Quantum Mechanics on a Poincaré Hyperboloid

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Abstract

We discuss the process to obtain the Poisson brackets among the phase space variables of the system of a free particle on a Poincaré Hyperboloid. We show that after quantization the Dirac bracket algebra forms the algebra of ISO(1,2). The representation of this algebra is explicitly analyzed.

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I. INTRODUCTION

A Poincaré hyperboloid, as a mathematical object, has many peculiarities appealing to physicists’ interests. It is a maximally symmetric and curved two-dimensional space. Unlike a sphere which is also a maximally symmetric and curved two-dimensional space, a Poincaré hyperboloid is an open space with a negative scalar curvature. A Poincaré hyperboloid can be pictured as an embedded manifold in a three-dimensional Minkowski space with the embedding equation \( x^2 + y^2 - z^2 + a^2 = 0 \). This embedded manifold is endowed with the induced metric which gives a negative Ricci scalar. Another aspect of a Poincaré hyperboloid is that all the Riemann surfaces with genus equal to a greater than two can be obtained by suitable identifications of points of this Poincaré hyperboloid\(^1\).

In this work, we analyze the quantum structure of a Poincaré hyperboloid. Starting from the Poisson brackets among the phase space variables of the three-dimensional Minkowski space and introducing the constraint \( x^2 + y^2 - z^2 + a^2 = 0 \), we derive the modified Poisson brackets among the phase space variables of a classical system of a free particle on a Poincaré hyperboloid. From these modified Poisson brackets we derive the Dirac brackets and show that the Dirac bracket algebra turns out to be the algebra of \( \text{ISO}(1,2) \), the Poincaré group on (1 + 2)-dimensional Minkowski space. This algebra has 6 generators \{\hat{x}, \hat{y}, \hat{z}, \hat{J}_1, \hat{J}_2, \hat{J}_3\}. There are also two constraints \( \hat{x}^2 + \hat{y}^2 - \hat{z}^2 = -a^2, \hat{x}\hat{J}_1 + \hat{y}\hat{J}_2 + \hat{z}\hat{J}_3 = 0 \) both of which are the Casimir operators of this algebra\(^2–4\). These two constraints reduce the dimension of the algebra from six to four which is the dimensionality of the phase space of the Poincaré hyperboloid.

We also give the representation of this algebra. Due to the 2nd constraint, the Hilbert space is spanned by the states with quantum numbers corresponding to the coordinates of the Poincaré hyperboloid. We explicitly derive the form of the Hamiltonian in this representation, which agrees with the known result\(^1, 5, 6\).

In the next section we consider a classical system of a free particle on a Poincaré hyperboloid. In section III we derive the Dirac bracket algebra and in section IV we give the representation of this algebra. Finally in section V we give a brief summary.
II. CLASSICAL MECHANICS ON A POINCARÉ HYPERBOLOID

We begin with a 3-dimensional Minkowski space $M^3$ with metric $g_{ij} = \text{diag}(1,1,-1)$. The coordinates of this space are $(x,y,z)$ and the metric is given by

$$ds^2 = dx^2 + dy^2 - dz^2.$$  \hfill (1)

The dynamics of a free particle in this space would be governed by the following Poisson bracket relations among the phase space variables of this theory

$$\{x^i, x^j\} = 0,$$  \hfill (2)

$$\{x^i, p_j\} = \delta^i_j,$$  \hfill (3)

$$\{p_i, p_j\} = 0,$$  \hfill (4)

and the Hamiltonian

$$H = \frac{1}{2m} p_i p_j g^{ij} = \frac{1}{2m} (p_x^2 + p_y^2 - p_z^2).$$  \hfill (5)

This Hamiltonian has no lower bound and the quantum version would be problematic.

We now introduce a constraint given by

$$C(x,y,z) = x^2 + y^2 - z^2 + a^2 = 0.$$  \hfill (6)

The sub-space satisfying this constraint is composed of two separate components. We choose the component with $z > 0$(FIG. 1). This space is called Poincaré hyperboloid. We choose

![FIG. 1: Poincaré Hyperboloid](image-url)
$(\theta, \phi)$ as the coordinates on this hyperboloid.

\[ x = a \cos \phi \sinh \theta, \quad (7) \]
\[ y = a \sin \phi \sinh \theta, \quad (8) \]
\[ z = a \cosh \theta. \quad (9) \]

The induced metric is given by

\[ ds^2_H = \tilde{g}_{ij} d\xi^i d\xi^j = a^2 \sinh^2 \theta d\phi^2 + a^2 d\theta^2. \quad (10) \]

Note that this metric is positive definite and the subspace is spacelike. Killing’s equation

\[ \nabla_i K_j + \nabla_j K_i = 0 \quad (11) \]

has three independent solutions.

\[ K_{(1)} = \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \coth \theta \frac{\partial}{\partial \phi}, \quad (12) \]
\[ K_{(2)} = -\cos \phi \frac{\partial}{\partial \theta} + \sin \phi \coth \theta \frac{\partial}{\partial \phi}, \quad (13) \]
\[ K_{(3)} = \frac{\partial}{\partial \phi}. \quad (14) \]

The dynamics of a particle on this hyperboloid is given by the following geodesic equations.

\[ \ddot{\xi}^i + \Gamma^i_{jk} \dot{\xi}^j \dot{\xi}^k = 0. \quad (15) \]

This equation can also be derived from the action

\[ S = \int \frac{1}{2} m \tilde{g}_{ij} \dot{\xi}^i \dot{\xi}^j dt. \quad (16) \]

Another way to describe the dynamics of a particle on the hyperboloid starts from the Poisson bracket relations given in eq\([24\). To derive the Poisson bracket relations among the phase space variables of the hyperboloid, we follow the standard procedure developed by Dirac\([7,10\). First we modify the Hamiltonian in eq\([5\) as

\[ \tilde{H} = H + \lambda C = \frac{p_x^2 + p_y^2 - p_z^2}{2m} + \lambda (x^2 + y^2 - z^2 + a^2) \quad (17) \]

and introduce $p_\lambda$ as the conjugate momentum of the Lagrange multiplier $\lambda$.

\[ \{\lambda, p_\lambda\} = 1. \quad (18) \]
The primary constraint $C_1 = p_\lambda = 0$ leads to a secondary constraint $C_2$ through

$$C_2 = \dot{C}_1 = \{C_1, \tilde{H}\} = 0,$$  \hspace{1cm} (19)

and we get $C_2 = -C$ with $C$ in eq(6). By imposing

$$\dot{C}_2 = \{C_2, \tilde{H}\} = 0,$$  \hspace{1cm} (20)

we get

$$C_3 = x^i p_i = -m \dot{C}_2 = 0.$$  \hspace{1cm} (21)

Similarly from

$$\dot{C}_3 = \{C_3, \tilde{H}\} = 0$$  \hspace{1cm} (22)

we get

$$C_4 = \tilde{H} + 2\lambda C_2 + \lambda a^2 = \frac{1}{2} \dot{C}_3 = 0.$$  \hspace{1cm} (23)

No further constraint appears because

$$\dot{C}_4 = 2\lambda \dot{C}_2 = -\frac{2\lambda}{m} C_3.$$  \hspace{1cm} (24)

We, therefore, have four constraints

$$C_1 = p_\lambda = 0,$$  \hspace{1cm} (25)

$$C_2 = z^2 - x^2 - y^2 - a^2 = 0,$$  \hspace{1cm} (26)

$$C_3 = x p_x + y p_y + z p_z = 0,$$  \hspace{1cm} (27)

$$C_4 = \tilde{H} + 2\lambda C_2 + \lambda a^2 = 0,$$  \hspace{1cm} (28)

to impose on the 8-dimensional phase space $\{(\lambda, x, y, z, p_\lambda, p_x, p_y, p_z)\}$. The subspace satisfying these constraints forms a 4-dimensional space which should be considered as the phase space of our system (a particle on a hyperboloid). The Poisson bracket relations should be modified in such a way that the four constraints are respected. Denoting the modified Poisson bracket by $\{\}_M$, the relation between $\{\}_M$ and the original Poisson bracket $\{}$ is given by

$$\{A, B\}_M = \{A, B\} - \{A, C_i\} M_i^j\{C_j, B\}$$  \hspace{1cm} (29)
where $M^{-1}$ is the inverse of the $4 \times 4$ matrix $M$ defined by

$$M_{ij} = \{C_i, C_j\}. \quad (30)$$

The matrix $M$ and its inverse $M^{-1}$ are as follows:

$$M = \begin{pmatrix} 0 & 0 & 0 & -a^2 \\ 0 & 0 & 2a^2 & 0 \\ 0 & -2a^2 & 0 & \frac{2p_i p_j}{m} \\ a^2 & 0 & -\frac{2p_i p_j}{m} & 0 \end{pmatrix}, \quad (31)$$

$$M^{-1} = \begin{pmatrix} 0 & \frac{p_i p_j}{ma} & 0 & \frac{1}{a^2} \\ -\frac{p_i p_j}{ma} & 0 & -\frac{1}{2a^2} & 0 \\ 0 & \frac{1}{2a^2} & 0 & 0 \\ -\frac{1}{a^2} & 0 & 0 & 0 \end{pmatrix}. \quad (32)$$

Note that the modified Poisson bracket relations automatically respect the constraints and we can safely put $C_i = 0$. The modified Poisson bracket relations are as follows:

$$\{x^i, x^j\}_M = 0, \quad (33)$$

$$\{x^i, p_j\}_M = \delta^i_j + \frac{1}{a^2} x^i x_j, \quad (34)$$

$$\{p_i, p_j\}_M = \frac{1}{a^2} (x_i p_j - x_j p_i). \quad (35)$$

Two variables $\lambda, p_\lambda$ disappear after solving the constraints $C_1$ and $C_4$ and we are left with only two constraints $C_2, C_3$ to be imposed on our 6 variables $(x^i, p_j)$. The Hamiltonian, after solving the constraints, regains its original form

$$H = \frac{1}{2m} (p_x^2 + p_y^2 - p_z^2). \quad (36)$$

This Hamiltonian on the constrained phase space has zero as its lower bound. In order to prove this, we solve constraints $C_1$ and $C_2$ and get

$$z = \sqrt{x^2 + y^2 + a^2}, \quad p_z = -\frac{x p_x + y p_y}{z}. \quad (37)$$

Substituting these into the Hamiltonian we obtain

$$H = \frac{p_x^2 + p_y^2}{2m} \left[ 1 - \frac{(x^2 + y^2) \cos^2 \theta}{x^2 + y^2 + a^2} \right]. \quad (38)$$
where the angle $\theta$ is defined by
\[ \cos \theta = \frac{x p_x + y p_y}{\sqrt{x^2 + y^2} \sqrt{p_x^2 + p_y^2}} \] (39)
which is bounded by $-1 \leq \cos \theta \leq 1$. It is obvious that the Hamiltonian in eq.(38) can never be negative and vanishes when $p_x = p_y = 0$.

The classical equation of motion is obtained by
\[ \dot{x}^i = \{x^i, H\}_M = \frac{p^i}{m}, \] (40)
\[ m \ddot{x}^i = \{p^i, H\}_M = \frac{p^2}{ma^2} x^i. \] (41)

We observe that $x^i \dot{x}^i = 0$, which assures the motion is confined in the hyperboloid and the force $F^i(= m \ddot{x}^i)$ is normal to the hyperboloid indicating that this force is the constraint force. It can be shown that the equation of motion is the geodesic on the hyperboloid.

Before quantizing this theory, we introduce new variables $J^i$ defined by
\[ J^i = \epsilon^{ijk} x^j p_k. \] (42)

Then $p_i$ can be obtained through
\[ p_i = \frac{1}{a^2} \epsilon_{ijk} x^j J^k. \] (43)

Here $\epsilon_{ijk}$ is defined by
\[ \epsilon_{123} = 1 = -\epsilon^{123}. \] (44)

With $(x^i, J^i)$ as new phase variables, we have
\[ \{x^i, x^j\}_M = 0, \] (45)
\[ \{J^i, x^j\}_M = -\epsilon^{ijk} x_k, \] (46)
\[ \{J^i, J^j\}_M = -\epsilon^{ijk} J_k, \] (47)

with constraints
\[ C_2 = -x^i x_i - a^2 = 0, \] (48)
\[ \bar{C}_3 = x^i J_i = 0. \] (49)

Note that $C_3 = x^i p_i = 0$ is automatically satisfied due to the relation in eq(43). New constraint $\bar{C}_3$ appears from the definition of $J^i$ (eq(42)). The Hamiltonian in these variables becomes
\[ H = \frac{J^i J_i}{2ma^2}. \] (50)
III. QUANTIZATION

The transition from classical to quantum physics is achieved by changing Poisson brackets to Dirac brackets as $[A, B] = i\hbar \{A, B\}_M$. We get

\[
\begin{align*}
[A, B] &= i\hbar \{A, B\}_M, \\
[\hat{x}_i, \hat{x}_j] &= 0, \\
[\hat{x}_i, \hat{x}_j] &= -i\hbar \epsilon^{ijk} \hat{x}_k, \\
[\hat{J}_i, \hat{x}_j] &= -i\hbar \epsilon^{ijk} \hat{J}_k.
\end{align*}
\]

We assume that $(\hat{x}^i, \hat{J}^j)$ are all hermitian. The constraints are

\[
\begin{align*}
\hat{C}_2 &= \hat{x}^i \hat{x}_i + a^2 = 0, \\
\hat{C}_3 &= \hat{x}^i \hat{J}_i = 0.
\end{align*}
\]

Two constraints are also hermitian. It can be checked that these constraints are compatible with the Dirac bracket relations. In other words,

\[
\begin{align*}
[\hat{x}^i, \hat{C}_i] &= [\hat{J}^i, \hat{C}_i] = 0.
\end{align*}
\]

The Hamiltonian is

\[
\hat{H} = \frac{\hat{J}^i \hat{J}_i}{2ma^2}
\]

Note here that momentum operator $\hat{p}_i$ is related to operators $\hat{x}^i, \hat{J}^j$ through

\[
\hat{p}_i = \frac{1}{a^2} \epsilon^{ijk} \hat{x}^j \hat{J}^k + \hat{J}_i \hat{x}_j = \frac{1}{a^2} \epsilon^{ijk} \hat{x}^j \hat{J}^k - \frac{i\hbar}{a^2} \hat{x}_i.
\]

The modification from eq.(43) is necessary to make $\hat{p}_i$ hermitian. The Dirac bracket relations among $(\hat{x}^i, \hat{p}_j)$ are as follows:

\[
\begin{align*}
[\hat{x}^i, \hat{x}^j] &= 0, \\
[\hat{x}^i, \hat{p}_j] &= i\hbar \delta^i_j + i\hbar \frac{1}{a^2} \hat{x}^i \hat{x}_j, \\
[\hat{p}_i, \hat{p}_j] &= i\hbar \frac{1}{a^2} (\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i).
\end{align*}
\]

We now observe that the algebra generated by $(\hat{x}^i, \hat{J}^j)$ is that of the Poincaré group $(ISO(1,2))$. The Casimir operators of this algebra are $\hat{x}_i \hat{x}^i$ and $\hat{x}_i \hat{J}^i$, which in our case are required to be $(-a^2)$ and zero respectively.
IV. REPRESENTATION

We choose \{\hat{x}^i\} as the generators of a maximal commuting subalgebra, and diagonalize them by the eigen kets \(|\vec{r}, q\rangle\) with q as extra quantum numbers to be fixed further.

\[
\hat{x}^i |\vec{r}, q\rangle = x^i |\vec{r}, q\rangle .
\]  

(62)

Because of the constraint \(\hat{x}_i \hat{x}^i + a^2 = 0\), the eigenvalue \(x^i\) should satisfy \(x^i x^i = -a^2\) and \(x^i\) corresponds to a point on the hyperboloid. The remaining three operators \{\hat{J}^i\} generate the Lorentz group \(SO(1,2)\) and this group acts on \(|\vec{r}, q\rangle\) in such a way that \(x^i\) undergoes the Lorentz transformation. In order to analyze how the extra quantum numbers \(q\) transform under the group action, we consider the eigen ket \(|(0,0,a), q\rangle\). We first note that the stability group of \((0,0,a)\) is the group generated by \(\hat{J}^3\). This subgroup, which is the group of the rotations around the z axis, is one-dimensional. It is represented by one real quantum number \(q\). We, therefore, have

\[
\exp \left( i \frac{\theta \hat{J}^3}{\hbar} \right) |(0,0,a), q\rangle = \exp \left( i \frac{\theta q}{\hbar} \right) |(0,0,a), q\rangle ,
\]  

(63)

or simply

\[
\hat{J}^3 |(0,0,a), q\rangle = q |(0,0,a), q\rangle .
\]  

(64)

We now impose the constraint \(\hat{x}^i \hat{J}_j = 0\) to determine the value of \(q\).

\[
\hat{x}^i \hat{J}_j |(0,0,a), q\rangle = a \hat{J}_3 |(0,0,a), q\rangle = aq |(0,0,a), q\rangle = 0 .
\]  

(65)

Therefore, the value of \(q\) should be zero and no other quantum number than \(\vec{r}\) appears in the representation. The representation space is spanned by \{|\vec{r}\}\rangle with \(\vec{r} = (x, y, z)\) on the hyperboloid and the coordinates \(x, y, z\) are given in eq(7-9). We introduce an inner product of two states \(|\vec{r}\rangle\) and \(|\vec{r}'\rangle\).

\[
<\vec{r}'|\vec{r}\rangle = \delta_{H}^{(2)}(\cosh \theta' - \cosh \theta) \delta(\phi' - \phi). 
\]  

(66)

Then the identity operator can be written as

\[
I = \int_H d^2V_H |\vec{r}\rangle <\vec{r}|, 
\]  

(67)

where \(d^2V_H = d(\cosh \theta) d\phi\) and the ranges of integration are \(0 \leq \cosh \theta < \infty\) and \(0 \leq \phi < 2\pi\).
With this inner product the representation space as Hilbert space is just the space of square integrable functions on the hyperboloid. A state $|\psi\rangle$ can be expressed as
\[
|\psi\rangle = \int_H d^2 V_H \bar{r} \langle \bar{r} | \psi \rangle = \int_H d^2 V_H \psi(\bar{r}) |\bar{r}\rangle.
\] (68)

Note here that $\psi(\bar{r})$ is a function of the coordinates of the hyperboloid.

We now proceed to find how $\hat{J}^i$ is represented in the basis $\{|\bar{r}\rangle\}$. Using eq.(52), we get
\[
\langle \bar{r}' | [\hat{J}^i, \hat{x}^j] | \psi \rangle = -i\hbar \epsilon^{ijk} x'_k \psi(\bar{r}).
\] (69)

The left hand side of this equation can be written as
\[
\text{LHS} = \int_H d^2 V_H \psi(\bar{r})(x^j - x'^j) < \bar{r}' | \hat{J}^i | \bar{r} \rangle.
\] (70)

The right hand side of eq.(69) can be written as
\[
\text{RHS} = -i\hbar \epsilon^{ijk} x'_k \int_H d^2 V_H \delta^j_i \delta^{(2)}(\bar{r}' - \bar{r}) \psi(\bar{r}).
\] (71)

In this expression, $\bar{r}$ and $\bar{r}'$ are the coordinates on the hyperboloid and the hyperbolic radius($r_H$) of $\bar{r}$ and $\bar{r}'$ are the same ($x^i x'_i = x^i x'_i = -a^2$).

\[
r_H(\bar{r}') = r_H(\bar{r}) = a.
\] (72)

We now extend the volume integral from two-dimensional hyperboloid $\int_H d^2 V_H$ to three-dimensional absolute future region of the origin $\int_M d^3 V$ and $r_H(\bar{r}) = r$. We have
\[
d^2 V_H \delta^{(2)}(\bar{r} - \bar{r}') = \int r^2 \frac{d^2 r}{a^2} d^2 V_H \delta^{(3)}(\bar{r} - \bar{r}') = \frac{1}{a^2} \int d^3 V \delta^{(3)}(\bar{r} - \bar{r}').
\] (73)

The extended function $\psi(\bar{r})$ is considered to be independent of the hyperbolic radius. Therefore, the right hand side of eq.(69) can be written as
\[
\text{RHS} = -i\hbar \epsilon^{ijk} x'_k \int d^3 V \delta^j_i \delta^{(3)}(\bar{r} - \bar{r}') \psi(\bar{r}).
\] (74)

From the identity
\[
(x^i - x'_0) \frac{\partial}{\partial x^j} \delta^{(3)}(\bar{r} - \bar{r}_0) = -\delta^j_i \delta^{(3)}(\bar{r} - \bar{r}_0),
\] (75)
the right hand side becomes
\[
\text{RHS} = -i\hbar \epsilon^{ijk} x'_k \int d^3 V (x^j - x'^j) \frac{\partial}{\partial x^i} \delta^{(3)}(\bar{r} - \bar{r}') \psi(\bar{r}).
\] (76)
We now integrate over the hyperbolic radius and obtain
\[ \text{RHS} = i \hbar \epsilon^{ikl} x^l \int_H d^2V_H (x^j - x'^j) \frac{\partial}{\partial x'^j} \delta^{(2)}(\vec{r} - \vec{r}') \psi(\vec{r}). \] (77)
Comparing this with eq. (70), we get
\[ < \vec{r}'|\hat{J}^i|\vec{r}> = i \hbar \epsilon^{ikl} x^l \frac{\partial}{\partial x^i} < \vec{r}'|\vec{r}> \] (78)
or
\[ < \vec{r}|\hat{J}^i|\psi > = i \hbar \epsilon^{ikl} x^l \frac{\partial}{\partial x^i} < \vec{r}|\psi>. \] (79)
Note that each \( \epsilon^{ikl} x^l \partial^i \) is a vector field on the hyperboloid for \( i = 1, 2, 3 \) and it contains no \( \frac{\partial}{\partial r} \) component. Explicit calculation gives us
\[ < \vec{r}|\hat{J}^1|\psi > = i \hbar \left( \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \coth \theta \frac{\partial}{\partial \phi} \right) \psi(\vec{r}), \] (80)
\[ < \vec{r}|\hat{J}^2|\psi > = i \hbar \left( -\cos \phi \frac{\partial}{\partial \theta} + \sin \phi \coth \theta \frac{\partial}{\partial \phi} \right) \psi(\vec{r}), \] (81)
\[ < \vec{r}|\hat{J}^3|\psi > = i \hbar \frac{\partial}{\partial \phi} \psi(\vec{r}), \] (82)
and we have
\[ \hat{J}^i = i \hbar K_{(i)} \] (83)
where \( K_{(i)} \) is defined in eq. (12-14). With these Schrödinger equation becomes
\[ < \vec{r}|\hat{H}|\psi_E > = -\frac{\hbar^2}{2ma^2} \left( \frac{1}{\sinh \theta} \frac{\partial}{\partial \theta} \sinh \theta \frac{\partial}{\partial \theta} + \frac{1}{\sinh^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \psi_E(\vec{r}) = E\psi_E(\vec{r}), \] (84)
and the solutions have been found [1, 5, 6].
\[ \hat{H}|\lambda,n> = E_{\lambda}|\lambda,n>. \] (85)
Here
\[ E_{\lambda} = \frac{\hbar^2}{2ma^2} (\lambda^2 + \frac{1}{4}) \] (86)
with \( \lambda \) being a non-negative real number and
\[ < \vec{r}|\lambda,n> = \psi^n_{\lambda}(\theta,\phi) = N_{\lambda}^n e^{i\phi} P_{i\lambda-\frac{1}{2}}^n (\cosh \theta) \] (87)
with \( n \in \mathbb{Z} \). The function \( P_{i\lambda}^n(x) \) is the conical function and the normalization factor \( N_{\lambda}^n \) is defined by
\[ N_{\lambda}^n = \left( \frac{2\pi}{\lambda \tanh(\pi \lambda)} \right)^{\frac{1}{2}} \frac{\Gamma(i\lambda + \frac{1}{2})}{\Gamma(i\lambda + m + \frac{1}{2})} \] (88)
[6, 11].
V. SUMMARY AND DISCUSSION

We derived the Poisson brackets among the phase space variables for the system of a free particle on the hyperboloid which is embedded in the 3-dimensional Minkowski space with $z$ as a time-like variable. The phase space variables are chosen to be $\{\hat{x}^i, \hat{J}^i\}$ with two constraints $x^i x_i - a^2 = 0$ and $x^i J_i = 0$. After quantization, we have shown that commutation relations among these variables given in eq.(51-53) from the algebra of ISO(1,2). The operators in the constraints, $\hat{x}^i \hat{x}_i$ and $\hat{x}^i \hat{J}_i$ are the Casimir operators of this algebra. We have also shown that the representation of this algebra with the Casimir operators fixed by the given constraints coincides with the previous results obtained by different method. We believe that own method can be used to analyze the system of a charged particle on the hyperboloid where background fields as applied.

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[1] N.L. Balazs, A. Voros, Phys. Rep. 143, Iss 3, 109 (1986).
[2] R. Jakiw, V. P. Nair, Phys. revD. 43, No. 6, 1933 (1991).
[3] Dmitri M Gitman and A L Shelepin, J. Phys. A Math. Gen. 30, 6093 (1997).
[4] Birne Binegar, J. Math. Phys. 23, 1511 (1982).
[5] A.C. Davis, A.J. Macfarlane and J.W. van Holten, Nucl. Phys. B216, 493 (1983).
[6] E N Argyres, C G Papadopoulos, E Papantonopoulos and K Tamvakis, J. Phys. A: Math. Gen. 22, 3577 (1989).
[7] P.A.M. Dirac. Lectures on Quantum Mechanics (Belfer Graduate School of Science, Yeshiva University, New York 1964).
[8] D. M. Gitman, I. V. Tyutin, Quantization of fields with Constraints (Springer-Verlag, Berlin, 1990).
[9] S. Weinberg, *The Quantum Theory of Fields Vol 1* (Cambridge University Press, 2005).

[10] S. Weinberg, *Lectures on Quantum Mechanics* (Cambridge University Press, 2012).

[11] Robin L, *Fonctions Spheriques de Legendre et Fonctions Spheroidnls* (Paris, Gauthier-Villars, 1957).