Quickest Bayesian and non-Bayesian detection of false data injection attack in remote state estimation

Akanshu Gupta Abhinava Sikdar Arpan Chattopadhyay

Abstract—In this paper, quickest detection of false data injection attack on remote state estimation is considered. A set of $N$ sensors make noisy linear observations of a discrete-time linear process with Gaussian noise, and report the observations to a remote estimator. The challenge is the presence of a few potentially malicious sensors which can start strategically manipulating their observations at a random time in order to skew the estimates. The quickest attack detection problem for a known linear attack scheme in the Bayesian setting with a Geometric prior on the attack initiation instant is posed as a constrained Markov decision process (MDP), in order to minimize the expected detection delay subject to a false alarm constraint, with the state involving the probability belief at the estimator that the system is under attack. State transition probabilities are derived in terms of system parameters, and the structure of the optimal policy is derived analytically. It turns out that the optimal policy amounts to checking whether the probability belief exceeds a threshold. Next, a generalised CUSUM based attack detection algorithm is proposed for the non-Bayesian setting where the attacker chooses the attack initiation instant in a particularly adversarial manner. It turns out that computing the statistic for the generalised CUSUM test in this setting relies on the same techniques developed to compute the state transition probabilities of the MDP. Numerical results demonstrate significant performance gain under the proposed algorithms against competing algorithms.

Index Terms—Secure estimation, CPS security, false data injection attack, quickest detection, Markov decision process.

I. INTRODUCTION

Networked estimation and control of physical processes and systems are indispensable components of cyber-physical systems (CPS) that involve integration of sensing, computation, communication and control to realize the ultimate combining of the physical systems and the cyber world. The applications of CPS are many-fold: intelligent transportation systems, smart grids, networked monitoring and control of industrial processes, environmental monitoring, disaster management, etc. These applications heavily depend on reliable estimation of a physical process or system via sensor observations collected over a wireless network [2]. However, malicious attacks on these sensors pose a major security threat to CPS. One such common attack is a denial-of-service (DoS) attack where the attacker attempts to block system resources (e.g., wireless jamming attack [3]). Contrary to DoS, we focus on false data injection (FDI) attacks which is a specific class of integrity or deception attacks, where the sensor observations are modified before they are sent to the remote estimator [4], [5]. The attacker can modify the information by breaking the cryptography of the packets or by physically manipulating the sensors (e.g., placing a cooler near a temperature sensor).

Recently, the problem of FDI attack and its countermeasures has received significant attention [6]. The literature in this area can be classified into two categories: (i) FDI on centralized systems, and (ii) FDI on distributed systems. In a distributed system, various entities such as sensors, estimators, controllers and actuators are connected via a multi-hop wireless network, and FDI on one such component propagates to other components over time via the network.

There has been a vast literature on FDI in centralized systems, particularly in the remote estimation setting. Such works include several attempts to characterize and design attack schemes: undetectable linear deception attack [7] in single sensor context, and also conditions for undetectable FDI attack [8]. The attack strategy to steer the control of CPS to a desired value under attack detection constraint is provided in [9]. Literature on attack detection considered a number of models and approaches; e.g., attack detection schemes for noiseless systems [10], comparing the observations from a few known safe sensors against potentially malicious sensor observations [11] to tackle the attack of [7], coding of output of sensors along with $\chi^2$ detector [12], Gaussian mixture model based detection and secure state estimation [13], and the innovation vector based attack detection and secure estimation schemes [14]. There have also been a number of works on secure estimation: see [15] for bounded noise case, [16] for sparsity models to characterize the location switching attack in a noiseless system and state recovery constraints, [17]...
for noiseless systems, [18]–[20] for linear Gaussian process and linear observation with Gaussian noise. Attack detection, estimation and control for power systems are addressed in [21]–[23]. Attack-resilient control under FDI for noiseless systems is discussed in [24].

Though relatively new, FDI on distributed systems is also increasingly being investigated; see [25] for attack detection and secure estimation, [26] for attack detection in networked control system via dynamic watermarking, [27] for distributed Krein space based attack detection in discrete time-varying systems, and [28] for distributed attack detection in power systems. On the other hand, attack design for distributed CPS is also being investigated; see [29] for linear attack design against distributed state estimation to push all nodes’ estimates to a desired target under a given attack detection constraint, [30] for attack design to maximize the network-wide estimation error via simple Gaussian noise addition, [31] for conditions for perfect attack in a distributed control system and design algorithms for perfect and non-perfect attacks, etc.

While there have been several schemes (such as [11] and the χ² detector) to detect FDI attack in a multi-sensor setting, the optimal attack detector is not yet known even for the centralized systems. Moreover, the popular χ² detector fails if the attacker judiciously injects false data in such a way that the innovation sequence of the Kalman state estimator remains constant over time. In light of this issue, our main contributions in this paper are as follows:

1) In a multi-sensor setting with some safe and some potentially unsafe sensors, for the linear attack of [7] with known attack parameters, we develop an optimal Bayesian attack detector that minimizes the mean delay in attack detection subject to a constraint on the false alarm probability. The problem is formulated as a partially observable Markov decision process (POMDP), and the optimal policy turns out to be a simple threshold policy on the belief probability that an attack has already been launched.

2) Though POMDP based Bayesian change detection techniques exist in the literature [32], our problem involves a far more complicated state that consists of the belief and the collection of past Kalman innovations; this occurs primarily because of the temporal dependence of innovation sequence after the attack and the uncertainty in the attack initiation instant. Also, we do not apply Shiryaev’s test directly because that would be optimal only for α → 0 and not necessarily for all α, where α is the probability of false alarm.

3) Computing the belief probability recursively is a challenging problem due to the complicated temporal dependence of the innovation sequence available to the estimator, and we solve this by modeling the post-attack state estimation process as a Kalman filter with a modified process and observation model. Our results in Section III explicitly show how to compute the true post-attack innovation distribution.

4) In order to reduce computational complexity, we propose a sub-optimal algorithm called QUICKDET whose structure is motivated by the optimal policy derived from the POMDP formulation. QUICKDET applies a constant threshold rule on the belief, while the POMDP formulation yields a time-varying threshold. We provide a simulation-based technique to optimize this constant threshold, by using tools from simultaneous perturbation stochastic approximation (SPSA [33]) and two timescale stochastic approximation [34]. Numerical results show that this sub-optimal algorithm significantly outperforms competing algorithms.

5) The Bayesian quickest FDI detection algorithm is also extended to the case where the attacker can launch an attack using multiple possible linear attack strategies.

6) In the non-Bayesian setting, we adapt the generalized CUSUM test from the literature for quickest detection of FDI. It turns out that the test statistic for generalized CUSUM in our problem can be computed by using the same tools developed in Section III.

The rest of the paper is organized as follows. The system model is described in Section II. Section III develops the necessary theory for calculating the state transition probabilities for the POMDP formulation. POMDP formulation for FDI attack detection in the Bayesian setting with known linear attack parameters is provided in Section IV. Quickest attack detection algorithm for the non-Bayesian setting is provided in Section V. Numerical results are provided in Section VI followed by the conclusions in Section VII. All proofs are provided in the appendices.

II. SYSTEM MODEL

In this paper, bold capital letters, bold small letters, and capital letters with calligraphic font will denote matrices, vectors and sets respectively. For any two square matrices $M_1$ and $M_2$, the block diagonalization operator is defined by

$\text{Blkdiag}(M_1,M_2) = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$. Also, the transpose of a matrix $M$ is denoted by $M'$.

A. Sensing and estimation model

We consider a set of sensors $\mathcal{N} = \{1, 2, \ldots, N\}$ sensing a discrete-time process $\{x_k\}_{k \geq 1}$ which is a linear Gaussian process with the following dynamics:

$$x_{k+1} = Ax_k + w_k$$

(1)

where $x_k \in \mathbb{R}^{q \times 1}$ is vector-valued, $A \in \mathbb{R}^{q \times q}$ is the process matrix, and $w_k \sim \mathcal{N}(0, Q)$ is the Gaussian process noise i.i.d. across $k$. The observation made by sensor $i$ at time $k$ is:

$$y_{k,i} = C_i x_k + v_{k,i}$$

(2)

where $C_i$ is a matrix of appropriate dimensions and $v_{k,i} \sim \mathcal{N}(0, R_i)$ is the Gaussian observation noise at sensor $i$ at time $k$, which is i.i.d. across $k$ and independent across $i$. We assume that $(A, Q^{\frac{1}{2}})$ is stabilizable and $(A, C_i)$ is detectable for all $i \in \mathcal{N}$. The complete observation from all sensors at time $k$ is denoted by $y_k = (y_{k,1}, y_{k,2}, \ldots, y_{k,N})'$ which can be written as:

$$y_k = Cx_k + v_k$$

(3)
where \( C = (C'_1, C'_2, \ldots, C'_N)' \) is the equivalent observation matrix and \( v_k \) is the zero mean Gaussian observation noise with covariance matrix \( \text{Blkdiag}(R_1, R_2, \ldots, R_N) \).

### B. Process estimation under no attack

For the model in (4), a standard Kalman filter [35] is used to estimate \( \hat{x}_k \) from \( \{ y_j \}_{j \leq k} \) in order to minimize the mean squared error (MSE):

\[
\hat{x}_{k+1|i} = A \hat{x}_k \\
P_{k+1|i} = AP_k A' + Q \\
K_{k+1} = P_{k+1|i} C' (C P_{k+1|i} C' + R)^{-1} \\
\hat{x}_{k+1} = \hat{x}_{k+1|i} + K_{k+1} (y_{k+1} - C \hat{x}_{k+1|i}) \\
P_{k+1} = (I - K_{k+1} C) P_{k+1|i},
\]

(4)

where \( \hat{x}_{k+1} = \mathbb{E}(x_{k+1}|y_0, y_1, \ldots, y_{k+1}) \) is the MMSE estimate and \( P_{k+1} \) is the error covariance matrix for this estimate. From [35], we know that \( \lim_{k \to \infty} P_{k+1|i} = P \) exists and is the unique fixed point to the Riccati equation which is basically the \( P_{k+1|i} \) iteration. Let us also define the innovation vector from sensor \( i \) at time \( k \) as \( z_{k,i} = y_{k,i} - C \hat{x}_{k|i} \), and the collective innovation as \( z_k = \sum_{i \in I} z_{k,i} \). It is well-known [35] that \( \{ z_k \}_{k \geq 1} \) is a zero-mean Gaussian sequence independent across time and whose steady-state covariance matrix is \( \Sigma_z = (C P C' + R) \). Hence, in most cases, the residue-based \( \chi^2 \) detector is used at the remote estimator's side for any attack or anomaly detection, which has the following mathematical form:

\[
\sum_{k=\tau-J+1}^\tau z_k' \Sigma_z^{-1} z_k \geq H_1
\]

(5)

where the null Hypothesis \( H_0 \) represents no attack or anomaly, and \( H_1 \) represents the presence of attack or anomaly. Here \( J \) is a pre-specified window size and \( \eta \) is a pre-specified threshold which can be tuned to control the false alarm probability.

### C. FDI attack

If a sensor \( i \) is under FDI attack, the observation sent to the remote estimator from this sensor becomes:

\[
y_{k,i} = C_i x_k + v_{k,i} + e_{k,i}
\]

(6)

where \( e_{k,i} \) is the false data injected by sensor \( i \) at time \( k \). Consequently, (3) is modified to:

\[
y_k = C x_k + v_k + e_k
\]

(7)

Let us recall linear attacks for the single sensor case [7] where, at time \( k \), the malicious sensor modifies the innovation as \( \tilde{z}_k = T z_k + b_k \), where \( T \) is a square matrix and \( b_k \sim N(0, \Sigma_b) \) is i.i.d. Gaussian random vector sequence. The authors of [7] had shown that \( \tilde{z}_k \sim N(0, \Sigma_{\tilde{z}}) \) under steady state, where \( \Sigma_{\tilde{z}} = T \Sigma_z T' + \Sigma_b \). Hence, by ensuring \( \Sigma_{\tilde{z}} = \Sigma_z \), one can preserve the distribution of \( \{ z_k \}_{k \geq 1} \) in the single sensor case, and hence the detection probability will remain same even under FDI, under the \( \chi^2 \) detector. It was also shown in [7] that inverting the sign of the innovation (i.e., \( T = -I \) and \( b_k = 0 \)) maximizes the MSE. Obviously, \( T = I \) and \( b_k = 0 \) imply that there is no attack.

In this paper, we assume that there is a set of sensors \( S \subset N \) which are safe, i.e., the sensor belonging to \( S \) can not be attacked. Let the set of potentially unsafe sensors be denoted by \( A \triangleq N - S \). Clearly, a general \( T \) matrix as the single sensor case will not be useful here; instead, we assume that \( T = \text{Blkdiag}(T_S, T_A) = \text{Blkdiag}(\text{Blkdiag}(T_{S}, T_A)) \) where, for \( i \in S \), the added noise \( b_{i,i} = 0 \) for all time \( k \geq 1 \), and, for \( i \in A \), we have \( b_{i,i} \sim N(0, \Sigma_{b,i}) \) i.i.d. across \( k \). Consequently, the modified innovation at node \( i \in A \) at time \( k \) is given by:

\[
\tilde{z}_{k,i} = T_i z_{k,i} + b_{k,i}
\]

(8)

We define \( z_{k,A} = \tilde{z}_{k,A}, y_{k,A}, y_{k,A}, C_A \) the components of \( z_k, \tilde{z}_k, y_k, C \) coming from sensors of \( A \); similar notation is used for the components corresponding to the safe sensors \( S \) also. We also define by \( \hat{x}_k \) the estimate at time \( k \) generated by a Kalman filter under FDI attack. The history available to the remote estimator at time \( k \) is denoted by \( H_k = \{ z_j : j \leq k \} \).

We also define \( \hat{x}_k \) as the estimate of \( x_k \) by applying a standard Kalman filter on \( H_k \). Also, let \( \tilde{z}_{k,A} = \mathbb{E}(\tilde{x}_k|H_k, a) \) denote the MMSE estimate of \( x_k \) given \( H_k \) and the event that there is no attack has already happened.

For the multi-sensor setting, [11] proposed an attack detection algorithm under the presence of a few known safe sensors. Unfortunately, the detection algorithm of [11] and other detection algorithms from the literature do not have any optimality proof. On the other hand, if the attacker ensures a constant \( \tilde{z}_k \) for all \( k \) such that \( \sum_{k=\tau-J+1}^\tau z_k' \Sigma_z^{-1} z_k < \eta \), then the \( \chi^2 \) detector fails. These inadequacies in the literature motivate the quickest attack detection problem formulation in this paper.

### III. Recovering true innovation and estimates from FDI attacked observations

In this section, we will develop a technique for computing \( \hat{x}_{k,A} \) under the assumption that the attack initiation instant \( t \) is known; this will also yield the statistics of \( \tilde{z}_k \). These results are necessary to calculate the state transition probabilities of the POMDP formulation in Section IV. These results also show that linear attack changes the distribution of innovations and estimates in a multi-sensor scenario with a few safe sensors. Throughout this section, we assume that \( T \) is known to the remote estimator.

#### A. Computing \( \tilde{x}_{k,A} \)

The innovation from the unsafe sensors for \( k \geq t \) is:

\[
\tilde{z}_{k,A} = T_A (y_{k,A} - C_A \hat{x}_{k-1}) + b_k \\
\Longrightarrow T_A^{-1} \tilde{z}_{k,A} = (y_{k,A} + T_A^{-1} b_k) - C_A \hat{x}_{k-1} \\
\tilde{y}_{k,A} = \hat{y}_{k,A} - C_A \hat{x}_{k-1}
\]

(9)

where \( \tilde{y}_{k,A} = y_{k,A} + T_A^{-1} b_k, k \geq t \).

It is important to note that, for \( k \geq t \), the estimator does not observe \( y_{k,A} \). However, after attack, the estimator can calculate \( \tilde{y}_{k,A} \) from (9), since the estimator knows \( \tilde{z}_{k,A} \). Hence,
we can define a new observation model for the estimator as follows:
\[
\begin{align*}
\tilde{y}_{k,S} &= C_S x_k + v_{k,S} = y_{k,S} \\
\tilde{y}_{k,A} &= C_A x_k + v_{k,A} + T_A^{-1} b_k \\
\implies \tilde{y}_k &= C x_k + \tilde{v}_k
\end{align*}
\] (10)

where the observation noise from sensors belonging to \( A \) becomes \( \tilde{v}_{k,A} \equiv v_{k,A} + T_A^{-1} b_k \), \( \tilde{y}_k \equiv \begin{bmatrix} \tilde{y}_{k,S} \\ \tilde{y}_{k,A} \end{bmatrix} \), \( \tilde{v}_k \equiv \begin{bmatrix} \tilde{v}_{k,S} \\ \tilde{v}_{k,A} \end{bmatrix} \), and the covariance matrix of \( \tilde{v}_k \) becomes \( \tilde{R} \equiv \begin{bmatrix} R_S & 0 \\ 0 & R_A + T_A^{-1}\Sigma_a T_A^{-1} \end{bmatrix} \).

It is to be noted that the estimator can use this model only for \( k \geq t \), when it knows that event \( a \) is true, i.e., an attack has already happened. Basically, for \( k \geq t \), the estimator can compute \( \hat{x}_{k|a} \) by using a standard Kalman filter under this new observation model. In this connection, we would also like to point out the connection between the innovation for this modified observation model and the innovation for the original observation model, both under attack:
\[
\begin{align*}
\tilde{z}_{k,S} &= y_{k,S} - C_S A \hat{x}_{k-1|a} \\
&= y_{k,S} - C_S A \hat{x}_{k-1} + C_S A (\hat{x}_{k-1} - \hat{x}_{k-1|a}) \\
\tilde{z}_{k,A} &= y_{k,A} - C_A \hat{x}_{k-1|a} - C_A (\hat{x}_{k-1} - \hat{x}_{k-1|a}) \\
&= T_A^{-1} \tilde{z}_{k,A} + C_A (\hat{x}_{k-1} - \hat{x}_{k-1|a})
\end{align*}
\] (11)

Now, \( \hat{x}_{k|a} \) can be computed via using standard Kalman filter equations:
\[
\begin{align*}
\hat{P}_{k|k-1} &= A \hat{P}_{k-1|k-1} A' + Q \\
\hat{K}_k &= \hat{P}_{k|k-1} C' (C \hat{P}_{k|k-1} C' + \tilde{R})^{-1} \\
\hat{x}_{k|a} &= A \hat{x}_{k-1|a} + \hat{K}_k \begin{bmatrix} \tilde{z}_{k,S} \\ \tilde{z}_{k,A} \end{bmatrix} \\
\hat{P}_k &= (I - \hat{K}_k C) \hat{P}_{k|k-1}
\end{align*}
\] (12)

Note that, since this modified observation model makes sense only for \( k \geq t \), the distribution of \( \tilde{z}_k \) for \( k \geq t \) depends on \( t \).

B. Computing the conditional distribution of \( \tilde{z}_k \)

From (11), we can write:
\[
\tilde{z}_{k,S} = \tilde{z}_{k,S} + C_S A (\tilde{x}_{k-1|a} - \tilde{x}_{k-1})
\] (13)

The innovation sequences \( \tilde{z}_{k,S} \) and \( \tilde{z}_{k,A} \) are Gaussian noise with zero mean and covariance \( C_S \hat{P}_{k|k-1} C_S' + \tilde{R}_S \) and \( C_A \hat{P}_{k|k-1} C_A' + \tilde{R}_A \) respectively. Hence, the conditional distribution of \( \tilde{z}_{k,S} \) will be \( N(C_S \hat{A} (\tilde{x}_{k-1|a} - \tilde{x}_{k-1}), C_S \hat{P}_{k|k-1} C_S' + \tilde{R}_S) \) given all necessary quantities involved, where \( \tilde{R}_S = R_S \). Using (11), conditional distribution of \( \tilde{z}_{k,A} \) is \( N(T_A \hat{C}_A \hat{A} (\tilde{x}_{k-1|a} - \tilde{x}_{k-1}), T_A \hat{C}_A \hat{P}_{k|k-1} C_A' + \tilde{R}_A) \).

IV. ATTACK DETECTION FOR KNOWN LINEAR ATTACK: BAYESIAN SETTING

In Section III we have shown that the distribution of the innovation changes under a linear attack. In this section, we provide a quickest change detection algorithm for detecting a linear attack with known \( T \) matrix. Motivated by the theory of \([36]\), we formulate the problem as a partially observable Markov decision process (POMDP, see \([37]\) Chapter 5) and derive the optimal attack detection policy.

A. Problem statement

We assume that the discrete time starts at \( k = 0 \), and the attack begins at a random time \( t \) which is modeled as a geometrically distributed random variable with mean \( \frac{1}{\theta} \), where \( \theta \in (0, 1) \) is known to the remote estimator. Clearly, \( \theta \) denotes the probability that, given that no attack has started up to time \( k \), the attacker launches an attack at time \( (k+1) \). This assumption allows us to formulate the quickest attack detection problem as a POMDP.

The two hypotheses considered here are the following:
\( H_0 \): there is no attack.
\( H_1 \): there is an attack.

At each \( k \geq 0 \), the remote estimator computes \( z_k \) (or \( \tilde{z}_k \) provided that there is an attack). At some (possibly random) time \( \tau \), the estimator decides to stop collecting observations \( y_k \) (or \( \tilde{y}_k \) provided that there is an attack), and declares that an attack has been launched on \( A \). This stopping time \( \tau \) is determined by a policy (a sequence of decision rules) \( \mu = \{ \mu_k \}_{k \geq 0} \), where \( \mu_k \) is a function that takes the history of observations available to the estimator at time \( k \) and decides whether to declare that an attack has been launched. Clearly, \( \tau < t \) denotes the event of a false alarm.

We seek to minimize the expected delay in detecting an attack subject to a constraint on the false alarm probability:
\[
\min_{\mu} \mathbb{E}_\mu[(\tau - t)^+] \quad \text{s.t.} \quad \mathbb{P}_\mu(\tau < t) \leq \alpha
\] (14)

This constrained problem can be relaxed using a Lagrange multiplier \( \lambda > 0 \) to obtain the following unconstrained problem:
\[
L(\mu) = \mathbb{E}_\mu[(\tau - t)^+] + \lambda \mathbf{1}_{\{\tau < t\}}
\] (15)

The following standard results tells us how to choose \( \lambda \):

**Theorem 1.** Let us consider \( \mu \) and its relaxed version \( \mu^* \). If there exists a \( \lambda^* \geq 0 \) and a policy \( \mu^*(\lambda^*) \) such that, (i) \( \mu^*(\lambda^*) \) is an optimal policy for \( L(\lambda^*) \) under \( \lambda^* \), and (ii) the constraint in \( \mu^* \) is met with equality under \( \mu^*(\lambda^*) \), then \( \mu^*(\lambda^*) \) is an optimal policy for the constrained problem \( L(\mu) \).

B. POMDP formulation

Let us define the belief probability of the estimator that an attack has been launched at or before time \( k \), as \( \pi_k = \)
\( \mathbb{P}(t \leq k | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k) \), where \( \tilde{z}_k = z_k \) if \( t > k \). Under this notation, (15) can be rewritten as:

\[
L(\mu) = \mathbb{E}_\mu \left[ \sum_{k=0}^{\tau-1} \mathbf{1}_{(t \leq k)} + \lambda(1 - \pi_t) \right] = \mathbb{E}_\mu \left[ \sum_{k=0}^{\tau-1} \pi_k + \lambda(1 - \pi_t) \right] \tag{16}
\]

For change detection, typically \( \pi_k \) is a sufficient statistic for decision-making at time \( k \), but that does not hold in our problem due to temporal dependence of the innovations after an attack is launched. Hence, we formulate a POMDP with state at time \( k \) given by \((\pi_k, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k)\), with the understanding that \( \tilde{z}_k = z_k \) if \( t > k \). The set of possible control actions is given by \( \mathcal{U} = \{0, 1\} \), where action 0 represents continuing to collect observations, and action 1 stands for stopping and declaring \( H_1 \).

Further, based on (16), the single stage cost at time \( k \) is:

\[
\begin{align*}
    c_k(\pi_k, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k, u_k) = \begin{cases} 
\pi_k & \text{if } u_k = 0 \\
\lambda(1 - \pi_k) & \text{if } u_k = 1
\end{cases}
\end{align*}
\]

### C. Recursive calculation of \( \pi_k \)

We need to compute \( \pi_k \) from \( \pi_{k-1} \) recursively, in order to be able to calculate state transitions. However, after attack, \( \tilde{z}_k \) sequence ceases to be i.i.d. across \( k \), which makes this recursive calculation non-trivial. Note that, the joint probability density of the innovations computed at the estimator:

\[
\begin{align*}
    p(\tilde{z}_1, \ldots, \tilde{z}_k) &= p(\tilde{z}_1) p(\tilde{z}_2 | \tilde{z}_1) \cdots p(\tilde{z}_k | \tilde{z}_1, \ldots, \tilde{z}_{k-1}) \\
    \text{Let us define } p_\ast(\tilde{z}_i) &= p(\tilde{z}_i | \mathcal{H}_i-1). \text{ Now, using Bayes rule, we can write:}
\end{align*}
\]

\[
\begin{align*}
    \pi_k &= \frac{p(\tilde{z}_1, \ldots, \tilde{z}_k | t \leq k) \times \mathbb{P}(t \leq k)}{p(\tilde{z}_1, \ldots, \tilde{z}_k)} \\
    &= \frac{\mathbb{P}(t \leq k) \Pi_{i=1}^{k-1} p_\ast(\tilde{z}_i | t \leq k) + \mathbb{P}(t > k) \Pi_{i=1}^{k-1} p_\ast(\tilde{z}_i | t > k)}{\beta_k} \\
    &= \frac{\beta_k}{\beta_k + 1}
\end{align*}
\]

where,

\[
\begin{align*}
    \beta_k &= \frac{\mathbb{P}(t \leq k) \Pi_{i=1}^{k-1} p_\ast(\tilde{z}_i | t \leq k)}{\mathbb{P}(t > k) \Pi_{i=1}^{k-1} p_\ast(\tilde{z}_i | t > k)} \\
    &= f_k(\theta) \frac{\mathbb{P}(t \leq k-1) p_\ast(\tilde{z}_k | t \leq k) \Pi_{i=1}^{k-1} p_\ast(\tilde{z}_i | t \leq k)}{\mathbb{P}(t > k-1) p_\ast(\tilde{z}_k | t > k) \Pi_{i=1}^{k-1} p_\ast(\tilde{z}_i | t > k)}
\end{align*}
\]

and,

\[
\begin{align*}
    f_k(\theta) &= \frac{1 - (1 - \theta)^k}{(1 - \theta)(1 - (1 - \theta)^{k-1})}
\end{align*}
\]

Now, note that:

\[
\begin{align*}
    \Pi_{i=1}^{k-1} p_\ast(\tilde{z}_i | t \leq k) &= p(\tilde{z}_1, \ldots, \tilde{z}_{k-1} | t \leq k) \\
    &= p(\tilde{z}_1, \ldots, \tilde{z}_{k-1} | t \leq k-1) \mathbb{P}(t \leq k-1 | t \leq k) \\
    &\quad + p(\tilde{z}_1, \ldots, \tilde{z}_{k-1} | t = k) \mathbb{P}(t = k | t \leq k) \\
    &= \frac{1 - (1 - \theta)^k - 1}{1 - (1 - \theta)^k} p(\tilde{z}_1, \ldots, \tilde{z}_{k-1} | t \leq k) \\
    &\quad + \frac{\theta(1 - \theta)^k}{1 - (1 - \theta)^k} \Pi_{i=1}^{k-1} p(z_i)
\end{align*}
\]

where the product form in the last line comes from the fact that innovations are i.i.d. before the attack is launched. Clearly, (20) allows us to calculate the numerator of (18). Similarly, in order to calculate the denominator of (18), we note that:

\[
\Pi_{i=1}^{k-1} p_\ast(\tilde{z}_i | t > k - 1) = \Pi_{i=1}^{k-1} p(z_i)
\]

This again holds because innovations are i.i.d. before the attack is launched.

Using (20) and (21) in (18) and upon simplification, we obtain:

\[
\begin{align*}
    \beta_k &= \frac{p_\ast(\tilde{z}_k | t \leq k)}{p_\ast(\tilde{z}_k | t > k)} \frac{\beta_{k-1}^{-1} + \theta}{(1 - \theta)} \\
    &= p_\ast(\tilde{z}_k | t \leq k) \frac{\beta_{k-1}^{-1} + \theta}{(1 - \theta)(1 - \pi_{k-1}) + \theta}
\end{align*}
\]

Obviously, \( \pi_k \) can be calculated from \( \beta_k \) using (17).

It is important to note that, using the results from Section III-B \( p_\ast(\tilde{z}_k | t \leq k) \) can be computed as:

\[
\sum_{i=0}^{k} p_\ast(\tilde{z}_k | t = i) \mathbb{P}(t = i | t \leq k)
\]

Here \( p_\ast(\tilde{z}_k | t = i) \) is basically the distribution of \( \tilde{z}_k \) which has already been calculated in Section III-B assuming that \( t = i \). However, for each \( i \in \{0, 1, \ldots, k\} \), we need to run a separate Kalman filter to calculate \( p_\ast(\tilde{z}_k | t = i) \), as described in III-B. On the other hand, \( p_\ast(\tilde{z}_k | t > k) \) is simply the unconditional distribution of \( \tilde{z}_k \) under no attack, since the innovations are independent of each other under no attack. Hence, results from Section III-B can be directly used to calculate \( \beta_k \) and hence \( \pi_k \) recursively.

### D. Bellman equation, value function and policy structure

Since the state transition is dependent on observation history, the optimal policy will be non-stationary, and the optimal value function will also be dependent on time. Let \( J_k^\ast(\pi_k, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k) \) denote the optimal cost-to-go starting from a state \((\pi_k, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k)\) at time \( k \), with \( J_k^\ast(\pi_0) = J_0^\ast \). Also, let \( \Psi_k \) be a function such that, given \( u_k = 0 \), we have \( \pi_{k+1} = \Psi_k(\pi_k, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k) \). Hence,

The Bellman equation for (15) is given by:

\[
\begin{align*}
    J_k^\ast(\pi_k, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k) &= \min_{\pi_k} \left( \lambda(1 - \pi_k) + \mathbb{E}[J_{k+1}^\ast(\Psi_k(\pi_k, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k), \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{k+1}) | \tilde{z}_1, \ldots, \tilde{z}_k] \right)
\end{align*}
\]

The first term in the minimization of (24) is the cost of stopping and declaring \( H_1 \), and the second term is the cost of continuing observation, which involves a single-stage cost \( \pi_k \) and an expected cost-to-go from the next step where the expectation is taken over the distribution of \( \tilde{z}_{k+1} \) conditioned on \( \tilde{z}_1, \ldots, \tilde{z}_k \).
Let us consider an $N$-horizon problem which is same as problem (15) except that, at time $N$, the detector must stop and declare $H_1$. Let the analogues of $J^*_k$ (for various values of $k$) and $J^*$ for this $N$-horizon problem be denoted by $J^{(N)*}_k$ and $J^{(N)*}$, respectively.

**Lemma 1.** $J^{(N)*}(\pi)$ is concave in $\pi \in [0,1]$.  

**Proof:** See Appendix A. Proof is based on outline from [39].

**Theorem 2.** $J^*(\pi)$ is concave in $\pi$.  

**Proof:** The proof follows from Lemma 1 and the fact that limit of a sequence of concave functions is concave. 

The next theorem describes the optimal policy for the unconstrained problem (15).

**Theorem 3.** The optimal policy for the constrained problem (15) is a threshold policy. At time $k$, if the state is $(\pi_k, \tilde{z}_{1:k}, \tilde{z}_{2:k}, \ldots, \tilde{z}_{k})$, then the optimal action is to stop and declare $H_1$ if $\pi_k > \Gamma_k(\tilde{z}_{1:k}) \in [0,1]$, and to continue collecting observations if $\pi_k < \Gamma_k(\tilde{z}_{1:k})$. If $\pi_k = \Gamma_k(\tilde{z}_{1:k})$, either action is optimal.

**Proof:** See Appendix B.

### E. Computational complexity and the QUICKDET algorithm

Due to the time-inhomogeneous state transition, problem (15) cannot be solved by standard techniques such as value iteration. On the other hand, the optimal threshold $\Gamma_k(\tilde{z}_{1:k})$ at time $k$ depends not only on $k$ but also the history of innovations $\tilde{z}_{1:k}$. Further, the presence of $J^*_{k+1}$ in the Bellman equation (24) makes even a fairly discretized version of the problem computationally very heavy to solve in an online fashion. These problems can be alleviated by setting a constant threshold, i.e., $\Gamma_k(\tilde{z}_{1:k}) = \Gamma$ for all $k \geq 1, \tilde{z}_{1:k}$. Definitely, this will yield a suboptimal solution, but as will be seen later, we can achieve much better performance than the competing algorithms in the literature by carefully choosing $\Gamma$.

1) The QUICKDET algorithm: Motivated by Theorem 3 along with the need for a constant threshold $\Gamma$ to reduce computational complexity, here we describe outline of an algorithm QUICKDET. The two key components of QUICKDET, apart from the threshold structure, are the choices of the optimal $\Gamma^*$ to minimize the objective in the unconstrained problem (15) within the class of stationary threshold policies, and $\lambda^*$ to meet the constraint in (14) with equality as per Theorem 1. These parameters are set via some off-line precomputation involving two timescale stochastic approximation [34], wherein we update $\lambda$ in the slower timescale, and $\Gamma$ in the faster timescale.

In the pre-computation phase, we generate various sample paths $P_n, P_{n+1}, \ldots$ of the process, observation and attack dynamics as per our discussed system model. The iterations start with some initial iterates $\Gamma(0)$ and $\lambda(0)$. For sample path $P_n$, a threshold policy with constant threshold $\Gamma(n)$ is used for attack detection on the state and modified observation sequence, and it is observed whether this policy generates a false alarm for sample path $P_n$; if it does not generate a false alarm, then the detection delay $t_n$ is recorded. Let $1_{FA}(n)$ and $1_D(n)$ be the indicators of false alarm and attack detection for sample path $P_n$.

Let $\{a(n)\}_{n \geq 0}$, $\{b(n)\}_{n \geq 0}$ and $\{\delta(n)\}_{n \geq 0}$ be three non-negative sequences satisfying the following properties: (i) $\sum_{n=0}^{\infty} a(n) = \sum_{n=0}^{\infty} b(n) = \infty$, (ii) $\sum_{n=0}^{\infty} a^2(n) < \infty, \sum_{n=0}^{\infty} b^2(n) < \infty$, (iii) $\lim_{n \to \infty} \frac{b(n)}{a(n)} = 0$, (iv) $\lim_{n \to \infty} \delta(n) = 0$, and (v) $\sum_{n=0}^{\infty} a^2(n) / \sum_{n=0}^{\infty} b^2(n) < \infty$. The first two conditions are standard requirements for stochastic approximation. Condition (iii) ensures the necessary timescale separation. The last two conditions are required for the convergence of the $\Gamma(n)$ update via stochastic gradient descent (SGD), adapted from the theory of simultaneous perturbation stochastic approximation (SPSA, see [33]).

Let $\Gamma^+(n) = \Gamma(n) + \delta(n)$ and $\Gamma^-(n) = \Gamma(n) - \delta(n)$ be two perturbations of $\Gamma(n)$ in opposite directions. Let $d_n^* = \tau_n 1_{D(n)} + \lambda(n) 1_{FA(n)} |\Gamma^+(n)|$ be the cost incurred along sample path $P_n$ if a threshold policy with a constant threshold $\Gamma^+(n)$ is used along sample path $P_n$; let us define $d_n^-$ similarly.

The following updates are made:

$$\Gamma(n+1) = [\Gamma(n) - a(n) \times d_n^+ - d_n^-] / 2\delta(n)$$

$$\lambda(n+1) = [\lambda(n) + b(n) \times (1_{FA(n)} - a(n))] / 2\delta(n)$$

The $\Gamma(n)$ update in (25) is a stochastic gradient descent algorithm used to minimize the expected cost along sample path $P_n$, for $\lambda = \lambda(n)$. This algorithm runs in a faster timescale. The $\lambda(n)$ update runs at a slower timescale to ensure that the false alarm probability equals $\alpha$. The faster timescale iterate $\Gamma(n)$ views the $\lambda(n)$ iterate as quasi-static, while the $\lambda(n)$ iterate views the $\Gamma(n)$ iterate as almost equilibrated. The iterates are projected onto desired intervals to ensure boundedness.

Using standard arguments, we can prove that $(\Gamma(n), \lambda(n)) \to \{(\Gamma, \lambda) \in [0,1] \times [0,\infty) : \mathbb{E}(\mathbb{1}_{FA}) = \alpha, \nabla_{\Gamma} \mathbb{E}(d)|_{\lambda} = 0\}$. In practice, if $(\Gamma(n), \lambda(n)) \to (\Gamma^*, \lambda^*)$, then this $\Gamma^*$ is used in the real attack detector on field.

### F. Extension to multiple attack matrices

1) Calculation of transition probabilities: Here we assume that the attack matrix $T$ is unknown but belongs to a known finite set $\mathcal{T}$, and that $I \in \mathcal{T}$. We assume an initial prior distribution on $\{T \in \mathcal{T} : T \neq I\}$. The detector maintains a belief probability $\pi^T_k$ for hypothesis $T$, and the total number of hypotheses is $|\mathcal{T}|$. In this case, the posterior belief probability of attack becomes $\pi^T_k = 1 - \pi^I_k$. Here, $\pi^T_k$ for $T \neq I$ is defined as the belief at time $k$ that the attack has already begun, and that the attacker uses $T$:

$$\pi^T_k = p(t \leq k, T|\tilde{z}_1, \ldots, \tilde{z}_k) = \frac{p(t \leq k|\tilde{z}_1, \ldots, \tilde{z}_k,T) p(T|\tilde{z}_1, \ldots, \tilde{z}_k)}{\pi^I_k}$$

where $\pi^I_k = \frac{\beta_k T}{\beta_{k|T+1}}$ (similar to (17)) is the conditional belief on attack having already started given that the attacker
uses $T$, and this $\beta_{k|T}$ can be calculated in the same way as in Section [IV.C] for each $T \in \mathcal{T}$. Posterior probability distribution over $T$, i.e., $p(T|\tilde{z}_1, \ldots, \tilde{z}_k, t \leq k)$ for $T \neq I$, can be calculated as given below:

$$p(T|\tilde{z}_1, \ldots, \tilde{z}_k) = \frac{\sum_{T \in \mathcal{T}} p(\tilde{z}_k|\tilde{z}_1, \ldots, \tilde{z}_{k-1}, T)p(T|\tilde{z}_1, \ldots, \tilde{z}_{k-1})}{\sum_{T \in \mathcal{T}} p(\tilde{z}_k|T)p(T|\tilde{z}_1, \ldots, \tilde{z}_{k-1})}$$

Here $p(T|\tilde{z}_1, \ldots, \tilde{z}_{k-1})$ is known through the iterative calculation, where the first iteration will depend on prior distribution on attack matrices $T \in \mathcal{T}$. The probability $p_c(\tilde{z}_k|T)$ is the distribution of innovation given the history and attack matrix $T$:

$$p_c(\tilde{z}_k|T) = p_c(\tilde{z}_k|T, t \leq k)p(t \leq k|\tilde{z}_1, \ldots, \tilde{z}_{k-1}, T) + p_c(\tilde{z}_k|T, t > k)p(t > k|\tilde{z}_1, \ldots, \tilde{z}_{k-1}, T) = p_c(\tilde{z}_k|T, t \leq k)(\pi_{k-1|T} + (1 - \pi_{k-1|T})\theta) + p_c(\tilde{z}_k|T, t > k)(1 - \pi_{k-1|T})(1 - \theta) \quad (26)$$

Now, by using (22), we can write:

$$\beta_{k|T} = \frac{p_c(\tilde{z}_k|T, t \leq k)(\pi_{k-1|T} + (1 - \pi_{k-1|T})\theta)}{p_c(\tilde{z}_k|T, t > k)(1 - \theta)(1 - \pi_{k-1|T})}$$

Hence,

$$\beta_{k|T}p_c(\tilde{z}_k|T, t > k)(1 - \theta)(1 - \pi_{k-1|T}) = p_c(\tilde{z}_k|T, t \leq k)(\pi_{k-1|T} + (1 - \pi_{k-1|T})\theta) \quad (27)$$

Finally, (26) can be simplified by substituting (27) resulting the following expression:

$$p_c(\tilde{z}_k|T) = p_c(\tilde{z}_k|T, t > k)(1 - \pi_{k-1|T})(1 + \beta_{k|T})(1 - \theta)$$

Now, $p_c(\tilde{z}_k|T, t > k)$ becomes the unconditional distribution of the innovation under no attack. On the other hand, $\pi_{k-1|T}$ is known from the previous iteration, and $\beta_{k|T}$ can be calculated as in Section [IV.C]. Thus, we can calculate $p_c(\tilde{z}_k|T)$ and consequently $p(T|\tilde{z}_1, \ldots, \tilde{z}_k)$ and $\pi_{k|T}$ for all $T \in \mathcal{T}$.

However, the time complexity of algorithm in multiple attack matrix case increases $|T|$ times compared to the binary hypothesis testing case, since we would need to maintain separate set of Kalman filters for each attack matrix $T$.

2) Policy structure: For (14), we can still compute the belief probability that the attack has begun:

$$\pi_k = \sum_{T \in \mathcal{T}} p(T|\tilde{z}_1, \ldots, \tilde{z}_k)$$

Similar to Theorem 3, the optimal Policy structure for multiple attack matrices will still be a threshold policy. We also note that, $\arg\max_{T \in \mathcal{T}} \pi_k$ yields the maximum likelihood (ML) detection of $T$, and $\arg\max_{T \in \mathcal{T}} \pi_k$ can be used for maximum a posteriori (MAP) detection of the attacker’s strategy.

V. ATTACK DETECTION IN THE NON-BAYESIAN SETTING

In this section, we consider the situation where the distribution of the attack initiation instant $t$ is not known. This eliminates the possibility of solving the problem via MDP formulation. Hence, we use the popular generalised CUSUM algorithm proposed by Lai [40] for change detection, and demonstrate how the results in Section [III-B] can be used in computing the generalised CUSUM statistic.

The basic CUSUM Algorithm was developed heuristically by Page [41], but was later analysed rigorously in [42], [43], [44] and [40]. In our non-Bayesian FDI detection setting, under no attack (i.e., $t = \infty$), the false alarm rate (FAR) is the inverse of the mean time to false alarm $\text{FAR}(\tau) = \frac{1}{E(\tau)}$.

With little abuse of notation where the stopping time $\tau$ also represents a stopping rule for the detector, we focus on the following set of detection/stopping rules:

$$D_\alpha = \{ \tau : \text{FAR}(\tau) \leq \alpha \} \quad (28)$$

Since finding a uniformly powerful test that minimizes detection delay over $D_\alpha$ is not possible, we study two important minimax formulations developed by Lorden [42] and Pollak [45].

Lordan’s Problem: For a given $\alpha$, find $\tau \in D_\alpha$ to minimize worst average detection delay WADD$(\tau)$ defined as:

$$\text{WADD}(\tau) = \sup_{n \geq 1} \mathbb{E}[n(\tau - n)^-|\tilde{z}_1, \ldots, \tilde{z}_{n-1}]$$

Pollak’s Problem: For a given $\alpha$, find $\tau \in D_\alpha$ to minimize cumulative average detection delay CADD$(\tau)$ defined as:

$$\text{CADD}(\tau) = \sup_{n \geq 1} \mathbb{E}[\tau - n|\tau \geq n]$$

It was proved in [46, Section IV] that $\text{WADD}(\tau) \geq \text{CADD}(\tau)$. Lai [40] studied both Lorden’s and Pollak’s problem in non-i.i.d. setting, and showed that under some additional condition, the basic CUSUM algorithm is asymptotically optimal as $\alpha \to 0$.

Let us define $p_c(\tilde{z}_k|t = k)$ (adapted to our problem setting where innovations are i.i.d. across time before attack), the associated log-likelihood ratio, and the corresponding running sum of the log-likelihood ratios as follows:

$$p_c(\tilde{z}_i|t = k) = \frac{p(\tilde{z}_i|\tilde{z}_0, \ldots, \tilde{z}_{i-1}, t = k)}{p(\tilde{z}_i|t = \infty)}$$

$$L_{i,k} = \log \left( \frac{p_c(\tilde{z}_i|t = k)}{p(\tilde{z}_i|t = \infty)} \right)$$

$$S_n = \max_{1 \leq k \leq n} \sum_{i=k}^{\infty} L_{i,k}$$

Here $S_{n,k}$ represents the cumulative sum of likelihood ratio up to time $n$ assuming that the attack started at $t = k$. Obviously, $S_{n,k}$ can be written recursively as follows:

$$S_{n,k} = \max \{0, S_{n-1,k} + \log \left( \frac{p_c(\tilde{z}_n|t = k)}{p(\tilde{z}_n|t = \infty)} \right) \}$$

Remark 1. It is important to note that, $p_c(\tilde{z}_i|t = k)$ for $i \geq k$ can be computed by using the theory developed in Section [III-B].
The generalized CUSUM algorithm for non i.i.d. observations involve the following stopping time:

$$\tau_g = \inf\{n \geq 1 : S_n \geq b\}$$

(32)

for some threshold \(b\). It was proved in [40] that, as \(\alpha \to 0\), under some regularity conditions,

$$\mathbb{E}_\infty[\tau_g] \geq e^b$$

$$CADD(\tau_g) \leq WADD(\tau_g) \leq \frac{b}{I}(1 + o(1))$$

as \(b \to \infty\)

for a constant \(I\). Hence, If we set \(b = |\log \alpha|\), then

$$\frac{FAR(\tau_g)}{WADD(\tau_g)} = \frac{\frac{1}{\mathbb{E}_\infty[\tau_g]} \leq \alpha}{(1 + o(1))}$$

Thus, \(\tau_g\) is first-order asymptotically optimal detection rule within \(D_\alpha\).

VI. NUMERICAL RESULTS

Since multiple sensor observations can be viewed as parts of a single imaginary sensor’s observation, we assume that there is one safe sensor and one unsafe sensor, and the sign of the innovation coming from the unsafe sensor is inverted, i.e., \(T_A = -I\). In this section, we first compare the performance of QUICKDET against three other detectors:

- **\(\chi^2\) detector**: This detector is as described in Section II except that \(\eta\) is optimized in an off-line pre-computation phase (as in QUICKDET) to meet the false alarm constraint with equality:

$$\eta(n + 1) = [\eta(n) + a(n) \times (1 - FA)(n) - \alpha]_0^\infty$$

The limit \(\eta^*\) of this iteration is used in the real detector on field. We choose \(J = 3\) in our simulation.

- **DET**: This is an adaptation of the DET algorithm in [18]. It requires the detector to run two separate parallel Kalman filters for the safe and unsafe sensors. Let \(\hat{x}_{k,S}\) and \(\hat{x}_{k,A}\) be the estimates declared by two blind Kalman filters using observations from the safe sensor and from the unsafe sensor, respectively. This detector declares an attack at time \(j\) if

$$\sum_{k=j-1}^{j} (\hat{x}_{k,A} - \hat{x}_{k,S})^\top \Sigma^{-1} (\hat{x}_{k,A} - \hat{x}_{k,S}) > \eta$$

where \(\eta\) can be optimized as in the \(\chi^2\) detector to meet the false alarm constraint with equality, and \(\Sigma\) is the steady state covariance matrix of \((\hat{x}_{k,A} - \hat{x}_{k,S})\) under no attack. We choose \(J = 3\) in our simulation.

- **SAFE**: This is the detection algorithm taken from [11], with the threshold optimized as before to meet the false alarm constraint.

We consider a process with dimension \(q = 2\). The \(A, Q, R_A\) and \(R_S\) matrices are all chosen to be

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0.5 \\ 0 & 1 \end{bmatrix}$$

The probability of launching an attack at a time, \(\theta\), is chosen to be 0.05, and we generate 10000 sample paths for the pre-computation phase of QUICKDET. Prior belief probability of attack \(\pi_0\) is set to 0 for all the sample paths.

Figure 2 compares the mean detection delay of QUICKDET, DET, \(\chi^2\) and SAFE detectors, for various values of false alarm probability \(\alpha\). We observe that QUICKDET outperforms all other detectors, and the performance margin is very high w.r.t. SAFE and \(\chi^2\) detectors. It is important to note that, though...
QUICKDET is a suboptimal detector motivated by the optimal detector of Theorem 3; it uses the knowledge of the matrix $T_A$ which the other three algorithms do not use. Hence, given the knowledge of $T_A$ (the attacker’s strategy), it is always better to use QUICKDET. It is also observed that DET’s performance is not far from that of QUICKDET, which means that DET can be used as a potential alternative to QUICKDET in case the knowledge of $T_A$ is not available a priori. These observations have been verified through numerous simulation experiments under various problem instances.

Since the probability of attack detection is 1 in all of our simulations, we do not compare ROC curves of the detectors. However, detection delay comparison of Figure 2 is a reasonable alternative to the ROC plots.

Figure 3 shows that the threshold $\Gamma$ in QUICKDET decreases with the false alarm constraint $\alpha$, which supports the intuition that a larger $\Gamma$ results in less frequent up-crossing of $\Gamma$ by the belief probability, and consequently a smaller false alarm probability.

Figure 4 compares the performance of generalised CUSUM test with the other three algorithms in terms of mean detection delay versus FAR in the non-Bayesian detection setting. It is observed that here again generalised CUSUM (G-CUSUM) significantly outperforms the other three algorithms.

VII. CONCLUSION

In this paper, we provided an algorithm for quickest detection of FDI attack on remote estimation in the Bayesian and non-Bayesian setting. Theoretical proof of optimality was provided wherever required, and numerical results demonstrated clear superiority of the proposed algorithms against other competing algorithms. However, there remain a number of open questions: (i) how to handle a nonlinear attack scheme? (ii) how to handle unknown attack strategy? (iii) how to perform distributed quickest detection in a multihop network setting? We plan to address these issues in our future research endeavours.

APPENDIX A

PROOF OF LEMMA

Under action $u_{k-1} = 0$ and for $\pi_{k-1} = \pi$, the recursion for $\pi$ can be written in a more elementary form as:

$$
\pi_{next} = \frac{p_c(\tilde{z}_k | t \leq k)(\pi + (1 - \pi)\theta)}{p_c(\tilde{z}_k | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{k-1})} 
$$

where $\tilde{z}_{1:k}$ represents $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k$. Since $\ldots d(\tilde{z}_k)$ is a linear operator and $E[\beta_k(\pi_{next})] = \int \Phi_k(\pi, \tilde{z}_1:k)d(\tilde{z}_k)$, it is sufficient to show that $\Phi_k(\pi, \tilde{z}_1:k)$ is concave in $\pi$. Since $J_k^{(N^*)}(\pi)$ is assumed to be concave in $\pi$, we can write:

$$
J_k^{(N^*)}(\pi) = \inf_{(x,y) \in V} (x + y)
$$

where $V = \{(x,y) \in \mathbb{R}^2 : x + y \geq J_k^{(N^*)}(\pi) \forall \pi \in [0,1]\}$. Hence,

$$
\Phi_k(\pi, \tilde{z}_{1:k}) = J_k^{(N^*)}(\pi) = \inf_{(x,y) \in V} (x + y)
$$

APPENDIX B

PROOF OF THEOREM

Let us define:

$$
g(\pi_k) = \lambda(1 - \pi_k)
$$

Clearly $g(0) = \lambda$ and $g(1) = 0$. Now note that,

$$
h_k(0, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k) = \pi_k + \mathbb{E}[J_{k+1}^*(\pi_k, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k)]
$$

where $\tilde{z}_{1:k}$ represents $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k$. Since $\lambda(1 - \pi_k) - g(1) > 0$ and $h_k(0, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_k) - g(0) < 0$ for all $k \geq 1$, by using the intermediate value theorem for continuous functions, we get that there exists $\Gamma_k(\tilde{z}_{1:k}) \in (0, 1)$ such that $h_k(\Gamma_k(\tilde{z}_{1:k}), \tilde{z}_{1:k}) = g(1)$. Further, since $h_k(\pi, \tilde{z}_{1:k}) - g(\pi)$ is a concave function, $h_k(1, \tilde{z}_{1:k}) - g(1) > 0$ and $h_k(0, \tilde{z}_{1:k}) - g(0) < 0$, the value of $\Gamma_k(\tilde{z}_{1:k})$ is unique for each $k \geq 1$. Hence the optimal stopping time $\tau^*$ is given by

$$
\tau^* = \inf\{k \geq 1 : \pi_k \geq \Gamma_k(\tilde{z}_{1:k})\}
$$

where $\Gamma_k(\tilde{z}_{1:k})$ is given by

$$
\Gamma_k(\tilde{z}_{1:k}) = \mathbb{E}[J_{k+1}^*(\pi_k+1(\Gamma_k(\tilde{z}_{1:k}), \tilde{z}_1, \ldots, \tilde{z}_k))]
$$

where $\Gamma_k(\tilde{z}_{1:k})$ is given by

$$
\Gamma_k(\tilde{z}_{1:k}) = \mathbb{E}[J_{k+1}^*(\Gamma_k(\tilde{z}_{1:k}), \tilde{z}_1, \ldots, \tilde{z}_k)]
$$

where $\lambda(1 - \Gamma_k(\tilde{z}_{1:k}))$
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