Space-time can be neither discrete nor continuous

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Abstract

We show that our recent Bohr-like approach to black hole (BH) quantum physics implies that space-time quantization could be energy-dependent. Thus, in a certain sense, space-time can be neither discrete nor continuous. Our approach permits also to show that the “volume quantum” of the Schwarzschild space-time increases with increasing energy during BH evaporation and arrives to a maximum value when the Planck scale is reached and the generalized uncertainty principle (GUP) prevents the total BH evaporation. Remarkably, this result does not depend on the BH original mass. The interesting consequence is that the behavior of BH evaporation should be the same for all Schwarzschild BHs when the Planck scale is approached.

To the memory of Stephen W. Hawking.

The search for a theory of quantum gravity (TQG) through BH physics started in the ’70s of last century with the famous papers of Bekenstein [1] and Hawking [2]. The famous formula of the Bekenstein-Hawking entropy [1, 2]

\[ S_{BH} = \frac{c^3 k_B A}{4G\hbar} \]

where \( S_{BH} \) stands for the BH entropy, \( A \) for the BH surface area (the event horizon), \( \hbar \) is the reduced Planck constant, \( c \) is the speed of light, \( k_B \) is the Boltzmann constant and \( G \) is the gravitational constant, is indeed considered very fundamental. In fact, on one hand it counts the BH effective degrees of freedom. On the other hand, it ties together notions from gravitation, thermodynamics and quantum theory. Hence, it includes all the 3 fundamental
constants in, and it is, in turn, considered an important window into the yet unknown TQG. In addition, the discovery by Hawking of BH radiation \[2\] represents the first, non-banal result of combining Einstein’s general relativity with Heisenberg’s uncertainty principle. After those pioneering works, an enormous amount of papers have been written and currently continue to be written on BH quantum physics. Today, there is indeed a large agreement on the idea that BHs should be highly excited states representing the fundamental bricks of the yet unknown TQG \[3\]. This idea represents a parallelism with QM. In fact, in the ’20s of last century, atoms were considered the fundamental bricks of QM. This analogy enables one to argue that the BH mass could have a discrete spectrum \[3\]. On the other hand, an immediate and natural question surfaces from such a parallelism. If one assumes that the BH should be the “nucleus” of the “gravitational atom”, then it is quite natural asking: What are the “electrons”? In a series of recent papers (see [4 - 6] and references within), an intriguing answer addressed such a question. The BH quasi-normal modes (QNMs) (which are the horizon’s oscillations in a semi-classical approach \[4\]), which are “triggered” by absorptions of external particles and by the emission of Hawking radiation, represent the “electrons” of that “gravitational atom”. In fact, in [4 - 6] it has been shown that the semi-classical evaporating Schwarzschild BH is the gravitational analogous of the historical, semi-classical hydrogen atom, introduced by Niels Bohr in 1913 \[7, 8\]. The idea underlying the results in [4 - 6] is founded on the non-thermal spectrum of Hawking radiation \[2\]. This indeed implies the countable character of subsequent emissions of Hawking quanta, which, in turn, generates an obvious and important correspondence between Hawking radiation \[2\] and the BH QNMs [4 - 6]. In the framework in [4 - 6], QNMs are seen as being the "electron states", jumping from a quantum level to another one. The analogy is completed when one identifies the energy "shells" of the “gravitational hydrogen atom” in terms of the absolute values of the quasi-normal frequencies [4 - 6]. Another remarkable result is that the BH information puzzle can be solved by considering the time evolution of the Bohr-like BH \[5, 6\]. In fact, BH evaporation results governed by a time-dependent Schrodinger equation, while subsequent emissions of Hawking quanta are entangled with the BH “electron states”, i.e. with the QNMs \[4 - 6\]. The results in [4 - 6] are completely consistent with previous literature. In particular, the result of Bekenstein on the area quantization \[10\] is in complete agreement with the works [4 - 6]. For the sake of completeness, we stress that our Bohr-like approach to BH quantum physics has been recently generalized to the Large AdS BHs in \[11\]. Hereafter, we will use Planck units \((G = c = k_B = \hbar = \frac{1}{4\pi\varepsilon_0} = 1)\) for the sake of simplicity. Then, for large values of the principal quantum number \(n\) (i.e. for excited BHs), the energy levels of the Schwarzschild BH, which is interpreted as the “gravitational hydrogen atom”, are given by [4 - 6]

\[
E_n \equiv |\omega_n| = M - \sqrt{M^2 - \frac{n}{2}}, \tag{2}
\]

where \(M\) is the initial BH mass and \(E_n\) represents the total energy emitted when the BH is excited at the level \(n\) [4 - 6]. The BH radiates a discrete amount of
energy in a quantum jump, and, for large values of \( n \), the process results independent of the other quantum numbers. This is in perfect agreement with the Correspondence Principle stated by Bohr in 1920 [12]. Bohr’s Correspondence Principle argues indeed that “transition frequencies at large quantum numbers should equal classical oscillation frequencies” [12]. In Bohr’s 1913 approach [7, 8], electrons only gain and lose energy through quantum jumps between different allowed energy shells. In each jump, the atom absorbs or emits radiation with an energy difference between the two involved levels which is given by the Planck relation (in standard units) \( E = hf \), \( (h \) is the Planck constant and \( f \) the frequency of the involved transition). In the current approach, the BH QNMs gain and lose energy through quantum jumps from one allowed energy shell to another with absorbed or emitted (Hawking) radiation. The energy difference between the two levels is given by [4 - 6]

\[
\Delta E_{n_1 \rightarrow n_2} \equiv E_{n_2} - E_{n_1} = M_{n_1} - M_{n_2} = \sqrt{M^2 - \frac{n_2^2}{4}} - \sqrt{M^2 - \frac{n_1^2}{4}},
\]

(3)

This equation governs the energy jump between two generic, allowed levels \( n_1 \) and \( n_2 > n_1 \). Such a jump is due to the emission of a particle having frequency \( \Delta E_{n_1 \rightarrow n_2} \). In Eq. (3), \( M_n \) is the residual mass of the BH excited at the level \( n \). It is given by the original BH mass minus the total energy emitted when the BH is excited at that level [4 - 6]. Hence, \( M_n = M - E_n \), and the jump between the two generic allowed levels depends only on the initial BH mass and on the two different values of the BH principal quantum number [4 - 6]. Instead, the case of an absorptions is governed by the equation [4 - 6]

\[
\Delta E_{n_2 \rightarrow n_1} \equiv E_{n_1} - E_{n_2} = M_{n_2} - M_{n_1} = \sqrt{M^2 - \frac{n_2^2}{4}} - \sqrt{M^2 - \frac{n_1^2}{4}} = -\Delta E_{n_1 \rightarrow n_2}.
\]

(4)

The analogy with Bohr’s hydrogen atom is finalized by the following intriguing remark. The interpretation of Eq. (2) is of a particle, the “electron” of the “gravitational atom”, which is quantized on a circle of length [4 - 6]

\[
L = 4\pi \left( M + \sqrt{M^2 - \frac{n}{2}} \right).
\]

(5)

This is exactly the analogous of the electron which travels in circular orbits around the nucleus in Bohr’s hydrogen atom [7, 8], and is also similar in structure to the solar system.

For the goals of this paper, the key point is the following. As we stressed above, in [4 - 6] we have shown that our results are in full agreement with the result of Bekenstein on the area quantization [10]. The area of the BH horizon is indeed quantized in units of the Planck length (\( l_p = 1.616 \times 10^{-33} \) cm is equal to one in Planck units) and Bekenstein has shown that the Schwarzschild BH area quantum is \( \Delta A = 8\pi l_p \). The analysis in [4 - 6] found the same result.
of Bekenstein for an energy jump among two allowed neighboring levels \( n \) and \( n - 1 \) as

\[
|\Delta A_n| = |\Delta A_{n-1}| = 8\pi.
\]

Thus, recalling that, in Schwarzschild BHs, the gravitational radius \( r_g \) is connected with the BH mass through the relation \( r_g \equiv 2M \) \([13, 14]\), one defines the gravitational radius associated to the BH quantum level \( n \) as

\[
r_g(n) \equiv 2M_n.
\]

From Eq. (3), one sees immediately that the variation of the gravitational radius due to an emission from the two levels \( n_1 \) and \( n_2 > n_1 \) is

\[
\Delta r_g(n_1 \rightarrow n_2) \equiv r_g(n_1) - r_g(n_2) = 2(E_{n_2} - E_{n_1}) =
\]

\[
= 2(M_n - M_{n_2}) = 2\left(\sqrt{M^2 - \frac{M^2}{n_2}} - \sqrt{M^2 - \frac{M^2}{n_1}}\right),
\]

while the variation of the gravitational radius due to an absorption from the two levels \( n_2 \) and \( n_1 \) is

\[
\Delta r_g(n_2 \rightarrow n_1) \equiv r_g(n_2) - r_g(n_1) = 2(E_{n_1} - E_{n_2}) =
\]

\[
= 2(M_{n_2} - M_{n_1}) = 2\left(\sqrt{M^2 - \frac{M^2}{n_1}} - \sqrt{M^2 - \frac{M^2}{n_2}}\right) = -\Delta r_g(n_1 \rightarrow n_2).
\]

Then, using Eqs. (6) and (8), or Eqs. (6) and (9), one finds immediately that the variation of the Schwarzschild “volume quantum”, corresponding to a transition between two neighboring levels \( n \) and \( n - 1 \), is

\[
\Delta V_{(n-1 \rightarrow n)} = \Delta V_{(n \rightarrow n-1)} \equiv |\Delta r_g(n \rightarrow n-1)||\Delta A_n| =
\]

\[
= |\Delta r_g(n \rightarrow n-1)||\Delta A_{n-1}| = 16\pi\left(\sqrt{M^2 - \frac{M^2}{n}} - \sqrt{M^2 - \frac{M^2}{n-1}}\right).
\]

We wrote “volume quantum” within inverted commas because the difference between two Schwarzschild radial coordinates is less than the correspondent physical proper distance \([13, 14]\). We also recall that the Schwarzschild radial coordinate is space-like and time-like outside and inside the BH event horizon, respectively, and that the horizon area is a proper area \([14]\).

Eq. (10) is intriguing because it shows that the variation of the Schwarzschild “spatial volume” of the external BH space-time, due to an emission/absorption of a particle, is not constant, but depends on the BH principal quantum number \( n \) (that means on the BH energy level), and on the BH initial mass. On the other hand, we recall that the singularity of the Schwarzschild radius is not a real physical singularity \([14]\). It is a coordinate singularity instead \([14]\). In other words, the space-time geometry is well behaved at the Schwarzschild radius \([14]\). This means that, if the variation of the Schwarzschild “spatial volume” is energy dependent, we can reasonably argue that also the variation of the proper spatial volume is energy dependent. Thus, space-time quantization seems to
be energy dependent. In other words, space-time can be considered as being neither discrete nor continuous.

Now, one recalls that BHs cannot emit more energy than their total mass. In addition, the total energy emitted by the BH cannot be imaginary [4 - 6]. Consequently, it must exist a maximum value of the principal quantum number \( n \) [4 - 6]. On the other hand, in [15] it has been shown that the GUP prevents the total BH evaporation in analogous way that the uncertainty principle prevents the hydrogen atom from the total collapse. In fact, the collapse is prevented by dynamics, rather than by symmetry, when the Planck scale is approached [15]. This fixes the maximum value of the principal quantum number \( n \) as [4 - 6]

\[
 n_{max} = 2(M^2 - 1). \tag{11}
\]

Let us compute the prime derivative of \( \Delta r_g(n_{-1\rightarrow n}) \) (that is the variation of the Schwarzschild radius due to an emission between two neighboring levels \( n \) and \( n - 1 \)) with respect to \( n \). One obtains

\[
 \frac{d}{dn} \left[ \Delta r_g(n_{-1\rightarrow n}) \right] = 4\pi \left( \frac{\sqrt{M^2 - \frac{n-1}{2}} - \sqrt{M^2 - \frac{n}{2}}}{\sqrt{(M^2 - n/2)(M^2 - (n-1)/2)}} \right). \tag{12}
\]

\[
 \frac{d}{dn} \left[ \Delta r_g(n_{-1\rightarrow n}) \right]
\]

is not defined when \( n = 2M^2 \) and \( n = 2M^2 + 1 \) which are values of \( n \) forbidden by the constraint (11). Analysing the sign of \( \frac{d}{dn} \left[ \Delta r_g(n_{-1\rightarrow n}) \right] \) one sees that it always positive for the permitted values of \( n \), which are \( 1 \ll n \leq n_{max} \). Thus, as \( \Delta r_g(n_{-1\rightarrow n}) \) is always positive (considering emissions, the Schwarzschild radius decreases), but we defined \( \Delta r_g(n_{1\rightarrow n_2}) \) in Eq. (8) as the difference between a longer and a smaller Schwarzschild radius in case of emissions), its value increases with increasing \( n \), and \( n_{max} \) is a maximum for \( \Delta r_g(n_{-1\rightarrow n}) \) given by

\[
 \Delta r_g(n_{max-1\rightarrow n_{max}}) = 2 \left( \sqrt{\frac{3}{2}} - 1 \right) \approx 0.449. \tag{13}
\]

As a consequence, also the value of the Schwarzschild “volume quantum” increases with increasing values of the BH energy level. Hence, the maximum value of the Schwarzschild “volume quantum” is obtained multiplying \( \Delta r_g(n_{max-1\rightarrow n_{max}}) \) by the Bekenstein area quantum, obtaining

\[
 \Delta V(n_{max-1\rightarrow n_{max}}) = \Delta V(n_{max-1\rightarrow n_{max}-1}) = \Delta V(n_{max-1\rightarrow n_{max}-1}) = 16\pi \left( \sqrt{\frac{3}{2}} - 1 \right) \approx 2.82. \tag{14}
\]

It is intriguing and remarkable that both of the results of Eqs. (13) and (14) do NOT depend on the BH original mass. In fact, this implies that the behavior of BH evaporation should be the same for ALL the Schwarzschild BHs when the Planck scale is approached.
Conclusion remarks

In this paper we have shown that our recent results in BH quantum physics [4 - 6] have the consequence that space-time quantization could be energy-dependent. Thus, the intriguing result is that, in a certain sense, space-time could be neither discrete nor continuous. Our analysis permitted also to show that the “volume quantum” of the Schwarzschild space-time increases with increasing energy during BH evaporation and arrives to a maximum value when the Planck scale is reached and the GUP prevents the total BH evaporation. As this result does not depend on the BH original mass, the remarkable consequence is that the behavior of BH evaporation should be the same for ALL the Schwarzschild BHs when the Planck scale is approached.

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