Parameterized complexity of machine scheduling: 15 open problems

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Abstract
Machine scheduling problems are a long-time key domain of algorithms and complexity research. A novel approach to machine scheduling problems are fixed-parameter algorithms. To stimulate this thriving research direction, we propose 15 interesting open questions in this area.

1 Introduction
Algorithms for machine scheduling problems form one of the core applications of combinatorial optimization. In those problems, we are generally given a finite set $J$ of jobs with certain characteristics, and we must find a schedule, which is an assignment of jobs to one or more machines that also may have their individual specifications. Typical characteristics of a job $j$ are its processing time $p_j \in \mathbb{N}$, its release date $r_j \in \mathbb{N}$, its due date $d_j \in \mathbb{N}$, or its importance reflected by a weight $w_j \in \mathbb{N}$. Machine characteristics typically include their speed or whether they are capable of processing a certain type of job. The jobs may be related to each other by precedence constraints, specifying that certain jobs must be scheduled before other jobs. Also, jobs may be required to be processed without preemption or preemption may be allowed.

Usually, we are not just seeking a feasible schedule respecting all constraints, but one that additionally optimizes some objective function. Classical objectives include the minimization of the makespan or the sum of weighted completion times (which is equivalent to minimizing the weighted average completion time). Since the inception of the field in the 1960s, thousands of research papers have been devoted to understanding the complexity of scheduling problems.

*Supported by ERC Starting Grant 306465 (BeyondWorstCase).
†Supported by the Russian Foundation for Basic Research, project 16-31-60007 mol_a_dk, and by the Ministry of Science and Education of the Russian Federation under the 5-100 Excellence Program.
A significant portion of investigated problems turned out to be NP-hard. In consequence, algorithm designers proposed algorithms for these problems that yield approximate solutions in polynomial time. In 1999, Schuurman and Woeginger [64] listed 10 of the most prominent open problems around polynomial-time approximation algorithms for NP-hard scheduling problems at that time.

Only recently, a different algorithmic approach has been put forward for solving NP-hard scheduling problems. While so far only few papers explicitly address the problem, there is a growing interest [9–12, 15, 21, 40, 43, 58, 59]: It is the design of fixed-parameter algorithms, which harness parameters $k_1, k_2, \ldots$ of a problem instance $I$ to solve it in $f(k_1, k_2, \ldots) \cdot n^{O(1)}$ time for some computable function $f$. While fixed-parameter algorithms are now a well-investigated area of algorithmics, their systematic application to scheduling problems has come about only recently. For further background on parameterized complexity, we refer to the recent monograph by Cygan et al. [24].

In the following, we summarize known results and list open problems regarding the parameterized complexity of scheduling problems on a single machine, on parallel identical machines, and in shop scheduling environments. We do not claim these problems to be the most important ones, but but believe that their resolution (in one way or the other) will lead to the discovery of new approaches both in parameterized complexity and scheduling theory, thus stimulating further research with both practical and theoretical significance.

In this sense, we hope that this work will be appealing and inspiring both to researchers with a scheduling background, as well as to researchers with a parameterized complexity background. For the latter, we exhibit some fixed-parameter tractability results that appeared before the advent of parameterized complexity theory and thus were not explicitly described as such.

2 Preliminaries

Machine scheduling problems. Throughout this work, we use the standard three-field notation of scheduling problems due to Graham et al. [36]. This allows us to denote many problems as a triple $\alpha|\beta|\gamma$, where $\alpha$ is the machine environment, $\beta$ are job characteristics and scheduling constraints, and $\gamma$ is the objective function.

In general, each job $j \in J$ may have a weight $w_j \in \mathbb{N}$ and the goal functions $\gamma$ that we minimize here are

— the makespan $C_{\text{max}} = \max_{j \in J} C_j$, where $C_j$ is the completion time of job $j$ in a schedule, and

— the weighted sum $\sum_{j \in J} w_j C_j$ of completion times, which is equivalent to minimizing the weighted average completion time, also known as mean flow time.

If each job $j \in J$ has a due date $d_j \in \mathbb{N}$, we also minimize

— the total weighted tardiness $\sum_{j \in J} w_j T_j$, where $T_j = \max\{0, d_j - C_j\}$, and
— the weighted number $\sum_{j \in J} w_j U_j$ of tardy jobs, where $U_j = 0$ if job $j$ is finished by its due date and $U_j = 1$ otherwise. This is equivalent to maximizing \textit{throughput} — the weighted number of jobs that get finished by their due dates.

Generally, we will drop the "$j \in J$" subscript under the sum and we will refer to the unit-weight variants of the scheduling problems by simply dropping the $w_j$ from the objective functions.

\textbf{Parameterized complexity.} The question addressed by parameterized complexity theory is which problems allow for fixed-parameter algorithms—algorithms that solve problems in $f(k) \cdot n^{O(1)}$ time, where $n$ is the input size, $k$ is some parameter of the input instance, and $f$ is an arbitrary computable function (superpolynomial for NP-hard problems). If a problem allows for such a fixed-parameter algorithm, then we say that the problem is \textit{fixed-parameter tractable parameterized by} $k$. FPT is the class of fixed-parameter tractable problems.

Note that fixed-parameter tractability parameterized by $k$ is a much stronger property than polynomial-time solvability for constant $k$: a fixed-parameter algorithm running in $O(2^k \cdot n)$ time is polynomial even for $k \in O(\log n)$. It may efficiently solve an NP-hard problem even on relatively large inputs. In contrast, an algorithm running in $O(n^k)$ time is polynomial for constant $k$, but impractical already for small input sizes.

Parameterized complexity also allows for provably effective polynomial-time data reduction. A \textit{kernelization algorithm} compresses any instance $I$ with parameter $k$ into an instance $I'$ with parameter $k'$ in polynomial time such that the size of $I'$ and $k'$ depends only on $k$ and such that $I$ is a yes-instance if and only if $I'$ is a yes-instance. The instance $I'$ is called a \textit{problem kernel}. It is well known that a decidable problem is fixed-parameter tractable parameterized by a parameter $k$ if and only if it has a problem kernel with respect to $k$. Problem kernels can form an important step in preprocessing; one thus tries to design kernels whose size is polynomial or even linear in the parameter.

To prove that a problem $\Pi$ is not fixed-parameter tractable parameterized by $k$, one shows that $\Pi$ is $W[1]$-hard or $W[2]$-hard parameterized by $k$, which is based on the widely accepted hypothesis that FPT $\neq W[1]$. In that case, one can aim to show that a problem $\Pi$ parameterized by $k$ lies in the class XP, that is, it can be decided in $n^{f(k)}$ time for some computable function $f$.

For background on fixed-parameter tractability we refer to the monograph by Cygan et al. [24].

\textbf{Approximation schemes.} Since we only consider minimization problems, we introduce approximation terminology only for minimization problems.

An $\alpha$-\textit{approximation} is a solution to an optimization problem that is at most $\alpha$ times worse than the optimum. A \textit{polynomial-time approximation scheme} (PTAS) is an algorithm that computes an $(1 + \varepsilon)$-approximation in polynomial time for any constant $\varepsilon > 0$. An \textit{efficient polynomial-time approximation scheme} (EPTAS) is an algorithm that computes an $(1 + \varepsilon)$-approximation in
time $f(1/\varepsilon) \cdot n^{O(1)}$ for an arbitrary computable function $f$ and $\varepsilon > 0$. Finally, a fully polynomial-time approximation scheme (FPTAS) is an algorithm that computes an $(1 + \varepsilon)$-approximation in time $(n + 1/\varepsilon)^{O(1)}$ for any $\varepsilon > 0$.

It is known that any problem having an EPTAS is fixed-parameter tractable parameterized by the cost of an optimal solution [20, Lemma 11]. Thus, showing $W[1]$-hardness of a problem parameterized by the optimum solution cost shows that a problem does neither have an EPTAS nor FPTAS unless FPT = W[1]. Moreover, it is known that strongly NP-hard optimization problems with polynomially bounded objective functions do not have FPTASes.

3 Single-machine problems

In this section, we study problems where each job $j$ has to be processed non-preemptively for a given amount $p_j \in \mathbb{N}$ of time (its processing time) on a single machine. The machine can process only one job at a time.

3.1 Total weighted completion time

In the problem $1|\text{prec}|\sum w_j C_j$, additionally to the processing time $p_j \in \mathbb{N}$ for each job $j$, we are given a weight $w_j \in \mathbb{N}$ for each job $j$ and a partial order on the set of jobs. A job may be started only after all of its predecessors are finished. The objective is to process each job $j$ non-preemptively for $p_j$ time, minimizing the weighted sum of completion times.

Ambühl and Mastrolilli [2] showed that $1|\text{prec}|\sum w_j C_j$ is a special case of Weighted Vertex Cover. Vertex Cover is one of the most well-studied problems in parameterized complexity theory, in particular in terms of problem kernels [26]. Therefore, it is interesting which kernelization algorithms carry over to $1|\text{prec}|\sum w_j C_j$. However, a problem kernel for $1|\text{prec}|\sum w_j C_j$ whose size is bounded by a function of the optimum is not interesting—the optimum is at least the weighted sum of all processing times and thus already bounds the input size without data reduction. Remarkably, Vertex Cover admits a (randomized) algorithm yielding a problem kernel with size polynomial in the difference between the optimum and a lower bound obtained by the relaxation of the natural ILP [47]. Since $1|\text{prec}|C_{\text{max}}$ is a special case of Weighted Vertex Cover, and thus allows for a likewise natural ILP formulation [2], the following question is interesting:

**Open Problem 1.** Does $1|\text{prec}|\sum w_j C_j$ allow for a problem kernel with size polynomial in the difference between the optimum and the lower bound obtained the relaxation of its Vertex Cover ILP?

We point out that any partial progress on this question would be interesting: be it a randomized kernelization algorithm or even a partial kernel that bounds only the number of jobs and not necessarily their weights or processing times.

The question is complicated by the fact that each vertex in the Weighted Vertex Cover instance created by Ambühl and Mastrolilli [2], and thus
each variable in the corresponding ILP, corresponds to a pair of jobs in the $1|\text{prec}|\Sigma w_j C_j$ instance, such that known data reduction rules for WEIGHTED VERTEX COVER do not allow for a straightforward interpretation in terms of jobs.

### 3.2 Throughput or weighted number of tardy jobs

In the problem $1|r_j|\Sigma w_j U_j$, additionally to a processing time $p_j \in \mathbb{N}$ for each job $j$, we are given a weight $w_j \in \mathbb{N}$, a release date $r_j \in \mathbb{N}$, and a due date $d_j \in \mathbb{N}$ for each job $j$. The goal is to process each job $jJ$ for $p_j$ time, starting no earlier than $r_j$ and minimizing the weighted number of tardy jobs.

According to Sgall [66], it is open whether there is a polynomial-time algorithm for $1|r_j,p_j\leq c|\Sigma U_j$, that is, for the unweighted variant and the processing times bounded by a constant $c$. Similarly, the NP-hardness of the weighted case for constant $c$ is open. Parameterized complexity can serve as an intermediate step towards resolving Sgall’s question:

**Open Problem 2.** Are $1|r_j|\Sigma U_j$ and $1|r_j|\Sigma w_j U_j$ W[1]-hard parameterized by the maximum processing time $p_{\text{max}}$?

Currently, even containment in the parameterized complexity class XP is open. The unweighted problem variant $1||\Sigma U_j$ without release dates is polynomial-time solvable, whereas the weighted problem $1||\Sigma w_j U_j$ without release dates is weakly NP-hard and solvable in pseudo-polynomial time [44, 54].

Fellows and McCartin [27] additionally considered the unweighted problem variant $1|\text{prec}, p_{ij}=1|\Sigma U_i$ without release dates, but where jobs have to be processed according to a partial order: a job can be started only after its predecessors have been finished. They showed that $1|\text{prec}, p_{ij}=1|\Sigma U_i$ is W[1]-hard parameterized by the number of tardy jobs, but fixed-parameter tractable if the partial order giving the precedence constraints has constant width. Herein, the width of a partial order is the size of a largest set of mutually incomparable jobs.

### 3.3 Forbidden start and end times

Machine scheduling problems with forbidden start and end times model the situation when an additional resource, subject to unavailability constraints, is required to start or finish a job. For example, this might be machine operators.

For makespan minimization on a single machine, Billaut and Sourd [13] gave an algorithm that runs in $n^{O(\tau^2)}$ time for $\tau$ forbidden start times and $n$ jobs; this was improved by Rapine and Brauner [61] to $n^{O(\tau)}$ time. For the high-multiplicity encoding of the input—given by binary numbers $n_t$ encoding the number of jobs having the same forbidden start and end times—Gabay et al. [31] show a polynomial-time algorithm if the number $\tau$ of forbidden times is constant. All of these results leave open the possibility for fixed-parameter tractability of the problem parameterized by $\tau$.

**Open Problem 3.** Is makespan minimization on a single machine with $\tau$ forbidden start and end times fixed-parameter tractable parameterized by $\tau$?
Chen et al. [22] consider the same problem with the objective of minimizing the total completion time.

4 Parallel identical machines

We now study problems where, instead of a single machine, we are given a number \( m \) of parallel identical machines (that is, of the same speed). Each job \( j \) has to be processed by exactly one machine for a given amount \( p_j \in \mathbb{N} \) of time and each machine can process only one job at a time.

4.1 Makespan

In the fundamental scheduling problem \( P||C_{\text{max}} \), the goal is to schedule \( n \) jobs with processing time \( p_j \in \mathbb{N} \) for each job \( j \) non-preemptively on \( m \) parallel identical machines, minimizing the makespan. Alon et al. [1] showed an EPTAS for \( P||C_{\text{max}} \), which implies that \( P||C_{\text{max}} \) is fixed-parameter tractable parameterized by the makespan \( C_{\text{max}} \) [20, Lemma 11].

This was improved by Mnich and Wiese [59] to a fixed-parameter algorithm for the smaller parameter \( p_{\text{max}} \), where \( p_{\text{max}} \) is the maximum processing time of any job; we generally expect \( p_{\text{max}} \ll C_{\text{max}} \). The running time of both algorithms is doubly-exponential in the parameter. An improved algorithm whose running time depends singly-exponentially on \( p_{\text{max}} \) was proposed by Knop and Koutecký [45]. Chen et al. [21] generalized the result of Mnich and Wiese [59] by showing that makespan minimization on parallel unrelated machines (each job \( j \) takes time \( p_{ji} \) on machine \( i \)) is fixed-parameter tractable parameterized by \( p_{\text{max}} \) and the rank of the processing time matrix \((p_{ji})\).

Despite all this progress, it remains unknown what is the parameterized complexity of \( P||C_{\text{max}} \) parameterized by the number \( \overline{p} \) of distinct processing times.

**Open Problem 4.** Is \( P||C_{\text{max}} \) fixed-parameter tractable parameterized by the number \( \overline{p} \) of distinct processing times?

Goemans and Rothvoß [34] show that \( P||C_{\text{max}} \) is polynomial-time solvable for constant \( \overline{p} \), that is, in XP, even when considering the succinct and natural encoding of the input when the number \( m \) of machines and the number \( n_j \) of jobs with processing time \( p_j \) are encoded in binary for each \( j \in \{1,\ldots,\overline{p}\} \). Consequently, if \( P||C_{\text{max}} \) is fixed-parameter tractable parameterized by \( \overline{p} \), then one should also try to find a fixed-parameter algorithm for this succinct encoding. One difficulty herein is outputting the schedule, whose obvious encoding requires at least \( m \) bits (to store how many jobs of each processing time are processed on each machine) and is therefore is neither polynomial in the input size nor bounded by a function of \( \overline{p} \). Such problems might be overcome in the framework of Brauner et al. [16].

4.2 Makespan and precedence constraints

In the problem \( P|\text{prec}|C_{\text{max}} \), the objective is non-preemptively scheduling \( n \) jobs with processing times \( p_j \in \mathbb{N} \) for each job \( j \) on \( m \) parallel identical machines
with minimum makespan. Additionally, a partial order imposes precedence constraints on the jobs: a job may be started only after all of its predecessors are finished.

Günther et al. [37] showed that that the special case $P2|\text{chains}|C_{\text{max}}$ with two machines is weakly NP-hard even when this partial order is the union of three total orders, that is, of three chains. This implies hardness for two machines and partial orders of width $w = 3$, where the width of a partial order is the size of its largest antichain—a set of pairwise incomparable jobs. Since this excludes fixed-parameter algorithms using the partial order width as parameter already on two machines, it has been tried to use the partial order width and the maximum processing time as parameters simultaneously.

However, Bodlaender and Fellows [15] showed that even the variant $P|\text{prec}, p_j=1|C_{\text{max}}$ with unit processing times is W[2]-hard parameterized by the partial order width and by the number of machines. Later, van Bevern et al. [11] showed that combining partial order width with maximum processing time does not even yield fixed-parameter algorithms on two machines. More precisely, they showed that even the two-machine variant $P2|\text{prec}, p_j\in\{1,2\}|C_{\text{max}}$ with processing times one and two is W[2]-hard parameterized by the partial order width. So is the three-machine variant $P3|\text{prec}, size_j\in\{1,2\}|C_{\text{max}}$ with unit processing times if each job may require up to two machines at the same time [11].

Further restricting this problem, one arrives at a long-standing open problem due to Garey and Johnson [32, OPEN8] of whether the three-machine variant $P3|\text{prec}, p_j=1|C_{\text{max}}$ with unit processing times is NP-hard or polynomial-time solvable. In fact, the complexity is open for any constant number of machines. Thus, as pointed out by van Bevern et al. [11], it would be surprising to show W[1]-hardness of this problem for any parameter, since this would exclude polynomial-time solvability unless FPT = W[1].

**Open Problem 5.** Is $P3|\text{prec}, p_j=1|C_{\text{max}}$ fixed-parameter tractable parameterized by the width $w$ of the partial order induced by the precedence constraints?

Note that a negative answer would basically answer the open question of Garey and Johnson [32] on whether the problem is polynomial-time solvable, whereas a positive answer would be in strong contrast to the W[2]-hardness of the slight generalizations $P|\text{prec}, p_j=1|C_{\text{max}}$, $P2|\text{prec}, p_j\in\{1,2\}|C_{\text{max}}$, and $P3|\text{prec}, size_j\in\{1,2\}|C_{\text{max}}$ considered by Bodlaender and Fellows [15] and van Bevern et al. [11].

### 4.3 Throughput or weighted number of tardy jobs

In the problem $P|r_j|\sum w_j U_j$, additionally to a processing time $p_j \in \mathbb{N}$ for each job $j$, we are given a weight $w_j \in \mathbb{N}$, a release date $r_j \in \mathbb{N}$, and a due date $d_j \in \mathbb{N}$ for each job $j$. The goal is to process each job $j$ non-preemptively for $p_j$ time on one of $m$ parallel identical machines, starting no earlier than $r_j$ and minimizing the weighted number of tardy jobs (or, equivalently, maximizing throughput).

Sgall [66] notes that even the problem variant $P|r_j, p_j=p|\sum w_j U_j$, where all jobs have the same processing time $p$, is far from trivial: it is known to be
Open Problem 6. Are \( P|r_j,p_j=p|\Sigma w_j U_j \) and \( P|r_j,p_j=p|U_j \) fixed-parameter tractable parameterized by the number \( m \) of machines?

A negative answer to this question would also be interesting since it is open whether these problems are even NP-hard if the number \( m \) of machines is part of the input. Notably, the special case \( P|p_j=p|\Sigma w_j U_j \) without release dates is polynomial-time solvable via the assignment problem but its variant \( P|p_j=p,\text{pttn}|\Sigma w_j U_j \) with allowed preemption is strongly NP-hard [17].

4.4 Interval scheduling

An important special case of maximizing the throughput on parallel identical machines is interval scheduling, where each job \( j \) has a weight \( w_j \), a fixed start time, and a fixed end time. The goal is to schedule a maximum-weight subset of jobs non-preemptively on \( m \) parallel identical machines. As always, each machine can process only one job at a time. Since we can interpret this problem as minimizing the total weight of jobs not meeting their due dates, where each job \( j \) has a processing time \( p_j \), a release date \( r_j \), and a due date \( d_j \) such that \( d_j - r_j = p_j \), we denote the problem as \( P|d_j - r_j = p_j|\Sigma w_j U_j \).

Arkin and Silverberg [3] show that this problem is solvable in \( O(n^2 \log n) \) time. However, they showed that the variant \( P|M_j,d_j - r_j = p_j|\Sigma w_j U_j \), where each job \( j \) can only be processed on a subset \( M_j \) of the \( m \) machines, is NP-hard and solvable in \( O(n^m+1) \) time.

Open Problem 7. Is the interval scheduling problem \( P|M_j,d_j - r_j = p_j|\Sigma w_j U_j \) fixed-parameter tractable parameterized by the number \( m \) of machines?

We point out that, if we only slightly relax the condition \( d_j - r_j = p_j \) to \( d_j - r_j \leq \lambda p_j \) for any constant \( \lambda > 0 \), then the problem is weakly NP-hard for \( m = 2 \) and strongly W[1]-hard parameterized by \( m \) [12], even when all jobs are allowed to be processed on all machines and only checking whether one can finish all jobs by their due date.

Using a construction due to Halldórsson and Karlsson [39], the problem \( P|M_j,d_j - r_j = p_j|\Sigma w_j U_j \) can be seen to be a special case of the Job Interval Selection problem introduced by Nakajima and Hakimi [60], where each job has multiple possible execution intervals and we have to select one execution interval for each job in order to process it: we model the machines by pairwise disjoint segments of the real line and each job has an interval in each segment belonging to a machine it can be processed on. Via this relation to Job Interval Selection, one can model \( P|M_j,d_j - r_j = p_j|\Sigma w_j U_j \) as Colorful Independent Set in an interval graph with at most \( mn \) intervals colored in \( n \) colors and having at most \( m \) intervals of each color [10], where the task is to find a maximum-weight independent set of intervals with mutually distinct colors.

Colorful Independent Set in interval graphs is fixed-parameter tractable parameterized by the number of colors in the solution [10], which implies that
\( P|M_j, d_j - r_j = p_j | \Sigma w_j U_j \) is fixed-parameter tractable parameterized by the number of jobs that can be scheduled in an optimal solution. To answer Open Problem 7, one could try to show that \textsc{Colorful Independent Set} in interval graphs is fixed-parameter tractable parameterized by the maximum number of intervals of any color.

4.5 Allowed preemption

We now present three problems where the parallel identical machines are allowed to preempt jobs and continue them at a later time.

In \( P|\text{pmtn}, r_j | \Sigma C_j \), each job \( j \) has a release date \( r_j \), a processing time \( p_j \) and the objective is to minimize the sum of completion times. The problem is NP-hard [25], yet its special case \( P|\text{pmtn}, r_j, p_j = p | \Sigma C_j \) with equal processing times is solvable in polynomial time [8].

In \( P|\text{pmtn}| \Sigma U_j \), each job \( j \) has a due date \( d_j \) and the objective is to minimize the number of jobs missing their due date. The problem is NP-hard [53], yet its special case \( P|\text{pmtn}, p_j = p | \Sigma U_j \) with equal processing times is solvable in polynomial time [7].

Finally, in \( P|\text{pmtn}| \Sigma T_j \), each job \( j \) has a due date \( d_j \) and the objective is to minimize the total tardiness \( \sum_{i \in J} T_j \). The problem is NP-hard [51], yet its special case \( P|\text{pmtn}, p_j = p | \Sigma T_j \) with equal processing times is solvable in polynomial time [50].

**Open Problem 8.** Are the above problems fixed-parameter tractable parameterized by the number \( \overline{p} \) of distinct processing times?

For a survey about similar problems, we refer to Kravchenko and Werner [50]. As of now, the complexity status of the weighted problem \( P2|\text{pmtn}, p_j = p | \Sigma w_j T_j \) where each job has constant processing time \( p \) is open. Herein, each job \( j \) has a weight and we minimize the weighted total tardiness. Parameterized hardness might serve as an intermediate step towards settling the complexity for fixed \( p \).

**Open Problem 9.** Is \( P2|\text{pmtn}, p_j = p | \Sigma w_j T_j \ W[1]-\text{hard parameterized by the processing time } p \text{ of all jobs?} \)

5 Shop scheduling

In shop scheduling problems, we are given a set \( M \) of machines and a set \( J \) of \( n \) jobs, where each job \( j \) consists of \( n_j \in \mathbb{N} \) operations. In the most general setting, the operations of each job are partially ordered: an operation of a job can only start once its preceding operations are completed. Processing an operation \( o_{ij} \) of job \( j \), where \( i \in \{1, \ldots, n_j\} \), requires a processing time \( p_{ji} \) on a certain machine \( \mu_{ji} \in M \). Each job can be processed by at most one machine at a time and each machine can process at most one operation at a time. The three most important classes of shop scheduling problems are open shop scheduling, job shop scheduling, and flow shop scheduling.
In open shop scheduling, the processing order of the operations of each job is free and must be decided by an algorithm. The only constraint is that each job has exactly one operation on each machine. When minimizing the makespan, the problem is denoted $O||C_{\text{max}}$.

In job shop scheduling, the operations of each job are totally ordered, each job may have a distinct total order on its operations. Herein, several operations may require processing on the same machine and not every job may have operations on all machines. When minimizing makespan, the problem is denoted as $J||C_{\text{max}}$.

Flow shop scheduling is a special case of job shop scheduling in which all jobs consist of the same set of operations with the same total order. When minimizing makespan, the problem is denoted as $F||C_{\text{max}}$.

Finally, in mixed shop scheduling, the operations of some jobs are totally ordered, and the order of operations of other jobs are free and must be determined by an algorithm.

### 5.1 Makespan

Kononov et al. [46] give a computational complexity dichotomy of $J||C_{\text{max}}$, $O||C_{\text{max}}$, and mixed variants into polynomial-time solvable cases and NP-hard cases depending on the maximum processing time $p_{\text{max}}$ of operations, the maximum number $n_{\text{max}}$ of operations per job, an upper bound on the makespan $C_{\text{max}}$, and the problem variant. All problems are NP-hard even if all of the listed parameters are simultaneously bounded from above by 4 [46, 70], which fully settles the parameterized complexity of $O||C_{\text{max}}$, $J||C_{\text{max}}$, and mixed variants with respect to these parameters.

However, in all hardness reductions of Kononov et al. [46], the number of machines is necessarily unbounded: when bounding both the number $m$ of machines and the makespan $C_{\text{max}}$, then shop scheduling problems are trivial, since the overall number of operations in the input is at most $m \cdot C_{\text{max}}$. This makes the parameters $p_{\text{max}}$ and $n_{\text{max}}$ interesting for fixed-parameter algorithms for shop scheduling problems with a fixed number of machines (or with the number of machines as an additional parameter).

The three-machine flow and open shop problems $F3||C_{\text{max}}$ and $O3||C_{\text{max}}$ are NP-hard [33, 35], but are not known to be so if the processing times are bounded from above by a constant.

### Open Problem 10

Are $O3||C_{\text{max}}$ or $F3||C_{\text{max}}$ fixed-parameter tractable parameterized by the maximum processing time $p_{\text{max}}$?

This question is likewise interesting for the three-machine no-wait flow shop problem $F3|$no-wait$|C_{\text{max}}$, where each operation of a job has to start immediately after its predecessor operation finished. The NP-hardness of $F3|$no-wait$|C_{\text{max}}$
was a long-standing open question [62]. However, for the three-machine job shop scheduling problem, the above question has a negative answer: even the special case $J3|p_{ij}=1|C_{\text{max}}$ with unit processing times is NP-hard [55].

**Open Problem 11.** Is $J3||C_{\text{max}}$ fixed-parameter tractable parameterized by the maximum processing time $p_{\text{max}}$ and the maximum number $n_{\text{max}}$ of operations per job?

5.2 Makespan with sequence-dependent setup times

When switching from processing one job to another, a sequence-dependent setup time on the machine may be required. In the following, we consider a variant $O|p_{ij}=1,bs_{pq}|C_{\text{max}}$ of open shop with unit processing times where the jobs are partitioned into $g$ batches and machines require a batch setup time $bs_{pq} \in \mathbb{N}$ when switching from jobs of batch $p$ to jobs of batch $q$. The goal is to minimize the makespan.

The case with unit processing times models applications where large batches of items have to be processed and processing individual items in each batch takes significantly less time than setting up the machine for the batch. Alternatively one can interpret this problem as an open shop problem where machines or experts have to travel between jobs in different locations in order to process them [4].

The problem $O|p_{ij}=1,bs_{pq}|C_{\text{max}}$ is NP-hard already for one machine and unit processing times as it generalizes the TRAVELING SALESPERSON problem. As shown by van Bevern and Pyatkin [9], the problem is solvable in $O(2^g g^2 + mn)$ time if each batch contains more jobs than there are machines. For the case where batches may contain less jobs, they showed that the problem is solvable in $O(n \log n)$ time if both the number $m$ of machines and the number $g$ of batches are constants.

**Open Problem 12.** Is $O|p_{ij}=1,bs_{pq}|C_{\text{max}}$ fixed-parameter tractable parameterized by the number $g$ of batches?

We point out that, for this problem, this would preferably be a fixed-parameter algorithm also for the high-multiplicity encoding, which compactly encodes in binary the number of jobs in each batch. The fixed-parameter algorithms of van Bevern and Pyatkin [9] also work in the high-multiplicity encoding as long as it is not required to output the schedule, which requires at least $mn$ space, but only to compute the minimum makespan.

5.3 Throughput or weighted number of tardy jobs

We now consider shop scheduling problems in which each job $j$ is equipped with a due date $d_j$ and a weight $w_j$ and we minimize weighted number of tardy jobs (or, equivalently, maximize throughput). Concretely, we will consider job shop scheduling $J||\Sigma w_j U_j$ and open shop scheduling $O||\Sigma w_j U_j$.

Since checking whether there is a schedule with makespan $L$ is equivalent to checking whether all jobs can meet a due date of $L$, all NP-hardness results from
the makespan minimization variants in Section 5.1 carry over to the throughput maximization variants. The throughput maximization variants turn out to be even harder: now job shop J2|p_{ij}=1,\Sigma w_jU_j with unit processing times is hard even on two machines \[49\] instead of three. If there are release dates, then even the unweighted case J2|p_{ij}=1, r_j|\Sigma U_j is NP-hard \[67\].

The situation is more relaxed for open shop scheduling: the unit-weight variant O|p_{ij}=1, r_j|\Sigma U_j with unit processing times and release dates is NP-hard for an unbounded number of machines \[48\]. The variant O|p_{ij}=1|\Sigma w_jU_j with unit processing times and without release dates is even fixed-parameter tractable parameterized by the number \(m\) of machines \[19\], whereas we only know that variant the Om|p_{ij}=1, r_j|\Sigma w_jU_j with release dates and a fixed number of machines \(m\) is polynomial-time solvable \[6\].

**Open Problem 13.** Is O|p_{ij}=1, r_j|\Sigma w_jU_j fixed-parameter tractable parameterized by the number \(m\) of machines?

It is known that the special case O|p_{ij}=1, r_j|C_{max} is polynomial-time solvable and that the problem O|p_{ij}=1|\Sigma w_jU_j is equivalent to the problem P|p_j=m, pmtnz, r_j|\Sigma w_jU_j of scheduling jobs with processing time \(m\) on \(m\) parallel identical machines with release dates and preemption allowed at integer times \[18\].

### 6 Capacitated lot sizing with few item sizes

The lot sizing problem is a planning problem in which one wishes to satisfy a time-varying demand \(d_t\) for a set of products over time periods \(t \in \{1, \ldots, T\}\). The objective is to determine periods where production will take place and the quantities that have to be produced in these periods. The total production should satisfy the demands while minimizing the total costs. In each time period \(t \in \{1, \ldots, T\}\), there are unit production costs \(p_t\) and inventory holding costs \(h_t\), which are generally functions depending on the amount of production and amount of inventory, respectively. Additionally, there are setup costs \(s_t\), which are incurred if a production process is started in period \(t\). This problem was introduced in 1958 by Wagner and Whitin \[69\] and Manne \[56\].

A significant amount of research in lot sizing is devoted to the capacitated version of the problem, which bounds the inventory size by a finite capacity \(c_t\) in each time period \(t\). This problem is NP-hard, even for a single item to be produced \[30\] and with linear production cost and holding cost functions \[14\].

On the one hand, Florian and Klein \[29\] in 1971 showed that the lot sizing problem can be solved in time \(O(T^4)\) when capacities are equal in all time periods; their algorithm was improved to \(O(T^3)\) by van Hoesel and Wagelmans \[68\].

On the other hand, for the special case of linear production cost and holding cost functions which do not depend on the time period \(t\), Baker et al. \[5\] in 1978 gave an algorithm that runs in time \(2^T \cdot |I|^{O(1)}\). Lambrecht and van der Eecken \[52\] extended the algorithm to concave cost functions. Finally, Chung et al. \[23\] gave a dynamic program for capacitated single-item lot sizing with
general cost functions, with the same asymptotic running time. Thus, the asymptotic running times of these fixed-parameter algorithms have not been improved since 1978.

**Open Problem 14.** Is there an algorithm for the capacitated single-item lot sizing problem with running time \( c^T \cdot n^{O(1)} \) for some \( c < 2 \)?

A similar question may be asked about the complexity of this problem when the number of capacities is bounded.

7 Parameterized approximation

Each EPTAS can be considered a fixed-parameter approximation algorithm that uses the solution quality \( \varepsilon \) as a parameter. Nothing prevents us from adding an additional parameter \( k \) to such an approximation algorithm in order to obtain fixed-parameter tractable approximation schemes (FPT-AS), running in \( f(k, 1/\varepsilon) \cdot n^{O(1)} \), for problems that are APX-hard [57]. While not necessarily being PTASes, an FPT-AS using the number \( m \) of machines as parameter and running in \( O(2^{m/\varepsilon} \cdot n) \) time might be practically more valuable than a PTAS running in \( O(n^{1/\varepsilon}) \) time or even an FPTAS running in \( O((n + 1/\varepsilon)^3) \) time.

Indeed, consider job and open shop scheduling problems: they are APX-hard [70] but also NP-hard for a constant number of machines [33, 35, 55] and many other constant parameters [46]. Thus, they are amenable neither to approximation schemes nor fixed-parameter algorithms. However, while having a constant number of machines does not help to get polynomial-time algorithms for these problems, it helps tremendously in getting PTASes [38, 42, 65]. Remarkably, these are not simply PTASes for a fixed number of machines running in time, say \( n^{O(m/\varepsilon)} \), but FPT-ASes for the parameters \( m \) and \( \varepsilon \):

Sevastianov and Woeginger [65] compute an \((1 + \varepsilon)\)-approximation in \( f(n, \varepsilon) + O(n \log n) \) time, Hall [38] computes an \((1 + \varepsilon)\)-approximation for \( F||C_{\text{max}} \) in \( f(m, \varepsilon) + n^{3.5} \) time, and Jansen et al. [42] compute a \((1 + \varepsilon)\)-approximation for \( J||C_{\text{max}} \) in \( f(m, m_{\text{max}}, \varepsilon) + O(n) \) time, where \( m_{\text{max}} \) is the maximum number of operations of a job. Remarkably, in all cases, the exponential part of the running time is not a factor to the input size, but only additive.

**Open Problem 15.** Is there an \((1 + \varepsilon)\)-approximation for \( J||C_{\text{max}} \) in \( f(m, \varepsilon) \cdot n^{O(1)} \) time?

Giving a negative answer to this question seems challenging: the obvious approach would be proving that \( J||C_{\text{max}} \) is \( W[1]\)-hard parameterized by \( C_{\text{max}} + m \). However, as discussed in Section 5.1, \( J||C_{\text{max}} \) is trivially fixed-parameter tractable parameterized by \( C_{\text{max}} + m \), so that this approach is inapplicable.

Finally, FPT-ASes were also known for parallel machine problems: Scharbrodt et al. [63] give an \((1 + \varepsilon)\)-approximation in \( f(m, \varepsilon) + O(n \log n) \) time for the variant of \( P||C_{\text{max}} \) where some jobs are already assigned to certain time slots on certain machines. For \( R||C_{\text{max}} \), Fishkin et al. [28] give a \((1 + \varepsilon)\)-approximation in \( O(n) + (\log m/\varepsilon)^m \) time. This result is refined by Jansen and Maack [41], who solve it in \( f(K, \varepsilon) + n^{O(1)} \), where \( K \) is the number of distinct machine types.
Acknowledgements. The first author expresses his gratitude towards Alexander Grigoriev for enlightening discussions and many helpful comments. The second author thanks Gerhard J. Woeginger for pointing out Open Problem 10.

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