The Web of Calabi–Yau Hypersurfaces
in Toric Varieties

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ABSTRACT
Recent results on duality between string theories and connectedness of their moduli spaces seem to go a long way toward establishing the uniqueness of an underlying theory. For the large class of Calabi–Yau 3-folds that can be embedded as hypersurfaces in toric varieties the proof of mathematical connectedness via singular limits is greatly simplified by using polytopes that are maximal with respect to certain single or multiple weight systems. We identify the multiple weight systems occurring in this approach. We show that all of the corresponding Calabi–Yau manifolds are connected among themselves and to the web of CICY’s. This almost completes the proof of connectedness for toric Calabi–Yau hypersurfaces.

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1. Introduction

The work on the connectedness of the moduli space of Calabi–Yau manifolds started with [1,2,3,4] where it was noted that different components of the moduli space meet along boundaries that correspond to singular manifolds. Since M.Reid’s conjecture [1] that the parameter space of 3-folds with vanishing first Chern class is connected, different steps have been taken in trying to make it a theorem. In [3,5] the connectedness of complete intersection Calabi–Yau manifolds (CICY’s) [6, 7, 8, 9] was proven. The second class of Calabi–Yau manifolds that has been constructed is made up of transverse hypersurfaces in weighted projective spaces [10, 11, 12]. The authors of [11, 12] constructed a list of 7555 weight vectors corresponding to these varieties. This class of hypersurfaces has been shown to be connected in [13, 14] by using toric geometry techniques which are similar to the ones used in this paper.

Since it was realized that to each dual pair of reflexive polyhedra of dimension four one can associate a family of Calabi–Yau manifolds embedded in the corresponding toric variety, an outstanding problem has been the explicit construction of these polytopes. Fortunately the proof of the connectedness of the moduli space of Calabi–Yau hypersurfaces does not require an explicit enumeration of all such families. The way toward a solution was paved by refs. [15, 16] which make it possible to construct a set of maximal reflexive polytopes in dimension four that contain all others. Proving that the respective families of Calabi–Yau manifolds are connected would imply the connectedness of the whole class of hypersurfaces.

In the classification program presented in [15, 16] the central role is played by the weight systems with the ‘interior point’ and the ‘span’ properties (see section 3). To each such weight system there corresponds a reflexive polyhedron, namely the associated maximal Newton polyhedron (MNP). The crucial observation of [15, 16] is that any reflexive polyhedron is a subpolyhedron of one of these or of certain MNPs defined by more than one weight system or of polyhedra that arise upon restriction of an MNP to some sublattice. In the present work we take another step towards the classification of reflexive polyhedra by identifying all possible combinations of weight systems that are relevant in this context. We will show that this set is connected; as we expect the set of polyhedra coming from sublattices to be rather small, this means that we have done most of the work necessary to show the connectedness of all toric Calabi–Yau hypersurfaces.
The correspondence between Calabi–Yau manifolds and reflexive polyhedra has been described in the work of Batyrev [17]. We consider varieties given by the zero locus of a polynomial \( p \) that contains all monomials satisfying certain constraints. The monomials are in one-to-one correspondence with the integer points of the polyhedron \( \Delta \). If \( \Delta \) has a property termed reflexivity, then there is a family \( \mathcal{M}(\Delta, \Delta^*) \) of Calabi–Yau hypersurfaces \( p = 0 \) in the toric variety \( V_{\Sigma^*} \) defined by a fan over (some triangulation of) the dual polytope \( \Delta^* \) [17].

The interplay between the analytic properties of Calabi–Yau manifolds and the geometry of reflexive polyhedra will be used to prove that the moduli space of these varieties is connected. Generalizing the concept of \( \mathcal{M}(\Delta, \Delta^*) \), we associate a family of hypersurfaces \( \mathcal{M}(\Delta_1, \Delta_2^*) \) even to non–dual pairs of reflexive polyhedra \( (\Delta_1, \Delta_2^*) \). If \( \Delta \) contains a reflexive subpolyhedron \( \delta \) then \( \mathcal{M}(\delta, \delta^*) \) is birational to the subfamily \( \mathcal{M}(\delta, \Delta^*) \) of \( \mathcal{M}(\Delta, \Delta^*) \). The moduli spaces of \( \mathcal{M}(\Delta, \Delta^*) \) and \( \mathcal{M}(\delta, \delta^*) \) overlap on the subfamily \( \mathcal{M}(\delta, \Delta^*) \) [18].

We emphasize that the singular manifolds where different regions of the moduli space touch have, in many cases, singularities different from the conifold type analyzed in [19, 20] and the physics associated with these spaces is not completely understood. The problem of determining the low energy effective theory of the Type II string compactified on an arbitrary singular variety, and describing the associated extremal transition, remains open. Current methods would demand that such an analysis be done on a case by case basis. We do not attempt this here.

In §2 we briefly review the basics of Calabi–Yau embeddings in toric varieties. After a summary of the methods and results of refs. [15, 16], we present our results on the classification of the combinations of weight systems in §3. The relevance of polyhedra being contained in one another is presented in §4 after which we illustrate the method with a two dimensional example in §5. The main steps of the computation that proves the connectedness are overviewed in §6. We present concluding remarks in §7. Two tables give information on data of some new Calabi–Yau hypersurfaces related to combinations of weight systems.

2. Calabi–Yau hypersurfaces in toric varieties

Probably the largest class of Calabi–Yau manifolds explicitly constructed up to now is represented by hypersurfaces in toric varieties. We first review some of the basic principles
underlying the interplay between reflexive polyhedra, toric varieties and the sections of the anticanonical bundle (sheaf) over these spaces.

We start with a reflexive polyhedron \( \Delta \) defined by its vertices belonging to a lattice \( M \). We recall that a reflexive polyhedron satisfies the following conditions: (i) it has integer vertices; (ii) it has only one interior point; (iii) the equation of any face of codimension 1 can be written in the form \( c_1 x^1 + \ldots + c_n x^n = 1 \), where the \( c_i \)'s are integers with no common divisor, the \( x^i \)'s are coordinates on \( M \), and \( n = \text{rank}(M) \). We define the polytope \( \Delta^* \) to be dual to \( \Delta \), i.e.

\[
\Delta^* = \{ y \in N_\mathbb{R} \mid \langle y, x \rangle \geq -1, \text{ for all } x \in \Delta \}, \tag{2.1}
\]

where \( N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R} \) is the real extension of \( N = \text{Hom}(M, \mathbb{Z}) \). Note that \( \Delta^* \) is itself reflexive.

The integer points of \( \Delta^* \cap N \) define the 1-dimensional cones \( \{ v_1, \ldots, v_N \} = \Sigma^1_{\Delta^*} \) of the fan \( \Sigma_{\Delta^*} \), which we assume to correspond to a maximal triangulation of the fan over the faces of \( \Delta^* \). The 1-dimensional cones span the vector space \( N_\mathbb{R} \) and satisfy relations of linear dependence

\[
\sum_l k^l_j v_l = 0, \quad k^l_j \geq 0. \tag{2.2}
\]

Following Cox [21], we can build a variety \( V_{\Sigma_{\Delta^*}} \) as the space \( \Phi^N \setminus Z_{\Sigma_{\Delta^*}} \) modulo the action of a group which is the product of a finite group and the torus \((\mathbb{C}^*)^N \). The action of the torus is defined by:

\[
(z_1, \ldots, z_N) \sim (\lambda^{k^1_j} z_1, \ldots, \lambda^{k^N_j} z_N), \quad j = 1, \ldots, N - n. \tag{2.3}
\]

\( Z_{\Sigma_{\Delta^*}} \) is an exceptional subset of \( \Phi^N \) defined as

\[
Z_{\Sigma_{\Delta^*}} = \bigcup_{\mathcal{I}} \{(z_1, \ldots, z_N) \mid z_i = 0 \text{ for all } i \in \mathcal{I}\}, \tag{2.4}
\]

where the union is taken over all index sets \( \mathcal{I} = \{i_1, \ldots, i_k\} \) such that \( \{v_{i_1}, \ldots, v_{i_k}\} \) do not belong to the same maximal cone in \( \Sigma_{\Delta^*} \). This depends explicitly on which triangulation of the fan over \( \Delta^* \) we have chosen. The elements of \( \Sigma^1_{\Delta^*} \) are in one-to-one correspondence with \( T \)-invariant divisors \( D_{v_i} \) on \( V_{\Sigma_{\Delta^*}} \). Knowing the embedding toric variety, \( V_{\Sigma_{\Delta^*}} \), we want to find the space of sections of the anticanonical sheaf. According to Batyrev [17], this is given
in terms of the polytope $\Delta$: The points of $\Delta \cap M$ are in one-to-one correspondence with monomials in the homogeneous coordinates $z_i$. A general polynomial determining a section of the anticanonical sheaf in $\mathcal{V}_\Sigma^*$ is given by

$$p = \sum_{x \in \Delta \cap M} c_x \prod_{l=1}^{N} z^{\langle \nu_l, x \rangle + 1}_l.$$  \hspace{1cm} (2.5)

The $c_x$ parametrize a family $\mathcal{M}_\Delta$ of Calabi-Yau hypersurfaces defined as the zero loci of $p$.

### 3. Minimal polyhedra and combinations of weight systems

Having in view that reflexive polyhedra encode properties of families of Calabi-Yau hypersurfaces in toric varieties, we now show how one might look for all inequivalent reflexive polyhedra. To this end we review and extend concepts and results from refs. [15, 16]. We recall that any polytope in $M_\mathbb{R}$ with the origin in its interior can be described by a set of inequalities $\langle n_i, x \rangle \geq -1$ with $n_i \in N_\mathbb{R}$. A complete and non-redundant description is provided if the set of $n_i$ corresponds to the set of vertices $V_i$ of the dual polytope. In particular, reflexivity of the polytope $\Delta$ implies that $V_i \in \mathbb{N}$. Of course $\Delta$ is a subset of any polyhedron defined only by a subset of the above mentioned inequalities.

The main idea of the classification program for reflexive polyhedra initiated in [15] was the introduction of so-called minimal polyhedra. These are polytopes defined by a collection of inequalities such that a polyhedron defined by any strict subset of this collection would be unbounded. In terms of the dual space this has the following meaning: if $Q$ is a minimal polyhedron (with respect to hyperplanes), then $Q^*$ is a polytope in $N_\mathbb{R}$ whose vertices are the $V_i$ involved in the description of $Q$, in such a way that the convex hull of any strict subset of these $V_i$ does not have the origin $0_N$ of $N_\mathbb{R}$ in its interior (i.e., $Q^*$ is minimal with respect to vertices). The possible shapes of these objects were classified in [15]. By considering triangulations, it is more or less easy to see that any minimal polyhedron must be the convex hull of the set of vertices of some simplices (possibly of lower dimension) which all contain $0_N$ in their interiors. The fact that subsets of such a set of simplices give rise to lower dimensional minimal polyhedra makes an iterative construction of all minimal polyhedra for rising dimensions possible. This construction was explained and applied to the cases with $n \leq 4$ in ref. [15], with the following results on the structure of $Q^*$.
In one dimension the only possibility is a line segment (‘1-simplex’) $V_1V_2$ with $0_N$ in its interior. In two dimensions there are the triangle $V_1V_2V_3$ and the parallelogram that is the convex hull of $V_1, V_2, V_1', V_2'$ such that $V_1V_2$ and $V_1'V_2'$ are 1-simplices with $0_N$ in their (1-d) interiors. The fact that simplices with $0_N$ in their interiors are the building blocks of minimal polytopes remains valid in arbitrary dimensions. Representing simplices by the numbers of their vertices, the results on the classification of minimal polyhedra can be summarized in the following way:

$n=1$: 2.

$n=2$: 3; 2+2.

$n=3$: 4; 3+2, 3+3; 2+2+2.

$n=4$: 5; 4+2, 3+3, 4+3; 4+4; 3+2+2, 3+3, 3+3+3; 2+2+2+2.

The underlining symbols indicate common vertices of simplices. For example, $3+3$ means the convex hull of $V_1, V_2, V_3, V_1', V_2'$, where $V_1V_2V_3$ and $V_1'V_2'V_3'$ are triangles with $0_N$ in their interiors, sharing the vertex $V_1$. In a similar way $4+4$ stands for two tetrahedra sharing two vertices and $3+3+3$ stands for three triangles sharing a single vertex. Denoting by $k$ the number of vertices of $Q^*$, the number of simplices in this description is always $k - n$.

Each of the simplices occurring above may be used to define a weight system $(q_i)$ that corresponds to the barycentric coordinates of $0_N$ with respect to the vertices of the simplex $\sum q_i = 1$, $\sum q_i V_i = 0_N$. These weight systems are the major tool for a convenient description of our original minimal polyhedra $Q \subset M_\mathbb{R}$. Recall that the most symmetric description of an $n$ dimensional standard simplex is as the convex hull of base vectors in $\mathbb{R}^{n+1}$ or, equivalently, the intersection of the positive hyperoctant in $\mathbb{R}^{n+1}$ with the hyperplane $\sum_{i=1}^{n+1} x^i = 1$. In a similar way, if $Q^*$ is a simplex giving rise to a weight system $(q_i)$, then $Q$ may be represented as the intersection of the positive octant in $\Gamma^{n+1}_\mathbb{R} \simeq \mathbb{R}^{n+1}$ with the hyperplane $\sum_{i=1}^{n+1} q_i x^i = 1$. For rational weights the latter equation defines an $n$ dimensional sublattice $\Gamma^n$ of $\Gamma^{n+1} \simeq \mathbb{Z}^{n+1}$. The $M$ lattice can be identified with a sublattice of $\Gamma^n$, and $0_M$ corresponds to $(1, \cdots, 1) =: 1$.

A similar prescription also works for minimal polyhedra that are not simplices. If $Q^*$ has $k > n + 1$ vertices, we may represent $Q$ as the intersection of the positive octant in

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Γ^n_k \simeq \mathbb{R}^k$ with $k - n$ hyperplanes of the type $\sum_{i=1}^k q_i x_i = 1$. For example, if $Q^*$ is of the type $3 + 3$, the 4-d polytope $Q$ is given by the positive octant in $\mathbb{R}^6$ intersected with two hyperplanes determined by the weight systems $(q_1, q_2, q_3, 0, 0, 0)$ and $(0, 0, 0, q'_1, q'_2, q'_3)$; if $Q^*$ is of the type $3 + 3\bar{3}$, the 3-d polytope $Q$ is given by the positive octant in $\mathbb{R}^5$ intersected with two hyperplanes determined by the weight systems $(q_1, q_2, q_3, 0, 0)$ and $(q'_1, 0, 0, q'_2, q'_3)$. Note that if the simplices in $Q^*$ have no vertex in common, then $Q$ is just the product of the simplices defined by the weight systems.

Given a weight system or a combination of weight systems, we call the convex hull of $Q \cap M$ the maximal Newton polyhedron $\Delta_{\text{max}}$ corresponding to the (combination of) weight system(s). The weight systems that play a role for the construction of reflexive polyhedra are just those with the property that their maximal Newton polyhedra (with respect to $M = \Gamma^n$) have 1 in their interiors. It is easy to see that a combination of weight systems can have this property only if each of the weight systems has it. If the simplices in $Q^*$ have no common points, the converse is also easily seen to be true. The weight systems with up to 5 weights which have this property have been classified in [16]. In the same paper it was also shown that for dimension $n \leq 4$ any maximal Newton polyhedron (whether with respect to a single weight system or a combination) with the interior point property is actually reflexive. Yet we do not need all of these weight systems for the classification program. Remembering that we assumed the $V_i$ to be vertices of $\Delta^*$, we see that the hyperplanes in $M\mathbb{R}$ dual to the $V_i$ should carry facets of $\Delta$. This can only happen if these hyperplanes are affinely spanned by points of $\Delta_{\text{max}}$. Again, for a combination of weight systems to have this property, it is a necessary condition that each of the systems involved has it; for direct products it is also sufficient.

Let us briefly summarize the existing classification methods and results on weight systems whose maximal Newton polyhedra have 1 in their interiors. The basic idea of the scheme of [16] is to reconstruct a weight system from the integer points it allows. Given a set $S = \{x_{(j)}\}$ of points in $\mathbb{Z}^{n+1}_{\geq 0}$ containing 1, it is relatively easy to find out whether this set allows any weight system $\tilde{(q_i)}$ with $\sum_{i=1}^{n+1} \tilde{q}_i x_{(j)}^i = 1$ for all $j$ and, if it exists, to find such a system. If $\tilde{(q_i)}$ has the interior point property, we can add it to our list of allowed systems. If we assume that there is another system $\tilde{(q_i)}$ compatible with all $x_{(j)} \in S$, then we may see the equation $\sum_{i=1}^{n+1} \tilde{q}_i x^i = 1$ as the equation of a hyperplane through 1 in the space $\Gamma^n$. 

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defined by \((q_i)\). For \(\Delta_{\text{max}}\) to have \(1\) in the interior, there must be a point ‘below’ this hyperplane, i.e. a point \(x_{\text{new}}\) fulfilling \(\sum_{i=1}^{n+1} \tilde{q}_i x_{\text{new}}^i < 1\). The number of such points is finite, so we may apply the same considerations to all of the sets \(S' = S \cup \{x_{\text{new}}\}\). Taking \(S = \{1\}\) as our starting point, it is possible to generate in this way all allowed weight systems because the new points are always affinely independent of the others (in \(\mathbb{Z}^{n+1}\)), implying that there is no need to go further than to the \(n^{\text{th}}\) recursion level in the algorithm.

Applying this algorithm to the cases with \(n + 1 \leq 5\) yields the following results: There is one interior point property system with 2 weights, namely \((1/2, 1/2)\). There are three such systems with 3 weights: \((1/3, 1/3, 1/3)\), \((1/2, 1/4, 1/4)\) and \((1/2, 1/3, 1/6)\). All of these systems have the span property. With four weights there are 95 such systems, 58 of them having the span property, and with five weights there are 184,026 systems among which 38,730 have the span property.

There are still three more steps to be taken for a complete classification of all reflexive polyhedra:

1. The identification of all combinations of weight systems that have both the interior point and the span property.
2. The identification of allowed sublattices such that the above properties are preserved.
3. The enumeration of all reflexive subpolyhedra of the maximal polyhedra.

Step (1) has been taken now. For direct products it is trivial because they have the interior point and the span property if and only if every factor has both properties. For other types of combinations we applied the following scheme: We first took all combinations of spanning weight systems. To write a computer program that creates all inequivalent combinations while at the same time avoiding any redundancy is an easy exercise in elementary combinatorics. Then we checked the combined system for the span property, using software developed for [16]. In a last step we then checked for the reflexivity of the maximal Newton polyhedron (which, by the theorem of [16], is equivalent to the interior point property). To this end we used software developed for [13] and independently checked the results with a different package [22].

We obtained the following results: For \(n = 3\) we got, in addition to the 58 single weight systems with 4 weights, the obvious 3 combinations of the type \(3+2\), 17 combinations of
the type 3+3 and of course the single combination 2+2+2. Altogether we have obtained 79 polyhedra in three dimensions which contain all reflexive 3d polyhedra. For \( n = 4 \) the numbers are:

\[
\begin{align*}
5: & \quad 38730; \quad 4+2: \quad 58, \quad 3+3: \quad 6, \quad 4+3: \quad 727, \quad 4+4: \quad 6365; \\
3+2+2: & \quad 3, \quad 3+3+2: \quad 17, \quad 3+3+3: \quad 36; \quad 2+2+2+2: \quad 1.
\end{align*}
\]

The total number of four dimensional polyhedra obtained in this way is 45,943. In table 1 we list the 426 new spectra that multiple weight systems yield in addition to the 10238 spectra\(^1\) that we obtained for single weight systems [23]. In table 2 we detail the results for the various types of combinations and give, in the last two columns, the number of new spectra (#new\(^{'}\)) and new mirror pairs of spectra (#sym\(^{'}\)) when also the orbifold results in weighted \( \mathbb{P}^4 \) are taken into account. Altogether we thus obtained 100 mirror pairs of spectra for which no reflexive polytopes were previously known.

Step (2) requires more work and its results will be reported elsewhere. It is likely that most if not all polyhedra defined by sublattices are subpolyhedra of the ones defined by the original lattices so we expect very few cases relevant to the connectedness proof are left out at this stage. Step (3) is not necessary for the present work which is why the number of polytopes that had to be effectively connected is manageabley small.

4. Connecting different regions of the moduli space

As mentioned in the introduction the final goal is to show that the moduli space of Calabi–Yau hypersurfaces (CYH) of dimension three in toric varieties is connected. Because of the one-to-one correspondence between subfamilies of CYH (with defined Hodge numbers) \( \mathcal{M} \) and pairs of reflexive polyhedra \((\Delta, \Delta^*)\), the task of showing that \( \mathcal{M}_1 \) is connected to \( \mathcal{M}_2 \) is reduced to that of showing there are \( \{\Delta_a, \Delta_b, \ldots, \Delta_z\} \) such that

\[
\Delta_1 \supseteq \Delta_a \supseteq \Delta_b \supseteq \cdots \Delta_z \supseteq \Delta_2, \tag{4.1}
\]

where \( \Delta_a \supseteq \Delta_b \) means that either \( \Delta_a \subset \Delta_b \) or \( \Delta_a \supset \Delta_b \). Let us examine the meaning of (4.1). Assume that \( \Delta_a \subset \Delta_b \). Then we have \( \Delta_a^* \supset \Delta_b^* \) and the fan \( \Sigma_{\Delta_a^*} \) has more

\(^1\) These can be found at [http://tph.tuwien.ac.at/~kreuzer/CY](http://tph.tuwien.ac.at/~kreuzer/CY)
1-dimensional cones (collectively labeled by $\Sigma_{a_1}^1$) than $\Sigma_{b_1}^1$. Then $\mathcal{V}_{\Sigma_{b_1}^1}$ can be deformed into $\mathcal{V}_{\Sigma_{b_1}^1}$ by refining $\Sigma_{b_1}^1$, that is, by “blowing up” $\mathcal{V}_{\Sigma_{b_1}^1}$. Hence we have a proper birational morphism of toric varieties $\mathcal{V}_{\Sigma_{a_1}^1} \rightarrow \mathcal{V}_{\Sigma_{b_1}^1}$ which is biholomorphic everywhere but on the exceptional sets associated with the divisors in $\Sigma_{a_1}^1 \setminus \Sigma_{b_1}^1$. The divisors introduced in $\mathcal{V}_{\Sigma_{b_1}^1}$ by this procedure will sometimes also intersect the family of Calabi–Yau hypersurfaces $\mathcal{M}_b$.

Returning to (4.1) assume for simplicity that a segment of the chain looks like this:

$$\Delta_b \supset \Delta_a \subset \Delta_c$$

(4.2)

This tells us that there are families of possibly singular hypersurfaces in both $\mathcal{V}_{\Sigma_{a_1}^1}$ and $\mathcal{V}_{\Sigma_{b_1}^1}$, $\mathcal{M}_b^\sharp$ and $\mathcal{M}_c^\sharp$ respectively, that can be blown up to become isomorphic to $\mathcal{M}_a$. It is conceivable that the complex structure deformations of $\mathcal{M}_b$ ($\mathcal{M}_c$) will result in a smooth manifold. In such a case we won’t deal with an extremal transition [24,25]. We do not change topologies. The $T$-divisors in $\Sigma_{a_1}^1 \setminus \Sigma_{b(c)}^1$ do not intersect the family $\mathcal{M}_{b(c)}^\sharp$ which is thus isomorphic with $\mathcal{M}_a$. We may express this as a diagram:

\[
\begin{align*}
\mathcal{M}_b : & \quad (\Delta_b, \Delta_b^*) \\
\mathcal{M}_b^\sharp : & \quad (\Delta_a, \Delta_b^*) \\
\mathcal{M}_a : & \quad (\Delta_a, \Delta_a^*) \\
\mathcal{M}_c : & \quad (\Delta_c, \Delta_c^*) \\
\mathcal{M}_c^\sharp : & \quad (\Delta_c, \Delta_c^*)
\end{align*}
\]

(4.3)

5. A lower dimensional example

We want to illustrate the theory with a 2-dimensional example\(^1\). We hope that by its simplicity it will make the procedure of the previous section more understandable.

\(^1\) A 4-d example was discussed in [13], mostly from the complex structure point of view.
Consider the triplet of reflexive polyhedra with their respective fans represented by dashed lines in Figure 1.

![Figure 1: Three pairs of reflexive polyhedra and the associated fans.](image)

By the methods of Section 2 we see that $\Delta_1^*$ corresponds to $\mathbb{P}^2_{[1,1,1]}$, while the general polynomial that describes a CYH is

$$p_1 = x_1^3 + x_2^3 + x_3^3 + x_1x_2x_3 + \cdots$$

$\Delta_2^*$ has 7 points which determine 6 1-dimensional cones in $\Sigma_{\Delta_2^*}^1$:

$$\{(1,0), (-1,0), (0,1), (-1,-1), (-1,1), (-1,2)\}$$

which we label $v_1, \ldots, v_6$. The general polynomial will be expressed in terms of 6 variables and will contain 7 monomials

$$p_2 = z_1^2z_3 + z_2^2z_3z_4^2z_5^2z_6^2 + z_1z_2z_3^2z_5^2z_6^3 + \cdots$$

The $(\mathbb{C}^*)^4$-action is read from the relations of linear dependence

$$v_1 + v_2 = 2v_1 + v_4 + v_5 = v_1 + v_3 + v_4 = 3v_1 + 2v_4 + v_6 = 0$$
between elements of $\Sigma_{\Delta_2}$ and is given by

$$(z_1, z_2, z_3, z_4, z_5, z_6) \rightarrow (\lambda \mu ^2 \nu \rho ^3 z_1, \lambda z_2, \nu z_3, \mu \nu \rho ^2 z_4, \mu z_5, \rho z_6).$$

We can define the following birational map between $\mathbb{P}^2_{[1,2,3]}$ and $\mathcal{V}_{\Sigma_{\Delta_2}}$

\begin{align*}
  z_1^2 z_3 &= y_4^2 \\
  z_2^2 z_4 z_5 &= y_2^3 \\
  z_2 z_3 z_4 z_5 z_6 &= y_1^6
\end{align*}

The same map is also birational between hypersurfaces in $\mathcal{V}_{\Sigma_{\Delta_2}}$ given by $p_2 = 0$ and hypersurfaces in $\mathbb{P}^2_{[1,2,3]}$ given by the zeroes of

$$p_2^* = y_1^6 + y_2^3 + y_3^2 + y_1 y_2 y_3 + \cdots$$

This is a dimensionally reduced example of a Calabi–Yau embedded in a toric variety described by a single weight vector: $(1, 2, 3)$.

Before going any further we warn the reader that in our 2-dimensional example no singularities will be encountered by specializing the defining polynomial. Singularities may be encountered in dimension 3 and higher. Of course all 1-dimensional Calabi–Yau manifolds are tori and all 2-d Calabi–Yau manifolds are K3 surfaces, so a transition can involve a change in Hodge numbers only if the ambient space has a dimension of 4 or more.

We can specialize the polynomial $p_1$ of the polyhedron $\Delta_1$ to $p_1^*$ by dropping the monomials $x_1^3, x_1^2 x_2, x_1 x_2^2$. For $p_1^* = 0$ to become isomorphic with $p_2 = 0$ in $\mathcal{V}_{\Sigma_{\Delta_2}}$ we have to blow up $\mathcal{V}_{\Sigma_{\Delta_1}}$.

![Figure 2: $\Sigma_2$ is a refinement of $\Sigma_1$.](image)

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Only one patch of \( \mathbb{P}^{2}_{[1,1,1]} \) will be affected in the process. The blow up in this case may be done “one divisor at a time” since there are reflexive polyhedra interpolating between \( \Delta_1 \) and \( \Delta_2 \) at each step.

![Diagram](image.png)

**Figure 3:** The successive blow-up of one of the smooth patches of \( \mathbb{P}^{2} \)

Since \( \Delta_2 \) is also contained in \( \Delta_3 \), by the same reasoning we show that the families \( M_{\Delta_2} \) and \( M_{\Delta_3} \) are connected. There is thus a continuous path between \( M_{\Delta_1} \) and \( M_{\Delta_3} \).

\[
M_{\Delta_1} \rightarrow M^x_{\Delta_1} \rightarrow M_{\Delta_2} \rightarrow M^x_{\Delta_3} \rightarrow M_{\Delta_3}
\]

6. **The Computation**

As we have seen in the previous Sections, the moduli spaces of two Calabi–Yau varieties defined by reflexive polyhedra \( \Delta_1 \) and \( \Delta_2 \) are connected if \( \Delta_1 \subset \Delta_2 \) or \( \Delta_2 \subset \Delta_1 \). By abuse of language, we will henceforth speak of “polyhedra being connected” instead of “moduli spaces of Calabi–Yau varieties defined by reflexive polyhedra being connected”. More generally, \( \Delta_1 \) and \( \Delta_k \) are called connected if there are reflexive polyhedra \( \Delta_2, \cdots, \Delta_{k-1} \) such that \( \Delta_i \) and \( \Delta_{i+1} \) are connected in the above sense for \( i = 1, \cdots, k - 1 \). The following statement, which is an immediate consequence of the results of [15,16] reviewed in Section 3, is the central point in our scheme for showing the connectedness of the moduli space.

*As any reflexive polyhedron is a subpolyhedron of some maximal Newton polyhedron defined by a weight system or combination of weight systems with the interior point and span property, perhaps with respect to some sublattice, and as these are always reflexive*
for \( n \leq 4 \), showing the connectedness of these maximal Newton polyhedra is sufficient for showing the connectedness of all reflexive polyhedra.

In the present work we provide a big step towards this goal by showing the connectedness of all maximal Newton polyhedra with respect to the maximal lattice \( \Gamma^n \), leaving the classification of sublattices and the proof of their connectedness to future work.

For \( n = 2 \) we can establish the connectedness of all reflexive polyhedra (even those obtained from sublattices) by noting that the set of maximal Newton polyhedra corresponding to a single weight system is just given by \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) of Fig. 1. In addition there is one MNP \( \Delta_4 \) coming from the only combination of weight systems possible for \( n = 2 \). It can be represented as the square \( |x_1| \leq 1, |x_2| \leq 1 \) in \( M_{\mathbb{R}} \) with \( M \simeq \mathbb{Z}^2 \). \( \Delta_1 \) and \( \Delta_3 \) allow for sublattices (w.r.t. \( \mathbb{Z}_3 \) and \( \mathbb{Z}_2 \), respectively). The reductions of these polyhedra are isomorphic to \( \Delta_1^* \) and \( \Delta_3^* \). The square \( \Delta_4 \) also admits a sublattice \( \mathbb{Z}_2 \), given by \( x_1 = x_2 \) mod 2, and again the reduction to the sublattice is isomorphic to the dual (\( \Delta_4^* \), in this case). Perhaps the simplest way of establishing connectedness is by noticing that any of our maximal polyhedra except \( \Delta_4^* \) contains either \( \Delta_1^* \) or \( \Delta_3^* \) and that these two polygons are both contained in \( \Delta_2 \simeq \Delta_2^* \) while \( \Delta_4^* \) is contained in most of the other maximal polyhedra.

For \( n = 3 \), the maximal Newton polyhedra corresponding to the types 3+2 and 2+2+2 are just prisms of height two over the maximal Newton polyhedra for \( n = 2 \), so their connectedness is established by the connectedness for \( n = 2 \). In order to show that they belong to the “web of \( n = 3 \) polyhedra”, we only have to show that one of them is connected with one of the other polyhedra. Showing the connectedness of the 58 polyhedra defined by a single weight system and the 17 polyhedra of type 3+3 of course requires more work and has to be done by computer. While it is highly probable that K3 hypersurfaces are connected at the level of polyhedra, we did not attempt to prove this because the connectedness of the whole class of K3 hypersurfaces follows trivially from the fact that this class contains only a single family, anyway.

In the 4-dimensional case which we are most interested in, we can again distinguish between maximal Newton polyhedra that have a direct product structure of the type 3+2+2 and 3+3, and therefore inherit the connectivity of two dimensional polyhedra, and the rest
of them which are of the types 4+2, 3+3+2, 2+2+2+2, 5, 4+3, 4+4 and 3+3+3. These last classes are too large to be dealt with manually, so we tackled the problem by computer.

The computer methods we are using are similar to the ones used in [13]. For the sake of completeness we will repeat part of the argument here and point out the differences. One possible approach is to try to identify all reflexive subpolyhedra (RSP) of each reflexive polyhedron (RP) that we try to connect. Since many of the RP’s have over 200 points this decomposition is computationally prohibitive.

Another approach is to limit the search to a certain subset of RSP and see if there is one contained in all the RP’s. This method also fails since there is no such magical RSP. To see this consider the minimal reflexive simplices that contain only 6 points. Since they cannot contain anything else they have to play the role of the magical RSP. There are nevertheless at least 3 minimal simplices in 4D that are inequivalent under the action of $GL(4,\mathbb{Z})$. Of course it is not surprising that there is no single RSP contained in every RP for $n = 4$. As we saw before, even in the much simpler case of $n = 2$ we needed the three RSP $\Delta^*_1$, $\Delta^*_3$ and $\Delta^*_4$ for showing connectedness.

For our proof we settled on the following strategy: identify as many 5-vertex irreducible RSP’s as possible. A 5-vertex irreducible polyhedron is a simplex that does not contain any other reflexive simplex. A set of 41 5-vertex irreducible simplices was generated in [13], ranging in size from 6 to 26 points. The list of 41 objects carries most of the burden of the connectedness proof: the great majority of MNP contain at least one member of the list and the 41 5-vertex irreducible simplices have already been proven to be connected [13]. Instead of looking at 6 or 7-vertex irreducible objects as in [13] we dealt with the remaining polyhedra in the following two ways.

First note that $\Delta_1 \subset \Delta_2$ implies $\Delta^*_2 \subset \Delta^*_1$. Suppose we want to show that $\Delta$ is connected to one of the 41 5-vertex irreducible simplices, despite the fact that it doesn’t contain any of them. The procedure is as follows. We form the dual of $\Delta$, $\Delta^*$, and search for a 5-vertex irreducible simplex. If it contains one, say $\Delta^*_i$, it follows that

$$\left(\Delta^*_i\right)^* \supset \Delta.$$  \hspace{1cm} (6.1)

Finally, we need to search $(\Delta^*_i)^*$ for a 5-vertex irreducible simplex. If we find one, then $(\Delta^*_i)^*$ is connected to all of the others, and by (6.1), so is $\Delta$. Applying this procedure to
the remaining 305 MNP (235 of type 5, 68 of type 4+4 and 2 of the type 3+3) we succeeded in connecting all but two of them.

These last two, both of type 5, were treated by literally chopping them into smaller pieces. We did this by searching for hyperplanes that slice the polyhedra into two parts, and then checking the one that contains the interior point for reflexivity. In this way, one of the two troublesome polyhedra was reduced to a previously connected polyhedron (by the method described in the previous paragraph). The other was reduced to a polyhedron that while not previously encountered, could be connected by the same method.

7. Concluding remarks

The moduli space of Calabi–Yau manifolds is naturally divided into parts $\mathcal{M}$ where each manifold in the corresponding family can be obtained from any other manifold in the same family by smooth variations of the complex and Kähler structures. From the mathematical point of view the issue is to show that all of these families are connected if boundary points of these distinct moduli spaces, which correspond to singular varieties, are included. Eventually it has to be checked that the physics of the involved singular transitions is smooth for the various string theories that can be compactified on the manifolds under consideration.

The families we were studying are the ones that refer to Calabi–Yau hypersurfaces in toric varieties. In this case each $\mathcal{M}$ contains one or more subspaces of the type $\mathcal{M}(\Delta, \Delta^*)$. We have established that all maximal polytopes defined by combined weight systems with respect to the canonical (maximal) lattices, and therefore all their subpolytopes, are connected. What still needs to be done in order to establish (mathematical) connectedness of all toric Calabi–Yau manifolds is to check connectedness of the maximal spanning polytopes that live on sublattices. We expect the number of new polytopes that arise in this way to be rather small, but it requires a considerable effort to construct all of them and we leave this task for future work.

A particular moduli space $\mathcal{M}$ may contain several "toric" submoduli spaces $\mathcal{M}(\Delta_i, \Delta_i^*)$ which will partly span different dimensions of $\mathcal{M}$.\footnote{Even though the correct dimensions of the homology groups can be calculated from the}
(not necessarily extremal) where in fact we have just changed from one toric description to another. This is always the case for one and two dimensional Calabi–Yau spaces (tori and K3 surfaces, respectively), but it may also happen for threefolds.

Assuming there is full equivalence between $IIA[M(\Delta, \Delta^*)]$ and $IIB[M(\Delta^*, \Delta)]$ string theories [19, 26, 27] we will in the following consider what happens when the connecting point is approached by blow-downs in the ambient space. Going from a family described by a polyhedron $\Delta_1$ to a family described by a subpolyhedron $\Delta_2 \subset \Delta_1$, we have to blow down one or more divisors in the ambient space $V_{\Delta_1}$. Depending on the geometries of $\Delta_1$ and $\Delta_2$, such a divisor may be blown down to a surface, a curve, or a point. Comparing this with general statements about boundaries of Kähler moduli spaces for Calabi–Yau threefolds [28, 29], we see that the blow–down of a single T–divisor in the ambient space can have the following effects on the threefold:

i) the family of hypersurfaces is unaffected.

ii) a 2-parameter family of 2-cycles shrinks to a curve of singularities.

iii) a 4-cycle shrinks to a point.

Let us start with discussing case (i). There are two essentially different ways in which a divisor in the ambient space may be blown down without affecting the hypersurface. One case is that the divisor does not intersect the hypersurface anyway. This happens when the divisor corresponds to a lattice point in the interior of a facet of $\Delta_1$. Since we are interested only in transitions induced by changes in the shapes of the polyhedra, this case will never characterize by itself a 3-fold transition. The other possibility is that a divisor is blown down to a surface that is precisely the intersection of the original divisor with the Calabi–Yau hypersurface. This is a higher dimensional analog of what happens in our example of section 5. Such a case was discussed in [30].

In the other two cases divisors of the Calabi–Yau hypersurfaces are blown down. As the canonical class will be affected in the process, we cannot go from one smooth Calabi–Yau manifold to another by this procedure.\footnote{3 We thank S. Katz for clarifying discussions on this issue.} When the transition involves a singular variety, combinatorial data, the lattice points in $(\Delta, \Delta^*)$ give us control on only part of the moduli space.
this is reached from several different subregions by either moving to the boundary of the complex structure or the Kähler class submoduli space. In these cases we deal with extremal transitions. Both singularities described by cases (ii) and (iii) occur at finite distance in the moduli space [31]. In case (ii) we expect to experience an enhanced nonabelian gauge symmetry [24, 32, 33, 34, 35, 36] if and when the singularities have appropriate properties while in case (iii) a generalization of the Argyres-Douglas phenomenon is possible with the appearance of dyonic massless hypermultiplets [37, 14].

In the majority of transitions more than one T-divisor has to be blown down to realize the transition. The superposition of the effects of individual blow-downs may lead in different situations to an interplay between cases (ii) and (iii). Moreover the curves of singularities might either develop their own singularities or may be shrunk to points by ulterior blow-downs, or entirely new and unexpected phenomena might occur.

We end by pointing out that the connectedness of the moduli space of 3-folds may have higher dimensional implications. It was suggested in [38] that by a degeneration of the fibers, Calabi–Yau 4-folds that are fibrations with fiber a Calabi–Yau 3-fold may be proven connected. This result is certainly true for the direct product spaces of the type $CY_4 = CY_3 \times P^1$ where extremal transitions between fibers are not constrained due to the double pyramidal shape of the polyhedron.

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Table 1: The 426 new spectra obtained from multiple weight systems

| $\chi$ | $h_{11}$ | $h_{21}$ |
|-------|---------|---------|
| -384  | 5       | 197     |
| -336  | 6       | 174     |
| -336  | 7       | 175     |
| -300  | 6       | 156     |
| -300  | 8       | 158     |
| -292  | 8       | 154     |
| -288  | 6       | 150     |
| -284  | 10      | 152     |
| -276  | 4       | 142     |
| -264  | 7       | 139     |
| -264  | 10      | 142     |
| -258  | 11      | 140     |
| -256  | 5       | 133     |
| -252  | 9       | 135     |
| -248  | 9       | 133     |
| -244  | 8       | 130     |
| -244  | 12      | 134     |
| -240  | 5       | 125     |
| -240  | 8       | 128     |
| -234  | 12      | 129     |
| -228  | 4       | 118     |
| -224  | 3       | 115     |
| -224  | 4       | 116     |
| -224  | 6       | 118     |
| -224  | 13      | 125     |
| -222  | 5       | 116     |
| -220  | 11      | 121     |
| -216  | 3       | 111     |
| -216  | 7       | 115     |
| -216  | 16      | 124     |
| -210  | 11      | 116     |
| -208  | 3       | 107     |
| -208  | 5       | 109     |
| -204  | 10      | 112     |
| -198  | 5       | 104     |
| -196  | 12      | 110     |
| -192  | 4       | 100     |
| -188  | 5       | 99      |
| -188  | 9       | 103     |
| -186  | 13      | 106     |

| $\chi$ | $h_{11}$ | $h_{21}$ |
|-------|---------|---------|
| -186  | 16      | 109     |
| -184  | 3       | 95      |
| -184  | 8       | 100     |
| -180  | 13      | 103     |
| -176  | 5       | 93      |
| -176  | 12      | 100     |
| -176  | 21      | 109     |
| -174  | 13      | 100     |
| -174  | 15      | 102     |
| -172  | 3       | 89      |
| -172  | 17      | 103     |
| -168  | 4       | 88      |
| -166  | 10      | 93      |
| -164  | 5       | 87      |
| -164  | 12      | 94      |
| -162  | 2       | 83      |
| -162  | 6       | 87      |
| -162  | 9       | 90      |
| -162  | 12      | 93      |
| -162  | 17      | 98      |
| -160  | 3       | 83      |
| -160  | 10      | 90      |
| -158  | 5       | 84      |
| -158  | 8       | 87      |
| -156  | 3       | 81      |
| -156  | 4       | 82      |
| -156  | 13      | 91      |
| -152  | 5       | 81      |
| -152  | 8       | 84      |
| -150  | 5       | 83      |
| -150  | 8       | 83      |
| -150  | 16      | 91      |
| -150  | 20      | 95      |
| -148  | 12      | 86      |
| -148  | 14      | 88      |
| -148  | 19      | 93      |
| -146  | 26      | 99      |
| -144  | 4       | 76      |
| -142  | 5       | 76      |
| -142  | 7       | 78      |

| $\chi$ | $h_{11}$ | $h_{21}$ |
|-------|---------|---------|
| -142  | 10      | 81      |
| -140  | 5       | 75      |
| -140  | 8       | 78      |
| -140  | 17      | 87      |
| -136  | 11      | 79      |
| -136  | 18      | 86      |
| -134  | 8       | 75      |
| -132  | 8       | 74      |
| -132  | 9       | 75      |
| -132  | 13      | 79      |
| -128  | 4       | 68      |
| -128  | 5       | 69      |
| -126  | 7       | 70      |
| -126  | 9       | 72      |
| -126  | 16      | 79      |
| -126  | 17      | 80      |
| -126  | 21      | 84      |
| -126  | 22      | 85      |
| -124  | 8       | 70      |
| -124  | 10      | 72      |
| -124  | 12      | 74      |
| -122  | 11      | 72      |
| -122  | 12      | 73      |
| -120  | 3       | 63      |
| -120  | 7       | 67      |
| -118  | 5       | 64      |
| -118  | 6       | 65      |
| -118  | 7       | 66      |
| -118  | 10      | 69      |
| -118  | 11      | 70      |
| -118  | 12      | 71      |
| -116  | 8       | 66      |
| -116  | 14      | 72      |
| -116  | 16      | 74      |
| -114  | 13      | 70      |
| -114  | 17      | 74      |
| -114  | 19      | 76      |
| -114  | 21      | 78      |
| -112  | 5       | 61      |
| -112  | 8       | 64      |
| -112  | 14      | 70      |
| -110  | 8       | 63      |
| -110  | 15      | 70      |
| -108  | 5       | 59      |
| -106  | 12      | 65      |
| -106  | 13      | 66      |
| -106  | 14      | 67      |
| -106  | 15      | 68      |
| -106  | 18      | 71      |
| -106  | 28      | 81      |
| -104  | 6       | 58      |
| -104  | 9       | 61      |
| -104  | 12      | 64      |
| -104  | 15      | 67      |
| -102  | 9       | 60      |
| -102  | 14      | 65      |
| -102  | 18      | 69      |
| -102  | 20      | 71      |
| -102  | 42      | 93      |
| -98   | 6       | 55      |
| -98   | 11      | 60      |
| -98   | 13      | 62      |
| -96   | 4       | 52      |
| -94   | 7       | 54      |
| -94   | 10      | 57      |
| -94   | 12      | 59      |
| -94   | 15      | 62      |
| -92   | 10      | 56      |
| -92   | 14      | 60      |
| -92   | 15      | 61      |
| -92   | 19      | 65      |
| -90   | 10      | 55      |
| -90   | 19      | 64      |
| -90   | 20      | 65      |
| -90   | 23      | 68      |
| -88   | 6       | 50      |
| -88   | 7       | 51      |
| -88   | 8       | 52      |
| -88   | 15      | 59      |
| \( \chi \) | \( h_{11} \) | \( h_{21} \) |
|-------|-------|-------|
| -86   | 14    | 57    |
| -86   | 16    | 59    |
| -86   | 19    | 62    |
| -84   | 7     | 49    |
| -82   | 12    | 53    |
| -82   | 15    | 56    |
| -80   | 7     | 47    |
| -80   | 12    | 52    |
| -80   | 13    | 53    |
| -78   | 6     | 45    |
| -78   | 7     | 46    |
| -78   | 8     | 47    |
| -78   | 9     | 48    |
| -78   | 10    | 49    |
| -78   | 12    | 51    |
| -78   | 13    | 52    |
| -78   | 15    | 54    |
| -78   | 16    | 55    |
| -78   | 19    | 58    |
| -78   | 21    | 60    |
| -78   | 25    | 64    |
| -76   | 8     | 46    |
| -76   | 14    | 52    |
| -76   | 15    | 53    |
| -76   | 24    | 62    |
| -76   | 27    | 65    |
| -74   | 9     | 46    |
| -74   | 11    | 48    |
| -74   | 12    | 49    |
| -74   | 14    | 51    |
| -74   | 17    | 54    |
| -74   | 21    | 58    |
| -70   | 13    | 48    |
| -70   | 15    | 50    |
| -70   | 18    | 53    |
| -70   | 20    | 55    |
| -68   | 6     | 40    |
| -68   | 8     | 42    |
| -68   | 11    | 45    |
| -68   | 13    | 47    |
| -68   | 14    | 48    |
| -68   | 15    | 49    |
| -68   | 17    | 51    |
| -68   | 30    | 64    |
| -66   | 8     | 41    |
| -66   | 12    | 45    |
| -66   | 13    | 46    |
| -66   | 15    | 48    |
| -66   | 19    | 52    |
| -66   | 22    | 55    |
| -66   | 30    | 63    |
| -66   | 43    | 76    |
| -64   | 12    | 44    |
| -64   | 18    | 50    |
| -62   | 11    | 42    |
| -62   | 12    | 43    |
| -62   | 14    | 45    |
| -62   | 15    | 46    |
| -62   | 16    | 47    |
| -62   | 17    | 48    |
| -60   | 8     | 38    |
| -58   | 12    | 41    |
| -58   | 14    | 43    |
| -58   | 16    | 45    |
| -58   | 18    | 47    |
| -58   | 21    | 50    |
| -58   | 22    | 51    |
| -58   | 24    | 53    |
| -58   | 26    | 55    |
| -56   | 10    | 38    |
| -56   | 15    | 43    |
| -54   | 13    | 40    |
| -54   | 14    | 41    |
| -54   | 16    | 43    |
| -54   | 23    | 50    |
| -52   | 10    | 36    |
| -50   | 13    | 38    |
| -50   | 18    | 43    |
| -50   | 20    | 45    |
| -50   | 28    | 53    |
| $\chi$ | $h_{11}$ | $h_{21}$ | $\chi$ | $h_{11}$ | $h_{21}$ | $\chi$ | $h_{11}$ | $h_{21}$ | $\chi$ | $h_{11}$ | $h_{21}$ |
|-------|--------|--------|-------|--------|--------|-------|--------|--------|-------|--------|--------|
| -10   | 33     | 38     | 10    | 22     | 17     | 26    | 51     | 38     | 58    | 43     | 14     |
| -10   | 35     | 40     | 10    | 24     | 19     | 30    | 31     | 16     | 58    | 44     | 15     |
| -10   | 38     | 43     | 10    | 26     | 21     | 30    | 36     | 21     | 58    | 51     | 22     |
| -8    | 15     | 19     | 10    | 28     | 23     | 30    | 58     | 43     | 62    | 43     | 12     |
| -8    | 16     | 20     | 10    | 30     | 25     | 34    | 30     | 13     | 62    | 48     | 17     |
| -6    | 31     | 34     | 10    | 31     | 26     | 34    | 35     | 18     | 62    | 58     | 27     |
| -6    | 34     | 37     | 10    | 32     | 27     | 34    | 36     | 19     | 66    | 52     | 19     |
| -4    | 43     | 45     | 10    | 35     | 30     | 34    | 37     | 20     | 74    | 45     | 8      |
| -2    | 19     | 20     | 10    | 37     | 32     | 34    | 44     | 27     | 78    | 63     | 24     |
| -2    | 20     | 21     | 10    | 41     | 36     | 34    | 48     | 31     | 78    | 72     | 33     |
| -2    | 21     | 22     | 14    | 40     | 33     | 38    | 33     | 14     | 86    | 56     | 13     |
| -2    | 22     | 23     | 18    | 31     | 22     | 38    | 34     | 15     | 86    | 58     | 15     |
| -2    | 25     | 26     | 18    | 34     | 25     | 38    | 36     | 17     | 86    | 66     | 23     |
| -2    | 26     | 27     | 20    | 31     | 21     | 38    | 39     | 20     | 94    | 53     | 6      |
| -2    | 30     | 31     | 22    | 24     | 13     | 38    | 43     | 24     | 94    | 59     | 12     |
| -2    | 35     | 36     | 22    | 26     | 15     | 38    | 56     | 37     | 94    | 68     | 21     |
| -2    | 48     | 49     | 22    | 28     | 17     | 46    | 34     | 11     | 98    | 58     | 9      |
| 2     | 30     | 29     | 22    | 29     | 18     | 46    | 35     | 12     | 98    | 70     | 21     |
| 2     | 39     | 38     | 22    | 34     | 23     | 46    | 36     | 13     | 110   | 63     | 8      |
| 2     | 40     | 39     | 22    | 43     | 32     | 46    | 41     | 18     | 122   | 71     | 10     |
| 6     | 24     | 21     | 22    | 48     | 37     | 46    | 42     | 19     | 122   | 78     | 17     |
| 6     | 26     | 23     | 26    | 26     | 13     | 46    | 43     | 20     | 126   | 68     | 5      |
| 6     | 27     | 24     | 26    | 29     | 16     | 46    | 51     | 28     | 126   | 72     | 9      |
| 6     | 33     | 30     | 26    | 32     | 19     | 50    | 33     | 8      | 134   | 75     | 8      |
| 6     | 41     | 38     | 26    | 34     | 21     | 50    | 36     | 11     | 148   | 101    | 27     |
| 6     | 47     | 44     | 26    | 40     | 27     | 50    | 48     | 23     | 160   | 112    | 32     |
Table 2: Results for multiple weight systems. 

#sym refers to comparing mirror pairs of spectra. 

#new' and #sym' give the respective numbers of new spectra as compared to single weight systems and abelian orbifolds of transversal weights [39].

| Weight systems | # poly | # spec | # new | # sym | # new' | # sym' |
|----------------|--------|--------|-------|-------|--------|--------|
| 4+4            | 6365   | 2078   | 381   | 101   | 332    | 96     |
| 4+3            | 727    | 485    | 73    | 9     | 60     | 7      |
| 4+2            | 58     | 56     | 8     | 0     | 6      | 0      |
| 3+3+3          | 36     | 29     | 5     | 0     | 4      | 0      |
| 3+3+2          | 17     | 17     | 4     | 0     | 4      | 0      |
| 3+3            | 6      | 6      | 3     | 1     | 3      | 1      |
| 3+2+2          | 3      | 3      | 1     | 0     | 1      | 0      |
| 2+2+2+2        | 1      | 1      | 1     | 0     | 1      | 0      |
| total          | 7213   | 2171   | 426   | 105   | 373    | 100    |
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