Adaptive Modified Function Projective Synchronization for Uncertain Complex Dynamic Networks With Multiple Time-Varying Delay Couplings Under Input Nonlinearity

JIE FANG,1,2 DAN-YING XU1, JUN-WEI SUN1,2 AND WEI WANG1,2

1 College of Electrical and Information Engineering, Zhengzhou University of Light Industry, Zhengzhou 450002, China
2 Henan Key Lab of Information-Based Electrical Appliances, Zhengzhou 450002, China

Corresponding author: Wei Wang (fang0511jie@zzuli.edu.cn)

This work was supported in part by the National Natural Science Foundation of China under Grant 61775198, in part by the Science and technology project of Henan Province under Grant 192102210083 and Grant 202102210317, in part by the Key scientific research projects of universities in Henan Province under Grant 20A413012, and in part by the Science and Technology Innovation Team Project of Henan Province under Grant 19IRTSTHN013.

ABSTRACT This paper is concerned with the modified function projective synchronization of the uncertain complex dynamic networks with both input nonlinearity and multiple time-varying delay couplings. Firstly, the model of the complex dynamic networks with sector nonlinear input, multiple time-varying delay couplings, model uncertainty and external disturbance is constructed. Then, the adaptive controller is formulated based on the Lyapunov stability theory and the matrix inequality theory, by which the network nodes of the driving system and the response system can realize modified function projective synchronization according to the corresponding function scaling factors. Finally, a four-dimensional hyperchaotic system is considered as the nodes of the complex dynamic networks to achieve numerical simulation. The corresponding theoretical proof and computer simulation are worked out to demonstrate the effectiveness and feasibility of the proposed scheme.

INDEX TERMS Complex dynamic networks, multiple time-varying delay couplings, input nonlinearity, modified function projection synchronization, adaptive control.

I. INTRODUCTION
Complex network is a large-scale network with complex topological structure and dynamic behavior, which exists widely in nature and society. The complex dynamic networks can represent almost any natural and man-made structure, and have been an active research topic because of their flexibility and versatility. It is well known that complex network usually consists of a series of coupled interconnected nodes, where each node is a dynamical system [1]–[4]. Chaotic system is a special kind of nonlinear system, which has the characteristics of initial value sensitivity, unpredictability, strong attractor and so on, so it is often used as the node of complex network in the research of complex network [5]–[6]. Synchronization is a typical and important dynamical behavior of complex dynamical networks. Complex network synchronization research is one of the most important research directions on complex networks [7]–[8]. A variety types of synchronization have been investigated and lots of valuable theoretical results have been gained, such as anti-synchronization, combination synchronization, finite-time synchronization, lag synchronization, impulsive synchronization, global synchronization, hybrid synchronization and so on [9]–[19].

Modified function projective synchronization is a more general synchronization form which means that the driving system realize synchronization with the response system by different functional scaling factors. The unpredictability of the functional scaling factor can additionally improve the reliability of secure communication [20]–[23]. Reference [24] investigated the cluster-modified function projective synchronization of a generalized linearly coupled network with...
asymmetric coupling and nonidentical dynamical nodes. Reference [25] investigated robust modified function projective lag synchronization between two nonlinear complex networks with different-dimensional nodes and disturbances. Reference [26] achieved modified function projective synchronization for two fractional-order complex dynamical networks with unknown parameters and unknown bounded external disturbances. Considering that time delays often occur in dynamical networks due to the finite speed of information propagation or processing, it is imperative to incorporate time delays into the models of complex dynamical networks [27]–[29]. However, the absolute constant delay may be scarce in the practical networks, it makes more sense to study complex networks with multiple time-varying delay couplings [30]–[33]. Reference [34] realized the modified function projective synchronization of complex dynamical network with multiple time-delay couplings and external disturbances. Reference [35] investigated modified function projective synchronization for complex dynamical networks with mixed time-varying and hybrid asymmetric coupling delays. Moreover, the time-varying coupling strength was also considered in [36]–[38].

On the other hand, in real control systems, actuators and sensors often have nonlinear characteristics such as dead-zone, backlash, and hysteresis. Owing to physical limitation, there usually exist nonlinearities in the control input. The effect of input nonlinearities usually results in control performance degeneration or instability in the controlled system. Hence, the existence of input nonlinearities cannot be ignored in the actual control system [39]–[42]. Sector nonlinear input refers to the system input within a sector which can represents a large class of input functions with nonlinearity. Sector nonlinear input exists in many actual control systems, such as air-breathing hypersonic vehicles, flexible robotic manipulator, mechanical connections, electric servo motors, magnetic levitation, etc. [43]–[47]. Therefore, it is of great significance to study the nonlinear system control with sector nonlinear input. Reference [48] designed a controller for a class of uncertain multi-input multi-output chaotic systems with both sector nonlinearities and dead-zones. Reference [49] achieved synchronization for a class of fractional-order chaotic systems with sector nonlinearities. What is more, there always exist model uncertainties, external noise disturbances and sector nonlinear input. Up to now, the research of sector nonlinear input is mainly concentrated on the chaotic system, and there is relatively little research on the complex network.

Motivated by the above discussion, an adaptive modified function projective synchronization strategy for a class of uncertain complex dynamic networks with sector nonlinear input and multiple time-varying delay couplings is studied in this paper. We first proposed a new complex dynamic network model which has sector nonlinear input, multiple time-varying delay couplings, model uncertainty and external disturbance. What is more, we investigated the function projective synchronization of two complex dynamic network based on the Lyapunov stability theory and the matrix inequality theory. The theoretical proof and simulation results are given to verify the effectiveness and feasibility of the synchronization scheme.

Compared with previous work, the main contribution of this study is: (a) Sector nonlinear input is introduced into the complex network, which makes the model get close to the engineering reality. (b) The effect of multiple time-varying delay couplings is considered in the modeling, which is like the real-world systems. (c) In the theoretical analysis, the Lyapunov stability theory and the matrix inequality theory are flexibly used to realize the modified function projective synchronization. (d) The controller does not include time delay terms in our work, so the proposed method is more general and realistic.

The rest of this paper is organized as follows: Section 2 introduces the new network model and some other preliminaries. In Section 3, the adaptive controller is designed to realize the modified function projective synchronization of the complex dynamic networks. In Section 4, a practical example is provided to demonstrate the effectiveness and feasibility of the proposed control method. Some conclusions and suggestions for future work are given in Section 5.

Notations. All the symbols are assumed to be standard and the matrices are with compatible dimensions in this paper. $\mathbf{R}^n$ represents an n dimensional Euclidean space; $\mathbf{R}^{m \times h}$ represents an $n \times h$ real matrix; $A^T$ is the transpose of the matrix $A$; $\| \cdot \|$ denotes the Euclidean norm of a vector; $I$ represents the unit matrix; $diag \{ \cdots \}$ represents a block-diagonal matrix; $\otimes$ represents the Kronecker product; $\lambda_{\text{max}} (Q)$ is the maximum eigenvalue of Matrix $Q$.

II. MODEL DESCRIPTION

We consider a complex dynamical network consisting of $N$ identical nodes, and the driving system can be described as follows:

$$
\dot{x}_i(t) = f_i(x_i(t)) + F_i(x_i(t)) \theta_i + \Delta f_i^w(x_i(t), t) + c_1(t) \sum_{l=0}^{m-1} \sum_{j=1}^{N} d_{ij}^l \Gamma_1 x_j(t - \tau_1(t)) + d_i^w(t) \\
= f_i(x_i(t)) + F_i(x_i(t)) \theta_i + \Delta f_i^w(x_i(t), t) + c_0(t) \sum_{j=1}^{N} d_{ij}^0 \Gamma_0 x_i(t - \tau_0(t)) \\
+ c_1(t) \sum_{j=1}^{N} d_{ij}^1 \Gamma_1 x_j(t - \tau_1(t)) + \cdots \\
+ c_{m-1}(t) \sum_{j=1}^{N} d_{ij}^{m-1} \Gamma_{m-1} x_j(t - \tau_{m-1}(t)) + d_i^w(t)
$$

(1)
For the driving system, the response system model can be written as follows:

\[
\dot{y}_i(t) = f_i(y_i(t)) + F_i(y_i(t)) (\theta_i + \Delta \theta_i) + \Delta f_i^w(y_i(t), t) + c_i(t) \sum_{j=1}^{N} d_{ij}^w \Gamma_1 y_j(t - \tau_i(t)) + d_i^s(t) + \phi_i(u_i(t)) \\
+ c_0(t) \sum_{j=1}^{N} d_{ij}^m \Gamma_0 y_j(t - \tau_0(t)) \\
+ c_{m-1}(t) \sum_{j=1}^{N} d_{ij}^{m-1} \Gamma_{m-1} y_j(t - \tau_{m-1}(t)) \\
+ d_i^s(t) + \phi_i(u_i(t))
\]  

where \( y_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \) and \( y_1(t) = (\dot{x}_{i1}(t), \dot{x}_{i2}(t), \ldots, \dot{x}_{in}(t))^T \), \( i = 1, 2, \ldots, N \) denote the state vector; \( f_i(\cdot) : R^n \rightarrow R^m \) is a known continuous nonlinear function; \( F_i(\cdot) : R^n \rightarrow R^{n \times h} \) is a matrix whose elements are continuous nonlinear functions; \( \theta_i \) is a \( h \times 1 \) dimension known constant parameter vector; \( \Delta \theta_i \) is the unknown parameter deviations; \( d_i^w(t) \) and \( d_i^s(t) \) are external disturbances; \( \Delta f_i^w(x_i(t), t) = [\Delta f_{i1}^w(x_{i1}(t), t), \Delta f_{i2}^w(x_{i2}(t), t), \ldots, \Delta f_{in}^w(x_{in}(t), t)] \) and \( \Delta f_i^w(x_i(t), t) = [\Delta f_{i1}^s(x_{i1}(t), t), f_{i2}(x_{i2}(t), t), \ldots, f_{in}(x_{in}(t), t)] \) represent model uncertainty of the system model; the complex network is divided into \( m \) subnetworks by \( \tau_i(t), \tau_0(t), i = 0, 1, 2, \ldots, m - 1 \) denotes different time-varying delays, and especially \( \tau_0(t) = 0 \) means that the coupling delay is 0; \( c_i(t) \) represents different coupling strengths, which can be constant or change with time; \( \Gamma_i \) is a matrix that describes the internal coupling of individual node; \( A_i = (a_{ij})_{N \times N} \) is the weight configuration matrix which represents the topological structure of the network. If nodes \( i \) and \( j \) have a connection, then \( a_{ij} = a_{ji} \neq 0, (i \neq j) \), otherwise, \( a_{ij} = a_{ji} = 0, (i \neq j) \), and the diagonal elements of matrix \( A_i \) are defined as \( a_{ii} = -\sum_{j=1, j \neq i}^{N} d_{ij}^w, (i = 1, 2, \ldots, N) \).

where \( \phi_i(u_i(t)) \) is a continuous nonlinear function with \( \phi_i(0) = 0 \). It belongs to a sector-like \([p_i, q_i]\), if there exist two non-negative numbers \( p_i \) and \( q_i \) such that \( p_i \leq \frac{\phi_i(u_i(t))}{u_i(t)} \leq q_i \) for \( u_i(t) \neq 0 \).

**Definition 1:** For complex network model (1) and (2), if there exists a continuously differentiable scaling function matrix \( M_i(t) = \text{diag} (m_1(t), m_2(t), \ldots, m_N(t)), i = 1, 2, \ldots, N \), such that (3) holds, it means that the driving system and the response system realize modified function projective synchronization.

\[
\lim_{t \to \infty} \| e_i(t) \| = \lim_{t \to \infty} \| y_i(t) - M_i(t) x_i(t) \| = 0, \\
i = 1, 2, \ldots, N
\]

where \( \| \cdot \| \) denotes the Euclidean norm of a vector, \( m_i(t) \) is a continuously differentiable scaling function.

**Assumption 1:** Model uncertainty \( \Delta f_i^w(x_i(t), t) \) and \( \Delta f_i^s(x_i(t), t) \) are bounded, and there exist positive constants \( \alpha_i^w, \alpha_i^s \), such that \( |\Delta f_i^w(x_i(t), t)| \leq \alpha_i^w, |\Delta f_i^s(x_i(t), t)| \leq \alpha_i^s \).

**Corollary 1:** Because \( m_i(t) \) is a continuously differentiable scaling function, there exists a positive constant \( \mu \), such that \( |m_i(t)| \leq \mu \). Under Assumption 1, there exists a positive constant \( \alpha_i \geq \alpha_i^w + \mu \alpha_i^s \), such that

\[
|\Delta f_i^w(x_i(t), t) - M_i(t) \Delta f_i^w(x_i(t), t)| \\
\leq |\Delta f_i^w(x_i(t), t)| + |M_i(t)| \Delta f_i^w(x_i(t), t) \\
\leq \alpha_i^w + |M_i(t)| \Delta f_i^w(x_i(t), t) \\
\leq \alpha_i^w + \mu \alpha_i^s \leq \alpha_i
\]

**Assumption 2:** External disturbance \( d_i^w(t) \) and \( d_i^s(t) \) are bounded, and there exist positive constants \( \beta_i^w, \beta_i^s \), such that \( |d_i^w(t)| \leq \beta_i^w, |d_i^s(t)| \leq \beta_i^s \).

**Corollary 2:** It follows from Corollary 1 that there exists a positive constant \( \beta_i \geq \beta_i^w + \mu \beta_i^s \), such that

\[
|d_i^w(t) - M_i(t) d_i^w(t)| \leq \beta_i
\]

**Assumption 3:** The time-varying coupling strength \( c_i(t) \) is bounded, and there exists a positive constant \( \beta_i \), such that \( |c_i(t)| \leq \beta_i \).

**Assumption 4:** \( \tau_i(t), i = 0, 1, 2, \ldots, m - 1 \), is a differentiable function with \( 0 \leq \dot{\tau}_i(t) \leq \epsilon < 1 \), so, we can get

\[
\frac{1 - \dot{\tau}_i(t)}{2(1 - \epsilon)} \geq \frac{1}{2}
\]

where \( 0 < \epsilon < 1 \) is positive constant.

This assumption is still satisfied if \( \tau_i(t) \) is zero or some other constants.

**Lemma 1** [34]: For any vectors \( X, Y \in R^n \) and a positive definite matrix \( Q \in R^{n \times n} \), the following matrix inequality holds: \( X^T Q Y \leq X^T Q Q^T X + Y^T Y \).

### III. CONTROLLER DESIGN

In order to achieve the modified function projective synchronization of the complex dynamic network model (1) and (2), we can design the adaptive controller as follows:

\[
u_i(t) = -\frac{1}{p_i} \left| [F_i(y_i(t)) \theta_i - M_i(t) F_i(y_i(t)) \theta_i] + [f_i(y_i(t)) - M_i(t) f_i(x_i(t)) - M_i(t) x_i(t)] + \left[ \hat{\alpha}_i + \hat{\beta}_i + F_i(y_i(t)) \Delta \hat{\theta}_i \right] \right| \\
+ \left[ \hat{\alpha}_i + \hat{\beta}_i + F_i(y_i(t)) \Delta \hat{\theta}_i \right] \sigma \left( e_i^T(t) \right)
\]

\[
\Delta \dot{\theta}_i = r_0 F_i^T(y_i(t)) e_i(t) \\
\dot{\theta}_i = r_1 e_i^T(t) \\
\dot{\beta}_i = r_2 e_i^T(t) \\
\dot{\alpha}_i = r_3 e_i^T(t) e_i(t)
\]
where \( r_0 > 0, r_1 > 0, r_2 > 0, r_3 > 0 \) are four positive constants; \( \Delta \tilde{\theta}_i \) is the estimated parameter for \( \Delta \theta_i; \tilde{\alpha}_i \) is the estimated parameter for \( \alpha_i; \hat{\beta}_i \) is the estimated parameter for \( \beta_i; \hat{k}_i \) is adaptive feedback control gain.

From Eq. (7), we can get \( u_i(t) = \xi_i \text{sgn} (e_i^T(t)) (\xi_i < 0) \), where

\[
\xi_i = -\frac{1}{p_i} \left[ \left| f_i (y_i(t)) \right| - M_i (t) F_i (x_i(t)) \right] \theta_i \\
+ \left| f_i (y_i(t)) - M_i (t) f_i (x_i(t)) - \tilde{M}_i (t) x_i(t) \right| \\
+ \left( \tilde{\alpha}_i + \hat{\beta}_i + F_i (y_i(t)) \Delta \tilde{\theta}_i + \hat{k}_i e_i(t) \right) \text{sgn} \left( e_i^T(t) \right)
\]

**Lemma 2:** Let \( u_i(t) = \xi_i \text{sgn} (e_i(t)) (\xi_i < 0) \), we can get \( e_i(t) \phi_i (u_i(t)) \leq p_i \xi_i |e_i(t)| \).

**Proof:** It follows from \( p_i \leq \frac{\Phi_i (u_i(t))}{u_i(t)} \leq q_i \) that \( p_i u_i^2(t) \leq u_i(t) \phi_i (u_i(t)) \leq q_i u_i^2(t) \).

When \( e_i(t) = 0 \), the equation obviously holds, that is \( e_i(t) \phi_i (u_i(t)) = p_i \xi_i |e_i(t)| \).

When \( e_i(t) \neq 0 \), substitute \( u_i(t) = \xi_i \text{sgn} (e_i(t)) \) into \( p_i u_i^2(t) \leq u_i(t) \phi_i (u_i(t)) \), we can get \( p_i \xi_i^2 |e_i(t)| \leq \xi_i \text{sgn} (e_i(t)) \phi_i (u_i(t)) \).

Use \( \text{sgn} (e_i(t)) \) instead of \( \text{sgn} (e_i(t)) \) to get \( p_i \xi_i^2 |e_i(t)| \leq \xi_i \text{sgn} (e_i(t)) \phi_i (u_i(t)) \).

Divide both sides by \( e_i^T(t) \) to get \( p_i \xi_i^2 |e_i(t)| |e_i(t)| \leq \xi_i \text{sgn} (e_i(t)) \phi_i (u_i(t)) \).

This completes the proof.

**Theorem 1:** For any initial conditions, if assumptions 1-4 are satisfied, the driving system (1) and the response system (2) can realize modified function projective synchronization with the controller (7) and adaptive laws (8)-(11).

**Proof:** From Definition 1, we have the error term:

\[
e_i(t) = y_i(t) - M_i(t) x_i(t)
\]

The time derivative of \( e_i(t) \) is:

\[
\dot{e}_i(t) = \dot{y}_i(t) - \dot{M}_i(t) x_i(t) - M_i(t) \dot{x}_i(t) \\
= f_i (y_i(t)) + F_i (y_i(t)) (\theta_i + \Delta \theta_i) \\
+ \Delta f_i^w (y_i(t)) + c_i (t) \sum_{j=0}^{m-1} d_i^{jw} \Gamma_i x_j(t - \tau_i(t)) \\
+ d_i^w (t) + \phi_i (u_i(t)) - M_i(t) f_i (x_i(t)) \\
- M_i(t) F_i (x_i(t)) \theta_i - M_i(t) \Delta f_i^w (x_i(t), t) \\
- \tilde{M}_i(t) c_i(t) \sum_{j=0}^{m-1} d_i^{jw} \Gamma_i x_j(t - \tau_i(t)) \\
- \tilde{M}_i(t) d_i^w(t) - \tilde{M}_i(t) x_i(t)
\]

Choose the following Lyapunov function:

\[
V(t) = \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) e_i(t) \\
+ \frac{1}{2 r_1} \sum_{i=1}^{N} (\tilde{\alpha}_i - \alpha_i)^2 \\
+ \frac{1}{2 r_2} \sum_{i=1}^{N} (\tilde{\beta}_i - \beta_i)^2 \\
+ \frac{1}{2} (1 - \varepsilon) \int_{t-\tau_i(t)}^{t} \sum_{i=1}^{N} e_i^T(\delta) e_i(\delta) d\delta \\
+ \frac{1}{2 r_3} \sum_{i=1}^{N} (\tilde{k}_i - k^*)^2
\]

where \( \Delta \tilde{\theta}_i = \Delta \theta_i - \Delta \theta_i \), and \( k^* \) is the positive constant to be designed later.

The time derivative of \( V(t) \) is:

\[
\dot{V}(t) = \sum_{i=1}^{N} e_i^T(t) \dot{e}_i(t) + \frac{1}{r_1} \sum_{i=1}^{N} (\tilde{\alpha}_i - \alpha_i) \dot{\tilde{\alpha}}_i \\
+ \frac{1}{2 r_2} \sum_{i=1}^{N} (\tilde{\beta}_i - \beta_i) \dot{\tilde{\beta}}_i + \frac{1}{2} (1 - \varepsilon) \sum_{i=1}^{N} e_i^T(t) e_i(t) \\
- \frac{1}{2} (\tilde{k}_i - k^*) \dot{\tilde{k}}_i + \frac{1}{2} (1 - \varepsilon) \sum_{i=1}^{N} (\tilde{k}_i - k^*)^2
\]

\[
\dot{V}(t) = \sum_{i=1}^{N} e_i^T(t) [f_i (y_i(t)) + F_i (y_i(t)) (\theta_i + \Delta \theta_i) \\
+ \Delta f_i^w (y_i(t)) + c_i (t) \sum_{j=0}^{m-1} d_i^{jw} \Gamma_i x_j(t - \tau_i(t)) + \phi_i (u_i(t)) \\
- M_i(t) f_i (x_i(t)) - M_i(t) F_i (x_i(t)) \theta_i \\
- M_i(t) \Delta f_i^w (x_i(t), t) \\
- M_i(t) c_i(t) \sum_{j=0}^{m-1} d_i^{jw} \Gamma_i x_j(t - \tau_i(t)) \\
- M_i(t) d_i^w(t) - M_i(t) x_i(t)] \\
+ \frac{1}{r_1} \sum_{i=1}^{N} (\tilde{\alpha}_i - \alpha_i) \dot{\tilde{\alpha}}_i + \frac{1}{r_2} \sum_{i=1}^{N} (\tilde{\beta}_i - \beta_i) \dot{\tilde{\beta}}_i \\
+ \frac{1}{2} (1 - \varepsilon) \sum_{i=1}^{N} e_i^T(t) e_i(t) \\
- \frac{1}{2} (\tilde{k}_i - k^*) \dot{\tilde{k}}_i + \frac{1}{2} (1 - \varepsilon) \sum_{i=1}^{N} (\tilde{k}_i - k^*)^2
\]

\[
\dot{V}(t) \leq \sum_{i=1}^{N} e_i^T(t) [f_i (y_i(t)) - M_i(t) f_i (x_i(t)) \\
- M_i(t) F_i (x_i(t)) \theta_i - F_i (y_i(t)) \Delta \theta_i] \\
+ \frac{1}{2} (1 - \varepsilon) \sum_{i=1}^{N} e_i^T(\delta) e_i(\delta) d\delta \\
+ \frac{1}{2 r_3} \sum_{i=1}^{N} (\tilde{k}_i - k^*)^2
\]
By Corollary 1 and 2, and substituting (9) and (10) into (17), we have

\[
\dot{V}(t) \leq \sum_{i=1}^{N} \left[ e_{i}^{T}(t) \left( \frac{c_{i}}{2} \sum_{l=1}^{m-1} \sum_{l=1}^{N} d_{ij}^{l} \Gamma_{ij}e_{j}(t - \tau_{l}(t)) \right) \right]
\]

\[
+ \frac{1}{2 (1 - \varepsilon)} \sum_{i=1}^{N} \sum_{l=1}^{m-1} e_{i}^{T}(t) e_{i}(t)
\]

\[
- \frac{1 - \dot{t}_{l}(t)}{2 (1 - \varepsilon)} \sum_{i=1}^{N} e_{i}^{T}(t - \tau_{l}(t)) e_{i}(t - \tau_{l}(t))
\]

\[
+ \frac{1}{r_{3}} \sum_{i=1}^{N} \left( \kappa_{i} - k^{*} \right) \dot{k}_{i}
\]

By Assumption 3 and Lemma 2, we have

\[
\dot{V}(t) \leq \sum_{i=1}^{N} \left[ e_{i}^{T}(t) \left( \frac{c_{i}}{2} \sum_{l=1}^{m-1} \sum_{l=1}^{N} d_{ij}^{l} \Gamma_{ij}e_{j}(t - \tau_{l}(t)) \right) \right]
\]

\[
+ \frac{1}{2 (1 - \varepsilon)} \sum_{i=1}^{N} \sum_{l=1}^{m-1} e_{i}^{T}(t) e_{i}(t)
\]

\[
- \frac{1 - \dot{t}_{l}(t)}{2 (1 - \varepsilon)} \sum_{i=1}^{N} e_{i}^{T}(t - \tau_{l}(t)) e_{i}(t - \tau_{l}(t))
\]

\[
+ \frac{1}{r_{3}} \sum_{i=1}^{N} \left( \kappa_{i} - k^{*} \right) \dot{k}_{i}
\]

Substituting $\xi_{i}$ into (19), we have

\[
\dot{V}(t) \leq \sum_{i=1}^{N} e_{i}^{T}(t) \left[ -F_{i}(y_{i}(t)) \Delta \theta_{i} + c_{i} \sum_{j=1}^{N} a_{ij}^{0} \Gamma_{0} e_{j}(t) \right]
\]

\[
+ \frac{1}{2 (1 - \varepsilon)} \sum_{i=1}^{N} \sum_{l=1}^{m-1} e_{i}^{T}(t) e_{i}(t)
\]

\[
- \frac{1 - \dot{t}_{l}(t)}{2 (1 - \varepsilon)} \sum_{i=1}^{N} e_{i}^{T}(t - \tau_{l}(t)) e_{i}(t - \tau_{l}(t))
\]

\[
+ \frac{1}{r_{3}} \sum_{i=1}^{N} \left( \kappa_{i} - k^{*} \right) \dot{k}_{i}
\]

By Assumption 4, and substituting (8) and (11) into (20), we have

\[
\dot{V}(t) \leq \sum_{i=1}^{N} e_{i}^{T}(t) \left[ c_{i} \sum_{j=1}^{N} a_{ij}^{0} \Gamma_{0} e_{j}(t) \right]
\]

\[
+ \frac{1}{2 (1 - \varepsilon)} \sum_{i=1}^{N} \sum_{l=1}^{m-1} e_{i}^{T}(t) e_{i}(t)
\]

\[
- \frac{1}{2 (1 - \varepsilon)} \sum_{i=1}^{N} e_{i}^{T}(t - \tau_{l}(t)) e_{i}(t - \tau_{l}(t))
\]

\[
- k^{*} \sum_{i=1}^{N} e_{i}^{T}(t) e_{i}(t)
\]

Let $e(t) = (e_{1}^{T}(t), e_{2}^{T}(t), \ldots, e_{N}^{T}(t))^{T} \in \mathbb{R}_{+}^{N}, P_{0} = (A_{0} \otimes \Gamma_{0}), P_{1} = (A_{1} \otimes \Gamma_{1}), \cdots, P_{l} = (A_{l} \otimes \Gamma_{l})$. 

\[127397\]
where \( \otimes \) represents the Kronecker product.

\[
\dot{V}(t) \leq c_i e^T(t) P_0e(t) + c_i \sum_{l=1}^{m-1} e^T(t) P_l e(t - \tau_l(t)) \\
+ \frac{1}{2(1-\varepsilon)} \sum_{l=1}^{m-1} e^T(t) e(t) \\
- \frac{1}{2} \sum_{l=1}^{m-1} e^T(t - \tau_l(t)) e(t - \tau_l(t)) \\
- k^* e^T(t) e(t) \tag{22}
\]

By Lemma 1, we have

\[
c_i e^T(t) P_l e(t - \tau_l(t)) \\
\leq \frac{1}{2} c_i^2 e^T(t) P_l P_l^T e(t) \\
+ \frac{1}{2} e^T(t - \tau_l(t)) e(t - \tau_l(t)) \tag{23}
\]

So, we have

\[
\dot{V}(t) \leq c_i e^T(t) P_0 e(t) + \frac{1}{2} c_i^2 \sum_{l=1}^{m-1} e^T(t) P_l P_l^T e(t) \\
+ \frac{1}{2} \sum_{l=1}^{m-1} e^T(t - \tau_l(t)) e(t - \tau_l(t)) \\
+ \frac{1}{2(1-\varepsilon)} \sum_{l=1}^{m-1} e^T(t) e(t) \\
- \frac{1}{2} \sum_{l=1}^{m-1} e^T(t - \tau_l(t)) e(t - \tau_l(t)) - k^* e^T(t) e(t) \tag{24}
\]

\[
\dot{V}(t) \leq e^T(t) \left[ c_i P_0 + \frac{1}{2} c_i^2 \sum_{l=1}^{m-1} P_l P_l^T \right] e(t) \\
+ \sum_{l=1}^{m-1} \frac{1}{2(1-\varepsilon)} e^T(t) e(t) - k^* e^T(t) e(t) \tag{25}
\]

\[
\dot{V}(t) \leq \lambda_{\text{max}} \left[ c_i P_0 + \frac{1}{2} c_i^2 \sum_{l=1}^{m-1} P_l P_l^T \right] e^T(t) e(t) \\
+ m - \frac{1}{2(1-\varepsilon)} - k^* \right] e^T(t) e(t) \tag{26}
\]

where \( \lambda_{\text{max}}(Q) \) is the maximum eigenvalue of Matrix \( Q \).

Therefore, by taking appropriate \( k^* \) such that

\[
k^* \geq \lambda_{\text{max}} \left[ c_i P_0 + \frac{1}{2} c_i^2 \sum_{l=1}^{m-1} P_l P_l^T \right] + \frac{m-1}{2(1-\varepsilon)} \tag{27}
\]

we can obtain \( \dot{V}(t) \leq 0 \).

According to Lyapunov stability theory, we can obtain \( e_i(t) \to 0 \) as \( t \to \infty \), which means that the modified function projective synchronization between the driving system (1) and the response system (2) is achieved. This completes the proof.

\[\text{Remark 1:} \text{If } m_1(t) = m_2(t) = \cdots = m_n(t) = m, \text{where } m(t) \text{ is a continuously differentiable scaling function, then the modified function projective synchronization problem will be changed into the function projective synchronization problem, and the controller (7) and adaptive laws (8)-(11) are also practicable.}\]

\[\text{Remark 2:} \text{If } m_1(t) = m_2(t) = \cdots = m_n(t) = m, \text{where } m \text{ is a constant, then the modified function projective synchronization problem will be changed into the projective synchronization problem, and the controller (7) and adaptive laws (8)-(11) are also practicable.}\]

\[\text{Remark 3:} \text{Time delay always varies in many practical applications, so it is difficult to measure the delay and implement the delay term. In our work, the controller does not include } \tau(t), \text{so the proposed method is more general and realistic.}\]

\[\text{IV. ILLUSTRATIVE EXAMPLE}\]

In this section, we present an example to illustrate the correctness of the results which obtained in this paper. We take a four-dimensional hyperchaotic system as reference node.

The four-dimensional hyperchaotic system is described as follows:

\[
\begin{align*}
\dot{v}_1 &= \rho_1 (v_2 - v_1) \\
\dot{v}_2 &= \rho_2 v_1 - v_1 v_3 + v_4 \\
\dot{v}_3 &= -\rho_3 v_3 + v_1 v_2 \\
\dot{v}_4 &= -\rho_4 v_1 \\
\end{align*}
\]

where \( v_1, v_2, v_3, v_4 \) are state variables and \( \rho_1, \rho_2, \rho_3, \rho_4 \) are real constants. The system is in a hyperchaotic state when \( \rho_1 = 35, \rho_2 = 35, \rho_3 = 3, \rho_4 = 8 \). The state phase diagram of this system is shown in figure 1.

Consider the complex dynamical network consisting of four four-dimensional hyperchaotic systems and two different time-varying delays, that is, \( N = 4, m = 3 \). The driving system and the response system have described as follows:

\[
\begin{bmatrix}
\dot{x}_{i1}(t) \\
\dot{x}_{i2}(t) \\
\dot{x}_{i3}(t) \\
\dot{x}_{i4}(t)
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 \\
-x_{i1}(t) x_{i3}(t) + x_{i4}(t) & x_{i1}(t) x_{i2}(t) & 0 & 0 \\
x_{i2}(t) - x_{i1}(t) & 0 & 0 & 0 \\
0 & x_{i1}(t) & 0 & 0 \\
0 & 0 & -x_{i3}(t) & 0 \\
0 & 0 & 0 & -x_{i1}(t)
\end{bmatrix}
\times
\begin{bmatrix}
\theta_{i1} \\
\theta_{i2} \\
\theta_{i3} \\
\theta_{i4}
\end{bmatrix}
\]

where \( x_{i1}, x_{i2}, x_{i3} \) are state variables and \( \theta_{i1}, \theta_{i2}, \theta_{i3}, \theta_{i4} \) are state variables.
where \( \theta_i = (\theta_{i1}, \theta_{i2}, \theta_{i3}, \theta_{i4})^T = (35, 35, 3, 8)^T, i = 1, 2, 3, 4. \)

In the numerical simulation, we set \( r_0 = 3, r_1 = 6, r_2 = 2, r_3 = 7, c_0 (t) = 0.01, c_1 (t) = 0.01 \sin t, c_2 (t) = 0.01 \cos t, \tau_1 (t) = \frac{\pi}{47}, \) and then \( \tau_1 (t) = \frac{2}{2+\pi^2}, \) and then \( \tau_2 (t) = \frac{2e}{2+e^2}, \) and then \( \tau_2 (t) = \frac{2e^2}{(2+e^2)^2} \in (0, \frac{1}{2}). \) And network inner-coupling matrix \( \Gamma_0 = \Gamma_1 = \Gamma_2 = I_{4 \times 4}. \)

The topological structure matrices \( A_0, A_1, A_2 \) are as follows:

\[
A_0 = \begin{pmatrix}
-2 & 1 & 0 & 1 \\
1 & -3 & 2 & 0 \\
0 & 2 & -3 & 1 \\
1 & 0 & 1 & -2
\end{pmatrix},
\]

\[
A_1 = \begin{pmatrix}
-3 & 0 & 1 & 2 \\
0 & -1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
2 & 1 & 0 & -3
\end{pmatrix},
\]
$A_2 = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix}$.

The model uncertainty $\Delta f_i(\cdot, t) = 0.01(\cdot)\sin(10t)$; the disturbance terms $d_{1i}^0(t) = 0.02\cos t$, $d_{1i}^0(t) = -0.02\cos t$;

$M(t) = \text{diag}(\sin t, \cos t, -\sin t, -\cos t)$; $\Delta \theta_1(t) = [\Delta \theta_1(t), \Delta \theta_2(t), \Delta \theta_3(t), \Delta \theta_4(t)]^T = [0.01 \sin t, -0.01 \sin t, 0.01 \cos t, -0.01 \cos t]^T$.

Take the nonlinear input as $\phi_i(u_i(t)) = [\phi_{1i}(u_{1i}(t)), \phi_{2i}(u_{2i}(t)), \phi_{3i}(u_{3i}(t)), \phi_{4i}(u_{4i}(t))]^T = [(0.6 + 0.3 \sin(u_{1i}(t))) u_{1i}(t), (0.6 + 0.3 \cos(u_{2i}(t))) u_{2i}(t), (0.6 - 0.3 \sin(u_{3i}(t))) u_{3i}(t), (0.6 - 0.3 \cos(u_{4i}(t))) u_{4i}(t)]^T$. 

**Figure 2.** The first dimensional states of the system nodes: (a) states of the driving system; (b) states of the response system; (c) synchronization error plots.

**Figure 3.** The second dimensional states of the system nodes: (a) states of the driving system; (b) states of the response system; (c) synchronization error plots.
The state of each nodes of the systems are expressed in dimensions, as shown in figure 2-5. Each of the corresponding state of the response system in figure 2-5(b) depicts a different behavior with the driving system in figure 2-5(a).

From figure 2-5(c), we can get all synchronization errors between the driving system and the response system can converging in neighborhood of zero by application of the proposed controller.
Remark 4: In the simulation process, the hyperbolic tangent function $\tanh(e_i(t))$ was used instead of the signum function $\text{sgn}(e_i(t))$ to reduce chattering.

V. CONCLUSIONS
In this paper, we introduced a more representative complex dynamical network model, which has sector nonlinear input, multiple time-varying delay couplings, model uncertainty and external disturbance. And the modified function projection synchronization of this model was investigated. The sufficient condition to realize the modified function projective synchronization of the driving system and the response system is strictly deduced. The robust controller does not include time delay terms, which makes the synchronization strategy more general and realistic. The simulation results demonstrated the correctness and superiority of the proposed control scheme. The work of this article provides a theoretical reference for the control and application of complex networks. In future work, we will focus on secure communication and information processing based on complex networks.

REFERENCES
[1] R. Yu, H. Zhang, Z. Wang, and Y. Liu, “Synchronization criterion of complex networks with time-delay under mixed topologies,” Neurocomputing, vol. 295, pp. 8–16, Jun. 2018.
[2] C. Zhang, X. Wang, S. Unar, and Y. Wang, “Finite-time synchronization of a class of nonlinear complex-valued networks with time-varying delays,” Phys. A, Stat. Mech. Appl., vol. 528, Aug. 2019, Art. no. 120985.
[3] W. Y. Duan, B. Z. Du, J. You, and Y. Zou, “Synchronization criteria for neutral complex dynamic networks with internal time-varying coupling delays,” Asian J. Control, vol. 15, no. 3, pp. 1385–1396, Sep. 2013.
[4] F. Nazarimehr, S. Panahi, M. Jalili, M. Perc, S. Jafari, and B. Ferecic, “Multivariable coupling and synchronization in complex networks,” Appl. Math. Comput., vol. 372, May 2020, Art. no. 124996.
[5] X. Qian, J. Qin, and J. Sun, “Adaptive modified function projective synchronization of chaotic complex dynamical networks with multiple intermittent coupling delays,” Appl. Comput. Math., vol. 19, no. 3, pp. 661–676, Sep. 2020.
[6] H. J. Yu and X. Y. Dong, “Synchronization of the composite network constructed by the Rössler chaotic system,” J. Shanghai Jiaotong Univ., vol. 52, no. 12, pp. 1559–1564, Dec. 2018.
[7] X. Wang and G. Chen, “A chaotic system with only one stable equilibrium,” Commun. Nonlinear Sci. Numer. Simul., vol. 17, no. 3, pp. 1264–1272, Mar. 2012.
[8] R. Cheng, M. Peng, J. Yu, and H. Li, “Synchronization for discrete-time complex networks with probabilistic time delays,” Phys. A, Stat. Mech. Appl., vol. 525, pp. 1088–1101, Jul. 2019.
[9] D. Ding, Z. Tang, Y. Wang, and Z. Ji, “Synchronization of nonlinearly coupled complex networks: Distributed impulsive method,” Chaos, Solitons Fractals, vol. 133, Apr. 2020, Art. no. 109620.
[10] X. Liu and Z. Li, “Finite time anti-synchronization of complex-valued neural networks with bounded asynchronous time-varying delays,” Neurocomputing, vol. 387, pp. 129–138, Apr. 2020.
[11] C. Ren, R. Nie, and S. He, “Finite-time positiveness and distributed control of Lipschitz nonlinear multi-agent systems,” J. Franklin Inst., vol. 356, no. 15, pp. 8080–8092, Oct. 2019.
[12] L. H. Zhao and J. L. Wang, “Lag $H_\infty$ synchronization and lag synchronization for multiple derivative coupled complex networks,” Neurocomputing, vol. 384, pp. 46–56, Apr. 2020.
[13] H. Leng and Z. Wu, “Impulsive synchronization of complex-variable network with distributed time delays,” Phys. A, Stat. Mech. Appl., vol. 536, Dec. 2019, Art. no. 122602.
[14] C. Ren and S. He, “Finite-time stabilization for positive Markovian jumping neural networks,” Appl. Math. Comput., vol. 365, Jun. 2020, Art. no. 124631.
[15] K. Guan, F. Tan, and J. Yang, “Global power synchronization of complex dynamical networks with proportional delay and impulsive effects,” Neurocomputing, vol. 366, pp. 23–34, Nov. 2019.
[16] P. Liu, H. Gu, Y. Kang, and J. Lü, “Global synchronization under PI/PD controllers in general complex networks with time-delay,” Neurocomputing, vol. 366, pp. 12–22, Nov. 2019.
[17] X. Song, J. Man, C. K. Ahn, and S. Song, “Finite-time dissipative synchronization for Markovian jump generalized inertial neural networks with reaction-diffusion terms,” IEEE Trans. Syst., Man, Cybern., Syst., early access, Dec. 27, 2019, doi: 10.1109/TSMC.2019.2958419.
[18] Z. Hao, W. Xing-yuan, Y. Peng-fei, and S. Yu-jie, “Combination synchronization and stability analysis of time-varying complex-valued neural networks,” Chaos, Solitons Fractals, vol. 131, Feb. 2020, Art. no. 109485.
[19] X. Song, M. Wang, S. Song, and Z. Wang, “Intermittent pinning synchronization of reaction–diffusion neural networks with multiple spatial diffusion couplings,” Neural Comput. Appl., vol. 31, no. 12, pp. 9279–9294, Dec. 2019.
[20] J. Mogambigai, M. Syed Ali, H. Alsulami, and M. S. Alhodaily, “Impulsive and pinning control synchronization of Markovian jumping complex dynamical networks with hybrid coupling and additive interval time-varying delays,” Commun. Nonlinear Sci. Numer. Simul., vol. 85, Jun. 2020, Art. no. 105215.
[21] S. Zheng, G. Dong, and Q. Bi, “Adaptive modified function projective synchronization of hyperchaotic systems with unknown parameters,” Commun. Nonlinear Sci. Numer. Simul., vol. 15, no. 11, pp. 3547–3556, Nov. 2010.
[22] J. Li, “Modified functional projective synchronization of the unidirectional and bidirectional hybrid connective star network with coupling time-delay,” Wuhan Univ. J. Natural Sci., vol. 24, no. 4, pp. 321–328, Aug. 2019.
[23] H. Tirandaz and A. Karmi-Mollaee, “Modified function projective feedback control for time-delay chaotic lhu system synchronization and its application to secure image transmission,” Optik, vol. 147, pp. 187–196, Oct. 2017.
[24] J. Liu and S. Liu, “Complex modified function projective synchronization of complex chaotic systems with known and unknown complex parameters,” Appl. Math. Model., vol. 48, pp. 440–450, Aug. 2017.
[25] S. Wang, “Cluster-modified function projective synchronisation of complex networks with asymmetric coupling,” Pramana, vol. 90, no. 2, Feb. 2018, Art. no. 25.
[26] C. Zhang, X. Wang, X. Ye, S. Zhou, and L. Feng, “Robust modified function projective lag synchronization between two nonlinear complex networks with different dimension nodes and disturbances,” ISA Trans., vol. 101, pp. 42–49, Jun. 2020.
[27] H. Du, “Modified function projective synchronization between two fractional-order complex dynamical networks with unknown parameters and unknown bounded external disturbances,” Phys. A, Stat. Mech. Appl., vol. 526, Jul. 2019, Art. no. 120997.
[28] Y. Z. Sun, W. Li, and J. Ruan, “Generalized outer synchronization between complex dynamical networks with time delay and noise perturbations,” Commun. Nonlinear Sci. Numer. Simul., vol. 18, no. 4, pp. 989–998, Apr. 2013.
[29] C. Zhang, Y. C. Lin, Y. L. Huang, and S.-Y. Ren, “Asymptotical equivalence and adaptive synchronization for robust synchronization and $H_\infty$ synchronization of complex dynamical networks with multiple time-delays,” Neurocomputing, vol. 289, pp. 241–251, May 2018.
[30] Z. Xu, X. Li, and P. Duan, “Synchronization of complex networks with time-varying delay of unknown bound via delayed impulsive control,” Neural Netw., vol. 125, pp. 224–232, May 2020.
[31] M. Syed Ali, M. Hymavathi, S. Senan, V. Shekher, and S. Arik, “Global asymptotic synchronization of impulsive fractional-order complex-valued memristor-based neural networks with time varying delays,” Commun. Nonlinear Sci. Numer. Simul., vol. 78, Nov. 2019, Art. no. 104869.
[32] C. Zhang, X. Wang, C. Luo, J. Li, and C. Wang, “Robust outer synchronization between two nonlinear complex networks with parametric disturbances and mixed time-varying delays,” Math. Probl. Eng., vol. 2018, no. 10, pp. 1707–1733, Feb. 2020.
[33] J. Fang, N. Liu, and J. Sun, “Adaptive modified function projective synchronization of uncertain complex dynamical networks with multiple time-delay couplings and disturbances,” Math. Problems Eng., vol. 2018, Jan. 2018, Art. no. 0384757.
Q. Hu, Y. Meng, C. Wang, and Y. Zhang, “Adaptive backstepping control,”

S. K. Pradhan and B. Subudhi, “Nonlinear control of a magnetic levitation system,”

S. Luo, S. Li, T. Phung, and J. Hu, “Chaotic behavior and adaptive control for the synchronization of the arch MEMS resonator with state constraint and sector input,”

X. Y. Wang and M. Liu, “Sliding mode control for the synchronization of master-slave chaotic systems with sector nonlinear input,”

K.-M. Chang, “Adaptive control for uncertain systems with sector-like bounded non-linear inputs,”

P. Niamsup, T. Botmart, and W. Weera, “Modified function projective synchronization of complex dynamical networks with mixed time-varying and asymmetric coupling delays via new hybrid pinning adaptive control,”

A. Boubellouta, F. Zouari, and A. Boulkroune, “Intelligent fuzzy controller for chaos synchronization of uncertain fractional-order chaotic systems with input nonlinearities,”

M. P. Aghababa, “Adaptive finite time controller design for switched systems with unknown parameters, mismatched uncertain terms, external disturbances and input perturbations,”

X. H. Chang and G.-H. Yang, “Nonfragile H∞ filtering of continuous-time fuzzy systems,”

X. Y. Wang and M. Liu, “Sliding mode control for the synchronization of master-slave chaotic systems with sector nonlinear input,”

JIE FANG received the B.S. degree in electrical engineering and the M.S. degree in electrician theory and new technology from Zhengzhou University, Zhengzhou, China, in 2002 and 2007, respectively, and the Ph.D. degree in control engineering from the Nanjing University of Aeronautics and Astronautics, Nanjing, China, in 2012. She is currently a Professor with the School of Electrical and Information Engineering, Zhengzhou University of Light Industry. Her research interests include nonlinear control theory and applications and complex network control.

DAN-YING XU was born in Henan, China, in 1997. She received the B.S. degree in electrical engineering from Henan Normal University, in 2017. She is currently pursuing the master's degree in electrical engineering with the Zhengzhou University of Light Industry. Her main research interests include control theory, complex dynamic networks, and secure communication.

JUN-WEI SUN received the M.S. degree from the Zhengzhou University of Light Industry, Zhengzhou, China, in 2011, and the Ph.D. degree from the Huazhong University of Science and Technology, Wuhan, China, in 2014. He is currently an Associate Professor with the School of Electrical and Information Engineering, Zhengzhou University of Light Industry. He has been involved in a wide range of research projects in the chaotic, neural networks, and memristors.

WEI WANG received the Ph.D. degree from Concordia University, Canada, in 2002. He is currently a Professor with the School of Electrical and Information Engineering, Zhengzhou University of Light Industry. His research interests include computer vision, artificial intelligence, tactile sensors, and tactile recognition.

* * *