The generalized Harer conjecture for the homology triviality

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Abstract
The classical Harer conjecture states the stable homology triviality of the canonical embedding $\phi : B_{2g+2} \to \Gamma_g$, which was proved by Song and Tillmann. The main part of the proof is to show that $B\phi^+ : B\Gamma_{\infty}^+ \to B\Gamma_{\infty}^+$, induced from $\phi$, is a double-loop space map. In this paper, we give a proof of the generalized Harer conjecture concerning the homology triviality for every regular embedding $\phi : B_n \to \Gamma_{g,k}$. The main strategy of the proof is to remove all the interchangeable subsurfaces from $S_{g,k}$ and collapse the new boundary components. Then, we obtain (the union of) covering spaces over a disk with $n$ marked points that we can analyze. The final goal is to show that the map $\Phi : C \to S$ induced by $B\phi : \text{Conf}_n(D) \to \mathcal{M}_{g,k}$ preserves the actions of the framed little 2-disks operad.

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1 | INTRODUCTION

Let $B_n$ be Artin’s braid group and $\Gamma_{g,k}$ the mapping class group of an orientable surface $S_{g,k}$ with genus $g$ and $k$ boundary components. There is a canonical embedding $\phi : B_{2g+2} \to \Gamma_g$, $\beta_i \mapsto \alpha_i$, where $\beta_i$ (1 $\leq$ $i$ $\leq$ 2$g$ + 1) are the usual generators of $B_{2g+2}$, and $\alpha_i$ are the Dehn twists along chain curves $a_i$ with $a_i \cap a_{i+1} = \{\ast\}$. The original Harer conjecture, raised in 1980’s, is that

$$\phi_* : H_*(B_{\infty}; \mathbb{Z}/2) \to H_*(\Gamma_{\infty}; \mathbb{Z}/2)$$

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is trivial. It has been well known that in order to show that this homomorphism is trivial, it suffices to show that $\phi_*$ preserves Araki–Kudo–Dyer–Lashof operations that come from the double-loop space structures [6]. We need to show that the map

$$B\phi^+ : B B^+_\infty \to B\Gamma^+_\infty$$

is a map of double-loop spaces. We will refer to this as Proposition X. As a matter of fact, Proposition X implies a much stronger fact (than the Harer conjecture) that $B\phi^+$ is homotopically trivial, because $B B^+_\infty \simeq \Omega^2 S^3 \simeq \Omega^2 \Sigma^2 S^1$ and every double-loop space map from $B B^+_\infty$ is determined by its restriction $S^1 \subset \Omega^2 \Sigma^2 S^1$ up to homotopy. Note that $B\Gamma^+_\infty$ is simply connected by Powell theorem. Hence, Proposition X implies a stronger version of the Harer conjecture:

$$\phi_* : H_i(B_\infty, R) \to H_i(\Gamma_\infty, R)$$

is trivial for all $i \geq 1$ and any constant coefficient $R$. This stronger version of Harer conjecture is called the homology triviality in this paper.

The original Harer conjecture (and its homology triviality) was proved by Song and Tillmann [11]. They constructed two monoidal 2-categories, one corresponding to braid groups and another to mapping class groups, and lifted the embedding $\phi$ to a monoidal 2-functor between them.

An alternative proof of Proposition X was suggested by Segal and Tillmann [10]. Their idea is to lift $\phi$ to the map $\Phi : \text{Conf}_{2g+2}(D) \to \mathcal{M}_{g,2}$, where $\text{Conf}_{2g+2}(D)$ denotes the configuration space of unordered $2g + 2$ distinct points on a disk $D$ and $\mathcal{M}_{g,2}$ denotes the moduli space on surface $S_{g,2}$. Note that $\text{Conf}_n(D) \simeq B\mathbb{B}_n$ and $\mathcal{M}_{g,2} \simeq B\Gamma_{g,2}$. They showed that $\Phi$ is compatible with the natural actions of the framed little 2-disks operad on configuration spaces and moduli spaces, which implies Proposition X. They interpreted the map $\phi$ as one induced by twofold branched covering over a disk with $2g + 2$ punctures. That is, a full Dehn twist on a surface is regarded as a lift, via twofold covering, of a half Dehn twist interchanging two points on a disk.

On the other hand, Kim–Song [9] found the construction of a nongeometric embedding $\phi : B_n \hookrightarrow \Gamma_{g,k}$ induced by threelfold covering, and proved the homology triviality. They [4, 9] and Ghaswala–McLeay [7] independently constructed an infinite family of nongeometric embeddings $\phi_d$ induced by $d$-fold ($d \geq 3$) branched coverings. In Section 2, we give a proof that every embedding induced by a covering is homologically trivial (Theorem 2.5).

It is natural to raise the question whether the Harer conjecture holds for every embedding. This question has been open for some years. In this paper, we prove that Harer conjecture holds for almost every embedding that is called a regular embedding. The definition of a regular embedding will be given in Definition 3.3. All known embeddings are regular. The following theorem is the main result of this paper.

**Theorem 4.1.** For every regular embedding $\phi : B_n \hookrightarrow \Gamma_{g,k}$,

$$\phi_* : H_i(B_\infty, R) \to H_i(\Gamma_\infty, R)$$

is trivial for all $i \geq 1$ and any coefficient $R$. 
The key idea of the proof is that a regular embedding, through a series of geometric surgeries, can be transformed into an embedding similar to that induced by $d$-fold branched covering.

Note that we are working on the stable range. Recall that as $g$ grows big enough, $H_i(\Gamma_{g,k}; R)$ stabilizes (Harer–Ivanov stability theorem). That is, we do not have to pay attention to the number of boundary components of the surface $S_{g,k}$.

To every embedding $\phi : B_n \hookrightarrow \Gamma_{g,k}$, there is a corresponding map $\Phi : \text{Conf}_n(D) \rightarrow \mathcal{M}_{g,k}$ by Earle–Eells theory. We show, by the same tactic as that of Segal and Tillmann [10], that the map between two algebras induced by $\Phi$ preserves the actions of the framed little 2-disk operad.

For a self-homeomorphism $f$ of a surface, we define the support $\text{supp}(f)$ to be the set of nonfixed points of $f$ (not the closure of the set of nonfixed points). Consider an embedding $\phi : B_n \hookrightarrow \Gamma_{g,k}$. Then, $S_{g,k}$ contains the union of all $T_i := \text{supp}(\tau_i)$, for $i = 1, \ldots, n - 1$. Because we are dealing with the surface of a huge genus, we may disregard the remaining fixed part of the surface (Harer–Ivanov stability theorem), that is, we may consider $S_{g,k}$ as only the union of all $T_i$. Each $T_i$ is a subsurface of $S_{g,k}$ and $\tau_i$ fixes the boundary components of $T_i$ pointwise and moves all interior points of $T_i$ while interchanging two identical subsurfaces (or points) $I_i$ and $I_{i+1}$ of $T_i$ by a ‘simple twist’ (Definition 3.2). We may also assume that all $T_i$ are homeomorphic to each other (Lemma 3.1) and connected. We define such an embedding to be a regular embedding (Definition 3.3).

In order to show that the $D$-algebra map (which will be defined in Section 2) induced by $\Phi : \text{Conf}_n(D) \rightarrow \mathcal{M}_{g,k}$ preserves the actions of the framed little 2 disks operad, we need to figure out the formula for the number of boundary components of the surface, because the action of the operad on the moduli spaces is a kind of ‘capping’ the holes of the surfaces by the framed little 2-disk operad.

In Section 2, we recall the construction of the embedding $\phi_d : B_n \rightarrow \Gamma_{g,k}$ induced by $d$-fold covering over a disk, and show that it satisfies the homology triviality by constructing the algebra map over the framed little 2-disk operad.

In Section 3, we introduce the definition of regular embedding and give a few examples of nongeometric embeddings that are all regular embeddings.

In Section 4, we prove the main theorem in three steps.

In Step I, we detach $I_i$ and $I_{i+1}$ from each $T_i$ (note that $I_{i+1} \subset T_i \cap T_{i+1}$) and let the holes (the boundary components of $I_i$ and $I_{i+1}$) collapse to points. Because all $I_j$ are identical, we may consider only $I_1$. Let $\overline{T}_1$ be the new surface obtained by the collapse of the boundary components of $I_1$ and $I_2$. Let $\overline{\tau}_1$ be the restriction of $\tau_1$ onto $\overline{T}_1$. Then, $\overline{\tau}_1$ acts on the $j$th component of $\overline{T}_1$ as interchanging two points $p_{1,j}$ and $p_{2,j}$ by so-called simple twist. In this step, we first consider only the case where $\overline{T}_1$ is connected.

In Step II, we show that $\overline{\tau}_1$ can be represented by a self-functor of a groupoid and the number of boundary components of $\overline{T}_1$ equals 1 or 2. That is, $\overline{\tau}_1$ is equivalent to the lift of half Dehn twist by $d$-fold covering over a disk with two points. Therefore, by the same reason as in Section 2, the homology triviality holds.

In Step III, we complete the proof of the theorem by counting the case where $\overline{T}_1$ is disconnected.
2 THE HOMOLOGY TRIVIALITY OF THE EMBEDDING $\phi_d$

Let $B_n$ be the braid group with the standard generators $\beta_i$ ($1 \leq i \leq n - 1$).

**Definition 2.1.** An embedding $\phi : B_n \hookrightarrow \Gamma_{g,k}$ is said to be **geometric** if it maps each $\beta_i$ to a Dehn twist in $\Gamma_{g,k}$.

There is a canonical geometric embedding $\phi : B_{2g+2} \hookrightarrow \Gamma_g, \beta_i \mapsto \alpha_i$, where $\alpha_i$ are the Dehn twists along chain curves $a_i$ with $a_i \cap a_{i+1} = \{\ast\}$. In [10], Segal and Tillmann viewed this embedding as being induced by a twofold branched covering over a disk. For $d \geq 3$, the embeddings $\phi_d$, induced by $d$-fold coverings, were constructed by Kim–Song, Ghaswala–McLeay, and Callegaro–Salvetti [3, 4, 7]. Each $\phi_d(\beta_i)$ is proved to be a product of $d-1$ Dehn twists, and hence, $\phi_d$ are all non-geometric for $d \geq 3$. This gives another answer to the question of Wajnryb [13] about the existence of non-geometric embedding.

Let $S_{g,b}(n)$ be the surface of genus $g$ with $b$ boundary components and $n$ marked points. Let $p : S_{g,b}(n) \rightarrow S_{0,1}(n)$ be a cyclic $d$-fold branched covering with $n$ branch points $p_1, \ldots, p_n$. A cyclic branched covering means that around the branch points, $p$ maps as $z \mapsto z^d$.

We may regard $S_{0,1}(n)$ as a full subcategory (subgroupoid) $D$ of the fundamental groupoid $\Pi_1(S_{0,1})$ whose set of objects is denoted by $\{p_0, p_1, \ldots, p_n, p_{n+1}\}$ where $p_0$ and $p_{n+1}$ are on the boundary component. Then, the morphisms of $D$ is generated by $\{e_0, e_1, \ldots, e_n\}$ where $e_i$ represents the homotopy class of a path from $p_i$ to $p_{i+1}$. Similarly, we can regard $S_{g,b}(n)$ as a full subcategory $E$ of $\Pi_1(S_{g,b})$ whose set of objects is $p^{-1}(p_0) \cup p^{-1}(p_{n+1}) \cup \{p_1, \ldots, p_n\}$, and the morphisms of $E$ are generated by the homotopy classes of paths $e_{i,j}$ corresponding to $e_i$ of $D$ through cyclic $d$-fold covering map $p$. A concrete construction can be found in [9] and [4].

The generators $\beta_i$ of $B_n$ are the half Dehn twists around the two points $p_i$ and $p_{i+1}$ in $S_{0,1}(n)$ as in Figure 1. The half Dehn twist $\beta_i$ on $S_{g,b}(n)$ can be expressed as a self-functor $\beta_i$ (abuse of notation) of the groupoid $D$ as follows:

\[
\beta_i : \begin{cases} 
 p_i &\mapsto p_{i+1} \\
 p_{i+1} &\mapsto p_i \\
 e_{i-1} &\mapsto e_{i-1} \cdot e_i \\
 e_i &\mapsto e_i^{-1} \\
 e_{i+1} &\mapsto e_i \cdot e_{i+1} 
\end{cases}
\]

for each $i = 1, \ldots, n - 1$. It fixes points and edges that do not appear in the list.
On the other hand, as a self-functor of the groupoid $E$, $\tilde{\beta}_i$ (abuse of notation) acts on points and edges as follows:

$$
\tilde{\beta}_i : \begin{cases}
  p_i &\mapsto p_{i+1} \\
  p_{i+1} &\mapsto p_i \\
  e_{i-1,j} &\mapsto e_{i-1,j} \cdot e_{i,j+1} \\
  e_{i,j} &\mapsto e_{i,j+1}^{-1} \\
  e_{i+1,j} &\mapsto e_{i,j} \cdot e_{i+1,j}.
\end{cases}
$$

**Remark 2.2.** The above groupoid argument can also be interpreted in terms of ribbon graphs. Let $G = (V, E)$ be the graph with the set of vertices $V = \{p_1, \ldots, p_n\}$ and the set of edges $E = \{e_{i,j}\}_{1 \leq i \leq n-1, 1 \leq j \leq d}$ where $e_{i,j}$ (abuse of notation) is an edge corresponding to the homotopy class of the path $e_{i,j}$. We view each $e_{i,j}$ as an edge between $p_i$ and $p_{i+1}$. The midpoint of the edge $e_{i,j}$ splits the edge into two parts, which are called the half edges. A half edge incident to $p_i$ is denoted by $e_{i,j}^-$, and a half edge incident to $p_{i+1}$ is denoted by $e_{i,j}^+$. Let $\mathcal{G}$ be the ribbon graph obtained from $G$ with the set of half-edges $H = \{e_{i,j}^-, e_{i,j}^+\}_{1 \leq i \leq n-1, 1 \leq j \leq d}$. The cyclic ordering on edges at $p_1$ is given by $(e_{1,1}^+, e_{1,2}^+, \ldots, e_{1,d}^+)$, and at $p_n$, it is given by $(e_{n-1,1}^-, e_{n-1,2}^-, \ldots, e_{n-1,d}^-)$. For $i \notin \{1, n-1\}$, the cyclic ordering at $p_i$ is given by $(e_{i-1,1}^-, e_{i,1}^+, e_{i-1,2}^-, e_{i,2}^+, \ldots, e_{i-1,d}^-, e_{i,d}^+)$. Note that the surface $S_{g,b}$ can be obtained as a geometric realization of this ribbon graph.

We can consider $\tilde{\beta}_i$ (again, abusing notation) as a self-homeomorphism of $S^{(n)}_{g,b}$, the geometric realization of the ribbon graph corresponding to the groupoid $E$, which fixes the boundary components pointwise and satisfies the following commutative diagram:

$$
\begin{array}{ccc}
S^{(n)}_{g,b} &\xrightarrow{\tilde{\beta}_i} & S^{(n)}_{g,b} \\
p \downarrow & & \downarrow p \\
S^{(n)}_{0,1} &\xrightarrow{\tilde{\beta}_i} & S^{(n)}_{0,1}.
\end{array}
$$

It was shown in [9] and [4] that the homomorphism $\phi_d : B_n \to \Gamma_{g,b}$, $\beta_i \mapsto \tilde{\beta}_i$, induced by $d$-fold cyclic branched covering over a disk with $n$ branch points, is well defined and injective by the Birman–Hilden theory. Also, it was shown that for $d$ odd, $\tilde{\beta}_i$ is lifted to a $\frac{1}{2d}$-Dehn twist, and for $d$ even, $\tilde{\beta}_i$ is lifted to a pair of $\frac{1}{d}$-Dehn twists. Figure 2 shows the lift of $\tilde{\beta}_i$ via a cyclic threefold branched covering, which is a $\frac{1}{6}$-Dehn twist. Figure 3 shows the lift of $\tilde{\beta}_i$ via a cyclic fourfold branched covering, which is a pair of $\frac{1}{4}$-Dehn twists. Note that a pair of $\frac{1}{2}$-Dehn twists is a full Dehn twist.

Now, let us show the homology triviality of the embedding $\phi_d$. For more details, see [9] and [4]. The main strategy is to show that the action of the framed little 2-disks operad is preserved by the map induced from $\phi_d$. This proves Proposition X.

Let $D$ be the unit closed disk $\{z \in \mathbb{C} | |z| \leq 1\}$, and $D_1, \ldots, D_k$ be $k$ disjoint, ordered copies of $D$. Define $D_k = \{f : D_1 \sqcup \cdots \sqcup D_k \to D | f|_{D_i}(x) = \alpha_i x + \beta_i\}$, where $\alpha_i \in \mathbb{C}^\times, \beta_i \in \mathbb{C}, f(D_i) \cap$
$f(D_j) = \phi$ for $i \neq j$, the space of smooth embeddings of $k$ disjoint ordered copies of $D$, that restrict on each $D_i$ by the composition of a translation and a multiplication by an element of $\mathbb{C}^\times$.

The spaces $\{D_n\}_{n \geq 0}$ form an operad $D$, called the framed little 2-disks operad, with the structure map

$$\gamma : D_k \times (D_{m_1} \times \cdots D_{m_k}) \to D_{\Sigma m_i}$$

given by composition of embeddings [9, 10].

Let $\text{Conf}_n(D)$ denote the configuration space of unordered $n$-tuples of distinct points in the interior of $D$. Recall that $\text{Conf}_n(D) \simeq B\, B_n$. Put $C_m = \text{Conf}_{dm}(D)\,(m \geq 0)$, and $C = \{C_m\}_{m \geq 0}$. Then
C is a $D$-algebra with the action

$$\gamma_C : D_k \times (C_{m_1} \times \cdots C_{m_k}) \to C_{\Sigma m_i}$$

defined by $(f; a_1, a_2, \ldots, a_k) \mapsto f(a_1 \cup a_2 \cup \cdots \cup a_k)$.

Let $M_{g,b}$ be the moduli space of Riemann surfaces of genus $g$ with $b$ boundary components. Recall that $M_{g,b} \simeq B\Gamma_{g,b}$ for $b \geq 1$. For the embedding $\phi_d : B_{dm} \hookrightarrow \Gamma_{g,d}$, we have $g = g(d,m) = (d^2m - md - 2d + 2)/2$ by the Riemann–Hurwitz formula. Let $S = \{S_m\}_{m \geq 0}$, where

$$S_m := \begin{cases} M_{g(d,m),d} & \text{for } m \geq 1 \\ M_{0,1} \sqcup \cdots \sqcup M_{0,1}(d \text{ times}) & \text{for } m = 0. \end{cases}$$

Each surface $T$ in $S_m$ has $d$ ordered parametrized boundary circles; let us call them $\partial_1 T, \ldots, \partial_d T$. For $f \in D_k$, and $j \in \{1, \ldots, d\}$, let $(P_f)_j = D \setminus f(D \cup \cdots \cup D) = S_{0,k+1}$. Also let us put $P_f = (P_f)_1 \cup \cdots \cup (P_f)_d$. We think of $(P_f)_j$ as a Riemann surface with $k+1$ parametrized boundary components.

Then, $S := \sqcup_{m \geq 0} S_m$ is a $D$-algebra, defined by maps

$$\gamma_S : D_k \times (S_{m_1} \times \cdots S_{m_k}) \to S_{\Sigma m_i}$$

$$(f; T_{m_1}, \ldots, T_{m_k}) \mapsto (P_f \cup T_{m_1} \cup \cdots \cup T_{m_k}) / \simeq,$$

where $\sqcup_i \partial_j T_{m_i}$ is identified with the interior boundaries of $(P_f)_j$ ($j = 1, \ldots, d$).

Remark 2.3. Regarding the $D$-algebra structure of $S$, we can think of it pictorially as capping off the boundary components. Let us place the surfaces $T_{m_i}$ from left to right as shown in the Figure 4. For fixed $j$, we glue the boundary components of $(P_f)_j$ with $\sqcup_i \partial_j T_{m_i}$. The resulting surface $K = (P_f \cup T_{m_1} \cup \cdots \cup T_{m_k}) / \simeq$ has extra genus, $(k-1)(d-1)$, that is, the total genus of $K$ is $g(d,m_1) + \cdots + g(d,m_k) + (k-1)(d-1)$ which equals $g(d,\Sigma m_i)$. This gives us the action of the framed little 2-disks operad, $\gamma_S : D_k \times (S_{m_1} \times \cdots S_{m_k}) \to S_{\Sigma m_i}$.

For $m \geq 0$, the embeddings $\phi_d : B_{dm} \hookrightarrow \Gamma_{g(d,m),d}$ naturally give $C_m \to S_m$. Hence, we have $D$-algebra map $\Phi_d$ induced by embeddings $\phi_d$. We have proved the following.

Theorem 2.4. $\Phi_d : C \to S$ is a map of $D$-algebras.

The map $\Phi_d$ induces a double-loop space map on their group completions, because maps of algebras over the framed little 2-disks operad group-complete to maps of double-loop spaces [10].

$$\Omega B(\sqcup_{m \geq 0} \mathbb{B}B_{dm}) \simeq \mathbb{Z} \times \mathbb{B}B^+_{\infty} \to \Omega B(\sqcup_{m \geq 0} \mathbb{B}\Gamma_{g(d,m),d}) \simeq \mathbb{Z} \times \mathbb{B}\Gamma^+_{\infty}.$$

That is, $B\phi_d^+ : \mathbb{B}B^+_{\infty} \to \mathbb{B}\Gamma^+_{\infty}$ is a double-loop space map (Proposition X), and hence, more strongly, a null-homotopic map. The following theorem is proved.
Theorem 2.5. Let \( \phi_d : B_n \hookrightarrow \Gamma \) be the embedding induced by \( d \)-fold branched covering (\( d \geq 2 \)). Then, for all \( i \geq 1 \),

\[
(\phi_d)_* : H_i(B_\infty; R) \to H_i(\Gamma_\infty; R)
\]

is trivial in any constant coefficient \( R \).

3 | REGULAR EMBEDDINGS

Let \( \phi : B_n \hookrightarrow \Gamma_{g,k} \) be an embedding, where each \( \beta_i \) is the standard generator interchanging the \( i \)th and \((i + 1)\)st points. Each \( \tau_i \) is taken to be a self-homeomorphism of the surface satisfying the braid relation \([\tau_i][\tau_{i+1}][\tau_i] = [\tau_{i+1}][\tau_i][\tau_{i+1}]\). We will consider \( \tau_i \) to be isotopic to a self-homeomorphism that interchanges (by braiding) two identical subparts of the surface \( S_{g,k} \).

Those interchanged subparts may be points or subsurfaces, not necessarily connected. Although the classification of embeddings from braid groups to mapping class groups is still widely open (e.g., [5]), all known embeddings are of this kind. Note that a standard generator of braid group is, in principle, characterized as interchanging two points by a half twist.

For a self-homeomorphism \( f \) of \( S_{g,k} \), define the support \( \text{supp}(f) \) to be the set of nonfixed points of \( f \). Since \( \phi(\beta_i) \) satisfy the braid relations, we may take the representative self-homeomorphism \( \tau_i \) of \( \phi(\beta_i) \) as a kind of simple twist interchanging two identical subsurfaces (or points) \( I_i \) and \( I_{i+1} \) on \( T_i = \text{supp}(\tau_i) \). Since all \( \tau_i \) will be taken to be conjugate to each other, we may assume that all \( T_i \) are homeomorphic to each other by the following lemma.
Lemma 3.1. Let $f$ be a self-map of $S_{g,k}$ and let $h$ be a self-homeomorphism of $S_{g,k}$. Then, we have

$$\text{supp}(f) \cong h^{-1}(\text{supp}(f)) = \text{supp}(h^{-1} \circ oh).$$

That is, the supports of two conjugate self-maps are homeomorphic.

Proof. Since $h$ is a homeomorphism, $\text{supp}(f) \cong h^{-1}(\text{supp}(f))$. We show that $h^{-1}(\text{supp}(f)) = \text{supp}(h^{-1} \circ oh)$:

$$x \in h^{-1}(\text{supp}(f)) \iff h(x) \in \text{supp}(f) \iff f(h(x)) \neq h(x) \iff x \in \text{supp}(h^{-1} \circ oh).$$

□

Take a representative $\tau_1$ of $\phi(\beta_1)$ and $\tilde{\delta}$ of $\phi(\delta)$, where $\delta = \beta_1 \cdots \beta_{n-1}$. Define $\tau_2, \ldots, \tau_{n-1}$ to be $\tau_i := \tilde{\delta}^{-1} \circ \tau_{i-1} \circ \tilde{\delta}$ inductively, then we have $\phi(\beta_i) = [\tau_i]$ for $i = 1, \ldots, n-1$.

Each atomic surface $T_i = \text{supp}(\tau_i)$ is homeomorphic to a compact surface whose boundary is fixed pointwise by $\tau_i$, while $\text{supp}(\tau_i)$ is an open subset of $S_{g,k}$. Because our goal is to show the homology triviality of $\phi$ in the stable range, we may assume that each $T_i$ is connected. For if $T_i$ consists of two or more components, considering the Harer–Ivanov stability theorem, we may take only one component and push another ones far away to the other side of the big surface. The self-homeomorphism $\tau_i$ is also assumed to be a simple twist that is defined as follows.

Definition 3.2. A self-homeomorphism $\tau_i$ of a connected compact orientable surface $T_i$ (with boundary) interchanging two identical subsurfaces (or points) $I_i$ and $I_{i+1}$ is called a simple twist if

(i) $T_i = \text{supp}(\tau_i)$ while the boundary of $T_i$ is fixed pointwise by $\tau_i$,

(ii) $\tau_i$ is not isotopic to a composition of two nontrivial self-homeomorphisms $f, g$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$. That is, $[\tau_i] = [f \circ g] = [g \circ f]$ implies that either $[f] = 1$ or $[g] = 1$,

(iii) if $[\tau_i] = [\gamma_i]^k$ for some self-homeomorphism $\gamma_i$, then $k = 1$ or $-1$.

Condition (ii) implies a kind of minimality of atomic surface $T_i$. Condition (iii) is necessary because for a pair of standard generators $\beta_1, \beta_2$ of $B_n$ satisfies the braid relation $\beta_1 \beta_2 \beta_1 = \beta_2 \beta_1 \beta_2$, but their odd products $\beta_1^{2i+1}, \beta_2^{2i+1} (i \geq 1)$ do not satisfy the braid relation, although they interchange two points by a twist (see Figure 5).

Definition 3.3. An embedding $\phi : B_n \hookrightarrow \Gamma_{g,k}, \beta_i \mapsto [\tau_i]$ is a regular embedding if
(i) let $T_1 = \text{supp}(\tau_1)$ and $\tau_1$ is chosen to be a simple twist on $T_1$ interchanging two identical subsurfaces (or points) $I_1$ and $I_2$ of $T_1$.
(ii) let $T_i = \text{supp}(\tau_i)$ and $\tau_i$ is defined by $\tilde{\delta}^{-1} \circ \tau_{i-1} \circ \tilde{\delta}$ where $\tilde{\delta}$ is a homeomorphism in the isotopy class $\phi(\delta)$, for $i = 2, ..., n-1$,
(iii) $S_{g,k} \cong \bigcup_{i=1}^{n-1} T_i$.

Recall that all $T_i$ are homeomorphic to each other, and $I_i \subset T_{i-1} \cap T_i$ for $i = 2, ..., n-1$ by Lemma 3.1.

Here, we give some examples of regular (nongeometric) embeddings.

**Example 3.4.** As seen in Section 2, there is an infinite family of nongeometric embeddings induced by the lifts of half Dehn twists on a disk with some marked points via $d$-fold covering map. Let $p : S_{g,b}^{(n)} \rightarrow S_{0,1}^{(n)}$ be a cyclic $d$-fold branched covering with $n$ branch points. Then, an embedding $\phi_d : B_n \hookrightarrow \Gamma_{g,b,1,1} \ni [\tau_i]$ is induced from this $d$-fold covering; where $\tau_i$ is obtained by a lift of half Dehn twist on a disk with two marked points via covering map $p$: for $p^{-1}(S_{0,1}^{(2)}) = S_{h,k}^{(2)}$, we have

$$
\begin{array}{ccc}
S_{h,k}^{(2)} & \xrightarrow{\tau_i} & S_{h,k}^{(2)} \\
p & & np \\
S_{0,1}^{(2)} & \xrightarrow{\beta_i} & S_{0,1}^{(2)}
\end{array}
$$

For $d = 2$, this embedding is the canonical geometric embedding (called Harer embedding). In the case $d \geq 3$, these embeddings are all nongeometric. These embeddings are all regular embeddings with the atomic surfaces $T_1 \cong p^{-1}(S_{0,1}^{(2)}) = S_{h,k}^{(2)}$, and each $I_i$ is a point.

**Example 3.5** ($B_n \hookrightarrow \Gamma_{n,g,1}$ [11]). The first example in Appendix of [11] is a nongeometric embedding $B_n \hookrightarrow \Gamma_{n,g,1}$ obtained by gluing $n$ copies of $S_{g,1}$ onto $S_{0,(n)+1}$ along the ordered parametrized boundary components (see Figure 6). The half Dehn twist $\tau_i : S_{0,(n)+1} \rightarrow S_{0,(n)+1}$ interchanging two parametrized boundary components can be extended to the homeomorphism interchanging
two copies of $S_{g,1}$. This gives a regular embedding

$$B_n \hookrightarrow \Gamma_{g,1} \hat{\xi}_{[1,\ldots,n]} B_n \hookrightarrow \Gamma_{g,1} \hat{\xi}_{[1,\ldots,n]} \Gamma_{0,(n)+1} \hookrightarrow \Gamma_{g,n+1}$$

since $B_n$ is a subgroup of ribbon braid group $RB_n := \mathbb{Z} \hat{\xi}_{[1,\ldots,n]} B_n \cong \Gamma_{0,(n)+1}$.

**Example 3.6** (Szepietowski’s construction [12]). Szepietowski constructed a nongeometric embedding $B_n \hookrightarrow \Gamma_{n-1,2}$ using nonorientable surface $N_{n,1}$ and the twofold covering map $p : S_{n-1,2} \to N_{n,1}$ induced by mapping two antipodal points of $S_{0,2}$ to one point of Möbius band. Starting with $S_{0,(n)+1}$, the disk with $n$ holes, we paste to each hole a Möbius band $N_{1,1}$ and get a nonorientable surface $N_{n,1}$. Then, we get a natural embedding

$$\varphi : B_n \hookrightarrow \mathcal{N}_{n,1},$$

where $\mathcal{N}_{n,1}$ denotes the mapping class group of $N_{n,1}$. The lift of homeomorphisms of $N_{n,1}$ to its double covering space $S_{n-1,2}$ induces an injection $L : \mathcal{N}_{n,1} \hookrightarrow \Gamma_{n-1,2}$. Then, the composite $L \circ \varphi : B_n \hookrightarrow \Gamma_{n-1,2}$ is a nongeometric regular embedding. The injectivity of the map $L$ can be shown by the result of Birman and Chillingworth [1].

**Example 3.7** (The mirror construction [2, 8]). The canonical injection $B_n = \Gamma_{0,1}^{(n)} \hookrightarrow \Gamma_{0,(n)+1}$ can be extended to an injection $\phi : B_n \hookrightarrow \Gamma_{n-1,2}$ by the mirror construction. From two sheets of disks with $n$ holes, by gluing them along the boundaries of the holes, we get a surface $S_{n-1,2}$ (see Figure 7).

This construction gives rise to an embedding $\phi : B_n \hookrightarrow \Gamma_{n-1,2}, \hat{\beta}_i \mapsto \tau_i$, where $\tau_i$ is called the pillar switching [8]. $\phi$ is obviously a regular embedding with the atomic surface $T_i \cong S_{1,2}$ and the interchangeable subsurface $I_i$ is the cylinder $S_{0,2}$. As a matter of fact, Szepietowski’s construction is essentially equal to this pillar switching.

**Example 3.8.** The second example in Appendix of [11] is a kind of mirror construction with some redundant part. A surface $S_{n+g-1,2}$ is obtained by gluing $S_{g,2}$ on a boundary of the mirror construction $S_{n-1,2}$, and a self-homeomorphism of $S_{n-1,2}$ can be extended to $S_{n+g-1,2}$ by the identity map. This induces a nongeometric embedding $B_n \hookrightarrow \Gamma_{n+g-1,2}$ which is not regular. However, if we reduce the target surface by considering only the union of the supports $T_i$, we can transform it into a regular embedding. We may cut off unnecessary part of the surface because we are mainly dealing with the stable homology triviality.
Note that we may have more general mirror constructions by taking the interchangeable subsurface $I_i$ as any surface of type $S_{g,2}$ (Figure 8).

More generally, we may have more than two sheets of disks with $n$ holes. If we have $m$ ($m \geq 3$) sheets of disks, then we get a mirror construction by gluing $n$ copies of $S_{g,m}$, which are interchangeable surfaces $I_i$, to these disks. That is, each hole of $S_{g,m}$ is glued to a hole of a disk. Call this type of mirror construction, Type II. The embedding induced by the mirror construction of this type is obviously regular.

There are even more general types of regular embeddings than these mirror constructions.

III. On each sheet of disks, the number of holes may not be necessarily all equal.

IV. Instead of sheets of disks, we may have any surfaces and any simple twists on them.

All of these general regular embeddings will be considered in the next section.

\section{The General Harer Conjecture}

All examples we have given in Section 3 are regular or can be transformed to be regular by restricting the target surface. In fact, it is not easy to imagine that there exists an embedding that is not (equivalent to) a regular embedding.

As explained in Remark 2.3, in order to construct the $D$-algebra map $\Phi : C \to S$, we may focus only on the number of boundary components of the surface with respect to a regular embedding $\phi : B_n \hookrightarrow \Gamma_{g,k}$, where $\beta_i$ is mapped to $[\tau_i]$.

\textbf{Theorem 4.1.} For every regular embedding $\phi : B_n \hookrightarrow \Gamma_{g,k}$,

$$\phi_* : H_i(B_n; R) \to H_i(\Gamma_{\infty}; R)$$

is trivial for all $i \geq 1$ and any coefficient $R$.

\textbf{Remark 4.2.} Note that the third condition of the definition of a regular embedding, which is $S_{g,k} \cong \bigcup_{i=1}^{n-1} T_i$, does not harm the generality of this theorem due to the Harer–Ivanov stability theorem.
Because we are dealing with stable homology of a big surface, we may disregard unnecessary part of the surface.

Proof. Let $\phi : B_n \to \Gamma_{g,k}$ be a regular embedding. Our strategy is that we first remove all the interchangeable subsurfaces from $S_{g,k}$ and collapse the new boundary components to points, then we eventually get a surface obtained from (the union of) covering spaces over a disk with some marked points. Then, we are able to control the number $k$ of boundary components with growing $n$; thus, we get a framed little 2-disks operad algebra.

Let us take a look at just $T_1 = \supp(\tau_1)$ with two interchangeable subsurfaces (or points) $I_1$ and $I_2$. Recall that all $T_i$ are identical for $i = 1, \ldots, n - 1$.

(Step I) Let us first detach $I_1$ and $I_2$ from $T_1$. Then $T_1 \setminus (I_1 \cup I_2)$ is not necessarily connected. We may assume that each connected component of $T_1 \setminus (I_1 \cup I_2)$ is homeomorphic to a surface with exactly two (new) interchangeable holes (boundary components). That is, we may assume that by detaching each $I_i$ from the surface, we have only one new hole in each component of the remaining surface, because in the case where there are two or more new holes, we can reduce them to one by increasing the genus (see Figure 9).

Although $T_1 \setminus (I_1 \cup I_2)$ is not necessarily connected, we first consider the case where it is connected. The disconnected case will be considered later.

Let the two holes of $T_1 \setminus (I_1 \cup I_2)$, generated by detaching $I_1, I_2$, collapse to two points. Denote this new atomic surface by $\tilde{T}_1$. Then, the embedding $\phi : B_n \to \Gamma_{g,k}$ collapses to another regular embedding $\phi' : B_n \to \Gamma_{g,k}^{(n)}$, where $S_{h,k} = \bigcup_{i=1}^{n-1} \tilde{T}_i$.

Now we consider $\phi' : B_n \to \Gamma_{h,k}^{(n)}, \beta_i \mapsto [\tilde{\tau}_i]$, where each $\tilde{\tau}_i$ is a self-homeomorphism of $\tilde{T}_i$ interchanging two marked points by a simple twist.

(Step II) We show that this embedding is equivalent (can be reduced) to that induced from ordinary $d$-fold ($d \geq 2$) covering map over a disk with $n$ marked points. For this, we will show that each $\tilde{T}_i$ can have only one or two boundary components and the action of $\tilde{\tau}_i$ is isotopic to that of a $\frac{1}{2d}$-Dehn twist for one boundary component, or a pair of $\frac{1}{d}$-Dehn twists for two boundary components.

Denote the atomic surface $\tilde{T}_1$ by $S_{h,b}^{(2)} (b \geq 1)$, that is, $\tau_1$ induces a simple twist $\tilde{\tau}_1 : S_{h,b}^{(2)} \to S_{h,b}^{(2)}$ interchanging two marked points $p_1$ and $p_2$, and fixing the boundary components pointwise. The mapping class group $\Gamma_{h,b}$ may be regarded as a subgroup of the automorphism group of $\pi_1(S_{h,b}) \cong F_{2h+b-1}$, that is, the isotopy class of a self-homeomorphism of $S_{h,b}$ is completely determined by its action on the fundamental group of the surface.
On the other hand, it is well known that closed surface $S_h$ can be expressed as a $(4h + 2)$-gon with two vertices and opposite sides identified (Figure 10). We may regard $S_{h,1} = S_h - D$ as the groupoid $\mathcal{G}$ with two vertices $p_1, p_2$ and $2h + 1$ edges $e_1, e_2, ..., e_{2h+1}$ from $p_1$ to $p_2$. $S_{h,b}$ has additional $b - 1$ edges $e_{2h+2}, ..., e_{2h+b}$ and a self-homeomorphism of $S_{h,b}$ is a self-functor fixing the loops along the boundary components.

Let us now figure out how $\overline{\tau}_1$ acts on $G \simeq S_{h,b}^{(2)}$ as a self-functor.

Case I. In the case of $b = 1$, a loop along the boundary (in counter clockwise direction) of the surface can be expressed as

$$\partial = e_1 \cdot e_2^{-1} \cdot \cdots \cdot e_{2h}^{-1} \cdot e_{2h+1} \cdot e_1^{-1} \cdot e_2 \cdot \cdots \cdot e_{2h+1}^{-1}.$$ 

$\overline{\tau}_1$ should move every edge of $G$ while fixing the boundary loop $\partial$. By the definition of simple twist, each edge should be moved to its adjacent edge, that is, if we have the twist in the counter clockwise direction in $G$, then $\overline{\tau}_1$ acts as:

$$e_1 \mapsto e_2^{-1}, \quad e_2^{-1} \mapsto e_3, \quad \cdots, \quad e_{2h}^{-1} \mapsto e_{2h+1}, \quad e_{2h+1} \mapsto e_1^{-1},$$

$$e_1^{-1} \mapsto e_2, \quad e_2 \mapsto e_3^{-1}, \quad \cdots, \quad e_{2h+1}^{-1} \mapsto e_1$$

because $p_1$ and $p_2$ should be interchanged only once (see Figure 12).

Case II. In the case of $b = 2$, the groupoid $G$ has an additional edge $e_{2h+2}$ to the groupoid of Case I (see Figure 11). Then, two boundary loops $\partial_1, \partial_2$ represented by

$$\partial_1 = e_1 \cdot e_2^{-1} \cdot \cdots \cdot e_{2h}^{-1} \cdot e_{2h+1} \cdot e_{2h+2}^{-1}, \quad \partial_2 = e_1^{-1} \cdot e_2 \cdot \cdots \cdot e_{2h} \cdot e_{2h+1}^{-1} \cdot e_{2h+2}$$
are supposed to be fixed. We may regard this groupoid as two identical \((2h + 2)\)-gons glued together along the edge \(e_{2h+2}\). As in Case I, the simple twist \(\tilde{\tau}_1\) moves each edge to its adjacent edge in each \((2h + 2)\)-gon.

Case III. If \(b \geq 3\), then we may add edges \(e_{2h+2}, e_{2h+3}, \ldots\) to the groupoid \(G\) in Case I. Then, moving each edge to its adjacent edge in one direction in each polygon causes a mismatch of sides, in particular, the continuity of \(\tilde{\tau}_1\) does not hold (see Figure 13).

In conclusion, the number of boundary components of the atomic surface \(\tilde{T}_1\) equals 1 or 2. As a matter of fact, the twist \(\tilde{\tau}_1\) on \(\tilde{T}_1\) is equivalent to that induced by a \(d\)-fold \((d = 2h + 1\) or \(2h + 2\)) cyclic branched covering over a disk with two marked points because the simple twist \(\tilde{\tau}_1\) maps \(e_i \mapsto e_i^{-1}\) where the indices read modulo \(d\). This can also be shown by the argument of ribbon graph automorphism (see Remark 2.2).

The embedding \(\phi' : B_n \to \Gamma_{g,k}^{(n)}\) is equivalent to that induced from \(d\)-fold covering over a disk with \(n\) marked points. Now we have

\[
g = \frac{1}{2}(dn - n - d - \gcd(d, n)) + 1 \quad \text{and} \quad k = \gcd(d, n)
\]

by the Riemann–Hurwitz formula. This means that if each \(\tilde{T}_i\) is connected, as we have seen in Section 2, we have a \(D\)-algebra map \(\Phi : C \to S\) induced by \(\phi\), where \(C_m = \text{Conf}_{dn}(D)\) and \(S_m = \mathcal{M}_{g,d(m),k}\). Therefore, \(B\phi^+ : B B_{\infty}^+ \to B \Gamma_{g,k}^+\) is a double-loop space map, which means that the embedding \(\phi : B_n \to \Gamma_{g,k}^{(n)}\) satisfies the homology triviality.

(Step III) Finally, consider the case where \(\tilde{T}_i\) is not connected. Let \(\tilde{T}_1 = S_1 \amalg \cdots \amalg S_m\), where \(S_i\) are the connected components. \(S_i\) are not necessarily homeomorphic to each other and the action of \(\tilde{\tau}_1\) on each \(S_i\) is not necessarily all equal. Each \(S_i\) \((1 \leq i \leq m)\) is induced from \(d_i\)-fold branched covering map, so it has one or two boundary components. Then, the embedding \(\phi : B_n \to \Gamma_{g,k}^{(n)}\) is
decomposed into

\[ B_n \xrightarrow{\Delta} \bigoplus_{i=1}^m B_n \xrightarrow{\bigoplus \phi'_i} \bigoplus_{i=1}^m \Gamma_{g, k_i}^{(n)} \xrightarrow{\gamma} \Gamma_{g, k}. \]

Here, \( \Delta \) is the diagonal map, each \( \phi'_i \) is the map obtained by \( d_i \)-fold covering (the same map as dealt in the connected case), \( \gamma \) is the map recovering the surfaces \( I_i \). Note that \( k = \sum_{i=1}^m k_i \) and each \( k_i \) is the number of boundary components of the surface obtained by \( d_i \)-fold covering.

Let \( d = \text{lcm}(d_1, \ldots, d_m) \). For a regular embedding \( \phi : B_n \to \Gamma_{g, k} \), by taking \( C_i = \text{Conf}_{d_i}(D) \) and \( S_i = \mathcal{M}_{g(i), k} \), we have the actions of the framed little 2-disks operad. Thus, we have a \( D \)-algebra map \( \Phi : C \to S \) as we have done in Section 2. This implies Proposition X and proves the homology triviality.

We believe that the regular embedding case is sufficiently general. First, note that all atomic surfaces \( \tilde{T}_i \) are homeomorphic to each other (Lemma 3.1). Then, \( \phi(\tilde{\beta}_i) \) should be represented by a self-homeomorphism interchanging two identical subparts of a surface, because the geometric realization of a standard generator of braid group is, in principle, to interchange two (identical) things by a twist. Furthermore, if we consider the map \( B \phi : \text{Conf}_n(D) \to \mathcal{M}_{g, k} \) corresponding to \( \phi \), then \( \hat{\beta}_i \) is represented by a loop (path) in \( \text{Conf}_n(D) \) that records a continuous movement interchanging two points. Similarly, \( B \phi(\beta_i) \) may be represented by a loop (path) in \( \mathcal{M}_{g, k} \) that records the continuous movement on a surface interchanging two identical parts. However, because this does not seem to provide a complete explanation why a regular case represents all cases, we now suggest a question.

**Question.** Is every embedding equivalent to a regular embedding?

This question may be specified as follows. Let \( \phi : B_3 \to \Gamma_{g, k} \) \( (g \gg 1, k \geq 1) \) be an embedding from braid group to mapping class group. Let \( \beta_1, \beta_2 \) be standard generators of \( B_3 \). Are there self-homeomorphisms \( \tau_1, \tau_2 \) that represent \( \phi(\tilde{\beta}_1), \phi(\tilde{\beta}_2) \), respectively, satisfying two conditions

(i) there are three disjoint homeomorphic (identical) subsurfaces (or points) \( I_1, I_2, I_3 \) of \( S_{g, k} \) (ii) \( \tau_1 \) interchanges \( I_1 \) and \( I_2 \), and \( \tau_2 \) interchanges \( I_2 \) and \( I_3 \) by a simple twist?

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