Guessing with a Bit of Help

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Abstract

What is the value of a single bit to a guesser? We study this problem in a setup where Alice wishes to guess an i.i.d. random vector, and can procure one bit of information from Bob, who observes this vector through a memoryless channel. We are interested in the guessing efficiency, which we define as the best possible multiplicative reduction in Alice’s guessing-moments obtainable by observing Bob’s bit. For the case of a uniform binary vector observed through a binary symmetric channel, we provide two lower bounds on the guessing efficiency by analyzing the performance of the Dictator and Majority functions, and two upper bounds via maximum entropy and Fourier-analytic / hypercontractivity arguments. We then extend our maximum entropy argument to give a lower bound on the guessing efficiency for a general channel with a binary uniform input, via the strong data-processing inequality constant of the reverse channel. We compute this bound for the binary erasure channel, and conjecture that Greedy Dictator functions achieve the guessing efficiency.

I. INTRODUCTION

In the classical problem of guessing, Alice wishes to learn the value of a discrete random variable (r.v.) $X$ as quickly as possible, by sequentially asking yes/no questions of the form “is $X = x$?”, until she gets it right. Alice’s guessing strategy, which is the ordering of the alphabet of $X$ according to which she states her guesses, induces a random guessing time. It is well known and simple to check that the optimal guessing strategy, which simultaneously minimizes all the positive moments of the guessing time, is to guess according to decreasing order of probability. Formally then, for any $s > 0$, the minimal $s$th-order guessing-time moment of $X$ is

$$G_s(X) := \mathbb{E}[\text{ORD}_X^s(x)],$$

where $\text{ORD}_X(x)$ returns the index of the symbol $x$ relative to the order induced by sorting the probabilities in a descending order, with ties broken arbitrarily. For brevity, we refer to $G_s(X)$ as the guessing-moment of $X$.

The guessing problem was first introduced and studied in an information-theoretic framework by Massey [1], who drew a relation between the average guessing time of an r.v. and its entropy, and was later explored more systematically by Arikan [2]. Several motivating problems for studying guesswork are fairness in betting games, computational complexity of lossy source coding and database search algorithms (see the introduction of [3] for a discussion), secrecy systems [4], [5], [6], crypt-analysis (password cracking) [7], [8], and computational complexity.

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of sequential decoding [2]. In [2], Arikan introduced the problem of guessing with side information, where Alice is in possession of another r.v. \( Y \) that is jointly distributed with \( X \). In that case, the optimal conditional guessing strategy is to guess by decreasing order of conditional probabilities, and hence the associated minimal conditional \( s \)-th-order guessing-time moment of \( X \) given \( Y \) is

\[
G_s(X|Y) := \mathbb{E} \left( \text{ORD}_{X|Y}(X \mid Y) \right),
\]

where \( \text{ORD}_{X|Y}(x \mid y) \) returns the index of \( x \) relative to the order induced by sorting the conditional probabilities of \( X \), given that \( Y = y \), in a descending order. Arikan showed that, as intuition suggests, side information reduces the guessing-moments [2, Corollary 1]

\[
G_s(X|Y) \leq G_s(X).
\]

Furthermore, he showed that if \( \{(X_i,Y_i)\}_{i=1}^n \) is an i.i.d. sequence, then [2, Prop. 5]

\[
\lim_{n \to \infty} \frac{1}{n} \log G_{1/s}^1(X^n|Y^n) = H_{1/s}(X_1 \mid Y_1),
\]

where \( H_\alpha(X \mid Y) \) is the Arimoto-Rényi conditional entropy of order \( \alpha \). The information-theoretic analysis of the guessing problem was further extended in multiple directions, such as allowing distortion in the guess [3], guessing under source uncertainty [9], improved bounds at finite blocklength [10] and an information-spectrum analysis [11], to name a few.

In the conditional setting described above, one may think of \( Y^n \) as side information observed by a "helper", say Bob, who then sends his observations to Alice. In this case, as in the problem of source coding with a helper [12], [13], it is more realistic to impose some communication constraints, and assume that Bob can only send a compressed description of \( Y^n \) to Alice. This question was recently addressed by Graczyk and Lapidoth [14], [15], who considered the case where Bob encodes \( Y^n \) at a positive rate using \( nR \) bits, before sending this description to Alice. They then characterized the best possible guessing-moments attained by Alice for general distributions, as a function of the rate \( R \). In this paper, we take this setting to its extreme, and attempt to quantify the value of a single bit in terms of reducing the guessing-moments, by allowing Bob to use only a one-bit description of \( Y^n \).

To that end, we define (in Section III) and study the guessing efficiency, which is the (asymptotically) best possible multiplicative reduction in the guessing-moments of \( X^n \) offered by observing a Boolean function \( f(Y^n) \), i.e., the minimal possible ratio \( G_s(X^n \mid f(Y^n))/G_s(X^n) \) as a function of \( s \), in the limit of large \( n \).

Characterizing the guessing efficiency appears to be a difficult problem in general. Here we mostly focus on the special case where \( X^n \) is uniformly distributed over the Boolean cube \( \{0,1\}^n \), and \( Y^n \) is obtained by passing \( X^n \) through a memoryless binary symmetric channel (BSC) with crossover probability \( \delta \). We derive two upper bounds and two lower bounds on the guessing efficiency in this case. The upper bounds, presented in Section III, are derived by analyzing the efficiency attained by two specific functions, Dictator and Majority. We show that neither of these functions is better than the other for all values of the moment order \( s \). The two lower bounds,
presented in Section IV are based on relating the guessing-moment to entropy using maximum-entropy arguments (generalizing [1]), and on Fourier-analytic techniques together with a hypercontractivity argument [16]. Several graphs illustrating the bounds are given in Section V. In Section VI we briefly discuss the more general case where $X^n$ is still uniform over the Boolean cube, but $Y^n$ is obtained from $X^n$ via a general binary-input, arbitrary-output channel. We generalize our entropy lower bound to this case using the strong data-processing inequality (SDPI) applied to the reverse channel (from $Y$ to $X$). We then discuss the case of the binary erasure channel (BEC), for which we also provide an upper bound by analyzing the Greedy Dictator function, namely where Bob sends the first bit that has not been erased. We conjecture that this function minimizes the guessing efficiency simultaneously at all erasure parameters and all moments $s$.

Related Work. Graczyk and Lapidoth [14], [15] considered the same guessing question in the case where Bob can communicate with Alice at some positive rate $R$, i.e., can use $nR$ bits to describe $Y^n$. This setup facilitates the use of large-deviation-based information-theoretic techniques, which allowed the authors to characterized the optimal reduction in the guessing-moments as a function of $R$. We note that this type of random-coding arguments cannot be applied in our external one-bit setup. Characterizing the guessing efficiency in the case of the BSC with a uniform input can also be thought of as a guessing variant of the most informative Boolean function problem introduced by Kumar and Courtade [17], who have asked about the maximal reduction in the entropy of $X^n$ obtainable by observing a Boolean function $f(Y^n)$. They have conjectured that a Dictator function, e.g. $f(y^n) = y_1$ is optimal simultaneously at all noise levels, see [18], [19], [20], [21] for some recent progress. We note that as in the guessing case, allowing Bob to describe $Y^n$ using $nR$ bits renders the problem amenable to an exact information-theoretic characterization [22]. In another related work [23], we have asked about the Boolean function $Y^n$ that maximizes the reduction in the sequential mean-squared prediction error of $X^n$, and have shown that the Majority function is optimal in the noiseless case, yet that there is no single function that is simultaneously optimal at all noise levels. Finally, in a recent work [24] the average guessing time using the help of a noisy version of $f(X^n)$, has been considered. By contrast, in this paper the noise is applied to the inputs of the function, rather than to its output.

II. PROBLEM STATEMENT

Let $X^n$ be an i.i.d. vector from distribution $P_X$, who is transmitted over a memoryless channel of conditional distribution $P_{Y|X}$. Bob observes $Y^n$ at the output of the channel, and can send one bit $f: \{0,1\}^n \rightarrow \{0,1\}$ to Alice, who in turn needs to guess $X^n$. Our goal is to characterize the best possible multiplicative reduction in guessing-moments offered by a function $f$, in the limit of large $n$. Precisely, we wish to characterize the guessing efficiency, defined as

$$\gamma_s(P_X, P_{Y|X}) := \limsup_{n \rightarrow \infty} \min_{f: \{0,1\}^n \rightarrow \{0,1\}} \frac{G_s(X^n | f(Y^n))}{G_s(X^n)}.$$ (5)

In this paper we are mostly interested in the case where $P_X = (1/2, 1/2)$, i.e., $X^n$ is uniformly distributed over $\{0,1\}^n$, and where the channel is a BSC with crossover probability $\delta \in [0, 1/2]$. With a slight abuse of notation,
we denote the guessing efficiency in this case by $\gamma_s(\delta)$. Before we proceed, we note the following simple facts.

**Proposition 1.** The following claims hold:

1) For $\gamma_s(\delta)$ the limit-supremum is a regular limit, achieved by a sequence of deterministic functions.

2) $\gamma_s(\delta)$ is a non-decreasing function of $\delta \in [0, 1/2]$ satisfying $\gamma_s(0) = 2^{-s}$ and $\gamma_s(1/2) = 1$, where $\gamma_s(0)$ is attained by any sequence of balanced functions.

Proof: See Appendix A.  

**III. Upper Bounds on $\gamma_s(\delta)$**

In this section we derive two upper bounds on the BSC guessing efficiency $\gamma_s(\delta)$, by analyzing two simple functions - the Dictator function and the Majority function. Let $a, b \in \mathbb{N}$, $a \leq b$ be given. The following sum will be useful for the derivations in the rest of the paper:

$$K_s(a, b) := \frac{1}{b - a} \sum_{i=a+1}^{b} i^s,$$

where we will abbreviate $K_s(b) := K_s(0, b)$. For a pair of sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ we will let $a_n \sim b_n$ mean $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

**Lemma 2.** Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be non-decreasing integer sequences such that $a_n < b_n$ for all $n$ and $\lim_{n\to\infty} (a_n + 1)/b_n = 0$. Then,

$$K_s(a_n, b_n) \sim \frac{2^n}{s+1}.$$  

Specifically, $G_s(X^n) = K_s(2^n) \approx \frac{2^n}{s+1}$.  

Proof: See Appendix A.  

**Theorem 3.** We have

$$\gamma_s(\delta) \leq (1 - 2\delta) \cdot 2^{-s} + 2\delta,$$

and guessing efficiency equal to the right-hand side (r.h.s.) can be achieved by a Dictator function.

Proof: Assume without loss of generality that $f(y^n) = y_1$. As $0 < \delta < 1/2$ it is easily verified that given $y_1$, the optimal guessing strategy is to first guess one of the $2^{n-1}$ vectors for which $x_1 = y_1$ (in an arbitrary order), and then guess one of the remaining $2^{n-1}$ vectors (again, in an arbitrary order). From symmetry, and Lemma 2

$$G_s(X^n \mid \text{Dict}(Y^n)) = G_s(X^n \mid Y_1 = 1)$$  

$$= (1 - \delta) \cdot K_s(2^{n-1}) + \delta \cdot K_s(2^{n-1}, 2^n)$$  

$$\approx (1 - \delta) \cdot \frac{2^{s(n-1)}}{s+1} + \delta \cdot \frac{2^{s(n-1)}}{s+1} \cdot (2^{s+1} - 1)$$  

$$= \frac{2^{s(n-1)}}{s+1} \cdot (1 - 2\delta + \delta \cdot 2^{s+1}).$$
The result then follows from (5) and Lemma 2.

We next consider the guessing efficiency of the Majority function.

**Theorem 4.** Let \( \beta := \frac{1-2\delta}{\sqrt{4\delta(1-\delta)}} \) and \( Z \sim \mathcal{N}(0,1) \). Then,

\[
\gamma_s(\delta) \leq 2 \cdot (s+1) \cdot \mathbb{E} \left[ Q(\beta Z) \cdot (1 - Q(Z))^s \right],
\]

where \( Q(\cdot) \) is the tail distribution function of the standard normal distribution, and guessing efficiency equal to the r.h.s. of (13) can be achieved by the Majority function.

**Proof:** We assume for simplicity that \( n \) is odd. The analysis for an even \( n \) is not fundamentally different. In this case, \( f(y^n) = \text{Maj}(y^n) = 1(\sum_{i=1}^{n} y_i > n/2) \), where \( 1(\cdot) \) is the indicator function. To evaluate the guessing-moment, we first need to find the optimal guessing strategy. To this end, we let \( W_H(x^n) \) be the Hamming weight of \( x^n \) and note that the posterior probability is given by

\[
\Pr(X^n = x^n \mid \text{Maj}(Y^n) = 1) = \frac{\Pr(\text{Maj}(Y^n) = 1 \mid X^n = x^n) \cdot \Pr(X^n = x^n)}{\Pr(\text{Maj}(Y^n) = 1)}
\]

\[
= 2^{1-n} \cdot \Pr \left( \sum_{i=1}^{n} Y_i > n/2 \mid X^n = x^n \right)
\]

\[
= 2^{1-n} \cdot \Pr \left( \sum_{i=1}^{n} Y_i > n/2 \mid W_H(X^n) = W_H(x^n) \right)
\]

\[
:= 2^{1-n} \cdot r_n(W_H(x^n)),
\]

where (16) follows from symmetry. Evidently, \( r_n(w) \) is an increasing function of \( w \in \{0,1,\ldots,n\} \). Indeed, \( \text{Bin}(n, \delta) \) be a binomial r.v. of \( n \) trials and success probability \( \delta \). Then, for any \( w \leq n-1 \), as \( \delta \leq 1/2 \),

\[
r_n(w + 1) \]

\[
= \Pr \left( \text{Bin}(w+1, 1-\delta) + \text{Bin}(n-w-1, \delta) > n/2 \right)
\]

\[
= \Pr \left( \text{Bin}(w, 1-\delta) + \text{Bin}(1, 1-\delta) + \text{Bin}(n-w-1, \delta) > n/2 \right)
\]

\[
\geq \Pr \left( \text{Bin}(w, 1-\delta) + \text{Bin}(1, \delta) + \text{Bin}(n-w-1, \delta) > n/2 \right)
\]

\[
= \Pr \left( \text{Bin}(w, 1-\delta) + \text{Bin}(n-w, \delta) > n/2 \right)
\]

\[
= r_n(w),
\]

where in each of the above probabilities, the summations is over independent binomial r.v.'s. Hence, we deduce that whenever \( \text{Maj}(Y^n) = 1 \) (resp. \( \text{Maj}(Y^n) = 0 \)) the optimal guessing strategy is by decreasing (resp. increasing) Hamming weight (with arbitrary order for inputs of equal Hamming weight).

We can now turn to evaluate the guessing-moment for the optimal strategy given Majority. Let \( M_{-1} = 0 \) and
\( M_w = M_{w-1} + \binom{n}{w} \) for \( w \in \{0, 1, \ldots, n\} \). From symmetry,

\[
G_s(X^n \mid \text{Maj}(Y^n)) = G_s(X^n \mid \text{Maj}(Y^n) = 1) = \sum_{w=0}^{n} \binom{n}{w} 2^{1-n} r_n(w) \sum_{i=M_{w-1}+1}^{M_w} i^s. \tag{24}
\]

Thus,

\[
G_s(X^n \mid \text{Maj}(Y^n)) \geq \sum_{w=0}^{n} \binom{n}{w} 2^{1-n} r_n(w) M_w^s - \sum_{w=0}^{n} \binom{n}{w} 2^{1-n} r_n(w) M_{w-1}^s \tag{25}
\]

where \( W, W' \sim \text{Bin}(n, 1/2) \) and independent. We next evaluate this expression using the central-limit theorem. To evaluate this expression asymptotically, we note that the Berry-Esseen theorem [25, Chapter XVI.5, Theorem 2] leads to (see, e.g., [23, proof of Lemma 15])

\[
r_n(w) = Q\left( \beta \cdot \frac{2}{\sqrt{n}} \left( \frac{n}{2} - w \right) \right) + a_{\delta} \frac{1}{\sqrt{n}}, \tag{28}
\]

for some universal constant \( a_{\delta} \). Using the Berry-Esseen central-limit theorem again, we have that \( \frac{2}{\sqrt{n}} (\frac{n}{2} - W') \xrightarrow{d} Z \), where \( Z \sim N(0, 1) \). Thus for a given \( w \),

\[
\Pr (W' \leq w - 1) = 1 - \Pr \left( \frac{2}{\sqrt{n}} (\frac{n}{2} - W') \geq \frac{2}{\sqrt{n}} (\frac{n}{2} - w - 1) \right) = 1 - Q \left( \frac{2}{\sqrt{n}} (\frac{n}{2} - w) \right) - a_{1/2} \frac{1}{\sqrt{n}}, \tag{29}
\]

where the last equality follows from the fact that \( |Q'(t)| \leq \frac{1}{\sqrt{2\pi}} \). Using the Berry-Esseen theorem once again, we have that \( \frac{2}{\sqrt{n}} (\frac{n}{2} - W') \xrightarrow{d} Z \), where \( Z \sim N(0, 1) \). Hence, Portmanteau’s lemma (e.g. [25, Chapter VIII.1, Theorem 1]), and the fact the \( Q(t) \) is continuous and bounded results in

\[
G_s(X^n \mid \text{Maj}(Y^n)) \geq 2^{sn+1} \cdot \mathbb{E} [Q(\beta N) \cdot (1 - Q(N))^s] + O \left( \frac{1}{n^{s/2}} \right). \tag{32}
\]

Similarly to (25), the upper bound

\[
G_s(X^n \mid \text{Maj}(Y^n)) \leq \sum_{w=0}^{n} \binom{n}{w} 2^{1-n} r_n(w) M_w^s, \tag{33}
\]

holds, and a similar analysis leads to an expression which asymptotically coincides with the r.h.s. of (32). The result then follows from (5) and Lemma 2.

We remark that the guessing efficiency of functions similar to Dictator and Majority, such as Dictator on \( k > 1 \)
inputs \((f(y^n) = 1\) if and only if \((y_1, \ldots, y_k) = 1^k)\), or unbalanced Majority \((f(y^n) = 1(\sum_{i=1}^n y_i > t)\) for some \(t)\) may also be analyzed in a similar way. However, numerical computations indicate that they do not improve the bounds obtained in Theorems [3] and [4] and thus their analysis is omitted.

IV. LOWER BOUNDS ON \(\gamma_s(\delta)\)

We derive two lower bounds on the BSC guessing efficiency \(\gamma_s(\delta)\), one based on maximum-entropy arguments, and the other based on Fourier-analytic arguments.

A. A Maximum-Entropy Bound

**Theorem 5.** We have

\[
\gamma_s(\delta) \geq e^{-1} \cdot \frac{s^{s-1} \cdot (s + 1)}{\Gamma(s)} \cdot 2^{-s(1-2\delta)^2}. \tag{34}
\]

**Proof:** With a standard abuse of notation, let us write the guessing-moment and the entropy as a function of the distribution. Consider the following maximum entropy problem [26, Ch. 12]

\[
\max_{P: G_s(P) = g} H(P), \tag{35}
\]

where it should be noted that the support of \(P\) is only restricted to be countable. Assuming momentarily that the entropy is measured in nats, it is easily verified (using the theory of exponential families [27, Ch. 3] or by standard Lagrangian duality [28, Ch. 5]), that the entropy maximizing distribution is

\[
P_\lambda(i) := \frac{\exp(-\lambda i^s)}{Z(\lambda)} \tag{36}
\]

for \(i \in \mathbb{N}_+\) where \(Z(\lambda) := \sum_{i=1}^{\infty} \exp(-\lambda i^s)\) is the partition function, and \(\lambda > 0\) is chosen such that \(G_s(P_\lambda) = g\). Thus, the resulting maximum entropy is given in a parametric form as

\[
H(P_\lambda) = \lambda G_s(P_\lambda) + \ln Z(\lambda). \tag{37}
\]

Evidently, if \(g = G_s(P_\lambda) \to \infty\) then \(\lambda \to 0\). In this case, we may approximate the partition function for \(\lambda \to 0\) by a Riemann integral. Specifically, by the monotonicity of \(e^{-\lambda i^s}\) in \(i \in \mathbb{N}\),

\[
Z(\lambda) = \sum_{i=1}^{\infty} e^{-\lambda i^s} \tag{38}
\]

\[
= \frac{1}{2} \left( \sum_{i=-\infty}^{\infty} \exp \left(- \left( \frac{|i|}{\lambda^{1/s}} \right)^s \right) - 1 \right) \tag{39}
\]

\[
\geq \frac{1}{2} \left( \int_{-\infty}^{\infty} \exp \left(- \left( \frac{|t|}{\lambda^{1/s}} \right)^s \right) dt - 1 \right) \tag{40}
\]

\[
= \frac{1}{s} \lambda^{-1/s} \cdot \Gamma \left( \frac{1}{s} \right) - \frac{1}{2}, \tag{41}
\]

\[\]
where we have used the definition of the Gamma function $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$ in the last equality\footnote{Further, by the convexity of $e^{-\lambda t}$ in $t \in \mathbb{R}$,}
\[
Z(\lambda) \leq \frac{1}{2} \left( \int_{-\infty}^{\infty} \exp \left( - \left( \frac{|t|}{\lambda^{1/s}} \right)^s \right) dt - 1 + e^{-\lambda} \right). \tag{42}
\]
Therefore
\[
Z(\lambda) = (1 + a_\lambda) \cdot \frac{1}{s} \lambda^{-1/s} \cdot \Gamma \left( \frac{1}{s} \right) \tag{43}
\]
where $a_\lambda \to 0$ as $\lambda \to 0$. In the same spirit,
\[
G_s(P_\lambda) = \sum_{i=1}^{\infty} t^s \cdot \frac{\exp(-\lambda t^s)}{Z(\lambda)} \tag{44}
\]

\[
= \int_0^\infty t^s \exp \left( - \left( \frac{|t|}{\lambda^{1/s}} \right)^s \right) dt + b_\lambda \tag{45}
\]

\[
= \frac{1}{s} \lambda^{-\frac{s}{s+1}} \cdot \Gamma \left( \frac{s+1}{s} \right) + b_\lambda \tag{46}
\]

\[
= \frac{1}{s} \lambda^{-\frac{s}{s+1}} \cdot \Gamma \left( \frac{1}{s} \right) + b_\lambda \tag{47}
\]

\[
= \frac{1}{s} \lambda^{-\frac{s}{s+1}} \cdot \Gamma \left( \frac{1}{s} \right) + \frac{1}{s} \lambda \cdot (1 + c_\lambda), \tag{48}
\]
where in (46), $b_\lambda \to 0$ as $\lambda \to 0$, in (47) we have used the identity $\Gamma(t+1) = t\Gamma(t)$ for $t \in \mathbb{R}_+$, and in (48), $c_\lambda \to 0$ as $\lambda \to 0$.

Returning to measure entropy in bits, we thus obtain that for any distribution $P$
\[
H(P) \leq \log \left( \frac{e^{1/s}}{s^{1/s}} \cdot G_s^{1/s}(P) \cdot \Gamma \left( \frac{1}{s} \right) \right) + o(1), \tag{49}
\]
and so
\[
G_s(P) \geq \Psi_s \cdot 2^{sH(P)} \cdot (1 + o(1)), \tag{50}
\]
where $\Psi_s := e^{-1} \cdot \frac{s^{\lambda^{-1}}}{\Gamma(\frac{1}{s})}$ and $o(1)$ is a vanishing term as $G_s(P) \to \infty$. In the same spirit, (50) holds whenever $H(P) \to \infty$.

We return to the Boolean helper problem. Using (50) once for the guessing-moment conditioning on $f(Y^n) = 0$, and once on $f(Y^n) = 1$ we get (see a detailed justification to (51) in Appendix A)
\[
G_s(X^n \mid f(Y^n)) \geq k_n \cdot \Psi_s \cdot \left[ \Pr(f(Y^n) = 0) \cdot 2^{sH(X^n \mid f(Y^n)=0)} + \Pr(f(Y^n) = 1) \cdot 2^{sH(X^n \mid f(Y^n)=1)} \right] \tag{51}
\]

\[
\geq k_n \cdot \Psi_s \cdot 2^{sH(X^n \mid f(Y^n))} \tag{52}
\]

\[
\geq k_n \cdot \Psi_s \cdot 2^{sH(X^n \mid f(Y^n))} \tag{53}
\]
\footnote{It can also be obtained by identifying the integral as an unnormalized generalized Gaussian distribution of zero mean, scale parameter $\lambda^{-1/s}$ and shape parameter $s$ \cite{29}.}
where $k_n \approx 1$ in (51), and (52) follows from Jensen’s inequality. The bound (53) is directly related to the Boolean function conjecture [17], and may be proved in several ways, e.g., using Mrs. Gerber’s Lemma [30 Th. 1], see [31 Section IV][22], [18].

Remark 6. In [1] the maximum-entropy problem was studied for $s = 1$. In this case, the maximum-entropy distribution is readily identified as the geometric distribution. The proof above generalizes that result to any $s > 0$.

Remark 7. In [18], the bound $H(X^n|f(Y^n)) \geq n - (1 - 2\delta)^2$ used in the proof of Theorem 5 above (see (53)) was improved for balanced functions, assuming $1/2(1 - 1/\sqrt{3}) \leq \delta \leq 1/2$. Using it here leads to an immediate improvement in the bound of Theorem 5. Furthermore, it is known [19] that there exists $\delta_0$ such that the most informative Boolean function conjecture holds for all $\delta_0 \leq \delta \leq 1/2$. For such crossover probabilities,

$$H(X^n|f(Y^n)) \geq n - 1 + h(\delta)$$

holds, and then Theorem 5 may be improved to

$$\gamma_s(\delta) \geq e^{-1} \cdot \frac{s^{s-1} \cdot (s + 1)}{\Gamma^s\left(\frac{1}{s}\right)} \cdot 2^{-s(1-h(\delta))}.$$ (55)

B. A Fourier-Analytic Bound

The second bound is based on Fourier analysis of Boolean functions [16], and so we briefly remind the reader of the basic definitions and results. To that end, it is convenient to assume that the binary alphabet is $\{-1, 1\}$ instead of $\{0, 1\}$. An inner product between two real-valued functions on the Boolean cube $f, g : \{-1, 1\}^n \rightarrow \mathbb{R}$ is defined as

$$\langle f, g \rangle := \mathbb{E}(f(X^n)g(X^n)),$$ (56)

where $X^n \in \{-1, 1\}^n$ is a uniform Bernoulli vector. A character associated with a set of coordinates $S \subseteq [n] := \{1, 2, \ldots, n\}$ is the Boolean function $x^S := \prod_{i \in S} x_i$, where by convention $x^\emptyset = 1$. It can be shown [16 Chapter 1] that the set of all characters forms an orthonormal basis with respect to the inner product (56). Furthermore,

$$f(x^n) = \sum_{S \subseteq [n]} \hat{f}_S \cdot x^S,$$ (57)

where $\{\hat{f}_S\}_{S \subseteq [n]}$ are the Fourier coefficients of $f$, given by $\hat{f}_S = \langle x^S, f \rangle = \mathbb{E}(X^S \cdot f(X^n))$. Plancharel’s identity then states that $\langle f, g \rangle = \mathbb{E}(f(X^n)g(X^n)) = \sum_{S \subseteq [n]} \hat{f}_S \hat{g}_S$. The $p$ norm of a function $f$ is defined as $\|f\|_p := [\mathbb{E}|f(X^n)|^p]^{1/p}$.

Letting the correlation parameter be defined as $\rho := 1 - 2\delta$, the noise operator is defined to be

$$T_\rho f(x^n) = \mathbb{E}(f(Y^n) | X^n = x^n).$$ (58)

The noise operator has a smoothing effect on the function, which is captured by the so-called hypercontractivity theorems. Specifically, we shall use the following version.
Theorem 8 (Theorem 8). Let \( f : \{-1, 1\}^n \to \mathbb{R} \) and \( 0 \leq \rho \leq 1 \). Then, \( \|T_\rho f\|_2 \leq \|f\|_{\rho^2+1} \).

Our Fourier-based bound is as follows:

Theorem 9. We have

\[
\gamma_s(\delta) \geq \max_{0 \leq \lambda \leq 1} \left[ 1 - \frac{(s + 1) \cdot (1 - 2\delta) \lambda}{\left[(1 + (1 - 2\delta)^2(1-\lambda))s + 1\right]^{\frac{1}{1 + (1 - 2\delta)^2}}} \right].
\] (59)

This bound can be weakened by the possibly sub-optimal choice \( \lambda = 1 \), which leads to the simpler and explicit bound:

Corollary 10. We have

\[
\gamma_s(\delta) \geq 1 - \frac{(s + 1) \cdot (1 - 2\delta)}{\sqrt{1 + 2s}}.
\] (60)

**Proof of Theorem 9** From Bayes law (recall that \( X_i \in \{-1, 1\} \))

\[
\Pr(X^n = x^n \mid f(Y^n) = b) = 2^{-(n+1)} \cdot \frac{1 + bT_\rho f(x^n)}{\Pr(f(Y^n) = b)},
\] (61)

and from the law of total expectation

\[
G_s(X^n \mid f(Y^n)) = \Pr(f(Y^n) = 1) \cdot G_s(X^n \mid f(Y^n) = 1) + \Pr(f(Y^n) = -1) \cdot G_s(X^n \mid f(Y^n) = -1).
\] (62)

Letting \( \hat{f}_\phi = \mathbb{E}f(X^n) \), and defining \( g = f - \hat{f}_\phi \), the first addend on the r.h.s. of (62) is given by

\[
\Pr(f(Y^n) = 1) \cdot G_s(X^n \mid f(Y^n) = 1) = 2^{-(n+1)} \sum_{x^n} \left(1 + \hat{f}_\phi + T_\rho g(x^n)\right) \cdot \text{ORD}^s_{T_\rho g}(x^n)
\] (63)

\[
= \frac{1 + \hat{f}_\phi}{2} \cdot \mathbb{E}\left(\text{ORD}^s_{T_\rho g}(X^n)\right) + \frac{1}{2}(T_\rho g, \text{ORD}^s_{T_\rho g})
\] (64)

\[
= \frac{1 + \hat{f}_\phi}{2} \cdot K_s(2^n) + \frac{1}{2}(T_\rho g, \text{ORD}^s_{T_\rho g})
\] (65)

\[
= \frac{1 + \hat{f}_\phi}{2} \cdot \ell_n \cdot \frac{2^{sn}}{s + 1} + \frac{1}{2}(T_\rho g, \text{ORD}^s_{T_\rho g}),
\] (66)

where in the last equality, \( \ell_n \approx 1 \) (Lemma 2). Let \( \lambda \in [0, 1] \), and denote \( \rho_1 := \rho^\lambda \) and \( \rho_2 = \rho^{1-\lambda} \). Then, the inner-product term in (66) may be upper bounded as

\[
\left| \langle T_\rho g, \text{ORD}^s_{T_\rho g} \rangle \right| = \left| \langle T_{\rho_1} g, T_{\rho_2} \text{ORD}^s_{T_{\rho_2} g} \rangle \right|
\] (67)

\[
\leq \|T_{\rho_1} g\|_2 \cdot \|T_{\rho_2} \text{ORD}^s_{T_{\rho_2} g}\|_2
\] (68)

\[
= \rho_1 \cdot \sqrt{1 - \hat{f}_\phi^2} \cdot \|T_{\rho_2} \text{ORD}^s_{T_{\rho_2} g}\|_2
\] (69)

\[
\leq \rho_1 \cdot \sqrt{1 - \hat{f}_\phi^2} \cdot \|\text{ORD}^s_{T_{\rho_2} g}\|_1 + \rho_2^2
\] (70)

\[
= \rho_1 \cdot \sqrt{1 - \hat{f}_\phi^2} \cdot \left(K_{\rho_2}(2^n)\right)^{1/(1 + \rho_2^2)}
\] (71)

\[
= \rho_1 \cdot \sqrt{1 - \hat{f}_\phi^2} \cdot \left(K_{\rho_2}(2^n)\right)^{1/(1 + \rho_2^2)} \cdot 2^{sn},
\] (72)
where (67) holds since $T_\rho$ is a self-adjoint operator, (68) follows from the Cauchy–Schwarz inequality. For (69) note that

$$||T_\rho g||^2_2 = \langle T_\rho g, T_\rho g \rangle$$

(73)

$$= \sum_{S \in [n]} \rho^{2|S|} \hat{g}_S^2$$

(74)

$$= \sum_{S \in [n] \setminus \phi} \rho^{2|S|} \hat{f}_S^2$$

(75)

$$\leq \rho^2 \cdot (1 - \hat{f}_\phi^2),$$

(76)

where (74) follows from Plancharel’s identity, (75) is since $\hat{g}_S = \hat{f}_S$ for all $S \neq \phi$, and $\hat{g}_\phi = 0$, and (76) follows from $\sum_{S \in [n]} \hat{f}_S^2 = ||f||^2 = Ef^2 = 1$. Equation (70) follows from Theorem 8 and in (72), $k_n \approx 1$. The second addend on the r.h.s. of (62) can be bounded in the same manner. Hence,

$$G_s(X^n | f(Y^n)) \geq \max_{0 \leq \lambda \leq 1} 2^{sn} \cdot \left[ \ell_n \cdot \frac{1}{s + 1} - \rho^\lambda \cdot \sqrt{1 - \hat{f}_\phi^2} \cdot \left[ k_n \cdot \frac{1}{(1 + \rho^{2(1-\lambda)})s + 1} \right]^{1/(1 + \rho^{2(1-\lambda)})} \right]$$

(77)

$$\geq \max_{0 \leq \lambda \leq 1} 2^{sn} \cdot \left[ \ell_n \cdot \frac{1}{s + 1} - \rho^\lambda \left[ k_n \cdot \frac{1}{(1 + \rho^{2(1-\lambda)})s + 1} \right]^{1/(1 + \rho^{2(1-\lambda)})} \right]$$

(78)

$$\rightarrow 2^{sn} \cdot \max_{0 \leq \lambda \leq 1} \left[ \frac{1}{s + 1} - \rho^\lambda \cdot \left[ (1 + \rho^{2(1-\lambda)})s + 1 \right]^{1/(1 + \rho^{2(1-\lambda)})} \right]$$

(79)

as $n \rightarrow \infty$. 

Figure 1. Bounds on $\gamma_s(\delta)$ for $s = 1$ (left) and $s = 5$ (right) and varying $\delta \in [0, 1/2]$. 

[Figure 1: Graphs showing bounds on $\gamma_s(\delta)$ for $s = 1$ and $s = 5$]
In Fig. 1 (resp. Fig. 2) the bounds on $\gamma_s(\delta)$ are plotted for fixed values of $s$ (resp. $\delta$). As for upper bounds, it can be found that when $s \lesssim 3.5$ Dictator dominates Majority (for all values of $\delta$), whereas for $s \gtrsim 4.25$ Majority dominates Dictator. For $3.5 \lesssim s \lesssim 4.25$ there exists $\delta'_s$ such that Majority is better for $\delta \in (0, \delta'_s)$ and Dictator is better for $\delta \in (\delta'_s, 1/2)$. Fig. 2 demonstrates the switch from Dictator to Majority as $s$ increases (depending on $\delta$).

As for lower bounds, first note that the conjectured maximum-entropy bound (55) was also plotted (see Remark 7). It can be observed that the maximum-entropy bound is better for low values of $\delta$, whereas the Fourier analysis bound is better for high values. As a function of $s$, it seems that the maximum-entropy bound (resp. Fourier-analysis bound) is better for high (resp. low) values of $s$. Finally, we mention that the maximizing parameter in the Fourier-based bound (Theorem 9) is $\lambda = 1$, and the resulting bound is as in (60). For values of $s$ as low as 10, the maximizing $\lambda$ may be far from 1, and in fact it continuously and monotonically increases from 0 to 1 as $\delta$ increases from 0 to 1/2.

VI. GUESSING EFFICIENCY FOR A GENERAL BINARY INPUT CHANNEL

In this section, we consider the guessing efficiency for general channels with a uniform binary input. The lower bound of Theorem 5 can be easily generalized for this case. To that end, consider the SDPI constant [32], [33] of the reverse channel $(P_Y, P_{X|Y})$, given by

$$\eta(P_Y, P_{X|Y}) := \sup_{Q_Y: Q_Y \neq P_Y} \frac{D(Q_X||P_X)}{D(Q_Y||P_Y)},$$

(80)
where $Q_X$ is the $X$-marginal of $Q_Y \circ P_{X|Y}$. As was shown in [34, Th. 2], the SDPI constant of $(P_Y, P_{X|Y})$ is also given by

$$\eta(P_Y, P_{Y|X}) = \sup_{P_W:W-Y-X, I(W;Y) > 0} \frac{I(W;X)}{I(W;Y)}. \tag{81}$$

**Theorem 11.** We have

$$\gamma_s(P_X, P_{Y|X}) \geq e^{-1} \cdot \frac{s^{s-1} \cdot (s + 1)}{\Gamma^s \left(\frac{1}{s}\right)} \cdot 2^{-s} \eta(P_Y, P_{Y|X}). \tag{82}$$

**Proof:** The proof follows the same lines of the proof of Theorem 5, up to (52), yielding

$$G_s(X^n | f(Y^n)) \geq k_n \cdot \Psi_s \cdot 2^{s[n - I(X^n;f(Y^n))]. \tag{83}$$

Now, let $W^{(n)}$ be such that $X^n - Y^n - W^{(n)}$ forms a Markov chain. Then,

$$\sup_{f:Y^n \rightarrow \{0,1\}} \frac{I(X^n;f(Y^n))}{I(Y^n;f(Y^n))} \leq \sup_{P_W:W^n|Y^n} \frac{I(X^n;W^{(n)})}{I(Y^n;W^{(n)})} \tag{84}$$

$$= \eta(P_Y^n, P_{X^n|Y^n}) \tag{85}$$

$$= \eta(P_Y, P_{X|Y}), \tag{86}$$

where (86) follows since the SDPI constant tensorizes (see [34] for an argument obtained by relating the SDPI constant to the hypercontractivity parameter, or [35, p. 5] for a direct proof). Thus, for all $f$

$$I(X^n;f(Y^n)) \leq \eta(P_Y, P_{X|Y}) \cdot I(Y^n;f(Y^n)) \tag{87}$$

$$\leq \eta(P_Y, P_{X|Y}) \cdot H(f(Y^n)) \tag{88}$$

$$\leq \eta(P_Y, P_{X|Y}). \tag{89}$$

Inserting (89) to (83). Therefore

$$G_s(X^n | f(Y^n)) \geq k_n \cdot \Psi_s \cdot 2^{s[n - \eta(P_Y, P_{X|Y})]}, \tag{90}$$

and using the definition of the guessing efficiency (5) completes the proof.

**Remark 12.** It is evident from (88) that if the helper is allowed to send $k$ bits, then the associated $k$-bit guessing efficiency is lower bounded by

$$\gamma_s(k)(\delta) \geq e^{-1} \cdot \frac{s^{s-1} \cdot (s + 1)}{\Gamma^s \left(\frac{1}{s}\right)} \cdot 2^{-s} \cdot k \cdot \eta(P_Y, P_{X|Y}). \tag{91}$$

**Remark 13.** The bound for the BSC case (Theorem 5) is indeed a special case of Theorem 11 as the reverse BSC channel is also a BSC with uniform input and the same crossover probability. For BSCs, it is well known that the SDPI constant is $(1 - 2\delta)^2 \tag{32}$ Th. 9].

Next, we consider in more detail the case where the observation channel is a BEC.
Figure 3. Bounds on $\gamma_s(\delta)$ for $s = 1$ varying $\epsilon \in [0, 1]$.

A. Binary Erasure Channel

Suppose that $Y^n \in \{0, 1, e\}^n$ is obtained from $X^n$ by erasing each bit independently with probability $\epsilon \in [0, 1]$. As before, Bob observes the channel output $Y^n$ and can send one bit $f : \{0, 1, e\}^n \rightarrow \{0, 1\}$ to Alice, who wishes to guess $X^n$. With a slight abuse of notation, the guessing efficiency (5) will be denoted by $\gamma_s(\epsilon)$.

To compute the lower bound of Theorem 11, we need to find the SDP I constant associated with the reverse channel, which is easily verified to be

$$P_{X|Y=y} = \begin{cases} y, & y = 0 \text{ or } y = 1 \\ \text{Ber}(1/2), & y = e \end{cases} \quad (92)$$

with an input distribution $P_Y = (\frac{1-\epsilon}{2}, \epsilon, \frac{1-\epsilon}{2})$. Letting $Q_Y(y) = q_y$ for $y \in \{0, 1, e\}$ yields $Q_X(x) = q_x + \frac{q_e}{2}$ for $x \in \{0, 1\}$. The computation of $\eta(P_Y, P_{X|Y})$ is now a simple three-dimensional constrained optimization problem. We plotted the resulting lower bound for $s = 1$ in Fig. 3.

Let us now turn to upper bounds, and focus for simplicity on the average guessing time, i.e., the guessing-moment for $s = 1$. To begin, let $S$ represent the set of indices of the symbols that were not erased, i.e., $i \in S$ if and only if $Y_i \neq e$. Any function $f : \{0, 1, e\}^n \rightarrow \{0, 1\}$ is then uniquely associated with a set of Boolean functions $\{f_S\}_{S \subseteq [n]}$, where $f_S : \{0, 1\}^{|S|} \rightarrow \{0, 1\}$ designates the operation of the function when $S$ is the set of non-erased symbols. We also let $\Pr(S) = (1 - \epsilon)^{|S|} \cdot \epsilon^{|S^c|}$ be the probability that the non-erased symbols have index set $S$. Then, the joint probability distribution is given by

$$\Pr(X^n = x^n, f(Y^n) = 1) = \Pr(X^n = x^n) \cdot \Pr(f(Y^n) = 1 | X^n = x^n)$$

$$= 2^{-n} \cdot \sum_{S \subseteq [n]} \Pr(S) \cdot \Pr(f(Y^n) = 1 | X^n = x^n, S) \quad (94)$$
\[ = 2^{-n} \cdot \sum_{S \subseteq [n]} \Pr(S) \cdot f_S(x^n), \quad (95) \]

and, similarly,
\[ \Pr(X^n = x^n, f(Y^n) = 0) = 2^{-n} \cdot \sum_{S \subseteq [n]} \Pr(S) \cdot (1 - f_S(x^n)) \]
\[ = 2^{-n} - 2^{-n} \cdot \sum_{S \subseteq [n]} \Pr(S) \cdot f_S(x^n). \quad (96) \]

Interestingly, for any given \( f \), the optimal guessing order given that \( f(Y^n) = 0 \) is reversed to the optimal guessing order when \( f(Y^n) = 1 \). Also apparent is that the posterior probability is determined by a mixture of \( 2^n \) different Boolean functions \( \{f_S\}_{S \subseteq [n]} \). This may be contrasted with the BSC case, in which the posterior is determined by a single Boolean function (though with noisy input).

A seemingly natural choice is a \textit{Greedy Dictator} function, for which \( f(Y^n) \) sends the first non-erased bit. Concretely, letting
\[ k(y^n) := \begin{cases} 
  n + 1, & y^n = e^n \\
  \min \{i : y_i \neq e\}, & \text{otherwise}
\end{cases}, \quad (98) \]

the \textit{Greedy Dictator} is defined by
\[ \text{G-Dict}(y^n) := \begin{cases} 
  \text{Ber}(1/2), & y^n = e^n \\
  y_k(y^n), & \text{otherwise}
\end{cases}, \quad (99) \]

where \( \text{Ber}(\alpha) \) is a Bernoulli r.v. of success probability \( \alpha \). From an analysis of the posterior probability (see Appendix A), it is evident that conditioned on \( f(Y^n) = 0 \), an optimal guessing order must satisfy that \( x^n \) is guessed before \( z^n \) whenever
\[ \sum_{i=1}^{n} \epsilon^{i-1} \cdot x_i \leq \sum_{i=1}^{n} \epsilon^{i-1} \cdot z_i, \quad (100) \]

which can be loosely thought of as comparing the "base 1/\epsilon expansion" of \( x^n \) and \( z^n \). Furthermore, when \( \epsilon \) is close to 1, then the optimal guessing order tends toward a \textit{minimum Hamming weight} rule (of maximum in case \( f = 1 \)).

The \textit{Greedy Dictator} function is "local optimal" when \( \epsilon \in [0, 1/2] \), in the following sense:

\textbf{Proposition 14.} If \( \epsilon \in [0, 1/2] \) then an optimal guessing order conditioning on \( \text{G-Dict}(Y^n) = 0 \) (resp. \( \text{G-Dict}(Y^n) = 1 \)) is lexicographic (reverse lexicographic). Also, given lexicographic (reverse lexicographic) order when the received bit is 0 (resp. 1), the optimal function \( f \) is \textit{Greedy Dictator}.

\textit{Proof:} See Appendix A.

The guessing efficiency of the \textit{Greedy Dictator} for \( s = 1 \) can be evaluated, and the analysis leads to the following upper bound:
Theorem 15. We have
\[ \gamma_1(\epsilon) \leq \frac{1}{2 - \epsilon}, \]  
(101)

and the r.h.s. above is achieved with equality by the Greedy Dictator for \( \epsilon \in [0, 1/2] \).

Proof: See Appendix A. □

The upper bound of Theorem 15 is plotted in Fig. 3. Based on Proposition 14 and numerical computations for moderate values of \( n \) we conjecture:

Conjecture 16. Greedy Dictator functions attain \( \gamma_s(\epsilon) \) for the BEC.

Supporting evidence for this conjecture includes the local optimality of Proposition 14 (although there are other locally optimal choices), as well as the following heuristic argument: Intuitively, Bob should reveal as much as possible regarding the bits he has seen and as little as possible regarding the erasure pattern. So, it seems reasonable to find a smallest possible set of balanced functions from which to choose all the functions \( f_S \), so that they coincide as much as possible. Greedy Dictator is a greedy solution to this problem: it uses the function \( x_1 \) for half of the erasure patterns, which is the maximum possible. Then, it uses the function \( x_2 \) for half of the remaining patterns, and so on. Indeed, we were not able to find a better function than G-Dict for small values of \( n \).

However, applying standard techniques in attempt to prove Conjecture 16 has not been fruitful. One possible technique is induction. For example, assume that the optimal functions for dimension \( n - 1 \) are \( f_S^{(n-1)} \). Then, it might be perceived that there exists a bit, say \( x_1 \), such that the optimal functions for dimension \( n \) satisfy \( f_S^{(n)} = f_S^{(n-1)} \) if \( x_1 \) is erased; in that case, it remains only to determine \( f_S^{(n)} \) when \( x_1 \) is not erased. However, observing (95), it is apparent that the optimal choice of \( f_S^{(n)} \) should satisfy two contradicting goals – on the one hand, to match the order induced by
\[ \sum_{S \subseteq \{n\}: 1 \notin S} \Pr(S) \cdot f_S(x^n) \]  
(102)
and on the other hand, to minimize the average guessing time of
\[ \sum_{S \subseteq \{n\}: 1 \in S} \Pr(S) \cdot f_S(x^n). \]  
(103)

It is easy to see that taking a greedy approach toward satisfying the second goal would result in \( f_S^{(n)}(x^n) = x_1 \) if \( 1 \in S \), and performing the recursion steps would indeed lead to a Greedy Dictator function. Interestingly, taking a greedy approach toward satisfying the first goal would also lead to a Greedy Dictator function, but one which operates on a cyclic permutation of the inputs (specifically, (99) applied to \((y_2^n, y_1)\)). Nonetheless, it is not clear that choosing \( \{f_S^{(n)}\}_{S:1 \in S} \) with some loss in the average guessing time induced by (103) could not lead to a gain in the second goal (matching the order of (102)) which outweighs that loss.

Another possible technique is majorization. It is well known that if one probability distribution majorizes another, then all the non-negative guessing-moments of the first are no greater than the corresponding moments of the second
Hence, one approach toward identifying the optimal function could be to try and find a function whose induced posterior distributions majorize the corresponding posteriors induces by any other functions with the same bias (it is of course not clear that such a function even exists). This approach unfortunately fails for the Greedy Dictator. For example, the posterior distributions induced by setting \( f_S \) to be Majority functions are not always majorized by those induces by the Greedy Dictator (although they seem to be "almost" majorized), e.g. for \( n = 5 \) and \( \epsilon = 0.4 \), even though the average guessing time of Greedy Dictator is lower. In fact, the guessing moments for Greedy Dictator seem to be better than these of Majority irrespective of the value of \( s \).

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**APPENDIX A**

**Proofs**

**Proof of Proposition 1:** The claim that random functions do not improve beyond deterministic ones follows directly from that property that conditioning reduces guessing-moment \[2\], Corollary 1]. Monotonicity follows from the fact that Bob can always simulate a noisier channel. Now, if \( \delta = 1/2 \) then \( X^n \) and \( Y^n \) are independent, and \( G_s(X^n \mid f(Y^n)) = G_s(X^n) \approx \frac{2^n}{s+1} \) for any \( f \) (Lemma 2). For \( \delta = 0 \), let \( \Pr(f(Y^n) = 1) := q \), and assume without loss of generality that \( q \leq 1/2 \). Then,

\[
G_s(X^n \mid f(Y^n)) = (1 - q) \cdot K_s((1 - q) \cdot 2^n) + q \cdot K_s(q \cdot 2^n)
\]

(104)

\[
= 2^{-n} \sum_{i=1}^{2^n} i^s + 2^{-n} \sum_{i=1}^{q \cdot 2^n} i^s
\]

(105)

\[
= 2^{-n} \left[ \sum_{i=1}^{2^n-1} i^s + \sum_{i=2^{n-1}+1}^{(1-q) \cdot 2^n} i^s + \sum_{i=1}^{2^{n-1}} i^s - \sum_{i=q \cdot 2^n}^{2^{n-1}} i^s \right]
\]

(106)

\[
\geq 2^{-(n-1)} \sum_{i=1}^{2^{n-1}} i^s
\]

(107)

\[
= K_s(2^{n-1}),
\]

(108)

with equality when \( q = 1/2 \). Thus, the minimal \( G_s(X^n \mid f(Y^n)) \) is obtained by any balanced function, and equal to \( K_s(2^{n-1}) \approx \frac{2^{(n-1)}}{s+1} \) (Lemma 2).

To prove that the limit in (5) exists, first note that

\[
G_s(X^{n+1}) = 2^{-(n+1)} \sum_{i=1}^{2^n+1} i^s
\]

(109)

The proof in \[24\] is only for \( s = 1 \), but it is easily extended to the general \( s > 0 \) case.
\[= 2^{-(n+1)} \sum_{i=1}^{2^n} (2i-1)s + (2i)s \] (110)

\[\geq 2^s \cdot 2^{-n} \sum_{i=1}^{2^n} (i-1)s \] (111)

\[= \ell_n \cdot 2^s \cdot 2^{-n} \sum_{i=1}^{2^n} i^s \] (112)

\[= \ell_n \cdot 2^s \cdot G_s(X^n), \] (113)

where

\[\ell_n := \frac{\sum_{i=1}^{2^n} (i-1)s}{\sum_{i=1}^{2^n} i^s}.\] (114)

Second, let

\[\gamma_s^{(n)}(\delta) := \min_{f:0,1^n \rightarrow \{0,1\}} \frac{G_s(X^n | f(Y^n))}{G_s(X^n)},\] (115)

and let \(\{f_n^*\}\) be a sequence of functions such that \(f_n^*\) achieves \(\gamma_s^{(n)}(\delta)\). Denote the order induced by the posterior \(\Pr(X^n = x^n | f_n^*(Y^n) = b)\) as \(\text{ORD}_{b,n,n}\), \(b \in \{0,1\}\), and the order induced by \(\Pr(X^{n+1} = x^{n+1} | f_n^*(Y^n) = b)\) as \(\text{ORD}_{b,n,n+1}\). As before (when breaking ties arbitrarily)

\[\text{ORD}_{b,n,n+1}(x^n, 0) = 2 \text{ORD}_{b,n,n}(x^n)\] (116)

and

\[\text{ORD}_{b,n,n+1}(x^n, 1) = 2 \text{ORD}_{b,n,n}(x^n) - 1 \leq 2 \text{ORD}_{b,n,n}(x^n).\] (117)

Thus,

\[G_s(X^{n+1} | f_{n+1}^*(Y^{n+1})) \leq G_s(X^{n+1} | f_n^*(Y^n))\] (118)

\[= \Pr(f_n^*(Y^{n+1}) = 0) \cdot G_s(X^{n+1} | f_n^*(Y^n) = 0) + \Pr(f_n^*(Y^n) = 1) \cdot G_s(X^{n+1} | f_n^*(Y^n) = 1)\] (119)

\[\leq \sum_{x^{n+1}} \Pr(X^{n+1} = x^{n+1}, f_n^*(Y^n) = 0) \cdot \text{ORD}_{0,n,n+1}^s(x^{n+1})\] (120)

\[+ \sum_{x^{n+1}} \Pr(X^{n+1} = x^{n+1}, f_n^*(Y^n) = 1) \cdot \text{ORD}_{1,n,n+1}^s(x^{n+1})\] (121)

\[\leq 2^s \cdot \sum_x \Pr(X^n = x^n, f_n^*(Y^n) = 0) \cdot \text{ORD}_{0,n,n}^s(x^n) + \Pr(X^n = x^n, f_n^*(Y^n) = 1) \cdot \text{ORD}_{1,n,n}^s(x^n)\] (122)

\[= 2^s \cdot G_s(X^n | f_n^*(Y^n)).\] (123)

Hence,

\[\gamma_s^{(n+1)}(\delta) \leq \ell_n^{-1} \cdot \gamma_s^{(n)}(\delta).\] (124)
To continue, we further explore $\ell_n$. Noting that we can start the summation in the numerator (114) from $i = 2$, and using (138) and (140) (proof of Lemma 2 below), we get

\begin{align*}
1 &\geq \ell_n \\
&\geq \frac{1}{s+1} \cdot \frac{2^{n(s+1)} - 1}{2^n - 1} \\
&\geq \frac{2^{n(s+1)} - 1}{(2^n + 1)^{s+1}} \\
&= \left( \frac{2^n}{2^n + 1} \right)^{s+1} - \frac{1}{2^{n(s+1)}} \\
&= \left( 1 + \frac{1}{2^n} \right)^{-(s+1)} - \frac{1}{2^{n(s+1)}} \\
&= 1 - \frac{(s + 1)}{2^n} + O \left( \frac{1}{2^{2n}} \right) \\
&= 1 - \frac{(s + 1)}{2^n} + O \left( \frac{1}{2^n \cdot \min\{1+s,2\}} \right).
\end{align*}

Thus, there exists $c, C > 0$ such that

\begin{align*}
\log \prod_{n=1}^{\infty} \ell_n^{-1} &= \sum_{n=1}^{\infty} \log \ell_n^{-1} \\
&\leq - \sum_{n=1}^{\infty} \log \left[ 1 - \frac{c}{2^n} \right] \\
&\leq C + \sum_{n=1}^{\infty} \frac{c}{2^n} + O \left( \frac{1}{2^{2n}} \right) \\
&< \infty,
\end{align*}

and consequently,

\begin{equation}
d_n := \prod_{j=n}^{\infty} \ell_j^{-1} \to 1
\end{equation}

as $n \to \infty$. Now, (123) implies that

\begin{equation}
e_n := d_n \cdot \gamma_s^{(n)}(\delta)
\end{equation}

is a non-increasing sequence which is bounded below by 0, and thus has a limit. Since $d_n \to 1$ as $n \to \infty$, $\gamma_s^{(n)}(\delta)$ also has a limit. 

Proof of Lemma 2: Due to monotonicity of $i^s$, standard bounds on sums using integrals lead to

\begin{align*}
K_s(a, b) &\leq \int_{a+1}^{b+1} \frac{t^s}{b-a} \cdot dt \\
&= \frac{1}{s+1} \left[ \frac{(b+1)^{s+1} - (a+1)^{s+1}}{b-a} \right]
\end{align*}
and

\[
K_s(a, b) \geq \int_a^b \frac{t^s}{b - a} \cdot dt = \frac{1}{s + 1} \cdot \frac{[b^{s+1} - a^{s+1}]}{b - a}.
\]  

(139)

(140)

The ratio between the upper and lower bound is

\[
\kappa_s(a, b) := \frac{(b + 1)^{s+1} - (a + 1)^{s+1}}{(s+1)^{s+1}}
\]

(141)

and clearly \(\kappa_s(a_n, b_n) \to 1\) given the premise of the lemma.

\[\square\]

**Proof of (51):** From the law of total expectation

\[
G_s(X^n | f(Y^n)) = \Pr(f(Y^n) = 0) \cdot G_s(X^n | f(Y^n) = 0) + \Pr(f(Y^n) = 1) \cdot G_s(X^n | f(Y^n) = 1).
\]  

(142)

Now, without loss of generality, we may assume that

\[
\Pr(f(Y^n) = 0) \cdot G_s(X^n | f(Y^n) = 0) \geq \Pr(f(Y^n) = 1) \cdot G_s(X^n | f(Y^n) = 1)
\]

(143)

for all \(n\), as otherwise one may consider \(1 - f\) which has the same guessing-moments as \(f\). As \(G_s(X^n | f(Y^n))\) is unbounded then \(G_s(X^n | f(Y^n) = 0)\) is unbounded too. Let \(\eta > 0\) be given. Utilizing (50), let \(g_\eta\) be such that \(G_s(P) \geq g_\eta\) ensures that

\[
G_s(P) \geq (1 - \eta) \cdot \Psi_s \cdot 2^{sH(P)}.
\]

(144)

Since \(G_s(X^n | f(Y^n) = 0)\) is unbounded by our assumption, we get from (50) that there exists \(n_0(\eta)\) such that

\[
G_s(X^n | f(Y^n) = 0) \geq (1 - \eta) \cdot \Psi_s \cdot 2^{sH(X^n | f(Y^n) = 0)}
\]

(145)

for all \(n \geq n_0(\eta)\). Furthermore, let \(N_+ := \{G_s(X^n | f(Y^n) = 1) \geq g_\eta\}\). If \(N_+\) is not empty then there exists \(n'_1(\eta)\) such that

\[
G_s(X^n | f(Y^n) = 1) \geq (1 - \eta) \cdot \Psi_s \cdot 2^{sH(X^n | f(Y^n) = 1)}
\]

(146)

for all \(n \geq n'_1(\eta)\) such that \(n \in N_+\). Thus,

\[
G_s(X^n | f(Y^n)) \geq (1 - \eta) \cdot \Psi_s \cdot \left[ \Pr(f(Y^n) = 0) \cdot 2^{sH(X^n | f(Y^n) = 0)} + \Pr(f(Y^n) = 1) \cdot 2^{sH(X^n | f(Y^n) = 1)} \right]
\]

(147)

for all \(n \geq \max\{n_0(\eta), n'_1(\eta)\}\) such that \(n \in N_+\).

Now, for \(\mathbb{N} \setminus N_+\), the guessing moment is bounded as \(G_s(X^n | f(Y^n) = 1) \leq g_\eta\). Evidently from (50) and the sentence that follows, if \(G_s(X^n | f(Y^n) = 1)\) is bounded then \(H(X^n | f(Y^n) = 1)\) is also bounded. Since \(H(X^n | f(Y^n) = 0)\) must be unbounded, there exists \(n''_1(\eta)\) such that

\[
G_s(X^n | f(Y^n)) \geq (1 - \eta/2) \cdot \Psi_s \cdot \left[ \Pr(f(Y^n) = 0) \cdot 2^{sH(X^n | f(Y^n) = 0)} + \Pr(f(Y^n) = 1) \cdot 2^{sH(X^n | f(Y^n) = 1)} \right]
\]

(148)
for all \( n \geq \max\{n_0(\eta), \ n_1^0(\eta)\} \) such that \( n \in \mathbb{N}\setminus\mathbb{N}_+ \).

\[ \text{Proof of (100):} \] Let us evaluate the posterior probability conditioned on \( \text{G-Dict}(Y^n) = 0 \). Since G-Dict is balanced, Bayes law implies that

\[
\Pr(X^n = x^n \mid \text{G-Dict}(Y^n) = 0) = 2^{-(n-1)} \cdot \Pr(G-Dict(Y^n) = 0 \mid X^n = x^n) \tag{149}
\]

\[
= 2^{-(n-1)} \cdot \sum_{i=1}^{n+1} \Pr(k(y^n) = i \mid X^n = x^n) \cdot \Pr(G-Dict(Y^n) = 0 \mid X^n = x^n, k(y^n) = i) \tag{150}
\]

\[
= 2^{-(n-1)} \cdot \left\{ \sum_{i=1}^{n} (1 - \epsilon) \epsilon^{i-1} \cdot 1 \{x_i = 0\} + \epsilon^n \cdot \frac{1}{2} \right\} \tag{151}
\]

This immediately leads to the guessing rule in (100). As stated in the beginning of Section VI, the guessing rule for \( \text{G-Dict}(Y^n) = 1 \) is on reversed order.

\[ \text{Proof of Proposition 14:} \] We denote the lexicographic order by \( \text{ORD}_{\text{lex}} \). Assume that \( \text{G-Dict}(Y^n) = 0 \) and that \( \text{ORD}_{\text{lex}}(x^n) \leq \text{ORD}_{\text{lex}}(z^n) \). Then, there exists \( j \in [n] \) such that \( x^{j-1} = z^{j-1} \) (where \( x^0 \) is the empty string) and \( x_j = 0 < z_j = 1 \). Then,

\[
\Pr(X^n = x^n \mid \text{G-Dict}(Y^n) = 0) - \Pr(X^n = z^n \mid \text{G-Dict}(Y^n) = 0)
\]

\[
= \epsilon^{j-1} + \sum_{i=j+1}^{n} \epsilon^{i-1} \cdot (z_i - x_i) \tag{152}
\]

\[
\geq \epsilon^{j-1} - \sum_{i=j+1}^{n} \epsilon^{i-1} \tag{153}
\]

\[
= \frac{\epsilon^{j-1}}{1 - \epsilon} \left( 1 - 2\epsilon + \epsilon^{n-j+1} \right) \tag{154}
\]

\[
\geq 0. \tag{155}
\]

This proves the first statement of the proposition. Now, let \( \text{ORD}_0 \) (\( \text{ORD}_1 \)) be the guessing order given that the received bit is 0 (resp. 1), and let \( f \) the Boolean function (which are not necessarily optimal). Then, from (97) and (95)

\[
G_1(X^n \mid f(Y^n))
\]

\[
= \sum_{x^n} \Pr(X^n = x^n, f(Y^n) = 0) \cdot \text{ORD}_0(x^n) + \Pr(X^n = x^n, f(Y^n) = 1) \cdot \text{ORD}_1(x^n) \tag{156}
\]

\[
= 2^{-n} \cdot \sum_{S \subseteq [n]} \Pr(S) \sum_{x^n} [(1 - f_S(x^n)) \cdot \text{ORD}_0(x^n) + f_S(x^n) \cdot \text{ORD}_1(x^n)] \tag{157}
\]

\[
= 2^{-n} \cdot \sum_{S \subseteq [n]} \Pr(S) \sum_{x^S} [(1 - f_S(x^n)) \cdot \text{ORD}_0(x^S || S) + f_S(x^n) \cdot \text{ORD}_1(x^S || S)] \tag{158}
\]

\[
\geq 2^{-n} \cdot \sum_{S \subseteq [n]} \Pr(S) \sum_{x^n} \min \{ \text{ORD}_0(x^S || S), \text{ORD}_1(x^S || S) \}, \tag{159}
\]
where for \( b \in \{0, 1\} \), the projected orders are defined as

\[
PORD_b(x^S \mid S) := \sum_{x^S} \text{ORD}_b(x^n). \tag{160}
\]

It is easy to verify that if \( \text{ORD}_0 (\text{ORD}_1) \) is the lexicographic (resp. revered lexicographic) order then the Greedy Dictator achieves \([159]\) with equality, due to the following simple property: If \( \text{ORD}_{\text{lex}}(x^n) < \text{ORD}_{\text{lex}}(z^n) \) then

\[
\sum_{x^S} \text{ORD}_{\text{lex}}(x^n) \leq \sum_{x^S} \text{ORD}_{\text{lex}}(z^n) \tag{161}
\]

for all \( S \in [n] \). This can be proved by induction over \( n \). For \( n = 1 \) the claim is easily asserted. Suppose it holds for \( n - 1 \), and let us verify it for \( n \). If \( 1 \in S \) then whenever \( \text{ORD}_{\text{lex}}(x^n) < \text{ORD}_{\text{lex}}(z^n) \)

\[
\sum_{x^S} \text{ORD}_{\text{lex}}(x^n) = \sum_{x^S} \text{ORD}_{\text{lex}}(x_1, x^n_2)
\]

\[
= x_1 \cdot 2^{n-1} + \sum_{x^S \setminus \{1\}} \text{ORD}_{\text{lex}}(x^n_2)
\]

\[
\leq z_1 \cdot 2^{n-1} + \sum_{x^S \setminus \{1\}} \text{ORD}_{\text{lex}}(z^n_2)
\]

\[
= \sum_{x^S} \text{ORD}_{\text{lex}}(z^n)
\]

where the inequality follows from the induction assumption and since \( x_1 \leq z_1 \). If \( 1 \not\in S \) then, similarly,

\[
\sum_{x^S} \text{ORD}_{\text{lex}}(x^n) = \sum_{x^S \setminus \{1\}} \left[ 2^{n-1} + 2 \cdot \text{ORD}_{\text{lex}}(z^n_2) \right]
\]

\[
\leq \sum_{z^S \setminus \{1\}} \left[ 2^{n-1} + 2 \cdot \text{ORD}_{\text{lex}}(z^n_2) \right]
\]

\[
= \sum_{z^S} \text{ORD}_{\text{lex}}(z^n).
\]

**Proof of Theorem 15**: We denote the lexicographic order by \( \text{ORD}_{\text{lex}} \). Then,

\[
G_1(X^n \mid G-\text{Dict}(Y^n)) = G_1(X^n \mid G-\text{Dict}(Y^n) = 0)
\]

\[
\leq \sum_{x^n} \text{Pr}(X^n = x^n \mid G-\text{Dict}(Y^n) = 0) \cdot \text{ORD}_{\text{lex}}(x^n)
\]

\[
= 2^{-(n-1)} \cdot \sum_{x^n} \sum_{i=1}^{n} (1 - \epsilon) \epsilon^{i-1} \cdot 1 \{x_i = 0\} \cdot \text{ORD}_{\text{lex}}(x^n) + \epsilon^n K_1(2^n)
\]

\[
= 2^{-(n-1)} \cdot (1 - \epsilon) \sum_{i=1}^{n} \epsilon^{i-1} \cdot \sum_{x^n} 1 \{x_i = 0\} \cdot \text{ORD}_{\text{lex}}(x^n) + \epsilon^n K_1(2^n)
\]

\[
= (1 - \epsilon) J_n + \epsilon^n K_1(2^n).
\]

(173)
where $J_1 := 1/2$ and for $n \geq 2$

$$J_n := 2^{-(n-1)} \sum_{i=1}^{n} \epsilon^{i-1} \cdot \sum_{x^n} \mathbb{1}\{x_i = 0\} \cdot \text{ORD}_{\text{lex}}(x^n)$$

$$= 2^{-(n-1)} \sum_{x^n} \mathbb{1}\{x_1 = 0, x_i = 0\} \cdot \text{ORD}_{\text{lex}}(x^n) + 2^{-(n-1)} \sum_{i=2}^{n} \epsilon^{i-1} \cdot \sum_{x^n} \mathbb{1}\{x_i = 0\} \cdot \text{ORD}_{\text{lex}}(x^n)$$

$$= K_1(2^{n-1}) + 2^{-(n-1)} \sum_{i=2}^{n} \epsilon^{i-1} \cdot \sum_{x^n} \mathbb{1}\{x_1 = 0, x_i = 0\} \cdot \text{ORD}_{\text{lex}}(x^n)$$

$$+ 2^{-(n-1)} \sum_{i=2}^{n} \epsilon^{i-1} \cdot \sum_{x^n} \mathbb{1}\{x_i = 0\} \cdot \text{ORD}_{\text{lex}}(x^n)$$

$$= K_1(2^{n-1}) + \epsilon J_{n-1} + 2^{n-2} \cdot \frac{\epsilon - \epsilon^n}{1 - \epsilon}.$$

So

$$J_n = K_1(2^{n-1}) + \epsilon \left[ K_1(2^{n-2}) + \epsilon J_{n-2} + 2^{n-3} \cdot \frac{\epsilon - \epsilon^{n-1}}{1 - \epsilon} \right] + 2^{n-2} \cdot \frac{\epsilon - \epsilon^n}{1 - \epsilon}$$

$$= K_1(2^{n-1}) + \epsilon K_{1}(2^{n-2}) + \epsilon^2 J_{n-2} + 2^{n-3} \cdot \frac{\epsilon^2 - \epsilon^n}{1 - \epsilon} + 2^{n-2} \cdot \frac{\epsilon - \epsilon^n}{1 - \epsilon}$$

$$= \sum_{i=1}^{n} \epsilon^{i-1} K_1(2^{n-i}) + \frac{1}{1 - \epsilon} \sum_{i=1}^{n} 2^{i-2} \cdot (\epsilon^{n-i+1} - \epsilon^n).$$

Hence,

$$G_1(X^n \mid \text{G-Dict}(Y^n)) \leq (1 - \epsilon) \sum_{i=1}^{n} \epsilon^{i-1} K_1(2^{n-i}) + \sum_{i=1}^{n} 2^{i-2} \cdot (\epsilon^{n-i+1} - \epsilon^n) + \epsilon^n K_1(2^n).$$

Noting that $K_1(M) = M/2 + 1/2$, we get

$$G_1(X^n \mid \text{G-Dict}(Y^n)) \leq 2^n \frac{1 - \epsilon}{\epsilon} \sum_{i=1}^{n} \left(\frac{\epsilon}{2}\right)^i + \frac{(1 - \epsilon)(1 - \epsilon^n)}{2\epsilon} + 2 \sum_{i=1}^{n} \left(\frac{\epsilon}{2}\right)^i \cdot \epsilon^{n+i} - \frac{1}{2} (2^n - 1) \epsilon^n + 2^{n-1} \epsilon^n + \frac{\epsilon^n}{2}$$

$$= \frac{1}{2 - \epsilon} \left(2^{n-1} + \frac{\epsilon^n}{2} (1 - \epsilon)\right) + \frac{(1 - \epsilon)(1 - \epsilon^n)}{2\epsilon}$$

$$\leq \frac{2^{n-1}}{2 - \epsilon}.$$
REFERENCES

[1] J. L. Massey, “Guessing and entropy,” in *IEEE International Symposium on Information Theory*, p. 204, 1994.

[2] E. Arikan, “An inequality on guessing and its application to sequential decoding,” *IEEE Transactions on Information Theory*, vol. 42, pp. 99–105, January 1996.

[3] E. Arikan and N. Merhav, “Guessing subject to distortion,” *IEEE Transactions on Information Theory*, vol. 44, pp. 1041–1056, March 1998.

[4] N. Merhav and E. Arikan, “The Shannon cipher system with a guessing wiretapper,” *IEEE Transactions on Information Theory*, vol. 45, pp. 1860–1866, June 1999.

[5] Y. Hayashi and H. Yamamoto, “Coding theorems for the Shannon cipher system with a guessing wiretapper and correlated source outputs,” *IEEE Transactions on Information Theory*, vol. 54, no. 6, pp. 2808–2817, 2008.

[6] M. K. Hanawal and R. Sundaresan, “The Shannon cipher system with a guessing wiretapper: General sources,” *IEEE Transactions on Information Theory*, vol. 57, no. 4, pp. 2503–2516, 2011.

[7] Y. Yona and S. Diggavi, “Password cracking: The effect of bias on the average guesswork of hash functions,” 2016. Available online: [http://arxiv.org/pdf/1608.02132.pdf](http://arxiv.org/pdf/1608.02132.pdf).

[8] M. M. Christiansen, K. R. Duffy, F. du Pin Calmon, and M. Méard, “Multi-user guesswork and brute force security,” *IEEE Transactions on Information Theory*, vol. 61, no. 12, pp. 6876–6886, 2015.

[9] R. Sundaresan, “Guessing under source uncertainty,” *IEEE Transactions on Information Theory*, vol. 53, pp. 269–287, January 2007.

[10] B. Serdar, “Comments on ’An inequality on guessing and its application to sequential decoding’,” *IEEE Transactions on Information Theory*, vol. 43, pp. 2062–2063, June 1997.

[11] M. K. Hanawal and R. Sundaresan, “Guessing revisited: A large deviations approach,” *IEEE Transactions on Information Theory*, vol. 57, pp. 70–78, January 2011.

[12] A. Wyner, “A theorem on the entropy of certain binary sequences and applications–II,” *IEEE Transactions on Information Theory*, vol. 19, pp. 772–777, June 1973.

[13] R. Ahlswede and J. Körner, “Source coding with side information and a converse for degraded broadcast channels,” *IEEE Transactions on Information Theory*, vol. 21, pp. 629–637, June 1975.

[14] R. Graczyk and A. Lapidoth, “Variations on the guessing problem,” in *IEEE International Symposium on Information Theory*, June 2018.

[15] R. Graczyk, “Guessing with a helper,” Master’s thesis, ETH Zurich, 2017.

[16] R. O’Donnell, *Analysis of Boolean functions*. Cambridge University Press, 2014.

[17] T. A. Courtade and G. R. Kumar, “Which Boolean functions maximize mutual information on noisy inputs?,” *IEEE Transactions on Information Theory*, vol. 60, pp. 4515–4525, August 2014.

[18] O. Ordentlich, O. Shayevitz, and O. Weinstein, “An improved upper bound for the most informative Boolean function conjecture,” May 2015. Available online: [http://arxiv.org/pdf/1505.05794v2.pdf](http://arxiv.org/pdf/1505.05794v2.pdf).

[19] A. Samorodnitsky, “On the entropy of a noisy function,” *IEEE Transactions on Information Theory*, vol. 62, pp. 5446–5464, October 2016.

[20] G. Kindler, R. O’Donnell, and D. Wittmer, “Remarks on the most informative function conjecture at fixed mean,” 2015. Available online: [http://arxiv.org/pdf/1506.03167.pdf](http://arxiv.org/pdf/1506.03167.pdf).

[21] J. Li and M. Méard, “Boolean functions: noise stability, non-interactive correlation, and mutual information,” 2018. Available online: [http://arxiv.org/pdf/1801.04462.pdf](http://arxiv.org/pdf/1801.04462.pdf).

[22] V. Chandar and A. Tchamkerten, “Most informative quantization functions,” tech. rep., 2014. Available online: [http://perso.telecom-paristech.fr/~tchamker/CTAT.pdf](http://perso.telecom-paristech.fr/~tchamker/CTAT.pdf).

[23] N. Weinberger and O. Shayevitz, “On the optimal Boolean function for prediction under quadratic loss,” *IEEE Transactions on Information Theory*, vol. 63, pp. 4202–4217, July 2017.
[24] A. Burin and O. Shayevitz, “Reducing guesswork via an unreliable oracle,” March 2017. Available online: http://arxiv.org/pdf/1703.01672.pdf

[25] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2. New York: John Wiley & Sons, 1971.

[26] T. M. Cover and J. A. Thomas, Elements of Information Theory. Wiley-Interscience, 2006.

[27] M. J. Wainwright and M. I. Jordan, “Graphical models, exponential families, and variational inference,” Foundations and Trends® in Machine Learning, vol. 1, no. 1–2, pp. 1–305, 2008.

[28] S. P. Boyd and L. Vandenberghe, Convex Optimization. Cambridge university press, 2004.

[29] S. Nadarajah, “A generalized normal distribution,” Journal of Applied Statistics, vol. 32, no. 7, pp. 685–694, 2005.

[30] A. Wyner and J. Ziv, “A theorem on the entropy of certain binary sequences and applications–I,” IEEE Transactions on Information Theory, vol. 19, pp. 769–772, November 1973.

[31] E. Erkip and T. M. Cover, “The efficiency of investment information,” IEEE Transactions on Information Theory, vol. 44, no. 3, pp. 1026–1040, 1998.

[32] R. Ahlswede and P. Gács, “Spreading of sets in product spaces and hypercontraction of the Markov operator,” The annals of probability, pp. 925–939, 1976.

[33] M. Raginsky, “Strong data processing inequalities and Φ–Sobolev inequalities for discrete channels,” IEEE Transactions on Information Theory, vol. 62, no. 6, pp. 3355–3389, 2016.

[34] V. Anantharam, A. Gohari, S. Kamath, and C. Nair, “On hypercontractivity and a data processing inequality,” in Proc. IEEE International Symposium Information Theory (ISIT), pp. 3022–3026, 2014.

[35] V. Anantharam, A. Gohari, S. Kamath, and C. Nair, “On maximal correlation, hypercontractivity, and the data processing inequality studied by Erkip and Cover,” 2013. Available online: http://arxiv.org/pdf/1304.6133.pdf