Quantum shells in a quantum space-time

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Abstract
We study the quantum motion of null shells in the quantum space-time of a black hole in loop quantum gravity. We treat the shells as test fields and use an effective dynamics for the propagation equations. The shells propagate through the region where the singularity was present in the classical black hole space-time, but is absent in the quantum space-time, eventually emerging through a white hole to a new asymptotic region of the quantum space-time. The profiles of the shells get distorted due to the quantum fluctuations in the Planckian region that replaces the singularity. The evolution of the shells is unitary throughout the whole process.

Keywords: null shells, loop quantum gravity, white holes, quantum space time

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(Some figures may appear in colour only in the online journal)

1. Introduction

The exact space of solutions of the equations of loop quantum gravity for vacuum, spherically symmetric space-times was recently found [1]. The breakthrough was due to the realization that a rescaling and linear combination of the constraints of canonical quantum gravity in the spherical case yields a constraint algebra that is a Lie algebra, allowing to complete the Dirac quantization of the model. The space of physical states can be found in closed form. It is based on one dimensional spin networks. The proximity of the nodes of the spin networks is limited by the condition of the quantization of the areas of the spheres of symmetry. The singularity that is inside classical black holes is replaced by a region where a description in terms of a semi-classical geometry is not possible and through it one propagates into another region of space-time. This confirms previous studies of the interior of Schwarzschild based on the isometry with Kantowski–Sachs [2]. Here we would like to analyze the propagation of
spherical null shells in the quantum black hole space-time. We will see that contrary to the behavior of quantum shells in more traditional treatments [3], an ingoing shell makes it through the quantum region and emerges on the other side of where the singularity used to be, via a white hole. The main difference with traditional treatments is that they consider self-gravitating shells and choose the wavefunction to vanish at \( r = 0 \) whereas that is not natural in the loop quantum gravity context since the quantum space-time is non-singular there and is continued beyond \( r = 0 \). This is in analogy with what is done in loop quantum cosmology, where the wavefunction is not required to vanish at the big bang.

Two types of quantum effects are present when propagating through the region where the singularity used to be: (1) the relative fluctuations of the Dirac observable that represents the metric of space-time are maximal; (2) effects of the quantization of area are also maximal giving rise to large discontinuities in the value of the Dirac observable that represents the metric when going from a point of the spin network to its closest neighbor. This induces random perturbations in the profile of the shells propagating through that region. We will also show that shells placed close to the horizon in a quantum state involving the superposition of ingoing and outgoing shells suffer a phenomenon analogous to Hawking radiation in that one shell falls into the black hole and is trapped and the other emerges, due to fluctuations in the position of the horizon. The evolution of the system is unitary if one considers both shells but unitarity is lost if one only considers the outgoing shell. We will treat the shells as test fields to study their propagation, and then consider their back reaction on the geometry. We will approximate the evolution through effective semi-classical equations, to avoid dealing with the true equations of motion that stem from the quantum space-time, that involve coefficients that change discretely in value from one link of the spin network to the next. The approximation is good except in the region where the singularity used to be where a more careful treatment will be needed to study the propagation. We only comment on some qualitative features of the propagation in that region.

2. Spherical shells on a spherical background

2.1. Classical theory

To treat the spherically symmetric background quantum space time we choose variables adapted to spherical symmetry, one is left with two pairs of canonical variables \( E^\theta, K_\theta \) and \( E^r, K_r \), that are related to the traditional canonical variables in spherical symmetry \( ds^2 = A^2 dx^2 + R^2 d\Omega^2 \) by

\[
A = \sqrt{E^\theta}, \quad P_\theta = -\sqrt{|E^\theta|} K_\theta, \quad R = \sqrt{|E^r|} \quad \text{and} \quad P_r = -2\sqrt{|E^r|} K_r - E^\theta K_\theta \sqrt{|E^\theta|},
\]

where \( P_\theta, P_r \) are the momenta canonically conjugate to \( A \) and \( R \) respectively, \( x \) is the radial coordinate and \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \). We will take the Immirzi parameter to one. After doing the rescaling and combination of the constraints discussed in [1] the total Hamiltonian becomes [3]

\[
H_T = \int dx \left[ -N' \left( -\sqrt{|E^r|} \left( 1 + K_\theta^2 \right) + \frac{(E^\theta)^2 \sqrt{|E^\theta|}}{4(E^\theta)^2} \right) + 2GM \right] + N \frac{(E^\theta)^2 \sqrt{|E^\theta|}}{(E^\theta)^2} \eta p \delta(x - r) + 2N \frac{K_\theta \sqrt{|E^\theta|}}{E^\theta} p \delta(x - r) + N \left[ - (E^\theta)' K_\theta + E^\theta K_\theta' - p \delta(x - r) \right]
\]  

(1)
with $r$ the position of the shell and $p$ its canonical momentum. The parameter $\eta = \pm 1$ is the sign of the momentum, depending on it one will have shells that are either ingoing or outgoing if one is outside the black hole. We will work perturbatively, taking the solution for the space-time of a black hole without a shell, putting a test shell on it and studying its back reaction on the geometry.

### 2.2. Quantization

We proceed to quantize the model. For the gravitational sector we consider the exact physical states found in [1]. On those states we define an evolving constant of the motion (a Dirac observable dependent on a (functional) parameters) that represents the matter Hamiltonian, following a procedure outlined in [4]. We assume the quantum states are a direct product of the gravitational states and the shell states. We take the expectation value of the shell Hamiltonian with respect to the gravitational states and this provides a classical Hamiltonian for matter that incorporates the corrections of the quantum space-time. We then proceed to quantize such Hamiltonian using traditional quantization techniques since it is a mechanical system. We will then study the back-reaction on the quantum space-time.

To construct the evolving constant of the motion for the Hamiltonian we first recall the expression of $\hat{E}^x$ in terms of an evolving constant of the motion $\hat{E}^x(z(x))|\tilde{g}, \tilde{k}, M\rangle = O(z(x))|\tilde{g}, \tilde{k}, M\rangle$ with $z(x)$ the (functional) parameter and the Dirac observable $\hat{O}$ is defined by $\hat{O}(z)|\tilde{g}, \tilde{k}, M\rangle = l_{\text{Planck}}^2\text{Int}(z)|\tilde{g}, \tilde{k}, M\rangle$, with $V$ the number of vertices in the spin network and $\text{Int}$ means integer part. In terms of $\hat{E}^x(z(x))$ (a Dirac observable), the matter part of the Hamiltonian constraint can be written as a Dirac observable itself

$$\hat{H}_\text{shell} = \sqrt{|\hat{E}^x|} \left( (\hat{E}^x)\eta + 2K_\phi \hat{E}^\phi \right) \left( \hat{E}^\phi \right)^2,$$

and $\hat{E}^\phi$ can be written in terms of $\hat{E}^x$ solving the Hamiltonian constraint

$$\hat{E}^\phi = \frac{\left( \hat{E}^x \right)^{-1}}{2\sqrt{1 + K_\phi^2 - 2GM/|\hat{E}^x|}}.$$

The Hamiltonian is an evolving constant of the motion parameterized by the (functional) parameters $z(x)$ (present in $\hat{E}^x$) and $K_\phi$. The first one is associated with spatial diffeomorphisms and the latter with the slicing of space-time chosen. One can now evaluate the expectation value of the Hamiltonian of the shell on the exact physical states of the gravitational theory. At this point it is good to choose the parameter $K_\phi$ of the shell Hamiltonian by fixing $K_\phi$ to be a (negative) function of $x$ that vanishes outside the black hole (starting with the first point of the spin network outside where the classical event horizon would have been) and is non-vanishing inside, its absolute value taking a maximum value where the singularity used to be and then diminishing as one goes past it and eventually becomes zero just outside the Cauchy horizon inside the black hole. These types of slicings are penetrating slicings akin to Eddington–Finkelstein coordinates and will allow us to follow the evolution of shells as they penetrate inside the black hole.

We will choose the states of the gravitational variables judiciously so they approximate well a classical space-time in regions far away from where the classical singularity used to be. This requires superpositions of states $|\tilde{g}, \tilde{k}, M\rangle$ such that the values of $\tilde{k}$ vary monotonously and without significant jumps between adjacent values (the values of $\tilde{k}$ determine the radial
coordinate up to a Planck scale) and that the spread in the values of $M$ is small compared to its value. Notice that the ADM mass is a Dirac observable and the values $\vec{k}$ are associated with the areas of the spheres of symmetry (also a Dirac observable). Both Dirac observables commute, and placing demands on their eigenvalues characterizes the states of the physical space of states. The minimum separation in radial coordinate in these states is given by $\ell_{\text{Planck}}/(2r)$ with $r$ the radial coordinate such that $4\pi r^2$ is the area of the surfaces of symmetry. In this idealized context, where one has an eternal black hole and no true dynamics, it could also be possible to choose a state that approximates a semi-classical geometry in the region of high curvature. It will have large discontinuities in the relative values of the metric components in that region, but well defined values. However, such a choice appears artificial. Instead we choose states that are superpositions and given the large (and different) discontinuities in the region of high curvature for each of the component states one does not have a semi-classical geometry well approximated there. As was discussed in reference [1], since the triad $E^r$ enters in all expressions in absolute value, one can extend the solution to negative values of $E^r$, which quantum mechanically corresponds to negative values of $k$. This gives rise to a new asymptotic region beyond where the singularity used to be, isometric to the geometry for $x > 0$. It includes a Cauchy horizon isometric to the event horizon. As the shells collapse, they will move through the region where the singularity used to be into the new asymptotic region.

![Figure 1. The evolution of a shell incoming towards the black hole from the exterior ($p < 0$), in an approximation where we consider a semi-classical geometry everywhere. The amplitude diminishes, but is non-vanishing in the Planckian region around $x = 0$ for all values of $t$. The portion at the left is the shell incoming into the black hole and the portion at the right the shell exiting through the whitehole. The behavior is completely symmetric around $x = 0$ as the effective model considered does not incorporate the fluctuations at the Planckian region.](image-url)
Taking the expectation value of the shell Hamiltonian on the exact physical states of vacuum gravity essentially determines the prefactor of $p$ in (2) entirely as a function $f(r, \eta)$ determined by $\zeta(r), K_\varphi(r)$ and $\eta$.

3. Motion of the shells on the quantum space-time and backreaction

From the quantum corrected matter Hamiltonian one can work out the equations of motion for the shell. For that it will be convenient to specify the parameters in the parameterized Dirac observables representing $E^x$ and $E^\varphi$. We have already chosen $K_\varphi$. We now choose a spin network such that the eigenvalues of $\hat{E}^x$ are approximated very well by $x^2$, therefore determining the (functional) parameter $\zeta(x)$. The condition on $E^x$ implies $N_r = 0$ and the diffeomorphism constraint can be solved for $K_x$. The condition on $K_\varphi$ implies that $N^r = 0$, so the lapse is a constant we call $N_0$. This allows to solve the Hamiltonian constraint for $E^\varphi$ and therefore the metric. Taking the Poisson bracket of $r$ and $p$ with the Hamiltonian we get

$$i = N_0 \sqrt{\left[ E^x(r) \right]} \frac{\left( (E^x)'(r) \eta + 2K_\varphi(r)E^\varphi(r) \right)}{(E^\varphi)^2},$$

(4)

$$p = -\frac{\partial i}{\partial r},$$

(5)

from which it follows that $ip$ is a constant and is the contribution of the shell to the Hamiltonian constraint up to a Dirac delta function. This allows to combine it with the other terms in the constraint and what one has is that the $2GM$ term in the Hamiltonian gets modified by a constant $ip$, times a Heaviside function $H(x - r)$, i.e. the energy of the shell contributes to the mass of the spacetime outside of the shell. This is the back-reaction of the
To try to mimic the effect of the fluctuations in the Planckian region on the shell we studied the evolution of the same shell in two quantum geometries that could be involved in a superposed state of the gravitational field and then superposed the results for the profiles of the shells. Panel (a) is the incoming pulse and in panel (b) we depict together the pulse evolved through the Planck regime using a single semi-classical geometry and using the superposition of the evolutions in two semi-classical geometries. The latter is an attempt to mimic the quantum fluctuations. As we see, the pulse is significantly distorted in its passage through the Planck region. Notice the significant decrease in amplitude in both cases, due to the pulse becoming very spread out while passing through the Planck region due to tidal forces. In addition to that, the fluctuations produce lasting distortions in the pulse after passage. The horizontal axis in centimeters and the mass scale is that of a solar sized black hole.

**Figure 3.** To try to mimic the effect of the fluctuations in the Planckian region on the shell we studied the evolution of the same shell in two quantum geometries that could be involved in a superposed state of the gravitational field and then superposed the results for the profiles of the shells. Panel (a) is the incoming pulse and in panel (b) we depict together the pulse evolved through the Planck regime using a single semi-classical geometry and using the superposition of the evolutions in two semi-classical geometries. The latter is an attempt to mimic the quantum fluctuations. As we see, the pulse is significantly distorted in its passage through the Planck region. Notice the significant decrease in amplitude in both cases, due to the pulse becoming very spread out while passing through the Planck region due to tidal forces. In addition to that, the fluctuations produce lasting distortions in the pulse after passage. The horizontal axis in centimeters and the mass scale is that of a solar sized black hole.
shell on the space-time. Notice that the variables \( r, p \) are continuous, but the coefficients of the equations of motion (4, 5) will change discretely as one traverses from one link of the spin network to the next through the evolution of the shell. In this paper we will not take this into account and treat the equations as a semi-classical approximation with the coefficients determined by a continuous metric, as we are choosing a quantum state in which the jumps in the values of variables from a link of the spin network to the next are small. We will discuss separately what happens in the region where the singularity used to be, where this approximation does not hold.

3.1. Shells with ingoing momentum

Depending on the sign of \( \eta \) the analysis changes. We start with \( \eta = -1 \) which corresponds an infalling shell from outside the black hole passing through the Planckian region inside and emerging in the white hole. One can study \( f(\eta, r) \) numerically with choices for \( z(r) \) and \( K_p \) as the ones we discussed that allow to cover the entire space-time. It turns out that it can be modeled by a function \( \epsilon = - \frac{f(r)}{r^2} - \frac{2}{2i} \) with \( r_0 \) a macroscopic quantity much smaller than the mean value of the Schwarzschild radius, since the exact \( f(r) \) is very small at the horizon, and \( \epsilon \) is small, of the order of Planck length. Notice that \( r \) now spans from \( \infty \) at \( \eta_0 \) all the way to \( -\infty \) with \( r = x = 0 \) the point where the singularity used to be in the classical solution.

We define a self-adjoint operator associated with the shell’s Hamiltonian and determine its spectrum

\[
-2i f(r)\Psi'(r) - if(r)\Psi(r) - 2E\Psi(r) = 0,
\]

with \( E \) its eigenvalue, which is positive as it is the correction of the mass due to the presence of the shell. Its eigenstates can be found \( \Psi_E(r) = C \exp \left( \int \frac{dr}{2} \frac{f(r)}{r^2} \right) \). The integral can be computed in closed form for the simple form of \( f(r) \) we introduced above

\[
\Psi_E(r) = \exp \left( - \frac{id}{n_0} \left( \frac{r^3}{3} + e^2 r \right) \right) \sqrt{\frac{2}{r^2} + e^2} \left( \frac{2\pi n_0^2}{r^2} \right). \]

These states are orthonormal

\[
\int_{-\infty}^{\infty} d\Psi_E d\Psi_E = \delta_{E,E} \quad \text{and they satisfy a closure relation, showing that the Hamiltonian is indeed self-adjoint.}
\]

Superposing such states with a Gaussian weight in \( E \), one obtains a time dependent wavefunction for the propagation of the shell, with a unitary evolution, as the one shown in figure 1. Notice that the incoming shell traverses the Planck region and will eventually emerge into the new asymptotically flat region that the eternal non-singular black hole solution in loop quantum gravity has after traversing a white hole. The eternal non-singular solution is symmetric around \( x = 0 \) and the horizon of the black hole is mirrored in \( x < 0 \) by a white hole. As we will see, when one considers a more realistic evolution through the Planck region, although the space-time is symmetric around \( x = 0 \) the evolution of the pulse will not be due to quantum fluctuations.

3.2. Shells with outgoing momentum

We also wish to consider the case of a shell with outgoing momentum \( (p > 0) \). The resulting shell would be outgoing in its motion \( (r > 0) \) if it is in the black hole exterior, but would still be ingoing in its motion, as expected, in the black hole interior. In this case we do not have a simple expression approximating the behavior of \( f(r) \) as before, but its behavior can be studied numerically. One now has that \( f(r) \) vanishes at the horizon. That means that the zero of energies for the shells is placed at the horizon. Shells with outgoing momentum inside the horizon will have negative energy and shells outside have positive energies [5]. This can be
easily seen in equation (5): $(E')' > 0$ everywhere, in this case $\eta = 1$ and $K_p = 0$ outside so one has the first term multiplied times $p > 0$ and the energy is positive. At the horizon one can check using the definition of $E^p$ that the two terms in (5) cancel each other. From there inward, their difference is negative as the modulus of $K_p$ grows. Shells with outgoing momentum inside the horizon behave like positive energy shells moving into the past. Notice that the foliation we are using, which behaves like Schwarzschild coordinates outside the black hole with $K_p = 0$ and behaves like Eddington–Finkelstein coordinates in the interior labels the hypersurfaces with the time of an observer at infinity. As such, the foliation cannot go beyond the Cauchy horizon in the interior of the black hole and the evolution of the shells seems to ‘stop’ there (see figure 2).

3.3. Effects of fluctuations in the Planck regime

Up to now we have made a treatment in terms of a semi-classical metric even in the Planckian region. Although states that approximate a semi-classical geometry everywhere do exist in our highly idealized description of an ever existing black hole, they are likely not the ones that occur in nature after gravitational collapse. More likely is that one will have superpositions of gravitational states and that implies that in the Planckian regions there will be large fluctuations, for instance, in the curvature. To try to mimic the effect of these fluctuations in the evolution of shells, we have studied the evolution of the same shell in two different quantum states of the gravitational that could enter in such superpositions and then superposed the evolutions. The results are shown in figure 3.

4. Discussion

One has interesting consequences when one considers superpositions of shells near the horizon. Consider outgoing shells. The ones that lie outside the horizon will make it to scri+. The ones inside the horizon, in spite of being ‘outgoing’ end up traveling through the Planckian region inside the black hole. Now, in a quantum space-time one expects fluctuations in the position of the horizon. The horizon carries out a random walk. After some time the fluctuations accumulate, and by page time could imply deviations in the radial coordinate position of the horizon of the order of Planck length [6]. A related discussion of fluctuations can be found in [7]. That sounds small, but due to the high blueshifts at the horizons, it implies very different evolutions for shells outside the horizon depending on where they lie with respect to the horizon. Therefore if one considers a shell of finite width obtained by superposing shells like the ones we consider in this paper, the various components will evolve in very different ways. This suggests a mechanism for the quantum space-time to produce appreciable effects in a region where one would have ordinarily not expected significant corrections due to quantum gravity. Note that although one would have expected a semi-classical metric at the horizon would describe well the situation, given that it is a region of low curvature, it is the fluctuations that give rise to the effect. This could be a potential source of breakdown of complementarity.

The fluctuations of the horizon might also imply that some of the shells in a superposition end up being inside the horizon and will therefore end up in the Planckian region even if they were outgoing shells ($p > 0$). This is similar to what happens in the pair production process of Hawking radiation. Notice that the shell inside the horizon contributes negatively to the mass of the black hole, as is commonly argued in the heuristic picture of Hawking radiation. The evolution of the complete wavepacket of shells is unitary provided one keeps track of the
shells inside the black hole. Looking at the evolution of the outside ones only, unitarity would appear lost.

It is worthwhile comparing our results with those obtained with more traditional quantizations of shells. Most efforts have concentrated on self-gravitating shells so the parallels are not easy to draw. However, in those proposals vanishing boundary conditions on the wavefunction are imposed at the origin. The resulting dynamics imply that the shells ‘bounce’ and behaviors like a superposition of white and black holes emerge, leading to the concept of ‘gray holes’ \[3, 8\]. None of these behaviors are observed here, largely in part due to the elimination of the singularity in loop quantum gravity. There have been studies of shells propagating on black holes but have not addressed in detail what happens at the singularity \[9\]. The fact that the shells emerge through a white hole may suggest that this is a way in which information could leak from an evaporating black hole, although our eternal black hole exact solution does not allow us to model that situation in detail. A conjectural scenario based on this observation is presented in \[6\].

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