Gaudin Model, KZ Equation, and Isomonodromic Problem on Torus

Kanehisa Takasaki
Department of Fundamental Sciences, Kyoto University
Yoshida, Sakyo-ku, Kyoto 606, Japan
E-mail: takasaki@yukawa.kyoto-u.ac.jp

Abstract

This paper presents a construction of isospectral problems on the torus. The construction starts from an SU($n$) version of the XYZ Gaudin model recently studied by Kuroki and Takebe in the context of a twisted WZW model. In the classical limit, the quantum Hamiltonians of the generalized Gaudin model turn into classical Hamiltonians with a natural $r$-matrix structure. These Hamiltonians are used to build a non-autonomous multi-time Hamiltonian system, which is eventually shown to be an isomonodromic problem on the torus. This isomonodromic problem can also be reproduced from an elliptic analogue of the KZ equation for the twisted WZW model. Finally, a geometric interpretation of this isomonodromic problem is discussed in the language of a moduli space of meromorphic connections.
1 Introduction

It has been argued for the last several years that isomonodromic problems are closely related to the Knizhnik-Zamolodchikov (KZ) equation [1]. One of the earliest observations is due to Reshetikhin [2]. Reshetikhin considered the Schlesinger equation, and concluded that the KZ equation may be viewed as a quantization of the Schlesinger equation. Harnad reformulated Reshetikhin’s observation in a Heisenberg picture [3]. Let us recall here that the Schlesinger equation is an isomonodromic problem on the Riemann sphere. Since the KZ equation on the Riemann sphere can be generalized to the Knizhnik-Zamolodchikov-Bernard (KZB) equation [4] on the torus, one will naturally expect that an associated isomonodromic problem should exist on the torus. Korotkin and Samtleben, indeed, derived such an isomonodromic problem from the KZB equation of the SU(2) WZW model [5]. Recently, Lavin and Olshanetsky proposed a general framework for this type of isomonodromic problems on a general compact Riemann surface [6].

We present below an SU(\(n\)) and “twisted” version of the isomonodromic problem of Korotkin and Samtleben. The word “twisted” means that our isomonodromic problem is related to the “twisted WZW model” recently studied by Kuroki and Takebe [7]. The method of construction, too, is considerably different. Korotkin and Samtleben start from a Hamiltonian formulation of the Chern-Simons theory (related to the ordinary untwisted WZW model), and formulate the isomonodromic problem in terms of a meromorphic connection induced on the torus. We follow a more direct approach that has been known for a class of isomonodromic problems on the Riemann sphere [8, 9]. This class of isomonodromic problems can be systematically derived from a class of isospectral problems as a non-autonomous analogue; of particular interest is the case where the isospectral problems are the so called Hitchin systems [10] and their generalizations to punctured Riemann surfaces [11]. As such an isospectral problem, we now take an SU(\(n\)) version of the XYZ Gaudin model considered by Kuroki and Takebe in the study of the twisted WZW model. Like the ordinary Gaudin (or “Calogero-Gaudin”) models [12, 13, 14], this generalized Gaudin model is a quantized Hitchin system in the sense of Beilinson and Drinfeld [15]. In the classical limit, mutually commutative Gaudin Hamiltonians turn into Poisson-commutative Hamiltonians with an \(r\)-matrix structure. We construct from these Hamiltonians a non-autonomous multi-time Hamiltonian system, then rewrite it...
into a Lax system of isomonodromic type, and finally confirm that this isomonodromic problem is linked, in the sense of Reshetikhin, with Etingof’s elliptic analogue of the KZ equation [16] (the “elliptic KZ equation” in the terminology of Kuroki and Takebe [7]). As shown by Kuroki and Takebe, this equation plays the role of the KZ equation for the twisted WZW model.

Isomonodromic problems on the torus (and more general compact Riemann surfaces) have also been studied by Okamoto [17], Iwasaki [18] and Kawai [19] by complex analytic and geometric methods. Although their work has been mostly focussed on scalar Fuchsian equations, their methods can be applied to matrix systems like ours. We shall show that this reveals a geometric origin of the Hamiltonian structure of our isomonodromic problem.

2 Generalized Gaudin Model

We now briefly review Kuroki and Takebe’s generalization of the XYZ Gaudin model to an SU(n) spin system [7].

Let us first introduce basic functions and matrices. Let X be the torus (elliptic curve) with modulus τ, i.e., \( X = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \). For integer indices \((a, b)\), let \( \theta_{ab}(z) \) denote the special theta functions

\[
\theta_{ab}(z) = \theta_{a+b+1/2, b+1/2}(z, \tau), \\
\theta_{\kappa\kappa'}(z, \tau) = \sum_{m \in \mathbb{Z}} \exp\left[ \pi i \tau (m + \kappa)^2 + 2\pi i (m + \kappa) (z + \kappa') \right].
\] (1)

Furthermore, let \( J_{ab} \) and \( J^{ab} \) be the \( n \times n \) matrices

\[
J_{ab} = g^a h^b, \quad J^{ab} = \frac{1}{n} J^{-1}_{ab},
\] (2)

where

\[
g = \text{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1}), \quad h = (\delta_{i-1, j}), \quad \omega = e^{2\pi \sqrt{-1}/n}.
\] (3)

Note that \( g \) and \( h \) obey the algebraic relation \( gh = \omega hg \). It is well known that these \( J \)’s give a basis of \( \text{su}(n) \) (over \( \mathbb{R} \)) and \( \text{sl}(n, \mathbb{C}) \) (over \( \mathbb{C} \)). \( J_{ab} \) and \( J^{ab} \) are “dual” bases in the sense that \( \text{Tr}(J_{ab} J^{cd}) = \delta_{ac} \delta_{bd} \).

These functions and matrices are building blocks of Belavin’s \( \mathbb{Z}_n \)-symmetric \( R \)-matrix.
and the associated r-matrix \([20]\). The R-matrix is given by

\[
R(\lambda) = \sum_{(a,b)\in \mathbb{Z}_n \times \mathbb{Z}_n} W_{ab}(\lambda, \eta) J_{ab} \otimes J^{ab},
\]

(4)

where

\[
W_{ab}(\lambda, \eta) = \frac{\theta_{[ab]}(\lambda + \eta)}{\theta_{[ab]}(\eta)}.
\]

(5)

These Boltzman weights are \(n\)-periodic in \(a\) and \(b\), thereby the summation is over \(\mathbb{Z}_n \times \mathbb{Z}_n\).

The \(r\)-matrix is the leading nontrivial part in the \(\eta\)-expansion of \(R(\lambda)\) at \(\eta = 0\):

\[
r(\lambda) = \sum_{(a,b)\neq (0,0)} w_{ab}(\lambda) J_{ab} \otimes J^{ab},
\]

(6)

where

\[
w_{ab}(\lambda) = \frac{\theta_{[ab]}(\lambda)\theta'_{[00]}(0)}{\theta_{[ab]}(0)\theta'_{[00]}(\lambda)},
\]

(7)

and the prime means \(\lambda\)-derivative, \(\prime = d/d\lambda\). The summation in \((a,b)\) is now over \(\mathbb{Z}_n \times \mathbb{Z}_n \ \backslash \ \{(0,0)\}\). We shall frequently omit showing this range explicitly (or, equivalently, obey the convention that \(w_{00}(\lambda) = 0\)). The \(r\)-matrix satisfies the classical Yang-Baxter equation

\[
[r^{(13)}(\lambda), r^{(23)}(\mu)] = -[r^{(12)}(\lambda - \mu), r^{(13)}(\lambda) + r^{(23)}(\mu)].
\]

(8)

Here, as usual, the superscript means in which part of the tensor product \(C^n \otimes C^n \otimes C^n\) the \(r\)-matrix acts nontrivially, e.g., \(r^{(12)}(\lambda) = \sum w_{ab}(\lambda) J_{ab} \otimes J^{ab} \otimes I\).

The generalized Gaudin model is a limit, as \(\eta \to 0\), of an inhomogeneous \(SU(n)\) spin chain with \(N\) lattice sites. The monodromy matrix of this inhomogeneous spin chain is given by

\[
T(\lambda) = L_N(\lambda - t_N) \cdots L_1(\lambda - t_1),
\]

(9)

where \(t_i\)’s are inhomogeneity parameters (which eventually play the role of time variables in our isomonodromic problem), and \(L_i\)’s are \(L\)-operators of the form

\[
L_i(\lambda) = \sum_{(ab)} W_{ab}(\lambda) J_{ab} \otimes \rho_i(J^{ab})
\]

(10)

that act on the tensor product \(C^n \otimes V_i\) of \(C^n\) and the representation space \(V_i\) of an irreducible representation \((\rho_i, V_i)\) of \(su(n)\). These \(L\)-operators satisfy the well known equation of “\(RLL = LLR\)” type with \(R\) being the above \(R\)-matrix. The monodromy
matrix thereby acts on $\mathbb{C}^n \otimes V$ ($V = V_1 \otimes \cdots \otimes V_N$). The leading nontrivial part in the $\eta$-expansion of $T(\lambda)$ at $\eta = 0$ can be written

$$T(\lambda) = \sum_{i=1}^{N} \sum_{(ab)} w_{ab}(\lambda - t_i) J_{ab} \otimes \rho_i(J^{ab}), \quad (11)$$

which satisfies the fundamental commutation relation

$$[T^{(1)}(\lambda), T^{(2)}(\mu)] = -[r(\lambda - \mu), T^{(1)}(\lambda) + T^{(2)}(\mu)]. \quad (12)$$

The superscript now stands for the component of the two $\mathbb{C}^n$’s in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes V$ in which the monodromy matrix acts nontrivially, e.g., $T^{(1)}(\lambda) = \sum_i \sum_{(ab)} w_{ab}(\lambda - t_i) J_{ab} \otimes I \otimes \rho_i(J^{ab})$.

Mutually commutative Hamiltonians $H_i$ ($i = 0, 1, \cdots, N$) of the generalized Gaudin model are defined by the relation

$$\frac{1}{2} \text{Tr}_{n \times n} T(\lambda)^2 = \sum_{i=1}^{N} C_i \mathcal{P}(\lambda - t_i) + \sum_{i=1}^{N} H_i \zeta(\lambda - t_i) + H_0, \quad (13)$$

where $\mathcal{P}(z)$ and $\zeta(z)$ are the Weierstrass functions with modulus $\tau$. Note that $C_i$’s are quadratic Casimir elements of the algebra generated by $\rho_i(J^{ab})$:

$$C_i = \frac{1}{2} \sum_{(ab)} \rho_i(J_{ab}) \rho_i(J^{ab}). \quad (14)$$

The $H_i$’s for $i = 1, \cdots, N$ can be written rather explicitly,

$$H_i = \sum_{j(\neq i)} \sum_{(ab)} w_{ab}(t_i - t_j) \rho_i(J_{ab}) \rho_j(J^{ab}), \quad (15)$$

but explicit expressions of $H_0$ become far complicated. There is a linear relation among these Hamiltonians: $\sum_{i=1}^{N} H_i = 0$.

3 Construction of Isomonodromic Problem on Torus

3.1 Classical limit and Poisson bracket

We now introduce the following matrix as a classical analogue of the $T$-operator of the generalized Gaudin model:

$$M(\lambda) = \sum_{i=1}^{N} \sum_{(ab)} w_{ab}(\lambda - t_i) J_{ab} A_i^{ab}. \quad (16)$$
$A_{i}^{ab}$’s are scalar functions of $t_1, \cdots, t_N$ and $\tau$. Our isomonodromic problem will be formulated as differential equations for these functions. (Note that $A_{i}^{ab}$’s depend on $t$ whereas $\rho_i(J^{ab})$’s do not. As we shall show later, this amounts to the difference of the Heisenberg and Schrödinger pictures in quantum mechanics.) We apply the same rule of raising and lowering the indices as the $J$-matrices to these coefficients. Thus, e.g., $A_{i,ab}$ are defined by

$$A_{i,ab} = n\omega^{ab}A_{i,-a,-b}$$

in accordance with the transformation rule for the $J$’s,

$$J_{i,ab} = n\omega^{ab}J_{i,-a,-b}.$$  

The commutation relations of $\rho_i(J^{ab})$’s can be now translated into Poisson commutation relations among the $A$-coefficients. This Poisson structure can be packed into the well known relation

$$\{M(\lambda) \otimes M(\mu)\} = -[r(\lambda - \mu), M(\lambda) \otimes I + I \otimes M(\mu)].$$  

(17)

The left hand side, as usual, is an abbreviation of $\sum \{M^{ab}(\lambda), M^{cd}(\mu)\} J_{ab} \otimes J_{cd}$, where $M^{ab}$ are the coefficients of the expansion $M = \sum M^{ab}J_{ab}$. In terms of the residue matrix

$$A_i = \text{Res}_{\lambda = t_i} M(\lambda) = \sum_{(ab)} J_{ab} A_i^{ab},$$  

(18)

the Poisson structure is nothing but the one induced from the Kirillov-Kostant Poisson structure on $\text{sl}(n, \mathbb{C})$. Symplectic leaves of this Poisson structure are the direct product $O_1 \times \cdots \times O_N$ of coadjoint orbits $O_i$ in $\text{sl}(n, \mathbb{C})$ on which $A_i$ is living. These symplectic leaves become the phase spaces of the following non-autonomous Hamiltonian system.

### 3.2 Non-autonomous Hamiltonian system

Hamiltonians of the classical Gaudin model can be obtained as follows:

$$\frac{1}{2} \text{Tr} M(\lambda)^2 = \sum_{i=1}^{N} C_i \mathcal{P}(\lambda - t_i) + \sum_{i=1}^{N} H_i \zeta(\lambda - t_i) + H_0.$$  

(19)

$C_i$’s are Casimir elements in the above Poisson algebra. The Hamiltonians $H_i$ for $i = 1, \cdots, N$, as in the quantum case, can be written explicitly:

$$H_i = \sum_{j(\neq i)} \sum_{(ab)} w_{ab}(t_i - t_j) A_{i,ab} A_i^{ab}.$$  

(20)

$H_0$ is also a quadratic form of $A_i^{ab}$’s, though its explicit expression is complicated.
It should be noted that $t_i$’s in these formulas are just parameters, not time variables, of the classical Gaudin system. The classical Gaudin system is a Hitchin system on the punctured torus $X \setminus \{t_1, \ldots, t_N\}$ [13, 14], thus $t_i$’s are nothing but the position of punctures. We now consider a non-autonomous multi-time Hamiltonian system with the same Hamiltonians, which gives a non-autonomous analogue of this Hitchin system.

The non-autonomous Hamiltonian system possesses the puncture coordinates $t_1, \ldots, t_N$ and the modulus $\tau$ as time variables. The equations with respect to the puncture coordinates are given by

$$\frac{\partial A_{ij}^{ab}}{\partial t_i} = \{A_{ij}^{ab}, H_i\}. \quad (21)$$

The equation with respect to the modulus, in contrast, turns out to take the somewhat strange form

$$\frac{\partial A_{ij}^{ab}}{\partial \tau} = \{A_{ij}^{ab}, (H_0 - \eta_1 \sum_{i=1}^N t_i H_i)/2\pi \sqrt{-1}\}, \quad (22)$$

where $\eta_1$ is the constant (depending only on $\tau$) that arises in the transformation law

$$\zeta(z + 1) = \zeta(z) + \eta_1$$

of the Weierstrass $\zeta$ function.

### 3.3 Lax representation

In order to confirm that the above non-autonomous Hamiltonian system is an isomonodromic problem, we now rewrite it into a Lax form.

A key is the following relation:

$$\left\{ M(\lambda), \frac{1}{2} \text{Tr} M(\mu)^2 \right\} = \left[ \sum_{i=1}^N \sum_{( ab )} w_{-a,-b}(\mu - \lambda)w_{ab}(\mu - t_i)J_{ab}A_i^{ab}, M(\mu) \right]. \quad (24)$$

This relation can be reduced to a collection of cubic relations among $w_{ab}$’s, and can be proven by the same function-theoretic method as the proof of the classical Yang-Baxter equation (which reduces to quadratic relations among $w_{ab}$’s).

The above formula can be used to rewrite the right hand side of the Hamiltonian equations into a matrix commutator. First, the residue of both hand sides of the above formula at $\mu = t_i$ immediately gives the relation

$$\{M(\lambda), H_i\} = -[A_i(\lambda), M(\lambda)], \quad (25)$$
where
\[ A_i(\lambda) = \sum_{(ab)} w_{ab}(\lambda - t_i) J_{ab} A_{i}^{ab}. \] (26)

(We have used the reflection property \( w_{-a,-b}(-\lambda) = -w_{ab}(\lambda) \) \[\text{of } w_{ab}\)'s, too.) The Poisson bracket with \( H_0 \) requires lengthy calculations, which eventually boil down to the relation
\[ \{ M(\lambda), H_0 \} = \left[ 4\pi \sqrt{-1} B(\lambda) - \eta_1 \sum_{i=1}^{N} \tau_i A_i(\lambda), M(\lambda) \right], \] (27)

where
\[ B(\lambda) = \sum_{i=1}^{N} \sum_{(ab)} Z_{ab}(\lambda - t_i) J_{ab} A_{i}^{ab}, \]
\[ Z_{ab}(\lambda) = \frac{w_{ab}(\lambda)}{4\pi \sqrt{-1}} \left( \frac{\theta'_{[ab]}(\lambda)}{\theta_{[ab]}(\lambda)} - \frac{\theta'_{[ab]}(0)}{\theta_{[ab]}(0)} \right). \] (28)

The above formula for \( \{ M(\lambda), M_0 \} \) is interesting in itself from two points of view. Firstly, this formula can be rewritten
\[ \left\{ M(\lambda), H_0 - \eta_1 \sum_{i=1}^{N} \tau_i H_i \right\} = \left[ 4\pi \sqrt{-1} B(\lambda), M(\lambda) \right], \] (29)
thus the previous strange linear combination of \( H_i \)'s in the Hamiltonian equation with respect to \( \tau \) emerges quite naturally. Secondly, the functions \( Z_{ab} \) are also basic constituents of the elliptic KZ equation of Etingof (see Section 4).

By this formula, the previous non-autonomous Hamiltonian system can be rewritten into the following Lax equations:
\[ \frac{\partial M(\lambda)}{\partial t_i} = -[A_i(\lambda), M(\lambda)] - \frac{\partial A_i(\lambda)}{\partial \lambda}, \]
\[ \frac{\partial M(\lambda)}{\partial \tau} = [2B(\lambda), M(\lambda)] + 2 \frac{\partial B(\lambda)}{\partial \lambda}. \] (30)

The second terms on the right hand side originate in differentiating \( w_{ab}(\lambda - t_j) \) in \( M(\lambda) \) by \( t_i \) and \( \tau \). The \( \tau \)-derivatives are converted into \( \lambda \)-derivatives by the heat equation
\[ \frac{\partial \theta_{[ab]}(\lambda)}{\partial \tau} = \frac{1}{4\pi \sqrt{-1}} \frac{\partial^2 \theta_{[ab]}(\lambda)}{\partial \lambda^2}. \] (31)
3.4 Isomonodromic Property

As usual, the above Lax equations are integrability conditions of a linear system:

\[
\left( \frac{\partial}{\partial \lambda} - M(\lambda) \right) Y(\lambda) = 0, \quad \left( \frac{\partial}{\partial t_i} + A_i(\lambda) \right) Y(\lambda) = 0, \quad \left( \frac{\partial}{\partial \tau} - 2B(\lambda) \right) Y(\lambda) = 0. \tag{32}
\]

The first equation is a matrix ODE on the torus with regular singular points at \( t_1, \cdots, t_N \).

The other equations may be interpreted as isomonodromic deformations of this ODE in the following sense.

Firstly, the solution of the ODE transforms along a small closed path around \( \lambda = t_i \) as:

\[
Y(\lambda) \rightarrow Y(\lambda) \Gamma_i. \tag{33}
\]

\( \Gamma_i \) represents the local monodromy around \( \lambda = t_i \). Similarly, the solution of the ODE transforms along the fundamental cycles \( \alpha : z \rightarrow z + 1 \) and \( \beta : z \rightarrow z + \tau \), respectively, as:

\[
Y(\lambda + 1) = h^{-1} Y(\lambda) \Gamma_\alpha, \quad Y(\lambda + \tau) = g Y(\lambda) \Gamma_\beta. \tag{34}
\]

The extra prefactors \( h^{-1} \) and \( g \) originate in the monodromy of \( M(\lambda) \) along those cycles, i.e., \( M(\lambda + 1) = h^{-1} M(\lambda) h \) and \( M(\lambda + \tau) = g M(\lambda) g^{-1} \). \( \Gamma_\alpha \) and \( \Gamma_\beta \) represent the global monodromy. “Isomonodromic deformations” means that these monodromy data are left invariant under deformations.

4 Relation to Elliptic KZ Equation of Twisted WZW Model

The elliptic KZ equation of Etingof [16] can be written

\[
\left( \kappa \frac{\partial}{\partial t_i} + \sum_{j \neq i} \sum_{(ab)} w_{ab}(t_i - t_j) \rho_i(J_{ab}) \rho_j(J^{ab}) \right) F(t_1, \cdots, t_N) = 0,
\]

\[
\left( \kappa \frac{\partial}{\partial \tau} + \sum_{i,j=1}^{N} \sum_{(ab)} Z_{ab}(t_i - t_j) \rho_i(J_{ab}) \rho_j(J^{ab}) \right) F(t_1, \cdots, t_N) = 0, \tag{35}
\]

where \( \kappa = k + n \), \( k \) is the level of the twisted WZW model, and the above equation characterizes \( N \)-point conformal blocks with the irreducible representations \( \rho_1, \cdots, \rho_N \) sitting at the marked points \( t_1, \cdots, t_N \).
Following Reshetikhin [2], we now add one more marked point at \(\lambda\) with the fundamental representation \((\mathbb{C}^n, \text{id})\). The associated \(N + 1\)-point elliptic KZ equation becomes

\[
\left(\kappa \frac{\partial}{\partial \lambda} + \sum_{i=1}^{N} \sum_{(ab)} w_{ab}(\lambda - t_i) J_{ab} \rho_i(J^{ab})\right) G(\lambda, t_1, \cdots, t_N) = 0,
\]

\[
\left(\kappa \frac{\partial}{\partial t_i} + \sum_{j \neq i} \sum_{(ab)} w_{ab}(t_i - t_j) \rho_i(J_{ab}) \rho_j(J^{ab}) \right.
\]

\[
\left. + \sum_{(ab)} w_{ab}(t_i - \lambda) \rho_i(J_{ab}) J^{ab} \right) G(\lambda, t_1, \cdots, t_N) = 0,
\]

\[
\left(\kappa \frac{\partial}{\partial \tau} + \sum_{i,j=1}^{N} \sum_{(ab)} Z_{ab}(t_i - t_j) \rho_i(J_{ab}) \rho_j(J^{ab}) \right.
\]

\[
\left. + \sum_{i=1}^{N} \sum_{(ab)} Z_{ab}(\lambda - t_i) J_{ab} \rho_j(J^{ab}) + \sum_{j=1}^{N} \sum_{(ab)} Z_{ab}(\lambda - t_j) J_{ab} \rho_j(J^{ab}) \right.
\]

\[
\left. + \sum_{(ab)} Z_{ab}(0) J_{ab} J^{ab} \right) G(\lambda, t_1, \cdots, t_N) = 0.
\] (36)

Let us now take an invertible “fundamental solution” \(F = \exp S\) of the \(N\)-point elliptic KZ equation, and consider equations for \(X = F^{-1} G\). This is a kind of “gauge transformation” to the \(N + 1\)-point elliptic KZ equation, and the outcome is the equations

\[
\left(\kappa \frac{\partial}{\partial \lambda} + \sum_{i=1}^{N} \sum_{(ab)} w_{ab}(\lambda - t_i) J_{ab} A_i^{ab}\right) X(\lambda, t_1, \cdots, t_N) = 0,
\]

\[
\left(\kappa \frac{\partial}{\partial t_i} - \sum_{(ab)} w_{ab}(\lambda - t_i) J_{ab} A_i^{ab}\right) X(\lambda, t_1, \cdots, t_N) = 0,
\]

\[
\left(\kappa \frac{\partial}{\partial \tau} + 2 \sum_{i=1}^{N} \sum_{(ab)} Z_{ab}(\lambda - t_i) J_{ab} A_i^{ab} + \sum_{(ab)} Z_{ab}(0) J_{ab} J^{ab} \right) X(\lambda, t_1, \cdots, t_N) = 0.
\] (37)

where

\[
A_i^{ab} = F(t_1, \cdots, t_N)^{-1} \rho_i(J^{ab}) F(t_1, \cdots, t_N).
\] (38)

We thus obtain almost the same equation as the isomonodromic linear system in the last section, but notice that there are a few discrepancies:

1. The last equation contains extra terms (a linear combination of \(J_{ab} J^{ab}\)).

2. The \(A_i^{ab}\)’s are not scalar functions, but operators on \(V = V_1 \otimes \cdots \otimes V_N\).

3. The above linear system contains an arbitrary parameter \(k\). The previous isomonodromic linear system can be reproduced by putting \(k = -1\).
The first discrepancy can be readily remedied, because the extra terms are scalar \((J_{ab}J^{ab} = I/n)\) and can be “gauged away” by a scalar gauge transformation \(X \mapsto e^{i(\tau)}X\) with a function \(f(\tau)\) of \(\tau\) only. This does not affect the other part of the above equations.

The second discrepancy is more essential. From the point of view of Reshetikhin \([2]\) and other people \([3, 5]\), the above system is a quantization of the isomonodromic problem in the last section. (This is parallel to the relation between quantum and classical Hitchin systems.) The operators \(A_{ab}^{i}\) depend on \(t's\) and \(\tau\), but inherit the same commutation relations as the \(J\)-matrices from \(\rho_i(J^{ab})\). The passage from \(\rho_i(J^{ab})\) to \(A_{ab}^{i}\) amounts to the change from the Schrödinger picture to the Heisenberg picture. “Classical limit” now means replacing these operators by functions on a phase space. This gives our isomonodromic problem.

The parameter \(\kappa\) plays a role in the limit towards the critical level \(k = -n\). In this limit, the (classical or quantum) isomonodromic problem in the above sense turns into an isospectral problem, i.e., a Hitchin system on the punctured Riemann surface \(X \setminus \{t_1, \ldots, t_N\}\). A more careful analysis of this limit leads to the so called “Whitham dynamics” of the spectral curve, which describes the isomonodromic problem as a slowly varying isospectral problem \([3, 21]\).

5 Geometric Origin of Hamiltonian Structure

Our isomonodromic problem may be reformulated in a geometric language at least in two different ways. One option is Levin and Olshanetsky’s framework \([1]\) based on Hamiltonian reduction. Another option is Iwasaki’s geometric framework \([18]\), which has been successfully applied by Kawai \([19]\) to a scalar isomonodromic problem on the torus. In this section, we follow the second approach and show a geometric origin of the somewhat involved Hamiltonian structure of our isomonodromic problem.

In the geometric approach of Iwasaki and Kawai, meromorphic linear ODE’s on a compact Riemann surface \(X\) are converted into meromorphic connections \(\nabla\) on a holomorphic vector bundle \(E\). All singular points are assumed to be regular singular points. One can then develop an analogue of the Kodaira-Spencer theory on the moduli space \(\mathcal{M}\) of meromorphic connections with a given constant set of local monodromy data around the regular singular points. The tangent spaces of this moduli space, i.e., infinitesimal
deformations leaving invariant the local monodromy, can be described by the language of twisted de Rham cohomologies with coefficients in the endomorphism bundle End\(E\).

A key of Iwasaki’s ideas is that the Poincaré-Lefschetz duality in those twisted de Rham cohomologies induces a closed 2-form (or a symplectic form) \(\Omega\) on the moduli space \(\mathcal{M}\). This closed 2-form describes the Hamiltonian structure of isomonodromic deformations. Kawai generalized these ideas of Iwasaki to a setting where the Riemann surface itself also deforms, and considered, as an example, the case of isomonodromic deformations of a second order Fuchsian ODE \(d^2y/d\lambda^2 = Q(\lambda)y\) on the torus. It is this generalization of Iwasaki’s framework by Kawai that we now attempt to apply to our problem.

It is rather straightforward to reformulate our problem into Kawai’s framework. In our case, the endomorphism bundle End\(E\) in twisted de Rham cohomologies is replaced by the Lie algebra bundle \(\text{sl}(n, \mathbb{C})^\mathfrak{t}\) of Kuroki and Takebe [7]. This Lie algebra bundle is obtained from the trivial bundle \(\text{sl}(n, \mathbb{C}) \times \tilde{X}\) over the universal covering \(\tilde{X} = \mathbb{C}\) of \(X\) by identifying \((A, \lambda) \sim (hAh^{-1}, \lambda + 1) \sim (g^{-1}Ag, \lambda + \tau)\). The closed 2-form \(\Omega\) is defined by the Poincaré-Lefschetz pairing \(\langle \delta_1 M, \delta_2 M \rangle\) of two infinitesimal deformations \(\delta_1 M(\lambda)\) and \(\delta_2 M(\lambda)\) of \(M(\lambda)\) that leave invariant the local monodromy around \(t_1, \cdots, t_N\):

\[
\Omega(\delta_1 M, \delta_2 M) = \langle \delta_1 M, \delta_2 M \rangle.
\]

Calculation of this pairing, as Kawai illustrated, can be reduced to a kind of residue calculus, and eventually yields the following expression of \(\Omega\):

\[
\Omega = \sum_{i=1}^N dH_i \wedge dt_i + d\left( H_0 - \eta \sum_{i=1}^N t_i H_i \right) \wedge \frac{d\tau}{2\pi \sqrt{-1}} - \sum_{i=1}^N \text{Tr} dB_i \wedge dA_i.
\]

Here \(A_i\) is the residue matrix of \(M(\lambda)\) at \(\lambda = t_i\), and \(dB_i\) is linked with \(dA_i\) as

\[
[A_i, dB_i] = dA_i.
\]

This result is almost parallel to Kawai’s result, but the third part on the right hand side does not exist in the case of Kawai’s isomonodromic problem for second order scalar ODE’s. This part is nothing but the Kirillov-Kostant symplectic form on the symplectic leaves (coadjoint orbits) of \(\text{sl}(n, \mathbb{C})\), on which the residue matrix \(A_i\) moves as a function of time variables. A similar expression of the fundamental 2-form for general isomonodromic problems is derived in Levin and Olshanetsky’s work [8] in the framework of Hamiltonian reduction.
The above expression of $\Omega$ also clearly shows the correspondence between the Hamiltonians and the time variables. In particular, one can see why we have to take the strange linear combination \( (H_0 - \sum_{i=1}^{N} t_i H_i) / 2\pi \sqrt{-1} \) of $H_i$’s in order to describe the isomonodromic deformations with respect to the modulus $\tau$.

## 6 Conclusion

The point of departure of our construction is the generalized Gaudin model of Kuroki and Takebe. Its classical limit provides a Poisson commutative set of Hamiltonians with an $r$-matrix structure. The isomonodromic problem is first formulated as a non-autonomous multi-time Hamiltonian system, then rewritten into a Lax representation. From the Lax representation, we have been able to see a link with the elliptic KZ equation of Etingof, which advocates Reshetikhin’s observation in a non-zero genus case. We have also found a geometric interpretation of the Hamiltonian structure of this isomonodromic problem in the framework of Iwasaki and Kawai.

Our isomonodromic problem is a generalization of the Schlesinger equation on the Riemann sphere. It should be stressed that constructing such an isomonodromic problem on the torus is by no means an easy task. This seems to be one of reasons that complex analytic and geometric studies on isomonodromic problems have been mostly focussed on scalar equations \[17, 18, 19\]. The approach from the KZ equation is rather suited to dealing with matrix equations, and deserves to be pursued further.

An interesting issue in this respect is to construct difference and $q$-difference analogues of isomonodromic problems from difference \[22\] and $q$-difference \[23, 24\] analogues of the KZ equation. For instance, Jimbo and Sakai constructed a $q$-difference analogue of the sixth Painlevé equation by a $q$-analogue of the standard method for isomonodromic problems \[23\]. It seems likely that their $q$-difference equation is linked with a $q$-difference KZ ($q$KZ) equation.

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