( t \).

Taking now \( \alpha(t) = \beta(t) \) we get the Lagrangian of the FRW scalar cosmology, for which it is
not difficult to derive the equations of motion. As in the static case, we can use \( \alpha \) as the new
independent variable and to derive the following first order equation for \( \chi(\psi) \equiv d\psi/d\alpha \equiv \psi'(\alpha)\):

\[
\frac{d\chi}{d\psi} = \frac{1}{2\chi(6 - \chi^2)} \left[ \chi + \frac{V_\psi}{V + 2\Lambda} \right]. \tag{33}
\]

If we could analytically solve this equation, we would derive the expression for \( \chi(\psi) \) and then, to
find \( \psi(\alpha) \), one should solve the equation \( \psi'(\alpha) = \chi(\psi(\alpha)) \). The last step must be to write and solve
the equation for \( \alpha(t) \):

\[
\dot{\alpha}^2(t) = \left[ V(\psi(\alpha)) + 2\Lambda \right] [6 - \psi'^2(\alpha)]^{-1}. \tag{34}
\]

All this looks nice but does not give the desired analytical expressions although the above formulae
may be useful for a qualitative analysis of some properties of possible solutions. To get approximate
analytical expressions the approach used for studying the static solutions is much more useful. Of
course, it can be applied mutatis mutandis to the scalar FRW cosmology but we omit this exercise
and will write only the solutions for the approximate Lagrangian of the EW model.

We first write the general cosmological EW Lagrangian supplemented by the minimally coupled
scalar field (that may represent either matter or inflaton). At first sight, the dimensional reduction
of the spherically symmetric Lagrangian [27]-[28] with the vector field must not differ from the
usual one used for the scalar cosmology and can be written as:

\[
\mathcal{L}^{(1)} = 6ke^{\alpha+\gamma} + e^{2\beta}[e^{-\alpha-\gamma}\hat{A}_1^2 - e^{-\alpha+\gamma}\mu^2A_1^2 - e^{\alpha+\gamma}(V + 2\Lambda) - e^{\alpha-\gamma}(2\beta^2 + 4\beta\dot{\alpha} - \dot{\psi}^2)], \tag{35}
\]

where all the fields depend on \( t \). Then, taking \( \alpha = \beta \), we apparently obtain a FRW type cosmology
with the vector field. However, unlike the scalar field, the two-dimensional vector field equations,

\[
\partial_0[e^{\alpha_0-\gamma_0+2\beta_1}\hat{A}_1] = -\mu^2e^{\alpha_0+\gamma_0+2\beta_1}A_1, \tag{36}
\]

\[
\partial_1[e^{\alpha_0-\gamma_0+2\beta_1}\hat{A}_1] = -\mu^2e^{3\alpha_0-\gamma_0+2\beta_1}A_0, \tag{37}
\]

do give additional constraints on \( \beta_1(r) \). The first equation does not depend on \( \beta_1 \), but
the second one requires either \( \beta_1'(r) = 0 \) or \( \beta_1'(r) = \text{const.} \) The second condition gives
\( A_0 \sim \hat{A}_1 \) and so (29) is incompatible with the isotropy condition \( \dot{\alpha} = \dot{\beta} \). Therefore, there remains only the first
case, \( \beta'(r) \equiv 0 \), from which it follows that \( k = \tilde{k} = 0 \). Although the constraint (29) is identically satisfied (as we suppose that \( \gamma' = 0 \)) it does not give the necessary isotropy condition \( \dot{\alpha} = \dot{\beta} \) that automatically emerges in the scalar cosmology case. As we’ll see in a moment, this condition cannot be exactly satisfied in the vector cosmology and can only be approximated.\(^{11}\)

Summarizing this discussion, we consider the vector plus scalar cosmology described by the Lagrangian (35) with \( k = \tilde{k} = 0 \) and \( A_1 \) being the \( A_z \) component of the four-dimensional vector field (it follows that the cosmology must be in general non-isotropic). To write the corresponding equations of motion in a most clear and compact form we introduce the temporal notation

\[
\rho \equiv \frac{1}{3}(\alpha + 2\beta), \quad \sigma \equiv \frac{1}{3}(\beta - \alpha), \quad A_{\pm} = e^{-2\rho + 4\sigma}(A^2 \pm \mu^2 e^{2\gamma} A^2), \quad \bar{V} \equiv V(\psi) + 2\Lambda.
\]

where \( A_1 \equiv A_z \equiv A \). Then the exact Lagrangian for EW plus scalar cosmology is:

\[
\mathcal{L}^{(1)} = e^{2\rho - \gamma}(\dot{\psi}^2 - 6\rho^2 + 6\dot{\sigma}^2) + e^{3\rho - \gamma} A_- - e^{3\rho + \gamma} \bar{V}(\psi).
\]

We see that \( A, \psi, \rho, \sigma \) are the dynamical variables and \( e^\gamma \) is the Lagrangian multiplier, variations of which give us the remaining energy constraint:

\[
\psi^2 - 6\rho^2 + 6\dot{\sigma}^2 + A_- + e^{2\gamma} \bar{V} = 0
\]

(40)

(the momentum constraint (29) is satisfied by construction). The other equations are:

\[
\ddot{A} + (\dot{\rho} + 4\dot{\sigma} - \dot{\gamma})\dot{A} + e^{2\gamma}\mu^2 A = 0,
\]

(41)

\[
4\dot{\rho} + 6\dot{\rho}^2 - 4\dot{\rho}\dot{\gamma} - 6\dot{\sigma}^2 + \frac{1}{3}A_- + \psi^2 - e^{2\gamma} \bar{V} = 0,
\]

(42)

\[
\ddot{\sigma} + 3\dot{\sigma}\dot{\rho} - \dot{\sigma}\dot{\gamma} - \frac{1}{3}A_- = 0.
\]

(43)

\[
\ddot{\psi} + (3\dot{\rho} - \dot{\gamma})\dot{\psi} + \frac{1}{2}e^{2\gamma} \ddot{\psi} = 0,
\]

(44)

These equations are much more complex than the equations of the scalar cosmology. They are not integrable in any sense and rather difficult for a qualitative analysis. Nevertheless, they are not much more difficult than the equations for the static solutions considered in Section 2.2 and we may apply the same approach to solving them in asymptotic regions. If the scalar field identically vanishes these equations are essentially identical to the equations for the static states.

They also would be greatly simplified if it were possible to neglect the \( \sigma \)-field. Unfortunately, it is evident that this is in general impossible because then \( A_- = 0 \) and the last condition is incompatible with the other equations. This means that the exact solutions of the EW model (even with many scalar fields minimally coupled to gravity) should be non-isotropic\(^{12}\). In the next subsection we consider the simplified model obtained by neglecting the anisotropy.

2.3.2 Simplified model

To get the simplified model we neglect Eq. (41) and take \( \sigma \equiv 0, \psi \equiv 0, V(\psi) \equiv 0 \). Then \( \rho = \alpha = \beta \) and the approximate effective Lagrangian (39) becomes\(^{13}\)

\[
\mathcal{L}_a = -6\dot{\sigma}^2 e^{3\alpha - \gamma} - 2\Lambda e^{3\alpha + \gamma} + \dot{A}^2 e^{\alpha - \gamma} - \mu^2 A^2 e^{\alpha + \gamma},
\]

(45)

\(^{11}\)This fact was not emphasized in the first version of this report. Moreover, the solutions derived there are only approximate solutions of the four-dimensional theory as clearly follows from the general equations written below. The author is thankful to V. Rubakov for calling his attention to this apparent inconsistency in the presentation of the results in the first version.

\(^{12}\) If one would introduce other scalar fields non-minimally coupled to gravity, this statement may become not valid. At this stage of investigation, we are not ready to add other vector fields or fields with the spin 1/2.

\(^{13}\) Above we completely neglected the dimensions of all the variables and omitted the gravitational constant. Here we only need to restore one of the dimensions supposing that \([r^{-2}] = [k] = [A] = [\mu^2] = [L^{-2}]\).
The corresponding equations of motion are the three equations \((40)-(42)\) with \(\sigma = \psi = V = 0\) and \(\rho = \alpha\). The first equation, \((40)\), is equivalent to vanishing of the Hamiltonian. Denoting \(f \equiv e^\alpha\) and taking the gauge fixing condition \(\gamma = 0\) (the ‘standard’ gauge) we have

\[ \mathcal{H}_0^\alpha \equiv f[-6f^2 + 2\Lambda f^2 + \dot{A}^2 + \mu^2 A^2] = 0. \]

Another useful gauge (the LC gauge) is \(\alpha = \gamma\). In this gauge, the effective Hamiltonian is:

\[ \mathcal{H}_1^\alpha \equiv -6f^2 + 2\Lambda f^4 + \dot{A}^2 + f^2\mu^2 A^2 = 0. \]

It is not difficult to write the equations independent of the gauge choice and we leave this as a simple exercise to the reader.

Let us write the equations of motion in the LC gauge \(\alpha = \gamma\). In analogy to the static case we write them in the first order form (the first equation is the definition of \(F\)),

\[ F \equiv \dot{\alpha} = \frac{\dot{f}}{f}, \quad \dot{F} + F^2 = \frac{2}{3}\Lambda f^2 + \frac{1}{6}\mu^2 A^2, \quad \dot{A} = B, \quad \dot{B} = -\mu^2 f^2 A, \]

and the Hamiltonian constraint is a simple polynomial function of \(f, F, A, B\):

\[ \mathcal{H}_1^\alpha = -6f^2F^2 + B^2 + 2\Lambda f^4 + \mu^2 f^2 A^2 = 0. \]

Similarly to our previous consideration of the static equations, we prefer to change the independent variable to \(\alpha \equiv \ln f\). It is convenient to introduce two new functions, \(\psi(\alpha)\) and \(G(\alpha)\),

\[ \psi'(\alpha) \equiv F'(\alpha)/F(\alpha), \quad G(\alpha) \equiv F^2(\alpha) - \frac{1}{3}\Lambda e^{2\alpha}, \]

and use the following equations (the prime denotes differentiation w.r.t. \(\alpha\)):

\[ G' + 2G = \frac{1}{3}\mu^2 A^2, \quad A'' + \psi A' + \mu^2 e^{2\alpha} F^{-2} A = 0. \]

Of course, instead of the first equation we can use the equivalent equation for \(F^2 = \mathcal{F}\) that directly follows from \((48)\):

\[ \mathcal{F}' + 2\mathcal{F} = \frac{4}{3}\Lambda e^{2\alpha} + \frac{1}{3}\mu^2 A^2. \]

Together with the constraint \((49)\), rewritten as

\[ \mathcal{H}_1^\alpha = -e^{2\alpha}[\mathcal{F}(6 - e^{-2\alpha} A^2) - 2\Lambda e^{2\alpha} - \mu^2 A^2] = 0, \]

equations \((51)\) form the complete system describing cosmology in the LC gauge. Note that the constraint \((49)\) is the integral of motion and thus it is sufficient to require that it vanishes just at one point, say, at \(t = 0\) or \(\alpha = -\infty\). To derive possible asymptotic behaviour of the solutions for \(|\alpha| \to \infty\) it is natural to expand \(A\) in powers of \(e^\alpha\) and to self consistently use the general solution of the first equation,

\[ G(\alpha) = \frac{1}{3}\mu^2 e^{-2\alpha} \int d\alpha A^2(\alpha)e^{2\alpha}, \]

with the relations for the expansion coefficients obtained from the second equation.

In this way we can find, step by step, the asymptotic expansion. In the asymptotic region \(\alpha \to -\infty\) we can then find the following possible asymptotic behaviour:

\[ A = \sum_{n=0}^{\infty} A_n e^{\alpha n}, \quad \mathcal{F} = e^{-2\alpha} \left[ C_\infty + \sum_{n=2}^{\infty} \mathcal{F}_n e^{\alpha n} \right], \quad \psi' = -1 + \sum_{n=2}^{\infty} n \psi_n e^{\alpha n}, \]

\[ ^{14} \text{Here we treat cosmological solutions independently of the static states and use somewhat different notation. For example, the normalization of } A \text{ and of } \mu^2 \text{ are slightly different.} \]
where $C_\infty$, $A_0$, $A_1$ are arbitrary constants. $A_n$, $F_n$ for $n \geq 2$ are derived recursively from (51), (54), and $\psi_n$ from definition (50). The first coefficients are:

$$A_1 = \pm \sqrt{6}, \quad A_2 = 0, \quad A_3 = -\frac{1}{6} \frac{\mu^2 A_0^2}{A_1 C_\infty}, \quad A_4 = \frac{\mu^2 A_0}{4C_\infty};$$

$$F_2 = \frac{1}{6} \mu^2 A_0^2, \quad F_3 = \frac{2}{9} \mu^2 A_0 A_1; \quad \psi_2 = \frac{1}{2C_\infty} F_2, \quad \psi_3 = \frac{1}{2C_\infty} F_3.$$

Thus we find the differential equation for the metric function $f(t)$ ('scale factor'):

$$\frac{d}{dt}(e^f) \equiv \dot{f} = \sqrt{C_\infty \left[ 1 + 2\psi_2 f^2 + 2\psi_3 f^3 + \ldots \right]^\frac{1}{2}},$$

and if we solve it we can find the vector field $A(t)$ by using (55), (56). Neglecting the third term in the r.h.s. it is easy to solve this equation finding the dependence of $f$ on $t$:

$$f(t) = \frac{\sqrt{6} C_\infty}{\mu A_0} \sinh \left[ \left( \frac{1}{6} \mu^2 A_0^2 \right)^{\frac{1}{2}} (t - t_0) \right].$$

At first sight, the exponential growth of $f(t)$ suggests a possibility of an inflation character of this solution. However, this is only the first approximation and we should take into account higher order terms to get a more solid conclusion. Moreover, we see that the qualitative character of the solutions essentially depends on the physical parameters $A_0$, $\mu^2$ on which at the moment we have no reliable information.

The discussed solution is not unique. Using the above equations we can derive another one, for which both $A$ and $F$ are finite for $\alpha \to -\infty$. To get it we take

$$G(\alpha) = \frac{1}{3} \mu^2 e^{-2\alpha} \int_{-\infty}^{\alpha} d\alpha A^2(\alpha) e^{2\alpha},$$

and then apply the above procedure. Then, using the expansions

$$A = \sum_{n=0}^{\infty} A_n e^{2n\alpha}, \quad F = \sum_{n=0}^{\infty} F_n e^{2n\alpha}, \quad F = \sum_{n=0}^{\infty} F_n e^{2n\alpha}$$

we can find that

$$F_0 \equiv F_0 = \frac{1}{6} \mu^2 A_0^2, \quad F_1 \equiv 2F_0 F_1 = \left[ \frac{1}{3} \Lambda + \frac{1}{6} \mu^2 A_0 A_1 \right],$$

$$A_1 = -\mu^2 A_0/4F_0^2 = -3/2A_0, \quad A_2 = -A_1(\mu^2 + 6F_1)/16F_0^2,$$

where now $A_0$ is the unique arbitrary constant (in the above solution we have one more constant $C_\infty$). Instead of Eq.(58) we now have the equation:

$$F \equiv \dot{f} = [F_0^2 + 2F_1 F_0^{-1} e^{2\alpha(t)} + \ldots]^{\frac{1}{2}},$$

which can easily be solved in this approximation:

$$f \equiv e^\alpha = 2e^{F_0 t}(1 - 2F_1 F_0^{-1} e^{2F_0 t})^{-1}.$$
The scale factor vanishes if \( t \to -\infty \). The parameter \( F_1 F_0^{-1} \) strongly depends on \( A_0, \Lambda, \) and \( \mu^2 \). It may be positive or negative, small or large:

\[
F_1 F_0^{-1} = \left( \Lambda - \frac{3}{4} \mu^2 \right) \left( \mu^2 A_0^2 \right)^{-1}.
\] (65)

We see that for the negative values of \( F_1 F_0^{-1} \) the scale factor \( f \) can grow with \( t \) up to the maximum value \( |F_0 F_1^{-1}| \) and then decrease to zero. For positive \( F_1 F_0^{-1} \) it blows up when the expression in the brackets vanishes, but then the approximation is certainly inapplicable. Thus, we must be very cautious in making definite conclusions basing on this simple result. The approximation \( (63) \) can only be reasonable when \( |f_1^{(2)}| f^2 \leq F_0^2 \). As \( F_0 \) and \( F_1 \) strongly depend on \( \Lambda \), on the absolutely unknown mass \( \mu \) and on the arbitrary constant \( A_0 \), it is not possible (at the moment) to make conclusive statements on the general properties of this solution though it depends on less parameters than the first one. Note only that both solutions are compatible with existence of a period of fast growing scale factor.

### 2.3.3 A generalization

The expansions for both cosmological solutions can generally be written as follows. Let us use the more convenient notation:

\[
\mathcal{F}(\alpha) \equiv F^2(\alpha) \equiv C_\infty e^{-2\alpha} + \sum_{n=0}^\infty \mathcal{F}_n e^{-n\alpha}, \quad A_n \equiv \sum_{l=0}^n A_l A_{n-l}, \quad \bar{\Lambda} \equiv \Lambda/3.
\] (66)

Then the general expression for \( \mathcal{F}_n \) can be written as

\[
\mathcal{F}_n = \mu^2 A_n / [3(n+2)] - k \delta_{n,0} + \bar{\Lambda} \delta_{n,2},
\] (67)

and Eq.(51) for \( A(\alpha) \equiv \sum_n A_n e^{n\alpha} \) rewritten as

\[
2 \mathcal{F} A'' + \mathcal{F}' A' + 2\mu^2 e^{2\alpha} A = 0,
\] (68)

defines the recurrence relation for \( n \geq 0 \) (\( A_n \equiv 0 \) for \( n = -1, -2 \)):

\[
2 C_\infty (n+1)(n+2) A_{n+2} + \sum_{m=1}^n m(n+m) \mathcal{F}_{n-m} A_m + 2\mu^2 A_{n-2} = 0.
\] (69)

Taking \( n = 0 \) we find that \( C_\infty A_2 = 0 \). The first solution discussed above is obtained when \( A_2 = 0 \), the second – when \( C_\infty = 0 \) (in the second case \( \mathcal{F}_{2n+1} = A_{2n+1} = 0 \) for all \( n \geq 0 \)). In both cases \( A_0 \) is arbitrary and other \( A_n \) are derived recursively from Eqs.(67), (69), taking into account that for the first solution \( A_1 \) is defined by the constraint \( (53) \), i.e. \( A_1 = \pm \sqrt{6} \).

Note that \( C_\infty > 0 \) but the signs of \( A_0 \) and of \( A_1 \) are not fixed. For this reason, the structure of the series expansion of \( A(\alpha) \) is rather complex. It follows that the behavior of \( \alpha(t) \) with growing time may change its character (for example, depending on the parameters \( C_\infty, A_0, \mu^2, \Lambda \), it may change the exponential growth to oscillating or even chaotic behavior). At the moment, the main unsolved problem is to derive the asymptotic behavior of \( A_n \) and \( \mathcal{F}_n \) for \( n \to \infty \). An educated guess is that the main terms are given by some powers of \( n \) and thus the expansions in powers of \( f = e^\alpha \) have a finite radius of convergence in the complex \( f \)-plane. If this is true, the radius of convergence and the precise position and nature of the corresponding singularity in the complex \( f \)-plane could give us a very important information on the solutions.

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18 This was proved for the analogous expansions near the horizons, see [22].
The dimensionless functions $F(\alpha)/\mu^2$ and $A(\alpha)$ of general solution depend on the dimensionless constants $\lambda \equiv \Lambda/\mu^2$, $\rho \equiv C_\infty/\mu^2$ and $A_0$. The dependence on $A_0$ is especially simple for the second solution. It is not difficult to prove (e.g., by induction) that

$$A_{2k}/A_0 = a_{2k}A_0^{-2k}, \quad F/F_0 = f_{2k}A_0^{-2k}, \quad F_0 \equiv \mu^2 A_0^3/6,$$

(70)

where $a_{2k}$, $f_{2k}$ depend only on $\lambda$. We thus have the expansions

$$A(\alpha)/A_0 = 1 + \sum_{k=1}^{\infty} a_{2k} e^{2k\tilde{\alpha}}, \quad F/F_0 = 1 + \sum_{k=1}^{\infty} f_{2k} e^{2k\tilde{\alpha}}, \quad \tilde{\alpha} \equiv \alpha - \ln A_0,$$

(71)

in which $a_{2k}$, $f_{2k}$ can be derived by the recurrence relations

$$f_{2n} = \frac{1}{3} \lambda \delta_1 n + \sum_{k=0}^{n} (n+1)^{-1} a_{2k} a_{2n-2k}, \quad \sum_{k=1}^{n} k(n+k) f_{2n-2k} a_{2k} + 3a_{2n-2} = 0.$$

(72)

It follows that the equation for $\alpha(t)$ can be rewritten as

$$\frac{d\tilde{\alpha}}{d\tau} = \left[ 1 + \sum_{k=1}^{\infty} f_{2k} e^{2k\tilde{\alpha}(\tau)} \right]^{\frac{1}{2}}, \quad \tau \equiv \mu A_0/\sqrt{6} t.$$

(73)

An interesting property of these expansions is the following: if the dimensionless parameter $A_0$ is large enough, $e^{\alpha}$ may remain small even for large positive $\alpha$ (thus the scale parameter $e^\alpha$ may be very large). This means that it may be possible to approximate the expansions by a small number of terms (in some domain of $t$ in which $e^{\alpha}$ is very large). This property is of interest for discussing inflation models. The first several terms in the expansions are easy to derive ($\lambda \equiv \Lambda/\mu^2$):

$$a_2 = -\frac{3}{2}, \quad a_4 = \frac{9}{8} \left[ \lambda - \frac{1}{4} \right], \quad a_6 = -\frac{5}{4} \lambda \left[ \lambda - \frac{21}{20} \right];$$

$$f_2 = 2 \left[ \lambda - \frac{3}{4} \right], \quad f_4 = \frac{3}{4} \left[ \lambda + \frac{3}{4} \right], \quad f_6 = -\frac{5}{8} \lambda \left[ \lambda - \frac{9}{20} \right] \left[ \lambda + \frac{3}{4} \right].$$

(74)

(75)

Our simplified Lagrangian [15] should be considered as a (mainly) mathematical model for a preliminary study of the EW theory. To really discuss its cosmological applications we must first find an approximation valid for high values of $f$ and then return to the complete set of the equations of motion, to which the simplified model is only a rather crude approximation. We also should not forget that it is absolutely necessary to include into consideration ‘ordinary’ matter before one can really discuss physical picture of the cosmological evolution.

3 Discussion

In this paper we briefly summarized the main ideas of the Einstein - Weyl model and presented its new interpretation, as well as some results obtained investigating its simplest solutions. We only considered the static spherically symmetric solutions and, in cosmology, only an artificially simplified homogeneous model. As we noted in [21], even small deviations from the spherical symmetry may result in a qualitatively different theory. In particular, if we consider axially symmetric configurations infinitesimally deviating from the spherically symmetric ones, we will find additional scalar fields in the vector dilaton gravity, which may be very important in cosmological considerations and in analysing black holes. We did not touch these problems here. Moreover, even in the spherically symmetric case our study is incomplete. In the static case, we have only proven that there may exist two horizons and derived the solutions close to the horizons. In cosmology, we have studied only the asymptotic behavior of the solutions in the simplified model.
As we mentioned above, we expect that the complete solutions should reveal some sort of chaotic behavior. To study these phenomena we must first carefully discuss the physical parameters of the theory. In the original formulation these are: the gravitational constant, the cosmological constant, and the vecton mass. In addition, the asymptotic boundary conditions introduce other parameters, the dependence on which is highly nontrivial. This does not allow us to make sound conclusions (or, even guesses) about the global behavior of the solutions derived in our essentially local approach. For example, if we try to glue together the left and the right asymptotic approximations, we will find that the gluing procedure is strongly dependent on the parameters that characterize the influence of the nonlinear terms in the equations, up to producing chaotic effects. This requires a very careful qualitative and numerical study of the equations. Of course, the most important task is taking into account the ‘ordinary’ matter.

Finally, we must admit that the vecton field is a rather unusual feature of the Einstein-Weyl model. I have found just a few papers in which a massive vector field is introduced in the cosmological context (see, e.g., [26]-[28] and references therein). Thus the unified model of dark energy and vecton dark matter considered here looks as fresh and new as it was in 1923. In addition, I wish to stress that the original Einstein Lagrangian (3) or (8) is more interesting and exciting than the simplified theory (12) (in particular, one may expect that in the original formulation there exists a relation among dimensional parameters that are arbitrary in the theory (12)). Unfortunately, the original theory is much more difficult to deal with and thus the prime goal must be the study of the simplified theory. In this paper I only give a sketch of how to begin such a study.

Additional comments

1. To better understand the brief exposure of the static solutions in Section 2.2 the reader is advised to consult our paper [22]. There we proved that the expansions of the solutions represent analytic functions that analytically depend on all the parameters. This means that the two horizons defined by Eq. (25) and corresponding to the same parameters $\hat{A}_0$, $\hat{B}_0$, $\hat{F}_0$, belong to the same static solution in spite of the fact that it is represented by two different expansions near the two horizons. Also, analyticity tells us that any solution can be analytically continued from one horizon to the other as well as to any regular point in the interval $0 < r < \infty$. The same remarks are also relevant to the cosmological solutions considered in Section 2.3. Recalling that by crossing the horizon we pass from the static to cosmological solutions one may even use the static coordinates for and alternative description of cosmologies (see, e.g., [29], [30]).

2. Recently, several authors considered massive vector models of inflation (see, e.g., [31]-[34] and references therein). As distinct from the EW model and from the first paper [27] on the vector inflation, they introduce some non-minimal couplings of the vecton to gravity and/or many vector fields. In the Einstein approach additional vector fields (as well as the inflaton - type scalar fields) can possibly be produced in some higher-dimensional version. It is not clear whether non-minimal couplings can be produced but, possibly, similar effects could be imitated in the really non-linear Eddington-Einstein type theory. There exist attempts to use the non-linear Eddington-Einstein type actions for description of dark energy and dark matter (see, e.g., [35] and references therein). Note also that, possibly, the anisotropic effects of the vecton field can be smeared out by inflation (see, e.g., [36] where the evolution of the universe with the primordial magnetic field was discussed).

3. In this paper, we did not discuss the possible interpretation of the vecton as a dark matter candidate. One reason for this is our inability to estimate its mass, even by order of magnitude. The other reason is that the vecton is interacting only with gravity and thus can be produced (in abundance) only in very high gravitational fields. This requires essentially quantum considerations that are beyond the scope of the present paper.

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19 In the second version of this paper I omitted two references in which the massive vector has only $A_0(t)$ component and thus $F_{ij} \equiv 0$. 

15
The connection \([5]\) is a special case of the general expression for the connection in affine spaces. The most general symmetric affine connection has the form (see, e.g., \([37]\)):

\[
\Gamma^m_{kl} = \frac{1}{2} \left[ s^{mn} (s_{nk,l} + s_{ln,k} - s_{kl,n}) + s^{mn} (s_{nk,l} + s_{ln,k} - s_{kl,n}) \right],
\]

where \(s_{kl}\) is an arbitrary symmetric tensor, \(s^{mn}\) is the inverse matrix to \(s_{kl}\), and \(s_{kl}^{mn}\) is an arbitrary tensor that is symmetric in \(k\) and \(l\). Both the Weyl and Einstein connections belong to the subclass for which \(s_{kl}^{mn}\) can be presented in the form:

\[
s_{kl}^{mn} = \alpha s_{kl} i^m + \beta (s_{nk} i_l + s_{ln} i_k).
\]

We may call it the Weyl-Einstein connection (defining the Weyl-Einstein spaces).

Inserting \((77)\) into \((76)\), we find:

\[
\Gamma^m_{kl} = \frac{1}{2} \left[ s^{mn} (s_{nk,l} + s_{ln,k} - s_{kl,n}) + \alpha (\delta^m_k i_l + \delta^m_l i_k) - (\alpha - 2\beta) s_{kl} i^m \right].
\]

Now it is easy to find that the Einstein connection \((5)\) corresponds to \(\alpha = -\beta = \frac{1}{3}\). The Weyl connection introduced in \([5]\) corresponds to \(\alpha = 1\), \(\beta = 0\). I could not find a discussion of the geometry of spaces with the Einstein connection in accessible literature. The geometry of the Weyl spaces is considered in \([8]\) \((D = 4\), Lorentzian signature\), and in \([37]\) \((D = 2, 3, 4\), Euclidean signature\).

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