Decays of the MSSM Higgs Bosons with Explicit CP Violation

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Abstract

We study Higgs boson decays in the minimal supersymmetric standard model where the tree–level CP invariance of the Higgs potential is explicitly broken by loop effects of soft CP–violating Yukawa interactions related to scalar quarks of the third generation. The scalar–pseudoscalar mixing among two neutral CP–even Higgs bosons and one CP–odd Higgs boson due to explicit CP violation modifies their tree–level couplings to fermions, to the $W^\pm$ and $Z$ bosons and to Higgs bosons themselves significantly. We analyze the phenomenological impact of explicit CP violation on the branching ratios of the neutral Higgs boson decays in detail and discuss how to directly confirm the existence of explicit CP violation through $\tau^+\tau^-$ and $t\bar{t}$ spin correlations in the decays of the neutral Higgs bosons into a tau–lepton pair and a top–quark pair.

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I. INTRODUCTION

Revealing the physical mechanism responsible for the breaking of the electroweak symmetry is one of the most crucial issues in particle physics. If the fundamental particles such as leptons, quarks and gauge bosons remain weakly interacting up to very high energies, the sector in which the electroweak symmetry is broken must contain one or more fundamental scalar Higgs bosons with masses of the order of the symmetry-breaking scale. However, if the Standard Model (SM) is embedded in a Grand Unified Theory (GUT) at high energies, the natural scale of the electroweak symmetry breaking would be expected close to the unification scale. Supersymmetry \cite{1} provides an elegant solution to this hierarchy problem and furthermore leads to an excellent agreement between the value of the electroweak mixing angle $\sin^2 \theta_W$ predicted by the unification of the gauge couplings and the experimentally measured value.

Of course, supersymmetry must be (softly) broken to be phenomenologically viable. In general, this breakdown introduces a large number of unknown parameters, many of which can be complex \cite{2}. CP–violating phases associated with sfermions of the first and, to a lesser extent, second generations are severely constrained by bounds on the electric dipole moments of the electron, neutron and muon. However, there have been several suggestions \cite{3–5} to evade these constraints without suppressing the CP–violating phases. One option is to make the first two generations of scalar fermions rather heavy so that one–loop EDM constraints are automatically evaded. As a matter of fact one can consider so–called effective SUSY models \cite{4} where de-couplings of the first and second generation sfermions are invoked to solve the SUSY FCNC and CP problems without spoiling the naturalness condition. Another possibility is to arrange for partial cancellations among various contributions to the electron and neutron EDM’s \cite{3}.

Following the suggestions that the phases do not have to be suppressed, many important works on the effects due to the CP phases in the minimal supersymmetric standard model (MSSM) have been already reported; the effects are very significant in extracting the parameters in the SUSY Lagrangian from experimental data \cite{3}, estimating dark matter densities and scattering cross sections and Higgs boson mass limits \cite{4,5}, CP violation in the $B$ and $K$ systems \cite{10}, and so on. In particular, it has been found \cite{11} that the Higgs–sector CP violation induced via loop corrections of soft CP–violating Yukawa interactions may drastically modify the couplings of the light neutral Higgs boson to the gauge bosons. As a result, the current experimental lower bound on the lightest Higgs boson mass may be dramatically relaxed up to a 60–GeV level in the presence of large CP violation in the Higgs sector of the MSSM.

Since the experimental observation of scalar Higgs particles and the detailed confirmation of their fundamental properties are crucial for our present understanding of the mechanism of the electroweak symmetry breaking, a very precise prediction of the production cross sections and of the branching ratios for the main decay channels is mandatory. So, several recent works \cite{9} have made the detailed predictions of the Higgs mass spectrum and their production cross sections in the MSSM with explicit CP violation. Along with the mass spectrum and cross sections, it is also very important to estimate the branching ratios of the main Higgs boson decays precisely. This indispensable requirement leads to our systematic study of the Higgs boson decays in the MSSM \cite{11} with explicit CP violation, where both
Higgs–boson masses and their couplings to fermions, gauge bosons are significantly affected by loop corrections of soft CP–violating Yukawa interactions to the Higgs sector. We provide a detailed quantitative estimate for the branching fractions of all possible two–body decay modes of three neutral Higgs bosons and investigate the possibility of measuring CP violation through the fermion spin correlations in the neutral Higgs decays into a tau–lepton pair and/or a top–quark pair.

The organization of the paper is as follows. In Section II we give a brief review for the explicit CP violation in the Higgs sector due to the significant contributions of radiative corrections from trilinear Yukawa couplings of the Higgs fields to scalar top and scalar bottom quarks. This review is mainly based on the work by Pilaftsis and Wagner [9]. The parts involving other supersymmetric particles such as charginos and neutralinos can contribute to the effective Higgs potential. But, the chargino and neutralino contributions are degraded in the present analysis with their masses taken to be of the order of a large SUSY breaking scale $M_{SUSY}$. Then, we introduce the general mass matrix dictating the mixing among three neutral Higgs states and the resulting Lagrangian describing the couplings of the Higgs bosons to fermions, gauge bosons, and Higgs bosons themselves. In Section III we discuss the effects of the CP phases on (almost) all the two–body Higgs boson decays in detail, which will give an important phenomenological impact on Higgs–boson searches. Section IV is devoted to investigating the possibility of detecting CP–violating effects in the neutral Higgs boson decays into tau–lepton pair or top–quark pair in which the subsequent decays of tau leptons and top quarks enable one to extract the information on the spin of the parent fermions; tau leptons and top quarks. Finally, Section V summarizes our findings and concludes.

II. CP VIOLATION IN THE MSSM HIGGS SECTOR

The MSSM introduces several new parameters in the theory that are absent from the SM and could, in principle, possess many CP–violating phases. Specifically, the new CP phases may come from the following parameters: (i) the higgsino mass parameter $\mu$, which involves the bilinear mixing of the two Higgs chiral superfields in the superpotential; (ii) the soft SUSY–breaking gaugino masses $M_a$ ($a = 1, 2, 3$), where the index $a$ stands for the gauge groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$, respectively; (iii) the soft bilinear Higgs mixing masses $m_{12}^2$, which is sometimes denoted as $B\mu$ in the literature; (iv) the soft trilinear Yukawa couplings $A_f$ of the Higgs particles to scalar fermions; and (v) the flavor mixing elements of the sfermions mass matrices. If the universality condition is imposed on all gaugino masses at the unification scale $M_X$, the gaugino masses $M_a$ have a common phase, and if the diagonal boundary conditions are added to the universality condition for the sfermion mass matrices at the GUT scale, the flavor mixing elements of the sfermions mass matrices vanish and the different trilinear couplings $A_f$ are all equal, i.e. $A_f = A$.

The conformal–invariant part of the MSSM Lagrangian has two global $U(1)$ symmetries; the $U(1)_Q$ Peccei–Quinn symmetry and the $U(1)_R$ symmetry acting on the Grassmann–valued coordinates. As a consequence, not all CP–violating phases of the four complex parameters $\{\mu, m_{12}^2, M_a, A\}$ turn out to be physical, i.e. two phases may be removed by redefining the fields accordingly [2]. Employing the two global symmetries, one of the
Higgs doublets and the gaugino fields can be rephased such that \( M_a \) and \( m_{12}^2 \) become real. In this case, \( \arg(\mu) \) and \( \arg(A) \) are the only physical CP–violating phases in the low–energy MSSM supplemented by universal boundary conditions at the GUT scale. Denoting the scalar components of the Higgs doublets \( H_1 \) and \( H_2 \) by \( H_1 = -i \tau_2 \Phi_1^+ \) (\( \tau_2 \) is the usual Pauli matrix) and \( H_2 = \Phi_2 \), the most general CP–violating Higgs potential of the MSSM can be conveniently described by the effective Lagrangian

\[
\mathcal{L}_V = \mu_1^2 (\Phi_1^+ \Phi_1) + \mu_2^2 (\Phi_2^+ \Phi_2) + m_{12}^2 (\Phi_1^+ \Phi_2) + m_{12}^2 (\Phi_2^+ \Phi_1) \\
+ \lambda_1 (\Phi_1^+ \Phi_1)^2 + \lambda_2 (\Phi_2^+ \Phi_2)^2 + \lambda_3 (\Phi_1^+ \Phi_1)(\Phi_2^+ \Phi_2) + \lambda_4 (\Phi_1^+ \Phi_2)(\Phi_2^+ \Phi_1) \\
+ \lambda_5 (\Phi_1^+ \Phi_2)^2 + \lambda_5^* (\Phi_2^+ \Phi_1)^2 + \lambda_6 (\Phi_1^+ \Phi_1)(\Phi_2^+ \Phi_2) + \lambda_6^* (\Phi_1^+ \Phi_2)(\Phi_2^+ \Phi_1) \\
+ \lambda_7 (\Phi_2^+ \Phi_2)(\Phi_1^+ \Phi_1) + \lambda_7^* (\Phi_2^+ \Phi_2)(\Phi_1^+ \Phi_1)_1. \\
\]  

(1)

In the Born approximation, the quartic couplings \( \lambda_{1,2,3,4} \) are solely determined by the gauge couplings and \( \lambda_{5,6,7} \) are zero. However, beyond the Born approximation, the quartic couplings \( \lambda_{5,6,7} \) receive significant radiative corrections from trilinear Yukawa couplings of the Higgs fields to scalar–top and scalar–bottom quarks. These parameters are in general complex and so lead to CP violation in the Higgs sector through radiative corrections. The explicit form of the couplings with the radiative corrections included can be found in Refs. \[9,13\].

It is necessary to determine the ground state of the Higgs potential to obtain physical Higgs states and their self–interactions. To this end we introduce the linear decompositions of the Higgs fields

\[
\Phi_1 = \left( \eta_1 e^{-i \phi_1} + \frac{\phi_1^+}{\sqrt{2}}(v_1 + \phi_1 + ia_1) \right), \quad \Phi_2 = e^{i \xi} \left( \phi_2^+ + \frac{\phi_2}{\sqrt{2}}(v_2 + \phi_2 + ia_2) \right),
\]

(2)

with \( v_1 \) and \( v_2 \) the moduli of the vacuum expectation values (VEVs) of the Higgs doublets and \( \xi \) their relative induced CP–violating phase. These VEVs and the relative phase can be determined by the minimization conditions on \( \mathcal{L}_V \), which can be efficiently performed by the so–called tadpole renormalization techniques \[1\]. It is always guaranteed that one combination of the CP–odd Higgs fields \( a_1 \) and \( a_2 \) \( (G^0 = \cos \beta a_1 - \sin \beta a_2) \) defines a flat direction in the Higgs potential and so it is absorbed as the longitudinal component of the \( Z \) boson. [Here, \( \sin \beta = v_2/\sqrt{v_1^2 + v_2^2} \) and \( \cos \beta = v_1/\sqrt{v_1^2 + v_2^2} \).] As a result, there exist one charged Higgs state and three neutral Higgs states that are mixed in the presence of CP violation in the Higgs sector. Denoting the remaining CP–odd state \( a = \sin \beta a_1 + \cos \beta a_2 \), the \( 3 \times 3 \) neutral Higgs–boson mass matrix describing the mixing between CP–even and CP–odd fields can be decomposed into four parts in the weak basis \( (a, \phi_1, \phi_2) \) :

\[
\mathcal{M}_0^2 = \begin{pmatrix} \mathcal{M}_P^2 & \mathcal{M}_{PS}^2 \\ \mathcal{M}_{SP}^2 & \mathcal{M}_S^2 \end{pmatrix},
\]

(3)

where \( \mathcal{M}_P^2 \) and \( \mathcal{M}_S^2 \) describe the CP–preserving transitions \( a \to a \) and \( (\phi_1, \phi_2) \to (\phi_1, \phi_2) \), respectively, and \( \mathcal{M}_{PS}^2 = (\mathcal{M}_{SP}^2)^T \) contains the CP–violating mixings \( a \leftrightarrow (\phi_1, \phi_2) \). The analytic form of the sub–matrices is given by
where \( \beta = \cos \beta \) and \( s_\beta = \sin \beta \) and the mass squared \( m_a^2 \) is given by

\[
m_a^2 = \frac{1}{s_\beta c_\beta} \left\{ \mathcal{R}(m_1^2 e^{i \xi}) + v^2 \left[ 2\mathcal{R}(\lambda_{5} e^{2i \xi}) s_\beta c_\beta + \frac{1}{2} \mathcal{R}(\lambda_{6} e^{i \xi}) c_\beta^2 + \frac{1}{2} \mathcal{R}(\lambda_{7} e^{i \xi}) s_\beta^2 \right] \right\}.
\]

(5)

Correspondingly, the charged Higgs-boson mass is given by

\[
m_{H^\pm}^2 = m_a^2 + \frac{1}{2} \lambda_4 v^2 - \mathcal{R}(\lambda_{5} e^{2i \xi}) v^2.
\]

(6)

Taking this very last relation between \( m_{H^\pm} \) and \( m_a \) into account, we can express the neutral Higgs–boson masses as functions of \( m_{H^\pm} \), \( \mu \), \( A_t \), \( A_b \), a common SUSY scale \( M_{\text{SUSY}} \), \( \tan \beta \) and the physical phase \( \xi \). However, with the chargino and neutralino contributions neglected, the radiatively induced phase \( \xi \) can be absorbed into the definition of the \( \mu \) parameter. Clearly, the CP–even and CP–odd states mix unless all of the imaginary parts of the parameters \( \lambda_{5,6,7} \) vanish. Since the Higgs–boson mass matrix \( \mathcal{M}_0^2 \) describing the scalar–pseudoscalar mixing is symmetric, we can diagonalize it by means of an orthogonal rotation \( O; O^T \mathcal{M}_0^2 O = \text{diag}(m_{H_1}^2, m_{H_2}^2, m_{H_3}^2) \) with the ordering of masses \( m_{H_1} \leq m_{H_2} \leq m_{H_3} \). The neutral Higgs–boson mixing affects the couplings of the Higgs fields to fermions, gauge bosons, and Higgs fields themselves as shown in the following. On the other hand, there could exist CP–violating vertex corrections, but they have been shown to be rather small \[14\] so that those effects are not included in the present work.

Firstly, the interactions of the neutral Higgs fields with SM fermions are described by the Lagrangian

\[
\mathcal{L}_{Hff} = -\frac{g}{2m_W} \mathcal{M}_0 \tilde{f} \left[v_f^i - i \tilde{R}_f^j a_f^i \gamma_5 \right] f H_i,
\]

(7)

\[
\tilde{R}_f^j = \begin{cases} \frac{c_\beta}{s_\beta} & \text{if } f = t, b, \tau \text{ because of their relatively large Yukawa couplings. We readily see that the effect of CP–violating Higgs mixing is to induce a simultaneous coupling of } H_i (i = 1, 2, 3) \text{ to CP–even and CP–odd fermion bilinears } \tilde{f} \tilde{f} \text{ and } \tilde{f} \gamma_5 \tilde{f} \text{. This can lead to a sizable phenomenon of CP violation in the Higgs decays into polarized top-quark or tau-lepton pairs } [14], \text{ which will constitute the topic of Section } [V].
\end{cases}
\]

Secondly, the couplings of the Higgs fields to \( W \) and \( Z \) bosons may be read off by the Lagrangian

\[
\mathcal{L}_{HWW} = \frac{g}{2} \mathcal{M}_0 \tilde{w} \left[ v_w^i - i \tilde{R}_w^j a_w^i \gamma_5 \right] w H_i,
\]

(8)

\[
\tilde{R}_w^j = \begin{cases} \frac{c_\beta}{s_\beta} & \text{for } f = t, b, \tau \text{ because of their relatively large Yukawa couplings. We readily see that the effect of CP–violating Higgs mixing is to induce a simultaneous coupling of } H_i (i = 1, 2, 3) \text{ to CP–even and CP–odd fermion bilinears } \tilde{f} \tilde{f} \text{ and } \tilde{f} \gamma_5 \tilde{f} \text{. This can lead to a sizable phenomenon of CP violation in the Higgs decays into polarized top-quark or tau-lepton pairs } [14], \text{ which will constitute the topic of Section } [V].
\end{cases}
\]
\[ \mathcal{L}_{HVV} = g m_W (c_\beta O_{2,4-i} + s_\beta O_{3,4-i}) H_i \left[ W_\mu^+ W^{-\mu} + \frac{1}{2c_W^2} Z_\mu Z^{\mu} \right], \] (9)

\[ \mathcal{L}_{HH^+H^0} = \frac{g}{2} (c_\beta O_{3,4-i} - s_\beta O_{2,4-i} + iO_{1,4-i}) W^{+\mu} \left( H_i \partial_\mu H^- \right) + \text{h.c.}, \] (10)

\[ \mathcal{L}_{HHZ} = \frac{g}{4c_W} \left[ O_{1,4-i}(c_\beta O_{3,4-j} - s_\beta O_{1,4-j}) - O_{1,4-j}(c_\beta O_{3,4-i} - s_\beta O_{1,4-i}) \right] Z^\mu \left( H_i \partial_\mu H_j \right). \] (11)

Note that the Z boson can only couple to two different Higgs particles. The reason is that Bose symmetry forbids any antisymmetric derivative coupling of a vector particle to two identical real scalar fields. On the other hand, the orthogonality of the mixing matrix \( O \) and the assumption \( \det O = 1 \), which we can take without loss of any generality, leads to the following important relation between the couplings of the neutral Higgs bosons to the gauge bosons:

\[ g_k = \epsilon_{ijk} g_{ij}, \] (12)

with the definitions: \( g_i = c_\beta O_{2i} + s_\beta O_{3i} \) and \( g_{ij} = O_{1i}(c_\beta O_{3j} - s_\beta O_{2j}) - O_{1j}(c_\beta O_{3i} - s_\beta O_{2i}) \).

One intermediate result from the relation and the unitarity constraint is that the knowledge of two \( g_i \) is sufficient to determine the whole set of couplings of the neutral Higgs to the gauge bosons [17]. As will be seen in Sec. III C, the above relation leads to a close correlation between \( \mathcal{B}(H_2 \rightarrow H_1 Z) \) and \( \mathcal{B}(H_3 \rightarrow VV) \) and it also leads to a similar close relation between \( \mathcal{B}(H_3 \rightarrow VV) \) and \( \mathcal{B}(H_2 \rightarrow H_1 Z) \).

Thirdly, the trilinear Higgs self–couplings also are affected by the radiative corrections. Their relevant interactions can be read off by the Lagrangian

\[ \mathcal{L}_{HHH} = v \left[ A_{ijk} H_i H_j H_k + B_i H_i H^+ H^- \right] \] (13)

where the totally symmetric \( A_{ijk} \) and the coefficients \( B_i \) are given by

\[ A_{ijk} = \sum_{\alpha\beta\gamma=1,2,3} O_{\alpha,4-i} O_{\beta,4-j} O_{\gamma,4-k} a_{\alpha\beta\gamma}, \quad B_i = \sum_{\alpha=1,2,3} O_{\alpha,4-i} b_{\alpha}, \] (14)

where the totally-symmetric coefficients \( a_{\alpha\beta\gamma} \) and the coefficients \( b_i \) can be directly obtained from taking every possible combination of third–order derivatives of the general Higgs potential \([\|] \) with respect to the fields \( \{a, \phi_1, \phi_2, H^\pm\} \). The coefficients, of which the form is presented in the Appendix, are functions of the quartic couplings \( \lambda_i \) (\( i = 1 \) to 7). As can be checked in the Lagrangian, the radiative corrections generate various new interaction vertices among the neutral Higgs fields and the charged Higgs fields, which are absent in the CP–invariant MSSM.

Finally, we emphasize that the size of CP violation due to radiative corrections to the Higgs potential is characterized by a dimensionless parameter \( \eta_{CP} \)

\[ \eta_{CP} = \frac{m_f^4}{v^4} \left( \frac{|\mu||A_f|}{32\pi^2 M_{SUSY}^2} \right) \sin \Phi, \] (15)

where \( \Phi = \arg(A_f \mu) + \xi \), i.e. the sum of three CP–violating phases. So, for \( |\mu| \) and \( |A_f| \) values larger than the SUSY–breaking scale \( M_{SUSY} \), the CP–violating effects can be significant.
III. HIGGS BOSON DECAYS

A. Higgs–boson masses

Higgs-boson masses are not determined a priori within the theory and their decay patterns depend strongly on the masses as well as several SUSY parameters. So, in order to determine the possible decay modes and branching ratios, it is necessary to investigate the change of the mass spectrum with respect to the SUSY parameters. As explained in Section II, the mass spectrum is determined by nine real quantities $\tan \beta$, $m_{H^\pm}$, $|\mu|$, $|A_t|$, $|A_b|$, $\Phi_\mu$, $\Phi_{A_t}$, $\Phi_{A_b}$, and $M_{\text{SUSY}}$. For simplicity, we assume in our numerical analysis for the following dimensionful parameters

$$|A_t| = |A_b| = 1 \text{ TeV}, \quad |\mu| = 2 \text{ TeV}, \quad M_{\text{SUSY}} = 0.5 \text{ TeV},$$

while we treat $m_{H^\pm}$ as a dimensionful free parameter. Noticing that the effects of the CP phases appear with the unique combination $\Phi = \arg(A_f \mu) + \xi$ and using the freedom of field redefinitions guaranteed only if chargino and neutralino contributions are neglected, we take for the phases

$$\Phi_\mu + \xi = 0, \quad \Phi_{A_t} = \Phi_{A_b} \equiv \Phi.$$  \hfill (17)

Then, we are left with two dimensionless free parameters $\tan \beta$ and $\Phi$. So, eventually, all the mass spectrums and couplings are determined completely by three parameters $\{\tan \beta, \Phi, m_{H^\pm}\}$.

Figure 1 shows the masses of three neutral Higgs bosons and one charged Higgs boson with respect to the charged Higgs mass $m_{H^\pm}$ for different values of the CP phase $\Phi$ with the fixed values for other SUSY parameters in Eq. (16). The value of $\tan \beta$ is taken to be 3 and an artificial experimental condition $m_{H_1} \geq 70$ GeV is imposed. The lower solid line is for the lightest Higgs boson $H_1$, the upper solid line for the heaviest Higgs boson $H_3$, and the dashed line for the intermediate Higgs boson $H_2$. Comparing the mass spectrums of the whole frames, we find the following interesting features for the Higgs boson masses:

- The mass of the lightest Higgs boson $H_1$ is always less than 130 GeV irrespective of the value of $\Phi$.
- The intermediate Higgs–boson mass is very sensitive to the CP–violating phase $\Phi$ and becomes very close to the lightest Higgs boson mass around $\Phi \sim 70^\circ$.
- The heaviest Higgs–boson is almost degenerate with the charged Higgs boson except the region with a small charged Higgs boson mass.
- Except the lightest Higgs boson mass, the other Higgs boson masses increase with the charged Higgs mass, which is treated as a free parameter.

One can also notice that the the minimum values of the allowed heavier Higgs masses under the constraint $m_{H_1} \geq 70$ GeV depend strongly on the CP phase. These features clearly suggest that while the decay pattern of the lightest neutral Higgs boson might be rather insensitive to the CP phase and the charged Higgs mass, the decay patterns of the other neutral Higgs bosons and the charged Higgs boson itself are very sensitive to them.
B. Higgs decay branching ratios

In most cases, the most important decay channels of the Higgs bosons are two–body decays to the heaviest particles because the Higgs couplings to the SM particles are proportional to their masses. Sometimes, because of the coupling enhancement below–threshold three–body decays of the Higgs bosons may also be important [19]. Nevertheless, in this work we concentrate on only two–body decay channels. On the other hand, if supersymmetric particles are light enough, Higgs bosons can decay into supersymmetric particles. However, since we have assumed a common SUSY scale of the order of 1 TeV, decays to sfermions, neutralinos, and charginos are not expected to play an important role in the Higgs mass range of a few hundred GeV which we are analyzing.

Let us now present the explicit form of each Higgs boson decay width. First of all, the decay width for the Higgs boson decays into a fermion–pair is given by

\[ \Gamma(H_i \rightarrow f \bar{f}) = \frac{N_c G_F}{4 \sqrt{2} \pi} m_f^2 m_{H_i} \beta_f \left( \frac{\beta_f^\prime}{R_{\beta_f^\prime}} \right)^2 \left[ (\beta_f^\prime v_f^a)^2 + (\bar{R}_{\beta_f^\prime}^\prime a_f^a)^2 \right], \]

where \( N_c = 3 \) and 1 for quarks and leptons, and \( \beta_f \) is the fermion velocity \( \sqrt{1 - \frac{4 m_f^2}{m_{H_i}^2}} \) in the rest frame of the Higgs boson \( H_i \). Among these fermionic decay channels, the bottom–quark and tau–lepton modes give most dominant contributions to the decay width of the lightest Higgs boson due to their relatively large Yukawa couplings. Numerically, the branching ratio of the bottom–quark channel is almost 90% and the tau–lepton mode is 10% in the almost whole range of the allowed lightest Higgs boson mass. Moreover, if the masses of the heavier Higgs bosons are smaller than the threshold of two \( W \)’s, they will be the main decay channels for the heavy states as well. As stated in the expression, the fermionic decay width is strongly dependent on the fermion mass so that the exact determination of the mass is required for the precise estimate of the branching ratio. Keeping this point in mind, we simply take \( m_b = 4.25 \) GeV and \( m_c = 1.25 \) GeV for our numerical analysis.

Secondly, the decay widths of the Higgs bosons into a gauge boson pair can be obtained by the Lagrangian for the Higgs couplings to the gauge bosons. The analytic form of these decay widths can be summarized as follows:

\[ \Gamma(H_i \rightarrow VV) = \frac{G_F m_{H_i}^3}{16 \sqrt{2} \pi} \delta_V \beta_V |c_\beta O_{2,4} + s_\beta O_{3,4} - i| \left( 1 - 4 \kappa_V^2 + 12 \kappa_V^4 \right), \]

where \( V = W^\pm \) or \( Z \), \( \delta_V \) is 2 for the \( W \) boson and 1 for the \( Z \) boson, \( \beta_V \) is the velocity of the gauge boson in the rest frame of the Higgs boson, and \( \kappa_V^2 = m_V^2/m_{H_i}^2 \). Note that two decay widths are of the same form and for \( m_{H_i} \gg m_Z \) the ratio of \( W^+W^- \) to \( ZZ \) rates approaches 2. Clearly, since the decay widths are proportional to the third power of the Higgs boson mass itself, these decay channels are expected to contribute dominantly to the decay widths of the heavy Higgs bosons. On the contrary, the partial width of the decay \( H_1 \rightarrow VV \) is strongly suppressed or vanishing due to kinematics, thus not playing a dominant role even if the decay channel with one off–shell gauge boson is considered. In the Born approximation, the intermediate Higgs boson is the pseudoscalar boson so that the decay \( H_2 \rightarrow ZZ \) is not allowed. However, beyond the Born approximation, this mass eigenstate will have a large portion of the light or heavy scalar Higgs states and as a result
there will be a large enhancement for the branching ratio for non–trivial values of the CP phase $\Phi$ in this specific decay channel unique to the CP–noninvariant theory.

Thirdly, the heavy Higgs–boson decays into a light Higgs boson and a gauge boson can be classified into the charged decays $H_i \rightarrow H^\pm W^\mp$ and the neutral decays $H_i \rightarrow H_j Z$. The width for the decay $H_i \rightarrow H^\pm W^\mp$ is given by

$$\Gamma(H_i \rightarrow H^\pm W^\mp) = \frac{G_F m_{H_i}^3}{8\sqrt{2}\pi} \left| c_\beta O_{3,4-i} - s_\beta O_{2,4-i} + i O_{1,4-i} \right|^2 \lambda^{3/2} \left( 1, \frac{m_W^2}{m_{H_i}^2}, \frac{m_{H^\pm}^2}{m_{H_i}^2} \right), \quad (20)$$

where $\lambda$ is the two–body phase space function; $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. Similarly, the width for the decay $H_i \rightarrow H_j Z$ is given by

$$\Gamma(H_i \rightarrow H_j Z) = \frac{G_F m_{H_i}^3}{8\sqrt{2}\pi} \left| O_{1,4-i}(c_\beta O_{3,4-j} - s_\beta O_{2,4-j}) - (i \leftrightarrow j) \right|^2 \lambda^{3/2} \left( 1, \frac{m_{H_j}^2}{m_{H_i}^2}, \frac{m_Z^2}{m_{H_i}^2} \right). \quad (21)$$

The branching ratios for the two–body decays $H_2 \rightarrow H_1 Z$ can be sizable in specific regions of the SUSY parameter space, especially, for small values of $\tan \beta$ and below the $tt$ thresholds.

Fourthly, the heavy Higgs bosons can decay into a pair of lighter scalars, $H_i \rightarrow H_j H_k$ and into a charged Higgs–boson pair $H_i \rightarrow H^+ H^-$, if they are kinematically allowed. The width of the latter decay channel is given by

$$\Gamma(H_i \rightarrow H^+ H^-) = \frac{v^2 |B_i|^2}{16\pi m_{H_i}} \beta_{H^\pm}, \quad (22)$$

with $\beta_{H^\pm}$ the velocity of the charged Higgs boson in the rest frame of the decaying neutral Higgs boson $H_i$ and $v^2 = v_1^2 + v_2^2$. On the other hand, the width for the former decay channel $H_i \rightarrow H_j H_k$ is given by

$$\Gamma(H_i \rightarrow H_j H_k) = \frac{9\eta_{jk}}{8\pi m_{H_i}} v^2 |A_{ijk}|^2 \beta_{ijk}, \quad (23)$$

where $\eta_{jk}$ is 2 for $j \neq k$ and 1 for $j = k$, and $\beta_{ijk} = \lambda^{1/2}(1, m_{H_j}^2/m_{H_i}^2, m_{H_k}^2/m_{H_i}^2)$.

Finally, the decays of the Higgs bosons into two gluons [20], which occur through quark one–loop diagrams may be sizable. As a matter of fact, the two–gluon partial width of the Higgs bosons is important because it is the main production mechanism for the Higgs boson production at the $pp$ collider LHC. Moreover, since gluons couple to Higgs bosons via heavy particle loops, the two–gluon widths are sensitive to heavy particle masses, standard and also supersymmetric, well above the Higgs masses themselves. It is a little involved but straightforward to calculate the width for the Higgs boson decay into two gluons, which can be expressed in the following compact form

$$\Gamma(H_i \rightarrow gg) = \frac{G_F \alpha_s^2 m_{H_i}^3}{16\sqrt{2}\pi^3} \sum_f \left[ \left( \frac{v_f}{R_f^3} \right)^2 |F(\tau)|^2 + \left( \frac{R_f^3}{R_f^3} \right)^2 |G(\tau)|^2 \right], \quad (24)$$

where $\tau = m_{H_i}^2/4m_f^2$, $f$ runs for all colored fermions, and the functions $F(\tau)$ and $G(\tau)$ are defined by
\begin{equation}
F(\tau) = \tau^{-1} \left[ 1 + (1 - \tau^{-1}) f(\tau) \right], \quad G(\tau) = \tau^{-1} f(\tau),
\end{equation}

with
\begin{equation}
f(\tau) = \begin{cases} 
\arcsin^2 \sqrt{\tau} & \text{for } \tau \leq 1, \\
-\frac{1}{4} \left[ \log \frac{\sqrt{\tau} + \sqrt{\tau-1}}{\sqrt{\tau} - \sqrt{\tau-1}} - i\pi \right]^2 & \text{for } \tau > 1.
\end{cases}
\end{equation}

The QCD radiative corrections \cite{21} to the two–gluon channel may be sizable and are built up by the exchange of virtual gluons, gluon radiation from the internal quark loop and the splitting of a gluon into two unresolved gluons or quark–antiquark pair. Although they are large enough to nearly double the partial width, we neglect the corrections in our numerical analyses because after all the two–gluon channel remains as a sub–dominant decay channel for all the neutral Higgs bosons. We note in passing that the two–photon channel and the other similar one–loop decay channels have much smaller branching ratios so that they are not included in our present discussion.

C. Numerical results on the branching ratios

In this section, we present a detailed numerical analysis of the total decay widths of each neutral Higgs boson, the branching ratios of each decay mode of the Higgs boson with respect to each Higgs–boson mass for various values of the CP phase \( \Phi \).

Figure 2 shows the total decay widths for the three neutral Higgs bosons with respect to each Higgs–boson mass; the left (right) solid line is for the lightest (heaviest) Higgs boson and the dashed line for the intermediate Higgs boson. The decay width \( \Gamma(H_1) \) is almost independent of the CP phase, but the other two total widths \( \Gamma(H_{2,3}) \) are strongly dependent on the CP phase at the lower ends of their allowed masses. The widths are rather small in size so that they may not be very hard to be determined at \( e^+e^- \) colliders, but they may be detectable at a \( \mu^+\mu^- \) collider with a very precise energy calibration.

Figure 3 shows the partial branching ratios of the lightest Higgs boson decays with respect to the mass \( m_{H_1} \) for several values of the CP phase \( \Phi \). Since the lightest Higgs boson mass is very strongly restricted to be less than 130 GeV, the main decay channels are into \( b\bar{b}, \tau^+\tau^- \), \( gg \), and \( c\bar{c} \). What can be immediately noted in Figure 3 is that the pattern of the branching ratios is (almost) independent of the CP phase. We have confirmed quantitatively that this feature is not much changed even if the possible off–shell decay channel \( H_1 \rightarrow W^+W^- \) is included. Numerically, we find that the CP phase \( \Phi \) tends to reduce the contribution from the off–shell decay mode.

On the contrary, as shown in Figure 4 the partial branching ratios for the \( H_2 \) decay channels with respect to the mass \( m_{H_2} \) are very sensitive to the CP phase \( \Phi \). The upper solid line in each figure frame denotes the sum of the branching ratios of the fermionic decay modes. The dot–dashed line is for the decay channel \( H_2 \rightarrow H_1 Z \), two dotted lines for the channels \( H_2 \rightarrow W^+W^- \) (upper line) and \( ZZ \) (lower line), respectively, and the dashed line for the decay channel \( H_2 \rightarrow H_1H_1 \). Finally, the lower solid line is for the two–gluon channel. Concerning those branching ratios, there are several interesting points worthwhile to be mentioned.
• Through the whole range of the allowed $H_2$ mass, the fermionic decay channel remains as one of the dominant channels. However, the decay channel $H_2 \rightarrow H_1 Z$ also becomes a dominant channel if the channel is open.

• The decay channel $H_2 \rightarrow t\bar{t}$ overwhelms all other decay channels as soon as it is allowed.

• If the CP phase $\Phi$ vanishes, the channel $H_2 \rightarrow H_1 H_1$ is prohibited, reflecting that the CP–odd state cannot decay into two identical CP–even states when the system is CP–invariant. However, the contribution of this decay mode becomes sizable or suppressed as the CP phase $\Phi$ varies from zero.

• The decay channels $H_2 \rightarrow W^+W^-$ and $H_2 \rightarrow ZZ$ are prohibited for the vanishing CP phase. But, they along with the channel $H_2 \rightarrow H_1 H_1$ can be also sizable and become the most dominant channel for some values of the CP phase $\Phi$.

These interesting features are partly because of a significant mixing between CP–even and CP–odd states. As shown clearly in Fig. 5 of the work by Pilaftsis and Wanger [9], the couplings of $H_2$ to $H_1 Z$ and, in particular, to $ZZ$ increases with the phase $\Phi$ up to $90^\circ$ so that they clearly enhance the decay channels $H_2 \rightarrow H_1 Z$. On the other hand, the dominance of the decay channel $H_2 \rightarrow H_1 H_1$ for a large mass range of the Higgs boson $H_2$ is mainly due to a large enhancement of the coupling strength of the $H_2 H_1 H_1$ vertex with a non–trivial phase. This distinct and unique feature may give a strong hint of the existence of a non–vanishing CP phase in the stop or sbottom sector.

When the CP phase is zero, the heaviest Higgs boson is the heavier CP–even state so that the decay channels $H_3 \rightarrow H_1 H_1$ and $H_3 \rightarrow VV$ ($V = W$ or $Z$) are among the dominant channels. This fact is clearly shown in Figure 5 for the partial branching ratios of various $H_3$ decay channels with respect to the mass $m_{H_3}$. In every frame of Figure 5 the upper solid line is for the sum of all available fermionic decay channels, two dotted lines for the decay channels into two gauge bosons, the dot–dashed line for the decay channel $H_3 \rightarrow H_1 Z$, the dashed line for the decay channel $H_3 \rightarrow H_1 H_1$ and finally the lower solid line for the two–gluon mode. These decays of the heaviest Higgs boson $H_3$ also exhibit several interesting features:

• Before the $t\bar{t}$ channel is open, the channels $H_3 \rightarrow H_1 H_1$ and $H_3 \rightarrow VV$ are dominant, while they are overwhelmed by the $t\bar{t}$ mode as soon as the latter mode is available. We note the similar behavior of $\mathcal{B}(H_3 \rightarrow VV)$ and $\mathcal{B}(H_2 \rightarrow H_1 Z)$ presented in Figure 4, a natural consequence from the relation (12) among the couplings of the neutral Higgs bosons to gauge bosons.

• The channel $H_3 \rightarrow H_2 H_1$ is always prohibited, which reflects that three Higgs mass eigenstates are originated from two CP–even states and one CP–odd state, and there do not exist any interactions between two different CP–even states and one CP–odd state.

• The channel $H_3 \rightarrow H_2 Z$ is not kinematically allowed for the whole mass range even though there exists the coupling for the decay mode.
The decay channel \( H_3 \rightarrow H_1 Z \) is prohibited for the vanishing CP phase, but the contribution of the decay channel to the total width becomes sizable as the CP phase varies from zero. Moreover, this mode overwhelms the other decay modes near the lower tail of the allowed \( H_3 \) mass for some nontrivial values of the CP phase. We note again the similar behavior of \( B(H_3 \rightarrow H_1 Z) \) and \( B(H_2 \rightarrow VV) \) presented in Figure 4.

All these interesting features can be understood by noting that the CP–even and CP–odd compositions of the heavy Higgs mass eigenstates are strongly dependent on the value of the CP phase \( \Phi \).

To recapitulate, the mass spectrum and the branching fractions of the lightest Higgs boson \( H_1 \) are not so sensitive to the CP–violating phase, but those of the heavy Higgs states are very sensitive to the phase. It means that the CP phase may cause in most cases a larger mixing of the CP–odd state with the heavy CP–even state than the light CP–even state, although the mixing with the light one also may be substantial, in particular for light charged Higgs boson masses. From our detailed analysis, it is clear that the CP phase will affect the phenomenology of the heavy Higgs–boson decays very significantly so that the heavy Higgs boson searches will depend very strongly on the CP phase as well as the other SUSY parameters.

### IV. MEASURING CP VIOLATION IN FERMIONIC HIGGS–BOSON DECAYS

In Section III, we have found that the branching ratios of the lightest Higgs boson decays are (almost) independent of the CP phase \( \Phi \), and we have noted that for a large value of \( \tan \beta \) the neutral Higgs boson masses and the branching ratios of the other neutral Higgs boson decays also become insensitive to the CP phase. However, the CP–violation effects caused by the loop corrections of soft CP–violating Yukawa couplings can be directly measured by an azimuthal CP–odd decay asymmetry in the decays \( H_i \rightarrow \tau^+ \tau^- \) and/or \( H_i \rightarrow t\bar{t} \) reflecting the correlations of two transversely polarized fermions. In this section, we estimate the size of the azimuthal CP–odd asymmetry in the MSSM with explicit CP violation through radiative corrections.

For the sake of discussion, we calculate the helicity amplitude explicitly for the Higgs boson decays \( H_i \rightarrow f\bar{f} \) by use of the so–called 2–component spinor technique \[22\]. The result is written as

\[
D_{\sigma\bar{\sigma}} = \frac{g m_f m_{H_i}}{2 m_W R^f_{\beta} \beta_f} \left[ \sigma \beta_f v^i_f + i \bar{R}^f_{\beta} a^i_f \right] \delta_{\sigma\bar{\sigma}},
\]

where \( \sigma \) and \( \bar{\sigma} \) are for the helicity of the fermions \( f \) and \( \bar{f} \), respectively, with \( f = \tau \) or \( t \). It is now clear that the helicity amplitude can be complex when both the scalar and pseudoscalar parts co-exist and the interference, signaling CP violation, can be extracted by adjusting the polarization of the final fermions. As a result, we obtain the following CP–odd asymmetry in terms of \( v^i_f, a^i_f \), and \( \bar{R}^f_{\beta} \):

\[
A^i_{\text{CP}} \equiv \frac{D_{--} D^{*}_{++}}{|D_{++}|^2 + |D_{--}|^2} = \frac{2 \beta_f R^f_{\beta} v^i_f a^i_f}{|\beta_f v^i_f|^2 + |\bar{R}^f_{\beta} a^i_f|^2}.
\]
In order to observe the CP violating spin–correlation of the top quarks or \( \tau^- \)-leptons, we need to look at the correlations among their decay products. The analysis power for the spins depends on each decay mode \([23]\). For example, if the top quark decays semi–leptonically, the spin analysis power is unity, but if it decays hadronically, the analysis power is about \(0.4\). Similarly, if the \( \tau^- \) decays semi–leptonically, the analysis power is \(\frac{-1}{3}\). And, the inclusive hadronic decays of the \( \tau^- \) is known to give the analysis power of \(0.42\). Certainly, if one can analyze the detailed structure of the hadronic decays, one can obtain better results.

Although we can do a more comprehensive numerical analysis for evaluating the possibility of detecting CP violation in the fermion decays, for simplicity we confine ourselves to estimating the size of the CP–odd asymmetry in the present work. Since the effectiveness of a decay channel in measuring CP violation directly depends on the number of events as well as the size of the CP–odd asymmetry, it is useful to define an effective CP–odd asymmetry as

\[
\hat{A}^i_{\text{CP}} = A^i_{\text{CP}} \sqrt{B(H_i \rightarrow \ell \ell)}.
\]

With this new effective CP–odd quantity, we can easily estimate the number of Higgs bosons required to see the CP–violation effect directly at 1–\(\sigma\) level. The number before including the detection efficiency and the polarization analysis powers is simply given by

\[
N_{H_i} = \frac{1}{(A^i_{\text{CP}})^2}.
\]

For a more realistic estimate, we refer to the work by Atwood and Soni of Ref. \([16]\).

The CP–odd asymmetry is strongly dependent on the value of \(\tan \beta\). In particular, the \(\tau^+\tau^-\) mode is expected to be greatly enhanced as \(\tan \beta\) increases. In this light, it is interesting to consider a large \(\tan \beta\) case as well, so we consider two values of \(\tan \beta = 3\) and \(30\) for comparison. The dependence of the CP–odd asymmetry on the charged Higgs boson mass \(m_{H^\pm}\) for several values of the CP phase in the decays \(H_{1,2,3} \rightarrow \tau^+\tau^-\) is present in Figures 6, 7, and 8, respectively, for two values of \(\tan \beta\); the solid line is for \(\tan \beta = 3\) and the dashed line for \(\tan \beta = 30\). We find that the CP–odd asymmetry in the decay \(H_1 \rightarrow \tau^+\tau^-\) is very sensitive to the charge Higgs boson mass \(m_{H^\pm}\) and can be significant near the lower tail of its allowed mass range for non–trivial values of the CP phase. Otherwise, the CP–odd asymmetry decreases as \(m_{H^\pm}\) increases. This is due to the suppression of the mixing matrix element \(O_{23}\) representing the composition of the CP–odd state in the lightest Higgs boson. Similarly, near the lower tail of the allowed charged Higgs boson mass, the CP–odd asymmetry for \(H_2 \rightarrow \tau^+\tau^-\) also is very sensitive to the charged Higgs boson mass. Otherwise, the CP–odd asymmetries remain constant instead of decreasing for the whole range of the charged Higgs boson mass, but they oscillate between -1 and 1 as the CP phase varies. The CP–odd asymmetry for the decay \(H_3 \rightarrow \tau^+\tau^-\) is (almost) independent of the charged Higgs boson mass, but it is also very sensitive to the CP–phase. On the other hand, the large value of the top–quark mass prohibits the lightest Higgs boson from decaying into a top–quark pair. So, we consider the decays \(H_{2,3} \rightarrow t\bar{t}\), of which the dependence on the charged Higgs boson mass \(m_{H^\pm}\) for several values of the CP phase is presented in Figures 9 and 10. Near the lower tail of the allowed charged Higgs boson mass, the CP–odd asymmetry for \(H_2 \rightarrow t\bar{t}\) is very sensitive to the charged Higgs boson mass as that for \(H_2 \rightarrow \tau^+\tau^-\). Otherwise, the
CP–odd asymmetry remains constant for the whole range of the charged Higgs boson mass, but it oscillates between -1 and 1 as the CP phase varies. The CP–odd asymmetry for the decay $H_3 \rightarrow t \bar{t}$ are almost independent of the charged Higgs boson mass, but it is also sensitive to the CP phase.

V. CONCLUSIONS

In this paper, we have performed a systematic study of the neutral Higgs boson decays in the MSSM with explicit CP violation, which is induced through loop corrections involving trilinear CP–violating couplings of the neutral Higgs bosons to scalar top and bottom quarks and can be sizable. Certainly, our analysis is far from complete in the sense that any below–threshold three–body decay channels and comprehensive QCD corrections, which may be significant, are not included. Therefore, a more complete work is needed and it is to be reported elsewhere.

The analysis on the branching ratios of the neutral Higgs boson decays shows that the decay pattern of the lightest neutral Higgs boson is almost independent of the CP phase, but the decay patterns of two other neutral Higgs bosons are very sensitive to the phase. Since the explicit CP violation is realized through the scalar–pseudoscalar mixing between the CP–odd Higgs state and, mainly, the heavy CP–even Higgs state, the decay patterns of two mass eigenstates are determined by the size of the mixing. Moreover, the dependence of each decay mode on the CP phase is different. Therefore, it is naturally expected that as the CP phase varies from zero, the branching ratios change a lot. In particular, for the intermediate Higgs boson (which is nothing but the CP–odd Higgs state in the CP invariant case) the channels $H_2 \rightarrow H_1 H_1$ and $H_2 \rightarrow VV$ ($V = W$ or $Z$), prohibited for the vanishing CP phase and so unique to the CP–noninvariant theory, can be sizable and become the most dominant channels for some nontrivial values of the CP phase. Furthermore, for the heaviest Higgs state (which is the heavier CP–even state in the CP invariant limit) the decay channel $H_3 \rightarrow H_1 Z$ prohibited for the vanishing CP phase may overwhelm the other possible decay channels near the lower tail of the allowed $H_3$ mass range for some nontrivial values of the CP phase.

We have also presented a numerical analysis of the CP–odd asymmetry that can be extracted through spin correlations of the fermions in the fermionic decays of the neutral Higgs bosons, $H_i \rightarrow \tau^+ \tau^-$ and $H_i \rightarrow t \bar{t}$. The CP–odd asymmetry is very sensitive to the value of $\tan \beta$, in particular, for the decay $H_1 \rightarrow \tau^+ \tau^-$. For a large value of $\tan \beta$, the branching ratio for the decay channel is enhanced and moreover the CP–odd asymmetry becomes significant for small Higgs boson masses. However, we have found that the CP–odd asymmetry for the Higgs boson $H_1$ decreases as the charged Higgs boson mass $m_{H^\pm}$ increases due to the suppression of $O_{23}$. Except the lower tail of the allowed charged Higgs boson mass, the CP–odd asymmetries for both $H_{2,3} \rightarrow \tau^+ \tau^-$ and $H_{2,3} \rightarrow t \bar{t}$ remain constant for the charged Higgs boson mass, but On the whole, the CP–odd asymmetry can be saturated to be 1 so that there is a great chance to see CP violation due to radiative corrections directly through the spin correlations of the tau leptons and/or top–quarks in the fermionic decays of any neutral Higgs boson.

To conclude, the analysis presented in this paper clearly demonstrates that the explicit
CP violation in the MSSM Higgs sector may modify the Higgs boson decays significantly so that it will have a great impact on Higgs searches, and give a great chance to measure the effects directly through the CP–odd spin correlations in the decays $H_i \to \tau^+\tau^-$ and/or $H_i \to t\bar{t}$.

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**APPENDIX**

In this Appendix, we present the explicit form of the coefficients $b_\alpha$ and the fully–symmetric trilinear Higgs self-couplings $a_{\alpha\beta\gamma}$, $(\alpha = 1, 2, 3)$ in terms of the quartic couplings $\lambda_i$ $(i = 1$ to $8)$, which are needed for the Higgs boson decays into a charged Higgs–boson pair and a neutral Higgs–boson pair, respectively. The coefficients describing the couplings of the neutral Higgs boson to a charged Higgs-boson pair in the weak eigenstates are given by

\[
b_1 = 2s_\beta c_\beta I(\lambda_5 e^{2i\xi}) - s_\beta^2 I(\lambda_6 e^{i\xi}) - c_\beta^2 I(\lambda_7 e^{i\xi}),
\]

\[
b_2 = 2s_\beta^2 c_\beta \lambda_1 + c_\beta^3 \lambda_3 - s_\beta^2 c_\beta \lambda_4 - 2s_\beta c_\beta R(\lambda_5 e^{2i\xi}) + s_\beta(s_\beta^2 - 2c_\beta^2) R(\lambda_6 e^{i\xi}) + s_\beta c_\beta^2 R(\lambda_7 e^{i\xi}),
\]

\[
b_3 = 2c_\beta^2 s_\beta \lambda_2 + s_\beta^3 \lambda_3 - c_\beta^2 s_\beta \lambda_4 - 2c_\beta s_\beta R(\lambda_5 e^{2i\xi}) + c_\beta s_\beta^2 R(\lambda_6 e^{i\xi}) + c_\beta(c_\beta^2 - 2s_\beta^2) R(\lambda_7 e^{i\xi}),
\]

and the symmetric coefficients $a_{\alpha\beta\gamma}$ describing the self–interactions of three neutral Higgs bosons by

\[
a_{111} = s_\beta c_\beta I(\lambda_5 e^{2i\xi}) - \frac{1}{2}s_\beta^2 I(\lambda_6 e^{i\xi}) - \frac{1}{2}c_\beta^2 I(\lambda_7 e^{i\xi}),
\]

\[
a_{112} = \frac{1}{3}\left[s_\beta^2 c_\beta \lambda_1 + \frac{1}{2}c_\beta^3(\lambda_3 + \lambda_4) - c_\beta(1 + s_\beta^2) R(\lambda_5 e^{2i\xi}) + \frac{1}{2}c_\beta s_\beta^3 R(\lambda_6 e^{i\xi}) + \frac{1}{2}c_\beta^2 s_\beta R(\lambda_7 e^{i\xi})\right],
\]

\[
a_{113} = \frac{1}{3}\left[c_\beta^2 s_\beta \lambda_2 + \frac{1}{2}s_\beta^3(\lambda_3 + \lambda_4) - s_\beta(1 + c_\beta^2) R(\lambda_5 e^{2i\xi}) + \frac{1}{2}s_\beta c_\beta^3 R(\lambda_6 e^{i\xi}) + \frac{1}{2}c_\beta s_\beta^2 R(\lambda_7 e^{i\xi})\right],
\]

\[
a_{122} = \frac{1}{3}\left[-s_\beta c_\beta I(\lambda_5 e^{2i\xi}) - \frac{1}{2}(1 + 2c_\beta) I(\lambda_6 e^{i\xi})\right],
\]

\[
a_{123} = \frac{1}{6}\left[-2I(\lambda_5 e^{2i\xi}) - c_\beta s_\beta(I(\lambda_6 e^{i\xi}) + I(\lambda_7 e^{i\xi}))\right],
\]

\[
a_{133} = \frac{1}{3}\left[-s_\beta c_\beta I(\lambda_5 e^{2i\xi}) - \frac{1}{2}(1 + 2s_\beta^2) I(\lambda_7 e^{i\xi})\right],
\]

\[
a_{222} = \frac{1}{3}\left[-c_\beta^2 I(\lambda_5 e^{2i\xi}) - \frac{1}{2}(1 + 2s_\beta^2) I(\lambda_6 e^{i\xi})\right],
\]

\[
a_{223} = \frac{1}{6}\left[-2I(\lambda_5 e^{2i\xi}) + c_\beta s_\beta(I(\lambda_6 e^{i\xi}) + I(\lambda_7 e^{i\xi}))\right],
\]

\[
a_{233} = \frac{1}{3}\left[-c_\beta^2 I(\lambda_5 e^{2i\xi}) - \frac{1}{2}(1 + 2c_\beta^2) I(\lambda_7 e^{i\xi})\right].
\]
\[ a_{222} = c_\beta \lambda_1 + \frac{1}{2} s_\beta R \left( \lambda_6 e^{i \xi} \right), \]
\[ a_{223} = \frac{1}{3} \left[ \frac{1}{2} s_\beta (\lambda_3 + \lambda_4) + s_\beta R \left( \lambda_5 e^{2i \xi} \right) + \frac{3}{2} c_\beta R \left( \lambda_6 e^{i \xi} \right) \right], \]
\[ a_{233} = \frac{1}{3} \left[ \frac{1}{2} c_\beta (\lambda_3 + \lambda_4) + c_\beta R \left( \lambda_5 e^{2i \xi} \right) + \frac{3}{2} s_\beta R \left( \lambda_7 e^{i \xi} \right) \right], \]
\[ a_{333} = s_\beta \lambda_2 + \frac{1}{2} c_\beta R \left( \lambda_7 e^{i \xi} \right). \]

(A2)

The remaining 17 coefficients of \( a_{\alpha \beta \gamma} \) can be obtained by the symmetry properties \( a_{\alpha \beta \gamma} = a_{\beta \gamma \alpha} = a_{\gamma \alpha \beta} = a_{\alpha \gamma \beta} = a_{\beta \alpha \gamma} \). We note that \( b_1, a_{111}, a_{122}, a_{123}, \) and \( a_{133} \) come only from the imaginary parts of one-loop corrections to the effective Higgs potential so that these coefficients vanish in the CP-invariant theory.
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FIG. 1. Higgs–boson masses with respect to the charged Higgs mass $m_{H^\pm}$ for different values of the CP phase $\Phi$ with the fixed values for other SUSY parameters in Eq. (16). The value of $\tan \beta$ is taken to be 3 and an artificial experimental condition $m_{H_1} \geq 70$ GeV is imposed. The lower solid line is for the lightest Higgs boson $H_1$, the upper solid line for the heaviest Higgs boson $H_3$, and the dashed line for the intermediate Higgs boson $H_2$. For reference, the charged Higgs mass is presented with the dot–dashed line.
FIG. 2. Total decay widths for the three neutral Higgs bosons with respect to each Higgs–boson mass for several values of the CP phase $\Phi$. The left (right) solid line is for the lightest (heaviest) Higgs boson and the dashed line for the intermediate Higgs boson.
FIG. 3. Partial branching ratios of the lightest Higgs boson decays with respect to the mass $m_{H_1}$ for several values of the CP phase $\Phi$. The upper solid line is for the channel $H_1 \rightarrow \bar{b}b$, the dashed line for the channel $H_1 \rightarrow \tau^+\tau^-$, the dotted line for the channel $H_1 \rightarrow \bar{c}c$ and the lower solid line for the two-gluon mode $H_1 \rightarrow gg$. 
FIG. 4. Partial branching ratios for the $H_2$ decays with respect to the mass $m_{H_2}$ for several values of the CP phase $\Phi$. The upper solid line in each figure frame is for the sum of the branching ratios of the fermionic decay modes. The dot–dashed line is for the decay channel $H_2 \rightarrow H_1 Z$, two dotted lines for the channels $H_2 \rightarrow W^+W^-$ (upper line) and $ZZ$ (lower line), respectively, and the dashed line for the decay channel $H_2 \rightarrow H_1H_1$. Finally, the lower solid line is for the two–gluon channel.
FIG. 5. Partial branching ratios for the $H_3$ decays with respect to the mass $m_{H_3}$ for several values of the CP phase $\Phi$. The upper solid line in each figure frame is for the sum of the branching ratios of the fermionic decay modes. The dot–dashed line is for the decay channel $H_3 \rightarrow H_1 Z$, two dotted lines for the channels $H_3 \rightarrow W^+W^-$ (upper line) and $ZZ$ (lower line), respectively, and the dashed line for the decay channel $H_3 \rightarrow H_1 H_1$. Finally, the lower solid line is for the two–gluon channel.
FIG. 6. The CP–odd effective asymmetry $\hat{A}_{\text{CP}}$ with respect to the charged Higgs boson mass $m_{H^\pm}$ for several values of the CP phase $\Phi$ in the decay $H_1 \to \tau^+\tau^-$. The solid line in each frame is for $\tan\beta = 3$ and the dashed line for $\tan\beta = 30$. 
FIG. 7. The CP–odd effective asymmetry $\hat{A}_{\text{CP}}^{i}$ with respect to the charged Higgs boson mass $m_{H^\pm}$ for several values of the CP phase $\Phi$ in the decay $H_2 \to \tau^+\tau^-$. The solid line in each frame is for $\tan \beta = 3$ and the dashed line for $\tan \beta = 30$. 
FIG. 8. The CP–odd effective asymmetry $\tilde{A}_{\text{CP}}$ with respect to the charged Higgs boson mass $m_{H^\pm}$ for several values of the CP phase $\Phi$ in the decay $H_3 \to \tau^+\tau^-$. The solid line in each frame is for $\tan\beta = 3$ and the dashed line for $\tan\beta = 30$. 
FIG. 9. The CP–odd effective asymmetry $\hat{A}_{\text{CP}}$ with respect to the charged Higgs boson mass $m_{H^\pm}$ for several values of the CP phase $\Phi$ in the decay $H_2 \to t\bar{t}$. The solid line in each frame is for $\tan \beta = 3$ and the dashed line for $\tan \beta = 30$. 
FIG. 10. The CP–odd effective asymmetry $\hat{A}_{\text{CP}}^i$ with respect to the charged Higgs boson mass $m_{H^\pm}$ for several values of the CP phase $\Phi$ in the decay $H_3 \rightarrow t\bar{t}$. The solid line in each frame is for $\tan \beta = 3$ and the dashed line for $\tan \beta = 30$. 