THE VALIDITY OF THE ANALOG OF THE RIEMANN HYPOTHESIS FOR SOME PARTS OF $\zeta(s)$ AND THE NEW FORMULA FOR $\pi(x)$

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ABSTRACT. An analog of the Riemann hypothesis is proved in this paper. Some new integral equations for the functions $\pi(x)$ and $R(x)$ follows. A new effect that is shown is that these function - with essentially different behavior - are the solutions of the similar integral equations.

This paper is the English version of the paper of reference [1].

1. The main result

1.1. Let (comp. [2], (7), (21); $2P\beta < \ln P_0$)

\begin{equation}
P = (\ln P_0)^{1 - \epsilon}, \quad \beta = \frac{\ln \Phi}{\ln 2}, \quad P_0 = \sqrt{\frac{T}{2\pi}},
\end{equation}

$0 < \epsilon$ is arbitrarily small and ($p$ is the prime)

\begin{equation}
\zeta_1(s) = \prod_{p \leq P} \sum_{k=0}^{\beta} \frac{1}{p^{sk}} = \sum_{n<P_0, p \leq P} \frac{1}{n^{s}} = \sum_{n<P_0} \frac{1}{n^{s}},
\end{equation}

\begin{equation}
\zeta_2(s) = \prod_{p \leq P} \sum_{k=0}^{\beta} \frac{1}{p^{(1-s)k}} = \sum_{n<P_0, p \leq P} \frac{1}{n^{1-s}} = \sum_{n<P_0} \frac{1}{n^{1-s}},
\end{equation}

\begin{equation}
\zeta_3(s) = \chi(s)\zeta_2(s)
\end{equation}

where (see [3], p. 16)

\begin{equation}
\chi(s) = \pi^{s^2} \frac{\Gamma \left(\frac{1-s}{2}\right)}{\Gamma \left(\frac{s}{2}\right)}, \quad s \neq 2k + 1, \quad k = 0, 1, 2, \ldots ,
\end{equation}

and $s = \sigma + it \in \mathbb{C}$. We define the function $\tilde{\zeta}(s)$ as follows

\begin{equation}
\tilde{\zeta}(s) = \tilde{\zeta}(s; P, \beta) = \zeta_1(s) + \zeta_3(s) =
\end{equation}

\begin{equation}
= \sum_{n<P_0} \frac{1}{n^{s}} + \chi(s) \sum_{n<P_0} \frac{1}{n^{1-s}}, \quad s \in \mathbb{C}, \quad s \neq 2k + 1.
\end{equation}

Since

\begin{equation}
\tilde{\zeta}(1-s) = \sum_{n<P_0} \frac{1}{n^{1-s}} + \chi(1-s) \sum_{n<P_0} \frac{1}{n^{s}},
\end{equation}

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and (see [3], p. 16) \( \chi(s)\chi(1-s) = 1 \), then
\[
\tilde{\zeta}(s) = \chi(s)\tilde{\zeta}(1-s), \quad s \in \mathbb{C}, \quad s \neq 2k + 1.
\]

**Remark 1.** The function \( \tilde{\zeta}(s) \) obeys the functional equation
\[
\tilde{\zeta}(s) = \chi(s)\tilde{\zeta}(1-s)
\]
hence, the zeros of \( \tilde{\zeta}(s) \) either lie on the critical line \( \sigma = \frac{1}{2} \) or occur in pairs symmetrical about this line.

1.2. Since (comp. [3], p. 79)
\[
\chi(\frac{1}{2} + it) = e^{-i2\vartheta(t)}
\]
then from (1.4) the formula
\[
e^{i\vartheta(t)}\tilde{\zeta}\left(\frac{1}{2} + it\right) = \sum_{n < P_0} e^{i(\vartheta(t) - t \ln n)} + \sum_{n < P_0} e^{-i(\vartheta(t) - t \ln n)} =
\]
\[
= 2 \sum_{n < P_0} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} = Z_1(t; P, \beta)
\]
follows. We have studied the zeros of \( Z_1(t) \), i.e. the zeros of \( \tilde{\zeta}(s) \), on the critical line in the paper [2]. Let
\[
D = D(T, H, K) = \{s : \sigma \in [-K, K], \ t \in [T, T + H]\},
\]
\[
K > 1, \ T > 0, \ H \leq \sqrt{T}.
\]

In this paper we prove the following theorem.

**Theorem.**

\[
(1.7) \quad \tilde{\zeta}(s) \neq 0, \ s \in D, \ \sigma \neq \frac{1}{2}
\]
for all sufficiently big \( T > 0 \), i.e. for \( \tilde{\zeta}(s) \), \( s \in D \), \( T \to \infty \) the analog of the Riemann hypothesis is true.

Let us remind the approximate functional equation of Riemann-Hardy-Littlewood ([3], p. 69)
\[
(1.8) \quad \zeta(s) = \sum_{n \leq \tau'} \frac{1}{n^s} + \chi(s) \sum_{n \leq \tau'} \frac{1}{n^{1-s}} + O(t^{-\frac{1}{2}}), \ \tau' = \sqrt{\frac{t}{2\pi}},
\]
and the Riemann-Siegel formula (comp. [1.5])
\[
e^{i\vartheta(t)}\zeta\left(\frac{1}{2} + it\right) = Z(t) = 2 \sum_{n \leq \tau'} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + O(t^{-\frac{1}{2}}) =
\]
\[
= 2 \sum_{n < P_0} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + O(T^{-\frac{1}{4}}) + O(HT^{-\frac{1}{4}}), \ t \in [T, T + H].
\]

**Remark 2.** The term the part of the function \( \zeta(s) \) is specified by the comparison of the formulae (1.4), (1.8). Next, the condition \( H \leq \sqrt{T} \) is related with (1.9).
2. The formulae for some parts of $\tilde{\zeta}(s)$

We have (see (1.2))

$$\zeta_1(s) = B_1(s)e^{i\psi_1(s)}, \quad B_1(s) = |\zeta_1(s)| > 0, \quad \sigma > 0,$$

where

$$B_1(s) = \prod_{p \leq P} |M_1(p; s, \beta)|, \quad \psi_1(s) = \sum_{p \leq P} \arg\{M_1(p; s, \beta)\},$$

and similarly,

$$\zeta_2(s) = B_2(s)e^{i\psi_2(s)}, \quad B_2(s) > 0, \quad \sigma < 1,$$

where

$$B_2(s) = \prod_{p \leq P} |M_2(p)|, \quad \psi_2(s) = \sum_{p \leq P} \arg\{M_2(p)\},$$

Next, we have (see [3], pp. 68, 79, 329)

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{-\frac{s}{2}} e^{-i2\theta(t)} \left\{1 + O\left(\frac{1}{t}\right)\right\},$$

i.e.

$$\chi(s) = |\chi(s)|e^{i\psi_3(s)}$$

where

$$|\chi(s)| = \left(\frac{t}{2\pi}\right)^{\frac{s}{2}} \left\{1 + O\right\} \quad \text{and} \quad |\chi(s)| > 0, \quad s \in D,$$

and

$$\psi_3(s) = -2\theta(t) + O\left(\frac{1}{t}\right), \quad T \to \infty.$$

Consequently, we obtain the following formulae

$$\tilde{\zeta}(s) = B_1(s)e^{i\psi(s)} + B_2(s)|\chi(s)|e^{i\psi_4(s)},$$

$$\psi_4(s) = \psi_2(s) + \psi_3(s), \quad a \in D \cap \{0 < \sigma < 1\}, \quad T \to \infty.$$

**Remark 3.** Let us remind that the formula (2.3) is connected with the Stirling’s formula for $\ln \Gamma(z)$, $z \in \mathbb{C}$ to which corresponds arbitrary fixed strip $-K \leq \sigma \leq K$ (comp. [3], p. 68).

3. The lemmas on $B_1(s), B_2(s)$

3.1. Let

$$D_1(\Delta) = \left\{s: \sigma \in \left[\frac{1}{2} + \Delta, 1 - \Delta\right], \quad t \in [T, T + H]\right\}, \quad \Delta \in \left(0, \frac{1}{4}\right).$$

The following lemma holds true.

**Lemma 1.**

$$\exp\left(-\frac{A}{\Delta} p_{\frac{1}{2} - \Delta}\right) < B_1(s) < \exp\left(\frac{A}{\Delta} p_{\frac{1}{2} - \Delta}\right), \quad s \in D_1(\Delta), \quad T \to \infty.$$
Proof. We have (see (2.1))

\[ |M_1| = \left| 1 - \frac{1}{p^{(\beta+1)\sigma}} \right| \left| 1 - \frac{1}{p^\sigma} \right|^{-1} = \]
\[ = \left\{ 1 + \frac{1}{p^{2(\beta+1)\sigma}} - \frac{2 \cos((\beta + 1)\varphi)}{p^{(\beta+1)\sigma}} \right\}^{\frac{1}{2}} \left\{ 1 + \frac{1}{p^{2\sigma}} - \frac{2 \cos \varphi}{p^\sigma} \right\}^{-\frac{1}{2}} = M_{11}M_{12} \]

where

\[ \varphi = t \ln p. \]

Next, we have (see (1.1))

\[ \ln M_{11} = \frac{1}{2} \ln \left\{ 1 + \mathcal{O} \left( \frac{1}{p^\sigma} \right) \right\} = \mathcal{O} \left( \frac{1}{p^\sigma} \right), \]

(3.3)

\[ M_{11} = \exp \left\{ \mathcal{O} \left( \frac{1}{p^\sigma} \right) \right\} \]

uniformly for \( \Delta \in (0, \frac{1}{4}) \), and since

\[ \frac{1}{2} + \Delta \leq \sigma \leq 1 - \Delta, \quad \frac{1}{p^{2\sigma}} \leq \frac{1}{p^{1+2\Delta}} < 1, \]

then

\[ \ln M_{12} = -\frac{1}{2} \ln \left( 1 - \frac{1}{p^{2\sigma}} \right) - \frac{1}{2} \ln \left( 1 - \frac{2p^\sigma}{p^{2\sigma} + 1} \cos \varphi \right) = \]
\[ = \frac{1}{p^\sigma} \cos \varphi + \mathcal{O} \left( \frac{1}{p^{2\sigma}} \right), \]

\[ M_{12} = \exp \left\{ \frac{1}{p^\sigma} \cos \varphi + \mathcal{O} \left( \frac{1}{p^{2\sigma}} \right) \right\}. \]

Hence (see (2.1)), we have

(3.4)

\[ B_1(s) = \exp \left\{ \sum_{p \leq p} \frac{1}{p^\sigma} \cos \varphi + \mathcal{O} \left( \sum_{p \leq p} \frac{1}{p^{2\sigma}} \right) \right\}, \quad s \in D_1(\Delta) \]

uniformly for \( \Delta \in (0, \frac{1}{4}) \). Since

\[ \Delta \leq 1 - \sigma \leq \frac{1}{2} - \Delta, \]

then

(3.5)

\[ \left| \sum_{p \leq p} \frac{1}{p^\sigma} \cos \varphi + \mathcal{O} \left( \sum_{p \leq p} \frac{1}{p^{2\sigma}} \right) \right| < A \sum_{p \leq p} \frac{1}{p^\sigma} < \frac{A}{1 - \sigma} p^{1-\sigma} < \frac{A}{\Delta} p^{\frac{2}{2\sigma} - \Delta}, \]

and from this (see (3.4)) we obtain (3.2). \( \square \)

3.2. The following lemma holds true

**Lemma 2.**

(3.6)

\[ \exp \left( -A p^{1-\Delta} \right) < B_2(s) < \exp \left( A p^{1-\Delta} \right), \quad s \in D_1(\Delta), \quad T \to \infty \]

if the condition

(3.7)

\[ \Delta \beta > \omega(T) \]

is fulfilled, where \( \omega(T) \) increases to \( \infty \) for \( T \to \infty \).
Proof. Since by \((3.7)\), \((\text{see } (2.2))\),
\[(1 - \sigma)(\beta + 1) \geq \Delta(\beta + 1) > \omega(T)\]
then putting \(1 - \sigma = \bar{\sigma}\), we obtain the formula
\[(3.8)\]

\[
B_2(s) = \exp \left\{ \sum_{p \leq P} \frac{1}{p^2} \cos \varphi + \mathcal{O} \left( \sum_{p \leq P} \frac{1}{p^{2\bar{\sigma}}} \right) \right\},
\]
(similarly to \((3.4)\)). Since (see \((3.1)\), comp. \((3.5)\); \(\Delta \leq \bar{\sigma} \leq \frac{1}{2} - \Delta\))
\[
\sum_{p \leq P} \frac{1}{p^\sigma} + \mathcal{O} \left( \sum_{p \leq P} \frac{1}{p^{2\bar{\sigma}}} \right) = \mathcal{O} \left( \frac{P^{1-\bar{\sigma}}}{1-\sigma} \right) = \mathcal{O}(P^{1-\Delta}),
\]
then we obtain \((3.6)\) from \((3.8)\). \(\square\)

Remark 4. The estimate \((3.6)\) is valid in somehow wider domain
\[
D_{1\dagger}(\Delta) = \left\{ s : \sigma \in \left[ \frac{1}{2}, 1 - \Delta \right], \ t \in [T, T + H] \right\}.
\]

4. The function \(\tilde{\zeta}(s)\) has no zero in the rectangle \(D_1(\Delta_0)\)

First off all (see \((2.4)\))

\[(4.1)\]

\[
|\chi(s)| < \frac{A}{P_{0\Delta}^2}, \ s \in D_1(\Delta).
\]

Next (see \((2.3)\), \((3.2)\), \((3.6)\))

\[
|\tilde{\zeta}(s)| \geq B_1(s) - |\chi(s)|B_2(s) > \exp \left( -\frac{A}{\Delta} P^{\frac{1}{2} - \Delta} \right) - \frac{A}{P_{0\Delta}^2} \exp(AP^{1-\Delta}) >
\]
\[
= \left\{ 1 - \frac{A}{P_{0\Delta}^2} \exp \left( AP + \frac{A}{\Delta} P^{\frac{1}{2} - \Delta} \right) \right\} \exp \left( -\frac{A}{\Delta} P^{\frac{1}{2} - \Delta} \right) >
\]
\[
= \left\{ 1 - A \exp \left( \frac{2A}{\Delta} P - 2\Delta \ln P_0 \right) \right\} \exp \left( -\frac{A}{\Delta} P^{\frac{1}{2} - \Delta} \right).
\]

Since
\[
2\Delta \ln P_0 - \frac{2A}{\Delta} P = \frac{2 \ln P_0}{\Delta} \left( \Delta^2 - \frac{P}{\ln P_0} \right)
\]
then we put (see \((1.1)\))

\[(4.3)\]

\[
\Delta_0 = \Delta_0(T, \varepsilon) = \left( \frac{2A}{\ln P_0} \right)^{\frac{1}{2}} = \sqrt{\frac{2A}{(\ln P_0)^2}}.
\]

Because (see \((1.1)\))

\[
\Delta_0 \beta > A_1(\ln P_0)^{\frac{1}{2}} \rightarrow \infty, \ T \rightarrow \infty.
\]
then the condition (3.7) is fulfilled. Hence, we obtain from (4.2) by (1.1) and (4.3) the estimate

\[ |\tilde{\zeta}(s)| > \frac{1}{2} \exp \left( -\frac{A}{\Delta_0} P^{\frac{1}{2} - \Delta_0} \right) > \exp \left( -\frac{A}{\Delta_0} P^{\frac{1}{2}} \right) = \exp \left( -\sqrt{\frac{A}{2}} \ln P_0 \right), \]

\[ s \in D_1(\Delta_0), \ T \to \infty. \]

Namely, we have the following lemma holds true.

**Lemma 3.**

\[ |\tilde{\zeta}(s)| > e^{-\sqrt{\frac{A}{2}} \ln P_0}, \ s \in D_1(\Delta_0), \ T \to \infty. \]

**Corollary 1.**

(4.4) \[ \tilde{\zeta}(s) \neq 0, \ s \in D_1(\Delta_0), \ T \to \infty. \]

5. **The function \( \tilde{\zeta}(s) \) has no zero in the rectangle \( D_2(\Delta_0) \)**

Let

\[ D_2(\Delta_0) = \{ s : \sigma \in [1 - \Delta_0, K], \ t \in [T, T + K] \}. \]

We remark that the formula (3.3) is valid for all \( \sigma \in [1 - \Delta_0, K] \), see the proof of the Lemma 1. Since in our case (comp. (3.3))

\[ \left| \sum_{p \leq P} \frac{1}{p^\sigma} \cos \varphi + O \left( \sum_{p \leq P} \frac{1}{p^{2\sigma}} \right) \right| < A \sum_{p \leq P} \frac{1}{p^{1 - \Delta_0}} < \frac{A}{\Delta_0} P^{\Delta_0}, \]

then we obtain the estimate (comp. (4.2))

(5.1) \[ \exp \left( -\frac{A}{\Delta_0} P^{\Delta_0} \right) < B_1(s) < \exp \left( \frac{A}{\Delta_0} P^{\Delta_0} \right) \]

for \( s \in D_2(\Delta_0), \ T \to \infty. \)

Next, for \( \zeta_2(s) \) we use the formula (see (1.2))

\[ \zeta_2(s) = \sum_{n<P_0} \frac{1}{n^{1-s}}. \]

First of all (see (1.1), (4.3) and (1.2) - the product formula for \( \zeta_2(s) \))

\[ \sum_{n<P_0} 1 = (\beta + 1) \pi(P) = \exp\{\pi(P) \ln(\beta + 1)\} < \]

(5.2) \[ < \exp \left( A(\epsilon) \frac{P}{\ln P} \ln P_0 \right) = \exp \{ A(\epsilon) (\ln P_0)^{1-\epsilon} \} < \exp(\Delta_0 \ln P_0) = P_0^{\Delta_0} \]

where we have used the upper estimate of Chebyshev for \( \pi(x) \). Next, (see (1.1), (2.3))

\[ |\chi(s)| < \frac{A}{P_0^{\sigma - 1}}, \ 1 - \Delta_0 \leq \sigma \leq K. \]

Now:
(A) in the rectangle
\[ D_{21}(\Delta_0) = D_2(\Delta_0) \cup \{1 - \Delta_0 \leq \sigma \leq 1\} \]
we have (see [1.2], (5.2); 1 - 2\Delta_0 \leq 2\sigma - 1 \leq 1)
\[ \zeta_3(s) = \chi(s)\zeta_2(s) = O\left(\frac{1}{P_0^{\sigma-1}} \sum_{n<P_0} \frac{1}{n^{1-\sigma}}\right) = \mathcal{O}\left(\frac{1}{P_0^{2\sigma-1}} \sum_{n<P_0} \right) = \]
\[ = O\left(\frac{1}{P_0^{1-2\Delta_0}} P_0^{\Delta_0} \right) = O\left(\frac{1}{P_0^{1-3\Delta_0}} \right), \]

(B) in the rectangle
\[ D_{22}(\Delta_0) = D_2(\Delta_0) \cup \{1 < \sigma \leq K\} \]
we have
\[ \zeta_3(s) = O\left(\frac{1}{P_0^{\sigma-1}} \sum_{n<P_0} \frac{1}{n^{1-\sigma}}\right) = \mathcal{O}\left(\frac{1}{P_0^{\sigma-1}} \sum_{n<P_0} \right) = \]
\[ = O\left(\frac{1}{P_0^{\sigma}} \sum_{n<P_0} \right) = O\left(\frac{1}{P_0^{1-\Delta_0}} \right). \]

Consequently (see [5.3], (5.4)), we have
\[ (5.5) \]
\[ \zeta_3(s) = O\left(\frac{1}{P_0^{1-3\Delta_0}} \right), \quad s \in D_2(\Delta_0). \]

Since (see [1.1], (4.3))
\[ \frac{A}{\Delta_0} P_0^{\Delta_0} = \frac{A}{\sqrt{2A_1}} (\ln P_0)^\tau (\ln P_0)^{(1-\epsilon)\Delta_0} < (\ln P_0)^{\frac{\tau}{2}}, \quad T \to \infty, \]
then (see [2.5], (5.1), (5.5)) we obtain in the domain \( D_2(\Delta_0) \)
\[ \left|\zeta(s)\right| \geq B_1(s) - \left|\zeta_3(s)\right| > \exp\left(-\frac{A}{\Delta_0} P_0^{\Delta_0}\right) - \frac{A}{P_0^{1-3\Delta_0}} = \]
\[ = \left\{1 - \exp\left[\frac{A}{\Delta_0} P_0^{\Delta_0} - (1 - 3\Delta_0) \ln P_0 + \ln A\right]\right\} \exp\left(-\frac{A}{\Delta_0} P_0^{\Delta_0}\right) > \]
\[ > \frac{1}{2} \exp\left[-(\ln P_0)^{\frac{\tau}{2}}\right] \exp\left[-(\ln P_0)^\tau\right], \quad T \to \infty, \]
i.e. the following lemma holds true.

**Lemma 4.**
\[ \left|\zeta(s)\right| > e^{-(\ln P_0)^\tau}, \quad s \in D_2(\Delta_0), \quad T \to \infty. \]

**Corollary 2.**
\[ (5.6) \]
\[ \zeta(s) \neq 0, \quad s \in D_2(\Delta_0), \quad T \to \infty. \]
6. Lemma on the difference of logarithms

Let

\[ D_3(\Delta_0) = \left\{ s : \sigma \in \left( \frac{1}{2}, \frac{1}{2} + \Delta_0 \right), \ t \in [T, T + H] \right\} \]

where

\[ \sigma = \frac{1}{2} + \delta, \ \delta \in (0, \Delta_0). \]

The following lemma holds true.

**Lemma 5.**

\[ \ln B_1(s) - \ln B_2(s) = O \left\{ \delta (\ln P_0) \right\}, \ s \in D_3(\Delta_0), \ T \to \infty. \]

**Proof.** We have (see (2.1), (2.2))

\[ \ln B_1(s) - \ln B_2(s) = Y_1 + Y_2 \]

where (\(|z| = |\bar{z}|\))

\[ Y_1 = \sum_{p \leq P} \left\{ \ln \left| 1 - \frac{p^{-(\beta+1)t}}{p^{(\beta+1)(\frac{1}{2} + \delta)}} \right| - \ln \left| 1 - \frac{p^{-(\beta+1)t}}{p^{(\beta+1)(\frac{1}{2} - \delta)}} \right| \right\}, \]

\[ Y_2 = \sum_{p \leq P} \left\{ \ln \left| 1 - \frac{p^{it}}{p^{2\delta}} \right| - \ln \left| 1 - \frac{p^{it}}{p^{2\delta + \delta}} \right| \right\}. \]

Let

\[ x = \frac{1}{p^\delta}, \ x \in \left[ \frac{p^\delta}{\sqrt{P}}, \frac{p^\delta}{\sqrt{P}} \right]. \]

It is clear that (see (1.1), (4.3))

\[ \delta \ln p = O(\Delta_0 \ln P) = O \left\{ \ln \ln P_0 \right\} \to 0, \ T \to \infty, \]

\[ \frac{p^\delta - p^{-\delta}}{\sqrt{P}} = O \left( \delta \frac{\ln P_0}{\sqrt{P}} \right). \]

Next, by the mean-value theorem

\[ \ln \left| 1 - \frac{p^{-(\beta+1)t}}{p^{(\beta+1)(\frac{1}{2} + \delta)}} \right| - \ln \left| 1 - \frac{p^{-(\beta+1)t}}{p^{(\beta+1)(\frac{1}{2} - \delta)}} \right| = \]

\[ \frac{p^{-\delta} - p^\delta}{\sqrt{P}} \frac{d}{dx} \left\{ \ln \left| 1 - x^{\beta+1} p^{-i(\beta+1)t} \right| \right\} \bigg|_{x=x_1}, \]

\[ x_1 = \frac{1}{p^\delta}, \ c \in \left( \frac{1}{2} - \delta, \frac{1}{2} + \delta \right). \]

Since (\( \varphi = t \ln p \))

\[ \ln \left| 1 - x^{\beta+1} p^{-i(\beta+1)t} \right| = \frac{1}{2} \ln \left( 1 + x^{2\beta + 2} - 2x^{\beta+1} \cos \{ (\beta + 1)\varphi \} \right), \]
Lemma 6.

Similarly, we obtain in the case $\sigma > \frac{1}{2}$, and consequently

$$Y_1 = O \left( \delta \sum_{p \leq P} \frac{\beta \ln p}{p^{\frac{1}{2} + \delta}} \right) = O \left( \frac{\beta \ln P}{2 \pi} \right),$$

(6.4)

Similarly, we obtain in the case $Y_2$

$$\ln \left| 1 - \frac{p^{it}}{p^{\frac{1}{2} + \delta}} \right| - \ln \left| 1 - \frac{p^{it}}{p^{\frac{1}{2} + \delta}} \right| = p^{-\delta} - p^{\delta} \frac{d}{dx} \ln (1 + x^2 - 2x \cos \varphi) \bigg|_{x=x_2}$$

where

$$\frac{d}{dx} \ln (1 + x^2 - 2x \cos \varphi) = \frac{2x - 2 \cos \varphi}{1 + x^2 - 2x \cos \varphi} = O(1),$$

because

$$1 + x^2 - 2x \cos \varphi \geq (1 - x)^2 > (1 - 2^{-\frac{1}{2} + \delta})^2 > (1 - 2^{-\frac{1}{2}})^2 > 0.$$ 

Thus, we have (see (1.1))

$$Y_2 = O \left( \delta \sum_{p \leq P} \frac{\ln p}{\sqrt{p}} \right) = O(\delta \sqrt{P}) = O \left( \delta (\ln P_0) \frac{\pi}{\ln 2} \right),$$

(6.5)

(see (1.1)), (31), (33)). Now, (6.2) follows from (6.3) by (6.4), (6.5).

7. More accurate formula for $\ln |\chi(s)|$

The following lemma holds true.

Lemma 6.

$$\ln |\chi(s)| = - \left( \sigma - \frac{1}{2} \right) \ln t + O \left( \frac{2\sigma - 1}{t} \right), \quad s \in D_3(\Delta_0), \; T \to \infty.$$ (7.1)

Proof. Since (see (1.3))

$$|\chi(s)| = \pi^{\sigma - \frac{1}{2}} \left| \frac{\Gamma \left( \frac{1+\sigma}{2} - it \right)}{\Gamma \left( \frac{1+\sigma}{2} + it \right)} \right| = \pi^{\sigma - \frac{1}{2}} G_1(\sigma, t),$$

where

$$G_1(\sigma, t) > 0, \; s \in D_3(\Delta_0), \; T \to \infty,$$

then

$$\ln |\chi(s)| = \left( \sigma - \frac{1}{2} \right) \ln \pi + \ln G_1(\sigma, t) = \left( \sigma - \frac{1}{2} \right) \ln \pi + G_2(\sigma, t),$$

and $G_2(\sigma, t)$ is the analytic function of the real variable $\sigma$ for arbitrary fixed $t$ if $s \in D_3(\Delta_0), \; T \to \infty$. Since

$$\left| \chi \left( \frac{1}{2} + it \right) \right| = 1$$
then
\[ G_2 \left( \frac{1}{2}, t \right) = 0, \]
and
\[ G_2(\sigma, t) = \left( \sigma - \frac{1}{2} \right) G_3(\sigma, t). \]

Now, (see (7.2))
\[ \ln \left| \chi(s) \right| = \left( \sigma - \frac{1}{2} \right) \left\{ \ln \pi + G_3(\sigma, t) \right\}, \ s \in D_3(\Delta_0), \ T \to \infty. \]

In the case (2.4) we have
\[ \ln \left| \chi(s) \right| = -\left( \sigma - \frac{1}{2} \right) \ln \frac{t}{2\pi} + G_4(\sigma, t), \ G_4(\sigma, t) = \mathcal{O} \left( \frac{1}{t} \right), \]
under the conditions (7.3) where \( G_4(\sigma, t) \) is the analytic function of the real variable \( \sigma \). Since
\[ G_4 \left( \frac{1}{2}, t \right) = 0 \]
(see (7.3), (7.4)) then
\[ G_4(\sigma, t) = \left( \sigma - \frac{1}{2} \right) G_5(\sigma, t). \]

Next, by (7.5), the orders of the functions
\[ G_4(\sigma, t), G_5(\sigma, t), \ s \in D_3(\Delta_0), \ T \to \infty \]
in the variable \( t \) are equal, i.e. (see (7.5))
\[ G_5(\sigma, t) = \mathcal{O} \left( \frac{1}{t} \right). \]

Now, the formula (7.6) follows from (7.5) by (7.6).

\[ \square \]

Remark 5. The formula (7.6) can be proved directly, of course.

8. **Proof of the Theorem**

8.1. Let
\[ \ln \Lambda(s) = \ln B_1(s) - \ln B_2(s) - \ln \left| \chi(s) \right|, \ s \in D_3(\Delta_0), \ T \to \infty. \]

Since (see (1.1), (7.1), \( \sigma = \frac{1}{2} + \delta \))
\[ \ln \left| \chi(s) \right| = \ln \frac{t}{2\pi} + \mathcal{O} \left( \frac{\delta}{T} \right) = -2\delta \ln P_0 + \mathcal{O} \left( \frac{\delta H}{T} \right) + \mathcal{O} \left( \frac{\delta}{T} \right) \]
then we obtain (see (1.0), (6.2), (8.1))
\[ \ln \Lambda(s) = 2\delta \ln P_0 + \mathcal{O} \left\{ \delta (\ln P_0) \frac{1}{\sqrt{T}} \right\} + \mathcal{O} \left( \frac{\delta}{\sqrt{T}} \right) = \]
\[ = \delta \left[ 2 \ln P_0 + \mathcal{O} \left\{ (\ln P_0) \frac{1}{\sqrt{T}} \right\} + \mathcal{O} \left( \frac{1}{\sqrt{T}} \right) \right] > \delta \ln P_0 > 0. \]

Consequently,
\[ \Lambda(s) > 1, \ s \in D_3(\Delta_0), \ T \to \infty, \]
and (see (2.5))

\[
|\zeta(s)| \geq B_1(s) - |\chi(s)|B_2(s) = |\chi(s)|B_2(s) \left( \frac{B_1(s)}{|\chi(s)|B_2(s)} - 1 \right) = |\chi(s)|B_2(s)|\Lambda(s) - 1| > 0, \ s \in D_3(\Delta_0), \ T \to \infty.
\]

(The inequality \(B_2(s) > 0, \ s \in D_3(\Delta_0), \ T \to \infty\) follows from Remark 4.) Now, by \((4.4), (5.6), (8.3)\) and from Remark 1, we have \((1.7)\).

8.2. As an addition to the Theorem we obtain an lower estimate for \(|\zeta(s)|, \ s \in D_3(\Delta_0)\). Namely, we have

Lemma 7.

\[(8.4) \quad |\zeta(s)| > \frac{1}{P_0} \sinh \left( \frac{\delta}{2} \ln P_0 \right), \ s \in D_3(\Delta_0), \ T \to \infty, \ \delta \in (0, \Delta_0).\]

Proof. Since (see (1.1), (3.6), Remark 4 and (6.1))

\[B_2(s) > \exp \left( -A\left( \ln P_0 \right)^{1-\epsilon} \right), \quad \Lambda(s) > \frac{1}{P_0} \sinh \left( \frac{\delta}{2} \ln P_0 \right), \]

and (see \((8.1), (8.2)\))

\[|\chi(s)| > P_0^{-\left(2+\epsilon\delta\right)}, \ A(s) > P_0^\delta\]

then (see \((8.3)\))

\[|\zeta(s)| > \frac{\exp \left( -A\left( \ln P_0 \right)^{1-\epsilon} \right)}{P_0^{(2+\epsilon\delta)}} \left( P_0^\delta - 1 \right) > 2 \exp \left( -A\left( \ln P_0 \right)^{1-\epsilon} \right) \frac{\delta}{P_0^{(2+\epsilon\delta)}} \sinh \left( \frac{\delta}{2} \ln P_0 \right) > \frac{1}{P_0} \sinh \left( \frac{\delta}{2} \ln P_0 \right),\]

i.e. \((8.4)\). \qed

9. A NEW PROPERTY OF THE FUNCTIONS \(\pi(x), R(x)\)

Let

\[D(\Delta_0) = D(\Delta_0, T, H, K) = \left\{ s : \sigma \in \left[ \frac{1}{2} + \Delta_0, K \right], \ t \in [T, T + H] \right\}, \ \Delta_0 = \frac{A}{\left( \ln P_0 \right)^2}.\]

We can prove the following

Formula 1.

\[(9.1) \quad \ln \zeta(s) = s \int_2^P \frac{\pi(x)}{x^{(x^s - 1)}} dx - \pi(P) \ln \left( 1 - \frac{1}{P^s} \right) + O \left( e^{-A^\beta} \right),\]

where \(s \in D(\Delta_0), \ T \to \infty\) and \(O \left( e^{-A^\beta} \right)\) is the estimate of

\[\sum_{p \leq P} \left( 1 - \frac{1}{p^{s(\beta + 1)}} \right) + \ln \left( 1 + \frac{\chi(s)\zeta(s)}{\zeta_1(s)} \right) = \Omega_1(s; P, \beta).\]

Since

\[\pi(x) = \int_0^x \frac{dx}{\ln v} + R(x) = U(x) + R(x),\]

then we obtain from \((9.1)\).
Formula 2.

\[
\ln \tilde{\zeta}(s) = s \int_2^P \frac{R(x)}{x(x^s - 1)} \, dx - R(P) \ln \left(1 - \frac{1}{P^s}\right) + \mathcal{O}\left(e^{-A\beta}\right),
\]

\[s \in D(\Delta_0), \quad T \to \infty\]

where \(\mathcal{O}\left(e^{-A\beta}\right)\) is the estimate of

\[
\Omega_1(s; P, \beta) + \Omega_2(s; P),
\]

and

\[
\Omega_2(s) = -s \int_2^P \frac{dx}{\ln x} \int_2^x \frac{dv}{v(v^s - 1)} - U(P) \ln \left(1 - \frac{1}{2^s}\right) = \sum_{n=0}^{\infty} \frac{1}{n+1} \int_2^P \frac{dx}{x^{n+1} \ln x} = \mathcal{O}\left(\frac{\sqrt{\ln T}}{T}\right).
\]

Thus, the following properties of the functions \(\pi(x), R(x)\) holds true:

(A) the function \(\pi(x), \quad x \in [2, P]\) is the solution of the integral equation (for every fixed \(s \in D(\Delta_0)\))

\[
\ln \tilde{\zeta}(s) = s \int_2^P \frac{\Phi(x)}{x(x^s - 1)} \, dx - \Phi(P) \ln \left(1 - \frac{1}{P^s}\right) + \Omega_1(s),
\]

(B) the function \(R(x), \quad x \in [2, P]\) is the solution of the perturbed integral equation

\[
\ln \tilde{\zeta}(s) = s \int_2^P \frac{\Phi(x)}{x(x^s - 1)} \, dx - \Phi(P) \ln \left(1 - \frac{1}{P^s}\right) + \Omega_1(s) + \Omega_2(s).
\]

Remark 6. Hence, we have a new property of the functions \(\pi(x), R(x)\): these functions are to solutions of the integral equations \((9.2)\) and \((9.3)\), respectively and the mentioned integral equations are close each to other. This property of \(\pi(x)\) and \(R(x)\) is fully missing in the theory of \(\pi(x), R(x)\) based on the Riemann zeta-function.

Let us remind that the behaviour of the functions \(\pi(x), R(x)\) is essentially different, \(\pi(x) \sim \frac{x}{\ln x}, \quad x \to \infty\), and \(R(x), \quad x \to \infty\) infinitely many times alternates its sign (Littlewood, 1914).

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