PÓLYA’S CONJECTURE FAILS FOR THE FRACTIONAL LAPLACIAN

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Abstract. The analogue of Pólya’s conjecture is shown to fail for the fractional Laplacian \((-\Delta)^{\alpha/2}\) on an interval in 1-dimension, whenever \(0 < \alpha < 2\). The failure is total: every eigenvalue lies below the corresponding term of the Weyl asymptotic.

In 2-dimensions, the fractional Pólya conjecture fails already for the first eigenvalue, when \(0 < \alpha < 0.984\).

Introduction. The Weyl asymptotic for the \(n\)-th eigenvalue of the Dirichlet Laplacian on a bounded domain of volume \(V\) in \(\mathbb{R}^d\) says that

\[
\lambda_n \sim (nC_d/V)^{2/d} \quad \text{as } n \to \infty,
\]

where \(C_d = (2\pi)^d/\omega_d\) and \(\omega_d = \text{volume of the unit ball in } \mathbb{R}^d\). In 1-dimension, “volume” means length and in 2-dimensions it means area, so that \(C_1 = \pi, C_2 = 4\pi\). Pólya suggested that the Weyl asymptotic provides more than a limiting relation. He conjectured that it gives a lower bound on each eigenvalue:

\[
\lambda_n \geq (nC_d/V)^{2/d}, \quad n = 1, 2, 3, \ldots
\]

He proved this inequality for tiling domains [18], but it remains open in general.

In this note, we deduce from existing results in the literature that the analogue of Pólya’s conjecture fails for the fractional Laplacian \((-\Delta)^{\alpha/2}\) on the simplest domain imaginable — an interval in 1-dimension. In 2-dimensions we show it fails on the disk and square, at least for some values of \(\alpha\).

Fractional Pólya conjecture. The fractional Laplacian \((-\Delta)^{\alpha/2}\) is a Fourier multiplier operator, with

\[
((-\Delta)^{\alpha/2}u)^{\wedge}(\xi) = |\xi|^\alpha \widehat{u}(\xi), \quad \alpha > 0,
\]

where the Fourier transform is defined by

\[
\widehat{u}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} u(x)e^{-ix\cdot\xi} \, dx.
\]

The fractional Laplacian is known to have discrete Dirichlet spectrum on the bounded domain \(\Omega \subset \mathbb{R}^d\), with weak eigenfunctions belonging to the fractional Sobolev space

\[
H_0^{\alpha/2}(\Omega) = \{ u \in H^{\alpha/2}(\mathbb{R}^d) : u = 0 \text{ a.e. on } \mathbb{R}^d \setminus \Omega \}.
\]
For further information on the fractional Sobolev space see [7]; for the fractional Laplacian see [15]; and for the variational formulation of the spectrum see [10].

Write $\lambda_n(\alpha)$ for the $n$-th eigenvalue of $(-\Delta)^{\alpha/2}$ on $\Omega$. The Weyl asymptotic (see [10, Theorem 3.1] and associated references) says that

$$\lambda_n(\alpha) \sim (nC_d/V)^{\alpha/d} \quad \text{as } n \to \infty. \quad (1)$$

Thus the fractional analogue of the Pólya conjecture is the assertion that

$$\lambda_n(\alpha) \geq (nC_d/V)^{\alpha/d}, \quad n = 1, 2, 3, \ldots.$$  

This inequality is what we shall disprove.

**Fractional Pólya conjecture fails for the unit interval, for all eigenvalues.**

In 1-dimension on an interval of length $L$, the conjecture says $\lambda_n(\alpha) \geq (n\pi/L)^{\alpha}$. Equality holds when $\alpha = 2$, the classical case of a vibrating string, but the equality is broken as soon as $\alpha$ drops below 2, according to the next theorem.

**Theorem 1 (Interval).** Suppose $\Omega = (0, L)$ is an interval in 1-dimension, and let $0 < \alpha < 2$. Then $\lambda_n(\alpha) < (n\pi/L)^{\alpha}$ for all $n$.

Hence the fractional Pólya conjecture fails on intervals, which contradicts a claim made about tiling domains (in all dimensions) in the literature [19]. See also our remark later in the paper about the square, which is a tiling domain in 2 dimensions.

**Proof.** The eigenvalues of the fractional Laplacian are known to be bounded above by powers of the usual Laplacian eigenvalues, with strict inequality:

$$\lambda_n(\alpha) < \lambda_n(2)^{\alpha/2}, \quad n = 1, 2, 3, \ldots,$$

whenever $0 < \alpha < 2$. See Proposition 3 and the discussion at the end of the paper.

On an interval in 1-dimension this last inequality says $\lambda_n(\alpha) < (n\pi/L)^{\alpha}$, which proves the theorem.

For an alternative proof when $\alpha = 1$ that provides more explicit estimates, we recall an estimate of Kulczycki, Kwaśnicki, Małecki and Stos [13, Theorem 6]. It implies for the interval of length $L = 2$ that

$$\lambda_n(1) < \frac{n\pi}{L} - \frac{\pi}{40}$$

whenever $n \geq 4$. When $n = 1, 2, 3$, those authors give the following numerical estimates [13, Section 11]:

$$\lambda_n(1) < \begin{cases} 1.16, & n = 1, \\ 2.76, & n = 2, \\ 4.32, & n = 3. \end{cases}$$

Their 12 digit estimates have been rounded up to 2 decimal places. The numerical estimates obviously satisfy $\lambda_n(1) < n\pi/L$ for $n = 1, 2, 3$, with $L = 2$.  

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A similarly explicit approach when \( \alpha \neq 1 \) proceeds through an asymptotic estimate of Kwaśnicki [14, Theorem 1], which asserts that for the interval of length \( L = 2 \),

\[
\lambda_n(\alpha) = \left( \frac{n \pi}{2} - \frac{(2 - \alpha) \pi}{8} \right)^\alpha + O\left( \frac{1}{n} \right).
\]

Rearranging, we find

\[
\lambda_n(\alpha) = \left( \frac{n \pi}{2} \right)^\alpha \left( 1 - \frac{\alpha(2 - \alpha)}{4n} + o(1/n) \right).
\]

Clearly the second factor on the right is less than 1 for all large \( n \), and so \( \lambda_n(\alpha) < (n \pi/L)^\alpha \) for all large \( n \). Thus once again we see Pólya’s conjecture fails for the fractional Laplacian. □

**Relation to Laptev’s inequality of Berezin–Li–Yau type.** Laptev [16, Corollary 2.3] extended Berezin’s eigenvalue sum inequality from the Laplacian to the fractional Laplacian, working on general domains and with an even more general class of operators. The resulting lower bound of “Li–Yau” form (see [10, formula (4.2)]) says for an interval in 1-dimension that

\[
\left( \frac{\pi}{L} \right)^{\alpha} n^{\frac{1}{1+\alpha}} \leq \sum_{k=1}^{n} \lambda_k(\alpha), \quad n = 1, 2, 3, \ldots.
\]

For more information, see Frank’s survey [10, Theorem 4.1], and the improvements by Yildirim–Yolcu and Yolcu [20, Theorem 1.4], who strengthened the inequality with a lower order term.

Combining this lower bound by Laptev with the upper bound on individual eigenvalues from Theorem 1 yields a two-sided bound, which in the special case \( \alpha = 1 \) has a particularly simple form:

\[
\frac{\pi}{2L} n^2 \leq \sum_{k=1}^{n} \lambda_k(1) < \frac{\pi}{2L} n(n+1), \quad n = 1, 2, 3, \ldots.
\]

**Fractional Pólya conjecture fails for the unit disk, for the first eigenvalue.** Take \( n = 1 \) and consider the unit disk in dimension \( d = 2 \), which has area \( \pi \). Then the corresponding term in the Weyl asymptotic (1) is \( (1 \cdot C_d/\pi)^{\alpha/2} = 2^\alpha \). The next theorem shows that the fractional Pólya conjecture fails already for the first eigenvalue of the disk, when \( \alpha \) is not too large.

**Theorem 2** (Disk). *For the unit disk, \( \lambda_1(\alpha) < 2^\alpha \) for all \( \alpha \in (0, 0.802) \).*

The theorem can be extended to \( \alpha \in (0, 0.984) \) provided one accepts a numerical plot as part of the proof; see part (iii) below.

**Proof.** We rely on several bounds from the literature for the unit ball in \( \mathbb{R}^d \).

(i) The first bound is the simplest, but handles only \( \alpha \in (0, 0.699) \). By work of Bañuelos and Kulczycki [2, Corollary 2.2],

\[
\lambda_1(\alpha) \leq \frac{2^{\alpha+1} \Gamma(\frac{d}{2} + 1) \Gamma(\frac{d}{2} + \alpha + 1)}{(d + \alpha) \Gamma(\alpha + 1) \Gamma(\frac{d}{2})} = \frac{2^{\alpha+1}(\alpha + 1)\Gamma(\frac{d}{2} + 1)^2}{\alpha + 2}.
\]
after substituting the dimension $d = 2$. Plotting this bound shows that $\lambda_1(\alpha) < 2^\alpha$ when $\alpha \in (0, 0.699)$. We will not justify this claim rigorously, since part (ii) below gives an analytic proof for an even larger interval of $\alpha$-values.

(ii) A somewhat stronger estimate by Dyda, Kuznetsov and Kwaśnicki, namely [9, formula (13)], says for $d = 2$ that

$$\lambda_1(\alpha) \leq \frac{2^{\alpha-1}(\alpha + 2)(7\alpha + 24)\Gamma(\frac{\alpha}{2} + 1)^2}{(\alpha + 4)(\alpha + 6)}.$$  

By plotting, we verify the desired inequality $\lambda_1(\alpha) < 2^\alpha$ on the larger interval $\alpha \in (0, 0.802)$. This inequality can be checked rigorously, as follows: to show the right side of (2) is less than $2^\alpha$ is equivalent to showing

$$2 \log \Gamma(\frac{\alpha}{2} + 2) - \log \frac{\alpha + 2}{\alpha + 24} - \log(\alpha + 4) - \log(\alpha + 6) + \log 2 < 0.$$  

Each term on the left is convex as a function of $\alpha$, and so it suffices to check that the left side equals 0 at $\alpha = 0$ and is negative at $\alpha = 0.802$, which is easily done.

(iii) To get the desired inequality for the interval $\alpha \in (0, 0.984)$, we apply an even stronger (and more complicated) bound of Dyda [8, Section 5]. It says for the unit ball that

$$\lambda_1(\alpha) \leq \frac{P - \sqrt{P^2 - QR}}{2R}$$  

where the quantities are defined (when $d = 2$) by

$$P = \frac{2^{\alpha-1} \pi^2(\alpha + 4)(\alpha^2 + 3\alpha + 6)\Gamma(\frac{\alpha}{2} + 1)^2}{(\alpha + 1)(\alpha + 3)(\alpha + 6)},$$  

$$Q = \frac{4^{\alpha+1} \pi^2(\alpha + 2)\Gamma(\frac{\alpha}{2} + 1)^4}{\alpha + 6},$$  

$$R = \frac{\pi^2(\alpha + 4)^2}{4(\alpha + 1)(\alpha + 2)^2(\alpha + 3)};$$  

the above formulation is taken from [9, formula (12)]. Substituting these values of $P, Q, R$ and then plotting as a function of $\alpha$ shows $\lambda_1(\alpha) < 2^\alpha$ when $\alpha \in (0, 0.984)$. We do not attempt an analytic proof of this last inequality.

The square. To disprove Pólya’s conjecture on the square $(-1, 1) \times (-1, 1)$ of side-length 2, it would suffice to show $\lambda_1(\alpha) < (C_2^2/4)^{\alpha/2} = \pi^{\alpha/2}$. Domain monotonicity of eigenvalues means it would be enough in fact to show the first eigenvalue of the unit disk (which lies inside the square) is less than $\pi^{\alpha/2}$. This last inequality can be verified when $\alpha < 0.417$ by using the estimate in (iii) above. The simpler bound in (ii) suffices for the square when $\alpha < 0.298$, while the bound in (i) is not good enough for any $\alpha$, for this purpose.

Hence in 2-dimensions, the fractional Pólya conjecture can fail even for a tiling domain, namely, the square.

Concluding discussion. We have shown that the analogue of Pólya’s conjecture fails for the fractional Laplacian. The conjecture is known to fail for another variant of the Laplacian too, the so-called magnetic Laplacian, by work of Frank, Loss and Weidl [12].
Thus any technique that might prove the original Pólya conjecture for the Dirichlet Laplacian must be rather special, because it must break down for both the magnetic Laplacian and the fractional Laplacian.

**Appendix. Spectral comparison**

**Theorem 1** depended on the fact that the eigenvalues of the fractional Laplacian are bounded above by powers of the classical Laplacian eigenvalues. We give a direct proof of this fact in the next Proposition, and then discuss earlier work. The proof relies on Jensen’s inequality and the Poincaré minimax characterization of eigenvalues, and it is new to the best of our knowledge.

**Proposition 3.** The function $\alpha \mapsto \lambda_n(\alpha)^{1/\alpha}$ is strictly increasing when $\alpha > 0$, for each $n \geq 1$. Hence $\lambda_n(\alpha) < \lambda_n(2)^{\alpha/2}$ when $0 < \alpha < 2$.

**Proof.** Suppose $0 < \alpha < \beta < \infty$. Take $u \in H^{\beta/2}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |u|^2 \, dx = 1$, so that $\int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \, d\xi = 1$ by Plancherel’s identity. Then

$$
\left( \int_{\mathbb{R}^d} |\xi|^\alpha |\hat{u}(\xi)|^2 \, d\xi \right)^{\beta/\alpha} < \int_{\mathbb{R}^d} |\xi|^\beta |\hat{u}(\xi)|^2 \, d\xi
$$

by Jensen’s inequality applied with the strictly convex function $t \mapsto t^{\beta/\alpha}$ and with measure $d\mu(\xi) = |\hat{u}(\xi)|^2 \, d\xi$, and where the inequality is shown to be strict by the following argument. If equality held then the equality conditions for Jensen would imply that $|\xi|^\alpha$ is constant $\mu$-a.e., meaning $\mu(\{\xi \neq c\}) = 0$ for some constant $c$. Also the sphere $|\xi| = c$ has $\mu$-measure zero, and so we conclude $\mu \equiv 0$ and hence $\hat{u} = 0$ a.e. with respect to Lebesgue measure. That contradiction shows that Jensen’s inequality must hold strictly.

Next, recall that the eigenvalues are characterized variationally [1, p. 97], with

$$
\lambda_n(\alpha) = \min_{S \in S_n(\alpha)} \max \left\{ \int_{\mathbb{R}^d} |\xi|^\alpha |\hat{u}|^2 \, d\xi : u \in S \text{ with } \int_{\mathbb{R}^d} |u|^2 \, dx = 1 \right\}
$$

for $\alpha > 0$, where $S_n(\alpha)$ is the collection of all $n$-dimensional subspaces of $H_0^{\alpha/2}(\Omega)$. The minimum is attained when $S$ is spanned by the first $n$ eigenfunctions of $(-\Delta)^{\alpha/2}$.

Choose $S \in S_n(\beta)$ to be the subspace of $H_0^{\beta/2}(\Omega)$ spanned by the first $n$ eigenfunctions of $(-\Delta)^{\beta/2}$. Then $S \in S_n(\alpha)$, just because $H_0^{\beta/2}(\Omega) \subset H_0^{\alpha/2}(\Omega)$, and so the variational characterization and strict Jensen inequality imply that

$$
\lambda_n(\alpha) \leq \max \left\{ \int_{\mathbb{R}^d} |\xi|^\alpha |\hat{u}|^2 \, d\xi : u \in S \text{ with } \int_{\mathbb{R}^d} |u|^2 \, dx = 1 \right\}
$$

$$
< \max \left\{ \left( \int_{\mathbb{R}^d} |\xi|^\beta |\hat{u}|^2 \, d\xi \right)^{\alpha/\beta} : u \in S \text{ with } \int_{\mathbb{R}^d} |u|^2 \, dx = 1 \right\}
$$

$$
= \lambda_n(\beta)^{\alpha/\beta},
$$

which completes the proof. \qed
Earlier work proved the non-strict inequality \( \lambda_n(\alpha) \leq \lambda_n(2)^{\alpha/2} \) for \( \alpha = 1 \) [2, Theorem 3.14], and for rational \( \alpha \in (0, 2) \) [5, Theorem 1.3], and for general \( \alpha \in (0, 2) \) [3, Theorem 3.4]. Further, \( \alpha \mapsto \lambda_n(\alpha)^{1/\alpha} \) is continuous [6, Theorem 1.3], [4, Example 5.1], and is increasing by work of Chen and Song [3, Example 5.4], while Proposition 3 shows it is strictly increasing.

A stronger result than Proposition 3 is true when \( 0 < \alpha < \beta = 2 \): the fractional Laplacian is bounded above as an operator by the \( \alpha/2 \)-th power of the Dirichlet Laplacian. References for the non-strict version of this operator inequality are in Frank’s survey paper [10, Theorem 2.3]. For the strict operator inequality, see the paper of Musina and Nazarov [17, Corollary 4].

Finally, Proposition 3 and its proof by Jensen’s inequality extend to eigenvalues of other families of operators, provided the corresponding Fourier multipliers are related by (strictly) convex transformations, just as \( |\xi|^\alpha \) is related to \( |\xi|^\beta \) by the transformation \( t \mapsto t^{\beta/\alpha} \). Additionally, the result extends from eigenvalues to the more general “inf–max” values defined by a variational formula in the case of non-discrete spectrum, although the inequality is no longer strict in that case.

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