Statistical interpretations of the null result of the KARMEN 2 experiment

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Abstract

Several possible statistical interpretations of the null result of the KARMEN 2 neutrino oscillation experiment are discussed with the aim of clarifying the implications of the fact that KARMEN 2 did not observe any of the expected background events. The formalism that allows to take into account the error of the expected mean background in a Poisson process with background is presented and applied to the statistical analysis of the KARMEN 2 null result. The possibility of ignoring the expected mean background calculated for the KARMEN 2 experiment is discussed and it is shown that the resulting exclusion curves are ultra-conservative, but may be a safe choice in such a controversial case. It is also shown that the sensitivity curves of neutrino oscillation experiments cannot be considered as exclusion curves. The implications of the different statistical analyses of the KARMEN 2 null result for the compatibility of the results of the KARMEN 2 and LSND experiments is discussed.

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I. INTRODUCTION

The KARMEN collaboration has reported recently [1] a null result of the KARMEN 2 experiment in the search for neutrino oscillations [2,3] in the channel $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$. This result is very interesting because the KARMEN neutrino oscillation experiment is sensitive to the same region in the plane of the neutrino mixing parameters $\sin^2 2\theta$ and $\Delta m^2$ as the LSND experiment [4] whose results provide an evidence in favor of $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ and $\nu_\mu \rightarrow \nu_e$ oscillations. Hence, the statistical interpretation of the KARMEN 2 null result (as well as that of the LSND result) is crucial in order to obtain an indication on the compatibility or incompatibility of the different results of the two experiments.

The KARMEN experiment [5] is located at at the ISIS spallation neutron facility of the Rutherford Laboratories (UK). The proton beam produced at ISIS is pulsed in time: two 100 ns wide proton pulses separated by 330 ns are produced every 20 ms. The proton pulses are directed on a target where they produce positive pions (the negative pions are absorbed in the source before decaying) which decay at rest according to the process

$$\pi^+ \rightarrow \mu^+ + \nu_\mu$$

$$\downarrow$$

$$e^+ + \nu_e + \bar{\nu}_\mu,$$

producing an equal number of $\nu_\mu$, $\nu_e$ and $\bar{\nu}_\mu$. The KARMEN experiment searches for $\bar{\nu}_e$ produced by $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations at a mean distance of 17.6 m. The time structure of the neutrino beam is important for the identification of the neutrino induced reactions in the KARMEN detector and for the effective suppression of the cosmic ray background. The KARMEN experiment started in 1990 and run until 1995 as KARMEN 1 [5]. In 1996 it was upgraded to KARMEN 2, eliminating the main cosmic ray induced background component in the search for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations [6].

So far the KARMEN 2 experiment measured no events [1], with an expected background of $2.88 \pm 0.13$ events. This null result has been analyzed with the following statistical methods:

**Bayesian Approach.** This method is recommended by the Particle Data Group [7,8] and has been used by the KARMEN collaboration [1]. The resulting upper limit for the mean $\mu$ of true neutrino oscillation events is 2.3 and the corresponding exclusion curve is reproduced in Fig. 1 (the solid curve passing through the filled squares).

**Unified Approach.** This method has been proposed recently by Feldman and Cousins [9] and has already been adopted by the Particle Data Group [8] as the new statistical standard for the calculation of frequentist confidence intervals and upper limits *with the correct coverage*. The Unified Approach is very attractive because it allows to construct classical confidence belts which “unify the treatment of upper confidence limits for null results and two-sided confidence intervals for non-null results”. On the other hand, the Unified Approach has the undesirable feature that when the number of observed events is smaller than the expected background, the upper limits for the mean $\mu$ of true neutrino oscillation events decreases rapidly when the background increases. Hence, by observing less events than the expected background an experiment can
establish a very stringent upper bound on $\mu$ even if it is not sensitive to such small values of $\mu$. This is what happens in the Unified Approach analysis of the KARMEN 2 null result: the upper limit for the mean $\mu$ of true neutrino oscillation events is 1.1 and the corresponding exclusion curve [4] reproduced in Fig. 1 (the solid curve passing through the filled circles) is very stringent, even if the KARMEN 2 experiment is not sensitive to the excluded region of the neutrino oscillation parameters (see Ref. [6]).

**New Ordering Approach.** This recently proposed method [10] is based on a new ordering principle for the construction of the classical confidence belt which has all the desirable properties of the one calculated with the Unified Approach and in addition minimizes the effect on the resulting confidence intervals of the observation of less background events than expected. Hence, it is appropriate for the statistical interpretation of the null result of the KARMEN 2 experiment. The resulting upper limit for the mean $\mu$ of true neutrino oscillation events in the KARMEN 2 experiment is 1.9 and the corresponding exclusion curve [10] is reproduced in Fig. 1 (the solid curve passing through the filled triangles).

The Unified Approach and the New Ordering Approach are reviewed in Section II. From Fig. 1 one can see that the Bayesian Approach and the Unified Approach yield rather different exclusion from the null result of the KARMEN 2 experiment. This difference is crucial in order to assess the compatibility of the KARMEN 2 null result with the positive result of the LSND experiment, whose 90% CL allowed region [4] is shown as the shadowed area in Fig. 1. The bayesian exclusion curve of KARMEN 2 is compatible with a large part of the LSND-allowed region, whereas the exclusion curve obtained with the Unified Approach is incompatible with almost all the LSND-allowed region. The exclusion curve obtained with the New Ordering Approach lies close to the bayesian exclusion curve and tends to support the compatibility of the KARMEN 2 and LSND results. This is a desirable achievement. Furthermore, it is important to emphasize that, as explained in Section II, *the New Ordering Approach gives a correct frequentist coverage as the Unified Approach.*

In view of the uncertainty of the physical meaning of the KARMEN 2 null result induced by the significant difference between the exclusion curve obtained in the Unified Approach on one hand and the exclusion curves obtained with the Bayesian Approach and with the New Ordering Approach on the other hand, it is interesting to explore other possibilities for the statistical interpretation of the KARMEN 2 null result.

In Sections III–V I discuss the following alternative statistical interpretations of the null result of the KARMEN 2 experiment, which take into account the problem of the non-observation of expected background events in the KARMEN 2 experiment:

**Larger Background Error.** Since no background events have been measured in the KARMEN 2 experiment, it is possible to doubt of the correctness of the calculated error for the mean expected background, $b = 2.88 \pm 0.13$. Hence, one can consider the possibility that the error has been underestimated, for example, by an order of magnitude. This approach is discussed in Section III, together with a presentation of the appropriate formalism.

**Unknown Background.** An extreme attitude towards the doubting of the correctness of the calculated mean expected background $b = 2.88 \pm 0.13$ is to ignore it and to assume
that the background is unknown. In this case the KARMEN 2 null measurement gives an estimation of the mean $\mu + b$ of signal plus background events. This approach is discussed in Section IV.

Sensitivity. The sensitivity of an experiment is defined as “the average upper limit that would be obtained by an ensemble of experiments with the expected background and no true signal” [9]. The sensitivities of the KARMEN 2 experiment calculated with the Unified Approach and with the New Ordering Approach are shown in Fig. [ ] (the solid curves passing through the empty circles and triangles, respectively). One could think of considering the sensitivity curve of the KARMEN 2 experiment as its exclusion curve. This approach is discussed and shown to be incorrect in Section V.

II. POISSON PROCESSES WITH BACKGROUND

The events measured in the KARMEN experiment follow a Poisson distribution. The probability to observe a number $n$ of events in a Poisson process consisting of signal events with mean $\mu$ and background events with known mean $b$ is

$$P(n|\mu; b) = \frac{1}{n!} (\mu + b)^n e^{-(\mu + b)}.$$ (2.1)

The classical frequentist method for obtaining the confidence interval for the unknown parameter $\mu$ is based on Neyman’s method to construct a confidence belt. This confidence belt is the region in the $n - \mu$ plane lying between the two curves $n_1(\mu; b, \alpha)$ and $n_2(\mu; b, \alpha)$ such that for each value of $\mu$

$$P(n \in [n_1(\mu; b, \alpha), n_2(\mu; b, \alpha)] \mid \mu; b) = \alpha,$$ (2.2)

where $\alpha$ is the desired confidence level. The two curves $n_1(\mu; b, \alpha)$ and $n_2(\mu; b, \alpha)$ are required to be monotonic functions of $\mu$ and can be inverted to yield the corresponding curves $\mu_1(n; b, \alpha)$ and $\mu_2(n; b, \alpha)$. Then, if a number $n_{\text{obs}}$ of events is measured, the confidence interval for $\mu$ is $[\mu_2(n_{\text{obs}}; b, \alpha), \mu_1(n_{\text{obs}}; b, \alpha)]$. This method guarantees by construction the correct coverage, i.e. the fact that the resulting confidence interval $[\mu_2(n_{\text{obs}}; b, \alpha), \mu_1(n_{\text{obs}}; b, \alpha)]$ is a member of a set of confidence intervals obtained with an ensemble of similar experiments that contain the true value of $\mu$ with a probability $\alpha$. Actually, in the case of a Poisson process, since $n$ is an integer, the relation (2.2) can only be approximately satisfied and in practice the chosen acceptance intervals $[n_1(\mu; b, \alpha), n_2(\mu; b, \alpha)]$ are the smallest intervals such that

$$P(n \in [n_1(\mu; b, \alpha), n_2(\mu; b, \alpha)] \mid \mu; b) \geq \alpha.$$ (2.3)

This choice introduces an overcoverage for some values of $\mu$ and the resulting confidence intervals are conservative. As emphasized in Ref. [9] conservativeness is an undesirable but unavoidable property of the confidence intervals in the case of a Poisson process (it
is undesirable because it implies a loss of power in restricting the allowed range for the parameter $\mu$.

The construction of Neyman’s confidence belt is not unique, because in general there are many different couples of curves $n_1(\mu; b, \alpha)$ and $n_2(\mu; b, \alpha)$ that satisfy the relation (2.2). Hence, an additional criterion is needed in order to define uniquely the acceptance intervals $[n_1(\mu; b, \alpha), n_2(\mu; b, \alpha)]$. The two common choices are

$$P(n < n_1(\mu; b, \alpha)|\mu; b) = P(n > n_2(\mu; b, \alpha)|\mu; b) = \frac{1 - \alpha}{2},$$

which leads to central confidence intervals and

$$P(n < n_1(\mu; b, \alpha)|\mu; b) = 1 - \alpha,$$

which leads to upper confidence limits. Central confidence intervals are appropriate for the statistical description of the results of experiments reporting a positive result, i.e. the measurement of a number of events significantly larger than the expected background. On the other hand, upper confidence limits are appropriate for the statistical description of the results of experiments reporting a negative result, i.e. the measurement of a number of events compatible with the expected background. However, Feldman and Cousins [9] noticed that switching from central confidence level to upper confidence limits or vice-versa on the basis of the experimental data (“flip-flopping”) leads to undercoverage for some values of $\mu$, which is a serious flaw for a frequentist method.

Feldman and Cousins [9] proposed an ordering principle for the construction of the acceptance intervals that is based on likelihood ratios and produces an automatic transition from central confidence intervals to upper limits when the number of observed events in a Poisson process with background is of the same order or less than the expected background, guaranteeing the correct frequentist coverage for all values of $\mu$. The acceptance interval for each value of $\mu$ is calculated assigning at each value of $n$ a rank obtained from the relative size of the ratio

$$R_{UA}(n|\mu; b) = \frac{P(n|\mu; b)}{P(n|\mu_{\text{best}}; b)},$$

where $\mu_{\text{best}} = \mu_{\text{best}}(n; b)$ (for a fixed $b$) is the non-negative value of $\mu$ that maximizes the probability $P(n|\mu; b)$:

$$\mu_{\text{best}}(n; b) = \max[0, n - b].$$

As emphasized in Ref. [9], “$R$ is a ratio of two likelihoods: the likelihood of obtaining $n$ given the actual mean $\mu$, and the likelihood of obtaining $n$ given the best-fit physically allowed mean”. For each fixed value of $\mu$, the rank of each value of $n$ is assigned in order of decreasing value of the ratio $R_{UA}(n|\mu; b)$: the value of $n$ which has bigger $R_{UA}(n|\mu; b)$ has rank one, the value of $n$ among the remaining ones which has bigger $R_{UA}(n|\mu; b)$ has rank two and so on. The acceptance interval for each value of $\mu$ is calculated by adding the values of $n$ in increasing order of rank until the condition (2.3) is satisfied.

The automatic transition from two-sided confidence intervals to upper confidence limits for $n \lesssim b$ is guaranteed in the Unified Approach by the fact that $\mu_{\text{best}}$ is always non-negative.
Indeed, since $\mu_{\text{best}}(n \leq b) = 0$, the rank of $n \leq b$ for $\mu = 0$ is one, implying that the interval $0 \leq n \leq b$ for $\mu = 0$ is guaranteed to lie in the confidence belt.

Although the Unified Approach solves brilliantly the problem of obtaining a transition with correct coverage from two-sided confidence intervals to upper confidence limits for $n \leq b$, it has the undesirable feature that when $n \leq b$ the upper bound $\mu_1(n; b, \alpha)$ decreases rather rapidly when $b$ increases and stabilizes around a value close to 0.8 for large values of $b$. From a physical point of view this is rather disturbing, because a stringent upper bound for $\mu$ obtained by an experiment which has observed a number of events significantly smaller than the expected background is not due to the fact that the experiment is very sensitive to small values of $\mu$, but to the fact that less background events than expected have been observed.

This is the case of the null result of the KARMEN 2 experiment, from which the Unified Approach yields an upper 90% CL confidence limit for the mean $\mu$ of neutrino oscillation events of 1.1 events. The corresponding exclusion curve in the plane of the neutrino mixing parameters $\sin^2 2\theta$ and $\Delta m^2$ is shown in Fig. 1 (the solid curve passing through the filled circles) and one can see that it is significantly more stringent than the exclusion curve obtained with the Bayesian Approach (the solid curve passing through the filled squares). The strictness of the Unified Approach exclusion curve is due to the non-observation of the expected background events and not to the sensitivity of the experiment (see the discussion in Ref. [10] and the sensitivity of the KARMEN experiment presented in Ref. [6]). This is clearly an undesirable result from a physical point of view, because the statistical interpretation of the data produces an exaggeratedly stringent result that could lead to incorrect physical conclusions.

The discrepancy between the Bayesian Approach exclusion curve and the Unified Approach exclusion curve is worrying for a physicist, because the Bayesian Approach exclusion curve is compatible with a large part of the LSND-allowed region (the shadowed area in Fig. 1), whereas the Unified Approach exclusion curve excludes almost all the LSND allowed region.

In Ref. [11] I have proposed a new ordering principle for the construction of a classical confidence belt which has all the desirable features of the one in the Unified Approach (i.e. an automatic transition with the correct coverage from two-sided confidence intervals to upper confidence limits when the observed number of events is of the order or less than the expected background) and in addition minimizes the decrease of the upper confidence limit for a given $n$ as the mean expected background $b$ increases. The new ordering principle is implemented as the Feldman and Cousins ordering principle in the Unified Approach, but for each value of $\mu$ the rank of each value of $n$ is calculated from the relative size of the likelihood ratio

$$R_{\text{NO}}(n|\mu; b) = \frac{P(n|\mu; b)}{P(n|\mu_{\text{ref}}; b)},$$

where the reference value $\mu_{\text{ref}} = \mu_{\text{ref}}(n; b)$ is taken to be the bayesian expected value for $\mu$:

$$\mu_{\text{ref}}(n; b) = \int_{0}^{\infty} \mu P(\mu|n; b) \, d\mu = n + 1 - \left( \sum_{k=0}^{n} \frac{k^k}{k!} \right) \left( \sum_{k=0}^{n} \frac{k^k}{k!} \right)^{-1}. \quad (2.9)$$

Here $P(\mu|n; b)$ is the bayesian probability distribution for $\mu$ calculated assuming a constant prior for $\mu \geq 0$ (see, for example, [11]):
\[ P(\mu|n; b) = (b + \mu)^n e^{-\mu} \left( n! \sum_{k=0}^{n} \frac{b^k}{k!} \right)^{-1}. \]  

(2.10)

The obvious inequality \( \sum_{k=0}^{n} k b^k / k! \leq n \sum_{k=0}^{n} b^k / k! \) implies that \( \mu_{\text{ref}}(n; b) \geq 1 \). Therefore, \( \mu_{\text{ref}}(n; b) \) represents a reference value for \( \mu \) that not only is non-negative, as desired in order to have an automatic transition from two-sided intervals to upper limits, but is even bigger or equal than one. This is a desirable characteristics in order to obtain a weak decrease of the upper confidence limit for a given \( n \) when the expected background \( b \) increases. Indeed, it has been shown in Ref. [10] that for \( n \lesssim b \) the upper bound \( \mu_1(n; b, \alpha) \) decreases rather weakly when \( b \) increases and stabilizes around a value close to 1.7 for large values of \( b \). This behaviour of \( \mu_1(n; b, \alpha) \) is more acceptable for the physical interpretation of experimental results than the behaviour of \( \mu_1(n; b, \alpha) \) in the Unified Approach.

The 90% CL upper confidence limit for the mean \( \mu \) of neutrino oscillation events following from the analysis of the KARMEN 2 null result with the New Ordering Approach is of 1.9 events and the corresponding exclusion curve in the plane of the neutrino mixing parameters \( \sin^2 2\theta \) and \( \Delta m^2 \) is shown in Fig. 1 (the solid curve passing through the filled triangles). One can see that this exclusion curve, although obtained with a method that guarantees the correct frequentist coverage as the Unified Approach, is significantly less stringent than the one obtained with the Unified Approach and lies close to the bayesian exclusion curve.

Hence, the New Ordering Approach has solved the apparent conflict between the frequentist and bayesian statistical interpretation of the null result of the KARMEN 2 experiment: by choosing an appropriate ordering principle in the construction of the confidence belt, the exclusion curve obtained with the frequentist method is in reasonable agreement with the one obtained with the Bayesian Approach.

Nevertheless, some concern still remain on the interpretation of the fact that the KARMEN 2 experiment did not observe any of the expected background events. Such concern will increase in the future if the KAREN 2 experiment will continue to observe significantly less events than those expected from the background. Hence, it is interesting to explore other possibilities for the statistical interpretation of the result of the KARMEN2 experiment. This is the aim of the following three Sections.

**III. LARGER BACKGROUND ERROR**

Since no background events have been measured in the KARMEN 2 experiment and the mean expected background is \( b = 2.88 \pm 0.13 \), it is possible to doubt of the correctness of the calculated error for the mean expected background. In this Section, I will investigate which would be the physical implications of the KARMEN 2 null result if the calculated error for the mean expected background has been underestimated by an order of magnitude, \( i.e. \) I will take \( b = 2.88 \pm 1.3 \). I will assume a normal probability distribution function for the mean expected background \( b \):

\[ f(b; \overline{b}, \sigma_b) = \frac{1}{\sqrt{2\pi} \sigma_b} \exp \left[ -\frac{(b - \overline{b})^2}{2\sigma_b^2} \right], \]  

(3.1)
with $\bar{b} = 2.88$ and $\sigma_b = 1.3$. For simplicity, since $\sigma_b \lesssim \bar{b}/3$, I will consider $b$ varying from $-\infty$ to $+\infty$, neglecting the small error introduced by considering negative values of $b$. This approximation allows an analytic solution of the integrals involved in the calculation.

If $\mu$ is the mean of true signal events, the probability $P(n|\mu; \bar{b}, \sigma_b)$ to observe $n$ events is given by

$$
P(n|\mu; \bar{b}, \sigma_b) = \int P(n|\mu; b) f(b; \bar{b}, \sigma_b) \, db,
$$

(3.2)

with the Poisson probability $P(n|\mu; b)$ given in Eq. (2.1). Hence, $P(n|\mu; \bar{b}, \sigma_b)$ can be written as

$$
P(n|\mu; \bar{b}, \sigma_b) = \frac{1}{n!} e^{-\mu} \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} I_k(\bar{b}, \sigma_b),
$$

(3.3)

with

$$
I_k(\bar{b}, \sigma_b) \equiv \frac{1}{\sqrt{2\pi} \sigma_b} \int_{-\infty}^{+\infty} b^k \exp \left[ -b - \frac{(b - \bar{b})^2}{2 \sigma_b^2} \right] \, db.
$$

(3.4)

The integral $I_k(\bar{b}, \sigma_b)$ can be written as

$$
I_k(\bar{b}, \sigma_b) = \exp \left[ -\bar{b} + \frac{\sigma_b^2}{2} \right] \sum_{j=0}^{k} \binom{k}{j} (\bar{b} - \sigma_b^2)^{k-j} \sigma_b^j m_j,
$$

(3.5)

where $m_j$ is the $j$th central moment of the normal distribution with unit variance,

$$
m_j \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} b^j e^{-b^2/2} \, db.
$$

(3.6)

Taking into account that $\int b e^{-b^2/2} \, db = -e^{-b^2/2}$, the integral in Eq. (3.6) can be calculated by parts, yielding

$$
m_j = \frac{j!}{(j/2)! \, 2^{j/2}}
$$

(3.7)

for $j$ even and $m_j = 0$ for $j$ odd.

From Eqs. (3.3), (3.5) and (3.7), for the probability of $n$ events one obtains

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1If $b$ is restricted to the interval $[0, +\infty)$ the exact normalization factor of $f(b; \bar{b}, \sigma_b)$ replacing $1/\sqrt{2\pi} \sigma_b$ in Eq. (3.1) is $1/N(\bar{b}, \sigma_b)$ with

$$
N(\bar{b}, \sigma_b) = \sqrt{\frac{\pi}{2}} \left[ 1 + \text{erf} \left( \frac{\bar{b}}{\sqrt{2} \sigma_b} \right) \right] \sigma_b.
$$

In this case $N(\bar{b}, \sigma_b)$ and the integral in Eq. (3.2) must be calculated numerically.
\[ P(n|\mu; \bar{b}, \sigma_b) = \exp \left[ -\left( \mu + \bar{b} \right) + \frac{\sigma_b^2}{2} \right] \sum_{k=0}^{n} \frac{\mu^{n-k}}{(n-k)!} J_k(\bar{b}, \sigma_b), \quad (3.8) \]

with

\[ J_k(\bar{b}, \sigma_b) \equiv \sum_{j=0}^{k/2} \frac{\left( \bar{b} - \sigma_b^2 \right)^{k-2j} \sigma_b^{2j}}{(k-2j)! j! 2^j}. \quad (3.9) \]

Equation (3.8) gives the formula for the probability \( P(n|\mu; \bar{b}, \sigma_b) \) to observe a number \( n \) of events in a Poisson process consisting in signal events with mean \( \mu \) and background events with known mean \( \bar{b} = \bar{b} \pm \sigma_b \), i.e. it replaces Eq.(2.1) if the error \( \sigma_b \) of the calculated mean background is not negligible (but \( \sigma_b \lesssim \bar{b}/3 \)). The construction of the confidence belt follows the same procedure described in Section II and now the confidence interval for \( \mu \) corresponding to a number \( n_{\text{obs}} \) of observed events is \([\mu_2(n_{\text{obs}}; \bar{b}, \sigma_b, \alpha), \mu_1(n_{\text{obs}}; \bar{b}, \sigma_b, \alpha)]\), i.e. it depends on \( \bar{b} \) and \( \sigma_b \). The acceptance intervals can be constructed following the same principles discussed in Section II, i.e. one can construct the confidence belt for central confidence intervals or upper confidence limits, or the confidence belt in the Unified Approach or in the New Ordering Approach. In the remaining part of this paper I will present the formalism for the implementation of the Unified Approach and of the New Ordering Approach and I will discuss the corresponding implications for the statistical interpretation of the KARMEN 2 null result.

The quantity \( \mu_{\text{best}}(n; \bar{b}, \sigma_b) \) in the Unified Approach is the value of \( \mu \) that maximizes \( P(n|\mu; \bar{b}, \sigma_b) \) and the acceptance interval for each value of \( \mu \) is calculated assigning at each value of \( n \) a rank obtained from the relative size of the ratio

\[ R_{\text{UA}}(n|\mu; \bar{b}, \sigma_b) = \frac{P(n|\mu; \bar{b}, \sigma_b)}{P(n|\mu_{\text{best}}; \bar{b}, \sigma_b)}. \quad (3.10) \]

The value of \( \mu_{\text{best}}(n; \bar{b}, \sigma_b) \) can be easily calculated by hand for \( n = 0, 1, 2 \), whereas for higher values of \( n \) it can be calculated numerically. The resulting confidence belt for \( \bar{b} = 2.88 \) and \( \sigma_b = 1.3 \) is plotted in Fig. 2 (the region between the two dashed lines), where it is compared with the confidence belt for \( \sigma_b = 0 \) (the region between the two solid lines), which practically coincides with the confidence belt for \( \sigma_b = 0.13 \). The confidence belt for \( \sigma_b = 1.3 \) is larger than the one for \( \sigma_b = 0 \) because the integral in Eq.(3.2) has the effect of flattening the probability \( P(n|\mu; \bar{b}, \sigma_b) \) as a function of \( n \) for fixed \( \mu \) with respect to \( P(n|\mu; b = \bar{b}) \) and this flattening effect increases with the size of \( \sigma_b \).

From Fig. 3 one can see that the upper limit \( \mu_1 \) for \( n = 0 \) is of 1.2 events, only slightly larger than the upper limit of 1.0 events obtained for \( \sigma_b = 0 \). Therefore, the exclusion curve that follow from the null result of the KARMEN 2 experiment if \( \sigma_b = 1.3 \) lies close to the solid curve passing through the filled circles in Fig. 3 (corresponding to \( \mu_1 = 1.1 \)).

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2In the Unified Approach of Ref. 11 \( \sigma_b = 0 \) but the upper limit \( \mu_1 \) for \( n = 0 \) is of 1.1 events because \( \mu_1 \) is forced to be a non-increasing function of \( \bar{b} \).
In the New Ordering Approach the acceptance interval for each value of $\mu$ is calculated assigning at each value of $n$ a rank obtained from the relative size of the ratio

$$R_{NO}(n|\mu; \bar{b}, \sigma_b) = \frac{P(n|\mu; \bar{b}, \sigma_b)}{P(n|\mu_{ref}; \bar{b}, \sigma_b)},$$

(3.11)

where the reference value $\mu_{ref} = \mu_{ref}(n; \bar{b}, \sigma_b)$ is the bayesian expected value for $\mu$. The value of $\mu_{ref}(n; \bar{b}, \sigma_b)$ can be calculated analytically. For the bayesian probability distribution function for $\mu$ with a constant prior,

$$P(\mu|n; \bar{b}, \sigma_b) = \frac{P(n|\mu; \bar{b}, \sigma_b)}{\int_{0}^{\infty} P(n|\mu; \bar{b}, \sigma_b) \, d\mu},$$

(3.12)

one obtains

$$P(\mu|n; \bar{b}, \sigma_b) = e^{-\mu} \left( \sum_{k=0}^{n} \frac{\mu^{n-k}}{(n-k)!} J_k(\bar{b}, \sigma_b) \right) \left( \sum_{k=0}^{n} J_k(\bar{b}, \sigma_b) \right)^{-1}.$$

(3.13)

Hence, the reference value $\mu_{ref}(n; \bar{b}, \sigma_b)$, which is the bayesian expected value for $\mu$, is given by

$$\mu_{ref}(n; \bar{b}, \sigma_b) = n + 1 - \left( \sum_{k=0}^{n} k \cdot J_k(\bar{b}, \sigma_b) \right) \left( \sum_{k=0}^{n} J_k(\bar{b}, \sigma_b) \right)^{-1}.$$

(3.14)

The resulting confidence belt in the New Ordering Approach for $\bar{b} = 2.88$ and $\sigma_b = 1.3$ is shown in Fig. [3] (the region between the two dashed lines), where it is compared with the confidence belt for $\sigma_b = 0$ (the region between the two solid lines), which practically coincides with the confidence belt for $\sigma_b = 0.13$. One can see that the upper limit for $\mu$ if $n_{obs} = 0$ is of 2.0 events, only slightly larger than the upper limit of 1.9 events obtained for $\sigma_b = 0$. Therefore, the KARMEN 2 exclusion curve for $\sigma_b = 1.3$ lies close to the solid curve passing through the filled triangles in Fig.[4] (for $\sigma_b = 1.3$ the ordinate of the exclusion curve for high $\Delta m^2$’s is $\sin^2 2\theta = 2.5 \times 10^{-3}$, compared with $\sin^2 2\theta = 2.3 \times 10^{-3}$ for $\sigma_b = 0$).

The conclusion that one can reach from the results presented in this Section is that the error of the calculated mean background has little effect on the exclusion curves obtained in the Unified Approach and in the New Ordering Approach (at least for $\sigma_b \lesssim \bar{b}/3$). Even if the error $\sigma_b$ of the mean background in the KARMEN 2 experiment is increased by an order of magnitude with respect to the calculated one ($\sigma_b = 0.13$), the exclusion curves in the Unified Approach and in the New Ordering Approach are practically equivalent to the corresponding curves calculated for $\sigma_b = 0$.

**IV. UNKNOWN BACKGROUND**

Since the KARMEN 2 experiment did not observe any event, with a mean expected background $\bar{b} = 2.88 \pm 0.13$, and even increasing by an order of magnitude the estimated error of the mean expected background does not help in solving the discrepancy among the exclusion
curves calculated with different statistical methods, one can consider the extreme possibility of doubting of the correctness of the calculated mean expected background and ignore it. In other words, one can assume that the background is unknown and that the KARMEN 2 null measurement gives an estimation of the mean $\mu + b$ of signal plus background events.

This approach leads to an upper confidence limit $(\mu + b)_1$ which depends only on the observed number of events. Since $b \geq 0$, the upper confidence limit $(\mu + b)_1$ for $\mu + b$ is also a conservative upper confidence limit for $\mu$. Although the derivation of a conservative upper confidence limit is generally an undesirable result, in controversial cases as that of the KARMEN 2 experiment it may be a safe choice.

Actually, in the implementation of the Unknown Background approach one has also to chose the method for building the confidence belt. I will consider the following four possibilities, whose confidence belts are plotted in Fig. 4:

**Standard confidence belt for upper confidence limits.** The resulting 90% CL confidence belt is the region below the solid line in Fig. 4. This method implies an upper 90% CL confidence limit of 2.3 events for $\mu + b$ if the number of observed events $n_{\text{obs}}$ is zero. Taking it as a conservative upper confidence limit for $\mu$ leads to an exclusion curve coinciding with the bayesian exclusion curve shown in Fig. 1. This exclusion curve is plotted in Fig. 5 (the solid curve passing through the filled squares).

**Standard confidence belt for central confidence intervals.** The resulting 90% CL confidence belt is the region between the two dashed lines in Fig. 4 and the 90% CL upper limit for $\mu + b$ is of 3.0 events if $n_{\text{obs}} = 0$. The corresponding exclusion curve is plotted in Fig. 5 (the solid curve passing through the dotted squares).

**Unified Approach.** The 90% CL confidence belt is the region between the two dash-dotted lines in Fig. 4 and the upper limit for $\mu + b$ is of 2.4 events for $n_{\text{obs}} = 0$. The corresponding exclusion curve is plotted in Fig. 5 (the solid curve passing through the dotted circles).

**New Ordering Approach.** The 90% CL confidence belt is the region between the two dash-dot-dotted lines in Fig. 4 and implies an upper limit of 2.6 events for $\mu + b$ if $n_{\text{obs}} = 0$. The corresponding exclusion curve is plotted in Fig. 5 (the solid curve passing through the dotted triangles).

One can see that the four exclusion curves in Fig. 5 obtained assuming an unknown background with the four different statistical methods listed above do not differ much. This is a good sign and in practice tends to support the exclusion curve obtained from the standard confidence belt for upper confidence limits, which coincides with the bayesian exclusion curve plotted in Fig. 1.

However, it must be noted that the choice to use the Unknown Background approach has been done on the basis of the result of the experiment. This procedure introduces some overcoverage, as in the flip-flopping policy discussed by Feldman and Cousins [9].

For example, one can consider the possibility to take $b = n_{\text{obs}}$ for $n_{\text{obs}} < 3$ and $b = 2.88$ for $n_{\text{obs}} \geq 3$. Using the standard confidence belts for upper confidence limits for $b = 0, 1, 2, 2.88$ shown in Fig. 4, one obtain the confidence belt below the thick solid line in Fig. 4. It is clear
from the figure that this policy leads to overcoverage for $\mu \lesssim 3.3$, that is the interesting region in practice. The value of the induced overcoverage is shown in Fig. 7 where I plotted the sum of the probabilities corresponding to the $n$‘s lying in the acceptance interval for a given $\mu$ of the confidence belt below the thick solid line in Fig. 6 as a function of $\mu$. One can see that for $\mu \lesssim 3.3$ this sum is even bigger than one! This is due to the fact that the probability has not been properly normalized to take into account that $b$ depends on $n$. (The fact that for $\mu \gtrsim 3.3$ the sum of the probabilities is bigger than 0.90 shows the unavoidable overcoverage in the case of a Poisson processes due to the discreteness of $n$. See, for example, Refs. [7–9].)

In conclusion of this Section, we have seen that the Unknown Background approach is allowed and the corresponding exclusion curves obtained with four different statistical methods do not differ much. In practice this result represents a support for the bayesian exclusion curve plotted in Fig. 1. However, it is necessary to emphasize that the Unknown Background approach is conservative for two reasons: 1) The upper confidence limit for the mean $\mu$ neutrino oscillation events is taken to be equal to the upper confidence limit for $\mu + b$, where $b$ is the unknown mean background events; 2) The choice to use the Unknown Background approach is done on the basis of the result of the experiment, introducing some overcoverage. Hence, we can consider the exclusion curves obtained with the Unknown Background approach as ultra-conservative.

V. SENSITIVITY

In their Unified Approach paper [9] Feldman and Cousins suggested that in the cases in which the measurement is less than the estimated mean background, the experimental collaboration should report also the sensitivity of the experiment, defined as “the average upper limit that would be obtained by an ensemble of experiments with the expected background and no true signal”. This is also recommended by the Particle Data Group [8].

From the definition, the sensitivity $\mu_s$ of an experiment measuring a Poisson process with an expected mean background $b$ with negligible error is given by

$$\mu_s(b, \alpha) = \sum_{n=0}^{\infty} \mu_1(n; b, \alpha) P(n|\mu = 0, b),$$

(5.1)

One can also calculate the coverage of the confidence belt below the thick solid line in Fig. 6 for different fixed values of $b$. In this case one obtains overcoverage for small values of $\mu$ if $b$ is large (for example, $\mu = 2$ and $b = 2.88$) and undercoverage for large values of $\mu$ if $b$ is small (for example, $\mu = 4$ and $b = 0$). However, this calculation is in contradiction with the assumption that $b$ is fixed by the procedure, i.e. $b = n$ for $n < 3$ and $b = 2.88$ for $n \geq 3$.

This formula can be easily generalized to the case of an expected mean background $b = \bar{b} \pm \sigma_b$ with non-negligible error $\sigma_b$: $\mu_s(\bar{b}, \sigma_b, \alpha) = \sum_{n=0}^{\infty} \mu_1(n; \alpha) P(n|\mu = 0, \bar{b}, \sigma_b)$, with $P(n|\mu; \bar{b}, \sigma_b)$ given by Eq.(3.2).
where $\alpha$ is the confidence level of the upper limits $\mu_1(n; b, \alpha)$. The sensitivity $\mu_s$ depends on the values of the upper limits $\mu_1(n; b, \alpha)$, which are different in the different methods for calculating the confidence belt (e.g., in the Unified Approach and in the New Ordering Approach).

The sensitivity of the KARMEN 2 experiment for $\alpha = 90\%$ is $\mu_s = 4.4$ [1] in the Unified Approach and $\mu_s = 4.7$ [10] in the New Ordering Approach. The corresponding sensitivity curves are shown in Fig. 1 (the solid curves passing through the empty circles and triangles, respectively).

One could think of considering the sensitivity curve of the KARMEN 2 experiment as its exclusion curve. However, this is wrong for the following reasons. From Eq. (5.1) it is clear that $\mu_s$ is the expected value of the upper limit $\mu_1$ under the hypothesis that $\mu = 0$. This is the only correct meaning of $\mu_s$. Even though the value of $\mu_s$ depends on the confidence level $\alpha$, the interval $[0, \mu_s]$ does not have a confidence level $\alpha$ from a frequentist point of view.

Notice that $\mu_s$ does not depend on any experimental result and can be calculated for real as well as hypothetical experiments. If the sensitivity curve corresponding to $\mu_s(b, \alpha)$ could be interpreted as an exclusion curve with confidence level $\alpha$, one could obtain exclusion curves (i.e., experimental results) without performing any experiment! This apparent paradox is solved by noting that $\mu_s$ is not calculated by assuming that the number of observed events is zero (all values of $n$ are taken into account in Eq. (5.1)) but assuming that the true value of $\mu$ is zero and if the true value of $\mu$ would be known it would be useless to do any experiment to measure it.

Each sensitivity curve in Fig. 1 is shifted to the right with respect to the corresponding exclusion curve, although it is calculated under the assumption that $\mu = 0$, because the sensitivity curve takes into account the possibility that an experiment with mean background $b$ can measure a number of events $n > b$, even if $\mu = 0$. For example, in the Unified Approach with $b = 2.88$ one has $\mu_1(n = 3; \alpha = 0.90) = 4.5$, $\mu_1(n = 4; \alpha = 0.90) = 5.7$, $\mu_1(n = 5; \alpha = 0.90) = 7.1$, and so on. On the other hand, since $\mu_1(n = 2; \alpha = 0.90) = 3.1$, an experiment measuring two events, in good agreement with the mean expected background $b = 2.88$, can draw an exclusion curve that is significantly more stringent than the sensitivity curve for $b = 2.88$.

Since the sensitivity curve of a neutrino oscillation experiment can be calculated before doing the experiment, it is useful in order to plan future experiments that cover approximately the region of interest in the plane of the neutrino mixing parameters $\sin^2 2\theta$ and $\Delta m^2$.

VI. CONCLUSIONS

The statistical interpretation of the null result of the KARMEN 2 neutrino oscillation experiment is rather problematic because no events were observed with a mean expected background of $2.88 \pm 0.13$ events [1]. The exclusion curves obtained with the Bayesian Approach and with the Unified Approach [3] are significantly different and yield contradicting indications on the compatibility of the KARMEN 2 result with the neutrino oscillation signal measured in the LSND experiment [4] (see Fig. 1). The analysis of the KARMEN 2 null result with the New Ordering Approach [10], which is a frequentist method with correct
coverage as the Unified Approach, yields an exclusion curve close to the one obtained with the Bayesian Approach. In this way, the undesirable discrepancy between frequentist and bayesian interpretations of the KARMEN 2 null result is removed. However, some concern still persist on the interpretation of the fact that none of the expected background events have been observed in the KARMEN 2 experiment. This concern will increase in the future if the KARMEN 2 experiment will continue to observe significantly less background events than expected. Hence, it is interesting to investigate other possibilities for the statistical interpretation of the KARMEN 2 null result.

In this paper I have discussed three alternative statistical interpretation of the KARMEN 2 null result: the Larger Background Error possibility in Section III, the Unknown Background possibility in Section IV and the Sensitivity possibility in Section V.

In Section III I have presented the formalism that allows to take into account the error of the expected mean background. However, the conclusion of this section is that taking into account the error of the expected mean background in the KARMEN 2 experiment, even if it is wrong by an order of magnitude, does not help in solving the problem of the statistical interpretation of the result of this experiment, because the resulting exclusion curves are practically equivalent to the ones obtained assuming no error for the expected mean background.

The Unknown Background approach discussed in Section IV gives ultra-conservative exclusion curves, which in the case of the KARMEN 2 experiment tend to support the exclusion curve obtained with the Bayesian Approach. Obtaining ultra-conservative exclusion curves is generally not desirable, but could be a safe choice in controversial cases as that of the KARMEN 2 experiment.

The possibility to consider a Sensitivity curve as an exclusion curve is discussed in Section V and is shown to be wrong. Since the sensitivity curve of a neutrino oscillation experiment can be calculated before doing the experiment, its usefulness lies in the possibility to plan future experiments in order to cover approximately the region of interest in the plane of the neutrino mixing parameters $\sin^2 2\theta$ and $\Delta m^2$.

Finally, I would like to remark that the direct comparison of exclusion curves and allowed regions obtained with different statistical methods does not have a precise statistical significance. Hence, such a comparison cannot be used to combine the results of different experiments or to infer with some known confidence level a contradiction between the results of different experiments when the comparison is done on the border of the exclusion curves and of the allowed regions. Hence, the comparison of the KARMEN 2 exclusion curves and the LSND-allowed region, which was obtained with a different statistical analysis (see Ref. [4]), must be done with great caution. In the future, if the KARMEN 2 experiment will continue to observe no neutrino oscillations, it will be possible to claim a contradiction between the results of the two experiments only when the exclusion curves obtained from the result of the KARMEN 2 experiment with different statistical methods will produce similar results and will lie well on the left of the region allowed by the results of the LSND experiment.
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FIGURES

Fig. 1. 90% CL exclusion curves in the plane of the neutrino oscillation parameters \( \sin^2 2\theta - \Delta m^2 \) corresponding to the null result of the KARMEN 2 experiment \([1]\). The solid curves passing through the filled squares, circles and triangles are obtained with the Bayesian Approach, the Unified Approach and the New Ordering Approach, respectively. The solid curves passing through the empty circles and triangles are the sensitivity curves obtained with the Unified Approach and the New Ordering Approach, respectively. The shadowed area is the region allowed at 90% CL by the results of the LSND experiment \([4]\) and the dashed, dash-dotted and dash-dot-dotted curves are the 90% CL exclusion curves of the Bugey \([13]\), BNL E776 \([14]\) and CCFR \([15]\) experiments, respectively.

Fig. 2. Confidence belts for 90% CL obtained with the Unified Approach for \( \bar{b} = 2.88 \) and \( \sigma_b = 0 \) (the region between the two solid lines) and \( \sigma_b = 1.3 \) (the region between the two dashed lines).

Fig. 3. Confidence belts for 90% CL obtained with the New Ordering Approach for \( \bar{b} = 2.88 \) and \( \sigma_b = 0 \) (the region between the two solid lines) and \( \sigma_b = 1.3 \) (the region between the two dashed lines).

Fig. 4. Confidence belts for 90% CL obtained in the case of unknown background (see Section IV) with the standard method for upper confidence limits (the region under the solid line), with the standard method for central confidence intervals (the region between the two dashed lines), with the Unified Approach (the region between the two dash-dotted lines) and with the New Ordering Approach (the region between the two dash-dot-dotted lines). The lower confidence limits \( (\mu + b)_2 \) calculated with the Unified Approach and the New Ordering Approach coincide for \( n < 3 \). All the confidence belts include the origin of the coordinates \( n, \mu + b \).

Fig. 5. 90% CL exclusion curves obtained in the case of unknown background (see Section IV) with the standard method for upper confidence limits (the solid curve passing through the filled squares), with the standard method for central confidence intervals (the solid curve passing through the dotted squares), with the Unified Approach (the solid curve passing through the dotted circles) and with the New Ordering Approach (the solid curve passing through the dotted triangles).

Fig. 6. Standard 90% CL confidence belts for upper confidence limits for \( b = 0, 1, 2, 2.88 \) (the region below the dashed, dotted, dash-dotted and dash-dot-dotted line, respectively). The region below the thick solid line is the confidence belt obtained taking \( b = n \) for \( n < 3 \) and \( b = 2.88 \) for \( n \geq 3 \).
Fig. 7. Sum of the probabilities corresponding the $n$’s lying in the acceptance interval for a given $\mu$ of the confidence belt below the thick solid line in Fig. 6 as a function of $\mu$. 
$\sin^2 2\theta$

$\Delta m^2$ (eV$^2$)

Figure 1
Unified Approach

\[ \sigma_b = 0 \]
\[ \sigma_b = 1.3 \]

Figure 2
\begin{figure}
\centering
\includegraphics[width=\textwidth]{new_ordering.pdf}
\caption{New Ordering}
\end{figure}
Figure 4

Unknown Background
Figure 6: Upper Confidence Limit with $b = n$ for $n < 3$ and $b = 2.88$ for $n \geq 3$
Figure 7

Upper Confidence Limit with $b = n$ for $n < 3$ and $b = 2.88$ for $n \geq 3$