ADAPTIVE TESTING ON A REGRESSION FUNCTION AT A POINT

BY TIMOTHY ARMSTRONG

Yale University

We consider the problem of inference on a regression function at a point when the entire function satisfies a sign or shape restriction under the null. We propose a test that achieves the optimal minimax rate adaptively over a range of Hölder classes, up to a log log \( n \) term, which we show to be necessary for adaptation. We apply the results to adaptive one-sided tests for the regression discontinuity parameter under a monotonicity restriction, the value of a monotone regression function at the boundary and the proportion of true null hypotheses in a multiple testing problem.

1. Introduction. We consider a Gaussian regression model with random design. We observe \( \{(X_i, Y_i)\}_{i=1}^n \) where \( X_i \) and \( Y_i \) are real valued random variables with \( (X_i, Y_i) \) i.i.d. and

\[
Y_i = g(X_i) + \varepsilon_i, \quad \varepsilon_i | X_i \sim N(0, \sigma^2(X_i)), \quad X_i \sim F_X,
\]

where \( F_X \) denotes the c.d.f. of \( X_i \). We are interested in hypothesis tests about the regression function \( g \) at a point, which we normalize to be zero. We impose regularity conditions on the conditional variance of \( Y_i \) and the distribution of \( X_i \) near this point: for some \( \eta > 0 \),

\[
\eta t \leq |F_X(t) - F_X(-t)| \leq t/\eta, \quad \eta \leq \sigma^2(x) \leq 1/\eta \quad \text{for } |x| < \eta, \; 0 < t < \eta.
\]

Note that this allows (but does not impose) that our point of interest, 0, may be on the boundary of the support of \( X_i \).

We consider the null hypotheses

\[
H_0 : \{ g | g(x) = 0 \ \text{all } x \in \text{supp}(X_1) \},
\]

\[
H_0 : \{ g | g(x) \leq 0 \ \text{all } x \in \text{supp}(X_1) \},
\]

where \( \text{supp}(X_1) \) denotes the support of the distribution \( F_X \), and the alternative \( H_1 : \{ g | g(0) \geq b, \ g \in \mathcal{F} \}, \) where \( \mathcal{F} \) imposes smoothness conditions on \( g \). In particular, we consider Hölder classes of functions with exponent \( \beta \leq 1 \):

\[
\mathcal{F} = \Sigma(\beta, L) \equiv \{ g | |g(x) - g(x')| \leq L|x - x'|^{\beta} \text{ all } x, x' \},
\]
where \( L > 0 \) and \( 0 \leq \beta \leq 1 \), so that the alternative is given by

\[
H_1 : g \in \mathcal{G}(b, L, \beta) \equiv \{ g | g(0) \geq b \text{ and } g \in \Sigma(L, \beta) \}.
\]

The focus on \( g(0) \) is a normalization in the sense that the results apply to inference on \( g(x_0) \) for any point \( x_0 \) by redefining \( X_i \) to be \( X_i - x_0 \), so long as the point of interest \( x_0 \) is known. We also consider cases where certain shape restrictions are imposed under the null and alternative.

For simplicity, we treat the distribution \( F_X \) of \( X_i \) and the conditional variance function \( \sigma^2 \) as fixed and known under the null and alternative. Thus we index probability statements with the function \( g \), which determines the joint distribution of \( \{ (X_i, Y_i) \}_{i=1}^n \). We note, however, that the tests considered here can be extended to achieve the same rates without knowledge of these functions, so long as an upper bound for \( \sup_x \sigma^2(x) \) is known or can be estimated.

It is known [see Lepski and Tsybakov (2000)] that the optimal rate for testing the null hypothesis (3) or (4) against the alternative \( H_1 \) when \( g \) is known to be in the Hölder class \( \Sigma(L, \beta) \) is \( n^{-\beta/(2\beta+1)} \). That is, for any \( \varepsilon > 0 \), there exists a constant \( C_* \) such that, for any \( \alpha \in (0, 1) \) and sequence of tests \( \phi_n \) with level \( \alpha \) under the null hypothesis (3),

\[
\limsup_n \inf_{g \in \mathcal{G}(C_* n^{-\beta/(2\beta+1)}, L, \beta)} E g \phi_n \leq \alpha + \varepsilon.
\]

Furthermore, using knowledge of \( \beta \), one can construct a sequence of tests \( \phi^*_n \) that are level \( \alpha \) for the null hypothesis (4) [and, therefore, also level \( \alpha \) for the null hypothesis (3)] such that, for any \( \varepsilon > 0 \), there exists a \( C^* \) such that

\[
\liminf_n \inf_{g \in \mathcal{G}(C^* n^{-\beta/(2\beta+1)}, L, \beta)} E g \phi^*_n \geq 1 - \varepsilon.
\]

We ask whether a single test \( \phi_n \) can achieve the rate in (5) simultaneously for all \( \beta \leq 1 \). Such a test would be called adaptive with respect to \( \beta \). We find that the answer is no, but that adaptivity can be obtained when the rate is modified by a \( \log \log n \) term, which we show is the necessary rate for adaptation. In particular, we show that for \( C_* \) small enough, any sequence \( \phi_n \) of level \( \alpha \) tests of (3) must have asymptotically trivial power for some \( \beta \) in the class \( \mathcal{G}(C_* (n/\log \log n)^{-\beta/(2\beta+1)}, L, \beta) \) in the sense that for any \( \beta \leq 1 \),

\[
\limsup_n \inf_{\beta \in [\beta, 1]} \inf_{g \in \mathcal{G}(C_* (n/\log \log n)^{-\beta/(2\beta+1)}, L, \beta)} E g \phi_n \leq \alpha.
\]

Furthermore, we exhibit a sequence of tests \( \phi^*_n \) that achieve asymptotic power 1 adaptively over the classes \( \mathcal{G}(C^* (n/\log \log n)^{-\beta/(2\beta+1)}, L, \beta) \) for \( C^* \) large enough, while being level \( \alpha \) for the null hypothesis (4): for any \( \varepsilon > 0 \),

\[
\liminf_n \inf_{\beta \in [\varepsilon, 1]} \inf_{g \in \mathcal{G}(C^* (n/\log \log n)^{-\beta/(2\beta+1)}, L, \beta)} E g \phi^*_n = 1.
\]
Our interest in testing at a point stems from several problems in statistics and econometrics in which a parameter is given by the value of a regression or density function at the boundary, and where the function can plausibly be assumed to satisfy a monotonicity restriction. This setup includes the regression discontinuity model and inference on parameters that are “identified at infinity,” both of which have received considerable attention in the econometrics literature; see, among others, Andrews and Schafgans (1998), Chamberlain (1986), Hahn, Todd and Van der Klaauw (2001), Heckman (1990). In the closely related problem where \( g \) is a density rather than a regression function, our setup covers the problem of inference on the proportion of null hypotheses when testing many hypotheses; see Storey (2002). We discuss these applications in Section 3. The results in this paper can be used to obtain adaptive one-sided confidence intervals for these parameters, and to show that they achieve the minimax adaptive rate.

Problems closely related to those considered here have been considered in the literature on asymptotic minimax bounds in nonparametric testing, and our results draw heavily from this literature. Here, we name only a few, and refer to Ingster and Suslina (2003), for a more thorough exposition of the literature. Typically, the goal in this literature is to derive bounds in problems similar to the one considered here, but with the alternative given by \( \{ \varphi(g) \geq b \} \cap F \), where \( \varphi(g) \) is some function measuring distance from the null and \( F \) a class of functions imposing smoothness on \( g \). Our problem corresponds to the case where \( \varphi(g) = g(0) \) and \( F = \Sigma(L, \beta) \), where we focus on adaptivity with respect to \( \beta \leq 1 \). Lepski and Tsybakov (2000) consider this problem for fixed \((L, \beta)\), and also consider the case where \( \varphi(g) \) is the \( \ell_\infty \) norm. Pouet (1999) considers \( \varphi(g) = g(0) \) with \( F \) given by a class of analytic functions satisfying certain restrictions. Dümbgen and Spokoiny (2001) consider the \( \ell_\infty \) norm and adaptivity over Hölder classes with respect to \((L, \beta)\) and find, in contrast to our case, that adaptivity can be achieved without a loss in the minimax rate (or, for adaptivity over \( L \), even the constant). In these papers, the optimal constants \( C^* \) and \( C_* \) are also derived in some cases. Spokoiny (1996) considers adaptivity to Besov classes under the \( \ell_2 \) norm and shows that, as we derive in our case, the minimax rate can be obtained adaptively only up to an additional \( \log \log n \) term. It should also be noted that the tests we use to achieve the minimax adaptive rate bear a close resemblance to tests used in other adaptive testing problems; see, for example, Donoho and Jin (2004), Fan (1996), as well as some of the papers cited above.

Our results can be used to obtain one-sided confidence intervals for a monotone function at the boundary of its support, which complements results in the literature on adaptive confidence intervals for shape restricted densities. Low (1997) shows that adaptive confidence intervals cannot be obtained without shape restrictions on the function. Cai and Low (2004) develop a general theory of adaptive confidence intervals under shape restrictions. Cai, Low and Xia (2013) consider adaptive confidence intervals for points on the interior of the support of a shape restricted density and show that, in contrast to our case, the adaptive rate can be
achieved with no additional log log term. Dümbgen (2003) considers the related problem of adaptive confidence bands for the entire function. Our interest in points on the boundary stems from the specific applications considered in Section 3.

2. Results. We first state the lower bound for minimax adaptation. All proofs are in Section 4. For the purposes of some of the applications, we prove a slightly stronger result in which $g$ may be known to be nonincreasing in $|x|$. Let $G_{|x| \downarrow}$ be the class of functions that are nondecreasing on $(-\infty, 0]$ and nonincreasing on $[0, \infty)$.

**Theorem 2.1.** Let $0 < \beta < \overline{\beta} \leq 1$ be given. There exists a constant $C_*$ depending only on $\beta, \overline{\beta}, L$ and the bounds on $F_X$ and $\sigma$ such that the following holds: Let $\phi_n$ be any sequence of tests taking the data $\{(X_i, Y_i)\}_{i=1}^n$ to a rejection probability in $[0, 1]$ with asymptotic level $\alpha$ for the null hypothesis (3),

$$\limsup_n E_0 \phi_n \leq \alpha.$$ 

Then

$$\limsup_n \inf_{\beta \in [\beta, \overline{\beta}]} \inf_{G_{|X| \downarrow} \cap G_{|x| \downarrow}} E_g \phi_n \leq \alpha.$$ 

Note that the results of the theorem imply the same results when the requirement that $g \in G_{|x| \downarrow}$ is removed from the alternative, or when the null is replaced by (4) with the possible requirement $g \in G_{|x| \downarrow}$.

We now construct a test that achieves the $(n/\log \log n)^{\beta/(2\beta+1)}$ rate. For $k \in \{1, \ldots, n\}$, let $\hat{g}_k$ be the $k$-nearest neighbor estimator of $g(0)$, given by

$$\hat{g}_k = \frac{1}{k} \sum_{|X_j| \leq |X(k)|} Y_j$$

where $|X(k)|$ is the $k$th least value of $|X_i|$ for $|X(k)| < \eta$, and $\hat{g}_k = 0$ otherwise, where $\eta$ is given in (2). Let

$$T_n = \max_{1 \leq k \leq n} \sqrt{k} \hat{g}_k,$$

and let $c_{\alpha,n}$ be the $1-\alpha$ quantile of $T_n$ under $g(x) = 0$ all $x$. Note that by the law of the iterated logarithm [applied to the $N(0, 1)$ variables $Y_i/\sigma(X_i)$ conditional on the $X_i$’s], $\limsup_n c_{\alpha,n}/\sqrt{\log \log n} \leq \sqrt{2} \sup_x \sigma(x)$. Let $\phi^*_n$ be the test that rejects when $T_n > c_{\alpha,n}$.

**Theorem 2.2.** The test $\phi^*_n$ given above has level $\alpha$ for the null hypothesis (4). Furthermore, there exists a constant $C^*$ such that, for all $\varepsilon > 0$,

$$\lim \inf_n \inf_{\beta \in [\varepsilon, 1]} \inf_{G_{(n/\log \log n)^{-\beta/(2\beta+1)}, L, \beta}} E_g \phi^*_n = 1.$$ 

From the proof of Theorem 2.2, it can be seen that we can take $C^* = \sup_{\beta \in (0, 1]} (\sqrt{2} \sup_x \sigma(x)^2 K^{3/2})^2/(2^\beta+1)^2$ where $2/K$ is a lower
bound on \((F_X(t) - F_X(-t))/t\). However, this does not answer the question of the best possible constant for the test \(\phi_n\), or whether another test could achieve a better constant. While we leave these questions for future research, we briefly discuss some conjectures. We conjecture that, under additional regularity conditions on the conditional variance \(\sigma(x)\) and distribution of the covariate \(X_i\), a sharp constant \(C(\beta, L)\) exists such that, for arbitrary \(\delta > 0\), Theorem 2.1 holds with \(C_*\) replaced by \((1 - \delta)C(\beta, L)\) and \(\phi_n\) replaced by a different test. A reasonable candidate for a test statistic to achieve the optimal constant would be a supremum over \(\beta\) of normalized estimates based on the optimal kernel given in Example 1 of Lepski and Tsybakov (2000), with the bandwidth calibrated appropriately for each \(\beta\). The conjectured behavior where minimax adaptive power goes to \(\propto\) or one on either side of a constant, where the constant does not depend on the size \(\alpha\) of the test, would be an instance of asymptotic degeneracy related to the phenomenon observed for the \(\ell_\infty\) case by Lepski and Tsybakov (2000) (in the nonadaptive setting) and Dümbgen and Spokoiny (2001) (for adaptivity with respect to \(L\)), and our conjecture is based partly on the fact that the tests and approximately least favorable distributions over alternatives used in our results have a similar structure to those used in the above papers.

3. Applications and extensions.

3.1. Inference on a monotone function at the boundary. We note that in the case where 0 is on the boundary of the support of \(X_i\), the results in the previous section give the optimal rate for a one-sided test concerning \(g(0)\) under a monotonicity restriction on \(g\). This can be used to obtain adaptive (up to a log log \(n\) term) one-sided confidence intervals for a regression function at the boundary, where the log log \(n\) term is necessary for adaptation. This can be contrasted to the construction of adaptive confidence regions for a monotone function on the interior of its support, in which case the log log \(n\) term is not needed; cf. Cai, Low and Xia (2013).

To form a confidence interval based on our test, we define \(T_\alpha(\theta_0) = \max_{1 \leq k \leq n} \sqrt{k}(\hat{g}_k - \theta_0)\), and form our confidence interval by inverting tests of \(H_0: g(0) \leq \theta_0\) based on \(T_\alpha(\theta_0)\) with critical value \(c_{\alpha,n}\) given above (the \(1 - \alpha\) quantile under \(g = 0\) and \(\theta_0 = 0\)). The confidence interval is then given by \([\hat{\alpha}, \infty)\) where \(\hat{\alpha} = \max_{1 \leq k \leq n}[\hat{g}_k - c_{\alpha,n}/\sqrt{k}]\), with \(k\) the largest value of \(k\) such that \(|X(k)| < \eta\). The following corollary to Theorems 2.1 and 2.2 shows that this CI achieves the adaptive rate.

**Corollary 3.1.** Let \(0 < \beta < \beta \leq 1\) be given. There exists a constant \(C_*\) depending only on \(\beta, \beta, L\) and the bounds on \(F_X\) and \(\sigma\) such that the following holds. Let \([\hat{\alpha}, \infty)\) be any sequence of one-sided CIs with asymptotic coverage \(1 - \alpha\) for \(g(0)\) when \(g \in G_{|x|^\perp}\): \(\liminf_n \inf_{g \in G_{|x|^\perp}} P_g(g(0) \in [\hat{\alpha}, \infty)) \geq 1 - \alpha\). Then
\[
\limsup_n \inf_{\beta \in [\beta, \beta]} \inf_{g \in \Sigma(\beta, L) \cap G_{|x|^\perp}} P_g(\hat{\alpha} > g(0) - C_*(n/\log \log n)^{-\beta/(2\beta+1)}) \leq \alpha.
\]
Furthermore, the CI [\(\hat{c}^*, \infty\)] given above has coverage of at least 1 - \(\alpha\) for \(g \in \mathcal{G}_{|x|}\), and there exists a \(C^*\) such that, for all \(\varepsilon > 0\),

\[
\lim_{n \to \infty} \inf_{\beta \in [\varepsilon, 1]} \inf_{g \in \Sigma(\beta, L) \cap \mathcal{G}_{|x|}} P_g(\hat{c}^* > g(0) - C^*(n / \log \log n)^{-\beta/(2\beta+1)}) = 1.
\]

The problem of inference on a regression function at the boundary has received considerable attention in the econometrics literature, where the problem is often termed identification at infinity; see, among others, Andrews and Schafgans (1998), Chamberlain (1986), Heckman (1990), Khan and Tamer (2010). In such cases, it may not be plausible to assume that the density of \(X_i\) is bounded away from zero or infinity near its boundary, and the boundary may not be finite [in which case we are interested in, e.g., \(\lim_{x \to -\infty} g(x)\)]. Such cases require one to relax the conditions on \(F_X\) in (2), which can be done by placing conditions on the behavior of \(u \mapsto g(F_X^{-1}(u))\). In the interest of space, however, we do not pursue this extension.

3.2. Regression discontinuity. Consider the regression discontinuity model

\[Y_i = m(X_i) + \tau I(X_i > 0) + \varepsilon_i, \quad \varepsilon_i | X_i \sim N(0, \sigma^2(X_i)), \quad X_i \sim F_X.\]

Here, we strengthen (2) by requiring that there exists some \(\eta > 0\) such that, for all \(|x| < \eta\) and \(0 < t < \eta\), the inequalities \(\eta t \leq F_X(t) - F_X(0) \leq t/\eta\), \(\eta t \leq F_X(0) - F(-t) \leq t/\eta\) and \(\eta \leq \sigma^2(x) \leq 1/\eta\) are satisfied. The regression discontinuity model has been used in a large number of studies in empirical economics in the last decade, and has received considerable attention in the econometrics literature; see Imbens and Lemieux (2008) for a review of some of this literature.

We are interested in inference on the parameter \(\tau\). Of course, \(\tau\) is not identified without constraints on \(m(X_i)\). We impose a monotonicity constraint on \(m\) and ask whether a one-sided test for \(\tau\) can be constructed that is adaptive to the Hölder exponent \(\beta\) of the unknown class \(\Sigma(L, \beta)\) containing \(m\). In particular, we fix \(\tau_0\) and consider the null hypothesis

(7) \[H_0 : \tau \leq \tau_0 \text{ and } m \text{ nonincreasing}\]

and the alternative

\[H_1 : (m, \tau) \in \mathcal{G}^{\text{rd}}(b, L, \beta)
\]

\[\equiv \{(m, \tau) | \tau \geq \tau_0 + b \text{ and } m \in \Sigma(L, \beta) \text{ nonincreasing}\}.\]

We extend the test of Section 2 to a test that is level \(\alpha\) under \(H_0\) and consistent against \(H_1\) when \(b = b_n\) is given by a \(\log \log n\) term times the fastest possible rate simultaneously over \(\beta \in [\varepsilon, 1]\), and we show that the \(\log \log n\) term is necessary for adaptation.

To describe the test, let \(\{(X_{i,1}, Y_{i,1})\}_{i=1}^{n_1}\) be the observations with \(X_i \leq 0\), and let \(\{(X_{i,2}, Y_{i,2})\}_{i=1}^{n_2}\) be the observations with \(X_i > 0\). Let \(\hat{g}_{1,k}\) be the \(k\)-nearest
neighbor estimator given in (6) applied to the sample with $X_i \leq 0$, and let $\hat{g}_{2,k}$ be defined analogously for the sample with $X_i > 0$. Let

$$T_{n, \tau}^{rd} = \max_{1 \leq k \leq n} \sqrt{k}(\hat{g}_{2,k} - \hat{g}_{1,k} - \tau).$$

Let $c_{n, \alpha}^{rd}$ be the $1 - \alpha$ quantile of $T_{n, \tau}^{rd}(0)$ when $m(x) = 0$ all $x$ and $\tau = 0$. The test $\phi_{n, \tau_0}^{rd}$ rejects when $T_{n, \tau}^{rd}(\tau_0) > c_{n, \alpha}^{rd}$.

The following corollary to Theorems 2.1 and 2.2 gives the optimal rate for adaptive testing in the regression discontinuity problem, and shows that the test $\phi_{n, \tau_0}^{rd}$ achieves it. Let $E_{m, \tau}$ denote expectation under $(m, \tau)$.

**Corollary 3.2.** Let $0 < \beta < \bar{\beta} \leq 1$ be given. There exists a constant $C^*$ depending only on $\beta, \bar{\beta}, L$ and the bounds on $F_X$ and $\sigma$ such that the following holds.

$$\limsup_{n} E_{m, \tau} \phi_{n} \leq \alpha.$$

Furthermore, the test $\phi_{n, \tau_0}^{rd}$ given above has level $\alpha$ for the null hypothesis (4), and there exists a constant $C^*$ such that, for all $\epsilon > 0$,

$$\lim_{n \to \infty} \inf_{\beta \in [\epsilon, 1]} \inf_{(m, \tau) \in G_{rd}(C^*(n/\log \log n)^{-\beta/(2\beta + 1)}, L, \beta)} E_{m, \tau} \phi_{n, \tau_0}^{rd} = 1.$$

3.3. **Inference on the proportion of true null hypotheses.** Motivated by an application to large scale multiple testing, we now consider a related setting in which we are interested in nonparametric testing about a density, rather than a regression function. We observe $p$-values $\hat{p}_i = \frac{1}{n}$ from $n$ independent experiments. The $p$-values follow the mixture distribution

$$\hat{p}_i \sim f_p(x) = \pi \cdot I(x \in [0, 1]) + (1 - \pi) \cdot f_1(x),$$

where $f_1$ is an unknown density on $[0, 1]$ and $\pi$ is the proportion of true null hypotheses. We are interested in tests and confidence regions for $\pi$, following a large literature on estimation and inference on $\pi$ in this setting; see, among others, Storey (2002), Donoho and Jin (2004), Meinshausen and Rice (2006), Cai, Jin and Low (2007) and additional references in Efron (2010).

Given observations from the density $f_p(x)$ with $f_1(x)$ completely unspecified, the best bounds that can be obtained for $\pi$ in the population are $\pi \in [0, \pi]$, where $\pi = \inf_{x \in [0, 1]} f_p(x)$. If the infimum is known to be taken at a particular location $x_0$, we can test the null hypothesis that $\pi \geq \pi_0$ against the alternative $\pi < \pi_0$ by testing the null

$$H_0 : f_p(x) \geq \pi_0 \quad \text{all } x.$$
against the alternative \( f_p(x_0) < \pi_0 \). In other words, we are interested in a version of the problem considered in Section 2, with the regression function \( g \) replaced by a density function \( f_p \). Inverting these tests over \( \pi_0 \), we can obtain an upper confidence interval for \( \pi \). Note that since the null hypothesis \( \pi(f_p) \geq \pi_0 \) is equivalent to the statement that there exists a \( \pi \geq \pi_0 \) such that \( f_p \) follows model (8) for some \( f_1 \), this can also be considered a test of the null \( \pi \geq \pi_0 \), and the CI can be considered a CI for \( \pi \).

Assuming the \( p \)-values tend to be smaller when taken from the alternative hypothesis, we can expect that \( f_1(x) \) is minimized at \( x = 1 \) so that \( f_p(x) \) will also be minimized at 1. Following this logic, Storey (2002) proposes a uniform kernel density estimator of \( f_p(1) \), which can be considered an estimator of \( \pi \) or of \( \pi \) itself. (In the latter case, the estimator provides an asymptotic upper bound, but is not, in general, consistent.) We now consider the related hypothesis testing problem with the null given in (9) and with the alternative

\[
H_1 : f_p \in G^\pi_0(b, L, \beta) \equiv \{ f | f_p(1) \leq \pi_0 - b \text{ and } f_p \in \Sigma(L, \beta) \},
\]

which allows for an upper confidence interval for \( \pi \) (and \( \pi \) itself). Under the maintained hypothesis that the infimum is taken at 1, the rate at which \( b = b_n \) can approach 0 with \( H_1 \) and \( H_0 \) being distinguished gives the minimax rate for inference on \( \pi \) when the density under the alternative is constrained to the Hölder class \( \Sigma(L, \beta) \).

To extend the approach of the previous sections to this model, let \( \hat{\pi}(\lambda) = \frac{1}{n(1-\lambda)} \sum_{i=1}^{n} I(\hat{p}_i > \lambda) \) be the estimate of \( \pi \) used by Storey (2002) for a given tuning parameter \( \lambda \). We form our test by searching over the tuning parameter \( \lambda \) after an appropriate normalization

\[
T_n(\pi_0) = \max_{0 \leq \lambda < 1} \sqrt{n(1-\lambda)}[\pi_0 - \hat{\pi}(\lambda)],
\]

where we write max since the maximum is obtained. We define our test \( \phi_n(\pi_0) \) of (9) to reject when \( T_n(\pi_0) \) is greater than the critical value \( c_{n, \alpha}(\pi_0) \), given by the \( 1 - \alpha \) quantile of \( T_n(\pi_0) \) under the distribution \( \pi_0 \cdot \text{unif}(0, 1) + (1 - \pi_0) \cdot \delta_0 \), where \( \delta_0 \) is a unit mass at 0 and \( \text{unif}(0, 1) \) denotes the uniform distribution on \( (0, 1) \).

We note that \( T_n(\pi_0) \) is related to the test statistics used by Donoho and Jin (2004) and Meinshausen and Rice (2006), and can be considered a version of their approach that searches over the larger, rather than smaller, \( p \)-values. Donoho and Jin (2004) set \( \pi_0 = 1 \) and consider alternatives where \( \pi \) is close to one and the remaining \( p \)-values come from a normal location model with the mean slightly perturbed, achieving a certain form of adaptivity with respect to the amount of deviation of \( \pi \) and the normal location under the alternative. Meinshausen and Rice (2006) consider estimation of \( \pi \) in related settings with \( \pi \) close to one; see also Cai, Jin and Low (2007) for additional results in this setting. In contrast, \( T_n(\pi_0) \) looks at the larger ordered \( p \)-values in order to achieve adaptivity to the smoothness of the distribution of \( p \)-values under the alternative in a setting where \( \pi \) may not be close to 1.

We now state the result giving the adaptive rate for the test \( \phi_n(\pi_0) \).
THEOREM 3.1. The test $\phi_n(\pi_0)$ is level $\alpha$ for (9). Furthermore, there exists a constant $C^*$ such that, for all $\epsilon > 0$,
\[
\lim_{n \to \infty} \inf_{\beta \in [\epsilon, 1]} \inf_{G_{\pi_0}(C^*(n/\log \log n)^{-\beta/(2\beta+1)}, L, \beta)} E_{f_\beta} \phi_n = 1.
\]

Given the close relation between nonparametric inference on densities and conditional means [cf. Brown and Low (1996), Nussbaum (1996)], a lower bound for this problem analogous to the one given in Theorem 2.1 for the regression problem seems likely. However, in the interest of space, we do not pursue such an extension.

4. Proofs.

4.1. Proof of Theorem 2.1. The following gives a bound on average power over certain alternatives, and will be used to obtain a bound on minimax power over certain alternatives conditional on $X_1, \ldots, X_n$. Note that the bound goes to zero as $M \to \infty$ for $C < 1$.

**Lemma 4.1.** Let $W_1, \ldots, W_N$ be independent under measures $P_0$ and $P_1, \ldots, P_N$, with $W_i \sim N(0, s_i^2)$ under $P_0$ and $W_i \sim N(m_{i,k}, s_i^2)$ under $P_k$. Let $M$ and $\overline{M}$ be integers with $1 \leq 2M < 2\overline{M} \leq N$, and let $M = \overline{M} - M + 1$. Let $\phi$ be a test statistic that takes the data to a rejection probability in $[0, 1]$. Suppose that for some $C$,
\[
|m_{i,k}/s_i| \leq C \sqrt{\log M}/\sqrt{k} \quad \text{all } i, k
\]
and that $m_{i,k} = 0$ for $i > k$. Then
\[
\frac{1}{M} \sum_{j=M}^{\overline{M}} E_{P_j} \phi - E_{P_0} \phi \leq \frac{1}{M}(MC^2 - 1) + \frac{2}{M(\sqrt{2} - 1)} C^2(\log M)MC^2/\sqrt{2}
\]
\[
\equiv B(C, M).
\]

**Proof.** We express the average power as the following sample mean of likelihood ratios under the null, following arguments used in, for example, Lepski and Tsybakov (2000):
\[
\frac{1}{M} \sum_{j=M}^{\overline{M}} E_{P_j} \phi - E_{P_0} \phi = \frac{1}{M} \sum_{j=M}^{\overline{M}} \int_{P_0} d_{P_j} \phi - E_{P_0} \phi
\]
\[
= E_{P_0} \left\{ \frac{1}{M} \sum_{j=M}^{\overline{M}} \exp \left( \sum_{i=1}^{N} (\mu_{i,j} Z_i - \mu_{i,j}^2/2) \right) - 1 \right\} \phi,
\]
where \( \mu_{i,j} = m_{i,2j}/s_i \) and \( Z_i \equiv W_i/s_i \) are independent \( N(0, 1) \) under \( P_0 \). By Cauchy–Schwarz, the above display is bounded by the square root of

\[
\frac{1}{M^2} \sum_{j=M}^M \sum_{\ell=M}^M E_{P_0} \left[ \exp \left( \sum_{i=1}^N (\mu_{i,j} Z_i - \mu_{i,j}^2/2) \right) - 1 \right]
\]

\[
\times \left[ \exp \left( \sum_{i=1}^N (\mu_{i,\ell} Z_i - \mu_{i,\ell}^2/2) \right) - 1 \right]
\]

\[
= \frac{1}{M^2} \sum_{j=M}^M \sum_{\ell=M}^M \left[ \exp \left( \sum_{i=1}^N (\mu_{i,j} \mu_{i,\ell}) \right) - 1 \right]
\]

\[
\leq \frac{1}{M^2} \sum_{j=M}^M \left[ \exp(C^2 \log M) - 1 \right]
\]

\[
+ \frac{2}{M^2} \sum_{j=M}^M \sum_{\ell=M}^M \left[ \exp(C^2 (\log M) 2^{-|j-\ell|/2}) - 1 \right],
\]

where the equality follows from using properties of the normal distribution to evaluate the expectation, and the last step follows by plugging in the bound

\[
C[\sqrt{\log M}/\sqrt{2^k}]I(i \leq 2^k)
\]

for \( \mu_{i,k} = m_{i,2^k}/s_{2^k} \). Using the fact that \( \exp(x) - 1 \leq x \cdot \exp(x) \), the inner sum of the second term can be bounded by

\[
\sum_{\ell=M}^M C^2 (\log M) 2^{-|j-\ell|/2} \exp(C^2 (\log M) 2^{-|j-\ell|/2})
\]

\[
\leq C^2 (\log M) \exp(C^2 (\log M)/\sqrt{2}) \sum_{k=1}^\infty 2^{-k/2}
\]

\[
= C^2 (\log M) M^{C^2/\sqrt{2}} \frac{1}{\sqrt{2} - 1}.
\]

Plugging this into (10) and taking the square root gives the claimed bound. □

Before proceeding, we recall a result regarding uniform convergence of empirical c.d.f.s.

**Lemma 4.2.** Let \( Z_1, \ldots, Z_n \) be i.i.d. real valued random variables with c.d.f. \( F_Z \). Then, for any sequence \( a_n \) with \( a_n n \to \infty \),

\[
\sup_{F(z) \geq a_n} \left| \frac{(1/n) \sum_{i=1}^n I(Z_i \leq z) - F_Z(z)}{F_Z(z)} \right| \to 0.
\]
Let $P_X$ denote the product measure on the $X_i$’s common to all distributions in the model, and let $A_n$ be the event that

$$\eta t/2 \leq \frac{1}{n} \sum_{i=1}^{n} I(|X_i| \leq t) \leq 2t/\eta \quad \text{for all } (\log n)/n < t < \eta.$$ \hfill (11)

We will use the fact that $P_X(A_n) \to 1$, which follows by plugging condition (2) into the conclusion of Lemma 4.2 for $Z_i = |X_i|$.

We now construct a function in $\mathcal{G}(b, L, \beta)$ for each $\beta \in [\beta, \beta]$ that along with Lemma 4.1, can be used to prove the theorem.

**Lemma 4.3.** For a given $L$, $\beta$, $n$ and $c$, define

$$g_{\beta, n, c}(x) = \max \{c[(\log \log n)/n]^{\beta/(2\beta+1)} - L|x|^\beta, 0\}.$$  

Let $0 < \beta < \beta$ be given. For small enough $c$, we have the following. For any sequence of tests $\phi_n$ taking the data into a $[0, 1]$ rejection probability,

$$\lim_{n \to \infty} \inf_{\beta \in [\beta, \beta]} [E_{g_{\beta, n, c}} \phi_n - E_0 \phi_n] = 0.$$  

**Proof.** Let $\hat{N}(\beta) = \hat{N}(\beta, X_1, \ldots, X_n) = \sum_{i=1}^{n} I(L|X_i|^\beta \leq c[(\log \log n)/n]^{\beta/(2\beta+1)}) = \sum_{i=1}^{n} I(|X_i| \leq (c/L)^{1/\beta}[(\log \log n)/n]^{1/(2\beta+1)})$. Let $\eta > 0$ satisfy condition (2). Letting $N(\beta) = \eta^{-1} \cdot n \cdot [(\log \log n)/n]^{1/(2\beta+1)}$, we have, for $(c/L) \leq 1$, $\hat{N}(\beta) \leq 2N(\beta)$ for all $\beta \in [\beta, \beta]$ on the event $A_n$ defined in (11). Note that $N(\beta)/\log n = \eta^{-1}(n/\log \log n)^{2\beta/(2\beta+1)}$ and $g_{\beta, n, c}(x) \leq c[(\log \log n)/n]^{\beta/(2\beta+1)}$ for all $x$, so that

$$g_{\beta, n, c}(x) \leq c[(\log \log n)/n]^{\beta/(2\beta+1)} = c\eta^{-1/2}[N(\beta)/\log \log n]^{-1/2}$$  

for all $x$.

Let $M_n = \lfloor \log_2[2N(\beta)] \rfloor$ and $\overline{M}_n = \lfloor \log_2[2N(\beta)] \rfloor$, and let $\beta_{k,n}$ be such that $k = 2N(\beta_{k,n})$ (so that $\beta \leq \beta_{k,n} \leq \overline{\beta}$ for $2M_n \leq k \leq 2\overline{M}_n$). Let $M_n = \overline{M}_n - M_n - 1$, and note that $M_n \geq (\log n)/K$ for a constant $K$ that depends only on $\beta$ and $\overline{\beta}$. Plugging these into the bound in (12) yields the bound

$$\frac{g_{\beta_{k,n}, n, c}(x)}{\sigma(x)} \leq \frac{c\eta^{-1/2}\sqrt{2k^{-1/2}[\log(KM_n)]^{1/2}}}{\inf_{|x| < \eta} \sigma(x)} \leq 2c\eta^{-1}k^{-1/2}[\log M_n]^{1/2},$$  

where the last inequality holds for large enough $n$. [The last equality uses the fact that $\inf_{|x| < \eta} \sigma(x) \geq \eta^{1/2}$ for $\eta$, satisfying condition (2).]

Since $\hat{N}(\beta) \leq 2N(\beta)$ for all $\beta \leq \beta \leq \overline{\beta}$ on the event $A_n$, we have, on this event, letting $X_{(i)}$ be the observation $X_i$ corresponding to the $i$th least value of $|X_i|$,
\( |g_{\beta_n,k,n,c}(X(i))| = 0 \) for \( i > 2N(\beta_n,k) = k \) for all \( \beta \leq \beta_n,k \leq \bar{\beta} \). Using this and the bound in (13), we can apply Lemma 4.1 conditional on \( X_1, \ldots, X_n \) to obtain, for any test \( \phi \),

\[
\frac{1}{M_n} \sum_{j=M_n}^{M_n} E_{g_{\beta_n,2j,n,c}}(\phi|X_1, \ldots, X_n) - E_0(\phi|X_1, \ldots, X_n) \leq B(2c\eta^{-1}, M_n)
\]

on the event \( A_n \) for large enough \( n \). Thus

\[
\lim_{n \to \infty} \inf_{\beta \in [\underline{\beta}, \bar{\beta}]} E_{g_{\beta,n,c}} \phi_n - E_0 \phi_n 
\]

\[
\leq \lim_{n \to \infty} \frac{1}{M_n} \sum_{j=M_n}^{M_n} E_{g_{\beta_n,2j,n,c}} \phi_n - E_0 \phi_n 
\]

\[
\leq \lim_{n \to \infty} E_{P_X} \frac{1}{M_n} \sum_{j=M_n}^{M_n} [E_{g_{\beta_n,2j,n,c}}(\phi|X_1, \ldots, X_n) - E_0(\phi|X_1, \ldots, X_n)] I(A_n)
\]

\[
+ [1 - P_X(A_n)]
\]

\[
\leq \lim_{n \to \infty} B(2c\eta^{-1}, M_n) + [1 - P_X(A_n)].
\]

This converges to zero for small enough \( c \). \( \square \)

Theorem 2.1 now follows from Lemma 4.3 since \( g_{\beta,n,c} \in G(c[(\log \log n)/n]^{\beta/(2\beta+1)}, L, \beta) \).

4.2. Proof of Theorem 2.2. For the test \( \phi_n^* \), we have, for \( (b/L)^{1/\beta} < \eta \),

\[
\inf_{g \in G(b,L,\beta)} E_g(\phi_n^*|X_1, \ldots, X_n) 
\]

\[
\geq \inf_{g \in G(b,L,\beta)} P_g \left\{ \frac{\sum_{|X_i| \leq (b/L)^{1/\beta} Y_i}{\sqrt{\sum_{i=1}^{n} I(|X_i| \leq (b/L)^{1/\beta})}} > c_{\alpha,n}|X_1, \ldots, X_n} \right\}.
\]

Under \( P_g \), the random variable \( \frac{\sum_{|X_i| \leq (b/L)^{1/\beta} Y_i}{\sqrt{\sum_{i=1}^{n} I(|X_i| \leq (b/L)^{1/\beta})}} \) in the conditional probability statement above is, conditional on \( X_1, \ldots, X_n \), distributed as a normal variable with mean

\[
\frac{\sum_{|X_i| \leq (b/L)^{1/\beta}} g(X_i)}{\sqrt{\sum_{i=1}^{n} I(|X_i| \leq (b/L)^{1/\beta})}} \geq b \frac{\sum_{i=1}^{n} I(|X_i| \leq (b/(2L))^{1/\beta})}{2 \sqrt{\sum_{i=1}^{n} I(|X_i| \leq (b/L)^{1/\beta})}}
\]

and variance

\[
\frac{\sum_{|X_i| \leq (b/L)^{1/\beta}} \sigma^2(X_i)}{\sum_{i=1}^{n} I(|X_i| \leq (b/L)^{1/\beta})} \leq \sup_x \sigma^2(x),
\]

where \( \sigma^2(X_i) \) is the variance of the random variable \( g(X_i) \) given the event \( |X_i| \leq (b/L)^{1/\beta} \).
where the lower bound on the mean holds for $g \in \mathcal{G}(b, L, \beta)$ by noting that for $g \in \mathcal{G}(b, L, \beta)$, $g(x) \geq b - L|x|^{\beta}$, so $g(x) \geq 0$ for $|x| \leq (b/L)^{1/\beta}$, and for $|x| \leq [b/(2L)]^{1/\beta}$, $g(x) \geq b - L[(b/(2L)]^{1/\beta} = b/2$. Let $K = 2\eta^{-1}$. On the event $A_n$ defined in (11) (which holds with probability approaching one), for $b$ in the appropriate range, the right-hand side of (14) is bounded from below by

$$\frac{b}{2} \cdot \frac{(1/K) \cdot n \cdot (b/(2L))^{1/\beta}}{\sqrt{K \cdot n \cdot (b/L)^{1/\beta}}} = \frac{1}{2K\sqrt{K}} \cdot 2^{-1/\beta} \cdot L^{-1/(2\beta)} \sqrt{n} b^{1+1/(2\beta)}.$$

For $b = c(n/\log \log n)^{-\beta/(2\beta+1)},$ this is

$$\frac{1}{2K\sqrt{K}} \cdot 2^{-1/\beta} \cdot L^{-1/(2\beta)} c^{1+1/(2\beta)} \sqrt{\log \log n},$$

and for large enough $n,$ this choice of $b$ is in the range where the bounds in (11), (14) and (15) can be applied for all $\beta \in [\varepsilon, 1].$ Thus on the event $A_n,$ we have for large enough $c$,

$$\inf_{\beta \in [\varepsilon, 1]} \inf_{g \in \mathcal{G}(c(n/\log \log n)^{\beta/(2\beta+1)}, L, \beta)} E_g(\Phi_n^\ast | X_1, \ldots, X_n)$$

$$\geq \inf_{\beta \in [\varepsilon, 1]} 1$$

$$- \Phi\left(\frac{c_{\alpha, n} - (1/(2K\sqrt{K})) \cdot 2^{-1/\beta} \cdot L^{-1/(2\beta)} c^{1+1/(2\beta)} \sqrt{\log \log n}}{\sup_x \sigma(x)}\right).$$

By the law of the iterated logarithm applied to the i.i.d. $N(0, 1)$ sequence $\{Y_i/\sigma(X_i)\}_{i=1}^n,$ we have $c_{\alpha, n} \leq C\sqrt{\log \log n}$ for large enough $n$ for any $C > \sqrt{2}\sup_x \sigma(x).$ For $c > \sup_{\beta \in [0, 1]}(\sqrt{2}\sup_x \sigma(x))2K^{5/2}2^{1/\beta} L^{1/(2\beta)} 2^{2\beta/(2\beta+1)},$ it follows that the above display converges to 0 as $n \to \infty.$ [Note that, while we have defined $K = 2\eta^{-1}$ where $\eta$ is used to bound both $\sigma(x)$ and $F_X(t)$ in (2), $K$ is used in this proof only in bounding $F_X$ from below and can therefore be taken to be any constant such that $2/K$ is a lower bound on $(F_X(t) - F_X(-t))/t$ near zero.] Since this bound holds on an event with probability approaching one, the result follows.

4.3. Proof of Corollary 3.1. The first display follows by Theorem 2.1 since $\phi_n = I(\hat{c} > 0)$ is level $\alpha$ for $H_0 : g = 0,$ and the display is bounded by

$$\limsup_n \inf_{\beta \in [\beta_*, \beta]} \inf_{g \in \mathcal{G}(c(n/\log \log n)^{-\beta/(2\beta+1)}, L, \beta)} E_g I(\hat{c} > 0).$$

For the second display, note that for any constant $a,$ the distribution of $T_n(a)$ under $g$ is the same as the distribution of $T_n(0)$ under the function $g - a$ that takes $t$ to $g(t) - a.$ Thus

$$P_g(\hat{c}^* > g(0) - b) = P_g(T_n(g(0) - b) > c_{\alpha, n}) = P_{g-g(0)+b}(T_n(0) > c_{\alpha, n}).$$

Since $g - g(0) + b$ is in $\mathcal{G}(b, L, \beta)$ for any $g \in \Sigma(L, \beta),$ the result follows from Theorem 2.2.
4.4. Proof of Corollary 3.2. The proof of the second part of the corollary (the extension of Theorem 2.2) is similar to the original proof and is omitted. To prove the first part of the corollary (the extension of Theorem 2.1), assume, without loss of generality, that \( \tau_0 = 0 \). Define \( \text{sgn}(X_i) \) to be \(-1\) for \( X_i \leq 0 \) and \( 1 \) for \( X_i > 0 \). Note that, for any function \( g \in G(b, L/2, \beta) \cap G_{\|x\|} \), the function \( m_\tau(x) = g(x) \cdot \text{sgn}(X_i) - 2g(0)I(X_i > 0) \) is in \( \Sigma(L, \beta) \) and is nonincreasing. [To verify Hölder continuity, note that, for \( x, x' \) with \( \text{sgn}(x) = \text{sgn}(x') \), \( |m_\tau(x) - m_\tau(x')| \leq |g(x) - g(x')| \) and, for \( x, x' \) with \( \text{sgn}(x) \neq \text{sgn}(x') \), \( |m_\tau(x) - m_\tau(x')| = |g(x) - g(0)| + |g(x') - g(0)| \leq (L/2)|x|^\beta + (L/2)|x'|^\beta \leq L|x - x'|^\beta \), where the last step follows since \( |x - x'| \geq x \lor x' \).]

Note that under \( m = m_\tau, \tau = 2g(0) \), the regression function is \( x \mapsto m_\tau(x) + 2g(0)I(x_i > 0) = g(x) \cdot \text{sgn}(X_i) \) so that \( \{Y_i \cdot \text{sgn}(X_i), X_i\}_{i=1}^n \) are distributed according to the original regression model (1) with the given function \( g \). Of course, for \( m(x) = 0 \) all \( x \) and \( \tau = 0 \), the regression function is 0 for all \( x \). Thus, for any level \( \alpha \) test \( \phi_n \) of \( (m, \tau) = (0, 0) \), we can construct a test \( \phi_n^* \) of (3) in the original model (1) that has identical power at the power in the regression discontinuity model at \( (m_\tau, 2g(0)) \) for any \( g \) with \( g \in G(b, L/2, \beta) \cap G_{\|x\|} \) for some \( b, L \) and \( \beta \). Since \( (m_\tau, 2g(0)) \in G_{\text{rd}}(2b, L, \beta) \) whenever \( g \in G(b, L/2, \beta) \cap G_{\|x\|} \) by the argument above, it follows that

\[
\inf_{\beta \in [\beta_0, \beta]} \inf_{(m, \tau) \in G_{\text{rd}}(c(n/\log \log n)^{-\beta/(2\beta + 1)}, L, \beta)} E_{(m, \tau)} \phi_n \\
\leq \inf_{\beta \in [\beta_0, \beta]} \inf_{g \in G(c(n/\log \log n)^{-\beta/(2\beta + 1)}, L/2, \beta) \cap G_{\|x\|}} E_g \phi_n^* ,
\]

which converges to zero for \( c \) small enough by Theorem 2.1.

4.5. Proof of Theorem 3.1. We first show that the distribution used to obtain the critical value is least favorable for this test statistic, so that the test does in fact have level \( \alpha \).

**Lemma 4.4.** The distribution \( \pi_0 = \pi_0 \cdot \text{unif}(0, 1) + (1 - \pi_0)\delta_0 \), where \( \delta_0 \) is a unit mass at 0, is least favorable for \( T_n(\pi_0) \) under the null \( \pi \geq \pi_0 \),

\[
P_{f_p}(T_n(\pi_0) > c) \leq P_{\pi_0}(T_n(\pi_0) > c) \quad \text{for } f_p \text{ defined by (8) with } \pi \geq \pi_0 .
\]

**Proof.** For \( \hat{p}_1, \ldots, \hat{p}_n \) drawn from \( f_p = \pi_0 \cdot \text{unif}(0, 1) + (1 - \pi_0)f_1 \), let \( q_1, \ldots, q_n \) be obtained from \( \hat{p}_1, \ldots, \hat{p}_n \) by setting all \( \hat{p}_i \)'s drawn from the alternative \( f_1 \) to 0. Then \( T_n(\pi_0) \) weakly increases when evaluated at the \( q_i \)'s instead of the \( \hat{p}_i \)'s, and the distribution under \( f_p \) of \( T_n(\pi_0) \) evaluated with the \( q_i \)'s is equal to the distribution under \( f_\pi \) of \( T_n(\pi_0) \) evaluated with the \( \hat{p}_i \)'s. \( \square \)

The result now follows from similar arguments to the proof of Theorem 2.2 after noting that \( c_{n, \alpha}(\pi_0) / \sqrt{\log \log n} \) is bounded as \( n \to \infty \); cf. Shorack and Wellner (2009), Chapter 16.
Acknowledgments. The author would like to thank Alexandre Tsybakov, Matias Cattaneo, Yuichi Kitamura and Tony Cai for helpful discussions, and the Associate Editor and two anonymous referees for helpful comments.

REFERENCES

ANDREWS, D. W. K. and SCHAFGANS, M. M. A. (1998). Semiparametric estimation of the intercept of a sample selection model. Rev. Econ. Stud. 65 497–517. MR1637898

BROWN, L. D. and LOW, M. G. (1996). Asymptotic equivalence of nonparametric regression and white noise. Ann. Statist. 24 2384–2398. MR1425958

CAI, T. T., JIN, J. and LOW, M. G. (2007). Estimation and confidence sets for sparse normal mixtures. Ann. Statist. 35 2421–2449. MR2382653

CAI, T. T. and LOW, M. G. (2004). An adaptation theory for nonparametric confidence intervals. Ann. Statist. 32 1805–1840. MR2102494

CAI, T. T., LOW, M. G. and XIA, Y. (2013). Adaptive confidence intervals for regression functions under shape constraints. Ann. Statist. 41 722–750. MR3099119

CHAMBERLAIN, G. (1986). Asymptotic efficiency in semiparametric models with censoring. J. Econometrics 32 189–218. MR0864926

DONOHO, D. and JIN, J. (2004). Higher criticism for detecting sparse heterogeneous mixtures. Ann. Statist. 32 962–994. MR2065195

DUMBGEN, L. (2003). Optimal confidence bands for shape-restricted curves. Bernoulli 9 423–449. MR1997491

DUMBGEN, L. and SPOKOINY, V. G. (2001). Multiscale testing of qualitative hypotheses. Ann. Statist. 29 124–152. MR1833961

EFRON, B. (2010). Large-Scale Inference: Empirical Bayes Methods for Estimation, Testing, and Prediction. Institute of Mathematical Statistics (IMS) Monographs 1. Cambridge Univ. Press, Cambridge. MR2724758

FAN, J. (1996). Test of significance based on wavelet thresholding and Neyman’s truncation. J. Amer. Statist. Assoc. 91 674–688. MR1395735

HAHN, J., TODD, P. and VAN DER KLAAUW, W. (2001). Identification and estimation of treatment effects with a regression-discontinuity design. Econometrica 69 201–209.

HECKMAN, J. (1990). Varieties of selection Bias. The American Economic Review 80 313–318.

IMBENS, G. W. and LEMIEUX, T. (2008). Regression discontinuity designs: A guide to practice. J. Econometrics 142 615–655. MR2416821

INGSTER, Y. I. and SUSLINA, I. A. (2003). Nonparametric Goodness-of-Fit Testing Under Gaussian Models. Lecture Notes in Statistics 169. Springer, New York. MR1991446

KHAN, S. and TAMER, E. (2010). Irregular identification, support conditions, and inverse weight estimation. Econometrica 78 2021–2042. MR2768989

LEPSKI, O. V. and TSYBAKOV, A. B. (2000). Asymptotically exact nonparametric hypothesis testing in sup-norm and at a fixed point. Probab. Theory Related Fields 117 17–48. MR1759508

LOW, M. G. (1997). On nonparametric confidence intervals. Ann. Statist. 25 2547–2554. MR1604412

MEINSHAUSEN, N. and RICE, J. (2006). Estimating the proportion of false null hypotheses among a large number of independently tested hypotheses. Ann. Statist. 34 373–393. MR2275246

NUSSBAUM, M. (1996). Asymptotic equivalence of density estimation and Gaussian white noise. Ann. Statist. 24 2399–2430. MR1425959

POUET, C. (1999). On testing nonparametric hypotheses for analytic regression functions in Gaussian noise. Math. Methods Statist. 8 536–549. MR1755899

SHORACK, G. R. and WELLNER, J. A. (2009). Empirical Processes with Applications to Statistics. SIAM, Philadelphia, PA.
SPOKOINY, V. G. (1996). Adaptive hypothesis testing using wavelets. *Ann. Statist.* **24** 2477–2498. MR1425962

STOREY, J. D. (2002). A direct approach to false discovery rates. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **64** 479–498. MR1924302

WELLNER, J. A. (1978). Limit theorems for the ratio of the empirical distribution function to the true distribution function. *Z. Wahrsch. Verw. Gebiete* **45** 73–88. MR0651392

Department of Economics
Yale University
30 Hillhouse Ave
New Haven, Connecticut 06511
USA
E-mail: timothy.armstrong@yale.edu