HASSE-WEIL ZETA FUNCTIONS OF $SL_2$-CHARACTER VARIETIES OF CLOSED ORIENTABLE HYPERBOLIC 3-MANIFOLDS

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Abstract. It is proved that the Hasse-Weil zeta functions of the canonical components of the $SL_2$ (PSL$_2$)-character varieties of closed orientable complete hyperbolic 3-manifolds of finite volume are equal to the Dedekind zeta functions of their trace fields (invariant trace fields). When the closed 3-manifold is arithmetic, the special value at $s = 2$ of the Hasse-Weil zeta function of the canonical component of the PSL$_2$-character variety is expressed in terms of the hyperbolic volume of the manifold up to rational numbers.

0. Introduction

For an orientable hyperbolic 3-manifold $M$ of finite volume the $SL_2(\mathbb{C})$-character variety $X(M)(\mathbb{C})$ of $M$ is the set of the characters of the representations of the fundamental group $\pi_1(M)$ into $SL_2(\mathbb{C})$. It is known that $X(M)(\mathbb{C})$ is an affine algebraic set over $\mathbb{Q}$, that is, it is the set of the common zeros of a finite number of polynomials with rational coefficients. Culler and Shalen have shown its importance in the study of 3-manifolds in [10] by constructing essential surfaces in the manifolds attached to the ideal points of the character varieties.

In this paper we establish the following results (the notation and the precise statements are explained below):

- The Hasse-Weil zeta function of the canonical component of the $SL_2$ (PSL$_2$)-character variety of a closed orientable hyperbolic 3-manifold of finite volume is equal to the Dedekind zeta function of the trace field (invariant trace field).
- The special value at $s = 2$ of the Hasse-Weil zeta function of the canonical component of the PSL$_2$-character variety of an arithmetic closed orientable hyperbolic 3-manifold of finite volume is expressed in terms of the hyperbolic volume of the closed 3-manifold up to a rational number.

Despite the importance of the $SL_2(\mathbb{C})$ (PSL$_2(\mathbb{C})$)-character variety of a 3-manifold, the algebro-geometric structure of the character variety is not known well and it does not seem to have simple structure. For instance, in general the dimension of the $SL_2(\mathbb{C})$-character variety does not behave nicely. Even if we consider the $SL_2(\mathbb{C})$-character variety of a hyperbolic knot complement in the 3-sphere it may have an irreducible component with arbitrary large dimension (cf. [23]). If $M$ is a complete hyperbolic 3-manifold with $n$ cusps, its complete hyperbolic structure is determined by the holonomy representation $\rho_M : \pi_1(M) \to PSL_2(\mathbb{C})$ (it is discrete, faithful, especially irreducible representation). By
taking a lift of \( \rho_M \) to \( \text{SL}_2(\mathbb{C}) \) there is an irreducible component (we call it a canonical component) of the character variety \( X(M)(\mathbb{C}) \) containing the character corresponding to the lift. It would contain the geometric information on the hyperbolic structure. In fact, it is proved by Thurston that the dimension of the canonical component \( X(M)(\mathbb{C})_0 \) of \( X(M)(\mathbb{C}) \) is equal to the number of cusps of \( M \). To retrieve further algebro-geometric properties of the canonical components of the character varieties of the hyperbolic 3-manifolds we will precisely determine them arithmetically and investigate their zeta functions.

For that purpose in what follows we will use the following terminology: For an orientable hyperbolic 3-manifold \( M \) of finite volume let \( X(M) \) be the \( \text{SL}_2 \)-character variety of \( M \), namely it is a unique affine reduced scheme of finite type over \( \mathbb{Q} \) such that the set of its \( \mathbb{C} \)-rational points is the \( \text{SL}_2(\mathbb{C}) \)-character variety \( X(M)(\mathbb{C}) \) of \( M \). In other words, there exist polynomials \( f_1, \ldots, f_r \) in \( \mathbb{Q}[T_1, \ldots, T_m] \) satisfying

\[
X(M) = \text{Spec} \left( \mathbb{A}(A := \mathbb{Q}[T_1, \ldots, T_m]/(f_1, \ldots, f_r)) \right)
\]

such that the set \( X(M)(\mathbb{C}) \) of \( \mathbb{C} \)-rational points

\[
X(M)(\mathbb{C}) = \text{Hom} \left( A, \mathbb{C} \right) = \{ (a_1, \ldots, a_m) \in \mathbb{C}^m \mid f_i(a_1, \ldots, a_m) = 0 \text{ for any } 1 \leq i \leq r \}
\]

is the \( \text{SL}_2(\mathbb{C}) \)-character variety of \( M \) in the usual sense in Topology. We denote by \( X_0(M) \) (resp. \( X(M)(\mathbb{C})_0 \)) an irreducible component of \( X(M) \) (resp. \( X(M)(\mathbb{C}) \)) containing the character corresponding to a lift of \( \rho_M \).

To determine the structure of the affine scheme \( X_0(M) \) and to define an action of \( H^1(\pi_1(M), C_2) \) on \( X_0(M) \), we will introduce another scheme \( X_0(M) \), which is considered as a model of \( X_0(M) \) for a closed hyperbolic 3-manifold \( M \).

Let \( X(M) \) be the moduli scheme of absolutely irreducible representations of the group ring \( \mathbb{Z}[\pi_1(M)] \) into Azumaya algebras (whose images are contained in norm 1 subgroups) with degree 2 studied by Procesi. When \( M \) is a complete hyperbolic 3-manifold we denote by \( X_0(M) \) an irreducible component of \( X(M) \) containing the image of the rational point corresponding to a lift of \( \rho_M \) (we call \( X_0(M) \) a canonical component of \( X(M) \) as well as the \( X(M) \)-case). In this paper we prove the following (for the definition of the zeta function see Section 1):

**Theorem 1** (Theorem 3.8). Let \( M \) be a closed orientable complete hyperbolic 3-manifold of finite volume. Then the Hasse-Weil zeta function \( \zeta(X_0(M), s) \) is equal to the Dedekind zeta function \( \zeta(K_M, s) \) of the trace field \( K_M \) up to rational functions in \( p^{-s} \) for finitely many prime numbers \( p \).

Here, the trace field \( K_M \) of \( M \) is the number field of finite degree over \( \mathbb{Q} \) generated by the traces of a lift of the holonomy representation \( \rho_M \) (which does not depend on the choice of a lift of \( \rho_M \)). As a corollary we obtain the following:

**Theorem 2** (Corollary 3.10). Let \( M \) be a closed orientable complete hyperbolic 3-manifold of finite volume. Then the reduced scheme \( X_0(M) \) is unique as a closed subscheme of \( X(M) \), which does not depend on the choice of a lift of the holonomy representation \( \rho_M : \pi_1(M) \to \text{PSL}_2(\mathbb{C}) \), and \( X_0(M) \) is isomorphic to the spectrum \( \text{Spec} K_M \) of the trace field \( K_M \). Therefore the Hasse-Weil zeta function \( \zeta(X_0(M), s) \) is equal to the Dedekind zeta function \( \zeta(K_M, s) \).
Since we could determine $X_0(M)$ it also is possible to determine the Hasse-Weil zeta function of $\text{PSL}_2$-character variety of a closed hyperbolic 3-manifold (in the sense of Heusener and Porti) as follows.

**Theorem 3** (Theorem 3.16). Let $M$ be an orientable closed hyperbolic 3-manifold of finite volume. Let $C_2 := \{ \pm 1 \}$ be the finite group of order 2 and $H^1(\pi_1(M), C_2) := \text{Hom}(\pi_1(M), C_2)$. Then $H^1(\pi_1(M), C_2)$ acts on $X_0(M)$ and the quotient scheme $\tilde{X}_0(M) := X_0(M)/H^1(\pi_1(M), C_2)$ is isomorphic to the spectrum $\text{Spec}(\text{Inv}K_M)$ of the invariant trace field $\text{Inv}K_M$. Therefore the Hasse-Weil zeta function $\zeta(\tilde{X}_0(M), s)$ is equal to the Dedekind zeta function $\zeta(\text{Inv}K_M, s)$ of the invariant trace field $\text{Inv}K_M$ of $M$.

Here the invariant trace field $\text{Inv}K_M$ is a subfield of the trace field $K_M$ generated by the traces of the squares of the image of the holonomy representation.

There is a one-to-one correspondence between the set of conjugacy classes of the lifts of the holonomy representation $\rho_M : \pi_1(M) \to \text{PSL}_2(\mathbb{C})$ and the cohomology group $H^1(\pi_1(M), C_2) = \text{Hom}(\pi_1(M), C_2)$. We can show (Lemma 3.12) that the cardinality of $H^1(\pi_1(M), C_2)$ is equal to $[K_M : \text{Inv}K_M]$. Thus we deduce the following:

**Theorem 4** (Corollary 3.17). Let $M$ be a closed oriented complete hyperbolic 3-manifold of finite volume. Then the number of canonical components $X(M)(\mathbb{C})_0$ of the $\text{SL}_2(\mathbb{C})$-character variety $X(M)(\mathbb{C})$ is equal to $[K_M : \text{Inv}K_M] = \#H^1(\pi_1(M), C_2)$.

For arithmetic 3-manifolds it is well-known as Borel’s formula (see Theorem 3.19) that the hyperbolic volumes are expressed, especially in terms of the special values at 2 of the Dedekind zeta functions of the invariant trace fields. Hence we obtain the following corollary.

**Theorem 5** (Corollary 3.20). Let $M$ be an arithmetic orientable closed hyperbolic 3-manifold. Then the special value $\zeta(\tilde{X}_0(M), 2)$ is expressed in terms of the hyperbolic volume $\text{Vol}(M)$ and $\pi$ as follows:

$$\zeta(\tilde{X}_0(M), 2) \sim_{\text{Q}} \frac{(4\pi^2)^{|\text{Inv}K_M : \mathbb{Q}| - 1}\text{Vol}(M)}{|\Delta_{\text{Inv}K_M}|^{3/2}},$$

where $\sim_{\text{Q}}$ means the equality holds up to a rational number.

For hyperbolic knot complements it is confirmed by some examples and is conjectured that the $A$-polynomials of hyperbolic knots would relate with the hyperbolic volumes by considering their Mahler measures (cf. [4], [3]). Unlike the closed 3-manifold case, the canonical component of the $\text{PSL}_2(\mathbb{C})$-character variety of any hyperbolic twist knot complement in the 3-sphere $S^3$ is isomorphic to the projective line $\mathbb{P}_\mathbb{C}^1$ ([19]). Thus it seems that we cannot expect a similar result on a relation between the hyperbolic volume and the special value of the zeta function of the $\text{PSL}_2$-character variety of a 1-cusped hyperbolic 3-manifold.

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1. Hasse-Weil zeta function

1.1. Hasse-Weil zeta function of a scheme. Here we review some basic facts on the Hasse-Weil zeta functions of schemes over \( \mathbb{Z} \). For details see [20].

In this subsection \( X \) is a scheme of finite type over \( \mathbb{Z} \). The dimension of \( X \) is the maximal length of a chain of closed irreducible subschemes of \( X \)
\[
X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X, \quad X_i \neq X_{i+1}.
\]

Let \( \overline{X} \) be the set of closed points of \( X \) and \( N(x) = |(k(x))| \), where \( k(x) \) is the residue field at \( x \in \overline{X} \). A point \( x \in X \) is a closed point if and only if its residue field \( k(x) \) is a finite field.

**Lemma 1.1.** There are only finitely many closed points of \( X \) which have the same isomorphic residue field.

**Proof.** Since \( X \) is of finite type over \( \mathbb{Z} \), we can reduce to the case where \( X \) is an affine scheme of finite type over \( \mathbb{Z} \). Hence it is enough to consider the case \( X = \text{Spec}(\mathbb{F}_p[X_1, \cdots, X_r]) \).

It follows from Zariski’s lemma that any maximal ideal of \( \mathbb{F}_p[X_1, \cdots, X_r] \) is generated by the elements \( f_1, \cdots, f_r \) such that \( f_i \) is in \( \mathbb{F}_p[X_1, \cdots, X_i] \) and \( f_i \) is irreducible in the quotient ring \( \mathbb{F}_p[X_1, \cdots, X_i]/(f_1, \cdots, f_i-1) \) for any \( 1 \leq i \leq r \). Then we see that there are finitely many maximal ideals in \( \mathbb{F}_p[X_1, \cdots, X_r] \) with the same residue field since there are only finitely many possibilities of the tuples of polynomials \( (f_1, \cdots, f_r) \) with given degree. \( \Box \)

Therefore the set \( X(\mathbb{F}_{p^n}) := \text{Hom}(\text{Spec}(\mathbb{F}_{p^n}), X) \) of \( \mathbb{F}_{p^n} \)-rational points of \( X \) is a finite set for any prime power \( p^n \). The Hasse-Weil zeta function \( \zeta(X, s) \) of \( X \) is defined by
\[
\zeta(X, s) := \prod_{x \in \overline{X}} \frac{1}{1 - N(x)^{-s}}.
\]

The function \( \zeta(X, s) \) converges absolutely on \( \Re(s) > \dim X \). Note that there is another expression of \( \zeta(X, s) \) as follows.

**Lemma 1.2.**
\[
\zeta(X, s) = \prod_{p \text{ prime}} Z(X, p, p^{-s}),
\]
where
\[
Z(X, p, T) = \exp \left( \sum_{n=1}^{\infty} \frac{|X(\mathbb{F}_{p^n})|}{n} T^n \right).
\]

**Proof.** By the definition we see that \( |X(\mathbb{F}_{p^n})| = \sum_{1 \leq r | p^n} a_r \), where \( a_r \) is the number of closed points \( x \in \overline{X} \) whose residue fields are isomorphic to \( \mathbb{F}_{p^n} \). Note that
\[
\exp \left( \sum_{n=1}^{\infty} \frac{1}{n} T^n \right) = (1 - T)^{-1}.
\]

Hence we have
\[
\zeta(X, s) = \prod_{p} \prod_{n=1}^{\infty} (1 - p^{-ns})^{-a_n} = \prod_{p} \prod_{n=1}^{\infty} \exp \left( \sum_{r=1}^{\infty} \frac{a_n}{r} (p^{-ns})^r \right).
\]
Therefore we have
\[
\prod_{p} \prod_{n=1}^{\infty} \exp \left( \sum_{r=1}^{\infty} \frac{a_n}{r} (p^{-ns})^r \right) = \prod_{p} \exp \left( \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_n}{r} (p^{-ns})^r \right)
\]
\[
= \prod_{p} \exp \left( \sum_{n=1}^{\infty} \sum_{1 \leq m/n \in \mathbb{Z}} \frac{na_n}{m} p^{-ms} \right) \quad \text{(put } m = nr)\]
\[
= \prod_{p} \exp \left( \sum_{m=1}^{\infty} \frac{\#X(F_p^n)}{m} p^{-ms} \right).
\]

\[\square\]

Let \(X_{\text{red}}\) be the reduced scheme of \(X\). Since \(X_{\text{red}}(\mathbb{F}_p^p) \rightarrow X(\mathbb{F}_p^p)\), we have \(\zeta(X_{\text{red}}, s) = \zeta(X, s)\).

**Example 1.3.** Let
\[
\zeta(K, s) = \prod_{0 \neq p \subset \mathbb{N}_K} (1 - N(p)^{-s})^{-1}
\]
be the Dedekind zeta function of a number field \(K/\mathbb{Q}\). Here we denote by \(\mathcal{O}_K\) the ring of integers of \(K\), \(p\) is a non-zero prime ideal of \(K\) and \(N(p) = \#(\mathcal{O}_K/p)\).

- (Dedekind zeta function)
  \[
  \zeta(\text{Spec } (\mathcal{O}_K), s) = \zeta(K, s).
  \]
  Especially \(\zeta(\text{Spec } (\mathbb{Z}), s) = \zeta(s)\) is the Riemann zeta function.

- (Affine space, Projective space) For the affine space \(\mathbb{A}^n_{\mathbb{Z}} = \text{Spec } \mathbb{Z}[T_1, \cdots, T_n]\) and the projective space \(\mathbb{P}^n_{\mathbb{Z}} = \text{Proj } \mathbb{Z}[T_0, \cdots, T_n]\)
  \[
  \zeta(\mathbb{A}^n_{\mathbb{Z}}, s) = \zeta(s - n).
  \]
  \[
  \zeta(\mathbb{P}^n_{\mathbb{Z}}, s) = \zeta(s - n)\zeta(s - (n - 1)) \cdots \zeta(s).
  \]

**Proposition 1.4.** Let \(K\) be a finite number field and \(\mathcal{O}_K\) the ring of integers of \(K\). Let \(\mathcal{O} \subset \mathcal{O}_K\) be an order of \(K\). Then \(\zeta(\text{Spec } (\mathcal{O}), s)\) is equal to \(\zeta(\text{Spec } (\mathcal{O}_K), s)\) up to rational functions in \(p^{-s}\) for finitely many prime numbers \(p\).

**Proof.** Since \(\mathcal{O}\) is an order of \(K\), it is of finite index in \(\mathcal{O}_K\). Note that there are bijective correspondence between the prime ideals of \(\mathcal{O}\) and those of \(\mathcal{O}_K\) lying on prime numbers \(p \nmid [\mathcal{O}_K : \mathcal{O}]\) (cf. [27], Example 3.2). Hence \(\zeta(\text{Spec } (\mathcal{O}), p, T)\) is equal to \(\zeta(\text{Spec } (\mathcal{O}_K), p, T)\) for any \(p \nmid [\mathcal{O}_K : \mathcal{O}]\). Therefore \(\zeta(\text{Spec } (\mathcal{O}), s)\) is equal to \(\zeta(\text{Spec } (\mathcal{O}_K), s)\) up to rational functions in \(p^{-s}\) for \(p \mid ([\mathcal{O}_K : \mathcal{O}])\).

\[\square\]

1.2. **Hasse-Weil zeta function of the character variety.** Since \(\text{SL}_2(\mathbb{C})\)-character variety \(X(M)(\mathbb{C})\) is an affine algebraic set over \(\mathbb{Q}\), there is a unique reduced affine scheme \(X(M)\) of finite type over \(\mathbb{Q}\) such that the set of its \(\mathbb{C}\)-rational points is isomorphic to \(X(M)(\mathbb{C})\). We will call \(X(M)\) the \(\text{SL}_2\)-character variety of \(M\). (For the existence of such scheme, see for instance [17], Lemma 3.2.6.)

Now we define the Hasse-Weil zeta function of the \(\text{SL}_2\)-character variety \(X(M)\) of a hyperbolic 3-manifold \(M\). It is defined by the Hasse-Weil zeta function in the preceding subsection in terms of a model of \(X(M)\).
Since $X(M)$ is an affine algebraic set over $\mathbb{Q}$ there exist polynomials $f_1, \cdots, f_r$ in $\mathbb{Q}[T_1, \cdots, T_m]$ such that $X(M)$ is written as

$$X(M) = \text{Spec } \mathbb{Q}[T_1, \cdots, T_m]/(f_1, \cdots, f_r).$$

By multiplying the above polynomials by some positive integer we can replace $f_1, \cdots, f_r$ by polynomials $f'_1, \cdots, f'_r$ in $\mathbb{Z}[T_1, \cdots, T_m]$. Let $X$ be the scheme defined by $f'_1, \cdots, f'_r$:

$$X = \text{Spec } \mathbb{Z}[T_1, \cdots, T_i]/(f'_1, \cdots, f'_r).$$

Then define $\zeta(X(M), s)$ by $\zeta(X, s)$.

**Proposition 1.5.** The function $\zeta(X(M), s)$ is well-defined up to rational functions in $p^{-s}$ for finitely many prime numbers $p$.

**Proof.** Given a system of polynomials $f_1, \cdots, f_r$ in $\mathbb{Q}[T_1, \cdots, T_m]$, let $N, M$ be positive integers which annihilate the denominators of the polynomials. Then it is obvious that the systems $(Nf)$ and $(Mf)$ have the same zero set in $\mathbb{F}_p$ for any prime $p \not| NM$ and $n \geq 1$. Hence the zeta functions defined by them are identical up to local factors for $p \not| NM$.

Take two systems of defining polynomials for $X(M)$, namely

$$X(M) = \text{Spec } \mathbb{Q}[T_1, \cdots, T_m]/(f_1, \cdots, f_r) \rightarrow \text{Spec } \mathbb{Q}[U_1, \cdots, U_n]/(g_1, \cdots, g_s),$$

where we can assume that $f_1, \cdots, f_r$ and $g_1, \cdots, g_s$ are integer coefficients. Now we have an isomorphism of $\mathbb{Q}$-algebras

$$\mathbb{Q}[T_1, \cdots, T_m]/(f_1, \cdots, f_r) \rightarrow \mathbb{Q}[U_1, \cdots, U_n]/(g_1, \cdots, g_s).$$

Let $\tilde{T}_i \in \mathbb{Q}[U_1, \cdots, U_n]$ (resp. $\tilde{U}_j \in \mathbb{Q}[T_1, \cdots, T_m]$) be a representative of the image of $T_i$ (resp. $U_j$) by the above isomorphism. If $\tilde{T}_i \in \mathbb{Q}[T_1, \cdots, T_m]$ (resp. $\tilde{U}_j \in \mathbb{Q}[U_1, \cdots, U_n]$) is the element obtained by substituting $\tilde{U}_j$ (resp. $\tilde{T}_i$) into $T_i$ (resp. $U_j$), we have

$$\tilde{T}_i \in T_i + (f_1, \cdots, f_r)\mathbb{Q}[T_1, \cdots, T_m], \quad \tilde{U}_j \in U_j + (g_1, \cdots, g_s)\mathbb{Q}[U_1, \cdots, U_n].$$

Hence there is a positive integer $N > 0$ such that

$$NT_i \in NT_i + (f_1, \cdots, f_r)\mathbb{Z}[T_1, \cdots, T_m], \quad NU_j \in NU_j + (g_1, \cdots, g_s)\mathbb{Z}[U_1, \cdots, U_n].$$

Let $\tilde{f}_i$ (resp. $\tilde{g}_j$) be the element obtained from $f_i$ (resp. $g_j$) by substituting $\tilde{T}_i$ (resp. $\tilde{U}_j$) into $T_i$ (resp. $U_j$). Then we have a matrix presentation

$$(\tilde{f}_1, \cdots, \tilde{f}_r) = (g_1, \cdots, g_s)A, \quad (\tilde{g}_1, \cdots, \tilde{g}_s) = (f_1, \cdots, f_r)B$$

for $A \in \mathbb{M}_{r,s}(\mathbb{Q}[U_1, \cdots, U_n])$ (resp. $B \in \mathbb{M}_{r,s}(\mathbb{Q}[T_1, \cdots, T_m])$). Let $M > 0$ be a positive integer of the l.c.m. of all the denominators of the coefficients of the elements in the above matrix presentations.

Now we see that, if $p$ is a prime number not dividing $NM$, then the above isomorphism induces an isomorphism

$$\mathbb{F}_p[T_1, \cdots, T_m]/(f_1, \cdots, f_r) \rightarrow \mathbb{F}_p[U_1, \cdots, U_n]/(g_1, \cdots, g_s)$$

which sends $T_i$ to $\tilde{T}_i$ and $U_j$ to $\tilde{U}_j$ respectively. This implies that the local zeta function $Z(f_1, \cdots, f_r, p, T)$ and $Z(g_1, \cdots, g_s, p, T)$ are equal for any $p \not| NM$. Therefore we have proved the proposition. □
2. Review of Moduli theory of Procesi

2.1. Universal representation ring and scheme. The universal representation ring $A_d(R)$ of a (non-commutative) ring $R$ is a commutative ring which parametrizes all the representations of $R$ with degree $d$ over commutative rings. Here we review its construction for an arbitrary associative ring (cf. [24], §1).

For any (non-commutative) ring $R$ it is written as $R = \mathbb{Z}\langle x_k \mid k \in S \rangle/\mathfrak{I}$, where $\mathbb{Z}\langle x_k \mid k \in S \rangle$ is the non-commutative polynomial ring of indeterminant $x_k$ with index set $S$ and $\mathfrak{I}$ a two-sided ideal of $\mathbb{Z}\langle x_k \mid k \in S \rangle$. Let $\mathbb{Z}[X^k_{ij}]$ be the (commutative) polynomial ring over $\mathbb{Z}$ (we write $\mathbb{Z}[X^k_{ij}]$ instead of $\mathbb{Z}[x_k]_{1 \leq i, j \leq d, k \in S}$ for short). Then we have the following canonical ring homomorphism

$$\rho : \mathbb{Z}\langle x_k \mid k \in S \rangle \rightarrow M_d(\mathbb{Z}[X^k_{ij}]) \quad x_k \mapsto (X^k_{ij})_{ij}. $$

Let $J$ be the two-sided ideal of $M_d(\mathbb{Z}[X^k_{ij}])$ generated by $\rho(\mathfrak{I})$. Then the ideal $J$ is written as $M_d(J)$, where $J$ is an ideal of $\mathbb{Z}[X^k_{ij}]$ defined by

$$J := \{ a \in \mathbb{Z}[X^k_{ij}] \mid a \text{ is an entry of some } M \in J \}. $$

Thus the above homomorphism induces the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{Z}\langle x_k \mid k \in S \rangle & \xrightarrow{\rho} & M_d(\mathbb{Z}[X^k_{ij}]) \\
\downarrow & & \downarrow \\
R & \xrightarrow{\rho_{d,R}} & M_d(A_d(R)),
\end{array}$$

where $A_d(R)$ is the quotient ring $\mathbb{Z}[X^k_{ij}]/J$. We call $\rho_{d,R}$ the universal representation of $R$ with degree $d$ and $A_d(R)$ the universal representation ring of $R$ with degree $d$.

**Proposition 2.1.** The covariant functor

$$\mathcal{R} : \text{(Comm.Rings)} \rightarrow \text{(Sets)}$$

$$A \mapsto \text{Hom}(R, M_d(A))$$

from the category of commutative rings into the category of sets is represented by $A_d(R)$, that is, we have $\text{Hom}(R, M_d(A)) \rightarrow \text{Hom}(A_d(R), A)$ for any commutative ring $A$.

**Proof.** Let $\rho : R \rightarrow M_d(A)$ be a representation of $R$ into $M_d(A)$. Define a ring homomorphism $f : \mathbb{Z}[X^k_{ij}] \rightarrow A$ by $f(X^k_{ij}) := (\rho(x_k))_{ij}$. Here $(\rho(x_k))_{ij}$ means the $(i,j)$-entry of the matrix $\rho(x_k)$. By the definition of $A_d(R)$, this induces a ring homomorphism $\tilde{f} : A_d(R) \rightarrow A$. It is easy to see that this correspondence induces a bijection between $\text{Hom}(R, M_d(A))$ and $\text{Hom}(A_d(R), A)$. \hfill $\square$

If $R$ is a finitely generated (non-commutative) ring, it is clear by construction that the universal representation ring $A_d(R)$ is a finitely generated $\mathbb{Z}$-algebra. We call the spectrum $X_d(R) := \text{Spec}(A_d(R))$ the universal representation scheme of $R$. If $R$ is the group ring $\mathbb{Z}[G]$ for a group $G$, we write $A_d(G)$ (resp. $X_d(G)$) for $A_d(R)$ (resp. $X_d(R)$).

Let $A'_d(G)$ be the quotient ring of $A_d(G)$ by the ideal generated by $\det(X^k_{ij}) - 1$ for all $k \in S$ and $\rho'_{d,G}$ the composite homomorphism of $\rho_{d,G} := \rho_{d,\mathbb{Z}[G]}$ and the projection
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\[ M_d(A_d(G)) \rightarrow M_d(A'_d(G)). \] Since \( \text{Hom}(\mathbb{Z}[G], M_d(A)) \) is identified with \( \text{Hom}(G, \text{GL}_d(A)) \), we also have the following:

**Proposition 2.2.** The covariant functor

\[ \mathcal{R} : (\text{Comm.Rings}) \rightarrow (\text{Sets}) \]

\[ A \mapsto \text{Hom}(G, \text{SL}_d(A)) \]

from the category of commutative rings into the category of sets is represented by \( A'_d(G) \), that is, we have \( \text{Hom}(G, \text{SL}_d(A)) \cong \text{Hom}(A'_d(G), A) \) for any commutative ring \( A \).

**Proof.** For a representation \( \rho : G \rightarrow \text{GL}_d(A) \), let \( f : A_d(G) \rightarrow A \) be the corresponding ring homomorphism associated with \( \mathbb{Z}[\rho] : \mathbb{Z}[G] \rightarrow M_d(A) \). It is obvious that \( \text{Im}(\rho) \) is contained in \( \text{SL}_d(A) \) if and only if \( f \) factors through \( A'_d(G) \). \( \square \)

2.2. **Moduli theory of Procesi.** Let \( R \) be a (non-commutative) associative ring. Here we briefly review the moduli theory of Procesi on absolutely irreducible representations of \( R \) into Azumaya algebras. For details, refer to the original paper (in particular \( \S 1, 2 \)) of Procesi ([24]) or \( \S 1, 2, 3 \) of [28] where the theory is discussed in a more general setting. Once we assume the theory of Procesi its \( \text{SL}_n \)-version (put restriction on the determinant) of the theory is immediately obtained (Theorem 2.15). First we collect some facts on Azumaya algebras. For the proofs, see for instance [16].

**Definition 2.3.** Let \( A \) be a commutative ring. We say that an \( A \)-algebra \( S \) is an Azumaya algebra of degree \( d \) if the following conditions are satisfied:

1. \( S \) is a finitely generated projective \( A \)-module of rank \( d^2 \),
2. the natural homomorphism \( S \otimes_A S^\circ \rightarrow \text{End}_A(S) \) given by \( s \otimes s' \mapsto (t \mapsto st s') \), is an isomorphism,

where \( S^\circ \) is the opposite ring of \( S \).

**Example 2.4.** The total matrix algebra \( M_d(A) \) is an Azumaya algebra of degree \( d \) over \( A \). If \( A \) is a field, an Azumaya algebra over \( A \) is just a central simple algebra over \( A \). Here a central simple algebra \( S \) over a field \( A \) is a finite dimensional \( A \)-algebra such that \( S \) has no non-trivial two sided ideal and the center \( C(S) \) is equal to \( A \).

Here we list some basic properties of Azumaya algebras.

**Proposition 2.5.** Let \( S \) be a finitely generated \( A \)-module and \( f : A \rightarrow B \) a ring homomorphism.

1. If \( S \) is projective, then \( S \otimes_A B \) is also projective. If \( f \) is faithfully flat, then the converse is also true.
2. If \( S \) is an Azumaya algebra of degree \( d \) over \( A \), then \( S \otimes_A B \) is also an Azumaya algebra of degree \( d \) over \( B \). If \( f \) is faithfully flat, then the converse is also true.

**Proposition 2.6.** Let \( S \) be a finitely generated \( A \)-module. Let \( a_1, \ldots, a_r \) be elements of \( A \) such that \( A = (a_1, \ldots, a_r)_A \). Then the canonical ring homomorphism \( A \rightarrow \prod_i A[1/a_i] \) is a faithfully flat homomorphism. Thus \( S \) is an Azumaya algebra of degree \( d \) over \( A \) if and only if \( S \otimes_A A[1/a_i] \) is an Azumaya algebra of degree \( d \) over \( A[1/a_i] \) for any \( i \).
Let \( S \) be an Azumaya algebra of degree \( d \) over a commutative ring \( A \). Then there is a faithfully flat homomorphism \( f : A \to C \) such that \( S \otimes_A C \) is isomorphic to \( M_d(C) \). (We call \( f \) a splitting of \( S \).)

**Remark 2.8.** Since \( f \) is faithfully flat, it is injective and the structure homomorphism \( A \to S \) is also injective. Moreover, \( S \) is identified with a subring of \( M_d(C) \) and \( S \cap C = A \) is the center of \( S \).

**Proposition 2.9.** Let \( S \) be an Azumaya algebra of degree \( d \) over \( A \). Then there is a surjective \( A \)-module homomorphism \( \text{Tr} := \text{Tr}_{S/A} : S \to A \). If \( A \to C \) is a splitting of \( S \), then \( \text{Tr} \otimes_A C : M_d(C) \to C \) is equal to the trace map of \( M_d(C) \). There is also a map \( N := N_{S/A} : S \to A \) such that the restriction on \( S^\times \), that is \( N|_{S^\times} : S^\times \to A^\times \) is a group homomorphism. If \( A \to C \) is a splitting of \( S \), then \( N \otimes_A C : M_d(C) \to C \) is equal to the norm map of \( M_d(C) \).

We call \( \text{Tr} \) (resp. \( N \)) the reduced trace (resp. reduced norm) on \( S \). Let \( T_d(R) \) be the subring of \( A_d(R) \) generated by \( \text{Tr}(\text{Im}(\rho_{d,R})) \) and \( S_d(R) \) the subring of \( M_d(A_d(R)) \) which is generated by \( T_d(R) \) and \( \text{Im}(\rho_{d,R}) \). For any \( d^2 \)-tuple \( r = (r_i)_{1 \leq i \leq d^2} \) of elements of \( R \), denote by \( d = d(r) \) the determinant \( \det(\text{Tr}(\rho_{d,R}(r_i)\rho_{d,R}(r_j))) \in T_d(R) \). We call \( d \) a discriminant of \( r \). Let \( S_{d}(R)[1/d] \) denote the localization \( S_{d}(R) \otimes_{T_d(R)} T_d(R)[1/d] \). Then \( (\rho_{d,R}(r_i))_{1 \leq i \leq d^2} \) is a \( T_{d}(R)[1/d] \)-basis of \( S_{d}(R)[1/d] \).

**Theorem 2.10** ([24], 2.2,Theorem). \( S_{d}(R)[1/d] \) is an Azumaya algebra of degree \( d \) over \( T_{d}(R)[1/d] \).

**Definition 2.11.** Let \( T_{d}(R) \) be the open subscheme of \( \text{Spec}(T_{d}(R)) \) covered by the affine open subschemes \( \text{Spec}(T_{d}(R)[1/d]) \), where \( d = d(r) \) runs through all the \( d^2 \)-tuples \( r = (r_i)_{1 \leq i \leq d^2} \) of elements of \( R \).

Note that if \( R \) is finitely generated over \( \mathbb{Z} \), then the scheme \( T_{d}(R) \) is of finite type over \( \mathbb{Z} \).

**Definition 2.12.** Let \( R \) be a (non-commutative) associative ring. Let \( S \) be an Azumaya algebra of degree \( d \) over \( A \). A ring homomorphism \( \rho : R \to S \) is called an absolutely irreducible representation of degree \( d \) over \( A \), if \( S \) is generated by \( \text{Im}(\rho) \) as an \( A \)-module. Two absolutely irreducible representations \( \rho_1 : R \to S_1 \) and \( \rho_2 : R \to S_2 \) over \( A \) are equivalent if there exists an \( A \)-algebra isomorphism \( f : S_1 \to S_2 \) such that \( \rho_2 = f \circ \rho_1 \).

**Remark 2.13.** Let \( k \) be a field. Let \( \rho : G \to \text{GL}_d(k) \) be a representation and \( k[\rho] : k[G] \to M_d(k) \) an associated ring homomorphism. It is known that \( \rho \) is absolutely irreducible (that is, the composition \( \rho : G \to \text{GL}_d(k) \to \text{GL}_d(\overline{k}) \) is irreducible for an algebraic closure \( \overline{k} \) of \( k \)) if and only if \( k[\rho] \) is absolutely irreducible in the above sense (cf. [2], §13, Prop. 5).

Let \( \mathcal{F}_{R,d} : (\text{Comm. Rings}) \to (\text{Sets}) \) be the functor which sends a commutative ring \( A \) to the set \( \mathcal{F}_{R,d}(A) \) of equivalence classes of absolutely irreducible representations of \( R \) of degree \( d \) over \( A \). Then the following result has been obtained by Procesi ([24]):

**Theorem 2.14** ([24], 2.2,Theorem). The functor \( \mathcal{F}_{R,d} \) is representable by the scheme \( T_{d}(R) \).

Here we only describe the correspondence between the sets \( \mathcal{F}_{R,d}(A) \) and \( T_{d}(R)(A) \) for any commutative ring \( A \). Let \( \rho : R \to S \) be an element of \( \mathcal{F}_{R,d}(A) \), i.e. (an isomorphism
class of) an absolutely irreducible representation of $R$ into an Azumaya algebra $S$ of degree $d$ over $A$. Then there is a faithfully flat ring homomorphism $A \hookrightarrow C$ such that $S \hookrightarrow S \otimes_A C \to M_d(C)$. Thus there is a unique ring homomorphism $f : A_d(R) \to C$ which induces a commutative diagram

$$
\begin{array}{ccc}
R & \to & S_d(R) \\
\downarrow & & \downarrow \\
S & \hookrightarrow & M_d(C).
\end{array}
$$

Since $S$ is an Azumaya algebra of degree $d$, there is a $d^2$-tuple $r = (r_i)_{1 \leq i \leq d^2}$ of elements of $R$ such that they generate $S$ over $A$. Put $\mathbf{d} = \mathbf{d}(r)$. Then $\det(\text{Tr}(r^i_r))$ is invertible in $A$. Thus $f : f : T_d(R) \to A$ induces $T_d(R)[1/\mathbf{d}] \to A$. This defines an $A$-rational point $\text{Spec}(A) \to \text{Spec}(T_d(R)[1/\mathbf{d}]) \to T_d(R)$.

Conversely, given an $A$-rational point $\text{Spec}(A) \to T_d(R)$, we have a ring homomorphism $T_d(R)[1/\mathbf{d}] \to A$ for some $\mathbf{d} = \mathbf{d}(r)$. Then we have an absolutely irreducible representation $\rho : R \to S_d(R)[1/\mathbf{d}] \to S_d(R)[1/\mathbf{d}] \otimes_{T_d(R)[1/\mathbf{d}]} A$ of degree $d$ over $A$, since $S_d(R)[1/\mathbf{d}]$ is an Azumaya algebra of degree $d$ over $T_d(R)[1/\mathbf{d}]$.

Now we put $R = \mathbb{Z}[G]$. For every Azumaya algebra $S$ over a commutative ring $A$, let $S^1$ be the kernel of the reduced norm $N_{S[A]} : S^* \to A^*$. Let

$$
\mathcal{F}_{G,d} : (\text{Comm. Rings}) \to (\text{Sets})
$$

be a functor which sends a commutative ring $A$ to the set of isomorphism classes of absolutely irreducible representations $\rho : R \to S$ of $R$ into Azumaya algebras over $A$ of degree $d$ such that $\rho(G)$ is contained in $S^1$. Note that $\mathcal{F}_{G,d}$ is a subfunctor of $\mathcal{F}_{\mathbb{Z}[G],d}$.

**Theorem 2.15.** The functor $\mathcal{F}_{G,d}$ is representable by a closed subscheme $\mathbb{T}_d(G)$ of $\mathbb{T}_d(G)$. (Here we write $\mathbb{T}_d(G)$ instead of $\mathbb{T}_d(R)$.)

**Proof.** Let $A'_{d}(G)$ be the quotient ring of $A_{d}(G)$ by the ideal generated by the elements $\det(\rho_{G}(g)) - 1$ as before. We denote by $T'_{d}(G)$ (resp. $S'_{d}(G)$) the subring of $A'_{d}(G)$ generated by the traces of $\text{Im}(\rho'_{d,g})$ (resp. the subring of $A'_{d}(G)$ generated by $T'_{d}(G)$ and $\text{Im}(\rho'_{d,G})$), which is a quotient ring of $T_d(G)$ (resp. $S_d(G)$). Let $\mathbb{T}'_d(G)$ be the closed subscheme of $\mathbb{T}_d(G)$ covered by the affine open subschemes $T'_{d}(G)[1/\mathbf{d}]$. Now we prove that this is the scheme which represents the functor $\mathcal{F}_{G,d}$. Let $\rho : R \to S$ be an element of $\mathcal{F}_{\mathbb{Z}[G],d}(A)$, i.e. (an isomorphism class of) an absolutely irreducible representation of $\mathbb{Z}[G]$ into an Azumaya algebra $S$ of degree $d$ over $A$. As we see above, there is a faithfully flat ring homomorphism $f : A_{d}(G) \to C$ where $C$ is a splitting of $S$ over $A$. By Theorem 2.14 we have a corresponding $A$-rational point $T_d(G)[1/\mathbf{d}] \to A$ of $\mathbb{T}_d(G)$ for a suitable discriminant $\mathbf{d}$. Now we have the following commutative diagram:

$$
\begin{array}{ccc}
M_d(A_d(G)) & \xhookrightarrow{\text{inj.}} & S_d(G) \\
\downarrow & & \downarrow \\
M_d(C) & \xhookrightarrow{\text{inj.}} & S_d(G)[1/\mathbf{d}] \\
\downarrow & & \downarrow \\
\mathbb{Z}[G] & \xrightarrow{\rho} & S \\
\downarrow & & \downarrow \\
\text{inj.} & & \text{isom.} \\
\mathbb{Z}[G] & \xrightarrow{\rho} & S_d(G)[1/\mathbf{d}] \otimes_{T_d(G)[1/\mathbf{d}]} A.
\end{array}
$$
Therefore we see that \( \rho \) is in \( \mathcal{F}_{G,d}(A) \) if and only if \( f_\rho : A_d(G) \to A \) factors through \( A'_d(G) \). Thus the statement follows.

\[ \square \]

**Corollary 2.16.** \( T_{\pi}^n(G)(\mathbb{C}) \to \mathcal{F}_{G,d}(\mathbb{C}) \) is equal to the set of conjugacy classes of irreducible representations of \( G \) into \( \text{SL}_d(\mathbb{C}) \). In particular, it is equal to the set of irreducible characters of \( \text{SL}_d(\mathbb{C}) \)-representations of \( G \) (cf. [22], Theorem 6.12).

Therefore we can regard \( T_{\pi}^n(\pi_1(M))(\mathbb{C}) \to \mathcal{F}_{\pi_1(M),2}(\mathbb{C}) \) as the open subset \( X_{\text{tr}}(M)(\mathbb{C}) \) consisting of all the irreducible characters of the \( \text{SL}_2(\mathbb{C}) \)-character variety \( X(M)(\mathbb{C}) \) of \( \pi_1(M) \) for a 3-manifold \( M \).

3. **Hasse-Weil zeta functions of character varieties**

3.1. **Hasse-Weil zeta functions of \( \text{SL}_2 \)-character varieties.** Let \( M \) be an orientable complete hyperbolic 3-manifold of finite volume and \( X(M) \) the moduli scheme \( T_{\pi_1}^n(\pi_1(M)) \) as in the previous section. Let \( X_0(M) \) be an irreducible component containing the point corresponding to a (fixed) lift \( \rho \) of the holonomy representation \( \rho_M : \pi_1(M) \to \text{PSL}_2(\mathbb{C}) \).

First we prove a result on the dimension of the canonical component corresponding to the following theorem for the \( \text{SL}_2(\mathbb{C}) \)-character variety.

**Theorem 3.1** (Thurston(cf. [29], [11])). Let \( M \) be an orientable complete hyperbolic 3-manifold of finite volume with \( n \) cusps. Then we have

\[ \dim X(M)(\mathbb{C})_0 = n. \]

**Corollary 3.2.** \( \dim X(M)(\mathbb{C})_0 = n \) implies \( \dim X_0(M) = n \).

**Proof.** Let \( L \) be the Galois closure of \( K_M \) over \( \mathbb{Q} \) in an algebraic closure \( \overline{\mathbb{Q}} \) of \( K_M \). Since the holonomy point is a smooth point of \( X(M)(\mathbb{C}) \) (when \( n = 0 \), it is clear. When \( n > 0 \), see [11], Appendix B), the point in \( X_0(M) \) is also smooth since it is an irreducible component of \( X(M) \). Let \( U \subset X_0(M) \) be the regular (smooth) locus of \( X_0(M) \), which is an open subset of \( X_0(M) \) containing the holonomy point. Let \( U_L := U \otimes_{\mathbb{Q}} L \) be the base change of \( U \). Note that \( U_L \) is dense in \( (X_0(M))_L := X_0(M) \otimes_{\mathbb{Q}} L \) since \( U \) is dense in \( X_0(M) \). Hence we see that \( \dim U_L = \dim(X_0(M))_L = \dim X_0(M) \). Therefore it is enough to show that \( \dim U_L = n \).

Note that, if the Galois group \( \text{Gal}(L/\mathbb{Q}) \) acts on the set of irreducible components (= connected components since \( U_L \) is smooth) of \( U_L \) transitively and \( \dim(U_L)_0 = n \) (where \( (U_L)_0 \) is an irreducible component of \( U_L \) containing the holonomy point), we see that \( \dim U_L = n \). It is proved as follows. Let \( \overline{\mathbb{Q}} \) be the algebraic closure of \( \mathbb{Q} \) containing \( L \). The scheme \( \overline{U} \) has an irreducible component (connected component) \( U_0 \) containing the holonomy point. Since the absolute Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts on \( \overline{U} \), the Galois image of the irreducible component \( U_0 \) consists of finite number of irreducible components of \( \overline{U} \). Note that \( \dim U_0 = n \) since \( \dim X(M)(\mathbb{C})_0 = n \). We can assume that all of the irreducible components (connected components) in the Galois image of \( U_0 \) are defined over a finite Galois extension field \( L' \) of \( \mathbb{Q} \) which contains \( L \).

Thus they are already decomposed over \( L \). Then we see from the descent theory that \( U_L \) is the Galois orbit of the irreducible component \( U_0' \) containing the holonomy point. Since \( (U_0')_{\overline{\mathbb{Q}}} = U_0' \) we have \( \dim U_0' = \dim U_0 = n \). Therefore \( \dim U_L = \dim U_0' = n \). Hence we have \( \dim X_0(M) = n \). \( \square \)
Lemma 3.3. \[ \dim X_0(M) = n \implies \dim X_0(M) = n + 1. \]

Proof. We can apply the same argument as in Corollary 3.2 for the generic fiber \( X_0(M) \otimes \mathbb{Q} \) as follows. Since \( X_0(M)(\mathbb{C}) \) is identified with \( X(M)_{\text{irr}}(\mathbb{C}) \), the point set in \( X_0(M)_{\overline{\mathbb{Q}}} := X_0(M) \otimes \overline{\mathbb{Q}} \) corresponding to the holonomy point in \( X(M)_{\text{irr}}(\mathbb{C}) \) is an irreducible component of \( X_0(M)_{\overline{\mathbb{Q}}} \). Note that \( \dim X(M)_{\text{irr}}(\mathbb{C}) = 0 \) by Theorem 3.1. Since there is an inclusion relation

\[
(X_0(M)_{\overline{\mathbb{Q}}})_0 \subset (X(M)_{\overline{\mathbb{Q}}})_0 \overset{\sim}{\rightarrow} X(M)_{\text{irr}}(\overline{\mathbb{Q}})_0,
\]

we have \( \dim (X_0(M)_{\overline{\mathbb{Q}}})_0 = n \). Thus we have \( \dim (X_0(M)_{\mathbb{Q}}) = n \) by the same argument as in the proof of Corollary 3.2.

By [14], Corol. (5.6.6) we have \( X_0(M)_{\text{red}} = \dim \text{Spec } (\mathbb{Z}) + \text{tr.deg}(R(X_0(M)_{\text{red}})/\mathbb{Q}) \). Here \( R(X_0(M)_{\text{red}}) \) is the function field of \( X_0(M)_{\text{red}} \) and \( \text{tr.deg}(R(X_0(M)_{\text{red}})/\mathbb{Q}) \) is the transcendental degree of \( R(X_0(M)_{\text{red}}) \) over \( \mathbb{Q} \). Note that

\[
\text{tr.deg}(R(X_0(M)_{\text{red}})/\mathbb{Q}) = \text{tr.deg}(R(X_0(M)_{\text{red}})/\overline{\mathbb{Q}}) = \text{tr.deg}(R(X_0(M)_{\text{red}})/\mathbb{C}) = \dim (X_0(M)_{\mathbb{Q}}).
\]

Therefore \( \dim X_0(M) = n + 1 \) since \( \dim (X_0(M)_{\mathbb{Q}}) = n \). \( \square \)

In what follows we assume that \( M \) is a closed orientable complete hyperbolic 3-manifold of finite volume. Note that \( \dim X_0(M) = 0 \) and \( \dim X_0(M) = 1 \) by Corollary 3.2 and Lemma 3.3.

Since \( X_0(M) \) is of finite type over \( \mathbb{Z} \) (\( \pi_1(M) \) is finitely generated), we see that \( X_0(M) \) is finite over \( \mathbb{Z} \). Thus the reduced scheme \( X_0(M)_{\text{red}} \) is equal to the scheme \( \text{Spec } \mathcal{O} \) for some integral domain of finite rank over \( \mathbb{Z} \). Let \( K \) be the quotient field of \( \mathcal{O} \). Note that \( K \) is a finite extension field of \( \mathbb{Q} \) and \( \mathcal{O} \) is contained in \( \mathcal{O}_K \), the ring of integers of \( K \).

Lemma 3.4. \( \mathcal{O} \) is an order of \( K \).

Proof. As we have seen in the proof of the previous lemma, we know that \( \dim (X_0(M) \otimes \mathbb{Q}) = 0 \). Therefore we have \( \dim (X_0(M)_{\text{red}} \otimes \mathbb{Q}) = 0 \) for \( X_0(M)_{\text{red}} = \text{Spec } \mathcal{O} \). Hence we have \( \dim \text{Spec } (\mathcal{O} \otimes \mathbb{Q}) = 0 \). Note that \( \mathcal{O} \otimes \mathbb{Q} \) is an integral domain contained in \( \mathcal{O}_K \otimes \mathbb{Q} = K \). Therefore \( \mathcal{O} \otimes \mathbb{Q} \) is a field. Note that this is the minimal field containing \( \mathcal{O} \), which is equal to \( K \). Thus \( \mathcal{O} \) is an order of \( K \). \( \square \)

Since \( \mathcal{O} \) is an order of \( K \), \([\mathcal{O}_K : \mathcal{O}]\) is of finite index. Note that there is a bijective correspondence between the prime ideals of \( \mathcal{O} \) and those of \( \mathcal{O}_K \) lying on prime numbers \( p \mid [\mathcal{O}_K : \mathcal{O}] \) (cf. [27]). Hence \( \zeta(\text{Spec } \mathcal{O}), s \) is equal to \( \zeta(\text{Spec } \mathcal{O}_K), s \) up to rational functions in \( p^{-s} \) for \( p \mid [\mathcal{O}_K : \mathcal{O}] \). Therefore we have the equality

\[
\zeta(X_0(M), s) = \zeta(X_0(M)_{\text{red}}, s) = \zeta(K, s)
\]

up to rational functions.

Now we would like to determine the number field \( K \). Suppose that the holonomy representation \( \rho_M : \pi_1(M) \rightarrow \text{PSL}_2(L) \) is defined over a number field \( L \) (for which we can take the trace field \( K_M \) or a quadratic extension of \( K_M \)). There is a commutative diagram

\[
\begin{array}{ccc}
\text{Spec } (L) & \longrightarrow & X_0(M) \otimes \mathbb{Q} = \text{Spec } (K) \\
\| & & \\
\text{Spec } (L) & \overset{p_p}{\longrightarrow} & X_0(M) = \text{Spec } (\mathcal{O}).
\end{array}
\]
Thus it is clear that $K \subset L$.

**Lemma 3.5.** We have $K_M \subset K \subset L$.

*Proof.* Let $A_{\text{univ}}(M) := A_{\text{univ}}'(\pi_1(M))$ be the universal representation ring of the $\text{SL}_2$-representations of $\pi_1(M)$ and $\rho_{\text{univ}} : \pi_1(M) \to \text{SL}_2(A_{\text{univ}}(M))$ the associated universal representation as in the previous section. Then the (lift of) holonomy representation $\rho : \pi_1(M) \to \text{SL}_2(L)$ factors through $\text{SL}_2(A_{\text{univ}}(M)) \to \text{SL}_2(L)$ induced by the homomorphism $f_M : A_{\text{univ}}(M) \to L$. Let $T_{\text{univ}}(M) := T_{\text{univ}}'(\pi_1(M))$ be the subring of $A_{\text{univ}}(M)$ generated by the trace of $\rho_{\text{univ}}$. Note that $f_M(\text{Tr} \rho_{\text{univ}}) = \text{Tr} \rho$. Therefore we see that $\mathbb{Q}(f_M(T_{\text{univ}}(M))) = K_M$.

The scheme $X(M)$ is an open subscheme of $\text{Spec}(T_{\text{univ}}(M))$ (which is of the form $\cup \text{Spec}(T_{\text{univ}}(M)[1/d])$). Since there is a commutative diagram as above, the homomorphism $f_M : T_{\text{univ}}(M) \to L$ corresponding to the lift $\rho$ of the holonomy representation factors through $K \to L$. Hence we see that $f_M(T_{\text{univ}}(M)) = \text{Tr} \rho \subset K \subset L$. Thus we have $K_M \subset K \subset L$. \hfill $\square$

**Lemma 3.6.** We can choose $L = K_M$.

*Proof.* Let $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ be a lift of the holonomy representation of $M$. Consider the associated absolutely irreducible representation $\mathbb{Z}[\rho] : \mathbb{Z}[\pi_1(M)] \to A$, where $A$ is an Azumaya algebra over $\mathbb{C}$. Then we see by [28], Proposition 2.7 that $K_M[\text{Im}(\rho)]$ is an Azumaya algebra over $K_M$ and its base change $K_M[\text{Im}(\rho)] \otimes_{K_M} \mathbb{C}$ is isomorphic to the Azumaya algebra $A$ over $\mathbb{C}$. This means that $\mathbb{Z}[\rho] : \mathbb{Z}[\pi_1(M)] \to A$ factors through the Azumaya algebra $K_M[\text{Im}(\rho)]$. Hence $\mathbb{Z}[\rho] : \mathbb{Z}[\pi_1(M)] \to K_M[\text{Im}(\rho)]$ is defined over the trace field $K_M$. This implies that $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ defines a $K_M$-rational point $\text{Spec} K_M \to X_0(M)$. Thus we can take $K_M$ as $L$. \hfill $\square$

**Remark 3.7.** In general the holonomy representation of an orientable hyperbolic 3-manifold $M$ of finite volume is defined over a finite number field. Namely there is an (at most) quadratic extension field $L$ of the trace field $K_M$ such that we can take $\rho_M : \pi_1(M) \to \text{PSL}_2(L)$ up to conjugacy (cf. [20], Corollary 3.2.4.). If $M$ is non-compact, namely if $M$ has a cusp then we can take $\rho_M : \pi_1(M) \to \text{PSL}_2(K_M)$ (cf. [20], Theorem 4.2.3).

Therefore we have proved the following:

**Theorem 3.8.** Let $M$ be an orientable closed hyperbolic 3-manifold of finite volume. Let $X_0(M)$ be an irreducible component of the moduli scheme $X(M)$ containing the point corresponding to a lift of the holonomy representation $\rho_M$. Then we have the equality $\zeta(X_0(M), s) = \zeta(K_M, s)$ up to rational functions in $p^{-s}$ for finitely many prime numbers $p$.

Let $X(M)$ be the $\text{SL}_2$-character variety of $M$ over $\mathbb{Q}$. Let $X_0(M)$ be an irreducible component of $X(M)$ containing the point corresponding to the holonomy character of $M$. Since $\dim X_0(M) = 0$, the (reduced) scheme $X_0(M)$ is written as $\text{Spec}(K')$, where $K'$ is a finite extension field of $\mathbb{Q}$.
Let $\rho : \pi_1(M) \to \text{SL}_2(L)$ be a lift of the holonomy representation of $M$. Here $L$ is the trace field $K_M$ or a quadratic extension field of $K_M$. Since the holonomy character $\chi_M = \text{Tr}\rho$ is an $L$-rational point of $X_0(M)$, we see that $K'$ is regarded as a subfield of $L$. Therefore we have $\zeta(X_0(M), s) = \zeta(K', s)$ for $K' \subset L$.

As we have seen, $X_0(M)_{\text{red}} = \text{Spec} (\emptyset)$, where $\emptyset$ is an order of $K_M$. Therefore $X_0(M)_{\text{red}} \otimes \mathbb{Q} = \text{Spec} (K_M)$. Note that the holonomy character $\chi_M$ defines a common zero of the minimal polynomials of $K_M$ and $K'$. Therefore $K_M$ and $K'$ are isomorphic each other. Hence we obtain the following:

**Lemma 3.9.** $X_0(M)_{\text{red}} \otimes \mathbb{Q}$ is isomorphic to $X_0(M)$.

This means that $X_0(M)_{\text{red}}$ is a model of $X_0(M)$. Hence we have

$$\zeta(X_0(M), s) = \zeta(X_0(M), s).$$

Consider another lift of the holonomy representation $\rho_M$ and let $X_0(M)' \subset X(M)$ be the canonical component of that lift. By the above argument both $X_0(M)$ and $X_0(M)'$ are isomorphic to $\text{Spec} K_M$. Hence the $K_M$-rational points defined by the characters of the lifts of the holonomy representation of $M$ are conjugate (more precisely, they are $\text{Gal}(K_M/\text{Inv}K_M)$-conjugate. See §3.2, especially Lemma 3.12. It implies that $X_0(M)(\mathbb{C}) = X_0(M)'(\mathbb{C})$ in $X(M)(\mathbb{C})$. Therefore we have $X_0(M) = X_0(M)'$ in $X(M)$. This means that the canonical component $X_0(M)$ does not depend on the choice of a lift of the holonomy representation $\rho_M$. Thus we have the following corollary.

**Corollary 3.10.** Let $M$ be an orientable closed hyperbolic 3-manifold of finite volume. Then the canonical component $X_0(M)$ is unique as a closed subscheme of $X(M)$, which does not depend on the choice of a lift of the holonomy representation, and $X_0(M)$ is isomorphic to the spectrum $\text{Spec} K_M$ of the trace field $K_M$. Therefore the Hasse-Weil zeta function $\zeta(X_0(M), s)$ is equal to the Dedekind zeta function $\zeta(K_M, s)$ of the trace field $K_M$.

**Remark 3.11.** As we have defined in Subsection 1.2, the Hasse-Weil zeta function of an algebraic set over $\mathbb{Q}$ in this paper is well-defined up to rational functions in $p^{-i}$ for finitely many prime numbers $p$. However the canonical component of the $\text{SL}_2$ ($\text{PSL}_2$)-character variety of a closed hyperbolic 3-manifold can be written as the spectrum of a number field (namely it is a smooth projective variety of dimension 0). Therefore we can take the unique maximal order, the ring of integers and the Hasse-Weil zeta function of the ring of integers is exactly the Dedekind zeta function of the number field. Thus we do not have to consider any ambiguity of rational functions in the descriptions of the Hasse-Weil zeta functions of $\text{SL}_2$ ($\text{PSL}_2$)-character varieties of the closed hyperbolic 3-manifolds.

**3.2. Hasse-Weil zeta functions of $\text{PSL}_2$-character varieties.** Let $M$ be a closed orientable hyperbolic 3-manifold of finite volume. Let $C_2 := \{\pm 1\}$ be the group of order 2 and let $H^1(\pi_1(M), C_2) = \text{Hom}(\pi_1(M), C_2)$. Then we can consider the group action of $H^1(\pi_1(M), C_2)$ on the canonical component $X_0(M)$ of $X(M)$ as follows.

Let $A$ be a commutative ring. For any element $\epsilon \in H^1(\pi_1(M), C_2)$ and $\rho : \mathbb{Z}[\pi_1(M)] \to S \in \mathcal{X}(M)(A)$ ($S$ is an Azumaya algebra of degree 2 over $A$) define the action of $H^1(\pi_1(M), C_2)$ on $\mathcal{X}(M)(A)$ by

$$\epsilon \cdot \rho(g) := \epsilon(g) \rho(g), \quad g \in \pi_1(M).$$
Note that for a ring homomorphism \( f : A \to B \) this action is compatible with the morphism \( f_* : \mathcal{X}(M)(A) \to \mathcal{X}(M)(B) \). Hence it naturally induces the group action of \( H^1(\pi_1(M), C_2) \) on the scheme \( \mathcal{X}(M) \), and on \( \mathcal{X}(M) \otimes \mathbb{Q} \).

Now we know by Lemma 3.9 that for any lift of the holonomy representation \( \rho_M : \pi_1(M) \to \text{PSL}_2(\mathbb{C}) \) the generic fiber \( X_0(M) \otimes \mathbb{Q} \) of a canonical component \( X_0(M) \) of \( \mathcal{X}(M) \) is isomorphic to \( X_0(M) \to \text{Spec} \ K_M \). Therefore the action of \( H^1(\pi_1(M), C_2) \) on \( \mathcal{X}(M) \) induces the action on the canonical component \( X_0(M) \to X_0(M) \otimes \mathbb{Q} \) of the \( \text{SL}_2 \)-character variety \( \mathcal{X}(M) \).

Note that the group \( H^1(\pi_1(M), C_2) \) is a finite group since \( \pi_1(M) \) is finitely generated. Hence there exists a quotient (reduced) scheme \( \overline{X}_0(M) := X_0(M)/H^1(\pi_1(M), C_2) \) of finite type over \( \mathbb{Q} \). Since there is a surjection \( X_0(M) \to \overline{X}_0(M) \) we see that the scheme \( \overline{X}_0(M) \) has dimension 0. Thus \( \overline{X}_0(M) \) is written as \( \text{Spec} \ K' \), where \( K' \subset K_M \) is a finite extension field of \( \mathbb{Q} \). We shall prove that \( K' \) is isomorphic to the invariant trace field \( \text{Inv} K_M \).

For any \( \epsilon \in H^1(\pi_1(M), C_2) \) the associated isomorphism of \( X_0(M) = \text{Spec} K_M \) is induced by the \( \mathbb{Q} \)-algebra isomorphism defined by \( \chi_\rho(g) \mapsto \epsilon(g)\chi_\rho(g) \), where \( \rho : \pi_1(M) \to \text{SL}_2(\mathbb{C}) \) is any (fixed) lift of the holonomy representation \( \rho_M \). Since the invariant trace field \( \text{Inv} K_M \) is generated by the elements \( \chi_\rho(g^2) \) for \( g \in \pi_1(M) \), we see that \( \epsilon \) is identity on the subfield \( \text{Inv} K_M \). Therefore the induced morphism \( X_0(M) = \text{Spec} K_M \to \text{Spec} \left( \text{Inv} K_M \right) \) is \( H^1(\pi_1(M), C_2) \)-invariant morphism. Thus there exists a unique morphism \( \overline{X}_0(M) \to \text{Spec} \left( \text{Inv} K_M \right) \) such that the composite morphism \( X_0(M) = \text{Spec} K_M \to \overline{X}_0(M) \to \text{Spec} \left( \text{Inv} K_M \right) \) is equal to the natural morphism \( X_0(M) \to \text{Spec} \left( \text{Inv} K_M \right) \). Hence we have the inclusion relation \( \text{Inv} K_M \subset K' \subset K_M \).

**Lemma 3.12.** Let \( M \) be an orientable closed hyperbolic 3-manifold of finite volume, \( K_M \) the trace field and \( \text{Inv} K_M \) the invariant trace field. Then \( K_M \) is an elementary abelian extension field of \( \text{Inv} K_M \) and its Galois group \( \text{Gal} (K_M/\text{Inv} K_M) \) is isomorphic to \( H^1(\pi_1(M), C_2) \).

If the finite group \( H^1(\pi_1(M), C_2) \) is written as

\[
H^1(\pi_1(M), C_2) \cong \pi_1(M)/\pi_1(M)^2 = \langle \overline{g}_1, \ldots, \overline{g}_r \rangle,
\]

then the trace field is expressed as

\[
K_M = \text{Inv} K_M(\chi_\rho(1), \ldots, \chi_\rho(r)),
\]

where \( \rho \) is a lift of the holonomy representation of \( M \).

**Proof.** Let \( \rho : \pi_1(M) \to \text{SL}_2(\mathbb{C}) \) be a lift of the holonomy representation of \( M \). Then the trace field \( K_M \) is \( \mathbb{Q}(\chi_\rho(g) \mid g \in \pi_1(M)) \) and the invariant trace field \( \text{Inv} K_M \) is \( \mathbb{Q}(\chi_\rho(g^2) \mid g \in \pi_1(M)) = \mathbb{Q}(\chi_\rho(1) \mid g \in \pi_1(M)) \). Thus it is obvious that \( K_M/\text{Inv} K_M \) is an elementary abelian extension. The group action of \( H^1(\pi_1(M), C_2) \) on \( X_0(M) \) induces a homomorphism

\[
H^1(\pi_1(M), C_2) \to \text{Gal} (K_M/\text{Inv} K_M); \quad \epsilon \mapsto \epsilon^*,
\]

where \( \epsilon^* \) is defined by \( \epsilon^*(\chi_\rho(g)) = \epsilon(g)\chi_\rho(g) \). On the other hand, take an element \( \sigma \) of \( \text{Gal} (K_M/\text{Inv} K_M) \). Note that any \( \chi_\rho(g) \) in \( K_M \) is a root of a monic quadratic polynomial over \( \text{Inv} K_M \). Hence there exists a unique element \( \epsilon_\sigma(g) \) which takes value in \( C_2 := \{ \pm 1 \} \) such that

\[
\sigma(\chi_\rho(g)) = \epsilon_\sigma(g)\chi_\rho(g).
\]
We know that for any \( g, h \in \pi_1(M) \) the elements \( \chi_\rho(g^2) \) and \( \chi_\rho(g)\chi_\rho(h)\chi_\rho(gh) \) are contained in \( \Inv K_M \) since there is an identity (cf. [20], §3.3.4, 3.3.5 or [15], §2.4)

\[
2\Tr(A)\Tr(B)\Tr(AB) = \Tr(A)^2\Tr(B)^2 + \Tr(AB)^2 - \Tr(AB^{-1})^2
\]

for any \( A, B \in \SL_2(\C) \). Therefore we have \( \epsilon_r(g^2) = 1 \) and

\[
\sigma(\chi_\rho(g)\chi_\rho(h)\chi_\rho(gh)) = \epsilon_r(g)\epsilon_r(h)\epsilon_r(gh)\chi_\rho(g)\chi_\rho(h)\chi_\rho(gh) = \chi_\rho(g)\chi_\rho(h)\chi_\rho(gh).
\]

Note that \( \chi_\rho(g)\chi_\rho(h)\chi_\rho(gh) \neq 0 \) since \( \pi_1(M) \) is torsion-free. Thus we have \( \epsilon_r(g)\epsilon_r(h)\epsilon_r(gh) = 1 \), namely \( \epsilon_r(gh) = \epsilon_r(g)\epsilon_r(h) \). Hence we deduce that \( \epsilon_r \) is an element of \( H^1(\pi_1(M), C_2) \) and it is the inverse of the previous homomorphism. Thus we see that \( \Gal(K_M/\Inv K_M) \) is isomorphic to \( H^1(\pi_1(M), C_2) \).

Since \( \chi_\rho(g)\chi_\rho(h)\chi_\rho(gh) \in \Inv K_M \) and \( \chi_\rho(g) \neq 0 \) for any \( g, h \in \pi_1(M) \) we see that \( \chi_\rho(gh) \) is contained in \( \Inv K_M(\chi_\rho(g), \chi_\rho(h)) \). Therefore if the finite group \( H^1(\pi_1(M), C_2) \) is written as

\[
H^1(\pi_1(M), C_2) = H^1(\pi_1(M)/\pi_1(M), C_2) \iso \pi_1(\mathcal{M})/\pi_1(\mathcal{M})^2 = \langle \overline{g}_1, \ldots, \overline{g}_r \rangle,
\]

we have \( K_M = \Inv K_M(\chi_\rho(g_1), \ldots, \chi_\rho(g_r)) \).

In particular, since \( H^1(\pi_1(M), C_2) = \Hom(\pi_1(M)^{\text{ab}}, C_2) \) we have the following corollary.

**Corollary 3.13.** Let \( M \) be an orientable closed hyperbolic 3-manifold of finite volume. Then the trace field \( K_M \) is equal to the invariant trace field if and only if the homology group \( H_1(M, \Z) \iso \pi_1(M)^{\text{ab}} \) has rank 0 and does not have 2-torsion.

**Remark 3.14.** If \( M \) is an \( r \)-component link in the 3-sphere \( S^3 \) then the abelianization of the fundamental group \( \pi_1(M)^{\text{ab}} := \pi_1(M)/[\pi_1(M), \pi_1(M)] \) is isomorphic to \( \Z^r \). Hence we have \( H^1(\pi_1(M), C_2) \iso C_2^r \). On the other hand, we know that the trace field is equal to the invariant trace field for any hyperbolic link in the 3-sphere ([20], Corollary 4.2.2). Therefore the above lemma does not hold for cusped hyperbolic 3-manifolds in general.

By the above lemma, we see that \( X_0(M) = \Spec K_M \to \Spec(\Inv K_M) \) is a Galois cover with Galois group \( H^1(\pi_1(M), C_2) \). Therefore the quotient scheme \( \overline{X}_0(M) = X_0(M)/H^1(\pi_1(M), C_2) \) is isomorphic to \( \Spec(\Inv K_M) \).

**Remark 3.15.** For a representation \( \overline{\rho} : \pi_1(M) \to \PSL_2(\C) \) its character \( \overline{\chi}_{\overline{\rho}} : \pi_1(M) \to \C \) is defined by \( \overline{\chi}_{\overline{\rho}}(g) := (\Tr \overline{\rho}(g))^2 \). If we denote by \( \overline{X}(M)(\C) \) the set of characters of representations \( \overline{\rho} : \pi_1(M) \to \PSL_2(\C) \) which lift to \( \SL_2(\C) \) there is an isomorphism between \( X(M)(\C)/H^1(\pi_1(M), C_2) \) and \( \overline{X}(M)(\C) \) ([15], Proposition 4.2). Since there is an isomorphism

\[
X(M)(\C)/H^1(\pi_1(M), C_2) \iso (X(M)/H^1(\pi_1(M), C_2))(\C)
\]

we can consider the quotient scheme \( \overline{X}_0(M) \iso \Spec \Inv K_M \) as the canonical component of the \( \PSL_2 \)-character variety of \( M \).

**Theorem 3.16.** Let \( M \) be an orientable closed hyperbolic 3-manifold of finite volume. Then the quotient scheme \( \overline{X}_0(M) := X_0(M)/H^1(\pi_1(M), C_2) \) is isomorphic to the spectrum \( \Spec(\Inv K_M) \) of the invariant trace field \( \Inv K_M \) and the Hasse-Weil zeta function \( \zeta(\overline{X}_0(M), s) \) is equal to the Dedekind zeta function \( \zeta(\Inv K_M, s) \) of the invariant trace field \( \Inv K_M \) of \( M \).
There is a one-to-one correspondence between the set of conjugacy classes of the lifts of the holonomy representation \( \rho_M : \pi_1(M) \to \text{PSL}_2(\mathbb{C}) \) and the cohomology group \( H^1(\pi_1(M), C_2) = \text{Hom}(\pi_1(M), C_2) \). By Lemma 3.12 we know that the cardinality of \( H^1(\pi_1(M), C_2) \) is equal to \([K_M : \text{Inv}K_M]\). Thus we obtain the following:

**Corollary 3.17.** Let \( M \) be a closed oriented complete hyperbolic 3-manifold of finite volume. Then the number of canonical components \( X(M)C_0 \) of the \( \text{SL}_2(\mathbb{C}) \)-character variety \( X(M)C_0 \) is equal to \([K_M : \text{Inv}K_M]\) = \(#H^1(\pi_1(M), C_2)\).

**Remark 3.18.** There is an exact sequence

\[ 1 \to \text{Isom}(\mathbb{H}^3) \to \text{Isom}(\mathbb{H}^3) \to \{ \pm 1 \} \to 1, \]

where \( \text{Isom}(\mathbb{H}^3) \) is the group of isometries of \( \mathbb{H}^3 \) and \( \text{Isom}_+(\mathbb{H}^3) \) is the subgroup consisting of orientation-preserving isometries. \( \text{Isom}_+(\mathbb{H}^3) \) is isomorphic to \( \text{PSL}_2(\mathbb{C}) \) and \( \text{Isom}(\mathbb{H}^3) \) is generated by \( \text{Isom}_+(\mathbb{H}^3) \) and the orientation-reversing isometry defined by the anti-holomorphic Möbius transformation on the Riemann sphere \( \hat{\mathbb{C}} \). Therefore there are two orientation-preserving isometric classes for an orientable hyperbolic 3-manifold of finite volume by Mostow-Prasad Rigidity (in other words, there are two \( \text{PSL}_2(\mathbb{C}) \)-conjugacy classes of discrete faithful representations of the fundamental group \( \pi_1(M) \) into \( \text{PSL}_2(\mathbb{C}) \) for an orientable hyperbolic 3-manifold \( M \) of finite volume, which are isomorphic by complex conjugation). Hence by the above result there are \( 2[K_M : \text{Inv}K_M] = 2\#H^1(\pi_1(M), C_2) \) canonical components in the \( \text{SL}_2(\mathbb{C}) \)-character variety \( X(M)C_0 \) for a closed orientable hyperbolic 3-manifold \( M \).

Let \( k \) be a number field with exactly one complex place and let \( A \) be a quaternion algebra over \( k \) which is ramified at all real places. Let \( \rho : A \to M_2(\mathbb{C}) \) be a \( k \)-embedding of \( A \) and \( \mathfrak{O} \) an (maximal) order of \( A \). Let \( \mathfrak{O}^1 \) be the subgroup of the unit group \( \mathfrak{O}^\times \) with reduced norm 1. A complete orientable hyperbolic 3-manifold of finite volume is called arithmetic when its fundamental group is commensurable with such \( P(\rho(\mathfrak{O}^1)) \), where \( P : \text{GL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C}) \) is the projection.

For arithmetic 3-manifolds it is well-known as Borel’s formula that the hyperbolic volumes are expressed, especially in terms of the special values at 2 of the Dedekind zeta functions of the invariant trace fields as follows.

**Theorem 3.19** (cf. [20], Theorem 11.1.3). Let \( k \) be a number field having exactly one complex place, \( A \) a quaternion algebra which ramifies at all real places and \( \mathfrak{O} \) a maximal order in \( A \). Let \( \mathfrak{O}^1 \) be the subgroup of \( \mathfrak{O}^\times \) of reduced norm 1 elements. Let \( \rho : A \to M_2(\mathbb{C}) \) be a splitting of \( A \) over \( k \) and denote by \( P(\mathfrak{O}^1) \) the projection of \( \mathfrak{O}^1 \) in \( \text{PSL}_2(\mathbb{C}) \). Then the hyperbolic volume of \( \mathbb{H}^3 / P(\mathfrak{O}^1) \) is

\[
\text{Vol}(\mathbb{H}^3 / P(\mathfrak{O}^1)) = \frac{4\pi^2|\Delta_k|^{3/2}\zeta(k, 2) \prod_{p \mid (\Delta_A)}(N(p) - 1)}{(4\pi^2)^{[k: \mathbb{Q}^1]}},
\]

Here \( \Delta_k \) (resp. \( \Delta(A) \)) is the discriminant of \( k \) (resp. \( A \)).

It is well-known that if \( M \) and \( M' \) are commensurable hyperbolic 3-manifolds of finite volume then the quotient \( \text{Vol}(M') / \text{Vol}(M) \) of their volumes is a rational number. Therefore we have the following corollary.
Corollary 3.20. Let $M$ be an arithmetic closed hyperbolic 3-manifold. Then the special value of $\zeta(X_0(M), s)$ at $s = 2$ is expressed in terms of the hyperbolic volume $\text{Vol}(M)$, the discriminant $\Delta_{\text{Inv}K_M}$ and $\pi$ as follows:

$$
\zeta(X_0(M), 2) \sim Q \times (4\pi^2)^{\text{Inv}K_M/\mathbb{Q}} - 1 \text{Vol}(M)
$$

where $\sim Q \times$ means the equality holds up to a rational number.

Remark 3.21. For a closed orientable hyperbolic 3-manifold of finite volume, the second cohomology group $H^2(\pi_1(M), \mathbb{C})$ is non-zero in general since the abelianization $\pi_1(M)^{ab}$ might have non-cyclic 2-torsion, namely there might be a representation $\rho : \pi_1(M) \rightarrow \text{PSL}_2(\mathbb{C})$ which never lifts to $\text{SL}_2(\mathbb{C})$ ([13], Lemma 2.3, see also [9]). Hence there might be a little difference in our case between the constructions of the $\text{PSL}_2(\mathbb{C})$-character varieties in the references González-Acuña-Montesinos-Amilibia [13], Boyer-Zhang [5], Long-Reid [18] and Heusener-Porti [15].

Remark 3.22. On the other hand, for any 1-cusped orientable hyperbolic 3-manifold $M$ of finite volume such that $H_1(M, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, it is known that $H^2(\pi_1(M), \mathbb{C}) = 0$ (for details, see [5], page 756). Hence there is no ambiguity on the definition of the $\text{PSL}_2(\mathbb{C})$-character variety of $M$. However, as it is obtained in [19] the canonical component of the $\text{PSL}_2(\mathbb{C})$-character variety of a hyperbolic twist knot complement in $S^3$ is the projective line $\mathbb{P}_C^1$. Therefore it seems we can not expect that the Hasse-Weil zeta function of the $\text{PSL}_2$-character variety of a 1-cusped orientable hyperbolic 3-manifold of finite volume would have information on the hyperbolic volume of the manifold.

By Mostow-Prasad Rigidity the isometric classes of orientable complete hyperbolic 3-manifolds of finite volume correspond bijectively to the isomorphism classes of the fundamental groups of the manifolds (more precisely the conjugacy classes of the holonomy representations). From the results in this paper we know that for a closed orientable hyperbolic 3-manifold $M$ of finite volume the canonical component $X_0(M)$ of the $\text{SL}_2$-character variety is determined by the trace field $K_M$. Therefore it is natural to ask whether the closed hyperbolic 3-manifold of finite volume is determined by the trace field. However, for a closed hyperbolic 3-manifold $M$, there are infinitely many non-isometric closed hyperbolic 3-manifolds in the commensurable class of $M$ since $\pi_1(M)$ is a torsion-free infinite group and is residually finite. Therefore there are infinitely many non-isometric closed hyperbolic 3-manifolds $M'$ which are commensurable with $M$ satisfying $\text{Inv}K_{M'} = \text{Inv}K_M$. Thus the trace field is not enough to distinguish the isometric class of a closed hyperbolic 3-manifold (the author would like to appreciate Alan Reid for answering this question).

Invariant trace fields have a characterization as number fields such that they have exactly one complex place. For such number fields it is known ([6], Corollary 1.4) that their isomorphism classes are determined by the Dedekind zeta functions. Namely two such number fields are isomorphic if and only if they are arithmetically equivalent. Hence it would be worth considering the following question.

Question 3.23. Are the trace fields of closed hyperbolic 3-manifolds isomorphic if and only if they are arithmetically equivalent?
4. Examples

Here we give some explicit examples of the defining polynomials of the \( \text{SL}_2(\mathbb{C}) \)-character varieties, holonomy representations and the trace fields of some closed arithmetic hyperbolic 3-manifolds of small volumes.

We followed the way in [13] to compute defining polynomials of the \( \text{SL}_2(\mathbb{C}) \)-character variety of a finitely presented group. After we have obtained defining polynomials of the character variety for each manifold, we have replaced those polynomials with simpler ones by computing their Gröbner basis and have found the common zeros of them by Maple. The polynomials presented here are the replaced ones. It is relatively not difficult to find the common zeros of the polynomials in an algebraic closure of each finite field \( \mathbb{F}_p \) once we know about the common zeros in \( \mathbb{C} \). Then we have determined the Weil-type and Hasse-Weil type zeta functions and the trace fields. (For closed hyperbolic 3-manifolds, by Corollary 3.10 it is enough to compute the trace field to obtain the zeta functions. Thus we include the description of the zeta function only in the Weeks manifold case.)

For computing an explicit form of the holonomy representation, since the fundamental groups in our examples are generated by two elements, we follow the way given in [8].

4.1. Weeks manifold case. The Weeks manifold \( M_W \) is obtained by \((5,1)\), \((5,2)\) Dehn surgeries on the Whitehead link complement. The Weeks manifold is the unique manifold up to isometry which has the smallest volume among all the orientable closed hyperbolic 3-manifolds ([12], [21]). Its fundamental group has the following presentation:

\[
\pi_1(M_W) \cong \langle a, b \mid w_1 = w_2 = 1 \rangle,
\]

where

\[
w_1 := ababaBa^2B, \quad w_2 := bababAb^2A
\]

for \( A := a^{-1}, B := b^{-1} \). The original 6 defining polynomials obtained by the method in [13] are quite complicated. However, by the theory of Gröbner basis, we can replace those polynomials by simpler ones. Here we only show those polynomials replaced by the Gröbner basis of them (which we calculated by the software Maple):

\[
f_1 = -2 + z + 4z^2 + 2z^3 - 4z^4 - z^5 + z^6 = (z - 2)(z^2 + z - 1)(z^3 - z - 1),
\]

\[
f_2 = -2 + 3z + 3z^2 - 4z^3 + 2y - 3yz - yz^2 - z^4 + z^5 + yz^3,
\]

\[
f_3 = -z - 3y + 4 - 4z^2 + z^4 - y^2 + y^3,
\]

\[
f_4 = -yz^2 + xz^2 - yz + xz + y - x,
\]

\[
f_5 = -x + z - 3z^3 + 2z^2 + z^5 + xy - yz - z^4 - y^2z + xy^2,
\]

\[
f_6 = -z^4 - 4 - xyz + z^3 + y^2 + x^2 + 4z^2 - 2z.
\]

Then the \( \text{SL}_2(\mathbb{C}) \)-character variety \( X(M_W)(\mathbb{C}) \) consists of the following points:

\[
\{(2,2,2)\},
\]

\[
\{(\alpha, \alpha, 2), (\alpha, 2, \alpha), (2, \alpha, \alpha) \mid \alpha^2 + \alpha - 1 = 0\},
\]

\[
\{(\alpha, -1 - \alpha, \alpha), (-1 - \alpha, \alpha, \alpha), (-1 - \alpha, -1 - \alpha, \alpha) \mid \alpha^2 + \alpha - 1 = 0\},
\]
Thus we see that \( \dim(X(M_w)(\mathbb{C})) = 0 \). The subset of \( X(M_w)(\mathbb{C}) \) consisting of reducible characters is the set of common zeros of the above polynomials and the polynomial \( x^2 + y^2 + z^2 - 4xyz - 4 \), which is equal to \( X(M_w)(\mathbb{C}) \) except \( \{(1 - \beta^2, 1 - \beta^2, \beta) \mid \beta^3 - \beta - 1 = 0\} \). Therefore the subset \( X(M_w)(\mathbb{C})_{\text{irr}} \) of \( X(M_w)(\mathbb{C}) \) consisting of irreducible characters is
\[
X(M_w)(\mathbb{C})_{\text{irr}} = \{(1 - \beta^2, 1 - \beta^2, \beta) \mid \beta^3 - \beta - 1 = 0\}.
\]

Now we can show that the set \( \text{Rep}_2(\pi_1(M_w))(\mathbb{k})/\text{PGL}_2(\mathbb{k}) \) of conjugacy classes of absolutely irreducible representations of \( \pi_1(M_w) \) into \( \text{SL}_2(\mathbb{C}) \) over an algebraically closed field \( \mathbb{k} \) consists of points of the form \( (1 - \beta^2, 1 - \beta^2, \beta) \), where \( \beta \) is a root of the polynomial \( f(T) = T^3 - T - 1 \) in \( \mathbb{k} \). Since \( X(M_w)(\mathbb{C})_{\text{irr}} \) contains a point corresponding to the holonomy character, the trace field \( K_{M_w} \) is equal to \( \mathbb{Q}[T]/(f) \). Its discriminant \( d_{K_{M_w}} \) is \(-23\) and the class number \( h_{K_{M_w}} = 1 \). Note that \( K_{M_w} \) is equal to the invariant trace field of the Weeks manifold since \( \pi_1(M_w) = \pi_1(M_w)^{(2)} \). The ring \( \mathbb{Z}[T]/(T^3 - T - 1) \subset K_{M_w} \) is equal to the ring of integers of \( K_{M_w} \). (We can check it by PARI-GP, for instance.) Hence the Hasse-Weil zeta function of the Weeks manifold \( M_w \) is written as follows:
\[
\zeta(X_0(M_w), s) = \zeta(X_0(M_w), s) = \zeta(\text{Spec } \mathbb{Z}[T]/(T^3 - T - 1), s) = \zeta(K_{M_w}, s).
\]

Since the holonomy representation is irreducible, if \( \rho : \pi_1(M_w) \to \text{SL}_2(\mathbb{C}) \) is a lift of the holonomy representation their images are expressed as
\[
\rho(a) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} y & 0 \\ r & y^{-1} \end{pmatrix}
\]
up to conjugation (cf. [25], lemma 7). Then the images \( \rho(w_1) \) and \( \rho(w_2) \) are expressed by the matrices \( W^1 = (w^1_{i,j}(x,y,r)) \) and \( W^2 = (w^2_{i,j}(x,y,r)) \), where \( w^1_{i,j}(x,y,r) \) and \( w^2_{i,j}(x,y,r) \) are polynomials in \( x, y, r \). We obtain the following solutions of these polynomials by Maple:
\[
(x,y,r) = \{(t,1,0), (t,x,-xy^5 + xy^4 + y^5 - 3xy^3 - y^4 + xy^2 + 3y^3 - 4xy + x + 2y)\}.
\]

Here \( t \) satisfies the equation \( t^4 + t^3 + t^2 + t + 1 = 0 \). \( y \) is a root of \( F(z) := z^6 - z^5 + 3z^4 - z^3 + 3z^2 - z + 1 \) and \( x \) is a root of \( T^2 + \alpha T + 1 \) for \( \alpha := y^5 - y^4 + 3y^3 - y^2 + 2y - 1 \). The representation defined by \( (x,y,r) = (t,1,0) \) is reducible. Thus we only need to consider the other case.

Since \( F(y) = y^6 - y^5 + 3y^4 - y^3 + 3y^2 - y + 1 = 0 \), we have \( y \alpha = -y^2 - 1 \). Hence \( \alpha = -(y + y^{-1}) \). Therefore \( x \) is either \( y \) or \( y^{-1} \).

When \( x = y \), we have \( r = -y^6 + 2y^5 - 4y^4 + 4y^3 - 4y^2 + 3y = \alpha + 2 = 2 - y - y^{-1} \). When \( x = y^{-1} \), we have \( r = y^5 - 2y^4 + 3y^3 - 3y^2 - 3y - 4 + y^{-1} = y^{-1}(y^6 - 2y^5 + 4y^4 - 3y^3 + 3y^2 - 4y + 1) = y^{-1}(-y^5 + y^4 - 2y^3 - 3y) = y^{-1}(-\alpha + y^3 - y^2 - y - 1) = y^2 + y^2 - y - y^{-1} \).

In each case we have \( \text{Tr} \rho([a,b]) = 2 - \left((y + y^{-1}) - 2\right)^2 \left((y + y^{-1}) + 1\right) \). Note that \( \rho \) is reducible if and only if \( \text{Tr} \rho([a,b]) = 2 \), which is equivalent to \( (y + y^{-1}) = 2 \) or \(-1 \). However we see from \( F(y) = 0 \) that \( y + y^{-1} \) is a root of \( T^3 - T^2 + 1 \). Therefore \( \rho \) is irreducible.
Note that the character $\chi$ of $\rho : \pi_1(M_W) \to \text{SL}_2(\mathbb{C})$ is determined by $(\chi(a), \chi(b), \chi(ab))$ since $\pi_1(M_W)$ is generated by two elements $a, b$. Now in each case we have
\[(\chi(a), \chi(b), \chi(ab)) = (y + y^{-1}, y + y^{-1}, (y + y^{-1})^2 - (y + y^{-1})).\]
Therefore they define the same representation up to conjugacy.

The element $y + y^{-1}$ is a root of the polynomial $T^3 - T^2 + 1$ which defines the trace field $K_{M_W}$ (we remark that we can replace $T^3 - T^2 + 1$ with $T^3 - T - 1$ appeared in the description of $X(M_W)(\mathbb{C})_{\text{tr}}$ by the change of variables). Hence there are two possibilities of irreducible representations whose traces are non-real numbers. Since each holonomy representation $\rho_M : \pi_1(M_W) \to \text{PSL}_2(\mathbb{C})$ has a unique lift, They are the lifts of the two holonomy representations of $M_W$ (which are not $\text{PSL}_2(\mathbb{C})$-conjugate but complex conjugate representations).

See §3.2 in [8] for more detailed explanation in another group presentation of the Weeks manifold case.

4.2. **Meyerhoff manifold case.** The Meyerhoff manifold $M_M$ is the complete orientable hyperbolic 3-manifold obtained by $(5, 1)$ Dehn surgery on the figure 8 knot complement. This is a unique arithmetic closed hyperbolic 3-manifold up to isometry with second smallest volume (for the arithmeticity, see [7]. For a proof of the second smallness of the volume, see [8]). Its fundamental group has the following presentation
\[\pi_1(M_M) \cong \gen{a, b | w_1 = w_2 = 1},\]
where
\[w_1 = aBAbABabb, \quad w_2 = aBabaaaaaabAB.\]
The following three polynomials define the $\text{SL}_2(\mathbb{C})$-character variety of $\pi_1(M_M)$:
\[x - z,\]
\[y + z^6 - 3z^5 - 2z^4 + 11z^3 - 3z^2 - 8z + 2,\]
\[z^7 - 4z^6 + z^5 + 13z^4 - 13z^3 - 6z^2 + 9z - 2.\]

Then the subset $X(M_M)(\mathbb{C})_{\text{tr}}$ consists of points of the form $(\alpha, 1 - \alpha - \alpha^2 + \alpha^3, \alpha)$, where $\alpha$ is a root of the polynomial $f(T) = T^4 - 3T^3 + T^2 + 3T - 1$. Therefore the trace field $K_{M_M}$ is $\mathbb{Q}[T]/(f)$, its ring of integers $\mathcal{O}_{K_{M_M}}$ is $\mathbb{Z}[T]/(f)$, its discriminant $d_{K_{M_M}}$ is $-283$ and the class number $h_{K_{M_M}} = 1$. Note that $K_{M_M}$ is isomorphic to the invariant trace field of the Meyerhoff manifold $M_M$.

We can compute an explicit description of a lift of the holonomy representation as well as the Weeks manifold case.

Let $\rho : \pi_1(M_M) \to \text{SL}_2(\mathbb{C})$ be a lift of the holonomy representation and put the images at $a, b$ as
\[\rho(a) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} y & 0 \\ r & y^{-1} \end{pmatrix}\]
up to conjugation. Then the possibilities of $(x, y, r)$ are $\{(t, 1, 0), (x, y, r)\}$. Here $t$ satisfies the equation $t^4 + t^3 + t^2 + t + 1 = 0$ and $x$ is a root of
\[F(z) := z^8 - 3z^7 + 5z^6 - 6z^5 + 7z^4 - 6z^3 + 5z^2 - 3z + 1.\]
If we put
\[\alpha = 3x^7 - 7x^6 + 10x^5 - 11x^4 + 13x^3 - 9x^2 + 8x - 4\]
Finally, equation 2, the possibilities of $(x$ is the invariant trace field. Hence $x + x^{-1}$ is a root of $T^4 - 3T^3 + T^2 + 3T - 1$. A simple computation shows that $y + y^{-1} = -\alpha = (x + x^{-1})^3 - (x + x^{-1})^2 - (x + x^{-1}) + 1$

and

$$r = (x^{-1} - x)y + x^{-1}(\alpha + x^2 + 1).$$

Therefore we have $(\text{Tr}\rho(a), \text{Tr}\rho(b), \text{Tr}\rho(ab)) = (x + x^{-1}, -\alpha, x + x^{-1})$.

Note that $f(T) = T^4 - 3T^3 + T^2 + 3T - 1$ is the minimal polynomial of the trace field $K_{M}$, which is also the invariant trace field of $M$. Hence if we take one of the two complex roots of $f(T)$ it defines a lift of the holonomy representation. Refer [7] for more detailed discussion in another group presentation of $\pi_1(M)$.

We give additional 3 examples of arithmetic closed 3 manifolds shortly.

**Example 4.1.** Let $M = \text{m010}(-1,2)$ in the list of SnapPea. This is the third smallest volume arithmetic closed orientable hyperbolic 3-manifold. The fundamental group has a group presentation

$$\pi_1(M) \cong \langle a, b \mid w_1 := aBab^3Babab, w_2 := ab^2A^2b^2aB = 1 \rangle.$$ 

The irreducible character variety $X(M)(\mathbb{C})_{\text{ irr}}$ is the zero set of the polynomial $f(T) := T^4 - 2T^2 + 4$. Thus $\mathbb{Q}[T]/f(T)$ is the trace field of $M$. We remark that the trace field is not equal to the invariant trace field since $\pi_1(M)^{ab} \cong \mathbb{Z}/6 \mathbb{Z} \oplus \mathbb{Z}/3 \mathbb{Z}$. In this case $\mathbb{Q}[T]/(T^2 - T + 1)$ is the invariant trace field.

For a representation $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ such that

$$\rho(a) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} y & 0 \\ r & y^{-1} \end{pmatrix},$$

the possibilities of $(x, y, r)$ are $(t, 1, 0), (x, y, r)$). Here $t$ satisfies the equation $(t^2 - t + 1)(t^2 + t + 1) = 0$. In the other case $x$ is a root of $F(z) := z^8 + 2z^6 + 6z^4 + 2z^2 + 1$ and $y$ satisfies the equation $2y^2 + 6y + 2 = 0$ for $\alpha = x^6 + 2x^4 + 5x^2$. Note that $y + y^{-1} = -\alpha/2 = (x + x^{-1})^2/2$. Finally $r$ is given by

$$r = 2^{-1}(x^7 + 2x^5 + 7x^3 + 2x) + (-x^7 - 2x^5 - 6x^3 - 3x)y$$
$$= 2^{-1}x^{-1}(x^8 + 2x^6 + 7x^4 + 2x^2) + x^{-1}y(-x^8 - 2x^6 - 6x^4 - 3x^2)$$
$$= 2^{-1}x^{-1}(x^8 - 1) + x^{-1}y(-x^2 + 1)$$
$$= 2^{-1}(x^3 - x^{-1}) + y(x^{-1} - x).$$

Hence we have

$$xy + (xy)^{-1} + r = x^{-1}(y + y^{-1}) + 2^{-1}(x^3 - x^{-1})$$
$$= 2^{-1}(-x^5 - x^3 - 5x - x^{-1}) \quad \text{(apply } y + y^{-1} = -\alpha/2\text{)}$$
$$= 2^{-1}(x^3 + x + x^{-1} + x^3) \quad \text{(use } F(x) = 0\text{)}$$
$$= 2^{-1}((x + x^{-1})^3 - 2(x + x^{-1})).$$
Therefore the character of \( \rho \) is determined by

\[
(\text{Tr} \rho(a), \text{Tr} \rho(b), \text{Tr} \rho(ab)) = (x + x^{-1}, (x + x^{-1})^2/2, ((x + x^{-1})^3 - 2(x + x^{-1}))/2).
\]

The trace field \( K_M = \mathbb{Q}[T]/(f(T)) \) is a quartic totally imaginary field and each root \( x + x^{-1} \) of \( f(T) \) defines one of the two lifts of the two complex conjugate holonomy representations \( \rho_M : \pi_1(M) \rightarrow \text{PSL}_2(\mathbb{C}) \).

**Example 4.2.** Put \( M = m003(-4,3) \) in the list of SnapPea. This is the fourth smallest volume arithmetic closed hyperbolic 3-manifold. A group presentation of \( \pi_1(M) \) is

\[
\pi_1(M) \cong \langle a, b \mid w_1 := a^2bAb^3Ab, w_2 := abaB^2a^2b = 1 \rangle.
\]

The irreducible character variety \( X(M)(\mathbb{C})_{\text{irr}} \) is defined by the polynomial \( f(T) := T^4 - T^3 - 2T^2 + 2T + 1 \). Thus \( \mathbb{Q}[T]/(f(T)) \) is the trace field of \( M \), and it also is the invariant trace field of \( M \).

The possibilities of \((x, y, r)\) are \((t, 1, 0), (x, y, r)\). Here \( t \) satisfies the equation \( t^4 + t^3 + t^2 + t + 1 = 0 \). In the other case \( y \) is a root of \( F(z) := z^8 - z^7 + 2z^6 - z^5 + 3z^4 - z^3 + 2z^2 - z + 1 \) and \( x \) satisfies the equation \( x^2 + ax + 1 = 0 \) for

\[
\alpha = -y^3 + y^2 - y = (y + y^{-1})^3 - (y + y^{-1})^2 - (y + y^{-1}) + 1.
\]

Finally \( r \) is written as

\[
r = x(-y^7 + y^6 - 2y^5 + y^4 - 3y^3 + y^2 - 3y + 1) + (-y^7 + y^6 - 2y^5 - 2y^3 - y) = x(-y + y^{-1}) + (-y^4 + y^3 - y^2 + y - 1 + y^{-1}).
\]

Hence we have

\[
xy + \text{(xy)}^{-1} - r = (x + x^{-1})y^{-1} + (-y^4 + y^3 - y^2 + y - 1 + y^{-1}) = y + y^{-1}.
\]

Therefore the lifts of the two holonomy representations \( \rho \) are determined by

\[
(\text{Tr} \rho(a), \text{Tr} \rho(b), \text{Tr} \rho(ab)) = -(y + y^{-1})^3 + (y + y^{-1})^2 + (y + y^{-1}) - 1, y + y^{-1}, y + y^{-1})
\]

for two complex roots \( y + y^{-1} \) of \( f(T) = T^4 - T^3 - 2T^2 + 2T + 1 \).

**Example 4.3.** Put \( M = m003(-3,4) \) in the list of SnapPea. It is the seventh smallest volume arithmetic closed hyperbolic 3-manifold.

\[
\pi_1(M) \cong \langle a, b \mid w_1 := ab^2aba^2b, w_2 := abABabABA^2b^2a^2BAb = 1 \rangle.
\]

The irreducible \( \text{SL}_2(\mathbb{C}) \)-character variety \( X(M)(\mathbb{C})_{\text{irr}} \) is defined by \( f(T) := T^6 - T^2 - 1 \). Thus \( \mathbb{Q}[T]/(f(T)) \) is the trace field of \( M \), and \( \mathbb{Q}[T]/(T^3 - T^2 + 1) \) is the invariant trace field. This is equal to the invariant trace field of the Weeks manifold.

If \( \rho : \pi_1(M) \rightarrow \text{SL}_2(\mathbb{C}) \) is a representation such that

\[
\rho(a) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} y & 0 \\ r & y^{-1} \end{pmatrix}
\]

then \((x, y, r)\) is determined as follows: \( x \) is a root of \( F(z) := z^{12} + 6z^{10} + 14z^8 + 17z^6 + 14z^4 + 6z^2 + 1 \) and \( y \) satisfies the equation

\[
y^2 + \alpha y + 1 = 0 \text{ for } \alpha = -x^{10} - 6x^8 - 14x^6 - 17x^4 - 13x^2 - 5 =
\]
\((x + x^{-1})^2 - 1 \). Finally \( r \) is written as
\[
 r = y(-x^{11} + 6x^9 - 14x^7 - 17x^5 - 14x^3 - 7x) + (6x^{11} + 34x^9 + 73x^7 + 79x^5 + 59x^3 + 17x)
\]
\[
= y(-x + x^{-1}) + (6x^{11} + 34x^9 + 73x^7 + 79x^5 + 59x^3 + 17x).
\]
Hence we have
\[
xy + (xy)^{-1} + r = (y + y^{-1})x^{-1} + (6x^{11} + 34x^9 + 73x^7 + 79x^5 + 59x^3 + 17x)
\]
\[
= 6x^{11} + 35x^9 + 79x^7 + 93x^5 + 76x^3 + 30x + 5x^{-1} \quad \text{(apply } y + y^{-1} = -\alpha\text{)}
\]
\[
= -(x^5 + x^{-5} + 5(x^3 + x^{-3}) + 9(x + x^{-1})) \quad \text{(use } F(x) = 0 \text{ repeatedly)}
\]
\[
= -(x + x^{-1})^5 + (x + x^{-1}).
\]
Since \( f(T) = T^6 - T^2 - 1 \) has two real roots and four complex roots, all the four lifts of the two holonomy representations \( \rho \) are determined by
\[
(\text{Tr} \rho(a), \text{Tr} \rho(b), \text{Tr} \rho(ab)) = (x + x^{-1}, 1 - (x + x^{-1})^2, -(x + x^{-1})^5 + (x + x^{-1}))
\]
for four complex roots \( x + x^{-1} \) of \( f(T) \).

|    | defining polynomial \( f \) of \( X(M)_{\text{tr}}(\mathbb{C}) \) |
|----|--------------------------------------------------|
| Weeks | \( T^3 - T - 1 \) |
| Meyerhoff | \( T^4 - 3T^3 + T^2 + 3T - 1 \) |
| m010 (-1,2) | \( T^4 - 2T^2 + 4 \) |
| m003 (-4,3) | \( T^4 - T^3 - 2T^2 + 2T + 1 \) |
| m004 (6,1) | \( T^6 - 7T^4 + 14T^2 - 4 \) |
| m003 (-3,4) | \( T^6 - T^2 - 1 \) |

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