Local time of an Ornstein–Uhlenbeck particle

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Received 20 November 2020
Accepted for publication 5 February 2021
Published 23 March 2021

Abstract. In this paper, we study the local time spent by an Ornstein–Uhlenbeck (OU) particle at some location till time \( t \). Using the Feynman–Kac formalism, the computation of the moment generating function (MGF) of the local time can be mapped to the problem of finding the eigenvalues and eigenfunctions of a quantum particle. We employ quantum perturbation theory to compute the eigenvalues and eigenfunctions in powers of the argument of the MGF which particularly help to directly compute the cumulants and correlations among local times spent at different locations. In particular, we obtain explicit expressions of the mean, variance, and covariance of the local times in the presence and in the absence of an absorbing boundary, conditioned on survival. In the absence of absorbing boundaries, we also study large deviations of the local time and compute exact asymptotic forms of the associated large deviation functions explicitly. In the second part of the paper, we extend our study of the statistics of local time of the OU particle to the case not conditioned on survival. In this case, one expects the distribution of the local time to reach a stationary distribution in the large time limit. Computations of such stationary distributions are known in the literature as the problem of first passage functionals. In this paper, we study the approach to this stationary state with time by providing a general formulation for evaluating the MGF. From this MGF, we compute the cumulants of the local time exhibiting the approach to the stationary values explicitly for a free particle and a OU particle.

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Our analytical results are verified and supported by numerical simulations.

**Keywords:** Brownian motion, correlation functions

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1. Introduction

The local time of a particle at a specified point in space is the total time spent by the particle at that location over a fixed duration of time [1–3]. Local times measured at different locations together can describe the spatiotemporal properties of the motion of the particle. In the context of reaction–diffusion processes, this quantity can be a key controlling factor of the reaction [2, 4–11]. Usually, the reaction centres are heterogeneously distributed over space [12] and quite often their distribution fluctuates over time. In such cases, the rate or the yield of a particular reaction would highly depend on the amount of time a reaction agent spends at these reaction centres [13–16]. These times are called residence times, and in the limit of small size of the reaction centres, the density of residence time spent is called the local time (density). In a slightly more complicated situation, one can think of a series of reactions which depend on each other. In such cases, the yield of these reactions may depend on the correlation between the residence times or local times spent at different locations. For example, the time spent by a diffusing agent near a boundary or a catalytic surface plays a crucial role in describing various surface-mediated phenomena [9–11].

In the physics and mathematics literature, local and residence times have been studied in detail in the context of Brownian functionals, both in presence and absence of external potentials, and with different types of boundary conditions [3, 17–23]. The local time spent at position $y$ by a Brownian particle moving in one dimension over a time duration $t$, denoted by $\ell_t(y)$, is written as a functional of its trajectory $x(\tau)$ [3, 24]:

$$\ell_t(y) = \int_0^t d\tau \delta(x(\tau) - y).$$  

(1)

Many properties of this quantity have been studied and analysed in various contexts over the years. Examples include diffusion in bounded domains [23], diffusion in a potential landscape [25], diffusion in a random potential landscape [24], diffusion on a graph [26], time averaged statistical observables [27], single file problem [28, 29], diffusion with reflected walls [30, 31], and in conditioned diffusions [32]. The local time has also been studied for random processes other than simple diffusion e.g., uniform empirical process [33], Brownian excursions [34] and for diffusion with stochastic resetting [35]. In fact, more general functionals like

$$T_t[x(\tau)] = \int_0^t d\tau U(x(\tau)),$$  

(2)

where $U(x)$ is an arbitrary function (with good properties), have been studied in different contexts by various methods [3, 23–25, 27–29, 36–39].

In this paper, we consider an over damped particle moving in one dimension in a viscous thermal medium, in the presence of an external potential $V(x)$. The equation of motion for such a particle is

$$\dot{x} = -V'(x) + \xi(t),$$  

(3)

where $\xi(t)$ is a mean zero Gaussian white noise with $\langle \xi(t)\xi(t') \rangle = 2D \delta(t - t')$. Here, $\langle \ldots \rangle$ represents the noise realisation average, and $D$ controls the strength of the noise which
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would also be the diffusion constant of the particle. We study the statistical properties of the local time \( \ell_t(y) \) spent by the particle at some position \( y \) in the presence of an external harmonic potential \( V(x) = \kappa x^2 / 2 \), with and without absorbing boundaries. For this choice of the potential \( V(x) \), the motion in equation (3) is known in the literature as the Ornstein–Uhlenbeck (OU) process [40, 41]. In particular, mapping the problem to an equivalent quantum problem through the Feynman–Kac formalism [42, 43], we use the techniques of quantum perturbation theory to compute various moments of \( \ell_t(y) \) as well as the equal time correlation \( \langle \ell_t(y)\ell_t(z) \rangle \) among local times spent at different locations. This method is similar to the method described in [23] where it was argued that within a bounded geometry, the cumulants of functionals like \( T_t[x(\tau)] \) in equation (2) are all linear in \( t \) for large \( t \). This implies that the typical fluctuations of \( T_t \) are of order \( O(\sqrt{t}) \) and are distributed according to a Gaussian.

A similar perturbation method, based on the spectral theory of non-Hermitian operators, has recently been applied and discussed in more general contexts such as the study of fluctuations in time-averaged statistical mechanics [27] and the single file problems [28, 29] where local time has been considered as a central object. In [27], general expressions for the mean, variance, and covariance of an arbitrary functional, written in terms of eigenfunctions and matrix elements have been obtained, and from the behaviour of these quantities at large time, the appearance of the Gaussian scaling distribution has been anticipated.

In this paper, applying quantum perturbation theory, we have obtained the mean, variance and covariance of the local time of an OU particle, expressed explicitly in terms of Hermite polynomials in the presence and in the absence of an absorbing boundary. From the large \( t \) behaviour of the moments, we also find that the typical fluctuations are Gaussian in the presence of a confining potential, as found in [23, 27]. However, in addition to the typical Gaussian distribution, we have computed the distribution of large deviations of the local time \( \ell_t(y) \) as well. These are described by appropriate large deviation functions (LDFs) on the infinite line and on the semi infinite line with a reflecting boundary. For reviews on large deviation theory, see [37, 44].

In the presence of absorbing boundaries, one usually studies statistics of path functionals such as \( T_t[x(\tau)] \) (defined above) in two ways. The first way is to study the distribution of \( T_t \) conditioned on survival over a given duration of time \( t \), i.e. over the ensemble of paths which survive till time \( t \). In the second way, known as the study of first passage functionals, one is interested in the distribution of \( T_t \) till the first passage time to the absorbing boundaries [3, 45]. In this paper, we formulate a new general method to study these functionals for an arbitrary time duration \( t \), not conditioned on survival, in the presence of a completely absorbing boundary. As it should, our formulation in the \( t \to \infty \) limit corresponds to the study of first passage functionals.

The problem of path functionals in the presence of absorbing boundaries may appear in various situations—for example, local time not conditioned on survival may appear in the context of reaction–diffusion processes inside a bounded domain with absorbing boundaries. Imagine catalytic reaction agents diffusing in a domain with heterogeneously distributed reaction centres. Once a particular agent reaches a centre, the corresponding reaction occurs. Such a catalytic reagent will continue to initiate reactions as long as it lives, i.e. is not absorbed at the boundary. It is easy to understand that the total amount

https://doi.org/10.1088/1742-5468/abe93d
of production from reactions till time $t$ would depend on the time spent by reagents at these centres. In this paper, we present a general formulation for studying path functionals (defined in equation (2)) not conditioned on survival and apply this formalism to compute the moments of $\ell_t(y)$ for an over damped particle evolving according to equation (3).

We briefly summarise our results along with a presentation of the organisation of the paper. In section 2, we follow the Feynman–Kac formalism to describe how the computation of the moment generating function (MGF) of the functional $T_t[x(\tau)]$, defined as $Q(k, t|x_0) = \langle e^{-k T_t[x(\tau)]} \rangle_{x_0}$ with $x_0$ being the starting position, can be mapped to the problem of finding the eigenvalues and eigenfunctions of a quantum particle inside an appropriate confining potential. This is a well known mapping that has been observed and employed in various contexts to compute distributions of different observables [3, 32, 36–38, 46, 47]. The Hamiltonian of the quantum particle naturally appears to be a sum of two parts, in which the part involving $k$ can be treated as a perturbation to the other part. Using this structure, we apply quantum perturbation theory to compute the eigenvalues and eigenfunctions as a series expansion in powers of $k$, which is useful to directly compute various moments and correlations. As mentioned earlier, this perturbation method has been discussed and applied previously in different contexts [23, 27].

To make this paper self-contained, we present the perturbation formalism for computing the moments of a general functional $T_t[x(\tau)]$ in section 2. In section 4.1, we apply this formulation to the functional $\ell_t(y)$, defined in equation (1), and obtain an exact, explicit expression for the mean $\langle \ell_t(y) \rangle$ (see equation (36)) for arbitrary $t$, $y$ and initial position $x_0$. We have also obtained exact, explicit expressions for the asymptotic variance $\langle \ell_t^2(y) \rangle_c$ and two-point correlation $\langle \ell_t(y)\ell_t(z) \rangle_c$ at large $t$ (see equations (37) and (38)). We use $\langle \ldots \rangle_c$ to represent cumulants and connected correlations.

For cases in which the effective quantum confining potential is such that the ground state is non-degenerate and gapped, we find that at large time $t$, the cumulants of $T_t[x(\tau)]$ (defined in equation (2)) grow linearly with $t$. This suggests that the distribution of $T_t[x(\tau)]$, when shifted by its mean and scaled by its standard deviation, approaches a Gaussian distribution at large $t$. However, this distribution describes only the typical fluctuations, not the large fluctuations. In section 3, we describe how by performing a saddle point calculation on the leading term in the spectral representation of the MGF, one can establish the large deviation principle satisfied by a general functional $T_t[x(\tau)]$. This method has been used in other contexts [36, 37, 46, 47]. One usually relates the LDF to the dominant eigenvalue, i.e. the ground state eigenvalue $\lambda_0(k)$ of the quantum problem through saddle point calculations, or equivalently, through a Legendre transform [44]. In section 4.2, we apply this method to the local time $\ell_t(y)$, where we study analytic properties of the lowest eigenvalue $\lambda_0(k)$ in different regimes of $k$ and use these properties to find different asymptotic behaviours of the associated LDF for two choices of boundary conditions (see equations (58) and (71)). In the mathematics literature, the LDF of functionals like $T_t[x(\tau)]$ (defined in equation (2)) can be computed from the large deviation theory developed by Donsker and Varadhan [44, 48]. The large deviations for local time can in principle be obtained from their theory for the empirical distribution [44]. However, in this paper, we use a different method to compute exact, explicit expressions of LDF in different asymptotic limits.
In the last part of the paper, starting in section 4.3, we study statistical properties of the local time in the presence of an absorbing boundary, but not conditioned on survival. We compute the local time $\ell_t(y)$ spent by the OU particle at $y$ till time $t$. It is clear that if the particle survives, its trajectory may contribute to $\ell_t(y)$, i.e. may visit the location $y$ till time $t$. On the other hand, if it gets absorbed at time $t' < t$, then its trajectory will not contribute to $\ell_t(y)$ after time $t'$. However, we compute the local time spent at $y$ till time $t$ without the knowledge if the particle has already been absorbed at the boundary. It is easy to realise that in the $t \to \infty$ limit the distribution of $\ell_t(y)$ would approach a time independent (stationary) distribution $P_{st}(\ell|y_0)$. We study this stationary distribution in section 4.3.1; in the literature, this is known as the problem of first passage functionals [3, 45]. In this paper, we also study the approach to this stationary state by formulating a general method in section 4.3.4 for computing the associated MGF at time $t$. We demonstrate this method by applying it to two cases: (i) free particle ($V(x) = 0$) and (ii) OU particle ($V(x) = \kappa x^2/2$). For these two cases, we compute the approach of the mean local time to its stationary value with time (see equations (119) and (126)).

2. Moments and cumulants using quantum perturbation theory

To compute the statistics of the path functional $T_t[x(\tau)]$, we define the MGF as [3]

$$Q(k, t|x_0) = \left\langle e^{-k T_t[x(\tau)]} \right\rangle_{x_0}. \tag{4}$$

The angle brackets in the above equation denote the expectation value over the distribution $P_{x_0}(T, t)$ of $T_t$, conditioned on survival of the particle which starts at $x_0$. Since the observable $T_t$ depends on the path as defined in equation (2), this average can be transformed into an average over paths surviving till time $t$, whose probability density can be obtained from the probability density of the noise realisation $\xi(\tau)$, given by

$$P(\{\xi(\tau)\}) = \frac{1}{\mathcal{N}} \exp \left(-\frac{1}{4D} \int_0^t \xi^2(\tau) d\tau \right) \tag{5}$$

where $\mathcal{N}$ is the normalisation constant. The MGF $Q(k, t|x_0)$ of $T_t[x(\tau)]$ can be written as

$$Q(k, t|x_0) = \left\langle e^{-k \int_0^t U(x(\tau)) d\tau} \right\rangle_{x_0} = \frac{Q(k, t|x_0)}{Q(0, t|x_0)} \tag{6}$$

where, using the standard Ito calculus and the usual connections to the Schrödinger equation in quantum mechanics, one can show that

$$Q(k, t|x_0) = \int_{x(0) = x_0} dx \int_{x(t) = x} \mathcal{D}[x(\tau)]$$

$$\times \exp \left( -\int_0^t \frac{d\tau}{4D} \left[ \left( \frac{dx}{d\tau} + V'(x) \right)^2 - 2DV''(x) + 4DkU(x) \right] \right) \tag{7}$$
and the corresponding cumulants can be obtained from

\[ \langle T_l^i \rangle_c = (-1)^l \frac{d^l}{dk^l} \log \left( \frac{Q(k, t|x_0)}{Q(0, t|x_0)} \right) \bigg|_{k \to 0}, \quad \text{for } l = 1, 2, 3, \ldots \]  

\[ = (-1)^l \frac{d^l}{dk^l} \log \left( \frac{Q(k, t|x_0)}{Q(0, t|x_0)} \right) \bigg|_{k \to 0}, \quad \text{for } l = 1, 2, 3, \ldots \]  

The operator \( H_k \) in the above equation is given by

\[ \hat{H}_k = -D \frac{d^2}{dx^2} + \mathcal{U}(x) + kU(x) \quad \text{with} \quad \mathcal{U}(x) = \left( \frac{V'(x)^2}{4D} - \frac{V''(x)}{2} \right), \]  

\[ V'(x) = dV(x)/dx, \quad \text{and} \quad V''(x) = d^2V(x)/dx^2 \]  

Note that the quantity \( Q(0, x_0, t) \) is actually the survival probability \( S(t|x_0) \) of the particle in the domain \( D \) till time \( t \), given that it started from \( x(0) = x_0 \). One has \( Q(0, t|x_0) = 1 \) when there is no absorbing boundary in the system, i.e., when the probability is conserved. Otherwise, in the presence of an absorbing boundary, \( Q(0, t|x_0) < 1 \).

The operator \( \hat{H}_k \) in equation (9) can be interpreted as the Hamiltonian of a quantum particle of mass \( m = \frac{1}{2D} \), moving in an effective potential given by \( \mathcal{U}(x) + kU(x) \) [3]. For appropriate choices of \( V(x) \) and \( U(x) \) this potential can be a confining potential, and for such choices we expect the Hamiltonian \( \hat{H}_k \) to have a discrete eigen spectrum. Let the eigenvalues and eigenfunctions of the 'quantum' Hamiltonian \( \hat{H}_k \), denoted in bra-ket notation by

\[ \hat{H}_k |\psi_{n,k}\rangle = \lambda_n(k) |\psi_{n,k}\rangle, \quad \text{where} \quad \langle x|\psi_{n,k}\rangle = \psi_{n,k}(x), \quad n = 0, 1, 2, \ldots \]  

be orthonormal eigenfunctions which satisfy the specified boundary conditions and \( \lambda_{n+1}(k) > \lambda_n(k) \geq 0 \forall n \). We can then represent equation (7) as

\[ Q(k, t|x_0) = \int dx \exp \left( \frac{-V(x)}{2D} \right) \left[ \sum_{n} e^{-\lambda_n(k)t} \psi_{n,k}(x) \psi_{n,k}^*(x_0) \right] \exp \left( \frac{V(x_0)}{2D} \right), \]  

where \( \psi_{n,k}^*(x_0) \) is the complex conjugate of \( \psi_{n,k}(x_0) \). In this way the problem of calculating the MGF of a given statistic gets mapped to the problem of finding the eigenvalues and eigenstates of a quantum particle in the corresponding one dimensional potential, for which a variety of methods exist.

From the Taylor series expansion of the MGF \( Q(k, t|x_0) \), one can compute moments of \( T_l \) at different order as

\[ \langle T_l^i \rangle = (-1)^l \frac{d^l}{dk^l} Q(k, t|x_0) \bigg|_{k \to 0} = \frac{(-1)^l}{Q(0, t|x_0)} \frac{d^l}{dk^l} Q(k, t|x_0) \bigg|_{k \to 0}, \quad \text{for } l = 1, 2, 3, \ldots \]  

and the corresponding cumulants can be obtained from

\[ \langle T_l^i \rangle_c = (-1)^l \frac{d^l}{dk^l} \log \left( \frac{Q(k, t|x_0)}{Q(0, t|x_0)} \right) \bigg|_{k \to 0}, \quad \text{for } l = 1, 2, 3, \ldots \]  

https://doi.org/10.1088/1742-5468/abe93d
In order to proceed, we note that the Hamiltonian \( \hat{H}_k \) in (9) is in the form
\[
\hat{H}_k = \hat{H}_u + \hat{h}_k \quad \text{where} \quad \hat{h}_k = kU(x).
\]
If the eigenvalues and eigenfunctions of \( \hat{H}_u \) are known, \( \lambda_n(k) \) and \( \psi_n(k) \) can be obtained considering \( kU(x) \) as the perturbation Hamiltonian. According to the ‘quantum’ perturbation theory, the perturbative expansions for the eigenstates and eigenvalues are (the superscript \( (0) \) denotes the unperturbed quantities) [50]
\[
\lambda_n(k) = \lambda_n^{(0)} + k \langle \psi_n^{(0)} | U | \psi_n^{(0)} \rangle + k^2 \sum_{m \neq n} \frac{\langle \psi_m^{(0)} U | \psi_n^{(0)} \rangle}{\lambda_n^{(0)} - \lambda_m^{(0)}} + \cdots
\]
\[
|\psi_n(k)\rangle = |\psi_n^{(0)}\rangle + k \sum_{m \neq n} \frac{\langle \psi_m^{(0)} U | \psi_n^{(0)} \rangle}{\lambda_n^{(0)} - \lambda_m^{(0)}} |\psi_m^{(0)}\rangle + k^2 \left[ \sum_{m \neq n} \frac{\langle \psi_m^{(0)} U | \psi_n^{(0)} \rangle}{\lambda_n^{(0)} - \lambda_m^{(0)}} \right] \sum_{p \neq n} \frac{\langle \psi_p^{(0)} U | \psi_n^{(0)} \rangle}{\lambda_n^{(0)} - \lambda_p^{(0)}} + \cdots
\]

We can substitute the above perturbative expansions of \( \lambda_n(k) \) and \( |\psi_n(k)\rangle \) in equation (11) and then in equations (12) and (13) to extract the required moments and cumulants of \( T_t \). In the above expansions, we have assumed the eigenvalues are non-degenerate. For degenerate eigenvalues one needs to use appropriate perturbation theory [50].

The above method can be extended straightforwardly to compute correlations between multiple statistics (for example, the local times at two different points). One can replace the term \( kU(x) \) in \( \hat{h}_k \) by \( kU_1 + kU_2 \) and then follow the same procedure. Extracting the coefficients of appropriate powers of \( k \) and \( k' \) from the MGF would provide the required moments up to a multiplicative constant. In fact, one can consider a vector of statistics defined by \( T[x(\tau)] \equiv \{T_1[x(\tau)], T_2[x(\tau)], \ldots, T_N[x(\tau)]\} \), along with a real vector \( \mathbf{k} \equiv \{k_1, k_2, \ldots, k_N\} \) of the same dimension \( N \). An extended MGF can be defined as \( Q(\mathbf{k}, t|x_0) \equiv \langle e^{-kT[x(\tau)]} \rangle \), from which one can get

\[
\langle T_{1}^{l_1} \ldots T_{r}^{l_r} \rangle = \frac{(-1)^{l_1 + l_2 + \cdots + l_r}}{S(t|x_0)} \frac{\partial^{l_1}}{\partial k_1^{l_1}} \cdots \frac{\partial^{l_r}}{\partial k_r^{l_r}} Q(k_1, k_2, \ldots, k_R, t|x_0) \bigg|_{\mathbf{k}=0}
\]
\[
\langle T_{1}^{l_1} \ldots T_{r}^{l_r} \rangle_c = (-1)^{l_1 + l_2 + \cdots + l_r} \frac{\partial^{l_1}}{\partial k_1^{l_1}} \cdots \frac{\partial^{l_r}}{\partial k_r^{l_r}} \log \left[ \frac{Q(k_1, k_2, \ldots, k_R, t|x_0)}{Q(0, t|x_0)} \right] \bigg|_{\mathbf{k}=0}
\]

where \( l_i = 0, 1, 2, \ldots \) with \( \sum_{i=1}^{r} l_i \neq 0 \) and \( 1 \leq r \leq N \). In section 4, we use the above-mentioned perturbative approach to obtain explicit expressions for the mean, variance and covariance of the local time spent by the OU particle at different locations.

https://doi.org/10.1088/1742-5468/abe93d
3. Distribution of $T_t[x(\tau)]$: typical and atypical fluctuations

While the perturbative approach allows us to evaluate a few lower order moments as well as cumulants of an arbitrary functional $T_t[x(t)]$ (of the form in equation (2)) in an external potential, the expressions obtained involve nested summations which may be difficult to evaluate. Note that to obtain an exact explicit expression for the $l$th moment, we need $l$th order perturbation theory. This approach is feasible for lower-order moments (such as mean, variance), but quickly becomes tedious and cumbersome for higher-order moments. However, for large $t$, it is easy to see that the leading term of cumulants of all order grows linearly with time, as follows: We first write the MGF $Q(k, t|x_0)$ in equation (11) as

$$Q(k, t|x_0) = \sum_n e^{-\lambda_n(k)t} g_n(k) f_n(k, x_0)$$

$$= e^{-\lambda_0(k)t} \left[ g_0(k) f_0(k, x_0) + e^{-\lambda_1(k)-\lambda_0(k)t} g_1(k) f_1(k, x_0) + \cdots \right]$$  \hspace{1cm} (18)

where

$$f_n(k, x_0) = \psi_{n,k}(x_0) \exp \left( \frac{V(x_0)}{2D} \right)$$

$$g_n(k) = \int dx \exp \left( -\frac{V(x)}{2D} \right) \psi_{n,k}(x).$$  \hspace{1cm} (19)

In the above, we have assumed that the eigenvalues inside a confining potential satisfy $0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \ldots$ even in the presence of perturbation. Using this expression in equation (13) and taking the large $t$ limit, we get

$$\langle T_t \rangle_c = (1)^l \left[ d^l \log \left( \frac{Q(k, t|x_0)}{Q(0, t|x_0)} \right) \right]_{k \to 0}$$

$$\simeq t \left[ (1)^{l+1} \left( \frac{\lambda_0(k)}{d^l} \right) \right]_{k \to 0}$$

$$+ \text{ constant term + terms exponentially small in } t$$  \hspace{1cm} (20)

for $l = 1, 2, 3, \ldots$. Hence, we observe that at large $t$, all cumulants in the leading order increase linearly. Consequently, the distribution of the scaled random variable $q = (T_t - \langle T_t \rangle)/\sqrt{\langle T_t^2 \rangle_c}$ approaches a Gaussian distribution, i.e. for large $t$ we have

$$P_{x_0}(T_t = T, t) \simeq \frac{1}{\sqrt{\langle T_t^2 \rangle_c}} \mathcal{G} \left( \frac{T - \langle T_t \rangle}{\sqrt{\langle T_t^2 \rangle_c}} \right), \quad \text{with } \mathcal{G}(q) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{q^2}{2} \right),$$  \hspace{1cm} (21)

as obtained in [23, 27]. The distribution in the above equation describes typical fluctuations of $T_t$ of order $O(\sqrt{t})$ around $\langle T_t \rangle$, but it does not describe the probabilities of rare fluctuations such as very large or small values of $T_t$ compared to $\langle T_t \rangle$. For this, we need to study the large deviation properties of $T_t$, which we present later.
The result in equation (21) is valid under the assumption that the eigenspectrum of the quantum Hamiltonian $\hat{H}_u$ has a gap above the non-degenerate ground state eigenvalue $\lambda_0(0)$, and that this remains true even after adding the perturbation $kU(x)$. Under these assumptions, the result in equation (21) can be easily generalised to the joint distribution of multiple observables $T_t[x(\tau)] = \{T_1[x(\tau)], T_2[x(\tau)], \ldots, T_N[x(\tau)]\}$. In the large $t$ limit, one can show that the joint distribution of the normalised observables $\zeta = \{\zeta_i\} = \{\frac{T_i - \langle T_i \rangle}{\sqrt{\langle T_i^2 \rangle_c}}\}$, obtained after proper shifting and scaling, is a multivariate Gaussian.

We now proceed to describe a non-perturbative method of finding the large-deviation functions of the functional $T_t[x(\tau)]$, defined in equation (2). As mentioned earlier, in the large $t$ limit, the 0th eigenvalue $\lambda_0(k)$ contributes at the leading order. Hence, we can write

$$ Q(k, t|x_0) \approx e^{-\lambda_0(k)t}g_0(k)f_0(k, x_0) $$

for large $t$, where $g_0(k)$ and $f_0(k, x_0)$ are given in equation (19). Observe that $Q(k, t|x_0)$ in equation (6) is basically the Laplace transform of $P_{x_0}(T, t)$ with respect to $T$. Hence, performing the inverse Laplace transform of $Q(k, t|x_0)$, one finds that

$$ P_{x_0}(T, t) \approx \frac{1}{S(t|x_0)} \int dk \exp \left( -t \left( -k \frac{T}{t} + \lambda_0(k) \right) \right) g_0(k)f_0(k, x_0) $$

for large $t$. In the above, $S(t|x_0)$ is the survival probability, given by

$$ S(t|x_0) = Q(0, t|x_0) = \sum_n e^{-t\lambda_n(0)}g_n^{(0)}f_n^{(0)}(x_0). $$

The integral in equation (23) can be performed using the steepest-descent method [51]. In order to do that, one needs to know $\lambda_0(k)$ for all $k$. The expression in equation (14) provides $\lambda_0(k)$ for small $k > 0$. While solving the eigenvalue problem in equation (10), one first writes the general solutions and then tries to fix the integration constants by satisfying the boundary conditions. Often, demanding non-zero and non-trivial values for the integration constants provides an equation involving $\lambda_0$ and $k$, solving which, one obtains the desired eigenvalues for each value of $k$. For example, in the case of the local time density $\ell_t(y)$ (in which we are mainly interested in this paper), this equation, for different boundary conditions, has the form

$$ G(\lambda_0, y) = k. $$

We will observe this fact in the next section, where we compute this relation explicitly for a harmonic confining potential with different boundary conditions.

In the steepest-descent method, the dominant contribution comes from the saddle point $k^*$ where the argument of the exponential in equation (23) is maximum, i.e.

$$ \left[ \frac{d\lambda_0(k)}{dk} \right]_{k=k^*} = w, \quad \text{where, } w = \frac{T}{t}. $$
Now, from equation (25), one finds \( \frac{d\lambda_0}{dk} = [\partial_{\lambda_0} G(\lambda_0, y)]^{-1} \). Hence, \( \lambda_0^*(w, y) = \lambda_0(k^*(w, y)) \) is obtained by solving

\[
[\partial_{\lambda_0} G(\lambda_0, y)]_{\lambda_0=\lambda_0^*} = w^{-1},
\]

which is then used to perform the saddle point integration in equation (23) to get

\[
P_{x_0}(T, t) \simeq \frac{P_{x_0}(T/t, t)}{S(t|x_0)},
\]

where

\[
P_{x_0}(w, t) = \exp \left\{ -t \left[ -wG(\lambda_0^*, y) + \lambda_0^* \right] g_0(G(\lambda_0^*, y), f_0(G(\lambda_0^*, y), x_0) \right\} \sqrt{2\pi t} \left| \frac{\partial^2 G(\lambda, y)}{(\partial_{\lambda} G(\lambda, y))^2} \right|_{\lambda = \lambda_0^*}.
\]

This is a general formula for the distribution of the local time density \( \ell_t \) (defined in equation (1)) spent at \( y \) by an overdamped particle moving inside a confining potential \( V(x) \).

The above discussion on saddle point computation, presented in the context of local time, can be straightforwardly generalised to other functionals of the form \( T_t[x(\tau)] \) (in equation (2)). For such functionals, the equivalent of equation (25) would be of different form, but would still provide a relation between \( \lambda_0 \) and \( k \). For small \( k \), this relation would provide an expansion of the form in equation (14). However, for arbitrary \( k \), one would be able to find \( \lambda_0 \) numerically or through asymptotic expansions. Such evaluations can then be used to find the saddle point for a given value of \( T_t[x(\tau)] \) and \( t \), which would finally provide a large deviation form similar to equation (29).

In the next section, we explicitly calculate \( \ell_t(y) \) for an OU particle with different boundary conditions.

### 4. Local time statistics in a harmonic potential

#### 4.1. Mean, variance and covariance

In this section, we apply the methods discussed and presented in sections 2 and 3 to study statistical properties of the local time functional (defined in equation (1)) for an OU particle, i.e. for \( V(x) = \kappa x^2/2, \kappa > 0 \). For this choice, the Hamiltonian \( \hat{H}_k \) in equation (9) becomes

\[
\hat{H}_k = \hat{H}_u + \hat{h}_k, \quad \text{where} \quad \hat{H}_u = -D \frac{\partial^2}{\partial x^2} + \frac{\kappa^2}{4D} x^2 - \frac{\kappa}{2} \text{ and } \hat{h}_k = k \delta(x - y).
\]

The delta function in the above appears from the definition of local time \( \ell_t(y) \) in equation (1). To compute the moments and cumulants, we will compute the characteristic function \( Q(k, t|x_0) \) defined in equation (8), explicitly as a series in powers of \( k \) using perturbation theory. Before going into that, we point out that in this case, because of the particular delta function form of the perturbation Hamiltonian \( \hat{h}_k \) the Laplace transform of the distribution \( P_{x_0}(\ell, t) \) of \( \ell \) with respect to \( t \) can be expressed in closed form in terms of the Green’s function of the unperturbed Hamiltonian [52].

https://doi.org/10.1088/1742-5468/abe93d
However, in this paper we apply perturbation theory in time domain explicitly, as it seems to be easily generalisable to study of local times at multiple locations. For example, if we are interested in local times at positions $y$ and $z$, then we should consider the perturbation Hamiltonian $\hat{H}_k = k\delta(x - y) + k'\delta(x - z)$. Similarly, to study local times at multiple locations one needs to modify the perturbation Hamiltonian accordingly.

Note that $\hat{H}_j$ is the Hamiltonian of a quantum harmonic oscillator, for which the eigenvalues and eigenfunctions can be obtained for specified boundary conditions. Once $\ell$, $\delta$, and $\delta'$s from which one would be able to find correlations and moments/cumulants from equation (18); for example, to compute the correlation between local times at two locations, $y$ and $z$, we have the following perturbation expansion of the MGF $Q(\mathbf{k}, t|x_0)$ (equation (18)):

$$Q(\mathbf{k}, t|x_0) = \sum_n e^{-t\left(\lambda_n^{(0)} + k\lambda_n^{(1)}(y) + k'\lambda_n^{(1)}(z) + k^2\lambda_n^{(2)}(y) + k'^2\lambda_n^{(2)}(z) + kk'\lambda_n^{(1,1)}(y,z) + \cdots\right)} \times \left[ g_n^{(0)}(y) + k'g_n^{(1)}(z) + k^2g_n^{(2)}(y) + k'^2g_n^{(2)}(z) + \cdots \right] \times \left[ f_n^{(0)}(x_0) + k'f_n^{(1)}(y|x_0) + k^2f_n^{(2)}(z|x_0) + k'^2f_n^{(2)}(y|x_0) \right] + kk'f_n^{(1,1)}(y,z|x_0) + \cdots,$$

where expressions for terms like $\lambda_n^{(0)}$, $\lambda_n^{(1)}$, $\ldots$; $g_n^{(0)}$, $g_n^{(1)}$, $\ldots$; and $f_n^{(0)}(x_0)$, $f_n^{(1)}(y|x_0)$, $\ldots$ can be obtained from equations (14) and (15) by replacing $kU$ by $\hat{H}_k = k\delta(x - y) + k'\delta(x - z)$. The explicit expressions are provided in appendix A. Substituting the expressions from equations (A.1)–(A.11) in equation (31), we get the expansion of $Q(\mathbf{k}, t|x_0)$ in powers of $k$ and $k'$ up to second order. Inserting this expansion in equation (16), one gets the first two moments and two-point correlations as

$$\langle \ell_1(y) \rangle = t \sum_n e^{-t\lambda_n^{(0)}(y)} g_n^{(0)} f_n^{(0)}(x_0) S(t|x_0)$$

$$- \sum_n e^{-t\lambda_n^{(0)}} \left[ g_n^{(1)}(y) f_n^{(0)}(x_0) + g_n^{(0)}(y) f_n^{(1)}(y|x_0) \right] S(t|x_0),$$

$$\langle \ell_1^2(y) \rangle = t^2 \sum_n e^{-t\lambda_n^{(0)}} \lambda_n^{(1)}(y)^2 g_n^{(0)} f_n^{(0)}(x_0) S(t|x_0)$$

$$- 2t \sum_n e^{-t\lambda_n^{(0)}} \left[ \lambda_n^{(1)}(y) g_n^{(0)} f_n^{(0)}(x_0) + \lambda_n^{(1)}(y) \left( g_n^{(0)} f_n^{(1)}(y|x_0) + g_n^{(1)}(y) f_n^{(0)}(x_0) \right) \right] S(t|x_0)$$

$$+ 2 \sum_n e^{-t\lambda_n^{(0)}} \left[ g_n^{(2)}(y) f_n^{(0)}(x_0) + g_n^{(0)} f_n^{(2)}(y|x_0) + g_n^{(1)}(y) f_n^{(1)}(y|x_0) \right] S(t|x_0).$$

https://doi.org/10.1088/1742-5468/abe93d J. Stat. Mech. (2021) 033218
and

\[ \langle \ell_t(y)\ell_t(z) \rangle = t^2 \sum_n e^{-\frac{\kappa}{2} n^2} f_{t,0}^{(1)}(x_0) g_{n,0}^{(2)}(y) g_{n,0}^{(2)}(z) - t \left[ \sum_n e^{-\frac{\kappa}{2} n^2} \frac{\lambda_{n,1}^{(2)}(y,z) g_{n,0}^{(2)} f_{t,0}^{(1)}(x_0)}{S(t|x_0)} \right] + \sum_n e^{-\frac{\kappa}{2} n^2} \left[ f_{t,0}^{(1)}(x_0) \left( g_{n,0}^{(2)}(y) \lambda_{n,1}^{(2)}(z) + g_{n,0}^{(2)}(z) \lambda_{n,1}^{(2)}(y) \right) + g_{n,0}^{(2)} f_{t,0}^{(1)}(y|x_0) \lambda_{n,1}^{(2)}(z) + f_{t,0}^{(1)}(y|x_0) g_{n,0}^{(2)}(z) \right] \]  

where \( S(t|x_0) \) is the survival probability, defined in equation (24). From these expressions, one can get the cumulants as \( \langle \ell_t(y)^2 \rangle_c = \langle \ell_t(y)^2 \rangle - \langle \ell_t(y) \rangle^2 \) and \( \langle \ell_t(y)\ell_t(z) \rangle_c = \langle \ell_t(y)\ell_t(z) \rangle - \langle \ell_t(y) \rangle \langle \ell_t(z) \rangle \). One could also compute these quantities directly by substituting the expression of \( Q(k,t|x_0) \) from equation (31) in equation (17). The above general expressions for the mean, variance, and covariance of the local time of an OU particle are similar to those obtained in [27]. In this paper, we provide more explicit expressions for these cumulants in the presence and in the absence of an absorbing boundary.

4.1.1. On infinite line. In this case, the eigenvalues and eigenfunctions of the bare Hamiltonian \( \hat{H}_u \) with the boundary conditions \( \psi_n(x) \to 0 \) as \( x \to \pm \infty \) are given by [53, 54]

\[ \lambda_{n,0}^{(0)} = n\kappa, \quad \text{and} \]

\[ \psi_{n,0}^{(0)}(x) = \langle x|\psi_n^{(0)} \rangle = \frac{1}{\sqrt{2^n n!}} \left( \frac{\kappa}{2\pi D} \right)^{\frac{1}{4}} e^{-\frac{x^2}{4\kappa}} H_n \left( \sqrt{\frac{\kappa}{2D}} x \right), \]

for \( n = 0, 1, 2, \ldots \) where \( H_n(z) \) is the Hermite polynomial of \( n \)th degree. Note that in this case, the survival probability \( S(t|x_0) = \sum_n e^{-\frac{\kappa}{2} n^2} g_{n,0}^{(2)} f_{t,0}^{(1)}(x_0) = 1 \) for any \( x_0 \), because \( g_{n,0}^{(1)} = 0, \lambda_{n,0}^{(0)} = 0, \text{ and } f_{t,0}^{(1)}(x_0) = 1 \).

Substituting the above eigenvalues and eigenfunctions in equations (A.1)–(A.11), we first find the functions \( \lambda_{n,0}^{(0)}, \psi_{n,0}^{(0)} \) and \( f_{n,0}^{(1)} \). These functions are then substituted in equations (32), (33), and (34) to find the first and second moments and the two point correlation \( \langle \ell_t(y)\ell_t(z) \rangle \). In particular, we find the following exact, explicit expression for the mean:

\[ \langle \ell_t(y) \rangle = \sqrt{\frac{\kappa}{2\pi D}} e^{-\frac{y^2}{4\kappa}} \left( t + \sum_{m=1}^{\infty} \frac{(1 - e^{-m\kappa t})}{m\kappa} H_m \left( \sqrt{\frac{\kappa}{2D}} y \right) H_m \left( \sqrt{\frac{\kappa}{2D}} x_0 \right) \right), \]

where \( x_0 \) is the point at which the OU particle starts its motion, and the local time is measured at the point \( y \). This expression is numerically verified in the leftmost panel of figure 1. Note from equation (36) that the average of the fraction of local time, \( \langle \ell/t \rangle \),
Figure 1. Comparison of numerical and analytical results for the mean local time $\langle \ell_t(y) \rangle$ (left) in equation (36), the variance $\langle \ell_t^2(y) \rangle_c$ (middle) in equation (37) and the covariance $\langle \ell_t(y) \ell_t(z) \rangle_c$ (right) in equation (38) for $y = 0$, $z = 1$ with $x_0 = 0$. We have chosen $D = 1/2$ and $\kappa = 1$. For the simulations, we used a forward Euler scheme with time-step $dt = 10^{-5}$, and took $10^5$ realizations.

converges in the large $t$ limit to the stationary position distribution of the OU particle. This is expected, since the particle dynamics is ergodic [55, 56].

The second cumulant and the two point correlations can also be evaluated from equations (33) and (34). Here, we do not present their expressions, as they are lengthy and involve multiple infinite series. Instead, we focus on the large $t$ limit, where one can make further simplifications. Moreover, we assume $D = 1/2$, $y = 0$ and $x_0 = 0$ to simplify the expression for the variance further. In the large $t$ limit, we find

$$\langle \ell_t(0)^2 \rangle_c \sim \frac{1}{2D} \left[ \frac{2}{\pi} \log 2 + \frac{A}{\kappa} \right],$$  \hspace{1cm} \text{where } A \approx 0.044 \hspace{1cm} (37)

$$\langle \ell_t(y) \ell_t(z) \rangle_c \sim \left[ \frac{1}{\pi D} e^{-\frac{\kappa}{4D} \frac{y^2}{D} - \frac{z^2}{D}} \sum_{m=1}^{\infty} \frac{H_m \left( \sqrt{\frac{2D}{\kappa}} y \right) H_m \left( \sqrt{\frac{2D}{\kappa}} z \right)}{m! m^2} \right] t + C_{\langle \ell\ell \rangle}(y, z|x_0),$$  \hspace{1cm} (38)

where $C_{\langle \ell\ell \rangle}(y, z|x_0)$ is given in equation (B.1). In the middle panel of figure 1, we compare equation (37) and in the right panel we compare equation (38) with simulations; we observe agreement in both cases.

4.1.2. On semi-infinite line: local time statistics conditioned on survival. In this case, we consider a particle starting from some position $x_0 > 0$, moving on the positive semi-axis, with an absorbing boundary at the origin ($x = 0$). The eigenfunctions of the bare Hamiltonian $\hat{H}_u$ need to satisfy the boundary conditions $\psi_n(x) \to 0$ as $x \to \infty$ and $\psi_n(x = 0) = 0$. The eigenvalues and eigenfunctions of the bare Hamiltonian $\hat{H}_u$ are then given by [53, 54]

$$\lambda_n^{(0)} = n \kappa$$

$$\psi_n^{(0)}(x) = \langle x | \psi_n^{(0)} \rangle = \frac{\sqrt{2}}{\sqrt{2^n n!} \left( \frac{\kappa}{2\pi D} \right)^{1/4}} e^{-x^2/2} \hat{H}_u \left( \sqrt{\frac{\kappa}{2D}} x \right)$$  \hspace{1cm} \text{for } n = 1, 3, 5, \ldots \hspace{1cm} (39)

https://doi.org/10.1088/1742-5468/abe93d
Note that in this case, the survival probability \( S(t|x_0) = Q(0, t|x_0) \neq 1 \) for \( x_0 > 0 \). In fact, we have

\[
S(t|x_0) = \sum_{n=1,3,5,\ldots} e^{-t\lambda_n^{(0)}(x_0)} g_n^{(0)}(x_0) = \sum_{n=1,3,5,\ldots} \sqrt{\frac{2}{2n!}} J_{0,n} H_n \left( \sqrt{\frac{\kappa}{2D}} x_0 \right) e^{-nx_0},
\]

where,

\[
J_{0,n} = \sqrt{\frac{2}{2n!}} \int_0^\infty du e^{-u^2} H_n(u).
\]

As in section 4.1.1, we substitute the above eigenvalues and eigenfunctions in equations (A.1)–(A.11). We find the functions \( \lambda_n^{(c)}, g_n^{(c)} \) and \( f_n^{(c)} \), and then substitute them in equations (32), (33), and (34) to find the first and second moments and the two point correlation \( \langle \ell_t(y) \ell_t(z) \rangle \). In particular, for large \( t \) we find the following explicit asymptotic expressions:

\[
\langle \ell_t(y) \rangle \simeq \frac{\kappa}{2\pi D} \frac{2\kappa y^2}{D} e^{-\frac{ny^2}{2D}} t + \frac{1}{\sqrt{\pi D}} y e^{-\frac{ny^2}{2D}} \sum_{m \neq 1 \atop m \text{ odd}} (-1)^{\frac{m-1}{2}} m! H_m \left( \sqrt{\frac{\kappa}{2D}} y \right) \frac{1}{2^m m! m (1-m)}
\]

\[
- \frac{2}{\sqrt{2\pi D}} \frac{y e^{-\frac{ny^2}{2D}}}{x_0} \sum_{m \neq 1 \atop m \text{ odd}} H_m \left( \sqrt{\frac{\kappa}{2D}} y \right) H_m \left( \sqrt{\frac{\kappa}{2D}} x_0 \right) \frac{1}{2^m m! (1-m)}
\]

\[
+ \text{terms decaying exponentially in } t
\]

\[
\langle \ell_t(y) \ell_t(z) \rangle_c \simeq -\frac{4\kappa y z}{\pi D^2} e^{-\frac{ny^2}{2D}} \frac{ny^2}{2D} \sum_{m \neq 1 \atop m \text{ odd}} H_m \left( \sqrt{\frac{\kappa}{2D}} y \right) H_m \left( \sqrt{\frac{\kappa}{2D}} z \right) \frac{1}{2^m m! (1-m)} t
\]

\[
+ C_{(\ell\ell)}^{(a)}(y, z|x_0) + \text{terms decaying exponentially in } t
\]

where \( C_{(\ell\ell)}^{(a)}(y, z|x_0) \) is given in equation (B.2). The variance \( \langle \ell_t^2(y) \rangle_c \) can be obtained from equation (43) by putting \( z = y \). These expressions for the mean, variance, and covariance are verified in figure 2 where we observe nice agreement.

### 4.2. Typical fluctuations and large deviations

To compute the probability of large deviations of local time \( \ell_t(y) \) at a location \( y \), we would apply the formalism described in section 3. This formalism has been used to compute LDFs for other choices of \( U(x) \) for Gaussian stationary Markov processes [36]. To proceed, we need to solve the following eigenvalue equation:

\[
\hat{H}_k \phi_{\lambda(k)}(x) = \lambda(k) \phi_{\lambda(k)}(x),
\]

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Figure 2. The analytical expressions for the mean, the variance, and the two point correlation given in equations (42) and (43) are verified by comparing them with numerical simulations. In the left panel, we plot the mean \( \langle \ell_t(y) \rangle \). In the middle panel, we plot the variance \( \langle \ell_t(y) \rangle_c \), and in the right panel, we plot the covariance \( \langle \ell_t(y) \ell_t(z) \rangle_c \) for \( y = 0 \) and \( z = 0.6 \). The other parameters of this plot are: \( x_0 = 0.9 \), \( \kappa = 1 \), and \( D = 1/2 \). For the simulations, we used a forward Euler scheme with time-step \( dt = 10^{-5} \). Averages were taken over \( \sim 10^5 \) surviving realizations.

with specified boundary conditions, where the Hamiltonian is given by equation (30). Performing the transformations

\[
\sqrt{\frac{\kappa}{D}} x = u, \quad \bar{\lambda}(k) = \frac{\lambda(k)}{\kappa} + \frac{1}{2}, \quad \Phi_\lambda(u) = \frac{\kappa}{D} \phi_\lambda(x), \quad \text{and } v = \sqrt{\frac{\kappa}{D}} y, \tag{45}
\]

the above equation becomes the well-known parabolic cylinder equation \([57, 58]\):

\[
\partial_u^2 \Phi_\lambda(u) - \left( \frac{u^2}{4} + a \right) \Phi_\lambda(u) = \frac{k}{\sqrt{\kappa D}} \delta(u - v) \Phi_\lambda(u) \quad \text{with } a = -\bar{\lambda}. \tag{46}
\]

The two linearly independent solutions of the homogeneous part of the equation are given by

\[
\mathcal{U}(a, u) = \mathcal{D}_{-a-1/2}(u), \quad \nu(a, u) = \frac{\Gamma[a + 1/2]}{\pi} \left( \sin(\pi a) \mathcal{D}_{-a-1/2}(u) + \mathcal{D}_{-a-1/2}(-u) \right). \tag{47}
\]

where \( \mathcal{D}_{-a-1/2}(u) \) is the parabolic cylinder function. In this case, the general solutions of equation (46) can also be written as

\[
\Phi_\lambda(u) = \begin{cases} 
A_1 \mathcal{D}_{\lambda-1/2}(u) + B_1 \mathcal{D}_{\lambda-1/2}(-u), & \text{for } u \geq v \\
A_2 \mathcal{D}_{\lambda-1/2}(u) + B_2 \mathcal{D}_{\lambda-1/2}(-u), & \text{for } u < v.
\end{cases} \tag{48}
\]

The constants \( A_{1,2} \) and \( B_{1,2} \) are determined by imposing continuity at \( u = v \),

\[
\Phi_\lambda(u)|_{u=v-} = \Phi_\lambda(u)|_{u=v+}, \tag{49}
\]

discontinuity of the first derivative at \( u = v \),

\[
[\partial_u \Phi_\lambda(u)]_{u=v+} - [\partial_u \Phi_\lambda(u)]_{u=v-} = \frac{k}{\sqrt{\kappa D}} \Phi_\lambda(v), \tag{50}
\]

https://doi.org/10.1088/1742-5468/abe93d
and the boundary conditions. In this section we consider two types of boundary conditions:

(A) The particle is allowed to move in the whole space $-\infty < x < \infty$ under the influence of the harmonic potential. The boundary conditions in this case are

$$\phi_\lambda(x \to \pm \infty) \to 0, \quad \Phi_\lambda(u \to \pm \infty) \to 0. \quad (51)$$

(B) The particle is allowed to move in the semi-infinite space $0 \leq x < \infty$ under the influence of harmonic potential with a reflecting boundary at $x = 0$. The boundary conditions in this case are

$$\partial_u \Phi_\lambda(u = 0) = 0 \quad \text{and} \quad \Phi_\lambda(u \to \infty) \to 0. \quad (52)$$

Let us first look at case (A).

4.2.1. Case (A): on infinite line. For this case, the boundary conditions in equation (51) imply $B_1 = 0$ and $A_2 = 0$. Now, using the continuity at $u = v$, we get

$$\Phi_\lambda(u) = C \begin{cases} D_{\lambda/\kappa-1/2}(-v) D_{\lambda/\kappa-1/2}(u), & \text{for } u \geq v \\ D_{\lambda/\kappa-1/2}(v) D_{\lambda/\kappa-1/2}(-u), & \text{for } u < v \end{cases} \quad (53)$$

where $C$ is a constant which can be fixed by requiring the eigenfunctions to be normalized. Inserting this expression for the eigenfunction in the discontinuity condition equation (50), we get

$$- \left[ \frac{D_{\lambda/\kappa+1}(v)}{D_{\lambda/\kappa}(v)} + \frac{D_{\lambda/\kappa+1}(-v)}{D_{\lambda/\kappa}(-v)} \right] = \frac{\lambda}{\kappa} \left[ \frac{D_{\lambda/\kappa-1}(v)}{D_{\lambda/\kappa}(v)} + \frac{D_{\lambda/\kappa-1}(-v)}{D_{\lambda/\kappa}(-v)} \right] = \frac{k}{\sqrt{\kappa D}}, \quad (54)$$

solving which we can get $\lambda$ for a given value of $k$. Note that this equation is exactly in the form of equation (25). Solving for the smallest eigenvalue $\lambda_0(k)$, and then performing the saddle point integration in equation (23), one finds the distribution equation (29) for large $t$. Note that in this case the survival probability is $S(t|x_0) = 1$ as the particle always survives. In figure 3(a), we plot the distribution of the local time, for times $t = 10$ and 20, at position $y = 0$ (i.e. $v = 0$) for a Brownian particle starting from $x_0 = 0$.

From the structure of the saddle point integration in equation (23), it is clear that the distribution $P_{x_0}(\ell, t)$ has a large deviation form

$$P_{x_0}(\ell, t) \asymp \exp \left( -t \ \mathbb{H} \left( \frac{\ell}{t} \right) \right), \quad (55)$$

where the LDF $\mathbb{H}(w)$ is obtained through the following Legendre transformation [44]:

$$\mathbb{H}(w) = \max_k (-kw + \lambda_0(k)) = -k^*(w) w + \lambda_0(k^*(w)), \quad (56)$$

where

$$\left( \frac{d \lambda_0(k)}{dk} \right)_{k=k^*} = w = \frac{\ell}{t}. \quad (57)$$

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Figure 3. (a) Distribution of the local time density spent, till times $t = 10$ and $20$ at position $y = 0$, by a Brownian particle moving in a harmonic potential over the full space. (b) The asymptotic behaviour of the LDF $H(w)$ in different regimes (given in equation (58)) is compared with the actual distribution given in equation (29) for $t = 200$. The behaviour at small $w$ is zoomed in the inset. For both the plots, $x_0 = 0$, $\kappa = 1$ and $D = 1$. Numerical simulations were performed using a forward Euler scheme with time-step $dt = 10^{-3}$, and data were averaged over $10^7$ realisations.

In mathematics literature, for diffusive processes, a general expression for the LDF $H(w)$ of $w = \ell/t$ (known as the empirical distribution) has been obtained by Donsker and Varadhan in terms of the generator of the process [32, 37, 44, 48]. From this result, one can in principle evaluate the LDF of $\ell$ and also compute LDFs of other functionals like $T_t[x(\tau)]$ (in equation (2)) through the contraction principle [44]. However, here, we apply a different method, based on analyzing the behaviour of $\lambda_0(k)$ in the $k \to 0^\pm$ and $k \to \pm \infty$ limits, to compute the LDF in different asymptotic limits explicitly.

Following this procedure, we find the following explicit asymptotic behaviours of the LDF

$$H(w) \approx \begin{cases} 
\frac{\kappa - 2\kappa}{2\sqrt{D\kappa}} & \text{for } w \to 0 \\
(\frac{w - \mu}{2\sigma^2})^2 - \frac{\Omega (w - \mu)^3}{\sigma^4} + O\left(\frac{|w - \mu|^4}{\sigma^4}\right) & \text{for } |w - \mu| \leq \sigma \\
Dw^2 - \frac{\kappa}{2} + \frac{\kappa^2}{8Dw^2} + O\left(\frac{1}{w^4}\right) & \text{for } w \to \infty
\end{cases}$$

(58)

where

$$\sigma = \sqrt{\frac{\ln 2}{\pi D}}, \quad \mu = \sqrt{\frac{\kappa}{2\pi D}}, \quad \Omega = \frac{D^2\kappa\sqrt{\pi}}{48\sqrt{2}\pi \ln 2} \left(36\left(\frac{\ln 2}{2}\right)^2 - \frac{\pi^2}{4}\right)$$

(59)

for $y = 0$. The asymptotic forms in equation (58) suggest that the typical fluctuations of $w = \ell/t$ are Gaussian with $\langle w \rangle = \langle \ell/t \rangle = \sqrt{\kappa/2\pi D}$ and variance $\langle (\Delta w)^2 \rangle = \sigma^2 = \ln 2/\pi D$. This means one can write $w = \langle w \rangle + \sigma \chi$ for $|w - \langle w \rangle| \lesssim \sigma$ where $\chi$ is a mean zero, unit variance Gaussian variable.
The asymptotic behaviours in equation (58) can be obtained as follows. First, we note that for $y = 0$, equation (54) simplifies to

$$\frac{D_{\lambda/\kappa+1}(v)}{D_{\lambda/\kappa}(v)} = \frac{k}{2\sqrt{\kappa D}}$$

(60)

where $\Gamma[x]$ is the gamma function. Now, for $k = 0$, it is easy to show that the solution for the smallest eigenvalue $\lambda_0$ of the above equation is zero. Hence, in the $k \to 0^+$ limit, we expect $\lambda_0(k)$ should also go to zero. This suggests that we expand the left-hand side of equation (60) in powers $\nu = \lambda_0/\kappa$ around $\nu = 0$. We get

$$\sqrt{2\pi\nu} + \sqrt{2\pi} \ln 2 + \nu^2 + \nu^3 \sqrt{2\pi} \left( \frac{(\ln 2)^2}{2} + \frac{\pi^2}{24} \right) + O(\nu^4) \approx \frac{k}{\sqrt{\kappa D}}, \quad \text{with} \quad \nu = \frac{\lambda_0}{\kappa}.$$  

Solving the above order by order in small $k$, we get

$$\lambda_0(k) \approx \sqrt{\frac{\kappa}{2\pi D}} k - \frac{\ln 2}{2\pi D} k^2 + k^3 \left( \frac{3\sqrt{2}(\ln 2)^2}{8\pi^2 D^2} - \frac{\sqrt{2\pi}}{96D^2} \right) + O(k^4) \quad \text{for} \quad k \to 0^+. \quad (61)$$

Using the above expression in equation (57), we get

$$\mathbb{H}(w) \approx \frac{(w - \mu)^2}{2\sigma^2} - \Omega(w - \mu)^3 + O(|w - \mu|^4) \quad \text{for} \quad |w - \mu| \leq \sigma, \quad (62)$$

where $\mu$, $\sigma$, and $\Omega$ are defined in equation (59). Note that $k \to 0^+$ provides the LDF for $\ell \to \sqrt{2\pi D}t$ from below. Now, let us look at the $k \to \infty$ limit along the real line. Note from equation (60) that as $k \to \infty$, the left-hand side also has to go to $\infty$ and that happens if $\lambda/\kappa \to 1$. Hence, we can look for a solution for $\lambda_0$ of the form $\lambda_0/\kappa = 1 - \epsilon$, such that $\epsilon \to 0$ as $k \to \infty$. Inserting this form in equation (60), expanding in powers of $\epsilon$, and solving order by order for $\epsilon$, we get $\epsilon = \sqrt{\frac{2\pi D}{\pi} \kappa - \frac{8\kappa D}{\pi k^2}} (1 - \ln 2) + O(k^{-3})$ which finally provides

$$\lambda_0(k) \approx \kappa (1 - \epsilon) = \kappa \left( 1 - 2\kappa \sqrt{\frac{2D}{\pi \kappa}} \frac{1}{k} + \frac{8\kappa D}{\pi k^2} (1 - \ln 2) + O(k^{-3}) \right) \quad \text{for} \quad k \to \infty. \quad (63)$$

Once again, using this form of $\lambda_0(k)$ in equation (57), we get

$$\mathbb{H}(w) \approx \kappa - 2\kappa \left( 2w \sqrt{\frac{2D}{\kappa \pi}} \right)^{1/2} + 2w (1 - \ln 2) \sqrt{\frac{2DK}{\pi \kappa}} + O(w^{3/2}) \quad \text{for} \quad w \to 0, \quad (64)$$

since $k \to \infty$ corresponds to the $w \to 0$ (i.e. $\ell \to 0$) limit of the saddle point result in equation (23).

From the above analysis, we observe that the range $0 < \kappa < \infty$ of $\lambda_0(k)$, provides information about the distribution $P_0(\ell, t)$ (or equivalently the LDF $\mathbb{H}(w)$) over the region $0 < \ell < \sqrt{2\pi D} t$. To obtain the LDF $\mathbb{H}(w)$ over the rest of the region
\[ \sqrt{\frac{\kappa}{2\pi D}} \ t < \ell < \infty, \] one needs to analytically continue the \( \lambda_0(k) \) to negative \( k \). If we write \( k = -|k| \) and \( \lambda = -|\lambda| \), then we get

\[
\sqrt{\frac{2D}{\kappa}} |\lambda| \frac{\Gamma \left( \frac{\kappa + |\lambda|}{2\kappa} \right)}{\Gamma \left( \frac{2\kappa + |\lambda|}{2\kappa} \right)} = |k|.
\]

(65)

Following the same procedure, we solve this equation in the \( |k| \to 0 \) and \( |k| \to \infty \) limits and find

\[
\lambda(k) \approx \begin{cases} 
\frac{\kappa}{2\pi D} k - \frac{\ln 2}{2\pi D} k^2 + k^3 \left( -\frac{\sqrt{2\pi}}{96 D^2} + \frac{3\sqrt{2}(\ln 2)^2}{8\pi^2 D^2} \right) + O(k^4) & \text{for } |k| \to 0 \\
\frac{\kappa^2}{4D} - \frac{\kappa}{2} + \frac{D\kappa^2}{2k^2} + O(k^{-4}) & \text{for } |k| \to \infty.
\end{cases}
\]

(66)

Now, substituting these asymptotic behaviours in equation (57), we get

\[
\mathbb{H}(w) \approx \begin{cases} 
\frac{(w - \mu)^2}{2\sigma^2} - \Omega(w - \mu)^3 + O(|w - \mu|^4) & \text{for } |w - \mu| \leq \sigma \\
D w^2 \left( \kappa^2 - \frac{\kappa}{2} + \frac{\kappa^2}{8Dw^2} + O(w^{-4}) \right) & \text{for } w \to \infty
\end{cases}
\]

(67)

where \( \mu, \sigma, \) and \( \Omega \) are defined in equation (59). The asymptotic behaviours of \( \mathbb{H}(w) \) in equation (58) are shown in figure 3(b). The above procedure can be developed further to find the next order terms systematically in each regime in equation (58). Similarly, the above analysis, although it is presented for \( y = 0 \), can be straightforwardly extended to other values of \( y \).

4.2.2. Case (B): semi-infinite space with a reflecting boundary at \( x = 0 \). In this section, we consider the case in which the particle is moving on the positive real axis with a reflecting boundary at the origin. We study the distribution of the local time spent by the particle at the origin. The eigenfunctions \( \phi_{\lambda} \), in this case, satisfy the boundary conditions in equation (52). Following the same procedure as in case (A), we get

\[
\Phi_{\lambda}(u) = C \begin{cases} 
D_{\lambda-1/2}(v) \left[ D_{\lambda-1/2}(u) + D_{\lambda-1/2}(-u) \right] & \text{for } 0 < u < v \\
D_{\lambda-1/2}(u) \left[ D_{\lambda-1/2}(v) + D_{\lambda-1/2}(-v) \right] & \text{for } u \geq v
\end{cases}
\]

(68)

with \( \lambda = \frac{\lambda}{\kappa} + \frac{1}{2} \).

where the constant \( C \) is determined by requiring the eigenfunctions to be normalized. The condition for finding the eigenvalues \( \lambda \), in this case, turns out to be

\[
\frac{D_{\lambda+1/2}(v) D_{\lambda-1/2}(-v) + D_{\lambda-1/2}(v) D_{\lambda-1/2}(-v)}{D_{\lambda-1/2}(v) \left[ D_{\lambda-1/2}(v) + D_{\lambda-1/2}(-v) \right]} = -\frac{k}{\sqrt{\kappa D}}
\]

(69)

where \( \lambda = \lambda/\kappa + 1/2 \) and \( v = \sqrt{\kappa/D} \) as given in equation (45). Once again, solving for the dominant eigenvalue \( \lambda_0(k) \), and then performing the saddle point integration in
equation (23) for large \( t \), one finds the distribution given in equation (29) which has the large deviation scaling form.

In this case also, it is possible to find the behaviour of the LDF in various asymptotic limits. We focus on the local time at the boundary, i.e. at \( v = 0 \). Computation of the local time at the boundary, known as the boundary local time, is important for various surface-mediated phenomena [9–11]. For \( v = 0 \), equation (69) simplifies to

\[
\frac{D_{\lambda/\kappa+1}(0)}{D_{\lambda/\kappa}(0)} = -\frac{k}{\sqrt{\kappa D}}. 
\]  

(70)

Note that the eigenvalue equation for this case is same as that obtained in equation (60) for the no-boundary case, but with \( k \) being replaced by \( 2k \). This implies that the distributions of the local time \( \ell(t) \) corresponding to the two boundary conditions (A) and (B) [see equations (51) and (52)] are related by \( P_{(B)}(0, \ell, t) = \frac{1}{2} P_{(A)}(0, \ell, t) \). The reason behind such a relation is simple: by symmetry, if \( x(\tau) \) is an OU process on the real line, then \( |x(\tau)| \) is an OU process on the positive real axis with reflections at the origin, where the harmonic potential has its minimum. Hence, the amount of time spent at a point \( y \) by the reflected process \( |x(\tau)| \) is same as the total amount of time spent by the original process \( x(\tau) \) at the points \( y \) and \(-y\). Since we are computing the local time at the origin \( y = 0 \) (where the reflecting boundary is placed), the local time spent at \( y = 0 \) in case (A) is half the local time spent at \( y = 0 \) in case (B). This can be understood by noting that for a given trajectory \( \{ x(\tau) : 0 \leq \tau \leq t \} \), the amount of time spent within a box \([-\epsilon, \epsilon]\) of infinitesimal size \( 2\epsilon \) in case (A) is same as the amount of time spent by the corresponding reflected trajectory \( \{|x(\tau)| : 0 \leq \tau \leq t\} \) in the box \([0, \epsilon]\) in case (B). Hence, dividing this time by the length of the box in the respective cases, one sees that the local time (density) in case (B) is twice that in case (A).

As a consequence, for the LDF \( \mathbb{H}_r(w) \) with a purely reflecting boundary at the origin, one gets \( \mathbb{H}_r(w) = \frac{w^2}{2} \). More explicitly, we get

\[
\mathbb{H}_r(w) \approx \begin{cases} 
\kappa - 2\kappa \left( w \sqrt{\frac{2D}{\kappa \pi}} \right)^{1/2} + w \left( 1 - \log 2 \right) \sqrt{\frac{2D\kappa}{\pi}} + O(w^{3/2}) & \text{for } w \to 0 \\
\frac{(w - \mu_r)^2}{2\sigma_r^2} - \Omega_r (w - \mu_r)^3 + O(|w - \mu_r|^4) & \text{for } |w - \mu_r| \leq \sigma_r \\
\frac{D}{4} w^2 - \frac{\kappa}{2} + \frac{\kappa^2}{2Dw^2} + O(w^{-4}) & \text{for } w \to \infty 
\end{cases} 
\]  

(71)

where,

\[
\sigma_r = 2\sigma = \sqrt{\frac{4 \ln 2}{\pi D}}, \quad \mu_r = 2 \mu = \sqrt{\frac{2\kappa}{\pi D}}, \quad \Omega_r = \frac{\Omega}{8} = \frac{D^2 \pi^2 \sqrt{\pi} \left( 36(\ln 2)^2 - \pi^2 \right)}{384 \sqrt{2D\kappa}(\ln 2)^3}.
\]  

(72)

These asymptotic behaviours of the LDF \( \mathbb{H}_r(w) \) in the presence of a purely reflecting boundary at \( x = 0 \), with \( y = 0 \), are verified in figure 4.
Figure 4. (a) Distribution of the local time density spent, till times \( t = 10 \) and 20 at position \( y = 0 \), by a Brownian particle moving in a harmonic potential on the positive real axis with a reflecting boundary at the origin. (b) The asymptotic behaviour of the LDF \( \Pi_r(w) \) of the distribution of the local time spent by the particle at \( y = 0 \) till time \( t = 200 \) in presence of a reflecting boundary at \( x = 0 \). The solid red line is computed from equation (29) and the points (solid discs) of different colours correspond to \( \Pi(w) \) in different regimes as given in equation (71). The parameters are \( x_0 = 0 \), \( \kappa = 1 \) and \( D = 1 \). Numerical simulations were performed using a forward Euler scheme with time-step \( dt = 5 \times 10^{-4} \), and data were averaged over \( 10^7 \) realisations.

It is important to mention that the relation \( \Pi_r(w) = \Pi \left( \frac{w}{\lambda} \right) \) holds only when the reflecting wall is placed at the origin, where the harmonic potential has its minimum; otherwise it breaks down. In such cases, one can follow the same procedure to solve the eigenvalue equation (44), except that now the derivative boundary condition in equation (52) has to be appropriately modified by equating the outward normal probability current across the boundary to zero in the corresponding Fokker–Planck equation. Explicitly, if the reflecting boundary were placed at a position \( x = r \), the relevant boundary conditions would be

\[
\partial_u \Phi_\lambda + \frac{u}{2} \Phi_\lambda \bigg|_{u = r \sqrt{\kappa/D}} = 0, \quad \text{and} \quad \Phi_\lambda(u \to \infty) \to 0. \tag{73}
\]

4.3. Local time statistics not conditioned on survival with a fully absorbing boundary at the origin

In the previous section, we studied the statistical properties of local time spent at location \( y \) by the particle, conditioned on survival till time \( t \) in presence of a fully absorbing boundary at the origin. As discussed in the introduction, often one needs to look at the statistics of \( \ell_t(y) \) (or more generally of \( T_t[x(\tau)] \)) not conditioned on survival. In such cases, one needs to take into account the contribution to \( \ell_t(y) \) from those trajectories which got absorbed before \( t \), in addition to the contribution from those which are still surviving at time \( t \). Let the MGF of \( \ell_t(y) \) in this case be denoted by \( Q(k, t \mid x_0) \), where \( x_0 > 0 \) is the initial position of the particle. Clearly, this quantity can be written as sum of two terms:

\[
Q(k, t \mid x_0) = Q(k, t \mid x_0) + Q_{fp}(k, t \mid x_0), \tag{74}
\]

https://doi.org/10.1088/1742-5468/abe93d
where $Q(k,t|x_0)$ represents the contribution from the trajectories surviving till time $t$ and is given by equation (7). The second term $Q_{fp}(k,t|x_0)$ represents the contribution from the trajectories which have already been absorbed at the origin before the observation time $t$. Of course, these trajectories contribute to $\ell_t(y)$ as long as they survive. As a result, the moments can be written as the sum of two parts:

$$\langle \ell_t(y)^n \rangle = \langle \ell_t(y)^n \rangle_s + \langle \ell_t(y)^n \rangle_{fp},$$

(75)

where the first term with subscript ‘$s$’ represents the contribution from the trajectories surviving till time $t$ (i.e. the contribution from $Q(k,t|x_0)$) and the second term with subscript ‘$fp$’ represents contribution from the trajectories absorbed before time $t$ (i.e. the contribution from $Q_{fp}(k,t|x_0)$).

In terms of the joint probability density $P_a(\ell,t|x_0)$ of the local time spent at $y$ till the first passage time $t_f$ (given that the particle started from $x_0$), the MGF $Q_{fp}(k,t|x_0)$ can be written as

$$Q_{fp}(k,t|x_0) = \int_0^\infty d\ell e^{-k\ell}P_{fp}(\ell,t|x_0) \quad \text{where} \quad P_{fp}(\ell,t|x_0) = \int_0^t dt\,P_a(\ell,t|t|x_0).$$

(76)

Note that the subscript ‘$fp$’ in $P_{fp}(\ell,t|x_0)$ denotes the distribution of local time $\ell_t(y)$ on the ensemble of trajectories which have crossed the origin before time $t$. In other words, $P_{fp}(\ell,t|x_0)d\ell$ represents the joint probability that the local time spent at location $y$ is within $\ell$ to $\ell+d\ell$ and $t_f \leq t$.

The Laplace transform of $P_a(\ell,t_f|x_0)$ with respect to both $\ell$ and $t_f$ is defined as

$$\bar{P}_a(k,q|x_0) = \int_0^\infty dt f e^{-qt} \int_0^\infty d\ell\,e^{-k\ell}P_a(\ell,t_f|x_0).$$

(77)

It is easy to see from equation (76) that the Laplace transform of $Q_{fp}(k,t|x_0)$ with respect to time, denoted by $\bar{Q}_{fp}(k,q|x_0)$ is related to $\bar{P}_a(k,q|x_0)$ as

$$\bar{Q}_{fp}(k,q|x_0) = \int_0^\infty dt\,e^{-qt}Q_{fp}(k,t|x_0) = \frac{\bar{P}_a(k,q|x_0)}{q}.$$  

(78)

Hence, at this stage, it is crucial to obtain $\bar{P}_a(k,q|x_0)$ explicitly. As shown in appendix C, this quantity satisfies the following differential equation:

$$D\frac{\partial^2}{\partial x_0^2}\bar{P}_a(k,q|x_0) - V'(x_0)\frac{\partial}{\partial x_0}\bar{P}_a(k,q|x_0) - (k\delta(x_0-y) + q)\bar{P}_a(k,q|x_0) = 0.$$

(79)

with boundary conditions

$$\lim_{x_0 \to 0} \bar{P}_a(k,q|x_0) = 1,$$

$$\lim_{x_0 \to \infty} \bar{P}_a(k,q|x_0) = 0$$

(80)

for $q > 0$. For $x_0 \to 0$, the first passage time $t_f \to 0$ and $\ell \to 0$, which (from equation (77)) implies $\lim_{x_0 \to 0} \bar{P}_a(k,q|x_0) = 1$. On the other hand, for $x_0 \to \infty$, the first passage time $t_f \to \infty$, which again (from equation (77)) implies $\lim_{x_0 \to \infty} \bar{P}_a(k,q|x_0) = 0$. 

https://doi.org/10.1088/1742-5468/abe93d

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4.3.1. Stationary distribution \( P_s(\ell|x_0) \). While the contribution to the cumulants from the first term \( Q(k,t|x_0) \) in equation (74) decays exponentially with time, the contribution from the second term saturates at large times. This implies the distribution of \( \ell \) becomes stationary at \( t \to \infty \). Denoting this stationary distribution by \( P_{st}(\ell|x_0) \), we have

\[
P_{st}(\ell|x_0) = P_{0p}(\ell, t \to \infty|x_0) = \int_0^\infty dt_1 P_a(\ell, t_1|x_0),
\]

where the subscript ‘st’ denotes stationary state. The Laplace transform of \( P_s(\ell|x_0) \) with respect to \( \ell \) is related to \( \tilde{P}_a(k,q|x_0) \) as

\[
\tilde{P}_{st}(k|x_0) = \int_0^\infty d\ell e^{-k\ell} P_s(\ell|x_0) = \tilde{P}_a(k,0|x_0).
\]

(82)

To compute \( \tilde{P}_s(k|x_0) \) we solve the following differential equation, obtained by putting \( q = 0 \) in equation (79):

\[
D \frac{\partial^2}{\partial x_0^2} \tilde{P}_{st}(k|x_0) - V'(x_0) \frac{\partial}{\partial x_0} \tilde{P}_{st}(k|x_0) - k \delta(x_0 - y) \tilde{P}_{st}(k|x_0) = 0.
\]

(83)

The boundary conditions are

\[
\lim_{x_0 \to 0} \tilde{P}_{st}(k|x_0) = 1
\]

\[
\lim_{x_0 \to \infty} \tilde{P}_{st}(k|x_0) < \infty \text{ (finite)}.
\]

(84)

The boundary condition at \( x_0 = 0 \) is same as given in equation (80) with \( q = 0 \), i.e. \( \lim_{x_0 \to 0} \tilde{P}_{st}(k|x_0) = 1 \) which can be understood as before. However the boundary condition \( x_0 \to \infty \) is different. In this limit, although (as we have noted earlier) \( t_1 \to \infty \), the function \( U(x) \), being \( \delta(x - y) \) does not make the local time necessarily diverge or approach zero. Hence, the appropriate boundary condition is that \( \tilde{P}_{st}(k|x_0) \) is finite for \( x_0 \to \infty \). Additionally, the presence of the delta function in equation (83) implies two extra conditions:

\[
\tilde{P}_{st}(k|x_0 \to y^+) = \tilde{P}_{st}(k|x_0 \to y^-) \quad \text{and}
\]

\[
\frac{\partial \tilde{P}_{st}(k|x_0)}{\partial x_0} \bigg|_{x_0 \to y^-} - \frac{\partial \tilde{P}_{st}(k|x_0)}{\partial x_0} \bigg|_{x_0 \to y^+} = k \tilde{P}_{st}(k|y).
\]

(85)

The differential equation (83) has been written and analysed in the general context of first passage Brownian functionals [3, 45]. In the following, we solve the differential equation (83) for two choices of \( V(x) \): (i) free particle \( (V(x) = 0) \) and (ii) OU particle \( (V(x) = \kappa x^2/2) \).

**Free particle** \( (V(x_0) = 0) \): in this case, equation (83) becomes

\[
D \frac{\partial^2 \tilde{P}_{st}(k|x_0)}{\partial x_0^2} = k \delta(x_0 - y) \tilde{P}_{st}(k|x_0).
\]

(86)
Solving the above with the boundary conditions in equations (84) and (85), we get

\[
\tilde{P}_{st}(k \mid x_0) = \begin{cases} 
1 - \frac{kx_0}{ky + D} & x_0 < y \\
\frac{D}{ky + D} & x_0 \geq y
\end{cases}
\]  

Taking the inverse Laplace transform (see equation (82)), we get

\[
P_{st}(\ell \mid x_0) = \delta(\ell) \left( 1 - \min\left(\frac{x_0, y}{y}\right) + \frac{D \min(x_0, y)}{y^2} \exp\left(-\frac{\ell D}{y}\right) \right).
\]  \hspace{1cm} \text{(88)}

The stationary values of the mean and variance of \(\ell\) at large time are given by

\[
\langle \ell(y) \rangle^{(st)} = \frac{\min(x_0, y)}{D}, \\
\langle \ell(y)^2 \rangle_c = \frac{\min(x_0, y)}{D^2} [2y - \min(x_0, y)].
\]  \hspace{1cm} \text{(89)}

**OU particle** (\(V(x_0) = \kappa x_0^2 / 2\)): in this case, equation (83) becomes

\[
D \frac{\partial^2 \tilde{P}_{st}(k \mid x_0)}{\partial x_0^2} - \kappa x_0 \frac{\partial \tilde{P}_{st}(k \mid x_0)}{\partial x_0} = k \delta(x_0 - y) \tilde{P}_{st}(k \mid x_0).
\]  \hspace{1cm} \text{(90)}

Solving the above with the boundary conditions in equations (84) and (85), we get

\[
\tilde{P}_{st}(k \mid x_0) = \begin{cases} 
1 - \frac{kK(y, x_0)}{1 + kK(y, y)} & x_0 < y \\
\frac{1}{1 + kK(y, y)} & x_0 \geq y
\end{cases}
\]

where

\[
K(y, z) = \sqrt{\frac{\pi}{2\kappa D}} \exp\left(-\frac{\kappa}{2D}\right).
\]  \hspace{1cm} \text{(91)}

Performing the inverse Laplace transform, we obtain

\[
P_{st}(\ell \mid x_0) = \left( 1 - \frac{K(y, \min(x_0, y))}{K(y, y)} \right) \delta(\ell) + \frac{K(y, \min(x_0, y))}{K^2(y, y)} \exp\left(-\frac{\ell K(y, y)}{K(y, y)}\right).
\]  \hspace{1cm} \text{(92)}

The stationary values of the mean and variance of \(\ell\) at large time are then given by

\[
\langle \ell(y) \rangle^{(st)} = K(y, \min(x_0, y)), \\
\langle \ell(y)^2 \rangle_c = K(y, \min(x_0, y)) [2K(y, y) - K(y, \min(x_0, y))].
\]  \hspace{1cm} \text{(93) \hspace{1cm} (94)}

Note that for \(\kappa \to 0\), the results in equations (92)–(94) match with the results of the free particle case (equations (88) and (89)), as expected.
4.3.2. Computation of $\tilde{Q}_{fp}(k, q|x_0)$ in equation (78). In this section, we focus on $Q_{fp}(k, t|x_0)$ which can be computed by solving the differential equation (79). We once again solve for two choices of $V(x)$: (i) free particle ($V(x) = 0$) and (ii) harmonic potential ($V(x) = \kappa x^2/2$).

**Free particle** ($V(x_0) = 0$): in this case, equation (79) becomes

$$D \frac{\partial^2}{\partial x_0^2} \tilde{P}_a(k, q|x_0) - (k \delta(x_0 - y) + q) \tilde{P}_a(k, q|x_0) = 0.$$  

(95)

Solving the above with the boundary conditions in equation (80), and then inserting that solution in equation (78), we get

$$\tilde{Q}_{fp}(k, q|x_0) = \frac{1}{q} \begin{cases} \exp \left( -\sqrt{\kappa} x_0 \right) + \frac{k}{\sqrt{D}} \exp \left( -\sqrt{\kappa} y \right) \sinh \left( \sqrt{\kappa} (y - x_0) \right) & x_0 < y \\ \frac{1}{q} \exp \left( -\sqrt{\kappa} y \right) \sinh \left( \sqrt{\kappa} x_0 \right) & x_0 \geq y. \end{cases}$$  

(96)

Note that for $q \to 0$, we get $\tilde{Q}_{fp}(k, q \to 0|x_0) = \frac{1}{q} \tilde{P}_s(k|x_0)$ as expected from equations (78) and (82), where $\tilde{P}_s(k|x_0)$ is given in equation (87).

**OU particle** ($V(x_0) = \kappa x_0^2/2$): in this case, equation (79) becomes

$$D \frac{\partial^2}{\partial x_0^2} \tilde{P}_a(k, q|x_0) - \kappa x_0 \frac{\partial}{\partial x_0} \tilde{P}_a(k, q|x_0) - (k \delta(x_0 - y) + q) \tilde{P}_a(k, q|x_0) = 0.$$  

(97)

Defining the function $\tilde{R}_a(k, q|x_0)$ as

$$\tilde{R}_a(k, q|x_0) = e^{-\frac{x_0^2}{\kappa D}} \tilde{P}_a(k, q|x_0),$$  

(98)

and performing the transformations $u_0 = \sqrt{\kappa} x_0$ and $v_0 = \sqrt{\kappa} y$, the above differential equation is converted to the parabolic cylinder equation, as in equation (46):

$$\frac{\partial^2}{\partial u_0^2} \tilde{R}_a(k, q|u_0) - \left( \frac{u_0^2}{4} + a \right) \tilde{R}_a(k, q|u_0) = \frac{k}{\sqrt{\kappa D}} \delta(u_0 - v_0) \tilde{R}_a(k, q|u_0)$$  

(99)

where $a = q/\kappa - 1/2$. The general solution of this equation is

$$\tilde{R}_a(k, q|u_0) = \begin{cases} A_1 u(a, u_0) + B_1 v(a, u_0), & u_0 < v_0 \\ A_2 u(a, u_0) + B_2 v(a, u_0), & u_0 \geq v_0 \end{cases}$$  

(100)

where $u(a, u_0)$ and $v(a, u_0)$ are the two independent solutions of the parabolic cylinder equations given in equation (47). The integration constants $A_{1,2}$ and $B_{1,2}$ are determined from the boundary conditions in equation (80), along with the continuity of $\tilde{R}(k, q|u_0)$ and the discontinuity of its derivative (with respect to $u_0$) across $u_0 = v_0$. These extra conditions appear due to the presence of the delta function in equation (99). After getting $\tilde{R}(k, q|u_0)$, we get $\tilde{P}_a(k, q|x_0)$ from equation (98). Inserting this solution in equation (78) we get

\[\text{https://doi.org/10.1088/1742-5468/abe93d}\]
Note that in the above, $a$ is a function of $q$, as defined after equation (99). In the $q \to 0$ limit, we use the properties given in appendix D and get $\tilde{Q}_{fp}(k, q \to 0|x₀) = \frac{1}{q} \tilde{P}_s(k|x₀)$, as expected from equations (78) and (82) where $\tilde{P}_s(k|x₀)$ is now given in equation (91).

4.3.3. Approach to the stationary values of the cumulants: time dependence at large $t$. The expressions in equations (92)–(94) provide the stationary values of the mean and variance of $\ell_t(y)$ for $V(x) = 0$ and $V(x) = \kappa x^2/2$ respectively. To understand the approach to these stationary values, one needs to expand $Q_{fp}(k, t|x₀)$ and $Q(k, t|x₀)$, given in equation (74), in powers of $k$. Using this expansion in equation (12), one can compute the moments of $\ell_t(y)$. As mentioned earlier, the first term $Q(k, t|x)$ in equation (74) represents the contribution from trajectories surviving till time $t$. This contribution to the MGF has been computed in section 4 for the harmonic potential and in appendix E for the free particle. The Laplace transform $\tilde{Q}_{fp}(k, t|x₀)$ of the second term $Q_{fp}(k, t|x₀)$ has been obtained in the previous section 4.3.2 for both cases. In particular, to get the mean of the local time, we find it convenient to first find the Laplace transform of the mean, defined as

$$\langle \tilde{\ell}_t(y) \rangle(q) = \int_0^\infty dt \ e^{-qt} \langle \ell_t(y) \rangle,$$

from the expansion of the Laplace transform (with respect to $t$) of the MGF in powers of $k$. After, that we perform the inverse Laplace transform to get the mean in the time domain. Below, we compute the mean local time $\ell_t(y)$ for the free particle and the harmonic case separately.

**Free particle:** we expand $\tilde{Q}_{fp}(k, q|x₀)$ in equation (96) and $\tilde{Q}(k, q|x₀)$ in equation (E.4) in powers of $k$ to obtain the Laplace transforms of moments of $\ell_t(y)$. In particular, for the mean we get

$$\langle \tilde{\ell}_t(y) \rangle(q) = \begin{cases} \frac{1}{q\sqrt{qD}} e^{-\sqrt{qD}y} \left[ \frac{1}{2} \frac{\sinh (\sqrt{qD}y) e^{-\sqrt{qD}y} - \sinh (\sqrt{qD}(y-x₀))}{\sinh (\sqrt{qD}(y-x₀))} \right], & x₀ < y \\ \frac{1}{q\sqrt{qD}} e^{-\sqrt{qD}y} e^{-\sqrt{qD}x₀} \sinh (\sqrt{qD}y) & x₀ \geq y \end{cases}$$

(103)
One can, in principle, perform the inverse Laplace transform with respect to $q$ to get an explicit exact expression of $\langle \ell_t(y) \rangle$. However, we are interested in the large $t$ limit, in which we find

$$\langle \ell_t(y) \rangle \simeq \frac{\min(x_0, y)}{D} - \frac{yx_0}{D\sqrt{\pi D t}},$$

(104)

We note that the asymptotic value of the above is the same as in equation (89). Following a similar procedure, one can compute higher order moments as well. **OU particle**: the contribution of the surviving trajectories to the mean of $\ell_t(y)$ can be obtained from the expansion of $Q(k,t|x_0)$ in equation (39), as

$$\langle \ell_t(y) \rangle_s = t \sum_{n=1,3,5,\ldots} e^{-\lambda_n^{(0)} t} \lambda_n^{(2)}(y) g_n^{(0)} f_n^{(0)}(x_0)$$

$$- \sum_{n=1,3,5,\ldots} e^{-\lambda_n^{(0)} t} \left[ g_n^{(1)}(y) f_n^{(0)}(x_0) + g_n^{(0)} f_n^{(1)}(y|x_0) \right],$$

(105)

where the expressions for $\lambda_n^{(0)}$, $\lambda_n^{(1)}$, $g_n^{(0)}$, $g_n^{(1)}$, $f_n^{(0)}(x_0)$, and $f_n^{(1)}(y|x_0)$ are given in equations (A.1)–(A.11), which have to be computed using the eigenvalues and eigenfunctions given in equation (39). In the leading order for large $t$, we get

$$\langle \ell_t(y) \rangle_s \simeq t e^{-\lambda_1^{(0)} t} \lambda_1^{(2)}(y) g_1^{(0)} f_1^{(0)}(x_0) \simeq t e^{-\kappa t} \frac{2\kappa^2}{\pi D^2} x_0 y^2 e^{-\frac{y^2}{2D}}.$$  

(106)

To find the contribution $\langle \ell_t(y) \rangle_{fp}$ from the trajectories absorbed before time $t$, we expand $\tilde{Q}_{fp}(k,q|x_0)$ in equation (101) to linear order in $k$, and then take the first derivative with respect to $k$ at $k \to 0$. Adding this contribution to $\langle \ell_t(y) \rangle_s$, we find that at large $t$ the leading time dependent contribution $t e^{-\kappa t}$ in $\langle \ell_t(y) \rangle_s$ gets exactly cancelled and one finally gets

$$\langle \ell_t(y) \rangle \simeq \langle \ell_t(y) \rangle^{(st)} - C(x_0, y, \kappa, D) e^{-\kappa t}$$

(107)

for both $x_0 > y$ and $x_0 \leq y$, where $C$ and $\langle \ell_t(y) \rangle^{(st)}$ are time-independent constants.

The cancellation mentioned above actually occurs for all the lower order terms $t e^{-\kappa n t}$ with $n = 3, 5, 7, \ldots$. This can be seen properly through an alternative formulation of the local time of an Ornstein–Uhlenbeck particle

4.3.4. General method. In order to compute $Q_{fp}(k,t|x_0)$, we define the MGF $Q_{t'}(k,t|x_0)$ at time $t$ which involves contribution of the trajectories which have survived till time $t' < t$, i.e. it does not include contributions from trajectories that have crossed the origin before time $t'$. Note that by definition, $Q_t(k,t|x_0) = Q(k,t|x_0)$. Hence, we have

$$Q_t(k,t|x_0) = Q(k,t|x_0) = \int_0^\infty dx \langle x | e^{-\tilde{H}_{t'} t} | x_0 \rangle e^{-\frac{V(x)}{2t'}} + \frac{V(x)}{2t'}$$

(108)
where $\hat{H}_k$ is given in equation (9) with eigenfunctions satisfying the boundary conditions $\psi_n(x)\to 0$ as $x\to\infty$ and $\psi_n(x=0)=0$. The absorbing boundary condition $\psi_n(x)=0$ at $x=0$ can equivalently be expressed by assuming that the external potential $U(x)$ is described by

$$U_0(x) = \begin{cases} \infty, & x < 0 \\ \frac{V'(x)^2}{4D} - \frac{V''(x)}{2}, & x \geq 0. \end{cases}$$

(109)

Still working with the ensemble of paths not absorbed till time $t$, we consider the MGF after a small interval $\Delta t$. The relevant MGF, $Q_t(k, t + \Delta t | x_0)$, is given by

$$Q_t(k, t + \Delta t | x_0) = \int_0^\infty dz \int_0^\infty dx \langle x | e^{-\hat{H}_k \Delta t} | x \rangle \langle x | e^{-H_k t} | x_0 \rangle e^{-\frac{V(z)}{2D} + \frac{V(0)}{2D}}$$

(110)

where $\hat{H}_k$ is a Hamiltonian similar to $\hat{H}_h$, but without the infinite potential over the negative real line, i.e. it is a Hamiltonian over full space with eigenfunctions satisfying the boundary conditions $\psi_n(x)\to 0$ as $x\to\pm\infty$. In the above, we have implicitly assumed that the time interval $\Delta t$ is small enough that we do not have to worry about multiple crossings of the origin within that interval. Note that $Q_t(k, t + \Delta t | x_0)$ represents the MGF at time $(t + \Delta t)$, which includes contributions only from trajectories that have survived till time $t$; we do not worry about whether they get absorbed between $t$ and $(t + \Delta t)$.

On the other hand, we have

$$Q_{t+\Delta t}(k, t + \Delta t | x_0) = \int_0^\infty dz \int_0^\infty dx \langle x | e^{-\hat{H}_k (t+\Delta t)} | x_0 \rangle e^{-\frac{V(z)}{2D} + \frac{V(0)}{2D}}$$

(111)

$$= \int_0^\infty dz \int_0^\infty dx \langle x | e^{-\hat{H}_k \Delta t} | x \rangle \langle x | e^{-H_k t} | x_0 \rangle e^{-\frac{V(z)}{2D} + \frac{V(0)}{2D}},$$

(112)

which represents the MGF at time $(t + \Delta t)$ that only includes contributions from trajectories surviving till time $(t + \Delta t)$. As a result, the difference

$$Q_t(k, t + \Delta t | x_0) - Q_{t+\Delta t}(k, t + \Delta t | x_0)$$

$$= \int_{-\infty}^\infty dz \int_0^\infty dx \langle x | e^{-\hat{H}_k \Delta t} | x \rangle \langle x | e^{-H_k t} | x_0 \rangle e^{-\frac{V(z)}{2D} + \frac{V(0)}{2D}}$$

$$- \int_0^\infty dz \int_0^\infty dx \langle x | e^{-\hat{H}_k \Delta t} | x \rangle \langle x | e^{-H_k t} | x_0 \rangle e^{-\frac{V(z)}{2D} + \frac{V(0)}{2D}}$$

(113)

represents the contribution to the MGF from those trajectories which have been absorbed at the origin (i.e. at the absorbing boundary) in the time interval $t$ to $t + \Delta t$. 

https://doi.org/10.1088/1742-5468/abe93d
Taking the $\Delta t \to 0$ limit on both sides, we get

$$\frac{dQ_{fp}(k, t|x_0)}{dt} = \int_0^\infty dz \langle z | \hat{H}_k e^{-\hat{H}_k t} | x_0 \rangle e^{-\frac{V(z)}{2D} + \frac{V(x_0)}{2D}}$$

$$- \int_{-\infty}^\infty dz \int_0^\infty dx \langle z | \hat{H}_k | x \rangle \langle x | e^{-\hat{H}_k t} | x_0 \rangle e^{-\frac{V(x)}{2D} + \frac{V(x_0)}{2D}}$$

(114)

which provides $Q_{fp}(k, t|x_0)$ upon integration over time from 0 to $t$. We get

$$Q_{fp}(k, t|x_0) = 1 - \int_0^\infty d\tau \int_{-\infty}^\infty dz \int_0^\infty dx \langle z | \hat{H}_k | x \rangle \langle x | e^{-\hat{H}_k \tau} | x_0 \rangle e^{-\frac{V(x)}{2D} + \frac{V(x_0)}{2D}},$$

(115)

which implies

$$Q(k, t|x_0) = 1 - \int_0^t d\tau \int_{-\infty}^\infty dz \int_0^\infty dx \langle z | \hat{H}_k | x \rangle \langle x | e^{-\hat{H}_k \tau} | x_0 \rangle e^{-\frac{V(x)}{2D} + \frac{V(x_0)}{2D}}.$$

(116)

Note that $Q(k, t \to 0|x_0) = 1$ as expected. Next, we compute $Q(k, t|x_0)$ for two cases, a free particle and an OU particle, as before.

**Free particle:** in this case, $\hat{V}(x) = 0$, and consequently $\langle z | \hat{H}_k | z' \rangle = -D \frac{d^2}{dz^2} \delta(z - z') + k \delta(y - z) \delta(z - z')$. Substituting this in equation (116), we get

$$Q(k, t|x_0) = 1 - k \int_0^t d\tau \langle y | e^{-\hat{H}_k \tau} | x_0 \rangle$$

$$= 1 - k \int_0^t d\tau \langle y | e^{-\hat{H}_k \tau} | x_0 \rangle + O(k^2)$$

(117)

where the free particle propagator $\langle y | e^{-\hat{H}_k \tau} | x_0 \rangle$ in the presence of an absorbing boundary at $x = 0$ is given by

$$\langle y | e^{-\hat{H}_k \tau} | x_0 \rangle = \frac{1}{\sqrt{4\pi D \tau}} \left( e^{-\frac{(y-x_0)^2}{4D \tau}} - e^{-\frac{(y+x_0)^2}{4D \tau}} \right).$$

(118)

From the coefficient of the term at $O(k)$ in equation (117), we correctly reproduce the result in equation (104), i.e.

$$\langle \ell_i(y) \rangle = \int_0^t d\tau \langle y | e^{-\hat{H}_k \tau} | x_0 \rangle$$

$$\approx \frac{\min(x_0, y)}{D} - \frac{y x_0}{D \sqrt{\pi} D t}$$

for large $t$. (119)
Figure 5. Numerical verification of the expressions of the mean ($\langle \ell_t(y) \rangle$) and variance ($\langle \ell^2_t(y) \rangle$) of the local time given in equations (119) and (120) unconditioned on survival of a free particle with an absorbing boundary at the origin. In both cases, we have taken $x_0 = 0.5$, $y = 0.3$, and $D = 1/2$. For the simulations, we used a forward Euler scheme with time-step $dt = 10^{-7}$, and took $10^5$ realizations.

It turns out that in this case one can compute the Laplace transform of all higher order moments $\langle \ell^k_t(y) \rangle(s)$ explicitly from which one can find their time dependence by performing inverse Laplace transforms. This calculation is presented in appendix F. In particular, we get the following explicit expression for the variance for large $t$:

$$\langle \ell^2_t(y) \rangle_c = \langle \ell^2_t(y) \rangle - \langle \ell_t(y) \rangle^2,$$

where

$$\langle \ell^2_t(y) \rangle_{t \gg 1} \approx \frac{2y \min(y, x_0)}{D^2} - \frac{2y^2 (x_0 + \min(y, x_0))}{D^2 \sqrt{\pi Dt}} + O(t^{-1}).$$

In figure 5, we show that the obtained expressions for the mean and variance match well with simulations.

**OU particle**: to avoid possible confusion, in this case, let us denote the eigenvalues and eigenfunctions of \( \hat{H}_k \) as $\lambda_n(k)$, $\psi_{n,k}(x)$ respectively, and of $\hat{H}_k$ as $\mu_n(k)$, $\phi_{n,k}(x)$ respectively. Note that $\mu_n(0)$ and $\phi_{n,0}(x)$ are given in equation (35) and $\lambda_n(0)$ and $\psi_{n,0}(x)$ are given in equation (39). Using the relevant completeness relations for $\psi_{n,k}(x)$ and $\phi_{n,k}(x)$ and performing the integration over time in equation (116), we get

$$Q(k, t|x_0) = 1 - \sum_{n=1,3,5,\ldots}^{\infty} \sum_{m=0,1,2,\ldots}^{\infty} \frac{\mu_m(k) [1 - e^{-\lambda_n(k)t}]}{\lambda_n(k)} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dx \phi_m(z) \psi^*_{n,k}(x) \psi_{n,k}(x_0) e^{-\frac{x^2}{4Dt} + \frac{x_0^2}{4Dt}}.$$  \hspace{1cm} (121)

It is straightforward to check that $Q(0, t|x_0) = 1$ for $x_0 > 0$. We observe that the MGF has the following form:

$$Q(k, t|x_0) = Q_{\infty}(k|x_0) + Q_d(k, t|x_0)$$  \hspace{1cm} (122)
where

\[ Q_\infty(k|x_0) = 1 - \sum_{n=1,3,5...}^{\infty} \frac{\mu_n(k)}{\lambda_n(k)} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dx \phi_{m,k}(z) \phi^*_m(x) \psi_n(x) \]

\times \psi^*_n(x_0) e^{-\frac{x^2}{2\sigma^2} + \frac{x^4}{4\lambda^2} },

\[ Q_d(k,t|x_0) = \sum_{n=1,3,5...}^{\infty} \frac{\mu_n(k)e^{-\lambda_n(k)t}}{\lambda_n(k)} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dx \phi_{m,k}(z) \phi^*_m(x) \]

\times \psi_n(x) \psi^*_n(x_0) e^{-\frac{x^2}{2\sigma^2} + \frac{x^4}{4\lambda^2} }. \hspace{1cm} (123)\]

Note that the first term with subscript \( \infty \) in equation (122) is actually \( Q_\infty(k|x_0) = Q(k,t \to \infty|x_0) \). The second term (with subscript \( d \)) consists of terms decaying exponentially in time.

This structure suggests that all the cumulants of \( \ell_t(y) \) become time-independent at large times. This is expected because the survival probability decreases with increasing time. As a result, the contribution to \( \ell_t(y) \) starts saturating with time. To compute the cumulants, one can expand \( Q_\infty(k,t|x_0) \) and \( Q_d(k,t|x_0) \) in equation (121) in powers of \( k \) using the perturbation expansions of the eigenvalues and eigenfunctions as described in section 2. The expansion of \( Q_\infty(k,t|x_0) \) would provide stationary values of the cumulants of \( \ell_t \). Recall that in section 4.3.1 we have computed stationary values of the mean and the variance using a different approach.

We now focus on computing the time dependence of the mean at large time. To get that, we expand \( Q_d(k,t|x_0) \) in equation (123) to linear order in \( k \) for large time \( t \). We get

\[ Q_d(k,t|x_0) \simeq k \left( \frac{2\pi D}{\kappa} \right)^{1/4} e^{\frac{x^2}{2\sigma^2}} \sum_{n=1,3,5...}^{\infty} e^{-n\kappa} \psi_n^{(0)}(x_0) n\kappa \]

\times \left[ \mu_0^{(1)} J_{0,n} + \sum_{m=1,2,3...}^{\infty} J_{m,n} \phi_m^{(0)}(y) \phi_0^{(0)}(y) \right] + O(k^2),

\[ \simeq k \left( \frac{2\pi D}{\kappa} \right)^{1/4} e^{\frac{x^2}{2\sigma^2}} \sum_{n=1,3,5...}^{\infty} e^{-n\kappa} \psi_n^{(0)}(x_0) n\kappa \]

\times \left[ \sum_{m=0,1,2...}^{\infty} J_{m,n} \phi_m^{(0)}(y) \phi_0^{(0)}(y) \right] + O(k^2), \hspace{1cm} (124)\]

where

\[ J_{m,n} = \int_{0}^{\infty} \psi_n^{(0)}(x) \phi_m^{(0)}(x) dx = \frac{1}{\sqrt{2^{m+1} \pi}} \sqrt{2} \int_{0}^{\infty} e^{-u^2} H_m(u) H_n(u) du. \hspace{1cm} (125)\]
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From this, one can readily find the approach of $\ell_t(y)$ to the stationary part given in equation (93). Adding the decaying part to the stationary part and then taking derivative with respect to $k$ at $k=0$, we get

$$
\langle \ell_t(y) \rangle = \left( \frac{2\pi D}{\kappa} \right)^{1/4} \frac{e^{x_0^2}}{\sqrt{\kappa}} \sum_{n=1,3,5,...} \sum_{m=0,1,2,...} \left( 1 - e^{-n\kappa t} \right) \frac{\psi_n^{(0)}(x_0)}{n\kappa} \left[ \sum_{m=0,1,2,...} J_{m,n} \phi_m^{(0)}(y) \phi_0^{(0)^*}(y) \right] .
$$

(126)

For large $t$, the above simplifies to

$$
\ell_t(y) \approx \langle \ell_t(y) \rangle^{(st)} - \left( \frac{2\pi D}{\kappa} \right)^{1/4} e^{-\kappa x_0^2} \frac{1}{\sqrt{\kappa}} \psi_1^{(0)}(x_0) \left[ \sum_{m=0,1,2,...} J_{m,1} \phi_m^{(0)}(y) \phi_0^{(0)^*}(y) \right] + O(e^{-3\kappa t}),
$$

(127)

which is in the form given in equation (107). Note that in this method, we get an alternative expression for $\langle \ell_t(y) \rangle^{(st)}$, apparently different from that given in equation (93):

$$
\langle \ell_t(y) \rangle^{(st)} = \left( \frac{2\pi D}{\kappa} \right)^{1/4} \frac{e^{x_0^2}}{\sqrt{\kappa}} \sum_{n=1,3,5,...} \frac{\psi_n^{(0)}(x_0)}{n\kappa} \left[ \sum_{m=0,1,2,...} J_{m,n} \phi_m^{(0)}(y) \phi_0^{(0)^*}(y) \right] .
$$

(128)

In figure 6, we have numerically checked that this series indeed converges to $\mathcal{K}(y, \min(x_0, y))$ (defined in equation (91)). The expression (given in equation (127)) for the mean local time unconditioned on survival is verified for two cases, $y < x_0$ and $y > x_0$, in figure 7.
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Figure 7. Numerical verification of the analytical expression of the mean local time \( \langle \ell_t(y) \rangle \) of an OU particle unconditioned on survival, given in equation (127), for \( y = 0.3 \) (left) and \( y = 1.5 \) (right). In both cases, we have taken \( x_0 = 0.9 \), \( \kappa = 1 \), and \( D = 1/2 \). For the simulations, we used a forward Euler scheme with time-step \( dt = 10^{-5} \), and took \( 10^5 \) realizations.

5. Conclusions

In this paper, we have studied the statistical properties of the local time \( \ell_t \) spent by an OU particle at different locations in the presence and in the absence of an absorbing boundary. Using the Feynman–Kac formalism, the MGF \( \mathcal{Q}(k, t| x_0) \) of \( \ell_t \) is written as an integral of the propagator of a quantum particle, for which the Hamiltonian naturally appears as the sum of a bare Hamiltonian and a perturbation. Exploiting this structure, we employ quantum perturbation theory to find the MGF as a power series in \( k \), from which one can directly compute cumulants of \( \ell_t \). This method is quite general, and can be applied for any functional \( T_t[x(\tau)] \) of the form given in equation (2) and an arbitrary confining potential \( V(x) \) (in equation (3)). We find that for choices of \( V(x) \) and \( U(x) \) such that the Hamiltonian \( \hat{H}_k \) with the effective potential \( U(x) + kU(x) \) in equation (9) has a non-degenerate and isolated ground state, all the cumulants of \( T_t[x(\tau)] \) increase linearly with time in the leading order. This observation leads us to predict a Gaussian distribution for the path functional, when shifted by its mean and scaled by its standard deviation. Since the Gaussian form of the distribution describes only the typical fluctuations, we study large deviations for large \( t \) using saddle point calculations. Such a calculation indicates the existence of a LDF \( \mathbb{H}(w) \) which would describe the large fluctuations. For the path functional \( \ell_t(y) \), we find different asymptotic behaviours of \( \mathbb{H}(w) \).

In the second part of the paper, we study statistical properties of \( \ell_t(y) \) in the presence of an absorbing boundary, but not conditioned on survival. In this case, we have found that the distribution of \( \ell_t(y) \) approaches a stationary distribution at large \( t \), as expected from the theory of first passage functionals [3]. In addition, we have shown how it approaches the stationary state by studying the MGF and computing the mean \( \ell_t(y) \). For this, we have developed a general formalism which shows how the distribution approaches the stationary state. In particular, we demonstrate this approach explicitly by computing the mean local time for two cases—a free particle and an OU particle.

https://doi.org/10.1088/1742-5468/abe93d
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We believe our results would be of interest to a broad community of physicists working in the areas of stochastic processes, biological processes and chemical kinetics. Our study can be naturally extended in several directions. For example, recently, there has been a lot of interest in active run-and-tumble particles which model the motion of a bacterium at the basic level. For such particles, the problem of computing the MGF can be mapped to the evaluation of the eigenvalues and eigenfunctions of a Hamiltonian similar to that of a relativistic quantum particle [59, 60]. It would be interesting to see how our results get modified for run-and-tumble particles. Another direction would be to allow the position of the absorbing boundary to vary. It has been shown that in such scenario, the spectrum of the effective quantum potential undergoes a gap closing transition which manifests as a freezing transition in the barrier crossing problem [61]. It would be interesting to see how our results modify under such a transition.

Acknowledgments

A K would like to acknowledge very stimulating discussions with Alain Comtet from which this project has originated. A K would also like to thank Prashant Singh for careful reading of the draft and for pointing out important mistakes. We thank Hugo Touchette for helpful suggestions and for drawing our attention to important references. We also thank Aljaž Godec for pointing out relevant references. A K acknowledges support of the Department of Atomic Energy, Government of India, under Project No. 12-R&D-TFR-5.10-1100. A K acknowledges support from DST, Government of India grant under Project No. ECR/2017/000634. GK acknowledges the Long-Term Visiting Students Program through which he visited ICTS, Bangalore, where most parts of the project were carried out.

Appendix A.

Here, we provide the expressions for terms like $\lambda_n^{(0)}$, $\lambda_n^{(1)}$, ...; $g_n^{(0)}$, $g_n^{(1)}$, ...; and $f_n^{(0)}(x_0)$, $f_n^{(1)}(y|x_0)$, ... appearing in equation (31).

\[
\lambda_n^{(1)}(y) = \int dx \psi_n^{(0)}(x)\delta(x - y)\psi_n^{(0)}(x) = |\psi_n^{(0)}(y)|^2
\]

(A.1)

\[
\lambda_n^{(2)}(y) = \sum_{m \neq n} \frac{\left|\psi_n^{(0)}(y)\psi_n^{(0)}(y)\right|^2}{\lambda_n^{(0)} - \lambda_m^{(0)}}
\]

(A.2)

\[
\lambda_n^{(1,1)}(y, z) = \sum_{m \neq n} \frac{\left[\psi_n^{(0)}(y)\psi_n^{(0)}(y)\psi_m^{(0)}(z)\psi_m^{(0)}(z) + \psi_m^{(0)}(y)\psi_m^{(0)}(y)\psi_n^{(0)}(z)\psi_n^{(0)}(z)\right]}{\lambda_n^{(0)} - \lambda_m^{(0)}}
\]

(A.3)

\[
f_n^{(0)}(x_0) = e^{\frac{V(x_0)}{2\hbar^2} \psi_n^{(0)}(x_0)}
\]

(A.4)

\[
f_n^{(1)}(y|x_0) = \sum_{m \neq n} \frac{\psi_n^{(0)}(y)\psi_n^{(0)}(y)}{\lambda_n^{(0)} - \lambda_m^{(0)}} e^{\frac{V(x_0)}{2\hbar^2} \psi_m^{(0)}(x_0)}
\]

(A.5)
\[ f_n^{(2)}(y|x_0) = \sum_{m \neq n} \left[ \sum_{p \neq n} \frac{\left[ \psi_n^{(0)}(y) \psi_p^{(0)}(y) \right]^2}{(\lambda_n - \lambda_m) (\lambda_n - \lambda_p)} - \frac{\left[ \psi_n^{(0)}(y) \psi_n^{(0)}(y) \right]^2}{(\lambda_n - \lambda_m)^2} \right] \times \psi_m^{(0)}(x_0) e^{-\frac{V_m}{2\beta}} \tag{A.6} \]

\[ f_n^{(1,1)}(y, z|x_0) = \sum_{m \neq n} \left[ \sum_{p \neq n} \frac{\left[ \psi_n^{(0)}(y) \psi_p^{(0)}(y) \psi_m^{(0)}(z) \psi_n^{(0)}(z) + \psi_n^{(0)}(y) \psi_n^{(0)}(y) \psi_m^{(0)}(z) \psi_n^{(0)}(z) \right]}{(\lambda_n - \lambda_m) (\lambda_n - \lambda_p)} \right] \times \psi_m^{(0)}(x_0) e^{-\frac{V_m}{2\beta}} \tag{A.7} \]

where \( V(x) = \kappa x^2 / 2 \). Similarly from equation (19) we get

\[ g_n^{(0)} = \int dx \ e^{-\frac{V(x)}{2\beta}} \psi_n^{(0)}(x), \tag{A.8} \]

\[ g_n^{(1)}(y) = \int dx \ e^{-\frac{V(x)}{2\beta}} \sum_{m \neq n} \frac{\psi_m^{(0)}(y) \psi_n^{(0)}(y)}{(\lambda_n - \lambda_m)} \psi_m^{(0)}(x) \tag{A.9} \]

\[ g_n^{(2)}(y) = \int dx \ e^{-\frac{V(x)}{2\beta}} \sum_{m \neq n} \left[ \sum_{p \neq n} \frac{\left[ \psi_m^{(0)}(y) \psi_n^{(0)}(y) \right]^2}{(\lambda_n - \lambda_m) (\lambda_n - \lambda_p)} \right] \times \psi_m^{(0)}(x) \tag{A.10} \]

\[ g_n^{(1,1)}(y, z) = \int dx \ e^{-\frac{V(x)}{2\beta}} \sum_{m \neq n} \left[ \sum_{p \neq n} \frac{\psi_p^{(0)}(y) \psi_n^{(0)}(y) \psi_m^{(0)}(z) \psi_n^{(0)}(z) + \psi_n^{(0)}(y) \psi_n^{(0)}(y) \psi_m^{(0)}(z) \psi_n^{(0)}(z)}{(\lambda_n - \lambda_m) (\lambda_n - \lambda_p)} \right] \times \psi_m^{(0)}(x). \tag{A.11} \]
Appendix B. Functions (independent of time) appearing in equations (38) and (43)

\[
C_{(\kappa)}(y, z | x_0) = \frac{1}{2D} \left[ \frac{1}{\kappa \pi} e^{-\frac{\kappa y^2}{2D}} \sum_{m \neq 0, l \neq 0} \frac{H_m(\sqrt{\frac{\kappa}{2D}} y) H_m(\sqrt{\frac{\kappa}{2D}} x_0)}{m! m^2 l! l^2} 
\right.

- \frac{1}{\kappa \pi} e^{-\frac{\kappa y^2}{2D}} \sum_{m \neq 0} \frac{H_m(\sqrt{\frac{\kappa}{2D}} y) H_m(\sqrt{\frac{\kappa}{2D}} x_0)}{m! m^2 l! l^2}

+ \frac{1}{\kappa \pi} e^{-\frac{\kappa z^2}{2D}} \sum_{m \neq 0} \frac{H_m(\sqrt{\frac{\kappa}{2D}} y) H_m(\sqrt{\frac{\kappa}{2D}} x_0)}{m! m^2 l! l^2}

- \frac{1}{\kappa \pi} e^{-\frac{\kappa z^2}{2D}} \sum_{m \neq 0} \frac{H_m(\sqrt{\frac{\kappa}{2D}} y) H_m(\sqrt{\frac{\kappa}{2D}} x_0)}{m! m^2 l! l^2}

- \frac{1}{\kappa \pi} e^{-\frac{\kappa y^2}{2D}} \sum_{m \neq 0} \frac{H_m(\sqrt{\frac{\kappa}{2D}} y) H_m(\sqrt{\frac{\kappa}{2D}} x_0)}{m! m^2 l! l^2}

\right]

(B.1)

\[
C_{(a)}(y, z | x_0) = \frac{y z^2 \sqrt{\kappa}}{\pi \sqrt{2D}^{5/2}} e^{-\frac{\kappa y^2}{2D}} \sum_{p \neq 1, p \text{ odd}} \frac{(-1)^{\frac{p-1}{2}} p!! H_p(\sqrt{\frac{\kappa}{2D}} y)}{(1-p)^2 p!^2 p!} - \frac{2}{\pi D^2} \frac{y z^2}{x_0} e^{-\frac{\kappa y^2}{2D}} - \frac{2}{\pi D^2} \frac{y z^2}{x_0} e^{-\frac{\kappa y^2}{2D}} - \frac{2}{\pi D^2} \frac{y z^2}{x_0} e^{-\frac{\kappa y^2}{2D}}

\times \sum_{m \neq 1, m \text{ odd}} \frac{H_m(\sqrt{\frac{\kappa}{2D}} y) H_m(\sqrt{\frac{\kappa}{2D}} x_0)}{2m! (1-m)^2} + \frac{y z^2 \sqrt{\kappa}}{\pi \sqrt{2D}^{5/2}} e^{-\frac{\kappa y^2}{2D}}

\times \sum_{p \neq 1, p \text{ odd}} \frac{(-1)^{\frac{p-1}{2}} p!! H_p(\sqrt{\frac{\kappa}{2D}} z)}{2p!^2 p! (1-p)^2} - \frac{2}{\pi D^2} \frac{y z^2}{x_0} e^{-\frac{\kappa y^2}{2D}} - \frac{2}{\pi D^2} \frac{y z^2}{x_0} e^{-\frac{\kappa y^2}{2D}}

\times \sum_{m \neq 1, m \text{ odd}} \frac{H_m(\sqrt{\frac{\kappa}{2D}} z) H_m(\sqrt{\frac{\kappa}{2D}} x_0)}{2m! (1-m)^2} - \frac{z}{\pi \sqrt{2D}^{5/2}} e^{-\frac{\kappa y^2}{2D}} - \frac{z}{\pi \sqrt{2D}^{5/2}} e^{-\frac{\kappa y^2}{2D}}

\times \sum_{p \neq 1, p \text{ odd}} \frac{(-1)^{\frac{p-1}{2}} p!! H_p(\sqrt{\frac{\kappa}{2D}} y)}{2p!^2 p! (1-p)^2} \sum_{l \neq 1, l \text{ odd}} \frac{H_l(\sqrt{\frac{\kappa}{2D}} y) H_l(\sqrt{\frac{\kappa}{2D}} z)}{(1-l)!} 2l!

+ \frac{2}{\pi \kappa D} \frac{z}{x_0} e^{-\frac{\kappa y^2}{2D}} \sum_{m \neq 1, m \text{ odd}} \frac{H_m(\sqrt{\frac{\kappa}{2D}} y) H_m(\sqrt{\frac{\kappa}{2D}} x_0)}{(1-m)! m^2 m!}

\]
In this section, we derive equation (79). Here, we present the derivation for a general functional $T_t^p$ of the form given in equation (2). The Laplace transform of the joint distribution $P_a(T, t| x_0)$ of $T_t^p[x(\tau)] = \int_0^t U(x) d\tau$ and the first-passage time $t_f = \int_0^{t_f} d\tau$ is given by

$$
\tilde{P}_a(k, q | x_0) = \int_0^\infty d\tau \int_0^\infty dT \ e^{-kT} e^{-q\tau} P_a(T, t| x_0)
$$

(C.1)

$$
= \left< e^{-kT} e^{-q\tau} \right>
$$

(C.2)

$$
= \left< e^0 \right>
$$

(C.3)

We split the interval $(0, t_f)$ into $(0, \Delta \tau) + (\Delta \tau, t_f)$. One can then write

$$
\tilde{P}_a(k, q | x_0) = \left< e^{\Delta \tau(-kU(x)-q)} \tilde{P}_a(k, q | x_0 + \Delta x) \right>_{\Delta x}
$$

(C.4)

$$
= \left< e^{\Delta \tau(-kU(x)-q)} \tilde{P}_a(k, q | x_0 + \Delta x) \right>_{\Delta x}
$$

(C.5)
where we average over all possible values of $\Delta x$, the distance the particle travels in the interval $(0, \Delta \tau)$.

$$
\bar{P}_a(k, q \mid x_0) = \left\langle e^{\Delta \tau(-kU(x_0)-q)} \bar{P}_a(k, q \mid x_0 + \Delta x) \right\rangle_{\Delta x} 
$$

$$
= \left\langle (1 - \Delta \tau (kU(x_0) + q)) \left( \bar{P}_a + \frac{\partial \bar{P}_a}{\partial x_0} \Delta x + \frac{1}{2} \frac{\partial^2 \bar{P}_a}{\partial x_0^2} \Delta x^2 \right) \right\rangle_{\Delta x}
$$

$$
= \bar{P}_a - \Delta \tau (kU(x_0) + q) \bar{P}_a + \frac{\partial \bar{P}_a}{\partial x_0} \langle \Delta x \rangle + \frac{1}{2} \frac{\partial^2 \bar{P}_a}{\partial x_0^2} \langle \Delta x^2 \rangle.
$$

From equation (3), we have $\langle \Delta x \rangle = -V'(x_0) \Delta \tau$ and $\langle \Delta x^2 \rangle = 2D \Delta \tau + O(\Delta \tau^2)$. Inserting these in equation (C.8) and taking the $\Delta \tau \to 0$ limit, we get

$$
D \frac{\partial^2}{\partial x_0^2} \bar{P}_a(k, q \mid x_0) - V'(x_0) \frac{\partial}{\partial x_0} \bar{P}_a(k, q \mid x_0) - (kU(x_0) + q) \bar{P}_a(k, q \mid x_0) = 0.
$$

The boundary conditions are

$$
\lim_{x_0 \to 0} \bar{P}_a(k, q \mid x_0) = 1 \quad \text{(C.10)}
$$

$$
\lim_{x_0 \to \infty} \bar{P}_a(k, q \mid x_0) = 0. \quad \text{(C.11)}
$$

For the local time $\ell_t(y)$, one needs to put $U(x_0) = \delta(x_0 - y)$ in equation (C.9).

**Appendix D. Some useful identities and relations**

$$
u \left( \frac{1}{2}, 0 \right) = e^{-\frac{x}{2}}, \quad \nu \left( -\frac{1}{2}, 0 \right) = 1
$$

$$
\nu \left( \frac{1}{2}, 0 \right) = 0, \quad \nu \left( \frac{1}{2}, z \right) = \frac{1}{\sqrt{\pi}} e^{\frac{z^2}{4}}, \quad \nu \left( -\frac{1}{2}, z \right) = e^{-\frac{z^2}{4}} \text{erfi} \left( \frac{v_0}{\sqrt{2}} \right)
$$

$$
\text{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dy e^{y^2}
$$

$$
\frac{d\nu(a, z)}{dz} = -\frac{z}{2} \nu(a, z) - \left(a + \frac{1}{2}\right) \nu(a + 1, z)
$$

$$
\frac{d\nu(a, z)}{dz} = \nu(a + 1, z) - \frac{z}{2} \nu(a, z)
$$

$$
\nu(a, z) \frac{d\nu(a, z)}{dz} - \nu(a, z) \frac{d\nu(a, z)}{dz} = \nu(a, z) \nu(a + 1, z) + \left(a + \frac{1}{2}\right) \nu(a, z) \nu(a + 1, z).
$$

(C.1)
Appendix E. Computation of $Q(k, t|x_0)$ for free particle

One can, in principle, compute the MGF $Q(k, t|x_0)$ using the perturbation expansion method given in section 2. However, since the eigenspectrum is continuous for $V(x) = 0$, the perturbation method is not especially advantageous over the usual backward Fokker–Planck method described in [3]. Below, we use the latter method to compute $Q(k, t|x_0)$. The backward Fokker–Planck equation satisfied by the Laplace transformed MGF $\tilde{Q}(k, q|x_0) = \int_0^\infty dt e^{-qt}Q(k, t|x_0)$ is

$$D \frac{\partial^2 \tilde{Q}}{\partial x_0^2} = k \delta(x_0 - y)\tilde{Q} + q\tilde{Q} - 1$$

(E.1)

with the boundary conditions

$$\tilde{Q}(k, q | x_0)_{x_0=0} = 0$$
$$\tilde{Q}(k, q | x_0)_{x_0=\infty} = 1.$$  

(E.2)

Due to the presence of the delta function in equation (E.1), we write its general solution as

$$\tilde{Q} = \begin{cases} 
C_1 e^{x_0\sqrt{\pi}} + C_2 e^{-x_0\sqrt{\pi}} + \frac{1}{q}, & x_0 < y \\
C_3 e^{x_0\sqrt{\pi}} + C_4 e^{-x_0\sqrt{\pi}} + \frac{1}{q}, & x_0 \geq y
\end{cases}$$

(E.3)

where $C_1, C_2, C_3$ and $C_4$ are constants. They can be determined from the boundary conditions in equation (E.2), along with the continuity of $\tilde{Q}(k, q | x_0)$ and the discontinuity of its derivative (with respect to $x_0$) across $x_0 = y$. These extra conditions appear due to the presence of the delta function in equation (E.1). Finding the constants, we finally get

$$\tilde{Q}(k, q | x_0) = \begin{cases} 
\frac{1}{q} + \frac{k}{q\sqrt{\pi}D} \left[ \frac{1 + \frac{1}{\sqrt{\pi}q}\sinh \left( \sqrt{\pi}q \right)}{1 + \frac{1}{\sqrt{\pi}q}\sinh \left( \sqrt{\pi}q \right)} e^{-\frac{1}{\sqrt{\pi}q}} - e^{-y\sqrt{\pi}} \right] e^{x_0\sqrt{\pi}} : x_0 < y , \\
\frac{1}{q} \left[ \frac{k}{\sqrt{\pi}qD} e^{-\sqrt{\pi}q} - \frac{1 + \frac{1}{\sqrt{\pi}q}\sinh \left( \sqrt{\pi}q \right)}{1 + \frac{1}{\sqrt{\pi}q}\sinh \left( \sqrt{\pi}q \right)} e^{-x_0\sqrt{\pi}} \right] e^{-y\sqrt{\pi}} : x_0 \geq y.
\end{cases}
$$

(E.4)

Performing the inverse Laplace transform with respect to $q$, one can get $Q(k, t|x_0)$. 

https://doi.org/10.1088/1742-5468/abe93d
Appendix F. Computation of large-time survival unconditioned moments for a free particle

By using
\[ \langle z | \hat{h}_k | z' \rangle = k \delta(y - z) \delta(z - z'), \tag{F.1} \]
we simplify equation (117) as
\[ Q(k, t|x_0) = 1 - k \int_0^t d\tau \langle y | e^{-\hat{H}_k \tau} | x_0 \rangle, \tag{F.2} \]
\[ = 1 - k \int_0^\infty d\tau \langle y | e^{-\hat{H}_k \tau} | x_0 \rangle + k \int_t^\infty d\tau \langle y | e^{-\hat{H}_k \tau} | x_0 \rangle, \tag{F.3} \]
\[ = C_k(y, x_0) + k A_k(y, x_0|t), \tag{F.4} \]
where we have defined
\[ A_k(y, x_0|t) \equiv \int_t^\infty d\tau \langle y | e^{-\hat{H}_k \tau} | x_0 \rangle, \tag{F.5} \]
\[ C_k(y, x_0) \equiv 1 - k \int_0^\infty d\tau \langle y | e^{-\hat{H}_k \tau} | x_0 \rangle = 1 - k A_k(y, x_0|0), \tag{F.6} \]
\[ G_k(y, x_0|\tau) \equiv \langle y | e^{-\hat{H}_k \tau} | x_0 \rangle. \tag{F.7} \]

We denote the Laplace transform of \( G_k(y, x_0|\tau) \) as
\[ \tilde{G}_k(y, x_0|s) = \int_0^\infty d\tau e^{-s\tau} G_k(y, x_0|\tau). \tag{F.8} \]
Since
\[ \frac{dA_k(y, x_0|t)}{dt} = -G_k(y, x_0|\tau), \tag{F.9} \]
we can write
\[ \tilde{A}_k(y, x_0|s) = \frac{A_k(y, x_0|0) - \tilde{G}_k(y, x_0|s)}{s}. \tag{F.10} \]

Using the above, we take the Laplace transform of both sides of equation (F.4) and write
\[ \tilde{Q}(k, s|x_0) = \frac{1 - k \tilde{G}_k(y, x_0|s)}{s}. \tag{F.11} \]

We now calculate \( \tilde{G}_k(y, x_0|\tau) \).
\[ \tilde{G}_k(y, x_0|s) = \int_0^\infty e^{-s\tau} \langle y | e^{-\hat{H}_u \tau} | x_0 \rangle e^{-s\tau} \langle y | e^{-\hat{h}_k} | x_0 \rangle = \langle y | \frac{1}{\hat{H}_u + s + \hat{h}_k} | x_0 \rangle, \tag{F.12} \]
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where $\hat{H}_u$ is the unperturbed Hamiltonian. We now define an operator corresponding to $\tilde{G}_k(s) = [\hat{H}_u + s + \hat{h}_k]^{-1}$, such that

$$
\tilde{G}_k(y, x_0|s) = \langle y | \tilde{G}_k(s) | x_0 \rangle .
$$

This operator can be simplified as

$$
\tilde{G}_k(s) = \frac{1}{\hat{H}_0 + s + \hat{h}_k}
$$

(F.14)

$$
= (\hat{H}_0 + s)^{-1} \left[ 1 + (\hat{H}_0 + s)^{-1} \hat{h}_k \right]^{-1}
$$

(F.15)

$$
= \tilde{G}_0(s) \left[ 1 - (\hat{H}_0 + s)^{-1} \hat{h}_k \left[ 1 + (\hat{H}_0 + s)^{-1} \hat{h}_k \right]^{-1} \right]
$$

(F.16)

$$
= \tilde{G}_0(s) - \tilde{G}_0(s) \hat{h}_k \tilde{G}_k(s).
$$

(F.17)

Using the above, we can write

$$
\tilde{G}_k(y, x_0|s) = \langle y | \tilde{G}_0(s) | x_0 \rangle - \int \int dz \, dz' \langle y | \tilde{G}_0(s) | z \rangle \langle z | \hat{h}_k | z' \rangle \langle z' | \tilde{G}_k(s) | x_0 \rangle
$$

(F.18)

$$
= \tilde{G}_0(y, x_0|s) - k \tilde{G}_0(y, y|s) \tilde{G}_k(y, x_0|s)
$$

(F.19)

$$
\implies \tilde{G}_k(y, x_0|s) = \frac{\tilde{G}_0(y, x_0|s)}{1 + k \tilde{G}_0(y, y|s)}.
$$

(F.20)

Putting the above in equation (F.11), we obtain

$$
\tilde{Q}(k, s|y, x_0) = \frac{1}{s} \left( 1 - \frac{k \tilde{G}_0(y, x_0|s)}{1 + k \tilde{G}_0(y, y|s)} \right)
$$

(F.21)

$$
= \frac{1}{s} \left[ 1 - k \tilde{G}_0(y, x_0|s) \sum_{n=0}^{\infty} (-k^n) \left( \tilde{G}_0(y, y|s) \right)^n \right]
$$

(F.22)

$$
= \frac{1}{s} \left[ 1 + \sum_{n=1}^{\infty} (-k^n) \tilde{G}_0(y, x_0|s) \left( \tilde{G}_0(y, y|s) \right)^{n-1} \right].
$$

(F.23)

From the above, it is clear that the Laplace transform of the $n$th moment of local time is given by

$$
\langle \ell_n(y) \rangle(s) = \frac{n!}{s} \tilde{G}_0(y, x_0|s) \left( \tilde{G}_0(y, y|s) \right)^{n-1}.
$$

(F.24)
To evaluate \( \tilde{G}_0(y, x_0|s) \) (which we will denote as \( G_s(y) \)), we note that this Green’s function satisfies

\[
\frac{\partial G_0(y, x_0|t)}{\partial t} = D \frac{\partial^2 G_0(y, x_0|t)}{\partial y^2} \quad \text{with} \quad \begin{align*}
G_0(y, x_0|t \to 0) &= \delta(y - x_0) \\
G_0(y \to 0, x_0|t) &= 0 \\
G_0(y \to \infty, x_0|t) &= 0.
\end{align*}
\] (F.25)

Taking the Laplace transform of both sides of the above \((t \to s)\), we write

\[
D \frac{\partial^2}{\partial y^2} \tilde{G}_0(y, x_0|s)(y) = s \tilde{G}_0(y, x_0|s)(y) - \delta(y - x_0).
\] (F.26)

The presence of the Dirac \( \delta \) in the above means that one should solve the above separately for \( y < x_0 \) and \( y > x_0 \), and then patch the two regions together by requiring continuity at \( y = x_0 \), along with a condition on the difference between the slopes on both sides. On performing these calculations, one obtains

\[
\tilde{G}_0(y, x_0|s) = \begin{cases}
\frac{1}{D} \sqrt{\frac{D}{s}} \exp \left( -\sqrt{\frac{s}{D}} x_0 \right) \sinh \left( \sqrt{\frac{s}{D}} y \right), & 0 \leq y \leq x_0 \\
\frac{1}{D} \sqrt{\frac{D}{s}} \exp \left( -\sqrt{\frac{s}{D}} y \right) \sinh \left( \sqrt{\frac{s}{D}} x_0 \right), & x_0 \leq y.
\end{cases}
\] (F.27)

The large-time limit corresponds to the limit \( s \to 0 \), in which case we can expand the above in a series and write

\[
\tilde{G}_0(y, x_0|s) = \min(y, x_0) - \frac{yx_0}{D \sqrt{D}} \sqrt{s} + O(s^{3/2}).
\] (F.28)

From equation (F.24), we see that taking the inverse Laplace transform of \( \tilde{G}/s \) should give us the mean local time at large time. The expression obtained is the same as given in equation (119).

Similarly, the Laplace-transform of the second moment is given by

\[
\langle \tilde{t}^2_1(y) \rangle = \frac{2}{s} \tilde{G}_0(y, x_0|s) \tilde{G}_0(y, x_0|s),
\] (F.29)

\[
\approx 2y \min(y, x_0) - \frac{2y^2 (x_0 + \min(y, x_0))}{D^2 \sqrt{\pi D t}} + O(1)
\] (F.30)

where we have used equation (F.28). Taking the inverse Laplace transform of the above, we obtain the following expression for the second moment:

\[
\langle \tilde{t}^2_1(y) \rangle_{t \gg 1} \approx \frac{2y \min(y, x_0)}{D^2} - \frac{2y^2 (x_0 + \min(y, x_0))}{D^2 \sqrt{\pi D t}} + O(t^{-1}).
\] (F.31)

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