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A Scheme-Driven Approach to Learning Programs from Input/Output Equations

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Abstract

We describe an approach to learn, in a term-rewriting setting, function definitions from input/output equations. By confining ourselves to structurally recursive definitions we obtain a fairly fast learning algorithm that often yields definitions close to intuitive expectations. We provide a PROLOG prototype implementation of our approach, and indicate open issues of further investigation.

Key words: inductive functional programming

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1 Introduction

This paper describes an approach to learn function definitions from input/output equations.\(^1\) In trivial cases, a definition is obtained by syntactical anti-unification of the given i/o equations. In non-trivial cases, we assume a structurally recursive function definition, and transform the given i/o equations into equations for the employed auxiliary functions. The latter are learned from their i/o equations in turn, until a trivial case is reached.

We came up with this approach in 1994 but didn’t publish it until today. In this paper, we explain it mainly along some learning examples, leaving a theoretical elaboration to be done. Also, we indicate several issues of improvement that should be investigated further. However, we provide at least a PROLOG prototype implementation of our approach.

In the rest of this section, we introduce the term-rewriting setting our approach works in. In Sect. 2, we define the task of function learning. In Sect. 3 and 4, we explain the base case and the inductive case of our approach, that is, how to learn trivial functions, and how to reduce learning sophisticated functions to learning easier functions, respectively. Section 5 sketches some ideas for possible extensions to our approach; it also shows its limitations. Some runs of our PROLOG prototype are shown in Appendix A.

\begin{figure}[h]
\begin{center}
\begin{tabular}{|c|}
\hline
1. nat ::= 0 | s(nat) & natural numbers \\
2. list ::= nil | nat::list & lists of natural numbers \\
3. tree ::= null | nd(tree, nat, tree) & binary trees of natural numbers \\
4. blist ::= nl | o(blist) | i(blist) & list of binary digits \\
\hline
\end{tabular}
\end{center}
\caption{Employed sort definitions}
\end{figure}

We use a term-rewriting setting that is well-known from functional programming: A sort can be defined recursively by giving its constructors. For example, sort definition 1, shown in Fig. 1, defines the sort \texttt{nat} of all natural numbers in 0-s notation. In this example, we use \texttt{0} as a nullary, and \texttt{s} as a unary constructor.

\begin{figure}[h]
\begin{center}
\begin{tabular}{|c|}
\hline
5. + : nat × nat → nat & addition of natural numbers \\
6. * : nat × nat → nat & multiplication of natural numbers \\
7. lgth : list → nat & number of elements of a list \\
8. app : list × list → list & concatenation of lists \\
9. size : tree → nat & number of elements of a binary tree \\
10. dup : nat → nat & duplicating a natural number \\
11. add : blist × blist → blist & addition of binary numbers (lists) \\
\hline
\end{tabular}
\end{center}
\caption{Employed function signatures}
\end{figure}

A sort is understood as representing a possibly infinite set of ground constructor terms,\(^2\) e.g. the

\(^1\) We will use henceforth “i/o equations” for brevity. We avoid calling them “examples” as this could cause confusion when we explain our approach along example sort definitions, example signatures, and example functions.

\(^2\) i.e. terms without variables, built only from constructor symbols
Figure 3. Example function definitions

sort \texttt{nat} represents the set \{0, s(0), s(s(0)), s(s(s(0))), \ldots\}. A function has a fixed signature; Fig. 2 gives some examples. The signature of a constructor can be inferred from the sort definition it occurs in, e.g. 0 : \rightarrow \texttt{nat} and s : \texttt{nat} \rightarrow \texttt{nat}. We don’t allow non-trivial equations between constructor terms, therefore, we have $T_1 = T_2$ iff $T_1$ syntactically equals $T_2$, for all ground constructor terms $T_1, T_2$.

A non-constructor function can be defined by giving a terminating ([DJ90, Sect.5.1, p.270]) term rewriting system for it such that its left-hand sides are sufficiently complete ([Gut77], [Com86], [DJ90, Sect.3.2, p.264]). Examples for function definitions are shown in Fig. 3.

Given some functions $f_1, \ldots, f_m$ defined by such a term rewriting system, for each $i$ and each ground constructor terms $T_1, \ldots, T_n$ we can find a unique ground constructor term $T$ such that $f_i(T_1, \ldots, T_n) = T$. We then say that $f_i(T_1, \ldots, T_n)$ evaluates to $T$.

Given a term $T$, we denote by $\text{vars}(T)$ the set of variables occurring in $T$.

2 The task of learning functions

The problem our approach shall solve is the following. Given a set of sort definitions, a non-constructor function symbol $f$, its signature, and a set of input/output equations for $f$, construct a term rewriting system defining $f$ such that it behaves as prescribed by the i/o equations. We say that we want to learn a definition for $f$, or sloppily, that we want to learn $f$, from the given i/o equations.

For example, given sort definition 1, signature 10, and the following input/output ground equations

\begin{align*}
22: \quad \text{dup}(0) &= 0 \\
23: \quad \text{dup}(s(0)) &= s(s(0)) \\
24: \quad \text{dup}(s(s(0))) &= s(s(s(s(0)))) \\
25: \quad \text{dup}(s(s(s(0)))) &= s(s(s(s(s(0))))))
\end{align*}

we are looking for a definition of $\texttt{dup}$ such that equations 22, 23, 24, and 25 hold. One such definition is

\begin{align*}
26: \quad \text{dup}(0) &= 0 \\
27: \quad \text{dup}(s(x)) &= s(s(\text{dup}(x)))
\end{align*}
We say that this definition covers the i/o equations 22, 23, 24, and 25. In contrast, a definition

\[
\begin{align*}
28: & \quad \text{dup}(0) = 0 \\
29: & \quad \text{dup}(s(x)) = s(s(x))
\end{align*}
\]

would cover i/o equations 22 and 23, but neither 24 nor 25. We wouldn’t accept this definition, since we are interested only in function definitions that cover all given i/o equations.

It is well-known that there isn’t a unique solution to our problem. In fact, given i/o equations \( f(L_1) = R_1, \ldots, f(L_n) = R_n \) and an arbitrary function \( g \) of appropriate domain and range, e.g. the function defined by

\[
f(x) = ( \text{if } x = L_1 \text{ then } R_1 \text{ elif } \ldots \text{ elif } x = L_n \text{ then } R_n \text{ else } g(x) \text{ fi })
\]

trivially covers all i/o equations. Usually, the “simplest” function definitions are preferred, with “simplicity” being some user-defined measure loosely corresponding to term size and/or case-distinction count, like e.g. in [Bur05, p.8] and [Kit10, p.77]. However, the notion of simplicity depends on the language of available basic operations. In the end, the notion of a “good” definition can hardly be defined more precisely than being one that meets common human prejudice. From our prototype runs we got the feeling that our approach often yields “good” definition in that sense.

3 Learning functions by anti-unification

One of the simplest ways to obtain a function definition is to syntactically anti-unify the given i/o equations.

Given i/o equations \( f(L_{11}, \ldots, L_{m_1}) = R_1 \)

\[
\vdots
\]

\( f(L_{1n}, \ldots, L_{mn}) = R_n ,
\]

let

\[
f(L_1, \ldots, L_m) = R
\]

be their least general generalization (lgg for short, see [Plo70,Plo71,Rey70]). If the variable condition \( \text{vars}(R) \subseteq \text{vars}(L_1) \cup \ldots \cup \text{vars}(L_m) \) holds, then the lgg will cover all \( n \) given i/o equations.

For example, assume we are to generate a definition for a unary function called \( g_2 \)

\[
\begin{align*}
30: & \quad g_2(0) = s(s(0)) \\
31: & \quad g_2(s(0)) = s(s(s(0))) \\
32: & \quad g_2(s(s(s(0)))) = s(s(s(s(s(s(0))))))
\end{align*}
\]

We obtain the lgg

\[
33: \quad g_2(x_{024}) = s(s(x_{024}))
\]

\[3\] We use common imperative notation here for sake of readability.

\[4\] The “invariance theorem” in Kolmogorov complexity theory (e.g. [LV08, p.105, Thm.2.1.1]) implies that \( \forall L_1, L_2 \exists c \forall x : |C_{L_1}(x) - C_{L_2}(x)| \leq c \), where the \( L_i \) range over Turing-complete algorithm description languages, \( c \) is a natural number, \( x \) ranges over i/o equation sets, and \( C_L(x) \) denotes the length of the shortest function definition, written in \( L \), that covers \( x \). This theorem is sometimes misunderstood to enable a language-independent notion of simplicity; however, it does not, at least for small i/o example sets.
As another example, we can generate a definition for a binary function called \( g_4 \) from the i/o equations:

\[
\begin{align*}
\text{34: } g_4(a, 0) &= s(0) \\
\text{35: } g_4(a, s(0)) &= s(s(0)) \\
\text{36: } g_4(a, s(s(0))) &= s(s(s(0))).
\end{align*}
\]

We obtain the lgg:

\[
\text{37: } g_4(a, y_{012}) = s(y_{012})
\]

which satisfies the variable condition. Hence when \( g_4 \) is defined by equation 37, it covers i/o equations 36, 35, and 34.

As a counter-example, the lgg of the above
dup i/o equations:

\[
\begin{align*}
\text{22: } \text{dup}(0) &= 0 \\
\text{23: } \text{dup}(s(0)) &= s(s(0)) \\
\text{24: } \text{dup}(s(s(0))) &= s(s(s(s(0)))) \\
\text{25: } \text{dup}(s(s(s(0)))) &= s(s(s(s(s(0)))))
\end{align*}
\]

is computed as:

\[
\text{38: } \text{dup}(x_{0123}) = x_{0246}
\]

which violates the variable condition, and thus cannot be used to reduce a term \( \text{dup}(T) \) to a ground constructor term, i.e. to evaluate \( \text{dup}(T) \).

The above anti-unification approach can be extended in several ways, they are sketched in Sect. 5.1. However, in all but trivial cases, an lgg will violate the variable condition, and we need another approach to learn a function definition.

### 4. Learning functions by structural recursion

For a function \( f \) that can’t be learned by Sect. 3, we assume a defining term rewriting system that follows a structural recursion scheme obtained from \( f \)'s signature and a guessed argument position.

For example, for the function \( \text{dup} \) with the signature given in 10 and the only possible argument position, 1, we obtain the schematic equations:

\[
\begin{align*}
\text{39: } \text{dup}(0) &= g_1 \\
\text{40: } \text{dup}(s(x)) &= g_2(\text{dup}(x))
\end{align*}
\]

where \( g_1 \) and \( g_2 \) are fresh names of non-constructor functions.

If we could learn appropriate definitions for \( g_1 \) and \( g_2 \), we could obtain a definition for \( \text{dup} \) just by adding equations 39 and 40. The choice of \( g_1 \) is obvious:

\[
\text{41: } 0 \overset{22}{=} \text{dup}(0) \overset{39}{=} g_1
\]

\[\text{5}\] Whenever applied to terms \( T_1, \ldots, T_m \) that don’t start all with the same function symbol, Plotkin’s \( lgg \) algorithm returns a variable that uniquely depends on \( T_1, \ldots, T_m \). We indicate the originating terms by an index sequence; e.g. \( y_{012} \) was obtained as \( \text{lgg}(0, s(0), s(s(0))) \).
In order to learn a definition for $g_2$, we need to obtain appropriate i/o examples for $g_2$ from those for $\text{dup}$. Joining equation 40 with $\text{dup}$’s relevant i/o equations yields three i/o equations for $g_2$:

$$
\begin{align*}
30 & \quad s(s(0)) \quad 23 & \quad \text{dup}(s(0)) \quad 40 & \quad g_2(\text{dup}(0)) \quad 22 & \quad g_2(0) \\
31 & \quad s(s(s(s(0)))) \quad 24 & \quad \text{dup}(s(s(0)))) \quad 40 & \quad g_2(\text{dup}(s(s(0)))) \quad 23 & \quad g_2(s(s(s(0)))) \\
32 & \quad s(s(s(s(s(s(0)))))) \quad 25 & \quad \text{dup}(s(s(s(s(s(0)))))) \quad 40 & \quad g_2(\text{dup}(s(s(s(s(s(0))))))) 24 & \quad g_2(s(s(s(s(s(s(0)))))))
\end{align*}
$$

A definition for $g_2$ covering its i/o examples 30, 31, and 32 has already been derived by anti-unification in Sect. 3 as

$$
g_2(x_{024}) = s(s(x_{024}))
$$

Altogether, we obtain the rewriting system

$$
\begin{align*}
39 & \quad \text{dup}(0) = g_1 \\
40 & \quad \text{dup}(s(x)) = g_2(\text{dup}(x)) \\
41 & \quad g_1 = 0 \\
33 & \quad g_2(x_{024}) = s(s(x_{024}))
\end{align*}
$$

as a definition for $\text{dup}$ that covers its i/o equations 22, 23, 24, and 25. Subsequently, this system may be simplified, by $\text{inlining}$, to

$$
\begin{align*}
39 & \quad \text{dup}(0) = 0 \\
40 & \quad \text{dup}(s(x)) = s(s(\text{dup}(x)))
\end{align*}
$$

which is the usual definition of the $\text{dup}$ function.

Returning to the computation of i/o equations for $g_2$ from those for $\text{dup}$, note that $g_2$’s derived i/o equations 30, 31, and 32 were necessary in the sense that they must be satisfied by each possible definition of $g_2$ that leads to $\text{dup}$ covering its i/o equations (23, 24, and 25). Conversely, $g_2$’s i/o equations were also sufficient in the sense that each possible definition of $g_2$ covering them ensures that $\text{dup}$ covers 23, 24, and 25, provided it covers 22:

- Proof of 23: $\text{dup}(s(0)) \quad 40 & \quad g_2(\text{dup}(0)) \quad 22 & \quad g_2(0) \quad 30 & \quad s(s(0))$
- Proof of 24: $\text{dup}(s(s(0)))) \quad 40 & \quad g_2(\text{dup}(s(s(0)))) \quad 23 & \quad g_2(s(s(0)))) \quad 31 & \quad s(s(s(s(0)))))$
- Proof of 25: $\text{dup}(s(s(s(s(s(0)))))) \quad 40 & \quad g_2(\text{dup}(s(s(s(s(s(0))))))) \quad 24 & \quad g_2(s(s(s(s(s(s(0))))))) \quad 32 & \quad s(s(s(s(s(s(s(0))))))))$

Observe that the above proofs are based just on permutations of the equation chains from 30, 31, and 32. Moreover, note that the coverage proof for $\text{dup}(s(T))$ relies on the coverage for $\text{dup}(T)$ already being proven. That is, the coverage proofs follow the employed structural recursion scheme. As for the base case, $g_1$’s coverage of 41 is of course necessary and sufficient for $\text{dup}$’s coverage of 22.

### 4.1 Non-ground i/o equations

As an example that uses i/o equations containing variables, consider the function $\text{lgth}$, with the signature given in 7. Usually, i/o equations for this function are given in a way that indicates that the particular values of the list elements don’t matter. For example, an i/o equation like $\text{lgth}(a::b::nil) = s(s(0))$ is seen much more often than $\text{lgth}(s(0)::0::nil) = s(s(0))$. Our approach
allows for variables in i/o equations, and treats them as universally quantified. That is, a
non-ground i/o equation is covered by a function definition iff all its ground instances are.

Assume for example we are given the i/o equations

\[
\begin{align*}
\text{lglth}(\text{nil}) &= 0 \\
\text{lglth}(a::\text{nil}) &= s(0) \\
\text{lglth}(a::b::\text{nil}) &= s(s(0)) \\
\text{lglth}(a::b::c::\text{nil}) &= s(s(s(0)))
\end{align*}
\]

Given the signature of lglth (see 7) and argument position 1, we obtain a structural recursion
scheme

\[
\begin{align*}
\text{lglth}(\text{nil}) &= \text{g}_3 \\
\text{lglth}(x::y) &= \text{g}_4(x, \text{lglth}(y))
\end{align*}
\]

Similar to the dup example, we get

\[
\begin{align*}
0 &= \text{lglth}(\text{nil}) = \text{g}_3
\end{align*}
\]

and we can obtain i/o equations for \( \text{g}_4 \) from those for \( \text{lglth} \):\(^6\)

\[
\begin{align*}
\text{s}(0) &= \text{lglth}(a::\text{nil}) = \text{g}_4(a, \text{lglth}(\text{nil})) = \text{g}_4(a, 0) \\
\text{s}(s(0)) &= \text{lglth}(a::b::\text{nil}) = \text{g}_4(a, \text{lglth}(b::\text{nil})) = \text{g}_4(a, s(s(0)))
\end{align*}
\]

Again, a function definition covering these i/o equation happens to have been derived by anti-
unification in Sect. 3:

\[
\begin{align*}
\text{g}_4(a, y_{012}) &= \text{s}(y_{012})
\end{align*}
\]

Altogether, equations 46, 47, 48, and 37 build a rewriting system for \( \text{lglth} \) that covers all its
given i/o equations. By subsequently inlining \( \text{g}_3 \)'s and \( \text{g}_4 \)'s definition, we obtain a simplified
definition for \( \text{lglth} \):

\[
\begin{align*}
\text{lglth}(\text{nil}) &= 0 \\
\text{lglth}(x::y) &= \text{s}(\text{lglth}(y))
\end{align*}
\]

which agrees with the usual one found in textbooks.

Similar to the ground case, \( \text{g}_4 \)'s derived i/o equations 34, 35, and 36 were necessary for \( \text{lglth} \)
covering its i/o equations. And as in the ground case, they are also sufficient:

\[
\begin{align*}
\text{Proof of 43:} & \quad \text{lglth}(a::\text{nil}) = \text{g}_4(a, \text{lglth}(\text{nil})) = \text{g}_4(a, 0) = \text{s}(0) \\
\text{Proof of 44:} & \quad \text{lglth}(a::b::\text{nil}) = \text{g}_4(a, \text{lglth}(b::\text{nil})) = \text{g}_4(a, s(s(0))) = \text{s}(s(0))) \\
\text{Proof of 45:} & \quad \text{lglth}(a::b::c::\text{nil}) = \text{g}_4(a, \text{lglth}(b::c::\text{nil})) = \text{g}_4(a, s(s(0))) = \text{s}(s(s(0)))
\end{align*}
\]

Again, renaming substitutions were used in the application of 43 and 44.

---

\(^6\) In the rightmost equation of each line, we employ a renaming substitution. For example, we apply
\{a \mapsto b, b \mapsto c\} to i/o equation 44 in line 36. For this reason, our approach wouldn’t work if \( a, b, c \)
were considered non-constructor constants rather than universally quantified variables.
4.2 Functions of higher arity

For functions with more than one argument, we have several choices of the argument on which to do the recursion. In these cases, we currently systematically try all argument positions\(^7\) in succession. This is feasible since

- our approach is quite simple, and hence fast to compute, and
- we have a sharp and easy to compute criterion (viz. coverage\(^8\) of all i/o examples) to decide whether recursion on a given argument was successful.

For the function +, with the signature given in 5, and argument position 2, we obtain the structural recursion scheme

\[
\begin{align*}
51: & \quad x + 0 = g_5(x) \\
52: & \quad x + s(y) = g_6(x, x + y).
\end{align*}
\]

Appendix A.1 shows a run of our PROLOG prototype implementation that obtains a definition for +. In Sect. 5.2, we discuss possible extensions of the structural recursion scheme, like simultaneous recursion.

4.3 Constructors with more than one recursion argument

When computing a structural recursion scheme, we may encounter a sort \(s\) with a constructor that takes more than one argument of sort \(s\). A common example is the sort of all binary trees (of natural numbers), as given in 3. The function size, with the signature given in 9, computes the size of such a tree, i.e. the total number of nd nodes. A recursion scheme for the size and argument position 1 looks like:

\[
\begin{align*}
53: & \quad \text{size}(\text{null}) = g_9 \\
54: & \quad \text{size}(\text{nd}(x, y, z)) = g_{10}(y, \text{size}(x), \text{size}(z)).
\end{align*}
\]

In App. A.2, we show a prototype run to obtain a definition for size.

4.4 General approach

In the previous sections, we have introduced our approach using particular examples. In this section, we sketch a more abstract and algorithmic description.

\(^7\) In particular, the recursive argument’s sort and the function’s result sort needn’t be related in any way, as the lgth example above demonstrates.

\(^8\) Checking if an i/o equation is covered by a definition requires executing the latter on the lhs arguments of the former. Our structural recursion approach ensures the termination of such computations, and establishes an upper bound for the number of rewrite steps. For example, \(g_2\) and \(g_4\), defined in 33 and 37, respectively, need one such step, while their callers dup and lgth, defined in 39,40 and 46,47, respectively, need a linear amount of steps. An upper-bound expression for learned functions’ time complexity remains to be defined and proven.
Given a function and its signature \( f : s_1 \times \ldots \times s_n \rightarrow s \), and given one of its argument positions \( 1 \leq i \leq n \), we can easily obtain a term rewriting system to define \( f \) by structural recursion on its \( i \)th argument. Assume in the definition of \( f \)'s \( i \)th domain sort \( s_i \) we have an alternative

\[
s_i ::= \ldots | c(s'_1, \ldots, s'_l) | \ldots,
\]

assume \( \{s'_{\nu(1)}, \ldots, s'_{\nu(m)}\} \neq s_i \) is the set of non-recursive arguments of the constructor \( c \), and \( s'_{\rho(1)} = \ldots = s'_{\rho(k)} = s_i \) are the recursive arguments of \( c \). Let \( g \) be a new function symbol. We build an equation

\[
f(x_1, \ldots, x_{i-1}, c(y_1, \ldots, y_l), x_{i+1}, \ldots, x_n) = \ldots =
\]

\[
f(x_1, \ldots, x_{i-1}, y_{\nu(1)}, x_{i+1}, \ldots, x_n,
\]

\[
f(x_1, \ldots, x_{i-1}, y_{\nu(2)}, x_{i+1}, \ldots, x_n)
\]

In a somewhat simplified presentation, we build the equation

\[
f(\ldots, c(y_1, \ldots, y_l), \ldots) = g(\ldots, f(\ldots, y_{\nu(1)}), \ldots, f(\ldots, y_{\nu(k)})\ldots).\]

From the i/o equations for \( f \), we often\(^9\) can construct i/o equations for \( g \): If we have an i/o equation that matches the above equation’s left-hand side, and we have all i/o equations needed to evaluate the recursive calls to \( f \) on its right-hand side, we can build an i/o equation for \( g \).

This way, we can reduce the problem of synthesizing a definition for \( f \) that reproduces the given i/o equations to the problem of synthesizing a definition for \( g \) from its i/o equations. As a base case for this process, we may synthesize non-recursive function definitions by anti-unification of the i/o equations.

It should be possible to prove that \( f \) covers all its i/o equations iff \( g \) covers its, under some appropriate conditions. We expect that a sufficient condition is that all recursive calls to \( f \) could be evaluated. At least, we could demonstrate this in the above \texttt{dup} and \texttt{lgth} example.

### 4.5 Termination

In order to establish the termination of our approach, it is necessary to define a criterion by which \( g \) is easier to learn from it i/o equations than \( f \) is from its. Term size or height cannot be used in a termination ordering; when proceeding from \( f \) to \( g \) they may remain equal, or may even increase, as shown in Fig. 4 for the \texttt{dup} vs. \texttt{g2} example.

However, the number of i/o equations decreases in this example, and in all other ones we dealt with. A sufficient criterion for this is that \( f \)'s i/o equations don’t all have the same left-hand side top-most constructor. However, the same criterion would have to be ensured in turn for \( g \), and it is not obvious how to achieve this.

\(^9\) Our construction isn’t successful in all cases. \ We give a counter-example in Sect. 5.3.
| Fct Eqn Lf Rg | Fct Eqn Lf Rg |
|----------------|----------------|
| 22 2 1         | g₁ 41 1 1     |
| 23 3 3         | 30 2 3       |
| dup 24 4 5     | g₂ 31 4 5   |
| 25 5 7         | 32 6 7     |

Figure 4. Left- and right-hand term sizes of i/o equations for \texttt{dup} and \texttt{g₂}

In any case, by construction of \(g\)'s i/o example from \(f\)'s, no new terms can arise.\(^{10}\) Even more, each term appearing in an i/o example for \(g\) originates from a right-hand side of an i/o example for \(f\). Therefore, our approach can’t continue generating new auxiliary functions forever, without eventually repeating the set of i/o equations. Our prototype implementation doesn’t check for such repetitions, however.

5 Possible extensions

In this section, we briefly sketch some possible extensions of our approach. Their investigation in detail still remains to be done.

5.1 Extension of anti-unification

In Sect. 3 we used syntactical anti-unification to obtain a function definition, as a base case of our approach. Several way to extend this technique can be thought of.

Set anti-unification It can be tried to split the set of i/o equations into disjoint subsets such that from each one an lgg satisfying the variable condition is obtained. This results in several defining equations. An additional constraint might be that each subset corresponds to another constructor symbol, observed at some given fixed position in the left-hand side terms.

Anti-unification modulo equational theory Another extension consists in considering an equational background theory \(E\) in anti-unification; it wasn’t readily investigated in 1994. See [Hei94b,Hei94a,Hei95] for the earliest publications, and [Bur05,Bur17] for the latest.

As of today, the main application of \(E\)-anti-unification turned out to be the synthesis of non-recursive function definitions from input/output equations [Bur17, p.3]. To sketch an example, let \(E\) consist just of definitions 12, 13, 14, and 15.

Assume the signature

\[
\text{sq} : \text{nat} \rightarrow \text{nat}
\]

\(^{10}\) except for the fresh left-hand side top function symbols
\[ \text{sq}(0) = 0 \]
\[ \text{sq}(s(0)) = s(0) \]
\[ \text{sq}(s(s(0))) = s(s(s(0))) \]
\[ \text{sq}(s(s(s(0)))) = s^2(0) \]
\[ \text{sq}(x_{0123}) = x_{0123} \times x_{0123} \]

Figure 5. Application of \(E\)-anti-unification to learn squaring

|   | \text{size} | \text{null} | \text{null} | 0 |
|---|-------------|-------------|-------------|---|
| 56| \text{size}(x) = 0 |
| 57| \text{size}(\text{nd}(x, y, z)) = f_1(y, \text{size}(x), \text{size}(z)) |
| 58| f_1(x, y, z) = s(z) |
| 59| f_1(x, s(y), z) = f_2(x, z, f_1(x, y, z)) |
| 60| f_2(x, y, z) = s(s(z)) |
| 61| f_2(x, s(y), z) = s(s(s(z))) |

Figure 6. Learned tree size definition for anti-unification depth 2, 3, and 4 and the i/o equations 56, 57, 58, and 59 of the squaring function. Applying syntactical anti-unification to the left-hand sides yields a variable \(x_{0123}\), and four corresponding substitutions. Applying constrained \(E\)-generalization [Bur05, p.5, Def.2] to the right-hand sides yields a term set that contains \(x_{0123} \times x_{0123}\) as a minimal-size member, see Fig. 5.

**Depth-bounded anti-unification** In many cases, defining equations obtained by syntactical anti-unification appear to be too particular. For example, \(s^4(0)\) and \(s^9(0)\) are generalized to \(s^4(x_{05})\), while being by 4 greater than something wouldn’t be the first choice for a common property of both numbers for most humans. As a possible remedy, a maximal depth \(d\) may be introduced for the anti-unification algorithm. Beyond this depth, terms are generalized by a variable even if all their root function symbols agree. Denoting by \(\text{lgg}_d(t_1, t_2)\) the result of an appropriately modified algorithm, it should be easy to prove that \(\text{lgg}_d(t_1, t_2)\) can be instantiated to both \(t_1\) and \(t_2\), and is the most special term with that property among all terms of depth up to \(d\). If \(d\) is chosen as \(\infty\), \(\text{lgg}_d\) and \(\text{lgg}\) coincide.

In our prototype implementation, we meanwhile built in such a depth boundary. Figure 6 compares the learned function definitions for \(\text{size}\) for \(d = 2, 3, 4\) (top to bottom). For example, for \(d = 2\), the —nonsensical— equation \(\text{size}(x) = 0\) is learned, while for \(d \geq 3\) the respective equation reads \(\text{size}(\text{null}) = 0\). Not surprisingly, for \(d = 2\) only one of the given 9 i/o equations is covered. For \(d \leq 1\), the attempt to learn defining equations for \(\text{size}\) fails.

For \(d = 4\), the learned equations agree with those for \(d = \infty\), and hence also with those for all intermediate depths. The prototype run for \(d = \infty\) is shown in App. A.2. Note that the
prototype simplifies equations by removing irrelevant function arguments. For this reason, \( f_{12} \) has only two arguments, while the corresponding function \( f_1 \) in Fig. 6 has three.

5.2 Extension of structural recursion

Some functions are best defined by simultaneous recursion on several arguments. As an example, consider the sort definition \( a \) with \( nl, o, \) and \( i \) denoting an empty list, a 0 digit, and a 1 digit, respectively. For technical reasons, such a list is interpreted in reversed order, e.g. \( o(i(i(nl))) \) denotes the number 6. The sum function \( add \), its signature shown in 11, may then be defined by the following rewrite system:

\[
\begin{align*}
61. \quad add(\ x, \ nl) & = x \\
62. \quad add(\ nl, \ y) & = y \\
63. \quad add(o(x),o(y)) & = o(add(x,y)) \\
64. \quad add(o(x),i(y)) & = i(add(x,y)) \\
65. \quad add(i(x),o(y)) & = i(add(x,y)) \\
66. \quad add(i(x),i(y)) & = o(inc(add(x,y))) \\
67. \quad inc : blist \rightarrow blist
\end{align*}
\]

where

\[
\begin{align*}
67. \quad inc : blist \rightarrow blist
\end{align*}
\]

is a function to increment a binary digit list. This corresponds to the usual hardware implementation, with \( inc \) being used for the carry.

It is obvious that this definition cannot be obtained from our simple structural recursion scheme from Sect. 4, neither by recurring over argument position 1 nor over 2. Instead, we would need recursion over both positions simultaneously, i.e. a scheme like

\[
\begin{align*}
68. \quad add(\ nl, \ nl) & = g_{15} \\
69. \quad add(\ nl, o(y)) & = g_{16}(y) \\
70. \quad add(\ nl, i(y)) & = g_{17}(y) \\
71. \quad add(o(x), \ nl) & = g_{18}(x) \\
72. \quad add(o(x),o(y)) & = g_{19}(add(x,y)) \\
73. \quad add(o(x),i(y)) & = g_{20}(add(x,y)) \\
74. \quad add(i(x), \ nl) & = g_{21}(x) \\
75. \quad add(i(x),o(y)) & = g_{22}(add(x,y)) \\
76. \quad add(i(x),i(y)) & = g_{23}(add(x,y))
\end{align*}
\]

An extension of our approach could provide such a scheme, additionally to the simple structural recursion scheme.

If we could prove that each function definition obtainable by the simple recursion scheme can also be obtained by a simultaneous recursion scheme, we needed only to employ the latter. This way, we would no longer need to guess an appropriate argument position to recur over; instead we could always recur simultaneously over all arguments of a given sort. Unfortunately, simultaneous recursion is not stronger than simple structural recursion. For example, the function \( app \) to concatenate two given lists can be obtained by simple recursion over the first argument...
(see 18, 19 in Fig. 3), but not by simultaneous recursion: \( \text{app}(w::x, y::z) = g_{24}(w, y, \text{app}(x, z)) \)
doesn’t lead to a sensible definition, for any choice of \( g_{24} \).

One possible remedy is to try simple structural recursion first, on any appropriate argument position, and simultaneous recursion next, on any appropriate set of argument positions. Alternatively, user commands may be required about which recursion to try on which argument position(s).

Another possibility might be to employ a fully general structural recursion scheme, like

\[
\begin{align*}
\text{app}(w::x, y::z) &= g_{24}(w, y, \text{app}(w::x, z), \text{app}(x, y::z), \text{app}(x, z)) \\
\text{add}(o(x), o(y)) &= g_{25}(\text{add}(o(x), y), \text{add}(x, o(y)), \text{add}(x, y))
\end{align*}
\]

In this scheme, calls for simple recursion over each position are provided, as well as for simultaneous recursion over each position set. A new symbol \( \Omega \), intended to denote an undefined term, could be added to the term language. When e.g. i/o equations are missing to compute \( \text{add}(o(x), y) \) for some particular instance, the first argument of \( g_{25} \) would be set to \( \Omega \) in the respective i/o equation. In syntactical anti-unification and coverage test, \( \Omega \) needed to be handled appropriately. This way, only one recursion scheme would be needed, and no choice of appropriate argument position(s) would be necessary. However, arities of auxiliary functions might grow exponentially.

### 5.3 Limitations of our approach

In this section, we demonstrate an example where our approach fails. Consider again the squaring function, its signature shown in 55, and consider again its i/o equations 56, 57, 58, and 59.

Since syntactical anti-unification as in Sect. 3 (i.e. not considering an equational background theory \( E \)) doesn’t lead to a valid function definition, we build a structural recursion scheme as in Sect. 4:

\[
\begin{align*}
\text{sq}(0) &= g_{11} \\
\text{sq}(s(x)) &= g_{12}(\text{sq}(x))
\end{align*}
\]

We get \( g_{11} = 0 \), and the following i/o equations for \( g_{12} \):

\[
\begin{align*}
\text{sq}(0) &\overset{57}{=} \text{sq}(s(0)) &\overset{80}{=} g_{12}(\text{sq}(0)) &\overset{56}{=} g_{12}(0) \\
\text{sq}(s(s(0)))) &\overset{58}{=} \text{sq}(s(s(0))) &\overset{80}{=} g_{12}(\text{sq}(s(0))) &\overset{57}{=} g_{12}(s(0)) \\
\text{sq}(s(s(s(0)))) &\overset{50}{=} \text{sq}(s(s(s(0)))) &\overset{80}{=} g_{12}(\text{sq}(s(s(0)))) &\overset{58}{=} g_{12}(s(s(s(s(0))))))
\end{align*}
\]

Observe that we are able to obtain i/o equations for \( g_{12} \) only on square numbers. For example, there is no obvious way to determine the value of \( g_{12}(s(s(s(0)))) \).

Syntactically anti-unifying \( g_{12} \)'s i/o equation still doesn’t yield a valid function definition. So we set up a recursion scheme for \( g_{12} \), in turn:

\[
\begin{align*}
\text{g}_{12}(0) &= g_{13} \\
g_{12}(s(x)) &= g_{14}(g_{12}(x))
\end{align*}
\]

13
Again, \( g_{13} = s(0) \) is obvious. Trying to obtain i/o equations for \( g_{14} \), we get stuck, since we don’t know how \( g_{12} \) should behave on non-square numbers:

\[
\begin{align*}
86: & \quad s(s(s(0)))) = g_{12}(s(0)) = g_{14}(g_{12}(0)) = g_{14}(s(0)) \\
87: & \quad ?? = g_{12}(s(s(0))) = g_{14}(g_{12}(s(0))) = g_{14}(s(s(s(0)))) \\
88: & \quad ?? = g_{12}(s(s(s(0)))) = g_{14}(g_{12}(s(s(0)))) = ?? = g_{14}(??) \\
89: & \quad s^9(0) = g_{12}(s(s(s(s(0)))))) = g_{14}(g_{12}(s(s(0)))) = ?? = g_{14}(??)
\end{align*}
\]

As an alternative, by applying \( g_{14} \) sufficiently often rather than just once, we can obtain:

\[
\begin{align*}
90: & \quad s^9(0) = g_{12}(s(s(s(s(s(0)))))) = g_{14}(g_{12}(s(s(s(0)))))) = g_{14}(g_{14}(g_{12}(s(s(0)))))) = g_{14}(g_{14}(g_{14}(g_{12}(s(s(0))))))) = g_{14}(g_{14}(s(s(s(s(s(0)))))))))
\end{align*}
\]

However, no approach is known to learn \( g_{14} \) from an extended i/o equation like \( g_{14} \), which determines \( g_{14} \) rather than \( g_{14} \) itself. In such cases, we resort to the excuse that the original function, \( sq \), isn’t definable by structural recursion.

A precise criterion for the class that our approach can handle is still to be found. It is not even clear that such a criterion can be computable. If not, it should still be possible to give computable necessary and sufficient approximations.
A Example runs of our prototype implementation

A.1 Addition of 0-s numbers

?- SgI = [ + signature [nat,nat] --> nat],
| SS = [ nat sortdef 0 ! s(nat)],
| ExI = [ 0 + 0 = 0,
| 0 + s(0) = s(0),
| 0 + s(s(0)) = s(s(0))],
| SD = [ nat sortdef 0 ! s(nat)],
| ExI = [ 0 + 0 = 0,
| 0 + s(0) = s(0),
| 0 + s(s(0)) = s(s(0))],
| run(+,SgI,SD,ExI).

+++++ Examples input check:
+++++ Example 1:
+++++ Example 2:
+++++ Example 3:
+++++ Example 4:
+++++ Example 5:
+++++ Example 6:
+++++ Example 7:
+++++ Example 8:
+++++ Examples input check done

FUNCTION SIGNATURES:

FUNCTION EXAMPLES:

FUNCTION DEFINITIONS:

?- SgI = [ + signature [nat,nat] --> nat],
| SS = [ nat sortdef 0 ! s(nat)],
| ExI = [ size signature [tree] --> nat,
| 0 + 0 = 0,
| 0 + s(0) = s(0),
| 0 + s(s(0)) = s(s(0))],
| SD = [ tree sortdef nl ! nd(tree,nat,tree),
| nat sortdef 0 ! s(nat)],
| ExI = [ size(nl) = 0,
| size(s(a(0))) = size(a(0)),
| size(s(a(s(0)))) = size(a(s(0)))],
| run(+,SgI,SD,ExI).

+++++ Examples output check:
+++++ Examples output check done

A.2 Size of a tree

?- SgI = [ size signature [tree] --> nat],
| SS = [ tree sortdef nl ! nd(tree,nat,tree),
| nat sortdef 0 ! s(nat)],
| ExI = [ size(nl) = 0,
| size(s(a(0))) = size(a(0)),
| size(s(a(s(0)))) = size(a(s(0)))],
| run(+,SgI,SD,ExI).
\[ \text{size}(\text{nd}(\text{n}1, \text{va}, \text{n}1)) = s(0), \]
\[ \text{size}(\text{nd}(\text{n}1, \text{va}, \text{vb}, \text{n}1)) = s(0), \]
\[ \text{size}(\text{nd}(\text{n}1, \text{va}, \text{nd}(\text{n}1, \text{vb}, \text{n}1))) = s(s(0)), \]
\[ \text{size}(\text{nd}(\text{n}1, \text{va}, \text{nd}(\text{n}1, \text{vb}, \text{vc}, \text{n}1))) = s(s(s(0))), \]
\[ \text{size}(\text{nd}(\text{n}1, \text{va}, \text{nd}(\text{n}1, \text{vb}, \text{vc}, \text{vd}, \text{n}1))) = s(s(s(s(0)))), \]

\]

\[ \text{run}(\text{size}, \text{Ex}1) \]

+++++ Examples input check:

+++++ Example 1:

+++++ Example 2:

+++++ Example 3:

+++++ Example 4:

+++++ Example 5:

+++++ Example 6:

+++++ Example 7:

+++++ Example 8:

+++++ Example 9:

Variable sorts:
\[ [\text{vd}: \text{nat}, \text{vc}: \text{nat}, \text{vb}: \text{nat}, \text{va}: \text{nat}] \]

+++++ Examples input check done

\[ \text{induce}(\text{size}) \]

. trying argument position: 1

. inducePos(\text{size}, 1, \text{nl})

. matching examples: \[ \text{size}(\text{nv}1) = 0 \]

. anti-unifier: \[ \text{size}(\text{nl}) = 0 \]

. inducePos(\text{size}, 1, \text{nd}(\text{tree}, \text{nat}, \text{tree}))

. matching examples: \[ \text{size}(\text{nd}(\text{nl}, \text{va}, \text{nd}(\text{nl}, \text{vb}, \text{nd}(\text{nl}, \text{vc}, \text{vd}, \text{nl})))) = s(s(s(s(0)))) \]

. new recursion scheme: \[ \text{size}(\text{nd}(\text{nv}10, \text{nv}9, \text{nv}11)) = f12(\text{nv}9, \text{size}(\text{nv}10), \text{size}(\text{nv}11)) \]

. derive new equation: \[ s(0) = \text{size}(\text{nd}(\text{nl}, \text{va}, \text{nd}(\text{nl}, \text{vb}, \text{nd}(\text{nl}, \text{vc}, \text{vd}, \text{nl})))) = f12(\text{va}, 0, 0) \]

. derive new equation: \[ s(s(0)) = \text{size}(\text{nd}(\text{nl}, \text{va}, \text{nd}(\text{nl}, \text{vb}, \text{nd}(\text{nl}, \text{vc}, \text{vd}, \text{nl})))) = f12(\text{vb}, 0, s(0)) \]

. derive new equation: \[ s(s(0)) = \text{size}(\text{nd}(\text{nl}, \text{va}, \text{nd}(\text{nl}, \text{vb}, \text{nd}(\text{nl}, \text{vc}, \text{vd}, \text{nl})))) = f12(\text{va}, 0, s(s(0))) \]

. derive new equation: \[ s(s(s(0))) = \text{size}(\text{nd}(\text{nl}, \text{va}, \text{nd}(\text{nl}, \text{vb}, \text{nd}(\text{nl}, \text{vc}, \text{vd}, \text{nl})))) = f12(\text{vb}, 0, s(s(0))) \]

. derive new equation: \[ s(s(s(s(0)))) = \text{size}(\text{nd}(\text{nl}, \text{va}, \text{nd}(\text{nl}, \text{vb}, \text{nd}(\text{nl}, \text{vc}, \text{vd}, \text{nl})))) = f12(\text{vb}, 0, s(s(s(0)))) \]

. induce(f12)

. . trying argument position: 1

. . inducePos(f12, 1, 0)

. . matching examples: \[ \]

. . . anti-unifier: \[ f12(\text{va}, 0, 0) = s(0) \]

. . inducePos(f12, 1, s(\text{nat}))

. . . matching examples: \[ \]

. . . . anti-unifier: \[ f12(\text{va}, 0, s(0)) = s(s(0)) \]

. . . inducePos(f12, 1, s(s(\text{nat})))

. . . . uncovered examples: \[ f12(\text{vb}, 0, s(s(0))) = s(s(s(0))) \]

. . . . new recursion scheme: \[ f12(\text{vb}, 0, s(v37)) = s(v37) \]

. . . derive new equation: \[ s(s(0)) = f12(\text{vb}, 0, s(0)) = f46(\text{vb}, 0, s(v37)) \]

. . . derive new equation: \[ s(s(s(0))) = f12(\text{vb}, 0, s(s(0))) = f46(\text{vb}, 0, s(s(v37))) \]

. . . derive new equation: \[ s(s(s(s(0)))) = f12(\text{vb}, 0, s(s(s(0)))) = f46(\text{vb}, 0, s(s(s(v37)))) \]

. . . induce(f46)

. . . . trying argument position: 1

. . . . inducePos(f46, 1, 0)

. . . . matching examples: \[ \]

. . . . . anti-unifier: \[ f46(\text{vb}, 0, 0) = s(0) \]

. . . . inducePos(f46, 1, s(\text{nat}))

. . . . . matching examples: \[ \]

. . . . . . anti-unifier: \[ f46(\text{vb}, 0, s(0)) = s(s(0)) \]

. . . . . inducePos(f46, 1, s(s(\text{nat})))

. . . . . . uncovered examples: \[ f46(\text{vb}, 0, s(s(0))) = s(s(0)), f46(\text{vb}, 0, s(s(s(0)))) = s(s(s(0))) \]

. . . . . . new recursion scheme: \[ f46(\text{vb}, 0, s(v63)) = s(v63) \]

. . . . . . derive new equation: \[ s(s(0)) = f46(\text{vb}, 0, s(0)) = f46(\text{vb}, 0, s(v63)) \]

. . . . . . derive new equation: \[ s(s(s(0))) = f46(\text{vb}, 0, s(s(0))) = f46(\text{vb}, 0, s(s(v63))) \]

. . . . . . derive new equation: \[ s(s(s(s(0)))) = f46(\text{vb}, 0, s(s(s(0)))) = f46(\text{vb}, 0, s(s(s(v63)))) \]

. . . . . . induce(f46)

. . . . . . . trying argument position: 1

. . . . . . . inducePos(f46, 1, 0)

. . . . . . . matching examples: \[ \]

. . . . . . . . anti-unifier: \[ f46(\text{vb}, 0, 0) = s(0) \]

. . . . . . . . inducePos(f46, 1, s(\text{nat}))

. . . . . . . . . matching examples: \[ \]

. . . . . . . . . . anti-unifier: \[ f46(\text{vb}, 0, s(0)) = s(s(0)) \]

. . . . . . . . . . inducePos(f46, 1, s(s(\text{nat})))

. . . . . . . . . . . matching examples: \[ \]

. . . . . . . . . . . . anti-unifier: \[ f46(\text{vb}, 0, s(s(0))) = s(s(s(0))) \]

. . . . . . . . . . . inducePos(f46, 1, s(s(s(\text{nat}))))

. . . . . . . . . . . . . all examples covered

. . . . . . . . . . . . . . induce(f46)

. . . . . . . . . . . . . . . matching examples: \[ \]

. . . . . . . . . . . . . . . . anti-unifier: \[ f46(\text{vb}, 0, s(s(0))) = s(s(s(0))) \]

. . . . . . . . . . . . . . . . inducePos(f46, 1, s(s(s(s(\text{nat}))))

. . . . . . . . . . . . . . . . . all examples covered

. . . . . . . . . . . . . . . . . . induce(f46)

. . . . . . . . . . . . . . . . . . . induce(f46)

. . . . . . . . . . . . . . . . . . . . induce(\text{size})

+++++ Examples output check:

+++++ Examples output check done

FUNCTION SIGNATURES:

\[ f46 \text{ signature } [\text{nat}, \text{nat}, \text{nat}] \rightarrow \text{nat} \]

\[ f12 \text{ signature } [\text{nat}, \text{nat}, \text{nat}] \rightarrow \text{nat} \]

FUNCTION EXAMPLES:

\[ \text{size}(\text{nl}) = 0 \]

\[ \text{size}(\text{nd}(\text{nl}, \text{va}, \text{nl})) = s(0) \]
FUNCTION DEFINITIONS:

```
size(nl) = 0
size(nd(v66,v67,v68)) = f12(size(v66),size(v68))
f12(0,v69) = s(v69)
f12(s(v70),v71) = f46(v71,f12(v70,v71))
f46(0,0) = s(s(0))
f46(s(v72),s(v73)) = s(s(s(v73)))
```

A.3 Reversing a list

```
A.3 Reversing a list

?- SgI = [rev signature [list] -> list],
   SD = [ list sortdef [] ! [nat|list],
   nat sortdef 0 ! s(nat)],
   ExI = [ rev([],[]) = [],
   rev([va]) = [va],
   rev([vb,va]) = [va,vb],
   rev([vc,vb,va]) = [va,vb,vc]],
   run(rev,SgI,SD,ExI).

+++++ Examples input check:
+++++ Example 1:
+++++ Example 2:
+++++ Example 3:
+++++ Example 4:
```

```
matching examples: []
no examples
inducePos(f25,3,[nat|list])
induce(f25)
inducePos(f9,2,[nat|list])
induce(f9)
inducePos(rev,1,[nat|list])
all examples covered
induce(rev)
Examples output check:
Examples output check done
FUNCTION SIGNATURES:

f25 signature [nat,nat,list]-->list
f9 signature [nat,list]-->list
rev signature [list]-->list

FUNCTION EXAMPLES:

rev([])=[]
rev([va])=[va]
rev([vb,va])=[va,vb]
rev([vc,vb,va])=[va,vb,vc]

FUNCTION DEFINITIONS:

rev([])=[]
rev([v39|v40])=f9(v39,rev(v40))
f9([v41],[v42],[v43])=[v42]
f9([v42],[v43],[v44])=f25(v43,f9([v42],[v44]))
f25([v41],[v45],[v46])=[v41,v45,v46]
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