Hierarchy of Higher Dimensional Integrable System

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Abstract. Integrable equations in (1 + 1) dimensions have their own higher order integrable equations, like the KdV, mKdV and NLS hierarchies etc. In this paper we consider whether integrable equations in (2 + 1) dimensions have also the analogous hierarchies to those in (1 + 1) dimensions. Explicitly is discussed the Bogoyavlenskii-Schiff (BS) equation. For the BS hierarchy, there appears an ambiguity in the Painlevé test. Nevertheless, it may be concluded that the BS hierarchy is integrable.

Short title: Hierarchy of Higher Dimensional Integrable System

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1. Introduction

The inverse scattering transform (IST) method is a powerful tool for the investigation of \((1+1)\) dimensional nonlinear differential equations. The extent of applicability of this method, however, depends on the dimension of space. The well-known \((2+1)\) dimensional integrable equations are the Kadomtsev-Petviashvili (KP), Davey-Stewarson (DS) equation and some new integrable equations which has been constructed by B.G.Konopelchenko and V.G.Dubrovsky \[1\]. In this paper we consider a new integrable equation in \((2+1)\) dimensions, the Bogoyavlenskii-Schiff equation (BS equation) \[2, 3\]. This equation is given by

\[
u_t + \Phi(u)\nu_z = 0, \tag{1}\]

where

\[
\Phi(u) = \frac{1}{4} \partial_x^2 + u + \frac{1}{2} u_x \partial_x^{-1} \tag{2}\]

is the recursion operator of KdV equation \[4\]. Potential form of this equation takes the form,

\[
\phi_{xt} + \frac{1}{4} \phi_{xxxz} + \phi_x \phi_{xz} + \frac{1}{2} \phi_{xx} \phi_z = 0 \quad (u \equiv \phi_x). \tag{3}\]

The BS equation has an infinite number of conservation laws \[5\], the Painlevé property \[3, 6\] and \(N\) soliton solutions \[7\]. That is, the BS equation is the \((2+1)\) dimensional integrable equation. For \((1+1)\) dimensional case \((z = x)\), i.e. KdV equation, we can construct the higher order equations of the KdV (the KdV hierarchy) with Lax operators and the conserved quantities \[8, 9\]. In this paper, we consider whether this construction of hierarchy is also valid for \((2+1)\) dimensional integrable equation or not. Namely the purpose of the present paper is to construct the BS hierarchy and to check its integrability through the Painlevé test.

In Sec. 2, we review the hierarchy of an integrable equation in \((1+1)\) dimensions and derive the Lax pair of the BS hierarchy. In Sec. 3, we investigate the integrability of the BS hierarchy obtained in the previous section. Sec. 4 is devoted to discussion mainly concerning with integrable equations in \((3+1)\) dimensions.

2. The Lax Representation of The BS Hierarchy

The potential KdV equation

\[
\phi_{xt} + \frac{1}{4} \phi_{xx} + \frac{3}{2} \phi_x \phi_{xx} = 0, \tag{4}\]

admits the Lax representation \[10\]

\[
[L, T] = 0. \tag{5}\]

\(L, T\) operators have the form

\[
L(x, t) = \partial_x^2 + \phi_x(x, t), \tag{6}\]

\[
T(x, t) = \left( L(x, t)^2 \right)_{+} + \partial_t = \partial_x L(x, t) + T'(x, t) + \partial_t, \tag{7}\]
where \((\ )_+\) means the part of \((\ )\) with non-negative powers of \(\partial_x\) and

\[
L(x,t)^2 = \partial_x + \frac{1}{2} \phi_x \partial_x^{-1} - \frac{1}{4} \phi_{xx} \partial_x^{-2} + \frac{1}{8} (\phi_{3x} - \phi_x^2) \partial_x^{-3} + 3 \left( \frac{1}{8} (\phi_x \phi_{xx} - \frac{1}{2} \phi_{4x}) \right) \partial_x^{-4} + \cdots, \tag{8}
\]

\[
T'(x,t) = \frac{1}{2} \phi_x \partial_x - \frac{1}{4} \phi_{xx}. \tag{9}
\]

The second equality in equation (7) is given for the later use of the BS equation and others.

An infinite set of equations,

\[
[L, T_n] = 0 \quad (n = 1, 2, 3, \cdots), \tag{10}
\]

constitute the potential KdV hierarchy and the equation for \(n\) is called the \((2n+1)\)th order potential KdV equation. The operator \(T_n\) has the form

\[
T_n(x,t) = \left( L(x,t) \frac{2n+1}{2} \right)_+ + \partial_t = \partial_x L(x,t)^n + T'_n(x,t) + \partial_t. \tag{11}
\]

For \(n = 2\) and \(n = 3\), equation (10) represents the 5th and the 7th order potential KdV equations, respectively:

\[
\phi_{xt} + \frac{1}{16} \phi_{6x} + \frac{5}{8} \left( \phi_x^3 + \frac{1}{2} \phi_{xx}^2 + \phi_x \phi_{xx} \right)_x = 0 \tag{12}
\]

and

\[
T'_2 = \frac{1}{2} \phi_x \partial_x^3 - \frac{1}{4} \phi_{xx} \partial_x^2 + \left( \frac{1}{8} \phi_{3x} + \frac{7}{8} \phi_x^2 \right) \partial_x - \frac{1}{16} \phi_{4x} - \frac{1}{8} \phi_x \phi_{xx} \tag{13}
\]

for \(n = 2\).

\[
\phi_{xt} + \frac{1}{64} \phi_{8x} + \frac{7}{64} \left( 5 \phi_x^4 + 10 \phi_x (\phi_{2x}^2 + \phi_x \phi_{3x}) + 2 \phi_x \phi_{5x} + 4 \phi_{xx} \phi_{4x} + 3 \phi_{3x}^2 \right)_x = 0 \tag{14}
\]

and

\[
T'_3 = \frac{1}{2} \phi_x \partial_x^5 - \frac{1}{4} \phi_{xx} \partial_x^4 + \left( \frac{1}{8} \phi_{3x} + \frac{11}{8} \phi_x^2 \right) \partial_x^3 - \left( \frac{1}{16} \phi_{4x} - \frac{9}{8} \phi_x \phi_{xx} \right) \partial_x^2 + \left( \frac{1}{32} \phi_{5x} - \frac{11}{32} \phi_x \phi_{3x} + \frac{19}{16} \phi_x^3 \right) \partial_x - \frac{1}{64} \phi_{6x} - \frac{9}{32} \phi_x \phi_{4x} - \frac{7}{16} \phi_{xx} \phi_{3x} + \frac{9}{32} \phi_x^2 \phi_{xx} \tag{15}
\]

for \(n = 3\). Equations (12) and (14) are integrable.

The Bogoyavlenski’s method is to modify the form of \(T\) operator (7) in \((1 + 1)\) dimensions to in \((2 + 1)\) dimensions as

\[
T(x,z,t) = \partial_z L(x,z,t) + \partial_t + T'(x,z,t), \tag{16}
\]
where
\[ T'(x, z, t) = \frac{1}{2} \partial_x \partial_x - \frac{1}{4} \partial_{xxz}. \] (17)

Namely we replace \( \partial_x L \) in equation (7) by \( \partial_z L \). Here \( L \) has the same form as equation (6), though \( \phi \) in this case, is a function of \( x, z \) and \( t \). Then, the Lax equation \( [L, T] = 0 \) becomes the BS equation \( \Box \). We can construct the \((2n + 1)\)th order BS hierarchy analogously to the potential KdV hierarchy:
\[ [L, T_n] = 0, \] (18)
where \( T_n \) have the forms
\[ T_n(x, z, t) = \partial_z L(x, z, t)^n + T'_n(x, z, t) + \partial_t. \] (19)

We could construct the same equations for the Lax equation \( [L, T_n] = 0 \) with other forms of the operator \( T_n(x, z, t) = L \cdots L \cdot (\partial_x L) \cdots L + T'_n(x, z, t) + \partial_t \). Here the \( m - 1 \) times \( n - m \) times operators \( T'_n(x, z, t) \) is different from \( T'_n(x, z, t) \), though we have the same equation after all. For \( n = 2 \) and \( n = 3 \), equations \( [L, T_n] = 0 \) represents the 5th and the 7th order BS equations respectively:

For \( n = 2 \), equation (18) becomes
\[ \phi_{xt} + \frac{1}{16} \phi_{xxxxxx} + \frac{1}{2} \phi_x \phi_{xxx} + \frac{3}{4} \phi_{x} \phi_{xxx} + \frac{1}{8} \phi_{xxxz} \phi_z + \frac{1}{2} \phi_{xxz} + \phi^2 \phi_{xz} \]
\[ + \frac{3}{4} \phi_x \phi_{xxz} \phi_z + \frac{1}{8} \phi_{xxz} \partial_x^{-1}(\phi^3_x) = 0, \] (20)
and
\[ T'_2 = \frac{1}{2} \partial_z \partial_x^3 - \frac{1}{4} \partial_x \partial_x^2 + \left( \frac{1}{8} \phi_{xxx} + \frac{3}{4} \partial_x^2 \phi_x + \frac{1}{8} \partial_{xxz}(\phi^2_x) \right) \partial_x - \frac{1}{16} \phi_{xxxz} \]
\[ - \frac{1}{2} \phi_x \phi_{xxz} + \frac{3}{8} \phi_{xxz} \phi_z. \] (21)

In this case, the above equation (20) is equivalent to that obtained by Bogoyavlenski\( \Box \) \( \Box \). For \( n = 3 \), equation (18) takes the form
\[ \phi_{xt} + \frac{1}{64} \phi_{xxxxxx} + \frac{3}{16} \phi_x \phi_{xxxz} + \frac{15}{32} \phi_{xxxz} \phi_{xxz} + \frac{5}{8} \phi_{xxxz} \phi_{xxxx} \]
\[ + \frac{9}{32} \phi_{x} \phi_{xxz} + \frac{3}{4} \phi_{x} \phi_{xxx} \phi_z + \frac{1}{32} \phi_{xxxz} \phi_z + \frac{9}{4} \phi_x \phi_{xxz} + \frac{9}{8} \phi_x \phi_{xxz} \phi_z \]
\[ + \frac{9}{8} \phi_{xxz} \phi_x + \frac{3}{16} \phi_{xxz} \phi_z + \frac{1}{2} \phi_x \phi_{xxx} \phi_z + \frac{5}{16} \phi_x \phi_{xxx} \phi_z + \frac{5}{16} \phi_{xxx} \phi_{xxz} \phi_z \]
\[ - \frac{1}{32} \phi_{xxz} \partial_x^{-1}(\phi^2_x) + \frac{1}{16} \phi_{xxxz} \partial_x^{-1}(\phi^3_x) + \phi^3_x \phi_x + \phi^2_x \phi_{xxz} \phi_z + \phi^2_x \phi_{xxz} \phi_z \]
\[ + \frac{1}{8} \phi_x \phi_{xxz} \partial_x^{-1}(\phi^3_x) + \frac{3}{16} \phi_x \phi_{xxz} \partial_x^{-1}(\phi_x) = 0, \] (22)
and
Equations (20) and (22) are, of course, reduced to the 5th and 7th order potential KdV equations in the case of $z = x$, respectively. The calculations for $n \geq 4$ can be performed similarly, but the procedure to determine $T'_n$ operators for larger $n$ becomes more complicated.

3. Application of The Painlevé Test

We have constructed the BS hierarchy in the previous section. In this section we proceed to check the integrability of the BS hierarchy through the Painlevé test in the sense of WTC [13, 14]. For the BS hierarchy, there appears an ambiguity in this test. Hence we give the detail of calculation, which may help to understand what it is. In order to eliminate the integral operator, we rewrite the 5th order BS equation (20) in the form of coupled systems,

\begin{equation}
\begin{aligned}
\phi_{xt} + \frac{1}{16} \phi_{xxxxz} + \frac{1}{2} \phi_x \phi_{xxz} + \frac{3}{4} \phi_x \phi_{xxz} + \frac{1}{2} \phi_{xxx} \phi_{xz} \\
+ \frac{1}{8} \phi_{xxx} \phi_x - \rho_{xx} \phi_x + \frac{3}{4} \phi_x \phi_{xx} \phi_x - \frac{1}{8} \phi_{xx} \rho_{xz} = 0,
\end{aligned}
\end{equation}

\begin{equation}
\rho_{xx} + \phi_x^2 = 0.
\end{equation}

We now consider a local Laurent expansion in the neighborhood of a non-characteristic singular manifold $\gamma(x, z, t) = 0, (\gamma_x \neq 0)$. Let us assume that the leading orders of the solutions of equations (24) and (25) have the forms

\begin{equation}
\phi = \phi_0 \gamma^\alpha, \quad \rho = \rho_0 \gamma^\beta.
\end{equation}

Here $\phi_0$ and $\rho_0$ are some analytic functions. Substituting (26) into (24) and (25), and equating the powers of the most dominant terms, we obtain

\begin{equation}
\alpha = -1, \quad \beta = -2
\end{equation}

with

\begin{equation}
\phi_0 (\phi_0 - 6 \gamma_x) (\phi_0 - 2 \gamma_x) = 0, \quad 6 \rho_0 + \phi_0^2 = 0.
\end{equation}

To find the resonances we now substitute the full Laurent expansion of the solutions

\begin{equation}
\phi = \sum_{j=0} \phi_j \gamma^{j-1}, \quad \rho = \sum_{j=0} \rho_j \gamma^{j-2},
\end{equation}

\begin{equation}
T'_3 = \frac{1}{2} \phi_x \frac{\partial}{\partial \phi_x} - \frac{1}{4} \phi_{xx} \frac{\partial}{\partial \phi_x} + \left( \frac{1}{8} \phi_{xxx} + \frac{5}{4} \phi_x \phi_z + \frac{1}{8} \phi_x^{-1} (\phi_x^2) \right) \frac{\partial}{\partial \phi_x}
\end{equation}

\begin{equation}
\begin{aligned}
&- \left( \frac{1}{16} \phi_{xxxxz} + 3 \frac{1}{4} \phi_x \phi_{xxz} - \frac{15}{8} \phi_{xxx} \phi_z \right) \frac{\partial}{\partial \phi_x} + \left( \frac{1}{32} \phi_{xxxxx} - \frac{5}{16} \phi_{xxx} \phi_{xz} \\
&+ \frac{3}{8} \phi_x \phi_{xxz} - \frac{1}{32} \phi_x^{-1} (\phi_x^2) \right) \frac{\partial}{\partial \phi_x} - \frac{1}{64} \phi_{xxxxxx} - \frac{3}{16} \phi_x \phi_{xxz} - \frac{1}{16} \phi_{xxx} \phi_{xxz} - \frac{3}{8} \phi_{xxx} \phi_{xz} \\
&+ \frac{15}{32} \phi_{xxxx} \phi_z - \frac{3}{4} \phi_x^2 \phi_x + \frac{15}{16} \phi_x \phi_{xx} \phi_z + \frac{3}{32} \phi_x \phi_{xx} \phi_x \phi_{xxz}.
\end{aligned}
\end{equation}
into equations (24) and (25). Rearranging equation (24) into terms of \( \gamma_j - 7 \) and the other higher powers of \( \gamma \), we obtain recurrence relations for \( \phi_j, \rho_j \),

\[
\left( (j-1)(j-2)(j-3)(j-4)(j-5)(j-6)\gamma^5_z \gamma_z \\
-10(j-1)(j-6)(j^2 - 7j + 16)\gamma^4_z \gamma_z \phi_0 \\
-108(j-1)(j-2)\gamma^5_z \gamma_z \rho_0 + 12(j-1)(j-6)\gamma^5_z \gamma_z \phi_0^2 \phi_j \\
-36(j-2)(j-3)\gamma^3_z \gamma_z \phi_0 \rho_j \right) \gamma^{j-7} = f_j. \tag{30}
\]

Similarly rearranging equation (25) into terms of \( \gamma_j - 4 \) and higher powers of \( \gamma \), we have

\[
(-2(j-1)\gamma^2 \phi_0 \phi_j + (j-2)(j-3)\gamma^2 \rho_j) \gamma^{j-4} = g_j. \tag{31}
\]

Here \( f_j \) and \( g_j \) are given in terms of \( \phi_l \) and \( \rho_l \) \((0 \leq l \leq j-1)\). Then we get three types of resonances:

\[
j = 0, 1, 1, 2, 3, 4, 5, 6, \tag{32}
\]

\[
j = -1, 1, 2, 2, 3, 5, 6, 8 \tag{33}
\]

and

\[
j = -3, -1, 1, 2, 3, 6, 8, 10. \tag{34}
\]

These correspond to the choices of solutions of equations (25),

\[
\phi_0 = 0, \quad \rho_0 = 0, \tag{35}
\]

\[
\phi_0 = 2\gamma_x, \quad \rho_0 = -(2/3)\gamma^2_x \tag{36}
\]

and

\[
\phi_0 = 6\gamma_x, \quad \rho_0 = -6\gamma^2_x. \tag{37}
\]

respectively. Here the first choice (35) does not include the universal resonance \( j = -1 \) and is discarded. If the recurrence relations are consistently satisfied at the resonances then the differential equations is said to possess the Painlevé property.

Succeeding coefficients \( \phi_j \) and \( \rho_j \) are determined from equations (30) and (31). However, from the consistency condition, they must include arbitrary functions at the resonances. To simplify the calculations, we use the reduced manifold ansatz of Kruskal\,[15]. That is, \( \gamma(x, z, t) = x + \delta(z, t) \) and \( \phi_j, \rho_j \) are function of \( z \) and \( t \).

Equations (30) and (31) must be satisfied in the respective powers of \( \gamma \). The lowest powers of \( \gamma \), \((\gamma^{-7}, \gamma^{-4})\) in equations (30), (31) give

\[
\phi_0(\phi_0 - 2)(\phi_0 - 6) = 0, \quad \rho_0 = -\frac{1}{6}\phi_0^2. \tag{38}
\]
which constitutes a set of equations (35), (36), and (37). Higher powers of $\gamma, (\gamma^{-7+k}, \gamma^{-4+k}$ with positive integer $k$) in equations (30), (31) lead us to the following consistency conditions.

$$(\gamma^{-6}, \gamma^{-3}) : \phi_0 \rho_1 \delta_z = 0, \quad \rho_1 = 0. \quad (39)$$

Hence $\phi_1$ is arbitrary.

$$(\gamma^{-5}, \gamma^{-2}) : \phi_0 \left[ (\phi_0 - 2) \phi_{1,z} + 2(\phi_0 - 5) \phi_2 \delta_z \right] = 0, \quad \phi_0 \phi_2 = 0. \quad (40)$$

These equations imply that $\phi_0$ is fixed as 2 and $\rho_2$ is arbitrary, which contradicts equation (37). So we must choose (33).

$$(\gamma^{-4}, \gamma^{-1}) : (3\phi_0 - 20) \phi_3 \delta_z = 0, \quad \phi_3 = 0, \quad (41)$$

and $\rho_3$ is arbitrary.

$$(\gamma^{-3}, \gamma^{0}) : 6(13\phi_4 - 3\rho_4) \delta_z + 8\delta_t - \rho_3 \delta_z = 0, \quad \rho_4 = 6\phi_4. \quad (42)$$

$$(\gamma^{-2}, \gamma^{1}) : 9(8\phi_5 - 3\rho_5) \delta_z + (6\phi_4 - \rho_4) \delta_z = 0, \quad \rho_5 = \frac{8}{3} \phi_5 \quad (43)$$

and

$$(\gamma^{-1}, \gamma^{2}) : 36(5\phi_6 - 3\rho_6) \delta_z + (8\phi_5 - 3\rho_5) \delta_z = 0, \quad \rho_6 = \frac{5}{3} \phi_6. \quad (44)$$

It follows from equations (43) and (44) that one of the two variables must be arbitrary in both pairs $(\phi_5, \rho_5)$ and $(\phi_6, \rho_6)$.

$$(\gamma^{0}, \gamma^{3}) : 12(13\phi_7 - 15\rho_7) \delta_z - (5\phi_6 + 4\rho_6) \delta_z - 6\phi_5 \phi_{1,z} = 0, \quad \rho_7 = \frac{6}{5} \phi_7. \quad (45)$$

$$(\gamma^{1}, \gamma^{4}) : \left[ (504\phi_8 - 540\rho_8) + 72\phi_4^2 \right] \delta_z + 3\phi_4(-18\rho_4 \delta_z + 8\delta_t - \rho_3 \delta_z) + 2(6\phi_7 - 5\rho_7) \delta_z = 0, \quad \rho_8 = \frac{14}{15} \phi_8 - \frac{3}{10} \phi_4^2. \quad (46)$$

That is, one of the two variables $(\phi_8, \rho_8)$ must be arbitrary. Problematic is the following point: $\phi_2$ at double root $j = 2$ has been fixed as 0 and the system lacks one more arbitrary function. However, $\phi_j$ and $\rho_j$ are determined consistently and, thus, dependent variables $\phi$ and $\rho$ have only movable poles. The apparently similar situation occurred in the system of coupled non-linear Schrödinger equation (NLS) [16]. Starting from NLS leaving the constant coefficients as free parameters in [16], Sahadevan et al. these parameters so as to include $N$ arbitrary functions at $N$-tuply degenerate resonance. However, there seems to exist an essential difference between their NLS and our 5th order BS equation since NLS does not allow the consistent Laurent expansion except for the special choices of constant coefficients above mentioned. In the 5th order BS equation there is no such room to adjust the number of arbitrary function, though
the 5th order BS equation has no inconsistency. We want to add one remark here. The coupled system of equations (24) and (25) is reduced to the 5th order potential KdV equation in the case of \( z = x \). Even in that case, the resonance structures (the positions and the number of arbitrary functions etc.) are quite the same as the 5th order BS equation. The 5th order potential KdV equation is manifestly integrable. From these facts we may conclude that the 5th order BS equation is also integrable.

Analogously, we have checked that the 7th order BS equation is integrable in the same sense.

Higher order equation can be constructed in several different ways. All the higher order BS equations\(^{[18]}\) are rewritten as

\[
 u_t + \Phi(u)^n u_x = 0
\]

by the use of the recursion operator of KdV equation\(^{[3]}\). Equation\(^{[17]}\) was conjectured to be integrable\(^{[3]}\) because the integrability arose from the existence of the recursion operator\(^{[17]}\). However, the recursion operator in \((2 + 1)\) dimensions is not so clear as in the case of \((1 + 1)\) dimensions. Explicit calculation in this section has uncovered a new feature and has shown that the BS hierarchy is integrable.

4. Discussion

In this paper, we have constructed the BS hierarchy\(^{[18]}\) using the Bogoyavlenskii’s method. This hierarchy is represented by the recursion operator\(^{[2]}\), and reduced to the potential KdV hierarchy by setting \( z = x \). The BS hierarchy\(^{[18]}\), \(^{[17]}\) has been proved to be integrable through the Painlevé test. The BS equation is an extension of the KdV equation like the KP equation is. However their extensions differ in algebraic structure, which is schematically depicted in Figure\(^{[4]}\).

\[\text{KdV} \xrightarrow{\text{BS}} \text{X}_3\]

\[\text{KP} \xrightarrow{\text{X}_1} \text{X}_2\]

\[\text{Figure 1.} \text{ Scheme of extensions of the KdV equation. There are two directional routes of extensions: One leads us to the BS equation and the other does to the KP equation. X equations are possible \((3 + 1)\) dimensional integrable equations obtained from further extensions.}\]

The potential KP equation is obtained from the potential KdV equation by the replacement,

\[
 L(x, t) \longrightarrow L'(x, y, t) = \partial_x^2 + \phi_x(x, y, t) + \partial_y \\
 \equiv L(x, y, t) + \partial_y,
\]

\[\text{Figure 1.} \text{ Scheme of extensions of the KdV equation. There are two directional routes of extensions: One leads us to the BS equation and the other does to the KP equation. X equations are possible \((3 + 1)\) dimensional integrable equations obtained from further extensions.}\]

The form of \( T(x, y, t) \) is

\[
 T(x, y, t) = \partial_x L(x, y, t) + T'(x, y, t) + \partial_t,
\]

\[\text{Figure 1.} \text{ Scheme of extensions of the KdV equation. There are two directional routes of extensions: One leads us to the BS equation and the other does to the KP equation. X equations are possible \((3 + 1)\) dimensional integrable equations obtained from further extensions.}\]
where

\[ T'(x, y, t) = \frac{1}{2} \phi_x \partial_x - \frac{1}{4} \phi_{xx} - \frac{3}{4} \partial_y. \]  

(50)

Here we have denoted an extended spatial coordinate as \( y \). These extensions are contrasted with those of the BS equation (16) in which explicit \( y \) (in that case \( z \)) dependence has appeared not in \( L \) but in \( T \).

These two extensions may be performed in the different spatial dimensions (i.e. \( y \neq z \)), which is described by \( X_1 \) in Figure. For instance, we consider the Lax equation (5) with the operators

\[ L'(x, y, z, t) = \partial^2_x + \phi_x(x, y, z, t) + \partial_y \equiv L(x, y, z, t) + \partial_y, \]  

(51)

\[ T(x, y, z, t) = \partial_z L(x, y, z, t) + T'(x, y, z, t) + \partial_t, \]  

(52)

where

\[ T'(x, y, z, t) = \frac{1}{2} \phi_z \partial_x - \frac{1}{4} \phi_{xz} - \frac{3}{4} \partial_x + c \partial_z^2 \]  

(53)

and \( c \) is a constant. However, the Lax equation (5) with operators (51) and (52) in \( (3 + 1) \) dimensions is reduced to \( (2 + 1) \) dimensional equation,

\[ \phi_{xt} + \frac{1}{4} \phi_{xxxz} + \phi_x \phi_{xz} + \frac{1}{2} \phi_{xx} \phi_z + c^2 \partial_x^{-1} \phi_{zzzz} = 0 \]  

(54)

with the condition,

\[ z = 2cy. \]  

(55)

This condition comes from the requirement that the coefficient of \( \partial_z \) should be vanished in the Lax equation and, therefore, seems to be indispensable. Equation (54) in \( (2 + 1) \) dimensions is the same as that obtained by Bogoyavlenskii. We have checked that equation (54) passes the Painlevé test. By the dependent variable transformation \( \phi = 2(\log \tau)_x \), equation (54) is transformed into the trilinear form,

\[ (36T_z^2T_x + T_z^4T_x^* + 8T_z^3T_x^2 + 9T_z^5)\tau \cdot \tau = 0. \]  

(56)

The operators \( T, T^* \) are defined by

\[ T_z^n f(z) \cdot g(z) \cdot h(z) \equiv (\partial_{z_1} + j \partial_{z_2} + j^2 \partial_{z_3})^n f(z_1)g(z_2)h(z_3)|_{z_1 = z_2 = z_3 = z}, \]  

(57)

where \( j \) is the cubic root of unity, \( j = \exp(2i\pi/3) \). \( T_z^* \) is the complex conjugate operator of \( T_z \) obtained by replacing \( (\partial_{z_1} + j \partial_{z_2} + j^2 \partial_{z_3}) \) by \( (\partial_{z_1} + j \partial_{z_2} + j \partial_{z_3}) \). Equation (54) coincides with the trilinear form coming from the quite different approach. Namely, Hietarinta, Grammaticos and Ramani derived this equation from the singularity analysis of trilinear equations. We have obtained \( N \) soliton solutions of equation (54) by the computer program Mathematica up to \( N = 5 \). However, we can not construct the hierarchy of equation (54). Namely we can not determine \( T'(x, y, z, t) \) operator in

\[ T(x, y, z, t) = \partial_z L(x, y, z, t)^n + T'(x, y, z, t) + \partial_t, \]  

(58)
so as to constitute Lax equation (5).

A little bit different formulation to reach X1 candidate was proposed by us,

\[
\left( u_t + \Phi(u)u_z \right)_x + \frac{3}{4}u_{yy} = 0. \tag{59}
\]

Unfortunately, equation (59) has a movable logarithmic branch point.

One of the examples for (3 + 1) dimensional X3 equations in Figure is obtained also by Bogoyavlenskii. However, it is not affirmative unless it is proved to be integrable.

The respective routes in Figure are not unique. Different formulations at each path are possible, though we can not specify which is correct at the present stage. Such specification enforces us to clarify the algebraic structures and integrability of the equations depicted in Figure and their hierarchies furthermore.

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