The role of slow manifolds in parameter estimation for a multiscale stochastic system with \( \alpha \)-stable Lévy noise

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This work is about parameter estimation for a fast-slow stochastic system with non-Gaussian \( \alpha \)-stable Lévy noise. When the observations are only available for slow components, a system parameter is estimated and the accuracy for this estimation is quantified by \( p \)-moment with \( p \in (1, \alpha) \), with the help of a reduced system through random slow manifold approximation. This method provides an advantage in computational complexity and cost, due to the dimension reduction in stochastic systems. To numerically illustrate this method, and to corroborate that the parameter estimator based on the reduced slow system is a good approximation for the true parameter value of the original system, a prototypical example is present.

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I. INTRODUCTION

Multi-scale stochastic dynamical systems are ubiquitous in engineering and science. For example, slow and fast surface dynamics often occur in an electrocatalytic oscillator in an interactive way\textsuperscript{1}; dynamics of gene regulatory networks\textsuperscript{2–4} usually evolve on notably different time scales, due to the fact that the production process of mRNA is faster than the protein dynamics. More specifically, the production of mRNA and proteins occur in an unpredictable and intermittent manner. These burst behaviors further contribute to variation or noise in individual cells or cell-to-cell interactions, which have been confirmed by a large number of observations from biological experiments. Such perturbations appear to be appropriate modeled by the non-Gaussian noise. In fact, non-Gaussian random influences are widely observed in many complex nonlinear systems\textsuperscript{5–8}. Thus, it is significant and desirable to investigate two-scale stochastic differential equations (SDEs) under non-Gaussian (in particular, Lévy type) fluctuations.

To make progress in understanding these complex dynamics, it is of a great importance to have a suitable tool for the reduction of such systems and their models to only their slow components, which is often essential for scientific computation and further analysis. The reduction method based on the random slow manifold is one of such effective tool\textsuperscript{9–11}.

We consider a type of reduction method for the multi-stochastic dynamical systems through the random slow manifolds in this paper. The theory of the random slow manifolds could serve as an effective tool for qualitative analysis of dynamical behaviors, as slow manifolds are geometric invariant structures in state space to examine or simplify stochastic dynamics\textsuperscript{11,12}. For fast-slow stochastic dynamical systems in the context of Gaussian random fluctuations, the random slow manifolds have been utilized to investigate effective filtering on a reduced slow system\textsuperscript{13}, provide an accurate estimate on system parameter\textsuperscript{14}, detect the stochastic bifurcation\textsuperscript{15} and understand certain chemical reactions\textsuperscript{16}. The study of the dynamics generated by SDEs with non-Gaussian Lévy noise is still in its infancy, but some interesting works are emerging\textsuperscript{8,17}. Under appropriate conditions, Yuan et al\textsuperscript{7} have obtained low dimensional reduction of fast-slow stochastic dynamical systems driven by $\alpha$-stable Lévy noise via random slow manifolds.

Parameter estimation issue is an important part of the overall multi-scale modeling strategy in a wide variety of applications. In general, when stochastic models are used to describe some certain phenomena, it is important to identify the unknown parameters in this model. For example, it’s of great interest to examine the change rate for low-risk bounds in financial markets\textsuperscript{6}. And we
are often interested in parameter (see \cite{4}), which represents the degradation or production rates of protein and mRNA. Recently, Zhang et al.\cite{8} have devised a parameter estimator for a multi-scale SDEs with Lévy noise by using stochastic averaging principle. In this paper, we develop a different parameter estimation method for multi-scale diffusions with non-Gaussian noise, with the help of the random slow manifolds.

In the present paper, we consider a fast-slow stochastic dynamical system under $\alpha$-stable Lévy noise, but the observations are only available for slow components. By focusing on the reduced slow system on the random slow manifold, we demonstrate that an unknown system parameter in the drift term of such systems could be estimated. And the accuracy for this estimation is quantified by $p$-moment with $p \in (1, \alpha)$. Instead of solving original stochastic systems, this estimation method offers a benefit of dimension reduction in quantifying parameters in stochastic dynamical system. Furthermore, we verify this method and search for the estimated parameter value numerically, with the help of the stochastic Nelder-Mead method for optimization\cite{18}. Finally, we make some remarks in section 5.

This paper is organized as follows. After recalling some facts about random slow manifold and its approximation in the context of Lévy random fluctuations in the next section, we devise a parameter estimator in section 3, by utilizing only observations on the slow component. Moreover, we establish the accuracy for this parameter estimator in terms of observation error and slow reduction error. And then in section 4, we illustrate our estimation method numerically in a specific example. Finally, we give some discussions and comments in a more biological context.

II. PRELIMINARIES

In this section, we recall some facts about Lévy motions, introduce the framework for our reduction method for parameter estimation and present some results on slow manifold and its approximation.

A. Lévy process\cite{19,20}

**Definition II.1.** A stochastic process $L_t$ is a Lévy process if

1. $L_0 = 0$ (a.s.);

2. $L_t$ has independent increments and stationary increments; and

3. $L_t$ has stochastically continuous sample paths, i.e. for every $s \geq 0$, $L_t \rightarrow L_s$ in probability,
as $t \to s$.

We now consider a special but important class of Lévy motions, the $\alpha$-stable Lévy motions, which are defined as follows.

**Definition II.2.** For $\alpha \in (0, 2)$, an $n$-dimensional symmetric $\alpha$-stable process $L^\alpha_t$ is a Lévy process with characteristic function

$$E[\exp(i\langle u, L^\alpha_t \rangle)] = \exp\{-C_1(n, \alpha)|u|^\alpha\}, \text{ for } u \in \mathbb{R}^n$$

with $C_1(n, \alpha) := \pi^{-\frac{1}{2}}\Gamma((1 + \alpha)/2)\Gamma(n/2)/\Gamma((n + \alpha)/2)$.

Here, $\Gamma$ is the Gamma function. A useful fact is that for $\alpha \in (0, 2)$, $E[|L^\alpha_0|^p]$ is finite according as $p \in (0, \alpha)$. We will quantify accuracy of estimation in terms of $p$-moment with $p \in (1, \alpha)$ in the next section.

**B. Framework**

We consider the parameter estimation on $\lambda$ in the parameter space $\Lambda$ which is a closed interval of $\mathbb{R}$ in the following multi-scale stochastic dynamical system

\[
\begin{align*}
\dot{x} & = \frac{1}{\varepsilon}Ax + \frac{1}{\varepsilon}\sigma x - \frac{1}{\sigma}L^\alpha_t, \quad x(0) = x_0 \in \mathbb{R}^n, \quad (II.1) \\
\dot{y} & = By + g(x, y, \lambda), \quad y(0) = y_0 \in \mathbb{R}^m. \quad (II.2)
\end{align*}
\]

The parameter $\varepsilon \ll 1$ represents the ratio of the two time scales. Here $A$ and $B$ are matrices, $f$, $g$ are nonlinear Lipschitz continuous functions with Lipschitz constant $L_f$ and $L_g$ respectively, $\sigma$ is the intensity of noise and $L^\alpha_t$ is a two-sided $\mathbb{R}^n$-valued symmetric $\alpha$-stable Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with index of the stability $1 < \alpha < 2$; refer to [7,19,20]. We remark that if $f$ and $g$ are only locally Lipschitz, but the corresponding deterministic system has a bounded absorbing set, we could obtain a modified system with globally Lipschitz drift by conducting a cut-off of the original system. Throughout the paper, we make the following hypotheses:

**(H1)** There exists positive constants $\beta$, $\gamma$ and $K$, such that for every $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, the following exponential estimates hold:

$$|e^{At}x|_{\mathbb{R}^n} \leq Ke^{-\gamma t}|x|_{\mathbb{R}^n}, \quad t \geq 0; \quad |e^{Bt}y|_{\mathbb{R}^m} \leq Ke^{\beta t}|y|_{\mathbb{R}^m}, \quad t \leq 0.$$

**(H2)** $\gamma > KL_f$. 

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Before using the low dimensional reduction of system (II.1)-(II.2) to estimate parameter $\lambda$, we give some results on random slow manifold and its approximation. We will treat the slow manifold under a driving flow $(\Omega, F, P, \theta)$.

**Definition II.3.** Let $(\Omega, F, P)$ be a probability space. And $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ is a flow on $\Omega$ which is defined as a mapping

$$\theta : \mathbb{R} \times \Omega \mapsto \Omega$$

satisfying

- $\theta_0 = \text{id}(\text{identity})$ on $\Omega$;
- $\theta_{t_1} \theta_{t_2} = \theta_{t_1 + t_2}$ for all $t_1, t_2 \in \mathbb{R}$;
- the mapping $(t, \omega) \mapsto \theta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \otimes F, F)$-measurable, where $\mathcal{B}(\mathbb{R})$ is the collection of Borels sets on the real line $\mathbb{R}$.

Now, introduce an auxiliary system

$$\dot{z} = \frac{1}{\varepsilon} Az + \frac{1}{\varepsilon} \int_{-\infty}^{t} e^{-\varepsilon \frac{A}{s} L_\alpha} \eta_t \omega, \zeta, \sigma \varepsilon \omega_\theta \eta \theta$$

So by (Lemma 3.1), there exists a random variable $\eta_t \omega$ such that $\eta_t \omega = \frac{1}{\varepsilon} \int_{-\infty}^{t} e^{-\varepsilon \frac{A}{s} L_\alpha} \eta_t \omega \zeta, \sigma \varepsilon \omega_\theta \eta \theta$ solves the above equation, where $\theta_t$ is the driving flow defined by $L_\alpha \theta \omega = L_\alpha \theta \omega - L_\alpha \theta \omega$. Set a random transformation

$$(\hat{x}, \hat{y}) = T(\omega, x, y) := \left( x - \sigma \eta^x(\omega), y \right),$$

and then $(\hat{x}(t), \hat{y}(t)) = T(\theta t \omega, x(t), y(t))$ satisfy the following system with random coefficients

$$\dot{\hat{x}}(t) = \frac{1}{\varepsilon} A \hat{x} + \frac{1}{\varepsilon} f(\hat{x}(t) + \sigma \eta^x(\theta t \omega), \hat{y}(t)),$$

$$\dot{\hat{y}}(t) = B \hat{y}(t) + g(\hat{x}(t) + \sigma \eta^x(\theta t \omega), \hat{y}(t), \lambda).$$

The following Lemma comes from (Theorem 4.3).

**Lemma II.1. (Random slow manifold).** Assume that $\varepsilon > 0$ is sufficiently small and (H1)-(H2) hold. Then the fast-slow system (II.1)-(II.2) has a càdlàg random slow manifold $M^\varepsilon(\omega) = \{h^\varepsilon(\zeta, \omega), \zeta \in \mathbb{R}^m\}$. Here, $h^\varepsilon(\zeta, \omega) : \mathbb{R}^m \to \mathbb{R}^n$ is a random nonlinear mapping expressed by

$$h^\varepsilon(\zeta, \omega) = \sigma \eta^\varepsilon(\omega) + \tilde{h}^\varepsilon(\zeta, \omega),$$

with $\tilde{h}^\varepsilon(\zeta, \omega)$ determined as follows,

$$\tilde{h}^\varepsilon(\zeta, \omega) = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{A}{s} \eta_t \omega, \zeta, \sigma \varepsilon \omega_\theta \eta \theta} f(\hat{x}(s, \omega, \zeta) + \sigma \eta^x(\theta_t \omega), \hat{y}(s, \omega, \zeta)) ds, \quad \zeta \in \mathbb{R}^m.$$
In fact, the graph of the random mapping \( \hat{h}_\varepsilon(\zeta, \omega) \) is the random slow manifold for the random system \( (II.4) - (II.5) \).

Next, through the slow manifold \( \mathcal{M}_\varepsilon(\omega) \) and by the same deduction as \( \text{(Corollary 4.1)} \), we could get a reduction system on \( \mathcal{M}_\varepsilon(\omega) \).

**Lemma II.2. (Reduced system on the random slow manifold).** Assume that \( \varepsilon > 0 \) is sufficiently small and \( (H1)-(H2) \) hold. Then through the slow manifold \( \mathcal{M}_\varepsilon(\omega) \), the system \( (II.1)-(II.2) \) can be reduced to a lower dimensional slow stochastic system

\[
\dot{y} = By + g(\hat{h}_\varepsilon(y, \theta_t \omega) + \sigma \xi(\omega), y, \lambda). \tag{II.6}
\]

where \( \xi(\omega) = \int_{-\infty}^{0} e^{-As} dL^a(\omega) \), through which \( \hat{\xi}(\theta_t \omega) = \int_{-\infty}^{t} e^{A(t-s)} dL^a(\omega) \) is the stationary solution of linear system \( \dot{z}(t) = Az + \hat{L}^a_t \).

Here, \( \xi(\omega) \) and \( \eta(\theta_t \omega) \) are identically distributed due to \( \eta(\theta_t \omega) \) and \( \xi(\omega) \) respectively in Lemma 3.2 by \( \xi \), together with the fact that \( \eta(\theta_t \omega) \) is identically distributed with \( \eta(\omega) \).

Finally, by time scaling for the system \( (II.4)-(II.5) \) and a singular perturbation method, we could get a small \( \varepsilon \) approximation for \( \hat{h}_\varepsilon \); see \( \text{7,9,22,23} \).

**Lemma II.3. (An approximate slow manifold).** Assume the hypotheses of Lemma II.2 to be valid. Then there exists an approximate random slow manifolds in distribution for the system \( (II.4)-(II.5) \). More precisely, \( \hat{h}_\varepsilon(\zeta, \omega) = \hat{h}^{(0)}(\zeta, \omega) + \varepsilon \hat{h}^{(1)}(\zeta, \omega) + O(\varepsilon^2) \), where

\[
\hat{h}^{(0)}(\zeta, \omega) = \int_{-\infty}^{0} e^{-As} f(\hat{x}^{(0)}(s) + \sigma \xi(\theta_t \omega), \zeta) ds, \tag{II.7}
\]

and

\[
\hat{h}^{(1)}(\zeta, \omega) = \int_{-\infty}^{0} e^{-As} \left[ f(\hat{x}^{(0)}(s) + \sigma \xi(\theta_t \omega), \zeta) B s \xi + \int_{0}^{s} g(\hat{x}^{(0)}(r) + \sigma \xi(\theta_t \omega), \zeta, \lambda) dr \right] \]
\[
+ f(\hat{x}^{(0)}(s) + \sigma \xi(\theta_t \omega), \zeta) \hat{x}^{(1)}(s) ds. \tag{II.8}
\]

Here, \( \hat{x}^{(0)}(t) \) and \( \hat{x}^{(1)}(t) \) solve the following random differential equations, respectively

\[
d\hat{x}^{(0)}(t) = A\hat{x}^{(0)}(t)dt + f(\hat{x}^{(0)}(t) + \sigma \xi(\theta_t \omega), \zeta) dt, \tag{II.9}
\]
\[
\hat{x}^{(0)}(0) = \hat{h}^{(0)}(0, \omega),
\]
\[
d\dot{x}^{(1)}(t) = \left[ A + f_s(\hat{x}^{(0)}(t) + \sigma \xi(\theta, \omega), \zeta) \right] \dot{x}^{(1)}(t) dt \\
+ f_s(\hat{x}^{(0)}(t) + \sigma \xi(\theta, \omega), \zeta) [B \dot{t} \zeta + \int_0^t g(\hat{x}^{(0)}(s) + \sigma \xi(\theta, \omega), \zeta, \lambda) ds] dt \tag{II.10}
\]
\[
\dot{x}^{(1)}(0) = \hat{h}^{(1)}(0, \omega).
\]

By combining Lemma [II.2] and Lemma [II.3], thus we have an approximated slow system
\[
\dot{y} = By + g(\hat{h}^\varepsilon(y, \theta, \omega) + \sigma \xi(\omega), y, \lambda), \tag{II.11}
\]
where \( \hat{h}^\varepsilon(y, \omega) = \hat{h}^{(0)}(y, \omega) + \varepsilon \hat{h}^{(1)}(y, \omega) \) is a first order approximation of \( \hat{h}^\varepsilon(y, \omega) \). This reduced system can be used to estimate parameter \( \lambda \) in the next section.

### III. PARAMETER ESTIMATION BASED ON A RANDOM SLOW MANIFOLD

In this section, we will estimate the unknown parameter \( \lambda \) in the system [II.1]-[II.2] based on the reduced slow system [II.6] or [II.11]. As we know, it is possible to make a good estimation of \( \lambda \), when observations are available for both components \( x \) and \( y \) (see \([24, 25]\)). In practice, it is often more feasible to observe slow variables than fast variables. So it’s necessary to develop a parameter method by utilizing the observations on slow component \( y \) only, which further reduces the computational complexity.

For the convenience of presentation, we introduce some notations. Denote the observation of the original slow-fast system [II.1]-[II.2] with actual system parameter value \( \lambda_0 \) by \((x_{\lambda_0}^b(t), y_{\lambda_0}^b(t))\), \( t \in [0, T] \), and the observation of the slow system [II.6] with parameter \( \lambda \) and initial value \( y_0 \) by \( y_0^S(t) \). Define the objective function \( F(\lambda) = \mathbb{E} \int_0^T |y_\lambda^S(t) - y_{\lambda_0}^b(t)|^p_{\mathbb{R}^n} dt \) with \( p \in (1, \alpha) \) and assume that there is a unique minimizer \( \lambda_E \in \Lambda \) such that \( F(\lambda_E) = \min_{\lambda \in \Lambda} F(\lambda) \). In fact, this \( \lambda_E \) is our parameter estimator. Besides, we further suppose that \( g(x, y, \lambda) \) is Lipschitz continuous with respect to \( x, y \) and \( \lambda \) with Lipschitz constant \( L_g \) and \( \nabla_{\lambda} g(x, y, \lambda) \) is also a continuous function of \( x, y \) and \( \lambda \). The following theorem can provide an error estimation for this parameter estimation method.

**Theorem III.1.** Set \( H(\lambda_0, \lambda_E) := \mathbb{E} \int_0^{t^*} e^{-Bt} \nabla_{\lambda} g(\hat{h}^\varepsilon(y_\lambda^S(t), \theta, \omega) + \sigma \xi(\omega), y_\lambda^S(t), \lambda') dt |_{\mathbb{R}^n} \), where \( \lambda' = \lambda_0 + \kappa (\lambda_E - \lambda_0) \) with \( \kappa \in (0, 1) \) and choose \( t^* \in [0, T] \) such that \( \int_0^{T} |y_{\lambda_0}^S(t) - y_{\lambda_0}^b(t)|_{\mathbb{R}^n} dt \geq T |y_{\lambda_0}^S(t^*) - y_{\lambda_0}^b(t^*)|_{\mathbb{R}^n} \). Assume that \( \varepsilon > 0 \) is sufficiently small. If \( H(\lambda_0, \lambda_E) > 0 \) for \( \lambda_E \in \Lambda \), then we obtain an
error estimation for $\lambda_E$:

$$|\lambda_0 - \lambda_E| < \frac{1}{H(\lambda_0, \bar{\lambda})} C^\frac{1}{p}(KT^{-\frac{1}{p}} + KL_a T^{\frac{p-1}{p}} + \frac{K^2 L_d L_f}{\sqrt{\gamma} - KL_f} T^{\frac{p-1}{p}}) \cdot \left[ \frac{C_1 \epsilon}{C p} (E|x_0 - \sigma \xi(\omega) - \hat{h}^e(0, \omega)|^p)^\frac{1}{p} + F(\lambda_E)^\frac{1}{p} \right].$$

i.e. $|\lambda_0 - \lambda_E|$ can be controlled by observation error $O((F(\lambda_E))^{\frac{1}{p}})$ and the error due to slow reduction $O(\epsilon)$. Here, $\bar{\lambda} \in \Lambda$ satisfies $H(\lambda_0, \bar{\lambda}) = \max_{\lambda_E \in \Lambda} H(\lambda_0, \lambda_E)$ and $C_1, c$ are positive constants.

**Proof.** For $p \in (1, \alpha)$,

$$\mathbb{E}|y^S_{\lambda_0}(t) - y^S_{\lambda_E}(t)|^p_{\mathbb{R}^m} \leq C(\mathbb{E}|y^S_{\lambda_0}(t) - y^{ob}_{\lambda_0}(t)|^p_{\mathbb{R}^m} + \mathbb{E}|y^S_{\lambda_0}(t) - y^{ob}_{\lambda_E}(t)|^p_{\mathbb{R}^m}) \quad (\text{III.1})$$

holds for some positive constant $C$. By integrating both sides with respect to time and Fubini’s theorem, we get

$$\int_0^T \mathbb{E}|y^S_{\lambda_0}(t) - y^S_{\lambda_E}(t)|^p_{\mathbb{R}^m} dt \leq C(F(\lambda_0) + F(\lambda_E)), \quad (\text{III.2})$$

where $F(\lambda) = \mathbb{E} \int_0^T |y^S_{\lambda_0}(t) - y^{ob}_{\lambda_0}(t)|^p_{\mathbb{R}^m} dt$ by definition.

According to $\mathcal{W}$ (Corollary 4.4), there exist positive constants $C_1$ and $c$ such that for every $t \geq 0$ and a.s. $\omega \in \Omega$, $|y^S_{\lambda_0}(t) - y^{ob}_{\lambda_0}(t)|_{\mathbb{R}^m} \leq C_1 e^{-\frac{t}{T}} |x_0 - \sigma \xi(\omega) - \hat{h}^e(0, \omega)|_{\mathbb{R}^m}$. Thus we have

$$F(\lambda_0) \leq \frac{C^p \epsilon}{c p} (E|x_0 - \sigma \xi(\omega) - \hat{h}^e(0, \omega)|^p)_{\mathbb{R}^m} \quad (\text{III.3})$$

Now we calculate the difference between $y^{ob}_{\lambda_0}(t)$ and $y^S_{\lambda_0}(t)$ to obtain

$$\dot{y}^S_{\lambda_0}(t) - \dot{y}^S_{\lambda_E}(t) = B(y^S_{\lambda_0}(t) - y^S_{\lambda_E}(t)) + [g(\hat{h}^e(y^S_{\lambda_0}(t), \theta_0), \sigma \xi(\omega), y^S_{\lambda_0}(t), \lambda_0)$$

$$- g(\hat{h}^e(y^S_{\lambda_0}(t), \theta_0), \sigma \xi(\omega), y^S_{\lambda_E}(t), \lambda_E)],$$

$$y^S_{\lambda_0}(0) - y^S_{\lambda_E}(0) = 0.$$
and then
\[
\int_0^\tau e^{-Bt} \{ g(\hat{y}_e^S(t), \theta_t \omega) + \sigma \xi(\omega), y_{AE}^S(t), \lambda_0) - g(\hat{y}_e^S(t), \theta_t \omega) + \sigma \xi(\omega), y_{AE}^S(t), \lambda_E) \} dt \\
= - \int_0^\tau e^{-Bt} \{ g(\hat{y}_e^S(t), \theta_t \omega) + \sigma \xi(\omega), y_{AE}^S(t), \lambda_0) - g(\hat{y}_e^S(t), \theta_t \omega) + \sigma \xi(\omega), y_{AE}^S(t), \lambda_0) \} dt \\
+ e^{-B\tau} \{ y_{AE}^S(t') - y_{AE}^S(t') \}.
\]

(III.4)

Furthermore, via taking norm on two sides of (III.4) and using the mean value theorem, it holds that
\[
|\lambda_0 - \lambda_E| \cdot \left| \int_0^\tau e^{-Bt} \nabla_\lambda g(\hat{y}_e^S(t), \theta_t \omega) + \sigma \xi(\omega), y_{AE}^S(t), \lambda') dt \right| \leq \int_0^\tau Ke^{-Bt} \| h_e^S(\lambda_0(t), \theta_t \omega) - h_e^S(\lambda_0(t), \theta_t \omega) \| dt \\
+ Ke^{-Bt} \| y_{AE}^S(t') - y_{AE}^S(t') \| dt
\]
\[
< KLg \int_0^T \| \hat{h}_e^S(\lambda_0(t), \theta_t \omega) - \hat{h}_e^S(\lambda_0(t), \theta_t \omega) \| dt + KLg(T^{p-1} \int_0^T \| y_{AE}^S(t) - y_{AE}^S(t) \| dt) \frac{1}{2}
\]
\[
+ K \| y_{AE}^S(t') - y_{AE}^S(t') \| dt,
\]
(III.5)

where \( \lambda' = \lambda_0 + \kappa(\lambda_E - \lambda_0) \) with \( \kappa \in (0, 1) \) and the last step is based on exponential estimation property, Lipschitz continuity of \( g \), together with Hölder inequality.

Note that \( t' \) satisfies \( \int_0^T \| y_{AE}^S(t') - y_{AE}^S(t') \| dt \geq T \| y_{AE}^S(t') - y_{AE}^S(t') \| dt \), thus by Hölder inequality
\[
\| y_{AE}^S(t') - y_{AE}^S(t') \| dt \leq \frac{1}{T} \int_0^T \| y_{AE}^S(t') - y_{AE}^S(t') \| dt \leq \left( \frac{1}{T} \int_0^T \| y_{AE}^S(t) - y_{AE}^S(t) \| dt \right)^{\frac{1}{2}}.
\]

(III.6)

As mentioned in Section 2, \( \hat{h}_e^S(\lambda_0(t), \theta_t \omega), y_{AE}^S(t) \) and \( \hat{h}_e^S(\lambda_0(t), \theta_t \omega), y_{AE}^S(t) \) satisfy the random system (II.4)-(II.5) with parameter \( \lambda_0, \lambda_E \) and same initial value \( (\hat{h}_e^S(\omega_0, \omega), y_0) \) respectively, due to the definition of slow manifold. Thus, the first term in (III.5) can be handled as follows: via the variation of constants formula, from
\[
\frac{d}{dt} \{ h_e^S(\lambda_0(t), \theta_t \omega) - h_e^S(\lambda_0(t), \theta_t \omega) \}
\]
\[
= \frac{1}{\varepsilon} \{ f(\hat{y}_e^S(t), \theta_t \omega) + \sigma \xi(\omega), y_{AE}^S(t)) - f(\hat{y}_e^S(t), \theta_t \omega) + \sigma \xi(\omega), y_{AE}^S(t)) \}
\]
\[
+ \frac{A}{\varepsilon} \{ h_e^S(\lambda_0(t), \theta_t \omega) - h_e^S(\lambda_0(t), \theta_t \omega) \}
\]
we obtain

\[
\left| \hat{h}^e(y^*_0(t), \theta, \omega) - \hat{h}^e(y^*_E(t), \theta, \omega) \right|_{\mathbb{R}^n} = \frac{1}{\varepsilon} \int_0^T e^{\frac{1}{\varepsilon} (t-s) \left[ f(\hat{h}^e(y^*_0(s), \theta, \omega), y^*_0(s)) - f(\hat{h}^e(y^*_E(s), \theta, \omega), y^*_E(s)) \right] ds} \leq \frac{KL_f}{\varepsilon} \int_0^T e^{\frac{1}{\varepsilon} (t-s) \left[ y^*_0(s) - y^*_E(s) \right]_{\mathbb{R}^m} + \left| \hat{h}^e(y^*_0(t), \theta, \omega) - \hat{h}^e(y^*_E(t), \theta, \omega) \right|_{\mathbb{R}^n} ds.
\]

By Gronwall’s inequality\(^{10}\), we have

\[
e^{\frac{1}{\varepsilon} (t-s) \left[ y^*_0(s) - y^*_E(s) \right]_{\mathbb{R}^m} + \left| \hat{h}^e(y^*_0(t), \theta, \omega) - \hat{h}^e(y^*_E(t), \theta, \omega) \right|_{\mathbb{R}^n} \leq \frac{KL_f}{\varepsilon} \int_0^T e^{\frac{1}{\varepsilon} (t-s) \left[ y^*_0(s) - y^*_E(s) \right]_{\mathbb{R}^m} e^{\frac{KL_f}{\varepsilon} (t-s)} ds,
\]

and thus

\[
\left| \hat{h}^e(y^*_0(t), \theta, \omega) - \hat{h}^e(y^*_E(t), \theta, \omega) \right|_{\mathbb{R}^n} \leq \frac{KL_f}{\varepsilon} \int_0^T \left| y^*_0(s) - y^*_E(s) \right|_{\mathbb{R}^m} e^{\frac{KL_f}{\varepsilon} (t-s)} ds.
\]

And by exchanging the order of integrals, we obtain that

\[
\int_0^T \int_0^T \left| y^*_0(s) - y^*_E(s) \right|_{\mathbb{R}^m} e^{\frac{-KL_f}{\varepsilon} (t-s)} dt ds = \int_0^T \int_0^T \left| y^*_0(s) - y^*_E(s) \right|_{\mathbb{R}^m} e^{\frac{-KL_f}{\varepsilon} (t-s)} ds dt < \frac{\varepsilon}{\gamma - KL_f} \int_0^T \left| y^*_0(s) - y^*_E(s) \right|_{\mathbb{R}^m} ds.
\]

Thus, by Hölder inequality, we obtain that

\[
\int_0^T \left| \hat{h}^e(y^*_0(t), \theta, \omega) - \hat{h}^e(y^*_E(t), \theta, \omega) \right|_{\mathbb{R}^n} dt \leq \frac{KL_f}{\gamma - KL_f} \int_0^T \left| y^*_0(s) - y^*_E(s) \right|_{\mathbb{R}^m} e^{\frac{-KL_f}{\varepsilon} (t-s)} ds dt \leq \frac{KL_f}{\gamma - KL_f} (T^{p-1} \int_0^T \left| y^*_0(t) - y^*_E(t) \right|_{\mathbb{R}^m}^p dt)^\frac{1}{p}
\]

By inserting (III.6) and (III.7) into (III.5), and taking expectation on two sides, and using Hölder inequality \(E(|XY|) \leq [E(|X|^p)]^{\frac{1}{p}} [E(|Y|^q)]^{\frac{1}{q}}\) where \(p, q\) satisfy \(\frac{1}{p} + \frac{1}{q} = 1\), together with (III.2), we have

\[
|\lambda_0 - \lambda_E| \cdot H(\lambda_0, \lambda_E) < (KT^{-\frac{1}{p}} + KL_f T^{\frac{p-1}{p}} + \frac{K^2 L_f L_g^{\frac{1}{p}}}{\gamma - KL_f} T^{\frac{p-1}{p}}) (E \int_0^T |y^*_0(t) - y^*_E(t)|_{\mathbb{R}^m}^p dt)^{\frac{1}{p}} \leq C^\frac{1}{p}(KT^{-\frac{1}{p}} + KL_f T^{\frac{p-1}{p}} + \frac{K^2 L_f L_g^{\frac{1}{p}}}{\gamma - KL_f} T^{\frac{p-1}{p}}) (F(\lambda_0)^{\frac{1}{p}} + F(\lambda_E)^{\frac{1}{p}}) \quad (\text{III.8})
\]
Here, $H(\lambda_0, \lambda_E) = E|\int_0^T e^{-Bt}\nabla_{\lambda g}(\hat{\theta}(y^S_{\lambda}(t), \theta_\omega) + \sigma \xi(\omega), y^S_{\lambda}(t), \lambda') dt|_{\mathbb{R}^n}$ where $\lambda' = \lambda_0 + \kappa(\lambda_E - \lambda_0)$ with $\kappa \in (0, 1)$.

Note that $\Lambda$ is a closed interval of $\mathbb{R}$ and $H(\lambda_0, \lambda_E) > 0$ for $\lambda_E \in \Lambda$, together with the fact that $H(\lambda_0, \lambda_E)$ is continuous with respect to $\lambda_E$ as $\nabla_{\lambda g}(x, y, \lambda)$ is a continuous function of $x, y$ and $\lambda$, then there exists a $\bar{\lambda} \in \Lambda$ such that $H(\lambda_0, \bar{\lambda}) = \max_{\lambda_E \in \Lambda} H(\lambda_0, \lambda_E)$. Hence by inserting (III.3) into (III.8), we obtain the desired error estimation for $\lambda_E$

$$|\lambda_0 - \lambda_E| < \frac{1}{H(\lambda_0, \bar{\lambda})}C_{\epsilon \xi}^2 (KT^{-\gamma} + KL_{\gamma}T^{\frac{\gamma}{\gamma}} + \frac{K^2 L_{\gamma} T_{\gamma}}{\gamma - KL_{\gamma} T_{\gamma}}) \cdot \left( C_{\epsilon \xi}(E|x_0 - \sigma \xi(\omega) - \hat{\theta}(y^0, \omega)|_{\mathbb{R}^n}^p)^{\frac{1}{\gamma}} + F(\lambda_E)^{\frac{1}{\gamma}} \right).$$

In addition, note that

$$\begin{align*}
|\hat{\theta}(y^S_{\lambda_0}(t), \theta_\omega) - \hat{\theta}(y^S_{\lambda_E}(t), \theta_\omega)|_{\mathbb{R}^n} &\leq |\hat{\theta}(y^S_{\lambda_0}(t), \theta_\omega) - \hat{\theta}(y^S_{\lambda_0}(t), \theta_\omega)|_{\mathbb{R}^n} + |\hat{\theta}(y^S_{\lambda_0}(t), \theta_\omega) - \hat{\theta}(y^S_{\lambda_E}(t), \theta_\omega)|_{\mathbb{R}^n} \\
&\quad + |\hat{\theta}(y^S_{\lambda_E}(t), \theta_\omega) - \hat{\theta}(y^S_{\lambda_E}(t), \theta_\omega)|_{\mathbb{R}^n},
\end{align*}$$

(III.9)

Thus, we also can use $\hat{\theta}(y(\omega), \omega)$ instead of $\hat{\theta}(\omega, \theta_\omega)$ to get an estimator with error which is also controlled by $O((F(\lambda_E))^{\frac{1}{\gamma}})$ and $O(\epsilon)$. The proof is complete.

□

We remark that using only observations on slow variables facilitates our method. It is often more feasible to observe slow variables than fast variables. In addition, our method reduces the computational complexity and has an important advantage in computational cost, since this slow system is lower dimensional than the original system. The results established here offer a benefit of dimension reduction in quantifying parameters in stochastic dynamical systems.

### IV. NUMERICAL EXPERIMENTS

In this section, we proceed an example in $\mathbb{R}^2$ to verify our parameter estimation method based on random slow manifolds.\(^{13}\)

Consider the following fast-slow stochastic system

$$\begin{align*}
\dot{x} &= -\frac{1}{\epsilon} x + \frac{1}{4\epsilon} \cos(y) + \sigma \epsilon^{-\frac{1}{2}} L_{\gamma}^\alpha, \quad x(0) = x_0 \in \mathbb{R}, \\
\dot{y} &= y + \frac{1}{4} \sin(\lambda x), \quad y(0) = y_0 \in \mathbb{R},
\end{align*}$$

(IV.1) (IV.2)
where $\lambda$ is a real unknown positive parameter, $A = -1$, $B = 1$, $f(x, y) = \frac{1}{4}\cos y$, $g(x, y, \lambda) = \frac{1}{4}\sin(\lambda x)$. It is easy to justify that $A$, $B$, $f$, $g$ satisfy (H1)-(H2) with $K = \gamma = \beta = 1$, $L_f = L_g = \frac{1}{4}$. Thus, the proposed method of this paper is applicable.

By the random transformation (II.3) and Lemma II.1 there exists $\hat{h}^e$ satisfying

$$\hat{h}^e(\zeta, \omega) = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{s} \frac{1}{4} \cos(\hat{y}(s, \omega, \zeta)) ds, \quad \zeta \in \mathbb{R}.$$ (IV.3)

In fact, $\hat{h}^e(\zeta, \omega)$ has an approximation $\hat{h}^e(\zeta, \omega) = \hat{h}^{(0)}(\zeta, \omega) + \varepsilon \hat{h}^{(1)}(\zeta, \omega)$ with error $O(\varepsilon^2)$ by the Lemma II.3. Here, $\hat{h}^{(0)}(\zeta, \omega) = \frac{1}{4}\cos \zeta$ and $\hat{h}^{(1)}(\zeta, \omega)$ has an explicit expression,

$$\hat{h}^{(1)}(\zeta, \omega) = \frac{1}{4}\zeta \sin \zeta - \frac{1}{16} \sin \zeta \int_{-\infty}^{0} e^{t} \int_{0}^{s} \sin \left(\frac{1}{4}\lambda \cos \zeta + \lambda \sigma \int_{-\infty}^{s} e^{-(s-r)} dL^\gamma_r(\omega) dt \right) ds dt.$$ 

So the approximated slow system is

$$\dot{y} = y + \frac{1}{4} \sin(\lambda [\hat{h}^e(y, \theta t \omega) + \sigma \xi(\omega)]),$$ (IV.4)

with $\xi(\omega) = \int_{-\infty}^{0} e^{s} dL^\gamma_s(\omega)$.

In the following numerical simulations, we use a stochastic Nelder-Mead method to estimate unknown parameter $\lambda$ in (IV.4). The main idea is as follows: we determine the estimated parameter value by minimizing the objective function $F(\lambda) = \mathbb{E} \sum_{i=1}^{K} \sum_{j=0}^{L} |y^{ij}_{ob} - y^{ij}_s(\lambda)|^p$, where $\{y^{ij}_{ob} : y^{ij}_s(t_i, \lambda_0), i = 0, 1, ..., L; j = 1, 2, ..., K\}$ are $K$ different only observations of the slow component $y$ from the original system with parameter value $\lambda_0$, and observations $\{y^{ij}_s : y^{ij}_s(t_i, \lambda), i = 0, 1, ..., L; j = 1, 2, ..., K\}$ are generated from reduced system corresponding to parameter $\lambda$. These data are available by using Euler-Maruyama method.

As shown in fig 1, we see that the reduced system on random slow manifold is a good approximation of the slow variable $y$ of the original system. By just about 10 iteration in the stochastic Nelder-Mead method, we get estimated parameter $\lambda_E \approx 0.9996$, which indicates that our estimator based on random slow manifolds is a good approximation for the true parameter value.

We remark that the parameter method established here can be used to examine complex physical or biological dynamics, although we illustrate this point by a simple two-dimensional example here. For instance, by the same deduction as here, the unknown parameter in Shimizu-Morioka model under stochastic fluctuations could be determined, see the reference in 26. The parameter in this physical model can capture stochastic bifurcation behaviors. In addition, we are often interested in parameter (see 4), which represents change rate for mRNA.
V. CONCLUSIONS AND DISCUSSION

We developed a parameter estimation method based on a fast-slow stochastic dynamical system by using the random slow manifolds. Instead of solving original systems, we can accurately estimate the unknown parameter only by the observation of the slow component, which offers a benefit of computational cost. The results established here can be used to examine biological dynamics, such as stochastic chemical kinetics, where we are more interested in the change rate for mRNA\(^{2-4}\).

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