Convergence to boundary values in the topology of the inductive limit of generalized holder spaces

O V Grober, T A Grober, A B Bychkov

Don State Technical University, 1 Gagarin square, Rostov-on-Don, 344000, Russia

E-mail: andrewbychkov@mail.ru

Abstract. Both the functions theory of a complex variable and the approximation theory are often used in modern engineering sciences. In the present work, the spaces of analytic functions inside the unit circle satisfying on the boundary a strong Holder condition for a rather wide class of the modulus of continuity are considered. The convergence question of functions from these spaces to their boundary values in the topology of the inductive limit is studied.

Introduction

Let the function $\omega (t)$, $t \geq 0$, be such that $\omega (t) > 0$ for $t > 0$, $\omega (0) = 0$, $\lim_{t \to 0^+} \omega (t) = 0$ and there is a constant $\gamma$ such that

$$\omega (t_1) \leq \gamma (t_2) \frac{\omega (t_2)}{t_2} \leq \gamma (t_1) \frac{\omega (t_1)}{t_1}, \quad \forall t_2 \geq t_1 > 0.$$

The garde of this function kind will be called the continuity modules, since it is known that it comprises all functions that are equivalent to some non-negative concave function on the semi-axis $[0, \infty)$ function that is continuous and is zero at zero.

Let us consider continuous function $f(x) \in C_{[-\pi; \pi]}$. We will say that $f(x)$ belongs to the grade $C_{\omega}[-\pi; \pi]$ if it satisfies (with respect to $\omega(t)$) the Holder condition

$$|f(x_1) - f(x_2)| = O\left(\omega\left(|x_1 - x_2|\right)\right),$$

where $x_1, x_2 \in [-\pi; \pi], |x_1 - x_2| \to 0$. The norm in the space $C_{\omega}[-\pi; \pi]$ is defined by the sum

$$f_{\omega} = f_c + |f|_{\omega}, \quad |f|_{\omega} = \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{\omega\left(|x_1 - x_2|\right)}.$$

It is well known that the grade $C_{\omega}[-\pi; \pi]$, equipped with the above-mentioned Holder norm, is a no separable Banach space.

Let us denote by $C_{\omega}^0[-\pi; \pi]$ the separable subspace of the space $C_{\omega}[-\pi; \pi]$ of functions satisfying on the mentioned segment the strong Holder condition.
\[ f(x_1) - f(x_2) = o(\omega(|x_1 - x_2|)), \quad x_1, x_2 \in [-\pi, \pi], \quad |x_1 - x_2| \to 0. \]

With \( AC_\omega \) we will denote the space of analytic functions in the circle \( |z| < 1 \) and satisfying in \( |z| \leq 1 \) the Holder condition
\[ |f(z_1) - f(z_2)| = O\left(\omega(|z_1 - z_2|)\right), \quad z_1, z_2 \in \{|z| \leq 1\} : |z_1 - z_2| \to 0. \]

Further, let \( AC^0_\omega \) be the subspace of \( AC_\omega \) functions satisfying in the circle \( |z| < 1 \) the strong Holder condition
\[ |f(z_1) - f(z_2)| = o(\omega(|z_1 - z_2|)), \quad z_1, z_2 \in \{|z| \leq 1\} : |z_1 - z_2| \to 0. \]

It is known that \( AC_\omega \) is a non separable Banach space, and \( AC^0_\omega \) is its separable subspace.

Later on, we will need to impose some restrictions on the modulus of continuity. If the condition \( t = o(\omega(t)) \) is satisfied when \( t \to 0 \), then we will say that \( \omega(t) \) belongs to the F grade. Besides, we will use the Zygmund conditions considered in [1]:
\[
\int_0^\delta \frac{\omega(t)}{t} \, dt = O\left(\omega(\delta)\right), \quad \delta \to 0; \quad (1)
\]
\[
\int_\delta^t \frac{\omega(t)}{t^2} \, dt = O\left(\frac{\omega(\delta)}{\delta}\right), \quad \delta \to 0. \quad (2)
\]

Following the article [2] and using the terminology from work [3], the function \( \omega \in F \) satisfying the conditions (1) and (2), will be assigned to the F grade, and will be called an admissible modulus of continuity. For modulus of continuity it is possible to enter an order relation: it is said that \( \omega_1(t) < \omega_2(t) \), if \( \omega_2(t) = o(\omega_1(t)) \) at \( t \to 0 \). Let \( \omega \in F \). The classical example: \( \omega_1(t) = t^\alpha \), \( \omega_2(t) = t^\beta \), \( \alpha < \beta \leq 1 \).

Banach spaces associated with Holder enhanced conditions were actively studied earlier. In [4] for derivative precompact \( S \subset R \) and a derivative dense in \( S \) countable set \( \{x_i\}_{i=0}^{\infty} \), the space basis \( C^0_\omega(S) \) is constructed interpolating at the nodes \( \{x_i\}_{i=0}^{\infty} \) \( \omega \in F \). Under the long-continued assumption of the function concavity \( \omega \), it is shown that this basis is monotonic. Under the stricter constraint \( \omega \in F \), a symmetric interpolating basis was found and a joint isomorphism \( C^0_\omega(S) \sim L_\infty, C^0_\omega(S) \sim c_0 \), [5] was constructed. In [6], the Banach space \( C^\delta_\omega(\mathbb{R}),|| \cdot ||_\omega \) was considered, where
\[
C^\delta_\omega(\mathbb{R}) = \left\{ f \in C^0_\omega(\mathbb{R}) \right\} \forall \varepsilon > 0 \exists l \left( f, \varepsilon \right) \in \mathbb{N} : f \left|_{[l^{-1}, l]} \right| < \varepsilon \}
\]

\( \omega \in \Phi \), and an interpolating basis is constructed in this space, consisting of piecewise linear functions. If \( \Delta \) is a Cantor set and \( \omega(t) \) satisfies (1) then we managed to construct a joint isomorphism \( C^0_\omega(\Delta) \sim L_\infty, C^0_\omega(\Delta) \sim c_0 \) by refusing the constraint (2) and a symmetric space basis \( C^0_\omega(\Delta) \) was constructed [5], [7]. The spaces of analytic functions associated with the generalized strengthened Holder conditions were also actively studied, for example, in [8] and [9].
Let \( \omega \in F \). We denote by \( C_{\omega e} \) the inductive limit of the sequence of Hölder spaces:

\[
C_{\omega e} = \lim_{\varphi \prec \omega, \varphi \in F} C_{\varphi}[-\pi; \pi].
\]

The space \( C_{\omega e} \) is so-called \( LN^* \)-space [10, p. 412] because Hölder spaces are enclosed into each other quite continuously [11, p. 271]. The space \( C_{\omega e} \) is a generalization of the space \( C_{\alpha e} \), for which \( \omega(t) = t^\alpha, 0 < \alpha < 1 \). In this special case the given function space was studied by V.P. Kondakov (on condition of \( f(0) = 0 \)). He established the isomorphism of this space to the space

\[
\lim_{\beta > \alpha} l_{\omega}^*(n^\beta).
\]

where \( l_{\omega}^*(n^\beta) \) is the well-known Köthe space

\[
l_{\omega}^*(n^\beta) = \left\{ \xi = (\xi_n) : \sup_n |\xi_n| n^\beta = |\xi_n| < \infty, \forall \beta < \alpha \right\}.
\]

Later V. P. Kondakov together with S. D. Umalatov obtained the generalization of this result for the case of several variables functions [12].

**Main results**

The following theorem [13] connects inductive limits of Hölder spaces and spaces of strong type.

**Theorem 1.** Let \( \omega \in F \). The presentation

\[
C_{\omega e} = \lim_{\varphi \prec \omega, \varphi \in F} C_{\varphi}^0[-\pi; \pi].
\]

is fair.

Let us note further an important property of analytic functions satisfying the generalized strong Hölder condition.

**Theorem 2.** Let \( \omega \in F \). In order \( f \left( r e^{i\theta} \right) \in AC_{\omega}^0 \) it is necessary and enough that \( f \left( r e^{i\theta} \right) \in AC_{\omega} \) and

\[
f \left( r e^{i\theta} \right) - f \left( e^{i\theta} \right) \rightarrow 0 \text{ at } r \rightarrow 1.
\]

Now we introduce the inductive limit of the sequence of Banach spaces of analytic functions

\[
AC_{\omega e} = \lim_{\varphi \prec \omega, \varphi \in F} AC_{\varphi}^0.
\]

Note that for \( \varphi_1 \prec \varphi_2 \), the space \( AC_{\varphi_1}^0 \) is compactly embedded in \( AC_{\varphi_2}^0 \).

Really, in view of that

\[
f_k \left( r e^{i\theta} \right)_{AC_{\varphi_1}^0} = f_k \left( e^{i\theta} \right)_{AC_{\varphi_2}^0},
\]
we conclude that the convergence of the sequence $\{f_k(re^{i\theta})\}$ in $AC^0_\varphi$ is equivalent to convergence of the sequence $\{f_k(e^{i\theta})\}$ in $C^0_\varphi$. But by $C^0_\varphi$ it is compactly embedded in $C^0_{\varphi_2}$ therefore $AC^0_\varphi$ is compactly embedded in $AC^0_{\varphi_2}$. We showed what $AC_{\varphi_1}$ is a $LN^*$-space.

The spaces considered above were studied by the authors in [13, 14].

The convergence in the space $AC_{\varphi_1}$ is implemented according to topology of an inductive limit as follows:

$$f_k(re^{i\theta}) \rightarrow f(re^{i\theta}) \text{ in } AC_{\varphi_1}, \quad \exists \varphi \prec \omega: f_k(e^{i\theta}) \in C^0_{\varphi}, \quad k = 1, 2, \ldots \quad \text{and}$$

$$f_k(e^{i\theta}) - f(e^{i\theta}) \rightarrow 0.$$

From this fact and theorems 1 and 2 follows

**Theorem 3.** Let $\omega \in \mathbb{F}$. If $f(re^{i\theta}) \in AC_{\varphi_1}$ then $f(re^{i\theta}) \rightarrow f(e^{i\theta})$ in the topology of the space $C_{\varphi_1}$ at $r \rightarrow 1$.

**Summary**

The present report is the continuation of our studies of various properties in function spaces connected with the strong Holder conditions with respect to the continuity moduli of a wide grade. First, authors were interested in the approximation questions and construction of isomorphisms and various bases, their property investigation. Earlier results were obtained in this direction [4, 5, 7, 8]. Some theorems managed to be generalized and transferred to some locally convex spaces that are projective limits of the studied Banach spaces [15, 16, 17]. Now authors try to extend the acquired ideas to the spaces of functions that are inductive limits of Holder spaces [14, 18]. This report is the next step in this direction. The authors’ nearest plans are consideration the Chazarov average

$$\sigma^I_n(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - t) K_n(t) dt,$$

function $f$ satisfying the generalized strengthened Holder condition, where

$$K_n(t) = \frac{1}{n} \left( \frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^2.$$

We are interested in the convergence question of Chazarov averages to their originating function in the topology of the inductive limit of Holder strengthened type spaces and the authors are inclined to assume that at the next CATPID-2020 conference they will report on the modulus of continuity conditions under which Chazarov average will converge to the function $f$ in the space $C_{\varphi_1}$.

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