Cardinality of Real Numbers and Set Axiomatization

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Abstract

The cardinality of real numbers is contingent upon the chosen axiomatic system and the specific models of set theory. According to the diagonal argument within the Zermelo-Fraenkel (ZF) framework, real numbers are non-denumerable. However, their precise cardinality remains a topic of debate, varying with different extensions of the ZF axioms. For example, under the inner-model axiom ‘\textit{V-ultimate L}’, the Continuum Hypothesis (CH) holds true. In contrast, it is negated if Martin’s Axiom is adopted. Additionally, there is an orthogonal view that assumes there are myriad mathematical universes, some in which the continuum hypothesis is true and others in which it is false, but all equally worth of exploring. This model-dependent viewpoint is underlined by the Löwenheim-Skolem theorem, which asserts that no theory can have only non-denumerable models. Accepting any form of axiomatization implies that terms like ‘\textit{denumerable}’ and ‘\textit{non-denumerable}’ are relative. This necessitates the acceptance of the relativization of cardinal numbers, as initially suggested by Skolem and Carnap.

To investigate this uncertainty in cardinality, we analyzed two analogous sets within the frameworks of the ZFC (ZF and Axiom of Choice (AC)) model and Wang’s Σ model, which is different from ZFC. The first set, denoted as \(S'\), comprises all pairs of algebraic numbers, and transcendental numbers inserted between each pair by a specifically defined function \(f\). This function places an arbitrarily chosen transcendental number between each algebraic pair. A key attribute of set \(S'\) in the ZF framework is that it encompasses all possible intervals with algebraic endpoints, defined by these algebraic pairs. This characteristic enables the construction of any Cauchy sequence representing any number using these intervals. By applying the AC, one can systematically identify within set \(S'\) the intervals constituting any given Cauchy sequence and coherently assemble them to represent any number, even though no individual sequence is explicitly defined within set \(S'\). Owing to the operational nature of function \(f\), every interval identified in these Cauchy sequences is assuredly non-empty. Consequently, in line with the Nested Interval Property (NIP), the intersections of these intervals are also non-empty, containing specific transcendental numbers as defined by the sequences. This leads to the remarkable
conclusion that set $S'$ is simultaneously complete (any number can be recognized in the set) and countable (a countable number of transcendental numbers is inserted in the countable set).

To expand upon this result, we examined an equivalent set that is also derived by inserting transcendental numbers between algebraic numbers. However, in this case, the transcendental numbers are not selected arbitrarily. Rather, they are systematically constructed and exactly defined by applying diagonal procedures on the sequence of listed numbers. This method is employed to create the layers of Wang’s $\Sigma$ model. In this novel approach, we present the first explicit example that is consistent with that model. The process begins with the sequence of all algebraic numbers representing the 0-th Wang’s layer. The 1-st layer is then constructed through diagonal procedures applied to the 0-th layer. These procedures yield transcendental numbers, which, in conjunction with the 0-th layer, form the first layer. Subsequent layers are similarly created, with each new layer arising from diagonal procedures made on the preceding layer and by adding these transcendental numbers created by these diagonal procedures to the preceding layer, thus progressively constructing Wang’s $\Sigma$ layered structure.

This process is finite and concludes precisely after $\omega$ steps, constrained by the atomic nature of real numbers. Upon examining the properties of the resulting set, and given its equivalence to set $S'$, we conclude that real numbers that can be constructively expressed are countable. This finding appears to challenge the traditional diagonal argument. However, it is important to recognize that this model does not encompass all real numbers, particularly those that are non-constructible. Nevertheless, the apparent contradiction persists, as the diagonal argument primarily concerns constructible numbers, since it is applied to well-defined numbers expressed by digital sequences.

The application of the AC was crucial in the construction of set $S'$; it allows for the selection of an arbitrary transcendental number for each pair of algebraic numbers. It is also essential in enabling the identification of any interval with algebraic endpoints and the identification of any specific sequence within set $S'$, leading to the conclusion that set $S'$ is complete and enumerable. While this result might resonate with constructivists who oppose the AC, our further exploration of the diagonal argument revealed elements that conflict with the NIP. This issue is not unique; it also emerges in other proofs of non-denumerability, as detailed in the article. This necessitates a more thorough reevaluation of the countability of real numbers.

Key words: Set axioms, Continuum hypothesis, Diagonal argument, Axiom of choice, Denumerability of numbers, V-ultimate $L$, Cardinal numbers

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Gödel conceived a small and constructible model universe called $L$ that is generated by starting with an empty set and iterating it to build bigger and bigger sets. In 1938, he put forward a new axiom for set theory: the axiom $V=L$. By considering the inner model for set theory in which $V=L$ is true, Gödel was able to prove the relative consistency of the Zermelo-Fraenkel (ZF) axioms plus the axiom of choice (ZFC), and the generalized Continuum Hypothesis (CH)\[1\]. Gödel briefly considered proposing to add $V=L$ to the accepted axioms for set theory as a sort of meaning stipulation, but he soon changed his mind. His later view was that $V=L$ is really false, even though it is consistent with set theory if set theory is itself consistent. He believed that CH is not true and that not every uncountable set of reals has the same size as $\mathbb{R}$, and that indeed $2^{\aleph_0} = \aleph_2$ is true\[2,3\]. “One may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor’s conjecture” \[5\]. Ever since it was established that CH is independent of the ZFC axioms by Gödel’s 1940 proof that the ZFC axioms cannot be used to disprove the continuum hypothesis, and Cohen’s 1963 proof to the contrary that they cannot be used to prove it\[4\], there have been efforts to add at least one new axiom to ZFC that would illuminate the structure of infinite sets. Currently, the two most promising candidates are the Ultimate-L axiom, or the statement that Ultimate-L is the universe of sets, which is an all-encompassing generalization of Gödel’s model universe of sets with cardinality $\aleph_1$; and Martin’s Maximum axiom\[6,7\], which asserts that anything conceived of using any forcing construction is a true mathematical entity, so long as the construction satisfies a certain consistency condition, compellingly pointing to $\aleph_2$ cardinality.

So, forcing axioms provide an avenue for generic extensions of the universe $V$ of all sets through an iterative process $V \subset V[g_0] \subset V[g_1] \subset \ldots \subset V[g_\alpha]$ of $V$ for any length $\alpha$, in such a way that the final model is itself a generic extension of $V$. Cohen used forcing to enlarge the continuum to include new reals beyond those of the model. But applying the forcing mechanism allows creating arbitrarily many reals, $\aleph_2$ or $\aleph_{315}$, etc., a cloud of universes, without the possibility to distinguish which is virtual and which is real, or even whether an object created that way really exists.

To address this ambiguity and establish the actual existence of specific objects rendered possible by Cohen’s method, various forcing axioms are introduced, such as Martin’s Maximum axiom and Woodin’s (*) axiom \[8\], that allow stating “for all $X$, there exists $Y$, such that $Z$” when referring to properties of sets within the domain.
However, there are inconsistencies between Martin’s and (*) axioms and between these axioms and ZFC [9,10,11]. For instance, the statement related to the (*) axiom, “for all sets of $\aleph_1$ reals, there exist reals not in those sets,” clearly contradicts the continuum hypothesis. Recent work [12] shows that Martin’s maximum++ (a technical variation of Martin’s maximum) implies Woodin’s (*) axiom, which amalgamates these two prominent axioms of set theory, both implying that there are $\aleph_2$ many real numbers.

Yet, there are still open questions. Woodin introduced the axioms (*)+ and (*)++, which apply to the full power set (the set of all subsets) of the reals, strengthening the (*) axiom [11]. But these axioms contradict Martin’s axiom in various models [11], questioning the existential statements about sets of reals in Martin’s maximum framework.

The problem can be further underlined by Löwenheim-Skolem theorem or the intimately related Gödel completeness theorem and its model-theoretic generalizations. It says that a suitable first-order theory (in countable language) has a countable model, that a set can be ‘nondenumerable’ in this relative sense and yet be denumerable ‘in reality’. As Skolem sums it up, “even the notions ‘finite’, ‘infinite’, ‘simply infinite sequence’ and so forth turn out to be merely relative within axiomatic set theory”, which means that the notion of countability is not absolute [13].

One of the consequences of the theorem is the existence of a countable model of set theory, which nevertheless must satisfy the sentence asserting that the real numbers are uncountable. The diagonal argument requires that $\mathbb{R}$ is nondenumerable in all models, that ZF set theory has only nondenumerable models. However, by the Löwenheim-Skolem theorem this is impossible, no theory can have only nondenumerable models. If a theory has a nondenumerable model, it must have denumerably infinite ones as well.

The non-denumerability is in contrast with predicates used in the diagonal argument, available only in a denumerably infinite multitude and yield no more than $\aleph_0$ subsets of the set of all integers, i.e., an enumeration of the continuum from outside the axiomatic system [14]. The enumeration from outside utilizes the structure of the system as a whole, which cannot be reached by operations within the system. There are one-to-one correspondences between $\mathbb{R}$ and $\mathbb{N}$, but they all lie outside the given model. It follows that $\mathbb{R}$ is nondenumerable in a relative sense. What is a ‘countable’ set from the point of view of one model may be an uncountable set from the point of view of another model. For any given nondenumerable theory model $M$, exist a countable model $M'$ of that same theory restricted to $M$, the totality that the variables of quantification range over ($M$) a proper subset of the $M'$. Therefore, it is not possible to rule out denumerable interpretation. Infinite sets may have different cardinal numbers and be nevertheless syntactically of the same cardinal number [14].
In a given axiomatic set theory $M$, there always exists a more comprehensive theory in which all infinite sets of $M$ prove equivalent, namely denumerable. Every non-denumerable set becomes denumerable in a higher system or in an absolute sense.

2 Discussion

Let us again consider Gödel’s axiom $V = L$ that all sets are constructible and examine the impact of the Löwenheim-Skolem theorem. As shown in [17], the theorem requires that ZF plus $V = L$ has an $\omega$-model which contains any given set of real numbers. It follows from the statement that for every real $s$, there is a countable $M'$ such that $M'$ is an $\omega$-model for ZF plus $V = L$ and $s$ is represented in $M'$, which follows from the theorem. So, by the Löwenheim-Skolem theorem, a model containing $s$ can satisfy ‘$s$ is constructible’ (because it satisfies ‘$V = L’$, and ‘$V = L’$ says everything is constructible) and be an $\omega$-model. In the examples below, we will construct such $\omega$-models.

Let us first follow the Löwenheim-Skolem theorem and expand ZF model with AC. While AC is independent of the ZF set theory, which means that both the axiom itself and its negation are consistent with ZF, AC expands ZF. It allows the construction of set $S'$, which is not possible without AC, and to recognize in the set $S'$ every possible Cauchy sequence representing any number, even if no sequence is ever explicitly defined. It also allows the construction of any sequences of any number from the intervals with algebraic endpoints that are present in the set, similarly to some kind of a variation of the ‘$V = L’ approach. To demonstrate this explicitly, in example 1, a countable set $S'$ will be considered, which contains all pairs of algebraic numbers and transcendental numbers generated by a function $f$ that places an arbitrarily selected transcendental number between each of these pairs of algebraic numbers. Then using AC, it will be shown that such a set is equivalent to the set of real numbers.

**Example 1.** Consider a closed interval $[a, b]$, where $a < b$ are both algebraic numbers. We define the set $S$, which consists of all possible closed intervals $I_n' = [a_n, b_n] \cap \mathbb{Q}$ within $[a, b]$, with the specification that $I_n'$ are considered in the context of containing only algebraic numbers, i.e., they are the intersection of the intervals $[a_n, b_n]$ with the set of all algebraic numbers, denoted as $\mathbb{Q}$, and will be denoted as $[a_n, b_n]'$. Here, $a_n$ and $b_n$ are algebraic numbers, and $a_n \leq b_n$. Thus, the set $S$ encompasses every such algebraic interval $I_n' = [a_n, b_n] \cap \mathbb{Q} = [a_n, b_n]' \subseteq [a, b]$.

Next, we define a function $f$ that, for each interval $I_n'$, selects an arbitrarily chosen transcendental number $t_n$, ensuring that $t_n$ falls within the algebraic
bounds of $I'_n$, i.e., $a_n \leq t_n \leq b_n$, and the function $f$ has to act on all intervals $I'_n$ in the set $S$. This setup positions $t_n$ as associated with the interval $I'_n$, conceptualizing the inclusion of $t_n$ in the space spanned by $I'_n$, despite $t_n$ not being an algebraic number. Formally, the function $f$ is described as follows:

$$f : S \to \mathbb{R} \setminus \mathbb{Q}, \text{ such that for all } I'_n, a_n \leq f(I'_n) \leq b_n, \text{ and } f(I'_n) = t_n \quad (1)$$

In addition, it is required that the function $f$ also satisfies: a) that it acts on all intervals $I'_n$ simultaneously, not consecutively. There is no difference in outcomes whether the function is acting on intervals successively or adding numbers simultaneously; the outcome must be the same since it depends only on the properties of intervals. However, this requirement is to avoid discussions such as whether the process of inserting numbers can be completed, and to ensure that function $f$ is not only acting on individual intervals $I'_n$, but also on the entire sequences of such intervals if such sequences exist in the set $S$. Thus, if in the set $S$ there are explicitly or implicitly defined sequences of nested intervals $I'_1 \supset I'_2 \supset I'_3 \supset \ldots$, when function $f$ is inserting numbers $t_n$ into each of these intervals, creating intervals $I''_n$, which in addition to algebraic numbers also contain numbers $t_n$, then the generated numbers $t_n$ must also satisfy the condition that there will be a number $t$ that fits into all such intervals when the intervals are nested and function $f$ by definition acted on all of them. Therefore the condition to insert numbers in all intervals at same time implies confrontation of the function $f$ with all intervals, with the entire sequence, which requires that there must be a number that is in the intersection of all of them since all intervals are overlapping and each interval $I''_n$ is nonempty. Therefore, when confronted with infinite sequences the function $f$ will also have to generate numbers $t_n$ that satisfy

$$I'_1 \supset I'_2 \supset I'_3 \supset \ldots = \bigcap_{n \in \mathbb{N}} \{I''_n\} = \{t\} \quad (2)$$

b) In addition, it is required that if a transcendental number generated by the function $f$ is in an interval $I''_n$, then it must also be contained in all superintervals that contain that interval, and c) that the function acts only on the intervals $I'_n$ that do not contain transcendental numbers and always generates a different transcendental number $t_n$. Additionally, the intervals $I'_n = [a_n, b_n] \cap \mathbb{Q}$ include all subintervals $[a_m, b_m] \cap \mathbb{Q}$ contained within $[a, b]$.

Let us denote such a created set of everywhere dense transcendental numbers $t_n$ as $T$, i.e.

$$T = \{f([a_n, b_n] \cap \mathbb{Q}) : \text{ for all } [a_n, b_n] \cap \mathbb{Q} \in [a, b]\} = \bigcup_{n \in \mathbb{N}} \{t_n\} \quad (3)$$

Adding set $T$ to set $S$ creates set $S'$ that contains all algebraic numbers, all
intervals $I'_n$ with algebraic endpoints, and also all transcendental numbers $t_n$ from set $T$, which are inserted by function $f$ in the $I'_n$ intervals.

$$S' = \bigcup_{n \in \mathbb{N}} \{a_n, t_n, b_n\} = \overline{\mathbb{Q}} \cup T$$  \hspace{1cm} (4)

The cardinality of set $S'$ and its relation to the cardinality of real numbers can then be formulated through the following theorem:

**Theorem 1:** Consider a closed interval $[a, b]$, where $a < b$ are algebraic numbers. Define a set $S$ to consist of all closed subintervals $I'_n = [a_n, b_n] \cap \mathbb{Q}$ within $[a, b]$, where $a_n$ and $b_n$ are algebraic endpoints satisfying $a_n \leq b_n$, and each $I'_n$ is populated exclusively by algebraic numbers. Concurrently, define the set $T$ to encompass only transcendental numbers $t_n$, produced by the function $f: S \rightarrow \mathbb{R} \setminus \mathbb{Q}$, which assigns a unique transcendental number $t_n = f(I'_n)$ to each interval $I'_n$, in such a manner that $a_n \leq t_n \leq b_n$ holds for all elements of $S$, and that the function $f$ has to act simultaneously on all elements, all $I'_n$, in the set $S$. This operation by $f$ effectively simultaneously augments each interval $I'_n$ with a transcendental number $t_n$, yielding enhanced intervals $I''_n$, which can then be conceptualized as containing both the original algebraic numbers and the newly added transcendental numbers $t_n$. Consequently, the augmented set $S'$, denoted as $S' = S \cup T$, is both denumerable and complete.

A simplified version of this theorem states: Construct a set $S$ that includes all pairs of algebraic numbers $\{a_n, b_n\}$ and simultaneously insert a single transcendental number $t_n$ between each pair, such that $t_n = f(\{a_n, b_n\})$ and $a_n \leq t_n \leq b_n$ for every pair, and do this for all pairs in $S$. This results in a new set $S'$ that is both complete and countable.

**Proof of Theorem 1.** The proof is based on a counterexample. Let us assume that there is a missing transcendental number $T_x$ that is different from all $t_n$ numbers generated by the function $f$ and therefore from all numbers in the set $S'$. It must be a transcendental number since all algebraic numbers are, by definition, included in the set $S'$. Then define that missing number $T_x$ with a sequence of nested closed intervals $[\alpha_n, \beta_n]$ with algebraic endpoints $\alpha_n$ and $\beta_n$, such that $T_x$ is defined by that sequence

$$[\alpha_1, \beta_1] \supset [\alpha_2, \beta_2] \supset [\alpha_3, \beta_3] \ldots \supset [\alpha_n, \beta_n]$$  \hspace{1cm} (5)

and it is the intersection of these intervals

$$T_x = \bigcap_{n \in \mathbb{N}} [\alpha_n, \beta_n].$$  \hspace{1cm} (6)

Using the AC, one can identify in the set $S'$ closed intervals $I''_n = [a_n, b_n] \cap$
\( \mathbb{Q} \cup T \) that will be denoted \([a_n, b_n]''\) that have the same endpoints as the closed intervals \([\alpha_n, \beta_n]\) in the sequence (5), i.e., closed intervals \(I_n'' = [a_n, b_n]''\) with \(a_n = \alpha_n\) and \(b_n = \beta_n\). Every closed interval in the sequence (5) can be recognized in the set \(S'\) because the set \(S'\) contains all closed intervals with algebraic endpoints, and therefore all intervals that are in the sequence (5). All these closed intervals \(I_n''\) can be recognized by AC and assembled together to create a sequence that also represents the number \(T_x\), i.e., the sequence of \(I_n''\) intervals

\[
[a_1, b_1]'' \supset [a_2, b_2]'' \supset [a_3, b_3]'' \ldots \supset [a_n, b_n]'' \ldots \tag{7}
\]

and \(T_x\) is the intersection of these \(I_n''\) intervals

\[
T_x = \bigcap_{n \in \mathbb{N}} I_n''. \tag{8}
\]

So, using the AC, one can recognize in the set \(S'\) the sequence (7) that is exactly the same as sequence (5), with closed intervals \(I_n''\) having the same endpoints as intervals in sequence (5). To demonstrate that the number \(T_x\) is not missing, that it is contained within the set \(S'\), it is sufficient to show that the intersection (8) of the closed intervals \(I_n''\) in sequence (7) is not empty and that it contains the number \(T_x\). To proceed further, let us first prove the following lemma.

**Lemma 1.** Let \(T_y\) be a real number defined as the limit of an infinite converging nested sequence of closed intervals \(I_n = [\alpha_n, \beta_n]\), satisfying the following conditions:

1. \(I_{n+1} \subseteq I_n\) for all \(n\),
2. \(|\beta_n - \alpha_n| \to 0\) as \(n \to \infty\), and
3. \(\alpha_n\) and \(\beta_n\) are algebraic for all \(n\).

The requirement to select a single number from each interval \(I_n\), and to continue this selection throughout all sequence intervals, implies that it is impossible to avoid choosing the number \(T_y\). This inevitability arises because, as \(n\) approaches infinity, all these overlapping intervals collectively represent the single number \(T_y\) on the real line.

**Proof of Lemma 1.** In a sequence of nested closed intervals, each chosen number at step \(n\) must reside within the intersection of all prior intervals, reflecting the sequence’s nested structure. As the sequence progresses, the intersection narrows, ultimately resulting in a single point, \(T_y\), as stipulated by the NIP. This convergence limits selection options exclusively to \(T_y\) when traversing the entire sequence. Therefore, it is impossible to avoid selecting \(T_y\), as it becomes the sole viable number within the intersection of all intervals,
given that each selected number must be from the intersection of intervals, which eventually contains only the single number \( T_y \). This outcome is evident when considering that the overlapping intervals collectively represent \( T_y \).

Dedekind previously considered a similar problem. To visualize the function \( f \), consider Dedekind’s cut \( A/B \), which also defines the number \( T_y \), with \( A \) and \( B \) as the disjoint components of the cut. The endpoints of the nested intervals \([a_n, b_n]\) serve as elements defining the cut \( A/B \), with \( a_n \) in component \( A \) and \( b_n \) in component \( B \). As \( n \) increases, the gap between \( a_n \) from \( A \) and \( b_n \) from \( B \) diminishes, reducing the distance between pairs \( a_n \) and \( b_n \) toward zero. This progressive narrowing necessitates that the only number fitting between every pair \( a_n \) and \( b_n \) is \( T_y \), the number that represents the gap created by the cut, as elucidated by Dedekind [18].

This outcome is also consistent with the Bolzano-Weierstrass (BW) and the NIP, since the intervals \([a_n, b_n]\) contract to a zero-size interval, a singleton, making \( T_y \) the only viable number within the intersection of all these intervals. Consequently, \( T_y \) is the only number that can be inserted within all these intervals. Since the function \( f \) must operate across all intervals, it necessarily generates \( T_y \) and incorporates it into the set \( S' \). This inclusion, as it will be shown, indicates that the set \( S' \) is complete, as \( f \), by generating numbers \( t_n \), effectively bridges all gaps between algebraic numbers.

Having proven this lemma, we can now return to proving that the intersection of the sequence (7) is not empty and indeed contains the number \( T_x \) from the set \( S' \). This conclusion directly follows from the lemma, which establishes that if the requirement is to select or insert a single number in each of the closed intervals of the sequence (5), then it is impossible to avoid inserting the number \( T_x \) in that sequence. The sequence (7) segments share the same endpoints as the segments of sequence (5); therefore, by the lemma, if it is required to insert a single number in each segment of that sequence, and to insert numbers in all segments, it is impossible to avoid inserting \( T_x \) since the segments in sequences (5) and (7) are identical, have the same endpoints. Consequently, if only the number \( T_x \) can be in the intersection of the sequence (5), then only \( T_x \) can be in the intersection of the sequence (7), as both intersections are identical, both sequences must have the same number \( T_x \) in their intersections. It follows by the lemma that the function \( f \) was compelled to insert the number \( T_x \) in the segments of sequence (7) when acted on all segments of that sequence. To satisfy the requirement of inserting numbers in all sequence segments, the function \( f \) necessarily inserted \( T_x \), because the nested and overlapping segments dictate that only \( T_x \) can be positioned among all of them. Thus, fulfilling the requirement to insert numbers in all sequence segments implies that the function \( f \) inserted \( T_x \) in the set \( S \) when acted on all segments in the set \( S \). This proves the theorem and establishes the completeness of the set \( S' \), given that an arbitrarily chosen number \( T_x \) is present in the intersection of
the sequence (7).

The AC allows the production of objects without explicitly defining them. To have a number to be included in the set $S'$, it is not necessary to define and explicitly write the sequence representing that number. The numbers are already inserted in the set $S'$ by the function $f$ and can be recognized or identified with AC. The AC enables recognition in the set $S'$ of all closed intervals of the sequence representing any specific real number. Consequently, to have comprised all real numbers in the set $S'$, it is not necessary to define all sequences of all real numbers. Instead, one can use AC to recognize all numbers in the set $S'$, by recognizing all segments of any specific sequence and that way identifying any sequence representing any real number. Since all segments in these sequences are nonempty, the intersections of these sequences are not empty; they contain numbers defined by the sequences. Nonempty intersections of these sequences necessitate that all real numbers are in the set $S'$. They are inserted in the set $S'$ by the function $f$. The function $f$ is required to act on all intervals $I'_n = [a_n, b_n]'$ that are in the set $S$, which also includes all intervals of the sequence (7), even if that sequence is not explicitly written because it is implicitly incorporated in the set $S'$, as can be confirmed by AC.

The crucial part of the proof is Lemma 1, which may require more elaboration. On one hand, it is evident that in an infinite converging sequence of algebraic pairs, if one is required to either insert or select a number between each pair, the act of inserting or selecting the number defined by that specific sequence of pairs cannot be avoided since that is the only number in the intersection of all pairs, as follows from either, Dedekind’s cut, the BW theorem, or the NIP. On the other hand, one might contend that between any two algebraic pairs, it is possible to insert a number representing their average value, which would not necessarily be a transcendental number, thus seemingly obviating the need to select or insert in the sequence intervals, at any stage, the number which is defined by the sequence. However, this argument overlooks the infinite structure of the sequence, leading to a fallacy. To clarify this misconception, let us examine the infinite sequence of closed intervals $[-1/n, 1/n]$.

**Lemma 2:** Let $\{I_n\}_{n=1}^\infty$ be an infinite sequence of nested closed intervals given as $I_n = [-\frac{1}{n}, \frac{1}{n}]$ for $n \in \mathbb{N}$. For any function $f : \mathbb{N} \to \mathbb{R}$ assigning numbers $f(n)$ to all intervals $I_n$, with the constraint that $f(n)$ must lie within $I_n$ for all $n$, the inevitability of selecting 0 by $f$ arises not from the individual choices at any finite step, but from the collective convergence of the sequence. This is because by NIP 0 is the only point remaining within all intervals as they converge, i.e., it is the only number that satisfies $-1/n < f(n) < 1/n$ for all $n$.

**Proof of Lemma 2:** The sequence $\{I_n\}_{n=1}^\infty$ contracts towards the center point
0 as \( n \) approaches infinity. By the NIP, the intersection of this infinite sequence of nested nonempty closed intervals is non-empty, containing only the number 0. As every selected number at any step \( n \) must come from the intersection of the first \( n \) intervals, and 0 is the only number in the intersection of all intervals, its selection is unavoidable when going through all intervals.

This implies that the request to insert numbers in all sequence intervals requires that a single number, 0, must be in the intersection. As a result, the number 0 must be inserted in intervals while fulfilling the imposed request, ensuring the presence of numbers in all intervals. The essence of the NIP, in combination with the contracting nature of the intervals towards 0, dictates that 0 is the only number that lies within the intersection of all intervals, i.e., it is the only number within all intervals, as \( n \) extends towards infinity. It is the only number that satisfies, \(-1/n < 0 < 1/n\) for all \( n \). Thus, generating the number 0 with the function \( f \) is unavoidable to satisfy the condition \( f(n) \in I_n \) across the entire sequence, underscoring the inevitability of selecting 0 when considering the sequence's collective behavior and convergence.

As explained earlier it also follows from the Dedekind’s cut \( A/B \) defined by the sequence’s segments \([-1/n, 1/n]\), where endpoints \(-1/n\) define left component \( A \) and endpoints \(-1/n\) form the right component \( B \). The ‘gap’ between \( A \) and \( B \) is precisely at 0, which separates component \( A \) that contains numbers less than 0 and \( B \) that contains numbers greater than 0. Since \( A \) has no greatest element and \( B \) has no least element (within \(-1/n \) and \( 1/n \) numbers), the cut they form corresponds to the real number that fills this gap, which is 0.

Thus, the Dedekind cut \( A/B \) demonstrates that 0 is the only number that can be situated between the two components without belonging to either, providing a unique real number that completes \( \mathbb{R} \) at this point. The construction illustrates that the requirement to insert a number between every pair \(-1/n \) and \( 1/n \) can be fulfilled only by inserting the number 0. There are no other numbers that can be between all pairs, except for the number 0.

This highlights 0 as the distinct number bridging the gap between the left and right components. This reflects the essence of Dedekind’s approach to real numbers: filling in the gaps among rational numbers to form the continuous real number line. Therefore, in the ‘gap’ created by this specific cut in the rational numbers \( 1/n \), 0 is identified as the only fitting number, exemplifying the completeness and continuity of \( \mathbb{R} \) through Dedekind’s construction \cite{18}.

In a sequence \( \left[ -\frac{1}{n}, \frac{1}{n} \right] \) numbers like \( \frac{1}{2n} \) are absent in the intersection of all intervals, having only 0 present in the intersection of all intervals. Excluding 0 (as in \( \left[ -\frac{1}{n}, 0 \right) \cup (0, \frac{1}{n}] \)) leads to an empty intersection. Each interval \( \left[ -\frac{1}{n}, 0 \right) \) contains \(-\frac{1}{n}\), but their intersection becomes empty when all intervals are considered. Similarly, numbers like \( \frac{1}{2n} \) are excluded in the limit, leaving only 0 in
the intersection. Thus, the requirement to place numbers in all sequence intervals cannot be fulfilled if the number 0 is not selected, since the intersection interval will be empty.

Each interval in the sequence \( \left[ -\frac{1}{n}, 0 \right) \) includes by definition the boundary point \(-\frac{1}{n}\), ensuring that there are no empty intervals. This implies that the intersection of the first \( n \) intervals remains nonempty, irrespective of the size of \( n \). Given that each interval occupies a finite \( n \)-th position in the sequence and contains at least the number \(-\frac{1}{n}\), one might assume that the intersection of all intervals, of the entire sequence, is also nonempty, because there is no empty intervals, each contains point \(-\frac{1}{n}\). But contrarily, when considering the entire sequence, this intersection becomes empty, as all \(-\frac{1}{n}\) are progressively excluded. Numbers like \(-\frac{1}{n}\) do not allow to complete traversal through all members of the sequence; there will always be sequence members beyond any given number \(-\frac{1}{n}\). The reason lies in the limit behavior, which differs from the behavior of individual sequence elements. Similarly, in the sequence \([-\frac{1}{n}, \frac{1}{n}]\), while numbers like \(\frac{1}{n}\) fit within individual intervals, they are excluded in the collective limit, leaving only 0 present in all intervals.

In the sequence \([-1/n, 1/n]\), no specific \(n\)-th interval mandates selecting or inserting 0 instead of \(\frac{1}{2n}\). However, when considering the collective scope of all intervals, only 0 remains in their intersection, necessitating its selection. Similarly, in the sequence \([-\frac{1}{n}, 0]\), no single \(n\) leads from a non-empty to an empty intersection. There is no specific \(n\)-th interval where the sequence intersection will change from non-empty to empty. It is the cumulative effect of all intervals as \(n\) approaches infinity that results in this outcome, highlighting the significance of considering the entire sequence collectively. When considering all intervals \([-1/n, 1/n]\) for every natural number \(n\), we are essentially looking for the intersection of these intervals. The intersection of a set of nested intervals like \(\left\{ [-1/n, 1/n] \right\}_{n=1}^{\infty} \) is the single point that is common to all intervals, and in this case, it is 0. Even when different numbers can be inserted in each interval, the requirement to insert numbers within every interval \([-1/n, 1/n]\) across the entire infinite set of such intervals, and the contracting nature of these intervals to a single point, which is 0, necessitates choosing number 0 since it is the only number that fits in every interval of that infinite set.

This is an unexpected result; the requirement imposed on the function \(f\) to insert a single transcendental number in each of the intervals with algebraic endpoints or between any two algebraic numbers is so compelling that it effectively fills all ‘gaps’ in the algebraic numbers, resulting in a continuum and yielding the complete set \(S'\). On the other hand, if transcendental numbers exist between every pair of algebraic numbers, then there can be no ‘gaps’. The existence of ‘gaps’ would imply the presence of algebraic pairs without the transcendental numbers between them, which is impossible given that the function \(f\) inserted transcendental numbers between every such pair of
algebraic numbers.

The reason for this is that in the set $S$, confronting infinite nested sequences, such as the sequence $(7)$, cannot be avoided, and there is a requirement that the function $f$ acts on all segments $I'_n$ in these sequences. Even if the function $f$ chooses an arbitrary number from each single interval, there is no freedom which number the function $f$ will select or generate when acting on all closed nested converging intervals of an infinite sequence. The function $f$ acts on all sequence intervals and always selects or generates numbers that are in the intersection; therefore, it must select or generate the number that will be in the intersection of all intervals when it is going through all of them. Only the number defined by the sequence satisfies that requirement.

It is essential to recognize that the function $f$ operates not only on individual intervals but also on the entire collections of intervals, encompassing every conceivable sequence that can be derived from these intervals. In the set $S$, although no particular sequence is predefined, the set $S$ contains all segments necessary to construct any given sequence. The specific arrangement of these segments within $S$, whether they form a sequence or not, is irrelevant. What matters is that they are included in $S$, and that the function $f$ acts on all of them.

Utilizing the AC, one can extract within $S$ the segments pertinent to any specific sequence and organize them to represent that sequence before the function $f$ operates on them. This organization leads to an overlap of the segments for a nested sequence, guaranteeing that, in a case where segments are converging and the sequence is infinite, a unique number defined by that sequence can only be placed within all these overlapping segments. However, it is not imperative to align and overlap these segments before $f$ operates on them since the outcome should be invariant under the segments' realignments, whether the segments are aligned and overlapping or merely present in the set unaligned, if the function $f$ must act on all of them.

The function $f$ is subjected to the same constraints in choosing numbers that fit all these segments, and the number defined by any specific sequence must be selected regardless of the segments' arrangements. Whether these specific segments representing a sequence are aligned in the form of that sequence before $f$ acts on them, the result must be identical since $f$ will encounter all the same segments of that sequence. The definition of $f$ mandates its action on all segments in $S$, including those which are not arranged or are arranged, representing a specific sequence, with sizes diminishing towards the transcendental number defined by that sequence. Therefore, the selection of this unique number becomes inevitable, as it is the only number that can exist in all such segments, whether they are aligned or not within the sequence. There is no need to require explicit alignments since their nested structure
makes theme effectively aligned.

The proof relies on the axiom of choice, which says if $T S$ is a disjointed set that does not contain the null set, the Cartesian product $P T S$ is different from the null set. In other words, the axiom maintains that its assumptions fulfilled, among the subsets of $U T S$ (or selection-set of $T S$) there is at least one whose intersection with each member of $T S$ is a unit-set. This ensures that after selecting all intervals $[a_n, b_n]$ in the set $S'$, which are the same as the intervals $[\alpha_n, \beta_n]$ in the sequence representing the missing number $T_x$, the intersection of these intervals will contain the single number $T_x$.

This unexpected result and its significance necessitate an independent proof of the completeness of the set $S'$, which is provided by the following theorem.

**Theorem 2:** Define $S'$ as the set comprising all algebraic numbers and only transcendental numbers uniquely determined by the function $f : S \rightarrow \mathbb{R} \setminus \overline{\mathbb{Q}}$, where $S$ includes every algebraic interval $I'_n = [a_n, b_n] \cap \overline{\mathbb{Q}}$ with algebraic endpoints satisfying $a_n \leq b_n$, and each interval $I'_n$ is defined such that it contains only algebraic numbers. For each such interval $I'_n$, the function $f$ assigns a transcendental number $f(I'_n) = t_n$, ensuring $t_n$ falls within $I'_n$ and belongs to the set $\mathbb{R} \setminus \overline{\mathbb{Q}}$, i.e., $t_n$ is not algebraic, and the function $f$ has to act on all intervals $I'_n$ in the set $S$. The set $S'$ constructed in this manner is everywhere dense in the sense of Dedekind, meaning it has no ‘holes’. Specifically, for every open interval within $\mathbb{R}$, there exists at least one element $t_n$ of the set $S'$ within each such interval. These $t_n$ elements act as ‘bridges’ between algebraic numbers, covering all possible transcendental ‘holes’ in the set $S'$.

**Proof of Theorem 2:** The proof is straightforward since if between every two algebraic numbers there is a transcendental number $t_n$, then pairs of algebraic numbers in a Dedekind cut cannot create a ‘hole’ in that set. A ‘hole’ would require that there are algebraic pairs defining the cut between which there is no $t_n$ transcendental number. However, let us formalize this proof. To prove the theorem, it will be shown that any cut $A/B$ in the set $S'$ does not create a ‘hole’ in $S'$ and therefore does not generate new elements in $S'$ since the first component $A$ will have the last element or the second component $B$ will have the first element, or both conditions will occur.

The cut $A/B$ will also make a cut in the set of algebraic numbers $\overline{\mathbb{Q}}$ in $S'$. This cut defines a well-defined number

$$t = (A \cap \overline{\mathbb{Q}})/(B \cap \overline{\mathbb{Q}})$$

(9)

The same cut $A/B$ splits intervals $I''_n = [a_n, b_n] \cap \overline{\mathbb{Q}} \cup T$ with algebraic endpoints $a_n, b_n$ in the set $S'$, positioning all endpoints $a_n$ within the cut component $A$, and all endpoints $b_n$ within component $B$. This configuration leads to an
infinite nested sequence of intervals $I''_n$, facilitated by the everywhere dense nature of $\mathbb{Q}$, with the sizes of the intervals contracting towards zero. The number $t$ is similarly defined by a sequence of these nested intervals with algebraic endpoints $a_n, b_n$

\[ [a_1, b_1]'' \supseteq [a_2, b_2]' \supseteq [a_3, b_3]'' \ldots [a_n, b_n]'' \supseteq [a_{n+1}, b_{n+1}]'' \supseteq \ldots \]  

(10)

Both numbers $t$, defined by the cut (9) in algebraic numbers, and $t$, defined by the sequence (10) of intervals $I''_n$, are the same because they are defined by the same cut $A/B$ in $\mathbb{Q}$ and therefore must be identical.

Since all algebraic intervals of the sequence (10) are nested and nonempty, because the function $f$ by definition inserts a transcendental number in each of them, they will contain numbers common to all intervals, as required by their nested structure. The same numbers appear in all nested intervals $[a_n, b_n]''$ because $[a_{n+1}, b_{n+1}]'' \subseteq [a_n, b_n]''$ for every $n$. In these nested intervals, any new point added by the construction in the interval $[a_{n+1}, b_{n+1}]'$ will also appear in the preceding intervals $[a_n, b_n]', [a_{n-1}, b_{n-1}]', [a_{n-2}, b_{n-2}]', \ldots, [a_1, b_1]'$. Consequently, there will be an intersection (an interval) that will contain these common numbers from the overlapping intervals. Because the sequence is infinite and the intervals are contracting, the interval representing the intersection of all intervals reduces to a zero-size interval with only one number remaining, the number $t$, defined by the sequence (10).

The number in the intersection, by definition, is one of the numbers from the intervals of the sequence (10). Therefore, it is one of the $t_n$ numbers created by the function $f$ when it acted on the intervals $[a_n, b_n]'$, since only the $t_n$ numbers are present in the segments of the sequence (10). Because the numbers generated by the function $f$ are elements of the set $S'$, the number $t$ must be from either component $A$ or component $B$ of the set $S'$.

It will be shown that if $t \in A$, then it is the last element of component $A$.

If $t$ is not the last element of component $A$, then there is

\[ t' \in A \text{ such that } t < t' \]  

(11)

The $t'$ cannot be an algebraic number because $t < t'$ and (9), which states that $t$ is a cut, will require that $t' \in (B \cap \mathbb{Q})$, and therefore $t' \in B$, which contradicts that $t' \in A$. If $t'$ is a transcendental number, then it represents a cut $C/D$ in the set of algebraic numbers $\overline{\mathbb{Q}}$, which creates a hole in $\mathbb{Q}$, and for this reason, the first component $C$ of that cut $t'$ does not have the last element.

Because $t < t'$ implies $(A \cap \overline{\mathbb{Q}})/(B \cap \overline{\mathbb{Q}}) < C/D$, it follows that $(A \cap \overline{\mathbb{Q}}) \subset C$. If
$t''$ is any number from the set $C \setminus A$ (because the set $S'$ is everywhere dense), then $t < t''$ requires $t''$ to be in $B$ (since $t'' \in S' \setminus A = B$).

Contrarily, because $t'' \leq t'$ (as $t'$ represents a cut in $\overline{Q}$ and is in $A$, as referenced in (11)), it requires that $t'' \in A$. However, having $t''$ in both $A$ and $B$ is not possible because $A$ and $B$ are disjoint components of the cut. Therefore, there can be no $t' \in A$ such that $t < t'$.

Thus, if $t \in A$, then it must be the last element of component $A$. Similarly, if $t \in B$, it can be shown in the same manner that $t$ is the first element of the second component $B$.

Therefore, any cut in the set $S'$ does not produce a hole or generate new elements, which substantiates Theorem 2.

**Explicit sequence of real numbers**

The properties of the set $S'$ clearly demonstrate that it supports the Löwenheim-Skolem theorem rather than the diagonal argument. This result, along with the Löwenheim-Skolem theorem, shows that our understanding of the nonde-numerability of real numbers consists in our knowing what it is for this to be proved and not in our grasp of a model.

The set $S'$, while complete and countable, does not provide an explicit sequence of all real numbers. This means that for a specific given number, it is not possible to identify its particular $n$-th position in the sequence. This limitation arises because the numbers are generated arbitrarily by the function $f$. For any given $n$-th position, $f$ may assign infinitely many numbers within its interval of operation. To address this, Example 2 considers a set equivalent to $S'$, which is also generated by inserting transcendental numbers between algebraic numbers. Contrary to the previous example, this method specifies all generated transcendental numbers at each step, thus providing a sequence in which numbers have clearly defined positions. This approach merges the concepts of Gödel’s Constructible Universe ($V = L$) with the earlier proposed Wang’s Σ model [19], which differs from ZF models. However, instead of using Wang’s original work [19], let us modify the model introduced in [20].

The model begins with a 0-th layer, which consists of some denumerable totality of objects, which may be taken to be, for instance, the positive integers or all finite sets built up from the empty set. In our next example, we will consider that the 0-th layer comprises algebraic numbers. Then, the first layer includes all the objects of the 0-th layer and, in addition, all those sets of these objects that correspond to conditions that contain bound variables ranging over objects of the 0-th layer, but no bound variables ranging over the first or higher layers. Generally, the $n+1$-st layer contains all the objects of the $n$-th layer, along with all such sets of these objects determined by conditions whose
bound variables range over objects of the \( n \)-th layer at most. The hierarchy of layers is continued beyond the finite ordinals. Layer \( \omega \), for instance, is the sum of all finite layers, and layer \( \omega + 1 \) contains, in addition, also such sets as are determined by conditions whose bound variables range over the objects of layer \( \omega \) at most. Since all the sets of layer \( \alpha \) are enumerable by a function \( E_\alpha \) ranging over entities of a layer, all sets of any subsystem \( \Sigma_\alpha \) of \( \Sigma \) are therefore enumerable. Therefore, this model requires that any infinite set can be enumerated in an appropriate partial system of \( \Sigma \). We consider here Wang's model, but the conception of cumulative layers (or orders) is not original with Wang [22,23], and there are also other attempts to realize it [21].

In the next example, we present our model. Its 0-th layer consists of a set of algebraic numbers. Layer one includes all numbers from the 0-th layer and all transcendental numbers created by simultaneously applying an infinite number of diagonal constructions on the 0-th layer. The layer \( n + 1 \) is obtained by applying \( \omega \) diagonal constructions on layer \( n \), which also incorporates all previous layers. More details are provided below.

**Example 2.** Let us consider an interval of numbers \([M, W]\), starting by writing a denumerable sequence of all algebraic reals \( A_\nu \) that belong to that interval, as specified in the corresponding array (12). Each number will be represented by an infinite binary sequence, even if it has a finite number of digits, with each digit \( a_{\nu \mu} \) being either 0 or 1:

\[
A_1 = (a_{11}, a_{12}, \ldots, a_{1\mu}, \ldots)
\]
\[
A_2 = (a_{21}, a_{22}, \ldots, a_{2\mu}, \ldots)
\]
\[
\vdots
\]
\[
A_\nu = (a_{\nu 1}, a_{\nu 2}, \ldots, a_{\nu \mu}, \ldots)
\]
\[
\vdots
\]

(12)

The sequence \( S_r \) contains only all algebraic numbers, and each algebraic number has a well-defined place within that sequence. Let us now use Cantor's diagonal construction to generate numbers not contained in this sequence. Such numbers are defined by sequences of digits \( d_{\nu 1}, d_{\nu 2}, \ldots, d_{\nu \mu}, \ldots \), where each \( d_{\nu \mu} \) is either 0 or 1, determined such that \( d_{\nu \nu} \neq a_{\nu \nu} \). From this sequence of \( d_{\nu \mu} \), we formulate the element \( D_1 = (d_{11}, d_{22}, \ldots, d_{\nu \nu}, \ldots) \), ensuring that \( D_1 \neq A_\nu \) for any index \( \nu \). Thus, a number different from all numbers in the sequence containing all algebraic numbers is created. Therefore, it must be a transcendental number. This will be the first transcendental number generated and will be incorporated into the sequence \( S_r \) as the first element, which shifts all elements in the original sequence from the \( n \)-th to the \( n + 1 \)-th place. This recursive construction will be repeated; it will be applied again on the
newly generated set $S_r$, generating in the same way the second transcendental number and placing it in the sequence $S_r$ in the first position, shifting all elements in $S_r$ again, then the third transcendental number, and so on. Please note that all numbers generated in this way are well-defined. All digits $d_{\nu\mu}$ of such generated transcendental numbers $D_1, D_2, \ldots, D_{\nu}, \ldots$ are defined and known. This is because all digits in the initial sequence $S_r$ that contains all algebraic reals are defined. In creating transcendental numbers by Cantor’s diagonal construction, there is no freedom because each diagonal element 0 must be switched to 1, or 1 to 0; one must apply $d_{\nu\mu} \neq a_{\nu\mu}$ for each diagonal element. Since in the initial sequence $S_r$ of algebraic numbers all $a_{\nu\mu}$ are known and fixed, all numbers $D_1, D_2, \ldots$ that are produced with the diagonal construction are defined.

To generate the first layer in our model, the construction will be modified so that in the first step, not only one but infinitely many transcendental numbers are generated. This is possible because we can use all decimal places $a_{11}, a_{12}, \ldots, a_{\mu}, \ldots$ simultaneously and apply an infinite number of diagonal constructions simultaneously in the first step. There will be a diagonal construction that starts with the first decimal place $a_{11}$, one that starts with the second decimal place $a_{12}$, one with the third $a_{13}$, and so on. New transcendental numbers will be created using an infinite number of diagonal constructions, each starting from a different decimal place. After step one, the first layer, denoted as $\Sigma_1$, will consist of an infinite sequence of $\aleph_0$ algebraic numbers from the 0-th layer and an infinite sequence of $\aleph_0$ transcendental numbers produced by an infinite number of diagonal constructions applied to the set of algebraic reals of the 0-th layer. These two sets, the 0-th layer and the transcendental numbers generated by diagonal constructions, will be combined into one sequence by alternating the first number from the 0-th layer with the first number from the list of created transcendental numbers, and so on, to create the first layer, the $\Sigma_1$ set. The construction from step one will be repeated, applying an infinite number of diagonal constructions to the sequence that represents the just created set $\Sigma_1$, creating an additional $\aleph_0$ transcendental numbers that will then be combined with set $\Sigma_1$ to create set $\Sigma_2$. This recursive construction will continue indefinitely, creating an additional $\aleph_0$ of newly created transcendental numbers each time.

However, this construction cannot continue indefinitely; it will stop after $\omega$ steps due to the atomistic structure of real numbers, which prevents endless division of an interval. The process stops exactly after $\omega$ divisions because an infinite division of intervals results in singletons that cannot be divided further. At each step of creating layers, infinite amounts of transcendental numbers are inserted between the algebraic numbers of the 0-th layer, dividing each segment defined by a pair of algebraic numbers further until it becomes a singleton, as follows from the BW theorem or the NIP.
In this way, this example is equivalent to Example 1. In both cases, transcendental numbers are generated and inserted between algebraic numbers, further subdividing the intervals into smaller and smaller segments. The model also represents a variant of forcing, and the forcing technique can be implemented here. Additionally, the diagonal argument cannot be used to create missing numbers not generated by this model because it is already utilized to generate new numbers. Any ‘missing’ number produced by diagonal construction is, in fact, just a number that belongs to this model.

To our knowledge, this is the first presentation of an explicit Wang’s model.

**Remarks to the non-denumerability proofs**

There are many variations of proofs of the uncountability of real numbers. However, fundamentally, all of them share a common basis: they revolve around infinite nested sequences of intervals. We will discuss the critical point that appears in all these proofs, the interpretation of the intersection of these intervals and the nature of the number that resides within this intersection. According to the NIP, the intersection of an infinite nested sequence of closed nonempty intervals must be a zero-size interval, a singleton, or a single number. This number must be one of the elements from the intervals of the sequence, which, in all uncountability proofs, are the numbers of the supposed denumerable sequence assumed to include all numbers. Contrarily, in all these proofs, that number is declared as the number that is outside the denumerable sequence, and its existence is used as the argument that the sequence does not contain all numbers. This discrepancy between the NIP, which posits that the number is a member of the sequence, and the uncountability proofs, which assert the opposite, will be elaborated upon, considering the proofs from 1873 and 1879, and the diagonal argument.

The proofs from 1873 and 1879 employ contradiction, and both begin by assuming that the real numbers in the interval \([\alpha, \beta]\) can be enumerated in a sequence:

\[
\omega_1, \omega_2, ..., \omega_\nu, ...
\]  

(13)

and then establish a contradiction by demonstrating that there exists a number \(\eta\) that is not included in (13). The 1873 article states that “\(\eta\) can be any number in the interval \([A, B]\), where \(A\) and \(B\) are limits \([\omega_\alpha, \omega_\beta]\) when \(\nu = \infty\)”, which is impossible because that is a zero-size interval, essentially a single number, and therefore no number \(\eta\) can reside within that interval. The same assertion appears in the 1879 article. The conclusion is that “the number \(\eta\) will be inside that interval, while all listed numbers will be outside”, which is incorrect because no number can reside inside a single number. These are significant objections that warrant further elaboration.
Let us refute the original proof from 1879 to illustrate the problem. Starting with the sequence (13) and an arbitrary interval $[\alpha, \beta]$, where $\alpha < \beta$, identified can be a real number $\eta$ that does not occur in the sequence (as a member of it).

Some of the numbers from the sequence (13) definitely occur within the interval $[\alpha, \beta]$. Among these, let $\omega_{k_1}$ be the number with the smallest index, and $\omega_{k_2}$ be the number with the next larger index. Let the smaller of these two numbers, $\omega_{k_1}$ and $\omega_{k_2}$, be denoted by $\omega^1_a$, and the larger by $\omega^1_b$. These two numbers define the first closed interval $[\omega^1_a, \omega^1_b]$. Within this interval, there are no numbers $\omega_\nu$ from sequence (13) for which $\nu \leq k_2$, as is immediately evident from the definition of the indices $k_1$ and $k_2$.

Similarly, let $\omega_{k_3}$ and $\omega_{k_4}$ be the two numbers from the sequence with the smallest indices that fall within the interior of the interval $[\omega^1_a, \omega^1_b]$. Let the smaller of these numbers, $\omega_{k_3}$ and $\omega_{k_4}$, be denoted by $\omega^2_a$, and the larger by $\omega^2_b$. It is evident that all numbers $\omega_\nu$ from the sequence (13), for which $\nu \leq k_4$, do not fall into the interior of the interval $[\omega^2_a, \omega^2_b]$.

The interval $[\omega^\nu_a, \omega^\nu_b]$ then lies within the interior of all preceding intervals, and it specifically relates to the sequence (13) in that all numbers $\omega_\nu$, for which $\nu \leq k_2\nu$, definitely do not lie within its interior.

Since the numbers $\omega^1_a, \omega^2_a, \omega^3_a, \ldots, \omega^\nu_a, \ldots$ are continually increasing in value while simultaneously being enclosed within the interval $[\alpha, \beta]$, they have a limit that will be denoted by $A$, such that: $A = \lim_{\nu \to \infty} \omega^\nu_a$.

The same applies to the numbers $\omega^1_b, \omega^2_b, \omega^3_b, \ldots, \omega^\nu_b, \ldots$, which are continually decreasing and also lie within the interval $[\alpha, \beta]$. Let us denote their limit by $B$, so that: $B = \lim_{\nu \to \infty} \omega^\nu_b$. The only interesting case to consider is when $A = B$, to demonstrate that the number: $\eta = A = B$ does not occur in sequence (13).

“If it were a member of the sequence, such as the $\nu$th, then one would have: $\eta = \omega_\nu$. But the latter equation is not possible for any value of $\nu$ because $\eta$ is in the interior of the interval $[\omega^\nu_a, \omega^\nu_b]$, but $\omega_\nu$ lies outside of it.”

The 1879 proof concludes at this point, while the 1873 proof includes an additional explanation: “For all $\nu, \eta \in (\omega^\nu_a, \omega^\nu_b)$ but $\omega_\nu \notin (\omega^\nu_a, \omega^\nu_b)$. Therefore, $\eta$ is a number in $[\alpha, \beta]$ that is not contained in (13)” [15][16].

In the concluding arguments of both proofs, two fundamental issues arise: the technical problem that the proposed construction of a number different from all listed numbers cannot be completed, and the logical problem that assumes if a number does not have an assigned specific $n$-th place in the sequence, it is not in the sequence. It will be demonstrated that the number proposed to be different from all others cannot be constructed as intended.
What is actually constructed is the number \( \eta \), which is not distinct from other numbers. The construction requires an additional step to differentiate it from the listed numbers, but this final step cannot be completed as proposed. Also, the constructed number \( \eta \) does not have a specific \( n \)-th place in the sequence due to the method of its construction; however, this does not imply that it is not part of the sequence. Let us elaborate on both of these points.

The technical problem lies in the statement: “But the latter equation is not possible for any value of \( \nu \) because \( \eta \) is in the interior of the interval \([\omega_\nu^a, \omega_\nu^b]\), but \( \omega_\nu \) lies outside of it. For all \( \nu, \eta \in (\omega_\nu^a, \omega_\nu^b) \) but \( \omega_\nu \not\in (\omega_\nu^a, \omega_\nu^b) \). Therefore, \( \eta \) is a number in \([\alpha, \beta]\) that is not contained in (13)” [15,16].

The problem arises because the constructed number \( \eta = A = B \) is actually from the sequence (13), which can be easily proven using the NIP. The subsequent step, which aims to distinguish \( \eta \) from the other sequence numbers by requiring that it lies within the open interval \((\omega_\nu^a, \omega_\nu^b) \subset [\omega_\nu^a, \omega_\nu^b]\), which eliminates the endpoints \( \omega_\nu^a \) and \( \omega_\nu^b \) in the limit when \( A = B \), cannot be completed.

The sequence elements \( \omega_\nu \) are organized to serve as endpoints in the intervals \([\omega_\nu^a, \omega_\nu^b]\). These intervals form a nested sequence:

\[
[\omega_1^a, \omega_1^b] \supset [\omega_2^a, \omega_2^b] \supset [\omega_3^a, \omega_3^b] \supset \ldots [\omega_\nu^a, \omega_\nu^b] \ldots
\]

that converges to the zero size interval \([A, B]\), which is the number \( \eta = A = B \) defined by that sequence. Each interval of this infinite sequence of nested, closed intervals is nonempty and contains \( \omega_\nu \) numbers from sequence (13), because that sequence is dense everywhere, as assumed by the construction. According to NIP the intersection of this sequence must contain a single number, which must be from sequence (13) since no other numbers are present in the sequence (14) intervals than those from sequence (13). Consequently, the number \( \eta = A = B \) in the intersection of sequence (14) is also a member of sequence (13). The sequence (14) concludes when the endpoints \( \omega_a^\nu \) and \( \omega_b^\nu \) converge, at which point \( A = B \), making \( \eta \) the sole element in all intervals of the sequence. Thus, \( \eta \) exists as an element of both sequences (14) and (13), because all numbers in sequence (14) are derived from sequence (13) and in each interval \([\omega_a^\nu, \omega_b^\nu]\) by construction there are \( \omega_\nu \) numbers. It is one of the numbers \( \omega_\nu \) from sequence (13), defined when ultimately \( \omega_a^\nu = \omega_b^\nu \) at the limit \( A = B \).

There is a misstep in proving that all numbers \( \omega_\nu \) are eliminated, leading to the incorrect conclusion that the number \( \eta \) is not included in sequence (13).

The problem with eliminating all numbers from sequence (13) using open intervals \((\omega_a^\nu, \omega_b^\nu)\) is as follows: For any finite \( \nu \), no interval \((\omega_a^\nu, \omega_b^\nu)\) can exclude
all $\omega_\nu$, while containing the number $\eta$, due to the density of sequence (13). This density implies that any interval $(\omega_\alpha^\nu, \omega_\beta^\nu)$ contains infinitely many numbers $\omega_\nu$ from sequence (13). Therefore, the sequence numbers $\omega_\nu$ must be eliminated only in the limit as $\nu$ approaches infinity. This is proposed to be achieved by designating all numbers $\omega_\nu$ from sequence (13) as the endpoints of the nested intervals $[\omega_\alpha^\nu, \omega_\beta^\nu]$ in sequence (14), and then eliminating them through the open intervals $(\omega_\alpha^\nu, \omega_\beta^\nu)$, which are structured to exclude all endpoints $\omega_\alpha^\nu$ and $\omega_\beta^\nu$, and thus all numbers $\omega_\nu$ in the sequences (13) and (14).

The claim is that in the limit as $\nu \to \infty$, where $\omega_\alpha^\nu = A$ and $\omega_\beta^\nu = B$, and when $A = B$, there exists an open interval $(A, B)$ that eliminates all endpoints $\omega_\alpha^\nu$, $\omega_\beta^\nu$, and is not empty, containing the number $A = B$. To quote: “because $\eta$ is in the interior of the interval $[\omega_\alpha^\nu, \omega_\beta^\nu]$, but $\omega_\nu$ lies outside of it. For all $\nu, \eta \in (\omega_\alpha^\nu, \omega_\beta^\nu)$ but $\omega_\nu \notin (\omega_\alpha^\nu, \omega_\beta^\nu)$.”

However, as explained above, the process concludes when the intervals $[\omega_\alpha^\nu, \omega_\beta^\nu]$ converge to a single point at $[A, B]$, forming a closed, zero-size interval that represents the intersection of sequence (14), a singleton, a single number. This number must reside within all intervals of sequence (14) and is derived from sequence (13).

The final step in the construction, required to make the number $\eta$ distinct from all listed numbers $\omega_\nu$, cannot be completed. This step involves using open intervals $(\omega_\alpha^\nu, \omega_\beta^\nu)$ to eliminate the endpoints of all closed intervals $[\omega_\alpha^\nu, \omega_\beta^\nu]$ and recognize within that open interval $(\omega_\alpha^\nu, \omega_\beta^\nu)$ the number $\eta$. While this procedure can be performed at any finite step $\nu$, it cannot be executed in the limit as $\nu \to \infty$. At this limit, it would require that the intersection $[A, B]$ includes an open interval $(A, B)$ that eliminates all $\omega_\nu$ by removing all endpoints $\omega_\alpha^\nu$, $\omega_\beta^\nu$, and that this open interval also contains the number $\eta = A = B$, purported to be distinct from all $\omega_\nu$.

However, according to the NIP, the intersection $[A, B]$, which is a singleton, a single number, cannot encompass an open interval $(A, B)$. Consequently, there can be no open interval $(A, B)$ that includes a number $\eta = A = B$ different from all numbers in sequence (13). Therefore, the constructed number $\eta$, is no different from all sequence’s numbers.

The logical problem arises from the statement: “$\eta = A = B$ does not occur in sequence (13). If it were a member of the sequence, such as the $\nu^{th}$, then one would have: $\eta = \omega_\nu$. ”

The problem is the misconception that the inability to assign a specific $\nu$-th place in a sequence to a number implies that the number is not in the sequence. This misconception stems from how the number $\eta$ is defined. It is identified by traversing an infinite number of sequence (14) elements, which was necessary to eliminate all $\omega_\nu$ numbers. While $\eta$ can still be one of the
Consider a sequence of all natural numbers: 1, 2, 3, . . . . Begin by rearranging it, moving the number one to the second position, resulting in 2, 1, 3, . . . . Then rearrange it again, this time placing number one in the 3rd position: 2, 3, 1, . . . . Continue this process of rearrangement an infinite number of times, which is feasible since the sequence is infinite. After an infinite number of rearrangements, it becomes impossible to specify the exact $\nu$-th position of the number one. However, it remains within the sequence; it has never been removed, only repositioned.

The situation with the missing number $\eta$, which is defined using an infinite procedure, is similar. After traversing all elements of the sequence (14), infinitely many of them, necessary to eventually eliminate all endpoints of the closed intervals $[\omega_\alpha, \omega_\beta]$, which can only occur in the limit as $\nu \to \infty$ when $\omega_\alpha = \omega_\beta$, $\eta$ is finally identified. Because $\eta$ emerges through an infinite process, going over an infinite number of sequence elements $\omega_\nu$, its $\nu$-th position cannot be specified. However, this does not imply that it is not a member of the sequence, as demonstrated by the given example of the permuted sequence of natural numbers.

One or sometimes both of these problems occur in all other proofs of the non-denumerability of real numbers. For instance, in the diagonal argument, the issue arises because the proposed construction cannot be completed in the limit, which is where the diagonal number is defined. Consequently, it becomes impossible to construct a number that is different from all listed sequence numbers, a point that will be demonstrated.

The diagonal argument, constructs the number $\eta$, designed to differ in the $n$-th decimal place from the $n$-th listed number $\omega_n$, implying that the list of numbers $\omega_n$ is incomplete. This concept can also be interpreted through nested intervals. To facilitate this, we redefine the diagonal number $\eta$ and the sequence of listed numbers $\omega_n$ using a framework of nested intervals instead of the traditional decimal expansion. In this representation, the diagonal argument ensures that at each finite step, $\eta$ differs from each $\omega_n$ by having sequences representing $\eta$ and $\omega_n$ differ in the $n$-th subinterval.

As will be demonstrated, the diagonal argument, when represented through nested intervals, conflicts with the common interpretation that the number $\eta$, which diverges from each $\omega_n$ at finite stages, can also be constructed in the limit, which must be confronted when considering all listed sequence numbers $\omega_n$. The number $\eta$ is fully defined only in the limit, after all its decimal places
in the decimal expansion or all intervals in the nested interval representation are determined, after traversing through all listed numbers. At any finite stage, the construction is feasible since each interval $I_n$ that represents $n$-th decimal place contains two disjoint subintervals, one $H_n$ assigned to $\eta$ and the other $\Omega_n$ to $\omega_n$. However, in the limit, the proposed construction of the diagonal number $\eta$ cannot be completed because it would contradict the NIP. This is because, in the limit, the intersection of intervals $I_n$ reduces to a singleton, a single number. Consequently, in the limit, it becomes impossible for there to be two distinct subintervals, one $H_n$ representing $\eta$ and another $\Omega_n$ representing $\omega_n$, to ensure that $\eta$ differs from all $\omega_n$. Instead, constructed is number $\eta$ that is not different from all other listed numbers. The final step, needed to make it different from all other numbers, which by the construction requires assigning $\eta$ a subinterval $H_n$ that is distinct from the subinterval $\Omega_n$ assigned to number $\omega_n$, cannot be completed. Thus, the proposed construction of $\eta$, achievable at all finite stages, cannot be completed because both $H_n$ and $\Omega_n$ represent the same decimal and therefore belong to the same intervals $I_n$, which at the limit have an intersection that is a singleton, not allowing disjunctive intervals $H_n$ and $\Omega_n$ as required by the construction.

The diagonal argument assumes that the conceptual process of constructing the number $\eta$ can theoretically be extended indefinitely. However, when this concept is translated into a practical framework, such as interval representation, significant implications arise at the limit, leading to the conclusion that such construction cannot be completed. The consideration of the limit is essential because the number $\eta$ is fully defined only in this context. While the diagonal argument, in its original formulation, ensures divergence at each finite stage without explicit concern for the limit, adopting an interval-based approach underscores the need to reconcile this finite-stage divergence with what occurs in the limit. Let us explore this explicitly.

Numbers $\eta$ and $\omega_n$ in decimal representation can be written as

$$0.a_1a_2a_3 \cdots a_n \cdots = \sum_{n=1}^{\infty} a_n 10^{-n} \tag{15}$$

where $a_n$ are 0,1,2,3,4,5,6,7,8,9.

In the interval representation, the decimal places $a_n$ are represented by nested intervals $I_n = [\alpha_n, \beta_n]$, where the sizes of the intervals $I_n = |\alpha_n - \beta_n| = 10^{-n}$ correspond to the $n$-th decimal place positions. These are specified by decimals $a_n$ in (15), ensuring that $a_n \in [\alpha_n, \beta_n]$. Each interval $[\alpha_n, \beta_n]$ represents the $n$-th decimal in (15), with $[\alpha_n, \beta_n] \supset [\alpha_{n+1}, \beta_{n+1}]$.

The numbers $\eta$ and $\omega_n$ are then defined as the intersections of an infinite
sequence of nested intervals

\[ I_1 \supset I_2 \supset I_3 \supset \ldots I_n \supset I_{n+1} \supset \ldots \]  \tag{16}

i.e., they are intersections of these \( I_n \) intervals

\[ \bigcap_{n \in \mathbb{N}} I_n. \]  \tag{17}

The number \( \eta \) constructed by the diagonal argument is defined such that each decimal \( a_n \) differs from the \( n^{th} \) decimal place of the corresponding number \( \omega_n \) in the list. Thus, \( a_n \neq \omega_{nn} \), where \( \omega_{nn} \) represents the \( n^{th} \) digit of the number \( \omega_n \) that occupies the \( n^{th} \) position in the listed sequence of \( \omega_n \) numbers.

In the nested interval representation, the equivalent expression stipulates that at each finite step \( n \), the process involves considering the intersection (17) of the first \( n \) intervals \( I_n \) representing the first \( n \) decimals of the numbers \( \eta \), and \( \omega_n \). Within these intersections, distinct subintervals \( H_n \) for \( \eta \) and \( \Omega_n \) for \( \omega_n \) are selected to ensure divergence. This divergence is created by choosing disjunctive subintervals to guarantee that \( \eta \) and \( \omega_n \) differ at the \( n^{th} \) stage, effectively making the numbers different at each \( n \)-th decimal place or \( n \)-th interval, because they will have different intersections (17):

\[ \eta = \bigcap_{i \in \mathbb{N}} H_i, \quad \text{and} \quad \omega_n = \bigcap_{i \in \mathbb{N}} \Omega_i. \]  \tag{18}

The diagonal argument assumes that the process of creating disjunctive subintervals \( H_i \) and \( \Omega_i \) can continue indefinitely, theoretically allowing for the construction of the number \( \eta \) different from all listed numbers \( \omega_n \). However, this seemingly straightforward assumption contradicts the NIP. At each finite stage in sequence (16), every interval \( I_n = [\alpha_n, \beta_n] \) has a finite size, with \( \alpha_n < \beta_n \) consistently satisfied. This ensures that each interval \( I_n \) contains at least two disjunctive subintervals \( H_n \) and \( \Omega_n \), necessary to construct number \( \eta \) such that it will be different from listed numbers \( \omega_n \). However, this intuitively natural requirement cannot be met when applied to the entire sequence (16) as its intersection interval (17), in the limit, has zero size. In the limit, as \( n \to \infty \), \( \alpha_n \to A \) and \( \beta_n \to B \), and \( A = B \), the intersection (17) of all intervals \( I_n = [\alpha_n, \beta_n] \), i.e., the intersection interval \([A, B]\), has zero size and contains only a single number. So, there is no possibility that in that intersection interval will exist two disjunctive subintervals \( H_n \) and \( \Omega_n \), which is required by diagonal argument to make the intersection (18) representing number \( \eta \) different from the intersections (19) representing a listed number \( \omega_n \). The intersection
theorem dictates that sequence (16) has a zero-size intersection with only one number in it. Intervals $H_n$ representing decimals $a_n$ in number $\eta$ and intervals $\Omega_n$ representing $\omega_n$ in $\omega$ for every $n$ belong to the same interval $I_n = [\alpha_n, \beta_n]$ because they represent the same $n$-th decimal position. Therefore, they must lie within the same intersection (17) of intervals $I_n$ at each step for any $n$. For every $n$ in the construction of numbers $\eta$ and $\omega$, the intersection of the first $n$ intervals $[\alpha_n, \beta_n]$ has non-zero size and contains disjunctive subintervals $H_n$ and $\Omega_n$, allowing $\eta$ to differ from the first $n$ listed numbers $\omega_n$ in their $n$-th decimal places. However, at these finite stages, number $\eta$ is not yet defined or constructed, regardless of how large $n$ is. Number $\eta$ is only defined or constructed in the limit as $n \rightarrow \infty$, when $\alpha_n \rightarrow A$ and $\beta_n \rightarrow B$. But in this limit, the intersection $[A, B]$ is a singleton containing only one number. The requirement at each $n$-th step to select two disjunctive subintervals $H_n \neq \Omega_n$ that make sequence intersections (18) and (19) different, cannot be fulfilled in the limit because the intersection (17) becomes a singleton, a single number. So, the number $\eta$, that is different from all listed numbers cannot be constructed since in the limit do not exist two disjunctive subintervals, required to make the number $\eta$ different from all listed $\omega_n$ numbers.

3 Conclusion

Discussed are differences between various models that generate sets of numbers and how the inclusion of specific axioms influences the cardinality of these generated sets. Considered are some open questions and discrepancies among the current leading models, particularly those represented by V - ultimate L and those based on Martin’s axioms. Additionally, two explicit examples of generating real numbers are considered. It is demonstrated that expanding the ZF model with the AC highlights uncertainties in determining the cardinality of infinite sets. The provided examples are consistent with the Löwenheim-Skolem theorem. It is shown that the AC facilitates the denumerability of real numbers, and issues are identified within proofs of non-denumerability. The second example is also the first sequence representing all real numbers and the first explicit example of Wang’s constructive model.

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