UNIVERSAL PROPERTIES OF CHIRAL SYMMETRY BREAKING

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Abstract

We discuss chiral symmetry breaking critical points from the perspective of PCAC, correlation length scaling and the chiral equation of state. A scaling theory for the ratio $R_\pi$ of the pion to sigma masses is presented. The Goldstone character of the pion and properties of the longitudinal and transverse chiral susceptibilities determine the ratio $R_\pi$ which can be used to locate critical points and measure critical indices such as $\delta$. We show how PCAC and correlation length scaling determine the pion mass’ dependence on the chiral condensate and lead to a practical method to measure the anomalous dimension $\eta$. These tools are proving useful in studies of the chiral transition in lattice QED and the quark-gluon plasma transition in lattice QCD.

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1. INTRODUCTION

Chiral symmetry phase transitions are particularly interesting because of their physical relevance and their theoretical constraints. The order parameter for the high temperature quark-gluon transition of QCD is the chiral condensate and considerable effort both in phenomenological studies and computer simulations are underway [1]. Non-compact lattice QED experiences a chiral transition at strong coupling and one needs new practical theoretical tools to understand the character of this transition [2]. Four Fermi models with N species of fermions in less than four dimensions have chiral transitions, and 1/N expansions indicate that these transitions correspond to ultra-violet fixed points of the renormalization group [3], [4]. These models illustrate the intimate relation between renormalizability, compositeness and hyperscaling in four Fermi and Yukawa models. Composite models of the Higgs mechanism which underlies the Standard Model typically rely on a chiral symmetry breaking phase transition to make composite mesons which can produce heavy W and Z particles. Although the original versions of this idea (Technicolor) copied the asymptotically free dynamics of QCD, other versions imagine theories which are strongly coupled at short distances [5]. Interesting models of a heavy top quark based on gauged Nambu Jona Lasinio models have been considered [6]. In this paper we shall be particularly interested in models where chiral symmetry breaking is a short distance phenomenon and we shall concentrate on non-asymptotically free dynamics.

Chiral symmetry breaking phase transitions have important simplifying features which will allow us to study them in several complementary ways. Our theoretical framework will be that of statistical mechanics and concepts borrowed from ferromagnetic phase transitions such as the equation of state, scaling variables, correlation length scaling and hyperscaling relations between critical indices, etc. will be used heavily here. This 'standard approach' gains far greater predictive power when current algebra relations following from the underlying continuous chiral symmetry of the system's Lagrangian are developed. The most familiar deduction of this approach is the Goldstone character and low energy theorems concerning the pion. But we shall present additional equally firm results here involving the chiral partner, the sigma. One of the goals of this work is to develop practical methods to extract critical indices from chiral symmetry breaking transitions and classify critical points as Gaussian (trivial) or non-Gaussian (nontrivial). We shall see that the ratio $R_\pi = M_\pi^2/M_\sigma^2$ and its dependence on coupling and bare fermion mass predicts the critical coupling and the critical index $\delta$. We shall relate $R_\pi$ to the transverse and longitudinal susceptibilities of the chiral phase transition and show that $R_\pi$ is completely determined by the theory's equation of state. The resulting formula has already proved its usefulness in studies of lattice QED [N=2] and we anticipate applications to the other models listed above. Most past work on lattice QED and its chiral symmetry breaking phase transition have concentrated on the order parameter $<\bar{\psi}\psi>$ and its equation of state [7]. Since these quantities are measured at nonvanishing bare fermion mass, the critical point must always be inferred and a rapid crossover between a symmetric and a broken phase occurs in raw simulation data. This obscures the critical point and the value of critical indices. However, since $R_\pi$ is determined by the same equation of state, any hypothesis one makes concerning $<\bar{\psi}\psi>$ and its equation of state must predict $R_\pi$ or be discarded. This fact should help us find the correct theoretical ideas describing the phase transition of lattice QED, and should be applicable to other models as well.

Following the same philosophy, we shall study $M_\pi$ as a function of $<\bar{\psi}\psi>$ itself. We shall see that...
combining correlation length scaling with the Goldstone description of the pion, yields particularly effective methods to find the critical coupling and the anomalous dimension $\eta$. Since $M_\pi$ and $\langle \bar{\psi} \psi \rangle$ are the easiest observables to measure in simulations and typically the least technical quantities to calculate in models, this approach to $\eta$ should prove quite useful. In addition, the anomalous dimension $\eta$ is central to discussions of nontriviality and thus lies at the heart of the issue of whether non-asymptotically free theories exist.

The rest of this paper is organized as follows. In Sec. 2 we introduce our general theoretical framework and discuss renormalization group trajectories, the chiral equation of state and the physics behind hyperscaling. In Sec. 3 we concentrate on $R_\pi$ and show that it is determined by the chiral equation of state and can be used, because of the Goldstone nature of the pion, to locate the critical coupling and measure the critical index $\delta$ in an (almost) model independent fashion. Sec. 4 considers the chiral equation of state and the constraints that PCAC places on it. In Sec. 5 the relation of the pion mass, and the chiral condensate is developed resulting in a novel way to find the critical point and the anomalous dimension $\eta$. Sec. 6 considers the fermion mass and the index $\delta$ and discusses the scaling region where the analysis of this paper applies. It also discusses non-asymptotically free dynamics and the breaking of scale invariance at the critical point. In Sec. 7 we illustrate some of our analyses in the context of two models, the four-fermi and the linear sigma model. Sec. 8 contains some brief concluding remarks.

2. RENORMALIZATION GROUP TRAJECTORIES AND HYPERSONCALING

The general idea of renormalizability is that a theory’s cutoff dependence can be absorbed into a finite set of bare parameters in such a way that its low energy physics is insensitive to the cutoff. Once this is done, it is possible to find the lines of constant physics (RG trajectories) in the bare parameter space. These lines are uniquely defined no matter what observable is taken. In this way low-energy quantities depend on each other and not on the cutoff. We will see that this is possible if hyperscaling is obeyed.

In our discussion we shall assume that the reader is familiar with traditional statistical mechanics topics of homogeneity, the Equation of State (EOS) of ferromagnets, correlation length scaling etc. We shall pass between the languages of field theory and statistical mechanics freely in much the same spirit as the textbook by C. Itzykson and J.-M. Drouffe [8].

The hyperscaling hypothesis claims that the only relevant scale in the critical region is the macroscopic correlation length $\xi$. If this hypothesis holds, it is possible to do dimensional analysis using this correlation length as a scale. As a consequence, all dimensionless observables, e.g. mass ratios and renormalized couplings, should be independent of $\xi$. The hyperscaling hypothesis can be stated as

$$F_{\text{sing}} = t^{2-\alpha} F(h/t^\Delta) \sim \xi^{-d}$$

where notation is standard ($F$ is the free energy, $h$ is the external magnetic field, $t$ is the deviation of the dimensionless temperature (coupling) from the critical point, etc.). All thermodynamic quantities are obtained by taking the derivatives of the free energy. In particular, the order parameter, defined as $\langle \phi \rangle = \partial F_{\text{sing}}/\partial h$, satisfies the Equation of State (EOS)
\[ <\phi> = t^{2-\alpha-\Delta} F'(h/t^\Delta) = t^\beta F'(h/t^\Delta) \tag{2.2} \]

This relation also defines the magnetic exponent: \( \beta = 2 - \alpha - \Delta \). Similarly, the susceptibility exponent, \( \gamma \), is obtained from

\[ \chi = \frac{\partial <\phi>}{\partial h} = t^{\beta-\Delta} F''(h/t^\Delta) \equiv t^{-\gamma} F''(h/t^\Delta) \tag{2.3} \]

i.e. \( \gamma = \Delta - \beta \).

For a given channel, the corresponding correlation length (= inverse mass) is defined by

\[ 2d\xi^2 = \int_\mathbb{R} x^2 <\phi(x)\phi(0)> \int_\mathbb{R} <\phi(x)\phi(0)> \tag{2.4} \]

If the field \( \phi \) develops a vacuum expectation value, then connected correlation functions should be taken in eq.(2.4). This will be understood throughout the paper. In the scaling region its behavior is given by

\[ \xi = t^{-\nu} g(h/t^\Delta) \tag{2.5} \]

Combined with eq.(2.1), it leads to the hyperscaling relation between the critical exponents: \( d\nu = 2 - \alpha \). The other relations between the exponents follow from the scaling form of the two-point function: \( <\phi(x)\phi(0)> = 1/|x|^{2d_{\phi}} f(x/\xi) \) [9].

Since hyperscaling is an important concept, we should explain its meaning and outline possible implications. It is generally believed that violations of hyperscaling lead to triviality. This is due to various inequalities between certain combinations of critical indices [10]. In simple models like scalar field theories and spin systems, the quantity that measures the violation of hyperscaling is the dimensionless renormalized coupling [11]. It is defined through the nonlinear susceptibility \( \chi^{(nl)} \)

\[ g_R = -\frac{\chi^{(nl)}}{\chi^{(nl)}} \quad \chi^{(nl)} = \frac{\partial^3 <\phi>}{\partial h^3} \tag{2.6} \]

The renormalized coupling is essentially the properly normalized connected four-point function. In terms of correlation functions, the nonlinear susceptibility is the zero-momentum projection of the four-point function

\[ \chi^{(nl)} = \int_{xyz} <\phi(x)\phi(y)\phi(z)\phi(0)>_{conn} \tag{2.7} \]

The \( \xi^d \) term in the denominator of eq.(2.6) is to account for an extra integration. By differentiating the free energy and using the EOS for the correlation length (eq.(2.5)), we arrive at

\[ g_R = t^{-2\Delta+\gamma+d\nu} H(h/t^\Delta) \tag{2.8} \]

We can trade \( t \) for the correlation length in eq.(2.4) and rewrite eq.(2.8) in terms of \( \xi \) as

\[ g_R = \xi^{(2\Delta-\gamma-d\nu)/\nu} H(h/t^\Delta) \tag{2.9} \]
In specific (ferromagnetic) models an estimate of the exponent can be made: since multi-spin correlations can not extend over a larger range than pair correlations, the following inequality holds [10]

$$2\Delta \leq \gamma + d\nu$$

(2.10)

Because $g_R$ is dimensionless, hyperscaling implies that it is independent of $\xi$ and the exponent in eq.(2.9) must vanish. In this case the renormalized coupling is a function of only one bare variable. Clearly, strict inequality in eq.(2.10) implies triviality. We mention that an equivalent way of stating the above inequality is $d\nu \geq 2 - \alpha$ [12]. It means that the singular part of the free energy vanishes at most as fast as required by hyperscaling i.e. $F_{\text{sing}} \geq \xi^{-d}$.

In a similar fashion the scaling of the mass ratios can be derived [13]. If hyperscaling is satisfied, the ratio of any particular pair of masses satisfies

$$R(t,h) = G(h/t^\Delta)$$

(2.11)

where $G(y)$ is a universal function. Comparing it with the expression for the renormalized coupling, we see that both observables depend on the same variable. One of the relations can be inverted to solve for the bare variable e.g. $h/t^\Delta = H^{-1}(g_R)$. This defines an RG trajectory for each value of $g_R$ and can be used to obtain the relation between the two observables $R = R(g_R)$. Note that this relation is independent of the bare parameters. The same manipulation can be done with two mass ratios. We note that the important point in the inversion is that both observables depend on just one bare variable, so that the inverse relation can always be found, at least in some regions of parameter space.

3. $\sigma - \pi$ SPLITTING, CRITICAL EXPOENTS AND EQUATION OF STATE

In this section we will relate the $\sigma - \pi$ splitting with other universal quantities like critical indices and universal amplitude ratios. Since the scalar and pseudoscalar propagators represent fluctuations of the order parameter, information about the $\sigma - \pi$ spectrum is contained in the equation of state. Before we consider the derivation, we briefly illustrate the connection between the behavior of $<\bar{\psi}\psi>$ and the $\sigma - \pi$ splitting. We use the spectral representation to express the order parameter

$$<\bar{\psi}\psi> = \int^{+\infty}_{-\infty} d\lambda \rho(\lambda) \frac{m}{\lambda^2 + m^2}$$

(3.1)

where $\rho(\lambda) \geq 0$ is the eigenvalue density of the Dirac operator. We define the two susceptibilities

$$\chi_\sigma = \int_x <\bar{\psi}(x)\bar{\psi}(0)> , \quad \chi_\pi = \int_x <\bar{\psi}\gamma_5\psi(x)\bar{\psi}\gamma_5\psi(0)>$$

(3.2)

which are the zero-momentum projections of the scalar and pseudoscalar propagators. Again, we emphasize that the scalar correlation function is understood as the connected one. These two susceptibilities are related to the order parameter via: $\chi_\sigma = \partial <\bar{\psi}\psi> /\partial m$ and the Ward identity $\chi_\pi = <\bar{\psi}\psi>/m$. (We will derive this relationship later in this section.) With this in mind, it is straightforward to show that these susceptibilities can be rewritten in terms of the spectral function as
\[ \chi_\sigma = \chi_\pi - \int_{-\infty}^{+\infty} d\lambda \rho(\lambda) \frac{2m^2}{(\lambda^2 + m^2)^2} \]  

(3.3)

The integral on the right hand side is positive leading to the following inequality \( \chi_\sigma \leq \chi_\pi \). Since the mass, defined as in eq.(2.4), is related to the susceptibility via \( M^2 = Z \chi^{-1} \), and since scalars and pseudoscalars renormalize in the same way, we obtain the following inequality (level ordering): \( M_\sigma^2 \leq M_\pi^2 \).

From eqs.(3.1) and (3.3) it is apparent that both, the order parameter and the \( \sigma-\pi \) splitting disappear in the chiral limit if the zero-mode is absent from the spectrum i.e. when \( \rho(0) = 0 \). Conversely, the presence of the zero-mode induces both a nonzero order parameter, \( \langle \bar{\psi}\psi \rangle = \pi \rho(0) \neq 0 \), and \( \sigma-\pi \) splitting. The same mechanism that leads to a nonvanishing value of the order parameter is responsible for lifting the degeneracy within the parity doublets.

The starting point in our analysis is the scaling form of the equation of state [8,9]. In the case of continuous symmetry the familiar form \( h = M^\delta f(t/M^{1/\beta}) \) becomes

\[ h_a = M_a M^{\delta-1} f(t/M^{1/\beta}) \]  

(3.4)

where, for simplicity, we denote the vector order parameter and external field by \( M_a \) and \( h_a \) respectively, and the modulus of the order parameter as \( M = \sqrt{M_a M_a} \). Our convention is that \( t > 0 \) corresponds to the broken phase. Eq.(3.4) is the gap equation for the order parameter. In the limit when \( h \to 0 \) we have

\[ 0 = M^\delta f(x) \]  

(3.5)

which has a nontrivial solution \( M \neq 0 \) only if the function \( f(x) \) has a positive zero \( f(x_0) = 0 \). The spontaneous magnetization is given by \( M = (t/x_0)^\beta \). This determines only the modulus of the order parameter (vacuum degeneracy), whereas the orientation and, thus, the particular Hilbert space is fixed by the direction of the external field. The function \( \tilde{f}(x/x_0) = f(x)/f(0) \) is the same for any theory within a given universality class. At the critical point \( x = 0 \) the response of the system is singular and is given by \( h = M^\delta f(0) > 0 \). In order to have no spontaneous magnetization in the symmetric phase \( f(x) \) should be positive on the negative x-axis.

The response to an external field \( h_a \) is given by the inverse susceptibility \( (\chi^{-1})_{ab} = \partial h_a/\partial M_b \) which can be obtained from the equation of state

\[ (\chi^{-1})_{ab} = \delta_{ab} M^{\delta-1} f(x) + (\delta - 1) \frac{M_a M_b}{M^2} M^{\delta-1} f(x) - \frac{x}{\beta} \frac{M_a M_b}{M^2} M^{\delta-1} f'(x) \]  

(3.6)

where \( x = t/M^{1/\beta} \). Eq.(3.6) can be rearranged into

\[ (\chi^{-1})_{ab} = \left( \delta_{ab} - \frac{M_a M_b}{M^2} \right) M^{\delta-1} f(x) + \frac{M_a M_b}{M^2} M^{\delta-1} \left( \delta f(x) - \frac{x}{\beta} f'(x) \right) \]  

(3.7)

in order to separate the susceptibility into transverse and longitudinal parts

\[ \chi_T^{-1} = M^{\delta-1} f(x), \quad \chi_L^{-1} = M^{\delta-1} \left( \delta f(x) - \frac{x}{\beta} f'(x) \right) \]  

(3.8)

The expression for the transverse susceptibility is the Ward identity which can be rewritten, after using the EOS, as
\[ h = M \chi_T^{-1} \] (3.9)

In the broken phase where \( M \neq 0 \), the divergence of the transverse susceptibility in the \( h \to 0 \) limit signals the appearance of massless modes (Goldstone bosons). At the critical point (\( x = 0 \)), eqs.(3.8) imply that

\[ \frac{\chi_L^{-1}}{\chi_T} = \delta \] (3.10)

This defines the exponent \( \delta \) as a measure of the relative strength of longitudinal and transverse responses of the system at the critical point. Eq.(3.10) can also be derived by noting that, at the critical point, \( M \sim h^{1/\delta} \), and using \( \chi_L = \partial M / \partial h \) and \( \chi_T = M / h \).

When the symmetry in question is chiral symmetry, eqs.(3.8) can be combined into a statement about the meson masses. The external field is replaced by the bare fermion mass \( m \) and the order parameter, \( M_a \), has components (\( <\bar{\psi}\psi> \), \( <\bar{\psi}i\gamma_5T^a\psi> \)) for the flavor group whose generators are \( T^a \). (The correct mapping to ensure dimensional consistency is \( <\bar{\psi}\psi> \to \Lambda^2 M \) and \( h \to m\Lambda^2 \). In the following we will work with dimensionless quantities, so we are allowed to take \( \Lambda = 1 \) from here on).

We introduce the bare mass in the standard way through \( m_{\bar{\psi}\psi} \), so that the ground state is parity invariant. Then, condensation occurs in the scalar channel and the only nonvanishing component of the order parameter is \( <\bar{\psi}\psi> \). The EOS, eq. (3.4), then reads

\[ m = <\bar{\psi}\psi>^\delta f(t/ <\bar{\psi}\psi>^{1/\beta}) \] (3.11)

The sigma and the pion are longitudinal and transverse modes, respectively. The Ward identity, eq.(3.9), now becomes

\[ <\bar{\psi}\psi> = m \int_x <\bar{\psi}i\gamma_5\psi(x)\bar{\psi}i\gamma_5\psi(0)> \] (3.12)

The masses are related to the corresponding susceptibilities via \( M_\pi^2 = Z_\pi \chi_\pi^{-1} \), \( M_\sigma^2 = Z_\sigma \chi_\sigma^{-1} \). Because of chiral symmetry, the wavefunction renormalization constants for the two modes are equal and the mass ratio (squared) is the susceptibility ratio i.e.

\[ R_\pi = M_\pi^2/M_\sigma^2 = \chi_T^{-1}/\chi_L^{-1} \] (3.13)

Thus, eqs.(3.8) implies

\[ \frac{1}{R_\pi(t, m)} = \delta - \frac{x f'(x)}{\beta f(x)}, \quad R_\pi(0, m) = \frac{1}{\delta} \] (3.14)

Eq.(3.14) is a stronger statement than the Goldstone theorem eqs.(3.9) or (3.12). Not only does it contain the above Ward identity, but it determines the splitting within the parity multiplet away from the chiral limit in both phases. Before we discuss the above result, we recall the EOS for the masses,

\[ M_\sigma(t, m) = t^\nu g_\sigma \left( \frac{m}{t^\Delta} \right), \quad \sigma = \pi, \quad M_\pi(t, m) = t^\nu g_\sigma \left( \frac{m}{t^\Delta} \right) \] (3.15)

While the equation for \( M_\sigma \) follows from hyperscaling, the same reasoning does not lead to the analogous expression for \( M_\pi \). Its form is fixed, however, by eq.(3.13) which states that the ratio \( R_\pi \) is a function of \( x \) and,
thus, of $m/t^\Delta$, implying eq.(3.15). The functions $g(y)$ are universal up to a multiplicative constant. Clearly, the Goldstone nature of the pion imposes a different boundary condition on $g_\pi(y)$ requiring $g_\pi(y) \sim y^{1/2}$ near the origin. The argument $m/t^\Delta$ is understood in the sense $t^\Delta \sim sgn(t)|t|^{\Delta}$ so that it changes sign as one goes through the critical point. This insures single-valuedness of $g(y)$ and the possibility of global inversion. Taking the mass ratio eliminates non-universal multiplicative factors as well as the exponent $\nu$ making it a universal function of $y = m/t^\Delta$ only

$$R_\pi(t, m) = \frac{M_2}{M_1} = G\left(\frac{m}{t^\Delta}\right)$$

(3.16)

In this equation, we have the low energy observable on one side and the bare parameters on the other. From eq.(3.14) it follows that

$$R_\pi(t = 0, m) = G(\infty) = \frac{1}{\delta}$$

(3.17)

There are two limiting values of $R_\pi$ that are fixed by chiral symmetry. In the broken phase ($t > 0$) pions are Goldstone bosons, so $R_\pi(t > 0, m = 0) = 0$. In the symmetric phase the sigma and pion degeneracy implies $R_\pi(t < 0, m = 0) = 1$. The ratio can never exceed unity since Euclidean propagators satisfy $|D_\pi(x)| \geq |D_\sigma(x)|$ which, in the large $|x|$ limit gives $R_\pi \leq 1$. As the bare mass increases, the ratio approaches the same value regardless of which phase it originates from. This is so simply because, as one increases the amount of explicit symmetry breaking, the dynamics becomes insensitive to the type of symmetry realization in the vacuum. Thus, if we fix $t$ and plot the curve $R_\pi(t, m)$ versus $m$ (Fig.1), in the broken phase $R_\pi = R_{>}$ will increase from zero while in the symmetric phase $R_\pi = R_{<}$ will decrease with $m$ starting from $R_{<} = 1$. Both families of curves approach $R_c = 1/\delta$ asymptotically from above (symmetric) and below (broken). Because of the scaling form of eq.(3.16), $R_\pi(t, m) = G(m/t^\Delta)$, it is clear that $R_c = G(\infty)$ is independent of $m$. So, we have traded small $t$ for large $m$. Since $0 \leq R_\pi \leq 1$, it follows that $R_{>}$ and $R_{<}$ curves have slopes with opposite signs. Thus, all the curves in the broken phase are monotonically increasing and lie below $R = R_c$ i.e. $R_{>}(m) < R_c$ and $R_{<}(m) > R_c$ for all $m$. Similarly, in the symmetric phase $R_{<}(m) > R_c$ and $R_{<}(m) < 0$. Therefore, the following inequality holds

$$R_{>}(m) < R_c < R_{<}(m)$$

(3.18)

which is saturated asymptotically for large values of $m$. Since $R_c = 1/\delta$, eq.(3.18) gives a bound on the value of $\delta$

$$\frac{1}{R_{<}(m)} < \delta < \frac{1}{R_{>}(m)}$$

(3.19)

Any curve $R_\pi(m)$ in the broken phase produces an upper bound on $\delta$ etc.. This bound improves as the ratio $m/t^\Delta$ becomes larger. The analysis of the mass ratio $R_\pi$ is consistent with the function $f(x)$ being positive semi-definite with only one zero at $x_0$ (Fig.2). A possible change in its monotonicty would imply the vanishing of its first derivative for at least one non-zero value of $x$. This would be in conflict with the physical behavior of $R_\pi$.

There are several reasons why this result is useful. Its application to data analysis produced by lattice simulations is obvious. Recent applications of this method to strongly coupled QED have been very successful
[14], giving more accurate estimates of the critical coupling and the exponent $\delta$ than those obtained with conventional methods on data samples of comparable statistics.

The reason for this can be easily understood. Instead of dealing with extrapolated data, as normally done when studying chiral symmetry breaking through $<\bar{\psi}\psi>$, here we determine both, the critical coupling and critical exponent $\delta$ from the raw data. Furthermore, the scaling form for the mass ratio is more accurate for larger values of $m/t$. This means that, instead of simulating at the critical point and looking at the $m\to 0$ limit, we can work away from it and use large masses without losing accuracy. The key point, from the point of view of numerical simulations, is the $m$-independence of $R_\pi(t, m)$ at the critical point, eq (3.17). This can be exploited by plotting $R_\pi(t, m)$ as a function of $\beta$ for different $m$ values. The spectral method can be used to get very accurate plots, which cross at the critical point, thus determining $\beta_c$. This is reminiscent of the techniques of Finite Size Scaling when applied to the Binder cumulant, as noticed also by Boyd et al. in ref. [15] where a similar method has been proposed and successfully applied to the QCD chiral transition in the strongly coupled phase.

4. PCAC, HYPERSCALING AND TRIVIALITY

Eq.(3.14) is a differential equation that determines the function $f(x)$ in terms of other universal quantities like critical exponents and mass ratios. Its solution requires one boundary condition which can be fixed at an arbitrary value of $x$. The formal solution for $f(x)$ is given by the form

$$f(x) = f(0) \exp \left( \beta \int_0^x \frac{dy}{y} \left( \delta - 1 \right) R_\pi \right)$$

which shows that the ratio $f(x)/f(0)$ is universal. In the symmetric phase, $x < 0$, $R_\pi \to 1$ in the chiral limit ($x \to -\infty$). Therefore, in this limit eq.(3.14) gives

$$\gamma = \beta(\delta - 1) = \lim_{x \to -\infty} \frac{xf'(x)}{f(x)}$$

where we used the scaling relation between the critical exponents: $\gamma = \beta(\delta - 1)$. The above equation implies that, for $x \to -\infty$, $f(x) \approx C(-x)^\gamma$. In general, the expansion of $f(x)$ in the symmetric phase has the form $f(x) = \sum_n a_n(-x)^{-2n\beta}$ [16].

Another constraint on the behavior of $f(x)$ is PCAC. It refers to the chiral limit in the broken phase. Eq.(3.14) can be rewritten as

$$R_\pi = \frac{\beta f(x)}{\delta \beta f(x) - xf'(x)}$$

which implies the vanishing of the pion mass in the chiral limit ($x \to x_0$). Now, we show how PCAC forces $x_0$ to be a first order zero. Assume that around $x_0$, $f(x)$ vanishes as

$$f(x) \approx a(x_0 - x)^\rho$$

In the chiral limit (in the broken phase) eq.(4.3) becomes
\[ R_\pi = \frac{\beta(x_0 - x)}{(\delta\beta - \rho)(x_0 - x) + \rho x_0} \tag{4.5} \]

so \( M_{\pi}^2 \) vanishes linearly. For \( x \approx x_0 \) the equation of state is

\[ \frac{m}{\langle \bar{\psi}\psi \rangle^3} = f(x) \approx a(x_0 - x)^\rho \tag{4.6} \]

Clearly, PCAC, which requires \( M_{\pi}^2 \sim m \), forces \( \rho = 1 \). Substituting eq.(4.6) into the expression for \( R_\pi \) (eq.(4.3)) gives

\[ R_\pi = \frac{\beta m}{(\delta\beta - 1)m + ax_0 \langle \bar{\psi}\psi \rangle^\delta} \tag{4.7} \]

As \( m \to 0 \) the leading contribution is

\[ R_\pi \approx \frac{\beta m}{ax_0 \langle \bar{\psi}\psi \rangle^\delta} \tag{4.8} \]

When compared with the PCAC relation, \( M_{\pi}^2 = 2m \langle \bar{\psi}\psi \rangle / f_\pi^2 \), eq.(4.8) gives

\[ \frac{f_\pi^2}{M_\sigma^2} = \frac{2x_0 a \langle \bar{\psi}\psi \rangle^{\delta+1}}{\beta M_\sigma^4} \tag{4.9} \]

This relates the pion decay constant to the physical scale. We can use the definition of the critical exponents to study the behavior of \( f_\pi \) in the scaling region e.g. \( \beta(\delta + 1) = 2\Delta - \gamma \). However, using the EOS for \( \langle \bar{\psi}\psi \rangle \) (eq.(3.11)) and the EOS for \( M_\sigma \) (eq.(3.15)) the above relation can be written as

\[ \frac{f_\pi^2}{M_\sigma^2} = M_\sigma^{(2\Delta - \gamma - 4\nu)/\nu} K(m/t^\Delta) \tag{4.10} \]

with \( K(y) \) being a universal function. The exponent is the same one that appears in the expression for the renormalized coupling (eq.(2.9)). Clearly, since the above ratio is dimensionless, hyperscaling implies the vanishing of the exponent. Conversely, its violation would imply that the above ratio is cutoff dependent. In theories with Yukawa couplings, triviality is intimately related to the issue of compositeness of the scalars. For a nontrivial continuum limit to exist, it is necessary for fermions to exchange composite scalars. This is contained in the Goldberger-Treiman relation \( g = M_F / f_\pi \), which, after recognizing that the pion radius scales as \( r_\pi \sim 1/f_\pi \) implies that the coupling vanishes in the limit where pions are pointlike \( (g \sim M_F r_\pi) \). In that sense, compositeness means that the pion decay constant has to scale as the physical mass scale. Pointlike structure, on the other hand, implies that \( f_\pi \) diverges in physical units. Although Baker’s inequality is rigorously proven only for ferromagnetic systems [10], it is interesting to note that it is valid in the above equation in all known models where \( f_\pi / M_\sigma \) either diverges or approaches a constant depending on whether pions are pointlike or not. The inequality \( \Delta \leq \gamma + 4\nu \) is in agreement with it.

5. PION MASS, ORDER PARAMETER AND ANOMALOUS DIMENSIONS

Nonvanishing anomalous dimensions and compositeness are intimately related. In addition, the compositeness condition itself is tied to the existence of a fixed point of the underlying theory. It is reflected in the
vanishing of the wave function renormalization constant associated with the composite degrees of freedom. Let us explain this in some detail. Consider a general field theory where a bound state $|B>,$ with binding energy $E_B < 0,$ appears. From general considerations [17], the wave function renormalization constant $Z$ can be expressed as

$$Z = \sum_{b,E} | <b, E|B>|^2 \quad (5.1)$$

where $|b, E>$ is the bare (elementary particle) state with energy $E$. Using standard techniques from quantum mechanics, it can be shown that $Z$ satisfies the following equation [17]

$$1 - Z = \int_0^\infty dE \frac{G^2(E)}{(E + |E_B|)^2} \quad (5.2)$$

with $G^2(E) > 0$ being the total decay rate of the state $|B>$, $G^2(E) = 2\pi \sum_b | <b, E|B>|^2$. As such $G(E)$ is proportional to an effective coupling constant. The compositeness condition, $Z = 0,$ implies two things: 1) The composite state has no projection in the space of bare states (eq.(5.1)) i.e. $<b|B>=0$ for any $|b>$, and 2) It is a sum rule that places an upper bound on the effective coupling, $G^2(E)$, (eq.(5.2)) and can be interpreted as a fixed point condition.

As the fixed point is approached, the order parameter, wave function renormalization constant, $Z$, and all the masses vanish. If there is a single correlation length, the vanishing of $Z$ is determined by the anomalous dimension, $\eta$, via $Z \sim \xi^{-\eta}$. Also, the order parameter, $<\phi>$ scales as $<\phi>\sim \xi^{-d_\phi}$, where $d_\phi = (d - 2 + \eta)/2$ is the scaling dimension of the field $\phi$. Combined with the standard scaling laws in the symmetric limit, $<\phi>\sim t^\beta$ and $\xi \sim t^{-\nu}$, this leads to the following scaling relation between the exponents

$$\frac{\beta}{\nu} = \frac{1}{2}(d - 2 + \eta) \quad (5.3)$$

The appearance of the anomalous dimensions guarantees the compositeness of the degrees of freedom involved, a necessary condition to produce a nontrivial continuum limit. Consider the pair of equations used in previous sections

$$m = \langle \bar{\psi}\psi \rangle^\delta f(x), \quad \chi^{-1}_{\pi} = \langle \bar{\psi}\psi \rangle^{-1} f(x) \quad (5.4)$$

We shall see how eqs.(5.4) restricts the dependence of $M_\pi$ on the order parameter in the chiral limit and show that this behavior is universal and is determined by the magnitude of the anomalous dimension $\eta$. As such, it contains information about the continuum limit of the theory and is capable of distinguishing between mean field and non-mean field behavior. This being the case, perhaps, the best way to start is to recall this dependence in the $\sigma$-model in four dimensions. The masses and EOS are given by

$$M_\pi^2 = -t + \frac{\lambda}{6} v^2, \quad M_\sigma^2 = -t + \frac{\lambda}{2} v^2, \quad h = v(-t + \frac{\lambda}{6} v^2) \quad (5.5)$$

where $v = <\sigma>$ is the order parameter. The universal function in this case is a straight line $f(x) = -x + \lambda/6$. In the chiral limit the order parameter is obtained from $f(x_0) = 0$ giving $v_0^2 = 6t/\lambda$. Since this is a mean field model, there is no wavefunction renormalization and $M_\pi^2 = \chi_{\pi}^{-1}$. Eq.(5.5) thus, reproduces the Ward Identity and EOS of eq.(5.4). For a fixed value of $t$, $M_\pi^2$ is a linear function of $v^2$ with different intersections
depending on the phase. The expression for the pion mass can be rewritten as $M_\pi^2 = (\lambda/6)(v^2 - v_0^2)$. Thus, as discussed in the previous section, the pion mass vanishes linearly as the chiral limit is approached. In the symmetric phase the $v \to 0$ limit results in a nonvanishing pion mass. Again, the chiral limit is approached linearly. The reason for such a simple behavior is the absence of wavefunction renormalization (vanishing anomalous dimensions). It is easily demonstrated that, as long as the identification $M_\pi^2 = \chi^{-1}_\pi$ can be made, eqs.(5.4) insures the linear dependence between $M_\pi^2$ and $<\bar{\psi}\psi>^2$. To show this, we recall the expansion of the universal function $f(x)$ around the chiral limit in the symmetric phase: $f(x) \sim |x|^{\gamma}$, which leads, in the limit $<\bar{\psi}\psi> \to 0$, to $M_\pi^2 \sim |t|^\gamma$. The general dependence of the pion mass on the order parameter for a mean field theory is summarized in Fig.3.

Since, for nonvanishing anomalous dimension $\eta > 0$, the wavefunction renormalization constant scales as well, it will give some curvature to $M_\pi^2$ near the origin both in the symmetric phase and at the critical point (Fig.4). This curvature will be a signal of nontrivial behavior of the theory and is related to the compositeness of the pions. This curvature is 1) determined by the magnitude of the anomalous dimensions and 2) is opposite from that induced by finite size effects. In this way, the above plot contains information about both the thermodynamic limit and nontriviality of the theory.

In general, the pion mass is related to the susceptibility as $M_\pi^2 = Z\chi^{-1}_\pi$. The wavefunction renormalization constant scales as $Z \sim \xi^{-\eta}$ and, since the order parameter scales as $<\bar{\psi}\psi> \sim \xi^{\beta/\nu}$, we have $Z \sim <\bar{\psi}\psi>^{\nu\eta/\beta}$. At the critical point, $t = 0$, $\chi^{-1}_\pi \sim <\bar{\psi}\psi>^{\delta-1}$. Thus,

$$M_\pi^2 \sim <\bar{\psi}\psi>^{\delta-1+\eta\nu/\beta} \hspace{1cm} (5.6)$$

The exponent can be expressed in a slightly different form using the scaling relations between the critical exponents: $\beta(\delta - 1) = \gamma$ and $\gamma = \nu(2 - \eta)$. This leads to

$$\delta - 1 + \frac{\eta\nu}{\beta} = \frac{1}{\beta}(\gamma + \eta\nu) = 2\left(1 - \frac{d - 4 + \eta}{d - 2 + \eta}\right) \hspace{1cm} (5.7)$$

In four dimensions, the expression in parenthesis is $1/(1 + \eta/2) < 1$. Thus, at the critical point we have

$$M_\pi^2 \sim (<\bar{\psi}\psi>^2)^{\nu/\beta}, \hspace{1cm} (t = 0) \hspace{1cm} (5.8)$$

The important point is that the slope of the curve is infinite near the origin because of the anomalous dimensions: $\beta/\nu = 1 + \eta/2$ in four dimensions.

In the symmetric phase, we use the asymptotic expansion of the universal function $f(x)$ for $x \to -\infty$ [16]:

$$f(x) = \sum_{n=0}^{\infty} a_n |x|^{\gamma - 2n\beta} \approx a_0 |x|^{\gamma} + a_1 |x|^{\gamma - 2\beta} \hspace{1cm} (5.9)$$

The pion susceptibility and its mass are determined from this expression giving

$$M_\pi^2 \sim <\bar{\psi}\psi>^{\delta-1+\eta\nu/\beta} \left( a_0 \frac{|t|^\gamma}{<\bar{\psi}\psi>^{\gamma/\beta}} + a_1 \frac{|t|^{\gamma-2\beta}}{<\bar{\psi}\psi>^{(\gamma-2\beta)/\beta}} \right) \hspace{1cm} (5.10)$$

The first term gives the leading contribution near the origin. To obtain its magnitude, we use once again $\beta(\delta - 1) = \gamma$. In that case the leading term gives
\[ M_\pi^2 \sim (\langle \bar{\psi} \psi \rangle^2)^{\eta \nu / 2 \beta}, \quad (t < 0) \]  

From the scaling relations it follows that the exponent in eq.(5.11) is less than 1:

\[
\frac{\eta \nu}{2 \beta} = 1 - \frac{d - 2}{d - 2 + \eta} < 1
\]  

Clearly, for vanishing \( \eta \), the leading term is constant and the second term in eq.(5.10) determines the curvature. The exponents in that case are given by mean field theory, so \( 1 / \beta = 2 \). Therefore, \( M_\pi^2 \) approaches some finite value linearly. The curvature, and the infinite slope come from the anomalous dimensions.

Finally, in the broken phase, \( t > 0 \), the order parameter is nonvanishing in the \( m \to 0 \) limit and the function \( f(x) \) vanishes for \( x = x_0 \). PCAC constrains this to be a first order zero i.e. \( f(x) \approx a(x_0 - x) \). Therefore, in the chiral limit we have

\[
M_\pi^2 = Z \chi^{-1}_\pi \sim \langle \bar{\psi} \psi \rangle^{\delta - 1 + \eta \nu / \beta} a(x_0 - x) \sim (\langle \bar{\psi} \psi \rangle - \langle \bar{\psi} \psi \rangle_0), \quad (t > 0)
\]  

and the pion mass vanishes linearly.

A remark about the apparent vanishing of the pion mass in the symmetric phase (in the chiral limit) should be made. At first glance this result is surprising, because symmetry considerations alone do not force its vanishing. In mean field theory where \( M_\pi^2 = \chi^{-1}_\pi \) everything is canonical and equation (5.10) tells us that \( M_\pi^2 \) is finite in the symmetric phase even for \( m = 0 \):

\[
\chi^{-1}_\pi \sim \langle \bar{\psi} \psi \rangle^{\delta - 1} (|t| / \langle \bar{\psi} \psi \rangle)^{1/\beta} \sim |t|^\gamma
\]  

The vanishing comes from the anomalous dimensions i.e. wavefunction renormalization! Let us analyze this point in more detail. The correlation length in Euclidean theory is

\[
2d\xi^2 = \frac{\int_x |x|^2 D(x)}{\int_x D(x)} \tag{5.15a}
\]

or, in momentum space

\[
2d\xi^2 = \left( \frac{1}{D^{-1}(k^2)} \frac{d D^{-1}(k^2)}{dk^2} \right)_{k^2=0} \tag{5.15b}
\]

This is a standard way of extracting the infra-red piece of the propagator that dominates at large separations. If we denote the self-energy corrections to the meson propagator as \( \Pi(k^2) \), the full propagator in momentum space reads

\[
D^{-1}(k^2) = k^2 + M_0^2 + \Pi(k^2) = k^2 + M_0^2 + \Pi(0) + k^2 \Pi'(0) + \Pi_2(k^2) \tag{5.16}
\]

This decomposition is made so that \( \Pi_2(k^2) = O(k^4) \) and the first two terms dominate the infra-red region. After identifying the \( k^2 \) term in eq.(5.15b) as the inverse wavefunction renormalization constant, \( Z^{-1} = 1 + \Pi'(0) \) and inverse susceptibility as \( \chi^{-1} = M_0^2 + \Pi(0) \), the meson propagator can be written as

\[
D^{-1}(k^2) = Z^{-1}(k^2 + Z \chi^{-1} + \Pi_2(k^2)) \approx Z^{-1}(k^2 + M^2 + O(k^4)) \tag{5.17}
\]
Thus, at low momenta, the propagator resembles that of a meson with mass $M^2 = Z\chi^{-1}$. This is the definition of the mass as given by eqs. (5.15). In this case analytic continuation to Minkowski space is standard and the pole structure is recovered. As is obvious from the decomposition in eq.(5.16), such manipulations assume a particular analytic structure of the composite propagators which is guaranteed only if the fermions are massive. Additional nonanalyticities in the form of branch cuts appear when fermions are massless as occurs in the chiral limit in the symmetric phase. The anomalous scaling of the scalar propagator is a consequence of these nonanalyticities – the $k^2$ term is absent. Rather, the leading low-momentum behavior is given by

$$D^{-1}(k^2) = k^{2-\eta} + C$$

The derivative diverges for $k^2 = 0$ giving zero mass as defined by eqs.(5.15), although there might be a pole in Minkowski space. Such a propagator does not have an exponential, but a power law decay. So, there are long-range correlations (“massless modes”), but the theory is not scale invariant. (Clearly, the case $\eta = 0$ is well behaved. In the broken phase there are no infra-red ”problems” of this sort because fermions are massive and eq.(5.15) is the appropriate definition of the correlation length.) Some attempts to confront this problem in a lattice study of three-dimensional Gross-Neveu model have been reported in [4].

When studying the theory in a finite volume, as is always done in lattice simulations, one has to insure that results are not obscured by finite size effects. Since different quantities have different sensitivity to finite size effects, the effective thermodynamic limit, therefore, must be monitored carefully through the consistency of certain relations. One set of such relations consists of the Ward Identity and the Equation of State. Being the lightest particle in the spectrum, the pion is most sensitive to finite size effects. On a finite lattice, the pion mass would tend to a higher value than in the thermodynamic limit. This would result in a change of sign in the curvature. In that sense, the plot $M^2_\pi$ vs $\langle \bar{\psi}\psi \rangle^2$ is also suited for controlling finite size effects in theories with nontrivial fixed points.

6. HEAVY QUARK LIMIT AND EXPONENT $\delta$

We have seen that $R_\pi = G(m/t^\Delta)$ with $G(\infty) = 1/\delta$. This means that, in the large-$m$ limit, $R_\pi \to 1/\delta$ (not necessarily 1). At first sight this looks puzzling since one expects that the nonrelativistic limit can be taken for very large constituent masses so that any meson mass should approach twice the constituent mass i.e. $M_{meson} \approx 2m$ which would necessarily imply $R_\pi \approx 1$. We should clarify what is meant by a large fermion mass: the bare mass is always much smaller than the cutoff and large $m$ means large compared to a typical mass scale, for example $M_\sigma$ in the chiral limit. In this way we are sure to stay in the scaling region where universality arguments hold and the RG trajectories are uniquely defined.

In theories that are not asymptotically free expansions in powers of $p/m$ are poorly behaved because the force between constituent quarks is strong at short distances forcing a large kinetic energy due to uncertainty relations. Thus, whatever the quark mass, it is never large compared to a typical kinetic energy. The composites in the scaling region are always relativistic. We note also that the bound states in the two phases are quite different from each other. In the symmetric phase, an increase in the bare mass is
compensated by a decrease in the coupling in order to keep the ratio unchanged. This is expected because the zero-point energy is reduced by increasing the mass and less attraction is needed to produce the same effect. In the broken phase, the opposite happens.

In theories like QCD this is not so because the force between the quarks is weak at short distances and the typical momenta could be small compared to the bare mass. Thus, in principle, one can still be in the scaling region and have a heavy quark limit. The most direct way of understanding the differences between asymptotically free and non-asymptotically free theories is to note that in the scaling region of the former \(|t| \ll 1\) means weak coupling \((g_c = 0, t = g^2)\), whereas for the latter \(|t| \ll 1\) means strong coupling.

We should add that \(\delta = 1\) has different physical origins in asymptotically free and non-asymptotically free theories. Since the exponent \(\delta\) gives the response of the system at the critical coupling: \(m = \langle \bar{\psi}\psi \rangle^{\delta} / t=0\), in asymptotically free theories the system responds as a free theory, so \(\delta = 1\). In that context, \(R_\pi \approx 1\) is a reflection of the fact that there is very weak binding which could not possibly distinguish between the scalars and pseudoscalars. In strongly coupled non-asymptotically free theories, tightly bound relativistic composites are formed in the scaling region. Their presence at high energies is the main difference relative to QCD-like theories where the only relevant degrees of freedom in the scaling region are quarks and gluons and where binding occurs in the infra-red regime. Because of the non-asymptotically free nature of the couplings, the ultra-violet asymptotics of the scalar correlation functions is not canonical – the theory has anomalous dimensions. In terms of the anomalous dimensions the exponent \(\delta\) in four dimensions is

\[
\delta = \frac{6 - \eta}{2 + \eta}
\]

Here, \(\delta = 1\) is a consequence of the large anomalous dimension, \(\eta = 2\). The meaning of this particular limit can be best understood if we write down the first two terms in the operator product expansion of the fermion propagator \[18\]

\[
S(p) \approx \frac{A}{p^4} + \frac{Bm}{p^{2+n/2}} + \frac{C < \bar{\psi}\psi >}{p^{4-n/2}} + \ldots
\]

As \(\eta \to 2\) i.e. \(\delta \to 1\), the system reacts to the bare mass the same way it would react to a change in the dynamical mass of its constituents. Thus, the persistence of \(R_\pi = 1\) away from the chiral limit in this case means that the system can not distinguish between bare and dynamical masses.

As a final remark, we use once again eq.(3.14) to argue the absence of the dilaton and scale invariance in theories with spontaneously broken chiral symmetry. Unlike previous treatments \[19\] our argument will require no knowledge of the composite propagators and is thus independent of the approximation scheme. Instead it follows from the scaling form of the EOS only. The idea of a dilaton as a Goldstone boson of spontaneously broken scale symmetry was introduced some time ago in ref.[20]. If the theory is scale invariant and if it breaks chiral symmetry spontaneously, it, at the same time, generates a scale, the fermion mass, and is no longer scale invariant. Since scale invariance is thus broken "spontaneously", one naively expects that there should be a corresponding Goldstone boson, a massless scalar, in the spectrum.

We use the scaling relation \(\beta = \gamma / (\delta - 1)\) to rewrite eq.(3.14) as

\[
R_\pi = \frac{1}{\delta - x(\delta - 1)/\gamma(f'(x)/f(x))}
\]
In general, it is clear that there has to be splitting between the $\sigma$ and $\pi$ i.e. there can be no massless scalar in the theory as long as chiral symmetry is realized in the Nambu Goldstone manner. An exception might only be the theory with $\delta = 1$. In that case $R_\pi = 1$ for any value of the bare parameters – not only are the parity partners degenerate, but they respond to symmetry breaking in the same way. This would be a very unusual realization of chiral symmetry: $\sigma$ and $\pi$ are indistinguishable for any finite $m$. In QCD, where $\delta = 1$, this situation is avoided by explicit violations of scaling due to quantum corrections. These scaling violations are the sole source of interaction and of the $\sigma - \pi$ splitting. In other cases e.g. strongly coupled QED with $\delta = 1$, the $\sigma - \pi$ degeneracy in the broken phase could not be reconciled with PCAC which requires $R_\pi \sim m$. Thus, if the current algebra relations are to be realized, scale invariance must be violated.

The entire content of PCAC is contained in these scaling violations. Consequently, the mass of the $\sigma$ comes solely from the scaling violation.

7. TWO EXAMPLES (SCALING PLOT)

To illustrate and complement the previous discussion, we analyse two simple examples: the four-fermi and linear $\sigma$ models. In addition to demonstrating realizations of the general ideas in these two models, we also discuss the mutual dependence of the mass ratios, or scaling plots, (as introduced in lattice spectroscopy in ref. [21]), and the heavy quark limit in these models. The scaling plot will prove especially suitable to argue the equivalence of the two models. (For notations and details of computations related to this Section see [13].)

First, we start with the four-fermi model. In the leading order in $1/N$ there is no distinction (on the technical level) between discrete and continuous chiral symmetry. We consider the continuous $U(1) \times U(1)$ model. For $2 < d < 4$ these models are renormalizable and nontrivial. To leading order, the critical exponents are $\beta = \nu = 1/(d - 2), \delta = d - 1, \eta = 4 - d$. The gap equation for the fermion self-energy $\Sigma$ is given by the tadpole contribution

$$\Sigma = m + 4g^2 < \bar{\psi}\psi > \quad (7.1)$$

or, explicitly, in terms of $\Sigma$,

$$cg^2\Sigma^{2-\epsilon} = t + \frac{m}{\Sigma} \quad (7.2)$$

where $\epsilon = 4 - d$ (not necessarily small, $0 < \epsilon < 2$) and $c = 4b/(2 - \epsilon)$, with $b = 2\Gamma(\epsilon/2)/(4\pi)^{d/2}$. We will express all the quantities in units of the momentum cutoff $\Lambda = 1$. The masses of the composites $M_\sigma, M_\pi$ are given by the one-loop diagrams [13]

$$M_\pi^2 = \frac{m}{\Sigma}Z, \quad M_\sigma^2 = M_\pi^2 + 4\Sigma^2 \quad (7.3)$$

with the wave function renormalization constant $Z = \Sigma' / bg^2$.

We first discuss the dependence of $M_\pi^2$ on the order parameter. In this case, the expectation value of the scalar field can be used to make contact with the $\sigma$-model i.e. $\Sigma = g < \sigma >$. Clearly, the relations in eq.(7.3) can be combined to give
\[ M^2_\pi = \frac{1}{b g^2} \Sigma_\eta (c g^2 \Sigma^{2-\eta} - t) \]  

(7.4)

which, in the limit \( \eta = 0 \) \((d = 4)\) reduces to the \( \sigma \)-model expression (eq.(5.5)). In particular, at the critical point, \( t = 0, \ M^2_\pi \sim \Sigma \). After recognizing that \( \beta = \nu \) in this model, we recover eq.(5.8). Similarly, in the symmetric phase, near the origin and for fixed \( t < 0 \), we obtain \( M^2_\pi \sim |t|(\Sigma^2)^{\nu/2} \) which is eq.(5.11). Thus, all the general features of this plot are clear from eq.(7.4): linear behavior in the broken phase, concavity in the symmetric phase and the vanishing of the pion mass in both phases. The last point is especially interesting considering the discussion of this problem in sect.5. One can illustrate this point further by calculating the pion propagator in the chiral limit in symmetric phase. Its form is given by [4]

\[ D^{-1}_\pi(k^2) = -t + C k^{2-\eta} \]  

(7.5)

which has a pole in the complex \( k^2 \)-plane (in Minkowski space), but gives a power law behavior of the Euclidean correlator.

As we remarked before, the general idea behind the universal behavior of a particular dimensionless observable is that, given its functional form \( R = G(m/\Lambda^2) \), one can invert this relation to find an RG trajectory and express another observable as \( R' = R'(R) \). We define two mass ratios: \( R_\pi = M^2_\pi/M^2_\sigma \) and \( R_F = 4 M^2_F/M^2_\sigma \) in terms of which the above equation becomes

\[ R_\pi = 1 - R_F \]  

(7.6)

In the chiral limit \( R_\pi = 0 \) and \( R_F = 1 \) i.e. \( M_\sigma = 2 M_F \) in the broken phase, whereas in the symmetric phase \( R_\pi \to 1 \) and \( R_F \to 0 \) as \( m \to 0 \). The curve \( R_\pi \) versus \( R_F \) is universal. The prediction of the four fermi theory (eq.7.6) is a straight line with unit slope. All the points on the curve that lie below \( R_\pi = 1/\delta \) belong to the broken phase and those above to the symmetric phase. All the physical points lie on the straight line eq.(7.6). Clearly, the naive heavy quark limit would require \( R_\pi = R_F = 1 \) which is completely missing from the plot. The points below the line \( R_\pi = 1/\delta \) are in the broken phase and at the critical point \( R_F = 1 - 1/\delta \). These two values correspond to the \( m \to \infty \) limit. Explicit calculation [13] gives for \( R_\pi = G(m/\Lambda^2) \)

\[ m/\Lambda^2 = \frac{4b}{g^{2\beta}} \left( \frac{R_\pi^{1/\beta} (1 - R_\pi)}{(c(1 - R_\pi) - 4b R_\pi)^\beta} \right) \]  

(7.7)

with \( b, c \) defined as before and satisfy \( 4b/c = 2 - \epsilon = 1/\beta \). Clearly, for large \( m \), the denominator on the right hand side vanishes giving

\[ R_\pi \to \frac{c}{c + 4b} = \frac{1}{\delta}. \]  

(7.8)

In the linear \( \sigma \)-model eq.(7.6) is slightly modified. The masses are given by

\[ M^2_\pi = -t + \frac{\lambda}{6} v^2, \quad M^2_\sigma = -t + \frac{\lambda}{2} v^2, \quad M^2_F = g^2 v^2 \]  

(7.9)

where \( t \) is the curvature of the scalar potential \((t > 0 \) corresponding to the broken phase\) and \( v = <\sigma > \). The equation of state is \( h = v M^2_\pi \). One can obtain the relation between the mass ratios from eq.(7.9). The analogue of eq.(7.6) in this case is
\[ R_\pi = 1 - \frac{\lambda}{12g^2} R_F \]  
\( (7.10) \)

Here, the universal curve is a straight line again, but with different slope determined by the ratio of the coupling constants. For a fixed value of \( \lambda/g^2 \) a different value of \( R_F \) emerges in the chiral limit. In that sense four-fermi is a special case of the \( \sigma \) model (as is well known) – for \( g^2 = 12\lambda \) they describe the same low energy physics. Since both mass ratios are low energy quantities, the slope should be in fact a ratio of renormalized couplings (of course, this is not visible in the MF treatment). In other words, the curve \( R_\pi = R_\pi(R_F) \) should have no knowledge of the bare parameters. In the four-fermi theories the renormalized couplings are independent of the bare four-fermi coupling \( G \) and the ratio is simply one. The reason behind \( M_\sigma = 2M_F \) is the relative magnitude of \( \lambda \) and \( g \). From eq.(7.9) we see that the magnitude of \( \lambda \) determines the size of the mass scale \( v \) (and meson masses), whereas \( g \) gives the magnitude of the fermion mass in units of \( v \). In four-fermi theory, unlike the \( \sigma \)-model, mesons are composite rather then elementary. The compositeness condition forces the ratio of the couplings to be such that the \( \sigma \) is a true bound state i.e. \( M_\sigma \leq 2M_F \). This requirement places a bound on the ratio \( g^2/\lambda \) and is intimately related to the compositeness of the \( \sigma \) and \( \pi \). If one looks closely at the two couplings \( \lambda \) and \( g \), two things become apparent. They appear in the lagrangian as \( g\sigma \bar{\psi}\psi \) and \( \lambda \sigma^4 \). If the \( \sigma \) is composite, then it renormalizes the same way as \( \bar{\psi}\psi \). Therefore, compositeness requires that \( g^2 \) and \( \lambda \) renormalize the same way. So, if \( \lambda_R = Z\lambda \), then \( g_R^2 = Zg^2 \) and \( \lambda_R/g_R^2 = \lambda/g^2 \). Thus, radiative corrections cancel and the slope of the plot is independent of them. This, in fact, is what must happen if the scalars are composites because the wavefunction renormalization vanishes. If one accepts the democratic principle that, in an interacting theory, everybody has to interact with everybody else (unless there are selection rules that forbid it), then in order for both couplings to be non-zero, they have to renormalize the same way. In the \( \sigma \)-model scalars are pointlike and the ratio \( g^2/\lambda \) can assume any value.

In the \( \sigma \)-model, the ratios are not constrained a priori since all the degrees of freedom are pointlike. Nevertheless, the large-\( m \) behavior is consistent with that of the four-fermi model indicating that compositeness is not the crucial ingredient here. Rather, it is chiral symmetry and the scaling of the mass ratios. In terms of the bare parameters, \( t, h \), the mass ratio reads

\[ \frac{h}{t^{3/2}} = \sqrt{\frac{24 R_\pi^2 (1 - R_\pi)}{\lambda (1 - 3R_\pi)^3}} \]  
\( (7.11) \)

which, apart from the different critical exponents, is the same as eq.(7.7) obtained for the four-fermi theory. From here it follows that, in the \( h \to \infty \) limit, \( R_\pi \to 1/3 = 1/\delta \) as it should.

It has been argued many times in the literature that either the \( \sigma \) model or the four-fermi model can be used to study the low energy regime of QCD. This is correct provided the bare masses are sufficiently small. These models are based on chiral symmetry and should be representative of QCD as long as the bound states that are studied are collective. Neither model is capable of describing atomic quarkonia like charmonium etc. In the heavy quark limit the predictions of these models should be qualitatively different from QCD. Whether or not they are suitable for the strange quark sector needs further investigation. It is interesting to note that an upper bound on the pseudo-Goldstone mass follows from the scaling plot eq.(7.10). In the broken phase, it is easy to see, that the following inequality holds
\[ M_{\pi} \leq \frac{2}{\sqrt{\delta - 1}} M_F \]  \hspace{1cm} (7.12)

In four dimension \((\delta = 3)\) gives \( M_{\pi} \leq \sqrt{2} M_F \). This bound is universal and in four dimensions holds to all orders in \( 1/N \) since the exponent \( \delta \) does not receive any corrections. The fact that an upper bound, \( m \to \infty \), on the \( \pi - \sigma \) mass ratio is given by \( 1/\delta \leq 1 \), shows that an explanation of this splitting as an effect of the standard spin-orbit interaction (à la atomic models) is problematic simply because the splitting survives even in the infinite mass limit.

8. CONCLUSIONS

In this paper we have illustrated how correlation length scaling and chiral symmetry together constrain and simplify the universal features of chiral symmetry breaking phase transitions. Perhaps the most useful results were 1. the explicit formula for \( R_\pi \) in terms of the universal function \( f \), 2. the relation between the pion mass, the chiral condensate, and the critical index \( \eta \), and 3. the simple properties of the universal function \( f \) itself. Within the context of the analysis of simulation data, each observation 1.-3. should lead to independent determinations of critical couplings and indices. Consistent results from all the methods should comprise convincing evidence that one has found universal features in the model of interest.

We have chosen to illustrate these ideas in several simple models such as mean field theory and an ultra-violet fixed point with power-law singularities. Other examples can also be considered and the general approach of this paper can be applied. In ref.[22], for example, we considered both a fixed point and a logarithmically trivial sigma model to describe the data of four flavor lattice QED. Although both models were able to fit the equation of state, only the fixed point model gave a consistent description of the spectroscopy data for \( R_\pi \). This exercise illustrates nicely how the use of PCAC and scaling in unison can help lead us to the correct theoretical scenario even when a comprehensive microscopic theory of a critical point is missing.

We look forward to developing the ideas of this paper further and applying them to other presently mysterious chiral phase transitions such as QCD at nonzero chemical potential and four fermi models with continuous chiral symmetries and few flavors in less than four dimensions.

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FIGURE CAPTIONS

1. $R_\pi(t, m)$ as a function of $m$ at fixed $t$ in the symmetric and broken phase. The horizontal line corresponds to the critical point $t = 0$.

2. Typical behaviour of the universal ratio $f(x)/f(0)$ as a function of $(x/x_0)$, $x$ being the scaled variable $t/\langle \bar{\psi}\psi \rangle^{1/\beta}$.

3. The linear dependence of $M_\pi^2$ on $\langle \bar{\psi}\psi \rangle^2$ for a mean field theory.

4. Expected behaviour of $M_\pi^2$ versus $\langle \bar{\psi}\psi \rangle^2$ in a theory with anomalous dimension.