Abstract

In this paper we show that classical notions from automata theory such as simulation and bisimulation can be lifted to the context of enriched categories. The usual properties of bisimulation are nearly all preserved in this new context. The class of enriched functors that correspond to functional bisimulations surjective on objects is investigated and appears to be “very close” to be open in the sense of Joyal and Moerdijk [4]. Seeing the change of base techniques as a convenient means to define process refinement/abstractions we give sufficient conditions for the change of base categories to preserve bisimularity. We apply these concepts to Betti’s generalized automata, categorical transition systems, and other exotic categories.

1 Introduction

In [10] it is shown that enrichments over a particular monoidal closed category may capture the notion of generalized metric spaces. Since these fundamental works, various mathematical objects have been successfully coded as enrichments. The long list includes sheaves [13], fibrations [1] and more recently again metric spaces and also quasi-uniform spaces [11] [12]. Betti introduced generalized automata qua enrichments in unpublished notes. Results regarding their minimal realization occur in [5]. In [6] it occurred the idea that $\mathcal{V}$-categories may represent abstract machines and the base $\mathcal{V}$ is seen a well structured set of computations. In this framework, refining the universe of computation becomes a change of base. This is the point of view that we develop in the present work.

We apply change-of-base techniques to define “good” notions of abstraction/refinement. Seeing $\mathcal{V}$-categories as processes performing their computations in $\mathcal{V}$, we claim that any reasonable change of base from $\mathcal{V}$ to $\mathcal{W}$ should preserve usual process equivalences. First we show that the notion of bisimulation generalizes to enrichments. Though the status of bisimulation in enriched categorical term remains unclear, it is pretty simple and behaves astonishingly well. To sum up all classical properties of the bisimulation of automata lift to enrichments. This fact yields a clear notion of “good” changes of base: the ones preserving bisimilarity. We investigate the conditions under which this happens and derive a sufficient one. Our work relies on recent results [2], [9] where changes of base are identified as two-sided enrichments. The latter are akin to geometric morphisms between categories of sheaves.
Eventually we treat a more elaborate example than Betti’s automata namely
the categorical transition systems as an illustration of our categorical machinery.

2 Enrichments over Bicategories

Enrichments over bicategories generalize the classical enriched category theory
over monoidal base categories (ECT). The latter being just one-object bicategories,
one could argue that both theories should formally be the same and it
is true to a certain extent. On the other hand, even classical ECT becomes less
so when the base category is not symmetric monoidal. In such a case, the best
one can hope for in order to have a well-behaved theory is biclosedness. It is a
rather unusual situation in mathematical practice which deals with enrichments
over \textbf{Ab} or over \textbf{sSet}, yet it seems to be standard in Computer Science where
computational paths are generally irreversible. It turns out that ECT over non-
symmetric monoidal categories is conveniently studied as a special case of ECT
over bicategories. In this paper we shall use enrichments over \textit{locally (pre)ordered}
bicategories. They are in fact very simple 2-categories and their simplicity of-
fers the pleasant fringe benefit that we can dispense with checking coherence
conditions. In this chapter we shall present briefly elements of the theory and
illustrates the relevance of these constructions with Betti’s automata and other
derivatives.

In what follows we shall denote by $\otimes$ the horizontal composition in a bicat-
ogy, written in the diagrammatic order.

**Definition 2.1** Let $\mathcal{V}$ be locally preordered \textit{bicategory}. An enrichment $\mathcal{A}$ over $\mathcal{V}$, also called a $\mathcal{V}$-category, is a set $\text{Obj}(\mathcal{A})$ along with mappings

$\quad (−)_+ : \text{Obj}(\mathcal{A}) \to \text{Obj}(\mathcal{V})$

and

$\quad \mathcal{A}(−, −) : \text{Obj}(\mathcal{A}) \times \text{Obj}(\mathcal{A}) \to \text{Ar}(\mathcal{V})$

such that

1. $\mathcal{A}(a, b) : a_+ \to b_+$ for all $a, b \in \text{Obj}(\mathcal{A})$;
2. $\text{id}_{a_+} \leq \mathcal{A}(a, a)$ for all $a \in \text{Obj}(\mathcal{A})$;
3. $\mathcal{A}(a, b) \otimes \mathcal{A}(b, c) \leq \mathcal{A}(a, c)$ for all $a, b, c \in \text{Obj}(\mathcal{A})$.

$\mathcal{A}$’s fiber $\mathcal{A}_x$ over $x$ is the set \{a $\in \text{Obj}(\mathcal{A})$ $| a_+ = x$\}

Given a $\mathcal{V}$-category $\mathcal{A}$, notice that

$\quad \text{Obj}(\mathcal{A}) = \prod_{x \in \mathcal{V}} \mathcal{A}_x$. 
Definition 2.2 Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories. A $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ is a map $f : \text{Obj}(\mathcal{A}) \to \text{Obj}(\mathcal{B})$ such that:

1. $(-)_+^\mathcal{A} = (-)_+^\mathcal{B} \circ f$;
2. $\mathcal{A}(a, b) \leq \mathcal{B}(fa, fb)$ for all $a, b \in \text{Obj}(\mathcal{A})$.

Given $\mathcal{V}$-functors $F, G : \mathcal{A} \to \mathcal{B}$, a $\mathcal{V}$-natural transformation from $F$ to $G$ is given by the family

$$\{ id_{a+} \leq B(fa, ga) \}_{a \in \text{Obj}(\mathcal{A})}$$

of 2-cells in $\mathcal{V}$.

Observe that there can be at most one $\mathcal{V}$-natural transformation from $F$ to $G$.

Definition 2.3 A bicategory $\mathcal{V}$ is biclosed if for any arrow $f : u \to v$ and any object $w$ of $\mathcal{V}$, the functors $- \otimes f : \mathcal{V}(w, u) \to \mathcal{V}(w, v)$ and $f \otimes - : \mathcal{V}(v, w) \to \mathcal{V}(u, w)$ are left adjoints. (This is exactly to say that $\mathcal{V}$ has all right Kan extensions and right liftings according to the terminology of [DaSt97]).

In this work, we shall mostly be concerned with quantaloids.

Definition 2.4 A quantaloid is a small, biclosed, locally cocomplete and locally ordered bicategory.

A familiar example of a quantaloid is $\text{Rel}$, the locally-ordered bicategory with

- objects: (small) sets;
- morphisms: $\text{Rel}(X, Y)$ is the set of the binary relations from $X$ to $Y$ ordered by inclusion;
- composition: the usual relational composition.

Proposition 2.1 Let $\mathcal{V}$ be a quantaloid. $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations form a locally preordered 2-category denoted $\mathcal{V}-\text{Cat}$.

Given a category $\mathcal{C}$, one can build a quantaloid $B(\mathcal{C})$ as follows. $B(\mathcal{C})$ has the same objects as $\mathcal{C}$. Given $c, c' \in \mathcal{C}$, the partial order $B(\mathcal{C})(c, c')$ is the powerset of $\mathcal{C}(c, c')$ ordered by inclusion. The horizontal composition

$$B(\mathcal{C})(c, c') \otimes B(\mathcal{C})(c', c'') \to B(\mathcal{C})(c, c'')$$

is pointwise and $id^B(\mathcal{C}) = \{ id^\mathcal{C} \}$. This bicategory admits right Kan extensions and right liftings.

Enrichments on $B(\mathcal{C})$ are lax functors from $\mathcal{C}$ to $\text{Rel}$. Suppose $\mathcal{A} \in B(\mathcal{C})-\text{Cat}$. There is a lax functor

$$F(\mathcal{C}) : \mathcal{C} \to \text{Rel} \quad c \mapsto \mathcal{A}_c$$

$$c \xrightarrow{f} c' \mapsto \{(a, a') \in \mathcal{A}_c \times \mathcal{A}_{c'} \mid f \in \mathcal{A}(a, a')\}$$
The construction reverses.

In order to emphasize the interpretation of $B(\mathbb{C})$-categories as computational devices, we use the automata-theoretic notation $a \xrightarrow{m} b$ for the arrow $m : a_+ \to b_+ \in A(a, b)$.

A $B(\mathbb{C})$-functor $F : \mathbb{A} \to \mathbb{B}$ is just a map $F$ from the states of $\mathbb{A}$ to those of $\mathbb{B}$, such that

1. $F(\mathbb{A}_c) \subseteq \mathbb{B}_c$ for all $c \in \mathbb{C}$;
2. $a \xrightarrow{m} a' \Rightarrow F(a) \xrightarrow{m} F(a')$ for all $a, a' \in \text{Obj}(\mathbb{A})$.

In other words, $F$ is a functional simulation. A $B(\mathbb{C})$-natural transformation $F \Rightarrow G : \mathbb{A} \to \mathbb{B}$ is given by the family of arrows

$$\left\{ F(a) \xrightarrow{id_a+} G(a) \right\}_{a \in \text{Obj}(\mathbb{A})}.$$

Recall that monoidal categories may be seen as one-object bicategories and monoidal functors correspond to lax functor between those. In the same way, quantales are one-object quantaloids. Seen as categories (more precisely as monoidal partial orders), quantales are complete and cocomplete.

Given a monoid $M$, the quantale of $M$-languages $C(M)$ (seen as a monoidal category) has objects the subsets of $M$ ordered by inclusion, its unit is the subset $\{id_M\}$. The tensor of $C(M)$ is the pointwise composition

$$L \otimes L' = \{ l \cdot l' \mid l \in L, l' \in L' \}$$

for $L, L' \in \wp(M)$. This quantale is generally not symmetric but always bi-closed, by construction. $C(M)$-enrichments were called generalized automata by Betti in his unpublished notes.

An extra motivation for considering enrichments over quantaloids rather than just on quantales comes from the following observation regarding slice categories.

Starting from a quantaloid $\mathcal{V}$ and a $\mathcal{V}$-category $\mathbb{A}$, the quantaloid $\mathcal{V}(\mathbb{A})$ is as follows. Its objects are those of $\mathbb{A}$. For any $a, b \in \text{Obj}(\mathbb{A})$, $\mathcal{V}(\mathbb{A})(a, b)$ is the preorder $\mathcal{V}(a_+, b_+) \downarrow \mathbb{A}(a, b)$. The composition of $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{V}(A)$ is the arrow of $\mathcal{V} f \otimes g \leq \mathbb{A}(a, b) \otimes \mathbb{A}(b, c) \leq \mathbb{A}(a, c)$. The identity in $a$ is the 2-cell $\text{id}_{a+} \leq \mathbb{A}(a, a) : a_+ \to a_+$.  

**Proposition 2.2** Given a quantaloid $\mathcal{V}$, a $\mathcal{V}$-category $A$, let $\mathcal{V}-\text{Cat} \downarrow \mathbb{A}$ denote slice 2-category over $\mathbb{A}$, then there is a 2-isomorphism

$$\mathcal{V}(\mathbb{A})-\text{Cat} \cong \mathcal{V}-\text{Cat} \downarrow \mathbb{A}$$
Actually the latter isomorphism is natural in \( \mathcal{A} \) in the following sense. Any \( \mathcal{V} \)-functor \( F : \mathcal{A} \to \mathcal{B} \) defines a 2-functor \( \mathcal{V}(F) : \mathcal{V}(\mathcal{A}) \to \mathcal{V}(\mathcal{B}) \), sending objects \( a \) to \( Fa \) and with components the embeddings
\[
\mathcal{V}(F)_{a,a'} : \mathcal{V}(a_+, a'_+) \downarrow \mathcal{A}(a, a') \to \mathcal{V}(a_+, a'_+) \downarrow \mathcal{B}(Fa, Fa').
\]
If there is a \( \mathcal{V} \)-natural \( F \Rightarrow G : \mathcal{A} \to \mathcal{B} \) which consists in our specific context in a collection of inequations
\[
\text{id}_{a_+} \leq \mathcal{B}(Fa, Ga) : a_+ \to a_+
\]
for all objects \( a \in \mathcal{A} \), then it is also a collection of arrows in \( \mathcal{V}(\mathcal{B}) \) from \( \mathcal{V}(F)(a) \) to \( \mathcal{V}(G)(a) \). This collection defines then a 2-natural transformation \( \mathcal{V}(F) \Rightarrow \mathcal{V}(G) \) as for all \( a, b \in \mathcal{A} \), \( - \circ \text{id}_a : \mathcal{V} \downarrow \mathcal{B}(Ga,Gb) \to \mathcal{V} \downarrow \mathcal{B}(Fa,Gb) \) and \( \text{id}_b \circ - : \mathcal{V} \downarrow \mathcal{B}(Fa,Gb) \) are the obvious embeddings. Now one may check that the assignments \( \mathcal{V}(\cdot) \) above define indeed a 2-functor \( \mathcal{V} \text{-Cat} \to 2 \text{-Cat} \).

We shall come back later on the above isomorphism when we treat the change of base bicategory.

Computationally, such slices can be seen as computational devices with interfaces, i.e. processes. Given a process \( \begin{pmatrix} X \\ \downarrow A \end{pmatrix} \), we refer to \( X \) as its implementation and to \( A \) as its interface. Put differently, the base (of the slice) \( A \) represents the part of \( A \) which is observable.

## 3 Bisimulation

With Betti’s automata in mind, we introduce now a quantaloid-enriched version of the well-known simulation and bisimulation for automata. These notions extend well to enrichments and the main results of the theory still hold in this more general setting.

In this section \( \mathcal{V} \) will denote a quantaloid.

Consider \( \mathcal{V} \)-enrichments \( \mathcal{A} \) and \( \mathcal{B} \). Let us first consider a relation \( R \) from \( \mathcal{A} \) to \( \mathcal{B} \), i.e. \( R \subseteq \text{Obj}(\mathcal{A}) \times \text{Obj}(\mathcal{B}) \), such that if \( (a, b) \in R \) then \( a_+ = b_+ \). \( R \) is a simulation from \( \mathcal{A} \) to \( \mathcal{B} \) if for all \( (a, b) \in R \),
\[
\forall a' \in \mathcal{A}, \quad \mathcal{A}(a, a') \leq \bigvee_{(a', b') \in R} \mathcal{B}(b, b')
\]
\( R \) is a bisimulation if and only if both \( R \) and \( R^{-1} \) are simulations.

For any \( \mathcal{V} \)-enrichments \( \mathcal{A} \) and \( \mathcal{B} \), any union of simulations (respectively bisimulations) from \( \mathcal{A} \) to \( \mathcal{B} \) is a simulation (respectively a bisimulation), therefore there exist a larger simulation and a larger bisimulation from \( \mathcal{A} \) to \( \mathcal{B} \). For \( a \in \mathcal{A} \),
and \( b \in B \) we say that \( b \) simulates \( a \) (respectively \( a \) bisimulates \( b \)) if there is a simulation from \( A \) to \( B \) (respectively a bisimulation) \( R \) with \((a, b) \in R\).

With the notation above, let \( i_a : B_a \to B \) denotes the full subcategory of \( B \) with objects those \( b \) with \((a, b) \in R\). Then to say that \( R \) is a simulation is to say that for any \((a, b) \in R\), \( A(a, -) \) is less than the colimit of \( B(b, -) \circ i_a : B_a \to V \).

Consider a map \( f : Obj(B) \to Obj(A) \) such that for all \( a \in Obj(A)\), \((f(a))_+ = a_+ \). It is a particular relation, say \( R \), from \( A \) to \( B \) and as such:

- \( R \) is a simulation from \( A \) to \( B \) if and only if

\[
(1) \forall b \in B, a' \in A, \ A(f(b), a') \leq \bigvee_{b' | f(b') = a'} B(b, b')
\]

- \( R^{-1} \) is a simulation if and only if

\[
(2) \forall b, b' \in B, \ B(b, b') \leq A(f(b), f(b')).
\]

Condition (2) is just that \( f \) defines a \( V \)-functor \( B \to A \). When (2) holds, (1) is equivalent then to

\[
(1') \forall b, a' \in A, \ A(f(b), a') = \bigvee_{b' | f(b') = a'} B(b, b').
\]

So that a functionnal bisimulation amounts to a \( V \)-functor satisfying (1').

With the notation above, let \( i_a : B_a \to B \) denotes the fiber of \( f \) over \( a \), that is the full subcategory of \( B \) with objects those \( b \) with \( f(b) = a \). Then to say that \( f \) satisfies (1') is to say that the representable module \( A(f(b), -) \) is given by the colimit

\[
A(f(b), a') = \text{colim}(B(b, -) \circ i'_a : B_a \to V).
\]

We shall say that:

- \( B \) simulates \( A \) when there exists a simulation from \( A \) to \( B \) such that

\[
\text{for all } a \in A, \text{ there exists } b \in B, (a, b) \in R.
\]

- \( A \) and \( B \) are bisimilar when there exists a bisimulation \( R \) from \( A \) to \( B \) such that

- for all \( a \in A \), there exists \( b \in B \) such that \((a, b) \in R\) and
- for all \( b \in B \), there exists \( a \in A \) such that \((a, b) \in R\).
In particular if \( f : A \to B \) is a functional bisimulation that is surjective on objects then \( A \) and \( B \) are bisimilar.

**Example 1** [Automata!]

Obviously our notions of simulation, bisimulation extends the classical ones for automata. For an alphabet \( \Sigma \), a relation of simulation, respectively bisimulation, between \( \Sigma \)-automata \( A \) and \( B \) is exactly a simulation, respectively a bisimulation between the \( C(\Sigma^*) \)-enrichments \( A \) and \( B \). Also morphism of \( \Sigma \)-automata \( f : A \to B \) is a functional bisimulation if and only if the corresponding \( C(\Sigma^*) \)-functor \( f \) is. Remember that these maps were called open in [3] as their whole class satisfies axioms \((A1) \)–\((A5)\) for open maps defined by Joyal and Moerdijk. The maps of the above kind are of particular importance in the classical theory of bisimulation for automata. As shown further it is also the case for the bisimulation of enrichments.

**Example 2** [Betti’s generalized automata]

Things are analogous for Betti’s automata. Let \( M \) be a monoid. Recall from section 2 that a generalized automaton is an enrichment over the quantale of languages \( C(M) \). A morphism of generalized automata \( f : A \to B \) is just \( C(M) \)-functor and it defines a surjective bisimulation if and only if for all \( a \in A \), for all \( b' \in B \), for all \( m \in M \),

\[
\text{If } f(a) \xrightarrow{m} b' \text{ then there exists } a' \in A \text{ such that } f(a') = b' \text{ and } a \xrightarrow{m} a'.
\]

One may also easily rephrase in automata theoretic terms, our notions of simulation and bisimulations for \( C(M) \)-enrichments.

**Examples 3,4** [preorders,metric spaces] In the context of pre-orders ([10]) a simulation relation from \( A \) to \( B \) is a relation \( R \subseteq A \times B \) such that if \( (a, b) \in R \) then for all \( a' \geq a \in A \), there exists a \( b' \geq b \) such that \( (a', b') \in R \). In the context of generalized metric spaces, a simulation relation of \( A \) by \( B \) is a relation \( R \subseteq A \times B \) such that if \( (a, b) \in R \) then for all \( a' \in A \), for all \( \epsilon > 0 \) there exists a \( b' \in B \) such that \( (a', b') \in R \) and \( A(a, a') \leq B(b, b') + \epsilon \). We leave to the reader to decode what bisimulations mean in those contexts.

**Example 5** [Simulation/bisimulation over \( \Delta \)] Via the correspondence \( \mathcal{V} \text{-} \text{Cat}(\Delta) \cong \mathcal{V} \text{-} \text{Cat} \downarrow \Delta \), simulation and bisimulation relations in \( \mathcal{V} \text{-} \text{Cat}(A) \) occurs as simulations/bisimulations over \( \Delta \). That is to say: a simulation over \( \Delta \) of the arrow \( f : B \to A \) by the arrow \( g : C \to A \) is a simulation \( R \) of \( B \) by \( C \) such that if \( (b, c) \in R \) then \( f(c) = g(b) \). Again we leave to the reader to define the bisimulation over \( \Delta \).

A few immediate remarks are in order. It is straightforward to check that simulations/bisimulations compose. Also given a \( \mathcal{V} \)-enrichment \( \Delta \), the diagonal \( \Delta_A \) on \( \text{Obj}(\Delta) \) is a bisimulation on \( A \) (i.e. from \( \Delta \) to \( \Delta \)) and also an equivalence. So that

**Proposition 3.1** Enrichments and simulation relations ordered by inclusion form a locally preordered 2-category.

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Also for any enrichments $A$, $B$ and $C$ if $B$ simulates $A$ and $C$ simulates $B$ then $C$ simulates $A$ so that

**Proposition 3.2** The relation of simularity is a preorder on $\mathcal{V}$–$\mathbf{Cat}$.

Along the same line, bisimulation relations do compose and as the inverse relation of a bisimulation is a bisimulation,

**Proposition 3.3** The bisimularity relation is an equivalence on $\mathcal{V}$–$\mathbf{Cat}$.

For any enrichment $A$, the set of bisimulations on $A$ - i.e. from $A$ to $A$ contains the diagonal, and is closed under composition, inverse and unions. So if $R$ is a bisimulation then the equivalence $\bar{R}$ generated by $R$ is again a bisimulation. Bisimulation relations on $A$ ordered by inclusion form a complete lattice $\text{Bisim}(A)$, Bisimulation equivalences on $A$ ordered by inclusion also form a complete lattice $\text{EqBis}(A)$ and the map $R \mapsto \bar{R}$ is an upper closure operation (i.e. for $\text{Bisim}(A)$ and $\text{EqBis}(A)$ seen as categories, one has a reflection situation: the inclusion $\text{EqBis}(A) \to \text{Bisim}(A)$ has as a left adjoint sending any $R$ to $\bar{R}$).

Consider now a bisimulation equivalence $\sim$ on $A$. It actually defines a “congruence” on $A$ in the following sense.

**Proposition 3.4** The quotient set $\text{Obj}(A)/\sim$ admits a $\mathcal{V}$-categorical structure $\tilde{A}$ and the quotient map $\text{Obj}(A) \to \text{Obj}(A)/\sim$ defines a $\mathcal{V}$-functor $A \to \tilde{A}$.

If $[a]$ denotes the equivalence class of $A$ under $\sim$, let for any $a, b \in A$,

$$\tilde{A}(a, b) = \bigvee_{b' \mid b' \sim b} A(a, b').$$

Then it is immediate that if $b \sim b'$ then for all $a \in A$, $\tilde{A}(a, b) = \tilde{A}(a, b')$. It also happens that if $a \sim a'$ then $\tilde{A}(a, -) = \tilde{A}(a', -)$. To see this let us suppose $a \sim a'$. So that, since $a'$ simulates $a$,

$$\forall b, \ A(a, b) \leq \bigvee_{b' \sim b} A(a', b')$$

So for any $b$,

$$\tilde{A}(a, b) = \bigvee_{b' \sim b} \tilde{A}(a, b') \leq \bigvee_{b' \sim b} \bigvee_{b'' \sim b'} A(a', b'') = \bigvee_{b'' \sim b} \tilde{A}(a', b'') = \tilde{A}(a', b).$$

Therefore it makes sense to define for $a, b \in A$, $\tilde{A}([a],[b])$ as $\tilde{A}(a', b')$ for whatever $a' \in [a]$ and $b' \in [b]$.

Now for any $a, b, c \in A$,
\[ \widehat{A}([a], [b]) \otimes \widehat{A}([b], [c]) = (\bigvee_{b' \sim b} \widehat{A}(a, b')) \otimes \widehat{A}([b], [c]) \]
\[ = \bigvee_{b' \sim b} ( \widehat{A}(a, b') \otimes \widehat{A}([b], [c])) \]
\[ = \bigvee_{b' \sim b} (\widehat{A}(a, b') \otimes \bigvee_{c' \sim c} \widehat{A}(b', c')) \]
\[ = \bigvee_{b' \sim b} \bigvee_{c' \sim c} (\widehat{A}(a, b') \otimes \widehat{A}(b', c')) \]
\[ \leq \bigvee_{c' \sim c} \widehat{A}(a, c') \]
\[ = \widehat{A}([a], [c]). \]

As for any \(a, b \in A\), it is immediate that \(A(a, b) \leq \widehat{A}([a], [b])\) then \(A\) defines a \(\mathcal{V}\)-category and the map \(a \mapsto [a]\) defines a \(\mathcal{V}\)-functor \(A \to \widehat{A}\). This map is surjective and defines actually a bisimulation from \(A\) to \(\widehat{A}\) as condition (1’) is satisfied by the very definition of \(\widehat{A}\).

We shall now relate bisimilarity to the existence of spans and cospans of surjective functional bisimulations. With cospans things are working well without extra assumptions.

**Proposition 3.5** \(A\) and \(B\) are bisimilar if and only if there exists a cospan of functional bisimulation \(\xymatrix{ & B \ar[dl] \ar[dr] & }\).

**Proof:** We know already that such a span implies that \(A\) and \(B\) are bisimilar. It remains to prove the converse. Suppose \(A\) and \(B\) bisimilar via \(R \subseteq A \times B\). If \(\sim\) denotes the equivalence bisimulation on \(A\) generated by \(R^{-1} \circ R\) and also the equivalence on \(B\) generated by \(R \circ R^{-1}\) then it happens that \(A/\sim \cong B/\sim\). To see this note that equivalence classes for \(\sim\) on \(A\) correspond sequences \(a_0, a_1, a_2, \ldots\) where \((a_i, b_i) \in R\) and \((a_{i+1}, b_i) \in R\) and that contains at least one \(a_i\) and one \(b_i\). Sequences as above also correspond bijectively to equivalence classes of \(\sim\) on \(B\). Let us write \(\sim\) again for the bijection above between classes \([a] \in \text{Obj}(A)/\sim\) and \([b] \in \text{Obj}(B)/\sim\). To see that the later defines a \(\mathcal{V}\)-equivalence one needs to show: \(A/\sim ([a], [a']) = B/\sim ([b], [b'])\) for any \(a, a', b, b'\) with \([a] \sim [b]\) and \([a'] \sim [b']\). Given such \(a, a', b, b'\), one may suppose \((a, b) \in R\) and \((a', b') \in R\) so that

\[
(\sim)\big((a), (a')\big) = \bigvee_{a'' \sim a} A(a, a'') \]
\[
\leq \bigvee_{a'' \sim a} \bigvee_{b'' \sim [b'] \sim [b]} B(b, b') \quad \text{since} \ (a, b) \in R \\
\leq \bigvee_{b'' \sim [b']} B(b, b') \quad \text{since} \ (a, b) \in R \\
= B/\sim ([b], [b']).
\]

To characterise the bisimilarity in terms of spans of surjective functional bisimulations we considered some extra assumptions on the base quantaloid. We shall call a quantaloid \(\mathcal{V}\) **locally distributive** when the local preorders \(\mathcal{V}(a, b)\) are distributive.
Proposition 3.6 If $V$ is locally distributive, $V$-categories $A$ and $B$ are bisimilar if and only if there exists a span of functional bisimulation

\[
\begin{array}{c}
A \\
\downarrow \\
\bullet \\
\downarrow \\
B
\end{array}
\]

The claimed result follows from the following two lemmas.

Lemma 3.1 For any quantaloid $V$, $V$-$\text{Cat}$ has pullbacks.

Proof: Given $V$-categories $A$ and $B$, their (cartesian) product $A \times B$ is as follows. Its set of objects is the subset cartesian product $\text{Obj}(A) \times \text{Obj}(B)$ of pairs $(a, b)$ with $a_+ = b_+$, its homsets are given by the formula

\[
(A \times B)((a, b), (a', b')) = A(a, a') \land B(b, b'),
\]

the compositions

\[
\mu_{(a, b), (a', b')} : (A \times B)((a, b), (a', b')) \otimes (A \times B)((a', b'), (a'', b'')) \leq (A \times B)((a, b), (a'', b''))
\]

are given by:

\[
\leq (A(a, a') \land B(b, b')) \otimes (A(a', a'') \land B(b', b''))
\]

\[
\leq (A(a, a'') \land B(b, b')).
\]

and the units by:

\[
I_{a_+} = I_{b_+} \leq (I_{a_+} \land I_{b_+}) \leq A(a, a) \land B(b, b).
\]

Now given a diagram in $V$-$\text{Cat}$,

\[
\begin{array}{ccc}
A & \xrightarrow{F} & C \\
\hspace{1cm} \downarrow \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \downarrow \hspace{1cm} \\
& \xrightarrow{G} & B
\end{array}
\]

its pullback is given by:

\[
\begin{array}{ccc}
A & \leftarrow & A \land C B \\
\hspace{1cm} \downarrow \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \downarrow \hspace{1cm} \\
& \xrightarrow{F} & B
\end{array}
\]

where $A \land C B$ is the subcategory of $A \land B$ generated by pairs $(a, b)$ with $F(a) = G(b)$ and the arrows are the obvious embeddings.

Lemma 3.2 If the quantaloid $V$ is locally distributive then pullback operation preserves the surjective functional bisimulation in $V$-$\text{Cat}$.

Proof: Consider the pullback diagram

\[
\begin{array}{ccc}
A \land C B & \xrightarrow{G} & B \\
\hspace{1cm} \downarrow \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \downarrow \hspace{1cm} \\
A & \xrightarrow{F} & C
\end{array}
\]

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where $G$ is a surjective functional bisimulation. Let $a \in A$, since $G$ is surjective, there exists $b \in B$ such that $G(b) = F(a)$ and $G(a, b) = a$. For such $a$ and $b$, given $a' \in A$,

\[
\begin{align*}
\bigvee_{(a', b') \mid G(a', b') = a'} (\mathbb{A} \land_C \mathbb{B})(a, b, (a', b')) \\
= \bigvee_{(a', b') \mid F(a') = G(b')} \mathbb{A}(a, a') \land \mathbb{B}(b, b') \\
= \mathbb{A}(a, a') \land \bigvee_{b' \mid F(a') = G(b')} \mathbb{B}(b, b') \text{ by distributivity} \\
= \mathbb{A}(a, a') \land \mathbb{C}(F(a), F(a')) \\
\geq \mathbb{A}(a, a').
\end{align*}
\]

Let $\mathcal{O}$ denotes the sets of functional bisimulations whose underlying maps on objects are surjections. This class of maps forms a subcategory of $\mathcal{V}$-$\text{Cat}$. It also satisfies a series of axioms (A1) to (A6) which are the same or quite similar to those stated by Joyal and Moerdijk [4]. We shall review these axioms pointing out the differences with the axioms for open maps.

**Proposition 3.7** (A1) $\mathcal{O}$ contains isomorphisms and is closed under composition.

This is straightforward. Remember that $f : A \to B$ is an isomorphism in $\mathcal{V}$-$\text{Cat}$ if and only if its underlying map on objects is one-to-one and $\forall a, a' \in A, \mathbb{A}(a, a') = \mathbb{B}(f(a), f(a'))$.

Note also that $f$ is a split epi in $\mathcal{V}$-$\text{Cat}$ if and only if its underlying map on objects is surjective and $\forall b \in B$, one may find a $s(b) \in A$ such that $f(s(b)) = b$ and for all $b, b' \in B$, $\mathbb{B}(b, b') = \mathbb{A}(s(b), s(b'))$. Given a split epi $f : A \to B$ such that for each $a \in A$, there exists a section $s$ for $f$ with value $a$ then $f$ belongs to $\mathcal{O}$, as in this case for such an $s$ for any $b \in B$,

\[
\mathbb{B}(fa, b) = \mathbb{A}(s(f(a)), s(b)) = \mathbb{A}(a, s(b)).
\]

As seen previously in 3.2

**Proposition 3.8** (A2 - stability axiom - preservation by pullback)

If $\mathcal{V}$ is locally distributive, in any pullback square

\[
\begin{array}{ccc}
A \land_C B & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{g} \\
A & \xrightarrow{f} & C
\end{array}
\]

if $g$ belongs to $\mathcal{O}$ then $\bar{g}$ also belong to $\mathcal{O}$.  

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Proposition 3.9 \((A3 - \text{descent axiom})\) In any pullback square

\[
\begin{array}{ccc}
A \land_C B & \xrightarrow{f} & B \\
\downarrow \bar{g} & & \downarrow g \\
A & \xrightarrow{f} & C
\end{array}
\]

if \(f\) is surjective on objects and satisfies the condition

\[ (\ast) \quad C(c, c') \leq \bigvee_{f(a) = c, f(a') = c'} \mathbb{A}(a, a') \quad \text{for all } c, c' \in C \]

and \(\bar{g}\) belongs to \(\mathbb{O}\) then \(g\) belongs to \(\mathbb{O}\).

Note that any \(f\) that is split epi satisfies the condition \((\ast)\) above. The descent axiom \((A3)\) in [4] differs from the one above on the point that \(f\) is there just required epi.

Proof: If \(f\) satisfies \((\ast)\) then for any \(b \in B\) and \(c \in C\),

\[
C(gb, c) \leq \bigvee_{f(a) = gb, f(a') = c} \mathbb{A}(a, a') \\
\leq \bigvee_{a' | f(a') = c} \bigvee_{b' | gb = b'} \mathbb{A}(a, a') \land \mathbb{B}(b, b') \\
\leq \bigvee_{b' | gb = c'} \mathbb{B}(b, b')
\]

\(\mathbb{V}-\text{Cat}\) has a terminal object 1 as follows. 1 has one point \(*_v\) per \(v \in \mathbb{V}\) with \((*_v)_+ = v\) and homs given by \(1(*_u, *_v) = \top_{u,v}\) the maximal element of \(\mathbb{V}(u, v)\). Also for any small collection \((\mathbb{A}_i)_{i \in I}\) of enrichments the coproduct \(\bigsqcup_{i \in I} \mathbb{A}_i\) exists, it has set of objects the disjoint union \(\bigsqcup_{i \in I} \text{Obj}(\mathbb{A}_i)\) and its hom is given by the formula: \((\bigsqcup_{i \in I} \mathbb{A}_i)(x, y) = \mathbb{A}_i(x, y)\) if both \(x\) and \(y\) belongs to \(\mathbb{A}_i\) or \(\bot_{x, y+}\) the least element of \(\mathbb{V}(x+, y+)\) otherwise.

Proposition 3.10 \((A4)\) For any set \(I\), the unique map \(\bigsqcup_{i \in I} 1 \to 1\) belongs to \(\mathbb{O}\).

Proof: The coproduct \(\bigsqcup_{i \in I} 1\) has as objects say the \(*_{v,i}\)'s where \(v\) ranges in \(\mathbb{V}\) and \(i \in I\) with \(\prod_{i \in I} (*_{u,i}, *_{v,i}) = \top_{u,v}\) if \(i = j\) and equals \(\bot_{u,v}\) otherwise. The unique map \(! : \prod_{i \in I} 1 \to 1\) sends any \(*_{v,i}\) to the unique object \(*_v\) in 1. Given any \(*_{u,i} \in \prod 1\),

\[
1(!(*_{u,i}, *_v)) = \top_{u,v} \\
\leq (\prod 1)(*_{u,i}, *_v) \\
= \bigvee_{x \in (\prod 1)!x = *_v} (\prod 1)(*_{u,i}, x).
\]

Proposition 3.11 For any family of arrows \(\mathbb{A}_i \to \mathbb{B}_i\) in \(\mathbb{O}\), their sum \(\bigsqcup_{i \in I} \mathbb{A}_i \to \mathbb{B}_i\) belongs to \(\mathbb{O}\).
Proof: Consider a family $f_i : A_i \to B_i$ in $O$. The sum $f = \bigsqcup_{i \in I} f_i$ sends $x \in A_i$ to $f_i(x) \in B_i$. Given $x \in A_i$ and $y \in B_j$, we have to show
\[
\left( \bigsqcup_{i \in I} B_i \right)(f_i(x), y) \leq \bigsqcup_{x' \mid f_i(x') = y} \left( \bigsqcup_{i \in I} A_i \right)(x, x').
\]
If $i \neq j$ then the left hand side term is $\bot$. If $i = j$, then the left hand side term is $B_i(f_i(x), y)$ that is less than $\bigvee_{x' \mid f_i(x') = y} A_i(x, x')$ as $f_i \in O$, that is less the right hand side term.

Proposition 3.12 (A6-Quotient axiom) In any commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{p} & B \\
\downarrow{g} & & \downarrow{f} \\
C & \xrightarrow{p} & B,
\end{array}
\]
if $g \in O$ and $p$ is surjective then $f \in O$.

Proof: Given a diagram as above with $g \in O$, for any $b \in B$ and $c \in C$, $C(f(b), c) = C(g(a), c)$ for some $a \in A$ with $p(a) = b$. Then
\[
C(f(b), c) \leq \bigvee_{a' \mid g(a') = c} A(a, a') \\
\leq \bigvee_{a' \mid g(a') = c} B(b, p(a')) \\
\leq \bigvee_{b' \mid f(b') = c} B(b, b')
\]

4 Change of Base

The study of the change of base for enrichments over bicategories gave rise to the concept of two-sided enrichments [2]. We will need some elements from of this theory (again in the particular and simpler case of enrichments over quantaloids). The main point here is the fact that all information about change-of-base is concealed in the tricategory $Caten$ to be introduced below. We shall revisit very quickly this simple tricategory admitting orders as 2-homs before to state the change of base theorems. We refer the reader to [9] for a detailed presentation of this material.

Let $\text{Span}$ be the bicategory of spans over $\text{Set}$ [Ben67]. Recall that the map $f : X \to Y$ in
is a 2-cell \( f : (X, l_X, r_X) \Rightarrow (Y, l_Y, r_Y) \) in \( \text{Span} \) provided it makes everything in sight commute. A 2-cell in \( \text{Span} \) is also called \textit{morphism of spans}. It is well known that a span is a left adjoint in \( \text{Span} \) precisely when its left leg is iso. It follows that the iso class of a left adjoint in \( \text{Span} \) has a span with identity left leg as a canonical representant. Right adjoints are characterized similarly, i.e. their right legs are iso.

\textbf{Caten}

\textit{Caten}'s objects are locally ordered bicategories. An arrow \( A : V \to W \) in \textit{Caten}, called a \textit{two-sided enrichment}, consists of

- a span

\[
\begin{array}{ccc}
\text{Obj}(A) & \xrightarrow{(-)_{-}} & \text{Obj}(V) \\
\downarrow \hspace{2cm} & & \downarrow \\
\text{Obj}(W) & \xleftarrow{(-)_{+}} & \text{Obj}(A) \\
\end{array}
\]

in \( \text{Set} \). Let \( \widehat{A} \overset{\text{def}}{=} \left( \text{Obj}(A), (-)_{-}, (-)_{+} \right) \);

- for all \( a, b \in \text{Obj}(A) \) a collection of functors (monotone maps)

\[
A_{a, b} : V(a_{-}, b_{-}) \to W(a_{+}, b_{+})
\]

such that

\[
V(a_{-}, b_{-}) \times V(b_{-}, c_{-}) \xrightarrow{\otimes^V} V(a_{-}, c_{-})
\]

\[
\begin{array}{c}
\leq \\
\downarrow \\
\end{array}
\]

\[
W(a_{+}, b_{+}) \times W(b_{+}, c_{+}) \xrightarrow{\otimes^W} W(a_{+}, c_{+})
\]

for any \( a, b, c \in A \) and such that

\[
1 \xrightarrow{id_{a_{-}}} V(a_{-}, a_{-})
\]

\[
\leq \\
\downarrow \\
\end{array}
\]

\[
1 \xrightarrow{id_{a_{+}}} W(a_{+}, a_{+})
\]
for any \( a \in A \).

For any two-sided enrichments \( \mathcal{V} \) and \( \mathcal{W} \), the bicategory \( \text{Caten}(\mathcal{V}, \mathcal{W}) \) is locally partially ordered. Let \( A, B \in \text{Caten}(\mathcal{V}, \mathcal{W}) \). A 2-cell \( f : A \Rightarrow B \) in \( \text{Caten} \) consists of a morphism of spans \( f : \widehat{A} \rightarrow \widehat{B} \) such that

\[
A_{a,b} \leq B_{f(a),f(b)} : \mathcal{V}(a-, b-) \rightarrow \mathcal{W}(a_+, b_+)
\]

for any \( a, b \in A \).

Vertical composition of 2-cells in \( \text{Caten} \) is determined by \( \text{Span} \)'s one. 2-cells in \( \text{Caten} \) are ordered as follows:

\[
f \leq g : A \Rightarrow B : \mathcal{V} \rightarrow \mathcal{W}
\]

when

\[
id_{a_+} \leq B_{f(a_+),f(a_-)} : a_+ \rightarrow a_+
\]

for any \( a \in A \).

Let \( A : \mathcal{U} \rightarrow \mathcal{V} \) and \( B : \mathcal{V} \rightarrow \mathcal{W} \) be two-sided enrichments. Their composite \( B \circ A : \mathcal{U} \rightarrow \mathcal{W} \) is given by

\[
\widehat{B} \circ \widehat{A} \overset{\text{def}}{=} \widehat{B \circ A}
\]

and by the functors \( B \circ A_{(a,b),(a',b')} \) being the composites

\[
\mathcal{U}(a_-, a'_-) \xrightarrow{A_{(a,a')}} \mathcal{V}(a_+, a'_-) = \mathcal{V}(b_-, b'_-) \xrightarrow{B_{(b,b')}} \mathcal{W}(b_+, b'_-)
\]

Horizontal composition of 2-cells in \( \text{Caten} \) is determined by \( \text{Span} \)'s one.

### Adjoint in Caten

Amongst the nice properties of \( \text{Caten} \) - proved in [2], (and detailed in [9] for locally preordered bases) - let us mention that the cartesian product of locally partially ordered bicategories extends straightforwardly to a pseudo functor \( \text{Caten} \times \text{Caten} \rightarrow \text{Caten} \) that makes \( \text{Caten} \) into a monoidal tricategory. With that monoidal structure \( \text{Caten} \) is biclosed. More importantly for our concern, the change of base bicategory, one has the following representability property:

**Proposition 4.1** For any quantaloid \( \mathcal{V} \), the 2-category \( \mathcal{V} \text{-Cat} \) is representable:

\[
\mathcal{V} \text{-Cat} \cong \text{Caten}(1, \mathcal{V})
\]

As expected, adjointness in \( \text{Caten} \) is by and large determined by what happens in \( \text{Span} \). A two-sided enrichment \( F : \mathcal{V} \rightarrow \mathcal{W} \) is a left adjoint provided

1. \( F \) is a 2-functor \( \mathcal{V} \rightarrow \mathcal{W} \) (so in particular \( \text{Obj} (A) = \text{Obj} (\mathcal{V}) \));
2. the functors \( F_{u,v} : \mathcal{V}(u,v) \rightarrow \mathcal{W}(Fu,Fv) \) have right adjoints \( G_{u,v} : \mathcal{W}(Fu,Fv) \rightarrow \mathcal{V}(u,v) \).
One may check that a for a lax functor $F : \mathcal{V} \to \mathcal{W}$ as above that satisfies condition 2, then $F$ satisfies condition 1. if and only if the collection of local adjoints $G_{u,v}$ satisfies the following two conditions:

1. $G_{u,v}(f) \otimes G_{v,t}(g) \leq G_{u,t}(f \otimes g)$ for all $u, v, t \in \mathcal{V}$ for all $f : F u \to F v$ and $g : F v \to F t$;

2. $\text{id}_u \leq G_{u,u} \circ \text{id}_{F u}$ for all $u$ of $\mathcal{V}$.

Using the representability for $\mathcal{V}$-categories and the fact that $\text{Caten}$ is a tricategory, one obtains a change of base theorem for enrichments over bicategories. Postcomposition with any two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$ yields a 2-functor $F \circ - : \text{Caten}(1, \mathcal{V}) \to \text{Caten}(1, \mathcal{W})$, and by the representability result a 2-functor $F@ : \mathcal{V}^\mathcal{V} \to \mathcal{W}^\mathcal{V}$ given by:

- $\text{Obj}(F@ A) = \{(a, x) \mid a \in \text{Obj}(A), x \in \text{Obj}(F), a_+ = x_\_\}$;
- $(a, x)^{F@ A} = x_+$;
- $F@ A ((a, x), (b, y)) = F_{x,y}(A(a, b))$.

For any $\mathcal{V}$-functor $f : A \to B$, $F@ (f)$ is the $\mathcal{W}$-functor with underlying map $(a, x) \mapsto (f a, x)$.

In the same way, any 2-cell $\sigma : F \to G$ of $\text{Caten}$ yields a 2-natural transformation $\sigma@ : F@ \to G@ : \mathcal{V}^\mathcal{V} \to \mathcal{W}^\mathcal{V}$.

**Theorem 4.1** There is a pseudo-functor $(-)@ : \text{Caten} \to 2\mathcal{V}^\mathcal{V}$ that sends a locally ordered bicategory $\mathcal{V}$ to the 2-category $\mathcal{V}^\mathcal{V}$. Its actions on two-sided enrichments and 2-cells are the ones described above.

Of particular interest to this investigation, one gets an adjoint pair $F@ \dashv G@$ in 2-$\mathcal{V}^\mathcal{V}$ from an adjoint pair $F \dashv G$ in $\text{Caten}$. (Recall from the previous section that a left adjoint in $\text{Caten}$ is given by a peculiar collection of local left adjoints)

**Theorem 4.2** Let $\mathcal{V}$ and $\mathcal{W}$ be quantaloids and $F : \mathcal{V} \to \mathcal{W}$ a 2-functor with local adjunctions $F_{u,v} \dashv G_{u,v}$ for all $u, v \in \text{Obj}(\mathcal{V})$. Then the 2-functor $F@$ admits a right 2-adjoint, denoted $F^@$ and given on objects by

- $\text{Obj}(F^@ B) = \{(b, v) \mid b \in \text{Obj}(B), v \in \text{Obj}(\mathcal{V}), b_+ = F v\}$;
- $(b, v)_+ = v$;
- $F^@ B ((b, v), (b', v')) = G_{v,v'}(B(b, b'))$.

**Example 1**

Betti’s automata will provide our first examples of change of base. Let $\mathbb{M}$ and $\mathbb{N}$ be monoids. A congruence relation $r : \mathbb{M} \to \mathbb{N}$ verifies
Note that congruences generalize the usual monoid morphisms \( \mathbb{M} \to \mathbb{N} \). For any such \( r \), there is a monoidal functor \( C(r) : C(\mathbb{M}) \to C(\mathbb{N}) \) taking \( L \subseteq \mathbb{M} \) to the direct image \( \{ n \mid (m, n) \in r \land m \in L \} \) of \( L \) by \( r \).

An important point is that the functor \( C(r) \) has a right adjoint, namely \( R(r) \) defined for \( K \subseteq \mathbb{N} \) by

\[
R(r)(K) = \{ m \in \mathbb{M} \mid \forall n \in \mathbb{N}, (m, n) \in r \Rightarrow n \in K \}
\]

This right adjoint fails to be monoidal for a general \( r \).

Consider now a monoid morphism \( f : \mathbb{M} \to \mathbb{N} \). Then for the relation \( r \subseteq \mathbb{N} \times \mathbb{M} \) given by \( (n, m) \in r \iff f(m) = n \), \( C(r) \) corresponds to the inverse image functor \( f^{-1} : C(\mathbb{N}) \to C(\mathbb{M}) \). So this functor has a right adjoint traditionally written \( \forall f \) and given by

\[
\forall f(L) = \{ n \mid \forall m \in \mathbb{M}, (f(m) = n \Rightarrow m \in L) \}
\]

for all \( L \subseteq \mathbb{M} \). In case when the relation \( r \subseteq \mathbb{M} \times \mathbb{N} \) is \( (m, n) \in r \iff n = f(m) \), \( C(r) \) corresponds to the left adjoint \( \exists f \) of \( f^{-1} : C(\mathbb{N}) \to C(\mathbb{M}) \). One may check that \( \exists f \) is strong (i.e. it preserves strictly the monoidal structure) and thus is a left adjoint in \( \text{Caten} \). Then \( (\exists f)_a \) is a left 2-adjoint by theorem 4.2.

**Example 2**

The previous example with monoids suggests an immediate generalization to categories (monoids with many points!). Define a congruence \( r \) between categories \( \mathbb{C} \) and \( \mathbb{D} \) - still denoted by \( r : \mathbb{C} \to \mathbb{D} \) - as a span

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{r_0} & \mathbb{D} \\
(-)_- & \searrow & (-)_+ \\
\mathbb{C} & & \mathbb{D}
\end{array}
\]

and a collection of relations \( r_{x,y} \subseteq \mathbb{C}(x-, y-) \times \mathbb{D}(x-, y-) \), \( x,y \) ranging in \( r_0 \) such that:

- \((id_{x-}, id_{x+}) \in r_{x,x}\) for all \( x \in r_0 \);
- if \((f, f') \in r_{x,y}\) and \((g, g') \in r_{y,z}\) then \((g \circ f, g' \circ f') \in r_{x,z}\) for all \( x,y,z \in r_0 \) and all \( f : x- \to y-, f' : x+ \to y+, g : y- \to z- \) and \( g' : y+ \to z+ \).

One obtains a two-sided enrichment \( B(r) : B(\mathbb{C}) \to B(\mathbb{D}) \) with \( \text{Obj}(B(r)) = r_0 \) and for objects \( x,y \) of \( \mathbb{D} \) for any \( L \subseteq \mathbb{C}(x-, y-) \),

\[
B(r)_{x,y}(L) = \{ g \mid (f, g) \in r_{x,y} \subseteq \mathbb{D}(x+, y+) \}.
\]

Each \( B(r)_{x,y} : B(\mathbb{C})(x-, y-) \to B(\mathbb{D})(x+, y+) \), as defined above, admits a right adjoint namely \( R(r)_{x,y} \) sending \( K \subseteq \mathbb{D}(x+, y+) \) to

\[
\{ l : x- \to y- \in \mathbb{C} \mid \forall k : x+ \to y+, (l, k) \in r_{x,y} \Rightarrow k \in K \} \subseteq \mathbb{C}(x-, y-).
\]
Nevertheless the data $R(r)_{x,y}$ for $x,y$ ranging in $r_0$ fails to define a two-sided enrichments in general.

Consider now a functor $f : C \to D$. Then for the congruence $D \to C$ defined by the relations $r_{x,y} \subseteq D(f(x), f(y)) \times C(x,y)$ defined by $(k,l) \in r_{x,y}$ if and only if $f(l) = k$, yields a two sided enrichment $B(r) : B(D) \to B(C)$ that we shall denote again $f^{-1}$. It has local right adjoints $R(r)_{x,y}$ still denoted $\forall x,y$ sending $L \subseteq C(x,y)$ to

$$\{ k : f(x) \to f(y) \mid \forall l : x \to y, f(l) = k \Rightarrow l \in L \}.$$ 

In case when $r$ denotes the inverse congruence $C \to D$, $B(r) : B(C) \to B(D)$ is a 2-functor that we shall denote $\exists f$, it has a local right adjoints given by the $R(r)_{x,y}$ above, so this is to say that $\exists f \dashv f^{-1}$ in Caten.

**Example 3**

We shall come back now to the isomorphism 2.2. For any $\mathcal{V}$-functor $f : \mathbb{A} \to \mathbb{B}$, the functor $\mathcal{V}(f) : \mathcal{V}(\mathbb{A}) \to \mathcal{V}(\mathbb{B})$ is actually a left adjoint in Caten. This is easy to see. The left adjoints to the $\mathcal{V}(f)_{a,b} : \mathcal{V}(a_+, b_+) \downarrow \mathbb{A}(a,b) \to \mathcal{V}(a_+, b_+) \downarrow \mathbb{B}(fa, fb)$ are the $G_{a,b}$ sending $y \leq \mathbb{B}(fa, fb)$ to $\min\{y, \mathbb{A}(a,b)\} \leq \mathbb{A}(a,b)$. Now for any $a, b, c \in \mathbb{A}$,

$$G_{a,b}(x) \otimes G_{b,c}(y) = \min\{x, \mathbb{A}(a,b)\} \otimes \min\{y, \mathbb{A}(b,c)\} \leq \min\{x \otimes y, \mathbb{A}(a,b) \otimes \mathbb{A}(b,c)\} \leq \min\{x \otimes y, \mathbb{A}(a,c)\} = G_{a,c}(x \otimes y)$$

which is the coherence condition 1. for the local right adjoints. The coherence condition 2. amounts to $id_{a_+} \leq \min\{id_{a_+}, \mathbb{A}(a,a)\}$ for all $a \in \mathbb{A}$, which clearly holds.

In the above situation the change of base $\mathcal{V}(f)_@ : \mathcal{V}(\mathbb{A}) - \text{Cat} \to \mathcal{V}(\mathbb{B}) - \text{Cat}$ corresponds exactly via the isomorphism 2.2 to the functor $\mathcal{V} - \text{Cat} \downarrow \mathbb{A} \to \mathcal{V} - \text{Cat} \downarrow \mathbb{B}$ given by composition with $f$, whereas the adjoint $\mathcal{V}(f)_@$ corresponds (via 2.2) to the pullback along $f$ functor $\mathcal{V} - \text{Cat} \downarrow \mathbb{B} \to \mathcal{V} - \text{Cat} \downarrow \mathbb{A}$.

## 5 Bisimulations and change of base

It seems natural to consider simulations/bisimulations upto change of base and to ask when the changes of base preserves/reflects bisimularity. We shall give a simple criterion for the preservation to happen.

**Proposition 5.1** Let $\mathcal{V}$ and $\mathcal{W}$ be quantaloids. Any 2-sided enrichment $F : \mathcal{V} \to \mathcal{W}$ with local right adjoints induces a change of base $F_@$ that preserves the class $\mathcal{O}$ (and thus bimularity).
Proof: Consider an open \( f : A \rightarrow B \) in \( \mathcal{V}-\text{Cat} \). If the underlying map of \( f \) is surjective then also is the underlying map of \( F_\emptyset(f) \). Then for any \((a, x) \in F_\emptyset(A)\) and \((b, y) \in F_\emptyset(B)\)

\[
F_\emptyset(B)(F_\emptyset(f)(a, x), (b, y)) = F_{x,y}(B(fa, b))
= F_{x,y}(\bigvee \{a' \in A | f(a') = b\} \emptyset(a, a'))
= \bigvee \{a' \in A | f(a') = b\} F_{x,y}(\emptyset(a, a')) \quad \text{since } F_{x,y} \text{ is left adjoint}
= \bigvee \{(a', y') \in F_\emptyset(A) | F_\emptyset(f)(a', y') = (b, y)\} F_\emptyset(B)((a, x), (a', y')).
\]

As a consequence of this any left adjoint \( F \) in \( \text{Cat} \) will induce a change of base \( F_\emptyset \) that preserves bisimularity. Our previous examples provides change of base preserving bisimilarity. First if \( f : C \rightarrow D \) is a functor then \( \exists f^{-1} \) (that is left adjoint) but also \( f^{-1} \), that has local right adjoints, will both induce change of bases preserving bisimilarity. For any \( \mathcal{V} \)-functor \( f : A \rightarrow B \), the \( \mathcal{V} \)-functor \( \mathcal{V}(f) : \mathcal{V}(A) \rightarrow \mathcal{V}(B) \) being left adjoint \( \mathcal{V}(f)_\emptyset \) preserves bisimilarity. Which is no suprise here as the bisimilarity in \( \mathcal{V}(A)-\text{Cat} \) correspond to the bisimilarity over \( A \) and its perservation by \( \mathcal{V}(f) \) is equivalent to the fact that any bisimilar arrows \( h, k \) over \( A \) yields bisimilar arrows \( h \circ f, k \circ f \) over \( B \).

6 Refinement of Specifications

In this section, we elaborate on an extended example illustrating the use of the categorical machinery introduced so far. We advocate that enriched categories are a convenient framework for the deployment of the so-called categorical transition systems \[\text{[14][15][8]}\], in the sense that coherence conditions are taken care of by the enriched structure. We then apply the rest of the machinery to study refinements of specifications in this framework.

Definition 6.1 Let \( K \) be a bicategory. The category \( //K \) is given by the data

1. Objects: normalized pseudo-functors from a free categories \( FG \) over graphs \( G \) to \( K \);

2. Morphisms: given \( s : FG \rightarrow K \) and \( t : FH \rightarrow K \) normal pseudo-functors, a morphism \( \alpha : p \rightarrow q \) is given by a graph morphism \( k : G \rightarrow H \) and a lax transformation \( \alpha : s \Rightarrow t \circ Fk \) with left adjoint components;

3. Composition: \( \beta \circ \alpha = (l \circ k, l \beta \circ \alpha) \) where \( \alpha : s \Rightarrow t \circ Fk \) and \( \beta : t \Rightarrow u \circ Fl \) while the vertical composition \( l \beta \circ \alpha \) is given by componentwise pasting.

Let \( \mathcal{T} \) be a category with finite limits and \( \text{Span}(\mathcal{T}) \) bicategory of spans over \( \mathcal{T} \). We call categorical transition systems or cts's over \( \mathcal{T} \) objects of \( //\text{Span}(\mathcal{T}) \). They are essentially generalized labelled transition systems where the labels are organized in \( \text{Span}(\mathcal{T}) \). As an example, consider the imperative program in-context
It gives rise to a pseudo-functor $p : \mathcal{F}\mathcal{G} \to \text{Span}(\text{Set})$ generated as follows

\[
\begin{array}{c}
\bullet \\
| \\
\downarrow a \\
| \\
\bullet \\
| \\
\downarrow w_1
\end{array}
\quad
\begin{array}{c}
\bullet \\
| \\
\downarrow b \\
| \\
\bullet \\
| \\
\downarrow w_2
\end{array}
\quad
\begin{array}{c}
\lambda x: \text{nat}. 20 \\
\downarrow \\
\text{nat} \\
\downarrow \\
\downarrow \{1, 2, \ldots\}
\end{array}
\quad
\begin{array}{c}
\lambda x: \text{nat}. x - 1 \\
\downarrow \\
\text{nat} \\
\downarrow \\
\downarrow \{0\}
\end{array}
\quad
\begin{array}{c}
\text{id} \\
\downarrow \\
\text{nat} \\
\downarrow \\
\downarrow \text{id}
\end{array}
\]

We see at hand of this example that the category $\mathbb{T}$ plays the rôle of the type theory underlying the computation performed by a cts. The states of a cts are labelled by $\mathbb{T}$’s objects i.e. types. These types are those of the variables in scope. The legs of the spans are labelled by terms, jointly representing a generalized transition relation. Cts’s allow a quite realistic modelling of imperative programs including communication over typed channels. Moreover, the view of a cts over $\mathbb{T}$ as an object of $\text{//Span}(\mathbb{T})$ offers a compact representation of such programs. This fact was for instance exploited in the design and the implementation of a deductive modelchecker [SpWo04] (indeed, the representations in question could be accomodated by the theorem prover PVS acting as a “logical back-end”). However, one has to cope with coherence conditions when it comes down to calculations.

We shall now propose an alternative view of a cts. It is particularly interesting when addressing the question of refinement of specifications that we shall interpret as a functor between categories of types $\mathbb{T} \to \mathbb{T}'$.

If $M \subseteq C_0$ is a collection of objects of a category $\mathbb{C}$ we write $M \downarrow$ the sieve (crible, right-ideal) generated by $M$.

**Definition 6.2** Let $\mathbb{T}$ be a category with finite limits. The quantaloid $S(\mathbb{T})$ is given by

- objects: $\text{Obj}(\mathbb{T})$;
- $S(\mathbb{T})(X,Y)$ is the set of generated cribles $M \downarrow$ for $M \subseteq \text{Span}(X,Y)$, ordered by inclusion;
- the horizontal composition is given by the formula
  
  $M \downarrow \circ N \downarrow = (M \circ N) \downarrow$
  
  where $\circ$ is the pointwise composition of set of spans.

A cts $p : \mathcal{F}\mathcal{G} \to \text{Span}(\mathbb{T})$ over $\mathbb{T}$ gives rise to an $S(\mathbb{T})$-category $\mathbb{A}_p$ as follows:
\(- \text{Obj}(\mathcal{A}_p)\) is the set of \(p\)'s states, i.e. \(G_0\);

\(- \mathcal{A}_p(x, y)\) is \(\{p(f) \mid f \in \mathcal{F}G(x, y)\} \downarrow \).

Now given a right exact functor \(F: \mathcal{A} \to \mathcal{B}\) between finitely complete categories, one defines an adjoint \(S(F): \mathcal{A} \to \mathcal{B}\) in \text{Caten} as follows. \(S(F)\)'s action is given on objects by \(S(F)(x) \overset{\text{def}}{=} F(x)\), and on arrows by

\[
S(F)_{x,y}: S(\mathcal{A})(x, y) \to S(\mathcal{B})(Fx, Fy)
\]

\[
M \mapsto \{(Fa, F(d_a), F(c_a)) \mid (a, d_a, c_a) \in M\} \downarrow
\]

It is easy to see that \(F\) is a 2-functor. For each \(x, y \in \mathcal{A}\), there is a “local” adjunction \(S(F)_{x,y} \dashv U_{x,y}\) with right adjoints given by

\[
U_{x,y}: S(\mathcal{B})(Fx, Fy) \to S(\mathcal{A})(x, y)
\]

\[
N \mapsto \{(a, d_a, c_a) \mid (F(a), F(d_a), F(c_a)) \in N\}
\]

Moreover in the case of an adjunction \(F \dashv G: \mathcal{A} \to \mathcal{B}\), \(S(G)\) (that is already left adjoint in \text{Caten}) also admits local left adjoints. To see this let \(\overline{t}: F(a) \to b\) denote the transposed of \(t: a \to G(b)\) across the adjunction, then for any \(x, y \in \mathcal{B}\), the left adjoint to \(S(G)_{x,y}\) is given by:

\[
R_{x,y}: S(\mathcal{A})(G(x), G(y)) \to S(\mathcal{B})(x, y)
\]

\[
N \mapsto \{(F(a), \overline{c_a}, d_a) \mid (a, c_a, d_a) \in N\} \downarrow
\]

According to the result above (bi)similar specifications of categorical transition systems will have (bi)similar refinements provided the refinement functor is left exact.

7 Concluding Remarks

We hope that the present work exhibits some pertinence of the interaction enriched category theory and simulation/bisimulation theory. Extending Betti’s work, we have shown that bicategorical enrichments over quantaloids can accommodate a wide spectrum of existing notions of automata and communicating processes including labelled transition systems, themselves a special case of Betti’s automata, and also categorical transition systems. We then presented a few new results about two-sided enrichments and exhibited applications measuring the impediments to the existence of good change-of-base homomorphims. After having introduced an appropriate notion of (bi)simulation for enrichments, “good” turned out to mean “bisimilarity-preserving”. We illustrated the notion with an example about refinements of specifications in the framework of categorical transitions systems.

All the material is indeed quite formal and it is precisely the whole point of the paper that it should be formal. In other words, our wish is that category theory is revelatory for structural properties.
This investigation represents only the beginning of a research with generalized automata and their properties as subject. We expect the framework flexible enough to accommodate more elaborated programming constructs. We also intend to introduce a notion of homotopy of paths within a generalized automaton. In a different perspective, we plan to study the notion of bisimulation itself in terms of homotopies [7].

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