Scattering on the Dirac magnetic monopole

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Abstract
We construct wave operators and a scattering operator for the scattering of a charged particle on the Dirac magnetic monopole. The analysis features a two-Hilbert-space approach in which the identification operator matches states of the same angular momentum.

Keywords Magnetic monopole · Scattering · Vector bundle

Mathematics Subject Classification 81U05

1 Introduction

We study the scattering of a single charged particle (an electron) by a magnetic monopole. The magnetic field of the monopole is described by a connection on a $U(1)$ vector bundle $E$ over $M = \mathbb{R}^3 - \{0\}$, the electron wave functions are sections of that vector bundle in $L^2(E)$, and there is a self-adjoint Hamiltonian $H$ on this space generating the dynamics. For the scattering problem, we identify asymptotic states which are elements of $L^2(\mathbb{R}^3)$ with dynamics generated by the free Hamiltonian $H_0$. Scattering is described by Møller wave operators which are proved to exist in a two-Hilbert-space formulation. They have the form

$$\Omega_\pm = \lim_{t \to \pm \infty} e^{iHt} J e^{-iH_0 t} \Psi$$

where $J : L^2(\mathbb{R}^3) \to L^2(E)$ identifies states of the same angular momentum. We also include results in which the monopole Hamiltonian $H$ is perturbed by a potential $V$ yielding a new Hamiltonian $H + V$.

This scattering problem has been previously treated by Petry [6], who takes asymptotic states which are also sections of $L^2(E)$ rather than $L^2(\mathbb{R}^3)$. Petry’s choice of the asymptotic dynamics is somewhat ad hoc and difficult to connect with the usual free dynamics. This means his scattering results are quoted in terms of “scattering into cones”, rather than the more standard asymptotic wave functions. Our two-Hilbert-space formulation allows for a more direct connection with the usual free dynamics.
space treatment puts the problem more in the mainstream of scattering theory. It opens the door for further developments such as the perturbing potential, which seems awkward in the Petry formulation.

The paper is organized as follows. In Sect. 2, we review the description of the monopole as a connection on a vector bundle. In Sect. 3, we review the definition of the free Hamiltonian \( H_0 \) in spherical coordinates. In Sect. 4, we give a detailed definition of the monopole Hamiltonian \( H \) also in spherical coordinates. It is a map on smooth sections of the vector bundle, and we show that it defines a self-adjoint operator on \( L^2(E) \). In Sect. 5, we find continuum eigenfunction expansions for the radial parts of both \( H_0 \) and \( H \), and in Sect. 6, this is used to give detailed estimates on the free dynamics. In Sect. 7, we prove the main result which is the existence of the wave operators. Finally, in Sect. 8 we extend the scattering result by including a potential.

2 The monopole

In spherical coordinates \((r, \theta, \phi)\), the magnetic field for a monopole of strength \( n \in \mathbb{Z}, \ n \neq 0 \), is the two-form (\( \star \) is the Hodge star operation)

\[
B = \star \frac{n}{r^2} dr = n \sin \theta d\theta d\phi. \tag{1}
\]

This is singular at the origin but otherwise is closed (\( dB = 0 \)) as required by Maxwell’s equations. However, it is not exact (\( B \neq dA \) for any \( A \)). If it were exact, the integral over the unit sphere would be zero, but \( \int_{|x|=1} B = 4\pi n \). Locally, one can take

\[
A = -n \cos \theta d\phi \tag{2}
\]

since then \( dA = B \). But this is singular at \( x_1 = x_2 = 0 \) as one can see from the representation in Cartesian coordinates

\[
A = n \frac{x_3}{|x|} \frac{x_2 dx_1 - x_1 dx_2}{x_1^2 + x_2^2}. \tag{3}
\]

This is a problem since we need the magnetic potential \( A \) to formulate the quantum mechanics.

The remedy is to introduce the vector bundle \( E \) defined as follows. First, it is a manifold and there is a smooth map \( \pi : E \to M = \mathbb{R}^3 - \{0\} \) such that each fiber \( E_x = \pi^{-1}(x) \) is a vector space isomorphic to \( \mathbb{C} \). Further, let \( U_\pm \) be an open covering of \( M \) defined for \( 0 < \alpha < \frac{1}{2}\pi \) as follows. First, in spherical coordinates and then in Cartesian coordinates

\[
U_+ = \{ x \in M : 0 \leq \theta < \frac{\pi}{2} + \alpha \} = \{ x \in M : 1 \geq \frac{x_3}{|x|} > \cos \left( \frac{\pi}{2} + \alpha \right) \}
\]

\[
U_- = \{ x \in M : \frac{\pi}{2} - \alpha < \theta \leq \pi \} = \{ x \in M : \cos \left( \frac{\pi}{2} - \alpha \right) > \frac{x_3}{|x|} \geq -1 \}. \tag{4}
\]
We require that in each region there is a trivialization (diffeomorphism)
\[ h_{\pm}: \pi^{-1}(U_{\pm}) \rightarrow U_{\pm} \times \mathbb{C} \tag{5} \]
such that for \( x \in U_{\pm} \) the map \( h_{\pm}: E_x \rightarrow \{x\} \times \mathbb{C} \) is a linear isomorphism. They are related by the transition function in \( U_+ \cap U_- \)
\[ h_+h_-^{-1} = e^{2in\phi} \tag{6} \]
where \( e^{2in\phi} \) acts on the second entry \( \mathbb{C} \). With the transition functions specified, \( E \) can be constructed as equivalence classes in \( M \times \mathbb{C} \) with \( (x, v) \sim (x, e^{2in\phi}v) \) if \( x \in U_+ \cap U_- \).

The connection is defined by a one-form \( A_{\pm} \) on \( U_{\pm} \). To compensate (6), they are related in \( U_+ \cap U_- \) by the gauge transformation
\[ A_{\pm} = A_- + 2n d\phi. \tag{7} \]
This is accomplished by taking instead of (2) and (3)
\[ A_{\pm} = -n(\cos \theta \mp 1) d\phi = n \left( \frac{x_3}{|x|} \mp 1 \right) \frac{x_2 dx_1 - x_1 dx_2}{x_1^2 + x_2^2}. \tag{8} \]
Each of these satisfies \( dA_{\pm} = B \), but now they have no singularity. Indeed, on \( U_+ \) we have for points with \( x_3 > 0 \)
\[ \left| \frac{x_3}{|x|} - 1 \right| \leq O \left( \frac{x_1^2 + x_2^2}{x_3^2} \right). \tag{9} \]
So, for fixed \( x_3 > 0 \) there is no singularity at \( x_1 = x_2 = 0 \). Points in \( U_+ \) with \( x_3 \leq 0 \) also have \( x_1^2 + x_2^2 > 0 \), so the singularity is avoided. Similarly, \( A_- \) has no singularity on \( U_- \).

Now, we can define a covariant derivative on sections of \( E \). A section of \( E \) is a map \( \psi: M \rightarrow E \) such that \( \pi(\psi(x)) = x \). The set of all smooth sections is denoted \( \Gamma(E) \). For \( f \in \Gamma(E) \), we define \( \nabla_k f \in \Gamma(E) \) by specifying that for \( x \in U_{\pm} \) if \( h_{\pm}(x) = (x, f_{\pm}(x)) \) then \( \nabla_k f \) satisfies \( h_{\pm}(\nabla_k f(x)) = (x, (\nabla_k f)_\pm(x)) \) where
\[ (\nabla_k f)_\pm = (\partial_k - i A_k^\pm) f_\pm. \tag{10} \]
Here, \( A_k^\pm \) are the components in \( A_{\pm} = \sum_k A_k^\pm dx_k \). This defines a section since in \( U_+ \cap U_- \) we have \( A_k^+ = A_k^- + 2n \partial \phi / \partial x_k \) so
\[ (\partial_k - i A_k^+) e^{2in\phi} = e^{2in\phi}(\partial_k - i A_k^-). \tag{11} \]
Thus, if \( f \) is a section, then \( f_+ = e^{2in\phi} f_- \), then \( (\nabla_k f)_+ = e^{2in\phi}(\nabla_k f)_- \), and hence, the pair \( (\nabla_k f)_\pm \) defines a section.
3 Free Hamiltonian

We first review the standard treatment of the free Hamiltonian. This will recall some facts we need and provide a model for the treatment of the monopole Hamiltonian. The free Hamiltonian on $L^2(\mathbb{R}^3)$ is minus the Laplacian:

$$H_0 = -\Delta = -\sum_i \partial_i \partial_i$$  \hspace{1cm} (12)

defined initially on smooth functions.

We study it as a quadratic form and begin by breaking it into radial and angular parts by

$$\langle f, H_0 f \rangle = \sum_i \| \partial_i f \|^2$$

$$= \sum_{i,j} (\partial_i f, \frac{x_i x_j}{|x|^2} \partial_j f) + \sum_{i,j} \left( \partial_i f, (\delta_{ij} - \frac{x_i x_j}{|x|^2}) \partial_j f \right)$$

$$= \left\| \frac{1}{|x|} (x \cdot \partial) f \right\|^2 + \sum_i \left\| \frac{1}{|x|} (x \times \partial_i) f \right\|^2.$$  \hspace{1cm} (13)

The last step follows from $(x \times \partial)_i = \sum_{jk} e_{ijk} x_j \partial_k$ and $\sum_i e_{ijk} e_{i\ell m} = \delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell}$. ($e_{ijk}$ is the Levi–Civita symbol.) The skew-symmetric operators $(x \times \partial)_i$ are recognized as a basis for the representation of the Lie algebra of the rotation group $SO(3)$ generated by the action of the group on $\mathbb{R}^3$. In quantum mechanics, the symmetric operators $L_i = -i (x \times \partial)_i$ are identified as the angular momentum. They satisfy the commutation relations $[L_i, L_j] = \sum_k e_{ijk} i L_k$ or $[L_1, L_2] = i L_3$, etc. Now, we have

$$\langle f, H_0 f \rangle = \left\| \frac{1}{|x|} (x \cdot \partial) f \right\|^2 + \sum_i \left\| \frac{1}{|x|} L_i f \right\|^2.$$  \hspace{1cm} (14)

Next, we change to spherical coordinates. The $|x|^{-1} (x \cdot \partial f)$ becomes $\partial f / \partial r$, and the $L_i$ becomes

$$L_1 = i \left( \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right)$$

$$L_2 = i \left( -\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right)$$

$$L_3 = -i \frac{\partial}{\partial \phi}.$$  \hspace{1cm} (15)

The Hamiltonian in spherical coordinates, still called $H_0$, has become

$$\langle f, H_0 f \rangle = \left\| \frac{\partial f}{\partial r} \right\|^2 + \sum_i \left\| \frac{1}{r} L_i f \right\|^2.$$  \hspace{1cm} (16)
The norms are now in the space
\[ \mathcal{H}_0 = L^2(\mathbb{R}^+ \times S^2, r^2 \, d\Omega) = L^2(\mathbb{R}^+, r^2 \, dr) \otimes L^2(S^2, d\Omega) \] (17)

where \( \mathbb{R}^+ = (0, \infty) \) and \( d\Omega = \sin \theta \, d\theta \, d\phi \) is the Haar measure on \( S^2 \). The \( L_i \) are symmetric in \( L^2(S^2, d\Omega) \), and after an integration by parts in the radial variable, we have
\[ H_0 f = -\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} f + \frac{L^2}{r^2}. \] (18)

Here, \( L^2 = L_1^2 + L_2^2 + L_3^2 \) is the Casimir operator for the representation of the Lie algebra of the rotation group on \( S^2 \). This is a case where it is equal to minus the Laplacian on \( S^2 \)
\[ L^2 = -\Delta_2 = -\left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \phi} \right) \] (19)
as can be checked directly.

The spectrum of \( L^2 \) on \( L^2(S^2, d\Omega) \) is studied by considering the joint spectrum of the commuting operators \( L^2, L_3 \). This is a standard problem in quantum mechanics. It is also the problem of breaking down the representation of the rotation group into irreducible pieces. Just from the commutation relations, one finds the \( L^2 \) can only have the eigenvalues \( \ell(\ell + 1) \) with \( \ell = 0, 1, 2, \ldots \) and that \( L_3 \) can only have integer eigenvalues \( m \) with \( |m| \leq \ell \). The corresponding normalized eigenfunctions are the spherical harmonics \( Y_{\ell,m}(\theta, \phi) \) and are explicitly constructed in terms of the Legendre polynomials. They satisfy
\[ L^2 Y_{\ell,m} = \ell(\ell + 1) Y_{\ell,m} \quad \ell \geq 0 \]
\[ L_3 Y_{\ell,m} = m Y_{\ell,m} \quad |m| \leq \ell. \] (20)

The spherical harmonics are complete so this gives the full spectrum of \( L^2, L_3 \) and yield a definition of corresponding self-adjoint operators.

Let \( \mathcal{K}_{0,\ell} \) be the \( 2\ell + 1 \)-dimensional eigenspace for the eigenvalue \( \ell(\ell + 1) \) of \( L^2 \). Then, \( \mathcal{K}_{0,\ell} \) is spanned by \( \{Y_{\ell,m}\}_{|m| \leq \ell} \). Then, we can write the Hilbert space as
\[ \mathcal{H}_0 = \bigoplus_{\ell=0}^{\infty} L^2(\mathbb{R}^+, r^2 \, dr) \otimes \mathcal{K}_{0,\ell} \] (21)

and on smooth functions in this space \( H_0 = \bigoplus_{\ell} (h_{0,\ell} \otimes I) \) where
\[ h_{0,\ell} = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell + 1)}{r^2}. \] (22)

We study the operator \( h_{0,\ell} \) further in Sects. 5 and 6.
4 Monopole Hamiltonian

The Hamiltonian for our problem is initially defined on smooth sections \( f \in \Gamma(E) \) by

\[
Hf = -\sum_{k=1}^{3} \nabla_k \nabla_k f.
\]  

(23)

We want to define it as a self-adjoint operator in \( L^2(E) \). In this section, we reduce it to a radial problem as for \( H_0 \). The treatment more or less follows Wu and Yang [8].

The Hilbert space \( L^2(E) \) is defined as follows. If \( x \in U_\pm \) and \( v \in E_x \), then \( h_\pm v = (x, v_\pm) \) and we define \( |v| = |v_\pm| \). This is unambiguous since if \( x \in U_+ \cap U_- \) then \( v_\pm \) only differ by a phase and so \( |v_+| = |v_-| \). Similarly, if \( v, w \in E_x \) we can define \( \overline{vw} \in \mathbb{C} \). The Hilbert space \( L^2(E) \) is all measurable sections \( f \) such that the norm \( \|f\|^2 = \int |f(x)|^2 dx \) is finite with \( (g, f) = \int \overline{g(x)f(x)} dx \).

The covariant derivative \( \nabla_k \) is skew-symmetric in this Hilbert space; hence, the Hamiltonian is symmetric. Indeed, if \( \text{supp} f, \text{supp} g \subset U_\pm \) and \( h_\pm f(x) = (x, f_\pm(x)) \), etc., then

\[
(g, \nabla_k f) = \int \overline{g_\pm(\partial_k - iA^\pm_k)f_\pm} = -\int (\partial_k - iA^\pm_k)g_\pm f_\pm = -(\nabla_k g, f). 
\]  

(24)

In the general case, we write a section \( f \) as a sum \( f(x) = f_+(x) + f_-(x) \) with \( \text{supp} f_\pm \subset U_\pm \).

Now, write the Hamiltonian as a quadratic form, and as in (13) break it into a radial and angular parts

\[
(f, Hf) = \sum_k \|\nabla_k f\|^2 = \left\| \frac{1}{|x|} (x \cdot \nabla) f \right\|^2 + \sum_k \left\| \frac{1}{|x|} (x \times \nabla) f \right\|^2 .
\]  

(25)

Now, there is a problem. The operators \( (x \times \nabla)_k \), although they have something to do with rotations, no longer give a representation of the Lie algebra of the rotation group. The commutators now involve extra terms \( [\nabla_j, \nabla_k] = -iF_{jk} \) where \( F_{jk} = \partial_j A^\pm_k - \partial_k A^\pm_j \) is the magnetic field. This is not special to the monopole but occurs whenever there is an external magnetic field. The resolution due to Fierz [2] is to add a term proportional to the field strength. Instead of \(-i(x \times \nabla)_k = (x \times -i\nabla)_k \), we define angular momentum operators by

\[
\mathcal{L}_k = (x \times -i\nabla)_k - n \frac{x_k}{|x|}.
\]  

(26)
These are symmetric and do satisfy the commutators $[\mathcal{L}_i, \mathcal{L}_j] = i \sum_k e_{ijk} \mathcal{L}_k$. This follows from commutators like

$$[\mathcal{L}_i, x_j] = i \sum_k e_{ijk} x_k$$

$$[\mathcal{L}_i, \nabla_j] = i \sum_k e_{ijk} \nabla_k.$$  \hspace{1cm} (27)

To give the idea, we show that $[\mathcal{L}_1, \nabla_2] = i \nabla_3$. We have

$$[\mathcal{L}_1, \nabla_2] = -i[(x \times \nabla)_1, \nabla_2] - n[x_1|x|^{-1}, \nabla_2]$$

$$= -i[x_2 \nabla_3 - x_3 \nabla_2, \nabla_2] - n x_1 x_2 |x|^{-3}$$

$$= i \nabla_3 - i x_2 [\nabla_3, \nabla_2] - n x_1 x_2 |x|^{-3}$$

$$= i \nabla_3 - x_2 F_{32} - n x_1 x_2 |x|^{-3}.$$ \hspace{1cm} (28)

But in $U_\pm$ we have $A_3^\pm = 0$ and taking $A_2^\pm$ from (8)

$$x_2 F_{32} = x_2 \partial_3 A_2^\pm$$

$$= x_2 n \partial_3 \left( x_3 \left| x \right| \pm 1 \right) \frac{-x_1}{x_1^2 + x_2^2}$$

$$= n x_2 \frac{|x|^2 - x_3^2}{|x|^3} \frac{-x_1}{x_1^2 + x_2^2}$$

$$= -n \frac{x_1 x_2}{|x|^3}.$$ \hspace{1cm} (29)

Thus, the second and third terms in (28) exactly cancel and hence the result.

Now, since $[(x \times \nabla)_k, n x_k |x|^{-1}] = 0$ and $x \cdot (x \times \nabla) = 0$, we have that

$$\mathcal{L}^2 = \sum_k \mathcal{L}_k^2 = \sum_k (x \times -i \nabla)^2_k + n^2.$$ \hspace{1cm} (30)

The gauge field has no radial component so $x \cdot \nabla = x \cdot \partial$ and $[(x \times -i \nabla)_k, |x|^{-1}] = 0$, so (25) becomes

$$(f, H f) = \left\| \frac{1}{|x|} (x \cdot \partial) f \right\|^2 + \left( f, \frac{1}{|x|^2} (\mathcal{L}^2 - n^2) f \right).$$ \hspace{1cm} (31)

Next, change to spherical coordinates. The vector bundle $\pi : E \to M$ becomes a vector bundle $\pi : E' \to \mathbb{R}^+ \times S^2$. With $U_\pm' \subset S^2$ defined as in (4), these have trivializations $h_\pm : \pi^{-1}(\mathbb{R}^+ \times U_\pm') \to (\mathbb{R}^+ \times U_\pm') \times \mathbb{C}$ which still have transition functions $h_+ h_-^{-1} = e^{2i \phi}$. The $|x|^{-1}(x \cdot \nabla f)$ becomes $\partial f / \partial r$, and the $\mathcal{L}_k$ becomes operators on $\Gamma(E')$ specified by saying that for $x \in \mathbb{R}^+ \times U_\pm'$, $(\mathcal{L}_k f)(x)$ satisfies

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\( h_\pm(\mathcal{L}_k f)(x) = (x, \mathcal{L}_k^\pm f_\pm(x)) \) where

\[
\begin{align*}
\mathcal{L}_1^\pm &= i \left( \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) - n \cos \phi \left( \frac{1 \mp \cos \theta}{\sin \theta} \right) \\
\mathcal{L}_2^\pm &= i \left( - \cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right) - n \sin \phi \left( \frac{1 \mp \cos \theta}{\sin \theta} \right) \\
\mathcal{L}_3^\pm &= -i \frac{\partial}{\partial \phi} \mp n.
\end{align*}
\]

(32)

Note that since \((\partial/\partial \phi)e^{2i\phi} = e^{2i\phi}\partial/\partial \phi + 2in\) we have in \(U'_+ \cap U'_-\) the required \(\mathcal{L}_i^+e^{2i\phi} = e^{2i\phi}\mathcal{L}_i^-\).

The Hamiltonian in spherical coordinates, still called \(H\), has become

\[
(f, Hf) = \left\| \frac{\partial f}{\partial r} \right\|^2 + \left( f, \frac{1}{r^2}(\mathcal{L}^2 - n^2) f \right)
\]

where now the norms and inner products are in \(\mathcal{H} = L^2(E', r^2 dr d\Omega)\). After an integration by parts, this implies

\[
Hf = -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r} + \frac{1}{r^2}(\mathcal{L}^2 - n^2).
\]

(34)

In fact, since the transition functions only depend on the angular variables, we can make the identification

\[
\mathcal{H} = L^2(\mathbb{R}^+, r^2 dr) \otimes L^2(\tilde{E}, d\Omega)
\]

(35)

where \(\tilde{E}\) is a vector bundle \(\pi : \tilde{E} \to S^2\) with trivializations \(h_\pm : \pi^{-1}(U'_\pm) \to U'_\pm \times \mathbb{C}\) which still satisfy \(h_+ h_-^{-1} = e^{2i\phi}\). Now, in (34) the \(\mathcal{L}^2 - n^2\) only acts on the factor \(L^2(\tilde{E}, d\Omega)\).

The joint spectrum of \(\mathcal{L}_2, \mathcal{L}_3\) has been studied by Wu and Yang [8]. The commutation relations again constrain the possible eigenvalues to \(\ell(\ell + 1)\) and \(|m| \leq \ell\). But now from (30) we have \(\mathcal{L}^2 \geq n^2\) so we must have \(\ell \geq |n|\). Only states with nonzero angular momentum exist on the monopole. Wu–Yang explicitly construct the eigenfunctions in terms of Jacobi polynomials. The normalized eigenfunctions \(Y_{n,\ell,m}(\theta, \phi)\) are sections of \(L^2(\tilde{E})\) called \textit{monopole harmonics}. They satisfy

\[
\begin{align*}
\mathcal{L}_2^2 Y_{n,\ell,m} &= \ell(\ell + 1)Y_{n,\ell,m} \quad \ell \geq |n| \\
\mathcal{L}_3 Y_{n,\ell,m} &= mY_{n,\ell,m} \quad |m| \leq \ell.
\end{align*}
\]

(36)

Explicitly, they are given in the trivializations on \(U'_\pm = U_\pm \cap S^2\) by

\[
Y_{n,\ell,m}(\xi, \phi) = \text{const}(1 - \xi)^{\frac{1}{2}\alpha}(1 + \xi)^{\frac{1}{2}\beta} P_{\ell+m}^{\alpha,\beta}(\xi) e^{i(m\pm n)\phi} \quad \xi = \cos \theta
\]

(37)
where \( \alpha = -n - m, \beta = n - m \) and \( P_{\ell+m}^{\alpha,\beta} \) are Jacobi polynomials given by

\[
P_{\ell+m}^{\alpha,\beta}(\xi) = \text{const}(1 - \xi)^{-\alpha}(1 + \xi)^{-\beta} \frac{d^{\ell+m}}{d\xi^{\ell+m}}(1 - \xi)^{\alpha+\ell+m}(1 + \xi)^{\beta+\ell+m}.
\] (38)

Completeness follows from the completeness of the Jacobi polynomials. Thus, the \( Y_{n,\ell,m} \) give the full spectrum of \( L^2, L_3 \) and yield a definition of these as self-adjoint operators.

Let \( K_{n,\ell} \) be the \( 2\ell + 1 \)-dimensional eigenspace in \( L^2(\tilde{E}, d\Omega) \) for the eigenvalue \( \ell(\ell + 1) \) of \( L^2 \). Then \( K_{n,\ell} \) is spanned by the \( \{ Y_{n,\ell,m} \}_{m \leq \ell} \). We now can write the Hilbert space as

\[
\mathcal{H} = \bigoplus_{\ell = |n|}^\infty L^2(\mathbb{R}^+, r^2 dr) \otimes K_{n,\ell}
\] (39)

and on smooth functions in this space \( H = \bigoplus_\ell (h_\ell \otimes I) \) where

\[
h_\ell = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell + 1) - n^2}{r^2}.
\] (40)

The operators \( h_\ell \) are essentially self-adjoint on \( C_0^\infty(\mathbb{R}^+) \). For an operator of this form, the condition is that the coefficient of the \( 1/r^2 \) term be \( \geq \frac{3}{4} \) (see Reed-Simon [4], p. 159–161 and earlier references). Here, we have \( \ell(\ell + 1) - n^2 \geq \ell \geq |n| \geq 1 \) which suffices. This means the repulsion from the \( 1/r^2 \) potential is strong enough to keep the particle away from the origin and a boundary condition at the origin is not needed.

(This is not the case for the free radial Hamiltonian \( h_{0,\ell} \) with \( n = 0, \ell = 0 \) in which case the \( 1/r^2 \) is absent and a boundary condition is needed. This case does not occur in this paper where \( \ell \geq 1 \).)

The self-adjoint \( h_\ell \) determines a unitary group \( e^{-ih_\ell t} \). This generates a unitary group on \( \mathcal{H} \), and we define \( H \) to be the self-adjoint generator. So, \( e^{-iHt} = \bigoplus_\ell (e^{-ih_\ell t} \otimes I) \).

### 5 Eigenfunction expansions

Domains of self-adjointness for \( h_{0,\ell} \) will be obtained by finding continuum eigenfunction expansions (c.f. Petry [6]). But we start with a more general operator

\[
h(\mu) = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\mu^2 - \frac{1}{4}}{r^2}
\] (41)

with \( \mu > 0 \). As is well-known, the continuum eigenfunctions have the form \( (kr)^{-\frac{1}{2}} J_\mu(kr) \) where \( J_\mu \) is the Bessel function of order \( \mu \) regular at the origin and we have

\[
h(\mu)((kr)^{-\frac{1}{2}} J_\mu(kr)) = k^2 ((kr)^{-\frac{1}{2}} J_\mu(kr)).
\] (42)
Expansions in the eigenfunctions are given by Fourier–Bessel transforms and we recall the relevant facts. (See, for example, Titchmarsh [7], where, however, results are stated with Lebesgue measure \( dr \) rather than \( r^2 \) employed here.) The transform

\[
\psi_\mu^\#(k) = \int_0^\infty (kr)^{-\frac{1}{2}} J_\mu(kr) \psi(r)r^2 dr
\]

(43)
defined initially for say \( \psi \) in the dense domain \( C_0^\infty(\mathbb{R}^+) \) satisfies

\[
\int_0^\infty |\psi_\mu^\#(k)|^2 k^2 dk = \int_0^\infty |\psi(r)|^2 r^2 dr
\]

(44)
and extends to a unitary operator from \( L^2(\mathbb{R}^+, r^2 dr) \) to \( L^2(\mathbb{R}^+, k^2 dk) \). It is its own inverse

\[
\psi(r) = \int_0^\infty (kr)^{-\frac{1}{2}} J_\mu(kr) \psi_\mu^\#(k) k^2 dk.
\]

(45)
Now, for \( \psi_\mu^\# \in C_0^\infty(\mathbb{R}^+) \) we have that \( \psi(r) \) is a smooth function and

\[
(h(\mu)\psi)(r) = \int_0^\infty (kr)^{-\frac{1}{2}} J_\mu(kr)(k^2 \psi_\mu^\#(k))k^2 dk.
\]

(46)
We use this formula to define \( h(\mu) \) as a self-adjoint operator with domain

\[
D(h(\mu)) = \{ \psi \in L^2(\mathbb{R}^+, r^2 dr) : k^2 \psi_\mu^\#(k) \in L^2(\mathbb{R}^+, k^2 dk) \}.
\]

(47)
The formula (46) provides the spectral resolution and so there is a unitary group

\[
(e^{-ih(\mu)t} \psi)(r) = \int_0^\infty (kr)^{-\frac{1}{2}} J_\mu(kr)e^{-i k^2 t} \psi_\mu^\#(k) k^2 dk.
\]

(48)
Now, if \( \mu = \ell + \frac{1}{2} \) then \( \mu^2 - \frac{1}{4} = \ell(\ell+1) \) and we have the free operator \( h_{0,\ell} \). Thus, with \( \psi^\# = \psi_{\ell+\frac{1}{2}}^\# \) the operator \( h_{0,\ell} \) is self adjoint on \( \{ \psi : k^2 \psi^\#(k) \in L^2(\mathbb{R}^+, k^2 dk) \} \).

The unitary group \( e^{-ih_{0,\ell}t} \) generates a unitary group on \( \mathcal{H}_0 \), and we define \( H_0 \) to be the self-adjoint generator. So, \( e^{-i H_0 t} = \bigoplus_\ell e^{-i h_{0,\ell} t} \otimes I \).

(Note also that if \( \mu = ((\ell + \frac{1}{2})^2 - n^2)^{\frac{1}{2}} \), then \( \mu^2 - \frac{1}{4} = \ell(\ell+1) - n^2 \) and we have the monopole operator \( h_\ell \). We do not use this representation here; see, however, [1].)

\section{6 The free dynamics}

We need more detailed control over the domain of operator \( h_{0,\ell} \) and the associated dynamics \( e^{-ih_{0,\ell} t} \). The eigenfunctions can be written

\[
\frac{1}{\sqrt{kr}} J_{\ell + \frac{1}{2}}(kr) = \sqrt{\frac{2}{\pi}} j_\ell(kr)
\]

(49)
\[\square\]
where $j_\ell(x)$ are the spherical Bessel functions which are given for $x > 0$ by

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell + \frac{1}{2}}(x) = (-x)^\ell \left( \frac{1}{x} \frac{d}{dx} \right)^\ell \frac{\sin x}{x}. \quad (50)$$

They are entire functions which are bounded for $x$ real and have the asymptotics

$$j_\ell(x) = \begin{cases} O(x^\ell) & x \to 0 \\ O(x^{-1}) & x \to \infty. \end{cases} \quad (51)$$

Now, we have the transform pair with $\psi^\# = \psi^\#_{\ell + \frac{1}{2}}$

$$\psi^\#(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty j_\ell(kr) \psi(r) r^2 dr \quad \psi(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty j_\ell(kr) \psi^\#(k) k^2 dk \quad (52)$$

which still define unitary operators.

**Lemma 1** If $\psi^\# \in C_0^\infty(\mathbb{R}^+)$, then for any $N$

$$\psi(r) = \begin{cases} O(r^\ell) & r \to 0 \\ O(r^{-N}) & r \to \infty. \end{cases} \quad (53)$$

Furthermore, $\psi$ is infinitely differentiable and the derivatives satisfy for any $N$

$$\psi^{(m)}(r) = \begin{cases} O(r^{\ell-m}) & r \to 0 \\ O(r^{-N}) & r \to \infty. \end{cases} \quad (54)$$

**Proof** In (52) $k$ is bounded above and below and so $j_\ell(kr)$ has asymptotics (51) in $r$, and hence, $\psi(r)$ satisfies (53) with $N = 1$.

To improve the long distance asymptotics, we use the identity

$$xj_\ell(x) = (\ell + 2) j_{\ell+1}(x) + xj'_{\ell+1}(x) \quad (55)$$

to write (52) as

$$\sqrt{\frac{\pi}{2}} \psi(r) = r^{-1} \int_0^\infty krj_\ell(kr)(k\psi^\#(k)) dk = r^{-1} \int_0^\infty (\ell + 2) j_{\ell+1}(kr)\left(k\psi^\#(k)\right) dk + \int_0^\infty j'_{\ell+1}(kr)\left(k^2\psi^\#(k)\right) dk. \quad (56)$$

The integral in the first term has the same form that we started with and we have an extra $r^{-1}$ in front, so the term is $O(r^{-2})$ as $r \to \infty$. After integrating by parts, the
second term can be written
\[
\frac{1}{r} \int_0^\infty \frac{d}{dk} j_{\ell+1}(kr)\left(k^2 \psi^{\#}(k)\right)dk = -\frac{1}{r} \int_0^\infty j_{\ell+1}(kr)\left(\frac{d}{dk}(k^2 \psi^{\#}(k))\right)dk.
\] (57)

Again the integral has the same form that we started with and there is an extra \(r^{-1}\) so the term is \(O(r^{-2})\). Thus, we have proved \(\psi(r) = O(r^{-2})\) as \(r \to \infty\). Repeating the argument gives \(\psi(r) = O(r^{-N})\) as \(r \to \infty\). Thus, (53) is established.

For the derivative, we use \(d/dr(j_\ell(kr)) = kr^{-1}d/dk(j_\ell(kr))\) and integration by parts to obtain
\[
\sqrt{\pi} \frac{d}{dr} \psi(r) = \int_0^\infty \frac{k}{r} \frac{d}{dk} j_\ell(kr)\psi^{\#}(k)k^2dk
\]
\[
= -\frac{1}{r} \int_0^\infty j_\ell(kr)\frac{d}{dk}\left(k^3 \psi^{\#}(k)\right)dk.
\] (58)

The integral is of the same form as we have been considering and so has the asymptotics (53). But we have an extra factor \(r^{-1}\) and so (54) is proved for \(m = 1\). Repeating the argument gives the general case. \(\Box\)

The free dynamics (48) is now expressed as
\[
(e^{-ih_0,\ell t}\psi)(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty j_\ell(kr)e^{-ik^2t}\psi^{\#}(k)k^2dk.
\] (59)

**Lemma 2** Let \(\psi^{\#} \in C_0^\infty(\mathbb{R}^+)\) and \(N > 0\). Then, there exists a constant \(C\) such that for \(0 < r \leq 1\), \(|t| \geq 1\)
\[
|e^{-ih_0,\ell t}\psi(r)| \leq Cr^\ell |t|^{-N}.
\] (60)

**Proof** In (59) \(k\) is bounded, hence \(j_\ell(kr) = O(r^\ell)\) as \(r \to 0\), and hence, \(|e^{-ih_0,\ell t}\psi(r)|\) is \(O(t^\ell)\) as \(r \to 0\) as in the previous lemma.

Now, in (59) we write
\[
e^{-ik^2t} = \frac{1}{-2ikt} \frac{d}{dk} e^{-ik^2t}
\] (61)

and then integrate by parts. This yields
\[
\sqrt{\frac{\pi}{2}} (e^{-ih_0,\ell t}\psi)(r)
\]
\[
= \frac{1}{2it} \int_0^\infty e^{-ik^2t} \frac{d}{dk} \left(j_\ell(kr) k\psi^{\#}(k)\right)dk
\]
\[
= \frac{1}{2it} \int_0^\infty e^{-ik^2t} \left(\kappa r j_\ell(kr) \psi^{\#}(k) + j_\ell(kr) \frac{d}{dk}(k\psi^{\#}(k))\right)dk
\]
\[
= \frac{1}{2it} \int_0^\infty e^{-ik^2t} \left(-kr j_{\ell+1}(kr) \psi^{\#}(k) + \ell j_\ell(kr) \psi^{\#}(k) + j_\ell(kr) \frac{d}{dk}(k\psi^{\#}(k))\right)dk.
\] (62)
Here, we used the identity
\[ xj'_\ell(x) = -xj_{\ell+1}(x) + \ell j_\ell(x). \] (63)

In the integral, each term has the same form we started with (possibly with an extra factor of \( r \)) and so are \( O(r^\ell) \). But we have gained a power of \( t^{-1} \) so this shows that \(|e^{-i\theta_0,\ell} \psi(r)| \leq O(r^\ell |t|^{-1})\) Repeating the argument gives the bound \( O(r^\ell |t|^{-N}) \). \( \square \)

7 Scattering

Now, we are ready to consider the scattering of a charged particle off a magnetic monopole. We use a two-Hilbert-space formalism which has been found useful elsewhere (see for example [5], p 34; the idea goes back to Kato [3]). Recall that the monopole Hilbert space is the space of sections
\[ \mathcal{H} = L^2(\mathbb{R}^+, r^2 dr) \otimes L^2(\tilde{E}, d\Omega) = \bigoplus_{\ell = |n|}^\infty L^2(\mathbb{R}^+, r^2 dr) \otimes K_{n,\ell} \] (64)

with dynamics \( e^{-iHt} \). The asymptotic space is
\[ \mathcal{H}_0 = L^2(\mathbb{R}^+, r^2 dr) \otimes L^2(S^2, d\Omega) = \bigoplus_{\ell = 0}^\infty L^2(\mathbb{R}^+, r^2 dr) \otimes K_{0,\ell} \] (65)

with dynamics \( e^{-iH_0t} \). To compare them, we need an identification operator \( J : \mathcal{H}_0 \to \mathcal{H} \). We define \( J \) by matching angular momentum eigenstates, taking account that for the monopole only states with \( \ell \geq |n| \) occur. Thus, we define \( J \) as a partial isometry by specifying
\[ J(\psi \otimes Y_{\ell,m}) = \begin{cases} \psi \otimes Y_{n,\ell,m} & \ell \geq |n| \\ 0 & 0 \leq \ell < n. \end{cases} \] (66)

The Møller wave operators are to be defined on \( \mathcal{H}_0 \) as
\[ \Omega_{\pm} \psi = \lim_{t \to \pm \infty} e^{iHt} J e^{-iH_0t} \psi \] (67)

if the limit exists. They vanish for \( \psi \) in the subspace of \( \mathcal{H}_0 \) with \( \ell < |n| \). The issue is whether they exist for \( \Psi \) in
\[ \mathcal{H}_{0,|n|} \equiv \bigoplus_{\ell = |n|}^\infty L^2(\mathbb{R}^+, r^2 dr) \otimes K_{0,\ell}. \] (68)
If they exist, then we have identified states with specified asymptotic form

$$e^{-iHt} \Omega_{\pm} \Psi \to Je^{-iH_0t} \Psi \quad \text{as } t \to \pm \infty.$$  

(69)

Only states with angular momentum $\ell(\ell+1)$, $\ell \geq 1$ occur in the asymptotics. Then, we can define a scattering operator

$$S = \Omega^*_+ \Omega_-$$

(70)

which maps $\mathcal{H}_{0, \geq |n|}$ to $\mathcal{H}_{0, \geq |n|}$.

The main result is:

**Theorem 1** The wave operators $\Omega_{\pm}$ exist.

**Proof** For $\Psi \in \mathcal{H}_{0, \geq |n|}$, we have $\|e^{iHt} Je^{-iH_0t} \Psi\| = \|\Psi\|$, so we can approximate $\Psi$ uniformly in $t$ and it suffices to prove the limit exists for $\Psi$ in a dense set. In fact, it suffices to consider $\Psi = \psi \otimes Y_{\ell, m}$ with $\psi \# \in C_0^\infty(\mathbb{R}^+)$ and $\ell \geq |n| \geq 1$ since finite sums of such vectors are dense. Since $e^{-iH_0t} \Psi = e^{-ih_{0, \ell}t} \Psi \otimes Y_{\ell, m}$ and $e^{iHt} \Psi = e^{ih_{\ell}t} \Psi \otimes Y_{n, \ell, m}$, the problem reduces to the existence in $L^2(\mathbb{R}^+, r^2dr)$ of

$$\lim_{t \to \pm \infty} e^{ih_{\ell}t} e^{-ih_{0, \ell}t} \psi \quad \psi \# \in C_0^\infty(\mathbb{R}^+).$$  

(71)

To analyze this we need to know that $e^{-ih_{0, \ell}t} \psi \in D(h_{\ell})$. It suffices to show that $\{\psi : \psi \# \in C_0^\infty(\mathbb{R}^+)\}$ is in $D(h_{\ell})$. By Lemma 1, this subspace is contained in the larger subspace

$$\mathcal{D} \equiv \{\psi \in C^2(\mathbb{R}^+) : \psi \text{ has asymptotics (54) for } m = 0, 1, 2\}$$

(72)

so it suffices to show $\mathcal{D} \subset D(h_{\ell})$. Note that with these asymptotics the derivatives are still in $L^2(\mathbb{R}^+, r^2dr)$. Indeed, with $m \leq 2$ the worst behavior as $r \to 0$ is $O(r^{-1})$ and this is still square integrable with the measure $r^2dr$. Thus, $h_{\ell}$ acting as derivatives is an operator on $\mathcal{D}$ and by integrating by parts it is symmetric. So we have a symmetric extension of the operator $h_{\ell}$ on $C_0^\infty(\mathbb{R}^+)$ and the latter is essentially self-adjoint. Thus $h_{\ell}$ on $\mathcal{D}$ is also essentially self-adjoint with the same closure. In particular $\mathcal{D} \subset D(h_{\ell})$.

Let $\Omega_{\ell} = e^{ih_{\ell}t} e^{-ih_{0, \ell}t}$. We now can compute

$$(\Omega_{\ell'} - \Omega_{\ell}) \psi = \int_t^{t'} \frac{d}{ds} \Omega_{s} \psi ds = \int_t^{t'} e^{ih_{\ell}s} (h_{\ell} - h_{0, \ell}) e^{-ih_{0, \ell}s} \psi.$$  

(73)

But

$$h_{\ell} - h_{0, \ell} = -\frac{n^2}{r^2} \equiv v(r)$$

(74)

and so

$$\| (\Omega_{\ell'} - \Omega_{\ell}) \psi \| \leq \int_t^{t'} \| v e^{-ih_{0, \ell}s} \psi \| ds.$$  

(75)
Now, it suffices to show that the function $t \to \|v e^{-i \theta_0 t} \psi\|$ is integrable to obtain a limit. We write $v = v_1 + v_2$ with supports respectively in $[0, 1]$ and $[1, \infty)$. Since $\psi^# \in C_0^\infty(\mathbb{R}^+)$ Lemma 2 says $|(e^{-i \theta_0 t} \psi)(r)| \leq O(r^\ell |t|^{-N})$ for any $N$ and so

$$\|v_1 e^{-i \theta_0 t} \psi\|^2 = \int_0^1 \frac{n^4}{r^4} |(e^{-i \theta_0 t} \psi)(r)|^2 r^2 dr \leq O(|t|^{-2N}) \int_0^1 r^{2\ell-2} dr \leq O(|t|^{-2N})$$

(76)

which suffices.

For the $v_2$ term we have

$$\|v_2 e^{-i \theta_0 t} \psi\|_{L^2(\mathbb{R}^+, r^2 dr)} = \|v_2 e^{-i \theta_0 t} \psi \otimes Y_{\ell,m}\|_{L^2(\mathbb{R}^+ \times S^2, r^2 dr d\Omega)} = \|v_2 e^{-iH_{0t}} \psi\|_{L^2(\mathbb{R}^+ \times S^2, r^2 dr d\Omega)} = \|v_2 e^{-iH_{0t}} \psi\|_{L^2(\mathbb{R}^3)}.$$  

(77)

In the last step, we have returned to Cartesian coordinates, so now $v_2 = v_2(|x|)$ and $e^{-iH_{0t}} \psi$ is the unitary transform of our $e^{-iH_{0t}} \psi$ defined in spherical coordinates.

We want to show that this time evolution is the usual time evolution defined with the Fourier transform. First, with $t = 0$, $\Psi$ has become $\Psi(x) = \psi(|x|) Y_{\ell,m}(x/|x|)$ with Fourier transform

$$\tilde{\Psi}(k) = (2\pi)^{-\frac{3}{2}} \int e^{ik \cdot x} \psi(|x|) Y_{\ell,m}(x/|x|) dx.$$  

(78)

There is a standard expansion of the complex exponential in spherical functions given by the distribution identity (with $k = |k|$)

$$e^{ik \cdot x} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell \frac{j_\ell(kr)}{r} Y_{\ell,m}(x/|x|) Y_{\ell,m}(x/|x|).$$  

(79)

Inserting this in (78) and changing back to spherical coordinates gives

$$\tilde{\Psi}(k) = 4\pi i^\ell \left( \int_0^\infty j_\ell(kr) \psi(r) r^2 dr \right) Y_{\ell,m}(k/|k|) = (2\pi)^{\frac{3}{2}} i^\ell \psi^#(k) Y_{\ell,m}(k/|k|).$$  

(80)

Now, replace $\psi$ by $e^{-i \theta_0 t} \psi$. Then $\psi^#(k)$ becomes $e^{-i k^2 t} \psi^#(k)$ and so $\tilde{\Psi}(k)$ becomes $e^{-i |k|^2 t} \tilde{\Psi}(k)$. Thus

$$(e^{-iH_{0t}} \psi)(x) = (2\pi)^{-\frac{3}{2}} \int e^{-ik \cdot x} e^{-i |k|^2 t} \tilde{\Psi}(k) dk$$  

(81)

which is the standard time evolution.
By Lemma 1, we have that \( \Psi \in L^1(\mathbb{R}^3, dx) \) since

\[
\| \Psi \|_1 = \int_{\mathbb{R}^3} |\psi(|x|) Y_{\ell,m}(x/|x|)| \, dx = \int_0^\infty |\psi(r)| r^2 \, dr \int_{S^2} |Y_{\ell,m}(\theta, \phi)| \, d\Omega < \infty.
\]

(82)

Thus \( \Psi \in L^1(\mathbb{R}^3, dx) \cap L^2(\mathbb{R}^3, dx) \) and in this case (81) has the well-known representation

\[
(e^{-iH_0 t} \Psi)(x) = (4\pi it)^{-\frac{3}{2}} \int e^{i|x-y|^2/4t} \psi(y) \, dy.
\]

(83)

This gives the estimate \( \| e^{-iH_0 t} \Psi \|_\infty \leq O(|t|^{-3/2}) \).

Now, \( v_2(|x|) \) is in \( L^2(\mathbb{R}^3, dx) \) (since \( \int_1^\infty r^{-4} r^2 \, dr < \infty \)) and so

\[
\| v_2 e^{-iH_0 t} \Psi \|_2 \leq \| v_2 \|_2 \| e^{-iH_0 t} \Psi \|_\infty \leq O(|t|^{-3/2})
\]

(84)

which gives the integrability in \( t \).

8 Perturbations

As an indication of the advantages of the present approach, we show that it can accommodate perturbations. Let \( V(|x|) \) be a smooth bounded spherically symmetric function on \( \mathbb{R}^3 \). This defines a multiplication operator on sections of the vector bundle. We consider instead of the Hamiltonian \( H \) the perturbed Hamiltonian \( H + V \). The corresponding radial Hamiltonian for angular momentum \( \ell \) is instead of (40)

\[
h_{\ell} + V = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \left[ \frac{\ell(\ell+1) - n^2}{r^2} + V(r) \right].
\]

(85)

As a bounded perturbation of \( h_{\ell} \) which is essentially self-adjoint on \( \mathcal{C}^\infty_0(\mathbb{R}) \), it is itself essentially self-adjoint on the same domain. Then, \( h_{\ell} + V \) generates a unitary group on \( L^2(\mathbb{R}^+, r^2 \, dr) \), hence a unitary group on the full Hilbert space \( \mathcal{H} \) and \( H + V \) is the generator so

\[
e^{-i(H+V)t} = \bigoplus_{\ell=|n|}^\infty (e^{-i(h_{\ell}+V)t} \otimes I).
\]

(86)

Theorem 2 Let \( V(|x|) \) be a smooth bounded spherically symmetric function in \( L^2(\mathbb{R}^3) \). Then the wave operators

\[
\Omega_{\pm}(V)\Psi = \lim_{t \to \pm \infty} e^{i(H+V)t} e^{-iH_0 t} \Psi
\]

exist.

Proof As in the proof of Theorem 1, the proof reduces to a demonstration that \( \|(v + V)e^{-ih_0,t} \Psi\| \) is integrable in \( t \) for \( \psi^\# \in \mathcal{C}^\infty_0(\mathbb{R}^+) \). We already know this for \( v = -n^2 / r \), so it suffices to consider \( \|Ve^{-ih_0,t} \psi\| \). We split \( V(r) = V_1(r) + V_2(r) \) with

\[\square\]
supports in $(0, 1]$ and $[1, \infty)$. The term $\| V_1 e^{-ih_0 \ell t} \psi \|$ is integrable as in (76); in fact it is easier since $V_1$ is bounded. For the $V_2$ term we follow the argument (77)–(84) and obtain the integrability from the condition $V_2 \in L^2$.

**Remark** One would like to relax the condition that $V$ be bounded near the origin. A key feature in our method is that $h_\ell + V$ should be essentially self-adjoint on $C_0^\infty (\mathbb{R}^+)$ and referring again to [4] this is true if the bracketed expression in (85) is greater than or equal to $\frac{3}{4}r^{-2}$ near zero. This expression is bounded below by $r^{-2} + V(r)$, so it suffices that

$$V_1(r) \geq -\frac{1}{4} \frac{1}{r^2}.$$  

(88)

With this hypothesis, the self-adjointness holds. If we also require that $V_1(r) = O(r^{-2})$ as $r \to 0$, as well as $V_2 \in L^2$, then the scattering estimates (76) and (84) still hold and the wave operators exist.

Note that the Coulomb potential $V(r) = \pm \frac{q}{r}$ satisfies the $V_1$ conditions; however the condition $V_2 \in L^2$ is violated. As in ordinary potential scattering, the remedy is to modify the free dynamics (see for example [5], p. 169). With this modification the wave operators would exist for the Coulomb potential as well.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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