Semidefinite programming is a fundamental tool in optimization and theoretical computer science. It has been extensively used as a black-box for solving many problems, such as embedding, complexity, learning, and discrepancy.

One natural setting of semidefinite programming is the small treewidth setting. The best previous SDP solver under small treewidth setting is due to [ZL18], which takes $n^{1.5} \cdot \tau^{0.5}$ time. In this work, we show how to solve a semidefinite programming with $n \times n$ variables, $m$ constraints and $\tau$ treewidth in $n \cdot \tau^{2\omega+1/2}$ time, where $\omega$ denotes the exponent of matrix multiplication. We give the first SDP solver that runs in time in linear in number of variables under this setting.

In addition, we improve the running time that solves a linear programming with $\tau$ treewidth from $n\tau^2$ [Ye20, DLY21] to $n\tau^{(\omega+1)/2}$. 

*yuzhougu@mit.edu*. MIT.
†zsong@adobe.com. Adobe Research.
1 Introduction

Semidefinite programming is one fundamental problem in optimization and theoretical computer science. It has many applications in computer science, such as verifying the robustness of neural network \cite{RSL18}, solving the discrepancy problems \cite{Ban10}, sparse matrix factorization \cite{CSTZ22}, sparsest cut \cite{ARV09}, 3-colorable graph \cite{KMS94}, terminal embeddings \cite{CN21}, quantum complexity theory \cite{JJUW11}, obtaining tight approximation ratio for MAXCUT \cite{GW95}, solving sum of squares programs \cite{BS16, FKP19}, and so on.

Mathematically, a semidefinite program can be defined as follows.

**Definition 1.1 (Semidefinite Programming (SDP)).** Given a collection of matrices $A_1, A_2, \cdots, A_m \in \mathbb{R}^{n \times n}$, and $b \in \mathbb{R}^m$, the goal is to find a matrix $X \in \mathbb{R}^{n \times n}$ such that

$$\min \quad C \cdot X$$
$$s.t. \quad A_i \cdot X = b_i \quad \forall i \in [m]$$
$$X \succeq 0$$

Here $C, X \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{m \times n^2}$, $b \in \mathbb{R}^m$.

Over the last century, many efforts have been put into optimizing the running time of semidefinite programming \cite{Sho77, YN76, Kha80, KTE88, NN89, Vai89, KM03, BV02, LSW15, JLSW20, JLSW20, JKL20, HJS22}. Based on how many iterations you need to solve semi-definite programming, the previous work of solving semi-definite programming can be splitted into three lines: cutting plane method \cite{LSW15, JLSW20}, log barrier function \cite{JKL20, HJS22}, volumetric/hybrid barrier function \cite{HJS22}.

Many graph problems can be written as semidefinite programs with certain structure. For example, the famous Goemans-Williamson algorithm \cite{GW95} relaxes MaxCut problem to a semidefinite program. When the underlying graph has low treewidth, the corresponding SDP also has small treewidth. While the MaxCut problem takes exponential time in treewidth \cite{LMS11} (under plausible hypothesis like Exponential Time Hypothesis \cite{IP01}), when treewidth is constant, it can be easily solved in linear time; on the other hand, before our work, it was not known whether MaxCut SDP can be solved in nearly linear time even when treewidth is constant. Another graph problem is Lovász theta function \cite{Lov79}, it is an important quantity in noisy communication and is naturally defined as a semidefinite program.

In practice, data sets on graphs often have small treewidth, see e.g. \cite[Table 1]{ZL18}, and \cite[Appendix B]{DLY21}. Therefore SDP with small treewidth is a problem of practical interest.

In this work, we consider SDP for which the treewidth is small. Formally,

**Definition 1.2 (Treewidth of SDP).** Given an SDP (Definition 1.1), we define its SDP graph as a graph $G = (V, E)$ where $V = [n]$ and

$$E(G) = \{(u, v) \in V \times V : C_{u,v} \neq 0\} \cup \bigcup_{i \in [m]} \text{supp}(A_i) \times \text{supp}(A_i)$$

where

$$\text{supp}(A) := \{u \in V : A_{u,*} \neq 0\}.$$

Treewidth of an SDP is defined as treewidth of the SDP graph.

Let $\tau$ denote the treewidth of the SDP graph. The best previous work on small treewidth semidefinite programming is due to \cite{ZL18}, which runs in $n^{1.5 + \tau}$ time. A natural question to consider is,
Can we solve SDP much faster under treewidth assumption, i.e., in particular can we solve SDP in nearly linear time in $n$?

In this work, we answer the above question in affirmative.

1.1 Our Results

1.1.1 Low-Treewidth SDP

We make the following linear independence assumption on our SDP. This assumption is standard in IPM literature, and is also present previous works [ZL18].

**Assumption 1.3 (Linear Independence).** The constraint matrices $A_1, \ldots, A_m \in \mathbb{R}^{n \times n}$ are linearly independent.

Under Assumption 1.3, the total number of constraints $m$ is bounded by $m \leq \tau n$.

For the standard semidefinite programming (without low-treewidth assumptions), the best previous algorithms are due to [JKL+20, HJS+22]. Those algorithms are based on dual-only central path algorithms. Due to technical reasons, even with small treewidth assumption, those techniques lead to algorithm with running time at least $\Omega(m^\omega)$,\(^1\) and cannot solve semidefinite programming with time only linear in $n$.

We state our main result as follows:

**Theorem 1.4 (Our Result on SDP, Informal Version of Theorem 6.1).** There is an algorithm that solves semidefinite programming (1) with accuracy $\epsilon > 0$ in

$$O(n \cdot \tau^{2\omega+0.5} \log(1/\epsilon))$$

time.

1.1.2 Low-Treewidth LP

Furthermore, we also improve the best previous low-treewidth LP result due to [DLY21] (which runs in $n\tau^2$ time). Before stating our result for LP, let us define treewidth of an LP.

**Definition 1.5 (Linear Programming (LP)).** Given matrix $A \in \mathbb{R}^{m \times n}$ and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, the goal is to find a vector $x \in \mathbb{R}^n$ such that

$$\min \quad c^\top x$$

$$\text{s.t.} \quad Ax = b$$

$$x \geq 0$$

**Definition 1.6 (Treewidth of LP).** Given an LP, we construct the LP dual graph where vertices are constraints, and there is an edge between two vertices if they share a common variable (with non-zero coefficients). Then treewidth of the LP is defined as the treewidth of its LP dual graph.

**Theorem 1.7 (Our Result on LP, Informal Version of Theorem 8.2).** There is an algorithm that solves linear programming (3) with accuracy $\epsilon > 0$ in

$$O(n \cdot \tau^{(\omega+1)/2} \log(1/\epsilon))$$

time.

\(^1\)Here $\omega$ denotes the exponent of matrix multiplication, i.e., multiplying an $n \times n$ matrix with another $n \times n$ matrix takes $n^\omega$ time. Currently, $\omega \approx 2.373$ [Wil12, AW21].
| Problem | Year | Authors | References | Time         |
|---------|------|---------|------------|--------------|
| LP      | 2020 | Ye      | [Ye20]     | $n^{1.25\text{poly}(\tau)}$ |
| LP      | 2021 | Dong, Lee, Ye | [DLY21] | $n^{\tau^2}$ |
| LP      | 2022 | This work | Theorem 1.7 | $n^{\tau(\omega+1)/2}$ |
| SDP     | 2018 | Zhang and Lavaei | [ZL18] | $n^{1.5\cdot0.5}$ |
| SDP     | 2022 | This work | Theorem 1.4 | $n^{2\omega+0.5}$ |

Table 1: Summary of results on low-treewidth LP and SDP.

1.1.3 Decomposable SDP

There are cases where Theorem 1.4 does not apply but we can still obtain an efficient SDP solving algorithm. One such case is summarized in the following theorem. To state the result, let us define the sparsity graph, a graph with potentially lower treewidth than the SDP graph (Definition 1.2).

Definition 1.8 (Sparsity graph [ZL18]). Given an SDP (Definition 1.1), we define its sparsity graph as a graph $G = (V, E)$ where $V = [n]$ and

$$E(G) = \{(u, v) \in V \times V : C_{u,v} \neq 0\} \cup \bigcup_{i \in [m]} \{(u, v) \in V \times V : A_{i,u,v} \neq 0\}. \tag{4}$$

Theorem 1.9 (Our Result on Decomposable SDP, Informal Version of Theorem 7.2). Given an SDP (1). Suppose we have a tree decomposition $T = \{T_1, \ldots, T_\ell\}$ of the sparsity graph $G$ (Definition 1.8). Suppose in addition that for every constraint $A_i$, we are given a connected (in the tree decomposition) set of bags $T_{i,1}, \ldots, T_{i,m_i}$ such that the union of these bags contains the support of the constraint, i.e.,

$$\{(u, v) \in V \times V : A_{i,u,v} \neq 0\} \subseteq \bigcup_{j \in [m_i]} (T_{i,j} \times T_{i,j}).$$

Let $\gamma_{\text{max}}$ be the maximum number of constraints a bag correspond to. Let $\tau$ be the maximum size of a bag. Then the SDP can be solved in time $\tilde{O}(n^{(\tau^2 + \gamma_{\text{max}})^\omega})$.

A special case for which the condition in Theorem 1.9 is satisfied is the “Network Flow Semidefinite Program” studied in [ZL18].

Definition 1.10 (Network Flow Semidefinite Program). A network flow SDP is an SDP (Definition 1.1) where for every constraint matrix $A_i$, there exists a vertex $k_i \in [n]$ such that $A_{i,u,v} \neq 0 \Rightarrow k_i \in \{u, v\}$.

[ZL18] gave an algorithm to solve network flow semidefinite programs in $\tilde{O}(n^{1.5\cdot(\tau^2 + d_{\text{max}}m_{\text{max}})^3})$ time, where $d_{\text{max}}$ is the maximum degree of the tree $T$ and $m_{\text{max}}$ is the maximum number of network flow constraints at any vertex (i.e., $m_{\text{max}} := \max_{k \in V} \# \{i \in [m] : k_i = k\}$).

Using Theorem 1.9, we can achieve almost linear running time for network flow semidefinite programs.

Corollary 1.11. The network flow semidefinite program can be solved in time $\tilde{O}(n^{0.5(\tau^2 + \gamma_{\text{max}})^\omega})$.

Note that for network flow SDPs, a trivial bound for $\gamma_{\text{max}}$ is $\gamma_{\text{max}} \leq \tau m_{\text{max}}$. 

3
1.2 Applications

In this section, we discuss three applications: MaxCut SDP, Lovasz Theta function, and low rank matrix completion.

1.2.1 MaxCut SDP and MaxkCut SDP

For a graph, a maximum cut is a cut whose size is at least the size of any other cut. That is, it is a partition of the graph’s vertices into two complementary sets $S$ and $T$, such that the number of edges between the set $S$ and the set $T$ is as large as possible. The problem of finding a maximum cut in a graph is known as the MaxCut Problem. This problem is NP-complete [Kar72], and also APX-hard [PY91]. The MaxCut problem is a natural generalization of MaxCut, where we would like to partition the set of vertices into $k$ disjoint parts, and maximize the number of edges between different parts. A Max2Cut problem is the same as a MaxCut problem.

MaxCut and MaxCut admit natural SDP relaxations.

**Definition 1.12 (MaxCut SDP [GW95] and MaxCut SDP [FJ97]).** Let $L_G$ be the weighted Laplacian matrix for a graph $G$ with $n$ vertices. There is a randomized algorithm to solve MaxCut with an approximation ratio of $1 - 1/k$ based on solving

$$\begin{align*}
\text{maximize} & \quad \frac{k - 1}{2k} L_G \cdot X \\
\text{subject to} & \quad X_{i,i} = 1, \quad \forall i \in [n] \\
& \quad X_{i,j} \geq -\frac{1}{k - 1}, \quad \forall (i, j) \in E(G) \\
& \quad X \succeq 0.
\end{align*}$$

The classic Goemans-Williamson 0.878-approximation algorithm [GW95] for MaxCut is recovered by setting $k = 2$ and removing the redundant constraint $X_{i,j} \geq -1$. This is the best possible approximation ratio under the Unique Games Conjecture [KKMO07].

In both the MaxCut relaxation and the MaxCut relaxation, note that the SDP graph (Definition 1.2) is the same as the given graph $G$. Therefore SDP treewidth $\tau_{\text{sdp}}$ is equal to treewidth of graph $G$.\footnote{To apply Theorem 1.4, we need a way to deal with inequality constraints. See Section 5.5 for a discussion.}

1.2.2 Lovász Theta Function

We discuss another famous application of SDP which is the Lovász theta function,

**Definition 1.13 (Lovasz Theta Function, [Lov79]).** Given a graph $G = (V, E)$ and $y_{i,j} \in \mathbb{R}$ for each $(i, j) \in E$. Let $n = |V|$. Let $\overline{E}$ denote the complement of $E$. Let $\overline{G} = (V, \overline{E})$. The Lovász number $\vartheta(G)$ of a graph $G$ is the optimal value to the following semidefinite program

$$\begin{align*}
\text{minimize} & \quad \begin{bmatrix} I & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix} \cdot X \\
\text{subject to} & \quad X_{i,j} = 0, \forall (i, j) \in \overline{E} \\
& \quad X_{n+1, n+1} = 1 \\
& \quad X \succeq 0.
\end{align*}$$
The SDP graph (Definition 1.2) is obtained by adding a new vertex \( o \) to the complement graph \( \overline{G} \) of input graph \( G \), with an edge \((i, o)\) for all \( i \in V \). Given a tree decomposition of \( \overline{G} \) with bag size \( \tau \), a tree decomposition of the SDP graph with bag size \( \tau + 1 \) can be obtained by adding the new vertex \( o \) into every bag. So SDP treewidth is small when \( G \) has small treewidth.

### 1.2.3 Low Rank Matrix Completion

In the low rank matrix completion problem, we are given a partially filled matrix \( B \), where entries \((i, j) \in \Omega\) are filled. The goal is to find a matrix \( X \) of smallest rank that agrees with \( B \) on \( \Omega \).

\[
\text{minimize } \text{rank}(X) \\
\text{subject to } X_{i,j} = B_{i,j}, \forall (i,j) \in \Omega.
\]

This problem is non-convex and in general difficult to solve. A popular relaxation [CR09] of the problem is to minimize the nuclear norm \( \| \cdot \|_* \) instead.

\[
\text{minimize } \|X\|_* \\
\text{subject to } X_{i,j} = B_{i,j}, \forall (i,j) \in \Omega.
\]

Furthermore, [CR09] rewrote the above formulation in the following equivalent form:

\[
\text{minimize } I \cdot Y \\
\text{subject to } Y_{i,n+j} = B_{i,j}, \forall (i,j) \in \Omega \\
Y \succeq 0
\]

where \( Y \in \mathbb{R}^{2n \times 2n} \).

Let \( G = ([n], \Omega) \) be the adjacency graph. The SDP graph (Definition 1.2) is as follows: for every edge \((i, j) \in \Omega\), the SDP graph contains the clique \( \{i, j, i+n, j+n\} \).

Given any tree decomposition of \( G \), we can define a tree decomposition of the SDP graph by replacing every bag \( S \) with \( \{i, i+n : i \in S\} \). So SDP treewidth is at most twice \( \text{tw}(G) \). When \( \text{tw}(G) \) is small, SDP treewidth is small.

### 1.3 Related Work

**Linear programming**  Linear programming is an old topic in computer science. Simplex algorithm [Dan47] is one of the most important algorithm in the history of linear programming, and it has exponential running time. Ellipsoid method reduced the running time to polynomial [Kha80], however, it is in practice slower than simplex method. The interior point method [Kar84] is a major breakthrough, because it has have theoretical polynomial running time and stably fast practical performance on real-world problems. Assuming \( d \) is the number of constraints and \( n \) is the number of variables, when \( d = \Omega(n) \), Karmarkar’s algorithm has a running time of \( O^*(n^{3.5}) \). The running time is further improved to \( O^*(n^3) \) in [Vai87, Ren88] and \( O^*(n^{2.5}) \) in [Vai89]. Recently, [CLS19] shows how to solve linear program in \( O(n^\omega + n^{2.5-\alpha/2} + n^{2+1/6}) \) time, where the \( \omega \) is the exponent of matrix multiplication, and \( \alpha \) is the dual exponent of matrix multiplication. Later, [LSZ19] generalize that algorithm to a more broader optimization problem called empirical risk minimization. [SY21] shows how to re-produce the result in [CLS19] via oblivious sketching matrices instead of non-oblivious sampling matrices. When the linear programming constraint matrix is tall and dense, [BLSS20] shows how to solve it in \( \tilde{O}(nd) + \text{poly}(d) \) time. For a linear program with small treewidth \( \tau \), the work [DLY21] shows how to solve it in \( \tilde{O}(n\tau^2) \) time.
Semidefinite programming In the previous SDP literature, there are two lines of SDP solvers: second order methods and first order methods. Second order methods usually have running time with logarithmic dependence on $\epsilon$, while first order methods usually have polynomial dependence on $\epsilon$.

Second order methods can be further split into several lines: first line is cutting plane methods, for example [LSW15, JLSW20]. The algorithm in [LSW15, JLSW20] can solve semidefinite programming in $O(m(mn^2 + m^2 + n^\omega))$ time. The second line is interior point methods (associated with certain self-concordant barrier functions), for example, [NN92, JKL+20, HJS+22] use log-barrier function, and [Ans00, HJS+22] use hybrid barrier function. The algorithm in [JKL+20] can solve SDP in $O(\sqrt{n}(mn^2 + n^\omega + n^\omega))$ time. Furthermore, [HJS+22] shows that as long as $m = \Omega(n^2)$, we can solve semidefinite programming in $O(m^\omega + m^{2+1/4})$ time.

The interior point method is a second-order algorithm. Second-order algorithms usually have logarithmic dependence on the error parameter $1/\epsilon$. First-order algorithms do not need to use second-order information, but they usually have polynomial dependence on $1/\epsilon$. There is a long list of work focusing on first-order algorithms [AK07, GH16, AZL17, CDST19, LP20, YTF+19, JY11, ALO16].

For a semidefinite program with bounded treewidth, the work [ZL18] shows how to solve it in $\tilde{O}(n^{1.5}\cdot \tau^{6.5})$ time.

2 Technique Overview

In this section, we explain the previous technique and summarize our new techniques.

- In Section 2.1, we briefly summarize the techniques in [ZL18] and explain the barrier of that algorithm.
- In Section 2.2, we provide an overview of an $n \cdot \text{poly}(\tau)$ time algorithm.
- In Section 2.3, we briefly analyze running time of the algorithm in previous section and explain the bottlenecks.
- In Section 2.4, we discuss how we improve the running time for bottlenecks, thus achieving an improved algorithm for low-treewidth SDP.
- In Section 2.5, we discuss how we achieve improved algorithm for low-treewidth linear program.

2.1 An Overview of Previous Techniques

In this section, we briefly summarize the technique in [ZL18].

[ZL18] uses interior point method to solve SDP. However, running IPM directly would be very expensive. The first step of their algorithm is to rewrite the SDP and reduce the number of (real) variables from $n^2$ to $n \cdot \text{poly}(\tau)$.

Variable reduction Suppose we have an SDP of form (1) with a tree decomposition of maximum bag size $\tau$.

[ZL18] creates another SDP of smaller number of variables, where every bag in the tree decomposition becomes one $\tau \times \tau$ PSD matrix, which corresponds to an $\tau \times \tau$ principal minor in the original PSD matrix.
\[
\min \sum_{j \in [b]} C_j \cdot X_j
\]  \tag{5}

s.t.  
\[A_i \cdot X_{j_i} = b_i \quad \forall i \in [m]\]
\[N_{i,j}(X_i) = N_{j,i}(X_j) \quad \forall (i,j) \in E(T)\]
\[X_j \succeq 0 \quad \forall j \in [b]\]

See Appendix 5 for a more detailed explanation of the program.

The new SDP has three types of constraints:

- \(A_i \cdot X_{j_i} = b_i\), which corresponds to linear constraints in the original SDP;
- \(N_{i,j}(X_i) = N_{j,i}(X_j)\), which says that when two minors overlap in the original PSD matrix, they should have the same value in overlapping positions;
- \(X_j \succeq 0\), meaning that each minor should be PSD.

[ZL18] proved that given any feasible solution to the new SDP, we can compute in \(O(n \tau^3)\) time a succinct representation of a feasible solution to the original SDP with the same objective value. The succinct representation is of form \(X = UU^\top\) where \(U \in \mathbb{R}^{n \times \tau}\) is a rectangular matrix.

After this reduction, we run IPM iteration directly.\(^3\)

[ZL18] uses interior point method to solve (5). For every step, we need to perform computation of the following kind

\[
\left( A_{lp} H_x A_{lp}^\top \right)^{-1} v
\]  \tag{6}

where

- \(A_{lp} = \begin{bmatrix} A \\ N \end{bmatrix} \in \mathbb{R}^{m_{lp} \times n_{lp}}\) is the constraint matrix (where \(m_{lp} = O(n_{lp})\)) (to be more precise, each of the first \(m_{sdp}\) rows corresponds one original SDP constraint, and rows after that correspond to overlap constraints),
- \(H_x\) is the Hessian matrix, a block-diagonal matrix.

Because of the low-treewidth assumption, these updates can be implemented efficiently, as discussed below.

**Number of iterations** Number of iterations we need to run is controlled by the number of variables and geometry of the primal space. (More precisely, number of iterations is square root of sum of self-concordance value of all blocks.) Because we can have \(O(n) \times \tau \times \tau\) matrix variables, each of them is in the PSD cone, which is is \(\tau\)-self-concordant, the total number of iterations is

\[\left( \# \text{ of matrix vars} \cdot \text{self-concordance parameter for PSD cone} \right)^{0.5} = \tilde{O}(\sqrt{n \tau}).\]

\(^3\)In fact, [ZL18] ran IPM on the dual SDP, because the maximum degree of a bag in the tree decomposition could be large. However, we can easily make the maximum degree \(O(1)\) by adding \(O(n)\) bags. This does not change the asymptotic running time but results in a simpler algorithm.
**Cost per iteration** In each step, we need to perform a computation of form Eq. (6). When the tree decomposition has $O(1)$ maximum degree, $A_{lp} H_x A_{lp}^\top$ is block-sparse and we can compute a Cholesky decomposition $A_{lp} H_x A_{lp}^\top = LL^\top$ is $O(n \cdot \text{poly}(\tau))$ time. Then $(A_{lp} H_x A_{lp}^\top)^{-1} v = L^{-\top} L^{-1} v$ can be computed in $O(n \cdot \text{poly}(\tau))$ time by performing multiplication from right to left.

Each of the matrix factors in the above equation can be computed in $\tilde{O}(n \tau^6)$ time, and contains $O(n \tau^4)$ non-zero entries, so cost per iteration is $\tilde{O}(n \tau^6)$.

**Putting it all together** We have

$$\#\text{iters} \cdot \text{(cost per iters)} = \tilde{O}(\sqrt{n \tau}) \cdot \tilde{O}(n \tau^6)$$

$$= \tilde{O}(n^{1.5 \tau^{6.5}})$$

So the overall running time of [ZL18]’s algorithm is $\tilde{O}(n^{1.5 \tau^{6.5}})$.

### 2.2 How to Get An $n \cdot \text{poly}(\tau)$ Algorithm?

[ZL18] solved (5) using IPM with $O(n^{0.5 \tau^{0.5}})$ iterations and $O(n \text{poly}(\tau))$ time per iteration. The number of iterations usually pays at least $\Omega(\sqrt{n})$ in the field of IPM methods due to known lower bound of barrier method.\(^4\) Therefore we seek to improve cost per iteration. Using robust IPM to solve linear program in tree-width setting, in each iteration, we only need to update those coordinates where there is a significant enough change, and this would reduce total number of coordinate updates to $O(n \text{poly}(\tau))$, giving an $O(n \text{poly}(\tau))$ algorithm.

More specifically, [DLY21] gave an $O(n \tau^2)$ algorithm for low-treewidth LP. Their central path equation is

$$s/t + w \nabla \phi(x) = \mu$$

$$Ax = b$$

$$A^\top y + s = c$$

where $x$ is the primal variable, $y$ is the dual variable, $s$ is the slack variable, and $\mu$ denotes the error. The central path is defined as the path of $(x, s)$ as $t$ goes from initial value to 0.

Then the Newton step is

$$\delta s/t + w H_x \delta x = \delta \mu$$

$$A \delta x = 0$$

$$A^\top \delta y + \delta s = 0$$

where $H_x = \nabla^2 \phi(x)$ is the Hessian.

We only need to follow the central path approximately, meaning that when computing the steps, we could use $\overline{x}$ instead of $x$, where $\overline{x}$ is a vector close enough to $x$. The IPM algorithm works roughly in the following sense.

- $t \leftarrow t_{\text{start}}$

- While $t \geq t_{\text{end}}$ do

\(^4\)[NN94] showed that any self-concordant barrier of $n$-dimensional simplex or hypercube must have self-concordance parameter at least $n$. This bound is tight, because [LY21] recently showed that there always exists an $n$-self-concordant barrier
\[
- \delta_t \leftarrow \arg\max_{\delta_t} \langle \nabla_{\delta_t} \Psi_{\lambda}(\|\mu_t\|_2), \mu + \delta_t \rangle \\
- \delta_x \leftarrow (H_\tau^{-1} - H_\tau^{-1} A_{lp}^\top (A_{lp} H_\tau^{-2} A_{lp}^\top)^{-1} A_{lp} H_\tau^{-1}) \delta_x \\
- \delta_s \leftarrow t A_{lp}^\top (A_{lp} H_\tau^{-1} A_{lp}^\top)^{-1} A_{lp} H_\tau^{-1} \delta_s \\
x \leftarrow x + \delta_x, \ s \leftarrow s + \delta_s \\
t \leftarrow t \cdot (1 - h)
\]

The parameters satisfy \( t_{\text{start}} / t_{\text{end}} = \text{poly}(n), \ h = \Theta(n^{-1/2}), \) so there are \( \tilde{O}(n^{1/2}) \) steps.

The general LP and SDP central path algorithm (without treewidth assumption) maintains \((AH_\tau^{-1} A^\top)^{-1}\) directly, since we don’t have the benefit of Cholesky decomposition (coming from treewidth assumption). This would lead to algorithms with cost-per-iteration super-linear in \( n \).

The key idea of \cite{DLY21} is that, instead of maintaining the pair \((x, s)\) on the central path explicitly, we maintain them implicitly (“multiscale coefficients”)

\[
x = \hat{x} + H_\tau^{-1/2} \beta_x c_x - H_\tau^{-1/2} \mathcal{W}^\top (\beta_x h + \epsilon_x) \\
s = \hat{s} + H_\tau^{-1/2} \mathcal{W}^\top (\beta_s h + \epsilon_s),
\]

where \( \mathcal{W} = L_\tau^{-1} AH_\tau^{-1/2} \) and \( \hat{x}, \hat{s}, \epsilon \) are sparsely changing vectors. Here \( L_\tau \) is the Cholesky factor in the Cholesky decomposition \( AH_\tau^{-1} A^\top = L_\tau L_\tau^\top \). Under a central path step and under a sparse update to \( \tau \), all terms in the multiscale representation (7) can be maintained efficiently (see Section 4.4.1). Therefore if the total number of coordinate updates is \( O(n) \), and the cost per coordinate is \( O(\text{poly}(\tau)) \), then the overall running time is \( O(n \text{poly}(\tau)) \).

Similar ideas can be applied to our treewidth SDP case. We maintain primal-dual pairs \((x_i, s_i)\), and let them follow the central path. In each step, we need to run approximate Newton step

\[
\delta_x = (H_\tau^{-1} - H_\tau^{-1} A_{lp}^\top (A_{lp} H_\tau^{-1} A_{lp}^\top)^{-1} A_{lp} H_\tau^{-1}) \delta_x \\
\delta_s = t A_{lp}^\top (A_{lp} H_\tau^{-1} A_{lp}^\top)^{-1} A_{lp} H_\tau^{-1} \delta_s,
\]

where

\begin{itemize}
  \item \( x = \left[ \text{vec}(x_1) \cdots \text{vec}(x_n) \right]^\top \in \mathbb{R}^{n_{lp}} \) is the primal variable,
  \item \( s = \left[ \text{vec}(s_1) \cdots \text{vec}(s_n) \right]^\top \in \mathbb{R}^{n_{lp}} \) is the slack variable,
  \item \( A_{lp} = \left[ \begin{array}{c}
  A \\
  N
  \end{array} \right] \in \mathbb{R}^{m_{lp} \times n_{lp}} \) is the constraint matrix (where \( m_{lp} = O(m_{lp}) \)) (to be more precise, each of the first \( m_{adp} \) rows corresponds one original SDP constraint, and rows after that correspond to overlap constraints),
  \item \( H_\tau = \nabla^2 \phi(\tau) \in \mathbb{R}^{n_{lp}} \) is the Hessian matrix for some potential function \( \phi \),
  \item \( \delta_t \in \mathbb{R}^{n_{lp}} \) is the step of the error vector \( \mu \) (see Section A for its definition).
\end{itemize}

We maintain the block version of the multiscale coefficients. We remark that one of the major difference for semidefinite programming compared to linear programming is that the Hessian matrix \( H_\tau \) is no longer a diagonal matrix. Nevertheless, the Hessian matrix is a block diagonal matrix. Therefore its inverse and square root can be computed in \( \text{poly}(\tau) \) time. When the tree decomposition has \( O(1) \) maximum degree, our program (5) has treewidth \( O(\tau_{lp}) = O(\tau^2) \) in the sense of Definition 4.1. Therefore we can still efficiently compute and maintain the Cholesky decomposition of one factor in the Newton step: \( A_{lp} H_\tau^{-1} A_{lp}^\top = L_\tau L_\tau^\top \).
To maintain the central path, we need to have a data structure which can support (1) central path steps (8), and (2) report coordinates on which \( x \) and \( \overline{y} \) differ too much.

The first task (central path steps) is maintained using ExactDS (Section 4.4.1). For the second task (finding important coordinates), a natural idea is to use Johnson–Lindenstrauss sketch (Lemma 3.6). Suppose we have a data structure which can answer range queries on \( \Phi \overline{y} \) and \( \Phi x \) (where \( \Phi \in \mathbb{R}^{r \times np} \) is the JL matrix), then we can efficiently find the coordinates that differ much.

However, because we maintain \((x, s)\) implicitly, maintaining \( \Phi x \) under central path steps and under change of multiscale representation and answering range queries is not a trivial task. We resolve this issue by using BlockBalancedSketch (Section 4.4.5), a block version of balanced sampling tree sketch in [DLY21]. Using BlockBalancedSketch and BlockVectorSketch (Section 4.4.4, an easy application of segment trees), we are able to implement BatchSketch (Section 4.4.3), a data structure for maintaining sketches of \((x, s)\).

Furthermore, even equipped with the sketching data structures, we still need an algorithm to maintain an approximation of the sketched vector. Because we are able to control the step size in the IPM algorithm, we know that \((x, s)\) is in fact slowly changing under \( \ell_2 \). Therefore, we are able to maintain a sparsely changing approximation of \((x, s)\). This is achieved using APPROXDS (Section 4.4.2). By removing a sampling step in the corresponding data structure in [DLY21], we fix a technical issue of [DLY21] (see Remark 4.19). APPROXDS uses BatchSketch as a subroutine.

Finally, by combining ExactDS and APPROXDS, we are able to maintain \((x, s)\) implicitly, and to maintain an approximation \((\overline{y}, \overline{s})\) explicitly. This gives our central path maintenance algorithm CentralPathMaintenance (Section 4.2).

### 2.3 Bottlenecks for Running Time

In this section we give a brief analysis of running time. The actual analysis is much more complicated, but the highlight the bottlenecks here.

In our central path maintenance data structure CentralPathMaintenance we regularly updates and reconstructs the data structure APPROXDS, because its running time is quadratic in the number of iterations between two reconstructs. (This is because after \( q \) iterations, \( \|x^{(q)} - x^{(0)}\|_2 \) is of order \( q^2 \), and \( \ell_\infty \) approximations of \( x^{(0)} \) and of \( x^{(0)} \) can differ in \( q^2 \) coordinates.) There are \( N = O(\sqrt{n_{sdp} \tau_{sdp} w}) \) iterations (where \( w \) is a parameter to be chosen, it does not affect running time as long as it is large enough) and we reconstruct the data structure after every \( q \) (a parameter to be chosen) iterations.

**Restart time:** Over the entire \( N \) iterations of the algorithm (Algorithm 1), we need to do \( N/q \) times reconstructions (restart). Each reconstruction takes \( npoly(\tau) \) time. Thus, the total number of restart time is \( O(N/q \cdot npoly(\tau)) \).

**Update time:** Between every two restarts, there are \( \tilde{O}(q^2) \) coordinate changes. So there are \( \tilde{O}(N/q \cdot q^2) = O(Nq) \) coordinate changes in total. Each coordinate change takes \( O(poly(\tau)) \) time to update. So the total cost for updates is \( O(Nq \cdot poly(\tau)) \).

Therefore, the final running time is a result of balancing the total restart time and update time.

In the following, we plug in the exact exponents of \( \tau \) and compute the running time.

**Restart** Restart can be splitted into two steps, one is initialization and the other is output. The most time-consuming step is computing Cholesky factorization in initialization.

We have \( n_{sdp} \) blocks, where each block is a \( \tau_{sdp} \times \tau_{sdp} \) matrix. By structure of our SDP, the Cholesky factor \( L_{\overline{y}} \) in \( A H_{\overline{y}}^{-1} A^T = LL^T \) is \( \tilde{O}(\tau_{lp}) \)-sparse (i.e., every column has \( \tilde{O}(\tau_{lp}) \) non-zero coordinates), where \( \tau_{lp} = \tau_{sdp}^2 \). So computing Cholesky factorization takes \( \tilde{O}(n_{lp} \tau_{lp}^2) = \tilde{O}(n_{sdp} \tau_{sdp}^6) \) time.
Algorithm 1: An informal version of our central path maintenance algorithm, Algorithm 2

1: Initialize ExactDS and ApproxDS
2: for $i = 1 \rightarrow N$ do
3:   if we have not restarted for $q$ iterations then
4:     /* Restart */
5:     Restart our data structure
6:   end if
7:   /* Move */
8:   Run one step of central path in ExactDS \hspace{1em} \triangleright\hspace{0.5em} Implicitly maintain $x$ and $s$
9:   Use ApproxDS to detect which coordinates of $(x, s)$ have heavy changes. \hspace{1em} \triangleright\hspace{0.5em} Compute update to $(\mathcal{x}, \mathcal{s})$
10:  Update ExactDS according to update in $(\mathcal{x}, \mathcal{s})$
11:  Update ApproxDS according to update in $(\mathcal{x}, \mathcal{s})$
12: end for

So

overall restart time = number of restarts \cdot cost per restart

\hspace{1em} = \tilde{O}(N/q) \cdot \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^6)
\hspace{1em} = \tilde{O}(n_{\text{sdp}}^{1.5} q^{-0.5} \tau_{\text{sdp}}^{6.5}).

Update: As described in Algorithm 1, update can be splitted into mainly four substeps: Central path (ExactDS) move, ApproxDS query (using sketching data structure BatchSketch), ExactDS update, and sketching data structure BatchSketch update.

- The central path move step takes $O(1)$ time per iteration because it only updates constant coefficients in the multiscale representation.
- ApproxDS query takes $\tilde{O}(qw^{-1})$ queries (on average) to the sketching data structure BatchSketch on average. Note that this also implies there are $\tilde{O}(qw^{-1})$ variable block changes per iteration. Every sketching data structure query takes $\tilde{O}(\tau_{\text{lp}}^2)$ time.
- For each variable block change in $\mathcal{x}$, updating ExactDS takes $\tilde{O}(\tau_{\text{lp}}^3) = \tilde{O}(\tau_{\text{sdp}}^6)$ time due to the cost of updating Cholesky factorization. This is because one variable block change in $\mathcal{x}$ leads to a rank $O(\tau_{\text{lp}})$-change in $AH_{\mathcal{x}}^{-1}A$, and a rank-1 change in $AH_{\mathcal{x}}^{-1}A$ takes $\tilde{O}(\tau_{\text{lp}}^2)$ to update the Cholesky factor $L_{\mathcal{x}}$. Therefore on average, every iteration takes $\tilde{O}(qw^{-1}n_{\text{sdp}}^6)$ time.
- For each variable block change in $\mathcal{x}$, updating BatchSketch takes $\tilde{O}(\tau_{\text{lp}}^3) = \tilde{O}(\tau_{\text{sdp}}^6)$ time due to the cost of updating Cholesky factorization. Therefore on average, every iteration takes $\tilde{O}(qw^{-1}n_{\text{sdp}}^6)$ time.

As analyzed above, on average, the update time for each iteration is $\tilde{O}(qw^{-1}n_{\text{sdp}}^6)$. So

overall update time = number of iters \cdot update cost per iter

\hspace{1em} = \tilde{O}(N) \cdot \tilde{O}(qw^{-1}n_{\text{sdp}}^6)
\hspace{1em} = \tilde{O}(n_{\text{sdp}}^{0.5} q^{-0.5} \tau_{\text{sdp}}^{6.5}).
Combining everything above, we have

\[ \text{overall running time} = \text{overall restart time} + \text{overall update time} \]
\[ = \tilde{O}(n_{\text{sdp}}^{1.5} w^{0.5} q^{-1} \tau_{\text{sdp}}^{6.5}) + \tilde{O}(n_{\text{sdp}}^{0.5} w^{-0.5} q \tau_{\text{sdp}}^{6.5}) \]
\[ = \tilde{O}(n_{\text{sdp}}^{6.5} \tau_{\text{sdp}}) \]

where in the last step we take \( w = \tau_{\text{sdp}}, q = n_{\text{sdp}}^{0.5} \).

### 2.4 Improve Running Time Bottleneck

In this section we discuss how to improve the running time bottleneck. As described in the last section, the most time consuming step is computing Cholesky factorization \( A^{1/2} A^\top = L \Sigma L^\top \) (which affects restart time), and updating Cholesky factorization under change of one variable block (which affects update time).

The way we achieve improvement is by using block structure in Cholesky-related computations. Previously, we use the property that every column \( L_{ij} \) is \( \tilde{O}(\tau_{\text{lp}}) \)-sparse, to achieve running time \( \tilde{O}(n_{\text{lp}}^2 \tau_{\text{lp}}^2) \) for computing the Cholesky factorization. In fact, \( L_{ij} \) has more structures. Because of our reduction to form (5), we can divide the indices of \( L_{ij} \) into \( \tilde{O}(n_{\text{sdp}}) \) blocks, where maximum block size is \( \tilde{O}(\tau_{\text{lp}}) \), and such that every block column of \( L_{ij} \) is \( \tilde{O}(1) \)-block sparse.

Using this block structure, we are able to devise an algorithm that computes the Cholesky factorization \( L_{ij} \) in \( \tilde{O}(n_{\text{sdp}} \tau_{\text{lp}}^2) = \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^{2.5}) \) time (Lemma 6.4). There are a few other computations in restart, which can all be improved to \( \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^{2.5}) \) time using this idea. Thus the time cost per restart becomes \( \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^{2.5}) \). We have

\[ \text{overall restart time} = \text{number of restarts} \cdot \text{cost per restart} \]
\[ = \tilde{O}(N/q) \cdot \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^{2.5}) \]
\[ = \tilde{O}(n_{\text{sdp}}^{1.5} w^{0.5} q^{-1} \tau_{\text{sdp}}^{2.5+0.5}). \]

Furthermore, we are able to achieve \( \tilde{O}(\tau_{\text{lp}}) = \tilde{O}(n_{\text{sdp}}^{2.5}) \) time for updating the Cholesky factorization \( L_{ij} \). There are a few other computations in update, which can all be improved to \( \tilde{O}(\tau_{\text{sdp}}^{2.5}) \) time per variable block update. Therefore update time for iteration becomes \( \tilde{O}(qw^{-1} \tau_{\text{sdp}}^{2.5}) \) on average. We have

\[ \text{overall update time} = \text{number of iters} \cdot \text{update cost per iter} \]
\[ = \tilde{O}(N) \cdot \tilde{O}(qw^{-1} n_{\text{sdp}}^{2.5}) \]
\[ = \tilde{O}(n_{\text{sdp}}^{0.5} w^{-0.5} q \tau_{\text{sdp}}^{2.5+0.5}). \]

Combining everything above, we have

\[ \text{overall running time} = \text{overall restart time} + \text{overall update time} \]
\[ = \tilde{O}(n_{\text{sdp}}^{1.5} w^{0.5} q^{-1} \tau_{\text{sdp}}^{2.5+0.5}) + \tilde{O}(n_{\text{sdp}}^{0.5} w^{-0.5} q \tau_{\text{sdp}}^{2.5+0.5}) \]
\[ = \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^{2.5+0.5}) \]

where in the last step we take \( w = \tau_{\text{sdp}}, q = n_{\text{sdp}}^{0.5} \).
2.5 Improve Running Time of Low-Treewidth LP

In this section, we discuss the technique we use to achieve a more efficient algorithm for LP with bounded treewidth.

Recall that the key idea of [DLY21] is, instead of maintaining the primal and slack variables on the (approximate) central path explicitly, we maintain a sparsely-changing representation (multiscale representation) of them. The central path maintenance algorithm has two main parts (1) restart (which contains initialize and output), and (2) update. In every iteration, if certain conditions are satisfied, we restart the data structure. Afterwards, we perform a central path move and maintain the relevant data structures ExactDS and ApproxDS. The final running time is achieved by balancing restart time and update time.

We improve restart time by devising improved algorithms for computations related to Cholesky factorization. One main step in initialization part is to compute the Cholesky factorization of the matrix $AH^{-1}A^\top = L_T L^\top_T$. Because of the treewidth assumption, the Cholesky factor $L_T$ is $\tilde{O}(\tau)$-column sparse, i.e., every column has at most $\tilde{O}(\tau)$ non-zero entries. Using this property, [DLY21] is able to compute the Cholesky factorization in $\tilde{O}(n\tau^2)$ time.

We can do better by utilization more properties of the matrix. The key observation is that we can divide the indices of $AH^{-1}A^\top$ into blocks of size $O(\tau)$, such that in the Cholesky factor is $\tilde{O}(1)$-block sparse, i.e., every block column has at most $\tilde{O}(1)$ non-zero block entries.

In this way, we can perform the Cholesky factorization algorithm on block level, achieving an algorithm (c.f. Lemma 6.4, 8.4) running in time

$$\#\text{blocks} \cdot (\#\text{column block sparsity})^2 \cdot \mathcal{T}\text{mat}(\text{block size})$$

$$= \tilde{O}(n/\tau) \cdot \tilde{O}(1) \cdot \mathcal{T}\text{mat}(\tau)$$

$$= \tilde{O}(n\tau^{\omega-1}).$$

Using similar ideas, we are able to improve other Cholesky-related computations, including Cholesky factor inverse (c.f. Lemma 8.7), product of Cholesky factor and a batch of vectors (c.f. Lemma 8.8), and so on. These are all bottleneck steps in the original [DLY21] algorithm. By combining all these improvements, we achieve improved running time for restart.

The overall running time is sum of restart time and update time. In the LP case, the number of iterations $N$ is $\sqrt{n}$. Let $q$ be the data structure restart threshold. Then overall restart time is given by

overall restart time $= \text{number of restarts} \cdot \text{cost per restart}$

$$= \tilde{O}(N/q) \cdot \tilde{O}(n\tau^{\omega-1})$$

$$= \tilde{O}(n^{1.5}q^{-1}\tau^{\omega-1}).$$

Overall update time is

overall update time $= \text{number of iters} \cdot \text{update cost per iter}$

$$= \tilde{O}(N) \cdot \tilde{O}(q\tau^2)$$

$$= \tilde{O}(Nq\tau^2).$$

Therefore

overall running time $= \text{overall restart time} + \text{overall update time}$
\[
\tilde{O}(n^{1.5} q^{-1} \tau^{\omega-1}) + \tilde{O}(n^{0.5} q \tau^2) = \tilde{O}(n^{\omega+1}/2)
\]

where the last step is by taking \( q = n^{0.5} \tau^{(\omega-3)/2} \).

**Acknowledgements**

The authors would like to thank Guanghao Ye and Lichen Zhang for useful discussions.
Roadmap.
Here, we provide an organization for the rest of the paper.

- In Section 3, we present a few basic definitions and results used in the paper.
- In Section 4, we present a general framework which covers both semidefinite programming and linear programming.
- In Section 5, we show how to apply our general framework to semidefinite programming solver to achieve an \(O(n \cdot \text{poly}(\tau))\) algorithm.
- In Section 6: we show how to further improve the running time of semidefinite programming solver to achieve an \(\tilde{O}(n^{\omega+0.5})\) time algorithm.
- In Section 7, we show how to solve a more general kind of low-treewidth SDP (which we call decomposable SDP) using our general framework.
- In Section 8, we show how to improve running time of low-treewidth linear programming to \(\tilde{O}(n^{\omega+1}/2)\).

3 Preliminaries
In Section 3.1, we define some basic notations. In Section 3.2, we define self concordant barrier. In Section 3.3, we define treewidth. In Section 3.4, we introduce some backgrounds for sketching matrices.

3.1 Basic Notions
For any positive integer \(n\), we use \([n]\) to denote the set \(\{1, 2, \cdots, n\}\). For a matrix \(A\), we use \(A^\top\) to denote the transpose of \(A\). For two matrices \(A\) and \(B\), we use \(A \bullet B\) and \((A, B)\) to denote the inner product between two matrices, i.e., \(\sum_{i,j} A_{i,j} B_{i,j}\). For a square matrix \(A\), we use \(\det(A)\) to denote the determinant of \(A\). We use \(\text{tr}(A)\) to denote the trace of matrix \(A\).

For two functions \(f, g\), we use the shorthand \(f \lesssim g\) (resp. \(\gtrsim\)) to indicate that \(f \leq C \cdot g\) (resp. \(\geq\)) for an absolute constant \(C\). We use \(f \approx g\) to mean \(cf \leq g \leq Cf\) for constants \(c, C\).

We use \(\omega\) to denote the exponent of matrix multiplication, i.e., multiplying an \(n \times n\) matrix with another \(n \times n\) matrix takes \(n^\omega\) time. Currently, \(\omega \approx 2.373\) [Wil12, AW21].

We will deal with block matrices and vectors a lot. Therefore we use a block-friendly notation. For a fixed block pattern \(N = \sum_{i \in [n]} n_i\), for \(x \in \mathbb{R}^N\), we denote \(x = (x_1, \ldots, x_n)\), where \(x_i \in \mathbb{R}^{n_i}\). Therefore \(x_i\) naturally denotes one block of \(x\). When we need to refer to coordinates of \(x\), we use \(x_{\text{coord } i}\).

**Definition 3.1 (Mixed Norm).** Let \(x \in \mathbb{R}^N\) be a vector with fixed block pattern \(N = \sum_{i \in [n]} n_i\). We define its \((p, q)\)-norm \(\|x\|_{p,q}\) as the \(q\)-norm of the vector \((\|x_i\|_p)_{i \in [N]}\).

\[
\|x\|_{p,q} := \|y\|_q, \text{ where } y = (\|x_1\|_p, \cdots, \|x_n\|_p) \in \mathbb{R}^n
\]

In particular \(\|x\|_{0,1} = \|x\|_0, \|x\|_{2,2} = \|x\|_2\).
3.2 Self-Concordant Barrier
We start with defining self concordant barrier,

**Definition 3.2** (Self-Concordant Barrier, [Nes03]). A function $\phi$ is a $\nu$-self-concordant barrier for a non-empty open convex set $K$ if $\text{dom}(\phi) = K$, $\phi(x) \to +\infty$ as $x \to \partial K$, and for any $x \in K$ and for any $u \in \mathbb{R}^n$

$$D^3\phi(x)[u, u, u] = 2\|u\|\nabla^2\phi(x) \quad \text{and} \quad \|\nabla\phi(x)\|_{(\nabla^2\phi(x))^{-1}} \leq \sqrt{\nu}. $$

A function $\phi$ is a self-concordant barrier if the first condition holds.

**Definition 3.3** (Log-Barrier for PSD Cone). For the PSD cone $K = S_n := \{X \in \mathbb{R}^{n \times n} : X \succeq 0\}$, we define the log barrier as $\phi(X) := -\log(\det(X))$.

**Lemma 3.4** ([NN94, HJS+22]). Log-barrier for the $n \times n$ PSD cone is $n$-self-concordant.

3.3 Treewidth

**Definition 3.5** (Tree decomposition and treewidth). Let $G = (V, E)$ be a graph, a tree decomposition of $G$ is a tree $T$ with $b$ vertices, and $b$ sets $J_1, \ldots, J_b \subseteq V$ (called bags), satisfying the following properties:

- For every edge $(u, v) \in E$, there exists $j \in [b]$ such that $u, v \in J_j$;
- For every vertex $v \in V$, $\{j \in [b] : v \in J_j\}$ is a non-empty subtree of $T$.

The treewidth of $G$ is defined as the minimum value of $\max\{|J_j| : j \in [b]\} - 1$ over all tree decompositions.

3.4 Sketching Matrices

**Lemma 3.6** (Johnson–Lindenstrauss transform, [JL84]). Let $\epsilon \in (0, 1)$ be the precision parameter. Let $\delta \in (0, 1)$ be the failure probability. Let $A \in \mathbb{R}^{m \times n}$ be a real matrix. Let $r = \epsilon^{-2}\log(mn/\delta)$. For $R \in \mathbb{R}^{r \times n}$ whose entries are i.i.d $\mathcal{N}(0, \frac{1}{r})$, the following holds with probability at least $1 - \delta$:

$$(1 - \epsilon)\|a_i\|_2 \leq \|Ra_i\|_2 \leq (1 + \epsilon)\|a_i\|_2, \forall i \in [m],$$

where for a matrix $A$, $a_i^\top$ denotes the $i$-th row of matrix $A \in \mathbb{R}^{m \times n}$.

3.5 Sparse Cholesky Decomposition

In this section we state a few basic results on sparse Cholesky decomposition. The following definition essentially come from [Sch82].

**Definition 3.7** (Elimination tree). Let $G$ be an undirected graph on $n$ vertices. An elimination tree $T$ is a rooted tree on $V(G)$ together with an ordering $\pi$ of $V(G)$ such that for any vertex $v$, its parent is the smallest (under $\pi$) element $u$ such that there exists a path $P$ from $v$ to $u$, such that $\pi(w) \leq \pi(v)$ for all $w \in P - u$.

**Lemma 3.8** ([Sch82]). Let $M$ be a PSD matrix and $T$ be an elimination tree of the adjacency graph of $M$ (i.e., $(i, j) \in E(G)$ iff $M_{i,j} \neq 0$) together with an elimination ordering $\pi$. Let $P$ be the permutation matrix $P_{i,v} = \{v = \pi(i)\}$. Then the Cholesky factor $L$ of $PMP^\top$ (i.e., $PMP^\top = LL^\top$) satisfies $L_{i,j} \neq 0$ only if $\pi(i)$ is an ancestor of $\pi(j)$. 
The following lemma is useful in proving elimination trees.

**Lemma 3.9.** Let $G$ be an undirected graph on $n$ vertices. Let $T$ be a rooted tree with $n$ vertices, satisfying the property that: for any path $u = v_1, \ldots, v_k = v$ with $(v_i, v_{i+1}) \in E(G)$ for all $i \in [k-1]$, then there exists $i \in [k]$ such that $v_i$ is an ancestor of both $u$ and $v$. Then $T$ together with any post-order traversal of $T$ is an elimination tree.

**Proof.** We prove that $T$ satisfies Definition 3.7. Let $v \in V(G)$ and $P$ be any path from $v$ to a vertex outside the subtree rooted at $v$. By assumption, there exists a vertex $u \in P$ which is an ancestor of $v$. Let $w$ be the parent of $v$. Then $\pi(u) \geq \pi(w) > \pi(v)$. Therefore $w$ is the smallest element reachable from $v$ using only elements before $v$. \qed

When any post-order traversal of $T$ works, we omit the choice of $\pi$ and say $T$ is an elimination tree.

**Lemma 3.10 ([GLN94]).** Under the setting of Lemma 3.8, if depth of $T$ is at most $\tau$, then the Cholesky decomposition $L$ can be computed in $O(n\tau^2)$ time.

**Lemma 3.11 ([DH99]).** Under the setting of Lemma 3.10, suppose we are already given the Cholesky decomposition $L$. Let $w \in \mathbb{R}^n$ be a vector such that $M + ww^\top$ has the same adjacency graph as $M$. Then we can compute $\Delta_L \in \mathbb{R}^{n \times n}$ such that $L + \Delta_L$ is the Cholesky factor of $M + ww^\top$ in $O(\tau^2)$ time.
4 General Treewidth Program Solver Framework

In this section we establish a general framework for solving treewidth LP/SDP and related problems. Our framework takes block size into consideration and do not assume various block size-related parameters are constant. Our framework takes in certain subroutine running time as parameters. This allows us to see which subroutines are running time bottlenecks. Our SDP and LP results follow from the general framework with minimal problem-specific results in addition.

We briefly describe the outline of this section.

- In Section 4.1, we present the definitions and backgrounds for this Section 4.
- In Section 4.2, we present the main data structure which is central path maintenance data structure.
- In Section 4.3, we present several computation-related lemmas which will be heavily used in our tasks.
- In Section 4.4, we present several data structures that are being used in central path maintenance, including ExactDS (Section 4.4.1), APPROXDS (Section 4.4.2), BATCHSKETCH (Section 4.4.3) and BLOCKVECTORSKETCH(Section 4.4.4).
- In Section 4.5, we prove correctness and running time of the central path maintenance data structure.
- In Section 4.6, we prove the main result (Theorem 4.3).

4.1 Main Statement

We consider programs of the following form.

\[
\begin{align*}
\min & \quad c^\top x \\
\text{s.t.} & \quad A_i x = b_i \quad \forall i \in [m] \\
& \quad x_i \in K_i \quad \forall i \in [n]
\end{align*}
\]  

(9)

where

- \( K_i \subset \mathbb{R}^{n_i} \) is a convex set.
- \( x = (x_1, \ldots, x_n) \in \mathbb{R}^{n_{lp}} \), where \( n_{lp} := \sum_{i \in [n]} n_i \).
- \( c = (c_1, \ldots, c_n) \in \mathbb{R}^{n_{lp}} \), where \( c_i \in \mathbb{R}^{n_i} \).
- \( A \in \mathbb{R}^{m_{lp} \times n_{lp}}, \quad b \in \mathbb{R}^{m_{lp}} \).

**Definition 4.1** (Treewidth of a general linear program). *Treewidth \( \tau_{lp} \) of Program (9) is defined as the treewidth (Definition 3.5) of its the LP dual graph, where the LP dual graph is defined as follows:

The vertex set is \([m_{lp}]\). There is an edge between \( i, j \in [m_{lp}] \) if and only if there exists \( k \in [n] \) such that \( A_{i,k} \neq 0, A_{j,k} \neq 0 \).

**Definition 4.2** (Parameters for General Treewidth Program). *We make the following assumptions on Program (9) and define relevant parameters. These parameters will have an impact to the final running time (Theorem 4.3).*
• Assume that for each $K_i$, we have a $\nu_i$-self-concordant barrier function $\phi_i$. Let $\nu_{\text{max}} := \max_{i \in [n]} \nu_i$.

• Assume that $\phi_i, \nabla \phi_i$ and $\nabla^2 \phi_i$ can be computed in $T_{H,i}$ time. Define $T_{H,\text{max}} := \max_{i \in [n]} T_{H,i}$ and $T_H := \sum_{i \in [n]} T_{H,i}$.

• Assume that we are given a tree decomposition of the LP dual graph, such that every bag has total size at most $\tau_{\text{lp}}$.

• Assume that we are given a block elimination tree (Definition 4.5) with $m$ vertices and maximum block-depth $\eta$. The block elimination tree partitions the constraints into $m$ blocks, each of size $m_i$ ($i \in [m]$), with $\sum_{i \in [m]} m_i = m_{\text{lp}}$.

• Assume that it takes $T_n$ time to perform matrix multiplication of two block diagonal matrices $\in \mathbb{R}^{n_{\text{lp}} \times n_{\text{lp}}}$ with block signature $(n_1, \ldots, n_n)$. We have $T_n = \sum_{i \in [n]} T_{\text{mat}}(n_i)$.

• Assume that it takes $T_m$ time to perform matrix multiplication of two block diagonal matrices $\in \mathbb{R}^{m_{\text{lp}} \times m_{\text{lp}}}$ with block signature $(m_1, \ldots, m_m)$. We have $T_m = \sum_{i \in [m]} T_{\text{mat}}(m_i)$.

• Assume that it takes $T_L$ time to compute Cholesky decomposition $AHA^{\top} = LL^{\top}$.

• Assume that it takes $T_{\Delta,\text{max}}$ time to update Cholesky decomposition, i.e., given $AHA^{\top} = LL^{\top}$ and $\Delta_H$ supported on a single diagonal block, computing $\Delta_L$ such that $A(H + \Delta_H)A^{\top} = (L + \Delta_L)(L + \Delta_L)^{\top}$.

• Assume that it takes $T_Z$ time to compute $L^{-1}v_i$ for all $i \in [m]$, where $v_i$ is supported on $\bigcup_{j \in P(i)} B_j$, where $P(i)$ is the set of ancestors of vertex $i$. This is used in Lemma 4.24.

• Assume that $R$ is the diameter of $K_i$. Assume that we are given an initial point $x$ such that $B(x, r) \subseteq K$. Assume that there exists $x$ such that $Ax = b$ and $B(x, r) \subseteq K$.

We are ready to state our main theorem.

**Theorem 4.3 (General Treewidth Program Solver).** Under assumptions in Definition 4.2, given any $0 < \epsilon \leq \frac{1}{2}$, we can find $x \in K$ with $Ax = b$ such that

$$c^{\top} x \leq \min_{Ax = b, x \in K} c^{\top} x + \epsilon \|c\|_2 R$$

in expected time

$$\tilde{O}(n^{0.5} \nu_{\text{max}}^{0.5} \cdot (T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\text{max}})^{0.5} \cdot (T_{\text{mat}}(n_{\text{max}}) + T_{\Delta,\text{max}} + T_{H,\text{max}} + \eta^2 m_{\text{max}}^2)^{0.5} \log(R/(r\epsilon)))$$

where $\tilde{O}$ hides polylog($n_{\text{lp}}$) terms.

We defer the proofs into Section 4.6.
### Table 2: Parameters needed to call Theorem 4.3. Their values are summarized in Table 3

| Notation | Meaning |
|----------|---------|
| \( n_{lp} \) | Total variable dimension |
| \( n \) | Number of variable blocks |
| \( n_i \) | Dimension of \( i \)-th variable block |
| \( n_{\text{max}} \) | Max dimension of a block |
| \( m_{lp} \) | Number of constraints |
| \( m \) | Number of constraint blocks |
| \( m_i \) | Size of \( i \)-th constraint block |
| \( m_{\text{max}} \) | Max size of constraint block |
| \( \tau_{lp} \) | Max size of a bag |
| \( \eta \) | Block height of elimination tree |
| \( \nu_i \) | Self-concordance of \( i \)-th block |
| \( \nu_{\text{max}} \) | Max self-concordance of a block |
| \( \text{nnz}(A) \) | Number of non-zero entries of \( A \) |
| \( T_n \) | Matrix mult time for \( n_{lp} \times n_{lp} \) block-diag matrix |
| \( T_m \) | Matrix mult time for \( m_{lp} \times m_{lp} \) block-diag matrix |
| \( T_H \) | Time to compute Hessian |
| \( T_{H,\text{max}} \) | Time to compute Hessian of one block |
| \( T_L \) | Time to compute Cholesky |
| \( T_{\Delta L,\text{max}} \) | Time to update Cholesky |

#### 4.2 Algorithm structure and CentralPathMaintenance

Our algorithm is a robust Interior Point Method (robust IPM). Details of the robust IPM will be given in Section A.

In robust IPM, we maintain the primal-dual solution pair \((x, s) \in \mathbb{R}^{n_{lp}} \times \mathbb{R}^{n_{lp}}\) on the robust central path. In addition, we maintain approximations \((\overline{x}, \overline{s}) \in \mathbb{R}^{n_{lp}} \times \mathbb{R}^{n_{lp}}\) which change sparsely. In each iteration, we implicitly perform update

\[
x \leftarrow x + (H_x^{-1} - H_{\overline{x}}^{-1}) A^\top (A H_{\overline{x}}^{-1} A^\top)^{-1} A H_{\overline{x}}^{-1} \delta_\mu(\overline{x}, \overline{s}, \overline{t}),
\]

\[
s \leftarrow s + t A^\top (A H_{\overline{x}}^{-1} A^\top)^{-1} A H_{\overline{x}}^{-1} \delta_\mu(\overline{x}, \overline{s}, \overline{t})
\]

and explicitly maintain \((\overline{x}, \overline{s})\) such that they remain close to \((x, s)\).

This task is handled by the CentralPathMaintenance data structure, which is the main data structure. The robust IPM algorithm (Algorithm 19,20) directly calls it in every iteration. This data structure is a generalization of CentralPathMaintenance data structure in [DLY21].

The CentralPathMaintenance data structure (Algorithm 2) has two main sub data structures, ExactDS (Algorithm 4 and Algorithm 5) and ApproxDS. ExactDS is used to to maintain \((x, s)\), and APPROXDS is used to monitor changes in \((x, s)\), and update \((\overline{x}, \overline{s})\) when necessary.

**Theorem 4.4.** Data structure CentralPathMaintenance (Algorithm 2) implicitly maintains the central path primal-dual solution pair \((x, s) \in \mathbb{R}^{n_{lp}} \times \mathbb{R}^{n_{lp}}\) and explicitly maintains its approximation \((\overline{x}, \overline{s}) \in \mathbb{R}^{n_{lp}} \times \mathbb{R}^{n_{lp}}\) using the following functions:

- **Initialize** \((x \in \mathbb{R}^{n_{lp}}, s \in \mathbb{R}^{n_{lp}}, t_0, q)\): (See Lemma 4.37) Initializes the data structure with initial primal-dual solution pair \((x, s) \in \mathbb{R}^{n_{lp}} \times \mathbb{R}^{n_{lp}}\), initial central path timestamp \(t_0\), and a
run-time tuning parameter $q$ in
\[ \tilde{O}(T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{lp} m_{max}) \]
time.

- **MultiplyAndMove($t$):** (See Lemma 4.38) It implicitly maintains
\[
\begin{align*}
x &\leftarrow x + H_\pi^{-1/2}(I - P_\pi)H_\pi^{-1/2} \delta_\mu(\pi, \bar{s}, \bar{t}) \\
s &\leftarrow s + t \cdot \Phi^{1/2} P_\pi H_\pi^{-1/2} \delta_\mu(\pi, \bar{s}, \bar{t})
\end{align*}
\]
where \( H_\pi := \nabla^2 \phi(\pi) \in \mathbb{R}^{m_{lp} \times m_{lp}}, P_\pi := H_\pi^{-1/2} A^\top (AH_\pi^{-1} A^\top)^{-1} A H_\pi^{-1/2} \in \mathbb{R}^{m_{lp} \times m_{lp}}, \) and \( \bar{t} \) is some earlier timestamp satisfying \( |t - \bar{t}| \leq c_1 \cdot \bar{t}. \)

It also explicitly maintains \((\pi, \bar{s}) \in \mathbb{R}^{m_{lp} \times m_{lp}}\) such that \( \|\pi_i - x_i\|_{\pi_i} \leq \pi \) and \( \|\pi_i - s_i\|_{\pi_i} \leq t \pi w_i \) for all \( i \in [n] \) with probability at least 0.9.

Assuming the function is called at most \( N \) times and \( t \) is monotonically decreasing from \( t_{max} \) to \( t_{min} \), the total running time is
\[
\tilde{O}(n^{0.5} \nu_{\max}^{0.5} \cdot (T_H + T_n + T_L + T_Z + \text{nnz}(A)) + \eta m_{lp} m_{max})^{0.5} \\
\cdot (T_{\text{mat}}(n_{\max}) + T_{\Delta L,\text{max}} + T_{H,\text{max}} + \eta^2 m_{\max}^2)^{0.5} \log(t_{max}/t_{min})).
\]
• OUTPUT: (Lemma 4.39) It computes \((x, s) \in \mathbb{R}^{np \times np}\) exactly and outputs them in
\[
O(mnz(A) + T_H + \eta T_m)
\]

time.

Correctness and running time analysis of CentralPathMaintenance is deferred to Section 4.5 and Section 4.6, after we establish properties of the sub data structures.

Algorithm 2 Central Path Maintenance. This is used in Algorithm 20.

1: data structure CentralPathMaintenance ⤷ Theorem 4.4
2: private : member
3: ExactDS exact ⤷ Algorithm 4, 5
4: APPROXDS approx ⤷ Algorithm 6
5: \(\ell \in \mathbb{N}\)
6: end members
7: procedure INITIALIZE\((x, s \in \mathbb{R}^{np}, t \in \mathbb{R}_+, \bar{\tau} \in (0, 1))\) ⤷ Lemma 4.37
8: exact.Initialize\((x, s, x, s, t)\) ⤷ Algorithm 4
9: \(\ell \leftarrow 0\)
10: \(w \leftarrow \nu_{\text{max}}, N \leftarrow \sqrt{n\nu_{\text{max}}w}\)
11: \(q \leftarrow \nu_{\text{max}}(T_H + T_L + \eta T_m + T_Z)^{0.5}(t_{\text{mat}}(n_{\text{max}}) + T_{\Delta L, \text{max}} + T_{H, \text{max}} + \eta^2 n_{\text{max}}^2)^{-0.5}\)
12: \(\epsilon_{\text{approx}, x} \leftarrow \tau, \zeta_x \leftarrow 2\alpha w^{-1/2}, \epsilon_{\text{approx}} \leftarrow \frac{1}{N}\)
13: \(\epsilon_{\text{approx}, s} \leftarrow \tau, \cdot w, \zeta_s \leftarrow 2\alpha w^{1/2}\)
14: approx.Initialize\((x, s, h, \epsilon_x, \epsilon_s, H_{\text{approx}}^{-1/2}, \bar{s}, \beta_x, \beta_s, q, &\text{exact}, \epsilon_{\text{approx}, x}, \epsilon_{\text{approx}, s}, \zeta_x, \zeta_s, \delta_{\text{approx}})\) ⤷ Parameters from \(x\) to \(\beta_s\) come from exact. &exact is pointer to exact
15: end procedure
16: procedure MULTIPLYANDMOVE\((t \in \mathbb{R}_+)\) ⤷ Lemma 4.38, 4.36
17: \(\ell \leftarrow \ell + 1\)
18: if \(|\bar{\ell} - t| > \bar{\ell} \cdot \epsilon_{\ell} \text{ or } \ell > q\) then
19: \(x, s \leftarrow \text{exact.OUTPUT}()\) ⤷ Algorithm 4
20: end if
21: INITIALIZE\((x, s, t, \bar{\tau})\) ⤷ Algorithm 4
22: end procedure
23: procedure OUTPUT() ⤷ Lemma 4.39
24: return exact.OUTPUT() ⤷ Algorithm 4
25: end procedure
26: end data structure

4.3 Block Elimination Tree and Computation-Related Lemmas

Our algorithm is based on efficient computations involving Cholesky factorization and the block elimination tree. In this section we introduce the block elimination tree and prove a few useful lemmas on running time of relevant computations. They will be used repeatedly in the running time analysis of our algorithm.
Definition 4.5 (Block elimination tree). Let $G$ be a graph with $m_{lp}$ vertices. A block elimination tree is a rooted tree with $m$ vertices, satisfying the following properties.

- The vertices correspond to a partition $[m_{lp}] = B_1 \cup \cdots \cup B_m$, where $|B_i| = m_i$.
- We can compute a permutation $P$ such that for any PSD matrix $M \in \mathbb{S}^{m_{lp}}$ supported on $G$ (i.e., $M_{ij} \neq 0$ only if $i = j$ or $(i, j) \in E(G)$), in the Cholesky factorization $PMP^\top = LL^\top$, the non-zero entries of $L_{E_i}$ are subsets of $\bigcup_{j \in P(i)} B_j$, where $P(i)$ is the set of ancestors of vertex $i$. In other words, the (block) non-zero pattern of the $i$-th (block) column is the set of vertices on the path from $i$ to the root.

Under the setting of Program 9, we always take $G$ to be the LP dual graph, and $M$ to be $AH_{\tau}^{-1}A^\top$.

The following lemma is the block version of Lemma 3.9.

Lemma 4.6. Let $G$ be an undirected graph on $m_{lp}$ vertices. Let $T$ be a tree whose vertices correspond to a partition of $[m_{lp}]$. Suppose $T$ satisfying the following property: for any path $u = v_1, \ldots, v_k = v$ such that for all $i \in [k - 1]$, there exists an edge connecting some pair of elements in $v_i$ and $v_{i+1}$ (i.e., there exists $a \in B_i$, $b \in B_j$ such that $(a, b) \in E(G)$), there exists $i \in [k]$ such that $v_i$ is an ancestor of both $u$ and $v$. Then $T$ is a valid block elimination tree of $G$.

Proof. We define another tree $T'$ with block pattern $(1, \ldots, 1)$. For every vertex $v$ in $T$, we replace it with a path (of an arbitrary ordering of elements in $v$). For all edges in $T$, we connect top element of the child and the bottom element of the parent.

In this way, $T'$ satisfies the property that for any path $u = v_1, \ldots, v_k = v$ for all $i \in [m_{lp}-1]$, then there exists $i \in [k]$ such that $v_i$ is an ancestor of both $u$ and $v$. By Lemma 3.9, $T'$ is a valid elimination tree. This implies that $T$ is a valid block elimination tree.

As in Definition 4.2, we assume that we are given a block elimination tree with (block) depth $\eta$. We can without loss of generality assume that the blocks are labeled in postorder, i.e., for any $i$ and $j \in P(i)$, we have $i < j$.

Given a block elimination tree, we can efficiently perform many computations related to the Cholesky decomposition, as shown in the following lemma.

Lemma 4.7. Assume that we are given a block elimination tree with block structure $(m_1, \ldots, m_m)$ and block depth $\eta$. Assume that we are given the Cholesky factorization $AH^\top A = LL^\top$ together with inverses of the diagonal blocks of $L$, i.e., $L_{i,i}^{-1}$ for all $i \in [m]$.

Then we have the following running time for matrix-vector multiplications.

(i) For $v \in \mathbb{R}^{m_{lp}}$, computing $Lv$, $L^\top v$, $L^{-1}v$, $L^{-\top}v$ takes $O(\eta m_{lp} m_{max})$ time.

(ii) For $v \in \mathbb{R}^{m_{lp}}$, computing $Lv$ takes $O(\|v\|_{2,0} \eta m_{max}^2)$ time.

(iii) For $v \in \mathbb{R}^{m_{lp}}$, computing $L^{-1}v$ takes $O(\|L^{-1}v\|_{2,0} \eta m_{max}^2)$ time.

(iv) For $v \in \mathbb{R}^{m_{lp}}$, if $v$ is supported on a path in the block elimination tree, then computing $L^{-1}v$ takes $O(\|v\|_{2,0} \eta m_{max}^2)$ time.

(v) For $v \in \mathbb{R}^{m_{lp}}$, given $H$, computing $H^{-1/2}v$, $H^{-1}v$, $H^{1/2}v$, $Hv$ takes $O(T_n)$ time.

(vi) For $v \in \mathbb{R}^{m_{lp}}$, computing $Av$ takes $O(\eta n z(A))$ time.

(vii) For $v \in \mathbb{R}^{m_{lp}}$, computing $A^\top v$ takes $O(\eta n z(A))$ time.
(vii) For $v \in \mathbb{R}^{mlp}$, computing $W^\top v$ takes $O(T_n + \eta m_{lp} m_{\max} + \text{nnz}(A))$ time, where $W = L^{-1}AH^{-1/2}$.

Proof. (i) Because every (block) column of $L$ is $\eta$-block sparse, every (coordinate) column of $L$ is $\eta m_{\max}$-coordinate sparse. So computing $Lv$ takes $\text{nnz}(L) \leq \eta m_{lp} m_{\max}$ time.

For $L^{-1}v$, let us consider Algorithm 3. Recall that we assume $L^{-1}_{i,i}$ are given. So the algorithm takes $\text{nnz}(L) + \sum_{i \in [m]} m_i^2 = \eta m_{lp} m_{\max}$ time.

Algorithm 3 Solving block triangular systems $Lx = v$

1: procedure SolveBlock()
2:     $x \leftarrow 0 \in \mathbb{R}^{mlp}$
3:     for increasing $j$ with $v_j \neq 0$
4:         $x_j \leftarrow L^{-1}_{j,j} v_j$
5:         $v \leftarrow v - L_{s,j} x_j$
6:     end for
7: end procedure

Statements about $L^\top v$ and $L^{-\top} v$ follow from the Transposition principle [Bor57].

(ii) If $v$ is supported on a single block $i \in [m]$, then computing $Lv$ takes $\sum_{j \in P(i)} m_j^2 = O(\eta m_{\max}^2)$ time by block column sparsity of $L$. So computing $Lv$ for general $v$ takes $O(\|v\|_2 \eta m_{\max}^2)$ time.

(iii) Consider Algorithm 3. By block column sparsity pattern of $L$, for every block $j \in [m]$ with $x_j \neq 0$, it takes $O(\eta m_{\max}^2)$ time to compute $x_j$ and update $v \leftarrow v - L_{s,j} x_j$. So overall running time is $O(\|L^{-1}v\|_2 \eta m_{\max}^2)$.

(iv) Corollary of (iii).

(v) It takes $\sum_{i \in [n]} T_{\text{mat}}(n_i) = T_n$ time to compute an eigendecomposition of $H$. Afterwards, it takes $\sum_{i \in [n]} T_{\text{mat}}(n_i) = O(T_n)$ time to compute the product.

(vi) It is obvious to see that computing $(Av)_i$ takes $\text{nnz}(A_{i,*})$ time. Summation over all the indices $i$, we know the running time is $\text{nnz}(A)$.

(vii) It is easy to see that computing $(A^\top v)_j$ takes $\text{nnz}(A_{*,j})$ time. Summation over all the indices $j$, we know the running time is $\text{nnz}(A)$.

(viii) $W^\top v = H^{-1/2} A^\top L^{-\top} v$. So the result follows from combining (i)(vii)(v).

Lemma 4.8. Assume that we are given a block elimination tree with block structure $(m_1, \ldots, m_m)$ and block depth $\eta$. Assume that we are given the Cholesky factorization $AHA^\top = LL^\top$ together with inverses of the diagonal blocks of $L$, i.e., $L_{i,i}^{-1}$ for all $i \in [m]$.

Then we have the following running time for matrix-vector multiplications, when we only need result for a subset of coordinates.

(i) Let $S$ be a path in the block elimination tree whose one endpoint is the root. For $v \in \mathbb{R}^{mlp}$, computing $(L^{-\top} v)_S$ takes $O(\eta^2 m_{\max}^2)$ time.
(ii) For $v \in \mathbb{R}^{m_{lp}}$, for $i \in [n]$, computing $(W^\top v)_i$ takes $O(n_{max}^2 + \eta^2 m_{max}^2)$ time, where $W = L^{-1}AH^{-1/2}$.

Proof. (i) We have $(L^{-\top} v)_S = e_i^\top L^{-\top} v = (v^\top L^{-1} e_S)^\top$. By column sparsity pattern of $L$, $L^{-1} e_S$ is supported on $S$. So $(L^{-\top} v)_S$ depends only on $v_S$. So we only need to compute $L^{-\top} S, S v_S$, which takes $O(\eta^2 m_{max}^2)$ time by Lemma 4.7(i).

(ii) By definition of the block elimination tree, for $j, k \in [m_{lp}]$, if there is an edge $(j, k)$ in the LP dual graph, then either $j$ and $k$ are in the same block, or they are in two different blocks, one is an ancestor of the other. Therefore, all constraints containing $i$ lie on a single path $S$. So $(W^\top v)_i = H_{i,i}^{-1/2} A_{i,i}^\top L^{-\top} v = H_{i,i}^{-1/2} A_{S,i}^\top (L^{-\top} v)_S$. So we can first compute $(L^{-\top} v)_S$, which takes $O(\eta^2 m_{max}^2)$ time by (i). Afterwards, it takes $O(n_{max}^2 + n_{max} \eta m_{max}) = O(n_{max}^2 + \eta^2 m_{max}^2)$ time to compute $H_{i,i} A_{S,i}^\top (L^{-\top} v)_S$.

We state a generic bound on $T_Z$ (recall Definition 7.1).

**Lemma 4.9.** $T_Z = O(\eta^2 mn_{max}^2)$.

Proof. Computing each $L^{-1} v_i$ takes $O(\eta^2 m_{max}^2)$ time by Lemma 4.7(iv). So computing $m$ of them takes $O(\eta^2 mn_{max}^2)$ time.

### 4.4 Data structures being used in CentralPathMaintenance

In Section 4.4, we present several data structures that are being used in central path maintenance, including:

- **ExactDS** (Section 4.4.1). This data structure implicitly maintains the primal-dual solution pair $(x, s)$. This data structure is directly used by CentralPathMaintenance.

- **ApproxDS** (Section 4.4.2). This data structure explicitly maintains the approximate primal-dual solution pair $(\overline{x}, \overline{s})$. This data structure is directly used by CentralPathMaintenance.

- **BatchSketch** (Section 4.4.3). This data structure maintains a sketch of $x$ and $s$, using BlockVectorSketch and BlockBalancedSketch. This data structure is used by ApproxDS.

- **BlockVectorSketch** (Section 4.4.4). This data structure maintains a sketch of a vector (with block pattern) under single-point updates. This data structure is used by BatchSketch.

- **BlockBalancedSketch** (Section 4.4.5). This data structure maintains a sketch of a vector of form $W^\top v$, under updates of $(\overline{x}, \overline{s})$. This data structure is used by BatchSketch.

#### 4.4.1 ExactDS

In this section, we present our ExactDS (Algorithm 4 and Algorithm 5). In Theorem 4.11, we provide our theoretical statement for Algorithm 4 and Algorithm 5. This is a block-based generalization of MultiscaleRepresentation data structure of [DLY21].
Definition 4.10 (Multiscale coefficients, Definition 6.4 in [DLY21]). At any step the of the robust central path with approximate primal-dual solution pair \((\overline{\tau}, \overline{s}) \in \mathbb{R}^{m_{ip}} \times \mathbb{R}^{m_{ip}}, \) we define
\[
\mathcal{W} := L_{\overline{\tau}}^{-1}AH_{\overline{\tau}}^{-1/2}, \quad h := L_{\overline{\tau}}^{-1}AH_{\overline{\tau}}^{-1} \cdot \delta_{\mu}(\overline{\tau}, \overline{s}, \overline{t})
\]
where \(H_{\overline{\tau}} = \nabla^2 \phi(\overline{\tau}) \in \mathbb{R}^{m_{ip} \times m_{ip}},\) and \(L_{\overline{\tau}} \in \mathbb{R}^{m_{ip} \times m_{ip}}\) is the lower Cholesky factor of \(AH_{\overline{\tau}}^{-1}A^\top \in \mathbb{R}^{m_{ip} \times m_{ip}}.\)

We can write the central path update using multiscale representation
\[
x \leftarrow x + H_{\overline{\tau}}^{-1/2} \beta_x c_x - H_{\overline{\tau}}^{-1/2} \mathcal{W}^\top (\beta_x h + \epsilon_x) \\
s \leftarrow s + t \cdot H_{\overline{\tau}}^{-1/2} \mathcal{W}^\top (\beta_s h + \epsilon_s)
\]

by maintaining the variables \(\overline{\tau}, \overline{s} \in \mathbb{R}^{m_{ip}},\) \(H_{\overline{\tau}} \in \mathbb{R}^{m_{ip} \times m_{ip}},\) \(L_{\overline{\tau}} \in \mathbb{R}^{m_{ip} \times m_{ip}},\) \(\hat{x}, \hat{s}, c_x \in \mathbb{R}^{m_{ip}},\) \(\epsilon_x, \epsilon_s, h \in \mathbb{R}^{m_{ip}},\) \(\beta_x, \beta_s \in \mathbb{R},\) \(\overline{\tau}, \overline{s} \in \mathbb{R}^n,\) \(\delta_{\mu} \in \mathbb{R}^n.\)

The data structure supports the following functions:

- **Initialize** \((x \in \mathbb{R}^{m_{ip}}, s \in \mathbb{R}^{m_{ip}}, \overline{\tau} \in \mathbb{R}^{m_{ip}}, \overline{s} \in \mathbb{R}^{m_{ip}}, \overline{t}):\) Initializes the data structure in \(O(T_H + T_L + \text{nnz}(A) + T_n + \eta m_{ip} m_{max})\) time, with initial value of the primal-dual pair \((x, s),\) its initial approximation \((\overline{\tau}, \overline{s}),\) and initial approximate timestamp \(\overline{t}.\)

- **Move()**: Moves \((x \in \mathbb{R}^{m_{ip}}, s \in \mathbb{R}^{m_{ip}})\) in \(O(1)\) time by updating its implicit representation.

- **Update** \((\delta_{\tau} \in \mathbb{R}^{m_{ip}}, \delta_{s} \in \mathbb{R}^{m_{ip}}):\) Updates the approximation pair \((\overline{\tau}, \overline{s})\) to \((\overline{\tau}_{\text{new}} = \overline{\tau} + \delta_{\tau} \in \mathbb{R}^{m_{ip}}, \overline{s}_{\text{new}} = \overline{s} + \delta_{s} \in \mathbb{R}^{m_{ip}})\) in \(O(T_{\text{mat}}(n_{\max}) + T_{\Delta_{\text{r}}, \text{max}} + T_{H, \text{max}} + \eta^2 m_{\text{max}}^2 ||\delta_{\tau}||_2, \eta^2 m_{\text{max}}^2 ||\delta_{s}||_2)\) time, and output the changes in variables \(\delta_h, \delta_{\epsilon_x}, \delta_{\epsilon_s}, \delta_{H_{\tau}^{-1/2} x}, \delta_{H_{\tau}^{-1/2} s}, \delta_{H_{\tau}^{-1/2} c_x}.\)

- **Output** : Output \(x\) and \(s\) in \(O(T_H + \text{nnz}(A) + \eta m_{ip} m_{\max})\) time.

- **Query** \((i \in [n]):\) Output \(x_i\) in \(O(n_{\max}^2 + \eta^2 m_{\max}^2)\) time. This function is used by APPROXDS.

- **Queries** \((i \in [n]):\) Output \(s_i\) in \(O(n_{\max}^2 + \eta^2 m_{\max}^2)\) time. This function is used by APPROXDS.

Proof of Theorem 4.11. By combining Lemma 4.12, 4.13, 4.15, 4.16. □
Algorithm 4 Variation of Algorithm 5 in page 29 in [DLY21]. This is used in Algorithm 2.

1: data structure ExactDS ▷ Theorem 4.11
2: members
3: \( T, \bar{T} \in \mathbb{R}^{m_p}, H_T \in \mathbb{R}^{m_p \times m_p}, L_T \in \mathbb{R}^{m_p \times m_p} \)
4: \( \tilde{T}, \bar{T}, c_x, \epsilon_x, \epsilon_s, h, \beta_x, \beta_s \in \mathbb{R} \)
5: \( \overline{T} \in \mathbb{R}, \delta_{\mu} \in \mathbb{R}^n \)
6: \( \overline{t} \in \mathbb{R}_+ \)
7: \( k \in \mathbb{N} \)
8: end members
9: procedure INITIALIZE\((x, s, T, \bar{T} \in \mathbb{R}^{m_p}, \overline{t} \in \mathbb{R}_+)\) ▷ Lemma 4.13
10: \( T \leftarrow T, \overline{T} \leftarrow \bar{T}, \overline{t} \leftarrow \overline{t} \)
11: \( \tilde{T} \leftarrow x, \bar{T} \leftarrow s, \epsilon_x \leftarrow 0, \epsilon_s \leftarrow 0, \beta_x \leftarrow 0, \beta_s \leftarrow 0 \)
12: \( H_T \leftarrow \nabla^2 \phi(T) \)
13: Find lower Cholesky factor \( L_T \) where \( L_T L_T^\top = AH_T^{-1}A^\top \)
14: INITIALIZE\(H(T, \bar{T}, H_T, L_T)\)
15: end procedure
16: procedure INITIALIZE\(H(T, \bar{T}, H_T, L_T)\)
17: for \( i \in [n] \) do
18: \( (\delta_{\mu})_i \leftarrow -\frac{\alpha \sinh(\frac{\lambda}{\gamma}(T, x, \overline{t}))}{\gamma(T, x, \overline{t})} \cdot \mu_i(T, x, \bar{T}, \overline{t}) \)
19: \( \overline{T} \leftarrow \overline{T} \cdot \overline{w}_i^{-1} \cosh^2(\frac{\lambda}{\gamma_i}(T, x, \bar{T}, \overline{t})) \)
20: end for
21: \( c_x \leftarrow H_T^{-1/2} \bar{T}_{\mu} \)
22: \( h \leftarrow L_T^{-1} A H_T^{-1} \overline{T}_{\mu} \)
23: end procedure
24: procedure MOVE() ▷ Lemma 4.14
25: \( \beta_x \leftarrow \beta_x + (\overline{T})^{-1/2} \)
26: \( \beta_s \leftarrow \beta_s + \overline{t} \cdot (\overline{T})^{-1/2} \)
27: return \( \beta_x, \beta_s \)
28: end procedure
29: procedure OUTPUT() ▷ Lemma 4.16
30: return \( \tilde{x} + H_T^{-1/2} \beta_x c_x - H_T^{-1/2} W^\top (\beta_x h + \epsilon_x), \bar{s} + H_T^{1/2} W^\top (\beta_s h + \epsilon_s) \)
31: end procedure
32: procedure QUERY\(x(i \in [n])\) ▷ Lemma 4.17
33: return \( \tilde{x}_i + H_T^{-1/2} \beta_x c_x, i - H_T^{-1/2} (W^\top (\beta_x h + \epsilon_x))_i \)
34: end procedure
35: procedure QUERY\(s(i \in [n])\) ▷ Lemma 4.17
36: return \( \tilde{s}_i + H_T^{1/2} (W^\top (\beta_s h + \epsilon_s))_i \)
37: end procedure
38: end data structure

Lemma 4.12. ExactDS correctly maintains an implicit representation of \((x, s)\), i.e., invariant
\[
\begin{align*}
x &= \tilde{x} + H_T^{-1/2} \beta_x c_x - H_T^{-1/2} W^\top (\beta_x h + \epsilon_x), \\
s &= \bar{s} + H_T^{1/2} W^\top (\beta_s h + \epsilon_s)
\end{align*}
\]
always holds.
Algorithm 5 EXACTDS Algorithm 4 continued.

1: data structure EXACTDS ▷ Theorem 4.11
2: procedure UPDATE($\overline{\delta}_x$, $\overline{\delta}_s$) ▷ Lemma 4.15
3: $\Delta H_{x} \leftarrow \nabla^2 \phi(\overline{x} + \delta_{x}) - H_{x} \triangleright \Delta H_{x}$ is non-zero only for diagonal blocks $i$ for which $\delta_{x,i} \neq 0$
4: UPDATE($\overline{\delta}_x$, $\overline{\delta}_s$, $\overline{\delta}_H$, $\overline{\delta}_L$)
5: Find $\Delta L_{x}$ where $(L_{x} + \Delta L_{x})(L_{x} + \Delta L_{x})^\top = A(H_{x} + \Delta H_{x})A^\top$
6: UPDATE($\Delta L_{x}$, $\Delta H_{x}$)
7: $\overline{x} \leftarrow x + \delta_{x}, \overline{s} \leftarrow s + \delta_{s}$
8: $H_{x} \leftarrow H_{x} + \Delta H_{x}, L_{x} \leftarrow L_{x} + \Delta L_{x}$
9: return $\delta_{h}, \delta_{x}, \delta_{s}, \delta_{H_{x}^{1/2}}, \delta_{L_{x}^{-1/2}}, \delta_{x}$
10: end procedure
11: procedure UPDATE($\overline{\delta}_x$, $\overline{\delta}_s$, $\overline{\delta}_H$)
12: $S \leftarrow \{i \in [n] \mid \delta_{x,i} \neq 0 \land \delta_{x,i} 
eq 0\}$
13: $\overline{\delta}_{s,i} \leftarrow 0$
14: for $i \in S$ do
15: Let $\gamma_i = \gamma_i(\overline{x}, \overline{s}, \overline{t}), \gamma_i^{\text{new}} = \gamma_i(\overline{x} + \delta_{x}, \overline{s} + \overline{\delta}_s, \overline{t}), \mu_i^{\text{new}} = \mu_i(\overline{x} + \delta_{x}, \overline{s} + \overline{\delta}_s, \overline{t})$
16: $\overline{\sigma}_i \leftarrow \overline{\sigma}_i - \sigma_i^2 \cdot w_i^{-1} \cosh^2(\frac{1}{w_i} \gamma_i) + \alpha^2 w_i^{-1} \cosh^2(\frac{\Delta_{x,i}^{\text{new}}}{w_i})$
17: $\delta_{\overline{\sigma}_i} \leftarrow -\alpha \sinh(\frac{1}{w_i} \gamma_i^{\text{new}}) \cdot \frac{1}{\gamma_i^{\text{new}}} \cdot \mu_i^{\text{new}} - \delta_{\sigma,i}$
18: end for
19: $\delta_{x} \leftarrow \Delta H_{x}^{-1/2} \cdot (\overline{\delta}_x + \delta_{\overline{\delta}_x}) + H_{x}^{-1/2} \cdot \delta_{\overline{\delta}_x}$
20: $\delta_{h} \leftarrow \Delta L_{x}^{-1} A(H_{x}^{-1} \cdot (\overline{\delta}_x + \delta_{\overline{\delta}_x}) + H_{x}^{-1} \cdot \delta_{\overline{\delta}_x})$
21: $\delta_{x} \leftarrow -\beta_x \delta_{h}$
22: $\delta_{x} \leftarrow -\beta_x \delta_{h}$
23: $\delta_x \leftarrow \beta_x (\Delta H_{x}^{-1/2} \cdot (\epsilon_x + \delta_{x}) - H_{x}^{-1/2} \delta_{x}) + \Delta H_{x}^{-1/2} \cdot L_{x}^\top \cdot (\beta_x h + \epsilon_x)$
24: $\delta_{s} \leftarrow -\Delta H_{x}^{-1/2} \cdot L_{x}^\top (\beta_x h + \epsilon_s)$
25: $\overline{\delta}_{\mu} \leftarrow \overline{\delta}_{\mu} + \delta_{\overline{\delta}_{\mu}}, c_x \leftarrow c_x + \delta_{c_x}, h \leftarrow h + \delta_{h}$
26: $\epsilon_x \leftarrow \epsilon_x + \delta_{c_x}, \epsilon_s \leftarrow \epsilon_s + \delta_{c_s}$
27: $\hat{x} \leftarrow \hat{x} + \delta_{\hat{x}}, \hat{s} \leftarrow \hat{s} + \delta_{\hat{s}}$
28: end procedure
29: procedure UPDATE($\Delta H$, $\Delta L$)
30: $\delta_x \leftarrow (H_{x} + \Delta H_{x})^{-1/2} \Delta H_{x}^{-1/2} \cdot A^\top L_{x}^{-1} \cdot (\beta_x h + \epsilon_x)$
31: $\delta_s \leftarrow - (H_{x} + \Delta H_{x})^{1/2} \Delta H_{x}^{-1/2} \cdot A^\top L_{x}^{-1} \cdot (\beta_x h + \epsilon_s)$
32: $\delta_{x} \leftarrow \Delta L_{x}^{-1} \cdot L_{x}^\top \cdot (\beta_x h + \epsilon_x)$
33: $\delta_{s} \leftarrow \Delta L_{x}^{-1} \cdot L_{x}^\top \cdot (\beta_x h + \epsilon_s)$
34: $\hat{x} \leftarrow \hat{x} + \delta_{\hat{x}}, \hat{s} \leftarrow \hat{s} + \delta_{\hat{s}}$
35: $\epsilon_x \leftarrow \epsilon_x + \delta_{c_x}, \epsilon_s \leftarrow \epsilon_s + \delta_{c_s}$
36: end procedure
37: end data structure

Proof. Initialize: Initialization satisfies the invariant because $\hat{x} = x, \hat{s} = s, \epsilon_x = \epsilon_s = 0, \beta_x = \beta_s = 0$. Furthermore, we correctly initialize $H_{x}, L_{x}, \overline{x}, \overline{\delta}_x, c_x = H_{x}^{-1/2} \delta_{\overline{\delta}_x}, h = L_{x}^{-1} A H_{x}^{-1} \delta_{\overline{\delta}_x}$.

Move: By inspecting terms with coefficient $\beta_x, \beta_s$, we see that we move in the correct direction and step size.

Update: We would like to prove that UPDATE does not change the value of $(x, s)$. First note
that $H_{\tau}$ and $L_{\tau}$, $\alpha$, $\delta_{\mu}$ are update correctly. The remaining updates are separated into two steps: Update $H$ and Update $W$.

**Step UpdateH:** Write $H_{\tau}^{\text{new}} := H_{\tau} + \Delta H_{\tau}$. Immediately after Algorithm 5, Line 24, we have

$$
\begin{align*}
c_x + \delta_{c_x} &= H_{\tau}^{-1/2} \delta_{\mu} + \Delta H_{\tau}^{-1/2} (\delta_{\mu} + \delta_{\mu_{\tau}}) + H_{\tau}^{-1/2} \cdot \delta_{\mu} \\
&= (H_{\tau}^{\text{new}})^{-1/2} (\delta_{\mu} + \delta_{\mu_{\tau}}) \\
h + \delta_h &= L_{\tau}^{-1} A H_{\tau}^{-1} \delta_{\mu} + L_{\tau}^{-1} A (\Delta H_{\tau}^{-1} \cdot (\delta_{\mu} + \delta_{\mu_{\tau}}) + H_{\tau}^{-1} \cdot \delta_{\mu}) \\
&= L_{\tau}^{-1} A (H_{\tau}^{\text{new}})^{-1} (\delta_{\mu} + \delta_{\mu_{\tau}}).
\end{align*}
$$

So $c_x$ and $h$ are updated correctly. Furthermore, immediately after Algorithm 5, Line 24, we have

$$
\begin{align*}
\delta_{\hat{x}} + (H_{\tau}^{\text{new}})^{-1/2} \beta_x (c_x + \delta_{c_x}) - H_{\tau}^{-1/2} c_x \\
- ((H_{\tau}^{\text{new}})^{-1/2} W^{\top} (\beta_x (h + \delta_h) + (\epsilon_x + \delta_{\epsilon_x})) - H_{\tau}^{-1/2} W^{\top} (\beta_x h + \epsilon_x)) \\
= \beta_x (-\Delta H_{\tau}^{-1/2} (c_x + \delta_{c_x}) + H_{\tau}^{-1/2} \delta_{c_x} + \Delta H_{\tau}^{-1/2} \cdot W^{\top} (\beta_x h + \epsilon_x) + (H_{\tau}^{\text{new}})^{-1/2} \beta_x (c_x + \delta_{c_x}) - H_{\tau}^{-1/2} c_x \\
- ((H_{\tau}^{\text{new}})^{-1/2} W^{\top} (\beta_x (h + \delta_h) + (\epsilon_x + \delta_{\epsilon_x})) - H_{\tau}^{-1/2} W^{\top} (\beta_x h + \epsilon_x)) \\
= - (H_{\tau}^{\text{new}})^{-1/2} W^{\top} (\beta_x \delta_h + \delta_{\epsilon_x}) = 0.
\end{align*}
$$

So $x$ is updated correctly, i.e., after Update $H$ finishes, we have

$$
x = \hat{x} + (H_{\tau}^{\text{new}})^{-1/2} \beta_x c_x - (H_{\tau}^{\text{new}})^{-1/2} W^{\top} (\beta_x h + \epsilon_x).
$$

Immediately after Algorithm 5, Line 24, we have

$$
\begin{align*}
\delta_{\hat{x}} + (H_{\tau}^{\text{new}})^{-1/2} W^{\top} (\beta_s (h + \delta_h) + (\epsilon_s + \delta_{\epsilon_s})) - H_{\tau}^{1/2} W^{\top} (\beta_s h + \epsilon_s) \\
= - \Delta H_{\tau}^{-1/2} \cdot W^{\top} (\beta_s h + \epsilon_s) \\
+ (H_{\tau}^{\text{new}})^{-1/2} W^{\top} (\beta_s (h + \delta_h) + (\epsilon_s + \delta_{\epsilon_s})) - H_{\tau}^{1/2} W^{\top} (\beta_s h + \epsilon_s) \\
= (H_{\tau}^{\text{new}})^{-1/2} W^{\top} (\beta_s \delta_h \delta_{\epsilon_s}) = 0.
\end{align*}
$$

So $s$ is updated correctly, i.e., after Update $H$ finishes, we have

$$
s = \hat{s} + (H_{\tau}^{\text{new}})^{-1/2} W^{\top} (\beta_s h + \epsilon_s).
$$

This proves correctness of Update $H$.

**Step UpdateW:** Let $H_{\tau}^{\text{new}} := H_{\tau} + \Delta H_{\tau}$ and $L_{\tau}^{\text{new}} := L_{\tau} + \Delta L_{\tau}$. Immediately after Algorithm 5, Line 33, we have

$$
\begin{align*}
\delta_{\hat{x}} - ((H_{\tau}^{\text{new}})^{-1} A^{\top} (L_{\tau}^{\text{new}})^{-1} (\beta_x h + (\epsilon_x + \delta_{\epsilon_x})) - ((H_{\tau}^{\text{new}})^{-1/2} H_{\tau}^{-1/2} A^{\top} L_{\tau}^{\top} (\beta_x h + \epsilon_x))) \\
= (H_{\tau}^{\text{new}})^{-1/2} \Delta H_{\tau}^{-1/2} A^{\top} L_{\tau}^{\top} (\beta_x h + \epsilon_x)
\end{align*}
$$

29
\[-((H^\text{new}_T)^{-1}A^\top(L^\text{new}_T)^{-\top}(\beta_s h + (\epsilon_s + \delta_{\epsilon_s})) - (H^\text{new}_H)^{-1/2}H^\text{new}_H^{-1/2}A^\top L^\text{new}_H^{-\top}(\beta_s h + \epsilon_s)))
\]
\[= (H^\text{new}_T)^{-1}A^\top(L^\text{new}_T)^{-\top} - (L^\text{new}_H)^{-\top}((H^\text{new}_H)^{-1/2}H^\text{new}_H^{-1/2}A^\top L^\text{new}_H^{-\top}(\beta_s h + \epsilon_s)))
\]
\[= 0.
\]
So \(x\) is updated correctly, i.e., after \(\text{UPDATE}_W\) finishes, we have
\[x = \hat{x} + (H^\text{new}_T)^{-1/2}\beta_s e_x - (H^\text{new}_H)^{-1}A^\top(L^\text{new}_H)^{-\top}(\beta_s h + \epsilon_s).
\]
Immediately after Algorithm 5, Line 33, we have
\[\delta_s + (A^\top(L^\text{new}_T)^{-\top}(\beta_s h + (\epsilon_s + \delta_{\epsilon_s}))) - ((H^\text{new}_H)^{-1/2}H^\text{new}_H^{-1/2}A^\top L^\text{new}_H^{-\top}(\beta_s h + \epsilon_s)))
\]
\[= - (H^\text{new}_T)^{-1/2}A^\top(L^\text{new}_T)^{-\top}(\beta_s h + \epsilon_s)
\]
\[+ (A^\top(L^\text{new}_H)^{-\top}(\beta_s h + (\epsilon_s + \delta_{\epsilon_s}))) - ((H^\text{new}_H)^{-1/2}H^\text{new}_H^{-1/2}A^\top L^\text{new}_H^{-\top}(\beta_s h + \epsilon_s)))
\]
\[= (A^\top((L^\text{new}_T)^{-\top} - L^\text{new}_H^{-\top})\beta_s h + \epsilon_s) + A^\top(L^\text{new}_H)^{-\top}\delta_{\epsilon_s}
\]
\[= 0.
\]
So \(s\) is updated correctly, i.e., after \(\text{UPDATE}_W\) finishes, we have
\[s = \hat{s} + A^\top(L^\text{new}_H)^{-\top}(\beta_s h + \epsilon_s).
\]

\[\square\]

**Lemma 4.13.** \(\text{EXACTDS.INITIALIZE}\) (Algorithm 4) runs in \(O(T_H + T_L + \text{nnz}(A) + T_n + \eta m_{ip} m_{\text{max}})\) time.

**Proof.** Computing \(H^T\) takes \(T_H\) time. Computing \(L^T\) takes \(T_L\) time.

It remains to analyze the running time of \(\text{INITIALIZE}_H\). Computing \(\mu_i(\overline{x}, \overline{s}, \overline{t})\) takes \(T_{H,i}\) time.
Computing \(\gamma_i(\overline{x}, \overline{s}, \overline{t})\) takes \(T_{\text{mat}}(n_i)\) time because it involves computing inverse of \(\nabla^2 \phi_i(x_i)\). So computing \(\delta_{\mu}\) and \(\overline{\tau}\) takes \(T_H + T_n\) time.

Computing \(c_s\) takes \(T_n\) time by Lemma 4.7(v). Computing \(h\) takes \(O(T_n + \eta m_{ip} m_{\text{max}} + \text{nnz}(A))\) time by Lemma 4.7(v)(vi)(i).

\[\square\]

**Lemma 4.14.** \(\text{EXACTDS.MOVE}\) (Algorithm 4) runs in \(O(1)\) time.

**Proof.** All steps in this procedure can be done in \(O(1)\) time.

\[\square\]

**Lemma 4.15.** Each call of \(\text{EXACTDS.UPDATE}\) (Algorithm 5) runs in
\[O((T_{\text{mat}}(n_{\max}) + T_{\Delta L, \max} + T_{H, \max} + \eta^2 m_{\text{max}}^2)(\|\delta_{\mu}\|_2, 0 + \|\delta_{\text{tau}}\|_2, 0))\]
time. Furthermore, \(\delta_{\mu}, \delta_{\epsilon_s}, \delta_{\epsilon_s}\) changes in \(O(\|\delta_{\mu}\|_2, 0 + \|\delta_{\epsilon_s}\|_2, 0)\) blocks, and \(\delta_{H^T, 1/2} \hat{\delta}_{s}, \delta_{L^T, 1/2} \hat{\delta}_{s}\) changes in \(O(\|\delta_{\mu}\|_2, 0 + \|\delta_{\epsilon_s}\|_2, 0)\) blocks.

**Proof.** Computing \(\Delta_{H^T}\) takes \(O(T_{H^T, \max} \|\delta_{\mu}\|_2, 0)\) time, and \(\text{nnz}(\Delta_{H^T}) = O(n_{\max}^2 \|\delta_{\mu}\|_2, 0)\).
Computing \(\Delta_{L^T}\) takes \(O(T_{\Delta L, \max} \|\delta_{\epsilon_s}\|_2, 0)\) time, and \(\text{nnz}(\Delta_{L^T}) = O(T_{\Delta L, \max} \|\delta_{\epsilon_s}\|_2, 0)\).
It remains to analyze two parts \(\text{UPDATE}_H\) and \(\text{UPDATE}_W\). We will analyze these two parts separately in the next a few paragraphs.

**Part 1.**

**UPDATE\(H\):**
Computing $\alpha_{\text{new}}$ and $\delta_\mu$ takes
\[ O(T_{\text{mat}}(n_{\text{max}}) \cdot (\|\delta\|_2, 0 + \|\delta\|_2, 0)) \]
time, and we have
\[ \|\delta_{\mu}\|_0 \leq \|\delta\|_2, 0 + \|\delta\|_2, 0. \]

Computing $\Delta_{H^{-1/2}}$, $\Delta_{H^{1/2}}$, $\Delta_{H^{-1}}$ takes $O(T_{\text{mat}}(n_{\text{max}})\|\delta\|_2, 0)$ time, and they all have nnz at most $O(n_{\text{max}}^2\|\delta\|_2, 0)$.

Computing $\delta_{ce}$ takes $\text{nnz}(\Delta_{H^{-1/2}}) + n_{\text{max}}^2\|\delta_{\mu}\|_0$ time.

Computing $\delta_h$ takes
\[
O(\text{nnz}(\Delta_{H^{-1/2}}) + n_{\text{max}}^2\|\delta_{\mu}\|_0 + \eta^2m_{\text{max}}^2\|\delta\|_2, 0)
\]
\[ = O((n_{\text{max}}^2 + \eta^2m_{\text{max}}^2)(\|\delta_{\mu}\|_2, 0 + \|\delta\|_2, 0)) \]
time by Lemma 4.8(i). Also, $\delta_h$ is supported on $O(\|\delta_x\|_2, 0)$ blocks, or $O(\eta\|\delta_x\|_2, 0)$ dimension.

We compute $\delta_x$ and $\delta_\nu$ from left to right. This takes
\[
O(\|\delta_{ce}\|_0 + \text{nnz}(\Delta_{H^{1/2}}) + \text{nnz}(\Delta_{H^{-1/2}}) + \eta^2m_{\text{max}}^2\|\delta_x\|_2, 0)
\]
\[ = O((n_{\text{max}}^2 + \eta^2m_{\text{max}}^2)(\|\delta_{\mu}\|_2, 0 + \|\delta\|_2, 0)) \]
time by Lemma 4.8(ii).

Remaining computations are all cheap.

**Part 2.**

**UPDATE $W$:**
Computing $\delta_x$ and $\delta_\nu$ takes
\[ O((\eta^2m_{\text{max}}^2 + T_{\text{mat}}(n_{\text{max}}))\|\delta\|_2, 0) \]
time by Lemma 4.7.

To compute $\delta_e$ and $\delta_{ce}$, we first compute $(L^{-\top}(\beta_e h + \epsilon_x))S$, where $S \in [m]$ is the row support of $\Delta_L$, which can be decomposed into at most $\|\delta\|_2, 0$ paths. This takes
\[ O(\eta^2m_{\text{max}}^2\|\delta\|_2, 0) \]
time by Lemma 4.8(i). So computing $\delta_{ce}$ and $\delta_e$ takes
\[ O(\text{nnz}(\Delta_L) + \eta^2m_{\text{max}}^2\|\delta\|_2, 0) \]
time.

Combining everything finishes the proof of running time.

For the claim on output sparsity, note that $\delta_h, \delta_{ce}, \delta_e$ change in $O(\|\delta\|_2, 0 + \|\delta\|_2, 0)$ paths, and that $\delta_{H^{1/2}}$, $\delta_{H^{-1/2}}$, $\delta_{H^{-1/2}}^{ce}$ change only in the support of $\delta_{\nu}$ and support of $\delta_{\nu}$.

**Lemma 4.16.** ExactDS.OUTPUT (Algorithm 4) runs in $O(\text{nnz}(A) + T_n + \eta m_{lp}m_{\text{max}})$ time and correctly outputs $(x, s)$.

**Proof.** Correctness is by Lemma 4.12. By Lemma 4.7(viii). 

**Lemma 4.17.** ExactDS.QUERY $x$ (Algorithm 4) and ExactDS.QUERYs (Algorithm 4) runs in $O(n_{\text{max}}^2 + \eta^2m_{\text{max}}^2)$ time and returns the correct answer.

**Proof.** Correctness is obvious. Running time follows from Lemma 4.8(ii).
4.4.2 APPROXDS

In this section we introduce APPROXDS, data structure for maintaining a sparsely-changing approximation of \((x, s)\).

**Algorithm 6** This is used in Algorithm 2.

```
1: data structure APPROXDS
2: private : members
3: \(\epsilon_{\text{apx}, x}, \epsilon_{\text{apx}, s}, \zeta_x, \zeta_s \in \mathbb{R}\)
4: \(\ell \in \mathbb{N}\)
5: BatchSketch bs \(\triangleright\) This maintains a sketch of \(H_{\mathbf{x}}^{1/2} x\) and \(H_{\mathbf{x}}^{-1/2} s\). See Algorithm 8 and 9.
6: ExactDS* exact \(\triangleright\) This is a pointer to the ExactDS (Algorithm 4, 5) we maintain in parallel to APPROXDS.
7: \(\bar{x}, \bar{s} \in \mathbb{R}^{n_p}\) \(\triangleright\) \((\bar{x}, \bar{s})\) is a sparsely-changing approximation of \((x, s)\). They have the same value as \((\mathbf{x}, \mathbf{s})\), but for these local variables we use \((\bar{x}, \bar{s})\) to avoid confusion.
8: end members
9: procedure Initialize \((x, s \in \mathbb{R}^{n_p}, h, \epsilon_x, \epsilon_s \in \mathbb{R}^{n_p}, H_{\mathbf{x}}^{1/2} x, H_{\mathbf{x}}^{-1/2} s, c_x \in \mathbb{R}^{n_p}, \beta_x, \beta_s \in \mathbb{R}, q \in \mathbb{N}\)
10: ExactDS* exact, \(\epsilon_{\text{apx}, x}, \epsilon_{\text{apx}, s}, \delta_{\text{apx}} \in \mathbb{R}\)
11: \(\ell \leftarrow 0, q \leftarrow q\)
12: \(\epsilon_{\text{apx}, x} \leftarrow \epsilon_{\text{apx}, x}, \epsilon_{\text{apx}, s} \leftarrow \epsilon_{\text{apx}, s}\)
13: bs.Initialize \((x, h, \epsilon_x, \epsilon_s, H_{\mathbf{x}}^{1/2} x, H_{\mathbf{x}}^{-1/2} s, c_x, \beta_x, \beta_s, \delta_{\text{apx}}/q)\) \(\triangleright\) Algorithm 8
14: exact \(\leftarrow\) exact
15: end procedure
16: procedure Update \((\delta_x \in \mathbb{R}^{n_p}, \delta_h, \delta_{\epsilon_x}, \delta_{s} \in \mathbb{R}^{n_p}, \delta_{H_{\mathbf{x}}^{1/2} x}, \delta_{H_{\mathbf{x}}^{-1/2} s}, \delta_{H_{\mathbf{c}}^{1/2} c_x} \in \mathbb{R}^{n_p})\)
17: bs.Update \((\delta_{\mathbf{x}}, \delta_h, \delta_{\epsilon_x}, \delta_s, \delta_{H_{\mathbf{x}}^{1/2} x}, \delta_{H_{\mathbf{x}}^{-1/2} s}, \delta_{H_{\mathbf{c}}^{1/2} c_x})\) \(\triangleright\) Algorithm 8
18: \(\ell \leftarrow \ell + 1\)
19: end procedure
20: procedure MoveAndQuery \((\beta_x, \beta_s \in \mathbb{R})\)
21: bs.Move \((\beta_x, \beta_s)\) \(\triangleright\) Do not update \(\ell\) yet
22: \(\delta_{\mathbf{x}} \leftarrow \text{QUERY}_x(\epsilon_{\text{apx}, x}/(2 \log q + 1))\) \(\triangleright\) Algorithm 7
23: \(\delta_{\mathbf{s}} \leftarrow \text{QUERY}_{s}(\epsilon_{\text{apx}, s}/(2 \log q + 1))\) \(\triangleright\) Algorithm 7
24: \(\bar{\mathbf{x}} \leftarrow \bar{x} + \delta_{\mathbf{x}}, \bar{s} \leftarrow \bar{s} + \delta_{\mathbf{s}}\)
25: return \((\delta_{\mathbf{x}}, \delta_{\mathbf{s}})\)
26: end procedure
27: end data structure
```

**Theorem 4.18.** Given parameters \(\epsilon_{\text{apx}, x}, \epsilon_{\text{apx}, s} \in (0, 1), \delta_{\text{apx}} \in (0, 1), \zeta_x, \zeta_s \in \mathbb{R}\) such that

\[
\|H_{\mathbf{x}}^{1/2}(t) x(t) - H_{\mathbf{x}}^{1/2}(t+1)\|_2 \leq \zeta_x, \quad \|H_{\mathbf{x}}^{-1/2}(t) s(t) - H_{\mathbf{x}}^{-1/2}(t+1)\|_2 \leq \zeta_s
\]

for all \(t \in \{0, \ldots, q - 1\}\), data structure APPROXDS (Algorithm 6 and Algorithm 7) supports the following operations:

- **Initialize** \((x, s \in \mathbb{R}^{n_p}, h, \epsilon_x, \epsilon_s \in \mathbb{R}^{n_p}, H_{\mathbf{x}}^{1/2} x, H_{\mathbf{x}}^{-1/2} s, c_x \in \mathbb{R}^{n_p}, \beta_x, \beta_s \in \mathbb{R}, q \in \mathbb{N}, \text{ExactDS}* exact, \epsilon_{\text{apx}, x}, \epsilon_{\text{apx}, s}, \delta_{\text{apx}} \in \mathbb{R})\): Initialize the data structure in

\[
\tilde{O}(T_H + T_n + T_L + r \cdot (n \cdot n_{\text{max}}^2 + T_Z + \text{nnz}(A)))
\]

\(\text{time.} \)
Algorithm 7 APPROXDS Algorithm 6 continued.

1: data structure APPROXDS \>\text{Theorem 4.18}
2: private:
3: procedure Queryx(\epsilon \in \mathbb{R})
4: \quad I \leftarrow 0
5: \quad for j = 0 \rightarrow \lfloor \log_2 \ell \rfloor \ do
6: \quad \quad if \ \ell \mod 2^j = 0 \ then
7: \quad \quad \quad I \leftarrow I \cup bs.Que\text{ry}_x(\ell - 2^j + 1, \epsilon) \>\text{Algorithm 9}
8: \quad \quad end if
9: \quad end for
10: \quad \delta_{\tilde{x}} \leftarrow 0
11: \quad for all \ i \in I \ do
12: \quad \quad x_i \leftarrow \text{exact}.Que\text{ry}_x(i) \>\text{Algorithm 4}
13: \quad \quad if \ \|\tilde{x}_i - x_i\| > \epsilon \ then
14: \quad \quad \quad \delta_{\tilde{x},i} \leftarrow x_i - \tilde{x}_i
15: \quad \quad end if
16: \quad end for
17: return \delta_{\tilde{x}}
18: end procedure

19: procedure Querys(\epsilon \in \mathbb{R})
20: \quad I \leftarrow 0
21: \quad for j = 0 \rightarrow \lfloor \log_2 \ell \rfloor \ do
22: \quad \quad if \ \ell \mod 2^j = 0 \ then
23: \quad \quad \quad I \leftarrow I \cup bs.Que\text{ry}_s(\ell - 2^j + 1, \epsilon) \>\text{Algorithm 9}
24: \quad \quad end if
25: \quad end for
26: \quad \delta_{\tilde{s}} \leftarrow 0
27: \quad for all \ i \in I \ do
28: \quad \quad s_i \leftarrow \text{exact}.Que\text{ry}_s(i) \>\text{Algorithm 4}
29: \quad \quad if \ \|\tilde{s}_i - s_i\| > \epsilon \ then
30: \quad \quad \quad \delta_{\tilde{s},i} \leftarrow s_i - \tilde{s}_i
31: \quad \quad end if
32: \quad end for
33: return \delta_{\tilde{s}}
34: end procedure
35: end data structure

- \text{MoveAndQuery}(\beta_x, \beta_s \in \mathbb{R}): Update values of \beta_x, \beta_s by calling BatchSketch.Move. This effectively moves \((x^{(\ell)}, s^{(\ell)})\) to \((x^{(\ell+1)}, s^{(\ell+1)})\) while keeping \(\tilde{x}^{(\ell)}\) unchanged.

Then return two sets \(L_x^{(\ell)}, L_s^{(\ell)} \subseteq [n]\) where
\[
L_x^{(\ell)} \supseteq \{ i \in [n] : \| H_{\tilde{x}^{(\ell)}}^{1/2} x^{(\ell)}_i - H_{\tilde{x}^{(\ell)}}^{1/2} x^{(\ell+1)}_i \|_2 \geq \epsilon_{apx,x} \},
\]
\[
L_s^{(\ell)} \supseteq \{ i \in [n] : \| H_{\tilde{x}^{(\ell)}}^{-1/2} s^{(\ell)}_i - H_{\tilde{x}^{(\ell)}}^{-1/2} s^{(\ell+1)}_i \|_2 \geq \epsilon_{apx,s} \},
\]
satisfying
\[
\sum_{0 \leq \ell \leq q-1} |L_x^{(\ell)}| = \tilde{O}(\epsilon_{apx,x}^2 q^2).
\]
\[\sum_{0 \leq \ell \leq q-1} |L^{(\ell)}_s| = \tilde{O}(\epsilon_{apx,s}^2 q^2).\]

For every query, with probability at least \(1 - \delta_{apx}/q\), the return values are correct.

Furthermore, total time cost over all queries is at most

\[\tilde{O}\left((\epsilon_{apx,s}^2 + \epsilon_{apx,s}^2)q^2 r(n_{max}^2 + \eta^2 m_{max}^2)\right).\]

- **UPDATE\((\delta_H, \delta_h, \delta_{e_s}, \delta_{e_s}, \delta_{H_{T}^{1/2,s}}, \delta_{H_{T}^{1/2,s}}, \delta_{H_{T}^{1/2,s}})\)**: Update sketches of \(H_{T_{(0)}}^{1/2}x^{(\ell+1)}\) and \(H_{T_{(1)}}^{1/2}s^{(\ell+1)}\) by calling BatchSketch.Update. This effectively moves \(x^{(\ell)}\) to \(x^{(\ell+1)}\) while keeping \((x^{(\ell+1)}, s^{(\ell+1)})\) unchanged. Then advance timestamp \(\ell\).

Each update costs

\[\tilde{O}\left((T_{H_{max}} + T_{\Delta_{L_{max}}} + T_{mat}(n_{max}) + r\eta^2 m_{max}^2)\|\delta_T\|_{2,0}ight.\]
\[+ rm_{max} \cdot (\|\delta_h\|_0 + \|\delta_{e_s}\|_0 + \|\delta_{e_s}\|_0)\]
\[+ rm_{max} \cdot (\|\delta_{H_{T}^{1/2,s}}\|_0 + \|\delta_{H_{T}^{1/2,s}}\|_0 + \|\delta_{H_{T}^{1/2,s}}\|_0))\]
\[\text{time.}\]

**Remark 4.19.** In [DLY21], Theorem 6.14 (their counterpart of Theorem 4.18) is not used properly, making their algorithm incorrect. For [DLY21, Theorem 6.14] to apply, they need \(\|H_{T_{(0)}}^{1/2}x^{(\ell)} - H_{T_{(1)}}^{1/2}x^{(\ell+1)}\|_2 \leq \zeta_{apx}\). However, in the proof of [DLY21, Theorem 6.1], they only prove \(\|H_{T_{(0)}}^{1/2}x^{(\ell)} - H_{T_{(1)}}^{1/2}x^{(\ell+1)}\|_2 \leq \zeta_{apx}\), and the change in \(T\) is ignored. Because [DLY21, Algorithm 8] uses sampling instead of top-down BFS as in Algorithm 9, the change in \(T\) could potentially make \(\|H_{T_{(0)}}^{1/2}x^{(\ell)} - H_{T_{(1)}}^{1/2}x^{(\ell+1)}\|_2\) very large, and make their algorithm unable to find large changes in \(x_i\). Similar issue also happens for \(H_{T_{(1)}}^{1/2}s\). We fix these issues by replacing their sampling step with a top-down BFS in BatchSketch, and considering change in \(T\) in the analysis of APPROXDS.

**Proof of Theorem 4.18. Correctness:** We prove that with high probability \((1 - 1/poly(m_{apx}))\), for all blocks \(i \in [n]\) not in supp\(\delta_T\), we have \(\|x^{(\ell)}_i - x^{(\ell+1)}_i\|_2 \leq \epsilon_{apx,x}\). Recall that \(x^{(\ell)}_i\) and \(x^{(\ell+1)}_i\) are equal for all \(t\) because updates \(\delta_T\) are always applied to \(x^{(\ell)}_i\).

Fix current timestamp \(\ell\) and a block \(i \in [n]\). Suppose that \(x^{(\ell)}_i\) is not updated at timestamp \(\ell\), i.e., \(x^{(\ell)}_i \neq x^{(\ell+1)}_i\). If \(i \in \mathcal{I}\) then we are done because it is checked that \(\|x^{(\ell)}_i - x^{(\ell+1)}_i\|_{\mathcal{I}} \leq \epsilon \leq \epsilon_{apx,x}\) at Algorithm 7 Line 13. Let \(\ell'\) be the last time where block \(i\) was in \(\mathcal{I}\) in Algorithm 7 Line 11, i.e., \(\ell' = \max\{t : t < \ell - 1, i \in \mathcal{I}\}\). Then we can divide the interval \([\ell', \ell]\) into \(k \leq 2 \log q\) intervals, i.e., we can find \(\ell' - t_0 \leq \cdots < \ell_k = \ell\) such \(k \leq 2 \log q\), and for every \(o \in [k]\), there exists \(j\) such that \(t_{o-1} = t_o - 2^j\) and \(t_o \mod 2^j = 0\). Then by our assumption on \(\epsilon\), we have

\[\|x^{(\ell'+1)}_i - x^{(\ell+1)}_i\|_{\mathcal{I}} \leq \epsilon\]

where \(\epsilon = \epsilon_{apx,x}/(2 \log q + 1)\), because whether or not \(x^{(\ell')}_i \neq x^{(\ell'+1)}_i\), this bound always holds. Furthermore, \(i \notin \mathcal{I}\) at time \(t_1, \ldots, t_k\), so

\[\|H_{T_{(0)}}^{1/2}x_{(t_0+1)} - H_{T_{(0)}}^{1/2}x_{(t_0+1)}\| \leq \epsilon,
\]
which implies
\[ \|x^{(t_0+1)}_i - x^{(t_0+1+1)}_i\|_{\mathcal{F}^{(t+1)}} \leq \epsilon \]
because \( \mathcal{F}^{(t+1)} = \cdots = \mathcal{F}^{(t)} \). So
\[
\|x^{(t+1)}_i - x^{(t+1)}_i\|_{\mathcal{F}^{(t)}} \leq \|x^{(t+1)}_i - x^{(t_0+1+1)}_i\|_{\mathcal{F}^{(t+1)}} + \sum_{o \in [k]} \|x^{(t_0+1)}_i - x^{(t_0+1)}_i\|_{\mathcal{F}^{(t+1)}}
\]
\[ \leq (k + 1)\epsilon \leq \epsilon_{apx,x}. \]
This proves that
\[ \|x^{(t+1)}_i - x^{(t+1)}_i\|_{\mathcal{F}^{(t+1)}} \leq \epsilon_{apx,x}. \]
Therefore
\[ L_x^{(t)} \supseteq \{ i \in [n] : \|H^{1/2}_{\mathcal{F}^{(t)}} x^{(t)}_i - H^{1/2}_{\mathcal{F}^{(t)}} x^{(t+1)}_i\|_2 \geq \epsilon_{apx,x} \}. \]
On the other hand, because Algorithm 7 Line 13, all elements \( i \in L_x^{(t)} \) satisfy
\[ \|x^{(t)}_i - x^{(t+1)}_i\|_{\mathcal{F}^{(t+1)}} \geq \epsilon. \]
So
\[
e^{-2} \sum_{0 \leq t \leq q-1} |L_x^{(t)}| \leq \sum_{0 \leq t \leq q-1} \sum_{i \in L_x^{(t)}} \|x^{(t)}_i - x^{(t+1)}_i\|_{\mathcal{F}^{(t+1)}}^2 \leq \sum_{i \in [n]} q \sum_{0 \leq t \leq q-1} \|x^{(t)}_i - x^{(t+1)}_i\|_{\mathcal{F}^{(t+1)}}^2 \leq q \sum_{0 \leq t \leq q-1} \|H^{1/2}_{\mathcal{F}^{(t)}} x^{(t)}_i - H^{1/2}_{\mathcal{F}^{(t)}} x^{(t+1)}_i\|_2 \leq q^2 \epsilon_{apx,x}^2. 
\]
This proves that
\[ \sum_{0 \leq t \leq q-1} |L_x^{(t)}| = O(\epsilon_{apx,x}^{-2} \epsilon_{apx,x}^2 q^2). \]
The proof for \( s \) is similar and omitted.

**Running time:**

**Initialize:** By Initialize part of Theorem 4.21.

**Update:** By Update part of Theorem 4.21.

**Query** \( x \) and **Query** \( s \): Let \( \mathcal{I}_x^{(t)}(\ell) \) be the set \( \mathcal{I} \) in Query \( x \) at timestamp \( \ell \). Then by Theorem 4.21, we have
\[
|\mathcal{I}_x^{(t)}(\ell)| = \sum_{0 \leq j \leq \lfloor \log_2 \ell \rfloor, 2^j \ell} O(\epsilon_{apx,x}^{-2}) \sum_{2^{j+1} \ell \leq t \leq \ell} \|H^{1/2}_{\mathcal{F}^{(t)}} x^{(t)}_i - H^{1/2}_{\mathcal{F}^{(t)}} x^{(t+1)}_i\|_2.
\]
\[ + \sum_{\ell-2^j+1 \leq t \leq \ell-1} \| \mathbf{r}^{(t)} - \mathbf{r}^{(t+1)} \|_2. \]

We have
\[
\sum_{0 \leq \ell \leq q-1} \sum_{0 \leq j \leq \lfloor \log_2 \ell \rfloor, 2^j \ell} O(\epsilon^{-2} 2^j \sum_{\ell-2^j+1 \leq t \leq \ell-1} \| H^{1/2}_{\mathbf{x}(t)} x^{(t)} - H^{1/2}_{\mathbf{x}(t+1)} x^{(t+1)} \|_2) = \tilde{O}(\epsilon^{-2} \zeta_2^2 q^2)
\]
and
\[
\sum_{0 \leq \ell \leq q-1} \sum_{0 \leq j \leq \lfloor \log_2 \ell \rfloor, 2^j \ell} O(\sum_{\ell-2^j+1 \leq t \leq \ell-1} \| \mathbf{x}(t) - \mathbf{x}^{(t+1)} \|_2) = \tilde{O}(\sum_{0 \leq \ell \leq q-1} L_x^{(\ell)}) = \tilde{O}(\epsilon^{-2} \zeta_2^2 q^2)
\]

because the dyadic intervals \([\ell-2^j+1, \ell]\) together cover the whole interval \([0, q-1]\] at most \(O(\log q) = \tilde{O}(1)\) times.

Therefore we have proved that
\[
\sum_{0 \leq \ell \leq q-1} \| I_x^{(\ell)} \| = \tilde{O}(\epsilon^2 \zeta_2^2 q^2) = \tilde{O}(\epsilon^2 \zeta_{apx.x}^2 \zeta_2 q^2).
\]

Then the running time bound for QUERYx follows from QUERY part of Theorem 4.11 and QUERYx part of Theorem 4.21.

The proof for \(s\) is similar and omitted. \(\square\)

### 4.4.3 BatchSketch

In this section we introduce **BatchSketch**, our data structure for maintaining a sketch of \(H^{1/2}_x x\) and \(H^{-1/2}_x s\).

For this task, we need another tree structure on the set of variable blocks.

**Definition 4.20 (Partition tree\textsuperscript{5}).** A partition tree \((S, \chi)\) of \(\mathbb{R}^n\) consists of a constant degree rooted tree \(S = (V, E)\) and a labeling of the vertices \(\chi : V \rightarrow 2^{[n]}\), such that

- \(\chi(\text{root}) = [n]\)
- If \(v\) is a leaf node of \(S\), then \(|\chi(v)| = 1\)
- For any node \(v\) of \(S\), the set \(\{\chi(c) : c \text{ is a child of } v\}\) forms a partition of \(\chi(v)\).

The partition tree does not need to (but can) have any relationship with the block elimination tree. The partition tree will need to satisfy \(O(1)\) maximum degree and \(\tilde{O}(1)\) depth. We will choose the partition tree in Section 4.4.5 so that the BlockBalancedSketch data structure can be efficiently maintained.

\textsuperscript{5}This is called a sampling tree in \cite{DLY21}. We changed the name because this data structure has little to do with sampling.
Algorithm 8 This is used by Algorithm 6 and 7.

1: data structure BatchSketch
2: memembers
3: $\Phi \in \mathbb{R}^{p \times n}$p
4: $S, \chi$ partition tree
5: $\ell \in \mathbb{N}$
6: BlockBalancedSketch sketch$W^T h$, sketch$W^T \epsilon_x$, sketch$W^T \epsilon_s$
7: BlockVectorSketch sketch$H_{\pi}^{1/2} \hat{x}$, sketch$H_{\pi}^{-1/2} \hat{s}$, sketch$e_x$
8: $\beta_x, \beta_s \in \mathbb{R}$
9: $(\text{history}[t])_{t \geq 0}$
10: end memembers

11: procedure Initialize($x \in \mathbb{R}^{m_p}$, $h, \epsilon_x, \epsilon_s \in \mathbb{R}^{m_p}$, $H_{\pi}^{1/2} \hat{x}, H_{\pi}^{-1/2} \hat{s}, c_x \in \mathbb{R}^{m_p}, \beta_x, \beta_s \in \mathbb{R}, \delta_{app} \in \mathbb{R}$)
12: Construct partition tree $(S, \chi)$ as in Definition 4.26
13: $r \leftarrow \Theta(\log^3(n_p) \log(1/\delta_{app}))$
14: Initialize $\Phi \in \mathbb{R}^{p \times n}$p with iid $N(0, \frac{1}{r})$
15: $\beta_x \leftarrow \beta_x, \beta_s \leftarrow \beta_s$
16: sketch$W^T h$.Initialize($S, \chi, \Phi, x, h$)
17: sketch$W^T \epsilon_x$.Initialize($S, \chi, \Phi, x, \epsilon_x$)
18: sketch$W^T \epsilon_s$.Initialize($S, \chi, \Phi, x, \epsilon_s$)
19: sketch$H_{\pi}^{1/2} \hat{x}$ Initialize($S, \chi, \Phi, H_{\pi}^{1/2} \hat{x}$)
20: sketch$H_{\pi}^{-1/2} \hat{s}$ Initialize($S, \chi, \Phi, H_{\pi}^{-1/2} \hat{s}$)
21: sketch$e_x$.Initialize($S, \chi, \Phi, c_x$)
22: $\ell \leftarrow 0$
23: Make snapshot history[$\ell$]
24: end procedure

25: procedure move($\beta_x, \beta_s \in \mathbb{R}$)
26: $\beta_x \leftarrow \beta_x, \beta_s \leftarrow \beta_s$
27: end procedure

28: procedure update($\delta_{\pi} \in \mathbb{R}^{m_p}, \delta_h, \delta_{\epsilon_x}, \delta_{\epsilon_s} \in \mathbb{R}^{m_p}, \delta_{H_{\pi}^{1/2} \hat{x}}, \delta_{H_{\pi}^{-1/2} \hat{s}}, \delta_{e_x} \in \mathbb{R}^{m_p}$)
29: sketch$W^T h$.Update($\delta_{\pi}, \delta_h$)
30: sketch$W^T \epsilon_x$.Update($\delta_{\pi}, \delta_{\epsilon_x}$)
31: sketch$W^T \epsilon_s$.Update($\delta_{\pi}, \delta_{\epsilon_s}$)
32: sketch$H_{\pi}^{1/2} \hat{x}$.Update($\delta_{H_{\pi}^{1/2} \hat{x}}$)
33: sketch$H_{\pi}^{-1/2} \hat{s}$.Update($\delta_{H_{\pi}^{-1/2} \hat{s}}$)
34: sketch$e_x$.Update($\delta_{e_x}$)
35: $\ell \leftarrow \ell + 1$
36: Make snapshot history[$\ell$]
37: end procedure
38: end data structure

Theorem 4.21. Data structure BatchSketch (Algorithm 8, 9) supports the following operations:

- Initialize($x \in \mathbb{R}^{m_p}, h, \epsilon_x, \epsilon_s \in \mathbb{R}^{m_p}, H_{\pi}^{1/2} \hat{x}, H_{\pi}^{-1/2} \hat{s}, c_x \in \mathbb{R}^{m_p}, \beta_x, \beta_s \in \mathbb{R}, \delta \in \mathbb{R}$): Initialize
Algorithm 9 BatchSketch  Algorithm 8 continued. This is used by Algorithm 6 and 7.

1: data structure BatchSketch
   \[ \triangleright \text{ Theorem } 4.21 \]
2: private:
3: procedure QueryxSketch\((v \in S)\)
   \[ \triangleright \text{ Return the value of } \Phi_{\chi(v)}(H_x^{1/2} x) \]
4: return \(-\beta_x \text{sketch}W^\top h.\text{QUERY}(v) - \text{sketch}W^\top \epsilon_x.\text{QUERY}(v) + \beta_x \text{sketch}_c x.\text{QUERY}(v)\)
   \[ \triangleright \text{ Algorithm } 10, 11 \]
5: end procedure
6: procedure QueryySketch\((v \in S)\)
   \[ \triangleright \text{ Return the value of } \Phi_{\chi(v)}(H_x^{-1/2} s) \]
7: return \(\beta_s \text{sketch}W^\top h.\text{QUERY}(v) + \text{sketch}W^\top \epsilon_s.\text{QUERY}(v) + \text{sketch}H_x^{-1/2} s.\text{QUERY}(v)\)
   \[ \triangleright \text{ Algorithm } 10, 11 \]
8: end procedure
9: public:
10: procedure Queryx\((\ell' \in \mathbb{N}, \epsilon \in \mathbb{R})\)
11: \(L_0 = \{\text{root}(S)\}\)
12: \(S \leftarrow \emptyset\)
13: for \(d = 0 \rightarrow \infty\) do
14: if \(L_d = \emptyset\) then
15: return \(S\)
16: end if
17: \(L_{d+1} \leftarrow \emptyset\)
18: for \(v \in L_d\) do
19: if \(v\) is a leaf node then
20: \(S \leftarrow S \cup \{v\}\)
21: else
22: for \(u\) child of \(v\) do
23: if \(||\text{QueryxSketch}(u) - \text{history}[\ell'].\text{QueryxSketch}(u)||_2 > 0.9\epsilon\) then
24: \(L_{d+1} \leftarrow L_{d+1} \cup \{u\}\)
25: end if
26: end for
27: end if
28: end for
29: end for
30: end procedure
31: procedure Queryy\((\ell' \in \mathbb{N}, \epsilon \in \mathbb{R})\)
32: Same as Queryx, except for replacing QueryxSketch in Line 23 with QueryySketch.
33: end procedure
34: end structure

the data structure in
\[ \tilde{O}(T_H + T_n + T_L + r \cdot (nn_{\max}^2 + T_Z + \text{nnz}(A))) \]
time, where \(r = \Theta(\log^3(m_p) \log(1/\delta))\).

- **Move(\(\beta_x, \beta_s \in \mathbb{R}\))**: Update values of \(\beta_x, \beta_s \in \mathbb{R}\) in \(O(1)\) time. This effectively moves \((x^{(\ell)}, s^{(\ell)})\) to \((x^{(\ell+1)}, s^{(\ell+1)})\) while keeping \(\hat{\pi}^{(\ell)}\) unchanged.

- **Update(\(\delta_{\pi} \in \mathbb{R}^{n_p}, \delta_h, \delta_{\epsilon_x}, \delta_{\epsilon_s} \in \mathbb{R}^{m_p}, \delta_{H_x^{-1/2} x}, \delta_{H_x^{-1/2} s}, \delta_{H_x^{-1/2} c_x} \in \mathbb{R}^{n_p}\))**: Update sketches of
\[ H_{\pi(t)}^{1/2} x^{(t+1)} \] and \[ H_{\pi(t)}^{−1/2} s^{(t+1)} \] by updating sketches of \( W^T h, W^T \epsilon_x, W^T \epsilon_s, H_{\pi}^{1/2} \hat{x}, H_{\pi}^{1/2} s, c_x \). This effectively moves \( x^{(t)} \) to \( x^{(t+1)} \) while keeping \( (x^{(t+1)}, s^{(t+1)}) \) unchanged. Then advance timestamp \( t \).

Each update costs
\[
\tilde{O}(T_{H_{\text{max}}} + T_{\Delta L_{\text{max}}} + T_{\text{mat}}(n_{\text{max}}) + r\eta^2 m_{\text{max}}^2) ||\delta_1||_{2,0} \\
+ r m_{\text{max}} \cdot (||\delta_h||_0 + ||\delta_\epsilon_x||_0 + ||\delta_\epsilon_s||_0) \\
+ r m_{\text{max}} \cdot (||\delta_{H^{1/2} \hat{x}}||_0 + ||\delta_{H^{-1/2} s}||_0 + ||\delta_\epsilon_s||_0))
\]
time.

- **QUERY \( x(\ell' \in \mathbb{N}, \epsilon \in \mathbb{R}) \):** Given timestamp \( \ell' \), return a set \( S \subseteq [n] \) where
  \[
  S \supseteq \{ i \in [n] : ||H_{\pi(\ell')}^{1/2} x_i^{(\ell')} - H_{\pi(\ell')}^{1/2} x_i^{(\ell+1)}||_2 \geq \epsilon \},
  \]
  and
  \[
  |S| = O(\epsilon^{-2}(\ell - \ell' + 1) \sum_{\ell' \leq t \leq \ell} ||H_{\pi(t)}^{1/2} x(t) - H_{\pi(t)}^{1/2} x^{(t+1)}||_2 + \sum_{\ell' \leq t \leq \ell-1} ||\pi(t) - \pi^{(t+1)}||_{2,0})
  \]
  where \( \ell \) is the current timestamp.

For every query, with probability at least \( 1 - \epsilon \), the return values are correct, and costs at most
\[
\tilde{O}(r\eta^2 m_{\text{max}} \cdot (\epsilon^{-2}(\ell - \ell' + 1) \sum_{\ell' \leq t \leq \ell} ||H_{\pi(t)}^{1/2} x(t) - H_{\pi(t)}^{1/2} x^{(t+1)}||_2 + \sum_{\ell' \leq t \leq \ell-1} ||\pi(t) - \pi^{(t+1)}||_{2,0}))
\]
running time.

- **QUERYs(\( \ell' \in \mathbb{N}, \epsilon \in \mathbb{R} \)):** Given timestamp \( \ell' \), return a set \( S \subseteq [n] \) where
  \[
  S \supseteq \{ i \in [n] : ||H_{\pi(\ell')}^{-1/2} s_i^{(\ell')} - H_{\pi(\ell')}^{-1/2} s_i^{(\ell+1)}||_2 \geq \epsilon \}
  \]
  and
  \[
  |S| = O(\epsilon^{-2}(\ell - \ell' + 1) \sum_{\ell' \leq t \leq \ell} ||H_{\pi(t)}^{-1/2} s(t) - H_{\pi(t)}^{-1/2} s^{(t+1)}||_2 + \sum_{\ell' \leq t \leq \ell-1} ||\pi(t) - \pi^{(t+1)}||_{2,0})
  \]
  where \( \ell \) is the current timestamp.

For every query, with probability at least \( 1 - \epsilon \), the return values are correct, and costs at most
\[
\tilde{O}(r\eta^2 m_{\text{max}} \cdot (\epsilon^{-2}(\ell - \ell' + 1) \sum_{\ell' \leq t \leq \ell} ||H_{\pi(t)}^{-1/2} s(t) - H_{\pi(t)}^{-1/2} s^{(t+1)}||_2 + \sum_{\ell' \leq t \leq \ell-1} ||\pi(t) - \pi^{(t+1)}||_{2,0}))
\]
running time.

**Remark 4.22 (Snapshot).** In our data structures, we use persistent data structure techniques (e.g., [DSST89]) to keep a snapshot of the data structure after every update. This allows us to support query to historical data. This incurs an \( O(\log n_{\text{ip}}) = \tilde{O}(1) \) multiplicative factor in all running times, which we ignore in our analysis.
Proof of Theorem 4.21. Correctness: Correctness of UPDATE follows from combining Lemma 4.23 and Lemma 4.29. In the following we focus on correctness of queries.

Let us analyze QUERYx. For $0 \leq t \leq q$, define $y(t) = H_{\mathbf{t}(t)}^{1/2} x(t)$, $z(t) = H_{\mathbf{t}(t)}^{1/2} x(t+1)$. By Johnson-Lindenstrauss (Lemma 3.6), for our choice of $r$, with probability $1 - \delta_{\text{apx}}/\poly(n_{\text{lp}})$, all sketches are 0.01-accurate, i.e.,

$$0.99 \leq \frac{\| \Phi_{X(u)} y(t')_{X(u)} - \Phi_{X(u)} z(t')_{X(u)} \|_2}{\| y(t')_{X(u)} - z(t')_{X(u)} \|_2} \leq 1.01$$

for all $u \in V(S)$.

If for some $i \in [n]$, $\| y_i(t') - z_i(t') \|_2 \geq \epsilon$, then all its ancestors $u$ in the partition tree all satisfy $\| y_{X(u)}(t') - z_{X(u)}(t') \|_2 \geq \epsilon$ and $\| \Phi_{X(u)} (y_{X(u)}(t') - z_{X(u)}(t')) \| \geq 0.9\epsilon$. So $i \in S$ with high probability. This proves that $S \geq \{ i \in [n] : \| y_i(t') - z_i(t') \|_2 \geq \epsilon \}$ with high probability.

Now let us bound the size of $S$. There are two types of elements in $S$. The first type comes from changes in $\overline{v}$. Let $T = \{ i : i \in S, \overline{v}_i(t') \neq \overline{v}(t+1) \text{ for some } t' \leq t \leq \ell - 1 \}$. Then $|T| \leq \sum_{0 \leq t \leq \ell - 1} \| \overline{v}(t) - \overline{v}(t+1) \|_2$.

The second type is blocks $i \in [n]$ where $\overline{v}$ does not change. Define $U = S - T$. For all $i \in [n]$ with $\| y_i(t') - z_i(t') \|_2 \leq 0.8\epsilon$, with high probability $i \notin S$. So with high probability, all $i \in S$ satisfy $\| y_i(t') - z_i(t') \|_2 \geq 0.8\epsilon$. Then we have

$$(0.8\epsilon)^2 |U| \leq \sum_{i \in U} \| z_i(t') - y_i(t') \|_2^2 \leq (\ell - \ell') + \sum_{0 \leq t \leq \ell} \sum_{i \in U} \| z_i(t') - y_i(t') \|_2^2 \leq (\ell - \ell') + \sum_{0 \leq t \leq \ell} \| z(t) - y(t) \|_2^2.$$ 

So $|U| = O(\epsilon^{-2} \sum_{0 \leq t \leq \ell} \| z(t) - y(t) \|_2^2)$. This proves that $|S| = |T| + |U| \leq O(\epsilon^{-2} \sum_{0 \leq t \leq \ell} \| z(t) - y(t) \|_2^2 + \sum_{0 \leq t \leq \ell - 1} \| \overline{v}(t) - \overline{v}(t+1) \|_2^2).$ 

Proof of correctness of QUERYs is similar and omitted.

Running time:

**Initialize:** Follows from Lemma 4.23 and Lemma 4.30.

**Update:** Follows from Lemma 4.23 and Lemma 4.31.

**QUERYx:** Let us bound the number of calls to QUERYxSKETCH. Let $C = \epsilon^{-2} \sum_{0 \leq t \leq \ell} \| z(t) - y(t) \|_2^2 + \sum_{0 \leq t \leq \ell - 1} \| \overline{v}(t) - \overline{v}(t+1) \|_2^2$ for simplicity. For every level $d$, there can be at most $O(C)$ vertices $u$ in level $d$ such that $\| y_{X(u)}(t') - z_{X(u)}(t') \| \geq \epsilon/2$ by similar discussion as the correctness analysis, because all such $u$'s have disjoint $X(u)$. So $L_d = O(C)$. By our assumption that the partition tree has $O(1)$ depth, the total size of all $L_d$ satisfies $\sum_{d \geq 0} |L_d| = O(C)$. So the number of calls to QUERYxSKETCH is at most $O(C)$.

Then the result follows from Lemma 4.23 and Lemma 4.35.

**QUERYs:** Proof is similar to QUERYx and is omitted. \qed
4.4.4 **BlockVectorSketch**

In this section we present BlockVectorSketch (Algorithm 10), data structure for maintaining a vector with block structure under sparse changes. This is generalization of VectorSketch in [DLY21].

**Lemma 4.23.** Given a partition tree \((S, \chi)\) of \(\mathbb{R}^n\), and a JL sketching matrix \(\Phi \in \mathbb{R}^{r \times n_p}\), the data structure BlockVectorSketch (Algorithm 10) maintains \(\Phi \chi(x(v))x(v)\) for all nodes \(v\) in the partition tree implicitly through the following functions:

- **Initialize** \((S, \chi, \Phi)\): Initializes the data structure in \(O(rn_p)\) time.
- **Update** \((\delta_x \in \mathbb{R}^{n_p})\): Maintains the data structure for \(x \leftarrow x + \delta_x\) in \(O(r\|\delta_x\|_0 \log n)\) time.
- **Query** \((v \in V(S))\): Outputs \(\Phi \chi(x(v))x(v)\) in \(O(r \log n)\) time.

Note that in Theorem 4.23, the running time does not depend on depth of the partition tree.

**Proof.** Correctness follows from from guarantees of segment trees.

Running time analysis is as follows.

- **Initialize**: Computing \(\Phi_i x_i\) for all \(i \in [n]\) takes \(O(rn_p)\) time. Afterwards, building \(T\) takes \(O(n)\) time.

- **Update**: For every modified coordinate, it takes \(O(r\|\delta_x\|_0 \log n)\) time to update \(T\). So total time cost is \(O(r\|\delta_x\|_0 \log n)\).

- **Query**: Takes \(O(r \log n)\) time because of segment tree query time.

\(\square\)

4.4.5 **BlockBalancedSketch**

In this section we present BlockBalancedSketch, data structure for maintaining a sketch of a vector of form \(W^\top h\), where \(W = L^{-1} A H^{-1/2}\) and \(h \in \mathbb{R}^{m_p}\) is a vector changing sparsely.

**Lemma 4.24.** Given the constraint matrix \(A\), a block elimination tree \(T\) with height \(\eta\), a JL matrix \(\Phi \in \mathbb{R}^{r \times n_p}\), and a partition tree \((S, \chi)\) constructed as in Definition 4.26 with height \(\tilde{O}(1)\), the data structure BlockBalancedSketch (Algorithm 11, 12, 13), maintains \(\Phi \chi(x(v)) (W^\top h) \chi(v)\) for each \(v \in V(S)\) through the following operations

- **Initialize** \((S, \chi, \Phi, \bar{x}, h)\): Initializes the data structure in

\[\tilde{O}(T_H + T_n + T_L + r \cdot (nn_{\max}^2 + T_Z + nnz(A)))\]

\(\tilde{O}\) time.

- **Update** \((\delta_{\bar{x}}, \delta_{h})\): Updates all sketches in \(S\) implicitly to reflect \((W, h)\) updating to \((W_{\text{new}}, h_{\text{new}})\) in

\[\tilde{O}((T_{H,\max} + T_{\Delta L,\max} + T_{\text{mat}}(n_{\max}) + rm_{\max})^2m_{\max}^2, 0 + rm_{\max}\|\delta_{h}\|_2, 0)\]

\(\tilde{O}\) time.

- **Query** \((v)\): Outputs \(\Phi \chi(x(v)) (W^\top h) \chi(v)\) in \(\tilde{O}(r^2m_{\max}^2)\) time.

41
Algorithm 10 This is used in Algorithm 8 and 9.

1: **data structure** BlockVectorSketch $\triangleright$ Lemma 4.23
2: **private: members**
3: $\Phi \in \mathbb{R}^{r \times n lp}$
4: partition tree $(S, \chi)$
5: $x \in \mathbb{R}^{n lp}$
6: Segment tree $T$ on $[n]$ with values in $\mathbb{R}$
7: **end members**
8: **procedure** INITIALIZE$(S, \chi)$: partition tree, $\Phi \in \mathbb{R}^{r \times n lp}, x \in \mathbb{R}^{n lp}$
9: $(S, \chi) \leftarrow (S, \chi)$
10: $\Phi \leftarrow \Phi$
11: $x \leftarrow x$
12: Order leaves of $S$ (variable blocks) such that every node $\chi(v)$ corresponds to a contiguous interval $\subseteq [n]$.
13: Build a segment tree $T$ on $[n]$ such that each segment tree interval $I \subseteq [n]$ maintains
14: $\Phi_{I} x_{I} \in \mathbb{R}$.
15: **end procedure**
16: **procedure** UPDATE$(\delta x \in \mathbb{R}^{n lp})$
17: for all $i \in [n lp]$ such that $\delta x_{\text{coord } i} \neq 0$ do
18: Let $j \in [n]$ be such that $i$ is in $j$-th block
19: Update $T$ at $j$-th coordinate $\Phi_{j} x_{j} \leftarrow \Phi_{j} x_{j} + \Phi_{\text{coord } i} \cdot \delta x_{\text{coord } i}$
20: $x_{\text{coord } i} \leftarrow x_{\text{coord } i} + \delta x_{\text{coord } i}$
21: **end for**
22: **end procedure**
23: **procedure** QUERY$(v \in V(S))$
24: Find interval $I$ corresponding to $\chi(v)$
25: return range sum of $T$ on interval $I$
26: **end procedure**
27: **end data structure**

The algorithm is a block version of [DLY21, Section 6.6.1]. We nevertheless present a full proof for completeness.

Let us first describe the main idea of the algorithm. Note that in BlockBalancedSketch, we have the freedom of choosing the partition tree $(S, \chi)$. Then this tree is used by BatchSketch and BlockVectorSketch. Therefore we can choose a partition tree which works well for our purpose.

Let us consider the operations BlockBalancedSketch needs to support. It needs to support updating $\overline{x}$ and $h$, and answering queries on a subtree $\chi(v)$. Change of one block in $\overline{x}$ leads to change of one path in the block elimination tree $T$, and change of one block in $h$ leads to change of one subtree in $T$. Therefore we essentially would like a data structure which supports subtree and path updates and subtree queries. With this in mind, it is natural to use heavy-light decomposition [ST81].

**Lemma 4.25** (Heavy-Light Decomposition [ST81]). Given a rooted tree $T$ with $m$ vertices, we can construct in $O(m)$ time an ordering $\pi$ of the vertices such that (1) every path in $T$ can be decomposed into $O(\log m)$ contiguous subsequences under $\pi$, and (2) every subtree in $T$ is a single contiguous subsequence under $\pi$. 
Algorithm 11 This is used in Algorithm 8 and 9.

1: **data structure** BlockBalancedSketch \(\triangleright\) Lemma 4.24
2: **private: members**
3: \(\Phi \in \mathbb{R}^{r \times n_p}\)
4: Partition tree \((S, \chi)\) with balanced binary tree \(B\)
5: \(t \in \mathbb{N}\)
6: \(h \in \mathbb{R}^{m_p}, \tau \in \mathbb{R}^{n_p}, H_{\tau} \in \mathbb{R}^{n_p \times n_p}\)
7: \(\{L[t] \in \mathbb{R}^{m_p \times m_p}\}_{t \geq 0}\)
8: \(\{J_v \in \mathbb{R}^{r \times m_p}\}_{v \in S}\)
9: \(\{Z_v \in \mathbb{R}^{r \times m_p}\}_{v \in B}\)
10: \(\{y_v^\triangledown \in \mathbb{R}^r\}_{v \in B}\)
11: \(\{t_v \in \mathbb{N}\}_{v \in B}\)
12: **end members**
13: **procedure** Initialize \((S, \chi : \text{partition tree}, \Phi \in \mathbb{R}^{r \times n_p}, \tau \in \mathbb{R}^{n_p}, h \in \mathbb{R}^{m_p})\) \(\triangleright\) Lemma 4.30
14: \((S, \chi) \leftarrow (S, \chi)\)
15: \(\Phi \leftarrow \Phi\)
16: \(t \leftarrow 0, h \leftarrow h\)
17: Compute \(H_{\tau} \leftarrow \nabla^2 \phi(\tau)\)
18: Find lower Cholesky factor \(L[t] \in AH_{\tau}^{-1}A^\top\)
19: **for** all \(v \in S\) **do**
20: \(J_v \leftarrow \Phi_{\chi(v)} H_{\tau}^{-1/2} A^\top\)
21: **end for**
22: **for** all \(v \in B\) **do**
23: \(Z_v \leftarrow J_v L[t]^{-\top}\)
24: \(y_v^\triangledown \leftarrow Z_v (I - I_{\Lambda(v)}) h\)
25: \(t_v \leftarrow t\)
26: **end for**
27: **end procedure**
28: **procedure** Query \((v \in S)\) \(\triangleright\) Lemma 4.35
29: **if** \(v \in S \setminus B\) **then**
30: **return** \(J_v \cdot L[t]^{-\top} h\)
31: **end if**
32: \(\Delta_{L[t]} \leftarrow (L[t] - L[t]_{t_v}) \cdot I_{\Lambda(v)}\)
33: \(\delta_{Z_v} \leftarrow - (L[t]_{t_v}^{-1} \cdot \Delta_{L} \cdot Z_v)^\top\)
34: \(Z_v \leftarrow Z_v + \delta_{Z_v}\)
35: \(\delta_{y_v^\triangledown} \leftarrow \delta_{Z_v} \cdot (I - I_{\Lambda(v)}) h\)
36: \(y_v^\triangledown \leftarrow y_v^\triangledown + \delta_{y_v^\triangledown}\)
37: \(t_v \leftarrow t\)
38: \(y_v^\triangledown \leftarrow Z_v \cdot I_{\Lambda(v)} \cdot h\)
39: **return** \(y_v^\triangledown + y_v^\triangledown\)
40: **end procedure**
41: **end data structure**

**Definition 4.26** (Construction of Partition Tree). We fix an ordering \(\pi\) of \([m]\) using the heavy-light decomposition (Lemma 4.25). We construct complete binary tree \(B\) with leaves \([m]\) and ordering \(\pi\).

To get a partition tree, we need to add leaves \([n]\) to \(B\). For every coordinate \(i \in [n]\), let \(\text{low}^\top (i)\) be any vertex \(v\) in \(T\) such that the support of \(A_{*,i}\) is contained in \(P^T(v)\). (Recall that for a block...
Algorithm 12 BlockBalancedSketch Algorithm 11 continued. This is used in Algorithm 8 and 9.

1: data structure BlockBalancedSketch
2: procedure Update($\delta X \in \mathbb{R}^{m \times n}$, $\delta h \in \mathbb{R}^{m \times p}$)
3:   for $i \in [n]$ where $\delta X_{\cdot i} \neq 0$
4:     Update($\delta X_{\cdot i}$)
5:   end for
6:   for all $\delta h_{\cdot i} \neq 0$
7:     $v \leftarrow \Lambda^o(i)$
8:     for all $u \in \mathcal{P}^B(v)$ do
9:       $y_u \leftarrow y_u + Z_u \cdot I(i) \cdot \delta h$
10:   end for
11: $h \leftarrow h + \delta h$
12: end procedure
13: end data structure

elimination tree, support of $A^\ast_{\cdot i}$ is contained in a path for any $i \in [n]$.) For any $j \in [m]$, we construct a complete binary tree with leaves $\{i \in [n] : \text{low}^T(i) = j\}$ and hang this tree under leaf $j$ in $\mathcal{B}$. This finishes the construction of a partition tree $(\mathcal{S}, \chi)$.

As a sanity check, height of $\mathcal{B}$ is $O(\log m)$, and height of each subtree under leaf $j \in \mathcal{B}$ is $O(\log n)$. So height of $\mathcal{S}$ is $O(\log m + \log n) = O(\log n_{lp}) = \tilde{O}(1)$. Also, from the definition, we see that construction of the partition tree is cheap, i.e., takes $O(\text{nnz}(A))$ time.

The following definitions come from [DLY21].

Definition 4.27. We make the following definitions. For $v \in \mathcal{B}$, define

\[
\Lambda(v) := \left( \bigcup_{i \in \overline{\chi}(v)} \mathcal{P}^T(i) \right) \cap \left( \bigcup_{i \in \mathcal{T} \setminus \overline{\chi}(v)} \mathcal{P}^T(i) \right),
\]

\[
\overline{\Lambda}(v) := \left( \bigcup_{i \in \overline{\chi}(v)} \mathcal{P}^T(i) \right) \setminus \left( \bigcup_{i \in \mathcal{T} \setminus \overline{\chi}(v)} \mathcal{P}^T(i) \right),
\]

where $\overline{\chi}(v)$ is the set of leaves in $\mathcal{B}$ which are descendants of $v$.

For $u \in \mathcal{T}$, define $\Lambda^o(u)$ be the lowest vertex $v \in \mathcal{B}$ such that $u \in \overline{\Lambda}(v)$. In other words, $\Lambda^o(u)$ is the lowest vertex $v \in \mathcal{B}$ such that $\overline{\Lambda}(v)$ contains $D^T(u)$ (set of descendants of $u$ in $\mathcal{T}$). Therefore $\Lambda^o(u)$ is well-defined.
Algorithm 13 BlockBalancedSketch Algorithm 11, 12 continued. This is used in Algorithm 8 and 9.

1: \textbf{data structure} BlockBalancedSketch \quad \triangleright \text{Lemma 4.24}
2: \textbf{private:}
3: \textbf{procedure} \text{UpdateL}(S \subseteq B, \Delta L_{\pi} \in \mathbb{R}^{m \times m}) \quad \triangleright \text{Lemma 4.33}
4: \quad \textbf{for all} \ v \in S \ \textbf{do}
5: \quad \quad \delta_{Z_v} \leftarrow -(L_{\pi}[t - 1] \cdot I_{\pi(t_v)} \cdot Z_v^T)^T
6: \quad \quad \delta_{Z_v} \leftarrow -(L_{\pi}[t - 1] \cdot \Delta L_{\pi} \cdot (Z_v + \delta_{Z_v})^T)^T
7: \quad \quad Z_v \leftarrow Z_v + \delta_{Z_v} + \delta_{Z_v}^T
8: \quad \quad y_v^T \leftarrow (\delta_{Z_v} + \delta_{Z_v}^T)(I - I_{\pi(v)})h
9: \quad \quad t_v \leftarrow t
10: \quad \textbf{end for}
11: \textbf{end procedure}
12: \textbf{private:}
13: \textbf{procedure} \text{UpdateH}(i \in [n], \Delta H_{\pi(i,i)} \in \mathbb{R}^{n 	imes n}) \quad \triangleright \text{Lemma 4.34}
14: \quad \text{Find} \ u \ \text{such that} \ \chi(u) = \{i\}
15: \quad \Delta H_{\pi}^{-1/2}(i,i) \leftarrow (H_{\pi(i,i)} + \Delta H_{\pi(i,i)})^{-1/2} - H_{\pi(i,i)}^{-1/2}
16: \quad \delta J_u \leftarrow \Phi_i \cdot \Delta H_{\pi}^{-1/2}(i,i) \cdot A^T
17: \quad \textbf{for all} \ v \in \mathcal{P}S(u) \ \textbf{do}
18: \quad \quad J_v \leftarrow J_v + \delta J_u
19: \quad \quad \textbf{if} \ v \in B \ \textbf{then}
20: \quad \quad \quad \delta_{Z_v} \leftarrow \delta J_u \cdot L_{\pi}[t_v]^T
21: \quad \quad \quad Z_v \leftarrow Z_v + \delta_{Z_v}
22: \quad \quad \quad \delta y_v^T \leftarrow \delta_{Z_v} \cdot (I - I_{\pi(v)}) \cdot h
23: \quad \quad \quad y_v^T \leftarrow y_v^T + \delta y_v^T
24: \quad \quad \textbf{end if}
25: \quad \textbf{end for}
26: \quad H_{\pi} \leftarrow H_{\pi} + \Delta H_{\pi(i,i)}
27: \textbf{end procedure}
28: \textbf{end data structure}

Lemma 4.28. For any \ v, \ \Lambda(v) \ is contained in the union of two paths in \ T. In particular, \ |\Lambda(v)| = O(\eta).

\textbf{Proof.} The order \ \pi \ in Lemma 4.25 \ is a pre-order traversal of \ T. Let \ u \ be the last vertex before \ \chi(v) \ under \ \pi, \ and \ w \ be the first vertex after \ \chi(v) \ under \ \pi. \ Then \ \Lambda(v) \ is contained in \ \mathcal{P}T(u) \cup \mathcal{P}T(w). \ \square

Lemma 4.29. BlockBalancedSketch correctly maintains a sketch of \ W^T h, and all query results are returned correctly.

\textbf{Proof.} We prove that the following invariant always holds after every call to BlockBalancedSketch.

\begin{align*}
J_v &= \Phi_{\chi(v)}H_{\pi}^{-1/2} A^T \quad \forall v \in S, \\
Z_v &= J_v L_{\pi}[t_v] - \tau \quad \forall v \in B, \\
y_v^T &= Z_v (I - I_{\pi(v)}) h \quad \forall v \in B,
\end{align*}

(10) (11) (12)
\[ 0 = (L[t] - L[t_v]) \cdot I_{\overline{\Phi}(v)} \quad \forall v \in \mathcal{B}. \tag{13} \]

**Initialize**: The invariants are clearly satisfied after initialization.

**Query**: If \( v \in \mathcal{S} \setminus \mathcal{B} \) then we compute \( J_v L_{\overline{\Pi}(v)}^{-1} h \) directly and the result is correct.

Now assume \( v \in \mathcal{B} \). We update \( t_v \leftarrow t \), and update \( Z_v \) and \( y_v^\top \) accordingly. We have that

\[
\delta_{Z_v} = -(L_{\overline{\Pi}(v)}^{-1}(L_{\overline{\Pi}(v)} - L_{\overline{\Pi}(t_v)})I_{\chi(u)} Z_v^\top)^	op
\]

\[
= -(L_{\overline{\Pi}(v)}^{-1}(L_{\overline{\Pi}(v)} - L_{\overline{\Pi}(t_v)}))(I - I_{\overline{\Phi}(v)}) - I_{\overline{\Phi}(v)} Z_v^\top)^	op
\]

because Invariant (13) and column sparsity of \( Z_v \). So

\[
Z_v^{\text{new}} = Z_v + \delta_{Z_v}
\]

\[
= Z_v - (L_{\overline{\Pi}(v)}^{-1}(L_{\overline{\Pi}(v)} - L_{\overline{\Pi}(t_v)})Z_v^\top)^	op
\]

\[
= J_v L_{\overline{\Pi}(t_v)}^{-1} - J_v (L_{\overline{\Pi}(t_v)}^{-1} - L_{\overline{\Pi}(v)}^{-1})
\]

\[
= J_v L_{\overline{\Pi}(t_v)}^{-1}. \]

So Invariant (11) is satisfied. Updating \( y_v^\top \) ensures that Invariant (12) is satisfied. Finally, the return value is correct because of definition of \( y_v^\top \) and \( y_v^\top \).

**Update**: We divide the proof into several steps. Correctness of UpdateB follows from correctness of UpdateL and UpdateH (which we will prove below).

**Correctness of UpdateH**: We update \( t_v, Z_v \) and \( y_v^\top \) for \( v \in S = \mathcal{P}^B(\Lambda^\circ(\text{low}^\top(i))) \). In other words, \( S \) is the set of all vertices \( v \) with \( \text{low}^\top(i) \in \overline{\Phi}(v) \). For any \( v \notin S \), \( L[t] \cdot I_{\overline{\Phi}(v)} \) is not changed.

So Invariant (13) is preserved for \( v \notin S \).

Fix \( v \in S \). In Algorithm 13, Line 5, we update \( t_v \) to \( t - 1 \). In Algorithm 13, Line 6, we update \( t_v \) to \( t \). By a similar computation as the one we did for Query, \( Z_v \) is updated correctly (i.e., Invariant (11) is preserved). This implies \( y_v^\top \) is updated correctly (i.e., Invariant (12) is preserved).

**Correctness of UpdateL**: We update \( J_v, Z_v, y_v^\top \) for all \( v \in \mathcal{P}^S(u) \) (where \( \chi(u) = \{i\} \)). Invariant (13) is preserved because \( t_v \) does not change. Invariant (10), (11), (12) are preserved by our choice of \( \delta_{J_v}, \delta_{Z_v}, \delta_{y_v^\top} \).

**Correctness of Algorithm 12**: Line 6 to Line 11: This part is “Updateh”. For \( i \in [m] \) with \( \delta_{h,i} \neq 0 \), we update \( y_v^\top \) for \( u \in \mathcal{B} \) such that \( i \in \overline{\Phi}(u) \). Recall that \( \mathcal{P}^B(\Lambda^\circ(i)) \) contains all vertices \( u \) such that \( i \in \overline{\Phi}(u) \). So Invariant (12) is satisfied.

**Lemma 4.30**: BlockBalancedSketch.Initialize (Algorithm 11) costs \( \tilde{O}(T_H + T_n + T_L + r \cdot (nn_{\text{max}}^2 + T_Z + \text{nnz}(A))) \) time.

**Proof**: Computing \( H_{\overline{\Pi}} \) takes \( T_H \) time. Computing \( H_{\overline{\Pi}}^{-1}, H_{\overline{\Pi}}^{-1/2} \) takes \( O(T_n) \) time. Computing \( L_{\overline{\Pi}(v)} \) takes \( T_L \) time.

Computation of \( J_v \): For all \( i \in [n] \), computing \( \Phi_i H_{\overline{\Pi}(v)}^{-1/2} \) takes \( r n_i^2 \) time. Support of \( \Phi_i H_{\overline{\Pi}(v)}^{-1/2} \) are all disjoint, so computing all \( \Phi_i H_{\overline{\Pi}(v)}^{-1/2} A^\top \) takes \( r \cdot \text{nnz}(A) \) time. Then we can compute all \( J_v \) by summing from bottom to up. Height of the tree is \( \tilde{O}(1) \), so every non-zero entry of \( \Phi_i H_{\overline{\Pi}(v)}^{-1/2} A^\top \) gets propagated \( \tilde{O}(1) \) times. So computing \( J_v \) takes

\[
\tilde{O}(r(\sum_{i \in [n]} n_i^2 + \text{nnz}(A))) = \tilde{O}(r(nn_{\text{max}}^2 + \text{nnz}(A)))
\]
time in total.

Computation of \(Z_v\): To compute \(Z_v\), we first compute \(Z_v\) for all leaves \(v \in \mathcal{B}\). This takes \(rT_Z\) time by assumption (Definition 4.2). Then we sum from bottom to up to compute \(Z_v\) for all \(v \in \mathcal{B}\). Because height of the partition tree is \(O(1)\), every non-zero entry in the leaves gets propagated \(\tilde{O}(1)\) times. So computing \(Z_v\) takes \(\tilde{O}(rT_Z)\) time in total.

Computation of \(y^v_u\): Takes \(O(\text{nnz}(Z)) = \tilde{O}(rT_Z)\) time.

Summing everything up we get the desired running time. \(\square\)

Lemma 4.31. BlockBalancedSketch.Update (Algorithm 12) costs
\[
\tilde{O}((T_{H,\max} + T_{\Delta L,\max} + \mathcal{T}_{\text{mat}}(n_{\max}) + r\eta^2 m_{\max}^2)\|\delta\|_{2,0} + rm_{\max}\|\delta_h\|_{2,0})
\]
time.

Proof. Each call to Update\(\overline{x}\) costs
\[
\tilde{O}(T_{H,\max} + T_{\Delta L,\max} + \mathcal{T}_{\text{mat}}(n_{\max}) + r\eta^2 m_{\max}^2)
\]
time by Lemma 4.32.

For each \(i \in [n]\) with \(\delta_{h,i} \neq 0\) and each \(v\), it takes \(O(rm_{\max})\) time to update \(y^v_u\). So the total time needed to update \(y^v_u\) is \(O(rm_{\max}\|\delta_h\|_{2,0})\). \(\square\)

Lemma 4.32. BlockBalancedSketch.Update\(\overline{x}\) (Algorithm 12) costs
\[
\tilde{O}(T_{H,\max} + T_{\Delta L,\max} + \mathcal{T}_{\text{mat}}(n_{\max}) + r\eta^2 m_{\max}^2)
\]
time.

Proof. Computing \(\Delta_{H,\overline{x}}\) takes \(O(T_{H,\overline{x}})\) time. Computing \(\Delta_{L,\overline{x}[t]}\) takes \(O(T_{\Delta L,\overline{x}[t]})\) time. Because \(S\) lies on a path in \(\mathcal{B}\), we have \(|S| = O(1)|.\) By Lemma 4.34 and Lemma 4.33, updating \(L\) costs \(O(r\eta^2 m_{\max}^2\log n)\) time and updating \(H\) costs \(O(\mathcal{T}_{\text{mat}}(n_{\max}) + r\eta^2 m_{\max}^2\log n)\) time. \(\square\)

Lemma 4.33. BlockBalancedSketch.Update\(L\) (Algorithm 13) costs \(O(r\eta^2 m_{\max}^2)\) time.

Proof. By Lemma 4.28, we have \(|\Lambda(v)| = O(\eta)|.\) So we can compute \((L_{\overline{x}[t-1]} - L_{\overline{x}[t_0]}) \cdot I_{\Lambda(v)}\) in \(O(\eta^2 m_{\max}^2)\) time by Lemma 4.7(ii). Then computing \((L_{\overline{x}[t-1]} - L_{\overline{x}[t_0]}) \cdot I_{\Lambda(v)} \cdot Z_v^\top\) takes \(O(r\eta^2 m_{\max}^2)\) time. Finally, computing \(\delta_{Z_v} = L_{\overline{x}[t-1]} - L_{\overline{x}[t_0]} \cdot Z_v^\top\) takes \(O(r\eta^2 m_{\max}^2)\) time by Lemma 4.7(iv). Analysis for \(\delta_{Z_v}^\top\) is the same.

Computing \(\delta_{y_v}^\top\) takes \(O(r\eta m_{\max})\) time by sparsity pattern of \(\delta_{Z_v} + \delta_{Z_v}^\top\).

Summing everything up we get the desired running time. \(\square\)

Lemma 4.34. BlockBalancedSketch.Update\(H\) (Algorithm 13) costs \(\tilde{O}(\mathcal{T}_{\text{mat}}(n_{\max}) + r\eta^2 m_{\max}^2)\) time.

Proof. Computing \(\Delta_{H,1/2}\) takes \(\mathcal{T}_{\text{mat}}(n_k)\) time. Computing \(\delta_{J_u}\) takes \(O(r\eta m_{\max}^2)\) by sparsity pattern of \(A\). Furthermore, (each row of) \(\delta_{J_u}\) is supported on a path in \(\overline{T}\). Therefore, for every \(v\), computing \(\delta_{Z_v}\) takes \(O(r\eta^2 m_{\max}^2)\) time by Lemma 4.7(iv). Afterwards, computing \(\delta_{y_v}\) takes \(O(r\eta m_{\max})\) time by sparsity pattern of \(\delta_{Z_v}\). Finally, \(|\mathcal{P}^\mathcal{S}(u)| = O(1)|,\) so the overall running time is \(\tilde{O}(\mathcal{T}_{\text{mat}}(n_{\max}) + r\eta^2 m_{\max}^2)\).

\(\square\)

Lemma 4.35. BlockBalancedSketch.Query (Algorithm 11) takes \(O(r\eta^2 m_{\max}^2)\) time.
Proof. If \( v \in S \setminus B \), then (each row of) \( J_v \) is supported on a path. So computing \( J_v L_{\pi}[t]^{-\top} h \) takes \( O(r\eta^2 m_{\max}^2) \) time by Lemma 4.7(iv).

Now suppose \( v \in B \). By Lemma 4.28, we have \( |\Lambda(v)| = O(\eta) \). So we can compute \( (L_{\pi}[t] - L_{\pi}[t_v]) \cdot I_{\Lambda(v)} \) in \( O(\eta^2 m_{\max}^2) \) time by Lemma 4.7(ii). Then computing \( \Delta L_{\pi} \cdot Z_v^\top \) takes \( O(r\eta^2 m_{\max}^2) \) time. Furthermore, \( \Delta L_{\pi} \) has columns supported on two paths. Therefore, computing \( L_{\pi}[t]^{-\top} \cdot \Delta L_{\pi} \cdot Z_v^\top \) takes \( O(r\eta^2 m_{\max}^2) \) time by Lemma 4.7(iv). So the total time needed to compute \( \delta_Z v \) is \( O(r\eta^2 m_{\max}^2) \).

Computing \( y_v^\Delta \) takes \( O(\eta m_{\max}) \) time by sparsity pattern of \( Z_v \). Computing \( y_v^\Delta \) time because \( |\Lambda(v)| = O(\eta) \).

Summing everything up we get the desired running time.

Combining everything we finish the proof of Lemma 4.24.

\( \square \)

\textbf{Proof of Lemma 4.24.} Combining Lemma 4.29, Lemma 4.30, Lemma 4.31, Lemma 4.35.

\( \square \)

\subsection{Analysis of CentralPathMaintenance}

The goal of this section is to prove Theorem 4.4.

We first prove correctness of CentralPathMaintenance.

\textbf{Lemma 4.36 (Correctness of CentralPathMaintenance).} \textit{Algorithm 2 implicitly}

\[
x \leftarrow x + H^{\gamma^2/2}(I - P_{\pi}) H^{-1/2} \delta_{\mu}(\pi, \pi, \pi)
\]

\[
s \leftarrow s + t \cdot H^{\gamma^2/2} P_{\pi} H^{-1/2} \delta_{\mu}(\pi, \pi, \pi)
\]

where \( H_{\pi} := \nabla^2 \phi(\pi) \in \mathbb{R}^{n_p \times n_p}, P_{\pi} := H_{\pi}^{-1/2} A^\top (AH_{\pi}^{-1} A^\top)^{-1} A H_{\pi}^{-1/2} \in \mathbb{R}^{n_p \times n_p} \), and \( \pi \) is some earlier timestamp satisfying \( |t - \pi| \leq \epsilon t \cdot \pi \).

It also explicitly maintains \( (\pi, \pi) \in \mathbb{R}^{n_p \times n_p} \) such that \( \|\pi_i - x_i\|_2 \leq \pi \) and \( \|\pi_i - s_i\|_2^\pi \leq \epsilon w \) for all \( i \in [n] \) with probability at least 0.9.

Proof. We correctly maintain a multiscale representation of \( (x, s) \) because of correctness of \texttt{exact.Update} (Lemma 4.12).

We show that \( \|\pi_i - x_i\|_2 \leq \pi \) and \( \|\pi_i - s_i\|_2^\pi \leq \epsilon w \) for all \( i \in [n] \) (c.f. Algorithm 20, Line 24).

\texttt{approx} maintains an \( \ell_2, \infty \)-approximation of

\[
H_{\pi}^{\gamma^2/2} x = H_{\pi}^{\gamma^2/2} \bar{x} + \beta z c_x - W^\top (\beta z h + \epsilon x).
\]

For \( \ell \leq q \), we have

\[
\|H_{\pi}^{\gamma^2/2} x^{(\ell+1)} - H_{\pi}^{\gamma^2/2} x^{(\ell)}\|_2 = \|H_{\pi}^{\gamma^2/2} (x^{(\ell+1)} - x^{(\ell)})\|_2 = w^{-1/2} \|\delta_{\sigma}^{(\ell)}\|_2
\]

\[
\leq \frac{9}{8} \alpha w^{-1/2}
\]

\[
\leq \zeta_x,
\]

where the second step follows from definition of \( \|\cdot\|_2 \), the third step follows from Lemma A.4, and the last step follows from our choice of \( \zeta_x \).

By Theorem 4.18, with probability at least \( 1 - \delta_{\text{apx}} \), \texttt{approx} correctly maintains \( \bar{x} \) such that

\[
\|\bar{x} - H_{\pi}^{\gamma^2/2} x\|_{2, \infty} \leq \epsilon_{\text{apx}, x} = \pi.
\]
We set \( \overline{x} = H_{\overline{x}}^{-1/2} \bar{x} \), so
\[
\|H_{\overline{x}}^{1/2}(\overline{x} - x)\|_{2,\infty} \leq \tau. \quad (14)
\]
Therefore
\[
\|\overline{x}_i - x_i\|_{\overline{x}_i} = \|H_{\overline{x}_i(i,i)}^{1/2}(\overline{x}_i - x_i)\|_2 \\
\leq \|H_{\overline{x}}^{1/2}(\overline{x} - x)\|_{2,\infty} \\
\leq \tau,
\]
where the first step follows from definition of \( \|\cdot\|_{\overline{x}_i} \), the second step follows from definition of \( \|\cdot\|_{2,\infty} \), the third step follows from (14).

The proof for \( s \) is similar. We have
\[
\|H_{\overline{x}}^{-1/2} \delta_s^{(t)}\|_2 \leq w^{1/2} \|\delta_s^{(t)}\|_{\overline{x}} \leq \frac{9}{8} \alpha \bar{t} w^{1/2} \leq \zeta_s,
\]
and
\[
\|H_{\overline{x}}^{-1/2}(s - \overline{x})\|_{2,\infty} \leq \epsilon_{\text{apx},s} = \frac{1}{2} \bar{t} \cdot \ell \cdot w.
\]

Now we prove running time claims in Theorem 4.4.

**Lemma 4.37** (INITIALIZE part of Theorem 4.4). CENTRALPATHMAINTENANCE.INITIALIZE (Algorithm 2) takes \( \widetilde{O}(T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_p m_{\text{max}}) \) time.

**Proof.** By Theorem 4.11 and Theorem 4.18.

**Lemma 4.38** (MULTIPLYANDMOVE part of Theorem 4.4). Total running time of MULTIPLYANDMOVE (Algorithm 2) is
\[
\widetilde{O}(q^{0.5} \ell^{0.5} \cdot (T_H + T_L + \eta T_m + T_Z)^{0.5} \\
\cdot (T_{\text{mat}}(n_{\text{max}}) + T_{\Delta L, \text{max}} + T_{H, \text{max}} + \eta^2 m_{\text{max}}^2)^{0.5} \log(t_{\text{max}}/t_{\text{min}})).
\]

**Proof.** The choice of parameters under modification of \( w \) is summarized in Table 4.

Between two restarts, the total size of \(|L_x|\) returned by approx.QUERY is bounded by \( \widetilde{O}(q^2 \zeta^2 / \epsilon_{\text{apx},x}^2) \) by Theorem 4.18. By plugging in \( \zeta_x = 2 \alpha w^{-1/2} \), \( \epsilon_{\text{apx},x} = \bar{t} \), we have
\[
\sum_{\ell \in [q]} |L_x^{(\ell)}| = \widetilde{O}(q^2 w^{-1}).
\]

Similarly, for \( s \), we have
\[
\sum_{\ell \in [q]} |L_s^{(\ell)}| = \widetilde{O}(q^2 \zeta_s^2 / \epsilon_{\text{apx},s}^2) \\
= \widetilde{O}(q^2 \cdot \Theta(\bar{t}^{1/2} \cdot \tilde{t} w^2)^2 / \Theta(\bar{t} \cdot \tilde{t} w)^2) \\
= \widetilde{O}(q^2 w^{-1}),
\]
where the first step is by Theorem 4.18, the second step is by our choice \( \zeta_s = 2 \alpha \bar{t} w^{1/2} \), \( \epsilon_{\text{apx},s} = \bar{t} \).

**Cost per iteration.**

By Theorem 4.11 and Theorem 4.18, in a sequence of \( q \) update/queries,
Table 4: Internal parameters of Theorem 4.3 and their values. Caller has no access to these variables. For simplicity, we ignore the $O()$ and $\Theta()$ in the table.

| Notation | Range | Meaning | Choice |
|----------|-------|---------|--------|
| $w$      | $\mathbb{R}_{>1}$ | weight | $\nu_{\max}$ |
| $N$      | $\mathbb{N}_{+}$ | number of central path steps | $\sqrt{\max(n_{\ell p})}$ |
| $q$      | $\mathbb{N}_{+}$ | number of steps before restart | See Eq. (15) |
| $r$      | $\mathbb{N}_{+}$ | JL dimension | $\log^3(n_{\ell p})$ |
| $\epsilon_{\text{apx},x}$ | $\in \mathbb{R}_{>0}$ | Theorem 4.18 approx parameter | $\tau$ |
| $\epsilon_{\text{apx},s}$ | $\in \mathbb{R}_{>0}$ | Theorem 4.18 approx parameter | $\tau_{tw}$ |
| $\delta_{\text{apx}}$ | $\in (0, 0.1)$ | Theorem 4.18 failure probability | $1/N$ |
| $\eta$   | $\in \mathbb{N}_{+}$ | elimination tree depth | $\log n$ |
| $\zeta_x$ | $\in \mathbb{R}_{>0}$ | $\ell_2$ step size | $2\alpha w^{-1/2}$ |
| $\zeta_s$ | $\in \mathbb{R}_{>0}$ | $\ell_2$ step size | $2\alpha tw^{1/2}$ |
| $\epsilon_t$ | $\in \mathbb{R}_{>0}$ | data structure restart threshold | $\tau \min\{1, w/\nu_{\max}\}$ |

- the total cost for update is
  
  \[
  \text{number of block updates} \cdot \text{time per block update} = \tilde{O}(q^2w^{-1}) \cdot (\mathcal{T}_{\text{mat}}(n_{\max}) + T_{\Delta L, \max} + T_{H, \max} + \eta^2m_{\max}^2),
  \]

- the total cost for query is $\tilde{O}(q^2w^{-1} \cdot \eta^2m_{\max}^2)$.

Therefore (amortized) cost per update is $\tilde{O}(qw^{-1}(\mathcal{T}_{\text{mat}}(n_{\max}) + T_{\Delta L, \max} + T_{H, \max} + \eta^2m_{\max}^2))$.

**Time for init/restart.**

We restart the data structure whenever $k > q$ or $|\bar{t} - t| > \epsilon_t$, so there are $O(N/q + \log(t_{\max}/t_{\min})\epsilon_t^{-1})$ restarts in total. By Theorem 4.11, Theorem 4.18, time cost per restart is

\[
\tilde{O}(T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\ell p} m_{\max})
\]

and the total initialization time is

\[
\text{number of restarts} \cdot \text{time per restart} = \tilde{O}((N/q + \log(t_{\max}/t_{\min})\epsilon_t^{-1}) \cdot (T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\ell p} m_{\max})).
\]

**Total time.**

The overall running time is

\[
\text{total} = \text{init/restart time} + N \cdot \text{cost per iter}
\]

\[
= \tilde{O}((N/q + \log(t_{\max}/t_{\min})\epsilon_t^{-1}) \cdot (T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\ell p} m_{\max})
\]

\[
+ N \cdot qw^{-1}(\mathcal{T}_{\text{mat}}(n_{\max}) + T_{\Delta L, \max} + T_{H, \max} + \eta^2m_{\max}^2))
\]

\[
= \tilde{O}((n^{0.5}/q + \log(t_{\max}/t_{\min}) \cdot (T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\ell p} m_{\max})).
\]
+ n^{0.5}q(T_{\text{mat}}(n_{\text{max}}) + T_{\Delta L,\text{max}} + T_{H,\text{max}} + \eta^2 m_{\text{max}}^2))
= \tilde{O}(n^{0.5} \nu_{\text{max}}^0 \cdot (T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\text{max}})^{0.5}
\cdot (T_{\text{mat}}(n_{\text{max}}) + T_{\Delta L,\text{max}} + T_{H,\text{max}} + \eta^2 m_{\text{max}}^2)^{0.5} \log(t_{\text{max}}/t_{\text{min}})).

The third step is by taking \( w = \nu_{\text{max}} \), \( N = \sqrt{n \nu_{\text{max}} w} \), \( \epsilon_t = \frac{1}{2^t} \). The fourth step is by taking
\[
q = n^{0.5} \nu_{\text{max}}^0 (T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\text{max}})^{0.5}
\cdot (T_{\text{mat}}(n_{\text{max}}) + T_{\Delta L,\text{max}} + T_{H,\text{max}} + \eta^2 m_{\text{max}}^2)^{-0.5}
\tag{15}
\]

Note that because the initialization time is bounded above by time for running \( n \) updates, we always have

\[ n(T_{\text{mat}}(n_{\text{max}}) + T_{\Delta L,\text{max}} + T_{H,\text{max}} + \eta^2 m_{\text{max}}^2) \geq (T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\text{max}}). \]

Therefore \( q \leq n^{0.5} \nu_{\text{max}}^0 \leq N \) and (15) is a valid choice for \( q \).

**Lemma 4.39** (Output part of Theorem 4.4). CENTRAL PATHMAINTENANCE.OUTPUT (Algorithm 2) takes \( O(\text{nnz}(A) + T_m) \) time.

**Proof.** The proof directly follows from Theorem 4.11.

### 4.6 Proofs of Main Result

In this section we combine everything and prove Theorem 4.4 and Theorem 4.3.

**Proof of Theorem 4.4.** Using Lemma 4.36, we finish the correctness part.

Using Lemma 4.37, we prove the running time for initialization,

\[ T_{\text{init}} = \tilde{O}(\text{nnz}(A) + T_H + T_L + \eta T_m). \]

Using Lemma 4.38, we prove the running time for multiply and move,

\[ T_{\text{multiply}} = \tilde{O}(n^{0.5} \nu_{\text{max}}^0 \cdot (T_H + T_L + \eta T_m + T_Z)^{0.5}
\cdot (T_{\text{mat}}(n_{\text{max}}) + T_{\Delta L,\text{max}} + T_{H,\text{max}} + \eta^2 m_{\text{max}}^2)^{0.5} \cdot \log(t_{\text{max}}/t_{\text{min}})). \]

By Lemma 4.39, we show the output time,

\[ T_{\text{output}} = O(\text{nnz}(A) + T_H + \eta T_m) \]

**Proof of Theorem 4.3.** So, the total running time is

\[ T_{\text{init}} + T_{\text{multiply}} + T_{\text{output}}
= \tilde{O}(n^{0.5} \nu_{\text{max}}^0 \cdot (T_H + T_L + \eta T_m + T_Z)^{0.5}
\cdot (T_{\text{mat}}(n_{\text{max}}) + T_{\Delta L,\text{max}} + T_{H,\text{max}} + \eta^2 m_{\text{max}}^2)^{0.5}
\cdot \log(R/(\epsilon r)))). \]

where we use Theorem 4.4 and Theorem A.1 (by setting \( t_{\text{max}}/t_{\text{min}} = R/(\epsilon r) \)).

\[ \]
5 Our First Result

In this section, we present an $\tilde{O}(n_{sdp} \cdot \text{poly}(\tau_{sdp}))$ algorithm for low-treewidth SDP. Outline of this section is as follows.

- In Section 5.1, we present the main result of this section, Theorem 5.2.
- In Section 5.2, we reduce our SDP to a form which can be handled by Theorem 4.3.
- In Section 5.3, we state and prove parameters needed to apply Theorem 4.3.
- In Section 5.4, we plug in all parameters and finish proof of Theorem 5.2.
- In Section 5.5, we discuss how to deal with inequality constraints.

5.1 Main Statement

We consider semidefinite program of form (1). Let us restate it here for clarity.

\[
\begin{align*}
\min \quad & C \cdot X \\
\text{s.t.} \quad & A_i \cdot X = b_i \quad \forall i \in [m_{lp}] \\
\quad & X \succeq 0
\end{align*}
\]

with $C, X \in \mathbb{R}^{n_{sdp} \times n_{sdp}}, A \in \mathbb{R}^{m_{sdp} \times n_{sdp}^2}, b \in \mathbb{R}^{m_{sdp}}$. (We add subscript $m_{sdp}$ to avoid confusion with parameters in Theorem 4.3.)

**Definition 5.1.** We make the following assumptions on program (16).

- Assume that constraint matrices $A_1, \ldots, A_{m_{sdp}} \in \mathbb{R}^{n_{sdp}^2}$ are linearly independent.
- Assume that we are given a tree decomposition (Definition 3.5) of the SDP graph (Definition 1.2) of (16) with maximum bag size $\tau_{sdp}$. \(^6\)
- Assume that any feasible solution $X \in \mathbb{S}^{n_{sdp}}$ satisfies $\|X\|_{\text{op}} \leq R$.
- There exists $Z \in \mathbb{S}^{n_{sdp}}$ such that $A \cdot Z = b$ and $\lambda_{\min}(Z) \geq r$.

**Theorem 5.2** (Our first result). Under assumptions in Definition 5.1, for any $0 < \epsilon \leq \frac{1}{2}$, there is an algorithm that outputs a solution $U \in \mathbb{R}^{n_{sdp} \times n_{sdp}}$ such that for $X = UU^\top$, we have $A \cdot X = b$ and

\[
C \cdot X \leq \min_{A \cdot X' = b, X' \in \mathbb{S}^{n_{sdp}}} C \cdot X + \epsilon \|C\|_2 R
\]

in expected time

\[
\tilde{O}(n_{sdp} \cdot \tau_{sdp}^{6.5} \log(R/(r\epsilon)))
\]

where $\tilde{O}$ hides $n^{o(1)}$ terms.

Our proof of Theorem 5.2 is in two steps. First we reduce program (16) to form (9), then we apply Theorem 4.3 by plugging in needed parameters.

\(^6\)If such a tree decomposition is not given, then using \cite{BGS22}, we can find a tree decomposition (Definition 3.5) with maximum bag size $\tilde{O}(\text{tw}(G))$, where $\text{tw}(G)$ is treewidth of the SDP graph $G$. 

52
5.2 Reduce Low-Treewidth SDP to General Treewidth Program

In this section we reduce program (16) satisfying Definition 5.1 to form (9).

Recall that we are given a tree decomposition $T, J_1, \ldots, J_n$ of the SDP graph with maximum bag size $\tau_{\text{sdp}}$, where $T$ is a tree on $n$ vertices and and $J_j$ are the bags.

Using Lemma 5.3, we can WLOG assume that $T$ has maximum degree $O(1)$.

**Lemma 5.3.** Given any tree decomposition $T$ with $n$ bags, we can construct another tree decomposition with at most $2n$ bags, with the same maximum bag size, and maximum degree at most 3.

**Proof.** For every vertex $j \in [n]$ with degree $d_j > 3$, we can replace it with $d_j - 1$ vertices, each of degree 3, with the corresponding bags equal to $J_j$. It is easy to verify that this is a valid tree decomposition, and that the total number of bags after this transformation is at most $2n$. \qed

By Lemma 5.4, to solve program (16), it suffices to solve the following program with fewer variables:

\[
\begin{align*}
\min & \quad \sum_{j \in [n]} C_j \cdot X_j \\
\text{s.t.} & \quad A_i \cdot X_{ji} = b_i \quad \forall i \in [m_{\text{sdp}}] \\
& \quad N_{i,j}(X_i) = N_{j,i}(X_j) \quad \forall (i,j) \in E(T) \\
& \quad X_j \succeq 0 \quad \forall j \in [n]
\end{align*}
\]

where $j_i$ is any number with $\text{supp} A_i \subseteq J_{j_i}$, and the $N$ constraints asserts consistency between different bags. Here $X_j \in \mathbb{R}^{|J_j| \times |J_j|}$, $A_i \in \mathbb{R}^{|J_i| \times |J_i|}$, $b \in \mathbb{R}^{m_{\text{sdp}}}$, $N_{i,j} \in \mathbb{R}^{|J_i \cap J_j|^2 \times |J_i|^2}$. (We view $X_i$ in $N_{i,j}(X_i)$ as a vector of length $\mathbb{R}^{|J_j|^2}$.) For all $j \in [n]$, $X_j$ corresponds to the $J_j \times J_j$ minor of $X$ (in (16)). By properties of the tree decomposition, we have $n = O(n_{\text{sdp}})$ and $|J_j| = \tilde{O}(\tau_{\text{sdp}})$ for all $j \in [n]$. So the total variable dimension of program (17) is

\[n_{\text{lp}} = \sum_{j \in [n]} |J_j|^2 = \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^2).\]

**Lemma 5.4 ([ZL18, Section 3.3]).** Under assumptions in Definition 5.1, the optimal value of program (16) is equal to the optimal value of program (17). Furthermore, given a solution $X_1, \ldots, X_n$ of program (17), there is an algorithm that takes $O(n_{\text{sdp}} \tau_{\text{sdp}}^3)$ time and outputs a matrix $U \in \mathbb{R}^{n_{\text{sdp}} \times \tau_{\text{sdp}}}$ such that for $X = U U^T$, $X$ is a solution of Eq. (1) with the same objective value.

Now we have reduced program (16) to program (17), which is of form (9). Let us compute the treewidth of program 17.

**Lemma 5.5.** We can construct a tree decomposition of the LP dual graph (Definition 4.1) with maximum bag size $\tau_{\text{lp}} = O(\tau_{\text{sdp}}^2)$ and at most $n$ blocks.

**Proof.** Let us define a bag decomposition for the LP dual graph. For every $j \in [n]$, we define a bag $B_j$ containing

- all type-A constraints $A_i$ with $j_i = j$ or $(j_i, j) \in T$, and
- all type-N constraints $N_{i,j}$ with $(i, j) \in T$.
We connect $B_j$ with $B_i$ if and only if $(i, j) \in T$.

Because of linear independence assumption, number of type-$A$ constraints in a block is at most $|J_i|^2$. Because $T$ has constant maximum degree, number of type-$N$ constraints in a block is at most $O(|J_i|^2)$ (recall that one constraint $N_{i,j}(X_i) = N_{j,i}(X_j)$ is in fact $|J_i \cap J_j|^2$ equations). So $|B_j| = O(|J_i|^2)$. The maximum bag size is $\tau_p = O(\max_j |J_j|^2) = O(\tau_s^{2})$.

Finally, we prove that $(T, B_1, \ldots, B_n)$ is a valid tree decomposition of the LP dual graph. For any two constraints sharing a variable $X_j$, they must both be in $B_j$. So the first condition in Definition 3.5 is satisfied. The second condition in Definition 3.5 is clearly satisfied. So $(T, B_1, \ldots, B_n)$ is a valid tree decomposition.

This enables solving program (17) using Theorem 4.3.

5.3 Choice of Parameters

In this section we present value of parameters needed to apply Theorem 4.3.

Lemma 5.6 (Choice of parameters in the first result). Under the setting of Theorem 5.2, we can choose the following set of parameters.

(i) $n = O(n_{\text{sdp}})$, $n_{\text{lp}} = O(n_{\text{sdp}} \tau_{\text{sdp}}^2)$, $n_{\max} = O(\tau_{\text{sdp}}^2)$.

(ii) $m = O(n_{\text{sdp}} \tau_{\text{sdp}}^2)$, $m_{\text{lp}} = O(n_{\text{sdp}} \tau_{\text{sdp}}^2)$, $m_{\max} = 1$.

(iii) $\tau_{\text{lp}} = O(\tau_{\text{sdp}}^2)$.

(iv) $\eta = \tilde{O}(\tau_{\text{sdp}}^2)$.

(v) $\nu_{\max} = O(\tau_{\text{sdp}})$.

(vi) $\text{nnz}(A) = O(n_{\text{sdp}} \tau_{\text{sdp}}^4)$.

(vii) $T_n = O(n_{\text{sdp}} \tau_{\text{sdp}}^2)$, $T_m = O(n_{\text{sdp}} \tau_{\text{sdp}}^2)$.

(viii) $T_H = O(n_{\text{sdp}} \tau_{\text{sdp}}^2)$, $T_{H,\text{max}} = O(\tau_{\text{sdp}}^2)$.

(ix) $T_L = \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^6)$.

(x) $T_Z = \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^6)$.

(xi) $T_{\Delta L,\max} = \tilde{O}(\tau_{\text{sdp}}^6)$.

Proof. (i) Values for $n, m_{\text{lp}}, n_{\max}$ have been discussed in Section 5.2.

(ii) By discussion in last section, we have $m_{\text{lp}} = O(n_{\text{sdp}} \tau_{\text{sdp}}^2)$. We choose block structure $(1, \ldots, 1)$, i.e., $m = m_{\text{lp}} = O(n_{\text{sdp}} \tau_{\text{sdp}}^2)$ and $m_{\max} = 1$.

(iii) Proved in Lemma 5.5.

(iv) By Lemma 5.7.

(v) We use the log barrier (Definition 3.3). By Lemma 3.4, we have $\nu_{\max} = O(\tau_{\text{sdp}})$.

(vi) Every $X_j$ is in at most $\tau_{\text{lp}} = O(\tau_{\text{sdp}}^2)$ constraints, so $\text{nnz}(A) = O(n_{\text{sdp}} \tau_{\text{sdp}}^2 \tau_{\text{lp}}) = O(n_{\text{sdp}} \tau_{\text{sdp}}^4)$.
Bounds on $T_n$ and $T_m$ are direct consequences of (i), (ii).

Computing Hessian of the log-barrier takes $O(\tau^{2\omega}_{\text{lp}})$ time. So $T_H = O(n_{\text{sdp}}\tau^{2\omega}_{\text{sdp}})$,

$T_{H,\max} = O(\tau^{2\omega}_{\text{sdp}})$.

By Lemma 3.10, we have $T_L = O(m_{\text{lp}}\eta^2) = \widetilde{O}(n_{\text{sdp}}\tau^6_{\text{sdp}})$.

By Lemma 4.9, we have $T_Z = O(2\eta mm_{\max}^2) = \widetilde{O}(n_{\text{sdp}}\tau^6_{\text{sdp}})$.

Updating $AH\tau A^\top = LL^\top$ under change of one block involves $O(n_{\max}) = O(\tau^2_{\text{sdp}})$ rank-1 updates, and each such update costs $O(\eta^2) = \widetilde{O}(\tau^4_{\text{sdp}})$ time by Lemma 3.11. So $T_{\Delta L,\max} = \widetilde{O}(\tau^6_{\text{sdp}})$.

Lemma 5.7. We can construct a block elimination tree with block structure $(1, \ldots, 1)$ (i.e., $m = m_{\text{lp}}$, $m_{\max} = 1$) and maximum block-depth $\eta = \widetilde{O}(\tau_{\text{lp}})$.

Proof. This basically follows from [BGHK95]. We present the construction here for completeness. Consider Algorithm 14. Because in each step we choose the centroid of $T$, the maximum recursion depth is at most $\widetilde{O}(1)$. Every bag $J_i$ has size $O(\tau_{\text{lp}})$, so maximum depth of the constructed tree is $\widetilde{O}(\tau_{\text{lp}})$.

It remains to prove that the constructed tree $T$ is a valid block elimination tree. By properties of a bag decomposition, the condition in Lemma 4.6 is satisfied. So $T$ is a valid block elimination tree.

Algorithm 14 Construct an Elimination Tree

\begin{algorithm}
1: \textbf{procedure} \textsc{ConstructElimTree}(T, J_1, \ldots, J_k) \\
2: \textbf{Input:} A tree decomposition of a certain graph \\
3: \textbf{Output:} Root of the constructed elimination tree \\
4: \textbf{if} $k \leq 1$ \textbf{then} \\
5: \hspace{1em} Construct a rooted path using an arbitrary ordering of $J_1$. Let $r$ be the root. \\
6: \hspace{1em} \textbf{return} $r$. \\
7: \textbf{end if} \\
8: \textbf{let} $i \in [k]$ be the centroid of $T$. \\
9: \textbf{let} $[k] = S' \cup S'' \cup \{i\}$ be a partition such that $|S'|, |S''| \leq \frac{2}{3}k$ \\
10: $r' \leftarrow \textsc{ConstructElimTree}(T', J_j - J_i : j \in S')$ where $T'$ is the spanning forest of $S'$. \\
11: $r'' \leftarrow \textsc{ConstructElimTree}(T'', J_j - J_i : j \in S'')$ where $T''$ is the spanning forest of $S''$. \\
12: Construct a rooted path using an arbitrary ordering of $J_i$. Let $r$ be the root, $u$ be the leaf. \\
13: Set $u$ as the parent of $r'$ and $r''$. \\
14: \textbf{return} $r$. \\
15: \textbf{end procedure}
\end{algorithm}

5.4 Proof of Theorem 5.2

In this section, we prove Theorem 5.2.
**Proof of Theorem 5.2.** According to Theorem 4.3, there is an algorithm solving SDP in

\[
O(n^{0.5} \tau \max^{0.5} \cdot (T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta \text{lp} m_{\max})^{0.5} \cdot (T_{\text{mat}}(n_{\max}) + T_{\Delta, \max} + T_{H, \max} + \eta^2 m_{\max}^2)^{0.5} \log(R/(r \epsilon))
\]

time.

It remains to compute the parameters \(T_H, T_L, \eta, T_m, m_{\max}, n_{\max}, T_{\Delta, \max}, T_{H, \max}\), the details can be found in Lemma 5.6.

Let \(A\) and \(B\) be defined as:

\[
A := T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta \text{lp} m_{\max},
B := T_{\text{mat}}(n_{\max}) + T_{\Delta, \max} + T_{H, \max} + \eta^2 m_{\max}^2.
\]

We can show that

\[
A = T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta \text{lp} m_{\max}
\]

\[
= O(n_{\text{sdp}} \tau^2) + T_n + T_L + T_Z + \text{nnz}(A) + \eta \text{lp} m_{\max}
\]

\[
= O(n_{\text{sdp}} \tau^2) + O(n_{\text{sdp}} \tau^2) + T_L + T_Z + \text{nnz}(A) + \eta \text{lp} m_{\max}
\]

\[
= O(n_{\text{sdp}} \tau^2) + O(n_{\text{sdp}} \tau^2) + T_Z + \text{nnz}(A) + \eta \text{lp} m_{\max}
\]

\[
= O(n_{\text{sdp}} \tau^2) + O(n_{\text{sdp}} \tau^2) + O(n_{\text{sdp}} \tau^2) + O(n_{\text{sdp}} \tau^2)
\]

\[
= O(n_{\text{sdp}} \tau^2) = (18)
\]

where the second step follows from Lemma 5.6(viii) \((T_H = O(n_{\text{sdp}} \tau^2))\) and the third step follows from Lemma 5.6(i) \((T_n = O(n_{\text{sdp}} \tau^2))\), the forth step follows from merging the terms, the fifth step follows from Lemma 5.6(ix) \((T_Z = O(n_{\text{sdp}} \tau^2))\), the sixth step follows from merging the terms, the seventh step follows from \(T_Z = O(n_{\text{sdp}} \tau^2)\), the eighth step follows from merging the terms, the ninth step follows Lemma 5.6(vi) \((\text{nnz}(A) = n_{\text{sdp}} \tau^4)\), and the tenth step follows from Lemma 5.6(iv) and Lemma 5.6(ii).

For the term \(B\), we have

\[
B = T_{\text{mat}}(n_{\max}) + T_{\Delta, \max} + T_{H, \max} + \eta^2 m_{\max}^2
\]

\[
= O(\tau^2) + T_{\Delta, \max} + T_{H, \max} + \eta^2 m_{\max}^2
\]

\[
= O(\tau^2) + O(\tau^3) + T_{H, \max} + \eta^2 m_{\max}^2
\]

\[
= O(\tau^2) + O(\tau^3) + O(\tau^2) + \tilde{O}(\tau^2)
\]

\[
= O(\tau^3)
\]

\[
= O(\tau^6_{\text{sdp}}).
\]

(19)
where the second step follows Lemma 5.6(i) \((n_{\text{max}} = \tau_{\text{lp}})\), and the third step follows from Lemma 5.6(xi) \((T_{\Delta},_{\text{max}} = O(\tau_{\text{lp}}^3))\), the forth step follows from Lemma 5.6(viii) \((T_H,_{\text{max}} = O(\tau_{\text{lp}}^\omega))\), the fifth step follows from Lemma 5.6(iv) \((\eta = \widetilde{O}(\tau_{\text{lp}})\) and \(m_{\text{max}} = O(1))\), and the last step follows from \(\tau_{\text{lp}} = \tau_{\text{sdp}}^2\).

Finally, we have

\[
\tilde{O}(n^{0.5} \nu_{\text{max}}^{0.5} \cdot A^{0.5} \cdot B^{0.5} \cdot \log(1/\epsilon))
\]

\[
= \tilde{O}(n_{\text{sdp}}^{0.5} \nu_{\text{sdp}}^{0.5} \cdot A^{0.5} \cdot B^{0.5} \cdot \log(1/\epsilon))
\]

\[
= \tilde{O}(n_{\text{sdp}}^{0.5} \nu_{\text{sdp}}^{0.5} \cdot (n_{\text{sdp}} \tau_{\text{sdp}}^6)^{0.5} \cdot (\tau_{\text{sdp}}^{0.5} \cdot \log(1/\epsilon)))
\]

\[
= \tilde{O}(n_{\text{sdp}}^{0.5} \nu_{\text{sdp}}^{0.5} \cdot \log(1/\epsilon))
\]

where the first step follows from \(n = n_{\text{sdp}}\) and Lemma 5.6(iv) \((\nu_{\text{max}} = O(\tau_{\text{sdp}}))\), the second step follows from \(A = \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^6)\) (see Eq. (18)) and \(B = \tilde{O}(\tau_{\text{sdp}}^6)\) (see Eq. (19)). Thus, we complete the proof.

5.5 Discussions on Inequality Constraints

In certain problems (e.g., MaxCut) there are inequality constraints. In this section we briefly discuss how to deal with inequality constraints.

For every inequality constraint of form \(A_j \bullet X_{J_i} \geq b_i\) (where \(J_i\) is a bag in the tree decomposition), we add a real variable \(v_i \in \mathbb{R}\) and replace the inequality constraint with \(A_j \bullet X_{J_i} - v_i = b_i\) and \(v_i \geq 0\).

This adds \(m_{\text{sdp}} = O(n_{\text{sdp}} \tau_{\text{sdp}})\) real variables in total. Let us discuss the variable block structure. For every \(j \in [n]\), consider the set \(S_j := \{i \in [m] : J_i = j\}\). We divide \(S_j\) into \(\lceil |S_j|/\tau_{\text{sdp}} \rceil\) blocks, each with \(O(\tau_{\text{sdp}})\) variables. In this way the number of variable blocks \(n\) increases by at most a constant factor, maximum size \(n_{\text{max}}\) of a variable block does not change, and \(\nu_{\text{max}}\) is still \(O(\tau_{\text{sdp}})\).

Number of constraints \(m_{\text{lp}}\) stays the same, and we can use the same block structure as in Lemma 5.7 or Lemma 6.3.

Therefore, the running time claims of Theorem 5.2 and Theorem 6.1 still hold in the presence of inequality constraints.
6 Our Second Result, An Improved Version of Our First Result

In this section, we improve our algorithm in Section 5 to get an $\tilde{O}(n_{sdp} \tau^{2\omega+0.5})$ time algorithm for low-treewidth SDP. Outline of this section is as follows.

- In Section 6.1, we present the main result of this section, Theorem 6.1.
- In Section 6.2, we state and prove parameters needed to apply Theorem 4.3.
- In Section 6.3, we develop improved algorithms for Cholesky-related computation.
- In Section 6.4, we plug in all parameters and finish proof of Theorem 6.1.

6.1 Main Statement

We work under the same setting as Section 5, where we are given program (17) with assumptions in Definition 5.1.

**Theorem 6.1** (Our second result). Under assumptions in Definition 5.1, for any $0 < \epsilon \leq \frac{1}{2}$, there is an algorithm that outputs a solution $U \in \mathbb{R}^{n_{sdp}}$ such that for $X = UU^\top$, we have $A \cdot X = b$ and

$$C \cdot X \leq \min_{A \cdot X' = b, X' \in S^{n_{sdp}}} C \cdot X + \epsilon \|C\|_2 R$$

in expected time

$$\tilde{O}(n \cdot \tau^{2\omega+0.5} \log(R/(r\epsilon)))$$

where $\tilde{O}$ hides $n^{o(1)}$ terms.

6.2 Choice of Parameters

In this section we present value of parameters needed to apply Theorem 4.3.

**Lemma 6.2** (Choice of parameters in the second result). Under the setting of Theorem 6.1, we can choose the following set of parameters.

(i) $n = O(n_{sdp})$, $m_{lp} = O(n_{sdp} \tau_{sdp}^2)$, $n_{\text{max}} = O(\tau_{sdp}^2)$.

(ii) $m = O(n_{sdp})$, $m_{lp} = O(n_{sdp} \tau_{sdp}^2)$, $m_{\text{max}} = O(\tau_{sdp}^2)$.

(iii) $\tau_{lp} = O(\tau_{sdp}^2)$.

(iv) $\eta = \tilde{O}(1)$.

(v) $\nu_{\text{max}} = O(\tau_{sdp})$.

(vi) $\text{nnz}(A) = O(n_{sdp} \tau_{sdp}^4)$.

(vii) $T_n = \tilde{O}(n_{sdp} \tau_{sdp}^{2\omega})$, $T_m = \tilde{O}(n_{sdp} \tau_{sdp}^{2\omega})$.

(viii) $T_H = \tilde{O}(n_{sdp} \tau_{sdp}^{2\omega})$, $T_{H,\text{max}} = \tilde{O}(\tau_{sdp}^{2\omega})$. 

58
(ix) \( T_L = \tilde{O}(n_{\text{sdp}}\tau_{\text{sdp}}^{2\omega}) \).

(x) \( T_Z = \tilde{O}(n_{\text{sdp}}\tau_{\text{sdp}}^{4}) \).

(xi) \( T_{\Delta L,\max} = \tilde{O}(\tau_{\text{sdp}}^{2\omega}) \).

Proof. 

(i) Same as Lemma 5.6(i).

(ii) Bound on \( m_{\text{lp}} \) is same as Lemma 5.6(ii). For choice of block structure (and thus \( m \) and \( m_{\text{max}} \)), see Lemma 6.3.

(iii) Same as Lemma 5.6(iii).

(iv) By Lemma 6.3.

(v) Same as Lemma 5.6(v).

(vi) Same as Lemma 5.6(vi).

(vii) Bounds on \( T_n \) and \( T_m \) are direct consequences of (i), (ii).

(viii) Same as Lemma 5.6(viii).

(ix) By Lemma 6.4, we have \( T_L = \tilde{O}(m\tau_{\text{lp}}^{\omega}) = \tilde{O}(n_{\text{sdp}}\tau_{\text{sdp}}^{2\omega}) \).

(x) By Lemma 4.9, we have \( T_Z = O(\eta^2 m m_{\text{max}}^{2}) = \tilde{O}(n_{\text{sdp}}\tau_{\text{sdp}}^{4}) \).

(xi) By Lemma 6.5, we have \( T_{\Delta L,\max} = \tilde{O}(\tau_{\text{lp}}^{\omega}) = \tilde{O}(\tau_{\text{sdp}}^{2\omega}) \).

Lemma 6.3. We can construct a block elimination tree with a block structure where \( m = O(n_{\text{sdp}}) \), \( m_{\text{max}} = O(\tau_{\text{lp}}) \), such that the maximum block-depth is \( \eta = \tilde{O}(1) \).

Proof. We run Algorithm 15, a block version of Algorithm 14. The difference is that, for every block, we create a new vertex containing all elements in the block, instead of constructing a path. In this way, the constructed block elimination tree \( T \) has block depth \( \tilde{O}(1) \). Every block in \( T \) is contained in an original block \( J_i \), so maximum block size is \( O(\tau_{\text{lp}}) \). Furthermore, number of vertices in \( T \) is at most number of original blocks, which is at most \( n = O(n_{\text{sdp}}) \) by Lemma 5.5.

It remains to prove that the constructed tree \( T \) is a valid block elimination tree. By properties of a bag decomposition, the condition in Lemma 4.6 is satisfied. So \( T \) is a valid block elimination tree.

6.3 Cholesky Decomposition Using Block Structures

In this section we discuss how to utilize the block elimination tree constructed in Lemma 6.3 to compute and update Cholesky decomposition faster.

Lemma 6.4. Let \( M \in \mathbb{R}^{m_{\text{lp}} \times m_{\text{lp}}} \) be a PSD matrix with block elimination tree \( T \), such that the coordinates have been re-ordered in post-traversal order of \( T \). Suppose that \( T \) has \( m \) vertices, has maximum depth \( \tilde{O}(1) \), and maximum block size \( O(\tau) \). Then we can compute the Cholesky factorization \( M = LL^\top \) in \( \tilde{O}(m\tau^{\omega}) \) time.
Algorithm 15 Construct a Block Elimination Tree

1: **procedure** \textsc{ConstructBlockElimTree}(T, J_1, \ldots, J_k)  
2: \hspace{1em} **Input:** A tree decomposition of a graph  
3: \hspace{1em} **Output:** Root of the constructed elimination tree  
4: \hspace{1em} if \( k \leq 1 \) then  
5: \hspace{2em} Let \( r \) be a single vertex containing block \( J_1 \).  
6: \hspace{2em} return \( r \).  
7: \hspace{1em} end if  
8: \hspace{1em} Let \( i \in [k] \) be the centroid of \( T \).  
9: \hspace{1em} Let \([k] = S' \cup S'' \cup \{i\}\) be a partition such that \(|S'|, |S''| \leq \frac{2}{3}k\)  
10: \hspace{1em} \( r' \leftarrow \textsc{ConstructBlockElimTree}(T', J_j - J_i : j \in S') \) where \( T' \) is the spanning forest of \( S' \).  
11: \hspace{1em} \( r'' \leftarrow \textsc{ConstructBlockElimTree}(T'', J_j - J_i : j \in S'') \) where \( T'' \) is the spanning forest of \( S'' \).  
12: \hspace{1em} Let \( r \) be a single vertex containing block \( J_i \).  
13: \hspace{1em} Set \( r \) as the parent of \( r' \) and \( r'' \).  
14: \hspace{1em} return \( r \).  
15: **end procedure**

Algorithm 16 Compute Cholesky factorization using Block Structure

1: **procedure** \textsc{BlockCholesky}(M, T)  
2: \hspace{1em} **Input:** PSD matrix \( M \) with block elimination tree \( T \)  
3: \hspace{1em} **Output:** Lower-triangular \( L \) such that \( M = LL^\top \)  
4: \hspace{1em} for \( j = 1 \) to \( m \) do  
5: \hspace{2em} \( L_{i,j} \leftarrow (M_{i,j} - \sum_{k \in \mathcal{D}(j) \setminus j} L_{j,k} L_{j,k}^\top)^{1/2} \)  
6: \hspace{3em} for \( i \in \mathcal{P}(j) \) do  
7: \hspace{4em} \( L_{i,j} \leftarrow (M_{i,j} - \sum_{k \in \mathcal{D}(j) \setminus j} L_{i,k} L_{j,k}^\top L_{j,k}^\top)^{-1} \)  
8: \hspace{3em} end for  
9: \hspace{2em} end for  
10: \hspace{1em} for \( j = 1 \) to \( m \) do  
11: \hspace{2em} Compute QR decomposition \( L_{i,j} = Q_j R_j \)  
12: \hspace{2em} \( L_{i,j} \leftarrow L_{i,j} Q_j^\top \) for \( i \in \mathcal{P}(j) \)  
13: \hspace{2em} end for  
14: **return** \( L \)  
15: **end procedure**

**Proof.** We run Algorithm 16.

**Correctness:** From the algorithm we can see \( L_{i,j} \neq 0 \) only when \( i \in \mathcal{P}(j) \). So for \( i, j \in [m], \) if \((LL^\top)_{i,j} \neq 0\), then either \( i \in \mathcal{P}(k) \) or \( j \in \mathcal{P}(i) \). WLOG assume that \( i \in \mathcal{P}(j) \). If \( j = i \), then Line 5 shows that \((LL^\top)_{i,i} = M_{i,i} \). If \( j \neq i \), then Line 7 shows that \((LL^\top)_{i,j} = M_{i,j} \). So \( LL^\top = M \).

Then we prove that the square root in Line 5 always exists. Let \( \mathcal{D}'(j) := \mathcal{D}(j) \setminus j \). Recall that

\[
L_{j,\mathcal{D}'(j)} L_{\mathcal{D}'(j),\mathcal{D}'(j)}^\top = M_{j,\mathcal{D}'(j)}.
\]

So

\[
M_{j,j} - L_{j,\mathcal{D}'(j)} L_{\mathcal{D}'(j),\mathcal{D}'(j)}^\top
\]
where the last step is by property of PSD matrix. So the square root in Line 5 can always be taken.

Finally, let us examine the effect of Line 12. Note that before Line 10, all $L_{j,j}$ are PSD matrices. Line 12 makes update $L_{j,j} \leftarrow L_{j,j}Q_{j}^{\top} = (Q_{j}L_{j,j})^{\top} = R_{j}^{\top}$, which makes $L_{j,j}$ a lower triangular matrix. So in the end $L$ is a lower triangular matrix as desired.

**Running time:** Because the block elimination tree has block depth $\tilde{O}(1)$, there are $\tilde{O}(m)$ triples $(i,j,k)$ with $i \in \mathcal{P}(j)$, $j \in \mathcal{P}(k)$. For each such triple, we take $\tilde{O}(\tau^{w})$ time to perform the corresponding computations. So computation before Line 9 takes $\tilde{O}(m\tau^{w})$ time. By [DDH07], computing QR decomposition of a matrix of size $\tau$ takes $\tilde{O}(\tau^{w})$ time. So computation starting from Line 10 takes $\tilde{O}(m\tau^{w})$ time. Therefore the whole algorithm runs in $\tilde{O}(m\tau^{w})$ time. □

---

**Algorithm 17** Update Cholesky factorization using Block Structure

```
1: procedure BLOCKCHOLESKYUPDATE($M, \mathcal{T}, \Delta_{M}, v$) ⊳ Lemma 6.5
2:    $L_{\text{new}}^{\text{new}} \leftarrow L$
3:     for $j \in \mathcal{P}(v)$ from bottom to up do
4:        $L_{i,j}^{\text{new}} \leftarrow (L_{i,j}L_{j,j}^{\top} + \Delta_{M_{i,j}}) - \sum_{k \in \mathcal{P}(v)} (L_{i,j}L_{i,k}^{\top} - L_{i,k}L_{j,k}^{\top})^{1/2}$
5:          for $i \in \mathcal{P}(j)$ do
6:              $L_{i,j}^{\text{new}} \leftarrow (L_{i,j}L_{j,j}^{\top} + \Delta_{M_{i,j}}) - \sum_{k \in \mathcal{P}(v)} (L_{i,j}L_{i,k}^{\top} - L_{i,k}L_{j,k}^{\top})(L_{j,j}^{\text{new}})^{-1}$
7:          end for
8:     end for
9:     for $j \in \mathcal{P}(v)$ from bottom to up do
10:        Compute QR decomposition $L_{j,j}^{\text{new}} = Q_{j}R_{j}$
11:     $L_{i,j}^{\text{new}} \leftarrow L_{i,j}^{\text{new}}Q_{j}^{\top}$ for $i \in \mathcal{P}(j)$
12: end for
13: return $(L_{\text{new}} - L)_{\mathcal{P}(v),v}$
14: end procedure
```

**Lemma 6.5.** Work under the setting of Lemma 6.4. In addition, assume that we already computed the Cholesky decomposition $M = LL^{\top}$. Suppose we perform an update $M \leftarrow M + \Delta_{M}$, where support of $\Delta_{M}$ is contained in the union of block row $v$ and block column $v$, for some vertex $v \in \mathcal{T}$. Then in $\tilde{O}(\tau^{w})$ time, we can compute $\Delta_{L}$ such that $M + \Delta_{M} = (L + \Delta_{L})(L + \Delta_{L})^{\top}$ is the Cholesky factorization of $M + \Delta_{M}$.

**Proof.** We run Algorithm 17.

**Correctness:** From Algorithm 16 we can see that only $L_{*,j}$ for $j \notin \mathcal{P}(v)$ are update. So we do not need to care about $L_{*,j}$ for $j \notin \mathcal{P}(v)$. Let $M_{\text{new}} = M + \Delta_{M}$. In Algorithm 17, Line 4, we have

\[
L_{i,j}^{\text{new}} = (L_{i,j}L_{j,j}^{\top} + \Delta_{M_{i,j}}) - \sum_{k \in \mathcal{P}(v)} (L_{i,j}L_{i,k}^{\top} - L_{i,k}L_{j,k}^{\top})^{1/2}
\]

\[
= (M_{i,j} + \Delta_{M_{i,j}}) - \sum_{k \in \mathcal{D}(j) \setminus \mathcal{P}(v)} L_{i,k}L_{j,k}^{\top} - \sum_{k \in \mathcal{P}(v) \setminus \mathcal{D}(j)} (L_{i,j}L_{i,k}^{\top} - L_{i,k}L_{j,k}^{\top})^{1/2}
\]

\[
= (M_{i,j}^{\text{new}} - \sum_{k \in \mathcal{D}(j) \setminus \mathcal{P}(v)} L_{i,j}L_{i,k}^{\top})^{1/2}.
\]
In Algorithm 17, Line 6, we have
\[
L_{i,j}^{\text{new}} = (L_{i,j}^T + \Delta_{M(i,j)}) - \sum_{k \in P(v) \cap (D(j))} (L_{i,k}^{\text{new}} L_{j,k}^T - L_{i,k} L_{j,k}^T))(L_{j,j}^{\text{new}})^{-T}
\]
Finally, we rotate the updated columns by orthogonal matrices to achieve lower triangular matrix.

**Running time:** There are \(\tilde{O}(1)\) tuples \((i,j,k)\) such that \(k \in P(v), j \in P(k), i \in P(j)\). For every such tuple, computation time is \(\tilde{O}(\tau^\omega)\). So total update time is \(O(\tau^\omega)\).

### 6.4 Proof of Theorem 6.1

**Proof of Theorem 6.1.** According to Theorem 4.3, there is an algorithm solving SDP in
\[
\tilde{O}(n^{0.5} \tau^{0.5} m_{\max} + T_H + T_n + T_L + T_Z + n\tau \eta m + \eta m_{\max})^{0.5} \cdot (\tau_{\text{mat}}(n_{\max}) + T_{\Delta L,\max} + T_{H,\max} + \eta^2 m_{\max}^{2})^{0.5} \log(R/\epsilon))
\]
time.

It remains to compute the parameters \(T_H, T_L, \eta, T_m, m_{\max}, n_{\max}, T_{\Delta L,\max}, T_{H,\max}\), the details can be found in Lemma 6.2.

Let \(A \) and \(B \) be defined as:
\[
A := T_H + T_n + T_L + T_Z + n\tau \eta m + \eta m_{\max},
B := \tau_{\text{mat}}(n_{\max}) + T_{\Delta L,\max} + T_{H,\max} + \eta^2 m_{\max}^{2}.
\]

We can show that
\[
A = T_H + T_n + T_L + T_Z + n\tau \eta m + \eta m_{\max}
= O(n_{\text{sdp}}^{2\omega} + T_n + T_L + T_Z + n\tau \eta m + \eta m_{\max})
= O(n_{\text{sdp}}^{2\omega} + n_{\text{sdp}}^{2\omega} + T_L + T_Z + n\tau \eta m + \eta m_{\max})
= O(n_{\text{sdp}}^{2\omega} + T_L + T_Z + n\tau \eta m + \eta m_{\max})
= O(n_{\text{sdp}}^{2\omega} + n_{\text{sdp}}^{2\omega} + T_Z + n\tau \eta m + \eta m_{\max})
= O(n_{\text{sdp}}^{2\omega} + n_{\text{sdp}}^{2\omega} + n\tau \eta m + \eta m_{\max})
= O(n_{\text{sdp}}^{2\omega} + n_{\text{sdp}}^{4} + \eta m_{\max})
= O(n_{\text{sdp}}^{2\omega} + n_{\text{sdp}}^{4} + \eta m_{\max})
= O(n_{\text{sdp}}^{2\omega} + n_{\text{sdp}}^{4} + \eta m_{\max})
= O(n_{\text{sdp}}^{2\omega} + n_{\text{sdp}}^{4} + \eta m_{\max})
\]
where the second step follows from Lemma 6.2(viii) \((T_H = O(n_{\text{sdp}}^{2\omega}))\) and the third step follows from Lemma 6.2(i) \((T_n = O(n_{\text{sdp}}^{2\omega}))\), the forth step follows from merging the terms, the
fifth step follows from Lemma 6.2(ix) \( T_L = \tilde{O}(\tau \omega_\text{sdp}) \), the sixth step follows from merging the terms, the seventh step follows from \( T_Z = O(n \tau_4 \omega_\text{sdp}) \), the eighth step follows from merging the terms, the ninth step follows Lemma 6.2(vi) \( \text{nnz}(A) = n \tau_4 \omega_\text{sdp} \), and the tenth step follows from Lemma 6.2(iv) and Lemma 6.2(ii).

For the term \( B \), we have

\[
B = T_{\text{mat}}(n_{\text{max}}) + T_{\Delta_{L,\text{max}}} + T_{H_{\text{max}}} + \eta^2 m_{\text{max}}^2 \\
= O(\tau_{\text{lp}}^\omega) + T_{\Delta_{L,\text{max}}} + T_{H_{\text{max}}} + \eta^2 m_{\text{max}}^2 \\
= O(\tau_{\text{lp}}^\omega) + O(\tau_{\text{lp}}^\omega) + T_{H_{\text{max}}} + \eta^2 m_{\text{max}}^2 \\
= O(\tau_{\text{lp}}^\omega) + O(\tau_{\text{lp}}^\omega) + O(\tau_{\text{lp}}^\omega) + \tilde{O}(\tau_{\text{lp}}^2) \\
= O(\tau_{\text{lp}}^\omega) \\
= O(\tau_{\text{sdp}}^\omega) \tag{21}
\]

where the second step follows Lemma 6.2(i) \( n_{\text{max}} = \tau_{\text{lp}} \), and the third step follows from Lemma 6.2(xi) \( T_{\Delta_{L,\text{max}}} = O(\tau_{\text{lp}}^\omega) \), the forth step follows from Lemma 6.2(viii) \( T_{H_{\text{max}}} = O(\tau_{\text{lp}}^\omega) \), the fifth step follows from Lemma 6.2(iv) \( \eta = \tilde{O}(1) \) and \( m_{\text{max}} = (\tau_{\text{lp}}) \), and the last step follows from \( \tau_{\text{lp}} = \tau_{\text{sdp}}^2 \).

Finally, we have

\[
\tilde{O}(n^{0.5} \nu_{\text{max}}^{0.5} \cdot A^{0.5} \cdot B^{0.5} \cdot \log(1/\epsilon)) \\
= \tilde{O}(n_{\text{sdp}}^{0.5} \cdot A^{0.5} \cdot B^{0.5} \cdot \log(1/\epsilon)) \\
= \tilde{O}(n_{\text{sdp}}^{0.5} \cdot (\tau_{\text{sdp}}^{2\omega})^{0.5} \cdot (\tau_{\text{sdp}}^{2\omega})^{0.5} \cdot \log(1/\epsilon)) \\
= \tilde{O}(n_{\text{sdp}}^{0.5} \cdot (\tau_{\text{sdp}}^{2\omega})^{0.5} \cdot \log(1/\epsilon))
\]

where the first step follows from \( n = n_{\text{sdp}} \) and Lemma 6.2(iv) \( \nu_{\text{max}} = O(\tau_{\text{sdp}}) \), the second step follows from \( A = \tilde{O}(n_{\text{sdp}}^{2\omega}) \) (see Eq. (20)) and \( B = \tilde{O}(\tau_{\text{sdp}}^{2\omega}) \) (see Eq. (21)).

Thus, we complete the proof.
7 Decomposable SDP

In this section we give a more general version of Theorem 6.1, which can handle more general decomposable SDPs. Outline of this section is as follows.

- In Section 7.1, we present the main result of this section, Theorem 7.2.
- In Section 7.2, we reduce decomposable SDPs to a form which can be handled by Theorem 4.3.
- In Section 7.3, we state and prove parameters needed to apply Theorem 4.3.
- In Section 7.4, we plug in all parameters and finish proof of Theorem 7.2.

7.1 Main Statement

We consider semidefinite programs of form (16). Instead of assumptions in Definition 5.1, we make the following assumptions on the program.

Definition 7.1. We make the following assumptions on program (16).

- Assume that constraint matrices $A_1, \ldots, A_{n_{\text{sdp}}} \in \mathbb{R}^{n_{\text{sdp}} \times n_{\text{sdp}}}$ are linearly independent.

- Assume that we are given a tree decomposition (Definition 3.5) $(T, J_1, \ldots, J_k)$ of the sparsity graph (Definition 1.8) of (16) with maximum bag size $\tau_{\text{sdp}}$. Note that here we work with the sparsity graph, not the SDP graph (Definition 1.2).

- Assume that for every constraint $A_i$, we have a connected (in the given tree decomposition) set $S_i$ of bags, such that the union of these bags contains the support of $A_i$, i.e.,

$$\{(u, v) \in [n_{\text{sdp}}] \times [n_{\text{sdp}}] : A_{i,u,v} \neq 0\} \subseteq \bigcup_{j \in S_i} J_j \times J_j.$$ 

Let $\gamma_{\text{max}}$ be the maximum number of constraints a bag correspond to.

- Assume that any feasible solution $X \in S^{n_{\text{sdp}}}$ satisfies $\|X\|_{\text{op}} \leq R$.

- Assume that there exists $Z \in S^{n_{\text{sdp}}}$ such that $A \bullet Z = b$ and $\lambda_{\min}(Z) \geq r$.

Theorem 7.2. Under assumptions in Definition 7.1, for any $0 < \epsilon \leq \frac{1}{2}$, there is an algorithm that outputs a solution $U \in \mathbb{R}^{n_{\text{sdp}} \times \tau_{\text{sdp}}}$ such that for $X = UU^\top$, we have $A \bullet X = b$ and

$$C \bullet X \leq \min_{A \bullet X' = b, X' \in S^{n_{\text{sdp}}}} C \bullet X + \epsilon \|C\|_2 R$$

in expected time

$$\tilde{O}(n \cdot \tau_{\text{sdp}}^{0.5} (\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^\omega)$$

where $\tilde{O}$ hides $n^{o(1)}$ terms.
7.2 Reduce Decomposable SDP to General Treewidth Program

In this section we reduce program (16) satisfying Definition 7.1 to form (9).

Let $T, J_1, \ldots, J_n$ be the given tree decomposition. Using Lemma 5.3, we can WLOG assume that $T$ has maximum degree $O(1)$.

Lemma 7.3 tells us that we can reduce program 16 to the following program.

\[
\begin{align*}
\min & \quad C \cdot X \\
\text{s.t.} & \quad A_i \cdot X = b_i \quad \forall i \in [m_{\text{sdp}}] \\
& \quad X_{J_i,J_i} \succeq 0 \quad \forall i \in [n]
\end{align*}
\]

Lemma 7.3 ([ZL18, Section 3.3]). Under assumptions in Definition 7.1, the optimal value of program (16) is equal to the optimal value of (22).

Furthermore, given a solution $X_1, \ldots, X_n$ of program (17), there is an algorithm that takes $O(n_{\text{sdp}}^3)$ time and outputs a matrix $U \in \mathbb{R}^{n_{\text{sdp}} \times n_{\text{sdp}}}$ such that for $X = UU^T$, $X$ is a solution of Eq. (1) with the same objective value.

Therefore, similar to Section 5.2, we decompose $X$ into $n$ smaller PSD variables $X_1, \ldots, X_n$, with $X_i \in \mathbb{S}^{|J_i|}$, satisfying overlapping constraints.

It remains to decompose the constraints $A_i$ to handle the new variables. Recall that we are given the set $S_i$. Because union of $(J_j \times J_j)_{j \in S_i}$ contains the support of $i$, we can find constraints $A_{i1}, \ldots, A_{i|S_i|}$ with $A_{i,j} \in \mathbb{R}^{J_j \times J_j}$ such that

\[A_i \cdot X = \sum_{j \in S_i} A_{i,j} \cdot X_{J_j,J_j}.
\]

We also decompose $C$ into $C_1, \ldots, C_n$ with $C_i \in \mathbb{R}^{J_i \times J_i}$ such that

\[C \cdot X = \sum_{j \in [n]} C_j \cdot X_{J_j,J_j}.
\]

In this way, we further reduce (22) into the following form.

\[
\begin{align*}
\min & \quad \sum_{j \in [n]} C_j \cdot X_j \\
\text{s.t.} & \quad \sum_{j \in S_i} A_{i,j} \cdot X_j = b_i \quad \forall i \in [m_{\text{sdp}}] \\
& \quad N_{i,j}(X_i) = N_{j,i}(X_j) \quad \forall (i, j) \in E(T) \\
& \quad X_j \succeq 0 \quad \forall j \in [n]
\end{align*}
\]

where $X_j \in \mathbb{R}^{|J_j| \times |J_j|}$, $A_{i,j} \in \mathbb{R}^{|J_j| \times |J_j|}$, $N$ constraints asserts consistency between different bags, $N_{i,j} \in \mathbb{R}^{|J_i \cap J_j|^2 \times |J_i|^2}$.

Program (23) is a form which can be handled by Theorem 4.3. Let us compute the treewidth of program (23).

Lemma 7.4. We can construct a treewidth decomposition of the LP dual graph (Definition 4.1) with maximum bag size $\tau_{\text{lp}} = O(\tau_{\text{sdp}}^2 + \gamma_{\text{max}})$ and at most $n$ blocks.

Proof. For every $j \in [n]$, we define a bag $B_j$ containing
• all type-\(A\) constraints \(A_i\) with \(j \in S_i\),
• all type-\(N\) constraints \(N_{i,j}\) with \((i, j) \in T\).

We connected \(B_i\) with \(B_j\) if and only if \((i, j) \in T\).

By assumption, number of type-\(A\) constraints in a bag is at most \(\gamma_{\text{max}}\). Number of type-\(N\) constraints in a bag is at most \(O(|J_j|^2)\) because \(T\) has constant maximum degree. So \(|B_j| \leq O(|J_j|^2 + \gamma_{\text{max}})\). The maximum bag size is \(\tau_p = O(\max_j |J_j|^2 + \gamma_{\text{max}}) = O(\tau_{\text{sdp}}^2 + \gamma_{\text{max}})\).

Finally, we prove that \((T, \bar{B}_1, \ldots, \bar{B}_n)\) is a valid tree decomposition of the LP dual graph. Recall that we assume that every \(S_i\) is connected in \(T\). So the second condition in Definition 3.5 is satisfied. For any two constraints sharing a variable \(X_j\), they must both be in \(B_j\). So the first condition in Definition 3.5 is satisfied. Therefore \((T, \bar{B}_1, \ldots, \bar{B}_n)\) is a valid tree decomposition.  \(\square\)

### 7.3 Choice of Parameters

In this section we present value of parameters needed to apply Theorem 4.3.

**Lemma 7.5** (Choice of parameters in the general result). Under the setting of Theorem 7.2, we can choose the following set of parameters.

(i) \(n = O(n_{\text{sdp}})\), \(n_{\text{lp}} = O(n_{\text{sdp}} \tau_{\text{sdp}}^2)\), \(n_{\text{max}} = O(\tau_{\text{sdp}}^2)\).

(ii) \(m = O(n_{\text{sdp}})\), \(m_{\text{lp}} = O(n_{\text{sdp}} (\tau_{\text{sdp}}^2 + \gamma_{\text{max}}))\), \(m_{\text{max}} = O(\tau_{\text{sdp}}^2 + \gamma_{\text{max}})\).

(iii) \(\tau_{\text{lp}} = O(\tau_{\text{sdp}}^2 + \gamma_{\text{max}})\).

(iv) \(\eta = \tilde{O}(1)\).

(v) \(\nu_{\text{max}} = O(\tau_{\text{sdp}})\).

(vi) \(\text{nnz}(A) = O(n_{\text{sdp}} \tau_{\text{sdp}}^2 (\tau_{\text{sdp}}^2 + \gamma_{\text{max}}))\).

(vii) \(\tau_{\text{lp}} = \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^2)\), \(T_m = \tilde{O}(n_{\text{sdp}} (\tau_{\text{sdp}}^2 + \gamma_{\text{max}})\omega)\).

(viii) \(T_H = \tilde{O}(n_{\text{sdp}} \tau_{\text{sdp}}^2)\), \(T_{H, \text{max}} = \tilde{O}(\tau_{\text{sdp}}^2)\).

(ix) \(T_L = \tilde{O}(n_{\text{sdp}} (\tau_{\text{sdp}}^2 + \gamma_{\text{max}})\omega)\).

(x) \(T_Z = \tilde{O}(n_{\text{sdp}} (\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^2)\).

(xi) \(T_{\Delta L, \text{max}} = \tilde{O}((\tau_{\text{sdp}}^2 + \gamma_{\text{max}})\omega)\).

**Proof.** (i) By discussion in Section 7.2.

(ii) For block structure, we apply Lemma 6.3. So \(m = O(n_{\text{sdp}})\), \(m_{\text{max}} = O(\tau_{\text{lp}}) = O(\tau^2 + \gamma_{\text{max}})\).

(iii) By Lemma 7.4.

(iv) By Lemma 6.3.

(v) Same as Lemma 5.6(v).

(vi) Every \(X_j\) is in at most \(\tau_p = O(\tau_{\text{sdp}}^2 + \gamma_{\text{max}})\) constraints, so \(\text{nnz}(A) = O(n_{\text{sdp}} \tau_{\text{sdp}}^2 \cdot \tau_p) = O(n_{\text{sdp}} \tau_{\text{sdp}}^2 (\tau_{\text{sdp}}^2 + \gamma_{\text{max}}))\).
(vii) Bounds on $T_n$ and $T_m$ are direct consequences of (i), (ii).

(viii) Same as Lemma 5.6(viii).

(ix) By Lemma 6.4, we have $T_L = \widetilde{O}(m \tau^\omega_{lp}) = \widetilde{O}(n_{\text{sdp}}(\tau^2_{\text{sdp}} + \gamma_{\max})^\omega)$.

(x) By Lemma 4.9, we have $T_Z = O(\eta m m^2_{\max}) = \widetilde{O}(n_{\text{sdp}}(\tau^2_{\text{sdp}} + \gamma_{\max})^2)$.

(xi) By Lemma 6.5, we have $T_{\Delta L,\max} = \widetilde{O}(\tau^\omega_{lp}) = \widetilde{O}((\tau^2_{\text{sdp}} + \gamma_{\max})^\omega)$.

\[ \square \]

7.4 Proof of Theorem 7.2

In this section, we prove Theorem 7.2.

Proof of Theorem 7.2. It remains to compute the parameters $T_H$, $T_L$, $\eta$, $T_m$, $m_{\max}$, $n_{\max}$, $T_{\Delta L,\max}$, $T_{H,\max}$, the details can be found in Lemma 6.2.

Let $A$ and $B$ be defined as:

\[ A := T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\max}, \]
\[ B := T_{\text{mat}}(n_{\max}) + T_{\Delta L,\max} + T_{H,\max} + \eta^2 m_{\max}^2. \]

We can show that

\[ A = T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\max} \]
\[ = O(n_{\text{sdp}} \tau^2_{\text{sdp}}) + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\max} \]
\[ = O(n_{\text{sdp}} \tau^2_{\text{sdp}}) + O(n_{\text{sdp}} \tau^2_{\text{sdp}}) + T_L + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\max} \]
\[ = O(n_{\text{sdp}} \tau^2_{\text{sdp}}) + O(n_{\text{sdp}} (\tau^2_{\text{sdp}} + \gamma_{\max})^\omega) + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\max} \]
\[ = O(n_{\text{sdp}} (\tau^2_{\text{sdp}} + \gamma_{\max})^\omega) + T_Z + \text{nnz}(A) + \eta m_{\text{lp}} m_{\max} \]
\[ = O(n_{\text{sdp}} (\tau^2_{\text{sdp}} + \gamma_{\max})^\omega) + O(n_{\text{sdp}} \tau^2_{\text{sdp}}) + \text{nnz}(A) + \eta m_{\text{lp}} m_{\max} \]
\[ = O(n_{\text{sdp}} (\tau^2_{\text{sdp}} + \gamma_{\max})^\omega) + O(n_{\text{sdp}} \tau^2_{\text{sdp}}) + \text{nnz}(A) + \eta m_{\text{lp}} m_{\max} \]
\[ = O(n_{\text{sdp}} (\tau^2_{\text{sdp}} + \gamma_{\max})^\omega) + O(n_{\text{sdp}} \tau^2_{\text{sdp}}) (\tau^2_{\text{sdp}} + \gamma_{\max})^\omega) + \eta m_{\text{lp}} m_{\max} \]
\[ = O(n_{\text{sdp}} (\tau^2_{\text{sdp}} + \gamma_{\max})^\omega) + O(n_{\text{sdp}} \tau^2_{\text{sdp}}) (\tau^2_{\text{sdp}} + \gamma_{\max})^\omega) + O(n_{\text{sdp}} (\tau^2_{\text{sdp}} + \gamma_{\max})^2) \]
\[ = O(n_{\text{sdp}} (\tau^2_{\text{sdp}} + \gamma_{\max})^\omega) \]

where the second step follows from Lemma 7.5(viii) ($T_H = O(n_{\text{sdp}} \tau^2_{\text{sdp}})$) and the third step follows from Lemma 7.5(i) ($T_n = O(n_{\text{sdp}} \tau^2_{\text{sdp}})$), the forth step follows from merging the terms, the fifth step follows from Lemma 7.5(ix) ($T_L = \widetilde{O}(\tau^2_{\text{sdp}} + \gamma_{\max})^\omega$), the sixth step follows from merging the terms, the seventh step follows from $T_Z = O(n_{\text{sdp}} (\tau^2_{\text{sdp}} + \gamma_{\max})^2)$, the eighth step follows from merging the terms, the ninth step follows Lemma 7.5(vi) ($\text{nnz}(A) = n_{\text{sdp}} \tau^2_{\text{sdp}}(\tau^2_{\text{sdp}} + \gamma_{\max})$), and the tenth step follows from Lemma 7.5(iv) and Lemma 7.5(ii).

For the term $B$, we have

\[ B = T_{\text{mat}}(n_{\max}) + T_{\Delta L,\max} + T_{H,\max} + \eta^2 m_{\max}^2 \]
\[ = O(\tau^2_{\text{sdp}}) + T_{\Delta L,\max} + T_{H,\max} + \eta^2 m_{\max}^2 \]
\[\begin{align*}
&= O(\tau_{\text{sdp}}^2) + O((\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^\omega) + T_{H,\text{max}} + \eta^2 m_{\text{max}}^2 \\
&= O(\tau_{\text{sdp}}^2) + O((\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^\omega) + O(\tau_{\text{sdp}}^2) + \eta^2 m_{\text{max}}^2 \\
&= O(\tau_{\text{sdp}}^2) + O((\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^\omega) + O(\tau_{\text{sdp}}^2) + O(\tau_{\text{sdp}}^4) \\
&= O((\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^\omega)
\end{align*}\]  
(25)  

where the second step follows Lemma 7.5(i) \((n_{\text{max}} = \tau_{\text{sdp}}^2)\), and the third step follows from Lemma 7.5(xi) \((T_{\Delta L,\text{max}} = O(\tau_{l_p}^\omega))\), the forth step follows from Lemma 7.5(viii) \((T_{H,\text{max}} = O(\tau_{l_p}^\omega))\), the fifth step follows from Lemma 7.5(iv) \((\eta = \tilde{O}(1)\) and \(m_{\text{max}} = (\tau_{\text{sdp}}^2)\).

Finally, we have

\[\begin{align*}
&\tilde{O}(n_{\text{sdp}}^0.5 \cdot A^0.5 \cdot B^0.5 \cdot \log(1/\epsilon)) \\
&= \tilde{O}(n_{\text{sdp}}^0.5 \cdot \tau_{\text{sdp}}^0.5 \cdot A^0.5 \cdot B^0.5 \cdot \log(1/\epsilon)) \\
&= \tilde{O}(n_{\text{sdp}}^0.5 \cdot A^0.5 \cdot \tau_{\text{sdp}}^0.5 \cdot (n_{\text{sdp}}(\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^\omega)^0.5 \cdot \log(1/\epsilon)) \\
&= \tilde{O}(n_{\text{sdp}}^0.5 \cdot (\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^\omega \cdot \log(1/\epsilon))
\end{align*}\]

where the first step follows from \(n = n_{\text{sdp}}\) and Lemma 7.5(iv) \((\nu_{\text{max}} = O(\tau_{\text{sdp}}))\), the second step follows from \(A = \tilde{O}(n_{\text{sdp}}(\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^\omega)\) (see Eq. (24)) and \(B = \tilde{O}((\tau_{\text{sdp}}^2 + \gamma_{\text{max}})^\omega)\) (see Eq. (25)).

Thus, we complete the proof. \(\square\)
8 Improving Running Time for Linear Programming

In this section, we discuss our result for linear programming,

- In Section 8.1, we present the main result in this Section, Theorem 8.2.
- In Section 8.2, we state and prove parameters needed to apply Theorem 4.3.
- In Section 8.3, we develop improved algorithms for Cholesky-related computation.
- In Section 8.4, we plug in all parameters and finish proof of Theorem 8.2.

8.1 Main Statement

We consider linear program of form (3). Let us restate it here for clarity.

\[
\begin{align*}
\text{min} & \quad c^\top x \\
\text{s.t.} & \quad Ax = b \\
& \quad \ell \leq x \leq u
\end{align*}
\]  \hspace{1cm} (26)

**Definition 8.1.** We make the following assumptions on program (26).

- Assume that the constraint matrix \( A \in \mathbb{R}^{m \times n} \) has full column rank.
- Assume that we are given a tree decomposition (Definition 3.5) of the LP dual graph (Definition 1.6) with maximum bag size \( \tau \).
- Assume that there exists \( x \) such that \( Ax = b \) and \( \ell + r \leq x \leq u - r \), for some \( r > 0 \).

**Theorem 8.2** (Our LP result, formal version of Theorem 1.7). Under assumptions in Definition 8.1, for any \( 0 < \epsilon \leq \frac{1}{2} \), there is an algorithm that outputs a solution \( x \in \mathbb{R}^n \) such that \( Ax = b, \ell \leq x \leq u, \) and

\[
c^\top x \leq \min_{Ax = b, \ell \leq x' \leq u, x' \in \mathbb{R}^n} c^\top x' + \epsilon LR
\]

where \( L = \|c\|_2, R = \|u - \ell\|_2 \), in expected time

\[
\tilde{O}(n \cdot \tau^{(\omega + 1)/2} \log(R/(r\epsilon))).
\]

8.2 Choice of Parameters

**Lemma 8.3** (Choice of parameters in the LP result). Under the setting of Theorem 8.2, we can choose the following set of parameters.

(i) \( n_{lp} = n, n_{\text{max}} = 1 \).
(ii) \( m = O(n), m_{lp} = O(n), m_{\text{max}} = 1 \).
(iii) \( \tau_{lp} = O(\tau) \).
(iv) \( \eta = \tilde{O}(\tau) \).
(v) \( \nu_{\text{max}} = O(1) \).
(vi) $\text{nnz}(A) = O(n\tau)$.
(vii) $T_n = O(n)$, $T_m = O(n)$.
(viii) $T_H = O(n)$, $T_{H,\text{max}} = O(1)$.
(ix) $T_L = O(n\tau^{\omega-1})$.
(x) $T_Z = O(n\tau^{\omega-1})$.
(xi) $T_{\Delta L,\text{max}} = O(\tau^2)$.

**Proof.**

(i) We let every variable block contain a single element.
(ii) Because $A$ is of full rank, $m \leq n$. We let every constraint block contain a single element.
(iii) The LP dual graph for program 9 and 26 are the same graphs.
(iv) Follows from Lemma 5.7.
(v) We use the log barrier $\phi_i(x_i) = -\log(u_i - x_i) - \log(x_i - l_i)$.
(vi) Every variable is contained in at most $\tau$ constraints, so $\text{nnz}(A) = O(n\tau)$.
(vii) Bounds on $T_n$ and $T_m$ are direct consequences of (i), (ii).
(viii) Computing a single Hessian takes $O(1)$ time.
(ix) Follows from Lemma 8.4.
(x) Follows from Lemma 8.6. There is one caveat: in the statement of Lemma 8.6, we use a block elimination tree $T_2$ constructed using Lemma 6.3. For definition of $T_Z$, we use block elimination tree $T_1$ constructed using Lemma 5.7. However by examining both algorithms, we see that every path in $T_1$ is contained in a path in $T_2$, and vice versa. So Lemma 8.6 can be applied here.
(xii) Follows from Lemma 3.11.

\[ \square \]

### 8.3 Improvement for Cholesky-Related Computation

In this section we prove Lemma 8.4 and Lemma 8.6.

In Lemma 8.3 we choose constraint block structure $(1, \ldots, 1)$ for the purpose of smaller updating time (e.g., $T_{\Delta L,\text{max}}$). Nevertheless, we can utilize another block structure to achieve faster initialization (e.g., $T_L$, $T_Z$).

**Lemma 8.4.** Let $M \in \mathbb{R}^{m \times m}$ be a PSD matrix. Let $G$ be its adjacency graph (i.e., there is an edge $(i, j)$ if and only if $M_{i,j} \neq 0$). Suppose we are given a tree decomposition of $G$ with maximum bag size $\tau$. Then we can compute the Cholesky factorization $M = LL^\top$ in $\tilde{O}(m\tau^{\omega-1})$ time.

**Proof.** Using Lemma 6.3, we can construct a block elimination tree $T$ with constant maximum degree, maximum depth $O(1)$ and maximum bag size $O(\tau)$. However, this block elimination tree could potentially have $\Omega(m)$ vertices. So we use Lemma 8.5 to compute a new block elimination tree $T'$ with number of blocks $O(m/\tau)$. Finally, we use Lemma 6.4 to compute Cholesky factorization using $T'$, which takes $\tilde{O}(m/\tau \cdot \tau^\omega) = \tilde{O}(m\tau^{\omega-1})$ time. \[ \square \]
Lemma 8.5. Given a block elimination tree \((\mathcal{T}, B_1, \ldots, B_h)\) with maximum degree \(O(1)\), maximum depth \(\tilde{O}(1)\), maximum block size \(\tau\), and total block size \(m\), we can construct a block elimination tree \((\mathcal{T}', B'_1, \ldots, B'_h)\) with maximum degree \(O(1)\), maximum depth \(\tilde{O}(1)\), maximum block size \(O(\tau)\), and \(O(m/\tau)\) blocks in total (i.e., \(b' = O(m/\tau)\)).

Proof. We perform a bottom up process to construct the new tree \(\mathcal{T}'\).

For each node \(v\) from bottom to up, if any of its children is in a block of size smaller than \(\tau\), then we combine their blocks with \(v\) (if there are multiple such children, then all of their blocks are merged).

In this way every block (except possibly the root block) has size at least \(\tau\). Because \(\mathcal{T}\) has \(O(1)\) maximum degree, every new block has \(O(\tau)\) size. So the number of blocks in \(\mathcal{T}'\) is \(O(m/\tau)\).

Because \(\mathcal{T}'\) preserves all ancestor-descendant relationships in \(\mathcal{T}\), condition in Lemma 3.9 is satisfied by \(\mathcal{T}'\). So \(\mathcal{T}'\) is still a block elimination tree.

The only remaining problem is that \(\mathcal{T}'\) could have nodes with large degree. For every node \(v\) with number of children \(c\) larger than 3, we replace this node with a perfect binary tree with \(c\) leaves, where the root node is \(v\), and all other nodes are empty. Then we link \(v\)’s original children to leaves of the perfect binary tree.

The number of added nodes is at most \(O(m/\tau)\). So this final tree satisfies all requirements. 

Lemma 8.6. Under the setting of Lemma 8.4, there is an algorithm to compute \(L^{-1}v_i\) for all \(i \in [m]\), where \(v_i\) is supported on a single path in \(\mathcal{T}\), in \(O(m\tau^{\omega-1})\) time.

Proof. By our construction in proof of Theorem 8.4, every block in \(\mathcal{T}'\) is the union of several blocks in \(\mathcal{T}\). For \(i \in [m]\), let \(b_i \in [b']\) be the block it belongs to in \(\mathcal{T}'\). Note that \(\mathcal{T}'\) preserves all ancestor-descendant relationships in \(\mathcal{T}\). So for every \(i \in [b]\), we have

\[
\bigcup_{j \in \mathcal{P}(i)} B_j \subseteq \bigcup_{j \in \mathcal{P}'(b_i)} B'_j
\]

So every path in \(\mathcal{T}\) is contained in a path in \(\mathcal{T}'\).

Therefore, we only need to solve the following problem: Compute \(L^{-1}v_i\) for \(i \in [m]\), where every \(v_i\) is supported on a single path in \(\mathcal{T}'\). Because \(\mathcal{T}'\) has maximum depth \(\tilde{O}(1)\), we can assume that every \(v_i\) is supported on a single block in \(\mathcal{T}'\) (with an \(\tilde{O}(1)\) factor loss in running time). Then the desired result follows from combining Lemma 8.7 and Lemma 8.8 on \(\mathcal{T}'\) and \(L\).

Lemma 8.7 (Inverse of Cholesky factor). Suppose we have a block elimination tree \((\mathcal{T}, B_1, \ldots, B_h)\) with maximum depth \(\tilde{O}(1)\), maximum block size \(O(\tau)\) and the total number of blocks is \(b = O(m/\tau)\). Let \(L \in \mathbb{R}^{m \times m}\) be a lower-triangular matrix compatible with \(\mathcal{T}\), i.e., for \(i, j \in [b]\), \(L_{i,j} \neq 0\) only if \(i \in \mathcal{P}(j)\). (Note that here \(L_{i,j}\) denotes the \(i\)-th block-row and \(j\)-th block-column.)

Then \(L^{-1}\) is also a lower-triangular matrix compatible with \(\mathcal{T}\), and we can compute \(L^{-1}\) in \(\tilde{O}(n\tau^{\omega-1})\) time.

Proof. Run Algorithm 18. Correctness is obvious. Let us focus on running time.

By induction, we can see that \(X_{j,k} \neq 0\) only when \(j \in \mathcal{P}(k)\). So we perform \(O(1)\) matrix multiplications for every tuple \((i, j, k)\) with \(i \in \mathcal{P}(j), j \in \mathcal{P}(k)\). Because maximum depth is \(\tilde{O}(1)\), number of such triples is \(\tilde{O}(m/\tau)\). So total running time is \(\tilde{O}(m/\tau \cdot \tau^{\omega-1}) = \tilde{O}(m\tau^{\omega-1})\).

Lemma 8.8. Work under the setting of Lemma 8.7. Let \(P\) be a lower-triangular matrix compatible with \(\mathcal{T}\). Let \(v_1, \ldots, v_m \in \mathbb{R}^m\) where support of every \(v_i\) is contained in a block. Then we can compute \(Pv_1, \ldots, Pv_m\) in \(\tilde{O}(n\tau^{\omega-1})\) time.
Algorithm 18

1: procedure BlockCholeskyInverse() \(\triangleright\) Lemma 8.7
2: \(V \leftarrow I, X \leftarrow 0\)
3: for \(j \leftarrow 1\) to \(b\) do
4: \(X_{j,*} \leftarrow L_{j,j}^{-1} V_{j,*}\)
5: \(V \leftarrow V - L_{*,j} X_{j,*}\)
6: end for
7: return \(X\)
8: end procedure

**Proof.** For \(j \in [b]\), let \(p_j\) be the number of vectors whose support is contained in \(j\)-th block. We combine these vectors into a matrix \(V\) of size \(|B_j| \times p_j\). To compute \(PV\), it suffices to compute \(P_{i,j} V\) for \(i \in \mathcal{P}(j)\). We use fast matrix multiplication to perform this task.

Overall running time is

\[
\sum_{j \in [b]} \sum_{i \in \mathcal{P}(j)} \tilde{O}(\text{mat}(|B_i|, |B_j|, p_j))
= \sum_{j \in [b]} \sum_{i \in \mathcal{P}(j)} \tilde{O}(\text{mat}(\tau, \tau, \tau + p_j))
= \sum_{j \in [b]} \tilde{O}(\text{mat}(\tau, \tau, \tau + p_j))
= \sum_{j \in [b]} \tilde{O}((\tau + p_j)\tau^{\omega-1})
= \sum_{j \in [b]} \tilde{O}(m\tau^{\omega-1})
\]

where in the last step we use that \(b = O(m/\tau)\) and \(\sum_{j \in [b]} p_j = O(m)\). \(\square\)

**8.4 Proof of Theorem 8.2**

**Proof.** According to Theorem 4.3, there is an algorithm solving LP in

\[
\tilde{O}(n^{0.5}, \nu_{\text{max}}^{0.5} \cdot (T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{lp} m_{\text{max}})^{0.5} \cdot (\text{mat}(n_{\text{max}}) + T_{\Delta_L, \text{max}} + T_{H, \text{max}} + \eta^2 m_{\text{max}}^2)^{0.5} \log(R/(r\epsilon)))
\]

time.

It remains to compute the parameters \(T_H, T_L, \eta, T_m, m_{\text{max}}, n_{\text{max}}, T_{\Delta_L, \text{max}}, T_{H, \text{max}}\), the details can be found in Lemma 8.3.

Let \(A\) and \(B\) be defined as:

\[
A := T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{lp} m_{\text{max}},
B := T_{\text{mat}}(n_{\text{max}}) + T_{\Delta_L, \text{max}} + T_{H, \text{max}} + \eta^2 m_{\text{max}}^2.
\]

We can show that

\[
A = T_H + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{lp} m_{\text{max}}
= O(n) + T_n + T_L + T_Z + \text{nnz}(A) + \eta m_{lp} m_{\text{max}}
\]

72
\[ O(n) + O(n\tau^{-1}) + T_L + T_Z + \text{nnz}(A) + \eta m_p m_{\text{max}} \]

\[ = O(n) + O(n\tau^{-1}) + \tilde{O}(n\tau^{-1}) + T_Z + \text{nnz}(A) + \eta m_p m_{\text{max}} \]

\[ = \tilde{O}(n\tau^{-1}) + T_Z + \text{nnz}(A) + \eta m_p m_{\text{max}} \]

\[ = \tilde{O}(n\tau^{-1}) + O(n\tau^{-1}) + \text{nnz}(A) + \eta m_p m_{\text{max}} \]

\[ = \tilde{O}(n\tau^{-1}) + O(n\tau^{-1}) + O(n\tau) + O(n\tau) \]

\[ = \tilde{O}(n\tau^{-1}) \] (27)

where the second step follows from Lemma 8.3(viii) \((T_H = O(n))\) and the third step follows from Lemma 8.3(vii) \((T_n = O(n\tau^{-1}))\), the fourth step follows from Lemma 8.3(ix) \((T_L = O(n\tau^{-1}))\), the fifth step follows from merging the terms, the sixth step follows Lemma 8.3(x), the seventh step follows from Lemma 8.3(vi), the eighth step follows from Lemma 8.3(iv) \((\eta = \tilde{O}(\tau), m_p = O(n), m_{\text{max}} = 1)\), and the last step follows from merging the terms.

For the term \(B\), we have

\[
B = T_{\text{mat}}(n_{\text{max}}) + T_{\Delta L, \text{max}} + T_{H, \text{max}} + \eta^2 m_{\text{max}}^2 \\
= O(1) + T_{\Delta L, \text{max}} + T_{H, \text{max}} + \eta^2 m_{\text{max}}^2 \\
= O(1) + O(\tau^2) + T_{H, \text{max}} + \eta^2 m_{\text{max}}^2 \\
= O(1) + O(\tau^2) + O(1) + \eta^2 m_{\text{max}}^2 \\
= O(1) + O(\tau^2) + O(1) + \tilde{O}(\tau^2) \\
= \tilde{O}(\tau^2) \] (28)

where the second step follows Lemma 8.3(i) \((n_{\text{max}} = O(1))\), and the third step follows from Lemma 8.3(xi) \((T_{\Delta L, \text{max}} = O(\tau^2))\), the fourth step follows from Lemma 8.3(viii) \((T_{H, \text{max}} = O(1))\), the fifth step follows from Lemma 8.3(iv) \((\eta = \tilde{O}(\tau)\) and \(m_{\text{max}} = O(1))\), and the last step follows from merging the terms.

Finally, we have

\[
\tilde{O}(n^{0.5}\nu_{\text{max}}^{0.5} \cdot A^{0.5} \cdot B^{0.5} \cdot \log(1/\epsilon)) \\
= \tilde{O}(n^{0.5} \cdot A^{0.5} \cdot B^{0.5} \cdot \log(1/\epsilon)) \\
= \tilde{O}(n^{0.5} \cdot (n\tau^{-1})^{0.5} \cdot (\tau^2)^{0.5} \cdot \log(1/\epsilon)) \\
= \tilde{O}(n\tau^{(\omega+1)/2}) \cdot \log(1/\epsilon))
\]

where the first step follows from Lemma 8.3(iv) \((\nu_{\text{max}} = O(1))\), the second step follows from \(A = \tilde{O}(n\tau^{-1})\) (see Eq. (27)) and \(B = \tilde{O}(\tau^2)\) (see Eq. (28)).

Thus, we complete the proof. 

References

[AK07] Sanjeev Arora and Satyen Kale. A combinatorial, primal-dual approach to semidefinite programs. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing (STOC)*, 2007.

[ALO16] Zeyuan Allen Zhu, Yin Tat Lee, and Lorenzo Orecchia. Using optimization to obtain a width-independent, parallel, simpler, and faster positive SDP solver. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 2016.

[Ans00] Kurt M Anstreicher. The volumetric barrier for semidefinite programming. *Mathematics of Operations Research*, 2000.

[ARV09] Sanjeev Arora, Satish Rao, and Umesh Vazirani. Expander flows, geometric embeddings and graph partitioning. *Journal of the ACM (JACM)*, 2009.

[AW21] Josh Alman and Virginia Vassilevska Williams. A refined laser method and faster matrix multiplication. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 522–539. SIAM, 2021.

[AZL17] Zeyuan Allen-Zhu and Yuanzhi Li. Follow the compressed leader: faster online learning of eigenvectors and faster mmwu. In *Proceedings of the 34th International Conference on Machine Learning (ICML)*, 2017.

[Ban10] Nikhil Bansal. Constructive algorithms for discrepancy minimization. In *2010 IEEE 51st Annual Symposium on Foundations of Computer Science*, pages 3–10. IEEE, 2010.

[BGHK95] Hans L Bodlaender, John R Gilbert, Hjálmtyr Hafsteinsson, and Ton Kloks. Approximating treewidth, pathwidth, frontsize, and shortest elimination tree. *Journal of Algorithms*, 18(2):238–255, 1995.

[BGS22] Aaron Bernstein, Maximilian Probst Gutenberg, and Thatchaphol Saranurak. Deterministic decremental sssp and approximate min-cost flow in almost-linear time. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1000–1008. IEEE, 2022.

[BLSS20] Jan van den Brand, Yin Tat Lee, Aaron Sidford, and Zhao Song. Solving tall dense linear programs in nearly linear time. In *52nd Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, 2020.

[Bor57] Jan Lourens Bordewijk. Inter-reciprocity applied to electrical networks. *Applied Scientific Research, Section A*, 6(1):1–74, 1957.

[BS16] Boaz Barak and David Steurer. Proofs, beliefs, and algorithms through the lens of sum-of-squares. *Course notes: http://www.sumofsquares.org/public/index.html*, 2016.

[BV02] Dimitris Bertsimas and Santosh Vempala. Solving convex programs by random walks. In *Proceedings of the thirty-fourth annual ACM symposium on Theory of computing (STOC)*, pages 109–115. ACM, 2002.
[CDST19] Yair Carmon, John C. Duchi, Aaron Sidford, and Kevin Tian. A rank-1 sketch for matrix multiplicative weights. In Conference on Learning Theory (COLT), pages 589–623, 2019.

[CLS19] Michael B Cohen, Yin Tat Lee, and Zhao Song. Solving linear programs in the current matrix multiplication time. In Proceedings of the 51st Annual ACM Symposium on Theory of Computing (STOC), 2019.

[CN21] Yeshwanth Cherapanamjeri and Jelani Nelson. Terminal embeddings in sublinear time. In FOCS, 2021.

[CR09] Emmanuel J Candès and Benjamin Recht. Exact matrix completion via convex optimization. Foundations of Computational mathematics, 9(6):717–772, 2009.

[CSTZ22] Sitan Chen, Zhao Song, Runzhou Tao, and Ruizhe Zhang. Symmetric sparse boolean matrix factorization and applications. In 13th Innovations in Theoretical Computer Science Conference (ITCS 2022). Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2022.

[Dan47] George B Dantzig. Maximization of a linear function of variables subject to linear inequalities. Activity analysis of production and allocation, 13:339–347, 1947.

[DDH07] James Demmel, Ioana Dumitriu, and Olga Holtz. Fast linear algebra is stable. Numerische Mathematik, 108(1):59–91, 2007.

[DH99] Timothy A Davis and William W Hager. Modifying a sparse cholesky factorization. SIAM Journal on Matrix Analysis and Applications, 20(3):606–627, 1999.

[DLY21] Sally Dong, Yin Tat Lee, and Guanghao Ye. A nearly-linear time algorithm for linear programs with small treewidth: a multiscale representation of robust central path. In Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing, pages 1784–1797, 2021.

[DSST89] James R Driscoll, Neil Sarnak, Daniel D Sleator, and Robert E Tarjan. Making data structures persistent. Journal of computer and system sciences, 38(1):86–124, 1989.

[FJ97] Alan Frieze and Mark Jerrum. Improved approximation algorithms for maxk-cut and max bisection. Algorithmica, 18(1):67–81, 1997.

[FKP19] Noah Fleming, Pravesh Kothari, and Toniann Pitassi. Semialgebraic Proofs and Efficient Algorithm Design. Foundations and Trends in Theoretical Computer Science, 2019.

[GH16] Dan Garber and Elad Hazan. Sublinear time algorithms for approximate semidefinite programming. Mathematical Programming, 158(1-2):329–361, 2016.

[GLN94] Alan George, Joseph Liu, and Esmond Ng. Computer solution of sparse linear systems. Oak Ridge National Laboratory, 1994.

[GW95] Michel X Goemans and David P Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. Journal of the ACM (JACM), 1995.
[HJS+22] Baihe Huang, Shunhua Jiang, Zhao Song, Runzhou Tao, and Ruizhe Zhang. Solving sdp faster: A robust ipm framework and efficient implementation. In FOCS, 2022.

[IP01] Russell Impagliazzo and Ramamohan Paturi. On the complexity of k-sat. Journal of Computer and System Sciences, 62(2):367–375, 2001.

[JJUW11] Rahul Jain, Zhengfeng Ji, Sarvagya Upadhyay, and John Watrous. QIP = PSPACE. Journal of the ACM (JACM), 2011.

[JKL+20] Haotian Jiang, Tarun Kathuria, Yin Tat Lee, Swati Padmanabhan, and Zhao Song. A faster interior point method for semidefinite programming. In FOCS, 2020.

[JL84] William B Johnson and Joram Lindenstrauss. Extensions of lipschitz mappings into a hilbert space. Contemporary mathematics, 26(189-206):1, 1984.

[JLSW20] Haotian Jiang, Yin Tat Lee, Zhao Song, and Sam Chi-wai Wong. An improved cutting plane method for convex optimization, convex-concave games and its applications. In STOC, 2020.

[MY11] Rahul Jain and Penghui Yao. A parallel approximation algorithm for positive semidefinite programming. In Proceedings of the 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science (FOCS), 2011.

[Kar72] Richard M Karp. Reducibility among combinatorial problems. In Complexity of computer computations, pages 85–103. Springer, 1972.

[Kar84] Narendra Karmarkar. A new polynomial-time algorithm for linear programming. In Proceedings of the sixteenth annual ACM symposium on Theory of computing (STOC), 1984.

[Kha80] Leonid G Khachiyan. Polynomial algorithms in linear programming. USSR Computational Mathematics and Mathematical Physics, 20(1):53–72, 1980.

[KKMO07] Subhash Khot, Guy Kindler, Elchanan Mossel, and Ryan O’Donnell. Optimal inapproximability results for max-cut and other 2-variable csps? SIAM Journal on Computing, 37(1):319–357, 2007.

[KM03] Kartik Krishnan and John E Mitchell. Properties of a cutting plane method for semidefinite programming. submitted for publication, 2003.

[KMS94] David Karger, Rajeev Motwani, and Madhu Sudan. Approximate graph coloring by semidefinite programming. In Proceedings 35th Annual Symposium on Foundations of Computer Science (FOCS). IEEE, 1994.

[KTE88] Leonid G Khachiyan, Sergei Pavlovich Tarasov, and I. I. Erlikh. The method of inscribed ellipsoids. Soviet Math. Dokl, 37(1):226–230, 1988.

[LMS11] Daniel Lokshtanov, Dániel Marx, and Saket Saurabh. Known algorithms on graphs of bounded treewidth are probably optimal. In Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms, pages 777–789. SIAM, 2011.

[Lov79] László Lovász. On the shannon capacity of a graph. IEEE Transactions on Information theory, 25(1):1–7, 1979.
Yin Tat Lee and Swati Padmanabhan. An $\tilde{O}(m/\epsilon^{3.5})$-cost algorithm for semidefinite programs with diagonal constraints. In Conference on Learning Theory (COLT), Proceedings of Machine Learning Research. PMLR, 2020.

Yin Tat Lee, Aaron Sidford, and Sam Chiu-wai Wong. A faster cutting plane method and its implications for combinatorial and convex optimization. In 56th Annual IEEE Symposium on Foundations of Computer Science (FOCS), 2015.

Yin Tat Lee, Zhao Song, and Qiuyi Zhang. Solving empirical risk minimization in the current matrix multiplication time. In Annual Conference on Learning Theory (COLT), 2019.

Yin Tat Lee and Man-Chung Yue. Universal barrier is n-self-concordant. Mathematics of Operations Research, 46(3):1129–1148, 2021.

Yurii Nesterov. Introductory lectures on convex optimization: A basic course, volume 87. Springer Science & Business Media, 2003.

Yurii Nesterov and Arkadi Nemirovski. Self-concordant functions and polynomial time methods in convex programming. preprint, central economic & mathematical institute, ussr acad. Sci. Moscow, USSR, 1989.

Yurii Nesterov and Arkadi Nemirovski. Conic formulation of a convex programming problem and duality. Optimization Methods and Software, 1(2):95–115, 1992.

Yurii Nesterov and Arkadi Nemirovski. Interior-point polynomial algorithms in convex programming, volume 13. Siam, 1994.

Christos H Papadimitriou and Mihalis Yannakakis. Optimization, approximation, and complexity classes. Journal of computer and system sciences, 43(3):425–440, 1991.

James Renegar. A polynomial-time algorithm, based on newton’s method, for linear programming. Mathematical Programming, 40(1-3):59–93, 1988.

Aditi Raghunathan, Jacob Steinhardt, and Percy S Liang. Semidefinite relaxations for certifying robustness to adversarial examples. In Advances in Neural Information Processing Systems (NeurIPS), pages 10877–10887, 2018.

Robert Schreiber. A new implementation of sparse gaussian elimination. ACM Transactions on Mathematical Software (TOMS), 8(3):256–276, 1982.

Naum Z Shor. Cut-off method with space extension in convex programming problems. Cybernetics and systems analysis, 13(1):94–96, 1977.

Daniel D Sleator and Robert Endre Tarjan. A data structure for dynamic trees. In Proceedings of the thirteenth annual ACM symposium on Theory of computing, pages 114–122, 1981.

Zhao Song and Zheng Yu. Oblivious sketching-based central path method for solving linear programming. In ICML, 2021.

Pravin M Vaidya. An algorithm for linear programming which requires $O((m+n)n^2 + (m+n)^{1.5}n)L$ arithmetic operations. In 28th Annual IEEE Symposium on Foundations of Computer Science (FOCS), 1987.
[Vai89] Pravin M Vaidya. A new algorithm for minimizing convex functions over convex sets. In *30th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 338–343, 1989.

[Wil12] Virginia Vassilevska Williams. Multiplying matrices faster than coppersmith-winograd. In *Proceedings of the forty-fourth annual ACM symposium on Theory of computing (STOC)*, pages 887–898. ACM, 2012.

[Ye20] Guanghao Ye. Fast algorithm for solving structured convex programs. *The University of Washington, Undergraduate Thesis*, 2020.

[YN76] David B Yudin and Arkadi S Nemirovski. Evaluation of the information complexity of mathematical programming problems. *Ekonomika i Matematicheskie Metody*, 12:128–142, 1976.

[YTF+19] Alp Yurtsever, Joel A. Tropp, Olivier Fercoq, Madeleine Udell, and Volkan Cevher. Scalable semidefinite programming, 2019.

[ZL18] Richard Y Zhang and Javad Lavaei. Sparse semidefinite programs with near-linear time complexity. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 1624–1631. IEEE, 2018.
Appendix

A Robust IPM Analysis

The goal of this section is to present some existing tools which give us a bound on the number of iterations of IPM. To really improve the total running time, we still need to improve the cost per iteration which is a major contribution of our work. Those discussions can be found in Section 4.

Let us begin with a roadmap for this section. In Section A.1, we present the main convergence statement. In Section A.2, we explain the the choice of step and present a useful lemma.

A.1 Main Statement

We consider program of form (9). The following theorem gives a bound on the number of iterations of a robust IPM algorithm.

**Theorem A.1 ([DLY21, Theorem A.1])**. Consider program of form (9). Assume we are given \( \nu_i \)-self-concordant barriers \( \phi_i : \mathcal{K}_i \to \mathbb{R} \). Furthermore, assume that the program satisfies the following properties.

- **Inner radius** \( r \): There exists a \( z \) such that \( Az = b \) and \( B(z, r) \subset \mathcal{K} \).
- **Outer radius** \( R \): We have \( \mathcal{K} \subset B(x, R) \) for some \( x \in \mathbb{R}^{n_{lp}} \).
- **Lipschitz constant** \( L \): \( \|c\|_2 \leq L \).

Let \( w \in \mathbb{R}_{\geq 1}^n \) be any weight vector, and \( \kappa = \sum_{i=1}^{n} w_i \nu_i \). For any \( \epsilon \in (0, 1/2] \), there is an algorithm that runs in \( O(\sqrt{\kappa} \log n \log(\frac{n_{lp} R}{\epsilon r})) \) iterations and output \( x \in \mathcal{K} \) and \( Ax = b \) such that:

\[
c^\top x \leq \min_{Ax = b, x \in \mathcal{K}} c^\top x + \epsilon LR.
\] (29)

The above Theorem A.1 provides the iteration bound for Algorithm 19. The overall running time largely depends on cost per iteration, which is decided by the implementation of CENTRALPATHMAINTENANCE. The high level framework of Algorithm 19 and 20 are the same as [DLY21], but the reduction to form (9) and the implementations of CENTRALPATHMAINTENANCE are different.

A.2 Definitions and Useful Lemmas

We define the following induced norms.

**Definition A.2 ([DLY21, Definition A.5])**. For each block \( \mathcal{K}_i \), we define

\[
\|v\|_{x_i} := \|v\|_{\nabla^2 \phi_i(x_i)} \\
\|v\|_{x_i}^* := \|v\|_{(\nabla^2 \phi_i(x_i))^{-1}}
\]

for \( v \in \mathbb{R}^{n_i} \).

For the whole domain \( \mathcal{K} = \prod_{i=1}^{n} \mathcal{K}_i \), we define

\[
\|v\|_x := \|v\|_{\nabla^2 \phi(x)} = \left( \sum_{i=1}^{n} w_i \|v_i\|_{x_i}^2 \right)^{1/2}
\]
Algorithm 19 Robust IPM algorithm. This decides the number of iterations but not the cost per iteration.

1: procedure RobustIPM($A \in \mathbb{R}^{m_p \times n_p}, b \in \mathbb{R}^{m_p}, c \in \mathbb{R}^{n_p}, (\phi_i : \mathbb{R}^{n_i} \to \mathbb{R})_{i \in [n]}, w \in \mathbb{R}^n$)
2: Input: Program of form (9) satisfying the assumptions in Theorem A.1.
3: Output: A solution $x$ satisfying the statement of Theorem A.1.
4: Let $\phi(x) = \sum_{i=1}^{n} w_i \phi_i(x_i)$, $L = \|c\|_2$, $\kappa = \sum_{i=1}^{n} w_i \mu_i$
5: Let $t = (n_p + \kappa)^5 \cdot \frac{LR}{L} \cdot \frac{R}{r}$ with $\delta = 1/128$
6: Compute $x_c = \arg \min_{x \in \mathbb{K}} c^\top x + t \cdot \phi(x)$ and $x_0 = \arg \min_{Ax=b} \|x - x_c\|_2$
7: Let $x = (x_c, 3R + x_0 - x_c, 3R)$ and $s = (-\nabla \phi(x_c), \frac{t}{3R + x_0 - x_c}, \frac{t}{3R})$
8: Let the new matrix $A^{\text{new}} = [A, A, -A]$, the new barrier and new weight

$$
\phi_i^{\text{new}} = \begin{cases} 
\phi_i, & \text{if } i \in [n_p], \\
-\log x, & \text{else}
\end{cases}
$$
and

$$
w_i^{\text{new}} = \begin{cases} 
w_i, & \text{if } i \in [n_p], \\
1, & \text{else}
\end{cases}
$$

9: $(x^{(1)}, x^{(2)}, x^{(3)}), (s^{(1)}, s^{(2)}, s^{(3)}) \leftarrow \text{CENTERING}(A^{\text{new}}, \phi^{\text{new}}, w^{\text{new}}, x, s, t, LR)$
10: $(x, s) \leftarrow (x^{(1)} + x^{(2)} - x^{(3)}, s^{(1)})$
11: $(x, s) \leftarrow \text{CENTERING}(A, \phi, w, x, s, LR, \frac{e}{4 \sum_i w_i \nu_i})$
12: return $x$
13: end procedure

and

$$
\|v\|_v^*: = \|v\|_{(\nabla^2 \phi(x))^{-1}} = \sum_{i=1}^{n} w_i^{-1} (\|v_i\|_{v_i}^*)^2)^{1/2}
$$
for $v \in \mathbb{R}^{n_p}$.

The central path is

$$
x(t) := \arg \min_{Ax=b} c^\top x + t \cdot \phi(x) \quad \text{with} \quad \phi(x) := \sum_{i=1}^{n} w_i \phi_i(x_i).
$$

Instead of following the path $x(t)$ exactly, we follow the path

$$
\begin{align*}
\frac{s}{t} + w \cdot \nabla \phi(x) &= \mu, \\
Ax &= b, \\
A^\top y + s &= c
\end{align*}
$$

where $\mu$ is close to 0 under $\|\cdot\|_v^*$. The norm of $\mu$ is controlled using the following potential function.

Definition A.3 (Potential function, [DLY21, Definition A.7]). For $i \in [n]$, define error at $i$-th variable block as

$$
\mu_i^t(x, s) := \frac{s_i}{t} + w_i \cdot \nabla \phi_i(x_i).
$$

Define $\gamma_i^t(x, s) := \|\mu_i^t(x, s)\|_{v_i}^*$. Define the soft-max function as

$$
\Psi_\lambda(r) := \sum_{i=1}^{n} \cosh(\lambda \frac{r_i}{w_i})
$$

80
for some $\lambda > 0$. Finally, the potential function is the soft-max of norm of the error at each variable block

$$\Phi^t(x, s) = \Psi_\lambda(\gamma^t(x, s)).$$

Since our goal is to decrease $\Phi(x, s) = \Psi_\lambda(\gamma)$, a natural choice is the steepest descent direction ([DLY21, Section A.4]):

$$\delta^*_\mu = \arg \min_{\|\delta_\mu\|_s = \alpha} \langle \nabla_\mu \Psi_\lambda(\|\mu_i\|_s^2), \mu + \delta_\mu \rangle$$ (30)
with step size $\alpha$.

Solving this gives

$$
\delta_{\mu,i}^*(x, s, t) = -\frac{\alpha \sinh(\frac{1}{w_i}\gamma_i(x, s))}{\gamma_i(x, s) \cdot (\sum_{j \in [n]} w_{ji}^{-1} \sinh^2(\frac{1}{w_j}\gamma_j(x, s)))^{1/2}} \cdot \mu_i(x, s).
$$

(In the actual algorithm, $\sinh$ in the denominator is replaced by $\cosh$ for easier analysis.) To move $\mu \in \mathbb{R}^{n_p}$ to $\mu + \delta \mu \in \mathbb{R}^{n_p}$ approximately, we take Newton step $\delta_x^*, \delta_s^* \in \mathbb{R}^{n_p}$:

$$
\frac{1}{t} \delta_s^* + \nabla^2 \phi(x) \delta_x^* = \delta_{\mu}(x, s),
A \delta_x^* = 0,
A^T \delta_y^* + \delta_s^* = 0.
$$

Using $H_x$ to denote $\nabla^2 \phi(x)$ and solve the above equations, we get

$$
\delta_x^* = H_x^{-1} \delta_{\mu} - H_x^{-1} A^T (AH_x^{-1} A^T)^{-1} A H_x^{-1} \delta_{\mu}(x, s, t),
\delta_s^* = tA^T (AH_x^{-1} A^T)^{-1} A H_x^{-1} \delta_{\mu}(x, s, t).
$$

This is the ideal IPM step. In robust IPM, we compute the steps using $(\overline{x}, \overline{s}, \overline{t})$, a sparsely changing approximation of $(x, s, t)$, giving

$$
\delta_x = H_{\overline{x}}^{-1} \delta_{\mu} - H_{\overline{x}}^{-1} A^T (AH_{\overline{x}}^{-1} A^T)^{-1} A H_{\overline{x}}^{-1} \delta_{\mu}(\overline{x}, \overline{s}, \overline{t}),
\delta_s = tA^T (AH_{\overline{x}}^{-1} A^T)^{-1} A H_{\overline{x}}^{-1} \delta_{\mu}(\overline{x}, \overline{s}, \overline{t}).
$$

We state a useful lemma for bounding the step size.

**Lemma A.4** ([DLY21, Lemma A.9]). The steps $\delta_x$ and $\delta_s$ satisfy

$$
\left( \sum_{i \in [n]} w_i \|\delta_{x,i} \|^2 \overline{x}_i \right)^{1/2} \leq \frac{9}{8} \alpha,
\left( \sum_{i \in [n]} w_i^{-1} (\|\delta_{s,i} \|^2 \overline{s}_i)^{1/2} \leq \frac{9}{8} \alpha \cdot t.
$$

82