Research Article

A Note on a Modified Cournot-Puu Duopoly

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The aim of this paper is to analyze a classical duopoly model introduced by Puu in 1991 when lower bounds for productions are added to the model. In particular, we prove that the complexity of the modified model is smaller than or equal to the complexity of the seminal one by comparing their topological entropies. We also discuss whether the dynamical complexity of the new model is physically observable.

1. Introduction

In this paper we study a model which is a modification of the well-known duopoly model introduced by Puu as follows (see [1]). Consider a market that consists of two firms which produce equivalent goods with isoelastic demand function:

\[ p = \frac{1}{q_1 + q_2}, \]

where \( q_i, i = 1, 2 \), are the outputs of each firm and \( p \) is the price and \( c_i, i = 1, 2 \), are the constant marginal costs. Under these assumptions, we see that both firms maximize their profits, given by \( \Pi_i = q_i / (q_1 + q_2) - c_i q_i, i = 1, 2 \), if

\[ q_1 = \sqrt{q_2^c_2 - q_2}, \]
\[ q_2 = \sqrt{q_2^c_1 - q_1}. \] (2)

The Cournot point, where both firms maximize their profits at the same time, is given by

\[ q_1^* = \frac{c_2}{(c_1 + c_2)^2}, \]
\[ q_2^* = \frac{c_1}{(c_1 + c_2)^2}. \] (3)

In addition, if \( q_i(t), i = 1, 2 \), are the production of both firms at time \( t \), then, under naive expectations on future, they plan the future production according to (2), and hence the dynamical model is given by

\[ q_1(t + 1) = f_1(q_2(t)) = \max \left\{ \sqrt{q_2^c_1 - q_2}, 0 \right\}, \]
\[ q_2(t + 1) = f_2(q_1(t)) = \max \left\{ \sqrt{q_1^c_2 - q_1}, 0 \right\}. \] (4)

The functions \( f_1 \) and \( f_2 \) are called reaction functions.

A detailed analysis of this model reveals that, when the firms are highly inhomogeneous, that is, when \( c_1 / c_2 \) or \( c_2 / c_1 \) are greater than 6.25, a paradoxical situation arises. We cite a sentence from [2]: “the disadvantages are that the model is no good for dealing with monopoly. As price and quantity are reciprocal, the revenue of a monopolistic firm would be constant, no matter how much the firm sells. On the other hand, any reasonable production cost function increases with output; so producing nothing is the best choice for lowering costs. With constant revenue, the obvious best choice is to actually produce nothing, so avoiding costs, and selling this nothing at an infinite price. The solution has no meaning in terms of substance.” Then, as is introduced in [2] “a more interesting model, satisfying the intuitive economic behavior, is that a state variable \( q_1 \) or \( q_2 \) can become very low, assuming...
a fixed low value, say $\varepsilon$, after which they can increase again."

Then, the reaction functions are as follows:

$$q_1(t + 1) = \frac{f_1(q_2(t))}{c_1} = \begin{cases} \sqrt{\frac{q_2(t)}{c_1}} - q_2(t), & \text{if } q_2(t) \leq \frac{1}{c_1}, \\ \varepsilon, & \text{if } q_2(t) > \frac{1}{c_1} \end{cases}$$

$$q_2(t + 1) = \frac{f_2(q_1(t))}{c_2} = \begin{cases} \sqrt{\frac{q_1(t)}{c_2}} - q_1(t), & \text{if } q_1(t) \leq \frac{1}{c_2}, \\ \varepsilon, & \text{if } q_1(t) > \frac{1}{c_2} \end{cases}$$

The new model is discontinuous, which makes its analysis more difficult than in the continuous case; for instance, the well-known period 3 implies that chaos is valid for continuous interval maps, but it is simple to show that it is not true in general for discontinuous maps. On the other hand, for some parameter values the production can be smaller than $\varepsilon$. For instance, we take $c_1 = 1$ and $c_2 = 6.25$. Then, the square $[1, 1/c_1] \times [0, 1/c_2]$ is invariant under the second iteration of the model and, inside this rectangle, the production of both firms can be as close to zero as we desire because the whole rectangle is an attractor for initial conditions inside it (see, e.g., [3]).

We propose a new model which keeps the idea of a minimal firm production as follows:

$$q_1(t + 1) = f_{c,\varepsilon}(q_2(t)) = \max \left\{ \sqrt{\frac{q_2(t)}{c_1}} - q_2(t), \varepsilon \right\},$$

$$q_2(t + 1) = f_{c,\varepsilon}(q_1(t)) = \max \left\{ \sqrt{\frac{q_1(t)}{c_2}} - q_1(t), \varepsilon \right\},$$

where $\varepsilon > 0$. Figure 1 shows the difference between the reactions functions for the discontinuous case and the model that we propose. Our aim is to analyze this new model and check what is the influence of the parameter $\varepsilon$ on it.

We will organize our paper as follows. Firstly, we will make a preliminary analysis of the model and explain how to reduce it to a one-dimensional model. Then, we will introduce notion of topological entropy that we will use to analyze it and, finally, we will show the result of our analysis for this model.

Figure 1: For $c = 1$ and $\varepsilon = 0.05$ we draw the graph of the discontinuous model (a) and our model (b) for the domain $[0, 2]$. (c) We make a zoom of the graph of our model for $q \in [0, 0.02]$ in order to show that it cannot be smaller than $\varepsilon$. 
We fix the model
\[ q_1(t + 1) = f_{c,\varepsilon}(q_2(t)) = \max \left\{ \sqrt{\frac{q_2(t)}{c_1}} - q_2(t), \varepsilon \right\}, \]
and denote that \( F_{c,\varepsilon} = (f_{c,\varepsilon}, f_{c,\varepsilon}) \), which initially depends on three parameters. It is well-known that the second iterate
\[ F^2_{c,\varepsilon} = (f_{c,\varepsilon} \circ f_{c,\varepsilon}) \times (f_{c,\varepsilon} \circ f_{c,\varepsilon}) \]
and therefore the composition of both maps \( f_{c,\varepsilon} \) and \( f_{c,\varepsilon} \) allows us to analyze some of the dynamic properties of \( F_{c,\varepsilon} \).

Below we summarize basic properties of \( f_{c,\varepsilon} \) which are analogous for \( f_{c,\varepsilon} \):

(i) The graph of \( f_{c,\varepsilon} \) attains its maximal value at 1/4\( c_1 \). In addition, \( f_{c,\varepsilon}(1/4c_1) = 1/4c_1 \). Here, we assume that \( \varepsilon \) is small enough since the values of \( \varepsilon \) which are greater than 1/4\( c_1 \) do not have sense.

(ii) For \( \varepsilon = 0 \), the map \( f_{c,\varepsilon} \) is positive on the interval \((0, 1/c_1)\).

(iii) For \( \varepsilon \geq 0 \), we denote the set \( P_\varepsilon = \{ q \in (0, 1/c_1) : f_{c,\varepsilon}(q) > \varepsilon \} \). If \( c_1 \geq c_2 \), then \( P_\varepsilon \subset P_{\varepsilon_2} \).

Changing the order of maps if necessary, we may assume that \( c_1 \geq c_2 \). Then
\[ \frac{1}{4c_1} \leq \frac{1}{4c_2} \]
and \( f_{c_2,\varepsilon} \circ f_{c_1,\varepsilon} \) attains its maximal value at
\[ f_{c_2,\varepsilon} \left( \frac{1}{4c_1} \right) = \frac{1}{2} \left( \frac{1}{c_1 c_2} - \frac{1}{4c_1} \right). \]
The map \( f_{c_2,\varepsilon} \circ f_{c_1,\varepsilon} \) is unimodal on \([0, 1/c_1]\): it has two monotone pieces, increasing on \((0, 1/4c_1)\) and decreasing otherwise. Moreover, the monotonicity is strict when it is not constant (see Figure 2).

Let \( \varphi(q_1, q_2) = (1/c_2)(q_1, q_2) \), which is a homeomorphism. Then
\[ \varphi^{-1} \circ F_{c_1,\varepsilon} \circ \varphi = \frac{1}{c_2} \left( \max \left\{ \frac{q_2}{c_1 c_2} - \frac{q_2}{c_2}, \varepsilon \right\}, \max \left\{ \frac{q_1}{c_1^2} - \frac{q_1}{c_2}, \varepsilon \right\} \right) \]
\[ = \left( \max \left\{ \frac{q_2}{c_1 c_2}, \varepsilon \right\}, \max \left\{ \frac{q_1}{c_1 c_2} - q_2, \varepsilon \right\} \right), \]
and therefore the maps \( F_{c_1,\varepsilon} \) and \( F_{c_1,\varepsilon} \) are conjugate. Then, from now on, we consider \( c_2 = 1, c_1 = c \geq 1, \varepsilon \geq 0 \), and \( F_{\varepsilon} = F_{c_1,\varepsilon} = (f_{\varepsilon}, f_{\varepsilon}) \).

The dynamics of \( F_{\varepsilon} \) is known (see [4]) and, in some cases, the dynamics of \( F_{c_1,\varepsilon} \) is analogous to that of \( F_{\varepsilon} \). We consider the map \( f_{\varepsilon} \circ f_{\varepsilon} \). Then, if
\[ \left( f_{\varepsilon} \circ f_{\varepsilon} \right) \left( \frac{1}{4c} \right) \leq q_{\varepsilon}, \]
where \( q_{\varepsilon} = \min \{ q \in (1/4c, 1/c) : (f_{\varepsilon} \circ f_{\varepsilon})(q) = f_{\varepsilon}(q) \} \), then the dynamics of \( f_{\varepsilon} \circ f_{\varepsilon} \) and \( f_{\varepsilon} \circ f_{\varepsilon} \) are the same because the dynamics of both maps are located in the interval \([0, 1/4c] \) and \([1/4c, 1/c] \). On the other hand, if we solve the equation
\[ (f_{\varepsilon} \circ f_{\varepsilon})(q) = f_{\varepsilon}(q) \]
we obtain
\[ q_{\varepsilon} = \frac{1 - 2c\varepsilon + \sqrt{1 - 4c\varepsilon}}{2c}, \]
and solving
\[ \left( f_{\varepsilon} \circ f_{\varepsilon} \right) \left( \frac{1}{4c} \right) = q_{\varepsilon}, \]
we get the solution
\[ ε = G(c) = \frac{1}{4} \left( -2 \sqrt{\frac{1}{c}} + \sqrt{\frac{1}{c} - \frac{4}{c} + 1} \right). \quad (16) \]

The graphic of \( G \) is shown in Figure 3.

Hence we see that for any \( ε \), there is a unique \( c_∗ > 1 \) such that (15) holds. If \( c ≤ c_∗ \), the dynamics of \( F_{c,ε} \) remains unaltered, while if \( c > c_∗ \), then changes may appear. Now, we will focus on the problem on whether the parameter \( ε \) influences the dynamics of \( F_{c,ε} \), when \( c > c_∗ \), and, more precisely, how the dynamics of the modified model is related to the model when \( ε = 0 \).

3. On Topological Entropy

In this section, we will study the topological entropy of the system (see [5] or [6]), finding those values for which its entropy is positive. It has been pointed out that positive entropy systems are chaotic in the sense of Li and Yorke (see [7] for definition and [8]). Below, we introduce Bowen's notion (see [6]) of topological entropy for a continuous map \( f : [a, b] → [a, b], a, b ∈ \mathbb{R}, a < b. \)

Fix \( δ > 0 \) and \( n ∈ \mathbb{N} \). A set \( E \subset [a, b] \) is said to be \((n, δ)\)-separated if for any \( x, y ∈ E, x ≠ y \), there is \( k ∈ \{0, 1, \ldots, n − 1\} \) such that \( |f^k(x) − f^k(y)| > δ \). Denote by \( s_n(δ, f) \) the cardinality of a maximal \((n, δ)\)-separated set. The topological entropy of \( f \) is given by
\[ h(f) = \lim_{δ → 0} \lim_{n → ∞} \frac{1}{n} \log s_n(δ, f). \quad (17) \]

Clearly, this notion is not useful when we want to compute the topological entropy for a one parameter family of continuous maps.

The computation of topological entropy for \( F_{c,ε} \), with \( c ∈ [1, 6.25] \), was made in [3] by using the algorithm described in [9]. When \( c > 6.25 \), the topological entropy is always log 2, although the chaotic behavior cannot be showed since almost any orbit seems to converge to \((0, 0)\), which is the paradoxical case that we can avoid with the new model.

Now, we will make use of it with a small modification of the algorithm from [9] as follows. The algorithm is based on several facts. The first one is that the topological entropy of the tent map family,
\[ t_ρ(x) = \begin{cases} px & \text{if } 0 ≤ x ≤ \frac{1}{2}, \\ p − px & \text{if } \frac{1}{2} ≤ x ≤ 1, \end{cases} \quad (18) \]
is \( h(t_ρ) = \log p \), for \( p ∈ [1, 2] \).

The second ingredient of the algorithm is the kneading sequence of an unimodal map \( f \) with maximum (also called turning point) at \( x_0 \). Let \( f^k \) denote the \( k \)th iterate of \( f \) and let \( k(f) = (k_1, k_2, k_3, \ldots) \) be the kneading sequence of \( f \) given by the rule
\[ k_i = \begin{cases} R & \text{if } f^i(x_0) > x_0, \\ C & \text{if } f^i(x_0) = x_0, \\ L & \text{if } f^i(x_0) < x_0. \end{cases} \quad (19) \]
We fix that \( L < C < R \). For two different unimodal maps \( f_1 \) and \( f_2 \), we say that their kneading sequences \( k(f_1) = (k_1, \ldots) \) and \( k(f_2) = (k_2, \ldots) \), we say that \( k(f_1) ≤ k(f_2) \) provided that there is \( m ∈ \mathbb{N} \) such that \( k_1 = k_2^1 \) for \( i < m \) and either an even number of \( k_i \)'s are equal to \( R \) and \( k_m ≷ k_m \) or an odd number of \( k_i \)'s are equal to \( R \) and \( k_m < k_m \). Then
\[ (i) \text{ if } k(f_1) ≤ k(f_2), \text{ then } h(f_1) ≤ h(f_2). \]
In addition, if \( k_m(f) \) denotes the first \( m \) symbols of \( k(f) \), then \( k_m(f_1) < k_m(f_2) \), then \( h(f_1) ≤ h(f_2) \).

So, the algorithm we are going to use for computing the topological entropy of our model reads as follows.

1. Fix \( ε > 0 \) (fixed accuracy) and \( m \) (the kneading sequence length).
2. Compute \( k_m(f) \), where \( f \) is a unimodal map.
3. Fix \( a = 1 \), \( b = 2 \), and \( s = (a + b)/2 \).
4. While \( b − a < ε \), do: if \( k_m(t_s) < k_m(f) \) then \( a = s \); otherwise \( b = s \).

When \( b − a < ε \), we take \( \log(a + b)/2 \) as the topological entropy of \( f \). Note that the algorithm can fail when \( k_m(t_s) \) and \( k_m(f) \) are not comparable. Often, this can be solved by increasing \( m \), but sometimes it is not possible to compare with any accuracy.

3.1. Computation of Topological Entropy of \( F_{c,ε} \). The second iterate of the map \( F_{c,ε} = (f_{c,ε}, f_ε) \) can be written as the product map
\[ F_{c,ε}^2 = (f_{c,ε} ∘ f_ε) × (f_ε ∘ f_{c,ε}). \quad (20) \]
If we denote by \( h(F_{c,ε}) \) the topological entropy of \( F_{c,ε} \) (we refer the reader to [10, Chapter 4] for checking the basic properties of topological entropy), we have that
\[ h(F_{c,ε}) = \frac{1}{2} h(F_{c,ε}^2) = \frac{1}{2} (h(F_{c,ε} ∘ f_ε) + h(f_ε ∘ f_{c,ε})). \quad (21) \]
By commutativity formula for topological entropy (see [11]), we get that $h(f_{c,\varepsilon} \circ f_{c}) = h(f_{c} \circ f_{c,\varepsilon})$, and therefore

$$h(F_{c,\varepsilon}) = h(f_{c,\varepsilon} \circ f_{c}) = h(f_{c} \circ f_{c,\varepsilon}); \quad (22)$$

that is, the topological entropy of the two-dimensional system can be computed from the associated one-dimensional systems.

Now, we fix the pair $(\varepsilon, c_{\varepsilon})$ and compute the topological entropy for these values. Recall that if $c < c_{\varepsilon}$ the topological entropy remains unaltered. Note also that the algorithm cannot be used when we have constant pieces in our maps $f_{c} \circ f_{c},$ and therefore for any $\varepsilon > 0$, $c_{\varepsilon}$ is the highest value of $c$ for which we can use the algorithm. With accuracy $10^{-4}$, we find that $c_{\varepsilon} = 6.1867 \cdots (\varepsilon = 0.000512067)$ is the lowest value of $c_{\varepsilon}$ providing positive topological entropy. Note that the value of $c_{\varepsilon}$ agrees with the smallest value of $c$ providing positive topological entropy when $\varepsilon = 0$. Figure 4 shows our computations.

Intuition tells us that when $c > c_{\varepsilon}$, topological entropy of $F_{c,\varepsilon}$ should not be higher than the topological entropy when $\varepsilon = 0$. The next result shows that it is true.

**Theorem 1.** Let $c > c_{\varepsilon}$. Then $h(f_{c} \circ f_{c,\varepsilon}) \leq h(f_{0} \circ f_{c,0}).$

**Proof.** When $c \geq 6.25$, we have that $h(f_{0} \circ f_{c,0}) = \log 2,$ which is the maximal value for the topological entropy of a unimodal map and therefore there is nothing to prove. So, we assume that $c < 6.25.$

Fix $\delta > 0$. Let $I_{c}$ be the maximal subinterval containing the turning point of $f_{c} \circ f_{c,\varepsilon}$ such that $(f_{c} \circ f_{c,\varepsilon})^{i}(I_{c}) = f_{c}(\varepsilon)$. Fix $n \in \mathbb{N}$, and for $0 \leq i < n,$ let $A_{i} = \{q \in [0,1/c] : (f_{c} \circ f_{c,\varepsilon})^{i}(q) = f_{c}(\varepsilon)\}$. Note that $A_{0} = (f_{c}(\varepsilon))$, $A_{1}$ consists of two constant intervals, and for $i \geq 2$ we have that $A_{i} = I_{c}$ if and only if $(f_{c} \circ f_{c,\varepsilon})^{i-1}(q) \in I_{c}.$ Let $A_{n} = [0,1/c] \setminus \bigcup_{i=1}^{n-1} A_{i}$, and let $E_{n}$ be an $(n, \delta/2)$-separated set of $f_{c} \circ f_{c,\varepsilon}$ contained in $A_{n}$ and with maximal cardinality. Now, let $E_{n}$ be $(\delta, n)$-separated sets with maximal cardinality contained in $A_{i}$, $0 \leq i < n$. Clearly, for any two points $q_{1}, q_{2} \in E_{n}$ there is $k < i$ such that $|((f_{c} \circ f_{c,\varepsilon})^{k}(q_{1}) - (f_{c} \circ f_{c,\varepsilon})^{k}(q_{2})| > \delta.$ There are $\bar{q}_{1}, \bar{q}_{2} \in E_{n}$ such that for $k = 0, \ldots, i-1$

$$\left| (f_{c} \circ f_{c,\varepsilon})^{k}(q_{1}) - (f_{c} \circ f_{c,\varepsilon})^{k}(q_{2}) \right| < \frac{\delta}{2}, \quad j = 1, 2, \quad (23)$$

because otherwise $E_{n}$ would not be maximal. Moreover, $\bar{q}_{1} \neq \bar{q}_{2}.$ Otherwise we would have that

$$\left| (f_{c} \circ f_{c,\varepsilon})^{k}(q_{1}) - (f_{c} \circ f_{c,\varepsilon})^{k}(q_{2}) \right| \leq \left| (f_{c} \circ f_{c,\varepsilon})^{k}(q_{1}) - (f_{c} \circ f_{c,\varepsilon})^{k}(\bar{q}_{1}) \right| + \left| (f_{c} \circ f_{c,\varepsilon})^{k}(\bar{q}_{1}) - (f_{c} \circ f_{c,\varepsilon})^{k}(q_{2}) \right| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

which is a contradiction. Then $\#E_{n} \leq \#E_{0} \leq \#E_{n}.$ Hence

$$\limsup_{n \to \infty} \frac{1}{n} \log s(\delta, n, f_{c} \circ f_{c,\varepsilon})$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \log ((n + 1) \cdot \#E_{n})$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log (\#E_{n})$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log s\left(\delta, n, f_{0} \circ f_{c,0}\right).$$

Taking limits when $\delta$ tends to zero, we conclude that

$$h(f_{c} \circ f_{c,\varepsilon}) \leq h(f_{0} \circ f_{c,0}). \quad (26)$$

**Remark 2.** Following the proof of Theorem 1 we may state the following result. Let $\varepsilon_{1} > \varepsilon_{2} > 0$ and let $c > \max\{c_{\varepsilon_{1}}, c_{\varepsilon_{2}}\}$. Then

$$h(f_{\varepsilon_{1}} \circ f_{c,\varepsilon_{1}}) \leq h(f_{\varepsilon_{2}} \circ f_{c,\varepsilon_{2}}) \leq h(f_{0} \circ f_{c,0}). \quad (27)$$
Hence, for \( \varepsilon > 0 \) and \( c > c_\varepsilon \), the topological entropy of \( f_\varepsilon \circ f_\varepsilon \) holds the inequalities
\[
h(f_\varepsilon \circ f_\varepsilon) \leq h(f_\varepsilon) \leq h(f_0 \circ f_0). \tag{28}
\]
In particular, when \( h(f_\varepsilon \circ f_\varepsilon) > 0 \), we have that the map \( f_\varepsilon \circ f_\varepsilon \) has also positive topological entropy and therefore it is chaotic in the sense of Li and Yorke. However, for \( \varepsilon > 0.000512067 \cdots \) we obtain zero topological entropy and therefore Theorem 1 is not useful. At least, we can state the following result that guarantees that topological entropy is positive for \( c > c_\varepsilon \), when \( \varepsilon > 0.000512067 \cdots \).

**Proposition 3.** Let \( \bar{q}_\varepsilon \) be the maximum of \( I_\varepsilon \), where \( I_\varepsilon \) is the maximal subinterval containing the turning point of \( f_\varepsilon \circ f_\varepsilon \) such that \( (f_\varepsilon \circ f_\varepsilon)^2(I_\varepsilon) = f_\varepsilon(I_\varepsilon) \). If \( (f_\varepsilon \circ f_\varepsilon)(f_\varepsilon(q_\varepsilon)) \leq \bar{q}_\varepsilon \), then \( h(f_\varepsilon \circ f_\varepsilon) > 0 \).

**Proof.** Consider \( I_1 = [f_\varepsilon(q_\varepsilon), \bar{q}_\varepsilon] \) and \( I_2 = [\bar{q}_\varepsilon, q_\varepsilon] \). Then \( [f_\varepsilon(q_\varepsilon), \bar{q}_\varepsilon] \cup [\bar{q}_\varepsilon, q_\varepsilon] = (f_\varepsilon \circ f_\varepsilon)((\bar{q}_\varepsilon, q_\varepsilon)) \) and \( [\bar{q}_\varepsilon, q_\varepsilon] \), then we construct the transition matrix
\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},
\]
where the element \( i, j \) in \( A \) is 1 if and only if \( I_I \subseteq (f_\varepsilon \circ f_\varepsilon)(I_\varepsilon) \).

By [10, Chapter 4], \( h(f_\varepsilon \circ f_\varepsilon) \geq \log(\rho(A)) = \log(1 + \sqrt{5})/2 \), which concludes the proof.

For instance, for \( c = 6.5 \) and \( \varepsilon = 10^{-8} \), we have that
\[
0.0580013 = (f_\varepsilon \circ f_\varepsilon)(f_\varepsilon(q_\varepsilon)) < \bar{q}_\varepsilon = 0.0600496, \tag{30}
\]
and then the above result guarantees that the topological entropy is positive.

Two questions still remain. The first one, which we cannot solve here, is how to compute with prescribed accuracy the topological entropy of \( f_\varepsilon \circ f_\varepsilon \) for \( c > c_\varepsilon \). The second one, which will be analyzed in the next subsection, is to study when such topological chaoticity can be observed in numerical simulations and we will show that, in all the examples we have considered, it never appears.

### 3.2. On Chaos and Further Discussions

As we pointed out in the introduction, positive topological entropy implies that the map (or the system) is chaotic in the sense of Li and Yorke (see [7]). We recall that \( F_{\varepsilon,\varepsilon} \) is Li-Yorke chaotic if there is an uncountable set \( S \subseteq [0, \infty)^2 \) such that, for any \( (x_1, y_1), (x_2, y_2) \in S, (x_1, y_1) \neq (x_2, y_2) \), it is held that
\[
\liminf_{n \to \infty} \left\| F_{\varepsilon,\varepsilon}^n(x_1, y_1) - F_{\varepsilon,\varepsilon}^n(x_2, y_2) \right\| = 0,
\]
\[
\limsup_{n \to \infty} \left\| F_{\varepsilon,\varepsilon}^n(x_1, y_1) - F_{\varepsilon,\varepsilon}^n(x_2, y_2) \right\| > 0. \tag{31}
\]
In [12] it was shown that \( F_{\varepsilon,\varepsilon} \) is Li-Yorke chaotic if and only if both maps \( f_\varepsilon \circ f_\varepsilon \) and \( f_\varepsilon \circ f_\varepsilon \) are also Li-Yorke chaotic. Although the existence of Li-Yorke chaotic maps with zero topological entropy is known even for continuous interval maps (see, e.g., [13]), such maps do not exist in Puu’s model for \( \varepsilon = 0 \) (see [3]).

However, even when \( \varepsilon = 0 \), the existence of topological chaos in the sense of Li and Yorke could not be observed in numerical simulations; that is, it would not be physically observable. We refer to [3] for an explanation of this fact when \( \varepsilon = 0 \). Here we focus on the new case when \( \varepsilon > 0 \).

Let \( \varepsilon > 0 \). The case \( c \leq c_\varepsilon \) is analogous to \( \varepsilon = 0 \), so again we refer to [3]. So, let \( c > c_\varepsilon \). Recall that \( I_\varepsilon \) is an interval containing the turning point of \( f_\varepsilon \circ f_\varepsilon \), such that \( (f_\varepsilon \circ f_\varepsilon)^2(I_\varepsilon) = \{ f_\varepsilon(\varepsilon) \} \).

If \( h(f_\varepsilon \circ f_\varepsilon) > 0 \), there is an uncountable scrambled set which will be contained in a (possibly with zero Lebesgue measure) invariant subset of \( f_\varepsilon \circ f_\varepsilon \). The chaotic behavior can be detected if \( f_\varepsilon(\varepsilon) \) belongs to such invariant set. In practice, we study the orbit of \( f_\varepsilon(\varepsilon) \) to analyze whether it is periodic or not. Clearly, when it is periodic we do not observe such chaotic behavior. We make simulations with \( c \in [6.1, 6.7] \), with step size 0.003 and \( \varepsilon \in [0, 0.0005] \) with step size 2.5 \times 10^{-6} obtaining that \( f_\varepsilon(\varepsilon) \) is periodic. In Figure 5 we show the different regions where the period of \( f_\varepsilon(\varepsilon) \) remains constant.

In addition, we fix several values of \( \varepsilon \) and vary \( c \in [6.18, 6.25] \), with step size 10^{-4}. Again, we always observe that the orbit of \( f_\varepsilon(\varepsilon) \) is periodic, although the periods can oscillate among a wide range of values, some of them being a power or two and some others not. When the period is not a power of two we can state that the map is chaotic due to the fact that in that case the topological entropy is positive (see, e.g., [10, Chapter 4]). In any case, the chaotic behavior cannot be observed. Figures 6-7 show the period of \( f_\varepsilon(\varepsilon) \) and bifurcation diagrams for some values of \( \varepsilon \). The reader should be advertised that they do not show any chaotic behavior.
Figure 6: (a) We show the period of $f_\epsilon(x)$ for $\epsilon = 10^{-5}$ when $c$ (X axis) ranges the interval $(c, 6.25)$. (b) We draw the bifurcation diagram of the orbit of $f_\epsilon(x)$. We compute 20,200 points and draw the last 200.

Figure 7: (a) We show the period of $f_\epsilon(x)$ for $\epsilon = 2 \cdot 10^{-4}$ when $c$ (X axis) ranges the interval $(c, 6.25)$. (b) We draw the bifurcation diagram of the orbit of $f_\epsilon(x)$. We compute 20,200 points and draw the last 200.

However, the situation is not so simple as numerical simulation shows. We recall briefly the notion of metric attractor introduced in [14]. Let $\varphi : [a, b] \rightarrow [a, b]$, with $a < b$ real numbers, be continuous. For $x \in [a, b]$, denote the set of limit points of the sequence $(\varphi^n(x))$ by $\omega(x, \varphi)$, which will be called the $\omega$-limit set of $x$ under $\varphi$. Recall that a metric attractor is a subset $A \subset [a, b]$ such that $\varphi(A) \subseteq A$, $\mathcal{B}(A) = \{x : \omega(x, \varphi) \subset A\}$ has positive Lebesgue measure, and there is no proper subset $A' \subset A$ with the same properties. Then, the following result shows that many nonexpected metric attractors can appear for our model.

**Proposition 4.** Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be unimodal with turning point $c$, such that $\varphi$ is strictly increasing on $(0, c)$ and strictly decreasing on $(c, \infty)$. Assume that $\varphi(0) = 0$, and $\varphi(x) > x$ for all $x \in (0, c)$. Let $x_0 > 0$ be such that $\varphi(x_0) = 0$, let $\epsilon > 0$, and define $\varphi_\epsilon(x) = \max\{\epsilon, \varphi(x)\}$. Assume that $\varphi(c) > x_0$. Let $\Omega$ be a $\omega$-limit set of $\varphi$ such that $\min \Omega > 0$. Then there is $\epsilon > 0$ such that $\Omega$ is a metric attractor of $\varphi_\epsilon$.

**Proof.** Let $y$ be such that its omega limit set $\omega(y, \varphi) = \Omega$. Take $\epsilon = y$. Then $\varphi_\epsilon(c) = y$ and since $\Omega$ is invariant by $\varphi$, we have that the $\omega$-limit set $\omega(c, \varphi_\epsilon) = \omega(y, \varphi) = \Omega$. Now, let $J$ be the interval such that $c \in J$ and $\varphi_\epsilon^2(J) = J$. Clearly $J$ has positive Lebesgue measure and then for all $x \in J$ we have that $\omega(x, \varphi_\epsilon) = \omega(c, \varphi_\epsilon)$ and therefore $\Omega$ is a metric attractor of $\varphi_\epsilon$.

Clearly, Proposition 4 can be applied to our model and so, a wide range of finite and infinite attractors, and not only stable periodic attractors, can appear. For instance, we take $c = 8$ and

$$\epsilon = \frac{185 \sqrt{17} - 609 - 8 \sqrt{9065 \sqrt{17} - 37009}}{14336}. \quad (32)$$

Then

$$\left(f_\epsilon \circ f_{c, \epsilon}\right)(\epsilon) = x_1 = \frac{2625 - 409 \sqrt{17} - 8 \sqrt{20041 \sqrt{17} - 71281}}{14336}.$$
\((f_\varepsilon \circ f_\varepsilon)(x_1) = x_2\)

\[
\frac{2625 - 409\sqrt{17} + 8\sqrt{20041\sqrt{17} - 71281}}{14336},
\]

and \((f_\varepsilon \circ f_\varepsilon)(x_2) = x_1\); that is, \(x_1\) is a periodic orbit of period 2 which is unstable since

\[
\|(f_\varepsilon \circ f_\varepsilon)'(x_1) \cdot (f_\varepsilon \circ f_\varepsilon)'(x_2)\| = 125.799. \tag{34}
\]

Moreover, since the orbit of any point \(q \in I_\varepsilon\) is mapped eventually to the orbit of \(x_1\), then the unstable periodic orbit is in fact a metric attractor since the Lebesgue measure of \(I_\varepsilon\) is obviously positive. In this example \(f_\varepsilon(\varepsilon)\) is a periodic orbit. However, if we choose

\[
\varepsilon = \left(213791 - 124103\sqrt{17}
+ 8\left(385 + 39\sqrt{17}\right)\sqrt{3409\sqrt{17} - 9401}\right)
\times \left(7168 \left(39\sqrt{17} - 6783 - 8\sqrt{3409\sqrt{17} - 9401}\right.
- 32 \left(14 \left(3199 - 39\sqrt{17}
+ 8\sqrt{3409\sqrt{17} - 9401}\right)\right)^{1/2}\right)^{-1}, \tag{35}
\]

then the orbit of \(x_1\) defined above is unstable and two-periodic but \(f_\varepsilon(\varepsilon)\) is no longer periodic. In this case, if we work with a computer with finite precision we would observe a periodic orbit of period 7, and therefore, computer simulations do not show the real dynamical behavior. It is clear that the above argument cannot be used for showing infinite attractors in numerical simulations because you need to know the exact value of a point in the \(\omega\)-limit set and this is impossible due to the square root which defines the model.

So, chaotic behavior is not shown in numerical simulations for two reasons. The first one is that \(\Omega\) in Proposition 4 has probably zero Lebesgue measure and since \(\varepsilon\) is closely related to the minimum value of \(\Omega\), the probability of finding such \(\varepsilon\) is zero. Even if a suitable \(\varepsilon\) is found, computer simulations do not show the real dynamical behavior due to round-off effects. In other words, the model is not stable under small perturbations at those parameter values providing chaotic attractors and the numerical simulations show only periodic orbits. In contrast, when the point \(f_\varepsilon(\varepsilon)\) is periodic, its orbit is superstable; that is, the derivative along its periodic orbit is zero and therefore is robust under small perturbations.

Remark 5. With a small variation, Proposition 4 and the following comments remain valid for the discontinuous model introduced in [2]. In fact, the discontinuous model can be written as \((f_0 \circ f_0)(q)\) if \(q \in [0,1/c]\), and \(\sqrt{\varepsilon} - \varepsilon\) otherwise.

Now, the arguments are simpler than in the continuous case. Namely, let \(q_0\) be such that the \(\omega\)-limit set \(\omega(q_0, f_\varepsilon \circ f_\varepsilon) = A \subset (0, 1/c)\). Clearly, if we choose \(\varepsilon\) such that \(f_\varepsilon(\varepsilon) = q_0\), then \(A\) is a metric attractor since it is the \(\omega\)-limit set of any point from \(I_\varepsilon\). Clearly, such metric attractor cannot be detected in computer simulations and therefore the nonobservability of chaos in computer simulations is not due to the fact that for all \(\varepsilon > 0\) the only possible attractor is the superstable periodic orbit of \(f_\varepsilon(\varepsilon)\). In particular, the sentence “we have now that almost all the trajectories are converging to a superstable cycle, whose period depends on the parameters values” stated without proof at the end of page 20 in [2] is false.

4. Conclusion

We have introduced a continuous duopoly model by making a slight modification of Puu’s duopoly in [1]. We analyze the dynamics of the model from three points of view, topological, physical, and numerical. From the topological point of view, the system is proved to be chaotic for a wide range of parameters. However, when \(c\) exceeds a critical value \(c_\varepsilon\), the chaotic behavior is not observed in numerical simulations. Although there are Milnor attractors (and hence physically observable), they are not observed in numerical simulations. Probably, this is due to the fact that they can appear for a zero Lebesgue measure set in the parameter space. Anyway, the combined approach of topological, physical, and numerical analysis shows rich dynamics.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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