Holographic Reformulation of String Theory on $\text{AdS}_5 \times S^5$ background in the PP-wave limit

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To the memory of Prof. Bunji Sakita

Abstract

The recent proposal on the correspondence between the $\mathcal{N} = 4$ super Yang-Mills theory and string theory in the Penrose limit of the $\text{AdS}_5 \times S^5$ geometry involves a few puzzles from the viewpoint of holographic principle, especially in connection with the interpretation of times. To resolve these puzzles, we propose to interpret the PP-wave strings on the basis of tunneling null geodesics connecting boundaries of the AdS geometry. Our approach predicts a direct and systematic identification of the S-matrix of Euclidean string theory in the bulk with the short-distance structure of correlation functions of super Yang-Mills theory on the AdS boundary, as an extension of the ordinary relation in supergravity-CFT correspondence. Holography requires an infinite number of contact terms for interaction vertices of string field theory and constrains their forms in a way consistent with supersymmetry.

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1. Introduction

Recently, the so-called parallel-plane (PP) wave limits of AdS geometries have attracted much attention. In particular, the authors of ref. [1] suggested an intriguing possibility of extracting all stringy degrees of freedom in this particular limit from the maximally supersymmetric Yang-Mills theory in a special large \( N \) limit, by identifying the string oscillation modes with local composite operators composed as products of large numbers (of order \( \sqrt{N} \)) of elementary fields.

In the original AdS/CFT correspondence, the basic postulate which realizes the holographic principle, namely, the correspondence between bulk gravity (closed string) theory and super Yang-Mills theory on the boundary is the relation [2]

\[
Z[\phi_0]_{\text{string/gravity}} = \langle \exp(-\int d^4x \sum_i \phi_0^i(x)O_i(x)) \rangle_{\text{ym}},
\]

This identifies the bulk partition function with boundary conditions of the form

\[
\lim_{z \to 0} \phi^i(z, x) = z^{4-\Delta_i} \phi_0^i(x),
\]

for bulk fields \( \phi^i \) to be the generating functional for correlation functions on the side of Yang-Mills theory living on the boundary where \( \{\phi_0^i(x)\} \) play the role of the source fields for local operators with definite conformal dimensions \( \{\Delta_i\} \). Although no rigorous derivation is known, the relation (1.1) can be interpreted as two different but physically equivalent descriptions of response of the system consisting of a large number (= \( N \)) of D3-branes under the influence of probe D3-branes which are put outside of the near horizon region of the system under consideration. § Namely, the left-hand side is the description from the viewpoint of gravity and the right-hand side is the one from the effective gauge theory: In the former the presence of probe D3-branes is encoded as the boundary conditions on the bulk fields, while in the latter the influence of the probe is represented as the external sources for SYM operators which couple with the bulk fields at the boundary.

If one trusts this relation, however, the arguments in ref. [1] which have been followed by many authors raise a couple of puzzles. First, as elucidated in ref. [4] and [5], the string theory on the PP wave geometry can be regarded as a semiclassical approximation

\[\text{§For a discussion elucidating this point, we refer the reader to [3].}\]
around special null geodesics as the trajectories of a string in its massless ground states with large angular momenta. The null geodesic considered in ref. [1] traverses a large circle on $S^5$, with large orbital angular momentum. The affine parameter $\tau$ along such a geodesic can be identified with the time parameter in the global coordinate system of AdS$_5$ ($R^4 = 4\pi g_s N$),

$$ds^2_{\text{ads}} = R^2(- \cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2), \quad (1.3)$$

which has no horizon. In the present paper, we always use the space-time coordinates in the string-frame metric with the string unit $\ell_s = 1$. It is easy to check that this geodesic never reaches the AdS boundary (corresponding to $z \to 0$ using the notation of Poincaré patch below) and also that it goes, within a finite interval with respect to global time, into the horizon of original D3 brane metric, described as the Poincaré patch of the AdS space:

$$ds^2_P = \frac{R^2 dz^2}{z^2} + \frac{dx_3^2 - dt^2}{R^2 z^2}. \quad (1.4)$$

Indeed, a typical null geodesic with large angular momentum $J$ of order $(g^2_{\text{YM}} N)^{1/2}$ along an $S^5$ direction whose location in the AdS$_5$ space is at $\rho = 0$ takes, in terms of the Poincaré coordinate, the form

$$z = \frac{1}{\cos \tau}, \quad t = R^2 \tan \tau. \quad (1.5)$$

This means that the PP wave geometry has no boundary\textsuperscript{*} at least in the original sense of holographic correspondence between bulk and boundary as signified by eq. (1.1). Thus we would not be able to apply the basic holographic relation in order to compute correlation functions of the Yang-Mills theory using the proposed correspondence between string states on the bulk and local operators on the Yang-Mills side.

Second puzzle is that on the Yang-Mills side the transverse directions of string include the time direction. Of course, the time direction of Yang-Mills theory does not in general coincide with the global time of bulk geometry. However, when one traverses outside the horizon of the D3-brane metric in the large $R$ limit, the global time $\tau$ and the time $t$ of D3-branes is proportional to each other, $R^2 \tau \sim t$, along a substantial part of the null geodesic

\textsuperscript{*}This does not exclude the possibility of boundaries for the PP-wave geometry in different sense. Our point is only that such boundaries, however, cannot be easily connected with the boundary in the sense of relation (1.1). For a (partial) list of other works on holography for the PP-wave geometry, see [6].
during a finite time (=D3-brane time) interval \( (\delta t \ll R^2, |\tau| \ll 1) \) characterized by the AdS radius \( R \). This is the regime where the trajectory becomes closest \( (\cos \tau \sim 1) \) to the AdS boundary. Therefore it is difficult to imagine that the Yang-Mills time direction turns into one of transverse directions for string oscillations, unless one stipulates that the regime where a closed string remains outside the horizon of D3 geometry could somehow be ignored. In fact, it is even more harder to imagine how information in the regime where the trajectory remains inside the D3-horizon could be related to the dynamics of boundary theory. We also emphasize that this problem exists irrespectively of whether we use Minkowskian or Euclidean metric (such as \( R^1 \times S^3 \)) on the boundary.

In connection with the problem of time, there is another puzzle. As was already mentioned above, even an infinite time interval with respect to the D3-brane metric corresponds to a finite interval with respect to the affine time by which the dynamics of strings propagating along the null geodesics is described. Thus the Yang-Mills correlators would then correspond to propagation of strings in the bulk in finite intervals with respect to affine time. However, in string theory, finite time transition amplitudes, in general, cannot be regarded as observables. Only meaningful observables are S-matrix elements describing transition amplitudes with infinite time intervals. It seems difficult to imagine how gauge invariant observables on Yang-Mills side are related to meaningful observables on string-theory side.

It is important to clarify these puzzles: Firstly, it is necessary to settle the issues whether and how holographic principle can be realized in the context of PP-wave geometry. Without establishing concrete relation between string-theory side and SYM side, we do not know how to use the conjectured relation between string theory and gauge theory. Secondly, if one wishes to extend the correspondence to other interesting cases, for instance, to general Dp-branes other than the D3 case, there is no corresponding thing to the global coordinate of the AdS geometry. Consequently, it becomes crucial to have correct interpretations of possible holographic relations, if any, by using only the regions outside the horizon of D-brane metrics. For instance, such possibility for 1+0 dimensional case may provide means of identifying the stringy degrees of freedom in Matrix theory.

In the present paper, we propose a new approach in which the basic idea behind the holographic relation (1.1) is kept as the fundamental premise for all our arguments. We show that this is indeed possible and that it predicts a direct relation between string
S-matrix in Euclidean formulation and gauge invariant Yang-Mills correlators, as an extension of the relation (1.1) in a suitable short-distance limit. This also provides a natural explanation from holography for the conjecture made in ref. [9], and shows how the latter conjecture should be generalized to higher-point interactions.

The plan of our paper is as follows. In the next section, we argue that semi-classical particle picture for the basic relation (1.1) should be based on tunneling null geodesics, instead of the real geodesics such as (1.5). It is shown that by doing this all of the above puzzles related to times are naturally resolved. In section 3, we demonstrate that the large $R$ limit of the world-sheet dynamics about the tunneling null geodesics can be described by the similar (but not identical) free massive two-dimensional field theory as the ordinary one based on a real null geodesic. This leads to a natural identification between the Euclideanized string S-matrix and the correlators of the operators identified in [1]. We believe that our proposal essentially solves the issue of holography for the PP-wave geometry. This is discussed in section 4. We show that the identification of the string S-matrix and the correlators provides a natural explanation of the conjectured relation between 3-point vertex of string field theory and the coefficients of operator-product expansion (OPE). Furthermore, our ansatz provides a basis for generalizing the correspondence to higher-point string amplitudes. We show, at the level of tree approximation, that holography essentially fixes the higher-point vertices of the type $1 \rightarrow n$ or $n \rightarrow 1$ of string-field theory in terms of OPE coefficients. Furthermore, it turns out that the constraints required by holography quite nicely conform to supersymmetry. The concluding section 5 will be devoted to further remarks. In Appendix, we briefly discuss the coordinate transformation associated with our Euclidean PP-wave limit, providing an independent derivation of a part of the results in section 3.

2. Bulk-boundary correspondence and tunneling null geodesics

In order to motivate our arguments in later sections, let us start from considering briefly the propagation of a massless scalar field in the background of $\text{AdS}_5 \times S^5$ geometry. For definiteness, we first assume Minkowskian metric. Then the field equation takes the form

$$\left(z^2 \partial_z^2 - 3z \partial_z + R^4 z^2 \omega^2 - J(J + 4)\right)\phi(z) = 0.$$  \hspace{1cm} (2.1)
We have already factorized the angular dependence along the large circle (parametrized by angle $\psi$ along a great circle of $S^5$) and the $R^{3,1}$ part by assigning, respectively, a definite large angular momentum $-i\partial_\psi \to J$ and a definite time-like momentum $\partial_t^2 \to \omega^2 (> 0)$. Now, in the approximation of local-field theory, what corresponds to the string picture is the point-particle approximation to the wave equation (2.1). In particular, the PP-wave limit corresponds to $J \propto R^2 \gg 1$. It is then natural to treat the wave equation by the WKB approximation along the trajectory of a particle. In the above factorized form, the WKB approximation is described as a simple one-dimensional problem by avoiding technical complications associated with higher-dimensional configuration space.

In discussing the correlation functions, we are interested in the bulk-boundary propagator, satisfying the boundary condition (1.2) when the bulk-point approaches to the boundary, $z \to 0$. Let us examine how this form emerges in the WKB approximation. Using the ansatz ($\mathcal{N} =$normalization factor)

$$\phi(z) \sim \mathcal{N}A(z) \exp iS(z), \quad (2.2)$$

where $S$ and $A$ are assumed to be of order $J$ and of order one, respectively, we have

$$z^2 \left( \frac{dS}{dz} \right)^2 - R^4 z^2 \omega^2 + J^2 = 0, \quad (2.3)$$

$$A(z) = J^{1/2} z^{3/2} \left( \frac{dS}{dz} \right)^{-1/2} \exp \left[ - 2iJ \int \frac{dz}{z^2} \left( \frac{dS}{dz} \right)^{-1} \right]. \quad (2.4)$$

From (2.3) which is nothing but the null condition, it is evident that there is no particle trajectory approaching to the boundary $z = 0$, as long as we assume real $S$ since it requires $z^2 \geq J^2/(\omega^2 R^4)$. This is of course the origin of one of the puzzles we have discussed above.

If we still wish to use particle picture, we are led to consider tunneling wave functions by assuming purely imaginary action, $S \to iS_E$, $\phi(r) \to NA(z) \exp -S_E(z)$. Then, (2.3) is replaced by

$$z^2 \left( \frac{dS_E}{dz} \right)^2 = J^2 \left( 1 - \frac{z^2 \omega^2 R^4}{J^2} \right), \quad (2.5)$$

and we can now take the near boundary limit $z \to 0$ limit,

$$S_E(z) \sim \pm J \log z, \quad A(z) \sim z^{2+2}, \quad (2.6)$$

which reproduces the behavior (1.2), provided

$$\Delta = J + 4 \quad \text{or} \quad - J, \quad (2.7)$$
corresponding to the well known mass-dimension relation \( m^2 = J(J + 4) = \Delta(\Delta - 4) \) for scalar field. The former positive solution \( \Delta = J + 4 \) represents the non-normalizable wave function which is responsible for the relation (1.1) and (1.2).

The negative solution would correspond to the tail of the normalizable wave function, which should perhaps be related to the ordinary real null geodesics described by (1.5). This might be a possible hint on how real geodesics may be related in some indirect way to the physics occurring at the boundary. However, in the present paper we do not pursue this possibility.

This simple exercise clearly shows that in order to reconcile the PP-wave limit with the holographic relation (1.1), we should consider tunneling trajectories by assuming purely imaginary momentum along the \( z \) direction. In fact, because of the existence of the potential barrier associated with large angular momentum, it was obvious from the beginning that the boundary-bulk connection for large KK momentum along \( S^5 \) must be actually a tunneling phenomenon. Let us therefore determine the trajectory of an imaginary null geodesic under this assumption. By replacing \( dS_E/dz \) by the \( z \)-momentum \( P_z = Jz^{-2}dz/d\tau \) with \( \tau \) being the affine parameter, we solve

\[
\frac{dz}{d\tau} = \pm z \sqrt{1 - \frac{R^4 \omega^2 z^2}{J^2}}, \tag{2.8}
\]

and obtain

\[
z = \frac{J}{R^2 \omega \cosh \tau}. \tag{2.9}
\]

(Our convention for other momenta is \( J = P_\psi = Jd\psi/d\tau, \omega = -P_t = J(R^4 z^2)^{-1}dt/d\tau \). See also section 3.) This describes a tunneling path, ‘tunneling null geodesic’, which starts from the boundary \( z = 0 \) at \( \tau = -\infty \), goes into the AdS space till the turning point \( z = J/R^2 \omega \), and finally comes back again to the boundary at \( \tau = +\infty \). Also using the momentum \( P_t = -J(R^4 z^2)^{-1}dt/d\tau = \omega \) along the time (target time) direction, we obtain

\[
r = \frac{J}{\omega} \tanh \tau. \tag{2.10}
\]

Here, in fact, since we made use of purely imaginary affine parameter \( \tau \rightarrow -i\tau \) but simultaneously kept the energy \( \omega \) and angular momentum \( J \) fixed, the target time and angle variables must also be understood to be Wick-rotated to the imaginary axes \( t \rightarrow -ir, \psi \rightarrow -i\psi \). The distance \(|r|\) with respect to Euclidean target time \( r \) on the boundary
is given by
\[ |r| = 2 \frac{J}{\omega}. \] (2.11)

Note that with the above double Wick rotation in the target space, the ordinary Minkowski null geodesic and the tunneling one are connected by the exchange \( \tau \leftrightarrow -i\tau \), as is easily seen by comparing the real and tunneling solutions.

These behaviors represent the classical particle picture for the boundary-to-boundary propagator which is relevant for computing 2-point correlation functions on the basis of the relation (1.1). We can indeed determine the factorized wave function \( \psi(x) = f(t) = \int d\omega \tilde{f}(\omega)e^{-i\omega t} \) corresponding to the propagator to the leading order with respect to large \( J \). Assuming large \( t \), the distance with respect to \( r \) is determined by the saddle-point equation
\[ \frac{\partial}{\partial \omega}(-\omega r + \log \tilde{f}(\omega)) = 0, \] (2.12)
after the Wick rotation. Comparing with (2.11), we get
\[ \tilde{f}(\omega) = \omega^{2J}, \] (2.13)
up to a normalization constant which is independent of \( \omega \). This gives the correct result for two-point correlators (\( \propto r^{-2\Delta} \)) of operators with conformal dimension \( \Delta \sim J \) to the leading order in the large \( J \) limit.

It is now manifest that in our picture the natural time direction along the geodesic is asymptotically orthogonal to the base space \((\mathbb{R}^{3,1} \to \mathbb{R}^4)\) of the boundary theory,
\[ z \to \frac{2J}{R^2 \omega} e^{-|\tau|} \text{ as } \tau \to \pm \infty. \] (2.14)

This also exhibits that the affine time \( \tau \) can be identified with the renormalization scale parameter of boundary theory, in conformity with the identification of its conjugate energy \( \partial_\tau \propto \pm \Delta \) \((\tau \to \pm \infty)\) with the conformal dimension of the boundary operators, apart from the angular part of the wave functions which contribute \( J \) for \( \partial_\tau \): \( e^{iJ\psi} \to e^{J\tau} \). If we identify the UV cutoff parameter as \( z \sim 1/\Lambda \) for \( z \to 0 \), the WKB solution (2.6) indicates that the asymptotic form of the boundary-to-boundary amplitude is proportional to
\[ \Lambda^{2J} \propto \omega^{2J} e^{2JT} = \tilde{f}(\omega)e^{2JT}, \] (2.15)
in conformity with (2.13), where \( 2T \to \infty \) is the time interval, with respect to the affine time \( \tau \) along the null geodesics, of propagation from boundary-to-boundary. Thus
the two-point correlators can essentially be identified with the classical part of two-point
S-matrix elements computed along the tunneling null geodesics. If we combine the radial
and angular parts together, the $J$ dependence cancels as $e^{2\Delta T - 2JT}$ to the leading order of
the present approximation.

It should also be noted that since both $J$ and $\omega$ are conserved, they can be assumed
to be always proportional to each other $J/\omega = \text{const}$. This leads to the fact that classical
tunneling paths with various different angular momenta can all be identified with one and
the same trajectory. In other words, we can assume the single tunneling trajectory for
interacting particles around this background: Particles can split and join but move along
the same trajectory at least in the classical approximation.

Now it is more or less clear that all the puzzles we have discussed may be solved
if we decide to formulate string theory in the PP-wave limit on the basis of the above
tunneling null geodesic. Our next task is then to examine whether string theory around
the tunneling null geodesic can be formulated in a well defined way.

3. String theory around the tunneling null geodesics

3.1 Derivation of the string action

We now study the type IIB string theory around the tunneling null geodesic (2.9). The
bosonic world-sheet action is in the standard conformal gauge for the world-sheet metric
($\ell_s = 1$)

$$
S_b = \frac{R^2}{2\pi} \int d\tau \int^{2\pi} d\sigma \frac{1}{2} \left[ z^{-2}(\partial z)^2 + z^{-2}(\partial x_4)^2 - \cos^2 \theta (\partial \psi)^2 + (\partial \theta)^2 + \sin^2 \theta (\partial \Omega_3)^2 \right],
$$

where we have performed the double Wick rotation for the target time and angle $\psi$
as discussed in the previous section. Hence, four-dimensional kinetic term $(\partial x_4)^2$ has
Euclidean signature, and the kinetic term corresponding to the angle $\psi$ has negative sign.
In the original signature, the $S^5$ metric is written as

$$
d s_5^2 = R^2 (\cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\Omega_3^2).
$$

Also we have rescaled the $R^4$ coordinates $x_i^4 \to R x_i^4$ in order to eliminate the factor $R^4$
such that the metric has $R^2$ as the global normalization factor. The fermionic part will
be considered later.
In the large $J \propto R^2$ limit, it is natural to choose the string-length parameter to be

$$\alpha = J/R^2 \ (> 0),$$

since it can be treated as a continuous parameter which is strictly conserved. Furthermore, it is convenient to set, in the notation of the previous section,

$$J = R^2 \omega,$$

or equivalently $\alpha = \omega$. Then the classical background we consider is

$$z = \frac{1}{\cosh \tau} \equiv z_0, \quad r = \tanh \tau \equiv r_0, \quad \psi = \tau = \psi_0,$$

(3.3)

with all other coordinates being zero, where we have denoted the direction of the trajectory projected on the boundary by $r$ (Euclidean target time), following the convention of the previous section. This is of course the classical solution to the above action.

Since the world-sheet time is also Wick-rotated, the signature of the world-sheet parametrization is Euclidean. In ordinary field theory, the existence of the negative metric would then make the theory ill-defined. In string theory, this difficulty is saved by the Virasoro constraints which can serve to eliminate the negative metric term. Apart from the fermion contribution, we have

$$z^{-2} (z^2 - z'^2 + \dot{x}_4^2 - x_4'^2) - \cos^2 \theta (\dot{\psi}^2 - \psi'^2) + \dot{\theta}^2 - \theta'^2 + \sin^2 \theta (\dot{\Omega}_3^2 - \Omega_3'^2) = 0,$$

(3.4)

$$z^{-2} (\dot{z} z' + \dot{x}_4 x_4') - \cos^2 \theta \dot{\psi} \psi' + \dot{\theta} \theta' + \sin^2 \theta \dot{\Omega}_3 \Omega_3' = 0.$$

(3.5)

For the classical solution (3.3), the first is nothing but the null condition and the second is trivially satisfied.

We now expand the action and the Virasoro constraints around the classical solution by making the usual decomposition

$$z = z_0 + \frac{1}{R} z^{(1)} + \frac{1}{R^2} z^{(2)} + \cdots, \quad r = r_0 + \frac{1}{R} r^{(1)} + \frac{1}{R^2} r^{(2)} + \cdots,$$

(3.6)

and similarly for other components whose classical parts are zero. The second order action is then

$$S_b^{(2)} = \frac{1}{4\pi} \int d\tau \int_0^{2\pi\alpha} d\sigma \left[ \frac{1}{z_0} (\partial z^{(1)})^2 + \frac{1}{z_0} (\partial x_4^{(1)})^2 + \frac{3(z^{(1)})^2}{z_0^2} (\dot{z}_0^2 + \dot{r}_0^2) \right]$$
Here we denoted the four of the first order coordinates of \( S^5 \) other than the \( \psi^{(1)} \) collectively by \( y^{(1),i} (i = 1, \ldots, 4) \). In the large \( R \) limit, 3rd and higher order terms does not contribute. The first order Virasoro constraints are

\[
-\frac{1}{z_0^3} z^{(1)} (\dot{z}_0^2 + \dot{r}_0^2) + \frac{1}{z_0^3} z^{(1)} (\dot{z}_0 \dot{z}^{(1)} + \dot{r}_0 \dot{r}^{(1)}) - \dot{\psi}_0 \dot{\psi}^{(1)} = 0, \tag{3.8}
\]

\[
\frac{1}{z_0^2} (\dot{z}_0 (z^{(1)})' + \dot{r}_0 (r^{(1)})') - \dot{\psi}_0 (\psi^{(1)})' = 0, \tag{3.9}
\]

which can serve to eliminate \( \dot{\psi}^{(1)}, (\psi^{(1)})' \). Fermionic degrees of freedom does not contribute to this order, since the classical solution for fermions is simply zero. It is easy to see that the second order Virasoro constraints can similarly determine the second order fluctuations \( \dot{\psi}^{(2)}, (\psi^{(2)})' \) in terms of other variables just as the longitudinal excitations are eliminated by the Virasoro constraints in flat space-time. Since they do not appear in the second order action (3.7), it is sufficient for our purpose to consider only the first order constraints.

Using (3.8) and (3.9) and the explicit form of the classical solution, the second order action is rewritten as

\[
S_b^{(2)} = \frac{1}{4\pi} \int d\tau \int_{0}^{2\pi \alpha} d\sigma \left[ (\partial z)^2 + z^2 + \sinh^2 \tau (\partial r)^2 + \cosh^2 \tau (\partial x_3)^2 + 2 \cosh \tau r \dot{z} + 2 \sinh \tau z \dot{r} + 2 \sinh \tau \partial z \partial r + (\partial y_4)^2 + (\partial y_4)^2 \right], \tag{3.10}
\]

where three coordinates of \( \mathbf{R}^4 \) other than ‘time’ \( r \) are separated and collectively denoted by \( x_3^i (i = 1, 2, 3) \). Note also that we have dropped the superscript (1) for notational brevity : \( z^{(1)} \to z, \text{ etc.} \). The explicit \( \tau \) dependence of this action can be eliminated, together with cross terms, by making the following field redefinition

\[
\frac{r + \frac{z}{\sinh \tau}}{\sinh \tau}, \quad \frac{x_3^i}{\cosh \tau}. \tag{3.11}
\]

By this redefinition, the variable \( z \) is also eliminated. By renaming \( (x_3^i, r) \) as \( x_4^i (i = 1, \ldots, 4) \), the bosonic action then takes the simple form

\[
S_b^{(2)} = \frac{1}{4\pi} \int d\tau \int_{0}^{2\pi \alpha} d\sigma \left[ (\partial x_4)^2 + x_4^2 + (\partial y_4)^2 + (\partial y_4)^2 \right]. \tag{3.12}
\]

Thus we arrived at the free massive world-sheet theory with Euclidean metric. It is also possible to arrive at the same result by directly making suitable coordinate transformation. See Appendix.
Let us now turn to the fermion action. In our quadratic approximation which becomes exact in the limit \( R^2 \rightarrow \infty \), the GS action using Euclidean signature (because of our Wick-rotation in \( \tau \)) for world-sheet metric is

\[
S_f^{(2)} = \frac{i}{2\pi} \int d\tau \int_0^{2\pi\alpha} d\sigma \left( \delta^{IJ} \delta^{ab} - i\epsilon^{ab} s^{IJ} \right) \theta^I \Gamma_0 \rho_a D_b \theta^J,
\]

(3.13)

with \( I, J = 1, 2 \) and \( s^{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Here, \( \rho_a \) is the projection of the Gamma matrices along the tangent space of the world sheet

\[
\rho_a = \Gamma_A E^A_M \partial_a x^M,
\]

(3.14)

and \( D_a \) is the similar projection of the extended covariant derivative \( D^I_M = \delta^{IJ} D_M - \epsilon^{IJ} \frac{1}{8!} F_5 \Gamma^M \) including the contribution from the self-dual 5-form RR field strength \( F_5 = \epsilon_5 + *\epsilon_5 \),

\[
D_a^{IJ} = \partial_a x^M D^I_M,
\]

which leads to

\[
D_a^{IJ} = \mathcal{D}_a \delta^{IJ} + i \epsilon^{IJ} \Gamma_5 \rho_a, \quad \Gamma_5 = i \Gamma_{01234}, \quad (\Gamma_5^2 = 1)
\]

(3.15)

\[
\mathcal{D}_a = \partial_a + \frac{1}{4} \omega_{AB} \partial_a x^M \Gamma_{AB},
\]

where \( \mathcal{D}_a = \partial_a x^M D_M \) denotes the standard spinor covariant derivative along the world-sheet. The target space-time geometry is the AdS_5 × S^5 with the double Wick rotation being assumed. For the corresponding discussion for the real null geodesic, see [7] and references therein. Hence, the Lorentz indices of the Dirac Gamma matrices must be understood with additional pure-imaginary factor when \( A, B, \ldots = 0 \) and \( A, B, \ldots = \psi \). It should be noted that the self-duality condition for 5-form field strength is preserved under the double Wick rotation and that, comparing with the Minkowski (with respect to both world-sheet and target space-time) action, the Levi-Civita symbols \( \epsilon^{ab}, \epsilon^{IJ} \) acquire pure-imaginary factor \( i \). The pure-imaginary factor for \( \epsilon^{IJ} \) originates from the RR-field strength \( F_5 \rightarrow i F_5 \). The spinor coordinate \( \theta^I \) \((I = 1, 2)\) are Majorana-Weyl spinor in the Minkowski metric. After the double Wick rotation, their components must in general be regarded as complex with the same number of independent components as in the case of ordinary Minkowski signature. Note also that the action does not involve complex conjugation and hence that this does not cause any trouble in Wick-rotating the fermion.
coordinates. In general path-integral formalism, as is well known, it is not meaningful to consider hermitian conjugation for fermions in Euclidean case in the same sense as in the ordinary Minkowski case.

Using the space-time metric along the classical solution (3.3), we find that the fermionic action is written as

$$S^{(2)}_f = \frac{i}{2\pi} \int d\tau \int_0^{2\pi\alpha} d\sigma \left[ \theta^I \Gamma_0 \rho_{\tau} (\partial_{\tau} + iz \Gamma_{Z0}) \theta^I - is^{IJ} \theta^I \Gamma_0 \rho_{\tau} \partial_{\sigma} \theta^J + \frac{1}{2} \epsilon^{IJ} \theta^I \Gamma_0 \rho_{\tau} \Gamma_{+} \rho_{\tau} \theta^J \right],$$

(3.16)

where

$$\rho_{\tau} = -z \sinh \tau \Gamma_{Z} + iz \Gamma_{0} + i\Gamma_{\psi},$$

(3.17)

which satisfies

$$\rho_{\tau}^2 = 0, \quad [D_{\tau}, \rho_{\tau}] = 0, \quad (D_{\tau} = \partial_{\tau} + iz \frac{1}{2} \Gamma_{Z0}).$$

(3.18)

The $\kappa$-symmetry transformation in the present approximation can be expressed as

$$\delta_{\kappa} \theta^I = \rho_{\tau} \kappa^I,$$

(3.19)

under which (3.24) is manifestly invariant. To simplify the fermion action further, let us make the following field redefinition

$$\theta^I \rightarrow e^{-i \frac{\beta}{2} \Gamma_{Z0}} \theta^I, \quad \theta^I \Gamma_{0} \rightarrow \theta^I \Gamma_{0} e^{+i \frac{\beta}{2} \Gamma_{Z0}},$$

(3.20)

such that

$$e^{i \frac{\beta}{2} \Gamma_{Z0}} D_{\tau} e^{-i \frac{\beta}{2} \Gamma_{Z0}} = \partial_{\tau}, \quad e^{i \frac{\beta}{2} \Gamma_{Z0}} \rho_{\tau} e^{-i \frac{\beta}{2} \Gamma_{Z0}} = \Gamma_{Z} + i\Gamma_{\psi} \equiv \sqrt{2} \Gamma_{-}.$$  

(3.21)

The left hand side of these equations are chosen in such a way that they coincide with their respective asymptotic forms in the limit $\tau \rightarrow -\infty$. This is achieved by setting

$$\sin \beta = \frac{1}{\cosh \tau}.$$  

(3.22)

To fix the $\kappa$-symmetry gauge, we adopt the gauge condition,

$$\Gamma_{+} \theta^I = 0, \quad \Gamma_{+} \equiv (\Gamma_{Z} - i\Gamma_{\psi})/\sqrt{2} = \Gamma^{\dagger}_{-}. $$

(3.23)

We remind the reader again that the above field redefinition and gauge condition are allowed because the fermionic coordinates are treated as complex with the same number.
of degrees of freedom as in the ordinary Minkowski case and also that the action does not involve complex conjugation.

Putting all these together, our final form of the second-order fermionic action is

\[ S_f^{(2)} = \frac{i}{2\pi} \int d\tau \int_{\sigma_0}^{2\pi} d\sigma \left[ \theta^I \Gamma_0 \partial_\tau \theta^I - i s^{IJ} \theta^I \Gamma_0 \partial_\sigma \theta^J - i \epsilon^{IJ} \theta^I \Gamma_0 \Gamma_- \Pi \theta^J \right], \quad (3.24) \]

with

\[ \Pi = i \Gamma_{0123}, \quad \Pi^2 = 1, \quad [\Pi, \Gamma_\pm] = 0 = \{\Pi, \Gamma_0\}, \quad \Pi^T = -\Pi, \quad \Pi^\dagger = \Pi. \quad (3.25) \]

This is the expected form of Euclidean action for free massive fermions as the superpartner of the bosonic action (3.12). Because of the coupling of RR fields in the mass term, the manifest global bosonic symmetry of the total action \( S = S_b^{(2)} + S_f^{(2)} \) is naively \( \text{SO}(4) \times \text{SO}(4) \times \mathbb{Z}_2 \), \( \mathbb{Z}_2 \) being associated with the discrete symmetry interchanging two bosonic \( \text{SO}(4) \) directions. Unlike the usual treatment based on the real null geodesics [8], four \( (i \times 0, 1, 2, 3) \) of the eight transverse directions can be identified manifestly with the base space of the boundary conformal theory of Euclidean signature including time direction.\(^{\text{II}}\) For this reason, our action is not exactly the Wick-rotated version of the action which we would obtain in the PP-wave limit from the real null geodesics, although the formal structure of both actions is very similar to each other. For example, our \( \Gamma_\pm = (\Gamma_z \mp i \Gamma_\psi) / \sqrt{2} \) are not the Wick-rotated version of the usual ones \( (\Gamma_0 \pm \Gamma_\psi) / \sqrt{2} \).

As for the global symmetry of the action, however, it is important to recognize that if we redefine the spinor coordinates as \( \theta^1 \to \theta^1, \theta^2 \to \Pi \theta^2 \), we can eliminate the matrix \( \Pi \) from the fermion action (3.24), since the kinetic term is invariant under this redefinition. Clearly, the continuous bosonic symmetry is now extended to \( \text{SO}(8) \) by treating \( (x^i, y^i) \) as an \( \text{SO}(8) \) vector and also by assuming that the \textit{redefined} spinor coordinates transform according to the usual transformation law, \( (\theta^I \to \exp(\omega^{ab} \Gamma_{ab}/4) \theta^I \equiv R \theta^I, \theta^I \Gamma_0 \to \theta^I \Gamma_0 R^{-1}) \) where \( a, b, \ldots \) run over the directions transversal to \( z, \psi \). Equivalently, if we insist on using the original spinor coordinates without making the redefinition, the \( \text{SO}(8) \) symmetry is hidden in such a way that \( \theta^1 \) and \( \theta^2 \) transform differently in general as given by

\[ \theta^1 \to R \theta^1, \quad \theta^2 \to \Pi R \Pi \theta^2. \quad (3.26) \]

\(^{\text{II}}\)In our convention, the other 4 transverse directions are \( (5, 6, 7, 8) \). The remaining longitudinal (‘light-like’) directions are \( z = 4, \psi = i \times 9 \).
The manifest SO(4)×SO(4) corresponds to ΠRΠ = R for R ∈ SO(4)×SO(4). Naively, we expect that the RR field strength \( F_5 = \epsilon_5 + *\epsilon_5 \) breaks the symmetry down to SO(4)×SO(4). However, the theory is actually SO(8) symmetric in the PP wave limit. The situation is that the presence of RR field is crucial for the emergence of the mass term for fermions, but that PP-wave limit is not fully sensitive with respect to its directions. The spectrum of fluctuations consists of an infinite number of SO(8) multiplets when the states are suitably relabeled. In the following, unless stated otherwise, we will use the manifestly SO(8) symmetric conventions, by renaming the 4+4 bosonic vector \((x,y)\) as \(x^i \ (i = 1, \ldots, 8)\) and by using the redefined fermion spinors \((\theta^1, \theta^2)\) after eliminating \(\Pi\). Once the world-sheet theory has SO(8) symmetry, it is reasonable to expect that the whole theory after second quantization should have the same symmetry.

### 3.2 Quantization

Let us next briefly treat the quantization of our Euclidean string theory. The appropriate mode expansion for bosons is, suppressing the SO(8) indices,

\[
x(\tau, \sigma) = \frac{1}{\sqrt{2\alpha}} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{E_n}} \left( e^{-E_n \tau + in\sigma/\alpha} a_n + e^{E_n \tau - in\sigma/\alpha} a_n^\dagger \right),
\]

\[
p(\tau, \sigma) = \frac{i}{2\pi} \frac{dx}{d\tau} = \frac{i}{2\pi \sqrt{2\alpha}} \sum_{n=-\infty}^{\infty} \sqrt{E_n} \left( - e^{-E_n \tau + in\sigma/\alpha} a_n + e^{E_n \tau - in\sigma/\alpha} a_n^\dagger \right),
\]

with

\[
E_n = \sqrt{1 + \frac{n^2}{\alpha^2}} = \sqrt{1 + \frac{R^2 n^2}{J^2}}.
\]

The canonical commutation relations are

\[
[x(\tau, \sigma), p(\tau, \sigma')] = i\delta(\sigma - \sigma'), \quad [a_n, a_m^\dagger] = \delta_{nm}.
\]

We emphasize that for nonzero \(\tau\) the field variables \(x(\tau, \sigma), p_\sigma(\tau, \sigma)\) are not self-adjoint:

\[
x(\tau, \sigma)^\dagger = x(-\tau, \sigma).
\]

However, we used the adjoint notation \(\dagger\) with understanding that after inverse Wick rotation these variables reduce to the standard ones.

For fermions, the mode expansions are

\[
\theta^1(\tau, \sigma) = \frac{1}{\sqrt{2\alpha}} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2E_n(E_n - \frac{n^2}{\alpha})}} \left( e^{-E_n \tau + in\sigma/\alpha} b_n + e^{E_n \tau - in\sigma/\alpha} \frac{i}{\sqrt{2}} \Gamma_+ \Gamma_0 b_n^\dagger \right),
\]
\[ \theta^2(\tau, \sigma) = \frac{i}{\sqrt{2\alpha}} \sum_{n=\infty}^{\infty} \frac{1}{\sqrt{2E_n(E_n + \frac{n}{\alpha})}} \left( e^{-E_n\tau + in\sigma/\alpha} b_n - e^{E_n\tau - in\sigma/\alpha} i \frac{1}{\sqrt{2}} \Gamma_0 b_n^\dagger \right). \] (3.33)

Because of massive nature, two spinors \( \theta^1, \theta^2 \) are not independent to each other after the imposition of the equations of motion. This is of course the most crucial difference between massive and massless fermions on the world sheet.

The fermionic canonical commutation relations are

\[
\{ \theta(\tau, \sigma), \Pi_\theta(\tau, \sigma') \} = P_+ \delta(\sigma - \sigma'), \tag{3.34}
\]

\[
\{ \theta(\tau, \sigma), \theta(\tau, \sigma') \} = 0 = \{ \Pi_\theta(\tau, \sigma), \Pi_\theta(\tau, \sigma') \}, \tag{3.35}
\]

\[
\{ b_n, b_m^\dagger \} = P_+ \delta_{nm}, \tag{3.36}
\]

where the new fermion variables are defined by

\[
\theta(\tau, \sigma) \equiv \theta^1(\tau, \sigma) + i\theta^2(\tau, \sigma), \quad \overline{\theta}(\tau, \sigma) \equiv \theta^1(\tau, \sigma) - i\theta^2(\tau, \sigma), \tag{3.37}
\]

with \( \Pi_\theta(\tau, \sigma) \equiv \frac{d\theta}{2\pi} - \overline{\theta} \). Here we suppressed spinor indices and the Weyl projection for notational brevity. All the spinor indices in the present paper should be understood as with the Weyl projection, \( \Gamma_\mu = h_- \Gamma_\mu h_+ \) or \( \Gamma_\mu = h_+ \Gamma_\mu h_- \) (\( h_\pm = (1 \mp \Gamma_{11})/2 \)), depending on the positions of \( \Gamma \) matrices. Remember that in our convention,

\[
(\Gamma^0 \Gamma_\pm)^T = \Gamma^0 \Gamma_\pm, \quad \Gamma^0_T = -\Gamma_0, \quad \Gamma^T_\pm = \Gamma_\pm, \quad (\Gamma^0 \Gamma_\pm \Gamma_{\mu\nu})^T = -\Gamma^0 \Gamma_\pm \Gamma_{\mu\nu}, \tag{3.38}
\]

and also that the spinor coordinates are further projected owing to the \( \kappa \)-gauge condition as

\[
P_+ b_n = b_n, \quad P_- b_n^\dagger = b_n^\dagger, \tag{3.39}
\]

with the projection operators

\[
P_\pm \equiv \frac{1}{2} \Gamma_\pm \Gamma_\mp, \quad \Gamma_\pm P_\pm = 0 = P_\pm \Gamma_\mp, \tag{3.40}
\]

satisfying

\[
P_\pm^T = P_\pm, \quad P_+ \Gamma_+ = \Gamma_+, \quad \Gamma_- P_+ = \Gamma_, \quad etc. \tag{3.41}
\]

Below, the \( \kappa \)-gauge projection will also be mostly suppressed. Note that \( P_\pm \) are hermitian and positive definite:

\[
P_\pm^\dagger = P_\pm = \frac{1}{2} \Gamma_\pm \Gamma_\mp. \tag{3.42}
\]
The conjugation property of the fermionic coordinates is

\[ \theta^I(\tau, \sigma)\dagger = -\frac{i}{\sqrt{2}} \Gamma_0 \theta^I(-\tau, \sigma). \] (3.43)

The total Hamiltonian is

\[ H = \frac{1}{2} \int_0^{2\pi} d\sigma \left( 2\pi p^2 + \frac{1}{2\pi} (x')^2 + \frac{1}{2\pi} x^2 + \frac{1}{2\pi} \theta \Gamma_0 \theta' + \frac{1}{2\pi} \bar{\theta} \Gamma_0 \bar{\theta}' + \frac{i}{\pi} \theta \Gamma_0 \bar{\theta} \right) : \]

\[ = \sum_n E_n \left( a_n^\dagger a_n + b_n^\dagger b_n \right), \] (3.44)

giving the Euclidean equations of motion \( x(\tau, \sigma) = e^{H\tau} x(0, \sigma)e^{-H\tau}, etc \). We can define Fock space as usual on the basis of the Fock vacuum \( |0\rangle \) and its conjugate, satisfying \( a_n|0\rangle = b_n|0\rangle = 0 = \langle 0|a_n^\dagger = \langle 0|b_n^\dagger \). The Hamiltonian is then self-adjoint in the usual sense and positive definite with eigenvalues

\[ E = \sum_n E_n \left( \sum_{i=1}^8 N_{i,n}^{\text{bose}} + \sum_{\alpha=1}^8 N_{\alpha,n}^{\text{fermi}} \right), \] (3.45)

where \( N_{i,n}^{\text{bose}} \) and \( N_{\alpha,n}^{\text{fermi}} \) are the numbers of excitations for eight bosonic directions and for eight independent fermionic directions, respectively. The theory is a perfectly well-defined Euclidean system with physical positivity (reflection positivity). The hidden SO(8) symmetry is manifest in the above final form of the Hamiltonian. To avoid confusion, it should be noted that the SO(8) symmetry cannot be interpreted as real rotations in the background space-time. For instance, the dilaton and axion are not singlet with respect to SO(4), although they are with respect SO(4)×SO(4).

The states where only the zero modes \( n = 0 \) are excited must correspond to the massless supergravity fields around the AdS background. Then, it is possible to interpret the eigenvalue of the Hamiltonian as the difference of field dimensions and angular momentum, \( E = \Delta - J \), by identifying the ground states as the chiral primary states of the supergravity multiplets with the given \( J \). This conforms to the semi-classical result in the previous section. For each \( J \), the states with only fermion zero modes are excited gives the basic 128 (bosons) + 128 (fermions) physical supergravity states. The infinite tower on these states formed by exciting bosonic zero modes correspond to orbital fluctuations around the \( \psi \) direction. Following ref. [1], it is natural to interpret the excitation energies \( \sqrt{1 + \frac{R^4n^2}{J^2}} - 1 \) of nonzero modes as the anomalous dimensions of gauge-field operators corresponding to general string fields, in the approximation where string loop effects are ignored.
It is more convenient to use $SO(8)$ spinor notations for Dirac matrices than the above 10-dimensional ones. We briefly indicate the expressions using them. The Hamiltonian and the canonical commutation relations are

$$
H = \frac{1}{2} \int_0^{2\pi} d\sigma : \left[ 2\pi p^2 + \frac{1}{2\pi} (x')^2 + \frac{1}{2\pi} x^2 - \frac{i}{2\pi} (\theta \theta' + \overline{\theta}\overline{\theta'}) + \frac{1}{\pi} \theta \theta' \right] : \tag{3.46}
$$

$$\{ \theta_a(\sigma), \overline{\theta}_b(\sigma') \} = 2\pi \delta_{ab} \delta(\sigma - \sigma'), \quad \theta(\tau, \sigma) \dagger = \overline{\theta}(-\tau, \sigma). \tag{3.47}
$$

If we wish to return to the convention with the spinor factor $\Pi$ before our redefinition $\theta_2 \to \Pi \theta_2$, the natural canonical spinor coordinates which we denote by $\psi, \overline{\psi}$ are related to our $\theta, \bar{\theta}$ by a canonical transformation breaking $SO(8)$ to $SO(4) \times SO(4)$,

$$
\psi = \frac{1}{2} (1 + \Pi) \theta + \frac{1}{2} (1 - \Pi) \overline{\theta}, \quad \overline{\psi} = \frac{1}{2} (1 - \Pi) \theta + \frac{1}{2} (1 + \Pi) \overline{\theta} \tag{3.48}
$$

where, in the standard $(8 \times 8)$ $SO(8)$ spinor notation for gamma matrices ($(\gamma^i \gamma^j + \gamma^j \gamma^i)_{ab} = 2\delta_{ab}$, $(\gamma^i T \gamma_j + \gamma^j T \gamma_i)_{\hat{a} \hat{b}} = 2\delta_{\hat{a} \hat{b}}$)

$$
\Pi_{ab} = \Pi_{ba} = (\gamma_1 \gamma_2 \gamma_3 \gamma_4)_{ab}, \quad \Pi_{\hat{a} \hat{b}} = \Pi_{\hat{b} \hat{a}} = (\gamma^1 T \gamma^2 \gamma^3 \gamma^4)_{\hat{a} \hat{b}}. \tag{3.49}
$$

In terms of these coordinates, the Hamiltonian takes the form

$$
H = \frac{1}{2} \int_0^{2\pi} d\sigma : \left[ 2\pi p^2 + \frac{1}{2\pi} (x')^2 + \frac{1}{2\pi} x^2 - \frac{i}{2\pi} (\psi \psi' + \overline{\psi}\overline{\psi'}) + \frac{1}{\pi} \psi \Pi \overline{\psi} \right] : \tag{3.50}
$$

The standard dynamical supersymmetry generators[8, 7], which only respect $SO(4) \times SO(4)$, are

$$
Q^-_{\hat{a}} = \int_0^{2\pi} d\sigma (p \cdot \gamma - \frac{i}{2\pi} x \cdot \gamma \Pi) \psi - \frac{1}{2\pi} x' \cdot \gamma \overline{\psi} \right]_{\hat{a}}, \tag{3.51}
$$

$$
\overline{Q}^-_{\hat{a}} = \int_0^{2\pi} d\sigma (p \cdot \gamma + \frac{i}{2\pi} x \cdot \gamma \Pi) \overline{\psi} - \frac{1}{2\pi} x' \cdot \gamma \psi \right]_{\hat{a}}, \tag{3.52}
$$

in terms of the spinor coordinates $\psi, \overline{\psi}$. Nontrivial part of the supersymmetry algebra is

$$
\{ Q^-_{\hat{a}}, \overline{Q}^-_{\hat{b}} \} = 2H \delta_{\hat{a} \hat{b}} + \sum_{(i,j) \in (1,2,3,4)} i(\gamma_{ij} \Pi)_{\hat{a} \hat{b}} J_{ij} - \sum_{(i,j) \in (5,6,7,8)} i(\gamma_{ij} \Pi)_{\hat{a} \hat{b}} J_{ij}, \tag{3.53}
$$

$$
J_{ij} = \int_0^{2\pi} d\sigma (x_i p_j - x_j p_i - \frac{1}{4\pi} i\psi \gamma_{ij} \overline{\psi}). \tag{3.54}
$$

More convenient representation of this algebra for our later purpose will be, suppressing spinor indices,

$$
\{ Q^-_1, Q^-_1 \} = \{ Q^-_2, Q^-_2 \} = 2H \tag{3.55}
$$
\[ \{Q_1^-, Q_2^-\} = -\sum_{(i,j)\in\{1,2,3,4\}} \gamma_{ij} \Pi J_{ij} + \sum_{(i,j)\in\{5,6,7,8\}} \gamma_{ij} \Pi J_{ij} \quad (3.56) \]

with
\[ Q_1^- \equiv \frac{1}{\sqrt{2}} (Q^- + \overline{Q}^-), \quad Q_2^- \equiv \frac{1}{\sqrt{2}} (Q^- - \overline{Q}^-). \quad (3.57) \]

The susy algebra does not respect the SO(8) symmetry of the world-sheet action and hence of the Hamiltonian. This suggests that supersymmetry may not be powerful enough in constraining the dynamics of the system. It is tempting to introduce fermionic symmetry generators respecting SO(8) as
\[ R^-_a = \int_0^{2\pi a} d\sigma \left[ \left(p - \frac{i}{2\pi} x\right) \cdot \gamma_{ab} \theta_b - \frac{1}{2\pi} x' \cdot \gamma_{ab} \overline{\theta}_b \right], \quad \overline{R}^-_a = \left(R^-_a\right)^\dagger, \quad (3.58) \]

using the manifestly SO(8) symmetric spinor coordinates \( \theta, \overline{\theta} \). Indeed we can check that these ‘pseudo’ susy generators commute with the Hamiltonian, and their anticommutators are
\[ \{ R^-_a, \overline{R}^-_b \} = 2\delta_{ab} H - i \gamma^{ij}_{ab} L^{ij}, \quad \{ R^-_a, R^-_b \} = 0, \quad (3.59) \]

with
\[ L^{ij} = \int_0^{2\pi a} d\sigma \left[ x^i p^j - x^j p^i + \frac{i}{4\pi} \theta \gamma^{ij} \overline{\theta} \right]. \quad (3.60) \]

Because of the plus sign of the fermionic contribution in (3.60), this algebra does not close with a finite number of generators. ** Alternatively, if we combine the standard susy generators with our SO(8), the algebra is extended to an infinite dimensional algebra. This suggests that we can have much stronger constraints on the dynamics of the system by combining SO(8) and susy than taking into account only the standard supersymmetry. In any case, it seems very important to further clarify the role of the hidden SO(8) symmetry.

4. Holography: Correspondence between string S-matrix and OPE

Once the free string theory is given, it should in principle be straightforward to construct superstring field theory following the procedure in the flat case. In the case of PP-wave limit on the real null geodesic, this task has been undertaken in [10]. We will pursue this subject in our Euclidean setting elsewhere. In the present work, instead of proceeding to such a direct construction of string field theory, we study the problem how the fundamental

**In the original version of the present paper, an erroneous statement has given with respect to this point.
relation (1.1) of holography can be realized on the basis of the Euclidean string field theory from a more general standpoint, without assuming explicit expression of the string-field Hamiltonian. Instead, we assume that the set of physical string states correspond, at the boundary, to a complete (in the sense of OPE) set of gauge-theory operators with definite conformal dimensions, as proposed in [1]. We are now initiating to study the general structure of ‘holographic’ string field theory.

4.1 Euclidean S-matrix

Let the string-field theory Hamiltonian be $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} \cdots \equiv \mathcal{H}^{(0)} + \mathcal{V}, \quad (4.1)$$

where $\mathcal{H}^{(0)}$ is the free Hamiltonian corresponding to the single-string Hamiltonian (3.44) and $\mathcal{H}^{(i)}$ ($i = 1, 2, \ldots$) are interaction vertices, expanded in powers of string coupling. If necessary we have to include some counter terms corresponding to renormalization effects when we study the string-loop effects. In terms of this Hamiltonian, the tunneling amplitudes after taking string interactions into account are essentially described by matrix elements of the transition operator

$$U(\tau_2, \tau_1) = \exp \left[ -\mathcal{H}(\tau_2 - \tau_1) \right], \quad (\tau_2 > \tau_1)$$

in the limit $\tau_2 = T \to \infty, \tau_1 = -T \to -\infty$. More precisely, the S-matrix is defined by multiplying the asymptotic transition operator for amputation of external lines,

$$S = \lim_{T \to \infty} e^{\mathcal{H}^{(0)}T} U(T, -T) e^{\mathcal{H}^{(0)}T}. \quad (4.2)$$

Euclidean Schrödinger equation corresponding to these transition operators is

$$\frac{d}{d\tau} |\Psi_{in}(\tau)\rangle = -\mathcal{H} |\Psi_{in}(\tau)\rangle. \quad (4.3)$$

Here we put the subscript ‘in’, since states obeying this equation are supposed to reduce to incoming asymptotic states in the limit $\tau \to -\infty$. For ‘out’ states corresponding to $\tau \to +\infty$, we have similarly

$$\frac{d}{d\tau} |\Psi_{out}(\tau)\rangle = \mathcal{H} |\Psi_{out}(\tau)\rangle. \quad (4.4)$$

These asymptotic states should correspond to the various composite operators as identified by the work [1]. Note that the sign here is chosen such that $\partial_\tau \sim \pm \Delta$ in the limit
\( \tau \to \pm \infty \) (See (2.15)). The choice of nonnormalizable boundary condition as discussed in section 3 corresponds to the behavior that \( |\Psi_{\text{in}}(-T)\rangle \) and \( |\Psi_{\text{out}}(T)\rangle \) exponentially increase in the limit \( T \to \infty \) for generic states other than the ground state.

To incorporate the asymptotic Hamiltonian \( \mathcal{H}^{(0)} \), we use the interaction representation:

\[
|\Psi_{\text{in}}^{I}(\tau)\rangle = e^{\mathcal{H}^{(0)}\tau} |\Psi_{\text{in}}(\tau)\rangle, \tag{4.5}
\]

satisfying

\[
|\Psi_{\text{in}}^{I}(\tau)\rangle = U_{+}(\tau) |\Psi_{\text{in}}(-T)\rangle, \tag{4.6}
\]

with

\[
U_{+}(\tau) = \mathcal{T} \exp \left( - \int_{-T}^{\tau} d\tau \mathcal{V}_{+}(\tau) \right), \quad \mathcal{V}_{+}(\tau) = e^{\mathcal{H}^{(0)}\tau} \mathcal{V} e^{-\mathcal{H}^{(0)}\tau}. \tag{4.7}
\]

Similarly, we define

\[
|\Psi_{\text{out}}^{I}(\tau)\rangle = e^{-\mathcal{H}^{(0)}\tau} |\Psi_{\text{out}}(\tau)\rangle, \tag{4.8}
\]

satisfying

\[
|\Psi_{\text{out}}^{I}(\tau)\rangle = U_{-}(\tau) |\Psi_{\text{out}}(+T)\rangle, \tag{4.9}
\]

with

\[
U_{-}(\tau) = \mathcal{T} \exp \left( - \int_{\tau}^{+T} d\tau \mathcal{V}_{-}(\tau) \right), \quad \mathcal{V}_{-}(\tau) = e^{-\mathcal{H}^{(0)}\tau} \mathcal{V} e^{\mathcal{H}^{(0)}\tau}. \tag{4.10}
\]

Here \( \mathcal{T} \) (\( \mathcal{T} \)) are (anti) time-ordering operators with respect to \( \tau \). The formal definition of S-matrix is then

\[
S = U_{-}(\tau) U_{+}(\tau) \equiv 1 + T, \tag{4.11}
\]

which satisfies

\[
\langle \Psi_{\text{out}}^{I}(\tau)|\Psi_{\text{in}}^{I}(\tau)\rangle = \langle \Psi_{\text{out}}(+T)|S|\Psi_{\text{in}}(-T)\rangle \\
= \langle \Psi_{\text{out}}(+T)|\mathcal{T} \exp \left( - \int_{-T}^{+T} d\tau \mathcal{V}_{+}(\tau) \right)|\Psi_{\text{in}}(-T)\rangle. \tag{4.12}
\]

Keep in mind that this definition amounts to normalizing two-point functions on the gauge-theory side as identity matrix. We here used the symbol \( T \) for the nontrivial part of the S-matrix, since we do not expect any confusion owing to this abuse of notations.

The perturbation expansion of the \( T \) matrix is

\[
\langle b|T|a\rangle = \lim_{T \to \infty} \left[ - \langle b|\mathcal{V}|a\rangle \int_{-T}^{T} d\tau e^{(E_{b}-E_{a})\tau} \\
+ \sum_{c} \langle b|\mathcal{V}|c\rangle \langle c|\mathcal{V}|a\rangle \int_{-T}^{T} d\tau e^{(E_{b}-E_{c})\tau} \int_{-T}^{T} d\tau_{1} e^{(E_{c}-E_{a})\tau_{1}} \right]
\]

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\[-\sum_{c} \sum_{d} \langle b | V | c \rangle \langle c | V | d \rangle \langle d | V | a \rangle \int_{-T}^{T} d\tau e^{(E_{b}-E_{c})\tau} \int_{-T}^{T} d\tau_{1} e^{(E_{c}-E_{d})\tau_{1}} \times \int_{-T}^{\tau_{1}} d\tau_{2} e^{(E_{d}-E_{a})\tau_{2}} + \cdots \]. \tag{4.13}

For notational simplicity we assumed that the states are eigenstates of energy. In general, we have to superpose them. Remember also that the states \( \langle b | \) and \( | a \rangle \) here are those in the interaction representation.

Now, we have to make a remark which will be crucial in our arguments below. In order to extract proper physical results from various infinite time integrals involved in this expression, it is necessary to carefully define the asymptotic \textit{in} and \textit{out} states using wave packet formalism. Remember, for instance, that the physical justification of the familiar \( i\epsilon \) prescription for ordinary Minkowskian S-matrix rests on the use of wave packets. In our case, because of confining harmonic potential in the transverse directions, it is not possible to make positions of wave packets far apart in the transverse directions. However, we can define asymptotic states along the \( \psi \) direction. Wave packet formalism is perfectly applicable to the present system by regarding it effectively as a \((1+1)\)-dimensional system. Namely, instead of considering states with a definite angular momentum \((J)\) and energy, we have to superpose states with various different angular momenta around \( J \) with appropriate weights. Then recalling that the exponent of the wave function with definite angular momentum is \( E(J)\tau - J\psi \), where \( \psi \) denotes of course the zero mode part which is not constrained by the Virasoro conditions (3.8) and (3.9), the velocity of a wave packet with average angular momentum \( J \) moving along the large circle corresponding to the angle \( \psi \) is determined by the saddle-point equation,\footnote{The reader might wonder how to justify the wave packet after Wick rotation. We understand this in the formal sense of saddle-point approximation in the large \( T \) limit: The classical trajectory is defined by Wick rotation. However, the superposition of states around the classical trajectory can be formed by integrating along appropriate directions in the complex \( J \)-plane, such that the saddle-point approximation is justified. Similar situation arises when we consider tunneling in quantum mechanics. Although there is no wave packet of the ordinary sense in the tunneling region, we can still talk about wave packets for initial and final states of tunneling processes.}

\[ v = R \frac{dE}{dJ}. \tag{4.14} \]

Note that this velocity is the correction to the velocity, with respect to the affine time \( \tau \), of classical solution which is \( R \) (=light velocity) in the same convention as we use here. Thus the relative velocities of wave packets are generically of order \( O((\alpha^{3}R)^{-1}) \),
except for strictly supergravity states for which $v = 0$. This means that two wave packets with a generic initial distance of order $O(R)$ (standard ‘macroscopic’ scale of the present background) requires a time interval of order $O(T) \sim O(R^2\alpha^3)$ to collide. Conversely, if wave packets made a collision at $\tau \sim 0$, they become free when average distances are of standard macroscopic scale after time passed as $\tau \sim O(R^2\alpha^3)$.

Now from known behavior of string scattering, we can assume that strings can interact ‘quasi-locally’, in the sense that the matrix elements of interaction vertices are nonvanishing only for the wave packets which come close up to string scale ($\sim 1$ by our choice of length unit) when they collide. Although in the extreme high-energy limits strings exhibit a characteristic non-local behavior signified by a space-time uncertainty relation [11], this assumption can be justified for generic finite-energy processes. We suppose that the initial ($|a\rangle$) and final ($\langle b|$) states in the interaction representation consist of wave packets such that the distances among the packets are in general of macroscopic order $R$. In order that the expression (4.13) has nonvanishing contribution, the wave packets, superposed with the energy factors $e^{\pm ET}$, must be such that the colliding wave packets come close to the order of string scale. There is a sign ambiguity here depending on how these states are prepared. We shall show that the choice of positive exponent $e^{ET}$ is appropriate to realize the basic holographic relation.

It is interesting to see what is the situation in the Schrödinger picture. Owing to the relations

$$e^{\mathcal{H}^{(0)}T}|\Psi_{in}^{I}(-T)\rangle = |\Psi_{in}(-T)\rangle, \quad e^{\mathcal{H}^{(0)}T}|\Psi_{out}^{I}(T)\rangle = |\Psi_{out}(T)\rangle,$$

our choice amounts to the assumption that the distances among wave packets in the Schrödinger picture are of order one at the boundary $\tau \sim \pm T$. In other words, in the Schrödinger picture, the interaction occurs mostly near the boundaries. Note that once a definite sign is chosen, amplitudes with the opposite sign would vanish since they cannot represent wave packets coming close to each other at the right time (in the interaction picture) and hence the matrix elements of string vertices would be zero. Remember that the difference of positive and negative exponents corresponds to the time duration of order $2T$. Therefore, if states are prepared in such a way that the wave packets come close for one sign, the states with opposite sign would represents wave packets whose relative positions are far apart from each other and hence the matrix elements would vanish.
The crucial implication of this assumption is that any initial $|a\rangle$ or final $\langle b|$ states involving more than one string can contribute to connected scattering amplitudes only when the energy exponents of all strings involved are positive. As we see later, this special feature, which is somewhat peculiar from the viewpoint of ordinary S-matrix, is related to the fact that our initial and final states should correspond to the multiple products of composite operators at the boundaries ($\tau = \pm T, \ T \to \infty$) of the single tunneling null geodesic. Therefore the initial and final states necessarily represent certain short-distance limits of correlation functions into two groups.

Armed with this general consideration, we can now discuss our main problem, holographic interpretation of the $T$-matrix elements (4.13). In the present paper, we restrict ourselves within the tree approximation.

4.2 3-point amplitude

The 3-point amplitudes come only from the first term in the expansion (4.13),

$$\frac{e^{(E_b-E_a)T} - e^{-(E_b-E_a)T}}{E_a - E_b} \langle b|H^{(1)}|a\rangle. \quad (4.15)$$

Because of the conservation of $\alpha$ (or $J$), only one of the states $\langle b|$ or $|a\rangle$ can be a two-particle state. According to the rule established in the previous subsection, the part contributing to the S-matrix in the large time limit is either

$$\frac{e^{(E_b-E_a)T}}{E_a - E_b} \langle b|H^{(1)}|a\rangle \quad \text{or} \quad \frac{e^{(E_a-E_b)T}}{E_b - E_a} \langle b|H^{(1)}|a\rangle, \quad (4.16)$$

depending on whether $b \ (1 \to 2)$ or $a \ (2 \to 1)$ is the two-particle state, respectively.‡‡

Thus, in terms of conformal dimensions, 3-point functions (namely, matrix elements of $H^{(1)}, \langle b|H^{(1)}|a\rangle = V_{ijk}$) in general take the form

$$V_{ijk} \frac{(\Delta_k - \Delta_i - \Delta_j)}{e^{(\Delta_i + \Delta_j - \Delta_k)T}}, \quad (4.17)$$

using light-cone 3-point vertex $V_{ijk}$ where the pair of indices $(i,j)$ designates the two-particle state, either $|a\rangle$ (initial state) or $\langle b|$ (final state), and $k$ being the single particle state. By relating the large time interval $T$ with the short-distance cutoff length by

$$e^{-T} \sim |x_i - x_j| \quad (4.18)$$

‡‡There is a subtlety here. The energy denominator $1/(E_a - E_b)$ becomes singular at $E_a = E_b$. However, we consider generic processes of wave packet states with $E_a \neq E_b$. If necessary, we can take the limit $E_a \to E_b$ after the limit of large $T$. This subtlety is related with the similar problem for ‘extremal’ correlators in the ordinary AdS-Gravity/CFT correspondence without the PP-wave limit. See, e.g., [12].
in conformity with the semiclassical result (2.15), this precisely corresponds to the short

\[ |x_i - x_j| \to 0 \ (T \to \infty) \]

distance limit of the 3-point function \( \langle O_i(x_i)O_j(x_j)O_k(x_k) \rangle \), provided the two-point functions are normalized to 1, which essentially amounts to setting \( |x_i - x_k| = |x_j - x_k| = 1 \) in this expression. Due to the assumption of the completeness of BMN operators with definite conformal dimensions corresponding to the transverse oscillations of string, the indices \((i, j, k, \ldots)\) of OPE coefficients behave essentially as the R-symmetry indices of \( \text{SO}(8) \), and the short distance OPE, which is a main assumption for the boundary theory, takes the form

\[ O_i(0)O_j(x) \sim \sum_k \frac{1}{|x|^{\Delta_i + \Delta_j - \Delta_k}} C_{ijk} O_k(0), \quad x \to 0. \] (4.19)

Thus the 3-point vertex is expressed in terms of the OPE coefficient as

\[ V_{ijk} = (\Delta_k - \Delta_i - \Delta_j)C_{ijk}. \] (4.20)

This form has been conjectured in [9], but now is directly explained as a consequence of holography. We note that some computations supporting this form of 3-point vertex have been reported in the case of the ordinary real-time formulation. For a (partial) list of such works, see [13] (See, however, the note added at the end of the present paper.). The explicit construction of string field theory vertex using the formalism developed in the previous section is left as a future work.

We can rewrite this as a commutation relation,

\[ \mathcal{H}^{(1)} = \pm[\mathcal{H}^{(0)}, \mathcal{C}]. \] (4.21)

We use this operator notation in the tree approximation in the sense that, in computing matrix elements of this relation, we only allow intermediate states which in terms of Feynman diagrams represents tree diagrams. The operator \( \mathcal{C} \) is defined such that its matrix elements give the OPE coefficient \( C_{ijk} \). No confusion should arise here since \( \mathcal{H}^{(0)} \) does not change the particle number. The sign in (4.21) is \(+(-)\) when \( a \) \((b)\) is the two-particle state, corresponding to \( 2 \to 1 \) \((1 \to 2)\) process. In the more standard notation, we would write \( \mathcal{C} \) as \( \frac{1}{2} (\Psi^\dagger)^2 \mathcal{C} \Psi \) or \( \frac{1}{2} \Psi^\dagger \mathcal{C} (\Psi)^2 \), corresponding to \( 1 \to 2 \) or \( 2 \to 1 \), respectively.

As has already been noticed in ref. [14] in a different context, this form is consistent with the susy algebra. By second-quantizing the algebra (3.56), we must have in the present approximation

\[ 2\mathcal{H}^{(1)} = \{ \mathcal{Q}^{(0)}_{1} , \mathcal{Q}^{(1)}_{1} \} + \{ \mathcal{Q}^{(1)}_{1} , \mathcal{Q}^{(0)}_{1} \} = \{ \mathcal{Q}^{(0)}_{2} , \mathcal{Q}^{(1)}_{2} \} + \{ \mathcal{Q}^{(1)}_{2} , \mathcal{Q}^{(0)}_{2} \}, \] (4.22)
and
\[ \{ Q^{-(0)}_1, Q^{-(1)}_2 \} + \{ Q^{-(0)}_2, Q^{-(1)}_1 \} = 0. \] (4.23)

Combining the lowest order algebra with (4.21), these relations determine the first order susy operators as
\[ Q^{-(1)}_1 = \pm [Q^{-(0)}_1, C], \quad Q^{-(1)}_2 = \pm [Q^{-(0)}_2, C] \] (4.24)

with the requirement of SO(4) \times SO(4) invariance,
\[ [J_{ij}, C] = 0, \] (4.25)

which comes from (4.23), as it should be since SO(4) \times SO(4) is a kinematical symmetry of the system. Of course, it is possible to require higher global symmetry SO(8).

4.3 4-point amplitude

We now have to study both the first and second terms in (4.13). Let us first consider the contribution from the second term which is after the integration
\[
\sum_c \langle b|H^{(1)}|c \rangle \langle c|H^{(1)}|a \rangle \\
\times \left[ e^{(E_b - E_a)T} + \frac{e^{(E_b - E_a)T}}{(E_a - E_b)(E_a - E_c)} + \frac{e^{(E_a + E_b - 2E_c)T}}{(E_a - E_b)(E_c - E_a)} \right].
\] (4.26)

Note that in our tree approximation the intermediate states |c⟩ contain only one particle which is not connected to initial or final states directly. In other words, if |c⟩ is a multiparticle state, such particles are ‘spectators’ except for a single particle corresponding to the internal line of 4-point Feynman diagram. There are two cases which we have to consider separately.

1. Both \(a \sim (i, j)\) and \(b \sim (k, \ell)\) are two-particle states. Our rule then tells us that the third term in (4.26) should correspond to a 4-point correlator
\[ \langle O_i(x_i)O_j(x_j)O_k(x_k)O_\ell(x_\ell) \rangle, \]
in the short-distance limit \(|x_i - x_j| \sim |x_k - x_\ell| \rightarrow e^{-T} \). This is indeed realized by the relation established in the previous subsection which shows that the third term is given as
\[
\sum_m C_{ijm} C_{mk\ell} e^{(\Delta_i + \Delta_j + \Delta_k + \Delta_\ell - 2\Delta_m)T} \left\langle b|C|c \right\rangle \left\langle c|C|a \right\rangle e^{(E_a + E_b)T} \sum_c e^{-2E_cT} \left\langle b|C|c \right\rangle \left\langle c|C|a \right\rangle, \] (4.27)
where \( m \) corresponds to the internal line. remembering that two-point functions are normalized to be 1 in our convention, this is the correct OPE result for the 4-point correlator, assuming that the transverse string states correspond to the complete set of gauge-theory operators with definite conformal dimensions.

We note that the case where the external lines in initial and final states cross does not contribute according to our rule, since in that case we would have negative exponents for some of the external lines owing to the energy factor \( e^{-2E_cT} \). It is a general rule that the intermediate state with energy factor \( e^{-2ET} \) must always be associated with a single internal line without any spectator strings. Even if we include such terms by relaxing the restriction in preparing the initial and final states, they would be nonleading contributions in the limit \( T \to \infty \). the amplitudes we are considering thus correspond generically to that in Fig. 1.

2. Either \( a \) or \( b \) is a three-particle state \((i, j, k)\). in this case, the second or first term, respectively, should contribute to the S-matrix. However, neither does take the precise structure expected from the short distance OPE of three operators, except for the energy exponent which is correct. For definiteness, let us consider the case where \( b \) is the 3-particle state \((i.e. \ 1 \to 3 \) process). Using the same notational convention introduced in the case of 3-point amplitude, the first term takes the form

\[
-\frac{e^{(E_b-E_a)T}}{E_a - E_b} \langle b | [\mathcal{H}^{(0)}, \mathcal{C}] |a \rangle. \tag{4.28}
\]

The energy exponent takes the required form \( E_b - E_a = \Delta_i + \Delta_j + \Delta_k - \Delta_\ell \).

The solution of this problem is that there must be the contribution from the 4-point vertex \( \mathcal{H}^{(2)} \) in this case. Indeed if we assume

\[
\mathcal{H}^{(2)} \Rightarrow -\mathcal{C}[\mathcal{H}^{(0)}, \mathcal{C}], \tag{4.29}
\]

for \( 1 \to 3 \) (from right to left) 4-point vertex, the total contribution to the present 4-point amplitude is

\[
-\frac{e^{(E_b-E_a)T}}{E_a - E_b} \langle b | (\mathcal{C}[\mathcal{H}^{(0)}, \mathcal{C}] + [\mathcal{H}^{(0)}, \mathcal{C}]\mathcal{C}) |a \rangle = e^{(E_b-E_a)T} \langle b | \mathcal{C}\mathcal{C} |a \rangle, \tag{4.30}
\]

as is required for the correct short distance OPE for the present case. To avoid confusion, we note that in the matrix-element form, the above expressions involving

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commutators are
\[
\langle b| \mathcal{C}[\mathcal{H}^{(0)}, \mathcal{C}]|a \rangle = \sum_{P(ijk)} \sum_{m} (E_m + E_k - E_\ell) C_{ijm} C_{mk\ell}, \quad (4.31)
\]
\[
\langle b|[\mathcal{H}^{(0)}, \mathcal{C}]|\mathcal{C}|a \rangle = \sum_{P(ijk)} \sum_{m} (E_i + E_j - E_m) C_{ijm} C_{mk\ell}, \quad (4.32)
\]
where the summation symbol \( \sum_{P(ijk)} \) should be understood as summation over inequivalent partition of \((i, j, k)\) into \(1+2\) groups. Note that because of the preservation of angular momentum we can always replace the energy \( E \) by conformal dimension \( \Delta \) in the commutator.

If we repeat the same argument for the case where \( a \) is the 3-particle state, the second term of (4.26) is now
\[
e^{(E_a - E_b)T} \frac{E_b - E_a}{E_b - E_a} \langle b| \mathcal{C}[\mathcal{H}^{(0)}, \mathcal{C}]|a \rangle. \quad (4.33)
\]
Hence the \( 3 \to 1 \) 4-point vertex must be \( \mathcal{H}^{(2)} \Rightarrow [\mathcal{H}^{(0)}, \mathcal{C}]|\mathcal{C}|a \), since it contributes to the 4-point amplitude as
\[
e^{(E_a - E_b)T} \frac{E_b - E_a}{E_b - E_a} \langle b|[\mathcal{H}^{(0)}, \mathcal{C}]|\mathcal{C}|a \rangle,
\]
and gives the correct 4-point amplitude \( e^{(E_a - E_b)T} \langle b|\mathcal{C}|a \rangle \), combining with (4.33).

Our analysis shows that to reproduce the OPE structure corresponding to two-group short distance limit of conformal operators it is necessary to add the second order interaction terms \( \mathcal{H}^{(2)} \) corresponding to \( 1 \to 3 \) and \( 3 \to 1 \) matrix elements. The one corresponding to \( 2 \to 2 \) is not necessary. Or more appropriately, we should say that the requirement of holography cannot directly fix the form of \( 2 \to 2 \) matrix element within the present tree approximation. This is essentially owing to our special choice in preparing the initial and final wave packet states, since according to our prescription the negative energy exponents associated with the multi-particle states can be ignored.

Our next task is to study whether the 4-point interaction vertex obtained above from the requirement of holographic correspondence is consistent with the susy algebra. The algebra to be satisfied is
\[
2\mathcal{H}^{(2)} = \{ \mathcal{Q}_1^{-1}, \mathcal{Q}_1^{-1} \} + \{ \mathcal{Q}_1^{-1}, \mathcal{Q}_1^{-2} \} + \{ \mathcal{Q}_1^{-2}, \mathcal{Q}_1^{-1} \}
\]
Figure 1: This shows a generic diagram we are considering with our choice of wave packet states.

\[
= \{Q_{2}^{-1}, Q_{2}^{-1}\} + \{Q_{2}^{-0}, Q_{2}^{-2}\} + \{Q_{2}^{-2}, Q_{2}^{-0}\},
\]
\[
\{Q_{1}^{-1}, Q_{2}^{-1}\} + \{Q_{1}^{-0}, Q_{2}^{-2}\} + \{Q_{2}^{-0}, Q_{1}^{-2}\} + \text{transposed expressions} = 0.
\]

It should be kept in mind that in dealing with these algebras in the present section we are considering only the matrix elements of the types, either \((1 \to n)\) or \((n \to 1)\), separately. In the present tree approximation, this restriction of matrix elements can be implemented consistently. Other types of matrix elements cannot be treated since we have not determined the vertices of general types. It is easy to check that all of these relations are satisfied for \(H^{(2)} = -C[H^{(0)}, C] (1 \to 3)\) or for \([H^{(0)}, C]C (3 \to 1)\) if we choose the susy generator of the present order as

\[
Q_{1,2}^{-2} = -C[Q_{1,2}^{-0}, C] \quad \text{or} \quad Q_{1,2}^{-2} = [Q_{1,2}^{-0}, C]C,
\]

respectively, providing that the SO(8) symmetry relation (4.25) is valid. The holography and supersymmetry conforms to each other in a quite miraculous way.

### 4.4 5-point amplitude

The integration in the third term of (4.13) gives

\[
\sum_{c,d} \langle b|\mathcal{H}^{(1)}|c\rangle \langle c|\mathcal{H}^{(1)}|d\rangle \langle d|\mathcal{H}^{(1)}|a\rangle \\
\times \left[ \frac{e^{(E_{b} - E_{a})T}}{(E_{a} - E_{b})(E_{a} - E_{d})(E_{a} - E_{c})} + \frac{e^{-(E_{b} - E_{a})T}}{(E_{b} - E_{a})(E_{b} - E_{d})(E_{b} - E_{c})} \right]
\]

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\[
\frac{e^{(E_b+E_a-2E_c)T}}{(E_c-E_b)(E_c-E_a)} + \frac{e^{(E_b+E_a-2E_d)T}}{(E_d-E_b)(E_d-E_a)(E_d-E_c)}.
\]

(4.37)

Comparing with the previous cases, we see that among these four terms the first or second term represent \((1 \rightarrow 4)\) or \((4 \rightarrow 1)\) process, respectively, and the 3rd or 4th term represent \((3 \rightarrow 2)\) or \((2 \rightarrow 3)\) process, respectively. Note in particular that in all other cases there arise negative exponents for multi-string initial or final states. Also, we have to take into account the contribution of higher-point vertices coming from second and first terms in (4.13).

We first consider the case \((1 \rightarrow 4)\). By adding the contributions of higher-point vertices, total amplitude is

\[
\frac{e^{(E_b-E_a)T}}{E_a-E_b} \left[ \sum_{c,d} \frac{\langle b|H^{(1)}|c\rangle \langle c|H^{(1)}|d\rangle \langle d|H^{(1)}|a\rangle}{(E_a-E_d)(E_a-E_c)} + \sum_{d} \frac{\langle b|H^{(2)}|d\rangle \langle d|H^{(1)}|a\rangle}{E_a-E_d} + \sum_{c} \frac{\langle b|H^{(1)}|c\rangle \langle c|H^{(2)}|a\rangle}{E_a-E_c} + \langle b|H^{(3)}|a\rangle \right].
\]

(4.38)

Substituting the previous results for \(H^{(1)}\) and \(H^{(2)}\), this expression is reduced to

\[
\frac{e^{(E_b-E_a)T}}{E_a-E_b} \langle b| \left( -[H^{(0)},C]CC - C[H^{(0)},C]C + H^{(3)} \right) |a\rangle.
\]

Thus we see that holography requires us to set

\[
H^{(3)} \Rightarrow -CC[H^{(0)},C],
\]

(4.39)

by which the \(1 \rightarrow 4\) amplitude takes the correct form \(e^{(E_b-E_a)T} \langle b|CC|a\rangle\).

The \(4 \rightarrow 1\) case is just the transpose (hermitian conjugate) of this process so that we find

\[
H^{(3)} \Rightarrow [H^{(0)},C]CC.
\]

(4.40)

As is always the case in the present paper, the multiplication symbol must be understood within the tree approximation. Note that if there are spectator strings they just pass through the matrix elements of \(C\) as identity. Thus these products represent ‘monotonic’ flow of either increasing or decreasing number of strings.

Let us next turn to the case \(2 \rightarrow 3\). In this case, the total amplitude is by taking into account the second order vertex \(H^{(2)}\),

\[
e^{(E_b+E_a)T} \left[ \sum_{c,d} \frac{e^{-2E_dT} \langle b|H^{(1)}|c\rangle \langle c|H^{(1)}|d\rangle \langle d|H^{(1)}|a\rangle}{(E_d-E_b)(E_d-E_a)(E_d-E_c)} + \sum_{d} \frac{e^{-2E_dT} \langle b|H^{(2)}|d\rangle \langle d|H^{(1)}|a\rangle}{(E_d-E_b)(E_d-E_a)} \right].
\]

(4.41)
On using the previous results, this is reduced to

\[- \sum_d e^{(E_b + E_a - 2E_d)T} \frac{\langle b| [\mathcal{H}^{(0)}, CC]|d \rangle}{(E_d - E_b)} \langle d|C|a \rangle.\]

This gives the correct expression \( \sum_d e^{(E_b + E_a - 2E_d)T} \langle b|CC|d \rangle \langle d|C|a \rangle \) for the \( 2 \to 3 \) correlation function in terms of the OPE coefficients. The transposed case \( 3 \to 2 \) can be treated in the same way. As in the case of 4-point case, the interaction vertices corresponding to \( 2 \to 3 \) and \( 3 \to 2 \) do not contribute to the leading short-distance behavior according to our prescription.

Finally, by examining the susy algebra to the 3rd order using the above results, we find that the 3rd order term,

\[ Q_{1,2}^{(3)} \to -CC[Q_{1,2}^{(0)}, C], \]

for \( 1 \to 4 \) and their hermitian conjugate \([Q_{1,2}^{(0)}, C]CC\) for \( 4 \to 1 \) satisfy all the required properties in the similar way as we have found for lower orders.

### 4.5 General case

We can continue this analysis to arbitrarily higher orders. In fact, we can easily guess the general forms from the above results. To reproduce the leading short-distance behaviors of the correlators, higher-interaction vertices (or contact terms) are necessary for the type \( 1 \to n + 1 \) and their transpose \( n + 1 \to 1 \) \((n \geq 1)\) and are given as

\[ H^{(n)}_{(1 \to n + 1)} = -C^{n-1}[H^{(0)}, C], \quad H^{(n)}_{(n+1 \to 1)} = [H^{(0)}, C]C^{n-1}. \]

We can prove this by a simple induction. It is sufficient to consider the case \( 1 \to n \). The general form of perturbative expansion of the T-matrix is

\[ e^{(E_b - E_a)T} \sum_{c_1, c_2, \ldots, c_{n-2}} \frac{\langle b| \mathcal{V}|c_{n-2}\rangle \langle c_{n-2}| \mathcal{V}|c_{n-3}\rangle \langle c_{n-3}| \mathcal{V}|c_{n-4}\rangle \cdots \langle c_1| \mathcal{V}|a\rangle}{(E_a - E_{c_{n-2}})(E_a - E_{c_{n-3}}) \cdots (E_a - E_{c_1})} \]

\[ \equiv e^{(E_b - E_a)T} \frac{\langle b| T_{n-1}|a \rangle}{E_a - E_b}, \]

where \( \mathcal{V} \) can be any vertices up to the order \( H^{(n-1)} \). Suppose that the sum in this expression is shown to be of the form

\[ T_{n-1} = -[H^{(0)}, C^{n-1}] \]
for the cases $n \leq N$ with the above form of the vertices. This means that

$$\langle b | T_N | a \rangle = \sum_{k=1}^{N-N-1} \sum_{c_{N-1}} \frac{\langle b | H^{(k)} | c_{N-1} \rangle \langle c_{N-1} | [H^{(0)}, C^{N-k}] | a \rangle}{E_a - E_{c_{N-1}}}$$

$$= \sum_{k=1}^{N-N-1} \sum_{c_{N-1}} \langle b | H^{(k)} | c_{N-1} \rangle \langle c_{N-1} | C^{N-k} | a \rangle.$$

Holography requires that this expression is equal to

$$-\langle b | [H^{(0)}, C^N] | a \rangle.$$

which requires $H^{(N)} = -C^{N-1}[H^{(0)}, C]$, completing the induction.

We now have a Hamiltonian for our Euclidean and ‘holographic’ string field theory, once the 3-point OPE coefficients $C$ are fixed, which is exact at least at the tree level approximation in computing holographic S-matrix elements of the type in Figure 1:

$$H = H^{(0)} - \frac{1}{1-C} [H^{(0)}, C] + [H^{(0)}, C] \frac{1}{1-C},$$

(4.44)

where the first interaction terms should be understood for the matrix elements $1 \rightarrow n+1$ and the second for $n+1 \rightarrow 1$.

The susy generators corresponding to the above form of the general interaction vertices are

$$Q^{-(n)}_{1,2(1\rightarrow n+1)} = -C^{n-1}[Q^{(0)}_{1,2}, C], \quad Q^{-(n)}_{1,2(n+1\rightarrow 1)} = [Q^{(0)}_{1,2}, C]C^{n-1}.$$

(4.45)

These expressions can be rewritten in such a way that the susy algebra is manifestly satisfied:

$$H = \frac{1}{1-C} H^{(0)}(1-C), \quad Q^{(0)}_{1,2} = \frac{1}{1-C} Q^{-(0)}_{1,2}(1-C),$$

(4.46)

for $1 \rightarrow n+1$ and their transpose for $n+1 \rightarrow 1$, respectively, for now $n \geq 0$. Remarkably, the full interacting operators of holographic string-field theory are obtained from the free theory by a similarity transformation composed of the OPE coefficients at least in the present classical (tree level) context. Of course the kinematical generators, which commute $C$ are not affected by this similarity transformations.

We emphasize that even if the interacting Hamiltonian is obtained by a similarity transformation, the system is not trivial. S-matrix is nontrivial in our approach by its fundamental assumption. There is no contradiction here. Remember that, already at the level of 3-point amplitude, the form of the vertex $[H^{(0)}, C]$, which is of course the
infinitesimal form of the above similarity transformation, would indicate vanishing of on-shell 3-point amplitude in ordinary Minkowski theory, because of energy conservation. This was not the case in our Euclidean theory, since we have the denominator of energy difference $1/(E_a - E_b)$, instead of delta function $\delta(E_a - E_b)$. What we have established for higher vertices are the extension of this phenomenon to higher orders.

4.6 Exact form of the Hamiltonian?

As has been already warned, for the matrix elements of the more general type as $m \to n$ ($m, n \neq 1$), our prescription cannot fix their forms, since their matrix elements of wave packet states vanish when they appear in the amplitudes with our prescription. So there are two options as for the existence of other types of interaction vertices.

1. There are no other types than those corresponding to $1 \to n$ or $n \to 1$.

2. There are other types as well. Holography alone is not sufficient to fix them.

The question is then what is the criterion in deciding the correct direction. A possible criterion which we can think of seems to require supersymmetry as a quantum symmetry beyond the tree approximation. The first option seems unlikely to be true, since it is difficult to satisfy the susy algebra with only those types of interaction vertices. An important task is now to try to construct the other terms assuming the possibility 2 above. This requires us to quantize the string field theory and hence is beyond the scope of the present paper. We hope to discuss this question elsewhere.

5. Concluding remarks

In the present paper, we have investigated the question how holographic principle for the string theory in AdS$_5 \times$S$^5$ background should be formulated in the context of PP-wave limit. We started from some puzzles involved in the important proposal made in [1]. We argued that all the puzzles mentioned are naturally resolved if we recall that the holographic correspondence as signified in the basic relation (1.1) between string theory and super Yang-Mills theory is correctly interpreted as a tunneling phenomena from the viewpoint of semiclassical approximation. This led us to consider a tunneling null geodesic as the basis for quantizing string theory to describe interactions of strings.
with large angular momenta along a particular direction of $S^5$. By assuming consistency of the string interaction with holography on the basis of tunneling picture, we found that the form of string vertices are strongly constrained in a way conforming to supersymmetry.

There are several points which are left untouched and/or remain to be further clarified. First, we have not discussed the effect of string loops. We hope that our formalism serves as a foundation for discussing higher genus effects and its connection with SYM. Second, we have not performed concrete comparison of our results on string S-matrix with higher-point correlators on the SYM side. In particular, in our formulation, the cutoff parameter in forming short distance OPE has been assumed always to be the single quantity $e^{-T}$ with one and the same normalization. However, on gauge-theory side, it is not trivial to achieve this in taking short-distance limit of general multiple products of operators in usual ways. For $1 \rightarrow n$ or $n \rightarrow 1$ up to, at least, $n = 5$, this can be achieved in flat 4-dimensional base space by setting all the distances equal for arbitrary pairs of $n$ points where the operators of initial or final states are located. However, for $n > 6$, such simple configurations are not possible, as long as we assume the flat metric. One possible way out of this problem might be to introduce nontrivial conformal metric for the base space of boundary theory and identify the cutoff parameter using the distances defined by metrics with some nontrivial Weyl factor, such that we can assume a single cutoff $e^{-T}$. In the limit $T \rightarrow \infty$ and $n \rightarrow \infty$, such metric would become very chaotic, but final physical results may show regular behaviors in the sense of our results. In the literature, some ambiguity of perturbative 4-point functions in taking short-distance limit has been reported [15]. We also mention that there have been some discussions [16] on the closure of OPE of BMN operators and its relevance to higher-point amplitudes. In view of our results, it is very important to develop a systematic method for constructing two-group short distance limits of multi-trace operators on gauge-theory side.

With respect to the possible ambiguity of normalization, we have to keep in mind the following subtlety. For string field-theory S-matrix, we have to sum over all different tree diagrams, while in OPE tree diagrams which are connected by channel duality must be equivalent and are not summed over, provided we have a complete set of operators. In our string-theory language, this ‘channel duality’ would be an automatic consequence. If these statements are justified, we have to normalize our initial and final states in comparing them with the CFT correlators, in such a way that the degeneracy caused by the sum over
dual channels are canceled. That amounts to dividing by the factor $(2n - 3)!!$ for each $n$-particle initial or final states, which is equal to the number of $(n + 1)$-point Feynman diagrams in $\phi^3$ field theory.

With the understanding of all these possible subtleties with respect to regularization and normalization, our results are essentially summarized by the following symbolic ‘reduction formula’ connecting string S-matrix $S^E(T)$ with sufficiently large Euclidean time duration $2T$ with two-group correlators of SYM,

$$
\langle i_1, i_2, \ldots, i_p | S^E(T) | j_1, j_2, \ldots, j_q \rangle_{\text{conn}} = \langle (\overline{O}_1 \overline{O}_2 \cdots \overline{O}_p)_T(1)(O_1 O_2 \cdots O_q)_T(0) \rangle_{\text{conn}},
$$

(5.47)

where the first and second groups of operators in the right-hand side are suitable short-distance limits of multiple products of BMN operators with negative and positive angular momentum, respectively, with cutoff length $e^{-T}$. One might wonder how about more general correlators with multi-centers. In principle, it should be possible to generalize our results to multi-center cases by introducing coherent states with respect to the zero-mode operators to shift the positions of wave packets along transverse directions. However, we cannot settle this question at this stage unless we have at hand systematic methods of dealing with short-distance limit as discussed above. All these are left for the future works.

To conclude, we add more remarks on some salient features in our arguments and results.

1. Wave packet for correlation functions:

We had to assume wave packets in order to extract S-matrix elements correctly such that they precisely reproduce the behaviors of CFT correlators defined on the boundary. The reader may wonder what wave packets mean for the correlators. For this question, we should recall that on the boundary we are treating nonstandard composite operators which involve infinitely large number of fields at the same space-time point in the form $\text{Tr}[\cdots Z^{J} \cdots](0)$. In our way of establishing the holographic correspondence of BMN operators with the S-matrix of Euclidean string-field theory, we have to further take short-distance products of such operators. $\text{Tr}[\cdots Z^{J_1} \cdots](0)\text{Tr}[\cdots Z^{J_2} \cdots](\epsilon) \cdots (\epsilon \sim e^{-T})$. These are certainly very singular objects. In particular, by rewriting symbolically as $Z = |Z|e^{i\psi}$, these are products of
‘plane waves’ $e^{ij\psi}$ which have infinite extensions in the configuration space of fields. Remember that since we are taking the limit $J \to \infty$ and treating $\alpha$ as a continuous parameter, the $\psi$ direction is now effectively noncompact. Then, it is a natural procedure to regularize the effect of the infinite extendedness in the configuration space of fields, by making wave packets even for correlators as we are forced to do in defining Euclidean S-matrix. It would be interesting to pursue this idea further. This might also be related to the resolution of possible ambiguity mentioned above.

2. Connection with holographic renormalization group:

In our formulation, the identification of time parameter $\tau$ along a tunneling null geodesic with $\Delta - J$ was completely manifest. In particular, the cutoff parameter $e^{-T}$ is identified with the radial coordinate $z \sim e^{-T}$ at the boundary. In this sense, the Schrödinger equation of string field theory can be interpreted as a version of holographic renormalization group equation. A very interesting problem now is whether it is possible to derive the same equation directly at the boundary using only the logic of super Yang-Mills theory. It is natural to expect that a Wilsonian approach to the renormalization group for the BMN operators would lead to such a derivation.

3. Universality of holographic correspondence:

An important question related to our approach is ‘How universal is the holographic correspondence?’ For example, we may ask, as has already been alluded to before, whether we can extract the string degrees of freedom in Matrix theory in a similar fashion. Since we have formulated the holographic correspondence entirely using the Poincaré patch of the AdS geometry, there is in principle no difficulty in extending our method to such cases. For instance, it would be interesting to see how the results found in [18] for supergravity modes on the basis of picture proposed in [19] are generalized to stringy degrees of freedom.

4. Exact solvability of quantum string theory:

We have presented strong evidence that holographic string theory has the simple Hamiltonian which is obtained by a simple similarity transformation from the free theory at least in the tree approximation for a special class of interaction vertices.
We may try to construct a fully quantized unitary operator for our string field theory at least formally as a generalization of our classical similarity transformation. This seems quite suggestive from the viewpoint of seeking for nonperturbative definition of string/M theory.

Note added

Responding to the comment from the referee, we would like to add a couple of remarks on the status of our prediction eq. (4.20) for further clarification. After the submission of the present paper, it has been pointed out in ref. [21] that some of the previously reported results on the properties of the prefactor of 3-point vertex were groundless. In particular, the claim of some works in [13] that the prefactor of 3-point vertex of the (real time) string field theory is equal to the difference of energies is not valid, because they are based on a sign error, as pointed out in [22]. However, this does not invalidate our prediction eq. (4.20) for several reasons:

1. Our claim is not that the prefactor itself is the difference of energies, but only that the correct 3-point function (OPE coefficient) on the gauge-theory side should take the form \( C_{ijk} = V_{ijk} / (\Delta_k - \Delta_i - \Delta_j) \) in terms of the string field vertex \( V_{ijk} \). For this, it is not necessary that the prefactor itself takes the form of energy difference.

2. In order to make comparison with computations on the gauge-theory side, we have to construct the Euclidean string field theory according to the formalism developed in section 3. In fact, from what we have discussed in section 4, the requirement of susy algebra in the approximation of classical string field theory is not sufficient to fix the interaction vertex uniquely, since we can always construct the 3-point interaction vertex formally such that it satisfies the first order form of the susy algebra, once a \( C \) preserving kinematical symmetries is given. The results would strongly depend on the choice of \( C \). For example, there is an ambiguity regarding to what extent possible kinematical symmetries (for instance, either \( \text{SO}(4) \times \text{SO}(4) \times \mathbb{Z}_2 \) or \( \text{SO}(8) \times \text{SO}(8) \)) are assumed. In the standard treatment, even \( \mathbb{Z}_2 \) symmetry is not taken into account as has been emphasized in [17]. Of course, for the result to be well behaved, the CFT coefficients should not diverge for the special cases when the energy is conserved. In other words, the 3-point vertex must have vanishing matrix elements for energy.
conserving processes. In the case of supergravity approximation, this is indeed satisfied before taking the plane-wave limit, as discussed in the literature in the name of ‘extremal’ correlators [12] as we have mentioned in the text. It seems that all the known results for CFT coefficients from perturbative computations on the Yang-Mills side are also consistent with this property [23].

3. On the gauge-theory side, it is known that we have to take into account operator mixing even at the lowest order level in order to maintain the standard form of the 3-point function, which is the basic assumption, eq. (4.19), in our arguments in section 4. In connection with this, we would like to mention a new proposal, appeared after the submission of the present paper, in ref. [23] for the prefactor such that it is indeed consistent with our prediction (4.20), compared with all the known perturbative results on gauge theory side, taking into account the operator mixing. More generally, however, we have to keep in mind an important question whether or not the logic of light-cone string field theory alone involves the first principle for constructing interacting string theory in curved space-time uniquely. Our consideration seems to indicate that the answer to this question is negative.

We are currently working on the explicit construction of string field theory which meets our criterion to be the ‘holographic string field theory’. We hope to report on this progress in a forthcoming paper.

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Appendix

In this appendix we show how the free massive world-sheet action derived in section 3 can
be obtained by first making a suitable coordinate transformation and then expanding the transformed action around the classical solution or the null-geodesic line.

We first choose the coordinate transformation so that one of the coordinates lies along the tunneling null-geodesic line of section 3, which makes it easy to expand the action around the solution since all coordinates of the solution are zero except for one along the geodesic. The desired transformation is given by

\[ z = \frac{1}{\cosh x^+}, \]  
\[ r = \tanh x^+ - x^- + \frac{X^2}{2} \tanh x^+ + \frac{\Phi^2}{2 \tanh x^+} + \frac{\Phi}{\sinh x^+}, \]  
\[ x^i = \frac{X^i}{\cosh x^+} \quad (i = 1, 2, 3), \]  
\[ \psi = x^+ + \frac{\Phi}{\sinh x^+}, \]  
\[ \theta = \Theta, \]

following the method of ref. [20].

After transforming the bosonic action by this map, we expand it around a classical solution \( x^+ = \tau \) and \( x^- = X^i = \Phi = \Theta = 0 \). With the coordinate expansion such as

\[ x^+ = \tau + \frac{1}{R} x^+_{(1)} + \frac{1}{R^2} x^+_{(2)} + \cdots, \quad x^- = \frac{1}{R} x^-_{(1)} + \frac{1}{R^2} x^-_{(2)} + \cdots, \quad \text{etc,} \]

we obtain the quadratic action

\[ S = \frac{1}{4\pi} \int d\tau \int_0^{2\pi} d\sigma \left[ -2 \partial x^+ \partial x^- + \Phi^2 + \sum_{i=1}^3 (X^i)^2 + \sum_{i=1}^4 (Y^i)^2 \right. \]  
\[ \left. + (\partial \Phi)^2 + \sum_{i=1}^3 (\partial X^i)^2 + \sum_{i=1}^4 (\partial Y^i)^2 + \frac{2 \cosh^2 \tau}{\sinh^2 \tau} \Phi \dot{x}^- - \frac{2 \cosh^2 \tau}{\sinh \tau} \partial \Phi \partial x^- + \cosh^2 \tau (\partial x^-)^2 \right], \]

where we have omitted the subscript and introduced the coordinates \( Y^i \) as the Cartesian coordinates defined by \((\partial \Theta)^2 + \sin^2 \Theta (\partial \Omega_3) \sim (\partial \Theta)^2 + \Theta^2 (\partial \Omega_3) = \sum_{i=1}^4 (\partial Y^i)^2 \).

The extra term appearing in (A.7) can be dropped by virtue of the Virasoro constraints, whose bosonic part is expanded to the quadratic order as

\[ -2 R \ddot{x}_{(1)} - 2 \ddot{x}_{(2)} - 2 \{ \dot{x}^+_{(1)} \dot{x}_{(1)} - (x^+_{(1)})' (x^-_{(1)})' \} + \Phi^2 + \sum_{i=1}^3 (X^i)^2 + \sum_{i=1}^4 (Y^i)^2 \]  
\[ + \dot{\Phi}^2_{(1)} - \Phi^2_{(1)} + \sum_{i=1}^3 \{ (\dot{X}^i)^2 - (X^i_{(1)})^2 \} + \sum_{i=1}^4 \{ (\dot{Y}^i)^2 - (Y^i_{(1)})^2 \} \]  
\[ + \frac{2 \cosh^3 \tau}{\sinh^2 \tau} \Phi_{(1)} \dot{x}_{(1)} - \frac{2 \cosh^2 \tau}{\sinh \tau} (\Phi_{(1)} \dot{x}_{(1)} - \Phi'_{(1)} x^-_{(1)}) + \cosh^2 \tau (\dot{x}^-_{(1)} - x^-_{(1)}) + O \left( \frac{1}{R} \right) = 0, \]
and

\[-2Rx_{(1)}' - 2x_{(2)}' - \dot{x}_{(1)}x_{(1)}' - x_{(1)}\dot{x}_{(1)} + \dot{\Phi}_{(1)}\Phi_{(1)} + \sum_{i=1}^{3} \dot{X}_{i} X_{i}' + \sum_{i=1}^{4} Y_{i} Y_{i}' \]

\[\quad + \frac{\cosh^3 \tau}{\sinh^2 \tau} \dot{\Phi}_{(1)} x_{(1)} - \frac{\cosh^2 \tau}{\sinh \tau} (\dot{\Phi}_{(1)} x_{(1)} + \dot{\Phi}_{(1)} \dot{x}_{(1)}) + \cosh \tau \dot{x}_{(1)} x_{(1)}' + O \left( \frac{1}{R} \right) = 0.\]  

As is easily seen, solving the above constraints recursively in terms of \(1/R\) gives \(\dot{x}_{(1)} = x_{(1)}' = 0\). Substituting them into (A.7) and redefining \((\Phi, X^i)\) as \(X_{(i)} = 1, \cdots, 4\), we finally obtain the the same free massive string action as (3.12):

\[S = \frac{1}{4\pi} \int d\tau \int_{0}^{2\pi/\alpha} d\sigma \left[ \sum_{i=1}^{4} (X^i)^2 + \sum_{i=1}^{4} (Y^i)^2 + \sum_{i=1}^{4} (\partial X^i)^2 + \sum_{i=1}^{4} (\partial Y^i)^2 \right]. \]  

(A.10)

We can repeat this analysis to fermions and find the same result as given in section 3.

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