MONODROMY GROUP FOR A STRONGLY SEMISTABLE PRINCIPAL BUNDLE OVER A CURVE, II

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Abstract. Let $X$ be a geometrically irreducible smooth projective curve defined over a field $k$. Assume that $X$ has a $k$–rational point; fix a $k$–rational point $x \in X$. From these data we construct an affine group scheme $G_X$ defined over the field $k$ as well as a principal $G_X$–bundle $E_{G_X}$ over the curve $X$. The group scheme $G_X$ is given by a $\mathbb{Q}$–graded neutral Tannakian category built out of all strongly semistable vector bundles over $X$. The principal bundle $E_{G_X}$ is tautological. Let $G$ be a linear algebraic group, defined over $k$, that does not admit any nontrivial character which is trivial on the connected component, containing the identity element, of the reduced center of $G$. Let $E_G$ be a strongly semistable principal $G$–bundle over $X$. We associate to $E_G$ a group scheme $M$ defined over $k$, which we call the monodromy group scheme of $E_G$, and a principal $M$–bundle $E_M$ over $X$, which we call the monodromy bundle of $E_G$. The group scheme $M$ is canonically a quotient of $G_X$, and $E_M$ is the extension of structure group of $E_{G_X}$. The group scheme $M$ is also canonically embedded in the fiber $\text{Ad}(E_G)_x$ over $x$ of the adjoint bundle.

1. Introduction

Let $X$ be a geometrically irreducible smooth projective curve defined over a field $k$ such that $X$ admits a $k$–rational point. Fix a $k$–rational point $x$ of $X$. From this data we construct a neutral Tannakian category $C_X$ defined over the field $k$ in the following way. The objects of $C_X$ are all maps $f$ from the rational numbers to the strongly semistable vector bundles over $X$ such that $f(\lambda) = 0$ for all but finitely many $\lambda \in \mathbb{Q}$, and if $f(\lambda) \neq 0$, then

$$\mu(f(\lambda)) := \frac{\text{degree}(f(\lambda))}{\text{rank}(f(\lambda))} = \lambda.$$ 

For any $f, f' \in C_X$, set

$$\text{Hom}(f, f') := \prod_{\lambda \in \mathbb{Q}} H^0(X, \text{Hom}(f(\lambda), f'(\lambda))).$$

We note that for any two vector bundles $V$ and $W$ over $X$, with

$$\mu(V) := \frac{\text{degree}(V)}{\text{rank}(V)} = \frac{\text{degree}(W)}{\text{rank}(W)} =: \mu(W),$$

we have $\mu(V \oplus W) = \mu(V)$. If $V$ and $W$ are also strongly semistable, then for any

$$\phi \in H^0(X, \text{Hom}(V, W)),$$

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either the homomorphism \( \phi \) is injective (respectively, surjective) or kernel(\( \phi \)) (respectively, cokernel(\( \phi \))) is a strongly semistable vector bundle with same degree/rank quotient as that of \( V \); the details are in Section 2. These properties imply that \( \mathcal{C}_X \) is an abelian category.

For any \( f, f' \in \mathcal{C}_X \), define their tensor product

\[
(f \otimes f')(\lambda) := \bigoplus_{z \in \mathbb{Q}} f(z) \otimes f'(\lambda - z),
\]

and define \( f^* \) by \( f^*(\lambda) = f(-\lambda)^* \). It can be shown that \( f \otimes f', f^* \in \mathcal{C}_X \). The object \( f_0 \in \mathcal{C}_X \), defined by \( f_0(\lambda) = 0 \) for \( \lambda \neq 0 \) and \( f_0(0) = \mathcal{O}_X \) (the structure sheaf of \( X \)), acts as the identity element for the tensor product operation on \( \mathcal{C}_X \).

Using the \( k \)–rational point \( x \in X \), we have a fiber functor on \( \mathcal{C}_X \) that sends any object \( f \) to the \( k \)–vector space \( \bigoplus_{z \in \mathbb{Q}} f(z)_x \).

All these operations together define a neutral Tannakian category over \( k \). Let \( \mathcal{G}_X \) denote the affine group scheme defined over \( k \) given by this neutral Tannakian category \( \mathcal{C}_X \).

Let \( \text{Vect}(X) \) denote the category of vector bundles over the curve \( X \). We have a covariant functor from \( \mathcal{C}_X \) to \( \text{Vect}(X) \) defined by

\[
\mathcal{C}_X \ni f \mapsto \bigoplus_{z \in \mathbb{Q}} f(z).
\]

This functor is compatible with the operations of tensor product, direct sum and dualization. Therefore, this functor defines a principal \( \mathcal{G}_X \)–bundle \( E_{\mathcal{G}_X} \) over \( X \).

The fundamental group scheme of \( X \) constructed in \([6, 7]\) is a quotient of \( \mathcal{G}_X \).

Let \( G \) be a linear algebraic group defined over the field \( k \) with the property that there is no nontrivial character of \( G \) which is trivial on the center of \( G \). Let \( Z_0(G) \) denote the maximal split torus contained in the reduced center of \( G \).

Take a strongly semistable principal \( G \)–bundle \( E_G \) over \( X \). Given a finite dimensional left \( G \)–module \( V \), consider the isotypical decomposition

\[
V = \bigoplus_{\chi \in Z_0(G)^*} V_{\chi}
\]

of the \( Z_0(G) \)–module \( V \), where \( Z_0(G)^* \) is the group of characters of \( Z_0(G) \). Since the actions of \( Z_0(G) \) and \( G \) on \( V \) commute, each \( V_{\chi} \) is a \( G \)–module. Let \( E_V \) (respectively, \( E_{V_{\chi}} \)) be the vector bundle over \( X \) associated to the principal \( G \)–bundle \( E_G \) for the \( G \)–module \( V \) (respectively, \( V_{\chi} \)). It can be shown that the vector bundle \( V_{\chi} \) is strongly semistable. Also, we have a homomorphism

\[
\delta_{E_G} : Z_0(G)^* \longrightarrow \mathbb{Q}
\]

that sends any character \( \chi' \) to \( \frac{\text{degree}(V'_{\chi'})}{\text{rank}(V'_{\chi'})} \) for some \( G \)–module \( V' \); it does not depend on the choice of \( V' \) (Corollary 3.3).
Therefore, for any finite dimensional left $G$–module $V$, we have an object $f_{E_G,V}$ of $C_X$ defined by

$$f_{E_G,V}(\lambda) := \bigoplus_{\{\chi \in Z_0(G)^*|\delta_{E_G}(\chi) = \lambda\}} E_{V\chi}.$$ 

We construct a subcategory of the Tannakian category $C_X$ by considering all objects of $C_X$ isomorphic to some subquotient of some $f_{E_G,V}$, where $V$ runs over all finite dimensional left $G$–modules. This subcategory gives a quotient group scheme $\mathcal{G}_X$, which we call the monodromy group scheme of $E_G$. Let $M$ denote the monodromy group scheme of $E_G$. Let $E_M$ be the principal $M$–bundle over $X$ obtained by extending the structure group of the principal $\mathcal{G}_X$–bundle $E_{\mathcal{G}_X}$. The details of these constructions are in Section 3.

In [1], the monodromy group scheme and the monodromy bundle were constructed under the extra assumptions that the base field is algebraically closed and the group $G$ is semisimple.

2. A universal Tannakian category for a pointed curve

Let $k$ be any field. Let $X$ be a geometrically irreducible smooth projective curve defined over $k$.

A vector bundle $W$ over $X$ is called semistable if for every subbundle $W' \subset W$ of positive rank, the inequality $\text{degree}(W')/\text{rank}(W') \leq \text{degree}(W)/\text{rank}(W)$ holds. We recall that the rational number $\frac{\text{degree}(W)}{\text{rank}(W)}$ is called the slope of $W$, and it is denoted by $\mu(W)$.

**Proposition 2.1** ([5], [4]). Let $\ell$ be a field extension of $k$. A vector bundle $W$ over $X$ is semistable if and only if the base change $W \otimes_k \ell$ over $X \times_k \ell$ is semistable.

The above proposition is proved in [5] under the assumption that $k$ is infinite (see [5, page 97, Proposition 3]), and it is proved in [4] under the assumption that $k$ is perfect (see [4, page 222]). We note that if $W$ is not semistable, then it is immediate that $W \otimes_k \ell$ is not semistable.

Consider the diagram

$$
\begin{array}{cccc}
X & \overset{\pi}{\longrightarrow} & X_1 & \overset{\varphi}{\longrightarrow} & X \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec}(k) & \overset{F_k}{\longrightarrow} & \text{Spec}(k)
\end{array}
$$

where $F_k$ is the Frobenius map of $k$ if the characteristic of the field $k$ is positive, and it is the identity map when the characteristic is zero, the square is Cartesian, and $\pi$ is the relative Frobenius map (see [9, page 118]). The composition $\varphi \circ \pi$ will be denoted by $F_X$.

A semistable vector bundle $W$ over $X$ is called strongly semistable if the iterated pull back

$$(F_X \circ \cdots \circ F_X)^n W$$

is semistable for all $n \geq 1$. 

Remark 2.2. Strongly semistable vector bundle are usually defined under the assumption that the base field is perfect. In view of Proposition 2.1, the above definition is compatible with it.

Let $\mathcal{C}_X$ denote the space of all maps $f$ from $\mathbb{Q}$ to the space of all strongly semistable vector bundles over $X$ satisfying the following two conditions:

- All but finitely many rational numbers are sent by $f$ to the vector bundle of rank zero.
- For any $\eta \in \mathbb{Q}$ with $f(\eta) \neq 0$,
  \[ \mu(f(\eta)) := \frac{\text{degree}(f(\eta))}{\text{rank}(f(\eta))} = \eta. \]

Therefore, $\mathcal{C}_X$ consists of finite collections of strongly semistable vector bundles over $X$ of distinct slopes. In other words, any element of $\mathcal{C}_X$ is of the form

\[ V := (V_{\lambda_1}, \cdots, V_{\lambda_n}), \]

where $\lambda_1 < \cdots < \lambda_n$ are finitely many (possibly empty) rational numbers, and for each $i \in [1, n]$, $V_i$ is a strongly semistable vector bundle over $X$ with $\mu(V_i) = \lambda_i$.

For any $f, f' \in \mathcal{C}_X$, define $f \oplus f' \in \mathcal{C}_X$ to be the function that sends any $\lambda \in \mathbb{Q}$ to the direct sum of vector bundles $f(\lambda) \oplus f'(\lambda)$. Since both $f(\lambda)$ and $f'(\lambda)$ are strongly semistable with

\[ \mu(f(\lambda)) = \mu(f'(\lambda)) = \lambda \]

provided $f(\lambda) \neq 0 \neq f'(\lambda)$, the vector bundle $f(\lambda) \oplus f'(\lambda)$ is also strongly semistable with

\[ \mu(f(\lambda) \oplus f'(\lambda)) = \lambda \]

provided $f(\lambda) \oplus f'(\lambda) \neq 0$.

If $V$ and $W$ are two strongly semistable vector bundles over $X$, then the vector bundle $V \otimes W$ is also strongly semistable; this follows from Remark 2.2 and [8, page 288, Theorem 3.23] (reproduced in Theorem 3.5). We further have

\[ \mu(V \otimes W) = \mu(V) + \mu(W). \]

This enables us to define the tensor product operation on $\mathcal{C}_X$ in the following way.

For any $f, f' \in \mathcal{C}_X$, define

\[ (f \otimes f')(\lambda) := \bigoplus_{z \in \mathbb{Q}} f(z) \otimes f'(\lambda - z). \]

If other words, if $f = V$ as in eqn. (2.2), then

\[ (f \otimes f')(\lambda) = (V_{\lambda_1} \otimes f'(\lambda - \lambda_1)) \bigoplus (V_{\lambda_2} \otimes f'(\lambda - \lambda_2)) \bigoplus \cdots \bigoplus (V_{\lambda_n} \otimes f'(\lambda - \lambda_n)). \]

Since both $f(c)$ and $f'(c)$ are zero except for finitely many $c$, eqn. (2.4) is a finite direct sum. Therefore, using eqn. (2.3) we conclude that $f \otimes f' \in \mathcal{C}_X$. 
For any \( f \in \mathcal{C}_X \), define the dual \( f^* \) of \( f \) to be the function from \( \mathbb{Q} \) to the strongly semistable vector bundles over \( X \) that sends any \( \lambda \in \mathbb{Q} \) to the dual vector bundle \( f(-\lambda)^* \). Clearly, we have \( f^* \in \mathcal{C}_X \).

For any \( f, f' \in \mathcal{C}_X \), a homomorphism from \( f \) to \( f' \) is a function

\[
\gamma : \mathbb{Q} \longrightarrow \bigoplus_{z \in \mathbb{Q}} H^0(X, \text{Hom}(f(z), f'(z)))
\]

such that \( \gamma(z) \in H^0(X, \text{Hom}(f(z), f'(z))) \) for all \( z \in \mathbb{Q} \). By \( \text{Hom}(f, f') \) we will denote the set of all homomorphisms from \( f \) to \( f' \). So

\[
\text{Hom}(f, f') := \prod_{\lambda \in \mathbb{Q}} H^0(X, \text{Hom}(f(\lambda), f'(\lambda))).
\]

A homomorphism \( \gamma \) from \( f \) to \( f' \) will be called an isomorphism if

\[
\gamma(z) : f(z) \longrightarrow f'(z)
\]
is an isomorphism for all \( z \in \mathbb{Q} \).

Let \( \phi : V \longrightarrow W \) be a homomorphism between strongly semistable vector bundles \( V \) and \( W \) over \( X \) with

\[
(2.5) \quad \mu(V) = \mu(W).
\]

Then it can be shown that either \( \phi \) is injective, or \( \text{kernel}(\phi) \) is a strongly semistable vector bundle over \( X \) with

\[
\mu(\text{kernel}(\phi)) = \mu(V).
\]

Indeed, if \( \phi \) is nonzero with \( \text{kernel}(\phi) \) a nonzero subsheaf of \( V \), then consider \( V/\text{kernel}(\phi) \). We note that \( V/\text{kernel}(\phi) \) is a quotient of \( V \) as well as a subsheaf of \( W \). Therefore, as \( V \) and \( W \) are semistable, we have

\[
(2.6) \quad \mu(V) \leq \mu(V/\text{kernel}(\phi)) \leq \mu(W).
\]

Now from eqn. (2.5) it follows that both the inequalities in eqn. (2.6) are equalities. Consequently, \( \mu(V) = \mu(\text{kernel}(\phi)) \). It also follows that \( V/\text{kernel}(\phi) \) is torsionfree, because the inverse image in \( V \), of the torsion part of \( V/\text{kernel}(\phi) \), has slope strictly greater than \( \mu(\text{kernel}(\phi)) \) if \( V/\text{kernel}(\phi) \) has torsion. Note that as \( V \) is semistable, it does not have any subsheaf with slope larger than \( \mu(V) \). Since \( V/\text{kernel}(\phi) \) is torsionfree, we conclude that \( \text{kernel}(\phi) \) is a subbundle of \( V \).

Similarly, either \( \phi \) is surjective or \( \text{cokernel}(\phi) := W/\phi(V) \) is a strongly semistable vector bundle over \( X \) with

\[
\mu(\text{cokernel}(\phi)) = \mu(W).
\]

(Replace \( \phi \) by its dual \( \phi^* \) in the above argument.) This enables us to define the kernel and the cokernel of any homomorphism between two objects in \( \mathcal{C}_X \).

Take any \( f, f' \in \mathcal{C}_X \) and any \( \gamma \in \text{Hom}(f, f') \). Consider the function from \( \mathbb{Q} \) to the space of all strongly semistable vector bundles over \( X \) that sends any \( \lambda \in \mathbb{Q} \) to the kernel of the homomorphism

\[
\gamma(\lambda) : f(\lambda) \longrightarrow f'(\lambda).
\]

This function defines an object of \( \mathcal{C}_X \), which we will call the kernel of \( \gamma \), and it will be denoted by \( \text{kernel}(\gamma) \). Similarly, consider the function from \( \mathbb{Q} \) to the space of all
strongly semistable vector bundles over \( X \) that sends any \( \lambda \in \mathbb{Q} \) to the cokernel of the homomorphism

\[
\gamma(\lambda) : f(\lambda) \longrightarrow f'(\lambda).
\]

The object of \( \mathcal{C}_X \) defined by this function will be denoted by \( \text{cokernel}(\gamma) \), and it will be called the *cokernel* of \( \gamma \).

**Remark 2.3.** The abelian category

\[
\mathcal{C}_X = \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{C}_X^\lambda,
\]

where \( \mathcal{C}_X^\lambda \) is the abelian category of strongly semistable vector bundles over \( X \) of slope \( \lambda \). The index \( \mathbb{Q} \) in the above direct sum acts as a weight that guides the tensor product and the dualization operations.

Take any closed point \( x \) of \( X \). The residue field will be denoted by \( k(x) \). For any vector bundle \( V \) over \( X \), the fiber of \( V \) over \( x \), which is a \( k(x) \)-vector space, will be denoted by \( V_x \). The category of finite dimensional vector spaces over the field \( k(x) \) will be denoted by \( k(x)\text{-mod} \).

We have a functor

\[
\omega : \mathcal{C}_X \longrightarrow k(x)\text{-mod}
\]

defined by

\[
f \longmapsto \bigoplus_{z \in \mathbb{Q}} f(z)_x.
\]

In other words, for any

\[
V := (V_{\lambda_1}, \ldots, V_{\lambda_n}) \in \mathcal{C}_X
\]

as in eqn. (2.2), we have \( \omega(V) = (V_{\lambda_1})_x \oplus \cdots \oplus (V_{\lambda_n})_x \).

Let \( \mathcal{O}_X \) be the trivial line bundle over \( X \) defined by the structure sheaf of \( X \). The object in \( \mathcal{C}_X \) defined by \( \mathcal{O}_X \) will also be denoted by \( \mathcal{O}_X \).

The triple \( (\mathcal{C}_X, \mathcal{O}_X, \omega) \) together form a Tannakian category over \( k \); see [10], [3] for Tannakian category. We recall that a Tannakian category over \( k \) is a rigid abelian tensor category \( \mathcal{C}' \) such that

- \( \text{End}(1) = k \), and
- there is a field extension \( k' \) of \( k \) and a \( k \)-linear fiber functor from \( \mathcal{C}' \) to the category of vector spaces over \( k' \).

Henceforth, we will assume that \( X \) admits a \( k \)-rational point. Fix a \( k \)-rational point \( x \) of \( X \).

Since \( k(x) = k \), the above Tannakian category defined by the triple \( (\mathcal{C}_X, \mathcal{O}_X, \omega) \) is a neutral Tannakian category. Hence they define an affine group scheme defined over \( k \) [3 page 130, Theorem 2.11], [71 Theorem 1.1], [10 Theorem 1].

Let \( \mathcal{G}_X \) denote the group scheme defined over \( k \) given by the neutral Tannakian category \( (\mathcal{C}_X, \mathcal{O}_X, \omega) \).

We note that the above neutral Tannakian category is \( \mathbb{Q} \)-graded (see [3 page 186]). We will now show that there is a tautological principal \( \mathcal{G}_X \)-bundle over \( X \).
Take any object

\[ V := (V_{\lambda_1}, \ldots, V_{\lambda_n}) \in C_X \]
as in eqn. (2.2). To it we associate the vector bundle \((V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n})\). In other words, we have a functor

\[ C_X \rightarrow \text{Vect}(X), \]
where \(\text{Vect}(X)\) as before is the category of vector bundles over \(X\), defined by

\[ (2.7) \quad f \mapsto \bigoplus_{z \in \mathbb{Q}} f(z). \]

Using [3, page 149, Theorem 3.2], this functor gives a principal \(G_X\)–bundle over \(X\). This principal \(G_X\)–bundle over \(X\) will be denoted by \(E_{G_X}\).

**Remark 2.4.** A vector bundle \(E\) over \(X\) is called *finite* if there are two distinct polynomials \(f_1, f_2 \in \mathbb{Z}[t]\) with nonnegative coefficients such that the vector bundle \(f_1(E)\) is isomorphic to \(f_2(E)\). Any finite vector bundle over \(X\) is strongly semistable of degree zero. A vector bundle \(V\) of degree zero over \(X\) is called *essentially finite* if there is a finite vector bundle \(E\) over \(X\) and a quotient bundle \(Q\) of \(E\) of degree zero such that the vector bundle \(V\) is a subbundle of \(Q\); see [6], [7]. Essentially finite vector bundles over \(X\) form a neutral Tannakian category, and the corresponding group scheme is called the *fundamental group scheme* of \(X\) [7]. Since any essentially finite vector bundle over \(X\) is strongly semistable, the fundamental group scheme of \(X\) is a quotient of the group scheme \(G_X\) constructed above.

### 3. Monodromy of Principal Bundles

Let \(G\) be a linear algebraic group defined over the field \(k\). Let \(Z'_0(G)\) denote the connected component, containing the identity element, of the reduced center of \(G\). We assume that \(G\) satisfies the following condition: there is no nontrivial character of \(G\) which is trivial on \(Z'_0(G)\). Let

\[ (3.1) \quad Z_0(G) \subset Z'_0(G) \]
be the (unique) maximal split torus contained in \(Z'_0(G)\). The above condition on \(G\) implies that there is no nontrivial character of \(G\) which is trivial on \(Z_0(G)\).

The subgroup \(Z_0(G)\) gives a decomposition of any \(G\)–module, which we will describe now.

Let \(V\) be a finite dimensional left \(G\)–module. Consider the action on \(V\) of the subgroup \(Z_0(G)\) in eqn. (3.1). Since \(Z_0(G)\) is a product of copies of \(\mathbb{G}_m\), the vector space \(V\) decomposes into a direct sum of one dimensional subspaces such that each of the one dimensional subspaces is preserved by the action of \(Z_0(G)\). Therefore, we obtain a finite collection of distinct characters of \(Z_0(G)\), say \(\chi_1, \ldots, \chi_m\), such that for each line \(\xi \subset V\) preserved by the action of \(Z_0(G)\) on \(V\), there is a character \(\chi_i\), for some \(i \in [1, m]\), such that \(Z_0(G)\) acts on \(\xi\) as scalar multiplications through the character \(\chi_i\).

For any \(i \in [1, m]\), let

\[ V_i \subset V \]
be the linear subspace on which $Z_0(G)$ acts as scalar multiplications through the character $\chi_i$. This subspace $V_i$ is preserved by the action of $G$ on $V$. Indeed, this follows immediately from the fact that the actions of $G$ and $Z_0(G)$ on $V$ commute. Therefore, we have a natural decomposition

$$V = V_1 \bigoplus \cdots \bigoplus V_m$$

(3.2)

of the $G$–module $V$. This is clearly the isotypical decomposition of the $Z_0(G)$-module $V$.

Let $Z_0(G)^*$ denote the group of all characters of $Z_0(G)$. We may reformulate the decomposition in eqn. (3.2) in the following way. Any finite dimensional left $G$–module has a natural decomposition

$$V = \bigoplus_{\chi \in Z_0(G)^*} V_\chi,$$

(3.3)

where $V_\chi \subset V$ is the subspace on which $Z_0(G)$ acts as scalar multiplications through the character $\chi$.

As before, let $X$ be a geometrically irreducible smooth projective curve defined over the field $k$.

**Lemma 3.1.** Take any character $\chi \in Z_0(G)^*$. Let $V$ and $W$ be two nonzero finite dimensional left $G$–modules such that $Z_0(G)$ acts on both $V$ and $W$ as scalar multiplications through the character $\chi$. Let $E_G$ be a principal $G$–bundle over $X$. Let $E_V$ (respectively, $E_W$) be the vector bundle over $X$ associated to the principal $G$–bundle $E_G$ for the $G$–module $V$ (respectively, $W$). Then

$$\mu(E_V) = \mu(E_W).$$

**Proof.** The $G$–module structures on $V$ and $W$ together induce a $G$–module structure on the vector space $\operatorname{Hom}(V,W) = V^* \otimes W$. The vector bundle over $X$ associated to $E_G$ for the $G$–module $\operatorname{Hom}(V,W)$ is evidently identified with the vector bundle $E_V^* \otimes E_W$.

Since $Z_0(G)$ acts on both $V$ and $W$ as scalar multiplications through the character $\chi$, the action of $Z_0(G)$ on the $G$–module $\operatorname{Hom}(V,W)$ is the trivial action. In particular, the action of $Z_0(G)$ on the one–dimensional $G$–module $\Lambda^{\text{top}} \operatorname{Hom}(V,W)$ is the trivial action.

By our assumption on $G$, the group $G/Z_0(G)$ does not admit any nontrivial character. Therefore, from the observation that $Z_0(G)$ acts trivially on the one–dimensional $G$–module $\Lambda^{\text{top}} \operatorname{Hom}(V,W)$ we conclude that the action of $G$ on $\Lambda^{\text{top}} \operatorname{Hom}(V,W)$ is the trivial action. This immediately implies that the line bundle over $X$ associated to the principal $G$–bundle $E_G$ for the $G$–module $\Lambda^{\text{top}} \operatorname{Hom}(V,W)$ is a trivial line bundle.

The line bundle over $X$ associated to the principal $G$–bundle $E_G$ for the $G$–module $\Lambda^{\text{top}} \operatorname{Hom}_k(V,W)$ is clearly identified with $\Lambda^{\text{top}} (E_V^* \otimes E_W)$. Since $\Lambda^{\text{top}} (E_V^* \otimes E_W)$ is a trivial line bundle, we have

$$\deg(\Lambda^{\text{top}} (E_V^* \otimes E_W)) = 0.$$ 

Now from the identity

$$\deg(\Lambda^{\text{top}} (E_V^* \otimes E_W)) = \deg(E_W) \operatorname{rank}(E_V) - \deg(E_V) \operatorname{rank}(E_W),$$
it follows that $\mu(E_V) = \mu(E_W)$. This completes the proof of the lemma. \qed

The following is a corollary of [3, page 139, Proposition 2.21].

**Corollary 3.2.** All characters of $Z_0(G)$ arise from the indecomposable representations of $G$. In other words, for any character $\chi$ of $Z_0(G)$, there is some nonzero finite dimensional indecomposable left $G$–module $V$ such that $Z_0(G)$ acts on $V$ as scalar multiplications through the character $\chi$.

If $E$ and $F$ are two vector bundles over $X$, then $\mu(E \otimes F) = \mu(E) + \mu(F)$. Therefore, Lemma 3.1 and Corollary 3.2 combine together to give the following corollary:

**Corollary 3.3.** Fix any principal $G$–bundle $E_G$ over $X$. Then there is a homomorphism to the additive group

$$\delta_{E_G} : Z_0(G)^* \longrightarrow \mathbb{Q}$$

that sends any character $\chi$ to $\frac{\deg(E_V)}{\rank(E_V)}$, where $V$ is a finite dimensional nonzero left $G$–module on which $Z_0(G)$ acts as scalar multiplications through the character $\chi$, and $E_V$ is the vector bundle over $X$ associated to the principal $G$–bundle $E_G$ for the $G$–module $V$.

**Definition 3.4.** Let $G$ be any affine group scheme defined over $k$. A principal $G$–bundle $E_G$ over a geometrically irreducible smooth projective curve $X$ will be called strongly semistable if for any indecomposable finite dimensional left $G$–module $V \in G$–mod, the vector bundle over $X$ associated to $E_G$ for $V$ is strongly semistable.

See [9, 8] for the definition of a (strongly) semistable principal bundle with a reductive group as the structure group. We will show that the above definition coincides with the usual definition when $G$ is reductive. For that we will need the following theorem.

**Theorem 3.5** (RR, Theorem 3.23). Let $H$ and $H'$ be reductive linear algebraic groups defined over $k$ and

$$\rho : H \longrightarrow H'$$

e a homomorphism of algebraic groups such that $\rho(Z'_0(H)) \subset Z'_0(H')$, where $Z'_0(H)$ (respectively, $Z'_0(H')$) is the connected component, containing the identity element, of the reduced center of $H$ (respectively, $H'$). Let $E_H$ be a strongly semistable principal $H$–bundle over $X$. Then the principal $H'$–bundle $E_{H'} := E_H(H')$ over $X$, obtained by extending the structure group of $E_H$ using $\rho$, is also strongly semistable.

Definition 3.4 is justified by the following lemma.

**Lemma 3.6.** Let $H$ be a reductive linear algebraic group and $X$ a geometrically irreducible smooth projective curve defined over $k$. A principal $H$–bundle $E_H$ over $X$ is strongly semistable if and only if for every indecomposable $H$–module $V$, the vector bundle $E_V = E_H(V)$ over $X$ associated to the principal $H$–bundle $E_H$ for $V$ is strongly semistable.

**Proof.** Let $E_H$ be a strongly semistable principal $H$–bundle over $X$. Take any indecomposable $H$–module $V$. Since $V$ is indecomposable, the group $Z'_0(H)$ acts on $V$ as scalar multiplications, where $Z'_0(H)$ as before is the connected component, containing the identity element, of the reduced center of $H$. Therefore, from Theorem 3.5 it follows that the associated vector bundle $E_V$ is strongly semistable.
To prove the converse, let $E_H$ be principal $H$–bundle over $X$ such that for every indecomposable $H$–module $V$, the vector bundle over $X$ associated to the principal $H$–bundle $E_H$ for $V$ is strongly semistable.

Let $\text{ad}(E_H)$ be the adjoint vector bundle for $E_H$. We recall that $\text{ad}(E_H)$ is the vector bundle over $X$ associated to the principal $H$–bundle $E_H$ for the adjoint action of $H$ on its Lie algebra $\mathfrak{h}$. To prove that the principal $H$–bundle $E_H$ is strongly semistable, it suffices to show that the vector bundle $\text{ad}(E_H)$ is strongly semistable. To see this, let

$$E_P \subset (F^*_X E_H)$$

be a reduction of structure group to a maximal parabolic subgroup $P \subset H$ that violates the semistability condition, or in other words, we have

$$(3.4) \quad \text{degree}((F^*_X \text{ad}(E_H))/\text{ad}(E_P)) < 0$$

(see [9], [8]). Consider the subbundle

$$\text{ad}(E_P) \subset \text{ad}((F^*_X \text{ad}(E_H)) = (F^*_X \text{ad}(E_H)).$$

Since $\text{degree}(\text{ad}(E_H)) = 0$ (as $\bigwedge^{\text{top}} \mathfrak{h}$ is a trivial $H$–module), from eqn. (3.4) it follows immediately that the subbundle $\text{ad}(E_P)$ contradicts the semistability condition of the vector bundle $(F^*_X \text{ad}(E_H))$.

Consider the Lie algebra $\mathfrak{h}$ as a $H$–module using the adjoint action. Note that $Z'_0(H)$ acts trivially on $\mathfrak{h}$. Let

$$(3.5) \quad \mathfrak{h} = \bigoplus_{i=1}^n V_i$$

be a decomposition of $\mathfrak{h}$ into a direct sum of indecomposable $H$–modules. Let $E_{V_i}$ denote the vector bundle over $X$ associated to the principal $H$–bundle $E_H$ for the $H$–module $V_i$. From eqn. (3.5) we have

$$\text{ad}(E_H) = \bigoplus_{i=1}^n E_{V_i}.$$ 

From the given condition on $E_H$ we know that $E_{V_i}$ is strongly semistable for all $i \in [1,n]$. As $Z'_0(H)$ acts trivially on $\mathfrak{h}$, and $H/Z'_0(H)$ being semisimple does not have any nontrivial character, the induced action of $H$ on the line $\bigwedge^{\text{top}} V_i$ is the trivial action. Therefore, $\bigwedge^{\text{top}} E_{V_i}$ is a trivial line bundle. In particular,

$$\text{degree}(E_{V_i}) = 0$$

for all $i \in [1,n]$.

Since $\text{ad}(E_H)$ is a direct sum of strongly semistable vector bundles of degree zero, we conclude that the vector bundle $\text{ad}(E_H)$ is strongly semistable. This completes the proof of the lemma. $\square$

Let $E_G$ be a strongly semistable principal $G$–bundle over $X$. To each $G$–module we will associate an object of the neutral Tannakian category $\mathcal{C}_X$ that we constructed in Section 2.
Let $V$ be a finite dimensional left $G$–module. First consider the natural decomposition into a direct sum of $G$–modules

$$ V = \bigoplus_{\chi \in Z_0(G)^*} V_{\chi} $$

constructed in eqn. (3.3). Let $E_{V_{\chi}}$ be the vector bundle over $X$ associated to the principal $G$–bundle $E_G$ for the above $G$–module $V_{\chi}$.

**Lemma 3.7.** The vector bundle $E_{V_{\chi}}$ is strongly semistable, and if $V_{\chi} \neq 0$, then

$$ \mu(E_{V_{\chi}}) = \delta_{E_G}(\chi), $$

where $\delta_{E_G}$ is the homomorphism constructed in Corollary 3.8.

**Proof.** Expressing $V_{\chi}$ as a direct sum of indecomposable $G$–modules, and using the fact that a direct sum of strongly semistable vector bundles of same slope remains strongly semistable, we conclude that the associated vector bundle $E_{V_{\chi}}$ is strongly semistable (see Definition 3.4).

From the definition of $\delta_{E_G}$ it follows immediately that $\mu(E_{V_{\chi}}) = \delta_{E_G}(\chi)$ if $V_{\chi} \neq 0$. This completes the proof of the lemma. □

Lemma 3.7 has the following corollary:

**Corollary 3.8.** For any $\lambda \in \mathbb{Q}$ and any $V \in G$–mod, the direct sum

$$ E^\lambda_G(V) := \bigoplus_{\{\chi \in Z_0(G)^* | \delta_{E_G}(\chi) = \lambda\}} E_{V_{\chi}} $$

is either zero, or it is a strongly semistable vector bundle with

$$ \mu(E^\lambda_G(V)) = \lambda. $$

Finally, consider the function $f_{E_G,V}$ from $\mathbb{Q}$ to the space of all vector bundles over $X$ defined by

$$ f_{E_G,V}(\lambda) := E^\lambda_G(V), $$

where $V \in G$–mod, and $E^\lambda_G(V)$ is defined in Corollary 3.8. From Corollary 3.8 it follows immediately that this function $f_{E_G,V}$ is an object of the category $C_X$ constructed in Section 2. Therefore, to each object of $G$–mod we have associated an object of $C_X$.

Our aim is to construct a Tannakian category out of the strongly semistable principal $G$–bundle $E_G$. Before that we will introduce some definitions.

Take any object $f$ in the category $C_X$. A sub–object of $f$ is an object $f'$ in $C_X$ such that for each $\lambda \in \mathbb{Q}$, the vector bundle $f'(\lambda)$ is a subbundle of the vector bundle $f(\lambda)$. If $f'$ is a sub–object of $f$, then the object of $C_X$ that sends any $\lambda \in \mathbb{Q}$ to the quotient vector bundle $f(\lambda)/f'(\lambda)$ will be called a quotient–object of $f$.

Let $E$ be a strongly semistable vector bundle over $X$ and $E'$ a nonzero proper subbundle of $E$ with

$$ \mu(E') = \mu(E). $$

Then $E'$ is strongly semistable, and furthermore, the quotient vector bundle $E/E'$ is also strongly semistable with $\mu(E/E') = \mu(E)$ if $E/E' \neq 0$. Therefore, for an object $f$ of the
category $C_X$, any quotient–object of $f$ also lie in $C_X$. Also, given a subbundle $V'_\lambda \subset f(\lambda)$ for each $\lambda \in \mathbb{Q}$, to check that the function

$$\lambda \mapsto V'_\lambda$$

is a sub–object of $f$, all we need to check that

$$\mu(V'_\lambda) = \lambda$$

for all $\lambda \in \mathbb{Q}$ with $V'_\lambda \neq 0$.

For any object $f$ of the category $C_X$, a sub-quotient of $f$ is a sub–object of some quotient–object of $f$.

Let $C_{EG}$ denote the subcategory of $C_X$ defined by all objects $f$ of $C_X$ such that there exists some $V \in G$–mod with the property that $f$ is isomorphic to a sub-quotient of $f_{EG,V}$, where $f_{EG,V}$ is the object of $C_X$ constructed from $V$ in eqn. (3.6). The morphisms remain unchanged. In other words, for any two objects $f$ and $f'$ in $C_{EG}$, the morphisms from $f$ to $f'$ are the morphisms from $f$ to $f'$ considered as objects of $C_X$.

It is straightforward to check that $C_{EG}$ is a neutral Tannakian subcategory of $C_X$. Therefore, the neutral Tannakian category $C_{EG}$ gives an affine group scheme defined over $k$.

**Definition 3.9.** The affine group scheme defined over $k$ given by the neutral Tannakian category $C_{EG}$ will be called the monodromy group scheme of $E_G$. The monodromy group scheme of $E_G$ will be denoted by $M$.

Since $C_{EG}$ is a Tannakian subcategory of $C_X$, the monodromy group scheme $M$ is a quotient of the group scheme $G_X$ constructed in Section 2 (see [3, Proposition 2.21]). Let

$$\phi_{EG} : G_X \rightarrow M$$

be the quotient map.

Just as we have the tautological $G_X$–bundle $E_{G_X}$ (see eqn. (2.7)), there is a tautological principal $M$–bundle over $X$.

**Definition 3.10.** Let $E_M$ denote the tautological principal $M$–bundle over $X$. This principal $M$–bundle $E_M$ will be called the monodromy bundle for $E_G$.

The principal $M$–bundle $E_M$ is evidently the one obtained by extending the structure group of the principal $G_X$–bundle $E_{G_X}$ using the homomorphism $\phi_{EG}$ in eqn. (3.7).

We will next show that there is a tautological embedding of the monodromy group scheme $M$ into the fiber, over the fixed $k$–rational point $x \in X$, of the adjoint bundle for $E_G$.

Let $Ad(E_G)$ be the adjoint bundle for the principal $G$–bundle $E_G$ over $X$. Let $Ad(E_G)_x$ be the fiber of $Ad(E_G)$ over the fixed $k$–rational point $x$ of $X$.

If $\omega$ is the fiber functor for the principal $G$–bundle $E_G$ over $X$, then the group $Ad(E_G)_x$ defined over $k$ represents the functor $\underline{\text{Aut}}^\otimes(\omega)$. Using Theorem 2.11 in [3, page 130], we get a natural homomorphism from the group scheme $G_X$ (constructed in Section 2) to $Ad(E_G)_x$. Let

$$\Phi(E_G) : G_X \rightarrow Ad(E_G)_x$$

be the tautological bundle.
be this natural homomorphism.

It is easy to see that the image of the homomorphism $\Phi(E_G)$ in eqn. \((3.8)\) coincides with the monodromy group scheme $M$ in Definition \((3.9)\).

Therefore, we have the following proposition:

**Proposition 3.11.** The monodromy group scheme $M$ for $E_G$ (introduced in Definition \((3.9)\)) is identified with the image of the homomorphism $\Phi(E_G)$ in eqn. \((3.8)\). In other words, the kernel of the homomorphism $\Phi(E_G)$ coincides with the kernel of the homomorphism $\phi_{E_G}$ in eqn. \((3.7)\).

There is a natural inclusion $M \hookrightarrow \text{Ad}(E_G)_x$ obtained from the fact that the quotients of $G_X$ for the two homomorphisms $\Phi(E_G)$ and $\phi_{E_G}$ coincide.

We will now investigate the behavior of the monodromy group and the monodromy bundle under the extensions of structure group.

Let
\begin{equation}
\rho : G \longrightarrow G_1
\end{equation}
be an algebraic homomorphism between linear algebraic groups defined over $k$. Let $Z_0(G_1)$ denote the (unique) maximal split torus contained in the reduced center of $G_1$. We assume the following:

- The group $G_1$ does not admit any nontrivial character which is trivial on $Z_0(G_1)$.
- The homomorphism $\rho$ in eqn. \((3.9)\) satisfies the condition
\begin{equation}
\rho(Z_0(G)) \subset Z_0(G_1).
\end{equation}

**Lemma 3.12.** Let $E_G$ be a strongly semistable principal $G$–bundle over $X$. Then the principal $G_1$–bundle $E_{G_1} := E_G(G_1)$, obtained by extending the structure group of $E_G$ using $\rho$ (defined in eqn. \((3.9)\)) is also strongly semistable.

**Proof.** Take an indecomposable $G_1$–module $V$. So $Z_0(G_1)$ acts on $V$ as scalar multiplications through a character. Therefore, using eqn. \((3.10)\) we know that $Z_0(G)$ acts on $V$ as scalar multiplications through a character. Since $E_G$ is strongly semistable, Corollary \((3.8)\) says that the associated vector bundle $E_G(V) = E_{G_1}(V)$ is strongly semistable. Hence $E_{G_1}$ is strongly semistable (see Definition \((3.4)\)).

Let $E_G$ be a strongly semistable principal $G$–bundle over $X$. Hence by Lemma \((3.12)\), the principal $G_1$–bundle $E_{G_1}$, obtained by extending the structure group of the principal $G$–bundle $E_G$ using the homomorphism $\rho$, is also strongly semistable. The homomorphism $\rho$ in eqn. \((3.9)\) induces a homomorphism of group schemes
\begin{equation}
\tilde{\rho} : \text{Ad}(E_G) \longrightarrow \text{Ad}(E_{G_1})
\end{equation}
over $X$.

**Lemma 3.13.** The monodromy group scheme $M_1 \subset \text{Ad}(E_{G_1})_x$ for $E_{G_1}$ is the image $\tilde{\rho}(x)(M)$, where $\tilde{\rho}(x)$ is the homomorphism in eqn. \((3.11)\) restricted to the $k$–rational point $x$ of $X$, and $M \subset \text{Ad}(E_G)_x$ is the monodromy group scheme of $E_G$ (see Proposition \((3.11)\)). Furthermore, the monodromy bundle $E_{M_1}$ for $E_{G_1}$ is the extension of structure
group of the monodromy bundle $E_M$ for $E_G$ by the homomorphism $M \rightarrow M_1$ obtained by restricting $\tilde{\rho}(x)$.

Proof. Take any $G_1$–module $V \in G_1$–mod. The $G$–module given by $V$ using the homomorphism $\rho$ in eqn. (3.9) will also be denoted by $V$. Consider the isotypical decomposition of the $Z_0(G)$–module $V$, and also consider the isotypical decomposition of the $Z_0(G_1)$–module $V$. The second decomposition is finer in the following sense. From eqn. (3.10) we get a homomorphism $\rho^*: Z_0(G_1)^* \rightarrow Z_0(G)^*$ of character groups. For any $\chi \in Z_0(G)^*$, the component of $V$ on which $Z_0(G_1)$ acts as scalar multiplications through $\chi$ is the direct sum
\[ \bigoplus_{\chi' \in (\rho^*)^{-1}(\chi)} V_{\chi'}, \]
where $V_{\chi'} \subset V$ is the subspace on which $Z_0(G_1)$ acts as scalar multiplications through $\chi'$.

Using this observation it follows that the neutral Tannakian category $\mathcal{C}_{E_G}$ for the principal $G_1$–bundle $E_G$ is a subcategory of the neutral Tannakian category $\mathcal{C}_{E_G}$ for $E_G$. Now the lemma follows from the constructions of the monodromy group scheme and the monodromy bundle and the criterion for surjectivity in [3, page 139, Proposition 2.21(a)]. □

4. SOME PROPERTIES OF THE MONODROMY GROUP SCHEME

As before, let $G$ be a linear algebraic group defined over $k$ which does not admit any nontrivial character trivial on $Z_0(G)$. Take a principal $G$–bundle $E_G$ over the curve $X$.

For a subgroup scheme $H \subset G$, let $(H \cap Z_0(G))_{\text{red}}$ denote the reduced intersection; let $(H \cap Z_0(G))_0$ denote the (unique) maximal split torus contained in the abelian group $(H \cap Z_0(G))_{\text{red}}$.

Definition 4.1. A reduction of structure group
\[ (4.1) \quad E_H \subset E_G \]
of $E_G$ to a subgroup scheme $H \subset G$ will be called balanced if for every character $\chi$ of $H$ trivial on $(H \cap Z_0(G))_0 \subset H$ (see the above definition), we have
\[ \deg(E_H(\chi)) = 0, \]
where $E_H(\chi)$ is the line bundle over $X$ associated to the principal $H$–bundle $E_H$ for the character $\chi$.

Remark 4.2. Since any character of $(H \cap Z_0(G))_{\text{red}}/(H \cap Z_0(G))_0$ is of finite order, if a character $\chi$ of $H$ is trivial on $(H \cap Z_0(G))_0$ and there is a positive integer $n$ such that the character $\chi^n$ of $H$ is trivial on $(H \cap Z_0(G))_{\text{red}}$. Therefore, a reduction $E_H \subset E_G$ as in Definition 4.1 is balanced if and only if for every character $\chi$ of $H$ trivial on $(H \cap Z_0(G))_{\text{red}}$ we have
\[ \deg(E_H(\chi)) = 0. \]
Since the quotient \((H \cap Z_0(G))/(H \cap Z_0(G))_{\text{red}}\) is a finite group scheme, if a character \(\chi\) of \(H\) is trivial on \((H \cap Z_0(G))_{\text{red}}\), then there is a positive integer \(n\) such that the character \(\chi^n\) of \(H\) is trivial on \(H \cap Z_0(G)\). Therefore, a reduction \(E_H \subset E_G\) as in Definition 4.3 is balanced if and only if for every character \(\chi\) of \(H\) trivial on \(H \cap Z_0(G)\) we have

\[
\text{degree}(E_H(\chi)) = 0.
\]

**Proposition 4.3.** Let \(E_G\) be a strongly semistable principal \(G\)–bundle over \(X\) and \(E_H \subset E_G\) a balanced reduction of structure group of \(E_G\) to a subgroup scheme \(H \subset G\). Then the principal \(H\)–bundle \(E_H\) over \(X\) is strongly semistable.

**Proof.** Take any indecomposable \(H\)–module \(W\). Let \(E_W = E_H(W)\) be the vector bundle over \(X\) associated to the principal \(H\)–bundle \(E_H\) for the \(H\)–module \(W\). We need to show that \(E_W\) is strongly semistable. For that it suffices to show that the vector bundle \(\text{End}(E_W)\) is strongly semistable. Indeed, if a subbundle \(F \subset (F^j_X)^*E_W\) contradicts the semistability condition of the vector bundle \((F^j_X)^*E_W\), then the subbundle

\[
((F^j_X)^*E_W/F)^* \bigotimes (F^j_X)^*E_W \subset ((F^j_X)^*E_W)^* \bigotimes (F^j_X)^*E_W = (F^j_X)^*\text{End}(E_W)
\]

contradicts the semistability condition of the vector bundle \((F^j_X)^*\text{End}(E_W)\).

Let

\[
\rho : H \rightarrow \text{GL}(\text{End}(W))
\]

be the homomorphism given by the action of \(H\) on \(\text{End}(W)\) induced by the action of \(H\) on \(W\). We note that \(\text{End}(E_W)\) is the vector bundle associated to the principal \(H\)–bundle \(E_H\) for the \(H\)–module \(\text{End}(W)\).

Since the \(H\)–module \(W\) is indecomposable, the group scheme \(H \cap Z_0(G)\) acts on \(W\) as scalar multiplications through a character of \(H \cap Z_0(G)\). Therefore, \(H \cap Z_0(G)\) acts trivially on \(\text{End}(W)\). In other words, \(\text{End}(W)\) is an \(H/(H \cap Z_0(G))\)–module.

As \(H\) is a subgroup scheme of \(G\), we have

\[
H/(H \cap Z_0(G)) \subset G/Z_0(G).
\]

Therefore, there is a \(G/Z_0(G)\)–module \(V\) such that the \(H/(H \cap Z_0(G))\)–module \(\text{End}(W)\) is a subquotient of the \(H/(H \cap Z_0(G))\)–module \(V\) (see [R page 139, Proposition 2.21(b)])

Let

\[
V \twoheadrightarrow Q
\]

be a quotient of the \(H/(H \cap Z_0(G))\)–module \(V\) such that \(\text{End}(W)\) is a submodule of the \(H/(H \cap Z_0(G))\)–module \(Q\).

Let

\[
V = \bigoplus_{i=1}^{\ell} V_i
\]

be a decomposition of the \(G/Z_0(G)\)–module \(V\) into a direct sum of indecomposable \(G/Z_0(G)\)–modules. For any \(i \in [1, \ell]\), let \(E_{V_i}\) be the vector bundle over \(X\) associated to the principal \(G\)–bundle \(E_G\) for the \(G\)–module \(V_i\) (the \(G/Z_0(G)\)–module \(V_i\) is considered as a \(G\)–module using the quotient map to \(G/Z_0(G)\)).
Since $E_G$ is strongly semistable, and the $G$–module $V_i$ is indecomposable, the associated vector bundle $E_{V_i}$ is strongly semistable. As $G/Z_0(G)$ does not admit any nontrivial characters, the induced action of $G$ on the line $\Lambda^\text{top} V_i$ is the trivial action. Therefore, the associated line bundle $\Lambda^\text{top} E_{V_i}$ is trivializable. In particular, we have
\[
\text{degree}(E_{V_i}) = 0.
\]
Since each $E_{V_i}$ is strongly semistable of degree zero, the vector bundle $E_V$ is also strongly semistable of degree zero.

Let $E_Q$ denote the vector bundle over $X$ associated to the principal $H$–bundle $E_H$ for the $H$–module $Q$ in eqn. (4.3) (the $H/(H \cap Z_0(G))$–module $Q$ is considered as an $H$–module using the quotient map to $H/(H \cap Z_0(G))$). Since $H \cap Z_0(G)$ acts trivially on the line $\Lambda^\text{top} Q$, and $E_H \subset E_G$ is a balanced reduction of structure group, we have
\[
\text{degree}(E_Q) = \text{degree}(E_H(\Lambda^\text{top} Q)) = 0,
\]
where $E_H(\Lambda^\text{top} Q)$ is the line bundle over $X$ associated to the principal $H$–bundle $E_H$ for the $H$–module $\Lambda^\text{top} Q$.

Since $Q$ is a quotient of the $H$–module $V$, the vector bundle $E_Q$ is a quotient of $E_V$. The vector bundle $E_V$ is strongly semistable of degree zero, and $E_Q$ is a quotient of it of degree zero. Hence the vector bundle $E_Q$ is also strongly semistable.

We recall that the $H$–module $\text{End}(W)$ is a submodule of the $H$–module $Q$. Therefore, the associated vector bundle $\text{End}(E_W)$ is a subbundle of $E_Q$. Since $E_Q$ is a strongly semistable vector bundle of degree zero, and $\text{End}(E_W)$ is a subbundle of it of degree zero, we conclude that the vector bundle $\text{End}(E_W)$ is strongly semistable.

We saw earlier that $E_W$ is strongly semistable if $\text{End}(E_W)$ is so. Therefore, the principal $H$–bundle $E_H$ is strongly semistable. This completes the proof of the proposition.

Let $E_G$ be a strongly semistable principal $G$–bundle over $X$. In Proposition 3.11 we saw that the monodromy group scheme $M$ (constructed in Definition 3.9) is canonically embedded in $\text{Ad}(E_G)_x$. For national convenience, we will denote by $\tilde{G}$ the group $\text{Ad}(E_G)_x$ defined over $k$. Let $E_{\tilde{G}}$ be the principal $\tilde{G}$–bundle over $X$ obtained by extending the structure group of the monodromy bundle $E_M$ (see Definition 3.10) using the inclusion of $M$ in $\tilde{G}$. Therefore,
\[
E_M \subset E_{\tilde{G}}
\]
is a reduction of structure group of $E_{\tilde{G}}$ to $M$.

Let $Z_0(\tilde{G})$ denote the unique maximal split torus contained in the reduced center of $\tilde{G}$.

**Theorem 4.4.** Assume that the group $\tilde{G} := \text{Ad}(E_G)_x$ does not admit any nontrivial character which is trivial on $Z_0(\tilde{G})$. Then the reduction of structure group in eqn. (4.5) is a balanced reduction of structure group of $E_{\tilde{G}}$ to $M$. In particular, the principal $M$–bundle $E_M$ is strongly semistable.

**Proof.** We first note that the quotient $M/(M \cap Z_0(\tilde{G}))$ is a subgroup scheme of
\[
G' := \tilde{G}/Z_0(\tilde{G})
\].
Therefore, any $M/(M \cap Z_0(\tilde{G}))$–module is a subquotient of some $G'$–module considered as a $M/(M \cap Z_0(G))$–module (see [3, page 139, Proposition 2.21(b)]). Let $\chi$ be a character of $M$ which is trivial on the group scheme $M \cap Z_0(\tilde{G})$. Let $L$ denote the line bundle over $X$ associated to the principal $M$–bundle $E_M$ for the character $\chi$. To prove that $E_M \subset E_{\tilde{G}}$ is a balanced reductive of structure group, it suffices to show that degree($L$) = 0 (see Remark [4.2]).

The one–dimensional $M$–module corresponding to $\chi$ will be denoted by $\xi$. Since $\chi$ is trivial on $M \cap Z_0(\tilde{G})$, the $M$–module $\xi$ is given by a $M/(M \cap Z_0(\tilde{G}))$–module. This $M/(M \cap Z_0(\tilde{G}))$–module will also be denoted by $\xi$. Let $V$ be a $G'$–module such that the $M/(M \cap Z_0(\tilde{G}))$–module $\xi$ is a subquotient of $V$ (we noted earlier that any $M/(M \cap Z_0(\tilde{G}))$–module is a subquotient of some $G'$–module).

Since $E_{\tilde{G}}$ is strongly semistable, the $\tilde{G}$–bundle $E_{\tilde{G}}$ is also strongly semistable. Let $E_V$ denote the vector bundle over $X$ associated to the principal $\tilde{G}$–bundle $E_{\tilde{G}}$ for the $\tilde{G}$–module $V$ (since $G'$ is a quotient of $\tilde{G}$, any $G'$–module is also a $\tilde{G}$–module). As $Z_0(\tilde{G})$ acts trivially on $V$, and the principal $\tilde{G}$–bundle $E_{\tilde{G}}$ is strongly semistable, the associated vector bundle $E_V$ is also strongly semistable (see Lemma [3.12]).

By our assumption, $G'$ does not admit any nontrivial character. Therefore, the one–dimensional $\tilde{G}$–module $\bigwedge_{\text{top}} V$ is a trivial $\tilde{G}$–module. Consequently, the line bundle over $X$ associated to the principal $\tilde{G}$–bundle $E_{\tilde{G}}$ for the one–dimensional $\tilde{G}$–module $\bigwedge_{\text{top}} V$ is a trivializable. Hence, we have

$$\text{degree}(E_V) = \text{degree}(\bigwedge_{\text{top}} E_V) = 0.$$ 

The earlier defined line bundle $L$ over $X$ is the one associated to the principal $M$–bundle $E_M$ for the $M$–module $\xi$. From the definition of the monodromy bundle $E_M$ it follows immediately that $L$ is an object of the category $C_X$ (defined in Section 2).

We recall that the $M$–module $\xi$ is a subquotient of the $M$–module $V$. This means that the object $L$ of $C_X$ is a subquotient of the object $E_V$ of $C_X$. On the other hand, $E_V$ is a strongly semistable vector bundle of degree zero. Therefore, we conclude that

$$\text{degree}(L) = 0.$$ 

Thus, $E_M \subset E_{\tilde{G}}$ is a balanced reduction of structure group of $E_{\tilde{G}}$ to $M$. Now from Proposition [4.3] it follows that the principal $M$–bundle $E_M$ is strongly semistable. This completes the proof of the theorem. \hfill \Box

**Remark 4.5.** Assume that the fiber of the principal bundle $E_G$, over the $k$–rational point $x$ of $X$, admits a rational point. If we fix a rational point in the fiber of $E_G$ over $x$, then $\tilde{G}$ gets identified with $G$, and the principal bundle $E_{\tilde{G}}$ gets identified with $E_G$.

If $E_H \subset E_G$ is a reduction of structure group, to a subgroup scheme $H \subset G$, of a principal $G$–bundle $E_G$ over $X$, then the adjoint bundle $\text{Ad}(E_H)$ is a subgroup scheme of the group scheme $\text{Ad}(E_G)$ over $X$.

**Theorem 4.6.** Let $E_G$ be a strongly semistable principal $G$–bundle over a geometrically irreducible smooth projective curve $X$ defined over $k$, where $G$ is a linear algebraic group.
defined over $k$ with the property that $G$ does not admit any nontrivial character which is trivial on $Z_0(G)$. Fix a $k$–rational point $x$ of $X$. Let $H \subset G$ be a subgroup scheme and $E_H \subset E_G$ a balanced reduction of structure group of $E_G$ to $H$. Then the image in $\text{Ad}(E_G)_x$ of the monodromy group scheme $M$ (image by the homomorphism in Proposition 3.11) is contained in the subgroup scheme $\text{Ad}(E_H)_x \subset \text{Ad}(E_G)_x$.

Proof. Take any indecomposable $H$–module $V$. Let $\chi$ be the character of $(H \cap Z_0(G))_0$ corresponding to the indecomposable $H$–module $V$ (see Definition 4.1). Let $E_V$ denote the vector bundle over $X$ associated to the principal $H$–bundle $E_H$ for the $H$–module $V$. To prove the theorem it suffices to show that $E_V$ is strongly semistable, and it is an object of the neutral Tannakian category $\mathcal{C}_{E_G}$. (We recall that the monodromy group $M$ is constructed from $\mathcal{C}_{E_G}$; see Definition 3.9.)

Since $E_H \subset E_G$ is a balanced reduction of structure group of $E_G$ to $H$, from Proposition 4.3 we know that the principal $H$–bundle $E_H$ is strongly semistable. As the $H$–module $V$ is indecomposable, and the principal $H$–bundle $E_H$ is strongly semistable, we conclude that the associated vector bundle $E_V$ is strongly semistable. Therefore, to complete the proof of the theorem we need to show that $E_V$ is an object of the neutral Tannakian category $\mathcal{C}_{E_G}$.

We recall that the group $(Z_0(G) \cap H)_0$ is a product of copies of the multiplicative group $\mathbb{G}_m$. Therefore, the inclusion

$$(Z_0(G) \cap H)_0 \hookrightarrow Z_0(G)$$

splits [2, §8.5, page 115, Corollary]. Also, in Lemma 3.2 we showed that any character of $Z_0(H)$ arises from an indecomposable $G$–module. Therefore, there is an indecomposable $G$–module $\hat{V}$ such that $(Z_0(G) \cap H)_0$ acts on $\hat{V}$ as scalar multiplication through the earlier defined character $\chi$ (the character through which $(Z_0(G) \cap H)_0$ acts on $V$).

Let $E_{\hat{V}}$ be the vector bundle over $X$ associated to the principal $G$–bundle $E_G$ for the $G$–module $\hat{V}$. Since the $G$–module $\hat{V}$ is indecomposable, the subgroup $Z_0(G)$ acts on $\hat{V}$ as scalar multiplications. As the principal $G$–bundle $E_G$ is strongly semistable, this implies that the associated vector bundle $E_{\hat{V}}$ is strongly semistable (see Corollary 3.3).

Since both the vector bundles $E_V$ and $E_{\hat{V}}$ are strongly semistable, the vector bundle

$$(4.6) \quad W := E_{\hat{V}}^* \otimes E_V$$

is also strongly semistable (see Theorem 3.5).

As both $E_{\hat{V}}$ and $W$ are strongly semistable, the vector bundle $E_{\hat{V}} \otimes W$ is also strongly semistable (see Theorem 3.5). Furthermore, the vector bundle $E_V$ is a subbundle of $E_{\hat{V}} \otimes W = \text{End}(E_{\hat{V}}) \otimes E_V$; note that $\mathcal{O}_X$ is subbundle of $\text{End}(E_{\hat{V}})$, and hence $E_V$ is a subbundle of $\text{End}(E_{\hat{V}}) \otimes E_V$. We also have

$$\mu(E_{\hat{V}} \otimes W) = \mu(E_V).$$

Therefore, to show that the strongly semistable vector bundle $E_V$ is an object of the Tannakian category $\mathcal{C}_{E_G}$, it suffices to show that the strongly semistable vector bundle $W$ is an object of the neutral Tannakian category $\mathcal{C}_{E_G}$. 

The group \((Z_0(G) \cap H)_0\) acts on both \(\hat{V}\) and \(V\) as multiplication by scalars through the character \(\chi\). This immediately implies that \((Z_0(G) \cap H)_0\) acts trivially on the \(H\)-module \(\hat{V} \otimes V\). Set
\[
H' := H/(Z_0(G) \cap H)_0.
\]
Therefore, \(\hat{V} \otimes V\) is an \(H'\)-module.

Set
\[
G' := G/(Z_0(G) \cap H)_0.
\]
Since \(H'\) is a subgroup scheme of \(G'\), there is a \(G'\)-module \(V'\) such that the \(H'\)-module \(\hat{V} \otimes V\) is a subquotient of \(V'\) considered as an \(H'\)-module (see [3, page 139, Proposition 2.21(b)]).

Let
\[
(4.7) \quad V' \rightarrow Q
\]
be a quotient of the \(H'\)-module \(V'\) such that the \(H'\)-module \(\hat{V} \otimes V\) is a submodule of \(Q\).

Let \(E_{V'}\) be the vector bundle over \(X\) associated to the principal \(G\)-bundle \(E_G\) for the \(G\)-module \(V'\). We will show that \(E_{V'}\) is strongly semistable of degree zero.

For that, express the \(H'\)-module \(V'\) as a direct sum of indecomposable \(H'\)-modules. Since \(E_H \subset E_G\) is a balanced reduction of structure group, the vector bundle over \(X\) associated to the principal \(H\)-bundle \(E_H\) for an indecomposable \(H'\)-module is strongly semistable of degree zero; see Definition 4.1 and Proposition 4.3. Therefore, \(E_{V'}\) is isomorphic to a direct sum of strongly semistable vector bundles of degree zero (corresponding to a decomposition of \(V'\) as a direct sum of indecomposable \(H'\)-modules). Hence \(E_{V'}\) is a strongly semistable vector bundle of degree zero. We also note that \(E_{V'}\) is an object of the Tannakian category \(\mathcal{C}_{E_G}\).

Consider the \(H'\)-module \(Q\) in eqn. (4.7). Let \(E_Q\) denote the vector bundle over \(X\) associated to the principal \(H\)-bundle \(E_H\) for the \(H\)-module \(Q\). Since the \(H\)-module \(Q\) is a quotient of \(V'\), the vector bundle \(E_Q\) is a quotient bundle of \(E_{V'}\). As the reduction \(E_H \subset E_G\) is balanced, we have degree(\(E_Q\)) = 0 (recall that \(Q\) is an \(H'\)-module). Since \(E_Q\) is a quotient bundle of degree zero of the strongly semistable vector bundle \(E_{V'}\) of degree zero, we conclude that \(E_Q\) is strongly semistable. Therefore, the vector bundle \(E_Q\) is also an object of the Tannakian category \(\mathcal{C}_{E_G}\).

Finally, the vector bundle \(W\) (defined in eqn. (4.0)) is a subbundle of \(E_Q\), because the \(H'\)-module \(\hat{V} \otimes V\) is submodule of \(Q\). Since the reduction \(E_H \subset E_G\) is balanced, and \(\hat{V} \otimes V\) is an \(H'\)-module, we conclude that degree(\(W\)) = 0. As \(E_Q\) is an object of the Tannakian category \(\mathcal{C}_{E_G}\), and \(E_Q\) is strongly semistable of degree zero, we conclude that the subbundle \(W \subset E_Q\) of degree zero is strongly semistable, and furthermore, \(W\) is an object of \(\mathcal{C}_{E_G}\). This completes this proof of the theorem.

\[\square\]

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