External vertices for crystals of type A

Ola Amara-Omari$^1$, Mary Schaps$^2$

$^1$ Partially supported by Ministry of Science, Technology and Space fellowship, at Bar-Ilan University, Ramat-Gan, Israel olaomari77@hotmail.com

$^2$ Bar-Ilan University, Ramat-Gan, Israel, mschaps@macs.biu.ac.il

Abstract

We show that a vertex in the reduced crystal $\hat{P}(\Lambda)$ is $i$-external for a residue $i$ if the defect is less than the absolute value of the $i$-component of the hub. We demonstrate the existence of a bound on the degree after which all vertices of a given defect $d$ are external in at least one $i$-string. Combining this with the Chuang-Rouquier categorification for the simple modules of the cyclotomic Hecke algebras of type $A$ and rank $e$, this would imply a version of Donovan’s conjecture for the cyclotomics. For $e = 2$, we calculate an approximation to this bound.

1 INTRODUCTION

For an affine Lie algebra of type $A$ and rank $e$, we take a dominant integral weight $\Lambda$ and let $V(\Lambda)$ with the corresponding highest weight module with set of weights $P(\Lambda)$. This can be made into a graph $\hat{P}(\Lambda)$ by adding edges labelled by residues $i = 0, 1, \ldots, e - 1$ determined by the operators $\tilde{e}_i$ and $\tilde{f}_i$ of the corresponding Kashiwara crystal $B(\Lambda)$. The Weyl group $W$ of the affine Lie algebra acts on this graph by reflecting $i$-strings. To each vertex there correspond various numerical invariants, including a defect, a content, a degree and a hub.

We first give a numerical criterion for a vertex of defect $d$ to be at the end of an $i$-string. We then show that there are only a finite number of vertices of any defect $d$ which are not at the high-degree end of an $i$-string for some $i$. Combining this with Chuang-Rouquier categorification of the cyclotomic Hecke algebras $H^n_\Lambda$, we conclude that there are only a finite number of Morita equivalence classes of blocks of cyclotomic Hecke algebras.

For the case $e = 2$, we give explicit formulae for all the invariants and an explicit bound on the degree after which all cyclotomic Hecke algebra blocks are Morita equivalent to blocks of lower degree.
2 DEFINITIONS AND NOTATION

Let \( \mathfrak{g} \) be the affine Lie algebra \( A^{(1)}_\ell \) as in [Ka] and let \( e = \ell + 1 \). Let \( C \) be the Cartan matrix, and \( \delta \) the null root. Let \( \Lambda \) be a dominant integral weight, let \( V(\Lambda) \) be the highest weight module with that highest weight, and let \( P(\Lambda) \) be the set of weights of \( V(\Lambda) \). Let \( Q \) be the \( \mathbb{Z} \)-lattice generated by the simple roots, \( \alpha_0, \ldots, \alpha_\ell \).

Let \( Q_+ \) be the subset of \( Q \) in which all coefficients are non-negative.

The weight space \( P \) of the affine Lie algebra has two different bases. One is given by the fundamental weights together with the null root, \( \Lambda_0, \ldots, \Lambda_{\ell-1}, \delta \), and one is given by \( \Lambda_0, \alpha_0, \ldots, \alpha_{\ell-1} \). We will usually use the first basis for our weights.

The Cartan matrix in type \( A \) is symmetric, and thus induces, through duality, a symmetric product on the weight space. As in [Kl] we define the defect of a weight \( \lambda = \Lambda - \alpha \) by

\[
\text{def}(\lambda) = \frac{1}{2}((\Lambda | \Lambda) - (\lambda | \lambda)) = (\Lambda | \alpha) - \frac{1}{2}(\alpha | \alpha).
\]

Since we are in a highest weight module, we always have \( (\Lambda | \Lambda) \geq (\lambda | \lambda) \), and the defect is in fact an integer for the affine Lie algebras of type \( A \) treated in this paper. The weights of defect 0 are those lying in the Weyl group orbit of \( \Lambda \).

Every weight \( \lambda \in P(\Lambda) \) has the form \( \Lambda - \alpha \), for \( \alpha \in Q_+ \). If \( \lambda = \sum_{i=0}^\ell c_i\alpha_i \), for all \( c_i \) non-negative, then the vector

\[
\text{cont}(\lambda) = (c_0, \ldots, c_\ell)
\]

is called the content of \( \lambda \).

Define

\[
\max(\Lambda) = \{ \lambda \in P(\Lambda) \mid \lambda + \delta \not\in P(\Lambda) \},
\]

and by [Ka], every element of \( P(\Lambda) \) is of the form \( \{ y + k\delta \mid y \in \max P(\Lambda), k \in \mathbb{Z}_{\geq 0} \} \). Let \( W \) denote the Weyl group

\[
W = T \rtimes \check{W}
\]

expressed as a semidirect product of a normal abelian subgroup \( T \) by the finite Weyl group given by crossing out the first row and column of the Cartan matrix. The elements of \( T \) are transformations of the form

\[
t_{\alpha}(\zeta) = \zeta + r\alpha - ((\zeta | \alpha) + \frac{1}{2}(\alpha | \alpha)r)\delta
\]
**Definition 2.1.** For any weight \( \lambda \) in the set of weights \( P(\Lambda) \) for a dominant integral weight \( \Lambda \), we let \( \text{hub}(\lambda) = (\theta_0, \ldots, \theta_{e-1}) \) be the hub of \( \lambda \), where
\[
\theta_i = \langle \lambda, h_i \rangle
\]

The hub is the projection of the weight of \( \lambda \) onto the subspace of the weight space generated by the fundamental weights \([Fa]\).

By the ground-breaking work of Chuang and Rouquier \([CR]\), the highest weight module \( V(\Lambda) \) can be categorified. The weight spaces lift to categories of representations of blocks of cyclotomic Hecke algebras, the basis vectors lift to simple modules, the Chevalley generators \( e_i, f_i \) lift to restriction and induction functors \( E_i, F_i \), and the simple reflections in the Weyl group lift to derived equivalences, which in a few important cases are actually Morita equivalences. It follows from the work of Scopes \([Sc]\) and generalizations by Chuang and Rouquier \([CR]\), that if we have a weight all of whose crystal elements are external, then acting on the block of the cyclotomic Hecke algebra by the Weyl group will produce a Morita equivalence.

### 3 External Vertices of the Reduced Crystal

The Kashiwara crystal, \( B(\Lambda) \) \([K1] \ [K2]\), is a basis for \( V(\Lambda) \) with some additional properties of which the only one of importance at the moment is the existence of operators \( \tilde{e}_i \) and \( \tilde{f}_i \) between basis elements. The set \( P(\Lambda) \) can be taken as the set of vertices of a graph \( \tilde{P}(\Lambda) \) which we will call the reduced crystal, as in \([AS]\) or \([BFS]\). Two vertices will be connected by an edge of residue \( i \) if there are two basis elements with those weights connected in the Kashiwara crystal by \( \tilde{e}_i \) or \( \tilde{f}_i \). A finite set of vertices connected by edges of residue \( i \) will be called an \( i \)-string.

**Definition 3.1.** We say that an element \( b \) of \( B(\Lambda) \) is \( i \)-external if for any \( i \) for which \( \theta_i > 0 \), then \( \tilde{e}_i(b) = 0 \) and if \( \theta_i < 0 \), then \( \tilde{f}_i(b) = 0 \)

**Lemma 3.1.** For a dominant integral weight \( \Lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_\ell\Lambda_\ell \) in a crystal \( B(\Lambda) \), if \( \eta = \Lambda - \alpha \) for \( \alpha \in Q_+ \) has a non-negative \( i \) component \( \theta_i = w \) in its hub, then

1. the defect of \( \lambda = \eta - k\alpha_i \) for \( 0 \leq k \leq w \) is \( \text{def}(\eta) + k(w-k) \).
2. The absolute values of the differences between the defects along the string for \( w \) odd are

\[ w-1, w-3, \ldots, 4, 2, 0, 2, 4, \ldots, w-3, w-1 \]

and for \( w \) even are

\[ w-1, w-3, \ldots, 3, 1, 3, \ldots, w-3, w-1 \]

**Proof.** 1. We simply compute the defect explicitly for the case of \( \alpha_i \),

\[
\text{def}(\lambda) = (\Lambda|\alpha + k\alpha_i) - \frac{1}{2}(\alpha + k\alpha_i|\alpha + k\alpha_i) \\
= (\Lambda|\alpha) - \frac{1}{2}(\alpha|\alpha) + (\Lambda|k\alpha_i) - (\alpha|k\alpha_i) - \frac{1}{2}(k\alpha_i|k\alpha_i) \\
= \text{def}(\eta) + k(a_i - (\alpha|\alpha_i)) - k^2 \\
= \text{def}(\eta) + k((\Lambda - \alpha, \alpha^\vee_i)) - k^2 \\
= \text{def}(\eta) + k(w-k)
\]

2. For \( 0 \leq k \leq w-1 \), we have the difference

\[
|(k+1)(w-k-1) - k(w-k)| = |kw - k^2 - k + w - k - 1 - kw + k^2| \\
= |(w-1) - 2k|
\]

\[\square\]

In a reduced crystal for \( \Lambda \), the \( i \)-string from \( \Lambda \) has length \( a_i \). The weights \( \Lambda - k\alpha_i \) necessarily lie in \( \text{max}(\Lambda) \) since \( \Lambda - k\alpha_i + \delta \) cannot lie in \( P(\Lambda) \) because the coefficient of \( \alpha_j \) is negative for every \( j \neq i \) and thus \( k\alpha_i - \delta \notin Q_+ \).

**Corollary 3.1.1.** In a crystal with \( \Lambda = \Lambda_0 + \cdots + \Lambda_\ell \), the defect of \( \lambda = \Lambda - k\alpha_i \) for \( 0 \leq k \leq a_i \) is \( k(a_i - k) \).

**Proof.** \( \text{def}(\Lambda) = 0 \) and \( \text{hub}(\Lambda) = [a_0, \ldots, a_\ell] \). \[\square\]

**Proposition 3.1.** Let \( \Lambda \) be a dominant integral weight. For any positive integer \( d \), a weight of defect \( d \) is \( i \)-external if \( d \leq |\theta_i| \).

**Proof.** A weight of defect \( d \) can only be an internal vertex of an \( i \) string, if the neighboring vertex of lower defect \( d' \) has \( i \)-component of absolute value \( w \) of the hub, and \( d - d' = w - 1 \). Since \( d' \geq 0 \) we have \( d \geq w - 1 \). Since the \( i \)-hub \( \theta_i \) is of absolute value \( w - 2 \) by Lemma 3.1.1, we have \( d \geq |\theta_i| + 1 \). Thus, whenever \( d \leq |\theta_i| \), the vertex must be \( i \)-external. \[\square\]
In order to show that for any $d$ there is a bound $N(d)$ on the degree $n$ such that every Morita equivalence class of blocks of $H_n^\Lambda$ appears in some degree $n \leq N$, we need to use the methods of [BFS]. In order to make our treatment self-contained, we review the necessary results. The set $\text{max}(\Lambda)$ is parametrized by an integral lattice $M$, which for type $A$ corresponds to $\mathbb{Z}^\ell$. We let $m = (m_1, m_2, \ldots, m_\ell)$ be an element of $M$. We then form an element of $P$ given by $\Lambda - (m_1 \alpha_1 + \cdots + m_\ell \alpha_\ell)$. By the main theorem of [BFS], there is an integer $s(m)$ such that

$$\eta_m = \Lambda - (m_1 \alpha_1 + \cdots + m_\ell \alpha_\ell) - s(m)\delta \in \text{max}(\Lambda)$$

We choose our simple roots such that $\alpha_0 = -\alpha_1 - \alpha_2 - \cdots - \alpha_\ell + \delta$. Solving this for $\delta$ and substituting, we get

$$\eta_m = \Lambda - (s(m)\alpha_0 + (s(m) + m_1)\alpha_1 + \cdots + (s(m) + m_\ell)\alpha_\ell)$$

From this we obtain the content

$$\text{cont}(\eta_m) = (s(m), (s(m) + m_1, \ldots, s(m) + m_\ell))$$

and, setting $m_0 = 0$, with indices taken modulo $e = \ell + 1$, the hub is

$$\text{hub}(\eta_m) = (2m_i - m_{i-1} - m_{i+1})_{i=0}^\ell$$

Thus the hubs are a linear transformation of the components of the content.

**Proposition 3.2.** For any defect $d$, there is a minimal degree $N(d)$ such that every occurrence of that defect in degree $n \geq N(d)$ is at the end of a string to a vertex of lower degree.

**Proof.** By Prop. 3.1, it suffices to show that we can find some degree such that for every hub in that and all greater degrees, there is a component $\theta_i$ of the hub satisfying $\theta_i \leq -d$, and then the vertex is an external vertex at the high-degree end of an $i$-string.

We first show that for any $d$, and any $i$, we can find an weight $\nu'_i$ in $\text{max}(\Lambda)$ such that $\theta_j \leq -d$ for all $j \neq i$. We first define $\nu_i$ to be $-(d + 1)$ for $j \neq i$, and $r + \ell(d + 1)$ for component $i$, which is a weight of level $r$. Let $\psi$ be the hub of $\Lambda - \nu_i$, which is of level $0$ since the level is additive. By Proposition 3.6 of [BFS], in order for $\psi$ to lie in $Q$, we need

$$\psi_1 + 2\psi_2 + \cdots + \ell\psi_\ell \equiv 0 \mod e$$

If it is congruent to $j$ with $j \neq 0$, then we can subtract 1 from $\psi_j$ and add 1 to $\psi_0$, getting $\psi'$ which is still of level 0 but satisfies the condition. We then define

5
\[ \nu' = \Lambda - \psi'. \]

With the appropriate \( \delta \)-shift, this is the hub of some element of \( \max(\Lambda) \) by Theorem 2.7 of [BFS].

This set of weights, \( \{\nu'_0, \ldots, \nu'_\ell\} \), transported to the lattice \( M \) from [BFS] of points in correspondence to \( \max(\Lambda) \), determine a simplex. Every face of the simplex lies on a hyperplane, and we claim that every point outside the simplex has at least one coordinate in the hub which is negative and less than or equal to \( -d \). If the face excludes \( \nu'_i \) for some \( i \neq 0 \), then since at every point defining the face, \( \theta_i = -d \) or \( \theta_i = -(d + 1) \). As shown before the proposition, \( \theta_i \) determines a hyperplane \( 2m_i - m_{i-1} - m_{i+1} = -d \). This hyperplane cuts the face only on its boundary. For \( i \neq 0 \), the value of \( \theta_i \) decreases on the side of the hyperplane not containing the simplex, since it is positive on \( \nu_i \).

If \( i = 0 \), then \( \theta_0 \) is determined by \( -m_1 - m_\ell \). By definition this is positive on \( \nu'_0 \) and negative at every corner of the 0-face of the simplex, so that all the points on the opposite side of the set with \( \theta_0 = -d \) have 0-component \( \theta_0 \leq -d \). We conclude that the finite set of points for which none of the hub components is less than or equal to \( -d \) must lie inside the simplex, with a finite maximum degree \( N(d) \).

Any vertex of defect \( d \) outside the simplex, since it has some \( i \) with \( \theta_i \leq -d \), must be \( i \)-external, so if the \( -d \) hub component occurs in component \( i \), we can reflect the \( i \)-string to a lower degree. Since \( \theta_i \) is negative, then \( b \) is at the high degree end of the \( i \)-string on which \( b \) is an external vertex, as desired.

\[ \square \]

**Corollary.** There are only a finite number of Morita equivalence classes of blocks of cyclotomic Hecke algebras \( H^n_\Lambda \), \( n \in \mathbb{Z}_+ \).

**Proof.** In [CR], not only did Chuang and Rouquier prove the categorification theorem for the cyclotomic Hecke algebras \( H^n_\Lambda \), they also prove an adjunction theorem. If \( \mu \) is a basis element of the crystal \( B(\Lambda) \) with weight \( \lambda \) which lies at the end of an \( i \)-string in the reduced crystal, and \( S \) is the corresponding simple module in the cyclotomic Hecke algebra, then there is an idempotent \( e \) in the blocks (of the same defect) corresponding to the image under the action of the Weyl group element \( s_i \) which cuts out a block corresponding to \( \mu \) under the derived equivalence. The induced module that one gets by going down the string is in fact \( w! \) copies of this block, so we get \( w! \) copies of each simple. A derived equivalence which gives a one-to-one correspondence
between simples is, by a result of Linckelmann \[\text{[Lin]}\], a Morita equivalence. To apply the Linckelmann result, we must know that the cyclotomic Hecke algebras of type $A$ are symmetric. This is apparently well-known. Malle and Mthas give a sketch of the proof in their introduction \[\text{[MM]}\].

\[\text{Figure 1. The reduced crystal for } e = 3, \Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2\]

In Figure 1, on the right, we give a truncation of the reduced, three-dimensional crystal, in which edges bounding a region with fixed $\delta$-shift are emphasized. Horizontal lines in the reduced crystal correspond to fixed degree. To the left is the two dimensional lattice $M$ corresponding to elements of $\max(\Lambda)$, in which the contours represent regions of corresponding to an interval of $\delta$-shifts. Outside of the triangle, every hub as a component $\leq -4$.

4 EXTERNAL VERTICES OF THE REDUCED CRYSTAL FOR $e = 2$

In the case of $e = 2$, the lattice is in one-to-one correspondence with the integers and we can give an explicit bound $N'(d)$ on the degree such that all Morita equivalence classes occur for degrees less than $N'(d)$. We begin by giving a description of the set $\max(\Lambda)$, with weights, hubs, defect and contents.

**Lemma 4.1.** Let $e = 2$ and $\Lambda = a_0\Lambda_0 + a_1\Lambda_1$, with $r = a_0 + a_1$. Set $b_0 = \left\lfloor \frac{a_0}{2} \right\rfloor$. Write the integers in the form $m = qr + u$, where $-b_0 \leq u \leq r - b_0 - 1$. Set

$$\eta_m = (a_0 + 2m)\Lambda_0 + (a_1 - 2m)\Lambda_1 + s(m)\delta,$$
where
\[ s(m) = \max(-u, 0, u - a_1) + q(a_1 - 2u) + q^2r. \]

Then
\[ \text{1. } \text{hub}(\eta_m) = [a_0 + 2m, a_1 - 2m], \]
\[ \text{2. } \text{def}(\eta_m) = \max(-u(a_0 + u), u(a_1 - u), (u - a_1)(r - u)), \]
\[ \text{3. } \max(\Lambda) = \{ \eta_m | m \in \mathbb{Z} \} \]
\[ \text{4. } \text{cont}(\eta_m) = (s(m), s(m) + m) \]
\[ \text{5. } \text{deg}(\eta_m) = 2s(m) + m. \]

Proof. Since \( e = 2 \), then \( \ell = e - 1 = 1 \). For the simple roots, we take
\[ \alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta \]
\[ \alpha_1 = -2\Lambda_0 + 2\Lambda_1 \]

1. The hub is just the pair of coefficients of the \( \Lambda_i \) in \( \eta_m \).

2. Let \( b_1 = \left\lfloor \frac{a_1}{2} \right\rfloor \). The set of dominant integral weights \( N_0 = P_+ \cap \max(\Lambda) \) lie on the two strings going out from \( \Lambda \). The 0-string has length \( a_0 \) and the 1-string has length \( a_1 \). They correspond to weights \( \eta_m \) for \( q = 0 \) and \(-b_0 \leq u \leq b_1\), have hubs \([a_0 - 2a_0, a_1 + 2a_0], \ldots, [a_0, a_1], \ldots, [a_0 + 2b_1, a_1 - 2b_1]\), and are obtained from \( \Lambda \) by adding \( u\alpha_1 \) out to half of each \( i \)-string. Each weight space along these strings has dimension 1 since there is a unique path leading to the vertex.

The defects are given by Lemma 3.1.1 as \(-u(a_0 + u)\) when \( u \) is non-positive and by \( u(a_1 - u)\) when \( u \) is non-negative.

The finite Weyl group \( \tilde{W} \) has two elements, the identity and \( s_1 \). As in [BFS], let \( N \) be the orbit of \( N_0 \) under \( \tilde{W} \). When we act on the dominant integral weights from the 1-string, we get the remaining elements of the 1-string. The defects will be preserved by the action of the Weyl group, which reflects the 1-string around its center, so we have defects for \(-b_0 \leq u \leq b_1\) given by \( \max(-u(a_0 + u), u(a_1 - u)) \), since each of these parabolas is non-positive where the other is non-negative.

To fill out \( N \), we act on the dominant integral weights in the half 0-string by \( s_1 \). This gives another \( b_0 \) elements of \( N \) which lie at the end of 1-strings. The hub of \( s_1\eta_u \) for \(-b_0 \leq u \leq -1\) is
\[ [a_0 + 2u, a_1 - 2u] - (a_1 - 2u)[-2, 2] = [a_0 + 2a_1 - 2u, -a_1 + 2u] \]

This has the same hub as \( \eta_m \) for \( m = a_1 - u \) and since acting by \( s_1 \) preserves \( \delta \), in order to show that \( s_1 \eta_u = \eta_{a_1-u} \) for \( -b_0 \leq u \leq -1 \) it remains to demonstrate that \( s(u) = s(a_1 - u) \). Since \( q = 0 \), we need to show that

\[
\max(-u, 0, u - a_1) = \max(-a_1 - u, 0, (a_1 - u) - a_1)
\]

Both are equal to \(-u\), since all the other terms being maximized are non-positive. Since action by the Weyl group preserves defect,

\[
\text{def}(\eta_{a_1-u}) = \text{def}(\eta_u) = \max(-u(a_0 - u), u(a_1 - u)) = -u(a_0 + u)
\]

If we substitute \( u = a_1 - u' \), where \( a_1 + 1 \leq u' \leq a_1 + b_0 \), we get a defect equal to \((u' - a_1)(a + a_1 - u')\). In the given range for \( u' \), both \(-u'(a_0 + u')\) and \(u'(a_1 - u')\) are negative, and similarly for \( b_0 \leq u \leq a_1 \), we have \((u - a_1)(a_0 + a_1 - u) < 0 \) so if we substitute \( r = a_0 + a_1 \) we get the desired formula for the case \( q = 0 \),

\[
\text{def}(\eta_u) = \max(-u(a_0+u), u(a_1-u), (u-a_1)(r-u)), -b_0 \leq u \leq r-b_0-1
\]

since \( r - b_0 - 1 = a_1 + b_0 \) if \( a_0 \) is odd and \( r - b_0 - 1 < a_1 + b_0 \) if \( a_0 \) is even.

3. Claim: \( \eta_{qr+u} = t_{-q\alpha_1}(\eta_u), -b_0 \leq u \leq r - b_0 - 1. \)

To demonstrate the claim, we simply calculate the action of \( t_{-q\alpha_1} \):

\[
t_{-q\alpha_1}(\eta_u) = \eta_u + r(-q\alpha_1) - ((\eta_u | -q\alpha_1) + \frac{1}{2}q^2(\alpha_1|\alpha_1)r)\delta
= (a_0 + 2u)\Lambda_0 + (a_1 - 2u)\Lambda_1 - \max(-u, 0, a_1 - u)\delta
- q\alpha_1 - ((a_1 - 2u)(-q) + q^2r)\delta
= (a_0 + 2(qr + u))\Lambda_0 + (a_1 - 2(qr + u))\Lambda_1
- (\max(-u, 0, a_1 - u) + q(a_1 - 2u) + q^2r)\delta
= \eta_{qr+u}
\]

Since defect is preserved by the action of \( T \), this shows that the defect is independent of \( q \).

The claim also demonstrates (3), since by [Ka], \( \max(\Lambda) = W \cdot N_0 \). Following [BFS], we want to shrink \( N \) to a fundamental region \( \bar{N} \) by
taking a single representative of each $T$ orbit. If $a_0$ is odd, then $N = \bar{N}$ because, as mentioned above, $r - b_0 - 1 = a_1 + b_0$ so no element of $N$ can be written with nonzero $q$. If, however, $a_0$ is even, so that $2b_0 = a_0$, then $a_1 + b_0 = r - b_0$, so

$$\eta_{a_1+b_0} = t_{-\alpha_1}(\eta_{a_1-b_0})$$

and we will not include $\eta_{a_1+b_0}$ in $\bar{N}$.

4. Letting $\text{cont}(\eta_m) = (c_0, c_1)$, then by the definition of $\alpha_0$, we get $c_0 = s(m)$. Substituting into the formula

$$\eta_m = \Lambda - c_0\alpha_0 - c_1\alpha_1$$

and projecting only the first component of the hub gives

$$a_0 + 2m = a_0 - 2s(m) + 2c_1$$

Eliminating $a_0$ and dividing by 2, we conclude that $c_1 = s(m) + m$, so

$$\text{cont}(\eta_m) = (s(m), s(m) + m)$$

5. The degree is the sum of the components of the content.

Example 1. As an example, consider the case of $a_0 = 2$ and $a_1 = 1$. The reduced crystal $\hat{P}(\Lambda)$ is given in Figure 2. To illustrate the lemma above, we give the results in tabular form.

| $m$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|-----|----|----|----|---|---|---|---|
| Hub | [-4,7] | [-2,5] | [0,3] | [2,1] | [4,-1] | [6,-3] | [8,-5] |
| Defect | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| Content | (4,1) | (2,0) | (1,0) | (0,0) | (0,1) | (1,3) | (2,5) |
| Degree | 5 | 2 | 1 | 0 | 1 | 4 | 7 |

We now use the information we have accumulated in the case of $e = 2$ to give an explicit upper estimate $N'(d)$ for $N(d)$, the bound after which the vertices are all external.
Proposition 4.1. Let $\Lambda$ be a dominant integral weight. For any positive integral $d$, the weights of defect $d$ are all external whenever $\deg(\eta) \geq N'(d)$ for

$$N'(d) = 2rq^2 + q(r + a_1)$$

where $q = \left\lfloor \frac{d + \max(a_0, a_1)}{2r} \right\rfloor$.

Proof. By Prop 3.1, weight $\eta$ of defect $d$ is an $i$-external vertex of an $i$ string $d \leq |\theta_i|$. We want this to hold for both $i$, and the negative $\theta_i$ always has the smaller absolute value. The possible hubs have the form $[a_0 + 2m, a_1 - 2m]$, so we want

$$d \leq 2|m| - a_0, m \leq 0$$
$$d \leq 2m - a_1, m \geq 0$$
This surely hold whenever

\[ |m| \geq \frac{d + \max(a_0, a_1)}{2} \]

We now convert this into an inequality for the degree. We substitute for \( s(m) \) to get

\[ \deg(\eta_m) = 2(\max(-u, 0, u - a_1) + q(a_1 - 2u) + q^2 r) + (qr + u) \]

If we don’t worry much about a sharp bound, we can take \( u = 0, m = qr, \) and get \( \max(-u, 0, u - a_1) = 0, \) so that

\[ \deg(\eta_m) = 2(q^2 r + q(r + a_1)) \]

The degree rises monotonically with \( n \), so to get our estimate for the bounding degree we take

\[ q = \left\lceil \frac{d + \max(a_0, a_1)}{2r} \right\rceil \]

and then we get

\[ N'(d) = 2q^2 r + q(r + a_1) \]

as desired

The bound \( N'(d) \) given in Proposition 4.1 is far from sharp. Let \( N(d) \) be the sharp bound, the degree from which the vertices of defect \( d \) are \( i \)-external for both \( i = 0 \) and \( i = 1 \) and we compare the two bounds in tabular form for Example 1.

| d   | 0  | 1  | 3  | 4  | 6  | 7  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|
| q   | 1  | 1  | 1  | 1  | 2  | 2  | 2  | 2  |
| \(N(d)\) | 0  | 2  | 5  | 7  | 12 | 15 | 22 | 25 |
| \(N'(d)\) | 13 | 13 | 13 | 13 | 38 | 38 | 38 | 38 |

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