Analysis and control of integro-differential Volterra equations with delays

Youness El Kadiri · Said Hadd · Hamid Bounit

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Abstract
We present a novel approach to address integro-differential systems incorporating state, input, and output delays. Our approach leverages product spaces and employs a boundary perturbation technique. Initially, we focus on state-delay equations, wherein we introduce a variation of constants formula for the mild solution. Additionally, we establish spectral properties using a characteristic equation. Subsequently, we extend our analysis to integro-differential systems affected by state, input, and output delays. Notably, we demonstrate the equivalence between such delay systems and regular free-delay systems within the Salamon–Weiss framework. This equivalence sheds valuable insights on the nature of the integro-differential system under consideration.

Keywords Integro-differential equations · Delay systems · Semigroup · Perturbation

1 Introduction

Integro-differential equations have been a subject of considerable interest among researchers for numerous years, resulting in a wealth of intriguing studies and contributions. Noteworthy works in this field include references such as [4, 18, 19, 21] and a host of other relevant sources. Within this class of evolution equations, it is...
particularly fascinating to observe the simultaneous presence of both differential and integral operators. This amalgamation of operators adds complexity and richness to the mathematical formulation and analysis of these equations.

In the initial part of this study, our focus lies in presenting a unified semigroup approach for addressing a specific class of integro-differential equations characterized by the presence of state delays. In fact, we consider the equation

\[
\begin{cases}
\dot{x}(t) = Ax(t) + \int_0^t a(t-s)Ax(s)ds + Lx + f(t), \\
x(0) = x, \
x_0 = \varphi
\end{cases},
\]

where \( A \) is a generator of a strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach space \( X \), \( x \in X \), \( \varphi \in L^p([-r, 0], X) \), \( a(\cdot) \in W^{1,p}(\mathbb{R}_+ \setminus \{0\}, \mathbb{C}) \), \( p \in (1, \infty) \), and \( L \) is a Riemann–Stieltjes integral of the form

\[
L\varphi := \int_{-r}^0 d\mu^L(\theta)\varphi(\theta), \quad \varphi \in W^{1,p}([-r, 0], X),
\]

with \( \mu^L : [-r, 0] \to \mathscr{L}(X) \) being a function of bounded variation which is continuous at 0, and has a total variation \( |\mu^L| \) (a positive Borel measure) satisfying \( |\mu^L|([-\epsilon, 0]) \to 0 \) as \( \epsilon \to 0 \). Here for each \( t \geq 0 \), the history function \( x_t : [-r, 0] \to X \) is defined by \( x_t(\theta) = x(t + \theta) \) for \( \theta \in [-r, 0] \).

It is worth noting that a comprehensive theory already exists for addressing the problem (1) in two specific cases: when \( a \equiv 0 \) (representing standard delay equations) and when \( L \equiv 0 \) (representing free-delay integro-differential equations). Extensive studies on standard delay equations can be found in references such as [1] and [10], while investigations on free-delay integro-differential equations are well-documented in works such as [5, 6], [7, Chap. 7], and [19–21]. In these cases, a commonly employed technique involves the utilization of product spaces and matrix operators, which is further detailed in references like [1] and [7, Chap. 7].

In our present study, we will also employ matrix operators and draw upon the feedback theory of infinite-dimensional linear systems [26]. These tools will facilitate our exploration of the well-posedness and enable the development of a spectral theory for the Eq. (1).

By considering the product space

\[
\mathcal{X} := X \times L^p(\mathbb{R}_+ \setminus \{0\}, X) \times L^p([-r, 0], X),
\]

we can reformulate the problem (1) as a standard Cauchy problem in \( \mathcal{X} \) of the form

\[
\dot{z}(t) = \mathfrak{A}_Lz(t), \quad z(0) = \left( \begin{array}{c}
x \\ f \\
\varphi
\end{array} \right), \quad t \geq 0,
\]

where \( \mathfrak{A}_L : D(\mathfrak{A}_L) \subset \mathcal{X} \to \mathcal{X} \) is defined in (22).

Based on this transformation, we define the well-posedness of the delay-integro-differential Eq. (1) as the existence of a strongly continuous semigroup \((\mathcal{T}_L(t))_{t \geq 0}\) on
\( \mathcal{L} \) generated by the operator \( \mathfrak{A}_L \). To establish that \( \mathfrak{A}_L \) acts as a generator on \( \mathcal{X} \), we employ the decomposition

\[
\mathfrak{A}_L = \mathfrak{A} + \mathcal{L},
\]

with the operators \( \mathcal{L} \) and \( \mathfrak{A} \) defined in (23) and (24), respectively. Firstly, we utilize a perturbation theorem from [14] to prove that the operator \( \mathfrak{A} \) generates a strongly continuous semigroup on \( \mathcal{X} \) (refer to Theorem 2). Secondly, based on regular linear systems, we demonstrate that the operator \( \mathcal{L} \) serves as a Miyadera–Voigt perturbation for \( \mathfrak{A} \). Finally, by applying [7, Chap. 3], we conclude that \( \mathfrak{A}_L \) acts as the generator of a strongly continuous semigroup on \( \mathcal{X} \).

Consider the operator \( L_0 \) defined in (12), and let \( L_{0,\Lambda} \) denote the Yosida extension of \( L_0 \) with respect to the left shift semigroup on \( L^p([-r, 0], X) \) (refer to (8) for the precise definition). We establish (refer to Theorem 5) that the solution to the aforementioned Cauchy problem (and hence the delay-integro-differential equation (1)) can be expressed as

\[
z(t) = \begin{pmatrix} x(t) \\ v(t, \cdot) \\ x_t \end{pmatrix}, \quad t \geq 0,
\]

where the function \( t \mapsto x(t) \) is given by the variation of parameters formula

\[
x(t) = R(t)x + \int_0^t R(t-s)(L_{0,\Lambda}x_s + f(s))ds,
\]

and \( v(t, \cdot) \) represents the solution of the boundary system (11). In this context, \( (R(t))_{t \geq 0} \) corresponds to the resolvent family associated with the free-delay integro-differential equation problem (see Definition (3)).

In terms of the spectral theory for the generator \( \mathfrak{A}_L \), we have established in Theorem 6 that for \( \lambda \in \mathbb{C}_0 \cap \rho(\mathfrak{A}^{b_0}) \cap \rho(\mathfrak{A}_0) \), the following equivalence holds:

\[
\lambda \in \rho(\mathfrak{A}_L) \iff \lambda \in \rho\left(1 + \hat{a}(\lambda)A + Le_\lambda\right),
\]

where \( \hat{a} \) represents the Laplace transform of \( a \), and \( \mathfrak{A}^{b_0} \) and \( \mathfrak{A}_0 \) correspond to the operators defined in (15). It is worth noting that when \( a(\cdot) \equiv 0 \), we obtain the same spectral result as presented in [1]. Similarly, when \( L \equiv 0 \), we identify the operators \( \mathfrak{A}_L \) and \( \mathfrak{A}^{b_0} \), enabling us to retrieve the well-known results on the spectrum of \( \mathfrak{A}^{b_0} \) as documented in [7, Chap. 7] and [21].

In the second part of this paper (refer to Sect. 5), we establish that integro-differential systems with state, input, and output delays (as given by equation (38)) constitute a subclass of infinite-dimensional linear systems in the Salamon-Weiss sense [26]. This expansion builds upon existing results for standard delay systems [11, 13], neutral systems [3], and extends the understanding of these systems. It is worth noting that the control theory of free-delay integro-differential equations with unbounded control.
operators was initially explored in [17], and has since been further investigated in references such as [2, 15, 16].

The paper is organized as follows: Section 2 provides a concise background on the feedback theory of regular linear systems in the Salamon-Weiss sense. Section 3 focuses on demonstrating the well-posedness of the problem (1). In Sect. 4, we present a comprehensive study of the spectral properties of state-delay integro-differential equations. The final section deals with the reformulation of integro-differential systems with delays in the state, control, and observation, into free-delay infinite-dimensional linear systems with unbounded control and observation operators.

Notation Throughout the paper we shall frequently use the following symbols. Let (X, ||·||) be a Banach space and γ a real constant. For p ∈ [1, ∞), we denote by $L^p_γ(\mathbb{R}_+, X)$ the space of functions $f : \mathbb{R}_+ \rightarrow X$, such that $t \mapsto e^{−γt} f(t)$ is Bochner integrable on $\mathbb{R}_+$. $W^{1,p}(\mathbb{R}_+, X)$ is the Sobolev space associated to $L^p(\mathbb{R}_+, X) := L^p_0(\mathbb{R}_+, X)$. Given two Banach spaces X and Y, we denote the space of bounded linear operators $X \rightarrow Y$ by $\mathcal{L}(X, Y)$, and for $G \in \mathcal{L}(X, Y)$, $\|G\|_{\mathcal{L}(X, Y)}$ means its operator norm (some time we only write $\|G\|$). We shall omit the subscripts if no confusion is possible. The identity operator will be denoted by I. The Laplace transform of a function $f \in L^1_γ(\mathbb{R}_+, X)$ is defined by $\hat{f}(\lambda) = \int_0^\infty e^{−\lambda t} f(t)dt$ for $\lambda > γ$. If $G : \mathbb{R}_+ \rightarrow \mathcal{L}(X, Y)$ is an operator valued function such that for each $x \in X$ the function $G(t)x$ is in $L^1_γ(\mathbb{R}_+, Y)$, then we define the linear operator $\hat{G}(\lambda)$ by $\hat{G}(\lambda)x = \int_0^\infty e^{−\lambda t} G(t)x dt$. Moreover, we shall use convolution of operator valued functions: Let $X, Y, Z$ be Banach spaces, and $G : \mathbb{R}_+ \rightarrow \mathcal{L}(X, Y)$ and $F : \mathbb{R}_+ \rightarrow \mathcal{L}(Y, Z)$ functions such that $G(t)x$ and $F(t)x$ are measurable for all $x \in X$ or Y, respectively, and $\|G(t)\|, \|F(t)\|$ are in $L^1_γ(\mathbb{R}_+, \mathbb{R})$. Then $F * G : \mathbb{R}_+ \rightarrow \mathcal{L}(X, Z)$ is defined by $(F * G)(t)x = \int_0^t F(t−s)G(s)x ds$ for $x \in X$. One can prove that $(\hat{F} * \hat{G}) = \hat{F} \cdot \hat{G}$. For a closed, linear operator $A : D(A) \subset X \rightarrow X$, the resolvent set of A is given by

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda − A : D(A) \subset X \rightarrow X \text{ is bijective}\},$$

and its spectrum is $\sigma(A) = \mathbb{C}\setminus \rho(A)$.

Let $A : D(A) \subset X \rightarrow X$ be a generator of a strongly continuous semigroup $T : (T(t))_{t \geq 0}$ on X. We define a new norm on X by $\|x\|_{−1} := \|R(\alpha, A)x\|$ for $x \in X$ and some (hence all) $\alpha \in \rho(A)$. The completion of X with respect to the norm $\|\cdot\|_{−1}$ is a Banach space denoted by $X_{−1}$. Moreover, the semigroup T can be extended to another strongly continuous semigroup $T_{−1} := (T_{−1}(t))_{t \geq 0}$ on $X_{−1}$, whose generator $A_{−1} : X \rightarrow X_{−1}$ is the extension of A to X. For more details on extrapolation theory we refer to [7, Chap. 2].

2 Background on regular linear systems

In this section, we will provide a brief overview of regular linear systems. For a more comprehensive understanding of this theory, we recommend referring to [26] for the case of Hilbert space, and [22] for the case of Banach spaces.
Let us consider Banach spaces $X$, $U$, and $Z$, where $Z$ is densely and continuously embedded in $X$. We denote $A_m : Z \to X$ as a closed operator on $X$, and $G : Z \to U$ as a linear operator, commonly known as the trace operator. Our focus lies on the boundary control problem described as follows:

$$\begin{align*}
\dot{z}(t) &= A_m z(t), \quad z(0) = x, \quad t > 0, \\
Gz(t) &= u(t), \quad t \geq 0,
\end{align*}$$

where $x \in X$ and $u \in L^p([0, +\infty), U)$. Under the given conditions, namely:

1. ($H1$) $A := A_m$ with the domain $D(A) = \ker(G)$ generates a strongly continuous semigroup $T = (T(t))_{t \geq 0}$,
2. ($H2$) $G$ is surjective,

it has been shown in [8] that the inverse $D_\lambda := (G|_{\ker(\lambda - A_m)})^{-1} \in \mathcal{L}(U, X), \quad \lambda \in \rho(A)$, exists. Furthermore, by selecting $B := (\lambda - A_m^{-1}) D_\lambda$ for $\lambda \in \rho(A)$, we obtain the relationship $A_m = (A_m^{-1} + BG)|_Z$.

Consequently, the boundary control problem (3) can be reformulated as a distributed system represented by the following equation:

$$\begin{align*}
\dot{z}(t) &= A_m^{-1} z(t) + Bu(t), \quad t \geq 0, \\
z(0) &= z.
\end{align*}$$

The mild solution of the distributed system (4) can be expressed as follows:

$$z(t) = T(t)x + \int_0^t T^{-1}(t-s)Bu(s)ds = T(t)x + \Phi^A_B u$$

for any $t \geq 0$, $x \in X$, and $u \in L^p([0, +\infty), U)$. It is important to note that this solution takes its values in $X_{-1}$. However, for practical purposes, it is more convenient to consider control operators $B \in \mathcal{L}(U, X_{-1})$ that ensure the solution $z(t) \in X$ for any $t \geq 0$, initial condition $z \in X$, and control function $u \in L^p([0, +\infty), U)$.

To guarantee this property, we require the existence of a $\tau > 0$ such that $\Phi^A_B \in X$ for any $u \in L^p([0, +\infty), U)$, as stated in [25]. Remarkably, this condition also implies that $\Phi^A_B \in \mathcal{L}(L^p([0, +\infty), U), X)$ for any $t \geq 0$, thanks to the closed graph theorem. When this condition holds, we refer to $B$ as an admissible control operator for $A$.

Now, let us consider the observation function given by

$$y(t) = Cz(t), \quad t \geq 0,$$
where \( t \mapsto z(t) \) represents the solution of (4), and \( C : Z \to U \) is a linear operator.

The system composed of Eqs. (4) and (6) is considered well-posed if the function \( t \mapsto y(t) \) can be extended to a function \( y \in L^p_{\text{loc}}([0, +\infty), U) \) satisfying the estimate

\[
\|y\|_{L^p([0, \alpha], U)} \leq c \left( \|x\| + \|u\|_{L^p([0, \alpha], U)} \right) \quad (7)
\]

for any \( x \in X, u \in L^p_{\text{loc}}([0, +\infty), U) \), and certain constants \( \alpha > 0 \) and \( c := c(\alpha) > 0 \).

In order to ensure the well-posedness of the system (4)–(6), we will introduce conditions on \( B \) and \( C \). Firstly, let us consider the operator

\[
C := C|_{D(A)} : L^p(D(A), U) \to L^p(U).
\]

We refer to \( C \) as an admissible observation operator for \( A \) if, for a chosen (or all) \( \alpha > 0 \), there exists a constant \( \gamma := \gamma(\alpha) > 0 \) such that

\[
\int_0^\alpha \|CT(t)x\|^p dt \leq \gamma^p \|x\|^p
\]

for all \( x \in D(A) \). It is worth noting that \( \lim_{\alpha \to 0} \gamma(\alpha) < \infty \), as described in [23, Prop. 4.3.3, page 132].

By referring to Theorem 4.5 and Proposition 4.7 of [24], it can be established that if \( C \) is an admissible observation operator for \( A \), then \( T(t)x \in D(C_{\Lambda}) \) for almost every \( t > 0 \) and all \( x \in X \). Additionally, the following inequality holds:

\[
\int_0^\alpha \|C_{\Lambda}T(t)x\|^p dt \leq \gamma^p \|x\|^p, \quad \forall x \in X,
\]

where \( C_{\Lambda} \) represents the Yosida extension of \( C \) with respect to \( A \). The Yosida extension is defined as follows:

\[
D(C_{\Lambda}) := \{ x \in X : \lim_{s \to +\infty} sC_{\Lambda}R(s, A)x; \text{exists in } U \}, \quad C_{\Lambda}x := \lim_{s \to +\infty} sC_{\Lambda}R(s, A)x. \quad (8)
\]

For further details, we refer to [24].

We will now introduce the space:

\[
W^{2,p}_{0,t}(U) := \left\{ u \in W^{2,p}([0, t], U) : u(0) = 0 \right\}, \quad t > 0,
\]

which is dense in \( L^p([0, t], U) \). Performing a straightforward integration by parts, we obtain \( \Phi_t^{A,B} u \in Z \) for any \( t \geq 0 \) and \( u \in W^{2,p}_{0,t}(U) \). Consequently, we can define the input-output map known as:

\[
(f^{A,B,C} u)(t) = C\Phi_t^{A,B} u, \quad t \geq 0, \quad u \in W^{2,p}_{0,t}(U).
\]
Utilizing the expression given in (5), for \( x \in D(A) \) and \( u \in W^{2,p}_{\tau} (U) \) \((\tau > 0)\), the output function \( y \) defined in (6) satisfies:

\[
y(t) = CT(t)x + (\mathbb{F}^{A,B,C}u)(t)
\]

for any \( t \in [0, \tau] \) and \( u \in L^p([0, \tau], U) \).

**Definition 1** Let \( A, B, C \), and \( \mathbb{F}^{A,B,C} \) be as described above. We define the operator triple \( (A, B, C) \) to be well-posed if the following conditions are satisfied:

(i) \( B \) is an admissible control operator for \( A \),
(ii) \( C \) is an admissible observation operator for \( A \), and
(iii) there exist \( \tau > 0 \) and \( \kappa > 0 \) such that

\[
\| \mathbb{F}^{A,B,C}u \|_{L^p([0,\tau], U)} \leq \kappa \| u \|_{L^p([0,\tau], U)}
\]

holds for all \( u \in W^{2,p}_{0,\tau} (U) \).

If the triple \( (A, B, C) \) is well-posed, the density of \( W^{2,p}_{0,\tau} (U) \) in \( L^p([0, \tau], U) \) allows us to extend \( \mathbb{F}^{A,B,C} \) to a bounded operator in \( \mathcal{L}(L^p([0, \tau], U)) \) for any \( \tau > 0 \). Moreover, the estimation of the observation (7) holds, confirming that the system (4)–(6) is indeed well-posed.

**Definition 2** A well-posed triple \( (A, B, C) \) is called regular (with feedthrough zero) if the following limit exists

\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau \left( \mathbb{F}^{A,B,C}(\mathbb{1}_{\mathbb{R}^+} \cdot v) \right)(\sigma) d\sigma = 0,
\]

for any \( v \in U \).

**Remark 1** (i) According to [26], in the case of a regular triple \( (A, B, C) \), we have \( \Phi_t^{A,B} u \in D(C_A) \) and \( (\mathbb{F}^{A,B,C}u)(t) = C_A \Phi_t^{A,B} u \) for almost every \( t \geq 0 \) and all \( u \in L^p_{\text{loc}}([0, +\infty), U) \). This implies that the state trajectory and the output function of the system (4)–(6) satisfy \( x(t) \in D(C_A) \) and \( y(t) = C_A z(t) \) for almost every \( t > 0 \), considering all initial states \( z(0) = x \in X \) and all inputs \( u \in L^p_{\text{loc}}([0, +\infty), U) \).

(ii) It is evident that if one of the operators \( B \) or \( C \) is bounded and the other is admissible for \( A \), then the operator triple \( (A, B, C) \) is regular.

We conclude this section by providing an example of a regular linear system, which will be frequently utilized in various proofs in the subsequent sections.

**Example 1** Let \( X \) be a Banach space and \( p > 1 \) a real number. It is known (see e.g. [7, Chap. 2]) that the family \( (S^X(t))_{t \geq 0} \) defined by

\[
(S^X(t)\varphi)(\theta) := \begin{cases} 0, & t + \theta \geq 0, \\
\varphi(t + \theta), & t + \theta \leq 0
\end{cases}
\]
for any \( f \in L^p([-r, 0], X), t \geq 0 \) and \( \theta \in [-r, 0] \), is a strongly continuous semigroup on \( L^p([-r, 0], X) \) (called the left shift semigroup). The generator of this semigroup is

\[
Q^X \varphi = \varphi', \quad D(Q^X) = \left\{ \varphi \in W^{1,p}([-r, 0], X) : \varphi(0) = 0 \right\}.
\] (10)

Now, consider the boundary system

\[
\begin{cases}
\frac{\partial v(t, \theta)}{\partial t} = \frac{\partial v(t, \theta)}{\partial \theta}, & t \geq 0, \theta \in [-r, 0], \\
v(0, \theta) = \varphi(\theta), & \theta \in [-r, 0], \\
v(t, 0) = x(t), & t \geq 0,
\end{cases}
\] (11)

We select \( Q_m := \frac{\partial}{\partial \theta} \) with maximal domain \( D(Q_m) = W^{1,p}([-r, 0], X) \), so \( Q^X = Q_m \) and \( D(Q^X) = \ker G \) with \( G = \delta_0 \) where \( \delta_0 f = f(0) \) is the Dirac operator. The Dirichlet operator \( d_\lambda \) associated to (11) is given by

\[
d_\lambda x = e^{\lambda x}, \quad \lambda \in \rho(Q^X) = \mathbb{C}, \quad x \in X.
\]

We put

\[
\beta^X := (\lambda - Q^X - 1)d_\lambda, \quad \lambda \in \mathbb{C}.
\]

The control maps associated with the control operator \( \beta^X \) are given by

\[
\Phi_t^{Q^X, \beta^X} u(\theta) = \begin{cases}
u(t + \theta), & -t \leq \theta \leq 0, \\0, & -r \leq \theta < -t,
\end{cases}
\]

for any \( t \geq 0 \) and \( u \in L^p([0, +\infty), X) \), see [13]. Thus the operator \( \beta^X \in \mathcal{L}(X, L^p([-r, 0], X)) \) is an admissible control operator for \( Q^X \). Now consider the operator \( L : W^{1,p}([-r, 0], X) \to X \) defined by (2) and define

\[
L_0 := L|_{D(Q^X)} \in \mathcal{L} \left( D(Q^X), X \right).
\] (12)

Then \( L_0 \) is an admissible observation operator for \( Q^X \), see [10, Lemma 6.2]. We select

\[
\left( \Phi_t^{Q^X, \beta^X, L_0} u \right)(t) = L\Phi_t^{Q^X, \beta^X} u, \quad u \in W^{2,p}_{0,t}(X).
\]

As in [13], we show that for any \( \tau > 0 \) and \( u \in W^{2,p}_{0,\tau}(X) \),
\[ \int_0^\tau \left\| \left( \mathcal{Q}^X, \beta^X, L_0 \right) u \right\|_p dt \leq \int_0^\tau \left( \int_{-\tau}^0 \| u(t + \theta) \| d|\mu^L|(|\theta|) \right)^p dt \]
\[ \leq \left( |\mu^L|([-\tau, 0]) \right)^{\frac{p}{q}} \int_0^\tau \int_{-\tau}^0 \| u(t + \theta) \|^p d|\mu^L|(|\theta|) dt \]
\[ \leq \left( |\mu^L|([-\tau, 0]) \right)^{\frac{p}{q}} \int_{-\tau}^0 \int_0^{\tau + \theta} \| u(\sigma) \|^p d\sigma d|\mu^L|(|\theta|) \]
\[ \leq \left( |\mu^L|([-\tau, 0]) \right)^{\frac{p}{q}} \int_0^\tau \| u(\sigma) \|^p d\sigma, \quad (13) \]
due to the Hölder inequality and Fubini’s theorem. This shows that the triple \((\mathcal{Q}^X, \beta^X, L_0)\) is well-posed. On the other hand, it is shown in [13, Theorem 3] that the triple \((\mathcal{Q}^X, \beta^X, L_0)\) is a regular linear system.

### 3 Well-posedness of the state-delay integro-differential equation

In this section, our focus lies on examining the well-posedness of the state-delay integro-differential Eq. (1). To provide a comprehensive understanding, we begin by revisiting important aspects related to the free-delay integro-differential equation. In fact, consider the following integro-differential equation

\[ \dot{x}(t) = Ax(t) + \int_0^t a(t-s)Ax(s)ds + f(t), \quad x(0) = x, \quad t > 0. \quad (14) \]

The solvability of (14) is intricately connected to the concept of strongly continuous resolvent families. These families are defined as follows:

**Definition 3** A family of bounded linear operators \((R(t))_{t \geq 0}\), is called resolvent family for the homogeneous free-delay integro-differential equation if the following three conditions are satisfied:

- For all \(x \in X\), \(R(t)x\) is continuous on \(\mathbb{R}^+\), \(R(0) = I\) (Identity operator).
- \(R(t)(D(A)) \subseteq D(A)\), for all \(t \geq 0\), for \(x \in D(A)\), \(AR(t)x\) is continuous and \(R(t)x\) is continuously differentiable on \(\mathbb{R}^+\).
- For all \(x \in D(A)\) and \(t \geq 0\), the following resolvent equations

\[ \dot{R}(t)x = AR(t)x + \int_0^t a(t-s)AR(s)xds, \]
\[ \dot{R}(t)x = R(t)Ax + \int_0^t R(t-s)a(s)Axds, \]

hold.
Generally, in the definition above, it is not necessary that $A$ is the generator of a semigroup. It is well-known (see Grimmer & Prüss [9]) that the homogeneous free-delay integro-differential system is well-posed if and only if it has a resolvent $R(t)$. In this situation, $x(t) = R(t)x$, $t \geq 0$ with $x \in X$ is the mild solution, which gives the unique classical solution if $x \in D(A)$. For more details on resolvent families we refer to the monograph by Prüss [21].

It is well-known that the assumptions $a(\cdot) \in W^{1,p}(\mathbb{R}_+, \mathbb{C})$ and $A$ generates a semigroup on $X$ imply that the free-delay integro-differential equation has a unique resolvent family $(R(t))_{t \geq 0}$, see e.g. [6, 21].

In the sequel, we will use matrices operators to solve the Eq. (14). We then select $X_0 := X \times L^p(\mathbb{R}_+, X)$ with norm \[ \| (x, f) \| := \| x \| + \| f \|_p. \]

In this space, we consider the following unbounded operator matrices

\[
\mathcal{A}_0 := \begin{pmatrix} A & \delta_0 \\ 0 & \frac{d}{ds} \end{pmatrix}, \quad \mathcal{A}^\delta_0 := \begin{pmatrix} A & \delta_0 \\ a(\cdot)A & \frac{d}{ds} \end{pmatrix},
\]

\[ D(\mathcal{A}^\delta_0) = D(\mathcal{A}_0) := D(A) \times W^{1,p}(\mathbb{R}_+, X), \]

where $\frac{d}{ds}$ is the first derivative with $D(\frac{d}{ds}) = W^{1,p}(\mathbb{R}_+, X)$. The following result is well known, see e.g. [7], [21, p.339] and [2].

**Theorem 1** The operator $\mathcal{A}^\delta_0$ generates a strongly continuous semigroup $(T^\delta_0(t))_{t \geq 0}$ on $\mathcal{X}_0$. Moreover, if we assume that $a(\cdot) \in W^{1,p}(\mathbb{R}_+, \mathbb{C})$, then

\[
T^\delta_0(t) = \begin{pmatrix} R(t) & \Upsilon(t) \\ * & * \end{pmatrix}, \quad t \geq 0,
\]

with

\[
\Upsilon(t) f := \int_0^t R(t-s)f(s)ds, \quad f \in L^p(\mathbb{R}_+, X).
\]

In particular $R(t)$ is exponentially bounded since $(T^\delta_0(t))_{t \geq 0}$ is so.

Let us now solve the state-delay integro-differential Eq. (1). The latter can be reformulated in $\mathcal{X}_0$ as the following problem

\[
\begin{cases}
\dot{\varphi}(t) = \mathcal{A}^\delta_0 \varphi(t) + \left( Lx_t \right), & t \geq 0, \\
\varphi(0) = (x, 0), & x_0 = \varphi.
\end{cases}
\]

This equation is not yet a Cauchy problem because we still have a delay term. In order to reformulate it as Cauchy problem, we need the following larger product state space

\[
\mathcal{X} := \mathcal{X}_0 \times L^p([-r, 0], X),
\]
and the new state

\[ z(t) := \begin{pmatrix} q(t) \\ x(t + \cdot) \end{pmatrix}, \quad t \geq 0. \]

(19)

By combining (11) and (17), we rewrite the problem (1) as the following boundary value problem

\[
\begin{aligned}
\dot{z}(t) &= \mathcal{A}_{m,L} z(t), \quad t \geq 0 \\
\dot{z}(0) &= z^0, \\
Gz(t) &= Mz(t), \quad t \geq 0,
\end{aligned}
\]

(20)

where the operator \( \mathcal{A}_{m,L} : \mathcal{X} \rightarrow \mathcal{X} \) is given by

\[
\mathcal{A}_{m,L} := \begin{pmatrix} \mathcal{A}^b_0 & L \\ 0 & 0 \end{pmatrix}, \quad \mathcal{X} := D(\mathcal{A}^b_0) \times W^{1,p}([0, 0], X),
\]

and \( G : \mathcal{X} \rightarrow X \) and \( M : \mathcal{X} \rightarrow X \) are the linear operators

\[
G := \begin{bmatrix} 0 & 0 & \delta_0 \end{bmatrix}, \quad M := \begin{bmatrix} I & 0 & 0 \end{bmatrix},
\]

and the initial state

\[
z^0 = \begin{pmatrix} x^0 \\ \psi \end{pmatrix}.
\]

To solve the problem (1), it suffices to solve the following Cauchy problem

\[
\begin{aligned}
\dot{z}(t) &= \mathcal{A}_L z(t), \quad t \geq 0, \\
\dot{z}(0) &= z^0,
\end{aligned}
\]

(21)

where

\[
\mathcal{A}_L := \mathcal{A}_{m,L}, \quad D(\mathcal{A}_L) := \{ z \in \mathcal{X} : Gz = Mz \}.
\]

(22)

This means that it suffices to show that the operator \( \mathcal{A}_L \) is the generator of a strongly continuous semigroup on \( \mathcal{X} \). Thus, we define

\[
\mathcal{A}_{m,0} := \begin{pmatrix} \mathcal{A}^b_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{L} := \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}
\]

(23)

on \( D(\mathcal{A}_{m,L}) = \mathcal{X} \), and

\[
\mathcal{A} := \mathcal{A}_{m,0},
\]
\[ D(\mathfrak{A}) := \{ z \in \mathcal{X}, \ Gz = Mz \}. \]  (24)

Observe that we have

\[ \mathfrak{A}_L = \mathfrak{A} + \mathcal{L}. \]

Next we follow the following strategy: We first prove that \( \mathfrak{A} \) is a generator on \( \mathcal{X} \) and second show that \( \mathcal{L} \) is a Miyadera–Voigt perturbation for \( \mathfrak{A} \).

Let us start by proving that \( \mathfrak{A} \) is a generator. In fact, define the operator

\[ \mathfrak{A} := \mathfrak{A}_{m, 0}, \text{ with domain } D(\mathfrak{A}) = \ker(G). \]

It is clear that the operator \( \mathfrak{A} \) generates a strongly continuous semigroup \( (\mathbb{T}(t))_{t \geq 0} \) on \( \mathcal{X} \) given by

\[ \mathbb{T}(t) \begin{pmatrix} x \\ f \\ \varphi \end{pmatrix} = \begin{pmatrix} T^0(t) \begin{pmatrix} x \\ f \end{pmatrix} \\ S^X(t) \varphi \end{pmatrix}, \ t \geq 0, \ \begin{pmatrix} x \\ f \\ \varphi \end{pmatrix} \in \mathcal{X}. \]  (25)

Now let us compute \( \mathbb{D}_\lambda \), the Dirichlet operator associated with \( G \) and \( \mathfrak{A}_{m, 0} \). To determine \( \mathbb{D}_\lambda \), it suffices to determine \( \ker(\lambda - \mathfrak{A}_{m, 0}) \), for \( \lambda \in \rho(\mathfrak{A}^0) \).

For \( \lambda \in \rho(\mathfrak{A}) = \rho(\mathfrak{A}^0) \),

\[ \ker(\lambda - \mathfrak{A}_{m, 0}) = \ker(\lambda - \mathfrak{A}^0) \times \ker(\lambda - Q_m) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \times \left\{ e_\lambda v : v \in X \right\}. \]

This implies that

\[ \mathbb{D}_\lambda v = \begin{pmatrix} 0 \\ 0 \\ e_\lambda v \end{pmatrix}, \ \lambda \in \rho(\mathfrak{A}^0), \ v \in X. \]

Now, let us consider the operator

\[ \mathcal{B} := (\lambda - \mathfrak{A}_-) \mathbb{D}_\lambda \in \mathcal{L}(X, X_-), \ \lambda \in \rho(\mathfrak{A}). \]  (26)

The facts mentioned above give the following variation of constants formula which will be useful later.

**Theorem 2** The operator \( \mathfrak{A} \) generates a strongly continuous semigroup \( (\Sigma(t))_{t \geq 0} \) on \( \mathcal{X} \) satisfying

\[ \Sigma(t)z^0 = \mathbb{T}(t)z^0 + \int_0^t \mathbb{T}_{-1}(t - s) \mathcal{B} M \Sigma(s)z^0 ds \]  (27)

for any \( z^0 \in \mathcal{X} \) and \( t \geq 0. \)
Proof For \( u \in L^p([0, \infty), X) \) and \( \lambda > 0 \) sufficiently large, using the Laplace-
transform we obtain

\[
(\mathcal{T}_1 \ast \mathcal{B}u)(\lambda) = R(\lambda, \mathcal{A}_1) \mathcal{B} \hat{u}(\lambda) = \mathcal{D}_\lambda \hat{u}(\lambda) = \begin{pmatrix}
0 \\
0 \\
\mathcal{D}_\lambda \hat{u}(\lambda)
\end{pmatrix}.
\]

Thus

\[
(\mathcal{T}_1 \ast \mathcal{B}u)(\lambda) = \begin{pmatrix}
0 \\
0 \\
\phi_{QX, \beta X} u(\lambda)
\end{pmatrix},
\]

where \( \phi_{QX, \beta X} \) is the control maps associated to the regular system \( (QX, \beta X, L_0) \) defined before in Example 1. By injectivity of Laplace-transform, we deduce that

\[
\int_0^t \mathcal{T}_1(t - s) \mathcal{B}u(s)ds = \begin{pmatrix}
0 \\
0 \\
\phi_{QX, \beta X} u(t)
\end{pmatrix} \in \mathcal{X}.
\]

It follows that \( \mathcal{B} \) is an admissible control operator for \( \mathcal{A} \). The fact that \( M \in \mathcal{L}(\mathcal{X}, X) \), \( (\mathcal{A}, \mathcal{B}, M) \) is a regular system with \( I : X \rightarrow X \) as an admissible feedback (see [26]). Therefore, the operator

\[
\mathcal{A}^{cl} := \mathcal{A}_1 + \mathcal{B}M
\]

\[
D(\mathcal{A}^{cl}) := \left\{ z \in \mathcal{X}, \quad \mathcal{A}^{cl} z \in \mathcal{X} \right\}
\]

generates a strongly continuous semigroup \( (T^{cl}(t))_{t \geq 0} \) on \( \mathcal{X} \) satisfying

\[
T^{cl}(t)z^0 = \mathcal{T}(t)z^0 + \int_0^t \mathcal{T}_1(t - s) \mathcal{B}MT^{cl}(s)z^0 ds
\]

for any \( z^0 \in \mathcal{X} \) and \( t \geq 0 \), see [26]. Finally, we mention that the operator \( \mathcal{A}^{cl} \) coincides with the operator \( \mathcal{A} \), due to [14, Theorem 4.1].

\[ \Box \]

Theorem 3 The operator

\[
\mathfrak{A}_L := \mathfrak{A} + \mathcal{L}, \quad D(\mathfrak{A}_L) := D(\mathfrak{A})
\]

generates a strongly continuous semigroup \( (\mathfrak{T}_L(t))_{t \geq 0} \) on \( \mathcal{X} \).
Proof To prove that $\mathfrak{A}_L$ is a generator it suffices to show that $\mathcal{L}$ is an admissible observation operator for $\mathfrak{A}$ (see [10, Theorem 2.1]). To this end, let us define the operator

$$\mathcal{L}_0 = \mathcal{L}|_{D(\mathfrak{A})} \in \mathcal{L}(D(\mathfrak{A}), \mathfrak{X}).$$

Let us first show that $\mathcal{L}_0$ is an admissible observation operator for $\mathfrak{A}$. For $\alpha > 0$ and $z^0 = (x, f, \varphi)^\top \in D(\mathfrak{A}) = D(\mathfrak{A}^{\Lambda_0}) \times D(QX)$, we have

$$\int_0^\alpha \|\mathcal{L}_0 \mathbb{T}(t)z^0\|^p dt = \int_0^\alpha \|L_0 S^X(t)\varphi\|^p dt,$$

where $L_0$ is the operator defined in Example 1, which is an admissible observation operator for $QX$. It follows from the above estimate that $\mathcal{L}_0$ is admissible for $\mathfrak{A}$. On the other hand, by using a similar argument as in [10, Lemma 6.3], one can see that the Yosida extension of $\mathcal{L}_0$ with respect to $\mathfrak{A}$ is explicitly given by

$$D(\mathcal{L}_0, \Lambda_1) = \mathfrak{X}_0 \times D(L_0, \Lambda_1), \quad \mathcal{L}_0, \Lambda_1 = \begin{pmatrix} 0 & L_0, \Lambda_1 \\ 0 & 0 \end{pmatrix}. \quad (28)$$

In addition for any $z^0 \in D(\mathfrak{A})$, we have

$$\int_0^\alpha \|\mathcal{L}_0, \Lambda_1 \mathbb{T}(t)z^0\|^p dt \leq c^p \|z^0\|^p \quad (29)$$

for a constant $c > 0$. According to [14, Lemma 3.6], we have

$$\mathfrak{X} \subset D(\mathcal{L}_0, \Lambda), \quad (\mathcal{L}_0, \Lambda)|_{\mathfrak{X}} = \mathcal{L}.$$

Thus for $z^0 \in D(\mathfrak{A})$,

$$\int_0^\alpha \|\mathcal{L} \mathbb{\Sigma}(t)z^0\|^p dt = \int_0^\alpha \|\mathcal{L}_0, \Lambda \mathbb{\Sigma}(t)z^0\|^p dt. \quad (30)$$

If we consider

$$u(t) := M \mathbb{\Sigma}(t)z^0, \quad t \geq 0,$$

then by using the proof of Theorem 2, we obtain

$$\int_0^t \mathbb{T}_{-1}(t-s) \mathcal{B} M T^{cl}(s)z^0 ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Phi_t^{QX, \rho^X} u.$$
Now from Example 1, \((Q^X, \beta^X, L_0)\) is regular and that \(\Phi^Q_{\cdot, \cdot} u \in D(L_{0,\Lambda})\) for a.e. \(t \geq 0\). Combining (28) and (31) we obtain

\[
\int_0^t \mathbb{T}_{-s}(t-s)\mathcal{B}M^{cl}(s)z^0 ds \in D(L_{0,\Lambda}),
\]

and

\[
\int_0^\alpha \|L_{0,\Lambda} \int_0^t \mathbb{T}_{-s}(t-s)\mathcal{B}M^{cl}(s)z^0 ds \|^p dt = \int_0^\alpha \|L_{0,\Lambda} \Phi^Q_{\cdot, \cdot} u \|^p dt \\
\leq \kappa^p \|u\|^p_{L^p([0,\alpha], X)} \\
\leq \tilde{\kappa}^p \|z^0\|^p,
\]

for a constant \(\tilde{\kappa} > 0\); due to the facts \(M\) is bounded and that \(\mathfrak{L}\) is exponentially bounded. Finally, using on the one side, (27) and (30), on the other side, the estimates (29) and (32), we obtain

\[
\int_0^\alpha \|\mathcal{L}\mathfrak{L}^0(t)z^0\|^p dt \leq \gamma^p \|z^0\|^p dt,
\]

for any \(z^0 \in D(\mathfrak{A})\) and some constant \(\gamma := \gamma(\alpha) > 0\). This ends the proof. \(\square\)

**Remark 2** Observe that \(M\mathbb{D}_\lambda = 0\) for any \(\lambda \in \rho(\mathfrak{A})\). It is shown in [14, Theorem 4.1] that for \(\lambda \in \rho(\mathfrak{A})\) we have

\[
\lambda \in \rho(\mathfrak{A}) \iff 1 \in \rho(\mathbb{D}_\lambda M) \iff 1 \in \rho(M\mathbb{D}_\lambda) = \mathbb{C}^*.
\]

Thus

\[
\rho(\mathfrak{A}) = \rho(\mathfrak{A}^{\mathbb{D}_0}) \subset \rho(\mathfrak{A}).
\]

Again by [14, Theorem 4.1], for \(\lambda \in \rho(\mathfrak{A})\), we have

\[
R(\lambda, \mathfrak{A}) = (I - \mathbb{D}_\lambda M)^{-1} R(\lambda, \mathfrak{A}).
\]

In the following result, we give a new expression of the solution of the Cauchy problem (21) by appealing only to semigroup \((\mathbb{T}(t))_{t \geq 0}\).

**Proposition 4** The mild solution \(z(\cdot)\) of the Cauchy problem (21) satisfies

\[
z(s) \in D(L_\Lambda), \text{ a.e. } s \geq 0,
\]

\[
z(t) = \mathbb{T}(t)z(0) + \int_0^t \mathbb{T}_{-s}(t-s)\mathcal{B}Mz(s) ds + \int_0^t \mathbb{T}(t-s)L_\Lambda z(s) ds
\]

for any \(t \geq 0\), \(z(0) \in X\), where \(L_\Lambda\) is the Yosida extension of \(L\) with respect to \(\mathfrak{A}\).
Proof Using [10, Theorem 5.1] and the fact that \( \mathcal{L} \) is an admissible observation operator for \( \mathfrak{A} \) (see the proof of the previous result), the solution of the Cauchy problem (21) satisfies \( z(s) \in D(\mathcal{L}_\Lambda) \) for a.e. \( s \geq 0 \), and

\[
z(t) = \mathcal{X}(t)z(0) + \int_0^t \mathcal{X}(t-s)\mathcal{L}_\Lambda z(s)ds,
\]

for any \( t \geq 0 \), \( z(0) \in X \).

For a sufficiently large \( \lambda > 0 \), the Laplace transform of \( z \) yields

\[
\hat{z}(\lambda) = R(\lambda, \mathfrak{A})z(0) + R(\lambda, \mathfrak{A})\mathcal{L}_\Lambda \hat{z}(\lambda)
= (I - \mathbb{D}_\lambda M)^{-1} [R(\lambda, \mathfrak{A})z(0) + R(\lambda, \mathfrak{A})\mathcal{L}_\Lambda \hat{z}(\lambda)].
\]

This in turn implies that

\[
\hat{z}(\lambda) = R(\lambda, \mathfrak{A})z(0) + \mathbb{D}_\lambda M \hat{z}(\lambda) + R(\lambda, \mathfrak{A})\mathcal{L}_\Lambda \hat{z}(\lambda).
\] (35)

Let

\[
\sigma(t) := \mathcal{I}(t)z(0) + \int_0^t \mathcal{I}^{-1}(t-s)\mathcal{B}Mz(s)ds + \int_0^t \mathcal{T}(t-s)\mathcal{L}_\Lambda z(s)ds, \quad t \geq 0.
\]

According to (26) and (35), the Laplace transform of \( \sigma \) satisfies

\[
\hat{\sigma}(\lambda) = R(\lambda, \mathfrak{A})z(0) + R(\lambda, \mathfrak{A})\mathcal{L}_\Lambda \hat{z}(\lambda)
= \hat{z}(\lambda),
\]

and the result follows by virtue of the injectivity the Laplace transform. \( \square \)

Note that for \( a(\cdot) \equiv 0 \), the resolvent family correspond to the semigroup generated by \( A \). Therefore the following result which establishes the relation between the solutions of (1) and (21), extends [13, Proposition 2] from delay differential equations to delay integro-differential one.

**Theorem 5** For any initial condition \( (x, f, \varphi)^\top \in \mathcal{D} \), there exists a unique mild solution \( x(\cdot) \) of (1) satisfying \( x_t \in D(L_{0, \Lambda}) \) for a.e. \( t \geq 0 \) and

\[
x(t) = R(t)x + \int_0^t R(t-s)(L_{0, \Lambda}x_s + f(s))ds, \quad t \geq 0.
\]

**Proof** Let \( z(0) = (x, f, \varphi)^\top \in \mathcal{D} \) and \( z(t) = (x(t), v(\cdot, \cdot), w(t))^\top \) be the solution of the Cauchy problem (21) associated to \( z(0) \). Appealing to Proposition 4 and by combining (34) together with (25), (31), (16) we deduce that \( w(t) = x_t \) and

\[
\begin{pmatrix}
x(t) \\
v(t, \cdot) \\
x_t
\end{pmatrix} = \begin{pmatrix}
T_{\delta_0}(t) x(0) \\
0 \\
0
\end{pmatrix} + \int_0^t \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} ds
+ \int_0^t \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} ds.
\]
\[
\begin{align*}
&= \left( \begin{pmatrix} R(t) & \Upsilon(t) & x & f \\ S^X(t) & \varphi \\
0 & 0 & \Phi^X & \beta^X \end{pmatrix} \right) + \\
&\quad + \int_0^t \left( \begin{pmatrix} R(t-s) & \Upsilon(t-s) & L_{0,\Lambda x} \\ 0 & 0 & \end{pmatrix} \right) ds.
\end{align*}
\]

Thus by equalizing the two first components of the above equality we get the required result.

\section*{4 Spectral theory}

In this section, we attempt to study the spectral theory for the generator \( \mathfrak{A}_L \) and we get some results extending those obtained in [1] and [7]. First of all, we require the following straightforward lemma.

\textbf{Lemma 1} For \( \lambda \in \rho(\mathfrak{A}^0) \), we have

\[
\lambda \in \rho(\mathfrak{A}_L) \iff 1 \in \rho(L R(\lambda, \mathfrak{A})).
\]

\textbf{Proof} For \( \lambda \in \rho(\mathfrak{A}^0) \), we write \( \lambda - \mathfrak{A}_L = \lambda - \mathfrak{A} - \mathbb{L} \). By Remark 2, we have \( \lambda \in \rho(\mathfrak{A}) \)

\[
\lambda - \mathfrak{A}_L = (1 - \mathbb{L} R(\lambda, \mathfrak{A}))(\lambda - \mathfrak{A}).
\]

This implies that \( \lambda \in \rho(\mathfrak{A}_L) \) if and only if \( (1 - \mathbb{L} R(\lambda, \mathfrak{A}))^{-1} \) exists.

From the previous results we immediately obtain the following result which characterises the spectrum of \( \mathfrak{A}_L \) and then extends [1, Lemma 4.1] from delay differential or integro-differential equations to delay integro-differential Eq. (1).

In the remainder of this section, we use the following notation: for any \( \alpha \in \mathbb{R} \), we select

\[
\mathbb{C}_\alpha := \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq \alpha \}.
\]

\textbf{Theorem 6} For \( \lambda \in \mathbb{C}_0 \cap \rho(\mathfrak{A}^0) \cap \rho(\mathfrak{A}) \), we have

\[
\lambda \in \rho(\mathfrak{A}_L) \iff \lambda \in \rho\left( (1 + \hat{a}(\lambda))A + Le_\lambda \right).
\]

\textbf{Proof} By virtue of Remark 2, we have \( \lambda \in \rho(\mathfrak{A}) \) and therefore

\[
R(\lambda, \mathfrak{A}) = (I - \mathbb{D}_\lambda M)^{-1} R(\lambda, \mathfrak{A})
\]

\[
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ e_\lambda & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} R(\lambda, \mathfrak{A}) & 0 \\ 0 & 0 & R(\lambda, Q^X) \end{pmatrix}.
\]
\[ \begin{pmatrix} R(\lambda, \mathfrak{A}^0) & 0 \\ (e_\lambda, 0) & R(\lambda, \mathfrak{A}^0) \end{pmatrix} \begin{pmatrix} 0 \\ R(\lambda, Q^X) \end{pmatrix} \].

Appealing to [7, Proposition 7.25], (see. [2]) leads to \( \frac{\lambda}{1 + \hat{a}(\lambda)} \in \rho(A) \) and by virtue of [2, Lemma 7 (ii)], we obtain

\[
R(\lambda, \mathfrak{A}^0) = \begin{pmatrix} H(\lambda) & H(\lambda)\delta_0 R(\lambda, \frac{d}{ds}) \\ R\left(\lambda, \frac{d}{ds}\right) V H(\lambda) R\left(\lambda, \frac{d}{ds}\right) V H(\lambda) R\left(\lambda, \frac{d}{ds}\right) + R\left(\lambda, \frac{d}{ds}\right) \end{pmatrix},
\]

where

\[ H(\lambda) = (\lambda - (1 + \hat{a}(\lambda)A)^{-1} \text{ and } V \in \mathcal{L}(D(A), W^{1,p}(\mathbb{R}_+, X)) \text{ is given by } (Vx)(t) := a(t)Ax \text{ for } x \in D(A) \text{ and } \text{a.e. } t > 0. \]

Therefore, direct computations implies

\[
R(\lambda, \mathcal{A}) = \begin{pmatrix} H(\lambda) & H(\lambda)\delta_0 R(\lambda, \frac{d}{ds}) \\ R\left(\lambda, \frac{d}{ds}\right) V H(\lambda) R\left(\lambda, \frac{d}{ds}\right) V H(\lambda) R\left(\lambda, \frac{d}{ds}\right) + R\left(\lambda, \frac{d}{ds}\right) \end{pmatrix}.
\]

and

\[
\mathcal{L} R(\lambda, \mathcal{A}) = \begin{pmatrix} Le_\lambda H(\lambda) & Le_\lambda H(\lambda)\delta_0 R\left(\lambda, \frac{d}{ds}\right) LR\left(\lambda, Q^X\right) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

By combining (36) and (37), we deduce that \( \lambda \in \rho(\mathcal{A}) \) if and only if \( 1 \in \rho\left(Le_\lambda H(\lambda)\right) \).

Rewriting

\[
(\lambda - (1 + \hat{a}(\lambda)A) - Le_\lambda = (1 - Le_\lambda H(\lambda))(\lambda - (1 + \hat{a}(\lambda))A)
\]

we conclude that

\[ \lambda \in \rho(\mathcal{A}) \iff \lambda \in \rho((1 + \hat{a}(\lambda))A + Le_\lambda). \]

\[ \square \]

**Remark 3** In the particular case of \( a(\cdot) \equiv 0 \), we obtain the known result which for \( \lambda \in \rho(\mathcal{A}^0) \)

\[ \lambda \in \rho(\mathcal{A}) \iff \lambda \in \rho(A + Le_\lambda). \]
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which coincides perfectly with the classic problem with delay (see [1, Lemma 4.1]). On the other hand, for the free-delay integro-differential equation we can identify $\rho(\mathfrak{A}_l)$ with $\rho(\mathfrak{A}_0)$. Thus for $\lambda \in \mathbb{C}_0 \cap \rho(\mathfrak{A}_0)$, we retrieve [7, Proposition 7.25] saying that

$$\lambda \in \rho(\mathfrak{A}_0) \iff \lambda \in \rho((1 + \hat{a}(\lambda))A).$$

5 Integro-differential systems with delays as regular Salamon–Weiss systems

In this section, we keep the same notation as in the previous sections. In addition, set the following Riemann–Stieltjes integrals

$$K \psi = \int_{-r}^{0} d\mu^K(\theta)\psi(\theta), \quad \mu^K : [-r, 0] \to \mathcal{L}(U, X)$$

$$C \varphi = \int_{-r}^{0} d\mu^C(\theta)\varphi(\theta), \quad \mu^C : [-r, 0] \to \mathcal{L}(X, Y),$$

$$D \psi = \int_{-r}^{0} d\mu^D(\theta)\psi(\theta), \quad \mu^D : [-r, 0] \to \mathcal{L}(U, Y),$$

for $\varphi \in W^{1,p}([-r, 0], X)$ and $\psi \in W^{1,p}([-r, 0], U)$, where $\mu^K$, $\mu^C$, and $\mu^D$ are functions of bounded variations assumed continuous on $[-r, 0]$ and vanishing at zero.

Now consider the state, input and output delay system

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + \int_{0}^{t} a(t-s)Ax(s)ds + Lx_t + Ku_t + f(t), \quad t \geq 0 \\
x(0) &= x, \quad x_0 = \varphi, \quad u_0 = \psi \\
y(t) &= Cx_t + Du_t, \quad t \geq 0,
\end{aligned}$$

(38)

for initial data $x \in X$, $\varphi \in L^p([-r, 0], X)$ and $\psi \in L^p([-r, 0], U)$, where $A, L, a(\cdot)$ and the history function $(t \mapsto x_t)$ are defined in the previous sections, $u \in L^p([-r, +\infty), U)$ is the control function, and $t \mapsto u_t$ is the control history function defined by $u_t(\theta) = u(t + \theta)$ for any $t \geq 0$ and $\theta \in [-r, 0]$.

On the space $L^p([-r, 0], U)$, we define the linear operator

$$(Q^U \psi)(\theta) := \psi(t + \theta), \quad D(Q^U) = \{ \psi \in W^{1,p}([-r, 0], U) : \psi(0) = 0 \}.$$

The operator $Q^U$ generates the left shift semigroup $(S^U(t))_{t \geq 0}$ on $L^p([-r, 0], U)$ defined by $(S^U(t)\psi)(\theta) = \psi(t + \theta)$ if $t + \theta \leq 0$ and zero if not, for all $t \geq 0$ and $\theta \in [-r, 0]$.

We select $K_0 := D(D(Q^U)) : D(Q^U) \rightarrow X$, $\beta^U := (\lambda - Q^U_{-1})e_\lambda$. 

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for any \( \lambda \in \mathbb{C} \). As in Example 1, the triple \((Q^U, \beta^U, K_0)\) is regular.

In the case of smooth control functions \( u \), we have the following results on the existence of the solution of the problem (38).

**Theorem 7** Let the initial conditions \( x \in X \), \( \varphi \in L^p([-r, 0], U) \), \( \psi \in D(Q^U) \), \( f \in L^p(\mathbb{R}^+, X) \) and a smooth control function \( u \in W^{2,p}((0, +\infty), U) \) with \( u(0) = 0 \). Then there exists a unique solution \( x(\cdot) \) of (38) satisfying \( x_t \in D(L_{0, \Lambda}) \) for a.e. \( t \geq 0 \) and

\[
x(t) = R(t)x + \int_0^t R(t-s)(L_{0, \Lambda}x_s + Ku_s + f(s))ds, \quad t \geq 0.
\]

**Proof** According to Example 1, the function \( t \mapsto u_t \) is the state of a regular linear system \((Q^U, \beta^U, K_0)\), which satisfies

\[
u_t = SU(t)\psi + \int_0^t SU(t-s)(-Q^U_{t-s}e_0)u(s)ds, \quad t \geq 0.
\]

As \( \psi \in D(Q^U) \) and \( u \in W^{2,p}((0, +\infty), U) \), an integration by parts shows that \( u_t \in W^{1, p}([-r, 0], U) \). Thus the function \( g(t) = Ku_t \) is well-defined for \( t \geq 0 \). On the other hand, by [14, Lemma 3.6], we have \( W^{1, p}([-r, 0], U) \subset D(K_{0, \Lambda}) \) and \( g(t) = K_{0, \Lambda}u_t \) for a.e. \( t \geq 0 \), where \( K_{0, \Lambda} \) is the Yosida extension of \( K_0 \) with respect to \( Q^U \). As \( t \mapsto K_{0, \Lambda}u_t \) is the extended output function of the regular linear system \((Q^U, \beta^U, K_0)\), we have \( g \in L^p_{loc}(\mathbb{R}^+, U) \). If we set

\[
\zeta(t) = \begin{pmatrix} g(t) \\ 0 \end{pmatrix}, \quad t \geq 0,
\]

then the problem (38) can be reformulated as

\[
\ddot{z}(t) = A_Lz(t) + \zeta(t), \quad z(0) = \begin{pmatrix} \frac{\lambda}{r} \\ \psi \end{pmatrix}, \quad t \geq 0,
\]

where \( z(\cdot) \) and \( A_L \) are given by (19) and (22), respectively. Now by using the same argument as in the proof of Proposition 4, one can see that the mild solution of the inhomogeneous Cauchy problem (40) is given by

\[
z(t) = \mathbb{T}(t)z(0) + \int_0^t \mathbb{T}(t-s)\mathcal{B}Mz(s)ds + \int_0^t \mathbb{T}(t-s)(\mathcal{L}_{\Lambda}z(s) + \zeta(s))ds,
\]

for \( t \geq 0 \), where \( (\mathbb{T}(t))_{t \geq 0} \) is the strongly continuous semigroup given by (25). The rest of the proof follows exactly as in the proof of Theorem 5. \( \square \)

From our discussion in the proof of Theorem 7, we adopt the following definition:

**Definition 4** Let the initial conditions \( x \in X \), \( \varphi \in L^p([-r, 0], X) \) and \( \psi \in L^p([-r, 0], U) \), and let \( f \in L^p(\mathbb{R}^+, X) \). A mild solution of the integro-differential
equation in (38) is a function $x : [-r, +\infty) \to X$ such that

$$x(t) = \begin{cases} R(t)x + \int_0^t R(t-s) \left( L_{0,A} x_s + K_{0,A} u_s + f(s) \right) ds, & t \geq 0 \\ \varphi(t), & a.e. t \in [-r, 0]. \end{cases}$$

(41)

The following result proves the relationship between smooth input solution and mild solution of the problem (38).

**Proposition 8** The mild solution of (38) is limit of a sequence of smooth input solutions of (38).

**Proof** Let $x(\cdot)$ be a mild solution of the problem (38) corresponding to the initial conditions $x \in X$, $\varphi \in L^p([-r, 0], X)$, the control function $u \in L^p_{loc}(\mathbb{R}^+, U)$ with $u_0 = \psi \in L^p([-r, 0], U)$, and the non-homogenous term $f \in L^p(\mathbb{R}^+, X)$. By density, we approximate $\psi$ and $u$ by sequences $(\psi_n)_n \subset D_{QU}$ and $(u^n)_n \in W^2_{0,loc}(\mathbb{R}^+, U)$, respectively. For any $t \geq 0$, define the function

$$u^n_t(\theta) = \begin{cases} u^n(t+\theta), & -t \leq \theta \leq 0 \\ \psi_n(t+\theta), & -r \leq \theta < -t. \end{cases}$$

According to Theorem 7, for any $n \in \mathbb{N}$, the following functions

$$x^n(t) = \begin{cases} R(t)x + \int_0^t R(t-s) \left( L_{0,A} x^n_s + K_{0,A} u^n_s + f(s) \right) ds, & t \geq 0, \\ \varphi(t), & t \in [-1, 0], \end{cases}$$

(42)

define smooth input solutions of the problem (38). Using the expression (41), we obtain

$$x^n(t) - x(t) = \int_0^t R(t-s) \left( L_{0,A} (x^n_s - x_s) + K_{0,A} (u^n_s - u_s) \right) ds, \quad t \geq 0. \quad (43)$$

Observe that

$$L_{0,A} (x^n_s - x_s) = \left( \mathbb{R}^{QX, \beta X, L_0} (x^n(\cdot) - x(\cdot)) \right)(s),$$

$$K_{0,A} (u^n_s - u_s) = K_{0,A} S^U(t)(\psi^n - \psi) + \left( \mathbb{R}^{QU, \beta U, K_0} (u^n(\cdot) - u(\cdot)) \right)(s)$$

for a.e. $s > 0$. Using properties of regular systems, for any $t > 0$, we have

$$r_{n,t} := \left\| K_{0,A} (u^n - u) \right\|_{L^p([0,t], X)} \leq \gamma_K \left( \left\| \psi^n - \psi \right\|_{L^p([-r,0], U)} + \left\| u - u^n \right\|_{L^p([0,t], U)} \right),$$
where \( \gamma_K = | \mu^K | ([-r, 0]) \). In particular,

\[
\lim_{n \to \infty} r_{n,t} = 0. \tag{44}
\]

We also have

\[
\| L_{0,\Lambda} (u^n_t - u_t) \|_{L^p([0,t], X)} \leq \gamma_L \| x^n(t) - x(t) \|_{L^p([0,t], X)}, \tag{45}
\]

where \( \gamma_L = | \mu^L | ([-r, 0]) \). Using Theorem 1, for any \( \omega > \omega_0 (\alpha \delta_0) \), there exists a constant \( M \geq 1 \) such that \( R(t) \leq Me^{ot} \) for any \( t \geq 0 \). Thus, by using standard argument, the Eq. (43), the inequality (45), and the fact that \( r_{n,t} \leq r_{n,\alpha} \) for any \( \alpha > 0 \) and \( t \in [0, \alpha] \), we have

\[
\| x^n(t) - x(t) \| \leq Me^{\alpha|\alpha|^{\frac{1}{q}}} \left( \gamma_L \| x^n(t) - x(t) \|_{L^p([0,t], X)} + r_{n,\alpha} \right),
\]

where \( q \) is a real number such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Now if we select \( c := Me^{\alpha|\alpha|^{\frac{1}{q}}} \), a constant depending on \( \alpha, M, \omega \) and \( q \), then we obtain

\[
\| x^n(t) - x(t) \| \leq 2^{p-1} c^p r_{n,\alpha}^p + 2^{p-1} (c\gamma_L)^p \int_0^t \| x^n(s) - x(s) \|^p ds.
\]

Therefore, by using the Gronwall lemma, we obtain

\[
\| x^n(t) - x(t) \| \leq 2^{p-1} c^p r_{n,\alpha}^p, \quad t \in [0, \alpha].
\]

This inequality together with (44) imply that \( x^n(t) \to x(t) \) as \( n \to \infty \), for any \( t \in [0, \alpha] \) and any \( \alpha > 0 \). This ends the proof. \( \square \)

In the rest of this section we will show that the delay system (38) is equivalent to a regular distributed linear system in the Salamon–Weiss sense (see Sect. 2 for the definition of such systems). In fact, as shown in the proof of Theorem 7, the problem (38) is reformulated as the following system

\[
\dot{z}(t) = \mathcal{A}_L z(t) + \begin{pmatrix} K_{0,\Lambda} u_t \\ 0 \end{pmatrix}, \quad z(0) = \begin{pmatrix} \frac{x}{\varphi} \\ \frac{f}{\varphi} \end{pmatrix}, \quad t \geq 0. \tag{46}
\]

In particular, the mild solution of the Eq. (46) is given by

\[
z(t) = \mathcal{S}_L(t) \begin{pmatrix} \frac{x}{\varphi} \\ \frac{f}{\varphi} \end{pmatrix} + \int_0^t \mathcal{S}_L(t - s) \begin{pmatrix} K_{0,\Lambda} u_s \\ 0 \end{pmatrix} ds. \tag{47}
\]

Let us introduce the new product state space

\[
\tilde{\mathcal{X}} = \mathcal{X} \times L^p([-r, 0], U),
\]
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where $\mathcal{X}$ is the product space defined in (18). Moreover, we select a new state function

$$w : t \in [0, +\infty) \mapsto w(t) = \begin{pmatrix} z(t) \\ u_t \end{pmatrix} \in \mathcal{X}.$$ 

In the upcoming sequel, we will demonstrate the validity of the assertion that the function $t \mapsto w(t)$ serves as the state solution for a well-posed and regular linear input-output system. Moreover, we will establish its profound connection to the integro-differential systems with delays presented in (38). To streamline our discussion, we introduce the notation $(SU, \Phi^U, \Psi^U, \Xi^U)$ to refer to the regular system derived from the regular triple $(Q^U, \beta^U, K_0)$. With this notation the input function is given by

$$u_t = SU(t)\psi + \Phi^U_t u, \quad t \geq 0, \quad \psi \in L^p([-1, 0], U).$$

Observe that for any $t \geq 0$,

$$z(t) = T_L(t) \begin{pmatrix} x \\ f \varphi \end{pmatrix} + M(t)\psi + N_t u, $$

where

$$M(t)\psi := \int_0^t T_L(t-s) \begin{pmatrix} (\Psi^U \psi)(s) \\ 0 \end{pmatrix} ds, $$

$$N_t u := \int_0^t T_L(t-s) \begin{pmatrix} (\Xi^U u)(s) \\ 0 \end{pmatrix} ds. $$

Thus the function $w$ can be rewritten as

$$w(t) = \tilde{T}(t) \begin{pmatrix} x \\ f \varphi \end{pmatrix} + \tilde{\Phi}_t u, $$

where

$$\tilde{T}(t) := \begin{pmatrix} T_L(t)M(t) \\ 0 \end{pmatrix}, \quad \text{and} \quad \tilde{\Phi}_t u := \begin{pmatrix} N_t u \\ \Phi^U_t u \end{pmatrix}. $$

A similar argument as in the proof of [11, Theorem 3.1] shows that the operators family $(\tilde{T}(t))_{t \geq 0}$ define a strongly continuous semigroup on $\tilde{\mathcal{X}}$ with generator

$$\tilde{\mathfrak{A}} = \begin{pmatrix} \mathfrak{A}_L & K \\ 0 & 0 \end{pmatrix}, \quad D(\tilde{\mathfrak{A}}) = D(\mathfrak{A}_L) \times D(Q^U).$$
Lemma 2 There exists an admissible control operator \( \tilde{B} \in L(U, \tilde{X}^-) \) for \( \tilde{A} \) such that the maps \( t \mapsto w(t) \) is the unique mild solution of the following control system

\[
\dot{w}(t) = \tilde{A}w(t) + \tilde{B}u(t), \quad w(0) = \left( \begin{array}{c} x \\ \psi \end{array} \right), \quad t \geq 0.
\]

Proof We know that (see Example 1)

\[
\| \Phi^U_t u \|_{L^p([0,t],U)} \leq \| u \|_{L^p([0,t],U)}
\]

for any \( t \geq 0 \) and \( u \in L^p(\mathbb{R}^+, U) \). Then by the Hölder inequality, we obtain

\[
\| \Phi u \|_{L^p([-r,0],U)} \leq c \| u \|_{L^p([0,t],U)}
\]

for some constant \( c = c(t) > 0 \). In addition, the fact that

\[
\Phi u = S(t) \Phi U u(t), \quad (\Phi u)(t + s) = (\Phi U u)(t) + (\Phi u)(t + s),
\]

implies that

\[
\tilde{A}u = \tilde{A} \tilde{T}_t u + \Phi u.
\]

According to [25], this functional equation shows that there exists an admissible control operator \( \tilde{B} \in L(U, \tilde{X}^-) \) for \( \tilde{A} \) such that

\[
\tilde{A}u = \tilde{T}_t u, \quad u \in L^p(\mathbb{R}^+, U). Thus the lemma follows from (48).
\]

Lemma 3 We select

\[
\tilde{C} := \begin{pmatrix} 0 & 0 & C & D \end{pmatrix} : D(\tilde{A}) \to Y.
\]

Then \( \tilde{C} \) is an admissible observation operator for \( \tilde{A} \).

Proof According to the expression the semigroup \( (\tilde{T}_t)_{t \geq 0} \), the fact that the operator \( D \in L(D(\tilde{Q}^U), Y) \) is admissible observation for \( \tilde{Q}^U \), and [10, Proposition 3.3] it suffices to show that the operator \( \mathcal{C} = (0 \ 0 \ C \ D) \in L(\tilde{A}) \) is an admissible observation operator for \( \tilde{A} \). To this end, as \( \tilde{A}_L = \tilde{A} + L \) and \( L \) is an admissible observation operator for \( \tilde{A} \), (see the proof of Theorem 3), by [12] it suffices to prove that \( \mathcal{C} \) is an admissible observation operator for \( \tilde{A} \). This is obvious due to the
expression of the semigroup \((T(t))_{t \geq 0}\) given in (25) and the fact that the operator \(C \in \mathcal{L}(D(Q), X)\) is an admissible observation operator for the left shift semigroup \((S(t))_{t \geq 0}\) on \(L^p([-r, 0], X)\).

The main result of this section is the following:

**Theorem 9** The triple \((\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})\) is regular on \(\tilde{\mathcal{X}}, U, Y\) and the integro-differential equation with state and input delays (38) is reformulated as the following input-output distributed linear system

\[
\begin{align*}
\dot{w}(t) &= \tilde{\mathcal{A}}w(t) + \tilde{\mathcal{B}}u(t), \quad w(0) = \begin{pmatrix} x \\ f \\ \varphi \end{pmatrix}, \quad t \geq 0, \\
y(t) &= \tilde{\mathcal{C}}w(t), \quad t \geq 0.
\end{align*}
\]

**Proof** Our first objective is to establish the well-posedness of the triple \((\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})\). To accomplish this, we can rely on the findings presented in Lemmas 2 and 3, which demonstrate that the operators \(\tilde{\mathcal{B}}\) and \(\tilde{\mathcal{C}}\) are admissible for \(\tilde{\mathcal{A}}\). However, in order to proceed, it becomes necessary to construct an input-output operator for the aforementioned triple. This necessitates the computation of the Yosida extension of \(\tilde{\mathcal{C}}\) for \(\tilde{\mathcal{A}}\). In fact, by taking the Laplace transform of \(\tilde{T}(t)\) we obtain

\[
R(\lambda, \tilde{\mathcal{A}}) = \begin{pmatrix}
R(\lambda, \mathcal{A}_L) & R(\lambda, \mathcal{A}_L) \left( \frac{KR(\lambda, Q_U)}{0} \right) \\
0 & 0 & R(\lambda, Q_U)
\end{pmatrix}, \quad \lambda \in \rho(\mathcal{A}_L).
\]

Let \((x \ f \ \varphi \ \psi)^\top \in \tilde{\mathcal{X}}\). For \(\lambda > 0\) sufficiently large we have

\[
\tilde{\mathcal{C}}\lambda R(\lambda, \tilde{\mathcal{A}}) \begin{pmatrix} x \\ f \\ \varphi \end{pmatrix} = \mathcal{C}\lambda R(\lambda, \mathcal{A}_L) \begin{pmatrix} x \\ f \\ \varphi \end{pmatrix} + \mathcal{C} R(\lambda, \mathcal{A}_L) \begin{pmatrix} K\lambda R(\lambda, Q_U) \psi \\ 0 \\ 0 \end{pmatrix} + D\lambda R(\lambda, Q_U)\psi.
\]

Observe that if \(\psi \in D(K_\Lambda)\), then

\[
\|\mathcal{C} R(\lambda, \mathcal{A}_L) \begin{pmatrix} K\lambda R(\lambda, Q_U) \psi \\ 0 \\ 0 \end{pmatrix}\| \\
\leq \|\mathcal{C} R(\lambda, \mathcal{A}_L)\| \left( \|K\lambda R(\lambda, Q_U)\psi - K_\Lambda \psi\| + \|K_\Lambda \psi\| \right).
\]

As \(\mathcal{C}\) is an admissible observation operator for \(\mathcal{A}_L\), it follows that \(\|\mathcal{C} R(\lambda, \mathcal{A}_L)\|\) goes to 0 when \(\lambda \to +\infty\). Then

\[
\lim_{\lambda \to +\infty} \|\mathcal{C} R(\lambda, \mathcal{A}_L) \begin{pmatrix} K\lambda R(\lambda, Q_U) \psi \\ 0 \\ 0 \end{pmatrix}\| = 0, \quad \forall \psi \in D(K_\Lambda).
\]
Then we have
\[ \Omega := D(\mathcal{E}_\Lambda) \times (D(K_\Lambda) \cap D(D_\Lambda)) \subset D(\mathcal{E}_\Lambda) \quad \text{and} \quad (\mathcal{E}_\Lambda)_{|\Omega} = (\mathcal{E}_\Lambda, D_\Lambda). \quad (49) \]

By [10, Proposition 3.3], for \( u \in L^p_{loc}(\mathbb{R}^+, U) \) we have
\[ N_t u \in D(\mathcal{F}_\Lambda) \quad \text{and} \quad \| \mathcal{F}_\Lambda N_t u \|_{L^p([0,\tau], Y)} \leq c \tau^{\frac{1}{p}} \| u \|_{L^p([0,\tau], X)} \]
\[ \leq c \tau^{\frac{1}{p}} |\mu| \left( [-\tau, 0] \right) \| u \|_{L^p([0,\tau], U)}, \]
for a.e. \( \tau \geq 0 \) where \( c > 0 \) is a constant and \( \frac{1}{p} + \frac{1}{q} = 1 \). On the other hand, the triple \((Q^U, \beta^U, K)\) and \((Q^U, \beta^U, D)\) are regular and having the same control maps. Then \( \Phi^U_t u \in D(K_\Lambda) \cap D(D_\Lambda) \) for a.e. \( \tau \geq 0 \). Moreover, the extended input-output operator \( \mathbb{F}^D \) of \((Q^U, \beta^U, D)\) is given by \((\mathbb{F}^D u)(\tau) = D_\Lambda \Phi^U_t u \) for almost every \( \tau \geq 0 \), and
\[ \| \mathbb{F}^D u \|_{L^p([0,\tau], Y)} \leq |\mu D| \left( [-\tau, 0] \right) \| u \|_{L^p([0,\tau], U)}, \]
Now according to (49), we have
\[ \tilde{\Phi}^U_t u \in \Omega \subset D(\mathcal{F}_\Lambda) \quad \text{and} \quad (\mathbb{F}^D u) := \mathcal{F}_\Lambda \tilde{\Phi}^U_t u = \mathcal{F}_\Lambda N_t u + (\mathbb{F}^D u)(\tau) \]
for a.e. \( \tau \geq 0 \) and all \( u \in L^p_{loc}(\mathbb{R}^+, U) \). Moreover
\[ \| \mathbb{F}^D u \|_{L^p([0,\tau], Y)} \leq c(\tau) \| u \|_{L^p([0,\tau], U)}, \quad \forall u \in L^p([0, \tau], U), \quad (50) \]
where
\[ c(\tau) := 2^{p-1} \left( c \tau^{\frac{1}{p}} |\mu| \left( [-\tau, 0] \right) + |\mu D| \left( [-\tau, 0] \right) \right) \longrightarrow 0. \]

Hence the triple \((\mathcal{F}, \tilde{\Phi}, \mathcal{F})\) is well-posed in \( \tilde{\mathcal{F}}, U, Y \). Let \( b \in U \) be a fixed control and set \( u_b(t) = b \) for any \( t \geq 0 \). By using the Hölder inequality and the estimate (50), we obtain
\[ \left\| \frac{1}{\tau} \int_0^\tau (\tilde{\mathbb{F}} u_b)(t) dt \right\| \leq c(\tau) \| b \|_U. \]

This implies that
\[ \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau (\tilde{\mathbb{F}} u_b)(t) dt = 0. \]
Thus the triple \((\mathcal{F}, \tilde{\Phi}, \mathcal{F})\) is regular in \( \tilde{\mathcal{F}}, U, Y \). \( \square \)

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