Faces of simplices of invariant measures for actions of amenable groups

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Abstract We extend the result of Downarowicz (Israel J Math 165:189–210, 2008) to the case of amenable group actions, by showing that every face in the simplex of invariant measures on a zero-dimensional dynamical system with free action of an amenable group $G$ can be modeled as the entire simplex of invariant measures on some other zero-dimensional dynamical system with free action of $G$. This is a continuation of our investigations from Frej and Huczek (Groups Geom Dyn 11:567–583, 2017), inspired by an earlier paper (Downarowicz in Israel J Math 156:93–110, 2006).

Keywords Invariant measure · Periodic measure · Choquet simplex · Amenable group · Group action · Symbolic system · Cantor space

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1 Introduction

Let $X$ be a Cantor space i.e. a compact, metrizable, zero-dimensional perfect space, and let $G$ be a countable amenable group acting on $X$ via homeomorphisms $\varphi_g$, $g \in G$. Amenability of $G$ means that there exists a sequence of finite sets $F_n \subset G$ (called a Følner sequence, or the sequence of Følner sets), such that for any $g \in G$ we have

$$\lim_{n \to \infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,$$

where $gF = \{gf : f \in F\}$, $|\cdot|$ denotes the cardinality of a set, and $\triangle$ is the symmetric difference. The action of $G$ is free if the equality $gx = x$ for any $g \in G$ and $x \in X$ implies that $g$ is the neutral element of $G$. It is well known that one can represent the system $(X, G)$ as an inverse limit $\lim_{\leftarrow} X_j \subset \prod_{j \in \mathbb{N}} X_j$ where each $X_j$ is a group subshift on finitely many symbols i.e. a closed $G$-invariant subset of some $\Lambda_j G$, $|\Lambda_j| < \infty$, with the action of $G$ defined by $gx(h) = x(hg)$. Indeed, if $\mathcal{U} = \{U_j : j \in \mathbb{N}\}$ is a base for topology in $X$ consisting of clopen sets then we define $\Lambda_i$ to be the set of all elements of the cover of $X$ by the sets of the form $V_1 \cap \cdots \cap V_i$, where either $V_j = U_j$ or $V_j = U^c_j$ for every $1 \leq j \leq i$. The space $X_i$ is an image of $X$ by the map $\pi_i : X \to \Lambda_i^G$ defined by the formula

$$\pi_i(x)(g) = \lambda_i \iff gx \in \lambda_i$$

The inverse system whose inverse limit is conjugate to $(X, G)$ is then given by the sequence of the spaces $X_i$ with bonding maps defined by coordinatewise inclusions. We will often refer to this inverse limit as a so called array system—an element of $X$ in this interpretation is a map $x(\cdot, \cdot)$ on $G \times \mathbb{N}$, where $x(\cdot, j) \in X_j$. We will call such a map an array and from now on we will assume that our system is in array representation. By an $(F, k)$-block we mean a map $B : F \times [1, k] \to \bigcup_j \Lambda_j$, where $F$ is a finite subset of $G$ (which will occasionally be called the shape of a block), $k$ is a positive integer and $[1, k]$ is an abbreviation for $\{1, \ldots, k\}$. If $E$ is a subset of the domain of a block $B$ then by $B[E]$ we will denote a restriction of $B$ to $E$. By abuse of the notation, we will mean by $|B|$ the cardinality of the shape of $B$. We will use the same letter to denote both a block and a cylinder set induced by this block—the exact meaning is always clear from the context. A block $B$ occurs in $X$ if $B$ is a restriction of some $x \in X$.

Let $K$ be an abstract metrizable Choquet simplex, i.e. it is a compact convex set of a locally convex metric vector space, such that for each $v \in K$ there is a unique Borel probability measure supported on the set of extreme points of $K$ with barycenter in $v$ (see [9] for an exhaustive course on the theory of Choquet simplices). Following [3] we define:

**Definition 1.1** 1. An assignment on $K$ is a function $\Phi$ defined on $K$ such that for each $p \in K$, the value of $\Phi(p)$ is a measure-preserving group action $(X_p, \Sigma_p, \mu_p, G_p)$, where $(X_p, \Sigma_p, \mu_p)$ is a standard probability space.
2. Two assignments $\Phi$ on $K$ and $\Phi'$ on $K'$ are equivalent if there exists an affine homeomorphism $\pi : K \rightarrow K'$ such that $\Phi(p)$ and $\Phi'(\pi(p))$ are isomorphic for every $p \in K$.

3. If $(X, G)$ is a continuous group action on a compact metric space $X$ then the set of all $G$-invariant measures supported by $X$, endowed with the weak* topology of measures, is a Choquet simplex, and the assignment by identity $\Phi(\mu) = (X, Bor_X, \mu, G)$ (where $Bor_X$ is the Borel sigma-field) is the natural assignment of $(X, G)$.

By a face of a simplex $S$ we mean a compact convex subset of $S$ which is a simplex itself and whose extreme points are also the extreme points of $S$. If $K$ is a face of a simplex $M_G(X)$ of all $G$-invariant probability measures on $X$ then by the identity assignment on $K$ we mean the restriction of the natural assignment on $M_G(X)$ to $K$.

In the current article we aim to prove the following:

**Theorem 1.2** Let $X$ be a Cantor system with free action of an amenable group $G$ and let $K$ be a face in the simplex $M_G(X)$ of $G$-invariant measures of $X$. There exists a Cantor system $Y$ with free action of $G$, such that the natural assignment on $Y$ is equivalent to the identity assignment on $K$.

In case of actions of $\mathbb{Z}$ the theorem was proved in [3] (even with weaker assumptions; see Sect. 4) and the key tool used there was approximation of an arbitrary ergodic measure by a block (periodic) measure, i.e. a measure supported on a finite orbit. Density of periodic measures in the set of all invariant measures is usually a desired property and was proved to be true in various cases, e.g. for systems with specification property (see [1]). In case of a one-dimensional subshift one can construct a periodic measure by choosing a block $B$ occurring in a system and uniformly distributing a probability mass on the orbit of a sequence obtained by periodic repetitions of $B$. Such a sequence need not be an element of a subshift (and the measure need not belong to its simplex of invariant measures), still it may give a useful approximation of a measure under consideration. For actions of groups other than $\mathbb{Z}$ (even $\mathbb{Z}^d$) this procedure usually cannot be performed, roughly saying, because of irregular shapes of blocks, and the notion of a block measure seems to be obscure. We devote the next section to implementing it in our setup, but before we proceed, we recall a few facts about Følner sequences.

In any amenable group there exists a Følner sequence with the following additional properties (see [6]):

1. $F_n \subset F_{n+1}$ for all $n$,
2. $e \in F_n$ for all $n$ ($e$ denotes the neutral element of $G$),
3. $\bigcup_{n \in \mathbb{N}} F_n = G$,
4. $F_n = F_n^{-1}$ for all $n$.

Following [10] we say that a Følner sequence $F_n$ is tempered if for some $C > 0$ and all $n$,

$$\bigcup_{k \leq n} F_k^{-1} F_{k+1} \leq C |F_{n+1}|.$$
Proposition 1.3 ([8]) Every Følner sequence $F_n$ has a tempered subsequence.

Standing assumption
Throughout this paper, we will assume that the Følner sequence which we use is tempered and has all the above properties.

We recall the pointwise ergodic theorem for amenable groups.

Theorem 1.4 ([8]) Let $G$ be an amenable group acting ergodically on a measure space $(X, \mu)$, and let $F_n$ be a tempered Følner sequence. Then for any $f \in L^1(\mu),$

$$\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) = \int f \, d\mu \quad a.e.$$ 

If $F$ and $A$ are finite subsets of $G$ and $0 < \delta < 1$, we say that $F$ is $(A, \delta)$-invariant if

$$\frac{|F \triangle AF|}{|F|} < \delta,$$

where $AF = \{af : a \in A, f \in F\}$. Clearly, if $F$ is $(A, \delta)$-invariant then it is also $(A, \delta')$-invariant for all $\delta' > \delta$. Moreover, if $F$ is simultaneously $(A, \delta)$-invariant and $(A', \delta')$-invariant then $F$ is $(A \cup A', \delta + \delta')$-invariant. Observe that if $A$ contains the neutral element of $G$, then $(A, \delta)$-invariance is equivalent to the simpler condition

$$|AF| < (1 + \delta)|F|.$$ 

It is not hard to observe that if $F$ is $(A, \delta)$-invariant then

$$||\{f \in F : Af \cap F^c \neq \emptyset\}|| < \delta |A||F| \quad (1.1)$$ 

and, equivalently,

$$||\{f \in F : Af \subset F\}|| > |F|(1 - \delta |A|) \quad (1.2)$$

If $(F_n)$ is a Følner sequence, then for every finite $A \subset G$ and every $\delta > 0$ there exists an $N$ such that for $n > N$ the sets $F_n$ are $(A, \delta)$-invariant.

Definition 1.5 For $S \subset G$ and a finite, nonempty $F \subset G$ denote

$$D_F(S) = \inf_{g \in G} \frac{|S \cap Fg|}{|F|}, \quad \overline{D}_F(S) = \sup_{g \in G} \frac{|S \cap Fg|}{|F|}. $$

If $(F_n)$ is a Følner sequence then we define two values

$$\underline{D}(S) = \limsup_{n \to \infty} D_{F_n}(S) \quad \text{and} \quad \overline{D}(S) = \liminf_{n \to \infty} \overline{D}_{F_n}(S),$$

which we call the lower and upper Banach densities of $S$, respectively.
Note that \( D(S) = 1 - D(G \setminus S) \). We recall the following standard fact:

**Lemma 1.6** Regardless of the set \( S \), the values of \( D(S) \) and \( \overline{D}(S) \) do not depend on the Følner sequence, the limits superior and inferior in the definition are in fact limits, and moreover

\[
\overline{D}(S) = \sup \{D_F(S) : F \subseteq G, F \text{ is finite} \} \quad \text{and} \quad D(S) = \inf \{\overline{D}_F(S) : F \subseteq G, F \text{ is finite} \} \geq D(S).
\]

For the proof see [5], Lemma 2.9.

**Lemma 1.7** Let \((X, G)\) be a Cantor system in the array representation and let \( \mu \) be an ergodic measure on \( X \). Denote by \( e \) the neutral element of \( G \). Let \( \varphi : X \to X \), \( \psi : X \to X \) be continuous maps which commute with the action of \( G \).

Then for \( \mu \)-almost every \( x \) then

\[
\mu(\{x \in X : \varphi(x)(e) \neq \psi(x)(e)\}) \leq \overline{D}(\{g \in G : \varphi(x)(g) \neq \psi(x)(g)\}).
\]

**Proof** Denote

\[
S(x) = \{g \in G : (\varphi(x))(g) \neq (\psi(x))(g)\}
\]

\[
B = \{x \in X : (\varphi(x))(e) \neq (\psi(x))(e)\}
\]

Note that \((\varphi(x))(g) = (g\varphi(x))(e) = \varphi(gx)(e)\) (and similarly for \( \psi \)), so \( g \in S(x) \) is equivalent to \( gx \in B \). Then,

\[
\overline{D}_F(S(x)) = \sup_{g \in G} \frac{|S(x) \cap F_n g|}{|F_n|}.
\]

\[
= \sup_{g \in G} \frac{1}{|F_n|} \sum_{h \in F_n g} \mathbb{1}_{S(x)}(h) \geq \frac{1}{|F_n|} \sum_{h \in F_n} \mathbb{1}_{S(x)}(h)
\]

\[
= \frac{1}{|F_n|} \sum_{h \in F_n} \mathbb{1}_{B}(hx).
\]

Taking the lower limits we obtain by Theorem 1.4 that \( \overline{D}(S(x)) \geq \mu(B) \).

**2 Block measures**

We will explicitly define a metric consistent with the weak* topology on the set of probability measures on \( X \), represented as an array system. First, let \( \mathcal{B}_k \) be the family of all blocks with domain \( F_k \times [1, k] \), occurring in \( X \), and let

\[
d_k(\mu, \nu) = \frac{1}{|\mathcal{B}_k|} \sum_{B \in \mathcal{B}_k} |\mu(B) - \nu(B)|.
\]
Now let
\[ d(\mu, v) = \sum_{k=1}^{\infty} \frac{1}{2^k} d_k(\mu, v). \]

Note that we may assume that \( B_k \) consists only of blocks which yield cylinders of positive measure for some ergodic measure \( \mu \).

**Lemma 2.1** Let \( X \) be an array system and let \((F_n)\) be a Følner sequence. For every \( t \in \mathbb{N} \) there exists \( \varepsilon_t > 0 \) such that if \( \mu \) and \( \nu \) are ergodic measures and \( x, y \) satisfy:

1. \( \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{1}_B(gx) = \mu(B) \) for all blocks \( B \) on \( F_j \times [1, j] \), where \( j \leq t + 1 \),
2. \( \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{1}_B( gy ) = \nu(B) \) for all blocks \( B \) on \( F_j \times [1, j] \), where \( j \leq t + 1 \),
3. \( \mathbb{D}(|\{ g \in G : x(g) \neq y(g) \}|) < \varepsilon_t \),

then \( d(\mu, \nu) < \frac{1}{2^t} \).

**Proof** We put \( \varepsilon_t = \frac{1}{2^{t+1}(1+|F_{t+1}|)} \). Let \( n \) be large enough to ensure that

1. \( \left| \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{1}_B(gx) - \mu(B) \right| < \varepsilon_t \) for all blocks \( B \) on \( F_j \times [1, j] \), where \( j \leq t + 1 \),
2. \( \left| \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{1}_B(gx) - \mathbb{1}_B(gy) \right| < \varepsilon_t \) for all blocks \( B \) on \( F_j \times [1, j] \), where \( j \leq t + 1 \),
3. \( |\{ g \in G : x(g) \neq y(g) \}| < 2\varepsilon_t |F_n| \).

Then, for \( B \) as above

\[
|\mu(B) - \nu(B)| \leq |\mu(B) - \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{1}_B(gx)| + \frac{1}{|F_n|} \sum_{g \in F_n} |\mathbb{1}_B(gx) - \mathbb{1}_B(gy)| + \frac{1}{|F_n|} \sum_{g \in F_n} |\mathbb{1}_B(gy) - \nu(B)|
\]
\[
< 2\varepsilon_t + \frac{1}{|F_n|} |\{ g \in G : x(g) \neq y(g) \}| \cdot |F_{t+1}|
\]
\[
< 2\varepsilon_t (1 + |F_{t+1}|) = \frac{1}{2^{t+1}}.
\]

Thus, for \( k \leq t + 1 \) we have

\[
d_k(\mu, \nu) = \frac{1}{|B_k|} \sum_{B \in B_k} |\mu(B) - \nu(B)| < \frac{1}{2^{t+1}}.
\]
and
\[ d(\mu, v) \leq \frac{1}{2^t} d_k(\mu, v) + \frac{1}{2^t+1} < \frac{1}{2^t}. \]

\[ \square \]

For the sake of convenience, we introduce a notion of “distance” between a block and a measure. Let \( B \) be a block occurring in \( X \), with domain \( F \times [1, k] \) for some \( F \subset G \) and \( k \in \mathbb{N} \). For any block \( C \) with domain \( F_j \times [1, j] \), where \( j \leq k \), we can define the frequency of \( C \) in \( B \) in the following way: let
\[ N_F(F_j) = \left| \left\{ g \in F : F_j g \subset F \right\} \right|, \]
\[ N_B(C) = \left| \left\{ g \in F : F_j g \subset F \text{ and } B[F_j g \times [1, j]] = C \right\} \right|, \]
(by the equality \( B[F_j g \times [1, j]] = C \) we understand that \( B(fg, i) = C(f, i) \) for all \( f \in F_j \) and \( i \in [1, j] \)).

If \( N_F(F_j) > 0 \) let
\[ \text{fr}_B(C) = \frac{N_B(C)}{N_F(F_j)}. \]
Otherwise let \( \text{fr}_B(C) = 0 \).

We say that \( A \) is a \((1-\delta)\)-subset of \( F \) if \( A \subset F \) and \( |A| \geq (1-\delta)|F| \). By a standard argument we can draw from the pointwise ergodic Theorem 1.4 the following corollary.

**Lemma 2.2** Let \( \mu \) be an ergodic measure on \( X \). For every \( \varepsilon \) and \( j \) we can find \( n \) and \( \eta \) such that if \( F \) is a \((1-\eta)\)-subset of \( F_m \), \( m \geq n \), then for some block \( C \) with domain \( F \times [1, m] \) occurring in \( X \) we have \( |\text{fr}_C(D) - \mu(D)| < \varepsilon \) for every block \( D \) with domain \( F_i \times [1, i] \), \( i = 1, \ldots, j \).

Moreover, \( n \) and \( \eta \) can be chosen so that the union of all blocks \( C \) not satisfying the approximation rule has measure smaller than \( \varepsilon \).

**Proof** Fix \( \varepsilon > 0 \) and \( j \in \mathbb{N} \). Taking in Theorem 1.4 \( f = 1_D \), where \( D \) is any fixed block with domain \( F_i \times [1, i] \), \( i = 1, \ldots, j \), we obtain that
\[ \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} 1_D(gx) = \mu(D) \]
for almost every \( x \). Since there are only finitely many blocks with such domain, we may assume that the above equality is satisfied simultaneously for all such blocks \( D \) on a subset of \( X \) having measure 1. Hence we can find \( n \) such that for every \( m \geq n \) the inequality
\[ \left| \frac{1}{|F_m|} \sum_{g \in F_m} 1_D(gx) - \mu(D) \right| < \frac{\varepsilon}{2} \quad (2.1) \]
holds, for all such \( D \), on a set of measure \( X_\varepsilon \) at least \( 1 - \varepsilon \). Pick \( x \) from this set. Additionally, increasing \( n \) we may demand that each \( F_m, m \geq n \), is \((F_j, \varepsilon_4(F_j))\)-invariant.

Fix \( m \geq n \) and let \( C \) be a block which appears in \( x \) on the domain \( F_m \). By \((1.2)\),

\[
|F_m| \geq N_{F_m}(F_i) > |F_m| \left(1 - \frac{\varepsilon}{4|F_j|}|F_i| \right) \geq |F_m| \left(1 - \frac{\varepsilon}{4}\right).
\]

Furthermore, using \((1.1)\)

\[
N_C(D) = \sum_{\{g \in F_m : F_g \subset F_m\}} \mathbb{1}_D(gx) \\
\leq \sum_{g \in F_m} \mathbb{1}_D(gx) \leq N_C(D) + |\{g \in F_m : F_i \cap F_m^c \neq \emptyset\}| \\
\leq N_C(D) + \frac{\varepsilon}{4|F_j|}|F_i|F_m| \leq N_C(D) + \frac{\varepsilon}{4}|F_m|.
\]

Hence,

\[
\frac{N_C(D) - \eta|F_m|}{N_{F_m}(F_i)} \leq \frac{1}{|F_m|} \sum_{g \in F_m} \mathbb{1}_D(gx) \leq \frac{N_C(D) + \frac{\varepsilon}{4}|F_m|}{|F_m|}
\]

implying that

\[
\text{fr}_C(D) - \frac{\varepsilon}{4} \leq \frac{1}{|F_m|} \sum_{g \in F_m} \mathbb{1}_D(gx) \leq \text{fr}_C(D) + \frac{\varepsilon}{4}.
\]

Combining it with \((2.1)\) we obtain \(|\text{fr}_C(D) - \mu(D)| < \frac{3\varepsilon}{4}\) for blocks \( C \) having domain exactly equal to \( F_m \times [1, m] \).

Now suppose that \( F \) is a \((1 - \eta)\)-subset of \( F_m \) and that \( C \) is a block which appears in \( x \) on the domain \( F \), while \( C' \) is \( x \) restricted to \( F_m \).

\[
\frac{N_{C'}(D) - \eta|F_m|}{N_{F_m}(F_i)} \leq \frac{N_C(D)}{N_F(F_i)} \leq \frac{N_{C'}(D)}{N_{F_m}(F_i) - \eta|F_m|}
\]

The left hand is greater than

\[
\frac{N_{C'}(D) - \eta\frac{N_{F_m}(F_i)}{1-\varepsilon/4}}{N_{F_m}(F_i)} \geq \text{fr}_{C'}(D) - \frac{\eta}{1 - \varepsilon/4}
\]

while the right hand is bounded from above by

\[
\frac{N_{C'}(D)}{N_{F_m}(F_i) - \eta\frac{N_{F_m}(F_i)}{1-\varepsilon/4}} \leq \text{fr}_{C'}(D) + \frac{\eta}{1 - \varepsilon/4 - \eta}.
\]
Hence
\[ |\text{fr}_{C'}(D) - \text{fr}_C(D)| < \frac{\eta}{1 - \varepsilon/4 - \eta} \]
which can be made smaller than \(\frac{\varepsilon}{4}\) by choosing appropriately small \(\eta\), irrespective of \(m, C\) and \(D\).

Finally, note that if \(|\text{fr}_C(D) - \mu(D)| < \varepsilon\) is not true for some \(D\) as above and \(C = x[F]\), \(F\) being a \((1 - \eta)\)-subset of \(F_m, m \geq n\), then \(x\) is not in \(X_\varepsilon\), as defined by 2.1, which means that the union of all \(C\) for which the approximation fails is less than \(\varepsilon\). \(\square\)

We can now define the distance between a block and a measure: let
\[ d_k(B, \nu) = \frac{1}{|B_k|} \sum_{D \in B_k} |\text{fr}_B(D) - \nu(D)|, \]
and let
\[ d(B, \nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} d_k(B, \nu). \]

**Remark 2.3** Let \(\varepsilon\) be a positive number. To ensure that \(d(B, \nu) < \varepsilon\) it is enough to verify that if \(j\) satisfies \(\sum_{k=j+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2j}\) then for any block \(D \in B_i\), where \(i = 1, \ldots, j\),
\[ |\text{fr}_B(D) - \nu(D)| < \frac{\varepsilon}{2j}. \]

Indeed, in this case we have \(d_k(B, \nu) < \frac{\varepsilon}{2j}\) and
\[ d(B, \nu) = \sum_{k=1}^{j} \frac{1}{2^k} d_k(B, \nu) + \sum_{k=j+1}^{\infty} \frac{1}{2^k} d_k(B, \nu) \]
\[ < \sum_{k=1}^{j} \frac{\varepsilon}{2j} + \sum_{k=j+1}^{\infty} \frac{1}{2^k} < \varepsilon. \]

**Lemma 2.4** For any \(\varepsilon > 0\) and any positive integer \(j\) there exists \(\delta\) such that if \(F\) is an \((F_j, \delta)\)-invariant set and \(B\) is a block with domain \(F \times [1, j]\), then there exists a probability measure \(\mu_B\) such that
\[ |\text{fr}_B(D) - \mu_B(D)| < \frac{\varepsilon}{2j} \]
for any block \(D \in B_i\), where \(i = 1, \ldots, j\).
Consequently, for any $\varepsilon > 0$ and sufficiently large $j$ there exists $\delta$ such that if $F$ is an $(F_j, \delta)$-invariant set and $B$ is a block with domain $F \times [1, j]$, then there exists a probability measure $\mu_B$ such that $d(B, \mu_B) < \varepsilon$.

**Proof.** Let us first observe that the second assertion follows from the first by Remark 2.3. Therefore, it suffices to prove the first statement.

Let $\Delta_j = \Lambda_1 \times \cdots \times \Lambda_j$. The full shift $\Delta_j$ is a Cantor set on which we have the uniform Bernoulli probability measure $\lambda$ which assigns equal measures $M_j$ to all cylinders with domain $F_j \times [1, j]$. We shall define $\mu_B$ by specifying its density $f_B$ with respect to $\lambda$. $f_B$ will be constant on cylinders with domain $F_j \times [1, j]$: on each such cylinder associated with a block $C$ let $f_B(x) = \frac{1}{M_j} f_{\mu_B}(C)$. Obviously $d_j(\mu_B, B) = 0$.

We will now estimate $d_i(\mu_B, B)$ for $i < j$. Let $D$ be any block from $B_i$. Let $C_j$ be the family of (distinct) blocks from $B_j$ such that $D = \bigcup_{C \in C_j} C$. We have:

$$
\mu_B(D) = \sum_{C \in C_j} \mu_B(C) = \sum_{C \in C_j} f_B(C),
$$

and we need to show that the latter quantity is close to $f_B(D)$. Since $F$ is $(F_j, \delta)$-invariant, the set $\{ g : F_jg \subset F \}$ is a $(1 - \delta |F_j|)$-subset of $F$ by (1.2) (note that $\{ g : F_jg \subset F \}$ is indeed a subset of $F$, because $F_j$ contains the neutral element). Consequently, the set $\{ g : F_jg \subset F \}$ also is a $(1 - \delta |F_j|)$-subset of $F$, being a superset of the former. For $C \in C_j$, let $F_C$ be the set of $g$ such that $F_jg \subset F$, $B[F_jg \times [1, j]] = C$ (and automatically $B[F_jg \times [1, i]] = D$). That way we can represent $\{ g : F_jg \subset F, B[F_jg \times [1, i]] = D \}$ as the following disjoint sum:

$$
\{ g : F_jg \subset F, B[F_jg \times [1, i]] = D \} = \bigcup_{C \in C_j} F_C \cup \{ g : F_jg \subset F, F_jg \cap F^c \neq \emptyset, B[F_jg \times [1, i]] = D \}.
$$

Taking cardinalities and dividing by $N_F(F_i)$, we obtain

$$
\text{fr}_B(D) = \sum_{C \in C_j} \frac{N_F(F_j)}{N_F(F_i)} \text{fr}_B(C)
$$

$$
+ \frac{1}{N_F(F_i)} \left| \{ g : F_jg \subset F, F_jg \cap F^c \neq \emptyset, B[F_jg \times [1, i]] = D \} \right|
$$

Since both $F_j$ and $F_i$ are $(1 - \delta |F_j|)$-subsets of $F$, we have $\frac{N_F(F_j)}{N_F(F_i)} \geq 1 - \delta |F_j|$ and

$$
\frac{1}{N_F(F_i)} \left| \{ g : F_jg \subset F, F_jg \cap F^c \neq \emptyset, B[F_jg \times [1, i]] = D \} \right|
$$

$$
\leq \frac{1}{N_F(F_i)} \left| \{ g \in F : F_jg \cap F^c \neq \emptyset \} \right|
$$

$$
\leq \frac{\delta |F_j| |F|}{(1 - \delta |F_j|) |F|} = \frac{\delta |F_j|}{1 - \delta |F_j|}.
$$
If \( \delta \) is small enough then the expression can be arbitrarily close to 0, while \( \frac{N_F(F_j)}{N_F(F_i)} \) can be arbitrarily close to 1, so we can assume that

\[
\left| \frac{\text{fr}_B(D) - \mu_B(D)}{\sum_{C \in C_j} \text{fr}_B(C_j)} \right| < \frac{\varepsilon}{2j}.
\]

Note that in the above lemma \( \delta \) may be as small as we want.

Corollary 2.5 Let \( X \) be a zero-dimensional dynamical system with the action of an amenable group \( G \) and let \( \mu \) be an ergodic measure on \( X \). For any \( \varepsilon > 0 \) and any sufficiently large \( j \) there exists \( \delta > 0 \) such that if \( F \) is a \( (1 - \delta) \)-subset of \( F_m \), where \( m \) is so large that \( F \) is \( (F_j, \delta) \)-invariant, then there is a block \( C \) occurring in \( X \), whose domain is \( F \) such that the measure \( \mu_C \) (as defined in Lemma 2.4) satisfies \( d(\mu_C, \mu) < \varepsilon \).

Proof Choose \( \delta \) and \( j \) from Lemma 2.4 with \( \varepsilon \) replaced by \( \varepsilon \). Assume also that \( \sum_{k=j+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2} \). By Lemma 2.2, we can find a block \( C \) on a \( (F_j, \delta) \)-invariant domain \( F \), such that \( \left| \frac{\text{fr}_C(D) - \mu(D)}{\sum_{C \in C_j} \text{fr}_B(C_j)} \right| < \frac{\varepsilon}{2} \) for any block \( D \) with domain \( F_i \times [1, i] \), \( i = 1, \ldots, j \). By Remark 2.3 we see that \( d(C, \mu) < \frac{\varepsilon}{2} \), and directly from Lemma 2.4 also \( d(C, \mu_C) < \frac{\varepsilon}{2} \), therefore \( d(\mu_C, \mu) < \varepsilon \). \( \square \)

For actions of \( \mathbb{Z} \), it is a well-known fact that if a sufficiently long block \( C \) is a concatenation of shorter blocks \( B_1, B_2, \ldots, B_n \) of equal length, then the probability measure \( \mu_C \) (which for actions of \( \mathbb{Z} \) can easily be assumed to be shift invariant) can be arbitrarily close to the arithmetic average of the measures \( \mu_B \). An analogous claim can be made for the action of any amenable group \( G \); however the lack of a natural way to decompose a subset of \( G \) into smaller sets requires the use of quasitilings.

Definition 2.6 A (static) quasitiling of a group \( G \) is a family \( \mathcal{T} \) of finite subsets of \( G \) (called tiles), for which there exist a family \( \mathcal{S}(\mathcal{T}) = \{S_1, S_2, \ldots, S_n\} \) of finite subsets of \( G \) (called shapes) and a family \( \mathcal{C}(\mathcal{T}) = \{C_1, C_2, \ldots, C_n\} \) of subsets of \( G \) (called centers), such that every \( T \in \mathcal{T} \) has a unique representation \( T = S_i c \) for some \( i \in \{1, \ldots, n\} \) and \( c \in C_i \).

Note that every quasitiling can be seen as a symbolic element \( T \in \{0, 1, \ldots, n\}^G \), such that \( \bar{T}(g) = i \) if \( g \in C_i \) for some \( i \), and \( \bar{T}(g) = 0 \) otherwise.

Definition 2.7 A quasitiling \( \mathcal{T} \) is:

1. disjoint, if the tiles are pairwise disjoint;
2. \( \alpha \)-covering, if the union of all tiles has lower Banach density at least \( \alpha \).
3. congruent with a quasitiling \( \mathcal{T}' \), if for any two tiles \( T \in \mathcal{T}, T' \in \mathcal{T}' \) we have either \( T \supset T' \) or \( T \cap T' = \emptyset \).

Any \((T, k)\)-block whose shape \( T \) belongs to a quasitiling \( \mathcal{T} \) will be called a \((\mathcal{T}, k)\)-block.
Let \((X, G)\) be a topological dynamical system. Suppose we assign to every \(x \in X\) a quasitiling \(T(x)\) of \(G\), with the same set of shapes \(S_1, \ldots, S_n\) for all \(x\). This induces a map \(x \mapsto T(x)\) which can be seen as a map from \((X, G)\) into \(\{0, 1, \ldots, n\}^G\) with the shift action. If such a map is a factor map (i.e. if it is continuous and commutes with the dynamics), we call it a dynamical quasitiling. A dynamical quasitiling is said to be disjoint and/or \(\alpha\)-covering, if \(T(x)\) has the respective property for every \(x\). Note that though the set of shapes is common, the collection of centers depends on \(x\) so \(C_i(x)\) become functions assigning to each \(x\) a subset of \(G\). We introduce the following new definition.

**Definition 2.8** We will say that a dynamical quasitiling \(T\) consisting of shapes \(S(T) = \{S_1, S_2, \ldots, S_n\}\) and centers \(C(T) = \{C_1, C_2, \ldots, C_n\}\) has restricted block distribution if for every \(x \in X\) any \((T, k)\)-block \(B\) which occurs in \(x\) on some domain \(S_i c\) may occur in \(x\) only on domains of this form for \(c \in C_i(x)\).

**Remark 2.9** Clearly, if \(T\) has restricted block distribution then for any block \(D\) occurring in \(x\) on some domain \(S_i c_0\) and any \(B\) being a block occurring in \(x\) on a (disjoint) union \(F\) of tiles we have

\[
fr_B(D) = \frac{1}{N_F(S)} |\{c \in C_i(x) \cap F : x(S_i c) = D\}|.
\]

**Lemma 2.10** For any \(\varepsilon > 0\) there exist \(j \in \mathbb{N}\) and \(\delta > 0\) such that if \(T\) is a disjoint quasitiling by \((F_j, \delta_j)\)-invariant sets, and \(C\) is a block with domain \(H \times [1, j]\) such that some disjoint union of tiles \(T_1, T_2, \ldots, T_n\) of \(T\) is a \((1 - \delta)\)-subset of \(H\), then the probability measure \(\mu_C\) is \(\varepsilon\)-close to the average of the measures associated with blocks over individual tiles, i.e. if we denote by \(B_i\) the block with domain \(T_i \times [1, j]\),

\[
d\left(\mu_C, \frac{1}{\sum_{i=1}^n |T_i|} \sum_{i=1}^n |T_i| \mu_{B_i}\right) < \varepsilon.
\]

**Proof** Applying Lemma 2.4, for any \(j\) there is \(\delta_j\) such that for any block \(B\) with domain \(F \times [1, j]\), where \(F\) is a \((F_j, \delta_j)\)-invariant set, and for any block \(D\) with domain \(F_i \times [1, i]\), \(i \leq j\), we have \(|fr_B(D) - \mu_B(D)| < \frac{\varepsilon}{F_j}\). Let \(H\) be a subset of \(G\) and let \(T\) be a quasitiling of \(G\) by \((F_j, \delta)\)-invariant sets for some \(\delta > 0\). Suppose that the union \(\bigcup_{i=1}^n T_i\) is a \((1 - \delta)\)-subset of \(H\) for some pairwise disjoint tiles \(T_1, T_2, \ldots, T_n\) belonging to \(T\). For every \(k \leq j\) let us define the set

\[
E_k = \{h \in H : \forall i \ F_k h \cap T_i^c \neq \emptyset\}
\]

Then

\[
|E_k| \leq \sum_{i=1}^n |\{h \in T_i : \forall i \ F_k h \cap T_i^c \neq \emptyset\}| + |H \setminus \bigcup_{i=1}^n T_i|
\]

\[
\leq \sum_{i=1}^n \delta |F_k| |T_i| + \delta |H| \leq \delta |H|(|F_k| + 1),
\]

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hence $N_H(F_k) \geq |H| - |E_k| \geq |H|(1 - \delta(1 + |F_k|))$. Clearly, we can demand that $\delta < \delta_j$ (further restrictions will follow). Note that since each $T \in \mathcal{T}$ is $(F_j, \delta)$-invariant, for any block $B$ whose domain is a tile of $\mathcal{T}$ the measure $\mu_B$ is well-defined.

Now, let $C$ be a block with domain $H \times [1, j]$, where $H$ is $(1-\delta)$-tiled by $T_1, \ldots, T_n$, and let $C[T_i] = B_i$. For any $k \leq j$ and for any block $D$ with domain $F_k \times [1, k]$ we have:

$$N_C(D) = \sum_{i=1}^{n} N_{B_i}(D) + N_{E_k}(D)$$

Therefore, using the triangle inequality,

$$\left| \text{fr}_C(D) - \frac{1}{\sum_{i=1}^{n} |T_i|} \sum_{i=1}^{n} \text{fr}_{B_i}(D) |T_i| \right|$$

$$= \left| \frac{\sum_{i=1}^{n} N_{B_i}(D) + N_{E_k}(D)}{N_H(F_k)} \right| - \frac{1}{\sum_{i=1}^{n} |T_i|} \sum_{i=1}^{n} \text{fr}_{B_i}(D) |T_i|$$

$$\leq \frac{N_{E_k}(D)}{N_H(F_k)} + \sum_{i=1}^{n} N_{B_i}(D) \cdot \left| \frac{1}{N_H(F_k)} - \frac{1}{\sum_{i=1}^{n} |T_i|} \right|$$

$$+ \frac{1}{\sum_{i=1}^{n} |T_i|} \cdot \sum_{i=1}^{n} \left( N_{B_i}(D) \left| 1 - \frac{|T_i|}{N_{T_i}(F_k)} \right| \right)$$

We can estimate that $\frac{N_{E_k}(D)}{N_H(F_k)} < \delta(|F_k| + 1)$, $\left| 1 - \frac{|T_i|}{N_{T_i}(F_k)} \right| \leq \frac{\delta |F_k|}{1-\delta |F_k|}$ and $\left| \frac{1}{N_H(F_k)} - \frac{1}{\sum_{i=1}^{n} |T_i|} \right| \leq \frac{\delta |F_k| + 2}{N_H(F_k)}$, so the whole expression can be made smaller than $\frac{\varepsilon}{8j}$ by appropriate choice of a small $\delta$.

Now, for every $B_i$ we have $\text{fr}_{B_i}(D)$ is approximately equal to $\mu_{B_i}(D)$ with error $\frac{\varepsilon}{8j}$, and this approximation is preserved by the weighted average we have obtained, therefore

$$\left| \text{fr}_C(D) - \frac{1}{\sum_{i=1}^{n} |T_i|} \sum_{i=1}^{n} |T_i| \mu_{B_i}(D) \right| < \frac{\varepsilon}{4j}$$

If $j$ is sufficiently large, Remark 2.3 implies that

$$d\left( C, \frac{1}{\sum_{i=1}^{n} |T_i|} \sum_{i=1}^{n} |T_i| \mu_{B_i} \right) < \frac{\varepsilon}{2},$$

and since $d(C, \mu_C) < \frac{\varepsilon}{2}$, we also have

$$d\left( \mu_C, \frac{1}{\sum_{i=1}^{n} |T_i|} \sum_{i=1}^{n} |T_i| \mu_{B_i} \right) < \varepsilon.$$
Remark 2.11 In the above lemma, we can increase \( j \) and decrease \( \delta \) without spoiling the approximation error \( \epsilon \), because if \( j' \geq j \) and \( \delta' \leq \delta \) then \((F_{j'}, \delta')\)-invariant set is also \((F_j, \delta)\)-invariant and a \((1 - \delta')\)-subset of any \( H \) is a \((1 - \delta)\)-subset of \( H \).

The next two lemmas concerning the existence of quasitilings were proved in [4].

Lemma 2.12 ([4], Corollary 3.5) Let \( G \) be an amenable group acting freely on a zero-dimensional metric space \( X \). For any \( \epsilon > 0 \), any finite \( K \subset G \) and any \( \delta > 0 \) there exists a disjoint, \((1 - \epsilon)\)-covering dynamical quasitiling \( T \) such that every shape of \( T \) is \((K, \delta)\)-invariant.

Lemma 2.13 ([4], Lemma 3.6) Let \( G \) be an amenable group acting freely on a zero-dimensional metric space \( X \) and let \( T_0 \) be any disjoint, dynamical quasitiling of \( G \). For any \( \epsilon > 0 \), any finite \( K \subset G \) and any \( \delta > 0 \) there exists a disjoint, \((1 - \epsilon)\)-covering dynamical quasitiling \( T_1 \) such that every shape of \( T_1 \) is \((K, \delta)\)-invariant, and every tile of \( T_0 \) is either a subset of some tile of \( T_1 \) or is disjoint from all such tiles.

The following lemma is analogous to the case of classical one-dimensional subshifts.

Lemma 2.14 For every \( \epsilon > 0 \) there exist \( J \) and \( \delta \) such that if for some \( j > J \) the set \( F \) is a \((F_{j}, \delta)\)-invariant and \( B \) is a \((F, j)\)-block, then \( d(B, \mathcal{M}_G(X)) < \epsilon \) and \( d(\mu_B, \mathcal{M}_G(X)) < \epsilon \).

Proof We will prove the assertion in the language of blocks, i.e. we will show that \( d(B, \mathcal{M}_G(X)) < \epsilon \). The assertion for measures will follow from Corollary 2.5.

Suppose that there is \( \epsilon > 0 \) such that for every \( J \) and \( \delta \) there exists an integer \( j > J \) and a \((F, j)\)-block \( B_{j, \delta} \) on domain \( D_{j, \delta} \times [1, j] \), which is \((F_{j}, \delta)\)-invariant and \( d(B_{j, \delta}, \mathcal{M}_G(X)) \geq \epsilon \). For \( \frac{\epsilon}{2} \) and every \( J \) we choose \( \delta = \delta_j \) via Lemma 2.4 with the additional requirement that \( \delta_j |F_j| < \frac{\epsilon}{2} \). We denote \( B_J = B_{J, \delta_j} \) and \( D_J = D_{J, \delta_j} \).

Let \( \mu \) be a limit point of the sequence \( \mu_{B_J} \). Clearly, \( d(\mu, \mathcal{M}_G(X)) \geq \epsilon/2 \).

We will obtain a contradiction by showing that \( \mu \) is \( G \)-invariant. It suffices to prove that \( \mu_C = \mu(g(C)) \) for every \((F_i, i)\)-block \( C \) and every \( g \in G \), where

\[
g(C) = \{ gx : x[F_i \times [1, i]] = C \} = \{ x : x[F_i g^{-1} \times [1, i]] = C \}
\]

Fix \( \gamma > 0 \), \( g \in G \) and a cylinder set \( C \) on \( F_i \times [1, i] \). Let \( J \) be large enough to ensure that

1. \( |\mu(C) - \mu_{B_J}(C)| + |\mu(g(C)) - \mu_{B_J}(g(C))| < \frac{\epsilon}{2} \) (note that \( C \) is clopen),
2. \( \frac{\epsilon}{4} < \frac{\gamma}{6} \),
3. The domain \( D_J \) is \((F_j, \delta_J)\)-invariant, \( j \geq J \), where \( F_J \supset F_i \cup F_i g^{-1} \cup \{g\} \).

Then,

\[
|\mu(C) - \mu(g(C))| \leq |\mu(C) - \mu_{B_J}(C)| + |\mu_{B_J}(C) - \mu_{B_J}(g(C))| + |\mu_{B_J}(g(C)) - \mu_{B_J}(g(C))| + |\mu_{B_J}(g(C)) - \mu(g(C))|
\]

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The first condition guarantees that the sum of the first and the last terms are less than \( \gamma \). The choice of \( B_J \) was made with use of Lemma 2.4, so both the second and the fourth summands are smaller than \( \varepsilon \). By the second requirement above, their sum is again less than \( \gamma \). We only need to show that the middle term is bounded by \( \gamma \).

Since \( D_J \) is \( (F_j, \delta_J) \)-invariant, it is also \( (F_i, \delta_J) \)-invariant and \( (F_i g^{-1}, \delta_J) \)-invariant, which implies that

\[
(1 - \delta_J |F_i|) |D_J| \leq N_{D_J}(F_i) \leq |D_J|
\]

\[
(1 - \delta_J |F_i|) |D_J| \leq N_{D_J}(F_i g^{-1}) \leq |D_J|.
\]

In particular,

\[
|N_{D_J}(F_i g^{-1}) - N_{D_J}(F_i)| \leq \delta_J |F_i||D_J|.
\]

Note also that \( B_J[F_i h \times [1, i]] = C \) if and only if \( B_J[(F_i g^{-1})(gh) \times [1, i]] = C \). Thus, an occurrence of \( C \) in \( B_J \) ‘at position \( h \)’ yields the occurrence of \( g(C) \) in \( B_J \) ‘at \( gh \)’, if only \( gh \in D_J \). By a similar argument, occurrences of \( g(C) \) force occurrences of \( C \). The number of pairs \( (h, gh) \) such that only one of these elements belongs to \( D_J \) is smaller than \( \delta_J |F_i||D_J| \), so

\[
|N_{B_J}(C) - N_{B_J}(g(C))| < \delta_J |F_i||D_J| \leq \delta_J |F_i| \frac{N_{D_J}(F_i)}{1 - \delta_J |F_i|}
\]

Hence

\[
|fr_{B_J}(C) - \frac{N_{B_J}(g(C))}{N_{D_J}(F_i)}| \leq \frac{\varepsilon}{1 - \varepsilon/J} < \frac{\gamma}{6}
\]

On the other hand,

\[
|\frac{N_{B_J}(g(C))}{N_{D_J}(F_i)} - fr_{B_J}(g(C))| \leq N_{B_J}(g(C)) \cdot \frac{|N_{D_J}(F_i g^{-1}) - N_{D_J}(F_i)|}{N_{D_J}(F_i) N_{D_J}(F_i g^{-1})}
\]

\[
\leq \frac{\delta_J |F_i|}{1 - \delta |F_i|} \leq \frac{\varepsilon}{1 - \varepsilon/J} < \frac{\gamma}{6},
\]

which ends the proof. \( \square \)

Finally, we prove our last tool.

**Lemma 2.15** Let \( X \) be a zero-dimensional dynamical system (in array form) with the shift action of an amenable group \( G \), and let \( K \) be a face in the simplex \( M_G(X) \). For any \( \delta > 0 \) and \( \varepsilon > 0 \) there exist \( \eta \) and \( j \) such that if \( T \) is a disjoint, \((1 - \eta)\)-covering dynamical quasitiling by \((F_j, \eta)\)-invariant sets, which has restricted block distribution and \( B \) denotes the family of all \((T, j)\)-blocks \( B \) such that \( d(B, K) > \delta \), then \( \sum_{B \in B} \mu(B) |B| \leq \varepsilon \) for every \( \mu \in K \) (by convention, the sum over the empty set is equal to zero).
Proof Fix $\delta > 0$ and $\epsilon > 0$. Let $\mathcal{F}$ denote the (closed) complement of the open $\delta/2$-ball around $K$ in $\mathcal{M}(X)$ (note that we use here the space of all probability measures, not the space of invariant measures). Obviously, $\{ \mu_B : B \in \mathcal{B} \} \subset \mathcal{F}$ for sufficiently large $j$. For every $\alpha$, consider the set $V_\alpha \subset \mathcal{M}(X)$ consisting of measures $\mu$ with the following property: if $\mu = \int_{\mathcal{M}(X)} v \, d\xi$, and $\xi$ is supported by the closed $\alpha$-neighborhood of $\mathcal{M}_G(X)$, then $\xi(\mathcal{F}) < \epsilon$.

**Claim 1** $V_\alpha$ is an open set.

We will prove that its complement $V_\alpha^c$ is closed. Let $\mu$ be the weak* limit of a sequence $\mu_k$ of elements of $V_\alpha^c$. Then $\mu_k = \int_{\mathcal{M}(X)} v \, d\xi_k$ for some $\xi_k$ supported by the closed $\alpha$-neighborhood of $\mathcal{M}_G(X)$ with $\xi_k(\mathcal{F}) \geq \epsilon$. The sequence $\xi_k$ has a subsequence which converges in the weak* topology to some measure $\xi$—let us assume that $\xi_k$ itself is already convergent. By the portmanteau lemma, $\xi(\mathcal{F}) \geq \lim_{k \to \infty} \xi_k(\mathcal{F}) \geq \epsilon$. By the same lemma, $\xi$ assigns to the closed $\alpha$-neighborhood of $\mathcal{M}_G(X)$ the value 1, so it is supported by this neighborhood.

The only thing left to show is the equality $\mu = \int_{\mathcal{M}(X)} v \, d\xi$. For any function $f \in C(X)$ the map $v \mapsto v(f) = \int f \, dv$ is a real continuous map of $\mathcal{M}(X)$. Therefore, by the definition of weak* convergence (used both in spaces $\mathcal{M}(X)$ and $\mathcal{M}(\mathcal{M}(X)))$,

$$\mu(f) = \lim_{k \to \infty} \mu_k(f) = \lim_{k \to \infty} \int_{\mathcal{M}(X)} v(f) \, d\xi_k = \int_{\mathcal{M}(X)} v(f) \, d\xi,$$

which is the desired equality.

**Claim 2** If $\alpha$ is small enough then $V_\alpha$ contains $K$.

If not then letting $\alpha$ tend to 0, we could find a measure in $K$ that is a barycenter of a distribution $\xi$ on $\mathcal{M}_G(X)$ with $\xi(\mathcal{F}) \geq \epsilon$. This is not possible.

Returning to the main proof, let $\gamma$ be small enough that the open $\gamma$-neighborhood of $K$ is contained in $V_\alpha$. Using Lemma 2.10 choose $\eta$ and $j$ to obtain the the error of approximation equal $\gamma/2$ for any $(F_j, \eta)$-quasitiling (i.e. in the lemma $\gamma/2$ and $\eta$ play the role of $\epsilon$ and $\delta$, respectively). Let $T$ be such a quasitiling. By Lemma 2.14 and Remark 2.11, making $j$ large enough, we can also assume that every block with domain $S \times [1, j]$ (where $S$ is a shape of $T$) that occurs in $X$ lies in the $\alpha$-neighborhood of the set of invariant measures on $X$. Note that the union $\bigcup B$ of the collection of all elements of $B$ (as defined in the statement of the lemma) is clopen, and thus the function $\mu \mapsto \mu(\bigcup B)$ is continuous on the set $\mathcal{M}(X)$. Suppose that $\mu$ is an ergodic measure in $K$ such that $\sum_{B \in B} \mu(B) |B| > \epsilon$. The function $v \mapsto \sum_{B \in B} v(B) |B|$ is continuous, therefore if $v$ is close enough to $\mu$, then $\sum_{B \in B} v(B) |B| > \epsilon$. In particular, by Corollary 2.5 we can find a block $C$ occurring in $X$, such that $d(\mu_C, \mu) < \frac{\epsilon}{2}$, and $\sum_{B \in B} \mu_C(B) |B| > \epsilon$. By Lemma 2.4 we can demand that $\text{fr}_C B$ approximates each $B \in B$ so well that also $\sum_{B \in B} \text{fr}_C(B) |B| > \epsilon$. Note that by the restricted block distribution, for elements $B$ of the tiling $\text{fr}_C(B)$ is derived by calculating only the appropriate elements of the tiling of $C$. We can also assume that the union of tiles of $T$ contained in the domain of $C$ is a $(1 - \eta)$-subset of $C$. By Lemma 2.10, $\mu_C$ is closer than $\frac{\epsilon}{2}$ to $v = \frac{1}{\sum_{i=1}^{n} |B_i|} \sum_{i=1}^{n} |B_i| \mu_{B_i}$, where $B_1, \ldots, B_n$ are all $(T, j)$-blocks occurring
in $C$ as elements of the tiling. For all $i$ such that $B_i \in \mathcal{B}$, we have $\delta_{\mu, B_i}(\mathcal{F}) = 1$, so for $\xi = \frac{1}{\sum_{i=1}^{n}|B_i|} \sum_{i=1}^{n}|B_i|\delta_{\mu, B_i}$ we have

$$
\xi(\mathcal{F}) \geq \frac{1}{\sum_{i=1}^{n}|B_i|} \sum_{i=1}^{n}|B_i| |B_i| \geq \sum_{B \in \mathcal{B}} |B| \text{fr}_{C}(B) > \varepsilon.
$$

Since $\nu$ is in $V_{a}$ (where such a decomposition should not exist), this is a contradiction. \hfill \Box

3 Proof of the main result

Proof (Proof of Theorem 1.2) Let $K$ be a face in $\mathcal{M}_{G}(X)$. Recall that $X$ is represented as an array system, i.e. it is a subset of $Z = \prod_{j \in \mathbb{N}} A_{j}^{\mathbb{G}}$, where $|A_{j}| < \infty$. For every $t \in \mathbb{N}$ we choose $\varepsilon_{t}$ so that the sequence is summable and that $2\varepsilon_{t}$ satisfies the hypotheses of Lemma 2.1.

By Lemma 2.12, we can construct a sequence $T_{i}$ of disjoint, $(1 - \frac{1}{t})$-covering, dynamical tilings of $X$, whose shapes are all $(F_{t}, \frac{1}{t})$-invariant subsets of $G$. Moreover, by Lemma 2.13 we can assume that every tile of $T_{i-1}$ is either a subset of some tile of $T_{i}$ or is entirely disjoint from all such tiles.

Having fixed the sequence $T_{i}$ we slightly change the array representation of $X$. We extend each alphabet $A_{j}$ to a new alphabet $A_{j}^{*}$, doubling the number of symbols by adding to it for every symbol $\lambda$ a copy of it with superscript $*$, namely $\lambda^{*}$. For every $x$ we then add the stars to symbols $x(g, t)$ where $g \in C(T_{i}(x))$. Thereby, we have defined a conjugate representation $X^{*}$ of $X$; we denote the conjugacy by $\psi$ and the corresponding map on the space of measures by $\Psi : \mathcal{M}(X) \rightarrow \mathcal{M}(X^{*})$, $\Psi(\mu) = \mu \circ \psi^{-1}$ (the map $\psi$ is continuous, because the quasitilings are dynamical; it is invertible, because removing stars we obtain the original system). The set $K^{*} = \Psi(K)$ is a face of the simplex $\mathcal{M}_{G}(X^{*})$, affinely homeomorphic to $K$. Each quasitiling $T_{i}$ is carried to a dynamical quasitiling $T_{i}^{*}$ of $X^{*}$ by the rule $T_{i}^{*}(\psi(x)) = T_{i}(x)$. Moreover, it gains the property of restricted block distribution, because symbols marked with stars prevent blocks occurring on tiles $Sc$ from occurring on positions not consistent with the tiling.

We will construct a sequence of maps $\phi_{i} : X^{*} \rightarrow Z^{*}, Z = \prod_{j \in \mathbb{N}} (A_{j}^{*})^{\mathbb{G}}$, which are all going to be invertible continuous maps commuting with the action of a group. Then we will prove that the sequence of maps $\Phi_{i} : \mathcal{M}(X^{*}) \rightarrow \mathcal{M}(Z^{*})$, $\Phi_{i}(\mu) = \mu \circ \phi_{i}^{-1}$, converges uniformly on $K^{*}$ to an affine homeomorphism $\Phi$, while $\phi_{i}$ converge pointwise on a set of full measure to a map establishing an isomorphism between $(X^{*}, \mu)$ and some $(Y, \Phi(\mu))$ for each $\mu \in K^{*}$ [hence also between $(X, \Psi^{-1}(\mu))$ and $(Y, \Phi(\mu))$].

The construction will be inductive: let $\phi_{0}$ be the identity map. Now, supposing we have constructed a map $\phi_{t-1}$, let $X_{t-1} = \phi_{t-1}(X^{*})$ (in particular, $X_{t-1}$ is conjugate to $X$). Please, note that we may treat any tiling $T_{i}^{*}$ as a tiling of $X_{t-1}$—we will use the same symbol to denote $T_{i}^{*}$ transported to $X_{t-1}$ by $\phi_{t-1}$. By Lemma 2.14 there exist
$J_I$ and $\delta_t > 0$ such that if $F$ is a $(F_k, \delta_I)$-invariant, $k \geq J_I$, and $B$ is a $(F, k)$-block occurring in $X_{t-1}$, then the distance between $\mu_B$ and $\mathcal{M}_G(X_{t-1})$ is less than $\varepsilon_t$.

For sufficiently large $n_t$ we can pick $k > J_I$ such that:

1. The tiling $T_{n_t}^*$ consists of tiles whose shapes are $(F_k, \delta_I)$-invariant.
2. For every shape $S$ of $T_{n_t}^*$ there exists a block $B_S$ with domain $S$, such that $\mu_{B_S}$ is closer than $\varepsilon_t$ to some $\mu \in \Phi_{t-1}(K^*)$ (by Corollary 2.5).
3. If $B$ denotes the family of all $(T_{n_t}^*, k)$-blocks $B$ such that $d(B, \Phi_{t-1}(K^*)) > \delta_t$, then $\sum_{B \in B} \mu(B) |B| < \varepsilon_t$ for every $\mu \in \Phi_{t-1}(K^*)$ (by Lemma 2.15).

We shall define an auxiliary map $\tilde{\phi}_t$ on $X_{t-1}$ as follows: for any $x \in X_{t-1}$ and any tile $T$ of $T_{n_t}^*(x)$, let $S$ be the shape of $T$ and let $B = x[T \times [1, k]]$. If the distance between $B$ and $\Phi_{t-1}(K^*)$ is more than $\delta_t$, replace $x[T \times [1, k]]$ with $B_S$. Otherwise, $\tilde{\phi}_t$ introduces no changes. By doing this for all $T \in T_{n_t}(x)$, we obtain a new array, $\tilde{\phi}_t(x)$.

Observe that if $x$ is in the support of any measure $\mu \in \Phi_{t-1}(K^*)$, then (by the third assumption) the union of tiles $T \in T_{n_t}(x)$ such that $x[T \times [1, k]]$ is a block distant by more than $\delta_t$ from $\Phi_{t-1}(K^*)$ has upper Banach density less than, say, $2\varepsilon_t$, therefore $\tilde{\phi}_t(x)$ differs from $x$ on a set of coordinates of density less than $2\varepsilon_t$. By Lemma 1.7 this means that the set of points $x \in X_{t-1}$ such that $\tilde{\phi}_t(x)$ differs from $x$ in column $e$ also has measure $\mu$ less than $2\varepsilon_t$ for any ergodic $\mu \in \Phi_{t-1}(K^*)$. The map $\Phi_{t-1}$ is affine so it takes ergodic measures to ergodic measures and for every $\mu \in K^*$ the ergodic decomposition of any $\Phi_{t-1}(\mu) \in \Phi_{t-1}(K^*)$ is induced by the ergodic decomposition of $\mu$. Thus this measure is less than $2\varepsilon_t$ for any $\mu \in \Phi_{t-1}(K^*)$.

Now let $\phi_t = \tilde{\phi}_t \circ \phi_{t-1}$. Since $\phi_t$ makes no changes in rows with indices $k$ and greater (and they allow us to determine the content of rows 0 through $k$), it is a conjugacy. Furthermore, let $X_t = \phi_t(X^*)$ and let $\nu$ be an ergodic measure in $\mathcal{M}_G(X_t) = \Phi_t(\mathcal{M}_G(X))$. By Corollary 2.5 for sufficiently large $n$ there is $x \in X_t$ such that $\mu_x[C]$, $C = F_n \times [1, k]$, is $\varepsilon_t$-close to $\nu$. By the construction of $\phi_t$, every $(T_{n_t}, k)$-block in $x$ is closer than $\varepsilon_t$ to some $\mu \in \Phi_{t-1}(K^*)$. If $F_n$ is a set sufficiently far in the Følner sequence, then $x[C]$ is a block that is close to being a concatenation of $T_{n_t}$ blocks (the union of tiles of $T_{n_t}$ contained in $F_n$ is a $(1-\delta_I)$-subset of $F_n$). Therefore, by Lemma 2.10 the measure $\mu_C$ differs by less than $\varepsilon_t$ from $1/\sum_{i=1}^n |B_i| \mu_{x[B_i]}$.

Since each $x[B_i]$ is $\varepsilon_t$-close to $\mu_{x[B_i]}$ the combination is $4\varepsilon_t$-close to measure in $\Phi_{t-1}(K^*)$.

We will show that the maps $\Phi_t$ converge uniformly on $K^*$. To this end, it suffices to uniformly estimate the distance between $\Phi_t(\mu)$ and $\Phi_{t-1}(\mu)$ for ergodic $\mu \in K^*$ by a summable sequence. By Lemma 2.15, for any $\mu \in K^*$ we have the estimate $\sum_{B \in B} (\Phi_{t-1}(\mu))(B) |B| < \varepsilon_t$, where $B$ denotes the family of all $T_{n_t}^*$-blocks $B$ such that $d(B, K^*) > \delta_t$. As we have already said, this implies that if $x \in X^*$, then the set of coordinates in $\phi_{t-1}(x)$ belonging to tiles of $T_{n_t}^*$ that are domains of blocks from $B$ has upper Banach density less than $2\varepsilon_t$. Since $\phi_t$ only makes any changes on these coordinates, $\phi_t(x)$ differs from $\phi_{t-1}(x)$ on a set of density less than $2\varepsilon_t$. If $x$ is in the support of some invariant measure $\mu$, then $\phi_{t-1}(x)$ and $\phi_t(x)$ are in the support of $\Phi_{t-1}(\mu)$ and $\Phi_t(\mu)$, respectively, and since the two points agree on a set of large upper Banach density, the measures are within distance less than $1/2$ (according to the choice of $\varepsilon_t$ with use of Lemma 2.1).
This uniform convergence, together with the fact that $\Phi_t(\mathcal{M}_G(X^*))$ is within the $2\varepsilon_t$-neighborhood of $\Phi_{t-1}(K^*)$, implies that $\Phi(\mathcal{M}_G(X^*)) \subset \Phi(K^*)$, and since the other inclusion is obvious, the two sets are equal.

Now, define the set $Y$ (which will support the desired assignment) as follows:

$$Y = \bigcup_{s=1}^{\infty} \bigcup_{t=s}^{\infty} X_t.$$

Observe that $Y$ is a closed, shift-invariant set, and that for any Følner set $F$ and any $k \in \mathbb{N}$ every block with domain $F \times [1, k]$ in $Y$ occurs in infinitely many of the sets $X_t$. It follows that every invariant measure on $Y$ can be approximated by invariant measures on the $X_t$'s, and thus the set of invariant measures on $Y$ is contained in $\Phi(\mathcal{M}_G(X^*)) = \Phi(K^*)$. The other inclusion is generally true: for any weakly* convergent sequence of measures $\mu_t$ supported by $X_t$, the limit measure $\mu$ is always supported by $\bigcap_{s=1}^{\infty} \bigcup_{t=s}^{\infty} X_t$. Therefore $\mathcal{M}_G(Y) = \Phi(K^*)$.

By Lemma 1.7 for every ergodic $\mu \in K^*$ the set of points $x \in X_{t-1}$ such that the column $x(e)$ is modified by $\phi_t$ has measure $\mu$ less than $2\varepsilon_t$, because $\tilde{\phi}_t$ commutes with the shift map and for any $x$ in the support of $\mu$ the set of modified coordinates has upper Banach density less than $2\varepsilon_t$. If this bound works for all ergodic measures it works for all measures in $K^*$. Since the sequence $\varepsilon_t$ is summable, the Borel-Cantelli lemma implies that for almost every $x \in X^*$ the columns $\phi_t(x)(e)$ are all equal from some point onwards. By shift-invariance, the same is true for $\phi_t(x)(g)$ for any $g$, so ultimately we conclude that if $\mu \in K^*$, then for $\mu$-almost every $x \in X^*$ every coordinate of $x$ is only changed finitely many times. This means that a limit point $\phi(x)$ is then well-defined, and this map $\phi$ is invertible (since every $\phi_t(x)$ retains the original contents of $x$ in the bottom row). In other words $\phi$ is an isomorphism between the measure-theoretic dynamical systems $(X^*, \mu)$ and $(Y, \Phi(\mu))$. □

4 Concluding remarks

Firstly, we note that we can strengthen Theorem 1.2 by combining it with theorem 1.2 of [7], obtaining the following version:

Theorem 4.1 Let $X$ be a Cantor system with free action of an amenable group $G$ and let $K$ be a face in the simplex $\mathcal{M}_G(X)$ of $G$-invariant measures of $X$. There exists a Cantor system $Y$ with minimal free action of $G$, such that the natural assignment on $Y$ is equivalent to the identity assignment on $K$.

Secondly, note that the result of this paper is not strictly a strengthening of the main theorem 4.1 in [3], since while we gain the result for actions of amenable groups, we add the requirement that the action be free, whereas the original result merely requires that the face in question contain no periodic measures. Unfortunately, it is very much unclear how the machinery used to deal with periodic points would transfer to the group case, which is why the matter of directly extending the result of [3] remains open.
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