PAULI OPERATORS AND THE $\overline{\partial}$-NEUMANN PROBLEM

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Abstract. We apply methods from complex analysis, in particular the $\overline{\partial}$-Neumann operator, to investigate spectral properties of Pauli operators.

1. Introduction

Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$ be a $C^2$-function. We consider the Schrödinger operators with magnetic field of the form

$$P_\pm = -\Delta_A \pm V,$$

also called Pauli operators, where

$$A = \frac{1}{2} \left( -\frac{\partial \varphi}{\partial y_1} \frac{\partial \varphi}{\partial x_1}, \ldots, -\frac{\partial \varphi}{\partial y_n} \frac{\partial \varphi}{\partial x_n} \right)$$

is the magnetic potential and

$$\Delta_A = \sum_{j=1}^n \left[ \left( -\frac{\partial}{\partial x_j} - \frac{i}{2} \frac{\partial \varphi}{\partial y_j} \right)^2 + \left( -\frac{\partial}{\partial y_j} + \frac{i}{2} \frac{\partial \varphi}{\partial x_j} \right)^2 \right],$$

and $V = \frac{1}{2} \Delta \varphi$; we wrote elements of $\mathbb{R}^{2n}$ in the form $(x_1, y_1, \ldots, x_n, y_n)$; we will identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, writing $(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$, this is mainly because we will use methods of complex analysis to analyze spectral properties of the above Schrödinger operators with magnetic field.

For $n = 1$, there is an interesting connection to Dirac and Pauli operators: recall the definition of $A$ in this case and define the Dirac operator $D$ by

$$(1.1) \quad D = (-i \frac{\partial}{\partial x} - A_1) \sigma_1 + (-i \frac{\partial}{\partial y} - A_2) \sigma_2 = A_1 \sigma_1 + A_2 \sigma_2,$$
where
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \]
Hence we can write
\[ D = \begin{pmatrix} 0 & A_1 - iA_2 \\ A_1 + iA_2 & 0 \end{pmatrix}. \]
We remark that \( i(A_2A_1 - A_1A_2) = B \) and hence it turns out that the square of \( D \) is diagonal with the Pauli operators \( P_\pm \) on the diagonal:
\[ D^2 = \begin{pmatrix} A_1^2 + i(A_2A_1 - A_1A_2) + A_2^2 & 0 \\ 0 & P_- \end{pmatrix}, \]
where
\[ P_\pm = \left( -i \frac{\partial}{\partial x} - A_1 \right)^2 + \left( -i \frac{\partial}{\partial y} - A_2 \right)^2 \pm B = -\Delta_A \pm B, \]
see [3] and [10].

Our aim is to investigate spectral properties of the Pauli operators \( P_\pm \). For this purpose we will use methods from complex analysis, the weighted \( \overline{\partial} \)-complex. We suppose that \( \varphi : \mathbb{C}^n \rightarrow \mathbb{R} \) is a plurisubharmonic \( C^2 \)-function.

Let
\[ L^2(\mathbb{C}^n, e^{-\varphi}) = \{ g : \mathbb{C}^n \rightarrow \mathbb{C} \text{ measurable} : \| g \|^2_{\varphi} = (g, g)_\varphi = \int_{\mathbb{C}^n} |g|^2 e^{-\varphi} d\lambda < \infty \}. \]

Let \( 1 \leq q \leq n \) and
\[ f = \sum_{|J|=q} f_j d\overline{z}_J, \]
where the sum is taken only over increasing multiindices \( J = (j_1, \ldots, j_q) \) and \( d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q} \) and \( f_j \in L^2(\mathbb{C}^n, e^{-\varphi}) \).

We write \( f \in L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \) and define
\[ \overline{\partial}f = \sum_{|J|=q} \sum_{j=1}^n \frac{\partial f_j}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J \]
for \( 1 \leq q \leq n-1 \) and
\[ \text{dom}(\overline{\partial}) = \{ f \in L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) : \overline{\partial}f \in L^2_{(0,q+1)}(\mathbb{C}^n, e^{-\varphi}) \}, \]
where the derivatives are taken in the sense of distributions.

In this way \( \overline{\partial} \) becomes a densely defined closed operator and its adjoint \( \overline{\partial}_\varphi^* \) depends on the weight \( \varphi \).

We consider the weighted \( \overline{\partial} \)-complex
\[ L^2_{(0,q-1)}(\mathbb{C}^n, e^{-\varphi}) \xrightarrow{\overline{\partial}} L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \xrightarrow{\overline{\partial}} L^2_{(0,q+1)}(\mathbb{C}^n, e^{-\varphi}) \]
and we set
\[ \Box^{(0,q)}_\varphi = \overline{\partial}_\varphi^* \overline{\partial} + \overline{\partial}_\varphi \overline{\partial}, \]

where
where
\[ \text{dom}(\Box^{(0,q)}) = \{ u \in \text{dom}(\partial) \cap \text{dom}(\partial_{\varphi}^*) : \partial u \in \text{dom}(\partial_{\varphi}^*), \partial_{\varphi}^* u \in \text{dom}(\partial) \}. \]

It turns out that \( \Box^{(0,q)} \) is a densely defined, non-negative self-adjoint operator, which has a uniquely determined self-adjoint square root \( (\Box^{(0,q)})^{1/2} \). The domain of \( (\Box^{(0,q)})^{1/2} \) coincides with \( \text{dom}(\partial) \cap \text{dom}(\partial_{\varphi}^*) \), which is also the domain of the corresponding quadratic form

\[ Q_{\varphi}(u, v) := (\partial u, \partial v)_{\varphi} + (\partial_{\varphi}^* u, \partial_{\varphi}^* v)_{\varphi}, \]

and \( \text{dom}(\Box^{(0,q)}) \) is a core of \( (\Box^{(0,q)})^{1/2} \), see for instance [4].

Next we consider the Levi matrix

\[ M_{\varphi} = \left( \frac{\partial^2 \varphi}{\partial z_j \partial z_k} \right)_{j,k=1}^n \]

and suppose that the lowest eigenvalue \( \mu_{\varphi} \) of \( M_{\varphi} \) satisfies

\[ \lim \inf_{|z| \to \infty} \mu_{\varphi}(z) > 0. \tag{1.2} \]

(1.2) implies that \( \Box^{(0,1)} \) is injective and that the bottom of the essential spectrum \( \sigma_e(\Box^{(0,1)}) \) is positive (Persson’s Theorem), see [6]. Now it follows that \( \Box^{(0,1)} \) has a bounded inverse, which we denote by

\[ N^{(0,1)} : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}). \]

Using the square root of \( N^{(0,1)} \) we get the basic estimates

\[ \|u\|_{\varphi}^2 \leq C(\|\partial u\|_{\varphi}^2 + \|\partial_{\varphi}^* u\|_{\varphi}^2), \tag{1.3} \]

for all \( u \in \text{dom}(\partial) \cap \text{dom}(\partial_{\varphi}^*) \), see [5] for more details.

In the following it will be important to know conditions on \( \varphi \) implying that the Bergman space of entire functions

\[ A^2(\mathbb{C}^n, e^{-\varphi}) := L^2(\mathbb{C}^n, e^{-\varphi}) \cap \mathcal{O}(\mathbb{C}^n) \]

is infinite dimensional. This space coincides with \( \ker \partial \), where

\[ \partial : L^2(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}). \]

If \( n = 1 \), we can use the following concept: let \( D(z, r) = \{ w : |z - w| < r \} \); a non-negative Borel measure \( \mu \) on \( \mathbb{C} \) is doubling, if there exists a constant \( C > 0 \) such that for any \( z \in \mathbb{C} \) and any \( r > 0 \)

\[ \mu(D(z, r)) \leq C \mu(D(z, r/2)). \tag{1.4} \]

It can be shown that

\[ \mu(D(z, 2r)) \geq (1 + C^{-3}) \mu(D(z, r)), \tag{1.5} \]

for each \( z \in \mathbb{C} \) and for each \( r > 0 \); in particular \( \mu(\mathbb{C}) = \infty \), unless \( \mu(\mathbb{C}) = 0 \) (see [9]).

Example: if \( p(z, \overline{z}) \) is a polynomial on \( \mathbb{C} \) of degree \( d \), then

\[ d\mu(z) = |p(z, \overline{z})|^a d\lambda(z), \quad a > -\frac{1}{d} \]

is a doubling measure on \( \mathbb{C} \), see [9].
Theorem 1.1. [2], [7] Let $\varphi : \mathbb{C} \rightarrow \mathbb{R}_+$ be a subharmonic $C^2$-function. Suppose that $d\mu = \Delta \varphi \, d\lambda$ is a non-trivial doubling measure. Then the weighted space of entire functions

$$A^2(\mathbb{C}, e^{-\varphi}) = \{ f \text{ entire} : \|f\|^2_\varphi = \int_\mathbb{C} |f|^2 e^{-\varphi} \, d\lambda < \infty \}$$

is of infinite dimension.

More general, in $\mathbb{C}^n$, Hörmanders $L^2$-estimates for the solution of the inhomogeneous Cauchy-Riemann equations yield

Theorem 1.2. [8], [5] Suppose that the lowest eigenvalue $\mu_\varphi$ satisfies

$$\lim_{|z| \to \infty} |z|^2 \mu_\varphi(z) = +\infty.$$  

Then the weighted space of entire functions

$$A^2(\mathbb{C}^n, e^{-\varphi}) = \{ f \text{ entire} : \|f\|^2_\varphi = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \, d\lambda < \infty \}$$

is of infinite dimension.

Concerning compactness of the $\overline{\partial}$-Neumann operator we have the following result:

Theorem 1.3. [5] Let $1 \leq q \leq n$. Suppose that the sum $s_q$ of the smallest $q$ eigenvalues of the Levi matrix $M_\varphi$ satisfies

$$\lim_{|z| \to \infty} s_q(z) = +\infty.$$  

Then the $\overline{\partial}$-Neumann operator

$$N_{\varphi}^{(0,q)} : L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \rightarrow L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$$

is compact.

The next result asserts that compactness percolates up the $\overline{\partial}$-complex.

Theorem 1.4. [5] Let $1 \leq q \leq n - 1$. Suppose that $N_{\varphi}^{(0,q)}$ is compact. Then $N_{\varphi}^{(0,q+1)}$ is also compact.

We will also consider special weight functions, the so-called decoupled weights, and, using the tensor product structure of the essential spectrum $\sigma_e(\Box_{\varphi}^{(0,q)})$ we get the following (see [1])

Theorem 1.5. Let $\varphi_j \in C^2(\mathbb{C}, \mathbb{R})$ for $1 \leq j \leq n$ with $n \geq 2$, and set

$$\varphi(z_1, \ldots, z_n) := \varphi_1(z_1) + \cdots + \varphi_n(z_n).$$

Assume that all $\varphi_j$ are subharmonic and such that $\Delta \varphi_j$ defines a nontrivial doubling measure. Then

(i) $\dim(\ker(\Box_{\varphi}^{(0,0)})) = \dim(A^2(\mathbb{C}^n, e^{-\varphi})) = \infty$, where $\Box_{\varphi}^{(0,0)} = \partial^*_\varphi \partial$,

(ii) $\ker(\Box_{\varphi}^{(0,q)}) = \{0\}$, for $q \geq 1$,

(iii) $N_{\varphi}^{(0,q)}$ is bounded for $0 \leq q \leq n$,

(iv) $N_{\varphi}^{(0,q)}$ with $0 \leq q \leq n - 1$ is not compact, and
(v) $N_\varphi^{(0,n)} = \overline{\partial} \partial \varphi$ is compact if and only if

$$\lim_{|z| \to \infty} \int_{B_1(z)} \text{tr}(M_\varphi) \, d\lambda = \infty,$$

where $B_1(z) = \{ w \in \mathbb{C}^n : |w - z| < 1 \}$.

2. Pauli operators

Now we apply the results on the weighted $\overline{\partial}$-Neumann operator to derive spectral properties of the Pauli operators and discuss some special examples.

**Theorem 2.1.** Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic $C^2$-function. Suppose that the smallest eigenvalue $\mu_\varphi$ of the Levi matrix $M_\varphi$ satisfies

$$(2.1) \quad \lim_{|z| \to \infty} \mu_\varphi(z) = \infty.$$  

Let

$$A = \frac{1}{2} \left( -\frac{\partial \varphi}{\partial y_1}, \frac{\partial \varphi}{\partial x_1}, \ldots, -\frac{\partial \varphi}{\partial y_n}, \frac{\partial \varphi}{\partial x_n} \right)$$

and $V = \frac{1}{2} \Delta \varphi$. Then the Pauli operator $P_- = -\Delta_A - V$ fails to have a compact resolvent, whereas the Pauli operator $P_+ = -\Delta_A + V$ has a compact inverse operator acting on $L^2(\mathbb{R}^{2n})$.

**Proof.** For the proof we first consider the complex Laplacian $\Box^{(0,0)} = \overline{\partial} \partial$, which acts on $L^2(\mathbb{C}^n, e^{-\varphi})$ at the beginning of the weighted $\overline{\partial}$-complex as a non-negative self-adjoint, densely defined operator, we take the maximal extension from $\mathcal{C}_0^\infty(\mathbb{C}^n)$, as $\Box^{(0,0)}$ is essentially self-adjoint, there is only one self-adjoint extension. For $f \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ we get

$$\Box^{(0,0)} f = \overline{\partial} \partial f = -\sum_{j=1}^n \left( \frac{\partial}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} \right) \frac{\partial f}{\partial \overline{z}_j}.$$ 

Now we apply the isometry

$$U_\varphi : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n, e^{-\varphi})$$

defined by $U_\varphi(g) = e^{\varphi/2} g$, for $g \in L^2(\mathbb{C}^n)$, and afterwards the isometry

$$U_{-\varphi} : L^2(\mathbb{C}^n, e^{-\varphi}) \to L^2(\mathbb{C}^n)$$

defined by $U_{-\varphi}(f) = e^{-\varphi/2} f$, for $f \in L^2(\mathbb{C}^n, e^{-\varphi})$. Hence we get

$$e^{-\varphi/2} \Box^{(0,0)}(e^{\varphi/2} g)$$

$$= \sum_{j=1}^n \left( -\frac{\partial^2 g}{\partial z_j \partial \overline{z}_j} + \frac{1}{2} \frac{\partial \varphi}{\partial z_j} \frac{\partial g}{\partial \overline{z}_j} - \frac{1}{2} \frac{\partial \varphi}{\partial \overline{z}_j} \frac{\partial g}{\partial z_j} + \frac{1}{4} \frac{\partial \varphi}{\partial z_j} \frac{\partial \varphi}{\partial \overline{z}_j} - \frac{1}{2} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_j} g \right),$$

and separating into real and imaginary part

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + \frac{\partial}{\partial y_j} \right)$$

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we obtain

\[ e^{-\varphi/2} \Box^{(0,0)}_{\varphi}(e^{\varphi/2}g) = \frac{1}{4}(-\Delta_A - V)g, \]

where

\[ A = \frac{1}{2} \left( -\frac{\partial \varphi}{\partial y_1}, \frac{\partial \varphi}{\partial x_1}, \ldots, -\frac{\partial \varphi}{\partial y_n}, \frac{\partial \varphi}{\partial x_n} \right) \]

and

\[ V = 2\text{tr}(M_{\varphi}) = \frac{1}{2} \Delta \varphi. \]

Since the kernel of \( \Box^{(0,0)}_{\varphi} \) coincides with the Bergman space \( A^2(\mathbb{C}^n, e^{-\varphi}) \) we get from (2.2) and the fact that \( A^2(\mathbb{C}^n, e^{-\varphi}) \) is infinite dimensional (see Theorem 1.2) that 0 is an eigenvalue of \( \Delta \varphi \). Hence \( \Box^{(0,0)}_{\varphi} \) fails to be with compact resolvent.

In order to show that the Pauli operator \( P_+ \) has a compact inverse we look at the end of the weighted \( \Box \)-complex.

Let \( u = u\, dz_1 \wedge \cdots \wedge dz_n \) be a smooth \((0, n)\)-form belonging to the domain of \( \Box^{(0,n)}_{\varphi} \). For \( 1 \leq j \leq n \) denote by \( K_j \) the increasing multiindex \( K_j := (1, \ldots, j - 1, j + 1, \ldots, n) \) of length \( n - 1 \). Then

\[ \overline{\partial}_{\varphi} u = \sum_{j=1}^{n} (-1)^{j+1} \left( \frac{\partial \varphi}{\partial z_j} u - \frac{\partial u}{\partial z_j} \right) \, d\zeta_{K_j}. \]

Hence

\[ \overline{\partial}_{\varphi} u = \sum_{j=1}^{n} \left( \frac{\partial \varphi}{\partial z_j} u - \frac{\partial u}{\partial z_j} \right) \, d\zeta_1 \wedge \cdots \wedge d\zeta_n \]

Conjugation with the unitary operator \( U_{\varphi} : L^2(\mathbb{C}^n, e^{-\varphi}) \to L^2(\mathbb{C}^n) \) of multiplication by \( e^{-\varphi/2} \) gives

\[ e^{-\varphi/2} \Box^{(0,n)}_{\varphi} e^{\varphi/2} g = \sum_{j=1}^{n} \left( -\frac{\partial^2 g}{\partial z_j \partial \bar{z}_j} - \frac{\partial \varphi}{2 \partial \bar{z}_j} \frac{\partial g}{\partial z_j} + \frac{1}{2} \frac{\partial^2 \varphi}{\partial z_j^2} \frac{\partial g}{\partial \bar{z}_j} + \frac{1}{2} \frac{\partial \varphi}{\partial \bar{z}_j} \frac{\partial g}{\partial z_j} + \frac{1}{2} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j} g + \frac{1}{2} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j} g \right), \]

where \( g \in L^2(\mathbb{C}^n) \) and we just wrote down the coefficient of the corresponding \((0, n)\)-form. This operator can be expressed by real variables in the form

\[ e^{-\varphi/2} \Box^{(0,n)}_{\varphi} e^{\varphi/2} g = \frac{1}{4}(-\Delta_A + V)g, \]

with

\[ \Delta_A = \sum_{j=1}^{n} \left( \left( -\frac{\partial}{\partial x_j} - \frac{i}{2} \frac{\partial \varphi}{\partial y_j} \right)^2 + \left( -\frac{\partial}{\partial y_j} + \frac{i}{2} \frac{\partial \varphi}{\partial x_j} \right)^2 \right), \]

and \( V = 2\text{tr}M_{\varphi} \). It follows that \(-\Delta_A + V\) is a Schrödinger operator on \( L^2(\mathbb{R}^{2n}) \) with magnetic vector potential

\[ A = \frac{1}{2} \left( -\frac{\partial \varphi}{\partial y_1}, \frac{\partial \varphi}{\partial x_1}, \ldots, -\frac{\partial \varphi}{\partial y_n}, \frac{\partial \varphi}{\partial x_n} \right). \]
where $z_j = x_j + iy_j$, $j = 1, \ldots, n$, and non-negative electric potential $V$ in the case where $\varphi$ is plurisubharmonic.

From (2.1) we get that $N_\varphi^{(0,1)}$ is compact (Theorem 1.3) and by Theorem 1.4 that $N_\varphi^{(0,n)}$ is compact. Finally (2.3) implies that the Pauli operator $P_+$ has a compact inverse. □

For decoupled weights $\varphi(z_1, \ldots, z_n) = \varphi_1(z_1) + \cdots + \varphi_n(z_n)$ even more can be said.

**Theorem 2.2.** Let $\varphi_j \in C^2(\mathbb{C}, \mathbb{R})$ for $1 \leq j \leq n$ with $n \geq 1$, and set

$$\varphi(z_1, \ldots, z_n) := \varphi_1(z_1) + \cdots + \varphi_n(z_n).$$

Assume that all $\varphi_j$ are subharmonic and such that $\Delta \varphi_j$ defines a nontrivial doubling measure.

Let

$$A = \frac{1}{2} \left( -\frac{\partial \varphi}{\partial y_1}, \frac{\partial \varphi}{\partial x_1}, \ldots, -\frac{\partial \varphi}{\partial y_n}, \frac{\partial \varphi}{\partial x_n} \right)$$

and $V = \frac{1}{2} \Delta \varphi$. Then the Pauli operator $P_- = -\Delta_A - V$ fails to have a compact resolvent, the Pauli operator $P_+ = -\Delta_A + V$ has a compact inverse if and only if

$$\lim_{|z| \to \infty} \int_{B_1(z)} \text{tr}(M_\varphi) \, d\lambda = \infty,$$

where $B_1(z) = \{ w \in \mathbb{C}^n : |w - z| < 1 \}$.

**Proof.** By Theorem 1.1 we obtain that $A^2(\mathbb{C}^n, e^{-\varphi})$ is infinite dimensional. So, $P_-$ fails to be with compact resolvent. The assertion about $P_+$ follows from Theorem 1.5. □

**Example:** For $\varphi(z_1, \ldots, z_n) = |z_1|^2 + \cdots + |z_n|^2$ both Pauli operators $P_-$ and $P_+$ fail to be with compact resolvent.

Finally, we get the following result for the Dirac operators (1.1).

**Theorem 2.3.** Let $n = 1$ and let $\varphi$ be a subharmonic $C^2$-function such that $\Delta \varphi$ defines a nontrivial doubling measure. Then the Dirac operator

$$D = \left( -i \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial \varphi}{\partial y} \right) \sigma_1 + \left( -i \frac{\partial}{\partial y} - \frac{1}{2} \frac{\partial \varphi}{\partial x} \right) \sigma_2,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

fails to be with compact resolvent.

**Proof.** By spectral analysis (see [5]) it follows that $D^2$ has compact resolvent, if and only if $D$ has compact resolvent. Suppose that $D$ has compact resolvent. Since

$$D^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix},$$

this would imply that both Pauli operators $P_-$ and $P_+$ have compact resolvent, contradicting Theorem 2.2. □
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