THE BICROSSED PRODUCT CONSTRUCTION FOR LOCALLY
COMPACT QUANTUM GROUPS

LEONID VAINERMAN

ABSTRACT. The cocycle bicrossed product construction allows certain free-
dom in producing examples of locally compact quantum groups. We give an
overview of some recent examples of this kind having remarkable properties.

1. INTRODUCTION

The initial motivation to introduce objects which are more general than locally
compact (l.c.) groups was to extend classical harmonic analysis on these groups,
including the Fourier transform and the Pontrjagin duality. Given an abelian l.c.
group $G$, the set $\hat{G}$ of its unitary continuous characters is again an abelian l.c. group
- the dual group of $G$. The Fourier transform maps functions on $G$ to functions on
$\hat{G}$, and the Pontrjagin duality theorem claims that the dual of $\hat{G}$ is isomorphic to
$G$. If $G$ is not abelian, the set of its characters is too small, and one should use
instead the set $\hat{G}$ of (classes of) its unitary irreducible representations and their
matrix coefficients. For compact groups, this leads to the Peter-Weyl theory; the
corresponding duality is due to T. Tannaka and M.G. Krein. In this case, $\hat{G}$ is not a
group, but carry a structure of a block-algebra or a Krein algebra \[18\]; however, this
structure allows to reconstruct the initial group. Such a non-symmetric duality was
later established by W.F. Stinespring for unimodular groups, and by P. Eymard
and T. Tatsuuma for general l.c. groups.

In order to restore the symmetry of the duality, G.I. Kac \[20\] introduced in
1961 a category of ring groups. A ring group is a Hopf-von Neumann algebra
$(M, \Delta, S)$, with the comultiplication $\Delta : M \to M \otimes M$ and the involutive antipode
$S : M \to M$, $S^2 = \text{id}$ equipped with a faithful normal trace $\varphi$ compatible with
$\Delta$ and $S$ and playing the role of a Haar measure. If $M$ is commutative (resp., $\Delta$
is co-commutative, i.e., $\sigma \circ \Delta = \Delta$, where $\sigma : a \otimes b \mapsto b \otimes a$ is the usual flip in
$M \otimes M$), this ring group can be identified with the algebra $L^\infty(G)$ (resp., group
von Neumann algebra $L(G)$), where $G$ is a unimodular group. Thus, unimodular
groups and their duals are embedded into this category, and the duality constructed
by Kac extended those of Pontrjagin, Tannaka-Krein and Stinespring.

The theory was completed in early 70-s by G.I. Kac and the author, and inde-
dependently by M. Enock and J.-M. Schwartz, in order to cover all locally compact
groups. They allowed $\varphi$ and $\varphi \circ S$ to be different weights on $M$ playing respectively
the role of left and right Haar measures (for ring groups $\varphi = \varphi \circ S$ was a trace), gave
appropriate axioms and extended the construction of the dual. These more general
objects are called Kac algebras \[15\]. L.c. groups and their duals can be viewed

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respectively as commutative and co-commutative Kac algebras, the corresponding duality covered all versions of duality for such groups.

Concrete examples of ring groups, which were neither ordinary groups nor their duals, were given in [22] - [24]. According to V.G. Drinfeld [14], these were the first concrete examples of what is now called quantum groups.

Quantum groups discovered by V.G. Drinfeld [14] and others gave new important examples of Hopf algebras, obtained by deformation of universal enveloping algebras and of function algebras on Lie groups. Their operator algebra versions did not fit into the Kac algebra theory, because their antipodes were neither involutive, nor bounded maps. This motivated strong efforts to construct a more general theory, which would be as elegant as that of Kac algebras but would also cover these new examples. Important steps in this direction were made by: S.L. Woronowicz [53] - [60] with his theory of compact quantum groups and a series of important concrete examples of compact and non-compact quantum groups; S. Baaj and G. Skandalis [2] - [4] with their fundamental concept of a multiplicative unitary; T. Masuda, Y. Nakagami and S.L. Woronowicz [36], [37] who gave a set of axioms of so called Woronowicz algebras; A. Van Daele [50], [51] who introduced an important notion of a multiplier Hopf algebra. Finally, the theory of l.c. quantum groups was proposed by J. Kustermans and S. Vaes [27] - [29].

A (von Neumann algebraic) l.c. quantum group is a collection $(M, \Delta, \varphi, \psi)$, where $M$ is a von Neumann algebra equipped with a co-associative comultiplication $\Delta : M \to M \otimes M$ and two normal semi-finite faithful (n.s.f.) weights $\varphi$ and $\psi$ - right and left Haar measures. The antipode is not explicitly present in this definition, but can be constructed from the above data. Kac algebras, compact and discrete quantum groups are special cases of a l.c. quantum group, and all important concrete examples of operator algebraic quantum groups fit into this framework. There is an equivalent $C^*$-algebraic version of a l.c. quantum group.

A number of "isolated" examples of l.c. quantum groups can be formulated in terms of generators of certain Hopf $*$-algebras and commutation relations between them. It is much harder to represent these generators as (typically, unbounded) operators acting on a Hilbert space, to give a meaning to the relations of commutation between them, to associate an operator algebra with the above system of operators and commutation relations and to construct comultiplication, antipode and invariant weights as applications related to this algebra. There is no general approach to these highly nontrivial problems, and one must design specific methods in each specific case (see, for example, [2], [25], [26], [39], [52], [53] - [60]).

A systematic approach to the construction of non-trivial Kac algebras has been proposed in [21]. Given two finite groups, $G_1$ and $G_2$, viewed as a co-commutative and a commutative Kac algebra, $(\mathcal{L}(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$ respectively, let us try to find a Kac algebra $(M, \Delta)$ which makes the sequence

\[(L^\infty(G_2), \Delta_2) \to (M, \Delta) \to (\mathcal{L}(G_1), \Delta_1)\]

exact. Kac explained that: 1) $(M, \Delta)$ exists if and only if $G_1$ and $G_2$ are subgroups of a group $G$ such that $G_1 \cap G_2 = \{e\}$ and $G = G_1G_2$. Equivalently, $G_1$ and $G_2$ must act on each other (as on sets), and these actions must be compatible in certain sense. Remark, that later on, such pairs of groups were also introduced by G.W. Mackey [31] and by M. Takeuchi [45]. Following [45], we will say that $G_1$ and $G_2$ form a matched pair. 2) To get all possible $(M, \Delta)$ (they are called extensions of $(L^\infty(G_2), \Delta_2)$ by $(\mathcal{L}(G_1), \Delta_1)$), one must find all possible 2-cocycles for the above
mentioned actions, compatible in certain sense. Then [21] gives explicit construction of \((M, \Delta)\) (the cocycle bicrossed product construction). The famous Kac-Paljutkin examples [22] - [24] are of this type.

Later on, both algebraic and analytic aspects of this construction were studied by S. Majid [32] - [35] who gave also a number of examples of operator algebraic quantum groups. The bicrossed product construction for multiplicative unitaries was done in [3]. A general theory of extensions of the form \((\mathbb{G}_1, \mathbb{G}_2)\), with l.c. quantum groups, which we don’t consider here).

In the recent years it became clear that this construction gives certain freedom in producing examples of l.c. compact quantum groups, see [5] - [7], [11] - [13], [17], [47] - [49]. This allows to construct such examples with prescribed special features. Some of these examples have quite remarkable and unexpected properties, and we discuss them briefly here. We recall in Section 2 the basic definitions and results on the von Neumann algebraic version of the l.c. quantum group theory and the cocycle bicrossed product construction. In Section 3 we discuss the problem of regularity for a multiplicative unitary related to a l.c. quantum group and give, following [5], a surprising example of a non-semi-regular l.c. quantum group. Section 4 is devoted to the examples of l.c. quantum groups whose von Neumann algebras are factors. Finally, we briefly discuss in Section 5 amenability and Kac exact sequence for l.c. quantum groups coming from the cocycle bicrossed product construction.

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### 2. Preliminaries

#### 2.1. Notations

Let us denote by \(B(H)\) the algebra of all bounded linear operators on a Hilbert space \(H\), by \(\otimes\) the tensor product of Hilbert spaces or von Neumann algebras and by \(\Sigma\) (resp., \(\sigma\)) the flip map on it. If \(H, K\) and \(L\) are Hilbert spaces and \(X \in B(H \otimes L)\) (resp., \(X \in B(H \otimes K), X \in B(K \otimes L)\)), we denote by \(X_{13}\) (resp., \(X_{12}, X_{23}\)) the operator \((1 \otimes \Sigma^*)(X \otimes 1)(1 \otimes \Sigma)\) (resp., \(X \otimes 1, 1 \otimes X\)) defined on \(H \otimes K \otimes L\). When \(H = H_1 \otimes H_2\) itself is a tensor product of two Hilbert spaces, we switch from the above leg-numbering notation with respect to \(H \otimes K \otimes L\) to the one with respect to the finer tensor product \(H_1 \otimes H_2 \otimes K \otimes L\), for example, from \(X_{13}\) to \(X_{124}\). There is no confusion here, because the number of legs changes.

Given a comultiplication \(\Delta\), denote by \(\Delta^\sigma\) the opposite comultiplication \(\sigma \Delta\). Our reference to the modular theory of weights on von Neumann algebras is [43]. Given a normal semi-finite faithful (n.s.f.) weight \(\theta\) on a von Neumann algebra \(N\), we denote:

\[
M_\theta^+ = \{x \in N^+ \mid \theta(x) < +\infty\}, \quad N_\theta = \{x \in N \mid x^* x \in M_\theta^+\} \quad \text{and} \quad M_\theta = \text{span} M_\theta^+.
\]

All l.c. groups considered in this paper supposed to be second countable.

#### 2.2. L.c. quantum groups [27] - [29]

A pair \((M, \Delta)\) is called a (von Neumann algebraic) l.c. quantum group when

- \(M\) is a von Neumann algebra and \(\Delta : M \to M \otimes M\) is a normal and unital \(*\)-homomorphism which is coassociative: \((\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta\).
There exist n.s.f. weights \( \varphi \) and \( \psi \) on \( M \) such that
- \( \varphi \) is left invariant in the sense that \( \varphi((\omega \otimes \text{id})\Delta(x)) = \varphi(x)\omega(1) \) for all \( x \in M^+_\psi \) and \( \omega \in M^+_\varphi \),
- \( \psi \) is right invariant in the sense that \( \psi((\text{id} \otimes \omega)\Delta(x)) = \psi(x)\omega(1) \) for all \( x \in M^+_\psi \) and \( \omega \in M^+_\varphi \).

Left and right invariant weights are unique up to a positive scalar.

Represent \( M \) on the GNS Hilbert space of \( \varphi \) and define a unitary \( W \) on \( H \otimes H \) by
\[
W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)((\Delta(b))(a \otimes 1)) \quad \text{for all } a, b \in N_\varphi.
\]
Here, \( \Lambda \) denotes the canonical GNS-map for \( \varphi \), \( \Lambda \otimes \Lambda \) the similar map for \( \varphi \otimes \varphi \).

One proves that \( W \) satisfies the pentagonal equation: \( W_{12}W_{13}W_{23} = W_{23}W_{12} \), and we say that \( W \) is a multiplicative unitary. The von Neumann algebra \( M \) and the comultiplication on it can be given in terms of \( W \) respectively as
\[
M = \{ (\text{id} \otimes \omega)(W) \mid \omega \in B(H)_+ \}^{-\sigma-\text{strong}^*}
\]
and \( \Delta(x) = W^*(1 \otimes x)W \), for all \( x \in M \). Next, the l.c. quantum group \( (M, \Delta) \) has an antipode \( S \), which is the unique \( \sigma \)-strongly* closed linear map from \( M \) to \( M \) satisfying \( (\text{id} \otimes \omega)(W) \in \mathcal{D}(S) \) for all \( \omega \in B(H)_+ \) and \( S(\text{id} \otimes \omega)(W) = (\text{id} \otimes \omega)(W^*) \) and such that the elements \( (\text{id} \otimes \omega)(W) \) form a \( \sigma \)-strong* core for \( S \). \( S \) has a polar decomposition \( S = R\tau_{-i/2} \), where \( R \) is an anti-automorphism of \( M \) and \( (\tau_t) \) is a strongly continuous one-parameter group of automorphisms of \( M \). We call \( R \) the unitary antipode and \( (\tau_t) \) the scaling group of \( (M, \Delta) \). We have \( \sigma(R \otimes R)\Delta = \Delta R \), so \( \varphi R \) is a right invariant weight on \( (M, \Delta) \) and we take \( \psi := \varphi R \).

Let us denote by \( (\sigma_t) \) the modular automorphism group of \( \varphi \). There exist a number \( \nu > 0 \), called the scaling constant, such that \( \psi \sigma_t = \nu^t \psi \) for all \( t \in \mathbb{R} \). Hence, we get the existence of a unique positive, self-adjoint operator \( \delta_M \) affiliated to \( M \), such that \( \sigma_t(\delta_M) = \nu^t \delta_M \) for all \( t \in \mathbb{R} \) and \( \psi = \varphi \delta_M \). The operator \( \delta_M \) is called the modular element of \( (M, \Delta) \). If \( \delta_M = 1 \) we call \( (M, \Delta) \) unimodular. The scaling constant can be characterized as well by the relative invariance \( \varphi \tau_t = \nu^{-t} \varphi \).

For the dual l.c. quantum group \( (\hat{M}, \hat{\Delta}) \) we have:
\[
\hat{M} = \{ (\omega \otimes \text{id})(W) \mid \omega \in B(H)_+ \}^{-\sigma-\text{strong}^*}
\]
and \( \hat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma \) for all \( x \in \hat{M} \). If we turn the predual \( M_* \) into a Banach algebra with product \( \omega \mu = (\omega \otimes \mu)\Delta \) and define
\[
\lambda : M_* \to \hat{M} : \lambda(\omega) = (\omega \otimes \text{id})(W),
\]
them \( \lambda \) is a homomorphism and \( \lambda(M_*) \) is a \( \sigma \)-strongly* dense subalgebra of \( \hat{M} \). A left invariant n.s.f. weight \( \hat{\varphi} \) on \( \hat{M} \) can be constructed explicitly and the associated multiplicative unitary \( W = \Sigma W^*\Sigma \).

Since \( (\hat{M}, \hat{\Delta}) \) is again a l.c. quantum group, we can introduce the antipode \( \hat{S} \), the unitary antipode \( \hat{R} \) and the scaling group \( (\hat{\tau}_t) \) exactly as we did it for \( (M, \Delta) \). Also, we can again construct the dual of \( (\hat{M}, \hat{\Delta}) \), starting from the left invariant weight \( \hat{\varphi} \). The bidual l.c. quantum group \( (\hat{M}, \hat{\Delta}) \) is isomorphic to \( (M, \Delta) \).

We denote by \( (\hat{\sigma}_t) \) the modular automorphism group of the weight \( \hat{\varphi} \). The modular conjugations of the weights \( \varphi \) and \( \hat{\varphi} \) will be denoted by \( J \) and \( \hat{J} \) respectively. Then it is worthwhile to mention that
\[
R(x) = Jx^*\hat{J}, \quad \text{for all } x \in M, \quad \text{and} \quad \hat{R}(y) = Jy^*J, \quad \text{for all } y \in \hat{M}.
\]
Let us mention important special cases of l.c. quantum groups.

a) **Kac algebras** ([15]). \((M, \Delta)\) is a Kac algebra if and only if \((\tau_t) = \text{id} and \sigma_t R = R \sigma_{-t}\) for all \(t \in \mathbb{R}\). Let \(\sigma'_t\) be the modular automorphism group of \(\psi\). Since \(\psi = \varphi R\), we get \(\sigma'_t R = R \sigma_{-t}\) for all \(t \in \mathbb{R}\). Hence \((M, \Delta)\) is a Kac algebra if and only if \((\tau_t) = \text{id} and \sigma' = \sigma\) or if and only if \(\delta_M\) is affiliated to the center of \(M\).

In particular, \((M, \Delta)\) is a Kac algebra if \(M\) is commutative. Then \((M, \Delta)\) is generated by a usual l.c. group \(G = M = L^\infty(G), (\Delta f)(g, h) = f(gh), (S f)(g) = f(g^{-1})\). \(\varphi(f) = \int f(g) dg\), where \(f \in L^\infty(G), g, h \in G\) and we integrate with respect to the left Haar measure \(dg\) on \(G\). The right invariant weight \(\psi\) is given by \(\psi(f) = \int f(g^{-1}) dg\). The modular element \(\delta_M\) is given by the strictly positive function \(g \mapsto \delta_G(g)\)^{-1}.

The von Neumann algebra \(L^\infty(G)\) acts on \(H = \mathbb{L}^2(G)\) by multiplication and

\[
(W_G \xi)(g, h) = \xi(g, g^{-1}h)
\]

for all \(\xi \in H \otimes H = \mathbb{L}^2(G \times G)\). Then \(\hat{M} = \mathcal{L}(G)\) is the group von Neumann algebra generated by the operators \((\lambda_g)_{g \in G}\) of the left regular representation of \(G\) and \(\hat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g\). Clearly, \(\hat{\Delta}^\text{op} := \sigma \hat{\Delta} = \hat{\Delta}\), so \(\hat{\Delta}\) is cocommutative.

b) A l.c. quantum group is called **compact** if \(\varphi(1) < +\infty\). A l.c. quantum group \((M, \Delta)\) is called **discrete** if \((\hat{M}, \hat{\Delta})\) is compact.

### 2.3. Cocycle crossed and bicrossed products.

An **action** of a l.c. quantum group \((M, \Delta)\) on a von Neumann algebra \(N\) is a normal, injective and unital \(*\)-homomorphism \(\alpha : N \to M \otimes N\) such that \((\text{id} \otimes \alpha)\alpha(x) = (\Delta \otimes \text{id})\alpha(x)\) for all \(x \in N\). This generalizes the definition of an action of a l.c. group \(G\) on a (\(\sigma\)-finite) von Neumann algebra \(N\), as a continuous map \(G \to \text{Aut} N : s \mapsto \alpha_s\), such that \(\alpha_{st} = \alpha_s \alpha_t\) for all \(s, t \in G\). Indeed, putting \(M = L^\infty(G)\), one can identify \(M \otimes N\) with \(L^\infty(G, N)\) and \(M \otimes M \otimes N\) with \(L^\infty(G \times G, N)\) and define the above homomorphism \(\alpha\) by \((\alpha(x))(s) = \alpha_{s^{-1}}(x)\). The fixed point algebra of an action \(\alpha\) is defined by \(N^\alpha = \{x \in N \mid \alpha(x) = 1 \otimes x\}\).

A **cocycle** for an action of a l.c. group \(G\) on a commutative von Neumann algebra \(N\) is a Borel map \(u : G \times G \to N\) such that \(\alpha_r(u(s, t)) u(r, st) = u(r, s) u(rs, t)\) nearly everywhere. Then, putting \(M = L^\infty(G)\), one can define a unitary \(U \in M \otimes M \otimes N\) by \(U(s, t) = u(t^{-1}, s^{-1})\) satisfying

\[
(\text{id} \otimes \text{id} \otimes \alpha)(U)(\Delta \otimes \text{id} \otimes \text{id})(U) = (1 \otimes U)(\text{id} \otimes \Delta \otimes \text{id})(U).
\]

The general definition of a cocycle action of a l.c. quantum group on a von Neumann algebra can be found in [15]. The **cocycle crossed product** \(G_{\alpha, \hat{\Delta}} \ltimes N\) is the von Neumann subalgebra of \(B(L^2(G)) \otimes N\) generated by

\[
\alpha(N) \text{ and } \{(\omega \otimes \text{id} \otimes \text{id})(\hat{W}) \mid \omega \in L^1(G)\},
\]

where \(\hat{W} = (W_G \otimes 1)U^\ast\). There exists a unique action \(\hat{\alpha}\) of \((\mathcal{L}(G), \hat{\Delta})\) on \(G_{\alpha, \hat{\Delta}} \ltimes N\) such that

\[
\hat{\alpha}(\alpha(x)) = 1 \otimes \alpha(x) \text{ for all } x \in N, (\text{id} \otimes \hat{\alpha})(\hat{W}) = W_{G, 12} \hat{W}_{134},
\]

and for any n.s.f. weight \(\gamma\) on \(N\), we can define the **dual** n.s.f. weight \(\hat{\gamma}\) on \(G_{\alpha, \hat{\Delta}} \ltimes N\):

\[
\hat{\gamma} = \gamma^{-1} (\hat{\varphi} \otimes \text{id} \otimes \text{id}) \hat{\alpha}.
\]

**Definition 2.1.** (see [3]) Let \(G, G_1\) and \(G_2\) be l.c. groups, and let a homomorphism \(i : G_1 \to G\) and an anti-homomorphism \(j : G_2 \to G\) have closed images and be homeomorphisms onto these images. Suppose that \(i(G_1) \cap j(G_2) = \{e\}\) and that
the complement of \( i(G_1) j(G_2) \) in \( G \) has measure zero. Then we call \((G_1, G_2 \subset G)\) a matched pair of l.c. groups.

**Remark 2.2.** The above mentioned group \( G \) is called a double crossed product of \( G_1 \) and \( G_2 \). The definition of a matched pair of general l.c. quantum groups was given in [13], and the double crossed product construction in this general case was studied in [7]. It is worthwhile to mention that this construction contains the famous Drinfeld double construction for quantum groups [14].

The map \( \theta : G_1 \times G_2 \to G : (g, s) \mapsto i(g) j(s) \) is automatically a Borel isomorphism, i.e., it induces an isomorphism between \( L^\infty(G_1 \times G_2) \) and \( L^\infty(G) \). Hence, this data allows to construct as follows two actions: \( \alpha \) of \( G_1 \) on \( M_2 = L^\infty(G_2) \) and \( \beta \) of \( G_2^{\text{op}} \) on \( M_1 = L^\infty(G_1) \), verifying certain compatibility relations.

Let \( \Omega \) be the image of \( \theta \) and define the Borel isomorphism
\[
\rho : G_1 \times G_2 \to \Omega^{-1} : (g, s) \mapsto j(s)i(g).
\]
So \( \mathcal{O} = \theta^{-1}(\Omega \cap \Omega^{-1}) \) and \( \mathcal{O}' = \rho^{-1}(\Omega \cap \Omega^{-1}) \) are Borel subsets of \( G_1 \times G_2 \), with complements of measure zero, and \( \rho^{-1} \theta \) is a Borel isomorphism of \( \mathcal{O} \) onto \( \mathcal{O}' \). For all \((g, s) \in \mathcal{O}\) define \( \beta_s(g) \in G_1 \) and \( \alpha_g(s) \in G_2 \) such that
\[
\rho^{-1}(\theta(g, s)) = (\beta_s(g), \alpha_g(s)).
\]
Hence we get \( j(\alpha_g(s)) \ i(\beta_s(g)) = i(g)j(s) \) for all \((g, s) \in \mathcal{O}\).

**Lemma 2.3.** (48, Lemma 4.8) Let \((g, s) \in \mathcal{O} \) and \( h \in G_1 \). Then \((hg, s) \in \mathcal{O} \) if and only if \((h, \alpha_g(s)) \in \mathcal{O} \), and in that case
\[
\alpha_{hg}(s) = \alpha_h(\alpha_g(s)) \quad \text{and} \quad \beta_s(hg) = \beta_{\alpha_g(s)}(h) \beta_s(g).
\]
Let \((g, s) \in \mathcal{O} \) and \( t \in G_2 \). Then \((g, ts) \in \mathcal{O} \) if and only if \((\beta_s(g), t) \in \mathcal{O} \) and in that case
\[
\beta_{ts}(g) = \beta_t(\beta_s(g)) \quad \text{and} \quad \alpha_g(ts) = \alpha_{\beta_s(g)}(t) \alpha_g(s).
\]
Finally, for all \( g \in G_1 \) and \( s \in G_2 \) we have \((g, e) \in \mathcal{O}, (e, s) \in \mathcal{O}, \) and
\[
\alpha_g(e) = e, \quad \alpha_s(e) = s, \quad \beta_s(e) = e \quad \text{and} \quad \beta_g(e) = g.
\]
This can be viewed as a definition of a matched pair of l.c. groups in terms of mutual actions.

The cocycles for the above actions can be introduced as measurable maps \( U : G_1 \times G_1 \times G_2 \to \mathbb{T} \) and \( V : G_1 \times G_2 \times G_2 \to \mathbb{T} \), where \( \mathbb{T} \) is the unit circle in \( \mathbb{C} \), satisfying
\[
U(g, h, \alpha_s(k)) U(gh, k, s) = U(h, k, s) U(g, hk, s),
\]

(2)
\[
\mathcal{V}(\beta_s(g), t, r) \mathcal{V}(g, s, tr) = \mathcal{V}(g, s, t) \mathcal{V}(g, ts, r),
\]
\[
\mathcal{V}(gh, s, t) \mathcal{U}(g, h, ts) = \mathcal{U}(g, h, s) \mathcal{U}(\beta_{\alpha_h(s)}(g), \beta_s(h), t) \mathcal{V}(g, \alpha_h(s), \alpha_{\beta_s(h)}(t)) \mathcal{V}(h, s, t)
\]
nearly everywhere. This gives a definition of a cocycle matched pair of l.c. groups.

Fixing a cocycle matched pair of l.c. groups \( G_1 \) and \( G_2 \), denoting \( H_t = L^2(G_t) \) \((t = 1, 2)\), \( H = H_1 \otimes H_2 \) and identifying \( U \) and \( V \) with unitaries in \( M_1 \otimes M_1 \otimes M_2 \) and in \( M_1 \otimes M_2 \otimes M_2 \) respectively, define unitaries \( W \) and \( \tilde{W} \) on \( H \otimes H \) by
\[
\tilde{W} = (\beta \otimes \text{id} \otimes \text{id})((W_{G_1} \otimes 1) U^\ast) (\text{id} \otimes \text{id} \otimes \alpha)(V(1 \otimes W_{G_2})) \quad \text{and} \quad W = \Sigma \tilde{W}^* \Sigma.
\]
On the von Neumann algebra $M = G_1 ∗_{\mathcal{U}} \ell^\infty(G_2)$, let us define a faithful $*$-homomorphism

$$\Delta : M \to B(H ⊗ H) : \Delta(z) = W^*(1 ⊗ z)W \quad (\forall z \in M)$$

and denote by $\varphi$ the dual weight of the canonical left invariant trace $\varphi_2$ on $\ell^\infty(G_2)$. Then, Theorem 2.13 of [48] shows that $(M, \Delta)$ is a l.c. quantum group with $\varphi$ as a left invariant weight, which we call the cocycle bicrossed product of $G_1$ and $G_2$. One can also show that its scaling constant is 1. The dual l.c. quantum group is $(\hat{M}, \hat{\Delta})$, where $M = G_2 ∗_{\mathcal{U}} \ell^\infty(G_1)$ and $\hat{\Delta}(z) = W^*(1 ⊗ z)\hat{W}$ for all $z \in M$.

One can get explicit formulas for the modular operators and conjugations of the left invariant weights, unitary antipodes, scaling groups and modular elements of both $(M, \Delta)$ and its dual in terms of $\alpha_g$, $\beta_s$, the cocycles and the modular functions $\delta, \delta_1$ and $\delta_2$ of the l.c. groups $G, G_1$ and $G_2$. These formulas imply

**Proposition 2.4.** The l.c. quantum group $(M, \Delta)$ is a Kac algebra if and only if

$$[\delta(i(g\beta_s(g)^{-1})), \delta_1(g^{-1}\beta_s(g))] \delta_2([\alpha_2(s)s^{-1}, \delta_1(g)]) = 1$$

**Corollary 2.5.** If $\alpha$ or $\beta$ is trivial, $(M, \Delta)$ and $(\hat{M}, \hat{\Delta})$ are Kac algebras.

**Corollary 2.6.** If both $\alpha$ and $\beta$ preserve modular functions and Haar measures, then $(M, \Delta)$ and $(\hat{M}, \hat{\Delta})$ are Kac algebras.

Remark that the conditions of this corollary are fulfilled if both groups are discrete since any discrete group is unimodular and its Haar measure is constant.

**Corollary 2.7.** If $(G_1, G_2 \subset G)$ is a fixed matched pair of l.c. groups and cocycles $\mathcal{U}$ and $\mathcal{V}$ satisfy (2), we get a cocycle bicrossed product $(M, \Delta)$. If one of these cocycle bicrossed products is a Kac algebra, then all of them are Kac algebras.

**Proof.** The necessary and sufficient conditions for $(M, \Delta)$ to be a Kac algebra in Proposition 2.4 are independent of $\mathcal{U}$ and $\mathcal{V}$. \qed

### 2.4. Extensions of l.c. groups

Recall that any normal $*$-homomorphism $\zeta : M_1 \to \hat{M}$ of l.c. quantum groups satisfying $\hat{\Delta}\zeta = (\zeta \otimes \zeta)\Delta_1$ generates two canonical actions: $\mu$ of $(M_1, \Delta_1)$ on $M$ and $\theta$ of $(\hat{M}_1, \hat{\Delta}_1^{\text{op}})$ on $M$ (see [48]). On a formal level, this means that $\zeta$ gives rise to a dual morphism $\hat{\zeta} : M \to \hat{M}_1$ and $\mu$ should be thought of as $\mu = (\hat{\zeta} \otimes \text{id})\Delta$, while $\theta$ should be thought of as $\theta = (\hat{\zeta} \otimes \text{id})\Delta^{\text{op}}$.

**Definition 2.8.** Let $G_i (i = 1, 2)$ be l.c. groups and let $(M, \Delta)$ be a l.c. quantum group. We call

$$(\ell^\infty(G_2), \Delta_2) \xrightarrow{\eta} (M, \Delta) \xrightarrow{\zeta} (\ell^\infty(G_1), \hat{\Delta}_1)$$

a short exact sequence, if

$$\eta : \ell^\infty(G_2) \to M \quad \text{and} \quad \hat{\zeta} : \ell^\infty(G_1) \to \hat{M}$$

are normal, faithful $*$-homomorphisms satisfying

$$\Delta\eta = (\eta \otimes \eta)\Delta_2 \quad \text{and} \quad \hat{\Delta}\zeta = (\zeta \otimes \zeta)\Delta_1,$$

and if $\eta(\ell^\infty(G_2)) = M^\theta$, where $\theta$ is the canonical action of $(\ell^\infty(G_1), \hat{\Delta}_1^{\text{op}})$ on $M$ generated by the morphism $\zeta$. Then we call $(M, \Delta)$ an extension of $G_2$ by $G_1$. 

The exactness of the sequence in the first, third and second term is reflected respectively by the faithfulness of \( \eta \) and \( \zeta \) and by the formula \( \eta(L^\infty(G_2)) = M^\theta \).

Given a cocycle matched pair of l.c. groups, one can check that their cocycle bicrossed product is an extension in the sense of Definition 2.2. Moreover, it belongs to a special class of extensions, called cleft extensions ([15], Theorem 2.8). This theorem and [14] also show that, whenever \((M, \Delta)\) is a cleft extension of \(G_2\) by \(G_1\), then \((G_1, G_2 \subset G)\) is a cocycle matched pair and \((M, \Delta)\) is isomorphic to their cocycle bicrossed product.

By definition, two extensions

\[
(L^\infty(G_2), \Delta_2) \xrightarrow{\eta} (M_a, \Delta_a) \xrightarrow{\zeta} (\mathcal{L}(G_1), \hat{\Delta}_1)
\]

and

\[
(L^\infty(G_2), \Delta_2) \xrightarrow{\eta} (M_b, \Delta_b) \xrightarrow{\zeta} (\mathcal{L}(G_1), \hat{\Delta}_1)
\]

are called isomorphic, if there is an isomorphism \( \pi : (M_a, \Delta_a) \rightarrow (M_b, \Delta_b) \) of l.c. quantum groups satisfying \( \pi \eta_a = \eta_b \) and \( \hat{\pi} \zeta_a = \zeta_b \), where \( \hat{\pi} \) is the canonical isomorphism of \((M_a, \Delta_a)\) onto \((M_b, \Delta_b)\) associated with \( \pi \).

Given a matched pair \((G_1, G_2 \subset G)\) of l.c. groups, any couple of cocycles \((U, V)\) satisfying 2 generates as above a cleft extension

\[
(L^\infty(G_2), \Delta_2) \xrightarrow{\eta} (M, \Delta) \xrightarrow{\zeta} (\mathcal{L}(G_1), \hat{\Delta}_1).
\]

The extensions given by two pairs of cocycles \((U_a, V_a)\) and \((U_b, V_b)\), are isomorphic if and only if there exists a measurable map \( \mathcal{R} \) from \(G_1 \times G_2\) to \(T\), satisfying

\[
U_b(g, h, s) = U_a(g, h, s) \mathcal{R}(h, s) \mathcal{R}(g, \alpha_h(s)) \mathcal{R}(gh, s),
\]

\[
V_b(g, s, t) = V_a(g, s, t) \mathcal{R}(g, s) \mathcal{R}(\beta_s(g), t) \mathcal{R}(g, ts)
\]

almost everywhere. Such pairs \((U_a, V_a)\) and \((U_b, V_b)\) will be called cohomologous.

The set of equivalence classes of cohomologous pairs of cocycles \((U, V)\) satisfying (2.2), exactly corresponds to the set \( \Gamma \) of classes of isomorphic extensions associated with \((G_1, G_2 \subset G)\).

The set \( \Gamma \) can be given the structure of an abelian group by defining

\[
\pi(U_a, V_a) \cdot \pi(U_b, V_b) = \pi(U_aU_b, V_aV_b),
\]

where \( \pi(U, V) \) denotes the equivalence class containing the pair \((U, V)\). The group \( \Gamma \) is called the group of extensions of \((L^\infty(G_2), \Delta_2)\) by \((\mathcal{L}(G_1), \hat{\Delta}_1)\) associated with the matched pair of l.c. groups \((G_1, G_2 \subset G)\). The unit of this group corresponds to the class of cocycles cohomologous to trivial. The corresponding extension is called split extension; all other extensions are called non-trivial extensions.

3. Regularity properties

3.1. General result. As we have seen, one can associate a multiplicative unitary with any l.c. quantum group. Vice versa, Baaj and Skandalis constructed in [2], [3] a couple of Hopf \(C^*\)-algebras in duality out of a given multiplicative unitary verifying certain regularity conditions. In order to discuss these conditions for multiplicative unitaries coming from the bicrossed product construction, let us present the \(C^*\)-algebraic version of the split extension (i.e., with trivial cocycles) and its dual.

Let us associate with a multiplicative unitary \(W\) acting on a Hilbert space \(H\), three natural algebras:

\[
S = [(\omega \otimes \text{id})(W)|\omega \in B(H)_s], \quad \hat{S} = [(\text{id} \otimes \omega)(W)|\omega \in B(H)_s],
\]
and \(|C(W)| = [(id \otimes \omega)(\Sigma W)]|\omega \in B(H)|\),
where \([\cdot]\) denotes norm closure. They are not \(C^*\)-algebras, in general.

**Definition 3.1.** (see [2], [3]) A multiplicative unitary \(W\) is called regular if \(|C(W)| = K(H)\), and semi-regular if \(|C(W)|\) contains \(K(H)\), the algebra of all compact operators on \(H\).

All \(W\) associated with Kac algebras, are regular [3], and all \(W\) associated with known "isolated" examples of l.c. quantum groups (see Introduction), are semi-regular; so it was quite a surprise to find an example of a l.c. quantum group whose multiplicative unitary is non-semi-regular (see below). If \(W\) is associated with a l.c. quantum group \((M, \Delta)\), then all \(|C(W)|\), \(S\) and \(\hat{S}\) are a \(C^*\)-algebras, and the comultiplications \(\Delta\) and \(\hat{\Delta}\) restrict nicely to morphisms \(S \rightarrow M(S \otimes S)\) and \(\hat{S} \rightarrow M(\hat{S} \otimes \hat{S})\) respectively, where \(M(A)\) is the multiplier \(C^*\)-algebra of a \(C^*\)-algebra \(A\).

If \((M, \Delta)\) is given by a bicrossed product construction out of a matched pair \((G_1, G_2 \subset G)\) of l.c. groups, then one can identify \(S\) and \(\hat{S}\) with \(C_0(G_2\backslash G) \rtimes \beta G_2\) and \(G_1 \rtimes \beta_1 C_0(G/G_1)\) respectively, where \(\tilde{\alpha}_g\) and \(\tilde{\beta}_s\) are the canonical continuous actions of \(G_1\) on \(G/G_1\) of \(G_2\) on \(G_2\backslash G\) respectively (\(C_0(X)\) denotes the \(C^*\)-algebra of continuous functions vanishing at infinity on a l.c. topological space \(X\) [5]). One can check that the measurable mutual actions \(\alpha_g\) and \(\beta_s\) of \(G_1\) and \(G_2\) are the restrictions of the above canonical continuous actions \(\tilde{\alpha}_g\) and \(\tilde{\beta}_s\) (topologies on \(G_1\) and \(G_2\backslash G\) and, respectively, on \(G_2\) and \(G/G_1\), are in general different).

Now we can formulate the main result of [5]:

**Theorem 3.2.** The multiplicative unitary \(W\) of the bicrossed product l.c. quantum group \((M, \Delta)\) is regular if and only if the map
\[
\theta : G_1 \times G_2 \rightarrow G : \theta(g, s) = gs
\]
is a homeomorphism of \(G_1 \times G_2\) onto \(G\). \(W\) is semi-regular if and only if \(\theta\) is a homeomorphism of \(G_1 \times G_2\) onto an open subset of \(G\) of full measure.

We will call the corresponding matched pairs of l.c. groups regular, semi-regular and non-semi-regular, respectively. To get an example of a regular matched pair, it suffices to take \(G\) discrete or in the form of a semi-direct product of \(G_1\) and \(G_2\), with closed subgroups \(G_1\) and \(G_2\). In both cases the bicrossed product l.c. quantum group is a Kac algebra, due to Corollaries 2.6 and 2.5 respectively.

**Remark 3.3.** Woronowicz [57] constructed a couple of Hopf \(C^*\)-algebras in duality out of a given multiplicative unitary under certain alternative conditions of manageability. All multiplicative unitaries associated with l.c. quantum groups, are manageable.

### 3.2. Matched pairs of Lie groups [49]

We consider Lie groups and Lie algebras over the field \(\mathbb{R}\) or \(\mathbb{C}\). A matched pair of Lie groups (i.e., when \(G\) is a Lie group), is always semi-regular:

**Proposition 3.4.** If, in Definition 2.1 \(G\) is a Lie group, then the map \(\theta\) has an open range \(\Omega\) and is a diffeomorphism of \(G_1 \times G_2\) onto \(\Omega\), where \(G_1\) and \(G_2\) are Lie groups under the identification with closed subgroups of \(G\).

The infinitesimal form of the last notion is as follows (see [35]):
Definition 3.5. We call \((g_1, g_2)\) a matched pair of Lie algebras, if there exists a Lie algebra \(g\) with Lie subalgebras \(g_1\) and \(g_2\) such that \(g = g_1 \oplus g_2\) as vector spaces.

These conditions are equivalent [35] to the existence of a left action \(\triangleright: g_2 \otimes g_1 \to g_1\) and a right action \(\triangleleft: g_2 \otimes g_1 \to g_2\), so that \(g_1\) is a left \(g_2\)-module and \(g_2\) is a right \(g_1\)-module and

\[
\begin{align*}
(1) \quad x \triangleright [a, b] &= [x \triangleright a, b] + [a, x \triangleright b] + (x \triangleleft a) \triangleright b - (x \triangleright b) \triangleleft a, \\
(2) \quad [x, y] \triangleleft a &= [x, y \triangleleft a] + [x \triangleleft a, y] + x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a),
\end{align*}
\]

for all \(a, b \in g_1\), \(x, y \in g_2\). Then, for \(g = g_1 \oplus g_2\) we have:

\[
[a \oplus x, b \oplus y] = ([a, b] + x \triangleright b - y \triangleright a) \oplus ([x, y] + x \triangleleft b - y \triangleleft a).
\]

Two matched pairs, \((g_1, g_2)\) and \((g'_1, g'_2)\), are called isomorphic if there is an isomorphism of the corresponding Lie algebras \(g\) and \(g'\) sending \(g_i\) onto \(g'_i\) \((i = 1, 2)\).

Let us explain the relation between the two notions of a matched pair.

Proposition 3.6. Let \((G_1, G_2 \subset G)\) be a matched pair of Lie groups. If \(g\) denotes the Lie algebra of \(G\), and if \(g_1\), resp. \(g_2\), are the Lie subalgebras corresponding to the closed subgroups \(i(G_1)\), resp. \(j(G_2)\), then \((g_1, g_2)\) is a matched pair of Lie algebras.

Proof. The fact that \(g = g_1 \oplus g_2\) as vector spaces follows from the fact that \(\theta\) is a diffeomorphism in the neighborhood of the unit element.

The converse problem, to construct a matched pair of Lie groups from a given matched pair \((g_1, g_2)\) of Lie algebras, is much more delicate. Let us start with

Proposition 3.7. Every matched pair of Lie algebras \(g_1 = g_2 = k\) can be exponentiated to a matched pair of Lie groups \((G_1, G_2 \subset G)\), where \(G_1, G_2\) are either \((k, +)\) or \((k \setminus \{0\}, \cdot)\), where \(k = \mathbb{C}\) or \(\mathbb{R}\).

Proof. a) The only two-dimensional complex Lie algebras are the abelian one and the one with generators \(X, Y\) and relation \([X, Y] = Y\). If \(g\) is abelian, the mutual actions of \(g_1\) and \(g_2\) on each other are trivial and exponentiation is a direct sum.

If \(g\) is generated by \([X, Y] = Y\), we either have that \(g_1\) or \(g_2\) equals to \(\mathbb{C}Y\), then one of the actions is trivial and \(G\) can be constructed as semi-direct product of the connected, simply connected Lie groups of \(g_1\) and \(g_2\), or both \(g_1, g_2 \neq \mathbb{C}Y\). In the latter case, there is, up to isomorphism, only one possibility, namely \(g_1 = \mathbb{C}X\), \(g_2 = \mathbb{C}(X + Y)\). Define on \(\mathbb{C} \setminus \{0\} \times \mathbb{C}\) the Lie group with product

\[
(t, s)(t', s') = (tt', s + ts').
\]

Define \(G_1 = G_2 = \mathbb{C} \setminus \{0\}\) with embeddings \(i(g) = (g, 0)\) and \(j(s) = (s, s - 1)\), we indeed get a matched pair of complex Lie groups with mutual actions

\[
\begin{align*}
\alpha_g(s) &= g(s - 1) + 1, \\
\beta_s(g) &= \frac{sg}{g(s - 1) + 1}.
\end{align*}
\]

b) The real case is completely analogous. \qed

Remark 3.8. The natural idea to take the connected, simply connected Lie group \(G\) of the corresponding \(g\) and its unique connected, closed subgroups \(G_1\) and \(G_2\) with tangent Lie algebras \(g_1\) and \(g_2\), respectively, fails because such \((G_1, G_2)\) is not necessarily a matched pair even if \(\dim g_1 = \dim g_2 = 1\). Indeed, if \(k = \mathbb{C}\), the connected simply connected Lie group \(G\) of \(g\) consists of all pairs \((t, s)\) with \(t, s \in \mathbb{C}\) and the product

\[
(t, s)(t', s') = (t + t', s + \exp(t)s'),
\]

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\[\alpha_g(s) = g(s - 1) + 1, \quad \beta_s(g) = \frac{sg}{g(s - 1) + 1}.\]
and its closed subgroups $G_1$ and $G_2$ corresponding to the decomposition $\mathfrak{g} = \mathbb{C}X \oplus \mathbb{C}(X+Y)$ above consist respectively of all pairs of the form $(g,0)$ and $(s,\exp(s)-1)$ with $g,s \in \mathbb{C}$. These groups do not form a matched pair because $G_1 \cap G_2 = \{(2\pi in,0)| n \in \mathbb{Z}\} = Z(G)$. So, it is crucial not to take $G$ simply connected.

Taking $g,t,s$ above real, we come to the example of a matched pair of real Lie groups from [18], Section 5.3 (see also [12] and [17]). Here $\mathfrak{g}$ is a real Lie algebra generated by $X$ and $Y$ subject to the relation $[X,Y] = Y$ and one considers the decomposition $\mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}(X+Y)$. Then, to get a matched pair of Lie groups, we consider $G$ as the variety $\mathbb{R} \setminus \{0\} \times \mathbb{R}$ with the product

$$(s,x)(t,y) = (st, sx + sy)$$

and embed $G_1 = G_2 = \mathbb{R} \setminus \{0\}$ by the formulas $i(g) = (g,0)$ and $j(s) = (s,s-1)$. Remark that here, it is impossible to take the connected component of the unity of the group of affine transformations of the real line as $G$, because it is easy to see that for its closed subgroups $G_1$ and $G_2$ corresponding to the above mentioned subalgebras, the set $G_1G_2$ is not dense in $G$. The corresponding multiplicative unitary is semi-regular, but not regular.

It is even possible that for a given matched pair of Lie algebras, $G_1 \cap G_2 \neq \{e\}$ for any corresponding pair of Lie groups, i.e., such a matched pair of Lie algebras cannot be exponentiated to a matched pair of Lie groups.

Example 3.9. Consider a family of complex Lie algebras $\mathfrak{g} = \text{span}\{X,Y,Z\}$ with $[X,Y] = Y$, $[X,Z] = \alpha Z$, $[Y,Z] = 0$, where $\alpha \in \mathbb{C} \setminus \{0\}$, and the decomposition $\mathfrak{g} = \text{span}\{X,Y\} \oplus \mathbb{C}(X+\alpha Z)$. The corresponding connected simply connected complex Lie group $H$ consists of all triples $(t,u,v)$ with $t,u,v \in \mathbb{C}$ and the product

$$(t,u,v)(t',u',v') = (t + t', u + \exp(t)u', v + \exp(\alpha t)v'),$$

and its closed subgroups $H_1$ and $H_2$ corresponding to the decomposition above consist respectively of all triples of the form $(t,u,0)$ and $(s,0,\exp(\alpha s)-1)$ with $t,u,v \in \mathbb{C}$. These groups do not form a matched pair because $H_1 \cap H_2 = \{(2\pi in,0,0)| n \in \mathbb{Z}\}$.

We claim that, if $1/\alpha \not\in \mathbb{Z}$ and if $G$ is any complex Lie group with Lie algebra $\mathfrak{g}$, such that $G_1,G_2$ are closed subgroups of $G$ with tangent Lie algebras $\mathfrak{g}_1$, resp. $\mathfrak{g}_2$, then $G_1 \cap G_2 \neq \{e\}$. Indeed, since the Lie group $H$ is connected and simply connected, the connected component $G^{(e)}$ of $e$ in $G$ can be identified with the quotient of $H$ by a discrete central subgroup. If $\alpha \not\in \mathbb{Q}$, the center of $H$ is trivial, so that we can identify $G^{(e)}$ and $H$. Under this identification, the connected components of $e$ in $G_1,G_2$ agree with $H_1,H_2$. Because $H_1 \cap H_2 \neq \{e\}$, our claim follows. If $\alpha = \frac{m}{n}$ for $m,n \in \mathbb{Z}\setminus\{0\}$ mutually prime, the center of $H$ consists of the elements $\{(2\pi nN,0,0)| N \in \mathbb{Z}\}$. Hence, the different possible quotients of $H$ are labeled by $N \in \mathbb{Z}$ and are given by the triples $(a,u,v) \in \mathbb{C}^3$, $a \not= 0$ and the product

$$\alpha \not\in \mathbb{Z}$$

$$\alpha \not= 0$$

The closed subgroups corresponding to $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are given by $(a,u,0)$ and $(b,0,b^{mN}-1)$ with $a,b,u \in \mathbb{C}$ and $a,b \not= 0$. The intersection of both subgroups is non-trivial whenever $mN \not= \pm 1$. This proves our claim.

Considering now the complex Lie algebras above as real Lie algebras with generators $X,iX,Y,iY,Z,iZ$ and the decomposition above as a decomposition of real Lie algebras, we get a matched pair of real Lie algebras which cannot be exponentiated to a matched pair of real Lie groups.
In the remaining case $\alpha = 1/n$ with $n \in \mathbb{Z} \setminus \{0\}$, we can consider the Lie group $G$ defined by Equation (4) with $m = N = 1$. Consider $G_1 = \mathbb{C} \setminus \{0\} \times \mathbb{C}$ with $(a, u)(a', u') = (aa', u + a_n u')$ and $G_2 = \mathbb{C} \setminus \{0\}$. Writing $i(a, u) = (a, u, 0)$ and $j(v) = (v, 0, v - 1)$, we get a matched pair of Lie groups with mutual actions

$$\alpha(a, u)(v) = a(v - 1) + 1 \quad \text{and} \quad \beta(v, a, u) = \left(\frac{vu}{a(v - 1) + 1}, \frac{u}{a(v - 1) + 1}\right).$$

However, any matched pair of real Lie algebras when one of them has dimension 1 and the other at most 2, can be exponentiated.

3.3. Cocycle matched pairs of Lie groups [49]. The usage of 2-cocycles gives much more concrete examples of l.c. quantum groups. We first explain the infinitesimal picture, i.e. how 2-cocycles for matched pairs of Lie algebras look like, how they are related to extensions and then discuss the problem of exponentiation.

Recall that a Lie bialgebra, according to V.G. Drinfeld, is a Lie algebra $g$ equipped with a Lie bracket $[\cdot, \cdot]$ and a Lie cobracket $\delta$, i.e., a linear map $\delta : g \to g \otimes g$ satisfying the co-anticommutativity and the co-Jacobi identity, that is:

$$(\text{id} - \tau)\delta = 0, \quad (\text{id} + \zeta + \zeta^2)(\text{id} \otimes \delta)\delta = 0,$$

where

$$\tau(u \otimes v) = v \otimes u, \quad \zeta(u \otimes v \otimes w) = v \otimes w \otimes u \quad (\text{for all } u, v, w \in g)$$

are the flip maps, and $[\cdot, \cdot], \delta$ are compatible in the following sense:

$$\delta[u, v] = [u, v_1] \otimes v_2 + v_1 \otimes [u, v_2] + [u_1, v] \otimes u_2 + u_1 \otimes [u_2, v].$$

Any Lie algebra (respectively, Lie coalgebra, i.e., vector space dual to a Lie algebra) is a Lie bialgebra with zero Lie cobracket (respectively, zero Lie bracket). The definition of a morphism of Lie bialgebras is obvious.

Given a pair of Lie algebras $(g_1, g_2)$, let look for a Lie bialgebra $g$ such that

$$g_2 \longrightarrow g \longrightarrow g_1$$

is a short exact sequence in the category of Lie bialgebras. This means precisely that $g$ has a sub-bialgebra with trivial bracket, which is an ideal and such that the quotient is a Lie bialgebra with trivial cobracket.

The theory of extensions in this framework has been developed in [35] and is similar to the theory of extensions of l.c. groups that we discussed above. Namely, for the existence of an extension $g$, it is necessary and sufficient that $(g_1, g_2)$ form a matched pair, and all extensions are bicrossed products with cocycles. We consider this theory as an infinitesimal version of the theory of extensions of Lie groups.

As we remember, for any matched pair of Lie algebras $(g_1, g_2)$, there are mutual actions $\triangleright : g_2 \otimes g_1 \to g_1$ and $\triangleleft : g_2 \otimes g_1 \to g_2$, compatible in a way explained above and such that, for all $a, b \in g_1$, $x, y \in g_2$, we have

$$[a \triangleright x, b \triangleright y] = ([a, b] + x \triangleright b - y \triangleright a) \oplus ([x, y] + x \triangleleft b - y \triangleleft a).$$

The definition of a pair of 2-cocycles on such a matched pair is given in [35], [38]. For our needs, it suffices to understand that these 2-cocycles are linear maps

$$U : g_1 \wedge g_1 \to g_2^*, \quad V : g_2 \wedge g_2 \to g_1^*$$

verifying certain 2-cocycle equations and compatibility equations that are infinitesimal forms of equations (4). The link between 2-cocycles on matched pairs of real Lie groups and those of real Lie algebras is given by
Proposition 3.10. Let \((G_1, G_2)\) be a matched pair of real Lie groups equipped with cocycles \(U\) and \(V\), which are differentiable around the unit elements, and let \((g_1, g_2)\) be the corresponding matched pair of real Lie algebras. Defining

\[
(U(X, Y), A) = -i(x \otimes y \otimes a - y \otimes x \otimes a)(U) \\
(V(A, B), X) = -i(a \otimes b \otimes x - b \otimes a \otimes x)(V)
\]

for \(X, Y \in g_1\) and \(A, B \in g_2\), we get a pair of cocycles on \((g_1, g_2)\).

Here \(\langle \cdot, \cdot \rangle\) is the duality between \(g_i\) and \(g_i^*\), and \(x, y, a, b\) denote the partial derivatives at \(e\) in the direction of the corresponding generator. The factor \(-i\) appears because for Lie groups \(U\) and \(V\) take values in \(\mathbb{T}\), and for real Lie algebras we consider 2-cocycles as real linear maps.

It is clear that these 2-cocycles \(U\) form a real vector space. If one of the Lie groups is 1-dimensional, the corresponding 2-cocycle is trivial, since it can be regarded as an antisymmetric, bilinear form on the corresponding Lie algebra.

We defined above the group of extensions for a matched pair of l.c. groups \((G_1, G_2)\) using the notion of cohomologous 2-cocycles. The same can be done for a matched pair of Lie algebras \([38]\). Two cocycles \(U_1\) and \(U_2\) are called cohomologous if \(U_1 - U_2\) is cohomologous to trivial. The quotient space of 2-cocycles modulo 2-cocycles cohomologous to trivial, with addition as the group operation, is called the group of extensions of the matched pair \((g_1, g_2)\).

In particular, the group of extensions for a matched pair of real Lie algebras of dimensions 1 and 2, is either trivial or \(\mathbb{R}\). To exponentiate these 2-cocycles, i.e., to construct the corresponding 2-cocycles on the level of Lie groups, is much more difficult. However, a complete classification of cocycle matched pairs of real Lie groups with \(\dim G \leq 3\) and at most 2 was obtained in \([49]\); the corresponding groups of extensions are either trivial, or \(\mathbb{R}\) or \(\mathbb{Z}\).

A series of examples of l.c. quantum groups was constructed in \([11]\) out of cocycle matched pairs of real 2-dimensional Lie algebras by their exponentiation.

3.4. Non-semi-regular l.c. quantum groups \([5]\). Let \(\mathcal{A} \neq 0\) be a l.c. ring with additive Haar measure \(\nu\), and let \(\mathcal{A}^\times\) be its group of invertible elements. Then \(\mathcal{A}^\times\) is a l.c. group by considering it as a closed subspace \(\{(a, b)ab = ba = 1\}\) of \(\mathcal{A} \times \mathcal{A}\), let \(\nu^\times\) be its Haar measure. Consider the group \(G = \{a, x | a \in \mathcal{A}^\times, x \in \mathcal{A}\}\) with multiplication \((a, x)(b, y) = (ab, x + ay)\), and its closed subgroups \(G_1 = \{(a, a - 1) | a \in \mathcal{A}^\times\}, G_2 = \{(b, 0) | b \in \mathcal{A}^\times\}\). The Haar measure of \(G\) is the product of \(\nu\) and \(\nu^\times\). Then \(G_1G_2 = \{(a, x) \in \mathcal{A}^\times \times \mathcal{A}| x + 1 \in \mathcal{A}^\times\}\). Hence, \((G_1, G_2)\) is a matched pair if and only if \(\mathcal{A}^\times\) has complement of measure \(\nu\) zero in \(\mathcal{A}\). Then one can prove

Proposition 3.11. The above matched pair is not regular. It is semi-regular if and only if \(\mathcal{A}^\times\) is open in \(\mathcal{A}\). If \((M, \Delta)\) is the corresponding bicrossed product l.c. quantum group, then the dual \((\hat{M}, \hat{\Delta})\) is isomorphic to the opposite quantum group \((M, \Delta^{op})\) and \(M \equiv \hat{M} \equiv B(L^2(\mathcal{A}^\times, \nu^\times))\).

Example 3.12. In order to construct an example of a non-semi-regular matched pair, let us choose the second countable ring \(\mathcal{A}\) in the following way. Let \(\mathcal{P}\) be a set of prime numbers such that

\[
\sum_{p \in \mathcal{P}} \frac{1}{p} < \infty.
\]
Define the restricted Cartesian product of l.c. fields $\mathbb{Q}_p$ of $p$-adic rational numbers relatively to the compact open subrings $\mathbb{Z}_p$ of the corresponding $p$-adic integers:

$$\mathcal{A} = \Pi_{p\in\mathcal{P}}(\mathbb{Q}_p, \mathbb{Z}_p).$$

It consists of sequences $(x_p) \in \mathbb{Q}_p$ which eventually belong to $\mathbb{Z}_p$. Equipped with the usual l.c. topology, $\mathcal{A}$ becomes a l.c. ring such that $\mathcal{A}^\times$ has complement of measure zero and empty interior in $\mathcal{A}$. The fact that $\mathcal{A}^\times$ has complement of measure zero in $\mathcal{A}$ follows from the Borel-Cantelli lemma: normalizing the Haar measure on $\mathbb{Q}_p$ in such a way that $\mathbb{Z}_p$ has measure one, observe that $\mathbb{Z}_p\setminus \mathbb{Z}_p^\times$ has measure $\frac{1}{p}$, which assumed to be summable over $p \in \mathcal{P}$.

Proposition 3.11 shows that the corresponding multiplicative unitary in not semi-regular. It is known that the $C^*$-algebra $S = \mathcal{A}^\times \rtimes C_0(\mathcal{A})$ is not of type I [8].

4. L.C. groups whose von Neumann algebras are factors

4.1. Motivation. Let $\mathcal{L}(G)$ be the von Neumann algebra generated by the operators $\lambda_g$, $g \in G$ of the left regular representation of a l.c. group $G$ or, equivalently, by the operators of the form $L_f = \int_G f(g) \lambda_g d\mu_G$, where $f \in L^1(G, \mu_G)$. All these operators act on the Hilbert space $L^2(G, \mu_G)$, where $\mu_G$ is a left Haar measure on $G$. It is equipped with the canonical n.s.f. weight defined by $\varphi(L_f) = f(e)$, for all functions from $L^1(G, \mu_G)$ continuous in the neutral element $e \in G$. This weight is a trace if and only if $G$ is unimodular.

If $G \neq \{e\}$ is compact, then $\mathcal{L}(G)$ is a direct sum of finite dimensional full matrix algebras and cannot be a factor (i.e., a von Neumann algebra with trivial center). Indeed, in this case one of the summands corresponding to the trivial representation of $G$ must be of dimension 1. But there exist non-compact groups $G$ with $\mathcal{L}(G)$ a factor of any type, in the sense of Murray-von Neumann classification, except for a full matrix algebra of finite dimension: $I_\infty$, $II_1$, $II_\infty$, $III_\lambda$, where $\lambda \in [0,1]$ (see, for example, [19]; the standard reference on the classification of type III factors and their invariants is [10]).

a) For the group $G = \mathbb{R} \rtimes \mathbb{R}^\times$ - semi-direct product with natural action of $\mathbb{R}^\times$ on $\mathbb{R}$ by multiplication, one has $\mathcal{L}(G) \simeq B(L^2(\mathbb{R}^\times))$ - the type $I_\infty$ factor.

b) Let $G$ be a discrete countable group with all nontrivial conjugacy classes infinite, for instance, $\mathbb{F}_n$ - the free group with $n \geq 2$ generators or $S_\infty$ - the group of permutations of the set of natural numbers such that any individual permutation permutes only finitely many numbers. Then $\mathcal{L}(G)$ equipped with the canonical finite trace $\varphi$ is a type $II_1$ factor (see, for example, [19]).

c) The Cartesian product of groups in a) and b) gives an example of $G$ such that $\mathcal{L}(G)$ is a type $II_\infty$ factor.

d) R. Godement showed that for $G = \mathbb{R}^2 \rtimes GL_2(\mathbb{Q})$ - semi-direct product with natural action of $GL_2(\mathbb{Q})$ on $\mathbb{R}^2$, $\mathcal{L}(G)$ is a non-hyperfinite type $III_1$ factor. Then C. Sutherland derived from this a series of examples of groups for which $\mathcal{L}(G)$ is a non-hyperfinite type $III_\lambda$ factor, for all $\lambda \in [0,1)$, and A. Connes constructed similar examples with hyperfinite factors, for $\lambda \in (0,1)$ (for all these results see [14]). Later on, for all $\lambda \in [0,1)$, examples of groups $G$ coming from number theory with $\mathcal{L}(G)$ a hyperfinite type $III_\lambda$ factor, were constructed in [9]. Moreover, one can construct such examples with specific properties of the invariant $T$ (see [14], [9]).
These results and certain freedom given by the bicrossed product construction in producing examples of l.c. quantum groups, allows us to ask if it is possible to construct such examples of \((M, \Delta)\) that both von Neumann algebras, \(M\) and \(\hat{M}\), are factors of all prescribed types, except for a full matrix algebra of finite dimension. Such quantum groups would be "as far as possible" from usual groups since, by definition, a factor is a von Neumann algebra with trivial center. We present below some partial results in this direction.

Since the cocycle bicrossed product construction gives \(M = G_1 \alpha,\beta \ltimes L^\infty(G_2)\) and \(\hat{M} = G_2 \alpha,\beta \ltimes L^\infty(\hat{G}_1)\), let us look when the centers of these algebras equal to \(\mathbb{C}1\). If \(\mathcal{U}\) and \(\mathcal{V}\) are trivial, one can use the result from \([11]\). Recall that a borelian measure \(\mu\), is called free if the stabilizer \(S_x := \{ \gamma \in \Gamma | gx = x \}\) equals to \(\{ e \}\) for \(\mu\)-almost all \(x \in X\). For such an action, \([11]\) Corollary 2.3 says that the center of \(\Gamma \ltimes L^\infty(X, \mu)\) is a factor if and only if \(\theta\) is also ergodic (i.e., the set of \(\theta\)-fixed points is of measure zero). Thus, it would be interesting to find matched pairs \((G_1, G_2)\) with both actions \(\alpha\) and \(\beta\) being free and ergodic (remark immediately that if \(G_1\) (resp., \(G_2\)) is discrete, the relations \(\alpha_g(e) = e, \beta_s(e) = e, \forall g \in G_1, s \in G_2\) show that \(\beta\) (resp., \(\alpha\)) is not free).

One instance of this kind is given by matched pairs with \(G_2 = g_0 G_1 g_0^{-1}\), where \(G_0\) is an element of \(G\), i.e., \(G_1\) and \(G_2\) are conjugated. Then \([5]\) Prop. 3.13 says that both actions \(\alpha\) and \(\beta\) are isomorphic to the action of \(G_2\) on itself by translations which is clearly free and ergodic. Moreover, in this case both \(M\) and \(\hat{M}\) are isomorphic to \(B(L^2(G_2))\), i.e., are type I\(_1\) factors.

In particular, for the matched pares introduced before Proposition \([5,11]\) \(G_1\) and \(G_2\) are conjugated by the element \((-1, -1)\). Another example (\([5]\), Ex. 4.6) is as follows. Let again \(A \neq 0\) be l.c. ring such that \(A^x\) has complement of measure \(\nu\) zero in \(A\). Take \(G = GL_2(A)\) and \(G_1 = \{(a_{ij})\}\) with \(a_{11} \in A^x, a_{12} \in A, a_{21} = 0, a_{22} = 1\), \(G_2 = \{(a_{ij})\}\) with \(a_{11} = 1, a_{12} = 0, a_{21} \in A, a_{22} \in A^x\). Clearly, \(G_1\) and \(G_2\) are conjugated.

4.2. l.c. quantum groups whose algebras are type II and type III factors \([17]\). Using the techniques close to that of \([5,3]\), one can show that a l.c. quantum group \((M, \Delta)\) such that \(M\) is a type II\(_1\) factor, is necessarily a compact quantum group in the sense of \([5,3]\), so that \(\hat{M}\) is necessarily a direct sum of finite dimensional full matrix algebras and cannot be a factor (one of the summands is generated by the counit and must be of dimension 1). Thus, the case when \(M\) or \(\hat{M}\) is a type II\(_1\) factor, must be excluded from consideration.

The following examples of l.c. quantum groups whose algebras are factors, are inspired by Example \([3,12]\). We use again second countable l.c. ring of the form

\[ A = \Pi'_{n \in \mathbb{N}}(\mathbb{Q}_{p_n}, \mathbb{Z}_{p_n}), \]

where \((p_n)\) is now an infinite sequence of prime numbers without repetitions, its group of invertible elements

\[ A^\times = \Pi'_{n \in \mathbb{N}}(\mathbb{Q}^\times_{p_n}, \mathbb{Z}^\times_{p_n}), \]

the group \(G = \{(a, x)|a \in A^\times, x \in A\}\) with multiplication \((a, x)(b, y) = (ab, x+ay)\), and the closed subgroup \(G_1 = \{(a, 0)|a \in A^\times\}\) of \(G\). But now we choose the second closed subgroup \(G_2\) of \(G\) in another way:

\[ G_2 = \{(a_n, (b_n)) \in G|a_n + b_np_n = 1, \forall n \in \mathbb{N}\}. \]
Then, checking the conditions of Theorem 4.2, one can prove that \((G_1, G_2 \subset G)\) is a matched pair of l.c. groups, which is semi-regular, but not regular. Thus, one can apply the bicrossed product construction and to get this way two l.c. compact quantum groups in duality, whose von Neumann algebras will be \(M = G_1 \times L^\infty(G_2)\) and \(\hat{M} = G_2 \times L^\infty(G_1)\). Moreover, one can show that both actions \(\alpha\) and \(\beta\) are free and ergodic, so that both \(M\) and \(\hat{M}\) are factors.

In order to determine their types, one constructs their explicit isomorphisms to so called infinite tensor products of type I factors, briefly, ITPFI factors (see, for example, [1]), of the form

\[
L = \otimes_{n \in \mathbb{N}} (B(l^2(\mathbb{N})), \varphi_n),
\]

where \(\varphi_n\) is a normal state on the type I factor \(B(l^2(\mathbb{N}))\) of the form \(\varphi_n(\cdot) = \text{tr}(\rho_n \cdot)\). Here \(\rho_n\) is a positive trace class operator characterized by the decreasing sequence of its eigenvalues. In the case of our \(M\), this sequence of eigenvalues is \(\lambda_{n,k}^M = (1 - p_n^{-1})p_{n-k}^{-1}, n, k \in \mathbb{N}\), and in the case of \(\hat{M}\) the corresponding sequence of eigenvalues \(\lambda_{n,k}^M, n, k \in \mathbb{N}\) also can be computed explicitly. Now we can use the following general result [1]:

**Proposition 4.1.** Let

\[
L = \otimes_{n \in \mathbb{N}} (M_n, \varphi_n),
\]

be an ITPFI factor, where \(M_n\) is a type \(I_{n^2}\) factor with \(n \in \mathbb{N} \cup \{\infty\}\), and let \(\varphi_n\) be a normal state on \(M_n\) of the form \(\varphi_n(\cdot) = \text{tr}(\rho_n \cdot)\). Here \(\rho_n\) is a positive trace class operator characterized by the decreasing sequence of its eigenvalues \(\lambda_{n,i}, i = 1, \ldots, n^2\). Then:

1. \(L\) is of type I if and only if

\[
\Sigma_n |1 - \lambda_{n,1}| < +\infty.
\]

2. \(L\) is of type II if and only if \(n^2 < +\infty\) for all \(n\) and

\[
\Sigma_{n,i} (n^2)^{\frac{1}{4}} - (\lambda_{n,i})^{\frac{1}{4}} |^2 < +\infty.
\]

3. If exists \(\delta > 0\) such that for all \(n\) we have \(\lambda_{n,i} \geq \delta\), then \(L\) is of type III if and only if

\[
\Sigma_{n,i} \inf\left\{\left|\frac{\lambda_{n,i}}{\lambda_{n,1}} - 1\right|, C\right\} < +\infty,
\]

for some constant \(C > 0\).

Comparing the above mentioned sequences of eigenvalues \(\lambda_{n,k}^M\) and \(\lambda_{n,k}^{\hat{M}}\), \(n, k \in \mathbb{N}\), with conditions of Proposition 4.1 and using the results of [17], one gets the following

**Theorem 4.2.** [17] For each infinite sequence \((p_n)\) of prime numbers without repetitions, the von Neumann algebras \(M\) and \(\hat{M}\) are hyperfinite factors. Then:

(i) Both \(M\) and \(\hat{M}\) have the same invariant \(T\) (in the sense of [18]).

(ii) The condition \(\Sigma_n \frac{1}{p_n} < +\infty\) is equivalent to the fact that \(A^\times\) has complement of measure zero in \(A\). In this case, \((M, \Delta)\) is of type \((I_{\infty}, I_{\infty})\).

(iii) The condition \(\Sigma_n \frac{1}{p_n} = +\infty\) is equivalent to the fact that \(\nu(A^\times)\) is zero. In this case, \((M, \Delta)\) is of type \((III, III)\).

Moreover, we have:

a) For each \(\lambda \in [0, 1]\), there exists such \((p_n)\) that both \(M\) and \(\hat{M}\) are of type \(III_\lambda\).
b) For each countable subgroup \( K \) of \( \mathbb{R} \) and a countable subset \( \Sigma \) of \( \mathbb{R} \setminus K \) there exists such \( (p_n) \) that that the corresponding invariant \( T \) contains \( K \) and does not intersect with \( \Sigma \).

Remark, that tensor products of \((M, \Delta)\) corresponding to the case (i) and their duals give examples of l.c. quantum groups with both factors of type II\(_\infty\).

5. Some other results

5.1. Amenability [13]. Let \((M, \Delta)\) be a l.c. quantum group. A state \( m \in M^* \) is said to be a left invariant mean on \((M, \Delta)\) if

\[
m((\omega \otimes \text{id})\Delta(x)) = m(x)\omega(1),
\]

for all \( \omega \in M_\ast \) and \( x \in M \). Similarly, one defines a right invariant mean. An invariant mean is both right and left invariant mean. One can show that a left invariant mean exists on \((M, \Delta)\) if and only if an invariant mean exists on it.

**Definition 5.1.** We call \((M, \Delta)\) amenable if there exists a left invariant mean on it. We say that \((M, \Delta)\) is coamenable if \((\hat{M}, \hat{\Delta})\) is amenable.

**Definition 5.2.** We call \((M, \Delta)\) strongly amenable if there exists a bounded counit on \((\hat{S}, \hat{\Delta})\), the reduced dual \( C^*\)-algebraic l.c. quantum group [27]. We say that \((M, \Delta)\) is strongly coamenable if \((\hat{M}, \hat{\Delta})\) is strongly amenable.

Strong amenability implies amenability; the two properties coincide for usual l.c. groups and for discrete l.c. quantum groups. It is not known if they coincide for general l.c. quantum groups, even for Kac algebras. The amenability for Kac algebras was studied in [16].

**Remark 5.3.** It was claimed in [16] that they coincide for Kac algebras, but Zh.-J. Ruan found a gap in the proof. He showed [10] that these properties coincide for discrete Kac algebras. Later on, E. Blanchard and S. Vaes proved the same for discrete quantum groups (unpublished), their proof uses the Powers-Störmer inequality. One can find another proof of the last result in [46].

According to [13], an extension of the form

\[
(M_2, \Delta_2) \to (M, \Delta) \to (\hat{M}_1, \hat{\Delta}_1),
\]

where \((M_1, \Delta_1)\), \((M_2, \Delta_2)\) are two l.c. quantum groups (see [18]) is amenable if and only if both \((M_1, \Delta_1)\) and \((M_2, \Delta_2)\) are amenable. The same holds in strongly amenable case, but only for split extensions. This allows to construct various examples. Given a matched pair of amenable l.c. groups, all the corresponding extensions are amenable, and the split extension is strongly amenable. All examples in [18] are of this kind.

Two concrete examples of non-amenable l.c. quantum groups were constructed in [13] using the techniques of [18]. First, taking \(G_1 = \mathbb{R}^2\) and \(G_2 = SL_2(\mathbb{R})\), one gets a matched pair of l.c. groups. Then one can show that the corresponding split extension is a Kac algebra, which is non-amenable, since \(G_2\) is known to be a non-amenable l.c. group.

Second, taking \(G_1 = \{(x, z) | x \in \mathbb{R}, x \neq 0, z \in \mathbb{C}\}\) with multiplication \((x, z)(y, u) = (xy, z + xu)\), and \(G_2\) a double cover of \(SU(1, 1)\), one gets again a matched pair of l.c. groups. Then one can show that the corresponding split extension is not a Kac algebra. This l.c. quantum group is non-compact, non-discrete and non-amenable, since \(G_2\) is known to be a non-amenable l.c. group.
5.2. Kac exact sequence \[9\]. We have seen above that one can associate an abelian group of extensions with any matched pair \((G_1, G_2 \subset G)\) of l.c. groups. In the case of finite groups, Kac constructed in \[21\] an exact sequence which allows to calculate the above group of extensions in terms of usual cohomology groups of \(G_1, G_2\) and \(G\) with coefficients in the trivial module \(\mathbb{T} = \{z \in \mathbb{C} | |z| = 1\}\). A similar exact sequence was constructed in \[9\] for cohomology of l.c. groups with coefficients in any Polish \(G\)-module \(A\), but here we discuss very briefly only the case \(A = \mathbb{T}\).

First of all, let us precise, which kind of group cohomology we deal with. Let us consider the cochain complex \((L(\Gamma^n, \mathbb{T}))_n\), where \(n = 0, 1, 2, ..., \Gamma = G_1, G_2, G\). \(\Gamma^n\) is the Cartesian product of \(n\) copies of \(\Gamma\); \(G^n = 1\) is just a single point, \(L(\Gamma^n, \mathbb{T})\) is the set of (equivalence classes of) Borel functions from \(\Gamma^n\) to \(\mathbb{T}\). The coboundary operator \(d : L(\Gamma^n, \mathbb{T}) \rightarrow L(\Gamma^{n+1}, \mathbb{T})\) will be defined as follows. Let us write face operators \(\partial_i : \Gamma^{n+1} \rightarrow \Gamma^n:\)

\[
\partial_i(g_0, ..., g_n) = \begin{cases} (g_1, ..., g_n) & \text{if } i = 0, \\ (g_0, ..., g_{i-1}g_i, ..., g_n) & \text{if } i = 1, ..., n, \\ (g_0, ..., g_{n-1}) & \text{if } i = n + 1. 
\end{cases}
\]

Then, consider \(d_i : L(\Gamma^n, \mathbb{T}) \rightarrow L(\Gamma^{n+1}, \mathbb{T}):\)

\[
(d_iF)(\bar{g}) = \begin{cases} g_0 \cdot F(\partial_0\bar{g}) & \text{if } i = 0, \\ F(\partial_i\bar{g}) & \text{if } i = 1, ..., n + 1,
\end{cases}
\]

where \(\bar{g} = (g_0, ..., g_n)\). Finally,

\[d = \Sigma_{i=0}^{n+1} (-1)^{i-1}d_i.
\]

By definition, the measurable cohomology of the l.c. group \(\Gamma\) with coefficients in \(\mathbb{T}\) is the cohomology of the above cochain complex.

Kac cohomology \(H(m.p., \mathbb{T})\) of the matched pair \((G_1, G_2 \subset G)\) is defined as cohomology of certain complex whose "building blocks" are \(L(\Gamma_{p,q}, \mathbb{T})\), the sets of (equivalence classes of) Borel functions from \(\Gamma_{p,q}\) to \(\mathbb{T}\), where \(\Gamma_{p,q}\) is a closed subspace of \(G_1^{p(q+1)} \times G_2^{p(p+1)}\) defined in an inductive way starting from \(\Gamma_{p,0} = G_1^p\); \(\Gamma_{0,q} = G_2^q\); \(p, q = 0, 1, 2, ...,\); typical example: \(\Gamma_{1,1} = \{(g, h, s, t) \in G_1 \times G_1 \times G_2 \times G_2 | sg = ht\}\). The coboundary operator of this complex is constructed in terms of the above \(d\) (see \[9\]). In particular, the group of extensions of the matched pair \((G_1, G_2 \subset G)\) is precisely the Kac 2-cohomology group \(H^2(m.p., \mathbb{T})\).

\[9\] Corollary 4.5 claims that this Kac cohomology \(H(m.p., \mathbb{T})\) satisfies the following long exact sequence:

\[
0 \rightarrow \mathbb{T} \rightarrow \mathbb{T} \oplus \mathbb{T} \rightarrow H^0(m.p., \mathbb{T}) \rightarrow H^1(G, \mathbb{T}) \rightarrow H^1(G_1, \mathbb{T}) \oplus H^1(G_2, \mathbb{T}) \rightarrow \\
H^1(m.p., \mathbb{T}) \rightarrow H^2(G, \mathbb{T}) \rightarrow H^2(G_1, \mathbb{T}) \oplus H^2(G_2, \mathbb{T}) \rightarrow H^2(m.p., \mathbb{T}) \rightarrow \\
H^3(G, \mathbb{T}) \rightarrow H^3(G_1, \mathbb{T}) \oplus H^3(G_2, \mathbb{T}) \rightarrow 
\]

Here \(H^k(\Gamma, \mathbb{T})\), \(k = 1, 2, ...\) are the above mentioned measurable cohomology groups of \(\Gamma = G_1, G_2, G\) with coefficients in \(\mathbb{T}\).

This Kac exact sequence shows that in order to calculate the group of extensions of the matched pair \((G_1, G_2 \subset G)\), it suffices to calculate \(H^n(\Gamma, \mathbb{T})\) for \(n = 2, 3\) and \(\Gamma = G_1, G_2, G\). This can be done for certain class of l.c. groups, using the techniques proposed by D. Wigner. In particular, in \[49\] we computed the groups of extensions of matched pairs of low dimensional Lie groups, passing to the corresponding matched pairs of Lie algebras, and then performing the exponentiation of
the results obtained on the Lie algebra level, as it was explained in Section 3. But this last operation is quite non-trivial, and it was precisely justified only in §6.

**Remark 5.4.** a) A cohomology theory for extensions of Lie algebras was developed by A. Masuoka in [38].

b) An example of calculation of the group of extensions for a matched pair of l.c. groups via the Kac measurable cohomology can be found in [12].

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DEPT. OF MATH., UNIVERSITY OF CAEN, B.P. 5186, 14032 CAEN CEDEX, FRANCE; E.mail: vainerma@math.unicaen.fr