Rotational invariance and the spin-statistics theorem.

Paul O’Hara

Dept. of Mathematics
Northeastern Illinois University
5500 North St. Louis Avenue
Chicago, Illinois 60625-4699.

email: pohara@neiu.edu

Abstract

In this article, the rotational invariance of entangled quantum states is investigated as a possible cause of the Pauli exclusion principle. First, it is shown that a certain class of rotationally invariant states can only occur in pairs. This is referred to as the coupling principle. This in turn suggests a natural classification of quantum systems into those containing coupled states and those that do not. Surprisingly, it would seem that Fermi-Dirac statistics follows as a consequence of this coupling while the Bose-Einstein follows by breaking it. In section 5, the above approach is related to Pauli’s original spin-statistics theorem and finally in the last two sections, a theoretical justification, based on Clebsch-Gordan coefficients and the experimental evidence respectively, is presented.

KEY WORDS: rotational invariance, bosons, fermions, spin-statistics.

1 INTRODUCTION

Rotational invariance in quantum mechanics is usually associated with spin-singlet states. In this article, after having first established a uniqueness theorem relating rotational invariance and spin-singlet states, a statistical
classification is carried out. In effect, it will be shown that, within the construct of the proposed mathematical model, rotationally invariant quantum states can only occur in pairs. These pairs will be referred to as isotropically spin-correlated states (ISC) and will be defined more precisely later on. This in turn will suggest a statistical classification procedure into systems containing paired states and those that do not. It will be shown that a system of \( n \) coupled and indistinguishable states obey the Fermi-Dirac statistic, while Bose-Einstein statistics will follow when the coupling is broken.

At the same time, it is important to point out that, although, these results can be made mathematically compatible with Pauli’s article of 1940 [11], nevertheless there are differences. In the usual quantum field theory approach, Klein-Gordon fields are first quantized and then it is shown that creation operators commute, as also do annihilation operators. Similarly, when the Pauli principle is applied to the Dirac field, it is found that the analogous operators anticommute. However, our approach is different. We begin, not by focusing on Hamiltonian fields but rather on the rotational properties of spin. From there we proceed to show that an antisymmetric wave function can be associated with rotationally invariant states (singlet states), while the symmetrical wave function can be associated with the absence of rotationally invariant states. Later these rotationally invariant states and non-rotationally invariant states are related to anticommutative spin operators and commutative spin operators, respectively. \textit{It follows that our usage of the terms fermions and bosons is not based on spin value but on rotational properties and correlations between quantum states.}

At first this may seem like a huge break with Pauli’s formulation in terms of anticommutator and commutator relationships and one may even object to giving new meanings to well established terms. However, on a closer analysis it will be found that mathematically, our results are compatible with Pauli’s approach, provided we redefine the spin angular momentum by \( S = nL \). Essentially, we are introducing a scaling factor into the definition of angular momentum which allows us to compare spin 1/2 particles (represented by \( n=1 \)) with spin 1 particles (represented by \( n=2 \)). Once, this rescaled spin operator is introduced, we will find that we are now free to apply the usual quantum field theory arguments to these rescaled operators. We will find, like Pauli, that antisymmetric wave functions can be
associated with anticommutative spin operators, while the symmetric wave function can be associated with commutative operators. However, because of our rescaled momentum operators, we will no longer be able to interpret the statistics in terms of 1/2 integer and integer values but rather in terms of rotationally and non-rotationally invariant states.

These results can perhaps be best summarized in terms of the following scheme:

Rotationally invariant pairs $\Rightarrow$ antisymmetric wave function $\Rightarrow$ anticommutator relations.

In contrast, Pauli’s approach gives:

1/2 integer spin $\Rightarrow$ antisymmetric wave function $\Rightarrow$ anticommutator relations.

The parallels are clear but so also are the differences. For this reason the reader should be attentive with the usage of the words fermions and bosons in this article. Our usage is in full agreement with the conventional usage if we focus on antisymmetric and symmetric quantum states. It is not the same if we focus on spin-value.

Finally, we turn to notation. Throughout the paper $\theta$ will represent a polar angle lying within a plane such that $0 \leq \theta < 2\pi$. Also denote $|\theta_j - \theta_i|$ by $\theta_{ij}$ and write $a.e. \theta$ for “$\theta$ almost everywhere”.

$|\psi_{1\ldots n}(\lambda_1, \ldots, \lambda_n)\rangle$ will represent an $n$-particle state, where $1\ldots n$ represent particles and $\lambda_1 \ldots \lambda_n$ represent the corresponding states. However, if there is no ambiguity oftentimes this state will be written in the more compact form $|\psi(\lambda_1, \ldots, \lambda_n)\rangle$ or more simply as $|\psi\rangle$.

$s_n(\theta)$ will represent the spin states of particle $n$ measured in direction $\theta$ where $s_n(\theta) = |\pm\rangle$. In the case of $\theta = 0$, replace $s_n(0)$ with $s_n$ or by $|+\rangle$ or $|-\rangle$ according to the context, where $+$ and $-$ represent spin up and spin down respectively.

Also let $s^\perp_n(\theta)$ denote the spin state orthogonal to $s_n(\theta)$.

The wedge product of $n$ 1-forms is given by:

$$a_1 \wedge \ldots \wedge a_n = \frac{1}{n!} \delta^{i_1\ldots i_n}_{i_1\ldots i_n} a^{i_1} \otimes \ldots \otimes a^{i_n}. \tag{1}$$

Specifically, $a^1 \wedge a^2 \wedge a^3 = \frac{1}{3!}(a^1 \otimes a^2 \otimes a^3 + a^2 \otimes a^3 \otimes a^1 + a^3 \otimes a^1 \otimes a^2 - a^2 \otimes a^1 \otimes a^3 - a^1 \otimes a^3 \otimes a^2 - a^3 \otimes a^2 \otimes a^1) = \det(a_1, a_2, a_3)e^1 \otimes e^2 \otimes e^3.$
2 A COUPLING PRINCIPLE

The concept of isotropically spin-correlated states (to be abbreviated as ISC) is now introduced. This definition is motivated by the probability properties of rotational invariance. Intuitively, \( n \) particles are said to be isotropically spin-correlated, if a measurement made in an arbitrary direction \( \theta \) on one of the particles allows us to predict with certainty the spin value of each of the other \( n - 1 \) particles for the same direction \( \theta \).

**Definition 1** Let \( H_1 \otimes H_2 \) be a tensor product of two 2-dimensional inner product spaces. Then \( |\psi\rangle \in H_1 \otimes H_2 \) is said to be rotationally invariant if
\[
(R_1(\theta), R_2(\theta)) |\psi\rangle = |\psi\rangle ,
\]
where each
\[
R_i(\theta) = \begin{bmatrix} \cos(c\theta) & \sin(c\theta) \\ -\sin(c\theta) & \cos(c\theta) \end{bmatrix}
\]
represents a rotation on the space \( H_i \) and \( c \) is a constant.\[4\]

**Definition 2** Let \( H_1, \ldots, H_n \) represent \( n \) 2-dimensional inner product spaces. \( n \) particles are said to be isotropically spin correlated (ISC) if

(1) for all \( \theta \) the two state \( |\psi_{ij}\rangle \in H_i \otimes H_j \) is rotationally invariant for all \( i, j \) where \( i \neq j \) and \( 1 \leq i, j \leq n \),

(2) for all \( \theta \) and each \( m \leq n \) the state \( |\psi\rangle \in H_1 \otimes \ldots \otimes H_m \) can be written as
\[
|\psi\rangle = \frac{1}{\sqrt{2}}[s_1(\theta) \otimes s_2(\theta) \ldots \otimes s_m(\theta) \pm s_1^-(\theta)s_2^-(\theta) \ldots \otimes s_m^-(\theta)]
\]

Note that it follows from the definition of ISC states that rotationally invariant states of the form
\[
|\psi\rangle = \frac{1}{2}(|+\rangle |+\rangle + |\rangle |\rangle - |\rangle |\rangle + |\rangle |\rangle - |\rangle |\rangle + |\rangle |\rangle)
\]
are excluded. In other words, the existence of ISC states means that if we measure the spin state \( s_1 \) then we have simultaneously measured the spin state for \( s_2 \ldots s_n \). It can also be shown by means of projection operators that
the state defined by equation (1) is the only state that can be projected onto the state $|\psi_{ij}\rangle \in H_i \otimes H_j$ for each $i, j$. This further highlights its significance.

Two examples of ISC states can be immediately given:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle |+\rangle + |\rangle |\rangle + |\rangle |\rangle + |\rangle |\rangle) \quad (5)$$

and

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle |\rangle - |\rangle |\rangle + |\rangle |\rangle). \quad (6)$$

However, what is not apparent is that these are the only ISC states permitted for a system of $n$-particles. This is now proven.

First note that in terms of the physics, the existence of ISC states means that once the spin-state of particle 1 is measured along an axis then the values of each of the other particle states are also immediately known along the same axis.

In effect, in order to show that such states exist only for $n=2$, it is sufficient to show that it is impossible to have three such particles. This follows, since the existence of $n$ ISC particles, presupposes the existence of $n-1$ such particles. Moreover, the proof also throws deeper understanding on the interpretation of Bell’s inequality.

In the interest of clarity, assume without loss of generality, that the three ISC particles are such that $s_1(\theta_i) = s_2(\theta_i) = s_3(\theta_i)$, for an arbitrary direction $\theta_i$. This means that the joint spin state for any two of them is given by equation (3). It follows that three spin measurements can be performed, in principle, on the three particle system, in the directions $\theta_i$, $\theta_j$, $\theta_k$. Let $(s_1(\theta_i), s_2(\theta_j), s_3(\theta_k))$ represent these observed spin values in the three different directions. Recall $s_n(\theta) = \pm$ for each $n$, which means that there exists only two possible values
for each measurement. Hence, for three measurements there are a total of 8 possibilities in total. In particular, following an argument of Wigner [13],

\[((+,+,+), (+,+,-), (+,-,-), (-,+,-), (+,-,+))\] (7)

implies

\[P\{(+,+,+), (+,+,-), (+,-,-), (-,+,-), (+,-,+))\} \leq P\{(+,+,+), (+,+,-), (+,-,-), (-,+,-), (+,-,+))\}. (8)\n
Therefore,

\[\frac{1}{2} \sin^2 \frac{\theta_{ki}}{2} \leq \frac{1}{2} \sin^2 \frac{\theta_{jk}}{2} + \frac{1}{2} \sin^2 \frac{\theta_{ij}}{2}, \] (9)

which is Bell’s inequality. Taking \(\theta_{ij} = \theta_{jk} = \frac{\pi}{3}\) and \(\theta_{ki} = \frac{2\pi}{3}\) gives \(\frac{1}{2} \geq \frac{3}{4}\), a contradiction. In other words, three particles cannot all be in the same spin state with probability 1, or, to put it another way, isotropically spin-correlated particles must occur in pairs.

**Remarks:** (i) If the ISC particles includes the singlet state (4), such that \((s_1(\theta_i) = +, s_2(\theta_i) = -, s_3(\theta_i) = +)\) (note same \(i\)), then regardless of distinguishability, the spin measurements in the three different directions \(\theta_i, \theta_j, \theta_k\) can be written as:

\[\{(+,+,-), (+,-,-), (-,+,-), (-,-,-), (+,+,+)\} \subset \{(+,+,-), (+,-,-), (-,+,-), (-,-,-), (+,+,+)\}. (10)\]

The previous argument can now be repeated as above.

(ii) Each of the previous arguments applies also to spin 1 particles, like the photons, provided full angle formulae are used to derive Bell’s inequality, instead of the half-angled formulae.

(iii) A more rigorous treatment of the above theorem can be found in [8].

### 3 PAULI EXCLUSION PRINCIPLE

The above results can be cast into the form of a theorem (already proven above) which will be referred to as the “coupling principle.”
Theorem 1 (The Coupling Principle) Isotropically spin-correlated particles must occur in PAIRS.

It follows from the coupling principle that multi-particle systems can be divided into two categories, those containing coupled particles and those containing decoupled particles. It now remains to show that a statistical analysis of these two categories, applied to indistinguishable particles, generates the Fermi-Dirac and Bose-Einstein statistics respectively.

First note that in the case of ISC particles the two rotationally invariant states (5) and (6) can be identified with each other by identifying a spin measurement of \( \pm \) on the second particle in the \( x \) direction with a spin measurement of \( \mp \) in the \( -x \) direction, in such a way as to maintain the rotational invariance. In other words, by replacing the second spinor with its spinor conjugate [3], the state \( |\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle |+\rangle + |\mp\rangle |\mp\rangle) \) can be written as \( |\psi(\pi)\rangle = \frac{1}{\sqrt{2}}(|+\rangle |-\rangle_{\pi} - |\mp\rangle |+\rangle_{\pi}) \), where the \( \pi \) subscript refers to the fact that the measurements on particles 1 and 2 are made in opposite directions, while maintaining the rotational invariance [3]. The state \( |\psi(\pi)\rangle \) shall be referred to as an improper singlet state. Furthermore, without loss of generality the above identification means it is sufficient to confine oneselfs to singlet states when discussing the properties of ISC particles. Whether or not improper singlet states (5), actually exist in nature remains an open question. They can be easily eliminated by including the azimuthal angle in the definition of the rotation operator, in which case only the regular singlet state would remain. However, we will continue to work with both, for if such states were actually discovered they would throw new light on the “handedness” problem and their existence might possibly be linked to parity violation.

It remains to show that the requirement of rotational invariance for ISC particles generates a Fermi-Dirac type statistic. Before doing so, it is important to emphasize that the Pauli principle has not been assumed but rather is being derived from the usual form of the principle (written in terms of the Slater determinant) by imposing orbital restrictions on the ISC states. The essential ideas are as follows: the existence of ISC particles means that from the coupling principle (above), the wave function can be written uniquely as a singlet state on the \( H_1 \otimes H_2 \) space. It then follows by imposing the additional requirement that ISC particles occur only in the
same orbital, that the usual singlet-state form of the wave function can be extended to the space $S_1 \otimes S_2$ where $S_i = L^2(\mathbb{R}^3) \otimes H_i$. Indeed, it would almost seem to be a tautology stemming from the definition of rotational invariance. Nevertheless, the manner in which the notion of ISC states extend to these spaces needs to be clarified, since it permits an extension of the results to the usual $L^2 \otimes H$ space associated with quantum mechanics and not just the more restricted spin spaces. Once this extension is made, proof by induction can be used to derive the usual form of the Pauli principle associated with the Slater determinant, for an $n$-particle system. Moreover, it is also worth noting that the existence of spin-singlet states in general, and not only in the same orbital, permits more general forms of the exclusion principle [8].

In what remains, let $s_n = s_n(\theta)$ represent the spin of particle $n$ in the direction $\theta$. Also, let $\lambda_n = (q_n, s_n)$ represent the quantum coordinates of particle $n$, with $s_n$ referring to the spin coordinate in the direction $\theta$ and $q_n$ representing all other coordinates. In practice, $\lambda_n = (q_n, s_n)$ will represent the coordinates of the particle in the state $\psi(\lambda_n)$ defined on the Hilbert space $S_n = L^2(\mathbb{R}^3) \otimes H_n$, where $H_n$ represents a two-dimensional spin space of the particle $n$. This distinction will also allow the ket to be written as: $|\psi(\lambda)\rangle = |\psi(q)\rangle s = |\psi(q)\rangle \otimes s$ where $s$ represents the spinor. With these distinctions made, the notion of orbital is now defined and a sufficient condition for obtaining the usual form of the Fermi-Dirac statistics within the context of our mathematical model is given.

**Definition 3** Two particles whose states are given by $|\psi(q_1, s_1)\rangle$ and $|\psi(q_2, s_2)\rangle$ respectively are said to be in the same $q$-orbital when $q_1 = q_2$.

The following lemma allows us to extend the results for ISC particles defined on the space $H_1 \otimes H_2$ to the larger space $S_1 \otimes S_2$. As mentioned above, the Pauli principle is not being assumed but rather is being deduced by invoking rotational invariance of the ISC particles. Conversely, if the rotational invariance condition is relaxed then the Pauli principle need not apply and as a result many particles can be in the same orbital.

**Lemma 1** Let

$$|\psi(\lambda_1, \lambda_2)\rangle = c_1 |\psi_1(\lambda_1)\rangle \otimes |\psi_2(\lambda_2)\rangle + c_2 |\psi_1(\lambda_2)\rangle \otimes |\psi_2(\lambda_1)\rangle$$  \hspace{1cm} (11)
represent an indistinguishable two particle system defined on the space $S_1 \otimes S_2$. If ISC states for a system of two indistinguishable and non-interacting particles occur only in the same q-orbital then the system of particles can be represented by the Fermi-Dirac statistics.

**Proof:** The general form of the non-interacting and indistinguishable two particle state is given by

$$|\psi(\lambda_1, \lambda_2)\rangle = c_1 |\psi_1(\lambda_1)\rangle \otimes |\psi_2(\lambda_2)\rangle + c_2 |\psi_1(\lambda_2)\rangle \otimes |\psi_2(\lambda_1)\rangle$$

(12)

$$= c_1 |\psi_1(q_1)\rangle s_1 \otimes |\psi_2(q_2)\rangle s_2 + c_2 |\psi_1(q_2)\rangle s_2 \otimes |\psi_2(q_1)\rangle s_1$$

(13)

where $c_1$, $c_2$ are constants for all $\lambda_1$ and $\lambda_2$. Let $q_1 = q_2$, then the particles are in the same q-orbital. By invoking the ISC condition, it follows from the coupling principle and the rotational invariance that $c_1 = -c_2$. Therefore,

$$|\psi(\lambda_1, \lambda_2)\rangle = \frac{1}{\sqrt{2}} \left[ |\psi_1(\lambda_1)\rangle \otimes |\psi_2(\lambda_2)\rangle - |\psi_1(\lambda_2)\rangle \otimes |\psi_2(\lambda_1)\rangle \right]$$

(14)

by normalizing the wave function. The result follows. QED

**Remarks:** (i) Singlet states composed of particles of spin $n$ can also be handled by the above theory. For example, the singlet state

$$|0, 0\rangle = \frac{1}{\sqrt{5}} (|2, -2\rangle - |1, -1\rangle + |0, 0\rangle - |1, 1\rangle + |2, 2\rangle)$$

(15)

can be decomposed into two irreducible ISC representations of the form $|2, -2\rangle - |2, 2\rangle$ and $|1, -1\rangle - |1, 1\rangle$ and the above theory can then be applied to these states. The remaining $|0, 0\rangle$ state can be treated as a bosonic state (see Section 4).

(ii) The above lemma also applies to improper singlet states, in other words to particles whose spin correlations are given by equation (5). This can be done by correlating a measurement in direction $\theta$ on one particle, with a measurement in direction $\theta + \pi$ on the other. In this case, the state vector for the parallel and anti-parallel measurements will be found to be:

$$|\psi(\lambda_1, \lambda_2)\rangle = \frac{1}{\sqrt{2}} \left[ |\psi(\lambda_1)\rangle \otimes |\psi_2(\pi)\rangle - |\psi(\lambda_2)\rangle \otimes |\psi_1(\pi)\rangle \right]$$

(16)
where the π expression in the above arguments refers to the fact that the measurement on particle two is made in the opposite sense to that of particle one.

(iii) If the coupling condition (rotational invariance) is removed then a Bose-Einstein statistic follows and is of the form:

\[ |\psi(\lambda_1, \lambda_2)\rangle = \frac{1}{\sqrt{2}} \left[ |\psi(\lambda_1)\rangle \otimes |\psi(\lambda_2)\rangle + |\psi(\lambda_2)\rangle \otimes |\psi(\lambda_1)\rangle \right]. \tag{17} \]

This will be discussed in more detail later. See, for example, Corollary 1 following Theorem 3.

Our next theorem gives the usual formulation of the Pauli exclusion principle for exchangeable particles.

**Theorem 2** *(The Pauli Exclusion Principle)* A sufficient condition for a state, representing \( n \)-cyclically permutable and non-interacting particles, defined on the space \( S_1 \otimes \ldots S_n \) to exhibit Fermi-Dirac statistics is that it contain spin-coupled \( q \)-orbitals.

**Remark:** A system of \( n \)-cyclically permutable particles will be referred to as \( n \) indistinguishable particles.

**Proof:** It is sufficient to work with three particles, but it should be clear that the argument can be extended by induction to an \( n \)-particle system. Consider a system of three indistinguishable particles, containing spin-coupled particles. Using the above notation and applying equation (14) of Lemma 1 to the coupled particles in the second line below, gives:

\[
|\psi(\lambda_1, \lambda_2, \lambda_3)\rangle = \frac{1}{\sqrt{3}} \left\{ |\psi(\lambda_1)\rangle \otimes |\psi(\lambda_2, \lambda_3)\rangle + |\psi(\lambda_2)\rangle \otimes |\psi(\lambda_3, \lambda_1)\rangle + |\psi(\lambda_3)\rangle \otimes |\psi(\lambda_1, \lambda_2)\rangle \right\}
\]

\[ = \frac{1}{\sqrt{3!}} \left\{ |\psi(\lambda_1)\rangle \otimes [|\psi(\lambda_2)\rangle \otimes |\psi(\lambda_3)\rangle - |\psi(\lambda_3)\rangle \otimes |\psi(\lambda_2)\rangle] + |\psi(\lambda_2)\rangle \otimes [|\psi(\lambda_3)\rangle \otimes |\psi(\lambda_1)\rangle - |\psi(\lambda_1)\rangle \otimes |\psi(\lambda_3)\rangle] + |\psi(\lambda_3)\rangle \otimes [|\psi(\lambda_1)\rangle \otimes |\psi(\lambda_2)\rangle - |\psi(\lambda_2)\rangle \otimes |\psi(\lambda_1)\rangle] \right\}
\]

\[ = \sqrt{3!} |\psi_1(\lambda_1)\rangle \wedge |\psi_2(\lambda_2)\rangle \wedge |\psi_3(\lambda_3)\rangle , \]

where \( \wedge \) represents the wedge product. Thus the wave function for the three indistinguishable particles obeys Fermi-Dirac statistics. The \( n \)-particle case
follows by induction. QED

For example, consider the case of an ensemble of $2n$ identical non-interacting particles with discrete energy levels $E_1, E_2, \ldots$, satisfying the Fermi-Dirac statistics as above then all occurrences of such a gas would necessarily have a twofold degeneracy in each of the discrete energy levels and the lowest energy would be given by

$$E = 2E_1 + 2E_2 + 2E_3 + \ldots + 2E_n.$$  \hspace{1cm} (18)

To conclude this section, note Lemma 1 and Theorem 2 express a Pauli type exclusion principle which follows naturally from the coupling principle, or equivalently the rotational invariance. In the next section it will be shown that if the rotational invariance is removed and replaced with the requirement that indistinguishable states be equally likely, then the system will obey a Bose-Einstein statistic.

4 BOSE-EINSTEIN STATISTICS

In the above discussion rotational invariance has played a key role in formulating a Fermi-Dirac statistic for multi-particle ISC systems as defined by definitions 1, 2 and 3. Indeed, from the perspective of this paper, it would seem to be the underlying cause of the Pauli exclusion principle. It now remains to investigate the statistics of multiparticle systems when this condition is relaxed. As noted previously the rotational invariance implies that ISC particles can be written in the form of a singlet state, either proper or improper. Moreover, the definition of indistinguishability (cf remark to Theorem 2) means that a uniform probability law is assigned to each orthonormal basis. Physically, this means that there is no bias in favor of any of the components of the permutable states. For example, if

$$|\psi(\lambda_1, \lambda_2)\rangle = a |\lambda_1\rangle \otimes |\lambda_2\rangle + b |\lambda_2\rangle \otimes |\lambda_1\rangle$$  \hspace{1cm} (19)

is permutable, then $|a|^2 = |b|^2$; otherwise if $|a| > |b|$ (respectively $|b| > |a|$) there would be a bias in favor of the state associated with $a$ (respectively $b$) which, together with the law of large numbers, could then be used to partially distinguish the states.
Theorem 3 Permutable states for a system of \( n \) non-interacting particles, defined on the space \( S_1 \otimes \ldots \otimes S_n \), obey either the Fermi-Dirac or the Bose-Einstein statistic.

Proof: Let

\[
\sigma |\psi(\lambda_1, \ldots, \lambda_n)\rangle = \sum_{i=1}^{n!} c_i |\psi(\sigma_i \lambda_1) \otimes \ldots \otimes |\psi(\sigma_i \lambda_n)\rangle
\]

where \( \sigma_i \) represents a permutation of the particles in the states \( \lambda_1, \ldots, \lambda_n \).

We now claim that if the system of indistinguishable particles are not in the Fermi-Dirac state then

\[
\sigma(\psi(\lambda_1, \ldots, \lambda_n)) = \frac{1}{\sqrt{n!}} \sum_{i=1}^{n!} |\psi(\sigma_i \lambda_1) \otimes \ldots \otimes |\psi(\sigma_i \lambda_n)\rangle .
\]

First note that if \( c_i = \frac{1}{\sqrt{n!}} \) and \( c_{i+1} = -\frac{1}{\sqrt{n!}} \) for each \( i \), then Fermi-Dirac statistics results. Hence, assume that there is not an exact pairing and that \( c_1 = \ldots c_i = \frac{1}{\sqrt{n!}} \) where either \( i > \frac{n!}{2} \) or \( i < \frac{n!}{2} \) and \( c_{i+1} = \ldots = c_{n!} = -\frac{1}{\sqrt{n!}} \). Then taking \( \lambda_1 = \lambda_2 \), many of the terms on the right hand side (in fact \( \min\{2i, 2(n! - i)\} \) terms) will cancel, leaving only the excess unpaired positive (negative) terms. If the remaining number of terms in the expansion is less than \( n! \) then \( |\psi(\lambda_1, \ldots, \lambda_n)\rangle \) is not invariant under the complete set of permutations, which is a contradiction. It follows that the number of terms must be \( n! \) and nothing vanishes. Hence \( |\psi(\lambda_1, \ldots, \lambda_n)\rangle \) exhibits Bose-Einstein statistics. The result follows. QED

It might be instructive to apply the above theorem to a three particle wave function that is not of the above type. Consider:

\[
|\psi(\lambda_1, \lambda_2, \lambda_3)\rangle = \frac{1}{\sqrt{3!}} \{|\psi(\lambda_1)\rangle \otimes |\psi(\lambda_2)\rangle \otimes |\psi(\lambda_3)\rangle + |\psi(\lambda_3)\rangle \otimes |\psi(\lambda_2)\rangle \rangle + |\psi(\lambda_2)\rangle \otimes |\psi(\lambda_3)\rangle \rangle + |\psi(\lambda_1)\rangle \rangle |\psi(\lambda_3)\rangle \rangle + |\psi(\lambda_1)\rangle \rangle |\psi(\lambda_2)\rangle \rangle + |\psi(\lambda_2)\rangle \rangle |\psi(\lambda_1)\rangle \rangle - |\psi(\lambda_2)\rangle \rangle |\psi(\lambda_1)\rangle \rangle\}
\]

On putting \( \lambda_1 = \lambda_2 \),

\[
|\psi(\lambda_1, \lambda_2, \lambda_3)\rangle = \frac{1}{\sqrt{3!}} \{|\psi(\lambda_1)\rangle \otimes |\psi(\lambda_2)\rangle \otimes |\psi(\lambda_3)\rangle + |\psi(\lambda_3)\rangle \otimes |\psi(\lambda_2)\rangle \rangle + |\psi(\lambda_2)\rangle \otimes |\psi(\lambda_3)\rangle \rangle + |\psi(\lambda_1)\rangle \rangle |\psi(\lambda_3)\rangle \rangle + |\psi(\lambda_1)\rangle \rangle |\psi(\lambda_2)\rangle \rangle + |\psi(\lambda_2)\rangle \rangle |\psi(\lambda_1)\rangle \rangle\}
\]
which is not invariant under permutations.

**Corollary 1** Let

$$|\psi(\lambda_1, \ldots, \lambda_n)\rangle = \frac{n!}{\prod_{i=1}^{n} c_i} |\psi(\sigma\lambda_1)\rangle \otimes \cdots \otimes |\psi(\sigma\lambda_n)\rangle,$$  \hspace{1cm} (22)

where $c_i$ are independent of $\lambda_i$, represent an indistinguishable $n$-particle system defined on the space $S_1 \otimes \ldots \otimes S_n$, such that no two states are ISC, then this system of particles can be represented by the Bose-Einstein statistics.

**Proof:** Normalizing the wave function and using indistinguishability gives $c_i^2 = c_j^2$, for each $i$ and $j$. If $c_i = -c_{i+1}$ then each q-orbital would be a spin-singlet state. But this is not so. Hence $c_i = c_j$ by the previous lemma and the result follows. QED

Denote the set of permutations that leave invariant the Bose-Einstein and Fermi-Dirac statistics by $s_n$ and $a_n$, respectively. It follows that certain types of mixed statistics can be now described. For example, the 2-electrons of a helium atom considered together with the 3-electrons in a lithium atom obey $a_2 \otimes a_3$ statistics, while the 5 electrons in the boron atom obey $a_5$ Fermi-Dirac statistics. The electrons in three different helium atoms obey $a_2 \otimes a_2 \otimes a_2$ if the helium atoms are considered distinguishable and $s_3 \circ (a_2 \otimes a_2 \otimes a_2)$ if the atoms are indistinguishable. Finally, if we consider collectively the $n$ distinct electrons in $n$ indistinguishable hydrogen atoms, then these $n$ electrons can be described with $s_n$ Bose-Einstein statistics.

5  **A CONTRAST WITH PAULI’S APPROACH**

It now remains to discuss the above mathematical results from the perspective of Pauli’s famous paper on spin-statistics [11] and in the overall context of the experimental evidence (discussed next section).

In Pauli’s paper the distinction between the two statistics is made by distinguishing between operators obeying commutator rules and those obeying anticommutator rules. Moreover, by defining the Lie algebra for spin
by $[L_i, L_j] = i \epsilon_{ijk} L_k$ fermions are identified with particles of half-integral spin. Microcausality then guarantees that particles of integral spin cannot be quantized as fermions and vice-versa. In contrast, the model presented in this paper suggests that rotational invariance, and not spin value, underlies Fermi-Dirac statistics.

Can the two points of view be reconciled? The answer is yes, provided we agree to slightly modify the Lie algebra by introducing a scaling parameter. Specifically, re-define the spin angular momentum operator by $S_i = n L_i$, where $n$ is an integer, and define the subsequent Lie Algebra by $[S_i, S_j] = i n \epsilon_{ijk} S_k$. Note immediately that the inclusion of the integer $n$ permits particles of integer spin as well as particles of 1/2-integer spin to be subjected to anticommutator relations and to obey Fermi-Dirac statistics. Specifically, when $n=1$ we obtain the usual relationship for spin 1/2 particles, whereas for $n=2$ we obtain the usual properties for a spin 1 photon and for $n=4$, the properties of the spin 2 gravitons.

Now, consider a two particle system $|\psi(\lambda_1)\rangle \otimes |\psi(\lambda_2)\rangle \in S_1 \otimes S_2$ where $S_1 = \mathcal{L}^2(\mathbb{R}^3) \otimes H_1$ and $S_2 = \mathcal{L}^2(\mathbb{R}^3) \otimes H_2$ respectively. Note that each ket $|\psi(\lambda)\rangle \in S$ can be written as $|\psi(q_1)\rangle \otimes s$, where $s$ is a spinor (page 8). Also, let $\vec{S}_1 = (S_i(q_1), S_j(q_1), S_k(q_1))$ and $\vec{S}_2 = (S_i(q_2), S_j(q_2), S_k(q_2))$ be spin operators defined on the Hilbert spaces $H_1$ and $H_2$ respectively. We have already seen (Cor. 1) that Bose-Einstein statistics follow when NO two states are ISC. If we further assume that non-ISC states are statistically independent of each other, then it would seem that their corresponding spin operators are best represented by the operators $S_i(q_1) \otimes I_2$ and $I_1 \otimes S_i(q_2)$, where $I_i$ represents an identity operator. It follows, trivially, that $[S_i(q_1) \otimes I_2, I_1 \otimes S_j(q_2)] = 0$, which means that in the case of Bose-Einstein statistics, spin operators must commute. On the other hand, in contrast to the triplet state, the singlet state defines a rotationally invariant state and obeys a Fermi-Dirac statistic. Once again, let $|\psi(\lambda_1, \lambda_2)\rangle \in S_1 \otimes S_2$ represent the spin-singlet state of two particles. Note that the perfect correlations between them allows us to put $H_1 = H_2 = H$ and to identify $\vec{S}_1$ and $\vec{S}_2$ as follows: Let $s_1(\theta)$ and $s_2(\theta)$ represent the spin states for particles 1 and 2 respectively, then for an arbitrary angle $\theta$ there exists a unit vector $\vec{n}(\theta)$ such that $\vec{S}_1.\vec{n}(\theta)(s_1(\theta)) = \pm s_1(\theta)$ if and only if $\vec{S}_2.\vec{n}(\theta)(s_2(\theta)) = \mp s_2(\theta)$. This relationship allows us to identify $s_2(\theta)$ with the orthogonal complement.
\( s^{-}_1(\theta) \) of \( s_1(\theta) \) and to put \( \vec{s}_1 = \vec{s}_2 \). Hence,

\[
\begin{align*}
\{S_i(q_1), S_j(q_2)\}_s &= \{S_i(q_1), S_j(q_1)\}_s \\
&= i\epsilon_{ijk} S_k(q_1)_s.
\end{align*}
\]

Consequently, Fermi-Dirac statistics imply spin operators must anticommutate and non-local events in the form of spin-singlet states need to be quantized according to the anticommutator rule. Moreover, since \( S_i(q_1)S_j(q_2) \neq 0 \) and \( S_i(q_1)S_j(q_2) + S_j(q_2)S_i(q_1) = 0 \), and the above identification is only valid for singlet states, it follows that bosons can never be fermions and fermions can never be bosons. Also as a special case, fields quantized by the anticommutator rule cannot be quantized with commutators.

In conclusion, note that Bose-Einstein statistics not only presuppose local and independent probability events, but also commuting spin operators. In contrast, Fermi-Dirac statistics presuppose singlet-states which define non-local interactions by way of perfect correlations and also imply anticommutating spin operators. Moreover, since bosons can never be fermions and vice-versa, it follows from the above that particles cannot be simultaneously in non-local and local states, cannot be simultaneously subjected to the rules of conditional and independent probability, cannot be simultaneously rotationally invariant and non-invariant. However, once a measurement is performed on the singlet state then the perfect correlation can be broken, and as a result the Fermi-Dirac state can be changed into a Bose-Einstein state; the singlet state can become two independent states. Finally, note that spin value is no longer the essential characteristic of the spin-statistics theorem.

6 CLEBSCH-GORDAN COEFFICIENTS

In this section we calculate Clebsch-Gordan coefficients for pairs of photons and pairs of deuterons to further justify the spin-statistics theorem, as presented in this paper.

Consider two photons. Let \( s = s_1 + s_2 \) represent their joint spin values, and denote their joint state by \( |llsm\rangle \). Then three possible states emerge
for $s = 2$:

\[ |2, 2\rangle = |1, 1\rangle \] (23)

\[ |2, 0\rangle = \langle 1, -1|2, 0\rangle |1, -1\rangle + \langle -1, 1|2, 0\rangle |1, 1\rangle \] (24)

\[ |2, -2\rangle = |1, -1\rangle \] (25)

which defines the triplet state. Also for $s = 0$ we obtain

\[ |0, 0\rangle = \langle 1, -1|0, 0\rangle |1, -1\rangle + \langle -1, 1|0, 0\rangle |1, 1\rangle \] (26)

which defines the singlet state. Note that the state represented by equation (24) can be calculated directly by means of C-G coefficients, by recalling that $S_i = nL_i$ and by observing that for $n = 2$

\[ \langle s, m|S^\pm S^\mp |s, m\rangle = 4 \langle s, m|L^\mp L^\pm |s, m\rangle , \] (27)

from which it follows that

\[ S^\pm |s, m\rangle = 2\hbar [(s \mp m)(s \pm m + 1)]^{1/2} |s, m \pm 2\rangle . \] (28)

In particular, if we set $S^- = S_1^- + S_2^-$ then

\[ 2L^- |2, 2\rangle = 2\hbar [(2 + 2)(2 - 2 + 1)]^{1/2} |2, 0\rangle \] (29)

and

\[ |2, 0\rangle = \frac{1}{\sqrt{2}} |1, 1\rangle + \frac{1}{\sqrt{2}} |1, -1\rangle . \] (30)

Also if we assume orthogonality of the different states then $\langle 2, 0|0, 0\rangle = 0$ implies

\[ |0, 0\rangle = \frac{1}{\sqrt{2}} |1, -1\rangle - \frac{1}{\sqrt{2}} |1, 1\rangle . \] (31)

The deuteron is likewise a spin-1 particle. However in this case, the spin 0 case can be observed. Moreover, a calculation of the C-G coefficients for a pair of deuterons says a lot about the probability weightings associated
with the $|1\rangle$, $|0\rangle$, $|-1\rangle$ states of an individual deuteron. Direct calculation gives:

\begin{align*}
|2, 2\rangle &= |1, 1\rangle \\
|2, 1\rangle &= \frac{1}{\sqrt{2}} |1, 0\rangle + \frac{1}{\sqrt{2}} |0, 1\rangle \\
|2, 0\rangle &= \sqrt{\frac{2}{3}} |0, 0\rangle + \frac{1}{\sqrt{6}} |1, -1\rangle + \frac{1}{\sqrt{6}} |-1, 1\rangle \\
|2, -1\rangle &= \frac{1}{\sqrt{2}} |-1, 0\rangle + \frac{1}{\sqrt{2}} |0, -1\rangle \\
|2, -2\rangle &= |-1, -1\rangle.
\end{align*}

On the other hand, the probabilities associated with the C-G coefficients of the states can be calculated directly from conditional probability theory, \footnote{Recall that for two events $A$ and $B$ defined on a finite sample space $S$, the conditional probability of $A$ given $B$ is denoted by $P(A|B)$ and $P(A|B) = P(A \cap B)/P(B)$ provided $P(B) \neq 0$.} provided the spectral distribution of an individual deuteron has a probability distribution of $1/4$, $1/2$, $1/4$ and not $1/3$, $1/3$, $1/3$, which is the current belief.

In particular, let $M_i$ where $i = 1, 2$ be a random variable associated with the spin of two independent deuterons such that

\begin{equation}
P(M_i = 1) = P(M_i = 0) = P(M_i = -1) = \frac{1}{3}.
\end{equation}

Let $M = M_1 + M_2$ be the sum of the spins. Note that $M$ too is a random variable with values $2, 1, 0, -1, -2$. Then the conditional distribution for the state $|2, 0\rangle$ associated with the two independent deuterons gives

\begin{align*}
P(M_1 = 0, M_2 = 0|M = 0) &= P(M_1 = 1, M_2 = -1|M = 0) \\
&= P(M_1 = -1, M_2 = 1|M = 0) = \frac{1}{3}.
\end{align*}

However, this distribution is clearly different from the C-G calculations which gives $(|0, 0|2, 0\rangle)^2 = \frac{2}{3}$, $(|1, -1|2, 0\rangle)^2 = \frac{1}{6}$, $(|-1, 1|2, 0\rangle)^2 = \frac{1}{6}$. On
the other hand, if
\[ P(M_i = 1) = P(M_i = -1) = \frac{1}{4} \quad P(M_i = 0) = \frac{1}{2} \]  \quad (40)
as suggested by Theorem 2, then direct calculation using conditional probability, shows that
\[ P(M_1 = 0, M_2 = 0 | M = 0) = \frac{2}{3} \]  \quad (41)
and
\[ P(M_1 = 1, M_2 = -1 | M = 0) = P(M_1 = -1, M_2 = 1 | M = 0) = \frac{1}{6}, \]  \quad (42)
which coincides with the C-G calculation.

7 EXPERIMENTAL EVIDENCE

The experimental justification for accepting the new form of the spin-statistics theorem would appear to come from a wide range of physical phenomena. First, note that the existence of photons in the spin-singlet state seems to support the above formulation. Secondly, we will argue that the new approach offers a more unified and coherent explanation of the phenomenon of paramagnetism. Thirdly, the existence of Cooper pairs in superconductivity can be explained as a specific instance of ISC particles. Fourthly, we discuss baryonic structure from the new perspective. Finally, in keeping with the tradition of theoretical physics, a prediction will be made about the probability distribution for the spin decomposition of a beam of ionized deuterons, a prediction which will distinguish it from the current theory.

(1) Rotational invariance demands the wave function for spin-singlet-state photons to be of the form \( |\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle |-\rangle - |-=\rangle |+\rangle) \). Since this is an antisymmetric state then by definition it obeys a Fermi-Dirac type statistic for the Hilbert space under discussion. Moreover, it has already been pointed out in Section 5 that singlet states, and not spin value, are essential to defining Fermi-Dirac statistics. Spin-singlet-state photons were
at the heart of the Aspect experiment [1] and hence their existence has already been verified. The exclusion principle is then a tautology in the sense that while photons are in a spin-singlet state then both of them cannot be in the same state. Note, however, that the fermionic state of photons can easily be destroyed by experiment and forced into a Bose-Einstein state. It is a trite (but nevertheless valid) application of the exclusion principle, as formulated in this paper.

(2) The theory of paramagnetism yields two different equations for the magnetic susceptibility, one given by the classical Langevin (Curie) function which makes no reference to the Pauli exclusion principle and the other which is derived as a direct application of the exclusion principle. It would appear that our formulation of the exclusion principle gives an equally apt understanding of the phenomenon and would appear to further clarify Pauli’s explanation, by focusing on the unique role of the non-spin-singlet states. Specifically, when the magnetic field is turned on, the spin up component of the spin-singlet state has its energy shifted down by $\mu B$ while the spin down component has its energy shifted up by $\mu B$ with the spins being aligned into parallel and anti-parallel states, resulting in a net contribution to the magnetic field of 0. Hence, the paired electrons contribute nothing to the magnetic susceptibility. The remaining unpaired electrons act in such a way that there is an excess of electrons in the spin up state over the spin down state, in order to maintain the common electrochemical potential. Specifically, if we let $g(\epsilon)$ be the electron density of available states per unit energy range then the total excess energy is given by $g(\epsilon_F)\mu B$, provided $\mu B << \epsilon_F$, which is Pauli’s result for paramagnetism. It should also be pointed out that from the perspective of Pauli’s version of the spin-statistics theorem, half-integral-spin particles such as electrons or gaseous-nitric-oxide (NO) molecules remain as fermions regardless of thermodynamic considerations or of the state they occupy. However, it is generally taken for granted [5] that as $kT >> \epsilon_f$ the Pauli principle no longer applies and the magnetic susceptibility is in this case best estimated by using the Boltzmann statistics. From the perspective of this article, this gives rise to the ambivalent situation of referring to particles as fermions, although they are no longer subjected to the Pauli exclusion principle. With our approach, this ambiguity is removed and a more natural and coherent explanation of the
transition from Fermi-Dirac to Boltzmann statistics is forthcoming. Essentially, the Boltzmann statistics emerge when the spin coupling which seems to be the underlying cause of the Fermi-Dirac statistics is first broken and the particles then move apart to become distinguishable and statistically independent. This breaking of the coupling occurs naturally when the temperature is raised, and they become distinguishable and independent when the distance separating the particles becomes large enough to overcome interactions between the particles. As a result, the particles obey Boltzmann statistics and “the Curie law applies to paramagnetic atoms in a low density gas, just as to well separated ions in a solid...”[5].

Admittedly, other viewpoints may be adequate and someone may believe that fermions remain so, even when the statistical constraints have been removed. However, the purpose of the above description is to show that an alternative consistent position, compatible with the claims of the paper, can be given.

(3) The existence of Cooper pairs as spin-singlet states in the theory of superconductors is another instance of the coupling principle at work. Moreover, the fact that $2n$ superconducting electrons exhibit the statistics of $n$ boson pairs and not the usual $a_{2n}$ Fermi-Dirac statistics [5], normally associated with the exclusion principle, again suggests that the current definition of bosons and fermions in terms of quantum number is inadequate. In contrast, this paper classifies particles into coupled or decoupled particles and then permits various statistics to emerge in accordance with the degree of indistinguishability that is imposed on the system. When complete indistinguishability is imposed on the system, then Fermi-Dirac or Bose-Einstein statistics will ensue according as to whether the system permits coupled (Theorem 2) or only decoupled particles (Cor 1), respectively. On the other hand, if complete indistinguishability is relaxed in favor of some type of partial indistinguishability (as with Cooper pairs), we obtain different types of mixed statistics. For example, if we denote by $s_n$ and $a_n$ the set of permutations that leave invariant the Bose-Einstein and Fermi-Dirac statistics, then as previously pointed out in Section 4, the 2-electrons in a helium atom taken together with the 3-electrons of a lithium atom obey $a_2 \otimes a_3$ statistics, in contrast to the 5 electrons in the boron atom that obey $a_5$ Fermi-Dirac statistics. Also, if we consider collectively the $n$ distinct elec-
trons in \( n \) indistinguishable hydrogen atoms, then these \( n \) electrons can be described with \( s_n \) Bose-Einstein statistics, since there is no electron pairing.

(4) Spin \( \frac{3}{2} \) baryons may be viewed as excited states of spin \( \frac{1}{2} \) baryons. In particular, from the perspective of the new approach, it is impossible by Theorem 1 that a spin \( \frac{3}{2} \) baryon be composed of three ISC quarks. It could be argued that it contains a pair of ISC quarks in an improper singlet state (page 7), with the third quark being uncorrelated with these two. However, unless there is experimental evidence to suggest otherwise, this seems unlikely. The other alternative is to view a spin \( \frac{3}{2} \) baryon as composed of three quarks with uncorrelated spin states (statistically independent), and to view the spin \( \frac{1}{2} \) baryon as composed of a pair of quarks in a singlet state. Moreover, the need of color to explain the structure of \( \Delta^{++} \) and \( \Omega^- \) particles now becomes both unnecessary and inadequate[10]. The coupling principle forbids the three quarks composing both the \( \Delta^{++} \) and \( \Omega^- \) particles to exist as ISC particles, but rather suggests that they are statistically independent. Color fails to fully address this issue. Of course, this does not preclude the use of color to give “colorless” baryons. [12]

(5) It is well known that the deuteron ion is in a spin-triplet state. Denote the possible observed spin values \( X \) by +1, 0, -1 respectively. Conventional quantum mechanics predicts that \( P(X = +1) = P(X = 0) = P(X = -1) = \frac{1}{3} \). On the other hand, if we assume that the absence of the spin-singlet state for deuteron ions means that the Bose-Einstein triplet state is composed of two independently distributed spin \( \frac{1}{2} \) particles then the model proposed in this paper predicts \( P(X = +1) = P(X = -1) = \frac{1}{4} \) and \( P(X = 0) = \frac{1}{2} \), using an argument based on Clebsch-Gordan coefficients [9]. This should be testable by passing a beam of neutral deuteron atoms (not molecules) through a Stern-Garlach apparatus.

Remark: Strictly speaking, the failure of particles to form a spin-singlet state does not necessarily mean that the subsequent spin values of the triplet state are governed by the laws of independent probability. It may mean that there is some type of dependent but non-deterministic relationship between the particles. This further highlights the importance of performing an experiment like that described above. If decoupled spin states imply statistical independence, then classification procedures become very simple.
On the other hand, if statistical independence fails to be observed then the Bose-Einstein type statistic would have to be further sub-classified.

8 CONCLUSION

In this paper a “spin-coupling principle” is derived which suggests a statistical classification of particles in terms of ISC states (spin-entangled pairs) and non ISC states. These ISC states appear to unify our understanding of atomic orbitals, covalent bonding, paramagnetism, superconductivity, baryonic structure and so on. In summary, subatomic particles seem to form entangled pairs whenever they are free to do so and there appears to be a universal principle at work, although the mechanism behind this coupling would need to be investigated further.

Secondly, in contrast to the current paper, Pauli’s version of the spin-statistics theorem imposes many other conditions on his particle system including Lorentz invariance, locality (“measurements at two space points with a space-like distance can never disturb each other”[11], charge and energy densities. However, the imposition of such extra conditions would seem to be unnecessary in the light of our current understanding of entanglement.

Finally, note that a connection between Bell’s inequality[2] and rotational invariance has been established.

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