A Randomized Nonlinear Rescaling Method in Large-Scale Constrained Convex Optimization

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Abstract

We propose a new randomized algorithm for solving convex optimization problems that have a large number of constraints (with high probability). Existing methods like interior-point or Newton-type algorithms are hard to apply to such problems because they have expensive computation and storage requirements for Hessians and matrix inversions. Our algorithm is based on nonlinear rescaling (NLR), which is a primal-dual-type algorithm by Griva and Polyak [Math. Program., 106(2):237-259, 2006]. NLR introduces an equivalent problem through a transformation of the constraint functions, minimizes the corresponding augmented Lagrangian for given dual variables, and then uses this minimizer to update the dual variables for the next iteration. The primal update at each iteration is the solution of an unconstrained finite sum minimization problem where the terms are weighted by the current dual variables. We use randomized first-order algorithms to do these primal updates, for which they are especially well suited. In particular, we use the scaled dual variables as the sampling distribution for each primal update, and we show that this distribution is the optimal one among all probability distributions. We conclude by demonstrating the favorable numerical performance of our algorithm.

1 Introduction

We consider the following constrained convex optimization problem:

\[ \mathbb{P} : \min_{x \in \mathbb{R}^n} \{ f(x) : g_i(x) \geq 0, \forall i \in \{1, 2, \ldots, m\} \}, \]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is convex, and \( g_i : \mathbb{R}^n \to \mathbb{R} \) are concave for all \( i = 1, 2, \ldots, m \). We are motivated by the large-scale case where \( m \gg 0 \).

There are many algorithms for solving constrained optimization problems. Our method is based on nonlinear rescaling (NLR) \[10, 38, 39\], which is a primal-dual-type algorithm with a \( Q \)-linear convergence rate \[10, 39\] (that can sometimes be improved to a \( Q \)-superlinear convergence rate \[10\]). In each iteration of this method, an unconstrained augmented Lagrangian is minimized for a given set of dual variables (the “primal update”). Then, the dual variables are updated based on the minimizer of this augmented Lagrangian (the “dual update”).

We propose a new computational scheme where we use randomized first-order algorithms to do the primal updates. In particular, when \( m \gg 0 \) is large, then the primal update amounts to solving a finite sum minimization problem with a large number of terms. Randomized first-order algorithms are specifically designed for this type of problems \[9, 10, 23, 43, 44, 48\]. Consequently, it is natural to combine these two ideas (NLR and randomized first-order algorithms).
1.1 Motivation

We motivate our study with some examples of large-scale constrained optimization problems. In radiation therapy treatment, beams of radiation are used to kill cancerous cells (known as intensity modulated radiotherapy treatment (IMRT) [2]). A fundamental predicament of IMRT is that it not only affects cancerous cells, but also neighboring healthy cells. Thus, in a treatment plan, the beams should target any tumors while limiting radiation exposure to healthy tissue. In the case of IMRT, the corresponding nonlinear optimization problem maximizes radiation to the tumors under constraints that limit the exposure to healthy cells. This problem consists of thousands of decision variables, such as beam angles, radiation intensity, as well as tens of thousands of constraints, which restrict the negative effects of radiation therapy on healthy tissue.

Sequential decision-making under uncertainty, and in particular, Markov decision processes (MDP) [11], are generally intractable due to the curse of dimensionality. Approximate dynamic programming (ADP) provides tractable techniques for getting good policies in MDPs, often through approximate linear programming (ALP) [7, 8, 29, 45]. ALP problems have a manageable number of variables, which are the weights for a given set of basis functions. In addition, there is a constraint for every state-action pair. Thus, the number of decision variables is relatively small but the number of constraints is huge. ALP has been successfully applied to inventory control [25], health care [42], revenue management [1], and queuing networks [7, 8].

Many risk-aware optimization problems [18, 33, 34], and in particular, many risk-constrained optimization models [11, 12, 17, 21, 22], are essentially semi-infinite programming (SIP) problems. SIP problems may be approximated by relaxed problems with finitely many constraints. However, the number of constraints must be very large in order to produce a good approximation. This phenomenon holds in general for finite relaxations of SIP problems.

1.2 Related Works

We summarize the literature on several methods for solving constrained optimization problems. We group these methods under the broad headings of deterministic and randomized methods.

Deterministic methods: First-order algorithms may be used for constrained optimization. A subgradient method for computing the saddle-points of a convex-concave function is proposed in [32], where near-optimal primal-dual solutions are obtained. The convergence rate of this method is $O\left(1/\sqrt{K}\right)$ for both the optimality gap and constraint violation, where $K$ is the number of iterations. A faster primal-dual algorithm based on the drift-plus-penalty method is proposed for general convex constrained optimization problems in [51] which has an $O\left(1/K\right)$ convergence rate. In addition, the dual subgradient method that averages the corresponding sequence of primal iterates can be applied to solve general convex constrained optimization problems with a sublinear convergence rate (see [30, 31, 52]).

Barrier methods use barrier functions (see [15] for the logarithmic barrier function and [6] for the inverse barrier function) to find a solution to convex inequality constrained problems by solving a sequence of unconstrained problems. As the barrier parameter goes to infinity in the unconstrained problem, the solution becomes a better approximation of the desired solution. However, the Hessian of the barrier function tends to infinity and becomes ill-conditioned, which makes the unconstrained minimization problem more difficult to solve.

The augmented Lagrangian method has been proposed to solve equality constrained problems (see [19, 40]). For inequality constrained problems, Polyak introduced the modified barrier method [36]. Similar to the augmented Lagrangian method, the modified barrier method explicitly uses dual variables to avoid ill-conditioning. The modified barrier method iteratively minimizes a modified barrier function with respect to its primal variables, and then it updates the dual variables. However, the modified barrier function is not defined for all real numbers and so its implementation can lead to numerical difficulties.
Later, NLR is proposed by Polyak and Teboulle as a generalization of the modified barrier method [37]. NLR transforms the set of constraint functions into an equivalent set by using a class of smoothing functions. NLR can solve inequality constrained problems by minimizing the Lagrangian of the equivalent problem, and then explicitly updating the dual variables [39]. NLR does not lead to an unbounded increase of the scaling parameter, and thus it avoids the ill-conditioning of the Hessian. Moreover, under the standard second order optimality conditions, NLR converges with a Q-linear rate for any fixed (but sufficiently large) scaling parameter [38].

The success of the primal-dual method for linear programming (see [26, 27, 53]) has stimulated interest in primal-dual methods for nonlinear programming (see [14, 46]). The best known primal-dual method is based on the path-following paradigm [46]. Another primal-dual method is based on NLR [16, 35]. This method can achieve a 1.5-Q-superlinear rate by increasing the scaling parameter in a carefully chosen way [16]. However, both approaches have expensive computation and storage requirements for Hessians and matrix inversions, which make them unsuitable for large-scale constrained optimization problems.

Very recently, inexact versions of the classical augmented Lagrangian method are developed to solve constrained optimization problems. In [24], a special class of convex optimization problems whose feasible regions consist of a simple compact convex set intersected with an affine manifold is considered. First-order methods are presented based on the inexact augmented Lagrangian method, where the subproblems are approximately solved by Nesterov's optimal method. General convex problems with both equality and inequality constraints are solved in [50] also using an inexact augmented Lagrangian method. Like [24], the primal subproblems in [50] are solved using Nesterov's optimal method.

**Randomized methods:** Randomized cutting plane algorithms have recently been developed for constrained optimization in [3, 4, 5, 29, 13]. The idea is to input a probability distribution over the constraints, randomly sample a modest number of constraints, and then solve the resulting relaxed problem. Intuitively, as long as a sufficiently large number of samples is drawn, the resulting randomized solution should violate only a small portion of the constraints and be nearly optimal.

In [25], a convex saddle-point reformulation is proposed to solve ALP problems. A proximal stochastic mirror descent method (PSMD) is developed which learns about regions of constraint violation via its dual update. PSMD returns a near-optimal solution and a lower bound on the cost of the optimal policy in a finite number of iterations with high probability. In [17], a first-order primal-dual algorithm based on Monte Carlo integration over the constraint index set is proposed to solve general convex SIP. Since the dual variables here are nonnegative measures on the constraint index set, a new prox function for nonnegative measures is needed which turns out to be a generalization of the classical Kullback-Leibler divergence.

In [20], an inexact primal-dual smoothing framework is developed for large-scale non-bilinear saddle point problems, in which randomized algorithms are used to solve the primal and dual subproblems. As an important application, this framework is applied to solve convex optimization problems with many constraints. In [49], a primal-dual stochastic gradient method is developed for problems with a stochastic objective and many functional constraints.

### 1.3 Main Contributions

We highlight the three main contributions of our present work as follows:

1. We do the primal updates for NLR in a new way using randomized first-order algorithms for unconstrained minimization of the augmented Lagrangian. We call our new algorithm 'Randomized Nonlinear Rescaling' (RanNLR). RanNLR supports any randomized first-order algorithm as its subroutine, e.g. SGD [43], and variance reduction methods (e.g. SVRG [23, 48], SAGA [9], SAG [44], Finito [10]).

2. We do adaptive random constraint sampling for the primal updates by constructing a probability distribution over the constraints based on the current dual variables (i.e., this distribution changes...
as the dual variables are updated). We show that random sampling from this distribution is optimal compared to any other possible sampling distribution.

3. We analyze the complexity of RanNLR required to obtain a solution within distance \( \varepsilon \) of the optimal solution of Problem \( \mathbb{P} \) when the objective \( f \) is strongly convex, with probability at least \( 1 - \delta \). If the primal update subroutine has a sublinear rate (e.g. SGD), then the overall complexity of our algorithm is \( \tilde{O}(1/(\varepsilon^2\delta)) \) (where \( \tilde{O}(\cdot) \) hides the \( \ln(1/\varepsilon) \) and \( \ln(1/\delta) \) factors). If the primal update subroutine has a linear rate (e.g. SVRG), then the overall complexity of our algorithm is \( O((\ln(1/\varepsilon))(2\ln(1/\varepsilon) + \ln(1/\delta))) \).

This paper is organized as follows. In Section 2 we review classical NLR. We then present the details of our new randomized NLR algorithm in Section 3. Our main results including the complexity analysis of RanNLR may be found in Section 4. Then, we present numerical experiments in Section 5 and conclude the paper in Section 6. Supporting technical results are gathered together in the Appendix.

**Notation.** Let \( \mathbb{N} \) be the set of natural numbers. For a positive integer \( n \in \mathbb{N} \), let \( [n] \triangleq \{1, 2, \ldots, n\} \) and \( [n]_0 \triangleq \{0, 1, \ldots, n-1\} \). For a real number \( x \), let \([x]\) be the ceiling of \( x \), i.e., the smallest integer greater than or equal to \( x \).

Let \( \mathbb{R}^n \) be \( n \)-dimensional Euclidean space. Let \( \mathbb{R}^n_+ \) and \( \mathbb{R}^n_{++} \) be the subsets of vectors in \( \mathbb{R}^n \) with nonnegative and strictly positive components, respectively. For a vector \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), define \( \|x\|_1 \triangleq \sum_{i=1}^n |x_i| \), \( \|x\|_2 \triangleq \sqrt{\sum_{i=1}^n x_i^2} \), and \( \|x\|_\infty \triangleq \max_{1 \leq i \leq n} |x_i| \). For vectors \( x, y \in \mathbb{R}^n \), \( \langle x, y \rangle \) denotes the Euclidean inner product.

Let \( \mathbb{R}^{n_1 \times n_2} \) be the set of matrices with dimension \( n_1 \times n_2 \). Let \( I_n \in \mathbb{R}^{n \times n} \) be the identity matrix. For a matrix \( A = (a_{ij})_{i \in [n_1], j \in [n_2]} \in \mathbb{R}^{n_1 \times n_2} \), let \( A^T \in \mathbb{R}^{n_2 \times n_1} \) be its transpose, and define the matrix norm \( \|A\|_2 \triangleq \|A\|_\infty \triangleq \max_{1 \leq i \leq n_1} \sum_{j=1}^{n_2} |a_{ij}| \) which is the maximum absolute row sum. For a vector \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \), define

\[
\text{diag}(a) \triangleq \begin{pmatrix}
a_1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & a_n
\end{pmatrix} \in \mathbb{R}^{n \times n}.
\]

For \( \eta > 0 \) and \( x \in \mathbb{R}^n \), let \( B_\eta(x) \triangleq \{z \in \mathbb{R}^n : \|z - x\|_\infty \leq \eta\} \) denote the Euclidean ball in \( \mathbb{R}^n \) with radius \( \eta \) in the \( \|\cdot\|_\infty \)-norm centered at \( x \). For a set \( C \subset \mathbb{R}^n \), let \( I_C \) denote the indicator function of the set \( C \). The projection operator \( \Pi_{\mathcal{X}} : \mathbb{R}^n \to \mathbb{R}^n \) (for a closed convex set \( \mathcal{X} \subset \mathbb{R}^n \)) is given by \( \Pi_{\mathcal{X}}[x] \triangleq \text{arg min}_{y \in \mathcal{X}} \|x-y\|_2 \), which always exists and is unique.

## 2 Nonlinear Rescaling (NLR)

We begin by reviewing the details of classical NLR, on which our present method is based. Classical NLR employs a nonlinear rescaling function scaled by \( N > 0 \) (hereafter called the "scaling parameter") to transform each constraint function. We keep \( N \) constant throughout the course of our algorithm, and we will see that the overall convergence rate depends on \( N \) \cite{16,38,39}. We detail the effect of \( N \) on the algorithm complexity and offer selection guidelines later, for now we just treat \( N \) as a constant.

Rescaling each constraint function gives a new problem that is equivalent to Problem \( \mathbb{P} \). Furthermore, the Lagrangian for this new problem can be viewed as an augmented Lagrangian for Problem \( \mathbb{P} \). In each iteration of NLR, we minimize this augmented Lagrangian for given dual variables, obtain a solution, and then use this solution to update the dual variables. The specific properties of the nonlinear rescaling function substantially affect both the global and local behavior of the overall algorithm (this phenomenon is characterized in Lemma \[4.15\]).

**Definition 2.1.** \cite{16} A *nonlinear rescaling function* \( \psi : \mathbb{R} \to \mathbb{R} \) is a twice continuously differentiable function such that: (i) \( \psi(0) = 0 \); (ii) \( \psi'(t) > 0 \) for all \( t \in \mathbb{R} \) and \( \psi'(0) = 1 \); (iii) \( \psi''(t) < 0 \) for all \( t \in \mathbb{R} \); (iv) \( \psi(t) \leq -at^2 \).
pseudo-code of RanNLR is summarized in Algorithm 1. We continue to let NLR by allowing the primal update to be done inexactly using a randomized first-order algorithm. The primal update in Eq. (2.1) is usually the bottleneck in NLR, especially when Eq. (2.2) can be expressed more compactly in vector notation as

\[ \Psi(t) = \begin{cases} \zeta_i(t), & t \geq \tau, \\ 0.5\zeta''_i(\tau)t^2 + (\zeta'_i(\tau) - \tau\zeta''_i(\tau))t + \zeta_i(\tau) - \tau\zeta'_i(\tau) + \tau^2\zeta''_i(\tau), & t \leq \tau, \end{cases} \]

for \( i = 1, 2, 3 \). We can directly verify that \( \psi_i \in \Psi \) for all \( i = 1, 2, 3 \).

For any \( \psi \in \Psi \) and scaling parameter \( N > 0 \), we define the smoothed optimization problem:

\[ \mathbb{P}_N : \min_{x \in \mathbb{R}^n} \{ f(x) : N^{-1}\psi(Ng_i(x)) \geq 0, \forall i \in [m] \}. \]

Problem \( \mathbb{P}_N \) is a convex optimization problem due to properties (i), (ii), and (iii) of Definition 2.1. Additionally, Problem \( \mathbb{P}_N \) is equivalent to Problem \( \mathbb{P} \) in the sense that both share the same feasible region, optimal solutions, and optimal values.

For Problem \( \mathbb{P}_N \), let \( \lambda \in \mathbb{R}_++^m \) be the dual variables corresponding to all \( m \) inequality constraints, and let \( \mathcal{L}_N(x, \lambda) \equiv f(x) - N^{-1}\sum_{i \in [m]} \lambda_i\psi(Ng_i(x)) \) be the Lagrangian (which is also an augmented Lagrangian for Problem \( \mathbb{P} \)). For fixed \( \lambda \in \mathbb{R}_++^m \), the “primal update” is to minimize \( x \mapsto \mathcal{L}_N(x, \lambda) \) which we denote as:

\[ \mathbb{P}_N(\lambda) : \min_{x \in \mathbb{R}^n} \mathcal{L}_N(x, \lambda). \]

Classical NLR proceeds as follows. We let \( k \geq 0 \) count iterations, \( \lambda^k = (\lambda^k_i)_{i \in [m]} \in \mathbb{R}_++^m \) denote the dual variables in iteration \( k \), and \( x^k \) denote the primal variables in iteration \( k \). In iteration \( k \), starting with \( \lambda^k \), we compute \( x^{k+1}_* (\lambda^k) \) by solving Problem \( \mathbb{P}_N(\lambda^k) \) exactly:

\[ x^{k+1}_* (\lambda^k) \in \arg \min_{x \in \mathbb{R}^n} \mathcal{L}_N(x, \lambda^k). \]  

Eq. (2.1) is the primal update of NLR, see [16, Eqs. (3.4)-(3.7)]. Next, we do the dual update:

\[ \lambda^{k+1}_i = \lambda^k_i \psi(Ng_i(x^{k+1}_* (\lambda^k))), \quad \forall i \in [m]. \]

Eq. (2.2) can be expressed more compactly in vector notation as \( \lambda^{k+1} = \lambda^k \psi(NG(x^{k+1})) \).

### 3 Randomized Nonlinear Rescaling (RanNLR)

The primal update in Eq. (2.1) is usually the bottleneck in NLR, especially when \( m \gg 0 \). RanNLR builds on NLR by allowing the primal update to be done inexactly using a randomized first-order algorithm. The pseudo-code of RanNLR is summarized in Algorithm 1. We continue to let \( k \) index the outer iterations of RanNLR, the same as for NLR.

We define, for all \( \lambda \in \mathbb{R}_++^m \), the terms:

\[ f_i^N(x; \lambda) \equiv f(x) - \|\lambda\|_1 N^{-1}\psi(Ng_i(x)), \quad \forall i \in [m]. \]
Algorithm 1 Randomized Nonlinear Rescaling $(N, x^0, \lambda^0, K, \epsilon, \delta)$

**Input:** Total number of iterations $K \geq 1$, scaling parameter $N > 0$, error tolerance $\epsilon > 0$, overall failure probability $\delta \in (0, 1)$.

**Initialize:** $x^0 \in \mathbb{R}^n$, $\lambda^0 \in \mathbb{R}^{m+}$. For $k = 0, 1, \ldots, K - 1$: 

- Use subroutine $A$ to compute $x^{k+1} \in \mathbb{R}^n$ such that $\|\nabla_{x} L_N(x^{k+1}, \lambda^k)\|_{\infty} \leq \epsilon$ with probability at least $(1 - \delta)^{1/K}$;

- Update $\lambda^{k+1}$ by Eq. (3.3).

**End**

Return: $x^K$.

Then, we may write Problem $P_N(\lambda)$ as an explicit finite sum minimization problem:

$$P_N(\lambda) \equiv \min_{x \in \mathbb{R}^n} \left\{ L_N(x, \lambda) \equiv \sum_{i \in [m]} \frac{\lambda_i}{\|\lambda\|_1} f_i^N(x; \lambda) \right\}. \tag{3.1}$$

The purpose of this reformulation is twofold. First, it absorbs the original objective function $f$ into the $m$ functions $\{f_i^N\}_{i \in [m]}$. Second, it shows how the dual variables determine an explicit probability distribution over the constraint index set $[m]$. Whenever we solve an instance of Problem $P_N(\lambda)$, we solve Problem (3.1) specifically.

### 3.1 The Subroutine $A$

We now let $A$ denote a general randomized first-order subroutine for solving Problem (3.1). Some specific examples of $A$ include: SGD [43], SVRG [23, 48], SAGA [9], SAG [44], and Finito [10]. These latter four variance reduction algorithms combine the advantages of full gradient descent and SGD to achieve a linear convergence rate in expectation while maintaining the low per-iteration cost of SGD.

The dual variables $\lambda \in \mathbb{R}^{m+}$ are fixed in each instance of Problem $P_N(\lambda)$. They enter into the sampling distribution of $A$ and weights of the finite sum minimization Problem (3.1). Define $P_+([m])$ to be the set of all probability distributions on $[m]$ with all positive components. The subroutine $A$ relies on a sampling distribution from $P_+([m])$. For easy reference, we denote this sampling distribution as $\varphi(\lambda) \triangleq (\lambda_i/\|\lambda\|_1)_{i \in [m]} \in P_+([m])$, and its components as $\varphi_i(\lambda) \triangleq \lambda_i/\|\lambda\|_1$ for all $i \in [m]$.

For each outer iteration $k$, $\lambda^k$ is fixed and we want to solve the corresponding Problem $P_N(\lambda^k)$. We let $t \geq 0$ index the inner iterations of $A$ applied to solve $P_N(\lambda^k)$. We also let $\{I_t\}_{t \geq 0}$ be a sequence of i.i.d. random variables drawn from $[m]$ according to $\varphi(\lambda^k)$. This sampling distribution is adaptive, it changes as $\lambda^k$ varies. We further explain our adaptive sampling scheme and its advantages in Subsection 4.5.

Our analysis requires $\varphi(\lambda^k) \in P_+([m])$ to hold for all $k \geq 0$. In fact, in our scheme, when we initialize with $\lambda^0 \in \mathbb{R}^{m+}$, then every subsequent dual iterate will remain in $\mathbb{R}^{m+}$.

In our implementation, we take $A$ to be SGD and SVRG. SGD has a sublinear convergence rate (in expectation), but its complexity does not depend on $m$. SVRG offers a linear convergence rate (in expectation), but its complexity does depend on $m$ since it does a full gradient update at the beginning of each epoch.
3.2 Relaxed Stopping Condition

In classic NLR, the primal update requires each instance of Problem $\mathbb{P}_N(\lambda^k)$ to be solved exactly. Since we are using a randomized subroutine, we will instead solve each instance of Problem $\mathbb{P}_N(\lambda^k)$ inexactly (with high probability). Specifically, we use the following relaxed stopping condition for the primal update:

$$\text{Find } x^{k+1} \in \mathbb{R}^n \text{ s.t. } \| \nabla_x \mathcal{L}_N(x, \lambda^k) \|_{\infty} \leq \epsilon,$$  \hfill (3.2)

for some small $\epsilon > 0$. Any $x^{k+1}$ satisfying Eq. (3.2) is a near-optimal solution of Problem $\mathbb{P}_N(\lambda^k)$. Once we have an $x^{k+1}$ satisfying Eq. (3.2), we update the dual variables in the usual way via:

$$\lambda^{k+1} = \lambda^k \psi^i(N G(x^{k+1})),$$  \hfill (3.3)

which is still deterministic.

**Remark 3.1.** Eq. (3.2) is different from the inexact stopping conditions in [38, Eq. (7.1)] and [16, Eq. (3.9)], which require $x^{k+1}$ to satisfy:

$$\| \nabla_x \mathcal{L}_N(x^{k+1}, \lambda^k) \|_{\infty} \leq a N^{-1} \| \lambda^k \psi^i(N G(x^{k+1})) - \lambda^k \|_{\infty},$$

for large enough $N > 0$ and some $a > 0$. Eq. (3.2) is more convenient for us because we can explicitly determine the number of iterations of $A$ required to meet this condition.

4 Main Results

We give the convergence analysis for RanNLR in this section. First, we gather all of the technical assumptions on $\mathbb{P}$ and $A$ in the following two subsections for easy reference. Then, we give our main results and proof, followed by a justification of our adaptive sampling scheme.

4.1 Assumptions on Optimization Problem

We begin with basic assumptions on Problem $\mathbb{P}$ itself.

**Assumption 4.1.** (i) (Solvability) There exists an optimal solution $x^*$ of Problem $\mathbb{P}$.

(ii) (Slater condition) There exists a Slater point $\tilde{x} \in \mathbb{R}^n$ such that $\kappa \triangleq \min_{i \in [m]} g_i(\tilde{x}) > 0$ (i.e., $g_i(\tilde{x}) \geq \kappa > 0$ for all $i \in [m]$).

Next, we make the following assumptions on the ingredients of Problem $\mathbb{P}$.

**Assumption 4.2.** (i) The objective function $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex with parameter $\mu_f > 0$ with respect to $\| \cdot \|_2$, i.e., for any $x_1, x_2 \in \mathbb{R}^n$, $f(x_1) \geq f(x_2) + \langle \nabla f(x_2), x_1 - x_2 \rangle + \mu_f \| x_1 - x_2 \|_2^2 / 2$. The objective function $f$ is Lipschitz continuous with parameter $L_f^{(0)} \geq 0$ with respect to $\| \cdot \|_{\infty}$, i.e., for any $x_1, x_2 \in \mathbb{R}^n$, $|f(x_1) - f(x_2)| \leq L_f^{(0)} \| x_1 - x_2 \|_{\infty}$. The Hessian $\nabla^2 f(\cdot)$ is Lipschitz continuous with parameter $L_f^{(2)} \geq 0$ with respect to $\| \cdot \|_{\infty}$, i.e., for any $x_1, x_2 \in \mathbb{R}^n$, $\| \nabla^2 f(x_1) - \nabla^2 f(x_2) \| \leq L_f^{(2)} \| x_1 - x_2 \|_{\infty}$. (ii) For all $i \in [m]$, $g_i : \mathbb{R}^n \to \mathbb{R}$ is concave and is Lipschitz continuous with parameter $L_{g_i}^{(0)} \geq 0$ with respect to $\| \cdot \|_{\infty}$ uniformly in $i \in [m]$, i.e., for any $x_1, x_2 \in \mathbb{R}^n$, $|g_i(x_1) - g_i(x_2)| \leq L_{g_i}^{(0)} \| x_1 - x_2 \|_{\infty}$ for all $i \in [m]$. For all $i \in [m]$, the Hessian $\nabla^2 g_i(\cdot)$ is Lipschitz continuous with parameter $L_{g_i}^{(2)} \geq 0$ with respect to $\| \cdot \|_{\infty}$ uniformly in $i \in [m]$, i.e., for any $x_1, x_2 \in \mathbb{R}^n$, $\| \nabla^2 g_i(x_1) - \nabla^2 g_i(x_2) \| \leq L_{g_i}^{(2)} \| x_1 - x_2 \|_{\infty}$ for all $i \in [m].
Similarly, Problem \( P \) always has a unique solution for any \( \lambda \in \mathbb{R}_{++}^m \) because its objective function \( L_N(x, \lambda) \) is strongly convex in \( x \) as well (also due to strong convexity of \( f \)).

The original Lagrangian for Problem \( P \) is defined by \( L(x, \lambda) \triangleq f(x) - \sum_{i \in [m]} \lambda_i g_i(x) \). The KKT conditions for an optimal solution \( x^* \) imply that there exists a nonnegative vector \( \lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \) such that
\[
\nabla_x L(x^*, \lambda^*) = 0, \quad \text{and} \quad \lambda^*_i g_i(x^*) = 0 \quad \text{for all} \quad i \in [m].
\]
Define the dual function \( d(\lambda) \triangleq \inf_{x \in \mathbb{R}^n} L(x, \lambda) \). Under Assumption 4.1(ii), for any \( \lambda \in \mathbb{R}_{++}^m \) we have \( \|\lambda^*\|_{\infty} \leq (f(\bar{x}) - d(\bar{\lambda}))/\kappa \) (see [31, Lemma 1]).

Under Assumption 4.1(ii), the KKT conditions have a solution. To characterize the KKT conditions, let \( I^* \triangleq \{ i \in [m] : g_i(x^*) = 0 \} \) denote the set of active constraints at \( x^* \). For brevity in notation, we define the vector-valued functions \( G : \mathbb{R}^n \rightarrow \mathbb{R}^m \) via \( G(x) \triangleq (g_i(x))_{i \in [m]} \), \( G_I : \mathbb{R}^n \rightarrow [I^*] \) via \( G_I(x) \triangleq (g_i(x))_{i \in I^*} \), and the Jacobian \( \nabla G_I : \mathbb{R}^n \rightarrow [I^*] \times n \) via \( \nabla G_I(x) \triangleq (\nabla g_i(x))_{i \in I^*} \).

We assume the following regularity condition holds at the optimal solution \( x^* \).

**Assumption 4.4.** We have \( \text{rank}(\nabla G_I(x^*)) = |I^*| \) and \( \lambda_i^* > 0 \) for all \( i \in I^* \), where \( |I^*| \) is the cardinality of \( I^* \).

**Remark 4.5.** In addition to Assumption 4.4, the standard second-order optimality sufficient conditions (see [16, 38, 39]) require that, for all \( y \neq 0 \) satisfying \( \nabla G_I(x^*)y = 0 \) (i.e., those vectors in the nullspace of \( \nabla G_I(x^*) \)), there exists a constant \( \rho > 0 \) such that
\[
\langle \nabla^2_{xx} L(x^*, \lambda^*) y, y \rangle \geq \rho \langle y, y \rangle.
\]
The condition in (4.1) holds with \( \rho = \mu_f \) by strong convexity.

Finally, we split the dual optimal vector \( \lambda^* \) into the active \( \lambda^*_A \) and inactive \( \lambda^*_I = (\lambda_i^*)_{i \in [m] \setminus I^*} = 0^{[m] \setminus I^*} \) parts. Define \( \Phi_N \triangleq \min \{ g_i(x^*) : i \in [m] \setminus I^* \} > 0, \) \( A_i^* \triangleq \text{diag}(\lambda_i^*) \), and
\[
\Phi_N(x^*, \lambda^*) \triangleq \begin{bmatrix}
\nabla_{xx} L(x^*, \lambda^*), & -\nabla G_I(x^*)^T \\
-\langle A_i^*, \nabla G_I(x^*) \rangle, & \psi'(0) N^{-1} I_{|I^*|}
\end{bmatrix}.
\]
From [38, p. 186] or [39, p. 442], we note that the inverse matrix \( \Phi_N(x^*, \lambda^*)^{-1} \) exists, and there is a (large enough) number \( N_0 > 0 \) and a number \( C_{\phi} > 0 \) such that
\[
\|\Phi_N(x^*, \lambda^*)^{-1}\| \leq C_{\phi}, \quad \forall N \geq N_0.
\]
From property (v) of the definition of \( \Psi \), there is a constant \( C_{\psi} > 0 \) such that
\[
\sum_{i \in [m] \setminus I^*} 4\psi'(N\sigma/2)\|\nabla g_i(x^*)\|_{\infty} \leq C_{\psi} N^{-1}.
\]
We also define a constant \( c_R \triangleq \max \{ 2d_1 \sigma^{-1}, 2(C_{\psi} - 2\psi''(0)) C_{\phi} \} \) for later use.

We pause to note that \( \sigma \) and \( C_{\phi} \) are unknown constants which are intrinsic to the NLR method. These unknown constants also appear generally in the NLR literature [16, 38]. In acknowledgement of these unknown constants, we emphasize that our upcoming main result gives the theoretical order of convergence of RanNLR.
4.2 Assumptions on the Subroutine \( \mathcal{A} \)

We need the following assumption for the convergence analysis of the subroutine \( \mathcal{A} \).

**Assumption 4.6.** (i) There exists a compact set \( \mathcal{X} \subseteq \mathbb{R}^n \) such that all primal iterates of inexact NLR lie within \( \mathcal{X} \).

(ii) For all \( \lambda \in \mathbb{R}^m_+ \) and \( i \in [m] \), the gradient \( \nabla_x f_i^N(\cdot;\lambda) \) is Lipschitz continuous with parameter \( L_N(\lambda) \) with respect to \( \| \cdot \|_2 \) over \( \mathcal{X} \).

The existence of such a compact set \( \mathcal{X} \subseteq \mathbb{R}^n \) will be confirmed by Lemma 4.16. Specifically, when the feasible region \( \{ x \in \mathbb{R}^n : g_i(x) \geq 0, \forall i \in [m] \} \) lies within a centered ball with radius \( \zeta \), as long as the scaling parameter \( N > 0 \) is sufficiently large and the error tolerance \( \epsilon > 0 \) in our inexact stopping condition Eq. (3.2) is small enough, then all primal iterates of inexact NLR will lie within the centered ball with radius \( 2\zeta \).

From the smoothness of objective and constraint functions in Assumption 4.2 and the continuity of \( \psi' \) and \( \psi'' \), we directly have

\[
\nabla_{xx} f_i^N(x, \lambda) = \nabla^2 f(x) - \| \lambda \|_1 (\psi''(Ng_i(x))N\nabla g_i(x)\nabla^T g_i(x) + \psi'(Ng_i(x))\nabla^2 g_i(x)), \forall i \in [m].
\]

Since \( \mathcal{X} \) is a compact set, we may take \( L_N(\lambda) = \sup_{x \in \mathcal{X}} \| \nabla_{xx} f_i^N(x, \lambda) \|_2 \).

Based on Assumption 4.6(i), we restrict the iterates to \( \mathcal{X} \) when we implement \( \mathcal{A} \) to solve Problem (3.1). In each iteration, the goal is for \( \mathcal{A} \) to return a nearly optimal solution to \( \mathbb{P}_N(\lambda) \) (based on our relaxed stopping condition) with a high probability. By the chain rule for the probability of the intersection of events, the primal updates in all iterations of RanNLR will then be nearly optimal with a high probability.

**Remark 4.7.** If we include nonnegativity constraints \( x_i \geq 0 \) for all \( i \in [n] \), then the augmented Lagrangian \( \mathcal{L}(x, \lambda) \) is strongly convex on any bounded set in \( \mathbb{R}^n \) (see [39] Lemma 2). In this case, we would not need strong convexity of \( f \) for our analysis to go through.

In the next assumption, we formalize the two possible cases for the convergence rate of \( \mathcal{A} \) for the distance to the optimal solution: the sublinear case (e.g. SGD) and the linear case (e.g. SVRG).

**Assumption 4.8.** Suppose \( F : \mathbb{R}^n \to \mathbb{R} \) is \( \mu \)-strongly convex with respect to \( \| \cdot \|_2 \), and the gradient \( \nabla F(\cdot) \) is \( L \)-Lipschitz continuous with respect to \( \| \cdot \|_2 \) over \( \mathcal{X} \). Let \( \{ y_t \}_{t \geq 0} \) be the iterates of \( \mathcal{A} \) computed to apply to \( \arg \min_{x \in \mathcal{X}} F(x) \), where \( y^* \) is the unique minimum of \( F \).

(i) *(Sublinear rate)* \( \mathbb{E} \| y_t - y^* \|_2^2 \leq O(1/t) \) for all \( t \geq 1 \). In particular, there exist \( A = A(\mu, L) > 0 \) and \( B = B(\mu, L) > 0 \) such that \( \mathbb{E} \| y_t - y^* \|_2^2 \leq \left( A \| y_0 - y^* \|_2^2 + B \right) / t \) for all \( t \geq 1 \).

(ii) *(Linear rate)* For some \( \alpha \in (0, 1) \) and all \( t \geq 1 \), \( \mathbb{E} \| y_t - y^* \|_2^2 \leq O(\alpha^t) \). In particular, there exist \( \zeta = \zeta(\mu, L) > 0 \) and \( \alpha = \alpha(\mu, L) \in (0, 1) \) such that \( \mathbb{E} \| y_t - y^* \|_2^2 \leq \zeta \alpha^t \| y_0 - y^* \|_2^2 \) for all \( t \geq 1 \).

To solve Problem \( \mathbb{P}_N(\lambda) \), we implement \( \mathcal{A} \) by sampling from \( \psi(\lambda) \in \mathbb{P}_+(\mathbb{R}^m) \) (the probability distribution scaled from the current dual variable). Let \( \mathbb{E}^{\psi(\lambda)}[\cdot] \) denote conditional expectation with respect to \( \psi(\lambda) \). We denote \( \mathcal{A}_{SGD} \) and \( \mathcal{A}_{SVRG} \) as SGD and SVRG with sampling distribution \( \psi(\lambda) \), respectively.

**Example 4.9.** *(SGD, see [39])* The iterates follow \( y_{t+1} = \Pi_{\mathcal{X}}[y_t - \gamma(\nabla f_i^N(y_t, \lambda)) + \xi_t] \), for all \( t \geq 0 \). Let \( M_B \triangleq \max_{i \in [m]} \| \nabla f_i^N(x^*(\lambda), \lambda) \|_2 \). By Remark B.5(i) and Theorem B.6 \( \mathcal{A}_{SGD} \) applied to \( \mathbb{P}_N(\lambda) \) has a sublinear convergence rate:

\[
\mathbb{E}^{\psi(\lambda)} \| y_t - x^*(\lambda) \|_2^2 \leq \frac{(6L_N(\lambda)/\mu_2^2 - 1)\|y_0 - x^*(\lambda)\|_2^2 + 8M_B^2/\mu_2^2}{t}, \forall t \geq 1.
\]
Example 4.10. (SVRG, see [23,18]) SVRG is an epoch-based algorithm, where each epoch consists of \( M \geq 1 \) inner iterations. At the beginning of epoch \( t \geq 1 \), we do a full gradient evaluation at the current iterate \( y_t \). Then, all of the inner iterations of this epoch follow (using \( l \geq 0 \) as the index for the inner iterations within an epoch of SVRG):

\[
\tilde{y}_{t+1} = \Pi_X \left[ \tilde{y}_t - \gamma \left( \nabla f^N_{i_t}(\tilde{y}_t, \lambda) - \nabla f^N_{i_t}(y_t, \lambda) + \sum_{i \in [m]} \varphi_i(\lambda) \nabla f^N_i(y_t, \lambda) \right) \right], \quad \forall l \in [M].
\]

At the end of epoch \( t \), we take \( y_{t+1} = \tilde{y}_M \) and begin the next epoch with a full gradient evaluation at \( y_{t+1} \). By Remark B.5(i) and Theorem B.7, we pick the constant step size \( \gamma_* = \frac{\mu_f^2}{(3+2M) L_N(\lambda)^2} \) to minimize the contraction factor and obtain the convergence rate across epochs in \( A_{SVRG} \):

\[
\mathbb{E}^\nu(\lambda) \left[ \|y_t - x^*(\lambda)\|_2^2 \right] \leq \left( 1 - \frac{\mu_f^2}{(3+2M) L_N(\lambda)^2} \right) \mathbb{E}^\nu(\lambda) \left[ \|y_{t-1} - x^*(\lambda)\|_2^2 \right], \quad \forall t \geq 1.
\]

Since the dual update is deterministic, all of the randomness in our overall algorithm comes from using \( A \) to do the primal updates. So, we only need to specify the required number of iterations of \( A \) in each iteration to achieve our overall tolerable error threshold. We formalize the notation for this sample complexity in the following assumption, which directly corresponds to Assumption 4.8.

Assumption 4.11. Choose \( K \geq 1, \epsilon > 0, \) and \( \delta \in (0, 1) \), then there exists \( \{J_k(K, \epsilon, \delta)\}_{k \in [K]_0} \) such that: if \( A \) is run for \( J \geq J_k(K, \epsilon, \delta) \) iterations for all \( k \in [K]_0 \), then \( x^{k+1} \) satisfies Eq. (3.2) for Problem \( \mathbb{P}_N(\lambda^k) \) with probability at least \((1 - \delta)^{1/K}\) for all \( k \in [K]_0 \).

We can determine the complexity \( \{J_k(K, \epsilon, \delta)\}_{k \in [K]_0} \) in Assumption 4.11 for any specific \( A \) from Assumption 4.8.

4.3 Convergence Analysis

Our complexity analysis is based on the following intuition. In the \( k \)-th iteration, the goal is for \( A \) to return a nearly optimal solution to \( \mathbb{P}_N(\lambda^k) \) (based on our relaxed stopping condition) with probability at least \((1 - \delta)^{1/K}\). By the chain rule for the probability of the intersection of events, the primal updates in all \( K \) iterations will then be nearly optimal with probability at least \( 1 - \delta \).

We now provide the overall complexity of RanNLR, which is the total number of inner iterations (in \( t \)) across all outer iterations (in \( k \in [K]_0 \)). We emphasize that this convergence result is with respect to the distance to the optimal solution of Problem \( \mathbb{P} \), i.e., \( \|x^K - x^*\|_\infty \).

Theorem 4.12. Suppose Assumptions 4.1, 4.2, and 4.4 hold. Let \( \{x^k\}_{k \geq 0} \) be produced by Algorithm 1. Choose \( \epsilon > 0 \) and \( \delta \in (0,1) \). Then, there exists \( N_L > 0 \) (independent of \( \epsilon \) and \( \delta \)) such that for all \( N > \max\{N_L, c_R\}, \epsilon = (1-c_RN^{-1})\epsilon/(4C_\Phi) \), and

\[
K = \left\lceil \ln \left( \frac{2 \|\lambda^0 - \lambda^\star\|_\infty/\epsilon}{\ln (N/c_R)} \right) \right\rceil,
\]

we have \( \|x^K - x^\star\|_\infty \leq \epsilon \) with probability at least \( 1 - \delta \).

(i) The overall complexity of Algorithm 1 is \( \tilde{O} \left( 1/(\epsilon^2 \delta) \right) \) if \( A \) satisfies Assumption 4.8(i).

(ii) The overall complexity of Algorithm 1 is \( O \left( \ln(1/\epsilon)(2\ln(1/\epsilon) + \ln(1/\delta)) \right) \) if \( A \) satisfies Assumption 4.8(ii).
The following result is an immediate consequence of our bound on $\|x^K - x^*\|_\infty$ (it follows by the Lipschitz continuity of the objective and constraint functions).

**Corollary 4.13.** Suppose Assumptions 4.1, 4.2, and 4.4 hold. Choose $\varepsilon > 0$ and $\delta \in (0, 1)$. Then, $x^K$ produced by Algorithm 1 satisfies $f(x^K) - f(x^*) \leq L(0)\varepsilon$ (optimality gap) and $g_i(x^K) \leq L_g(0)\varepsilon$ for all $i \in [m]$ (constraint violation), with probability at least $1 - \delta$.

Remark 4.14. If the scaling parameter $N$ is large, then the required $K$ will be small. However, making $N$ larger also makes the condition number of Problem $P_N(\lambda)$ larger, which results in a slower rate of convergence for the subroutine $\mathcal{A}$. For larger $N$, the required number of outer iterations $K$ will be smaller, but the required number of inner iterations for the subroutine $\mathcal{A}$ will be larger. Thus, there is a trade-off in the required number of outer iterations versus inner iterations through the selection of $N$.

4.4 Proof of Theorem 4.12

Given dual variables $\lambda \in \mathbb{R}^m_+$, the next primal-dual pair $(\hat{x}, \hat{\lambda})$ determined by inexact NLR is generated by Eqs. (3.2)–(3.3). Lemma 4.15 below on the one-step error is a modification of [10, Proposition 1] to account for our new stopping criterion Eq. (3.2).

**Lemma 4.15.** Suppose Assumptions 4.1, 4.2, and 4.4 hold. For dual variables $\lambda \in \mathbb{R}^m_+$ and sufficiently small error tolerance $\varepsilon > 0$, there exists $N_L > 0$ independent of $\varepsilon > 0$, such that for all $N \geq N_L$, we have

$$ \max \left\{ \|\hat{x} - x^*\|_\infty, \|\hat{\lambda} - \lambda^*\|_\infty \right\} \leq 2C\varepsilon + cR^{-1}\|\lambda - \lambda^*\|_\infty. $$

The next result shows that all primal iterates $\{x^k\}_{k \geq 1}$ of inexact NLR are bounded.

**Lemma 4.16.** Suppose Assumptions 4.1, 4.2, and 4.4 hold. For dual variables $\lambda^0 \in \mathbb{R}^m_+$ and sufficiently small error tolerance $\varepsilon > 0$, there exists $N_L > 0$ independent of $\varepsilon > 0$, such that for all $N \geq N_L$, we have

$$ \|x^k - x^*\|_\infty \leq 2C\varepsilon/(1 - cR^{-1}) + cR^{-1}\|\lambda^0 - \lambda^*\|_\infty, \quad \forall k \geq 1. $$

We can now provide the required number of iterations $\{J_k(K, \varepsilon, \delta)\}_{k \in [K]}$ of each call to $\mathcal{A}$ at different iterations of Algorithm 1, so that Eq. (3.2) is satisfied with probability at least $(1 - \delta)^{1/K}$.

We first consider the case where $\mathcal{A}$ has a sublinear convergence rate. Define $J_0^{SL}(K, \varepsilon, \delta)$ as

$$ \max \left\{ \frac{L_N(\lambda^0)^2 \left( nA \left( \|x^0 - x^*\|^2_\infty + (cR^{-1})2\|\lambda^0 - \lambda^*\|^2_\infty \right) + B \right)}{(1 - \varepsilon/(1 - \delta)^{1/K})^2}, 1 \right\}, $$

and define $J_k^{SL}(K, \varepsilon, \delta)$ as

$$ \max \left\{ \frac{L_N(\lambda^k)^2 \left( nA \left( 1 + (cR^{-1})2 \right) \left( 2C\varepsilon/(1 - cR^{-1}) + (cR^{-1})^k\|\lambda^0 - \lambda^*\|^2_\infty + B \right) \right)}{(1 - \varepsilon/(1 - \delta)^{1/K})^2}, 1 \right\}, $$

for each $k \in [K - 1]$.

**Lemma 4.17.** Suppose $\mathcal{A}$ satisfies Assumption 4.8(i).

(i) At iteration $k = 0$, if we run $\mathcal{A}$ for $J_0^{SL}(K, \varepsilon, \delta)$ iterations, then the output $x^1$ satisfies Eq. (3.2) with probability at least $(1 - \delta)^{1/K}$.

(ii) At iteration $k \geq 1$, suppose that the solution returned by $\mathcal{A}$ satisfies Eq. (3.2) in all previous iterations. If we run $\mathcal{A}$ for $J_k^{SL}(K, \varepsilon, \delta)$ iterations, then the output $x^{k+1}$ satisfies Eq. (3.2) with probability at least $(1 - \delta)^{1/K}$. 

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Now we treat the case where \( \mathcal{A} \) has a linear convergence rate. Define \( J^L_{i0}(K, \epsilon, \delta) \) as

\[
\max \left\{ \left[ \ln \left( \frac{n\zeta N(\lambda^0)^2 (\|x^0 - x^*\|_2^2 + (c_R N^{-1})^2 \|\lambda^0 - \lambda^*\|_\infty^2)}{(1 - (1 - \delta)^{1/K}) \epsilon^2} \right) / \ln(1/\alpha) \right], 1 \right\},
\]

and define \( J^L_k(K, \epsilon, \delta) \) as

\[
\max \left\{ \left[ \ln \left( \frac{n\zeta N(\lambda^k)^2 (1 + (c_R N^{-1})^2) (2C\Phi/(1 - c_R N^{-1}) + (c_R N^{-1})^K \|\lambda^0 - \lambda^*\|_\infty^2)}{(1 - (1 - \delta)^{1/K}) \epsilon^2} \right) / \ln(1/\alpha) \right], 1 \right\},
\]

for each \( k \in [K - 1] \). Using the same argument as Lemma 4.17, we have the following complexity result for the present case.

**Lemma 4.18.** Suppose \( \mathcal{A} \) satisfies Assumption 4.8(ii).

(i) At iteration \( k = 0 \), if we run \( \mathcal{A} \) with \( J^L_{i0}(K, \epsilon, \delta) \) iterations, then the output \( x^1 \) satisfies Eq. (3.2) with probability at least \( (1 - \delta)^{1/K} \).

(ii) At iteration \( k \geq 1 \), suppose that the solution returned by \( \mathcal{A} \) satisfies Eq. (3.2) in all previous iterations. If we run \( \mathcal{A} \) for \( J^L_k(K, \epsilon, \delta) \) iterations, then the output \( x^{k+1} \) satisfies Eq. (3.2) with probability at least \( (1 - \delta)^{1/K} \).

Based on Lemmas 4.17 and 4.18, we can now provide the proof of Theorem 4.12. If Eq. (3.2) holds for all iterations \( k \in [K]\), then iterating Lemma 4.15 shows:

\[
\|x^K - x^*\|_\infty \leq (1 - (c_R N^{-1})^K) 2C\Phi/(1 - c_R N^{-1}) + (c_R N^{-1})^K \|\lambda^0 - \lambda^*\|_\infty.
\]

Plugging in \( \epsilon = (1 - c_R N^{-1}) \delta/(4C\Phi) \) and \( K = \lfloor \ln \left( 2 \|\lambda^0 - \lambda^*\|_\infty / \delta \right)/\ln(N/c_R) \rfloor = O(\ln(1/\epsilon)) \) into the above inequality, and noting that \( 1 - (c_R N^{-1})^K < 1 \), we see that \( \|x^K - x^*\|_\infty \leq \epsilon \).

We estimate the complexity of Algorithm 1 as follows. First suppose \( \mathcal{A} \) has a sublinear convergence rate. By Lemma 4.17, we know that if \( \mathcal{A} \) is run for \( J^L_{i0}(K, (1 - c_R N^{-1}) \delta/(4C\Phi)) \) iterations, then it returns an approximately optimal solution satisfying Eq. (3.2) with \( \epsilon = (1 - c_R N^{-1}) \delta/(4C\Phi) \) with probability at least \( (1 - \delta)^{1/K} \). Using the chain rule for the probability of the intersection of events, the solutions \( x^k (k \in [K]) \) returned from \( \mathcal{A} \) satisfy Eq. (3.2) with \( \epsilon = (1 - c_R N^{-1}) \delta/(4C\Phi) \) for all \( K \) iterations with probability at least \( 1 - \delta \). Clearly, \( J^L_{i0}(K, (1 - c_R N^{-1}) \delta/(4C\Phi), \delta) = O\left( K/(\epsilon^2 \delta) \right) \) for all \( k \in [K]\) and \( K = O(\ln(1/\epsilon)) \). Therefore, the overall complexity of Algorithm 1 is \( O\left( \ln^2(1/\epsilon)(1/(\epsilon^2 \delta)) \right) \).

Now suppose that \( \mathcal{A} \) has a linear convergence rate. Clearly, \( J^L_k(K, (1 - c_R N^{-1}) \delta/(4C\Phi), \delta) = O\left( \ln(K)/(\epsilon^2 \delta) \right) \) for all \( k \in [K]\) and \( K = O(\ln(1/\epsilon)) \). Using the same argument as above along with Lemma 4.18, we see that the overall complexity of Algorithm 1 is \( O\left( \ln(1/\epsilon)(2 \ln(1/\epsilon) + \ln(1/\delta)) \right) \).

### 4.5 Optimal Sampling from \( \varphi(\lambda) \)

In our implementation of RanNLR, the subroutine \( \mathcal{A} \) samples from \( \varphi(\lambda) \in \mathcal{P}_+([m]) \) when solving each instance of Problem \( \mathcal{P}_N(\lambda) \). For the purpose of comparison, let \( \mathcal{Q} = (q_i)_{i \in [m]} \) be any other probability distribution in \( \mathcal{P}_+([m]) \), and let \( \mathbb{E}^\mathcal{Q} [\cdot] \) denote conditional expectation with respect to \( \mathcal{Q} \). We use \( \mathcal{A}^\mathcal{Q} \) to denote the subroutine \( \mathcal{A} \) where \( \mathcal{Q} \) is used to sample \( \{I_i\}_{i \geq 0} \) instead of \( \varphi(\lambda) \). In particular, we denote \( \mathcal{A}^\mathcal{Q}_{\text{SGD}} \) and \( \mathcal{A}^\mathcal{Q}_{\text{SVRG}} \) as SGD and SVRG with a generic sampling distribution \( \mathcal{Q} \), respectively.

We want to compare the performance of \( \mathcal{A} \) (which samples from \( \varphi(\lambda) \)) with \( \mathcal{A}^\mathcal{Q} \). The key difference in performance between \( \mathcal{A} \) and \( \mathcal{A}^\mathcal{Q} \) is captured by the constant \( r^\mathcal{Q} \equiv \sum_{i \in [m]} \varphi_i(\lambda)^2/q_i \).
Lemma 4.19. Fix $\lambda \in \mathbb{R}^n_+$ and let $Q \triangleq (q_i)_{i \in [m]} \in \mathcal{P}_+([m])$. Then $r^Q \geq 1$, and equality holds if and only if $q_i = \varphi_i(\lambda)$ for all $i \in [m]$.

Proof. Define the random variable $X$ such that $\Pr(X = \varphi_i(\lambda)/q_i) = q_i$, for all $i \in [m]$. By Jensen’s inequality, we have

$$r^Q = \sum_{i \in [m]} q_i (\varphi_i(\lambda)/q_i)^2 = \mathbb{E}[X^2] \geq (\mathbb{E}[X])^2 = 1,$$

since $\mathbb{E}[X] = \sum_{i \in [m]} q_i \varphi_i(\lambda)/q_i = 1$. Equality holds if and only if $q_i = \varphi_i(\lambda)$ for all $i \in [m]$. \hfill $\square$

When solving Problem $\mathbb{F}_N(\lambda)$, it is optimal to sample from $\varphi(\lambda)$ for both SGD and SVRG.

Example 4.20. (SGD with generic sampling) Fix $\lambda \in \mathbb{R}^n_+$ and let $M_B \triangleq \max_{i \in [m]} \|\nabla f_i^N(x^*(\lambda), \lambda)\|_2$. By Theorem B.6, the convergence rate of $A^Q_{\text{SGD}}$ applied to $\mathbb{F}_N(\lambda)$ is:

$$\mathbb{E}^Q \left[ \|y_t - x^*(\lambda)\|_2^2 \right] \leq (A^Q_{\text{SGD}} \|y_0 - x^*(\lambda)\|_2^2 + B^Q_{\text{SGD}})/t, \quad \forall t \geq 1,$$  \hfill (4.7)

where $A^Q_{\text{SGD}} \triangleq 2(1 + 2r^Q)\|\nabla f_0(x^*(\lambda)\|_2^2 / \mu_f^2 - 1$ and $B^Q_{\text{SGD}} \triangleq 8r^Q M_B^2 / \mu_f^2$. For comparison, the convergence rate of $A^Q_{\text{SGD}}$ satisfies Eq. (4.7) with $A^Q_{\text{SGD}} \triangleq 6\|\nabla f_0(x^*(\lambda)\|_2^2 / \mu_f^2 - 1$ and $B^Q_{\text{SGD}} \triangleq 8M_B^2 / \mu_f^2$. Since $r^Q \geq 1$, we must have $A^Q_{\text{SGD}} \geq A^Q_{\text{SGD}}$ and $B^Q_{\text{SGD}} \geq B^Q_{\text{SGD}}$ (for any other $Q$). Thus, if we use any distribution other than $\varphi(\lambda)$ to sample $\{I_t\}_{t \geq 0}$, then the constants in the sublinear rate will increase (and become worse).

Example 4.21. (SVRG with generic sampling) Fix $\lambda \in \mathbb{R}^n_+$ and a constant step-size $\gamma_* = \frac{\mu_f}{(1 + 2r^Q + 4M^2_\|\nabla f_0(x^*(\lambda)\|_2^2 / \mu_f^2}$.

By Theorem B.7, the convergence rate of $A^Q_{\text{SVRG}}$ applied to $\mathbb{F}_N(\lambda)$ is:

$$\mathbb{E}^Q \left[ \|y_t - x^*(\lambda)\|_2^2 \right] \leq (\alpha^Q_{\text{SVRG}}) \mathbb{E}^Q \left[ \|y_0 - x^*(\lambda)\|_2^2 \right],$$  \hfill (4.8)

where $\alpha^Q_{\text{SVRG}} \triangleq 1 - r^Q\|\nabla f_0(x^*(\lambda)\|_2^2/(3 + 2M^2_\|\nabla f_0(x^*(\lambda)\|_2^2 / \mu_f^2$. For comparison, the convergence rate of $A^Q_{\text{SVRG}}$ satisfies Eq. (4.8) with contraction factor $\alpha^Q_{\text{SVRG}} \triangleq 1 - \frac{r^Q\|\nabla f_0(x^*(\lambda)\|_2^2}{(3 + 2M^2_\|\nabla f_0(x^*(\lambda)\|_2^2 / \mu_f^2}$. Since $r^Q \geq 1$, we must have $\alpha^Q_{\text{SVRG}} \geq \alpha^Q_{\text{SVRG}}$. Thus, if we use any distribution other than $\varphi(\lambda)$ to sample $\{I_t\}_{t \geq 0}$, then the contraction factor will increase (and become worse).

5 Numerical Experiments

In this section, we present two case studies to illustrate the effectiveness and behavior of RanNLR. The first one is a simple case adapted from [28], where we choose the primal-dual type algorithm from [51] as the baseline for comparison. This baseline algorithm theoretically achieves an $O(1/K)$ convergence rate in terms of the optimality gap and constraint violation. The second one is an inventory control problem adapted from [26], where we compare our algorithm with a commercial solver (Gurobi 9.0) in the task of solving an LP with one million constraints.

We use the following nonlinear rescaling function:

$$\psi(t) \triangleq \begin{cases} 1 - e^{-t}, & t \geq -0.5, \\ -0.5e^{0.5t^2} + 0.5e^{0.5t} + 1 - \frac{5}{8}e^{0.5}, & t \leq -0.5, \end{cases}$$  \hfill (5.1)

for our implementation of RanNLR.
Table 1: Simulation results

| Algorithm  | Stepsize $\gamma$ | Iteration $K$ | $N$ | Epoch no. $M$ | $\epsilon$ | Obj. | Relative gap | CPU time (s) |
|------------|-------------------|---------------|-----|--------------|----------|------|--------------|-------------|
| Baseline   | 0.0001            | 30000         | -   | -            | -        | 3.231| 0.3%         | 2.07        |
| PDSVRG     | 0.0001            | 62            | 100 | 20           | 0.0001   | 3.221| <0.01%       | 0.10        |
| PDSVRG     | 0.0001            | 4             | 1000| 400          | 0.0001   | 3.221| <0.01%       | 0.02        |

Remark 5.1. In the NLR literature, Newton’s method is used to do the primal update [16, 35]. The corresponding numerical results demonstrate the “hot start” phenomenon, where only a few updates and very few (often just one) Newton steps per update are required. It can be expected that our algorithm will also experience this “hot start” phenomenon. After a few updates, the optimizer obtained from $A$ is always in the neighborhood of the next one, and so we require fewer and fewer iterations of $A$ to do the primal updates with an extra digit of accuracy.

Remark 5.2. It is hard to use a universal nonlinear rescaling function for all problems. Therefore, we may need to tune the rescaling function for different problems. However, as we will see in Subsection 5.2, we can simply normalize Eq. (5.1) with division by a positive number $\beta$ to control the range of the values of the constraint functions. In this way, we can just tune $\beta$ rather than redesign the entire nonlinear rescaling function.

5.1 A Simple Case

This case is adapted from a semi-infinite programming problem in [28], where we discretize the constraint index set over a uniform grid and construct the following approximate problem:

$$\min (x_1 - 2)^2 + (x_2 - 0.2)^2$$

s.t.

$$\frac{5\sin\left(\pi \sqrt{\frac{i}{m}}\right)}{1 + (\frac{i}{m})^2} x_1^2 - x_2 \leq 0, \quad \forall i \in [m],$$

$$x_1 \in [-1, 1], x_2 \in [0, 0.2].$$

The optimal solution is $x = (0.20523677, 0.2)$ and the optimal value is 3.221. We set $m = 10,000$ and compare the performance of RanNLR (with SVRG as the subroutine, henceforth denoted RanNLR-A_SVRG) with the baseline algorithm taken from [51]. The results are presented in Table 1 and Figure 1.

We run the baseline algorithm for 30,000 iterations and see the achieved objective value is 3.231, with a relative optimality gap of 0.3%. However, RanNLR-A_SVRG converges to the optimal value 3.221 within 62 iterations, with a relative optimality gap of only 0.01%. In addition, the CPU time of RanNLR-A_SVRG (0.10 seconds) is faster than the baseline algorithm (2.07 seconds). We also explore the effect of changing the scaling parameter $N$. When $N = 1,000$ and the number of epochs used in SVRG is 400, the algorithm converges to a nearly optimal solution in merely 4 iterations and the required CPU time decreases to 0.02 seconds.

5.2 Inventory Control with Approximate Linear Programming

In this subsection, we present a numerical study adapted from [25] for a single-product inventory control problem with partially backlogged demand and zero lead time. Let $s \in S \subset [-10, 10]$ and $a \in A \subset [0, 20]$ denote the on-hand inventory (with negative values indicating backlogged orders) and the order quantity, respectively. We discretize the state/action space with a precision of 0.02 so that $S \equiv$
Let $D$ denote the stochastic customer demand, which is assumed to have a discrete sample space $\mathcal{D} \triangleq \{0, 0.02, \ldots, 9.98, 10\}$. The probability density for $D$ is given by $p(d) = \int_{d-0.01}^{d+0.01} f_D(x) \, dx$ for each $d \in \mathcal{D}$, where $f_D(\cdot)$ is the density function of a truncated normal distribution on $[0, 10]$ with mean 5 and standard deviation 2. The state transitions are given by $s' \triangleq \min(\max(s + a - D, l), \bar{u})$, where $l = -10$ and $\bar{u} = 10$ denote the lower/upper bounds of the state, respectively.

The cost function for each state-action pair $(s, a) \in S \times A$ is

$$c(s, a) \triangleq c_p a + c_h E[(s')_+] + c_b E[(s')_+] + c_d E[(s + a - D - \bar{u})_+] + c_l E[(l - s - a + D)_+]$$

where $(c_p, c_h, c_b, c_d, c_l) = (20, 2, 10, 10, 100)$ are the cost coefficients. The discount factor is $\gamma = 0.95$ in this example.

Let $V : S \to \mathbb{R}$ be the value function of the MDP, and let $q(\cdot)$ be a given probability mass function on $S$ (in the numerical study, we choose $q(\cdot)$ to be uniform on $S$). The above infinite time horizon MDP can be solved by the following linear programming problem:

$$\max_V \mathbb{E}_q[V(s)] \quad (5.3)$$

subject to $V(s) - \gamma \mathbb{E}_D[V(s')|s, a] \leq c(s, a), \forall (s, a) \in S \times A$.

The above problem optimizes over $V(\cdot)$, which is an intractable problem when the state space is large or infinite.

Alternatively, a more tractable method approximates the value function $V$ via a linear combination of basis functions $\phi_b(\cdot)$, i.e., the approximate linear programming (ALP) method. The ALP formulation of the
above problem is:
\[
\max_{\theta} \sum_{b=1}^{B} \theta_b \mathbb{E}_q [\phi_b (s)] \\
\text{s.t.} \sum_{b=1}^{B} \theta_b (\phi_b (s) - \gamma \mathbb{E}_D [\phi_b (s') | s, a]) - c (s, a) \leq 0, \quad \forall (s, a) \in \mathcal{S} \times \mathcal{A},
\]
where \(\phi_b (\cdot)\) are the basis functions chosen to approximate the value functions of the MDP, and \(\theta_b\) are the weights of the basic functions. Problem (5.4) is a large-scale linear programming problem with \(|\mathcal{S}| \times |\mathcal{A}| = 1,002,001\) constraints (equal to the number of state-action pairs). We choose \(B = 2\) and \((\phi_1, \phi_2) = (1, s)\) for our numerical study.

We first solve Problem (5.4) via RanNLR with SGD (henceforth denoted RanNLR-\(\text{ASGD}\)). We normalize the constraints of Problem (5.4) by dividing the constraint functions by \(\beta > 0\) to get:
\[
\left( \sum_{b=1}^{B} \theta_b (\phi_b (s) - \gamma \mathbb{E}_D [\phi_b (s') | s, a]) - c (s, a) \right) / \beta \leq 0, \quad \forall (s, a) \in \mathcal{S} \times \mathcal{A},
\]
so that the values of the constraints range from \([-1, 1]\). We take \(\beta = 600\) for this problem instance. Then, we transform the constraints via the nonlinear rescaling function Eq. (5.1). In addition, we check the termination condition for SGD every 1000 iterations (instead of every iteration) because calculating \(\|\nabla x L_N (x^{k+1}, \lambda^k)\|_\infty\) can be expensive when the number of constraints is large.

Note that we need to calculate the expectations \(\mathbb{E}_D [\cdot]\) for all \((s, a) \in \mathcal{S} \times \mathcal{A}\) in the constraints of Problem (5.4), which can be time-consuming if we calculate them on-the-fly. Instead, we calculate the expectations and store the values in a matrix, and call the values whenever we need them. Therefore, the CPU time listed in Table 2 excludes the calculation time of \(\mathbb{E}_D [\cdot]\) for all three algorithms.

The optimal value of Problem (5.4) is 2146.94, as provided by Gurobi 9.0. The simulation results for this case are presented in Table 2. We choose \(N = 1000\) and \(\epsilon = 1\). We see that RanNLR-\(\text{ASGD}\) finds a near-optimal solution in 30 outer iterations, and the number of required iterations for the subroutine \(\text{ASGD}\) ranges from 1000 to 9000. In addition, the CPU time for RanNLR-\(\text{ASGD}\) is 11.0 seconds, while the commercial solver takes 79.3 seconds.

We also test the performance of RanNLR-\(\text{AVRG}\) on this problem instance. We see from Figure 2 that the convergence rates of RanNLR-\(\text{ASVRG}\) and RanNLR-\(\text{ASGD}\) are similar. However, RanNLR-\(\text{ASVRG}\) is more expensive per iteration and so its CPU time is longer compared to that of RanNLR-\(\text{ASGD}\).

We plot the probability distribution \(\wp (\lambda)\) for the 5th and 80th outer iterations of RanNLR-\(\text{ASGD}\) for \((N = 1000)\) in Figure 3. This probability distribution is flat in the 5th iteration, but it concentrates on a certain subset of the constraints as the algorithm proceeds. This phenomenon may help explain why our algorithm can converge to a near-optimal solution by using several thousand samples when there are more than one million constraints.
Figure 2: Convergence analysis with $N = 1000$ and $\epsilon = 1$

Figure 3: Probability distribution with adaptive sampling
6 Conclusion

In this paper, we develop RanNLR to solve convex optimization programs (with high probability) when the number of constraints is very large. For a tolerance $\varepsilon > 0$ and an overall failure probability $\delta \in (0, 1)$, we provide the complexity analysis for obtaining a solution within $\varepsilon$ of the optimal solution to Problem $\mathbb{P}$ with probability at least $1 - \delta$. The core of RanNLR is the use of randomized first-order algorithms to do the primal updates. Due to the special structure of the finite sum minimization problem in the primal updates, we can leverage on the success of these randomized first-order algorithms to achieve significant computational savings as suggested by our experiments.

We briefly remark on the connection between our results and the most closely related works. In [16, 35], Newton’s method is used to do the primal update. In this case, Hessian computations and matrix inversions are required in every iteration, and the associated computational and storage requirements are between $O((n + m)^2)$ and $O((n + m)^3)$, which is prohibitive when $m \gg 0$. In addition, the methods in [51] and [52] are based on the average of the sequence of primal iterates and the convergence rate is $O(1/K)$ for both the optimality gap and the constraint violation. In contrast, RanNLR only uses the last primal iterate.

In future research, we will extend RanNLR to semi-infinite programming problems. Here we will identify a tractable family of sampling distributions, and combine this family with the efficient randomized first-order algorithms that have been demonstrated here.

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A Proofs of Technical Lemmas

A.1 Proofs of Lemmas 4.15 and 4.16

Recall that

\[
\Phi_N(x^*, \lambda^*) \triangleq \left[ \nabla_{xx}L(x^*, \lambda^*), -\nabla_{x}G_T^*(x^*), \psi''(0)^{-1}N^{-1}I_{I^*}, \right].
\]

Without loss of generality, assume that the active constraints \(I^*\) correspond to the first \(|I^*|\) components of \(\lambda\). Then define the following vectors: \(\Delta \lambda \triangleq \hat{\lambda} - \lambda^* = (\Delta \lambda_{I^*}, \Delta \lambda_{[m]\setminus I^*})\), \(\Delta x \triangleq \hat{x} - x^*\), \(\Delta z_{I^*} \triangleq (\Delta x, \Delta \lambda_{I^*})\), and \(\Delta z \triangleq (\Delta x, \Delta \lambda)\). Moreover, we define a mapping \(h_N : \mathbb{R}^n \times \mathbb{R}^{n-|I^*|} \rightarrow \mathbb{R}\) as \(h_N(x, \lambda_{[m]\setminus I^*}) \triangleq \sum_{i \in [m]\setminus I^*} \lambda_i \psi'(N g_i(x)) \nabla g_i(x)\). For all \(i \in I^*\), define a mapping \(m^N_i : \mathbb{R}^n \rightarrow \mathbb{R}\) where \(m_i^N(x) \triangleq \psi'(N g_i(x)) - 1\). Then, define \(m_i^N(\hat{x}) \triangleq (m_i^N(\hat{x}))_{i \in I^*}, E_i^N(\hat{x}) \triangleq \text{diag}(m_i^N(\hat{x}))\), and

\[
a_N(\hat{x}, \lambda) \triangleq \left[ \nabla_{x}L_N(\hat{x}, \lambda) + h_N(\hat{x}, \lambda_{[m]\setminus I^*}), -\psi''(0)^{-1}N^{-1}(I_{I^*} + E_{I^*}(\hat{x}, N))(\lambda_{I^*} - \lambda_{I^*}) \right].
\]

The following lemma helps to bound \(\|\Delta z_{I^*}\|_\infty\).

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Lemma A.1. Suppose Assumptions 4.1, 4.2, and 4.3 hold. For any \( \lambda \in \mathbb{R}_{m+}^n \) and sufficiently large \( N \), we have
\[
\Phi_N(x^*, \lambda^*) \Delta z_i = (a_N(\hat{x}, \lambda) + r(\Delta z_i)),
\]
where \( r : \mathbb{R}^{n+|I^*|} \rightarrow \mathbb{R}^{n+|I^*|} \) and there exists \( R > 0 \) such that \( \|r(\Delta z_{I^*})\|_\infty \leq R \|\Delta z_{I^*}\|_\infty^2/2 \).

Proof. Fix an arbitrary small \( \eta > 0 \). For sufficiently small \( \epsilon > 0 \) and sufficiently large \( N \), we have \( \hat{x} \in B_\eta(x^*) \) and \( \lambda = \lambda_\psi'(NG(\hat{x})) \in B_\eta(\lambda^*) \), due to properties (i), (ii), (iv), and (v) of the nonlinear rescaling function class \( \Psi \).

Due to Assumption 4.2, we have by Taylor’s theorem
\[
\nabla f(\hat{x}) = \nabla f(x^*) + \nabla^2 f(x^*) \Delta x + r_0(\Delta x),
\]
where \( r_0(0) = 0, r_1(0) = 0, \)
\[
\|r_0(\Delta x)\|_\infty \leq L_i^{(2)} \|\Delta x\|_\infty^2/2, \quad \text{and} \quad \|r_1(\Delta x)\|_\infty \leq L_i^{(2)} \|\Delta x\|_\infty^2/2, \quad \forall i \in [m].
\]
Then
\[
\nabla_x \mathcal{L}_N(\hat{x}, \lambda) = \nabla f(\hat{x}) - \sum_{i \in [m]} \lambda_i \psi'(NG(\hat{x})) \nabla g_i(\hat{x})
\]
\[
= \nabla f(\hat{x}) - \sum_{i \in [m]} \lambda_i \nabla g_i(\hat{x})
\]
\[
= \nabla f(\hat{x}) - \sum_{i \in I^*} (\lambda_i^* + \Delta \lambda_i) \nabla g_i(\hat{x}) - h_N(\hat{x}, \lambda_{[m]\backslash I^*}).
\]

Using Eqs. 4.2-4.3 and the KKT conditions at the optimal solution \( x^* \), we have
\[
\nabla_x \mathcal{L}_N(\hat{x}, \lambda) = \nabla f(x^*) + \nabla^2 f(x^*) \Delta x + r_0(\Delta x) - \sum_{i \in I^*} (\lambda_i^* + \Delta \lambda_i) (\nabla g_i(x^*) + \nabla^2 g_i(x^*) \Delta x + r_i(\Delta x))
\]
\[
- h_N(\hat{x}, \lambda_{[m]\backslash I^*})
\]
\[
= \nabla f(x^*) - (\lambda_i^* - \lambda_i) \nabla g_i(x^*) + \frac{\nabla^2 f(x^*) - \sum_{i \in I^*} \lambda_i^* \nabla^2 g_i(x^*)}{(\lambda_i^* - \lambda_i)} \Delta x - \nabla G_{I^*}(x^*)^T \Delta \lambda_{I^*}
\]
\[
+ r^{(1)}(\Delta z_{I^*}) - h_N(\hat{x}, \lambda_{[m]\backslash I^*})
\]
\[
= \nabla_x L(x^*, \lambda^*) \Delta x - \nabla G_{I^*}(x^*)^T \Delta \lambda_{I^*} + r^{(1)}(\Delta z_{I^*}) - h_N(\hat{x}, \lambda_{[m]\backslash I^*}),
\]
where \( r^{(1)}(\Delta z_{I^*}) \triangleq r_0(\Delta x) - \sum_{i \in I^*} \Delta \lambda_i \nabla^2 g_i(x^*) \Delta x - \sum_{i \in I^*} (\lambda_i^* + \Delta \lambda_i) r_i(\Delta x) \). From Eq. 4.4, we have \( r^{(1)}(0) = 0 \) and there exists \( L_i^{(1)} > 0 \) such that
\[
\|r^{(1)}(\Delta z_{I^*})\|_\infty \leq L_i^{(1)} \|\Delta z_{I^*}\|_\infty^2/2.
\]

Using the fact that \( \lambda_i - \lambda_i = \lambda_i m_i^N(\hat{x}) \), we have \( \lambda_i^* m_i^N(\hat{x}) - \Delta \lambda_i = (1 + m_i^N(\hat{x})) (\lambda_i^* - \lambda_i) \) for all \( i \in I^* \), i.e.,
\[
A_i^* m_i^N(\hat{x}) - \Delta \lambda_i = (I_{I^*} + E_i^N(\hat{x})) (\lambda_i^* - \lambda_i).
\]
Moreover, \( m_i^N(\hat{x}) = m_i^N(x^*) + (\nabla m_i^N(x^*), \Delta x) + r_i^{(m)}(\Delta x) \) for all \( i \in I^* \), where \( r_i^{(m)}(0) = 0 \). Due to the fact \( \hat{x} \in B_\eta(x^*) \) for small enough \( \eta > 0 \), and by property (v) of the nonlinear rescaling function class \( \Psi \), there exists \( L_i^{(m)} > 0 \) such that
\[
|r_i^{(m)}(\Delta x)| \leq L_i^{(m)} \|\Delta x\|_\infty^2/2, \quad i \in I^*.
\]
Noting that \( g_i(x^*) = 0 \) for all \( i \in I^* \) and using property (ii) of \( \Psi \), we have
\[
m_i^N(x^*) = \psi'(Ng_i(x^*)) - 1 = \psi'(0) - 1 = 0, \quad i \in I^*,
\]
\[
\nabla m_i^N(x^*) = \psi''(Ng_i(x^*))N\nabla g_i(x^*) = \psi''(0)N\nabla g_i(x^*), \quad i \in I^*.
\]
Therefore, we have
\[
m_i^N(\hat{x}) = \psi''(0)N(\nabla g_i(x^*), \Delta x) + r_i^m(\Delta x), \quad i \in I^*. \tag{A.9}
\]

From Eqs. (A.7)–(A.9), we obtain
\[
-A_i^\eta, \nabla G_{1^*}(x^*) \Delta x + \psi''(0)^{-1}N^{-1}\Delta \lambda_{1^*} = -\psi''(0)^{-1}N^{-1}(I_{[I^*]} + E_i^N(\hat{x})) (\lambda_{1^*} - \lambda_{I^*}) + r^{(2)}(\Delta x), \tag{A.10}
\]
where \( r^{(2)}(\Delta x) \equiv \psi''(0)^{-1}N^{-1}A_i^\eta, r_i^m(\Delta x), r_i^m(\Delta x) \equiv (r_i^m(\Delta x))_{i \in I^*}, \) and \( r_i^m(0) = 0 \). From Eq. (A.8), we note that there exists \( L^{(2)} > 0 \) such that
\[
\|r^{(2)}(\Delta x)\|_\infty \leq L^{(2)} \|\Delta x\|^2 / 2. \tag{A.11}
\]
Rearranging Eq. (A.5) and Eq. (A.10), we have \( \Phi_N(x^*, \lambda^*) \Delta z_{I^*} = a_N(\hat{x}, \lambda) + r(\Delta z_{I^*}) \), where
\[
r(\Delta z_{I^*}) \equiv \left[ -r^{(1)}(\Delta z_{I^*}) \right].
\]
From Eqs. (A.6) and (A.11), we have that there exists \( R > 0 \) such that \( \|r(\Delta z_{I^*})\|_\infty \leq R \|\Delta z_{I^*}\|^2 / 2. \]

Next, we are going to give the proofs of Lemmas 4.15 and 4.16.

**Proof of Lemma 4.15.** Fix an arbitrary small \( \eta > 0 \). For sufficiently small \( \epsilon > 0 \) and sufficiently large \( N \), we have \( \hat{x} \in B_\eta(x^*) \) and \( \hat{\lambda} = \lambda \psi'/(NG(\hat{x})) \in B_\eta(\lambda^*) \), due to properties (i), (ii), (iii), and (iv) of the nonlinear rescaling function class \( \Psi \). Noting that \( g_i(x^*) \geq \sigma \) for all \( i \in [m] \setminus I^* \), \( \hat{x} \in B_\eta(x^*) \) for small enough \( \eta > 0 \) and property (v) of \( \Psi \), we obtain that
\[
\hat{\lambda}_i = \lambda_i \psi'/(Ng_i(\hat{x})) \leq 2\lambda_i \psi'/(Ng_i(x^*)) \leq 2\lambda_i \psi'/(N\sigma) \leq 2d_1 \sigma^{-1}N^{-1}\lambda_i, \quad i \in [m] \setminus I^*.
\]
Hence we have
\[
\|\hat{\lambda}_{[m]\setminus I^*} - \lambda_{[m]\setminus I^*}\|_\infty \leq 2d_1 \sigma^{-1}N^{-1}\|\lambda_{[m]\setminus I^*} - \lambda_{[m]\setminus I^*}\|_\infty. \tag{A.12}
\]

In the following, we will estimate \( \|a_N(\hat{x}, \lambda)\|_\infty \) and then estimate \( \|\Delta z_{I^*}\|_\infty \) based on Eq. (A.11). First, from the fact that \( g_i(x^*) \geq \sigma \) for all \( i \in [m] \setminus I^* \), \( \hat{x} \in B_\eta(x^*) \) for small enough \( \eta > 0 \), the Lipschitz continuity of \( \nabla g_i(\cdot) \), and properties (i) and (iii) of \( \Psi \), we have
\[
\|h_N(\hat{x}, \lambda)\|_\infty \leq \sum_{i \in [m] \setminus I^*} 4\lambda_i \psi'(N\sigma/2) \|\nabla g_i(x^*)\|_\infty.
\]
From property (v) of \( \Psi \), there is a constant \( C_\Psi > 0 \) such that \( \sum_{i \in [m] \setminus I^*} 4\lambda_i \psi'(N\sigma/2) \|\nabla g_i(x^*)\|_\infty \leq C_\Psi N^{-1} \). Therefore,
\[
\|h_N(\hat{x}, \lambda)\|_\infty \leq C_\Psi N^{-1}\|\lambda_{[m]\setminus I^*} - \lambda_{[m]\setminus I^*}\|_\infty \leq C_\Psi N^{-1}\|\lambda - \lambda^*\|_\infty. \tag{A.13}
\]
Moreover,
\[
I_{[I^*]} + E_{I^*}^N(\hat{x}) = \text{diag}((\psi'(Ng_i(\hat{x})))_{i \in I^*}).
\]

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Using the fact that \( \hat{x} \in B_{\eta}(x^*) \) for small enough \( \eta > 0 \), \( g_i(x^*) = 0 \) for all \( i \in I^* \), the continuity of \( \psi''(\cdot) \), \( \psi'(0) = 1 \) from property (ii), and \( \psi''(0) < 0 \) from property (iii) of \( \Psi \), we have

\[
\| -\psi''(0)^{-1}N^{-1} \left(1_{J^{l*}_1} + E_{I^*_1}^N(\hat{x}) \right) (\lambda^*_{J^*} - \lambda_{J^*}) \| \leq -2\psi''(0)^{-1}N^{-1} \|\lambda - \lambda^*\|_{\infty}.
\] (A.14)

From Eqs. (3.2), (A.13), and (A.14), we obtain

\[
\|a_N(\hat{x}, \lambda)\|_{\infty} \leq \epsilon + (C_{\Psi} - 2\psi''(0)^{-1}) N^{-1} \|\lambda - \lambda^*\|_{\infty}.
\]

From Eqs. (4.2) and (A.1), we have for sufficiently large \( N \geq N_0 \),

\[
\|\Delta z_{I^*}\|_{\infty} \leq \|\Phi_N(x^*, \lambda^*)^{-1}\| \left(\|a_N(\hat{x}, \lambda)\|_{\infty} + \|r(\Delta z_{I^*})\|_{\infty} \right)
\leq C_{\Phi} \left( \epsilon + (C_{\Psi} - 2\psi''(0)^{-1}) N^{-1} \|\lambda - \lambda^*\|_{\infty} + R \|\Delta z_{I^*}\|_{\infty} \right)^{2}/2 \right).
\]

After rearrangement of the quadratic in \( \|\Delta z_{I^*}\|_{\infty} \), we obtain

\[
\|\Delta z_{I^*}\|_{\infty} \leq \left(1 - \sqrt{1 - 2C_{\Phi}^2 \epsilon (C_{\Psi} - 2\psi''(0)^{-1}) N^{-1} \|\lambda - \lambda^*\|_{\infty}} \right) / (C_{\Phi} R).
\]

For sufficiently large \( N \), we have

\[
1 - 2C_{\Phi}^2 \epsilon (C_{\Psi} - 2\psi''(0)^{-1}) N^{-1} \|\lambda - \lambda^*\|_{\infty} \leq \sqrt{1 - 2C_{\Phi}^2 \epsilon (C_{\Psi} - 2\psi''(0)^{-1}) N^{-1} \|\lambda - \lambda^*\|_{\infty}},
\]

thus

\[
\|\Delta z_{I^*}\|_{\infty} \leq 2C_{\Phi} \epsilon + 2(C_{\Psi} - 2\psi''(0)^{-1})C_{\Phi} N^{-1} \|\lambda - \lambda^*\|_{\infty}.
\] (A.15)

From inequalities (A.12) and (A.15), and \( c_R = \max \{2d_1 \sigma^{-1}, 2(C_{\Psi} - 2\psi''(0)^{-1})C_{\Phi} \} \), we have

\[
\max \left\{ \|\hat{x} - x^*\|_{\infty}, \|\lambda - \lambda^*\|_{\infty} \right\} \leq 2C_{\Phi} \epsilon + c_R N^{-1} \|\lambda - \lambda^*\|_{\infty}.
\] (A.16)

Proof of Lemma 4.16. For all \( k \geq 1 \), we iteratively apply Eq. (A.16) to obtain

\[
\|x^k - x^*\|_{\infty} \leq (1 - (c_R N^{-1})^k) 2C_{\Phi} \epsilon / (1 - c_R N^{-1}) + (c_R N^{-1})^k \|\lambda^0 - \lambda^*\|_{\infty}
\leq 2C_{\Phi} \epsilon / (1 - c_R N^{-1}) + c_R N^{-1} \|\lambda^0 - \lambda^*\|_{\infty}.
\]

A.2 Proof of Lemma 4.17

We divide this proof into two parts.

(i) At iteration \( k = 0 \), we obtain \( x^1 \) from running \( A \) with \( J_0^{SL}(K, \epsilon, \delta) \) iterations. We show that \( x^1 \)
Moreover, using the definition of $\lambda_0$ in Eq. (4.3), we have
\[ P \left( \| \nabla_x \mathcal{L}_N(x^1, \lambda^0) \|_\infty > \epsilon \right) \leq P \left( \| \nabla_x \mathcal{L}_N(x^1, \lambda^0) \|_2 > \epsilon \right) \]
\[ \leq P \left( \| x^1 - x^*_1(\lambda^0) \|_2 > \epsilon / L_N(\lambda^0) \right) \]
\[ = \mathbb{E} \left[ \| x^1 - x^*_1(\lambda^0) \|_2^2 / (\epsilon / L_N(\lambda^0))^2 \right] \]
\[ \leq A \left( \| x^0 - x^* \|_\infty^2 + \| x^*_1(\lambda^0) - x^* \|_\infty^2 \right) + B \]
\[ \leq \frac{nA \left( \| x^0 - x^* \|_\infty^2 + \| x^*_1(\lambda^0) - x^* \|_\infty^2 \right) + B}{J_{0}^{SL}(K, \epsilon, \delta)(\epsilon / L_N(\lambda^0))^2}, \] (A.17)

where (a) follows from $\| . \|_\infty \leq \| . \|_2$, (b) holds due to $\nabla_x \mathcal{L}_N(\cdot, \lambda^0)$ is Lipschitz continuous with parameter $L_N(\lambda^0)$, (c) follows from Markov’s inequality, (d) is due to Assumption 4.8(i), (e) follows from triangle inequality, (f) is due to $\| x \|_2^2 \leq n \| x \|_\infty^2$ for any $x \in \mathbb{R}^n$.

Now, using $\| x^*_1(\lambda^0) - x^* \|_\infty \leq cR N^{-1} \| \lambda^0 - \lambda^*_\infty \|$ from Theorem 4.15 and the definition of $J_{0}^{SL}(K, \epsilon, \delta)$ in Eq. (4.3), we have $P \left( \| \nabla_x \mathcal{L}_N(x^1, \lambda^0) \|_\infty > \epsilon \right) \leq \left( 1 - (1 - \delta)^{1/K} \right)$. (ii) At iteration $k_0 \geq 1$, suppose the solution returned from the subroutine $\mathcal{A}$ satisfies Eq. (3.2) in all previous iterations, we run $\mathcal{A}$ with $J_{k_0}^{SL}(K, \epsilon, \delta)$ iterations at the current iteration. In the following, we show that $x^{k_0+1}$ satisfies Eq. (3.2) with probability at least $(1 - \delta)^{1/K}$. First, using the same argument as in inequality (A.17), we have
\[ P \left( \| \nabla_x \mathcal{L}_N(x^{k_0+1}, \lambda^{k_0}) \|_\infty > \epsilon \right) \leq \frac{nA \left( \mathbb{E} \left[ \| x^{k_0} - x^* \|_\infty^2 \right] + \mathbb{E} \left[ \| x^{k_0+1}(\lambda^{k_0}) - x^* \|_\infty^2 \right] \right) + B}{J_{k_0}^{SL}(K, \epsilon, \delta)(\epsilon / L_N(\lambda^{k_0}))^2}. \]

Now, due to the assumption that the solution returned from the subroutine $\mathcal{A}$ satisfies Eq. (3.2) in all previous iterations, we iteratively apply Theorem 4.15 to obtain
\[ \| x^{k_0} - x^* \|_\infty \leq (1 - (cR N^{-1})^{k_0}) \| x^{k_0} - x^* \|_\infty \leq (1 - (cR N^{-1})^{k_0}) \| x^{k_0} - x^* \|_\infty \leq (1 - (cR N^{-1})^{k_0}) \| x^{k_0} - x^* \|_\infty \leq \left( 1 - (1 - \delta)^{1/K} \right). \]

Moreover, using the definition of $J_k(K, \epsilon, \delta)$ in Eq. (4.4), we have $P \left( \| \nabla_x \mathcal{L}_N(x^{k_0+1}, \lambda^{k_0}) \|_\infty > \epsilon \right) \leq \left( 1 - (1 - \delta)^{1/K} \right)$. 

**B Randomized First-order Algorithms with Generic Sampling**

We recall that Problem $\mathbb{P}_N(\lambda)$ can be rewritten as:
\[ \min_{x \in \mathcal{X}} \left\{ \mathcal{L}_N(x, \lambda) = \sum_{i \in [m]} (\lambda_i / \| \lambda \|_1) f_i^N(x; \lambda) \right\}. \] (B.1)
As stated in Subsection 4.1, for any fixed $\lambda \in \mathbb{R}_+^m$, the primal update $\min_{x \in \mathcal{X}} \mathcal{L}_N(x, \lambda)$ has a unique solution because $f(\cdot)$ is strongly convex.

We can rewrite the optimality condition of Problem (B.1) in a more convenient form using the monotone operators $B_i(x, \lambda) \triangleq \nabla_x f_i^N(x; \lambda) = \nabla f(x) - \|\lambda\|_1 \psi'(N g_i(x)) \nabla g_i(x)$ for all $i \in [m]$. Then, the unique solution of Problem (B.1) satisfies the monotone inclusion:

$$0 \in \partial I_{\mathcal{X}}(x) + B(x, \lambda),$$

where $B(x, \lambda) \triangleq \sum_{i \in [m]} \varphi_i(\lambda) B_i(x, \lambda)$. We have $\langle B(x, \lambda) - B(y, \lambda), x - y \rangle \geq \mu_f \|x - y\|_2^2$ for all $x, y \in \mathbb{R}^n$ by strong convexity of $f$. We recall that by Assumption 4.6, $L_N(\lambda) > 0$ is an upper bound on the Lipschitz constant of $B_i(\cdot, \lambda)$ for all $i \in [m]$ over $\mathcal{X}$.

We adopt a generic framework for $\mathcal{A}^Q$ which encompasses unbiased first-order methods like SGD, SVRG, SAGA, etc. Let $\{y_t\}_{t \geq 0}$ denote the iterates of $\mathcal{A}^Q$, and let $\varphi^t = \{\varphi_i^t\}_{i \in [m]}$ be a collection of auxiliary variables (which serve as proxies for past gradient evaluations).

Suppose that $\{I_t\}_{t \geq 0}$ are i.i.d. following a probability distribution $Q = (q_i)_{i \in [m]} \in \mathcal{P}_+([m])$. We define additional operators $A_i(x, \lambda) \triangleq \varphi_i(\lambda) B_i(x, \lambda)/q_i$ for all $i \in [m]$. We will later need the fact that each $A_i(\cdot, \lambda)$ is $\varphi_i(\lambda)L_N(\lambda)/q_i$-Lipschitz continuous for all $i \in [m]$. Moreover, we define a gradient estimator

$$\mathcal{G}(y_t, \varphi^t, I_t) \triangleq A_t(y_t, \lambda) - \varphi^t_t + \sum_{i \in [m]} q_i \varphi_i^t_i, \quad \forall t \geq 0,$$

which is an unbiased estimator of $B(y_t, \lambda)$ (see the formal statement in Lemma B.1).

The primal sequence $\{y_t\}_{t \geq 0}$ of $\mathcal{A}^Q$ is updated according to:

$$y_{t+1} = \Pi_{\mathcal{X}}[y_t - \gamma_t \mathcal{G}(y_t, \varphi^t, I_t)], \quad \forall t \geq 0,$$

for step-sizes $\{\gamma_t\}_{t \geq 0}$ and the auxiliary variables are updated according to some generic scheme:

$$\varphi^{t+1} = \mathcal{U}_t(y_t, \varphi^t, I_t), \quad \forall t \geq 0.$$

Let $\mathcal{F}_t \triangleq \sigma\left(y_0, \varphi^0, I_0, \ldots, y_{t-1}, \varphi^{t-1}, I_{t-1}, y_t, \varphi^t\right)$ denote the history of $\mathcal{A}^Q$ up to iteration $t$, which forms a filtration.

### B.1 Basic Properties

We first confirm that this construction of $\mathcal{G}(y_t, \varphi^t, I_t)$ is an unbiased estimator.

**Lemma B.1.** Fix $\lambda \in \mathbb{R}_+^m$ and $Q = \mathcal{P}_+([m])$, then $\mathbb{E}^Q[\mathcal{G}(y_t, \varphi^t, I_t) \mid \mathcal{F}_t] = B(y_t, \lambda)$ for all $t \geq 0$.

The expected distance to the solution $x^*(\lambda)$ of Eq. (B.2) contracts after each iteration, depending on the conditional variance of $\mathcal{G}(y_t, \varphi^t, I_t)$.

**Lemma B.2.** Fix $\lambda \in \mathbb{R}_+^m$ and $Q = \mathcal{P}_+([m])$, and let $\{y_t\}_{t \geq 0}$ be produced by Eq. (B.4) using $\mathcal{A}^Q$. Then, for all $t \geq 0$,

$$\mathbb{E}^Q \left[ \|y_{t+1} - x^*(\lambda)\|_2^2 \mid \mathcal{F}_t \right] \leq (1 - 2\gamma_t \mu_f + \gamma_t^2 L_N(\lambda)^2) \|y_t - x^*(\lambda)\|_2^2 + \gamma_t^2 \mathbb{E}^Q \left[ \|\mathcal{G}(y_t, \varphi^t, I_t) - B(y_t, \lambda)\|_2^2 \mid \mathcal{F}_t \right].$$
Proof. Recall that \( x^*(\lambda) \) is the unique solution of Eq. \( (B.2) \), i.e., \( x^*(\lambda) = \Pi_X [x^*(\lambda) - \gamma B(x^*(\lambda), \lambda)] \). It follows that:

\[
\|y_{t+1} - x^*(\lambda)\|_2^2 \\
= \|\Pi_X[y_t - \gamma_t \mathcal{G}(y_t, \varphi^t, I_t)] - \Pi_X[x^*(\lambda) - \gamma B(x^*(\lambda), \lambda)]\|_2^2 \\
\leq \|(y_t - \gamma_t \mathcal{G}(y_t, \varphi^t, I_t)) - (x^*(\lambda) - \gamma_t B(x^*(\lambda), \lambda))\|_2^2 \\
= \|(y_t - x^*(\lambda)) - \gamma_t (B(y_t, \lambda) - B(x^*(\lambda), \lambda)) - \gamma_t (\mathcal{G}(y_t, \varphi^t, I_t) - B(y_t, \lambda))\|_2^2 \\
= \|y_t - x^*(\lambda)\|_2^2 + \gamma_t^2 \|B(y_t, \lambda) - B(x^*(\lambda), \lambda)\|_2^2 + \gamma_t^2 \|\mathcal{G}(y_t, \varphi^t, I_t) - B(y_t, \lambda)\|_2^2 \\
- 2\gamma_t\langle B(y_t, \lambda) - B(x^*(\lambda), \lambda), y_t - x^*(\lambda) \rangle \\
+ 2\gamma_t^2 \langle B(y_t, \lambda) - B(x^*(\lambda), \lambda), \mathcal{G}(y_t, \varphi^t, I_t) - B(y_t, \lambda), y_t - x^*(\lambda) \rangle,
\]

where the inequality follows from non-expansiveness of the projection operator. Since \( \mathcal{G}(y_t, \varphi^t, I_t) \) is a (conditionally) unbiased estimator of \( B(y_t, \lambda) \) by Lemma B.1, we have

\[
\mathbb{E}_Q \left[ \|y_{t+1} - x^*(\lambda)\|_2^2 \mid F_t \right] \\
\leq \|y_t - x^*(\lambda)\|_2^2 + \gamma_t^2 \|B(y_t, \lambda) - B(x^*(\lambda), \lambda)\|_2^2 + \gamma_t^2 \mathbb{E}_Q \left[ \|\mathcal{G}(y_t, \varphi^t, I_t) - B(y_t, \lambda)\|_2^2 \mid F_t \right] \\
\leq (1 - 2\gamma_t \mu_f + \gamma_t^2 L_N(\lambda)^2) \|y_t - x^*(\lambda)\|_2^2 + \gamma_t^2 \mathbb{E}_Q \left[ \|\mathcal{G}(y_t, \varphi^t, I_t) - B(y_t, \lambda)\|_2^2 \mid F_t \right],
\]

where the second inequality is due to strong monotonicity of \( B(\cdot, \lambda) \) and \( L_N(\lambda) \)-Lipschitz continuity of \( B(\cdot, \lambda) \).

We can upper bound the conditional variance of \( \mathcal{G}(y_t, \varphi^t, I_t) \).

**Lemma B.3.** Fix \( \lambda \in \mathbb{R}^n_+ \) and \( Q \in \mathcal{P}_+(|m|) \), and let \( \{y_t\}_{t \geq 0} \) be produced by Eq. \( (B.4) \) using \( A^Q \). Then, for all \( t \geq 0 \),

\[
\mathbb{E}_Q \left[ \|\mathcal{G}(y_t, \varphi^t, I_t) - B(y_t, \lambda)\|_2^2 \mid F_t \right] \leq 2 \left( r_Q L_N(\lambda)^2 \|y_t - x^*(\lambda)\|_2 + \sum_{i \in [m]} q_i \|\varphi_i - B_i(x^*(\lambda), \lambda)\|_2^2 \right).
\]

**Proof.** Using \( \mathbb{E}_Q [A_{I_t}(y_t, \lambda) \mid F_t] = \sum_{i \in [m]} \lambda_i B_i(y_t, \lambda) = B(y_t, \lambda) \) and \( \mathbb{E}_Q [\varphi_{I_t} \mid F_t] = \sum_{i \in [m]} q_i \varphi_i \), we have

\[
\mathbb{E}_Q \left[ \|\mathcal{G}(y_t, \varphi^t, I_t) - B(y_t, \lambda)\|_2^2 \mid F_t \right] = \mathbb{E}_Q \left[ \|A_{I_t}(y_t, \lambda) - \varphi_{I_t} - \mathbb{E}_Q [A_{I_t}(y_t, \lambda) - \varphi_{I_t} \mid F_t]\|_2^2 \mid F_t \right] \\
\leq \mathbb{E}_Q \left[ \|A_{I_t}(y_t, \lambda) - \varphi_{I_t}\|_2^2 \mid F_t \right],
\]

because the conditional variance is bounded by the conditional second moment. We further bound the term \( \mathbb{E}_Q \left[ \|A_{I_t}(y_t, \lambda) - \varphi_{I_t}\|_2^2 \mid F_t \right] \) as follows:

\[
\mathbb{E}_Q \left[ \|A_{I_t}(y_t, \lambda) - \varphi_{I_t}\|_2^2 \mid F_t \right] = \mathbb{E}_Q \left[ \|(A_{I_t}(y_t, \lambda) - A_{I_t}(x^*(\lambda), \lambda)) - (\varphi_{I_t} - A_{I_t}(x^*(\lambda), \lambda))\|_2^2 \mid F_t \right] \\
\leq 2 \mathbb{E}_Q \left[ \|A_{I_t}(y_t, \lambda) - A_{I_t}(x^*(\lambda), \lambda)\|_2^2 + \|\varphi_{I_t} - A_{I_t}(x^*(\lambda), \lambda)\|_2^2 \mid F_t \right] \\
\leq 2 \left( r_Q L_N(\lambda)^2 \|y_t - x^*(\lambda)\|_2 + \sum_{i \in [m]} q_i \|\varphi_i - A_i(x^*(\lambda), \lambda)\|_2^2 \right),
\]

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where the first inequality follows from the fact that \((a - b)^2 \leq 2(a^2 + b^2)\), and the second inequality is due to the \(\varphi_i(\lambda)L_N(\lambda)/q_i\)-Lipschitz continuity of each \(A_i(\cdot, \lambda)\) for all \(i \in [m]\).

Combining Lemmas \[B.2\] and \[B.3\] gives the following result.

**Corollary B.4.** Fix \(\lambda \in \mathbb{R}^m_+\) and \(Q \in \mathcal{P}_+([m])\), and let \(\{y_t\}_{t \geq 0}\) be produced by Eq. \[B.4\] using \(A_Q\). Then, for all \(t \geq 0\),

\[
\mathbb{E}^Q \left[ \|y_{t+1} - x^*(\lambda)\|^2 \right] \\
\leq (1 - 2\gamma_t \mu_f + (1 + 2r^Q) \gamma_t^2 L_N(\lambda)^2) \mathbb{E}^Q \left[ \|y_t - x^*(\lambda)\|^2 \right] + 2\gamma_t^2 \mathbb{E}^Q \left[ \sum_{i \in [m]} q_i \|\varphi_i^t - A_i(x^*(\lambda), \lambda)\|^2 \right].
\]

**Remark B.5.** (i) If we take \(Q = \varphi(\lambda)\), then \(q_i = \varphi_i(\lambda)\) for all \(i \in [m]\) and \(r^Q = \sum_{i \in [m]} \varphi_i(\lambda) = 1\).

(ii) If \(Q\) is uniform, then \(q_i = 1/m\) for all \(i \in [m]\) and \(r^Q = \sum_{i \in [m]} m \varphi_i(\lambda)^2 = m\|\lambda\|^2/\|\lambda\|^2\).

### B.2 SGD with Generic Sampling

SGD has a sublinear convergence rate in expectation.

**Theorem B.6.** Let \(M_B \triangleq \max_{i \in [m]} \|B_i(x^*(\lambda), \lambda)\|_2\), and suppose \(A_Q^Q_{SGD}\) is run with decreasing stepsizes \(\gamma_t = \frac{2}{\mu_f(t + 2(1 + 2r^Q)L_N(\lambda)^2/\mu_f)}\) for all \(t \geq 0\). Then,

\[
\mathbb{E}^Q \left[ \|y_t - x^*(\lambda)\|^2 \right] \leq \frac{(2(1 + 2r^Q)L_N(\lambda)^2/\mu_f^2 - 1)\|y_0 - x^*(\lambda)\|^2 + 8r^Q M_B^2/\mu_f^2}{t}, \quad \forall t \geq 1.
\]

**Proof.** Using Corollary \[B.3\] for all \(t \geq 0\) we have

\[
\mathbb{E}^Q \left[ \|y_{t+1} - x^*(\lambda)\|^2 \right] \leq (1 - 2\gamma_t \mu_f + (1 + 2r^Q) \gamma_t^2 L_N(\lambda)^2) \mathbb{E}^Q \left[ \|y_t - x^*(\lambda)\|^2 \right] + 2\gamma_t^2 r^Q M_B^2.
\]

With the above choice of stepsizes \(\{\gamma_t\}_{t \geq 0}\), we have

\[
1 - 2\gamma_t \mu_f + (1 + 2r^Q) \gamma_t^2 L_N(\lambda)^2 = 1 - \gamma_t \mu_f - (1 + 2r^Q) \gamma_t L_N(\lambda)^2 \left( \frac{\mu_f}{(1 + 2r^Q)L_N(\lambda)^2} - \gamma_t \right) \leq 1 - \gamma_t \mu_f.
\]

Therefore, for the constant \(c \triangleq 2(1 + 2r^Q)L_N(\lambda)^2/\mu_f^2\), we can write

\[
\mathbb{E}^Q \left[ \|y_{t+1} - x^*(\lambda)\|^2 \right] \leq (1 - c \gamma_t) \mathbb{E}^Q \left[ \|y_t - x^*(\lambda)\|^2 \right] + 2\gamma_t^2 r^Q M_B^2 \frac{\mu_f}{\mu_f^2(t + c)^2} M_B^2.
\]

Applying the above inequality recursively gives

\[
\mathbb{E}^Q \left[ \|y_t - x^*(\lambda)\|^2 \right] \leq \frac{(c - 1)(c - 2)}{t(t - 1)(t - 2 + c)} \|y_0 - x^*(\lambda)\|^2 + \frac{8r^Q M_B^2}{\mu_f^2(t - 1 + c)(t - 2 + c)} \sum_{l=0}^{t-1} \frac{l + c - 1}{l + c}
\]

\[
\leq \left( (c - 1) \|y_0 - x^*(\lambda)\|^2 + 8r^Q M_B^2/\mu_f^2 \right)/t,
\]

where the last inequality follows from the fact that \(\frac{t(c - 1)}{t + c} \leq 1\) for all \(l \in [t]_0\), \(t(c - 2) \leq (t - 1 + c)(t - 2 + c)\) and \(t^2 \leq (t - 1 + c)(t - 2 + c)\).
B.3 SVRG with Generic Sampling

SVRG has a linear convergence rate in expectation. For a constant step-size $\gamma_t = \gamma$ for all $t \geq 0$, we let $
abla_{SVRG}^Q(\gamma) \triangleq 1 - 2\gamma(1 + 2rQ + 2Mr^2)L_N(\lambda)^2$ be the contraction factor for the iterates of SVRG.

**Theorem B.7.** Suppose $A_{SVRG}^Q$ is run with a constant stepsize $\gamma < \frac{2\mu}{(1 + 2rQ + 2Mr^2)L_N(\lambda)^2}$. Then $\nabla_{SVRG}^Q(\gamma) \in (0, 1)$ and

$$\mathbb{E}^Q \left[ \|y_j - x^*(\lambda)\|^2 \right] \leq \nabla_{SVRG}^Q(\gamma)^j \mathbb{E}^Q \left[ \|y_0 - x^*(\lambda)\|^2 \right], \quad \forall j \geq 1.$$

**Proof.** Recall that we denote the inner iterates as $\{\tilde{y}_l\}_{l \geq 0}$, where $M_j \leq l < M(j + 1)$ correspond to epoch $j \geq 1$. In SVRG, the proxies $\{\phi_i^l\}_{i \in [m]}$ are only updated at the beginning of each epoch. Since $\{\phi_i^l\}_{i \in [m]} = \{A_i(\tilde{y}_M, \lambda)\}_{i \in [m]}$ for $M_j \leq l < M(j + 1)$, we have:

$$\sum_{i \in [m]} q_i \|\phi_i^l - A_i(x^*(\lambda), \lambda)\|^2 = \sum_{i \in [m]} q_i \|A_i(\tilde{y}_M, \lambda) - A_i(x^*(\lambda), \lambda)\|^2 \leq rQ L_N(\lambda)^2 \|\tilde{y}_M - x^*(\lambda)\|^2,$$

where the inequality is due to the $\phi_i(\lambda)L_N(\lambda)/q_i$-Lipschitz continuity of each $A_i$ for all $i \in [m]$. Using Corollary B.4 and defining $\overline{\mu} \triangleq 1 - 2\gamma(1 + 2rQ + 2Mr^2)L_N(\lambda)^2$, we immediately have that for all $M_j \leq l < M(j + 1)$,

$$\mathbb{E}^Q \left[ \|\tilde{y}_{l+1} - x^*(\lambda)\|^2 \right] \leq \overline{\mu} \mathbb{E}^Q \left[ \|\tilde{y}_l - x^*(\lambda)\|^2 \right] + 2\gamma^2 rQ L_N(\lambda)^2 \mathbb{E}^Q \left[ \|\tilde{y}_M - x^*(\lambda)\|^2 \right]. \quad (B.6)$$

If the stepsize satisfies $\gamma < \frac{2\mu}{(1 + 2rQ + 2Mr^2)L_N(\lambda)^2}$, then both $\nabla_{SVRG}^Q(\gamma), \overline{\mu} \in (0, 1)$. By recursively applying inequality (B.6), we obtain:

$$\mathbb{E}^Q \left[ \|\tilde{y}_{M(j+1)} - x^*(\lambda)\|^2 \right] \leq \overline{\mu}^M \mathbb{E}^Q \left[ \|\tilde{y}_{M(j)} - x^*(\lambda)\|^2 \right] + 2\gamma^2 rQ L_N(\lambda)^2 \mathbb{E}^Q \left[ \|\tilde{y}_M - x^*(\lambda)\|^2 \right] \sum_{i=0}^{M-1} \overline{\mu}^i \leq (\overline{\mu} + 2M\gamma^2 rQ L_N(\lambda)^2) \mathbb{E}^Q \left[ \|\tilde{y}_{M(j)} - x^*(\lambda)\|^2 \right] = \nabla_{SVRG}^Q(\gamma) \mathbb{E}^Q \left[ \|\tilde{y}_{M(j)} - x^*(\lambda)\|^2 \right]. \quad (B.7)$$

The desired result then follows by recursively applying inequality (B.7).