CONVERGENCE OF SPECTRAL DECOMPOSITIONS
OF HILL OPERATORS WITH TRIGONOMETRIC
POLYNOMIAL POTENTIALS

PLAMEN DJAKOV AND BORIS MITYAGIN

Abstract. We consider the Hill operator
\[ Ly = -y'' + v(x)y, \quad 0 \leq x \leq \pi, \]
subject to periodic or antiperiodic boundary conditions, with potentials \( v \) which are trigonometric polynomials with nonzero coefficients, of the form
(i) \( ae^{-2ix} + be^{2ix}, \)
(ii) \( ae^{-2ix} + Be^{4ix}, \)
(iii) \( ae^{-2ix} + Ae^{-4ix} + be^{2ix} + Be^{4ix}. \)

Then the system of eigenfunctions and (at most finitely many) associated functions is complete but it is not a basis in \( L^2([0, \pi], \mathbb{C}) \) if \( |a| \neq |b| \) in the case (i), if \( |A| \neq |B| \) and neither \( -b^2/4B \) nor \( -a^2/4A \) is an integer square in the case (iii), and it is never a basis in the case (ii) subject to periodic boundary conditions.

Keywords: Hill operators, Riesz bases, trigonometric polynomial potentials

2000 Mathematics Subject Classification: 47E05, 34L40, 34L10.

1. Introduction

Convergence of spectral decompositions of ordinary differential operators with various boundary conditions \( bc \) is a classical area of research and has a long history – see the monographs [26, 21, 11, 17].

In the present paper we consider the Hill operators \( L = L_{bc}(v) \) with smooth \( \pi \)-periodic potentials \( v \)
\[ Ly = -y'' + v(x)y, \quad 0 \leq x \leq \pi, \tag{1.1} \]
subject to periodic (\( Per^+ \)) or antiperiodic (\( Per^- \)) boundary conditions:
\[ Per^\pm : \quad y(\pi) = \pm y(0), \quad y'(\pi) = \pm y'(0). \]

See basics and details in [13].

---

B. Mityagin acknowledges the hospitality of Sabanci University, May–June, 2009.
Of course, if $v$ is real-valued, then $L_{\text{Per}^\pm}(v)$ is a self-adjoint operator with a discrete spectrum. The system of its eigenfunctions

$$\Phi = \{\varphi_k : L\varphi_k = \lambda_k \varphi_k, \|\varphi_k\| = 1\}$$

is orthonormal, and the spectral decompositions

$$f = \sum_k \langle f, \varphi_k \rangle \varphi_k$$

converge (unconditionally) in $L^2([0,\pi])$ for every $f \in L^2([0,\pi])$.

If $v$ is a complex-valued potential the picture becomes more complicated. If the boundary conditions are strictly regular then the system of eigenfunctions and associated functions (SEAF) is a Riesz basis in $L^2([0,\pi])$ as it has been shown in \cite{7, 8, 10, 18}; see more details and history in \cite{19, 20}. However, $\text{Per}^+, \text{Per}^-$ are regular but not strictly regular boundary conditions. In this case properly chosen two-dimensional block-decompositions do converge as it has been shown by A. Shkalikov \cite{23, 24, 25} (even in a more general context of ordinary differential operators of higher order). For certain classes of potentials, there have been given sufficient and necessary conditions on whether blocks could be split into (one-dimensional) eigenfunction decompositions \cite{16, 2, 15, 27}. Maybe, in 2006 A. Makin \cite{14} and the authors \cite{3, Thm 71} gave first examples of such potentials that SEAF for periodic or antiperiodic boundary conditions is NOT a basis in $L^2([0,\pi])$ even though all but finitely many eigenvalues are simple. The existence of such potentials indirectly follows from the recent results in \cite{9} as well.

We will extend many constructions and results of SEAF divergence to 1D Dirac operators in an oncoming paper \cite{6}.

In this paper we analyze low degree trigonometric polynomials and show that the spectral decompositions of $L_{\text{Per}^\pm}$ diverge if we exclude some exceptional values of coefficients of these polynomials.

For example, if

$$v(x) = ae^{-2ix} + be^{2ix}, \quad a, b \in \mathbb{C} \setminus \{0\},$$

the SEAF decompositions converge if and only if $|a| = |b|$.

In Section 2 we give the necessary preliminaries and prove a general criterion (in terms of the Fourier coefficients of the potential $v$, see Theorem 1) which says whether the SEAF is (or is not) a basis in $L^2([0,\pi])$. Our constructions from \cite{14} are used in an essential way when analyzing SEAF related to trigonometric potentials in Sections 3–5.
2. Preliminary results

It is well known that the spectra of the operators $L_{Per}^\pm$ are discrete, and the following localization formulas hold (see, for example, [4, Prop 1]):

\[(2.1) \quad Sp(L_{Per}^\pm) \subset \Pi_N \cup \bigcup_{n>N, n \in \Gamma_\pm} D_n, \quad \#(Sp(L_{Per}^\pm) \cap D_n) = 2,\]

where $D_n = n^2 + D$, $D = \{z : |z| < 1\}$, $\Gamma_+ = 2\mathbb{N}$, $\Gamma_- = 2\mathbb{N} - 1$,

\[(2.2) \quad \Pi_N = \{z = x + iy \in \mathbb{C} : |x| < (N + 1/2)^2, |y| < N\}, \quad N = N(v).\]

In either case the spectral block decompositions

\[(2.3) \quad g = S_N g + \sum_{n>N, n \in \Gamma_\pm} P_n g, \quad \forall g \in L^2([0, \pi]),\]

where

\[(2.4) \quad S_N = \frac{1}{2\pi i} \int_{\partial \Pi_N} (z - L_{Per}^\pm)^{-1} dz, \quad P_n = \frac{1}{2\pi i} \int_{|z-n^2|=1} (z - L_{Per}^\pm)^{-1} dz,\]

converge unconditionally in $L^2([0, \pi])$. This is true even if the $\pi$-periodic potential $v$ is singular, i.e., $v \in H_{loc}^{-1}(\mathbb{R})$, as A. Savchuk and A. Shkalikov showed [22]. An alternative proof is given in [5].

We are going to provide in Theorem 1 below sufficient conditions which guarantee for large enough $n$ that each disc $D_n$ contains exactly two simple eigenvalues, and a criterion when the two-dimensional spectral blocks in (2.3) could be split into one-dimensional spectral blocks so that to get an unconditional basis in $L^2([0, \pi])$.

We shall use the following notations (compare with [4]). For each $n \in \mathbb{N}$ a walk $x$ from $-n$ to $n$ or from $n$ to $-n$ is defined through its sequence of steps

\[(2.5) \quad x = (x(t))_{t=1}^{\nu+1}, \quad 1 \leq \nu = \nu(x) < \infty,\]

where, respectively,

\[(2.6) \quad \sum_{t=1}^{\nu+1} x(t) = 2n \quad \text{or} \quad \sum_{t=1}^{\nu+1} x(t) = -2n.\]

A walk $x$ is called admissible if its vertices $j(t) = j(t, x)$ given, respectively, by

\[(2.7) \quad j(0) = -n \quad \text{or} \quad j(0) = +n\]
and
\[(2.8) \quad j(t) = -n + \sum_{t=1}^{t} x(i) \quad \text{or} \quad j(t) = n + \sum_{t=1}^{t} x(i), \quad 1 \leq t \leq \nu + 1,\]
satisfy
\[(2.9) \quad j(t) \neq \pm n \quad \text{for} \quad 1 \leq t \leq \nu.\]

Let \(X_n\) and \(Y_n\) be, respectively, the set of all admissible walks from \(-n\) to \(n\) and the set of all admissible walks from \(n\) to \(-n\). For each walk \(x \in X_n\) or \(x \in Y_n\) we set
\[(2.10) \quad h(x; z) = \frac{\prod_{t=1}^{k+1} V(x(t))}{\prod_{t=1}^{k}[n^2 - j(t)^2 + z]}\]
where \(V(m), \ m \in 2\mathbb{Z}\) are the Fourier coefficients of the potential \(v(x)\) with respect to the system \(e^{imx}, \ m \in 2\mathbb{Z}\). We set also
\[(2.11) \quad B^+(n, z) = \sum_{x \in X_n} h(x, z), \quad B^-(n, z) = \sum_{x \in Y_n} h(x, z).\]

**Theorem 1.** Suppose \(v \in L^2([0, \pi])\). If
\[(2.12) \quad B^+(n, 0) \neq 0, \quad B^-(n, 0) \neq 0\]
and
\[(2.13) \quad \exists c > 0 : \quad c^{-1}|B^+(n, 0)| \leq |B^+(n, z)| \leq c|B^+(n, 0)| \quad \forall z \in D\]
for all sufficiently large even \(n\) (if \(bc = \text{Per}^+\)) or odd \(n\) (if \(bc = \text{Per}^-\)), then

(a) there is \(N = N(v)\) such that for \(n > N\) the operator \(L_{\text{Per}^\pm}(v)\) has exactly two simple periodic (for even \(n\)) or antiperiodic (for odd \(n\)) eigenvalues in the disc \(D_n = n^2 + D\);

(b) a system of normalized eigenfunctions and associated functions of \(L_{\text{Per}^\pm}(v)\) is a Riesz basis in \(L^2([0, \pi])\) if and only if
\[(2.14) \quad 0 < \alpha := \inf_{n>N} \frac{|B^-(n, 0)|}{|B^+(n, 0)|} \quad \text{and} \quad \beta := \sup_{n>N} \frac{|B^-(n, 0)|}{|B^+(n, 0)|} < \infty,\]
where we take \(\inf\) and \(\sup\) over even \(n\) if \(bc = \text{Per}^+\) and over odd \(n\) if \(bc = \text{Per}^-\).

**Remarks.** 1) Notice, that by (a) the SEAF of \(L_{\text{Per}^\pm}(v)\) has at most finitely many associated functions.

2) To avoid any confusion, let us emphasize that in Theorem 1 are stacked together two independent theorems: one for the case of periodic boundary conditions \(\text{Per}^+\) (where we consider only even \(n\)), and another one for the case of antiperiodic boundary conditions \(\text{Per}^-\) (where we consider only odd \(n\)).
Proof. By the spectra localization formulas (2.1) the operator $L_{Per}^{\pm}(w)$ has, for each $n > N,$ two periodic (for even $n$) or antiperiodic (for odd $n$) eigenvalues in the disc $n^2 + D$ (counted with multiplicity). Moreover, by [3] (see Lemma 21 and Section 2.2, in particular, formula (2.23) and the three lines which follow), the number $\lambda = n^2 + z,$ $z \in D,$ is an eigenvalue of $L_{Per}^{\pm}(v)$ if and only if $z$ satisfies the basic equation

$$\left(z - a(n, z; v)\right)^2 = B^+(n, z; v)B^-(n, z, v), \quad z \in D,$$

where $a(n, z; v), B^\pm(n, z; v)$ are analytic functions of $z$ and $v$ defined for $|Re z| < n.$ Next we show that for large enough $n$ the equation (2.15) has exactly two roots in $D$ if counted with multiplicity.

In view of [3, Prop 28] we have

$$|a(n, z, v)| \leq \frac{C}{n}, \quad |B^\pm(n, z, v) - V(\pm 2n)| \leq \frac{C}{n} \text{ if } z \in D,$$

where $C = C(\|v\|)$ and $V(\pm 2n)$ are the $\pm 2n$-th Fourier coefficients of the potential $v.$

Consider the family of potentials $w = tv,$ $t \in [0, 1].$ Since $V(\pm 2n) \to 0$ as $n \to \infty,$ the inequalities (2.16) imply

$$\sup_{\|z\| \leq 1} |a(n, z, tv)| \to 0, \quad \sup_{\|z\| \leq 1} |B^\pm(n, z; tv)| \to 0 \quad \text{as } n \to \infty$$

uniformly for $t \in [0, 1].$

Consider the function

$$F_n(z, t) = \left(z - a(n, z; tv)\right)^2 - B^+(n, z; tv)B^-(n, z, tv), \quad t \in [0, 1].$$

In view of (2.17), for large enough $n,$ the function $F_n(z, t)$ does not vanish on the unit circle $\partial D.$ Therefore, the number of zeroes of the equation (2.15) considered with $w = tv$ is given by

$$\mathcal{N}(t) = \frac{1}{2\pi i} \int_{\partial D} \frac{F'_n(\zeta, t)}{F_n(\zeta, t)} d\zeta.$$

Since the function $\mathcal{N}(t), \, t \in [0, 1],$ is continuous and takes integer values, it is a constant, so we have $\mathcal{N}(1) = \mathcal{N}(0).$ On the other hand, for zero potential the basic equation is reduced to $z^2 = 0,$ i.e., $\mathcal{N}(0) = 2.$ Thus, for sufficiently large $n,$ say $n > N_1$ the equation (2.15) has exactly two roots in $D,$ counted with multiplicities.

So, we have proved for $n > N_1$ that $\lambda = n^2 + z,$ $z \in D,$ is a periodic or antiperiodic value of algebraic multiplicity 2 if and only if $z$ is a double root of (2.15). Thus, the number $\lambda = n^2 + z,$ $z \in D,$ is a periodic or antiperiodic value of algebraic multiplicity 2 if and only if
\( z \) satisfies the system of the equation (2.15) and
\[
2(z - a(n, z)) \left(1 - \frac{d}{dz}a(n, z)\right) = \frac{d}{dz} \left(B^+(n, z)B^-(n, z)\right).
\]
Therefore, Part (a) of the theorem will be proved if we show that there are at most finitely many \( n \) such that the system (2.15), (2.18) has a solution \( z \in D \).

If \( z(n) \in D \) is a root of (2.15), then by (2.17)
\[
|z(n)| \leq |a(n, z(n))| + |B^+(n, z(n))B^-(n, z(n))|^{|1/2} \to 0.
\]
Therefore, there is \( \tilde{N}_1 = \tilde{N}_1(v) > N_1 \) such that
\[
|z(n)| \leq 1/2, \quad n > \tilde{N}_1.
\]
Suppose that \( n > \tilde{N}_1 \) and \( z^*_n \in D \) satisfies the system (2.15), (2.18).

By the Cauchy inequality for the first derivative, the first inequality in (2.16) implies
\[
\left|\frac{d}{dz}a(n, z^*_n)\right| \leq 2C/n,
\]
while (2.13) and (2.20) yield
\[
\left|\frac{d}{dz}B^+(n, z^*_n)B^-(n, z^*_n)\right| \leq 2 \sup_{|z| \leq 1} |B^+(n, z)B^-(n, z)| \leq 2c^2 |B^+(n, 0)B^-(n, 0)| \leq 2c^4 |B^+(n, z^*_n)B^-(n, z^*_n)|.
\]
By (2.15), we have \( |z^*_n - a(n, z^*_n)| = |B^+(n, z^*_n)B^-(n, z^*_n)|^{1/2} \). Therefore, by (2.21) and (2.22), the equation (2.18) implies
\[
2|B^+(n, z^*_n)B^-(n, z^*_n)|^{1/2}(1 - 2C/n) \leq 2n^4 |B^+(n, z^*_n)B^-(n, z^*_n)|,
\]
so it follows that
\[
1 - 2C/n \leq c^4 \left|B^+(n, z^*_n)B^-(n, z^*_n)\right|^{1/2}.
\]
By (2.16), the right-hand side of the latter inequality tends to zero, so that inequality fails for large enough \( n \). Hence, increasing if necessary \( N_2 \), we obtain for \( n > N_2 \) that the operator \( L_{Per^\pm}(v) \) has no double periodic or antiperiodic eigenvalues, i.e., (a) holds.

Next we prove part (b) of the theorem. In view of (2.12)–(2.14), for large enough \( n \) the analytic functions \( B'(n, z) \) and \( B^-(n, z) \) do not vanish if \( z \in D \). Therefore, there are appropriate branches of \( \log z \) (which depend on \( n \) and the choice of \( \pm \)) defined on a neighborhood of \( B^\pm(n, D) \). We set
\[
\Log^\pm (B^\pm(n, z)) = \log |B^\pm(n, z)| + i\varphi^\pm_n(z);
\]
then
\begin{equation}
B^\pm(n, z) = |B^\pm(n, z)| e^{i\varphi^\pm_n(z)} \quad \forall z \in D, \ n \geq N_2(v)
\end{equation}
and the square root $\sqrt{B^+(n, z)B^-(n, z)}$ is well defined by
\begin{equation}
\sqrt{B^+(n, z)B^-(n, z)} = |B^+(n, z)B^-(n, z)|^{1/2} e^{i\varphi^+_n(z) + \varphi^-_n(z)}.
\end{equation}

Let us mention that the functions $\varphi^\pm_n$ are uniformly Lipschitz on $\frac{1}{2}D$; more precisely
\begin{equation}
|\varphi^\pm_n(z_1) - \varphi^\pm_n(z_2)| \leq 2c^2 |z_1 - z_2| \quad \text{for } z_1, z_2 \in \frac{1}{2}D.
\end{equation}
Indeed, from (2.13) and the Cauchy inequality for the first derivative it follows, for $|z| < 1/2$, that
\[
\left| \frac{d}{dz} \log^+ (B^\pm(n, z)) \right| = \frac{1}{|B^\pm(n, z)|} \left| \frac{d}{dz} B^\pm(n, z) \right| \\
\leq \frac{c}{|B^\pm(n, 0)|} \cdot 2 \sup_{|z| \leq 1} |B^\pm(n, z)| \leq \frac{c}{|B^\pm(n, 0)|} \cdot 2c |B^\pm(n, 0)| = 2c^2.
\]

Now the basic equation (2.15) splits into the following two equations
\begin{align}
(2.26) & \quad z = \zeta^+_n(z) := a(n, z) + \sqrt{B^+(n, z)B^-(n, z)}, \\
(2.27) & \quad z = \zeta^-_n(z) := a(n, z) - \sqrt{B^+(n, z)B^-(n, z)}.
\end{align}
For large enough $n$, each of the equations (2.26) and (2.27) has exactly one root in the disc $D$. Indeed, in view of (2.16), the Cauchy inequality for the first derivative implies
\[
\sup_{|z| \leq 1/2} |d\zeta^+_n/dz| \to 0 \quad \text{as } n \to \infty.
\]
Therefore, for large enough $n$ each of the functions $\zeta^+_n$ is a contraction on the disc $\frac{1}{2}D$, which implies that each of the equations (2.26) and (2.27) has at most one root in the disc $\frac{1}{2}D$.

On the other hand, by Part (a) and (2.20), for large enough $n$ the basic equation has two simple roots in $\frac{1}{2}D$ and no root on $D \setminus \frac{1}{2}D$, which implies that each of the equations (2.26) and (2.27) has exactly one root in the disc $\frac{1}{2}D$ and no root on $D \setminus \frac{1}{2}D$.

For large enough $n$, let $z_1(n)$ (respectively $z_2(n)$) be the only root of the equation (2.26) (respectively (2.27)) in the unit disc $D$. Let $f = f(n)$ and $g = g(n)$ be corresponding unit eigenvectors of the operator $L = L_{Per^\pm}$, i.e., $\|f(n)\| = \|g(n)\| = 1$ and
\[Lf(n) = (n^2 + z_1(n))f(n), \quad Lg(n) = (n^2 + z_2(n))g(n).\]
Let $P_n$ be the Riesz projections defined by (2.4), and let $P^0_n$ be the Riesz projections associated with the free operator. We have (e.g., see Proposition 11 in [3])

$$\dim P_n = \dim P^0_n = 2, \quad \|P_n - P^0_n\| \leq C/n.$$

Each of the projections $P_n$, $n > N$, could be written as a sum of one-dimensional projections on the subspaces generated by $f(n)$ and $g(n)$ so that

$$P_n = P^1_n + P^2_n, \quad P^1_n P^2_n = P^2_n P^1_n = 0.$$

An elementary calculation shows that

$$\|P^1_n\| = \|P^2_n\| = (1 - |\langle f(n), g(n) \rangle|^2)^{-1/2}.$$

Therefore, the system of normalized eigenfunctions and associated functions will be a Riesz basis if and only if

$$\limsup_{n \to \infty} |\langle f(n), g(n) \rangle| < 1.$$

We set

$$f^0(n) = P^0_n f(n), \quad g^0(n) = P^0_n g(n).$$

From (2.28) it follows

$$\|f(n) - f^0(n)\| = \|(P_n - P^0_n)f(n)\| \leq \|P_n - P^0_n\| \leq C/n$$

and $\|g(n) - g^0(n)\| \leq C/n$, $|\langle f(n) - f^0(n), g(n) - g^0(n) \rangle| \leq C/n^2$. Since

$$\|f(n)\|^2 = \|f^0(n)\|^2 + \|f(n) - f^0(n)\|^2$$

and

$$\langle f(n), g(n) \rangle = \langle f^0(n), g^0(n) \rangle + \langle f(n) - f^0(n), g(n) - g^0(n) \rangle,$$

we get

$$\|f^0(n)\|, \|g^0(n)\| \to 1, \quad \limsup_{n \to \infty} |\langle f(n), g(n) \rangle| = \limsup_{n \to \infty} |\langle f^0(n), g^0(n) \rangle|.$$

Then, by [3, Lemma 21] (see formula (2.4)), $f^0(n)$ is an eigenvector of the matrix

$$\begin{pmatrix}
    a(n, z_1) & B^+(n, z_1) \\
    B^-(n, z_1) & a(n, z_1)
\end{pmatrix}$$

corresponding to its eigenvalue $z_1 = z_1(n)$, i.e.,

$$\begin{pmatrix}
    a(n, z_1) - z_1 & B^+(n, z_1) \\
    B^-(n, z_1) & a(n, z_1) - z_1
\end{pmatrix} f^0(n) = 0.$$

Therefore, $f^0(n)$ is proportional to the vector

$$\left(1, \frac{z_1 - a(n, z_1)}{B^+(n, z_1)} \right)^T.$$

Taking into account (2.23), (2.24) and (2.26) we obtain

$$f^0(n) = \frac{\|f^0(n)\|}{\sqrt{1 + \frac{B^-(n, z_1)}{B^+(n, z_1)}}} \left(\frac{B^-(n, z_1)}{B^+(n, z_1)}\right)^{1/2} \frac{1}{e^{\frac{1}{2} \varphi^+_n(z_1) - \varphi^-_n(z_1)}}.$$
In an analogous way, from (2.23), (2.24) and (2.27) it follows

\begin{equation}
(2.32) \quad g^0(n) = \frac{\|g^0(n)\|}{\sqrt{1 + \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right|}} \left( - \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right|^{1/2} e^{i\psi_n} \right).
\end{equation}

Now, (2.31) and (2.32) imply

\begin{equation}
\langle f^0(n), g^0(n) \rangle = \|f^0(n)\| \|g^0(n)\| \sqrt{\frac{1 - \sqrt{\left| \frac{B^{-}(n,z_1)}{B^{+}(n,z_1)} \right| \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right|}}{1 + \left| \frac{B^{-}(n,z_1)}{B^{+}(n,z_1)} \right| \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right|}} e^{i\psi_n}
\end{equation}

where

\begin{equation}
\psi_n = \frac{1}{2} \left( [\varphi_n^{-}(z_1(n)) - \varphi_n^{+}(z_1(n))] - [\varphi_n^{-}(z_2(n)) - \varphi_n^{+}(z_2(n))] \right).
\end{equation}

In view of (2.19) we have \( z_1(n) \to 0 \) and \( z_2(n) \to 0 \) as \( n \to \infty \), so by (2.25) it follows

\begin{equation}
(2.33) \quad \psi_n \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}

We have

\begin{equation}
|\langle f^0(n), g^0(n) \rangle|^2 = \|f^0(n)\| \|g^0(n)\| \cdot \Pi_n,
\end{equation}

where

\begin{equation}
\Pi_n = \frac{1 + \left| \frac{B^{-}(n,z_1)}{B^{+}(n,z_1)} \right| \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right| - 2 \sqrt{\left| \frac{B^{-}(n,z_1)}{B^{+}(n,z_1)} \right| \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right|}}{\left( 1 + \left| \frac{B^{-}(n,z_1)}{B^{+}(n,z_1)} \right| \right) \left( 1 + \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right| \right)} \cos \psi_n.
\end{equation}

If (2.14) fails, then there is a subsequence \( n_k \to \infty \) such that

\begin{equation}
\left| \frac{B^{-}(n_k, 0)}{B^{+}(n_k, 0)} \right| \to 0 \quad \text{or} \quad \left| \frac{B^{-}(n_k, 0)}{B^{+}(n_k, 0)} \right| \to \infty,
\end{equation}

which implies, in view of (2.13), \( \Pi_{n_k} \to 1 \). Therefore, by (2.30),

\begin{equation}
limit_{n \to \infty} \{ \langle f(n), g(n) \rangle \} = 1,
\end{equation}

i.e., (2.29) fails, so the system of normalized eigenfunctions and associated functions is not a (Riesz) basis.

Suppose (2.13) holds. From (2.33) it follows \( \cos \psi_n > 0 \) for large enough \( n \), so taking into account that \( \|f^0(n)\|, \|g^0(n)\| \leq 1 \), we obtain

\begin{equation}
|\langle f^0(n), g^0(n) \rangle|^2 \leq \Pi_n \leq \frac{1 + \left| \frac{B^{-}(n,z_1)}{B^{+}(n,z_1)} \right| \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right|}{\left( 1 + \left| \frac{B^{-}(n,z_1)}{B^{+}(n,z_1)} \right| \right) \left( 1 + \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right| \right)} \leq \delta < 1,
\end{equation}

where

\begin{equation}
\delta = \frac{1 + \left| \frac{B^{-}(n,z_1)}{B^{+}(n,z_1)} \right| \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right|}{\left( 1 + \left| \frac{B^{-}(n,z_1)}{B^{+}(n,z_1)} \right| \right) \left( 1 + \left| \frac{B^{-}(n,z_2)}{B^{+}(n,z_2)} \right| \right)}.
\end{equation}
where
\[ \delta = \max \left\{ \frac{1 + xy}{(1 + x)(1 + y)} : \frac{\alpha}{c^2} \leq x, y \leq c^2 \beta \right\}. \]

Now (2.30) implies that (2.29) holds, hence the system of normalized eigenfunctions and associated functions is a (Riesz) basis in \( L^2([0, \pi]) \). The proof is complete. \( \square \)

In the next sections we consider the following three families of trigonometric polynomial potentials
\begin{align*}
(2.34) & \quad v(x) = ae^{-2ix} + be^{2ix}, \\
(2.35) & \quad v(x) = ae^{-2ix} + Be^{4ix}, \\
(2.36) & \quad v(x) = ae^{-2ix} + Ae^{-4ix} + be^{2ix} + Be^{2ix}
\end{align*}
and give conditions when the SEAF is a basis, in terms of the coefficients of these polynomials (see, respectively, Sections 3–5).

In all cases we consider in Sections 3–5 a special role is played by forward and backward walks. We say that \( x \) is a forward (respectively, backward) walk if all steps are positive, \( x(t) > 0 \) (respectively, negative, \( x(t) < 0 \)). Let \( X_n^+ \) and \( Y_n^- \) be, respectively, the set of all admissible forward walks and the set of all admissible backward walks.

**Lemma 2.** If \( \xi \in X_n^+ \) or \( \xi \in Y_n^- \), then for large enough \( n \) and \( |z| \leq 1 \)
\[ h(\xi, z) = h(\xi, 0)(1 + \tau_n), \quad |\tau_n| \leq \frac{4 \log n}{n}. \]

**Proof.** By (2.10),
\[ \tau_n = \frac{h(\xi, z)}{h(\xi, 0)} - 1 = \prod_{t=1}^{\nu} \frac{n^2 - j(t)^2}{n^2 - j(t)^2 + z} - 1 = e^{-w_n} - 1, \]
where \( w_n = \sum_{t=1}^{\nu} \log \left( 1 + \frac{z}{n^2 - j(t)^2} \right) \). Since \( \xi \in X_n^+ \) or \( \xi \in Y_n^- \), all vertices \( j(t) = j(t, \xi) \) are distinct, \(-n < j(t) < n\) for \( 1 \leq t \leq \nu \). Therefore, by the inequality \( |\log(1 + \zeta)| \leq \sum_{k=1}^{\infty} |\zeta|^k \leq 2|\zeta| \) for \( |\zeta| \leq 1/2 \), for large enough \( n \) it follows
\[ |w_n| \leq \sum_{t=1}^{\nu} \frac{2|z|}{n^2 - j(t)^2} \leq \sum_{k=1}^{n-1} \frac{2}{n^2 - (-n + 2k)^2} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \leq \frac{2 \log n}{n} \leq \frac{1}{2}. \]

On the other hand, if \( |w| \leq 1/2 \) then \( |e^{-w} - 1| \leq \sum_{k=1}^{\infty} |w|^k \leq 2|w| \), which implies (2.37). \( \square \)
3. Potential \( v = ae^{-2ix} + be^{2ix} \)

We follow the notations and definitions of walks, steps, vertices and functions \( h, B^\pm \) given in (2.5)–(2.11). The Fourier coefficients of the potential \( v = ae^{-2ix} + be^{2ix} \) are

\[
(3.1) \quad V(-2) = a, \quad V(2) = b, \quad V(m) = 0 \text{ for } m \neq \pm 2.
\]

Let us focus on \( B^+(n, z) \). We say that a walk \( x \) is \( v\)-admissible, if \( x \) is admissible and its steps are equal to \( \pm 2 \). If \( x \) has \( p \) steps equal to 2 and \( q \) steps equal to \(-2 \), then

\[
(3.2) \quad 2p - 2q = 2n, \quad \text{so} \quad p = n + q,
\]

and

\[
(3.3) \quad p + q = \nu + 1.
\]

We set

\[
(3.4) \quad X_n(q) = \{ v\text{-admissible } x \in X_n \text{ with } q \text{ steps} = -2 \}.
\]

Notice, that every \( v\)-admissible walk from \(-n\) to \( n\) has vertices only between \(-n\) and \( n\), and we have \( x(1) = x(2) = 2 \). If

\[
(3.5) \quad i = \min \{ t : x(t) \cdot x(t+1) < 0 \},
\]

then

\[
(3.6) \quad x(t) = 2 \quad \text{if} \quad 1 \leq t \leq i, \quad x(i+1) = -2.
\]

We perform a ”surgery” on \( x \) by removing the steps \( x(i) \) and \( x(i+1) \) and constructing a walk \( \xi \in M^+(q-1) \) such that

\[
(3.7) \quad \nu(\xi) = \nu - 2, \quad \nu = \nu(x),
\]

and

\[
(3.8) \quad \xi(t) = \begin{cases} 
  x(t) & \text{for } 1 \leq t \leq i - 1 \\
  x(t+2) & \text{for } i \leq t \leq \nu - 1.
\end{cases}
\]

Then

\[
(3.9) \quad j(t, \xi) = \begin{cases} 
  j(t, x) & \text{for } 1 \leq t \leq i - 1 \\
  j(t+2, x) & \text{for } i \leq t \leq \nu - 2.
\end{cases}
\]

Now we have

\[
(3.10) \quad h(x, z) = \frac{V(x(i))V(x(i+1))}{(n^2 - j(i, x)^2 + z)(n^2 + j(i + 1, x)^2 + z)} \times h(\xi, z).
\]

With \( c = |ab| \) the identity (3.10) implies for \( |z| \leq 1 \)

\[
(3.11) \quad \forall x \in X_n(q) \ \exists \xi \in X_n(q-1) : \ |h(x, z)| \leq \frac{c}{n^2}|h(\xi, z)|.
\]
Repeating the same procedure \( q \) times we come to the inequality
\[
|h(x, z)| \leq \left( \frac{c}{n^2} \right)^q |h(\xi^*, z)|, \quad \exists \xi^* \in X_n(0), \quad |z| \leq 1.
\]
But \( X_n(0) \) has only one element, and its only walk \( \xi^* \) has its steps, \( n \) of them, equal to 2, so
\[
j(t, \xi^*) = -n + 2t, \quad 0 \leq t \leq n.
\]
We evaluate \( h(\xi^*, 0) \) and estimate \( h(\xi^*, z) \) below.

Let us notice that by (3.2)–(3.4)
\[
\#X_n(q) \leq \left( p + q \right)^q = \binom{n + 2q}{q} \leq \begin{cases} 2^{3q} & \text{if } q > n \\ \frac{1}{q!}(3n)^q & \text{if } q \leq n. \end{cases}
\]
Therefore,
\[
\sum_{q \geq 1} \sum_{x \in X_n(q)} |h(x, z)| \leq \sigma_1(n) \cdot |h(\xi^*, z)|,
\]
where
\[
\sigma_1(n) = \sum_{q \geq 1} \binom{n + 2q}{q} \left( \frac{c}{n^2} \right)^q = \sum_{q=1}^{n} + \sum_{n+1}^{\infty} \frac{1}{q!} \left( \frac{3nc}{n^2} \right)^q + \sum_{q=n+1}^{\infty} \left( \frac{8c}{n^2} \right)^q \leq \frac{3c}{n} e^{3c/n} + \left( \frac{8c}{n^2} \right)^n \frac{1}{1 - 8c/n^2} = O(1/n).
\]
Thus, for \( |z| \leq 1 \) we obtain
\[
B^+(n, z) = h(\xi^*, z) + \sum_{q \geq 1} \sum_{x \in X_n(q)} h(x, z) = h(\xi^*, z)(1 + O(1/n)).
\]
By Lemma 2 we have \( h(\xi^*, z) = h(\xi^*, 0)(1 + O(\log n/n)) \), which leads to the following.

**Lemma 3.**
\[
B^+(n, z) = h(\xi^*, 0)(1 + O(\log n/n)), \quad |z| \leq 1.
\]

The structure (3.13) of \( \xi^* \) makes possible to evaluate \( h(\xi^*, 0) \) explicitly.

**Lemma 4.**
\[
\sigma_1(n) = 4(b/4)^n[(n - 1)!]^2.
\]

**Proof.** Indeed, by (2.10), (3.1) and (3.13), it follows that
\[
h(\xi^*, 0) = b^n \prod_{t=1}^{n-1} \left[ n^2 - (-n + 2t)^2 \right]^{-1} = b^n \left( \prod_{t=1}^{n-1} 2t(2n - 2t) \right)^{-1} =
\]
\[ b^n 4^{-n+1} \left( \prod_{t=1}^{n-1} t \right)^{-2} = 4(b/4)^n [(n - 1)!]^{-2}. \]

\[ \square \]

Lemmas 3 and 4 imply the following.

**Proposition 5.**

(3.19) \( B^+(n, z) = 4(b/4)^n [(n - 1)!]^{-2}(1 + O(\log n/n)), \quad |z| \leq 1. \)

To evaluate \( B^-(n) \) we need to change forward walks to backward walks, \( b \) to \( a \), etc., which leads to the following

**Proposition 6.**

(3.20) \( B^-(n, z) = 4(a/4)^n [(n - 1)!]^{-2}(1 + O(\log n/n)), \quad |z| \leq 1. \)

The formulas (3.19) and (3.20) yield (2.12) and (2.13), so Part (a) of Theorem 1 implies that all but finitely many of the eigenvalues of the operators \( L_{\text{Per}}^\pm \) are simple. Moreover, the following holds.

**Theorem 7.** Let \( \{ \varphi_k \} \) be a system of eigenfunctions and associated functions of the operator

(3.21) \[ -d^2/dx^2 + ae^{-2ix} + be^{2ix}, \quad a, b \neq 0, \]

subject to periodic (\( \text{Per}^+ \)) or antiperiodic (\( \text{Per}^- \)) boundary conditions. Then the spectral decomposition

(3.22) \[ f = \sum c_k(f) \varphi_k \]

converges (unconditionally) in \( L^2([0, \pi]) \) for each \( f \in L^2([0, \pi]) \) if and only if

(3.23) \[ |a| = |b|. \]

**Proof.** If \( bc = \text{Per}^+ \) we use the formulas (3.19) and (3.20) for even \( n \), while for antiperiodic boundary conditions \( bc = \text{Per}^- \) we use the same formulas with odd \( n \). By Propositions 5 and 6

(3.24) \[ \frac{B^-(n, z)}{B^+(n, z)} = \frac{a^n}{b^n} \left( 1 + O \left( \frac{\log n}{n} \right) \right). \]

If \( |a| = |b| \neq 0 \), then (2.14) holds, so by Theorem 1 the system \( \{ \varphi_k \} \) is an unconditional basis in \( L^2([0, \pi]) \).

If \( |a| \neq |b| \), then (2.14) fails, so Theorem 1 implies that the system \( \{ \varphi_k \} \) is not a basis in \( L^2([0, \pi]) \).

\[ \square \]

For Examples (2.35) and (2.36) we use the same general scheme but technical details in estimations of \( B^\pm(n, z) \) become more complicated and interesting.
4. Potential $v = ae^{-2ix} + Be^{4ix}$.

In this case

\[(4.1) \quad V(-2) = a, \quad V(4) = B; \quad V(j) = 0 \text{ for } j \neq -2, 4.\]

There is no symmetry in the structure of $v$-admissible forward and backward walks (i.e., one cannot transform a forward part into a backward one or vice versa by replacing positive steps with the same size negative steps). Therefore, we need to evaluate $B^-(n)$ and $B^+(n)$ separately.

Now we consider only periodic boundary conditions $Per^+$, so $n$ is even (see in Section 6 comments about the case $bc = Per^-$).

1. First we estimate $B^-(n, z)$, $n = 2m$. We consider $v$-admissible walks $x$ from $n$ to $-n$; then

\[(4.2) \quad x(t) = -2 \text{ or } 4, \quad 1 \leq t \leq \nu + 1, \quad \sum_{1}^{\nu+1} x(t) = -2n,\]

where $\nu = \nu(x)$. If $p$ is the number of steps equal to 4 and $q$ is the number of steps equal to -2, then we have

\[(4.3) \quad -2q + 4p = -2n = -4m, \quad \text{so } q - 2p = n = 2m.\]

Then $q$ should be even, say $q = 2r$, and we have

\[(4.4) \quad r = p + m, \quad p + q = \nu + 1.\]

If $p = 0$ then we have $q = n$, and there is only one walk $\xi_*$ from $n$ to $-n$ with $n$ steps equal to $-2$. We want to compare, for any walk $x$, $h(x, z)$ with $h(\xi_*, z)$. To this end we do a surgery of $x$ by removing at once a triple of consecutive steps $-2, -2, +4$ and get a walk $\tilde{x}$ with $p - 1$ steps equal to 4 and $q - 2$ steps equal to $-2$. After that we estimate the ratio $|h(x, z)|/|h(\tilde{x}, z)|$, and proceed further with another surgery, and so on.

Let us denote by $Y_n(p)$ the set of all $v$-admissible walks from $n$ to $-n$ having $p$ steps equal to 4. Suppose $x \in Y_n(p)$, $n > 5$. Then $x$ has a triple of consecutive steps $-2, -2, 4$. Indeed, one can easily see that

\[x(1) = x(2) = x(3) = -2,\]

because otherwise $n$ would be an intermediate vertex with necessity, which is not possible for admissible walks by (2.9). Set

\[(4.5) \quad i = \min\{t : x(t + 1) = 4\}\]
and define \( \bar{x} \in Y_n(p - 1) \) as
\[
(4.6) \quad \bar{x}(t) = \begin{cases} 
  x(t) = -2, & 1 \leq t \leq i - 2 \\
  x(t + 3), & i - 1 \leq t \leq \nu - 2.
\end{cases}
\]
Then
\[
h(x, z) = \frac{V(x(i - 1))V(x(i))V(x(i + 1))}{\prod_{j=1}^{n^2} ((n^2 - j(t, x))^2 + z)} \times h(\bar{x}, z),
\]
so with \( c = |a^2B| \) it follows
\[
(4.7) \quad \forall x \in Y_n(p) \exists \bar{x} \in Y_n(p - 1) : |h(x, z)| \leq \frac{c}{n^3} |h(\bar{x}, z)|, \quad |z| \leq 1.
\]
Repeating the same procedure \( p \) times we obtain the inequality
\[
(4.8) \quad |h(x, z)| \leq \left( \frac{c}{n^3} \right)^p |h(\xi^*, z)| \quad \forall x \in Y_n(p), \quad |z| \leq 1,
\]
where \( \xi^* \) is the only walk of \( Y_n(0) \). We have
\[
(4.9) \quad j(t, \xi^*) = n - 2t, \quad 0 \leq t \leq n.
\]
Let us notice that by (4.3)
\[
(4.10) \quad \#Y_n(p) \leq \left( \frac{p + q}{p} \right) = \left( \frac{3p + 2m}{p} \right).
\]
Therefore, by (4.8) and (4.10) it follows that
\[
(4.11) \quad \sum_{p \geq 1} \sum_{x \in Y_n(p)} |h(x; z)| \leq \sigma_2(n) \cdot |h(\xi^*, z)|,
\]
where
\[
(4.12) \quad \sigma_2(n) = \sum_{p \geq 1} \left( \frac{3p + 2m}{p} \right) \left( \frac{c}{n^3} \right)^p = O(1/n^2).
\]
Indeed, since
\[
\left( \frac{3p + 2m}{p} \right) \leq \begin{cases} \frac{25p}{p} & \text{if } p > m, \\
\frac{1}{p!}(5m)^p & \text{if } p \leq m,
\end{cases}
\]
we have
\[
\sigma_2(n) \leq \sum_{p=1}^{m} + \sum_{p=m+1}^{\infty} \leq \sum_{p=1}^{\infty} \frac{1}{p!} \left( \frac{5c}{8m^2} \right)^p + \sum_{p=m+1}^{\infty} \left( \frac{4c}{m^3} \right)^p
\]
\[
\leq \frac{5c}{8m^2} e^{\frac{5c}{8m^2}} + \left( \frac{4c}{m^3} \right)^{m+1} \frac{1}{1 - \frac{4c}{m^3}} = O(1/n^2).
\]

Now we obtain, for $|z| \leq 1$,
\begin{equation}
B^-(n, z) = h(\xi_*, z) + \sum_{p \geq 1} \sum_{x \in \mathcal{Y}_n(p)} h(x; z) = h(\xi_*, z)(1 + O(1/n^2)).
\end{equation}

By Lemma 2, $h(\xi_*, z) = h(\xi_*, 0)(1 + O(\log n/n))$, which leads to the following.

**Lemma 8.**
\begin{equation}
B^-(n, z) = h(\xi_*, 0)(1 + O(\log n/n)), \quad |z| \leq 1.
\end{equation}

By (4.9), we evaluate $h(\xi_*, 0)$ (compare with Lemma 4).

**Lemma 9.**
\begin{equation}
h(\xi_*, 0) = 4(a/4)^n[(n - 1)!]^{-2}.
\end{equation}

In view of Lemmas 8 and 9 the following holds.

**Proposition 10.**
\begin{equation}
B^-(n, z) = 4(a/4)^n[(n - 1)!]^{-2} (1 + O(\log n/n)), \quad |z| \leq 1.
\end{equation}

2. To estimate $B^+(n, z)$ we need to consider $\nu$-admissible walks from $-n$ to $n$. Let $X_n(q)$ be the set of all such walks that have $q$ steps equal to $-2$. Notice, that $X_n(q) = \emptyset$ if $q$ is odd because (compare with (4.2))
\begin{equation}
-2q + 4p = 2n = 4m \quad \text{so} \quad q + 2m = 2p.
\end{equation}

For even $q$, say $q = 2r$, every $x \in X_n(q)$ has $p = (2n + 2q)/4 = m + r$ steps of length 4. The number of elements of $X_n(q)$ could be estimated as
\begin{equation}
\#X_n(q) \leq \binom{p + q}{q} = \binom{m + 3r}{2r} \leq \begin{cases} 2^{4r} & \text{if } r > m, \\ \left(\frac{1}{(2r)!}\right)(4m)^{2r} & \text{if } r \leq m. \end{cases}
\end{equation}

**Lemma 11.** For large enough $n$,
\begin{equation}
\forall x \in X_n(q) \exists \tilde{x} \in X_n(q - 2) : \quad |h(x, z)| \leq (c/n)^{5/2} \cdot |h(\tilde{x}, z)|, \quad |z| \leq 1,
\end{equation}
where the constant $c > 0$ depends on $a$ and $B$.

**Proof.** Fix $x \in X_n(q)$; then one of the following two cases holds.

**Case 1.** There are three consecutive steps $x(i_1), x(i_1 + 1), x(i_1 + 2)$ with zero sum, i.e., $x(i_1) + x(i_1 + 1) + x(i_1 + 2) = 0$;

**Case 2.** There is no triple of consecutive steps with zero sum, i.e., two steps equal to $-2$ and one step equal to $+4$ (any order).

In Case 1 we set
\begin{equation}
\tilde{x}(t) = \begin{cases} x(t), & 1 \leq t \leq i_1 - 1, \\ x(t + 3), & i_1 \leq t \leq \nu - 2. \end{cases}
\end{equation}
Then each vertex of \( \tilde{x} \) is a vertex of \( x \) but \( x \) has in addition the vertices \( j(i_1, x), j(i_1 + 1, x), j(i_1 + 2, x) \). Therefore, it follows that

\[
(4.20) \quad h(x, z) = \frac{V(x(i_1))V(x(i_1 + 1))V(x(i_1 + 2))}{\prod_{t=i_1}^{i_1+2} [n^2 - j(t, x)^2 + z]} \cdot h(\tilde{x}, z),
\]

so

\[
(4.21) \quad |h(x, z)| \leq \frac{|a^2 B|}{n^3} \cdot |h(\tilde{x}, z)| \quad \text{if} \quad |z| \leq 1.
\]

In Case 2, set

\[
(4.22) \quad i_1 = \min\{t : x(t) = -2\}, \quad i_2 = \min\{t > i_1 : x(t) = -2\}
\]

and

\[
(4.23) \quad \tilde{x}(t) = \begin{cases} 
  x(t) = 4, & 1 \leq t \leq i_1 - 1, \\
  x(t + 2) = 4, & i_1 \leq t \leq i_2 - 3, \\
  x(t + 3), & i_2 - 2 \leq t \leq \nu - 2.
\end{cases}
\]

Notice that \( i_2 - i_1 \geq 3 \) (otherwise we are in Case 1). Moreover, from (4.22) and (4.23) it follows that

\[
\tilde{j}(t, x) = \begin{cases} 
  j(t, x) & 1 \leq t \leq i_1 - 1, \\
  j(t + 2, x) + 2 & i_1 \leq t \leq i_2 - 3, \\
  j(t + 2, x) & i_2 - 2 \leq t \leq \nu - 2.
\end{cases}
\]

Therefore, \( \frac{h(x, z)}{h(\tilde{x}, z)} = P_1(z) \cdot P_2(z) \), where

\[
P_1(z) = \frac{a^2 B}{[n^2 - j(i_1, x)^2 + z][n^2 - j(i_1 + 1, x)^2 + z][n^2 - j(i_2 - 1, x)^2 + z]}
\]

and

\[
P_2(z) = \prod_{t=1}^{i_2-3} \frac{n^2 - (-n + 4t)^2 + z}{n^2 - (-n + 4t - 2)^2 + z}
\]

Obviously, we have \( |P_1(z)| \leq a^2 B/n^3 \). On the other hand,

\[
P_2(z) \leq P_2(0)(1 + 4 \log n/n) = 2\sqrt{n}(1 + 4 \log n/n)
\]

by Lemma 2 and Lemma 12 below. Thus, Lemma 11 holds with \( c = 2|a^2 B| \).

\[\square\]

**Lemma 12.**

\[
(4.24) \quad \prod_{t=i}^{j} \frac{n^2 - (-n + 4t)^2}{n^2 - (-n + 4t - 2)^2} \leq 2\sqrt{n}, \quad 1 \leq i \leq j < n/2.
\]
Proof. Since
\[ \frac{n^2 - (-n + 4t)^2}{n^2 - (-n + 4t - 2)^2} = \frac{4t(2n - 4t)}{(4t - 2)(2n - 4t + 2)} \leq \frac{2t}{2t - 1}, \]
the product in (4.24) does not exceed \( \prod_{i=1}^{n} \frac{2t}{2t - 1} \). Since \( \frac{2t}{2t - 1} \leq \sqrt{\frac{t}{(t - 1)}} \) for \( t > 1 \), we obtain
\[ \prod_{i=1}^{n} \frac{2t}{2t - 1} \leq 2 \sqrt{\frac{2}{3}} \cdots \sqrt{\frac{n}{n - 1}} = 2\sqrt{n}. \]
□

Now let us find the asymptotics of \( B^+(n, z) \). If we iterate (4.18) \( r \) times then it follows
(4.25) \[ |h(x, z)| \leq (2c/n)^{5r/2} \cdot |h(\xi^*, z)| \quad \forall x \in X_n(q), \quad |z| \leq 1, \]
where \( \xi^* \) is the only walk in \( X_n(0) \); all its steps are equal to 4, so
(4.26) \[ j(t, \xi^*) = -n + 4t, \quad 0 \leq t \leq m. \]
By (4.17) and (4.18),
(4.27) \[ \sum_{r \geq 1} \sum_{x \in X_n(2r)} |h(x, z)| \leq \sigma_3(n) \cdot |h(\xi^*, z)|, \]
where
\[ \sigma_3(n) \leq \sum_{r=1}^{\infty} \left( \frac{m + 3r}{2r} \right) \left( \frac{c}{n} \right)^{5r/2} \leq \sum_{r=1}^{m} + \sum_{r>m} \]
\[ \leq \sum_{r=1}^{\infty} \frac{1}{(2r)!} \left( \frac{4n^2c^{5/2}}{n^{5/2}} \right)^r + \sum_{r=m+1}^{\infty} 2^{4r} \left( \frac{c}{n} \right)^{5r/2} \]
\[ \leq \frac{2c^{5/2}}{\sqrt{n}} \exp(4c^{5/2}/\sqrt{n}) + O \left( \frac{16c^{5/2}}{n^{5/2}} \right) = O(1/\sqrt{n}). \]

Therefore, since \( |B^+(n, z) - h(\xi^*, z)| \) does not exceed the left-hand side of (4.27), we obtain, in view of Lemma 2, the following.

Lemma 13.
(4.28) \[ B^+(n, z) = h(\xi^*, 0)(1 + O(1/\sqrt{n})), \quad |z| \leq 1. \]

Next we evaluate \( h(\xi^*, 0) \).

Lemma 14.
(4.29) \[ h(\xi^*, 0) = 16(B/16)^m((m - 1)!)^{-2}. \]
Proof. By (4.26),
\[ h(\xi^*, 0) = B^m \cdot P^{-1}, \]
where
\[ P = \prod_{1}^{m-1} \left[ n^2 - (-n + 4t)^2 \right] = \prod_{1}^{m-1} 4t(4m - 4t) = 16^{m-1}((m - 1)!)^2. \]
This proves (4.29). □

Lemmas [13] and [14] imply the following (compare with Proposition [10]).

**Proposition 15.** For even \( n = 2m \)

\[ B^+(n, z) = 16(B/16)^m((m - 1)!)^{-2}(1 + O(1/\sqrt{n})). \]

11. Now we apply Theorem [1] and obtain the following.

**Theorem 16.** (a) If

\[ v = ae^{-2ix} + Be^{4ix}, \quad a, B \neq 0, \]
then all but finitely many of the eigenvalues of the operator \( L_{Per^+} \) are simple.

(b) If \( (\psi_k) \) is a system of eigenfunctions and associated functions of the operator \( L_{Per^+}(v) \), then this system is complete in \( L^2([0, \pi]) \) but it is not a basis in \( L^2([0, \pi]) \).

Proof. In view of (4.16) and (4.30), the conditions (2.12) and (2.13) in Theorem [2] hold for even \( n \). Therefore, by Part (a) of Theorem [1] the operator \( L_{Per^+} \) has at most finitely many multiple eigenvalues.

Let \( \{\psi_k\} \) be a system of normalized eigenfunctions and associated functions of the operator \( L_{Per^+} \). By (4.16) and (4.30),

\[ \lim_{n \text{ even}} \frac{B^-(n, 0)}{B^+(n, 0)} = 0, \]

so the condition (2.14) fails. Thus, by Part (b) of Theorem [1] the system \( \{\psi_k\} \) is not a basis in \( L^2([0, \pi]) \). This completes the proof. □

5. Potential \( v = ae^{-2ix} + be^{2ix} + Ae^{-4ix} + Be^{4ix} \)

Now we analyze trigonometric polynomials with four nonzero coefficients of the form

\[ v = ae^{-2ix} + be^{2ix} + Ae^{-4ix} + Be^{4ix}. \]

Since the set

\[ \{k : V(k) \neq 0\} = \{-2, -4, 2, 4\} \]

\[ B^+(n, z) = 16(B/16)^m((m - 1)!)^{-2}(1 + O(1/\sqrt{n})). \]
is symmetric, it is enough to find the asymptotics of \( B^+(n, z) \) in terms of the coefficients \( a, b, A, B \). Then we may obtain the asymptotics of \( B^-(n) \) just by exchanging the roles of \( a, A \) and \( b, B \).

In our paper [4], we found the asymptotic behavior of the spectral gaps of one-dimensional Schrödinger operator with a two term potential \( v = a \cos 2x + b \cos 4x \), where \( a \) and \( b \) are real and nonzero. There, an essential part of the analysis is related to the asymptotic behavior of the sums \( \sum_{x \in X_n} h(x, z) \), so the techniques or even explicitly stated results from [4, Section 5] give us tools to obtain asymptotics for \( B^+(n) \).

Let \( X_n \) be the set of all walks \( x \) from \(-n\) to \( n \) that are \( v \)-admissible, i.e., \( x(t) \in \{-2, -4, 2, 4\} \) (2.9) hold, and we have

\[
\sum_{1}^{\nu+1} x(t) = 2n,
\]

and let \( X_n^+ \) be the set of all \( v \)-admissible forward walks from \(-n\) to \( n \).

In the case analyzed in Sections 3 (i.e., when \( v = a e^{-2i} + b e^{2i} \)) there was only one forward walk. But now we have many such walks; more precisely, if \( A(n) \) is the number of solutions of (5.3) with \( x(t) = 2 \) or \( 4 \), then \( A(1) = 1 \), \( A(2) = 2 \) and \( A(n+1) = A(n) + A(n-1) \), so

\[
\#X_n^+ = A(n) = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}
\]

(Fibonacci numbers, see [1, Sect. 4.1]).

2. For convenience, we change the parameters in (5.1) by setting

\[
A = -\alpha^2, \quad a = -2\tau\alpha, \quad B = -\beta^2, \quad b = -2\sigma\beta.
\]

In these notations the following statement (which is proven in [4]) holds.

**Lemma 17.** For even \( n \)

\[
\sum_{\xi \in X_n^+} h(\xi, 0) = \frac{4(\beta/2)^n}{((n-1)!)^2} \prod_{i=1}^{n/2} [\sigma^2 - (2i - 1)^2],
\]

and for odd \( n \)

\[
\sum_{\xi \in X_n^+} h(\xi, 0) = -\frac{4(\beta/2)^n}{((n-1)!)^2} \sigma \prod_{i=1}^{(n-1)/2} [\sigma^2 - (2i)^2].
\]
Product representations of \( \cos t \) and \( \sin t \) show that (5.6) and (5.7) could be rewritten as

\[
\sum_{\xi \in X^+_n} h(\xi, 0) = \frac{4(i\beta/2)^n}{((n-2)!!)^2} \cos \left( \frac{\pi \sigma}{2} \right) (1 + O(1/n))
\]

for even \( n \), and

\[
\sum_{\xi \in X^+_n} h(\xi, 0) = \frac{4i(i\beta/2)^n}{((n-2)!!)^2} \frac{2}{\pi} \sin \left( \frac{\pi \sigma}{2} \right) (1 + O(1/n))
\]

for odd \( n \).

By Lemma 2, for every \( \xi \in X^+_n \) we have

\[
|h(\xi, z) - h(\xi, 0)| \leq \frac{4 \log n}{n} |h(\xi, 0)|, \quad |z| \leq 1.
\]

These inequalities enable us to consider \( z = 0 \) instead of \( z \) in our analysis of \( B^+(n, z) \).

3. Now we are dealing with the difficulties brought by the huge size of \( X^+_n \) (see (5.4)).

For \( \xi \in X^+_n \), let \( X_{n, \xi} \) denote the set of all walks \( x \in X_n \setminus X^+_n \) such that each vertex \( j(t, \xi) \) is a vertex of \( x \) also, i.e., \( j(s, \xi) = j(t, x) \) for some \( t_s \). Then we have

\[
X_n \setminus X^+_n = \bigcup_{\xi \in X^+_n} X_{n, \xi}.
\]

Indeed, for \( x = (x(t))_{t=1}^{\nu+1} \in X_n \) define \( t_0 = 0 \) and

\[
t_{s+1} = \min \{ t > t_s : j(t, x) > j(t_s, x) \}, \quad 0 \leq s < \tilde{\nu},
\]

where

\[
\tilde{\nu} = \min \{ s : j(t, x) = n - 4 \text{ or } n - 2 \}.
\]

Define \( \xi \) by the formula

\[
\xi(s) = \begin{cases} j(t, x) - j(t_{s-1}, x) & \text{for } 1 \leq s \leq \tilde{\nu}, \\ n - j(t_{\tilde{\nu}}, x) & \text{for } s = \tilde{\nu} + 1. \end{cases}
\]

Then \( \xi \in X^+_n \), and by the construction \( x \in X_{n, \xi} \).

For \( \xi \in X^+_n \) and \( m \in \mathbb{N} \), let \( X_{n, \xi, m} \) be the set of walks \( x \in X_{n, \xi} \) such that \( x \) has \( m \) more steps than \( \xi \), i.e.,

\[
X_{n, \xi, m} = \{ x \in X_{n, \xi} : \nu(x) - \nu(\xi) = m \}.
\]
Then we have

\[ X_{n,\xi} = \bigcup_{m=1}^{\infty} X_{n,\xi,m}. \]  

For \( \xi \in X_n^+ \) and any \( m \)-tuple \( I = (i_1, \ldots, i_m) \) of integers \( i_\beta \in n + 2\mathbb{Z} \setminus \{\pm n\} \), let \( X_{n,\xi}(I) \) be the set of all walks \( x \) with \( \nu(\xi) + 1 + m \) steps such that \( I = (i_1, \ldots, i_m) \) and the sequence of the vertices of \( \xi \) are complementary subsequences of the sequence of the vertices of \( x \). Then

**Lemma 18.** In the above notations, we have

\[ \#X_{n,\xi}(I) \leq 5^m \quad \forall I = (i_1, \ldots, i_m). \]

This is Lemma 12 in [4].

The following is an analogue of Lemma 13 in [4].

**Lemma 19.** There exists \( n_1 \) such that for \( n \geq n_1 \)

\[ \sum_{x \in X_{n,\xi}} |h(x, z)| \leq \frac{K \log n}{n} |h(\xi, z)| \quad \forall \xi \in X_n^+, \quad |z| \leq 1, \]

where

\[ K = 40C^2, \quad C = 1 + \max(|a|, |b|, |A|, |B|) \frac{\min(|a|, |b|, |A|, |B|)}{m}. \]

**Proof.** In view of (5.16), it is enough to show that

\[ \sum_{x \in X_{n,\xi,m}} |h(x, z)| \leq \left( \frac{K \log n}{n} \right)^m |h(\xi, z)|, \]

with \( K \) and \( C \) defined by (5.19). Indeed, if (5.20) holds, then with \( n_1 \) chosen so that \((K \log n)/n \leq 1/2 \) we would have

\[ \sum_{X_{n,\xi}} \frac{|h(x, z)|}{|h(\xi, z)|} \leq \sum_{m=1}^{\infty} \left( \frac{K \log n}{n} \right)^m \leq \frac{K \log n}{n}, \quad n > n_1, \]

which implies (5.18).

To prove (5.20), we use the inequality

\[ \sum_{x \in X_{n,\xi,m}} \sum_{I \in X_{n,\xi}(I)} |h(x, z)| \leq \sum_{x \in X_{n,\xi,m}} |h(x, z)|, \]

where the first sum is taken over all \( m \)-tuples \( I \) of integers \( i_\beta \in n + 2\mathbb{Z}, \ i_\beta \neq \pm n \). Fix such \( m \)-tuple \( I = (i_1, \ldots, i_m) \); then for every \( x \in X_{\xi}(I) \)

\[ \frac{|h(x, z)|}{|h(\xi, z)|} = \frac{\prod_{t=1}^{\nu} V(x_t)}{\prod_{\alpha=1}^{\nu} V(\xi_{\alpha})} \times \frac{1}{\prod_{t=1}^{m} (n^2 - i_{\beta}^2 + z)}. \]
We can split the first factor \( P \) as

\[
(5.22) \quad P = \prod_{\alpha=1}^{\tilde{\nu}} \left( \frac{1}{V(\xi_{\alpha})} \prod_{1+t_{\alpha}-1}^{t_{\alpha}} V(x(t)) \right) \equiv \prod_{\alpha=1}^{\tilde{\nu}} r(\alpha).
\]

Let \( d_{\alpha} = t_{\alpha} - t_{\alpha-1} \); then \( \sum_{\alpha} (d_{\alpha} - 1) = m \). If \( d_{\alpha} = 1 \), then the ratio \( r(\alpha) \) in (5.22) equals 1. Otherwise \( d_{\alpha} \geq 2 \), so, by the inequality \( d_{\alpha} \leq 2(d_{\alpha} - 1) \), it follows that

\[
|r(\alpha)| \leq C^{d_{\alpha}} \leq (C^2)^{d_{\alpha}-1}, \quad \alpha = 1, \ldots, \tilde{\nu},
\]

which implies

\[
|P| = \prod_{\alpha=1}^{\tilde{\nu}} r(\alpha) \leq (C^2)^{\sum(d_{\alpha}-1)} = C^{2m}.
\]

Therefore, taking into account that

\[
|n^2 - i^2 + z|^{-1} \leq 2|n^2 - i^2| \quad \text{if} \quad i \neq \pm n, \quad |z| \leq 1
\]

we obtain

\[
(5.23) \quad \frac{|h(x, z)|}{|h(\xi, z)|} \leq \frac{(2C^2)^m}{|n^2 - i_1^2| \cdots |n^2 - i_m^2|}.
\]

Now by Lemma [18]

\[
(5.24) \quad \sum_{X_n, \xi(t)} \frac{|h(x, z)|}{|h(\xi, z)|} \leq \frac{(10C^2)^m}{|n^2 - i_1^2| \cdots |n^2 - i_m^2|},
\]

and by (5.21) and the elementary inequality

\[
\sum_{i \neq \pm n} \frac{1}{|n^2 - i^2|} \leq \frac{4 \log n}{n} \quad \text{for} \quad n \geq 10
\]

it follows that

\[
(5.25) \quad \sum_{x \in X_n, \xi, m} \frac{|h(x, z)|}{|h(\xi, z)|} \leq \sum_{t_1, \ldots, t_m \neq \pm n} \frac{(10C^2)^m}{|n^2 - i_1^2| \cdots |n^2 - i_m^2|}
\]

\[
\leq (10C^2)^m \left( \frac{4 \log n}{n} \right)^m = \left( \frac{40C^2 \log n}{n} \right)^m.
\]

Thus (5.20) holds, which completes the proof of Lemma [19] \( \square \)

4. Now we are going to complete the proof of the main result of this section (compare with Step 5 and 6, pp. 187–190, in [4]).
Proposition 20. If $\tau, \sigma$ given by (5.5) are not integers then for $|z| \leq 1$

$B^+(n, z) = 4(i\beta/2)^n \frac{\cos\left(\frac{\pi\sigma}{2}\right)\left(1 + O\left(\frac{\log n}{n}\right)\right)}{((n-2)!!)^2},$

$B^-(n, z) = 4(i\alpha/2)^n \frac{\cos\left(\frac{\pi\tau}{2}\right)\left(1 + O\left(\frac{\log n}{n}\right)\right)}{((n-2)!!)^2},$

for even $n$, and

$B^+(n, z) = 4i\frac{(i\beta/2)^n}{((n-2)!!)^2} \frac{2}{\pi} \sin\left(\frac{\pi\sigma}{2}\right) \left(1 + O\left(\frac{\log n}{n}\right)\right)$

$B^-(n, z) = 4i\frac{(i\alpha/2)^n}{((n-2)!!)^2} \frac{2}{\pi} \sin\left(\frac{\pi\tau}{2}\right) \left(1 + O\left(\frac{\log n}{n}\right)\right),$

for odd $n$, with nonzero $\alpha, \beta, \tau, \sigma \in \mathbb{C}$ defined in (5.5).

Proof. By symmetry of (5.2), it is enough to prove only the estimates for $B^+(n, z)$.

From (5.11) and (5.18) it follows that

$\sum_{x \in X_n \setminus X_n^+} |h(x, z)| \leq \sum_{\xi \in X_n^+} \sum_{x \in X_n \setminus X_n^+} |h(x, z)| \leq K \frac{\log n}{n} \sum_{\xi \in X_n^+} |h(\xi, z)|.$

Since $B^+(n, z) = \sum_{x \in X_n} h(x, z)$ and $|h(\xi, z)| \leq (1 + 4 \log n/n)|h(\xi, 0)|$ due to Lemma 2, the inequality (5.30) implies

$|B^+(n, z) - \sum_{\xi \in X_n^+} h(x, z)| \leq 2K \frac{\log n}{n} \sum_{\xi \in X_n^+} |h(\xi, 0)|.$

On the other hand, (5.10) implies

$\left| \sum_{\xi \in X_n^+} h(x, z) - \sum_{\xi \in X_n^+} h(x, 0) \right| \leq 4 \frac{\log n}{n} \sum_{\xi \in X_n^+} |h(\xi, 0)|.$

Therefore, we have

$|h(\xi, 0)| = \frac{\prod_{t=1}^{\nu+1} |V(\xi(t))|}{\prod_{t=1}^{\nu} (n^2 - j(t, \xi)^2)} = \frac{\prod_{t=1}^{\nu+1} W(\xi(t))}{\prod_{t=1}^{\nu} (n^2 - j(t, \xi)^2)} \equiv h_w(\xi, 0),$
where \( w \) is the potential \( w = |a| e^{-2ix} + |b| e^{2ix} + |A| e^{-4ix} + |B| e^{4ix} \) with only nonzero Fourier coefficients

\[(5.32) \quad W(-2) = |a|, \quad W(2) = |b|, \quad W(-4) = |A|, \quad W(4) = |B|.\]

Of course, now (5.5) is replaced by

\[(5.33) \quad |A| = -\tilde{\alpha}^{2}, \quad |a| = -2\tilde{\tau} \tilde{\alpha}, \quad |B| = -\tilde{\beta}^{2}, \quad b = -2\tilde{\sigma} \tilde{\beta},\]

with

\[(5.34) \quad \tilde{\alpha} = i|\alpha|, \quad \tilde{\tau} = i|\tau|, \quad \tilde{\beta} = i|\beta|, \quad \tilde{\sigma} = i|\sigma|,\]

\( \alpha, \beta, \tau, \sigma \) coming from (5.5).

Thus, we obtain,

\[(5.35) \quad \sum_{\xi \in X_n^+} |h(\xi, 0)| = \frac{4(|\beta|/2)^n}{((n-2)!!)^2} \cosh \left( \frac{\pi|\sigma|}{2} \right) \left( 1 + O \left( \frac{1}{n} \right) \right)\]

for even \( n \), and

\[(5.36) \quad \sum_{\xi \in X_n^+} |h(\xi, 0)| = \frac{4(|\beta|/2)^n}{((n-2)!!)^2} \frac{2|\sigma|}{\pi} \sinh \left( \frac{\pi|\sigma|}{2} \right) \left( 1 + O \left( \frac{1}{n} \right) \right)\]

for odd \( n \).

Now we continue to analyze \( B^+(n, z) \). In view of (5.8), (5.9), (5.35) and (5.36) we have

\[(5.37) \quad \left| \sum_{\xi \in X_n^+} h(\xi, 0) \right| = \left( \sum_{\xi \in X_n^+} |h(\xi, 0)| \right) \cdot R^+ \cdot \left( 1 + O \left( \frac{1}{n} \right) \right),\]

where

\[(5.38) \quad R^+ \begin{cases} \text{for even } n, & \cos \frac{\pi|\sigma|}{2} \cosh \frac{\pi|\sigma|}{2} \\ \text{for odd } n, & \sin \frac{\pi|\sigma|}{2} \sinh \frac{\pi|\sigma|}{2} \end{cases}\]

Therefore, by (5.31) and (5.37) it follows that

\[(5.39) \quad \left| B^+(n, z) - \sum_{\xi \in X_n^+} h(\xi, 0) \right| \leq M \frac{\log n}{n} \left| \sum_{\xi \in X_n^+} h(\xi, 0) \right|,\]

where \( M = (2K + 4)/R^+ \), so if \( R^+ \neq 0 \), we have

\[(5.40) \quad B^+(n, z) = \left( \sum_{\xi \in X_n^+} h(\xi, 0) \right) \left( 1 + O \left( \frac{\log n}{n} \right) \right).\]

The condition \( R^+ \neq 0 \) holds in both \( Per^\pm \) cases if and only if \( \sigma \) is not an integer. By (5.8) and (5.9), we know the sum in the right-hand side of (5.40). This completes the proof of Proposition 20.
Remark. In analysis of $B^-(n,z)$ as an analog of $R^+$ we would have

$$R^- = \frac{|\cos \frac{\pi \tau}{2}|}{\cosh \frac{\pi |\tau|}{2}} \quad \text{for even } n, \quad R^- = \frac{|\sin \frac{\pi \tau}{2}|}{\sinh \frac{\pi |\tau|}{2}} \quad \text{for odd } n.$$ 

In terms of the coefficients $a, b, A, B$ the condition “$\tau, \sigma$ are not integers” in Proposition 20 holds if and only if neither $-b^2/(4B)$, nor $-a^2/(4A)$ is an integer square.

8. Now by the general scheme given in Theorem 1, we obtain the following.

**Theorem 21.** Consider the Hill operator $L_{Per\pm}(v)$, where

$$v = ae^{-2ix} + Ae^{-4ix} + be^{2ix} + Be^{4ix}$$

with $a, b, A, B \neq 0$ and

$$\text{neither } -b^2/(4B), \text{ nor } -a^2/(4A) \text{ is an integer square.}$$

All eigenvalues of $L_{Per\pm}(v)$ but finitely many are simple; the system $\Phi = \{\varphi_k\}$ of eigenfunctions and associated functions is complete. $\Phi$ is an (unconditional) basis in $L^2([0, \pi])$ if and only if $|A| = |B|$.

**Proof.** In view of (5.43), we may apply Proposition 20. Then, (5.26) and (5.27) imply the conditions (2.12) and (2.13) in Theorem 1 for even $n$, and (5.28) and by (5.29) imply (2.12) and (2.13) for odd $n$. Therefore, by Part (a) of Theorem 1 each of the operators $L_{Per^+}$ and $L_{Per^-}$ has at most finitely many multiple eigenvalues.

Let $\Phi = \{\varphi_k\}$ be a system of normalized eigenfunctions and associated functions of the operator $L_{Per^+}$. Then by (5.26) and (5.27) we have

$$\frac{|B^-(n,0)|}{|B^+(n,0)|} = \left|\frac{A}{B}\right|^{n/2} \frac{|\cos \frac{\pi \tau}{2}|}{\cos \frac{\pi |\tau|}{2}} \left(1 + O\left(\frac{n \log n}{n}\right)\right), \quad n \text{ even.}$$

Therefore,

$$\lim_{n \text{ even}} \frac{|B^-(n,0)|}{|B^+(n,0)|} = \begin{cases} 0 & |A| < |B| \\ \infty & |A| > |B| \\ |A| = |B| \end{cases}$$

so the condition (2.14) fails if $|A| \neq |B|$ and holds if $|A| = |B|$. Thus, by Part (b) of Theorem 1 if $|A| \neq |B|$ the system $\Phi_k$ is not a basis in $L^2([0, \pi])$, and if $|A| = |B|$ then $\Phi_k$ is an unconditional basis in $L^2([0, \pi])$.

In the same way, the conditions (5.28) and (5.29) imply the theorem for antiperiodic boundary conditions $Per^-$. This completes the proof.
6. Comments, conclusion

1. In Section 4 we consider only periodic boundary conditions in the case of potentials \( v(x) = ae^{-2ix} + Be^{4ix} \). In the case of antiperiodic boundary conditions we need to analyze \( B^\pm(n, z) \) for odd \( n \). It turns out that most of the estimates done in Section 4 can be carried on for odd \( n \) as well. But the crucial fact

\[
B^+(n, z) = h(\xi^*, 0) \left(1 + O(\log n/n)\right), \quad n \text{ even,}
\]

(see (4.28), (4.29), (4.30)) does not have a reasonable analog if \( n \) is odd. This observation and attempts to follow the scheme which was successful for even \( n \) are interesting because they lead to some combinatorial problems and maybe give some hints how the case \( bc = \text{Per}^- \) could be studied.

Now, for an odd \( n = 2m + 1 \) we write formulas that are analogous to (4.2)–(4.4). Let \( x = (x(t))_{t=1}^{\nu+1} \) be a \( v \)-admissible walk from \(-n\) to \( n \) with \( x(t) \in \{-2, 4\} \). We denote by \( p \) and \( q \), respectively, the number of steps equal to 4 and the number of steps equal to \(-2\). Then \( 4p - 2q = 2n = 2(2m + 1) \), so we have

\[
2p = 2m + 1 + q, \quad p + q = \nu + 1.
\]

Now \( q \) is odd, say \( q = 2r + 1 \), and \( q = 1 \) is the minimal possible value of \( q \).

Let \( X^+_n(q) \) denote the set of all admissible walks with \( q \) steps equal to \(-2\). By repeating the constructions of Section 4 one may prove the following statements.

**Lemma 22.** If \( r > 0 \), then

\[
\sum_{x \in X^+_n(2r+1)} |h(x, z)| \leq \left(\frac{c}{n^{5/2}}\right)^r \sum_{\xi \in X^+_n(1)} |h(\xi, z)|.
\]

**Lemma 23.** For large enough \( n \)

\[
\left|B^+(n, z) - \sum_{\xi \in X^+_n(1)} h(\xi, z)\right| \leq \frac{c}{n^{5/2}} \sum_{\xi \in X^+_n(1)} |h(\xi, z)|, \quad |z| \leq 1.
\]

**Lemma 24.** For large enough \( n \)

\[
\sum_{\xi \in X^+_n(1)} h(\xi, z) - \sum_{\xi \in X^+_n(1)} h(\xi, 0) \leq 4\frac{\log n}{n} \sum_{\xi \in X^+_n(1)} |h(\xi, 0)|
\]
So far it is OK. But

\[(6.5) \quad \#X_n^+(1) = m + 2,\]

and in order to apply (6.3) and (6.4) we need to evaluate

\[H_0^* = \sum_{\xi \in X_n^+(1)} |h(\xi, 0)| \quad \text{and} \quad H_0 = \sum_{\xi \in X_n^+(1)} h(\xi, 0)\]

and be sure that \(H_0 \neq 0\).

We can evaluate \(H^*\) and \(H_0\) (see Proposition 25) but \(H_0 = 0\) – see (6.15).

Any walk \(\xi \in X_n^+(1)\) has only one step equal to \(-2\) but that step could appear on the left of \(-n\) (denote that walk by \(\xi^-\)), on the right of \(n\) (denote that walk by \(\xi^+\)) and anywhere between \(-n\) and \(n\) (denote the set of all such walks by \(\hat{X}_n^+(1)\)). With \(p = m + 1\) the numerator in \(h(\xi, 0)\) is equal to \(ab^{m+1}\) for every \(\xi \in M^+(1)\), so we can assume in the calculations which follow that \(a = b = 1\). Then the sum \(H_0\) has two negative terms, namely

\[(6.6) \quad h(\xi^-, 0) = h(\xi^+, 0) = 1/P,\]

where, with \(n = 2m + 1\),

\[(6.7) \quad P = (n^2 - (-n-2)^2) \prod_{\tau=0}^{m-1} [n^2 - (-n+2+4\tau)^2] = -2(2n+2) \prod_{\tau=0}^{m-1} (2+4\tau)(4m-4\tau)\]

\[= -8(m+1)8^m m!(2m-1)!! = -8(m+1)4^m(2m)!\]

Therefore,

\[(6.8) \quad h(\xi^-, 0) + h(\xi^+, 0) = - \frac{1}{m+1} \cdot \frac{1}{4m+1} \cdot \frac{1}{(2m)!}\]

The remaining walks \(\xi \in \hat{X}_n^+\) give a sum of positive terms of the form \((P(t)P(s))^{-1}\) with \(s = m - t + 1\), where

\[(6.9) \quad P(t) = \prod_{\tau=1}^{t} [n^2 - (-n + 4\tau)^2] = \prod_{\tau=1}^{t} 4\tau(4m + 2 - 4\tau) =
\]

\[8^t t! \prod_{\tau=0}^{t} [2(m - \tau + 1) - 1] = 8^t t! \frac{(2m-1)!!}{(2(m-t)-1)!!} \cdot \frac{2^{m-t}(m-t)!}{(2(m-t))!!} \cdot \frac{(2m)!!}{2^m m!}\]

\[= 4^t t! \frac{(2m)!}{m!} \cdot \frac{(m-t)!}{(2(m-t))!} \cdot \frac{(2m)!}{m!}\]

Then with \(t + s = m + 1\), \(1 \leq t \leq m\), we have

\[(6.10) \quad P(s) = 4^{m+1-t}(m+1-t)! \frac{(t-1)!}{(2(t-1))!} \cdot \frac{(2m)!}{m!}\]
and

\begin{equation}
(6.11) \quad P(t)P(s) = 4^{m+1} \binom{2m}{m} (2m)! \cdot \frac{t}{(2^{t-1})_{t-1}} \cdot \frac{m+1-t}{(2^{m-t})_{m-t}}.
\end{equation}

Next, we use Catalan numbers (see [11, Section 4.5, (4.5.1) and (4.5.2)] or [12, pp. 117, (14.10)–(14.12)])

\begin{equation}
(6.12) \quad C_k = \frac{1}{k} \binom{2k-2}{k-1}, \quad k \geq 1,
\end{equation}

and the fundamental recurrence for Catalan numbers

\begin{equation}
(6.13) \quad C_{k+1} = \sum_{i=1}^{k} C_i C_{k+1-i}.
\end{equation}

In view of (6.12) and (6.13), we obtain

\begin{equation}
(6.14) \quad \sum_{t=1}^{m} \left( \frac{P(t)P(m+1-t)}{(2m)!} \right) = \frac{1}{4^{m+1}} \cdot \frac{1}{m+1} \cdot \frac{1}{(2m)!} \left[ C_{m+1} \sum_{t=1}^{m} C_t C_{m+1-t} \right]
\end{equation}

Now (6.8) and (6.14) imply the following.

**Proposition 25.** *In the above notations,*

\begin{equation}
(6.15) \quad H_0^* = (2 \cdot 4^m (m+1)(2m))^{-1}, \quad H_0 = 0.
\end{equation}

With $H_0 = 0$ we cannot find the asymptotic of $B^+(n)$ by applying the same scheme which was successful in Sections 3-5. Notice that in Section 5 we have the same difficulties in the case of exceptional values of the coefficients of $v$. There $R^+$ and $R^-$ are analogs of $H_0$ (see (5.37), (5.38) and (5.41)). More precisely, if $v$ is given by (5.42) then

\begin{equation}
H_0^+ = \sum_{\xi \in X^+} h(\xi, 0) = 0, \quad \text{if} \quad \begin{cases} n \text{ is even and } \cos \frac{\pi}{2} \sigma = 0, \\ n \text{ is odd and } \sin \frac{\pi}{2} \sigma = 0, \end{cases}
\end{equation}

where $\sigma^2 = -b^2/(4B)$, and

\begin{equation}
H_0^- = \sum_{\xi \in Y^-} h(\xi, 0) = 0, \quad \text{if} \quad \begin{cases} n \text{ is even and } \cos \frac{\pi}{2} \tau = 0, \\ n \text{ is odd and } \sin \frac{\pi}{2} \tau = 0, \end{cases}
\end{equation}

where $\tau^2 = -a^2/(4A)$. 


2. In this paper we consider only operators on the interval \([0, \pi]\). But let us mention that F. Gesztesy and V. Tkachenko results \[9, \text{Remark 8.10}\] together with our examples from Sections 3–5 show that the Schrödinger operators

\[ Ly = -y'' + v(x)y, \quad x \in \mathbb{R}, \]

with potentials

1. \( v = ae^{-2ix} + be^{2ix}, \quad a, b \neq 0, |a| \neq |b|; \)
2. \( v = ae^{-2ix} + be^{4ix}, \quad a, b \neq 0; \)
3. \( v = ae^{-2ix} + Ac^{-4ix} + be^{2ix} + Be^{4ix}, \quad a, b, A, B \neq 0, |A| \neq |B|, \)

are not spectral operators of scalar type.

References

[1] P. J. Cameron, Combinatorics: topics, techniques, algorithms, Cambridge University Press, 1994.
[2] N. Dernek and O. A. Veliev, On the Riesz basisness of the root functions of the nonself-adjoint Sturm-Liouville operator. Israel J. Math. 145 (2005), 113–123.
[3] P. Djakov and B. Mityagin, Instability zones of periodic 1D Schrödinger and Dirac operators (Russian), Uspehi Mat. Nauk 61 (2006), no 4, 77–182 (English: Russian Math. Surveys 61 (2006), no 4, 663–766).
[4] P. Djakov and B. Mityagin, Asymptotics of instability zones of the Hill operator with a two term potential. J. Funct. Anal. 242 (2007), no. 1, 157–194.
[5] P. Djakov and B. Mityagin, Bari-Markus property for Riesz projections of Hill operators with singular potentials, Contemporary Mathematics 481 (2009), 59–80, AMS, Functional Analysis and Complex Analysis, editors A. Aytuna, R. Meize, T. Terzioglu, D. Vogt.
[6] P. Djakov and B. Mityagin, A criterion for convergence of spectral decompositions of 1D Dirac operators, in preparation.
[7] N. Dunford, A survey of the theory of spectral operators, Bull. Amer. Math. Soc. 64 (1058), 217–274.
[8] N. Dunford, J. Schwartz, Linear Operators, Part III, Spectral Operators, Wiley, New York, 1971.
[9] F. Gesztesy and V. Tkachenko, A criterion for Hill operators to be spectral operators of scalar type, Journal d’Analyse Mathematique, 107 (2009), 287–353.
[10] G. M. Keselman, On the unconditional convergence of eigenfunction expansions of certain differential operators, Izv. Vyssh. Uchebn. Zaved. Mat. 39 (2) (1964), 82–93 (Russian).
[11] B. M. Levitan and I. S. Sargsjan, Sturm-Liouville and Dirac operators. Translated from the Russian. Mathematics and its Applications (Soviet Series), 59. Kluwer Academic Publishers Group, Dordrecht, 1991.
[12] J. H. van Lint and R. M. Wilson, A course in Combinatorics, Cambridge University Press, 1992.
[13] W. Magnus and S. Winkler, "Hill equation", Interscience Publishers, John Wiley, 1969.
[14] A. S. Makin, On the convergence of expansions in root functions of a periodic boundary value problem, Dokl. Akad. Nauk 406 (2006), no. 4, 452–457.

[15] A. S. Makin, On the basis property of system of root functions of regular boundary value problems for the Sturm–Liouville operator, Differ. Uravn. 42 (2006), no. 12, 1646–1656, 1727 (russian); English transl.: Differ. Equ. 42 (2006), no. 12, 1717–1728.

[16] X. R. Mamedov, N. B. Kerimov, On Riesz basisness of root functions of regular boundary problems, Matem. Zamet., 64 (1998), 558–563.

[17] V. A. Marchenko, Sturm–Liouville Operators and their Applications, Kyiv, Naukowa Dumka, 1977 (Russian); English transl.: in Oper. Theory Adv. Appl., vol. 22, Birkhaeuser, Boston, 1986.

[18] V. P. Mikhailov, On Riesz bases in $L^2(0,1)$, Dokl. Akad. Nauk SSSR 144 (1962), 981–984 (Russian)

[19] A. Minkin, Equiconvergence theorems for differential operators. Functional analysis, 4. J. Math. Sci. (New York) 96 (1999), no. 6, 3631–3715.

[20] A. Minkin, Resolvent growth and Birkhoff-regularity. J. Math. Anal. Appl. 323 (2006), no. 1, 387–402.

[21] M. A. Naimark, Linear Differential Operators, Moscow, Nauka, 1969 (Russian) (English transl.: Part 1, Elementary Theory of Linear Differential Operators, Ungar, New York, 1967; Part 2: Linear Differential Operators in Hilbert Space, Ungar, New York, 1968)

[22] A. M. Savchuk and A. A. Shkalikov, Sturm–Liouville operators with distribution potentials (Russian) Trudy Mosk. Mat. Obs. 64 (2003), 159–212; English transl. in Trans. Moscow Math. Soc. 2003, 143–192.

[23] A. A. Shkalikov, The basis property of eigenfunctions of an ordinary differential operator, Uspekhi Mat. Nauk 34 (1979), 235–236 (Russian)

[24] A. A. Shkalikov, On the basisness property of eigenfunctions of ordinary differential operators with integral boundary conditions, Vestnik Mosk. Univ., ser. 1, Math. & Mech., 6 (1982), 12–21.

[25] A. A. Shkalikov, Boundary value problems for ordinary differential equations with a parameter in the boundary conditions, Trudy Sem. I. G. Petrovskogo 9 (1983), 190–229 (Russian); English transl.: J. Sov. Math. 33 (6) (1986), 1311–1342.

[26] E. C. Titchmarsh, Eigenfunction Expansions associated with Second-Order Differential Equations, Part II, Oxford University Press, Oxford, 1958.

[27] O. A. Veliev, A. A. Shkalikov, On Riesz basisness of eigenfunctions and associated functions of periodic and anti-periodic Sturm–Liouville problems, Matem. Zamet., 85 (2009), 671–686.

Sabanci University, Orhanli, 34956 Tuzla, Istanbul, Turkey

E-mail address: djakov@sabanciuniv.edu

Department of Mathematics, The Ohio State University, 231 West 18th Ave, Columbus, OH 43210, USA

E-mail address: mityagin.1@osu.edu