Semiclassical Strings on Curved Branes

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Abstract: We study semiclassical strings in the near horizon geometry of certain curved branes. We investigate the rigidly rotating strings in the near horizon geometry of NS5-branes wrapped on $AdS_3 \times S^3$ and in the presence of background NS-NS flux. We study several string solutions corresponding to giant magnon, single spike and more general folded strings for the fundamental string in this background. We comment that in the S-dual background the situation changes drastically.

Keywords: AdS-CFT correspondence, Bosonic Strings.
1. Introduction

According to AdS/CFT duality [1],[2],[3] quantum closed string states in AdS should be dual to quantum Super Yang-Mills (SYM) states on the boundary. More precisely, this duality implies the equality between the AdS energy $E$ of quantum closed string states (as function of effective string tension $T$ and other quantum numbers like the angular momenta $J_i$ on the sphere) and the dimension $\Delta$ of the corresponding local SYM operators. Though the state-operator matching is extremely difficult, but has been tractable in certain limits, such as the large angular momentum limit, on both sides of the duality [4],[5]. Further, it was observed that $\mathcal{N}=4$ SYM in planar limit can be described by an integrable spin chain model where the anomalous dimension of the gauge invariant operators were found in [6],[7],[8],[9],[10],[11],[12]. In the dual picture, it was noticed that the string theory is integrable in the semiclassical limit as well, see for example [13],[14],[15],[16],[17], hence providing further insight into the AdS/CFT duality. However apart from few ‘solvable’ examples of AdS/CFT, in many cases the exact nature of the boundary operators is not known, and hence it is interesting to make calculations in the gravity side and then look for possible operators on the boundary by invoking the duality map. The study of rigidly rotating strings in semiclassical approximation in the gravity side has been one of the interesting areas of research in the last few years. In this connection a large number of rotating and pulsating string solutions have been studied in $\text{AdS}_5 \times S^5$, $\text{AdS}_4 \times \text{CP}^3$, orbifolded and in the near horizon geometry of certain nonlocal string theory backgrounds, see for example, [18],[19],[20],[21],[22],[23],[24],[25],[26],[27],[28],[29],[30],[31],[32],[33],[34],[35],[36],[37],[38],[39],[40],[41],[42],[43],[44],[45],[46],[47],[48],...
However, more recently, the integrability of the classical string motion in curved p-brane background has been explored in \cite{50} in an attempt to see whether the full string equations of motion is integrable. It is shown that moving away from the throat geometry or in other words switching on the brane charges actually destroyed the string integrability. Though the point like string equations are in complete agreement with the integrability, the equations describing an extended string in the complete D-brane background, the integrable structure is lost. It is definitely interesting to look for string equations of motion in connection with integrability in various other situations.

Further to understand AdS/CFT like dualities in more general backgrounds, arising out of near horizon geometries of various branes in supergravities it is also interesting to look for classical string equations of motion in the gravity side and make statements about the integrability. It might help us in making some observations about the dual theory which is apriori less understood. Branes solutions with curved worldvolumes have widespread applications in string theory and black holes. In the past they have been used to identify the non-perturbative states of strings in lower dimensions in various string compactifications. The curved brane solutions can be constructed from the elementary solutions of the NS-NS sector which are associated with conformally invariant sigma model \cite{51,52}. Indeed a large class of solutions have been constructed in \cite{53} by using various dualities in string theory. We would like to study semiclassical strings in some of these backgrounds. Specifically we wish to study rigidly rotating strings in the near horizon geometry of stack of NS5-brane with $AdS_3 \times S^3$ worldvolume. We study the most general form of the string equation of motion and solve for the giant magnon and spiky like strings. We further study a few general pulsating strings in this background. Finally, we make some comments regarding F-string in the S-dual background, namely the nature of the solutions to the F-string equations of motion on a D5-brane wrapped on $AdS_3 \times S^3$. We remark that the possible non appearance of the giant magnon or spike like solution is perhaps due to the non integrability of the classical string equations of motion in the D5-brane background.

The rest of the paper is organized as follows. In section-2, we study rigidly rotating strings on NS5-brane wrapped on $AdS_3 \times S^3$ space. We find two limiting cases corresponding to giant magnon and single spike solutions for the string in this background. We present the regularized dispersion relations among various conserved charges corresponding to the string motion. Section-3 is devoted to the study of pulsating strings in this background. In section 4, we make some remarks about the string motion in the D5-brane wrapped on $AdS_3 \times S^3$ and conclude.

2. Rotating String on Curved NS5-branes

We start with the solutions presented in \cite{53} that correspond to intersecting $NS1-NS1'-NS5-NS5'$ branes in supergravity. The details of this background is given by the following form of the metric, 2-form Neveu-Schwarz (NS) field strength and dilaton \cite{53}

$$ds^2 = g_1^{-1}(x,y)(-dt^2 + dz^2) + H_5(x)dx^n dx^n + H_5'(y)dy^m dy^m, \quad (2.1)$$

$$dB = dg_1^{-1} \wedge dt \wedge dz + \star dH_5 + \star dH_5',$$
where
\[ e^{2\phi} = \frac{H_5(x)H_5(y)}{g_1(x, y)}, \]

and
\[ [H'_5(y)\partial^2_y + H_5(x)\partial^2_x]g_1(x, y) = 0. \tag{2.2} \]

A particular solution is given by
\[ g_1(x, y) = H_1(x)H'_1(y), \tag{2.3} \]

with \( H_{1,5} = 1 + \frac{Q_{1,5}}{y^2} \), and \( H'_1 = 1 + \frac{Q'_1}{y^2} \), where \( Q_{1,5} \) etc correspond to the charges of F1 and NS5-brane respectively. The above solution corresponds to the so called ”dyonic string” generalization of the \( 5_{NS} + 5_{NS} \) solution in supergravity.

To continue further let us choose \( H'_1 = 1 \), in which the solution is just a direct product of \( NS1 + NS5 \) configuration and another \( NS5- \) brane. Further, in the near horizon limit as \( x \to 0 \), the metric becomes,
\[ ds^2 = \frac{y^2}{Q_1}(-dt^2 + dz^2) + Q_1 \frac{dx^2}{x^2} + Q_1 d\Omega_3^2 + H'_1(y)(dy^2 + y^2 d\Omega_3^2) \tag{2.4} \]

where we have set for simplicity \( H_1 = H_5 \). The resulting sigma model describes a curved NS5-brane wrapped on \( AdS_3 \times S^3 \) and defines an exact CFT. We are interested in studying rigidly rotating string on this NS5-brane in the near horizon geometry. In the near horizon limit, \( y \to 0 \), we get the following form of the metric and NS-NS B-field.

\[ ds^2 = Q'_5(- \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\varphi^2 + d\Omega_3^2) + Q'_5(\frac{dy^2}{y^2} + d\Omega_3^2) \tag{2.5} \]

with
\[ d\Omega_3^2 = d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + \cos^2 \theta_1 d\psi_1^2, \]

and
\[ d\Omega_3^2 = d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2 + \cos^2 \theta_2 d\psi_2^2. \]

The above metric is further supported by a NS-NS two form field given by
\[ B = 2Q'_5 \sin^2 \theta_2 d\phi_2 \wedge d\psi_2. \]

Note that for convenience we have set \( Q_1 = Q'_5 \). This background is also associated with an appropriate form of the dilaton whose explicit form will not be needed here. To proceed further we make the following change of variables \( \chi = \ln y \). The final form of metric and background field now takes the form
\[ ds^2 = Q'_5(- \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\varphi^2 + d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + \cos^2 \theta_1 d\psi_1^2 \]
\[ + d\chi^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2 + \cos^2 \theta_2 d\psi_2^2), \quad B_{\phi_2 \psi_2} = 2Q'_5 \sin^2 \theta_2. \tag{2.6} \]

Note that we have used a different parameter to represent the \( AdS_3 \times S^3 \) subspace. We start by writing down the Polyakov action of the F-string in the above background,
\[ S = \frac{-\sqrt{\lambda}}{4\pi} \int d\sigma d\tau [\sqrt{-g} \alpha^M \gamma^{\alpha \beta} g_{\alpha \beta} \partial_M X^M \partial_N X^N - e^{\alpha \beta} \partial_\alpha X^M \partial_\beta X^N b_{MN}], \tag{2.7} \]
where the ’t Hooft coupling $\sqrt{\lambda} = Q_5$, $\gamma^{\alpha\beta}$ is the worldsheet metric and $e^{\alpha\beta}$ is the antisymmetric tensor defined as $e^{\tau\sigma} = -e^{\sigma\tau} = 1$. Under conformal gauge (i.e. $\sqrt{-\gamma^{\alpha\beta}} = \eta^{\alpha\beta}$) with $\eta^{\tau\tau} = -1$, $\eta^{\sigma\sigma} = 1$ and $\eta^{\tau\sigma} = \eta^{\sigma\tau} = 0$, the Polyakov action in the above background takes the form,

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[ -\cosh^2 \rho(t'^2 - t^2) + \rho'^2 - \dot{\rho}^2 + \sinh^2 \rho(\varphi'^2 - \varphi^2) + \theta_1'^2 - \dot{\theta}_1^2 \\
+ \sin^2 \theta_1(\phi_1'^2 - \dot{\phi}_1^2) + \cos^2 \theta_1(\psi_1'^2 - \dot{\psi}_1^2) + \chi^2 - \chi^2 + \theta_2'^2 - \dot{\theta}_2^2 + \sin^2 \theta_2(\phi_2'^2 - \dot{\phi}_2^2) \\
+ \cos^2 \theta_2(\psi_2'^2 - \dot{\psi}_2^2) - 4 \sin^2 \theta_2(\phi_2\psi_2 - \phi_2\psi_2') \right],$$

where ‘dots’ and ‘primed’ denote the derivative with respect to $\tau$ and $\sigma$ respectively. For studying the rigidly rotating strings we choose the following ansatz,

$$\rho = \rho(y), \quad t = \tau + h_1(y), \quad \varphi = \mu(\tau + h_2(y)), \quad \theta_1 = \theta_1(y), \quad \phi_1 = \nu_1(\tau + g_1(y)), \quad \psi_1 = \omega_1(\tau + f_1(y)), \quad \theta_2 = \theta_2(y), \quad \phi_2 = \nu_2(\tau + g_2(y)), \quad \psi_2 = \omega_2(\tau + f_2(y)), \quad \chi = \kappa \tau . \tag{2.9}$$

where $y = \sigma - \nu \tau$. Variation of the action with respect to $X^M$ gives us the following equation of motion

$$2\partial_\alpha(\eta^{\alpha\beta}\partial_\beta X^N g_{KN}) - \eta^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \partial_K g_{MN} - 2\partial_\alpha(e^{\alpha\beta}\partial_\beta X^N b_{KN}) + e^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \partial_K b_{MN} = 0 , \tag{2.10}$$

and variation with respect to the metric gives the two Virasoro constraints,

$$g_{MN}(\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N) = 0 , \quad g_{MN}(\partial_\tau X^M \partial_\sigma X^N) = 0 . \tag{2.11}$$

Next we have to solve these equations by the ansatz we have proposed above in eqn. (2.9). Solving for $t, \varphi$ we get,

$$\frac{\partial h_1}{\partial y} = \frac{1}{1 - v^2} \left[ \frac{c_1}{\cosh^2 \rho} - v \right], \quad \frac{\partial h_2}{\partial y} = \frac{1}{1 - v^2} \left[ \frac{c_2}{\sinh^2 \rho} - v \right] . \tag{2.12}$$

Substituting these for $\rho$ equation we get,

$$(1 - v^2)\frac{\partial^2 \rho}{\partial y^2} = \sinh \rho \cosh \rho \left[ (1 - \frac{c_1^2}{\cosh^4 \rho}) - \mu^2 (1 - \frac{c_2^2}{\sinh^4 \rho}) \right] ;$$

$$(1 - v^2) \left( \frac{\partial \rho}{\partial y} \right)^2 = (1 - \mu^2) \sinh^2 \rho + \frac{c_1^2}{\cosh^2 \rho} - \frac{\mu^2 c_2^2}{\sinh^2 \rho} + c_3 , \tag{2.13}$$

where $c_1$, $c_2$ and $c_3$ are integration constants as well. Similarly solving for $\phi_1$ and $\psi_1$ equations we get,

$$\frac{\partial g_1}{\partial y} = \frac{1}{1 - v^2} \left[ \frac{c_4}{\sin^2 \theta_1} - v \right], \quad \frac{\partial f_1}{\partial y} = \frac{1}{1 - v^2} \left[ \frac{c_5}{\cos^2 \theta_1} - v \right] . \tag{2.14}$$
Substituting these for \( \theta_1 \) equation we get,

\[
(1 - v^2)^2 \left( \frac{\partial \theta_1}{\partial y} \right)^2 = (\omega_1^2 - \nu_1^2) \sin^2 \theta_1 - \frac{\nu_1^2 c_4}{\sin^2 \theta_1} - \frac{\omega_1^2 c_5^2}{\cos^2 \theta_1} + c_6,
\]

where \( c_4, c_5 \) and \( c_6 \) are integration constants. Again solving for \( \phi_2 \) and \( \psi_2 \) equations we get,

\[
\frac{\partial g_2}{\partial y} = \frac{1}{1 - v^2} \left( \frac{c_7 c - 2 \omega_2}{\nu_2} - v \right), \quad \frac{\partial f_2}{\partial y} = \frac{1}{1 - v^2} \left( \frac{c_8}{\omega_2} - \frac{2 \nu_2}{\omega_2} - v \right).
\]

Substituting these in \( \theta_2 \) equation we get,

\[
(1 - v^2)^2 \left( \frac{\partial \theta_2}{\partial y} \right)^2 = 3(\nu_2^2 - \omega_2^2) \sin^2 \theta_2 - \frac{c_7^2}{\sin^2 \theta_2} - \frac{c_8^2}{\cos^2 \theta_2} + c_9,
\]

where \( c_7, c_8 \) and \( c_9 \) are integration constants as well. Now the Virasoro constraint \( T_{\tau \sigma} = 0 \) gives

\[
(1 - v^2)^2 \left[ \left( \frac{\partial \rho}{\partial y} \right)^2 + \left( \frac{\partial \theta_1}{\partial y} \right)^2 + \left( \frac{\partial \theta_2}{\partial y} \right)^2 \right] = \cosh^2 \rho - \mu^2 \sinh^2 \rho - \nu_1^2 \sin^2 \theta_1 - \omega_1^2 \cos^2 \theta_1 - \nu_2^2 - \omega_2^2 - 3 \nu_2^2 \cos^2 \theta_2 - 3 \omega_2^2 \sin^2 \theta_2 \\
+ \frac{c_1^2}{\cosh^2 \rho} - \mu^2 c_2^2 - \frac{\nu_1^2 c_4}{\sin^2 \theta_1} - \frac{\omega_1^2 c_5^2}{\cos^2 \theta_1} - \frac{c_7^2}{\sin^2 \theta_2} - \frac{c_8^2}{\cos^2 \theta_2} + 4(\omega_2 c_7 + \nu_2 c_8) \\
+ \frac{1 + v^2}{\nu_2} (-c_1 + \mu^2 c_2 + \nu_1^2 c_4 + \omega_1^2 c_5 + \nu_2 c_7 + \omega_2 c_8 - 2 \nu_2 \omega_2) \tag{2.17}
\]

Further, the Virasoro constraint \( T_{\tau \tau} + T_{\sigma \sigma} = 0 \) gives

\[
(1 - v^2)^2 \left[ \left( \frac{\partial \rho}{\partial y} \right)^2 + \left( \frac{\partial \theta_1}{\partial y} \right)^2 + \left( \frac{\partial \theta_2}{\partial y} \right)^2 \right] = \cosh^2 \rho - \mu^2 \sinh^2 \rho - \nu_1^2 \sin^2 \theta_1 - \omega_1^2 \cos^2 \theta_1 - \nu_2^2 - \omega_2^2 - 3 \nu_2^2 \cos^2 \theta_2 - 3 \omega_2^2 \sin^2 \theta_2 \\
+ \frac{c_1^2}{\cosh^2 \rho} - \mu^2 c_2^2 - \frac{\nu_1^2 c_4}{\sin^2 \theta_1} - \frac{\omega_1^2 c_5^2}{\cos^2 \theta_1} - \frac{c_7^2}{\sin^2 \theta_2} - \frac{c_8^2}{\cos^2 \theta_2} + 4(\omega_2 c_7 + \nu_2 c_8) \\
+ \frac{4v}{1 + v^2} (-c_1 + \mu^2 c_2 + \nu_1^2 c_4 + \omega_1^2 c_5 + \nu_2 c_7 + \omega_2 c_8 - 2 \nu_2 \omega_2) - \frac{(1 - v^2)^2 \kappa^2}{1 + v^2}. \tag{2.18}
\]

Subtracting the above two equations we get the following relation among various parameters,

\[-c_1 + \mu^2 c_2 + \nu_1^2 c_4 + \omega_1^2 c_5 + \nu_2 c_7 + \omega_2 c_8 - 2 \nu_2 \omega_2 + \kappa^2 v = 0. \tag{2.19}\]

In what follows we will look at the two limiting cases corresponding to giant magnon and single spike solutions for the string in the curved NS5-brane near horizon background.

2.1 Limiting cases

Recall, we have from (2.15)

\[
\left( \frac{\partial \theta_1}{\partial y} \right)^2 = \frac{1}{(1 - v^2)^2} \left[ (\omega_1^2 - \nu_1^2) \sin^2 \theta_1 - \frac{\nu_1^2 c_4}{\sin^2 \theta_1} - \frac{\omega_1^2 c_5^2}{\cos^2 \theta_1} + c_6 \right].
\]
\( \frac{\partial \theta}{\partial y} \to 0 \) as \( \theta_1 \to \frac{\pi}{2} \) implies \( c_5 = 0 \) and \( c_6 = \nu_1^2 c_4^2 + \nu_1^2 - \omega_1^2 \), substituting this in the above equation we get,

\[
\frac{\partial \theta_1}{\partial y} = \frac{\sqrt{\nu_1^2 - \omega_1^2}}{1 - v_1^2} \cot \theta_1 \sqrt{\sin^2 \theta_1 - \alpha_1^2} ,
\]

where \( \alpha_1^2 = \frac{\nu_1^2 c_2^2}{\nu_1^2 - \omega_1^2} \). Further, we have from (2.16)

\[
\left( \frac{\partial \theta_2}{\partial y} \right)^2 = \frac{1}{(1 - v_2^2)^2} \left[ 3(\nu_2^2 - \omega_2^2) \sin^2 \theta_2 - \frac{c_2^2}{\sin^2 \theta_2} - \frac{c_8^2}{\cos^2 \theta_2} + c_9 \right] .
\]

Similarly, \( \frac{\partial \alpha}{\partial y} \to 0 \) as \( \alpha_2 \to \frac{\pi}{2} \) implies \( c_8 = 0 \) and \( c_9 = 3(\omega_2^2 - \nu_2^2) + c_7^2 \). Substituting this in the above equation we get

\[
\frac{\partial \theta_2}{\partial y} = \frac{\sqrt{3(\omega_2^2 - \nu_2^2)}}{1 - v_2^2} \cot \theta_2 \sqrt{\sin^2 \theta_2 - \alpha_2^2} ,
\]

where \( \alpha_2^2 = \frac{c_2^2}{3(\omega_2^2 - \nu_2^2)} \). Substituting the values of \( \frac{\partial \theta_0}{\partial y} \) and \( \frac{\partial \theta_2}{\partial y} \) with \( c_5 = c_8 = 0 \) in first Virasoro constraint (2.17) we get,

\[
(1 - v^2)^2 \left( \frac{\partial \rho}{\partial y} \right)^2 = 1 + (1 - \mu^2) \sinh^2 \rho + \frac{c_1^2}{\cosh^2 \rho} - \frac{\mu^2 c_2^2}{\sinh^2 \rho} - \alpha_3^2 ,
\]

where

\[
\alpha_3^2 = \nu_1^2 (1 + c_4^2) + \frac{1}{\nu_2^2} \left\{ c_1 - \mu^2 c_2 - \nu_1^2 c_4 - \kappa^2 v \right\}^2 - \left\{ \kappa^2 (1 + v^2) + \kappa^2 v (1 + \frac{4\omega_2}{\nu_2}) + \nu_2^2 - 2\nu_2 \omega_2 \right\} \]

(2.23)

In limit \( \frac{\partial \phi}{\partial y} \to 0 \) as \( \rho \to 0 \) implies \( c_2 = 0 \) and \( c_1 = \alpha_3^2 - 1 \). Hence

\[
\frac{\partial \rho}{\partial y} = \frac{1 - \mu^2}{1 - v^2} \tanh \rho \sqrt{\cosh^2 \rho + \alpha_1^2} ,
\]

(2.24)

where \( \alpha_1^2 = \frac{1 - \alpha_3^2}{1 - \mu^2} \). Looking at the symmetry of the background of the near horizon of NS5-branes, a number of conserved charges can be constructed as follows

\[
E = - \int \frac{\partial L}{\partial t} d\sigma = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{1 - v^2} \int (\cosh^2 \rho - c_1 v) d\sigma ,
\]

\[
S = \int \frac{\partial L}{\partial \phi} d\sigma = \frac{\sqrt{\lambda}}{2\pi} \frac{\mu}{1 - v^2} \int \sinh^2 \rho d\sigma ,
\]

\[
J_1 = \int \frac{\partial L}{\partial \phi_1} d\sigma = \frac{\sqrt{\lambda}}{2\pi} \frac{\nu_1}{1 - v^2} \int (\sin^2 \theta_1 - c_4 v) d\sigma ,
\]

\[
J_2 = \int \frac{\partial L}{\partial \psi_1} d\sigma = \frac{\sqrt{\lambda}}{2\pi} \frac{\omega_1}{1 - v^2} \int \cos^2 \theta_1 d\sigma ,
\]

\[
K_1 = \int \frac{\partial L}{\partial \phi_2} d\sigma = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{1 - v^2} \int (3\nu_2 \cos^2 \theta_2 - 3\nu_2 - c_7 v) d\sigma ,
\]
\[ K_2 = \int \frac{\partial L}{\partial \dot{\psi}_2} d\sigma = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{1 - v^2} \int (4\omega_2 + 2\nu_2 v - 2c_7 - 3\omega_2 \cos^2 \theta_2) d\sigma , \]

\[ P = \int \frac{\partial L}{\partial \dot{\chi}} d\sigma = \frac{\sqrt{\lambda}}{2\pi} \kappa \int d\sigma . \]  

(2.25)

Also we have the following relation among various integration constants

\[ c_7 = \frac{1}{\nu_2} [c_1 - \nu_1^2 c_4 - \kappa^2 v + 2\nu_2 \omega_2] \]  

(2.26)

2.2 Single spike

Let us look at various solutions to the string equations of motion derived in the last section with appropriate choice of integration constant. First, we choose \( c_1 = c_4 = v \). Now the conserved quantities become,

\[ E = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{1 - v^2} \int (\cosh^2 \rho - v^2) d\sigma, \]  

(2.27)

\[ S = \frac{\sqrt{\lambda}}{2\pi} \frac{\mu}{1 - v^2} \int \sinh^2 \rho d\sigma, \]

\[ J_1 = \frac{\sqrt{\lambda}}{2\pi} \frac{\nu_1}{1 - v^2} \int (\sin^2 \theta_1 - v^2) d\sigma, \]

\[ J_2 = \frac{\sqrt{\lambda}}{2\pi} \frac{\omega_1}{1 - v^2} \int \cos^2 \theta_1 d\sigma, \]

\[ K_1 = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{1 - v^2} \int (3\nu_2 \cos^2 \theta_2 - 3\nu_2 - c_7 v) d\sigma, \]

\[ K_2 = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{1 - v^2} \int (4\omega_2 + 2\nu_2 v - 2c_7 - 3\omega_2 \cos^2 \theta_2) d\sigma, \]

\[ P = \frac{\sqrt{\lambda}}{2\pi} \kappa \int d\sigma . \]

Also the relation among the integration constants now becomes

\[ c_7 = \frac{1}{\nu_2} [v(1 - \nu_1^2 - \kappa^2) + 2\nu_2 \omega_2] . \]  

(2.28)

It is straightforward to see that the among various conserved charges we get the following relations,

\[ E - \frac{S}{\mu} = \frac{J_1}{\nu_1} + \frac{J_2}{\omega_1} \]  

(2.29)

and

\[ \frac{K_1}{\nu_2} + \frac{K_2}{\omega_2} = \frac{1}{1 - v^2} \kappa \left[ \frac{\omega_2 + 2\nu_2 v}{\omega_2} - \frac{(\omega_2 v + 2\nu_2)(2\nu_2 \omega_2 + v(1 - \nu_1^2 - \kappa^2))}{\nu_2^2 \omega_2} \right] P . \]

(2.30)
To find the explicit relation among various conserved charges which looks like the spiky string, we now write the explicit expression of the conserved charges. Now

\[ J_1 = \frac{\sqrt{\lambda}}{\pi} \frac{\nu_1}{\sqrt{\nu_1^2 - \omega_1^2}} \left[ (1 - v^2) \int_0^{\pi} \sin \theta_1 d\theta_1 \cos \theta_1 \sqrt{\sin^2 \theta_1 - \alpha_1^2} \right]^{\arcsin(\alpha_1)} - \int_0^{\pi} \sin \theta_1 \cos \theta_1 d\theta_1 \sqrt{\sin^2 \theta_1 - \alpha_1^2}. \]

(2.31)

\( J_1 \) diverges, but on regularization we get,

\[ (J_1)_{reg} = \frac{\sqrt{\lambda}}{\pi} \frac{\nu_1}{\sqrt{\nu_1^2 - \omega_1^2}} \sqrt{1 - \alpha_1^2}. \]

(2.32)

On the other hand \( J_2 \) is finite and is written as

\[ J_2 = -\frac{\sqrt{\lambda}}{\pi} \frac{\omega_1}{\sqrt{\nu_1^2 - \omega_1^2}} \sqrt{1 - \alpha_1^2}. \]

(2.33)

Similarly, \( K_1 \) and \( K_2 \) both diverge, however the regularized expressions are given by

\[ (K_1)_{reg} = -\frac{\sqrt{\lambda}}{\pi} \frac{3\nu_2}{\sqrt{3(\omega_2^2 - \nu_2^2)}} \sqrt{1 - \alpha_2^2}, \]

(2.34)

and

\[ (K_2)_{reg} = \frac{\sqrt{\lambda}}{\pi} \frac{3\omega_2}{\sqrt{3(\omega_2^2 - \nu_2^2)}} \sqrt{1 - \alpha_2^2}. \]

(2.35)

Now the angle difference between the end points of the string is given by

\[ \Delta \phi_1 = \nu_1 \int_{-\infty}^{\infty} dy \frac{\partial g_1}{\partial y} = -2 \arccos(\alpha_1), \]

(2.36)

which implies \( \alpha_1 = \cos \frac{\Delta \phi_1}{2} \). However, \( \Delta \phi_2 = \nu_2 \int_{-\infty}^{\infty} dy \frac{\partial g_2}{\partial y} \) diverge, but the regularized expression is given by

\[ (\Delta \phi_2)_{reg} = -2 \arccos(\alpha_2), \]

(2.37)

which implies \( \alpha_2 = \cos \frac{(\Delta \phi_2)_{reg}}{2} \). In terms of \( \Delta \phi_1 \) and \( (\Delta \phi_2)_{reg} \) we can express,

\[ (J_1)_{reg} = \frac{\sqrt{\lambda}}{\pi} \frac{\nu_1}{\sqrt{\nu_1^2 - \omega_1^2}} \sin \frac{\Delta \phi_1}{2}, \quad J_2 = -\frac{\sqrt{\lambda}}{\pi} \frac{\omega_1}{\sqrt{\nu_1^2 - \omega_1^2}} \sin \frac{\Delta \phi_1}{2}, \]

(2.38)

and they satisfy the relation,

\[ (J_1)_{reg} = \sqrt{J_2^2 + \left( \frac{\lambda}{\pi^2} \right) \sin^2 \frac{\Delta \phi_1}{2}}. \]

(2.39)

This relation looks precisely like the single spike dispersion relation with two spins on \( R \times S^3 \) [33]. Now,

\[ (K_1)_{reg} = -\frac{\sqrt{\lambda}}{\pi} \frac{3\nu_2}{\sqrt{3(\omega_2^2 - \nu_2^2)}} \sin \frac{(\Delta \phi_2)_{reg}}{2}, \quad (K_2)_{reg} = \frac{\sqrt{\lambda}}{\pi} \frac{3\omega_2}{\sqrt{3(\omega_2^2 - \nu_2^2)}} \sin \frac{(\Delta \phi_2)_{reg}}{2}, \]

(2.40)
and they satisfy the relation,

\[
(K_2)_{\text{reg}} = \sqrt{(K_1^2)_{\text{reg}} + 3 \left( \frac{\lambda}{\pi^2} \right) \sin^2 \left( \frac{\Delta \phi_2}_{\text{reg}} \right) . }
\]  

(2.41)

We wish to mention that due to the presence of the background $B$-field in the metric which is essentially the volume form of the three sphere in the transverse space, we get a factor of 3 in the dispersion relation in (2.41) as compared to (2.39). We also have energy $E$ and spin $S$ of AdS space as conserved quantities, which are diverging. However the regularized expressions are given by

\[
E_{\text{reg}} = \left( \frac{S}{\mu} \right)_{\text{reg}} = -\frac{\sqrt{\lambda}}{\pi} \frac{\sqrt{1 + \alpha^2}}{\sqrt{1 - \mu^2}} .
\]  

(2.42)

So they satisfy

\[
E_{\text{reg}} - \left( \frac{S}{\mu} \right)_{\text{reg}} = 0 .
\]  

(2.43)

The regularized spin can be rewritten as,

\[
\frac{S_{\text{reg}}}{\mu} = \sqrt{S_{\text{reg}}^2 + \frac{\lambda}{\pi^2} (1 + \alpha^2)} .
\]  

(2.44)

The time difference $\Delta t$ between the end point of the string can be defined as,

\[
\Delta t = \int_{-\infty}^{\infty} \frac{\partial h_1}{\partial y} dy = -\frac{2v}{\sqrt{1 - \mu^2}} \left\{ \int_0^\infty \frac{\sinh \rho d\rho}{\cosh \rho \sqrt{\cosh^2 \rho + \alpha^2}} \right\} ,
\]  

(2.45)

which is finite and is given by,

\[
(\Delta t) = -\frac{2v}{\sqrt{\alpha^2^2} - 1} \arcsin \left( \frac{\sqrt{\alpha^2^2 - 1}}{\sqrt{1 - \mu^2}} \right) ,
\]  

(2.46)

which implies

\[
\frac{\sqrt{\alpha^2^2 - 1}}{\sqrt{1 - \mu^2}} = -\sin \left( \frac{\Delta t \sqrt{\alpha^2^2 - 1}}{2v} \right) .
\]

In terms of $\Delta t$ we can express,

\[
E_{\text{reg}} = \frac{S_{\text{reg}}}{\mu} = \sqrt{S_{\text{reg}}^2 + \left( \frac{\lambda}{\pi^2} \right) \cos^2 \left( \frac{\Delta t \sqrt{\alpha^2^2 - 1}}{2v} \right) .}
\]  

(2.47)
2.3 Magnon Case

In this case, let us choose \( c_1 = c_4 = \frac{1}{v} \). Then the conserved quantities become,

\[
E = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{1 - v^2} \int \sinh^2 \rho d\sigma ,
\]

\[
S = \frac{\sqrt{\lambda}}{2\pi} \frac{\mu}{1 - v^2} \int \sinh^2 \rho d\sigma ,
\]

\[
J_1 = -\frac{\sqrt{\lambda}}{2\pi} \frac{\nu_1}{1 - v^2} \int \cos^2 \theta_1 d\sigma ,
\]

\[
J_2 = \frac{\sqrt{\lambda}}{2\pi} \frac{\omega_1}{1 - v^2} \int \cos^2 \theta_1 d\sigma ,
\]

\[
K_1 = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{1 - v^2} \int (3\nu_2 \cos^2 \theta_2 - 3\nu_2 - c_7 v) d\sigma ,
\]

\[
K_2 = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{1 - v^2} \int (4\omega_2 + 2\nu_2 v - 2c_7 - 3\omega_2 \cos^2 \theta_2) d\sigma ,
\]

\[
P = \frac{\sqrt{\lambda}}{2\pi} \frac{\kappa}{1 - v^2} \int d\sigma .
\]

\[ (2.48) \]

Also the relation among the integration constants now become

\[
c_7 = \frac{1}{\nu_2} \left[ \frac{1}{v} (1 - \nu_1^2) - \kappa^2 v + 2\nu_2 \omega_2 \right] .
\]

\[ (2.49) \]

Among the conserved quantities we get the following relations,

\[
E - S = \frac{J_1}{\nu_1} + \frac{J_2}{\omega_1} = 0 ,
\]

\[ (2.50) \]

and

\[
\frac{K_1}{\nu_2} + \frac{K_2}{\omega_2} = \frac{1}{1 - v^2} \frac{1}{\nu_2} \left[ \frac{\omega_2 + 2\nu_2 v}{\omega_2} - \frac{(\omega_2 v + 2\nu_2)(2\nu_2 \omega_2 - \kappa^2 v + \frac{1}{v} (1 - \nu_1^2))}{\nu_2^2 \omega_2} \right] P .
\]

\[ (2.51) \]

The explicit expression of spin \( S \) associated with \( \text{AdS} \) is diverging, but the regularized form is,

\[
\frac{S_{\text{reg}}}{\mu} = -\frac{\sqrt{\lambda} \sqrt{1 + \alpha_1^2}}{\pi} \frac{1}{\sqrt{1 - \mu^2}}
\]

\[ (2.52) \]

This can be rewritten as,

\[
\frac{S_{\text{reg}}}{\mu} = \sqrt{\left( S_{\text{reg}}^2 + \frac{\lambda}{\pi^2} (1 + \alpha_1^2) \right)}
\]

\[ (2.53) \]

The time difference \( \Delta t \) between the end point of the string can be defined as,

\[
\Delta t = \int_{-\infty}^{\infty} \frac{\partial h_1}{\partial y} dy
\]

\[
= \frac{2}{\sqrt{1 - \mu^2}} \left[ \left( \frac{1}{v} - v \right) \int_0^{\infty} \frac{\cosh \rho d\rho}{\sinh \rho \sqrt{\cosh^2 \rho + \alpha_1^2}} - \frac{1}{v} \int_0^{\infty} \frac{\sinh \rho d\rho}{\cosh \rho \sqrt{\cosh^2 \rho + \alpha_1^2}} \right],
\]

\[ (2.54) \]
which diverges, however the regularized $\langle \Delta t \rangle_{\text{reg}}$ is,

$$\langle \Delta t \rangle_{\text{reg}} = -\frac{2}{v\sqrt{\alpha_3^2 - 1}} \arcsin \left( \sqrt{\alpha_3^2 - 1} \right),$$

(2.55)

which implies

$$\frac{\sqrt{\alpha_3^2 - 1}}{\sqrt{1 - \mu^2}} = -\sin \left( \frac{\Delta t_{\text{reg}} v \sqrt{\alpha_3^2 - 1}}{2} \right).$$

In terms of $\langle \Delta t \rangle_{\text{reg}}$ we can express,

$$\frac{S_{\text{reg}}}{\mu} = \sqrt{S_{\text{reg}}^2 + \left( \frac{\lambda}{\pi^2} \right) \cos^2 \left( \frac{\Delta t_{\text{reg}} v \sqrt{\alpha_3^2 - 1}}{2} \right)}.$$

(2.56)

Again the angle difference $\Delta \phi_1$ is defined as,

$$\Delta \phi_1 = \nu_1 \int_{-\infty}^{\infty} \frac{\partial g_1}{\partial y} dy$$

$$= \frac{2\nu_1}{\sqrt{\nu_1^2 - \omega_1^2}} \left[ \frac{1}{v} \int_{\arcsin(\alpha_1)}^{\infty} \frac{\cos \theta_1 d\theta_1}{\sin \theta_1 \sqrt{\sin^2 \theta_1 - \alpha_1^2}} + \left( \frac{1}{v} - v \right) \int_{\arcsin(\alpha_1)}^{\infty} \frac{\sin \theta_1 d\theta_1}{\cos \theta_1 \sqrt{\sin^2 \theta_1 - \alpha_1^2}} \right],$$

(2.57)

diverges. After excluding the divergence part, we get the regularized $\Delta \phi_1$,

$$\langle \Delta \phi_1 \rangle_{\text{reg}} = -2 \arcsin(\alpha_1),$$

(2.58)

which implies $\alpha_1 = -\sin \frac{\langle \Delta \phi_1 \rangle_{\text{reg}}}{2}$. The angular momentum $J_2$ is given by,

$$J_2 = -\frac{\sqrt{\lambda}}{\pi} \frac{\omega_1}{\sqrt{\nu_1^2 - \omega_1^2}} \cos \frac{\langle \Delta \phi_1 \rangle_{\text{reg}}}{2},$$

(2.59)

which can be rewritten as,

$$\frac{J_2 \nu_1}{\omega_1} = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2 \frac{\langle \Delta \phi_1 \rangle_{\text{reg}}}{2}} = \frac{\langle J_2 \rangle_{\text{reg}}}{\omega_1}.$$ 

(2.60)

Therefore we can write the giant magnon dispersion relation as,

$$\left( E - J_1 \right)_{\text{reg}} = \frac{S_{\text{reg}}}{\mu} + \frac{\langle J_2 \rangle_{\text{reg}}}{\omega_1}$$

$$= \sqrt{S_{\text{reg}}^2 + \frac{\lambda}{\pi^2} \cos^2 \left( \frac{\Delta t_{\text{reg}} v \sqrt{\alpha_3^2 - 1}}{2} \right)} + \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2 \frac{\langle \Delta \phi_1 \rangle_{\text{reg}}}{2}}.$$ 

(2.61)

One may like to find further relations among the other charges such as $K_1$, $K_2$ and $\Delta \phi_2$ which are confined to the transverse space of the NS-brane.
3. Folded String

In this section we wish to study some string solutions which are pulsating in the background of the near horizon geometry of the curved NS5-branes and also contain some extra angular momentum. To study folded strings on this background we choose the following ansatz,

\[
\rho(\sigma) = \rho(\sigma + 2\pi), \quad t = \kappa \tau, \quad \varphi = \mu_1 \tau, \quad \chi = \mu_2 \tau, \quad (3.1)
\]

\[
\theta_1(\sigma) = \theta_1(\sigma + 2\pi), \quad \phi_1(\sigma) = \phi_1(\sigma + 2\pi), \quad \psi_1 = \omega_1 \tau, \quad (3.1a)
\]

\[
\theta_2(\sigma) = \theta_2(\sigma + 2\pi), \quad \phi_2(\sigma) = \phi_2(\sigma + 2\pi), \quad \psi_2 = \omega_2 \tau .
\]

The Polyakov action of the string, in the conformal gauge and with these ansatz, becomes,

\[
S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[ \cosh^2 \rho \dot{\rho}^2 + \rho'^2 - \sinh^2 \rho \dot{\varphi}^2 + \sin^2 \theta_1 \dot{\phi}_1^2 - \cos^2 \theta_1 \dot{\psi}_1^2 - \dot{\chi}^2 + \theta_2'^2 + \sin^2 \theta_2 \phi_2'^2 - \cos^2 \theta_2 \psi_2^2 + 4 \sin^2 \theta_2 \dot{\phi}_2 \dot{\psi}_2 \right]. \quad (3.2)
\]

Solving for \( \rho \), \( \theta_1 \) and \( \theta_2 \) equations we get,

\[
\rho'' = \sinh \rho \cosh \rho (\kappa^2 - \mu_1^2) ,
\]

\[
\theta_1'' = \sin \theta_1 \cos \theta_1 (\phi_1'^2 + \omega_1^2) ,
\]

\[
\theta_2'' = \sin \theta_2 \cos \theta_2 (\phi_2'^2 + \omega_2^2 + 4\omega_2 \phi_2) .
\]

(3.3)

Again solving for \( \phi_1 \) and \( \phi_2 \) equations, we get,

\[
\frac{d}{d\sigma} (\phi_1' \sin^2 \theta_1) = 0 , \quad \frac{d}{d\sigma} (\phi_2' \sin^2 \theta_2) + 4\omega_2 \sin \theta_2 \cos \theta_2 \frac{d\theta_2}{d\sigma} = 0 . \quad (3.4)
\]

Integrating these two, we get

\[
\phi_1' = \frac{c_1}{\sin^2 \theta_1} , \quad \phi_2' = \frac{c_2}{\sin^2 \theta_2} - 2\omega_2 ,
\]

(3.5)

where \( c_1 \) and \( c_2 \) are integration constants. Substituting the values of \( \phi_1' \) and \( \phi_2' \) in (3.3) equations of motion and integrating them we get,

\[
\rho'^2 = (\kappa^2 - \mu_1^2) \sinh^2 \rho + c_3 ,
\]

\[
\theta_1'^2 = -\frac{c_1^2}{\sin^2 \theta_1} + \omega_1^2 \sin^2 \theta_1 + c_4 ,
\]

\[
\theta_2'^2 = -\frac{c_2^2}{\sin^2 \theta_2} - 3\omega_2^2 \sin^2 \theta_1 + c_5 ,
\]

(3.6)

where \( c_3, c_4 \) and \( c_5 \) are integration constants as well. Now from the Virasoro constraints, we get the following relation among various integration constants,

\[
-\kappa^2 - \mu_2^2 + \omega_1^2 - \omega_2^2 + 4\omega_2 \omega_2 + c_3 + c_4 - c_5 = 0 . \quad (3.7)
\]
The conserved quantities in these case are given by,

\[
E = \frac{\sqrt{\lambda \kappa}}{2\pi} \int_{0}^{2\pi} d\sigma \cosh^2 \rho ,
\]
\[
S = \frac{\sqrt{\lambda \mu_1}}{2\pi} \int_{0}^{2\pi} d\sigma \sinh^2 \rho
\]
\[
J = \frac{\sqrt{\lambda \omega_1}}{2\pi} \int_{0}^{2\pi} d\sigma \cos^2 \theta_1 ,
\]
\[
K = \frac{\sqrt{\lambda}}{2\pi} \int_{0}^{2\pi} d\sigma (4\omega_2 - 3\omega_2 \cos^2 \theta_2 - 2c_2) ,
\]
\[
P = \sqrt{\lambda \mu_2} .
\]

(3.8)

We can choose \(c_1 = c_2 = 0\), so that we can express our result in terms of elliptic functions as is the usual practice. In what follows, we wish to study few subset of pulsating solutions.

**3.1 For \(\theta_1 = \theta_2 = 0\)**

In this section we wish to studying the string which pulsates in the AdS\(_3\) subspace and which contains extra charges due to the transverse motion of the string along the radial direction. We define the energy and spin density as

\[
\mathcal{E} = \frac{E}{\sqrt{\lambda}} = \frac{\kappa}{2\pi} \int_{0}^{2\pi} d\sigma \cosh^2 \rho ,
\]
\[
\mathcal{S} = \frac{S}{\sqrt{\lambda}} = \frac{\mu_1}{2\pi} \int_{0}^{2\pi} d\sigma \sinh^2 \rho .
\]

(3.9)

Hence they satisfy the relation

\[
\frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}}{\mu_1} = 1
\]

or,

\[
\mathcal{E} = \kappa + \frac{\kappa}{\mu_1} \mathcal{S}
\]

(3.10)

Also we can define,

\[
\mathcal{J} = \frac{J}{\sqrt{\lambda}} = \omega_1 , \quad \mathcal{K} = \frac{K}{\sqrt{\lambda}} = \omega_2 , \quad \mathcal{P} = \frac{P}{\sqrt{\lambda}} = \mu_2 .
\]

(3.11)

Now we have,

\[
\rho' = \frac{d\rho}{d\sigma} = \sqrt{c_3 + (\kappa^2 - \mu_1^2) \sinh^2 \rho}
\]
\[
\int_{0}^{2\pi} d\sigma = 4 \int_{0}^{\rho_0} \frac{d\rho}{\sqrt{c_3 + (\kappa^2 - \mu_1^2) \sinh^2 \rho}}
\]

(3.12)

where \(\rho_0\) corresponds to maximum value \(\sinh \rho_0\). Solving this we get,

\[
\sqrt{\mu_1^2 - \kappa^2} = \frac{2}{\pi} K(q) ,
\]

(3.13)
where $K(q)$ is the elliptic function of first kind with argument $q = \sqrt{\frac{c_3}{\kappa^2 - \mu_1^2}}$. Also we have,

$$\mathcal{E} = \frac{\kappa}{2\pi} \int_0^{2\pi} d\sigma \cosh^2 \rho = \frac{4\kappa}{2\pi} \int_0^{\rho_0} \frac{d\rho \cosh^2 \rho}{\sqrt{c_3 + (\kappa^2 - \mu_1^2) \sinh^2 \rho}}.$$  \hspace{1cm} (3.14)

Solving this integration we get,

$$\mathcal{E} = \frac{2\kappa}{\pi} \frac{E(q)}{\sqrt{\mu_1^2 - \kappa^2}},$$  \hspace{1cm} (3.15)

where $E(q)$ is the elliptic function of second kind. Combining the two equations (3.13) and (3.15) we get

$$\kappa^2 = \left( \frac{K(q)}{E(q)} \mathcal{E} \right)^2,$$

$$\mu_1^2 = \left( \frac{K(q)}{E(q)} \mathcal{E} \right)^2 + \frac{4}{\pi^2} (K(q))^2.$$  \hspace{1cm} (3.16)

Similar solutions were found in [54].

### 3.2 For $\rho = \theta_2 = 0$

In this section we wish to study strings that pulsate in one of the $S^3$ and at the same time have extra charges due to the transverse motion of the string in the radial direction of the NS5-brane. Note that the extra charges appear because of the translational symmetry along the $\xi$ direction of the original background. For this case we have,

$$\mathcal{E} = \kappa, \mathcal{S} = 0, \mathcal{K} = \omega_2, \mathcal{P} = \mu_2.$$  \hspace{1cm} (3.17)

We also have,

$$\frac{d\theta_1}{d\sigma} = \sqrt{c_4 + \omega_1^2 \sin^2 \theta_1},$$  \hspace{1cm} (3.18)

which implies

$$\omega_1 = -\frac{2}{\pi} K(r),$$  \hspace{1cm} (3.19)

where $K(r)$ is the elliptic function of first kind with the argument $r = \sqrt{\frac{c_4}{\omega_1}}$ and

$$\mathcal{J} = \frac{\omega_1}{2\pi} \int_0^{2\pi} d\sigma \cos^2 \theta_1,$$  \hspace{1cm} (3.20)

which implies

$$\mathcal{J} = -\frac{2}{\pi} E(r),$$  \hspace{1cm} (3.21)

where $E(r)$ is the elliptic function of second kind. Combining the two equations (3.19) and (3.21) we get,

$$\omega_1^2 = \left( \frac{K(r)}{E(r)} \mathcal{J} \right)^2.$$  \hspace{1cm} (3.22)

Similar solutions have been found in [54].
3.3 For $\rho = \theta_1 = 0$

This is an interesting case, where not only the string pulsates in one of the $S^3$, it also has extra charges due to the transverse motion of the string in the radial direction and furthermore there is a non-zero $B$-which contributes the equations of motion. Hence the fundamental string would know the presence of such field through the energy-spin relationship. For this case we have,

$$\mathcal{E} = \kappa, \mathcal{S} = 0, \mathcal{J} = \omega_1, \mathcal{P} = \mu_2.$$  \hspace{1cm} (3.23)

We also have,

$$\frac{d\theta_2}{d\sigma} = \sqrt{c_5 - 3\omega_2^2 \sin^2 \theta_2},$$  \hspace{1cm} (3.24)

which implies

$$\omega_1 = \frac{2}{\sqrt{3\pi}} K(s)$$  \hspace{1cm} (3.25)

where $K(s)$ is the elliptic function of first kind with the argument $s = \frac{\sqrt{c_5}}{\sqrt{3}\omega_2}$ and

$$K = \frac{\omega_2}{2\pi} \int_0^{2\pi} d\sigma (4 - 3\cos^2 \theta_2)$$ \hspace{1cm} (3.26)

$$= \frac{4\omega_2}{2\pi} \left[ 4 \int_0^{\frac{\pi}{2}} \frac{d\theta_2}{\sqrt{c_5 - 3\omega_2^2 \sin^2 \theta_2}} - 3 \int_0^{\frac{\pi}{2}} \frac{d\theta_2 \cos^2 \theta_2}{\sqrt{c_5 - 3\omega_2^2 \sin^2 \theta_2}} \right],$$

which implies

$$K = \frac{2}{\sqrt{3\pi}} [4K(s) - 3E(s)],$$ \hspace{1cm} (3.27)

where $E(s)$ is the elliptic function of second kind. Combining these two equations we can write,

$$K = \left[ 4 - 3 \frac{E(s)}{K(s)} \right] \omega_2,$$ \hspace{1cm} (3.28)

or,

$$\omega_2^2 = \frac{K^2}{\left[ 4 - 3 \frac{E(s)}{K(s)} \right]^2}.$$  \hspace{1cm} (3.29)

4. Discussion and Conclusion

In this paper we have studied semiclassical strings in the near horizon geometry of curved NS5-branes, namely on the NS5-branes with $AdS_3 \times S^3$ worldvolume. We have found the most general solutions of the equations of motion of the probe fundamental string in this background and found out solutions corresponding to giant magnon, single spike and furthermore the pulsating strings. We have found out the dispersion relation among various conserved charges and compare them with the existing ones. The novelty of these solutions is that they contain the information about the background NS-NS field. Further, the presence of the charge, $P$, in the dispersion relation reflects the fact that the motion of
the string in the radial direction $\xi$ in the near horizon geometry of NS5-branes is free. In the spirit of the non-integrability of the classical strings in the generic $p$-brane background, one can try to investigate the fundamental string equations of motion in the S-dual background, i.e. the D5-brane background wrapped on $AdS_3 \times S^3$, which is presented in this paper. The details of the background is given by [53]. By looking at the classical integrability of the string solutions presented in this paper in the NS5-brane background, one might be tempted to believe that similar solution would appear because of the $AdS_3 \times S^3 \times S^3$ structure of the parent background (in the absence of any brane charges). But we notice that while solving for the equations of motion in the D5-brane background, it is not possible to find simple or similar solution corresponding to the usual giant magnon and single spike strings. This is perhaps a hint to believe that F-string equations of motion are non-integrable in the D5-brane background. However, we wish to remark that the background solutions for the NS5-brane and D5-brane are similar being related by S-duality, but this S-duality does not act on classical string solutions in these backgrounds. Therefore, classical string solutions would indeed be very different. Hence in D5-brane background case there is no reason to expect integrability of probe fundamental string equations. It would perhaps be interesting to study D1-brane equations of motion in the D5-brane background and look for exact solutions in the context of integrability. We wish to come back to this issue in future.

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