OPTIMAL HEIGHT AND SHAPE OF A BUILDING WITH EXTERNAL AND INTERNAL TRAFFIC

Masashi Miyagawa
University of Yamanashi

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Abstract This paper presents an analytical model for determining the height and shape of a building. The average travel distances of external and internal traffic are obtained for a multi-story building with rectangular floors. The analytical expressions for the average distances demonstrate how the number of floors and the total floor area affect the travel distance in the building. The optimal number of floors that minimizes the average distance is then obtained. The effects of the shape of floors and the locations of the escalator and entrance on the average distance and the optimal number of floors are also examined. The result shows that a one-story building can be optimal if the total floor area is small and the internal traffic is dominant and that the diamond floor is superior to the square and rectangular floors.

Keywords: Transportation, building design, average distance, rectilinear distance, total floor area

1. Introduction

The height and shape of a building affect the ease of movement inside the building. If the total floor area is constant, as the height of a building increases and the size of floors decreases, the average horizontal travel time (by walking) decreases but the average vertical travel time (by escalator or elevator) increases. Examining the relationship between the height and shape of a building and the travel time will help designers to achieve travel efficient buildings.

The optimal height and shape of a building have been addressed from economic and environmental perspectives [3, 11, 12]. Several studies have focused on the travel distance (time) in a building. Sullivan [15] examined the effect of the horizontal and vertical travel costs on the optimal height of a building. Johnson [5] found the optimal number and shape of floors that minimize the average travel time. Koshizuka [8] derived the distribution of distances between two uniformly distributed points in a multi-story building. This distribution was extended by Koshizuka [9] to include the waiting time and speed of the elevator. Taguchi and Koshizuka [17] determined the optimal distribution of residential and transportation areas that minimizes the total travel time in a building. The optimal height of a city consisting of many buildings has also been studied [6, 7, 10, 16].

In this paper, we obtain the optimal height and shape of a building that minimize the average travel distance in the building. Since the average travel distance represents the travel efficiency in a building, this is an optimization from an efficiency point of view. Minimizing the maximum travel distance, which is important for the evacuation from a building in an emergency, is left for future research. Although the effect of the height on the average travel distance was examined by Johnson [5] and Koshizuka [8], we extend their works in the following respects. First, we incorporate external traffic to deal with the case where
either origin or destination is outside the building. In fact, external traffic is dominant for residential buildings. Second, as the shape of floors, we consider not only a rectangle but also a diamond, which is a square rotated by 45°. As will be shown in the present paper, the diamond floor is superior to the square and rectangular floors. Finally, we examine how the locations of the escalator and entrance of the building affect the average distance and the optimal height.

The remainder of this paper is organized as follows. The next section develops a model of a $n$-story building. Section 3 obtains the optimal number of floors that minimizes the average distance in the building. Sections 4, 5, and 6 examine the effects of the shape of floors, the location of the escalator, and the location of the entrance, respectively. The final section presents concluding remarks.

2. Model

Consider a $n$-story building, as shown in Figure 1. All floors are represented as rectangles with side lengths $a$ and $b$. Let $A = nab$ be the total floor area in the building and $W = a/b$ be the aspect ratio of the rectangular floor. The side lengths are then expressed in terms of $A$ and $W$ as follows:

$$a = \sqrt{AW \frac{n}{n}}, \quad b = \sqrt{\frac{A}{nW}}.$$  (2.1)

Let $h$ be the floor height. The entrance of the building is located at a corner of the first floor $O$.

![Figure 1: n-story building](image)

The traffic in the building is classified into two types. If either origin or destination is outside the building, the traffic is called external traffic. If both origin and destination are inside the building, the traffic is called internal traffic. Examples of external and internal traffic in office buildings are commuting to the workplace and visiting other departments, respectively.

The travel distance on the same floor is measured as the rectilinear distance. The rectilinear distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is defined as $|x_1 - x_2| + |y_1 - y_2|$. The rectilinear distance has frequently been used as an approximation for the travel distance in a building [1, 13, 14]. The travel between different floors is only allowed by the escalator at the center of the floors $(a/2, b/2)$. The vertical distance is multiplied by $\alpha$ ($> 1$) because the speed of vertical travel is lower than that of horizontal travel. Koshizuka [8] used the
value $\alpha = 5$, assuming that the average walking speed is $4$ km/h and the speed of the escalator is $800$ m/h. The travel time is then proportional to the travel distance.

To obtain the average travel distance in the building, we provide three average distances in a rectangle with side lengths $a$ and $b$, as shown in Figure 2. The average rectilinear distance between a corner of the rectangle and a uniformly distributed point in the rectangle (Figure 2a) is

$$r_a = \frac{1}{ab} \int_0^b \int_0^a (x + y) \, dx \, dy$$

$$= \frac{1}{2} (a + b). \quad (2.2)$$

The average rectilinear distance between the center of the rectangle and a uniformly distributed point in the rectangle (Figure 2b) is

$$r_b = \frac{1}{4} (a + b), \quad (2.3)$$

which is obtained by replacing $a$ and $b$ in Equation (2.2) with $a/2$ and $b/2$, respectively. The average rectilinear distance between two uniformly distributed points in the rectangle (Figure 2c) is

$$r_c = \frac{1}{a^2} \int_0^a \int_0^a |x_1 - x_2| \, dx_1 \, dx_2 + \frac{1}{b^2} \int_0^b \int_0^b |y_1 - y_2| \, dy_1 \, dy_2$$

$$= \frac{1}{3} (a + b). \quad (2.4)$$

3. Optimal Number of Floors

We first obtain the average distance of external traffic. Assume that origin is at the entrance of the building and destination is uniformly distributed in the building. The uniform distribution serves as a basis for further analysis with more realistic distributions. If destination is on the first floor, from Equation (2.2), the average distance is

$$r_1 = \frac{1}{2} (a + b). \quad (3.1)$$

If destination is on the $k$-th ($k = 2, 3, \ldots, n$) floor, the average distance is the sum of the distance from the entrance to the escalator at the center of the first floor, the vertical
distance to the $k$-th floor, and the average distance from the escalator to destination on the $k$-th floor. From Equation (2.3), the average distance on the $k$-th floor is $(a + b)/4$. The average distance to destination on the $k$-th floor is

$$r_k = \frac{1}{2}(a + b) + \alpha(k - 1)h + \frac{1}{4}(a + b). \quad (3.2)$$

The average distance of external traffic is then given by

$$R^E = \frac{1}{n} \sum_{k=1}^{n} r_k = \frac{3n - 1}{4n} (a + b) + \frac{n - 1}{2} \alpha h, \quad (3.3)$$

which under Equation (2.1) reduces to

$$R^E = \frac{3n - 1}{4n\sqrt{n}} \sqrt{A} \left( \sqrt{W} + \frac{1}{\sqrt{W}} \right) + \frac{n - 1}{2} \alpha h. \quad (3.4)$$

The average distance of external traffic for the square floor ($W = 1$) is shown in Figure 3a, where $h = 4, \alpha = 5$, as suggested by Koshizuka [8]. As the number of floors $n$ increases, the average distance first decreases and then increases. This is because the increase in the number of floors reduces the horizontal travel distance but increases the vertical travel distance. The number of floors that minimizes the average distance is $n^* = 3$ for $A = 10000$, $n^* = 4$ for $A = 20000$, and $n^* = 5$ for $A = 30000$. The optimal number of floors is shown in Figure 3b as a function of total floor area. It can be seen that the optimal number of floors increases with the total floor area. If the total floor area is small, a one-story building is optimal. The optimal number of floors also depends on the floor height $h$. If $h = 3$, the optimal number of floors is $n^* = 4$ for $A = 10000$, $n^* = 5$ for $A = 20000$, and $n^* = 6$ for $A = 30000$.

![Figure 3: (a) Average distance of external traffic; (b) Optimal number of floors](image)

We next obtain the average distance of internal traffic. Assume that both origin and destination are uniformly distributed in the building, as Johnson [5] and Koshizuka [8, 9]. If origin and destination are on the same floor, from Equation (2.4), the average distance is

$$r_1 = \frac{1}{3}(a + b). \quad (3.5)$$
If origin and destination are \( k \) \((k = 1, 2, \ldots, n - 1)\) floors apart, the average distance is

\[
r_k = \frac{1}{4}(a + b) + akh + \frac{1}{4}(a + b). \tag{3.6}
\]

The numbers of combinations for the above two cases are \( n \) and \( 2(n - k) \), respectively. The average distance of internal traffic is then given by

\[
R^I = \frac{1}{n^2} \left\{ nr_1 + \sum_{k=1}^{n-1} 2(n-k)r_k \right\} = \frac{3n-1}{6n}(a + b) + \frac{n^2-1}{3n} \alpha h = \frac{3n-1}{6n\sqrt{n}} \sqrt{A} \left( \sqrt{W} + \frac{1}{\sqrt{W}} \right) + \frac{n^2-1}{3n} \alpha h, \tag{3.7}
\]

[8]. The average distance of internal traffic for the square floor is shown in Figure 4a. The number of floors that minimizes the average distance is \( n^* = 1 \) for \( A = 10000 \), \( n^* = 4 \) for \( A = 20000 \), and \( n^* = 5 \) for \( A = 30000 \). The optimal number of floors is shown in Figure 4b as a function of total floor area. Note that a two-story building cannot be optimal. If \( h = 3 \), the optimal number of floors is \( n^* = 4 \) for \( A = 10000 \), \( n^* = 5 \) for \( A = 20000 \), and \( n^* = 6 \) for \( A = 30000 \).

![Figure 4: (a) Average distance of internal traffic; (b) Optimal number of floors](image)

The average distance in the building is expressed as the weighted sum of the average distances of external and internal traffic

\[
R = \beta R^E + (1 - \beta) R^I, \tag{3.8}
\]

where \( \beta (0 \leq \beta \leq 1) \) is a weight of external traffic. The average distance for the square floor is shown in Figure 5. The average distance increases with the weight of external traffic. If \( A = 10000 \), the optimal number of floors is \( n^* = 1 \) for \( 0 \leq \beta < 0.446 \) and \( n^* = 3 \) for \( 0.446 \leq \beta < 1 \). If the weight of external traffic is small, that is, internal traffic is dominant, a one-story building can be optimal. If \( A = 20000 \), the optimal number of floors is \( n^* = 4 \) irrespective of \( \beta \). The optimal number of floors is shown in Figure 6 as a function of total floor area. It can be seen that \( \beta \) has a little effect on the optimal number of floors.
4. Effect of Floor Shape

In this section, we examine the effect of the shape of floors on the average distance and the optimal number of floors. Four shapes of floors are considered: square \((W = 1)\), rectangle \((W = 2, 3)\), and diamond, which is a square rotated by \(45^\circ\), as shown in Figure 7.

The three average distances in a diamond with side length \(a\) are obtained as follows. The average rectilinear distance between a corner of the diamond and a uniformly distributed point in the diamond is

\[
r_a = \frac{2\sqrt{2}}{3}a \approx 0.943a.
\]  

The average rectilinear distance between the center of the diamond and a uniformly distributed point in the diamond is

\[
r_b = \frac{\sqrt{2}}{3}a \approx 0.471a.
\]  

The average rectilinear distance between two uniformly distributed points in the diamond is

\[
r_c = \frac{7\sqrt{2}}{15}a \approx 0.660a,
\]
The average distances of external and internal traffic for the diamond floor are then

\[ R^E = \frac{\sqrt{2}(5n-1)}{6n} a + \frac{n-1}{2} \alpha h \]

\[ = \frac{\sqrt{2}(5n-1)}{6n\sqrt{n}} \sqrt{A} + \frac{n-1}{2} \alpha h, \tag{4.4} \]

\[ R^I = \frac{\sqrt{2}(10n-3)}{15n} a + \frac{n^2-1}{3n} \alpha h \]

\[ = \frac{\sqrt{2}(10n-3)}{15n\sqrt{n}} \sqrt{A} + \frac{n^2-1}{3n} \alpha h, \tag{4.5} \]

respectively. The average distances of external and internal traffic for the four shapes of floors with \( A = 30000 \) are shown in Figure 8. Note that both the average distances for the diamond floor are smaller than those for the square and rectangular floors. Note also that the average distances for the rectangular floor increase with the aspect ratio \( W \). The number of floors that minimizes the average distances of external and internal traffic is \( n^* = 4 \) for the diamond floor and \( n^* = 5 \) for the square and rectangular floors. The optimal number of floors thus depends on the shape of floors.

5. Effect of Escalator Location

In this section, we examine the effect of the location of the escalator on the average distance and the optimal number of floors. Let \((u, v) (0 \leq u \leq a, 0 \leq v \leq b)\) be the location of the
escalator. From Equation (2.2), the average rectilinear distance between the escalator at \((u, v)\) and a uniformly distributed point in the rectangle is

\[
\begin{align*}
r_b &= \frac{1}{ab} \left\{ uv \frac{u + v}{2} + u(b - v) \frac{u + b - v}{2} + (a - u)v \frac{a - u + v}{2} + (a - u)(b - v) \frac{a - u + b - v}{2} \right\} \\
&= \frac{u^2}{a} + \frac{v^2}{b} - (u + v) + \frac{1}{2}(a + b).
\end{align*}
\]

(5.1)

The average distances of external and internal traffic are then

\[
\begin{align*}
R^E &= \frac{n - 1}{n} \left( \frac{u^2}{a} + \frac{v^2}{b} \right) + \frac{1}{2}(a + b) + \frac{n - 1}{2} \alpha h \\
&= \frac{n - 1}{\sqrt{nA}} \left( \frac{u^2}{\sqrt{W}} + \frac{v^2}{\sqrt{W}} \right) + \frac{\sqrt{A}}{2n} \left( \sqrt{W} + \frac{1}{\sqrt{W}} \right) + \frac{n - 1}{2} \alpha h, \\
R^I &= \frac{2(n - 1)}{n} \left\{ \frac{u^2}{a} + \frac{v^2}{b} - (u + v) \right\} + \frac{3n - 2}{3n} (a + b) + \frac{n^2 - 1}{3n} \alpha h \\
&= \frac{2(n - 1)}{\sqrt{nA}} \left( \frac{u^2}{\sqrt{W}} + \frac{v^2}{\sqrt{W}} - (u + v) \right) + \frac{3n - 2}{3n\sqrt{n}} \sqrt{A} \left( \sqrt{W} + \frac{1}{\sqrt{W}} \right) + \frac{n^2 - 1}{3n} \alpha h,
\end{align*}
\]

(5.2)

(5.3)

respectively. The locations of the escalator that minimize the average distances of external and internal traffic are \(u^* = 0, v^* = 0\) and \(u^* = a/2, v^* = b/2\), respectively. The location of the escalator that minimizes the average distance in the building (3.8) is

\[
\begin{align*}
&u^* = \frac{1 - \beta}{1 - \beta} a, \quad v^* = \frac{1 - \beta}{1 - \beta} b.
\end{align*}
\]

(5.4)

The average distances of external and internal traffic for three locations of the escalator for the square floor with \(A = 20000\) are shown in Figure 9. The number of floors that minimizes the average distance of external traffic is \(n^* = 4\) irrespective of the location of the escalator, whereas the number of floors that minimizes the average distance of internal traffic is \(n^* = 6\) for \(u = 0, v = 0\), \(n^* = 5\) for \(u = a/2, v = 0\), and \(n^* = 4\) for \(u = a/2, v = b/2\).

![Figure 9](image-url)
6. Effect of Entrance Location

In this section, we examine the effect of the location of the entrance on the average distance and the optimal number of floors. Let \((s,0)\) \((0 \leq s \leq a/2)\) be the location of the entrance. From Equation (2.2), the average rectilinear distance between the entrance at \((s,0)\) and a uniformly distributed point in the rectangle is

\[
r_a = \frac{1}{ab} \left\{ sb \left( s + b \right) + (a - s) \left( a - s + b \right) \right\} = \frac{a + b}{2} + \frac{s^2}{a} - s.
\] (6.1)

The average distance of external traffic is then

\[
R_E = \frac{3n - 1}{4n} (a + b) + \frac{s^2}{an} - s + \frac{n - 1}{2} \alpha h = \frac{(3n - 1)}{4n \sqrt{n}} \sqrt{A} \left( \sqrt{W} + \frac{1}{\sqrt{W}} \right) + \frac{s^2}{\sqrt{nAW}} - s + \frac{n - 1}{2} \alpha h.
\] (6.2)

The location of the entrance that minimizes the average distance of external traffic is \(s^* = a/2\). The average distance of external traffic for three locations of the entrance for the square floor is shown in Figure 10. If \(A = 10000\), the number of floors that minimizes the average distance of external traffic is \(n^* = 3\) for \(s = 0\), \(n^* = 1\) for \(s = a/4\), \(n^* = 2\) for \(s = a/2\). If \(A = 20000\), the optimal number of floors is \(n^* = 4\) for \(s = 0\) and \(n^* = 3\) for \(s = a/4, a/2\).

![Figure 10: Location of entrance and average distance: (a) A = 10000; (b) A = 20000](image)

7. Conclusions

This paper has developed an analytical model for determining the height and shape of a building. The average distances of external and internal traffic in a multi-story building have been obtained for rectangular and diamond floors. The analytical expressions for the average distances demonstrate how the number of floors, the total floor area, the shape of floors, and the locations of the escalator and entrance affect the travel distance in the building. These relationships provide a fundamental understanding of the travel distance in a building.

The model provides a useful framework for analyzing the height and shape of a building as follows. First, the average distance helps designers to estimate the height of a new
building to achieve a certain level of travel efficiency. Note that the estimation requires neither much data nor the computation of the average distance for various combinations of the parameters. The effects of the shape of floors and the locations of the escalator and entrance give an insight into the floor plan design. Second, the optimal number of floors can be used to evaluate the height of existing buildings, thereby assisting decisions about the regulations of the building height and the total floor area. Finally, the model supplies building blocks for planning a city with many buildings.

The present model can be extended in future research. First, not only the average distance but also the maximum distance should be considered. Second, multiple escalators and entrances should be incorporated. Finally, introducing floors of different sizes would be interesting. Using smaller floors for upper floors than lower floors can reduce the travel time, as discussed by Johnson [5].

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Masashi Miyagawa  
Department of Regional Social Management  
University of Yamanashi  
4-4-37 Takeda, Kofu  
Yamanashi 400-8510, Japan  
E-mail: mmiyagawa@yamanashi.ac.jp