From a \((p, 2)\)-Theorem to a Tight \((p, q)\)-Theorem

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Abstract

A family \(\mathcal{F}\) of sets is said to satisfy the \((p, q)\)-property if among any \(p\) sets of \(\mathcal{F}\) some \(q\) have a non-empty intersection. The celebrated \((p, q)\)-theorem of Alon and Kleitman asserts that any family of compact convex sets in \(\mathbb{R}^d\) that satisfies the \((p, q)\)-property for some \(q \geq d + 1\), can be pierced by a fixed number (independent of the size of the family) \(f_d(p, q)\) of points. The minimum such piercing number is denoted by \(\text{HD}_d(p, q)\). Already in 1957, Hadwiger and Debrunner showed that whenever \(q > \frac{d-1}{d} p + 1\) the piercing number is \(\text{HD}_d(p, q) = p - q + 1\); no tight bounds on \(\text{HD}_d(p, q)\) were found ever since. While for an arbitrary family of compact convex sets in \(\mathbb{R}^d, d \geq 2\), a \((p, 2)\)-property does not imply a bounded piercing number, such bounds were proved for numerous specific classes. The best-studied among them is the class of axis-parallel boxes in \(\mathbb{R}^d\), and specifically, axis-parallel rectangles in the plane. Wegner (Israel J Math 3:187–198, 1965) and (independently) Dol’nikov (Sibirsk Mat Ž 13(6):1272–1283, 1972) used a \((p, 2)\)-theorem for axis-parallel rectangles to show that \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds for all \(q \geq \sqrt{2} p\). These are the only values of \(q\) for which \(\text{HD}_{\text{rect}}(p, q)\) is known exactly. In this paper we present a general method which allows using a \((p, 2)\)-theorem as a bootstrapping to obtain a tight \((p, q)\)-theorem, for classes with Helly number 2, even without assuming that the sets in the class are convex or compact. To demonstrate the strength of this method, we show that \(\text{HD}_{d, \text{box}}(p, q) = p - q + 1\) holds for all \(q > c^d \log^{d-1} p\), and in particular, \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds for all \(q \geq 7 \log_2 p\) (compared to \(q \geq \sqrt{2} p\), obtained by Wegner and Dol’nikov more than 40 years ago). In addition, for several classes, we present improved \((p, 2)\)-theorems, some of which can be used as a bootstrapping to obtain tight \((p, q)\)-theorems. In particular, we show that any class \(\mathcal{G}\) of compact convex sets in \(\mathbb{R}^d\) with Helly number 2 admits a \((p, 2)\)-theorem with piercing number \(O(p^{2d-1})\), and thus, satisfies \(\text{HD}_G(p, q) = p - q + 1\), for a universal constant \(c\).
1 Introduction

1.1 Helly’s Theorem and (p,q)-Theorems

The classical Helly’s theorem says that if in a family of compact convex sets in $\mathbb{R}^d$ every $d + 1$ members have a non-empty intersection then the whole family has a non-empty intersection.

For a pair of positive integers $p \geq q$, we say that a family $F$ of sets satisfies the $(p, q)$-property if $|F| \geq p$, none of the sets in $F$ is empty, and among any $p$ sets of $F$ there are some $q$ with a non-empty intersection. A set $P$ of points is called a transversal (or alternatively, a piercing set) for $F$ if it has a non-empty intersection with every member of $F$. The minimal size of a transversal for $F$ is denoted by $\tau(F)$.

Throughout the paper, we consider families $F$ of sets that belong to some underlying class $G$. In this language, Helly’s theorem considers the class $G$ of compact convex sets in $\mathbb{R}^d$, and states that any $F \subset G$ that satisfies the $(d + 1, d + 1)$-property, has a singleton transversal (alternatively, can be pierced by a single point).

In general, $d + 1$ is clearly optimal in Helly’s theorem, as any family of $n$ hyperplanes in a general position in $\mathbb{R}^d$ satisfies the $(d, d)$-property but cannot be pierced by fewer than $n/d$ points. However, for numerous specific classes $G$, a $(d', d')$-property for some $d' < d + 1$ is already sufficient for a family $F \subset G$ to imply piercing by a single point. The minimal number $d'$ for which this holds is called the Helly number of the class. For example, the class of axis-parallel boxes in $\mathbb{R}^d$ has Helly number 2.

In 1957, Hadwiger and Debrunner [16] proved the following generalization of Helly’s theorem:

**Theorem 1.1** (Hadwiger–Debrunner Theorem [16]) For all $p \geq q \geq d + 1$ such that $q > \frac{d-1}{d} p + 1$, any family of compact convex sets in $\mathbb{R}^d$ that satisfies the $(p, q)$-property can be pierced by $p - q + 1$ points.

**Remark 1.2** The bound in Theorem 1.1 is tight. Indeed, any family of $n$ sets which consists of $p - q$ pairwise disjoint sets and $n - (p - q)$ copies of a set that is disjoint from them, satisfies the $(p, q)$-property but cannot be pierced by fewer than $p - q + 1$ points.

We note that throughout this paper, by saying that some $(p, q)$-theorem is “tight”, we mean that the piercing number asserted by the theorem cannot be improved, like in this remark.

Hadwiger and Debrunner conjectured that while for general $p \geq q \geq d + 1$, a transversal of size $p - q + 1$ is not guaranteed, a $(p, q)$-property does imply a bounded-size transversal. This conjecture was proved only 35 years later, in the celebrated $(p, q)$-theorem of Alon and Kleitman.
Theorem 1.3  (Alon–Kleitman (p, q)-Theorem [3]) For any triple of positive integers \( p \geq q \geq d + 1 \), there exists an integer \( s = s(p, q, d) \) such that if \( \mathcal{F} \) is a family of compact convex sets in \( \mathbb{R}^d \) satisfying the \((p, q)\)-property, then there exists a transversal for \( \mathcal{F} \) of size at most \( s \).

The smallest value \( s \) that works for \( p \geq q > d \) is called “the Hadwiger–Debrunner number” and is denoted by \( \text{HD}_d(p, q) \). For various specific classes, a stronger \((p, q)\)-theorem can be obtained. For any such class \( \mathcal{G} \), we denote the minimal \( s \) that works for any family \( \mathcal{F} \subset \mathcal{G} \) by \( \text{HD}_d\mathcal{G}(p, q) \). In particular, we write \( \text{HD}_{\text{rect}}(p, q) \) and \( \text{HD}_{\text{d-box}}(p, q) \) for the Hadwiger–Debrunner numbers of the classes of axis-parallel rectangles in the plane and of axis-parallel boxes in \( \mathbb{R}^d \), respectively.

The \((p, q)\)-theorem has a rich history of variations and generalizations. To mention a few: In 1997, Alon and Kleitman [4] presented a simpler proof of the theorem (that leads to a somewhat weaker quantitative result). Alon et al. [2] proved in 2002 a “topological” \((p, q)\)-theorem for finite families of sets which are a good cover (i.e., the intersection of every subfamily is either empty or contractible), and Bárány et al. [5] obtained in 2014 colorful and fractional versions of the theorem.

The size of the transversal guaranteed by the \((p, q)\)-theorem is huge, and a large effort was invested in proving better bounds on \( \text{HD}_d(p, q) \), both in general and in specific cases. The most recent general result, by the authors and Tardos [20], shows that for any \( \epsilon > 0 \), \( \text{HD}_d(p, q) \leq p - q + 2 \) holds for all \((p, q)\) such that \( p > p_0(\epsilon) \) and \( q > p^\frac{d-1}{d} + \epsilon \). Yet, no exact values of the Hadwiger–Debrunner numbers are known except for those given in the Hadwiger–Debrunner theorem. In fact, even the value \( \text{HD}_2(4, 3) \) is not known, the best bounds being \( 3 \leq \text{HD}_2(4, 3) \leq 13 \) (obtained by Kleitman et al. [22] in 2001).

1.2 \((p, 2)\)-Theorems and Their Applications

As mentioned above, while no general \((p, q)\)-theorems exist for \( q \leq d \), such theorems can be proved for various specific classes. Especially desirable are \((p, 2)\)-theorems, which relate the packing number, \( \nu(\mathcal{F}) \), of the family \( \mathcal{F} \) (i.e., the maximum size of a subfamily all of whose members are pairwise disjoint) to its piercing number, \( \tau(\mathcal{F}) \) (i.e., the minimal size of a piercing set for the family \( \mathcal{F} \)), as having the \((p, 2)\)-property is equivalent to having packing number fewer than \( p \).

In the last decades, \((p, 2)\)-theorems were proved for numerous classes. In particular, in 1991 Károlyi [19] proved a \((p, 2)\)-theorem for the class of axis-parallel boxes in \( \mathbb{R}^d \), guaranteeing piercing by \( O(p \log^{d-1} p) \) points. Kim et al. [21] proved in 2006 that any family of translates of a fixed convex set in \( \mathbb{R}^d \) that satisfies the \((p, 2)\)-property can be pierced by \( 2^{d-1}d^d(p - 1) \) points; five years later, Dumitrescu and Jiang [10] obtained a similar result for homothets of a convex set in \( \mathbb{R}^d \). In 2012, Chan and Har-Peled proved a \((p, 2)\)-theorem for families of pseudo-discs in the plane ([7], Theorem 4.6), with a piercing number linear in \( p \). Two years ago, Govindarajan and Nivasch [14] showed that any family of convex sets in the plane in which among any \( p \) sets there is a pair that intersects on a given convex curve \( \gamma \), can be pierced by \( O(p^8) \) points.

In 2004, Matoušek [24] showed that classes with bounded dual VC-dimension have a bounded fractional Helly number, and therefore, any such class admits a \((p, q)\)-
theorem. Recently, Pinchasi [25] drew a similar relation between the union complexity and the fractional Helly number. Each of these results implies a \((p, 2)\)-theorem for certain classes, using the proof technique of the Alon–Kleitman \((p, q)\)-theorem.

Besides their intrinsic interest, \((p, 2)\)-theorems serve as a tool for obtaining other results. One such result is an improved Ramsey Theorem. Consider, for example, a family \(\mathcal{F}\) of \(n\) axis-parallel rectangles in the plane. The classical Ramsey theorem implies that \(\mathcal{F}\) contains a subfamily of size \(\Omega(\log n)\), all whose elements are either pairwise disjoint or pairwise intersecting. As was observed by Larman et al. [23], the aforementioned \((p, 2)\)-theorem for axis-parallel rectangles [19] allows obtaining an improved bound of \(\Omega(\sqrt{n/\log n})\). Indeed, either \(\mathcal{F}\) contains a subfamily of size \(\lceil\sqrt{n/\log n}\rceil\) all whose elements are pairwise disjoint, and we are done, or \(\mathcal{F}\) satisfies the \((p, 2)\)-property with \(p = \lceil\sqrt{n/\log n}\rceil\). In the latter case, by the \((p, 2)\)-theorem, \(\mathcal{F}\) can be pierced by \(O(p \log p) = O(\sqrt{n \log n})\) points. The largest among the subsets of \(\mathcal{F}\) pierced by a single point contains at least \(\Omega\left(\frac{n}{\sqrt{n \log n}}\right) = \Omega(\sqrt{n/\log n})\) rectangles, and all its elements are pairwise intersecting.

Another result that can be obtained from a \((p, 2)\)-theorem is an improved \((p, q)\)-theorem; this will be described in detail below.

### 1.3 \((p, 2)\)-Theorems and \((p, q)\)-Theorems for Axis-Parallel Rectangles and Boxes

The \((p, q)\)-problem for the class of axis-parallel boxes is almost as old as the general \((p, q)\)-problem, and was studied almost as thoroughly (see the survey of Eckhoff [11]). It was posed in 1960 by Hadwiger and Debrunner [17], who proved that any family of axis-parallel rectangles in the plane that satisfies the \((p, q)\)-property, for \(p \geq q \geq 2\), can be pierced by \(\left(p - \frac{q + 2}{2}\right)\) points. Unlike the \((p, q)\)-problem for general families of convex sets, in this problem a finite bound on the piercing number was known from the very beginning, and the research goal has been to improve the bounds on these numbers, denoted \(\text{HD}_{\text{rect}}(p, q)\) for rectangles and \(\text{HD}_{d, \text{box}}(p, q)\) for boxes in \(\mathbb{R}^d\).

For rectangles and \(q = 2\), the quadratic upper bound on \(\text{HD}_{\text{rect}}(p, 2)\) was improved to \(O(p \log p)\) by Wegner (unpublished), and independently, by Károlyi [19]. The best currently known upper bound, which follows from a recursive formula presented by Fon-Der-Flaass and Kostochka [13], is

\[
\text{HD}_{\text{rect}}(p, 2) \leq p \lceil \log_2 p \rceil - 2^\lceil \log_2 p \rceil + 1, \tag{1}
\]

for all \(p \geq 2\). On the other hand, it is known that the “optimal possible” answer \(p - q + 1 = p - 1\) fails already for \(p = 4\). Indeed, Wegner [30] showed that \(\text{HD}_{\text{rect}}(4, 2) = 5\), and by taking \(\lceil p/3 \rceil - 1\) pairwise disjoint copies of his example, one obtains a family of axis-parallel rectangles that satisfies the \((p, 2)\)-property but cannot be pierced by fewer than \(\approx 5p/3\) points.

Wegner [30] conjectured that \(\text{HD}_{\text{rect}}(p, 2)\) is linear in \(p\), and is possibly even bounded by \(2p - 3\). While Wegner’s conjecture is believed to hold (see [11,15]), no improvement of the bound (1) was found so far.

For rectangles and \(q > 2\), Hadwiger and Debrunner showed that the tight bound \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds for all \(q \geq p/2 + 1\). Wegner [30] and
(independently) Dol’nikov [9] presented recursive formulas that allow leveraging a \((p, 2)\)-theorem for axis-parallel rectangles into a tight \((p, q)\)-theorem. Applying some of these formulas along with the Hadwiger–Debrunner quadratic upper bound on \(\text{HD}_{\text{rect}}(p, 2)\), Dol’nikov showed that \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds for all \(2 \leq q \leq p < (q+1)/2\). Applying the formulas along with the improved bound (1) on \(\text{HD}_{\text{rect}}(p, 2)\), Schell (see also [11]) obtained (by a computer-aided computation) upper bounds on the minimal \(p\) such that \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds, for all \(q \leq 12\). These values suggest that \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds already for \(q = \Omega(\log p)\). However, it appears that the method in which Dol’nikov proved a tight bound in the range \(p < (q+1)/2\) does not extend to show a tight bound for all \(q = \Omega(\log p)\) (even if (1) is employed), and in fact, no concrete improvement of Dol’nikov’s result was presented (see the survey [11]).

Dol’nikov [9] claimed that if \(\text{HD}_{\text{rect}}(p, 2)\) is linear in \(p\) as conjectured by Wegner [30], then one can deduce that \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds for all \(q \geq c\), for some constant \(c\). Eckhoff [11] wrote that the proof of this claim presented in [9] is flawed, but it is plausible that the claim does hold. On the other direction, nothing is known about the minimal \(q\) for which \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) may hold; in particular, it is not impossible that the optimal bound \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds already for \(q = 3\).

For axis-parallel boxes in \(\mathbb{R}^d\), the aforementioned recursive formula of [13] implies the bound \(\text{HD}_{\text{box}}(p, 2) \leq O(p \log^{d-1} p)\). While it is believed that the correct upper bound is \(O(p)\), the result of [13] was not improved ever since; the only advancement is a recent result of Chudnovsky et al. [8], who proved an upper bound of \(O(p \log \log p)\) on the piercing number of any family of axis-parallel boxes that satisfies the \((p, 2)\)-property and the additional condition that for each pair of intersecting boxes, a corner of one is contained in the other.

1.4 Our Results

From \((p, 2)\)-theorems to tight \((p, q)\)-theorems

The main result of this paper is a general method for leveraging a \((p, 2)\)-theorem into a tight \((p, q)\)-theorem, applicable to classes with Helly number 2. Interestingly, the method does not assume that the sets in the class are convex or compact.

**Theorem 1.4** For any \(m \in \mathbb{N}\), there exists \(c' = c'(m)\) such that the following holds. Let \(\mathcal{G}\) be a class of sets in \(\mathbb{R}^d\) with Helly number 2. Assume that for all \(2 \leq p \in \mathbb{N}\) we have \(\text{HD}_{\mathcal{G}}(p, 2) \leq pf(p)\), where \(f : [2, \infty) \to [1, \infty)\) is a differentiable function of \(p\) that satisfies \(f'(p) \geq \frac{\log_2 e}{p}\) and \(f'(p) \leq \frac{m}{p}\) for all \(p \geq 2\). Denote by \(T_{c'}(p) = T_{c'}(p, f) = \min\{q : q \geq 2c \cdot f(2p/q)\}\). Then for any \(p \geq q \geq 2\) such that \(q \geq T_{c'}(p)\), we have \(\text{HD}_{\mathcal{G}}(p, q) = p - q + 1\).

While the condition on the function \(f(p)\) looks a bit “scary”, it actually holds for any function \(f\) whose growth rate, as expressed by its derivative \(f'(p)\) and by the derivative of its logarithm \((\log f(p))' = \frac{f'(p)}{f(p)}\), is between the growth rates of
\( f(p) = \log_2 p \) and \( f(p) = p^m \) (where \( m \) can be any integer, and \( c' \) in the assertion depends on it), including all cases needed in the current paper.

The first application of our general method, which is of an independent interest, is the following theorem for the class of axis-parallel rectangles in the plane, obtained using (1) as the basic \((p, 2)\)-theorem and employing some local refinements.

**Theorem 1.5** \( \text{HD}_{\text{rect}}(p, q) = p - q + 1 \) holds for all \( q \geq 7 \log_2 p \).

**Remark 1.6** Theorem 1.5 improves significantly on the aforementioned best previous result of Wegner [30] and Dol’nikov [9], that obtained the tight result \( \text{HD}_{\text{rect}}(p, q) = p - q + 1 \) only for \( q \geq \sqrt{2p} \).

Another corollary is a tight \((p, q)\)-theorem for axis-parallel boxes in \( \mathbb{R}^d \):

**Theorem 1.7** \( \text{HD}_{d-\text{box}}(p, q) = p - q + 1 \) holds for all \( q > c \log^{d-1} 2p \), where \( c \) is a universal constant.

In the proofs of Theorems 1.4, 1.5 and 1.7 we deploy the following observation of Wegner and Dol’nikov:

**Observation 1.8** (Wegner and Dolnikov) Let \( G \) be a class of sets that has Helly number 2. Let \( p \geq q \geq 2 \) and let \( \mathcal{F} \subset G \) be a family that satisfies the \((p, q)\)-property. Denote the packing number of \( \mathcal{F} \) by \( \lambda = \nu(\mathcal{F}) \). Then there exists \( \mathcal{F}' \subset \mathcal{F} \) such that:

1. \( \mathcal{F}' \) satisfies the \((p - \lambda, q - 1)\)-property.
2. We have
   \[
   \tau(\mathcal{F}) \leq \tau(\mathcal{F}') + \lambda - 1 \leq \text{HD}_G(p - \lambda, q - 1) + \lambda - 1. \tag{2}
   \]

We use an inductive process in which (2) is applied as long as \( \lambda \) is sufficiently large. To treat the case where \( \lambda \) is small, we make use of a combinatorial argument, based on a variant of a “combinatorial dichotomy” presented by the authors and Tardos [20], which first leverages the \((p, 2)\)-theorem into a “weak” \((p, q)\)-theorem, and then uses that \((p, q)\)-theorem to show that if \( \lambda \) is “small” then \( \tau(\mathcal{F}) < p - q + 1 \).

**From \((2, 2)\)-theorems to \((p, 2)\)-theorems**

It is natural to ask, under which conditions existence of a \((2, 2)\)-theorem implies existence of a \((p, 2)\)-theorem for all \( p > 2 \).

While in general, existence of a \((2, 2)\)-theorem does not imply existence of a \((p, 2)\)-theorem (see an example in Sect. 4), we prove such an implication for several classes. Our first result here concerns classes with Helly number 2.

**Theorem 1.9** Let \( G \) be a class of compact convex sets in \( \mathbb{R}^d \) that has Helly number 2. Then \( \text{HD}_G(p, 2) \leq p^{2d-1}/2^{d-1} \), and consequently, \( \text{HD}_G(p, q) = p - q + 1 \) holds for all \( q > cp^{1-1/(d-1)} \), where \( c = c(d) \) is a constant depending only on the dimension \( d \).

The second result only assumes the existence of a \((2, 2)\)-theorem (where the piercing set may contain more than one point).
**Theorem 1.10** Let $G$ be a class of compact convex sets in $\mathbb{R}^d$ that admits a $(2, 2)$-theorem (i.e., there exists $c = c(G)$ such that any $F \subseteq G$ that satisfies the $(2, 2)$-property, can be pierced with $c$ points). Then:

1. $G$ admits a $(p, 2)$-theorem for piercing with a bounded number $s = s(p, d)$ of points.
2. If $d = 2$, then $\text{HD}_G(p, 2) = O(p^8 \log^2 p)$.
3. If $d = 2$ and $G$ has a bounded VC-dimension (see [29]), then $\text{HD}_G(p, 2) = O(p^4 \log^2 p)$.

Note that any class $G$ of sets in the plane for which any family $F \subseteq G$ has a sub-quadratic union complexity, admits a $(2, 2)$-theorem and has a bounded VC-dimension. Therefore, Theorem 1.10-3 implies that any such class $G$ satisfies $\text{HD}_G(p, 2) = O(p^4 \log^2 p)$. This significantly improves over the bound $\text{HD}_G(p, 2) = O(p^{16})$ that was obtained under the same assumptions in [20].

**1.5 Organization of the Paper**

Since the proof of our main theorem is somewhat technically involved, we first demonstrate our method for leveraging a $(p, 2)$-theorem into a tight $(p, q)$-theorem for the specific case of axis-parallel rectangles, by proving Theorem 1.5, in Sect. 2. The method is presented in its full generality in Sect. 3, where we prove Theorem 1.4 and use it to deduce Theorem 1.7. The proofs of Theorems 1.9 and 1.10 are presented in Sect. 4. We conclude the paper with a discussion and open problems in Sect. 5.

**2 From $(p, 2)$-Theorems to Tight $(p, q)$-Theorems for Axis Parallel Rectangles**

In this section we present the proof of Theorem 1.5. Before presenting the proof in Sect. 2.4, we briefly present the Wegner–Dol’nikov argument (parts of which we use in our proof) in Sect. 2.1, provide an outline of our method in Sect. 2.2, and prove two preparatory lemmas in Sect. 2.3.

**2.1 The Wegner–Dol’nikov Method**

As mentioned in the introduction, Wegner and (independently) Dol’nikov leveraged the Hadwiger–Debrunner $(p, 2)$-theorem for axis-parallel rectangles in the plane, which asserts that $\text{HD}_{\text{rect}}(p, 2) \leq \binom{p}{2}$, into a tight $(p, q)$-theorem, asserting that $\text{HD}_{\text{rect}}(p, q) \leq p - q + 1$ holds for all $p \geq q \geq 2$ such that $p < \binom{q + 1}{2}$. The heart of the Wegner–Dol’nikov argument is the aforementioned Observation 1.8. We present its proof for sake of completeness.

**Proof of Observation 1.8** The weaker assertion that there exists $F' \subseteq F$ such that $F'$ satisfies the $(p - \lambda, q - 1)$-property and $\tau(F) \leq \tau(F') + \lambda$, holds trivially, and does not even require the assumption that $G$ has Helly number 2. Indeed, if $S$ is a subfamily
of $\mathcal{F}$ of pairwise disjoint sets with $|S| = \lambda$, then $\mathcal{F} \setminus S$ satisfies the $(p - \lambda, q - 1)$-property. As $S$ clearly can be pierced by $\lambda$ points, we obtain $\tau(\mathcal{F}) \leq \tau(\mathcal{F} \setminus S) + \lambda$, and hence, $\mathcal{F} \setminus S$ is the required subfamily $\mathcal{F}'$.

To get the improvement by 1, let $S$ be a subfamily of $\mathcal{F}$ of pairwise-disjoint sets whose cardinality is $\lambda = v(\mathcal{F})$. Denote $\mathcal{F}' = \mathcal{F} \setminus S$, and let $T$ be a transversal of $\mathcal{F}'$ of size $\tau(\mathcal{F}')$. Take an arbitrary $x \in T$, and consider the subfamily $\mathcal{X} = \{A \in \mathcal{F}' : x \in A\}$ (i.e., the sets in $\mathcal{F}'$ pierced by $x$). By the maximality of $S$, each $A \in \mathcal{X}$ intersects some $B \in S$. Hence, we can write $\mathcal{X} = \bigcup_{B \in S} \mathcal{X}_B$, where $\mathcal{X}_B = \{A \in \mathcal{X} : A \cap B \neq \emptyset\}$. Observe that for each $B$, the set $\mathcal{X}_B \cup \{B\}$ is pairwise-intersecting. Indeed, any $A, A' \in \mathcal{X}$ intersect in $x$, and all elements of $\mathcal{X}_B$ intersect $B$. Therefore, by the assumption on $\mathcal{F}$, each $\mathcal{X}_B \cup \{B\}$ can be pierced by a single point. Since $\mathcal{X} = \bigcup_{B \in S} \mathcal{X}_B$, this implies that there exists a transversal $T'$ of $\mathcal{X} \cup S$ of size $|S| = \lambda$. Now, the set $(T \setminus \{x\}) \cup T'$ is a transversal of $\mathcal{F}$ with $\tau(\mathcal{F}') + \lambda - 1$ points. Therefore, we have $\tau(\mathcal{F}) \leq \tau(\mathcal{F}') + \lambda - 1$, and so, $\mathcal{F}'$ is the desired subfamily of $\mathcal{F}$. 

\[ \square \]

**Remark 2.1** We note that a generally similar argument of dividing the family $\mathcal{F}$ into a “good” subfamily that satisfies a “stronger” property (like the $(p - \lambda, q - 1)$-property in Observation 1.8) and a small “bad” subfamily that does not admit a “good” property (like the independent set in Observation 1.8 which clearly cannot be pierced by fewer than $\lambda$ points) appears also in the improved bound on the Hadwiger–Debrunner number for general convex sets presented in [20]. While in [20], the bound on the piercing number is $p - q + 2$, Observation 1.8 leads to the optimal piercing number $p - q + 1$, as shown below. The advantage of Observation 1.8 is the “improvement by 1” step, which reduces the piercing number by 1; this step cannot be applied for general families of compact convex sets, since it relies on the fact that the class of axis-parallel rectangles has Helly number 2.

Using Observation 1.8, Wegner and Dol’nikov proved the following theorem, which we will use in our proof below.

**Theorem 2.2** ([9, Thm. 2]; [30]) For any $p \geq q \geq 2$ such that $p < \binom{q + 1}{2}$, we have $\text{HD}_{\text{rect}}(p, q) = p - q + 1$.

**Proof** The proof is by induction. The induction basis is $q = 2$: for this value, the assertion is relevant only for $p = 2$, and we indeed have $\text{HD}_{\text{rect}}(2, 2) = 1 = 2 - 2 + 1$ as asserted.

For the inductive step, let $\mathcal{F}$ be a family of axis-parallel rectangles that satisfies the $(p, q)$-property. Denote $\lambda = v(\mathcal{F})$. Note that $\mathcal{F}$ satisfies the $(\lambda + 1, 2)$-property. Thus, if $\binom{\lambda + 1}{2} \leq p - q + 1$ then we have $\tau(\mathcal{F}) \leq p - q + 1$ by the aforementioned Hadwiger–Debrunner $(p, 2)$-theorem for axis-parallel rectangles [17]. On the other hand, if $\binom{\lambda + 1}{2} > p - q + 1$ then it can be checked (by a direct but somewhat tedious calculation) that $p - \lambda < \binom{q}{2}$, so by the induction hypothesis we have $\text{HD}_{\text{rect}}(p - \lambda, q - 1) = (p - \lambda) - (q - 1) + 1$. By Observation 1.8, this implies $\tau(\mathcal{F}) \leq \text{HD}_{\text{rect}}(p - \lambda, q - 1) + \lambda - 1 = p - q + 1$, as asserted. 

\[ \square \]
2.2 Outline of Our Method

Instead of leveraging the Hadwiger–Debrunner \((p, 2)\)-theorem into a \((p, q)\)-theorem as was done by Wegner and Dol’nikov, we would like to leverage the stronger bound \(\text{HD}_{\text{rect}}(p, 2) \leq p \log_2 p\) which follows from (1). We want to deduce that \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds for all \(q \geq 7 \log_2 p\).

Basically, we would like to perform an inductive process similar to the process applied in the proof of Theorem 2.2. Let \(\mathcal{F}\) be a family of axis-parallel rectangles in the plane. As above, put \(\lambda = v(\mathcal{F})\). If \(\lambda\) is “sufficiently large” (namely, if \(q - 1 \geq 7 \log_2(p - \lambda)\)), we apply the recursive formula \(\tau(\mathcal{F}) \leq \text{HD}_{\text{rect}}(p - \lambda, q - 1) + \lambda - 1\) from Observation 1.8 and use the induction hypothesis to bound \(\text{HD}_{\text{rect}}(p - \lambda, q - 1)\). Otherwise, we would like to use the improved \((p, 2)\)-theorem to deduce that \(\mathcal{F}\) can be pierced by at most \(p - q + 1\) points.

However, since we want to prove the theorem in the entire range \(q \geq 7 \log_2 p\), in order to apply the induction hypothesis to \(\text{HD}_{\text{rect}}(p - \lambda, q - 1)\), \(\lambda\) must be at least linear in \(p\) (specifically, we need \(\lambda \geq 0.1 p\), as is shown below). Thus, in the “otherwise” case we have to show that if \(\lambda < 0.1 p\), then \(\mathcal{F}\) can be pierced by at most \(p - q + 1\) points. If we merely use the fact that \(\mathcal{F}\) satisfies the \((\lambda + 1, 2)\)-property and apply the improved \((p, 2)\)-theorem, instead of the Hadwiger–Debrunner \((p, 2)\)-theorem applied in the proof of Theorem 2.2, we only obtain that \(\mathcal{F}\) can be pierced by \(O(p \log p)\) points—significantly weaker than the desired bound \(p - q + 1\).

Instead, we use a more complex procedure, partially based on the following observation, that was presented in [20] under the name “combinatorial dichotomy”, and can be viewed as a generalization of the easy part of Observation 1.8 (see first paragraph of the proof of Observation 1.8 which is equivalent to applying Observation 2.3 with \(p' = \lambda\) and \(q' = 2\)).

**Observation 2.3** Let \(\mathcal{F}\) be a family that satisfies the \((p, q)\)-property. For any \(p' \leq p, q' \leq q\) such that \(q' \leq p'\), either \(\mathcal{F}\) satisfies the \((p', q')\)-property, or there exists \(S \subset \mathcal{F}\) of size \(p'\) that does not contain an intersecting \(q'\)-tuple. In the latter case, \(\mathcal{F} \setminus S\) satisfies the \((p - p', q - q' + 1)\)-property.

First, we use Observation 2.3 to leverage the \((p, 2)\)-theorem by an inductive process into a “weak” \((p, q)\)-theorem that guarantees piercing with \(p - q + 1 + O(p)\) points, for all \(q = \Omega(\log p)\). We then show that if \(\lambda < 0.1 p\) then \(\mathcal{F}\) can be pierced by at most \(p - q + 1\) points, by combining the weak \((p, q)\)-theorem, another application of Observation 2.3, and a lemma which exploits the size of \(\lambda\).

**Remark 2.4** We note that aforementioned stronger bound \(\text{HD}_{\text{rect}}(p, 2) \leq p \log_2 p\) of [13] implies that if \(\lambda < c \frac{p}{\log p}\) for a sufficiently small constant \(c\), then \(\mathcal{F}\) can be pierced by at most \(p - q + 1\) points. Using this instead of the inequality \((\frac{\lambda + 1}{2}) \leq p - q + 1\) yielded by the Hadwiger–Debrunner \((p, 2)\)-theorem [17], one can apply the inductive step of Wegner and Dol’nikov (i.e., the proof strategy of Theorem 2.2 above) to deduce that \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds for all \(q \geq C \log^2 p\), for a sufficiently large \(C\). (This follows from a straightforward, but a somewhat tedious, calculation.) Using our method, we obtain a tight \((p, q)\)-theorem in the significantly wider range \(q \geq 7 \log_2 p\).
2.3 The Two Main Lemmas Used in the Proof

Our first lemma leverages the \((p, 2)\)-theorem \(\text{HD}_{\text{rect}}(p, 2) \leq p \log_2 p\) into a weak \((p, q)\)-theorem, using Observation 2.3.

Lemma 2.5  For any \(c > 0\) and for any \(p \geq q \geq 2\) such that \(q \geq c \log_2 p\), we have

\[
\text{HD}_{\text{rect}}(p, q) \leq p - q + 1 + \frac{2p}{c}.
\]

**Proof**  First, assume that both \(p\) and \(q\) are powers of 2. Let \(\mathcal{F}\) be a family of axis-parallel rectangles in the plane that satisfies the \((p, q)\)-property. We perform an inductive process with \(\ell = (\log_2 q) - 1\) steps, where we set \(\mathcal{F}_0 = \mathcal{F}\) and \((p_0, q_0) = (p, q)\), and in each step \(i\), we apply Observation 2.3 to a family \(\mathcal{F}_{i-1}\) that satisfies the \((p_{i-1}, q_{i-1})\)-property, with \((p', q') = (\frac{p_{i-1}}{2}, \frac{q_{i-1}}{2})\) which we denote by \((p_i, q_i)\).

Consider Step \(i\). By Observation 2.3, either \(\mathcal{F}_{i-1}\) satisfies the \((p_i, q_i) = (\frac{p_{i-1}}{2}, \frac{q_{i-1}}{2})\)-property, or there exists a “bad” set \(S_i\) of size \(\frac{p_{i-1}}{2}\) without an intersecting \(\frac{q_{i-1}}{2}\)-tuple, and the family \(\mathcal{F}_{i-1} \setminus S_i\) satisfies the \((\frac{p_{i-1}}{2}, \frac{q_{i-1}}{2} + 1)\)-property, and in particular, the \((\frac{p_{i-1}}{2}, \frac{q_{i-1}}{2})\)-property. In either case, we are reduced to a family \(\mathcal{F}_i\) (either \(\mathcal{F}_{i-1}\) or \(\mathcal{F}_{i-1} \setminus S_i\)) that satisfies the \((p_i, q_i)\)-property, to which we apply Step \(i + 1\).

At the end of Step \(\ell\) we obtain a family \(\mathcal{F}_\ell\) that satisfies the \((2p/q, 2)\)-property. (Note that the ratio between the left term and the right term remains unchanged along the way.) By the \((p, 2)\)-theorem, \(\mathcal{F}_\ell\) can be pierced by \(2p/\log_2(2p/q)\) points. As \(p \geq \max(c \log_2 p, 2)\), this implies that \(\mathcal{F}_\ell\) can be pierced by

\[
\frac{2p}{q} \log_2 \left(\frac{2p}{q}\right) \leq \frac{2p}{q} \log_2 p \leq \frac{2p}{c}
\]

points.

In order to pierce \(\mathcal{F}\), we also have to pierce the “bad” sets \(S_i\). In the worst case, in each step we have a bad set, and so we have to pierce \(S = \bigcup_{i=1}^\ell S_i\). The size of \(S\) is \(|S| = \frac{p}{2} + \frac{p}{4} + \cdots + 2 + 1 = p - 1\). Since any family that satisfies the \((p, q)\)-property also satisfies the \((p - k, q - k)\)-property for any \(k\), the family \(S\) contains an intersecting \((q - 1)\)-tuple, which of course can be pierced by a single point. Hence, \(S\) can be pierced by \((p - 1) - (q - 1) + 1 = p - q + 1\) points. Therefore, in total \(\mathcal{F}\) can be pierced by \(p - q + 1 + 2p/c\) points, as asserted.

Now, we have to deal with the case where \(p, q\) are not necessarily powers of 2, and thus, in some of the steps either \(p_{i-1}\) or \(q_{i-1}\) or both are not divisible by 2. It is clear from the proof presented above that if we can define \((p_i, q_i)\) in such a way that in both cases (i.e., whether there is a “bad” set or not), we have \(\frac{p_i}{q_i} \leq \frac{p_{i-1}}{q_{i-1}}\), and also the total size of the bad sets (i.e., \(|S|\)) is at most \(p - 1\), the assertion can be deduced as above (as the ratio between the left term and the right term only decreases). We show that this can be achieved by a proper choice of \((p_i, q_i)\) and a slight modification of the steps described above. Let
\[(p', q') = \left(\left\lfloor \frac{p_{i-1}}{2} \right\rfloor, \left\lceil \frac{q_{i-1}}{2} \right\rceil \right).\]

If \(F_{i-1}\) satisfies the \((p', q')\)-property, we define \(F_i = F_{i-1}\) and \((p_i, q_i) = (p', q')\). Otherwise, there exists a “bad” set \(S_i\) of size \(p'\) that does not contain an intersecting \(q'\)-tuple, and the family \(F_{i-1} \setminus S_i\) satisfies the
\[(p_{i-1} - p', q_{i-1} - q' + 1) = \left(\left\lfloor \frac{p_{i-1}}{2} \right\rfloor, \left\lceil \frac{q_{i-1}}{2} \right\rceil + 1 \right)\]
property. In this case, we define \(F_i = F_{i-1} \setminus S_i\) and \((p_i, q_i) = (p_{i-1} - p', q_{i-1} - q' + 1)\).

In the former case, we have
\[\frac{p_i}{q_i} = \frac{\left\lfloor \frac{p_{i-1}}{2} \right\rfloor}{\left\lceil \frac{q_{i-1}}{2} \right\rceil} \leq \frac{p_{i-1}}{q_{i-1}}.\]

To see that \(\frac{p_i}{q_i} \leq \frac{p_{i-1}}{q_{i-1}}\) holds also in the latter case, note that we have
\[\left\lfloor \frac{p_{i-1}}{2} \right\rfloor \leq \frac{p_{i-1} + 1}{2} \quad \text{and} \quad \left\lceil \frac{q_{i-1}}{2} \right\rceil + 1 \geq \frac{q_{i-1} + 1}{2},\]
and thus,
\[\frac{p_{i-1}}{q_{i-1}} \leq \frac{(p_{i-1} + 1)/2}{(q_{i-1} + 1)/2} = \frac{p_{i-1} + 1}{q_{i-1} + 1} \leq \frac{p_{i-1}}{q_{i-1}},\]
where the last inequality holds since \(p_{i-1} \geq q_{i-1}\). Furthermore, it is easy to see that \(|S| \leq p - 1\) holds also with respect to the modified definition of the \(S_i\)’s. Hence, the proof indeed can be completed, as above.

Our second lemma is a simple upper bound on the piercing number of a family that satisfies the \((p, 2)\)-property. We shall use it to show that if \(\nu(F)\) is “small”, then we can save “something” when piercing large subsets of \(F\).

**Lemma 2.6** Any family \(F\) of \(m\) sets that satisfies the \((p, 2)\)-property can be pierced by \(\left\lfloor \frac{m + p - 1}{2} \right\rfloor\) points.

**Proof** We perform the following simple recursive process. If \(F\) contains a pair of intersecting sets, pierce them by a single point and remove both of them from \(F\). Continue in this fashion until all remaining sets are pairwise disjoint. Then pierce each remaining set by a separate point.

As \(F\) satisfies the \((p, 2)\)-property, the number of sets that remain in the last step is at most \(p - 1\) if \(m - (p - 1)\) is even and at most \(p - 2\) otherwise. In the former case, the resulting piercing set is of size at most \(\frac{m - (p - 1)}{2} + (p - 1) = \frac{m + p - 1}{2}\). In the latter case, the piercing set is of size at most \(\frac{m - (p - 2)}{2} + (p - 2) = \frac{m + p - 2}{2}\). Hence, in both cases the piercing set is of size at most \(\left\lfloor \frac{m + p - 1}{2} \right\rfloor\), as asserted. \(\square\)
Remark 2.7 The assertion of Lemma 2.6 is tight, as for a family $\mathcal{F}$ composed of $m - p + 2$ lines in general position in the plane and $p - 2$ pairwise-disjoint segments that do not intersect any of the lines, we have $|\mathcal{F}| = m$, $\mathcal{F}$ satisfies the $(p, 2)$-property, and $\mathcal{F}$ clearly cannot be pierced by fewer than $\left\lfloor \frac{m+p-1}{2} \right\rfloor$ points.

Corollary 2.8 Let $\mathcal{F}$ be a family of sets, and put $\lambda = \nu(\mathcal{F})$. Then any subset $S \subset \mathcal{F}$ can be pierced by at most $\left\lfloor \frac{|S| + \lambda}{2} \right\rfloor$ points.

The corollary follows from the lemma immediately, as any such family $\mathcal{F}$ satisfies the $(\lambda + 1, 2)$-property.

2.4 Proof of Theorem 1.5

Now we are ready to present the proof of our main theorem, in the specific case of axis-parallel rectangles in the plane. Let us rephrase its statement.

Theorem 1.5 Let $\mathcal{F}$ be a family of axis-parallel rectangles in the plane. If $\mathcal{F}$ satisfies the $(p, q)$-property, for $p \geq q \geq 2$ such that $q \geq 7 \log_2 p$, then $\mathcal{F}$ can be pierced by $p - q + 1$ points.

Remark 2.9 We note that the parameters in the proof (e.g., the values of $(p', q')$ in the inductive step) were chosen in a sub-optimal way, that is however sufficient to yield the assertion with the constant 7. (The straightforward choice $(p', q') = (0.5p, 0.5q)$ is not sufficient for that.) The constant can be further optimized by a more careful choice of the parameters; however, it seems that in order to reduce it below 6, a significant change in the proof is needed.

Proof of Theorem 1.5 The proof is by induction.

Induction Basis One can assume that $q \geq 37$, as for any smaller value of $q$, there are no $p$'s such that $7 \log_2 p \leq q \leq p$. For $q = 37$, the theorem is only relevant for the pairs $(p, q) = (37, 37), (38, 37), (39, 37)$, and in these cases we clearly have $HD_{\text{rect}}(p, q) = p - q + 1$ by the Hadwiger–Debrunner theorem (i.e., Theorem 1.1 above). Generally speaking, this is a sufficient basis, since in the inductive step, the value of $q$ is reduced by 1 every time. However, in the proof of the inductive step we would like to assume that both the parameters $p, q$ are “sufficiently large”; hence, we use Theorem 2.2 as the induction basis in order to cover a larger range of small $(p, q)$ values.

We observe that for $q \leq 70$, all relevant $(p, q)$ pairs (i.e., all pairs for which $7 \log_2 p \leq q \leq p$) satisfy $p \leq \left(\frac{q + 1}{2}\right)$. Hence, in this range we have $HD_{\text{rect}}(p, q) = p - q + 1$ by Theorem 2.2. Therefore, we may assume that $q > 70$; we also may assume that $q < \sqrt{2p}$ (as otherwise, the assertion follows from Theorem 2.2), and thus, we have $p > 2450$ and

$$\frac{p}{q} > \frac{q}{2} > \frac{70}{2} = 35.$$ 

Inductive Step Put $\lambda = \nu(\mathcal{F})$. By Observation 1.8, we have $\tau(\mathcal{F}) \leq HD_{\text{rect}}(p - \lambda, q - 1) + \lambda - 1$. We want $\lambda$ to be sufficiently large, such that if $(p, q)$ lies in the range
covered by the theorem (i.e., if \( q \geq 7 \log_2 p \)), then \((p - \lambda, q - 1)\) also lies in the range covered by the theorem (i.e., \(q - 1 \geq 7 \log_2 (p - \lambda)\)). Note that the condition \( q \geq 7 \log_2 p \) is equivalent to \(2^{q/7} \geq p\), which implies \(2^{(q-1)/7} = 2^{q/7-1} \geq \frac{1}{2}p\). Hence, if \( \lambda \geq 0.1p \) then \(q - 1 \geq 7 \log_2 (p - \lambda)\), and so we can deduce from the induction hypothesis that

\[
\tau(F) \leq \text{HD}_{\text{rec}}(p - \lambda, q - 1) + \lambda - 1 \\
\leq (p - \lambda) - (q - 1) + 1 + (\lambda - 1) = p - q + 1,
\]

as asserted. Therefore, it is sufficient to prove that \(\tau(F) \leq p - q + 1\) holds when \(\lambda < 0.1p\).

Under this assumption on \(\lambda\), we apply Observation 2.3 to \(F\), with \((p', q') = ([0.62p], 0.5q)\). We have to consider two cases:

**Case 1** \(F\) satisfies the \((p', q')\)-property. By the assumption on \((p, q)\), we have \(q \geq 7 \log_2 p\), and thus, \(0.5q \geq 3.5 \log_2 p \geq 3.5 \log_2 [0.62p]\). Hence, by Lemma 2.5,

\[
\tau(F) \leq \text{HD}_{\text{rec}} ([0.62p], 0.5q) \leq 0.62p - 0.5q + 1 + \frac{2}{3.5} \cdot 0.62p \\
< 0.975p - 0.5q + 1 \leq p - q + 1,
\]

where the last inequality holds because we may assume \(q \leq 0.05p\), since \(\frac{p}{q} > 35\) as was mentioned above. Thus, \(F\) can be pierced by at most \(p - q + 1\) points, as asserted.

**Case 2** \(F\) does not satisfy the \((p', q')\)-property. In this case, there exists a “bad” subfamily \(S\) of size \(p' = [0.62p]\) that does not contain an intersecting 0.5-rectangle, and the family \(F\setminus S\) satisfies the \(([0.38p], 0.5q)\)-property.

To pierce \(F\setminus S\), we use Lemma 2.5. Like above, we have \(0.5q \geq 3.5 \log_2 [0.38p]\), whence by Lemma 2.5,

\[
\tau(F\setminus S) \leq \text{HD}_{\text{rec}} ([0.38p], 0.5q) \\
\leq 0.39p - 0.5q + 1 + \frac{2}{3.5} \cdot 0.39p < 0.613p - 0.5q + 1,
\]

where the second inequality holds since we may assume \(p \geq 100\) (as was mentioned above), and thus, \([0.38p] \leq 0.39p\).

To pierce the “bad” subfamily \(S\), we use Corollary 2.8, which implies that \(S\) can be pierced by \(\left\lfloor \frac{1}{2} (|S| + \lambda) \right\rfloor \leq \frac{1}{2} (0.62p + 0.1p) = 0.36p\) points. Therefore, in total \(F\) can be pierced by \((0.613p - 0.5q + 1) + 0.36p < 0.975p - 0.5q + 1\) points. Since we may assume \(q \leq 0.05p\) (like above), this implies that \(F\) can be pierced by \(p - q + 1\) points. This completes the proof of the theorem. \(\square\)
3 From \((p, 2)\)-Theorems to Tight \((p, q)\)-Theorems for Families with Helly Number 2

In this section we present the proof of Theorem 1.4, which allows leveraging a \((p, 2)\)-theorem into a tight \((p, q)\)-theorem, for classes \(\mathcal{G}\) that satisfy \(\text{HD}_\mathcal{G}(2, 2) = 1\). In order to be concrete and avoid some cumbersome calculations, we present the proof in the case \(m = 5\); it will be apparent from the proof that essentially the same argument holds for any \(m \in \mathbb{N}\), with the appropriate choice of \(c'(m)\). The proof builds upon the simpler proof of Theorem 1.5 presented in Sect. 2. For the sake of completeness we present it in full generality. We present not only the parts of the proof that do not differ from their “rectangles-case” counterparts; these include the proof outline.

This section is organized as follows. First we outline the proof in Sect. 3.1. Then we present several lemmas required for the proof in Sect. 3.2, and the proof itself in Sect. 3.3. We deduce Theorem 1.7 from Theorem 1.4 in Sect. 3.4.

3.1 Proof Outline

Let \(\mathcal{G}\) be a class that satisfies \(\text{HD}_\mathcal{G}(2, 2) = 1\). In order to leverage a \((p, 2)\)-theorem for \(\mathcal{G}\) into a tight \((p, q)\)-theorem we would like to perform an inductive process similar to the process Wegner and Dol’nikov applied in the proof of Theorem 2.2. Let \(\mathcal{F} \subseteq \mathcal{G}\) be a family that satisfies the \((p, q)\)-property. Put \(\lambda = \nu(\mathcal{F})\). If \(\lambda\) is “sufficiently large”, we apply the recursive formula \(\tau(\mathcal{F}) \leq \text{HD}_\mathcal{G}(p - \lambda, q - 1) + \lambda - 1\) from Observation 1.8 and use the induction hypothesis to bound \(\text{HD}_\mathcal{G}(p - \lambda, q - 1)\). Otherwise, we would like to use the \((p, 2)\)-theorem to deduce that \(\mathcal{F}\) can be pierced by at most \(p - q + 1\) points.

Since we allow \(q\) to be as small as roughly \(\log p\), and as we want to apply the induction hypothesis to \(\text{HD}_\mathcal{G}(p - \lambda, q - 1)\), the largest possible \(\lambda\) we will have to deal with is linear in \(p\). Thus, in the “otherwise” case we have to show directly that if \(\lambda < c' p\) for a sufficiently small constant \(c'\), then \(\mathcal{F}\) can be pierced by at most \(p - q + 1\) points. If we merely use the fact that \(\mathcal{F}\) satisfies the \((\lambda + 1, 2)\)-property and apply the \((p, 2)\)-theorem, we only obtain that \(\mathcal{F}\) can be pierced by \(c' p \cdot f(c' p)\) points – significantly weaker than the desired bound \(p - q + 1\).

Instead, we use a more complex procedure, based on Observation 2.3 presented above. First, we use Observation 2.3 to leverage the \((p, 2)\)-theorem by an inductive process into a “weak” \((p, q)\)-theorem that guarantees piercing with \(p - q + 1 + O(p)\) points, for all \(q = \Omega(T_{100}(p))\), where \(T_c(p) = \min\{q : q \geq 2c \cdot f(2p/q)\}\). We then show that if \(\lambda < c' p\) for a sufficiently small absolute constant \(c'\), then \(\mathcal{F}\) can be pierced by at most \(p - q + 1\) points, by combining the weak \((p, q)\)-theorem, another application of Observation 2.3, and a lemma which exploits the size of \(\lambda\). (Note that while for \(f(p) \gg \log p\) we may apply the inductive step also when \(\lambda \ll p\), we actually apply it only when \(\lambda \geq c' p\). The reason is that our argument for the “otherwise” case, which proves that if \(\lambda < c' p\) for a sufficiently small constant \(c'\), then \(\mathcal{F}\) can be pierced by at most \(p - q + 1\) points, holds also for \(f(p) \gg \log p\); hence, there is no loss in applying the inductive step only for higher values of \(\lambda\).)
In addition, we have to handle the induction basis: while in the proof of Theorem 2.2, Dol’nikov could use the case \( p = q = 2 \) as the induction basis, our assertion applies only to significantly larger values of \( q \). Hence, we will have to guarantee that for the “minimum relevant” value of \( q \), for all “relevant” values of \( p \) (i.e., all values of \( p \) such that \( p \geq q \geq T_{100}(p) \)) we have \( \text{HD}_G(p, q) = p - q + 1 \). We shall deduce this from another result of Dol’nikov presented below.

### 3.2 Lemmas Used in the Proof

The first lemma we use is a weak \((p, q)\)-theorem, that can be obtained from a \((p, 2)\)-theorem using Observation 2.3. This lemma is a straightforward generalization of Lemma 2.5, and thus its proof is omitted.

**Lemma 3.1** Let \( G \) be a class of sets in \( \mathbb{R}^d \) and let \( c > 0 \). Assume that for all \( 2 \leq p \in \mathbb{N} \) we have \( \text{HD}_G(p, 2) = pf(p) \), where \( f : [2, \infty) \to [1, \infty) \) is a monotone increasing function of \( p \). Let \( T_c(p) = \min\{q : q \geq 2c \cdot f(2p/q)\} \). Then for any \( q \geq T_c(p) \), we have

\[
\text{HD}_G(p, q) \leq p - q + 1 + \frac{p}{c}.
\]

The second result we use is an easy extension of the classical Hadwiger–Debrunner theorem, obtained by Dol’nikov [9].

**Proposition 3.2** ([9, Thm. 1]) Let \( G \) be a class that satisfies \( \text{HD}_G(2, 2) = 1 \). Then for any \( p \geq q \geq 2 \) such that \( p \leq 2q - 2 \), we have \( \text{HD}_G(p, q) = p - q + 1 \).

The third lemma we use concerns several properties of the function \( T_c(p) \).

**Claim 3.3** Let \( f : [2, \infty) \to [1, \infty) \) be an increasing function of \( p \), let \( c > 0 \), and let \( T_c(p) = \min\{q : q \geq 2c \cdot f(2p/q)\} \). Then:

1. For each \( p \), the condition \( q \geq 2c \cdot f(2p/q) \) holds for all \( T_c(p) \leq q \leq p \).
2. \( T_c(p) \) is a non-decreasing function of \( p \).
3. If, in addition, \( f \) satisfies \( f'(p) \geq \frac{\log_e 2}{p} \) for all \( p \geq 1 \), then:

   (a) For all \( k \in \mathbb{N} \), all \( c \geq \frac{1}{2} \log_2(k+1)/k \) and all \( p \) such that \( k \leq T_c(p-1) \leq p-1 \), we have \( T_c(2p-1) \geq T_c(p-1) + 1 \).

   (b) For all \( k \in \mathbb{N} \), all \( 0 < \alpha < \frac{k-1}{k} \), all \( c \geq \frac{1}{2} \log((k-1)/(\alpha k)) \) and all \( p \) such that \( k \leq T_c(\alpha p) \leq \alpha p \), we have \( T_c(p) \geq T_c(\alpha p) + 1 \).

**Proof** Properties 1, 2 follow immediately from the definition of \( T_c(p) \) and the assumption that \( f \) is increasing. To prove (3a), consider some specific value of \( p \) and denote \( T_c(p-1) = q_0 \). By the definition of \( T_c(p-1) \), we have \( q_0 - 1 < 2c \cdot f\left(\frac{2(p-1)}{q_0-1}\right) \) (which will imply \( T_c(2p-1) \geq q_0 + 1 \) by Property 1). It is clearly sufficient to show that for any \( p \) for which \( q_0 = T_c(p-1) \geq k \), we have

\[
f\left(\frac{2(2p-1)}{q_0}\right) - f\left(\frac{2(p-1)}{q_0-1}\right) \geq \frac{1}{2c}.
\]
By the assumption on the derivative \( f' \), for any \( t > t' \) we have

\[
\int_{t'}^t f'(x) \, dx \geq \int_{t'}^t \frac{\log_2 e}{x} \, dx = \log_2 \left( \frac{t}{t'} \right).
\]

Hence,

\[
f \left( \frac{2(2p - 1)}{q_0} \right) - f \left( \frac{2(p - 1)}{q_0 - 1} \right) \geq \log_2 \left( \frac{2(2p - 1)(q_0 - 1)}{2q_0(p - 1)} \right) \geq \log_2 \left( \frac{2(q_0 - 1)}{q_0} \right) \geq \log_2 \left( \frac{2(k - 1)}{k} \right),
\]

where the last inequality holds since \( q_0 \geq k \) by assumption. Since \( \frac{1}{2} \leq \log_2 \left( \frac{2(k - 1)}{k} \right) \) by the assumption on \( c \), this completes the proof of (3a). The proof of (3b) is almost identical to the proof of (3a), with a general \( \alpha \) instead of 1/2, and thus is omitted. \( \square \)

As was outlined in Sect. 3.1, in order to apply the inductive step of the proof, we have to assume that \( \lambda = \nu(\mathcal{F}) \) is “sufficiently large”. Specifically, in the inductive step we move from a \((p', q')\)-property to a \((p' - \lambda, q' - 1)\)-property. We assume that \((p', q')\) lies in the range covered by the theorem, i.e., that \( q' \geq T_c(p') \), and want to deduce that \((p' - \lambda, q' - 1)\) also lies in the range covered by the theorem, i.e., that \( q' - 1 \geq T_c(p' - \lambda) \). It is clearly sufficient to take \( \lambda = \lambda(p') \) such that

\[ T_c(p' - \lambda) \leq T_c(p') - 1. \tag{3} \]

Our fourth lemma states how large should \( \lambda \) be in order to guarantee this, for the particular choice \( c = 100 \) that we use in Theorem 1.4.

**Lemma 3.4** Assume that \((p', q')\) lies in the range covered by Theorem 1.4, i.e., that \( q' \geq T_{100}(p') \). Then \( q' - 1 \geq T_{100}(0.99p') \), and thus, \((0.99p', q' - 1)\) lies in the range covered by Theorem 1.4 as well.

**Proof** Denoting \( \lambda(p') = \beta p' \), we can apply Claim 3.3 (3b) with \( \alpha = 1 - \beta \) to deduce that (3) holds for all \( c \geq \frac{1}{2} \log_{(k-1)/((1-\beta)k)} 2 \), provided that \( T_c((1 - \beta)p) \geq k \) and \( 1 - \beta < \frac{k-1}{k} \).

In the special case \( c = 100 \), we may take \( k = 200 \) (as \( T_{100}(p) = \min\{q : q \geq 200 f(2p/q)\} \geq 200 f(2) \geq 200 \) for any \( p \)). Hence, we may take \( \alpha \) to be any number in \((0, \frac{199}{200})\) such that \( 100 \geq \frac{1}{2} \log_{199/200(1-\beta)} 2 \). In particular, \( \alpha = 0.99 \) works. \( \square \)

### 3.3 Proof of Theorem 1.4

We are ready to prove our main theorem. As written above, we state the theorem in the case \( m = 5 \), with the concrete value \( c'(m) = c'(5) = 100 \).

**Theorem 1.4** (for \( m = 5 \)) Let \( \mathcal{G} \) be a class of sets in \( \mathbb{R}^d \) with Helly number 2. Assume that for all \( 2 \leq p \in \mathbb{N} \) we have \( \text{HD}_\mathcal{G}(p, 2) \leq pf(p) \), where \( f : [2, \infty) \to [1, \infty) \) is a...
differentiable function of $p$ that satisfies $f'(p) \geq \frac{\log_2 e}{p}$ and $\frac{f'(p)}{f(p)} \leq \frac{5}{x}$ for all $p \geq 2$. Denote $T_c(p) = \min\{q : q \geq 2c \cdot f(2p/q)\}$. Then for any $p \geq q \geq 2$ such that $q \geq T_{100}(p)$, we have $HD_G(p, q) = p - q + 1$.

**Proof** By induction. We start with the inductive step, and leave the induction basis for the end.

Let $F \subset G$ be a family that satisfies the $(p, q)$-property. Put $\lambda = v(F)$. By Observation 1.8, we have $\tau(F) \leq HD_G(p - \lambda, q - 1) + \lambda - 1$. If $\lambda \geq 0.01p$, then by Lemma 3.4, the pair $(p - \lambda, q - 1)$ satisfies the assumption of the theorem, and thus, by the induction hypothesis we have $HD_G(p - \lambda, q - 1) = (p - \lambda) - (q - 1) + 1$, whence $\tau(F) = p - q + 1$ as asserted. Therefore, it is sufficient to prove that if $\lambda < 0.01p$, then $F$ can be pierced by at most $p - q + 1$ points.

We apply Observation 2.3 to $F$, with $(p', q') = (\frac{5}{2}p, \frac{5}{2})$. (For the sake of simplicity, we assume that $p, q$ are divisible by 3 and 2, respectively. It will be apparent that this does not affect the proof.) We have to consider two cases:

**Case 1** $F$ satisfies the $(p', q')$-property. Note that by the assumption on $\frac{f'(p)}{f(p)}$, for any $p \geq 2$ we have

$$
\ln(f(4p/3)) - \ln(f(p)) = \int_p^{4p/3} \ln(f(t))' dt = \int_p^{4p/3} \frac{f'(t)}{f(t)} dt \leq \int_p^{4p/3} \frac{5}{t} dt = 5 \ln(4/3) < \ln 5,
$$

and thus, $f(4p/3) < 5f(p)$. By the assumption on $(p, q)$, we have $q \geq 200f(2p/q)$, and thus,

$$
\frac{q}{2} \geq 100f\left(\frac{2p}{q}\right) > 20f\left(\frac{4p/3}{q}\right) = 20f\left(\frac{2 \cdot 2p/3}{q/2}\right).
$$

By the definition of $T_c$, this implies $\frac{q}{2} \geq T_{10}(\frac{2p}{q})$. Therefore, we can apply Lemma 3.1 to deduce

$$
\tau(F) \leq HD_G(p', q') = HD_G\left(\frac{2}{3}p, \frac{q}{2}\right) \leq \frac{2}{3}p - \frac{1}{2}q + 1 + \frac{1}{10} \cdot \frac{2}{3}p < 0.74p - 0.5q + 1.
$$

This shows that $F$ can be pierced by fewer than $p - q + 1$ points, if we may assume $0.26p \geq 0.5q$, or equivalently, $q \leq 0.52p$.

To see that we indeed may assume this, note that by Proposition 3.2, $HD_G(p, q) = p - q + 1$ holds whenever $q \geq \frac{p}{2} + 1$. Our theorem applies only for $q \geq 200$ (as for any “relevant” pair $(p, q)$ we have $q \geq 200f(2p/q) \geq 200 \cdot 1$), and in this range, $(q > 0.52p) \Rightarrow (q > 0.5p + 1)$. Thus, either the above argument implies $\tau(F) < p - q + 1$, or Proposition 3.2 implies $\tau(F) = p - q + 1$, and either way we are done.
Case 2 $\mathcal{F}$ does not satisfy the $(p', q')$-property. In this case, there exists a “bad” subfamily $S$ of size $p' = \frac{2p}{3}$ that does not contain an intersecting $q'$-tuple, and the family $\mathcal{F} \setminus S$ satisfies the $\left(\frac{p}{3}, \frac{q}{2}\right)$-property.

To pierce the family $\mathcal{F} \setminus S$ we use Lemma 3.1. By the monotonicity of $f$, we have $\frac{q}{2} \geq 100f\left(\frac{2p}{q}\right) \geq 100f\left(\frac{2(p/3)}{q/2}\right)$, and thus, $\frac{q}{2} \geq T_{50}\left(\frac{p}{3}\right)$. Hence, Lemma 3.1 implies

$$\tau(\mathcal{F} \setminus S) \leq HD_G\left(\frac{p}{3}, \frac{q}{2}\right) \leq \frac{p}{3} - \frac{q}{2} + 1 + \frac{1}{50} \cdot \frac{p}{3} = 0.34p - 0.5q + 1,$$

whence $\mathcal{F} \setminus S$ can be pierced by $0.34p - 0.5q + 1$ points.

To pierce the “bad” subfamily $S$, we use Lemma 2.6, which implies that $S$ can be pierced by

$$\left\lfloor \frac{|S| + \lambda}{2} \right\rfloor \leq \frac{p}{3} + 0.005p \leq 0.34p.$$

Therefore, $\mathcal{F}$ can be pierced by $0.68p - 0.5q + 1$ points. As in Case 1, we may argue that either $0.68p - 0.5q + 1 < p - q + 1$ and we are done, or $q \geq \frac{p}{2} + 1$ and then we are done by Proposition 3.2. This completes the inductive step.

To conclude the proof, we need the induction basis. The idea is to show that for

$$q_0 = \min\{q : \text{there exists a “relevant” pair } (p, q)\}$$

(where “relevant” means a pair $(p, q)$ that belongs to the range covered by the theorem), for all relevant pairs $(p, q_0)$ we have $p \leq 2q_0 - 2$, and thus $HD_G(p, q_0) = p - q_0 + 1$ holds by Proposition 3.2. This is a sufficient basis, since in the inductive process, $q$ is decreased by 1 in each step, and so we will eventually reduce to $q = q_0$, for which the assertion holds. Note that we cannot move from $(p, q)$ to $(p', q - 1)$ such that $p' < q - 1$, since this would mean that the family contains an independent set of size $\geq p - q + 2$; completing such a set to $p$ elements by adding $\leq q - 2$ arbitrary elements, we obtain a subfamily of $\mathcal{F}$ of size $p$ without an intersecting $q$-tuple, a contradiction.

By Claim 3.3-2, $T_c(p)$ is increasing in $p$, and thus, if for some $q$ there exists a $p$ such that $q \geq T_c(p)$, then we also have $q \geq T_c(q)$. Hence, for each $q$, the smallest $p$ for which $(p, q)$ lies in the range covered by the theorem is $q$ itself. Consequently, the smallest $q$ for which there exists a “relevant” $(p, q)$ is equal to the smallest $p$ for which there exists a “relevant” $(p, q)$. Denote this value by $q_0$. By its minimality, we have $q_0 - 1 < T_{100}(q_0 - 1)$. Therefore, by Claim 3.3 (3a) we have $q_0 < T_{100}(2q_0 - 1)$. (Note that in order to apply the claim, we need $c \geq \frac{1}{2}\log_2(k-1)/k$, where $k$ is a lower bound on $T_c(q_0 - 1)$. This indeed holds for $c = 100$, as we can take $k = 200$ as a lower bound, as mentioned above.) As $T_c(p)$ is increasing in $p$, this implies that $\{p : T_{100}(p) = q_0\} \subset\{q_0, q_0 + 1, \ldots, 2q_0 - 2\}$. Therefore, by Proposition 3.2, we have $HD_{\mathcal{F}}(p, q_0) = p - q_0 + 1$ for all “relevant” $(p, q_0)$. This completes the proof of the induction basis, and hence the proof of the theorem. 

□
3.4 Proof of Theorem 1.7

We conclude this section with the simple deduction of Theorem 1.7 from Theorem 1.4. Let us recall the statement of the theorem.

**Theorem 1.7** \( HD_{d}\text{-box}(p, q) = p - q + 1 \) holds for all \( q > c \log_{2}^{d-1}(p) \), where \( c \) is a universal constant.

**Proof** The \((p, 2)\)-theorem for axis-parallel boxes in \( \mathbb{R}^{d} \) \cite{19} yields \( HD_{d}\text{-box}(p, 2) \leq O(p \log_{2}^{d-1}(p)) \), which means that \( HD_{d}\text{-box}(p, 2) \leq pf(p) \) holds for \( f(p) = c' \log_{2}^{d-1}(p) \) (where \( c' \) is a universal constant). For this \( f(p) \), we have \( T_{100}(p) \leq 200c' \log_{2}^{d-1}(p) \).

Hence, the assertion of Theorem 1.7 will follow from Theorem 1.4, once we verify that \( f(p) \) satisfies the conditions of the theorem. The condition regarding \( f'(p) \) is clearly satisfied: we have \( f'(p) = c'(d - 1) \log_{2}^{d-2}(p) \log_{2} e/p \geq \log_{2} e/p \), for all \( p \geq 2 \) and \( c' \geq 1 \). As for the condition regarding \( f''(p)/f(p) \), we observe that in the proof of Theorem 1.4, this condition is applied only for values of \( p \) for which there exists a “relevant” pair \((p, q)\), and thus, it is sufficient to show that it holds for all such values. We have

\[
\frac{f'(p)}{f(p)} = \frac{c'(d - 1) \log_{2}^{d-2}(p) \log_{2} e}{c' p \log_{2}^{d-1}(p)} = \frac{(d - 1) \log_{2} e}{p \log_{2} p},
\]

and so we have to show that

\[
\frac{(d - 1) \log_{2} e}{p \log_{2} p} \leq \frac{5}{p}. \tag{4}
\]

This indeed holds in all the required range, since if there exists a “relevant” pair \((p, q)\) then \( p \geq 200 \log_{2}^{d-1}(p) \), and thus, \( \log_{2} p \geq (d - 1) \log_{2} \log_{2} p \geq d - 1 \), which clearly implies (4). This completes the proof. \( \square \)

4 From \((2, 2)\)-Theorems to \((p, 2)\)-Theorems

As was mentioned in the introduction, in general, the existence of a \((2, 2)\)-theorem (and even Helly number 2) does not imply the existence of a \((p, 2)\)-theorem.

An example of this phenomenon is the class \( G \) of all axis-parallel boxes (with no restriction on the dimension). It is easy to show that \( G \) has Helly number 2. On the other hand, we show now that \( G \) does not admit a \((3, 2)\)-theorem, using an example presented by Fon-der-Flaass and Kostochka \cite{13} in a slightly different context.

The example uses a classical result of Erdős \cite{12} which asserts that for any \( m \in \mathbb{N} \), there exists an \( m \)-chromatic triangle-free graph \( G_{m} \) on \( n(m) = O(m^{2} \log^{2} m) \) vertices.

Since any graph on \( n \) vertices can be represented as the intersection graph of a family of axis-parallel boxes in \( \mathbb{R}^{\lfloor n/2 \rfloor} \), the complement graph \( \overline{G}_{m} \) of \( G_{m} \) can be represented as the intersection graph of some family \( \mathcal{F} \) of axis-parallel boxes in \( \mathbb{R}^{n'} \), for \( n' = \lceil n/m \rceil = O(m^{2} \log^{2} m) \). The family \( \mathcal{F} \) satisfies the \((3, 2)\)-property, since if some three elements of \( \mathcal{F} \) are pairwise disjoint, then the intersection graph of \( \mathcal{F} \) contains an empty triangle, and this cannot happen since the intersection graph \( \overline{G}_{m} \).
is the complement of a triangle-free graph. On the other hand, $\mathcal{F}$ cannot be pierced by fewer than $m$ points, as any transversal of $\mathcal{F}$ of size $k$ induces a partitioning of the vertices of $G_m$ into $k$ cliques, which in turn yields a $k$-coloring of $G_m$ (which was assumed to be $m$-chromatic). Therefore, we have $HD_{G}(3, 2) \geq m$ although the Helly number of $\mathcal{G}$ is 2.

In this section we prove Theorem 1.9 which asserts that for any class of convex sets with Helly number 2, a $(2, 2)$-theorem does imply a $(p, 2)$-theorem, and consequently, a tight $(p, q)$-theorem for a large range of $q$’s. In addition, we prove Theorem 1.10 which provides $(p, 2)$-theorems for classes of compact convex sets that admit a $(2, 2)$-theorem, even if the piercing number guaranteed by the $(2, 2)$-theorem is larger than 1 (and thus, the class does not have Helly number 2). We note, however, that in these cases the $(p, 2)$-theorem cannot be leveraged to a tight $(p, q)$-theorem using the method we presented in Sect. 3.

### 4.1 Proof of Theorem 1.9

Let us recall the assertion of the theorem:

**Theorem 1.9** Let $\mathcal{G}$ be a class of compact convex sets in $\mathbb{R}^d$ that has Helly number 2. Then $HD_{G}(p, 2) \leq p^{2d-1}/2^{d-1}$, and consequently, $HD_{G}(p, q) = p - q + 1$ holds for all $q > cp^{d-1}/2^{d-1}$, where $c = c(d)$ is a constant depending only on the dimension $d$.

The “consequently” part follows immediately from the $(p, 2)$-theorem via Theorem 1.4. Hence, we only have to prove the $(p, 2)$-theorem.

Let us present the proof idea first. The proof goes by induction on $d$. Given a class $G_d$ of compact convex sets in $\mathbb{R}^d$ that has Helly number 2, and a family $\mathcal{F} \subset G_d$ that has the $(p, 2)$-property, we take $S$ to be a maximum (with respect to size) pairwise-disjoint subfamily of $\mathcal{F}$, and consider the intersections of other sets of $\mathcal{F}$ with the elements of $S$. We observe that by the maximality of $S$, each set $A \in \mathcal{F} \setminus S$ intersects at least one element of $S$, and thus, we may partition $\mathcal{F}$ into three subfamilies: $S$ itself, the family $\mathcal{U}$ of sets in $\mathcal{F} \setminus S$ that intersect only one element of $S$, and the family $\mathcal{M} \subset \mathcal{F} \setminus S$ of sets that intersect at least two elements of $S$.

We show (using the maximality of $S$ and the $(2, 2)$-theorem on $G_d$) that $\mathcal{U} \cup S$ can be pierced by $p - 1$ points. As for $\mathcal{M}$, we represent it as a union of families: $\mathcal{M} = \bigcup_{C, C' \in S} \mathcal{X}_{C, C'}$, where each $\mathcal{X}_{C, C'}$ consists of the elements of $F \setminus S$ that intersect both $C$ and $C'$. We use a geometric argument to show that each $\mathcal{X}_{C, C'}$ corresponds to a family $\mathcal{Y}_{C, C'} \subset \mathbb{R}^{d-1}$ which satisfies the $(p, 2)$-property and is included in a class $G_{d-1}$ of compact convex sets in $\mathbb{R}^{d-1}$ that has Helly number 2. This allows us to bound the piercing number of $\mathcal{Y}_{C, C'}$ by the induction hypothesis, and consequently, to bound the piercing number of $\mathcal{X}_{C, C'}$. Adding up the piercing numbers of all $\mathcal{X}_{C, C'}$’s and the piercing number of $\mathcal{U} \cup S$ completes the inductive step.

**Proof of Theorem 1.9** By induction on $d$.

**Induction Basis** For any class $\mathcal{G}$ of compact convex sets in $\mathbb{R}^1$, by the Hadwiger–Debrunner theorem [16] we have $HD_{G}(p, 2) = p - 2 + 1 < p = p^{2d-1}/2^{d-1}$, and so the assertion holds.
**Inductive Step** Let \( G_d \) be a class of compact convex sets in \( \mathbb{R}^d \) that has Helly number 2 and let \( F \subset G_d \) be a family that has the \((p, 2)\)-property. Let \( S \) be a maximum (with respect to size) pairwise-disjoint subfamily of \( F \). We may assume \(|S| = p - 1\).

By the maximality of \( S \), each set \( A \in F \setminus S \) intersects at least one element of \( S \). Moreover, any two sets \( A, B \in F \) that intersect the same \( C \in S \) and do not intersect any other element of \( S \) are intersecting, as otherwise, the subfamily \( S \cup \{A, B\} \setminus \{C\} \) would be a pairwise-disjoint subfamily of \( F \) that is larger than \( S \), a contradiction. Hence, for each \( C_0 \in S \), the subfamily

\[
\mathcal{X}_{C_0} = \{A \in F : \{C \in S : A \cap C \neq \emptyset\} = \{C_0\}\} \cup \{C_0\}
\]

satisfies the \((2, 2)\)-property, and thus, can be pierced by a single point by the assumption on \( G_d \). Therefore, denoting \( U = \{A \in F : |\{C \in S : A \cap C \neq \emptyset\}| = 1\} \), all sets in \( U \cup S \) can be pierced by at most \( p - 1 \) points.

Let \( M \subset F \) be the family of all sets in \( F \) that intersect at least two elements of \( S \). For each \( C, C' \in S \), let

\[
\mathcal{X}_{C, C'} = \{A \in F \setminus S : A \cap C \neq \emptyset \land A \cap C' \neq \emptyset\}.
\]

(Note that the elements of \( \mathcal{X}_{C, C'} \) may intersect other elements of \( S \).)

Let \( H \subset \mathbb{R}^d \) be a hyperplane that strictly separates \( C \) from \( C' \). Consider the class

\[
G_{d-1} = G_{d-1}(C, C', H) = \{A \cap H : (A \in G_d) \land (A \cap C \neq \emptyset) \land (A \cap C' \neq \emptyset)\}.
\]

It is clear that \( G_{d-1} \) consists of compact convex subsets of \( H \), that can be viewed as residing in \( \mathbb{R}^{d-1} \). Consider the family \( \mathcal{Y}_{C, C'} \subset G_{d-1} \) defined by \( \mathcal{Y}_{C, C'} = \{A \cap H : A \in \mathcal{X}_{C, C'}\} \).

**Claim 4.1** The class \( G_{d-1} \) has Helly number 2, and the family \( \mathcal{Y}_{C, C'} \subset G_{d-1} \) satisfies the \((p, 2)\)-property.

**Proof** To prove the claim, we observe that \( A \cap H, A' \cap H \in G_{d-1} \) intersect if and only if \( A \) and \( A' \) intersect. Indeed, assume \( A \cap A' \neq \emptyset \). The family \( \{A, A', C\} \) satisfies the \((2, 2)\)-property, and hence, can be pierced by a single point by the assumption on \( G_d \).

Thus, \( A \cap A' \) contains a point of \( C \). For the same reason, \( A \cap A' \) contains a point of \( C' \). Therefore, \( A \cap A' \) contains points on the two sides of the hyperplane \( H \). However, \( A \cap A' \) is convex, and so, \((A \cap A') \cap H \neq \emptyset \), which means that \((A \cap H)\) and \((A' \cap H)\) intersect. The other direction is obvious.

It is now clear that since \( \mathcal{X}_{C, C'} \subset F \) satisfies the \((p, 2)\)-property, then \( \mathcal{Y}_{C, C'} \) satisfies the \((p, 2)\)-property as well. Moreover, let \( T = \{A_1 \cap H, A_2 \cap H, A_3 \cap H, \ldots\} \subset G_{d-1} \) be pairwise-intersecting. The corresponding family \( \tilde{T} = \{C, A_1, A_2, A_3, \ldots\} \) is pairwise-intersecting, and thus, can be pierced by a single point by the assumption on \( G_d \). Thus, \((A_1 \cap A_2 \cap A_3 \cap \cdots) \cap C \neq \emptyset \). For the same reason, \((A_1 \cap A_2 \cap A_3 \cap \cdots) \cap C' \neq \emptyset \). Since \( A_1 \cap A_2 \cap A_3 \cap \cdots \) is convex, this implies that \((A_1 \cap A_2 \cap A_3 \cap \cdots) \cap H \neq \emptyset \), or equivalently, that the family \( T \) can be pierced by a single point. Therefore, \( G_{d-1} \) satisfies \( \text{HD}_{G_{d-1}}(2, 2) = 1 \), as asserted.

\( \square \)
Claim 4.1 allows us to apply the induction hypothesis to $Y_C, C'$, to deduce that it can be pierced by fewer than $p^{2d-3}/2^{d-2}$ points. Since $S$ contains only $\binom{p}{2}$ pairs $(C, C')$, and since any set in $M$ belongs to at least one of the $X_C, C'$, this implies that $M$ can be pierced by fewer than $\binom{p-1}{2} \cdot p^{2d-3}/2^{d-2}$ points. As $U \cup S$ can be pierced by $p-1$ points as shown above, $F$ can be pierced by fewer than $(p-1) < p^{2d-1}/2^{d-1}$ points. This completes the proof. □

4.2 Proof of Theorem 1.10

Let us restate Theorem 1.10 in a more precise form.

**Theorem 1.10** (Precise formulation) Let $G$ be a class of compact convex sets in $\mathbb{R}^d$ such that $\text{HD}_G(2, 2) = t$. Then:

1. We have
   $$\text{HD}_G(p, 2) = \tilde{O}\left(4^{pd} \frac{(p/2)^t-1}{(p/2)^t-d}\right).$$
   In particular, $G$ admits a $(p, 2)$-theorem for piercing with a bounded number $s = s(p, d, t)$ of points.

2. If $d = 2$, we have $\text{HD}_G(p, 2) = O_1(p^8 \log^2 p)$.

3. If $d = 2$ and the VC-dimension of $G$ is $k$, then $\text{HD}_G(p, 2) = O_{1,k}(p^4 \log^2 p)$.

Two remarks are due at this point.

**Remark 4.2** The difference between the general case (Part 1 of the theorem) and the planar case (Parts 2,3 of the theorem) looks surprisingly huge. We do not know whether any of these results are tight; however, the difference is well-explained by the proof method. While in the proof of Parts 2,3 we use a Ramsey-type theorem for families of convex sets in the plane of Larman et al. [23] (Theorem 4.4 below) in which the “Ramsey number” $R(k)$ is polynomial in $k$, in the general case we have to resort to the classical Ramsey theorem in which $R(k)$ is exponential in $k$. This is in a sense necessary, since Tietze [28] and (independently) Besicovitch [6] showed that any graph can be represented as the intersection graph of a family of convex sets in $\mathbb{R}^3$, which implies that no “Ramsey theorem for convex sets in $\mathbb{R}^d$ for $d \geq 3$ can improve over the classical Ramsey theorem.

**Remark 4.3** As mentioned in the introduction, Matoušek [24] showed that classes of sets with dual VC-dimension $k$ have fractional Helly number at most $k + 1$. This allows deducing that such classes admit a $(p, k)$-theorem, using the proof technique of the Alon–Kleitman $(p, q)$-theorem. This result (which applies in a much more general setting than Part 3 of our theorem) does not imply our theorem, since it yields a $(p, 2)$-theorem only for classes with dual VC-dimension 1, while our theorem applies whenever the VC-dimension is bounded.
The proof of the theorem is a combination of three tools.

The first is a Ramsey-type theorem. Recall that the classical Ramsey theorem \([26]\) asserts that for any \(k\), there exists \(R(k)\) such that any graph on \(R(k)\) vertices contains either a complete subgraph on \(k\) vertices or an empty subgraph on \(k\) vertices. Ramsey showed that \(R(k) \leq \binom{2k-2}{k-1} \leq 4^k\). As the best currently known upper bound is not much lower, we use the upper bound \(R(k) \leq 4^k\) for the sake of simplicity.

The Ramsey theorem implies that any family of \(R(k)\) sets contains either a pairwise-intersecting subfamily of size \(k\) or a pairwise-disjoint subfamily of size \(k\). Larmeanet \([23]\) showed that for families of compact convex sets in the plane, a significantly better result can be achieved.

**Theorem 4.4** \([23]\) Let \(\mathcal{F}\) be a family of \(k\ell^d\) compact convex sets in the plane. Then \(\mathcal{F}\) contains either \(k\) pairwise-intersecting sets or \(\ell\) pairwise-disjoint sets.

The second result is a quantitative bound for the Alon–Kleitman \((p, q)\)-theorem obtained in \([20, Theorem 1.3]\):

**Theorem 4.5** \([20]\) We have

\[
HD_d(p, q) \leq \begin{cases} 
O \left( p^{d \frac{q-1}{q-d}} \log^{cd^3} p \right) & \text{for all } q \geq d + 1; \\
\tilde{O} \left( p + \left( \frac{p}{q} \right)^d \right) & \text{if } q \geq \log p.
\end{cases}
\]

(c) Furthermore, for \(d = 2\), the bound in (b) can be replaced by \(p - q + O \left( \left( \frac{p}{q} \right)^2 \log^2 \left( \frac{p}{q} \right) \right)\).

The third result is the \(\epsilon\)-net theorem for families with a bounded VC-dimension. Let us recall a few definitions.

For a set system \((U, R)\), where \(U\) is a set of points and \(R \subset \mathcal{P}(U)\) is a set of ranges (or alternatively, a hypergraph \((U, R)\) in which \(U\) is the set of vertices and \(R\) is the set of hyperedges), we say that a set \(Y \subset U\) is shattered by \(R\) if every subset of \(Y\) can be obtained as the intersection of \(Y\) with some range \(e \in R\). The VC-dimension of \(R\) is the maximal size of a set \(Y\) that is shattered by \(R\). Similarly, for a class \(G\), the VC-dimension of \(G\) is the maximal size of a set of points that is shattered by elements of \(G\). For example, any set of three non-collinear points in the plane can be shattered by halfplanes, but no four points can be. Hence, the VC-dimension of the class of all halfplanes in the plane is 3. This notion was introduced by Vapnik and Chervonenkis \([29]\).

An \(\epsilon\)-net of \((U, R)\) is a subset \(S \subset U\), such that any range \(e \in R\) that contains at least \(\epsilon\)-fraction of the elements of \(U\), intersects \(S\).

The \(\epsilon\)-net theorem of Haussler and Welzl \([18]\) asserts the following:

**Theorem 4.6** (The \(\epsilon\)-net theorem \([18]\)) Let \((U, R)\) be a range space of VC-dimension \(k\), let \(A\) be a finite subset of \(U\) and suppose \(0 < \epsilon, \delta < 1\). Let \(N\) be a set obtained by \(m\) random independent draws from \(A\), where

\[
m \geq \max \left( \frac{4}{\epsilon} \log \frac{2}{\delta}, \frac{8k}{\epsilon} \log \frac{8k}{\epsilon} \right).
\]
Then $N$ is an $\varepsilon$-net for $A$ with probability at least $1 - \delta$.

In particular, any family with VC-dimension $k$ admits an $\varepsilon$-net of size $O_k\left(\frac{1}{k} \log \frac{1}{\varepsilon}\right)$.

Now we are ready to present the proof of the theorem.

**Proof of Theorem 1.10 Part 1** As any family that satisfies the $(p, q)$-property, clearly satisfies the $(p', q)$-property for all $p' \geq p$ (if it contains at least $p'$ sets), and as we are not interested in constant factors, we may assume that $p$ is larger than any prescribed constant; in particular, we may assume $p \geq (d + 1)t$. Let $\mathcal{F} \subset \mathcal{G}$ be a family that has the $(p, 2)$-property. We claim that $\mathcal{F}$ satisfies the $(4p, \lceil p/t \rceil)$-property.

To prove this, let $S$ be a subfamily of $\mathcal{F}$ of size $4p$. We have to show that $S$ contains an intersecting $\lceil p/t \rceil$-tuple.

By the Ramsey theorem [26], either $S$ contains $p$ pairwise intersecting sets, or else it contains $p$ pairwise disjoint sets. The latter is impossible since $\mathcal{F}$ satisfies the $(p, 2)$-property. Hence, $S$ contains a pairwise intersecting subfamily $T$ of size $p$. As $T$ satisfies the $(2, 2)$-property, by the assumption on $\mathcal{G}$ it can be pierced by $t$ points. The largest among the subsets of $T$ pierced by a single point is of size $\geq \lceil p/t \rceil$, and so, $S$ contains an intersecting $\lceil p/t \rceil$-tuple, as asserted.

Since $\lceil p/t \rceil \geq d + 1$ by assumption, we can apply Theorem 4.5 (a) to $\mathcal{F}$ to deduce that

$$\tau(\mathcal{F}) = \tilde{O}\left((4p)^d \frac{(\lceil p/t \rceil - 1)}{(\lceil p/t \rceil - d)}\right) = \tilde{O}\left(4p^d \frac{(\lceil p/t \rceil - 1)}{(\lceil p/t \rceil - d)}\right),$$

completing the proof.

**Part 2** As in the proof of Part 1, we may assume that $p$ is sufficiently large so that $p/\log p > 5t$. Let $\mathcal{F} \subset \mathcal{G}$ be a family that has the $(p, 2)$-property. Applying the same argument as in Part 1, with Theorem 4.4 instead of the Ramsey theorem, we deduce that $\mathcal{F}$ satisfies the $(p^5, \lceil p/t \rceil)$-property.

Since $p/t \geq \log(p^5)$ by assumption, we can apply to $\mathcal{F}$ Theorem 4.5 (c) to deduce that

$$\tau(\mathcal{F}) = O\left(p^5 + \left(\frac{p^5}{p/t}\right)^2 \left(\log \left(\frac{p^5}{p/t}\right)\right)^2\right) = O(p^8 \log^2 p),$$

completing the proof.

**Part 3** Let $\mathcal{F} \subset \mathcal{G}$ be a family that has the $(p, 2)$-property. From Theorem 4.4 we can deduce that for any $q$, $\mathcal{F}$ satisfies the $(tp^4q, q)$-property. By [20, Proposition 2.3], this implies that there exists a point that pierces a $\Omega\left(\frac{q}{(tp^4q)^{q/(q-1)(q-2)}}\right)$-fraction of the sets in $\mathcal{F}$. By the proof method of [20, Proposition 2.6] (with a different value of $\beta$), this implies that $\mathcal{F}$ can be pierced by $f(\beta)$ points, where $\beta = \Omega\left(\left((tp^4q)^{-q/(q-1)(q-2)}\right)\right)$ and $f(\beta)$ is the size of the minimal weak $\varepsilon$-net guaranteed by the weak $\varepsilon$-net theorem [1] in the plane for $\varepsilon = \beta$.

Since by assumption, the VC-dimension of $\mathcal{F}$ is bounded by $k$, $\mathcal{F}$ admits an $\varepsilon$-net of size $O_k\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$ by Theorem 4.6. Hence, we can replace the application of the weak
\(\varepsilon\)-net theorem in the above argument with an application of Theorem 4.6. Substituting \(q = \log p\), which is easily seen to be (roughly) optimal, we obtain
\[
\tau(F) = O_k \left( (t^4 \log p)^{\frac{\log p - 1}{\log p - 2}} \cdot \log \left( (t^4 \log p)^{\frac{\log p - 1}{\log p - 2}} \right) \right) = O_{k,i}(p^4 \log^2 p),
\]
as asserted.

\(\square\)

5 Discussion and Open Problems

A central problem left for further research is whether Theorem 1.4 which allows leveraging a \((p, 2)\)-theorem into a \((p, q)\)-theorem, can be extended to the cases \(\text{HD}_G(p, 2) = pf(p)\) where \(f(p) \ll \log p\) or \(f(p)\) being super-polynomial in \(p\). It seems that super-polynomial growth rates can be handled with a slight modification of the argument (at the expense of replacing \(T_{c'}(p)\) with some worse dependence on \(p\)). For a sub-logarithmic growth rate, it seems that the current argument does not work, since the inductive step requires the packing number of \(F\) to be extremely large, and so, Lemma 2.6 allows reducing the piercing number of the “bad” family \(S\) only slightly, rendering Lemma 2.5 insufficient for piercing \(F\) with \(p - q + 1\) points in total. Extending the method for sub-logarithmic growth rates will have interesting applications. For instance, it will immediately imply that any family of axis-parallel boxes that satisfies the \((p, q)\)-property for some \(q = \Omega(\log \log p)\) and the additional condition that for each pair of intersecting boxes, a corner of one is contained in the other, can be pierced by \(p - q + 1\) points, following the work of Chudnovsky et al. [8]. Furthermore, it will imply that if the class of axis-parallel rectangles admits a \((p, 2)\)-theorem with the size of the piercing set linear in \(p\) (as conjectured by Wegner [30]), then \(\text{HD}_{\text{rect}}(p, q) = p - q + 1\) holds for all \(q \geq c\) for a constant \(c\). As remarked by Eckhoff [11], this was claimed by Dol’nikov [9], but with a flawed argument.

Another open problem is whether the method can be extended to classes \(G\) that admit a \((2, 2)\)-theorem, but satisfy \(\text{HD}_G(2, 2) > 1\). Such an improvement would allow transformation into tight \((p, q)\)-theorems of the \((p, 2)\)-theorems presented in Sect. 1.3, such as the \((p, 2)\)-theorem for pseudo-discs of Chan and Har-Peled [7]. (Note that one cannot hope to obtain the lowest possible value \(\text{HD}_G(p, q) = p - q + 1\) for such classes in general, as already for \(p = q = 2\) the assumption says \(\text{HD}_G(2, 2) > 1 = 2 - 2 + 1\).)

A main obstacle here is that in this case, Observation 1.8 does not apply, and instead, we have the bound \(\tau(F) \leq \text{HD}_G(p - \lambda, q - 1) + \lambda\). While the bound is only slightly weaker, it precludes us from using the inductive process of Wegner and Dol’nikov, as in each application of the inductive step we have an “extra” point.

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