Evolving Network With Different Edges

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Abstract

We proposed an evolving network model constituted by the same nodes but different edges. The competition between nodes and different links were introduced. Scale free properties have been found in this model by continuum theory. Different network topologies can be generated by some tunable parameters. Simulation results consolidate the prediction.

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I. INTRODUCTION

For the past few years, the interesting properties of complex network, which can be found in many fields, such as biology, sociology, physics, and so on, have attracted a number of people to explore. Great contribution including the idea and analysis of random networks\[1\], small world networks such as the WS network\[7, 8, 9\] has been made to the modeling process\[1, 4, 5, 6\]. However, real world networks such as Internet, Movie actor and science collaboration graph(see \[7, 19\] all show a power-law degree distribution property, which can not be explained by the two models mentioned above. A scale free model\[15, 16\] that satisfies a lot of real systems and led many subsequent researchers to further studying has been found by Barabasi and Albert in 1999. On the other hand, this model suffers a drawback that the exponent of the power law is always fixed, while in real networks it varies. Further study has made it explicit that preferential attachment plays a key role in deciding whether the evolving network shows scaling property as expected\[15, 16, 24, 25, 26\]. In 2000, an exact solution to the degree distribution of BA model was obtained by Dorogovtsev, Mendes and Samukhin\[26\]. In their paper, different initial attractiveness is used, to change the exponent of the power-law curve.

Network models are applied to the description of the complex human interactions, known as social networks. Enlightened by the fact that there are complicated relationships between the elements of social networks, researchers brought multiple-type vertex rather than single one, into network models\[27\]. Weighted edges as represented by kinds of social relationships are also introduced in this area\[28\]. Both models show scaling properties. It is challenging to find more fundamental network structures.

Similarly, our model can be understood from another perspective. One way, but not the exclusive one, to understand our concept is to consider the Internet as a network, which comprised by online websites as nodes. We set the virtual connections, e.g. the transportation forms of information between different websites rather than physical wires and cables, as edges between the nodes. It is clear that there are various kinds of communication ways between two websites( HTTP, FTP ). So to speak, different kinds of links need to be applied to growing networks, where competitions exist between different links as well as the nodes. It is attractive to discover the topologies of the whole network and different sub-networks containing only one kind of the two edges, and to consider the relation between different
edges. In the model we construct, the whole network is somewhat like the partial preference BA model. The significant difference is that we add some tunable parameters into the preferential attachment, which makes it possible to alter the exponent of the power-law degree distribution and the shift constant of degree distribution function. Our work begins with the discussion of a scale free network growing like the BA model with a change of the preferential attachment. This is a special example of the growing networks put forward by Dorogovstev and Mendes in 2000 \cite{26} in which they add a constant in the preferential attachment term. In our main work, the tunable parameters in the model play different roles in different edge preferential attachments, and are regarded as a weight to discriminate different attractiveness of different links in the evolving network. The research produces some interesting results: the whole network and the two sub-networks all evolve just like the ordinary scale free network, while the two kinds of links connected to a certain node always differ. Besides, the degree distributions of the two sub-networks are unequal.

II. MODEL AND THEORETICAL APPROACH

At the beginning, we revise the BA model only a bit and get different expressions of the degree distribution function. Before introducing our findings, it is necessary to take a look at the work by Dorogovstev and Mendes in 2000 \cite{26}. Their contribution mainly lies with the modulation of the original BA model and the way they solved the problem, well known as the 'Master Equation Approach'. In their paper, the preferential attachment in BA model is no longer fixed; instead, changeable initial attractiveness, which is given to each site, together with incoming links of the site, determines the probability whether a new link will point to this site. Under long-time limitation, the exponent of the degree distribution varies from 2 to $\infty$, depending on the initial attractiveness. In this paper we do not intend to discuss too many complicated forms of preferential attachments deriving from the BA model. We barely make a simple and clear revision, and attain concise and approximately precise results, with the application of continuum theory.

The preferential attachment in BA model is changed in this way: the possibility of the degree increase of a site is proportional to $\prod(k_i) = m \frac{k_i + f}{\sum(k_i + f)} (k_i$ is the degree of site i, while $f$ is a given constant). This model may be considered as a special example of the model generalized by Dorogovstev and Mendes if one takes $A^{(0)}$ in their model as $m + f$ in our
By making use of master equation to solve this problem, the degree distribution was found to have the exponent \(- (2 + a)\) where \(a = \frac{A^{(0)}}{m}\). A linear shift \(ma = A^{(0)}\) is found in the distribution function

\[
P(q) \approx (1 + a) \frac{\Gamma[(m + 1)a + 1]}{\Gamma(ma)} (q + ma)^{-(2+a)}
\]  

We here apply continuum theory, rather than master equation, to solve the problem we raise. It is not difficult to draw the following equation:

\[
\frac{\partial k_i}{\partial t} = m \frac{k_i + f}{\sum (k_i + f)}
\]  

where \(m\) is the number of links every time the network increases. The solution is quite simple: as \(t\) approaches infinity, the degree distribution follows this equation

\[
P(k) = (2 + \frac{f}{m})(m + f)^{2 + \frac{f}{m}}(f + k)^{-(3 + \frac{f}{m})}
\]  

We compare the degree functions (1) and (3), taking the power term into account, and we find that the two results are the same in essence. In our derivation the tunable parameter is \(f\), while in D&M’s it’s the initial attractiveness \(A^{(0)}\). By transforming \(A^{(0)}\) to \(m + f\) or inversely, the two functions show the same property. This transformation is reasonable. If we adopt D&M’s way in our model, \(A^{(0)}\) should be the initial degree a site has when it enters the network, which is exactly \(m + f\). In D&M’s paper, the exponent \(\gamma = 2 + a\) varies from 2 to \(\infty\) while in our model, \(\gamma = 3 + \frac{f}{m}\). \(f\) can be taken from \(-m\) to \(\infty\). Therefore, the range is the same. The advantage by using continuum theory is that no special function appears, which makes it relatively easy to solve the two-link networks we conceive in the following content. Another difference between our model and D&M’s model is that we consider the network as an undirected network while they consider the network as directed.

The following is our new proposal based on the idea of a new kind of network. This network, which contains two kinds of edges, is very different from the networks we have studied till now. If we consider the different links as identical, the model should come back to partial preferential attachment BA case. To reach this point briefly, we give different linear preferential attachments to two kinds of edges respectively. Of course, modulatory parameters are added to make our model changeable to fit the real networks.

*The model:* At each time interval, a new node adds to the network and connects to the old nodes with \(m\) edges. We divide the \(m\) edges into two types: one is called \(X\) edge(or \(X\)
link) and the other $Y$ edge(or $Y$ link). The probability that node $i$ in the original network makes a connection to the new added node follows such preferential attachments:

$$\frac{\partial x_i}{\partial t} = \frac{m x_i + y_i + f + gy_i}{2(\sum x_j + y_j + f)}$$ (4)

$$\frac{\partial y_i}{\partial t} = \frac{m x_i + y_i + f - gy_i}{2(\sum x_j + y_j + f)}$$ (5)

where $x_i$ represents the number of $X$-edges node $i$ has connected at time $t$; $y_i$ represents the number of $Y$-edges.

Here, we give linear asymmetric expressions in $X$ and $Y$ edge aiming to represent the linear relations in two networks. However, we do not assume this to be exactly the reflection of real world networks, it is a step forward in the direction of finding out complex relations between networks that share same nodes but different links. The parameter $g$ here plays the key role in determining the relation between $X$ and $Y$ network: When $g$ is very small, it can be neglected so that the two networks evolve in the same way– they just behave as the same network. When $g$ approaches infinity, say, some relatively large number, the above expressions show that new edges will always be added as $X$ edges while the number of $Y$ edges could only remain constant instead of increasing. In general, $g$ is used to determine the difference of the increasing rates of the two different edges.

The number of total edges node $i$ connects satisfies the following equation:

$$\frac{\partial (x_i + y_i)}{\partial t} = \frac{m x_i + y_i + f}{\sum x_j + y_j + f}$$ (6)

We have studied this kind of network before, $x_i + y_i = k_i$ and degree distribution follows equation (3).

Firstly, we discuss $Y$ sub-network. Because we add $m$ edges at every time step, so

$$\sum (x_i + y_i + f) = (2m + f)t$$ (7)

According to continuum theory, equation (5) became

$$\frac{\partial y_i}{\partial t} = \frac{m(m + f)(\frac{t}{t_i})^{\frac{m}{2(m + f)}} - gy_i}{2(2m + f)t}$$ (8)

Its solution is

$$y_i = \frac{m + f}{2 + g} \left( \frac{t}{t_i} \right)^{\frac{m}{2(m + f)}} + Const \cdot t^{-\left(\frac{ma}{2(2m + f)}\right)}$$ (9)
When \( t \) approaches infinity, \( t^{-\left(\frac{mg}{2m} \right)} \) approaches 0 and \( y_i \) becomes

\[
y_i \approx \frac{m + f \cdot t}{2 + g \cdot t_i}^{\frac{m}{2m + f}} \tag{10}
\]

Then we obtain

\[
P_y(y_i < y) = P(t_i > y^{-\left(2 + \frac{f}{m}\right)}\left(\frac{m + f}{2 + g}\right)^{\left(2 + \frac{f}{m}\right)t_i})
= 1 - \frac{1}{m_0 + t} y^{-\left(2 + \frac{f}{m}\right)}\left(\frac{m + f}{2 + g}\right)^{\left(2 + \frac{f}{m}\right)t}
\]

For large \( t \), \( P(y_i) \) reads

\[
P_y(y) = \frac{\partial P(y_i < y)}{\partial y} \approx \left(2 + \frac{f}{m}\right)\left(\frac{m + f}{2 + g}\right)^{\left(2 + \frac{f}{m}\right)y^{-\left(3 + \frac{f}{m}\right)}}
\]

Up to now, we have got the mathematical expression of degree distribution of \( Y \) sub-network, i.e. equation \((12)\). The degree distribution is power law with the same exponent as that of total degree distribution but without shift \( f \). Similarly, we proceed in \( X \) sub-network.

According to equation \((11)\) and \((10)\), we get

\[
x_i = k_i - y_i
= \frac{1 + g}{2 + g} (m + f)\left(\frac{t}{t_i}\right)^{\frac{m}{2m + f}} - f - Const \cdot t^{-\left(\frac{mg}{2m + f}\right)}
\]

Under long-time limitation, we neglect \( Const \cdot t^{-\left(\frac{mg}{2m + f}\right)} \), and obtain

\[
x_i = \frac{1 + g}{2 + g} (m + f)\left(\frac{t}{t_i}\right)^{\frac{m}{2m + f}} - f
\]

The same as what we dealt with \( Y \) sub-network, the degree distribution of \( X \) sub-network becomes

\[
P_x(x) = \left(2 + \frac{f}{m}\right)\left[\frac{1 + g}{2 + g} (m + f)\right]^{\left(2 + \frac{f}{m}\right)^{\left(2 + \frac{f}{m}\right)(x + f)^{-\left(3 + \frac{f}{m}\right)}}}
\]

It is a power-law distribution with shift \( f \). The preferential attachments \((11)\) and \((15)\) are not the same. Therefore, the number of \( x \) edges does not equal to the number of \( y \) edges. From the calculations above, we get the ratios of total degree to \( x \) degree and \( y \) degree

\[
P_{x+y}(k-f) : P_x(k-f) : P_y(k)
= 1 : \left(\frac{1 + g}{2 + g}\right)^{2 + \frac{f}{m}} : \left(\frac{1}{2 + g}\right)^{2 + \frac{f}{m}}
\]

the subscript \( x + y \) in \( P_{x+y}(k) \) means the degree in total network.
III. NUMERICAL SIMULATIONS

We made computer simulations of the model. The results accorded with the theoretical predictions quite well at most parameters ranges. Figure 1(a), (b) and (c) give simulations of networks with size \(N = 1000000\), \(m = 5\) and \(f = -2, 2, 5\) respectively. With log-log scale, the main part of the distributions give slope lines in the figure. The slopes of linear fit approximately equal to the theoretical results \((\gamma = 2.6, 3.4, 4\) respectively). The prediction of the ratios of \(x\) degree to \(y\) degree are also confirmed by numerical simulations (see figure 2). One should notice that in the figures, we had made shifts for \(x\) degree distributions and total degree distributions from \(P_x(k)\) and \(P_{x+y}(k)\) to \(P_x(k-f)\) and \(P_{x+y}(k-f)\) respectively. Therefore, their log-log plots behave as straight lines parallel to those of \(y\) degree.

However, we found that when we take small \(g\), the simulations are always different from theoretical predictions. With small \(g\), when \(f\) is positive, the component of \(x\) degree is larger than theoretical value while the component \(y\) degree accords with theoretical value (see figure 3(a)); and when \(f\) is negative, the component of \(y\) degree is larger than theoretical value while the component of \(x\) degree accords with theoretical value (see figure 3(b)). In theoretical calculus, at infinite \(t\) limitation, we neglect the last term of equation (9). When \(g\) is small, decay of this term is slow, and actually in numerical simulations, \(t\) is not infinite. Therefore, the neglect of this term make the bias of simulation results from predictions at small \(g\).

We also looked at the fluctuation of \(x\) degree \(x_k\) to theoretical value as a function of total degree \(k\). It illustrates how the competition generates heterogeneity in edge composition of each vertex. We calculated the relative standard deviation respectively to theoretical value. The fluctuation decreases quickly with the increase of total degree nearly in a power law way (see figure 4). We did not calculate the standard deviation of \(x_k/y_k\) because \(y\) degree may be zero for some vertexes.

IV. DISCUSSIONS

The mathematical expressions of degree distribution of both \(Y\) sub-network and \(X\) sub-network provide deep insight into the dynamics of evolving systems. We build a competitive environment where not only nodes but also different types of links compete. This model
can reflect many properties of social network. A newly added node has a fixed number of $m$ adding edges. However, other nodes decided how many $X$ edges and $Y$ edges there would be. Obviously, other nodes compete for these $m$ edges by what they already have. For instance, let $X$-edge denotes financial relationship between individual persons, and $Y$ represents other connection. Rich people tend to have more financial relationship with other people, while interestingly, more financial links signifies his richness. But when we focus on other links between people, rich people are not necessarily to be so lucky. Due to the limitation of personal capability, time, devotion, one can not have infinite connections with others. Therefore, it is at the expense of less $Y$ edges to get more $X$ edges, and vice versa. The above study shows the relationship between $X$ and $Y$ edges.

An important character of this model is that the degree distribution of $X$ sub-network shows a linear shift while $Y$ sub-network does not. Simulations confirm this character, though the exact value of the shift may has some errors, due to the application of mean-field approximation. The difference of linear shift between $X$ and $Y$ sub-network comes from different preferential attachment.

In the following, we will discuss how the variables $g$, $f$ and $m$ work on in our theory.

$m$ is always thought as a fixed number in a certain network, for we can identify at least vaguely how many links are added in each time interval. The value of $f$ varies from $-m$ to $\infty$, but if $f$ is too large both of total degree distribution and $x$ degree distribution approach exponential increment,

$$
(k + f)^{-(3 + \frac{f}{m})} = e^{-3 - \frac{f}{m} + (k + f)} = e^{-3 - \frac{f}{m} + (k + f)}\approx f^{-(3 + \frac{f}{m})} e^{-x(3 + \frac{f}{m})}
$$

(17)

And if $f >> m$, the characteristic degree of total network and $X$ sub-network is $m$. On the other hand, when $f$ is taken a bit large, the actual probability to connect $Y$ links will be so small that in the end most of the nodes have small $y$ degree, while the rest have very large $y$ degree.

Note that in the several derivations above, we neglect the term $Const \cdot t^{-(\frac{mg}{2(2m+f)})}$ for time $t$ is considered approaching to infinite. However, this ideal condition can not be reached to apply the model, i.e. $t$ will always be finite actually, even considerable large. So problem rises that if $g$, $f$ and $m$ are chosen or set so that the component $\frac{mg}{2(2m+f)}$ is small enough and the whole term becomes an unneglectable factor to the degree distribution. The derivation
above may not hold true. But if we take this term into account, we will find ourself awkward in searching a precise solution for the function of degree distribution.

The most intriguing parameter in the parameters is $g$. Apparently, from above discussion, $g$ should not be chosen too small to make the term $t^{(-\frac{mg}{2(2m+f)})}$ unneglectable. The simulations showed that small $g$ make the degree distributions depart from predictions. $g$ should not be too large, as well. A very large $g$ in finite time lead the network to have few $Y$ links also, as we have found by computer simulation. Based on above reasons, $g$ is preferred to be such a number that is neither too large, nor too small. Here we only use vague words "large" and "small", for these limitations are due to the finity of time a network evolves and the finity of nodes and links it contains. To say a bit more accurate, we need $g$ to be big enough, with a view to $t$, to make the term $t^{(-\frac{mg}{2(2m+f)})}$ small enough to be neglected; but $g$ should not be too larger to make $\frac{\partial x_i}{\partial t} = m \frac{x_i + y_i + f + gy_i}{2(\sum x_j + y_j + f)}$ too larger than $\frac{\partial y_i}{\partial t} = m \frac{x_i + y_i + f - gy_i}{2(\sum x_j + y_j + f)}$, which leads the network having few $Y$ edges.

For a given node with degree $k$ at given time, the probability $p_x(k)$ that how many $X$ edges it has follows equation (4). The figure 4 shows the fluctuation decreases quickly with $k$. It seems that the distribution of fluctuation should follows the binomial distribution. However, here the degree of a node is varying with time. It increased from $m$ to $k$ and $p_x$ varied synchronously. It results in the fact that the fluctuation of the number of $X$ edges is much smaller than that of the binomial distribution.

Finally we discuss directed network and find some difference between this kind of network and what we have studied. We give general results below. Every time step, we introduce $m$ directed edges. The whole degree has a $m$ increment (see [26]).

$$\sum (k_i + f) = (m + f) t$$

using the same approaches, we get

$$P_{x+y}(k) = (1 + \frac{f}{m})^{f+\frac{1}{m+\frac{1}{2}}} (k + f)^{-(2+\frac{1}{m})}$$

(18)

And the degree distributions of $X$ subnetwork and $Y$ subnetwork are

$$P_x(k) = \left(1 + \frac{f}{m}\right) \left[\frac{f(2 + g) - 1}{2 + g}\right]^{(1 + \frac{1}{m})} (k + f)^{-(2 + \frac{1}{m})}$$

(19)

$$P_y(k) = \left(1 + \frac{f}{m}\right) \left(\frac{1}{2 + g}\right)^{(1 + \frac{1}{m})} k^{-(2 + \frac{1}{m})}$$

(20)
The ratios of total degree to $X$ degree and $Y$ degree are

$$P_{x+y}(k - f) : P_x(k - f) : P_y(k)$$

$$= 1 : \left( \frac{f(2 + g) - 1}{f(2 + g)} \right)^{1 + \frac{f}{m}} : \left( \frac{1}{f(2 + g)} \right)^{1 + \frac{f}{m}}$$

Compared with the results of undirected network [16], one can find that proportion of $Y$ edges of directed network is larger than that of undirected network.

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FIG. 1: The distribution of total degree, X degree and Y degree. The size of the network \(N = 1000000\) and \(m = 5\). The lines are linear fits of the main part of the data. (a) \(f = -2, g = 3\). The slopes of the lines are \(\gamma_k = -2.5, \gamma_x = -2.5\) and \(\gamma_y = -2.6\). The prediction is \(\gamma = -2.6\). (b) \(f = 2, g = 1\). The slopes of the lines are \(\gamma_k = -3.3, \gamma_x = -3.4\) and \(\gamma_y = -3.4\). The prediction is \(\gamma = -3.4\). (a) \(f = 5, g = 2\). The slopes of the lines are \(\gamma_k = -3.8, \gamma_x = -4.0\) and \(\gamma_y = -3.9\). The prediction is \(\gamma = -4\).
FIG. 2: The ratios of the distribution $X$ degree to the distribution of $Y$ degree with respective to different $g$. The curves are theoretical predictions. The size of the network $N = 1000000$ and $m = 5$. (a) $f = -2$; (b) $f = 2$; (c) $f = 5$. 
FIG. 3: The degree distribution of total degree, $x$ degree and $y$ degree with small $g = 0.1$. The size of the network $N = 1000000$ and $m = 5$. (a) $f = -2$. (b) $f = 2$. 
FIG. 4: The relative standard deviation of $X$ degree respective to total degree. The size of the network $N = 1000000$. 