Spectroscopy of SU(4) gauge theory with two flavors of sextet fermions

Thomas DeGrand,1 Yuzhi Liu,1 Ethan T. Neil,1,2 Yigal Shamir,3 and Benjamin Svetitsky3,4

1Department of Physics, University of Colorado, Boulder, CO 80309, USA
2RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA
3Raymond and Beverly Sackler School of Physics and Astronomy, Tel Aviv University, 69978 Tel Aviv, Israel
4Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

We present a first look at the spectroscopy of SU(4) gauge theory coupled to two flavors of Dirac fermions in the two-index antisymmetric representation, which is a real representation. We compute meson and diquark masses, the pseudoscalar and vector meson decay constants, and the masses of six-quark baryons. We make comparisons with large-$N_c$ expectations.

I. INTRODUCTION

The limit of QCD-like theories in which the number of colors $N_c$ is taken to infinity [1, 2] has a long history of applications to QCD phenomenology. Despite the large size of the expansion parameter $1/N_c = 1/3$, large-$N_c$ models of some QCD quantities have been quite successful [3–9]. There is also broad interest in the subject outside of direct application to QCD. Many extensions of the Standard Model, such as composite Higgs models, contain “QCD-like” sectors, built from some non-Abelian gauge group and fermions in various gauge representations. In addition, such theories can also appear as “hidden sectors” from grand unified theories or string theories [10–12], leading to potential dark matter candidates.

Nonperturbative properties of these systems, such as the symmetry breaking pattern, the value of the fermion condensate, or that of some decay amplitude, often feature prominently in their phenomenology. Often, large-$N_c$ extrapolations based on experimental QCD data are the only analytic way to estimate their values. Lattice studies can provide a rigorous, non-perturbative way to calculate strongly-coupled properties in a particular theory of choice. However, since the space of interesting models remains vast and poorly constrained, an interesting alternative approach is to use lattice input in order to constrain and study the large-$N_c$ expansion itself, allowing more general statements to be made.

There is a substantial body of work on pure gauge theories at large $N_c$, summarized in a recent review [13]. There are a number of studies of gauge theories coupled to fundamental-representation fermions. Meson spectroscopy in these systems shows reasonable extrapolations in these systems shows reasonable

A possible ultraviolet completion is any confining gauge theory with 5 Majorana fermions in some real representation. There is a lively literature associated with this class of theory (see for example [24, 32–35]). In this alternative expansion, fermion loops are not suppressed at large-$N_c$, leading to somewhat different predictions for QCD quantities in the large-$N_c$ expansion; a recent review with emphasis on properties of baryons is given in [36]. Interesting equivalences to supersymmetric Yang-Mills theory in the $N_c \to \infty$ limit have been observed for theories with AS2 fermions [37–40], related to the orientifold equivalence between gauge theories with all two-index representations in the large-$N_c$ limit [41–43]. Finally, the behavior of this class of theories at finite baryon density has attracted some theoretical interest [44, 45]. Since the AS2 representation of SU(4) is real, lattice simulations of this theory at finite baryon density can be performed with no sign problem.

Members of this family also play a role in composite Higgs studies. For example, the Littlest Higgs model [46] relies on the non-linear sigma model SU(5)/SO(5).
tation. The most economical way to realize this scenario is an SU(4) gauge theory, where the AS2 representation is real. The SU(5)/SO(5) sigma model is also central to the more recent composite-Higgs models of Refs. [47–49]. In particular, Ref. [48] makes the case why the SU(4) theory with AS2 fermions is the most attractive candidate within this approach, whereas Ref. [49] elaborates on the phenomenology of this composite-Higgs model.\(^1\)

B. SU(4) with fermions in the two-index antisymmetric representation

In this paper, we study the SU(4) theory with two flavors of AS2 Dirac fermions (equivalent to four Majorana fermions). This theory can be simulated using the standard Hybrid Monte Carlo (HMC) algorithm, whereas study of the theory with 5 Majorana fermions would require the more expensive rational HMC algorithm. We compare spectroscopy of this theory to already-published fundamental-representation spectroscopy for \(N_c = 3, 5, 7\), as well as to large-\(N_c\) expectations. We do this at a single value of the bare gauge coupling, in the confined and chirally broken phase of our lattice action.

All previous large-\(N_c\) studies using lattice gauge theory that we are aware of were performed in the quenched approximation. For study of the large-\(N_c\) limit with AS2 fermions, quenching is less justified than for fermions in the fundamental representation, since the AS2 fermion loops remain important as \(N_c \to \infty\). The SU(4) simulations we present therefore use dynamical fermions.

We compare our results to the SU(3) gauge theory, where the AS2 representation is simply the \(3\). For this comparison, we use our own dynamical-fermion data sets in the SU(3) theory, roughly matched in simulation volume and lattice spacing to previously published quenched ones. The dynamical fermion masses in the simulations are not so light that there are appreciable differences with our quenched SU(3) data set, but of course dynamical data sets are better justified than quenched ones.

The comparison of \(N_c = 4\) to \(N_c = 3\) allows a first, though limited, test of large-\(N_c\) scaling for AS2 fermions. It is limited because we only have two \(N_c\)'s, and so it is hard to make more than a qualitative statement: that large-\(N_c\) arguments do or do not seem to work. To see why this is so, consider the case of baryon spectroscopy. In large-\(N_c\) the mass of a baryon is characterized by a set of terms, proportional to powers of \(N_c\) and to other invariants. Specifically, the baryon mass including dependence on the angular momentum is given in lowest order by the rotor formula [17, 20, 50],

\[
M_B\langle J \rangle = m_0 N_b + B \frac{J(J+1)}{N_b}.
\]

\(N_b\) is the number of quarks in the baryon, and \(m_0\) can be interpreted as a constituent quark mass. \(N_b = N_c\) for fundamental-representation fermions and \(N_b = 6\) for SU(4) AS2. [The general formula for AS2 fermions is \(N_b = N_c(N_c - 1)/2\).] The parameters \(m_0\) and \(B\) depend on the fermion mass, and they also contain hidden \(N_c\) dependence: for example, \(m_0 = m_0 + (1/N_c)m_{01} + (1/N_c^2)m_{02} + \ldots\). The terms in the expansion, such as \(m_{01}\), are expected to have some "typical hadronic size." Comparisons at different \(N_c\)'s must somehow be matched in quark mass. With only two \(N_c\)'s, one can only hope to see the leading \(1/N_c\) correction, and even that only from fits with no degrees of freedom.

Now we proceed to the body of the paper. Section II collects some useful group theoretic results about AS2 fermions in the SU(4) gauge theory, and their special symmetries. We present the lattice action and a new discretization issue in Sec. III. The choice of parameters used in these simulations was made after a scan of the bare parameter space (bare gauge coupling and hopping parameter). This scan revealed some of the phase structure of this system, which we present in Sec. IV. We describe our methods for obtaining spectra in Sec. V, and display tables of the resulting meson and baryon spectra. We then plot these results and offer comparisons among the SU(4) AS2 theory, the SU(3) theory, and quenched SU(3), SU(5), and SU(7) theories: for mesons in Sec. VI and for baryons in Sec. VII. Finally, Sec. VIII makes some phenomenological observations, summarizes our results, and suggests future directions.

II. GROUP THEORY AND SYMMETRIES

In this section we discuss the symmetry aspects of AS2 fermions in SU(4), specializing to \(N_f = 2\). In Sec. II A we review some basic properties of real and pseudoreal representations, and how they are reflected in symmetries of the Wilson–Dirac operator and meson/diquark propagators.

In Sec. II B we turn to global symmetries. The pattern of chiral symmetry breaking in SU(4) AS2 is different from that in SU(3) gauge theories with fundamental-representation fermions, because the AS2 fermions live in a real representation of the gauge group. The usual breaking pattern, \(SU(N_f) \times SU(N_f) \to SU(N_f)\), is replaced by \(SU(2N_f) \to SO(2N_f)\) [51–53]. There are \(2N_f^2 + N_f - 1\) Nambu–Goldstone bosons (NGBs), nine in all for \(N_f = 2\). A consequence of reality is that, in addition to meson \((\bar{q}q)\) and baryon states, there are also diquark states. Symmetries associated with the fermions’ reality means that all diquark correlators are identical to corresponding meson ones. For example, the diquarks are needed to fill out the NGB multiplets. The nine NGBs

\(^1\) The models of Refs. [48, 49] require fermions in the fundamental representation in addition to the AS2 ones, in order to give the top quark a mass via the partial-compositeness scenario.
The global symmetry of the \( N_f = 2 \) AS2 theory is thus \( \text{SU}(2N_f) = \text{SU}(4) \). After dynamical symmetry breaking, the unbroken symmetry is \( \text{SO}(4) \). We elaborate on this symmetry breaking pattern, focusing on how the two invariant \( \text{SU}(2) \) subgroups of \( \text{SO}(4) \) are realized. As an example, we classify the 9 Nambu–Goldstone bosons under the unbroken symmetry. In Sec. II C we recall the equivalence between the \( \text{AS}^2 \) representation of \( \text{SU}(4) \) and the vector representation of \( \text{SO}(6) \). We use this equivalence to introduce the color-singlet state of 6 \( \text{AS}^2 \) and the vector representation of \( \text{SO}(6) \). We begin by recalling how basis states of the \( \text{AS}^2 \) irreducible representation are built from color basis states.

The matrix \( S \) is defined as follows. A real or pseudoreal irreducible representation is self-conjugate, meaning that there is a quadratic form \( S \) such that for two vectors \( a \) and \( b \) the product \( a^T S b \) is a singlet. Demanding invariance under \( a = U a' \), \( b = U b' \), we find

\[
U^T S U = S, \tag{2.4}
\]

which implies that the Hermitian generators \( T_a \) satisfy

\[
T_a^T S = T_a S = -S T_a. \tag{2.5}
\]

The entries of \( S \) are real, \( S^* = S \), and it satisfies \( S^{-1} = S^T \). For a real representation \( S \equiv R = R^T \) is symmetric, whereas for a pseudoreal representation \( S \equiv P = -P^T \) is antisymmetric. For the \( \text{AS}^2 \) representation of \( \text{SU}(4) \), it is realized as

\[
S_{ij} = \sum_{k<l} \epsilon_{ijkl} \psi_{[kl]}. \tag{2.6}
\]

(Note that \( \epsilon_{ijkl} = \epsilon_{ijlk} \), so \( S \) is symmetric as it should be for a real representation.)

The matrix \( C \) occurring in Eq. (2.3) is the usual charge-conjugation matrix, which satisfies \( C \gamma_{\mu} = -\gamma_{\mu}^T C \), and \( C^{-1} = C^\dagger = C^T = -C \). We recall that charge-conjugation symmetry acts as

\[
\psi \rightarrow C \tilde{\psi}^T, \tag{2.7a}
\]

\[
\bar{\psi} \rightarrow \psi^T C, \tag{2.7b}
\]

\[
A_{\mu} \rightarrow -A^*_{\mu} \quad (\text{continuum}), \tag{2.7c}
\]

\[
U_{\mu} \rightarrow U^*_{\mu} \quad (\text{lattice}). \tag{2.7d}
\]

We are now ready to derive the identity (2.3). Consider any fermion action that is invariant under the charge-conjugation symmetry (2.7) when the fermions belong to a complex representation. If we now take the fermions to be in a real representation \( (S \equiv R) \), then the fermion action will be invariant under the following discrete symmetry:

\[
\psi \rightarrow R C \tilde{\psi}^T, \tag{2.8a}
\]

\[
\bar{\psi} \rightarrow \psi^T C R, \tag{2.8b}
\]

\[
A_{\mu} \rightarrow A_{\mu} \quad (\text{continuum}), \tag{2.8c}
\]

\[
U_{\mu} \rightarrow U_{\mu} \quad (\text{lattice}). \tag{2.8d}
\]

Thanks to the reality condition (2.4), the inclusion of \( R \) in the fermions’ transformation rule makes up for the fact that the gauge field does not transform. For a pseudoreal representation \( (S \equiv P) \), the discrete symmetry is

\[
\psi \rightarrow P C \tilde{\psi}^T, \tag{2.9a}
\]

\[
\bar{\psi} \rightarrow -\psi^T C P. \tag{2.9b}
\]

2 The Euclidean rules (2.7a) and (2.7b) are consistent with the Minkowskian relation \( \psi = \tilde{\psi} \gamma_0 \), where we have identified \( \gamma_0 \equiv \gamma_4 \).
We may apply the transformation (2.8) [or (2.9)] to a single Dirac fermion. This is unlike the usual charge conjugation (2.7), which acts on the gauge field as well and must be applied to all fields simultaneously. For both real and pseudoreal representations, it follows that the (lattice) Dirac operator satisfies the identity

\[ SCD^T S^{-1} C^{-1} = -SCD^T S^{-1} C = D, \]  

and Eq. (2.3) follows.

We comment in passing that for Wilson fermions, \( \gamma_5 D^1 \gamma_5 = D \). Together with (the Hermitian conjugate of) Eq. (2.10) this implies

\[ S \gamma_5 CD^* S \gamma_5 C = -D, \]

and hence that the fermion determinant is real.

**B. The unbroken SO(4) symmetry and baryon number**

In a gauge theory with \( N_f \) Dirac fermions in a real representation, the global symmetry is SU(2\( N_f \)). After chiral symmetry breaking, the unbroken symmetry is SO(2\( N_f \)). These statements are most obvious when the theory is formulated in terms of Majorana fermions. Invariance under the transformation (2.8) allows each Dirac field to be broken up into two Majorana fields, with no mixing in the action as long as there are no mass terms. The number of independent Majorana (or Weyl) fields is \( N_{\text{Max}} = 2N_f \), making the global symmetry SU(2\( N_f \)). The fermion condensate is a Majorana-fermion bilinear which, for a real representation, is symmetric in its color indices. As the expectation value of a scalar operator, it is antisymmetric in its spin indices, and so it must be symmetric in its (Majorana) flavor indices. It then follows that the unbroken symmetry is SO(2\( N_f \)) [51–53].

Since we elect to work with two AS2 Dirac fermions (instead of four Majorana fermions), we should understand how the SO(4) unbroken symmetry is realized on them. SO(4) is doubly covered by SU(2) \( \times \) SU(2). We will now work out how the two SU(2) groups act on our Dirac fermions. As we will see, one of the SU(2) groups may be identified with isospin, while the baryon number symmetry becomes a subgroup of the other SU(2).

We start with the observation that SO(4) is the symmetry group of the three-sphere \( S^3 \), which in turn can be identified with the SU(2) \( \times \) SU(2) manifold via \( \hat{x} = x_4 + i \sum_{a=1}^3 x_a \sigma_a \), where \( \sigma_a \) are the Pauli matrices and \( \sum_{a=1}^4 x_a^2 = 1 \). The product group SU(2) \( \times \) SU(2) is then realized as \(^3\)

\[ \hat{x} \to g \hat{x} h^1, \quad g, h \in \text{SU}(2). \]  

In order to keep track of the SU(2) transformation properties it is convenient to rearrange the four real coordinates into two complex ones. We choose \( \phi_1 = x_4 + i x_3 \), \( \phi_2 = -x_2 + i x_1 \), so that

\[ \hat{x} = \begin{pmatrix} \phi_1 - \phi_2^* \\ \phi_2 - \phi_1^* \end{pmatrix}, \quad -\hat{x}^\dagger = \begin{pmatrix} \phi_1^* - \phi_2^* \\ \phi_2^* - \phi_1^* \end{pmatrix}. \]  

(The minus sign in front of \( \hat{x}^\dagger \) is introduced for convenience below.)

The transformation properties under left-multiplication are now obvious. The left column of the \( \hat{x} \) matrix is an SU(2) doublet \((\phi_1, \phi_2)\). Denoting this doublet as \( \Phi_{\alpha} \), the right column is \( \Phi_{\alpha}' = \epsilon_{\alpha\beta} \Phi_{\beta}^* \), which again transforms in the fundamental representation of SU(2).

Next, in order to obtain the behavior under right-multiplication we consider the left-action of \( h = -\hat{x}^\dagger \). We read off the right-multiplication doublets: \((-\phi_1^*, \phi_2)\) from the left column of \(-\hat{x}^\dagger\), and \((-\phi_2^*, -\phi_1)\) from its right column. The left- and right- doublets are related by interchanging \( \phi_1 \) with \(-\phi_1^*\).

We now turn to our AS2 theory. The role of real coordinates is played by Majorana fermions, whereas that of complex coordinates is played by Dirac fermions. What takes the place of complex conjugation is the transformation (2.8). We may arrange our two Dirac fermions, \( u \) and \( d \), as well as their anti-fermions, in complete analogy with Eq. (2.13),

\[ \Psi = \begin{pmatrix} u & -RCd^T \\ d & RCu^T \end{pmatrix}. \]  

Motivated by this arrangement we will refer to the left-multiplication SU(2) as isospin symmetry, and to the right-multiplication SU(2) as custodial symmetry. It goes without saying that the two SU(2)’s play a similar role, and the only “preference” for the left-multiplication doublets is in our notation. The isospin and custodial symmetries get interchanged by \( u \leftrightarrow -RCu^T \), which is basically the discrete symmetry (2.8) applied to the \( u \) quark only.\(^4\)

Let us take a closer look at the custodial-symmetry generator \( \sigma_3 \). With reference to Eq. (2.13), its action on the second row of \( \hat{x} \), which is the multiplet \((\phi_2, \phi_1^*)\), is \( \delta \phi_2 = \phi_2 \) and \( \delta \phi_1^* = -\phi_1^* \), or \( \delta \phi_1 = \phi_1 \). Thus \( \phi_1 \) and \( \phi_2 \) transform with the same phase. A translation to the language of Eq. (2.14) is that the custodial \( \sigma_3 \) is just the baryon number. In a two-flavor theory of complex-representation fermions, the unbroken symmetries are isospin and the U(1) of baryon number. In our case, the U(1) is enlarged to a second SU(2) that we call the custodial symmetry, whose two other generators thus raise or lower the baryon number.

\(^3\) The product-group elements \( g = -1, h = 1 \), and \( g = 1, h = -1 \), coincide when they act on \( \hat{x} \). Hence SU(2) \( \times \) SU(2) is a double covering of SO(4).

\(^4\) We are free to add minus signs on the right-hand sides of Eqs. (2.8a) and (2.8b) simultaneously.
Now that we have understood the unbroken symmetry structure, let us consider a few simple applications. As a first exercise, one can show that the transformation (2.8), when applied to the $u$ and $d$ fields simultaneously, is in fact an element of SO(4). Indeed, consider $\Psi \rightarrow -i\sigma_2 \Psi i\sigma_2$, which is a simultaneous rotation in isospin and custodial SU(2). This is just $u \rightarrow R\bar{u}^T$, and the same for $d$.

We next turn to the NGBs. Start with the familiar triplet of pions: $d\gamma_5 u$, $\bar{u}\gamma_5 d$, and $\bar{u}\gamma_5 u - d\gamma_5 d$. Now let us apply a custodial rotation of the form $\exp(i\theta\sigma_1)$. Then $\bar{u}\gamma_5 d$ rotates into a linear combination of itself, of $d^T R\gamma_5 d$, and of $\bar{u}\gamma_5 R\bar{u}^T$. The last two are respectively a diquark and an anti-diquark, each belonging to an isospin-1 multiplet. It follows that there are indeed 9 NGBs, which fall into 3 isospin triplets: one made of diquarks, one of anti-diquarks, and one of quark-antiquark pairs.

C. SU(4) ↔ SO(6) correspondence and the six-quark baryon

In this subsection we first work out in detail the identification between the AS2 representation of SU(4) and the vector representation of SO(6). This allows us to construct a fully antisymmetric color wave function for six AS2 fermions, which will be common to all our baryon states.

In Sec. II A we labeled the components of the AS2 representation by an index pair. We can alternatively introduce a single index $a = 1,\ldots,6$, with the correspondence $\psi_1 = \psi_{[12]}$, $\psi_2 = \psi_{[13]}$, $\psi_3 = \psi_{[34]}$. In the $\psi_a$ basis the matrix $R$ of Sec. II A takes the explicit form

$$R = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \quad (2.15)$$

The inner product of two AS2 spinors,

$$\sum_{i<j} \sum_{k<l} \epsilon_{ijkl} \chi_{[ij]} \chi_{[kl]} = \psi_a R_{ab} \chi_b = \psi^T R \chi, \quad (2.16)$$

is SU(4)-invariant by virtue of Eq. (2.4).

We can recover the standard formulation of SO(6) by applying a U(6) basis transformation to the AS2 states. As a first step, we permute the basis elements and multiply one of them by a minus sign, bringing $R$ to the block diagonal form

$$R = \begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_1 & 0 \\
0 & 0 & \sigma_1
\end{pmatrix}. \quad (2.17)$$

For a further change of basis, we note that

$$\sigma_1 = z^2, \text{ with } \tau = \begin{pmatrix}
z \\
z^* \\
z
\end{pmatrix}, \quad (2.18)$$

where $z = (1 + i)/2$. We denote by $Q$ the $6 \times 6$ matrix with three blocks of the matrix $\tau$ along the main diagonal. Upon performing the basis change

$$\psi \rightarrow \psi' = Q \psi, \quad (2.19)$$

the inner product becomes

$$\psi^T R \chi \rightarrow \psi^T Q^T R Q \chi = \psi^T \chi. \quad (2.20)$$

The inner product has now taken its standard SO(6) form. Under the same basis change, the AS2 SU(4) generators transform as

$$T_a \rightarrow QT_a Q^T. \quad (2.21)$$

Using the properties of the $R$ and $Q$ matrices and Eq. (2.5) it follows that

$$(QT_a Q^T)^T = Q^T T_a^T Q = Q^T R T_a^T R Q = -Q T_a Q^T. \quad (2.22)$$

In the new basis, the generators are antisymmetric (and purely imaginary), as required for the standard basis of SO(6).

As an application of the above, we can show that the fully antisymmetric six-quark wave function

$$B = \epsilon_{a_1 a_2 \ldots a_6} \psi_{a_1} \psi_{a_2} \cdots \psi_{a_6}, \quad (2.23)$$

is gauge invariant (we suppress flavor indices). To prove this, start from

$$B' = \epsilon_{a_1 a_2 \ldots a_6} \psi_{a_1} \psi'_{a_2} \cdots \psi'_{a_6}, \quad (2.24)$$

where the $\psi'$ basis was introduced in Eq. (2.19). This operator is clearly gauge invariant, because in the $\psi'$ basis the SU(4) elements are mapped to orthogonal SO(6) matrices, and the epsilon tensor in Eq. (2.24) is the invariant 6-dimensional tensor. Going back to the original basis we have

$$B' = \epsilon_{a_1 a_2 \ldots a_6} (Q \psi)_{a_1} (Q \psi)_{a_2} \cdots (Q \psi)_{a_6} \quad (2.25)$$

The matrix $Q$ is unitary, and so it leaves invariant the epsilon tensor, up to a factor of $\det Q = -i$. It follows that $B' = iB$, and hence $B$ is gauge invariant as well. We use the fully antisymmetric color wave function (2.23) in the construction of all baryon operators.

D. Diquarks, tetraquarks, and baryons

In constructing states with baryon number, we note first that a color singlet state has to be made of an even number of quarks. Thus we begin with diquarks. As we have seen, the real color representation of the quarks

---

5 The inner product is invariant under SU(4) [equivalently SO(6)] transformations, not under general U(6) transformations.
leads to the conclusion that diquarks are degenerate with mesons. Their color wave function involves the inner product (2.16)
\[ D = \psi_f^d R_{ab} \psi_b^g, \]  
(2.26) where \( f, g \) stand for the spin and flavor indices. Because \( R \) is symmetric, diquarks have a symmetric color wave function. Viewed through the prism of the nonrelativistic quark model, which puts the two quarks in an s-wave, the product of their spin and isospin wave functions must then be antisymmetric. For (pseudo)scalars, the spin wave function is antisymmetric, and the isospin wave function should be symmetric. Those that are NGBs have \( I = 1 \), as seen above. States with higher angular momentum are, of course, also possible. These would include the diquark analogs of (axial) vector and tensor mesons.

The only way to construct a color-singlet tetraquark state is by pairwise contraction of the color indices,
\[ T = (\psi_f^d R_{ab} \psi_b^g)(\psi_g^h R_{cd} \psi_d^h). \]  
(2.27)
One can permute the spin-flavor indices to derive a total of three pairwise coupling schemes. Linear combinations of these schemes will have mixed symmetry under color, but each term will still factor into two color-singlet diquarks. Moreover, by applying an \( RC \) transformation to one quark flavor at a time, one finds that the tetraquarks are degenerate with \( \bar{\psi} \psi \bar{\psi} \psi \) and \( \bar{\psi} \psi \bar{\psi} \bar{\psi} \) states. It is an open question as to whether the tetraquark states in this theory will be meson and diquark scattering states, or bound states; we will not study them further here.

The first baryonic state that cannot be factored into smaller color-singlet components is the six-quark baryon written in Eq. (2.23). It differs essentially from the various pairwise contractions in that it is fully antisymmetric in color. This makes it similar to the baryon of QCD, and indeed similar to baryons made of fundamental-representation quarks for any \( N_c \). Bolognesi [32] has argued that this is the correct baryonic state for studying the large-\( N_c \) limit of gauge theories with AS2 quarks. In general, he finds that baryons made of \( N_b = N_c(N_c-1)/2 \) constituents in the AS2 color representation fit well into a Skyrminon picture. While our construction of the wave function (2.23) relies on special properties of the \( N_c = 4 \) theory, Bolognesi has given an existence proof for a fully antisymmetric, gauge invariant color wave function for any \( N_c \).

### E. Interpolating fields

A lattice simulation needs interpolating fields with an appropriate set of quantum numbers. As noted above, since it is fully antisymmetric under exchange, the AS2 color wave function (2.23) is similar to the color wave function of baryons made of fundamental representation fermions. The multiplet patterns are therefore similar as well. The construction of baryon correlators was discussed in detail in a previous work by one of us [27]. Here we give a brief synopsis.

A convenient set of interpolating fields for baryons are operators which create nonrelativistic quark model trial states. They are diagonal in a \( \gamma_0 \) basis. In the case at hand, a generic two-flavor baryon interpolating field made out of \( k \) up quarks and \( 6 - k \) down quarks can be written as
\[ O_B = c_1 \cdots c_6 \psi^{s_1} \cdots \psi^{s_6} u^{s_1} \cdots u^{s_6} d^{s_{k+1}} \cdots d^{s_6}, \]  
(2.28) where summation over all color and spin indices is implied. (We are free to put all the \( u \)'s to the left of all the \( d \)'s.) The \( C \)'s are an appropriate set of Clebsch–Gordan coefficients. The spin wave function of each quark type, \( u \) or \( d \), must be totally symmetric.

Next we may take linear combinations of the \( O_B \)'s to construct operators with definite isospin quantum numbers. For states built of two flavors of quarks all in the same spatial wave function, multiplets are locked in equal values for angular momentum \( J \) and isospin \( I \). Thus we have states with \( I = J = 3, 2, 1, \) and \( 0 \).

The two-baryon correlator must include all nonzero contractions of creation operators at the source and annihilation operators at the sink. For each flavor, this gives a determinant of quark propagators. These must be summed over all the ways that colors can be apportioned between the quarks. For the analog of the \( \Delta^{++} \), the state with \( I = J = J_3 = N_b/2 = 3 \), this is a single term. The number of terms increases rapidly as the angular momentum decreases, raising the computational cost of the calculation. (This was an issue for the \( N_c = 7 \) baryons of Ref. [27].) Fortunately, \( N_b = 6 \) is not too large and the calculation always remains manageable.

With baryon number as its third generator, our baryons are highest-weight states of the custodial symmetry. In this paper, we are content with studying these states, and we do not consider the six-quark states with smaller baryon number that would be needed to fill in multiplets of the custodial symmetry.

### III. Lattice Action

We define the lattice theory with the usual Wilson plaquette gauge action and with Wilson–clover fermions. The fermion action uses gauge connections defined as normalized hypercubic (nHYP) smeared links [55–57]. The gauge coupling is set by the parameter \( \beta = 2N_c/g_0^2 \). We take the two Dirac flavors to be degenerate, with common bare quark mass introduced via the hopping parameter \( \kappa = (2m_q/a + 8)^{-1} \). As is appropriate for nHYP smearing [58], we fix the clover coefficient at its tree level value, \( c_{SW} = 1 \).
nHYP smearing introduces a new type of discretization error, peculiar to the real representation of the matter field. Our prescription for smearing the fermion’s gauge connection begins with applying the nHYP formulas [57] to the fundamental gauge link, and then the resulting fat link \( V_{ik} \) is promoted to the AS2 representation via Eq. (2.2). The problem is that \( V_{ik} \) is in fact an element of \( U(N_c) \), not \( SU(N_c) \), viz.,

\[
V_{ik} = e^{i\theta} U_{ik},
\]

where both the SU(4) part \( U_{ik} \) and the U(1) phase \( \theta \) are determined by our smearing recipe. Having its origin in the smearing formulas, this U(1) phase is a discretization determined by our smearing recipe. Having its origin in only restored in the continuum limit. For \( \kappa > \kappa_c \) the NGBs become massless when we try breaking pattern of the two-flavor continuum theory and the U(1) phase \( \theta \) fails to satisfy the reality condition (2.4). This in turn leads to violation of relations like Eq. (2.3).

We can gauge the severity of this discretization error by looking at violations of Eq. (2.3). Comparing meson and diquark propagators calculated on single configurations, we have found differences in the third significant digit. Similar effects are seen in the eigenvalue spectrum of the Wilson–clover operator. To the extent that this error creeps into the generation of configurations, there is no cause for concern.

Nonetheless, the breaking of the symmetry (2.4) in the observables is annoying. A way to fix this problem is to replace the AS2 fat link \( V'_{[ij][kl]} \) obtained from Eq. (2.2) with

\[
V'_{[ij][kl]} = \frac{1}{2} \left( V + SV^*S \right)_{[ij][kl]},
\]

before calculating observables. (This can be regarded as a partial quenching, since the correction here is applied only to the valence fermions; one may use \( V' \) for the sea fermions as well, but since we had already generated ensembles without this correction we chose not to do so.)

The new AS2 link \( V'_{[ij][kl]} \) satisfies Eq. (2.4) by construction, at the price of being slightly non-unitary. We compared spectroscopy with and without this correction for a \( 12^3 \times 24 \) data set at one of our parameter values \((\beta = 9.6, \kappa = 0.1285)\). The differences turned out to lie well under one standard deviation. We conclude that the discretization error and the partial quenching (3.2) are benign.

Wilson fermions break chiral symmetry explicitly. In the familiar case of a complex representation, the symmetry breaking pattern of the two-flavor continuum theory is \( SU(2)_L \times SU(2)_R \rightarrow SU(2)_V \). With Wilson fermions, the breaking of \( SU(2)_L \times SU(2)_R \) becomes explicit, and only \( SU(2)_V \) (and, of course, baryon number) is a good symmetry. While the NGBs become massless when we tune \( \kappa \) to its critical value \( \kappa_c \), full chiral symmetry is only restored in the continuum limit. For \( \kappa > \kappa_c \) one enters the Aoki phase [59, 60], where one of the NGB fields condenses, and \( SU(2)_V \) is broken spontaneously.

In our model, the spontaneous symmetry breaking \( SU(4) \rightarrow SO(4) \) of a real representation turns into explicit breaking with Wilson fermions. Only SO(4) is a good symmetry on the lattice, and the full SU(4) flavor symmetry is only recovered in the continuum limit. For \( \kappa < \kappa_c \), the NGBs discussed in Sec. II—mesons and diquarks—acquire a mass. For \( \kappa > \kappa_c \), again one expects to find an Aoki phase. The case of 5 AS2 Majorana fermions (relevant for the SU(5)/SO(5) non-linear sigma model mentioned in the introduction) was recently studied using chiral Lagrangian techniques in Ref. [61].

### IV. PHASE DIAGRAM

As preparation for spectroscopy, we have to find couplings in the confining and chirally broken phase. The phase diagram of Wilson fermion actions in the \( (\beta, \kappa) \) plane can be complicated, depending on the fermion content and the specific action used [59–63]. Figure 1 shows the phase diagram we have observed for the SU(4) AS2 action considered in this paper. The curves shown indicate:

1. \( \kappa_c(\beta) \), the critical value of the hopping parameter where the quark mass \( m_q \) vanishes.
2. \( \kappa_b(\beta) \), the curve of the thermal phase transition. Its location shifts with the lattice size, and two lattice sizes are indicated.
3. \( \kappa_b(\beta) \), the curve of a bulk phase transition that does not move with lattice size.

We discuss each in turn.

#### A. \( \kappa_c \) determination

We define the quark mass through the axial Ward identity (AWI), which relates the divergence of the axial current \( A^a_\mu = \bar{\psi} \gamma_\mu \gamma_5 (r^a/2) \psi \) to the pseudoscalar density \( P^a = \bar{\psi} \gamma_5 (r^a/2) \psi \). At zero three-momentum we have

\[
\partial_\mu \sum_x \langle A^a_\mu (x, t) O^a \rangle = 2m_q \sum_x \langle P^a (x, t) O^a \rangle.
\]

where \( O^a \) is a source, here taken to be a smeared “Gaussian shell” source. The critical \( \kappa_c(\beta) \) line is determined through the vanishing of the quark mass \( m_q \). As noted in Fig. 1, we use several lattice sizes \( N_x \times N_t \). When \( N_t > N_x \), \( t \) labels the usual temporal direction, but when \( N_t < N_x \), we choose one of the spatial directions to be \( t \) in Eq. (4.1) from correlators taken along one of the spacial directions of the lattice (so that the sum over \( x \) in Eq. 4.1 includes two directions with periodic fermion boundary conditions and one antiperiodic direction).
moves to weaker coupling as $N_{12}$ from asymptotic freedom. Z theory symmetry is broken only to AS2 fermions, the $Z_2$ is a true confinement phase transition in our theory.\[\begin{aligned}\langle L \rangle = 0 \text{ in the low-temperature phase, while in the high-temperature phase} \langle L \rangle \text{ orders along the real axis. Typical}\end{aligned}\]

scatter plots of the Polyakov loop in the two phases are shown in the left panel of Fig. 3. The average Polyakov loop as a function of $\kappa$ at four different $\beta$ values for a $16^3 \times 8$ volume are shown in the right panel of Fig. 3.

The $\kappa_t(\beta)$ lines for two different volumes, $16^3 \times 8$ and $12^3 \times 6$, are shown in Fig. 1 as red lines. The transition moves to weaker coupling as $N_t$ increases, as expected from asymptotic freedom.

C. $\kappa_b$ determination

In addition to the temperature-dependent deconfinement lines $\kappa_t(\beta)$, our system exhibits another transition line $\kappa_b(\beta)$. Its presence is signaled by discontinuities in several observables, notably the average plaquette and the quark mass $m_q$. We find that the position of the discontinuity is independent of volume. This is a bulk transition associated with the particular lattice action we use; most likely it has nothing to do with continuum physics. The mechanism that triggers the bulk transition is not clear to us. Similar behavior has been observed in other lattice actions, when the number of fermionic degrees of freedom is large [59, 60, 62, 63]. A similar bulk transition has been observed in studies of the SU(4) pure-gauge theory at $\beta \sim 10.2$ [65].

We have already seen, in the left panel of Fig. 2, a discontinuity in the quark mass $m_q$ at $\kappa \approx 0.127$ as we scan at $\beta = 9.6$. The right panel of Fig. 2 shows the average plaquette values for $\beta = 9.6$ in three different volumes. All the plaquette data show a sudden jump at the same value of $\kappa$.

The transition weakens in the large-$\beta$/small-$\kappa$ region and appears to show only a smooth crossover at $\beta \approx 10.0$. The $\kappa_b(\beta)$ line, determined on two different volumes, $16^3 \times 8$ and $12^3 \times 6$, is sketched in Fig. 1 in blue. Further work is needed to understand the origin of this peculiar bulk transition. For the current study, however, we only need to make sure that our simulation is on the weak-coupling (large-$\beta$) side of this transition so that it has a direct connection to continuum physics.

V. SPECTROSCOPY

A. SU(4) AS2

Referring to the phase diagram, Fig. 1, we chose to simulate the SU(4) AS2 theory at $\beta = 9.6$ for a range of hopping parameter values $0.127 < \kappa < 0.130$, between the bulk transition and $\kappa_c$. Our simulation volumes were all $16^3 \times 32$, and the resulting spectra show that our chosen $\kappa$ values kept us in the confining phase, $\kappa < \kappa_t$. Gauge-field updates used the HMC algorithm with a multi-level Omelyan integrator, including one level of mass preconditioning for the fermions; integration parameters were adjusted to maintain acceptance rates on the order of 70-80%. Gauge configurations were saved to disk every 10 updates. The simulations are summarized in Table I.

The coupling $\beta = 9.6$ gives a lattice spacing that is neither too large nor too small. For comparison with other theories, we fix the lattice spacing using the shorter version [66] of the Sommer [67] parameter $r_1$, defined in terms of the force $F(r)$ between static quarks: $r^2 F(r) = -1.0$ at $r = r_1$. The real-world value is $r_1 = 0.31$ fm [68], and thus Table I shows that our lattice spacings would correspond to a length scale of approximately 0.1 fm in
FIG. 2: Left: Quark mass $m_q$ as a function of $\kappa$ in different volumes at $\beta = 9.6$. Right: Average plaquette as a function of $\kappa$ in different volumes at $\beta = 9.6$.

FIG. 3: Left: Scatter plots of the Polyakov loop in the two different phases on $16^3 \times 8$ lattices at $\beta = 9.6$. Right: Average Polyakov loop on $16^3 \times 8$ lattices for different $\beta$ and $\kappa$ values. The jump of the average Polyakov loop values for each $\beta$ value signals a finite-temperature transition.
QCD. For later comparison, we plot both Sommer parameters for our simulations in Fig. 4.

In addition to the simulations listed in Table I, we used the \( \kappa = 0.129 \) lattices as a set of configurations on which we computed partially quenched (PQ) spectroscopy with four values of the valence quark mass, \( \kappa_V = 0.1295, 0.130, 0.1305, 0.131 \). These data sets used the full complement of \( \kappa = 0.129 \) configurations. Of course, their lattice spacing is the same as that of the \( \kappa = 0.129 \) set.

The correlation functions whose analysis produced our spectroscopy used propagators constructed in Coulomb gauge, whose sources were Gaussians. We used \( \bar{p} = 0 \) point sinks. We collected sets for several different values of the width \( R_0 \) of the source. These correlation functions are not variational since the source and sink are different. We begin each fit with a distance-dependent effective mass \( m_{\text{eff}}(t) \), defined to be \( m_{\text{eff}}(t) = \log C(t)/C(t+1) \) consistent with open boundary conditions for the correlator \( C(t) \). Because our sources and sinks are not identical, \( m_{\text{eff}}(t) \) can approach its asymptotic value from above or below. We mixed data with different values of \( R_0 \) to produce correlators with relatively flat mass \( m_{\text{eff}}(t) \), which we then used in a full analysis involving fits to a wide range of \( t \)'s. For more detail see Ref. [27].

Our resulting data are shown in Tables II, III, and IV. Table II shows the AWI mass and meson spectra and decay constants. The number labels the angular momentum of the fermion fields. The conversion to continuum numbers and (6.2) below, are given with lattice normalization for the different angular-momentum states are identical, the uncertainty in the mass difference is usually smaller than the naïve combination of uncertainties on the individual masses. These fits are over the range \( 5 \leq t \leq 10 \).

We also measured the masses of the \( J = 0 \) and \( J = 1 \) diquarks using nonrelativistic quark model interpolating fields, diquark analogs of the operators we used for baryons. Their masses are, as expected, degenerate with those of their mesonic partners.

Tables III and IV give the baryon masses and mass differences. These are computed together: a jackknife average of correlated, single-exponential fits to all four masses is performed and the differences are collected. This insures that the average mass difference is indeed the difference of the average masses. Since the data sets for the different angular-momentum states are identical, the uncertainty in the mass difference is usually smaller than the naïve combination of uncertainties on the individual masses. These fits are over the range \( 5 \leq t \leq 10 \). We have checked that fits over nearby \( t \) ranges are consistent within uncertainties with these results. We omit results for \( \kappa_V = 0.131 \) because the uncertainties in the baryon masses, especially the \( J = 0 \) baryon, are very large.

### B. SU(3) fundamental

We also generated a data set for SU(3) gauge fields coupled to \( N_f = 2 \) fermions in the fundamental representation. We did this for two (related) reasons. First, the SU(4) data sets include dynamical fermions, and so we felt that our comparison to \( N_c = 3 \) ought to be dynamical-to-dynamical. Second, all previous large-\( N_c \) comparisons were of quenched data sets. While quench-

---

**Table I: Parameters of the SU(4) AS2 simulations.** All are at coupling \( \beta = 9.6 \), in volume \( 16^3 \times 32 \).

| \( \kappa \) | configurations | \( r_1/a \) |
|----------|----------------|--------|
| 0.128    | 146            | 2.50(1)|
| 0.1285   | 140            | 2.78(2)|
| 0.129    | 200            | 2.97(2)|
| 0.1292   | 161            | 3.22(3)|

**Table II: AWI mass and meson spectra and decay constants from dynamical SU(4) AS2 simulations.** \( f_{PS} \) and \( f_{V} \) have lattice normalization.

| \( \kappa \) | \( aM_B(3) \) | \( aM_B(2) \) | \( aM_B(1) \) | \( aM_B(0) \) |
|----------|---------------|---------------|---------------|---------------|
| 0.1280   | 3.134(43)     | 3.055(32)     | 2.972(30)     | 2.923(32)     |
| 0.1285   | 2.608(35)     | 2.513(27)     | 2.442(25)     | 2.389(23)     |
| 0.1290   | 2.297(21)     | 2.212(17)     | 2.147(16)     | 2.113(16)     |
| 0.1292   | 2.046(24)     | 1.990(22)     | 1.948(22)     | 1.920(18)     |
| 0.1295*  | 2.179(26)     | 2.075(20)     | 2.002(20)     | 1.972(18)     |
| 0.1300*  | 2.094(37)     | 1.964(35)     | 1.902(29)     | 1.848(28)     |
| 0.1305*  | 1.984(77)     | 1.854(52)     | 1.826(95)     | 1.732(57)     |

**Table III: Baryon masses from dynamical SU(4) AS2 simulations.** The number labels the angular momentum of the state: \( M_B(3) = M_B(J = 3) \).

| \( \kappa \) | \( a\Delta M_{23} \) | \( a\Delta M_{13} \) | \( a\Delta M_{03} \) |
|----------|-----------------|-----------------|-----------------|
| 0.1280   | 0.079(29)       | 0.162(34)       | 0.210(34)       |
| 0.1285   | 0.095(26)       | 0.166(33)       | 0.219(34)       |
| 0.1290   | 0.086(16)       | 0.151(17)       | 0.185(19)       |
| 0.1292   | 0.056(16)       | 0.098(22)       | 0.126(12)       |
| 0.1295*  | 0.104(21)       | 0.177(22)       | 0.207(23)       |
| 0.1300*  | 0.131(36)       | 0.192(37)       | 0.247(37)       |
| 0.1305*  | 0.131(66)       | 0.159(117)      | 0.253(80)       |

*Partially quenched: same gauge configurations as \( \kappa = 0.129 \).
FIG. 4: Sommer parameters $r_0$ and $r_1$ from the dynamical SU(3) and SU(4) data sets [panels (a) and (b), respectively].

| $\kappa$ | configurations | $r_1/a$ |
|----------|----------------|---------|
| 0.125    | 100            | 2.95(2) |
| 0.126    | 100            | 3.08(3) |
| 0.1265   | 100            | 3.11(3) |
| 0.127    | 100            | 3.23(3) |
| 0.1272   | 100            | 3.30(3) |

TABLE V: Parameters of the SU(3) simulations. All are at coupling $\beta = 5.4$, in volume $16^3 \times 32$.

We have presented our results for meson and baryon spectra and also for meson decay constants in the SU(4) AS2 and SU(3) theories in Sec. V. In this section and the next we will plot and rescale them for comparison with each other and with the quenched SU($N_c$) theories, $N_c = 3, 5$, and 7.

VI. COMPARISONS: MESONS

We plot the data for the pseudoscalar and vector meson masses in Fig. 5. To set the scale, we use the Sommer parameter $r_1$, and for the quark mass we use the lattice-
FIG. 5: Meson spectroscopy. On the left, the squared pseudoscalar mass scaled by $r_1^2$, on the right, $r_1$ times the vector meson mass. The abscissa is $r_1$ times the AWI quark mass. The data sets are: black squares for quenched SU(3) fundamentals, black diamonds for quenched SU(5) fundamentals, black octagons for quenched SU(7) fundamentals, red crosses for SU(4) AS2; the fancy diamonds are the PQ data. Finally, the blue squares are SU(3) with two dynamical, fundamental flavors.

FIG. 6: Two ways to match bare parameters: panel (a) $(m_{PS}/m_V)^2$ vs $\kappa$, and panel (b) $r_1 m_{AWI}$ vs $\kappa$. The data sets are: black squares for quenched SU(3) fundamentals, black diamonds for quenched SU(5) fundamentals, black octagons for quenched SU(7) fundamentals, red crosses for SU(4) AS2. Finally, the blue squares are SU(3) with two dynamical, fundamental flavors.
corrections to scale with $r_0 = 0.56$, $0.40$, and $0.29–0.32$ as the three ratios. We plot Fig. 7.

Masses seem to depend on the quark mass in a way that is reasonably independent of $N_c$ and representation. We are, of course, interested in checking for the lack of perfect scaling with $N_c$. To do this, we must somehow match data at different $N_c$'s. This is an inherently ambiguous procedure, but let us make the attempt. We know that hadron masses depend monotonically on the quark mass. We can compare results at the same values of the quark mass by selecting data at constant $(m_{PS}/m_V)^2$—this is a quantity which is roughly linear in the quark mass—or we can use the AWI quark mass itself, rendered dimensionless by multiplication by $r_1$. These comparisons are shown in Fig. 6. For both quantities, the theories can be matched over almost the entire range of $N_c$. We now select matching points for which we have many data sets, of course, interested in checking for the lack of perfect scaling with $N_c$. To do this, we must somehow match data at constant $(m_{PS}/m_V)^2 = 0.54–0.56$, octagons for $(m_{PS}/m_V)^2 = 0.40$, and squares for $(m_{PS}/m_V)^2 = 0.29–0.32$. The blue symbols are the dynamical SU(3) data and the red symbols, the SU(4) AS2 data. Black symbols show quenched fundamental results.

We display several data sets together. The new ones are the SU(4) AS2 sets, shown in red (crosses for the full dynamical sets and fancy diamonds for the partially quenched ones), and the dynamical SU(3) sets (blue squares). The black squares, diamonds, and octagons are previously published data from quenched simulations with $N_c = 3, 5, 7$ with fundamental fermions [27]. Masses seem to depend on the quark mass in a way that is reasonably independent of $N_c$ and representation. We are, of course, interested in checking for the lack of perfect scaling with $N_c$. To do this, we must somehow match data at different $N_c$'s. This is an inherently ambiguous procedure, but let us make the attempt. We know that hadron masses depend monotonically on the quark mass. We can compare results at the same values of the quark mass by selecting data at constant $(m_{PS}/m_V)^2$—this is a quantity which is roughly linear in the quark mass—or we can use the AWI quark mass itself, rendered dimensionless by multiplication by $r_1$. These comparisons are shown in Fig. 6. For both quantities, the theories can be matched over almost the entire range of $\kappa$.

We now select matching points for which we have many data sets. Thus we choose to use $(m_{PS}/m_V)^2 = 0.54–0.56, 0.40, \text{and } 0.29–0.32$ as the three ratios. We plot $r_1 m_V$ as a function of $1/N_c$, since we expect the leading corrections to scale with $1/N_c$. The result is shown in Fig. 7.

It appears that the systems connected by the original 't Hooft large-$N_c$ scaling argument—fundamental fermions—show smaller $1/N_c$ variation than the AS2 systems over the range of $N_c$ shown. This is, of course, a soft conclusion because there are only two AS2 data sets, $N_c = 3$ and 4, to compare. While we could plausibly make an extrapolation of the fundamental data for $1/N_c \to 0$, we cannot do that for the AS2 results with only two points. We note that the AS2 data with $N_c = 4$ and the fundamental data with $N_c = 7$ show roughly the same shift compared to $N_c = 3$. This is seen to be the case for all quark mass values (see Fig. 5).

B. Decay constants

We define the pseudoscalar decay constant $f_{PS}$ through the matrix element

$$\langle 0|\bar{u}\gamma_5 d|PS \rangle = m_{PS} f_{PS} \tag{6.1}$$

so $f_{PS} \approx 132 \text{ MeV}$, while the vector meson decay constant $f_V$ of state $V$ is defined as

$$\langle 0|\bar{u}\gamma_i d|V \rangle = m_V^2 f_V \epsilon_i, \tag{6.2}$$

where $\epsilon$ is a polarization vector. With clover fermions in the usual $(\kappa)$ normalization, a continuum matrix element (carrying dimension $D$) is defined to be

$$\langle \bar{\psi}\Gamma\psi \rangle_{\text{cont}} = \left(1 - \frac{3}{4} \frac{\kappa}{\kappa_c}\right) Z_{\Gamma}\langle \bar{\psi}\Gamma\psi \rangle_{\text{latt}} a^D, \tag{6.3}$$

and in perturbation theory for fermions in representation $R$ the one-loop renormalization factor is

$$Z_{\Gamma} = 1 + \frac{g^2 C_2(R)}{16\pi^2} z_{\Gamma} + \cdots. \tag{6.4}$$

$z_{\Gamma}$ for nHYP clover fermions is recorded in Ref. [69] as $-1.28$ for the vector current and $-1.30$ for the axial current. In the usual tadpole-improved analysis, one might take the coupling from the lowest-order expression for the plaquette, using the fundamental representation Casimir,$$

$$-\text{Tr} \frac{U_P}{N_c} = g^2 \frac{C_2(F)}{4}. \tag{6.5}$$

For the quenched data sets, the plaquette values (1.787, 2.858 and 3.976) give $g^2 C_2(F) = 2–2.26$. In principle, we should run the scale of the coupling from its value for the plaquette, $q^* a = 3.41$, to the values computed in Ref. [69], $q^* a \approx 1.7$, but the combination of coupling and $z_{\Gamma}$ is so small for nHYP fermions that, in all cases, $Z_{\Gamma}$ is within one percent of unity.

A first determination of $\kappa_c$ was described in Sec. IV. Figure 4 shows that the lattice spacing is rather strongly dependent on $\kappa$ at fixed $\beta$, so one would not expect a naive extrapolation of, say, $am_\psi$ or $(am_{PS})^2$ as a linear function of $\kappa$ would perform particularly well. In fact,
it does not; we can imagine doing fits to all four data points, or to the lightest three. Since the fits have a nonzero number of degrees of freedom, we can evaluate their quality. It is poor.

Instead, we focus on the dimensionless quantities $r_1m_q$ and $r_1^2m_{PS}^2$. A comparison of critical hopping parameters from the fits is shown in Fig. 8 for four possibilities, all with a linear fit:

1. From $r_1m_q$ with all four mass values ($\chi^2 = 24.5$ with 2 degrees of freedom (dof));
2. From $r_1m_q$ with the lowest three mass values ($\chi^2 = 2.4$ with 1 dof);
3. From $r_1^2m_{PS}^2$ with all four mass values ($\chi^2 = 8.6$ with 2 dof);
4. From $r_1^2m_{PS}^2$ with with the lowest three mass values ($\chi^2 = 7$ with 1 dof).

The estimates of $\kappa_c$ are all quite close. More importantly, the uncertainty in the rescaling between lattice and continuum-normalized matrix elements due to different choices of $\kappa_c$ is under half a per cent at any of the quark masses in our data sets. The plots below assume $\kappa_c = 0.13122$ and $Z = 1$.

The partially quenched data sets, at fixed $\beta$ and sea quark $\kappa$, should all have the same lattice spacing. We should be able to find a “valence $\kappa_c$” just by fitting $am_q$ or $(am_{PS})^2$ to a straight line. This we do, finding $\kappa_c = 0.13137$.

The dynamical SU(3) data sets have $\kappa_c = 0.12838(9)$ from a linear fit of $r_1m_q$ in $\kappa$. The fit is stable with a $\chi^2$ below 1.1 per degree of freedom for the lowest five masses (or fewer).

We collect our results for $f_{PS}$ and $f_V$ in Figs. 9 and 10. We have rescaled all fundamental-representation data by $\sqrt{3/N_c}$, and we rescaled the AS2 data by $(3/N_c)$, to remove the leading expected large-$N_c$ scaling, leaving the residual. The dynamical SU(3) data sets agree with the previously-presented quenched sets (at the relatively heavy quark masses where they overlap), and the trend of remarkable $\kappa_c$ scaling for the fundamental-representation data contrasts with the AS2 data sets, where the shift from $N_c = 3$ to 4 is about twenty per cent.

It is unclear whether the observed discrepancy would persist to larger values of $N_c$ in the AS2 expansion, or if it is peculiar to the SU(2$N_f$) $\rightarrow$ SO(2$N_f$) pattern of chiral symmetry breaking of the $N_c = 4$ theory. The slope of the rescaled $r_1f_{PS}$ with respect to $r_1m_q$ is roughly 50% larger for the SU(4) AS2 results. Next-to-leading order chiral perturbation theory predicts a larger contribution by a factor of 2 from the low-energy constant $L_4$ for the SU(4) AS2 data [70]; however, $L_4$ itself is usually taken to be small or even zero in QCD, since it is suppressed at large $N_c$ (with fundamental fermions) by the OZI rule [71].

The OZI rule, which follows from suppression of quark loops, does not hold for the AS2 expansion [72] and so we might expect a larger slope for $f_{PS}$ vs $m_q$ in any AS2 theory compared to the conventional expansion at large $N_c$. Results at larger values of $N_c$ with AS2 fermions would shed light on this discrepancy.

VII. COMPARISONS: BARYONS

Our baryon data are shown in Fig. 11. Unlike mesons, baryon masses depend strongly on $N_c$ and representation. Fundamental representation data with $N_c = 3$, 5, and 7 make that point. In the figure, quenched data are shown in black while the blue points are the SU(3) dynamical-fermion data. Again, we scale the lattice masses by $r_1$ and plot the data versus the AWI quark mass. The SU(4) AS2 masses are shown in red, with octagons for dynamical data sets and fancy diamonds for the partially quenched ones. We have used the same symbols for all states, regardless of their angular momentum, but have connected the states with the same $J$ by lines. The masses of all the states (all $N_c$, all representations) are ordered in angular momentum so that higher $J$ lies higher. Of course, the masses in each set come from the same

---

7 Recent global analyses of the low-energy constants in QCD [73] indicate that $L_4$ is not necessarily small, despite the expected large-$N_c$ suppression.
It was observed that the variation in the data was... fundamental data was discussed in Ref. [28].

The situation for the quark mass... shown in the figure. Repeating these fits for all masses, we can plot the quark mass dependence of $m_0$ and $B$. This is shown in Fig. 13.

Some drift is observed in Fig. 13, especially in $m_0$, shown in the left-hand panel. This situation for the quenched fundamental data was discussed in Ref. [28]. It was observed that the variation in the data was (noisily) consistent with a $1/N_c$ contribution to $m_0$; that is, $m_0(N_c) = m_{00} + m_{01}/N_c + \ldots$ where $m_{00}$ and $m_{01}$ were of comparable, “typical QCD” size. Considering $N_c = 3$ fundamental as $N_c = 3$ AS2, we only have two AS2 points to compare. Clearly, we cannot say anything of statistical significance. We observe, however, that $m_0$ for $N_c = 4$ AS2 is roughly equal to $m_0$ for $N_c = 5$ fundamentals, so modeling $m_0(N_b) = m_{00} + m_{01}/N_b + \ldots$ would give an AS2 $m_{01}$ of roughly the same size as the fundamental value.

The situation for $B$ is less clear cut: $B$ comes from small mass differences. Certainly, the $N_c = 5$ and 7 fundamental $B$ data and the $N_c = 4$ AS2 $B$ data lie on a common line slightly separated from the $N_c = 3$ data. This is in qualitative agreement with large-$N_c$ expectations, $B(N_b) = B_0 + B_1/N_b + \cdots$.

Overall, both $m_0$ and $B$ are of “typical hadronic size” since $1/r_1 \sim 635$ MeV and $r_1 m_0$ and $r_1 B$ are order unity. However, they have rather different dependence on the quark mass. In the Skyrme picture, $B$ is the inverse of the moment of inertia, scaled by $N_b$, so that $B$ should be proportional to $1/m_0$. In a quark model with hyperfine interactions mediated by gluons, $B$ is basically a product of color magnetic moments for the quarks, and for heavy quarks, the magnetic moment scales inversely with the quark mass. This suggests $B \propto 1/m_0^2$. A log-log plot of $B$ versus $1/m_0$ certainly looks like a power law, with an exponent near unity. This is shown for $N_c = 3$ and 4 in Fig. 14 and for the quenched fundamental data in Fig. 15.

The overall dependence of the baryon mass $M_B$ on the quark mass $m_q$ is also interesting to study, since it may be used with the Feynman-Hellmann theorem to determine the baryonic matrix element of the scalar density, if one defines

$$f_q(B) \equiv \frac{m_q}{M_B} \frac{\partial M_B}{\partial m_q} = \frac{m_q}{M_B} (B|\bar{\psi}\psi|B). \quad (7.1)$$

Multiplying by the ratio $m_q/M_B$ cancels the renormalization of the quark mass and gives a dimensionless ratio. For the lowest-lying baryon, this quantity would determine the cross section for direct detection through Higgs exchange in the context of a composite dark matter model [30], in conjunction with the same quantity defined for matrix elements of the proton and neutron [74].
To determine the scalar matrix element, we carry out a linear fit to the quantity $r_1 M_B$ as a function of $r_1 m_q$; the resulting slope is then multiplied by $m_q/M_B$ at each data point. To suppress possible finite-volume systematic errors, only points with $m_{PS} N_s \gtrsim 4$ are used in the fit; this excludes a small fraction of our data.

Results of this analysis are plotted in Fig. 16. Since in the limit $m_q \to \infty$ we expect $M_B \sim N_b m_q$, the quantity $f^{(6)}_q$ should approach 1 in the heavy-quark limit and 0 in the chiral limit. No definitive conclusion about the scaling of the AS2 points with $N_c$ can be drawn, due to the large uncertainties, but the functional dependence is broadly consistent with the fundamental-representation points, and with other lattice results for SU(2) and SU(4) theories with relatively heavy quark masses [30, 75].

VIII. CONSEQUENCES AND CONCLUSIONS

Comparisons of SU(4) AS2 spectroscopy and matrix elements with fundamental fermion data reveal regularities anticipated by large-$N_c$ arguments. Meson masses scale with quark mass in a nearly $N_c$ independent way. The change in meson mass at fixed quark mass from $N_c = 3$ to 4 is about the same size as the change in fundamental fermion masses from $N_c = 3$ to 7. Scaling of decay constants seems to be less exact for AS2 fermions than for fundamental ones, but the shift from $N_c = 3$ to $N_c = 4$ exceeds large-$N_c$ counting expectations by no more than twenty per cent.

Baryon spectroscopy with AS2 fermions seems to fit well into the large-$N_c$ framework, matching well to the rotor spectrum and giving parameters $m_0$ and $B$ which are qualitatively not that different from the ones used to describe fundamental-representation spectroscopy at similar values of the quark mass.

More complete comparisons of large-$N_c$ regularities for AS2 fermions require several $N_c$’s. This allows one to consider a further comparison: With $N_c \neq 4$, AS2 fermions live in complex representations. The pattern of chiral symmetry breaking is then identical to that of ordinary QCD. It could be that for the behavior of chirally-sensitive observables such as $f_{PS}$, $N_c = 4$ is an outlier compared to $N_c \neq 4$.

As far as we know, no dynamical simulations of gauge plus fermionic systems on volumes large enough for spectroscopy with $N_c > 4$ have ever been performed. But at heavier quark masses, quenching effects are not large. Perhaps it would be appropriate to imagine a first round of quenched simulations with AS2 fermions. Studies of mesonic properties could be done with modest resources. With $N_b = N_c (N_c - 1)/2$ quarks in a baryon, they are bosons for $N_c = 5$ (with 10 constituents) and oscillate between fermion and boson at larger $N_c$. But the number of terms in the wave function grows rapidly with $N_c$. Even for $N_c = 5$, the combinatorics of the lower-$J$ correlators seem quite daunting.
The six-quark AS2 baryons are almost certainly unstable against decay in the chiral limit, since they can fall apart into three diquarks. Measuring this decay width would be a difficult task in a lattice simulation (as it is for all decay widths). One could observe it indirectly in spectroscopic studies, in the following way: A lattice correlator always couples to all states with the appropriate quantum numbers, and behaves with $t$ as

$$C(t) = \sum_n A_n \exp(-M_n t). \quad (8.1)$$

Presumably, an interpolating field such as the one we use couples strongly to a single baryon and weakly to a three diquark state. The three-diquark state would have a small $A_n$ and a small mass, so it would only appear at large $t$.

As an example, consider any of our baryons with $I = J > 0$. Di-quark NGBs have $I = 1$, and so the decay into 3 such NGBs is allowed by isospin conservation. However, the NGBs have $J = 0$, meaning that the baryon’s angular momentum will have to be converted into an orbital motion. This leads to a kinematic suppression of the decay. The same applies to the $I = J = 0$ baryon: The isospin state of the three NGBs is antisymmetric, so they will have to be in a spatially antisymmetric state that perforce contains orbital angular momentum. Of course, knowing that the decay actually occurs as a strong-interaction process might be sufficient for phenomenology. For example, in the chiral limit of a model like this, the lightest baryon would probably not be a good dark matter candidate because it would decay into massless NGBs.

As a different generalization of the model we have studied in this paper, we could keep the four-color gauge group as well as the AS2 representation, and vary the number of Majorana fermions $N_{\text{Maj}}$. Whenever $N_{\text{Maj}}$ is even we may instead use $N_f = N_{\text{Maj}}/2$ Dirac fermions. Reality of the representation then gives rise to a special kind of conjugation symmetry, Eq. (2.8). This discrete symmetry interchanges Dirac fermions with their antifermions, and hence it does not commute with baryon number. As a result, baryon number gets embedded into the enlarged, non-Abelian (unbroken) flavor symmetry of the theory, $\text{SO}(N_{\text{Maj}})$.

As mentioned in the introduction, of particular interest for phenomenology is the $N_{\text{Maj}} = 5$ theory, whose low-energy effective theory is the SU(5)/SO(5) non-linear sigma model. We have begun a detailed study of this model that we hope to report on in the future.

---

8 Similar statements apply to pseudoreal representations.
FIG. 14: $B$ vs $1/m_0$ from the rotor formula (1.1); black diamonds from quenched SU(3), blue squares from full SU(3). The SU(4) data are shown as red octagons for the dynamical sets and fancy diamonds for the partially quenched set.

Acknowledgments

T. D. would like to thank Richard Lebed for correspondence and conversations. B. S. thanks the University of Colorado for hospitality, as well as the Yukawa Institute for Theoretical Physics at Kyoto University. This work was supported in part by the U. S. Department of Energy, and by the Israel Science Foundation under Grant no. 449/13. Brookhaven National Laboratory is supported by the U. S. Department of Energy under contract DE-SC0012704. Computations were performed using USQCD resources at Fermilab and on the University of Colorado theory group’s cluster. Our computer code is based on version 7 of the publicly available code of the MILC collaboration [76].

FIG. 15: For comparison, $B$ vs $1/m_0$ from the rotor formula (1.1) from the quenched fundamental data sets of Ref. [27]: $N_c = 3, 5, \text{ and } 7$ data sets are squares, diamonds and octagons.

FIG. 16: The quantity $f_q^{(B)}$ defined in Eq. (7.1), plotted vs the ratio $(m_{PS}/m_{V})^2$. Data shown include quenched fundamental SU(3), SU(5) and SU(7) (black squares, diamonds, octagons), dynamical SU(3) (blue squares), and dynamical SU(4) AS2 (red crosses). We also plot in purple results from [30] for quenched fundamental SU(4), for bare gauge coupling $\beta = 11.5$ (fancy diamonds) and $\beta = 12.0$ (fancy crosses).
[65] D. Barkai, M. Creutz and K. J. M. Moriarty, Nucl. Phys. B 225, 156 (1983).
[66] C. W. Bernard, T. Burch, K. Orginos, D. Toussaint, T. A. DeGrand, C. E. DeTar, S. A. Gottlieb and U. M. Heller et al., Phys. Rev. D 62, 034503 (2000) [hep-lat/0002028].
[67] R. Sommer, Nucl. Phys. B 411, 839 (1994) [arXiv:hep-lat/9310022].
[68] A. Bazavov et al., Rev. Mod. Phys. 82, 1349 (2010) [arXiv:0903.3598 [hep-lat]].
[69] T. A. DeGrand, A. Hasenfratz and T. G. Kovacs, Phys. Rev. D 67, 054501 (2003) [hep-lat/0211006].
[70] J. Bijnens and J. Lu, JHEP 0911, 116 (2009) [arXiv:0910.5424 [hep-ph]].
[71] J. Gasser and H. Leutwyler, Nucl. Phys. B 250, 465 (1985).
[72] A. Cherman and T. D. Cohen, JHEP 0612, 035 (2006) [hep-th/0607028].
[73] J. Bijnens and G. Ecker, Ann. Rev. Nucl. Part. Sci. 64, 149 (2014) [arXiv:1405.6488 [hep-ph]].
[74] P. Junnarkar and A. Walker-Loud, Phys. Rev. D 87, no. 11, 114510 (2013) [arXiv:1301.1114 [hep-lat]].
[75] W. Detmold, M. McCullough and A. Pochinsky, Phys. Rev. D 90, 114506 (2014) [arXiv:1406.4116 [hep-lat]].
[76] http://www.physics.utah.edu/~detar/milc/