THE STOCHASTIC HEAT EQUATION AS THE LIMIT OF A STIRRING DYNAMICS PERTURBED BY A VOTER MODEL

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ABSTRACT. We prove that in dimension $d \leq 3$ a modified density field of a stirring dynamics perturbed by a voter model converges to the stochastic heat equation.

1. INTRODUCTION

While the equilibrium fluctuations of the density field are well understood since Brox and Rost [3] and Chang [5] (cf. references and comments in Chapter 11 of [15]), nonequilibrium fluctuations are considered to be one of the main open problems in the theory of hydrodynamic limit of interacting particle systems.

For almost three decades, no progress has been made in this subject. The few known results were restricted to one dimension and their proofs either relied on special features of the dynamics, such as duality or integrability of certain quantities, or required strong estimates, such as a logarithmic Sobolev inequality in Chang and Yau [6].

With the recent developments in the theory of non-linear stochastic partial differential equations, this problem became even more interesting. Indeed, to define solutions of these non-linear equations, it has been proposed to smooth the noise by convolving it with a smooth kernel and then to show that, after a renormalization, the limit exists and does not depend on the kernel of the convolution (cf. [10, 9, 4] and references therein).

Since interacting particle systems possess a built-in noise, it is natural to expect that the density fields converge to the normalized solutions of the SPDEs derived in the theories mentioned above. This question has attracted much attention recently and many problems remain unsolved [1, 18, 11, 21, 23, 17].

In this article, we pursue in this direction of research by considering the fluctuations of a gradient exclusion dynamics perturbed by a voter model. One of the novelties lies in the definition of the density field, which is not normalized by the square root of the degrees of freedom; and on the non-conservative noise which appears in the limiting equation. Indeed, even if the stochastic PDE which describes the asymptotic behavior of the density fluctuation is linear, the noise is non-conservative, in contrast to most of the previous results [15].

Our aim is to show that the scaling limit of the density fluctuation field of this dynamics is described by a version of the stochastic heat equation

$$\partial_t X = \Delta X + \xi,$$ 

(1.1)

where $\xi$ is a space-time white-noise.
In both models, the exclusion process and the voter dynamics, on the diffusive time-scale the density of particles evolve according to the solution of a linear parabolic PDE. The hydrodynamic behavior of the voter model has been derived by Presutti and Spohn in [19], and we refer to [15] for references on the corresponding result for exclusion dynamics.

We consider here the exclusion process on the diffusive time-scale and the voter model evolving on a slower time-scale. This dynamics has two absorbing states: the empty configuration and the full one. Nevertheless, as the voter part evolves in a slower scale, the global evolution can be understood as a small perturbation of the exclusion process, and, starting from a state close to an equilibrium state of the exclusion dynamics, the homogeneous Bernoulli product measures, one expects that at a later time the state of process remains close to the equilibrium state of the exclusion dynamics. One of the main results of this article provides a quantitative estimate for this closeness.

The main obstacle in the proof of the fluctuations lies in the replacement of a space-time average of cylinder functions by a space-time average of the density of particles. This is the so-called Boltzmann-Gibbs principle.

In equilibrium, this replacement is derived using a classic bound on the variance of an additive functional of a Markov process [15, Proposition A1.6.1]. In non-equilibrium this tool is not available and one has to rely on entropy bounds of the state of the process with respect to a reference measure. In our context, the aforementioned Bernoulli product measures.

To obtain such bounds we rely on the approach introduced recently by Jara and Menezes [12, 13], which improved the estimate on the entropy production obtained by Yau [22] in the context of interacting particles systems. These bounds are sharp enough to permit the derivation of the Boltzmann-Gibbs principle (and the tightness of the density fluctuation field) in dimension $d \leq 3$.

2. Notation and results

Denote by $T_n^d = (\mathbb{Z}/n\mathbb{Z})^d$, $n \in \mathbb{N} = \{1, 2, \ldots \}$, the $d$-dimensional discrete torus with $n^d$ points. Here and below, we use blue color to indicate the first appearance of a new notation. We consider a particle system which describes voters with a binary opinion, 0 or 1, evolving on $T_n^d$.

Let $\Omega_n = \{0, 1\}^{T_n^d}$ be the state space. Elements of $\Omega_n$ are represented by the Greek letters $\eta = (\eta_x : x \in T_n^d)$ and $\xi$. Hence, $\eta_x = 1$ if the voter at $x$ for the configuration $\eta$ has the opinion 1.

Let $L^V_n$ be the generator of the voter model in $\Omega_n$:

$$(L^V_n f)(\eta) = \sum_{x \in T_n^d} \sum_{y \in T_n^d: \|y-x\|=1} (\eta_y - \eta_x)^2 \left[ f(\sigma^x \eta) - f(\eta) \right]$$

for all $f : \Omega_n \to \mathbb{R}$. In this formula, the second sum is carried over all neighbours $y$ of $x$: $y \in T_n^d$, $\|x - y\| = 1$, and $\| \cdot \|$ stands for the $\ell^1$ norm: $\|(z_1, \ldots, z_d)\| = \sum_{1 \leq j \leq d} |z_j|$. Moreover, $\sigma^x \eta$ represents the configuration obtained from $\eta$ by flipping the value of $\eta_x$:

$$\sigma^x \eta = \begin{cases} 
\eta_z, & z \neq x, \\
1 - \eta_x, & z = x.
\end{cases}$$
Denote by \( \{e_1, \ldots, e_d\} \) the canonical basis of \( \mathbb{R}^d \). Let \( c_j : \{0,1\}^{2d} \to \mathbb{R}, 1 \leq j \leq d \), be strictly positive cylinder functions (functions which depend only on a finite number of variables \( \eta_x \)):

\[
c_j(\eta) \geq c_0 > 0
\]  

(2.1)

for all \( \eta \in \{0,1\}^{2d}, 1 \leq j \leq d \). Assume that \( c_j \) does not depend on the variables \( \eta_0 \) and \( \eta_{e_j} \) and that the following gradient conditions are in force. For each \( j \), there exist cylinder functions \( h_{j,k}, 1 \leq k \leq d \), such that

\[
c_j(\eta_1) (\eta_{e_j} - \eta_0) = \sum_{k=1}^{d} \{ (h_{e_k} h_{j,k})(\eta) - h_{j,k}(\eta) \}.
\]  

(2.2)

In this formula, \( \{\tau_z : z \in \mathbb{Z}^d\} \) represents the group of translations acting on the configurations:

\[
(\tau_{x\eta})_z = \eta_{x+z}, \quad x, z \in \mathbb{Z}^d, \quad \eta \in \{0,1\}^{2d}.
\]  

(2.3)

Denote by \( L_n^S \) the generator of the speed-change, symmetric exclusion process given by

\[
(L_n^S f)(\eta) = \sum_{x \in \mathbb{T}_n^d} \sum_{j=1}^{d} c_j(\tau_{x\eta}) \{ f(\sigma^{x,y} \eta) - f(\eta) \}.
\]  

(2.4)

In this formula, \( \sigma^{x,y} \eta \) represents the configuration of particles obtained from \( \eta \) by exchanging the values of \( \eta_x \) and \( \eta_y \):

\[
(\sigma^{x,y} \eta)_z = \begin{cases} 
\eta_z, & z \neq x, \ y, \\
\eta_x, & z = y, \\
\eta_y, & z = x,
\end{cases}
\]

and \( \{\tau_x : x \in \mathbb{T}_n^d\} \) the translations acting on \( \Omega_n \). The summation and translation in (2.4) now have to be understood modulo \( n \). We used the same notation for translations acting on \( \Omega_n \) and on \( \{0,1\}^{2d} \), but the context will make clear which one we are referring to.

In the special case where \( c_j(\eta) = 1 \) for all \( 1 \leq j \leq d \), we recover the symmetric simple exclusion process on \( \mathbb{T}_n^d \), whose generator, denoted by \( L_n^E \), to stress this particular case, can be written as

\[
(L_n^E f)(\eta) = \sum_{x \in \mathbb{T}_n^d} \sum_{y \in \mathbb{T}_n^d, \|y-x\|=1} \eta_x (1 - \eta_y) \left[ f(\eta^{x,y}) - f(\eta) \right].
\]

Denote by \( \nu_\rho^n, 0 \leq \rho \leq 1 \), the Bernoulli product measure on \( \Omega_n \) with density \( \rho \). This is the product measure whose marginals are Bernoulli distributions with parameter \( \rho \). A straightforward computation shows that these measures satisfy the detailed balance conditions for the speed-change exclusion process because the cylinder functions \( c_j \) are assumed not to depend on \( \eta_0, \eta_{e_j} \). In particular, they are stationary for this dynamics.

Fix a sequence of positive numbers \( \{a_n : n \in \mathbb{N}\} \) such that \( \lim_{n \to \infty} a_n = \infty \) and \( \lim_{n \to \infty} a_n/n^2 = 0 \). Let \( (\eta^n(t) ; t \geq 0) \) be the \( \Omega_n \)-valued, continuous-time Markov chain whose generator, denoted by \( L_n \), is given by

\[
L_n = n^2 L_n^S + a_n L_n^V.
\]

We call this process the voter model with stirring [7, 8].
Denote by $D([0,T], \Omega_n)$, $T > 0$, the set of right-continuous trajectories $\xi : [0,T] \rightarrow \Omega_n$ with left-limits, endowed with the Skorohod topology. For a probability measure $\mu_n$ on $\Omega_n$, denote by $\mathbb{P}_\mu$ the measure on $D([0,T], \Omega_n)$ induced by the Markov chain $\eta^n(t)$ and the initial distribution $\mu_n$.

It can be verified that the measures $\nu^n_\rho$ are not invariant with respect to $L_n$. Actually, the only extremal measures are the singletons supported on the empty and the full configurations. However, as the Bernoulli product measures are stationary for the exclusion dynamics, and since the exclusion generator is accelerated by $n^2$, while the voter one is accelerated by $a_n$, and $a_n/n^2 \rightarrow 0$, we expect the law of the voter model with stirring to be close to a Bernoulli product measure if the initial state is close to this measure.

Our aim is to study the density fluctuations of this model, when the process starts from a measure close to a product Bernoulli measure $\nu^n_\rho$.

2.1. The density fluctuation field. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the continuous one-dimensional torus, and $\mathbb{T}^d$ be its $d$-dimensional version. Denote by $C(\mathbb{T}^d)$ the space of continuous, real-valued functions on $\mathbb{T}^d$ and by $C^k(\mathbb{T}^d)$, $1 \leq k \leq \infty$, the space of real-valued functions on $\mathbb{T}^d$ with continuous $k$-th derivatives. Elements of $C(\mathbb{T}^d)$ are represented by the letters $F, G$.

Denote by $L^2(\mathbb{T}^d)$ the space of complex-valued, square-integrable, measurable functions on $\mathbb{T}^d$ endowed with the usual scalar product, represented by $\langle \cdot, \cdot \rangle$. Let $\mathcal{H}_r$, $r > 0$, be the Hilbert space generated by the functions in $C^\infty(\mathbb{T}^d)$ with the scalar product $\langle \cdot, \cdot \rangle_r$ defined by

$$
\langle F, G \rangle_r = \sum_{m \in \mathbb{Z}^d} \gamma^r_m F_m \overline{G_m},
$$

where $\overline{z}$ represents the complex conjugate of $z \in \mathbb{C}$, $\gamma^r_m := 1 + \|m\|^2$, $F_m = \langle F, \phi_m \rangle$, and $\phi_m(x) = \exp\{2\pi i x \cdot m\}$. The sum is finite because $F, G$ belong to $C^\infty(\mathbb{T}^d)$. Clearly $\mathcal{H}_r \subset \mathcal{H}_s$ for $r \geq s$.

Denote by $\mathcal{H}_{-r}$, $r > 0$, the dual space of $\mathcal{H}_r$. Elements of $\mathcal{H}_{-r}$ are represented by the letters $X, Y$. For $X$ in $\mathcal{H}_{-r}$ and $F$ in $\mathcal{H}_r$, $X(F)$ can be represented as

$$
X(F) = \sum_{m \in \mathbb{Z}^d} X(\phi_m) F_m .
$$

Denote by $X^n_\rho$ the random element of $\mathcal{H}_{-1}$ defined by

$$
X^n_\rho(F) = \frac{1}{\sqrt{n^d a_n}} \sum_{x \in \mathbb{T}^d_n} F(x/n)(\eta^n_x(t) \rho - \rho) , \quad F \in C^\infty(\mathbb{T}^d) .
$$

This formula defines a $\mathcal{H}_{-r}$-valued process $\{X^n_\rho(t) : t \geq 0\}$. The exact order $r$ will be specified in the statements of the theorems. We call this process the density fluctuation field.

For a cylinder function $f : \{0,1\}^{2^d} \rightarrow \mathbb{R}$, denote by $\tilde{f} : [0,1] \rightarrow \mathbb{R}$ the function defined by

$$
\tilde{f}(\rho) = \mathbb{E}_{\nu_\rho} \left[f(\eta)\right] . \quad (2.5)
$$

In this formula, $\nu_\rho$ represents the Bernoulli product measure on $\{0,1\}^{2^d}$ with density $\rho$. Note that $\tilde{f}$ is a polynomial. Its first derivative is represented by $\tilde{f}'$. 


Recall the definition of the cylinder functions $h_{j,k}$ introduced in (2.2). Denote by $\mathcal{A}$ the second-order linear elliptic operator defined by
\[
\mathcal{A} F = \sum_{j,k=1}^{d} \hat{h}_{j,k}^\prime(\rho) \partial^2_{x_j,x_k} F ,
\]
for functions $F$ in $C^2(\mathbb{T}^d)$. Let $(P_t : t \geq 0)$ be the semigroup associated to the operator $\mathcal{A}$.

The functions $\phi_m$ are eigenvectors of the operator $\mathcal{A}$,
\[
\mathcal{A} \phi_m = -\lambda(m) \phi_m , \quad \text{where } \lambda(m) = 4 \pi^2 m^1 \mathbb{H} m .
\]
In this formula, $\mathbb{H} = (\mathbb{H}_{j,k})_{1 \leq j,k \leq d}$ stands for the symmetric matrix whose entries are given by $\mathbb{H}_{j,k} = (1/2) \{ \hat{h}_{j,k}^\prime(\rho) + \hat{h}_{k,j}^\prime(\rho) \}$ and $m^1$ for the transpose of $m$.

Denote by $D([0,T], \mathcal{H}_r)$ the space of $\mathcal{H}_r$-valued, right-continuous functions with left-limits, endowed with the Skorohod topology, and by $C([0,T], \mathcal{H}_r)$ the space of continuous functions endowed with the uniform topology.

Denote by $H_n(\mu | \nu)$ the relative entropy of the probability measure $\mu$ with respect to $\nu$ on $\Omega_n$:
\[
H_n(\mu | \nu) = \sup \left\{ \int_{\Omega_n} f d\mu - \log \int_{\Omega_n} e^{f} d\nu \right\},
\]
where the supremum is carried over all functions $f : \Omega_n \to \mathbb{R}$.

It is known [15, Theorem A1.8.3] that
\[
H_n(\mu | \nu) = H_n(f) := \int f \log f \, d\nu
\]
if $\mu$ is absolutely continuous with respect to $\nu$, and $f$ represents the Radon-Nikodym derivative $d\mu/d\nu$. Otherwise, $H_n(\mu | \nu) = \infty$.

**Theorem 2.1.** Suppose that $d = 1$ or $2$, and fix $0 < \rho < 1$, $T > 0$ and $r > (3d + 5)/2$. Assume that $(a_n : n \geq 1)$ is a sequence such that $a_n \rightarrow \infty$, $a_n \leq \sqrt{\log n}$. Let $\mu_n$ be a sequence of probability measures on $\Omega_n$ such that $\lim_{n \rightarrow \infty} a_n^{-1} H_n(\mu_n | \nu^n) = 0$. Then, the sequence of probability measures $\mathbb{Q}_n := \mathbb{P}_{\mu_n} \circ (X^n)^{-1}$ on $D([0,T], \mathcal{H}_r)$ converges weakly to the measure induced by the solution of the equation
\[
\begin{cases}
\partial_t X_t = \mathcal{A} X_t + \sqrt{4d} \chi(\rho) \xi_t, \\
X_0 = 0.
\end{cases}
\]
In this formula, $\chi(\rho) = \rho(1 - \rho)$ is the static compressibility of the stirring dynamics and $\xi$ is a standard space-time white noise.

In dimension 3, we are not able to prove the convergence of the process $X_t^n$ but only of its time integral. Let $X^n_t = \int_0^t X^n_s \, ds$.

**Theorem 2.2.** Suppose that $d = 3$, and fix $0 < \rho < 1$, $T > 0$ and $r > 8$. Assume that $(a_n : n \geq 1)$ is a sequence such that $a_n \rightarrow \infty$, $a_n \leq \sqrt{\log n}$. Let $\mu_n$ be a sequence of probability measures on $\Omega_n$ such that $\lim_{n \rightarrow \infty} a_n^{-1} H_n(\mu_n | \nu^n) = 0$. Then, under the measure $\mathbb{P}_{\mu_n}$, the finite-dimensional distributions of $X^n_t$ converge to the ones of the solution of (2.9). Moreover, the sequence of probability
measures $\mathbb{Q}_n := \mathbb{P}_{\mu_n} \circ (X^n)^{-1}$ on $C([0, T], \mathcal{H}_{-r})$ converges weakly to the measure induced by the time-integral of the solution of the equation (2.9).

**Remark 2.3.** The condition $a_n \leq \sqrt{\log n}$ is not optimal. We just need that $e^{C_0 a_n / \sqrt{n}} \to 0$ for all $C_0 > 0$. We also do not claim that the choice of $r$ is optimal.

**Remark 2.4.** Note that the fluctuation at time $0$, $X^n_0$, vanishes in the limit. The process $X^n_t$ is built exclusively by the noise.

**Remark 2.5 (The time-scales).** The large-time behavior of solutions of the stochastic heat equation (1.1) can be described as follows. Looking at the Fourier decomposition of $X_t$, the constant mode evolves like a standard Brownian motion; the other modes converge to a Gaussian free field. In particular, the law of $X_t$ is sensibly different from the spatial white noise, which is the scaling limit of the product Bernoulli measure. Therefore, the product Bernoulli measures are not a good approximation for the law of the particle system. The idea is that at small time-scales, the exclusion dynamics is dominant and the law of the particle system looks like a product Bernoulli measure of corresponding density. At larger time-scales, the voter dynamics is dominant and the law of the system is governed by the stochastic heat equation. The idea is to use the exclusion dynamics to average out some interaction terms; the scales at which we need to average these terms determine the range of scales at which our methods work. A worst-case scenario computation forces us to consider a very narrow region of parameters for the voter dynamics.

**Remark 2.6.** The result is restricted to dimensions $d \leq 3$ for the following reason. As is long known, the crux of the proof of the convergence of the density fluctuation fields lies in the so called Boltzmann-Gibbs principle, which permits the replacement of average of cylinder functions by their projections on the density field. The proof of this result relies on a bound on the entropy production, presented in the next subsection, which holds in all dimensions. This bound, however, is not strong enough in dimension $d \geq 4$ to yield the Boltzmann-Gibbs principle.

**Remark 2.7.** Usually the gradient condition requires that the jump rates $c_j$, $1 \leq j \leq d$, fulfill the following assumption. For each $j$, there exist cylinder functions $g_{j,p}$ and finitely-supported signed measures $m_{j,p}$, $1 \leq p \leq n_j$, such that

$$c_j(\eta)(\eta_{e_j} - \eta_0) = \sum_{p=1}^{n_j} \sum_{y \in \mathbb{Z}^d} m_{j,p}(y) (\tau_y g_{j,p})(\eta), \quad \sum_{y \in \mathbb{Z}^d} m_{j,p}(y) = 0 \quad (2.10)$$

for all $1 \leq p \leq n_j$. However, if conditions (2.10) are in force, then there exist cylinder functions $h_{j,k}$ for which (2.2) hold.

**Proof:** Fix $1 \leq j \leq d$ and consider the formula (2.10) for $c_j(\eta)(\eta_0 - \eta_{e_j})$. We omit $j$ from the notation from now on. As $\sum_{y \in \mathbb{Z}^d} m_p(y) = 0$ for all $p$, we can write this sum as

$$\sum_{p=1}^{n} \sum_{y \in \mathbb{Z}^d} m_p(y) \{ (\tau_y g_p)(\eta) - g_p(\eta) \}.$$
Fix $y$ such that $m_p(y) \neq 0$. Consider a path $0 = z_0, z_1, \ldots, z_{\|y\|} = y$ such that $\|z_{i+1} - z_i\| = 1$ for $0 \leq i < \|y\|$. With this notation,

$$\tau_y g_p - g_p = \sum_{i=0}^{\|y\|-1} \left( \tau_{z_{i+1}, g_p} - \tau_{z_i, g_p} \right).$$

Since $\|z_{i+1} - z_i\| = 1$, there exists $1 \leq k \leq d$ such that $z_{i+1} - z_i = \pm e_k$.

If $z_{i+1} - z_i = e_k$, let $g_{p,i} := \tau_{z_i, g_p}$ so that $\tau_{z_{i+1}, g_p} - \tau_{z_i, g_p} = \tau_{e_k, g_p} = \tau_{e_k, g_{p,i}} - g_{p,i}$. In contrast, if $z_{i+1} - z_i = -e_k$, let $g_{p,i} := -\tau_{z_{i+1}, g_p}$ so that $\tau_{z_{i+1}, g_p} - \tau_{z_i, g_p} = \tau_{e_k, g_{p,i}} - g_{p,i}$. With this notation,

$$\tau_y g_p - g_p = \sum_{i=0}^{\|y\|-1} \left( \tau_{e_k(p,i), g_p} - g_{p,i} \right).$$

Note that $g_{p,i}$ and $k(p, i)$ depend on $y$ but this fact has been omitted from the notation.

To complete the proof of the remark, it remains to fix $1 \leq \ell \leq d$ and define $h_\ell$ as

$$h_\ell = \sum_{p=1}^{n} \sum_{y \in \mathbb{Z}^d} m_p(y) \sum_{i} g_{p,i},$$

where the sum over $i$ is carried over all indices $i$ such that $k(p, i) = \ell$. $\square$

The main tool in the proof of Theorem 2.1 and 2.2 is an a priori bound on the entropy production of the process $\eta^n(t)$.

### 2.2. The entropy estimate.

Let $(S^n(t) : t \geq 0)$ be the semigroup of the voter model with stirring. Thus, $\mu S^n(t)$ represents the distribution at time $t$ of the process $\eta^n(\cdot)$ starting from the probability measure $\mu$.

It is well known that, for any initial distribution $\mu$, the relative entropy of $\mu S^n(t)$ with respect to a stationary measure decreases in time [15].

As stressed above, the Bernoulli product measures are not stationary for the voter model with stirring. Nevertheless, Theorem 2.8 states that the relative entropy of $\mu S^n(t)$ with respect to a product Bernoulli measure $\nu^\mu_n$ does not grow too fast. More precisely, for a sequence of probability measures $(\mu_n : n \geq 1)$ on $\Omega_n$, denote by $H_n(t)$ the relative entropy of $\mu_n S^n(t)$ with respect to a Bernoulli measure $\nu^\mu_n$:

$$H_n(t) = H_n \left( \mu_n S^n(t) \mid \nu^\mu_n \right).$$

**Theorem 2.8.** Fix a sequence of probability measures $(\mu_n : n \geq 1)$ on $\Omega_n$. Then, there exists a finite constant $C_0 = C_0(\rho)$ such that

$$H_n(t) \leq C_0 a_n \left\{ H_n(t) + R_d(n) \right\}$$

for all $t \geq 0$ and all $n \geq 1$. In this formula, $R_d(n)$ represents the sequence given by

$$R_d(n) = \begin{cases} \sqrt{a_n} & \text{for } d = 1, \\ a_n \log n & \text{for } d = 2, \\ a_n n^{d-2} & \text{for } d \geq 3. \end{cases}$$
It follows from the previous result and Gronwall’s lemma that
\[
H_n(t) \leq \left\{ H_n(0) + R_d(n) \right\} e^{C_0 a_n t}
\]  
(2.11)
for all \( t \geq 0 \).

**Remark 2.9.** Assume that \( a_n \leq \sqrt{\log n} \) and fix \( \kappa > 0, \ T > 0 \). There exists \( n_0 = n_0(\rho, \kappa, T) \) such that \( e^{C_0 a_n t} \leq n^\kappa \) for all \( n \geq n_0 \) and all \( 0 \leq t \leq T \). In particular, \( H_n(t) \leq \left\{ H_n(0) + R_d(n) \right\} n^\kappa \) for all \( 0 \leq t \leq T, \ n \geq n_0 \).

The article is organized as follows. In Section 3, we present a proof of Theorem 2.1 in the linear case (\( c_j(\eta) = 1 \)) to highlight the main steps of the argument. In Section 4, we present a sketch of the proofs of Theorems 2.1 and 2.2. In Section 5, we prove Theorem 2.8. The proof of this result is independent from the rest of the paper. In Section 6, we prove the Boltzmann-Gibbs principle. In Section 7, we prove the tightness of the sequences \( \mathbb{P}_{\nu^n} \circ (X^n)^{-1} \) in dimension 1 and 2, and the one of \( \mathbb{P}_{\nu^n} \circ (X^n)^{-1} \) in dimension 3. In Section 8, we compute the limits of the finite-dimensional distributions of these processes, completing the proofs of Theorems 2.1 and 2.2. In Appendix A, we present the entropy bounds used in the article, and, in Appendix B, some general results on continuous-time Markov chains. Finally, in Appendix C, we provide a decomposition of a cylinder function as the sum of polynomials of fixed degree.

3. **Proof of Theorem 2.1 in the linear case**

In this section we present a sketch of proof of Theorem 2.1 in the **linear case**, where \( c_j(\eta) = 1 \) for every \( n \in \mathbb{N}, \ \eta \in \Omega_n \) and \( j \in \{1, \ldots, d\} \).

By Assertion 4.1, for every \( F \in C^\infty(T^d) \), the process \( \{M^n_t(F); t \geq 0\} \) given by
\[
M^n_t(F) := X^n_t(F) - X^n_0(F) - (1 + a_n n^{-2}) \int_0^t X^n_s(\Delta_n F) \, ds, \quad t \geq 0 ,
\]  
(3.1)
is a martingale, where \( \Delta_n F \) denotes the discrete Laplacian of \( F \). By (7.7), the quadratic variation of this martingale is equal to
\[
\langle M^n_t(F) \rangle = \int_0^t \Gamma^n_s(F) \, ds ,
\]
where \( \Gamma^n_s(F) = \Gamma^n(F; \eta^n(s)) \) and
\[
\Gamma^n_s(F; \eta) = \frac{1}{a_n n^d} \sum_{j=1}^d \sum_{x \in \mathbb{Z}^d} (\eta_{x+j} - \eta_x)^2 [(\nabla_{n,j} F)(x/n)]^2
\]
\[  + \frac{1}{n^d} \sum_{x \in \mathbb{Z}^d} \sum_{y \parallel x-y \parallel = 1} F(x/n)^2 (\eta_y - \eta_x)^2 .
\]
The first sum is bounded by \( a_n^{-1} \|\nabla F\|_\infty^2 \), and therefore it converges to 0 as \( n \to \infty \).

If we were able to show that
\[
\lim_{n \to \infty} \int_0^t \frac{1}{n^d} \sum_{x \in \mathbb{Z}^d} \sum_{y \parallel x-y \parallel = 1} F(x/n)^2 (\eta^\eta(s) - \eta^\eta(s))^2 \, ds = 4d\rho(1 - \rho)t \|F\|^2 ,
\]  
(3.2)
then the proof of Theorem 2.1 would follow from the martingale characterization of solutions of (1.1) and a tightness argument. The factor $2\rho(1 - \rho)$ corresponds to the expectation of $(\eta_y - \eta_x)^2$ under the Bernoulli product measure $\nu^\rho_n$.

Here is an heuristic derivation of (3.2). At small scales, the dynamics is dominated by the exclusion part, and therefore the law of the process is close to $\nu^\rho_n$. Since the function $(\eta_y - \eta_x)^2$ is local, at first order it can be replaced by its average with respect to $\nu^\rho_n$. This argument can be made rigorous combining the entropy estimate stated in Theorem 2.8 and Lemma 7.4.

On the other hand, the reasoning presented in Section 7.4 shows that the martingale process $\{M^n_t; t \geq 0\}$, defined in (3.1), is tight for every dimension $d \geq 1$. Tightness of $\{X^n_t; t \geq 0\}$ can be derived from this result by the linear relation (3.1) between them. This proves Theorem 2.1.

It remains to discuss the proof of Theorem 2.8. Denote by $f^n_t$ the density of $\mu_n, S^n(t)$ with respect to $\nu^\rho_n$. Yau’s inequality states that

$$H^n_t(\eta) \leq -n^2 I_n(f^n_t) + a_n E_{\mu_n}[V(\eta^n(t))],$$

where

$$I_n(f) := \int \sum_{x \in \mathbb{T}_0^d} \sum_{j=1}^d [\sqrt{f(\sigma^{x,x+e_j}\eta)} - \sqrt{f(\eta)}]^2 d\nu^\rho_n$$

and

$$V(\eta) := \frac{2}{\rho(1 - \rho)} \sum_{x \in \mathbb{T}_0^d} \sum_{j=1}^d (\eta_x - \rho)(\eta_{x+e_j} - \rho).$$

The term $a_n E_{\mu_n}[V(\eta^n(t))]$ can be thought of as an entropy production term, which is responsible for the growth of $H_n(t)$. Recall that in the long run, the law of $\eta^n(t)$ is far from $\nu^\rho_n$, and this term puts this intuition in more quantitative grounds. We can think about $I_n(f^n_t)$ as an energy term that can be used to gain regularity on the entropy production term. The idea is to replace the local product $(\eta_x - \rho)(\eta_{x+e_j} - \rho)$ by the square of the averaged density of particles over a mesoscopic box, at the expenses of the energy term $I_n(f^n_t)$. After this replacement lemma, the remaining term can be estimated in terms of concentration inequalities and the entropy $H_n(t)$. The estimate of Theorem 2.8 follows by Gronwall’s inequality.

In the linear case Theorem 2.1 holds in every dimension $d \geq 1$ due to the linearity of the relation (3.1).

In the non-linear case, in view of Assertion 4.1, the martingale $M^n_t(F)$ is not expressed in terms of the density field $X^n_t$. Therefore, one needs to replace a local function average by a function of the fluctuation field $\{X^n_t; t \geq 0\}$. This is the content of the Boltzmann-Gibbs principle, stated in Theorem 4.3. A proof of this principle for non-equilibrium systems remained open for almost 30 years, and it was recently achieved in [13]. The proof of this result uses Theorem 2.8 as a starting point for a multiscale scheme, and it is the most technical part of this article. Note that it is the Boltzmann-Gibbs which restricts the validity of Theorem 2.1 to dimensions $d \leq 3$. 


4. OUTLINE OF THE PROOFS OF THEOREMS 2.1 AND 2.2

We present in this section the main steps of the proof of Theorems 2.1 and 2.2 in the non-linear case.

We first decompose the density field as the sum of a martingale and integral processes. Fix \( r > 0 \) and denote by \( M^n_t \) the \( \mathcal{H}_r \)-valued process defined by

\[
M^n_t := X^n_t - X^n_0 - \int_0^t L^n_s X^n_s ds, \quad F \in C^\infty(\mathbb{T}^d).
\]  

(4.1)

By [15, Lemma A.5.1], the process \( M^n_t(F) \) is a martingale for each \( F \) in \( C^\infty(\mathbb{T}^d) \).

We turn to the integral term.

**Assertion 4.1.** For every function \( F \) in \( C^\infty(\mathbb{T}^d) \),

\[
L^n_s X^n_s = \frac{1}{\sqrt{a_n h^d}} \sum_{j,k=1}^d \sum_{x \in \mathbb{T}^d} \{ h_{j,k}(\tau_x \eta) - \tilde{h}_{j,k}(\rho) \} (\Delta^n_{j,k} F)(x/n)
\]

\[+ \frac{a_n}{h^d} \frac{1}{\sqrt{a_n h^d}} \sum_{x \in \mathbb{T}^d} \{ \eta_x - \rho \} (\Delta_n F)(x/n). \]

In this formula, \( \tilde{h}_{j,k}(\rho) \) has been introduced in (2.5),

\[
(\Delta^n_{j,k} F)(x/n) = n^2 \left\{ F\left( \frac{x + e_j}{n} \right) - F\left( \frac{x}{n} \right) - F\left( \frac{x + e_k - e_j}{n} \right) + F\left( \frac{x - e_k}{n} \right) \right\},
\]

and \( (\Delta_n F)(x/n) = \sum_{j=1}^d (\Delta^n_{j,j} F)(x/n) \).

**Proof:** An elementary computation yields that

\[
L^n_s X^n_s = \frac{n^2}{\sqrt{a_n h^d}} \sum_{x \in \mathbb{T}^d} \sum_{j=1}^d c_{j}(\tau_x \eta) \{ \eta_x - \eta_x + e_j \} [F(x + e_j)/n) - F(x/n)]
\]

\[+ \frac{a_n}{h^d} \frac{1}{\sqrt{a_n h^d}} \sum_{x \in \mathbb{T}^d} \sum_{y \in \mathbb{T}^d} \{ \eta_x - \eta_y \}^2 [1 - 2\eta_x] F(x/n),
\]

(4.2)

where the sum over \( y \) is carried over all neighbours of \( x \).

Apply the gradient condition (2.2) to replace in the first sum on the right-hand side \( c_{j}(\eta) \{ \eta_0 - \eta_{e_j} \} \) by \( \sum_{j} h_{j,k}(\eta) - h_{j,k}(\tau_x \eta) \). In this difference, replace \( h_{j,k}(\eta) \) by \( h_{j,k}(\eta) - \tilde{h}_{j,k}(\rho) \). Finally, sum by parts to get that the first term on the right-hand side of the previous equation is equal to

\[
\frac{1}{\sqrt{a_n h^d}} \sum_{j,k=1}^d \sum_{x \in \mathbb{T}^d} \{ h_{j,k}(\tau_x \eta) - \tilde{h}_{j,k}(\rho) \} (\Delta^n_{j,k} F)(x/n).
\]

We turn to the second sum on the right-hand side of (4.2). Write \( 1 - 2\eta_x \) as \( (1 - \eta_x) - \eta_x \), note that \( [\eta_x - \eta_y]^2 (1 - \eta_x) = \eta_y (1 - \eta_x) \) and \( [\eta_x - \eta_y]^2 \eta_x = \eta_x (1 - \eta_y) \), to conclude that \( [\eta_x - \eta_y]^2 [1 - 2\eta_x] = \eta_y - \eta_x \). Hence, a summation by parts yields that the second term on the right-hand side of (4.2) is equal to

\[
\frac{a_n}{h^d} \frac{1}{\sqrt{a_n h^d}} \sum_{x \in \mathbb{T}^d} \sum_{j=1}^d \{ \eta_x - \rho \} (\Delta^n_{j,j} F)(x/n).
\]

This completes the proof of the assertion. \( \square \)
Recall the definition of the differential operator $\mathcal{A}$, introduced in (2.6), and the definition of the projection operators $\Pi^1_\rho$, $\Pi^\perp_\rho$, introduced in Assertion C.1. Write $L_n X^n(F)$ as

$$L_n X^n(F) = R^n(F) + B^n(F) + X^n(AF), \quad (4.3)$$

where

$$R^n(F) = \frac{1}{\sqrt{a_n n^d}} \sum_{j,k=1}^d \sum_{x \in \mathbb{T}^d_n} \{ h_{j,k}(\tau x \eta) - \tilde{h}_{j,k}(\rho) \} \{ (\Delta^2_{j,k} F) - (\partial^2_{x_j,x_k} F) \}(x/n)$$

$$+ \frac{a_n}{n^2} \frac{1}{\sqrt{a_n n^d}} \sum_{x \in \mathbb{T}^d_n} \{ \eta_x - \rho \} (\Delta_n F)(x/n)$$

$$+ \frac{1}{\sqrt{a_n n^d}} \sum_{j,k=1}^d \sum_{x \in \mathbb{T}^d_n} (\Pi^1_\rho h_{j,k})(\tau x \eta) (\partial^2_{x_j,x_k} F)(x/n),$$

and

$$B^n(F) = \frac{1}{\sqrt{a_n n^d}} \sum_{j,k=1}^d \sum_{x \in \mathbb{T}^d_n} (\Pi^\perp_\rho h_{j,k})(\tau x \eta) (\partial^2_{x_j,x_k} F)(x/n).$$

In view of (4.1) and (4.3), the process $X^n_t$ can be decomposed as

$$X^n_t(F) = X^n_0(F) + M^n_t(F) + \int_0^t R^n_s(F) \, ds + \int_0^t B^n_s(F) \, ds + \int_0^t X^n_s(AF) \, ds, \quad (4.4)$$

for $F \in C^\infty(\mathbb{T}^d)$.

We examine each term of the decomposition (4.4) separately. We start with $X^n_0$.

**Lemma 4.2.** Fix $0 < \rho < 1$, and let $\mu_n$ be a sequence of probability measures on $\Omega_n$ such that $\lim_{n \to \infty} a_n^{-1} H_n(\mu_n \| \nu^\rho) = 0$. Then,

$$\lim_{n \to \infty} \mathbb{E}_{\mu_n}[\|X^n_0\|_{2,r}^2] = 0$$

provided $r > d/2$.

**Proof.** By definition of the norm $\| \cdot \|_{2,r}$, we have to show that

$$\lim_{n \to \infty} \sum_{m \in \mathbb{Z}^d} \gamma^{-r}_m \mathbb{E}_{\mu_n}\left[ \left( \frac{1}{\sqrt{a_n n^d}} \sum_{x \in \mathbb{T}^d_n} \varphi_m(x/n) [\eta_x - \rho] \right)^2 \right] = 0,$$

where $\varphi_m(x) \in \{ \cos(2\pi x \cdot m), \sin(2\pi x \cdot m) \}$. We consider the cosine case, the other one being identical. By the entropy inequality, the expectation in the previous equation is bounded by

$$\frac{1}{A} H_n(\mu_n \| \nu^\rho) + \frac{1}{A} \log \mathbb{E}_{\mu_n}\left[ e^{(A/a_n) X_n(m)^2} \right],$$

where $X_n(m) = n^{-d/2} \sum_{x \in \mathbb{T}^d_n} \varphi_m(x/n) [\eta_x - \rho]$, and $A$ is arbitrary. By Corollary 4.5 below, there exist finite constants $0 < c_0 < C_0 < \infty$ such that

$$\mathbb{E}_{\mu_n}[\|X^n_0\|_{2,r}^2] \leq 2 \left( \frac{1}{A} H_n(\mu_n \| \nu^\rho) + \frac{C_0}{a_n} \right) \sum_{m \in \mathbb{Z}^d} \gamma^{-r}_m \quad (4.5)$$

provided $A < c_0 a_n$. Choose $A = c_0 a_n/2$ to complete the proof, since $\gamma^{-r}_m$ is summable in $m$. □
Let \((R_t^n : t \geq 0)\) be the \(H_{-r}\)-valued process given by \(R_t^n(F) = \int_0^t R^n_s(F) \, ds\), for \(F \in C^\infty(\mathbb{T}^d)\), \(t > 0\). Recall from the statement of Theorem 2.8 the definition of the sequences \(R_d(n)\). In Lemma 7.2, we prove that in dimension \(d \leq 3\), for any sequence of measures \(\mu_n\) such that 
\[
\lim_{n \to \infty} \mathbb{E}_{\mu_n} \left[ \sup_{0 \leq t \leq T} \| R_t^n \|_{r}^2 \right] = 0
\]  
provided \(r > 3 + (d/2)\).

The next result, the so-called Boltzmann-Gibbs principle, derived by Brox and Rost [3] in the context of equilibrium fluctuations, asserts that the local fields \(\{a_n, n^d\}^{-1/2} \sum_{x \in \mathbb{T}^d} G(x/n) [f(\tau_x n)(t) - \bar{f}(\rho)]\) are projected on the density field. It reads as follows. Denote by \(C_j^k(R_+ \times \mathbb{T}^d)\), \(j, k \geq 0\), the set of continuous functions \(G : R_+ \times \mathbb{T}^d \to \mathbb{R}\) which have \(j\) continuous derivatives in time and \(k\) continuous derivatives in space.

**Theorem 4.3** (Boltzmann-Gibbs principle). Assume that \(d \leq 3\) and fix \(0 < \rho < 1\). Let \(\mu_n\) be a sequence of probability measures on \(\Omega_n\) such that \(H_n(\mu_n \| \nu_\rho^n) \leq R_d(n)\). Then,
\[
\lim_{n \to \infty} \mathbb{E}_{\mu_n} \left[ \int_0^t \frac{1}{\sqrt{a_n n^d}} \sum_{x \in \mathbb{T}^d} G(s, x/n) (\Xi_{\rho}(f)(\tau_x n)(s)) \, ds \right] = 0 ,
\]
for all \(t > 0\), functions \(G \in C_{0,1}(R_+ \times \mathbb{T}^d)\), and cylinder functions \(f : \{0, 1\}^{2d} \to \mathbb{R}\). In this formula, \(\Xi_{\rho}\) stands for the operator \(\Pi_{\rho}\), introduced in \((C.3)\).

The proof of this result is given in Section 6, where quantitative bounds are provided. We show in \((6.1)\) that this statement holds with the absolute value inside the time-integral for \(\Xi_{\rho} = \Pi_{\rho}^1\). This result is a simple consequence of a summation by parts and the entropy estimate. The real challenging part is to prove Theorem 4.3 for \(\Xi_{\rho} = \Pi_{\rho}^{2}\). We stated this result with \(\Xi_{\rho} = \Pi_{\rho}\) for historical reasons and to stress that the dynamics projects averages of cylinder functions on the density field [since \((\Pi_{\rho} f)(\eta) = f(\eta) - \bar{f}(\rho) - \bar{f}'(\rho)(\eta_0 - \rho)\)].

In view of \((4.4)\), \((4.6)\) and Theorem 4.3, the process \(X_t^n\) can be written as
\[
X_t^n(F) = M_t^n(F) + \int_0^t X^n(\Delta F) \, ds + E_n(t) ,
\]  
where \(E_n(t)\) is an error term which vanishes as \(n \to \infty\). Therefore, besides tightness of the process, the proofs of Theorems 2.1 and 2.2 reduces to showing that the martingale part, \(M_t^n\), converges to a white-noise. This is the content of Section 8.

**Concentration inequalities.** We conclude this section recalling some results on subgaussian random variables. A mean-zero random variable \(X\) is said to be \(\sigma^2\)-subgaussian if \(E[\exp\{\theta X\}] \leq \exp\{\sigma^2 \theta^2 / 2\}\) for all \(\theta \in \mathbb{R}\).

By [12, Proposition B.1], if \(X\) is a \(\sigma^2\)-subgaussian random variable,
\[
E[e^{aX^2}] \leq e^{8\sigma^2 a^2} \tag{4.8}
\]
for all \(0 < a < 1/4\sigma^2\).

According to Hoeffding’s inequality [2, Lemma 2.2], a mean-zero random variable taking values in the interval \([a, b]\) is \([b - a]^2/4\)-subgaussian. Next lemma follows from this result.
Lemma 4.4. Let $X_1, \ldots, X_p$ be independent mean-zero random variables, and suppose that $X_j$ takes values in the interval $[a_j, b_j]$. Then $\sum_{1 \leq j \leq p} X_j$ is $A$-subgaussian, where $A = (1/4) \sum_{1 \leq j \leq p} (b_j - a_j)^2$.

This result provides an estimate in the context of the voter model with stirring.

Corollary 4.5. Fix a cylinder function $f$ and a function $F : \mathbb{T}^d_n \to \mathbb{R}$. Then there exist constants $0 < c_0 < C_0 < \infty$, depending only on the cylinder function $f$, such that

$$\log E_{\nu_{\rho}} \left[ \exp a \left\{ \frac{1}{\sqrt{n}^d} \sum_{x \in \mathbb{T}^d_n} \left[ f(\tau_x \eta) - \tilde{f}(\rho) \right] F_x \right\}^2 \right] \leq C_0 a \| F \|_{\infty}^2,$$

for all $0 < a < c_0 \| F \|_{\infty}^2$.

Proof: Let $p \geq 1$ be the smallest integer such that $\Xi_p := \{-p, \ldots, p\}^d$ contains the support of the cylinder function $f$. In particular, under the product measure $\nu_{\rho}$, the random variables $\tau_x f$ and $\tau_y f$ are independent if $y - x \notin \Xi_{2p+1}$.

Let $q = 2p + 1$, and write

$$\sum_{x \in \mathbb{T}^d_n} \left\{ f(\tau_x \eta) - \tilde{f}(\rho) \right\} F_x = \sum_{z \in \Xi_q} \sum_{y} \left\{ f(\tau_{z+qy} \eta) - \tilde{f}(\rho) \right\} F_{z+qy},$$

where the second sum on the right-hand side is performed over all $y \in \mathbb{Z}^d$ such that $z + qy \in \{0, \ldots, n-1\}^d$.

By Schwarz and Hölder inequalities, the expression on the left-hand side of the statement of the corollary is bounded above by

$$\frac{1}{(2q + 1)^d} \sum_{z \in \Xi_q} \log E_{\nu_{\rho}} \left[ \exp a(2q + 1)^{2d} \left\{ \frac{1}{\sqrt{n}^d} \sum_{y} \left\{ f(\tau_{z+qy} \eta) - \tilde{f}(\rho) \right\} F_{z+qy} \right\}^2 \right].$$

By Lemma 4.4, under the measure $\nu_{\rho}$, $n^{-d/2} \sum_{y} \left\{ f(\tau_{z+qy} \eta) - \tilde{f}(\rho) \right\} F_{z+qy}$ is an $A$-subgaussian random variable, where $A = \| f \|_{\infty}^2 \| F \|_{\infty}^2$. Thus, for $a < 1/(4(2q + 1)^{2d} \| f \|_{\infty}^2 \| F \|_{\infty}^2)$, by (4.8), the previous expression is less than or equal to

$$8 a \| f \|_{\infty}^2 \| F \|_{\infty}^2,$$

as claimed. \[ \square \]

5. Proof of Theorem 2.8

In this section, we prove Theorem 2.8. We start with Yau’s inequality, stated in Proposition 5.1 below. This inequality estimates the rate of grow of the relative entropy $H_n(t)$ by two terms: a negative term, which we interpret as an energy by analogy with the theory of parabolic differential equations, and an expectation of the form $a_n \mathbb{E}_{\mu_n} [V(\eta^n(t))]$ for some explicit function $V$, see (5.2), which we call the entropy production. Then we aim to replace $V$ by a function $V^\ell$, see (5.7) on which we introduce spatial averages over mesoscopic boxes of size $\ell$. This replacement has a cost that can be estimated by the energy term and a renormalized function, see Lemma 5.5. The integrals of both the renormalized average and $V^\ell$ are then estimated by the entropy $H_n(t)$ by means of a concentration inequality, see Lemma 5.7 and equations (5.13), (5.14). Theorem 2.8 then follows by Gronwall’s inequality.
The statement of the first result requires some notation. Denote by $I_n$ the large deviations rate functional given by

$$I_n(f) := -\int (L_n^S \sqrt{f}) \sqrt{f} \nu_n^\rho(d\eta). \quad (5.1)$$

As $c_j$ does not depend on the variables $\eta_0$, $\eta_e$, an elementary computation yields that

$$I_n(f) = \frac{1}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}_n^d} c_j(\tau_x \eta) \left[ \sqrt{f(\sigma_x \tau_x \eta)} - \sqrt{f(\eta)} \right]^2 \nu_n^\rho.$$

Let $L_n^{S*}$, $L_n^V$ be the adjoints of the generators $L_n^S$, $L_n^V$ in $L^2(\nu_n^\rho)$, respectively. Thus, for all $f, g \in L^2(\nu_n^\rho),$

$$\int (L_n^B f) g \ d\nu_n^\rho = \int f (L_n^{B*} g) \ d\nu_n^\rho,$$

for $B \in \{S, V\}$.

Since the Bernoulli measures $\nu_n^\rho$ satisfy the detailed balance conditions for the exclusion dynamics, $L_n^{S*} = L_n^S$. On the other hand, an explicit computation yields that

$$(L_n^{V*} h)(\eta) = \sum_{x \in \mathbb{T}_n^d} \sum_{y : |y - x| = 1} \left\{ \eta_x \eta_y \frac{1 - \rho}{\rho} + (1 - \eta_x) (1 - \eta_y) \frac{\rho}{1 - \rho} \right\} h(\sigma_x \eta) - \sum_{x \in \mathbb{T}_n^d} \sum_{y : |y - x| = 1} (\eta_x - \eta_y)^2 h(\eta)$$

for all functions $h : \Omega_n \to \mathbb{R}$.

Denote by $1 : \Omega_n \to \mathbb{R}$ the function which is constant equal to 1, and by $V$ the function $L_n^{V*} 1$. Note that $V$ would vanish if $\nu_n^\rho$ were invariant for $L_n^V$ because in this case $L_n^{V*}$ would be the generator of a Markov chain. Thus, in a vague sense, $V = L_n^{V*} 1$ indicates how far is $\nu_n^\rho$ from the stationary state for $L_n^V$. It follows from the explicit formula for $L_n^{V*}$ that

$$V(\eta) := (L_n^{V*} 1)(\eta) = \sum_{x \in \mathbb{T}_n^d} \sum_{y : |y - x| = 1} \omega_x \omega_y = 2 \sum_{j=1}^d \sum_{x \in \mathbb{T}_n^d} \omega_x \omega_{x+e_j}, \quad (5.2)$$

where

$$\omega_x := \frac{\eta_x - \rho}{\sqrt{\rho(1 - \rho)}}, \quad x \in \mathbb{T}_n^d. \quad (5.3)$$

Notice that $\{\omega_x ; x \in \mathbb{T}_n^d\}$ is an orthonormal family with respect to the measure $\nu_n^\rho$.

Recall the definition of the relative entropy $H_n(t)$, introduced above the statement of Theorem 2.8.

**Proposition 5.1.** Fix a probability measure $\mu_n$ on $\Omega_n$, and let $f_n^\rho$, $t \geq 0$, be the density of $\mu_n S_n^\rho(t)$ with respect to $\nu_n^\rho$, 

$$f_n^\rho := \frac{d \mu_n S_n^\rho(t)}{d \nu_n^\rho}.$$
Then,
\[ H'_{n}(t) \leq -2n^2I(f^n_{t}) + a_n \int V f^n_{t} \, dv^\rho, \]
for all \( t \geq 0 \).

**Proof.** By [15, equation (A1.9.1)], the density \( f^n_{t} \) solves the equation
\[ \frac{d}{dt}f^n_{t} = (n^2 L^S_{n} + a_n L^V_{n}) f^n_{t}. \]  
(5.4)

On the other hand, by (2.8),
\[ H_{n}(t) = H_{n}(f^n_{t}) = \int f^n_{t} \, \log f^n_{t} \, dv^\rho. \]

By the relative entropy bound [15, Theorem A1.9.2] and (5.4),
\[ H'_{n}(t) \leq -2n^2I(f^n_{t}) + a_n \int (L^V_{n} \log f^n_{t}) f^n_{t} \, dv^\rho, \]  
(5.5)
where \( I \) is the functional introduced in (5.1).

Since \( \log r \leq r - 1 \) for \( r > 0 \),
\[ (L^V_{n} \log f^n_{t}) f^n_{t} \leq L^V_{n} f^n_{t}. \]
The second term on the right-hand side of (5.5) is thus bounded by
\[ a_n \int L^V_{n} f^n_{t} \, dv^\rho = a_n \int (L^V_{n} \cdot 1) f^n_{t} \, dv^\rho = a_n \int V f^n_{t} \, dv^\rho, \]
as claimed. \( \square \)

Let \( m_\ell, \ell \geq 1 \), be the uniform measure on the cube \( \Lambda_\ell := \{0, 1, \ldots, \ell - 1\}^d \),
\[ m_\ell(z) := \frac{1}{\ell^d} \chi_{\Lambda_\ell}(z), \]
where \( \chi_A \) stands for the indicator of the set \( A \).
Let \( m_\ell^{(2)} \) be the convolution of \( m_\ell \) with itself:
\[ m_\ell^{(2)}(z) = \sum_{y \in \mathbb{T}^d_n} m_\ell(y) m_\ell(z - y), \]
Notice that \( m_\ell^{(2)} \) is supported on the cube \( \Lambda_{2\ell - 1} \).

Recall the definition of the variables \( \omega_x \) introduced in (5.3) and keep in mind that \( \omega_x \) depends on \( \eta_x \). Denote by \( \omega^\ell_x \) the average of \( \omega_{x+z} \) with respect to the measure \( m_\ell^{(2)} \):
\[ \omega^\ell_x = \sum_{y \in \mathbb{T}^d_n} m_\ell^{(2)}(y) \omega_{x+y} = \sum_{y \in \Lambda_{2\ell - 1}} m_\ell^{(2)}(y) \omega_{x+y}, \]  
(5.6)
and let \( V_\ell : \Omega_n \to \mathbb{R} \) be given by
\[ V_\ell(\eta) := 2 \sum_{j=1}^{d} \sum_{x \in \mathbb{T}^d_n} \omega_x \omega^\ell_{x+e_j}. \]  
(5.7)
A change of variables yields that
\[ V_\ell(\eta) = 2 \sum_{j=1}^{d} \sum_{x \in \mathbb{T}^d_n} \left( \sum_{y \in \Lambda_\ell} m_\ell(y) \omega_{x-y} \right) \left( \sum_{z \in \Lambda_\ell} m_\ell(z) \omega_{x+e_j+z} \right). \]  
(5.8)
Notice that the averages are performed over disjoint sets due to the definition of \( m_t \): for every \( x \) and \( j \), the sets \( \{ x \in \Lambda_t \} \) and \( \{ x + e_j \} \) are disjoint.

Let \( (g_d(n) : n \geq 1) \) be the sequence defined by

\[
g_d(n) = \begin{cases} n & \text{if } d = 1 \\ \log n & \text{if } d = 2 \\ 1 & \text{if } d \geq 3. \end{cases}
\]  

(5.9)

**Proposition 5.2.** There exists a finite constant \( C_1(\rho) \), depending only on \( \rho \) and the constant \( c_0 \) introduced in (2.1), such that

\[
a_n \int \{ V(\eta) - V_\ell(\eta) \} f \, d\nu^n_\rho \leq \delta n^2 I_n(f) + \frac{C_1(\rho)}{\delta n^2} \{ H_n(f) + (n/\ell)^d \}
\]

for every \( 1 \leq \ell < n/4 \), \( \delta > 0 \) and density \( f \) with respect to \( \nu^n_\rho \).

The proof of this proposition is divided in several steps.

**Integration by parts.** For \( x \in T^d_n \), \( 1 \leq j \leq d \), let \( I_{x,x+e_j} \) be the functional \( I_n \) restricted to the bond \( \{ x, x + e_j \} \):

\[
I_{x,x+e_j}(h) = \frac{1}{2} \int c_2(\tau_\eta) \left\{ \sqrt{h(\sigma^{x,x+e_j} \eta)} - \sqrt{h(\eta)} \right\}^2 \, d\nu^n_\rho,
\]

\( h : \Omega_n \to \mathbb{R} \). The proof of the next result is omitted, being similar to the one of [16, Lemma 3.1]. (It is enough to perform the change of variables \( \xi = \sigma^{x,x+e_j} \eta \) to write \( h(\eta) [\eta_{x+e_j} - \eta_x] f(\eta) \) as \((1/2) h(\eta) [\eta_{x+e_j} - \eta_x] [f(\eta) - f(\sigma^{x,x+e_j} \eta)]\), and to apply Schwarz inequality). Recall the definition of the constant \( c_0 \) introduced in (2.1).

**Lemma 5.3.** Fix \( x \in T^d_n \), \( 1 \leq j \leq d \) and \( h : \Omega_n \to \mathbb{R} \) such that \( h(\sigma^{x,x+e_j} \eta) = h(\eta) \) for all \( \eta \in \Omega_n \). Then,

\[
\int h[\eta_{x+e_j} - \eta_x] f \, d\nu^n_\rho \leq \frac{\beta}{2} I_{x,x+e_j}(f) + \frac{1}{2 c_0 \beta} \int h^2 f \, d\nu^n_\rho
\]

for all \( \beta > 0 \) and density \( f : \Omega_n \to [0, \infty) \) with respect to \( \nu^n_\rho \).

**Flows.** Let \( G \) be a finite set. For probability measures \( \mu \) and \( \nu \) on \( G \), a function \( \Phi : G \times G \to \mathbb{R} \) is called a flow connecting \( \mu \) to \( \nu \) if

1. \( \Phi(x, y) = -\Phi(y, x) \), for all \( x, y \in G \);
2. \( \sum_{y \in G} \Phi(x, y) = (\mu(x) - \nu(x)) \), for all \( x \in G \).

The next result is [12, Corollary 3.12]. Recall the definition of the sequence \( g_d(\ell) \) introduced in (5.9).

**Lemma 5.4.** There exist a finite constant \( C_\ell \), depending only on the dimension \( d \), and, for all \( \ell \geq 1 \), a flow \( \Phi_\ell \) connecting the Dirac measure at the origin to the measure \( m^{(2)}_\ell \). This flow is supported in \( \Lambda_{2\ell-1} \) and on nearest-neighbour bonds.

That is,

\[
\Phi_\ell(x, y) = 0
\]

if \( \| y - x \| \neq 1 \) or if \( \{ x, y \} \not\subseteq \Lambda_{2\ell-1} \). Moreover,

\[
\sum_{j=1}^d \sum_{x \in T^d_n} \Phi_\ell(x, x + e_j)^2 \leq C_\ell g_d(\ell).
\]
Consider the partial order \( \prec \) on \( \mathbb{Z}^d \) defined by \((x_1, \ldots, x_d) \prec (y_1, \ldots, y_d)\) if \( x_j \leq y_j \) for all \( 1 \leq j \leq d \). Fix a subset \( A = \{x_k : k \in J\} \) of \( \mathbb{Z}^d \), where \( J \subset \mathbb{N} \) and \( x_k = (x_k,1,\ldots,x_k,d) \). A point \( x_k \) in \( A \) is said to be maximal if \( x_k \prec x_j \) entails that \( x_k = x_j \).

Every finite subset \( A \) of \( \mathbb{Z}^d \) has at least one maximal element. Fix a finite subset \( A \) of \( \mathbb{Z}^d \) with at least two elements, \( A = \{x_1, \ldots, x_p\}, p \geq 2 \). Denote by \( x_A \) a maximal element of \( A \), and let \( A_* = A \setminus \{x_A\} \).

Recall the definition of the average \( \omega_x^f \) introduced in (5.6). For a finite subset \( A \) of \( \mathbb{Z}^d \), let

\[
\omega_A = \prod_{x \in A} \omega_x^f.
\]

**Lemma 5.5.** Fix a finite subset \( A \) of \( \mathbb{Z}^d \) with at least two elements, a function \( G : \mathbb{T}_n^d \to \mathbb{R}, b_n \in \mathbb{R} \) and \( \ell \geq 1 \). Let

\[
W = \sum_{x \in \mathbb{T}_n^d} G_x \omega_{x+A_*} \left( \omega_{x+x_A} - \omega_{x+x_A} \right).
\]

Then,

\[
b_n \int W \ f \ d\nu_n^\rho \leq \frac{\beta}{2} n^2 I_n(f) + \frac{b_n^2}{2 c_0 n^2} \chi(\rho) \sum_{k=1}^{d} \sum_{x \in \mathbb{T}_n^d} (H_{k,x}^{(\ell)})^2 f \ d\nu_n^\rho
\]

for all \( \beta > 0 \), density \( f \) with respect to \( \nu_n^\rho \) and \( n > 4\ell \). In this formula, \( \chi(\rho) = \rho(1 - \rho) \) is the static compressibility of the exclusion process, introduced in the statement of Theorem 2.1, and

\[
H_{k,x}^{(\ell)} = \sum_{(y,y+e_k) \in \Lambda_{2\ell-1}} \Phi_\ell(y, y+e_k) G(x-x_A-y) \omega_{x-x_A-y+A_*},
\]

where \( \Phi_\ell \) is the flow introduced in Lemma 5.4.

**Proof:** Assume without loss of generality that \( x_A = 0 \) [Otherwise, in the sum defining \( W \) change variables to \( x' = x + x_A \)]. This means that the origin is a maximal point in \( A \). Let \( B = A_* \), and rewrite \( W \) as

\[
W(\eta) = \sum_{x \in \mathbb{T}_n^d} G_x \omega_{x+B} \sum_{y \in \Lambda_{2\ell-1}} \omega_{x+y} \left\{ \delta_0(y) - m_\ell(2)(y) \right\},
\]

where \( \delta_0 \) stands for the Dirac measure concentrated at \( 0 \).

Denote by \( \Phi_\ell \) the flow introduced in Lemma 5.4. In the previous equation, we may replace \( \Lambda_{2\ell-1} \) by \( \mathbb{Z}^d \). This simplifies the summation by parts performed below. After this replacement, since the flow connects \( \delta_0 \) to \( m_\ell(2) \), it is anti-symmetric and supported on nearest-neighbour bonds, the sum over \( y \) becomes

\[
\sum_{y \in \mathbb{Z}^d} \sum_{z: \|z\|=1} \Phi_\ell(y, y+z)
\]

\[
= \sum_{k=1}^{d} \sum_{y \in \mathbb{Z}^d} \omega_{x+y} \left\{ \Phi_\ell(y, y+e_k) - \Phi_\ell(y, -e_k, y) \right\}.
\]

Performing a summation by parts, this last sum becomes

\[
\sum_{k=1}^{d} \sum_{y \in \Lambda_{2\ell-1}} \Phi_\ell(y, y+e_k) \left\{ \omega_{x+y} - \omega_{x+y+e_k} \right\}.
\]
As $\Phi_y$ is supported on $\Lambda_{2^l-1}$, we may restrict the sum over $y$ to the set of all points in $\mathbb{Z}^d$ such that $\{y, y + e_k\} \subset \Lambda_{2^l-1}$.

Perform a change of variables $x' = x + y$ to conclude that

\[
W(\eta) = \sum_{k=1}^{d} \sum_{x \in \mathbb{Z}^d} \{ \omega_x - \omega_{x+e_k} \} \sum_{(y, y+e_k) \subset \Lambda_{2^l-1}} \Phi_y(y, y + e_k) G_{x-y} \omega_{x-y+B}
\]

\[
= \sum_{k=1}^{d} \sum_{x \in \mathbb{Z}^d} \{ \omega_x - \omega_{x+e_k} \} H_{k,x}(\eta)
\]

where $H_{k,x}(\eta)$ has been introduced in the statement of the lemma. Since the origin is a maximal point of $A$, the support of $H_{k,x}(\eta)$ is disjoint from $\{x, x + e_k\}$ in the sense that the indices $z$ of $\omega_z$ which appear in the definition of $H_{k,x}(\eta)$ are different from $x$ and $x + e_k$. In particular,

\[
H_{k,x}(\sigma_{x,x+e_k} \eta) = H_{k,x}(\eta).
\]

Fix a density $f : \Omega_{\eta} \to [0, \infty)$ with respect to $\nu_{\rho}^n$. In view of the formula for $W(\eta)$, by Lemma 5.3 and (5.11),

\[
b_n \int W f \, d\nu_{\rho}^n \leq \frac{\beta n^2}{2} \sum_{k=1}^{d} \sum_{x \in \mathbb{Z}^d} I_{x,x+e_k}(f) + \frac{b_n^2}{2 \epsilon_0 \beta n^2 \chi(\rho)} \sum_{k=1}^{d} \sum_{x \in \mathbb{Z}^d} (H_{k,x}(\eta))^2 f \, d\nu_{\rho}^n
\]

for all $\beta > 0$, as claimed.

Recall from (5.2), (5.7) the definitions of the functions $V$, $V_\ell$. Lemma 5.5 with $b_n = a_n$, $G = 2$, $A = \{0, e_j\}$, $x = e_j$, $\beta = 2\delta/d$ yields the next result.

**Corollary 5.6.** For all $n$ large enough and density $f$ with respect to $\nu_{\rho}^n$,

\[
a_n \int \{ V(\eta) - V_\ell(\eta) \} f \, d\nu_{\rho}^n
\]

\[
\leq \frac{\delta n^2 I_n(f)}{4 \epsilon_0 \delta n^2 \chi(\rho)} \sum_{j,k=1}^{d} \sum_{x \in \mathbb{Z}^d} (H_{j,k,x}(\eta))^2 f \, d\nu_{\rho}^n
\]

(5.12)

for all $\delta > 0$, where

\[
H_{j,k,x}(\eta) = 2 \sum_{(y, y+e_k) \subset \Lambda_{2^l-1}} \Phi_y(y, y + e_k) \omega_{x-y-e_j}.
\]

**Proof of Proposition 5.2.** In view of Corollary 5.6, we have to estimate the second term on the right-hand side of (5.12).

By the entropy inequality, the second term of this expression is bounded by

\[
\frac{1}{\gamma} H_n(f) + \frac{1}{\gamma} \log \int \exp \left\{ \frac{d^2 a_n^2 \gamma}{4 \epsilon_0 \delta n^2 \chi(\rho)} \sum_{j,k=1}^{d} \sum_{x \in \mathbb{Z}^d} (H_{j,k,x}(\eta))^2 \right\} \, d\nu_{\rho}^n
\]

(5.13)

for all $\gamma > 0$. Under the measure $\nu_{\rho}^n$, the variables $H_{j,k,x}(\eta), H_{l,m,y}(\eta)$ are independent if $\|x - y\| \geq 2d \ell$. Hence, rewriting the sum over $x$ as $C \ell^d$ sums of terms spaced by $2d \ell$ and applying Hölder’s inequality [see the Proof of Lemma 6.1.8]
in [15] for a detailed presentation of this step] yield that the second term of the previous expression is bounded by

\[ \frac{C_0}{\gamma L} \sum_{j,k=1}^{d} \sum_{x \in \mathbb{T}_n^d} \log \int \exp \left\{ \frac{C_0 a_n^2 \gamma \ell^d}{\delta n^2} \chi(\rho) (H(t)^{y,k,x})^2 \right\} \, d\nu^a \tag{5.14} \]

for some finite constants \( C_0 \) depending only on the dimension.

By Lemma 4.4, \( H(t)^{y,k,x} \) is a \( \sigma_f^2 \)-subgaussian random variable, where

\[ \sigma_f^2 = \frac{C_0}{\chi(\rho)} \sum_{y,y+e_k \in \Lambda_{2t-1}} \Phi(y,y+e_k)^2. \]

By Lemma 5.4, \( \sigma_f^2 \leq C_0 g_d(\ell)/\chi(\rho) \). Therefore, by (4.8), there exists a finite constant \( C_1(\rho) \) for which the sum (5.14) is bounded by

\[ C_1(\rho) \delta^{-1} a_n^2 n^{d-2} g_d(\ell), \]

for all \( \gamma \) such that \( C_1(\rho)(a_n/n)^2 \ell^d g_d(\ell) \gamma < \delta/4 \).

Choose \( \gamma^{-1} = C_1(\rho)(a_n/n)^2 \ell^d g_d(\ell) \delta^{-1} \), where the value of the constant \( C_1(\rho) \) has changed, to conclude that (5.13) is bounded above by

\[ \frac{C_1(\rho) a_n^2 \ell^d g_d(\ell)}{\delta n^2} H_n(f) + C_1(\rho) \delta^{-1} a_n^2 n^{d-2} g_d(\ell) \]

\[ = \frac{C_1(\rho) a_n^2 \ell^d g_d(\ell)}{\delta n^2} \left\{ H_n(f) + (n/\ell)^d \right\}, \]

as claimed. \( \square \)

The next result is the main step in the estimation of \( V_t \). Recall that \( \chi(\rho) = \rho(1 - \rho) \). For a finite subset \( B \) of \( \mathbb{Z}^d, \ell \geq 1 \) and a function \( G : \mathbb{T}_n^d \rightarrow \mathbb{R} \), let

\[ M_t(x) = \sum_{y \in \Lambda_{\ell}} m_t(y) G(x-y) \omega_{x-y+B}. \]

Note that we omit from the notation the dependence of \( M_t(x) \) on the set \( B \).

**Lemma 5.7.** There exists a finite constant \( C_0 \), depending only on the dimension and the set \( B \), such that

\[ \int \sum_{x \in \mathbb{T}_n^d} M_t(x)^2 \, d\nu^a \leq C_0 \frac{\|G\|_\infty^2}{\chi(\rho)|B|} \left\{ H_n(f) + (n/\ell)^d \right\} \]

for all density \( f \) with respect to \( \nu^\rho \) and all \( n > 4\ell \geq 4 \).

**Proof.** Let

\[ W(\eta) = \sum_{x \in \mathbb{T}_n^d} M_t(x)^2. \]

By the entropy inequality,

\[ \int W f \, d\nu^a \leq \frac{1}{\gamma} H_n(f) + \frac{1}{\gamma} \log \int e^{\gamma W} \, d\nu^a \]

for all \( \gamma > 0 \). Repeating the argument presented below (5.13), we bound the second term of this expression by

\[ \frac{C_0}{\gamma L} \sum_{x \in \mathbb{T}_n^d} \log \int \exp \left\{ C_0 \gamma \ell^d M_t(x)^2 \right\} \, d\nu^a \tag{5.15} \]
If $B$ were a singleton, under the measure $\nu^n_\rho$, the variables $\omega_{x-y+B}$ would be independent. Since this may not be the case, we divide the sum further. Let $p \geq 1$ be the smallest integer such that $B \subset \Xi_p := \{-p, \ldots, p\}^d$, and rewrite $M_\ell(x)$ as

$$M_\ell(x) = \sum_{z \in \Xi_p} \sum_{w \in \Lambda_\ell(z)} m_\ell(z + pw) G(x - z - pw) \omega_{x-z-pw+B} := \sum_{z \in \Xi_p} M_\ell(z, x),$$

where the second sum is performed over all $w \in \mathbb{Z}^d$ such that $z + pw \in \Lambda_\ell$. Now, for each fixed $x$ and $z$, the variables $\{\omega_{x-z-pw+B} : w \in \Lambda_\ell(z)\}$ are independent. Apply Hölder’s inequality once more to bound (5.15) by

$$\sum_{z \in \Xi_p} \sum_{w \in \Lambda_\ell(z)} m_\ell(z + pw) G(x - z - pw) \omega_{x-z-pw+B} \leq C_0 \gamma \ell^d M_\ell(z, x)^2,$$

where the value of the constant $C_0$ has changed.

By definition of $m_\ell$, $\sum_{y \in \Lambda_\ell} m_\ell(y)^2 \leq C_0 \ell^{-d}$ for some finite constant $C_0$. Hence, by Lemma 4.4, under the measure $\nu^n_\rho$, $M_\ell(z, x)$ is a $\sigma_\ell^2$-subgaussian random variable, where $\sigma_\ell^2 = C_0 \|G\|_\infty^2/\chi(\rho)^{|B|} \ell^d$ for some finite constant $C_0$.

Therefore, taking $\gamma = c_0 \chi(\rho)^{|B|}/\|G\|_\infty$ for some positive constant $c_0$, depending only on the dimension, by (4.8), the sum (5.16) is bounded by

$$C_0 \frac{\|G\|_\infty^2}{\chi(\rho)^{|B|} \ell^d}.$$

To complete the proof of the lemma, it remains to recollect the previous estimates.

**Corollary 5.8.** There exists a finite constant $C_1(\rho)$, depending only on $\rho$ and on the dimension, such that

$$\int V_\ell(\eta) \, d\nu^n_\rho \leq C_1(\rho) \{ H_n(f) + (n/\ell)^d \}$$

for all density $f$ with respect to $\nu^n_\rho$.

**Proof:** The proof is similar to the one of Proposition 5.2. In view of (5.8), by Young’s inequality $ab \leq (1/2)a^2 + (1/2)b^2$, $V_\ell(\eta)$ is bounded by $V_\ell^{(1)}(\eta) + V_\ell^{(2)}(\eta)$, where

$$V_\ell^{(1)}(\eta) := \frac{1}{2} \sum_{j=1}^{d} \sum_{x \in \Xi_\ell^d} \left( \sum_{y \in \Lambda_\ell} m_\ell(y) \omega_{x-y} \right)^2,$$

and $V_\ell^{(2)}$ is similar to $V_\ell^{(1)}$, with the average inside the square replaced by $\sum_{z \in \Lambda_\ell} m_\ell(z) \omega_{x+y+z}$.

To complete the proof, it remains to apply Lemma 5.7 with $G = 1$. 

The next result follows from Proposition 5.2 and Corollary 5.8.

**Lemma 5.9.** There exists a finite constant $C_1(\rho)$, depending only on $\rho$, $c_0$ and the dimension, such that

$$a_n \int V f \, d\nu^n_\rho \leq \delta n^2 I_n(f) + C_1(\rho) a_n \left\{ 1 + \frac{a_n \ell^d g_d(\ell)}{\delta n^2} \right\} \{ H_n(f) + (n/\ell)^d \}$$

for all $\delta > 0$ and density $f$ with respect to $\nu^n_\rho$. 

Proof of Theorem 2.8. By Proposition 5.1 and Lemma 5.9 with \( \delta = 1 \) and \( f = f^n \), \( \mathcal{H}_n(t) \) is bounded by

\[
C_1(\rho) a_n \left\{ 1 + \frac{a_n \ell_n d(\ell)}{n^2} \right\} \left\{ \mathcal{H}_n(f^n) + (n/\ell)^d \right\}
\]

for some finite constant \( C_1(\rho) \).

At this point, the natural choice is \( \ell = \ell_n \) so that \( \ell_n d(\ell_n) = n^2/a_n \). Thus, define the sequence \( (\ell_n : n \geq 1) \) by

\[
\ell_n = \begin{cases} 
\frac{n/\sqrt{a_n}}{n} & \text{in } d = 1, \\
\frac{n^2}{a_n \log n} & \text{in } d = 2, \\
\frac{n^2/a_n}{n} & \text{in } d \geq 3.
\end{cases}
\]

(5.17)

With these choices, the previous expression is bounded by

\[
C_1(\rho) a_n \left\{ \mathcal{H}_n(f^n) + R_d(n) \right\},
\]

as claimed. \( \square \)

6. The Boltzmann-Gibbs Principle

We prove in this section Theorem 4.3. Recall the definition of the operators \( \Pi^1_{\rho} \) and \( \Pi^2_{\rho} \) from the statement of Assertion C.1. We prove Theorem 4.3 for \( \Xi_{\rho} = \Pi^1_{\rho} \) in Lemma 6.1, and for \( \Xi_{\rho} = \Pi^2_{\rho} \) in Proposition 6.2. Throughout this section, \( f^n_t, t \geq 0 \), represents the density of the measure \( \mu_n S^n(t) \) with respect to \( \nu^n_\rho \), where \( \mu_n \) is a probability measure on \( \Omega \).

**Lemma 6.1.** Fix a function \( G \in C^{0,1}(\mathbb{R}_+ \times \mathbb{T}^d) \) and a sequence of measures \( \mu_n \) on \( \Omega \). Then,

\[
\mathbb{E}_{\mu_n} \left[ \int_0^t \frac{1}{\sqrt{a_n}} n^d \left| \sum_{x \in \mathbb{T}^d} G(s, x/n) \left[ \eta^n_x(s) - \eta^n_{x+e_j}(s) \right] \right| \, ds \right]
\]

\[
\leq \frac{1}{\gamma} \int_0^t \mathcal{H}_n(f^n_s) \, ds + \frac{t}{\gamma} \log 2 + \frac{3 \gamma t}{a_n n^2} \sup_{0 \leq s \leq t} \| \partial_{x_j} G(s) \|_\infty
\]

for every \( 1 \leq j \leq d, t > 0, 0 \leq \gamma \leq \sqrt{a_n n^{d+2}/\sup_{0 \leq s \leq t} \| \partial_{x_j} G(s) \|_\infty}, \) \( n \geq 1 \).

**Proof.** Let \( W(s, \eta) = \{a_n n^d\}^{-1/2} \sum_{x \in \mathbb{T}^d} G(s, x/n) - G(s, x - e_j/n) \left[ \eta^n_x - \rho \right] \). By the entropy inequality, the expectation appearing in the statement of the lemma is bounded by

\[
\frac{1}{\gamma} \int_0^t \mathcal{H}_n(f^n_s) \, ds + \int_0^t \frac{1}{\gamma} \log \int e^{\gamma |W(s)|} \, d\nu^n_\rho \, ds
\]

for every \( \gamma > 0 \).

As \( \exp[|a|] \leq e^a + e^{-a} \) and \( e^b + e^{-b} \leq 2 \max\{e^b, e^{-b}\} \), by linearity of the expectation,

\[
\frac{1}{\gamma} \log \int e^{\gamma |W(s)|} \, d\nu^n_\rho \leq \frac{\log 2}{\gamma} + \max_{b = \pm 1} \frac{1}{\gamma} \log \int e^{b \gamma W(s)} \, d\nu^n_\rho.
\]
We estimate the second term with $b = 1$, as the argument applies to $b = -1$. As $\nu^\gamma_n$ is a product measure,
\[
\frac{1}{\gamma} \log \int e^{\gamma W(s)} \, d\nu^\gamma_n = \frac{1}{\gamma} \sum_{x \in \mathbb{T}^d_n} \log \int e^{b_n(s) (\eta_x - \rho)} \, d\nu^\gamma_n
\]
where $b_n(s) = \gamma [G(s, x/n) - G(s, (x - e_j)/N)] / \sqrt{a_n n^d}$. Since $e^a \leq 1 + a + a^2 e[a]$, $E_{\nu^\gamma_n} [\eta_x - \rho] = 0$, and $\log(1 + b) \leq b$, the previous expression is bounded by
\[
\frac{3 \gamma \|\partial_x G(s)\|_\infty^2}{a_n n^d}
\]
provided $\gamma \|\partial_x G(s)\|_\infty / \sqrt{a_n n^{d+2}} \leq 1$.

To complete the proof of the lemma, it remains to recollect the previous estimates. \hfill \Box

Assume that $a_n \leq \sqrt{\log n}$ and that the sequence of measures $\mu_n$ on $\Omega_n$ satisfies the bound $H_n(\mu_n | \nu^\rho_n) \leq R_d(n)$. Fix $\kappa > 0$ and $T > 0$. By Remark 2.9, there exists $n_0$ such that $H_n(\mu_n) \leq R_d(n) n^\kappa$ for all $0 \leq t \leq T$, $n \geq n_0$.

Choose $\gamma_n = \sqrt{R_d(n) n^{\kappa+\kappa}} a_n / \sup_{0 \leq s \leq t} \|\partial_x G(s)\|_\infty$. By definition of $R_d(n)$, $\gamma_n$ satisfies the bound required in the lemma for all $n \geq n_0$. With this choice,
\[
\mathbb{E}_{\mu_n} \left[ \int_0^t \frac{1}{\sqrt{a_n n^d}} \left| \sum_{x \in \mathbb{T}^d_n} G(s, x/n) \left[ \eta^\gamma_n(s) - \eta^\rho_n x + x_j(s) \right] \right| \, ds \right]
\leq C_0 t \sqrt{\frac{R_d(n)}{a_n n^2}} \sup_{0 \leq s \leq t} \|\partial_x G(s)\|_\infty
\]
for every $1 \leq j \leq d$, $0 < t \leq T$, $n \geq n_0$. By definition of $R_d(n)$, this expression vanishes as $n \to \infty$ provided $d \leq 3$.

We turn to the proof of Theorem 4.3 in the case where $\Xi = \Pi^{\pm 2}_\rho$.

**Proposition 6.2.** Fix $0 < \rho < 1$ and a finite subset $A$ of $\mathbb{Z}^d$ with at least two elements. Then, there exists a finite constant $C_1 = C_1(\rho, A)$, depending only on the dimension, the cardinality of the set $A$ and the density $\rho$ such that
\[
\mathbb{E}_{\mu_n} \left[ \left| \int_s^t \sum_{x \in \mathbb{T}^d_n} G_x(u) \omega_{A+x}(u) \, du \right| \right] \leq C_1 \left\{ \frac{1 + H_n(f^\gamma_n)}{a_n} + c(G) h_n(s, t) \right\}
\]
for all $t > s \geq 0$, bounded function $G : \mathbb{R} \times \mathbb{T}^d_n \to \mathbb{R}$, probability measure $\mu_n$ and $n \geq 1$. On the right-hand side, $c(G) = 1 + \sup_{0 \leq u \leq t} \|G(u)\|_\infty^2$, $h_n(s, t) = \int_s^t H_n(f^\gamma_n) \, du + (t - s) (n/\ell_n)^d$, and $(\ell_n : n \geq 1)$ is the sequence introduced in (5.17).

Note that $(n/\ell_n)^d = R_n(d)$. In this proposition, replacing the function $G$ by $\gamma G$, dividing the corresponding estimate by $\gamma$ and optimizing over $\gamma > 0$ yield the next result.
Corollary 6.3. Fix \( 0 < \rho < 1 \) and a finite subset \( A \) of \( \mathbb{Z}^d \) with at least two elements. Then, there exists a finite constant \( C_1 = C_1(\rho, A) \), depending only on the dimension, the set \( A \) and the density \( \rho \) such that

\[
E_{\mu_n} \left[ \left| \int_{s}^{t} \sum_{x \in \mathbb{T}_n^d} G_x(u) \omega_{A+x}(u) \, du \right| \right] \leq C_1 \sup_{s \leq u \leq t} \| G(u) \|_{\infty} \sqrt{\left( \frac{1 + H_n(f^n_s)}{a_n} \right) + H_n(s, t)} \mathbb{H}_n(s, t),
\]

for all \( t > s \geq 0 \), \( G : \mathbb{R}_+ \times \mathbb{T}_n^d \to \mathbb{R} \), probability measure \( \mu_n, n \geq 1 \).

The proof of Proposition 6.2 is divided in several steps. Recall the definition of the constant \( c_0 \) introduced in (2.1), and, from Lemma 5.5, the decomposition of a finite subset \( A \) of \( \mathbb{Z}^d \) as \( A = A_0 \cup \{ x_A \} \). Recall, furthermore, the definition of the functions \( V, V_t \) introduced in (5.2), (5.7), respectively, and the ones of \( H_{k,x_n}^{(t)}, H_{j,k,x_n}^{(t)} \) presented in Lemma 5.5 and Corollary 5.6, respectively. Note that \( H_{k,x_n}^{(t)} \) depends on time, because in Proposition 6.2 the function \( G \) depends on time. Let \( \mathcal{T} = \mathcal{T}_{G,A,t,\rho} : \mathbb{R}_+ \times \mathcal{T}_n^d \times \Omega_n \to \mathbb{R} \) and \( \Psi = \Psi_{G,A,t,\rho} : \mathbb{R}_+ \times \mathcal{T}_n^d \times \Omega_n \to \mathbb{R} \) be given by

\[
\mathcal{T}(u, x, \eta) = G_x(u) \omega_{x+A_0} \{ \omega_{x+x_A} = \omega^\ell \},
\]

\[
\Psi(u, x, \eta) := \frac{a_n}{2c_0 n^2 \chi(\rho)} \sum_{k=1}^{d} (H_{k,x_n}^{(t)}(u))^2 + \sum_{j=1}^{d} \omega_x \omega^\ell \omega_{x+e_j} + \frac{d_a n}{8 c_0 n^2 \chi(\rho)} \sum_{j,k=1}^{d} (H_{j,k,x_n}^{(t)})^2.
\]

We denote below \( \mathcal{T}(u, x, \eta), \Psi(u, x, \eta) \) by \( \mathcal{T}_x(u, \eta), \Psi_x(u, \eta) \), respectively.

In Lemma 6.4 below, we estimate the expectation of

\[
\left| \int_{s}^{t} \sum_{x \in \mathcal{T}_n^d} \mathcal{T}_x(u, \eta^n(u)) \, du \right| - \int_{s}^{t} \sum_{x \in \mathcal{T}_n^d} \Psi_x(u, \eta^n(u)) \, du,
\]

in Lemma 6.6 and equation (6.3) the expectation of

\[
\int_{s}^{t} \left\{ \left| \sum_{x \in \mathcal{T}_n^d} G_x(u) \omega_{x+A_0} \omega^\ell \omega_{x+x_A}(u) \right| + \sum_{j=1}^{d} \sum_{x \in \mathcal{T}_n^d} \omega_x \omega^\ell \omega_{x+e_j}(u) \right\} \, du,
\]

and in Lemma 6.7 and equation (6.6) the expectation of

\[
\int_{s}^{t} \sum_{x \in \mathcal{T}_n^d} \left\{ \sum_{j=1}^{d} [H_{k,x_n}^{(t)}(u, \eta^n(u))]^2 \right\} + \sum_{j,k=1}^{d} [H_{j,k,x_n}^{(t)}(\eta^n(u))]^2 \, du.
\]

Proposition 6.2 follows from these bounds

Lemma 6.4. For all \( t > s > 0 \) and \( n \geq 1 \),

\[
E_{\mu_n} \left[ \left| \int_{s}^{t} \sum_{x \in \mathcal{T}_n^d} \mathcal{T}_x(u, \eta^n(u)) \, du \right| - \int_{s}^{t} \sum_{x \in \mathcal{T}_n^d} \Psi_x(u, \eta^n(u)) \, du \right] \leq \frac{\log 2}{a_n} + \frac{1}{a_n} H_n(f^n_s).
\]

Proof. Rewrite the expectation on the left-hand side as

\[
E_{\mu_n(s)} \left[ \left| \int_{0}^{t-s} \sum_{x \in \mathcal{T}_n^d} \mathcal{T}_x(s+u, \eta^n(u)) \, du \right| - \int_{0}^{t-s} \sum_{x \in \mathcal{T}_n^d} \Psi_x(s+u, \eta^n(u)) \, du \right],
\]


where $\mu_n(s) = \mu_n S^n(s)$. By the entropy inequality, this expression is bounded above by
\[
\frac{1}{\gamma} H_n(f^n_s) + \frac{1}{\gamma} \log \mathbb{E}_{\nu^n_\rho} \left[ \exp \gamma \left( \int_0^{t_s} \sum_{x \in \mathbb{T}^d_n} \Upsilon_x(s + u, \eta^n(u)) \, du \right) \right] \geq \frac{1}{\gamma} H_n(f^n_s) + \frac{1}{\gamma} \log \mathbb{E}_{\nu^n_\rho} \left[ \exp \gamma \left( \int_0^{t_s} \sum_{x \in \mathbb{T}^d_n} \Psi_x(s + u, \eta^n(u)) \, du \right) \right],
\]
for all $\gamma > 0$, where $t_s = t - s$. As $e^{a|u|} \leq e^a + e^{-a}$ and $e^{b + e^{-b}} \leq 2 \max\{e^b, e^{-b}\}$, by the linearity of the expectation, the second term of this expression is less than or equal to
\[
\frac{1}{\gamma} \log 2 + \max_{b = \pm 1} \frac{1}{\gamma} \log \mathbb{E}_{\nu^n_\rho} \left[ \exp \gamma \left( \int_0^{t_s} \sum_{x \in \mathbb{T}^d_n} \{ b \Upsilon_x(s+u, \eta^n(u)) - \Psi_x(s+u, \eta^n(u)) \} \, du \right) \right].
\]

We estimate the second term for $b = 1$; the same argument applies to $b = -1$. By Corollary B.2 below, the second term of this expression is bounded by
\[
\int_0^{t_s} \sup_f \left\{ \int W(s + u) \, d\nu^n_{\rho} + \frac{a_n}{2\gamma} \int V \, d\nu^n_{\rho} - \frac{n^2}{\gamma} I_n(f) \right\} \, du,
\]
where the supremum is carried over all densities $f$ with respect to $\nu^n_{\rho}$, and $W(s + u) = \sum_{x \in \mathbb{T}^d_n} \{ \Upsilon_x(s + u) - \Psi_x(s + u) \}$. Choosing $\gamma = a_n$, the previous supremum becomes
\[
\frac{1}{a_n} \int_0^{t_s} \sup_f \left\{ a_n \int \left[ W(s + u) + \left( \frac{1}{2} V \right) \right] \, d\nu^n_{\rho} - n^2 I_n(f) \right\} \, du. \tag{6.2}
\]

By definition of $W(s + u)$, $a_n \{ W(s + u) + (1/2) V \}$ is equal to
\[
a_n \sum_{x \in \mathbb{T}^d_n} G_x(s + u) \omega_x + \omega^e_x + \omega^f_x - \omega_x \right\} - \frac{a_n}{2} \sum_{k=1}^{d} \sum_{x \in \mathbb{T}^d_n} (H^{(e)}_{k,x}(s + u))^2 + \frac{1}{2} \left\{ a_n \left[ V(\eta) - V_l(\eta) \right] - \frac{d a_n^2}{4} \sum_{j,k=1}^{d} \sum_{x \in \mathbb{T}^d_n} (H^{(f)}_{j,k,x}(s + u))^2 \right\}.
\]
Thus, by Lemma 5.5, with $\beta = 1$ and $b_n = a_n$, and Corollary 5.6, with $\delta = 1$, the expression inside braces in (6.2) is less than or equal to 0. This completes the proof of the lemma. \hfill $\square$

**Remark 6.5.** The previous result holds in any dimension and for any sequence $a_n$. It is a consequence of Feynman-Kac formula and the integration by parts stated in Lemma 5.3.

In view of the decomposition carried out just after the statement of Proposition 6.2, it remains to estimate sums involving the cylinder functions $\omega_x + \omega^e_x + \omega^f_x$ and sums of squares of averages with respect to the flow $\Phi$. The next lemma handles the first type of terms.

**Lemma 6.6.** Fix $0 < \rho < 1$ and a finite subset $A$ of $\mathbb{Z}^d$ with at least two elements. Then, there exists a constant $C_1 = C_1(A, \rho)$, depending only on the density $\rho$, the
dimension and the set $A$, such that
\[
\mathbb{E}_{\mu_n} \left[ \int_s^t \left| \sum_{x \in \mathbb{T}_n^d} G_x(u) \omega_{x+A}(u) \omega_{x+\Lambda}(u) \right| \, du \right] 
\leq C_1(\rho) \sup_{s \leq u \leq t} \|G(u)\|_\infty \left\{ \int_s^t H_n(f^n_u) \, du + (t-s)(n/\ell)^d \right\}
\]
for all functions $G : \mathbb{R}_+ \times \mathbb{T}_n^d \to \mathbb{R}$, $t > s \geq 0$, measure $\mu_n$ on $\Omega_n$ and $n \geq \ell \geq 1$.

Proof: Assume, without loss of generality, that $x_A = 0$, so that the origin is a maximal element of $A$, and let $B = A_\ast$. A change of variables, similar to the one performed in \eqref{eq:change_of_variables}, yields that
\[
\sum_{x \in \mathbb{T}_n^d} G_x(u) \omega_{x+B} \omega_x = \sum_{x \in \mathbb{T}_n^d} M^{(1)}_\ell(u, x) M^{(2)}_\ell(x),
\]
where
\[
M^{(1)}_\ell(u, x) = \sum_{y \in \Lambda_\ell} m_\ell(y) G_{x-y}(u) \omega_{x-y+B}, \quad M^{(2)}_\ell(x) = \sum_{z \in \Lambda_\ell} m_\ell(z) \omega_{x+z}.
\]

The expectation appearing in the statement of the lemma is equal to
\[
\int_s^t du \int \left| \sum_{x \in \mathbb{T}_n^d} M^{(1)}_\ell(u, x) M^{(2)}_\ell(x) \right| f^n_u \, d\nu^n
\leq \frac{1}{2} \sum_{i=1}^{\ell} \gamma^{3-2i} \int_s^t du \int \sum_{x \in \mathbb{T}_n^d} M^{(i)}_\ell(u, x)^2 f^n_u \, d\nu^n
\]
for all $\gamma > 0$. We applied here Young’s inequality $2ab \leq \gamma a^2 + \gamma^{-1}b^2$.

By Lemma 5.7 and optimizing over $\gamma > 0$, the previous expression is bounded by
\[
C_1 \sup_{s \leq u \leq t} \|G(u)\|_\infty \left\{ \int_s^t H_n(f^n_u) \, du + (t-s)(n/\ell)^d \right\}
\]
for some finite constant $C_1 = C_1(A, \rho)$, as claimed. \hfill \square

The same argument yields that
\[
\mathbb{E}_{\mu_n} \left[ \int_s^t \left| V_\ell(\eta^n(u)) \right| \, du \right] \leq C_1(\rho) \left\{ \int_s^t H_n(f^n_u) \, du + (t-s)(n/\ell)^d \right\}. \quad \text{(6.3)}
\]

Recall the definition of $H_{k,\ell}(u, \eta)$, introduced in the statement of Lemma 5.5.

Lemma 6.7. Fix $0 < \rho < 1$, and a finite subset $A$ of $\mathbb{Z}^d$ with at least two elements. Then, there exists a finite constant $C_1 = C_1(A, \rho)$, depending only on the dimension, the density $\rho$ and the number of elements of the set $A$, such that
\[
\mathbb{E}_{\mu_n} \left[ \int_s^t \sum_{x \in \mathbb{T}_n^d} \left[ H_{k,\ell}^{(l)}(u, \eta^n(u)) \right]^2 \, du \right] 
\leq C_1 \sup_{s \leq u \leq t} \|G(u)\|_\infty^2 \ell^d \phi_d(\ell) \left\{ \int_s^t H_n(f^n_u) \, du + (t-s)(n/\ell)^d \right\}
\]
for all $1 \leq k \leq d$, $t > s \geq 0$, function $G : \mathbb{R}_+ \times \mathbb{T}_n^d \to \mathbb{R}$, probability measure $\mu_n$ on $\Omega_n$ and $n \geq 1$. 


**Proof.** Let $U(u, \eta) = \sum_{x \in \mathbb{Z}^d} \left[ H_{k,x}^{(\ell)}(u, \eta) \right]^2$. By the entropy inequality, the expectation appearing in the statement of the lemma is bounded by

$$
\int_s^t \left\{ \frac{1}{\gamma} H_n(f_u^n) + \frac{1}{\gamma} \log \int e^{\gamma U(u)} \, d\nu^u_\rho \right\} \, du
$$

for every $\gamma > 0$.

We turn to the second term. Fix $s \leq u \leq t$, and recall the definition of $H_{k,x}^{(\ell)}(u)$ and let $B = A_s$. We repeat here the decomposition performed in the proof of Lemma 5.7. If $B$ were a singleton, under the measure $\nu^u_\rho$, the variables $\{\omega_{y+B} : y \in \Lambda_{2t-1}\}$ would be independent. Since this may not be the case, we divide the sum further. Let $p \geq 1$ be the smallest integer such that $B \subset \Xi_p := \{-p, \ldots, p\}^d$, and rewrite $H_{k,x}^{(\ell)} = H_{k,x}^{(\ell)}(u)$ as

$$
H_{k,x}^{(\ell)} = \sum_{z \in \mathbb{Z}^d} \sum_{w \in \Lambda_d(z)} \Phi(z + pw, z + pw + e_k) G_u(x - x_A - z - pw) \omega_{x - x_A - z - pw + B} ,
$$

where the second sum is performed over all $w \in \mathbb{Z}^d$ such that $z + pw$ and $z + pw + e_k$ belong to $\Lambda_{2t-1}$. Now, for each fixed $x$, the variables $\{\omega_{x - x_A - z - pw + B} : w \in \Lambda_d(z)\}$ are independent. Rewrite this sum as

$$
H_{k,x}^{(\ell)}(u) =: \sum_{z \in \Xi_p} M_t(u, z, x)
$$

Applying the arguments presented in the proof of Proposition 5.2 [after equation (5.13)] and Lemma 5.7 [after equation (5.15)], we obtain that the second term inside the braces of (6.4) is bounded by

$$
\frac{C_0}{\gamma^d} \sum_{x \in \mathbb{Z}^d} \sum_{z \in \Xi_p} \log \int \exp \left\{ C_0 \gamma \ell^d M_t(u, z, x)^2 \right\} \, d\nu^u_\rho ,
$$

for some finite constant $C_0$.

At this point, the proof of the lemma is similar to the one of Proposition 5.2. By Lemma 4.4, $M_t(u, z, x)$ is a $\sigma^2$-subgaussian random variable, where

$$
\sigma^2_t = \frac{\|G(u)\|_\infty^2}{\chi(\rho) |B|} \sum_{\{y, y + e_k\} \subset \Lambda_{2t-1}} \Phi(y, y + e_k)^2 .
$$

By Lemma 5.4, $\sigma^2_t \leq \frac{\|G(u)\|_\infty^2}{\chi(\rho) |B|} g_d(\ell) / \chi(\rho) |B|$. Let $\gamma = \gamma_\rho$ be given by the identity

$$
C_0 \sup_{0 \leq s \leq t} \frac{\|G(u)\|_\infty^2}{\chi(\rho) |B|} \ell^d g_d(\ell) = \frac{1}{4} .
$$

By (4.8), the sum (6.5) is bounded by

$$
\frac{C_0}{\gamma^d} \sum_{x \in \mathbb{Z}^d} \sum_{z \in \Xi_p} \log \int \exp \left\{ C_0 \gamma \ell^d M_t(u, z, x)^2 \right\} \, d\nu^u_\rho
$$

for some finite constant $C_0$.

Hence, (6.4) is less than or equal to

$$
C_1(A, \rho) \sup_{s \leq u \leq t} \|G(u)\|_\infty^2 \ell^d g_d(\ell) \int_s^t H_n(f_u^n) \, du + C_1(A, \rho) \sup_{s \leq u \leq t} \|G(u)\|_\infty^2 \, t n^d g_d(\ell)
$$

for some finite constant $C_1(\rho)$, depending only on the dimension, on the density $\rho$ and the number of elements of the set $A$, as claimed. $\square$
The same arguments exposed in the proof of Lemma 6.7 yield that there exists a finite constant $C_1(\rho)$, depending only on the dimension, the density $\rho$ and the number of elements of the set $A$, such that

$$E_\mu_n \left[ \int_s^t \sum_{x \in \mathbb{T}_n^d} \left[ H_{j,k,x}^{(\ell)}(\eta^n(u)) \right]^2 \, du \right] \leq C_1(\rho) \ell^d g_d(\ell) \left\{ \int_s^t H_n(f_n^u) \, du + (t-s)(n/\ell)^d \right\}$$

(6.6)

for all $1 \leq j,k \leq d$, $t > s \geq 0$, probability measure $\mu_n$ on $\Omega_n$ and $n \geq 1$.

**Proof of Proposition 6.2.** The result follows from the decomposition presented just below the statement of the proposition, Lemmata 6.4, 6.6, 6.7, equations (6.3), (6.6), and the definitions of $g_d(\ell)$, $\ell_n$ given in (5.9) and (5.17), respectively. The dependence on $G$ of the constant provided by the proof is $1 + c + c^2$, where $c = \sup_{s \leq u \leq t} \|G(u)\|_\infty$, and this quantity is bounded by $2(1 + c^2)$.

The term $\ell^d g_d(\ell)$ which appears in Lemma 6.7 and equation (6.6) cancels with the term $a_n/n^2$ which appears in the definition of $\Psi_x(s)$ provide we choose $\ell_n$ as in (5.17). \hfill $\square$

**Proof of Theorem 4.3.** In view of the decomposition of $\Pi_{\rho f}$ presented in Assertion C.1, the Boltzmann-Gibbs principle follows from (6.1), Corollary 6.2 with $s = 0$, estimate (2.11) and the definition (5.17) of the sequence $\ell_n$. \hfill $\square$

7. Tightness

Throughout this section, $0 < \rho < 1$ and $T > 1$ are fixed. Recall that we denote by $Q_n$ the probability measure on $D([0,T], \mathcal{H}_{-\rho})$, $r \geq 1$, induced by the process $X_t^n$ and the initial measure $\mu_n$. The first main result of this section reads as follows.

**Proposition 7.1.** Assume that $d = 1$ or 2, and fix $r > (3d+5)/2$. Let $\mu_n$ be a sequence of probability measures on $\Omega_n$ such $\lim_{n \to \infty} a_n^{-1} H_n(\mu_n | \nu_n^n) = 0$. Then, the sequence of probability measures $Q_n$ is tight. Moreover, every limit point is concentrated on continuous paths.

The proof of Proposition 7.1 is carried out by showing that each of the processes on the right-hand side of (4.4) is tight. Note that the only one which is not continuous in time is the martingale process $M_t^n$. Recall the definition of $R^n$ and $B^n$ given below (4.3). The process $R^n$ can be written as the sum of $R^{\Delta,n}$, $R^{j,k,n}$ and $\Pi^{j,k,n}(F)$, $1 \leq j,k \leq d$, where for $F \in C^\infty(\mathbb{T}^d)$,

$$R^{j,k,n}(F) = \frac{1}{\sqrt{a_n n^d}} \sum_{x \in \mathbb{T}_n^d} \left\{ h_{j,k}(\tau_x \eta) - \hat{h}_{j,k}(\rho) \right\} \left\{ (\Delta_{j,k}^n F) - (\partial^2_{x_j,x_k} F) \right\}(x/n),$$

$$R^{\Delta,n}(F) = \frac{a_n}{n^d} \frac{1}{\sqrt{a_n n^d}} \sum_{x \in \mathbb{T}_n^d} \{ \eta_x - \rho \} (\Delta_n F)(x/n),$$

$$\Pi^{j,k,n}(F) = \frac{1}{\sqrt{a_n n^d}} \sum_{x \in \mathbb{T}_n^d} (\Pi_{\rho}^1 h_{j,k})(\tau_x \eta) (\partial^2_{x_j,x_k} F)(x/n).$$
On the other hand, the process $B^n$ can be written as the sum of the components $B^{j,k,n}$, where $B^{j,k,n}(F)$ is given by

$$B^{j,k,n}(F) = \frac{1}{\sqrt{a_n n^d}} \sum_{x \in \mathbb{T}^d} (\Pi_\rho^{j+2} h_{j,k}(\tau_x \eta)) (\partial_x^2 \phi_{j,k,n} F)(\sigma_x) ,$$

for $F \in C^\infty(\mathbb{T}^d)$. The projections $\Pi_\rho^j$ and $\Pi_\rho^{j+2}$ are introduced in Assertion C.1.

Denote by $R^n_t$, $B^n_t$ the $\mathcal{H}^{-1}$-valued process defined as the time-integrals of the processes $R^n$ and $B^n$, respectively:

$$R^n_t(F) = \int_0^t \left\{ R^n_s, + \sum_{j,k=1}^d R^{j,k,n}(F) + \sum_{j,k=1}^d \Pi^{j,k,n}_s(F) \right\} ds ,$$

$$B^n_t(F) = \sum_{j,k=1}^d \int_0^t B^{j,k,n}_s(F) ds .$$

Note that $B^n_t$ contains terms of degree two or larger in the terminology of Appendix C.

### 7.1 Tightness of the process $R^n_t$

The process $R^n_t$ has been expressed as the sum of three terms. Consider the last one and fix $1 \leq j, k \leq d$. By definition of $\Pi_\rho^j$ and by a summation by parts,

$$\Pi^{j,k,n}_t(F) = \frac{1}{n} \frac{1}{\sqrt{a_n n^d}} \sum_{x \in \mathbb{T}^d} \sum_{z \in A_{j,k}} c_{j,k}(z) \left( D^n_x \partial_x^2 F \right)(x/n) [\eta_x - \rho]$$

for some finite subset $A_{j,k}$ of $\mathbb{Z}^d$ and real numbers $c_{j,k}(z)$, $z \in A_{j,k}$, which depend only on the cylinder function $h_{j,k}$. In this formula, for a function $J$ in $C(\mathbb{T}^d)$,

$$(D^n_x J)(x/n) = n \left[ J([x - z]/n) - J(x/n) \right] .$$

Note the additional factor $1/n$ in (7.1) which appeared from the summation by parts.

For $k \geq 0$ and a function $F$ in $C^\infty(\mathbb{T}^d)$, denote by $\|F\|_{C^k(\mathbb{T}^d)}$ the $C^k(\mathbb{T}^d)$-norm of $F$:

$$\|F\|_{C^k(\mathbb{T}^d)} = \sum_{|i| \leq k} \|D_i F\|_{C^k(\mathbb{T}^d)} ,$$

where the sum is carried over multi-indices $i = (i_1, \ldots, i_d) \in \mathbb{N}^d$ and $D_i F = \partial_{i_1} \ldots \partial_{i_d} F$.

By the explicit formula for $R^n$ presented below (4.3), (7.1) and Lemma A.1, there exists a finite constant $C_0$, depending only on the cylinder functions $h_{j,k}$, such that

$$E_{\mu_n} \left[ \int_0^t R^n_s(G_s)^2 \right] \leq \frac{C_0}{a_n n^2} \left\{ \int_0^t H_n(f^n_s) ds + t \right\} \sup_{0 \leq s \leq t} \|G_s\|_{C^3(\mathbb{T}^d)} ,$$

for all $0 < t \leq T$, smooth function $G : [0, T] \times \mathbb{T}^d \to \mathbb{R}$, probability measure $\mu_n$ and $n \geq 1$.

In the next lemma, we assume that $H_n(\mu_n \mid \nu^n_\rho) \leq a_n + R_d(n)$. In dimension $d \geq 2$, the term $a_n$ is insignificant being much smaller than $R_d(n)$, but in dimension 1 it is much larger than $R_d(n)$.
Lemma 7.2. Assume that \( d \leq 3 \). Fix \( r > 3 + (d/2) \) and a sequence of measures \( \mu_n \) such that \( H_n(\mu_n | \nu_0) \leq a_n + R_d(n) \) for all \( n \geq 1 \). Then,

\[
\lim_{n \to \infty} \mathbb{E}_{\mu_n} \left[ \sup_{0 \leq t \leq T} \| R^\mu_t \|_{-r}^2 \right] = 0 .
\]

Proof. By definition of the norm \( \| \cdot \|_{-r} \), the expectation appearing in the statement of the lemma is bounded by

\[
\sum_{m \in \mathbb{Z}^d} \gamma_m^{-r} \mathbb{E}_{\mu_n} \left[ \sup_{0 \leq t \leq T} \| R^\mu_t(\phi_m) \|^2 \right] .
\]

By Schwarz inequality, this sum is bounded by

\[
T \sum_{m \in \mathbb{Z}^d} \gamma_m^{-r} \mathbb{E}_{\mu_n} \left[ \int_0^T \| R^\mu_t(\phi_m) \|^2 \, ds \right] .
\]

By (7.2), this expression is less than or equal to

\[
\frac{C_0 T}{\alpha_n n^2} \left\{ \int_0^T H_n(f^\mu_s) \, ds + T \right\} \sum_{m \in \mathbb{Z}^d} \gamma_m^{-r} \| m \|^6 .
\]

To complete the proof, it remains to recall the estimate (2.11). \( \square \)

7.2. Hölder spaces and Kolmogorov-Čentsov theorem. The proof of the tightness of the process \( B^\mu_t \) is based on the Kolmogorov-Čentsov theorem stated below for convenience. We refer to [14, Section 2.4, Problem 4.11] or [20, Proposition 7] for a proof. These results apply to distribution-valued processes since they depend only on estimates of the norms.

Fix \( T_0 > 0 \) and \( r > 0 \). Denote by \( C^\alpha([0, T_0], \mathcal{H}_{-r}) \), \( 0 < \alpha < 1 \), the Hölder space of \( \mathcal{H}_{-r} \)-valued, continuous functions endowed with the norm

\[
\| X \|_{C^\alpha} = \sup_{0 \leq t \leq T_0} \| X_t \|_{-r} + \sup_{0 \leq t < s \leq T_0} \frac{\| X_t - X_s \|_{-r}}{|t - s|^{\alpha}} .
\]

Note that this topology is stronger than the uniform topology of \( C([0, T_0], \mathcal{H}_{-r}) \).

Theorem 7.3. Fix \( T_0 > 0 \), \( 0 < \alpha < 1 \) and \( r > 0 \). A sequence of probability measures \( \mathbb{M}_n \) on \( C^\alpha([0, T_0], \mathcal{H}_{-r}) \) is tight if

(i) There exist a constant \( a > 0 \) such that \( \sup_{n \geq 1} \mathbb{M}_n \left[ \| X_0 \|_{-r}^2 \right] < \infty \);

(ii) There exist a finite constant \( C_0 \) and positive constants \( b > 0 \), \( c > 0 \) such that \( c/b > \alpha \) and \( \sup_{n \geq 1} \mathbb{M}_n \left[ \| X_t - X_s \|_{-r}^2 \right] \leq C_0 |t - s|^{1+\alpha} \) for all \( 0 \leq s, t \leq T_0 \).

We used here the same notation \( \mathbb{M}_n \) to represent a probability on \( C^\alpha([0, T_0], \mathcal{H}_{-r}) \) and the expectation with respect to this probability.

7.3. Tightness of the process \( B^\mu_t \) for \( d = 1, 2 \). Recall the definition of the \( \mathcal{H}_{-r} \)-valued process \( B^\mu_t \) introduced at the beginning of this section. Note that this process is continuous in time.

Fix \( 1 \leq j, k \leq d \). By definition of the operator \( \Pi_{j,k}^{\mu,2} \), there exists finite collection \( E_{j,k} \) of subsets of \( \mathbb{Z}^d \) with at least two elements such that

\[
(\Pi_{j,k}^{\mu,2} \delta_j)(\eta) = \sum_{D \in E_{j,k}} \epsilon_{j,k,D}(\rho) \omega_D ,
\]

where \( \epsilon_{j,k,D}(\rho) =\)
where $\omega_D$ has been introduce in (5.10).

In particular, to prove the tightness of the process $B_t^\rho$, it is enough to prove this property for the processes $B_t^{i,j,A,n}$, $A$ a finite subset of $\mathbb{Z}^d$ with at least two elements, where

$$B_t^{i,j,A,n}(F) := \int_0^t B_s^{A,n}(\partial_{x_i,x_j} F) \, ds$$

and

$$B_t^{A,n}(G) := \frac{1}{\sqrt{a_n} n^d} \sum_{x \in T_n^d} G_x \omega_{x+A}(t)$$

(7.3)

for $G : T_n^d \to \mathbb{R}$.

Fix $1 \leq i, j \leq d$ until the end of this subsection. We omit these indices from the notation hereafter and represent $B_t^{i,j,A,n}$, $B_t^{i,j,A,n}$ simply as $B_t^{A,n}$, $B_t^{A,n}$, respectively. The tightness of the process $B_t^{A,n}$ relies on the next estimate.

**Lemma 7.4.** Fix a finite subset $A$ of $\mathbb{Z}^d$. Then, there exists a finite constant $C_0 = C_0(\rho, A)$ such that

$$\mathbb{E}_{\mu_n} \left[ \left| \sum_{x \in T_n^d} F_x \omega_{x+A}(t) \right|^\alpha \right] \leq C_0 n^{\alpha d/2} \|F\|_\infty^n \left( 1 + H_n(f^n) \right)^{\alpha/2}$$

for all $1 \leq \alpha \leq 2$, $t > 0$, $F : T_n^d \to \mathbb{R}$, probability measure $\mu_n$, and $n \geq 1$. Here, $f^n_t$ stands for density of $\mu_n S^n(t)$ with respect to a Bernoulli measure $\nu^n_{\rho}$.

**Proof.** Fix $1 \leq \alpha \leq 2$. By Hölder inequality, the expectation is bounded by

$$\mathbb{E}_{\mu_n} \left[ \left\{ \sum_{x \in T_n^d} F_x \omega_{x+A}(t) \right\}^2 \right]^{\alpha/2} = \mathbb{E}_{\mu_n} \left[ \left\{ \sum_{x \in T_n^d} F_x \omega_{x+A} \right\}^2 \right]^{\alpha/2}.$$

By the entropy inequality, this expectation without the exponent $\alpha/2$ is less than or equal to

$$\frac{1}{\gamma} H_n(f^n) + \frac{1}{\gamma} \log \mathbb{E}_{\nu^n_{\rho}} \left[ \exp \left\{ \sum_{x \in T_n^d} F_x \omega_{x+A} \right\}^2 \right]$$

(7.4)

for all $\gamma > 0$.

Consider the second term. The function $\omega_A$ can be written as $i_{\leq j \leq 0} C_j(\rho) \{ f_j(\eta) - \tilde{f}_j(\rho) \}$, where each $f_j$ is a cylinder function. Hence, by Schwarz inequality [to move the sum over $j$ out of the square], Hölder’s inequality [to move the sum over $j$ out of the exponential], and Corollary 4.5, the second term of the previous displayed equation is bounded above by

$$C_0(\rho, A) \|F\|_\infty^n n^d$$

provided $\gamma n^d \|F\|_\infty^n < c_0(\rho, A)$. Here $0 < c_0 < C_0 < \infty$ are constants which depend on $A$ and $\rho$ only. Therefore, setting $\gamma n^d \|F\|_\infty^n = c_0/2$ yields that (7.4) is bounded by

$$C_0 n^d \|F\|_\infty^n \left\{ 1 + H_n(f^n) \right\},$$

which completes the proof of the lemma because $\alpha \leq 2$ [so that $C_0^{\alpha/2} \leq C_0$]. □

The next result is a simple consequence of Lemma 7.4 and Hölder’s inequality.
Corollary 7.5. Fix a finite subset $A$ of $\mathbb{Z}^d$. Then, there exists a finite constant
\[ C_0 = C_0(\rho, A) \] such that
\[
\mathbb{E}_\mu \left[ \int_s^t \sum_{x \in \mathbb{Z}^d} F_x(u) \omega_{x+A}(u) \, du \right]^\alpha \leq C_0 n^{\alpha d/2} |t-s|^\alpha \sup_{s \leq u \leq t} \|F(u)\|_\infty^\alpha \sup_{s \leq u \leq t} \left( 1 + H_n(f^n_u) \right)^{\alpha/2}
\]
for all $1 \leq \alpha \leq 2$, $0 < s < t$, $F: [0, T] \times \mathbb{T}_n^d \to \mathbb{R}$, probability measure $\mu_n$ and $n \geq 1$.

The proof of the tightness of the process $B_t^{A,n}$ relies on the following estimates. Recall the definition of the process $B_t^{A,n}$ introduced in (7.3).

Lemma 7.6. Fix $T > 0$, a finite subset $A$ of $\mathbb{Z}^d$ with at least two elements and $\delta > 0$. Fix also $0 < \theta < 1$ and $1 < \alpha \leq 2$. Let $\{\mu_n : n \geq 1\}$ be a sequence of probability measures such that $H_n(\mu_n | \nu^n_0) \leq a_n + R_d(n)$. Then, there exist a finite constant $C_1 = C_1(\rho, A, T)$ and an integer $n_0 = n_0(\delta, T)$ such that
\[
\mathbb{P}_\mu \left[ \int_s^t B_u^{A,n}(F(u)) \, du \right] > \lambda \leq C_1 \frac{1}{\lambda^\beta} \Omega_{s,t}^n \sup_{0 \leq u \leq T} \|F(u)\|_\infty^\gamma
\]
for all $\lambda > 0$, $0 \leq s < t \leq T$, $n \geq n(\delta, T)$ and $F: [0, T] \times \mathbb{T}_n^d \to \mathbb{R}$. In this formula,
\[
\Omega_{s,t}^n = \frac{n^\gamma \{a_n + R_d(n)\}^\gamma}{(a_n n^d)^{\beta/2}} |t-s|^{-(\beta/2) + (1-\theta)/2}
\]
and $\beta = \theta + (1-\theta)\alpha$, $\gamma = \theta + \alpha(1-\theta)/2$ and $R_d(n)$ is the sequence introduced in Theorem 2.8.

Proof: Let $p$ denote the probability appearing in the statement of the lemma, and write it as $p^\theta p^{1-\theta}$. Apply Chebyshev inequality to both terms to obtain that the previous integral is less than or equal to
\[
\frac{1}{\lambda^\beta} \mathbb{E}_{\mu_n} \left[ \int_s^t B_u^{A,n}(F(u)) \, du \right]^\theta \mathbb{E}_{\mu_n} \left[ \int_s^t B_u^{A,n}(F(u)) \, du \right]^{1-\theta}, \tag{7.5}
\]
where $\beta = \theta + (1-\theta)\alpha$.

By Corollary 6.3, the first expectation (including the exponent $\theta$) is bounded by
\[
C_1(A, \rho) \frac{1}{(a_n n^d)^{\theta/2}} \sup_{s \leq u \leq t} \|F(u)\|_\infty^\theta \left\{ \frac{1 + H_n(f^n_u)}{a_n} + \mathbb{H}_n(s, t) \right\}^{\theta/2},
\]
where $\mathbb{H}_n(s, t) = \int_s^t H_n(f^n_u) \, dr + (t-s)(n/\ell_n)^d$. Fix $\delta > 0$. Since $a_n \leq \sqrt{\log n}$, there exists $n(\delta, T) \geq 1$ such that, by (2.11) and (5.17), $\mathbb{H}_n(s, t) \leq n^\delta (t-s) \{a_n + R_d(n)\}$ for all $n \geq n(\delta, T)$, $0 \leq s \leq t \leq T$. We estimate $|1 + H_n(f^n_u)|/a_n + \mathbb{H}_n(s, t)$ by $3n^\delta (1 + T) \{a_n + R_d(n)\}$ and the second $\mathbb{H}_n(s, t)$ by $n^\delta (t-s) \{a_n + R_d(n)\}$. Hence, the expression appearing in the previous displayed equation is bounded by
\[
C_1(A, \rho) (1 + T) \sup_{0 \leq u \leq T} \|F(u)\|_\infty^\theta n^\delta \{a_n + R_d(n)\}^{\theta} (a_n n^d)^{\theta/2} |t-s|^{\theta/2}
\]
for all $0 \leq s, t \leq T$, $n \geq n(\delta, T)$. We estimated $(1 + T)^{\theta/2}$ by $1 + T$, as $\theta < 1$. 

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By Corollary 7.5, the second expectation in (7.5) (including the exponent $1 - \theta$) is bounded by
\[ C_1(\rho, A) \left| t - s \right|^{\alpha(1 - \theta)} \sup_{0 \leq u \leq t} \| F(u) \|_\infty^{\alpha(1 - \theta)} \sup_{0 \leq u \leq T} \left( 1 + H_n(f_u^n) \right)^{\alpha(1 - \theta)/2}. \]
Thus, by the bound on $H_n(f_u^n)$ presented in the previous paragraph, this expression is bounded by
\[ C_1(\rho, A, T) \left| t - s \right|^{\alpha(1 - \theta)} \sup_{0 \leq u \leq t} \| F(u) \|_\infty^{\alpha(1 - \theta)} n^{\delta \alpha(1 - \theta)/2} \{ a_n + R_d(n) \}^{\alpha(1 - \theta)/2} \]
for all $0 \leq s, t \leq T, n > n(\delta, T)$.

Putting together the previous estimates yields that (7.5) is bounded above by
\[ C_1(\rho, A, T) \frac{1}{\lambda^d} \frac{n^\gamma \{ a_n + R_d(n) \} \gamma}{(a_n n^d)^{\theta/2}} \left| t - s \right| \left( \frac{\theta}{2} + \alpha(1 - \theta) \right) \sup_{0 \leq u \leq t} \| F(u) \|_\infty^\beta \]
for all $n \geq n(\delta, T)$, where $\gamma = \theta + \alpha(1 - \theta)$ and, recall, $\beta = \theta + \alpha(1 - \theta)$. This is the assertion of the lemma.

The previous bound provides two estimates. The first one will be used later in the proof of the tightness of the process $X^n_t$ in dimension $d = 3$. The second one is needed in the proof of the tightness of the process $B^n_t$ in dimensions $d = 1, 2$. Recall the definition of $\Omega_{s,t}$ introduced in the previous lemma.

**Corollary 7.7.** Fix a finite subset $A$ of $\mathbb{Z}^d$ with at least two elements and $\delta > 0$. Fix also $v > 1$, $0 < \theta < 1$ and $1 < \alpha \leq 2$ such that
\[ \beta := \theta + \alpha(1 - \theta) > v. \]
Let $(\mu_n : n \geq 1)$ be a sequence of probability measures such that $H_n(\mu_n | \nu^n_\rho) \leq a_n + R_d(n)$. Then, there exist a finite constant $C_1 = C_1(\rho, A, T)$ and an integer $n_0 = n_0(\delta, T)$ such that
\[ \mathbb{E}_{\mu_n} \left[ \left| \int_s^t B_{u,n}^a(F(u)) \, du \right|^\beta \right]^{\beta/v} \leq C_1 \left( \frac{\beta}{\beta - v} \right)^{\beta/v} \Omega_{s,t}^n \sup_{0 \leq u \leq T} \| F(u) \|_\infty^\beta \]
for all $0 \leq s < t \leq T, n \geq n(\delta, T)$ and $F : [0, T] \times \mathbb{T}^d \rightarrow \mathbb{R}$.

**Proof:** Fix $0 \leq s \leq t$, and write
\[ \mathbb{E}_{\mu_n} \left[ \left| \int_s^t B_{u,n}^a(F(u)) \, du \right| \right] = \int_0^\infty \mathbb{E}_{\mu_n} \left[ \left| \int_s^t B_{u,n}^a(F(u)) \, du \right| > \kappa^{1/v} \right] \, d\kappa. \]
Fix $a > 0$. We estimate the above probability by 1 on the interval $0 \leq \kappa \leq a$. On the other hand, by Lemma 7.6 with $\lambda = \kappa^{1/v}$, the previous integral restricted to the interval $[a, \infty)$, is bounded above by
\[ C_1(\rho, A, T) \Omega_{s,t}^n \sup_{0 \leq u \leq t} \| F(u) \|_\infty^{\beta} \int_a^\infty \frac{1}{\kappa^{\beta/v}} \, d\kappa \]
for all $n \geq n(\delta, T)$, where, recall, $\beta = \theta + \alpha(1 - \theta)$. As $\beta > v$, this expression is equal to $C_1 (b - 1)^{-1} a^{1 - b} \Omega_{s,t}^n \sup_{0 \leq u \leq t} \| F(u) \|_\infty^{\beta}$, where $b = \beta/v > 1$.

Up to this point, we proved that the expectation appearing in the statement of the corollary is bounded by $a + C_1 (b - 1)^{-1} a^{1 - b} \Omega_{s,t}^n \sup_{0 \leq u \leq t} \| F(u) \|_\infty^{\beta}$ for all $a > 0$. It remains to optimize over $a$ to complete the proof. \( \square \)
Recall that $1 \leq i, j \leq d$ are fixed and that $B_t^{i,n}$ stand for $B_t^{i,j,A,n}.$

**Corollary 7.8.** Fix a finite subset $A$ of $\mathbb{Z}^d$ with at least two elements and $\delta > 0$. Fix also $\nu > 1$, $0 < \theta < 1$ and $1 < \alpha \leq 2$ such that

$$\beta := \theta + \alpha (1 - \theta) > \nu.$$  

Finally, fix $r > 0$ such that

$$r > \frac{d+5}{2} + \frac{d}{\beta}.$$  

Let $(\mu_n : n \geq 1)$ be a sequence of probability measures such that $H_n(\mu_n | \nu^n) \leq a_n + R_d(n)$. Then, there exist a finite constant $C_1 = C_1(\rho, A, T, \nu, \theta, \alpha, r)$ and an integer $n_0 = n_0(\delta, T)$ such that

$$\mathbb{P}_{\mu_n} \left[ \left\| B_t^{A,n} - B_t^{A,n} \|^{-r} \right\|^{\beta / \nu} \leq C_1 \Omega_{s,t} \right]$$

for all $0 \leq s < t \leq T$, $n \geq n_0$.

**Proof:** Fix $0 \leq s < t \leq T$, and write

$$\mathbb{E}_{\mu_n} \left[ \left\| B_t^{A,n} - B_t^{A,n} \|^{-r} \right\|^{\beta / \nu} \right] = \int_0^\infty \mathbb{P}_{\mu_n} \left[ \left\| B_t^{A,n} - B_t^{A,n} \|^{-r} > \kappa^{1/\nu} \right\| \right] \, d\kappa.$$

Fix $a > 0$ and estimate the above probability by $1$ on the interval $[0, a]$.

We turn to the interval $[a, \infty)$. Let $\{w_m : m \in \mathbb{Z}^d\}$ be the sequence given by $w_m = \omega (1 + \|m\|)^{-(d+1)}$, where $\omega$ is the normalizing constant chosen so that

$$\sum_{m \in \mathbb{Z}^d} w_m = 1.$$  

(7.6)

The probability on the right-hand side of the first displayed equation of the proof is equal to

$$\mathbb{P}_{\mu_n} \left[ \sum_{m \in \mathbb{Z}^d} \gamma_{m}^{-r} \left\| B_t^{A,n}(\phi_m) - B_t^{A,n}(\phi_m) \right\|^2 > \kappa^{2 / \nu} \right]$$

$$\leq \sum_{m \in \mathbb{Z}^d} \mathbb{P}_{\mu_n} \left[ \left\| B_t^{A,n}(\phi_m) - B_t^{A,n}(\phi_m) \right\| > \kappa^{1 / \nu} \{w_m \gamma_m^r \}^{1/2} \right].$$

By Lemma 7.6 with $\lambda = \kappa^{1 / \nu} \{w_m \gamma_m^r \}^{1/2}$, $F(u) = \partial_{x_i,x_j}^2 \phi_m$, since $\|\partial_{x_i,x_j}^2 \phi_m\|_\infty \leq C_0 \|m\|^2$, the previous expression is bounded above by

$$C_1(\rho, A, T) \Omega_{s,t} \frac{1}{\kappa^{2 / \nu}} \sum_{m \in \mathbb{Z}^d} \frac{1}{(w_m \gamma_m^r)^{\beta / 2}} \|m\|^{2\beta}$$

for all $n \geq n(\delta, T)$. As $r > (d + 3) / 2 + (d / \beta)$, the sum over $m$ is finite.

Up to this point, we proved that the expectation appearing in the statement of the corollary is bounded by $a + C_2(b - 1) n^{1 - b} \Omega_{s,t}$ for all $a > 0$. Here, $b = \beta / \nu > 1$ and $C_2$ is a constant which depends on $\rho$, $A$, $T$ and also on $\theta, \alpha, \nu, r$ [through $\beta$]. It remains to optimize over $a$ to complete the proof. $\square$

Fix a finite subset $A$ of $\mathbb{Z}^d$ with at least two elements and $r > 0$. Let $(\mu_n : n \geq 1)$ be a sequence of probability measures such that $H_n(\mu_n | \nu^n) \leq a_n + R_d(n)$. Denote by $Q_{\rho}^{A,B}$ the measure on $D([0,T], \mathcal{H}_{-r})$ induced by the process $B_t^{A,n}$ and the measure $\mathbb{P}_{\mu_n}$. 

Remark 7.10. Unfortunately, the previous argument does not apply to the dimension $d = 3$. The reason is the term $1/a_n$ in the estimate stated in Proposition 6.2, which does not depend on time. In particular, for $|t - s|$ very small, the estimate is bad.

This term $1/a_n$, independent of time, gives rise to the power $1/2$ in the expression $|t - s|^{\theta/2}$ when we estimate the first expectation in equation (7.5). In Proposition 6.2, if, we had, instead of $1/a_n$, a term which decreases linearly in time, as $t \to 0$, we would get $|t - s|^\theta$ in equation (7.5), instead of $|t - s|^{\theta/2}$, and the proof of the tightness could be carried out in dimension 3 taking $\theta = 2/3 + \epsilon$ in Corollary 7.8.

7.4. Tightness of the martingale $M^n_t$. By [15, Lemma A.5.1],

$$M^n_t(F)^2 - \int_0^t \{L_nX^n_s(F)^2 - 2X^n_s(F) L_nX^n_s(F)\} \, ds$$

is a martingale. Denote by $\Gamma^n(F)$ the expression inside braces. A straightforward computation yields that

$$\Gamma^n(F) = \frac{1}{a_n n^d} \sum_{j=1}^d \sum_{x \in T^n_a} e_j(\tau_x \eta) [\eta_{x+e_j} - \eta_x]^2 \left[|\nabla_{n,j}F|(x/n)\right]^2$$

$$+ \frac{1}{n^d} \sum_{x \in T^n_a} \sum_y F(x/n)^2 \left[\eta_y - \eta_x\right]^2,$$

(7.7)

where $\nabla_{n,j}F$ stands for the discrete partial derivative given by $(\nabla_{n,j}F)(x) = n[F((x+e_j)/n) - F(x/n)]$. Thus, $|\Gamma^n(F)|$ is bounded by $C_0 \{\|F\|^2_\infty + \|\nabla F\|^2_\infty\}$ for some finite constant $C_0$, and

$$\mathbb{E}_n[\mathbb{E}^n[M^n_t(F)^2]] \leq C_0 t \{\|F\|^2_\infty + \|\nabla F\|^2_\infty\}$$

(7.8)

for all $t > 0$. 
Lemma 7.11. Fix $d \geq 1$ and $r > 1 + (d/2)$. Then, there exists a finite constant $C_0$ such that
\[
\limsup_{n \to \infty} \sup_{\eta \in \Omega_n} \mathbb{E}_\eta^n \left[ \sup_{0 \leq t \leq T} \| M^n_t \|_{-r}^2 \right] \leq C_0 T.
\]
Moreover,
\[
\lim_{p \to \infty} \limsup_{n \to \infty} \sup_{\eta \in \Omega_n} \mathbb{E}_\eta^n \left[ \sup_{0 \leq t \leq T} \sum_{\|m\| \geq p} \gamma_m^{-r} \| M^n_t(\phi_m) \|^2 \right] = 0.
\]

Proof. By the formula for the $\mathcal{H}_{-r}$-norm and since $\sup_t \sum_m a_m(t) \leq \sum_m \sup_t a_m(t)$, the first expectation is bounded by
\[
\sum_{m \in \mathbb{Z}^d} \gamma_m^{-r} \mathbb{E}_\eta^n \left[ \sup_{0 \leq t \leq T} \| M^n_t(\phi_m) \|^2 \right].
\]
Since $M^n_t(\phi_m)$ is a martingale for each $m \in \mathbb{Z}^d$, by Doob’s inequality, this sum is bounded by
\[
4 \sup_{\eta \in \Omega_n} \sum_{m \in \mathbb{Z}^d} \gamma_m^{-r} \mathbb{E}_\eta^n \left[ \| M^n_T(\phi_m) \|^2 \right].
\]
By (7.8) and by definition of $\phi_m$, this expression is less than or equal to
\[
C_0 T \sum_{m \in \mathbb{Z}^d} \gamma_m^{-r} (1 + \|m\|^2).
\]
This proves the first assertion of the lemma since $r > 1 + (d/2)$. The proof of the second one is similar.

Fix $r > 0$, a sequence of probability measures $\mu_n$ on $\Omega_n$, and denote by $\mathbb{Q}^M_n$ the measure on $D([0, T], \mathcal{H}_{-r})$ induced by the martingale $M^n_t$ and the measure $\mathbb{P}^n_{\mu_n}$.

Lemma 7.12. Fix $d \geq 1$ and $r > 1 + (d/2)$. The sequence of measures $\mathbb{Q}^M_n$ is tight in $D([0, T], \mathcal{H}_{-r})$. Moreover, all limit points are concentrated on continuous trajectories.

Proof. According to [15, Lemma 11.3.2], we have to show that
\[
\lim_{A \to \infty} \limsup_{n \to \infty} \mathbb{P}^n_{\mu_n} \left[ \sup_{0 \leq t \leq T} \| M^n_t \|_{-r} > A \right] = 0
\]
for some $r' < r$ and that for every $\epsilon > 0$,
\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}^n_{\mu_n} \left[ \sup_{s,t} \| M^n_t - M^n_s \|_{-r} > \epsilon \right] = 0
\]
where the supremum is carried over all $0 \leq s, t \leq T$ such that $|t - s| \leq \delta$.

The first condition follows from the first assertion of Lemma 7.11. By the second assertion of this lemma, to prove the second condition, it is enough to show that for all $m \in \mathbb{Z}^d$, $\epsilon > 0$,
\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}^n_{\mu_n} \left[ \sup_{s,t} \| M^n_t(\phi_m) - M^n_s(\phi_m) \| > \epsilon \right] = 0.
\]
The proof of this statement is similar to the one of [15, Lemma 11.3.7] and left to the reader. Note that the proof of [15, Lemma 11.3.7] uses that the process is in equilibrium only at the top of page 303. In the present context we can recall that there is at most one particle per site to derive a similar bound out of equilibrium.
7.5. **Tightness of the process $I^r_t$ in dimension 1 and 2.** Denote by $I^r_t$ the process defined by

$$I^r_t(F) = \int_0^t X^r_s(AF) \, ds, \quad F \in C^\infty(T^d).$$

The main result of this section asserts that the process $I^r_t$ is tight in $D([0,T],\mathcal{H}_{-r})$ for $r > (3d + 5)/2$. The proof is based on a representation of $I^r_t$.

Recall that we denote by $(P_t : t \geq 0)$ the semigroup associated to the operator $A$. Fix $0 < t \leq T$. For $F$ in $C^\infty(T^d)$, let $F_{s,t} = P_{t-s}F$, $0 \leq s \leq t$. Denote by $M^{r,t}_s(F)$, $0 \leq s \leq t$, the martingale defined by

$$M^{r,t}_s(F) := X^n_s(F_{s,t}) - X^n_0(F_{0,t}) - \int_0^s (\partial_u + L_n)X^n_u(F_{s,t}) \, du,$$

for $F \in C^\infty(T^d)$. Taking $s = t$ yields a representation for $X^n_t(F)$:

$$X^n_t(F) = X^n_t(F_{0,t}) + M^{r,t}_s(F) + \int_0^t (\partial_u + L_n)X^n_u(F_{s,t}) \, du. \quad (7.10)$$

Since $\partial_u X^n_u(F_{s,t}) = -X^n_u(AF_{s,t})$, by (4.3),

$$(\partial_u + L_n)X^n_u(F_{s,t}) = R^n_u(F_{s,t}) + B^n_u(F_{s,t}),$$

where $R^n$ and $B^n$ have been introduced just below (4.3). In particular, the tightness of the process $I^r_t$ follows from the tightness of the processes $I^{p,n}_t$, $1 \leq p \leq 4$, where

$$I^{1,n}_t(F) := \int_0^t X^n_0(P_sAF) \, ds, \quad I^{2,n}_t(F) := \int_0^t M^{r,n}_s(AF) \, ds,$$

$$I^{3,n}_t(F) := \int_0^t \int_0^s R^n_u((AF)_{u,s}) \, du \, ds, \quad I^{4,n}_t(F) := \int_0^t \int_0^s B^n_u((AF)_{u,s}) \, du \, ds.$$

Consider the process $I^{1,n}_t$. Recall from (2.7) that $\phi_m$ is an eigenvector of $A$ so that $P_sA\phi_m = -\lambda(m)e^{-\lambda(m)s}\phi_m$ and $X^n_0(P_sA\phi_m) = -\lambda(m)e^{-\lambda(m)s}X^n_0(\phi_m)$. In particular, by definition, of the $\mathcal{H}_{-r}$-norm, for $0 \leq s \leq t$,

$$\|I^{1,n}_t - I^{1,n}_s\|_{-r}^2 = \sum_{m \in \mathbb{Z}^d} \lambda_m^{-r} \int_s^t e^{-\lambda(m)u} \, du \|X^n_0(\phi_m)\|^2,$$

so that $\|I^{1,n}_t - I^{1,n}_s\|_{-r}^2 \leq C_0(t-s)^2 \|X^n_0\|^2_{-r+1}$ for some finite constant $C_0$. Therefore, by (4.5), for $r > (d/2) + 1$,

$$E_{\mu_n}\left[ \sup_{|t-s| \leq \delta} \|I^{1,n}_t - I^{1,n}_s\|_{-r}^2 \right] \leq C_0(\rho) \delta^2 \frac{1}{\alpha_n} (H_n(\mu_n | \nu_n) + C_0).$$

By definition of the Skorohod topology, the tightness of the process $I^{1,n}_t$ in $D([0,T],\mathcal{H}_{-r})$ for $r > 1 + (d/2)$ follows from this estimate and the fact that $I^{1,n}_0 = 0$. This estimate also yields that all limit points are concentrated on continuous trajectories.

We turn to the process $I^{3,n}_t$. Since $\phi_m$ is an eigenvector for $A$, a change of the order of summations yields that

$$I^{3,n}_t(\phi_m) = -\int_0^t R^n_u(\phi_m)\left( 1 - e^{-\lambda(m)|t-u|} \right) \, du.$$


Hence, by definition of the norm in $\mathcal{H}_{-r}$ and Schwarz inequality,
$$\| I^{3,n}_t \|_{-r}^2 \leq t \int_0^t \| R^n_s \|_{-r}^2 \, ds$$
for all $t > 0$, $r > 0$. In particular, by the proof of Lemma 7.2, for $r > 3 + (d/2)$, as $n \to \infty$ the process $I^{3,n}$ vanishes in $D([0,T], \mathcal{H}_{-r})$.

We turn to the process $I^{4,n}_t$. Similar to $I^{3,n}_t$, we have that
$$I^{4,n}_t(\phi_m) = - \int_0^t B^n_u(\phi_m)\{1 - e^{-\lambda(m)|t-u|}\} \, du.$$ 
We may therefore repeat the arguments presented in Subsection 7.3 to prove that, in dimension 1 and 2, the process $I^{4,n}$ is tight in $D([0,T], \mathcal{H}_{-r})$ and that all limit points are concentrated on continuous trajectories provided $r > (3d+5)/2$.

Finally, we consider the process $I^{2,n}_t$. The computation, performed at the beginning of Subsection 7.4, yields that the predictable quadratic variation of the martingale $M^{n,t}(F)$, denoted by $(M^{n,t}(F))_s$, is given by
$$(M^{n,t}(F))_s = \int_0^s \Gamma^n_u(F_u,t) \, du,$$ (7.12)
where $\Gamma^n(F)$ has been introduced in (7.7).

By Schwarz inequality and since $\phi_m$ is an eigenvector of $A$ associated to the eigenvalue $\lambda(m)$, for $m \in \mathbb{Z}^d$,
$$\| I^{2,n}_t(\phi_m) \|^2 \leq \lambda(m)^2 t \int_0^t \| M^{n,a}_u(\phi_m) \|^2 \, du.$$ 

It follows from the two previous displayed equations that for every $m \in \mathbb{Z}^d$, $T > 0$,
$$\sup_{n \geq 1} \mathbb{E}_q^n \left[ \sup_{0 \leq s \leq T} \| I^{2,n}_s(\phi_m) \| \right] \leq C_0 T \lambda(m)^2 \int_0^T \mathbb{E}_q^n \left[ \int_0^u \Gamma^n_u(\phi_m) \, dv \right] \, du.$$ 

At the beginning of the proof of Lemma 7.11, we showed that $\mathbb{E}_q^n \left[ \sup_{0 \leq s \leq T} \| I^{2,n}_s \|^2_{-r} \right]$ is bounded by a weighted sum of terms like the one appearing on the left-hand side of the previous inequality. At this point, we may repeat the arguments presented in Subsection 7.4 to deduce that the process $I^{2,n}_t$ is tight in $D([0,T], \mathcal{H}_{-r})$ and that all limit points are concentrated on continuous trajectories provided $r > 3 + (d/2)$.

We have proved the following result. Let $\mu_n$ be a sequence of probability measures on $\Omega_n$. Denote by $Q^n_\mu$ the measure on $D([0,T], \mathcal{H}_{-r})$ induced by the process $I^{n}_t$ and the measure $\mathbb{P}_{\mu_n}$.

**Lemma 7.13.** Assume that $d = 1$ or 2. Let $\mu_n$ be a sequence of probability measures on $\Omega_n$ such $\lim_{n \to \infty} a_{n}^{-1} H_n(\mu_n | \nu^0_\rho) = 0$. The sequence of probability measures $Q^n_\mu$, is tight in $D([0,T], \mathcal{H}_{-r})$ and all its limit points are concentrated on continuous trajectories provided $r > (3d+5)/2$.

7.6. **Proof of Proposition 7.1.** Fix $r > (3d+5)/2$ and assume that $d = 1$ or 2. The tightness of the measure $Q_n$ in $D([0,T], \mathcal{H}_{-r})$ and the fact that all its limit points are concentrated on continuous trajectories follow from the decomposition (4.4), Lemma 7.2, Corollary 7.9 and Lemmata 7.12 and 7.13. \qed
Remark 7.14. The proof does not hold in dimension 3 only because we are not able to prove that the \( \mathcal{H}_{-r} \)-valued process \( B^n_t \) is tight in \( C([0, T], \mathcal{H}_{-r}) \). See Remark 7.10.

7.7. Tightness in dimension 3. In dimension 3, we prove that the time integral of \( X^n_t \) is tight under the measure \( \mathbb{P}_{\mu_n} \). We start with a bound in \( L^p \) for \( p > 1 \). Throughout this subsection, \( \mu_n \) is a sequence of probability measures on \( \Omega_n \) such \( \lim_{n \to \infty} \alpha_n^{-1} H_n(\mu_n | \nu^\alpha) = 0 \).

Lemma 7.15. For every \( p < 4/3 \), there exists a finite constant \( C_0 = C_0(\rho, A, T, p) \) such that
\[
\mathbb{E}_{\mu_n} \left[ \| X^n_t(F) \|^p \right] \leq C_0 \| F \|_{C^3(\mathbb{T}^d)}^p
\]
for all \( 0 \leq t \leq T, F \in C^\infty(\mathbb{T}^d), n \geq 1 \).

Proof: In view of (7.10) and (7.11), we have to estimate four terms. The first one is easy. By the proof of Lemma 4.2, and since
\[
\mathbb{E}_{\mu_n} \left[ X^n_0(P_t F)^2 \right] \leq \frac{C_0}{\alpha_n} \left( H_n(\mu_n | \nu^\rho) + 1 \right) \| F \|_\infty^2.
\]
for some finite constant \( C_0 \) depending only on \( \rho \). By hypothesis, this expression vanishes as \( n \to \infty \). Note that this estimate holds also in dimension 1 and 2.

The martingale term is also simple to estimate. By (7.12) and (7.7),
\[
\mathbb{E}_{\mu_n} \left[ \| M^n_{0,t}(F) \|^2 \right] \leq C_0 \left\{ \frac{1}{\alpha_n} \sup_{0 \leq s \leq t} \| \nabla P_s F \|_\infty^2 + \| F \|_\infty^2 \right\}
\]
for some finite constant \( C_0 \) depending only on the dimension. As \( P_s \) commutes with the gradient, the right-hand side is bounded by \( C_0 \| F \|_{C^4(\mathbb{T}^d)}^2 \).

We turn to the term \( R^n \). By Schwarz inequality and (7.2),
\[
\mathbb{E}_{\mu_n} \left[ \left( \int_0^t R^n_s(P_{t-s} F) \ ds \right)^2 \right] \leq \frac{C_0 T}{\alpha_n} \left\{ \int_0^t H_n(f^n_s) \ ds + t \right\} \sup_{0 \leq s \leq t} \| P_s F \|_{C^3(\mathbb{T}^d)}^2
\]
for a finite constant \( C_0 \) which depends only on the cylinder functions \( h_{1,k} \). As \( P_s \) commutes with the spatial derivatives, \( \sup_{0 \leq s \leq t} \| P_s F \|_{C^3(\mathbb{T}^d)}^2 \leq \| F \|_{C^3(\mathbb{T}^d)}^2 \).

Hence, by (2.11), for every \( \delta > 0 \),
\[
\mathbb{E}_{\mu_n} \left[ \left( \int_0^t R^n_s(P_{t-s} F) \ ds \right)^2 \right] \leq \frac{C_0(T)}{n^{4-\delta}} \| F \|_{C^3(\mathbb{T}^d)}^2
\]
for all \( n \) large enough.

It remains to consider the term \( B^n \). In Corollary 7.7, fix \( \epsilon > 0 \) small, and let \( s = 0, F(u) = P_{t-u} F, \alpha = 2, \delta = \epsilon/2, \theta = (2/3) + \epsilon, v = (4/3) - 2\epsilon \). With this choice, \( \beta = (4/3) - \epsilon > v \), and there exists a constant \( C_1 = C_1(\rho, A, T) \) such that
\[
\mathbb{E}_{\mu_n} \left[ \left( \int_0^t B^n_s N P_{t-s} F \ ds \right)^v \right] \leq C_1 \frac{\beta}{\beta - v} \left( \Omega_T^n \right)^{v/\beta} \sup_{0 \leq s \leq T} \| P_s F \|_{C^3(\mathbb{T}^d)}^v
\]
where
\[
\Omega_T^n = \frac{n^{\delta} R_d(n)}{\alpha_n n^{\delta}} T^2.
\]
Here, we used that \( \gamma = 1, T > 1 \) (assumed at the beginning of this section) and \( \theta > 0 \). As \( d = 3 \), by definition of \( \theta, \delta \) and \( R_d(n) \), the \( n \)-dependent constant multiplying \( T^2 \) is equal to \( \alpha_n^{(2/3)-(\epsilon/2)} n^{-\epsilon} \), which vanishes as \( n \to \infty \).
To complete the proof of the lemma, it remains to apply Hölder’s inequality to the first three terms to derive $L^p$ estimates from $L^2$ ones. □

Recall that $X^n_t$ is the $\mathcal{H}_{-r}$-valued process defined by

$$X^n_t(F) = \int_0^t X^n_s(F) \, ds, \quad F \in C^\infty(\mathbb{R}^d).$$

**Corollary 7.16.** Fix $T > 0$, $r > (3d + 7)/2$, $0 < \vartheta < 1/4$. Let $\mu_n$ be a sequence of probability measures on $\Omega_n$ such that $\lim_{n \to \infty} a^{-1}_n H_n(\mu_n | \nu^n_0) = 0$. Then, the sequence of probability measures $P^{\mu_n} \circ (X^n)^{-1}$ on $C^\infty([0,T], \mathcal{H}_{-r})$ is tight.

**Proof.** The proof is based on verifications of Theorem 7.3 hypotheses. The first condition of Theorem 7.3 is in force because $X^n_0 = 0$. The second one follows from (7.14) below. Fix $q$ so that $(q - 1)/q = \vartheta$ and note that $1 < q < 4/3$. We claim that there exists a finite constant $C_0$, depending on $\rho$, $T$, $q$ and $r$ such that

$$E_{\mu_n} \left[ \|X^n_t - X^n_s\|^q_{-r} \right] \leq C_0 |t - s|^q$$

(7.14)

for all $0 \leq s < t \leq T$.

The proof of this claim is similar to the one of Corollary 7.8. Fix $0 < s \leq t \leq T$, $1 < q < p < 4/3$, and write

$$E_{\mu_n} \left[ \|X^n_t - X^n_s\|^q_{-r} \right] = \int_0^\infty P_{\mu_n} \left[ \|X^n_t - X^n_s\|_{-r} \geq \kappa^{1/q} \right] d\kappa.$$

Fix $a > 0$ and estimate the above probability by 1 on the interval $[0,a]$.

We turn to the interval $[a,\infty)$. Recall the definition of the sequence $\{w_m : m \in \mathbb{Z}^d\}$ introduced in (7.6). The probability on the right-hand side of the previous displayed equation is equal to

$$P_{\mu_n} \left[ \sum_{m \in \mathbb{Z}^d} \gamma_m^{r^{-1}} \|X^n_t(\phi_m) - X^n_s(\phi_m)\|^2 > \kappa^{2/q} \right] \leq \sum_{m \in \mathbb{Z}^d} P_{\mu_n} \left[ \|X^n_t(\phi_m) - X^n_s(\phi_m)\|^p > \kappa^{p/q} \{w_m \gamma_m^{r^{-1}}\}^{p/2} \right].$$

By Chebyshev, followed by Hölder inequality and by Lemma 7.15, as $\|\phi_m\|_{C^3(\mathbb{R}^d)} \leq C_0 |m|^3$, this sum is bounded by

$$C_1(\rho, A, T, p) \frac{1}{\kappa^{p/q}} \sum_{m \in \mathbb{Z}^d} \frac{|m|^{3p}}{(w_m \gamma_m^{r^{-1}})^{p/2}} |t - s|^p.$$

As $r > (3d + 7)/2 > (d + 7)/2 + (d/p)$, the sum over $m$ is finite.

Up to this point, we proved that the expectation appearing in (7.14) is bounded by $a + C_2(b - 1)^{-1} a^{1-b} |t - s|^p$ for all $a > 0$. Here, $b = p/q > 1$ and $C_2$ is a constant which depends on $\rho$, $T$, $p$, $q$ and $r$. It remains to optimize over $a$ to derive the bound (7.14). □

**8. PROOF OF THEOREMS 2.1 AND 2.2**

The proof is divided in two parts. In the previous section, we proved that the sequence $X^n$, $X^n$ are tight in dimension $d \leq 2$ and $d = 3$, respectively. Proposition 8.6 below characterizes the limit points of the sequence $P_{\mu_n} \circ (X^n)^{-1}$ in dimension $d \leq 2$, and Lemma 8.7 the ones of the sequence $P_{\mu_n} \circ (X^n)^{-1}$ in $d = 3$. [Further text]
Throughout this section \( T > 0 \) and \( 0 < \rho < 1 \) are fixed, and, unless otherwise stated, all results hold for \( d \leq 3 \).

We start with a simple consequence of the entropy inequality. The proof of this result is similar to the one of Lemma 6.1 and left to the reader.

**Lemma 8.1.** Fix a cylinder function \( f \). Then, there exists a finite constant \( C_0 = C_0(f, \rho) \) such that

\[
\mathbb{E}_{\mu_n} \left[ \int_0^t \left| \sum_{x \in \mathbb{T}^d_n} J_x [ f(\tau_x \eta^n(s)) - \hat{f}(\rho) ] \right| \, ds \right] \\
\leq \int_0^t \left\{ \frac{1}{\gamma} [H_n(f^n_s) + \log 2] + C_0 \gamma n^d \| J \|_\infty e^{C_0 \gamma \| J \|_\infty} \right\} \, ds
\]

for every function \( J : \mathbb{T}^d_n \to \mathbb{R}, t > 0, n \geq 1, \gamma > 0 \).

Fix a function \( F \in C^\infty(\mathbb{T}^d), \) and recall the definition of the process \( R^n(F) \), introduced in (4.3). It is expressed as the sum of three terms. To estimate the first one, in the previous lemma, set \( J_x = \{a_n n^d\}^{-1/2} \{ (\Delta_{j,k}^n_F) - (\partial^2_{j,k} F) \} (x/n), \) \( f = h_{j,k}, \gamma = \sqrt{R_n(d) a_n n}. \) To estimate the second one, let \( J_x = \{a_n n^d\}^{-1/2} (\Delta^n_F) (x/n), \) \( f(\eta) = \eta_0, \gamma = \sqrt{R_n(d) a_n n^2}. \) For the last one, in view of the formula for \( \Pi_1 \), presented in Assertion C.1, perform an integration by parts and set \( J_x = \{a_n n^d\}^{-1/2} \{ (\partial^2_{j,k} F)(x + z)/n - (\partial^2_{j,k} F)(x/n) \}, f = \eta_0, \gamma = \sqrt{R_n(d) a_n n}. \) In this formula, \( z \) is the element of \( \mathbb{Z}^d \) which appears in the formula of \( \Pi_1 h_{j,k}. \) Putting together the two estimates yield the following result.

**Corollary 8.2.** Consider a sequence of measures \( \mu_n \) on \( \Omega_n \) such that \( H_n(\mu_n | \nu^n) \leq R_d(n) \). For every function \( F \in C^\infty(\mathbb{T}^d), t > 0, \)

\[
\lim_{n \to \infty} \mathbb{E}_{\mu_n} \left[ \int_0^t |R^n(F)| \, ds \right] = 0.
\]

Recall the definition of the martingale \( M^n_t(F), F \in C^\infty(\mathbb{T}^d), \) introduced in (4.1). Let \( H : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R} \) be a smooth function, and denote by \( H_t : \mathbb{T}^d \to \mathbb{R} \) the function given by \( H_t(x) = H(t, x), \) and by \( M^n_t(H) \) the value of the martingale \( M^n_t(F) \) for \( F = H_t. \)

**Lemma 8.3.** For every \( t > 0, \)

\[
M^n_t(H_t) = X^n_t(H_t) - X^n_0(H_0) - \int_0^t \left\{ L_n X^n_s(H_s) + X^n_s(\partial_s H_s) \right\} \, ds + \int_0^t M^n_s(\partial_s H_s) \, ds.
\]

**Proof:** Fix \( t > 0 \) and let \( F = H_t. \) By definition of the martingale \( M^n_t(F), \)

\[
M^n_t(H_t) = X^n_t(H_t) - X^n_0(H_t) - \int_0^t L_n X^n_s(H_s) \, ds.
\]

Writing \( H_t \) as \( H_u + \int_{[u,t]} \partial_v H_v \, dv, \) for \( u = 0 \) and \( u = s, \) yields that the right-hand side is equal to

\[
X^n_t(H_t) - X^n_0(H_0) - \int_0^s X^n_0(\partial_s H_s) \, ds - \int_0^t L_n X^n_s(H_s) \, ds - \int_0^t \int_s^t \, ds L_n X^n_s(\partial_u H_u).
\]
Change the order of the integrals in the last term. The sum of this integral with the third term is equal to
\[ -\int_0^t \left\{ X^n_0 (\partial_s H_s) + \int_0^s L_n X^n_u (\partial_s H_s) \, du \right\} \, ds = \int_0^t \left\{ M^n_s (\partial_s H_s) - X^n_s (\partial_s H_s) \right\} \, ds. \]

This completes the proof of the lemma. \( \square \)

This lemma provides a representation of the martingale \( M^{n,t} \), introduced in (7.9), in terms of the martingales \( M^n \). Fix \( 0 < t \leq T \), and a function \( F \) in \( C^\infty(\mathbb{T}^d) \). Let \( H(s, x) = (P_{t-s} F)(x), 0 \leq s \leq t \). By (7.9) and Lemma 8.3, since \( H_t = F \) and \( \partial_s H_s = \rho \),
\[ M^{n,t}(F) = M^n(F) + \int_0^t M^n_s (P_{t-s} A F) \, ds. \] (8.1)

Denote by \( \langle \cdot, \cdot \rangle \) the scalar product in \( L^2(\mathbb{T}^d) \):
\[ \langle F, G \rangle = \int_{\mathbb{T}^d} F(x) G(x) \, dx. \]

**Lemma 8.4.** Fix \( r > 1 + (d/2) \) and a sequence of probability measures \( \mu_n \) on \( \Omega_n \), such that \( H_n(\mu_n | \nu^0) \leq R_n(d) \). Then, the sequence of measures \( Q_n^M = \mathbb{P}_{\mu_n} \circ (M^n)^{-1} \) on \( D([0, T], \mathcal{H}_r) \) converges to the centered Gaussian random field whose covariances are given by
\[ Q_n^M \left[ M_s(F) M_s(G) \right] = 4 \, d \chi(\rho) \langle s \wedge t, F, G \rangle, \quad F, G \in C^\infty(\mathbb{T}^d). \] (8.2)

**Proof:** We proved in Subsection 7.4 that the sequence \( Q_n^M \) is tight. It remains to check the uniqueness of limit points. Denote by \( Q^M \) one of them and assume, without loss of generality, that the sequence \( Q_n^M \) converges to \( Q^M \).

Fix a function \( F \) in \( C^\infty(\mathbb{T}^d) \). By (7.8), the martingale \( M^n_s(F) \) is uniformly bounded in \( L^2(\mathbb{P}^n_{\mu_n}) \). Therefore, under the measure \( Q^M \), \( M_s(F) \) is a martingale.

Recall the definition of \( \Gamma^n(F) \), introduced in (7.7). We have seen in Subsection 7.4 that
\[ M^{(2),n}_s(F) := M^n_s(F)^2 - \int_0^t \Gamma^n_s(F) \, ds \]
is a martingale. On the one hand, by Lemma 8.1 [with, for the main term, \( J_x = x^{-d} F(x)/n^2 \), \( f(\eta) = \sum [\eta_y - \eta_0]^2, \gamma = \sqrt{\gamma n^2} \)] and Theorem 2.8, under the hypotheses of the lemma,
\[ \lim_{n \to \infty} \mathbb{E}_{\mu_n} \left[ \int_0^t \left| \Gamma^n_s(F) - 4 \, d \chi(\rho) \| F \|^2 \right| \, ds \right] = 0. \]

On the other hand, by (B.2), the martingale \( M^{(2),n}_s(F) \) is uniformly bounded in \( L^2(\mathbb{P}^n_{\mu_n}) \). Therefore, under the measure \( Q^M \), \( M_s(F)^2 - 4 \, d \chi(\rho) \| F \|^2 t \) is a martingale. In particular, for each \( F \in C^\infty(\mathbb{T}^d) \), \( M_s(F) \) is a time-change of Brownian motion, and \( M_s(F) \) a Gaussian random variable.

To complete the proof of the lemma, it remains to compute the covariance of \( M_s(F) \) and \( M_s(G) \) through polarization. \( \square \)

Note that the process \( M_t \) can be represented in terms of a space-time white noise, denoted by \( \{ \xi(t, x) : t \in \mathbb{R}, x \in \mathbb{T}^d \} \), as
\[ M_s(F) = \sqrt{4 \, d \chi(\rho)} \int_0^s ds \int_{\mathbb{T}^d} dxF(x) \xi(s, x). \] (8.3)
Corollary 8.5. Fix $0 < t \leq T$ and a real-valued function $F$ in $C^0,\infty([0, t] \times \mathbb{T}^d)$. Under the hypotheses of the lemma, the sequence of continuous processes

$$M^n_s = \int_0^s M^n_u(F_u) \, du, \quad 0 \leq s \leq t,$$

converges in $D([0, t], \mathbb{R})$ to the centered Gaussian process $\int_0^t M_u(F_u) \, du$.

Proof: This result follows from the lemma and the continuity of the function $\Psi : D([0, t], H_{-r}) \to D([0, t], \mathbb{R})$ given by $\Psi(M)_s = \int_{[0,s]} M_u(F_u) \, du$. □

Fix $0 < t \leq T$ and recall from (7.9) the definition of the $H_{-r}$-valued process $M^{n,t}_s$, $0 \leq s \leq t$. By (8.1), $M^{n,t}_s$ can be expressed in terms of the processes $M^n_t$ and $M^n_u$, where the last one has been introduced in Corollary 8.5. The continuity argument used in the proof of this corollary yields the following result.

Fix $p \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_p \leq T$ and functions $F_j \in C^\infty(\mathbb{T}^d)$, $1 \leq j \leq p$. The random vector $(M^{n,t}_{t_1}(F_1), \ldots, M^{n,t}_{t_p}(F_p))$ converges in distribution to a centered Gaussian random vector, characterized by its covariances which are given by

$$\text{Cov} \left( M^{t_j}_{t_j}(F_j), M^{t_k}_{t_k}(F_k) \right) = 4 \, \chi(\rho) \int_0^{t_j} \langle P_{t_j-s}F_j, P_{t_k-s}F_k \rangle \, ds \quad (8.4)$$

for $t_j < t_k$. A similar result holds for the process $X^n$. In the proof of Lemma 8.8 appears a computation similar to the one leading to the previous identity.

Proposition 8.6. Let $\mu_n$ be a sequence of probability measures on $\Omega_n$, such that $\lim_{n \to \infty} a_n^{-1} H_n(\mu_n | \mu^n_p) = 0$. Fix $p \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_p \leq T$ and functions $F_j \in C^\infty(\mathbb{T}^d)$, $1 \leq j \leq p$. Under the measure $\mathbb{P}_{\mu_n}$, the random vector $(X^n_{t_1}(F_1), \ldots, X^n_{t_p}(F_p))$ converges in distribution to a centered Gaussian random vector whose covariances are given by

$$\text{Cov} \left( X_{t_1}(F), X_{t_2}(G) \right) = 4 \, \chi(\rho) \int_0^{t_1} \langle P_{t_1-s}F, P_{t_2-s}G \rangle \, ds$$

for every $0 \leq t_1 \leq t_2 \leq T$, $F, G \in C^\infty(\mathbb{T}^d)$.

Proof: Fix $F \in C^\infty(\mathbb{T}^d)$, $0 < t \leq T$, and recall from (7.10) that

$$X^n_t(F) = X^n_t(F_{0,t}) + M^{n,t}_t(F) + \int_0^t R^n_s(F_{s,t}) \, ds + \int_0^t B^n_s(F_{s,t}) \, ds.$$ 

By Theorem 4.3 and Corollary 8.2, as $n \to \infty$, the last two terms converge to 0 in $L^1(\mathbb{P}_{\mu_n})$. By (7.13), $X^n_t(F_{0,t})$ converges to 0 in $L^2(\mathbb{P}_{\mu_n})$. Hence, $X^n_t(F) - M^{n,t}_t(F)$ converges to 0 in $L^1(\mathbb{P}_{\mu_n})$. It remains to compute the asymptotic behavior of the finite-dimensional distributions of $M^{n,t}_t$. This has been done in (8.4), which completes the proof of the proposition. □

The previous result identifies the limit points of the sequence $Q_n = \mathbb{P}_{\mu_n} \circ (X^n)^{-1}$. A similar argument, relying on the computation of the limit of the Fourier transform of linear combinations of the random variables $X^{t_j}_{t_j}(F_j)$ [based on the observation that $X^n_t(F) - M^{n,t}_t(F)$ converges to 0 in $L^1(\mathbb{P}_{\mu_n})$, that $M^{n,t}_t(F)$ can be expressed in terms of the martingale $M^n_t$, and that this later process converges], yields the following result.
Lemma 8.7. Let \( \mu_n \) be a sequence of probability measures on \( \Omega_n \) such that 
\[
\lim_{n \to \infty} a_n^{-1} H_n(\mu_n | \nu_p^n) = 0.
\]
Fix \( p \geq 1, 0 \leq t_1 < t_2 < \cdots < t_p \leq T \) and functions \( F_j \in C^\infty(T^d), 1 \leq j \leq p \). Under the measure \( \mathbb{P}_{\mu_n} \), the random vector 
\( \{X^n_k(F_1), \ldots, X^n_k(F_p)\} \) converges in distribution to a centered Gaussian random vector. The covariances are the ones obtained by integrating the covariances (8.4).

The process \( X_t \) can also be represented in terms of the space-time white noise \( \{\xi(t, x) : t \in \mathbb{R}, x \in T^d\} \) introduced in (8.3):
\[
X_t(F) = 2 \sqrt{d \chi(\rho)} \int_0^t ds \int_{T^d} dx \langle P_{t-s} F \rangle(x) \xi(s, x).
\]

The last result of this section states that the process \( X_t \) solves the stochastic differential equation (2.9). Define the \( \mathcal{H}_-\)-valued process \( M_t \) by
\[
M_t(F) := X_t(F) - \int_0^t X_s(\mathcal{A} F) \, ds,
\]
\( 0 \leq t \leq T, F \in C^\infty(T^d) \).

Lemma 8.8. The processes \( M \) is the centered Gaussian random field whose covariances are given by (8.2).

Proof. Since \( M \) is a centered Gaussian random field, it is enough to compute its covariances. Fix \( 0 \leq s \leq t \leq T \) and \( F, G \in C^\infty(T^d) \). By definition of \( M \) and Proposition 8.6, \( \{4d \chi(\rho))^{-1} \mathcal{Q}[M_t(F) M_s(G) \} \) is equal to
\[
\int_0^s du \langle P_{t-u} F, P_{s-u} G \rangle - \int_t^s du \int_0^{u:s} dv \langle P_{u-v} \mathcal{A} F, P_{s-v} G \rangle
\]
\[- \int_0^t du \int_0^s dv \langle P_{t-v} F, P_{u-v} G \rangle
\]
\[+ \int_0^t du_1 \int_0^s du_2 \int_0^{u_1:s} dv \langle P_{u_1-v} \mathcal{A} F, P_{u_2-v} G \rangle.
\]

Denote by \( I_{ij} \), \( 1 \leq j \leq 4 \), the \( j \)-th term in this expression. Recall that \( \mathcal{A} \) and \( P_t \) are symmetric operators in \( L^2(T^d) \). Let \( c(r) = \langle F, G \rangle = \langle F, P_r G \rangle, \) \( r \geq 0 \). As \( \mathcal{A} P_{t-s} = -(d/ds) P_{t-s} \), a straightforward computation yields that
\[
I_1 = \frac{1}{2} \int_{t-s}^{t+s} c(r) \, dr,
I_2 = \frac{1}{2} \left\{ \int_0^s c(r) \, dr + \int_0^{t-s} c(r) \, dr - \int_{t-s}^{t+s} c(r) \, dr \right\},
I_3 = \frac{1}{2} \left\{ \int_0^t c(r) \, dr - \int_{t-s}^{t+s} c(r) \, dr \right\},
I_4 = s \langle F, G \rangle + \frac{1}{2} \left\{ - \int_0^s c(r) \, dr + \int_s^{2s} c(r) \, dr \right\}
\]
\[+ \frac{1}{2} \left\{ - \int_0^s c(r) \, dr - \int_{t-s}^{t+s} c(r) \, dr + \int_t^{t+s} c(r) \, dr \right\}.
\]

Denote by \( I_{ij} \) the \( p \)-th term of \( I_j \). Observe that \( I_{2,1} + I_{4,2} = 0, I_{3,1} + I_{4,5} = 0, I_{3,2} + I_{4,6} = 0 \) and \( [I_{1} + I_{2,2}] + [I_{2,3} + I_{4,3} + I_{4,4}] = 0 \). Hence,
\[
\mathcal{Q}[M_t(F) M_s(G)] = 4d \chi(\rho) s \langle F, G \rangle,
\]
as claimed. \( \square \)
Proof of Theorems 2.1 and 2.2. In dimensions \( d = 1 \) and \( 2 \), the convergence of the processes \( X_t^n \) follows from the tightness proved in Section 7 and from the characterization of the limit points established in Proposition 8.6. By Lemma 8.8, \( X \) solves the stochastic differential equation (2.9).

In dimensions \( d = 3 \), the asymptotic behavior of the finite-dimensional distributions of \( X^n \) has been computed in Proposition 8.6. The tightness of the processes \( X^n \) has been established in Corollary 7.16, and uniqueness of limit points in Lemma 8.7.

\[ \square \]

Appendix A. Entropy Estimates

We present in this section some estimates, based on the entropy inequality, used repeatedly in the article. A function \( J : [0, \infty) \times T_n^d \to \mathbb{R} \) is sometimes represented by \( J_s(x) \).

Lemma A.1. Fix a cylinder function \( h \). There exists a finite constant \( C_0 \), depending only on \( h \), such that

\[
\mathbb{E}_{\mu_n} \left[ \int_0^t \left( \frac{1}{\sqrt{n^d}} \sum_{x \in T_n^d} J_x(s) \left[ (\tau_x h)(\eta_n^s) - \tilde{h}(\rho) \right] \right)^2 \, ds \right] \\
\leq C_0 \left\{ \int_0^t H_n(f_s^n) \, ds + t \right\} \sup_{0 \leq s \leq t} \|J(s)\|_{\infty}^2 ,
\]

for all \( t > 0 \), smooth function \( J : [0,t] \times T_n^d \to \mathbb{R} \), probability measure \( \mu_n \) and \( n \geq 1 \).

Proof: By the entropy inequality, the expectation appearing in the statement of the lemma is bounded by

\[
\int_0^t \frac{1}{\gamma_s} \left\{ H_n(f_s^n) + \log \mathbb{E}_{\nu^n} \left[ \exp \gamma_s \left\{ \frac{1}{\sqrt{n^d}} \sum_{x \in T_n^d} J_x(s) \left[ (\tau_x h)(\eta_n^s) - \tilde{h}(\rho) \right] \right\}^2 \right] \right\} \, ds
\]

for every \( \gamma_s > 0 \). Apply Corollary 4.5 with \( f = h, F_x = J_x(s), a = \gamma_s = c_0/2\|J(s)\|_{\infty}^2 \) to conclude that the previous expression is bounded by

\[
C_0 \left\{ \int_0^t H_n(f_s^n) \, ds + t \right\} \sup_{0 \leq s \leq t} \|J(s)\|_{\infty}^2 ,
\]

for some finite constant \( C_0 \) which depends only on \( h \). This complete the proof. \( \square \)

It follows from this result and Schwarz inequality that there exists a finite constant \( C_0 \), depending only on \( h \), such that

\[
\mathbb{E}_{\mu_n} \left[ \left( \int_0^t \frac{1}{\sqrt{n^d}} \sum_{x \in T_n^d} J_x(s) \left[ (\tau_x h)(\eta_n^s) - \tilde{h}(\rho) \right] \, ds \right)^2 \right] \\
\leq C_0 t \left\{ \int_0^t H_n(f_s^n) \, ds + t \right\} \sup_{0 \leq s \leq t} \|J(s)\|_{\infty}^2 ,
\]

for all \( t > 0 \), smooth function \( J : [0,t] \times T_n^d \to \mathbb{R} \), probability measure \( \mu_n \) and \( n \geq 1 \).
APPENDIX B. FINITE STATE MARKOV CHAINS

We present in this section some general results on continuous-time Markov chains which we could not find in the literature.

Let $E$ be a finite state-space, and $X(t)$ an $E$-valued, continuous-time Markov chain. Denote its generator by $L$:

$$(Lh)(x) = \sum_{y \in E} r(x, y) \left( h(y) - h(x) \right).$$

We start with a Feynman-Kac formula. For a probability measure $\mu$ on $E$, let $\Gamma_\mu(h,h)$ be the functional given by

$$\Gamma_\mu(h,h) = \frac{1}{2} \sum_{x,y \in E} \mu(x) r(x,y) \left( h(y) - h(x) \right)^2$$

for $h : E \to \mathbb{R}$. This functional is sometimes called the “carré du champs”. In the case where the process is reversible with respect to $\mu$, $\Gamma_\mu$ coincides with the Dirichlet form:

$$\Gamma_\mu(h,h) = -\int (Lh) \, d\mu.$$

The next result is an extension of [15, Lemma A.7.2], as it does not require the measure $\mu$ to be stationary for the process $X$. This result appears as Lemma 3.5 in [12]. We provide a slightly different proof, based on the one of [15, Lemma A.7.2].

Lemma B.1. For every function $W : \mathbb{R}_+ \times E \to \mathbb{R}$, probability measure $\mu$ on $E$ and $t > 0$,

$$\log \mathbb{E}_\mu \left[ e^{\int_0^t W(s,X(s)) \, ds} \right] \leq \int_0^t \sup_f \left\{ \int W(s) f \, d\mu + \frac{1}{2} \int (Lf) \, d\mu - \Gamma_\mu(\sqrt{f}, \sqrt{f}) \right\} \, ds,$$

where $W(s) = W(s, \cdot)$, and the supremum is carried over all densities $f$ with respect to $\mu$.

Proof: By the proof of [15, Lemma A.7.2], the left-hand side of the inequality appearing in the statement of the lemma is bounded by

$$\int_0^t \sup_h \left\{ \int W(s) h^2 \, d\mu + \int (Lh) \, h \, d\mu \right\} \, ds,$$

where the supremum is carried over all functions $h : E \to \mathbb{R}$ such that $\int h^2 \, d\mu = 1$.

A straightforward computation yields that

$$\frac{1}{2} \int Lh^2 \, d\mu - \int (Lh) \, h \, d\mu = \Gamma_\mu(h,h).$$

Therefore, the previous integral is equal to

$$\int_0^t \sup_h \left\{ \int W(s) h^2 \, d\mu + \frac{1}{2} \int Lh^2 \, d\mu - \Gamma_\mu(h,h) \right\} \, ds,$$

Since $\Gamma_\mu(|h|, |h|) \leq \Gamma_\mu(h,h)$, we may restrict the supremum to non-negative functions $h$. At this point, to complete the proof of the lemma it remains to replace $h$ by $\sqrt{f}$. \hfill \Box

In the case of the voter model with stirring, the previous result provides the following bound.
Corollary B.2. For every function $W : \mathbb{R}_+ \times \Omega_n \to \mathbb{R}$, $0 < \rho < 1$ and $t > 0$,
\[
\log \mathbb{E}_{\nu^n_\rho} \left[ e^{\int_0^t W(s, \eta^n(s)) \, ds} \right] \leq \int_0^t \sup_f \left\{ \int W(s) \, f \, d\nu^n_\rho + \frac{a_n}{2} \int V \, f \, d\nu^n_\rho - n^2 I_n(f) \right\} \, ds,
\]
where the supremum is carried over all densities $f$ with respect to $\nu^n_\rho$ and $V$ is the function introduced in (5.2).

Proof. We have to estimate the right-hand side of the formula appearing in the statement of Lemma B.1 with $L = L_n$. On the one hand, as the measure $\nu^n_\rho$ is invariant for the exclusion dynamics,
\[
\int L_n f \, d\nu^n_\rho = a_n \int L_n^V f \, d\nu^n_\rho = a_n \int (L_n^{V,1}) f \, d\nu^n_\rho = a_n \int V f \, d\nu^n_\rho,
\]
by definition of $V$.

On the other hand, since all terms of $\Gamma_{\nu^n_\rho}$ are non-negative, disregarding the ones associated to the voter dynamics yields that
\[
\Gamma_{\nu^n_\rho}(\sqrt{f}, \sqrt{f}) \geq n^2 I_n(f).
\]
in view of the explicit formulae for $I_n$ and $\Gamma_{\nu^n_\rho}$. This completes the proof of the corollary. □

Martingales. Denote by $\Gamma_k$, $k = 2, 3, 4$, the operators defined by
\[
\Gamma_2(h) = Lh^2 - 2 hLh, \quad \Gamma_3(h) = Lh^3 - 3 hLh^2 + 3 h^2 Lh, \quad \Gamma_4(h) = Lh^4 - 4 hLh^3 + 6 h^2 Lh^2 - 4 h^3 Lh.
\]
A straightforward computation yields that
\[
\Gamma_k(h) = \sum_{x,y \in E} r(x, y) \left[ h(y) - h(x) \right]^k.
\]

Fix a function $h : E \to \mathbb{R}$. It is well known that
\[
M_t(h) := h(X(t)) - h(X(0)) - \int_0^t (Lh)(X(s)) \, ds, \quad \text{ (B.1)}
\]
\[
M_t^{(2)}(h) := M_t(h)^2 - \int_0^t (\Gamma_2 h)(X(s)) \, ds
\]
are martingales. The next lemma provides a formula for the quadratic variation of $M_t^{(2)}(h)$.

Lemma B.3. Fix a function $h : E \to \mathbb{R}$. Let
\[
A(s) = (\Gamma_4 h)(X(s)) + 4 M_s(h) (\Gamma_3 h)(X(s)) + 6 M_s(h)^2 (\Gamma_2 h)(X(s)).
\]
Then,
\[
M_t(h)^4 - \int_0^t A(s) \, ds
\]
is a martingale which vanishes at $t = 0$. In particular, the quadratic variation of the martingale $M_t^{(2)}(h)$, denoted by $\langle M^{(2)}(h) \rangle_t$, is given by

$$\langle M^{(2)}(h) \rangle_t = \int_0^t (\Gamma_1 h)(X(s)) \, ds + 2 \left( \int_0^t (\Gamma_2 h)(X(s)) \, ds \right)^2 + 4 M_t(h) \int_0^t (\Gamma_3 h)(X(s)) \, ds + 4 M_t^{(2)}(h) \int_0^t (\Gamma_2 h)(X(s)) \, ds .$$

Proof. The proof of this result relies on a long computation. As in the proof of [15, Lemma A.5.1], the unique ingredients are integration by parts and the fact that the integral of a predictable process with respect to a martingale is a martingale. Since it is an identity, details are left to the reader. \qed

We apply Lemma B.3 to the voter model with stirring. Fix a function $F$ in $C^\infty(T^d)$. Let $h : \Omega_n \to \mathbb{R}$ be given by $h(\eta) = X^n(F)$. Then, there exists a finite constant $C_0$ such that $|\Gamma_k(h)| \leq C_0 \{ \|F\|_k^k + \|\nabla F\|_k^k \}$ for $2 \leq k \leq 4$, $n \geq 1$. In particular, by Lemma B.3 and by definition of the martingale $M^n_t(F)$, introduced in (4.1),

$$\mathbb{E}^n_\eta[M^n_t(F)^4] \leq C_0 \int_0^t \mathbb{E}^n_\eta[c_4 + c_3 | M^n_t(F) | + c_2 M^n_t(F)^2] \, ds ,$$

where $c_k = \|F\|_k^k + \|\nabla F\|_k^k$. Hence, by (7.8) and Young's inequality, there exists a finite constant $C_0$ such that

$$\mathbb{E}^n_\eta[M^n_t(F)^4] \leq C_0 T^2 \{ \|F\|^4_\infty + \|\nabla F\|^4_\infty \} \quad (B.2)$$

all $0 \leq t \leq T$, $\eta \in \Omega_n$, $n \geq 1$.

APPENDIX C. DECOMPOSITION OF CYLINDER FUNCTIONS

Throughout this section, $0 < \alpha < 1$ is fixed, and $0 \leq \rho \leq 1$ is a parameter whose value may change. Consider a cylinder function $f : \{0, 1\}^Z \to \mathbb{R}$. Denote by $A \subset Z^d$ its support: $f(\eta) = f(\eta_z : z \in A)$. In particular, there exist constants $c_B, B \subset A$, such that

$$f(\eta) = \sum_{B \subset A} c_B \eta_B ,$$

where $\eta_\emptyset = 1$, $\eta_B = \prod_{x \in B} \eta_x$ and the sum is performed over all subsets $B$ of $A$. Note that the constants $c_B$’s may depend on $\alpha$: $f(\eta) = \eta_0 - \alpha$ is an admissible cylinder function. Recall the definition of the function $\tilde{f} : [0, 1] \to \mathbb{R}$ introduced in (2.5). With this notation, for $0 \leq \rho \leq 1$,

$$\tilde{f}(\rho) = \sum_{B \subset A} c_B \rho^{|B|} \quad \text{and} \quad \tilde{f}'(\rho) = \sum_{B \subset A, B \neq \emptyset} c_B |B| \rho^{|B|-1} . \quad (C.1)$$

Let $\xi_D^\rho = 1$, $\xi_D^\rho = \prod_{x \in D}(\eta_x - \rho)$, $D$ a finite subset of $Z^d$. Since

$$\eta_B = \sum_{D \subset B} \rho^{|B|-|D|} \xi_D^\rho ,$$

we may rewrite $f(\eta)$ as

$$f(\eta) = \sum_{D \subset A} \xi_D^\rho \sum_{B : D \subset B \subset A} c_B \rho^{|B|-|D|} . \quad (C.2)$$
The cylinder function \( f \) is said to have \emph{degree} \( n \geq 0 \) in \( L^2(\nu) \) if there exists a finite collection of subsets \( D \) of \( \mathbb{Z}^d \) with cardinality \( n \) and real numbers \( c_d \) such that

\[
f(\eta) = \sum_D c_D \xi_D^n.
\]

Denote by \( \Pi_\rho \) the operator given by

\[
(\Pi_\rho f)(\eta) := f(\eta) - \tilde{f}(\rho) - \hat{f}(\rho) (\eta_0 - \rho).
\]

**Assertion C.1.** Fix a cylinder function \( f(\eta) = \sum_{B \subset A} c_B \eta_B \) whose support is contained in a finite subset \( A \) of \( \mathbb{Z}^d \). Then, for every \( 0 < \rho < 1 \),

\[
(\Pi_\rho f)(\eta) = (\Pi^1_\rho f)(\eta) + (\Pi^{+2}_\rho f)(\eta),
\]

where

\[
(\Pi^1_\rho f)(\eta) = \sum_{z \in A} c_{\{z\}} (\eta_z - \eta_0) \sum_{B \subset A, B \ni z} c_B \rho^{|B|-1},
\]

\[
(\Pi^{+2}_\rho f)(\eta) = \sum_{D \subset A, \ |D| \geq 2} \xi_D^n \sum_{B \subset D \subset C \subset A} c_B \rho^{|B|-|D|}.
\]

Note that \( \Pi^1_\rho f \) corresponds to the terms of degree 1 of \( \Pi_\rho f \), and \( \Pi^{+2}_\rho f \) to the ones of degree greater than or equal to 2 in \( L^2(\nu) \).

**Proof of Assertion C.1.** By (C.1) and (C.2) \( f(\eta) - \tilde{f}(\rho) \) is equal to

\[
\sum_{z \in A} c_{\{z\}} \xi_{\{z\}}^0 + \sum_{B \subset A, \ |B| \geq 2} c_B \left\{ \sum_{z \in B} \rho^{|B|-1} \xi_{\{z\}}^0 + \sum_{D \subset B, \ |D| \geq 2} \rho^{|B|-|D|} \xi_D^n \right\}.
\]

and

\[
\hat{f}(\rho) \ (\eta_0 - \rho) = \hat{f}(\rho) \xi_{\{0\}}^0 = \sum_{z \in A} c_{\{z\}} \xi_{\{0\}}^0 + \sum_{B \subset A, \ |B| \geq 2} c_B |B| \rho^{|B|-1} \xi_{\{0\}}^0.
\]

Thus,

\[
(\Pi_\rho f)(\eta) = \sum_{z \in A} c_{\{z\}} (\eta_z - \eta_0) + \sum_{B \subset A, \ |B| \geq 2} c_B \rho^{|B|-1} \sum_{z \in B} (\eta_z - \eta_0)
\]

\[
+ \sum_{B \subset A, \ |B| \geq 2} c_B \rho^{|B|-|D|} \xi_D^n.
\]

To complete the proof, it remains to change the order of summations. \( \Box \)

**Remark C.2.** The above computation shows that the term \( \tilde{f}(\rho) \) removes the constants from \( f(\eta) \), while the expression \( \hat{f}(\rho) \ (\eta_0 - \rho) \) transforms the terms of degree 1 of \( f(\eta) \) [that is, the expressions \( c_{\{z\}} \xi_{\{z\}}^0 \)] in gradients of the form \( c_{\{z\}} (\eta_z - \eta_0) \).

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