Variational Approaches as Fractional Differential Equations Along Theoretical and Numerical Examples

HUSSEIN AL-OBAIDI\(^a\)   RYADH AL-OBAIDI\(^b\)

\(^a\)-Educational Directorate Bavalon, Ministry of Education   E-mail address: has_bx@yahoo.com

\(^b\)- Al-Mustaqbal University College, Hillah, Babil, Iraq   E-mail address: raidh.h.salm\(\bar{a}\)n@mustaqbal-college.edu.iq

Abstract. This paper deals along the solutions numerical and presentence estimates As fractional equation as differential class, while the problem nonlinear part admits some distinct hypotheses. In particular, As precise parameter localization, the non-0 solution presentence is recognized requiring the sub linearity nonlinear part at infinity and origin. Furthermore, theoretical and numerical examples of applications are provided.

1. Introduction

Fractional differential equations (FDEs) are a good tool As modeling of many events in different fields of science and engineering such as chemistry, electrochemistry, electromagnetic, mechanics, electricity, biology, polymer rheology, economics, control theory, regular thermodynamics variation, image and signal processing, aerodynamics, wave propagation, complex medium electrodynamics, biophysics, phenomena of blood flow, damping and visco-elasticity, etc. [5, 9, 11, 12, 16, 19].

We also cite the papers [6, 7, 8, 17, 18] Since systems as fractional was investigated. As of [17, 18], via variation approaches and theory using critical point the multiple solutions presentence As fractional nonlinear differential equations coupled systems was explored. As of [8], utilizing Principle of Ricceri's Variational, the 1 weak solution presentence As class as fractional differential systems was debated. As of [7], engaging Principle of Ricceri's Variational, the infinite weak solutions numisr presentence As impulsive differential fractional systems class was assured. As of [6], utilizing variation approaches and theory using critical point, the multiplicity solutions results As an impulsive fractional class as differential systems was explored.
In this paper, we are attached in the presence results and numerical estimates of solutions As the following nonlinear fractional boundary value problem

\[
(D_\gamma) \quad \frac{\partial^\gamma}{\partial t^\gamma} \left( \frac{\partial^\alpha}{\partial t^\alpha} u(t) \right) = \gamma f(t, u(t), \lambda \in D) = \lambda \equiv D_\lambda \beta \begin{pmatrix} D_1 \end{pmatrix}_\alpha [u] \lambda \in [0,1] \\
\begin{pmatrix} f \end{pmatrix} = 0, 0 \\
\begin{pmatrix} u \end{pmatrix}(0) = u(1) = 0 \\
\end{pmatrix}
\]

Since \( \alpha \in (0,1], \lambda > 0, f: [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \) is an \( L^1 \)- Caratheodory function. The main result of this paper is the investigate of variational and numerical result As the problem \((D_\lambda)\). Also, we direct the reader to \([3, 4, 1]\) As few associated results at this field.

2. Preliminaries

At the current part, we focus on many definitions as basic, lemmas, notations, and propositions utilized throughout the current work.

Defining 2.1 ([11]). Suppose \( a, b \in \mathbb{R} \) and AC \( ([a; b]) \) is the absolutely functions continuous space on \([a; b] \). As \( 0 < \alpha < 1, f \in AC([a, b]) \) right and left CaReplaces fractional derivatives as fractional are distinct via:

\[
\frac{\partial^\alpha}{\partial t^\alpha} f(t) \equiv \frac{\partial^\alpha}{\partial t^\alpha} f(t) = aD_{a}^\alpha f(t) + \frac{1}{\Gamma(1-\alpha)} \int_{a}^{b} (t-s)^{-\alpha} f'(s) ds \\
\]

And

\[
\frac{\partial^\alpha}{\partial t^\alpha} f(t) \equiv \frac{\partial^\alpha}{\partial t^\alpha} f(t) = cD_{b}^\alpha f(t) = -aD_{b}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{b} (s-t)^{-\alpha} f'(s) ds \\
\]

Since \( \Gamma(\alpha) \) is the function of gamma. Notice such if \( \alpha = 1, \frac{\partial^\alpha}{\partial t^\alpha} f(t) = f'(t), \frac{\partial^\alpha}{\partial t^\alpha} f(t) = f'(t) \).

For creating appropriate spaces of function and Replace on theory as point of critical for exploring the solutions presence As the problem \((D_\lambda)\) we need the essential following findings and notations that will is utilized in launching our key results.

Suppose \( 0 < \alpha \leq 1, 1 < p < \infty, \) and \( E_{0}^{\alpha, p}(0,1) \) is the space of Banach that is \( C_0^\infty \) closure \(([0,1])\) respecting to norm

\[
\|u\|_{E_0^{\alpha, p}(0,1)} = \|D_{a}^\alpha u(t)\|_{L^p(0,1)} + \|D_{b}^\alpha u(t)\|_{L^p(0,1)} \\
\]

It is a recognized element where \( E_{0}^{\alpha, p}(0,1) \) is a Banach space as separable and reflexive \([10, Proposition 3.1]\). Shortly, \( E_{0}^{\alpha, 2} = E^{\alpha} \) is space of Hilisrt of product as inner.

\[
(u, v)_{\alpha} = \int_{0}^{1} (D_{a}^\alpha u(t)D_{b}^\alpha v(t) + u(t)v(t)) dt \\
\]

and the norm
\[ \|u\|_{0,\infty}^2 = \int_0^1 (|\mathcal{D}_t u(t)|^2 + a(t)|u(t)|^2)dt \]

**Remark 2.1** ([10]). When \( \varpi > \frac{1}{2} \) As any \( u \in E^\varpi \) we have

\[ \|u\|_{\infty} \leq k \left( \int_0^1 |\mathcal{D}_t u(t)|^2 dt \right)^{1/2} = k\|u\|_{0,\varpi}, \]

Along \( k = \frac{1}{f(\varpi)^{2\varpi-1}} \)

Here, we offer l solution being classicals and weak definition where problem \( (D_\varpi) \) as following:

**Defining 2.2.** A function \( u \in E^\varpi \) is supposed as problem weak solution \( (D_\varpi) \), if as any \( v \in E^\varpi \),

\[ \int_0^1 (\mathcal{D}_t \mathcal{D}_t^\varpi u(t))(\mathcal{D}_t \mathcal{D}_t^\varpi v(t))dt = \lambda f(t, u(t))v(t)dt \]

At the present, we utilize [13, 15] as following for to establishing the results.

Setting

(2.1) \( \rho(r) = \sup_{v \in M^{-1}(r, +\infty)} \left. \frac{\sup_{u \in M^{-1}(\varpi, +\infty)} N(v) - N(u)}{M(v) - r} \right|_{M(v) - r} \)

(2.2) \( \beta(r_1, r_2) = \inf_{v \in M^{-1}(r_1, r_2)} \left. \frac{\sup_{u \in M^{-1}(r_1, r_2)} N(v) - N(u)}{r_2 - M(v)} \right|_{M(v) - r_2} \)

And

(2.3) \( \rho_2(r_1, r_2) = \sup_{v \in M^{-1}(r_1, r_2)} \left. \frac{N(v) - \sup_{u \in M^{-1}(\varpi, +\infty)} N(u)}{M(v) - r_1} \right|_{M(v) - r_1} \)

In evidence chief results, we will Replace on the coming two formulas.

**Formula 2.1.** [2, formula 5.1] Suppose \( X \) is an actual space of Banach and Suppose \( M, N: X \to \mathbb{R} \) is 2 Gateaux continuous differentiable functions. Adopt that there present \( r_1, r_2 \in \mathbb{R} \) along \( r_1 < r_2 \), as

\[ \beta(r_1, r_2) < \rho_2(r_1, r_2) \]

Since \( \beta \) and \( \rho_2 \) are offered via (2.2) and (2.3), and As every

\[ \lambda \in \left( \frac{1}{\rho_2(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)} \right) \]
the function \( J_\lambda = M - \lambda N \) satisfies \([r_1]PS^{[r_2]}\)-case (see [13, 15]). So as all

\[
\lambda \in \left( \frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)} \right)
\]

There exists \( u_0, \lambda \in M^{-1}(r_1, r_2) \) so \( J_\lambda(u_0, \lambda) \leq J_\lambda(u) \) so all \( u \in M^{-1}(r_1, r_2) \) and \( \lambda(u_0, \lambda) = 0 \).

**Formula 2.2.** [2, Corolary 5.1] Suppose \( X \) is an actual space of Banach and Suppose \( M, N: X \rightarrow \mathbb{R} \) is 2 Gateaux continuously function being differentiable. Replace

\[
\beta^* := \lim_{r \rightarrow +\infty} \inf_{\lambda \in M^{-1}(-\infty, r)} \frac{\sup N(u)}{r}
\]

And adopt that there is \( \bar{r} \in \mathbb{R} \) so \( \rho(\bar{r}) > \beta^* \). Since \( \rho \) is offered via (2.1). Furthermore, adopt that as every \( \lambda \in \left( \frac{1}{\rho(\bar{r})}, \frac{1}{\beta^*} \right) \) the function \( J_\lambda = M - \lambda N \) satisfies \([r_1]PS^{[r_2]}\)-case \( r > \bar{r} \). So there is \( r_2 > \bar{r} \). So as every \( \lambda \in \left( \frac{1}{\rho(\bar{r})}, \frac{1}{\beta^*} \right) \), there is \( u_0, \lambda \in M^{-1}(r_1, r_2) \) so \( J_\lambda(u_0, \lambda) \leq J_\lambda(u) \) as all \( u \in M^{-1}(\bar{r}, r_2) \) and \( \lambda'(u_0, \lambda) = 0 \).

Corresponding to the function \( f \) we introduce the function \( F: [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) as

\[
F(t, \xi) := \int_0^1 f(t, \xi) \, dx
\]

As all \( \xi \in \mathbb{R} \) We need the following Proposition As presentence our main results.

**Proposition 2.3.** [1] Suppose \( S: E^\infty \rightarrow (E^\infty)^* \) is the operator defined via

\[
s(u)(v) = \int_0^1 (D^\infty_t u(t))(D^\infty_t v(t)) \, dt
\]

As any \( u, v \in E^\infty \). So, \( S \) admits as inverse of continuousity on \((E^\infty)^*\).

### 3. Solutions as Variational

In this section, we Asmulate our main results. As this we Replace

\[
A(\alpha) = \frac{1}{\Gamma^2(1-\alpha)} \frac{6 \alpha^2 - 19 \alpha + 16}{(1-\alpha)(2-\alpha)(3-2\alpha)4^{4-2\alpha}}
\]

And \( l := k\sqrt{A(\alpha)} \)

Furthermore, As any two nonnegative constants \( \gamma \) and \( \sigma \) along \( \gamma \neq \sigma \), we set

\[
b_\gamma(\sigma) = \frac{\int_0^l \max \{F(t, \xi)\} \, dt - \int_0^3 F(t, \sigma) \, dt}{\gamma^2 - \sigma^2}.
\]

We signify via F all continuous functions class \( f: \mathbb{R} \rightarrow \mathbb{R} \) satisfy in case as follow:

- there present 2 non-negative constants \( a_1, a_2 \) so
\[
(3.2) \quad |f(t,x)| \leq a_1 + a_2|x|^{p-1} \quad \text{as } x \in \mathbb{R}.
\]

**Formula 3.1.** Adopt that \( f \in F \) and there present 3 actual constants \( \gamma_1, \gamma_2 \) and \( \sigma \) so
\[
0 < \gamma_1 < k\sigma l < \gamma_2 \quad \text{and} \quad b_{\gamma_2}(\sigma) < b_{\gamma_1}(\sigma) \quad \text{so as every parameter}
\]
\[
\lambda \in \left( \frac{1}{2k^2b_{\gamma_2}(\sigma)}, \frac{1}{2k^2b_{\gamma_1}(\sigma)} \right).
\]
the problem \((D_\lambda)\) has at minimum 1 non-01 solution being classical \( u_0, \lambda \in E^\infty \) so \( 2\gamma_1 < \|u_0, \lambda\| < 2\gamma_2 \)

Evidence. We will use formula 2.1. Suppose \( X := E^\infty \) and think through the functionals \( M, N: X \to \mathbb{R} \) definite via
\[
(3.4) \quad M(u) := \frac{1}{2}\|u\|_{X,\infty}^2 \quad \text{and} \quad M(u) := \int_0^1 F(t,u(t))dt.
\]
Via (3.4), the functional \( M: X \to \mathbb{R} \) is coercive. In contrast, \( M \) and
\( N \) are Gateaux continuously differentiable. Extra accurately, we have
\[
M'(u)(v) = \int_0^1 (\xi D^\gamma u(t))(\xi D^\gamma v(t))dt
\]
\[
N'(u)(v) = \int_0^1 f(t,u(t))v(t)dt
\]
As any \( v \in X \). Make \( \lambda > 0 \) A critical functional point \( J\lambda = M - \lambda N \)
is a function \( u \in X \) so \( M'(u)(v) - \lambda N'(u)(v) = 0 \) As any \( v \in X \)
Thus, the critical functional points \( J\lambda \) are weak solution solutions of \((D_\lambda)\) anyway, suppose we perceive that \( M(0) = N(0) = 0 \). Furthermore, via electing Choose \( r_1 = \frac{\gamma_1}{2k^2} \)
and \( r_2 = \frac{\gamma_2}{2k^2} \). At this point Suppose \( u \in \emptyset^{-1}(\infty, r_1) \), owing to (3.4), we have that in consideration, one has
\[
M^{-1}(\infty, r_1) = \{ u \in X; M(u) < r_1 \} \subseteq \{ u \in X; |u| < \gamma_1 \}
\]
and via same argument as above,
\[
M^{-1}(\infty, r_1) \subseteq \{ u \in X; |u| < \gamma_2 \}
\]
Thus, due to the case \( H \)
\[
\sup_{u \in M^{-1}(\infty, r_1)} N(u) \leq \int_0^1 \sup_{|\xi| \leq r_2} F(t, \xi)dt
\]
And
\[\sup_{u \in M^{-1}(-\infty, r_2)} N(u) \leq \int_0^1 \sup_{|\xi| \leq r_2} F(t, \xi) dt\]

At this point we define \(w_\sigma\) via \(4\sigma t\) as \(E \in \left[0, \frac{1}{2}\right]\), \(\sigma\) as \(t \in \left[\frac{1}{2}, \frac{3}{2}\right]\) and \(4\sigma(1 - t)\) as \(t \in \left(\frac{3}{2}, 1\right]\) Clearly, \(w_\sigma \in X\). Obviously, 1 has

\[\|w_\sigma\|_{a, \infty}^2 = A(\infty)\sigma^2 + \int_0^1 a(t) \|w_n(t)\| dt \leq A(\infty)\sigma^2\]

and via considering (3.4) we have

\[(3.5) \quad M(w_\sigma) \leq \frac{A(\infty)}{2}\sigma^2 \quad \text{and} \quad N(w_\sigma) \geq \int_0^1 f(t, \sigma) dt\]

Taking (3.3) in consideration, via computation being direct, one has \(r_1 < M(w_\sigma) < r_2\)

On the other hand,

\[\beta(r_1, r_2) \leq \frac{\sup_{u \in M^{-1}(-\infty, r_1)} N(u) - N(w_\sigma)}{r_2 - M(w_\sigma)} \leq \frac{\int_0^1 \sup_{|\xi| \leq r_2} F(t, \xi) dt - \int_2^3 F(t, \sigma) dt}{\frac{3}{2}} \gamma_1^2 - A(\infty)k^2\sigma^2\]

And

\[\rho_2(r_1, r_2) \geq \frac{N(w_\sigma) - \sup_{u \in M^{-1}(-\infty, r_1)} N(u)}{M(w_\sigma) - r_1} \geq 2k^2 \frac{\int_0^1 \sup_{|\xi| \leq r_1} F(t, \xi) dt - \int_2^3 F(t, \sigma) dt}{\frac{3}{2}} \gamma_1^2 - A(\infty)k^2\sigma^2\]

Thus, via using the notation (3.1), from (3.5), it follows that \(\beta(r_1, r_2) \leq k^2b_{r_2}(\sigma)\) and \(\rho_2(r_1, r_2) \geq k^2b_{r_1}(\sigma)\). Finally, the postulation (3.3) yields \(\beta(r_1, r_2) < \rho_2(r_1, r_2)\). At this end, from the foregoing functional \(M\) is Gateaux continuously differentiable whereas via 2.3 Proposition that discloses inverse as continuous on \(X^*\), the functional \(M\) is Gateaux continuously differentiable whose derivative Gateaux is compact and as \(f \in F\) the functional \(M - N^{[r_1]} P^{[r_2]}\) -case as all \(r_1\) and \(r_2\) along \(r_1 < r_2 < +\infty\) There As, via formula 2.1, As every

\[\lambda \in \left(\frac{1}{2k^2b_{r_1}(\sigma)}, \frac{1}{2k^2b_{r_2}(\sigma)}\right)\]

The functional \(f_\lambda\) has at minimum 1 critical point \(u_0, \lambda\) so \(r_1 < M(u_0, \lambda) < r_2\) which is \(2\gamma_1 < \|u_0, \lambda\| < 2\gamma_2\). Such ends the evidence.

Remark 3.1. The results of formula 3.1 hold if case (3.2) is put instead via the sub-linear at postulation of infinity i.e. \(\lim_{|t| \to \infty} \frac{f(t)}{|t|} = 0\).

At this point, we give a particular case of Formula 3.1.
Formula 3.2. Adopt that \( f \in F \) and there present 2 positive constants and \( \sigma \) along \( \gamma < \sigma \) : so
\[
\frac{\int_0^1 \sup_{|t| \leq \gamma} F(t, x) dt - \int_{\frac{3}{4}}^1 F(t, \sigma) dt}{\gamma^2 - l^2 \sigma^2} < \frac{\int_{\frac{3}{4}}^1 F(t, \sigma) dt}{l^2 \sigma^2}
\]

So as every parameter
\[
\lambda \in \left( \frac{1}{2k^2} \int_0^1 \frac{\gamma^2 - l^2 \sigma^2}{l^2 \sigma^2}, \frac{1}{2k^2} \int_0^1 \sup_{|t| \leq \gamma} F(t, \sigma) dt \right)
\]

The problem \((D_\lambda)\) has at minimum 1 non-0 l solution being classical \( u_0, \lambda \in E^\infty \) so \( \|u_0, \lambda\| < 2\gamma \)

Evidence. Taking \( \gamma_1 = 0 \) and \( \gamma_2 = \gamma \) and in mind having \((3.1)\), we get \( b_\gamma(\sigma) < b_0(\sigma) \) Thus, Formula 3.1 confirms the conclusion.

At this point, we offer an application of Formula 2.2 which will be used later to get problem multiple solutions \((D_\lambda)\).

Formula 3.3. Adopt that \( f \in F \) and there present two positive constants \( \tilde{\gamma} \) and \( \tilde{\sigma} \) along \( \gamma < \tilde{\sigma} \) so as every \( \lambda > \tilde{\lambda} \) so
\[
\tilde{\lambda} := \frac{1}{2k^2} \int_0^1 \frac{\tilde{\gamma}^2 - l^2 \tilde{\sigma}^2}{l^2 \tilde{\sigma}^2} \sup_{|t| \leq \gamma} F(t, \sigma) dt - \int_{\frac{3}{4}}^1 F(t, \sigma) dt
\]

the problem \((D_\lambda)\) has at minimum 1 non-trivial l solution being classical \( u_0, \tilde{\lambda} \in E^\infty \) so \( \|u_0, \tilde{\lambda}\| < 2\tilde{\gamma} \)

Evidence. Take \( X = E^\infty \) and Replace \( I_\lambda = M - \lambda N \) Since \( M \) and \( N \) are offered as in \((3.4)\). The \( M \) and \( N \) functionals fulfill all requested postulations in Formula 2.2. Replace, \( \tilde{\gamma} := \frac{1}{2k^2} \tilde{\gamma}^2 \). From [2, Proposition 1], the functional \( J_\lambda \) fulfills \( PS^{[r]} \) -case as all \( r \) along \( r > \tilde{r} \) debating as in the Evidence of Formula 3.1, we get
\[
p(\tilde{r}) \geq \frac{N(w_o) - \sup_{u \in M^{x^*} (\tilde{r}, \infty)} N(u)}{m(w_o)} \geq 2k^2 \frac{\int_0^1 \sup_{|t| \leq \gamma} F(t, x) dt - \int_{\frac{3}{4}}^1 F(t, \sigma) dt}{\tilde{\gamma}^2 - l^2 \tilde{\sigma}^2}
\]

Thus, from our postulation it is following which \( p(\tilde{r}) > 0 \). Thus, it follows from Formula 2.2 along \( X^* = 0 \) as every \( \lambda > \tilde{\lambda} \), the functional \( J_\lambda \) discloses at minimum 1 local minimum \( u_0, \tilde{\lambda} \in E^\infty \) so \( M(u_0, \tilde{\lambda}) < \tilde{\gamma} \) that is just \( \|\tilde{u}_0, \tilde{\lambda}\| > \tilde{\gamma}^{\frac{1}{c_2}} \). Therefore, the conclusion is gotten.

The result as follow is a straight formula 3.2 consequence.
Formula 3.4. Adopt that \( f \in F \) and

\[
(3.6) \quad \lim_{\xi \to 0^+} \frac{F(t,\xi)}{\xi^2} = +\infty
\]

Furthermore, suppose \( \gamma > 0 \)

\[
\lambda_\gamma^* := \left( \frac{1}{2k^2} \sup_{t \leq \gamma} \frac{\gamma^2}{F(t,\xi)} \right).
\]

So as any \( \lambda \in (0, \lambda_\gamma^*) \), problem \( (D_0) \) admits at minimum 1 non-0 l solution being classical \( u_0, \lambda \in E^\infty \) solve \( ||u_0,\lambda|| < 2\gamma \).

Evidence. Make \( \lambda \in (0, \lambda_\gamma^*) \). From (3.6) there presents a constant \( \sigma > 0 \) along \( \gamma > \sigma l \) so

\[
\frac{1}{2k^2} \int_0^l \frac{\sigma^2}{F(t,\sigma)} dt < \lambda < \frac{1}{2k^2} \int_0^l \frac{\gamma^2 - \sigma^2}{F(t,\gamma)} dt - \int_0^l \frac{\sigma^2}{F(t,\sigma)} dt
\]

Thus, via Formula 3.2, problem \( (D_\lambda) \) has at minimum 1 non-0 l solution being classical \( u_0, \lambda \) so \( ||u_0,\lambda|| < 2\gamma \).

**Remark 3.2.** Suppose that \( g \in G \) so the plotting \( \lambda \to J_\lambda(u_0,\lambda) \) is negative and declining strictly in \( (0, \lambda_\gamma^*) \).

**Remark 3.3.** Generally, Formula 3.4 confirms that if \( g \in G \) fulfills (3.6), so as any parameter \( \lambda \) is long to the actual interval \( A_\theta := (0, \lambda_\gamma^*) \) since

\[
\lambda^* := \left( \frac{1}{2k^2} \max_{\lambda > 0} \frac{\gamma^2}{F(t,\gamma)} \right).
\]

The problem \( (D_\lambda) \) has at minimum 1 non-0 solution \( u_0, \lambda \in E^\infty \).

**Remark 3.4.** We note that, in particular, if \( f \) is sub-linear at infinity along respect to the 2nd variable, formula 3.4 confirms that problem \( (D_\lambda) \) disclose at minimum 1 non-0 l solution being classical as any positive parameter \( \lambda \).

Furthermore, at the current case, the solution geted is non-0, whereas the direct classical method which accepted in the current context, confirms the presentence at minimum 1 solution that may is 0.

**Remark 3.5.** A watchful formula 3.4 evidence analysis confirms that results quiet stay actual if case (3.6) is put instead via the additional general postulation

\[
\lim_{\xi \to 0^+} \sup_{t \leq \xi} \frac{F(t,\xi)}{\xi^2} = +\infty
\]

Furthermore, the preceding asymptotic case at 0 can is put instead via the following

\[
\lim_{\xi \to 0^+} \sup_{t \leq \xi} \frac{F(t,\xi)}{\xi^2} = +\infty
\]
Via the above remarks, we get results as follow:

**Formula 3.5.** Suppose \( \lim_{\xi \to 0^+} \frac{F(t, \xi)}{\xi} = +\infty \) and \( \lim_{\xi \to +\infty} \frac{F(t, \xi)}{\xi} = 0 \). So there presents \( \lambda^* > 0 \) so as any \( \lambda \in (0, \lambda^*) \), the problem \((D_\lambda)\) has at minimum 1 non-0 l solution being classically \( u_0, \lambda \in E^\infty \). Furthermore, we have

\[
\int_0^1 (|\xi| D_t^\infty u(t)|^2)\,dt \to 0 \quad as \quad \lambda \to 0^+
\]

and the charting \( \lambda \to \int_0^1 (|\xi| D_t^\infty u(t)|^2)\,dt \leq \int_0^1 (\int_0^{t_0 \lambda} f(t, x)\,dx)\,d\lambda \) is negative and declining strictly in \((0, \lambda^*)\).

The latest result of this section is the following three presentence formula, that is a consequence of Formulas 3.2 and 3.3.

**Formula 3.6.** Adopt that \( g(0) \neq 0 \) and there present four positive constants \( \gamma, \sigma, \bar{\gamma}, \bar{\sigma} \) along \( \bar{\gamma} < \bar{\sigma} \leq \sigma < \gamma \) so

\[
\int_0^1 \sup_{|t| \leq \gamma} F(t, \xi)\,dt < \int_0^3 F(t, \sigma)\,dt
\]

And \( \int_0^1 \sup_{|t| \leq \gamma} F(t, \xi)\,dt < \int_0^3 F(t, \sigma)\,dt \)

hold, and

\[
(3.7) \quad \frac{\int_0^1 \sup_{|t| \leq \gamma} F(t, \xi)\,dt}{\gamma^2} < \frac{\int_0^1 \sup_{|t| \leq \gamma} F(t, \xi)\,dt < \int_0^3 F(t, \sigma)\,dt}{\gamma^2 - \bar{\sigma}^2} < \frac{\gamma^2}{\bar{\sigma}^2}
\]

is fulfilled. So As every

\[
\lambda \in \Lambda = \left( \max_{|t| \leq \gamma} \left\{ \lambda, \frac{\gamma^2 - \bar{\sigma}^2}{\int_0^1 \sup_{|t| \leq \gamma} F(t, \xi)\,dt < \int_0^3 F(t, \sigma)\,dt}, \int_0^1 \sup_{|t| \leq \gamma} F(t, \xi)\,dt \right\} \right);
\]

the problem \((D_\lambda)\) has at minimum three 1 solution being classically \( u_0, \lambda ; \bar{u}_0, \lambda ; \bar{u}_0, \bar{\lambda} \), so \( \|u_0, \lambda\| < \frac{\gamma}{c_1} \) and \( \|\bar{u}_0, \lambda\| < \frac{\gamma}{c_2} \).

**Evidence.** 1st, in (3.7) view, we obtain \( \Lambda \neq 0 \) then, Mark \( \lambda \in \Lambda \) using Formula 3.2, there is a positive solution being classical \( u_0, \lambda \) i.e., \( \|u_0, \lambda\| < \frac{\gamma}{c_1} \) that is a local associated functional as minimum \( J_\lambda \) whereas formula 3.3 confirms a weak solution \( \bar{u}_0, \lambda \) so \( \|\bar{u}_0, \lambda\| < \frac{\gamma}{c_2} \) which is a local minimum As \( J_\lambda \). Arguing as in the formula 3.1 evidence, we
notice that the functional $I_\lambda$ is coercive, so it fulfills the (PS)-case. Thus, the conclusion drives from the formula of mountain pass as offered via Pucci and Serrin (see [14]).

### 4. Theoretical and Numerical Examples

**Example 4.1.** Consider the problem

$$\frac{\chi D_t^{0.75}}{D_t^{0.75}} \left( \frac{\chi D_t^{0.75}}{D_t^{0.75}} u(t) \right) = \lambda tf(u(t)), \text{ a.e } t \in [0,1]$$

Cases as boundary $u(0) = u(1) = 0$ Since $f(x) = e^x \text{ as } x \in (-\infty,1], f(x) = e^{\sin(\frac{\pi}{2}x)}$ as $x \in (-1,1)$ and $f(x) = e^{-\cos(\pi x)}$ as $x \in [1,\infty)$. A calculation being direct illustrates that $k = \frac{3}{\sqrt{3e(\frac{\pi}{2})}}$,

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{\xi} = \lim_{\xi \to 0^+} \frac{e^{\sin(\frac{\pi}{2}\xi)}}{\xi} = +\infty$$

$$\lim_{\xi \to +\infty} \frac{F(\xi)}{\xi} = \lim_{\xi \to 0^+} \frac{e^{-\cos(\pi \xi)}}{\xi} = 0.$$

At this point, via electing $\gamma = 1$ we obviously obseve that all formula postulations 3.4 are fulfilled. Therefore, using formula 3.4 and remark 3.1 as any $\lambda \in \left( 0, \frac{1}{3e} I^2 \left( \frac{\pi}{2} \right) \right)$, the problem has at minimum $1$ non-0 weak solution $u_0, \lambda \in E^\times$ so $\|u_0, \lambda\| < 2$.

**Example 4.2.** Consider the following fractional differential equation

$$\frac{\chi D_t^{0.75}}{D_t^{0.75}} \left( \frac{\chi D_t^{0.75}}{D_t^{0.75}} u(t) \right) = -u(t), \ t \in [0,1]$$

$$u(0) = u(1) = 0$$

The numerical solution of (4.1) As different values of are plotted in Fig.1.

![Figure 1. The dynamics](image_url)

**Example 4.3.** Consider the problem
(4.2) \( \mathcal{D}^{0.5}_t \left( \mathcal{D}^{0.5}_t u(t) \right) = \cos t - e^{t^2} \sin t, \ t \in [0,1] \)

\[ u(0) = u(1) = 0 \]

Fig.2 shows the numerical approximate solution of (4.2) As several values Of \( \gamma \).

![Figure 2. The dynamics](image)

**Example 4.4.** We solve the following equation

(4.3) \( \mathcal{D}^{\gamma}_t \left( \mathcal{D}^{\gamma}_t u(t) \right) = -t^u(t) \tan h \left( -\sin u(t) + t \right), \ t \in [0,1] \)

\[ u(0) = u(1) = 0 \]

Fig.3 depicts the numerical simulations of (4.3) As several values of several fractional orders of \( \gamma \) on \( t \in [0,1] \).

![Figure 3. The dynamics](image)
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