Continuity in right semitopological groups

Evgenii Reznichenko

Department of General Topology and Geometry, Mechanics and Mathematics Faculty,
M. V. Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 199991 Russia

Abstract

Groups with a topology that is in consistent one way or another with the algebraic structure are considered. Classical groups with a topology are topological, paratopological, semitopological, and quasitopological groups. We also study other ways of matching topology and algebraic structure. The minimum requirement in this paper is that the group is a right semitopological group (such groups are often called right topological groups). We study when a group with a topology is a topological group; research in this direction began with the work of Deane Montgomery and Robert Ellis. (Invariant) semi-neighborhoods of the diagonal are used as a means of study.

Keywords: Baire space, nonmeager space, topological group, semitopological group, paratopological group, quasitopological group, feebly continuous map, quasi continuous map, generalization of Baire space,

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1. Introduction

Since the 1936 paper [1] of Montgomery and the 1957 paper [2] of Ellis, the following problem has been studied:

Problem 1. Let a topology be given on a group $G$ that is consistent with the group structure of $G$. Under what conditions on the topology and the algebraic structure does the consistency improve? When is $G$ a topological group?

We study the following connections between topology and group structure:

$(O_p)$ multiplication $(g, h) \mapsto gh$ in a group is continuous;

$(O_s)$ multiplication in a group is separately continuous;

$(O_i)$ the operation of taking the inverse element $g \mapsto g^{-1}$ is continuous;

$(O_l)$ left shifts $\lambda_g : G \to G, x \mapsto gx$ are continuous for all $g \in G$;

Email address: erezn@inbox.ru (Evgenii Reznichenko)
(O_r) right shifts $\rho_g : G \rightarrow G, x \mapsto xg$ are continuous for all $g \in G$.

Recall that a group $G$ is called

- **semitopological if** $O_s$ holds;
- **right semitopological if** $O_r$ holds;
- **left semitopological if** $O_l$ holds;
- **paratopological if** $O_p$ holds;
- **quasitopological if** $O_s$ and $O_i$ hold;
- **topological if** $O_p$ and $O_i$ holds.

In some papers, left (right) semitopological groups are also called left (right) topological groups.

Montgomery [1] proved that a (locally) complete metrizable semitopological group is a paratopological group and a Polish semitopological group is a topological group. Ellis [2] proved that a locally compact Hausdorff paratopological group is a topological group.

Later, the study of the continuity of operations in groups was continued in many papers (see [3] and references therein). The most studied objects in this area are semitopological, paratopological and topological groups. The most important class of these is the class of topological groups. The main question in this field is: when is a group with a topology a topological group?

Groups with a structures weaker than that of semitopological groups have been studied as well [4, 5, 6, 7, 8, 9] are studied also.

Papers devoted to right topological compact groups contain also results on strengthening continuity in such groups. If $G$ is a right topological compact group in which the multiplication $(g, h) \mapsto gh$ is continuous at the identity of $G$, then the inversion map $g \mapsto g^{-1}$ is continuous at the identity group $G$ [10].

Theorem 15(1) extends this theorem to Baire spaces in the wide class $\mathcal{D}_d$ defined in Section 9. The class $\mathcal{D}_d$ includes, for example, locally pseudo-compact spaces and Baire metric spaces.

A right topological group $G$ is called **admissible** if there is a dense subset $S$ of $G$ such that $x \mapsto zx$ is continuous for each $z$ in $S$. We write “CHART” for “compact Hausdorff admissible right topological”. Let $G$ be a CHART group. Any of the following conditions implies that $G$ is a topological group.

(C1) $G$ is metrizable (Theorem 2.1 [11]).

(C2) $G$ is first-countable (Remark after Proposition 1.7 [12]).

(C3) $G$ is Fréchet (Corollary 8.8 [8]).

(C4) $G$ is tame (Theorem 8.7 [8], see also Theorem 3.3 [9]).

(C5) The multiplication of $G$ is feebly continuous. (Proposition 3.2 [9], see also Corollary 2 (2)).
Note that the group $G$ in $(C_2)$, $(C_2)$, and $(C_3)$ is metrizable because the compact first-countable and Fréchet topological groups are metrizable (Corollary 4.2.2 of [13]). Theorem 17 (2) strengthens $(C_1)$ and Theorem 3.3 [7]. Theorem 15 (2) extends $(C_3)$ to Baire spaces in the class $\mathcal{D}_d$. We also obtain the following conditions for a CHART group to be a topological group:

$$(G_6)$$ $G$ has countable $\pi$-character (for example, $G$ is a compact space with countable tightness) (Corollary 2 (3));

$$(G_7)$$ (MA) $w(G) < 2^\omega$ (Corollary 3).

Note that condition $(G_6)$ is weaker than $(G_3)$.

To strengthen continuity in groups, the Baire property is needed. There exists a countable metrizable paratopological nontopological group, for example, a countable dense subgroup of the Songenfrey line. In this paper, we study how the Baire-type properties in [13] and [15] affect continuity in topological groups.

In [15], Baire-type properties were introduced and studied with the help of semineighborhoods of the diagonal; see Section 5. In right semitopological groups, there are invariant diagonal semineighbourhoods, with the help of which some algebraic-topological properties of right semitopological groups are defined.

2. Definitions and notation

The sign $:=$ will be used for equality by definition.

In what follows, we suppose given a group $G$. Usually, we will denote the identity of $G$ as $e$. For $g \in G$ we denote

$$\lambda_g : G \to G, \; x \mapsto gx,$$

$$\rho_g : G \to G, \; x \mapsto xg,$$

left and right shifts in $G$. We denote by $m$ and $i$ multiplication and inversion in the group:

$$m : G \times G \to G, \; (g, h) \mapsto gh,$$

$$i : G \to G, \; g \mapsto g^{-1}.$$

Usually, it is assumed that there is some topology on the group $G$. In this case we denote by $\mathcal{N}_e$ the family of open neighborhoods of $e$.

The family of all subsets of a set $X$ is denoted by $\text{Exp}(X)$. The family of all nonempty subsets of the set $X$ is denoted by $\text{Exp}_*(X)$: $\text{Exp}_*(X) := \text{Exp}(X) \setminus \{\emptyset\}$.

If $B$ is a subset of the $aA$ then we denote by $B^c = A \setminus B$ the complement to $A$. We use this notation in situations where it is clear from the context which set $A$ is meant.

We define an indexed set $x = (x_\alpha)_{\alpha \in A}$ as a function on $A$ such that $x(\alpha) = x_\alpha$ for $\alpha \in A$. If the elements of an indexed set $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ are themselves sets,
then $\mathcal{X}$ is also called an indexed family of sets; $\mathcal{X}$ is a function on $A$: $\mathcal{X}(\alpha) = X_\alpha$ for $\alpha \in P$.

We denote by $\text{Aut}(X)$ the set of all homeomorphisms of the space $X$ onto itself.

A subset $M$ of a topological space $X$ is called locally dense (nearly open or preopen) if $M \subset \text{Int} \ M$.

Let $M \subset X$. If $M$ is the union of a countable number of nowhere dense sets, then $M$ is called a meager set. Non-meager sets are called sets of the second Baire category or nonmeager sets. A subset of $M$ is called residual (comeager) if $X \setminus M$ is a meager set.

A space $X$ is called a space of the first Baire category or a meager space if the set $X$ is of the first category in the space $X$. A space $X$ is called a space of the second Baire category or a nonmeager space if $X$ is not a meager space.

A space in which every residual set is dense is called a Baire space. A space is nonmeager if and only if some open subspace of it is a Baire space.

A family $\mathcal{N}$ of nonempty subsets of $X$ is called a $\pi$-net if for any open nonempty $U \subset X$ there exists $M \in \mathcal{N}$ such that $M \subset U$.

A $\pi$-network consisting of open sets is called a $\pi$-base.

A subset $U \subset X$ is called regular open if $U = \text{Int} \ U$.

A space $X$ is called quasi-regular if for every nonempty open $U \subset X$ there exists a nonempty open $V \subset X$ such that $V \subset U$.

A space $X$ is called semiregular if $X$ has a base consisting of regular open sets.

A family $\mathcal{B}$ of open nonempty sets in $X$ is called an outer base of $M \subset X$ if $M \subset U$ for each $U \in \mathcal{B}$ and for each open $W \supset M$ there exists $U \in \mathcal{B}$ such that $M \subset U \subset W$.

We denote by $\beta \omega$ the space of ultrafilters on $\omega$, the Stone-Čech extension of the discrete space $\omega$. We denote by $\omega^* = \beta \omega \setminus \omega$ the set of nonprincipal ultrafilters.
Let \((x_n)_{n \in \omega}\) be a sequence of points in the space \(X\) and let \(p \in \omega^*\) be a nonprincipal ultrafilter. A point \(x \in X\) is called the \(p\)-limit of a sequence \((x_n)_{n \in \omega}\) if \(\{n \in \omega : x_n \in U\} \in p\) for any neighborhood \(U\) of the point \(x\). We will write \(x = \lim_p x_n = \lim_p (x_n)_{n \in \omega}\) for the \(p\)-limit \(x\).

A space \(X\) is said to have countable pseudocharacter if each point of \(X\) is a set of type \(G_\delta\).

A space \(X\) is submetrizable if there exists a continuous injective mapping of \(X\) into a metrizable space.

3. Weakenings of continuity

We will use several continuity relaxations following [7]. A mapping of topological spaces \(f : X \to Y\) is called

\[
\begin{align*}
\text{continuous} & \quad \{ x \in \text{Int} f^{-1}(V) \} \\
\text{nearly continuous} & \quad \{ x \in \text{Int} f^{-1}(V) \} \\
\text{quasi-continuous} & \quad \{ x \in \text{Int} f^{-1}(V) \} \\
\text{semi-precontinuous} & \quad \{ x \in \text{Int} f^{-1}(V) \} \\
\text{feebly continuous} & \quad \{ \text{Int} f^{-1}(V) \neq \emptyset \}
\end{align*}
\]

for any neighborhood \(V\) of the point \(f(x)\). If the properties under consideration are satisfied at every point, then we obtain the definition: the mapping \(f\) is called

\[
\begin{align*}
\text{continuous} & \quad \{ f^{-1}(V) \subset \text{Int} f^{-1}(V) \} \\
\text{nearly continuous} & \quad \{ f^{-1}(V) \subset \text{Int} f^{-1}(V) \} \\
\text{quasi-continuous} & \quad \{ f^{-1}(V) \subset \text{Int} f^{-1}(V) \} \\
\text{semi-precontinuous} & \quad \{ f^{-1}(V) \subset \text{Int} f^{-1}(V) \} \\
\text{feebly continuous} & \quad \{ \text{Int} f^{-1}(V) \neq \emptyset \}
\end{align*}
\]

for any open nonempty \(V \subset X\).

We call a mapping \(f\) a feebly homeomorphism if \(f\) is a bijection and the mappings \(f\) and \(f^{-1}\) are feebly continuous.

We call a mapping \(f\) a quasi homeomorphism if \(f\) is a bijection and the mappings \(f\) and \(f^{-1}\) are quasi continuous.

Let \(X, Y, Z\) be three topological spaces, and let \(f : X \times Y \to Z\) be a function from \(X \times Y\) into \(Z\). Recall that \(f\) is called quasi-continuous with respect to the second variable [13] at \((x, y)\) if for every open neighborhood \(W\) of \(f(x, y)\) and every open neighborhood \(U \times V\) of \((x, y)\), there are an open neighborhood \(V'\) of \(y\) and a nonempty open set \(U' \subset U\) such that \(f(U' \times V') \subset W\). Quasi-continuity with respect to the second variable plays an important role in the theory of separate vs. joint continuity. It is called strong quasi-continuity in [19] and [20], where it is applied to the study of the problem when a semitopological group is a topological group.

Similarly, define that \(f\) is called quasi-continuous with respect to the first variable at \((x, y)\) if for every open neighborhood \(W\) of \(f(x, y)\) and every open
neighborhood $U \times V$ of $(x, y)$, there are an open neighborhood $U'$ of $x$ and a nonempty open set $V' \subset V$ such that $f(U' \times V') \subset W$.

Let us list the connections between the topology and the group structure of the group $G$.

(O₁) holds (O₆) and (O₇), that is, $G$ is a topological group:

(O₆) multiplication $m$ in a group is continuous in $(e, e)$;

(O₇) $m$ is quasi-continuous in the first coordinate in $(e, e)$;

(O₈) $m$ is quasi-continuous in $(e, e)$;

(O₉) $m$ is feebly continuous in $(e, e)$;

(O₁₀) the operation of taking the inverse element $i$ is feebly continuous;

(O₁₁) $i$ is continuous at the identity $e$;

(O₁₂) $i$ is almost continuous in $e$;

(O₁₃) $i$ is quasi-continuous in $e$;

(O₁₄) $i$ is semi-continuous in $e$;

(O₁₅) $i$ is feebly continuous in $e$;

(O₁₆) left shifts of $\lambda_g$ are feebly continuous for any $g \in G$;

(O₁₇) there exists a dense $H \subset G$ such that $\lambda_g$ are feebly continuous for any $g \in H$;

(O₁₈) there exists a dense $H \subset G$ such that $\lambda_g$ are feebly homeomorphisms for any $g \in H$;

(O₁₉) left shifts of $\lambda_g$ are quasi continuous for any $g \in G$;

(O₂₀) there exists a dense $H \subset G$ such that $\lambda_g$ are quasi continuous for any $g \in H$;

(O₂₁) there exists a dense $H \subset G$ such that $\lambda_g$ are quasi homeomorphisms for any $g \in H$;

(O₂₂) there is a dense $H \subset G$ such that $\lambda_g$ are continuous for any $g \in H$;

(O₂₃) there is a dense $H \subset G$ such that $\lambda_g$ are homeomorphisms for any $g \in H$;

We set

$$\Lambda_f(G) := \{ g \in G : \lambda_g \text{ is feebly continuous} \}, \quad \Lambda_f^*(G) := \{ g \in G : g, g^{-1} \in \Lambda_f(G) \},$$

$$\Lambda_q(G) := \{ g \in G : \lambda_g \text{ is quasi continuous} \}, \quad \Lambda_q^*(G) := \{ g \in G : g, g^{-1} \in \Lambda_q(G) \},$$

$$\Lambda(G) := \{ g \in G : \lambda_g \text{ is continuous} \}, \quad \Lambda^*(G) := \{ g \in G : g, g^{-1} \in \Lambda(G) \}.$$

Easy to check
Statement 1. The sets $\Lambda^*_f(G), \Lambda^*_q(G), \Lambda^*(G)$ are subgroups of the group $G$. The following alternative definitions of the listed properties are true: $(O_l) \Lambda(G) = G; (O_{fl}) \Lambda_f(G) = G; (O_{ql}) \Lambda_q(G) = G; (O_{dl}) \Lambda(G) = G; (O_{dl^*}) \Lambda_f(G) = G; (O_{dq}) \Lambda_q(G) = G; (O_{dl^*}) \Lambda_f(G) = G; (O_{dq^*}) \Lambda_q(G) = G$.

Let us give an alternative definition of some of the listed properties. Let $\mathcal{T}^*$ be all nonempty open subsets of $G$.

$(O_{pe})$ for any $U \in \mathcal{N}_c$ there exists $V \in \mathcal{N}_c$ so $V^2 \subset U$;
$(O_{sqpe})$ for any $U \in \mathcal{N}_c$ there are $V \in \mathcal{N}_c$ and $W \in \mathcal{T}^*$ so $VW \subset U$;
$(O_{fpe})$ for any $U \in \mathcal{N}_c$ there are $V \in \mathcal{T}^*$ and $W \in \mathcal{T}^*$ so $VW \subset U$;

$(O_{fi}) \text{ Int } U^{-1} \neq \emptyset$ for any $U \in \mathcal{T}^*$;
$(O_{ie}) e \in \text{ Int } U^{-1}$ for any $U \in \mathcal{N}_c$;
$(O_{nie}) e \in \text{ Int } U^{-1}$ for any $U \in \mathcal{N}_c$;
$(O_{qie}) e \in \text{ Int } U^{-1}$ for any $U \in \mathcal{N}_c$;

$(O_{sie}) e \in \text{ Int } U^{-1}$ for any $U \in \mathcal{N}_c$;
$(O_{fie}) \text{ Int } U^{-1} \neq \emptyset$ for any $U \in \mathcal{N}_c$;

$(O_{l}) gU \in \mathcal{N}_c$ for any $U \in \mathcal{T}^*$ and $g \in G$;
$(O_{r}) Ug \in \mathcal{N}_c$ for any $U \in \mathcal{T}^*$ and $g \in G$;

$(O_{fl}) \text{ Int } gU \neq \emptyset$ for any $U \in \mathcal{T}^*$ and $g \in G$;

$(O_{ql}) g \in \text{ Int } U^{-1}g$ for any $U \in \mathcal{T}^*$ and $g \in G$.

Definition 1. For

\{x_1, x_2, ..., x_n\} \subset \{p, s, i, l, r, pe, qpe, sqpe, fpe, ie, nie, fie, qie, sie, f1, dfl, ql, dl, dfl^*, dq1, dql, dl^*\}

we say that a group $G$ is $O(x_1, x_2, ..., x_n)$-topological if the condition $(O_{x_i})$ is satisfied for all $i = 1, 2, ..., n$.

Topological groups are $O(p, i)$-topological groups, and other introduced classes of groups with topology can be defined similarly.

Proposition 1. Below, $(A) \rightarrow (B)$ means that $O(A)$-topological group is $O(B)$-topological group and $(A) \leftrightarrow (B)$ means that $G$ is a $O(A)$-topological group if
and only if \( G \) is a \( O(B) \)-topological group.

\[
\begin{align*}
&(p, i) \leftrightarrow (t) \quad (l, r) \leftrightarrow (s) \quad (s, pe) \leftrightarrow (p) \quad (s, ie) \leftrightarrow (s, i) \\
&(l, i) \leftrightarrow (r, i) \leftrightarrow (s, i) \\
&(p) \rightarrow (s) \quad (p) \rightarrow (pe) \rightarrow (sqpe) \quad (i) \rightarrow (ie) \rightarrow (fie) \\
&(l) \rightarrow (fl) \rightarrow (dfl) \\
&(ie) \rightarrow (nie) \rightarrow (sie) \quad (ie) \rightarrow (qie) \rightarrow (sie) \quad (qie) \rightarrow (fie) \\
&(dfl^*, fie, r) \rightarrow (fl, fi) \quad (dfl^*, pe, r) \rightarrow (sqpe) \\
&(p, sie) \rightarrow (t) \quad (dfl^*, ie, pe, r) \rightarrow (t)
\end{align*}
\]

**Proof.** Most of these statements are either trivial or widely known (see [7] and [13]). Let us prove new nontrivial assertions.

\((p, sie) \rightarrow (t)\). Theorem 2.3 from [7], Lemma 1.2 from [21].

\((dfl^*, fie, r) \rightarrow (fl, fi)\). Let us show that \((dfl^*, fie, r) \rightarrow (fi)\). Let \( U \in T^* \). Take \( g \in \Lambda^*_f(G) \cap U \). From

\[
U^{-1} = g^{-1}(Ug^{-1})^{-1}
\]

it follows that \( \text{Int} \, U^{-1} \neq \emptyset \). Let us show \((fl)\). We have \( hU = (U^{-1}h^{-1})^{-1} \) for any \( h \in G \). Since \((fi, r)\), then \( \text{Int} \, hU \neq \emptyset \).

\((dfl^*, ie, pe, r) \rightarrow (t)\). It follows from the previous subsection that \( G \) is a \( O(fl) \)-topological group. By virtue of \((s, pe) \leftrightarrow (p)\) and \((s, ie) \leftrightarrow (s, i)\), it suffices to prove that \( G \) is a \( O(l) \)-topological group. Let \( g \in G \) and \( U \in T^* \).

You need to check that \( gU \) is open. Let \( q \in U \). It suffices to check that \( gg \in \text{Int} \, gU \), which is equivalent to \( g \in \text{Int} \, gV \), where \( V = Uq^{-1} \in \mathcal{N}_c \). There is \( W \in \mathcal{N}_c \) for which \( WW^{-1} \subset V \). Let \( s \in \text{Int} \, gW \) and \( w = g^{-1}s \in W \). There is \( S \in \mathcal{N}_c \) so \( Ss \subset gW \). Then \( s = gw \), \( g = sw^{-1} \) and

\[
Sg = Ssw^{-1} \subset gWw^{-1} \subset gWW^{-1} \subset gV.
\]

Hence \( g \in \text{Int} \, gV \).

\((dfl^*, pe, r) \rightarrow (sqpe)\). Let \( U \in \mathcal{N}_c \). Then \( WVx \subset U \) for some \( x \in G \), \( W \in T^* \) and \( V \in \mathcal{N}_c \). Let \( y \in \Lambda^*_f(G) \cap W \). Then \( Sy \subset W \) for some \( S \in \mathcal{N}_c \).

For some \( z \in \text{Int} \, yV \) and \( Q \in \mathcal{N}_c \), \( Qz \subset yV \). We get \( SQzx \subset U \).

4. **\( R \)-topological groups**

Let us call the group \( G \)

- \( R \)-semitopological if \( G \) is a \( O(r) \)-topological group, that is, \( G \) is a right semitopological group;
- \( R \)-paratopological if \( G \) is a \( O(pe, r) \)-topological group;
- \( R \)-quasitopological if \( G \) is a \( O(ie, r) \)-topological group;
- \( R \)-topological if \( G \) is a \( O(ie, pe, r) \)-topological group.

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Let $N$ be a real-valued function on $G$. We shall call $N$ a prenorm \cite[(Section 3.3)]{13} on $G$ if the following conditions are satisfied for all $x, y \in G$:

(PN1) $N(e) = 0$;
(PN2) $N(xy) \leq N(x) + N(y)$;
(PN3) $N(x^{-1}) = N(x)$.

If, in addition, the condition

(PN4) $N(x) \neq 0$ for $x \neq e$,

then $N$ is called norm. In \cite{22} the prenorm is called the pseudonorm.

**Statement 2** (Proposition 3.3.1 and Proposition 3.3.2 from \cite{13}). Let $N$ be a prenorm on the group $G$. Then for $x, y \in G$

- $N(x) \geq 0$;
- $|N(x) - N(y)| \leq N(xy^{-1})$.

**Statement 3.** Let $(U_n)_{n\in\omega}$ be a sequence of subsets of the group $G$ such that $U_n = U_n^{-1}$ and $U_{n+1}^2 \subset U_n$ for $n \in \omega$.

1. There is a prenorm $N$ on $G$ such that
   
   \[
   \{ x \in G : N(x) < 1/2^n \} \subset U_n \subset \{ x \in G : N(x) \leq 2/2^n \}
   \]
   
   for $n \in \omega$.

2. If $G$ is an $R$-semitopological group and $e \in \text{Int} U_n$ for $n \in \omega$, then the prenorm $N$ is continuous.

**Proof.** Condition (1) is actually proved in Lemma 3.3.10 from \cite{13}. Let us prove (2). Let $y \in G$ and $\varepsilon > 0$. There is $n \in \omega$ for which $2/2^n < \varepsilon$. Then $y \in \text{Int} U_n y$. Let $x \in U_n y$. Then $xy^{-1} \in U_n$ and

\[
|N(x) - N(y)| \leq N(xy^{-1}) \leq 2/2^n < \varepsilon.
\]

For the prenorm $N$ we denote

\[
B_N := \{ g \in G : N(g) < 1 \}.
\]

**Proposition 2.** Let $G$ be an $R$-topological group and $U \in \mathcal{N}_e$. Then $e \in B_N \subset U$ for some continuous prenorm $N$ on $G$.

**Proof.** There exists $(U_n)_{n\in\omega}$ there is a sequence of subsets of the group $G$ such that $U = U_0$, $V_n \in \mathcal{N}_e$ and $V_{n+1}^2 \subset V_n$ for $n \in \omega$. Let $U_n = V_n \cap V_n^{-1}$. The prenorm $N$ from Statement 3 is the desired one.
Denote
\[ d_N(x, y) := N(xy^{-1}) \] and
\[ B_N(x, r) := \{ y \in G : d_N(x, y) < r \} \]
for \( x, y \in G \) and \( r > 0 \). The function \( d_N \) is a right-invariant pseudometric on \( G \) and \( d_N \) is continuous if the prenorm \( N \) is continuous. The right-handed metric \( d \) defines the prenorm \( N_d: N_d(x) = d(e, x) \). Note that \( d = d_{N_d} \). Thus, there is a naturally one-to-one correspondence between prenorms and right-invariant metrics on \( G \), and continuous right-invariant pseudometrics correspond to continuous prenorms.

A family \( \{ d_\alpha : \alpha \in A \} \) of pseudometrics on a space \( X \) defines the topology of the space \( X \) if the family of sets open with respect to \( d_\alpha \) for \( \alpha \in A \) form a prebase \( X \).

**Proposition 3.** Let \( G \) be a topological group. A group \( G \) is an \( R \)-topological group if and only if the topology of \( G \) is given by a family of right-invariant pseudometrics.

**Proof.** Let \( G \) be an \( R \)-topological group. It follows from Proposition 2 that \( U \in \mathcal{N}_e \) has a continuous prenorm \( N_U \) for which \( B_{N_U} \subset U \). Let \( d_U = d_{N_U} \). The family \( \{ d_U : U \in \mathcal{N}_e \} \) of continuous right-invariant metrics defines the topology \( G \).

Let \( \{ N_\alpha : \alpha \in A \} \) be a family of continuous prenorms such that \( \{ d_{N_\alpha} : \alpha \in A \} \) is a family of right-invariant metrics defining the \( G \) topology. Right shifts \( \rho_y \) are continuous in \( G \). Let \( U \in \mathcal{N}_e \). There is \( \varepsilon > 0 \) and a finite \( B \subset A \) such that \( \bigcap_{b \in B} B_{N_b}(e, \varepsilon) \subset U \). Let \( V = \bigcap_{b \in B} B_{N_b}(e, \varepsilon/2) \). Then \( V \in \mathcal{N}_e \), \( V = V^{-1} \) and \( V^2 \subset U \).

**Proposition 4.** Let \( G \) be a \( T_0 \) \( R \)-topological group.

1. If \( G \) is a first countable space, then there exists a continuous right-invariant metric \( d \) on \( G \) that defines a topology on \( G \).

2. If \( G \) is a space of countable pseudocharacter, then there exists a continuous right-invariant metric \( d \).

**Proof.** Let \( (W_n)_{n \in \omega} \) be a sequence of neighborhoods of unity, which in case (1) forms a base in \( e \) and \( \{ e \} = \bigcap n \in \omega W_n \) in case (2). There is a sequence \( (V_n)_{n \in \omega} \) of neighborhoods of unitsuch that \( V_{n+1}^2 \subset V_n \subset W_n \) for \( n \in \omega \). Let \( U_n = U_n \cap U_n^{-1} \) for \( n \in \omega \). Let \( N \) be the prenorm as in Statement 3 and \( d = d_N \). \( \square \)

**Theorem 1.** Let \( G \) be a \( T_0 \) \( R \)-topological group.

1. A group \( G \) is metrizable if and only if \( G \) is first countable.

2. A group \( G \) is submetrizable if and only if \( G \) has a countable pseudocharacter.
Theorem 1 is a generalization of the Birkhoff-Kakutani theorem, see Theorem 3.3.12 and Theorem 3.3.16 from [13].

Proposition 1 implies

**Theorem 2.** Let $G$ be an $R$-topological group. If the set $\Lambda_f^*(G)$ dense in $G$, then $G$ is a topological group.

**Example 1.** Let us describe Example (d) from [23]. Let $T = \{ z \in \mathbb{C} : |z| = 1 \}$, $\varphi : T \to T$ be a discontinuous automorphism of the group $T$ such that $\varphi^2 = id_T$. We set $H = \{ id_T, \varphi \}$ with discrete topology, $G = T \times H$, $G$ as a set, and the space is homeomorphic to $T \times H$. Recall the definition of multiplication in a semidirect product $G$:

$$(t_1, h_1) \cdot (t_2, h_2) = (t_1 h_1(t_2), h_1 h_2).$$

The group $G$ is a compact metrizable right semitopological group. The subgroup $T$ is normal and clopen in $G$. Since $T$ is a topological group, $G$ is an $O(ie, pe)$-topological group. Finally, $G$ is a metrizable compact $R$-topological group which is not a topological group.

5. Properties of spaces defined by semi-neighborhoods of the diagonal

5.1. Semi-neighborhoods of the diagonal

Let $X$ be a set, $P, Q \subseteq X \times X$. Denote

$$P(x) := \{ y \in X : (x, y) \in P \},$$

$$P(M) := \{ y \in X : \text{exists } x \in M \text{ so } (x, y) \in P \} = \bigcup_{x \in M} P(x),$$

$$P|_M := P \cap (M \times M),$$

$$P^{-1} := \{ (x, y) : (y, x) \in P \},$$

$$P \circ Q := \{ (x, y) : \text{there is } z \in X \text{ so } (x, z) \in P \text{ and } (z, y) \in Q \}$$

for $x \in X$ and $M \subseteq X$.

**Definition 2.** Let $X$ be a space, $P \subseteq X \times X$. We call a set $P$ semiopen if every $P(x)$ is open. We call $P$ a diagonal semi-neighborhood if $x \in \text{Int } P(x)$ for all $x \in X$. We denote by $\mathcal{S}(X)$ the family of all semi-neighborhoods of the diagonal of the space $X$.

**Remark 1.** In [24], in the definition of a semi-neighborhood and a diagonal, it was assumed that the set $P(x)$ is open, that is, $P$ was assumed to be semi-open. The concept of a semi-open semi-neighborhood of the diagonal essentially coincides with the concept of [25] neighbourhood assignment: a map $P : X \to \mathcal{T}$ is called a neighbourhood assignment if $x \in P(x)$ and $P(x)$ is open to $x \in X$.

A set $P$ is a semi-neighborhood of the diagonal if and only if $P$ contains some semi-open semi-neighborhood of the diagonal.
If $X$ is a topological space, then we denote
$$P^v := \{(x, y) \in X \times X : y \in P(x)\}.$$ It is clear that $P^v(x) = P(x)$ for $x \in X$.

On $\text{Exp}(\mathcal{G}(X))$ we introduce an order relation and an equivalence relation related to it. For $P, Q \subset \mathcal{G}(X)$ we set
- $P \preceq Q$ if and only if for any $Q \in Q$ there exists $P \in P$ such that $P \subset Q$;
- $P \sim Q$ if and only if $P \preceq Q$ and $Q \preceq P$.

Note that if $P \supset Q$, then $P \preceq Q$. For $P \subset \mathcal{G}(X)$ we denote
$$P^+(P) := \{R \in \mathcal{G}(X) : P \subset R, \text{ for some } P \in S\}.$$ For $P \subset \mathcal{G}(X)$ we denote
$$P(P) := P,$$ $$P^+(P) := \{Q \circ P : Q \in \mathcal{G}(X) \text{ and } P \in P\},$$ $$P^v(P) := \{P^v : P \in P\},$$ $$P^c(P) := \{P : P \in P\},$$ $$P^+_v(P) := P^+(P^+_v(P)).$$

**Proposition 5** (Proposition 11 [15]). *Let $X$ be a space. For $P \subset \mathcal{G}(X)$*

- $P \preceq P^+(P) \preceq P^+_v(P)$
- $P \preceq P^v(P) \preceq P^c(P) \preceq P^+_v(P)$

Let $\lambda$ be an ordinal, $P_\alpha \subset X \times X$ for $\alpha < \lambda$. By induction on $\beta < \gamma$ we define the sets $p_\beta(S_\beta), p_\beta^c(S_\beta) \subset X \times X$, where $S_\beta = (P_\alpha)_{\alpha < \beta}$.

**β = 0.** We set
$$p_0(S_0) := p_0^c(S_0) := \{\emptyset\}.$$ **β = 1.** We set
$$p_1(S_1) := P_0, \quad p_1^c(S_1) := P_0.$$ **1 < β ≤ γ.** We set
$$p_\beta(S_\beta) := \begin{cases} Q_\beta, & \text{if } \beta \text{ is cardinal limit} \\ Q_\beta \circ P_\beta', & \text{if } \beta = \beta' + 1 \end{cases}$$
$$p_\beta^c(S_\beta) := \begin{cases} Q_\beta^c, & \text{if } \beta \text{ is cardinal limit} \\ Q_\beta^c \circ P_\beta', & \text{if } \beta = \beta' + 1 \end{cases}$$
where
$$Q_\beta = \bigcup_{\alpha < \beta} p_\alpha(S_\alpha) \quad \text{and} \quad Q_\beta^c = \bigcup_{\alpha < \beta} p_\alpha^c(S_\alpha).$$
For $\mathcal{P} \subset \mathcal{S}(X)$ we denote
\[
\begin{align*}
P_\gamma(\mathcal{P}) &:= \{ p_\gamma(S) : S \in \mathcal{P} \}, \\
P^c_\gamma(\mathcal{P}) &:= \{ p^c_\gamma(S) : S \in \mathcal{P} \}, \\
\mathcal{P}|_Y &:= \{ P|_Y : P \in \mathcal{P} \}
\end{align*}
\]
for $Y \subset X$.

**Proposition 6** (Proposition 12 \[15\]). Let $X$ be a space, $\mathcal{P} \subset \mathcal{S}(X)$, and $1 < \alpha < \beta$ ordinals. Then
\[
\begin{align*}
P_+\gamma(\mathcal{P}) &\preceq P_\gamma(\mathcal{P}) \preceq P^c_\gamma(\mathcal{P}) \preceq P^c_\beta(\mathcal{P}) \\
P^c_\gamma(\mathcal{P}) &\preceq P^c_\alpha(\mathcal{P}) \preceq P^c_\beta(\mathcal{P}) \\
P^c_\gamma(\mathcal{P}) &\preceq P^c_\gamma(\mathcal{P}) \preceq P^c_\alpha(\mathcal{P}) \preceq P^c_\beta(\mathcal{P})
\end{align*}
\]

### 5.2. Normal square functors

A functor $Q$ that associates a topological space $X$ with a family $Q(X) \subset \text{Exp}_a(X \times X)$ of nonempty subsets of $X \times X$ is called the *square functor*. A square functor $Q$ is called a *normal square functor* (NFS) if the following conditions are satisfied:

\(Q_1\) if $f : X \to Y$ is a homeomorphism, then
\[
Q(Y) = \{(f \times f)(P) : P \in Q(X)\};
\]

\(Q_2\) if $U$ is an open nonempty subset of $X$, then
\[
Q(U) = Q(X)|_U;
\]

\(Q_3\) if $S \in Q(X)$, $S \subset Q \subset X \times X$, then $Q \in Q(X)$;

\(Q_4\) if $S \in Q(X)$ then $\overline{S} = X \times X$.

We introduce an order relation on normal square functors. For NFS $\mathfrak{Q}$ and $\mathfrak{R}$, $\mathfrak{Q} \preceq \mathfrak{R}$ is true if and only if $Q(X) \supseteq R(X)$ for any space $X$.

Let us define NFS which we will use. Let $k \in \{d, h, v, s, a\}$, $X$ be a space, $S \in \text{Exp}_a(X \times X)$. Then $S \in \mathfrak{Q}_k(X)$ if and only if the condition $(L_k)$:

\(L_d\) $\overline{S} = X \times X$;

\(L_v\) $\overline{S(x)} = X$ for any $x \in X$;

\(L_h\) $\overline{S^{-1}}(x) = X$ for any $x \in X$;

\(L_s\) $M \times X \subset S$ for some $M \subset X = \overline{M}$;

\(L_a\) $S = X$. 

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Proposition 7 (Proposition 21 [15]). Let \( k \in \{d, h, v, s, a\} \), \( X \) be a space and 
\( S \in \text{Exp}_k(X \times X) \). Then \( S \in \Omega_k(X) \) if and only if for any \( x \in X \) and any 
neighborhood \( U \subset X \) of \( x \) the condition \((L'_k)\) is satisfied:

\[
\begin{align*}
(L'_d) \quad & S(U) = X; \\
(L'_v) \quad & S(x) = X; \\
(L'_h) \quad & S(U) = X; \\
(L'_s) \quad & S(z) = X \text{ for some } z \in U; \\
(L'_a) \quad & S(x) = X.
\end{align*}
\]

Proposition 8 (Proposition 22 [15]). For any NFS \( \Omega \)
\[
\Omega_d \preceq \Omega \preceq \Omega_a, \\
\Omega_h \preceq \Omega_s.
\]

We introduce an increment operation on NFS: for NFS \( \Omega \) we define NFS \( \Omega^+ \). Let \( X \) be a space, \( Q \in \text{Exp}_k(X \times X) \). Then \( Q \in \Omega^+(X) \) if and only if 
\( P \circ S \subset Q \) for some \( P \in \mathcal{G}(X) \) and \( S \in \Omega(X) \).

Proposition 9 (Proposition 23 [15]).
\[
\Omega_v \preceq \Omega^+_d \\
\Omega_a = \Omega^+_s = \Omega^+_h
\]

5.3. \( \Delta(P; \Omega) \)-Baire spaces

Definition 3. Let \( \Omega \) be a normal square functor, \( (X, \mathcal{T}) \) a space, \( \mathcal{T}^* = \mathcal{T} \setminus \{\emptyset\} \), 
\( \mathcal{P} \subset \mathcal{G}(X) \). Let us call \( X \)

- \( \Delta(P; \Omega) \)-nonmeager (\( \Delta(P; \Omega) \)-nonmeager) space if for any \( P \in \mathcal{P} \) there 
  exists \( V \in \mathcal{T}^* \), so \( P|_V \in \Omega(V) \);
- \( \Delta(P; \Omega) \)-Baire space (\( \Delta(P; \Omega) \)-Baire space) if for any \( P \in \mathcal{P} \) and any \( U \in \mathcal{T}^* \) there exists an open nonempty \( V \subset U \), so the condition \( P|_V \in \Omega(V) \) 
is satisfied.

Proposition 10 (Proposition 30 [15]). Let \( X \) be a space, \( \mathcal{P} \subset \mathcal{G}(X) \). In the 
diagrams below, the arrow 
\[
(F)_k \rightarrow (G)_l
\]
means that \( F, G : \text{Exp}(\mathcal{G}(X)) \rightarrow \text{Exp}(\mathcal{G}(X)) \) are mappings \( k, l \in \{d, v, h, s, a\} \)
and terms

1. If \( X \) is a \( \Delta(F(\mathcal{P}; \Omega_k)) \)-nonmeager space, then \( X \) is \( \Delta(G(\mathcal{P}; \Omega_l)) \)-nonmeager 
   space and
2. if \( X \) is a \( \Delta(F(\mathcal{P}; \Omega_k)) \)-Baire space, then \( X \) is \( \Delta(G(\mathcal{P}; \Omega_l)) \)-Baire space.
5.4. $\Delta(\Omega)$-Baire spaces

**Definition 4.** Let $\Omega$ be a normal square functor, $X$ be a space, $\gamma$ be an ordinal, $\mathcal{P} = \mathcal{G}(X)$. Let $P^K : \text{Exp}(\mathcal{G}(X)) \rightarrow \text{Exp}(\mathcal{G}(X))$ be one of the mappings considered in the 5.1 section:

$$P^K \in \{P, P^e, P^v, P^c, P^+, P^c_+, P^-_\gamma, P^-_\gamma\}.$$ 

We say that $X$ is a $k\Delta^l(\Omega)$-Baire ($k\Delta^l(\Omega)$-nonmeager) space if $X$ is $\Delta(\mathcal{P}^K(\mathcal{G}(X)); \Omega)$-Baire ($\Delta(\mathcal{P}^K(\mathcal{G}(X)); \Omega)$-nonmeager) space. For

$$\hat{\Delta} \in \{\Delta, \Delta^c, \Delta^v, \Delta^c, \Delta^+, \Delta^c, \Delta^+, \Delta^\gamma, \Delta^c \gamma\}$$
we have defined $\tilde{\Delta}(\Omega)$-Baire ($\tilde{\Delta}(\Omega)$-nonmeager) spaces. For
\[ k \in \{d, h, v, s, a\} \]
we say $X$ is a $\tilde{\Delta}_k$-Baire ($\tilde{\Delta}_k$-nonmeager) space if
\[ X \text{ is a } \tilde{\Delta}(Q)_{\Delta_k} \text{-Baire (} \tilde{\Delta}(Q)_{\Delta_k} \text{-nonmeager) space. For} \]
\[ \tilde{\Delta} \in \{\Delta_k, \epsilon\Delta_k, \sigma\Delta_k, \Delta_k^+, \sigma\Delta_k^+, \Delta_k^\gamma, \sigma\Delta_k^\gamma\} \]
we have defined $\tilde{\Delta}$-Baire ($\tilde{\Delta}$-nonmeager) spaces. Also, if the subscript is not written, $d$ is implied: $\Delta_{\Delta_d}$-nonmeager ($\Delta_{\Delta_d}$-Baire) is $\Delta_{\Delta_d}$-nonmeager ($\Delta_{\Delta_d}$-Baire). A separate direct definition for the most important classes of spaces: the space $X$ is called
\begin{align*}
(1) & \quad \Delta$-nonmeager ($\Delta$-Baire) if $X$ is $\Delta(S(X); \Omega_d)$-nonmeager ($\Delta(S(X); \Omega_d)$-Baire); \\
(2) & \quad \Delta_h$-nonmeager ($\Delta_h$-Baire) if $X$ is $\Delta(S(X); \Omega_h)$-nonmeager ($\Delta(S(X); \Omega_h)$-Baire); \\
(3) & \quad \Delta_s$-nonmeager ($\Delta_s$-Baire) if $X$ is $\Delta(S(X); \Omega_s)$-nonmeager ($\Delta(S(X); \Omega_s)$-Baire); \\
(4) & \quad \Delta^\gamma$-nonmeager ($\Delta^\gamma$-Baire) if $X$ is $\Delta(P_\gamma(S(X)); \Omega_d)$-nonmeager ($\Delta(P_\gamma(S(X)); \Omega_d)$-Baire); \\
(5) & \quad \epsilon\Delta^\gamma$-nonmeager ($\epsilon\Delta^\gamma$-Baire) if $X$ is $\Delta(P_\gamma(S(X)); \Omega_d)$-nonmeager ($\Delta(P_\gamma(S(X)); \Omega_d)$-Baire);
\end{align*}

Remark 2. The spaces that are here called "$\Delta$-nonmeager" were called "$\Delta$-Baire" in papers [24, 3, 26].

Proposition 11 (Proposition 33 [15]). Let $X$ be a space, $\Omega$ be NFS and $n \in \omega$. Consider the condition
\begin{align*}
(*) & \quad 1. \text{A space } X \text{ is } \tilde{\Delta}(\Omega)$-nonmeager if and only if some nonempty open } U \subset X \text{ is } \tilde{\Delta}(\Omega)$-Baire. \\
& \quad 2. \text{If the space } X \text{ is homogeneous, then } X \text{ is } \tilde{\Delta}(\Omega)$-nonmeager if and only if } X \text{ is } \tilde{\Delta}(\Omega)$-Baire. \\
(1) & \quad \text{If } \tilde{\Delta} \in \{\Delta, \Delta^n\} \text{ then } (*) \text{ holds.} \\
(2) & \quad \text{If the space } X \text{ is quasiregular and } \tilde{\Delta} \in \{\Delta, \Delta^n, \sigma\Delta, \epsilon\Delta, \sigma\Delta^+, \epsilon\Delta^+\} \text{, then } (*) \text{ holds.}
\end{align*}

Theorem 3 (Theorem 4 [15]). Let $\mathcal{P}_k$ ($\mathcal{P}_c$) be the smallest class of spaces that
\begin{itemize}
\item contains $p$-spaces and strongly $\Sigma$-spaces;
\item is closed under arbitrary (countable) products;
\end{itemize}
• is closed under taking open subspaces.

For $\tilde{\Delta} \in \{\Delta_s, \Delta_h, \Delta\}$, if a regular Baire (nonmeager) space $X$ belongs to the class of spaces described in $(\tilde{\Delta})$, then $X$ is $\tilde{\Delta}$-Baire ($\tilde{\Delta}$-nonmeager).

$(\Delta_s)$ $\sigma$-spaces.

$(\Delta_h)$ is the smallest class of spaces that
– contains $\Sigma$-spaces and $w:\Delta$-spaces;
– is closed under products by spaces from the class $\mathcal{P}_c$;
– is closed under taking open subspaces.

$(\Delta)$ is the smallest class of spaces that
– contains $\Sigma$-spaces, $w:\Delta$-spaces, and feebly compact spaces;
– is closed under products by spaces from the class $\mathcal{P}_k$;
– is closed under taking open subspaces.

5.5. CDP-Baire spaces

Let $X$ be a space, $\mathcal{G}$ be a family of open subsets of $X$. Denote

$$B(X, \mathcal{G}) := \{x \in X : \text{family}$$

$$\{\text{st}(x, \gamma) : \gamma \in \mathcal{G} \text{ and } x \in \bigcup \gamma\}$$

$$\text{is the base at point } x\}$$

$$\text{dev}(X) := \min\{|\mathcal{G}| : \mathcal{G} \text{ open cover family } X$$

$$\text{and } X = B(X, \mathcal{G})\}.$$ 

Let $Y \subset X, \mathcal{N}$ be the family of all nowhere dense subsets of $X$. If $Y \not\subset \bigcup \mathcal{N}$ then put $\text{Nov}(Y, X) := \infty$, otherwise

$$\text{Nov}(Y, X) := \min\{|\mathcal{L}| : \mathcal{L} \subset \mathcal{N} \text{ and } Y \subset \bigcup \mathcal{L}\},$$

$$\text{Nov}(X) := \text{Nov}(X, X).$$

**Definition 5.** Let $X$ be a space.

• We call a space a CDP-space if $\text{dev}(X) < \text{Nov}(X)$, that is, if there exists a family $\mathcal{P}$ of open coverings of the space $X$ such that $|\mathcal{P}| < \text{Nov}(X)$ and $B(X, \mathcal{P}) = X$.

• We call a space a CDP$_0$-space if there exists a family $\mathcal{P}$ of open partitions of $X$ such that $|\mathcal{P}| < \text{Nov}(X)$ and $B(X, \mathcal{P}) = X$.

• We call a space $X$ CDP-nonmeager if there exists a family $\mathcal{G}$ of open families in $X$ such that $|\mathcal{G}| < \text{Nov}(Y, X)$ for $Y = B(X, \mathcal{P})$. 
• We call a space $X$ CDP-Baire if every nonempty open subset of $X$ is a CDP-nonmeager space.

In the article [7] one studies spaces for which $\text{dev}(X) < \text{Nov}(X)$, that is, CDP-spaces. Metrizable nonmeager spaces are CDP-spaces. Examples of non-metrizable CDP-spaces can be obtained using Martin’s axiom (MA).

**Proposition 12** (Proposition 38 [15]). (MA) Let $\tau < 2^\omega$ be an infinite cardinal, $X$ be an absolute $G_\tau$ space with countable Suslin number and $\text{dev}(X) \leq \tau$, for example $X = \mathbb{R}^\tau$. Then $X$ is a CDP-space.

**Corollary 1.** (MA) Let $X$ be a ccc compact space and $w(X) < 2^\omega$. Then $X$ is a CDP-space.

Clearly, a CDP-space is a CDP$_0$-space.

**Theorem 4** (Theorem 5 [15]). Let $X$ be a semiregular space. The following conditions are equivalent

1. $X$ is a CDP-nonmeager space;
2. some nonempty open subset $U \subset X$ contains a dense CDP-space $Y \subset U \subset \overline{Y}$;
3. some nonempty open subset $U \subset X$ contains a dense CDP$_0$-space $Y \subset U \subset \overline{Y}$;

**Theorem 4** implies

**Proposition 13** (Proposition 41 [15]). Let $X$ be a semiregular space. If $X$ has a metrizable nonmeager subspace, then $X$ is a CDP-nonmeager space.

5.6. $\Delta_s$-Baire spaces

Known examples of $\Delta_s$-Baire spaces are obtained using the following proposition

**Proposition 14** (Proposition 42 [15]). Let $X$ be a space. If $X$ is CDP-nonmeager (CDP-Baire), then $X$ is $\Delta_s$-nonmeager ($\Delta_s$-Baire).

[15] lists other subclasses of $\Delta_s$-nonmeager spaces, but it is not known whether these classes contain non-CDP-nonmeager spaces.

Propositions [13] and [14] imply

**Proposition 15.** Let $X$ be a space. If $X$ contains a nonmeager metrizable subspace, then $X$ is $\Delta_s$-nonmeager.

Under the assumption MA+$\neg$-CH, there exists a $\Delta_s$-nonmeager space without first-countable points: Propositions [14] and [12] imply that the space $\mathbb{R}^{\omega_1}$ is $\Delta_s$-nonmeager.

**Proposition 16.** Let $(X, \mathcal{T})$ be a space, $\mathcal{T}' = \mathcal{T}\setminus\{\emptyset\}$. The following conditions are equivalent:
(1) $X$ is a $\Delta_s$-nonmeager space.

(2) Let $W : \mathcal{T}^* \to \mathcal{T}^*$ be a mapping such that

$$W(V) \subset W(U) \subset U$$

for $V, U \in \mathcal{T}^*$, $V \subset U$. Then there exists $x \in X$ such that $x \in W(U)$ for every $U \in St(x, \mathcal{T}^*)$.

Proof. (1) $\implies$ (2). Let us assume the opposite. Then there exists a semi-open neighborhood $P$ of the diagonal such that $x \not\in W(P(x))$ for all $x \in X$. Since $X$ is a $\Delta_s$-nonmeager space, there exists $U \in \mathcal{T}^*$ for which $M = \{x \in U : U \subset P(x)\}$ is dense in $U$. Let $x \in W(U) \cap M$. Then $x \in U \subset W(P(x))$. Contradiction.

(2) $\implies$ (1). Let us assume the opposite. Then there exists a semi-open diagonal neighborhood $P$ such that $M(U) = \{x \in U : U \subset P(x)\}$ is not dense in $U$ for all $U \in \mathcal{T}^*$. Let $W(U) = U \setminus M(U)$. Conditions (2) are satisfied for $W$. Then there exists $x \in X$ for which $x \in W(U)$ for every $U \in St(x, \mathcal{T}^*)$. Then $x \in W(U)$ for $U = P(x)$. Hence $x \not\in M(U)$ and $U \not\subset P(x)$. A contradiction, since $U = P(x)$.

Theorem 5. Let $X$ be a $\Delta_s$-nonmeager space and let $f : X \to X$ be a feebly homeomorphism. Then for some $x \in X$ the mapping $f$ is continuous in $x$ and $f^{-1}$ is continuous in $f(x)$.

Proof. Let $\mathcal{T}^*$ be all open nonempty subsets of $X$. We set

$$W(U) = \text{Int} f^{-1}(\text{Int} f(U))$$

for $U \in \mathcal{T}^*$. Since $f$ is a feebly homeomorphism, $W(U) \neq \emptyset$ and $W$ satisfy the conditions of Proposition 16. Then there exists $x \in X$ such that $x \in W(U)$ for every $U \in St(x, \mathcal{T}^*)$. The point $x$ is the desired one.

6. Properties of groups defined by invariant semi-neighborhoods of the diagonal

In this section, $(G, \mathcal{T})$ is a right semitopological group, $\mathcal{T}^* = \mathcal{T} \setminus \{\emptyset\}$, $\mathcal{N}_e$ is a family of open neighborhoods of the identity, $\tilde{\mathcal{N}}_e$ is a family of neighborhoods of the identity that are not necessarily open.

6.1. Diagonal Semi-Neighborhoods in Groups

A set $P \subset G \times G$ is called right-invariant if

$$\{(xg, yg) : (x, y) \in P\} = P$$

for all $g \in G$. For $M \subset G$ we set

$$\mathfrak{N}(M) := \{(x, gx) : g \in M \text{ and } x \in G\}.$$
If $P = \mathcal{R}(M)$, then $P(g) = Mg$. The set $\mathcal{R}(M)$ is right-invariant. The mapping $\mathcal{R}$ establishes a one-to-one correspondence between subsets of $G$ and right-invariant subsets of $G \times G$. If $P \subset G \times G$ is a right-invariant subset, then $P = \mathcal{R}(P(e))$. Denote

$$M^d := \bigcap \{MV : V \in \mathcal{N}_e\},$$
$$M^h := \bigcap \{MV : V \in \mathcal{N}_e\}.$$

**Statement 4.** Let $L, M \subset G$. Then

1. $(x, y) \in \mathcal{R}(M)$ if and only if $yx^{-1} \in M$;
2. $(\mathcal{R}(M))^{-1} = \mathcal{R}(M^{-1})$;
3. $\overline{\mathcal{R}(M)} = \mathcal{R}(M^d)$;
4. $\mathcal{R}(M) \circ \mathcal{R}(L) = \mathcal{R}(LM)$.

**Proof.**

1. $(x, y) \in \mathcal{R}(M) \iff y = gx$ for some $g \in M \iff yx^{-1} \in M$.
2. $(x, y) \in (\mathcal{R}(M))^{-1} \iff (y, x) \in \mathcal{R}(M) \iff xy^{-1} \in M \iff (x, y) \in \mathcal{R}(M^{-1})$.
3. $(x, y) \in \overline{\mathcal{R}(M)} \iff (Vx \times Vy) \cap \mathcal{R}(M) \neq \emptyset$ for any $V, U \in \mathcal{N}_e \iff$ $MUx \cap Vy \neq \emptyset$ for any $V, U \in \mathcal{N}_e \iff y \in MUx$ for any $U \in \mathcal{N}_d x$.
4. $(x, y) \in \mathcal{R}(M) \circ \mathcal{R}(L) \iff (x, z) \in \mathcal{R}(M)$ and $(z, y) \in \mathcal{R}(L)$ for some $z \in G \iff z \in Mx$ and $y \in Lz$ for some $z \in G \iff y \in LMx$.

Denote

$$\mathcal{G}_{y}(G) := \{P \in \mathcal{G}(G) : G \text{ is a right invariant subset}\}.$$

The mapping $\mathcal{R}$ establishes a bijection between $\overline{\mathcal{N}}_e$ and $\mathcal{G}_{y}(G)$.

On $\text{Exp}(\overline{\mathcal{N}}_e)$ we introduce an order relation and an equivalence relation related to it. For $\mathcal{U}, \mathcal{V} \subset \overline{\mathcal{N}}_e$ we put

- $\mathcal{U} \preceq \mathcal{V}$ if and only if for any $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $U \subset V$;
- $\mathcal{U} \sim \mathcal{V}$ if and only if $\mathcal{U} \preceq \mathcal{V}$ and $\mathcal{V} \preceq \mathcal{U}$.

Note that if $\mathcal{U} \supset \mathcal{V}$, then $\mathcal{U} \preceq \mathcal{V}$. For $\mathcal{U} \subset \overline{\mathcal{N}}_e$ we denote

$$N^e(\mathcal{U}) := \{W \in \overline{\mathcal{N}}_e : U \subset W, \text{ for some } U \in \mathcal{U}\}.$$

**Proposition 17.** For $\mathcal{U}, \mathcal{V} \subset \overline{\mathcal{N}}_e$

- $\mathcal{U} \preceq \mathcal{V}$ if and only if $N^e(\mathcal{U}) \supset N^e(\mathcal{V})$;
- $\mathcal{U} \sim \mathcal{V}$ if and only if $N^e(\mathcal{U}) = N^e(\mathcal{V})$.  

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For $\mathcal{U} \subset \tilde{\mathcal{N}_e}$ we denote

\[
N(\mathcal{U}) := \mathcal{U}, \\
N^+(\mathcal{U}) := \{UV : V \in \tilde{\mathcal{N}_e} \text{ and } U \in \mathcal{U}\}, \\
N^v(\mathcal{U}) := \{U : U \in \mathcal{U}\}, \\
N^c(\mathcal{U}) := \{U^d : U \in \mathcal{U}\}, \\
N^c_\gamma(\mathcal{U}) := N^v(N^+(\mathcal{U})).
\]

It follows from the definitions

**Proposition 18.** Let $X$ be a space. For $\mathcal{U} \subset \tilde{\mathcal{N}_e}$

- $\mathcal{U} \not\leq N^+(\mathcal{U}) \not\leq N^c_\gamma(\mathcal{U})$
- $\mathcal{U} \not\leq N^v(\mathcal{U}) \not\leq N^c(\mathcal{U}) \not\leq N^c_\gamma(\mathcal{U})$

Let $\lambda$ be an ordinal, $M_\alpha \subset G$ for $\alpha < \lambda$. By induction on $\beta < \gamma$ we define the sets $n_\beta(\mathcal{S}_\beta), n^c_\beta(\mathcal{S}_\beta) \subset G$, where $\mathcal{S}_\beta = (M_\alpha)_{\alpha < \beta}$.

- $\beta = 0$. We set $n_0(\mathcal{S}_0) := n^c_0(\mathcal{S}_0) := \{\emptyset\}$.
- $\beta = 1$. We set $n_1(\mathcal{S}_1) := M_0, \quad n^c_1(\mathcal{S}_1) := \mathcal{M}_0$.
- $1 < \beta \leq \gamma$. We set

\[
n^c_\beta(\mathcal{S}_\beta) := \begin{cases} Q_\beta, & \text{if } \beta \text{ is cardinal limit} \\ M_\beta Q_\beta, & \text{if } \beta = \beta' + 1 \end{cases}
\]

where $Q_\beta = \bigcup_{\alpha < \beta} n_\alpha(\mathcal{S}_\alpha)$ and $Q^c_\beta = \bigcup_{\alpha < \beta} n^c_\alpha(\mathcal{S}_\alpha)$.

For $\mathcal{U} \subset \tilde{\mathcal{N}_e}$ we denote

\[
N(\mathcal{U}) := \{n_\gamma(\mathcal{S}) : \mathcal{S} \in \mathcal{U}\}, \\
N^c(\mathcal{U}) := \{n^c_\gamma(\mathcal{S}) : \mathcal{S} \in \mathcal{U}\}.
\]

From the clause [18] and the definitions it follows

**Proposition 19.** Let $\mathcal{U} \subset \tilde{\mathcal{N}_e}$ and $1 < \alpha < \beta$ be ordinals. Then

\[
N^+(\mathcal{U}) \not\leq N^c(\mathcal{U}) \not\leq N^c_\alpha(\mathcal{U}) \not\leq N^c_\beta(\mathcal{U}) \not\leq N^c_\gamma(\mathcal{U}) \\
N^c(\mathcal{U}) = N^c_\alpha(\mathcal{U}) \not\leq N^c_\beta(\mathcal{U}) \not\leq N^c_\gamma(\mathcal{U}) \\
N^c(\mathcal{U}) \not\leq N^c_\alpha(\mathcal{U}) \not\leq N^c_\beta(\mathcal{U}) \\
N^c(\mathcal{U}) \not\leq N^c_\alpha(\mathcal{U}) \not\leq N^c_\beta(\mathcal{U}) \not\leq N^c_\gamma(\mathcal{U}).
\]
For $\mathcal{B} \subset \mathcal{N}_e$, $\mathcal{B} \sim \mathcal{N}_e$ if and only if $\mathcal{B}$ is a base in $e$.

**Statement 5.** Let $G$ be an $R$-paratopological group.

1. $\mathcal{N}_\gamma(\mathcal{N}_e) \sim \mathcal{N}_e$ for $0 < \gamma \leq \omega$.

2. If $G$ is a regular space, then $\mathcal{N}_\gamma(\mathcal{N}_e) \sim \mathcal{N}_e$ for $0 < \gamma < \omega$.

**Proof.** (1) Let $U \in \mathcal{N}_e$. There is $\mathcal{S} = (U_n)_{n<\lambda} \subset \mathcal{N}_e$, so $U_0^2 \subset U$ and $U_{n+1}^2 \subset U_n$ for $n < \lambda$. Then $V = n_\gamma(\mathcal{S}) \in \mathcal{N}_\gamma(\mathcal{N}_e)$ and $V \subset U$.

(2) Let us prove by induction on $\gamma$. For $\gamma = 1$ the assertion is obvious. Let $\gamma = n + 1$ and $U \in \mathcal{N}_e$. Then $\mathcal{V}_d^2 \subset \mathcal{V}_3^3 \subset U$ for some $V \in \mathcal{N}_e$. By the inductive hypothesis $S \subset V$ for some $S \in \mathcal{N}_\gamma(\mathcal{N}_e)$. Then $Q = \mathcal{V}_S^d \subset \mathcal{N}_\gamma(\mathcal{N}_e)$ and $Q \subset U$.

From the definitions and Statement [4] it follows

**Proposition 20.** Let $\mathcal{U} \subset \mathcal{N}_e$, $\mathcal{P} = \mathcal{N}(\mathcal{U})$ and $\gamma$ be an ordinal. Then

- $\mathcal{P}^\gamma(\mathcal{P}) = \mathcal{N}(\mathcal{P}^\gamma(\mathcal{U}))$,
- $\mathcal{P}^\gamma(\mathcal{U}) = \mathcal{N}(\mathcal{P}^\gamma(\mathcal{U}))$.

6.2. $g\Delta(\mathcal{U}; \Omega)$-Baire groups

**Definition 6.** Let $\Omega$ be the normal square functor, $\mathcal{U} \subset \mathcal{N}_e$. Let us call the group $G$ a $g\Delta(\mathcal{U}; \Omega)$-Baire $(g\Delta(\mathcal{U}; \Omega)$-Baire) group if $G$ is a $\Delta(\mathcal{N}(\mathcal{U}); \Omega)$-Baire space.

Any element of $P \in \mathcal{N}(\mathcal{U})$ is a right-invariant subset and $\rho_g \times \rho_g(P|_W) = P|_{\rho_g(W)}$ for $g \in G$ and $W \in \mathcal{T}^*$. So if $P|_W \in \mathcal{O}(W)$ then $P|_{Wg} \in \mathcal{O}(Wg)$. We get that $G$ is a $\Delta(\mathcal{N}(\mathcal{U}); \Omega)$-Baire space if and only if $G$ is a $\Delta(\mathcal{N}(\mathcal{U}); \Omega)$-nonmeager space.

It follows from the definitions

**Proposition 21.** Let $\Omega, \mathcal{R}$ be NFS, $\mathcal{U}, \mathcal{V} \subset \mathcal{N}_e$. Suppose $\mathcal{O} \not\cong \mathcal{R}$ and $\mathcal{U} \not\cong \mathcal{V}$. If $G$ is a $g\Delta(\mathcal{U}; \Omega)$-Baire group, then $G$ is a $g\Delta(\mathcal{V}; \mathcal{R})$ is a Baire group.

Similar to the Proposition [10] it checks

**Proposition 22.** Let $\mathcal{U} \subset \mathcal{N}_e$. If in the diagram from Proposition [10] replace $\mathcal{P}$ with $\mathcal{N}$, then arrow

$$(F)_k \rightarrow (G)_l$$

means that $F, G : \text{Exp}(\mathcal{N}_e) \rightarrow \text{Exp}(\mathcal{N}_e)$ are mappings, $k, l \in \{d, v, h, s, a\}$ and the following condition is satisfied:

if $G$ is a $g\Delta(\mathcal{F}(\mathcal{U}); \Omega_k)$-Baire group, then $G$ is a $g\Delta(\mathcal{G}(\mathcal{U}); \Omega_l)$-Baire group.
Proposition 23. Let $G$ be an $R$-semitopological group, $M \subset G$. Then

$$M^h = M^{-1}.$$ 

Proof. Let $x \in G$. Then $x \in M^h$ if and only if for any $V \in N_e \; x \in MV \iff x^{-1} \in V^{-1}M^{-1} \iff VX^{-1} \cap M^{-1} \neq \emptyset$, that is, $x^{-1} \in M^{-1}$ and $x \in M$.

Proposition 24. Let $U \in \tilde{N}_e$ and $W \in T^*$. For $k \in \{d, v, h, s, a\}$, $\mathfrak{R}(U) |_W \in \Omega_k(W)$ if and only if the condition $(L^k_d)$:

$(L^d_g)$ $WW^{-1} \subset U^d$;

$(L^g_g)$ $WW^{-1} \subset U$;

$(L^d_g)$ $WW^{-1} \subset U^d$ or equivalently $WW^{-1} \subset U^h$;

$(L^g_g)$ $WM^{-1} \subset U$ for some $M \subset W \subset \tilde{M}$;

$(L^d_g)$ $WW^{-1} \subset U$.

Proof. $(L^d_g)$ For $x, y \in W, (x, y) \in \mathfrak{R}(U)$ if and only if $yx^{-1} \in U$.

$(L^g_g)$ For $x \in M, \{x\} \times W \in \mathfrak{R}(U)$ if and only if $Wx^{-1} \in U$.

$(L^d_g)$ $W \subset U^d$ for all $x \in W$ if and only if $WW^{-1} \subset U^{-1}$. The Proposition 23 implies that $WW^{-1} \subset U^{-1}$ is equivalent to $WW^{-1} \subset U^h$.

$(L^g_g)$ $W \subset U^d$ for all $x \in W$ if and only if $WW^{-1} \subset U$.

$(L^d_g)$ Follows from $\tilde{\mathfrak{R}}(U) = \mathfrak{R}(U^d)$ and $(L^g_g)$.

6.3. $g\Delta(\Omega)$-Baire groups

Definition 7. Let $\Omega$ be a normal square functor, $G$ be an $R$-topological group, $\gamma$ be an ordinal, $U = \tilde{N}_e$. Let $N^k_\gamma : \text{Exp}(\tilde{N}_e) \rightarrow \text{Exp}(\tilde{N}_e)$ be one of the mappings considered in the 6.1 section:

$$N^k_\gamma \in \{N, N^*, N^+, N^o, N^+, N^+, N^+, N^+\}.$$ 

We say $G$ is a $k\{g\Delta(\Omega)\}$-Baire group if $G$ is a $g\Delta(N^k_\gamma(\tilde{N}_e); \Omega)$-Baire group. For

$$\Delta \in \{g\Delta, \overline{g}\Delta, g\Delta, g^+\Delta, g\Delta^+, g\Delta^\gamma, \overline{g}\Delta^\gamma\}$$

we have defined $\Delta(\Omega)$-Baire groups. For

$$k \in \{d, h, v, s, a\}$$

we say $G$ is a $\Delta_k$-Baire group if $G$ is a $\Delta(\Omega_k)$-Baire group. For

$$\Delta \in \{g\Delta_k, \overline{g}\Delta_k, g\Delta_k, g^+\Delta_k, g\Delta^+_k, g\Delta^\gamma_k, \overline{g}\Delta^\gamma_k\}$$

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we have defined \( \tilde{\Delta} \)-Baire groups. Also, if the subscript is not written, \( d \) is implied: \( \hat{g} \Delta^1 \)-tober is \( \hat{g} \Delta^+ \)-tober. A separate direct definition for the most important classes of groups: the group \( G \) is called

- \( g \Delta \)-Baire if \( G \) is \( g \Delta(\tilde{\mathcal{N}}_c; \Omega_d) \)-Baire;
- \( g \Delta_h \)-Baire if \( G \) is \( g \Delta(\tilde{\mathcal{N}}_c; \Omega_h) \)-Baire;
- \( g \Delta_s \)-Baire if \( G \) is \( g \Delta(\tilde{\mathcal{N}}_c; \Omega_s) \)-Baire;
- \( g \Delta_v \)-Baire if \( G \) is \( g \Delta(\tilde{\mathcal{N}}_c; \Omega_v) \)-Baire;
- \( g \Delta_a \)-Baire if \( G \) is \( g \Delta(\tilde{\mathcal{N}}_c; \Omega_a) \)-Baire.

Since \( N^+_c(\tilde{\mathcal{N}}_c) = N_2(\tilde{\mathcal{N}}_c) \), the classes of \( g \Delta^+(\Omega) \)-Baire and \( g \Delta^2(\Omega) \)-Baire groups coincide.

Since \( \mathfrak{g}(\tilde{\mathcal{N}}_c) = \mathfrak{S}_g(G) \subset \mathfrak{S}(G) \), Proposition 20 implies

**Proposition 25.** Let \( \Omega \) be a normal square functor, \( G \) be an \( R \)-topological group, \( \gamma \) be an ordinal,

\[
\mathcal{N}_m^\gamma \in \{ \mathcal{N}, \mathcal{N}^c, \mathcal{N}^v, \mathcal{N}^s, \mathcal{N}_+^c, \mathcal{N}_+^v, \mathcal{N}_+^s \}.
\]

If \( G \) is an \( \hat{\gamma} \Delta^m(\Omega) \)-Baire space, then \( G \) is an \( \hat{\gamma} g \Delta^m(\Omega) \)-Baire group.

Let

\[
\hat{\gamma} g \Delta^m_k \in \{ g \Delta_k, \hat{g} \Delta_k, \hat{g} g \Delta_k, \hat{c} g \Delta_k, g \Delta^+_k, \hat{c} g \Delta^+_k, g \Delta^\gamma_k, \hat{c} g \Delta^\gamma_k \}
\]

where \( k \in \{ d, h, v, s, a \} \). If \( G \) is an \( \hat{\gamma} \Delta^m_k \)-Baire space, then \( G \) is an \( \hat{\gamma} g \Delta^m_k \)-Baire group.

From Proposition 24 it follows

**Proposition 26.** Let

\[
\tilde{\Delta} \in \{ g \Delta, g \Delta_h, g \Delta_s, g \Delta_v, g \Delta_a \}.
\]

A group \( G \) is \( \tilde{\Delta} \)-Baire if for any \( U \in \mathcal{N}_c \) there exists \( W \in \mathcal{N}_c \) such that the condition (\( \tilde{\Delta} \)) is satisfied:

\[
(g \Delta) \; W W^{-1} \subset \mathcal{U}^d;
\]
\[
(g \Delta_h) \; W W^{-1} \subset \mathcal{U}^{-1} \text{ or equivalently } W W^{-1} \subset \mathcal{U}^h;
\]
\[
(g \Delta_s) \; W M^{-1} \subset U \text{ for some } M \subset W \subset \mathcal{M}.
\]
\((g\Delta_v) W W^{-1} \subseteq \overline{U}\);
\((g\Delta_a) W W^{-1} \subseteq U\).

**Proposition 27.** Let \(\gamma\) be an ordinal and
\[\hat{\Delta} \in \{g\Delta_\gamma, g\Delta_\gamma^\gamma, g\Delta_\alpha\}\].

A group \(G\) is \(\hat{\Delta}\)-Baire if for any \(S = (W_\alpha)_{\alpha<\gamma} \in \mathcal{N}_{\hat{\gamma}}\) there exists \(W \in \mathcal{N}_{\hat{\gamma}}\) so the condition \((\hat{\Delta})\) is satisfied:

\((g\Delta_\gamma) W W^{-1} \subseteq n_{\gamma}(\overline{S})^d\);
\((g\Delta_\gamma^\gamma) W W^{-1} \subseteq n_{\gamma}(S)\);
\((g\Delta_\alpha) W W^{-1} \subseteq n_{\gamma}(S)\).

From Proposition 10 it follows

**Proposition 28.** In the diagrams below, the arrow
\[A \rightarrow B\]
means that if \(G\) is an \(A\)-Baire group, then \(G\) is a \(B\)-Baire group.

\[
\begin{array}{ccc}
g\Delta_a & \rightarrow & g\Delta_v \\
\downarrow & & \downarrow \\
g\Delta_v & & g\Delta_h \\
& & \nearrow \ \\
& & g\Delta \\
\end{array}
\]

7. Continuity in \(R\)-semitopological groups

**Theorem 6.**

(1) A \(R\)-semitopological group \(G\) is an \(R\)-topological group if and only if \(G\) is \(g\Delta_a\)-Baire.

(2) A \(R\)-quasitopological group \(G\) is \(g\Delta_v\)-Baire if and only if \(G\) is \(g\Delta_h\)-Baire.

(3) A semiregular \(R\)-semitopological group \(G\) is an \(R\)-topological group if and only if \(G\) is \(g\Delta_v\)-Baire.

(4) A semiregular \(R\)-quasitopological group \(G\) is an \(R\)-topological group if and only if \(G\) is \(g\Delta_h\)-Baire.

(5) A semiregular \(R\)-paratopological group \(G\) is an \(R\)-topological group if and only if \(G\) is \(g\Delta\)-Baire.
(6) Let \( \gamma \leq \omega \). A \( R \)-paratopological group \( G \) is an \( R \)-topological group if and only if \( G \) is \( g\Delta^\gamma \)-Baire.

(7) Let \( \gamma < \omega \). A regular \( R \)-paratopological group \( G \) is an \( R \)-topological group if and only if \( G \) is \( "g\Delta^\gamma " \)-Baire.

Proof. (1) Follows from Proposition 26 (\( g\Delta^\alpha \)).
(2) Follows from Proposition 26 (\( g\Delta^\alpha \) and \( g\Delta^\beta \)).
(3) Let \( U \in \mathcal{N}_e \) be a regular open set. Proposition 26 (\( g\Delta^\alpha \)) implies that \( WW^{-1} \subset \overline{U} \) for some \( W \in \mathcal{N}_e \). Since \( WW^{-1} \) is open, \( WW^{-1} \subset Int \overline{U} = U \).
(4) Follows from (2) and (3).
(5) Let \( U \in \mathcal{N}_e \) be a regular open set. For some \( V \in \mathcal{N}_e \), \( V^2 \subset U \). Proposition 26 (\( g\Delta \)) implies that \( WW^{-1} \subset \overline{V^d} \subset \overline{V^2} \subset \overline{U} \) for some \( W \in \mathcal{N}_e \). Then \( WW^{-1} \subset Int \overline{U} = U \).
(6) Follows from Proposition 27 (\( g\Delta^\alpha \)) and Statement 5 (1).
(7) Follows from Proposition 27 (\( g\Delta^\gamma \)) and Statement 5 (2).

Theorem 7. Let \( G \) be an \( R \)-semitopological \( g\Delta^\alpha \)-Baire group. Then \( \Lambda^r_f(G) \) is a topological group and \( \lambda_g \) is a homeomorphism for all \( g \in \Lambda^r_f(G) \).

Proof. The set \( H = \Lambda^r_f(G) \) is a subgroup of \( G \). Let \( g \in H \). Let us show that \( \lambda_g \) is a homeomorphism. For \( U \in \mathcal{T}^* \) we denote

\[ \tilde{U} = Int g^{-1} Int gU = Int \lambda_g^{-1}(Int \lambda_g(U)) \]

Note that \( \tilde{U} \in \mathcal{T}^* \), \( \tilde{U} \subset U \subset \overline{U} \), \( \overline{\tilde{U}x} = \tilde{U}x \) for \( x \in G \) and \( \overline{V} \subset \overline{\tilde{W}} \) for \( V \subset U \), \( V \in \mathcal{T}^* \). Assume that \( \lambda_g \) is not a homeomorphism. Then \( x \notin \tilde{U}x = \tilde{U}x \) for some \( x \in G \) and \( U \in \mathcal{T}^* \). Then \( e \notin \tilde{U} \).

Lemma 1. Let \( U \in \mathcal{N}_e \). Then \( WM^{-1} \subset U \) for some \( W \in \mathcal{T}^* \) and \( M \subset W \subset \overline{M} \) so \( e \in M \) and \( W \subset U \).

Proof. Since \( G \) is a \( g\Delta^\alpha \)-Baire group, it follows from Proposition 24 that \( W_{*,M^{-1}} \subset U \) for some \( W_* \in \mathcal{N}_e \) and \( M_* \subset W \subset \overline{M_*} \). Let \( W = W_* \cap U \) and \( M = M_* \cup \{e\} \).

Lemma 1 implies that \( WM^{-1} \subset U \) for some \( W \in \mathcal{N}_e \), \( W \subset U \) and \( M \subset W \subset \overline{M} \). Since \( \tilde{W} \) is open and dense in \( W \) and \( M \) is dense in \( W \), there exists \( y \in \tilde{W} \cap M \). Since \( W_{y^{-1}} \subset U \), then

\[ e \in \tilde{W}y^{-1} \subset \tilde{W}_{y^{-1}} \subset \tilde{U}. \]

A contradiction with the fact that \( e \notin \tilde{U} \).

The group \( H \) is a semitopological group.

Lemma 2. Let \( M \subset G \), \( W \in \mathcal{N}_e \). Then \( H \cap \overline{M} \subset MW^{-1} \).

Proof. Let \( x \in H \cap \overline{M} \). Then \( xW \cap M \neq \emptyset \). Hence \( x \in MW^{-1} \). \( \square \)
Let us prove (1). Theorem 7 implies that $WM^{-1} \subset U$ for some $W \in \mathcal{N}_e$ and $M \subset W \subset \mathcal{M}$. Lemma 2 implies that
\[ H \cap W \subset H \cap \mathcal{M} \subset MW^{-1} \subset U^{-1}. \]

Let us show that $H$ is a paratopological group, that is, for $U \in \mathcal{N}_e$ there exists an $S \in \mathcal{N}_e$ such that $(H \cap S)^2 \subset U$. Lemma 1 implies that $WM^{-1} \subset U$ for some $W \in \mathcal{N}_e$ and $M \subset W \subset \mathcal{M}$. Also $V M^{-1} \subset W$ for some $V \in \mathcal{N}_e$ and $L \subset V \subset \mathcal{V}$ so $e \in L$ and $V \subset W$. Then $MLV^{-1} \subset U^{-1}$.

Lemma 2 implies that $H \cap \mathcal{M}L \subset MLV^{-1} \subset U^{-1}$. Let $W_s = W \cap H$ and $V_s = V \cap H$. Since $\mathcal{M}L \subset \mathcal{M}L$ and $W_s \subset \mathcal{M}$, then $W_sL \subset \mathcal{M}L$. Since $V_s \subset \mathcal{L}$ and $W_s \subset H$, then $W_sV_s \subset W_sL \subset \mathcal{M}L$. We get $W_sV_s \subset U^1$. Since $V_s \subset W_s$, then $(V \cap H)^2 \subset U^{-1}$. Since $H$ is a quasitopological group, $S \cap H \subset V^{-1}$ for some $S \in \mathcal{N}_e$. Then $(H \cap S)^2 \subset U$.

Theorem 7 implies

**Theorem 8.** Let $G$ be a $R$-semitopological $O(fl)$-topological $g\Delta_s$-Baire group. Then $G$ is a topological group.

**Lemma 3.** Let $G$ be an $R$-topological group, $M \subset G$, $W \in \mathcal{N}_e$. Then $\mathcal{M} \subset W^{-1}M$.

**Proof.** Let $x \in \mathcal{M}$. Then $Wx \cap M \neq \emptyset$. Therefore, $x \in W^{-1}M$. \qed

**Theorem 9.** Let $G$ be a $R$-semitopological $O(df\ell^*)$-topological $g\Delta_s$-Baire group. Then

1. $G$ is a $g\Delta_s$-Baire group;
2. if $G$ is semiregular, then $G$ is a topological group.

**Proof.** Condition (2) follows from condition (1), Theorem 6 (3) and Theorem 2. Let us prove (1). Theorem 7 implies that $H = \Lambda^*(G)$ is dense in $G$ and $H$ is a topological group.

**Lemma 4.** Let $U \in \mathcal{N}_e$. There exists $V \in \mathcal{N}_e$ such that $\mathcal{V}^2 \subset U$.

**Proof.** There is $V \in \mathcal{N}_e$ such that $V_s = V^{-1}$ and $V^2 \subset U$, where $V_s = V \cap H$. Then $V_s \subset \mathcal{U}$ and $V_s \subset \mathcal{V}$. Since $V_s = V$, then $\mathcal{V}^2 \subset U$. \qed

By Proposition 26 $(g\Delta_s)$, it suffices to show that for $U \in \mathcal{N}_e$ there exists a $W \in \mathcal{N}_e$ such that $WW^{-1} \subset U$. Lemma 4 implies that there exist $V, S \in \mathcal{N}_e$ such that $\mathcal{V}^2 \subset U$ and $\mathcal{S}^2 \subset \mathcal{V}$. Then $S^{-1}S^{-1} \subset \mathcal{V}^{-1}$. Lemma 3 implies $S^{-1} \subset \mathcal{V}^{-1}$. Since $H$ is dense in $G$ and $H$ is a topological group, $e \in \operatorname{Int} S^{-1}$. Let $W = V \cap \operatorname{Int} S^{-1}$. Then $W \in \mathcal{N}_e$, $W^{-1} \subset \mathcal{V}$ and $W \subset V$. We get $WW^{-1} \subset \mathcal{V}^2 \subset U$. \qed

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Lemma 5. Let \( G \) be a \( R \)-semitopological \( O(\text{sqpe}) \)-topological group, \( U \in \mathcal{T}^* \) and \( M \subset G = \overline{M} \). For any positive \( n \in \omega \) there are \( x \in M \) and \( V \in \mathcal{N}_c \), so \( V^n x \subset U \).

**Proof.** Let us prove it by induction on \( n \).

\( n = 1 \). Let \( x \in U \cap M \) and \( V = Ux^{-1} \).

\( n > 1 \). By the induction hypothesis, there are \( x_* \in U \cap M \) and \( V_* \in \mathcal{N}_c \) for which \( V_*^{n-1} x_* \subset U \). Let \( h \in V_* \). Since \( G \) is a \( O(\text{sqpe}) \)-topological group, there are \( g \in V_* x_* h^{-1} \) and \( S \in \mathcal{N}_c \) such that \( S^2 g \subset V_* x_* h^{-1} \). Let \( x \in Sgh \cap M \) and

\[
V = Sghx^{-1} \cap S \cap V_*.
\]

Then \( V^2 x \subset SSGh \subset V_* x_* \) and \( V^nx \subset V_*^{n-1} x_* \subset U \). \( \square \)

**Theorem 10.** Let \( G \) be a \( R \)-semitopological \( O(df^{*},f\text{pe}) \)-topological \( g\Delta \)-Baire group.

1. If \( G \) is a \( \pi \)-semiregular space, then \( G \) is a \( g\Delta \)-Baire group;

2. If \( G \) is a semiregular space, then \( G \) is a topological group.

**Proof.** The group \( H = \Lambda_{f}^{*}(G) \) is dense in \( G \). Proposition 1 ((\( df^{*},f\text{pe},r \)) \( \to (\text{sqpe}) \)) implies that \( G \) is a \( O(\text{sqpe}) \)-topological group, that is, multiplication in \( G \) quasi-continuous in the first coordinate in \((e,e)\).

**Lemma 6.** \( \text{Int} S^{-1} \neq \emptyset \) for any regular open \( S \in \mathcal{T}^* \).

**Proof.** Lemma \( \square \) implies that there exists \( g \in S \cap H \) and \( U \in \mathcal{N}_c \), so \( U^2 g \subset S \).

Proposition 26 \((g\Delta)\) implies that \( WW^{-1} \subset \overline{U}^d \) for some \( W \in \mathcal{N}_c \). Since \( \overline{U}^d \subset \overline{U}^2 \), then

\[
WW^{-1} g \subset \overline{U}^d g \subset \overline{U}^2 g \subset \mathfrak{F}.
\]

Since \( S \) is a regular open subset of \( G \) and \( WW^{-1} g \) is open, then \( WW^{-1} g \subset S \). Then \( g^{-1} WW^{-1} \subset S^{-1} \). Since \( g \in H \), then \( \text{Int} g^{-1} WW^{-1} \neq \emptyset \). Hence \( \text{Int} S^{-1} \neq \emptyset \). \( \square \)

Condition (2) follows from condition (1) and Theorem \( \square \). Let us prove (1). Since \( G \) is a \( \pi \)-semiregular space, it follows from the Lemma \( \square \) that \( G \) is a \( O(fi) \)-topological group. Proposition \( \square \) ((\( df^{*},fi,e, \)) \( \to (fi,fi) \)) implies that \( G \) is a \( O(fi,fi) \)-topological group. So the group \( G \) is a \( O(fi,\text{sqpe},fi,r) \)-topological group.

Let \( U \in \mathcal{N}_c \). Let us show that \( \text{Int} \overline{U}^d \subset \overline{U} \). Let us assume the opposite. Then \( S = \text{Int} \overline{U}^d \setminus \overline{U} \neq \emptyset \). Lemma \( \square \) implies that \( q \in S \) and \( Q \in \mathcal{N}_c \) existsuch that \( Q^2 q \subset S \). Since \( G \) is a \( O(fi) \)-topological group, \( \text{Int}(Qq)^{-1} \neq \emptyset \) and \( Vg^{-1} \subset \text{Int}(Qq)^{-1} \) for some \( g \in Qq \) and \( V \in \mathcal{N}_c \). We get \( gV^{-1} \subset Qq \) and \( QgV^{-1} \subset Q^2 q \subset S \). Since \( QgV^{-1} \cap U = \emptyset \), then \( Qg \cap UV = \emptyset \) and \( g \notin UV \subset \overline{U}^d \). A contradiction with the fact that \( g \in S \subset \overline{U}^d \).

We have shown that \( \text{Int} \overline{U}^d \subset \overline{U} \). Proposition 26 \((g\Delta)\) implies that \( WW^{-1} \subset \overline{U}^d \) for some \( W \in \mathcal{N}_c \). Since \( WW^{-1} \in \mathcal{T}^* \), then \( WW^{-1} \subset \text{Int} \overline{U}^d \subset \overline{U} \). \( \square \)
Proposition 29. Let $G$ be a $R$-semitopological $O(sqpe)$-topological $g\Delta_h$-Baire group. Then $G$ is a quasi-regular space.

Proof. Let $U \in \mathcal{T}^*$. Lemma 5 implies that $V^3g \subset U$ for some $g \in U$ and $V \in \mathcal{T}^*$. Since $V^h \subset V^2$, it follows from Proposition 26 ($g\Delta_h$) that $WW^{-1} \subset V^h \subset V^2$ for some $W \in \mathcal{N}_c$. Then $SVg \subset U$, where $S = WW^{-1}$. Since $S = S^{-1}$, it follows from the Lemma 3 that $V \subset SV$. Hence $Vg \subset U$.

Theorem 11. Let $G$ be a $R$-semitopological $O(dfll*, fpe)$-topological $g\Delta_h$-Baire group. Then

1) $G$ is a $g\Delta_v$-Baire group;

2) if $G$ is a semiregular space, then $G$ is a topological group.

Proof. Condition (1) follows from Theorem 10 (1) and Proposition 29. Condition (2) follows from condition (1) and Theorem 6. Also, condition (2) follows from Theorem 10 (2). □

Theorem 12. Let $G$ be a semitopological $O(dfl*, fie)$-topological $g\Delta_h$-Baire group. Then

1) $G$ is a $g\Delta_v$-Baire group;

2) if $G$ is a semiregular space, then $G$ is a topological group.

Proof. Condition (2) follows from condition (1) and Theorem 6. Let us prove (1). Proposition 24 ($dfll*, fie, r$) implies that $G$ is a $O(fli)$-topological group. The group $H = \Lambda^*(G)$ is dense in $G$. Let $U \in \mathcal{N}_c$.

Lemma 7. $\text{Int} \overline{U}^h \subset \overline{U}$.

Proof. Let us assume the opposite. Then $S = \text{Int} \overline{U}^h \setminus \overline{U} \neq \varnothing$. Since $G$ is a $O(fli)$-topological group, $\text{Int} S^{-1} \neq \varnothing$. Then $gQ^{-1} \subset \text{Int} S^{-1}$ for some $g \in H \cap S$ and $Q \in \mathcal{N}_c$. Then $Q^{-1} \subset \overline{Q}^{-1} \subset g^{-1}S$. Proposition 26 ($g\Delta_h$) implies that $V = WW^{-1} \subset \overline{Q}^{-1}$ for some $W \in \mathcal{N}_c$. Then $gV \subset S$ and $gV \cap U = \varnothing$. Hence $g \notin UV \subset \overline{U}^h$. A contradiction with $g \in S \subset \overline{U}^h$.

Proposition 26 ($g\Delta_h$) implies that $WW^{-1} \subset \overline{U}^h$ for some $W \in \mathcal{N}_c$. Then $WW^{-1} \subset \text{Int} \overline{U}^h \subset \overline{U}$.

8. Groups with quasicontinuous multiplication

Let us define the topological games $G(y_*, Y)$ and $\overline{G}(y_*, Y)$ for the space $Y$ and $y \in Y$ [27, 28]. Players $\alpha$ and $\beta$ are playing. On the $n$th move, player $\alpha$ chooses

- open neighborhood $W_n \subset Y$ of point $y_*$ in game $G(y_*, Y)$;
- is an open non-empty set $W_n \subset Y$ in the game $\overline{G}(y_*, Y)$. 29
Player $\beta$ chooses $y_n \in W_n$. Player $\alpha$ wins if $y_* \in \{y_n : n \in \omega\}$.

A point $x \in X$ is called a $W$-point ($\tilde{W}$-point) if if player $\alpha$ has a winning strategy in the game $G(y_*, Y)$ ($\tilde{G}(y_*, Y)$). A space $X$ is called a $W$-space ($\tilde{W}$-space) if every point in $Y$ is a $W$-point ($\tilde{W}$-point) [27 28].

Any $W$-space is a $\tilde{W}$-space. Property $W$ is inherited by subspaces, property $\tilde{W}$ is inherited by dense subspaces.

Recall the definition of the topological Banach-Mazur game $BM(X)$ for the space $X$. Denote $U_{-1} = X$. Players $\alpha$ and $\beta$ are playing. On the $n$th move, player $\beta$ chooses an open non-empty set $V_n \subset U_{n-1}$, player $\alpha$ chooses an open non-empty set $U_n \subset V_n$. Player $\alpha$ wins if $\bigcap_{n \in \omega} V_n = \bigcap_{n \in \omega} U_n \neq \emptyset$.

The Banach-Mazur theorem [29] says that if the player $\beta$ has a winning strategy, then the space $X$ is not Baire.

A mapping $\Phi : X \times Y \to Z$ is called $KC$-continuous if the mappings $\Phi(\cdot, y_*) : X \to Z, x \mapsto \Phi(x, y_*)$ is quasicontinuous and the mappings $\Phi(x_*, \cdot) : Y \to Z, y \mapsto \Phi(x_*, y)$ are continuous for all $(x_*, y) \in X \times Y$.

**Proposition 30.** Let $X, Y, Z$ be a regular space, the map $\Phi : X \times Y \to Z$ be $KC$-continuous, the point $y_* \in Y$ be a $\tilde{W}$-point, and the space $X$ Baire. Then the mapping $\Phi$ is quasi-continuous at every point $(x_*, y_*)$ for all $x_* \in X$.

**Proof.** Assume the contrary. Then the mapping $\Phi$ is not quasi-continuous at the point $(x_*, y_*)$ for some point $x_* \in X$. There is a neighborhood $S \subset X \times Y$ of $(x_*, y_*)$ and a neighborhood $O \subset Z$ of $\Phi(x_*, y_*)$ so that $\Phi(S') \not\subset O$ for any non-empty $S' \subset S$.

Let $A$ and $A'$ be neighborhoods of the point $\Phi(x_*, y_*)$ such that $A \subset \overline{A} \subset A' \subset \overline{A'} \subset O$. Let $B = Z \setminus \overline{A'}$. Then $\overline{A} \cap \overline{B} = \emptyset$ and $\Phi(S') \cap B \neq \emptyset$ for any non-empty open $S' \subset S$. That is, $S \subset \Phi^{-1}(B)$. Take open non-empty open $U \subset X$ and $V \subset Y$ such that $y_* \in V$, $U \times V \subset S$ and $\Phi(U \times \{y_*\}) \subset A$.

Let us construct a strategy $s_3^\alpha$ for player $\beta$ in the Banach-Mazur game $BM(X)$. At the same time, we will play an auxiliary game in the game $\tilde{G}(y_*, Y)$, in which the player $\alpha$ follows the winning strategy $p_\alpha$. We can assume that in the strategy $p_\alpha$ the player chooses an open subset $V$. Let $s_\alpha$ be some strategy of player $\alpha$ in the game $BM(X)$. We will also construct an auxiliary strategy $p_3^\beta$ of player $\beta$ in the game $\tilde{G}(y_*, Y)$. The strategies $s_\alpha$ and $p_\alpha$ are given, the strategies $s_3^\alpha$ and $p_3^\beta$ will be built.

During construction, the following conditions will be met:

$(C_n)$ $\Phi(U_n \times \{y_n\}) \cap B \neq \emptyset$;
$(D_n)$ $\Phi(V_{n+1} \times \{y_n\}) \subset B$.

0th step:

$(s_\beta^*) V_0 = V$;

$(s_\alpha) U_0 \subset V_0$;

$(p_\alpha) W_0 \subset V$;
\( (p^*_\alpha) \) \( y_0 \in W_0 \) is chosen in such a way that the condition \((C_0)\) is satisfied.

\textbf{n}th step:

\( (s^*_\beta) \) since the condition \((C_{n-1})\) is satisfied and the mapping \( \Phi \) is KC-continuous, there exists a non-empty open \( V_n \subset U_{n-1} \) so that the condition \((D_{n-1})\) is satisfied;

\( (s_\alpha) \) \( U_n \subset V_n \);

\( (p_\alpha) \) \( W_n \subset V \);

\( (p^*_\beta) \) \( y_n \in W_n \) is chosen such that \((C_n)\) is satisfied.

Let’s check that player \( \beta \) won the game \( BM(X) \). Assume the opposite, that is, there exists \( x \in \bigcap_{n \in \omega} V_n \). From \((D_n)\) it follows that \( \Phi(x, y_n) \in B \) for \( n \in \omega \). Since the strategy \( p_\alpha \) is winning, then \( y_* \in \{y_n : n \in \omega\} \). Since the function \( \Phi(x, \cdot) \) is continuous, \( \Phi(x, y_*) \in B \). Since \( \Phi(U \times \{y_*\}) \subset A \), then \( \Phi(x, y_*) \in A \). A contradiction with the fact that \( \widetilde{A} \cap B = \emptyset \).

We have shown that the strategy \( s^*_\beta \) is winning for player \( \beta \) in the game \( BM(X) \). Hence the space \( X \) is not Baire. Contradiction.

**Proposition 31.** Let \( G \) be a \( O(dql, r) \)-topological Baire regular group and \( \Lambda_q(G) \) contain a dense \( \tilde{\mathcal{W}} \)-space. Then \( G \) is a \( O(qpe) \)-topological group.

**Proof.** Let \( X \) be a \( \tilde{\mathcal{W}} \)-space dense in \( \Lambda_q(G) \). Since \( G \) is homogeneous, we can assume that \( e \in X \). Let’s put

\[ \Phi : X \times G \to G, (x, g) \mapsto gx. \]

The mapping \( \Phi \) is KC-continuous. Proposition [30] implies that \( \Phi \) is quasicontinuous. Since \( X \) is dense in \( G \), the multiplication \( m \) is quasicontinuous in \((e, e)\). \( \square \)

Proposition [31] implies the following statement.

**Proposition 32.** Let \( G \) be a \( O(ql, r) \)-topological Baire regular group and \( G \) contain a dense \( \tilde{\mathcal{W}} \)-space. Then \( G \) is a \( O(qpe) \)-topological group.

**Proposition 33.** A regular space with a countable \( \pi \)-character is a \( \tilde{\mathcal{W}} \)-space.

**Proof.** Let \( X \) be a space with countable \( \pi \)-character, \( x_* \in X \), \( (W_n)_{n \in \omega} \) be a countable \( \pi \)-base in \( x_* \). The winning strategy for player \( \alpha \) in the game \( \tilde{G}(x_*, X) \) is that on the \( n \)th move player \( \alpha \) chooses \( W_n \) from the \( \pi \)-base. \( \square \)

**Proposition 34.** Let \( G \) be a \( O(dql, r) \)-topological Baire regular group with countable \( \pi \)-character. Then \( G \) is a \( O(qpe) \)-topological group.

**Proof.** The countable \( \pi \)-character is inherited by dense subspaces. Hence \( \Lambda_q(G) \) has a countable \( \pi \)-character. Next, apply Proposition [31]. \( \square \)
9. Main results

In this section, we formulate corollaries from the Section 7.

Let \( G \) be a right semitopological group. A set \( P \subseteq G \times G \) is called a semi-neighborhood of the diagonal if \( P(x) = \{ y \in G : (x, y) \in P \} \) is a neighborhood (not necessarily open) of a point \( x \) in \( G \). For a neighborhood \( U \in \mathcal{N}_c \) of unity, \( \mathcal{R}(U) = \{(x, y) \in G^2 : yx^{-1} \in U\} \) is a semi-neighborhood of the diagonal.

For \( \Delta \in \{ \Delta, \Delta_g, \Delta_s \} \), a space \( G \) (right semitopological group \( G \)) is \( \Delta \)-nonmeager (\( g\Delta \)-Baire) if for any semi-neighborhood of the diagonal \( P \) (semi-neighborhoods of the diagonal of the form \( P = \mathcal{R}(U) \), where \( U \) is a neighborhood of unity) there exists an open non-empty \( W \subset G \), so that the condition \( (\Delta) \) is satisfied:

\[(\Delta) \ W \times W \subset P \cap (W \times W);\]
\[(\Delta_g) \ W \subset \{ x : (x, y) \in P \} \text{ for all } y \in W;\]
\[(\Delta_s) \ W \subset \{ x : \{ x \} \times W \subset P \}.\]

Let \( \mathcal{P}_k \ (\mathcal{P}_c) \) be the smallest class of spaces that

- contains \( p \)-spaces and strongly \( \Sigma \)-spaces;
- is closed under arbitrary (countable) products;
- is closed under taking open subspaces.

Let \( \mathcal{D}_d \) be the smallest class of spaces that

- contains \( \Sigma \)-spaces, \( w\Delta \)-spaces, and feebly compact spaces;
- is closed under products by spaces from the class \( \mathcal{P}_k \);
- is closed under taking open subspaces.

Let \( \mathcal{D}_h \) be the smallest class of spaces that

- contains \( \Sigma \)-spaces and \( w\Delta \)-spaces;
- is closed under products by spaces from the class \( \mathcal{P}_c \);
- is closed under taking open subspaces.

Let \( \mathcal{D}_s \) be the class of semiregular spaces that contain a metrizable nonmeager subset. Baire semiregular spaces from the following classes belong to this class:

- \( \sigma \)-spaces, spaces with a countable network;
- developable spaces.

In [15] \( \Gamma_{\sigma,d}^{\mathcal{D}_{\sigma}} \)-nonmeager, \( \Gamma_{p,d}^{\mathcal{D}_{p}} \)-nonmeager and \( \Gamma_{f}^{\mathcal{D}_{f}} \)-nonmeager spaces are introduced and studied.
Theorem 13. Let \( G \) be a \( R \)-topological group.

\((g\Delta)\) A right semitopological group is \( g\Delta \)-Baire if it is a \( \Delta \)-nonmeager space. The class of \( \Delta \)-nonmeager spaces contains \( \Gamma_{o,l}^{\text{O}_D} \)-nonmeager spaces, which contain regular Baire spaces from the class \( D_d \).

\((g\Delta_h)\) A right semitopological group is \( g\Delta_h \)-Baire if it is a \( \Delta_h \)-nonmeager space. The class of \( \Delta_h \)-nonmeager spaces contains \( \Gamma_{p,l}^{\text{O}_D} \)-nonmeager spaces, which contain regular Baire spaces from the class \( D_h \).

\((g\Delta_s)\) A right semitopological group is \( g\Delta_s \)-Baire if it is \( \Delta_s \)-nonmeager or CDP-nonmeager space. The class of \( \Delta_s \)-nonmeager spaces contains \( \Gamma_{p,l}^{\text{B}_M} \)-nonmeager spaces. The classes of \( \Delta_s \)-nonmeager and CDP-nonmeager spaces contain the class \( D_s \).

Proof. \((g\Delta)\) It follows from Proposition 26 that an right semitopological group is \( g\Delta \)-Baire if \( G \) is a \( \Delta \)-nonmeager space. Proposition 36 (1) \([15]\) implies that \( \Gamma_{o,l}^{\text{O}_D} \)-nonmeager spaces are \( \Delta \)-nonmeager. It follows from the Theorem 4 \([15]\) that the class of \( \Gamma_{o,l}^{\text{O}_D} \)-nonmeager spaces contains regular Baire spaces from the class \( D_d \).

\((g\Delta_h)\) It follows from Proposition 26 that an right semitopological group is \( g\Delta_h \)-Baire if \( G \) is a \( \Delta_h \)-nonmeager space. Proposition 36 (2) \([15]\) implies that \( \Gamma_{p,l}^{\text{O}_D} \)-nonmeager space is \( \Delta_h \)-nonmeager. It follows from the Theorem 4 \([15]\) that the class of \( \Gamma_{p,l}^{\text{O}_D} \)-nonmeager spaces contains regular Baire spaces from the class \( D_h \).

\((g\Delta_s)\) It follows from Proposition 26 that an right semitopological group is \( g\Delta_s \)-Baire if \( G \) is a \( \Delta_s \)-nonmeager or CDP-nonmeager space. Proposition 36 (3) \([15]\) implies that \( \Gamma_{p,l}^{\text{B}_M} \)-nonmeager space is \( \Delta_s \)-nonmeager. It follows from the Theorem 4 \([15]\) that the class of \( \Gamma_{p,l}^{\text{B}_M} \)-nonmeager spaces contains regular Baire spaces from the class \( D_s \). Proposition 14 implies that CDP-nonmeager spaces are \( \Delta_s \)-nonmeager spaces. Proposition 13 implies that the spaces in the class \( D_s \) are CDP-nonmeager.

Theorem 14 (Theorem 2). Let \( G \) be a \( R \)-topological group. If the set \( \Lambda^*_f(G) \) dense in \( G \), then \( G \) is a topological group.

Lemma 8. Let \( G \) be a compact Hausdorff right semitopological group. Then \( \Lambda(G) = \Lambda^*(G) \).

Proof. Clearly, \( \Lambda(G) \supset \Lambda^*(G) \). Let \( g \in \Lambda(G) \). The mapping \( \lambda_g : G \to G \) is a bijective continuous mapping of compact Hausdorff spaces. Hence \( \lambda_g \) is a homeomorphism and the mapping \( (\lambda_g)^{-1} = \lambda_{g^{-1}} \) is continuous, i.e. \( g^{-1} \in \Lambda^*(G) \).

Theorem 15. Let \( G \) be a \( g\Delta \)-Baire right semitopological group.

(1) If \( G \) is an \( R \)-paratopological group, then \( G \) is an \( R \)-topological group.
(2) If \( G \) is a semiregular space, the set \( \Lambda^*_f(G) \) is dense in \( G \) and multiplication \( m : G \times G \to G, \ (g, h) \mapsto gh \) is feebly continuous in \((e, e)\), then \( G \) is a topological group.

(3) If \( G \) is a regular space, the set \( \Lambda^*_q(G) \) is dense in \( G \) and \( \Lambda^*_q(G) \) contains a dense \( W \)-space, then \( G \) is a topological group.

Proof. Condition (1) follows from Theorem 6 (5). Condition (2) follows from Theorem 10. Condition (3) follows from (2) and Proposition 31. \( \square \)

Corollary 2. Let \( G \) be a regular right semitopological group.

(1) If \( G \) is a product of \( \breve{C}ech \) complete first-countable spaces (for example, \( G \) is homeomorphic to \( \mathbb{R}^\tau \)) and \( \lambda_g \) is quasi-continuous for every \( g \in G \), then \( G \) is a topological group.

(2) [9, Proposition 3.2] If \( G \) is a CHART group with feebly continuous multiplication, then \( G \) is a topological group.

(3) If \( G \) is a CHART group with countable \( \pi \)-character (for example, \( G \) is a compact space with countable tightness), then \( G \) is a metrizable topological group.

Proof. (1) The product of \( \breve{C}ech \) complete spaces is a Baire space, and \( \breve{C}ech \) complete spaces are \( p \)-spaces. Theorem 15 implies that \( G \) is a \( \breve{g}\Delta \)-Baire group. The product \( G \) contains the \( \Sigma \)-product \( S \) of first-countable spaces. [27, Theorem 4.6] implies that \( S \) is a \( W \)-space and hence a \( \breve{W} \)-space. Theorem 15 (3) implies that \( G \) is a topological group.

(2) Lemma \( \square \) implies that \( \Lambda(G) = \Lambda^*(G) \subset \Lambda^*_q(G) \) and \( \Lambda^*_q(G) \) is dense in \( G \). Theorem 15 (2) implies that \( G \) is a topological group.

(3) Compact spaces with countable tightness have countable \( \pi \)-character [30]. It follows from Proposition 34 that \( G \) is a \( O(qpe) \)-topological group and, therefore, multiplication in \( G \) is feebly continuous. It follows from (2) that \( G \) is a topological group. Topological groups with countable \( \pi \)-character are first-countable [13, Proposition 5.2.6]. The Birkhoff–Kakutani theorem [13, Theorem 3.3.12] implies that \( G \) is metrizable. \( \square \)

Theorem 16. Let \( G \) be a semiregular \( g\Delta_h \)-Baire right semitopological group.

(1) If \( G \) is an \( R \)-quasitopological group, then \( G \) is an \( R \)-topological group.

(2) If the set \( \Lambda^*(G) \) is dense in \( G \) and the inverse mapping \( i : G \to G, \ g \mapsto g^{-1} \) is feebly continuous in \( e \), then \( G \) is a topological group.

Proof. Condition (1) follows from the Theorem 6 (2) and (4). Condition (2) follows from Theorem 12. \( \square \)

Theorem 17. Let \( G \) be an \( g\Delta_s \)-Baire right semitopological group.

(1) If \( \lambda_g \) is feebly continuous for any \( g \in G \), then \( G \) is a topological group.
(2) If $G$ is a semiregular space and the set $\Lambda_f^* (G)$ dense in $G$, then $G$ is a topological group.

Proof. Condition (1) follows from Theorem 8. Condition (2) follows from Theorem 9. □

Corollary 3. (MA) Let $G$ be a CHART group and $w(G) < 2^\omega$. Then $G$ is a topological group.

Proof. CHART groups have a right-invariant Haar measure [31, 32]. Hence $G$ is a ccc space. Corollary 1 implies that $G$ is a CDP-space. Hence $X$ is a CDP-nonmeager space. Theorem 13 and 17 (2) implies that $G$ is a topological group. □

10. Examples and questions

Example 2 (Example 1). Metrizable compact $R$-topological non-topological group.

Example 3. Let $G = \mathbb{R}$ with a topology whose open sets are $U \setminus P$, where $U$ is open in $\mathbb{R}$ and $P$ is nowhere dense in $\mathbb{R}$. The group $G$ is a quasitopological $O(supe)$-topological quasi-regular Hausdorff $g\Delta_n$-Baire $g\Delta_h$-Baire $\Delta$-Baire non-$\Delta_h$-nonmeager group which is not a topological group.

Example 4. Let $G = \mathbb{R}^2$ with a topology whose base point $(x_*, y_*) \in G$ form sets of the form

\[(U \cap \{(x, y) \in G : y > y_*\}) \cup \{(x_*, y_*)\},\]

where $U$ is an open neighborhood $(x_*, y_*)$ in the standard topology $\mathbb{R}^2$ [7]. The group $G$ is a Hausdorff quasi-regular non-regular paratopological group.

Example 5 (Example 2.13 [7]). There exists a Baire and metric left semitopological group $G$ which is not a right semitopological group, and whose inversion is nearly continuous but not feebly continuous. The group $G$ is $\Delta_s$-Baire.

Example 6. Let $G$ be a pseudocompact Boolean quasitopological group that is not a topological group (Example 3 [15]). The group $G$ is a Tychonoff $\Delta$-Baire non-$g\Delta_h$-Baire group.

Problem 2. Let $G$ be a (semi)regular $R$-semitopological group. Which of the following conditions imply that $G$ is an $R$-topological group?

1. The group $G$ is $g\Delta$-Baire ($\Delta$-Baire, pseudocompact) and the multiplication of $m$ in $G$ is quasi (feebly) continuous (in $(e, e)$).

2. The group $G$ is $g\Delta_h$-Baire ($\Delta_h$-Baire, countably compact, compact) and the multiplication of $m$ in $G$ is quasi (feebly) continuous (in $(e, e)$).

3. The group $G$ is $g\Delta_s$-Baire ($\Delta_s$-Baire, metrizable Baire) and the multiplication of $m$ in $G$ is quasi (feebly) continuous (in $(e, e)$).
(4) The group \( G \) is \( g\Delta h \)-Baire (\( \Delta h \)-Baire, countably compact, compact) and the operation of taking the inverse of \( i \) in \( G \) is quasi (feebly) continuous (in \( e \)).

(5) The group \( G \) is \( g\Delta s \)-Baire (\( \Delta s \)-Baire, metrizable Baire) and the operation of taking the inverse of \( i \) in \( G \) is quasi (feebly) continuous (in \( e \)).

**Problem 3.** Let \( G \) be a (semi)regular \( g\Delta h \)-Baire (\( \Delta h \)-Baire, countably compact, compact) and \( R \)-semitopological group. Which of the following conditions imply that \( G \) is a topological group?

1. \( G = \Lambda_f(G) \).
2. \( G = \Lambda_q(G) \).

A space \( X \) is called weakly pseudocompact if there exists a compact Hausdorff extension \( bX \) of the space \( X \) in which the space \( X \) is \( G_\delta \)-dense, i.e. \( X \) intersects any nonempty \( G_\delta \) subset of \( bX \). It is clear that the product of weakly pseudocompact spaces is weakly pseudocompact; in particular, the product of pseudocompact spaces is weakly pseudocompact.

**Problem 4.** Let \( G \) be a (semi)regular semitopological group. Which of the following conditions imply that \( G \) is a topological group?

1. The group \( G \) is \( g\Delta h \)-Baire (\( \Delta h \)-Baire).
2. The group \( G \) is paratopological and \( G \) is weakly pseudocompact (product of pseudocompact spaces, product of two pseudocompact spaces) (Question 1).
3. The group \( G \) is feebly compact and belongs to one of the following classes of spaces: separable; countable tightness; \( k \)-space (Problem 3.5).

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