Spatially homogeneous solutions of the Vlasov-Nordström-Fokker-Planck system

José Antonio Alcántara Felix
Department of Applied Mathematics
University of Granada
Granada, Spain
jaaf@correo.ugr.es

Simone Calogero
Department of Mathematics
Chalmers Institute of Technology
Gothenburg, Sweden
calogero@chalmers.se

Stephen Pankavich
Department of Applied Mathematics and Statistics
Colorado School of Mines
Golden, CO USA
pankavic@mines.edu

Abstract
The Vlasov-Nordström-Fokker-Planck system describes the evolution of self-gravitating matter experiencing collisions with a fixed background of particles in the framework of a relativistic scalar theory of gravitation. We study the spatially-homogeneous system and prove global existence and uniqueness of solutions for the corresponding initial value problem in three momentum dimensions. Additionally, we study the long time asymptotic behavior of the system and prove that even in the absence of friction, solutions possess a non-trivial asymptotic profile. The admissible future attractors are given by spatially-homogeneous solutions of the associated ultra-relativistic Fokker-Planck equation evaluated at finite time. Finally, in the case of spherically symmetric initial data, we obtain an explicit representation for the asymptotic behavior.

Keywords: Vlasov-Nordström, Fokker-Planck equation, spatially homogeneous, global existence, ultra-relativistic, long time behavior

1. Introduction

The Vlasov-Nordström-Fokker-Planck (VNFP) system has been introduced in [2] as a simplified model for the relativistic diffusion dynamics of self-gravitating particle systems. In the absence of
Spacetime is curved by the action of the gravitational forces and is given by the manifold \((\mathbb{R}^3, g_{\mu\nu})\) constants, i.e., the speed of light and the gravitational constant \(G\), have been set equal to one. The physical interpretation of a solution of (1)-(2) as follows. Spacetime is curved by the action of the gravitational forces and is given by the manifold \((\mathbb{R}^4, g)\), where \(g\) is the conformally Minkowskian metric \(g = \exp(2\phi)\eta\). In the collisionless case (i.e., for \(\sigma = 0\)), the VNFP system reduces to the Nordström-Vlasov system [7, 8], a toy model for the full general relativistic Einstein-Vlasov system [3]. In contrast to the collisionless case, particles undergoing diffusion no longer move along the geodesics of spacetime. Their trajectories are defined through the system of stochastic differential equations naturally associated to the Fokker-Planck equation (1) via Ito’s formula.

A consistent theory for the diffusion dynamics of particle systems in General Relativity has been proposed in [9], but due to the well-known complexity of the Einstein field equations, it seems wise to deal first with the analysis of the system (1)-(2). The VNFP system already captures some of the essential features of relativistic gravitational systems undergoing diffusion: the hyperbolic character of the field equation, the invariance under Lorentz transformations, and the space-time dependence of the diffusion matrix. These features distinguish the model under study from the Vlasov-Poisson-Fokker-Planck system, which is the non-relativistic analogue of the VNFP system [5, 6, 10, 11, 12, 16, 20]. While the non-relativistic problem has been investigated for a long time, the interest on relativistic diffusion models has only recently started to increase [2, 9, 13, 14, 15, 17, 18, 21].

In this paper we make a further simplification by restricting the discussion to spatially homogeneous solutions \(f = f(t, p), \phi = \phi(t)\), for which the VNFP system becomes (setting \(\sigma = 1\))

\[
\partial_t f = e^{2\phi} \partial_p \left( \frac{e^{2\phi} \delta^{ij} + p^i p^j}{\sqrt{e^{2\phi} + |p|^2}} \partial_p f \right),
\]

\[
\partial_t \phi = -e^{2\phi} \int_{\mathbb{R}^3} \frac{f}{\sqrt{e^{2\phi} + |p|^2}} dp,
\]

where \(f = f(t, x, p)\) is the particle density in phase space, \(\phi = \phi(t, x)\) is the Nordström gravitational potential generated by the particles, and \(\sigma > 0\) is the diffusion constant. The remaining physical constants, i.e., the speed of light \(c\), the mass \(m\) of the particles, and the gravitational constant \(G\), have been set equal to one. The physical interpretation of a solution of (1)-(2) is as follows. Spacetime is curved by the action of the gravitational forces and is given by the manifold \((\mathbb{R}^4, g)\), where \(g\) is the conformally Minkowskian metric \(g = \exp(2\phi)\eta\). In the collisionless case (i.e., for \(\sigma = 0\)), the VNFP system reduces to the Nordström-Vlasov system [7, 8], a toy model for the full general relativistic Einstein-Vlasov system [3]. In contrast to the collisionless case, particles undergoing diffusion no longer move along the geodesics of spacetime. Their trajectories are defined through the system of stochastic differential equations naturally associated to the Fokker-Planck equation (1) via Ito’s formula.

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this mechanism in a simpler context, consider the non-autonomous heat equation

\[ \partial_t u = \lambda(t) \Delta_x u, \quad t > 0, \ x \in \mathbb{R}^3, \]  

(5)

where \( \lambda(t) \) is a smooth positive function integrable on \((0, \infty)\). Upon introducing the change of variables \( \tau(t) = \int_0^t \lambda(s) \, ds \), equation (5) transforms into the standard, autonomous heat equation. It follows that the solution of (5) with initial datum \( u(0, x) = u_0(x) \) is given in terms of the heat kernel by

\[ u(t, x) = \frac{1}{(4\pi \tau(t))^{3/2}} \int_{\mathbb{R}^3} u_0(y) e^{-\frac{|x-y|^2}{4\tau(t)}} \, dy. \]

Hence, as \( t \to \infty \),

\[ u(t, x) \sim \frac{1}{(4\pi \tau_{\infty})^{3/2}} \int_{\mathbb{R}^3} u_0(y) e^{-\frac{|x-y|^2}{4\tau_{\infty}}} \, dy, \quad \text{where} \quad \tau_{\infty} = \lim_{t \to \infty} \tau(t) = \int_0^\infty \lambda(s) \, ds < \infty, \]

i.e., the solution has a non-trivial asymptotic profile.

This paper proceeds as follows. In the next section, we state and prove a global existence and uniqueness theorem for (3)-(4) and derive estimates needed within the associated proof. Since the differential operator in the right side of (3) is not uniformly elliptic and has time dependent coefficients, the standard theory for parabolic equations does not apply in our case and we shall need to rely on stochastic methods to prove existence of solutions. Section 3 is then devoted to the study of the time asymptotic behavior of solutions as described above. Finally, details of more technical computations are included within an appendix.

2. Global existence and uniqueness

We begin by fixing some notation. Given \( T > 0 \) we denote by \( C_T \) any positive constant that depends only on the initial data and the time interval \([0, T]\). If \( \sup_{T>0} C_T < \infty \), we denote \( C_T \equiv C \).

The diffusion matrix in the Fokker-Planck equation (3) will be denoted by \( D[\phi] \) with entries

\[ D^{ij}[\phi] = \frac{e^{2\phi} \delta^{ij} + p^i p^j}{\sqrt{e^{2\phi} + |p|^2}}. \]

(6)

In the appendix we collect some identities and estimates satisfied by \( D[\phi] \) that are used throughout the paper. We use the index summation rule, which means that an index appearing twice in an expression, once in the lower and once in the upper position, is summed over \{1, 2, 3\}, e.g.,

\[ A^{ij} B_{jk} = \sum_{j,k=1}^3 A^{ij} B_{jk}. \]

Moreover, indexes are raised and lowered with the Kronecker symbol, e.g., \( D^i_j = D^{ik} \delta_{kj} \). The Banach space \( X \) is defined throughout as

\[ X = \{ g : \mathbb{R}^3 \to \mathbb{R} : g \in L^1 \cap L^2, \nabla g \in L^2 \} = L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), \]

with the usual notation for Lebesgue and Sobolev spaces. The ultimate purpose of this section is to prove the following global existence and uniqueness theorem for the spatially homogeneous VNFP system (3)-(4).
Theorem 2.1. Given $\phi_{\text{in}}, \psi_{\text{in}} \in \mathbb{R}$ and $f_{\text{in}} \in X$ such that $f_{\text{in}} \geq 0$ a.e., and

$$
\int_{\mathbb{R}^3} |p|(f_{\text{in}} + |\nabla f_{\text{in}}|^2) \, dp < \infty,
$$

there exists a unique solution $f \in L^\infty((0, \infty); X)$, $\phi \in C^1((0, \infty)) \cap W^{2, \infty}_{\text{loc}}((0, \infty))$ of (1)-(2) such that $f(0, p) = f_{\text{in}}(p)$ and $(\phi(0), \phi(0)) = (\phi_{\text{in}}, \psi_{\text{in}})$. Moreover $f \geq 0$ a.e. and there exist constants $\alpha, \beta, C > 0$ such that the following estimates hold

$$
\|f(t)\|_{L^1(\mathbb{R}^3)} = \|f_{\text{in}}\|_{L^1(\mathbb{R}^3)}, \quad \|f(t)\|_{L^2(\mathbb{R}^3)} \leq \|f_{\text{in}}\|_{L^2(\mathbb{R}^3)},
$$

$$
\|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} |p|(f + |\nabla_p f|^2) \, dp < C,
$$

$$
-C - \alpha t \leq \phi(t) \leq C - \beta t, \quad |\dot{\phi}(t)| < C, \quad -C e^{-\alpha t} \leq \ddot{\phi}(t) \leq 0.
$$

Finally, if $\nabla^2 f_{\text{in}} \in L^2(\mathbb{R}^3)$, then $\nabla^2 f \in L^\infty((0, T); L^2(\mathbb{R}^3))$, for all $T > 0$, with the growth estimate

$$
\|\nabla^2 f(t)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t).
$$

We split the proof of Theorem 2.1 into several subsections.

2.1. The Nordström equation

Assume first that $0 \leq f \in C((0, \infty); L^1(\mathbb{R}^3))$ is given and consider the Cauchy problem for the Nordström field equation

$$
\begin{align*}
\ddot{\phi}(t) &= -H_f(t, \phi), \quad t > 0, \tag{12a} \\
\phi(0) &= \phi_{\text{in}}, \quad \dot{\phi}(0) = \psi_{\text{in}}, \tag{12b}
\end{align*}
$$

with

$$
H_f(t, \phi) = e^{2\phi} \int_{\mathbb{R}^3} \frac{f(t, p)}{\sqrt{e^{2\phi} + |p|^2}} \, dp. \tag{12c}
$$

Since the function $x \to e^{2x}/\sqrt{e^{2x} + |p|^2}$ is convex and monotonically increasing, we have

$$
|H_f(t, \phi_1) - H_f(t, \phi_2)| \leq \partial_\phi H_f(t, \phi_*)|\phi_2 - \phi_1| = e^{2\phi_*}|\phi_2 - \phi_1| \int_{\mathbb{R}^3} \frac{f(t, p) e^{2\phi_*} + 2|p|^2}{(e^{2\phi_*} + |p|^2)^{3/2}} \, dp
\leq 2\|f(t)\|_{L^1(\mathbb{R}^3)} e^{\phi_*}|\phi_2 - \phi_1|,
$$

where $\phi_* = \max\{\phi_1, \phi_2\}$. Letting $\psi_0 = \dot{\phi}, y = (\phi, \psi)$ and $F(t, y) = (y_2, H_f(t, y_1))$, equation (12a) becomes $y = F(t, y)$. From the regularity of $f$ and the estimate (13), the function $F$ is continuous in $t > 0$ and locally Lipschitz in $y$, uniformly in the time variable. It follows by Picard’s theorem that the Cauchy problem (12) has a unique local classical solution. Moreover by straightforward estimates we obtain

$$
-\mathcal{K}_f(t) e^{\phi(t)} \leq -H_f(t, \phi) = \ddot{\phi}(t) \leq 0, \tag{14a}
$$

$$
\psi_{\text{in}} - \mathcal{K}_f(t) \int_0^t e^{\phi(s)} \, ds \leq \dot{\phi}(t) \leq \psi_{\text{in}}, \tag{14b}
$$

$$
\psi_{\text{in}} t + \phi_{\text{in}} - \mathcal{K}_f(t) \int_0^t \int_0^s e^{\phi(\tau)} \, d\tau \, ds \leq \phi(t) \leq \psi_{\text{in}} t + \phi_{\text{in}}, \tag{14c}
$$

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where
\[ K_f(t) = \sup_{s \in (0,t)} \| f(s) \|_{L^1(\mathbb{R}^3)}, \] (15)

These estimates imply that \( \phi \in W^{2,\infty}((0,T)) \) and
\[ \| \phi \|_{W^{2,\infty}((0,T))} \leq CTK_f(T), \] (16)
for all \( T > 0 \). Hence we have proved

**Proposition 2.1.** The Cauchy problem \([12]\) has a unique global solution \( \phi \in C^2((0,\infty)) \). Moreover the solution satisfies the estimates \([14]-[16]\), for all \( t \in [0,T] \) and \( T > 0 \).

### 2.2. The linear Fokker-Planck equation

In this section we assume that \( \phi \in C^2((0,\infty)) \cap W^{1,\infty}_{\text{loc}}([0,\infty)) \) is given and consider the Cauchy problem for the linear Fokker-Planck equation
\[
\begin{align*}
\partial_t f &= e^{2\phi} \partial_p (D^{ij} \phi \partial_j f), \quad t > 0, \ p \in \mathbb{R}^3, \\
f(0, p) &= f_{\text{in}}(p), \quad p \in \mathbb{R}^3,
\end{align*}
\] (17a)

where we recall that the diffusion matrix \( D^{ij} \phi \) is defined by \([6]\). The purpose of this subsection is to prove the following result.

**Proposition 2.2.** Given \( 0 \leq f_{\text{in}} \in C^2_c(\mathbb{R}^3) \), there exists a unique, positive, classical solution of the Cauchy problem \([17]\). Moreover \( f \) satisfies
\[
\begin{align*}
\| f(t) \|_{L^1(\mathbb{R}^3)} &= \| f_{\text{in}} \|_{L^1(\mathbb{R}^3)}, \\
\| f(t) \|_{L^q(\mathbb{R}^3)} &\leq \| f_{\text{in}} \|_{L^q(\mathbb{R}^3)}, \text{ for all } q > 1, \\
\| \nabla_p f(t) \|_{L^2(\mathbb{R}^3)} &+ \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma (f + |\nabla_p f|^2) \, dp \leq C \exp \left( C \int_0^t Q_\phi(s) \, ds \right) \\
\| \nabla_p^2 f(t) \|_{L^2(\mathbb{R}^3)} &\leq C \left( 1 + \int_0^t \| \nabla_p f(s) \|_{L^2(\mathbb{R}^3)} \, ds \right) \exp \left( C \int_0^t e^{\phi(s)} \, ds \right),
\end{align*}
\] (18, 19, 20)

where, denoting \( (z)_+ = \min(0,z) \),
\[
Q_\phi(t) = e^{\phi(t)} + (\dot{\phi})_+(t).
\] (21)

**Proof.** For the proof of existence we employ methods from the theory of stochastic differential equations and diffusion processes described in \([4]\). To adhere with the formulation used in \([4]\), we define the functions
\[
\tilde{f}(t, p) = f(-t, p), \quad \tilde{\phi}(t) = \phi(-t), \quad t < 0, \ p \in \mathbb{R}^3,
\]
in terms of which the Cauchy problem \([17]\) becomes
\[
\begin{align*}
\partial_t \tilde{f} + \mathcal{D} \tilde{f} &= 0, \quad t < 0, \ p \in \mathbb{R}^3, \\
\tilde{f}(0, p) &= f_{\text{in}}(p), \quad p \in \mathbb{R}^3,
\end{align*}
\] (22a, 22b)
where $\mathcal{D}$ is the differential operator

$$\mathcal{D} = d^i \partial_{p^i} + \frac{1}{2} b^{ij} \partial_{p^i} \partial_{p^j},$$

and

$$d^i(t, p) = e^{2\phi} \partial_{p^i}(D^{ij}[\phi]) = \frac{3e^{2\phi} p^i}{\sqrt{e^{2\phi} + |p|^2}}, \quad b^{ij}(t, p) = 2e^{2\phi} D^{ij}[\phi]. \quad (23)$$

Let $G$ denote the square root of $b$, i.e., the unique positive definite matrix such that $b = G \cdot G^T$. It can be verified that

$$G^{ij} = \frac{e^{\phi}}{(e^{2\phi} + |p|^2)^{1/4}} \left[ \frac{\phi^g \delta^{ij} + \frac{p^i p^j}{e^{\phi} + \sqrt{e^{2\phi} + |p|^2}}}{e^{\phi}} \right]. \quad (24)$$

Now let $T > 0$ be fixed. Note that $|d(t, p)| + |G(t, p)| \leq C_T(1 + |p|)$, for $t \in [-T, 0]$. Additionally, we show in the appendix that the first and second derivatives of $d$ and $G$ with respect to $p$ are bounded. These estimates are exactly those required to apply [4, Th. 9.4.4]. Hence for any $t \in [-T, 0]$, the system of stochastic differential equations

$$dP = b(s, P) \, ds + G(s, P) \cdot dW, \quad (25)$$

with $dW$ denoting the standard Wiener process, admits a unique solution $P(s; x, t)$, $t \leq s \leq 0$ satisfying $P(t; x, t) = p$. Moreover, the Feynman-Kac formula

$$\tilde{f}(t, p) = \mathbb{E}[f_n(P(0, p; t))]$$

is a classical positive solution of the Cauchy problem [22] in the interval $[-T, 0]$. We recall that in the theory of stochastic differential equations, equation (22a) is the backward Kolmogorov equation associated to the system (25). In conclusion, transforming back to the original variables $(f, \phi)$, we have found a classical solution $f(t, p)$ of (17) defined for all $t > 0$. Next, we show that classical solutions satisfy the estimates [18]. Let $\xi \in C^\infty_c([0, \infty))$ be a non-increasing function such that

$$\xi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases}$$

and define $\xi_n(p) = \xi \left( \frac{|p|}{n} \right)$, for $p \in \mathbb{R}^3$ and $n \in \mathbb{N}$, $n \geq 1$. Then $\xi_n \in C^\infty_c(\mathbb{R}^3)$ is a cut-off function satisfying $0 \leq \xi_n \leq 1$, $\xi_n(p) = 1$ if $|p| \leq n$, and $\xi_n(p) = 0$ if $|p| \geq 2n$. We clearly have $|\nabla_p \xi_n| \leq C/n$ and $|\Delta_p \xi_n| \leq C/n^2$. By direct computation we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} \xi_n f^q \, dp = -q(q - 1)e^{2\phi} \int_{\mathbb{R}^3} \xi_n f^{q - 2} D^{ij}[\phi] \partial_{p^i} f \partial_{p^j} f \, dp$$

$$+ e^{2\phi} \int_{\mathbb{R}^3} f^q \left( (\partial_{p^i} D^{ij}[\phi]) \partial_{p^i} \xi_n + D^{ij}[\phi] \partial_{p^i} \partial_{p^j} \xi_n \right) dp, \quad (26)$$

for all $q \geq 1$. By the positivity of $D$, the first term in the right side of (26) is non-positive. From the properties of the cutoff function, the term in square brackets in the last integral satisfies

$$[\ldots] \leq \frac{C_T}{n}, \quad \text{for all } t \in [0, T] \text{ and all } T > 0.$$
Hence using Grönwall’s inequality, the identity \[26\] gives
\[
\|f(t)\|_{L^1(\mathbb{R}^3)} \leq C_T.
\]
Substituting again in \[26\], we get the inequalities
\[
\|f_{in}\|_{L^1(\mathbb{R}^3)} - \frac{C_T}{n} \leq \|f(t)\|_{L^1(\mathbb{R}^3)} \leq \|f_{in}\|_{L^1(\mathbb{R}^3)} + \frac{C_T}{n}, \quad \|f(t)\|_{L^1(\mathbb{R}^3)} \leq \|f_{in}\|_{L^1(\mathbb{R}^2)} + \frac{C_T}{n}.
\]
Taking the limit as \(n \to \infty\) proves \[18\]. The uniqueness statement of the proposition follows by applying the estimate on \(\|f(t)\|_{L^2(\mathbb{R}^3)}\) to the difference of two solutions. As to the estimates \[19\]-\[20\], we present for brevity only a formal proof; the computations can be made rigorous by introducing the cut-off function \(\xi\) as above. We then compute
\[
\frac{d}{dt} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma^2 f dp = 2\gamma e^{2\phi} \dot{\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma-1} f dp + e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma \partial_p (D_{ij}[\phi] \partial_p f) dp
\]
\[
\leq C(\dot{\phi}) + \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma f dp + e^{2\phi} \int_{\mathbb{R}^3} \partial_p \{D_{ij}[\phi] \partial_p [(e^{2\phi} + |p|^2)^\gamma]\} f dp,
\]
where \((\cdot)_+\) denotes the positive part. Bounding the bracketed portion \{\ldots\} of the second term, we find
\[
\nabla_p \cdot \{\ldots\} = 4\gamma(\gamma - 1/2)(e^{2\phi} + |p|^2)^{\gamma-3/2}|p|^2 + 6\gamma(e^{2\phi} + |p|^2)^{\gamma-1/2} \leq Ce^{-\phi}(e^{2\phi} + |p|^2)^\gamma,
\]
and thus obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma f dp \leq C(e^{\phi} + (\phi)_+) \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma f dp,
\]
which by Grönwall’s inequality implies
\[
\int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma f dp \leq C \exp \left(C \int_0^t Q_\phi(s) \, ds\right).
\]
As to the estimate on \(\nabla_p f\), we compute
\[
\frac{d}{dt} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla_p f|^2 dp = 2\gamma e^{2\phi} \dot{\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma-1}|\nabla_p f|^2 dp
\]
\[
+ 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma \nabla_p f \cdot \nabla_p (\partial_p (D_{ij}[\phi] \partial_p f) \right) dp. \tag{27}
\]
In the integral in \((\ast)\) we first integrate by parts in the variable \(p^i\) and then, after straightforward calculations, we obtain
\[
(\ast) = I_1 + I_2 + I_3 + I_4,
\]
where
\[
I_1 = -2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma D_{ij}[\phi] \partial_p \nabla_p f \cdot \partial_p \nabla_p f dp
\]
\[
I_2 = -4\gamma e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma A^{jk}[\phi] \partial_p f \partial_p \nabla_p f dp
\]
\[
I_3 = 2\gamma e^{2\phi} \int_{\mathbb{R}^3} \nabla_p \cdot (p(e^{2\phi} + |p|^2)^{\gamma-1/2}) |\nabla_p f|^2 dp
\]
\[
I_4 = -2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma \partial_p \partial_p \nabla_p f (\partial_p D_{ij}[\phi]) \partial_p f dp
\]
and

\[ A^i_k[\phi] = \frac{p_i \partial \rho_k D^{ij}}{e^{2\phi} + |p|^2} = \frac{\delta^i_k |p|^2}{(e^{2\phi} + |p|^2)^{3/2}}. \]

By the positivity of \( D \) and \( A \) we have \( I_1 + I_2 \leq 0 \). In the integral \( I_3 \) we compute

\[ \nabla_p \cdot (p(e^{2\phi} + |p|^2)^{-1/2}) = 3(e^{2\phi} + |p|^2)^{-1/2} + (2\gamma - 1)(e^{2\phi} + |p|^2)^{-3/2}|p|^2 \leq C e^{-\phi}(e^{2\phi} + |p|^2)^\gamma \]

and thus the integral \( I_3 \) is bounded by

\[ I_3 \leq Ce^\phi \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla_p f|^2 \, dp. \]

The integral \( I_4 \) requires some further work. Integrating by parts the \( p^k \) derivative in \( \partial_{p^k} f \), we obtain

\[ I_4 = 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma (\Delta_p D^{ij}[\phi]) \partial_{p^i} f \partial_{p^j} f \]

\[ + 4\gamma e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma B^{ij} \partial_{p^i} f \partial_{p^j} f \]

\[ + 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma \partial_{p^i} f (\partial_{p^k} D^{ij}) \partial_{p^j} \partial_{p^k} f \, dp, \]

where

\[ B^{ij} = \frac{p \cdot \nabla_p D^{ij}[\phi]}{(e^{2\phi} + |p|^2)^\gamma} = \frac{2p_ip_j}{(e^{2\phi} + |p|^2)^{3/2}} - \frac{|p|^2}{(e^{2\phi} + |p|^2)^2} D^{ij}[\phi], \]

\[ \Delta_p D^{ij}[\phi] = \frac{1}{e^{2\phi} + |p|^2} \left( 2\delta^{ij} \frac{p_ip_j}{(e^{2\phi} + |p|^2)^2} + \frac{3e^{2\phi}}{(e^{2\phi} + |p|^2)^2} \right) D^{ij}[\phi]. \]

By the symmetry of \( D \), the last integral equals \(-I_4\) and thus we have obtained

\[ I_4 = I_{4A} + I_{4B}, \]

where

\[ I_{4A} = e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma (\Delta_p D^{ij}[\phi]) \partial_{p^i} f \partial_{p^j} f \, dp \]

\[ I_{4B} = 2\gamma e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma B^{ij} \partial_{p^i} f \partial_{p^j} f \, dp. \]

Using the bounds

\[ B^{ij} x_i x_j \leq Ce^{-\phi}|x|^2, \quad \Delta_p D^{ij}[\phi] x_i x_j \leq Ce^{-\phi}|x|^2, \text{ for all } x \in \mathbb{R}^3, \]

we have

\[ I_4 \leq Ce^\phi \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla_p f|^2 \, dp. \]

Collecting the estimates we find that the term \((*)\) in [27] satisfies

\[ (*) \leq Ce^\phi \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla_p f|^2 \, dp, \quad (28) \]
An application of Grönwall’s inequality completes the proof of (19). To prove (20) we use that $g_k = \partial_{\rho^k} f$ satisfies, for all $k = 1, 2, 3$,

$$\partial_t g_k = e^{2\phi} \partial_{\rho^i} (D^{ij} \partial_{\rho^j} g_k) + e^{2\phi} \partial_{\rho^i} [(\partial_{\rho^k} D^{ij}) g_j]$$

and thus

$$\frac{d}{dt} \int_{\mathbb{R}^3} \nabla_p g^k \cdot \nabla_p g_k = 2e^{2\phi} \int_{\mathbb{R}^3} \nabla_p g^k \cdot \nabla_p \left[ \partial_{\rho^i} (D^{ij} \partial_{\rho^j} g_k) \right] dp + 2e^{2\phi} \int_{\mathbb{R}^3} \nabla_p g^k \cdot \nabla_p \left[ (\partial_{\rho^k} D^{ij}) g_j \right] dp = II + III.$$  

The term $II$ is the same as $(*)_\gamma=0$ in (27) with $f$ replaced by $g_k$ and thus by (28) it satisfies the bound

$$II \leq Ce^{\phi} \int_{\mathbb{R}^3} \nabla_p g^k \cdot \nabla_p g_k dp \leq Ce^{\phi} \|
abla_p f(t)\|_{L^2(\mathbb{R}^3)}.$$  

Expanding the term $III$ in (29) we obtain

$$III = 2e^{2\phi} \int_{\mathbb{R}^3} \nabla_p g^k \cdot \nabla_p (\partial_{\rho^i} \partial_{\rho^k} D^{ij}) g_j dp + 2e^{2\phi} \int_{\mathbb{R}^3} \partial_{\rho^i} \partial_{\rho^k} D^{ij} \nabla_p g^k \cdot \nabla_p g_j dp + 2e^{2\phi} \int_{\mathbb{R}^3} \partial_{\rho^i} D^{ij} \nabla_p g^k \cdot \nabla_p \partial_{\rho^k} g_j dp$$

$$= III_1 + III_2 + III_3 + III_4.$$  

In $III_4$ we integrate by parts in the $p^i$ derivative acting on $g_j$ and obtain

$$III_4 = -2e^{2\phi} \int_{\mathbb{R}^3} \partial_{\rho^i} \partial_{\rho^k} D^{ij} \nabla_p g^k \cdot \nabla_p g_j dp - 2e^{2\phi} \int_{\mathbb{R}^3} \partial_{\rho^i} D^{ij} \nabla_p g^k \cdot \nabla_p \partial_{\rho^k} g_j dp$$

$$= III_{4A} + III_{4B}.$$  

Note that $III_2 + III_{4A} = 0$; in $III_{4B}$ we integrate by parts in the $p^k$ derivative within $g_k = \partial_{\rho^k} f$, so that

$$III_{4B} = 2e^{2\phi} \int_{\mathbb{R}^3} \Delta_p D^{ij} \nabla_p g_i \cdot \nabla_p g_j dp + 2e^{2\phi} \int_{\mathbb{R}^3} \partial_{\rho^k} D^{ij} \nabla_p g_i \cdot \nabla_p \partial_{\rho^k} g_j dp.$$  

By the symmetry of $D$, the second term in the latter equation equals $-III_{4B}$, hence

$$III_{4B} = e^{2\phi} \int_{\mathbb{R}^3} \Delta_p D^{ij} \nabla_p g_i \cdot \nabla_p g_j.$$  

Substituting (33) into (32) and then returning to (31) we obtain

$$III = 2e^{2\phi} \int_{\mathbb{R}^3} \nabla_p (\partial_{\rho^i} \partial_{\rho^k} D^{ij}) \cdot (\nabla_p g^k) g_j dp$$

$$+ 2e^{2\phi} \int_{\mathbb{R}^3} \nabla_p (\partial_{\rho^k} D^{ij}) \cdot \nabla_p g^k \partial_{\rho^k} g_j dp$$

$$+ e^{2\phi} \int_{\mathbb{R}^3} \Delta_p D^{ij} \nabla_p g_i \cdot \nabla_p g_j dp.$$  

(34)
We show in the appendix that
\[ |\partial_p \partial_p D_{ij}| \leq C e^{-\phi}, \quad |\nabla_p (\partial_p \partial_p D_{ij})| \leq C e^{-2\phi}. \] (35)

Using these estimates within (34) we get
\[ III \leq C \int_{\mathbb{R}^3} |\nabla_p g_k| |g_j| \, dp + C e^\phi \int_{\mathbb{R}^3} (|\nabla_p g_k| + |\nabla_p g_i|) |\nabla_p g_j| \, dp \]
\[ \leq C \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)} \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)} + C e^\phi \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)}^2, \] (36)

Substituting the bounds (30) and (36) into (29) we obtain
\[ \frac{d}{dt} \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)}^2 \leq C \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)} \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)} + C e^\phi \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)}^2 \]
and therefore
\[ \frac{d}{dt} \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)} \leq C \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)} + C e^\phi \|\nabla_p f(t)\|_{L^2(\mathbb{R}^3)}, \]
which by Grönwall’s inequality gives (20). \(\square\)

2.3. Existence and uniqueness

By a simple density argument we can assume that \(f_{in} \in C^2_c(\mathbb{R}^3)\). We fix \(T > 0\) and consider the sequence \((f_n, \phi_n)\) defined iteratively as follows. For \(n = 0\) we set \((f_0, \phi_0) = (f_{in}, \phi_{in})\). Assuming that the pair \((f_n, \phi_n)\) is given, we define \((f_{n+1}, \phi_{n+1})\) as the unique solution of the equations
\[ \partial_t f_{n+1} = e^{2\phi_n} \partial_p (D_{ij}|\phi_n| \partial_p f_{n+1}), \quad \phi_{n+1} = -e^{2\phi_n} \int_{\mathbb{R}^3} \frac{f_{n+1}}{\sqrt{e^{2\phi_{n+1}} + |p|^2}} \, dp, \]
with initial data \(f_{n+1}(0, p) = f_{in}(p), \ (\phi_{n+1}(0), \phi_{n+1}(0)) = (\phi_{in}, \psi_{in})\). It follows by induction and Propositions 2.1 and 2.2 that the sequence \((f_n, \phi_n)\) consists of smooth functions. Moreover
\[ \|f_n(t)\|_{L^1(\mathbb{R}^3)} = \|f_{in}\|_{L^1(\mathbb{R}^3)}, \]
and therefore the function \(K_{f_n}(t)\) given by (15) is equibounded along the sequence \(f_n\). Thus, by (16),
\[ \|\phi_n\|_{W^{2,\infty}((0,T))} \leq C_T. \]
We infer that the function \(Q_{\phi_n}(t)\) given by (21) is equibounded along the sequence \(\phi_n\). Hence, by (19),
\[ \|\nabla_p f_n(t)\|_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} |p| (f_n + |\nabla_p f_n|^2) \, dp \leq C_T, \quad \text{for all } t \in [0, T]. \]
It follows that there exist
\[ f \in L^\infty((0,T); H^1(\mathbb{R}^3)), \quad \phi \in W^{2,\infty}((0,T)) \]
and a subsequence, still denoted \((f_n, \phi_n)\), such that
\[ f_n \to f, \text{ in } L^2((0,T) \times \mathbb{R}^3), \quad \phi_n \rightharpoonup \phi, \text{ in } W^{2,\infty}(0,T), \quad \text{as } n \to \infty. \]
By a standard diagonal sequence argument, we can choose \((f_n, \phi_n)\) to be independent of \(T > 0\). Moreover

\[
f_n(t, \cdot) \to f(t, \cdot) \text{ in } H^1(\mathbb{R}^3) \quad \text{for all } t \in [0, T].
\]

By compactness, we may extract a subsequence such that \(f_n(t, \cdot)\) converges strongly in \(L^2(\mathbb{R}^3)\) and \((\phi_n, \phi_n)\) converges uniformly on \([0, T]\) (which implies in particular that \(\phi \in C^1\)). It is clear that this convergence is strong enough to pass to the limit in the equations and conclude that \((f, \phi)\) is a solution of the spatially homogeneous VNFP system \(3\)–\(4\). Moreover it is easy to show that \(f_n(t, \cdot) \to f(t, \cdot)\) in \(L^1(\mathbb{R}^3)\) (up to subsequences), and so in particular that the mass of \(f\) is preserved. In fact the sequence \(f_n\) is bounded in \(L^2(\mathbb{R}^3)\) and it is tight, because \(|p| f_n\) is bounded in \(L^1(\mathbb{R}^3)\). Hence weak convergence in \(L^1(\mathbb{R}^3)\) of \(f_n(t, \cdot)\) follows by the Dunford-Pettis theorem.

Next, we prove the uniqueness statement of Theorem 2.1. We do so by deriving a homogeneous Grönwall’s type inequality on the difference of two solutions with the same initial data. For brevity we limit ourselves to a formal derivation assuming all the regularity of solutions necessary for the computations which follow. However, after regularizing with a mollifying test function \(\xi \in C^\infty_c((0, T) \times \mathbb{R}^3)\) of the form \(\xi(t, p) = \theta(t)\mu(p)\) for an appropriate choice of \(\theta\) and \(\mu\), one may work with only the proven regularity of solutions and make the proof completely rigorous (an example of an application of this argument can be found for instance in \([6]\)). Let \((f_i, \phi_i), i = 1, 2,\) be two regular solutions with the same initial data. We compute

\[
\frac{d}{dt} \int_{\mathbb{R}^3} (f_1 - f_2)^2 \, dp = 2 \int_{\mathbb{R}^3} (f_1 - f_2) [e^{2\phi_1} \partial_p(D^{ij}[\phi_1]\partial_p f_1) - e^{2\phi_2} \partial_p(D^{ij}[\phi_2]\partial_p f_2)] \, dp
\]

\[
= 2e^{2\phi_1} \int_{\mathbb{R}^3} (f_1 - f_2) \partial_p(D^{ij}[\phi_1]\partial_p f_1 - f_2) \, dp
\]

\[
+ 2 \int_{\mathbb{R}^3} (f_1 - f_2) \partial_p([e^{2\phi_1} D^{ij}[\phi_1] - e^{2\phi_2} D^{ij}[\phi_2]] \partial_p f_2) = I_1 + I_2.
\]

By the positivity of \(D\), the first integral satisfies

\[
I_1 = -2e^{2\phi_1} \int_{\mathbb{R}^3} D^{ij}[\phi_1] \partial_p(f_1 - f_2) \partial_p(f_1 - f_2) \, dp \leq 0.
\]

In the second integral we apply the bound

\[
|e^{2\phi_1} D^{ij}[\phi_1] - e^{2\phi_2} D^{ij}[\phi_2]| \leq C_T(1 + |p|) |\phi_1 - \phi_2|,
\]

which follows by straightforward estimates (see the appendix). After an integration by parts, we find

\[
I_2 = -2 \int_{\mathbb{R}^3} \partial_p(f_1 - f_2) \partial_p f_2 (e^{2\phi_1} D^{ij}[\phi_1] - e^{2\phi_2} D^{ij}[\phi_2]) \, dp
\]

\[
\leq C_T |\phi_1 - \phi_2| \int_{\mathbb{R}^3} (1 + |p|)(|\nabla_p f_1| |\nabla_p f_2| + |\nabla_p f_2|^2) \leq C_T |\phi_1 - \phi_2|,
\]

where we used the fact that \(\sqrt{|p|} \nabla_p f_k \in L^\infty((0, T); L^2(\mathbb{R}^3))\) and Cauchy-Schwarz. Hence, we have the estimate

\[
\|f_1(t) - f_2(t)\|_{L^2(\mathbb{R}^3)}^2 \leq C_T \int_0^t |\phi_1(s) - \phi_2(s)| \, ds.
\]
Now we compute

\[
\phi_1 - \phi_2 = - \int_0^t \int_0^s \int_{\mathbb{R}^3} \left( \frac{f_1 e^{2\phi_1}}{\sqrt{e^{2\phi_1} + |p|^2}} - \frac{f_2 e^{2\phi_2}}{\sqrt{e^{2\phi_2} + |p|^2}} \right) \, dp \, d\tau \, ds
\]

\[
= \int_0^t \int_0^s \int_{\mathbb{R}^3} f_1 \left( \frac{e^{2\phi_1}}{\sqrt{e^{2\phi_1} + |p|^2}} - \frac{e^{2\phi_2}}{\sqrt{e^{2\phi_2} + |p|^2}} \right) \, dp \, d\tau \, ds
\]

\[
+ \int_0^t \int_0^s \int_{\mathbb{R}^3} \frac{e^{2\phi_1}}{\sqrt{e^{2\phi_1} + |p|^2}} (f_1 - f_2) \, dp \, d\tau \, ds
\]

\[= I_3 + I_4.\]

In the first integral we simply use the Mean Value Theorem so that

\[
\left| \frac{e^{2\phi_1}}{\sqrt{e^{2\phi_1} + |p|^2}} - \frac{e^{2\phi_2}}{\sqrt{e^{2\phi_2} + |p|^2}} \right| \leq C_T |\phi_1 - \phi_2|.
\]

Therefore, as \( f \in L^\infty((0, T); L^1(\mathbb{R}^3)) \),

\[I_3 \leq C_T \int_0^t \|\phi_1 - \phi_2\|_{L^\infty((0, s))} \, ds.\]

For \( I_4 \) we use an interpolation estimate so that for all \( R > 0 \)

\[
\int_{\mathbb{R}^3} \frac{(f_1 - f_2)}{\sqrt{e^{2\phi_2} + |p|^2}} \, dp \leq \frac{1}{R} \int_{|p| \geq R} (|f_1| + |f_2|) \, dp + \int_{|p| \leq R} |f_1 - f_2| \, dp |p|
\]

\[\leq C + C\|f_1(t) - f_2(t)\|_{L^2(\mathbb{R}^3)} \sqrt{R}
\]

\[\leq C\|f_1(t) - f_2(t)\|^{2/3}_{L^2(\mathbb{R}^3)},\]

where for the last inequality we chose \( R = \|f_1(t) - f_2(t)\|^{-2/3}_{L^2(\mathbb{R}^3)} \). Hence, collecting the estimates on \( I_3, I_4 \) we obtain

\[|\phi_1(t) - \phi_2(t)| \leq C_T \int_0^t \left( |\phi - \phi_2|_{L^\infty((0, s))} + \sup_{\tau \in (0, s)} \|f_1(\tau) - f_2(\tau)\|^{2/3}_{L^2(\mathbb{R}^3)} \right) \, ds \tag{39}\]

Combining (38) and (39) we have, for all \( T > 0 \),

\[
\sup_{t \in (0, T)} \|f_1(t) - f_2(t)\|^{2}_{L^2(\mathbb{R}^3)} + \|\phi_1 - \phi_2\|_{L^\infty((0, T))} \leq C_T \int_0^T \left( \sup_{s \in (0, t)} \|f_1(s) - f_2(s)\|^{2/3}_{L^2(\mathbb{R}^3)} + \|\phi_1 - \phi_2\|_{L^\infty((0, t))} \right) \, dt
\]

and conclude that \( f_1 = f_2 \) and \( \phi_1 = \phi_2 \) a.e. on \([0, T) \times \mathbb{R}^3\) by Gronwall’s inequality.

2.4. Uniform estimates

To complete the proof of Theorem 2.1 we must establish the estimates (9)-(10) and the last statement on the Hessian \( \nabla^2_p f \) of the solution. To this purpose we first notice that since \( \phi \) is decreasing, the limit

\[\phi_\infty = \lim_{t \to \infty} \phi(t)\]

exists.
Lemma 2.1. \( \dot{\phi}_\infty < 0 \).

Proof. Let
\[ M = \| f(t) \|_{L^1(\mathbb{R}^3)} = \| f_n \|_{L^1(\mathbb{R}^3)}, \quad \mathcal{E}(t) = \int f \sqrt{e^{2\phi} + |p|^2} \, dp + \frac{1}{2} \dot{\phi}^2 \] (40)
be the mass and the energy of the solution constructed in Section 2.3. By Hölder’s inequality,
\[ M^2 \leq \left( \int_{\mathbb{R}^3} f \sqrt{e^{2\phi} + |p|^2} \, dp \right) \left( \int_{\mathbb{R}^3} f \sqrt{e^{2\phi} + |p|^2} \, dp \right) \leq \left( \int_{\mathbb{R}^3} \sqrt{e^{2\phi} + |p|^2} \, dp \right) \mathcal{E}(t). \] (41)
Now, by a direct formal computation we have
\[ \dot{\mathcal{E}}(t) = 3 e^{2\phi} \int_{\mathbb{R}^3} f \, dp, \]
whence
\[ \mathcal{E}(t) = \mathcal{E}(0) + 3M \int_0^t e^{2\phi(s)} \, ds. \] (42)
The previous identity holds for the solution constructed in the previous section, as it follows by applying the above formal calculation to the sequence \((f_n, \phi_n)\) and then passing to the (strong) limit resulting as \( n \to \infty \). Using (42) in (41), we arrive at
\[ \int_{\mathbb{R}^3} f \sqrt{e^{2\phi} + |p|^2} \, dp \geq \frac{M^2}{\mathcal{E}(0) + 3M \int_0^t e^{2\phi(s)} \, ds}. \]
Utilizing this inequality yields
\[ \ddot{\phi} = -e^{2\phi} \int f \sqrt{e^{2\phi} + |p|^2} \, dp \leq - \frac{M^2 e^{2\phi}}{\mathcal{E}(0) + 3M \int_0^t e^{2\phi(s)} \, ds} - \frac{M}{3} \frac{d}{dt} \log \left[ \mathcal{E}(0) + 3M \int_0^t e^{2\phi(s)} \, ds \right]. \]
Whence
\[ \dot{\phi}(t) \leq \dot{\phi}(0) - \frac{M}{3} \log \left[ \mathcal{E}(0) + 3M \int_0^t e^{2\phi(s)} \, ds \right] + \frac{M}{3} \log \mathcal{E}(0). \] (43)
If \( \dot{\phi}_\infty \geq 0 \), then \( \phi \) is positive for all \( t \in [0, \infty) \). Hence the right side of (43) tends to \(-\infty\) as \( t \to \infty \), a contradiction. Thus \( \dot{\phi}_\infty < 0 \) must hold.

The previous lemma easily yields the desired estimates. In fact, since \( \dot{\phi}_\infty < 0 \) and \( \dot{\phi} \) is decreasing, there exists \( t_0 \geq 0 \) such that \( \dot{\phi}(t) < \dot{\phi}(t_0) < 0 \), for all \( t \geq t_0 \). Hence
\[ \phi(t) = \phi(t_0) + \int_{t_0}^t \dot{\phi}(s) \, ds \leq \left( \phi(t_0) + |\dot{\phi}(t_0)|t \right) - |\dot{\phi}(t_0)|t, \]
and therefore \( \dot{\phi}(t) \leq C - \beta t \) holds for some \( \beta, C > 0 \). Using this within (14) yields (10). Finally, since \( Q_\phi(t) = e^{\phi(t)} \), for \( t \geq t_0 \), we have
\[ \int_0^\infty Q_\phi(t) \, dt \leq C \left( 1 + \int_0^\infty e^{-\beta t} \, dt \right) < C, \]
and thus the estimates (9), (11) follow from (19), (20). This concludes the proof of Theorem 2.1.
3. Long time limit of the particle density

The results in this section do not require $\phi$ to be a solution of the Nordström field equation, but only assume that $\phi$ is smooth and satisfies

$$\lim_{t \to +\infty} \phi(t) = -\infty \quad \text{and} \quad \int_0^\infty e^{2\phi(t)} \, dt < \infty,$$

which holds of course for the solutions of the VNFP system considered in the previous section. In order to study the asymptotic behavior of solutions to VNFP, we first consider a reduced equation which arises by setting $\phi \equiv -\infty$, or $e^{2\phi} \equiv 0$ within the diffusion matrix $D[\phi]$. This is motivated by the previous result that $e^{2\phi(t)} \to 0$ as $t \to \infty$ for solutions of VNFP, and hence one expects the asymptotic behavior of the density $f$ to mimic that of a solution to the reduced equation.

3.1. The ultra-relativistic Fokker-Planck equation

We begin by investigating solutions to the ultra-relativistic Fokker-Planck equation

$$\partial_t g = \partial_{p^i} \left( D^{ij}[\infty] \partial_{p^j} g \right), \quad t > 0, \quad p \in \mathbb{R}^3. \quad (44)$$

where

$$D^{ij}[\infty] = \frac{p^i p^j}{|p|}.$$ 

Proposition 3.1. For $g_{in} \in L^2(\mathbb{R}^3)$ there exists a unique global solution $g \in L^\infty((0, \infty); L^2(\mathbb{R}^3))$ of (44) such that $g(0, p) = g_{in}(p)$. Moreover if the initial datum is spherically symmetric, i.e., $g_{in}(p) = g_{in}(|p|)$ for some $g_{in} : [0, \infty) \to [0, \infty)$, then the solution $g$ is also spherically symmetric and it is given by $g(t, p) = \hat{g}(t, |p|)$, where

$$\hat{g}(t, q) = e^{-q t} \int_0^\infty g_{in}(z) z e^{-z t} I_2 \left[ \frac{2 \sqrt{q} \sqrt{z} t}{t} \right] \, dz, \quad (45)$$

and $I_\alpha[x]$ denotes the $\alpha$th modified Bessel function of the first kind [1].

**Proof.** We only sketch the proof of existence and uniqueness as the main argument is the same used to prove Proposition 2.2. We begin by approximating the ultrarelativistic diffusion matrix $D[-\infty]$ with $D[\log \epsilon]$, i.e., we first consider the equation

$$\partial_t g = \partial_{p^i} \left( \epsilon^2 \delta^{ij} + \frac{p^i p^j}{|p|^2} \partial_{p^j} g \right), \quad \epsilon > 0. \quad (46)$$

As in the proof of Proposition 2.2, we may interpret (46) as the backward Kolmogorov equation for a suitable stochastic differential equation obtained via Itô's formula. The properties of the diffusion matrix $D[\log \epsilon]$ and the results in [1] ensure that the Cauchy problem for the stochastic differential equation is well-posed and that a classical solution $g_\epsilon$ of (46) with initial datum $g_{in} \in C^2(\mathbb{R}^3)$ is given by the Feynman-Kac formula. Moreover $g_\epsilon$ satisfies the estimate

$$\|g_\epsilon(t)\|_{L^2(\mathbb{R}^3)} \leq \|g_{in}\|_{L^2(\mathbb{R}^3)}. \quad (47)$$

It follows that there exists $g \in L^\infty((0, \infty); L^2(\mathbb{R}^3))$ and a subsequence (not re-labeled) $g_\epsilon$ such that

$$g_\epsilon \to g, \quad \text{in } L^2((0, T) \times \mathbb{R}^3), \quad \text{as } \epsilon \to 0, \quad \text{for all } T > 0,$$
and it is straightforward to verify that \( g \) solves (44) in the sense of distributions. Uniqueness follows by applying the estimates \( \|g(t)\|_{L^2(\mathbb{R}^3)} \leq \|g_{in}\|_{L^2(\mathbb{R}^3)} \) to the difference of two solutions with the same initial datum.

We now turn to the proof of the claims concerning spherically symmetric solutions. First we show that the spatial operator defined by

\[
L u = \partial_{p^i} \left( \frac{p^i p^j}{|p|} \partial_{p^j} u \right)
\]

is invariant under rotation. Let \( Q = [q^{ij}] \) be an orthogonal matrix and assume the functions \( v(p) = w(z) \) where \( z = Qp \). Then, computing derivatives yields

\[
\partial_{p^i} v(t, p) = q^{ki} \partial_{z^k} w(t, Qp).
\]

Using this and the result \( |z| = |Qp| = |p| \), we find

\[
L v = 3 \frac{p^i}{|p|} \partial_{p^i} v + \frac{p^i p^j}{|p|} \partial_{p^i} \partial_{p^j} v
\]

\[
= 3 \frac{q^{ki} p^i}{|p|} \partial_{z^k} w + \frac{q^{ki} p^i q^{lj} p^j}{|p|} \partial_{z^k} \partial_{z^l} w
\]

\[
= 3 \frac{z^k}{|z|} \partial_{z^k} w + \frac{z^k z^l}{|z|} \partial_{z^k} \partial_{z^l} w
\]

\[
= L w.
\]

With the invariance property and the uniqueness of solutions to the Cauchy problem for (44), it follows that spherically symmetric initial data launch spherically symmetric solutions.

Before proving the next portion of the theorem we derive a representation formula for solutions of the spherically symmetric heat equation in six dimensions, which will subsequently be of importance. Let \( u(t, x) \) be the solution of the Cauchy problem

\[
\begin{align*}
\partial_t u &= \Delta u, \quad t > 0, \quad x \in \mathbb{R}^6, \\
u(0, x) &= u_{in}(x), \quad x \in \mathbb{R}^6
\end{align*}
\]  \hspace{1cm} (48a)

that is

\[
u(t, x) = \frac{1}{(4\pi t)^3} \int_{\mathbb{R}^6} e^{-\frac{|x-y|^2}{4t}} u_{in}(y) \, dy.
\]  \hspace{1cm} (49)

Assuming spherical symmetry, i.e., \( u(t, x) = u(t, r) \) with \( r = |x| \), the heat equation (48a) becomes

\[
\partial_t u = \partial_r^2 u + \frac{5}{r} \partial_r u.
\]  \hspace{1cm} (50)

Letting \( u(0, x) = u_{in}(r) \) and passing to hyperspherical coordinates in the integral on the right side of (49) we obtain

\[
u(t, r) = \frac{e^{-\frac{r^2}{2}}}{(4\pi t)^3} \int_{\mathbb{R}^6} e^{-\frac{(|y|^2 - 2y \cdot x)}{4t}} u_{in}(|y|) \, dy
\]

\[
= \frac{8\pi^2}{3} \frac{e^{-\frac{r^2}{4}}}{(4\pi t)^3} \int_0^\infty u_{in}(s) e^{-\frac{s^2}{4t}} \int_0^\pi \exp \left( \frac{rs \cos \theta}{2t} \right) \sin^4 \theta \, d\theta \, ds.
\]  \hspace{1cm} (51)
Figure 1: Numerical depiction of a spherically symmetric solution $g(t, q)$ of the ultrarelativistic Fokker-Planck equation (44). The picture shows the sections $t = \text{const.}$ of $g$ from $t = 0.1$ (top curve) until $t = 0.5$ (bottom curve). The initial datum is taken to be $g_{in}(q) = e^{-q}$.

Evaluating the angular integral gives

$$ \int_0^\pi \exp \left( \frac{rs \cos \theta}{2t} \right) \sin^4 \theta \, d\theta = 12\pi \left( \frac{t}{rs} \right)^2 I_2 \left[ \frac{rs}{2t} \right], $$

where $I_\alpha[x]$ denotes the $\alpha$th modified Bessel function of the first kind [1]. Substituting this expression into (51) we obtain

$$ u(t, r) = \frac{1}{2} e^{-r^2/4t} \int_0^\infty u_{in}(s) e^{-s^2/4t} I_2 \left[ \frac{rs}{2t} \right] ds. $$

Let us now return to the proof of the formula (45). Assume that the initial datum is spherically symmetric, namely $g_{in}(p) = g_{in}(q)$ where $q = |p|$. Then, the corresponding solution must be spherically symmetric, and equation (44) for spherically symmetric solutions $g(t, p) = g(t, q)$ takes the form

$$ \partial_t g = q \partial^2_q g + 3 \partial_q g. $$

A straightforward calculation shows that if $u(t, r)$ solves (50), then $g(t, q) = u(t, 2\sqrt{q})$ solves (53). Hence, with the substitution $r = 2\sqrt{q}$ and the change of variables $s = 2\sqrt{z}$ in (52), the solution of the 3-dimensional ultrarelativistic Fokker-Planck equation (44) with initial datum $g_{in}(p) = g_{in}(|p|)$ is given by

$$ g(t, q) = \frac{e^{-q}}{tq} \int_0^\infty g_{in}(z) e^{-z} I_2 \left[ \frac{2\sqrt{q}}{t} \sqrt{z} \right] \, dz, $$

as claimed.

Hence, we have obtained an explicit form for solutions to (44) in the case of spherically symmetric initial data. Figure 1 contains a numerical depiction of such a solution $g(t, q)$ for a few specific
times. With the preceding result, we can answer the analogous question for the ultrarelativistic system with a scalar field using a simple change of variables.

**Corollary 3.1.** The solution of
\[ \partial_t h = e^{2\phi} \partial_p \left( \frac{p^i p^j}{|p|} \partial_p h \right), \]
with initial datum \( h(0, p) = h_{\text{in}}(p) \) is given by
\[ h(t, p) = g(\tau(t), p), \]
where
\[ \tau(t) = \int_0^t e^{2\phi(s)} \, ds. \]
Here \( g(t, p) \) is the solution of \( \text{(44)} \) with the same initial datum. In particular, if \( h_{\text{in}}(p) = h_{\text{in}}(|p|) \) then \( h(t, p) = h(t, |p|) \) where
\[ h(t, q) = g(\tau(t), q) = \frac{e^{-q}}{\tau(t)} \int_0^\infty h_{\text{in}}(z) z e^{-\frac{z}{\tau(t)}} I_2 \left[ \frac{2\sqrt{q}}{\tau(t)} \sqrt{z} \right] \, dz. \]

**Proof.** The result follows by rescaling time to account for the gravitational potential. Let
\[ \tau(t) = \int_0^t e^{2\phi(s)} \, ds \]
and make the change of variables \( h(t, p) = g(\tau(t), p) \). Then, \( \partial_t h = e^{2\phi} \partial_\tau g \), hence the unknown function \( g(\tau, p) \) satisfies the parabolic equation
\[ \partial_\tau g = \partial_p \left( \frac{p^i p^j}{|p|} \partial_p g \right) \]
and any solution must be of the form \( (55) \). Finally, the representation of solutions arising from spherically symmetric initial data follows in view of Proposition 3.1.

**Proposition 3.2.** If \( h_{\text{in}}(p) = h_{\text{in}}(q) \) where \( q = |p| \), then
\[ h(t, p) = h(t, q) \sim \frac{e^{-q}}{q \tau_{\infty}} \int_0^\infty h_{\text{in}}(z) z e^{-\frac{z}{\tau_{\infty}}} I_2 \left[ \frac{2\sqrt{q}}{\tau_{\infty}} \sqrt{z} \right] \, dz \quad \text{as} \quad t \to \infty, \]
both pointwise and in \( L^\gamma(\mathbb{R}^3) \) for any \( \gamma \in [1, \infty) \), where
\[ \tau_{\infty} = \lim_{t \to \infty} \tau(t) = \int_0^\infty e^{2\phi(s)} \, ds. \]

**Proof.** We shall use the following bounds satisfied by the modified Bessel functions \( I_2 \): there exists \( C > 0 \) such that
\[ I_2(x) \leq C x^2 \]
for $x \leq 1$, while for $x \geq 1$ we have
\[ I_2[x] \leq Cx^{-1/2}e^x \leq Ce^x. \]

It follows that the integrand function in the right side of (56) satisfies
\[ h_{in}(z)e^{-\frac{\sqrt{q}}{\tau(t)}\sqrt{z}} \leq C h_{in}(z)e^{-\frac{\sqrt{q}}{\tau(t)}(\sqrt{z}-\sqrt{q})^2}. \]

Because $\tau(t)$ is increasing and $\tau_{\infty}$ is finite, if we take $t \geq 1$ then $\tau(t)$ is bounded above and below by positive constants. Hence the right side of the previous equation is dominated uniformly in $t \geq 1$ by a function that is integrable in $z \in (0, \infty)$. Applying Lebesgue’s Dominated Convergence Theorem we have
\[ \lim_{t \to \infty} h(t, q) = e^{-\frac{q}{\tau_{\infty}}} \int_0^{\infty} h_{in}(z)z e^{-\frac{\sqrt{q}}{\tau_{\infty}}I_2} \left( 2\frac{\sqrt{q}}{\tau(t)}\sqrt{z} \right) dz. \]

Similarly, in order to prove the convergence in $L^\gamma(\mathbb{R}^3)$, it suffices to show that $h(t, p)$ is dominated by a function $w \in L^\gamma(\mathbb{R}^3)$ for $t$ large and apply the Dominated Convergence Theorem in Lebesgue spaces. Thus, using Cauchy-Schwarz we find
\[
\int_0^{\infty} h_{in}(z)z e^{-\frac{\sqrt{q}}{\tau_{\infty}}I_2} \left( 2\frac{\sqrt{q}}{\tau(t)}\sqrt{z} \right) dz \leq C \left( \int_0^{1/q} z^2 e^{-2z} I_2 \left[ \sqrt{qz} \right]^2 dz + \int_{1/q}^{\infty} z^2 e^{-2z} I_2 \left[ \sqrt{qz} \right]^2 dz \right)^{1/2}
\]

Within the first integral, $\sqrt{qz} \leq 1$ and thus
\[
\int_0^{1/q} z^2 e^{-2z} I_2 \left[ \sqrt{qz} \right]^2 dz \leq C q^2 \int_0^{1/q} z^4 e^{-2z} dz
\]
\[
\leq C q^2 \Gamma(5) \leq C q^2
\]

Similarly, $\sqrt{qz} \geq 1$ within the second integral and we find
\[
\int_{1/q}^{\infty} z^2 e^{-2z} I_2 \left[ \sqrt{qz} \right]^2 dz \leq C \int_{1/q}^{\infty} z^2 e^{-2z} e^{2\sqrt{qz}} dz
\]
\[
\leq C \int_{1/q}^{\infty} z^2 e^{-2z} e^z dz
\]
\[
\leq C e^q \int_{1/q}^{\infty} z^2 e^{-z} dz
\]
\[
\leq C e^q e^{-1/2q}
\]

Again, using the boundedness of $\tau(t)$, the preceding estimates imply
\[ h(t, q) \leq C \frac{e^{-q}}{q} \left( q^2 + e^q e^{-1/2q} \right)^{1/2} \leq C \left( qe^{-q} + \frac{e^{-1/4q}}{q} e^{-q/2} \right) \leq Ce^{-q/2}
\]
for all $t \geq 1$. Therefore, we find that $h(t, p)$ is uniformly dominated by $e^{-|p|/2} \in L^\gamma(\mathbb{R}^3)$ for every $\gamma \in [1, \infty)$, and the result follows.\[\square\]
3.2. Long time behavior of the particle density

With the results of the previous section, we may finally derive the asymptotic behavior of the particle density for solutions to (3)-(4).

**Theorem 3.1.** Let \((f, \phi)\) be the solution of (3)-(4) given by Theorem 2.1 with initial data \((f_{\text{in}}, \phi_{\text{in}}, \psi_{\text{in}}) \in X \times \mathbb{R}^2\) satisfying (7) and \(\nabla^2 f_{\text{in}} \in L^2(\mathbb{R}^3)\). Let \(h \in L^\infty((0, \infty); L^2(\mathbb{R}^3))\) be the solution of (55) given by Proposition 3.1 with initial datum \(h(0, p) = f_{\text{in}}(p)\), for every \(p \in \mathbb{R}^3\). Then, there exists \(t_0, C, \lambda > 0\) such that

\[
\| (f - h)(t) \|_2 \leq Ce^{-\lambda t}
\]

for \(t \geq t_0\).

**Proof.** To prove the result, we estimate the difference of the densities in \(L^2\) weighted by the potential. Taking a time derivative, integrating by parts, and using the positive-definite nature of the diffusion matrix, we first find

\[
\frac{1}{2} \frac{d}{dt} \int (f - h)^2 \, dp = \int (f - h)e^{2\phi} \left[ \partial_{\rho'} \left( \frac{e^{2\phi} \delta^{ij} + p^i p^j}{\sqrt{e^{2\phi} + |p|^2}} \partial_{\rho'} f - \frac{p^i p^j}{|p|} \partial_{\rho'} h \right) \right] \, dp
\]

\[
= -e^{2\phi} \int \partial_{\rho'} (f - h) \left( \frac{e^{2\phi} \delta^{ij} + p^i p^j}{\sqrt{e^{2\phi} + |p|^2}} \partial_{\rho'} f - \frac{p^i p^j}{|p|} \partial_{\rho'} h \right) \, dp
\]

\[
= -e^{2\phi} \int \partial_{\rho'} (f - h) D^{ij} [\phi] \partial_{\rho'} (f - h) \, dp
\]

\[
\leq -e^{2\phi} \int \partial_{\rho'} (f - h) (D^{ij} [\phi] - D^{ij} [-\infty]) \partial_{\rho'} f \, dp
\]

\[
= e^{2\phi} \int (f - h) \partial_{\rho'} [(D^{ij} [\phi] - D^{ij} [-\infty]) \partial_{\rho'} f] \, dp.
\]

Then, we incorporate the potential term and use this estimate, so that

\[
\frac{1}{2} \frac{d}{dt} \left( e^{-2\phi} \| (f - h)(t) \|_2^2 \right) = -e^{-2\phi} \| (f - h)(t) \|_2^2 + \frac{1}{2} e^{-2\phi} \frac{d}{dt} \| (f - h)(t) \|_2^2
\]

\[
\leq -e^{-2\phi} \| (f - h)(t) \|_2^2 + \int (f - h) \partial_{\rho'} [(D^{ij} [\phi] - D^{ij} [-\infty]) \partial_{\rho'} f] \, dp.
\]

Integrating and using the fact that \(f\) and \(h\) possess the same initial data, we find

\[
e^{-2\phi(t)} \| (f - h)(t) \|_2^2 \leq \int_0^t \left[ -\phi(s)e^{-2\phi(s)} \| (f - h)(s) \|_2^2
\]

\[
+ \int (f - h) \partial_{\rho'} [(D^{ij} [\phi] - D^{ij} [-\infty]) \partial_{\rho'} f] \, dp \right] \, ds.
\]

As shown in the appendix, the diffusion operator satisfies

\[
| D^{ij} [\phi] - D^{ij} [-\infty] | \leq Ce^\phi
\]

(57)
and

$$|\partial_{\mu} (D^{ij}[\phi] - D^{ij}[-\infty])| \leq C.$$  \hfill (58)

With this, we find

$$e^{-2\phi(t)}\| (f - h)(t) \|_2^2 \leq \int_0^t \left[ |\dot{\phi}(s)| e^{-2\phi(s)}\| (f - h)(s) \|_2^2 + C\int |f - h| \left( e^{\phi(s)}|\partial_{\mu'} \partial_{\mu} f| + |\partial_{\mu'} f| \right) \right] ds \leq \int_0^t \left[ |\dot{\phi}(s)| e^{-2\phi(s)}\| (f - h)(s) \|_2^2 + C\| (f - h)(s) \|_2 \left( e^{\phi(s)}\| \nabla_p f(s) \|_2 + \| \nabla_p f(s) \|_2 \right) \right] ds \leq \int_0^t \left[ |\dot{\phi}(s)| e^{-2\phi(s)}\| (f - h)(s) \|_2^2 + C\left(1 + se^{\phi(s)}\right) \right] ds \leq Ct + \int_0^t |\dot{\phi}(s)| e^{-2\phi(s)}\| (f - h)(s) \|_2^2 ds.$$ where we have used (10) in the last inequality. Letting $A(t) = e^{-2\phi(t)}\| (f - h)(t) \|_2^2$, this becomes

$$A(t) \leq Ct + \int_0^t |\dot{\phi}(s)| A(s) \ ds$$

Since $\dot{\phi}(s)$ is decreasing with $\dot{\phi}(t) < 0$ for $t$ sufficiently large, we find

$$\int_0^t |\dot{\phi}(s)| ds = -|\phi_0| - \phi(t)$$

for large $t$. Thus, by Gronwall’s inequality, we have

$$A(t) \leq Ct \exp \left( \int_0^t |\dot{\phi}(s)| \ ds \right) \leq Cte^{-\phi(t)}$$

By the definition of $A(t)$ this yields

$$\| (f - h)(t) \|_2^2 = A(t)e^{2\phi(t)} \leq Cte^{\phi(t)}$$

Finally, using (10) the exponential decay follows.

\[\square\]

**Appendix: Properties of the diffusion matrix**

In this appendix we collect some properties of the diffusion matrix

$$D^{ij}[\phi] = \frac{e^{2\phi} \delta^{ij} + p^i p^j}{\sqrt{e^{2\phi} + |p|^2}}$$

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and other quantities which are used in the main body of the paper. We begin proving (35). Clearly

$$|D^{ij}[\phi]| \leq \sqrt{e^{2\phi} + |p|^2}.$$  

The first derivatives of $D$ are given by

$$\partial_{p^k} D^{ij} = \frac{\delta^i_k p^j + \delta^j_k p^i}{\sqrt{e^{2\phi} + |p|^2}} - \frac{D^{ij} p_k}{e^{2\phi} + |p|^2},$$

and therefore

$$|\partial_{p^k} D^{ij}| \leq C.$$

Moreover

$$\partial_{p^l} (\partial_{p^k} D^{ij}) = -3 \frac{\delta^i_k p_l}{(e^{2\phi} + |p|^2)^{3/2}} - 3 \frac{\delta^j_k p_l + \delta_{kl} p^j}{(e^{2\phi} + |p|^2)^{3/2}} + 9 \frac{p^j p_k p_l}{(e^{2\phi} + |p|^3)^{5/2}},$$

and each term in the right hand side is bounded in modulus by $Ce^{-\phi}$, which proves the first estimate in (35). Furthermore

$$\partial_{p^l} (\partial_{p^k} D^{ij}) = -3 \frac{\delta^i_k p_l}{(e^{2\phi} + |p|^2)^{3/2}} - 3 \frac{\delta^j_k p_l + \delta_{kl} p^j}{(e^{2\phi} + |p|^2)^{3/2}} + 9 \frac{p^j p_k p_l}{(e^{2\phi} + |p|^3)^{5/2}},$$

and each term in the right hand side is bounded in modulus by $Ce^{-\phi}$, which proves the second estimate in (35).

We next justify (57) and (58). It seems a derivative bound is too crude for the first inequality, so after some algebra we find

$$|D^{ij}[\phi] - D^{ij}[-\infty]| = \left| \frac{e^{2\phi} \delta^{ij} + p^i p^j}{\sqrt{e^{2\phi} + |p|^2}} - \frac{p^i p^j}{|p|} \right|$$

$$\leq \frac{e^{2\phi}}{\sqrt{e^{2\phi} + |p|^2}} \left| \delta^{ij} - \frac{p^i p^j}{|p|(|p| + \sqrt{e^{2\phi} + |p|^2})} \right|$$

$$\leq \frac{Ce^{2\phi}}{\sqrt{e^{2\phi} + |p|^2}}$$

and the result follows. Similarly, for (58) we differentiate to find

$$|\partial_{p^l} (D^{ij}[\phi] - D^{ij}[-\infty])| = \left| \frac{3e^{2\phi} p^j}{|p| \sqrt{e^{2\phi} + |p|^2} \left( |p| + \sqrt{e^{2\phi} + |p|^2} \right)} \right| \leq C$$

since $|p| + \sqrt{e^{2\phi} + |p|^2} \geq \sqrt{e^{2\phi} + |p|^2} \geq e^\phi$.

In order to justify the computation (37), we first consider the function

$$L(\psi) = e^{2\psi} D^{ij}[\psi]$$

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and compute
\[ L'(\psi) = \frac{e^{2\psi}}{\sqrt{e^{2\psi} + |p|^2}} \left[ 3e^{2\psi} \delta_{ij} + p^i p^j \right] + \frac{|p|^2 e^{2\psi}}{(e^{2\psi} + |p|^2)^{3/2}} \left[ e^{2\psi} \delta_{ij} + p^i p^j \right]. \]

Thus, for all bounded \( \psi \)
\[ |L'(\psi)| \leq e^{2\psi} e^{-\psi} \cdot 3e^{2\psi} + e^{2\psi} |p|^{-1} |p^i p^j| + |p|^2 e^{2\psi} \left( e^{\psi} |p| \right)^{-3/2} e^{2\psi} + |p|^2 e^{2\psi} |p|^{-3} |p^i p^j| \]
\[ \leq 4e^{3\psi} + e^{2\psi} |p|^{1/2} e^{\psi/2} + 2e^{2\psi} |p| \]
\[ \leq C(1 + |p|). \]

Additionally, we must verify the smoothness of coefficients within the system of stochastic differential equations (25) in order to arrive at the existence of a unique solution, where the vector \( d \) and the matrix \( G \) are given in [23]-[24]. We first compute
\[ \partial_{\rho^i} d^i(t, p) = \frac{3e^{2\phi}}{(e^{2\phi} + |p|^2)^{-3/2}} \left( \delta_{ij} e^{2\phi} + \delta_{ij} |p|^2 - p^i p^j \right) \]
So, estimating we find for every \( i, j \)
\[ |\partial_{\rho^i} d^i(t, p)| \leq C e^{2\phi} (e^{2\phi} + |p|^2)^{-3/2} \left[ e^{2\phi} + |p|^2 \right] \]
\[ \leq C e^{2\phi} (e^{2\phi} + |p|^2)^{-1/2} \]
\[ \leq C_T \]
and derivatives are uniformly bounded. Second derivatives can be computed and bounded similarly.

Next, we find
\[ \partial_{\rho_k} G^{ij}(t, p) = e^{\phi} \left( e^{2\phi} + |p|^2 \right)^{-5/4} \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right)^{-2} \right[ -\frac{1}{2} p_k p^i p^j e^{\phi} - \frac{3}{2} p_k p^i p^j \sqrt{e^{2\phi} + |p|^2} \]
\[ + (e^{2\phi} + |p|^2) \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right) (\delta_k p^j + \delta_k p^i) - \frac{1}{2} e^{\phi} \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right)^2 p_k \delta_{ij} \]
and thus for every \( i, j, k \)
\[ |\partial_{\rho_k} G^{ij}(t, p)| \leq C e^{\phi} \left( e^{2\phi} + |p|^2 \right)^{-5/4} \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right)^{-2} \]
\[ \left[ |p|^3 \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right) + |p| \left( e^{2\phi} + |p|^2 \right) \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right) + e^{\phi} |p| \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right)^2 \right] \]
\[ \leq C |p| e^{\phi} \left( e^{2\phi} + |p|^2 \right)^{-5/4} \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right)^{-2} \]
\[ \left[ \left( e^{2\phi} + |p|^2 \right) \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right) + e^{\phi} \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right)^2 \right] \]
\[ \leq C |p| e^{\phi} \left( e^{2\phi} + |p|^2 \right)^{-1/4} \left( e^{\phi} + \sqrt{e^{2\phi} + |p|^2} \right)^{-1} + C |p| e^{2\phi} \left( e^{2\phi} + |p|^2 \right)^{-5/4} \]
\[ = I + II. \]
The first term satisfies
\[ I \leq C e^{\tilde{\phi}} \left( e^{2\tilde{\phi}} + |p|^2 \right)^{-1/4} \leq C e^{\tilde{\phi}/2}. \]

Alternatively, the second term can be estimated for \(|p| \leq e^{\tilde{\phi}}\) as
\[ II \leq C e^{\tilde{\phi}} e^{2\tilde{\phi}} e^{-5\tilde{\phi}/2} \leq C e^{\tilde{\phi}/2} \]
and for \(|p| \geq e^{\tilde{\phi}}\) as
\[ II \leq C |p| e^{2\tilde{\phi}} \left( |p| e^{\tilde{\phi}} \right)^{-5/4} \leq C |p|^{-1/4} e^{3\tilde{\phi}/4} \leq C e^{\tilde{\phi}/2}. \]

Combining the estimates, we use (10) to conclude
\[ |\partial_{p_k} G^{ij}(t, p)| \leq C e^{\tilde{\phi}/2} \leq C_T \]
for all \(i, j, k\). Hence, derivatives are uniformly bounded in the time interval \([-T, 0]\). As before, second derivatives can be bounded using similar estimates.

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