Boundary-value problems for the squared Laplace operator

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Summary. - The squared Laplace operator acting on symmetric rank-two tensor fields is studied on a (flat) Riemannian manifold with smooth boundary. Symmetry of this fourth-order elliptic operator is obtained provided that such tensor fields and their first (or second) normal derivatives are set to zero at the boundary. Strong ellipticity of the resulting boundary-value problems is also proved. Mixed boundary conditions are eventually studied which involve complementary projectors and tangential differential operators. In such a case, strong ellipticity is guaranteed if a pair of matrices are non-degenerate. These results find application to the analysis of quantum field theories on manifolds with boundary.

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1. - Introduction

Mathematicians and physicists have become familiar, along the years, with the key role played by the operators of Dirac and Laplace type in the investigation of elliptic operators on Riemannian manifolds. For example, it is by now well known that the symbol of the Dirac operator is a generator of all elliptic symbols on (closed) Riemannian manifolds [1], and deep results in index theory have been found by looking at non-local boundary conditions for operators of Dirac type [1,2]. More recently, local supersymmetry and quantum supergravity have led to the consideration of local boundary conditions for the Dirac operator [1,3–5], while theoretical models relevant for quantum chromodynamics rely on mathematical structures weaker than the standard spin-structures where, again, a Dirac operator is found to be quite essential [1,6]. This analysis leads, in turn, to a deeper understanding of the geometry and topology of four-manifolds [6–8]. The operators of Laplace type, on the other hand, occur naturally in the consideration of quantized gauge theories (Abelian and non-Abelian) and Euclidean quantum gravity [5,9], and many efforts are devoted, within that framework, to the evaluation of the one-loop semiclassical approximation in quantum field theory, with the help of heat-kernel and ζ-function methods [4,5,10,11].

Nevertheless, there is still room left for the analysis of many other classes of differential operators on manifolds. In particular, we are here concerned with the so-called conformally covariant operators [12,13]. They arise in the course of studying the behaviour of field theories and differential geometric objects under conformal rescalings of the background metric $g$. More precisely, if the conformal rescaling of $g$ is written in the form $g_\omega = e^{2\omega}g$, a conformally covariant operator $Q$ satisfies, by definition, the transformation property

$$Q_\omega = e^{-(m+4)\omega/2} Q(\omega = 0) e^{(m-4)\omega/2},$$

(1.1)

where $m$ is the dimension of the background Riemannian geometry. On compact Riemannian manifolds without boundary, the recent investigations have focused on the structure of anomalies [14] and on the application to the effective-action formalism [12,13]. For yet other classes of higher order differential operators, obtained by composition of operators of Laplace type, much insight has been gained into the structure of the associated Green functions [15,16]. Even more recently, however, fourth-order elliptic operators have been studied on Riemannian four-manifolds with boundary, motivated by the path-integral quantization programme in gauges which are invariant under conformal rescalings [1,17–19]. Of course, even independently of the particular physical motivation, once that such operators have been studied for a first time in the presence of boundaries, the analysis
of a broader class of examples appears both natural and desirable, to improve the understanding of some key features never studied before. For this purpose, bearing in mind all motivations mentioned so far, but leaving aside, for the time being, the immediate physical applications, we here consider the squared Laplace operator on flat Riemannian four-manifolds with smooth boundary, with its action on symmetric rank-two tensor fields on \((M,g)\):

\[ \Box^2 \equiv g^{ab} g^{cd} \nabla_a \nabla_b \nabla_c \nabla_d, \quad (1.2) \]

\[ \Box^2 : C^\infty(V,M) \to C^\infty(V,M). \quad (1.3) \]

With our notation, \(\nabla\) is the connection on the vector bundle, \(V\), of symmetric rank-two tensor fields on \(M\). On the bundle \(V\), we consider the particular DeWitt super-metric given by

\[ E^{ab \ cd} \equiv \frac{1}{2} \left( g^{ac} g^{bd} + g^{ad} g^{bc} \right) + \alpha g^{ab} g^{cd}, \quad (1.4) \]

with inverse

\[ E^{-1}_{ab \ cd} = \frac{1}{2} \left( g_{ac} g_{bd} + g_{ad} g_{bc} \right) - \frac{\alpha}{(1 + \alpha m)} g_{ab} g_{cd}. \quad (1.5) \]

Since \(M\) is taken to be \(m\)-dimensional, with \(m \neq 2\), our form of the DeWitt super-metric is non-singular, and is positive-definite for all \(\alpha > -\frac{1}{m}\).

The step leading to the definition (1.2) is simple but non-trivial, and hence deserves further comments. Indeed, in Euclidean quantum gravity at one loop, it is necessary to add to the Einstein–Hilbert action a gauge-averaging term, to ensure that the resulting gauge-field operator \(P_{ab \ cd}\) admits a Green function and has a non-degenerate leading symbol [5,9]. In particular, the de Donder gauge-averaging term makes it possible to turn \(P_{ab \ cd}\) into an operator of Laplace type (also called minimal), and in flat space one finds

\[ P_{ab \ cd} h_{cd} = -\Box h_{ab}, \quad (1.6) \]

where

\[ \Box \equiv g^{cd} \nabla_c \nabla_d = \nabla^e \nabla_e. \quad (1.7) \]

Thus, the proof of symmetry of \(P_{ab \ cd}\) is reduced to proving that the operator \(\Box\) defined in (1.7) is symmetric. In a flat Riemannian background, an analysis along similar lines leads to the consideration of the operator (1.2) starting from the composition of the operator \(P_{ab \ cd}\) with itself, i.e.

\[ P_{ab \ cd} P_{cd \ ef} h_{ef} = \Box^2 h_{ab}, \quad (1.8) \]
and hence the object of interest is actually the $\Box^2$ operator (although we are leaving aside the problem of deriving the full gauge-field operator on manifolds with boundary for curvature-squared theories of gravity).

Section 2 studies the inner product on the space of smooth and symmetric rank-two tensor fields on $M$, and performs the integrations by parts which are necessary to understand whether $\Box^2$ can be realized as a symmetric operator with suitable boundary conditions. A first set of boundary conditions, which make it possible to achieve the desired symmetry, are obtained in sect. 3. A second set of admissible boundary conditions in instead derived in sect. 4. Section 5 proves that the resulting boundary-value problems are strongly elliptic, and extends such a proof to the case of mixed boundary conditions for the squared Laplacian. Concluding remarks are presented in sect. 6.

2. - Inner product and integration by parts

In our paper, we use the following definition of inner product:

$$\langle \eta, h \rangle \equiv \int_M \eta_{ab} E^{ab\,cd} h_{cd} \sqrt{g} \, d^m x, \tag{2.1}$$

where $E^{ab\,cd}$ is the DeWitt super-metric (1.4), and $\eta_{ab}, h_{ab}$ are any two sections of the vector bundle of smooth, symmetric rank-two tensor fields over the $m$-dimensional Riemannian manifold $(M, g)$. The definition (2.1) gives rise to an inner product if $\alpha + 1/m > 0$, because by virtue of (1.4) one finds

$$\langle \eta, \eta \rangle = \int_M \left[ \eta_{ab} \eta^{ab} + \alpha \left( \eta^a_a \right)^2 \right] \sqrt{g} \, d^m x, \tag{2.2}$$

and, after using the general decomposition of $\eta_{ab}$, the coefficient of $\left( \eta^a_a \right)^2$ in the integrand (2.2) is found to be $\alpha + 1/m$. Hereafter we shall therefore restrict $\alpha$ to be greater than $-1/m$.

If $\Box^2$ were a symmetric operator, the scalar products $\langle \eta, \Box^2 h \rangle$ and $\langle \Box^2 \eta, h \rangle$ should be equal for all $\eta, h$ in the domain of $\Box^2$. In our paper, the above difference reads (hereafter, we assume $M$ is four-dimensional)

$$I(\eta, h) \equiv \langle \eta, \Box^2 h \rangle - \langle \Box^2 \eta, h \rangle$$

$$= \int_M \left[ \eta_{ab} \nabla^c \nabla_e \nabla^d \nabla_d \left( h^{ab} + \alpha g^{ab} h \right) \right] \sqrt{g} \, d^m x$$
where we have assumed that $g^{00} = 1$, $g^{0i} = 0$, $n^p = (1, 0, 0, 0)$ is the unit normal to the boundary, and $\gamma$ is the determinant of the induced three-metric on $\partial M$. Of course, $h$ and $\eta$ in the integrand (2.3) denote the traces $h \equiv g^{ab} h_{ab}$, $\eta \equiv g^{ab} \eta_{ab}$. It is now convenient to define, for $\phi_{ab} = \eta_{ab}, h_{ab}$,

$$F^{00}(\phi) \equiv \phi^{00} + \alpha g^{00} \varphi = \varphi^{00} + \alpha \varphi,$$

$$F^{0i}(\phi) \equiv \phi^{0i} + \alpha g^{0i} \varphi = \varphi^{0i},$$

$$F^{ij}(\phi) \equiv \phi^{ij} + \alpha g^{ij} \varphi,$$

so that Eq. (2.3) is expressed in the more convenient form

$$\left( \eta, \square^2 h \right) - \left( \square^2 \eta, h \right)$$

$$= \int_{\partial M} n^p \left[ \eta_{00} \nabla_p F^{00}(h) - h_{00} \nabla_p F^{00}(\eta) \right] \sqrt{\gamma} \, d^3 x$$

$$+ 2 \int_{\partial M} n^p \left[ \eta_{0i} \nabla_p F^{0i}(h) - h_{0i} \nabla_p F^{0i}(\eta) \right] \sqrt{\gamma} \, d^3 x$$

$$+ \int_{\partial M} n^p \left[ \eta_{ij} \nabla_p F^{ij}(h) - h_{ij} \nabla_p F^{ij}(\eta) \right] \sqrt{\gamma} \, d^3 x,$$

where we have used the well known commutation property of covariant derivatives in a background whose Riemann curvature vanishes. Our aim is now to use integration by parts so as to remove the third-order derivatives occurring in Eq. (2.7). For this purpose, we use repeatedly the Leibniz rule. The analysis of the first boundary integral in Eq. (2.7) suggests considering (hereafter, $K_{ij}$ is the extrinsic-curvature tensor of the boundary, with trace $K$)

$$\nabla_p \left( n^p \eta_{00} \square F^{00}(h) \right) = K \eta_{00} \square F^{00}(h) + n^p (\nabla_p \eta_{00}) \square F^{00}(h)$$

$$+ n^p \eta_{00} \nabla_p F^{00}(h),$$
jointly with

\[
\nabla_c \left( n^p (\nabla_p \eta_{0i}) \nabla^c F^{00}(h) \right) = K^c_p (\nabla_p \eta_{0i}) \nabla^c F^{00}(h) \\
+ n^p (\nabla_c \nabla_p \eta_{0i}) \nabla^c F^{00}(h) + n^p (\nabla_p \eta_{0i}) \square F^{00}(h).
\]

(2.9)

Moreover, the analysis of the second boundary integral in Eq. (2.7) suggests using the identity

\[
\nabla_p \left( n^p \eta_{0i} \square F^{0i}(h) \right) = K \eta_{0i} \square F^{0i}(h) + n^p (\nabla_p \eta_{0i}) \square F^{0i}(h) \\
+ n^p \eta_{0i} \square \nabla_p F^{0i}(h),
\]

(2.10)

and

\[
\nabla_c \left( n^p (\nabla_p \eta_{0i}) \nabla^c F^{0i}(h) \right) = K^c_p (\nabla_p \eta_{0i}) \nabla^c F^{0i}(h) \\
+ n^p (\nabla_c \nabla_p \eta_{0i}) \nabla^c F^{0i}(h) + n^p (\nabla_p \eta_{0i}) \square F^{0i}(h).
\]

(2.11)

Furthermore, the consideration of the third boundary integral in Eq. (2.7) makes it convenient to use the identities

\[
\nabla_p \left( n^p \eta_{ij} \square F^{ij}(h) \right) = K \eta_{ij} \square F^{ij}(h) + n^p (\nabla_p \eta_{ij}) \square F^{ij}(h) \\
+ n^p \eta_{ij} \square \nabla_p F^{ij}(h),
\]

(2.12)

\[
\nabla_c \left( n^p (\nabla_p \eta_{ij}) \nabla^c F^{ij}(h) \right) = K^c_p (\nabla_p \eta_{ij}) \nabla^c F^{ij}(h) \\
+ n^p (\nabla_c \nabla_p \eta_{ij}) \nabla^c F^{ij}(h) + n^p (\nabla_p \eta_{ij}) \square F^{ij}(h).
\]

(2.13)

If the boundary of \( M \) is smooth, one has \( \partial \partial M = \emptyset \). Moreover, the integral vanishes over zero-measure sets. It is hence possible to use Eqs. (2.8)–(2.13) and the Stokes theorem to re-express Eq. (2.7) in the form

\[
(\eta, \square^2 h) - (\square^2 \eta, h) \\
= \int_{\partial M} \left[ -K \left( \eta_{00} \square F^{00}(h) - h_{00} \square F^{00}(\eta) \right) \\
+ K^j_i \left( (\nabla_j \eta_{00}) \nabla^i F^{00}(h) - (\nabla_j h_{00}) \nabla^i F^{00}(\eta) \right) \\
+ n^p \left( (\nabla_c \nabla_p \eta_{00}) \nabla^c F^{00}(h) - (\nabla_c \nabla_p h_{00}) \nabla^c F^{00}(\eta) \right) \\
- \left( n^p (\nabla_p \eta_{00}) \nabla^c F^{00}(h) - n^p (\nabla_p h_{00}) \nabla^c F^{00}(\eta) \right) \right].
\]
\[
- 2K \left( \eta_{0i} \Box F^{0i}(h) - h_{0i} \Box F^{0i}(\eta) \right) \\
+ 2K_{ij} \left( (\nabla_j \eta_{0i}) \nabla^l F^{0i}(h) - (\nabla_j h_{0i}) \nabla^l F^{0i}(\eta) \right) \\
+ 2n^p \left( (\nabla_c \nabla_p \eta_{0i}) \nabla^c F^{0i}(h) - (\nabla_c \nabla_p h_{0i}) \nabla^c F^{0i}(\eta) \right) \\
- K \left( \eta_{ij} \Box F^{ij}(h) - h_{ij} \Box F^{ij}(\eta) \right) \\
+ K_{i}^r \left( (\nabla_r \eta_{ij}) \nabla^l F^{ij}(h) - (\nabla_r h_{ij}) \nabla^l F^{ij}(\eta) \right) \\
+ n^p \left( (\nabla_c \nabla_p \eta_{ij}) \nabla^c F^{ij}(h) - (\nabla_c \nabla_p h_{ij}) \nabla^c F^{ij}(\eta) \right) \right] \sqrt{\gamma} d^3 x. \tag{2.14}
\]

3. - Boundary conditions: first option

We know from the work of ref. [19], which studied the action of the squared Laplace operator on scalar functions \( f \in C^\infty(M) \), that one can impose boundary conditions where \( f \) and its normal derivative vanish at the boundary. This “doubling” of the boundary conditions, with respect to the analysis of the Laplacian, is clearly understood if one thinks of the eigenvalue problem. In other words, given a spectral resolution of a fourth-order elliptic operator, one deals with fourth-order eigenvalue equations which admit four linearly independent integrals. If one were to fix just the eigenfunctions at the boundary, one would not get enough equations to determine the coefficients of linear combination in the equation

\[
f_\lambda(x) = C_{1,\lambda} f_{1,\lambda}(x) + C_{2,\lambda} f_{2,\lambda}(x) + C_{3,\lambda} f_{3,\lambda}(x) + C_{4,\lambda} f_{4,\lambda}(x)
\]

for the eigenfunction belonging to the eigenvalue \( \lambda \). At a deeper level, as shown in ref. [19], one has to integrate by parts to prove that suitable boundary conditions exist for which the operator \( \Box^2 \) is (essentially) self-adjoint (see Eq. (6.1)).

In our problem, the technical details are more elaborated, since we study the action of \( \Box^2 \) on smooth, symmetric rank-two tensor fields on a flat Riemannian manifold with smooth boundary (e.g. the Euclidean four-ball), but Eq. (2.14) can be used to derive all admissible sets of boundary conditions. First, we consider a scheme where half of the boundary conditions consist of requiring that all components of \( \eta_{ab} \) and \( h_{ab} \), both spatial and normal, should vanish at the boundary. Thus, we require that

\[
\left[ \eta_{00} \right]_{\partial M} = \left[ h_{00} \right]_{\partial M} = 0, \tag{3.1}
\]
\[\left[\eta_{0i}\right]_{\partial M} = \left[h_{0i}\right]_{\partial M} = 0,\]  
(3.2)
\[\left[\eta_{ij}\right]_{\partial M} = \left[h_{ij}\right]_{\partial M} = 0.\]  
(3.3)

By virtue of (3.1)–(3.3) one then finds that the following covariant derivatives vanish at the boundary of \(M\) (see (3.20) and (3.21)):

\[\left[\nabla_j \eta_{00}\right]_{\partial M} = \left[\nabla_j h_{00}\right]_{\partial M} = 0,\]  
(3.4)
\[\left[\nabla_j \eta_{0i}\right]_{\partial M} = \left[\nabla_j h_{0i}\right]_{\partial M} = 0,\]  
(3.5)
\[\left[\nabla_k \eta_{ij}\right]_{\partial M} = \left[\nabla_k h_{ij}\right]_{\partial M} = 0.\]  
(3.6)

At this stage, only the third, sixth and ninth line give non-vanishing contributions to the integrand in Eq. (2.14). It is hence appropriate to write them explicitly, bearing in mind, from sect. 2, that the normal to the boundary takes the form \(n^p = (1,0,0,0)\). In other words, one has (with \(c,p\) ranging from 0 through 3, and \(i,k\) ranging from 1 through 3)

\[n^p(\nabla_c \nabla_p \eta_{00}) \nabla^c F^{00}(h) = (\nabla_0 \nabla_0 \eta_{00}) \nabla^0 F^{00}(h) + (\nabla_k \nabla_0 \eta_{00}) \nabla^k F^{00}(h),\]  
(3.7)
\[n^p(\nabla_c \nabla_p \eta_{0i}) \nabla^c F^{0i}(h) = (\nabla_0 \nabla_0 \eta_{0i}) \nabla^0 F^{0i}(h) + (\nabla_k \nabla_0 \eta_{0i}) \nabla^k F^{0i}(h),\]  
(3.8)
\[n^p(\nabla_c \nabla_p \eta_{ij}) \nabla^c F^{ij}(h) = (\nabla_0 \nabla_0 \eta_{ij}) \nabla^0 F^{ij}(h) + (\nabla_k \nabla_0 \eta_{ij}) \nabla^k F^{ij}(h),\]  
(3.9)

and another triple of identities, which also contribute to the third, sixth and ninth line of (2.14), and are obtained by interchanging the roles of \(\eta_{ab}\) and \(h_{ab}\) in (3.7)–(3.9). The right-hand sides of (3.7)–(3.9) are linear combinations of products of covariant derivatives of \(\eta_{ab}\) and \(h_{ab}\). Thus, it is sufficient to set to zero at the boundary only one of the two functions occurring in the product. For example, to avoid setting to zero on \(\partial M\) the second normal derivatives of \(\eta_{00}, \eta_{0i}\) and \(\eta_{ij}\) (and hence of \(h_{00}, h_{0i}\) and \(h_{ij}\) as well) one can require that

\[\left[\nabla_0 \eta_{00}\right]_{\partial M} = \left[\nabla_0 h_{00}\right]_{\partial M} = 0,\]  
(3.10)
\[\left[\nabla_0 \eta_{0i}\right]_{\partial M} = \left[\nabla_0 h_{0i}\right]_{\partial M} = 0,\]  
(3.11)
\[\left[\nabla_0 \eta_{ij}\right]_{\partial M} = \left[\nabla_0 h_{ij}\right]_{\partial M} = 0,\]  
(3.12)
\[\left[\nabla^0 F^{00}(h)\right]_{\partial M} = \left[\nabla^0 F^{00}(\eta)\right]_{\partial M} = 0,\]  
(3.13)
\[
\begin{align*}
\left[ \nabla^0 F^{0i}(h) \right]_{\partial M} &= \left[ \nabla^0 F^{0i}(\eta) \right]_{\partial M} = 0, \\
\left[ \nabla^0 F^{ij}(h) \right]_{\partial M} &= \left[ \nabla^0 F^{ij}(\eta) \right]_{\partial M} = 0.
\end{align*}
\] (3.14)

Note that we do not get an overdetermined problem, because, by virtue of (2.4)–(2.6) and (3.10)–(3.12), Eqs. (3.13)–(3.15) are satisfied. Now it is helpful to consider an example, and for this purpose we choose the Euclidean four-ball, whose metric may be locally cast in the form

\[
g = d\tau \otimes d\tau + \tau^2 c_{ij} dx^i \otimes dx^j,
\] (3.16)

where the radial coordinate \( \tau \) lies in the closed interval \([0, a]\), with \( a \) the radius of the three-sphere boundary, and \( c_{ij} \) is the metric on a unit three-sphere, with local coordinates \( \{x^i\} \). One then finds (Latin indices run here from 1 through 3)

\[
\nabla_0 h_{00} = \frac{\partial h_{00}}{\partial \tau},
\] (3.17)

\[
\nabla_0 h_{0i} = \frac{\partial h_{0i}}{\partial \tau} - \frac{1}{\tau} h_{0i},
\] (3.18)

\[
\nabla_0 h_{ij} = \frac{\partial h_{ij}}{\partial \tau} - \frac{2}{\tau} h_{ij},
\] (3.19)

\[
\nabla_k h_{00} = \frac{\partial h_{00}}{\partial x^k} - \frac{2}{\tau} h_{0k},
\] (3.20)

\[
\nabla_k h_{0i} = \frac{\partial h_{0i}}{\partial x^k} - \Gamma^l_{ki} h_{00} - \frac{1}{\tau} h_{ik} + \frac{1}{\tau} g_{ik} h_{00},
\] (3.21)

with \( \Gamma \) used to denote the connection coefficients, and hence

\[
\nabla_k \nabla_0 h_{00} = \frac{\partial}{\partial x^k} \frac{\partial h_{00}}{\partial \tau} - \frac{1}{\tau} \left( \frac{\partial h_{00}}{\partial x^k} \frac{\partial h_{00}}{\partial \tau} + 2 \frac{\partial h_{00}}{\partial x^k} - \frac{4}{\tau} h_{00} \right),
\] (3.22)

\[
\nabla_k \nabla_0 h_{0i} = \frac{\partial}{\partial x^k} \left( \frac{\partial h_{0i}}{\partial \tau} - \frac{1}{\tau} h_{0i} \right) - \frac{1}{\tau} \left( \frac{\partial h_{0i}}{\partial x^k} - \Gamma^l_{ki} h_{00} \right) - \Gamma^l_{ik} \left( \frac{\partial h_{0i}}{\partial x^l} - \frac{1}{\tau} h_{0i} \right) + \frac{1}{\tau} g_{ik} \left( \frac{\partial h_{00}}{\partial \tau} - \frac{1}{\tau} h_{00} \right)
- \frac{1}{\tau} \left( \frac{\partial h_{ik}}{\partial \tau} - \frac{3}{\tau} h_{ik} \right),
\] (3.23)
\[ \nabla_k \nabla_0 h_{ij} = \frac{\partial}{\partial x^k} \left( \frac{\partial h_{ij}}{\partial \tau} - \frac{2}{\tau} h_{ij} \right) - \frac{1}{\tau} \nabla_k h_{ij} \]
\[ + \frac{2}{\tau} g_{k(i} \left( \frac{\partial}{\partial \tau} - \frac{1}{\tau} \right) h_{j)0} - 2\Gamma^l_{k(i} \left( \frac{\partial}{\partial \tau} - \frac{2}{\tau} \right) h_{j)l}. \] (3.24)

The joint effect of Eqs. (3.1)–(3.3), (3.7)–(3.12) is then that the right-hand side of Eq. (2.14) vanishes if the following boundary conditions hold:

\[ h_{ab} \mid_{\partial M} = \eta_{ab} \mid_{\partial M} = 0 \quad \forall a, b = 0, 1, 2, 3, \] (3.25)
\[ \left[ \frac{\partial h_{ab}}{\partial \tau} \right] \mid_{\partial M} = \left[ \frac{\partial \eta_{ab}}{\partial \tau} \right] \mid_{\partial M} = 0 \quad \forall a, b = 0, 1, 2, 3. \] (3.26a)

One then deals with 10 boundary conditions on \( h_{ab}, \eta_{ab} \) and 10 boundary conditions on their normal derivatives, bearing in mind the symmetry of these rank-two tensor fields. This is a generalization of the boundary conditions obtained in ref. [19] for the squared Laplacian acting on smooth functions, and once more the number of boundary conditions is exactly doubled, with respect to the analysis of the Laplacian. Last, a covariant form of Eq. (3.26a) is easily obtained from (3.10)–(3.12), i.e.

\[ [n^p \nabla_p h_{ab}] \mid_{\partial M} = [n^p \nabla_p \eta_{ab}] \mid_{\partial M} = 0 \quad \forall a, b = 0, 1, 2, 3. \] (3.26b)

The normal components of \( \varphi_{ab} = h_{ab} \) or \( \eta_{ab} \) are given by

\[ \varphi_{00} = \varphi_{ab} n^a n^b, \] (3.27)
\[ \varphi_{0i} = \varphi_{ai} n^a, \] (3.28)

and the spatial components are obtained, instead, by applying a projection operator, i.e.

\[ \varphi_{ij} = \Pi_{ij}^{cd} \varphi_{cd}, \] (3.29)

where, on defining [20]

\[ q^b_a \equiv \delta^b_a - n_a n^b, \] (3.30)

one has [20]

\[ \Pi_{ab}^{cd} \equiv q^c_{(a} q^d_{b)}. \] (3.31)

The tensor field \( q^b_a \) is the standard projector of tensor fields over the bounding surface, and is, by construction, orthogonal to the unit normal vector, in that \( q^b_a n_b = n_a - n_a = 0 \).
4. - Boundary conditions: second option

After writing down the identities (3.7)–(3.9), one can also make the alternative choice, according to which (cf. (3.10)–(3.15))

\[
\begin{align*}
\left[ \nabla_0 \nabla_0 \eta_{00} \right]_{\partial M} &= \left[ \nabla_0 \nabla_0 h_{00} \right]_{\partial M} = 0, \\
\left[ \nabla_0 \nabla_0 \eta_{0i} \right]_{\partial M} &= \left[ \nabla_0 \nabla_0 h_{0i} \right]_{\partial M} = 0, \\
\left[ \nabla_0 \nabla_0 \eta_{ij} \right]_{\partial M} &= \left[ \nabla_0 \nabla_0 h_{ij} \right]_{\partial M} = 0, \\
\left[ \nabla^k F^{00} (h) \right]_{\partial M} &= \left[ \nabla^k F^{00} (\eta) \right]_{\partial M} = 0, \\
\left[ \nabla^k F^{0i} (h) \right]_{\partial M} &= \left[ \nabla^k F^{0i} (\eta) \right]_{\partial M} = 0, \\
\left[ \nabla^k F^{ij} (h) \right]_{\partial M} &= \left[ \nabla^k F^{ij} (\eta) \right]_{\partial M} = 0.
\end{align*}
\]

(4.1) (4.2) (4.3) (4.4) (4.5) (4.6)

Now the boundary conditions (3.1)–(3.3), jointly with the definitions (2.4)–(2.6), imply that Eqs. (4.4)–(4.6) are identically satisfied. In particular, on the Euclidean four-ball, by virtue of the formulae

\[
\begin{align*}
\nabla_0 \nabla_0 h_{00} &= \frac{\partial^2 h_{00}}{\partial \tau^2}, \\
\nabla_0 \nabla_0 h_{0i} &= \frac{\partial^2 h_{0i}}{\partial \tau^2} - \frac{2}{\tau} \frac{\partial h_{0i}}{\partial \tau} + \frac{2}{\tau^2} h_{0i}, \\
\nabla_0 \nabla_0 h_{ij} &= \frac{\partial^2 h_{ij}}{\partial \tau^2} - \frac{4}{\tau} \frac{\partial h_{ij}}{\partial \tau} + \frac{6}{\tau^2} h_{ij},
\end{align*}
\]

(4.7) (4.8) (4.9)

the boundary conditions (3.1)–(3.3) and (4.1)–(4.3) lead to 10 homogeneous Dirichlet conditions, jointly with 10 boundary conditions for a linear combination of first- and second-order partial derivatives, i.e.

\[
\begin{align*}
\left[ \frac{\partial^2 \eta_{00}}{\partial \tau^2} \right]_{\partial M} &= \left[ \frac{\partial^2 h_{00}}{\partial \tau^2} \right]_{\partial M} = 0, \\
\left[ \frac{\partial^2 \eta_{0i}}{\partial \tau^2} - \frac{2 \partial \eta_{0i}}{\tau} \frac{\partial}{\partial \tau} \right]_{\partial M} &= \left[ \frac{\partial^2 h_{0i}}{\partial \tau^2} - \frac{2 \partial h_{0i}}{\tau} \frac{\partial}{\partial \tau} \right]_{\partial M} = 0, \\
\left[ \frac{\partial^2 \eta_{ij}}{\partial \tau^2} - \frac{4 \partial \eta_{ij}}{\tau} \frac{\partial}{\partial \tau} \right]_{\partial M} &= \left[ \frac{\partial^2 h_{ij}}{\partial \tau^2} - \frac{4 \partial h_{ij}}{\tau} \frac{\partial}{\partial \tau} \right]_{\partial M} = 0.
\end{align*}
\]

(4.10) (4.11) (4.12)
Of course, the boundary conditions (4.1)–(4.3) can also be expressed in a covariant form by a single equation, i.e.

\[
\left[ n^p n^q \nabla_p \nabla_q \eta_{ab} \right]_{\partial M} = \left[ n^p n^q \nabla_p \nabla_q h_{ab} \right]_{\partial M} = 0 \quad \forall a, b = 0, 1, 2, 3. \tag{4.13}
\]

As in sect. 3, the number of boundary conditions is doubled with respect to the analysis of the Laplacian, and Eqs. (3.1)–(3.3) and (4.1)–(4.3) provide the generalization of yet another set of boundary conditions derived, in ref. [19], when the action of the squared Laplacian on smooth functions is considered.

5. - Strong ellipticity

If a differential operator is studied on a Riemannian manifold \( M \) with smooth boundary \( \partial M \), the ellipticity is obtained upon proving that the leading symbol is elliptic in the interior of \( M \), and that a unique solution exists of the eigenvalue equation for the leading symbol, subject to a decay condition at infinity and to suitable boundary conditions. For the squared Laplace operator defined in Eq. (1.2), the leading symbol

\[
\sigma_L \left( \Box^2; x, \xi \right) = |\xi|^4 I = g^{\mu \nu} g^{\rho \sigma} \xi_\mu \xi_\nu \xi_\rho \xi_\sigma I,
\]

with \( \xi \in T^*(M) \), is clearly elliptic. In fact, for any Riemannian metric \( g \), the leading symbol in (5.1) is non-degenerate for \( \xi \neq 0 \). Moreover, for any \( \lambda \in \mathbb{C} - \mathbb{R}_+ \), one has

\[
\det \left( \sigma_L \left( \Box^2; x, \xi \right) - \lambda \right) = \left( |\xi|^4 - \lambda \right)^{\dim(V)} \neq 0. \tag{5.2}
\]

The left-hand side of Eq. (5.2) vanishes if and only if both \( \xi \) and \( \lambda \) vanish. This completes the proof of ellipticity of \( \sigma_L(\Box^2) \) in the interior of \( M \).

Now we have to define the condition of strong ellipticity, which involves in a crucial way the boundary conditions. For this purpose, we consider again the leading symbol \( \sigma_L \left( \Box^2; \hat{x}, r, \zeta, \omega \right) \), with \( \{ \hat{x}^k \} \) local coordinates on \( \partial M \), \( r \) the normal geodesic distance to the boundary, \( \zeta_j \) as components of a cotangent vector on the boundary, and \( \omega \) a real parameter. We then set \( r = 0 \) and replace \( \omega \) by \(-i\partial_r\), and consider the eigenvalue equation for \( \sigma_L \), i.e. [9,10]

\[
\sigma_L \left( \Box^2; \hat{x}, r = 0, \zeta, -i\partial_r \right) \varphi(r) = \lambda \varphi(r). \tag{5.3}
\]
Note that, in writing down Eq. (5.3), we have not doubled the number of arguments in the leading symbol. In the presence of boundaries, the \( m \) local coordinates \( x \) are split into \( m - 1 \) local coordinates on the boundary, jointly with the normal geodesic distance (which, of course, vanishes on the boundary). Similarly, the \( m \) coordinates \( \xi_\mu \) are split into \( m - 1 \) coordinates on \( T^*(\partial M) \), appropriate for cotangent vectors, jointly with a real parameter \( \omega \). Hence one has (with \( |\zeta|^2 \equiv \zeta_i \zeta^i \))

\[
\sigma_L \left( \square^2; \{ \hat{x}^k \}, r = 0, \{ \zeta_j \}, \omega \right) = |\zeta|^4 + \omega^4 + 2\omega^2|\zeta|^2.
\]

Last, to obtain the left-hand side of Eq. (5.3), one has to replace \( \omega \) by \(-i\partial_r\) as we said before [10], and this leads to the fourth-order differential operator

\[
\sigma_L \left( \square^2; \{ \hat{x}^k \}, r = 0, \{ \zeta_j \}, \omega = -i\partial_r \right) = \frac{\partial^4}{\partial r^4} - 2|\zeta|^2 \frac{\partial^2}{\partial r^2} + |\zeta|^4.
\]

Thus, Eq. (5.3) should be viewed as an ordinary differential equation involving the field \( \varphi \) which is a smooth section of the vector bundle \( V \) over the product manifold \( \partial M \times \mathbb{R}_+ \).

By definition, for a given form of the boundary operator \( B \), the boundary-value problem \( \left( \square^2, B \right) \) is strongly elliptic with respect to the cone \( C - \mathbb{R}_+ \) if there exists a unique solution of Eq. (5.3) with \( \lambda \in C - \mathbb{R}_+ \), for all \( (\zeta, \lambda) \neq (0, 0) \), subject to the asymptotic condition

\[
\lim_{r \to \infty} \varphi(r) = 0,
\]

and to the boundary condition

\[
\sigma_g (B)(\hat{x}, \zeta) \psi(\varphi) = \psi',
\]

for all \( \psi' \). With a standard notation, \( \psi(\varphi) \) denotes the boundary data, usually arranged in the form of a column vector consisting of the field and its normal derivative(s) evaluated at the boundary. The boundary conditions originally imposed read (cf. (3.25) and (3.26), or (3.25) and (4.13))

\[
B \psi(\varphi) = 0 \text{ at } \partial M,
\]

where, in our case, the boundary operator takes the form

\[
B = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
\]
Unlike the case of mixed boundary conditions for the Laplacian, no projectors occur in Eq. (5.7), because we are setting to zero on the boundary the whole of $\varphi$ and its first (or second) normal derivative. The graded leading symbol $\sigma_g(B)$ of $B$, is defined according to the rule [10]

$$
\sigma_g(B)_{ij} \equiv \sigma_L(B)_{ij} \text{ if } \text{ord}(B_{ij}) = j - i, \quad (5.8a)
$$

$$
\sigma_g(B)_{ij} \equiv 0 \text{ if } \text{ord}(B_{ij}) < j - i, \quad (5.8b)
$$

where “ord” denotes the order of the $B_{ij}$ component of $B$ as a differential operator on the boundary, subject to the restriction

$$
\text{ord}(B_{ij}) \leq j - i. \quad (5.9)
$$

By virtue of (5.7)–(5.9), the graded leading symbol of the boundary operator is again the identity matrix

$$
\sigma_g(B) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad (5.10)
$$

in the boundary-value problems of Secs. III and IV, whereas $\psi(\varphi)$ consists of the column vectors (here, the symbol $;N$ coincides with $\nabla_0$ used in the previous sections)

$$
\psi(\varphi) = \begin{pmatrix} [\varphi]_{\partial M} \\ [\varphi;N]_{\partial M} \end{pmatrix}
$$

or

$$
\psi(\varphi) = \begin{pmatrix} [\varphi]_{\partial M} \\ [\varphi;NN]_{\partial M} \end{pmatrix}
$$

respectively, with $\psi'$ any pair of boundary data [9]

$$
\psi' = \begin{pmatrix} \psi'_0 \\ \psi'_1 \end{pmatrix}.
$$

After having defined in detail the strong ellipticity setting, we can perform the next step, i.e. the solution of Eq. (5.3) jointly with the conditions (5.4) and (5.5). Indeed, Eq. (5.3) reads

$$
\left[ \frac{d^4}{dr^4} - 2|\zeta|^2 \frac{d^2}{dr^2} + |\zeta|^4 \right] \varphi(r) = \lambda \varphi(r). \quad (5.11)
$$

Equation (5.11) is solved by $\varphi(r) = e^{\alpha r}$, with $\alpha$ given by the roots of the algebraic equation

$$
\alpha^4 - 2\alpha^2|\zeta|^2 + |\zeta|^4 - \lambda = 0, \quad (5.12)
$$
i.e.

\[ \alpha_1 = +\sqrt{\mid\zeta\mid^2 + \sqrt{\lambda}}, \]  
\[ \alpha_2 = +\sqrt{\mid\zeta\mid^2 - \sqrt{\lambda}}, \]  
\[ \alpha_3 = -\sqrt{\mid\zeta\mid^2 + \sqrt{\lambda}}, \]  
\[ \alpha_4 = -\sqrt{\mid\zeta\mid^2 - \sqrt{\lambda}}. \]  

Comparison with the case of the Laplace operator [10] shows that one obtains strong ellipticity provided that \( \pm\sqrt{\lambda} \in \mathcal{C} - \mathbb{R}_+ \), which yields

\[ \lambda \in (\mathcal{C} - \mathbb{R}_+) \cap \mathcal{C} = \mathcal{C} - \mathbb{R}_+. \]  

Among the four values of \( \alpha \), only \( \alpha_3 \) and \( \alpha_4 \) fulfill the condition (5.4). The exponentially decaying solution picked out by (5.4) reads therefore (\( \chi_1 \) and \( \chi_2 \) being some parameters)

\[ \varphi(r) = \chi_1 e^{-\rho_1 r} + \chi_2 e^{-\rho_2 r}, \]  

where

\[ \rho_1 \equiv +\sqrt{\mid\zeta\mid^2 + \sqrt{\lambda}}, \]  
\[ \rho_2 \equiv +\sqrt{\mid\zeta\mid^2 - \sqrt{\lambda}}. \]  

One then finds

\[ \varphi\bigg|_{r=0} = \chi_1 + \chi_2, \]  
\[ \frac{d\varphi}{dr}\bigg|_{r=0} = -\rho_1 \chi_1 - \rho_2 \chi_2, \]  
\[ \frac{d^2\varphi}{dr^2}\bigg|_{r=0} = \rho_1^2 \chi_1 + \rho_2^2 \chi_2. \]  

Thus, if the boundary conditions (3.25) and (3.26) are imposed, the condition (5.5) leads to

\[ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} [\varphi]_{\partial M} \\ [\varphi;\mathcal{N}]_{\partial M} \end{pmatrix} = \begin{pmatrix} \psi'_0 \\ \psi'_1 \end{pmatrix}, \]  

which implies, by virtue of (5.21) and (5.22),

\[ A\chi = \psi', \]
where $\chi$ is the column vector $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$, and $A$ is the matrix
\[
A \equiv \begin{pmatrix} 1 & 1 \\ -\rho_1 & -\rho_2 \end{pmatrix}.
\] (5.26)

Now a unique solution $\chi = A^{-1}\psi'$ exists of the linear system (5.25), because
\[
\det(A) = \rho_1 - \rho_2 \neq 0.
\] (5.27)

Explicitly, one finds
\[
\chi_1 = \frac{\psi'_1 + \rho_2 \psi'_0}{\rho_2 - \rho_1},
\] (5.28)
\[
\chi_2 = -\frac{\psi'_1 + \rho_1 \psi'_0}{\rho_2 - \rho_1}.
\] (5.29)

Moreover, if the boundary conditions (3.25) and (4.13) are imposed, the condition (5.5) leads to
\[
\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} [\varphi]_{\partial M} \\ [\varphi_{,NN}]_{\partial M} \end{pmatrix} = \begin{pmatrix} \psi'_0 \\ \psi'_2 \end{pmatrix},
\] (5.30)
and this implies, by virtue of (5.21) and (5.23),
\[
\tilde{A} \chi = \psi',
\] (5.31)
where $\tilde{A}$ is the matrix
\[
\tilde{A} \equiv \begin{pmatrix} 1 & 1 \\ \rho_2 & \rho_1 \end{pmatrix}.
\] (5.32)

A unique solution $\chi = \tilde{A}^{-1}\psi'$ exists of the linear system (5.31), because
\[
\det(\tilde{A}) = \rho_2^2 - \rho_1^2 = -2\sqrt{\lambda} \neq 0.
\] (5.33)

The explicit calculation yields (cf. (5.28) and (5.29))
\[
\chi_1 = \frac{\psi'_2 - \rho_2^2 \psi'_0}{\rho_1^2 - \rho_2^2},
\] (5.34)
\[
\chi_2 = -\frac{\psi'_2 - \rho_1^2 \psi'_0}{\rho_1^2 - \rho_2^2}.
\] (5.35)
So far our examples of strong ellipticity have been almost straightforward. A more
relevant case is obtained on considering mixed boundary conditions for the squared Lapla-
cian, with boundary operator involving also tangential derivatives. For this purpose, we
assume that the boundary operator in (5.7) is replaced by

\[ B = \begin{pmatrix} \Pi & 0 \\ \Lambda^k & I - \Pi \end{pmatrix}, \quad (5.36) \]

where \( k \) is an integer \( \geq 1 \), \( \Lambda \) is a first-order tangential differential operator on the boundary:

\[ \Lambda : C^\infty(W_0, \partial M) \to C^\infty(W_0, \partial M), \]

and \( \Pi \) is a self-adjoint projector. With a standard notation, \( W \) is the bundle of boundary
data over \( \partial M \), given by the direct sum

\[ W = W_0 \oplus W_1, \quad (5.37) \]

with \( W_j \) representing normal derivatives of order \( j \). In the previous case, we had \( W \)
expressed by (5.37) or, instead, by

\[ W = W_0 \oplus W_2. \quad (5.38) \]

The projector \( \Pi \) maps each \( W_k \) sub-space into itself. The form of \( \Lambda \) which leads to tan-
gential derivatives in the boundary conditions is

\[ \Lambda = (I - \Pi) \left[ \frac{1}{2} \left( \Gamma^i \tilde{\nabla}_i + \tilde{\nabla}_i \Gamma^i \right) + S \right] (I - \Pi), \quad (5.39) \]

where \( \Gamma^i \) are endomorphism-valued vector fields on \( \partial M \), and \( S \) is an endomorphism of
the vector bundle \( W_0 \). Following ref. 9, \( \Gamma^i \) and \( S \) are taken to be anti-self-adjoint and
self-adjoint, respectively, and annihilated by the projector \( \Pi \) from the left and from the
right. By virtue of the assumption (5.36), the graded leading symbol of the boundary
operator is (here \( T \equiv \Gamma^i \zeta_j \))

\[ \sigma_g(B) = \begin{pmatrix} \Pi & 0 \\ (iT)^k & I - \Pi \end{pmatrix}, \quad (5.40) \]

and hence, when (5.37) holds, the boundary condition (5.5) takes the form

\[ \begin{pmatrix} \Pi & 0 \\ (iT)^k & I - \Pi \end{pmatrix} \begin{pmatrix} \chi_1 + \chi_2 \\ -\rho_1 \chi_1 - \rho_2 \chi_2 \end{pmatrix} = \begin{pmatrix} \psi'_0 \\ \psi'_1 \end{pmatrix}, \quad (5.41) \]
where

\[ \psi'_0 = \Pi(\gamma_1 + \gamma_2) \quad (5.42) \]
\[ \psi'_1 = (I - \Pi)(-\rho_1 \gamma_1 - \rho_2 \gamma_2) \quad (5.43) \]

This means that the right-hand side of Eq. (5.41) is a smooth section of an auxiliary vector bundle \( W' \) over \( \partial M \), where any \( \psi' \in C^\infty(W',\partial M) \) is given by \( \psi' = P\tilde{\psi} \), with

\[ P \equiv \begin{pmatrix} \Pi & 0 \\ 0 & I - \Pi \end{pmatrix} \quad (5.44) \]

and \( \tilde{\psi} \in C^\infty(W,\partial M) \). The boundary operator in (5.36) is related to the operator \( P \) defined in (5.44) by the equation

\[ B = PL \quad (5.45) \]

where

\[ L = \begin{pmatrix} I & 0 \\ \Lambda^k & I \end{pmatrix} \quad (5.46) \]

In other words, since we deal with a squared Laplacian, we allow for a number of tangential derivatives greater than the number considered in the case of the Laplacian, and we try to impose mixed boundary conditions, so that projectors (rather than identity operators) occur in the boundary operator. Since \( \tilde{\psi} \) differs in general from \( \psi \), we have constants \( \gamma_1 \) and \( \gamma_2 \) in (5.42) and (5.43) instead of \( \chi_1 \) and \( \chi_2 \) (cf. (5.18)).

Thus, for all \( j = 1, 2 \), if the matrix \((iT)^k - I\rho_j \) is non-singular we may use the identity

\[ \chi_j = \Pi\chi_j + (I - \Pi)\chi_j \quad (5.47) \]

to solve the system (5.41) in the form

\[ \Pi\chi_j = \Pi\gamma_j \quad (5.48) \]
\[ (I - \Pi)\chi_j = (I\rho_j - (iT)^k)^{-1} \left[ (iT)^k\Pi + (I - \Pi)\rho_j \right] \gamma_j \quad (5.49) \]

A necessary and sufficient condition to obtain a unique solution is of course the non-degeneracy of the matrix \((I\rho_j - (iT)^k)\), i.e.

\[ \det \left[ I\rho_j - (iT)^k \right] \neq 0. \quad (5.50) \]
Note now that (see (5.19) and (5.20))

\[ I\rho_1 - (iT)^k = I\sqrt{|\zeta|^2 + \sqrt{\lambda}} - (iT)^k, \]  

\[ (5.51) \]

\[ I\rho_2 - (iT)^k = I\sqrt{|\zeta|^2 - \sqrt{\lambda}} - (iT)^k, \]  

\[ (5.52) \]

and such matrices are never singular if the condition (5.17) is satisfied and \( iT \) is self-adjoint.

6. - Concluding remarks

Our paper has analyzed the squared Laplace operator \( \Box^2 \) acting on symmetric rank-two tensor fields \( \varphi_{ab} \) on (flat) Riemannian manifolds with boundary. Its original contributions, of technical nature, are as follows.

(i) Symmetry of \( \Box^2 \) is achieved provided that both \( \varphi_{ab} \) and its normal derivative \( n^p \nabla_p \varphi_{ab} \), or \( \varphi_{ab} \) and the second normal derivative \( n^p n^q \nabla_p \nabla_q \varphi_{ab} \), are set to zero at the boundary.

(ii) The resulting boundary-value problems are strongly elliptic with respect to the cone \( C - \mathbb{R}_+ \).

(iii) Strong ellipticity with respect to \( C - \mathbb{R}_+ \), in the case of mixed boundary conditions including tangential derivatives, has also been proved. Interestingly, no restriction involving \( T \), and hence the matrices \( \Gamma^j \), is obtained unlike the case of an operator of Laplace type [9].

Of course, in the case of symmetric rank-two tensor fields the identity operator in Eq. (5.25) reads actually \( \delta^c_{(a} \delta^d_{b)} \), but apart from such minor details, all calculations in Sec. V prove indeed that the boundary-value problem \( (\Box^2, B) \) is strongly elliptic with respect to the cone \( C - \mathbb{R}_+ \), on considering the vector bundle of symmetric rank-two tensor fields over \( M \). Our proof is simple but of some interest, because it clearly shows the role played by fourth-order operators in doubling the number of linearly independent solutions of the eigenvalue equation (5.3) for the leading symbol. Strong ellipticity is crucial to ensure the existence of the asymptotic expansions used in the theory of heat-kernel asymptotics [10]. From the point of view of quantum field theory, this means that the one-loop semiclassical approximation is well defined and can be explicitly evaluated [19].

We find it appropriate to stress once more that the double integration by parts used to derive Eq. (2.14) is necessary to recover the correct number (and form) of boundary conditions for a fourth-order elliptic operator like \( \Box^2 \). For example, its simplest (but still
useful) form, i.e. the operator $B \equiv \frac{d^4}{dx^4}$ on a closed interval of the real line, satisfies the identity [19]

$$(Bu, v) - (u, B^\dagger v) = \left[ \frac{d^3 u^*}{dx^3} v \right]_0^1 - \left[ \frac{d^2 u^*}{dx^2} \frac{dv}{dx} \right]_0^1 + \left[ \frac{du^*}{dx} \frac{d^2 v}{dx^2} \right]_0^1 - \left[ \frac{u^*}{dx} \frac{d^3 v}{dx^3} \right]_0^1,$$  

(6.1)

where $u$ is a vector in the domain of $B$, $v$ is a vector in the domain of the adjoint $B^\dagger$ of $B$, and we use the definition of inner product [19]

$$(u, v) \equiv \int_0^1 u^*(x)v(x)dx.$$  

(6.2)

Domains of self adjointness of $\frac{d^4}{dx^4}$ are therefore, in particular, the set of functions belonging to $AC^4[0,1]$ (this is the set of functions in $L^2[0,1]$ whose weak derivatives up to third order are absolutely continuous in $[0,1]$, which ensures that the weak derivatives, up to fourth order, are Lebesgue summable in $[0,1]$, and that all $u$ in the domain are of class $C^4$ on $[0,1]$) and satisfying the boundary conditions [19] (cf. (3.25) and (3.26))

$$u(0) = u(1) = 0,$$  

(6.3)

$$u'(0) = u'(1) = 0,$$  

(6.4)

or Eq. (6.3) jointly with [19] (cf. (3.25) and (4.13))

$$u''(0) = u''(1) = 0.$$  

(6.5)

Thus, a complete correspondence can be established between the boundary-value problem for the squared Laplace operator in one dimension [19] and the more elaborated case studied in our paper.

It now appears both interesting and necessary to study a scheme more general than the one where $\eta_{ab}, h_{ab}$ and their first or second normal derivatives are set to zero at the boundary. For this purpose, one may start again from Eq. (2.14), but requiring that the integrand, as a whole, should vanish at $\partial M$. Such an integrand $\sigma(\eta, h)$ may be re-expressed as

$$\sigma(\eta, h) = \left[ -Kh_{ab}\nabla_c + n^P(\nabla_c \nabla_p h_{ab}) + K^P_c(\nabla_p h_{ab}) \right] \nabla^c F^{ab}(\eta)$$

$$- \left[ -K\eta_{ab}\nabla_c + n^P(\nabla_c \nabla_p \eta_{ab}) + K^P_c(\nabla_p \eta_{ab}) \right] \nabla^c F^{ab}(h).$$  

(6.6)
The non-trivial problem, however, is to understand how to derive the boundary operator (5.36) from the analysis of (6.6). In the previous section, the introduction of (5.36) was too simplified, leaving aside the problem of integrating by parts in the action. If it were possible to achieve this, it would then remain to be seen whether such a kind of generalized boundary conditions for $\Box^2$ can be derived from an invariance principle, as is indeed the case for the Laplacian itself [1,5,9], upon requiring invariance under infinitesimal gauge transformations. Non-local boundary conditions for $\Box^2$ might also be studied with some profit, following the recent attempts to consider a non-local formulation of Euclidean quantum gravity based on integro-differential boundary conditions [21]. Last, but not least, the resulting heat-kernel asymptotics should be thoroughly developed, to supplement the recent, encouraging progress in the case of generalized boundary-value problems for operators of Laplace type [22–25]. All this adds evidence in favour of the problems of quantum field theory and spectral geometry being able to lead to a deeper vision in modern mathematical physics [1].

∗ ∗ ∗

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