Seymour’s Second Neighborhood Conjecture for Subsets

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Abstract

Seymour conjectured that every oriented simple graph contains a vertex whose second neighborhood is at least as large as its first. In this note, we put forward a conjecture that we prove is actually equivalent: every oriented simple graph contains a subset of vertices \( S \) whose second neighborhood is at least as large as its first.

This subset perspective gives several quick results about the original conjecture: for example, a digraph on \( n \) vertices with minimum degree less than \( \sqrt{2n - \frac{1}{2}} \) is sure to satisfy the second neighborhood conjecture.

Given a vertex \( v \), let \( d^+_1(v) \) and \( d^+_2(v) \) be the size of its first and second neighborhoods respectively. A digraph is \( m \)-free if there is no directed cycle on \( m \) or fewer vertices. Let \( \lambda_m \) be the largest value such that every \( m \)-free graph contains a vertex \( v \) with \( d^+_2(v) \geq \lambda_md_1(v) \). The second neighborhood conjecture implies \( \lambda_m = 1 \) for all \( m \geq 2 \). Liang and Xu provided lower bounds for all \( \lambda_m \), and showed that \( \lambda_m \to 1 \) as \( m \to \infty \). We improve on Liang and Xu’s bound for \( m \geq 3 \), again using this subset perspective.

Keywords: Seymour’s Second Neighborhood Conjecture, \( m \)-free, digraphs

1 Introduction

Unless otherwise noted, all digraphs in this paper are oriented simple graphs, and thus do not contain loops or two-cycles. We will also assume they are strongly connected. We will use \( V(D) \) to denote the set of vertices of a digraph \( D \).

Given a digraph \( D \) and vertices \( u \) and \( v \), let \( d(u,v) \) be the length of the shortest directed path from \( u \) to \( v \). Let \( N^+_k(v) \), the set of \( k \)th out-neighbors, be all vertices \( u \) such that \( d(v,u) = k \). We will focus on \( N^+_1(v) \) and \( N^+_2(v) \), and we note that these are disjoint. We will use \( N^-_1(v) \) and \( N^-_2(v) \) to refer to the sets of first and second in-neighbors, defined analogously to out-neighbors. If not specified, the term neighbors refers to first out-neighbors. Let \( d^+_k(v) = |N^+_k(v)| \) and \( d^-_k(v) = |N^-_k(v)| \). If \( d^+_1(v) \leq d^+_2(v) \), we will call \( v \) a Seymour vertex.

Seymour made the following conjecture, which has become known as Seymour’s Second Neighborhood Conjecture.

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Conjecture 1 (Seymour, see [3]). Every oriented simple graph contains a Seymour vertex.

We will use SNC to refer to this conjecture throughout this note.

The SNC, along with related conjectures of Caccetta and Häggvist [1] and the Hoang and Reed [6], have remained open for decades and seem to be very difficult. (See Sullivan [4] for a nice summary of results and conjectures related to the Caccetta-Häggvist conjecture.) In this note, we introduce a new, related conjecture. For a set of vertices $S$, let $N^+_k(S)$ be all vertices $u$ such that $\min_{s \in S} d(s, u) = k$, and note that $S, N^+_1(S), N^+_2(S), \text{ etc. are all disjoint.}$ Define $d^+_k(S) = |N^+_k(S)|$.

Conjecture 2. Every oriented simple graph contains a set of vertices $S$ such that $d^+_1(S) \leq d^+_2(S)$.

Note that Conjecture 2 is clearly weaker than the SNC, since if there is a Seymour vertex $v$, then we can simply let $S = \{v\}$ and the Conjecture 2 follows. We prove Conjecture 2 is actually equivalent to the SNC. This follow from a lemma we prove in Section 2. There may be some hope that Conjecture 2 is easier to prove than the SNC: for example, Conjecture 2 has an easy proof for regular graphs (see Proposition 7), a case that has received much attention but has yet to yield a proof for the SNC.

Since $N^+_1(S)$ is a cut set of the graph, it is possible Conjecture 2 is related to the isoperimetric method of Hamidoune. Using the isoperimetric method, Hamidoune [5] proved the SNC for vertex-transitive graphs, and later Lladó [8] proved the SNC for $r$-out-regular graphs of connectivity $r - 1$.

In attempt to make progress on the SNC, Chen, Shen, and Yuster [2] asked the question: What is the largest $\lambda$ such that one could prove the existence of a vertex $v$ such that

$$d^+_2(v) \geq \lambda d^+_1(v). \quad (1)$$

They proved this approximate form of the conjecture for $\lambda$ the real root of the equation $2x^3 + x^2 - 1 = 0$, with $\lambda \approx 0.6573\ldots$. They also claimed that $\lambda \approx 0.67815\ldots$ was achievable with similar methods.

A digraph is $m$-free if it has no directed cycles with length at most $m$. One can then ask the Chen, Shen, and Yuster question in regards to this restricted set of digraphs. Let $\lambda_m$ be the largest value such that every $m$-free digraph has a vertex $v$ where $d^+_2(v) \geq \lambda_m d^+_1(v)$. The second neighborhood conjecture implies $\lambda_m = 1$ for all $m \geq 2$. Zhang and Zhou [9] showed $\lambda_3 \geq 0.6751$. Liang and Xu [7] improved this and extended the result for all $m$, showing that $\lambda_m$ is greater than the only real root in the interval $(0, 1)$ of the polynomial

$$2x^3 - (m - 3)x^2 + (2m - 4)x - (m - 1).$$

This implies $\lambda_3 \geq 0.6823\ldots$ which improved the Zhang and Zhou result for $\lambda_3$. The value of $\lambda_4$ was $0.7007\ldots$, and in general, $\lambda_m \rightarrow 1$ as $m \rightarrow \infty$. Using this subset approach, we improve the Liang and Zu result for all $m \geq 3$.

Theorem 3. An $m$-free digraph contains a vertex $v$ with $d^+_2(v) \geq \lambda_m d^+_1(v)$, where $\lambda_m$ is the unique positive real root of

$$x^m + x^{m-1} = 1$$
For 2-free digraphs, we get the golden ratio of $\lambda_2 \geq .6180\ldots$, which is not as good as the Chen, Shen, and Yuster result. However, our $\lambda_3 \geq .7548\ldots$, which is the best known result, and $\lambda_4 \geq 0.8191\ldots$, which is the best known result. Note that this goes to 1 faster than the Liang and Xu result: the Liang and Xu result grows like $1 - \sqrt{2} \frac{1}{\sqrt{m}} + o\left(\frac{1}{\sqrt{m}}\right)$, while our result grows like $1 - \ln(2) \frac{1}{m} + o\left(\frac{1}{m}\right)$.

## 2 Main Lemma

We say $D$ is a $\lambda$ counterexample (to the SNC) if $d^+_2(v) < \lambda d^+_1(v)$ for all vertices $D$. We say $D$ is an edge-minimal $\lambda$ counterexample if one cannot remove edges to create a smaller $\lambda$ counterexample. Given two subsets of vertices $S$ and $T$ in a digraph $D$, let $NO^D_S(T)$ be all the neighbors of $T$ outside of $S \cup T$.

**Lemma 4.** Let $D$ be an edge-minimal $\lambda$ counterexample to the SNC, and let $S$ be any proper subset of the vertices of $D$. Then $d^+_2(S) < \lambda d^+_1(S)$. In other words, $|NO^D_S(N^+_1(S))| \leq \lambda|N^+_1(S)|$.

**Proof.** Choose a subset of vertices $T \subset N^+_1(S)$ to be maximal such that $\lambda |T| \geq |NO^D_S(T)|$, or $T = \emptyset$ if no such $T$ exists. If $T = N^+_1(S)$, then $\lambda |T| = \lambda |N^+_1(S)| \geq |NO^D_S(T)| = |NO^D_S(N^+_1(S))|$ and we are done. So assume $T \not\subset N^+_1(S)$. Set $T' = N^+_1(S) - T$. Let $S'$ be $S \cup T \cup NO_S(T)$.

Now create a new graph $D'$ equal to $D$ but with all arcs from $S$ to $T'$ removed. I claim that $D'$ is an $\lambda$ counterexample to the SNC, contradicting the minimality of $D$. Suppose $D'$ is not an $\lambda$ counterexample, and that it has $v$ such that $d^+_2(v) \geq \lambda d^+_1(v)$. Since we only removed outgoing arcs from vertices in $S$, $v$ must be in $S$.

Now $v$ may have lost some neighbors $A \subseteq T'$ that did not become second neighbors. Also, $v$ may have lost some neighbors $B \subseteq T'$ that did become second neighbors. Set $C = NO^D_S(A \cup B)$, and note that every vertex in $C$ is a second neighbor that $v$ lost. ($v$ may have also lost other second neighbors, but we are only concerned with those in $NO^D_S(A \cup B)$). See Figure 1 for a diagram of some of these sets. We know that $v$ went from $d^+_2(v) < \lambda d^+_1(v)$ to $d^+_2(v) \geq \lambda d^+_1(v)$, we have that $\lambda(|A| + |B|) > |C| - |B|$. That is, $\lambda$ times the number of neighbors lost must be at least the number of second neighbors lost.

Consider the effect of adding $A$ and $B$ to $T$ to create $T_2$. Since $B$ consists of second neighbors of $v$ in $D'$, but we removed all arcs from $S$ to $B$, it must be the case that the vertices of $B$ are second neighbors of $v$ through $T$. In other words, the vertices of $B$ lie inside $NO^D_S(T)$. Consider the difference in the two sets.

![Figure 1: A diagram of some of the sets used in the proof of Lemma 4.](image-url)
$NO^D_S(T_2)$ and $NO^D_S(T_2)$. By adding $A$ and $B$ to $T$, we gain $C$ as new neighbors in $NO^D_S(T_2)$ and no others, and we lose elements of $B$ as elements of $NO^D_S(T)$ (and perhaps some others). Hence, $|NO^D_S(T_2)| \leq |NO^D_S(T)| + |C| - |B|$. By assumption, $|NO^D_S(T)| \leq \lambda |T|$, and we also have $|C| - |B| < \lambda (|A| + |B|)$. Hence
\[
|NO^D_S(T_2)| \leq |NO^D_S(T)| + |C| - |B| < \lambda (|T| + \lambda (|A| + |B|)) = \lambda |T_2|
\]
But this contradicts the maximality of $T$.

\section{Quick Consequences of the Lemma}

Lemma \ref{lem:maximality} leads to two quick corollaries regarding the SNC itself, both of which use the lemma with $\lambda = 1$.

**Corollary 5.** Given a graph $G$ with girth $g$, if $g > \delta(G)$, then $G$ satisfies the SNC.

**Proof.** Let $D$ be a minimum counterexample to the SNC such that $g > \delta(G)$. By removing edges, the girth can only increase and the minimum degree can only decrease, so we will still have $g > \delta(G)$ for any proper spanning subgraph of $D$. That means $D$ is a minimum counterexample to the SNC, which means Lemma \ref{lem:maximality} applies.

Let $v$ be a vertex of minimum degree. Applying Lemma \ref{lem:maximality} with $\lambda = 1$ and $S = \bigcup_{k=1}^{g-1} N^+_1(v)$, we have that $d^+_k(v) > d^+_k(v)$ for $k = 1, 2, 3, \ldots$. Since $d^+_1(v) = \delta(G)$, and each neighborhood is smaller than the last, there are only $\delta(G)$ non-empty neighborhoods of $v$. One of these neighborhoods must contain $v$, and hence there is a cycle of length at least $\delta(G)$, a contradiction.

**Corollary 6.** Given a graph $G$ such that $\delta^+(D) < \sqrt{2n} - \frac{1}{2}$, $G$ satisfies the SNC.

**Proof.** Let $D$ be a minimum counterexample to the SNC such that $\delta^+(D) < \sqrt{2n} - 1/2$. Note that this implies $(\delta^+(D)+1) < n$. By removing edges, the minimum degree can only decrease, so we will still have $(\delta^+(D)+1) < n$ for any proper spanning subgraph of $D$. That means $D$ is a minimum counterexample to the SNC, which means Lemma \ref{lem:maximality} applies.

Let $v$ be a vertex of minimum degree. Similar to the previous corollary, we have that $d^+_k(v) > d^+_k(v)$ for $k = 1, 2, 3, \ldots$. Since $d^+_1(v) = \delta(G)$, and each neighborhood is smaller than the last, there are at most
\[
\delta^+(D) + (\delta^+(D) - 1) + (\delta^+(D) - 2) + \cdots + 1 = \binom{\delta^+(D) + 1}{2} < n
\]
vertices in the graph, a contradiction.

An in-regular graph is a graph such that $|N^-_1(v)|$ is the same for all $v$. Here we show that Conjecture \ref{conj:in-regular} is true in case of in-regular graphs. Note that this proof unfortunately does not translate to the SNC since in-regular graphs are not closed under removal of edges, and therefore Lemma \ref{lem:maximality} does not help.

**Proposition 7.** Given an in-regular digraph $D$ without loops or multiple edges, there exists a subset of vertices $S$ such that $d^+_1(S) \leq d^+_2(S)$.
Proof. Consider a counterexample $D$ to this proposition. $D$ would also be a counterexample to the SNC, and hence for every vertex $v$, we have $d_1^{-}(v) > d_2^{-}(v)$. Since $\sum_{v \in V(D)} d_1^{-}(v) = \sum_{v \in V(D)} d_1^{+}(v)$ and $\sum_{v \in V(D)} d_2^{+}(v) = \sum_{v \in V(D)} d_2^{-}(v)$, we know there exists at least one vertex $v$ such that $d_1^{-}(v) > d_2^{-}(v)$. Let $\mathcal{V}$ be the set of all vertices such that $d_1^{-}(v) > d_2^{-}(v)$.

For any $v \in \mathcal{V}$, set $S_v = V(D) \setminus (N_1^{-}(v) \cup N_2^{-}(v))$. Since $D$ is a counterexample, we know that $d_1^{-}(S_v) > d_2^{-}(S_v)$. Since $N_1^{+}(S_v) \subseteq N_2^{-}(v)$, there is no way $N_1^{+}(S_v) \cup N_2^{+}(S_v)$ covers all the vertices in $N_1^{-}(v) \cup N_2^{-}(v)$. Therefore, there must be some vertex $u$ in $N_1^{-}(v)$ such that $N_1^{-}(u) \cup N_2^{-}(u) \subseteq N_1^{-}(v) \cup N_2^{-}(v)$.

If $u \in \mathcal{V}$, then we can apply the same argument and get a $u'$ such that $u'$ first and second in-neighborhoods are contained in $u$'s first and second in-neighborhoods. By repeating this argument, eventually we find a $u^*$ whose first and second in-neighborhoods are contained in the first and second in-neighborhoods of $v$, but $u^* \notin \mathcal{V}$. So $N_1^{-}(u^*) \cup N_2^{-}(u^*) \subseteq N_1^{-}(v) \cup N_2^{-}(v)$. However, since $D$ is in-regular, we have $|N_1^{-}(u^*)| = |N_1^{-}(v)|$, and $N_2^{-}(u^*) \supseteq N_2^{-}(v)$, and so this containment is a contradiction. \qed

4 Approximate Second Neighborhood for $m$-free digraphs

Let $d(u,v)$ be the length of the shortest directed path from $u$ to $v$. For purposes of this section $d(v,v)$ is not zero but instead the length of the shortest cycle from $v$ to itself. For a vertex $v$, let eccentricity $e(v)$ is the distance to the farthest vertex:

$$e(v) = \max_{u \in V(D)} d(v,u).$$

The radius $\text{rad}(D)$ of a digraph $D$ is the minimum eccentricity:

$$\text{rad}(D) = \min_{v \in V(D)} e(v).$$

The reverse radius $\overleftarrow{\text{rad}}(D)$ is the radius of the reverse of $D$, and may be completely different from $\text{rad}(D)$. However, note that an $m$-free digraph has radius reverse radius at least $m + 1$, since at the very least every vertex is a distance of $m + 1$ from itself.

**Theorem 8.** Any digraph of reverse radius $r \geq 3$ has a vertex $v$ such that $d_2^{+}(v) \geq \lambda d_1^{+}(v)$ for $\lambda$ a real number between 0 and 1 satisfying

$$\lambda^{r+1} + \lambda^{r-2} = 1.$$

**Proof.** Let $D$ be a minimum counterexample with $d_2^{+}(v) < \lambda d_1^{+}(v)$ for all vertices $v$. Then by Lemma 7, we know that it is the case that for every subset of vertices $S$, $d_2^{+}(S) < \lambda d_1^{+}(S)$. This, as we have seen, implies that $d_{i+1}^{+}(v) < \lambda d_i^{+}(v)$, provided $d_i^{+}(v)$ is nonzero. One can then show that this implies that $d_{i-1}^{+}(v) < \lambda^{-i+1} d_i^{+}(v)$, and hence $d_i^{+}(v) > \frac{1}{\lambda^{-i+1}} d_{i-1}^{+}(v)$.  

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Let \( A_v = \bigcup_{i=1}^{r-2} N_i^+(v) \). We see

\[
|A_v| = \sum_{i=1}^{r-2} d_i^+(v) > \sum_{i=1}^{r-2} \frac{1}{\lambda^{r-1-i}} d_{r-1}^+(v) = \frac{1 - \lambda^{r-2}}{\lambda^{r-2}(1 - \lambda)} d_{r-1}^+(v)
\]

If we set \( \gamma = \frac{\lambda^{r-2}(1 - \lambda)}{1 - \lambda^{r-2}} \), then we see for every vertex \( v \), \( d_{r-1}^+(v) < \gamma |A_v| \).

Define \( B_v = \bigcup_{i=1}^{r-2} N_i^-(v) \). Since \( \sum_{v \in V} d_i^+(v) = \sum_{v \in V} d_i^-(v) \), we see that on average \( B_v \) is the same size as \( A_v \), and \( d_{r-1}^-(v) \) is on average the same size as \( d_{r-1}^+(v) \). Therefore, there must exist a particular vertex \( v \) such that \( d_{r-1}^-(v) < \gamma |B_v| \).

Since \( r \) is the reverse radius of \( D \), \( N_r^-(v) \) is non-empty. Setting \( S^* = N_r^-(v) \), we see \( N_1^+(S^*) \subseteq N_{r-1}^-(v) \), and therefore \( |N_1^+(S^*)| \leq \gamma |B_v| \). By repeated use of \( |N_k^+(S)| < \lambda |N_k^+(S)| \) for appropriate \( S \), we see that \( |N_2^+(S^*)| < \lambda |B_v|, |N_3^+(S^*)| < \lambda^2 |B_v|, \) etc., and in general \( |N_k^+(S^*)| < \lambda^{k-1} |B| \).

Eventually, these \( N_k^+(S^*) \) must cover \( B_v \). Therefore,

\[
\bigcup_{i=2}^{\infty} |N_i^+(S^*)| \geq |B_v| \\
\sum_{i=1}^{\infty} \lambda^i |B_v| > |B_v| \\
\sum_{i=1}^{\infty} \lambda^i \gamma > 1 \\
\frac{\lambda}{1 - \lambda} \gamma > 1 \\
\frac{\lambda}{1 - \lambda}, \frac{\lambda^{r-2}(1 - \lambda)}{1 - \lambda^{r-2}} > 1 \\
\lambda^{r-1} > 1 - \lambda^{-2}
\]

This gives the result. \( \Box \)

Applying Theorem 8 to \( m \)-free digraphs gives Theorem 3.

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