POISSON SPACING STATISTICS FOR VALUE SETS OF POLYNOMIALS

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Abstract. If $f$ is a polynomial with integer coefficients and $q$ is an integer, we may regard $f$ as a map from $\mathbb{Z}/q\mathbb{Z}$ to $\mathbb{Z}/q\mathbb{Z}$. We show that the distribution of the (normalized) spacings between consecutive elements in the image of these maps becomes Poissonian as $q$ tends to infinity along any sequence of square free integers such that the mean spacing modulo $q$ tends to infinity.

1. Introduction

Let $f$ be a polynomial with integer coefficients. Given an integer $q$, we may regard $f$ as a map from $\mathbb{Z}/q\mathbb{Z}$ to $\mathbb{Z}/q\mathbb{Z}$, and the image of this map will be denoted the image of $f$ modulo $q$. The purpose of this paper is to investigate the distribution of spacings between consecutive elements in the image of $f$ modulo $q$ as $q$ tends to infinity along square free integers. The main emphasis will be placed on the highly composite case, i.e., by letting $q$ tend to infinity in such a way that the number of prime factors of $q$ also tends to infinity.

The case $f(x) = x^2$ and $q$ prime was investigated by Davenport. In [6, 7] he proved that the probability of two consecutive squares being spaced $h$ units apart tends to $2^{-h}$ as $q \to \infty$. We may interpret this as if spacings between squares modulo prime $q$ behave like gaps between heads in a sequence of fair coin flips.

The case $f(x) = x^2$ and $q$ highly composite was studied by Rudnick and the author in [14, 13]. If we let $\omega(q)$ be the number of distinct prime factors of $q$, then the number of squares modulo $q$ equals $\prod_{p|q} \frac{p+1}{2}$, and the average spacing between the squares is given by

$$s_q = \frac{q}{\prod_{p|q} \frac{p+1}{2}} = 2^{\omega(q)} \prod_{p|q} \frac{p}{p+1}.$$
Hence \( s_q \to \infty \) as \( \omega(q) \to \infty \), so we would expect that the probability of two squares being 1 unit apart vanishes as \( \omega(q) \to \infty \), and it is thus natural to normalize so that the mean spacing is one. A natural statistical model for the spacings is then given by looking at random points in \( \mathbb{R}/\mathbb{Z} \); for independent uniformly distributed numbers in \( \mathbb{R}/\mathbb{Z} \), the normalized spacings are said to be Poissonian. In particular, the distribution \( P(s) \) of spacings between consecutive points is that of a Poisson arrival process, i.e., \( P(s) = e^{-s} \), and the joint distribution of \( l \) consecutive spacings is a product \( l \) independent exponential random variables (see [8]). Using Davenport’s result together with the heuristic that “primes are independent”, it is seems reasonable to expect that the distribution of the normalized spacings between squares modulo \( q \) becomes Poissonian in the limit \( s_q \to \infty \), and the main result of [14] is that this is indeed the case for squarefree \( q \) (the general case is treated in [13].)

What can be said about more general polynomials \( f \in \mathbb{Z}[x] \)? For \( p \) prime, let

\[
\Omega_p := \{ t \in \mathbb{Z}/p\mathbb{Z} : t = f(x) \text{ for some } y \in \mathbb{Z}/p\mathbb{Z} \}
\]

be the image of \( f \) modulo \( p \). Given \( k \geq 2 \) and integers \( h_1, h_2, \ldots, h_{k-1} \), let

\[
N_k((h_1, h_2, \ldots, h_{k-1}), p) := |\{ t \in \Omega_p : t + h_1, \ldots, t + h_{k-1} \in \Omega_p \}|
\]

be the counting function for the number of \( k \)-tuples of elements in the image of the form \( t, t + h_1, \ldots, t + h_{k-1} \). Letting \( s_p := p/|\Omega_p| \) denote the average gap modulo \( p \), the “probability” of an element being in the image is \( 1/s_p \). Thus, if the conditions \( t \in \Omega_p, t + h_1 \in \Omega_p, \ldots, t + h_{k-1} \in \Omega_p \) are independent, we would expect \( N_k((h_1, h_2, \ldots, h_{k-1}), p) \) to be of size \( p/s_p^k \), and a natural analogue of Davenport’s result is then that

\[
N_k((h_1, h_2, \ldots, h_{k-1}), p) = p/s_p^k + o(p)
\]

as \( p \to \infty \) provided that \( 0, h_1, \ldots, h_{k-1} \) are distinct modulo \( p \). In [10] Granville and the author proved that

\[
N_k((h_1, h_2, \ldots, h_{k-1}), p) = p/s_p^k + O_{f,k}(\sqrt{p})
\]

holds if \( f \) is a Morse polynomial and \( 0, h_1, \ldots, h_{k-1} \) are distinct modulo \( p \). Using this, Poisson spacings for the image of Morse polynomials in the highly composite case follows from the following criteria (see [10], Theorem 1): Assume that there exists \( \epsilon > 0 \) such that for each integer \( k \geq 2 \),

\[
N_k((h_1, h_2, \ldots, h_{k-1}), p) = \frac{p}{s_p^k} (1 + O_k((1 - s_p^{-1})p^{-\epsilon}))
\]
provided that $0, h_1, h_2, \ldots, h_{k-1}$ are distinct mod $p$. If $s_p = p^{o(1)}$ for all primes $p$, then the spacings modulo $q$ become Poisson distributed as $s_q$, the mean spacing modulo $q$, tends to infinity.

What about non-Morse polynomials? Rather surprisingly, it turns out that (1) does not hold for all polynomials. For example, in [10] it was shown that for $f(x) = x^4 - 2x^2$,

$$N_2(h, p) = \begin{cases} 
2/3 \cdot \frac{p}{s_p^2} + O(\sqrt{p}) & \text{if } h \equiv \pm 1 \mod p, p \equiv 1 \mod 4 \\
4/3 \cdot \frac{p}{s_p^2} + O(\sqrt{p}) & \text{if } h \equiv \pm 1 \mod p, p \equiv 3 \mod 4 \\
\frac{p}{s_p^2} + O(\sqrt{p}) & \text{if } h \not\equiv \pm 1, 0 \mod p
\end{cases}$$

Hence the assumptions in (3) are violated. However, we can prove that (2) holds for most values of $(h_0, h_1, \ldots, h_{k-1})$:

**Theorem 1.** Let $p$ be a prime and let

$$R_p := \{ f(\xi) : f'(\xi) = 0, \xi \in \mathbb{F}_p \}.$$ 

be the set of critical values modulo $p$. If the sets $R_p, R_p - h_1, R_p - h_2, \ldots, R_p - h_{k-1}$ are pairwise disjoint,

$$N_k((h_1, h_2, \ldots, h_{k-1}), p) = p/s^k + O_f(k^2)$$

In other words, the analogue of Davenport's result holds for all but $O(p^{k-2})$ elements in $(\mathbb{Z}/p\mathbb{Z})^{k-1}$. Allowing for overlap between two translates of the set of critical values, we also have the following weaker upper bound on $N_k((h_1, h_2, \ldots, h_{k-1}), p)$:

**Proposition 2.** Let $p$ be a prime. There exists a constant $C_0 < 1$, only depending on $f$, with the following property: if the sets

$$(R_p \cup R_p - h_1, R_p - h_2, \ldots, R_p - h_{k-1})$$

are pairwise disjoint and $h_1 \not\equiv 0 \mod p$, then

$$N_k((h_1, h_2, \ldots, h_{k-1}), p) \leq C_0 s^{-k-1} \cdot p + O_f(k^2)$$

It turns out that these two results are enough to obtain Poisson spacings in the highly composite case. However, rather than studying

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1In particular, the spacing distribution for the image of such polynomials is *not* consistent with the coin flip model! (That is, independent coin flips where the probability of heads is given by $|\Omega_p|/p$.)

2In the case $f(x) = x^2$ this condition is equivalent to $0, h_1, \ldots, h_{k-1}$ being distinct modulo $p$. However, for general polynomials (including the case of Morse polynomials), the two conditions are *not* equivalent.
the spacings directly, we proceed by determining the \textit{k-level correlation functions}. Let
\[ \Omega_q := \{ t \in \mathbb{Z}/q\mathbb{Z} : t = f(x) \text{ for some } x \in \mathbb{Z}/q\mathbb{Z} \} \]
be the image of \( f \) modulo \( q \), let
\[ s_q := q/|\Omega_q| \]
be the mean spacing modulo \( q \), and given \( h = (h_1, h_2, \ldots, h_{k-1}) \in \mathbb{Z}^{k-1} \), put
\[ N_k(h, q) := |\{ t \in \Omega_q : t + h_1, t + h_2, \ldots, t + h_{k-1} \in \Omega_q \}| \]
For \( X \subset \mathbb{R}^{k-1} \), the \textit{k-level correlation function} is then given by
\[ R_k(X, q) := \frac{1}{|\Omega_q|} \sum_{h \in s_q X \cap \mathbb{Z}^{k-1}} N_k(h, q) \]

The main result of this paper is then the following:

\textbf{Theorem 3.} Let \( q \) be square free, \( k \geq 2 \) an integer, and let \( X \subset \mathbb{R}^{k-1} \) be a convex set with the property that \((x_0, x_1, \ldots x_{k-1}) \in X\) implies that \( x_i \neq x_j \) if \( i \neq j \). Then the \textit{k-level correlation function} of the image of \( f \) modulo \( q \) satisfies
\[ R_k(X, q) = \text{vol}(X) + O_{f,k} \left( s_q^{-1/2+o(1)} + C_0^\omega(q)(1-o(1)) \right) \]
as \( s_q \to \infty \), where \( C_0 < 1 \) is the constant given in Proposition 2.

Using a standard inclusion-exclusion argument (see \cite{14}, appendix A for details), this implies that the spacing statistics are Poissonian. In particular we have the following:

\textbf{Theorem 4.} For \( q \) square free, the limiting (normalized) spacing distribution\(^3\) of the image of \( f \) modulo \( q \) is given by \( P(t) = \exp(-t) \) as \( s_q \to \infty \). Moreover, for any integer \( k \geq 2 \), the limiting joint distribution of \( k \) consecutive spacings is a product \( \prod_{i=1}^k \exp(-t_i) \) of \( k \) independent exponential variables.

\(^3\)By normalized spacings we mean the following: with \( 0 \leq x_1 < x_2 < \cdots < x_{|\Omega_q|} < q \) being integer representatives of the image of \( f \) modulo \( q \), the spacings between consecutive elements are defined to be \( \Delta_i = x_{i+1} - x_i \) for \( 1 \leq i < |\Omega_q| \), and \( \Delta_{|\Omega_q|} = x_1 - x_{|\Omega_q|} + q \). The normalized spacings are then given by \( \tilde{\Delta}_i := \Delta_i/s_q \).
1.1. **Some remarks on the mean spacing.** We note that the only way for which \( s_p = 1 \) for all primes \( p \) is if \( f(x) \) is of degree one. However, there are nonlinear polynomials \( f \) such that \( s_p = 1 \) for infinitely many primes. For example, if \( f(x) = x^3 \) and we take \( q \) to be a product of primes \( p \equiv 2 \mod 3 \), then \( s_p = 1 \) for all \( p | q \), and \( s_q = \prod_{p|q} s_p = 1 \) clearly does not tend to infinity. On the other hand, if \( \deg(f) > 1 \), there is always a positive density set of primes \( p \) such that \( s_p > 1 \). Moreover, if \( f \) is not a permutation polynomial\(^4\) modulo \( p \), Wan has shown \[15\] that

\[
\begin{align*}
|\Omega_p| & \leq p - \frac{p - 1}{\deg(f)}.
\end{align*}
\]

Thus, for primes \( p \) such that \( s_p > 1 \), \( s_p \) is in fact uniformly bounded away from 1.

It is also worth noting that Birch and Swinnerton-Dyer have shown \[1\] that for \( f \) Morse, \( |\Omega_p| = c_f \cdot p + O_f(\sqrt{p}) \) where \( c_f < 1 \) only depends on the degree of \( f \), hence \( s_p = 1/c_f + O(p^{-1/2}) \) for all \( p \), and thus \( s_q \to \infty \) as \( \omega(q) \to \infty \).

1.2. **Related results.** There are only a few other cases for which Poisson spacings have been proven. Notable examples are Hooley’s result \[11\] \[12\] on invertible elements modulo \( q \) under the assumption that the average gap \( s_q = q/\phi(q) \) tends to infinity, and the work by Cobeli and Zaharescu \[3\] on spacings between primitive roots modulo \( p \), again under the assumption that the average gap \( s_p = (p - 1)/\phi(p - 1) \) tends to infinity. Recently, Cobeli, Vâjâitu, and Zaharescu \[2\] extended Hooley’s results and showed that subsets of the form \( \{ x \mod q : x \in I_q, x^{-1} \in J_q \} \) have limiting Poisson spacings if the intervals \( I_q, J_q \) have large lengths (more precisely, that \( |I_q| \in \left[q^{1-(2/9(\log \log q)^{1/2})}, q\right] \) and \( |J_q| \in \left[q^{1-1/(\log \log q)^2}, q\right] \) as \( q \) tends to infinity along a subsequence of integers such that \( q/\phi(q) \to \infty \).

1.3. **Acknowledgements.** The author would like to thank Juliusz Brzeziński, Andrew Granville, Moshe Jarden, Zeév Rudnick, and Thomas J. Tucker for helpful discussions.

2. **Proof of Theorem \[11\]**

Given a polynomial \( f \in \mathbb{F}_p[x] \) and \( k \) distinct elements \( h_0 = 0, h_1, h_2, \ldots, h_{k-1} \in \mathbb{F}_p \), we wish to count the number of \( t \in \mathbb{F}_p \) for which there exists

\(^4f \) is said to be a permutation polynomial modulo \( p \) if \( |\Omega_p| = p \).
Moreover, given \( \tau \) an isomorphism between \( G \) and \( \tilde{G} \), define a proof.

Let \( L \) be the field of constants for \( x \).

In order to study this, put

\[
h := (h_1, h_2, \ldots, h_{k-1})
\]

and let \( X_{k,h} \) be the affine curve defined by

\[
X_{k,h} := \{ f(x_0) = t, \ f(x_1) = t + h_1, \ldots, f(x_{k-1}) = t + h_{k-1} \},
\]

and let \( \mathbb{F}_p[X_{k,h}] \) be the coordinate ring of \( X_{k,h} \). We then have

\[
N_k((h_1, h_2, \ldots, h_{k-1}), \Omega_p) = |\{ m \in \mathbb{F}_p[t] : \Omega|m \text{ for some degree one prime } \Omega \in \mathbb{F}_p[X_{k,h}] \}|
\]

In order to estimate the size of this set, we will use the Chebotarev density theorem, made effective via the Riemann hypothesis for curves, for the Galois closure of \( \mathbb{F}_p[X_{k,h}] \). Thus, let \( Y_{k,h} \) be the curve whose function field \( \mathbb{F}_p(Y_{k,h}) \) corresponds to the Galois closure of the extension \( \mathbb{F}_p(X_{k,h})/\mathbb{F}_p(t) \).

We begin with the case \( k = 1 \). Given \( h \in \mathbb{F}_p \), define a polynomial \( F_h \in \mathbb{F}_p[x,t] \) by

\[
F_h(x,t) := f(x) - (t + h).
\]

Since the \( t \)-degree of \( F_h \) is one, \( F_h \) is irreducible, and thus \( K_h = \mathbb{F}_p[x,t]/F_h(x,t) \) is a field. Let \( L_h \) be the Galois closure of \( K_h \), and let \( G_h = \text{Gal}(L_h/\mathbb{F}_p(t)) \) be the Galois group of the field extension \( L_h/\mathbb{F}_p(t) \). By allowing for worse constants in the error terms, we may assume that \( p > n \), so that all field extensions are separable, and no wild ramification can occur.

The following Lemma shows that \( G_h \) and \( L_h \cap \overline{\mathbb{F}_p} \) are independent of \( h \).

**Lemma 5.** Let \( h \in \mathbb{F}_p \). Then \( G_h \cong G_0 \) and \( L_h \cap \overline{\mathbb{F}_p} = L_0 \cap \overline{\mathbb{F}_p} \).

**Proof.** Define a \( \mathbb{F}_p \)-linear automorphism \( \sigma : \mathbb{F}_p[t] \rightarrow \mathbb{F}_p[t] \) by \( \sigma(t) = t + h \). Since \( \sigma(F_0) = F_h \) we may extend \( \sigma \) to an isomorphism \( \sigma' : L_0 \rightarrow L_h \).

Moreover, given \( \tau \in G_0 \), \( \sigma' \tau (\sigma')^{-1} \in G_h \), the map \( \tau \rightarrow \sigma' \tau (\sigma')^{-1} \) gives an isomorphism between \( G_0 \) and \( G_h \).

Let \( l_0 = L_0 \cap \overline{\mathbb{F}_p} \) and let \( l_h = L_h \cap \overline{\mathbb{F}_p} \). Since \( l_0/\mathbb{F}_p \) is normal, \( l_0 = \sigma'(l_0) \subset K_h \cap \overline{\mathbb{F}_p} = l_h \), and the same argument for \( (\sigma')^{-1} \) gives that \( l_h \subset l_0 \), hence \( l_h = l_0 \). \( \square \)

Thus

\[
l := L_0 \cap \overline{\mathbb{F}_p}
\]

is the field of constants for \( L_h \) for any \( h \in \mathbb{F}_p \). Arguing as in the proof of Lemma 5 we obtain:
Lemma 6. Let $H_h := \text{Gal}(L_h/l(t))$. Then $H_h \cong H_0$.

Our next goal is to obtain a criterion for linear disjointness for the field extensions $L_h/l(t)$ as $h$ varies.

Lemma 7. Let $E_1, E_2$ be finite extensions of $\mathbb{F}_p(t)$, both having the same constant field $l$, and degree smaller than $p$. If $E_1/l(t)$ and $E_2/l(t)$ have disjoint finite ramification, then $E_1 \cap E_2 = l(t)$.

Proof. Let $E = E_1 \cap E_2$. By the assumption, $E/l(t)$ can only ramify at infinity. Moreover, the ramification must be tame. With $g_E$ denoting the genus of $E$, the Riemann-Hurwitz genus formula now gives

\[-2 \leq 2(g_E - 1) = [E : l(t)]2(0 - 1) + \sum_\mathcal{P} (e(\mathcal{P}/\infty) - 1) \deg(\mathcal{P})\]

\[= -2[E : l(t)] + [E : l(t)] - \sum_\mathcal{P} \deg(\mathcal{P}) < -[E : l(t)]\]

and thus $[E : l(t)] < 2$. □

We now easily obtain the desired criteria for linear disjointedness.

Proposition 8. If the sets $R_p, R_p - h_1, R_p - h_2, \ldots, R_p - h_j$ are pairwise disjoint, then the field extensions $L_0/l(t), L_{h_1}/l(t), \ldots, L_{h_j}/l(t)$ are linearly disjoint.

Proof. Since $L_h$ is the Galois closure of $K_h$, both extensions, relative $\mathbb{F}_p(t)$, ramify over the same primes. The assumption of pairwise disjointness of $R_p, R_p - h_1, \ldots, R_p - h_j$ means that there is no common finite ramification among the fields $L_0, L_{h_1}, \ldots, L_{h_j}$, hence any intersection of composites of the fields must, by Lemma 7 and Lemma 5, equal $l(t)$ □

If $G = \text{Gal}(E/\mathbb{F}_p(t))$ is the Galois group of an extension $E/\mathbb{F}_p(t)$ with constant field $l$, define (following Cohen [4, 5])

\[G^* := \{\sigma \in G : \sigma|_{l(t)} = \text{Frob}(l(t)/\mathbb{F}_p(t))\}\]

where $\text{Frob}(l(t)/\mathbb{F}_p(t))$ is the canonical generator of $\text{Gal}(l(t)/\mathbb{F}_p(t))$ given by $x \rightarrow x^p$.

Let $L^k = \mathbb{F}_p(Y_{k,h})$ be the compositum of the fields $L_{h_0}, L_{h_1}, \ldots, L_{h_{k-1}}$. For $k \geq 2$, define a conjugacy class $\text{Fix}_{k,h} \subset \text{Gal}(L^k/\mathbb{F}_p(t))^*$ by

$\text{Fix}_{k,h} := \{\sigma \in \text{Gal}(L^k/k(t))^* :$

$\sigma$ fixes at least one root of $F_{h_i}$ for $i = 0, 1, \ldots, k - 1\}$
For $k = 1$ we define (note that there is no dependence on $h$) a conjugacy class $\text{Fix}_1 \subset \text{Gal}(L^1/\mathbb{F}_p(t))^*$ by

$$\text{Fix}_1 := \{ \sigma \in \text{Gal}(L^1/k(t))^* : \sigma \text{ fixes at least one root of } f(x) - t \}$$

Then, taking into account $O_{k, f}(1)$ ramified primes, we have

$$(8) \quad N_k(h, p) =$$

$$= |\{ m \in \mathbb{F}_p[t] : \deg(m) = 1, \exists \mathcal{M}|m, \mathcal{M} \subset \mathbb{F}_p[Y_{k, h}], \text{Frob}(\mathcal{M}|m) \in \text{Fix}_{k, h}\}| + O_{k, f}(1)$$

where $\text{Frob}(\mathcal{M}|m) \in \text{Gal}(L^k/\mathbb{F}_p(t))$ denotes the Frobenius automorphism. Applying the Chebotarev density theorem (e.g., see [9], Proposition 5.16), we obtain

$$N_k(h, p) = \frac{|\text{Fix}_{k, h}|}{|\text{Gal}(L^k/l(t))|} \cdot p + O_{k, f}(\sqrt{p})$$

Our next goal is to determine $|\text{Fix}_{k, h}|/|\text{Gal}(L^k/l(t))|$. 

**Lemma 9.** Given $k \geq 2$, define

$$C_k(h, p) := \frac{|\text{Fix}_{k, h}|}{|\text{Gal}(L^k/l(t))|},$$

and

$$C_1(p) := \frac{|\text{Fix}_1|}{|\text{Gal}(L^1/l(t))|}.$$ 

Assume that $R_0, R_p - h_1, \ldots, R_p - h_{k-1}$ are pairwise disjoint. Then $C_k(h, p) = C_1(p)^k$ where $C_1(p) = 1/s_p + O_f(p^{-1/2})$.

**Proof.** For simplicity, we consider only the case $k = 2$, and for ease of notation, let $h = (h_1) = (h)$.

The action of $\text{Gal}(L^2/\mathbb{F}_p(t))$ on the roots of $F_0$ and $F_h$ allows us to identify $\text{Gal}(L^2/\mathbb{F}_p(t))$ and $\text{Gal}(L^2/l(t))$ with subgroups of $S_n \times S_n$. Moreover, since $L_0$ and $L_h$ are linearly disjoint over $l(t)$ and have isomorphic Galois groups, we may identify $\text{Gal}(L^2/l(t)) \cong H_0 \times H_h$ with a subgroup of $S_n \times S_n$ in such a way that

$$H_0 \cong H' \times 1 \subset S_n \times 1 \subset S_n \times S_n$$

and

$$H_h \cong 1 \times H' \subset 1 \times S_n \subset S_n \times S_n$$

where $H' \cong H_0 \cong H_h$ and $H'$ is a subgroup of $S_n$.

Define a $\mathbb{F}_p$-linear map $\tau : \mathbb{F}_p(t) \to \mathbb{F}_p(t)$ by $\tau(t) = t + h$, and extend it to a map from $L_0$ to $L_h$. Given $\mu_1 \in G_0^*$, let $\mu_2 = \tau \mu_1 \tau^{-1}$. Clearly $\mu_2 \in G_h$, and since $\text{Gal}(l(t)/\mathbb{F}_p(t)) \cong \text{Gal}(l/\mathbb{F}_p)$ is abelian, $\mu_1|_{l(t)} = \mu_2|_{l(t)}$ and hence $\mu_2 \in G_h^*$. Let us consider the possible extensions of $\mu_1, \mu_2$ to $L^2$. After making a fixed, but arbitrary choice, of extensions
we find that all pairs extensions are of the form \((\delta \mu_1, \gamma \mu_2)\) where \(\delta \in H_h\) and \(\gamma \in H_0\). Now, for any such pair of extensions, we have
\[
\delta \mu_1 (\gamma \mu_2)^{-1} = \delta \mu_1 \mu_2^{-1} \gamma^{-1} \in \text{Gal}(L^2/l(t))
\]
But since \(\text{Gal}(L^2/l(t)) \cong H_0 \times H_h\) we may choose \(\gamma\) and \(\delta\) in such a way that \(\delta \mu_1 \mu_2^{-1} \gamma^{-1} = 1\). In other words, it is possible to choose \(\tilde{\mu}_1, \tilde{\mu}_2\) so that \(\tilde{\mu}_1 = \tilde{\mu}_2\).

Thus, there is an extension of \(\mu \in G_0^*\) to an element \(\tilde{\mu}\) of \(\text{Gal}(L^2/\mathbb{F}_p(t))^*\) in such a way that \(\tilde{\mu}\) embeds diagonally when regarded as an element of \(S_n \times S_n\), i.e., there exists \(\sigma \in S_n\) such that \(\tilde{\mu}\) corresponds to \((\sigma, \sigma) \in S_n \times S_n\).

Now, all elements of \(\text{Gal}(L^2/\mathbb{F}_p(t))^*\), regarded as elements of \(S_n \times S_n\), must be of the form \((\delta \sigma, \gamma \sigma) \in S_n \times S_n\) where \(\delta, \gamma \in H'\). In particular, if we let \(H'' \subset H'\) be the set of elements \(\delta\) such that \(\delta \sigma\) has at least one fix point, we find that
\[
C_2(h, p) = \frac{|H''|^2}{|\text{Gal}(L^2/l(t))|^2} = \frac{|H''|^2}{|\text{Gal}(L^1/l(t))|^2} = C_1(p)^2
\]
since \(\text{Gal}(L^2/l(t)) \cong H_0 \times H_h\) and \(H_h \cong H_0 = \text{Gal}(L^1/l(t))\).

Finally, we note that
\[
|\Omega_p| = p/s_p = |\{ t \in \mathbb{F}_p \text{ for which there exists } x \in \mathbb{F}_p \text{ such that } f(x) = t \}|
\]
\[
= C_1(p) \cdot p + O_f(\sqrt{p})
\]
and thus \(C_1(p) = 1/s_p + O_f(\sqrt{p})\). \(\square\)

3. Proof of Proposition 2

We will begin by giving a proof for the case \(k = 2\), and then show how the general case can be reduced to this case. We will be using the same notation as in the proof of Theorem 1 and, by allowing worse constants in the error terms as before, we may assume that \(p > \deg(f)\).

3.1. The case \(k = 2\). We start by showing that the field extensions \(K_0, K_h\) are disjoint if \(h \neq 0\).

Lemma 10. Let \(f \in \mathbb{F}_p[x]\) be a polynomial of degree smaller than \(p\). Then the affine curve defined by
\[
\{ x, y : f(x) = f(y) + h = 0 \}
\]
is absolutely irreducible if \(h \neq 0 \mod p\).
Proof. Let \( x, y \) be roots of \( f(x) = t \) and \( f(y) = t + h \) where \( t \) is transcendental over \( \mathbb{F}_p \). If \( \overline{\mathbb{F}}_p(x) \) and \( \overline{\mathbb{F}}_p(y) \) are not linearly disjoint over \( \overline{\mathbb{F}}_p(t) \) then, by Lüroth’s theorem, \( \overline{\mathbb{F}}_p(x) \cap \overline{\mathbb{F}}_p(y) = \overline{\mathbb{F}}_p(u) \) for some \( u \not\in \overline{\mathbb{F}}_p(t) \). Hence there exists non-constant rational functions \( g_1 \) and \( g_2 \) such that \( u = g_1(x) = g_2(y) \), and a rational function \( q \), of degree less than \( p \), such that \( q(u) = t \). However, since \( t = f(x) = q(u) = q(g_1(x)) \) and \( f \) is a polynomial, \( q \) and \( g_1 \) must be of a special form: either \( q \) and \( g_1 \) are both polynomials, or \( g_1(x) = c_1 + c_2/b(x) \) where \( c_1, c_2 \) are constants, \( b(x) \) is a polynomial, and \( q(u) = \sum_{i=0}^{t} a_i/(u - c_1)^i \). In the latter case, we can replace \( u \) by \( \tilde{u} = c_2/(c_1 - u) \), and hence we may assume that \( q \) and \( g_1 \) are in fact both polynomials. Similarly, since \( q(g_2(y)) = t = f(y) - h \), we may assume that \( g_2 \) is a polynomial as well.

Now, since \( t \) is transcendental, so is \( y \) and therefore \( q(g_2(y)) = t = f(y) - h \) implies that \( q(g_2(x)) = f(x) - h \). Thus

\[
q(g_1(x)) - q(g_2(x)) = h
\]

and hence \( g_1(x) - g_2(x) \) must divide \( h \), which can only happen if \( g_1(x) = g_2(x) + C \) for some constant \( C \neq 0 \). Thus

\[
q(g_2(x) + C) - q(g_2(x)) = h
\]

and hence

\[
q'(g_2(x) + C)g_2'(x) - q'(g_2(x))g_2'(x) = 0
\]

which, since \( g_2 \) is non-constant, implies that

\[
q'(g_2(x) + C) = q'(g_2(x))
\]

Therefore, if \( g_2(\alpha) = \beta \) where \( q'(\beta) = 0 \) we find that \( q'(\beta + C) = q'(\beta) = 0 \), and more generally, that \( q'(\beta + lC) = 0 \) for \( l = 0, 1, \ldots, p-1 \), which is impossible since the degree of \( q \) is smaller than \( p \).

Thus, the two fields \( \mathbb{F}_p(x, t)/(f(x) - t) \) and \( \mathbb{F}_p(y, t)/(f(y) - t - h) \) are linearly disjoint over \( \mathbb{F}_p(t) \) and hence \( f(x) - (f(y) + h) \), when regarded as a polynomial over \( \mathbb{F}_p(y) \), is irreducible. \( \square \)

We are now ready to give a proof for Proposition 2 in the case \( k = 2 \).

**Lemma 11.** There exists \( C_0 < 1 \), only depending on \( f \), with the following property: for all sufficiently large \( p \) for which \( f \) is not a permutation polynomial modulo \( p \),

\[
C_2((h), p) \leq C_0/s_p
\]

if \( h \not\equiv 0 \mod p \).
Proof. For $f$ fixed there are only finitely many possibilities for $\text{Gal}(L^2/F_p(t))$, hence $C_2((h),p) = |\text{Fix}_2(h)|/|\text{Gal}(L^2/l(t))|$ can only take finitely many values. Thus, since $C_2((h),p) \leq C_1(p) = 1/s_p + O_f(p^{-1/2})$ it is enough to show that $C_2((h),p) = C_1(p)$ can only happen for finitely many primes $p$.

Given $a \in F_p$, let $M(a) = |\{x \in F_p : f(x) = a\}|$. Then

$$|\{x, y \in F_p : f(x) = f(y) + h\}| = \sum_{a \in F_p} M(a)M(a + h)$$

On the other hand, by Lemma 10, the curve defined by $f(x) = f(y) + h$ is absolutely irreducible, and hence the Riemann hypothesis for curves gives that

$$|\{x, y \in F_p : f(x) = f(y) + h\}| = p + O_f(\sqrt{p})$$

We have

$$|\{a : M(a) > 0\}| = |\{a : M(a - h) > 0\}| = |\text{Image}(f)| = p/s_p$$

Thus, if

$$N_2(h,p) = |\{a \in F_p : M(a) > 0, M(a - h) > 0\}| =$$

$$C_2(h,p) \cdot p + O_f(\sqrt{p}) = C_1(p) \cdot p + O_f(\sqrt{p}) = \frac{1}{s_p} \cdot p + O_f(\sqrt{p})$$

then, since $|\{a : M(a - h) > 0\}| = |\text{Image}(f)| = p/s_p$, we have

$$|\{a \in F_p : M(a) = 0, M(a - h) > 0\}| = O_f(\sqrt{p})$$

Therefore

$$p + O_f(\sqrt{p}) = \sum_{a \in F_p} M(a)M(a - h)$$

$$\geq \sum_{a \in F_p : M(a) = 1} M(a - h) + 2 \sum_{a \in F_p : M(a) > 1} M(a - h)$$

$$= \sum_{a \in F_p : M(a) > 0} M(a - h) + \sum_{a \in F_p : M(a) > 1} M(a - h)$$

$$= \sum_{a \in F_p} M(a - h) + \sum_{a \in F_p : M(a) > 1} M(a - h) - \sum_{a \in F_p : M(a) = 0} M(a - h)$$

$$= p + \sum_{a \in F_p : M(a) > 1} M(a - h) - O_f(\sqrt{p})$$

and thus

$$\sum_{a \in F_p : M(a) > 1} M(a - h) = O_f(\sqrt{p})$$
Hence
\[ |\{a \in \mathbb{F}_p : M(a) > 1, M(a - h) > 0\}| = O_f(\sqrt{p}) \]
and we similarly obtain that
\[ |\{a \in \mathbb{F}_p : M(a) > 0, M(a - h) > 1\}| = O_f(\sqrt{p}) \]
But then
\[ p + O_f(\sqrt{p}) = \sum_{a \in \mathbb{F}_p} M(a)M(a - h) \]
\[ = |\{a \in \mathbb{F}_p : M(a) = M(a - h) = 1\}| + O_f(\sqrt{p}) \]
In other words, \( M(a) = 1 \) for all but \( O_f(\sqrt{p}) \) elements, which, by Wan’s result (see (6), section 1.1), can only happen if \( f \) is bijection once \( p \) is sufficiently large.

\[ \square \]

3.2. The case \( k > 2 \). As usual, we use the convention that \( h_0 = 0 \). Arguing as in the proof of Lemma 7, we find that the field extensions
\[ (L_{h_0}, L_{h_1}), L_{h_2}/l(t), \ldots, L_{h_{k-2}}/l(t), L_{h_{k-1}}/l(t) \]
are linearly disjoint since they have disjoint ramification. Hence there is an isomorphism
\[ \text{Gal} \left( L_{h_0}, L_{h_1}, \ldots, L_{h_{k-1}}/l(t) \right) \]
\[ \simeq \text{Gal} \left( L_{h_0}, L_{h_1}/l(t) \right) \times \text{Gal} \left( L_{h_2}/l(t) \right) \times \ldots \times \text{Gal} \left( L_{h_{k-1}}/l(t) \right) \]
Putting \( h' = (h_0, h_1) \) and arguing as in Lemma 9, we find that
\[ \frac{|\text{Fix}_{k, h}|}{|\text{Gal}(L^k/l(t))|} = \frac{|\text{Fix}_{2, h'}|}{|\text{Gal}(L_{h_0}L_{h_1}/l(t))|} \cdot \frac{1}{s_p^{k-2}} = C_2(h', p) \cdot \frac{1}{s_p^{k-2}}. \]
By Lemma 11 \( C_2(h', p) \leq C_0/s_p \) and the proof is complete.

4. Proof of Theorem 8

For \( h \in \mathbb{Z}^{k-1} \) fixed, it follows immediately from the Chinese Remainder Theorem that \( N_k(h, q) \) is multiplicative in \( q \). The following Lemma shows that we may assume that \( q \) is a product of primes \( p \) for which \( f \) is not a permutation polynomial modulo \( p \), and hence that \( s_p \) is uniformly bounded away from 1 for all \( p|q \).

Lemma 12. Given a square free integer \( q \), write \( q = q_1q_2 \) where
\[ q_1 = \prod_{p|q, |\Omega_p| < p} p, \quad q_2 = \prod_{p|q, |\Omega_p| = p} p \]
Then
\[ R_k(X, q) = R_k(X, q_1) \]

**Proof.** If \( p \mid q_2 \) we have \( s_p = p/|\Omega_p| = 1 \) and \( N_k(h, p) = p \) for all \( h \in \mathbb{Z}^{k-1} \). Thus \( s_q = s_{q_1} \cdot s_{q_1} = s_{q_1} \), and since for \( h \) fixed, \( N_k(h, q) \) is multiplicative, we find that \( N_k(h, q) = N_k(h, q_1) \cdot q_2 \). Thus
\[
R_k(X, q) = \frac{1}{|\Omega_q|} \sum_{h \in s_q X \cap \mathbb{Z}^{k-1}} N_k(h, q) = \frac{q_2}{|\Omega_{q_1}| \cdot |\Omega_{q_2}|} \sum_{h \in s_q X \cap \mathbb{Z}^{k-1}} N_k(h, q_1)
\]
\[
= \frac{1}{|\Omega_{q_1}|} \sum_{h \in s_{q_1} X \cap \mathbb{Z}^{k-1}} N_k(h, q_1) = R_k(X, q_1)
\]
\[ \square \]

We also note the following easy consequence of Theorem 1.

**Lemma 13.** Let \( l \) be the largest integer such that \( R_p - h_{i_1}, R_p - h_{i_2}, \ldots, R_p - h_{i_l} \) are pairwise disjoint for some choice of indices \( 0 \leq i_1, i_2, \ldots, i_l \leq k - 1 \) (with the usual convention that \( h_0 = 0 \)). Then
\[
N_k((h_1, h_2, \ldots, h_{k-1}), p) \leq p/s_l + O_{f, k}(\sqrt{p})
\]

**Proof.** If \( \{h'_1, h'_2, \ldots, h'_{l-1}\} \) is a subset of \( \{h_1, h_2, \ldots, h_{k-1}\} \) then trivially
\[
N_k((h_1, h_2, \ldots, h_{k-1}), p) \leq N_l((h'_1, h'_2, \ldots, h'_{l-1}), p)
\]
and the Lemma follows from Theorem 1. \[ \square \]

**4.1. Some remarks on affine sets.** We will partition \( \mathbb{Z}^{k-1} \) according to the size of the bounds on \( N_k(h, q) = \prod_{p \mid q} N_k(h, p) \) given by Theorem 1 and Proposition 2. In order to do this, we need to introduce some notation: By an *affine set* \( L \subset \mathbb{Z}^{k-1} \) we mean an integer translate of a lattice \( L' \subset \mathbb{Z}^{k-1} \). We then define the rank, respectively discriminant, of \( L \) as the rank, respectively discriminant of \( L' \). Similarly, we define \( \text{codim}(L) \) as \( k - 1 \) minus the rank of \( L \).

Let \( R \) be the set of critical values of \( f \), i.e.,
\[
R := \{ f(\xi) : f'(\xi) = 0, \xi \in \overline{Q} \}
\]
and recall that \( R_p = \{ f(\xi) : f'(\xi) = 0, \xi \in \overline{F_p} \} \) is the set of critical values of \( f \) modulo \( p \). Let
\[
\tilde{R} := R - R = \{ \alpha - \beta : \alpha, \beta \in R \},
\]
put
\[
\tilde{R}_\infty := \tilde{R} \cap \mathbb{Z},
\]
\footnote{By the discriminant of \( L' \subset \mathbb{Z}^{k-1} \) we mean the index of \( L' \) in \( \mathbb{Z}^{k-1} \).}
and let
\[ \tilde{R}_p := (R_p - R_p) \cap \mathbb{F}_p. \]
If \( R_p + h_i \cap R_p + h_j \neq \emptyset \), then \( h_i - h_j \in \tilde{R}_p \), so the affine sets to be considered will be given by equations of the form
\[ h_i - h_j = r, \quad r \in \tilde{R}_\infty \]
(9)
or congruences of the form
\[ h_i - h_j \equiv r_p \pmod{p}, \quad r_p \in \tilde{R}_p \]
(10)
We note that the bounds given by Theorem 1 and Proposition 2 only depend on the congruence class of \( h \), but we will treat the case of equality separately since \( N_k(h, p) \) will be large for all \( p \) if \( h \) satisfies an equation of the form (9).

To ensure that the equations defining the affine sets are independent, we will need the following notions: Given
\[ E \subset \{(i, j) : 0 \leq i < j \leq k - 1\} \]
we may associate a graph \( G(E) \) on the set of vertices \( \{0, 1, \ldots, k - 1\} \) by regarding \( E \) as the set of edges, i.e., two nodes \( i, j \) are connected by an edge if and only if \( (i, j) \in E \). Let
\[ \mathcal{AG} := \{ E \subset \{(i, j) : 0 \leq i < j \leq k - 1\} : G(E) \text{ is acyclic.} \} \]
be the collection of edge sets whose associated graphs are acyclic.

Given \( E \in \mathcal{AG} \) and a map \( \alpha : E \rightarrow \tilde{R}_\infty \), define an affine set
\[ L(E, \alpha) := \{ h \in \mathbb{Z}^{k-1} : h_i - h_j = \alpha((i, j)) \text{ for all } (i, j) \in E. \} \]
(with the usual convention that \( h_0 = 0 \)). Note that \( G(E) \) acyclic implies that the equations defining \( L(E, \alpha) \) are independent. Further, given \( E \in \mathcal{AG} \), let
\[ \mathcal{L}(E) := \{ L(E, \alpha) \text{ where } \alpha \text{ ranges over all maps } \alpha : E \rightarrow \tilde{R}_\infty \} \]
be the collection of affine sets defined by independent relations between \( h_i \) and \( h_j \) for all \( (i, j) \in E \). We note that \( \mathcal{L}(\emptyset) \) contains exactly one element, namely the full lattice \( L(\emptyset, -) = \mathbb{Z}^{k-1} \). Moreover, if \( L \in \mathcal{L}(E) \), then (since we assume that \( E \in \mathcal{AG} \)) \( \text{codim}(L) = |E| \), and if \( h \in L \), then Proposition 2 will, for all \( p \) if \( h \) satisfies additional equations, i.e., if \( h \in L' \) for some \( L' \in \mathcal{L}(E') \) such that \( E' \supseteq E \).

Given \( L(E, \alpha) \in \mathcal{L}(E) \), let
\[ L^\times(E, \alpha) := \{ h \in L(E, \alpha) : h \notin L(E', \alpha') \text{ for all } E' \supseteq E, \alpha' : E' \rightarrow \tilde{R}_\infty \} \]
In particular, if \( h \in L^\times(E, \alpha) \), the components of \( h \) satisfy exactly \( |E| \) independent equations of the form \( h_i - h_j = r_{ij} \) where \( r_{ij} \in \tilde{R}_\infty \).

We also need to keep track of similar relations, modulo \( p \), between the components of \( h \). Thus, given \( E_p \in \mathcal{A}G \) and \( \alpha_p : E_p \to \tilde{R}_p \), define an affine set

\[
L_p(E_p, \alpha_p) := \{ h \in \mathbb{Z}^{k-1} : h_i - h_j \equiv \alpha_p((i, j)) \mod p \text{ for all } (i, j) \in E_p \}.
\]

We note that the rank of \( L_p(E_p, \alpha_p) \) is \( k - 1 \) and that the discriminant of \( L_p(E_p, \alpha_p) \) is \( p^{|E_p|} \), and if \( h \in L_p(E_p, \alpha_p) \), then Proposition 2 will at best give the bound

\[
N_k(h, p) \leq C_0 \frac{p}{s_p k^{-|E_p|}} + O_{f,k}(\sqrt{p}).
\]

Now, given \( E \in \mathcal{A}G \), let

\[
L_p(E) := \{ L_p(E_p, \alpha_p) : E_p \in \mathcal{A}G, \alpha_p : E_p \to \tilde{R}_p, E_p \cap E = \emptyset, E_p \cup E \in \mathcal{A}G \}
\]

and for \( L_p \in L_p(E) \), let

\[
L_p^\times := \{ h \in L_p : h \notin L'_p \text{ for all } L'_p \in L_p(E'_p), E'_p \supsetneq E_p \}
\]

If \( h \in L^\times \cap L_p^\times \) for \( L \in \mathcal{L}(E) \) and \( L_p = L_p(E_p, \alpha_p) \in \mathcal{L}_p(E) \), then \((h_0, h_1, \ldots, h_{k-1}) = h \) satisfies exactly \( |E| \) independent equations of the form \( h_i - h_j = r_{ij} \) where \( r_{ij} \in \tilde{R}_\infty \), and exactly \( |E_p| \) independent congruences of the \( h_i - h_j \equiv r'_{ij} \mod p \) where \( r'_{ij} \in \tilde{R}_p \), and furthermore, there is no overlap between the equations and congruences. The reason for keeping track of equalities and congruences separately is that if \( h \in L \) for \( L \in \mathcal{L}(E) \) and \( |E| > 0 \), then the bounds given on \( N_k(h, p) \) given by Proposition 2 allows \( N_k(h, p) \) to deviate quite a bit from its mean value for all \( p \). On the other hand, if we let \( c \) be the product of primes \( p|q \) for which the bounds are bad because of congruence conditions, rather than equalities, then we can bound the size of \( c \) (see Lemma 14). We can now partition \( \mathbb{Z}^{k-1} \) according to the size of the bounds on \( N_k(h, p) \) given by Theorem 1 and Proposition 2.

**Lemma 14.** Let \( L = L(E, \alpha) \), \( L_p = L_p(E_p, \alpha_p) \in \mathcal{L}_p(E) \), and assume that \( h \in L^\times \cap L_p^\times \). If \( |E| + |E_p| = 0 \), then

\[
N_k(h, p) = s_p^{-k} \cdot p + O_{k,f}(p^{1/2})
\]

whereas if \( k > |E| + |E_p| > 0 \), then

\[
N_k(h, p) \leq C_0 \cdot s_p^{|E|+|E_p|-k} \cdot p + O_{k,f}(p^{1/2})
\]

where \( C_0 < 1 \) is as in Proposition 2.
Lemma 15. and Lemma 14, we obtain the following:

Proof. The first assertion follows immediately from Theorem since $R_p + h_i \cap R_p + h_j \neq \emptyset$ implies that $h_i - h_j \in R_p$.

For the second assertion, we argue as follows: Since $h = (h_1, h_2, \ldots, h_{k-1}) \in L^\times \cap L_p^\times$ there are indices $i_1, i_2, \ldots, i_{k-\mid E \mid - \mid E_p \mid}$ such that $h_{i_1} \neq h_{i_2}$ and

$$(R_p - h_{i_1} \cup R_p - h_{i_2}), R_p - h_{i_2}, \ldots, R_p - h_{i_{k-\mid E \mid - \mid E_p \mid}}$$

are pairwise disjoint. Putting

$$h' = (h_{i_2} - h_{i_1}, h_{i_3} - h_{i_1}, \ldots, h_{i_{k-\mid E \mid - \mid E_p \mid}} - h_{i_1}),$$

the result follows from the bound for $N_k(h', p)$ given by Proposition. 

However, partitioning $\mathbb{Z}^{k-1}$ according to the size of $N_k(h, p)$ for individual prime factors $p | q$ is not quite enough; we need to partition $\mathbb{Z}^{k-1}$ according to the size of $N_k(h, q) = \prod_{p \mid q} N_k(h, p)$. Thus, let

$$\mathcal{L}_c(E) := \{L \cap (\cap_{p \mid c} L_p) : L \in \mathcal{L}(E), \forall p \mid c L_p \in \mathcal{L}_p(E) \setminus L_p(\emptyset, -)\}$$

(40) (where $L_p(\emptyset, -) \in \mathcal{L}_p(E)$ is the maximal lattice, i.e., $L_p(\emptyset, -) = \mathbb{Z}^{k-1}$) and given

$$L_c = L \cap (\cap_{p \mid c} L_p) \in \mathcal{L}_c(E)$$

let

$$L_c^\times := L^\times \cap (\cap_{p \mid c} L_p^\times) \cap (\cap_{p \mid c} L_p^\times(\emptyset, -))$$

We can now partition $\mathbb{Z}^{k-1}$ into subsets $L_c^\times$, where $L_c \in \mathcal{L}_c(E)$, $E \in \mathcal{A}_G$, and $c \mid q$. Moreover, as an immediate consequence of the definitions and Lemma 14 we obtain the following:

Lemma 15. Assume that $L_c = L \cap (\cap_{p \mid c} L_p(E_p, \alpha_p)) \in \mathcal{L}_c(E)$ and that $h \in L_c^\times$. If $p \nmid c$, then

$$N_k(h, p) = s_p^{-k} \cdot p + O_{k,f}(p^{1/2}).$$

If $p \mid c$, then

$$N_k(h, p) \leq C_0 \cdot s_p^{\mid E \mid + \mid E_p \mid - k} \cdot p + O_{k,f}(p^{1/2}).$$

where $C_0 < 1$ is as in Proposition 2.

Using the previous Lemma we can now bound sums of the form $\sum_{n \in s_q X \cap L_c^\times} N_k(h, q)$. 

Lemma 16. If

$$L_c = L \cap (\cap_{p \mid c} L_p(E_p, \alpha_p)) \in \mathcal{L}_c(E),$$

then

$$\mid \{h \in s_q X \cap L_c^\times \} \mid \leq \mid \{h \in s_q X \cap L_c \} \mid \ll_{k,f,X} s_q^{k-\mid E \mid - 1} + s_q^{k-\mid E \mid - 2}$$
Moreover, if \( h \in L_c^\times \), then

\[
\frac{N_k(h, q)}{q/s_q} \ll \prod_{p|c} \left( \frac{s_p^{(E)} + |E_p|}{s_p^{k-1}} + O_{k,f}(p^{-1/2}) \right) \cdot \prod_{p|\frac{q}{c}} (C_0 \cdot \frac{s_p^{(E)}}{s_p^{k-1}} + O_{k,f}(p^{-1/2}))
\]

In particular,

\[
\sum_{h \in s_q X \cap L_c^\times} \frac{N_k(h, q)}{q/s_q} \ll s_c^{k-1} C_0^{-\omega(c)} \left( \frac{1}{s_q} + \frac{1}{c} \right) \cdot C_0^{\omega(q)} \cdot \prod_{p|q} \left( 1 + O_{k,f}(p^{-1/2}) \right)
\]

**Proof.** The first assertion follows from the Lipschitz principle\(^\text{6}\) (e.g., see Lemma 16 in [14]) since \( L_c \) is a translate of a lattice with discriminant (relative \( L \)) divisible by \( c \). The second assertion follows from Lemma 15.

Thus

\[
\sum_{h \in s_q X \cap L_c^\times} \frac{N_k(h, q)}{q/s_q} \ll \prod_{p|c} \left( s_p^{(E)} + O_{k,f}(p^{-3/2}) \right) \cdot \prod_{p|\frac{q}{c}} (C_0 + O_{k,f}(p^{-1/2})) + \frac{1}{s_q} \prod_{p|c} (s_p^{(E)} + O_{k,f}(p^{-1/2})) \cdot \prod_{p|\frac{q}{c}} (C_0 + O_{k,f}(p^{-1/2}))
\]

\[
\ll \frac{C_0^{-\omega(c)} (s_c^{k-1} + \frac{s_c^{k-1}}{s_q}) \cdot C_0^{\omega(q)} \cdot \prod_{p|q} \left( 1 + O_{k,f}(p^{-1/2}) \right)}{c}
\]

\( \square \)

Since the bound in (11) is not useful for large \( c \), we will also need the following:

**Lemma 17.** Let \( d \) be the degree of the field extension \( \mathbb{Q}(\tilde{R})/\mathbb{Q} \). If \( L_c \in L_c(E) \) for some \( E \in AG \) and \( s_q X \cap L_c^\times \neq \emptyset \) then

\[
c \ll X_{\tilde{R}} |s_q^{d(\frac{k}{2})}| |\tilde{R}|.
\]

Moreover, there exist a constant \( D \), only depending on \( k \) and \( f \), such that

\[
|L_c(E)| \ll_{k,f} D^{\omega(c)}.
\]

\(^\text{6}\)Actually, we have to be a little careful: if we embed \( L \) into \( \mathbb{Z}^{k-1-|E|} \) and apply the Lipschitz principle, there is an implicit constant in the bound that will depend on \( L \). However, the estimate is uniform since \( L \) only can be choosen in \( O_k(1) \) ways.
Proof. We first assume that all elements of $\tilde{R}$ are algebraic integers. Let $B$ be the ring of integers in $\mathbb{Q}(\tilde{R})$. For each prime $p|q$ chose a prime $\mathfrak{p}_p \subset B$ lying above $p$, so that we may regard any element in $\tilde{R}_p$ as the image of an element in $\tilde{R}$ under the reduction map $B \to B/\mathfrak{p}_p$.

For $0 \leq i < j \leq k - 1$, $r \in \tilde{R}$, and $h \in L_c^\times$, let

$$\gamma_{i,j,r}(h) = \prod_{p : h_i - h_j \equiv r \mod \mathfrak{p}_p} p$$

Then $c$ divides

$$\prod_{0 \leq i < j \leq k - 1 \atop r \in \mathbb{R} : h_i - h_j \neq r} \gamma_{i,j,r}(h)$$

Since $h_i - h_j - r \equiv 0 \mod \mathfrak{p}_p$ for all $p$ dividing $\gamma_{i,j,r}$, we find that $\gamma_{i,j,r}$ divides $N_{\mathbb{Q}(\tilde{R})}(h_i - h_j - r)$. Moreover, if $h \in s_q X$, then $|h_i - h_j| \ll_X s_q$, thus

$$N_{\mathbb{Q}(\tilde{R})}(h_i - h_j - r) \ll_{f,X} s_q^d$$

and hence

$$c \leq \prod_{0 \leq i < j \leq k - 1 \atop r \in \mathbb{R} : h_i - h_j \neq r} N_{\mathbb{Q}(\tilde{R})}(h_i - h_j - r) \ll_{k,f,X} s_q^d \tilde{R}^{(k)}$$

(Note that $N_{\mathbb{Q}(\tilde{R})}(h_i - h_j - r) \neq 0$ since $h_i - h_j - r \neq 0$).

In case $\tilde{R}$ contains elements that are not algebraic integers, we can find an integer $m$, only depending on $\tilde{R}$, such that all elements of $m \cdot \tilde{R} = \{m \cdot r : r \in \tilde{R}\}$ are algebraic integers, and apply the above argument to $m \cdot \tilde{R}$ and $m h$ (for primes $p$ not dividing $m$, but since $c$ is square free this just makes the constant worse by a power of $(c,m) \leq m$, which is $O(1)$.)

The second assertion follows upon noting that there are $O_{k,f}(1)$ possible choices of $E_p$ and $\alpha_p$ for each $p|c$.

4.2. Conclusion. We can now write $\mathbb{Z}^{k-1}$ as a disjoint union of sets $L^\times$ where $L$ ranges over all elements in $\bigcup_{E \in \mathcal{A}G} \mathcal{L}(E)$, and hence $R_k(X,q)$ equals

$$\sum_{h \in s_q X \cap \mathbb{Z}^{k-1}} N_k(h, q) = \sum_{E \in \mathcal{A}G} \sum_{L \in \mathcal{L}(E)} \sum_{h \in s_q X \cap L^\times} N_k(h, q)$$

(Note that $N_{\mathbb{Q}(\tilde{R})}(h_i - h_j - r) \neq 0$ since $h_i - h_j - r \neq 0$).
The term corresponding to $E = \emptyset$ in (12) will give the main contribution (note that if $E = \emptyset$, then $L = L_\infty(E, -) = \mathbb{Z}^{k-1}$.) Let

$$X' := \{h \in X : h_i - h_j \not\in \tilde{R}_\infty \text{ for } 0 \leq i < j \leq k - 1\}$$

where we as usual use the convention that $h_0 = 0$. Then

$$s_q X \cap L^\times = s_q X' \cap \mathbb{Z}^{k-1}$$

Note that $X'$ is just $\mathbb{R}^{k-1}$ with some hyperplanes removed, so if $X$ is convex, we can write $X'$ as a finite union of convex sets. We now rewrite (12) as follows:

$$\frac{1}{|\Omega_q|} \sum_{h \in s_q X \cap \mathbb{Z}^{k-1}} N_k(h, q) = \sum_{h \in s_q X' \cap \mathbb{Z}^{k-1}} N_k(h, q) + \text{Error}_1$$

where

$$\text{Error}_1 := \frac{1}{|\Omega_q|} \sum_{E \in \mathcal{A}G, |E| > 0} \sum_{L \in \mathcal{L}(E)} \sum_{h \in s_q X \cap L^\times} N_k(h, q)$$

and the main term is given by

$$(13) \quad \sum_{h \in s_q X' \cap \mathbb{Z}^{k-1}} N_k(h, q)$$

We begin by showing that $\text{Error}_1 = o(1)$ as $\omega(q) \to \infty$.  

Lemma 18. As $\omega(q) \to \infty$,

$$\text{Error}_1 = \frac{1}{|\Omega_q|} \sum_{E \in \mathcal{A}G, |E| > 0} \sum_{L \in \mathcal{L}(E)} \sum_{h \in s_q X \cap L^\times} N_k(h, q) \ll C_0^{\omega(q)(1-o(1))}.$$ 

Proof. Given $E \in \mathcal{A}G$ with $|E| > 0$, we find that

$$(14) \quad \frac{1}{|\Omega_q|} \sum_{L \in \mathcal{L}(E)} \sum_{h \in s_q X \cap L^\times} N_k(h, q)$$

$$= \frac{1}{q/s_q} \sum_{c|q} \sum_{L \in \mathcal{L}_c(E)} \sum_{h \in s_q X \cap L^\times} N_k(h, q)$$

which, by Lemmas 16 and 17 is

$$(15) \quad \ll C_0^{\omega(q)} \prod_{p|q} (1 + O(p^{-1/2})) \sum_{c|q} D^{\omega(c)s_c^{k-1}C_0^{-\omega(c)}} \left(\frac{1}{s_q} + \frac{1}{c}\right)$$

for $c < s_q^{a(\frac{c}{k})/k}$.  

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Now,
\[
\sum_{c \in \mathbb{L}_q} D^{\omega(c)} s_c^{k-1} C_0^{-\omega(c)} \frac{1}{c} \ll \prod_{p|q} \left(1 + O(1/p)\right)
\]
and, for any \(\delta > 0\),
\[
\frac{1}{s_q} \sum_{c \in \mathbb{L}_q} D^{\omega(c)} s_c^{k-1} C_0^{-\omega(c)} \ll \left(\frac{1}{s_q} \prod_{p|q} \left(1 + O(1/p^\delta)\right)\right) \ll \frac{1}{s_q} \prod_{p|q} \left(1 + O(p^{-1/2})\right) = C_0^{\omega(q)} \prod_{p|q} \left(1 + O(p^{-1/2})\right)
\]
Thus, taking \(\delta = 1/(2d(\frac{k}{2})|R|)\), we find that (16) is
\[
\ll C_0^{\omega(q)} \prod_{p|q} \left(1 + O(p^{-1/2})\right) \cdot \left(\frac{1}{s_q^{1/2-o(1)}} + \prod_{p|q} \left(1 + O(p^{-1})\right)\right)
\]
\[
= C_0^{\omega(q)} \prod_{p|q} \left(1 + O(p^{-1/2})\right) = C_0^{\omega(q)(1-o(1))}
\]
Since there are \(O(1)\) possible choices of \(L \in \mathcal{L}(E)\) for \(E\) fixed, and \(E\) ranges over a finite number of subsets, we find that (14) is \(C_0^{\omega(q)(1-o(1))}\).

We proceed by rewriting the main term in terms of a divisor sum. For \(p\) prime and \(h \in \mathbb{Z}^{k-1}\), let
\[
\varepsilon_k(h, p) = \frac{s_p^{k-1} \cdot N_k(h, p)}{|\Omega_p|} - 1,
\]
so that we may write
\[
N_k(h, p) = \frac{|\Omega_p|}{s_p^{k-1}} \left(1 + \varepsilon_k(h, p)\right)
\]
(recall that \(s_p = |\Omega_p|\).) Further, for \(d > 1\) a square free integer, put
\[
\varepsilon_k(h, d) = \prod_{p|d} \varepsilon_k(h, p)
\]
and, to make \(\varepsilon_k\) multiplicative in the second parameter, set \(\varepsilon_k(h, 1) = 1\) for all \(h\). Since \(N_k(h, q)\) is multiplicative, we then have
\[
(16) \quad N_k(h, q) = \prod_{p|q} \frac{1}{s_p^{k-1}} \frac{|\Omega_p|}{s_p^{k-1}} \left(1 + \varepsilon_k(h, p)\right) = \frac{|\Omega_q|}{s_q^{k-1}} \sum_{d|q} \varepsilon_k(h, d)
\]
The following Lemma shows that the average of $\varepsilon_k(h, d)$, over a full set of residues modulo $d$, equals zero if $d > 1$.

**Lemma 19.** If $d > 1$ then
\[
\sum_{h \in (\mathbb{Z}/d\mathbb{Z})^{k-1}} \varepsilon_k(h, d) = 0
\]

**Proof.** Since $\varepsilon_k(h, d)$ is multiplicative it is enough to show that
\[
\sum_{h \in (\mathbb{Z}/p\mathbb{Z})^{k-1}} \varepsilon_k(h, p) = 0
\]
for $p$ prime, and because
\[
N_k(h, p) = \frac{1}{s_p^{k-1}} |\Omega_p| (1 + \varepsilon_k(h, p))
\]
it is enough to show that
\[
\sum_{h \in (\mathbb{Z}/p\mathbb{Z})^{k-1}} N_k(h, p) = \frac{1}{s_p^{k-1}} |\Omega_p| p^{k-1} = |\Omega_p|^k
\]
But \[\sum_{h \in (\mathbb{Z}/p\mathbb{Z})^{k-1}} N_k(h, p)\] equals the number of $k$-tuples of elements from $\Omega_p$, and hence \[\sum_{h \in (\mathbb{Z}/p\mathbb{Z})^{k-1}} N_k(h, p) = |\Omega_p|^k.\]

We will also need the following bound:

**Lemma 20.** We have
\[
\sum_{h \in (\mathbb{Z}/d\mathbb{Z})^{k-1}} |\varepsilon_k(h, d)| \ll d^{k-3/2+o(1)}
\]

**Proof.** Since the sum is multiplicative in $d$, it is enough to show that
\[
\sum_{h \in (\mathbb{Z}/p\mathbb{Z})^{k-1}} |\varepsilon_k(h, p)| \ll p^{k-3/2}
\]
for $p$ prime. By Theorem \[|\varepsilon_k(h, p)| \ll p^{-1/2}\] for all but $O(p^{k-2})$ residues modulo $p$, and for the remaining residues we have $|\varepsilon_k(h, p)| = O_{k, f}(1)$. Thus
\[
\sum_{h \in (\mathbb{Z}/p\mathbb{Z})^{k-1}} |\varepsilon_k(h, p)| \ll p^{k-1-1/2} + p^{k-2} \ll p^{k-3/2}
\]

We now find that the main term \[\text{(13)}\] equals
\[
\frac{1}{|\Omega_q|} \sum_{h \in s_qX' \cap \mathbb{Z}^{k-1}} N_k(h, q) = \frac{1}{s_q^{k-1}} \sum_{d|q} \sum_{h \in s_qX' \cap \mathbb{Z}^{k-1}} \varepsilon_k(h, d)
\]
\begin{align*}
&= \frac{1}{s_q^{k-1}} \sum_{h \in s_q X' \cap \mathbb{Z}^{k-1}} \sum_{d \mid q \mid d > 1} \varepsilon_k(h, d) + \text{Error}_2 \\
\text{where} \\
\text{Error}_2 := \frac{1}{s_q^{k-1}} \sum_{d \mid q \mid d > 1} \sum_{h \in s_q X' \cap \mathbb{Z}^{k-1}} \varepsilon_k(h, d)
\end{align*}

and the modified main term is

\begin{align*}
\frac{1}{s_q^{k-1}} \sum_{h \in s_q X' \cap \mathbb{Z}^{k-1}} 1 &= \frac{1}{s_q^{k-1}} \left( \text{vol}(s_q X') + O(s_q^{k-2}) \right) \\
&= \text{vol}(X) + O(1/s_q).
\end{align*}

We conclude by showing that Error_2 = o(1) as \( s_q \to \infty \).

**Lemma 21.** As \( s_q \to \infty \), we have

\begin{equation}
\text{Error}_2 = \frac{1}{s_q^{k-1}} \sum_{d \mid q \mid d > 1} \sum_{h \in s_q X' \cap \mathbb{Z}^{k-1}} \varepsilon_k(h, d) \ll s_q^{-1/2 + o(1)}
\end{equation}

**Proof.** In order to show that Error_2 is small, we split the divisor sum in two parts according to the size of \( d \).

**Small d:** We first consider \( d \leq s_T^q \) where \( T \in (0, 1) \) is to be chosen later. A point \( h \in s_q X' \cap \mathbb{Z}^{k-1} \) is contained in a unique cube \( C_{h,d} \subset \mathbb{R}^{k-1} \) of the form

\( C_{h,d} = \{(x_1, x_2, \ldots, x_{k-1}) : dt_i \leq x_i < d(t_i+1), t_i \in \mathbb{Z}, i = 1, 2, \ldots, k-1\} \)

We say that \( h \in s_q X' \cap \mathbb{Z}^{k-1} \) is a \textit{d-interior} point of \( s_q X' \) if \( C_{h,d} \subset s_q X' \), and if \( C_{h,d} \) intersects the boundary of \( s_q X' \), we say that \( h \) is a \textit{d-boundary point} of \( s_q X' \).

By Lemma 19, the sum over the \textit{d-interior} points is zero, and hence

\begin{equation}
\frac{1}{s_q^{k-1}} \sum_{d \mid q \mid d > 1} \sum_{h \in s_q X' \cap \mathbb{Z}^{k-1}} \varepsilon_k(h, d) \ll s_q^{-1/2 + o(1)}
\end{equation}

Since \( s_q X' \) is a union of convex sets, the number of cubes \( C_{h,d} \) intersecting the boundary of \( s_q X' \) is \( \ll (s_q/d)^{k-2} \), and hence is

\begin{equation*}
\ll \frac{1}{s_q^{k-1}} \sum_{d \mid q \mid d > 1} (s_q/d)^{k-2} \sum_{h \in (\mathbb{Z}/d\mathbb{Z})^{k-1}} |\varepsilon_k(h, d)|
\end{equation*}
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\[ (19) = \frac{1}{s_q} \sum_{d \mid q} \frac{1}{d^{k-2}} \sum_{h \in (\mathbb{Z}/d\mathbb{Z})^{k-1}} |\varepsilon_k(h, d)| \]

which by Lemma 20 is, for any \( \alpha > 1/2, \)
\[ \ll \frac{1}{s_q} \sum_{d \mid q} d^{1/2 - \alpha + o(1)} \ll s_q^{\alpha T - 1 + o(1)} \]
\[ \ll \sum_{d \mid q} d^{-\epsilon} = \prod_{p \mid q} (1 + p^{-\epsilon}) = s_q^{o(1)} \]

if \( \epsilon > 0 \) (recall that \( s_p \) is assumed to be uniformly bounded away from 1 and \( s_q = \prod_{p \mid q} s_p. \))

\textit{Large } d: \ We now consider
\[ (20) \frac{1}{s_q} \sum_{d \mid q} \sum_{h \in s_q X \cap \mathbb{Z}^{k-1}} \varepsilon_k(h, d) \]

Given \( h \) and \( d, \) let \( c \) be the largest divisor of \( d \) such that \( h \in L_c \) for some \( L_c \subseteq L_c(L) \). Then
\[ \varepsilon_k(h, d) \ll \frac{s_c^{k-1}}{(d/c)^{1/2 - o(1)}} \]

by Lemma 15. Hence, for \( E \in \mathcal{A}G \) fixed,
\[ \sum_{L \in L(E)} \sum_{h \in s_q X \cap L^c} \varepsilon_k(h, d) \ll \sum_{c \mid d} \sum_{L_c \in L_c(E)} \sum_{h \in s_q X \cap L_c^c} |\varepsilon_k(h, d)| \]
\[ \ll \sum_{c \mid d} s_c^{k-1} \sum_{L_c \in L_c(E)} \sum_{h \in s_q X \cap L_c^c} 1 \]

which by Lemmas 16 and 17 is
\[ (21) \ll s_q^{k-1} \cdot d^{-1/2 + o(1)} \cdot \sum_{c \mid d} s_c^{k-1} c^{1/2 - o(1)} D^\omega(c) \left( \frac{1}{c} + \frac{1}{s_q} \right) \]

Now,
\[ \sum_{c \mid d} s_c^{k-1} c^{1/2 - o(1)} D^\omega(c) \ll \sum_{c \mid d} c^{-1/2 + o(1)} \ll s_q^{o(1)} \]
and similarly

\[
\frac{1}{s_q} \sum_{c \mid d, c \ll s_q^{1/2}} s_q^{k-1} c^{1/2-o(1)} D^{o(c)} \ll \frac{1}{s_q} \sum_{c \mid d, c \ll s_q^{1/2}} c^{1/2-o(1)}
\]

Thus (20) is

\[
(22) \ll \frac{s_q^{k-1}}{s_q^{k-1}} \sum_{d \mid q, d > s_q^T} \left( \frac{s_q^{o(1)}}{d^{1/2-o(1)}} + \frac{1}{s_q d^{1/2-o(1)}} \sum_{c \mid d, c \ll s_q^{1/2}} c^{1/2-o(1)} \right) = s_q^{o(1)} \sum_{d \mid q, d > s_q^T} d^{-1/2+o(1)} + \frac{1}{s_q} \sum_{d \mid q, d > s_q^T} \frac{1}{d^{1/2-o(1)}} \sum_{c \mid d, c \ll s_q^{1/2}} c^{1/2-o(1)}
\]

Now, for any \( \beta \in (0, 1/2) \),

\[
\sum_{d \mid q, d > s_q^T} d^{-1/2+o(1)} \ll \sum_{d \mid q} d^{-1/2+o(1)} \left( \frac{d}{s_q^T} \right)^\beta \ll s_q^{-\beta T} \sum_{d \mid q} d^{\beta-1/2+o(1)} \ll s_q^{-\beta T+o(1)}.
\]

Similarly, for any \( \gamma > 0 \),

\[
\sum_{c \mid d, c \ll s_q^{1/2}} c^{1/2-o(1)} \ll \sum_{c \mid d, c \ll s_q^{1/2}} c^{1/2-o(1)} \ll \sum_{c \mid d, c \ll s_q^{1/2}} c^{1/2-o(1)} \ll s_q^{\gamma d_s(k)} |\tilde{R}| \sum_{c \mid d, c \ll s_q^{1/2}} c^{1/2-o(1)} \ll s_q^{\gamma d_s(k)} |\tilde{R}| d^{1/2-\gamma+o(1)}
\]

and thus

\[
\sum_{d \mid q, d > s_q^T} \frac{1}{d^{1/2-o(1)}} \sum_{c \mid d, c \ll s_q^{1/2}} c^{1/2+o(1)} \ll \sum_{d \mid q, d > s_q^T} c^{1/2+o(1)} \ll \sum_{d \mid q, d > s_q^T} c^{1/2+o(1)} \ll s_q^{\gamma d_s(k)} |\tilde{R}| + o(1)
\]

Hence (22) is

\[
\ll s_q^{-\beta T+o(1)} + s_q^{-1+\gamma d_s(k)} |\tilde{R}| + o(1) \ll s_q^{-1/2+o(1)}
\]

if we take \( T = 1-o(1), \beta = 1/(2T) - o(1), \) and \( \gamma = 1/(2d_s(k)|\tilde{R}|) \). Thus, with \( \alpha = 1/2 + o(1) \) (to bound the contribution from small \( d \)),
we find that
\[
\text{Error}_2 = \frac{1}{s_q^{k-1}} \sum_{d|q} \sum_{d \geq 1} \varepsilon_k(h, d) \ll s_q^{-1/2 + o(1)}
\]

\[\square\]

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