COMPACTNESS FOR THE $\overline{\partial}$ - NEUMANN PROBLEM - A FUNCTIONAL ANALYSIS APPROACH.

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Abstract.
We discuss compactness of the $\overline{\partial}$-Neumann operator in the setting of weighted $L^2$-spaces on $\mathbb{C}^n$. For this purpose we use a description of relatively compact subsets of $L^2$-spaces. We also point out how to use this method to show that property (P) implies compactness for the $\overline{\partial}$-Neumann operator on a smoothly bounded pseudoconvex domain and mention an abstract functional analysis characterization of compactness of the $\overline{\partial}$-Neumann operator.

1. Introduction.

In this paper we continue the investigations of [HaHe] concerning existence and compactness of the canonical solution operator to $\overline{\partial}$ on weighted $L^2$-spaces over $\mathbb{C}^n$. Let $\varphi : \mathbb{C}^n \to \mathbb{R}^+$ be a plurisubharmonic $C^2$-weight function and define the space

$$L^2(\mathbb{C}^n, \varphi) = \{ f : \mathbb{C}^n \to \mathbb{C} : \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda < \infty \},$$

where $\lambda$ denotes the Lebesgue measure, the space $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$ and the space $L^2_{(0,2)}(\mathbb{C}^n, \varphi)$ of $(0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$. Let

$$\langle f, g \rangle_{\varphi} = \int_{\mathbb{C}^n} f \overline{g} e^{-\varphi} d\lambda$$

denote the inner product and

$$\|f\|_{\varphi}^2 = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda$$

the norm in $L^2(\mathbb{C}^n, \varphi)$.

We consider the weighted $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n, \varphi) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\mathbb{C}^n, \varphi) \xrightarrow{\overline{\partial}} L^2_{(0,2)}(\mathbb{C}^n, \varphi),$$

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1
where $\overline{\partial}_\varphi$ is the adjoint operator to $\bar{\partial}$ with respect to the weighted inner product. For $u = \sum_{j=1}^n u_j dz_j \in \text{dom}(\overline{\partial}_\varphi)$ one has

$$\overline{\partial}_\varphi u = -\sum_{j=1}^n \left( \frac{\partial}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} \right) u_j.$$ 

The complex Laplacian on $(0, 1)$-forms is defined as

$$\Box_\varphi := \overline{\partial} \overline{\partial}^* + \overline{\partial}_\varphi \partial,$$

where the symbol $\Box_\varphi$ is to be understood as the maximal closure of the operator initially defined on forms with coefficients in $C_0^\infty$, i.e., the space of smooth functions with compact support.

$\Box_\varphi$ is a selfadjoint and positive operator, which means that

$$\langle \Box_\varphi f, f \rangle_\varphi \geq 0, \text{ for } f \in \text{dom}(\Box_\varphi).$$

The associated Dirichlet form is denoted by

$$(1.1) \quad Q_\varphi(f, g) = \langle \overline{\partial} f, \overline{\partial} g \rangle_\varphi + \langle \overline{\partial}_\varphi f, \overline{\partial}_\varphi g \rangle_\varphi,$$

for $f, g \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_\varphi)$. The weighted $\overline{\partial}$-Neumann operator $N_\varphi$ is - if it exists - the bounded inverse of $\Box_\varphi$.

We indicate that $f \in \text{dom}(\overline{\partial}_\varphi)$ if and only if

$$\sum_{j=1}^n \left( \frac{\partial f_j}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} f_j \right) \in L^2(\mathbb{C}^n, \varphi)$$

and that forms with coefficients in $C_0^\infty(\mathbb{C}^n)$ are dense in $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_\varphi)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|_\varphi^2 + \|\overline{\partial}_\varphi f\|_\varphi^2)^{1/2}$ (see [GaHa]).

Now we suppose that the lowest eigenvalue $\mu_\varphi$ of the Levi - matrix

$$M_\varphi = \left( \frac{\partial^2 \varphi}{\partial z_j \partial z_k} \right)_{j,k}$$

of $\varphi$ satisfies

$$\liminf_{|z| \to \infty} \mu_\varphi(z) > 0, \quad (*)$$

and mention the Kohn-Morrey formula:

$$(1.2) \quad \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi u\|_\varphi^2 = \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial z_k} \right|^2 e^{-\varphi} d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial z_k} u_j \overline{u}_k e^{-\varphi} d\lambda$$

from which we get

$$(1.3) \quad \int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial z_k} u_j \overline{u}_k e^{-\varphi} d\lambda \leq \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi u\|_\varphi^2,$$
hence for a plurisubharmonic weight function $\varphi$ satisfying (*), there is a $C > 0$ such that

$$\|u\|_\varphi^2 \leq C(\|\partial u\|_\varphi^2 + \|\partial^*_\varphi u\|_\varphi^2)$$

for each $(0,1)$-form $u \in \text{dom}(\partial) \cap \text{dom}(\partial^*_\varphi)$.

For the proof see [FS], [GaHa] or [Str].

Now it follows that there exists a uniquely determined $(0,1)$-form $N_{\varphi} u \in \text{dom}(\partial) \cap \text{dom}(\partial^*_\varphi)$ such that

$$\langle u, v \rangle_\varphi = Q_{\varphi}(N_{\varphi} u, v) = \langle \overline{\partial} N_{\varphi} u, \overline{\partial} v \rangle_\varphi + \langle \overline{\partial^*_\varphi} N_{\varphi} u, \overline{\partial^*_\varphi} v \rangle_\varphi,$$

and that

$$\|\overline{\partial} N_{\varphi} u\|_\varphi^2 + \|\overline{\partial^*_\varphi} N_{\varphi} u\|_\varphi^2 \leq C_1 \|u\|_\varphi^2$$

which means that

$$N_{1,\varphi} : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \longrightarrow \text{dom}(\partial) \cap \text{dom}(\partial^*_\varphi)$$

is continuous in the graph topology, as well as

$$\|N_{\varphi} u\|_\varphi^2 \leq C_2(\|\overline{\partial} N_{\varphi} u\|_\varphi^2 + \|\overline{\partial^*_\varphi} N_{\varphi} u\|_\varphi^2) \leq C_3 \|u\|_\varphi^2,$$

where $C_1, C_2, C_3 > 0$ are constants. Hence we get that $N_{\varphi}$ is a continuous linear operator from $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into itself (see also [ChSh]).

We will give a new proof of the main result in [HaHe] using a direct approach, see [B], Corollaire IV.26, where two conditions are given which imply that a subset of an $L^2$-space is relatively compact. The first of these conditions will correspond to Gårding’s inequality (see for instance [F], [GaHa]), and the second condition corresponds to our assumption on the lowest eigenvalue of the Levi matrix $M_\varphi$.

We indicate how to use this method to show that property (P) implies compactness for the $\overline{\partial}$-Neumann operator on a smoothly bounded pseudoconvex domain $\Omega \subset \subset \mathbb{C}^n$ and finally mention an abstract necessary and sufficient condition for the $\overline{\partial}$-Neumann operator to be compact.

2. Weighted Sobolev spaces

Now we define an appropriate Sobolev space and prove compactness of the corresponding embedding, for related settings see [BDH], [Jo], [KM].

**Definition 2.1.** Let

$$W^Q_{\varphi} = \{ u \in L^2_{(0,1)}(\mathbb{C}^n, \varphi) : \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial^*_\varphi} u\|_\varphi^2 < \infty \}$$

with norm

$$\|u\|_{Q_{\varphi}} = (\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial^*_\varphi} u\|_\varphi^2)^{1/2}.$$

**Remark:** $W^Q_{\varphi}$ coincides with the form domain $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial^*_\varphi})$ of $Q_\varphi$ (see [Ga], [GaHa]).
Proposition 2.2. Suppose that the weight function $\varphi$ is plurisubharmonic and that the lowest eigenvalue $\mu_\varphi$ of the Levi - matrix $M_\varphi$ satisfies

$$\lim_{|z| \to \infty} \mu_\varphi(z) = +\infty. \quad (**$$

Then the embedding

$$j_\varphi : \mathcal{W}_Q^\varphi \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$$

is compact.

Proof. For $u \in \mathcal{W}_Q^\varphi$ we have by \[1.3\]

$$\|\partial u\|_\varphi^2 + \|\partial^* u\|_\varphi^2 \geq \langle M_\varphi u, u \rangle_\varphi.$$  

This implies

(2.1)  

$$\|\partial u\|_\varphi^2 + \|\partial^* u\|_\varphi^2 \geq \int_{\mathbb{C}^n} \mu_\varphi(z) |u(z)|^2 e^{-\varphi(z)} d\lambda(z).$$

We show that the unit ball in $\mathcal{W}_Q^\varphi$ is relatively compact in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. For this purpose we use the following lemma, see for instance [B] Corollaire IV.26.

Lemma 2.3. Let $A$ be a bounded subset of $L^2(\mathbb{C}^n, \varphi)$. Suppose that

(i) for each $\epsilon > 0$ and for each $R > 0$ there exists $\delta > 0$ such that

$$\|\tau_h f - f\|_{L^2(\mathbb{B}_R, \varphi)} < \epsilon$$

for each $h \in \mathbb{C}^n$ with $|h| < \delta$ and for each $f \in A$, where $\tau_h f(z) = f(z + h)$ and $\mathbb{B}_R = \{z \in \mathbb{C}^n : |z| < R\}$;

(ii) for each $\epsilon > 0$ there exists $R > 0$ such that

$$\|f\|_{L^2(\mathbb{C}^n \setminus \mathbb{B}_R, \varphi)} < \epsilon$$

for each $f \in A$.

Then $A$ is relatively compact in $L^2(\mathbb{C}^n, \varphi)$.

Remark 2.4. Conditions (i) and (ii) are also necessary for $A$ to be relatively compact in $L^2(\mathbb{C}^n, \varphi)$ (see [B]).

First we show that condition (i) of Lemma 2.3 is satisfied in our situation. Let $u = \sum_{j=1}^n u_j \, dz_j$ be a $(0,1)$-form with coefficients in $\mathcal{C}^\infty_0$. For each $u_j$ and for $t \in \mathbb{R}$ and $h = (h_1, \ldots, h_n) \in \mathbb{C}^n$ let

$$v_j(t) := u_j(z + th).$$

Note that

$$|v'_j(t)| \leq |h| \left[ \sum_{k=1}^n \left( \left| \frac{\partial u_j}{\partial x_k} (z + th) \right|^2 + \left| \frac{\partial u_j}{\partial y_k} (z + th) \right|^2 \right) \right]^{1/2},$$

where $z_k = x_k + iy_k$, for $k = 1, \ldots, n$. By the fact that

$$u_j(z + h) - u_j(z) = v_j(1) - v_j(0) = \int_0^1 v'_j(t) \, dt$$

we can now estimate for $|h| < R$

$$\int_{\mathbb{B}_R} |\tau_h u_j(z) - u_j(z)|^2 e^{-\varphi(z)} \, d\lambda(z) = \int_{\mathbb{B}_R} |\tau_h (\chi_R u_j)(z) - \chi_R u_j(z)|^2 e^{-\varphi(z)} \, d\lambda(z)$$

\[
\leq |h|^2 \int_{\mathbb{B}_R} \left[ \sum_{k=1}^n \left( \left| \frac{\partial (\chi_R u_j)}{\partial x_k} (z + th) \right|^2 + \left| \frac{\partial (\chi_R u_j)}{\partial y_k} (z + th) \right|^2 \right) dt \right] e^{-\varphi(z)} \, d\lambda(z)
\]

\[
\leq C_{R,\varphi} |h|^2 \int_{\mathbb{B}_{3R}} \sum_{k=1}^n \left( \left| \frac{\partial (\chi_R u_j)}{\partial x_k} (z) \right|^2 + \left| \frac{\partial (\chi_R u_j)}{\partial y_k} (z) \right|^2 \right) e^{-\varphi(z)} \, d\lambda(z)
\]

for $j = 1, \ldots, n$ where $\chi_R$ is a $C^\infty$ cutoff function which is identically 1 on $\mathbb{B}_{2R}$ and zero outside $\mathbb{B}_{3R}$ and by Gårding’s inequality for $\mathbb{B}_{3R}$ (see [ChSh], [F], [GaHa])

\[
\|\chi_R u\|_{\varphi,1}^2 \leq C'_{\varphi,R} \left( \|\overline{\partial}(\chi_R u)\|_{\varphi}^2 + \|\overline{\partial} \varphi (\chi_R u)\|_{\varphi}^2 + \|\chi_R u\|_{\varphi}^2 \right)
\]

\[
\leq C''_{\varphi,R} \left( \|\partial u\|_{\varphi}^2 + \|\partial \varphi u\|_{\varphi}^2 + \|u\|_{\varphi}^2 \right)
\]

we can control the last integral by the norm $\|u\|_{Q,\varphi}^2$. Since we started from the unit ball in $\mathcal{W}_Q^\varphi$ we get that condition (i) of Lemma 2.3 is satisfied.

Condition (ii) of Lemma 2.3 is satisfied for the unit ball of $\mathcal{W}_Q^\varphi$ since we have

$$\int_{\mathbb{C}^n \setminus \mathbb{B}_R} |u(z)|^2 e^{-\varphi(z)} \, d\lambda(z) \leq \int_{\mathbb{C}^n \setminus \mathbb{B}_R} \frac{\mu_\varphi(z)|u(z)|^2}{\inf\{\mu_\varphi(z) : |z| \geq R\}} e^{-\varphi(z)} \, d\lambda(z).$$

So formula (2.1) together with assumption (**) shows that

$$\int_{\mathbb{C}^n \setminus \mathbb{B}_R} |u(z)|^2 e^{-\varphi(z)} \, d\lambda(z) \leq \frac{\|u\|_{Q,\varphi}^2}{\inf\{\mu_\varphi(z) : |z| \geq R\}} < \epsilon,$$

if $R$ is big enough.

\[\square\]

We are now able to give a short proof of the main result in [HaHe] or [GaHa]

**Proposition 2.5.** Let $\varphi$ be a plurisubharmonic $C^2$-weight function. If the lowest eigenvalue $\mu_\varphi(z)$ of the Levi-matrix $M_\varphi$ satisfies (**), then $N_\varphi$ is compact.

**Proof.** By Proposition 2.2 the embedding $\mathcal{W}_Q^\varphi \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ is compact. The inverse $N_\varphi$ of $\square_\varphi$ is continuous as an operator from $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into $\mathcal{W}_Q^\varphi$, this follows from 1.4. Therefore we have that $N_\varphi$ is compact as an operator from $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into itself.

\[\square\]

Now notice that

$$N_\varphi : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \rightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$$

can be written in the form

$$N_\varphi = j_\varphi \circ j_\varphi^*,$$
where

\[ j_\varphi^* : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \rightarrow \mathcal{W}^{Q_\varphi} \]

is the adjoint operator to \( j_\varphi \) (see [Str]).

This means that \( N_\varphi \) is compact if and only if \( j_\varphi \) is compact and summarizing the above results we get the following

**Proposition 2.6.** Let \( \varphi : \mathbb{C}^n \rightarrow \mathbb{R}^+ \) be a plurisubharmonic \( C^2 \)-weight function. The \( \overline{\partial} \)-Neumann operator

\[ N_\varphi : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \rightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi) \]

is compact if and only if for each \( \epsilon > 0 \) there exists \( R > 0 \) such that

\[ \| u \|_{L^2_{(0,1)}(\mathbb{C}^n \setminus B_R, \varphi)} < \epsilon \]

for each \( u \in \mathcal{W}^{Q_\varphi} \) with

\[ \| \overline{\partial} u \|_\varphi^2 + \| \overline{\partial}^* u \|_\varphi^2 \leq 1. \]

### 3. Smoothly bounded pseudoconvex domains and properties (P) and (\( \tilde{P} \))

Let \( \Omega \subset\subset \mathbb{C}^n \) be a smoothly bounded pseudoconvex domain. \( \Omega \) satisfies property (P), if for each \( M > 0 \) there exists a neighborhood \( U \) of \( \partial \Omega \) and a plurisubharmonic function \( \varphi_M \in C^2(U) \) such that

\[ \sum_{j,k=1}^{n} \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(p) t_j \overline{t}_k \geq M \| t \|^2, \]

for all \( p \in \partial \Omega \) and for all \( t \in \mathbb{C}^n \).

\( \Omega \) satisfies property (\( \tilde{P} \)) if the following holds: there is a constant \( C \) such that for all \( M > 0 \) there exists a \( C^2 \) function \( \varphi_M \) in a neighborhood \( U \) (depending on \( M \)) of \( \partial \Omega \) with

(i) \[ \sum_{j=1}^{n} \left| \frac{\partial \varphi_M}{\partial z_j}(z) t_j \right|^2 \leq C \sum_{j=1}^{n} \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k \]

and

(ii) \[ \sum_{j=1}^{n} \frac{\partial \varphi_M}{\partial z_j}(z) t_j \overline{t}_k \geq M \| t \|^2, \]

for all \( z \in U \) and for all \( t \in \mathbb{C}^n \).

In [C] Catlin showed that condition (P) implies compactness of the \( \overline{\partial} \)- operator \( N \) on \( L^2_{(0,1)}(\Omega) \) and McNeal ([McN]) showed that property (\( \tilde{P} \)) also implies compactness of the \( \overline{\partial} \)- operator \( N \) on \( L^2_{(0,1)}(\Omega) \). It is not difficult to show that property (P) implies property (\( \tilde{P} \)), see for instance [Str].

We can now use a similar approach as in section 2 to prove Catlin’s result. For this purpose we use the following version of lemma 2.3.
Lemma 3.1. Let $A$ be a bounded subset of $L^2(\Omega)$. Suppose that

(i) for each $\epsilon > 0$ and for each $\omega \subset \subset \Omega$ there exists $\delta > 0$, $\delta < \text{dist}(\omega, \Omega^c)$ such that

$$\|\tau_h f - f\|_{L^2(\omega)} < \epsilon$$

for each $h \in \mathbb{C}^n$ with $|h| < \delta$ and for each $f \in A$,

(ii) for each $\epsilon > 0$ there exists $\omega \subset \subset \Omega$ such that

$$\|f\|_{L^2(\Omega \setminus \omega)} < \epsilon$$

for each $f \in A$.

Then $A$ is relatively compact in $L^2(\Omega)$.

Remark 3.2. Conditions (i) and (ii) are also necessary for $A$ to be relatively compact in $L^2(\Omega)$.

In order to show that the unit ball in $\text{dom}(\partial) \cap \text{dom}(\partial^*)$ in the graph norm $f \mapsto (\|\partial f\|^2 + \|\partial^* f\|^2)^{1/2}$ satisfies condition (i) of 3.1 we remark that Gårding’s inequality holds for $\omega \subset \subset \Omega$ (see section 2). To verify condition (ii) we use property (P) and the following version of the Kohn-Morrey formula

$$\int_{\Omega} \sum_{j,k=1}^{n} \frac{\partial^2 \phi_M}{\partial z_j \partial z_k} u_j \overline{u}_k e^{-\phi_M} \, d\lambda \leq \|\partial u\|_{\phi_M}^2 + \|\partial^* u\|_{\phi_M}^2,$$

here we used that $\Omega$ is pseudoconvex, which means that the boundary terms in the Kohn-Morrey formula can be neglected. Now we point out that the weighted $\partial$- complex is equivalent to the unweighted one and that the expression $\sum_{j=1}^{n} \frac{\partial^2 \phi_M}{\partial z_j} u_j$ which appears in $\partial \phi_M u$, can be controlled by the complex Hessian $\sum_{j,k=1}^{n} \frac{\partial^2 \phi_M}{\partial z_j \partial z_k} u_j \overline{u}_k$, which follows from the fact that property (P) implies property ($\tilde{\text{P}}$) (see [Str]). Of course we also use that the weight $\phi_M$ is bounded on $\Omega \subset \subset \mathbb{C}^n$. In this way the same reasoning as in section 2 shows that property (P) implies condition (ii) of lemma 3.1. Therefore condition (P) gives that the unit ball of $\text{dom}(\partial) \cap \text{dom}(\partial^*)$ in the graph norm $f \mapsto (\|\partial f\|^2 + \|\partial^* f\|^2)^{1/2}$ is relatively compact in $L^2_{(0,1)}(\Omega)$ and hence that the $\partial$-Neumann operator is compact.

Now let

$$j : \text{dom}(\partial) \cap \text{dom}(\partial^*) \hookrightarrow L^2_{(0,1)}(\Omega)$$

denote the embedding. It follows from [Str] that

$$N = j \circ j^*.$$

Hence $N$ is compact if and only if $j$ is compact, where $\text{dom}(\partial) \cap \text{dom}(\partial^*)$ is endowed with the graph norm $f \mapsto (\|\partial f\|^2 + \|\partial^* f\|^2)^{1/2}$.

Proposition 3.3. Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Let $B$ denote the unit ball of $\text{dom}(\partial) \cap \text{dom}(\partial^*)$ in the graph norm $f \mapsto (\|\partial f\|^2 + \|\partial^* f\|^2)^{1/2}$.

The $\partial$- Neumann operator $N$ is compact if and only if $B$ as a subset of $L^2_{(0,1)}(\Omega)$ satisfies the following condition:

for each $\epsilon > 0$ there exists $\omega \subset \subset \Omega$ such that

...
\[ \|f\|_{L^2_{(0,1)}(\Omega \setminus \Omega)} < \epsilon \]

for each \( f \in \mathcal{B} \).

This follows from the above remarks about the embedding \( j \) and the fact that the two conditions in 3.1 are also necessary for a bounded set in \( L^2 \) to be relatively compact. For a localized version of the above result see [Sa].

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