On the universal Gaussian behavior of Driven Lattice Gases at short-times

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The dynamic and static critical behaviors of driven and equilibrium lattice gas models are studied in two spatial dimensions. We show that in the short-time regime immediately following a critical quench, the dynamics of the transverse order parameters, auto-correlations, and Binder cumulant are consistent with the prediction of a Gaussian, i.e., non-interacting, effective theory, both for the equilibrium lattice gas and its nonequilibrium counterparts. Such a “super-universal” behavior is observed only at short times after a critical quench, while the various models display their distinct behaviors in the stationary states, described by the corresponding, known universality classes.

I. INTRODUCTION

The search for universal behavior, which unites a class of systems in terms of some common collective properties, lies at the very heart of Statistical Physics. Both in its static and dynamic manifestations, universality emerges in large systems of interacting degrees of freedom close to a critical point, when they display a behavior which is actually independent of their microscopic features. This critical behavior is usually dictated by symmetry properties, or conservation laws; critical phenomena occurring in various systems having the same symmetries belong to the same universality class. Beyond its numerous and celebrated manifestations in equilibrium universality, universality plays an important role also in the dynamical relaxation of nonequilibrium systems, ranging from diffusive and reaction-diffusion systems to surface growth.

Remarkably, universality emerges not only in systems that are close to their stationary state, but also far from it, i.e., during the early stages of the relaxation process, when the correlation length of the fluctuations of the relevant order parameter is still very small compared to the system size. This fact often translates in the observation of novel critical exponents, but also in the possibility to measure the equilibrium and dynamical critical exponents which characterize the stationary state from the observation of this nonequilibrium relaxation, with a substantial reduction of the numerical costs.

Classifying and characterizing nonequilibrium universality classes remain a challenge in Statistical Physics. Investigations of lattice models are very useful in this respect: their simplicity makes them amenable to numerical, and sometimes analytical studies, yet they often show rich and novel physical phenomena. Lattice gases, which describe stochastic hopping of particles on a lattice, belong to one such class of models which has been extensively used to explore critical phenomena in and out of equilibrium. These systems often show a continuous transition to an ordered state, where the particles cluster together, at a certain critical temperature. Such transitions are relevant in various physical situations including binary mixtures, driven diffusive systems, viscoelastic fluids, vehicular traffic and active matter.

The critical behavior characterizing the phase transitions in the various lattice gas models depends on the symmetries of their specific dynamics: equilibrium, driven and randomly driven lattice gases therefore belong to different universality classes. All these models have one common feature though: the density of particles is locally conserved by the dynamics. Such a conservation law strongly constrains and slows the dynamics down, hence different dynamical behaviors are expected and observed compared to non-conserved models. While considerable amount of work has been devoted to study the critical behaviors of the latter, both in and out of equilibrium, much less attention has been given to the short-time dynamics of the former.

In this work we show that, remarkably, a sort of “super-universality”, which unites the different lattice gas models, emerges in the short-time regime after a critical quench, irrespective of their specific critical behavior. In this regime, in fact, the dynamical behavior of certain observables which can be considered the natural order parameters for these transitions are described by a non-interacting (Gaussian) effective theory. In particular, we will focus on the behavior of “transverse” observables in the driven lattice gas, the randomly driven lattice gas and the equilibrium lattice gas in two spatial dimensions. Despite the fact that features like the driving or the spatial anisotropy introduce a relevant perturbations in the lattice gases which change entirely the critical properties of the system, the short-time behavior of these natural observables is independent of these features. The peculiar properties of specific universality classes are recovered, for all observables, only at longer times.

The presentation is organized as follows: In Sec. I we recall the equilibrium and driven lattice gas models and define certain relevant observables. A brief discussion of the different effective field theories introduced in the past in order to study the critical behaviors of these models is presented in Sec. II. Based on a Gaussian theory, the dynamical behavior of the transverse order parameters...
and auto-correlation of one of them are computed. This section elaborates and substantially extends the analysis of Ref. [20]. In Sec. [IV] we compare the results obtained from Monte Carlo simulations in the short-time regime with the predictions of a Gaussian effective theory for both the driven and equilibrium lattice gases. The time evolution of the Binder cumulant starting from various initial states is also studied in the various models. Section [V] is devoted to the study of the stationary state behavior of the conserved lattice gases. We conclude with some general remarks in Sec. [VI].

II. THE MODELS

We consider a periodic $d$-dimensional hyper-cubic lattice with size $V = L_{||} \times L_{\perp}^{d-1}$. The generic $i$-th site of the lattice can be either empty or occupied by a particle with a corresponding occupation number $n_i = 0, 1$. The particles interact via a nearest-neighbour Ising Hamiltonian,

$$
\mathcal{H}(C) = -4 \sum_{\langle i,j \rangle} n_i n_j,
$$

which depends on the configuration $C = \{n_1, n_2, \ldots, n_V\}$. We consider the case of a half-filled lattice, i.e., the total number $\sum_i n_i$ of particles is fixed to be $V/2$.

The equilibrium Lattice Gas (LG) dynamics consists of jump attempts of randomly chosen particles to one of its unoccupied neighbouring sites with the Metropolis rate $w(\Delta \mathcal{H}) = \min\{1, e^{-\beta \Delta \mathcal{H}}\}$, where $\beta = 1/T$ is the inverse temperature and $\Delta \mathcal{H}$ is the change in energy due to the proposed jump; see Fig. 1 for a schematic representation. The dynamics conserves the total number of particles in the system. The choice of the rate function $w$ ensures that the dynamics satisfies detailed balance and therefore the system eventually reaches the equilibrium state characterized by the usual Gibbs measure $P(C) \propto e^{-\beta \mathcal{H}(C)}$.

In the thermodynamic limit, the system undergoes a continuous phase transition at a critical temperature $T_{\text{DLG}}\parallel$, from a disordered state to a phase-separated one where the particles cluster together: Fig. 2 (left panel) shows a typical low-temperature configuration of the LG, in which the jumps along (opposite to) the field are always accepted (rejected).

In the thermodynamic limit the LG also shows a "phase" transition at the critical temperature $T_{\text{DLG}}\parallel$, which surprisingly increases upon increasing $E$, saturating at a finite value $T_{\text{DLG}}\parallel$. For $T < T_{\text{DLG}}\parallel$ the system shows a phase-separated state where the particles cluster in a single strip aligned with the direction of the external field; a typical low-temperature configuration of the RDLG is shown in Fig. 2 (central panel).

An important variant of the DLG is the Randomly Driven Lattice Gas (RDLG) where the field $E$ randomly changes its sign at each attempted move. Although this dynamics breaks the detailed balance condition, no particle current flows through the system in the stationary state of the RDLG, in contrast to the DLG. Also for the RDLG, we consider the case $E \to \infty$.

The RDLG also undergoes a continuous transition to a phase-separated state below a critical temperature $T_{\text{RDLG}}\parallel$. The low-temperature stationary state of the RDLG looks similar to that of the IDLG, the interface of the formed strip being aligned with the direction of the field $E$; see the right panel in Fig. 2.

Both the DLG and the RDLG show remarkable properties, such as generic long-range correlations in the disordered state and strong anisotropy in space [12]. As a consequence of this anisotropy, finite-size scaling analysis have to be performed at a fixed anisotropic aspect ratio

$$
S_\Delta = \frac{L_{||}}{L_{\perp}^{1+\Delta}},
$$

where the anisotropy exponent $\Delta$ controls the degree of anisotropy in the model. While $\Delta = 0$ for the equilibrium LG, field-theoretical studies in two spatial dimen-
sions conclude that $\Delta = 2$ for the DLG and the IDLG \[22, 23\], and $\Delta \approx 1$ for the RDLG [19].

In the anisotropic IDLG and RDLG, the presence of the field naturally introduces the distinction between what we refer to as “transverse” and “longitudinal” observables. A transverse observable is obtained as a spatial average along the direction of the field and is thus insensitive to spatial fluctuations along the field. Analogously, one can also define longitudinal observables by averaging along the orthogonal direction. In the equilibrium LG such a distinction is entirely arbitrary. However, we can always fix a direction in space as being the preferred one, and define “transverse” and “longitudinal” observables with respect to it.

The structure of the ordered state in the driven lattice gases (see Fig. 2) naturally leads to the choice of order parameters which are transverse in nature. One such typical transverse observable used to characterize the onset of order in these systems is the so-called anisotropic order parameter $m$, which is related to the average amplitude of the first non-zero transverse Fourier mode of the spatial density of the particles. To define it precisely, let us consider a $d = 2$ dimensional lattice of size $V = L_x \times L_y$ and associate, to each site $(x, y)$, a “spin variable” $\sigma_{xy} = 2n_{xy} - 1$ which takes values $\pm 1$. The relevant quantity is its Fourier transform

$$\tilde{\sigma}(k_x, k_y) = \sum_{x=0}^{L_x-1} \sum_{y=0}^{L_y-1} e^{i(k_x x + k_y y)} \sigma_{xy}$$

where, due to the periodic boundary conditions, the allowed longitudinal and transverse momenta are

$$\langle k_x, k_y \rangle = \left( \frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y} \right),$$

with integers $0 \leq n_x \leq L_x$ and $0 \leq n_y \leq L_y$. The half-filling condition on particle density implies that the total “magnetization” $\sum_{xy} \sigma_{xy}$ vanishes and in turn, $\tilde{\sigma}(0, 0) = 0$.

The anisotropic order parameter $m$ is defined as the statistical average of the absolute value of the first non-zero transverse mode $\mu = \tilde{\sigma}(0, 2\pi/L_y)$, i.e., as

$$m = \langle |\mu| \rangle / V,$$

where $\langle \cdot \rangle$ denotes the statistical average. In order to characterize the dynamical behavior it is also instructive to consider the temporal auto-correlation function $C_m$ of the anisotropic order parameter, i.e.,

$$C_m(s, t) = \frac{1}{V} \left[ \langle |\mu(s)| \mu(t) \rangle - \langle |\mu(s)| \rangle \langle |\mu(t)| \rangle \right].$$

An alternative observable used to detect the onset of an ordered phase in these systems is the average absolute value $O$ of the magnetization along the field direction, which was proposed and studied in Ref. [9],

$$O = \frac{1}{V} \sum_{y=0}^{L_y-1} \left\langle \sum_{x=0}^{L_x-1} \sigma_{xy} \right\rangle.$$  

Both $O$ and $m$ are expected to be non-zero in the ordered phase, where the particles cluster together to form a single strip aligned along the direction of the drive, although their stationary values are expected to be different. On the other hand, both $m$ and $O$ vanish in the disordered phase in the thermodynamic limit thus providing two alternative definitions of an order parameter.

The alternative order parameter $O$ is also a transverse observable, as it can be expressed as a sum of transverse modes,

$$O = \frac{1}{V^2} \sum_{y=0}^{L_y-1} \left\langle \sum_{x=0}^{L_x-1} \sum_{k_x, k_y} e^{-i(k_x x + k_y y)} \tilde{\sigma}(k_x, k_y) \right\rangle$$

$$= \frac{1}{V} \left\langle \sum_{n_x=1}^{L_x-1} \tilde{\sigma}(0, 2\pi n_y/L_y) \right\rangle.$$  

The last equality follows from the condition $\tilde{\sigma}(0, 0) = 0$ and the fact that the expectation value on the first line does not depend on $y$. 

III. MESOSCOPIC DESCRIPTION: FIELD THEORETICAL APPROACH

The critical behavior of the lattice gas models can be understood based on effective, mesoscopic field-theoretical descriptions of their dynamics [2]. Near criticality, the evolution of the coarse-grained local spin density φ(x,t) is expected to be governed by a Langevin equation which takes into account the relevant interactions specific to each universality class. We briefly recall some basic facts about the effective theories corresponding to the lattice gases discussed here.

Equilibrium Lattice Gas: The phase transition in the equilibrium lattice gas belongs to the Ising universality class which is characterized by the standard φ^4 theory [17]. Its dynamics, in the case of conserved order parameter φ, is described by the corresponding Langevin equation (known as Model B) [2] [3],

\[ \partial_t \phi = \alpha [ (\tau - \nabla^2) \nabla^2 \phi ] + u \nabla^2 \phi^3 - \nabla \cdot \xi \]  (9)

where \( \tau \) measures the distance from the critical point, \( u > 0 \) is the interaction strength, \( \alpha \) is a positive constant and \( \xi \) is a while noise with \( \langle \xi_i(x,t) \xi_j(x', t') \rangle \propto \delta_{ij} \delta^d(x-x') \delta(t-t') \). The resulting critical behavior and exponents are known exactly in \( d = 2 \) while the upper critical spatial dimensionality \( d_c \) is 4.

Driven Lattice Gas: The mesoscopic description of the driven lattice gas was developed independently by Janssen and Schmittmann [22] and Leung and Cardy [23]. This theory, henceforth referred to as JSLC theory, differs from the LG in two respects: the external field introduces a new interaction term and induces strong anisotropy. Correspondingly, the Langevin equation describing the evolution of the coarse-grained spin density in the near-critical DLG (and IDLG) is given by

\[ \partial_t \phi = \alpha [ (\tau - \nabla^2) \nabla^2 \phi ] + u \nabla^2 \phi^3 - \nabla \cdot \xi + \nabla \cdot \nabla \phi \]  (10)

where \( \nabla \) represents the coarse-grained driving field, while \( \nabla \parallel \) and \( \nabla \perp \) denote spatial derivatives orthogonal and parallel to the driving directions, respectively. The presence of an additional relevant interaction term \( \nabla \cdot \nabla \phi^3 \) and spatial anisotropy cause the critical behavior to change compared to that of the Ising universality class. Also in this case, the exponents are known exactly in all spatial dimensions \( d \geq 2 \) up to the upper critical dimension \( d_c = 5 \); of primary importance for the purpose of the present study is the anisotropy exponent \( \Delta = 2 \) in two spatial dimensions which shall intervene in the following analysis. The specific form of the interaction term causes the behavior of \( \phi \) at varying parallel wavevector \( k_\parallel \) to be effectively described by a non-interacting theory and therefore its transverse fluctuations are expected to be described by a simple Gaussian theory discussed below [24].

Randomly Driven Lattice Gas: The Langevin equation takes a different form compared to Eq. (10) when the driving field changes sign randomly, i.e., in the case of the RDLG; the particle current is no longer relevant, but anisotropy continues to be a significant factor, resulting in the effective equation

\[ \partial_t \phi = \alpha (\tau - \nabla^2 \phi)^2 - \nabla \cdot \xi \]  (11)

In turn, this results in yet another universality class, different from both LG and DLG; the critical exponents are known in terms of a series expansion around the upper critical dimension \( d_c \) = 3 [19, 25]. In addition, the anisotropy exponent \( \Delta \approx 1 \) differs from that of the DLG in \( d = 2 \).

Gaussian effective theory: The Gaussian or non-interacting theory describes a fluctuating field in the absence of non-linear interactions. The corresponding Langevin equation for a system with locally conserved field can be obtained by setting \( u = 0 \) in Eq. (9),

\[ \partial_t \phi = \alpha (\tau - \nabla^2 \phi)^2 - \nabla \cdot \xi \]  (12)

Irrespective of the fact that the phase transitions in the three different models, namely LG, DLG and RDLG belong to three different universality classes, the short-time dynamical behaviors of certain transverse observables, after a quench to the critical point, turn out to be very similar in all these models. In fact, as discussed in Ref. [20], transverse modes in all the lattice gas models show a behavior at short times which is consistent with a free theory, i.e., the distribution of transverse modes is effectively Gaussian. Some of the results of this section have already been briefly anticipated in Ref. [20]; in the following we also provide additional details of that analysis.

In particular, our objective is to determine the temporal behavior of the order parameters introduced in Sec.

| Theory   | JSLC | RDLG | LG |
|----------|------|------|----|
| \( \Delta \) | 2    | 3    | 4  |
| \( \beta \) | 1/2  | 0.315 | 1/8 |
| \( \nu \)   | 1/2  | 0.626 | 1  |
| \( \eta \)  | 0    | 0.016 | 1/4 |
| \( \xi \)   | 4    | 3.984 | 15/4 |
for a model system which is described by an effective Gaussian theory. In order to do so we need to look at the time evolution of the transverse modes \( \sigma_k \) (defined in Eq. (11)) which is obtained by taking the Fourier transform of the Langevin equation (12). However, since we are interested in lattice models, the spatial gradients in that equation have to be interpreted as being defined on a lattice. Consequently, the amplitude \( \sigma_k \) of the transverse mode \( k = (0, k_\perp) \) evolves according to,

\[
d\frac{d\tilde{\sigma}_k(t)}{dt} = -\gamma_k \tilde{\sigma}_k(t) + i\hat{k} \tilde{\eta}_k(t)
\]

where \( \hat{k} = 2\sin(k/2) \) is the lattice momentum and

\[
\gamma_k = \alpha(\tau + \hat{k}^2)\hat{k}^2.
\]

As mentioned above, \( \alpha \) is a coarse-grained diffusion constant, possibly depending on the lattice parameters and \( \tau \) measures the distance from the critical point. Additionally, \( \tilde{\eta} \) is a white noise in momentum space, obtained by taking the Fourier transform of the noise in real space, and is also delta correlated, with

\[
\langle \tilde{\eta}_k(t)\tilde{\eta}_k(t') \rangle = 2\alpha T_\eta L_\parallel L_\perp \delta(k + k')\delta(t - t'),
\]

where the normalization factor \( T_\eta \) signifies an “effective temperature” associated with the noise in terms of which the fluctuation-dissipation theorem (26) is effectively satisfied when looking at correlations and response functions of the transverse fluctuations. Note that the noise strength in momentum space is proportional to the volume of the lattice because of the discrete nature of the allowed momenta.

Let us consider the case in which the system is initially in a disordered configuration corresponding to a high temperature, so that \( \tilde{\sigma}_k(t = 0) = 0 \) for all transverse modes \( k \). For this initial condition Eq. (13) has the solution

\[
\tilde{\sigma}_k(t) = i\hat{k} \int_0^t ds \tilde{\eta}_k(s) e^{-\gamma_k(t-s)}.
\]

This leads to a Gaussian behavior, i.e., the \( k \)-th Fourier mode has a Gaussian probability distribution \( P \) at any time \( t \),

\[
P(\tilde{\sigma}_k(t)) = N_k(t) \exp \left[ -\frac{\tilde{\sigma}_k(t)^2}{L_\parallel L_\perp G_\perp(t,k)} \right],
\]

where \( G_\perp(t,k) \) is the transverse propagator,

\[
G_\perp(t,k) = \frac{1}{L_\parallel L_\perp} \langle |\tilde{\sigma}_k(t)|^2 \rangle
\]

and \( N_k(t) = [\pi L_\parallel L_\perp \tilde{G}_\perp(t,k)]^{-1} \) is the normalization. The transverse propagator is easily computed from Eq. (16),

\[
G_\perp(t,k) = \alpha T_\eta \hat{k}^2 \Big( 1 - e^{-2\gamma_k t} \Big).
\]

The anisotropic order parameter \( m(t) \) (defined in Eq. (5)) can be calculated easily from Eqs. (19) and (17),

\[
m(t) = \frac{2\pi N_k(t)}{L_\parallel L_\perp} \int_0^\infty dt r^2 \exp \left[ -\frac{r^2}{L_\parallel L_\perp G_\perp(t,k)} \right] = \frac{\pi}{4} G_\perp(t,k_1)
\]

where \( k_1 \equiv 2\pi/L_\perp \) indicates the first non-zero mode allowed in the transverse direction.

We are particularly interested in the dynamical behavior of \( m(t) \) in the short-time regime, i.e., immediately after the critical quench, when the system is far from reaching its stationary state. In this regime, one can expand the exponential in Eq. (19) and keep only the linear term in \( t \), finding, for any \( \tau \),

\[
G_\perp(t,k_1) = 2\alpha T_\eta \hat{k}_1^2 t + O(t^2).
\]

In the thermodynamic limit, i.e., for large \( L_\perp \), we have \( \hat{k}_1 = 2\sin(k_1/2) \simeq 2\pi/L_\perp \). To the leading order in \( t \), then, Eq. (20) implies,

\[
m(t) \simeq \sqrt{2\pi^3} T_\eta \frac{t}{L_\parallel L_\perp},
\]

for \( t \ll L_\perp^4 \). This spells a clearer meaning to the term short-time regime: this behavior is expected to hold up to a time which is much shorter than the time-scale set by the system size.

At longer times, instead, \( m(t) \) approaches a stationary value \( m_S \) which can also be obtained from Eq. (20). In particular, at the critical point \( \tau = 0 \),

\[
m_S \equiv \lim_{t \to \infty} m(t) = \sqrt{T_\eta L_\perp/16\pi L_\parallel},
\]

which depends only on the isotropic aspect ratio \( L_\perp/L_\parallel \).

In order to predict the behavior of the order parameter \( O \), defined in Eq. (8), we first note that as each single mode \( \tilde{\sigma}_k \) (see Eq. (13)) is a stochastic variable with a Gaussian distribution, their sum in Eq. (8) also has a Gaussian distribution. Accordingly,

\[
O(t) = \sqrt{\frac{\pi}{4} L_\perp L_\parallel} D(t),
\]

where \( D(t) \) is the sum of the transverse propagators of the modes appearing in Eq. (5), i.e.,

\[
D(t) = \sum_{n_\perp = 1}^{L_\perp - 1} G_\perp(t, \frac{2\pi n_\perp L_\perp}{L_\parallel}).
\]

For sufficiently large \( L_\perp \) one can take the continuum limit of this expression and the sum over \( n_\perp \) is replaced by a momentum integral; at the critical point \( \tau = 0 \), one then finds,

\[
D(t) \simeq T_\eta L_\perp \int_{-\pi/2\pi}^{\pi} \frac{dk}{2\pi} \frac{1 - e^{-2\alpha T_\eta k^2}}{k^2}
\]
\[
\frac{T_\eta L_\perp}{\pi^2} \left\{ (2\alpha t)^{1/4} \left[ \Gamma \left( \frac{3}{4} \right) - \Gamma \left( \frac{3}{4}, 2\alpha t \pi^2 \right) \right] + e^{-2\alpha t \pi^2} - 1 \right\},
\]
where \(\Gamma(x)\) is the Gamma function and \(\Gamma(x, y)\) is the incomplete Gamma function; see, e.g., Eq. 8.2.2 in Ref. [27]. In particular, for large enough \(\alpha \gg 1\),
\[
D(t) \approx \frac{T_\eta L_\perp}{\pi} \Gamma \left( \frac{3}{4} \right) (2\alpha t)^{1/4}.
\]
Accordingly, from Eq. (25), \(O\) grows, in this intermediate time regime, as
\[
O(t) \sim \frac{t^{1/8}}{2L_\parallel^{1/2}} \sqrt{T_\eta (2\alpha t)^{1/4} \Gamma \left( \frac{3}{4} \right)},
\]
\(i.e., O(t) \sim t^{1/8}\) upon increasing \(t\). We emphasize here that the limits \(L_\perp \to \infty\) and \(t \to \infty\) do not commute. To obtain the stationary value \(O_S\) of \(O\), one can perform a direct summation in Eq. (25) with \(G_L\) given by Eq. (19), and get,
\[
\lim_{t \to \infty} D(t) = \frac{T_\eta}{12} \left( \frac{L_\perp}{2} - 1 \right).
\]
Accordingly, from Eq. (25), assuming \(L_\perp \gg 1\),
\[
O_S \equiv \lim_{t \to \infty} O(t) = \frac{T_\eta \pi}{48} \frac{L_\perp}{L_\parallel}.
\]
One comment is in order here. As we will show in the next Section, the short-time behavior of the order parameters \(m\) and \(O\) predicted on the basis of the effective Gaussian theory (in Eqs. (22) and (28)) holds in driven lattice gases irrespective of the system size and of any specific geometrical aspect ratio of the lattices [20]. On the other hand, the stationary state, reached at larger times, is different for the various lattice gases and it is only for the specific case of IDLG that the JSCL theory predicts a Gaussian behavior of transverse modes, also in the stationary state. Consequently, the behaviors of \(m\) and \(O\), in the stationary state, as predicted by the Gaussian theory (in Eqs. (23) and (29), respectively) are expected to hold only for the IDLG assuming the appropriate anisotropic scaling.

The auto-correlation \(C_m(s, t)\) of the order parameter \(m\), defined in Eq. (3), can also be easily calculated within the Gaussian model discussed here. The joint distribution of \(\hat{\sigma}_k(s)\) and \(\hat{\sigma}_k(t)\) following from Eq. (16) is nothing but a multi-variate Gaussian distribution
\[
P[\hat{\sigma}_k(s), \hat{\sigma}_k(t)] = \frac{1}{4\pi^2 D} \exp \left\{ -\frac{1}{2D} \left[ \lambda_{\perp} |\hat{\sigma}_k(s)|^2 + \lambda_{\parallel} |\hat{\sigma}_k(t)|^2 - 2\lambda_{\parallel} \text{Re} \left[ \hat{\sigma}_k(s) \hat{\sigma}_k^*(t) \right] \right] \right\}
\]
where \(\ast\) denotes the complex conjugate, and
\[
\lambda_{\parallel, \perp} = \langle |\text{Re} \hat{\sigma}_k(t_1)| |\text{Re} \hat{\sigma}_k(t_2)| \rangle
\]
with \(D = \lambda_{ss} \lambda_{tt} - \lambda_{st}^2 > 0\). To obtain the auto-correlation \(C_m(s, t)\) of the lowest mode \(\mu\) with \(k = k_1\), we need to compute a double spherical integral,
\[
\langle |\mu(s)| \mu(t) \rangle = \frac{1}{4\pi^2 D} \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 r_1^2 r_2^2 \times \exp \left\{ -\frac{1}{2D} \left[ \lambda_{\perp} r_1^2 + \lambda_{\parallel} |r_1|^2 - 2\lambda_{\parallel} r_1 r_2 \cos(\theta_2 - \theta_1) \right] \right\}
\]
\[
= \sqrt{\lambda_{\parallel} \lambda_{ss}} \left[ 2E(y) - (1 - y) K(y) \right]
\]
where \(y = \frac{\lambda_{\parallel}}{\lambda_{\parallel} + \lambda_{tt}}\). Here \(K(x)\) and \(E(x)\) are the Legendre’s complete Elliptic integrals of the first and second kind, respectively; see Sec. 19.2 in Ref. [27].

In the short-time regime where \(s < t < \xi_k^{-1}\), one has \(\lambda_{ss} \lambda_{tt} \approx (a T_\eta V)^2 k^4 s t\) and \(\lambda_{st}^2 / (\lambda_{tt} \lambda_{ss}) \approx s/t\). Moreover, for small \(x\),
\[
K(x) = \frac{\pi}{2} + \frac{\pi x}{8} + O(x^2),
\]
\[
E(x) = \frac{\pi}{2} - \frac{\pi x}{8} + O(x^2).
\]
Combining Eq. (32) with Eq. (33) and using Eq. (20) yields the connected correlation function (defined in Eq. (10)); to the leading order in \(s/t\),
\[
C_m(s, t) = a T_\eta k^2 \frac{\pi}{8} \left( \frac{s}{t} \right)^{3/2}.
\]
This behavior is expected to hold in the short-time regime, i.e., for \(s/t \lesssim 1\).

A useful indicator of deviation from the Gaussian behavior is the so-called Binder cumulant \(g\) [29]. Its appropriate definition for systems with conserved order parameter has been proposed in Ref. [30].
\[
g = 2 - \frac{\langle |\mu|^4 \rangle}{\langle |\mu|^2 \rangle^2},
\]
where \(\mu\) is defined before Eq. (5). For a Gaussian field, \(\langle |\mu|^4 \rangle = 2\langle |\mu|^2 \rangle^2 = 2V^2 C_\parallel\), and thus the Binder cumulant vanishes. Its possible finite value is therefore a good measure of the deviation from a Gaussian behavior.

In the following Secs. IV and V we compare the predictions of the Gaussian theory with the results of numerical simulations in the three different lattice gas models, both in the short-time regime and in the stationary state.

**IV. **THE SHORT-TIME REGIME

We perform Monte Carlo simulations to determine the dynamical behavior of the order parameters \(m\) and \(O\) and their auto-correlations in all the three lattice gas models introduced above, namely, LG, IDLG, and RDLG. The simulations are done on two-dimensional rectangular lattices of size \(L_\parallel \times L_\perp\) where \(\parallel\) and \(\perp\) denote the directions
parallel and transverse to the driving field in IDLG and RDLG, and arbitrary directions in LG. Periodic boundary conditions are assumed in both the spatial directions. Each Monte Carlo step, which sets the unit of time, consists of $V = L_L L_L$ attempted jumps.

In each case, the system is prepared initially in a disordered configuration corresponding to the stationary state at $T \to \infty$ in which both the order parameters $m$ and $O$ vanish. The time evolution is studied at the critical temperature $T_c$, which is different for the three models with $T_{c,\text{IDLG}} = 2.269$ [21], $T_{c,\text{RDLG}} = 3.20$ [10] and $T_{c,\text{LG}} = 3.15$ [20], (see also Sec. V B below for the determination of $T_{c,\text{LG}}$) respectively. In Sec. V C below we will also consider the time evolution of the Binder cumulant starting from different initial conditions and study how this affects the short-time Gaussian behavior.

### A. Evolution of the order parameters

The behavior of the order parameters $m$ and $O$ agree very well with the predictions of the Gaussian theory in Eqs. (22) and (25), the only exception being the case of $O$ in LG [20]. This can be seen in Fig. 3 where we compare $m(t)$ and $O(t)$ for different models for the same system size. In the short-time regime the curves corresponding to IDLG and RDLG are almost identical with $m(t) \sim t^{1/2}$ and $O(t) \sim t^{1/8}$. For LG, instead, $O(t) \sim t^{1/10}$ while the anisotropic order parameter $m(t)$ still shows a $t^{1/2}$ growth, consistent with a Gaussian behavior (see Ref. [20] for more details).

The Gaussian theory provides a way to determine the normalization constants $\alpha$ and $T_\eta$ independently. From a fit of the curves of $m(t)$ and $O(t)$ according to Eqs. (22) and (25) in the short-time regime (excluding possible lattice effects for very small $t$) one can determine the combinations $\alpha T_\eta$ and $\alpha^{1/4} T_\eta$ respectively. These values along with the individual estimates of $\alpha$ and $T_\eta$ obtained using them are reported in Table I. The values of $\alpha$ and $T_\eta$ for the IDLG and the RDLG are very close, consistent with their behavior as seen in Fig. 3. We have also checked that these values do not depend significantly on the system size. For the LG, instead, only $m$ follows the Gaussian prediction and we can determine the combination $\alpha T_\eta$ only, not the individual parameters and this estimate of $\alpha T_\eta$ (see lowest row on Table I) differs considerably from those for the driven lattice gases.

It is interesting to note that $T_\eta/T_c$ is very close to unity for both IDLG and RDLG. This suggests that the dynamics of the lowest transverse modes at short-times is not only ruled by an effective Gaussian model leading to a linear Langevin equation, but also that this dynamics occurs as in an equilibrium system at the same temperature as that ruling the particle transitions on the lattice transversely to the driving field.

### B. Auto-correlation of the order parameter

Close to a phase transition, the temporal auto-correlation of the order parameter also typically carries the signature of the universal critical behavior [31]. This has fact been used in the literature to distinguish between different universality classes of driven lattice gases by studying, e.g., the particle density auto-correlation of the IDLG [10]. In this view, it is interesting to explore the behavior of the auto-correlation of the anisotropic order parameter $m$ for the various lattice gas models and compare it with the prediction of the Gaussian theory.

To this end, we measure the auto-correlation $C_m(s,t)$ (defined in Eq. (4)) of the order parameter $m$ in the short-time regime after a critical quench for all the three models using Monte Carlo simulations. Figure 7 shows plots of $C_m(s,t) t^\zeta$ as a function of $s/t$ for IDLG (panel (a)), RDLG (panel (b)) and LG (panel (c)) where $\zeta$ is the exponent obtained from the best collapse of the data. In particular, we obtain $\zeta = 0.96(2)$ for the IDLG, $\zeta = 0.96(2)$ for the RDLG and $\zeta = 0.95(2)$ for the LG. All these three values agree rather well with the prediction $\zeta = 1$ of the Gaussian theory, see Eq. (24). Moreover, the behaviors of the scaled curves is also consistent with the Gaussian theory in all the cases, showing a growth $\sim (s/t)^{1/2}$ upon increasing $s/t$ (dashed red lines in Fig. 3). Accordingly, we conclude that it is not possible to distinguish between the different lattice gas models even on the basis of the the auto-correlation of the anisotropic order parameter $m$ in the short-time regime.

|       | $\alpha T_\eta$ | $\alpha^{1/4} T_\eta$ | $\alpha$  | $T_\eta$ | $T_\eta/T_c$ |
|-------|-----------------|----------------------|----------|----------|-------------|
| IDLG  | 0.23264         | 1.6485               | 0.0745   | 3.166    | 0.989       |
| RDLG  | 0.23374         | 1.62275              | 0.0755   | 3.096    | 0.983       |
| LG    | 0.06987         | –                    | –        | –        | –           |

TABLE II. Values of normalization factors $T_\eta$ and $\alpha$ as obtained from the temporal growth of $m(t)$ and $O(t)$ in the different lattice gas models. The system size $L_L \times L_L$ used to determine these quantities are 1024 × 64 for the IDLG, 144 × 48 for the RDLG, and 128 × 64 for the LG.
However, it is to be noted that other two-time quantities like the density auto-correlation, which cannot be expressed as a function of the transverse modes only, can be successfully used in order to discriminate the different models even in the short-time regime, as it has been demonstrated in Ref. [10]. This fact clearly shows that, in the presence of a local conservation law, an attentive choice of observables is necessary in order to be able to distinguish between different universality classes and that some choices turn out to be inadequate at short-times in spite of the fact that they naturally appear as being bona fide order parameters.

C. Binder cumulant: dependence on the initial condition

The Binder cumulant $g$ is an effective measure of Gaussian behavior or deviation therefrom. Beyond its widespread applications in equilibrium statistical physics, it has also been used in the context of nonequilibrium lattice gases in order to characterize the stationary state behavior [25, 26]. More recently, $g$ has been used to show that the dynamical behavior of the first non-trivial transverse mode is well described by a Gaussian theory up to a time which scales as $L^3$ in all the three different lattice gas models [26]. However, the stationary value of the Binder cumulant conclusively distinguishes between these three universality classes. It is therefore natural to ask what is the origin of the observed “super-universal” Gaussian behavior in the short-time regime and, via the analysis of the behavior of $g$, to investigate how much of it depends on the specific choice of the initial condition, chosen to be disordered in Ref. [25]. Accordingly, in the following we explore the dynamical behavior of the Binder cumulant starting from the various initial conditions depicted schematically in Fig. 5. In particular, we consider the following configurations:

I. Disordered configuration: This corresponds to a typical configuration at high temperature, as the particles are distributed randomly; we ensure that the magnetization on each row is exactly zero so that $m$ vanishes in this state. This initial condition is the one used to study the behavior of the order parameters in Sec. IV.

II. Column-ordered configuration: This initial condition resembles a phase-separated state but the interface is orthogonal to the direction of the field and hence the anisotropic order parameter $m$ vanishes. Note that this configuration corresponds to one of the two equivalent low-temperature configurations of the LG on a square lattice.

III. Mixed-ordered configuration: For this initial condition the particles are arranged on the lattice in order to form a checker board pattern, the top right and bottom left sub-rectangles are the only ones being occupied. Also in this configuration the order parameter $m$ vanishes.
IV. Row-ordered configuration: Here we start from the phase separated state, with the interface being parallel to the direction of the drive. In the case of the LG, this is taken to be the $x$-direction, mimicking the ordered configuration in the driven cases. This configuration corresponds to a finite non-zero value of $m$.

The Binder cumulant $g$ is computed, as discussed in Sec. III with reference to the first transverse mode, according to Eq. (35).

Figure 6 shows plots of the time evolution of the Binder cumulant $g$ starting from these various initial configurations for all the three lattice gas models, at the corresponding critical temperatures. Although the three initial conditions I, II, and III all correspond to a vanishing value of the order parameter $m$, the particle distributions in space are very different in the three cases. However, in each case, after an initial transient there is an intermediate regime where the transverse fluctuations are Gaussian, as indicated by the vanishingly small value of $g$ (see the light orange, dark green, and purple curves in Fig. 6(a)). This observation reinforces the idea that, at criticality, the short-time evolution of the transverse modes of the lattice gases is indeed governed by a Gaussian dynamics as in Eq. (16) as long as the initial configuration of the lattice is a not-ordered one, i.e., with a vanishing initial value of the order parameter.

Note that, the LG, in contrast with the IDLG and the RDLG, shows a more pronounced initial nonzero stretch. Also, for the LG with the column initial condition II, the onset of growth of the Binder is marked by an unexpected dip. These features bear signature of the fact that the LG is, in some way, “less Gaussian” than the driven lattice gases. In fact, it is rather surprising that the Binder cumulant shows a vanishingly small value for a considerably long time for the LG (see Fig. 6(c)), because, as it is well known, it is actually described by an interacting $\phi^4$ theory characterized by a non-vanishing stationary value of the cumulant $g$ [17].

In order to understand the short-time behavior of the Binder cumulant in the LG model we perform a perturbative analysis for the $\phi^4$ theory around the Gaussian fixed point. We calculate the evolution of $g$ for a small interaction strength $\nu$, as defined in Eq. (9). It turns out that the growth of $g$ is slowed down by a factor of $k^4$ compared to the non-conserved case. Consequently, for the first transverse mode with the smallest value of $k$, $g$ appears to be vanishingly small; see Appendix A for the details.

The configuration IV corresponds to an ordered state, and in this case, for all the models considered, the Binder cumulant $g$ starts from unity and monotonically decreases towards the stationary value (uppermost, light brown curves in the plots of Fig. 6). In the long-time limit, as expected, $g$ attains the same stationary value irrespective of the initial conditions, depending only on the specific model. This is clearly shown by all curves in Fig. 6.

V. THE STATIONARY STATE

The stationary state of the lattice gas models bears the signatures of the specific universality class, displaying different behavior for the three different models considered in this study. In the case of the IDLG, also the stationary behavior of transverse observables is described by the Gaussian theory in the limit of large system size, as predicted by the corresponding JSLC theory [22, 23]. However, this is not the case for LG and RDLG, the stationary properties of which are significantly different from those predicted by a Gaussian theory. In the following we discuss the stationary behavior of the order parameters $m$ and $O$ in the driven lattice gases and compare them with the Gaussian behavior.

A. Stationary values of the order parameters

The predictions of the Gaussian theory for the stationary values of $m$ and $O$ are reported in Eqs. (23) and (29), respectively. Accordingly, at the critical temperature, these transverse observables in the IDLG should
reach the stationary values,

\[ m_S = \sqrt{\frac{T_c L_\perp}{16\pi L_\parallel}}, \quad \text{and} \quad O_S = \sqrt{\frac{T_c \pi L_\perp}{48 L_\parallel}}. \tag{36} \]

which depend on the geometry of the lattice only via the isotropic aspect ratio \( L_\perp / L_\parallel \).

An alternative way to predict the finite-size behavior of \( m_S \) and \( O_S \) is to use the scaling theory which demands that, at the critical point, the order parameter vanishes as

\[ m_S \sim L_\perp^{-\beta/\nu}, \tag{37} \]

upon increasing the system size \( L_\perp \). To connect the prediction of the scaling theory with that of the Gaussian theory, we need to express the behavior of \( m_S \) as a function of the isotropic aspect ratio \( L_\perp / L_\parallel \). In order to do so, we remember that the finite-size scaling of the driven lattice gases has to be performed at a fixed anisotropic aspect ratio \( L_\perp / L_\parallel \). In order to distinguish between the IDLG and RDLG, the Binder cumulant attains a stationary value at the critical point, which does not depend on the system size. Here we use this fact in order to determine the proper value of \( \Delta \) for the different models.

The finite-size scaling behavior of the Binder cumulant is also widely used to determine the value of the critical temperature in various equilibrium systems [20]. The method relies on the fact that, for certain systems, including LG, the Binder cumulant attains a stationary value at the critical point, which does not depend on the system size. Here we use this fact in order to determine the critical temperature of the RDLG. Figure 6 shows the plot of the stationary value \( g_S \) of the Binder cumulant as a function of the temperature \( T \) for a set of lattice sizes \( L_\parallel \times L_\perp \) with fixed \( S_\Delta = 2^{-2} \) and \( \Delta = 1 \).

### B. Stationary values of the Binder cumulant

An effective and direct way of distinguishing between the different universality classes of lattice gas models is to investigate the Binder cumulants in the stationary state [20, 21]. In the thermodynamic limit, the stationary value of Binder cumulant defined in Eq. (25) vanishes at the critical temperature in the case of the IDLG (consistently with a Gaussian behavior) while it converges to a value independent of the system size for the RDLG and the LG. Once again, these scaling behaviors are observed as long as the finite-size scaling is performed at fixed \( S_\Delta \), with the proper value of \( \Delta \) for the different models.

The finite-size scaling behavior of the Binder cumulant is also widely used to determine the value of the critical temperature in various equilibrium systems [20]. The method relies on the fact that, for certain systems, including LG, the Binder cumulant attains a stationary value at the critical point, which does not depend on the system size. Here we use this fact in order to determine the critical temperature of the RDLG. Figure 8 shows the plot of the stationary value \( g_S \) of the Binder cumulant as a function of the temperature \( T \) for different geometries, but with a fixed \( S_\Delta = 2^{-2} \) for \( \Delta = 1 \). The crossing point of the curves provides an accurate estimate of the critical temperature \( T_\text{c}^{\text{RDLG}} = 3.150(5) \). This value has been used throughout this work and also in Ref. [20].
VI. CONCLUSIONS

In this work we have investigated the critical behavior of conserved lattice gas models, both driven and undriven, and the possibility to describe it with an effective Gaussian theory. The three models studied here, namely, the infinitely strongly driven lattice gas (IDLG), the randomly driven lattice gas (RDLG) and the equilibrium lattice gas (LG), belong to three different universality classes. However, in Ref. [20] it was shown that in the short-time dynamics after a critical quench, all these conserved lattice gases behave in a similar way, which is consistent with the transverse modes being described by a Gaussian theory. Hence the dynamics of transverse observables in this regime cannot be used to infer the universality classes of the different models. Here we elaborate and substantially extend the ideas and results anticipated in Ref. [20], providing additional analytical results and numerical evidence.

The phase transitions in these conserved driven lattice gas models are characterized by considering the behavior of the order parameters which are “transverse” in nature, meaning that they are insensitive to fluctuations along the direction of the drive. The dynamics of two of such order parameters $m$ and $O$ (see Eqs. (5) and (5)), and the auto-correlation of $m$ (see Eq. (6)) are then predicted analytically assuming that the transverse modes are effectively described by a Gaussian theory. These predictions of the Gaussian theory compared with the results of Monte Carlo simulations from which it emerges that the dynamical behavior of both the order parameters $m$ and $O$ and the auto-correlation of $m$ agree very well with the Gaussian theory, in all the three different models, with the sole exception of $O$ in LG (see Fig. 3 and 4). On this basis a unified short-time behavior emerges for the driven and undriven lattice gases, irrespective of the fact that their critical behaviors actually belong to different universality classes.

The Gaussian theory also provides a way to determine normalization constants which appear to be arbitrary in the usual field-theoretic description. We extract the effective coarse-grained diffusion constant $\alpha$ and the effective temperature $T_\eta$ associated with the noise from the short-time behavior of the two order parameters in the driven lattice gases. The value of $\alpha$ and $T_\eta$, for the IDLG and the RDLG, turn out to be almost equal. For the LG, however, one can only determine the product $\alpha T_\eta$, and the corresponding value turns out to be quite different from that for IDLG and RDLG.

To investigate the origin of the short-time Gaussian behavior we have studied the dynamical evolution of the Binder cumulant $g$ starting from various initial conditions, including the fully disordered one. It appears that, as long as the order parameter vanishes in the initial configuration, the Binder cumulant signals a Gaussian-like behavior for a considerably long duration as it remains very close to zero (see Fig. 5).

We have also studied the behavior of $m$, $O$ and $g$ in these models in the stationary state. In contrast to the short-time regime, the stationary state bears the signatures of the specific critical behavior. In fact, the stationary behavior of $g$ is very different in the three different lattice gases investigated here. We exploit the finite-size behavior of the stationary value of $g$ in the RDLG in order to determine accurately the corresponding critical temperature. However, it turns out that, even in the stationary state, the dependence of $m$ and $O$ on the isotropic aspect ratio does not discriminate between the IDLG and RDLG universality classes.

In summary, we have shown that the short-time behavior of the transverse observables in the lattice gas models, both driven and undriven, is described by an effective Gaussian theory irrespective of them belonging to different universality classes. The origin of this “super-universal” behavior in the short-time regime may be related with the presence of a local density conservation in all these systems, which slows down the dynamics considerably [20]. However, the various models display their distinct critical behaviors in the stationary states. Our work emphasizes the importance of the choice of order parameters, particularly in the presence of conservation laws.

Appendix A: Perturbative calculations

The critical behavior of the equilibrium LG is known to be described by an isotropic $\phi^4$ effective theory. However, the behavior of the Binder cumulant $g$ measured in Monte Carlo simulations (see Fig. 5) is seemingly consistent with the Gaussian theory in the short-time regime following a critical quench. To understand this surprising fact, we perform a first-order perturbative calculation of the Binder cumulant for the LG. For simplicity, we assume the lattice to be sufficiently large to replace it with a continuum.

The time evolution of the coarse-grained spin-field $\phi(x,t)$ in LG is governed by the Langevin equation (9) where $u$ denotes the strength of the perturbation. To linear order in $u$,

$$\phi(x,t) = \phi^0(x,t) + u \phi^1(x,t)$$  \hspace{1cm} (A1)

where $\phi^0(x,t)$ is the solution of Eq. (9) with $u = 0$ (Gaussian) and $\phi^1(x,t)$ is the perturbative correction. It is useful to recall that the Fourier transform $\hat{\phi}^0_k(t)$ of $\phi^0(x,t)$ is the continuum version of Eq. (10), i.e.,

$$\hat{\phi}^0_k(t) = ik \int_0^t ds \eta_k(s)e^{-\gamma_k(t-s)}$$  \hspace{1cm} (A2)

where $\gamma_k = \alpha k^2(x+k^2)$ (see Eq. (14) with $\hat{k} \to k$), and $\eta_k$ is the white noise on the continuum with $\langle \eta_k(t)\eta_k(t') \rangle = 2\alpha T_\eta(2\pi)^d \delta(k+k')\delta(t-t')$. The time evolution of the Fourier transform $\hat{\phi}^0_k(t)$ of the linear correction $\phi^1(x,t)$ follows from Eq. (6) which takes the form

$$\frac{d}{dt} \hat{\phi}^0_k(t) = -\gamma_k \hat{\phi}^0_k(t) - \alpha k^2 f_k(t),$$  \hspace{1cm} (A3)
where $f_k(t)$ is the Fourier transform of $[\phi^0(x,t)]^3$, i.e.,

$$f_k(t) = \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} \phi^0_{k-k_1}(t) \phi^0_{k-k_1}(t) \phi^0_{k}(t). \quad (A4)$$

Eq. (A3) is can be solved and yields,

$$\phi^0_k(t) = -\alpha k^2 \int_0^t ds \ e^{-\gamma_s(t-s)} f_k(s). \quad (A5)$$

To compute the Binder cumulant of the $k$-th mode, as defined in Eq. (A5), we need to evaluate the second and fourth moment of $\phi^0_k(t)$. For the Gaussian theory, the Binder cumulant vanishes while, to the leading order in $u$ it takes the value

$$g = \left[ \frac{4\delta_2}{\langle |\phi^0_k|^2 \rangle^2} - \frac{\delta_4}{\langle |\phi^0_k|^4 \rangle^2} \right] \quad (A6)$$

where the coefficients $\delta_2$ and $\delta_4$ are correlations between the Gaussian field $\phi^0$ with the linear correction $\phi^1$:

$$\delta_2 = \langle \phi^0_k \phi^1_k \rangle + \langle \phi^0_{-k} \phi^0_k \rangle$$

$$\delta_4 = 2[\langle \phi^0_k \phi^0_{-k} \phi^1_{k'} \rangle + \langle \phi^0_k \phi^0_{-k} \phi^0_{-k} \phi^1_{k'} \rangle]. \quad (A7)$$

Here all the fields $\phi^0_k$ and $\phi^1_k$ are evaluated at the same time $t$. Since $\phi^1_k$ contains product of three $\phi^0_k$, $\delta_2$ and $\delta_4$ are four- and six-point correlations of the Gaussian field which can be evaluated via Wick’s theorem. The two contributions in $\delta_4$ are connected by a $k \to -k$ exchange and it is straightforward to see that

$$\delta_4 = 4\langle |\phi^0_k|^2 \rangle^2 \delta_2 - 4\bar{g}, \quad (A8)$$

where $\bar{g}$ contains the contributions from the “connected” terms, i.e., terms in which each $\phi^0_k(t)$ is contracted with a $\phi^0_k(s)$ for a time $t > s$. There are six such connected contributions which can be obtained explicitly from Eqs. (A4) and (A5). It is easy to see that they all contribute the same and we finally get

$$\bar{g} = 6\alpha k^2 \int_0^t ds \ e^{-\gamma_s(t-s)} \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} \times \langle \phi^0_k(t) \phi^0_{k-k_1}(s) \rangle \langle \phi^0_k(t) \phi^0_{k-k_1}(s) \rangle \langle \phi^0_k(t) \phi^0_{k}(s) \rangle. \quad (A9)$$

The auto-correlation of the Gaussian field $\phi^0_k(t)$ is directly obtained from Eq. (A2),

$$\langle \phi^0_k(t) \phi^0_{k}(s) \rangle = \alpha T_s \frac{(2\pi)^d k^2}{\gamma_k} e^{-\gamma_k(t-s)} (1 - e^{-2\gamma_k}) \delta(k+k')$$

for $t > s$. Using Eq. (A10), the momentum integrals in Eq. (A9) can be calculated, and, at the critical point (i.e., with $\gamma_k = \alpha k^2$) we get,

$$\bar{g} = \frac{6\alpha T_s^3 (2\pi)^d}{k^4} \int_0^t ds \ e^{-\gamma_s(t-s)} \left( e^{\alpha k^4 s} - e^{-\alpha k^4 s} \right)^3$$

$$= 6T_s^3 (2\pi)^d \left[ \frac{1}{4} + \frac{1}{2} e^{-6\alpha k^4 t} + 3 e^{-4\alpha k^4 t} \left( \frac{1}{4} + \alpha k^4 t \right) - \frac{3}{2} e^{-2\alpha k^4 t} \right]. \quad (A11)$$

Finally, combining Eqs. (A6), (A8), and (A11) we get the linear correction to the Binder cumulant for a $\phi^4$ theory,

$$g = \frac{4u\bar{g}}{\langle |\phi^0_k|^2 \rangle^2} = \frac{24uT_s}{(2\pi)^d k^4 (1 - e^{-2\alpha k^4 t})^2} \left[ \frac{1}{4} + \frac{1}{2} e^{-6\alpha k^4 t}ight.$$

$$+ 3 e^{-4\alpha k^4 t} \left( \frac{1}{4} + \alpha k^4 t \right) - \frac{3}{2} e^{-2\alpha k^4 t} \right]. \quad (A12)$$

We are particularly interested in the behavior of $g$ in the short-time regime, which can be obtained by expanding the exponential in Eq. (A12) and by keeping the lowest order terms in $t$. It turns out that the Binder cumulant grows quadratically upon increasing $t$,

$$g \sim k^4 t^2 + O(t^3). \quad (A13)$$

In order to appreciate the role of the local conservation of $\phi$ it is useful to repeat this calculation for the non-conserved field, in which case $\gamma_k = \alpha k^2$ at the critical point. Following the same steps, one finds,

$$g \sim t^2 + O(t^3). \quad (A14)$$

This lowest order perturbative calculation is strictly valid around the upper critical dimensionality $d_c = 4$ of the model. However, the qualitative feature that is important here is the fact that in the conserved case in Eq. (A13) the growth of $g$ is reduced by a factor of $k^4$ compared to the non-conserved case in Eq. (A14). For the first mode on a large lattice of linear size $L$, $k \sim 1/L$ and hence $g$ appears to be vanishingly small for the LG. This heuristic calculation provides a way to see how drastically conservation can alter the dynamical behavior of a system.

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