Fundamental solutions of the
Knizhnik-Zamolodchikov equation of one variable
and the Riemann-Hilbert problem
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Abstract
In this article, we derive multiple polylogarithms from multiple zeta values by using a recursive Riemann-Hilbert problem of additive type. Furthermore we show that this problem is regarded as an inverse problem for the connection problem of the KZ equation of one variable, so that the fundamental solutions to the equation are derived from the Drinfel’d associator by using a Riemann-Hilbert problem of multiplicative type. The solvability condition for this inverse problem is given by the duality relations for the Drinfel’d associator.

1 Introduction
In [OiU2], we showed that the polylogarithms \( \text{Li}_k(z) \) are characterized by the inversion formula of polylogarithms

\[
\text{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + \text{Li}_{2,1,\ldots,1}(1-z) = \zeta(k),
\]

which is viewed as a recursive Riemann-Hilbert problem of additive type. Generalizing this scheme, we give a characterization of the multiple polylogarithms of one variable \( \text{Li}_{k_1,\ldots,k_r}(z) \).

The multiple polylogarithms of one variable are holomorphic functions on \( |z| < 1 \) determined by the Taylor expansions

\[
\text{Li}_{k_1,\ldots,k_r}(z) = \sum_{n_1 > \cdots > n_r > 0} \frac{z^{n_1}}{n_1^{k_1} \cdots n_r^{k_r}}
\]

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where \( r \geq 1, k_1, \ldots, k_r \geq 1 \). By using iterated integrals (2.12), they can be expressed as

\[
\text{Li}_{k_1, \ldots, k_r}(z) = \int_0^z \left( \frac{dz}{z} \right)^{k_1-1} \frac{dz}{1-z} \cdots \left( \frac{dz}{z} \right)^{k_r-1} \frac{dz}{1-z},
\]

(1.3)

and can be continued onto \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) as many-valued analytic functions along the integral path, where \( \mathbb{P}^1 \) denotes the Riemann sphere.

The generating function (2.19) of the multiple polylogarithms yields a fundamental solution of the Knizhnik-Zamolodchikov equation (the KZ equation, for short) of one variable. This is an ordinary differential equation

\[
\frac{dG}{dz} = \left( \frac{X_0}{z} + \frac{X_1}{1-z} \right) G
\]

(1.4)

defined on the moduli space

\[
\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}.
\]

(1.5)

Here \( X_0, X_1 \) are generators of the free Lie algebra \( \mathfrak{X} = \mathbb{C}\{X_0, X_1\} \), which is a Lie algebra derived from the lower central series of the fundamental group of \( \mathcal{M}_{0,4} \). The 1-forms \( \xi_0 = \frac{dz}{z} \) and \( \xi_1 = \frac{dz}{1-z} \) are considered as dual variables of \( X_0, X_1 \) and generate a shuffle algebra \( S = S(\xi_0, \xi_1) \). This algebra describes iterated integrals of forms consisting of \( \xi_0 \) and \( \xi_1 \).

Moreover the connection matrix between the fundamental solution \( \mathcal{L}^{(0)}(z) \) of (1.4) normalized at \( z = 0 \) (see Section 2, Proposition 6) and the fundamental solution \( \mathcal{L}^{(1)}(z) \) normalized at \( z = 1 \) is given by the Drinfel’d associator \( \Phi_{\text{KZ}} \);

\[
\left( \mathcal{L}^{(1)}(z) \right)^{-1} \mathcal{L}^{(0)}(z) = \Phi_{\text{KZ}}.
\]

(1.6)

The elements \( \mathcal{L}^{(0)}, \mathcal{L}^{(1)} \) and \( \Phi_{\text{KZ}} \) are grouplike elements of \( \tilde{U}(\mathfrak{X}) = \mathbb{C}\langle\langle X_0, X_1\rangle\rangle \), which denotes the non-commutative formal power series algebra of the variables \( X_0, X_1 \).

The Drinfel’d associator \( \Phi_{\text{KZ}} \) is expressed as the generating function (2.33) of the multiple zeta values

\[
\zeta(k_1, k_2, \ldots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.
\]

(1.7)

In \([\text{OKU}]\), it was shown that the connection relation (1.6) is equivalent to the system of the functional relations (3.1) among extended multiple polylogarithms

\[
\sum_{w=w} \text{Li}(\tau(u); 1-z) \text{Li}(v; z) = \zeta(\text{reg}^{10}(w)),
\]

(1.8)

referred to as “the generalized inversion formulas.” Here \( w \) denotes a word of \( \xi_0 \) and \( \xi_1 \). This system contains the inversion formulas (1.1) of polylogarithms as the special case of \( w = \xi_0^{k_1-1} \xi_1 \).

In this article, we consider an inverse problem for the generalized inversion formulas (1.8), and show that this problem is nothing but a recursive Riemann-Hilbert problem (or a Plemelj-Birkhoff decomposition) of additive type \([\text{Bi}], \text{M}, \text{P}\). Under the duality relations of multiple zeta values

\[
\zeta(\text{reg}^{10}(w)) = \zeta(\text{reg}^{10}(\tau(w))),
\]

(1.9)
for any word \( w \) of \( \xi_0, \xi_1 \), and a certain asymptotic condition, this problem has a unique solution (Theorem 11).

Furthermore this result can be interpreted as an inverse problem for the connection problem of the KZ equation (1.6). Thus one can establish the existence and uniqueness of a solution \( F^{(0)}(z) \) and \( F^{(1)}(z) \) to the equation

\[
\left( F^{(1)}(z) \right)^{-1} F^{(0)}(z) = \Phi_{KZ} \tag{1.10}
\]

under certain assumptions (Theorem 13). In other words, the solutions \( L^{(0)}(z) \) and \( L^{(1)}(z) \) are completely characterized by the Riemann-Hilbert problem of multiplicative type (1.10).

This article is organized as follows: In Section 2, we prepare some terminologies about free Lie algebras and shuffle algebras, and survey the connection problem of the KZ equation of one variable due to [OkU] and [OiU1]. In Section 3, we consider in details the generalized inversion formulas. We prove the generalized inversion formulas independently of the connection relation (1.6), and show that the consistency condition for the generalized inversion formulas is given by (1.9). In Section 4, we formulate and solve the recursive Riemann-Hilbert problem of additive type corresponding to the inverse problem for the generalized inversion formulas. Finally, in Section 5, we consider the inverse problem for the connection problem of the KZ equation and show that it has a unique solution.

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2 The connection problem of the KZ equation of one variable

Let \( \mathcal{X} = \mathcal{C}\langle X_0, X_1 \rangle \) be the free Lie algebra generated by \( X_0 \) and \( X_1 \), and \( \mathcal{U} = \mathcal{U}(\mathcal{X}) \) be the universal enveloping algebra of \( \mathcal{X} \). \( \mathcal{U} \) is the non-commutative polynomial algebra \( \mathcal{C}(X_0, X_1) \) generated by \( X_0, X_1 \) with the unit element \( \mathbf{I} \).

The algebra \( \mathcal{U} \) has a co-commutative Hopf algebra structure by the following coproduct \( \Delta \), the counit \( \varepsilon \) as algebra morphisms and the antipode \( \rho \) as an anti-automorphism:

\[
\Delta(X_i) = \mathbf{I} \otimes X_i + X_i \otimes \mathbf{I}, \quad \varepsilon(X_i) = 0, \quad \rho(X_i) = -X_i.
\]

The Hopf algebra \( \mathcal{U} \) has the grading defined by the length of words;

\[
\mathcal{U} = \bigoplus_{s=0}^{\infty} \mathcal{U}_s. \tag{2.1}
\]

We also denote by \( \mathcal{\tilde{U}} = \mathcal{\tilde{U}}(\mathcal{X}) \) the completion of \( \mathcal{U} \) with respect to this grading. \( \mathcal{\tilde{U}} \) is the non-commutative formal power series algebra \( \mathcal{C}(\langle X_0, X_1 \rangle) \).
Let $S = S(\xi_0, \xi_1)$ be the shuffle algebra generated by the 1-forms

$$\xi_0 = \frac{dz}{z}, \quad \xi_1 = \frac{dz}{1 - z}. \quad (2.2)$$

This is the non-commutative polynomial algebra $\mathbb{C} \langle \xi_0, \xi_1 \rangle$ generated by $\xi_0, \xi_1$ with the shuffle product $\shuffle$. The shuffle product is defined recursively as

$$w \shuffle 1 = 1 \shuffle w = w,$$

$$(\xi_i, w_1) \shuffle (\xi_i, w_2) = \xi_i (w_1 \shuffle (\xi_i, w_2)) + \xi_i ((\xi_i, w_1) \shuffle w_2),$$

where $1$ is the unit element of $\mathbb{C} \langle \xi_0, \xi_1 \rangle$ (that is, $1$ stands for the empty word) and $w, w_1, w_2$ are words of $\mathbb{C} \langle \xi_0, \xi_1 \rangle$.

By virtue of $[R]$, $S$ is an associative commutative algebra and has a Hopf algebra structure by the coproduct

$$\Delta^*(\xi_{i_1} \cdots \xi_{i_r}) = \sum_{k=0}^{r} \xi_{i_1} \cdots \xi_{i_k} \otimes \xi_{i_{k+1}} \cdots \xi_{i_r}$$

(we regard $\xi_{i_1} \cdots \xi_{i_0}$ (at $k = 0$) and $\xi_{i_{r+1}} \cdots \xi_{i_r}$ (at $k = r$) as 1), the counit $\varepsilon^*(\xi_{i_1} \cdots \xi_{i_r}) = 0$ and the antipode $\rho^*(\xi_{i_1} \cdots \xi_{i_r}) = (-1)^r \xi_{i_r} \cdots \xi_{i_1}$.

The shuffle algebra

$$S = \bigoplus_{s=0}^{\infty} S_s \quad (2.3)$$

is also a graded Hopf algebra with the grading defined by the length of words. The dual of this Hopf algebra is the algebra $\tilde{U}$ defined above with respect to the pairing

$$\langle \xi_{i_1} \cdots \xi_{i_r}, X_{j_1} \cdots X_{j_s} \rangle = \begin{cases} 1 & (r = s, i_k = j_k \text{ for } 1 \leq k \leq r), \\ 0 & \text{(otherwise).} \end{cases}$$

In what follows, we denote conveniently the sum over all words $w$ in $S$ by $\sum_{w \in S}$, and the dual element of $w$ by the capital letter $W$ (that is, for $w = \xi_{i_1} \cdots \xi_{i_r} \in S$, the capitalization $W$ stands for $X_{i_1} \cdots X_{i_r} \in U$).

The following lemmas are basic and will be used in Section 5.

**Lemma 1.** A $\tilde{U}$-valued function

$$F(z) = \sum_{w \in S} f(w; z)W$$

is grouplike if and only if $f(w; z)$ is a shuffle homomorphism.

**Lemma 2.** If a $\tilde{U}$-valued function

$$F(z) = \sum_{w \in S} f(w; z)W$$

is grouplike, the reciprocal of $F(z)$ is written as

$$(F(z))^{-1} = \sum_{w \in S} f(w; z)\rho(W) = \sum_{w \in S} f(\rho^*(w); z)W. \quad (2.4)$$
We denote by $S^0$ and $S^{10}$ the subalgebras of $S$ defined as

$$S^0 = C1 + S\xi_1,$$
$$S^{10} = C1 + \xi_0 S\xi_1.$$ 

$S$ has polynomial ring structures as follows:

**Proposition 3** ([R]). $S$ is a polynomial algebra of $\xi_0$ whose coefficients are in $S^0$, and is a polynomial algebra of $\xi_0, \xi_1$ whose coefficients are in $S^{10}$. That is,

$$S = S^0[\xi_0] = S^{10}[\xi_0, \xi_1].$$ \tag{2.5}

We define the regularizing maps $\text{reg}^0$ and $\text{reg}^{10}$ as picking up the constant terms of a word with respect to the polynomial representations (2.5):

$$\text{reg}^0: S = S^0[\xi_0] \to S^0, \quad u = \sum w_j \xi_j^0 \mapsto w_0 (w_j \in S^0),$$

$$\text{reg}^{10}: S = S^{10}[\xi_0, \xi_1] \to S^{10}, \quad u = \sum \xi_i^1 \xi_j^0 \mapsto w_0 (w_{ij} \in S^{10}).$$

The maps $\text{reg}^0$ and $\text{reg}^{10}$ are shuffle homomorphisms and are calculated by the following lemma. This lemma was implicitly shown in [IKZ].

**Lemma 4** ([IKZ]). (i) For a word $u \in S^0$, we have

$$u\xi_i^0 = \sum_{j=0}^{n} \text{reg}^0(u\xi_i^{n-j}) \xi_j^0, \tag{2.6}$$

$$\text{reg}^0(u\xi_i^0) = \sum_{j=0}^{n} (-1)^j (u\xi_i^{n-j}) \xi_j^0. \tag{2.7}$$

(ii) For a word $u \in S^{10}$, we have

$$\xi_i^m u\xi_i^0 = \sum_{i=0}^{m} \sum_{j=0}^{n} \xi_i^j \xi_j^0 \text{reg}^{10}(\xi_i^{m-i} u\xi_i^{n-j}) \xi_j^0, \tag{2.8}$$

$$\text{reg}^{10}(\xi_i^m u\xi_i^0) = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} \xi_i^j \xi_j^0 (\xi_i^{m-i} u\xi_i^{n-j}) \xi_j^0. \tag{2.9}$$

Let $D_0$ and $D_1$ be domains on $C$ defined by

$$D_0 = C \setminus \{z = x \mid x \geq 1\}, \tag{2.10}$$

$$D_1 = C \setminus \{z = x \mid x \leq 0\}. \tag{2.11}$$

For a word $w = \xi_{i_1} \cdots \xi_{i_r} \in S^0$ (that is, $i_r = 1$), we define an iterated integral

$$\int_0^z w = \begin{cases} 1, & (w = 1), \\ \int_0^z \xi_{i_1} \left( \int_0^z \xi_{i_2} \cdots \xi_{i_r} \right) & (\text{otherwise}) \end{cases} \tag{2.12}$$

recursively and extend it to $S^0$ as a linear map.
For \( z \in D_0 \) and \( w = \xi_1 \cdots \xi_r \in S^0 \), we define the multiple polylogarithms of one variable \( \text{Li}(w; z) \) as
\[
\text{Li}(w; z) = \int_0^z w = \int_0^z \xi_1 \cdots \xi_r. \tag{2.13}
\]
These are holomorphic functions on \( D_0 \) and have the Taylor expansions
\[
\text{Li}(\xi_0^{-k_1} \cdots \xi_0^{-k_r} \xi_1; z) = \sum_{n_1, \ldots, n_r > 0} \frac{z^{n_1}}{n_1^{k_1} \cdots n_r^{k_r}} \quad (|z| < 1), \tag{2.14}
\]
which coincides with \( \text{Li}_{k_1, \ldots, k_r}(z) \). (In what follows, we consider the multiple polylogarithms only of one variable, so we omit “of one variable.”)

We extend \( \text{Li}(w; z) \) to \( S^0 \) as a linear map, and further extend it to \( S \) as follows: For a word \( w = \sum w_j \xi_0^{j} \) \((w_j \in S^0)\), we set an extended multiple polylogarithm by
\[
\text{Li}(w; z) = \sum \text{Li}(w_j; z) \log^{j} z. \tag{2.15}
\]
Here \( \log z \) is defined as the principal value on \( D_1 \). These extended multiple polylogarithms are holomorphic on \( D_0 \cap D_1 \), and the map
\[
\text{Li}(\cdot; z) : S \rightarrow \mathbb{C}, \quad w \mapsto \text{Li}(w; z) \quad (z \in D_0 \cap D_1) \tag{2.16}
\]
is a shuffle homomorphism.

**Lemma 5** ([HPH] [OR]). The extended multiple polylogarithms satisfy the following recursive differential relations:
\[
\frac{d}{dz} \text{Li}(\xi_0 w; z) = \frac{\text{Li}(u; z)}{z}, \quad \frac{d}{dz} \text{Li}(\xi_1 u; z) = \frac{\text{Li}(u; z)}{1 - z}. \tag{2.17}
\]

We observe that the extended multiple polylogarithms can be continued onto \( P^1 \setminus \{0, 1, \infty\} \) as many-valued analytic functions along the integral paths of \( \text{Li}(w; z) \).

If \( w = \xi_0^{-k_1} \cdots \xi_0^{-k_r} \xi_1 \in S^{10} \), the limit of \( \text{Li}(w; z) \) as \( z \) tends to 1 in \( D_0 \cap D_1 \) converges and defines multiple zeta value:
\[
\zeta(w) = \lim_{z \in D_0 \cap D_1} \text{Li}(w; z) = \sum_{n_1, \ldots, n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}. \tag{2.18}
\]
The multiple zeta values \( \zeta(w) \) are denoted by \( \zeta(k_1, \ldots, k_r) \) as usual.

Under these notations, the specific solution to the equation (1.4) can be written as follows:

**Proposition 6** ([OiU1], [OkU]). The KZ equation of one variable (1.4) has the solution \( \mathcal{L}^{(0)}(z) \) which satisfies the asymptotic condition
\[
\mathcal{L}^{(0)}(z) = \hat{\mathcal{L}}^{(0)}(z) z^{X_0},
\]
where \( \hat{\mathcal{L}}^{(0)}(z) \) is holomorphic at \( z = 0 \) and \( \hat{\mathcal{L}}^{(0)}(0) = I \).
The solution $L^{(0)}(z)$ is uniquely characterized by this condition and is a grouplike element of $\tilde{U}$.

Furthermore the solution $L^{(0)}(z)$ is expressed as

$$L^{(0)}(z) = \sum_{w \in S} Li(w; z) W \quad (2.19)$$

$$= \left( \sum_{w \in S} Li(\text{reg}^0(w); z) W \right) z^{X_0} \quad (2.20)$$

$$= (1 - z)^{-X_1} \left( \sum_{w \in S} Li(\text{reg}^1(w); z) W \right) z^{X_0}. \quad (2.21)$$

We call the solution $L^{(0)}(z)$ the fundamental solution of (1.4) normalized at $z = 0$.

We also refer to the solution $L^{(1)}(z) = \hat{L}^{(1)}(z)(1 - z)^{-X_1}$, where $\hat{L}^{(1)}(z)$ is holomorphic at $z = 1$, $\hat{L}^{(1)}(1) = I$, as the fundamental solution normalized at $z = 1$.

With respect to the transformation $t : z \mapsto 1 - z$, we introduce the automorphism $t^*$ on $S$ by

$$t^*(\xi_0) = -\xi_1, \quad t^*(\xi_1) = -\xi_0, \quad (2.23)$$

which is the pull back induced from $t$, and also introduce the automorphism $t_*$ on $\mathcal{U}$ by

$$t_*(X_0) = -X_1, \quad t_*(X_1) = -X_0, \quad (2.24)$$

which is the dual map of $t^*$. Let $\tau : S \to S$ be an anti-automorphism defined by $\tau = t^* \circ \rho^*$, that is,

$$\tau(\xi_0) = \xi_1, \quad \tau(\xi_1) = \xi_0. \quad (2.25)$$

Furthermore put $T = t_* \circ \rho$ which is an anti-automorphism on $\mathcal{U}$ satisfying

$$T(X_0) = X_1, \quad T(X_1) = X_0. \quad (2.26)$$

Under these notations, the regularizing map for $S^1 = C1 + S\xi_0$

$$\text{reg}^1 : S = S^1[\xi_1] \to S^1, \quad u = \sum w_j \xi_j \mapsto w_0 \quad (2.27)$$

is written as

$$\text{reg}^1 = t^* \circ \text{reg}^0 \circ t^*. \quad (2.28)$$

Note that, for a $\tilde{U}$-valued function $F(z) = \sum_{w \in S} f(w; z) W$, we have

$$T(F(z)) := \sum_{w \in S} f(w; z) T(W) = \sum_{w \in S} f(\tau(w); z) W. \quad (2.29)$$
Using the transformation $t$ and the automorphism $t_*$, the fundamental solution $L^{(1)}$ of the KZ equation (1.4) normalized at $z = 1$ is written as

$$L^{(1)}(z) = \sum_{w \in S} \text{Li}(w; 1 - z) t_*(W)$$

(2.30)

$$= \left( \sum_{w \in S} \text{Li}(\text{reg}^0(w); 1 - z) t_*(W) \right) (1 - z)^{-X_1}$$

(2.31)

$$= z^{X_0} \left( \sum_{w \in S} \text{Li}(\text{reg}^{10}(w); 1 - z) t_*(W) \right) (1 - z)^{-X_1}.$$  

(2.32)

The connection relation between $L^{(0)}$ and $L^{(1)}$ is described as follows:

**Proposition 7 ([D] [OKU]).** (i) The connection matrix between $L^{(0)}(z)$ and $L^{(1)}(z)$ is given by the Drinfel’d associator

$$\Phi_{KZ} = \Phi_{KZ}(X_0, X_1) = \sum_{w \in S} \zeta(\text{reg}^{10}(w)) W.$$  

(2.33)

That is, the connection formula reads

$$L^{(0)}(z) = L^{(1)}(z) \Phi_{KZ}.$$  

(2.34)

(ii) The connection formula (2.34) is equivalent to the system of relations

$$\sum_{w \in w = w'} \text{Li}(\tau(u); 1 - z) \text{Li}(v; z) = \zeta(\text{reg}^{10}(w))$$

(2.35)

for all words $w \in S$.

We call the relations (2.35) the generalized inversion formulas for extended multiple polylogarithms.

This proposition follows from the representation (2.21), (2.32), Lemma 2 and the definition (2.18) of the multiple zeta values.

### 3 The generalized inversion formulas for the extended multiple polylogarithms

According to Proposition 7, the generalized inversion formulas (2.35) is equivalent to the connection problem of the KZ equation of one variable. However we can show these formulas independently of the connection problem of the KZ equation.

**Proposition 8.** For any word $w \in S$, the generalized inversion formula

$$\sum_{w \in w = w'} \text{Li}(\tau(u); 1 - z) \text{Li}(v; z) = \zeta(\text{reg}^{10}(w))$$

(3.1)

holds.
To prove this, it is enough to show the following lemma. This lemma also plays a key role to prove Theorem 11 in Section 4.

**Lemma 9.** (i) We have

\[
\frac{d}{dz} \left( \sum_{uv=w} \operatorname{Li}(\tau(u);1-z) \operatorname{Li}(v;z) \right) = 0. \tag{3.2}
\]

(ii) For any word \( w \) in \( S \), we have

\[
\lim_{z \to \mathbb{D}_0 \cap \mathbb{D}_1, w' = w} \sum_{z \in \mathbb{D}_0 \cap \mathbb{D}_1} \operatorname{Li}(\tau(u);1-z) \operatorname{Li}(v;z) = \zeta(\text{reg}^{10}(w)), \tag{3.3}
\]

\[
\lim_{z \to \mathbb{D}_0 \cap \mathbb{D}_1, w' = w} \sum_{z \in \mathbb{D}_0 \cap \mathbb{D}_1} \operatorname{Li}(\tau(u);1-z) \operatorname{Li}(v;z) = \zeta(\text{reg}^{10}(\tau(w))). \tag{3.4}
\]

**Proof.** [1] We represent the differential recursive relations (2.17) in terms of the exterior derivative with respect to the variable \( z \);

\[
d\operatorname{Li}(\xi_i;w;z) = \xi_i \operatorname{Li}(w,z) \quad (i = 0, 1).
\]

From this, it follows that

\[
d\operatorname{Li}(\tau(\xi_i)w;1-z) = -\xi_i \operatorname{Li}(w,1-z) \quad (i = 0, 1).
\]

Hence, for a word \( w = \xi_{i_1} \cdots \xi_{i_r} \), we have

\[
d \left( \sum_{uv=w} \operatorname{Li}(\tau(u);1-z) \operatorname{Li}(v;z) \right)
\]

\[
= d \left( \sum_{k=0}^{r} \operatorname{Li}(\tau(\xi_{i_1} \cdots \xi_{i_k});1-z) \operatorname{Li}(\xi_{i_{k+1}} \cdots \xi_{i_r};z) \right)
\]

\[
= \xi_{i_1} \operatorname{Li}(\xi_{i_2} \cdots \xi_{i_r};z) + \sum_{k=1}^{r-1} \left( -\xi_{i_k} \operatorname{Li}(\tau(\xi_{i_1} \cdots \xi_{i_k-1});1-z) \operatorname{Li}(\xi_{i_{k+1}} \cdots \xi_{i_r};z) \right.
\]

\[
+ \xi_{i_{k+1}} \operatorname{Li}(\tau(\xi_{i_1} \cdots \xi_{i_k});1-z) \operatorname{Li}(\xi_{i_{k+2}} \cdots \xi_{i_r};z) \right) - \xi_i \operatorname{Li}(\tau(\xi_{i_1} \cdots \xi_{i_{r-1}});1-z)
\]

\[
= \xi_{i_1} \operatorname{Li}(\xi_{i_2} \cdots \xi_{i_r};z) - \xi_{i_1} \operatorname{Li}(\xi_{i_2} \cdots \xi_{i_r};z) + \xi_i \operatorname{Li}(\tau(\xi_{i_1} \cdots \xi_{i_{r-1}});1-z) - \xi_i \operatorname{Li}(\tau(\xi_{i_1} \cdots \xi_{i_{r-1}});1-z)
\]

\[
= 0.
\]

For a word \( w = \xi_0^0 \) or \( w = \xi_1^0 \), the both sides of (3.3) and (3.4) are zero.

For a word \( w = \xi_1^k w' \), \( w' \in S^{10} \), we will prove (3.3). One can similarly prove (3.4).

For \( w = \xi_1^k w' \), \( w' \in S^{10} \), we have

\[
\sum_{uv=w} \operatorname{Li}(\tau(u);1-z) \operatorname{Li}(v;z)
\]

\[
= \sum_{i=0}^{k} \operatorname{Li}(\tau(\xi_{i_1}^k);1-z) \operatorname{Li}(\xi_{i_1}^{k-1} w';z) + \sum_{uv=w' \neq 1} \operatorname{Li}(\tau(\xi_1^k u);1-z) \operatorname{Li}(v;z). \tag{3.5}
\]
For the second term of the right hand side of (3.5), by putting \( u = \zeta_0 u' \), we obtain
\[
\text{Li}(\tau(\zeta_1^k u) ; 1 - z) \text{Li}(v; z) = \text{Li}(\tau(\zeta_1^k \zeta_0 u') ; 1 - z) \text{Li}(v; z)
\]
\[
= \text{Li}(\tau(u') \zeta_1^k \zeta_0) ; 1 - z) \text{Li}(v; z)
\]
\[
= \sum_{s=0}^{k} \text{Li}(\text{reg}^0(\tau(u') \zeta_1^k \zeta_0^{s-k}) ; 1 - z) \frac{\log^s(1 - z)}{s!} \text{Li}(v; z).
\]

Since \( \text{reg}^0(\tau(u') \zeta_1^k \zeta_0^{s-k}) \neq 1 \) so that
\[
\text{Li}(\text{reg}^0(\tau(u') \zeta_1^k \zeta_0^{s-k}) ; 1 - z) = O(1 - z) \quad (z \to 1),
\]
and \( \text{Li}(v; z) \) diverges at most of logarithmic order as \( z \to 1 \), we have
\[
\text{Li}(\tau(\zeta_1^k u) ; 1 - z) \text{Li}(v; z) \to 0 \quad (z \to 1). \tag{3.6}
\]

Next, we consider the first term of the right hand side of (3.5). By using Lemma 4, we have
\[
\sum_{i=0}^{k} \text{Li}(\tau(\xi_1^i) ; 1 - z) \text{Li}(\xi_1^{k-i} w' \xi_0^l; z)
\]
\[
= \sum_{i=0}^{k} \text{Li}(\xi_1^i ; 1 - z) \text{Li}(\xi_1^{k-i} w' \xi_0^l; z)
\]
\[
= \sum_{i=0}^{k} \sum_{p=0}^{i} \sum_{q=0}^{l} \text{Li}(\xi_1^i ; 1 - z) \text{Li}(\xi_1^p ; z) \text{Li}(\text{reg}^{10}(\xi_1^{k-i-p} w' \xi_0^{l-q}) ; z) \text{Li}(\xi_0^q ; z)
\]
\[
= \sum_{i=0}^{k} \sum_{p=0}^{i} \sum_{q=0}^{l} \frac{\log^i(1 - z)(- \log(1 - z))^p}{i!} \frac{\log^q z}{q!} \text{Li}(\text{reg}^{10}(\xi_1^{k-i-p} w' \xi_0^{l-q}) ; z)
\]
\[
= \sum_{r=0}^{l} \left( \sum_{i+p=r} \frac{1}{i!} \frac{(-1)^p}{p!} \right) \log^r(1 - z) \text{Li}(\text{reg}^{10}(\xi_1^{k-r} w' \xi_0^{l-q}) ; z) \frac{\log^q z}{q!}. \tag{3.7}
\]
From the identity
\[
\sum_{i+p=r} \frac{1}{i!} \frac{(-1)^p}{p!} = \begin{cases} 1 & (r = 0), \\ 0 & (r \neq 1), \end{cases} \tag{3.8}
\]
it follows that
\[
\text{Li}(\text{reg}^{10}(\xi_1^k w' \xi_0^q) ; z) \frac{\log^q z}{q!} \to \text{Li}(\text{reg}^{10}(\xi_1^k w' \xi_0^q) ; 1) = \zeta(\text{reg}^{10}(\xi_1^k w' \xi_0^q)) \quad (z \to 1). \tag{3.9}
\]

We should observe that the generalized inversion formulas are overdetermined in the following sense: Replacing \( w \) to \( \tau(w) \) in (3.1), we have
\[
\sum_{w=\tau(w)} \text{Li}(\tau(u) ; 1 - z) \text{Li}(v; z) = \zeta(\text{reg}^{10}(\tau(w))).
\]
Furthermore, in this formula, replace $z$ to $1 - z$ and put $u = \tau(v')$, $v = \tau(u')$. Then we obtain
\[ \sum_{u'v' = w} \text{Li}(v'; z) \text{Li}(u'; 1 - z) = \zeta(\text{reg}^{10}(\tau(w))). \tag{3.10} \]

The left hand side of (3.10) coincides with the left hand side of (3.1), so that we need the consistency condition for (3.1) and (3.10).

**Proposition 10.** From consistency conditions for the generalized inversion formulas, we have
\[ \zeta(\text{reg}^{10}(w)) = \zeta(\text{reg}^{10}(\tau(w))) \tag{3.11} \]
for any word $w$ in $S$.

The relations (3.11) are referred to as the duality relations for multiple zeta values.

## 4 The Riemann-Hilbert problem of additive type for multiple polylogarithms

In this section, we solve the recursive Riemann-Hilbert problem of additive type corresponding to the inverse problem for the generalized inversion formulas (3.1).

First note that the generalized inversion formulas (3.1) read
\[ \text{Li}(\tau(w); 1 - z) + \text{Li}(w; z) = \zeta(\text{reg}^{10}(w)) - \sum_{u,v \neq 1} \text{Li}(\tau(u); 1 - z) \text{Li}(v; z) \]
\[ \leftrightarrow \text{Li}(\text{reg}^{0}(\tau(w)); 1 - z) + \text{Li}(\text{reg}^{0}(w); z) = \zeta(\text{reg}^{10}(w)) - \sum_{u,v \neq 1} \text{Li}(\tau(u); 1 - z) \text{Li}(v; z) \]
\[ - \text{Li}(\tau(w) - \text{reg}^{0}(\tau(w)); 1 - z) + \text{Li}(w - \text{reg}^{0}(w); z). \tag{4.1} \]

The equation says that the right hand side, which is holomorphic on $D_0 \cap D_1$, decomposes to the sum of $\text{Li}(\text{reg}^{0}(\tau(w)); 1 - z)$ and $\text{Li}(\text{reg}^{0}(w); z)$, which are holomorphic on $D_0$ and $D_1$ respectively. Moreover the length of words appeared in the expansion (2.15) of the right hand side are less than the length of the word $w$. Hence this decomposition is considered as a recursive Riemann-Hilbert problem of additive type. Since the generalized inversion formulas are over-determined as was mentioned in the previous section, we need certain consistency conditions to solve the recursive Riemann-Hilbert problem to this decomposition.

**Theorem 11.** Under the duality relations (3.11) for multiple zeta values, there exist uniquely $\mathfrak{w}$-homomorphisms $f^{(0)}(\bullet; z), f^{(1)}(\bullet; z) : S \to \mathbb{C}$, which satisfy
\[ f^{(0)}(\xi_0; z) = \log z, \quad f^{(1)}(\xi_0; z) = \log(1 - z), \tag{4.2} \]
and the following three conditions:
(i) For any word $w \in S$, $f^{(0)}(\text{reg}_0^0(w); z)$ and $f^{(1)}(\text{reg}_0^0(w); z)$ are holomorphic on $D_0$ and $D_1$ respectively, and enjoy the functional equations

$$\sum_{u \cdot v = w} f^{(1)}(\tau(u); z)f^{(0)}(v; z) = \zeta(\text{reg}_0^0(w)) \quad (z \in D_0 \cap D_1). \quad (4.3)$$

(ii) For any word $w \in S$, $f^{(0)}(\text{reg}_0^0(w); z)$ and $f^{(1)}(\text{reg}_0^0(w); z)$ satisfy the asymptotic condition

$$\frac{d}{dz} f^{(i)}(\text{reg}_0^0(w); z) \to 0 \quad (z \to \infty, \ z \in D_i). \quad (4.4)$$

(iii) For any word $w \in S$, $f^{(0)}(\text{reg}_0^0(w); z)$ satisfies the normalizing condition

$$f^{(0)}(\text{reg}_0^0(w); 0) = 0. \quad (4.5)$$

The solutions $f^{(i)}(\bullet; z)$ are expressed in terms of extended multiple polylogarithms as follows;

$$f^{(0)}(w; z) = \text{Li}(w; z), \quad (4.6)$$

$$f^{(1)}(w; z) = \text{Li}(w; 1 - z). \quad (4.7)$$

Proof. We show that $f^{(i)}(w; z)$ are uniquely determined by using induction on the length of a word $w$.

First, in the case of $w = \xi_0$, the equation (4.3) reads

$$f^{(1)}(\xi_1; z) + f^{(0)}(\xi_0; z) = 0. \quad (4.8)$$

Therefore we obtain

$$f^{(1)}(\xi_1; z) = -f^{(0)}(\xi_0; z) = -\log z = \text{Li}(\xi_1; 1 - z). \quad (4.9)$$

In a similar fashion, in the case of $w = \xi_1$ we have

$$f^{(0)}(\xi_1; z) = -f^{(1)}(\xi_0; z) = -\log(1 - z) = \text{Li}(\xi_1; z). \quad (4.10)$$

Since $f^{(0)}$ is a shuffle homomorphism, we have

$$f^{(0)}(\xi^r_1; z) = \frac{1}{r!} f^{(0)}(\xi^w_r; z) = \frac{1}{r!} \text{Li}(\xi^w_r; z) = \text{Li}(\xi^r_1; z). \quad (4.11)$$

Similarly we have

$$f^{(1)}(\xi^r_1; z) = \text{Li}(\xi^r_1; 1 - z). \quad (4.12)$$

Next we assume that $f^{(0)}(w'; z) = \text{Li}(w'; z)$ and $f^{(1)}(w'; z) = \text{Li}(w'; 1 - z)$ for words $w'$ whose length is less than that of the word $w$. Under the assumption of induction, the equation (4.3) becomes

$$f^{(1)}(\tau(w); z) + \sum_{k=1}^{r-1} \text{Li}(\tau(\xi_{i_1} \cdots \xi_{i_k}); 1 - z) \text{Li}(\xi_{i_{k+1}} \cdots \xi_{i_r}; z) + f^{(0)}(w; z) = \zeta(\text{reg}_0^0(w)) \quad (4.13)$$
for \( w = \xi_1, \ldots, \xi_r \).

Now we show \( f^{(1)}(w; z) = \text{Li}(w; 1 - z) \) and \( f^{(0)}(w; z) = \text{Li}(w; z) \) by using (4.14), (4.15) and (4.13). According to Lemma 9 (i), we have

\[
d f = \sum_{k=1}^{r-1} \frac{d}{dz} \left( \text{Li}(\xi_k, \ldots, \xi_r; 1 - z) \text{Li}(\xi_{k+1}, \ldots, \xi_r; z) \right)
\]

Thus the differentiation of (4.13) leads to the equation

\[
d f^{(0)}(w; z) - d \text{Li}(w; z) = -d f^{(1)}(\tau(w); z) + d \text{Li}(\tau(w); 1 - z). \quad (4.14)
\]

On the other hand, we apply the decomposition \( w = \sum_{j=0}^{r} w_j \xi_j^0, w_j \in S^0 \) (see Proposition 3 and notice that \( w_0 = \text{reg}^0(w) \) to \( f^{(0)}(w; z) \) and \( \text{Li}(w; z) \). Then, using the hypothesis of induction, we have

\[
f^{(0)}(w; z) = f^{(0)}(\text{reg}^0(w); z) + \sum_{j=0}^{r-1} \text{Li}(w_j; z) \xi_j^0; z), \quad (4.15)
\]

\[
\text{Li}(w; z) = \text{Li}(\text{reg}^0(w); z) + \sum_{j=0}^{r-1} \text{Li}(w_j; z) \xi_j^0; z). \quad (4.16)
\]

In the same way, we also have

\[
f^{(1)}(\tau(w); 1 - z) = f^{(1)}(\text{reg}^0(\tau(w)); 1 - z) + \sum_{j=0}^{r-1} \text{Li}(\tau(w_j); 1 - z) \xi_j^0; 1 - z), \quad (4.17)
\]

\[
\text{Li}(\tau(w); 1 - z) = \text{Li}(\text{reg}^0(\tau(w)); 1 - z) + \sum_{j=0}^{r-1} \text{Li}(\tau(w_j); 1 - z) \xi_j^0; 1 - z). \quad (4.18)
\]

Therefore the equation (4.13) reduces to

\[
d f^{(0)}(\text{reg}^0(w); z) - d \text{Li}(\text{reg}^0(w); z) = -d f^{(1)}(\text{reg}^0(\tau(w)); z) + d \text{Li}(\text{reg}^0(\tau(w)); 1 - z). \quad (4.19)
\]

Since the left hand side (resp. right hand side) of (4.19) is holomorphic on \( D_0 \) (resp. \( D_1 \)), the both side of (4.19) are entire functions. By (4.17), due to Liouville’s theorem, we obtain

\[
d f^{(0)}(\text{reg}^0(w); z) - d \text{Li}(\text{reg}^0(w); z) = 0,
\]

\[
d f^{(1)}(\text{reg}^0(\tau(w)); z) - d \text{Li}(\text{reg}^0(\tau(w)); 1 - z) = 0.
\]

Thus the functions \( f^{(0)}(w; z) \) and \( f^{(1)}(w; z) \) are determined as

\[
f^{(0)}(\text{reg}^0(w); z) = \text{Li}(\text{reg}^0(w); z) + c^{(0)}(\text{reg}^0(w)),
\]

\[
f^{(1)}(\text{reg}^0(\tau(w)); z) = \text{Li}(\text{reg}^0(\tau(w)); 1 - z) + c^{(1)}(\text{reg}^0(w)),
\]
where \( c^{(0)}(\text{reg}^0(w)) \) and \( c^{(1)}(\text{reg}^0(w)) \) are integration constants. Applying (4.15)\( \sim \) (4.18) again, we also have

\[
\begin{align*}
\hat{f}^{(0)}(w; z) &= \text{Li}(w; z) + c^{(0)}(\text{reg}^0(w)), \\
\hat{f}^{(1)}(\tau(w); z) &= \text{Li}(\tau(w); 1 - z) + c^{(1)}(\text{reg}^0(w)).
\end{align*}
\]

Finally, we determine the integral constants \( c^{(0)}(\text{reg}^0(w)) \) and \( c^{(1)}(\text{reg}^0(w)) \). By (4.5), \( c^{(0)}(\text{reg}^0(w)) = 0 \) is clear. Substituting (4.20) and (4.21) to (4.3), we have

\[
\sum_{u \neq w} \text{Li}(\tau(u); 1 - z) \text{Li}(v; z) + c^{(1)}(\text{reg}^0(w)) = \zeta(\text{reg}^{10}(w)).
\]

In this relation, letting \( z \to 1 (z \in \mathcal{D}_0 \cap \mathcal{D}_1) \), we obtain, from Lemma 9 (3.3),

\[
\zeta(\text{reg}^{10}(\tau(w))) + c^{(1)}(\text{reg}^0(w)) = \zeta(\text{reg}^{10}(w)).
\]

On the other hand, from (3.4), we have

\[
\zeta(\text{reg}^{10}(\tau(w))) + c^{(1)}(\text{reg}^0(w)) = \zeta(\text{reg}^{10}(w)).
\]

By virtue of the duality relations (3.11), we can consistently determine

\[
c^{(1)}(\text{reg}^0(w)) = 0.
\]

We have thus completed the proof of this theorem.

\[\square\]

5 \hspace{1em} The Riemann-Hilbert problem corresponding to the KZ equation of one variable

In this section, we consider the Riemann-Hilbert problem as the inverse problem for the connection problem of the KZ equation of one variable (2.34) and discuss the relationship between this Riemann-Hilbert problem and Theorem 11.

The connection formula (2.34) can be written as

\[
\left(\hat{\mathcal{L}}^{(1)}(z)\right)^{-1} \hat{\mathcal{L}}^{(0)}(z) = (1 - z)^{-X_1} \Phi_{\text{KZ}} z^{-X_0},
\]

where \( \hat{\mathcal{L}}^{(0)}(z) \) is holomorphic on \( \mathcal{D}_0 \) and \( \hat{\mathcal{L}}^{(1)}(z) \) is holomorphic on \( \mathcal{D}_1 \). The equation (5.1) can be interpreted as the decomposition formula of the right hand side, which is holomorphic on \( \mathcal{D}_0 \cap \mathcal{D}_1 \), to the multiplication of the function holomorphic on \( \mathcal{D}_0 \) and the function holomorphic on \( \mathcal{D}_1 \). In this sense, the equation (5.1) is a Riemann-Hilbert problem of multiplicative type.

The Riemann-Hilbert problem corresponding to the KZ equation is to find \( \hat{\mathcal{U}} \)-valued functions \( \hat{F}^{(0)}(z) \) and \( \hat{F}^{(1)}(z) \) which are holomorphic on \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \) respectively, are grouplike, and satisfy the relation

\[
\left(\hat{F}^{(1)}(z)\right)^{-1} \hat{F}^{(0)}(z) = (1 - z)^{-X_1} \Phi_{\text{KZ}} z^{-X_0}
\]

in \( \mathcal{D}_0 \cap \mathcal{D}_1 \). Applying Theorem 11, we can find a solution to this Riemann-Hilbert problem.

Before stating the theorem, we have to establish the following proposition.
Proposition 12 ([OkU]). The duality relations (3.11) of the multiple zeta values are equivalent to the duality of the Drinfel’d associator
\[ \Phi_{\text{KZ}}(X_0, X_1) = \left( \Phi_{\text{KZ}}(-X_1, -X_0) \right)^{-1}. \] (5.3)

Proof. Since the Drinfel’d associator is a grouplike element, we have
\[ \left( \Phi_{\text{KZ}}(-X_1, -X_0) \right)^{-1} = \left( \sum_{w \in S} \zeta(\text{reg}^{10}(w)) t_\ast(W) \right)^{-1} = \sum_{w \in S} \zeta(\text{reg}^{10}(w)) T(W) = \sum_{w \in S} \zeta(\text{reg}^{10}(\tau(w))) W. \]
Hence (5.3) is equivalent to (3.11).

Now we formulate and solve the Riemann-Hilbert problem.

Theorem 13. Under the duality of the Drinfel’d associator (5.3), there exist uniquely the \( \hat{U} \)-valued functions \( \hat{F}^{(0)}(z) \) and \( \hat{F}^{(1)}(z) \) defined by
\[ \hat{F}^{(0)}(z) = \sum_{w \in S} \hat{f}^{(0)}(\text{reg}^0(w); z) W, \] (5.4)
\[ \hat{F}^{(1)}(z) = \sum_{w \in S} \hat{f}^{(1)}(\text{reg}^1(w); z) W, \] (5.5)
which are holomorphic on \( D_0 \) and \( D_1 \) respectively, are grouplike, and enjoy the following conditions:

(i) \( \hat{F}^{(0)}(z) \) and \( \hat{F}^{(1)}(z) \) satisfy the functional equation
\[ \left( \hat{F}^{(1)}(z) \right)^{-1} \hat{F}^{(0)}(z) = (1 - z)^{-X_1} \Phi_{\text{KZ}} z^{-X_0}. \] (5.6)

(ii) \( \hat{F}^{(0)}(z) \) and \( \hat{F}^{(1)}(z) \) satisfy the asymptotic condition
\[ \frac{d}{dz} \hat{F}^{(i)}(z) \to I \quad (z \to \infty, z \in D_i). \] (5.7)

(iii) \( \hat{F}^{(0)}(z) \) satisfies the normalizing condition
\[ \hat{F}^{(0)}(0) = I. \] (5.8)

Then the functions
\[ \hat{F}(z) z^{X_0}, \quad \hat{F}^{(1)}(z) (1 - z)^{-X_1}, \]
give the fundamental solutions of the KZ equation of one variable normalized at \( z = 0 \) and \( 1 \) respectively.
Proof. We reduce this problem to Theorem 11. First, since \( \hat{F}^{(0)}(z) \) and \( \hat{F}^{(1)}(z) \) are grouplike, by virtue of Lemma 1 and 2, the functions \( \hat{f}^{(i)}(\bullet; z) \) are regarded as shuffle homomorphisms from \( S^i \) to \( C \), and the reciprocal of \( \hat{F}^{(1)}(z) \) is given by

\[
\left( \hat{F}^{(1)}(z) \right)^{-1} = \sum_{w \in S} \hat{f}^{(1)}(\text{reg}^1 \circ \rho^*(w); z) W.
\]

Put

\[
f^{(0)}(w; z) = \sum_{j=0}^{n} \hat{f}^{(0)}(\text{reg}^0(w\xi^n_{i-1}); z) \frac{\log z}{j!}, \quad (5.9)
\]

\[
f^{(1)}(w; z) = \sum_{j=0}^{n} \hat{f}^{(1)}(t^* \circ \text{reg}^0(w\xi^n_{i-1}); z) \frac{\log(1-z)}{j!}, \quad (5.10)
\]

for \( w \in S^0 \). Since both \( f^{(0)}(\bullet; z) \) and \( f^{(1)}(\bullet; z) \) are shuffle homomorphisms on \( S \), we have \( f^{(0)}(\xi_0; z) = \log z, \ f^{(1)}(\xi_0; z) = \log(1-z) \), and

\[
f^{(0)}(w; z) = \hat{f}^{(0)}(w; z), \quad f^{(1)}(w; z) = \hat{f}^{(1)}(t^*(w); z)
\]

hold for all words \( w \in S^0 \). Since \( \hat{F}^{(1)}(z) \) are holomorphic on \( D_i \), we see that \( f^{(i)}(w; z) \) are holomorphic on \( D_i \) for all words \( w \in S^0 \).

In this situation, it is clear that the asymptotic condition (5.7) is equivalent to (4.4), and is also clear that the normalizing condition (5.8) is equivalent to (4.5).

Finally we show that (5.6) and (4.3) are equivalent under these conditions above. By definition, we have

\[
\hat{F}^{(0)}(z) X_0 = \left( \sum_{w \in S^0} \sum_{n=0}^{\infty} \hat{f}^{(0)}(w; z) W X^n_0 \right) \left( \sum_{j=0}^{\infty} \frac{\log^j z}{j!} X^j_0 \right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{w \in S^0} \sum_{j=0}^{n} \hat{f}^{(0)}(w; z) \frac{\log^j z}{j!} W X^n_0
\]

\[
= \sum_{w \in S^0} f^{(0)}(w; z) W X^n_0
\]

\[
= \sum_{w \in S} f^{(0)}(w; z) W.
\]
Using \( \text{reg}^1 = t^* \circ \text{reg}^0 \circ t^* \), \( \tau = t^* \circ \rho^* \) and (2.29), we also have

\[
(1 - z)^X_1 \left( \hat{F}^{(1)}(z) \right)^{-1} = \left( \sum_{j=0}^{\infty} \frac{\log^j(1 - z)}{j!} X_1^j \right) \left( \sum_{w \in S^0} \sum_{n=0}^{\infty} \hat{f}^{(1)}(t \circ \text{reg}^0(w); z) W X_0^n \right)
\]

\[
= \left( \sum_{j=0}^{\infty} \frac{\log^j(1 - z)}{j!} T(X_0^j) \right) \left( \sum_{w \in S^0} \sum_{n=0}^{\infty} \hat{f}^{(1)}(t^* \circ \text{reg}^0(w); z) T(W X_0^n) \right)
\]

\[
= \sum_{w \in S^0} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \hat{f}^{(1)}(t^* \circ \text{reg}^0(w); z) \frac{\log^j(1 - z)}{j!} T(W X_0^n)
\]

\[
= \sum_{w \in S^0} \sum_{n=0}^{\infty} f^{(1)}(w; z) T(W X_0^n)
\]

Thus the equation (5.6) can be written as

\[
\left( \sum_{u \in S} f^{(1)}(\tau(u); z) U \right) \left( \sum_{v \in S} f^{(0)}(v; z) V \right) = \Phi_{KZ} = \sum_{w \in S} \zeta(\text{reg}^0(w)) W.
\]

The coefficient of \( W \) of this equation is the equation (5.13).

From Proposition 12, we can apply Theorem 11 and have

\[
f^{(0)}(w; z) = \text{Li}(w; z), \quad f^{(1)}(w; z) = \text{Li}(w; 1 - z).
\]

Therefore,

\[
\hat{F}^{(0)}(z) = \sum_{w \in S} \text{Li}(\text{reg}^0(w); z) W,
\]

\[
\hat{F}^{(1)}(z) = \sum_{w \in S} \text{Li}(t^* \circ \text{reg}^1(w); 1 - z) W = \sum_{w \in S} \text{Li}(\text{reg}^0(w); 1 - z) t_*(W) \quad (5.12)
\]

are unique solutions to this Riemann-Hilbert problem. The last claim follows from (2.20) and (2.31).

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