Efficient pairwise tomography: reconstructing multiplex quantum networks in many-body systems

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We introduce the concept of pairwise tomography networks to characterise quantum properties in many-body systems and demonstrate an efficient protocol to measure them experimentally. Pairwise tomography networks are generators of multiplex networks where each layer represents the graph of a relevant quantifier such as, e.g., concurrence, quantum discord, purity, quantum mutual information, or classical correlations. We propose a measurement scheme to perform two-qubit tomography of all pairs showing exponential improvement in the number of qubits \(N\) with respect to previously existing methods. We illustrate the usefulness of our approach by employing it in diverse domains playing a key role in quantum technologies, namely, Dicke states, decoherence models, and critical spin chains.

I. INTRODUCTION

The identification, characterisation and measurement of quantum properties in complex many-body systems is one of the greatest challenges of modern quantum physics. We are currently approaching a paradigm shift: on the one hand the quantum technology revolution is gaining speed as larger and more sophisticated quantum computers [1], simulators [2] and communication networks [3] are being developed and commercialised. On the other hand, we are acquiring increasing evidence of the presence and role of quantumness in real complex systems, such as biological [4] and condensed matter systems [5].

We are now more than ever in need of novel multidisciplinary methods, drawing on recent advances in complex network science, to tackle the formidable task of describing emergent and collective behaviour of quantum systems of increasing size and complexity [6]. The potential benefits of such a merging of disciplines would be remarkable. They would give us powerful tools to investigate questions such as: does quantumness play a functional role in biological systems? How can we optimise navigation and data transmission in the future quantum communication networks? How to best engineer the quantum internet? How to simulate complex new materials? How can we use quantum computers to design new chemical reactions and for drug discovery?

The results presented in this article combine techniques and tools of complex network science, quantum information, quantum measurement theory, and condensed matter physics. We introduce a new powerful concept able to capture, describe and visualise at once a class of quantum and classical properties in \(N\)-qubit systems, and we present an efficient measurement scheme to experimentally observe such properties. We focus on pairwise quantities, that is, those that can be computed from the reduced two-qubit density operators obtained by tracing out the remaining \(N - 2\) qubits. Our main result is twofold. We firstly demonstrate in full generality how to perform pairwise tomography for all \(N(N-1)/2\) pairs of qubits with only \(\mathcal{O}(\log N)\) measurement settings. This constitutes an exponential improvement with respect to the expected scaling, which is polynomial in \(N\).

Secondly, we introduce the concept of quantum tomography multiplex, i.e., multilayer networks where the nodes are the qubits and, in every layer, the weighted links represent some (classical or quantum) pairwise quantity that can be directly obtained from the tomographic data. This results in a single mathematical object containing information about pairwise entanglement, mutual information, classical correlations, von Neumann entropy, quantum discord, or any other two-body quantifier which might be useful to characterise the many-body state, as well as the correlations between them.

We illustrate the potential and usefulness of quantum tomography multiplexes with a number of examples, ranging from the properties of some Dicke states, to correlation networks in open quantum systems, to spin chains displaying critical behaviour. We expect our results to become essential diagnostic and characterization tools not only in the field of quantum technologies, e.g., for quantum communication networks, quantum simulations, quantum computation, and quantum metrology and sensing, but also in quantum biology and quantum chemistry. Indeed, it is reasonable to assume that, in general, the characterisation of a complex state involving hundreds of qubits might be unfeasible unless a statistical perspective is taken, very much in the spirit of how the field of classical complex networks describes large complex structures.

The article is structured as follows. In Sec. [1] we present the measurement scheme to reconstruct the pair-
wise tomography network, and we prove that it scales logarithmically with \( N \). In Sec. III we apply our method to reconstruct and characterise the quantum and classical properties of W and GHZ states, system-environment states in an open quantum systems scenario, and the ground states of the Ising and XX spin chains in a transverse magnetic field.

II. MEASUREMENT SCHEME FOR EFFICIENT PAIRWISE TOMOGRAPHY

The tomographic reconstruction of the quantum state of two qubits \( i \) and \( j \) requires the measurement of the nine correlators of the form \( \langle \sigma^{(i)}_a \otimes \sigma^{(j)}_b \rangle \), where \( \sigma_a \) and \( \sigma_b \) represent Pauli matrices with \( a \) and \( b \) taking values \( x, y \), and \( z \). Therefore, characterising all pairwise density matrices in a system of \( N \) qubits involves measuring \( 9N(N-1)/2 \) observables. However, if all qubits can be locally measured in any desired basis in every experimental realisation, it is possible to arrange the measurements in such a way that a much smaller number of measurement settings is needed. For instance, a simple parallelization scheme, in which one measures all non-overlapping pairs of qubits at once, reduces the number of measurement settings by a factor \( \lfloor N/2 \rfloor \), thus bringing the number of required measurement setups to \( \mathcal{O}(N) \).

In this section, we introduce a measurement scheme that allows us to obtain all these observables using only \( \mathcal{O}(\log N) \) copies of the state. First, notice that all the correlators \( \langle \sigma^{(i)}_x \otimes \sigma^{(j)}_b \rangle \) can be obtained via a single measurement setting in which all qubits are projected in the \( x \) basis. The correlators in which the two qubits are measured in different bases require more careful thinking.

Our measurement scheme relies on the assignment of three different labels, \( a, b, \) and \( c \), to each qubit. These three labels are then taken to represent measurement bases for each qubit, in such a way that any two different letters represent two different directions, \( x, y \), or \( z \). By letting these three letters run over all the six possible orderings of measurement bases, it is guaranteed that all the non-trivial correlators for any two qubits with different letters will be covered. However, no non-trivial correlators are measured for those pairs with equal letters. Hence, in this scheme, we aim at finding the minimal set of qubit labelings such that all pairs of qubits are covered.

Let us assume, without loss of generality, that each qubit is indexed by a different integer between 0 and \( N - 1 \). These integers can be represented in base three using only \( \lceil \log_3 N \rceil \) digits, each of which can only take three different values. Our strategy is thus simple: we use \( \lceil \log_3 N \rceil \) labelings, indexed by \( l = 1, \cdots, \lceil \log_3 N \rceil \), such that in labeling \( l \) each qubit \( i \) is assigned the letter \( a, b, \) or \( c \), depending on the value of its \( l \)-th digit in the base-three representation of its index. Since any two different qubits have distinct indices, their base-three representation must have at least one different digit, so there will be at least one labeling in which their non-trivial correlators will be measured. Furthermore, it is clear that it is not possible to find a smaller number of labelings in which any two qubits are covered at least once; indeed, this would imply that one could assign a different string of length \( M \leq \lceil \log_3 N \rceil - 1 \), each of them containing only letters \( a, b, \) and \( c \), to each qubit. However, there are only \( 3^M \leq 3^{\lceil \log_3 N \rceil - 1} \) such strings, while \( N > 3^{\lceil \log_3 N \rceil - 1} \). Figure 3 illustrates the different labelings, as well as the pairs of qubits covered by each of them, for \( N = 20 \) qubits.

Overall, the required number of different measurement settings is

\[ 6 \lceil \log_3 N \rceil + 3. \]
This means that, for example, in the case of $N = 53$ qubits (the largest quantum computer available to date), we need only 27 measurement settings, as opposed to the 477 settings needed with the naïve parallel approach.

III. QUANTUM TOMOGRAPHY MULTIPLEX IN N-QUBIT SYSTEMS

Once all the measurements in the scheme have been performed, we can reconstruct the so-called pairwise tomography network, in which every pair of qubits is assigned its corresponding reduced density operator reconstructed from the tomographic data. This network can then be unfolded into a quantum tomography multiplex [7, 8], a multilayer network involving the qubits as nodes in which, in every layer, edges represent a different pairwise quantity.

In this work, we focus on six such quantities, namely mutual information, classical correlations, quantum discord [9], entanglement (measured via concurrence [10]), von Neumann entropy, and purity; to assign an edge between two qubits $i$ and $j$ in any of those layers, we simply compute the corresponding quantity from their reduced density matrix. For the classical correlations and quantum discord, non-symmetric quantities that depend on the choice of the measured qubit, we show the values obtained by performing the measurement on the qubit with the smallest index.

One should bear in mind that, in general, it is possible to obtain non-zero values in correlation-related quantities as a consequence of mere fluctuations due to the finite amount of experimental data. However, in order to unveil the complex topological structure of the correlations of a given state, we filter out those links whose numerical value can be regarded as statistically irrelevant. To assess which connections are statistically significant, we apply a simple criterion: we first reconstruct the quantum tomography multiplex of a fully separable state and, from it, we compute the mean and standard deviation of the weights in each layer, e.g. concurrence, mutual information, etc. With these values, we can then consider as statistically significant those links whose value is larger than the mean plus five standard deviations [11].

In this section, we present applications of the pairwise tomography multiplex in various fields of quantum technologies and condensed matter physics. All the multiplexes included in what follows have been obtained through a Qiskit [12] implementation of our measurement scheme [13], either in simulated experiments using Qiskit’s QASM Simulator or in real experiments in the IBM Q Experience device ibmq_ourense. The tomographic reconstruction of the two-qubit density operator is done using Qiskit’s tool, which employs a maximum-likelihood method proposed in Ref. [14].

A. Dicke and GHZ states

We begin by considering a class of paradigmatic entangled states which have been extensively studied in the literature: the Dicke states. An $N$-qubit Dicke state with $k$ excitations is defined as

$$|D^N_k\rangle = \frac{1}{\sqrt{{N\choose k}}} \sum_j P_j\{|1\rangle^\otimes k \otimes |0\rangle^{\otimes (N-k)}\}$$

(2)
FIG. 3. Tomography multiplex for a 5-qubit GHZ state ($|00000\rangle + |11111\rangle)/\sqrt{2}$. The upper plots are simulated, while the lower plots are run on a real IBM Q device (ibmq ourense). 15 different measurement circuits were run, with 8192 shots per circuit, the maximum allowed by the device. The results were filtered as explained in Sec. III: the tomography multiplex of the separable state was obtained experimentally on the same device.

where $\sum_j P_j \{ \}$ denotes the sum over all possible permutations of the positions of excitations in the state. Dicke states have gained widespread attention due to their usefulness in quantum metrology [15], quantum game theory [16], quantum networks [17], and, interestingly, they have been proven useful for combinatorial optimization problems with hard constraints [18, 19]. They have been experimentally implemented in a variety of physical platforms, from trapped ions [20] to cold atoms [21, 22], from superconducting qubits [24, 25] to photons [17, 26]. Remarkably, Dicke states of over 200 qubits have been recently created in a solid-state platform [27].

Here we consider, as an initial example of our measurement scheme, the lowest order ($k = 1$) Dicke state, known as W state. The entanglement of this highly symmetric state is very robust against particle loss: indeed, the state remains entangled even if any $N-2$ parties lose the information about their particle [28, 29]. This makes it particularly useful for quantum communication purposes as well as for robust quantum memories. In an $N$-qubit state, each randomly picked pair of qubits possesses the same amount of entanglement, their concurrence being equal to $2/N$. Their corresponding entanglement network is therefore a fully connected graph, as shown in Fig. 2, where we also present, as a simple benchmarking example, the pairwise tomography multiplex comprising concurrence, discord, classical correlations, mutual information, von Neumann entropy and purity for a 7-qubit W state. As expected, due to the symmetry of the state, all quantities generate fully connected graphs. The thick-
ness and color of the connecting links in the network give us information on the numerical value of the given quantifier. This tells us, for example, that quantum correlations, as measured by discord, are dominant with respect to classical correlations.

As a second simple benchmarking example, we consider another celebrated multipartite entangled state: the GHZ state $\ket{\text{GHZ}} = (\ket{0}^{\otimes N} + \ket{1}^{\otimes N})/\sqrt{2}$. This state has very different properties compared to the W state, and in fact rather opposite characteristics when considering specifically pairwise concurrence. Indeed, the GHZ state does not have any pairwise entanglement, despite being maximally entangled. The two-qubit reduced states, $\frac{1}{2}(\ket{00} + \ket{11})$, have zero discord and maximum classical correlations.

$N$-qubit GHZ states are easy to prepare experimentally on a gate-based quantum computer, as only one Hadamard and $N$ CNOT gates are required. This allows us to verify our scheme on a real noisy intermediate-scale quantum (NISQ) device, such as the IBM Q Experience computers. We developed a tool for IBM’s library Qiskit [12] that allows us to prepare the measurement circuits described in Sec. [1] available on GitHub.com [13]. We then prepared a 5-qubit GHZ state on the ibmq_ourense device and performed pairwise tomography. The result is shown in the lower plot of Fig. [3] where it is compared with the simulation (upper plot). While the concurrence and discord networks correspond to the theoretical ones, the effects of the decoherence induced by the noise substantially affect all other layers. We see that both classical correlations and mutual information are diminished, while pairwise entropy and purity take higher and lower values, respectively.

**B. Decoherence in open quantum systems**

The multiplex representation of quantum states can also be useful for understanding the relation between an open quantum system and its quantum environment, as well as among the different parts of the latter. In order to illustrate this, we apply our machinery to the simulation of a collisional model in which a system qubit decoheres as a result of the interaction with other ancillary qubits at random times. In particular, we assume that each ancilla collides only once and at a time exponentially distributed with rate $\lambda/n$, where $n$ is the number of ancillae, and that the interaction between the system and an ancilla, driven by the Hamiltonian $H_I = \frac{1}{2}\sigma^z \otimes \sigma^z$, can be considered instantaneous, resulting in the unitary transformation $U_\theta = e^{-i\frac{1}{2}\sigma^z \otimes \sigma^z}$, where $\theta = \lim_{t \to \infty} \tau t$ denotes the interaction strength and $\tau$ is the duration of the collision. Furthermore, we will consider the states of the system and an ancilla to be $\ket{+}_S$ and $\ket{0}_A$, respectively, before the collision.

It has been recently shown that this simple model can lead to the decoherence of the system even if the total state of system and ancillae remains fully separable at all times as a consequence of the randomness in the collision times [30]. However, to illustrate the potential of the multiplex representation, we will consider entangling interactions in the current manuscript, as they lead to more complex quantum states.

We further give a quantum origin to the randomness in the collision times through the introduction of $n$ emitters, initially in the excited state, which relax to their ground state emitting an ancilla that immediately col-
FIG. 5. Ising ground state for external magnetic field parameter $B = 0.89$. Concurrence is maximal between nearest neighbour spins and decreasing with the spin distance. Because of finite statistics, the values of concurrence between third-nearest neighbours become indistinguishable from fluctuations of non-entangled pairs, and the corresponding link is therefore filtered out. All other quantities generate fully connected graphs with discord being the least sensitive to spin-spin separation.

lides with the system qubit. Hence, if the initial state of the system is $|\psi_0\rangle_S$, the total state for $n = 1$ at time $t$ is given by $\sqrt{e^{-\lambda t}}|1\rangle_a \otimes |0\rangle_a \otimes |\psi_0\rangle_S + \sqrt{1-e^{-\lambda t}}|0\rangle_a \otimes U_\theta(|0\rangle_a \otimes |\psi_0\rangle_S)$; the generalisation for $n > 1$ is straightforward.

Although this dynamical process exhibits several interesting regimes as time evolves, we only focus on the long-time one here. In Fig. 4 we show the multiplex of the corresponding state for $N = 9$ (that is, with 4 emitter-ancilla pairs), at time $\lambda t = 1000$, and with entangling interaction strength $\theta = 2\pi/3$. The resulting multiplex network exhibits a complex structure from which it is easy to identify the role of every qubit, i.e. system, emitter, or ancilla, in the dynamics. The concurrence layer reveals that the system qubit is indeed entangled with all the ancillae but not to the emitters. Interestingly, despite the lack of entanglement between the different ancillae, these are nevertheless correlated, both at the classical and at the quantum levels, with non-zero classical correlations and discord (and, consequently, mutual information). Finally, the connectivity of the emitters reveals that, as expected at long times, they are in the ground state. This is consistent with the total lack of correlations with any other qubits and with the fact that the four emitters form a strongly connected clique in the purity layer; also, their connections are even deemed statistically irrelevant in the entropy layer.

C. Ising model

The quantum Ising model in a transverse field is one of the most studied condensed matter systems, and it has been experimentally simulated in a number of scenarios. It describes a spin $1/2$ chain with nearest neighbours interaction subjected to a transverse magnetic field, the Hamiltonian being

$$H = -\sum_{i=1}^N \sigma_i^z \sigma_{i+1}^z + B \sigma_i^x,$$

where $B$ is the magnetic field, and where we assume periodic boundary conditions. For $|B| > 1$ the ground state is in the paramagnetic phase, since the transverse field dominates with respect to the spin-spin interaction. For $|B| < 1$ the ground state is ferromagnetic, while at $B = \pm 1$ the system exhibits a quantum phase transition.

The entanglement properties of this model have been extensively studied in the literature and, specifically, it is known that pairwise entanglement is maximal between nearest neighbors and then decreases monotonically as the distance between the spins increases. This is indeed revealed in the first layer of the multiplex network for the ground state of the Ising model displayed in Fig. 5. The multiplex network reveals at a glance that, even when the value of entanglement between distant pairs is very small (as for spins 3 and 6, e.g.) and therefore not visible in the graph, all other correlations are present, both quantum and classical. Interestingly, discord seems to be less sensitive to distance than entanglement, and classical correlations are stronger between nearest neighbours and then more or less of the same intensity for all other spins. Finally, from entropy and purity we see that nearest neighbour pairs are less entangled with the rest of the spin chain, which is consistent with the fact that they show higher pairwise entanglement.
FIG. 6. Upper panel: concurrence network of the XX ground state of an \( N = 9 \) spin chain for different values of \( k \). Lower panel: pairwise multiplex for the ground state in the \( k = 2 \) zone showing the differences in the two-spin properties between bulk pairs, e.g., spins 5 and 6, and edge pairs, e.g. spins 1 and 2.

D. XX model

The full power of the pairwise tomography multiplex can be appreciated for systems displaying non trivial and non-homogeneous pairwise correlations. A perfect example is the spin-1/2 XX chain in a magnetic field, whose ground state possesses nontrivial topological order. In this model, the quasi-long-range order manifests itself in the formation of entangled edge states, with spins at the edge of the chain sharing entanglement quite differently from bulk spins \[35\].

The Hamiltonian of the spin chain is given by

\[
H = - \sum_{i=1}^{N} \frac{1}{2} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + B \sigma_i^z, \tag{4}
\]

with \( B \) the magnetic field. The system is characterized at finite size by an instability of the ground state determined by a sequence of energy level crossings as the magnetic field is varied. This gives rise to sudden jumps in pairwise entanglement which behaves non analytically. Following Ref. \[35\], we indicate with \( k \) the number of crossings, and with \( B_{k+1} < B < B_k \) the corresponding regions of magnetic field, where \( B_k = \cos(\pi k/(N + 1)) \) are the critical values of \( B \).

Pairwise entanglement jumps are very well captured by our concurrence networks (see Fig. 6), whose topology exhibits dramatic changes as we pass through different \( k \) zones. The concurrence networks also clearly illustrate the difference in the entanglement of spin pairs at the bulk of the chain with respect to the edge. Specifically, one sees immediately the formation of entangled edge
states (see, e.g., $k = 5$), indicating the onset of long-range order in the system.

Let us now look at the pairwise tomography multiplex. Comparing the concurrence and entropy layers, we observe a somewhat expected and yet interesting phenomenon: there is a correlation between the weight of the edge connecting two qubits $i$ and $j$ in the entropy layer and their corresponding strengths $s_i$ and $s_j$ in the concurrence layer, where the strength of a node in a weighted network is defined as the sum of the weights of the edges intersecting it. This effect is especially visible by comparing the pairs $(3, 7)$ and $(1, 9)$. This correlation is a consequence of the fact that the pairwise state of two qubits that are highly entangled with other qubits is highly mixed. A similar anti-correlation can be observed between concurrence and the parity layers. Finally, the discord, classical correlations and mutual information graphs are fully connected, showing that, even if pairwise entanglement is not present, other types of quantum and classical correlations are small but non-vanishing.

IV. CONCLUSIONS

We have introduced a new powerful concept for the characterization of quantum and classical properties in many-body systems: the quantum tomography multiplex. We have demonstrated that the full pairwise tomography network generating the multiplex can be efficiently reconstructed experimentally and we presented a measurement scheme showing exponential improvement with respect to the known scaling.

Applications of pairwise tomography networks to the investigation of quantum and classical properties of paradigmatic states, such as the W and GHZ states, as well as the ground states of strongly correlated many-body systems have been presented. These examples show that, through our new representation of two-body quantum systems, we can gain new insights about the physical properties of complex quantum systems.

We believe that our results bridge the gap between complex network science and quantum many-body physics, and as such they pave the way to a much sought cross-fertilization between these fields. This in turn may be a key ingredient for the emergence of a new approach to answer both fundamental and applicative questions in quantum biology, quantum chemistry, quantum technologies, and condensed matter physics.

Note – The measurement scheme, its implementation on the Qiskit platform as well as the majority of the results were developed and obtained during the 2019 hackathon event called Qiskit Camp Europe organized by IBM Q in Mürren, Switzerland. The implementation is accessible from the GitHub repository at [13]. We later became aware of two recent preprints that propose similar schemes for reducing the number of measurements required for $k$-wise tomography [36, 37]. However, it should be stressed that, although our scheme is less general in that we only consider pairwise (as opposed to $k$-wise) tomography to construct the multiplex representations, it has a better scaling due to the fact that we label the qubits using three letters instead of two. As a consequence, we only require $6 \lceil \log_3 N \rceil + 3$, instead of $6 \lceil \log_2 N \rceil + 3$, measurement settings.

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