COUNTING THE IDEALS OF GIVEN CODIMENSION OF THE ALGEBRA OF LAURENT POLYNOMIALS IN TWO VARIABLES

CHRISTIAN KASSEL AND CHRISTOPHE REUTENAUER

Abstract. We establish an explicit formula for the number $C_n(q)$ of ideals of codimension $n$ of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of Laurent polynomials in two variables over a finite field $\mathbb{F}_q$ of cardinality $q$. This number is a palindromic polynomial of degree $2n$ in $q$. Moreover, $C_n(q) = (q - 1)^2 P_n(q)$, where $P_n(q)$ is another palindromic polynomial; the latter is a $q$-analogue of the sum of divisors of $n$, which happens to be the number of subgroups of $\mathbb{Z}^s$ of index $n$.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of cardinality $q$ and $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ be the algebra of Laurent polynomials in two variables with coefficients in $\mathbb{F}_q$. Our main aim is to give a formula for the number $C_n(q)$ of ideals of codimension $n$ of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$. Our main result is the following.

Theorem 1.1. For each integer $n \geq 1$ we have

$$C_n(q) = \sum_{\lambda \vdash n} (q - 1)^{2v(\lambda)} q^{n - \ell(\lambda)} \prod_{i=1, \ldots, t \mid d_i \geq 1} \frac{q^{2d_i} - 1}{q^2 - 1},$$

where the sum runs over all partitions $\lambda$ of $n$. The expression $C_n(q)$ is a monic polynomial of degree $2n$ in the variable $q$ with integer coefficients. Moreover, the polynomial $C_n(q)$ is divisible by $(q - 1)^2$.

The notation $\ell(\lambda), v(\lambda), d_i$ appearing in the formula will be explained in Section 3.1. The proof of the theorem will be given in Section 5.3; it relies on a parametrization by Conca and Valla [6] of the affine cells in the Ellingsrud–Strømme decomposition of the Hilbert scheme of $n$ points on the affine plane.

Note that since $C_n(q)$ is divisible by $(q - 1)^2$, we may define for each $n \geq 1$ a unique polynomial $P_n(q)$ by

$$C_n(q) = (q - 1)^2 P_n(q),$$

which clearly implies $C_n(1) = 0$ for all $n \geq 1$. Table 1 (resp. Table 2) at the end of the paper displays the polynomials $C_n(q)$ (resp. the polynomials $P_n(q)$) for $n \leq 12$.

Theorem 1.1 has two interesting consequences. The first one concerns the polynomials $P_n(q)$. Let us state it.
Corollary 1.2. For each $n \geq 1$ the polynomial $P_n(q)$ is a monic polynomial of degree $2n - 2$ with integer coefficients and we have

$$P_n(1) = \sigma(n) = \sum_{d|n; d \geq 1} d.$$  

As is well known, the sum $\sigma(n)$ of positive divisors of $n$ is equal to the number of subgroups of index $n$ of the free abelian group $\mathbb{Z}^2$ of rank two. Thus Theorem 1.1 and Corollary 1.2 imply that the number of ideals of codimension $n$ of the Laurent polynomial algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$, i.e., of the algebra of the group $\mathbb{Z}^2$, is, up to the factor $(q - 1)^2$, a $q$-analogue of $n$ of the number of subgroups of index $n$ of $\mathbb{Z}^2$.

A similar phenomenon had been observed by Bacher and the second-named author in [3]: up to a power of $q - 1$, the number of right ideals of codimension $n$ of the algebra $\mathbb{F}_q[F_2]$ of the rank two free group $F_2$ is a $q$-analogue of the number of subgroups of index $n$ of $F_2$. Actually it was this observation that prompted us to compute the number of ideals of codimension $n$ of the algebra $\mathbb{F}_q[\mathbb{Z}^2]$ of the free abelian group $\mathbb{Z}^2$, i.e., of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$.

In a similar context, the following holds.

(a) By [8] (see also Section 3.1 below) the number of ideals of codimension $n$ of the polynomial algebra $\mathbb{F}_q[x, y]$, which is the algebra of the free abelian monoid $\mathbb{N}^2$, is a $q$-analogue of the number $\rho(n)$ of partitions of $n$; as is well known, the latter is equal to the number of ideals of the monoid $\mathbb{N}^2$ whose complement is of cardinality $n$.

(b) In a non-commutative setting, by [20, 2], the number of right ideals of codimension $n$ of the algebra $\mathbb{F}_q\langle x, y \rangle$ is a $q$-analogue of the number of right ideals of the free monoid $\langle x, y \rangle^*$ whose complement is of cardinality $n$.

(c) It may be shown that the number of right ideals of codimension $2$ of the algebra $\mathbb{F}_q[F_3]$ of the rank three free group $F_3$ is equal to

$$q^2(q - 1)^3 ((q + 1)^3 - 1).$$

The last factor is obviously a $q$-analogue of $2^3 - 1 = 7$, which is the number of subgroups of index $2$ of $F_3$.

We conjecture the number of right ideals of codimension $2$ of the algebra $\mathbb{F}_q[F_r]$ of the free group $F_r$ with $r$ generators to be of the form $q^r(q - 1)^j ((q + 1)^r - 1)$ for some non-negative integers $i, j$; the last factor is then a $q$-analogue of the number $2^r - 1$ of subgroups of index $2$ of $F_r$. More generally, we expect the number of right ideals of codimension $n$ of $\mathbb{F}_q[F_r]$, up to a power of $q - 1$, to be a $q$-analogue of the number of subgroups of index $n$ of $F_r$ (see also the conclusion of [3]).

Remark 1.3. The commutative algebra $L_r = \mathbb{F}_q[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$ of Laurent polynomials in $r$ variables ($r \geq 3$) provides a distinct contrast with the cases discussed above. We can show that the number of right ideals of codimension $2$ of $L_r$, which is the algebra of the free abelian group $\mathbb{Z}^r$, is equal to $(q - 1)^2 R_r(q)$, where

$$R_r(q) = \frac{1}{2} \left((q + 1)^r + (q - 1)^r\right) + \frac{q^r - 1}{q - 1} - 1.$$  

The latter is a $q$-analogue of $R_r(1) = 2^r - 1 + r - 1$. Now the number of subgroups of index $2$ of $\mathbb{Z}^r$ is equal to $2^r - 1$, which is different from $R_r(1)$ when $r \geq 3$.

\footnote{By a $q$-analogue of an integer $r$ we mean a polynomial $P(q)$ in the variable $q$ such that $P(1) = r$.}
The second consequence of Theorem 1.1 expresses the generating function of the polynomials \( C_n(q) \) as a nice infinite product.

**Corollary 1.4.** (a) We have

\[
1 + \sum_{n \geq 1} \frac{C_n(q)}{q^n} t^n = \prod_{i \geq 1} \frac{1}{1 - (q + q^{-1}) t^i + t^{2i}}.
\]

(b) The polynomials \( C_n(q) \) and \( P_n(q) \) are palindromic.

The previous infinite product shows up in [9, p. 10] (see for instance Equations (9.2) and (10.1)) and probably in other papers on basic hypergeometric series; in an algebraic geometry context it appears in [16, Th. 4.1.3], where it is equal to the generating function of the \( E \)-polynomials of the punctual Hilbert schemes of the complex two-dimensional torus (see details in Section 6.3 below).

Using Corollary 1.4 we gave explicit expressions for the coefficients of the polynomials \( C_n(q) \) and \( P_n(q) \) in the companion paper [18] (see Theorems 1.1 and 1.2 in loc. cit.). We obtained a rather striking positivity result, namely the coefficients of \( P_n(q) \) are all non-negative integers. For the sake of completeness we recall our formulas for the coefficients of the polynomials \( C_n(q) \) and \( P_n(q) \) in Appendix A.

The paper is organized as follows. Section 2 is devoted to some preliminaries: we first recall the one-to-one correspondence between the ideals of the localization \( S^{-1}A \) of an algebra \( A \) and certain ideals of \( A \); we also count tuples of polynomials subject to certain constraints over a finite field.

In Section 3 we recall Conca and Valla’s parametrization of the affine cells in a decomposition of the Hilbert scheme of \( n \) points in the plane; these cells are indexed by the partitions of \( n \). We show how to deduce a parametrization of the cells in the induced decomposition of the Hilbert scheme of \( n \) points in a Zariski open subset of the plane.

In Section 4 we apply the techniques of the preceding section to compute the number of ideals of codimension \( n \) of \( \mathbb{F}_q[x, y, y^{-1}] \). In passing we give a criterion (Proposition 4.1) which will also be used in the proof of Theorem 1.1.

In Section 5 we define what we call an invertible Gröbner cell, which is a Zariski open subset of the corresponding affine cell, and compute its cardinality over a finite field. We derive a proof of Theorem 1.1.

The proofs of Corollary 1.4 of and of Corollary 1.2 are given in Section 6.

In Appendix A we briefly recall the results on the coefficients of \( C_n(q) \) and \( P_n(q) \) we obtained in [18].

2. Preliminaries

We fix a ground field \( k \). By algebra we mean an associative unital \( k \)-algebra. In this paper all algebras are assumed to be commutative.

2.1. Ideals in localizations. Let \( A \) be a (commutative) algebra, \( S \) a multiplicative submonoid of \( A \) not containing 0, and \( S^{-1}A \) the corresponding localization of \( A \). We assume that the canonical algebra map \( i : A \to S^{-1}A \) is injective (this is the case, for instance, when \( A \) is a domain).

Recall the well-known correspondence between the ideals of \( S^{-1}A \) and those of \( A \) (see [4] Chap. 2, § 2, no 4–5, [7] Prop. 2.2)].
(a) For any ideal \( J \) of \( S^{-1}A \), the set \( i^{-1}(J) = J \cap A \) is an ideal of \( A \) and we have \( J = i^{-1}(J)S^{-1}A \). The map \( J \mapsto i^{-1}(J) \) is an injection from the set of ideals of \( S^{-1}A \) to the set of ideals of \( A \).

(b) An ideal \( I \) of \( A \) is of the form \( i^{-1}(J) \) for some ideal \( J \) of \( S^{-1}A \) if and only if for all \( s \in S \) the endomorphism of \( A/I \) induced by the multiplication by \( s \) is injective.

Given an integer \( n \geq 1 \), a \( n \)-codimensional ideal of \( A \) is an ideal such that \( \dim_k A/I = n \). For such an ideal, the previous condition (b) is then equivalent to: for all \( s \in S \), the endomorphism of \( A/I \) induced by the multiplication by \( s \) is a linear isomorphism.

We leave the proof of the following lemma to the reader.

**Lemma 2.1.** If \( J \) is a finite-codimensional ideal of \( S^{-1}A \), then the canonical algebra map \( i: A \to S^{-1}A \) induces an algebra isomorphism

\[
A/i^{-1}(J) \cong (S^{-1}A)/J.
\]

It follows that there is a bijection between the set of \( n \)-codimensional ideals of \( S^{-1}A \) and the set of \( n \)-codimensional ideals \( I \) of \( A \) such that for all \( s \in S \), the endomorphism of \( A/I \) induced by the multiplication by \( s \) is a linear isomorphism. The latter assertion is equivalent to \( s \) being invertible modulo \( I \), that is the image of \( s \) in \( A/I \) being invertible.

The following criterion will be used in Sections 4 and 5.

**Lemma 2.2.** Let \( A \) be a commutative algebra. For any \( s \in A \), let \( p: A \to A/(s) \) be the natural projection onto the quotient algebra of \( A \) by the ideal generated by \( s \). If \( I \) is an ideal of \( A \), then \( s \) is invertible modulo \( I \) if and only if \( p(I) = A/(s) \).

**Proof.** If \( s \) is invertible modulo \( I \), then there exists \( t \in A \) such that \( st - 1 \in I \). Hence, \( p(1) \) belongs to \( p(I) \), which implies \( p(I) = A/(s) \). Conversely, if \( p(I) = A/(s) \), then \( p(1) = p(u) \) for some \( u \in I \). Hence \( 1 - u \in (s) \), which means that there is \( t \in A \) such that \( 1 - u = st \). Thus, \( st \equiv 1 \mod I \). \( \Box \)

### 2.2. Counting polynomials over a finite field.

In this subsection we assume that \( k = \mathbb{F}_q \) is a finite field of cardinality \( q \). We shall need the following in Section 5.

**Proposition 2.3.** Let \( d, h \) be integers \( \geq 1 \) and \( Q_1, \ldots, Q_h \in \mathbb{F}_q[x] \) be coprime polynomials. The number of \((h+1)\)-tuples \((P, P_1, \ldots, P_h)\) satisfying the three conditions

(i) \( P \) is a degree \( d \) monic polynomial with \( P(0) \neq 0 \),
(ii) \( P_1, \ldots, P_h \) are polynomials of degree \( < d \), and
(iii) \( P \) and \( P_1 Q_1 + \cdots + P_h Q_h \) are coprime,

is equal to

\[
(q - 1)^2 q^{(h-1)d} \frac{q^{2d} - 1}{q^2 - 1}.
\]

Before giving the proof, we state and prove two auxiliary lemmas.

**Lemma 2.4.** Let \( R \) be a finite commutative ring and \( a_1, \ldots, a_h \in R \) such that \( a_1 R + \cdots + a_h R = R \). For any \( b \in R \), the number of \( h \)-tuples \((x_1, \ldots, x_h) \in R^h \) such that \( a_1 x_1 + \cdots + a_h x_h = b \) is equal to \((\text{card } R)^h \).
Proof. The map \((x_1, \ldots, x_h) \mapsto a_1 x_1 + \cdots + a_h x_h\) is a homomorphism \(R^h \to R\) of additive groups. Since it is surjective, the number of \(h\)-tuples satisfying the above condition is equal to the cardinality of its kernel, which is equal to \(\text{card } R^h / \text{card } R = (\text{card } R)^{h-1}\). \(\square\)

**Lemma 2.5.** Let \(d \geq 1\) be an integer. The number of couples \((P, Q) \in \mathbb{F}_q[y]^2\) such that \(P\) is a degree \(d\) monic polynomial with \(P(0) \neq 0\), \(Q\) is of degree \(< d\), and \(P\) and \(Q\) are coprime is equal to

\[
c_d = (q - 1)^2 \frac{q^{2d} - 1}{q^2 - 1}.
\]

**Proof.** This amounts to counting the number of couples \((P, z)\), where \(P \in \mathbb{F}_q[y]\) is a degree \(d\) monic polynomial not divisible by \(y\) and \(z\) is an invertible element of the quotient ring \(\mathbb{F}_q[y]/(P)\).

Expanding \(P\) into a product of irreducible polynomials and using the Chinese remainder lemma, we have

\[
1 + \sum_{d \geq 1} c_d t^d = \prod_{P \text{ irreducible } P \neq y} \left( 1 + \sum_{k \geq 1} \text{card} \left( \mathbb{F}_q[y]/(P) \right) \times t^k \deg(P) \right),
\]

where the product is taken over all irreducible polynomials of \(\mathbb{F}_q[y]\) different from \(y\) and where \(\deg(P)\) denotes the degree of \(P\). First observe that for any irreducible polynomial \(P \in \mathbb{F}_q[y]\) the group \((\mathbb{F}_q[y]/(P))^\times\) of invertible elements of \(\mathbb{F}_q[y]/(P)\) is of cardinality \(q^{\deg(P)} - q^{(k-1)\deg(P)}\); indeed, there are \(q^{\deg(P)}\) polynomials of degree \(< k \deg(P)\) and \(q^{(k-1)\deg(P)}\) of them are divisible by \(P\), hence not invertible in \(\mathbb{F}_q[y]/(P)\). Consequently,

\[
1 + \sum_{d \geq 1} c_d t^d = \prod_{P \text{ irreducible } P \neq y} \left( 1 + \left( 1 - q^{-\deg(P)} \right) \sum_{k \geq 1} (qt)^{k \deg(P)} \right)
\]

\[
= \prod_{P \text{ irreducible } P \neq y} \left( 1 + \left( 1 - q^{-\deg(P)} \right) \frac{(qt)^{\deg(P)}}{1 - (qt)^{\deg(P)}} \right)
\]

\[
= \prod_{P \text{ irreducible } P \neq y} \frac{1 - t^{\deg(P)}}{1 - (qt)^{\deg(P)}}.
\]

On one hand the infinite product \(\prod_{P \text{ irreducible } P \neq y} (1 - t^{\deg(P)})^{-1}\) is equal to the zeta function \(Z_{\mathbb{A}^1 \setminus \{0\}}(t)\) of the affine line minus a point. On the other,

\[
Z_{\mathbb{A}^1 \setminus \{0\}}(t) = \frac{Z_{\mathbb{A}^1}(t)}{Z_{\{0\}}(t)} = \frac{1 - t}{1 - qt},
\]

Therefore,

\[
1 + \sum_{d \geq 1} c_d t^d = \frac{1 - qt}{1 - q^2 t} \bigg/ \frac{1 - t}{1 - qt} = \frac{(1 - qt)^2}{(1 - t)(1 - q^2 t)}.
\]

Subtracting 1 from both sides, we obtain

\[
\sum_{d \geq 1} c_d t^d = (q - 1)^2 \frac{t}{(1 - t)(1 - q^2 t)},
\]
from which it is easy to derive the desired formula for $c_d$. □

Proof of Proposition 2.3. We have to count the number of those $(h + 2)$-tuples $(P, Q, P_1, \ldots, P_h)$ such that $P$ is a degree $d$ monic polynomial with $P(0) \neq 0$, $Q$ is a polynomial of degree $< d$ and coprime to $P$, each polynomial $P_i$ is of degree $< d$, and $\sum_{i=1}^h P_iQ_i \equiv Q$ modulo $P$.

By Lemma 2.3, the number of couples $(P, Q)$ satisfying these conditions is equal to $(q - 1)^2 (q^{2d} - 1)/(q^2 - 1)$. Since card $\mathbb{F}_q[y]/(P) = q^d$, by Lemma 2.4 we have $q^{d(h-1)}$ choices for the $h$-tuples $(P_1, \ldots, P_h)$. The number we wish to count is the product of the two previous ones. □

3. The Hilbert scheme of points in a Zariski open subset of the plane

Let $k$ be a field. As is well known, the ideals of codimension $n$ of an affine $k$-algebra $A$ are in bijection with the $k$-points of the Hilbert scheme parametrizing finite subschemes of colength $n$ of the spectrum of $A$. For instance the ideals of codimension $n$ of the polynomial algebra $k[x, y]$ are in bijection with the $k$-points of the Hilbert scheme $\text{Hilb}^n(k^2)$ of $n$ points on the affine plane. Similarly, the ideals of codimension $n$ of the Laurent polynomial algebra $k[x, y, x^{-1}, y^{-1}]$ are in bijection with the $k$-points of the Hilbert scheme $\text{Hilb}^n((\mathbb{A}^1_2 \setminus \{0\}) \times (\mathbb{A}^1_2 \setminus \{0\}))$ of $n$ points on the two-dimensional torus, which is a Zariski open subset of the plane.

In this paragraph we prove that the Hilbert scheme of $n$ points in a Zariski open subset of the plane is an open subscheme of the Hilbert scheme of $n$ points in the plane, and show how to determine it explicitly.

3.1. Parametrizing the finite-codimensional ideals of $k[x, y]$. Computing the homology of Hilbert scheme $\text{Hilb}^n(k^2)$, Ellingsrud and Strømme [8] showed that it has a cellular decomposition indexed by the partitions $\lambda$ of $n$, each cell $C_\lambda$ being an affine space of dimension $n + \ell(\lambda)$, where $\ell(\lambda)$ is the length of $\lambda$.

It follows that, in the special case when $k = \mathbb{F}_q$ is a finite field of cardinality $q$, the number $A_n(q)$ of ideals of $\mathbb{F}_q[x, y]$ of codimension $n$ is finite and given by the polynomial

$$A_n(q) = \sum_{\lambda \vdash n} q^{\ell(\lambda)},$$

(3.1)

where the sum runs over all partitions $\lambda$ of $n$ (we indicate this by the notation $\lambda \vdash n$ or by $|\lambda| = n$). The polynomial $A_n(q)$ clearly has non-negative integer coefficients, its degree is $2n$, and $A_n(1) = p(n)$ is equal to the number of partitions of $n$ (for more on the polynomials $A_n(q)$, see Remark 4.7).

For our purposes we need an explicit description of the affine cells $C_\lambda$. We use a parametrization due to Conca and Valla [6]. Let us now recall it.

Given a positive integer $n$, there is a well-known bijection between the partitions of $n$ and the monomials ideals of codimension $n$ of $k[x, y]$. The correspondence is as follows: to a partition $\lambda$ of $n$ we associate the sequence

$$0 = m_0 < m_1 \leq \cdots \leq m_t$$

of integers counting from right to left the boxes in each column of the Ferrers diagram of $\lambda$; we have $m_1 + \cdots + m_t = n$. Then the associated monomial ideal $I_\lambda^0$ is given by

$$I_\lambda^0 = (x_1^{m_1}y^{m_1}, \ldots, x_1^{m_t}y^{m_t}).$$

(3.2)
tion on two cases:

3.2. degree of polynomial algebra \( k \) of this integer is equal to the number of distinct values of the sequence \( \lambda \).

We have \( m_1 = m_1 > 0 \).

Later we shall also need the integer

\[
\nu(\lambda) = \text{card} \{ i = 1, \ldots, t \mid d_i \geq 1 \}.
\]

This integer is equal to the number of distinct values of the sequence \( m_1 \leq \ldots \leq m_t \).

Note that \( \nu(\lambda) \geq 1 \); moreover, \( \nu(\lambda) = 1 \) if and only if the partition is "rectangular", i.e. \( m_1 = \cdots = m_t \) (> 0).

Let \( T_3 \) be the set of \( (t + 1) \times t \)-matrices \( (p_{i,j}) \) with entries in the one-variable polynomial algebra \( k[y] \) satisfying the following conditions: \( p_{i,j} = 0 \) if \( i < j \), the degree of \( p_{i,j} \) is less than \( d_j \) if \( i \geq j \) and \( d_j \geq 1 \), and \( p_{i,j} = 0 \) for all \( i \) if \( d_j = 0 \). The set \( T_3 \) is an affine space whose dimension is \( n + \ell(\lambda) \).

Now consider the \( (t + 1) \times t \)-matrix

\[
M_d = \begin{pmatrix}
\begin{array}{ccccccc}
p_{1,1} - x & y^2 + p_1 & 0 & 0 & \cdots & 0 & 0 \\
p_{2,1} - x & y^2 + p_2 & 0 & 0 & \cdots & 0 & 0 \\
p_{3,1} & y^2 + p_3 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
p_{i-1,1} & p_{i-1,1} & p_{i-1,2} & y^{d_i-1} & + & p_{i-1} & 0 & 0 & \cdots & 0 \\
p_{i,1} & p_{i,1} & p_{i,2} & p_{i,3} & \cdots & p_{i-1,1} - x & y^{d_i} + p_i & 0 & \cdots & 0 \\
p_{i+1,1} & p_{i+1,1} & p_{i+1,2} & p_{i+1,3} & \cdots & p_{i-1,1} - x & y^{d_i} + p_i & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
p_{1,1} & p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{i-1,1} & p_{i-1} & p_{i} & p_{i+1} & \cdots & y^{d_i} + p_i \\
p_{i+1,1} & p_{i+1,1} & p_{i+1,2} & p_{i+1,3} & \cdots & p_{i+1,1} - x & p_{i+1} & p_{i+2} & p_{i+3} & \cdots & p_{i+1} - x
\end{array}
\end{pmatrix}
\]

where for simplicity we set \( p_{i,j} = p_{i,i} \).

By [6, Th. 3.3] the map sending the polynomial matrix \( (p_{i,j}) \in T_3 \) to the ideal \( I_d \) of \( k[x,y] \) generated by all \( t \)-minors (the maximal minors) of the matrix \( M_d \) is a bijection of \( T_3 \) onto \( C_d \). These minors are polynomial expressions with integer coefficients in the coefficients of the \( p_{i,j} \)’s.

3.2. Localizing. Let \( S \) be a multiplicative submonoid of \( k[x,y] \) not containing 0.

We assume that \( S \) has a finite generating set \( \Sigma \). In the sequel we shall concentrate on two cases: \( \Sigma = \{ y \} \) (in Section 3) and \( \Sigma = \{ x, y \} \) (in Section 5).

It follows from Section 3 that the set of \( n \)-codimensional ideals of the localization \( S^{-1} k[x,y] \) can be identified with the subset of \( \text{Hilb}^n(A^2_k) \) consisting of the \( n \)-codimensional ideals \( I \) of \( k[x,y] \) such that for all \( s \in S \), the endomorphism \( \mu_s \) of \( k[x,y]/I \) induced by the multiplication by \( s \) is a linear isomorphism. The latter is equivalent to \( \det \mu_s \neq 0 \) for all \( s \in \Sigma \).
By the considerations of Section 3.1, the set of \( n \)-codimensional ideals of the algebra \( S^{-1}k[x, y] \) is the disjoint union
\[
\bigcup_{\lambda \vdash n} C^\Sigma_{\lambda},
\]
where \( C^\Sigma_{\lambda} \) is the Zariski open subset of the affine Gröbner cell \( C_\lambda \) consisting of the points satisfying \( \det \mu_s \neq 0 \) for all \( s \in \Sigma \).

Consequently, the Hilbert scheme \( \text{Hilb}^n(\text{Spec}(S^{-1}k[x, y])) \) parametrizing sub-schemes of colength \( n \) in \( \text{Spec}(S^{-1}k[x, y]) \) is an open subscheme of \( \text{Hilb}^n(\mathbb{A}^2_k) \), hence an open subscheme of \( \text{Hilb}^n(\mathbb{P}^2_k) \). Since by [10,12] the latter is smooth and projective, \( \text{Hilb}^n(\text{Spec}(S^{-1}k[x, y])) \) is a smooth quasi-projective variety.

The endomorphism \( \mu_s \) (resp. \( \mu_y \)) of \( k[x, y]/I \) induced by the multiplication by \( x \) (resp. by \( y \)) can be expressed as a matrix in the basis \( \mathcal{B}_I \). Observe that the entries of such a matrix are polynomial expressions with integer coefficients in the coefficients of the \( p_{ij} \)'s. Therefore, if any \( s \in \Sigma \) is a linear combination with integer coefficients of monomials in the variables \( x, y \), then the Hilbert scheme \( \text{Hilb}^n(\text{Spec}(S^{-1}k[x, y])) \) is defined over \( \mathbb{Z} \) as a variety.

In particular, the Hilbert schemes \( \text{Hilb}^n(\mathbb{A}^1_k \times (\mathbb{A}^1_k \setminus \{0\})) \) and \( \text{Hilb}^n((\mathbb{A}^1_k \setminus \{0\})^2) \) are smooth quasi-projective varieties defined over \( \mathbb{Z} \).

**Example 3.1.** Let \( \lambda \) be the unique self-conjugate partition of 3. In this case, \( t = 2, m_1 = 1, m_2 = 2 \), hence \( d_1 = d_2 = 1 \). The corresponding matrix \( M_\lambda \), as in (3.5), is
\[
M_\lambda = \begin{pmatrix}
y + a & 0 \\
-b - x & y + d \\
c & -e - x
\end{pmatrix},
\]
where \( a, b, c, d, e \) are scalars. The associated Gröbner cell \( C_\lambda \) is a 5-dimensional affine space parametrized by these five scalars. The ideal \( I_\lambda \) is generated by the maximal minors of the matrix, namely by \((b-x)(e-x)-c(y+d),(e-x)(y+a)\), and \((y+a)(y+d)\). It follows that modulo \( I_\lambda \) we have the relations
\[
x^2 \equiv (b+e)x + cy + (cd - be), \quad xy \equiv -ax + ey + ae, \quad y^2 \equiv -(a+d)y - ad.
\]

In the basis \( \mathcal{B}_I = \{ x, y, 1 \} \) the multiplication endomorphisms \( \mu_s \) and \( \mu_y \) can be expressed as the matrices
\[
\mu_s = \begin{pmatrix}
b + e & -a & 1 \\
c & e & 0 \\
(cd - be) & ae & 0
\end{pmatrix} \quad \text{and} \quad \mu_y = \begin{pmatrix}
a & 0 & 0 \\
e & -(a+d) & 1 \\
(ae) & -ad & 0
\end{pmatrix}.
\]
We have \( \det \mu_s = e(ac - cd + be) \) and \( \det \mu_y = -ad^2 \).

It follows from the above computations that, if for instance \( \Sigma = \{ x, y \} \), then \( C^\Sigma_{\lambda} \) is the complement in the affine space \( \mathbb{A}^2_k \) of the union of the three hyperplanes \( a = 0, d = 0, e = 0 \) and of the quadric hypersurface \( ac - cd + be = 0 \).

4. **The punctual Hilbert scheme of the complement of a line in an affine plane**

In this section we apply the considerations of the previous section to the case \( \Sigma = \{ y \} \). Here \( S \) is the multiplicative submonoid of \( k[x, y] \) generated by \( y \) and \( S^{-1}k[x, y] = k[x, y, y^{-1}] = k[x][y, y^{-1}] \).

By Section 3.2, the Hilbert scheme \( \text{Hilb}^n(\mathbb{A}^1_k \times (\mathbb{A}^1_k \setminus \{0\})) \), that is the set of \( n \)-codimensional ideals of \( k[x, y, y^{-1}] \), is the disjoint union over the partitions \( \lambda \) of \( n \)
of the sets $C^y_\lambda$, where $C^y_\lambda$ consists of the ideals $I \in C_\lambda$ such that $y$ is invertible in $k[x,y]/I$. We call $C^y_\lambda$ the semi-invertible Gröbner cell associated to the partition $\lambda$.

4.1. A criterion for the invertibility of $y$. Let $p_y : k[x,y] \to k[x]$ be the algebra map sending $x$ to itself and $y$ to 0. Then by Lemma 22, the set $C^y_\lambda$ consists of the ideals $I \in C_\lambda$ such that $p_y(I) = k[x]$.

Recall from Section 3.1 that $I_\lambda$ is generated by the maximal minors of the matrix $M_\lambda$ of (3.5), namely by the polynomials $f_0(x,y), \ldots, f_t(x,y)$, where we define $f_i(x,y)$ to be the determinant of the $t \times t$-matrix obtained from $M_\lambda$ by deleting its $(i+1)$-st row. Then the ideal $p_y(I_\lambda)$ can be identified with the ideal of $k[x]$ generated by the polynomials $f_0(x,0), \ldots, f_t(x,0) \in k[x]$ obtained by setting $y = 0$. We need to determine under what conditions this ideal is equal to the whole algebra $k[x]$.

Recall the entries of the matrix $M_\lambda$ and particularly the polynomials $p_{i,j}$, and $p_t = p_{t,i} \in k[y]$. Let $a_{i,j} = p_{i,j}(0)$ be the constant term of $p_{i,j}$. As above, we set $a_0 = a_t = a_t(0)$. Note that $a_j = 1$ and $a_{i,j} = 0$ for $i \neq j$ whenever $d_j = 0$.

Then $f_0(x,0), \ldots, f_t(x,0)$ are the maximal minors of the matrix

$$M'_\lambda = \left( \begin{array}{cccccccc} a_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{2,1} - x & a_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} - x & a_3 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1} & 0 & 0 & \cdots & 0 \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i-1} - x & a_i & 0 & \cdots & 0 \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,1} & a_{i+1,2} - x & a_{i+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{t,1} & a_{t,2} & a_{t,3} & \cdots & a_{t-1} & a_t & a_{t+1} & \cdots & a_t \\ a_{t+1,1} & a_{t+1,2} & a_{t+1,3} & \cdots & a_{t+1,1} & a_{t+1,2} & a_{t+1,3} & \cdots & a_{t+1} - x \end{array} \right).$$

To be precise, $f_i(x,0)$ is the determinant of the square matrix obtained from $M'_\lambda$ by deleting its $(i+1)$-st row.

The criterion we need is the following.

**Proposition 4.1.** We have $p_y(I_\lambda) = k[x]$ if and only if $a_j \neq 0$ for all $i = 1, \ldots, t$ such that $d_i \geq 1$.

**Proof.** Since $a_0 = 1$ when $d_0 = 0$, it is equivalent to prove that $p_y(I_\lambda) = k[x]$ if and only if $a_1 a_2 \cdots a_t \neq 0$.

Set $I_\lambda = p_y(I_\lambda) \subset k[x]$. The condition $a_1 a_2 \cdots a_t \neq 0$ is sufficient. Indeed, the last polynomial, $f_t(x,0)$, is the determinant of a lower triangular matrix whose diagonal entries are the scalars $a_i$; hence, $f_t(x,0) = a_1 a_2 \cdots a_t$. Thus, if $f_t(x,0)$ is non-zero, then $I_\lambda = k[x]$.

To check the necessity of the condition, we will prove that for each $i = 1, \ldots, t$, the vanishing of the scalar $a_i$ implies that the ideal $I_i$ is contained in a proper ideal generated by a minor of $M'_\lambda$.

If $a_1 = 0$, then $f_1(x,0) = \cdots = f_t(x,0) = 0$ since these are determinants of matrices whose first row is zero. It follows that $I_\lambda$ is the principal ideal generated by the characteristic polynomial $f_0(x,0)$, which is of degree $t \geq 1$. Hence, $I_\lambda$ is a proper ideal of $k[x]$. 


Let now \( i \geq 2 \). If for \( k \geq i \), we delete the \((k + 1)\)-st row of \( M_i^\lambda \), we obtain a lower block-triangular matrix of the form

\[
\begin{pmatrix}
M_1 & 0 \\
* & M_2^{(k)}
\end{pmatrix},
\]

where \( M_1 \) is the square submatrix of \( M_i^\lambda \) corresponding to the rows 1, \ldots, \( i \) and to the columns 1, \ldots, \( i \); this is a lower triangular matrix whose diagonal entries are \( a_1, \ldots, a_i \). Consequently, if \( a_i = 0 \), then \( f_k(x, 0) = 0 \) for all \( k \geq i \).

Under the same condition \( a_i = 0 \), if we delete the \((k + 1)\)-st row of \( M_i^\lambda \) for \( k < i \), then we obtain a lower block-triangular matrix of the form

\[
\begin{pmatrix}
M_1^{(k)} & 0 \\
* & M_2
\end{pmatrix},
\]

where \( M_2 \) is the square submatrix of \( M_i^\lambda \) corresponding to the rows \( i + 1, \ldots, t + 1 \) and to the columns \( i, \ldots, t \):

\[
M_2 = \begin{pmatrix}
a_{i+1,i} - x & a_{i+1} & \cdots & 0 & 0 \\
a_{i+2,i} & a_{i+2,i+1} - x & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{t,i} & \cdots & \cdots & a_{t,t-1} - x & a_t \\
a_{t+1,i} & a_{t+1,i+1} & \cdots & a_{t+1,t-1} & a_{t+1,t} - x
\end{pmatrix}.
\]

Consequently, the polynomials \( f_k(x, 0) \) for \( k < i \) are all divisible by the determinant of \( M_2 \). Thus, \( I_x \) is contained in the ideal generated by \( \det(M_2) \), which is a characteristic polynomial of degree \( t - i + 1 \). Since \( t - i + 1 \geq 1 \) for all \( i = 1, \ldots, t \), we have \( I_x \neq k[x] \).

As an immediate consequence of Section 3.2 and of Proposition 4.1 we obtain the following.

**Corollary 4.2.** The set of \( n \)-codimensional ideals of \( k[x, y, y^{-1}] \) is the disjoint union

\[
\bigcup_{\lambda \vdash n} C_\lambda^y,
\]

where \( C_\lambda^y \) is the complement in the affine Gröbner cell \( C_\lambda \) of the union of the hyperplanes \( a_i = 0 \) where \( i \) runs over all integers \( i = 1, \ldots, t \) such that \( d_i \geq 1 \).

**4.2. On the number of finite-codimensional ideals of \( k[x, y, y^{-1}] \).** Recall the positive integer \( v(\lambda) \) defined by (3.4).

**Proposition 4.3.** Let \( k = F_q \). For each partition \( \lambda \) of \( n \), the set \( C_\lambda \) is finite and its cardinality is given by

\[
\text{card } C_\lambda^y = (q - 1)^{v(\lambda)} q^{n + \ell(\lambda) - v(\lambda)}.
\]

**Proof.** By Corollary 4.2, the set \( C_\lambda^y \) is parametrized by \( n + \ell(\lambda) \) parameters subject to the sole condition that \( v(\lambda) \) of them are not zero. \( \square \)

**Corollary 4.4.** For each integer \( n \geq 1 \), the number \( B_n(q) \) of \( n \)-codimensional ideals of \( F_q[x, y, y^{-1}] \) is equal to \((q - 1) q^n B_n^q(q)\), where

\[
B_n^q(q) = \sum_{\lambda \vdash n} (q - 1)^{v(\lambda) - 1} q^{\ell(\lambda) - v(\lambda)}.
\]
Note that $B_n^p(q)$ is a polynomial in $q$ since $v(\lambda) \geq 1$ and $\ell(\lambda) \geq v(\lambda)$ for all partitions. It is of degree $n-1$ and has integer coefficients. The coefficients of $B_n^p(q)$ may be negative, as one can see in Table 3 at the end of the paper.

Remark 4.5. Let $v_n$ be the valuation of the polynomial $B_n^p(q)$, i.e. the maximal integer $r$ such that $q^r$ divides $B_n^p(q)$. We conjecture that $v_n = 0$, 1, or 2, and that the infinite word $v_1v_2v_3\ldots$ is equal to $0\sigma_0 p^n$.

Let us now give a product formula for the generating function of the sequence of polynomials $B_n^p(q)$ and an arithmetical interpretation for two values of $B_n^p(q)$.

Theorem 4.6. (a) Let $B_n(q)$ be the number of ideals of $\mathbb{F}_q[x, y, y^{-1}]$ of codimension $n$. We have

$$1 + \sum_{n \geq 1} \frac{B_n(q)}{q^n} t^n = \prod_{i \geq 1} \frac{1 - t^i}{1 - qt^i}.$$ 

(b) Let $B_n^p(q)$ be the polynomial $B_n^p(q) = (q-1)^{-1}q^{-n}B_n(q)$. It has integer coefficients and satisfies

$$B_n^p(1) = \sigma_0(n),$$

where $\sigma_0(n)$ is the number of divisors of $n$, and

$$B_n^p(-1) = \begin{cases} (-1)^{k-1} & \text{if } n = k^2 \text{ for some integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (a) Since an analogous proof will be used in Remark 4.7 and in Section 6.2, we give here a detailed proof. Let $X$ be a set and $M$ be the free abelian monoid on $X$ ($X$ is called a basis of $M$). We say that a function $\varphi : M \to R$ from $M$ to a ring $R$ is multiplicative if $\varphi(puv) = \varphi(p)\varphi(u)\varphi(v)$ for all couples $(u, v) \in M^2$ of words having no common basis element. Under this condition, it is easy to check the following identity:

$$\sum_{w \in M} \varphi(w) = \prod_{x \in X} \left(1 + \sum_{e \geq 1} \varphi(x^e)\right).$$

Now, identifying each partition with its planar diagram, we consider a partition $\lambda$ as a union of rectangular partitions $i^e_i$, with $e_i$ parts of length $i$, for $e_i \geq 1$ and distinct $i \geq 1$, which we denote by the formal product $\lambda = \prod_{i \geq 1} i^{e_i}$. Thus the set of partitions is equal to the free abelian monoid on $X = \mathbb{N}\setminus\{0\}$. Before we apply (4.1), let us remark that $|\lambda| = \sum_i e_i$ and $\ell(\lambda) = \sum_i e_i$. Moreover, the multisets $\{e_i | i \geq 1\}$ and $\{d_i | i \geq 1\}$ are equal (recall that the integers $d_i$ are those associated with $\lambda$ in (3.3)); therefore, $v(\lambda) = \sum_i 1 = \text{card}\{i | e_i \geq 1\}$. 

The function $\lambda \mapsto \mathrm{card} \, C_\lambda s^{[d]}$ computed in Proposition 4.3 is clearly multiplicative. Applying (4.1), we obtain

$$1 + \sum_{n \geq 1} B_n(q)s^n = 1 + \sum_{|d| \geq 1} \mathrm{card} \, C_{\lambda} s^{[d]}$$

$$= \prod_{i \geq 1} \left( 1 + \sum_{e \geq 1} (q - 1)q^{i e + e - 1}s^{i e} \right)$$

$$= \prod_{i \geq 1} \left( 1 + (q - 1)q^{-1} \sum_{e \geq 1} (q^{i + 1}s)^e \right)$$

$$= \prod_{i \geq 1} \left( 1 + (q - 1)q^{-1} \frac{q^{i + 1}s}{1 - q^{i + 1}s} \right)$$

$$= \prod_{i \geq 1} \left( 1 - q^{i + 1}s \right) + (q - 1)q^{q} \frac{1}{1 - q^{i + 1}s}$$

Finally replace $s$ by $q^{-1}t$.

(b) To compute $B_n^\sigma(1)$ we use the formula of Corollary 4.4. Since the values at $q = 1$ of $(q - 1)^{v(\lambda)} - 1$ is 1 if $v(\lambda) = 1$ and 0 otherwise and since $v(\lambda) = 1$ if and only if $m_1 = \cdots = m_t = d$, in which case $dt = n$, we have

$$B_n^\sigma(1) = \sum_{dt = n} 1 = \sum_{d|n} 1 = \sigma_0(n).$$

For $B_n^\sigma(-1)$ we use the infinite product expansion of Item (a): replacing $B_n(q)$ by $(q - 1)q^q B_n^\sigma(q)$, we obtain

$$1 + \sum_{n \geq 1} (q - 1)B_n^\sigma(q) t^n = \prod_{i \geq 1} \frac{1 - t^i}{1 - qt^i}.$$

Setting $q = -1$ yields

$$1 - 2 \sum_{n \geq 1} B_n^\sigma(-1) t^n = \prod_{i \geq 1} \frac{1 - t^i}{1 + t^i}.$$

Now recall the following identity of Gauss (see [9, (7.324)] or [17, 19.9 (i)]):

$$\prod_{i \geq 1} \frac{1 - t^i}{1 + t^i} = \sum_{k \in \mathbb{Z}} (-1)^k t^k. \quad (4.2)$$

It follows that

$$1 - 2 \sum_{n \geq 1} B_n^\sigma(-1) t^n = 1 + 2 \sum_{k \geq 1} (-1)^k t^k,$$

which allows us to conclude.
Remark 4.7. The results of Theorem 4.6 should be compared to the following ones concerning the number $A_n(p)$ of ideals of $\mathbb{F}_q[x,y]$ of codimension $n$. Proceeding as in the proof of Theorem 4.6 we deduce from (3.1) that

$$1 + \sum_{n \geq 1} A_n(q)s^n = \prod_{i \geq 1} \frac{1}{1 - q^{i+1}s^i}.$$ 

Setting $q = -1$, we have

$$1 + \sum_{n \geq 1} A_n(-1)s^n = \prod_{i \geq 1} \frac{1}{1 - (-1)^{i+1}s^i} = \prod_{m \geq 1} \frac{1}{(1 - s^{2m-1})(1 + s^{2m})}. \quad (4.3)$$

Multiplying by $\prod_{m \geq 1} (1 + s^{2m})^{-1}$ both sides of the Euler identity

$$\prod_{m \geq 1} \frac{1}{1 - s^{2m-1}} = \prod_{i \geq 1} (1 + s^i)$$

(see [17], (19.4.7)), we deduce that the right-hand side of (4.3) is equal to the infinite product

$$\prod_{m \geq 1} (1 + s^{2m-1}).$$

Thus by [1], Table 14.1, p. 310 or [17], (19.4.4), the value $A_n(-1)$ is equal to the number of partitions of $n$ with unequal odd parts. Note that $A_n(1)$ is equal to the number of partitions of $n$. See Table 4 at the end for a list of the polynomials $A_n(q)$ ($1 \leq n \leq 12$).

5. Invertible Grobner cells

Let $\text{Hilb}^n((\mathbb{A}_k^1\setminus\{0\})^2)$ be the Hilbert scheme parametrizing finite subschemes of colength $n$ of the two-dimensional torus, i.e. of the complement of two distinct intersecting lines in the affine plane. Its $k$-points are in bijection with the set of ideals of $k[x,y,x^{-1},y^{-1}]$ of codimension $n$. By Section 3.2 this set of ideals is the disjoint union over the partitions $\lambda$ of $n$ of the sets $C_{x,y}^\lambda$, where $C_{x,y}^\lambda$ consists of the ideals $I \in C_{\lambda}$ such that both $x$ and $y$ are invertible in $k[x,y]/I$. We call $C_{x,y}^\lambda$ the invertible Grobner cell associated to the partition $\lambda$.

When the ground field is finite, so is $C_{x,y}^\lambda$. The aim of this section is to compute the cardinality of $C_{x,y}^\lambda$ when $k = \mathbb{F}_q$.

5.1. The cardinality of an invertible Grobner cell. Recall the non-negative integers $d_1, \ldots, d_t$ defined by (3.3) and the positive integer $v(\lambda)$ defined by (3.4). We now give a formula for $\text{card} C_{x,y}^\lambda$.

Theorem 5.1. Let $k = \mathbb{F}_q$, $n$ an integer $\geq 1$ and $\lambda$ a partition of $n$. Then

$$\text{card} C_{x,y}^\lambda = (q - 1)^{2v(\lambda)} q^{n - \ell(\lambda)} \prod_{d_i \geq 1} \frac{q^{2d_i} - 1}{q^{2d_i}}.$$

The theorem will be proved in Section 5.3. It has the following straightforward consequences.

\footnote{See Sequence A000700 in [19].}
\footnote{See Sequence A000041 in [19].}
Corollary 5.2. Let $k = \mathbb{F}_q$ and $\lambda$ be a partition of $n$.

(a) $\text{card } C^{xy}_\lambda$ is a monic polynomial in $q$ with integer coefficients; it is of degree $n + \ell(\lambda)$.

(b) The polynomial $\text{card } C^{xy}_\lambda$ is divisible by $(q - 1)^2$. The quotient

$$P_\lambda(q) = \frac{\text{card } C^{xy}_\lambda}{(q - 1)^2}$$

is a monic polynomial in $q$ with integer coefficients and of degree $n + \ell(\lambda) - 2$.

(c) If the partition $\lambda$ is rectangular, i.e., if $v(\lambda) = 1$, in which case $d_2 = \cdots = d_t = 0$ and $d = d_1$ is a divisor of $n$, then

$$P_\lambda(q) = q^{n/d} \frac{q^{2d} - 1}{q^2 - 1} = q^{n/d} \left(1 + q^2 + \cdots + q^{2d-2}\right).$$

In this case, $P_\lambda(1) = d$.

(d) If $v(\lambda) \geq 2$, then $P_\lambda(q)$ is divisible by $(q - 1)^2$, and $P_\lambda(1) = 0$.

Remark 5.3. The polynomials $P_\lambda(q)$ may have negative coefficients. For instance, if $\lambda$ is the partition of 4 corresponding to $t = 2$, $d_1 = 1$, $d_2 = 2$, then

$$P_\lambda(q) = q^5 - 2q^4 + 2q^3 - 2q^2 + q.$$

The rest of the section is devoted to the proof of Theorem 5.1.

5.2. A criterion for the invertibility of $x$. In Section 4, we introduced the algebra map $p_x : k[x, y] \to k[x]$ sending $x$ to itself and $y$ to 0. Similarly, let $p_x : k[x, y] \to k[x]$ be the algebra map sending $x$ to 0 and $y$ to itself. Then by Lemma 2.2, the set $C^{xy}_\lambda$ consists of the ideals $I \in C_\lambda$ such that $p_x(I) = k[y]$ and $p_y(I) = k[x]$. We already have a criterion for $p_y(I) = k[x]$ (see Proposition 4.1). We shall now give a necessary and sufficient condition for $p_x(I)$ to be equal to $k[y]$.

Resuming the notation of Section 4, we see that $p_x(I)$ can be identified with the ideal of $k[y]$ generated by the polynomials $f_0(0, y), \ldots, f_t(0, y) \in k[y]$ obtained from the polynomials $f_0(x, y), \ldots, f_t(x, y)$ by setting $x = 0$. The polynomials $f_0(0, y), \ldots, f_t(0, y)$ are the maximal minors of the matrix

$$M^y_\lambda = \begin{pmatrix}
\gamma^{p_1} + p_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
p_{2,1} & \gamma^{p_2} + p_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
p_{3,1} & p_{3,2} & \gamma^{p_3} + p_3 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
p_{t,1} & p_{t,2} & p_{t,3} & \cdots & p_{t-1,1} & y^{p_t} + p_t & 0 & 0 & \cdots & 0 \\
p_{t+1,1} & p_{t+1,2} & p_{t+1,3} & \cdots & p_{t+1,1} & p_{t+1,t} & y^{p_{t+1}} + p_{t+1} & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
p_{t+1,t-1} & p_{t+1,t} & p_{t+1,t} & \cdots & p_{t+1,t-1} & p_{t+1,t} & p_{t+1,t} & y^{p_{t+1}} + p_{t+1} & 0 & \cdots & 0 \\
p_{t+1,1} & p_{t+1,2} & p_{t+1,3} & \cdots & p_{t+1,t-1} & p_{t+1,t} & p_{t+1,t} & p_{t+1,t} & y^{p_{t+1}} + p_{t+1} & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}$$

obtained from the matrix $M^y_\lambda$ of (5.5) by setting $x = 0$.

Let $\mu_i$ be the determinant of the submatrix $M_i$ of $M^y_\lambda$ corresponding to the rows $(i + 1), \ldots, (t + 1)$ and to the columns $i, \ldots, t$. We have $\mu_t = p_{t+1,t}$ and

$$\mu_i = \begin{vmatrix}
p_{t+i+1} & y^{p_{t+i+1}} + p_{t+i+1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
p_{t,i+1} & p_{t+1,i+1} & \cdots & y^{p_t} + p_t \\
p_{t+1,i} & p_{t+1,t+1} & \cdots & p_{t+1,t} \\
\end{vmatrix}$$

for $i = t, t-1, \ldots, 1$.
Lemma 5.4. If $1 \leq i < t$. Expanding $\mu_i$ along its first column, we obtain

\begin{equation}
\mu_i = \sum_{j=1}^{t-i+1} p_{i+j} q_{i+j} i, \tag{5.1}
\end{equation}

where

\begin{equation}
q_{i+j} = \begin{cases}
\mu_{i+1} & \text{if } j = 1, \\
(-1)^{i-1} (y^d+1 + p_{i+1}) \cdots (y^d+j-1 + p_{i+j-1}) \mu_{i+j} & \text{if } 1 < j < t-i+1, \\
(-1)^{t-i} (y^d+1 + p_{i+1}) \cdots (y^d+t-1)(y^d+p_t) & \text{if } j = t-i+1.
\end{cases} \tag{5.2}
\end{equation}

Observe also that

\begin{equation}
f_i(0, y) = \begin{cases}
\mu_1 & \text{if } i = 0, \\
(y^d+p_1) \cdots (y^d+p_i) \mu_{i+1} & \text{if } 1 \leq i < t, \\
(y^d+p_1) \cdots (y^d+p_t) & \text{if } i = t. \tag{5.3}
\end{cases}
\end{equation}

Lemma 5.4. If $1 \leq i \leq j \leq t$, then $\mu_i$ belongs to the ideal $(\mu_j, y^d + p_j)$ generated by $\mu_j$ and $(y^d + p_j)$.

Proof. The case $i = j$ is obvious. Otherwise, consider the matrix $M_i$ whose determinant is $\mu_i$; the column of $M_i$ containing the entry $y^d + p_j$ can be written as the sum of a column containing only the entry $y^d + p_j$, the other entries being zero, and of a column whose top entry is zero and the bottom ones form the first column of the matrix $M_j$ whose determinant is $\mu_j$. Therefore by the multilinearity property of determinants, $\mu_i$ is the sum of a determinant which is a multiple of $\mu_j$ and of another determinant which is a multiple of $\mu_j$; indeed, this second determinant is block-triangular with one diagonal block equal to $\mu_j$. \hfill \square

Here is our criterion for the invertibility of $x$.

Proposition 5.5. We have $p_x(I_A) = k[\gamma]$ if and only if $y^d + p_i$ and $\mu_i$ are coprime for all $i = 1, \ldots, t$.

Proof. (a) Let us first check that the above condition is sufficient. The fact that $y^d + p_i$ and $\mu_i$ are coprime implies that by (5.3) the gcd of $f_i(0, y)$ and of $f_{i-1}(0, y)$ is $(y^d+i+1) \cdots (y^d+i+1)$. Now the gcd of the latter and of $f_{i-2}(0, y)$ is $(y^d+i+1) \cdots (y^d+i+1)$ in view of the fact that $y^d+i+1$ and $\mu_{i-1}$ are coprime. Repeating this argument, we find that the gcd of $f_0(0, y), \ldots, f_i(0, y)$ is 1, which implies that $p_x(I_A) = k[\gamma]$.

(b) Conversely, suppose that $y^d + p_j$ and $\mu_j$ are not coprime for some $j$, i.e., $(\mu_j, y^d + p_j) \neq k[\gamma]$. By (5.3) and Lemma 5.4, $f_0(0, y), \ldots, f_{i-1}(0, y)$ belong to the ideal $(\mu_j, y^d + p_j)$. On the other hand, again by (5.3), the remaining polynomials $f_j(0, y), \ldots, f_t(0, y)$ are divisible by $y^d + p_j$, hence belong to $(\mu_j, y^d + p_j)$. Therefore, $p_x(I_A) \subseteq (\mu_j, y^d + p_j) \neq k[\gamma]$. \hfill \square

For the proof of Theorem 5.1, we shall also need the following result.

Lemma 5.6. If $y^d + p_j$ and $\mu_j$ are coprime for all $j > i$, then the polynomials $q_{i+1,i}, \ldots, q_{t+1,i}$ of (5.2) are coprime.
Proof. Proceeding as in Part (a) of the proof of Proposition 5.5 and using (5.2), one shows by descending induction on \( j \) that the gcd of \( q_{j+1,i}, \ldots, q_{t+1,i} \) is
\[
(y^{d_{i+1}} + p_{i+1}) \cdots (y^{d_j} + p_j).
\]
In particular, for \( j = i + 1 \), the gcd of \( q_{i+1,i}, \ldots, q_{t+1,i} \) is \((y^{d_{i+1}} + p_{i+1})\). The conclusion follows from this fact together with the coprimality of \((y^{d_{i+1}} + p_{i+1})\) and of \(q_{i+1,i} = \mu_{i+1} \).

5.3. Proof of Theorem 5.1. By Propositions 3.1 and 5.5 it is enough to count the entries of the matrix \(M_\lambda \) over \( \mathbb{F}_q[y] \) such that \( p_i(0) \neq 0 \) and \( y^{d_i} + p_i \) and \( \mu_i \) are coprime for all \( i = 1, \ldots, t \). We consider these conditions successively for \( i = t, t-1, \ldots, 1 \).

Assume first that all integers \( d_1, \ldots, d_t \) are non-zero. For \( i = t \), \( y^{d_t} + p_t \) is a monic polynomial of degree \( d_t \) with non-zero constant term, \( \mu_t = p_{t+1,t} \) is of degree \( < d_t \), and both polynomials are coprime. It follows from Lemma 2.3 (or from Proposition 5.5 with \( d = d_t \) and \( h = 1 \)) that we have \((q-1)^2(q^{2d_t-1} - 1)/(q^2 - 1)\) possible choices for the last column of \(M_\lambda \).

For \( i = t-1 \), it follows from (5.1) that \( \mu_{i-1} = P_1 Q_1 + P_2 Q_2 \), where \( Q_1 = q_{i+1,i-1} \) and \( Q_2 = -q_{i+1,i-1} \), which are coprime by Lemma 5.6. \( P_1 = p_{i+1,i-1} \) and \( P_2 = p_{i+1,i-1} \), which are both polynomials of degree \( < d_{i-1} \). The polynomial \( P = y^{h-1} + p_{i-1} \) is monic of degree \( d_{i-1} \) with non-zero constant term, and \( Q = \mu_{i-1} = P_1 Q_1 + P_2 Q_2 \) is coprime to \( P \) by the coprimality condition. It then follows from Proposition 2.3 applied to the case \( d = d_{i-1} \) and \( h = 2 \) that there are
\[
(q-1)^2 q^{2d_{i-1} - 1}/q^2 - 1
\]
possible choices for the \((t-1)\)-st column of \(M_\lambda \).

In general, the polynomial \( P = y^{d_t} + p_t \) is monic of degree \( d_{t-1} \) with non-zero constant term, and is assumed to be coprime to \( Q = \mu_t = \sum_{j=1}^{t+1} p_{i+j,i} q_{i+j,i} \). By Lemma 5.6 the polynomials \( q_{i+1,i}, \ldots, q_{i+1,i} \) are coprime. Applying Proposition 2.3 to the case \( d = d_i \) and \( h = t + 1 - i \), we see that there are
\[
(q-1)^2 q^{d_i-t}/q^2 - 1
\]
possible choices for the \(i\)-th column of \(M_\lambda \).

In the end we obtain a number of possible entries for \(M_\lambda \) equal to
\[
\prod_{i=1}^{t} (q-1)^2 q^{d_i-t}/q^2 - 1 = q^{n-\ell(\lambda)} \prod_{i=1}^{t} (q-1)^2 q^{2d_i}/q^2 - 1
\]
since \( \ell(\lambda) = \sum_{i=1}^{t} d_i \) and \( n = |\lambda| = \sum_{i=1}^{t} (t - i + 1) d_i \). We have thus proved the theorem when all \( d_1, \ldots, d_t \) are non-zero.

Let \( E \) be the subset of \( \{1, \ldots, t\} \) consisting of those subscripts \( i \) for which \( d_i = 0 \). (Note that 1 does not belong to \( E \) since \( d_1 > 0 \).) Assume now that \( E \) is non-empty and set \( t' = t - \text{card} \ E \). By assumption \( t' < t \). For any positive integer \( i \leq t' \), let \( d_i' = d_i \) be equal to the \(i\)-th non-zero \( d_i \). The integers \( d_1', d_2', \ldots, d_t' \) are positive.

Recall that if \( i \in E \), then the \(i\)-th column of the matrix \(M_\lambda \) is zero except for the \((i,i)\)-entry which is 1. Permuting rows and columns, we may rearrange \(M_\lambda \) into a
matrix $M'_i$ of the form

$$M'_i = \begin{pmatrix} M_v & 0 \\ N & I_{t-t'} \end{pmatrix},$$

where $I_{t-t'}$ is an identity matrix of size $(t-t')$. The $(t'+1) \times t'$-matrix $M_v$ is of the form (3.5) with $t$ replaced by $t'$, the sequence $d_1, \ldots, d_t$ by the shorter sequence $d'_1, \ldots, d'_{t'}$, and the partition $\lambda$ by the partition $\nu$ associated to the sequence $d'_1, \ldots, d'_{t'}$.

Let $f'_i$ be the determinant of the square matrix obtained from $M'_i$ by deleting its $(i+1)$-st row. It is clear that up to sign and to reordering the maximal minors $f'_0, \ldots, f'_{t'}$ of $M'_i$ are the same as those of $M_i$. In view of the special form of $M'_i$, observe that

$$f'_i = \begin{cases} f_i^{\nu} & \text{if } 0 \leq i \leq t', \\ 0 & \text{if } t' < i \leq t, \end{cases}$$

where $f_i^{\nu}$ is the determinant of the $t' \times t'$-matrix obtained from $M_v$ by deleting its $(i+1)$-st row.

The number of possible entries of $M_i$, which is the same as the number of possible entries of $M'_i$, is then the product of the number of possible entries of $N$, which is a power of $q$, and of the number of possible entries of $M_v$. Since $d'_1, \ldots, d'_{t'}$ are positive, by the first part of the proof, we know that the number of possible entries of $M_v$ is the product of a power of $q$ by

$$\prod_{i=1}^{t'} (q-1)^2 \frac{q^{2d'_i} - 1}{q^2 - 1}.$$

In other words, the number of possible entries of $M_i$ is

$$q^c \prod_{i=1}^{t'} (q-1)^2 \frac{q^{2d'_i} - 1}{q^2 - 1}$$

for some non-negative integer $c$. Now since the invertible Gröbner cell $C_{\lambda}^{x,y}$ is a Zariski open subset of the affine Gröbner cell $C_{\lambda}$, the degree of the previous polynomial in $q$ must be the same as the degree of the cardinal of $C_{\lambda}$, which is $q^{\nu(x,\lambda)}$ by Section 5.4. This suffices to establish that $c = n - \ell(\lambda)$ and to complete the proof of the theorem.

### 5.4. Proof of Theorem 5.1

By our remark at the beginning of Section 5 the number $C_n(q)$ of ideals of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of codimension $n$ is given by

$$C_n(q) = \sum_{\lambda \vdash n} \text{card } C_{\lambda}^{x,y},$$

where $C_{\lambda}^{x,y}$ is the invertible Gröbner cell associated to the partition $\lambda$. The equality in Theorem 5.1 follows then from the formula for card $C_{\lambda}^{x,y}$ given in Theorem 5.1.

By Corollary 5.2(a) $C_{\lambda}^{x,y}$ is a monic polynomial which has integer coefficients and whose degree is $n + \ell(\lambda)$. Therefore, $C_n(q)$ has integer coefficients and its degree is $\max\{n + \ell(\lambda) \mid \lambda \vdash n\}$. Now $\ell(\lambda)$ is maximal if and only if $\lambda \vdash n$, in which case $\ell(\lambda) = n$. Therefore $C_n(q)$ is monic and its degree is $2n$.

Since $\nu(\lambda) \geq 1$, it follows from the formula in Theorem 5.1 that card $C_{\lambda}^{x,y}$ is divisible by $(q-1)^2$ for each invertible Gröbner cell. Therefore, the polynomial $C_n(q)$ is divisible by $(q-1)^2$. 
6. Proofs of the Corollaries

We now start the proofs of Corollary 1.2 and of Corollary 1.4.

6.1. Proof of Corollary 1.2. Since $C_n(q)$ and $(q-1)^2$ are both monic with integer coefficients, so is $P_n(q)$. The latter is the sum over all partitions of $n$ of the polynomials $P_\lambda(q)$ (introduced in Corollary 5.2(b)). By Corollary 5.2(c)--(d), we have $P_\lambda(1) = 0$ if $\nu(\lambda) \geq 2$ and, if $\nu(\lambda) = 1$, then $\lambda$ is of the form $r^t$, where $dt = n$, in which case $P_\lambda(1) = d$. The desired formula for $P_n(1)$ follows.

6.2. Proof of Corollary 1.4. As in the proof of Theorem 4.6 we consider each partition $\lambda$ as a union of rectangular partitions $r^t$, with $e_i$ parts of length $i$, for $e_i \geq 1$ and distinct $i \geq 1$. Recall that $|\lambda| = \sum_i i e_i$, $\ell(\lambda) = \sum_i e_i$, and $\nu(\lambda) = \sum_i 1$. To indicate the dependence of $e_i$ on $\lambda$, we write $e_i = e_i(\lambda)$. We then obtain the following statement.

Proposition 6.1. We have the infinite product expansion

$$1 + \sum_\lambda \text{card } C_\lambda \cdot s_1^{e_1(\lambda)} s_2^{e_2(\lambda)} \cdots = \prod_{i \geq 1} \frac{(1 - q^i s_i)^2}{(1 - q^{i+1} s_i)(1 - q^{i-1} s_i)}.$$  

Proof. Proceeding as in the proof of Theorem 4.6 and using Theorem 5.1, we deduce that the left-hand side is equal to

$$1 + \sum_\lambda \prod_{i \geq 1} (q - 1)^2 \frac{q^{2e_i} - 1}{q^2 - 1} q^{e_i} s_i^{e_i},$$

which in turn is equal to

$$\prod_{i \geq 1} \left(1 + \frac{(q - 1)^2}{q^2 - 1} \sum_{e_i \geq 1} ((q^{i+1} s_i)^{e_i} - (q^{i-1} s_i)^{e_i})\right).$$

$$= \prod_{i \geq 1} \left(1 + \frac{(q - 1)^2}{q^2 - 1} \left(\frac{q^{i+1} s_i}{1 - q^{i+1} s_i} - \frac{q^{i-1} s_i}{1 - q^{i-1} s_i}\right)\right)$$

$$= \prod_{i \geq 1} \left(1 + \frac{(q - 1)^2}{q^2 - 1} \frac{(q^2 - 1)q^{i-1} s_i}{(1 - q^{i+1} s_i)(1 - q^{i-1} s_i)}\right)$$

$$= \prod_{i \geq 1} \left(1 + \frac{(q - 1)^2 q^{i-1} s_i}{(1 - q^{i+1} s_i)(1 - q^{i-1} s_i)}\right)$$

$$= \prod_{i \geq 1} \frac{(1 - q^i s_i)^2}{(1 - q^{i+1} s_i)(1 - q^{i-1} s_i)}.$$  

□

Proof of Corollary 1.4. (a) Replace $s_i$ by $(t/q)^i$ in Proposition 6.1, use (5.4) and Theorem 5.1, and observe that $(1 - q t^i)(1 - q^{-1} t^i) = 1 - (q + q^{-1}) t^i + t^{2i}$.

(b) The infinite product is clearly invariant under the transformation $q \leftrightarrow q^{-1}$, thus, $C_n(q^{-1}) = q^{-2n} C_n(q)$. Together with $\deg C_n(q) = 2n$, this implies that $C_n(q)$ is palindromic. The polynomial $P_n(q)$ is palindromic as a quotient of two palindromic polynomials. □
6.3. An alternative proof of Corollary[14](a). After we made public a first version of this article, we learnt of an alternative geometric approach to the polynomials $C_n(q)$. Indeed, Götsche and Soergel determined the mixed Hodge structure of the punctual Hilbert schemes of any smooth complex algebraic surface (see [11] Th. 2). Applying their result to the Hilbert scheme $H^n_q = \text{Hilb}^n(\mathbb{C}^* \times \mathbb{C}^*)$ of $n$ points of the complex two-dimensional torus, Hausel, Letellier and Rodriguez-Villegas observed in [16] Th. 4.1.3 that the compactly supported mixed Hodge polynomial $H_c(H^n_q; q, u)$ of $H^n_q$ fits into the equality of formal power series

\begin{equation}
1 + \sum_{n \geq 1} H_c(H^n_q; q, u) \frac{t^n}{q^n} = \prod_{i \geq 1} \frac{(1 + u^{2i+1}t^2)}{(1 - u^{2i+2}q^it^2)(1 - u^{2i}q^{-1}t^2)}.
\end{equation}

Setting $u = -1$ in (6.1), we obtain an infinite product expansion for the generating function of the $E$-polynomial $E(H^n_q; q) = H_c(H^n_q; q, -1)$ of $H^n_q$, namely

\begin{equation}
1 + \sum_{n \geq 1} E(H^n_q; q) \frac{t^n}{q^n} = \prod_{i \geq 1} \frac{(1 - t^i)^2}{1 - (q + q^{-1})t^i + t^{2i}}.
\end{equation}

Now, $H^n_q$ is strongly polynomial-count in the sense of Nick Katz (see [13] Appendix), probably a well-known fact (which also follows from the computations in the present paper). Therefore, by [13] Th. 6.1.2 the $E$-polynomial counts the number of elements of $H^n_q$ over the finite field $\mathbb{F}_q$, which is also the number $C_n(q)$ of ideals of codimension $n$ in $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$. Thus (6.2) implies the equality of Corollary[14](a).

Remark 6.2. In the same vein as above, there is a geometric explanation of the palindromicity of the polynomials $C_n(q)$. In [5] de Cataldo, Hausel, Migliorini observed that any diffeomorphism between $\mathbb{C}^* \times \mathbb{C}^*$ and the cotangent bundle $E \times \mathbb{C}$ of the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i]$ induces a linear isomorphism of graded vector spaces between the cohomology groups of the corresponding Hilbert schemes: $H^*(H^n_q, \mathbb{Q}) \cong H^*(\text{Hilb}^n(E \times \mathbb{C}), \mathbb{Q})$. This isomorphism does not preserve the mixed Hodge structures, as the one on the right-hand side is pure whereas the one on the left-hand side is not. Nevertheless, such an isomorphism identifies the weight filtration on $H^*(H^n_q, \mathbb{Q})$ with the perverse Leray filtration on $H^*(\text{Hilb}^n(E \times \mathbb{C}), \mathbb{Q})$ associated to the natural projective map from $\text{Hilb}^n(E \times \mathbb{C})$ to the $n$-th symmetric product of $\mathbb{C}$ induced by the projection of $E \times \mathbb{C}$ on the second factor. The perverse Leray filtration is “palindromic” as a consequence of the relative hard Lefschetz theorem for the map above (see [5] Th. 4.1.1 and Th. 4.3.2).

Note that Hausel, Letellier and Rodriguez-Villegas observed a similar palindromicity for the $E$-polynomial of certain character varieties and termed it “curious Poincaré duality” in [15] Cor. 5.2.4] (see also [13] Cor. 3.5.3, [14] Cor. 1.4).

Remark 6.3. The natural action of the group $\mathbb{C}^* \times \mathbb{C}^*$ on itself induces an action on the Hilbert scheme $H^n_q$. Consider the GIT quotient $\tilde{H}_c^n = H^n_c // (\mathbb{C}^* \times \mathbb{C}^*)$. Using [13] Th. 2.2.12 and [15] Sect. 5.3, we see that the $E$-polynomial of $\tilde{H}_c^n$ is given by

\begin{equation}
E(\tilde{H}_c^n; q) = E(H^n_c; q)/(q - 1)^2 = C_n(q)/(q - 1)^2 = P_n(q).
\end{equation}

Recall from the introduction (see also the appendix below) that the coefficients of $P_n(q)$ are all non-negative. Therefore, $\tilde{H}_c^n$ provides an example of a polynomial-count variety with odd cohomology and a counting polynomial with non-negative
coefficients. This implies non-trivial cancellation for the mixed Hodge numbers of $\hat{H}_C^n$. No similar positivity phenomenon was observed for the character varieties investigated by Hausel, Letellier and Rodriguez-Villegas.

**Appendix A. The coefficients of the polynomials $C_n(q)$ and $P_n(q)$**

We now state the results of the companion paper [18] on the coefficients of the polynomials $C_n(q)$ and $P_n(q)$.

Since $C_n(q)$ and $P_n(q)$ are palindromic of respective degrees $2n$ and $2n - 2$, we may expand $C_n(q)$ and $P_n(q)$ as follows:

$$C_n(q) = c_{n,0} q^n + \sum_{i=1}^{n} c_{n,i} \left( q^{n+i} + q^{n-i} \right),$$

where $c_{n,0}, c_{n,1}, c_{n,2} \ldots$ are integers, and

$$P_n(q) = a_{n,0} q^{n-1} + \sum_{i=1}^{n-1} a_{n,i} \left( q^{n+i-1} + q^{n-i+1} \right),$$

where $a_{n,0}, a_{n,1}, a_{n,2} \ldots$ are integers.

By Theorem 1.1 of [18] the coefficients $c_{n,i}$ of $C_n(q)$ are given by the following formulas: (a) For the central coefficients $c_{n,0}$ we have

$$c_{n,0} = \begin{cases} 
2 (-1)^k & \text{if } n = k(k + 1)/2 \text{ for some integer } k \geq 1, \\
0 & \text{otherwise}.
\end{cases}$$

(b) For the non-central coefficients ($i \geq 1$) we have

$$c_{n,i} = \begin{cases} 
(-1)^k & \text{if } n = k(k + 2i + 1)/2 \text{ for some integer } k \geq 1, \\
(-1)^{k-1} & \text{if } n = k(k + 2i - 1)/2 \text{ for some integer } k \geq 1, \\
0 & \text{otherwise}.
\end{cases}$$

Note that in Item (b) the first two conditions are mutually exclusive.

As for the coefficients of $P_n(q)$, the coefficient $a_{n,i}$ is by [18] Th. 1.2 equal to the number of divisors $d$ of $n$ such that

$$\frac{i + \sqrt{2n + i^2}}{2} < d \leq i + \sqrt{2n + i^2}.$$

It follows that all coefficients $a_{n,i}$ of $P_n(q)$ are non-negative integers.

**Acknowledgement**

We are grateful to Olivier Benoist, François Bergeron, Mark Haiman, Emmanuel Letellier and Luca Migliorini for useful discussions, and to Pierre Baumann for suggesting the proof of Lemma 2.5.

The second-named author is grateful to the Université de Strasbourg for the invited professorship which allowed him to spend the month of June 2014 at IRMA; he was also supported by NSERC (Canada).
Table 1. The polynomials $C_n(q)$

| $n$ | $C_n(q)$ |
|-----|----------|
| 1   | $q^2 - 2q + 1$ |
| 2   | $q^4 - q^3 - q + 1$ |
| 3   | $q^6 - q^5 - q^4 + 2q^3 - q^2 - q + 1$ |
| 4   | $q^8 - q^7 - q + 1$ |
| 5   | $q^{10} - q^9 - q^7 + q^6 + q^4 - q^2 - q + 1$ |
| 6   | $q^{12} - q^{11} + q^7 - 2q^6 + q^5 - q + 1$ |
| 7   | $q^{14} - q^{13} - q^{10} + q^9 + q^3 - q^4 - q + 1$ |
| 8   | $q^{16} - q^{15} - q + 1$ |
| 9   | $q^{18} - q^{17} - q^{13} + q^{12} + q^{11} - q^{10} - q^8 + q^7 + q^6 - q^5 - q + 1$ |
| 10  | $q^{20} - q^{19} - q^{11} + 2q^{10} - q^9 - q + 1$ |
| 11  | $q^{22} - q^{21} - q^{16} + q^{15} + q^7 - q^6 - q + 1$ |
| 12  | $q^{24} - q^{23} + q^{15} - q^{14} - q^{10} + q^9 - q + 1$ |

References

[1] T. M. Apostol, *Introduction to analytic number theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[2] R. Bacher, C. Reutenauer, *The number of right ideals of given codimension over a finite field*, Noncommutative birational geometry, representations and combinatorics, 1–18, Contemp. Math., 592, Amer. Math. Soc., Providence, RI, 2013.

[3] R. Bacher, C. Reutenauer, *Number of right ideals and a q-analogue of indecomposable permutations*, Canad. J. Math. 68 (2016), no. 3, 481–503.

[4] N. Bourbaki, *Algèbre commutative*, Herman, Paris 1961 (English translation: *Commutative algebra*, Chapters 1–7, Springer-Verlag, Berlin, 1989).

[5] M. A. de Cataldo, T. Hausel, L. Migliorini, *Exchange between perverse and weight filtration for the Hilbert schemes of points of two surfaces*, J. Singul. 7 (2013), 23–38.

[6] A. Conca, G. Valla, *Canonical Hilbert-Burch matrices for ideals of $k[x,y]$, Michigan Math. J. 57 (2008), 157–172.

[7] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Grad. Texts in Math., 150, Springer-Verlag, New York, 1995.

[8] G. Ellingsrud, S. A. Stromme, *On the homology of the Hilbert scheme of points in the plane*, Invent. Math. 87 (1987), no. 2, 343–352.

[9] N. J. Fine, *Basic hypergeometric series and applications*, Mathematical Surveys and Monographs, 27, Amer. Math. Soc., Providence, RI, 1988.

[10] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math 90 (1968), 511–521.

[11] L. Göttsche, W. Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. 296 (1993), no. 2, 235–245.

[12] A. Grothendieck, *Techniques de construction et théorèmes d’existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Exp. No. 221, 249–276, W. A. Benjamin, New York-Amsterdam, 1966.

[13] T. Hausel, F. Rodriguez-Villegas, *Mixed Hodge polynomials of character varieties. With an appendix by Nicholas M. Katz*, Invent. Math. 174 (2008), no. 3, 555–624.

[14] T. Hausel, E. Letellier, F. Rodriguez-Villegas, *Topology of character varieties and representation of quivers*, C. R. Math. Acad. Sci. Paris 348 (2010), no. 3–4, 131–135.

[15] T. Hausel, E. Letellier, F. Rodriguez-Villegas, *Arithmetic harmonic analysis on character and quiver varieties*, Duke Math. J. 160 (2011), no. 2, 323–400.
Table 2. The polynomials $P_n(q)$

| $n$ | $P_n(q)$ | $P_n(1)$ |
|-----|----------|----------|
| 1   | $q^2 + q + 1$ | 1        |
| 2   | $q^3 + q^2 + q + 1$ | 3        |
| 3   | $q^4 + q^3 + q^2 + q + 1$ | 4        |
| 4   | $q^5 + q^4 + q^3 + q^2 + q + 1$ | 7        |
| 5   | $q^6 + q^5 + q^4 + q^3 + q^2 + 3q + 1$ | 6        |
| 6   | $q^{10} + q^9 + q^8 + q^7 + q^6$ | 12       |
| 7   | $q^{12} + q^{11} + q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$ | 8        |
| 8   | $q^{14} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8$ | 15       |
| 9   | $q^{16} + q^{15} + q^{14} + q^{13} + q^{12} + q^{11} + q^{10} + q^9$ | 13       |
| 10  | $q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8 + q^7 + q^6$ | 18       |
| 11  | $q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15}$ | 12       |
| 12  | $q^{22} + q^{21} + q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + 2q^{13} + 2q^{12} + 2q^{11} + 2q^{10} + 2q^9 + 2q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$ | 28       |

[16] T. Hausel, E. Letellier, F. Rodríguez-Villegas, Arithmetic harmonic analysis on character and quiver varieties II, Adv. Math. 234 (2013), 85–128.
[17] G. H. Hardy, E. M. Wright, An introduction to the theory of numbers, 3rd ed., Clarendon Press, Oxford, 1954.
[18] C. Kassel, C. Reutenauer, Complete determination of the zeta function of the Hilbert scheme of $n$ points on a two-dimensional torus, [arXiv:1610.07793](http://arxiv.org/abs/1610.07793).
[19] The On-Line Encyclopedia of Integer Sequences, published electronically at [http://oeis.org](http://oeis.org).
[20] M. Reineke, Cohomology of noncommutative Hilbert schemes, Algebr. Represent. Theory 8 (2005), no. 4, 541–561.

CHRISTIAN KASSEL: UNIVERSITÉ DE STRASBOURG, CNRS, IRMA UMR 7501, F–67000 STRASBOURG, FRANCE
E-mail address: kassel@math.unistra.fr
URL: [wwwirma.u-strasbg.fr/~kassel](http://wwwirma.u-strasbg.fr/~kassel)

CHRISTOPHE REUTENAUER: MATHEMÁTIQUES, UNIVERSITÉ DU QUÉBEC À MONTRÉAL, MONTRÉAL, CP 8888, succ. CENTRE VILLE, CANADA H3C 3P8
E-mail address: reutenauer.christophe@uqam.ca
URL: [www.lacim.uqam.ca/~christo](http://www.lacim.uqam.ca/~christo)
## COUNTING IDEALS

### Table 3. The polynomials $B_n(q)$

| $n$ | $B_n(q)$                             | $B_n(1)$ | $B_n(-1)$ |
|-----|--------------------------------------|----------|-----------|
| 1   | 1                                    | 1        | 1         |
| 2   | $q + 1$                              | 2        | 0         |
| 3   | $q^2 + q$                            | 2        | 0         |
| 4   | $q^3 + q^2 + q$                      | 3        | -1        |
| 5   | $q^4 + q^3 + q^2 - 1$                | 2        | 0         |
| 6   | $q^5 + q^4 + q^3 + q^2$              | 4        | 0         |
| 7   | $q^6 + q^5 + q^4 + q^3 - q - 1$      | 2        | 0         |
| 8   | $q^7 + q^6 + q^5 + q^4 + q^3 - q$    | 4        | 0         |
| 9   | $q^8 + q^7 + q^6 + q^5 + q^4 - q^2 - q$ | 3     | 1         |
| 10  | $q^9 + q^8 + q^7 + q^6 + q^5 - q^2 - q$ | 4     | 0         |
| 11  | $q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 - q^3 - 2q^2 - q$ | 2     | 0         |
| 12  | $q^{11} + q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 - q^3 - q^2 + 1$ | 6     | 0         |

### Table 4. The polynomials $A_n(q)$

| $n$ | $A_n(q)$                             | $A_n(1)$ | $A_n(-1)$ |
|-----|--------------------------------------|----------|-----------|
| 1   | $q^2$                                | 1        | 1         |
| 2   | $q^4 + q^3$                          | 2        | 0         |
| 3   | $q^6 + q^5 + q^4$                    | 3        | 1         |
| 4   | $q^8 + q^7 + 2q^6 + q^5$             | 5        | 1         |
| 5   | $q^{10} + q^9 + 2q^8 + 2q^7 + q^6$   | 7        | 1         |
| 6   | $q^{12} + q^{11} + 2q^{10} + 3q^9 + 3q^8 + q^7$ | 11     | 1         |
| 7   | $q^{14} + q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 3q^9 + q^8$ | 15     | 1         |
| 8   | $q^{16} + q^{15} + 2q^{14} + 3q^{13} + 5q^{12} + 5q^{11} + 4q^{10} + q^9$ | 22     | 2         |
| 9   | $q^{18} + q^{17} + 2q^{16} + 3q^{15} + 5q^{14} + 6q^{13} + 7q^{12} + 4q^{11} + q^{10}$ | 30     | 2         |
| 10  | $q^{20} + q^{19} + 2q^{18} + 3q^{17} + 5q^{16} + 7q^{15} + 9q^{14} + 8q^{13} + 5q^{12} + q^{11}$ | 42     | 2         |
| 11  | $q^{22} + q^{21} + 2q^{20} + 3q^{19} + 5q^{18} + 7q^{17} + 10q^{16} + 11q^{15} + 10q^{14} + 5q^{13} + q^{12}$ | 56     | 2         |
| 12  | $q^{24} + q^{23} + 2q^{22} + 3q^{21} + 5q^{20} + 7q^{19} + 9q^{18} + 13q^{17} + 15q^{16} + 12q^{15} + 6q^{14} + q^{13}$ | 77     | 3         |