Multi-dimensional Rational Bubbles and fat tails *

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Abstract

We extend the model of rational bubbles of \cite{Blanchard, 1979} and \cite{Blanchard and Watson, 1982} to arbitrary dimensions $d$: a number $d$ of market time series are made linearly interdependent via $d \times d$ stochastic coupling coefficients. We first show that the no-arbitrage condition imposes that the non-diagonal impacts of any asset $i$ on any other asset $j \neq i$ has to vanish on average, i.e., must exhibit random alternative regimes of reinforcement and contrarian feedbacks. In contrast, the diagonal terms must be positive and equal on average to the inverse of the discount factor. Applying the results of renewal theory for products of random matrices to stochastic recurrence equations (SRE), we extend the theorem of \cite{Lux and Sornette, 1999} and demonstrate that the tails of the unconditional distributions associated with such $d$-dimensional bubble processes follow power laws (i.e., exhibit hyperbolic decline), with the same asymptotic tail exponent $\mu < 1$ for all assets. The distribution of price differences and of returns is dominated by the same power-law over an extended range of large returns. This small value $\mu < 1$ of the tail exponent has far-reaching consequences in the non-existence of the means and variances. Although power-law tails are a pervasive feature of empirical data, the numerical value $\mu < 1$ is in disagreement with the usual empirical estimates $\mu \approx 3$. It, therefore, appears that generalizing the model of rational bubbles to arbitrary dimensions does not allow us to reconcile the model with these stylized facts of financial data. The non-stationary growth rational bubble model seems at present the only viable solution \cite{Sornette, 2001}.

1 The model of rational bubbles

\cite{Blanchard, 1979} and \cite{Blanchard and Watson, 1982} originally introduced the model of rational expectations (RE) bubbles to account for the possibility, often discussed in the empirical literature and by practitioners, that observed prices may deviate significantly and over extended time intervals from fundamental prices. While allowing for deviations from fundamental prices, rational bubbles keep a fundamental anchor point of economic modelling, namely that bubbles must obey the condition of rational expectations. In contrast, recent works stress that investors are not fully rational,

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or have at most bounded rationality, and that behavioral and psychological mechanisms, such as herding, may be important in the shaping of market prices [Thaler, 1993, Shefrin, 2000, Shleifer, 2000]. However, for fluid assets, dynamic investment strategies rarely perform over simple buy-and-hold strategies [Malkiel, 1999], in other words, the market is not far from being efficient and little arbitrage opportunities exist as a result of the constant search for gains by sophisticated investors. Here, we shall work within the conditions of rational expectations and of no-arbitrage condition, taken as useful approximations. Indeed, the rationality of both expectations and behavior often does not imply that the price of an asset be equal to its fundamental value. In other words, there can be rational deviations of the price from this value, called rational bubbles. A rational bubble can arise when the actual market price depends positively on its own expected rate of change, as sometimes occurs in asset markets, which is the mechanism underlying the models of [Blanchard, 1979] and [Blanchard and Watson, 1982].

In order to avoid the unrealistic picture of ever-increasing deviations from fundamental values, [Blanchard and Watson, 1982] proposed a model with periodically collapsing bubbles in which the bubble component of the price follows an exponential explosive path (the price being multiplied by \( a_t = \bar{a} > 1 \)) with probability \( \pi \) and collapses to zero (the price being multiplied by \( a_t = 0 \)) with probability \( 1 - \pi \). It is clear that, in this model, a bubble has an exponential distribution of lifetimes with a finite average lifetime \( \pi / (1 - \pi) \). Bubbles are thus transient phenomena. The condition of rational expectations imposes that \( \bar{a} = 1/\delta \), where \( \delta \) is the inverse of the discount factor. In order to allow for the start of new bubbles after the collapse, a stochastic zero mean normally distributed component \( b_t \) is added to the systematic part of \( B_t \). This leads to the following dynamical equation

\[
X_{t+1} = a_t X_t + b_t,
\]

(1)

where, as we said, \( a_t = \bar{a} \) with probability \( \pi \) and \( a_t = 0 \) with probability \( 1 - \pi \). There is a huge literature on theoretical refinements of this model and on the empirical detectability of RE bubbles in financial data (see [Camerer, 1989] and [Adam and Szafarz, 1992], for surveys of this literature).

Recently, [Lux and Sornette, 1999] studied the implications of the RE bubble models for the unconditional distribution of prices, price changes and returns resulting from a more general discrete-time formulation extending (1) by allowing the multiplicative factor \( a_t \) to take arbitrary values and be i.i.d. random variables drawn from some non-degenerate probability density function (pdf) \( P_a(a) \). The model can also be generalized by considering non-normal realizations of \( b_t \) with distribution \( P_b(b) \) with \( E[b_t] = 0 \), where \( E[\cdot] \) is the expectation operator. Since in (1) the bubble \( X_t \) denotes the difference between the observed price and the fundamental price, the "bubble" regimes refer to the cases when \( X_t \) explodes exponentially under the action of successive multiplications by factor \( a_t, a_{t+1}, \ldots \) with a majority of them larger than 1 but different, thus adding a stochastic component to the standard model of [Blanchard and Watson, 1982].

Provided \( E[\ln a] < 0 \) (stationarity condition) and if there is a number \( \mu \) such that \( 0 < E[|b|^\mu] < +\infty \), such that

\[
E[|a|^\mu] = 1
\]

(2)

and such that \( E[|a|^\mu \ln |a|] < +\infty \), then the tail of the distribution of \( B \) is asymptotically (for large \( X \)'s) a power law [Kesten, 1973, Goldie, 1999]

\[
P_X(X) \, dX \approx \frac{C}{|X|^{1+\mu}} \, dX ,
\]

(3)

with an exponent \( \mu \) given by the real positive solution of (2). Rational expectations require in addition that the bubble component of asset prices obeys the "no free-lunch" condition

\[
\delta \cdot E[X_{t+1}] = X_t
\]

(4)
where $\delta$ is the discount factor $< 1$. Condition (4) with (1) imposes the condition
\[ E[a] = \frac{1}{\delta} > 1, \]
on the distribution of the multiplicative factors $a_t$. Since the function $E[|a|^\mu]$ is convex, [Lux and Sornette, 1999] showed that this automatically enforces $\mu < 1$. It is easy to show that the distribution of price differences has the same power law tail with the exponent $\mu < 1$ and the distribution of returns is dominated by the same power-law over an extended range of large returns [Lux and Sornette, 1999]. Although power-law tails are a pervasive feature of empirical data, these characterizations are in strong disagreement with the usual empirical estimates which find $\mu \approx 3$ [de Vries, 1994, Lux, 1996, Pagan, Guillaume et al., 1997, Gopikrishnan et al., 1998]. [Lux and Sornette, 1999] concluded that exogenous rational bubbles are thus hardly reconcilable with some of the stylized facts of financial data at a very elementary level.

2 Generalization of rational bubbles to arbitrary dimensions

2.1 Introduction

In reality, there is no such thing as an isolated asset. Stock markets exhibit a variety of interdependences, based in part on the mutual influences between the USA, European and Japanese markets. In addition, individual stocks may be sensitive to the behavior of the specific industry as a whole to which they belong and to a few other indicators, such as the main indices, interest rates and so on. [Mantegna, 1999, Bonanno, Lillo and Mantegna, 2001] have indeed shown the existence of a hierarchical organization of stock interdependences. Furthermore, bubbles often appear to be not isolated features of a set of markets. For instance, [Flood et al., 1984] tested whether a bubble simultaneously existed across the nations, such as Germany, Poland, and Hungary, that experienced hyperinflation in the early 1920s. Coordinated bubbles can sometimes be detected. One of the most prominent example is found in the market appreciations observed in many of the world markets prior to the world market crash in Oct. 1987 [Barro et al., 1989]. Similar intermittent coordination of bubbles have been detected among the significant bubbles followed by large crashes or severe corrections in Latin-American and Asian stock markets [Johansen and Sornette, 2000]. It is therefore desirable to generalize the one-dimensional RE bubble model (1) to the multi-dimensional case. One could also hope a priori that this generalization would modify the result $\mu < 1$ obtained in the one-dimensional case and allow for a better adequation with empirical results. As we shall show, this turns out to be ill-born for reasons that we shall explain in details.

The simplest such generalization is to consider two assets $X$ and $Y$ with prices respectively equal to $X_t$ and $Y_t$ at time $t$, evolving according to
\begin{align*}
X_{t+1} &= a_t X_t + b_t Y_t + \eta_t \\
Y_{t+1} &= c_t X_t + d_t Y_k + \epsilon_t
\end{align*}
where $a_t$, $b_t$, $c_t$ and $d_t$ are drawn from some multivariate probability density function. The two additive noises $\eta_t$ and $\epsilon_t$ are also drawn from some distribution function with zero mean. The diagonal case $b_t = c_t = 0$ for all $t$ recovers the previous one-dimensional case with two uncoupled bubbles, provided $\eta_t$ and $\epsilon_t$ are independent.

Rational expectations require that $X_t$ and $Y_t$ obey both the “no-free lunch” condition (4), i.e.,
\[ \delta \cdot E[X_{t+1}] = X_t \quad \text{and} \quad \delta \cdot E[Y_{t+1}] = Y_t. \]
With (3), this gives
\begin{align*}
(E[a_t] - \delta^{-1}) X_t + E[b_t] Y_t &= 0, \\
E[c_t] X_t + (E[d_t] - \delta^{-1}) Y_t &= 0,
\end{align*}
with (1),
where we have used $E[\eta_t] = E[\epsilon_t] = 0$. The two equations (8,9) must be true for all times, i.e. for all values of $X_t$ and $Y_t$ visited by the dynamics. This imposes $E[b_t] = E[c_t] = 0$ and $E[a_t] = E[d_t] = \delta^{-1}$. We are going to retrieve this result more formally in the general case.

### 2.2 General formulation

A generalization to arbitrary dimensions lead to the following stochastic random equation (SRE)

$$X_t = A_t X_{t-1} + B_t$$

where $(X_t, B_t)$ are $d$-dimensional vectors. Each component of $X_t$ can be thought of as the price of an asset above its fundamental price. The matrices $(A_t)$ are identically independent distributed $d \times d$-dimensional stochastic matrices. We assume that $B_t$ are identically independent distributed random vectors and that $(X_t)$ is a causal stationary solution of (10). Generalizations introducing additional arbitrary linear terms at larger time lags such as $X_{t-2}, ...$ can be treated with slight modifications of our approach and yield the same conclusions. We shall thus confine our demonstration on the SRE of order 1, keeping in mind that our results apply analogously to arbitrary orders of regressions.

To formalize the SRE in a rigorous manner, we introduce in a standard way the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)$. Here $\mathbb{P}$ represents the product measure $\mathbb{P} = \mathbb{P}_X \otimes \mathbb{P}_A \otimes \mathbb{P}_B$, where $\mathbb{P}_X$, $\mathbb{P}_A$ and $\mathbb{P}_B$ are the probability measures associated with $(X_t)$, $(A_t)$ and $(B_t)$. We further assume as is customary that the stochastic process $(X_t)$ is adapted to the filtration $(\mathcal{F}_t)$.

In the following, we denote by $|\cdot|$ the Euclidean norm and by $||\cdot||$ the corresponding norm for any $d \times d$-matrix $A$

$$||A|| = \sup_{|x|=1} |Ax|.$$  \hfill (11)

In the next section, we formalize the “no-free lunch” condition for the SRE (10) and show that its entails in particular that the spectral radius (larger eigenvalue) of $E[A_t]$ must be equal to the inverse of the discount factor, hence it must be larger than 1. In section 4, we synthesize the main results on the renewal theory of SRE of the type (10) and show that the condition imposing that the exponent of the power law tails be larger than 1 implies that the spectral radius of $E[A_t]$ must be smaller than 1. By the reverse logic, the “no-free lunch” condition automatically enforces our main result that the exponent is less than 1.

### 3 The no-free lunch condition

#### 3.1 No-free lunch condition under the risk-neutral probability measure

The “no-free lunch” condition is equivalent to the existence of a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that, for all self-financing portfolio $P_t$, $\frac{S_t}{S_0}$ is a $\mathbb{Q}$-martingale, where $S_{0,t} = \prod_{i=0}^{t-1} \delta_i^{-1}$, $\delta_i = (1 + r_i)^{-1}$ is the discount factor for period $i$ and $r_i$ is the corresponding risk-free interest rate.

It is natural to assume that, for a given period $i$, the discount rates $r_i$ are the same for all assets. In frictionless markets, a deviation for this hypothesis would lead to arbitrage opportunities. Furthermore, since the sequence of matrices $(A_t)$ is i.i.d. and therefore stationary, this implies that $\delta_i$ or $r_i$ must be constant and equal respectively to $\delta$ and $r$.

Under those conditions, we have the following proposition :
Proposition 1

The stochastic process

\[ X_t = A_t X_{t-1} + B_t \]  

satisfies the no-arbitrage condition if and only if

\[ \mathbb{E}_Q[A] = \frac{1}{\delta} I_d. \]  

The proof is given in the Appendix A.

The condition (13) imposes some stringent constraints on admissible matrices \( A_t \). Indeed, while \( A_t \) are not diagonal in general, their average must be diagonal. This implies that the off-diagonal terms of the matrices \( A_t \) must take negative values, sufficiently often so that their averages vanish. The off-diagonal coefficients quantify the influence of other bubbles on a given one. The condition (13) thus means that the average effect of other bubbles on any given one must vanish. It is straightforward to check that, in this linear framework, this implies an absence of correlation between the different bubble components

\[ \mathbb{E}[X^{(k)}X^{(\ell)}] = 0 \quad \text{for any} \ k \neq \ell. \]  

In constrast, the diagonal elements of \( A_t \) must be positive in majority in order for

\[ \mathbb{E}_P[A_{ii}] = \delta_i^{-1} \quad \text{for all} \ i's, \ 	ext{to hold true.} \]  

In fact, on economic grounds, we can exclude the cases where the diagonal elements take negative values. Indeed, a negative value of \( A_{ii} \) at a given time \( t \) would imply that \( X_i^{(i)} \) abruptly change sign between \( t-1 \) and \( t \), what does not seem to be a reasonable financial process.

3.2 The no-free lunch condition under historic probability measure

The historical \( \mathbb{P} \) and risk-neutral \( \mathbb{Q} \) probability measures are equivalent. This means that there exists a non-negative matrix \( h(\theta) = (h_{ij}(\theta_{ij})) \) such that, for each element indexed by \( i, j \), we have

\[ \mathbb{E}_P[A_{ij}] = \mathbb{E}_Q[h_{ij} \cdot A_{ij}] = h_{ij}(\theta_{ij}^0) \cdot \mathbb{E}_Q[A_{ij}] \quad \text{for some} \ \theta_{ij}^0 \in \mathbb{R}. \]  

The second equation comes from the well known result:

\[ \int f(\theta) \cdot g(\theta) \ d\mu(\theta) = g(\theta_0) \cdot \int f(\theta) \ d\mu(\theta) \quad \text{for some} \ \theta_0 \in \mathbb{R}. \]  

We thus get

\[ \mathbb{E}_P[A_{ij}] = 0 \quad \text{if} \ i \neq j \]  

\[ \mathbb{E}_P[A_{ij}] = \frac{1}{\delta^{(i)}} \quad \text{if} \ i = j, \]  

where the \( \delta^{(i)} \) can be different. We can thus write

\[ \mathbb{E}_P[A] = \tilde{\delta}^{-1}, \quad \text{where} \ \tilde{\delta}^{-1} = \text{diag}[\delta^{(1)}^{-1}, \ldots, \delta^{(d)}^{-1}]. \]  

Appendix B gives a proof showing that \( \delta^{(i)}^{-1} \) is indeed the genuine discount factor for the \( i \)-th bubble component.

4 Renewal theory for products of random matrices

In the following, we will consider that the random \( d \times d \) matrices \( A_t \) are invertible matrices with real entries. We will denote by \( GL_d(\mathbb{R}) \) the group of these matrices.
4.1 Definitions

Definition 1 (Feasible matrix)
A matrix \( M \in GL_d(\mathbb{R}) \) is \( \mathbb{P} \)-feasible if there exists an \( n \in \mathbb{N} \) and \( M_1, \ldots, M_n \in \text{supp}(\mathbb{P}) \) such that \( M = M_1 \cdots M_n \) and if \( M \) has a simple real eigenvalue \( q(M) \) which, in modulus, exceeds all other eigenvalue of \( M \).

Definition 2
For any matrix \( M \in GL_d(\mathbb{R}) \) and \( M' \) its transpose, \( MM' \) is a symmetric positive definite matrix. We define \( \lambda(M) \) the square root of the smallest eigenvalue of \( MM' \).

4.2 Theorem
We extend the theorem 2.7 of Davis et al. [Davis et al., 1999], which synthetized Kesten’s theorems 3 et 4 in [Kesten, 1973], to the case of real valued matrices. The proof of this theorem is given in [Le Page, 1983].

Theorem 1
Let \( (A_n) \) be an i.i.d. sequence of matrices in \( GL_d(\mathbb{R}) \) satisfying the following set of conditions:

\[ H_1 : \text{for some } \epsilon > 0, \ E_{\mathbb{P}_A} \left[ ||A||^\epsilon \right] < 1, \]

\[ H_2 : \text{For every open } U \subset S_{d-1} \text{ (the unit sphere in } \mathbb{R}^d \text{) and for all } x \in S_{d-1} \text{ there exists an } n \text{ such that} \]

\[ \Pr \left\{ \frac{xA_1 \cdots A_n}{||xA_1 \cdots A_n||} \in U \right\} > 0 . \]  \hspace{1cm} (20)

\[ H_3 : \text{The group } \{ \ln |q(M)|, M \text{ is } \mathbb{P}_A \text{-feasible} \} \text{ is dense in } \mathbb{R}. \]

\[ H_4 : \text{for all } r \in \mathbb{R}^d, \Pr\{A_1r + B_1 = r\} < 1. \]

\[ H_5 : \text{There exists a } \kappa_0 > 0 \text{ such that} \]

\[ E_{\mathbb{P}_A} \left[ (||A_1||)^{\kappa_0} \right] \geq 1. \]  \hspace{1cm} (21)

\[ H_6 : \text{With the same } \kappa_0 > 0 \text{ as for the previous condition, there exists a real number } u > 0 \text{ such that} \]

\[ \begin{cases} E_{\mathbb{P}_A} \left[ (\sup\{||A_1||, ||B_1||\})^{\kappa_0 + u} \right] < \infty, \\
E_{\mathbb{P}_A} \left[ ||A_1||^{-u} \right] < \infty. \end{cases} \]  \hspace{1cm} (22)

Provided that these conditions hold,

- there exists a unique solution \( \kappa_1 \in (0, \kappa_0] \) to the equation

\[ \lim_{n \to \infty} \frac{1}{n} \ln E_{\mathbb{P}_A} \left[ ||A_1 \cdots A_n||^{\kappa_1} \right] = 0, \]  \hspace{1cm} (23)

- If \( (X_n) \) is the stationary solution to the SRE in (14) then \( X \) is regularly varying with index \( \kappa_1 \). In other words, the tail of the marginal distribution of each of the components of the vector \( X \) is asymptotically a power law with exponent \( \kappa_1 \).
4.3 Comments on the theorem

4.3.1 Intuitive meaning of the hypotheses

A suitable property for an economic model is the stationarity, i.e. the solution $X_t$ of the SRE (10) does not blow up. This condition is ensured by the hypothesis $(H1)$. Indeed, $E_{\mathbb{P}_A} [\ln ||A||] < 0$ implies that the Lyapunov exponent of the sequence $\{A_n\}$ of i.i.d. matrices is negative [Davis et al., 1999]. And it is well known, that the negativity of the Lyapunov exponent is a sufficient condition for the existence of a stationary solution $X_t$.

However, this condition can lead to a too fast decay of the tail of the distribution of $\{X\}$. This phenomenon is prevented by $(H5)$ which means intuitively that the multiplicative factors given by the elements of $A_t$ sometimes produce an amplification of $X_t$. In the one-dimensional bubble case, this condition reduces to the simple rule that $a_t$ must sometimes be larger than 1 so that a bubble can develop. Otherwise, the power law tail will be replaced by an exponential tail.

So, $(H1)$ and $(H5)$ keep the balance between two opposite objectives: to obtain a stationary solution and to observe a fat tailed distribution for the process $(X_t)$.

Another desirable property for the model is the ergodicity: we expect the price process $(X_t)$ to explore the entire space $\mathbb{R}^d$. This is ensured by hypothesis $(H2)$ and $(H4)$: hypothesis $(H2)$ allows $X_t$ to visit the neighborhood of any point in $\mathbb{R}^d$, and $(H4)$ forbids the trajectory to be trapped at some points.

Hypothesis $(H3)$ and $(H6)$ are more technical ones. The hypothesis $(H6)$ simply ensures that the tails of the distributions of $A_t$ and $B_t$ are thinner than the tail created by the SRE (10), so that the observed tail index is really due to the dynamics of the system and not to an ill-posed problem where the tail distribution of $A_t$ or $B_t$ is fat enough to dominate the large deviations behavior of the process. The hypothesis $(H3)$ expresses some kind of aperiodicity condition.

4.3.2 Intuitive meaning of (23)

The equation (23) determining the tail exponent $\kappa_1$ reduces to (2) in the one-dimensional case, which is simple to handle. In the multi-dimensional case, the novel feature is the non-diagonal nature of the multiplication of matrices which does not allow in general for an explicit equation similar to (2).

4.4 Constraint on the tail index

The first conclusion of the theorem above shows that the tail index $\kappa_1$ of the process $(X_t)$ is driven by the behavior of the matrices $(A_t)$. We will then state a proposition in which we give a majoration of the tail index.

Proposition 2

A necessary condition to have $\kappa_1 > 1$ is that the spectral radius of $E_{\mathbb{P}_A} [A]$ be smaller than 1:

$$\kappa_1 > 1 \implies \rho(E_{\mathbb{P}_A} [A]) < 1.$$ (24)

The proof is given in the Appendix C.

This proposition, put together with Proposition 1 above, will allow us to derive our main result.
5 Consequences for rational expectation bubbles

We have seen in section 3 from Proposition 1 that, as a result of the no-arbitrage condition, the spectral radius of the matrix \( E \[ A \] \) is greater than 1. As a consequence, by application of the converse of Proposition 2, we find that the tail index \( \kappa_1 \) of the distribution of \((X)\) is smaller than 1. This result does not rely on the diagonal property of the matrices \( E[A_t] \) but only on the value of the spectral radius imposed by the no-arbitrage condition.

This result generalizes to arbitrary \( d \)-dimensional processes the result of [Lux and Sornette, 1999]. As a consequence, \( d \)-dimensional rational expectation bubbles linking several assets suffer from the same discrepancy compared to empirical data as the one-dimensional bubbles. It would therefore appear that exogenous rational bubbles are hardly reconcilable with some of the most fundamental stylized facts of financial data at a very elementary level.

A possible remedy has recently been suggested [Sornette, 2001] which involves an average exponential growth of the fundamental price at some return rate \( r_f > 0 \). With the condition that the price fluctuations associated with bubbles must on average grow with the mean market return \( r_f \), it can be shown that the exponent of the power law tail of the returns is no more bounded by 1 as soon as \( r_f \) is larger than the discount rate \( r \) and can take essentially arbitrary values. It would be interesting to investigate the interplay between inter-dependence between several bubbles and this exponential growth model.
APPENDIX A: proof of Proposition 1 on the no-arbitrage condition

Let $P_t$ be the value at time $t$ of any self-financing portfolio:

$$P_t = W_t'X_t,$$  \hfill (25)

where $W_t' = (W_1, ..., W_d)$ is the vector whose components are the weight of the different assets and the prime denotes the transpose. The no-free lunch condition reads:

$$P_t = \delta \cdot E_Q[P_{t+1}|\mathcal{F}_t] \quad \forall\{P_t\}_{t \geq 0}. \hfill (26)$$

Therefore, for all self-financing strategies $(W_t)$, we have:

$$W_{t+1}' \left\{ E_Q[A] - \frac{1}{\delta}I_d \right\} X_t = 0 \quad \forall X_t \in \mathbb{R}^d, \hfill (27)$$

where we have used the fact that $(W_{t+1})$ is $(\mathcal{F}_t)$-measurable and that the sequence of matrices $\{A_t\}$ is i.i.d.

The strategy $W_t' = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in $i^{th}$ position), for all $t$, is self-financing and implies:

$$(a_{i1}, a_{i2}, \ldots, a_{ii} - \frac{1}{\delta}, \ldots, a_{id}) \cdot (X_{t}^{(1)}, X_{t}^{(2)}, \ldots, X_{t}^{(i)}, \ldots, X_{t}^{(d)})' = 0,$$ \hfill (28)

for all $X_t \in \mathbb{R}^d$. We have called $a_{ij}$ the $(i, j)^{th}$ coefficient of the matrix $E_Q[A]$. As a consequence,

$$(a_{i1}, a_{i2}, \ldots, a_{ii} - \frac{1}{\delta}, \ldots, a_{id}) = 0 \quad \forall i,$$ \hfill (29)

and

$$E_Q[A] = \frac{1}{\delta}I_d. \hfill (30)$$

The no-arbitrage condition thus implies: $E_Q[A] = \frac{1}{\delta}I_d$.

We now show that the converse is true, namely that if $E_Q[A] = \frac{1}{\delta}I_d$ is true, then the no-arbitrage condition is verified. Let us thus assume that $E_Q[A] = \frac{1}{\delta}I_d$ hold. Then:

$$E_Q[P_{t+1}|\mathcal{F}_t] = E_Q[W_{t+1}' \cdot X_{t+1}|\mathcal{F}_t] = W_{t+1}' - E_Q[A_{t+1}X_t + B_{t+1}|\mathcal{F}_t] = W_{t+1}' \cdot E_Q[A_{t+1}|\mathcal{F}_t] \cdot X_t = \frac{1}{\delta}W_{t+1}' \cdot X_t. \hfill (31)$$

The condition that the portfolio is self-financing is $W_{t+1}'X_t = W_t'X_t$, which means that the weights can be rebalanced a priori arbitrarily between the assets with the constraint that the total wealth at the same time remains constant. We can thus write thus:

$$E_Q[P_{t+1}|\mathcal{F}_t] = \frac{1}{\delta}W_{t+1}' \cdot X_t = \frac{1}{\delta}P_t. \hfill (32)$$

Therefore, the discounted process $\{P_t\}$ is a $Q$-martingale.
APPENDIX B: proof that $\delta^{(i)}_{-1}$ is the discount factor for the $i$-th bubble component in the historical space

Here, we express the no-free lunch condition in the historical space (or real space). The condition we will obtain is the so-called “Rational Expectation Condition”, which is a little bit less general than the condition detailed in the previous appendix A.

Given the prices $\{X^{(i)}_{k}\}_{k \leq t}$ of an asset, labeled by $i$, until the date $t$, the best estimation of its price at $t + 1$ is $E_p[X^{(i)}_{t+1}|F_t]$. So, the RE condition leads to

$$\frac{E_p[X^{(i)}_{t+1}|F_t] - X^{(i)}_{t}}{X^{(i)}_{t}} = r^{(i)}_{t}, \quad (39)$$

where $r^{(i)}_{t}$ is the return of the asset $i$ between $t$ and $t + 1$. As previously, we will assume in what follows that $r^{(i)}_{t} = r_{t}$ is time independent. Thus, the rational expectation condition for the assets $i$ reads

$$X^{(i)}_{t} = \delta^{(i)} \cdot E_p[X^{(i)}_{t+1}|F_t], \quad (40)$$

$$= \delta^{(i)} \cdot E_p[X^{(i)}_{t+1}|F_t], \quad (41)$$

where $\delta^{(i)}$ is the discount factor.

A priori, each asset has a different return. Thus, introducing the vector $\tilde{X}_t$ whose $i$th component is $X^{(i)}_{t} \delta^{(i)}_{-1}$, we can summarize the rational expectation condition as

$$\tilde{X}_t = E_p[X_{t+1}|F_t]. \quad (42)$$

Again, we evaluate the conditional expectation of (10), and using the fact that $\{A_t\}$ are i.i.d., we have

$$E_p[X_t|F_{t-1}] = E_p[A]X_{t-1}. \quad (43)$$

This equation together with (42), leads to

$$\tilde{X}_{t-1} = E_p[A]X_{t-1}, \quad (44)$$

which can be rewritten

$$\left( E_p[A] - \delta_{-1} \right) X_{t-1} = 0, \quad (45)$$

where $\delta_{-1} = \text{diag}[\delta^{(1)}_{-1} \ldots \delta^{(d)}_{-1}]$ is the matrix whose $i$th diagonal component is $\delta^{(i)}_{-1}$ and 0 elsewhere.

The equation (45) must be true for every $X_{t-1} \in \mathbb{R}^d$, thus

$$E_p[A] = \delta_{-1}, \quad (46)$$

which is the result announced in section (3.2).
APPENDIX C: proof of Proposition 2 on the condition $\kappa_1 < 1$

First step: Behavior of the function

$$f(\kappa) = \lim_{n \to \infty} \frac{1}{n} \ln E_{\mathbb{P}_A} \left| \| A_n \cdots A_1 \| \right|^\kappa_1$$

in the interval $[0, \kappa_0]$.

In [Kesten, 1973], Kesten shows that the function $f$ has the following properties:

- $f$ is continuous on $[0, \kappa_0]$,
- $f(0) = 0$ and $f(\kappa_0) > 0$,
- $f'(0) < 0$ (this results from the stationarity condition),
- $f$ is convex on $[0, \kappa_0]$.

Thus, there is a unique solution $\kappa_1$ in $(0, \kappa_0)$ such that $f(\kappa_1) = 0$. In order to have $\kappa_1 > 1$, it is necessary that $f(1) < 0$, or using the definition of $f$:

$$\lim_{n \to \infty} \frac{1}{n} \ln E_{\mathbb{P}_A} \left| A_n \cdots A_1 \right| < 0.$$  \hfill (48)

The qualitative shape of the function $f(\kappa)$ is shown in figure 1.

Second step:

The operator $\| \cdot \|$ is convex:

$$\forall \alpha \in [0, 1] \text{ and } \forall (A, C) \text{ } d \times d\text{-matrices}, \ |\alpha A + (1-\alpha)C| \leq \alpha \|A\| + (1-\alpha)\|C\| .$$  \hfill (49)

Thanks to Jensen’s inequality, we have

$$E_{\mathbb{P}_A} \| A_n \cdots A_1 \| \geq \| E_{\mathbb{P}_A} [A_n \cdots A_1] \|.$$  \hfill (50)

The matrices $(A_n)$ being i.i.d., we obtain

$$E_{\mathbb{P}_A} \| A_n \cdots A_1 \| \geq \| \{ E_{\mathbb{P}_A} [A] \}^n \|. $$  \hfill (51)

Now, let consider the normalized-eigenvector $x_{\text{max}}$ associated with the largest eigenvalue

$$\lambda_{\text{max}} \equiv \rho(E_{\mathbb{P}_A} [A]) ,$$  \hfill (52)

where $\rho(E_{\mathbb{P}_A} [A])$ is the spectral radius of $A$. By definition,

$$\| \{ E_{\mathbb{P}_A} [A] \}^n \| \geq \| E_{\mathbb{P}_A} [A] \| x_{\text{max}} = \lambda_{\text{max}}^n .$$  \hfill (53)

Then

$$\lim_{n \to \infty} \frac{1}{n} \ln E_{\mathbb{P}_A} \| A_n \cdots A_1 \| \geq \lim_{n \to \infty} \frac{1}{n} \ln \rho(E_{\mathbb{P}_A} [A])^n = \ln \rho(E_{\mathbb{P}_A} [A]) .$$  \hfill (54)

Now, suppose that $\rho(E_{\mathbb{P}_A} [A]) \geq 1$. We obtain

$$f(1) = \lim_{n \to \infty} \frac{1}{n} \ln E_{\mathbb{P}_A} \| A_n \cdots A_1 \| \geq 0,$$  \hfill (55)

which is in contradiction with the necessary condition (48).

Thus,

$$f(1) < 0 \implies \rho(E_{\mathbb{P}_A} [A]) < 1.$$  \hfill (56)
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Figure 1: Schematic shape of the function $f(\kappa)$ defined in (47).