Static brane–like vacuum solutions
in $D \geq 5$ dimensional spacetime
with positive ADM mass but no horizon

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ABSTRACT

We describe static, brane–like, solutions to vacuum Einstein’s equations in $D = n + m + 2$ dimensional spacetime with $m \geq 2$ and $n \geq 1$. These solutions have positive ADM mass but no horizon. The curvature invariants are finite everywhere except at $r = 0$ where $r$ is the radial coordinate in the $m + 1$ dimensional space. The presence of $n \geq 1$ extra dimensions is crucial for these properties. Such solutions may be naturally anticipated if Mathur’s fuzzball proposal for black holes is correct.
1. Introduction

We consider a $D = n + m + 2$ dimensional spacetime with $m \geq 2$. The $n$ dimensional space may be taken to be compact and toroidal, or to be $\mathbb{R}^n$. We assume a static, brane–like ansatz for the line element, hence the metric components are independent of time and the $n$ dimensional coordinates, and depend only on the radial coordinate $r$ of the $m + 1$ dimensional space.

We study the solutions to vacuum Einstein’s equations $R_{MN} = 0$ where $R_{MN}$ is the $D$ dimensional Ricci tensor. The solutions are given by

$$ds^2 = -e^{2a_0 F}dt^2 + \sum_i e^{2a_i F} dx^i dx^i + \frac{dr^2}{f} + r^2 d\Omega_m^2$$

where $i = 1, 2, \ldots, n$ and $(a_0, a_i)$ are constants obeying $a_0 + \sum_i a_i = \frac{1}{2}$. The solutions are required to be asymptotically Minkowskian with positive ADM mass. Einstein’s equations can be solved in a closed form for $F(f)$ and $r(f)$ which, however, is cumbersome to analyse. But it turns out that the qualitative properties of the evolution of $f(r)$ and $e^{F(r)}$ can be understood without this closed form. We find the following results.

For $K = a_0^2 + \sum a_i^2 - \frac{1}{4} = 0$, we get $e^{F} = f = 1 - \frac{M_\infty}{M_\infty}$, where $M_\infty > 0$ is a constant proportional to ADM mass. Standard black $n$–brane solution then follows for $2a_0 - 1 = a_i = 0$; for other values of $(a_0, a_i)$ satisfying $K = 0$, one obtains solutions studied e.g. in [1].

For $K > 0$, the solutions have novel properties. For these solutions, as $r$ decreases from $\infty$ to $0$:

- $f(r)$ decreases from $1$, reaches a minimum $f_0 > 0$ at some $r_0$, increases to $1$ again at some $r = r_1 < r_0$, and then increases to $\infty$ in the limit $r \to 0$. In particular, $f(r)$ always remains positive and non zero.

- $e^{F(r)}$ decreases monotonically from $1$ to $0$.

- Define a ‘mass’ function $M(r) = r^{m-1} (1 - f)$. As $r$ decreases from $\infty$ to $0$, $M(r)$ decreases from $M_\infty$ but remains positive for $r_1 < r$, vanishes at $r_1$, becomes negative for $r < r_1$, and decreases to $-\infty$ in the limit $r \to 0$. The evolution of $M(r)$ from $M_\infty$ to $-\infty$ is monotonic.
These features are in contrast to the standard Schwarzschild or black 
$n$–brane solution where, as $r$ decreases from $\infty$ to $0$, $M(r) = M_\infty$
remains constant and $e^F = f = 1 - \frac{M_\infty}{r}$ decreases from $1$, vanishes
at some $r_h$, and decreases to $-\infty$ in the limit $r \to 0$.

Note that $f$ and $e^F$ and, hence, all metric components remain non zero
and finite for $0 < r \leq \infty$. This implies that there is no horizon
and that the curvature invariants are all finite for $0 < r \leq \infty$. As $r \to 0$
we have $f \to \infty$ and $e^F \to 0$. In this limit, the tidal forces also diverge and there is
a curvature singularity.

The presence and the role of the $n$–dimensional space is crucial for these
properties of the solutions. The absence of the $n$–dimensional space, or
the trivialty of its metric, means that $a_i = K = 0$ which leads to the
standard Schwarzschild, or black $n$–brane, solution. We assume that $a_i \neq
0$ generically and, further, that $K > 0$ which then lead to the solutions
described here.

We discuss how the $K > 0$ solutions may be naturally anticipated if one
assumes that Mathur’s fuzzball proposal [6] – [10] for black holes is correct.

This paper is organised as follows. In section 2, we present the equations
and write them in a convenient form. In section 3, we analyse the equations
and describe the evolution of $f$, $e^F$, and $M$. In section 4, we discuss the
physical relevance of the $K > 0$ solutions, and conclude in Section 5.

2. Einstein’s equations in vacuum

Consider a $D = n + m + 2$ dimensional spacetime with $m \geq 2$. Consider
static, brane–like ansatz for the line element $ds$ given by
\[ ds^2 = -e^{2\psi} dt^2 + \sum_i e^{2\lambda^i} (dx^i)^2 + e^{2\lambda} dr^2 + e^{2\sigma} d\Omega_m^2 \]
where $d\Omega_m$ is the standard line element on an $m$ dimensional unit sphere,
$i = 1, 2, \cdots, n$, and $(\psi, \lambda^i, \lambda, \sigma)$ are all functions of $r$ only. Such an ansatz is

\[^1\]Solutions with negative masses in the interior and with no horizon occur in [2, 3] which
study the back reaction of Hawking radiation in four dimensional spacetime. Solutions
with a central singularity and with no horizon occur in [4, 5] which study static solutions
with incoming radiation matching the outgoing one; the solutions in [4] can be matched
onto negative mass Schwarzschild solutions.
suitable for describing intersecting brane configurations of string/M theory. We take the $n$ dimensional space described by $x^i$ coordinates to be compact and toroidal but it may also be taken to be $\mathbb{R}^n$.

Let $\Lambda = \psi + m \sigma + \sum_i \lambda^i$. The vacuum Einstein’s equations $\mathcal{R}_{MN} = 0$, where $\mathcal{R}_{MN}$ is the $D$ dimensional Ricci tensor, then give

\[
\Lambda^2 - (\psi^2_r + m \sigma^2_r + \sum_i (\lambda^i_r)^2) = m (m - 1) e^{2\lambda - 2\sigma}
\]

\[
\psi_{rr} + (\Lambda_r - \lambda_r) \psi_r = 0
\]

\[
\lambda^i_{rr} + (\Lambda_r - \lambda_r) \lambda^i_r = 0
\]

\[
\sigma_{rr} + (\Lambda_r - \lambda_r) \sigma_r = (m - 1) e^{2\lambda - 2\sigma}
\]

where $r-$subscripts denote derivatives with respect to $r$. Equations (3) and (4) imply that

\[
\psi = a_0 F, \quad \lambda^i = a_i F
\]

where $a_0$ and $a_i$ are constants and the function $F(r)$ is defined by

\[
F_{rr} + (\Lambda_r - \lambda_r) F_r = 0 \quad \Rightarrow \quad e^{\Lambda - \lambda} F_r = (m - 1) \mathcal{N},
\]

with $\mathcal{N}$ an integration constant. Choose $e^\sigma = r$ so that $r$ denotes the physical size of the $m$ sphere. Then

\[
\Lambda = m \ln r + A F, \quad A = a_0 + \sum_i a_i.
\]

Further, replace $\lambda(r)$ by an equivalent function $f(r)$ and also define a ‘mass’ function $M(r)$ analogous to that in the study of stars, as follows:

\[
e^{-2\lambda} = f(r) = 1 - \frac{M(r)}{r^{m-1}}.
\]

The $D$ dimensional line element $ds$ is now given by

\[
ds^2 = -e^{2a_0 F} dt^2 + \sum_i e^{2a_i F} (dx^i)^2 + \frac{dr^2}{f} + r^2 d\Omega^2_{m-1}.
\]

Equations (2), (5), and (7) now give, after some rearrangements,

\[
2A f (r F_r) = (m - 1) (1 - f) + \frac{K}{m} (r F_r)^2
\]

\[
2A f (r F_r) = 2(m - 1) (1 - f) - r f_r
\]

\[
e^{2A F} f (r F_r)^2 = \frac{(m - 1)^2 \mathcal{N}^2}{r^{2(m-1)}}
\]
where \( K = a_0^2 + \sum_i a_i^2 - A^2 \). From equations (11) and (12), we have
\[
\frac{K}{m} f (r F_r)^2 = (m - 1) (1 - f) - r f_r .
\]
(14)
We set \( 2A = 1 \) with no loss of generality since this just amounts to defining \( F \) by \( F = 2 (\psi + \sum_i \lambda^i) \). Thus, we have \(^2\)
\[
A = a_0 + \sum_i a_i = \frac{1}{2} , \quad K = a_0^2 + \sum_i a_i^2 - \frac{1}{4} .
\]
(15)
Note that if \( a_i \) do not all vanish then, generically, \( K \neq 0 \). \(^3\) Now using equation (12) for \((r F_r)^2\) in equations (14) and (13), we have
\[
\frac{K}{m} (2(m - 1) (1 - f) - r f_r)^2 = f ((m - 1) (1 - f) - r f_r) ,
\]
(16)
which is to be solved for \( f(r) \), and
\[
e^F = \frac{(m - 1)^2 N^2 f}{r^{2(m-1)} (2(m - 1) (1 - f) - r f_r)^2} .
\]
(17)
Thus, once \( f(r) \) is known, \( e^F \) and the line element \( ds \) are completely determined. In order to obtain asymptotically Minkowskian solutions with positive ADM mass, we require that, in the limit \( r \to \infty \),
\[
e^F \to 1 , \quad f(r) \to 1 - \frac{M_\infty}{r^{m-1}} , \quad M_\infty = \text{const} > 0 .
\]
(18)
Note that the condition on \( e^F \) implies that \( N^2 = M_\infty^2 \), and that the condition on \( f \) implies that \( r f_r \to (m - 1)(1 - f) \) irrespective of whether
\(^2\)More generally, one may also consider \( \sigma = ln r + c\ F \) and \( e^{-2(\lambda - c F)} = f(r) \). Then \( A, K, \) and the condition \( 2A = 1 \) are replaced by \( A = a_0 + mc + \sum_i a_i , \quad K = a_0^2 + mc^2 + \sum_i a_i^2 - A^2 , \) and \( 2(A - c) = 1 \).
\(^3\)Indeed, we have \( -\frac{n}{4(n+1)} \leq K \leq \infty \) which can be derived as follows. Let \( \vec{a} = (a_0, a_1, \ldots, a_n) \) and \( \vec{1} = (1, 1, \ldots, 1) \) be two \((n+1)\)-component vectors. Then \( A = \vec{a} \cdot \vec{1} , \quad A^2 = (n+1)|\vec{a}|^2 \cos^2 \theta , \) and \( K = |\vec{a}|^2 - A^2 \). The inequality follows since \( |\vec{a}|^2 = \frac{A^2}{(n+1) \cos \theta} \geq \frac{A^2}{(n+1)} \) and \( A = \frac{1}{2} \).
\( M_\infty \) is positive or negative. Also, in the limit \( r \to \infty \), it follows from equations that (11) – (13)

\[
\begin{align*}
f &= 1 - \frac{M_\infty}{r^{m-1}} + \frac{(m - 1) K}{m} \left( \frac{M_\infty}{r^{m-1}} \right)^2 + \frac{(m - 1) K}{2m} \left( \frac{M_\infty}{r^{m-1}} \right)^3 + \cdots \\
e^F &= 1 - \frac{M_\infty}{r^{m-1}} + \frac{(m - 1) K}{6m} \left( \frac{M_\infty}{r^{m-1}} \right)^3 + \cdots \quad (19)
\end{align*}
\]

where \( \cdots \) denote terms of \( \mathcal{O} \left( r^{-4(m-1)} \right) \). ADM mass is given, using the asymptotic form of \( e^F \), by

\[
M_{ADM} = \frac{m \omega_m}{16\pi G_D} M_\infty \left( 1 - \frac{2(m - 1)}{m} \sum_i a_i \right) \quad (20)
\]

where \( \omega_m \) is the volume of the \( m \) dimensional unit sphere and \( G_D \) the \( D \) dimensional Newton’s constant. The last term can be ensured to be \(< 1\) by choosing \( (\sum_i a_i) \) to be sufficiently small. The expression for \( M_{ADM} \) is obtained by using an effective \( m + 2 \) dimensional metric in Einstein frame, and also by using the formula given in [11] with a modification: the formula given there applies to the case where \( \lambda_1 = \cdots = \lambda^n \). The term equivalent to \( n\lambda^1 \) there is replaced by \( \sum_i \lambda^i \) when \( \lambda^i \)’s are unequal.

Equations (16) and (17) for \( f \) and \( e^F \) may be written in a more convenient form. Define a new variable \( R \) and a constant \( b \) by

\[
R = r^{m-1} \quad \text{and} \quad b = \frac{4(m - 1) K}{m} \quad . \quad (21)
\]

Then \( r(\ast)_r = (m - 1) R(\ast)_R \), where \( R \)-subscripts denote derivatives with respect to \( R \), and equations (16) and (17) become

\[
b \left( 2(1 - f) - Rf_R \right)^2 = 4f \left( 1 - f - Rf_R \right) \quad (22)
\]

and

\[
e^F = \frac{M_\infty^2 f}{R^2 \left( 2(1 - f) - Rf_R \right)^2} \quad . \quad (23)
\]

Equation (18) now means that, in the limit \( R \to \infty \),

\[
e^F \to 1 \quad , \quad f(R) \to 1 - \frac{M_\infty}{R} \quad , \quad M_\infty = \text{const} > 0 \quad . \quad (24)
\]
The quadratic equation (22) can be solved for $Rf_R$. The resulting expressions for $Rf_R$ and $e^F$ are given, after a little algebra, by

$$Rf_R = \frac{2 (1 - f) (f - f_0)}{f - f_0 \pm \sqrt{\alpha f (f - f_0)}}$$

(25)

and

$$e^F = \frac{M^2_\infty (\sqrt{f - f_0} \pm \sqrt{\alpha f})^2}{4 \alpha R^2 (1 - f)^2}$$

(26)

where square roots are always to be taken with a positive sign and

$$\alpha = \frac{1}{1 + b} , \quad f_0 = 1 - \alpha = \frac{b}{1 + b} .$$

(27)

Among the ± signs in equation (25) for $Rf_R$, and correspondingly in equation (26) for $e^F$, + sign is to be chosen in the limit $R \to \infty$ so that, for any $\alpha > 0$ , \footnote{Note that the inequalities on $K$ given in footnote 3 and the definitions $b = \frac{4(m-1)K}{m}$ and $\alpha = \frac{1}{1+b}$ imply that $-\frac{n(m-1)}{m(n+1)} \leq b \leq \infty$ and $\frac{n(m+1)}{m(n+1)} \geq \alpha \geq 0$.} one has $Rf_R \to 1 - f$ in that limit. This is easily checked since $f \to 1$ and $f - f_0 \to \alpha$. This branch choice also gives $Rf_R = 1 - f$ in the limit $b \to 0$, equivalently $\alpha \to 1$. This is also easily checked since $f_0 \to 0$ in the limit $\alpha \to 1$.

Defining a new variable $h$ by $\sqrt{f - f_0} = \epsilon_h \sqrt{\alpha} h$ where $\epsilon_h = Sgn h$, equation (25) becomes

$$\frac{dR}{R} = dh \frac{\epsilon_h \sqrt{1 - \alpha + \alpha h^2}}{1 - h^2} ,$$

(28)

which can be integrated, and thus $R(h)$ obtained, in a closed form. But this closed form involves $ln$ and $Sinh^{-1}$ terms; it is difficult to invert it to obtain $h(R)$; and its analysis is cumbersome even in special limits. Hence, we work with equation (25) itself.

3. Analysis of solutions

\footnote{We have taken the solution to the quadratic equation $\hat{a}x^2 + \hat{b}x + \hat{c} = 0$ in the form $x = \frac{-\hat{b} \pm \sqrt{\hat{b}^2 - 4\hat{a} \hat{c}}}{2\hat{a}}$ which is more convenient here.}
If there are no compact directions, i.e. if \( n = 0 \), or if \( a_i = 0 \) for all \( i \) then we have \( 2a_0 = 1 \) and \( K = 0 \). But \( K = 0 \) for other choices of \((a_0, a_i)\) also, see equation (15). If \( K = 0 \) then \( b = 0 \) and equation (22), together with the boundary conditions (24), implies that

\[
1 - f - R f_R = M_R = 0 \implies M(R) = M_\infty
\]

(29)

and, hence, \( f = 1 - \frac{M_\infty}{R} \). It then follows from equation (23) that \( e^F = f \). Schwarzchild or black \( n \)-brane solution follows when \( 2a_0 = 1 \), and \( n = 0 \) or \( a_i = 0 \), but there are solutions for other values of \((a_0, a_i)\) which satisfy \( 2A - 1 = K = 0 \). Such solutions, including also the parameter \( c \) mentioned in footnote 2, have been used in [1] to generate, following the methods of [12] – [14], the multi parameter solutions studied in [15] – [19] in the context of non BPS branes and tachyon condensation.

\[
K > 0 \quad \text{case} : \quad b > 0 \quad , \quad \alpha < 1
\]

From now onwards, we assume that \( a_i \) do not all vanish and that \( K \neq 0 \), hence \( b \neq 0 \). Equation (22) implies that \( M_R = 1 - f - R f_R \neq 0 \) and, hence, the mass function \( M(R) = R (1 - f) \) is non trivial. We now study the solutions \( f(R) \) to the equation

\[
R f_R = \frac{2 \, (1 - f) \, (f - f_0)}{f - f_0 + \sqrt{\alpha \, f \, (f - f_0)}}.
\]

(30)

We have chosen the positive square root branch, for reasons explained below equation (27). Assuming that \( f(R) \to 1 - \frac{M_\infty}{R} \) in the limit \( R \to \infty \), with \( M_\infty > 0 \) a constant, we study the behaviour of \( f(R) \) as \( R \) decreases from \( \infty \).

For \( K < 0 \), we are unable to find \( f(R) \) for all \( R \), with \( M_\infty > 0 \). Solutions exist with \( M_\infty < 0 \) which, however, are likely to be of no physical interest. Therefore, we study only the \( K > 0 \) case here.

Consider the \( K > 0 \) case. For \( n \geq 2 \), \( K > 0 \) can be ensured by choosing \( a_i \) such that \( a_i \) do not all vanish but \( \sum_i a_i = 0 \). In this case, it follows that

\[
K > 0 \quad \text{case} : \quad b > 0 \quad , \quad \alpha < 1
\]
\[ a_0 = \frac{1}{2}, \quad K = \sum_i a_i^2 > 0, \quad M_{\text{ADM}} > 0 \] always, and
\[ ds^2 = -e^F dt^2 + \sum_i e^{2a_i F} (dx^i)^2 + \frac{dr^2}{f} + r^2 d\Omega_m^2, \quad (31) \]

see equations (15), (20), and (10). Now \( b > 0 \) since \( K > 0 \), and it follows from equation (27) that \( \alpha < 1 \) and \( f_0 = 1 - \alpha > 0 \). We now have from equation (30) that \( R f_R > 0 \) and, hence, \( f_R > 0 \) for \( f_0 < f < 1 \). Therefore, as \( R \) decreases from \( \infty \), the function \( f(R) \) continuously decreases from 1.

**Evolution of \( f(R) \) near \( R_0 \) where \( f(R_0) = f_0 \)**

Let \( f(R) = f_0 > 0 \) at \( R = R_0 \). As \( R \) approaches \( R_0 \) from above, \( i.e. \) as \( R \to R_{0+} \), it follows from equation (30) that
\[ R f_R \to 2 \sqrt{\frac{\alpha}{1-\alpha}} \sqrt{f - f_0} \to 0_+ . \quad (32) \]

Further, using \( R f_{RR} = (R f_R)_R - f_R \), and after a little algebra, it follows that, as \( R \to R_{0+} \),
\[ R f_{RR} \to \frac{2}{R_0} \frac{\alpha}{1-\alpha} > 0 . \quad (33) \]

This implies that, as one goes from \( R > R_0 \) to \( R < R_0 \), the derivative \( f_R \) goes from positive to negative values, becoming zero and changing sign at \( R_0 \). Hence, the function \( f(R) \) decreases for \( R > R_0 \), reaches a minimum \( f_0 > 0 \) at \( R_0 \), and then starts to increase for \( R < R_0 \).

Now note that the expression inside the square root in equation (30) can be written as
\[ \alpha f(f - f_0) = (f - f_0)^2 + (1 - \alpha) (1 - f) (f - f_0) > (f - f_0)^2 , \quad (34) \]
the last inequality being valid as long as \( f_0 < f < 1 \), which is true near \( R_0 \) since \( 1 > f \geq f_0 \) there. Therefore, one has to choose the negative square root branch for \( R < R_0 \) in order to accomodate the change of sign of \( f_R \) at \( R_0 \). Hence, for \( R < R_0 \), we have
\[ R f_R = \frac{2 (1-f) (f - f_0)}{f - f_0 - \sqrt{\alpha f (f - f_0)}} . \quad (35) \]
Note that, as \( R \to R_0 \) and \( f \to f_0 \), the above equation implies that
\[
Rf_R \to -2 \sqrt{\frac{\alpha}{1-\alpha}} \sqrt{f-f_0} \to 0^-.
\] (36)

The evolution of \( f(R) \) near \( R_0 \) is similar to that of a particle trajectory \( x(t) \) near a turning point. Let the particle velocity be \( \dot{x} = -\sqrt{2(E-V(x))} \), in an obvious notation. \( V(x_0) = E \) near a turning point \( x_0 \) and, as \( x \to x_{0+} \), \( E-V(x) \propto (x-x_0) \) generically. As \( x \to x_{0+} \), the particle velocity \( \dot{x} \) approaches zero. But its acceleration \( \ddot{x} \) remains finite, non zero, and positive. Hence \( \dot{x} \) changes sign at \( x_0 \) and becomes \( \dot{x} = +\sqrt{2(E-V(x))} \), and the trajectory \( x(t) \) reverses its path.

**Evolution of \( f(R) \) near \( R_1 \) where \( R_1 < R_0 \) and \( f(R_1) = 1 \)**

As \( R \) decreases below \( R_0 \), \( f \) increases above \( f_0 \) since \( f_R < 0 \) for \( R < R_0 \). Let \( f(R_1) = 1 \) and \( R_1 < R_0 \). Consider the limit where \( R \to R_1 \) and \( g = 1-f \to 0 \). Noting that \( f-f_0 = \alpha - g \) and
\[
\sqrt{\alpha f (f-f_0)} = \alpha - \frac{(1+\alpha)g}{2} + \mathcal{O}(g^2),
\]
it follows from equation (35) that, in the limit \( g \to 0 \), we have
\[
Rf_R = -\frac{4\alpha}{1-\alpha} + \mathcal{O}(g) < 0.
\] (37)

Note that the above expression is valid for both signs of \( g \) in the limit \( g \to 0 \); equivalently for both \( f < 1 \) and \( f > 1 \) cases in the limit \( f \to 1 \). Thus, \( f_R(R_1) \) remains negative and non zero which implies that as \( R \) approaches \( R_1 \) and decreases further, the function \( f \) approaches 1 and increases further.

In equation (35) for \( Rf_R \), the numerator is positive for \( f < 1 \) and negative for \( f > 1 \). However, the denominator which is negative for \( f < 1 \) becomes positive for \( f > 1 \) since \( \alpha f(f-f_0) < (f-f_0)^2 \) for \( f > 1 \), see equation (34). Hence, \( Rf_R \) is negative for both \( f < 1 \) and \( f > 1 \). This is also clear from equation (37) since it is valid for both signs of \( g \) in the limit \( g \to 0 \). In particular, it follows that \( Rf_R < 0 \) and \( f > 1 \) for \( R < R_1 \).

**Evolution of \( f(R) \) in the limit \( f \to \infty \)**

10
As $R$ decreases below $R_1$, $f$ increases above 1. Consider the limit $f \gg 1$. It then follows from equation (35) that

$$R_f R \simeq - \frac{2f}{1 - \sqrt{\alpha}} \implies f \simeq \left(\frac{\text{const}}{M_\infty^2 R}\right)^{\frac{2}{1 - \sqrt{\alpha}}}$$

and, hence, that $f \to \infty$ as $R \to 0$.

To summarise: the evolution of $f(R)$ for $b > 0$ is as follows. As $R$ decreases from $\infty$ to 0, $f(R)$ decreases from 1, reaches a minimum $f_0 = 1 - \alpha > 0$ at $R = R_0$, increases to 1 again at $R = R_1 < R_0$, and then increases to $\infty$ as $f \sim R^{-\frac{2}{1 - \sqrt{\alpha}}}$ in the limit $R \to 0$.

**Evolution of $e^F$**

The evolution of $e^F$ can be easily read off from equations (23) and (26), which we reproduce below:

$$e^F = \frac{M_\infty^2 f}{R^2 (2(1 - f) - Rf_R)^2} \quad \text{for} \quad 0 < R < \infty$$

$$= \frac{M_\infty^2 (\sqrt{f - f_0} + \sqrt{\alpha f})^2}{4 \alpha R^2 (1 - f)^2} \quad \text{for} \quad R_0 < R < \infty$$

$$= \frac{M_\infty^2 (\sqrt{f - f_0} - \sqrt{\alpha f})^2}{4 \alpha R^2 (1 - f)^2} \quad \text{for} \quad 0 < R < R_0.$$  \hspace{1cm} (41)

The behaviour of $e^F$ in the limit $R \to \infty$ is given by equation (19). It can be checked that $e^F$ remains non zero and finite for $0 < R < \infty$, in particular at $R_0$ and $R_1$; and that, in the limit $R \to 0$ where $f \gg 1$,

$$e^F \sim \frac{M_\infty^2}{R^2 f} \sim \left(\frac{M_\infty}{R}\right)^{\frac{2}{1 - \sqrt{\alpha}}} \to 0.$$ \hspace{1cm} (42)

It can be shown that $F_R \neq 0$ for $R < \infty$. If $F_R = 0$ then it follows from equation (12), and then from equation (22), that

$$2(1 - f) - Rf_R = 0 = 1 - f - Rf_R \implies 1 - f = Rf_R = 0.$$
This is the case at $R = \infty$. From the evolution of $f(R)$, we have $f_R = 0$ but $1 - f = 1 - f_0 = \alpha \neq 0$ at $R = R_0$, and $1 - f = 0$ but $Rf_R \neq 0$ at $R = R_1$. Thus, besides at $R = \infty$, we see from the evolution of $f(R)$ that $1 - f$ and $Rf_R$ do not both vanish and, hence, that $F_R$ cannot vanish. The asymptotic behaviour of $e^F$ given in equations (19) and (42) in the limits $R \to \infty$ and $R \to 0$ then implies that $e^F$ decreases monotonically from 1 to 0 as $R$ decreases from $\infty$ to 0.

It can further be shown that $e^F$ always remains $< f$. Note from equations (19) that $e^F < f$ in the limit $R \to \infty$. It also follows, using equations (12) and (30), that $f(RF_R - Rf_R) > 0$, and hence $F_R > f_R$, for $R_0 < R < \infty$ where $f_0 < f < 1$. It then follows that $e^F < f$ for $R_0 < R < \infty$. Since $e^F$ continues to decrease and $f$ increases above $f_0$ for $R < R_0$, it follows that $e^F < f$ for all $R$.

### Evolution of the mass function $M(R)$

The mass function is defined by $M(R) = R(1 - f)$. Since $M_R = 1 - f - Rf_R$, it follows from equation (22) that $M_R = 0$ and $M(R)$ is constant if and only if $b = 0$. For $b > 0$, it follows from the evolution of $f(R)$ that $M(R)$ is a positive constant $= M_\infty$ at $R = \infty$, remains positive for $R_1 < R < \infty$, vanishes at $R = R_1$, becomes negative for $R < R_1$, and, in the limit $R \to 0$ where $f \gg 1$,

$$M(R) \sim -RF \sim -R^{-\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}} \to -\infty.$$

If $b > 0$ then it follows, for the same reasons as in the case of $F_R$, that $M_R = 1 - f - Rf_R$ cannot vanish for $R < \infty$. Its asymptotic behaviour described above then implies that $M(R)$ decreases monotonically from a positive constant $M_\infty$ to $-\infty$ as $R$ decreases from $\infty$ to 0. We point out here that solutions with negative masses in the interior also occur in [2, 3] which study the back reaction of Hawking radiation in four dimensional spacetime; and in [4] which study static solutions with incoming radiation matching the outgoing one if such solutions are matched onto negative mass Schwarzschild ones.

### Summary of the solutions
In summary, we have the $D = n + m + 2$ dimensional metric components, with $m \geq 2$ and corresponding to static brane–like solutions, given by

$$-g_{tt} = e^{2a_0 F}, \quad g_{ii} = e^{2a_i F}, \quad g_{rr} = \frac{1}{f}$$

which are all functions of $R = r^{m-1}$, with $r$ denoting the physical size of the $m$ sphere. The solutions are all required to have positive ADM mass and the asymptotic behaviour given in equation (18) in the limit $r \to \infty$. We also have

$$A = a_0 + \sum_i a_i = \frac{1}{2}, \quad K = a_0^2 + \sum_i a_i^2 - \frac{1}{4}$$

and the definitions $b = \frac{4(m-1)K}{m}$ and $\alpha = \frac{1}{1+b}$. The standard Schwarzschild solution follows for $2a_0 - 1 = a_i = 0$. For other values of $(a_0, a_i)$ but with $b = 0$, there exist more general solutions. In all these solutions, the metric components vanish or diverge at a non zero, finite value of $R = R_h$, which is either a regular horizon or, possibly, a curvature singularity depending on the values of $(a_0, a_i)$. In all these solutions, the mass function $M(R) = R (1 - f)$ remains constant.

We assume that $b \neq 0$ generically. Then $M(R)$ is non trivial and cannot remain constant. Further assuming that $b > 0$, we have described the evolution of $f$, $e^F$, and $M$. Note that $f$ and $e^F$ and, hence, all metric components remain non zero and finite for $0 < R \leq \infty$. This implies that there is no horizon, and that the curvature invariants are all finite, for $0 < R \leq \infty$.

Consider the limit $R \to 0$. In this limit, we have $f \gg 1$ and $e^F \to 0$. Consider the Riemann tensor components $R_{abcd} = e^M e^N e^P e^Q R_{MNPQ}$ in local tangent frame coordinates. For the $D$ dimensional metric given by equation (1), the non vanishing components of $R_{abcd}$ are given by

$$R_{rr'rr'} = -\delta_{r'r'} e^{-2\lambda} \left( \lambda_{rr}' + (\lambda_r'^{-} - \lambda_r^-) \lambda_r'^{-} \right)$$  \hspace{1cm} (43)

$$R_{rarb} = -h_{arb} e^{-2\lambda} (\sigma_{rr} + (\sigma_r - \lambda_r) \sigma_r)$$  \hspace{1cm} (44)

$$R_{rr'kk'} = (\delta_{r'r'} \delta_{kk'} - \delta_{rr'} \delta_{kk'}) e^{-2\lambda} \left( \lambda_r'^{-} \lambda_k'^{-} \right)$$  \hspace{1cm} (45)

$$R_{araj'b} = -\delta_{a'a'} h_{ab} e^{-2\lambda} \lambda_r'^{-} \sigma_r$$  \hspace{1cm} (46)

$$R_{abcd} = e^{-2\sigma} \rho_{abcd}(h) + (h_{ab} h_{cd} - h_{ac} h_{bd}) e^{-2\lambda} \sigma_r^2$$  \hspace{1cm} (47)
where \( i' = (0, i) \), \( \lambda' = (\psi, \lambda) \), \( h_{ab} \) is the metric on the \( m \) dimensional unit sphere given by \( d\Omega_m^2 = h_{ab}d\theta^a d\theta^b \), and \( \rho_{abcd}(h) \) is the corresponding Riemann tensor. It can now be seen that, in the limit \( R \to 0 \), we have

\[
R_{abcd} \sim \frac{f}{r^2} \to \infty \tag{48}
\]

This implies that the tidal forces diverge and there is a curvature singularity in the limit \( R \to 0 \).

Note that the presence and the role of the \( n \)-dimensional space is crucial for these properties of the solutions. The absence of the \( n \)-dimensional space, or the triviality of its metric, means that \( a_i = b = 0 \), thus leading to the standard Schwarzschild or black \( n \)-brane solution. We have assumed that \( a_i \neq 0 \) generically and, further, that \( b > 0 \) which then lead to the present solutions.

4. Physical relevance of the solutions

Physical relevance of the present solutions can be naturally motivated and, indeed, such solutions may be naturally anticipated if one assumes that Mathur’s fuzzball proposal for black holes is correct. See [6] – [10] for a review of this proposal. Broadly speaking, according to this proposal, the black hole entropy arises due to the microstates of M theory objects, equivalently string theory objects, which are typically bound states of intersecting brane configurations with a large number of low energy excitations living on them.

For example, an effective four dimensional black hole may be described by a \( 22'55' \) configuration which consists of two sets of \( M2 \) branes and two sets of \( M5 \) branes, intersecting according to BPS rules.

According to the fuzzball picture, the spacetime described by such brane configurations is indistinguishable from that of black holes at large distances, typically larger than \( \mathcal{O}(1) \) times the Schwarzschild radius. At shorter distances, the spacetime is different from that of black holes and, in particular, has no horizon.

If this picture is correct then it should be possible to construct a star, modelling its M theory brane constituents by appropriate matter sources. At large distances, it should appear as a spherically symmetric four dimensional (more generally, \((m + 2)\) dimensional) star; should have a finite radius, be
stable, and have no horizon irrespective of how high its mass \( M \) is; and the thermodynamics of its constituents should give an entropy \( \propto M^2 \).

Technically, one constructs the interior of the star and, at its surface, matches the interior solution onto vacuum solutions. If the matching vacuum solution is the standard Schwarzschild one then, for any choice of matter sources that the author can think of, it seems impossible to obtain a star solution with the above properties. Also, such a matching seems to miss a crucial ingredient: that, at a fundamental level, both the spacetime and the constituents of the star are higher dimensional and this higher dimensionality is likely to play an important role.

In [20, 21, 22], we had studied early universe using 22'55' intersecting brane configuration. Starting with a eleven dimensional universe, we found that, at later times, the seven toroidal brane directions cease to expand or contract and stabilise to constant sizes; and, in the limit \( t \to \infty \), the corresponding metric components \( e^{\lambda^i} \to e^{v^i} (1 + c(t)) \) where \( v^i \) and \( \delta > 0 \) are constants and \( |c(t)| \) is finite. This results in an effectively four dimensional expanding universe. The tailing-off behaviour of \( e^{\lambda^i} \) suggests that, in the context of stars also, the internal directions are likely to have non trivial \( r \) dependence in the limit \( r \to \infty \).

This line of reasoning is what led us to study the higher dimensional vacuum solutions, in particular to study the general solutions with non trivial dependence of \( e^{\lambda^i} \). It turned out that such solutions exist indeed, with the properties described in this paper. The early universe study mentioned above also suggests that stars whose exterior solutions are similar to the ones presented here may form in a physical collapse, and that one has to carefully take into account the higher dimensional nature of the constituents.

5. Conclusions

Finding the more general vacuum solutions is only a beginning. It is important to actually construct both equilibrium and collapsing star solutions, study their stability, thermodynamic entropy, and other properties.

Also, it will be interesting to generalise the present solutions to include rotation and charges. One may also start from the present solutions and, using the techniques of \( e.g. \) [12, 13, 14, 1], generate string and M theory brane solutions.
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