Entanglement Assisted Covert Communication and its Coding Limits over Bosonic Channels

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Abstract

We investigate the quantum-secure covert communication capabilities of lossy thermal noise bosonic channels, the quantum-mechanical model for many practical channels. We determine the expressions for the capacity-like constants governing covert communication: \( L_{\text{no-EA}} \) when Alice and Bob only share a classical secret, and \( L_{\text{EA}} \) when they benefit from the entanglement assistance (EA). While we confirm our prior conjecture for \( L_{\text{no-EA}} \), we find that EA alters the fundamental scaling law for covert communication: instead of \( L_{\text{no-EA}} \sqrt{n} \) limit on the number of reliable covert bits transmissible over \( n \) channel uses, EA allows \( L_{\text{EA}} \sqrt{n} \log n \) covert bits to be reliably transmitted over \( n \) channel uses. We also argue that a recent EA optical transceiver design proposal yields \( \approx \frac{L_{\text{EA}}}{2} \sqrt{n} \log n \) reliable covert bits in \( n \) optical modes.

I. Introduction

In contrast to the standard information security methods (e.g., encryption, information-theoretic secrecy, and quantum key distribution (QKD)) that protect the transmission’s content from unauthorized access, covert, or low probability of detection/intercept (LPD/LPI), communication...
[1]–[3] prevents adversarial detection of transmissions in the first place. Covertness requirement constrains the transmission power averaged over the blocklength $n$ to $\propto 1/\sqrt{n}$, where the power is either measured directly in Watts [1], [2] and mean photon number [4], [5] output by a physical transmitter, or indirectly as the frequency of non-zero symbol transmission over the discrete classical [6], [7] and quantum [8], [9] channels.

For many channels, including classical additive white Gaussian noise (AWGN) [1], [2] and discrete memoryless channels (DMCs) [6], [7], the power constraint prescribed by the covertness requirement imposes the square root law (SRL): no more than $L\sqrt{n}$ covert bits can be transmitted reliably in $n$ channel uses, where $L$ is a constant corresponding to the “capacity” of a covert channel. Attempting to transmit more results in either detection by the adversary with high probability as $n \to \infty$, or unreliable transmission. Even though the capacity of such channels is zero (since $\lim_{n \to \infty} L\sqrt{n}/n = 0$), SRL allows reliable transmission of a significant number of covert bits for large $n$.

To date, the focus has been on classical covert communication. However, quantum mechanics governs the fundamental laws of nature, and quantum information theory [10] is required to determine the ultimate limits of any communications system. Here we focus on the lossy thermal noise bosonic channel depicted in Fig. [1] and formally described in Section [II-B] which is a quantum-mechanical description of many practical channels (including optical, microwave, and radio frequency (RF)). It is parametrized by the power coupling (transmissivity) $\eta$ between the transmitter Alice and the intended receiver Bob, and the mean photon number $\bar{n}_B$ per mode injected by the thermal environment, where a single spatial-temporal-polarization mode is our fundamental transmission unit. We call a covert communication system quantum-secure when it is robust against an adversary Willie that not only knows the transmission parameters (including the start time, center frequency, duration, and bandwidth) but also has access to all the transmitted photons that are not captured by Bob and arbitrary quantum information processing resources (e.g., joint detection measurement, quantum memory, and quantum computer). Our assumptions correspond to the “gold standard” of security from the QKD literature, except that we also
require excess noise that is not under Willie’s control (e.g., the unavoidable thermal noise from the blackbody radiation at the center-wavelength of transmission and the receiver operating temperature). This assumption is not only well-grounded in practice, but also necessary for covertness (unlike in QKD), as the transmissions cannot be hidden from quantum-capable Willie that fully controls noise on the channel [4, Th. 1], [11]. Finally, we assume that Alice and Bob share a resource that is inaccessible by Willie. Not only does this enable covertness irrespective of channel conditions, but also substantially increases the number of reliably-transmissible covert bits when the resource is an entangled quantum state.

In our recent work [5] we develop an expression for the maximum mean photon number per mode $\bar{n}_S$ that is covert under the aforementioned conditions:

$$\bar{n}_S = \frac{\sqrt{\delta c_{cov}}}{\sqrt{n}},$$

where $\delta$ parametrizes the desired covertness, $n$ is the total number of modes, and

$$c_{cov} = \frac{\sqrt{2\eta \bar{n}_B(1 + \eta \bar{n}_B)}}{1 - \eta}.$$  (2)

We note that, in the absence of exact values for $\bar{n}_B$ and $\eta$, they can be lower-bounded for free-space bosonic channels via Plank law and diffraction-limited propagation model, respectively. The constant (2) can be achieved by modulating coherent states using continuous-valued complex
Gaussian distribution or more practical quadrature phase shift keying (QPSK). However, [5] provides only the prescription for maintaining covertness of a transmission, leaving open the question of how much information can thus be covertly transmitted.

We answer this question here by examining the coding limits for covert communication over the bosonic channels. It is straightforward to obtain the converse results using the standard channel capacity theorems with the constraint in (1). However, the dependence of mean photon number per mode $\bar{n}_S$ on the blocklength $n$ in (1) complicates both classical and quantum achievability by rendering invalid the conditions for employing standard results such as the asymptotic equipartition property. Classical results [6], [7] overcome this issue using the information spectrum methods. While the quantum information spectrum approaches [12], [13] predate their classical counterparts [14], [15], until recently their application has been limited to channels with output quantum states living in the finite-dimensional Hilbert space, which is not the case for bosonic channels. Here, we prove the achievability that matches the converse by employing the lower bound on the second-order coding rate that is based on the new quantum union bound [16].

Our main contribution is the analysis of covert communication described in Fig. 2 with and without an entangled resource state shared by Alice and Bob. Since entanglement assistance (EA) gain manifests only when $\bar{n}_S \to 0$ while $\bar{n}_B > 0$, we expect it to benefit covert communication, since that is its natural regime. However, we find that EA, in fact, alters its fundamental scaling law! Using the asymptotic notation defined in Sec. III-A:

1) We show that without EA, the SRL holds in its standard form: the total number of reliably-transmissible covert bits $M = L_{no-EA} \sqrt{n} + o(\sqrt{n})$. The expression for the constant $L_{no-EA}$ confirms our conjecture in [5, Sec. V]. In fact, our second-order expression has a form similar to classical [17]: $M \geq L_{no-EA} \delta \sqrt{n} + K_{no-EA} \Phi^{-1}(\epsilon) n^{1/4} + O(\ln n)$, where $\epsilon$ is the average decoding error probability and $\Phi^{-1}(x)$ is the inverse Gaussian cumulative
Fig. 2. Covert communication over lossy thermal noise bosonic channel. Alice has a lossy thermal noise bosonic channel depicted in Fig. 1 to legitimate receiver Bob and adversary Willie. Alice and Bob share a bipartite resource state $\hat{\rho}_{S^mR^m}$ that is not accessible by Willie and may or may not be entangled. The solid arrows trace its evolution. Alice uses her share of $\hat{\rho}_{S^mR^m}$ in $S$ systems to encode message $x$ with blocklength $n$ code, and chooses whether to transmit it using $E_{A \rightarrow BW}^{(n,\overline{n})} n$ times. Willie observes his channel from Alice to determine whether she is quiet (null hypothesis $H_0$) or not (alternate hypothesis $H_1$). Covert communication system must ensure that any detector Willie uses is close to ineffective (i.e., a random guess between the hypotheses), while allowing Bob to reliably decode the message (if one is transmitted).

We also show that QPSK modulation achieves the same constants $L_{\text{no-EA}}$ and $K_{\text{no-EA}}$ as the Gaussian modulation, making it optimal for both the reliability and quantum-secure covertness.

2) We show that, with EA, the scaling law becomes $M = L_{\text{EA}} \sqrt{n} \log n + o(\sqrt{n} \log n)$ and derive the expression for the optimal constant $L_{\text{EA}}$ as well as the second-order expression. While a receiver structure that achieves $L_{\text{EA}}$ is an open problem, there exists a design \cite{18} that achieves the $O(\sqrt{n} \log n)$ scaling law, albeit with a constant $\approx \frac{L_{\text{EA}}}{2}$.

We summarize our results in the next section, stating the theorems that we prove in the subsequent sections. In Section III we present the mathematical prerequisites, including the asymptotic notation, and the formal definitions of covertness and reliability. In Sections IV and V we prove the converse and achievability results, respectively. We conclude with discussion of future work in Section VI.

\footnote{Note that $\Phi^{-1}(x) < 0$ for $0 < x < \frac{1}{2}$.}
II. RESULTS

A. The notion of capacity for covert communication

In classical and quantum information theory [10], [19], the classical channel capacity is measured in bits per channel use and is expressed as $C = \liminf_{n \to \infty} \frac{M}{n}$, where $M$ is the total number of reliably-transmissible bits in $n$ channel uses. On the other hand, the power constraint imposed by covert communication implies that $M = o(n)$ and that the capacity of the covert channel is zero. Inspired by [7], we regularize the number of covert bits that are transmitted reliably without entanglement assistance (EA) by $\sqrt{n}$ and $\sqrt{n} \log n$ with EA\(^2\) instead of $n$. This approach allows us to state the Definitions 1 and 2 of capacity for covert communication without and with EA and derive the results that follow. For consistency with [7], we also normalize the capacity by the covertness parameter $\delta$, which we discuss in Section III-C.

B. Non-EA covert communication

We define the capacity of covert communication over the bosonic channel when Alice and Bob do not have access to a shared entanglement source as follows:

**Definition 1** (Covert capacity without EA). The capacity of covert communication without entanglement assistance is:

$$L_{\text{no-EA}} \doteq \liminf_{n \to \infty} \frac{M_{\text{no-EA}}}{\sqrt{\delta n}},$$

(3)

where $M_{\text{no-EA}}$ is the number of covert bits that are reliably transmissible in $n$ channel uses (modes), and $\delta$ parametrizes the desired covertness.

The following theorem provides the expression for $L_{\text{no-EA}}$:

**Theorem 1.** The covert capacity of the bosonic channel without entanglement assistance is $L_{\text{no-EA}} = c_{\text{cov}} c_{\text{rel, no-EA}}$, where $c_{\text{cov}}$ is defined in (2) and $c_{\text{rel, no-EA}} = \eta \log \left(1 + \frac{1}{(1-\eta)\bar{B}}\right)$.

\(^2\)Our fundamental information unit is a bit and $\log x$ indicates the binary logarithm, while $\ln x$ is the natural logarithm.
Theorem 1 confirms the expression for the capacity of covert communication over the bosonic channel conjectured in [5, Sec. V]. We prove the converse of Theorem 1 in Section IV and its achievability in Section V-B. For the latter we employ the recent results on finite-blocklength bounds for infinite-dimensional quantum channels [16] to prove the following lemma:

**Lemma 1.** There exists a sequence of random codes with covertness parameter $\delta$, blocklength $n$, size $2^M$, and average error probability $\epsilon$ that satisfy:

$$M_{\text{no-EA}} \geq L_{\text{no-EA}} \sqrt{\delta n} + K_{\text{no-EA}} \Phi^{-1}(\epsilon)n^{1/4} + O(\ln n),$$

(4)

where $K_{\text{no-EA}} = \sqrt{c_{\text{cov}} \sqrt{\delta} (1 + 2(1 - \eta)\bar{n}_B) c_{\text{rel,no-EA}}}$. 

C. EA covert communication

EA is well-known to increase the communication channel capacity [20], [21]. However, in most settings of practical interest (including optical communication where noise level is low $\bar{n}_B \ll 1$ and microwave/RF communication where signal power is high $\bar{n}_S \gg 1$), the gain over the non-EA Holevo capacity is at most a factor of two. The only scenario with a significant gain is when $\bar{n}_S \to 0$ while $\bar{n}_B > 0$ [18, App. A]. This corresponds precisely to the covert communication setting. In fact, entanglement assistance alters the fundamental square root scaling law for covert communication, resulting normalization by $\sqrt{n} \log n$ instead of $\sqrt{n}$ in:

**Definition 2** (EA covert capacity). The capacity of covert communication with entanglement assistance is:

$$L_{\text{EA}} \triangleq \lim inf_{n \to \infty} \frac{M_{\text{EA}}}{\sqrt{\delta n \log n}},$$

(5)

where $M_{\text{EA}}$ is the number of covert bits that are reliably transmissible in $n$ channel uses (modes), and $\delta$ parametrizes the desired covertness.

The following theorem provides the expression for $L_{\text{EA}}$: 


Theorem 2. The covert capacity of the bosonic channel with entanglement assistance is \( L_{\text{EA}} = c_{\text{cov}} c_{\text{rel,EA}} \), where \( c_{\text{cov}} \) is defined in (2) and \( c_{\text{rel,EA}} = \frac{n}{2(1+(1-\eta)\bar{n}_B)} \).

Thus, while availability of quantum resources such as entanglement shared between Alice and Bob and joint detection receiver at Bob does not affect \( \bar{n}_S \), it impacts dramatically the amount of information that can be covertly conveyed. We prove the converse of Theorem 2 in Section IV and its achievability in Section V-C. As in the corresponding proof of Theorem 2 for achievability we use [16] to show that the following lemma holds:

Lemma 2. There exists a sequence of random codes with covertness parameter \( \delta \), blocklength \( n \), size \( 2^M \), and average error probability \( \epsilon \) that satisfy:

\[
M_{\text{EA}} \geq L_{\text{EA}} \sqrt{\delta n} \log n + K_{\text{EA}} \Phi^{-1}(\epsilon) n^{1/4} \log n + O(\ln n),
\]

where \( K_{\text{EA}} = \sqrt{c_{\text{cov}} \sqrt{\delta c_{\text{rel,EA}}}} \).

The result in [16] used in the proof of Lemma 2 employs a construction that does not correspond to any known receiver structure. In fact, despite the EA enhancement of classical communication capacity being known for over two decades [20], [21], a strategy to achieve the full gain has eluded us until recent work on the structured receiver for EA communication in [18]. We argue that the receiver design [18] achieves EA covert capacity \( \approx \frac{L_{\text{EA}}}{2} \).

III. PREREQUISITES

A. Asymptotic notation

We use the standard asymptotic notation [22, Ch. 3.1], where \( f(n) = O(g(n)) \) denotes an asymptotic upper bound on \( f(n) \) (i.e. there exist constants \( m, n_0 > 0 \) such that \( 0 \leq f(n) \leq mg(n) \) for all \( n \geq n_0 \)) and \( f(n) = o(g(n)) \) denotes an upper bound on \( f(n) \) that is not asymptotically tight (i.e. for any constant \( m > 0 \), there exists constant \( n_0 > 0 \) such that \( 0 \leq f(n) < mg(n) \) for all \( n \geq n_0 \)). We note that \( f(n) = O(g(n)) \) is equivalent to \( \limsup_{n \to \infty} | \frac{f(n)}{g(n)} | \leq \infty \) and \( f(n) = o(g(n)) \) is equivalent to \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \).
B. Channel model

We focus on a single mode lossy thermal noise bosonic channel $\mathcal{E}_{A\rightarrow BW}^{(\eta, \bar{n}_B)}$ in Fig. 1. It describes quantum-mechanically the transmission of a single (spatio-temporal-polarization) mode of the electromagnetic field at a given transmission wavelength (such as optical or microwave) over linear loss and additive Gaussian noise (such as noise stemming from blackbody radiation). Here we introduce the bosonic channel briefly, deferring the details to [23]–[26]. The attenuation in Alice-to-Bob channel is modeled by a beamsplitter with transmissivity (fractional power coupling) $\eta$. The input-output relationship between the bosonic modal annihilation operators of the beamsplitter, $\hat{b} = \sqrt{\eta} \hat{a} + \sqrt{1-\eta} \hat{e}$, requires the “environment” mode $\hat{e}$ to ensure that the commutator $[\hat{b}, \hat{b}^\dagger] = 1$, and to preserve the Heisenberg uncertainty law of quantum mechanics. On the contrary, classical power attenuation is described by $b = \sqrt{\eta} a$, where $a$ and $b$ are complex amplitudes of input and output mode functions. Bob captures a fraction $\eta$ of Alice’s transmitted photons, while Willie has access to the remaining $1-\eta$ fraction. We model noise by mode $\hat{e}$ being in a zero-mean thermal state $\hat{\rho}_{\bar{n}_B}$, expressed in the coherent state (quantum description of ideal laser light) and Fock (photon number) bases as follows:

$$\hat{\rho}_{\bar{n}_B} = \frac{1}{\pi \bar{n}_B} \int_{\mathbb{C}} \exp \left[ -|\alpha|^2 \right] d^2 \alpha |\alpha\rangle \langle \alpha| = \sum_{k=0}^{\infty} t_k(\bar{n}_B) |k\rangle \langle k|, \quad (7)$$

where

$$t_k(\bar{n}) = \frac{\bar{n}^k}{(1 + \bar{n})^{k+1}} \quad (8)$$

and $\bar{n}_B$ is the mean photon number per mode injected by the environment.

Consider the covert communication framework depicted in Fig. 2. Our fundamental transmission unit is the field mode described above. We assume that $n = 2TW$ modes are available to Alice and Bob, where $TW$ is the number of orthogonal temporal modes, which is the product of the transmission duration $T$ (in seconds) and the optical bandwidth $W$ (in Hz) of the source around its center frequency, and the factor of two corresponds to the use of both orthogonal polarizations. We denote by $\mathcal{E}_{A\rightarrow B}^{(\eta, \bar{n}_B)}$ the channel from Alice to Bob (ignoring the output to Willie).
Alice attempts to communicate reliably to Bob without detection by Willie. Next we discuss the constraints imposed by the covertness requirement, and then define reliability.

C. Quantum-secure Covertness

In order to be quantum-secure, covert communication system has to prevent the detection of Alice’s transmission by Willie who has access to all transmitted photons that are not received by Bob and arbitrary quantum resources. Thus, the quantum state $\hat{\rho}_W$ observed by Willie when Alice is transmitting has to be sufficiently similar to the product thermal state $\hat{\rho}_B^{\otimes n}$ describing the noise observed when she is not. In [5] we investigate the impact of this requirement on Alice’s modulation scheme. We employ quantum relative entropy (QRE) as a measure of similarity between states, which is defined as follows:

**Definition 3.** The quantum relative entropy between quantum states $\hat{\rho}$ and $\hat{\sigma}$ is:

$$D(\hat{\rho}||\hat{\sigma}) = \text{Tr} [\hat{\rho} \log \hat{\rho} - \hat{\rho} \log \hat{\sigma}] .$$  \hspace{1cm} (9)

This yields the following covertness criterion:

**Criterion 1.** A system is covert if, for any $\delta > 0$, $D\left(\hat{\rho}_W^{\otimes n} || \hat{\rho}_B^{\otimes n}\right) \leq \frac{\delta}{\log e}$ for $n$ large enough.

Arbitrarily small $\delta > 0$ implies that the performance of a quantum-optimal detection scheme is arbitrarily close to that of a random coin flip through quantum Pinsker’s inequality [10, Th. 10.8.1]. The properties of QRE are highly attractive for mathematical proofs, and, indeed, a classical version of Criterion 1 has been used to analyze covert communication over classical DMCs and AWGN channels [6], [7]. We discuss our choice of Criterion 1 as well as significance of QRE in [5, Sec. II.B]. The optimal mean photon number that satisfies Criterion 1 is given by [1] [5, Th. 1]. It is achievable using a continuous complex-valued Gaussian distribution [4, Th. 2], or by discrete quadrature phase shift keying (QPSK) [5, Th. 2]. We assume that Alice and Bob share a bipartite resource state $\hat{\rho}^{S \otimes R^n}$ with systems $S$ at Alice and $R$ at Bob. Willie
cannot access $\hat{\rho}^{S_m R_m}$ (though we assume that he knows how it is generated), which enables covert communication irrespective of the channel conditions.

While quantum resources such as entanglement shared between Alice and Bob or quantum states lacking a semiclassical description (e.g., squeezed light) do not improve signal covertness, the quantum methodology in Criterion 1 allows covertness without assumptions of adversary’s limits other than the laws of physics. However, the square root scaling in (1) holds even when Willie uses readily-available devices such as noisy photon counters [4, Th. 5], with a constant larger than $c_{cov}$. Nevertheless, here we show that the availability of quantum resources—specifically entanglement assistance (EA)—allows more bits to be transmitted reliably using the signals that meet the covertness constraint of Criterion 1.

D. Reliability

In this work, we characterize number of bits $M$ that Alice can transmit to Bob using the system depicted in Figure 2 while satisfying the following reliability criterion in addition to the covertness condition of Criterion 1:

**Criterion 2.** A system is reliable if, for any $\epsilon > 0$, $\frac{1}{2M} \sum_{n=1}^{2M} P(\bar{x} \neq x) \leq \epsilon$ for $n$ large enough.

Alice and Bob decode and encode using their respective systems $S$ and $R$ of the shared resource state $\hat{\rho}^{S_m R_m}$. While Alice has no access to Bob’s systems $R$ and Bob to Alice’s systems $S$, $S$ and $R$ are correlated. This correlations can be either classical or quantum: the latter enables entanglement-assisted (EA) communication. Bob may employ joint detection (entangling) measurement across $n$ modes and his share of resource state to decode. We now proceed to the proofs of Theorems 1 and 2.

IV. CONVERSE PROOFS

Before proving the converses of Theorems 1 and 2, we present the common construction elements for both proofs. Alice desires to transmit $x$, one of $2^M$ equally-likely $M$-bit messages,
covertly to Bob. She encodes $x$ using an operator $M_{S^m \rightarrow A^n}^{(x)}$ that takes $m$ systems $S$ of the resource state $\hat{\rho}^{S^m R^m}$ to $\hat{\rho}_x^{A^n R^m} = M_{S^m \rightarrow A^n}^{(x)}(\hat{\rho}^{S^m R^m})$. Criterion 1 is satisfied only if the mean photon number of the transmitted state $\frac{1}{n} \text{Tr} \left[ \hat{n}^{\otimes n} \hat{\rho}_x^{A^n R^m} \right] \leq \bar{n}_S$, where $\bar{n}_S$ is defined in (1), $\hat{n} = \hat{a} \hat{a}^\dagger$ is the photon number operator, $\hat{\rho}_x^{A^n R^m} = \text{Tr}_{R^m} \left[ \hat{\rho}_x^{A^n R^m} \right]$, and $\hat{\rho}^A = \text{Tr}_B \left[ \hat{\rho}^{AB} \right]$ denotes the partial trace over system $B$. Suppose that there exists a sequence of encoders $M_{S^m \rightarrow A^n}^{(x)}$ and corresponding decoders such that the probability of decoding error $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Denote by $X^n$ and $Y^n$ the random variables corresponding to Alice’s message and Bob’s decoding of it. By Fano’s inequality [19, Th. 2.10.1]:

$$M(1 - \epsilon_n) - 1 \leq I(X^n; Y^n),$$

(10)

where the mutual information $I(X^n; Y^n)$ between $X^n$ and $Y^n$ depends on whether the resource state $\hat{\rho}^{S^m R^m}$ is classically correlated or entangled.

**Proof (Th. 1 converse):** When $\hat{\rho}^{S^m R^m}$ is a classically-correlated state, $I(X^n; Y^n)$ is bounded by the Holevo capacity of $E^{(\eta, \bar{n}_B)}_{A \rightarrow B}$ [27]:

$$I(X^n; Y^n) \leq nC_\chi(\bar{n}_S; \eta, \bar{n}_B),$$

(11)

where $C_\chi(\bar{n}_S; \eta, \bar{n}_B) = g(\eta \bar{n}_S + (1 - \eta)\bar{n}_B) - g((1 - \eta)\bar{n}_B)$ and

$$g(x) \equiv (1 + x) \log(1 + x) - x \log x.$$

(12)

Using Taylor series expansion of $C_\chi(\bar{n}_S; \eta, \bar{n}_B)$ around $\bar{n}_S = 0$ in (11), and substituting $\bar{n}_S$ given in (1) yields:

$$M_{\text{no-EA}}(1 - \epsilon_n) - 1 \leq \sqrt{n}\delta_{\text{cov}} \log \left( 1 + \frac{1}{(1 - \eta)\bar{n}_B} \right).$$

(13)

Dividing both sides of (13) by $\sqrt{n}\delta$ and taking the limit yields the converse.

**Proof (Th. 2 converse):** When $\hat{\rho}^{S^m R^m}$ is an entangled state, $I(X^n; Y^n)$ is bounded by the entanglement-assisted capacity $C_{EA}(\bar{n}_S; \eta, \bar{n}_B)$ of $E^{(\eta, \bar{n}_B)}_{A \rightarrow B}$:

$$M_{\text{EA}}(1 - \epsilon_n) - 1 \leq nC_{EA}(\bar{n}_S; \eta, \bar{n}_B),$$

(14)
where \( C_{EA}(\bar{n}_S; \eta, \bar{n}_B) \) is [21]:

\[
C_{EA}(\bar{n}_S; \eta, \bar{n}_B) = g(\bar{n}_S) + g(\eta \bar{n}_S + (1 - \eta)\bar{n}_B) - g(A_+) - g(A_-),
\]

(15)

\( A_\pm = \frac{D\pm(1-\eta)(\bar{n}_B-\bar{n}_S)}{2} \), \( D = \sqrt{(\bar{n}_S + 1\eta \bar{n}_S + (1 - \eta)\bar{n}_B)^2 - 4\eta \bar{n}_S(\bar{n}_S + 1)} \), and \( g(x) \) is defined in (12). Note that, when \( \bar{n}_S \to 0 \), the dominant term in (15) is \(-\bar{n}_S \log \bar{n}_S\). Substituting \( \bar{n}_S \) given in (1) into (14), dividing both sides by \( \sqrt{n} \log n \), and taking the limit yields the converse.

V. ACHIEVABILITY PROOFS

A. Preliminaries

In our achievability proofs we specialize the lower bounds on the reliable information transmission over the block of \( n \) channel uses to the analysis of covert communication. These finite-blocklength results were recently extended to infinite-dimensional quantum systems [16], thus allowing their application to the bosonic channel considered herein. Our analysis takes inspiration from the classical approach in [7]. Before proceeding to the proofs, we describe the common elements.

Consider the Alice-to-Bob part of the covert communication setup in Fig. 2 within the standard classical-quantum communication framework where Alice and Bob have access to a bipartite resource state \( \hat{\rho}^{S^m R^m} \) occupying \( m \) systems \( S \) at Alice and \( R \) at Bob. Correlations between parts of \( \hat{\rho}^{S^m R^m} \) in systems \( S \) and \( R \) can either be classical or quantum: in the former case \( \hat{\rho}^{SR} \) is a classical-quantum state, while in the latter it is entangled and allows EA. Alice encodes a message \( x \) by acting on \( m \) systems \( S \) of \( \hat{\rho}^{S^m R^m} \) with an encoding operator \( \mathcal{M}_{S^m \to A^n}^{(x)} \), transforming the resource state to \( \hat{\rho}_x^{A^n R^m} = \mathcal{M}_{S^m \to A^n}^{(x)}(\hat{\rho}^{S^m R^m}) \). Transmission of the resulting \( n \) systems \( A \) over a quantum channel \( \mathcal{N}_{A^n \to B^n} \) results in state \( \hat{\rho}_x^{B^n R^m} = \mathcal{N}_{A^n \to B^n}(\mathcal{M}_{S^m \to A^n}^{(x)}(\hat{\rho}^{S^m R^m})) \) being available for Bob to measure and estimate message \( x \). We note that neither Alice nor the channel act on systems \( R \). Here \( n \) corresponds to the number of channel uses and \( m \gg n \) is proportional to the message set size. Furthermore, our quantum channel is memoryless \( \mathcal{N}_{A^n \to B^n} = \left( \mathcal{E}_{A \to B}^{(\eta \bar{n}_B)} \right)^{\otimes n} \) and we assume a product resource state: \( \hat{\rho}^{S^m R^m} = (\hat{\rho}^{SR})^{\otimes n} \). The encoder is memoryless in the
sense of acting on individual systems $S$, and discarding some of them: $\mathcal{M}_{S\rightarrow A}^{(X)}(\rho_{SmRm}) = \text{Tr}_X \left( \mathcal{M}_{S\rightarrow A}^{(X)}(\rho_{SmRm}) \right)$ where $\mathcal{X}$ is the set of $m - n$ discarded systems.

In order to proceed with our lemmata, we need to define the QRE variance $V(\hat{\rho}||\hat{\sigma})$:

**Definition 4.** Quantum relative entropy variance (QREV) between quantum states $\hat{\rho}$ and $\hat{\sigma}$ is:

$$V(\hat{\rho}||\hat{\sigma}) = \text{Tr} \left( \hat{\rho} \left( \log \hat{\rho} - \log \hat{\sigma} - D(\hat{\rho}||\hat{\sigma}) \right)^2 \right).$$ (16)

The finite blocklength capacity of a memoryless classical-quantum channel described above is as follows:

**Lemma 3.** There exists a scheme that, given a resource state $(\rho_{SR})^\otimes m$, allows the transmission of $M$ bits over $n$ uses of memoryless quantum channel $N_{A\rightarrow B}$ with decoding error probability $\epsilon$ for a sufficiently large $n$ and $m$, such that:

$$M \geq nD(\rho_{BR}||\rho_B \otimes \rho_R) + \sqrt{nV(\rho_{BR}||\rho_B \otimes \rho_R) \Phi^{-1}(\epsilon)} + O(\log n),$$ (17)

where $\Phi^{-1}(x)$ is the inverse Gaussian distribution function discussed in Section II-A, $D(\hat{\rho}||\hat{\sigma})$ is the QRE defined in (9), and $V(\hat{\rho}||\hat{\sigma})$ is the QRE variance defined in (16).

**Proof:** Lemma follows from application of [16, Eq. (5.11), and App. A] and either [16, Corr. 8] [28, Sec. 3] when $\rho_{SR}$ is classical-quantum or [16, Th. 6] when $\rho_{SR}$ is entangled.

We now specialize Lemma 3 to covert communication.

**B. Achievability without entanglement assistance**

The proof of the achievability of Theorem 1 follows from Lemma 1 which we prove first.

**Proof (Lemma 1):** Alice and Bob generate a random codebook $C = \{c(x), x = 1, \ldots, 2^M\}$ mapping $M$-bit input blocks to $n$-symbol codewords. Each $c(x) \in Q^n$ is generated according to $p(c) = \prod_{k=1}^{n} p_q(c_k)$, where $Q = \{a, ja, -a, -ja\}$ is the quadrature phase shift keying (QPSK) alphabet and $p_q(y) = \frac{1}{4}$ is the uniform distribution over it. We set $a = \sqrt{\bar{n}S}$, where $\bar{n}S$ is defined in (1). We assume that $C$ is shared between Alice and Bob prior to the start of transmission, and
unknown and we derive the Taylor series expansions at $\bar{\chi}$ where von Neumann entropy is $\{\text{ble}\}$. Thus, Alice transmits maximum mean photon number that ensures covertness [5, Th. 2].

The above construction corresponds to a shared resource state $\rho^{\text{sym}}_{\text{RM}} = (\hat{\rho}^{SR})^{\otimes m}$ containing the entire random codebook $C$ modulated by coherent states, where $\hat{\rho}^{SR} = \sum_{y \in Q} p_q(y) |y\rangle \langle y|^{A} \otimes |y\rangle \langle y|^{R}$ is a classical-quantum state, system $A$ is in a coherent state $|y\rangle^{A}$, and system $R$ is in one of the orthonormal states $|y\rangle^{R}$ corresponding to QPSK symbol index. Alice’s encoder then just selects the systems $S$ that correspond to the $n$-symbol codeword for message $x$, and discards the rest. Bob employs sequential decoding strategy described in [16 Corr. 8] and [28 Sec. 3].

Since the propagation of a coherent state $|\alpha\rangle$ through a thermal noise lossy bosonic channel $\mathcal{E}_{A \rightarrow B}^{(\eta,\bar{n}_B)}$ induces a displaced thermal state $\hat{\rho}_{(1-\eta)n_B}(\eta\alpha) = \hat{\rho}_{T}(\alpha)$ in Bob’s output port, the received state is $\hat{\rho}_{BR} = \sum_{y \in Q} p_q(y) \hat{\rho}_{T}^{B}(y) \otimes |y\rangle \langle y|^{R}$, where displaced thermal states $\{\hat{\rho}_{T}^{B}(y), y \in Q\}$ form an ensemble corresponding to transmission of QPSK symbols. Letting $\hat{\rho}_{B}^{T} \equiv \sum_{y \in Q} p_q(y) \hat{\rho}_{T}^{B}(y)$,

$$D\left(\rho^{BR}\|\rho^{B} \otimes \rho^{R}\right) = \chi \left(\{p_q(y), \hat{\rho}_{T}^{B}(y)\}\right) = S\left(\hat{\rho}_{B}^{T}\right) - \sum_{y \in Q} p_q(y) S\left(\hat{\rho}_{T}^{B}(y)\right)$$

$$V\left(\rho^{BR}\|\rho^{B} \otimes \rho^{R}\right) = V_{\chi} \left(\{p_q(y), \hat{\rho}_{T}^{B}(y)\}\right) = \sum_{y \in Q} p_q(y) \left[V\left(\hat{\rho}_{T}^{B}(y)\|\rho^{B}\right) + D\left(\hat{\rho}_{T}^{B}(y)\|\rho_{B}\right)\right] - \left[\chi \left(\{p_q(y), \hat{\rho}_{T}^{B}(y)\}\right)\right]^{2},$$

where von Neumann entropy is

$$S(\hat{\rho}) = - \text{Tr}[\hat{\rho} \log \hat{\rho}],$$

while $\chi \left(\{p(x), \hat{\rho}_{x}\}\right)$ and $V_{\chi} \left(\{p(x), \hat{\rho}_{x}\}\right)$ are Holevo information and its variance for ensemble $\{p(x), \hat{\rho}_{x}\}$. The closed-form expressions for $\chi \left(\{p_q(y), \hat{\rho}_{T}^{B}(y)\}\right)$ and $V_{\chi} \left(\{p_q(y), \hat{\rho}_{T}^{B}(y)\}\right)$ are unknown and we derive the Taylor series expansions at $\bar{n}_S = 0$ in Appendices A-B and A-C:

$$\chi \left(\{p_q(y), \hat{\rho}_{T}^{B}(y)\}\right) = \bar{n}_S c_{\text{rel, no-EA}} + O(\bar{n}_B^2)$$

$$V_{\chi} \left(\{p_q(y), \hat{\rho}_{T}^{B}(y)\}\right) = (1 + 2(1 - \eta)\bar{n}_B) c_{\text{rel, no-EA}} + O(\bar{n}_B^2).$$
Substituting $\bar{n}_S$ defined in (1) and invoking Lemma 3 yields the proof.

**Remark:** Using the Gaussian ensemble of coherent states $G = \left\{ e^{-|\alpha|^2/\bar{n}_S} \pi^{\bar{n}_S}, |\alpha\rangle \right\}$ as Alice’s input (corresponding to a random Gaussian codebook) requires an additional continuity argument in the version of Lemma 3 for classical-quantum states. However, since $G$ achieves the Holevo capacity of the bosonic channel, we compare the information ones for QPSK modulation in (21) and (22) to the corresponding quantities for $G$. We confirm that well-known fact [29] that QPSK modulation achieves the Holevo capacity in the low signal-to-noise ratio (SNR) regime in (21), and show that both modulation schemes have the same first term in the Holevo information variance expansion by calculating the latter for $G$ in Appendix B-B. Thus, QPSK modulation has the same finite blocklength performance as $G$ in the low SNR regime.

The proof of Theorem 1 is a straightforward application of Lemma 1:

**Proof (Th. 1):** Dividing both sides of (4) by $\sqrt{n}\delta$ and taking the limit yields the proof.

The fundamental limit in Theorem 1 cannot be exceeded when only classical secret is available without weakening the adversary. However, next we show how the scaling law can be improved using the shared quantum resource.

**C. Achievability with entanglement assistance**

The construction of the codebook in the proof of Lemma 2 is very similar to that in Lemma 1. However, quantum correlations yield significant improvement in decoding that is most pronounced in the covert regime.

**Proof (Lemma 2):** Let the resource state be a tensor product \( (|\psi\rangle^S)^\otimes m \) of a two-mode squeezed vacuum (TMSV) states such that $m = n2^M$ and

\[
|\psi\rangle^S = \sum_{k=1}^\infty \sqrt{t_k(\bar{n}_S)} |k\rangle^S |k\rangle^R,
\]

where $\bar{n}_S$ and $t_k(\bar{n})$ are defined in (1) and (8), respectively. A codebook is formed by assigning each message to $n$ TMSV states. Alice transmits $n$ modes corresponding to message $x$ from
her part of \(|\psi\rangle^R \otimes m\), discarding the rest. Willie has no access to Bob’s system \(R\). Since \(\text{Tr}_R |\psi\rangle \langle \psi|^R = \hat{n}_S\) is a thermal state, Criterion 1 is satisfied by setting \(\bar{n}_S\) as in (1) [4, Th. 2].

Bob uses sequential decoding from [16, Th. 6]. By [21], \(D (\rho^{BR} || \rho^B \otimes \rho^R) = C_{EA}(\bar{n}_S; \eta, \bar{n}_B)\) given in (15). The limit yielding \(L_{EA}\) is identical to that in the proof of the converse in Section IV. We use the symplectic matrix formalism to derive the expression for \(V(\rho^{BR} || \rho^B \otimes \rho^R)\) in Appendix B-C and note that its dominant term is \(\bar{n}_S \log_2 \bar{n}_S\). The following limit, evaluated using L’Hôpital’s rule, allows us to invoke Lemma 3 and complete the proof:

\[
\lim_{n \to \infty} nV (\rho^{BR} || \rho^B \otimes \rho^R)_{\bar{n}_S=\bar{n}_Cov/\sqrt{n}} = c_{cov} \sqrt{\delta_{rel}}. 
\]

(24)

**Proof (Th. 1):** Result follows directly from Lemma 2.

**Remark:** The sequential decoding strategy used by Bob in the proof of Lemma 2 does not correspond to any known receiver architecture. However, [18] combines insights from the sum-frequency generation receiver proposed for a quantum illumination radar [30] and the Green Machine receiver proposed for attaining superadditive communication capacity over the bosonic channel [31] to obtain a structured receiver design that realizes the EA scaling gain at low SNR. We employ the approximation [18, App. A.2] of this receiver’s achievable rate (in bits/mode):

\[
R_{sr} \approx \frac{\eta \bar{n}_S \gamma}{2(1 + (1 - \eta)\bar{n}_B)} \left[ \log \frac{w}{\bar{n}_S} - \log \left[ \ln \frac{w}{\bar{n}_S} \right] - g \left[ \frac{2(1 - \eta)\bar{n}_B(1 + (1 - \eta)\bar{n}_B)}{M \eta \gamma} \right] \right], 
\]

(25)

where \(\gamma = 1 - e^{-2(1+\eta)\bar{n}_B} \), \(w = \frac{4(1+(1-\eta)\bar{n}_B)}{M \eta \gamma + 4(1-\eta)\bar{n}_B + (1+\eta)\bar{n}_B}\), \(g(x)\) is defined in (12), and \(M \geq 1\) is a receiver design parameter. Fixing Bob’s receiver makes Alice-to-Bob channel a classical DMC, allowing us to follow the achievability approach in [7] almost exactly and obtain the following approximation to its EA covert capacity:

\[
L_{EA, sr} \approx \frac{\eta \gamma c_{cov}}{4(1 + (1 - \eta)\bar{n}_B)} \approx \frac{L_{EA}}{2}, 
\]

(26)

where the second approximation is valid when \(\bar{n}_B \gg 1\). Evolving the receiver [18] to achieve \(L_{EA}\) is an ongoing work.
VI. DISCUSSION

We derived the ultimate limits for quantum-secure covert communication over the bosonic channel both when only classical secret is available as a resource to Alice and Bob, and when they have access to entangled photon source. We confirmed our conjecture from [5] in the former case. Since entanglement assistance is known to particularly benefit the low-SNR regime, we expected it to improve covert capacity. However, we were surprised to have it alter the fundamental scaling law for covert communication from $O(\sqrt{n})$ to $O(\sqrt{n} \log n)$ covert bits reliably transmissible in $n$ channel uses. While here we focused on the bosonic channel, our work paves the way to the eventual characterization of covert communication over arbitrary quantum channels.

Many important questions remain. The resource state $(\hat{\rho}^{BR})^{\otimes m}$ employed in the proofs of Lemmas 1 and 2 is quite large: $m = n 2^M$. While the proofs in [16] rely on such a large state, one could conceivably extend the quantum channel resolvability approach [12], [13] to reduce $m$ as was done in [6] for classical covert communication. Furthermore, this work motivates the development of structured receivers for EA communication beyond [18]. Finally, quantum-secure covert active sensing problems [32], [33] need re-visiting in the light of our results. While EA yields no scaling improvement of estimation error for a single parameter, we conjecture a substantial benefit in multiple-parameter covert sensing problems.

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A. Preliminaries

In order to prove Theorem 1, we must characterize the behavior of the Holevo information and its variance as a function of the transmitted mean photon number per mode $\bar{n}_S$. However, there are no known closed-form expressions for the Holevo information or its variance for QPSK. Thus, we prove Theorem 1 using Taylor’s theorem:

**Lemma 4** (Taylor’s theorem). If $f(x)$ is a function with $k + 1$ continuous derivatives on the interval $[v, w]$, then

$$f(w) = f(v) + f'(v)(w - v) + \ldots + \frac{f^{(k)}(v)}{k!}(w - v)^k + R_{k+1}(w)$$

where $f^{(k)}(x)$ denotes the $k^{th}$ derivative of $f(x)$, and the Lagrange form of remainder is $R_{k+1}(w) = \frac{f^{(k+1)}(\xi)}{(k+1)!}(w - v)^{k+1}$ with $\xi$ satisfying $v \leq \xi \leq w$.

To evaluate the Taylor series expansion, we use the following lemmas where $\hat{A}(t)$ and $\hat{B}(t)$ are non-singular operators parameterized by $t$, and where $\hat{I}$ is the identity operator.

**Lemma 5** ([34, Th. 6]). $\frac{d}{dt} \ln \hat{A}(t) = \int_0^1 ds \left[s\hat{A}(t) + (1 - s)\hat{I}\right]^{-1} \frac{d\hat{A}(t)}{dt} \left[s\hat{A}(t) + (1 - s)\hat{I}\right]^{-1}$.

**Lemma 6** ([34, lemma in Sec. 4]). $\frac{d}{dt} \hat{B}^{-1}(t) = -\hat{B}^{-1}(t)\frac{d\hat{B}(t)}{dt} \hat{B}^{-1}(t)$.

B. Holevo information for quadrature phase shift keying

Here, we derive the Taylor expansion of Holevo information defined in (18) for QPSK at the displacement term $u = 0$. Setting $u = 0$ in (18) yields

$$\chi \left(p_d(y), \hat{\rho}^B_T(y)\right)\bigg|_{u=0} = \hat{\rho}_{\bar{n}_T} \log \hat{\rho}_{\bar{n}_T} - \hat{\rho}_{\bar{n}_T} \log \hat{\rho}_{\bar{n}_T} = 0,$$

(27)

where $\hat{\rho}_{\bar{n}_T}$ is the zero mean thermal state defined in [7], with $\bar{n}_T = (1 - \eta)\bar{n}_B$. 

APPENDIX A

TAYLOR SERIES EXPANSION OF HOLEVO INFORMATION AND ITS VARIANCE FOR QPSK MODULATION
Von Neumann entropy is invariant under unitary transformations. Since displacement is a unitary, \( S(\hat{\rho}_T^B(y)) = S(\hat{\rho}_{\vec{y}T}^B) \), implying that the derivatives with respect to \( u \) are zero. We evaluate the derivatives of the remaining von Neumann entropy term \( S(\hat{\rho}_T^B) \), given by its definition in (20). Using Lemma 5, the first derivative of \( S(\hat{\rho}_T^B) \) with respect to \( u \) is as follows:

\[
\frac{dS(\hat{\rho}_T^B)}{du} = \text{Tr} \left[ -\frac{d\hat{\rho}_T^B}{du} \log \hat{\rho}_T^B - \frac{\hat{\rho}_T^B}{\ln(2)} \int_0^1 ds \hat{\sigma}_1^{-1}(s) \frac{d\hat{\rho}_T^B}{du} \hat{\sigma}_1^{-1}(s) \right],
\]

(28)

where \( \hat{\sigma}_1(s) = s\hat{\rho}_T^B + (1 - s)I \). The derivatives of \( \hat{\rho}_T^B(u), \hat{\rho}_T^B(ju), \hat{\rho}_T^B(-u), \) and \( \hat{\rho}_T^B(-ju) \) are as follows [35, Ch. VI, Eq. (1.31)]:

\[
\frac{d\hat{\rho}_T^B(u)}{du} = \tilde{n}_T^{-1} \left( (\hat{a} - u)\hat{\rho}_T^B(u) + \hat{\rho}_T^B(u)(\hat{a}^\dagger - u) \right),
\]

(29)

\[
\frac{d\hat{\rho}_T^B(ju)}{du} = \tilde{n}_T^{-1} \left( (j\hat{a} + u)\hat{\rho}_T^B(ju) - \hat{\rho}_T^B(ju)(j\hat{a}^\dagger - u) \right),
\]

(30)

\[
\frac{d\hat{\rho}_T^B(-u)}{du} = \tilde{n}_T^{-1} \left( (\hat{a} + u)\hat{\rho}_T^B(-u) + \hat{\rho}_T^B(-u)(\hat{a}^\dagger + u) \right),
\]

(31)

\[
\frac{d\hat{\rho}_T^B(-ju)}{du} = \tilde{n}_T^{-1} \left( (j\hat{a} - u)\hat{\rho}_T^B(-ju) - \hat{\rho}_T^B(-ju)(j\hat{a}^\dagger + u) \right),
\]

(32)

where \( \hat{a}^\dagger \) and \( \hat{a} \) denote the creation and annihilation operators, respectively. Summing over \( \mathcal{Q} \) gives us

\[
\sum_{y \in \mathcal{Q}} p_q(y) \frac{d\hat{\rho}_T^B(y)}{du} = \frac{d\hat{\rho}_T^B}{du} = \frac{1}{4} \left( \frac{d\hat{\rho}_T^B(u)}{du} + \frac{d\hat{\rho}_T^B(ju)}{du} + \frac{d\hat{\rho}_T^B(-u)}{du} + \frac{d\hat{\rho}_T^B(-ju)}{du} \right).
\]

(33)

Setting \( u = 0 \) in (33) gives us

\[
\frac{d\hat{\rho}_T^B}{du} \bigg|_{u=0} = 0.
\]

(34)
Since both terms in (28) are zero when $u = 0$, $\left. \frac{dS(\hat{\rho}^B)}{du} \right|_{u=0} = 0$. Using Lemma 6, the second derivative of $S(\hat{\rho}^B)$ with respect to $u$ is as follows:

$$\frac{d^2 S(\hat{\rho}^B)}{du^2} = -\text{Tr} \left[ -2 \frac{d \hat{\rho}^B}{du} \frac{1}{\ln(2)} \int_0^1 ds \hat{\sigma}_1^{-1}(s) \frac{d \hat{\rho}^B}{du} \hat{\sigma}_1^{-1}(s) 
+ 2 \frac{\hat{\rho}^B}{\ln(2)} \int_0^1 ds \hat{\sigma}_1^{-1}(s) \frac{d \hat{\rho}^B}{du} \hat{\sigma}_1^{-1}(s) \right].$$

Setting $u = 0$ in (35) and removing terms containing $\frac{d \hat{\rho}^B}{du} \bigg|_{u=0}$ gives us

$$\frac{d^2 S(\hat{\rho}^B)}{du^2} \bigg|_{u=0} = -\text{Tr} \left[ -\frac{\hat{\rho}_{\tilde{n}_T}}{\ln(2)} \int_0^1 ds \hat{\sigma}_0^{-1}(s) \frac{d^2 \hat{\rho}^B}{du^2} \bigg|_{u=0} \hat{\sigma}_0^{-1}(s) - \frac{d^2 \hat{\rho}^B}{du^2} \bigg|_{u=0} \right].$$

where $\hat{\sigma}_0(s) = s \hat{\rho}_{\tilde{n}_T} + (1-s)I$. Setting $u = 0$ in $\frac{d \hat{\rho}^B}{du}$ gives us

$$\left. \frac{d^2 \hat{\rho}^B}{du^2} \right|_{u=0} = 2 \frac{1}{\tilde{n}_T^2} \left( \hat{a} \hat{\rho}_{\tilde{n}_T} \hat{a}^\dagger \right) - 2 \frac{1}{\tilde{n}_T} \left( \hat{\rho}_{\tilde{n}_T} \right).$$

Substitution of (37) into (36) gives us

$$\left. \frac{d^2 S(\hat{\rho}^B)}{du^2} \right|_{u=0} = -\text{Tr} \left[ -\frac{\hat{\rho}_{\tilde{n}_T}}{\ln(2)} \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{a} \hat{\rho}_{\tilde{n}_T} \hat{a}^\dagger \hat{\sigma}_0^{-1}(s) + 2 \frac{\hat{\rho}_{\tilde{n}_T}}{\ln(2)} \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{\rho}_{\tilde{n}_T} \hat{\sigma}_0^{-1}(s) 
- \left( 2 \frac{1}{\tilde{n}_T^2} \left( \hat{a} \hat{\rho}_{\tilde{n}_T} \hat{a}^\dagger \right) - 2 \frac{1}{\tilde{n}_T} \left( \hat{\rho}_{\tilde{n}_T} \right) \right) \right].$$

Note that $\hat{\sigma}_0(s)$ is diagonal in the Fock state basis, implying:

$$\hat{\sigma}_0^{-1}(s) = \sum_{k=0}^{\infty} (s \tau_k + (1-s))^{-1} |k\rangle \langle k|,$$

where $\tau_k = t_k(\tilde{n}_T)$, defined in (8). Now,

$$\int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{a} \hat{\rho}_{\tilde{n}_T} \hat{a}^\dagger \hat{\sigma}_0^{-1}(s) = \int_0^1 ds \sum_{k=0}^{\infty} \tau_k (s \tau_k + (1-s))^{-2} |k\rangle \langle k| = \hat{I},$$

$$\int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{a} \hat{\rho}_{\tilde{n}_T} \hat{a}^\dagger \hat{\sigma}_0^{-1}(s) = \int_0^1 ds \sum_{k=0}^{\infty} (k + 1) \tau_{k+1} (s \tau_k + (1-s))^{-2} |k\rangle \langle k|$$

$$= -\frac{\tilde{n}_T}{1 + \tilde{n}_T} \sum_{k=0}^{\infty} (k + 1) |k\rangle \langle k|,$$
since \( \int_0^1 ds (sq + (1 - s))^2 = \frac{1}{q} \) for \( q > 0 \). Thus, the traces of the first two terms in (38) cancel and we are left with

\[
\frac{d^2 S(\hat{\rho}^B)}{du^2} \bigg|_{u=0} = \text{Tr} \left[ -\frac{2}{\tilde{n}_T^2} \left( \hat{a} \hat{\rho}_{\tilde{n}_T} \hat{a}^\dagger \right) \log \hat{\rho}_{\tilde{n}_T} + \frac{2}{\tilde{n}_T} \left( \hat{\rho}_{\tilde{n}_T} \right) \log \hat{\rho}_{\tilde{n}_T} \right].
\]

(42)

The first term in (42) can be written in the Fock state basis as

\[
-\frac{2}{\tilde{n}_T^2} \left( \hat{a} \hat{\rho}_{\tilde{n}_T} \hat{a}^\dagger \right) \log \hat{\rho}_{\tilde{n}_T} = -\frac{2}{nt^2} \sum_{k=0}^\infty (k + 1) \tau_k \log \tau_k |k\rangle \langle k|
\]

(43)

\[
= -\frac{2}{\tilde{n}_T^2} \left[ \log \tilde{n}_T \sum_{k=0}^\infty k(k + 1) \tau_k |k\rangle \langle k| - \log(1 + \tilde{n}_T) \sum_{l=0}^\infty (l + 1)^2 \tau_{l+1} |l\rangle \langle l| \right]
\]

(44)

Taking the trace and evaluating the sums gives us

\[
\text{Tr} \left[ -\frac{2}{\tilde{n}_T^2} \left( \hat{a} \hat{\rho}_{\tilde{n}_T} \hat{a}^\dagger \right) \log \hat{\rho}_{\tilde{n}_T} \right] = -\frac{2}{\tilde{n}_T^2} \left[ 2\tilde{n}_T^2 \log \tilde{n}_T - 2\tilde{n}_T^2 \log(1 + \tilde{n}_T) - \tilde{n}_T \log(1 + \tilde{n}_T) \right]
\]

(46)

\[
= -\frac{2}{\tilde{n}_T^2} \left[ 2\tilde{n}_T \log \left( \frac{\tilde{n}_T}{1 + \tilde{n}_T} \right) - \log(1 + \tilde{n}_T) \right].
\]

(47)

The second term in (42) can be written in the Fock state basis as

\[
\frac{2}{\tilde{n}_T} \left( \hat{\rho}_{\tilde{n}_T} \right) \log \hat{\rho}_{\tilde{n}_T} = \frac{2}{\tilde{n}_T} \left[ \sum_{k=0}^\infty \tau_k \log \tau_k |k\rangle \langle k| \right]
\]

(48)

\[
= \frac{2}{\tilde{n}_T} \left[ \log \tilde{n}_T \sum_{k=0}^\infty k \tau_k |k\rangle \langle k| - \log(1 + \tilde{n}_T) \sum_{l=0}^\infty (l + 1) \tau_{l+1} |l\rangle \langle l| \right].
\]

(49)

Taking the trace and evaluating the sums gives us

\[
\text{Tr} \left[ \frac{2}{\tilde{n}_T} \left( \hat{\rho}_{\tilde{n}_T} \right) \log \hat{\rho}_{\tilde{n}_T} \right] = \frac{2}{\tilde{n}_T} \left[ \tilde{n}_T \log \tilde{n}_T - \tilde{n}_T \log(1 + \tilde{n}_T) - \log(1 + \tilde{n}_T) \right]
\]

(50)

\[
= \frac{2}{\tilde{n}_T} \left[ \tilde{n}_T \log \left( \frac{\tilde{n}_T}{1 + \tilde{n}_T} \right) - \log(1 + \tilde{n}_T) \right].
\]

(51)

Summing (47) and (51) yields:

\[
\frac{d^2 S(\hat{\rho}^B)}{du^2} \bigg|_{u=0} = 2 \log \left( 1 + \frac{1}{\tilde{n}_T} \right).
\]

(52)
Thus, the first non-zero term in the Taylor series expansion of the Holevo information is
\[
\frac{1}{2!} \frac{d^2 \chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)}{du^2} \bigg|_{u=0} = \log \left( 1 + \frac{1}{n_T} \right). \tag{53}
\]

C. Holevo information variance for quadrature phase shift keying

Now we derive the Taylor expansion of Holevo information variance defined in (19) for QPSK at the displacement term \( u = 0 \). The first two derivatives of \( [D(\hat{\rho}_T^B(y)||\hat{\rho}^B)]^2 \) with respect to displacement \( u \) are as follows:
\[
\frac{d}{du} \left( D(\hat{\rho}_T^B(y)||\hat{\rho}^B) \right)^2 = 2 \frac{dD(\hat{\rho}_T^B(y)||\hat{\rho}^B)}{du} D(\hat{\rho}_T^B(y)||\hat{\rho}^B),
\]
\[
\frac{d^2}{du^2} \left( D(\hat{\rho}_T^B(y)||\hat{\rho}^B) \right)^2 = 2 \frac{d^2D(\hat{\rho}_T^B(y)||\hat{\rho}^B)}{du^2} D(\hat{\rho}_T^B(y)||\hat{\rho}^B) + 2 \left( \frac{dD(\hat{\rho}_T^B(y)||\hat{\rho}^B)}{du} \right)^2. \tag{55}
\]

Since \( \frac{d(D(\hat{\rho}_T^B(y)||\hat{\rho}^B))^2}{du} \bigg|_{u=0} = \frac{d^2(D(\hat{\rho}_T^B(y)||\hat{\rho}^B))^2}{du^2} \bigg|_{u=0} = 0 \), \( [D(\hat{\rho}_T^B(y)||\hat{\rho}^B)]^2 \) contributes nothing to the first two terms of the Taylor series. The first two derivatives of \( \chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)^2 \) with respect to displacement \( u \) are as follows:
\[
\frac{d}{du} \left[ \chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)^2 \right] = 2 \frac{d\chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)}{du} \chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right), \tag{56}
\]
\[
\frac{d^2}{du^2} \left[ \chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)^2 \right] = 2 \frac{d^2\chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)}{du^2} \chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right) + 2 \left( \frac{d\chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)}{du} \right)^2. \tag{57}
\]

Note that \( \frac{d\chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)^2}{du} \bigg|_{u=0} = \frac{d^2\chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)^2}{du^2} \bigg|_{u=0} = 0 \). Thus, \( \chi \left( \{ p_q(y), \hat{\rho}_T^B(y) \} \right)^2 \) does not contribute to the first two terms of the Taylor series. Next, we evaluate the derivatives of \( V(\hat{\rho}_T^B(y)||\hat{\rho}^B) \), given by
\[
V(\hat{\rho}_T^B(y)||\hat{\rho}^B) = \text{Tr} \left[ \hat{\rho}_T^B(y) \left( \log \hat{\rho}_T^B(y) - \log \hat{\rho}^B - D(\hat{\rho}_T^B(y)||\hat{\rho}^B) \right)^2 \right]. \tag{58}
\]
Let \( \hat{R} = \log \hat{\rho}_B^T(y) - \log \hat{\rho}^B - D(\hat{\rho}_B^T(y)||\hat{\rho}^B) \) be the term inside the square in the QRE variance \( V(\hat{\rho}_B^T(y)||\hat{\rho}^B) \). Note that

\[
\hat{R}_{|u=0} = 0. \quad (59)
\]

Taking the derivative of \( V(\hat{\rho}_B^T(y)||\hat{\rho}^B) \) with respect to \( u \) gives us

\[
\frac{dV(\hat{\rho}_B^T(y)||\hat{\rho}^B)}{du} = \text{Tr} \left[ \hat{\rho}_T^B(y) \left( \frac{\hat{\rho}_B^T(y) d\hat{R}}{du} + \frac{d\hat{R}}{du} \hat{R} \right) \right]. \quad (60)
\]

Setting \( u = 0 \),

\[
\frac{dV(\hat{\rho}_B^T(y)||\hat{\rho}^B)}{du} \bigg|_{u=0} = 0, \quad (61)
\]

since \( \hat{R}_{|u=0} = 0 \). Taking another derivative with respect to \( u \) gives us

\[
\frac{d^2V(\hat{\rho}_B^T(y)||\hat{\rho}^B)}{du^2} = \text{Tr} \left[ \hat{\rho}_T^B(y) \left( 2 \left( \frac{d\hat{R}}{du} \right)^2 + \hat{R} \frac{d^2\hat{R}}{du^2} + \frac{d^2\hat{R}}{du^2} \hat{R} \right) \right] \left( \frac{\hat{\rho}_B^T(y) d\hat{R}}{du} + \frac{d\hat{R}}{du} \hat{R} \right) + \frac{d\hat{\rho}_B^T(y)}{du} \left( \frac{\hat{\rho}_B^T(y) d\hat{R}}{du} + \frac{d\hat{R}}{du} \hat{R} \right) + \frac{d^2\hat{\rho}_B^T(y)}{du^2} \hat{R}^2 \right]. \quad (62)
\]

Setting \( u = 0 \),

\[
\frac{d^2V(\hat{\rho}_B^T(y)||\hat{\rho}^B)}{du^2} \bigg|_{u=0} = \text{Tr} \left[ 2 \hat{\rho}_T^B(y) \left( \frac{d\hat{R}}{du} \bigg|_{u=0} \right)^2 \right]. \quad (63)
\]

Using Lemma 5, we find that the derivative of \( \hat{R} \) with respect to \( u \) is

\[
\frac{d\hat{R}}{du} = -\frac{1}{\ln(2)} \int_0^1 ds \hat{\sigma}_{\hat{\rho}_B^{-1}}(s) \frac{d\hat{\rho}_B^T(y)}{du} \hat{\sigma}_{\hat{\rho}_B^{-1}}(s) - \frac{1}{\ln(2)} \int_0^1 ds \hat{\sigma}_{\hat{\rho}_T^{-1}}(s) \frac{d\hat{\rho}_B^T(y)}{du} \hat{\sigma}_{\hat{\rho}_T^{-1}}(s) - \frac{dD(\hat{\rho}_B^T(y)||\hat{\rho}^B)}{du}. \quad (65)
\]

Setting \( u = 0 \),

\[
\frac{d\hat{R}}{du} \bigg|_{u=0} = \frac{1}{\ln(2)} \int_0^1 ds \hat{\sigma}_{\hat{\rho}_B^{-1}}(s) \frac{d\hat{\rho}_B^T(y)}{du} \bigg|_{u=0} \hat{\sigma}_{\hat{\rho}_B^{-1}}(s), \quad (66)
\]
since \( \frac{d\hat{\rho}_B}{du} \bigg|_{u=0} = 0 \) and \( \frac{dD(\hat{\rho}_B(y)||\hat{\rho}_B)}{du} \bigg|_{u=0} = 0 \). Substituting this term into \( (64) \) and expanding gives us

\[
\frac{d^2V(\hat{\rho}_T^B(y)||\hat{\rho}_B)}{du^2} \bigg|_{u=0} = \text{Tr} \left[ 2 \hat{\rho}_{\tilde{n}_T} \frac{1}{(\ln(2))^2} \int_0^1 ds \hat{\sigma}_0^{-1}(s) \frac{d\hat{\rho}_T^B(y)}{du} \right] \hat{\sigma}_0^{-1}(s) \int_0^1 ds \hat{\sigma}_0^{-1}(s) \frac{d\hat{\rho}_T^B(y)}{du} \right] \hat{\sigma}_0^{-1}(s). 
\]

(67)

Summing over \( Q \) is needed to calculate the quantum relative entropy variance term in the Holevo information variance:

\[
\sum_{y \in Q} \frac{d^2V(\hat{\rho}_T^B(y)||\hat{\rho}_B)}{du^2} \bigg|_{u=0} = \frac{8}{\bar{n}_T^2(\ln(2))^2} \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{\rho}_{\tilde{n}_T} \hat{\sigma}_0^{-1}(s) \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{\rho}_{\tilde{n}_T} \hat{\sigma}_0^{-1}(s)
\]

\[
+ \frac{8}{\bar{n}_T^2(\ln(2))^2} \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{\rho}_{\tilde{n}_T} \hat{\sigma}_0^{-1}(s) \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{\rho}_{\tilde{n}_T} \hat{\sigma}_0^{-1}(s). 
\]

(68)

Since \( \int_0^1 ds (s q + (1 - s))^{-1} (s r + (1 - s))^{-1} = \frac{\ln(q)}{q - r} \) for \( q, r > 0 \) and \( q \neq r \),

\[
\int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{\rho}_{\tilde{n}_T} \hat{\sigma}_0^{-1}(s) = \int_0^1 ds \sum_{k=0}^{\infty} \frac{\tau_k \sqrt{k} |k\rangle \langle k - 1|}{(s \tau_k + (1 - s))(s \tau_{k-1} + (1 - s))}
\]

\[
= \bar{n}_T \ln(1 + \frac{1}{\bar{n}_T}) \sum_{k=0}^{\infty} \sqrt{k} |k\rangle \langle k - 1|, 
\]

(70)

\[
\int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{\rho}_{\tilde{n}_T} \hat{\sigma}_0^{-1}(s) = \int_0^1 ds \sum_{k=0}^{\infty} \frac{\tau_k \sqrt{k} |k - 1\rangle \langle k|}{(s \tau_{k-1} + (1 - s))(s \tau_k + (1 - s))}
\]

\[
= \bar{n}_T \ln(1 + \frac{1}{\bar{n}_T}) \sum_{k=0}^{\infty} \sqrt{k} |k - 1\rangle \langle k|. 
\]

(71)
Using (71) and (73), we find the first term of (69) as:

\[
\frac{8}{\hat{n}_T^2 (\ln(2))^2} \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{\rho}_{n_T} \hat{a}^{-1} \hat{\sigma}_0^{-1}(s) \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{a}^{-1} \hat{\sigma}_0^{-1}(s) \\
= \frac{8}{(\ln(2))^2} \left( \ln \left( 1 + \frac{1}{\hat{n}_T} \right) \right)^2 \sum_{l=0}^{\infty} \tau_l \sum_{k=0}^{\infty} \sqrt{k} \sum_{m=0}^{\infty} \sqrt{m} |m-1\rangle \langle m| \\
= \frac{8}{(\ln(2))^2} \left( \ln \left( 1 + \frac{1}{\hat{n}_T} \right) \right)^2 \sum_{k=0}^{\infty} k \tau_k |k\rangle \langle k| \\
= \frac{8}{(\ln(2))^2} \hat{n}_T \left( \ln \left( 1 + \frac{1}{\hat{n}_T} \right) \right)^2 .
\]

(74)

Similarly, the second term of (69) is:

\[
\frac{8}{\hat{n}_T^2 (\ln(2))^2} \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{a} \hat{\sigma}_0^{-1}(s) \int_0^1 ds \hat{\sigma}_0^{-1}(s) \hat{a}^{-1} \hat{\sigma}_0^{-1}(s) \\
= \frac{8}{(\ln(2))^2} (1 + \hat{n}_T) \left( \ln(1 + \frac{1}{\hat{n}_T}) \right)^2 .
\]

(75)

Thus,

\[
\sum_{y \in Q} \frac{d^2 V(\hat{\rho}_T^B(y)||\hat{\rho}_T^B)}{du^2} \bigg|_{u=0} = 8(1 + 2\hat{n}_T) \left( \ln \left( 1 + \frac{1}{\hat{n}_T} \right) \right)^2 .
\]

(76)

Since the only other term in the definition of the Holevo information variance that contributes here is \( p_q(y) \), the first non-zero term in the Taylor series is:

\[
\frac{1}{2!} \frac{d^2 V_{\chi}(\{(p_q(y), \hat{\rho}_T^B(y))\})}{du^2} \bigg|_{u=0} = (1 + 2\hat{n}_T) \left( \ln \left( 1 + \frac{1}{\hat{n}_T} \right) \right)^2 .
\]

(77)

(78)

**APPENDIX B**

**CALCULATION OF QUANTUM RELATIVE ENTROPY VARIANCE**

**A. Preliminaries**

In this appendix we employ the symplectic formalism to derive the quantum relative entropy variance (QREV). We also analyze the asymptotic behavior of the QREV for small \( \bar{n}_S \).
The QREV $V(\hat{\rho} \| \hat{\sigma})$ between two quantum Gaussian states $\hat{\rho}$, $\hat{\sigma}$ with first moments $\tilde{\mu}_\rho$, $\tilde{\mu}_\sigma$ and covariance matrices $\Sigma_\rho$, $\Sigma_\sigma$, respectively, is \[36]:

$$V(\hat{\rho} \| \hat{\sigma}) = \frac{1}{2} \text{Tr} \left[ \Delta \Sigma_\rho \Delta \Sigma_\rho \right] + \frac{1}{8} \text{Tr} \left[ \Delta \Omega \Delta \Omega \right] + \delta^T G_\sigma \Sigma_\rho G_\sigma \delta,$$

(79)

where $\Delta$ is the difference of the Gibbs matrices $\Delta = G_\rho - G_\sigma$, $\delta = \tilde{\mu}_\rho - \tilde{\mu}_\sigma$, and $\Omega = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{pmatrix}$, is the symplectic matrix in the $qqpp$ representation ($n$ is the number of modes, $I_{n \times n}$ is the $n \times n$ identity, and $0_{n \times n}$ is an $n \times n$ matrix whose every element is 0). The relation of a Gibbs matrix to its corresponding covariance matrix (CM) is

$$G_\rho = -2 \Omega S_\rho \left[ \text{arccoth} \left( 2D_\rho \right) \right] \otimes^2 S^T_\rho \Omega,$$

where $S_\rho$ are the symplectic eigenvectors of $\Sigma_\rho$, $D_\rho = \text{diag} \left( \lambda_1, \ldots, \lambda_n, \lambda_1, \ldots, \lambda_n \right)$ with $\lambda_i$ the symplectic eigenvalues. We note that $\text{arccoth} = \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right)$, $x \in (-\infty, -1) \cup (1, +\infty)$ and for that reason we will work with the natural logarithm in this appendix. After the QREV is obtained one can convert it to base 2 logarithm by dividing with $(\ln 2)^2$.

We note that a phase shifting does not play any role in calculating the QREV in the following case: $\hat{\sigma}$ is a thermal product state and the rotation is applied in one of the modes of $\hat{\rho}$. Taking into account that the phase shifting corresponds to an orthogonal symplectic matrix $S_\phi$, the property of symplectic matrices $S^T_\phi \Omega S_\phi = \Omega$, and the cyclic permutation property of trace, one can see that (79) remains the same if we used $\Sigma_\rho$ or $\Sigma_\rho(\phi) = S_\phi \Sigma_\rho S^T_\phi$.

We close this section by reminding the reader that a matrix $S$ is symplectic if it is real and satisfies $S \Omega S^T = S^T \Omega S = \Omega$. Also the symplectic eigenvalues $\lambda_i$ of any CM of a quantum mechanical system should be $\lambda_i \geq 1/2$.

**B. QREV for the non-EA case**

Consider the ensemble of Gaussian single-mode thermal states $\{\hat{\rho}^{\tilde{y}}_E\}_{\tilde{y} \in \mathbb{R}^2}$ with CM,

$$\Sigma_{\text{th}} = \left( \bar{n}_B + \frac{1}{2} \right) I_{2 \times 2},$$

(80)

and first moments,

$$\tilde{d} = W \tilde{y} + \tilde{\nu},$$

(81)
where \( \bar{n}_B \) is the mean number of thermal photons, \( W \) is a \( 2 \times 2 \) matrix, and \( \bar{y}, \bar{v} \) are \( 2 \)-dimensional vectors. The prior classical distribution \( p_Y(y) \) of this ensemble is Gaussian,

\[
p_Y(y) = \frac{\exp\left(-\frac{1}{2}(\bar{y} - \mu)^T \Sigma^{-1} (\bar{y} - \mu)\right)}{2\pi \sqrt{\det\Sigma}},
\]

where \( \mu \) are the first moments and \( \Sigma \) is the CM,

\[
\Sigma = \bar{n}_S I_{2 \times 2}
\]

The formula for the QREV of the ensemble \( \{p_Y(y), \hat{\rho}_E\} \) has been derived in [37, Definition 1, Proposition 2],

\[
V_{Y||E}(\bar{n}_S, \bar{n}_B) = \frac{1}{2} \text{Tr} \left[ (\Delta \Sigma_{\text{th}})^2 \right] + \frac{1}{8} \text{Tr} \left[ (\Delta \omega)^2 \right] + \frac{1}{2} \text{Tr} \left[ W \Sigma W^T G_E \Sigma_{\text{th}} G_E \right] + \frac{1}{2} \text{Tr} \left[ (W \Sigma W^T G_E)^2 \right]
\]

where,

\[
\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
G = -2\omega \text{arccoth}(2\Sigma_{\text{th}})\omega,
\]

\[
\Sigma_E = \Sigma_{\text{th}} + 2W\Sigma W^T,
\]

\[
G_E = -2\omega \text{arccoth}(2\Sigma_E)\omega,
\]

\[
\Delta = G_E - G.
\]

For \( W = I_{2 \times 2} \) (and \( \bar{v} = 0 \), even though \( \bar{v} \) does not play any role in the calculation of the variance), i.e., we assume that the first moments of the Gaussian states are equal to the random vector \( \bar{y} \),
and using (80), (83), (84), (85), (86), (87), (88), (89), we find,

\[
V_{Y\|E}(\bar{n}_S, \bar{n}_B) = \bar{n}_B(\bar{n}_B + 1) \left( \ln \frac{\bar{n}_B + 1}{\bar{n}_B} \right)^2 - 2\bar{n}_B(\bar{n}_B + 1) \ln \left( \frac{\bar{n}_B + 1}{\bar{n}_B} \right) \ln \left( \frac{\bar{n}_B + \bar{n}_B + 1}{\bar{n}_B + \bar{n}_B} \right)
+ \bar{n}_B(\bar{n}_B + 1) \left( \ln \frac{\bar{n}_B + \bar{n}_S + 1}{\bar{n}_B + \bar{n}_S} \right)^2 + \left( \bar{n}_B + \frac{1}{2} \right) \bar{n}_S \left( \ln \frac{\bar{n}_B + \bar{n}_S + 1}{\bar{n}_B + \bar{n}_S} \right)^2
+ \bar{n}_S^2 \left( \ln \frac{\bar{n}_B + \bar{n}_S + 1}{\bar{n}_B + \bar{n}_S} \right)^2.
\]

(90)

Taylor series expansion of (90) at \( \bar{n}_S = 0 \) yields:

\[
V_{Y\|E}(\bar{n}_S, \bar{n}_B) = \left( \bar{n}_B + \frac{1}{2} \right) \left( \ln \frac{\bar{n}_B + 1}{\bar{n}_B} \right)^2 \bar{n}_S + O(\bar{n}_S^2)
\]

(91)

C. QREV for the EA case

Here, the upper mode of a two-mode squeezed state (TMSV) is phase modulated (which we will not consider in our QREV calculation as per the discussion in App. B-A). Then, the same mode goes through a thermal lossy channel of transitivity \( \eta \) and mean thermal photon number \( \bar{n}_B \). The lower mode remains the same. The mean photon number per mode of the TMSV is \( \bar{n}_S \). The two-mode output state of this setup, denoted by \( \hat{\rho} \), has zero displacements (no displacements are involved in the TMSV nor the evolution of the state) and its CM is:

\[
\Sigma_{\hat{\rho}} = \begin{pmatrix}
    w_{11} & w_{12} & 0 & 0 \\
    w_{12} & w_{22} & 0 & 0 \\
    0 & 0 & w_{11} & -w_{12} \\
    0 & 0 & -w_{12} & w_{22}
\end{pmatrix},
\]

(92)

where,

\[
w_{11} = \left( \bar{n}_B + \frac{1}{2} \right) (1 - \eta) + \left( \bar{n}_S + \frac{1}{2} \right) \eta,
\]

(93)

\[
w_{12} = \sqrt{\eta \bar{n}_S (\bar{n}_S + 1)},
\]

(94)

\[
w_{22} = \bar{n}_S + \frac{1}{2}.
\]

(95)
The CM (92) is found from the CM of the TMSV,

\[ \Sigma_{\text{TMSV}} = \begin{pmatrix}
\bar{n}_S + \frac{1}{2} & \sqrt{\bar{n}_S(\bar{n}_S + 1)} & 0 & 0 \\
\sqrt{\bar{n}_S(\bar{n}_S + 1)} & \bar{n}_S + \frac{1}{2} & 0 & 0 \\
0 & 0 & \bar{n}_S + \frac{1}{2} & -\sqrt{\bar{n}_S(\bar{n}_S + 1)} \\
0 & 0 & -\sqrt{\bar{n}_S(\bar{n}_S + 1)} & \bar{n}_S + \frac{1}{2}
\end{pmatrix}, \quad (96) \]

by applying \( X\Sigma_{\text{TMSV}}X^T + Y = \Sigma_{\rho} \), where the matrices \( X, Y \) describe the thermal loss channel which is applied to the upper mode of the TMSV,

\[
X = \text{diag} \left( \sqrt{\eta}, 1, \sqrt{\eta}, 1 \right), \quad (97)
\]

\[
Y = \text{diag} \left( (1-\eta) \left( \bar{n}_B + \frac{1}{2} \right), 0, (1-\eta) \left( \bar{n}_B + \frac{1}{2} \right), 0 \right). \quad (98)
\]

The task is to compute the QREV \( V(\hat{\rho} \parallel \hat{\sigma}) \), where \( \hat{\rho} \) is a Guassian state with zero displacement and CM given in (92) and \( \hat{\sigma} \) is the product of two thermal states with zero displacement and CM given by (92) with all correlations (off-diagonal elements) set to zero,

\[
V_{\sigma} = \begin{pmatrix}
w_{11} & 0 & 0 & 0 \\
0 & w_{22} & 0 & 0 \\
0 & 0 & w_{11} & 0 \\
0 & 0 & 0 & w_{22}
\end{pmatrix}. \quad (99)
\]

The key calculations to achieve the aforesaid task are to find the symplectic spectrum of \( \Sigma_{\rho} \) and \( \Sigma_{\sigma} \). That means, to find symplectic matrices \( S_{\rho}, S_{\sigma} \) and diagonal matrices \( D_{\rho}, D_{\sigma} \) as defined in App. B-A such that \( \Sigma_{\rho,\sigma} = S_{\rho,\sigma} (D_{\rho,\sigma} \oplus D_{\rho,\sigma}) S_{\rho,\sigma}^T \). For the CM \( \Sigma_{\sigma} \) the situation is trivial; \( \Sigma_{\sigma} \) is already in the symplectic diagonal form, with \( S_{\sigma} = I \) and symplectic eigenvalues \( w_{11}, w_{22} \geq 1/2 \).

For \( \Sigma_{\rho} \) we find the symplectic eigenvalues,

\[
\lambda_1 = \frac{1}{2} \left( \sqrt{(w_{11} + w_{22})^2 - 4w_{12}^2} + (w_{11} - w_{22}) \right), \quad (100)
\]

\[
\lambda_2 = \frac{1}{2} \left( \sqrt{(w_{11} + w_{22})^2 - 4w_{12}^2} - (w_{11} - w_{22}) \right) \quad (101)
\]
and the symplectic eigenvectors (organized in a symplectic matrix),

\[
S_\rho = \begin{pmatrix}
s_+ & s_- & 0 & 0 \\
-s_- & -s_+ & 0 & 0 \\
0 & 0 & s_+ & -s_- \\
0 & 0 & s_- & -s_+
\end{pmatrix},
\] (102)

where,

\[
s_\pm = \frac{1}{2} \left( w \pm \frac{1}{w} \right),
\] (103)

\[
w = \frac{\sqrt{w_{11} - 2w_{12} + w_{22}}}{\sqrt{(w_{11} + w_{22})^2 - 4w_{12}^2}}.
\] (104)

Using (100), (101), (103), and (104), one can verify indeed the following two things: The matrix \( S_\rho \) is symplectic, i.e., \( S_\rho \Omega S_\rho^T = S_\rho^T \Omega S_\rho = \Omega \) and \( S_\rho D_\rho S_\rho^T = \Sigma_\rho \), where \( D_\rho = \text{diag}(\lambda_1, \lambda_2, \lambda_1, \lambda_2) \).

Therefore, we have found the symplectic diagonal form of \( \Sigma_\rho \).

We are now ready to apply (79) and find the variance \( V(\hat{\rho} \| \hat{\sigma}) := V_{\text{EA}}(w_{11}, w_{12}, w_{22}) = V_{\text{EA}}(\eta, \bar{n}_S, \bar{n}_B) \),

\[
V_{\text{EA}}(w_{11}, w_{12}, w_{22}) = \sum_{i=1}^{9} r_i,
\] (105)

where,

\[
r_1 = \left( 4w_{11}^2 - 1 \right) \text{arccoth}^2(2w_{11})
\] (106)

\[
r_2 = \left( 4w_{22}^2 - 1 \right) \text{arccoth}^2(2w_{22})
\] (107)

\[
r_3 = \left( 2w_{11}^2 + 2(w_{11} - w_{22})\sqrt{(w_{11} + w_{22})^2 - 4w_{12}^2} \right) \text{arccoth}^2(2\lambda_1)
\] (108)

\[
r_4 = \left( 2w_{11}^2 + 2(w_{11} - w_{22})\sqrt{(w_{11} + w_{22})^2 - 4w_{12}^2} \right) \text{arccoth}^2(2\lambda_2)
\] (109)

\[
r_5 = 8w_{12}^2 \text{arccoth}(2w_{11}) \text{arccoth}(2w_{22})
\] (110)
\[ r_6 = A(w_{11}, w_{12}, w_{22}) \text{arccoth}(2w_{11}) \text{arccoth}(2\lambda_1) \]  
\[ r_7 = \left( A(w_{11}, w_{12}, w_{22}) - 2 + 8w_{11}^2 - 8w_{12}^2 \right) \text{arccoth}(2w_{11}) \text{arccoth}(2\lambda_2) \]  
\[ r_8 = \left( A(w_{22}, w_{12}, w_{11}) - 2 + 8w_{22}^2 - 8w_{12}^2 \right) \text{arccoth}(2w_{22}) \text{arccoth}(2\lambda_1) \]  
\[ r_9 = A(w_{22}, w_{12}, w_{11}) \text{arccoth}(2w_{22}) \text{arccoth}(2\lambda_2), \] 

where,
\[
A(w_{11}, w_{12}, w_{22}) = 1 - 4(w_{11}^2 - w_{12}^2) + \frac{w_{11} + w_{22} - 4w_{11}(w_{11}^2 - 3w_{12}^2) - 4w_{22}(w_{11}^2 + w_{12}^2)}{\sqrt{(w_{11} + w_{22})^2 - 4w_{12}^2}}
\] (115)

and \(A(w_{22}, w_{12}, w_{11})\) is given by (115) by interchanging \(w_{11}\) with \(w_{22}\).

Using (93), (94), (95), and (106)-(114) we find that,
\[
\lim_{\bar{n}_S \to 0} V_{EA}(\eta, \bar{n}_S, \bar{n}_B) = 0
\] (116)
as expected (the QREV of two identical states must be zero). By observation, the leading terms of \(V_{EA}(\eta, \bar{n}_S, \bar{n}_B)\) scale as \(\bar{n}_S(\ln \bar{n}_S)^2\). Indeed, we have,
\[
\lim_{\bar{n}_S \to 0} \frac{V_{EA}(\eta, \bar{n}_S, \bar{n}_B)}{\bar{n}_S(\ln \bar{n}_S)^2} = \frac{\eta}{(1 - \eta)\bar{n}_B + 1}
\] (117)
and higher order denominators return 0 in the same limit. Keeping terms proportional to \(\bar{n}_S\), \(\bar{n}_S \ln \bar{n}_S\), and \(\bar{n}_S(\ln \bar{n}_S)^2\) we find the approximate behavior of \(V_{EA}(\eta, \bar{n}_S, \bar{n}_B)\) for \(\bar{n}_S \ll 1\),
\[
V_{EA}(\eta, \bar{n}_S \ll 1, \bar{n}_B) = \frac{\eta}{(1 - \eta)\bar{n}_B + 1} \left[ \ln \frac{(1 - \eta)\bar{n}_B + 1}{(1 - \eta)\bar{n}_B} \right]^2 \bar{n}_S
- \frac{2\eta}{(1 - \eta)\bar{n}_B + 1} \ln \left[ \frac{(1 - \eta)\bar{n}_B + 1}{(1 - \eta)\bar{n}_B} \right] \ln \bar{n}_S
- \frac{2(1 - \eta)(\bar{n}_B + 1)}{(1 - \eta)\bar{n}_B + 1} \bar{n}_S \ln \bar{n}_S \ln \frac{(1 - \eta)(\bar{n}_B + 1)\bar{n}_S}{(1 - \eta)\bar{n}_B + 1}
+ \frac{(1 - \eta)(\bar{n}_B + 1)}{(1 - \eta)\bar{n}_B + 1} \bar{n}_S \ln \frac{(1 - \eta)(\bar{n}_B + 1)\bar{n}_S}{(1 - \eta)\bar{n}_B + 1}
+ \bar{n}_S(\ln \bar{n}_S)^2.
\] (118)
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