ANALYTICAL RESULTS ON THE UNSTEADY ROTATIONAL FLOW OF FRACTIONAL-ORDER NON-NEWTONIAN FLUIDS WITH SHEAR STRESS ON THE BOUNDARY

MUHAMMAD MANSHA GHALIB
Department of Mathematics and statistics, The University of Lahore
Lahore, Pakistan

AZHAR ALI ZAFAR
Department of Mathematics, Govt. College University
Lahore, Pakistan

ZAKIA HAMMOUCH∗
Faculty of Sciences and Techniques Errachidia
Moulay Ismail University, Morocco

MUHAMMAD BILAL RIAZ
Department of Mathematics, University Of Management and Technology
Lahore, Pakistan

KHURRAM SHABBIR
Department of Mathematics, Govt. College University
Lahore, Pakistan

Abstract. The objective of this paper is to study the unsteady rotational flow of some non Newtonian fluids with Caputo fractional derivative through an infinite circular cylinder by means of the finite Hankel and Laplace transform. The novelty of the work is that motion is produced by applying tangential force not a specific but general function of time on the boundary. Initially the cylinder is at rest and after time $t_0 = 0^+$ it begins to rotate about its axis with an angular velocity $\tau_0 g(t)$. The obtained solutions of velocity field and shear stress have been presented under series form in terms of generalized $G$-function, satisfying all imposed initial and boundary conditions. The corresponding solutions can be easily particularized to give similar solutions from existing literature for Oldroyd-B fluids, Maxwell fluids, Second grade fluids and Newtonian fluids with/without fractional derivatives performing similar motions.

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* Corresponding author: Zakia Hammouch, email: hammouch.zakia@gmail.com.
1. Introduction. Nowadays, fractional calculus is very interesting topic for scientific research. The power of some fractional differential operators such as Riemann-Liouville [39], Caputo [16, 17, 18], Riesz [12], Caputo-Fabrizio [13, 40] and Atangana-Baleanu[14, 1, 2, 22] is their nonlocal properties which is not found in integer order differential operators. Also fractional derivatives supply memory description and hereditary properties of several materials, therefore several applications of fractional modeling of various phenomena and analytical/numerical methods were recently investigated in many papers [6, 5, 7].

The Oldroyd-B fluid model introduced by J. G. Oldroyd in [29] is very important among the fluids of rate type due to its special behavior. Also, this model contains the characteristics of Newtonian fluid model and Maxwell fluid model as special cases. The Oldroyd-B fluid model [30] considers the memory effects and elastic effects exhibited by a large class of fluids, such as the biological and polymeric liquids. Guillope and Saut [15] and Fontelos and Friedman [10] established the stability, existence and uniqueness results for some shearing flows of such fluids. Exact solutions for some simple flows of Oldroyd-B fluids were presented by many authors, see for example, Rajagopal and Bhatnagar [32], Hayat et al. [19, 20].

Recently, various problems regarding flows of Oldroyd-B fluids through cylindrical domains have been studied. Singh and Varshney [41] have considered the unsteady laminar flow of an electrically conducting Oldroyd-B fluid through a circular cylinder boundary by permeable bed under the influence of an exponentially decreasing pressure gradient in porous medium. Burdujan [4] studied Taylor-Couette flows of the Oldroyd-B fluid with fractional derivatives within the annular region between two infinitely coaxial circular cylinders due to a time dependent axial tension given on the surface of the inner cylinder. The unsteady unidirectional transient flow of Oldroyd-B fluid with time-fractional derivatives, in an annular domain, produced by a constant pressure gradient and a translation with constant velocity of the inner cylinder was studied by Mathur and Khandelwal [26]. Liu et al. [24] studied some helical flows of an Oldroyd-B fluid with time-fractional derivatives, between two infinite concentric oscillating cylinders and within an infinite circular oscillating cylinder. Most existing solutions in the literature correspond to problems with boundary conditions on the velocity. There are several practical problems with specified force on the boundary [36, 37, 42]. For example in [36], Renardy has studied the motion of a Maxwell fluid across a strip bounded by parallel plates and proved that, in order to formulate a well posed problem it is necessary to impose the boundary conditions on the stresses at the inflow boundary. In [37], Renardy explained how well posed boundary value problems can be formulated using boundary conditions on stresses. Waters and King [44] are among the first specialists who used the shear stress on the boundary to find exact solutions for motions of rate type fluids. Other interesting problems regarding flows of non-Newtonian fluids, in various geometry or boundary conditions, can be found in the references [21, 23, 3, 28, 11]. Our goal is to investigate unsteady rotational flows of Oldroyd-B fluids with/ without fractional derivatives in an infinite circular cylinder. In the present discussion the governing equation of the flow is related to the azimuthal tension and the boundary conditions on the shear stress are considered. The flow of the fluid is due to rotation of the cylinder around its axis, under the shear stress \( \tau_o g(t) \), given on the boundary. Finally, the obtained solutions can be easily particularized to give similar solutions for Oldroyd-B, Maxwell, second grade
2. Problem formulation. The constitutive equations for an Oldroyd-B fluid [29] are

\[ T = -pI + S, \] (1)
\[ S + \lambda \left( \frac{dS}{dt} - LS - SL^T \right) = \mu \left( A + \lambda_r \left( \frac{dA}{dt} - LA - AL^T \right) \right). \] (2)

Consider an infinite circular cylinder of radius \( R \). At \( t = 0 \), the cylinder and fluid are at rest. After time \( t > 0 \), the cylinder begins to turn about its axis as the result of a time dependent torque per unit length. We infer velocity and shear stress of the form

\[ V = V(r, t) = \omega(r, t)\hat{e}_\theta, \quad S = S(r, t). \] (3)

As the fluid and the cylinder are at rest initially, we have

\[ \omega(r, 0) = 0, \quad S(r, 0) = 0. \] (4)

From equation (2) using Eqs. (3) and (4), we get \( S_{rr} = S_{rz} = S_{z\theta} = S_{zz} = 0 \), along with the following meaningful partial differential equation [9].

\[ \left( 1 + \lambda \frac{\partial}{\partial t} \right) \tau(r, t) = \mu \left( 1 + \lambda_r \frac{\partial}{\partial r} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{4}{r^2} \right) \omega(r, t). \] (5)

where \( \tau(r, t) = S_{\theta r}(r, t) \) is one of the nonzero component of extra stress tensor. The balance of linear momentum in the absence of body forces and no pressure gradient along the flow reduces to

\[ \rho \frac{\partial \omega(r, t)}{\partial t} = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \tau(r, t). \] (6)

By eliminating \( \omega(r, t) \) between equations (5) and (6), we get the following governing equation for the shear stress [9]

\[ \left( 1 + \lambda \frac{\partial}{\partial t} \right) \frac{\partial \tau(r, t)}{\partial t} = \nu \left( 1 + \lambda_r \frac{\partial}{\partial r} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{4}{r^2} \right) \tau(r, t). \] (7)

The suitable initial and boundary conditions are

\[ \omega(r, 0) = 0, \quad \tau(r, 0) = 0, \quad \frac{\partial \tau(r, t)}{\partial t} \bigg|_{t=0} = 0, \quad \tau(R, t) = \tau_o g(t), \] (8)

where \( \tau_o \) is a constant having dimension of shear stress and \( g \) is a continuous function of time with \( g(0) = 0 \). By introducing the following dimensionless quantities

\[ t^* = \frac{\nu t}{R^2}, \quad \lambda^* = \frac{\nu \lambda}{R^2}, \quad \lambda_r^* = \frac{\nu \lambda_r}{R^2}, \quad \tau^* = \frac{\tau}{\tau_o}, \quad r^* = \frac{r}{R}, \quad \omega^* = \frac{\mu}{R \tau_o} \omega. \]

Eqs. (7)–(9) after eliminating the * notation becomes

\[ \frac{\partial \omega(r, t)}{\partial t} = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \tau(r, t). \] (9)
\[ \left( 1 + \lambda \frac{\partial}{\partial t} \right) \frac{\partial \tau(r, t)}{\partial t} = \left( 1 + \lambda_r \frac{\partial}{\partial r} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{4}{r^2} \right) \tau(r, t). \] (10)

\[ \tau(r, 0) = 0, \quad \tau(r, t) \rightarrow 0 \quad as \quad r \rightarrow \infty, \quad \tau(1, t) = g(t). \]
The corresponding fractional model analogous to Eq. (10) is characterized by

\[ (1 + \lambda D^\beta_t) \frac{\partial \tau(r,t)}{\partial t} = \left( 1 + \lambda r D^\beta_r \right) \left( \frac{\partial^2 \tau}{\partial r^2} + \frac{1}{r} \frac{\partial \tau}{\partial r} - \frac{4}{r^2} \right) \tau(r,t). \]  

Eq. (11)

The physical explanations of this generalization is that this fractional partial differential equation proves a damping of shear stress, whereas the generalization of Eq. (9) has no justification. Therefore, this equation must be kept in the classical form.

In the Eq. (11) \( D^\beta_t f \) is a Caputo fractional derivative [31] defined as

\[ D^\beta_t f(t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\beta} \, d\tau, & 0 < \beta < 1 \\
\beta = 1. \end{cases} \]

3. Calculation of shear stress. By applying Laplace transform [27] on Eq. (11), we have

\[ (q + \lambda q^{\alpha+1}) \tilde{\tau}(r,q) = \left( 1 + \lambda r q^{\beta} \right) \left( \frac{\partial^2 \tilde{\tau}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{\tau}}{\partial r} - \frac{4}{r^2} \right) \tilde{\tau}(r,q). \]

\[ \tilde{\tau}(r,0) = 0, \quad \tilde{\tau}(1,q) = G(q). \]

Now by applying Hankel transform [25] we get

\[ (q + \lambda q^{\alpha+1}) \tilde{\tau}_H(r_n, q) = \left( 1 + \lambda r_n q^{\beta} \right) \left( -r_n J_1(r_n) \tilde{\tau}_H(1,q) - r_n^2 \tilde{\tau}_H(r_n, q) \right). \]

\[ (q + \lambda q^{\alpha+1}) \tilde{\tau}_H(r_n, q) + \left( 1 + \lambda r_n q^{\beta} \right) r_n^2 \tilde{\tau}_H(r_n, q) = -r_n \left( 1 + \lambda r_n q^{\beta} \right) J_1(r_n) \tilde{\tau}_H(1,q). \]

Taking into account the boundary conditions, we have

\[ \tilde{\tau}_H(r_n, q) = \frac{1 + \lambda r_n q^{\beta}}{q + \lambda q^{\alpha+1} + (1 + \lambda r_n q^{\beta}) r_n^2} \left[ -r_n J_1(r_n) G(q) \right]. \]

or

\[ \tilde{\tau}_H(r_n, q) = G(q) \left[ \frac{-r_n J_1(r_n)}{r_n^2} + \frac{r_n J_1(r_n)}{r_n^2} \sum_{k=0}^\infty \sum_{m=0}^k (-1)^k \frac{k!}{(k-m)!m!} \lambda^m r_n^2 \beta^{k-m} \left( q^{\alpha+1} \right) \right]. \]

Using the inverse Laplace transform, it yields

\[ \tau_H(r_n,t) = \frac{-J_1(r_n)}{r_n} g(t) + \frac{J_1(r_n)}{r_n} \sum_{k=0}^\infty \sum_{m=0}^k (-1)^k \frac{k!}{(k-m)!m!} \lambda^m r_n^2 \beta^{k-m} \left( q^{\alpha+1} \right) \int_0^t g(u) \int_0^{t-u} \delta(t - \sigma - u) G_{\alpha,\beta-k-1,k}(-\lambda^{-1}, \sigma) d\sigma du, \]  

where \( G \) is the generalized \( G \) function defined in [33]. Now applying inverse Hankel transform, we get

\[ \tau(r,t) = r^2 g(t) + 2 \sum_{n=1}^\infty \frac{J_2(nn)}{r_n J_1(r_n)} \sum_{k=0}^\infty \sum_{m=0}^k (-1)^k \frac{k!}{(k-m)!m!} \lambda^m r_n^2 \beta^{k-m} \left( q^{\alpha+1} \right) \int_0^t g(u) \int_0^{t-u} \delta(t - \sigma - u) G_{\alpha,\beta-k-1,k}(-\lambda^{-1}, \sigma) d\sigma du, \]

which is the expression for non-dimensional shear stress.
4. Calculation of velocity. From Eq. (9), we have
\[ \frac{d\omega(r,t)}{dt} = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \tau(r,t). \] (19)

By substituting the value of shear stress from equation (16) we have
\[ \frac{d\omega(r,t)}{dt} = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \left( r^2 g(t) + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} J_2(rr_n)(-1)^k \lambda_n^m(r_n^2)^k \right. \]
\[ \left. \times \int_0^t g(u) \int_0^{t-u} \delta(t-s-u)G_{\alpha,m\beta-k-1,k}(-\lambda^{-1},\sigma)\sigma \sigma du \right). \] (20)

Applying Laplace transform and using initial conditions Eq. (18) gives
\[ q(\tilde{\omega}(r,q)) = 4rG(q) \]
\[ + J_1(rr_n)2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^k \lambda_n^m(r_n^2)^k \frac{(k-m)!m!\lambda^k}{(k-m)!m!\lambda^k} \left[ G(q) \left( \frac{q^{m-k-1}}{\lambda^{-1} + q^k} \right) \right]. \] (21)

Finally, taking Laplace inverse transform, we obtain
\[ \omega(r,t) = \frac{4r}{\Gamma(\alpha)} \int_0^t g(u)du + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} J_1(rr_n) \frac{(-1)^k \lambda_n^m}{J_1(r_n) (k-m)!m!\lambda^k (r_n^2)^k} \]
\[ \int_0^t g(t-u)G_{\alpha,\beta m-k-1,k}(-\lambda^{-1},u)du \] (22)
the expression of non-dimensional velocity.

5. Special cases.

5.1. Case-I : - \( g(t) = t \). Putting the \( g(t) = t \) into equations (16) and (20), we obtain the expressions of shear stress and velocity.
\[ \tau(r,t) = rt^2 + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} J_2(rr_n) \frac{(-1)^k \lambda_n^m}{J_1(r_n) (k-m)!m!\lambda^k (r_n^2)^k} \]
\[ \frac{\lambda_n^m(r_n^2)^k}{\lambda^k} \int_0^t u \int_0^{t-u} \delta(t-s-u)G_{\alpha,\beta m-k-1,k}(-\lambda^{-1},\sigma)\sigma \sigma du, \] (23)
and
\[ \omega(r,t) \]
\[ = \frac{2rt^2}{\Gamma(\alpha)} + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} J_1(rr_n) \frac{(-1)^k \lambda_n^m}{J_1(r_n) (k-m)!m!\lambda^k (r_n^2)^k} G_{\alpha,\beta m-k-3,k}(-\lambda^{-1},t). \] (24)

5.2. Case-II : - \( g(t) = te^t \). Putting \( g(t) = te^t \) into equations (16) and (20), we obtain the expressions of shear stress and velocity profile.
\[ \tau(r,t) = rt^2e^t + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} J_2(rr_n) \frac{(-1)^k \lambda_n^m}{J_1(r_n) (k-m)!m!\lambda^k (r_n^2)^k} \]
\[ \frac{\lambda_n^m(r_n^2)^k}{\lambda^k} \int_0^t (u)e^u \int_0^{t-u} \delta(t-s-u)G_{\alpha,\beta m-k-1,k}(-\lambda^{-1},\sigma)\sigma \sigma du, \] (25)
and
\[
\omega(r, t) = \frac{4r}{\Gamma(\alpha)} ((t - 1)e^t + 1) + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{J_1(rr_n)}{J_1(r_n)} \frac{(-1)^k k! \lambda^m}{(k-m)! m! \lambda^k (r_n^2)^k} \int_0^t (t-u)e^{(t-u)} G_{\alpha,m\beta-k-1,k}(-\lambda^{-1},u)du. \tag{26}
\]

5.3. Case-III : - \( g(t) = \sin(\omega t) \). Putting \( g(t) = \sin(\omega t) \) into equations (16) and (20), we obtain the expressions of shear stress and velocity.

\[
\tau(r, t) = r^2 \sin(\omega t) + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{J_2(rr_n)}{r_n J_1(r_n)} (-1)^k \frac{k!}{(k-m)! m!} \lambda^m \frac{(r_n^2)^k}{\lambda^k} \int_0^t \sin(\omega u) \int_0^{t-u} \delta(t - \sigma - u) G_{\alpha,m\beta-k-1,k}(-\lambda^{-1},\sigma)d\sigma du, \tag{27}
\]

and

\[
\omega(r, t) = \frac{4r}{\omega \Gamma(\alpha)} \left[1 - \cos(\omega t) \right] + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{J_1(rr_n)}{J_1(r_n)} \frac{(-1)^k k! \lambda^m}{(k-m)! m! \lambda^k (r_n^2)^k} \frac{1}{\lambda^k} \int_0^t \sin(\omega u) \int_0^{t-u} \delta(t - \sigma - u) G_{\alpha,m\beta-k-1,k}(-\lambda^{-1},u)du. \tag{28}
\]

Eqs. (25) and (26) for \( \alpha, \beta \to 1 \) as expected equivalent to those obtained by Rauf et al. [45] (Eq. (28), (32)).

5.4. Case-IV : - \( g(t) = \frac{t^{a-1}}{\Gamma a} \). Putting the value of \( g(t) = \frac{t^{a-1}}{\Gamma a} \) into equations (16) and (20), we obtain the expressions of shear stress and velocity.

\[
\tau(r, t) = r^2 \frac{t^{a-1}}{\Gamma a} + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{J_2(rr_n)}{r_n J_1(r_n)} (-1)^k \frac{k!}{(k-m)! m!} \lambda^m \frac{(r_n^2)^k}{\lambda^k} \frac{1}{\Gamma a} \int_0^t \frac{u^{a-1}}{\Gamma a} \int_0^{t-u} \delta(t - \sigma - u) G_{\alpha,m\beta-k-1,k}(-\lambda^{-1},\sigma)d\sigma du, \tag{29}
\]

and

\[
\omega(r, t) = \frac{4rt^a}{\Gamma(\alpha)\Gamma(a+1)} + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{J_1(rr_n)}{J_1(r_n)} \frac{(-1)^k k! \lambda^m}{(k-m)! m! \lambda^k (r_n^2)^k} \frac{t^a}{\Gamma a} \int_0^t \frac{(t-u)^{a-1}}{\Gamma a} G_{\alpha,m\beta-k-1,k}(-\lambda^{-1},u)du. \tag{30}
\]

These last solutions for \( a = \delta \), as expected are equivalent to those obtained by Zafar et al. [38] (Eqs.(26), (29)). Moreover, for \( a = 1 \), in Eqs (27)and (28), we recover the results by Riaz et al. [38] (Eqs. (16), (21)).
6. Limiting cases.

6.1. Ordinary Oldroyd-B fluid. Making \( \alpha \to 1 \) and \( \beta \to 1 \) into equations (16) and (20), we obtain the expressions of shear stress and velocity field for ordinary Oldroyd-B fluid

\[
\tau_{OOB}(r, t) = r^2 g(t) + 2 \sum_{n=1}^{\infty} \frac{J_2(\rho r_n)}{r_n J_1(r_n)} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^k k!}{(k - m)!m!} \frac{\lambda^m (r_n^2)^k}{r_n^k} \int_0^t g(u) \int_0^{t-u} \delta(t - \sigma - u)G_{1,m-k-1,k}(-\lambda^{-1}, \sigma) d\sigma du,
\]

and

\[
\omega_{OOB}(r, t) = 4r \int_0^t g(u) du + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{J_1(\rho r_n)}{J_1(r_n)} \sum_{m=0}^{k} \frac{(-1)^k k! \lambda^m (r_n^2)^k}{(k - m)!m! \lambda^{k}} \int_0^t g(t - u)G_{1,m-k-1,k}(-\lambda^{-1}, \sigma) d\sigma du.
\]

If we take \( g(t) = H(t) \) in Eqs. (29) and (30) we recover solutions for rotational flow of an Oldroyd-B fluid generated by a circular cylinder that applies a constant couple to the fluid as obtained by Fetecau et al. [46].

6.2. Maxwell fluid with fractional derivative. Making \( \lambda_r \to 0 \) and \( \beta \to 1 \) in equations (16) and (20), we obtain the expressions of shear stress and velocity field for the Maxwell fluid with fractional derivatives

\[
\tau_{MFF}(r, t) = r^2 g(t) + 2 \sum_{n=1}^{\infty} \frac{J_2(\rho r_n)}{r_n J_1(r_n)} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^k k!}{(k - m)!m!} \frac{\lambda^m (r_n^2)^k}{r_n^k} \int_0^t g(u) \int_0^{t-u} \delta(t - \sigma - u)G_{1,m-k-1,k}(-\lambda^{-1}, \sigma) d\sigma du,
\]

and

\[
\omega_{MFF}(r, t) = \frac{4r}{\Gamma(\alpha)} \int_0^t g(u) du + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{J_1(\rho r_n)}{J_1(r_n)} \sum_{m=0}^{k} \frac{(-1)^k k! \lambda^m (r_n^2)^k}{(k - m)!m! \lambda^{k}} \int_0^t g(t - u)G_{1,m-k-1,k}(-\lambda^{-1}, \sigma) d\sigma du.
\]

For \( g(t) = H(t) \frac{t^{\delta-1}}{\Gamma(\delta)} \) we recover the results as obtained by Zafar et al. in [34] (Eqs. (3.26), (3.29))

6.3. Ordinary Maxwell fluid. By making \( \alpha \to 1 \) in equations (31) and (32), we get corresponding expressions of shear stress and velocity field for ordinary Maxwell fluid

\[
\tau_{OMF}(r, t) = r^2 g(t) + 2 \sum_{n=1}^{\infty} \frac{J_2(\rho r_n)}{r_n J_1(r_n)} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^k k!}{(k - m)!m!} \frac{\lambda^m (r_n^2)^k}{r_n^k} \int_0^t g(u) \int_0^{t-u} \delta(t - \sigma - u)G_{1,m-k-1,k}(-\lambda^{-1}, \sigma) d\sigma du,
\]
and

$$\omega_{OMF}(r, t) = 4r \int_{0}^{t} g(u)du + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{J_{1}(rr_{n})}{J_{1}(r_{n})} \frac{(-1)^{k}k!\lambda_{r}^{m}}{(k-m)!m!}(r_{n}^{2})^{k} \int_{0}^{t} g(t-u)G_{1,m-k-1,k}(-\lambda^{-1}, u)du.$$  (36)

6.4. **Fractional second grade fluid.** Making $\lambda = 0$ and $\alpha \to 1$ into equations (13) and (17), lengthy but straightforward calculations leads to the following expressions of shear stress and velocity field for fractional second grade fluid model performing similar motions

$$\tau_{FSGF}(r, t) = r^{2}g(t) + 2 \sum_{n=1}^{\infty} \frac{J_{2}(rr_{n})}{r_{n}J_{1}(r_{n})} \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^{k} \frac{k!\lambda_{r}^{m}}{(k-m)!m!}(r_{n}^{2})^{k} \frac{1}{\Gamma(k-m\beta + 1)} \int_{0}^{t} g(t-u)u^{k-m\beta-1}du.$$  (37)

and

$$\omega_{FSGF}(r, t) = 4r \int_{0}^{t} g(u)du + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^{k} \frac{k!\lambda_{r}^{m}}{(k-m)!m!}(r_{n}^{2})^{k} \frac{1}{\Gamma(k-m)} \int_{0}^{t} g(t-u)u^{k-m-1}du.$$  (38)

Here again by customizing the values of general function $g(t)$ we could recover several results from the literature for case of rotational flows of second grade fluid with fractional derivatives for example results obtained by Raza [35] could be obtain from our last results by taking $g(t) = G\sin \omega t$. Moreover, by taking $g(t) = At^{2}$ into Eqs. (35) and (36), the results of Raza et al. [35] are recovered.

6.5. **Ordinary second grade fluid.** Making $\beta \to 1$ into equations (35) and (36), we get expressions of shear stress and velocity field for the case of ordinary second grade fluid model

$$\tau_{OSGF}(r, t) = r^{2}g(t) + 2 \sum_{n=1}^{\infty} \frac{J_{2}(rr_{n})}{r_{n}J_{1}(r_{n})} \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^{k} \frac{k!\lambda_{r}^{m}}{(k-m)!m!}(r_{n}^{2})^{k} \frac{1}{\Gamma(k-m)} \int_{0}^{t} g(t-u)u^{k-m-1}du.$$  (39)

and

$$\omega_{OSGF}(r, t) = 4r \int_{0}^{t} g(u)du + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^{k} \frac{k!\lambda_{r}^{m}}{(k-m)!m!}(r_{n}^{2})^{k} \frac{1}{\Gamma(k-m+1)} \int_{0}^{t} g(t-u)u^{k-m}du.$$  (40)
6.6. **Newtonian fluid.** Making $\lambda_r \to 1$ in equation (37) and (38), we get

$$\tau_{NF}(r,t) = r^2 g(t) + 2 \sum_{n=1}^{\infty} \frac{J_2(\eta r_n)}{\eta J_1(\eta r_n)} \sum_{k=0}^{\infty} (-1)^k (r_n^2)^k \frac{1}{\Gamma(k+1)} \int_0^t g(t-u) u^{k-1} du,$$

(41)

$$\omega_{NF}(r,t) = 4r \int_0^t g(u) du + 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{J_1(\eta r_n)}{J_1(\eta r_n)} (-1)^k (r_n^2)^k \frac{1}{\Gamma(k+1)} \int_0^t g(t-u) u^{k-1} du.$$  

(42)

By setting $g(t) = H(t)$ into (39) and (40), we recovered the solutions obtained in [46] (Eqs. (22) and (23)).

7. **Conclusion.** The rotational flows of non Newtonian fluids with fractional derivatives are studied. The fluid fills an infinite circular cylinder and time dependent shear stress as the general function of time is applied tangentially on the boundary of the cylinder. The expressions of shear stress and velocity field are obtained in terms of generalized G-functions. The results satisfy the imposed initial and boundary conditions. Several results from existing literature are recovered from our general solutions by either customizing the values of the time dependent shear stress or by using the interpolation property of the fractional order parameters. In order to avoid repetition, graphs for the analysis of the boundary conditions and material parameters are not included as they already exists in literature. Hence the problem of rotational flows for some non Newtonian fluids with/ without non integer derivatives and Newtonian fluid with shear stress on the boundary is completely solved.

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E-mail address: mansha303426@yahoo.com
E-mail address: azharalizafar@gmail.com
E-mail address: hammouch.zakia@gmail.com
E-mail address: Bilesehole@gmail.com
E-mail address: khurramsms@gmail.com