Pair approximation for the $q$-voter model with independence on multiplex networks

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The $q$-voter model with independence is investigated on multiplex networks with fully overlapping layers in the form of various complex networks corresponding to different levels of social influence. Detailed studies are performed for the model on multiplex networks with two layers with identical degree distributions, obeying the LOCAL&AND and GLOBAL&AND spin update rules differing by the way in which the $q$-lobbies of neighbors within different layers exert their joint influence on the opinion of a given agent. Homogeneous pair approximation is derived for a general case of a two-state spin model on a multiplex network and its predictions are compared with results of Monte Carlo simulations of the above-mentioned $q$-voter model with independence for a broad range of parameters. As the parameter controlling the level of agents’ independence is changed ferromagnetic phase transition occurs which can be first- or second-order, depending on the size of the lobby $q$. Details of this transition, e.g., position of the critical points, depend on the topology and other features, e.g., the mean degree of nodes of the layers. If the mean degree of nodes in the layers is substantially larger than the size of the $q$-lobby good agreement is obtained between numerical results and theoretical predictions based on the homogeneous pair approximation concerning the order and details of the ferromagnetic transition. In the case of the model on multiplex networks with layers in the form of homogeneous Erdős-Rényi and random regular graphs as well as weakly heterogeneous scale-free networks this agreement is quantitative, while in the case of layers in the form of strongly heterogeneous scale-free networks it is only qualitative. If the mean degree of nodes is small and comparable with $q$ predictions of the homogeneous pair approximation are in general even qualitatively wrong.

I. INTRODUCTION

Studies of interacting systems on complex, possibly heterogeneous networks constitute an important area of research in contemporary statistical physics [1,2]. In particular, they comprise investigation of critical phenomena in nonequilibrium models for, e.g., contact processes and epidemic spreading [3], synchronization [4] and the opinion formation in contemporary statistical physics [1,2]. In particular, they comprise investigation of critical phenomena in nonequilibrium models for, e.g., contact processes and epidemic spreading [3], synchronization [4] and the opinion formation in contemporary statistical physics [1,2].

Recent it has been realized that even more complex and heterogeneous structures occur frequently in social systems which has prompted interest in the study of interacting systems on "networks of networks" [26]. In this context much attention was devoted to multiplex networks (MNs) which consist of a fixed set of nodes connected by various sets of edges called layers [26–28]. Interacting systems on MNs exhibit rich variety of collective behavior and critical phenomena. For example, percolation transition [29,30], cascading failures [31], threshold cascades [32,33], diffusion processes [34,35], epidemic spreading [36] and phase transitions in the equilibrium Ising model [37,38] and related Ashkin-Teller model [39] in a non-equilibrium majority vote model [40] were studied on MNs. Also the $q$-voter model with independence [16] and the $q$-neighbor Ising model [20–23] were studied on MNs with layers in the form of complete graphs. For the two latter models various opinion update rules were assumed as generalizations of the rules for the corresponding models on (monoplex) networks in order to take into account that agents interact with separately chosen lobbies within different layers, which correspond to different levels of social influence. It was shown...
that the effect of network multiplicity on the critical behavior (e.g., on the range of parameters for the occurrence of the discontinuous FM transition) depends on the model and on the assumed opinion update rule and is quantitatively well described in the MF approximation.

The aim of this paper is to investigate the $q$-voter model with independence on MNs with layers in the form of complex networks rather than complete graphs. This requires going beyond the MF approximation. Thus, in this paper PA is extended to a general case of two-state spin systems with up-down symmetry on MNs with layers in the form of complex networks and with various spin update rules. This PA is applied to the $q$-voter model with independence and the resulting theoretical predictions concerning the FM transition are compared with results of Monte Carlo (MC) simulations of the model on MNs with layers with different degree distributions, such as random regular graphs (RRGs), Erdős-Rényi graphs (ERGs) \cite{6, 7, 11} and scale-free (SF) networks \cite{6, 7, 42}. For the sake of brevity the problem under study is considered here under several simplifying assumptions. First, as natural generalizations of the opinion update rule for the model on monoplex networks, the corresponding rules for the model on MNs are assumed to have different AND forms \cite{10, 22, 33}: the agent changes opinion if interaction with every lobby from every layer suggest change (LOCAL&AND rule), or interaction with a set of all neighbors from all lobbies from all layers suggests change (GLOBAL&AND rule). Then, only the simplest homogeneous PA is considered in which all nodes and edges are treated as equivalent and their possible heterogeneity is neglected \cite{10, 17, 19, 23}. Finally, detailed calculations are performed only for the case of MNs with two independently generated layers (so-called dupplex networks) with identical degree distributions and with full overlap (with each node belonging to both layers).

Under the above-mentioned assumptions the model under study exhibits qualitatively similar behavior as that on MNs with layers in the form of complete graphs \cite{10}. However, details of the first- or second-order FM transition observed in MC simulations depend significantly on the topology of the networks forming the layers of the MN. In general, better agreement between numerical and theoretical results based on the homogeneous PA occurs for the model on MNs with layers with high density of edges. Then, as expected, in the case of MNs composed of layers with negligible heterogeneity, e.g., RRGs and ERGs, the details of the FM transition are quantitatively well captured by the theory based on the homogeneous PA. Besides, it is shown that this theory yields reasonably good agreement with numerical results also for the model on MNs with weakly heterogeneous SF layers with finite second moment of the degree distributions. In the case of MNs with strongly heterogeneous layers this agreement is much worse and only qualitative; possibility of formulation of heterogeneous PA \cite{11, 24, 25} for the model on such MNs, which could better reproduce results of MC simulations, is only briefly discussed here. Agreement between numerical and theoretical results substantially deteriorates for the model on MNs with low density of edges, and in this case predictions of the homogeneous PA can be quantitatively wrong even in the case of MNs with layers with negligible heterogeneity. Finally, it should be emphasized that the $q$-voter model with independence is considered here only as a relatively simple example and extension of the derived PA to other models on MNs with similar opinion (spin) update rules is straightforward.

II. THE MODEL

A. The $q$-voter model with independence on multiplex networks

The $q$-voter model with independence is a sort of stochastic spin model for the opinion formation with random sequential updating \cite{15, 17}. Let us first describe the model on a monoplex network. In this model agents represented by spins $s_i = \pm 1$, $i = 1, 2, \ldots, N$ with two states corresponding to opposite opinions are located in $N$ nodes of the network, and the edges correspond to possible interactions between them. The dynamics of the model is defined by the spin flip rate which depends on a parameter $p$ ($0 \leq p \leq 1$) determining the degree of stochasticity ("social temperature") in the model. This stochasticity manifests itself as agents' independence in decision making. At each elementary time step an agent and a subset of $q$ her neighbors (a $q$-lobby) are chosen randomly; the neighbors are chosen without repetitions. Then, the opinion of the agent is updated according to the following rule. With probability $1 - p$ the agent acts as a conformist and with probability $p$ acts independently. In the case of conformity the agent changes opinion, and the spin flips, if and only if the opinions of all members of the $q$-lobby are identical and opposite to that of the agent. In the case of independence the agent changes opinion, and the spin flips, with probability $1/2$, independently of the opinions of the members of the $q$-lobby. Otherwise, the opinion of the agent remains unchanged. Hence, the elementary time step corresponds to the opinion update of one agent. This procedure is repeated until all agents update their opinions, which corresponds to one MCSS; thus, duration of the elementary time step is $\Delta t = 1/N$.

The $q$-voter model with independence on a MN consists of agents represented by two-state spins $s_i = \pm 1$, $i = 1, 2, \ldots, N$ located in the nodes of the MN which interact with independently chosen $q$-lobbies from every layer, each $q$-lobby being a subset of $q$ agent's neighbors within one layer. It should be emphasized that in the MN there is only one
set of nodes, while sets of edges corresponding to different layers are generated separately. Thus, the agents located in the nodes present the same opinion \( s_i \) to their neighbors within each layer (the agents are non-schizophrenic). The dynamics of the model is again defined by the spin-flip rate which is a generalization of that for the model on monoplex networks. In particular, in this paper only AND generalizations of the spin-update rule are considered in which, in general, it is assumed that a node is activated if a sufficiently large fraction of its neighbors in every layer are active \([33]\). Besides, different assumptions concerning the status of independence of the agents (GLOBAL on all layers or LOCAL on each layer separately) can be made. The above-mentioned assumptions lead to the LOCAL&AND or GLOBAL&AND spin update rules which are described below.

In both LOCAL&AND and GLOBAL&AND cases the spins are updated randomly and sequentially. At each elementary time step an agent is chosen randomly. Then, \( q \)-lobbies of her neighbors, one lobby per each layer of the MN, are chosen randomly and independently. The neighbors belonging to a \( q \)-lobby within a given layer are chosen without repetitions, however, due to the topology of connections within the MN (see Sec. II.B) it can happen that the same node belongs to two or more \( q \)-lobbies within different layers. In the LOCAL&AND case the agent first interacts with each \( q \)-lobby separately: with probability \( 1 - p \) she acts as a conformist and tends to change opinion if and only if the opinions of all members of the \( q \)-lobby are identical and opposite to her opinion, and with probability \( p \) she acts independently and tends to change opinion with probability \( 1/2 \). Then, if and only if for every layer of the MN the agent tends to change opinion the spin eventually flips; otherwise the opinion of the agent remains unchanged. Thus, in the LOCAL&AND case the spin-flip rate is a product of the rates for the \( q \)-lobby model with independence on monoplex networks corresponding to subsequent layers of the MN. In the GLOBAL&AND case all \( q \)-lobbies from all layers of the MN are first aggregated into a single lobby. Then, with probability \( 1 - p \) the agent acts as a conformist and changes opinion, i.e., the spin flips, if and only if the opinions of all members of the latter lobby are identical and opposite to her opinion, and with probability \( p \) the agent acts independently and changes opinion, i.e., the spin flips, with probability \( 1/2 \). Otherwise, the opinion of the agent remains unchanged. Thus, in the GLOBAL&AND case the spin-flip rate is equal to the rate for the \( q \)-voter model with independence on an aggregate monoplex network composed of all layers of the MN with proportionally rescaled size of the \( q \)-lobby.

### B. The models for multiplex networks

As mentioned in Sec. I a MN consists of a fixed set of \( N \) nodes which are connected by many separately generated sets of edges called layers \([26, 28]\). For simplicity, in this paper MNs with only two layers denoted as \( G^{(A)}, G^{(B)} \) are considered (so-called duplex networks); theoretical approach of Sec. III can be easily extended to the case with more than two layers. Thus, each node is characterized by two degrees \( k^{(A)}, k^{(B)} \) defined as the numbers of edges attached to it within the respective layer. Moreover, it is assumed that the layers are independently generated complex networks with the degree distributions \( P^{(A)}(k), P^{(B)}(k) \) and with mean degrees of nodes \( \langle k^{(A)} \rangle, \langle k^{(B)} \rangle \). Only MNs with fully overlapping layers are considered, such that each node has non-zero degree within each layer, \( k^{(A)} > 0, k^{(B)} > 0 \). Thus, the joint degree distribution is \( P^{(A)}(k^{(A)}, k^{(B)}), \ P^{(B)}(k^{(A)}, k^{(B)}) \). Detailed theoretical calculations and MC simulations are performed for the \( q \)-voter model with independence on MNs with layers characterized by identical degree distributions \( P^{(A)}(k^{(A)}) = P^{(B)}(k^{(B)}) \), thus with \( \langle k^{(A)} \rangle = \langle k^{(B)} \rangle \). In particular, the layers under study can be complex networks in the form of RRGs, ERGs and heterogeneous SF networks.

RRG is a sort of random graph with degree distribution \( P(k) = \delta_{k,k_0} \) and mean degree \( \langle k \rangle = k_0 \), with \( N \) randomly connected nodes, each with the same degree \( k_0 \). ERG is a sort of random graph with \( N \) nodes and binomial degree distribution \( P(k) = \binom{N-1}{k} \frac{\hat{p}^k (1-\hat{p})^{N-1-k}}{N^{k}} \) with \( \langle k \rangle = N\hat{p} \) \([6, 7, 41]\). SF network is characterized by a power degree distribution \( P(k) \propto k^{-\gamma}, \gamma > 2 \), for \( k > k_{\text{min}} \), and \( P(k) = 0 \) otherwise, with \( \langle k \rangle = (\gamma - 1)k_{\text{min}}/ (\gamma - 2) \) \([6, 7, 42]\). ERGs can be efficiently constructed by randomly connecting \( N \) nodes with \( N \langle k \rangle / 2 \) edges. In turn, an efficient method to generate RRGs and SF networks is to apply the Configuration Model \([33]\). Using the above-mentioned methods the two layers \( G^{(A)}, G^{(B)} \) are generated independently. Care is taken to avoid multiple connections between nodes within each layer. However, since layers are generated independently it is possible that a given pair of nodes is connected by two edges, one from the layer \( G^{(A)} \) and the other from \( G^{(B)} \). Hence, in the process of MC simulation of the \( q \)-voter model with independence on a MN it can happen that at a certain elementary time step the same neighbor of a chosen agent belongs to both her \( q \)-lobbies, one within the layer \( G^{(A)} \) and the other within \( G^{(B)} \).
III. THEORY

A. General formulation of the pair approximation

Theoretical description of the $q$-voter model with independence and $q$-neighbor Ising model on MNs with layers in the form of complete graphs based on the MF approximation yields quantitative agreement with results of MC simulations concerning both the order of the transition from the PM to the FM phase and the position of the critical point [16, 22]. However, when the above-mentioned models are studied on monoplex networks a more accurate PA is necessary in order to take into account the effect of the network topology (e.g., the degree distribution and the mean degree of nodes) on the properties of the phase transition [17, 19, 23]. Hence, in this paper PA is used for theoretical investigation of the $q$-voter model with independence on MNs with two layers in the form of complex networks. In order to make our approach more widely applicable in this subsection a general case of a two-state spin model with up-down symmetry is considered, and application of the results to the $q$-voter model with LOCAL&AND or GLOBAL&AND spin update rules is presented in Sec. III.B. Henceforth in this subsection $\hat{\gamma}$ denotes a parameter which controls the degree of stochasticity in the general model (in the case of the $q$-voter model this is the parameter $p$ characterizing the degree of independence of agents).

As mentioned in Sec. I only homogeneous PA is considered neglecting the possible effect of heterogeneity of the layers on the properties of the phase transitions observed in the model, which can be particularly strong in the case of layers in the form of SF networks with $2 < \gamma < 3$. Within this approach macroscopic quantities characterizing the model are concentrations of spins with orientation up and of active bonds connecting nodes occupied by spins with opposite orientations. In the case of a model on a MN with two fully overlapping layers a single quantity $c$ is enough to characterize the concentration of spins with orientation up (normalized to the number of nodes $N$), but two quantities $\theta^A_\uparrow$, $\phi^A_\uparrow$ characterizing the concentrations of active bonds within different layers (normalized to the total numbers of edges $N(k_i^A)/2$ ($N(k_i^B)/2$) within the layer $G^A$ ($G^B$) are necessary. The rate equations for the concentrations $c$, $\theta^A_\uparrow$, $\phi^A_\uparrow$ can be obtained in a similar way as, e.g., in Ref. [17] in the case of a model on a monoplex network, thus the derivation is only outlined here.

In the framework of the homogeneous PA the probabilities that a randomly selected spin with orientation $j \in \{\uparrow, \downarrow\}$ has within the layer $G^A$ ($G^B$) a neighbor with opposite orientation are independent of the degrees of nodes and are denoted by $\theta^A_j$, $\phi^A_j$. It can be easily shown that

$$\theta^A_\uparrow = \frac{b^A}{2c}, \quad \phi^A_\uparrow = \frac{b^A}{2(1-c)},$$

$$\theta^B_\uparrow = \frac{b^B}{2c}, \quad \phi^B_\uparrow = \frac{b^B}{2(1-c)}.$$

Hence, the concentration of spins up $c$ obeys the Master equation

$$\frac{\partial c}{\partial t} = \gamma_+ \left( c, \theta^A_\uparrow, \phi^B_\uparrow, \hat{\gamma}_+ \right) - \gamma_- \left( c, \theta^A_\uparrow, \phi^B_\uparrow, \hat{\gamma}_- \right),$$

where $\gamma_+$ ($\gamma_-$) are rates of spin flips in the direction up (down) averaged over all nodes.

Each flip of spin located in the node with degrees $k^A$, $k^B$ causes changes in the concentrations of active bonds $b^A$, $b^B$ by

$$\Delta b^A \left( i^A \mid k^A \right) = \frac{2}{N(k_i^A)} \left( k^A - 2i^A \right),$$

$$\Delta b^B \left( i^B \mid k^B \right) = \frac{2}{N(k_i^B)} \left( k^B - 2i^B \right),$$

where $i^A$ ($i^B$) is the number of active bonds attached to this node within the layer $G^A$ ($G^B$), $0 \leq i^A \leq k^A$, $0 \leq i^B \leq k^B$. The spin flip rate $f \left( i^A, i^B, \hat{\gamma} \mid k^A, k^B \right)$ is also a function of the concentrations of active bonds within layers, and its form characterizes the particular model. Denoting by $P \left( j, i^A, i^B \mid k^A, k^B \right)$ conditional probability that the spin with orientation $j \in \{\uparrow, \downarrow\}$ has $i^A$, $i^B$ neighboring spins with opposite orientation within the layers $G^A$, $G^B$, respectively, provided that it is located in the node with degrees $k^A$, $k^B$, it is easily obtained that the average changes of the concentrations $b^A$, $b^B$ at each elementary time step are

$$\Delta b^A = \sum_{j \in \{\uparrow, \downarrow\}} \sum_{k^A, k^B} P \left( k^A, k^B \right) \sum_{i^A=0}^{k^A} \sum_{i^B=0}^{k^B} P \left( j, i^A, i^B \mid k^A, k^B \right).$$
\[ \frac{\partial b^{(A)}}{\partial t} = \frac{2}{\langle k^{(A)} \rangle} \sum_{j \in \{\uparrow, \downarrow\}} \sum_{k^{(A)}, k^{(B)}} c_j P \left( k^{(A)}, k^{(B)} \right) \times \sum_{i^{(A)} = 0}^{k^{(A)}} \sum_{i^{(B)} = 0}^{k^{(B)}} B_{k^{(A)}, i^{(A)}} \left( \theta^{(A)}_j \right) B_{k^{(B)}, i^{(B)}} \left( \theta^{(B)}_j \right) \times f \left( i^{(A)}, i^{(B)}, \hat{p} \middle| k^{(A)}, k^{(B)} \right) \left( k^{(A)} - 2i^{(A)} \right), \]

and a complementary equation for the concentration \( b^{(B)} \) which can be obtained from Eq. \( \eqref{eq:transient} \) by replacing \( \langle k^{(A)} \rangle \) by \( \langle k^{(B)} \rangle \) and \( k^{(A)} - 2i^{(A)} \) by \( k^{(B)} - 2i^{(B)} \).

A final form of the above system of equations depends on the joint degree distribution \( P \left( k^{(A)}, k^{(B)} \right) \) and the form of the spin flip rate. A particularly simple situation occurs if the layers of the MN are independently generated networks with degree distributions \( P \left( k^{(A)} \right), P \left( k^{(B)} \right) \) which yields the joint degree distribution \( P \left( k^{(A)}, k^{(B)} \right) = P \left( k^{(A)} \right) P \left( k^{(B)} \right) \), and if the model obeys the LOCAL&AND spin update rule so that the spin-flip rate is a product of rates for the model on two monoplex networks corresponding to the two layers, \( f \left( i^{(A)}, i^{(B)}, \hat{p} \middle| k^{(A)}, k^{(B)} \right) = f \left( i^{(A)}, \hat{p} \middle| k^{(A)} \right) f \left( i^{(B)}, \hat{p} \middle| k^{(B)} \right) \). Under these two assumptions the summations over \( k^{(A)} \), \( k^{(B)} \) as well as over \( i^{(A)} \), \( i^{(B)} \) in Eq. \( \eqref{eq:transient} \) can be performed separately which yields

\[ \frac{\partial b^{(A)}}{\partial t} = \frac{2}{\langle k^{(A)} \rangle} \sum_{j \in \{\uparrow, \downarrow\}} c_j \times \sum_{k^{(A)}} P \left( k^{(A)} \right) \sum_{i^{(A)} = 0}^{k^{(A)}} B_{k^{(A)}, i^{(A)}} \left( \theta^{(A)}_j \right) f \left( i^{(A)}, \hat{p} \middle| k^{(A)} \right) \left( k^{(A)} - 2i^{(A)} \right), \]

and a complementary equation for \( b^{(B)} \) which can be obtained from Eq. \( \eqref{eq:transient} \) by exchanging the superscripts \( A, B \).

### B. Application to the case of the q-voter model with independence on multiplex networks

In this subsection general equations of Sec. III.A for the concentrations of spins with orientation up and of the active bonds are written explicitly for the q-voter model with independence on MNs with two layers.

Let us start with the model with the LOCAL&AND spin update rule. For the q-voter model with independence on a monoplex network corresponding to the layer \( G^{(A)} \) of the MN the spin flip rate is

\[ f \left( i^{(A)}, p \middle| k^{(A)} \right) = (1 - p) \prod_{j=1}^{q} \frac{(i^{(A)} - j + 1)}{(k^{(A)} - j + 1)} + \frac{p}{2} \]

\[ = (1 - p) \frac{i^{(A)}! (k^{(A)} - q)!}{k^{(A)}! (i^{(A)} - q)!} + \frac{p}{2}. \]

\[ \eqref{eq:q-voter} \]
The formula for \( f(i^{(B)}, p | k^{(B)}) \) can be obtained from Eq. \([8]\) by changing the superscript \( A \) into \( B \). The rates \( \gamma^+ \), \( \gamma^- \) are

\[
\gamma^+ (c, \theta^+_1, \theta^+_1, p) = (1 - c) \left[ (1 - p) \theta^+_1 q + \frac{p}{2} \right], \\
\gamma^- (c, \theta^+_1, \theta^+_1, p) = c \left[ (1 - p) \theta^+_1 q + \frac{p}{2} \right].
\]

(9)

Substituting Eq. \([8]\) in Eq. \([2]\) as well as Eq. \([8]\) in Eq. \([7]\) and in the latter case performing summations over \( k^{(A)} \), \( k^{(B)} \) as in Ref. \([12]\) yields the following system of rate equations for the macroscopic quantities \( c, b^{(A)}, b^{(B)} \),

\[
\frac{dc}{dt} = \gamma^+ (c, \theta^+_1, \theta^+_1, p) - \gamma^- (c, \theta^+_1, \theta^+_1, p), \\
\frac{db^{(A)}}{dt} = \frac{2}{\langle k^{(A)} \rangle} \sum_{j \in \{\uparrow, \downarrow\}} c_j \left[ (1 - p) \theta^+_1 q + \frac{p}{2} \right] \\
\times \left\{ (1 - p) \theta^+_1 \left[ (\langle k^{(A)} \rangle - 2q - 2(\langle k^{(A)} \rangle - q) \theta^+_1) + \frac{p}{2} \langle k^{(A)} \rangle (1 - 2\theta^+_1) \right] \right\},
\]

(10)

and a complementary equation for \( b^{(B)} \) which can be obtained from Eq. \([10]\) by exchanging the superscripts \( A \) and \( B \).

Further simplification of the above system of equations can be achieved if the two layers are independently generated networks with identical degree distributions \( P(k^{(A)}) = P(k^{(B)}) \) and mean degrees of nodes \( \langle k^{(A)} \rangle = \langle k^{(B)} \rangle = \langle k \rangle \).

Due to the symmetry of Eq. \([10]\) and the complementary equation for \( b^{(B)} \) a solution exists with equal active bond concentrations within both layers, \( b^{(A)} = b^{(B)} = b \) (and thus with \( \theta^+_1 = \theta^+_1 = \theta_j, j \in \{\uparrow, \downarrow\} \)). Then the two rate equations for the active bond concentrations can be replaced with a single equation for \( b \). Hence, in this case the model on a MN is described by only two macroscopic quantities \( c, b \) obeying the equations

\[
\frac{dc}{dt} = (1 - c) \left[ (1 - p) \theta^+_1 q + \frac{p}{2} \right] - c \left[ (1 - p) \theta^+_1 q + \frac{p}{2} \right]^2, \\
\frac{db}{dt} = \frac{2}{\langle k \rangle} \sum_{j \in \{\uparrow, \downarrow\}} c_j \left[ (1 - p) \theta^+_1 q + \frac{p}{2} \right] \\
\times \left\{ (1 - p) \theta^+_1 \left[ (\langle k \rangle - 2q - 2(\langle k \rangle - q) \theta^+_1) + \frac{p}{2} \langle k \rangle (1 - 2\theta^+_1) \right] \right\}.
\]

(11)

In particular, for \( p = 0 \) the above system of equations reduces to

\[
\frac{dc}{dt} = (1 - c) \theta^+_1 q - c \theta^+_1 q, \\
\frac{db}{dt} = \frac{2}{\langle k \rangle} \sum_{j \in \{\uparrow, \downarrow\}} c_j \theta^+_1 q \left[ (\langle k \rangle - 2q - 2(\langle k \rangle - q) \theta^+_1) \right].
\]

(12)

It can be easily verified that in the framework of the PA the above q-voter model with \( p = 0 \) is equivalent to the 2q-voter model on an aggregate monoplex network with mean degree \( 2\langle k \rangle \) being a superposition of the two layers. However, this is not the case for \( 0 < p \leq 1 \).

In the case of the GLOBAL\&AND spin update rule the respective spin-flip rate cannot be written as a product of the rates evaluated separately for each layer. For the model on MNs with two layers it takes a form

\[
f \left( i^{(A)}, i^{(B)}, p | k^{(A)}, k^{(B)} \right) = (1 - p) \prod_{j=1}^q \left( i^{(A)} - j + 1 \right) \prod_{j'=1}^q \left( i^{(B)} - j' + 1 \right) + \frac{p}{2} \langle k \rangle \prod_{j=1}^q \left( k^{(A)} - j + 1 \right) \prod_{j'=1}^q \left( k^{(B)} - j' + 1 \right) \\
= (1 - p) \frac{i^{(A)}! (k^{(A)} - q)! i^{(B)}! (k^{(B)} - q)!}{k^{(A)}! (k^{(A)} - q)! k^{(B)}! (k^{(B)} - q)!} + \frac{p}{2}.
\]

(13)

The rates \( \gamma^+ \), \( \gamma^- \) are

\[
\gamma^+ (c, \theta^+_1, \theta^+_1, p) = (1 - c) \left[ (1 - p) \theta^+_1 q \theta^+_1 q + \frac{p}{2} \right], \\
\gamma^- (c, \theta^+_1, \theta^+_1, p) = c \left[ (1 - p) \theta^+_1 q \theta^+_1 q + \frac{p}{2} \right].
\]

(14)
Assuming again that layers of the MN are independently generated so that \( P(k^{(A)},k^{(B)}) = P(k^{(A)}) P(k^{(B)}) \), substituting Eq. (14) in Eq. (2) as well as Eq. (13) in Eq. (6) and in the latter case performing summations over \( k^{(A)}, k^{(B)} \) as in Ref. [17] yields the following system of rate equations for the macroscopic quantities \( c, b^{(A)}, b^{(B)}, \)

\[
\frac{\partial c}{\partial t} = \gamma^+ \left( c, \theta^{(A)}_+, \theta^{(B)}_+, p \right) - \gamma^- \left( c, \theta^{(A)}_-, \theta^{(B)}_-, p \right),
\]

\[
\frac{\partial b^{(A)}}{\partial t} = \frac{2}{\langle k^{(A)} \rangle} \sum_{j \in \{↑,↓\}} c_j 
\times \left\{ (1-p) \theta^{(A)q}_j \langle k^{(A)} \rangle - 2q - 2 \left( \langle k^{(A)} \rangle - q \right) \theta^{(A)}_j \right\} + \frac{p}{2} \langle k^{(A)} \rangle \left( 1 - 2\theta^{(A)}_j \right) \right\},
\]

and a complementary equation for \( b^{(B)} \) which can be obtained from Eq. (15) by exchanging the superscripts \( A \) and \( B \).

In the case of independently generated layers with identical degree distributions \( P(k^{(A)}) = P(k^{(B)}) \) and \( \langle k^{(A)} \rangle = \langle k^{(B)} \rangle = \langle k \rangle \) the solution with \( b^{(A)} = b^{(B)} = b \) exists which obeys a system of equations

\[
\frac{\partial c}{\partial t} = (1-c) \left( (1-p) \theta^{2q}_+ + \frac{p}{2} \right) - c \left( (1-p) \theta^{2q}_- + \frac{p}{2} \right),
\]

\[
\frac{\partial b}{\partial t} = \frac{2}{\langle k \rangle} \sum_{j \in \{↑,↓\}} c_j \left\{ (1-p) \theta^{2q}_j \langle k \rangle - 2q - 2 \left( \langle k \rangle - q \right) \theta_j \right\} + \frac{p}{2} \langle k \rangle \left( 1 - 2\theta_j \right) \right\}.
\]

It can be easily verified that in the framework of the PA the above \( q \)-voter model with independence in a whole range of \( p, 0 \leq p \leq 1 \), is equivalent to the \( 2q \)-voter model with independence on an aggregate monoplex network with mean degree \( 2\langle k \rangle \) being a superposition of the two layers of the MN.

### IV. RESULTS AND DISCUSSION

In this section results of MC simulations of the mentioned \( q \)-voter models with independence on MNs with layers in the form of complex networks are presented and compared with predictions of the analytic approach based on the PA. Both layers have identical degree distributions. In particular, layers in the form of RRG, ERG and SF networks with various power law exponents \( \gamma \), covering both homo- and heterogeneous networks are considered. The order parameter for the model is the magnetization \( m = \frac{1}{N} \sum_{i=1}^{N} s_i \). Simulations are performed under a reasonable condition \( q \leq \langle k \rangle \); if for a given node the degree within a layer is smaller than \( q \) the whole neighborhood is assumed to form the \( q \)-lobby.

In general, numerical evolution of the models under consideration revealed existence of two phases: the PM phase with \( m = 0 \) and the FM phase with \( m \neq 0 \) with continuous or discontinuous transition from the former to the latter with decreasing parameter \( p \). The overall scenario of this transition for the models on all kinds of investigated MNs, with both LOCAL&AND and GLOBAL&AND spin update rules is the same, only details such as the critical values of the control parameter \( p \) differ. For small values of \( q \) the transition from FM to PM phase with decreasing \( p \) is continuous and becomes discontinuous above a certain value of \( q \), with hysteresis loop which size depends on the topology of the layers, in particular on \( \langle k \rangle \). In order to determine stability regions of both phases, the order parameter for the model is the magnetization \( m = \frac{1}{N} \sum_{i=1}^{N} s_i \). Simulations are performed under a reasonable condition \( q \leq \langle k \rangle \); if for a given node the degree within a layer is smaller than \( q \) the whole neighborhood is assumed to form the \( q \)-lobby.

Analysis of the theoretical equations resulting from the PA presented in section III.B is performed using numerical methods of solving systems of algebraic equations applied to Eq. (11) and (16) in order to find their equilibria and to determine their stability, as well as a Dormand-Prince order 4/5 Runge-Kutta method to obtain time evolution of the concentrations \( c, b \). Equilibria of the systems of equations (11) and (16) are solutions of equations \( \partial b/\partial t = 0, \partial c/\partial t = 0 \). Different stable equilibria correspond to different thermodynamic phases of the models under study. In general, numerical results and predictions of the PA show quantitative or at least qualitative (e.g., concerning the order of the FM transition) agreement for the model on MNs with the mean degree of nodes within layers \( \langle k \rangle \) large enough, in particular for \( \langle k \rangle \gg q \). Otherwise, for small \( \langle k \rangle \) comparable with \( q \), numerical and theoretical results differ substantially. Thus, below the two above-mentioned regimes are discussed separately.
Let us start with the case with large $\langle k \rangle$, so that $\langle k \rangle \gg q$. First, results obtained from the homogeneous PA for this case are outlined; their overall form is the same for both LOCAL&AND and GLOBAL&AND spin update rules. Depending on the parameters $\langle k \rangle$, $q$ two different kinds of bifurcation diagrams of the systems of equations Eq. (11) and (16) are obtained as the parameter $p$ is varied, typical of the continuous or discontinuous phase transition. In both cases at high $p$ the only stable fixed point is $c = 1/2$ ($m = 0$), $b \leq 1/2$, corresponding to the PM phase. In the scenario corresponding to the continuous transition to the FM phase as $p$ is decreased this point loses stability via a supercritical pitchfork bifurcation at $p = p_c$ and for $p < p_c$ a pair of stable equilibria with $c > 1/2$ ($m > 0$), $b < 1/2$, or $c < 1/2$ ($m < 0$), $b < 1/2$, exists, corresponding to the FM phase with positive or negative magnetization, respectively. In the scenario corresponding to the discontinuous transition as $p$ is decreased two pairs of stable and unstable equilibria appear via two saddle-node bifurcations taking place simultaneously at $p = p^{(2)}_c$. For $p^{(1)}_c < p < p^{(2)}_c$ the two above-mentioned stable equilibria, one with $c > 1/2$ ($m > 0$), $b < 1/2$, and the other with $c < 1/2$ ($m < 0$), $b < 1/2$, corresponding again to the FM phase with positive or negative magnetization, respectively, coexist with the stable equilibrium with $c = 1/2$ ($m = 0$), $b \leq 1/2$, corresponding to the PM phase; the basins of attraction of the three stable equilibria are separated by stable manifolds of the two unstable equilibria. Eventually at $p = p^{(1)}_c$ the fixed point corresponding to the PM phase loses stability via a subcritical pitchfork bifurcation by colliding with the above-mentioned pair of unstable equilibria, and for $p < p^{(1)}_c$ the only two stable fixed points are those corresponding to the FM phase. Hence, $p^{(1)}_c$, $p^{(2)}_c$ correspond to the lower and upper critical value of $p$ for the first-order transition, respectively, and for $p^{(1)}_c < p < p^{(2)}_c$ stable PM and FM phases coexist. It should be noted that the values of $p_c$ for
FIG. 2. GLOBAL&AND rule on duplex network. (a): Concentration $c$ vs. degree of stochasticity parameter $p$ according to the PA for fixed value of $\langle k \rangle = 40$. Stable fixed points denoted with solid line, unstable fixed points - with dashed line ($q = 2..7$ from right to left). (b-d) Magnetization $m$ vs. parameter $p$ - numerical results for the ERG (black dots), SF with $\gamma = 3$ (blue circles) and SF with $\gamma = 2.5$ (red triangles) compared with the PA results for different values of $q$.

the continuous and $p_c^{(1)}$ for the discontinuous transition to the FM phase can be obtained analytically by means of linear stability analysis of the PM fixed point of Eq. (11) or (16) with $c = 1/2$ ($m = 0$) (see Appendix).

Provided that $\langle k \rangle \gg q$, according to the homogeneous PA the models under consideration exhibit first-order phase transition to FM phase for $q \geq 5$ in the case of LOCAL&AND spin update rule and for $q \geq 3$ for the GLOBAL&AND rule; otherwise, the transition is second-order. This is illustrated in Fig. 1(a) and Fig. 2(a), respectively, for the case of the model on MNs with layers with relatively large mean degree of nodes $\langle k \rangle$. The above-mentioned minimal values of the size of the $q$-lobby for the occurrence of the discontinuous FM transition agree with those in the appropriate models on MNs with layers in the form of complete graphs [10]. However, MC simulations reveal that the details of the continuous or discontinuous FM transition depend on the topology of the layers. Comparison of the theoretical approach with numerical simulations of the model on MNs with various topologies of the layers, presented in Fig. 1 (b-d) and Fig. 2 (b-d), shows very good quantitative agreement between theory and simulations for layers in the form of homogeneous networks, in particular ERG, RRG (results not shown on plots due to almost full overlap with those for ERG) and weakly heterogeneous SF networks with exponent $\gamma \geq 3$ (i.e., with finite second moment of the degree distribution). Both the order of the transition and the width of the possible hysteresis loop are predicted correctly by the PA. In contrast, numerical results for MNs with layers in the form of strongly heterogeneous SF networks with exponent $2 < \gamma < 3$ significantly differ from the theoretical predictions, which is illustrated in in Fig. 1 (b-d) and Fig. 2 (b-d) for $\gamma = 2.5$. The discrepancies become even bigger for smaller values of $\gamma$ (not shown on plots) which manifests itself with shrinking, or even practically disappearing, hysteresis loop. Nevertheless, despite quantitative
FIG. 3. Comparison between the lower ($p_c^{(1)}$) and upper ($p_c^{(2)}$) critical values of parameter $p$ for the LOCAL&AND rule model with (a) $q = 4$, (c) $q = 5$, (e) $q = 6$, obtained from MC simulations of the model on RRGs (symbols) and from PA (lines). Magnetization $m$ vs. parameter $p$ - numerical results for the RRG (connected black dots) compared with the PA results for: (b) $q = 4$, (d) $q = 5$, (f) $q = 6$, $\langle k \rangle = 6$. Differences, the order of the phase transition is predicted correctly by the homogeneous PA, thus there is qualitative agreement between theoretical and numerical results. Provided that $\langle k \rangle \gg q$ in the case of MNs with homogeneous or weakly heterogeneous layers agreement between numerical and theoretical results based on the homogeneous PA is very good in a broad range of the mean degrees of nodes $\langle k \rangle$. This is illustrated in Fig. 3 where location of the critical point or points in the case of second- and first-order FM transition, respectively, is shown as a function of $\langle k \rangle$ for the model with LOCAL&AND spin update rule with different sizes of the $q$-lobby. The position of the critical point $p_c$ in the case of the second-order transition (Fig. 3(a))
and of the critical points $p_c^{(1)}$, $p_c^{(2)}$ in the case of the first-order transition (Fig. 3(c,e)) is very well predicted directly from simulations of Eq. (11). In the case of the critical point $p_c$ for the second-order transition and the lower critical point $p_c^{(1)}$ for the first-order transition this prediction coincides with the analytic result of Eq. (26) in the Appendix. The values of $p_c$ or $p_c^{(1)}$ and $p_c^{(2)}$, as well as the width of the possible hysteresis loop increase with $\langle k \rangle$ and saturate at maximum values for $\langle k \rangle \to \infty$, with the topology of the layers approaching that of a complete graph. This proves that the homogeneous PA captures all details and provides quantitatively correct description of the FM transition in the $q$-voter model with independence on MNs with homogeneous or weakly heterogeneous layers with large enough mean degrees of nodes within layers. This description is significantly improved in comparison with that based on the MF approximation which results do not depend even on such basic feature of the layers as the mean degree of nodes [16]. In contrast, as mentioned above, in the case of MNs with strongly heterogeneous layers the homogeneous PA, which does not assume any particular topology of the network, is insufficient to describe quantitatively the details of the FM transition in the $q$-voter model under study.

As mentioned in Sec. III.B in the framework of PA the $q$-voter model with independence on MNs with the GLOBAL\&AND spin update rule is equivalent to the 2$q$-voter model on an aggregate monoplex network being a superposition of the two layers. This can be also seen from comparison of numerical results in Fig. 2(b, d) for the model on a MN with layers in the form of ERG with the corresponding results in Fig. 4 for the model on an appropriate monoplex network. This equivalence has already been observed in the $q$-voter model on MNs with layers in the form of complete graphs [16].

Let us in turn consider the models on MNs with layers with the mean degree of nodes $\langle k \rangle$ small and comparable with the size of the $q$-lobby. In this case predictions of the homogeneous PA and results of MC simulations differ substantially even for the models on MNs with layers in the form of homogeneous ERG or RRG. These differences are illustrated in Fig. 3 for the model with the LOCAL\&AND spin update rule. Again, MC simulations reveal that for $q \leq 4$ the model undergoes continuous (Fig. 3(a,b)) and for $q \geq 5$ discontinuous FM transition (Fig. 3(c-f)). For $q = 4$ and small $\langle k \rangle$ the PA correctly predicts the occurrence of the continuous FM transition, but the predicted critical value $p_c$ significantly exceeds that obtained numerically (Fig. 3(a,b)), in contrast with what is observed for larger $\langle k \rangle$. For $q = 5$ both the lower and upper critical values $p_c^{(1)}$, $p_c^{(2)}$ obtained from the MC simulations decrease fast with decreasing $\langle k \rangle$, and for $\langle k \rangle < 10$ the numerical value of $p_c^{(1)}$ becomes much lower than that predicted by the PA (Fig. 3(c)). Nevertheless, for $\langle k \rangle > 6$ the theory correctly predicts that the transition is first-order, and the numerical value of $p_c^{(2)}$ agrees with that predicted from the PA (Fig. 3(c)). However, for $\langle k \rangle \leq 6$ the PA incorrectly predicts that the lower and upper critical values $p_c^{(1)}$, $p_c^{(2)}$ merge, the transition is second-order and occurs at higher $p_c$ than observed in simulations (Fig. 3(c,d)).

Discrepancies between the theory based on the PA and results of the MC simulations are even more significant for $q = 6$ (Fig. 3(e,f)). For example, for $\langle k \rangle = 7$ the PA predicts that with decreasing $p$ the PM fixed point at $m = 0$ loses stability at $p = p_c^{(1)}$, but in a narrow interval of the parameter $p$ just below $p_c^{(1)}$ another stable fixed point appears corresponding to a phase with partial FM ordering characterized by small but non-zero value of the magnetization. Only as $p$ is further decreased the latter fixed point loses stability and a discontinuous transition to the usual FM
phase with \( m \approx 1 \) occurs (Fig. 3(f)). This complex scenario is not confirmed by the MC simulations where usual discontinuous jump of the magnetization is observed with decreasing \( p \), typical of the first-order FM transition to a highly ordered phase, at a critical value \( p_c^{(1)} \) much lower than that predicted theoretically (Fig. 3(f)). In contrast, for increasing \( p \) the theory correctly predicts that the transition from the FM to the PM phase is discontinuous and the predicted upper critical value \( p_c^{(2)} \) agrees well with that obtained from the MC simulations (Fig. 3(e,f)). For \( q \geq 7 \) and small \( \langle k \rangle \) such substantial differences between the numerical and theoretical scenarios for the FM transition in the model under study do not occur. Nevertheless, the examples above show that the theory based on the homogeneous PA, though more elaborate than that based on a simple MF approximation, can also fail for the \( q \)-voter model with independence on MNs with homogeneous layers with low density of edges (similar observation was reported for the case of the \( q \)-voter model on monoplex networks [17]).

V. SUMMARY AND CONCLUSIONS

In this paper the \( q \)-voter model with independence, which is a sort of model for the opinion formation, was studied on MNs with two fully overlapping layers in the form of complex networks with identical degree distributions, corresponding to different levels of social influence. The presence of the two layers was taken into account by assuming the LOCAL&AND or GLOBAL&AND rules for the opinion update of the agents, which differ by the way in which the lobby of neighbors influencing the opinion of a given agent is formed. Both in theoretical investigations based on the homogeneous PA and in MC simulations FM phase transition was observed as the parameter \( p \) controlling the level of agents’ independence was changed. This transition can be first- or second-order, depending on the size of the \( q \)-lobby, and the details of the transition depend on the topology of the underlying MN. The homogeneous PA was derived for a more general case of a two-state spin model with up-down symmetry on MNs and applied to the above-mentioned \( q \)-voter model. Good agreement was obtained between predictions of this PA and results of MC simulations for the model on MNs with layers with moderate and large mean degrees of nodes \( \langle k \rangle \), in particular significantly larger than the size of the \( q \)-lobby. Then theoretical and numerical results show good quantitative agreement for the model on MNs with layers in the form of homogeneous ERGs and RRGs and weakly heterogeneous SF layers. For the model on MNs with strongly heterogeneous SF layers this agreement is only qualitative, e.g., the order of the transition is predicted correctly. Theoretical and numerical results diverge and can differ even qualitatively for mean degrees of nodes \( \langle k \rangle \) small and comparable with \( q \), independently of the degree of heterogeneity of the layers forming the MN. Results obtained in this paper can be easily extended to other models, e.g., the \( q \)-neighbor Ising model, as well as to models on MNs with more than two layers and layers with different topologies.

It can be seen that the systems of equations (11) and (16) for the concentrations \( c, b \) obtained in the homogeneous PA depend only on the mean degree of nodes \( \langle k \rangle \) within each layer of the MN rather than on the precise form of the degree distribution \( P(k) \). This is probably the main source of quantitative discrepancy between theoretical and numerical results for the width and location of the hysteresis loop in the case of the first-order transition to the FM phase in the \( q \)-voter model with independence on MNs with strongly heterogeneous SF layers with \( \gamma < 3 \). It can be expected that predictions based on some form of heterogeneous PA will yield better agreement with results of MC simulations. Such predictions can be obtained, e.g., by extending the general formulation of the heterogeneous PA for systems on monoplex networks [24, 25] to the case of MNs, as it was done in Ref. [40] for the majority-vote model on MNs. It should be mentioned that also the cases of the \( q \)-voter model with independence on MNs with partial overlap and with correlations between degrees of nodes within different layers can be relatively easily studied in the framework of the above-mentioned extension.

APPENDIX

In this Appendix stability of the PM phase for the \( q \)-voter model with independence on a MN with two layers with identical degree distributions is investigated. For the LOCAL&AND and GLOBAL&AND spin update rules the PM phase corresponds to the fixed point of the two-dimensional systems of equations, Eq. (11) and Eq. (16), respectively, characterized by \( c = 1 - c = 1/2 \). By performing linear stability analysis it is shown that with decreasing \( p \) the PM fixed point loses stability at \( p = p^\ast \) \((0 < p^\ast < 1)\); in the case of the second-order transition to the FM phase \( p^\ast \) corresponds to the critical value \( p_{c}^{(1)} \), while in the case of the first-order transition it corresponds to the lower critical value \( p_{c}^{(1)} \).

Let us start with the LOCAL&AND spin update rule and denote the right-hand sides of the system of equations (11) as...
\[ A(c, b) = (1 - c) \left[ (1 - p)\theta^q + \frac{p}{2} \right]^2 - c \left[ (1 - p)\theta^q + \frac{p}{2} \right]^2 \]
\[ B(c, b) = \frac{2}{\langle k \rangle} \sum_{j \in \{1, 4\}} c_j \left[ (1 - p)\theta^q + \frac{p}{2} \right] \times \left\{ (1 - p)\theta^q \left[ \langle k \rangle - 2q - 2(\langle k \rangle - q)\theta_j \right] + \frac{p}{2} \langle k \rangle (1 - 2\theta_j) \right\}. \] (17)

At fixed points there is \( A(c, b) = 0, B(c, b) = 0 \). The (stable or unstable) PM fixed point exists for a whole range of the parameter \( p \), i.e., for \( 0 < p < 1 \). Since at this point \( c = 1/2 \), from the condition \( A(c, b) = 0 \) follows that \( \theta^q = \theta, \) where \( 0 < \theta < 1 \) depends on \( p \), i.e., the position of the PM fixed point is \( (c = 1/2, b = \theta) \). Then, since \( 0 < p < 1 \) there is \( (1 - p)\theta^q + \frac{p}{2} > 0 \), for \( c = 1/2 \) from the condition \( B(c, b) = 0 \) follows that
\[ (1 - p)\theta^q \left[ \langle k \rangle - 2q - 2(\langle k \rangle - q)\theta \right] = -\frac{p}{2} \langle k \rangle (1 - 2\theta). \] (18)

Solution of this nonlinear equation yields the value of \( \theta \) at the PM fixed point.

Stability of the PM fixed point can be determined from the eigenvalues of the Jacobian matrix of the right-hand sides of Eq. (17) at the fixed point. For this purpose let us first evaluate
\[
\begin{align*}
\frac{\partial \theta^q}{\partial c} & \bigg|_{c = \frac{1}{2}, b = \theta} = q \theta^q \left( \frac{1}{1 - c} \right) \bigg|_{c = \frac{1}{2}, b = \theta} = 2q\theta^q, \\
\frac{\partial \theta^q}{\partial b} & \bigg|_{c = \frac{1}{2}, b = \theta} = \frac{q \theta^q}{2(1 - c)} \bigg|_{c = \frac{1}{2}, b = \theta} = q \theta^q, \\
\frac{\partial \theta^q}{\partial c} & \bigg|_{c = \frac{1}{2}, b = \theta} = -\frac{q \theta^q}{c} \bigg|_{c = \frac{1}{2}, b = \theta} = -2q\theta^q, \\
\frac{\partial \theta^q}{\partial b} & \bigg|_{c = \frac{1}{2}, b = \theta} = \frac{q \theta^q}{2c} \bigg|_{c = \frac{1}{2}, b = \theta} = q \theta^q. 
\end{align*}
\] (19)

Using the above formulae it is easily obtained that
\[ \frac{\partial A}{\partial b} \bigg|_{c = \frac{1}{2}, b = \theta} = 0. \] (20)

Thus, the Jacobian at the PM fixed point has an overall form
\[ \begin{bmatrix}
\frac{\partial A}{\partial c} \bigg|_{c = \frac{1}{2}, b = \theta} & 0 \\
\frac{\partial B}{\partial c} \bigg|_{c = \frac{1}{2}, b = \theta} & \frac{\partial B}{\partial \theta} \bigg|_{c = \frac{1}{2}, b = \theta}
\end{bmatrix}, \] (21)
and its eigenvalues are \( \lambda_1 = \frac{\partial A}{\partial c} \bigg|_{c = \frac{1}{2}, b = \theta}, \lambda_2 = \frac{\partial B}{\partial \theta} \bigg|_{c = \frac{1}{2}, b = \theta} \). The PM fixed point loses stability as with decreasing \( p \) one of the eigenvalues changes sign from negative to positive. After some transformations it is obtained that
\[ \lambda_1 = \left[(1 - p)\theta^q + \frac{p}{2}\right] \left[ 2(1 - p)(2q - 1)\theta^q - p \right], \] (22)
and, taking into account Eq. (18),
\[ \lambda_2 = \frac{2}{\langle k \rangle} \left[(1 - p)\theta^q + \frac{p}{2}\right] \left\{ (1 - p)q \theta^q - \left[ \langle k \rangle - 2q - 2(\langle k \rangle - q)\theta \right] - 2(1 - p)\theta^q \langle \langle k \rangle - q \rangle - p \langle k \rangle \right\}. \] (23)
Concerning \( \lambda_2 \), multiplying both sides of Eq. (18) by \( q \) and dividing by \( \theta \) and then inserting the result in Eq. (23) it is finally obtained that
\[ \lambda_2 = -\frac{2}{\langle k \rangle} \left[(1 - p)\theta^q + \frac{p}{2}\right] \left[ \frac{pq \langle k \rangle}{\theta}(1 - 2\theta) + 2(1 - p)\theta^q \langle \langle k \rangle - q \rangle + p \langle k \rangle \right]. \] (24)
Thus, for $0 < p < 1$, $0 \leq \theta \leq 1/2$, $q \leq \langle k \rangle$ there is $\lambda_2 < 0$. Hence, the PM fixed point with decreasing $p$ loses stability when $\lambda_1$ crosses zero. Since $(1 - p)\theta^q + \frac{p}{2} > 0$ this happens at $p = p^*$ such that

$$2(1 - p^*)(2q - 1)\theta^{*q} - p^* = 0,$$

i.e., for

$$p^* = \frac{2(2q - 1)\theta^{*q}}{1 + 2(2q - 1)\theta^{*q}}.$$

where $\theta^*$ denotes the value of $\theta$ at the critical point. Substituting $p = p^*$ in Eq. (18) yields finally

$$\theta^* = \frac{\langle k \rangle - 1}{2\langle k \rangle - 1}.$$

For $\langle k \rangle \to \infty$, i.e., for the case of layers in the form of fully connected graphs, there is $\theta^* \to 1/2$ and

$$p^* = \frac{2q - 1}{2q - 1 + 2^{q - 1}}.$$

Comparison of the above result for $q = 2$ ($p^* = 0.6$) and $q = 3$ ($p^* = 5/9 = 0.555\ldots$) with Fig. 9 and Fig. 16(b) of Ref. [16] (when the model shows continuous phase transition) and for $q = 5$ ($p^* = 0.36$) with Fig. 14 and Fig. 16(b) of Ref. [16] (when the model shows first-order phase transition) shows good agreement of Eq. (28) with predictions based on the MF approximation and with results of MC simulations of the respective $q$-voter model with independence.

In the case of the GLOBAL&AND spin update rule the stability analysis of the PM fixed point of the system of equations (16) can be performed in a similar way, which yields at the critical point

$$p^* = \frac{2(2q - 1)\theta^{*2q}}{1 + 2(2q - 1)\theta^{*2q}},$$

$$\theta^* = \frac{\langle k \rangle - 1}{2\langle k \rangle - 1}.$$

The same result can be obtained by replacing $\langle k \rangle \to 2\langle k \rangle$, $q \to 2q$ in Eq. (34) of Ref. [17], which yields the value of $p^*$ for the $q$-voter model with independence on a monoplex network. This is in agreement with the fact that the system of equations (16) describing the $q$-voter model with GLOBAL&AND spin update rule on a MN with two layers can be obtained from that describing the $q$-voter model on a monoplex network by making the above-mentioned replacement (Sec. III.B). Again, for $\langle k \rangle \to \infty$ there is $\theta^* \to 1/2$ and

$$p^* = \frac{2q - 1}{2q - 1 + 2^{2q - 1}}.$$

Comparison of the above result for $q = 2$ ($p^* = 0.2727\ldots$) with Fig. 7 of Ref. [16] (when the model shows continuous phase transition) shows good agreement of Eq. (30) with predictions based on the MF approximation and with results of MC simulations of the respective $q$-voter model with independence.

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