Generation of binary words avoiding alternating patterns

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Abstract

In this paper we propose an algorithm to generate binary words with no more 0’s than 1’s having a fixed number of 1’s and avoiding the pattern (10)j1 for any fixed j ≥ 1. We will prove that this generation is exhaustive, that is, all such binary words are generated.

Keywords: Binary words, Pattern avoiding, Exhaustive generation.

1 Introduction

The problem of determining the appearance of a fixed pattern in long sequences of observation is relevant in many scientific problems.

For example, in the area of computer network security, the detection of intrusions, which become increasingly frequent, is very important. Intrusion detection is primarily concerned with the detection of illegal activities and acquisitions of privileges that cannot be detected by information flow and access control models. There are several approaches to intrusion detection, but recently this subject has been studied in relation to pattern matching (see [1, 6, 9]).

This leads to the study of the construction of particular words avoiding a given pattern in an alphabet Σ. The present paper aims to be a contribution in this direction.

Let $F \subset \{0,1\}^*$ be the set of binary words $\omega$ such that $|\omega|_0 \leq |\omega|_1$, for any $\omega \in F$, $|\omega|_0$ and $|\omega|_1$ corresponding to the number of 0’s and 1’s in the word $\omega$, respectively. In this paper we study the construction of the subset $F[p] \subset F$ of binary words excluding a given pattern $p = p_0 \ldots p_{\ell-1} \in \{0,1\}^\ell$, that is a word $\omega \in F[p]$ if and only if it does not contain a sequence of consecutive indices $i, i+1, \ldots, i+\ell-1$ such that $\omega_i, \omega_{i+1} \ldots \omega_{i+\ell-1} = p_0p_1 \ldots p_{\ell-1}$.

If we consider the set of binary words without any restriction, the defined language is regular and we can refer to using classical results (see, e.g., [7, 8, 10]). When the restriction to words with no more 0’s than 1’s is valid, the language $F[p]$ is not a regular one and it becomes more difficult to deal with. For example, in order to generate the language $F[p]$ for each forbidden pattern $p$ an “ad hoc” grammar should be defined. Our aim is to determine a constructive algorithm suggesting a more unified approach which makes it possible to generate all binary words in the class $F[p]$.

In this paper we show how to obtain all binary words belonging to $F$ and avoiding the pattern $p = (10)^j1$, for any fixed $j ≥ 1$.

We [3] introduced an algorithm for the construction of all binary words in $F$ having a fixed number of 1’s and excluding those containing the forbidden pattern $1^{j+1}0^j$, for any fixed $j ≥ 1$. That algorithm generates all the words in $F$ then eliminates those containing the forbidden pattern. Basically, the construction marks in an appropriate way the forbidden patterns in the

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words and generates \(2^C\) copies of each word having \(C\) forbidden patterns such that the \(2^{C-1}\) instances containing an odd number of marked forbidden pattern are annihilated by the other \(2^{C-1}\) instances containing an even number of marked forbidden patterns. For example, the words 00110110 and 01101110, containing two copies of the forbidden pattern \(p = 110\), (the marked forbidden patterns are over-lined) are eliminated by the words 00110110 and 00110110, respectively.

This is possible since no prefix of \(p = 1^j10^j\) is also a suffix of \(p\), that is the forbidden patterns do not overlap and so they are univocally identified inside the words.

Then, the algorithm in [3] cannot be used to generate the words in \(F[p]\) when \(p = (10)^j1\) since the forbidden patterns may overlap inside the words. For example, in \(\omega = 110101010\) there are two overlapping copies of the forbidden pattern \(p = (10)^j1\). So, we propose a new algorithm that generates right the words in \(F\) avoiding the forbidden pattern \(p = (10)^j1\), for any fixed \(j \geq 1\).

The paper is organized as follows. In Section 2 we give some basic definitions and notation. In particular, we recall how every binary word can be represented as a path on the Cartesian plane.

In Section 3 we give a construction, according to the number of 1’s, for the set of binary words excluding the pattern \(p = (10)^j1\), for any fixed \(j \geq 1\), and such that the number of 0’s in each word is inferior to or the same as the number of 1’s.

In Section 4 we prove that the construction given in Section 3 allows us to obtain an exhaustive and univocal generation of such binary words having \(n\) 1’s.

## 2 Basic definitions and notation

Let \(F \subset \{0,1\}^*\) be the set of binary words \(\omega\) such that \(|\omega|_0 \leq |\omega|_1\), for any \(\omega \in F\), \(|\omega|_0\) and \(|\omega|_1\) corresponding to the number of 0’s and 1’s in the word \(\omega\), respectively. In this paper we study the construction of the subset \(F[p] \subset F\) of binary words excluding a given pattern \(p = (10)^j1\), for any fixed \(j \geq 1\).

Given \(|\omega| = |\omega|_0 + |\omega|_1\) the length of \(\omega \in F\), we denote by \(\omega^h\), \((h > 0)\), the word with length \(h \cdot |\omega|\) obtained by linking \(\omega\) to itself \(h\) times, that is \(\omega^h = \underbrace{\omega \cdots \omega}_{h}\) and \(\omega^0 = \varepsilon\), \(\varepsilon\) being the empty word.

Each word \(\omega \in F\) can be naturally represented as a path on the Cartesian plane by associating a rise (or up) step, defined by \((1,1)\) and indicated by \(x\), with each bit 1 in \(\omega\) and a fall (or down) step, defined by \((1,-1)\) and indicated by \(\bar{x}\), with each bit 0 in \(\omega\). For example, the word \(\omega = 11011010010000101111\) is represented by the path \(\gamma = xxxxxxxx\) (see Figure 1). An up-down step is the sequence \(x\bar{x}\).

![Figure 1: The path representing \(\omega = 11011010010000101111\)](image)

From now on, we refer interchangeably to words or their graphical representation on the Cartesian plane, that is paths. So by \(F[p]\) we denote both the set of pattern \(p\) avoiding binary words and the set of corresponding paths.

In the rest of this paper, a path is defined as:

- **primitive** if it begins and ends at ordinate 0 and remains strictly above the x-axis,
- *positive* if it begins at ordinate 0 and remains above or on the x-axis,

- *negative* if it begins and ends at ordinate 0 and remains below or on the x-axis (remark that a negative path in \( F \) necessarily ends at ordinate 0),

- *strongly negative* if it begins and ends at ordinate -1 and remains below or on the line \( y = -1 \),

- *underground* if it ends with a negative suffix.

The *complement* of a path \( \varphi \) is the path \( \varphi^c \) obtained from \( \varphi \) by switching rise and fall steps.

## 3 A construction for the set \( F^p \)

In this section we show the constructive algorithm to generate the set \( F^p \), \( p = (x\bar{x})^j x = (10)^j 1 \) for any fixed \( j \geq 1 \), according to the number of rise steps, or equivalently to the number of 1’s. Given a path \( \omega \in F^p \) with \( n \) rise steps, we generate a given number of paths in \( F^p \) with \( n + h \) rise steps, \( 1 \leq h \leq j \), by means of constructive rules. The number and the shape of the generated paths depend on the ordinate \( k \) of the endpoint of \( \omega \) and on its suffix. With regard to \( k \), we can point out three cases: \( k = 0 \), \( k = 1 \) and \( k \geq 2 \), while as for the suffix we consider whether it is equal to \( (x\bar{x})^j \) or not. When \( k = 0 \), we must pay attention also to the case in which \( \omega \) is an underground path ending with the pattern \( (x\bar{x})^{j-1} x \).

As we will show further on, for each \( \omega \in F^p \) such that \( k = 0 \) or \( k \geq 2 \), the generating algorithm produces two or more positive paths and one underground path with \( n + h \) rise steps, \( 1 \leq h \leq j \), while, when \( k = 1 \), it produces only one positive path with \( n + h \) rise steps. Let us denote by \( \omega^{|k} \) a path with endpoint at ordinate \( k \).

The generating algorithm of the class \( F^p \) with \( p = (x\bar{x})^j x = (10)^j 1 \), for any fixed \( j \geq 1 \), is described in the following sections. The constructive rules related to the special cases in which the suffix of \( \omega \) is \( (x\bar{x})^j \) or \( (x\bar{x})^{j-1} x \) are described in Sections 3.2 and 3.3 respectively, while in Section 3.1 we examine all the other simple cases.

The starting point of the algorithm is the empty word \( \varepsilon \).

### 3.1 Simple cases

In this section we describe the constructive rules to be applied when the suffix of \( \omega \) is neither \( (x\bar{x})^j \) nor \( (x\bar{x})^{j-1} x \). We point out three cases for the ordinate \( k \) of the endpoint of \( \omega \): \( k = 0 \), \( k = 1 \) and \( k \geq 2 \).

\( k = 0 \). A path \( \omega \in F^p \), with \( n \) rise steps and such that its endpoint has ordinate 0, generates, for any \( h \), \( 1 \leq h \leq j \), three paths with \( n + h \) rise steps: a path ending at ordinate 1 by adding to \( \omega \) a rise step and a sequence of \( h - 1 \) up-down steps; a path ending at ordinate 0 by adding to \( \omega \) a rise step, a sequence of \( h - 1 \) up-down steps and a fall step, and an underground path obtained by the one generated in the previous step and mirroring on x-axis its rightmost primitive suffix.

Figure 2 shows the above described operations; the number above the right arrow corresponds to the value of \( h \). Both in this figure and in the following ones we consider \( j = 4 \), that is \( p = (x\bar{x})^4 x = (10)^4 1 \).
Figure 2: The paths generated by $\omega_0$

Therefore

$$\omega_0 \Rightarrow \begin{cases} 
\omega_0 x (x\bar{x})^{h-1} \\
\omega_0 \bar{x} (\bar{x}x)^{h-1} \\
\omega_0 \bar{x} (\bar{x}x)^{h-1} \bar{x}
\end{cases} \tag{1}$$

$k = 1$. A path $\omega \in F^p$, with $n$ rise steps and such that its endpoint has ordinate 1, generates, for any $h$, a path with $n + h$ rise steps with endpoint at ordinate 2 obtained by adding to $\omega$ a rise step and a sequence of $h - 1$ up-down steps (see Figure 3).

Figure 3: The paths generated by $\omega_1$

Therefore

$$\omega_1 \Rightarrow \omega_1 x (x\bar{x})^{h-1} \tag{2}$$

$k \geq 2$. A path $\omega \in F^p$, with $n$ rise steps and such that its endpoint has ordinate $k$, $k \geq 2$, generates, for any $h$, $k + 2$ paths with $n + h$ rise steps: a path ending at ordinate $(k + 1)$ by adding to $\omega$ a rise step and a sequence of $h - 1$ up-down steps; $k - 1$ paths ending at ordinate $(k - 1), (k - 2), \ldots, (1)$, respectively, by adding to $\omega$ a rise step, a sequence of $m$, $2 \leq m \leq k$, fall steps and a sequence of $h - 1$ up-down steps; a path ending at ordinate 0 by adding to $\omega$ a rise step, a sequence of $k$ fall steps, a sequence of $h - 1$ up-down steps.
and a fall step, and an underground path which will be described in Section 3.4. Figure 4 shows the above described operations.

![Figure 4: The paths generated by \( \omega_k \), \( k \geq 2 \)](image)

Therefore

\[
\omega_k \Rightarrow \begin{cases} 
\omega_k x (x\overline{x})^{h-1} \\
\omega_k x (x\overline{x})^m (x\overline{x})^{h-1} \\
\omega_k x (x\overline{x})^k (x\overline{x})^{h-1} \overline{x}
\end{cases} \quad 2 \leq m \leq k \tag{3}
\]

At this point it is clear that:

1. when the path \( \omega \) ends with the suffix \( (x\overline{x})^j \) the paths obtained by means of the constructions (1), (2) and (3) contain the forbidden pattern \( p = (x\overline{x})^j x \). So, we will act as described in Section 3.2.

2. when \( \omega \) is an underground path ending with the pattern \( (x\overline{x})^{j-1} x \), some paths generated by means of the above constructions might contain the forbidden pattern \( p = (x\overline{x})^j x \). So, we will follow a different procedure described in Section 3.3.

### 3.2 Paths ending with \( (x\overline{x})^j \)

Even when the path \( \omega \) ends with the suffix \( (x\overline{x})^j \), the number and the shape of the generated paths depend on the ordinate \( k \) of the endpoint of \( \omega \). Let \( q = (x\overline{x})^j \) be the suffix of \( \omega \).

\( k = 0 \). A path \( \omega \in F[p] \), with \( n \) rise steps and such that its endpoint has ordinate 0, generates, for any \( h, 1 \leq h \leq j \), three paths with \( n + h \) rise steps (see Figure 5): a path ending at ordinate 1, by inserting a sequence of \( h - 1 \) up-down steps and a rise step on the left of \( q \); a path ending at ordinate 0, by inserting a sequence of \( h - 1 \) up-down steps and a rise step on the left of \( q \) and adding a fall step at the end of \( \omega \), and an underground path, obtained by mirroring on \( x \)-axis the rightmost primitive suffix of the path generated at the previous step. Therefore

\[
\omega|_{0q} \Rightarrow \begin{cases} 
\omega_0 (x\overline{x})^{h-1} x q \\
\omega_0 (x\overline{x})^{h-1} x q \overline{x} \\
\omega_0 (x\overline{x})^{h-1} \overline{x} \overline{x} (x\overline{x})^{j-1} xx
\end{cases} \tag{4}
\]
Figure 5: The paths generated by $\omega_{0}(x\bar{x})^j$

$k = 1$. A path $\omega \in F^{[p]}$, with $n$ rise steps and such that its endpoint has ordinate 1, generates, for any $h$, a path with $n + h$ rise steps with endpoint at ordinate 2, obtained by inserting a sequence of $h - 1$ up-down steps and a rise step on the left of the suffix $\bar{\rho}$ (see Figure 6).

Therefore

$$\omega_{1}\bar{\rho} \Rightarrow \omega_{1}(x\bar{x})^{h-1}x\bar{\rho}$$

(5)

Figure 6: The paths generated by $\omega_{1}(x\bar{x})^j$

$k \geq 2$. A path $\omega \in F^{[p]}$, with $n$ rise steps and such that its endpoint has ordinate $k$, $k \geq 2$, generates, for any $h$, $k + 2$ paths with $n + h$ rise steps (see Figure 7): a path ending at ordinate $(k + 1)$, by inserting a sequence of $h - 1$ up-down steps and a rise step on the left of the suffix $\bar{\rho}$; $k - 1$ paths ending at ordinate $(k - 1), (k - 2), \ldots, (1)$, respectively, by inserting a sequence of $h - 1$ up-down steps, a rise step and a sequence of $m, 2 \leq m \leq k$, fall steps on the left of $\bar{\rho}$; a path ending at ordinate 0, by inserting a sequence of $h - 1$ up-down steps, a rise step and a sequence of $k$ fall steps on the left of $\bar{\rho}$, and then adding a fall step at the end of $\omega$, and an underground path which will be described in Section 3.4.

Therefore
\[
\omega_{jk} \Rightarrow \begin{cases} 
\omega |_{kj} (\bar{x}x)^{h-1}x & 
\omega |_{kj} (\bar{x}x)^{h-1}x(\bar{x}x)^m & 2 \leq m \leq k 
\omega |_{kj} (\bar{x}x)^{h-1}x(\bar{x}x)^k \end{cases}
\]  

Figure 7: The paths generated by \(\omega |_{kj} (\bar{x}x)^j\), \(k \geq 2\)

3.3 Paths ending with \((\bar{x}x)^j\)

The paths \(\omega \in \mathcal{F}[b]\) ending on the \(x\)-axis with the sequence \((\bar{x}x)^j\) have the following shape

\[
\omega |_{0} = \mu \bar{x} \eta x (\bar{x}x)^j
\]

where \(\mu\) is a path ending on the \(x\)-axis and \(\eta\) is either the empty path \(\varepsilon\) or is a strongly negative path.

The constructions applied to paths ending at ordinate 0 described in (1) (see Figure 2) can be used even for the paths ending with the sequence \((\bar{x}x)^j\), when \(h \geq 2\), or to generate the paths ending at ordinate 1 or on the \(x\)-axis with a positive suffix, when \(h = 1\). Nevertheless, when \(h = 1\), by applying the construction, we obtain an underground path which contains the forbidden pattern \(p = (\bar{x}x)^jx\).

Therefore if the path ends with the sequence \((\bar{x}x)^j\) and \(h = 1\), in order to generate the underground path we proceed as follows. Two cases must be taken into consideration.

1) \(\mu\) does not end with a peak \(x\bar{x}\). The underground path generated from \(\omega |_{0} = \mu \bar{x} \eta x (\bar{x}x)^j\) is obtained by adding the path \(\bar{x}x\) to \(\omega |_{0}\), mirroring on \(x\)-axis the rightmost suffix \((\bar{x}x)^j\) of \(\omega |_{0}\bar{x}x\) and shifting the sequence \((\bar{x}x)^j\) between \(\mu\) and the sub-path \(\bar{x} \eta x\).

So the path \(\omega |_{0} = \mu \bar{x} \eta x (\bar{x}x)^j\) generates the underground path \(\mu (\bar{x}x)^j \bar{x} \eta x\) (see Figure 8). It should be noticed that this construction applies to \(\omega\) even if \(\mu = \varepsilon\).

2) \(\mu\) ends with a peak \(x\bar{x}\). When the path \(\mu\) ends with a peak \(x\bar{x}\), that is \(\mu = \mu' \bar{x} \bar{x}\), the insertion of the sequence \((\bar{x}x)^j\) between \(\mu\) and the sequence \(\bar{x} \eta x\) produces the forbidden pattern \(p = (\bar{x}x)^j\). Let us consider the following subcases: \(\eta \neq \varepsilon\) and \(\eta = \varepsilon\).
2.1) $\eta \neq \varepsilon$. The underground path is obtained by performing on $\omega_0 = \mu' x \bar{x} \eta x (\bar{x}x)^{j-1}$ the following operations: shifting the rightmost peak $x \bar{x}$ of $\mu$ to the right of the sub-path $\bar{x} \eta x$, mirroring on $x$-axis the sequence $(\bar{x}x)^{j-1}$ and adding to such path the steps $\bar{x} x$.

So, when $h = 1$, the underground path with negative suffix generated by $\omega_0 = \mu' x \bar{x} \eta x (\bar{x}x)^{j-1}$ is $\mu' \bar{x} \eta x (\bar{x}x)^{j-1}$ (see Figure 9).

2.2) $\eta = \varepsilon$. In this case, the underground path obtained by means of the construction described in 2.1) is $\omega' = \mu' x \bar{x} (x \bar{x})^j \bar{x} x$ and it contains the forbidden pattern $p = (x \bar{x})^j x$ if $\mu'$ ends with the sequence $(\bar{x}x)^j$ or with the sequence $\bar{x} \eta' x (\bar{x}x)^{j-1}$, where $\eta'$ is a non-empty strongly negative path. Let us take the longest suffix of $\omega_0 = \mu' x \bar{x} (\bar{x}x)^j$ into account so that $\omega_0 = \varphi \nu_1 \nu_2 \ldots \nu_k$, where

$$
\begin{align*}
\nu_1 &= \bar{x} \lambda x (\bar{x}x)^{j-1} x \bar{x} \\
\nu_i &= (\bar{x}x)^{j} x \bar{x} & 1 < i < k \\
\nu_k &= (\bar{x}x)^{j} x \bar{x}
\end{align*}
$$

and $\lambda$ is the empty path or is a strongly negative path. Every sequence $\nu_i$, $1 \leq i \leq k$, will be changed into $\bar{\nu}_i$ in the following way:

2.2.1) if $\varphi$ is a path that does not end with a peak $x \bar{x}$, then

$$
\begin{align*}
\bar{\nu}_1 &= (x \bar{x})^{j} \bar{x} \lambda x \\
\bar{\nu}_i &= (x \bar{x})^{j} \bar{x} x & 1 < i < k \\
\bar{\nu}_k &= (x \bar{x})^{j} \bar{x} x
\end{align*}
$$

and the underground path generated by $\omega_0$ is $\varphi \bar{\nu}_1 \ldots \bar{\nu}_k$ (see Figure 10);

2.2.2) if $\varphi$ ends with a peak $x \bar{x}$, that is $\varphi = \varphi' x \bar{x}$, then

$$
\begin{align*}
\bar{\nu}_1 &= \bar{x} \lambda x (x \bar{x})^{j} \bar{x} x \\
\bar{\nu}_i &= (x \bar{x})^{j} \bar{x} x & 1 < i < k \\
\bar{\nu}_k &= (x \bar{x})^{j} \bar{x} x
\end{align*}
$$

and the underground path generated by $\omega_0$ is $\varphi' \bar{\nu}_1 \ldots \bar{\nu}_k$ (see Figure 11).
3.4 The underground path generated by $\omega_{|k}$

Now let us describe how to obtain the underground path generated by $\omega_{|k}$, $k \geq 2$.

For any $h$, $1 \leq h \leq j$, let $\omega' = \nu \varphi$ be the path obtained from $\omega_{|k}$ and ending on the $x$-axis with a positive suffix, $\varphi$ is the rightmost suffix in $\omega'$ which is primitive.

If the path $\varphi^c$ does not contain the forbidden pattern $p$, the underground path generated by $\omega_{|k}$ is $\nu \varphi^c$.

If the path $\varphi^c$ contains the forbidden pattern $p$, we must apply a swap operation $\Phi$ in order to obtain a path $\varphi_1 = \Phi(\varphi^c)$ avoiding the forbidden pattern. The underground path generated by $\omega_{|k}$ is $\nu \varphi_1$.

Before describing the $\Phi$ operation on $\varphi^c$, let us consider the following proposition.

**Proposition 1.** Let $\mu \in F^{[p]}$ a primitive path; $\mu^c$ contains the forbidden pattern $p = (x\bar{x})^j x$ if and only if $\mu$ contains the pattern $p' = (\bar{x} \bar{x})(x\bar{x})^j \bar{x}$.

From Proposition 1 it follows that, if $\varphi^c$ contains the forbidden pattern $p$, then it is preceded and followed by at least a rise step.

Operation $\Phi$ must generate a path $\varphi_1$ avoiding the forbidden pattern $p = (x\bar{x})^j x$ and such that $\varphi_1 \notin F^{[p]}$; in this way $\varphi_1$ is not the complement of any path in $F^{[p]}$. The path $\varphi_1 = \Phi(\varphi^c)$ is obtained in the following way:

i) consider the straight line $r$ from the beginning of the pattern $p = (x\bar{x})^j x$ and let $t_1$ be the rightmost point in which $r$ intersects $\varphi^c$ on the left of $p$ such that $t_1$ is preceded by at least two fall steps;

ii) let $\delta_2 = (x\bar{x})^m$, $0 \leq m < j$, the subsequence on the right of $t_1$, followed by at least a fall step;
iii) \textit{swap} the initial subsequence $\delta_1 = (x\bar{x})^j$ of $\varphi$ and $\delta_2$. Let us remark that $\delta_2$ cannot be equal to $(x\bar{x})^j$ as $\varphi$ does not contain the forbidden pattern $p = (x\bar{x})^jx$ (see Figure 12a).

When $m = 0$, that is $\delta_2$ is the empty word, we simply insert $\delta_1$ into $t_1$ (see Figure 12b).

Operation $\Phi$ is applied to each forbidden pattern in $\varphi^c$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{Some examples of the $\Phi$ operation, $p = (x\bar{x})^3x$}
\end{figure}

\begin{proposition}
Let $\varphi_1 = \Phi(\varphi^c)$, then $\varphi_1^c \in F \setminus F[p]$.
\end{proposition}

\begin{proof}
The $\Phi$ operation transforms the subsequence $\varrho_1 = (\bar{x})^m \delta_2 \bar{x}$, $(m \geq 2)$, of $\varphi^c$ into the subsequence $\varrho_2 = (\bar{x})^m \delta_1 \bar{x} = (\bar{x})^m (x\bar{x})^j \bar{x}$ of $\varphi_1$. The complement of $\varrho_2$ is

$\varrho_2^c = (x)^m (\bar{x}x)^j x = (x)^{m-1} (x\bar{x})^j x x$

So $\varphi_1^c$ contains the forbidden pattern $p = (x\bar{x})^jx$. \hfill $\square$

\begin{proposition}
Let $\mu \in F \setminus F[p]$ a primitive path such that $\mu^c \in F[p]$. Then there exists a path $\eta \in F[p]$ such that $\mu^c = \Phi(\eta^c)$.
\end{proposition}

\begin{proof}
If $\mu \in F \setminus F[p]$ and $\mu^c \in F[p]$ then $\mu^c$ contains the pattern $\bar{x}\bar{x}(x\bar{x})^j \bar{x}$; we apply to $\mu^c$ the following operation $\Phi^{-1}$:

i) consider the straight line $r$ from the end of the pattern $(x\bar{x})^j$ and let $t_2$ be the leftmost point where $r$ intersects $\mu^c$ on the right of $(x\bar{x})^j$ such that $t_2$ is followed by at least two rise steps;

ii) let $\delta_2 = (x\bar{x})^m$, $0 \leq m < j$, the subsequence on the left of $t_2$, preceded by at least a rise step;

iii) swap the subsequence $(x\bar{x})^j$ and $\delta_2$. When $m = 0$, that is $\delta_2$ is the empty word, we simply insert $(x\bar{x})^j$ into $t_2$. \hfill $\square$

Figure 13 shows the initial steps of the generating algorithm of the paths corresponding to words in $F[p]$, $p = (x\bar{x})^2x = (10)^21$.

Let us remark that, following the above constructions, given a path $\omega$, the number of generated paths depends only on the ordinate of endpoint of $\omega$. 

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Figure 13: The initial steps of the generating algorithm of the paths corresponding to words in $F^{|p|}$, 
$p = (x\bar{x})^2x = (10)^21$. Dotted lines are related to $h = 2$

So, the complete generating algorithm can be briefly described by the succession rule (7) (for more details on succession rules see [2, 4, 5])

\[
\begin{align*}
(0) & \rightarrow (0)(0)(1) & 1 \leq h \leq j \\
(1) & \rightarrow (2) & 1 \leq h \leq j \\
(k) & \rightarrow (0)(0)(1) \cdots (k-1)(k+1) & 1 \leq h \leq j, \ k \geq 2
\end{align*}
\]

where each number corresponds to the ordinate of the endpoint of a path. The zero in the first line in (7) is associated with the empty path. The second line in (7) is associated with operations (1) and (4), the third line is associated with operations (2) and (5), and the last line describes the construction when the endpoint has ordinate $k \geq 2$, underground path included.
4 Exhaustive generation

In this section we prove that the construction described in Section 3 allows to generate the class $F[p]$ exhaustively for any fixed forbidden pattern $p = (10)^j$, $j \geq 1$, in the sense that all the words in $F[p]$ with $n$ 1's, $n \geq 0$, can be generated.

**Theorem 1.** Given a fixed forbidden pattern $p = (10)^j$, $j \geq 1$, the construction described in Section 3 generates all the paths with $n$, $n \geq 0$, rise steps representing the binary words in $F[p]$ with $n$ 1's.

Let $\omega \in F[p]$, then

$$\omega = \varphi_0 \varphi_1 \varphi_2 \ldots \varphi_s$$

that is, $\omega$ is made of $s + 1$ sub-paths such that:

- $\varphi_0$ is the empty path $\varepsilon$,
- $\varphi_i$, $1 \leq i < s$ is a path in $F[p]$ beginning from and ending on the $x$-axis,
- $\varphi_s$ is a path in $F[p]$ beginning from the $x$-axis with endpoint at ordinate $k \geq 0$.

The proof of Theorem 1 is obtained by induction on the number of sub-paths.

**Proof.** The empty path $\varepsilon$ is generated as the starting point of the algorithm. Let us assume that all the possible sub-paths $\varphi_1 \varphi_2 \ldots \varphi_i$ of $\omega$ are generated. We prove that the algorithm generates the sub-path $\varphi_1 \varphi_2 \ldots \varphi_i \varphi_{i+1}$ for any path $\varphi_{i+1} \in F[p]$.

Let $\varphi_i$ be a path that does not end with the pattern $(xx)^j$. In this case the path $\varphi_i$ may be either positive or negative.

If $\varphi_{i+1}$ is a positive path we have to prove that all the positive paths are generated and this will be demonstrated in Section 4.1. In the case of a negative path, denoting $d$ the largest value of its absolute ordinate, let us remark that:

- the negative paths with $d = 1$ are $(\bar{x}x)^\ell$, $1 \leq \ell \leq j$, and they are generated by iterating the construction (1);
- negative paths with $d = 2$ are $\bar{x}(\bar{x}x)^\ell x$, $1 \leq \ell \leq j$, and they are generated by means of (1) when $1 \leq \ell < j$, or by means of (1) when $j = \ell$;
- let $\gamma$ be a negative path with $d > 2$; if $\gamma^c \in F[p]$ then $\gamma$ is the underground path generated by a positive path with endpoint at ordinate $k \geq 2$, (see Section 3.3), otherwise $\gamma = \Phi(\eta^c)$ for a positive path $\eta$ in $F[p]$ with endpoint at ordinate $k \geq 2$ (see Proposition 3).

Note that when $j = 1$ and $\varphi_i$ ends with the pattern $\bar{x}x$, the only possible negative path $\varphi_{i+1}$ with $d = 2$ is $\bar{x}xxx$ and it is generated by applying the construction (1) to the path $\varphi_i x\bar{x}$.

When the suffix of $\varphi_i$ is the pattern $(\bar{x}x)^\ell$, $\varphi_{i+1}$ must be a negative path and the path $\varphi_1 \varphi_2 \ldots \varphi_i \varphi_{i+1}$ is the underground path obtained by means of the construction described in case 1) in Section 3.3 (see Figure 8).

In the same way, when the sub-path $\varphi_i \varphi_{i+1}$ is of type

$$(\bar{x}x)^\ell \bar{x}x((\bar{x}x)^\ell \bar{x}x)^r,$$  
$r \geq 1$

or

$$\bar{x}x((\bar{x}x)^\ell \bar{x}x)^r,$$  
$r \geq 1$
where $\lambda$ is the empty path $\varepsilon$ or is a strongly negative path, then it is generated by the constructions described in case 2) in Section 3.2 (see Figures 9, 11 and 11).

Then, if we show that all the possible positive paths are generated, then we can claim that Theorem 1 is proved.

Moreover, we observe that for each path $\omega$ in $F^{[p]}$ with $n$ rise steps there exists one and only one path $\omega'$ in $F^{[p]}$ with $n - h$ rise steps, $1 \leq h \leq j$, such that $\omega$ is obtained from $\omega'$ by means of the construction described in Section 3.

This assertion is a direct consequence of the construction, since the actions described are univocally determined. □

4.1 Positive paths

In this section we prove that all the positive paths with $n$ rise steps are generated by means of the construction described in Section 3. In the sequel of this section we analyze only positive paths.

The proof is obtained by induction on $n$. There are only two paths with $n = 1$ rise step, that is $x$ and $x\bar{x}$, and they are generated by means of construction (1) applied to the empty path $\varepsilon$. Let us assume that all the paths with $n' < n$ rise steps are generated; we will prove that all the paths with $n$ rise steps are generated.

Note that, following the construction given in [2], a path with $n$ rise steps can be obtained from a Dyck path $\omega$ with $n - 1$ rise step by inserting one rise step in each point at ordinate $i$ of its last descent followed by $q$ fall steps, $0 \leq q \leq i + 1$. We will prove that all the paths obtained so are also generated following the constructions given in the above sections. Let us denote by $\omega_q$ a paths ending with $q$ fall steps.

Let $m$ be the number of fall steps in the last descent of $\omega$. First of all, we note that for any value of $m$ the paths obtained by inserting a rise step in the point at ordinate 0 are generated by means of the constructions (1) or (4) applied to the path $\omega$. Here we give the proof for the case with $m > 2$, distinguishing three cases: $i = 1$, $(1 < i < m - 1) \lor (i = m)$ and $i = m - 1$. The analogous and simple cases $m = 1$ and $m = 2$ are left to the reader.

- $i = 1$. Let $\omega = \gamma_{m-1}x(\bar{x})^m$. The insertion of a rise step in the point at ordinate 1 gives three paths:

  - $\omega_0 = \gamma_{m-1}x(\bar{x})^{m-1}x$, which is generated by means of (2) applied to the prefix $\gamma_{m-1}x(\bar{x})^{m-1}$ of $\omega$;
  - $\omega_1 = \gamma_{m-1}x(\bar{x})^{m-1}x\bar{x}$ and $\omega_2 = \gamma_{m-1}x(\bar{x})^{m-1}x\bar{x}\bar{x}$, which are the paths with endpoints at ordinate 1 and 0, respectively. If $j > 1$, then $\omega_1$ and $\omega_2$ are generated by the construction (3), where $k = m - 1$, applied to the path $\gamma_{m-1}$ with $h = 2$, otherwise, if $j = 1$, they are generated by means of (6) with $k = m - 1$ and $h = 1$ applied to the path $\gamma_{m-1}x\bar{x}$.

- $(1 < i < m - 1) \lor (i = m)$. The insertion of a rise step in the point at ordinate $i$ gives $i + 2$ paths $\omega_q$, $0 \leq q \leq i + 1$. The paths $\omega_q$ with $q \neq 1$ are all the positive paths generated by means of (3) with $k = i$ and $h = 1$ applied to the prefix of $\omega$ of length $|\omega| - i$. The path $\omega_1 = \gamma_{m-1}x(\bar{x})^{m-1}x\bar{x}$ is the path with endpoint at ordinate $i$. When $j > 1$, $\omega_1$ is generated by means of construction (3) with $k = m - 1$ and $h = 2$ applied to the path $\gamma_{m-1}$, while, when $j = 1$, it is generated by means of (6), with $k = m - 1$ and $h = 1$, applied to the path $\gamma_{m-1}x\bar{x}$.
• $i = m - 1$. The insertion of a rise step in the point at ordinate $m - 1$ generates $m + 1$ paths $\omega_q$, $0 \leq q \leq m$. The paths $\omega_q$ with $q \neq 1$ are all the positive paths generated by means of (3) with $k = m - 1$ and $h = 1$ applied to the prefix of $\omega$ of length $|\omega| - m + 1$.

As far as the generation of $\omega_1$ is concerned, we have to distinguish three cases:

1. If $\omega = \gamma_{m-2}(x\bar{x})^{\ell-1}(\bar{x})^{m-1}$, $1 \leq \ell - 1 < j$, then $\omega_1 = \gamma_{m-2}(x\bar{x})^\ell$. If $\ell < j$, then $\omega_1$ is the path with endpoint at ordinate $(k + 1)$ generated by means of (3) with $k = m - 2$ and $h = \ell + 1$, applied to the path $\gamma_{m-2}$. If $\ell = j$, then $\omega_1$ is the path with endpoint at ordinate $(k + 1)$ generated by means of (3) with $k = m - 2$ and $h = 1$ applied to the path $\gamma_{m-2}(x\bar{x})^\ell$. Note that, when $m = 3$, the endpoint of the prefix $\gamma_{m-2}$ has ordinate 1, and the path $\omega_1$ is obtained applying the construction (2) (or (5)) instead of (3) (or (7)).

2. If $\omega = \gamma_{m+m'-2}(x\bar{x})^{m'}(\bar{x})^{m-1}$, $1 \leq \ell - 1 < j$ and $m' > 2$, then $\omega_1 = \gamma_{m+m'-2}(x\bar{x})^{m'}$. If $\ell < j$, then $\omega_1$ is the path with endpoint at ordinate $(k + m' + 1)$ generated by means of (3) with $k = m + m' - 2$ and $h = \ell + 1$ applied to the path $\gamma_{m+m'-2}$. If $\ell = j$, then $\omega_1$ is generated by means of (4) with $k = m + m' - 2$ and $h = 1$ applied to the path $\gamma_{m+m'-2}(x\bar{x})^\ell$.

3. If $\omega = \gamma_{m-2}(x\bar{x})^r(x\bar{x})^{\ell-1}(\bar{x})^{m-1}$, $0 \leq r < j$ and $1 \leq \ell - 1 < j$, then $\omega_1 = \gamma_{m-2}(x\bar{x})^r$. If $\ell < j$, then $\omega_1$ is the path with endpoint at ordinate $(k + 1)$ generated by means of (3), with $k = m - 2$ and $h = \ell + 1$ applied to the path $\gamma_{m-2}(x\bar{x})^r$. If $\ell = j$, then $\omega_1$ is generated by means of (4), with $k = m - 2$ and $h = r + 1$ applied to the path $\gamma_{m-2}(x\bar{x})^\ell$.

\[\square\]

5 Conclusions and further developments

In this paper we propose an algorithm for the construction of particular binary words, according to the number of 1’s, excluding a fixed pattern $p = (10)^j1$, $j \geq 1$.

Successive studies should take into consideration binary words avoiding different forbidden patterns both from an enumerative and a constructive point of view.

Moreover, it would be interesting to study words avoiding patterns which have a different shape, that is not only patterns consisting of a sequence of rise and fall steps. This could be the first step in the study of a possible universal generating algorithm for pattern avoiding words.

Another interesting field of study is to determine a sort of invariant class of avoiding patterns that is the paths $p_1, p_2, \ldots, p_l$ such that $|F[p_1]| = |F[p_2]| = \cdots = |F[p_l]|$ with consequent bijective problems.

One could also consider a forbidden pattern on an arbitrary alphabet and investigate words avoiding that pattern, or study words avoiding more than one pattern and the related combinatorial objects, considering various parameters.

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