Non-finitely based monoids

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Abstract

We present a method for proving that a semigroup is non-finitely based and find some new sufficient conditions under which a monoid is non-finitely based. Our method also gives a short proof to the result of [23].

Keywords: Finite Basis Problem, Semigroups, Monoids

1 Introduction

An algebra is said to be finitely based (abbreviated to FB) if there is a finite subset of its identities from which all of its identities may be deduced. Otherwise, an algebra is said to be non-finitely based (abbreviated to NFB). The famous Tarski’s Finite Basis Problem asks if there is an algorithm to decide when a finite algebra is finitely based. In 1996, R. McKenzie [10] solved this problem in the negative showing that the classes of FB and inherently not finitely based finite algebras are recursively inseparable. (A locally finite algebra is said to be inherently not finitely based (INFB) if any locally finite variety containing it is NFB.) It is still unknown whether the set of FB finite semigroups is recursive although a very large volume of work is devoted to this problem (see the surveys [19, 21]). In contrast with McKenzie’s result, a powerful description of the INFB finite semigroups has been obtained by M. Sapir [12, 13].

In 1968, P. Perkins [11] found the first sufficient condition under which a monoid (semigroup with an identity element) is NFB. By using this condition, he constructed the first two examples of finite NFB semigroups. The first example was the 6-element Brandt monoid and the second example was the 25-element monoid obtained from the set of words $W = \{abtba, atbab, abab, aat\}$ by using the following construction attributed to Dilworth.

Let $\mathcal{A}$ be an alphabet and $W$ be a set of words in the free monoid $\mathcal{A}^*$. Let $S(W)$ denote the Rees quotient over the ideal of $\mathcal{A}^*$ consisting of all words that are not subwords of words in $W$. For each set of words $W$, the semigroup $S(W)$ is a monoid with zero whose nonzero elements are the subwords of words in $W$. Evidently, $S(W)$ is finite if and only if $W$ is finite.
In [3], M. Jackson proved that the varieties generated by \( S(\{at_1abt_2b\}) \) and \( S(\{abt_1at_2b, at_1bt_2ab\}) \) are limit varieties in a sense that each of these varieties is NFB while each proper monoid subvariety of each of these varieties is FB. In [5], E. Lee gave the affirmative answer to the question of Jackson from [3] by proving that these varieties are the only limit varieties generated by finite aperiodic monoids with central idempotents and later, in [6], he proved that these varieties are the only limit varieties generated by aperiodic monoids with central idempotents. In [23], W. Zhang gave the affirmative answer to the second question posed by Jackson in [3] by proving that the varieties generated by \( S(\{at_1abt_2b\}) \) and \( S(\{abt_1at_2b, at_1bt_2ab\}) \) are not the only limit varieties in a wider class of varieties of all aperiodic monoids. She proved it by showing that a certain seven-element monoid \( L^1 \) is non-finitely based (see more details below). Investigation of the monoid \( L^1 \) was suggested to her by E. Lee because the variety generated by the monoid \( L^1 \) contains neither \( S(\{at_1abt_2b\}) \) nor \( S(\{abt_1at_2b, at_1bt_2ab\}) \).

We say that a semigroup \( S \) is non-finitely based by a set of identities \( \Sigma \) if \( S \) satisfies all the identities in \( \Sigma \) but \( \Sigma \) can not be derived from any finite set of identities of \( S \). Recall that axiomatic rank of an algebra is the minimal number of variables \( m \) so that all of its identities may be deduced from its identities in \( m \) variables. It is well-known that every locally finite variety of algebras of finite axiomatic rank is finitely based. This implies that if a locally finite semigroup is non-finitely based by a set of identities \( \Sigma \) then there is no bound on the number of variables involved in the identities from \( \Sigma \).

The seven-element monoid \( L^1 \) mentioned above is obtained by adjoining an identity element to the semigroup \( L = \langle a, b \mid aa = a, bb = b, aba = 0 \rangle \). In [22], W. Zhang and Y. Luo proved that semigroup \( L \) is non-finitely based by the set of identities \( \{yx_n^2x_n^2 \ldots x_n^2y \approx yx_n^2x_n^2 \ldots x_n^2y \mid n > 1 \} \). In [7], E. Lee generalized this result into a sufficient condition under which a semigroup is non-finitely based by a set of identities \( \{yx_n^kx_n^2 \ldots x_n^ky \approx yx_n^kx_n^2 \ldots x_n^ky \mid n > 1 \} \) for \( k > 1 \).

Throughout this article, elements of a countable alphabet \( \mathcal{A} \) are called variables and elements of the free semigroup \( \mathcal{A}^+ \) are called words. If \( \{x_1, x_2, \ldots, x_n\} \) is a set of variables then we denote \( [Xn] = x_1x_2 \ldots x_n \) and \( [nX] = x_nx_{n-1} \ldots x_1 \). The mentioned result of [23] says that monoid \( L^1 \) is non-finitely based by the set of identities \( \{[Xn][Yn][nX][nY] \approx [Yn][Xn][nY][nX] \mid n > 1 \} \).

In this article we present a method (see Lemma [11] below) that can be used for proving that a semigroup is non-finitely based. We use this method to find nine new sufficient conditions under which a monoid is non-finitely based (see Theorems [2.4] and [4.6] below). Theorem [2.3] gives us a sufficient condition under which a monoid is non-finitely based by the set of identities \( \{[Xn][Yn][nX][nY] \approx [Yn][Xn][nY][nX] \mid n > 1 \} \). The proof of Theorem [2.3] yields a short proof to the result of Zhang that monoid \( L^1 \) is non-finitely based (see Corollary [2.3] below).

A word \( u \) is said to be an isoterm \( ([11]) \) for a semigroup \( S \) if \( S \) does not satisfy any nontrivial identity of the form \( u \approx v \). The part of Theorem [4.6] which is encoded in the first row of Table [1] can be decoded into the following statement: if \( S \) is a monoid that satisfies the identity \( xx[Yn][nY] \approx [Yn][nY]xx \) for each \( n > 1 \) and
the word $xyyx$ is an isoterm for $S$ then $S$ is non-finitely based. This sufficient condition for the non-finite basis property of monoids can be deduced easily from the proofs of some interesting results obtained in 1970’s independently by J. Isbell and L. Shneerson.

In 1970, J. Isbell (11) proved that the variety of groups defined by the identity $xxyy \approx yyxx$ generates the variety of monoids $\mathcal{M}$ which is non-finitely based by the set of identities $\{xx[Yn][nY] \approx [Yn][nY]xx \mid n > 1\}$. His proof was based on the fact that the identity $xxyy \approx yyxx$ is the only nontrivial identity whose left-hand side has length at most 4 and which holds in the group of integers $\mathbb{Z}$ and in the symmetric group $S_3$ simultaneously. This property of the monoid $\mathbb{Z} \times S_3$ is equivalent to the property that the word $xyyx$ is an isoterm for $\mathbb{Z} \times S_3$ and consequently, for $\mathcal{M}$.

In 1972, L. Shneerson (see [20] for exact references) described all semigroups and monoids with one defining relation satisfying nontrivial identities and noticed that the monoid $M = \langle a, b \mid abba = 1 \rangle$ represents a unique (up to isomorphism) example of a one-relator monoid which is non-cyclic group without free submonoids of rank 2. A couple of years later, he proved that monoid $M$ is non-finitely based by the set of identities $\{xx[Yn][nY] \approx [Yn][nY]xx \mid n > 1\}$. An examination of the Shneerson’s proof also shows that the only property of the monoid $M$ which is responsible for $M$ being non-finitely based by the set of identities $\{xx[Yn][nY] \approx [Yn][nY]xx \mid n > 1\}$, is that the word $xyyx$ is an isoterm for $M$.

The NFB monoid $M$ is also interesting because it is finitely based as a group. The mentioned results of Shneerson together with his other results about monoids with one defining relation were published in English in [20]. In particular, it is proved there that if a semigroup or monoid with one defining relation has a finite axiomatic rank then it is finitely based. Also, the article [20] contains the first example of a NFB semigroup $S$ so that the monoid $S^1$ obtained by adjoining an identity element to $S$ is finitely based. Recently [8], E. Lee found a way to construct finite semigroups with this property.

Theorem 4.6 will be used in articles [16, 17]. If a variable $t$ occurs exactly once in a word $u$ then we say that $t$ is linear in $u$. If a variable $x$ occurs more than once in a word $u$ then we say that $x$ is non-linear in $u$. Articles [16, 17] contain some algorithms that recognize FB semigroups among certain finite monoids of the form $S(W)$. In particular, in article [16], we show how to recognize FB semigroups among the monoids of the form $S(W)$ where $W$ consists of a single word with at most two non-linear variables.

Let $\sim_S$ denote the fully invariant congruence on $A^*$ corresponding to a semigroup $S$ and $[u]_S$ denote the equivalence class of $\sim_S$ containing the word $u$. We say that an equivalence class $[u]_S$ is in $k > 0$ variables if the word $u$ depends on $k$ variables. We observe that the original sufficient condition for the non-finite basis property of monoids found by Perkins in [14], the many sufficient conditions contained in articles [2, 3, 4, 7, 9, 13] and the nine sufficient conditions that we exhibit in this article all are of the following form:

- If a monoid $S$ satisfies all identities in a set of identities $\Sigma$ and for some $m > 0$...
the equivalence classes of $\sim_S$ in at most $m$ variables satisfy certain restrictions $\mathcal{R}$ then monoid $S$ is non-finitely based by $\Sigma$.

For example, the sufficient condition for the non-finite basis property of monoids encoded in the first row of Table I below is also of this form where $\Sigma = \{xx[Yn][nY] \approx [Yn][nY]xx \mid n > 1\}$, $m = 2$ and those restrictions $\mathcal{R}$ on the equivalence classes of $\sim_S$ in at most two variables are saying that the equivalence class $[xxyyx]_S$ is a singleton.

We observe that Lemma 4.1 can be used to prove or reprove all known sufficient conditions of this form for the non-finite basis property of semigroups. We say that an equivalence class $[u]_S$ is with at most two non-linear variables if the word $u$ contains at most two non-linear variables. Finally, we observe that in all known sufficient conditions of this form those restrictions $\mathcal{R}$ are put on the equivalence classes of $\sim_S$ with at most two non-linear variables.

2 A method for proving that a semigroup is non-finitely based

If some variable $x$ occurs $n \geq 0$ times in a word $u$ then we write $\text{occ}_u(x) = n$ and say that $x$ is $n$-occurring in $u$. The set $\text{Cont}(u) = \{x \in \mathcal{A} \mid \text{occ}_u(x) > 0\}$ of all variables contained in a word $u$ is called the content of $u$.

We use $\iota_u^i x$ to refer to the $i$th from the left occurrence of $x$ in $u$. We use $\ell_u^i x$ to refer to the last occurrence of $x$ in $u$. The set $\text{OccSet}(u) = \{\iota_u^i x \mid x \in \mathcal{A}, 1 \leq i \leq \text{occ}_u(x)\}$ of all occurrences of all variables in $u$ is called the occurrence set of $u$. The word $u$ induces a (total) order $<_u$ on set $\text{OccSet}(u)$ defined by $\iota_u^i x <_u \iota_u^j y$ if and only if the $i$th occurrence of $x$ precedes the $j$th occurrence of $y$ in $u$. For example, $\text{OccSet}(xyx) = \{1y_1x, 1xy_2y, 2xy_1x\}$ together with the order $<_{xyx}$ is an algebra $(\text{OccSet}(xyx), <_{xyx})$ isomorphic to the three-element totally ordered set: $(1y_1x, 1xy_2y, 2xy_1x)$.

If $u$ and $v$ are two words then $l_{u,v}$ is a map from $\{\iota_u^i x \mid x \in \text{Cont}(u), i \leq \text{occ}_u(x)\}$ to $\{\iota_v^j x \mid x \in \text{Cont}(v), j \leq \text{occ}_v(x)\}$ defined by $l_{u,v}(\iota_u^i x) = \iota_v^i x$. Note that $l_{u,u}$ is the identity map of $\text{OccSet}(u)$.

If $X \subseteq \text{OccSet}(u)$ then we say that set $X$ is left-stable in an identity $u \approx v$ if the map $l_{u,v}$ is defined on $X$ and is an isomorphism of the (totally) ordered sets $(X, <_{u})$ and $(l_{u,v}(X), <_{v})$. Otherwise, we say that set $X$ is left-unstable in $u \approx v$. For example, set $\{2(xy_1y)\}$ is left-unstable in the identity $xxxyy \approx xxy$ but set $\{1(xy_1x), 2(xy_2x), 1(xy_2y)\}$ is left-stable in this identity. Dually, by introducing the maps $r_{u,v}$ that count occurrences of variables from right to left, one can define a right-stable set. For example, set $\{1(xy_1) x, 1(xy_2) y\}$ is left-stable but right-unstable in the identity $xy \approx xxy$.

If $u$, $v$ and $w$ are three words then we define $(l_{u,v} \circ l_{v,w})$ as the map from $X = \{\iota_u^i x \mid x \in \text{Cont}(u), i \leq \text{occ}_u(x)\}$ to $Y = \{\iota_w^i x \mid x \in \text{Cont}(w), i \leq \text{occ}_w(x)\}$ by $(l_{u,v} \circ l_{v,w})(x) = l_{v,w}(l_{u,v}(x))$ for
each \( x \in X \). Evidently, the maps \((l_{u,v} \circ l_{v,w})\) and \(l_{u,w}\) coincide on \( X \) and \((l_{u,v} \circ l_{v,u})\) is the identity map on \( \{iux | x \in \text{Cont}(u), i \leq \min(\text{occ}_u(x), \text{occ}_v(x))\} \). The following fact can be easily verified.

**Fact 2.1.** Let \( u, v \) and \( w \) be words and

\[
X \subseteq \{iux | x \in \text{Cont}(u), i \leq \min(\text{occ}_u(x), \text{occ}_v(x))\}.
\]

If the map \( l_{u,v} \) is an isomorphism from \((X, <_u)\) to \((l_{u,v}(X), <_v)\) and the map \( l_{v,w} \) is an isomorphism from \((l_{u,v}(X), <_v)\) to \((l_{u,w}(X), <_w)\) then the map \( l_{u,w} \) is an isomorphism from \((X, <_u)\) to \((l_{u,w}(X), <_w)\).

If \( \mathcal{A} \) is a countable alphabet then a substitution \( \Theta : \mathcal{A} \rightarrow \mathcal{A}^+ \) is a homomorphism of the free semigroup \( \mathcal{A}^+ \). If \( S \) is a finitely based semigroup, then it is always possible to choose a finite basis \( \Sigma \) for \( S \) in such a way so that a derivation of any identity \( U \approx V \) of \( S \) from \( \Sigma \) is a sequence of words \( U = U_1 \approx U_2 \approx \cdots \approx U_t = V \) and substitutions \( \Theta_1, \ldots, \Theta_{t-1} : \mathcal{A} \rightarrow \mathcal{A}^+ \) so that for each \( i = 1, \ldots, t - 1 \) we have \( U_i = \Theta_i(u_i) \) and \( U_{i+1} = \Theta_i(v_i) \) for some identity \( u_i \approx v_i \in \Sigma \). (Since the derivation described in the previous sentence is not standard, see [19] Section 9) for an explanation why one can always choose a finite basis \( \Sigma \) in this way.)

**Lemma 2.2.** Let \( S \) be a semigroup.

Suppose that for each \( n \) large enough one can find an identity \( U_n \approx V_n \) of \( S \) in at least \( n \) variables so that for some set \( X \subseteq \text{OccSet}(U_n) \) each of the following conditions is satisfied:

(i) \( X \) is left-unstable in \( U_n \approx V_n \);

(ii) If \( U \in [U_n]_S \) and set \( X \) is left-stable in \( U_n \approx U \) then for every identity \( U \approx V \) of \( S \) in less than \( n/2 \) variables and every substitution \( \Theta : \mathcal{A} \rightarrow \mathcal{A}^+ \) so that \( \Theta(u) = U \) the set \( l_{U_n,u}(X) \) is left-stable in \( U \approx \Theta(v) \).

Then semigroup \( S \) is non-finitely based.

**Proof.** Working toward a contradiction, assume that \( S \) is finitely based by a set of identities \( \Sigma \) that involves less than \( m \) variables for some \( m > 0 \). By our assumption, one can find an identity \( U_{2m} \approx V_{2m} \) of \( S \) in \( 2m \) variables and a set \( X \subseteq \text{OccSet}(U_{2m}) \) so that Condition (ii) is satisfied.

Since \( S \) is finitely based by \( \Sigma \), there is a sequence of words \( U_{2m} = W_1 \approx W_2 \approx \cdots \approx W_l = V_{2m} \) and substitutions \( \Theta_1, \ldots, \Theta_{l-1} : \mathcal{A} \rightarrow \mathcal{A}^+ \) so that for each \( i = 1, \ldots, l - 1 \) we have \( W_i = \Theta_i(u_i) \) and \( W_{i+1} = \Theta_i(v_i) \) for some identity \( u_i \approx v_i \in \Sigma \).

Condition (ii) implies that set \( X \) is left-stable in \( W_1 \approx W_2 \), set \( l_{W_1,W_2}(X) \) is left-stable in \( W_2 \approx W_3 \), \ldots, set \( l_{W_1,W_{i-1}}(X) \) is left-stable in \( W_{i-1} \approx W_i \). In view of Fact 2.1 set \( X \) is left-stable in \( U_{2m} = W_1 \approx W_l = V_{2m} \). To avoid a contradiction with Condition (i) we must assume that \( S \) is non-finitely based. \( \square \)

Let \( \Theta : \mathcal{A} \rightarrow \mathcal{A}^+ \) be a substitution so that \( \Theta(u) = U \). Then \( \Theta \) induces a map \( \Theta_u \) from \( \text{OccSet}(u) \) to subsets of \( \text{OccSet}(U) \) as follows. If \( 1 \leq i \leq \text{occ}_u(x) \) then \( \Theta_u(ux) \) denotes the set of all elements of \( \text{OccSet}(U) \) contained in the subword of \( U \) of the form \( \Theta(x) \) that corresponds to the \( i \)th occurrence of variable \( x \) in \( u \). For example,
if \( \Theta(x) = ab \) and \( \Theta(y) = bab \) then \( \Theta_{xy2}(2xy2)x) = \{3(abbabab)a, 4(abbabab)b\} \). Evidently, for each \( x \in \text{OccSet}(u) \) the set \( \Theta_u(x) \) is an interval in \( (\text{OccSet}(u), <_u) \). Now we define a function \( \Theta_u^{-1} \) from \( \text{OccSet}(u) \) to \( \text{OccSet}(u) \) as follows. If \( c \in \text{OccSet}(u) \) then \( \Theta_u^{-1}(c) = d \) so that \( \Theta_u(d) \) contains \( c \). For example, \( \Theta_u^{-1}(3(abbabab)a) = 2(xy2)x \).

It is easy to see that if \( U = \Theta(u) \) then function \( \Theta_u^{-1} \) is a homomorphism from \( (\text{OccSet}(U), <_U) \) to \( (\text{OccSet}(u), <_u) \), i.e., for every \( c, d \in \text{OccSet}(U) \) we have \( \Theta_u^{-1}(c) \leq_u \Theta_u^{-1}(d) \) whenever \( c <_U d \). The following fact can be easily verified.

**Fact 2.3.** Let \( u \) be a word and \( \Theta : \mathfrak{A} \to \mathfrak{A}^+ \) be a substitution so that \( \Theta(u) = U \).

If \( \Theta_u^{-1}(ux) = (juy) \) for some \( x \in \text{Cont}(U) \) and \( y \in \text{Cont}(u) \) then \( \text{occ}_u(y) \leq \text{occ}_U(x) \) and \( j \leq i \).

If \( \mathfrak{X} \) is a set of variables then we write \( u(\mathfrak{X}) \) to refer to the word obtained from \( u \) by deleting all occurrences of all variables that are not in \( \mathfrak{X} \) and say that the word \( u \) deletes to the word \( u(\mathfrak{X}) \). We say that a set \( X \subseteq \text{OccSet}(u) \) is left-stable in \( u \) with respect to a semigroup \( S \) if \( X \) is left-stable in any identity of \( S \) of the form \( u \approx v \). For simplicity, we sometimes write \( x \in U y \) instead of \( i_U x <_U y \). If a variable \( t \) is linear (1-occurring) in a word \( u \) then we use \( ut \) to denote the only occurrence of \( t \) in \( u \). An identity \( u \approx v \) is called regular if \( \text{Cont}(u) = \text{Cont}(v) \). Now we illustrate how to use Lemma 2.2.

**Theorem 2.4.** Let \( S \) be a monoid that satisfies the following conditions:

(i) For each \( n \) large enough, \( S \) satisfies the identity \( U_n = [X_n][Y_n][nx][ny] \approx [yn][nx][ny][xn] = V_n \);

(ii) For each \( m + c > 0 \), the set \( \{1x, t\} \) is left-stable in \( x^m tx^c \) with respect to \( S \);

(iii) For each \( m, c > 1 \) the equivalence class \( [x^m y^c]_S \) contains only words of the same form;

(iv) For each \( m, c > 0 \) and \( d > 1 \) the equivalence class \( [x^m y^d x^c]_S \) contains only words of the same form.

Then monoid \( S \) is non-finitely based.

**Proof.** First, notice that Condition (iii) implies that \( S \) satisfies only regular identities. We need the following property of the equivalence class \( [U_n]_S \).

**Claim 1.** Let \( n > 2, 1 < i \leq n \) and \( U \in [U_n]_S \). If \( 1ux_i <_U 1uy_2 \) then \( 1ux_{i+1} <_U 1uy_2 \).

**Proof.** Working toward a contradiction, assume that for some \( i > 1 \) we have \( 1x_i <_U 1y_2 \) but \( y_2 <_U 1x_{i+1} \). Then \( 1x_i <_U 1y_2 <_U 1x_{i+1} <_U (x_{i+1}) \).

Where is \( \ell y_2 \)? Since \( U_n(y_2, x_{i+1}) = (x_{i+1})(y_2)(x_{i+1})(y_2) \), in order to avoid a contradiction to Conditions (iiii), we conclude that \( (1x_{i+1}) <_U \ell y_2 \).

Now assume that \( \ell x_{i+1} <_U \ell y_2 \). Since \( U_n(y_1, y_2) = y_1y_2y_2y_1 \), Condition (iv) implies that \( U(y_1, y_2) = y_1^my_2^dy_1^c \) for some \( m, c > 0 \) and \( d > 1 \). So, we have \( (1y_1) <_U (1y_2) <_U (1x_{i+1}) <_U (\ell y_2) <_U (\ell y_1) \). Now Condition (iv) implies that \( U(y_1, x_{i+1}) = y_1^mx_{i+1}^dy_1^c \) for some \( m, c > 0 \) and \( d > 1 \). Now since \( U_n(y_1, x_{i+1}) = \)
\[(x_{i+1})(y_1)(x_{i+1})(y_1),\] to avoid a contradiction to Condition (iv) we conclude that \(e y_2 \not< \mathcal{U} e x_{i+1}\). So, we have that \((x_i) \not< \mathcal{U} (y_1) \not< \mathcal{U} (x_{i+1}) \not< \mathcal{U} e x_{i+1}\).

Since \(\mathcal{U} \{ x_{i+1}, x_{i+1} x_{i+1} \} \) to avoid a contradiction to Condition (iv), we must assume that \((x_i) \not< \mathcal{U} (y_1) \not< \mathcal{U} (x_{i+1}) \not< \mathcal{U} e x_{i+1}\). Now since \(\mathcal{U} \{ x_{i-1}, x_i \} = x_{i-1} x_i x_{i-1}\), to avoid a contradiction to Condition (iv), we must assume that \(U(x_{i-1}, y_2) = x_{i-1} y_2^1 \mathcal{U} x_{i-1}\) for some \(m, c > 0\) and \(d > 1\). Now since \(\mathcal{U} \{ x_{i-1}, y_2 \} = (x_{i-1})(y_2)(x_{i-1})(y_2)\), this gives us a contradiction to Condition (iv) that we can do nothing to avoid.

Now take \(n > 2\) and consider the identity \(\mathcal{U} \approx \mathcal{V} n\) and set \(X = \{ x_n, x_2, \mathcal{U} n y_2 \}\). Evidently, set \(X\) is left-unstable in \(\mathcal{U} \approx \mathcal{V} n\).

Let us check the second condition of Lemma 2.2. Let \(\mathcal{U} \in \mathcal{U} \{ \mathcal{U} \} S\) be a word so that set \(X = \{ \mathcal{U} x_n, \mathcal{U} n y_2 \}\) is left-stable in \(\mathcal{U} \approx \mathcal{U} \), i.e. \(\mathcal{U} x_2 \not< \mathcal{U} \mathcal{U} y_2\).

Since for each \(1 < i < n - 1\) we have \(\mathcal{U} \{ x_i, x_{i+1} \} = x_i x_{i+1} x_i\), Condition (iv) implies the following property of the word \(\mathcal{U} \).

\(\mathcal{P}\) For each \(1 < i < n - 1\) we have \(\mathcal{U} (x_i, x_{i+1}) = x_i^m \mathcal{U} _{i}^p x_{i+1}^q\) for some \(m, q > 0\) and \(p > 1\).

Property (P) and Claim 1 imply that \((\mathcal{U} x_2) \not< \mathcal{U} (\mathcal{U} x_3) \not< \mathcal{U} \cdots \not< \mathcal{U} (\mathcal{U} x_n) \not< \mathcal{U} (\mathcal{U} y_2)\).

Let \(\mathcal{U} \) be a word in less than \(n/2\) variables so that \(\Theta(\mathcal{U}) = \mathcal{U}\) for some substitution \(\Theta : \mathcal{A} \mapsto \mathcal{A}^+\). Since the word \(\mathcal{U}\) has less than \(n/2\) variables, for some \(c \in \text{OccSet}(\mathcal{U})\) and \(2 < i < n - 1\) both \(\mathcal{U} x_i\) and \(\mathcal{U} x_{i+1}\) are contained in \(\Theta(\mathcal{U})(c)\). Then property (P) implies that \(c\) must be the only occurrence of a linear variable \(t\) in \(\mathcal{U}\).

Since \(\Theta^{-1}(\mathcal{U})\) is a homomorphism from \((\text{OccSet}(\mathcal{U}), <)\) to \((\text{OccSet}(\mathcal{U}), <)\), we have that \(\Theta^{-1}(\mathcal{U} x_2) \leq \mathcal{U} (u t) \leq \mathcal{U} (\mathcal{U} y_2)\).

Now let \(\mathcal{U} \approx \mathcal{V} \) be an arbitrary identity of \(S\) and \(\mathcal{V} = \Theta(\mathcal{V})\). In view of Fact 2.3 for some \(x, x' \in \text{Cont}(\mathcal{U})\) and \(y, y' \in \text{Cont}(\mathcal{V})\) we have \(\mathcal{U} x = \Theta(\mathcal{U} x_2), \mathcal{U} y = \Theta^{-1}(\mathcal{U} y_2), \mathcal{V} x' = \Theta^{-1}(\mathcal{V} x_2)\) and \(\mathcal{V} y' = \Theta^{-1}(\mathcal{V} y_2)\). Then we have \(\mathcal{U} x \leq \mathcal{U} (u t) \leq \mathcal{U} (\mathcal{U} y) \leq \mathcal{U} (\mathcal{U} y'\).

Now in view of Condition (ii) we have that \((1 v x) \leq v (1 v x) \leq v (1 v y)\) and \((1 v t) \leq v (1 v t) \leq v (1 v y)\).

Therefore, \(\Theta^{-1}(\mathcal{V} x_2) = (1 v x) \leq v (1 v t) = v (1 v y') = \Theta^{-1}(\mathcal{V} y_2)\) and, consequently, \((1 v x_2) < v (1 v y_2)\). This means that set \(\mathcal{U} n, \mathcal{U} (X) = \{ \mathcal{U} x_2, \mathcal{U} y_2 \}\) is left-stable in \(\mathcal{U} \approx \mathcal{V}\). Therefore, monoid \(S\) is non-finitely based by Lemma 2.2.

**Corollary 2.5.** Let \(L^1\) denote the monoid obtained by adjoining an identity element to the semigroup \(L = \langle a, b \mid aa = a, bb = b, aba = 0 \rangle\) of order six. Then \(L^1\) is non-finitely based.

**Proof.** (i) According to Lemma 10 in 2.3, the monoid \(L^1\) satisfies \(\mathcal{U} \approx \mathcal{V} n\) for each \(n \geq 2\).

(ii) First notice that since \((ab)^2 = 0\) and \(ab \neq 0\), the word \(t\) is an isoterm for \(L^1\). Now Condition (ii) can be checked easily by substituting \(x \rightarrow b\) and \(t \rightarrow a\).

Conditions (iii)-(iv) can be checked easily by substituting \(x \rightarrow b\) and \(y \rightarrow a\). So, by Theorem 2.2 the monoid \(L^1\) is non-finitely based. \(\square\)
3 Interrelations between words induced by monoids

Notice that the sufficient condition for the non-finite basis property of monoids given by Theorem 2.4 is of the same form as we described in the end of the Introduction. Assertions (ii)-(iv) of Theorem 2.4 put certain restrictions on the equivalence classes of \( \sim_S \) in two variables, and the proof of Theorem 2.4 is done by analyzing how these restrictions affect the equivalence classes corresponding to words in arbitrary many variables.

Our goal for the rest of this article is to prove eight similar statements in a similar way (see Theorem 4.6 below). Each of these statements is encoded in one of the eight rows of Table 1 by a set of words \( W \) in the left column and a system of identities \( \{ U_n \approx V_n \mid n > 1 \} \) in the right column.

In order to achieve this goal, we study how restrictions on certain equivalence classes modulo \( \sim_S \) affect other equivalence classes modulo \( \sim_S \).

If \( W \) and \( W' \) are two sets of words then we write \( W \preceq W' \) if for any monoid \( S \) each word in \( W' \) is an isoterm for \( S \) whenever each word in \( W \) is an isoterm for \( S \).

It is easy to see that the relation \( \preceq \) is reflexive and transitive, i.e. it is a quasi-order on sets of words. If \( W \preceq W' \preceq W \) then we write \( W \sim W' \). The relations \( \preceq \) and \( \sim \) we extend to individual words. Say, if \( u \) and \( v \) are two words then \( u \sim v \) means \( \{ u \} \sim \{ v \} \). Also, if \( W \) is a set of words and \( u \) is a word then \( W \preceq u \) means \( W \preceq \{ u \} \).

This quasi-order \( \preceq \) provides us with a compact way to express how restrictions on certain equivalence classes modulo \( \sim_S \) affect other equivalence classes modulo \( \sim_S \). For example, it is easy to see that for every \( n > 1 \) we have \( xy \preceq x_1x_2\ldots x_n \). If \( u \) is a subword of a word \( v \) then evidently, we have \( v \preceq u \). The following fact will be used often in this article as well as in articles [15, 16, 17].

**Fact 3.1.** If \( xtx \) is an isoterm for a monoid \( S \), then

(i) the words \( xt_1yxt_2y \) and \( xt_1xyt_2y \) can only form an identity of \( S \) with each other;

(ii) the words \( xyt_1xt_2y \) and \( yxt_1xt_2y \) can only form an identity of \( S \) with each other;

(iii) the words \( xt_1yt_2xy \) and \( xt_1yt_2yx \) can only form an identity of \( S \) with each other.

**Proof.** (i) If \( S \) satisfies an identity \( xt_1yxt_2y \approx u \) then since \( xtx \preceq xy \) we have \( u(t_1, t_2) = t_1t_2 \). Since \( xtx \) is an isoterm for \( S \) we have that \( u(t_1, t_2, x) = xt_1xt_2 \) and \( u(t_1, t_2, y) = t_1yt_2y \). Therefore, the words \( xt_1yxt_2y \) and \( xt_1xyt_2y \) can only form an identity of \( S \) with each other.

The proofs of Parts (ii) and (iii) are similar. 

Since each of the six words considered in Fact 3.1 is less than the word \( xtx \) in the order \( \preceq \), Fact 3.1 immediately implies the following statement that will be often used without a reference.

**Fact 3.2.** \( xt_1xyt_2y \sim xt_1yxt_2y, xyt_1xt_2y \sim yxt_1xt_2y \) and \( xt_1yt_2xy \sim xt_1yt_2yx \).
We say that a set of variables $X$ is stable in an identity $u \approx v$ if $u(X) = v(X)$. Otherwise, we say that set $X$ is unstable in $u \approx v$. In particular, a variable $x$ is stable in $u \approx v$ if and only if $occ_u(x) = occ_v(x)$. We say that a set of variables $X$ is stable in a word $u$ with respect to $S$ if set $X$ is stable in every identity of $S$ of the form $u \approx v$. The following fact can be easily verified.

**Fact 3.3.** Let $S$ be a monoid and $u$ be a word.

(i) If the word $x$ is an isoterm for $S$ then $S$ satisfies only regular identities.

(ii) If the word $x^m$ is an isoterm for $S$ then every variable $x$ with $occ_u(x) \leq m$ is stable in $u$ with respect to $S$.

(iii) If the word $xy$ is an isoterm for $S$ then the set of all linear variables in $u$ is stable in $u$ with respect to $S$.

If $X \subseteq \text{OccSet}(u)$ then $\text{Cont}(X)$ denotes the set of all variables involved in $X$. We say that a set $X \subseteq \text{OccSet}(u)$ is stable in $u$ with respect to $S$ if $X$ is left-stable in $u$ with respect to $S$ and each variable in $\text{Cont}(X)$ is stable in $u$ with respect to $S$. If $\{c,d\} \subseteq X \subseteq \text{OccSet}(u)$, then we say that pair $\{c,d\}$ is adjacent in $X$ if there is no element $e \in X$ so that $c <_u e <_u d$. If $X = \text{OccSet}(u)$ then we simply say that pair $\{c,d\}$ is adjacent in $u$. We say that a pair of variables $\{x,y\}$ is adjacent in $u$ if for some $c,d \in \text{OccSet}(u(x,y))$ the pair $\{c,d\}$ is adjacent in $u$. The following two facts can be easily verified and will be sometimes used without a reference.

**Fact 3.4.** Let $S$ be a monoid and $u$ be a word. The following conditions are equivalent:

(i) $u$ is an isoterm for $S$;

(ii) The set $\text{Cont}(u)$ is stable in $u$ with respect to $S$;

(iii) The set $\text{OccSet}(u)$ is stable in $u$ with respect to $S$;

(iv) Each adjacent pair in $\text{OccSet}(u)$ is stable in $u$ with respect to $S$;

(v) Each adjacent pair in $\text{Cont}(u)$ is stable in $u$ with respect to $S$.

**Fact 3.5.** Let $u \approx v$ be an identity.

(i) If a set $X \subseteq \text{OccSet}(u)$ is stable in $u \approx v$, then every subset of $X$ is also stable in $u \approx v$.

(ii) If a set $X \subseteq \text{Cont}(u)$ is stable in $u \approx v$, then every subset of $X$ is also stable in $u \approx v$.

(iii) If a set $X \subseteq \text{Cont}(u)$ is stable in $u \approx v$, then $\text{OccSet}(u(X))$ is also stable in $u \approx v$.

We use letter $t$ with or without subscripts to denote linear (1-occurring) variables. If we use letter $t$ several times in a word, we assume that different occurrences of $t$ represent distinct linear variables. The word $x_1y_1x_2y_2\ldots x_ny_n$ is denoted by $[XYn]$. We use $U^t$ ($^tU$) to denote the word obtained from a word $U$ by inserting a linear letter after (before) each occurrence of each variable in $U$. For example, $[Zn]^t = z_1t_1z_2t_2\ldots z_nt_n$.

The next three lemmas are needed only to prove Theorem 4.6 below. We start with a statement related to the first row in Table \ref{table:definitions}. 

\[9\]
Fact 3.4. \(1 \leq t \) to \(S\) so that \(t\) is linear in \(u\).

Proof. Let \(S\) be a monoid so that the word \(xyyx\) is an isoterm for \(S\). Let \(u\) be a word that satisfies the following conditions:

(i) some variables \(x, y \in \text{Cont}(u)\) occur at most twice in \(u\);

(ii) \((1u)x < u(1uy)\);

(iii) If both variables \(x\) and \(y\) occur twice in \(u\) then \(u\) contains a linear variable \(t\) so that \((1uy) < u(2uy)\) and \((2ux) < u(2ut)\).

Then set \(\{1ux, 1uy\}\) is stable in \(u\) with respect to \(S\).

Lemma 3.6. Let \(S\) be a monoid so that the word \(xyyx\) is an isoterm for \(S\). Let \(u\) be a word that satisfies the following conditions:

(i) some variables \(x, y \in \text{Cont}(u)\) occur at most twice in \(u\);

(ii) \((1u)x < u(1uy)\);

(iii) If both variables \(x\) and \(y\) occur twice in \(u\) then \(u\) contains a linear variable \(t\) so that \((1uy) < u(2uy)\) and \((2ux) < u(2ut)\).

Then set \(\{1ux, 1uy\}\) is stable in \(u\) with respect to \(S\).

Proof. Let \(u \approx v\) be an identity of \(S\). Since the word \(xx\) is an isoterm for \(S\), we have that \(\text{occ}_u(x) = \text{occ}_v(x)\) and \(\text{occ}_u(y) = \text{occ}_v(y)\).

If both \(x\) and \(y\) are linear in \(u\) then set \(\{1ux, 1uy\}\) is stable in \(u \approx v\) by Fact 3.3.

If one variable among \(x\) and \(y\) is linear in \(u\), then set \(\{1ux, 1uy\}\) is stable in \(u \approx v\) because \(xyyx \leq \{txt, xxt, txx\}\).

If both \(x\) and \(y\) occur twice in \(u\) then either \(u(x, y, t) = xxyty\) or \(u(x, y, t) = xytxy\). In both cases, set \(\{1ux, 1uy\}\) is stable in \(u \approx v\).

The next lemma is related to the second row in Table 1.

Lemma 3.7. \(\{ytxxy, ytxxy\} \leq z_1t_1z_2t_2 \ldots z_n t_n x x z_1 \ldots z_n\).

Proof. Let \(S\) be a monoid so that the words \(ytxxy\) and \(ytxxy\) are isoterms for \(S\). Denote \(u = z_1t_2z_3 \ldots \). Since \(xx\) is an isoterm for \(S\) and every variable occurs in \(u\) at most twice, every variable in \(\text{Cont}(u)\) is stable in \(u\) with respect to \(S\). Since \(ytxxy \leq \{txt, xtx, txx\}\), each pair of variables that involves a linear variable is stable in \(u\) with respect to \(S\). Since for each \(1 \leq i < j \leq n\) we have \(u(z_i, z_j, t_i, t_n) = z_tz_2t_3t_nz_3z_j \sim ytxtyx \sim ytxxy \leq ytxxy\), for each \(1 \leq i < j \leq n\) the pair \((z_i, z_j)\) is stable in \(u\) with respect to \(S\). Since for each \(1 \leq i \leq n\) we have \(u(x, z_i, t_i) = z_tz_2t_3t_nz_3z_j \sim ytxxy\), for each \(1 \leq i \leq n\) the pair \((x, z_i)\) is stable in \(u\) with respect to \(S\). The rest follows from Fact 3.4.
The next lemma is related to the third row in Table 1.

**Lemma 3.8.** \(xtyxty \preceq [ZPn]^4[ZQn][PRn]^4[Qn] = \)
\[ (z_1t_1z_2t_2z_3t_3 \cdots z_nt_n)(z_1q_1z_2q_2 \cdots z_nq_n)(p_1r_1p_2r_2 \cdots p_nr_n)(tq_1tr_1tq_2tr_2 \cdots q_nr_n). \]

*Proof.* Let \(S\) be a monoid so that the word \(xtyxty\) is an isoterm for \(S\). Denote \(u = [ZPn]^4[ZQn][PRn]^4[Qn]\).

Notice that if \(x\) is a nonlinear variable in \(u\) then for some linear variable \(t\) we have \(u(x,t) = xtx\). This implies that each pair of variables that involves a linear variable is stable in \(u\) with respect to \(S\).

Now for each \(1 \leq i \leq n\) we have \(u(z_i,q_i) = z_itz_itq_i, u(q_i,z_{i+1}) = z_{i+1}tq_i z_{i+1}tq_i, u(q_n,p_1) = p_1tq_np_1tq_n, u(p_i,r_i) = p_itp_ir_ir_i, u(r_i,p_{i+1}) = p_{i+1}tr_ip_{i+1}tr_i\). In view of Fact 3.2(ii), all these words are isoterm for \(S\). Since every adjacent pair of variables is stable in \(u\) with respect to \(S\), the word \(u\) is an isoterm for \(S\) by Fact 3.3. \(\square\)

### 4 Some sufficient conditions for a monoid to be non-finitely based

The following statement generalizes Lemma 2.2 slightly and can be proved in a similar way.

**Lemma 4.1.** Let \(S\) be a semigroup.

Suppose that for each \(n\) large enough one can find an identity \(U_n \approx V_n\) of \(S\) in at least \(n\) variables and two sets \(X \subseteq \text{OccSet}(U_n)\) and \(X \subseteq \text{Cont}(U_n)\) so that each of the following conditions is satisfied:

(i) Either set \(X\) is left-unstable in \(U_n \approx V_n\) or some variable \(x \in X\) is unstable in \(U_n \approx V_n\):

(ii) If \(U \in [U_n]_S\), each variable \(x \in X\) is stable in \(U_n \approx U\) and set \(X\) is left-stable in \(U_n \approx U\) then for every identity \(u \approx v\) of \(S\) in less than \(n/2\) variables and every substitution \(\Theta : A \rightarrow A^+\) so that \(\Theta(u) = U\), each variable \(x \in X\) is stable in \(U \approx \Theta(v)\) and set \(\text{Inv}_U \cup \Theta(X)\) is left-stable in \(U \approx \Theta(v)\).

Then semigroup \(S\) is non-finitely based.

If we take set \(X\) to be the empty set then Lemma 4.1 turns into Lemma 2.2. If we take \(X = \text{OccSet}(U_n)\) and \(X = \text{Cont}(U_n)\) then in view of Fact 3.4(i) \(\leftrightarrow (iii))\), Lemma 4.1 immediately implies the following commonly used statement.

**Corollary 4.2.** Let \(S\) be a semigroup. Suppose that for each \(n\) large enough one can find a word \(U_n\) in at least \(n\) variables so that \(U_n\) is not an isoterm for \(S\) but every word \(w\) in less than \(n/2\) variables is an isoterm for \(S\) whenever \(\Theta(w) = U_n\) for some substitution \(\Theta : A \rightarrow A^+\). Then semigroup \(S\) is non-finitely based.

If \(\Theta : A \rightarrow A^+\) is a substitution and \(W\) is a set of variables then we define \(\Theta^{-1}(W) := \{x \in A \mid \text{Cont}(\Theta(x)) \cap W \neq \emptyset\}\). For example, if \(\Theta(x) = abc, \Theta(y) = bab\) and \(\Theta(z) = bb\) then \(\Theta^{-1}(\{a,c\}) = \{x,y\}\). If \(y\) is a variable then we write \(\Theta^{-1}(y)\) instead of \(\Theta^{-1}(\{y\})\).
Lemma 4.3. Let $u \approx v$ and $U \approx V$ be two identities so that for some substitution $\Theta : A \to A^+$ we have that $\Theta(u) = U$ and $\Theta(v) = V$.

Then a set of variables $\mathcal{Y}$ is stable in $U \approx V$ whenever the set $\Theta^{-1}(\mathcal{Y})$ is stable in $u \approx v$.

Proof. Denote $\mathcal{X} = \Theta^{-1}(\mathcal{Y})$. Since $\mathcal{X}$ is stable in $u \approx v$, we have that $u(\mathcal{X}) = v(\mathcal{X})$. Define a substitution $\Theta' : A \to A^+$ by $\Theta'(x) = \Theta(x)(\mathcal{Y})$. Then $U(\mathcal{Y}) = \Theta'(u(\mathcal{X})) = \Theta'(v(\mathcal{X})) = V(\mathcal{Y})$. Therefore, set $\mathcal{Y}$ is stable in $U \approx V$. $\square$

The next statement can be easily verified.

Fact 4.4. Let $u$ be a word and $\Theta : A \to A^+$ be a substitution. Suppose that variable $x$ occurs twice in $U = \Theta(u)$.

Then either set $\Theta^{-1}(x)$ contains two variables $t_1$ and $t_2$ so that $\operatorname{occ}_u(t_1) = \operatorname{occ}_u(t_2) = 1$, $\Theta^{-1}_u(1Ux) = ut_1$ and $\Theta^{-1}_u(2Ux) = ut_2$ or set $\Theta^{-1}(x)$ contains variable $x$ so that $\operatorname{occ}_u(x) = 2$, $\Theta^{-1}_u(1Ux) = 1ux$ and $\Theta^{-1}_u(2Ux) = 2ux$.

Lemma 4.5. Let $S$ be a monoid so that the word $xy$ is an isoterm for $S$ and $S$ satisfies an identity $u \approx v$. Suppose that for some substitution $\Theta : A \to A^+$, some variable $x$ occurs twice in both $U = \Theta(u)$ and $V = \Theta(v)$. Then $l_{u,v}(\Theta^{-1}_u(1Ux)) = \Theta^{-1}_v(1Vx)$ and $l_{u,v}(\Theta^{-1}_u(2Ux)) = \Theta^{-1}_v(2Vx)$.

Proof. In view of Fact 4.3, there are only two possibilities for set $\Theta^{-1}(x)$ and word $u$.

First, suppose that $u$ contains two variables $t_1$ and $t_2$ so that $\operatorname{occ}_u(t_1) = \operatorname{occ}_u(t_2) = 1$, $\Theta^{-1}_u(1Ux) = ut_1$ and $\Theta^{-1}_u(2Ux) = ut_2$. Since the word $xy$ is an isoterm for $S$ this implies that $u(t_1, t_2) = v(t_1, t_2)$. Since both $\Theta(t_1)$ and $\Theta(t_2)$ contain $x$, the only possibility for word $v$ is that $\Theta^{-1}_v(1Vx) = vt_1$ and $\Theta^{-1}_v(2Vx) = vt_2$.

Now assume that $u$ contains variable $x$ so that $\operatorname{occ}_u(x) = 2$, $\Theta^{-1}_u(1Ux) = 1ux$ and $\Theta^{-1}_u(2Ux) = 2ux$. Since the word $x$ is an isoterm for $S$, we have $\operatorname{occ}_v(x) = 2$. Since $\Theta(x)$ contains $x$, we have $\Theta^{-1}_v(1Vx) = 1vx$ and $\Theta^{-1}_v(2Vx) = 2vx$.

In any case we have $l_{u,v}(\Theta^{-1}_u(1Ux)) = \Theta^{-1}_v(1Vx)$ and $l_{u,v}(\Theta^{-1}_u(2Ux)) = \Theta^{-1}_v(2Vx)$. $\square$

The word obtained from a word $u$ by deleting all occurrences of all variables in a set $\mathcal{X}$ is denoted by $D_X(u)$. If set $\mathcal{X}$ contains only one variable $x$ then we simply write $D_x(u)$.

Theorem 4.6. Each row in Table 1 encodes a sufficient condition under which a monoid is non-finitely based. More precisely, each row in Table 1 encodes the following statement: if $S$ is a monoid that satisfies the identity $U_n \approx V_n$ for each $n > 1$ and every word in set $W$ is an isoterm for $S$ then $S$ is non-finitely based.

Proof. Row 1. Here $W = \{xyyx\}$ and $U_n = xx[YN][nY] \approx [YN][nY]xx = V_n$. Let $S$ be a monoid so that the word $xyyx$ is an isoterm for $S$ and $S$ satisfies the identity $U_n \approx V_n$ for each $n > 1$.
Claim 2. If $U \in [U_n]_S$ then $U$ satisfies the following properties:

(P1) $\text{occ}_U(x) = 2$;
(P2) $D_x(U) = [Yn][nY] = y_1y_2 \ldots y_{n-1}y_ny_n-1 \ldots y_2y_1$;
(P3) If $1u_x < u 1 uy_1$ then $2ux < u 2uy_n$.

Proof. Property (P1) follows from the fact that the word $xx$ is an isoterm for $S$ and $\text{occ}_{U_n}(x) = 2$.

Property (P2) follows from the fact that

$D_x(U_n) = [Yn][nY] = y_1y_2 \ldots y_{n-1}y_ny_n-1 \ldots y_2y_1 \succeq xy yx$.

Let us verify Property (P3). If $1u_x < u 1 uy_1$ but $2uy_n < u 2x$, then in view of Property (P2) we would had $U(x, y_n) = xy y_n x \neq x x y_n y_n = U_n(x, y_n)$. Since the word $x y_n y_n x$ is an isoterm for $S$, we must assume that $U$ satisfies Property (P2). \(\square\)

Now take $n > 10$ and $X = \{1u_x, 1u_y, y_1\}$. Evidently, set $X = \{1u_x, 1u_y, y_1\}$ is left-unstable in $U_n \approx V_n$.

Let us check the second condition of Lemma 2.2. Let $U \in [U_n]_S$ be a word so that set $X = \{1u_x, 1u_y, y_1\}$ is left-stable in $U_n \approx U$, i.e. $1u_x < u 1 uy_1$.

Property (P2) implies that

$1u_x < u 1 uy_1 < u 2uy_n < u \cdots < u 2uy_2 < u 2uy_1$.

Properties (P2) and (P3) imply that

$1u_x < u 2ux < u 2uy_n < u \cdots < u 2uy_2 < u 2uy_1$.

Let $u$ be a word in less than $n/2$ variables so that $\Theta(u) = U$ for some substitution $\Theta: \mathfrak{A} \to \mathfrak{A}^+$. Since the word $u$ has less than $n/2$ variables, for some $c \in \text{OccSet}(u)$ and $2 < i < n - 1$ both $2uy_{i+1}$ and $2uy_i$ are contained in $\Theta(u)(c)$. Then property (P2) implies that $c$ must be the only occurrence of a linear variable $t$ in $u$.

Since $\Theta^{-1}_u$ is a homomorphism from $(\text{OccSet}(U), < U)$ to $(\text{OccSet}(u), < u)$, we have that $\Theta^{-1}_u(1u_x) \leq u \Theta^{-1}_u(1uy_1) \leq u (ut) \leq u \Theta^{-1}_u(2uy_1)$ and

$\Theta^{-1}_u(1ux) \leq u \Theta^{-1}_u(2ux) \leq u (ut) \leq u \Theta^{-1}_u(2uy_1)$.

Now $\Theta^{-1}_u(1ux)$ and $\Theta^{-1}_u(1uy_1)$ are occurrences of some variables $x$ and $y$ in $u$. By Fact 2.4 variables $x$ and $y$ occur at most twice in $u$ and if each of them occurs twice in $u$ then $1u_x = \Theta^{-1}_u(1ux), 2ux = \Theta^{-1}_u(2ux), 1uy = \Theta^{-1}_u(1uy_1)$ and $2uy = \Theta^{-1}_u(2uy_1)$.

Now let $u \approx v$ be an arbitrary identity of $S$ and $V = \Theta(v)$. Since the word $u(x, y, t)$ satisfies all conditions of Lemma 3.6 the set $\{1ux, 1uy\}$ is left-stable in $u \approx v$, i.e. $(1uv) \leq v (1uv)$. Then Lemma 4.5 implies that $(1uvx) < v (1uvy_1)$. This means that the set $l_{U_n} U(X) = \{1ux, 1uyy_1\}$ is left-stable in $U \approx V$. Therefore, monoid $S$ is non-finitely based by Lemma 2.2.

Row 2. Here $W = \{xy x y, y x x y\}$ and $U_n = [Zn]^t y x x [Zn]^t y \approx [Zn]^t x y x [Zn]^t y = \approx V_n$. Let $S$ be a monoid so that each word in set $W$ is an isoterm for $S$ and $S$ satisfies the identity $U_n \approx V_n$ for each $n > 1$. If $U \in [U_n]_S$ then $U$ satisfies the following properties:

(P1) $\text{occ}_U(y) = 2$;
(P2) $D_y(U) = [Zn]^t x x [Zn]$.

(Property (P1) follows from the fact that the word $xx$ is an isoterm for $S$ and $\text{occ}_{U_n}(y) = 2$. Property (P2) follows from the fact that $D_y(U_n) = [Zn]^t x x [Zn] = z_1z_2 \ldots z_n t x x z_1 z_2 \ldots z_n$ and Lemma 5.7.)
Now take \( n > 10 \) and \( X = \{1u_n x, 1u_n y\} \). Evidently, set \( X = \{1u_n x, 1u_n y\} \) is left-unstable in \( U_n \approx V_n \).

Let us check the second condition of Lemma \( 22 \). Let \( U \in [U_n]_S \) be a word so that set \( X = \{1u_n x, 1u_n y\} \) is left-stable in \( U_n \approx U \), i.e. \( 1u_n y <_U 1u_n x \).

Let \( u \) be a word in less than \( n/2 \) variables so that \( \Theta(u) = U \) for some substitution \( \Theta : A \to A^+ \). Since the word \( u \) has less than \( n/2 \) variables, for some \( c \in \text{OccSet}(u) \) and \( 2 < i < n - 1 \) both \( 2u z_i \) and \( 2u z_{i+1} \) are contained in \( \Theta_u(c) \). Then property (P2) implies that \( c \) must be the only occurrence of a linear variable \( t \) in \( u \).

Since \( \Theta_u^{-1} \) is a homomorphism from \((\text{OccSet}(U),<_U)\) to \((\text{OccSet}(u),<_u)\), we have that \( \Theta_u^{-1}(1u y) \leq_u \Theta_u^{-1}(1u x) \leq_u \Theta_u^{-1}(2u x) \leq_u (u t) \leq_u \Theta_u^{-1}(2u y) \).

Now \( \Theta_u^{-1}(1u x) \) and \( \Theta_u^{-1}(1u y) \) are occurrences of some variables \( x \) and \( y \) in \( u \). By Fact \( \text{4.4} \), variables \( x \) and \( y \) occur at most twice in \( u \) and if each of them occurs twice in \( u \) then \( 1u x = \Theta_u^{-1}(1u x) \), \( 2u x = \Theta_u^{-1}(2u x) \), \( 1u y = \Theta_u^{-1}(1u y) \) and \( 2u y = \Theta_u^{-1}(2u y) \).

Now let \( u \approx v \) be an arbitrary identity of \( S \) and \( V = \Theta(v) \). If either \( x \) or \( y \) is linear in \( u \) then set \( \{1u x, 1u y\} \) is left-stable in \( u \approx v \) because \( y x y t \leq \{x, x t, x t x, x t x t\} \).

If both \( x \) and \( y \) occur twice in \( u \) then the set \( \{1u x, 1u y\} \) is left-stable in \( u \approx v \) because \( u(x, y, t) = y x y t \) is an isoterm for \( S \). Since the set \( \{1u x, 1u y\} \) is left-stable in \( u \approx v \), we have \( (1v x) \leq_v (1v y) \). Then Lemma \( 4.5 \), implies that \( (1v x) <_V (1v y) \). This means that the set \( I_{U_n, U}(X) = \{1u x, 1u y\} \) is left-stable in \( U \approx V \). Therefore, monoid \( S \) is non-finitely based by Lemma \( 22 \).

Row 3. Here \( W = \{x t x y t y\} \) and \( U_n = \{\text{ZP} n x, \text{ZQ} n y[\text{PR} n] y'[\text{QR} n] \approx [\text{ZP} n]' x[\text{ZQ} n]' y[\text{PR} n]' y'[\text{QR} n] \approx \text{V}_n \}. \) Let \( S \) be a monoid so that the word \( x t x y t y \) is an isoterm for \( S \) and \( S \) satisfies the identity \( U_n \approx V_n \) for each \( n > 1 \). If \( U \in [U_n]_S \) then \( U \) satisfies the following property:

\[
(P) \quad D_{\{x, y\}}(U) = [\text{ZP} n]'[\text{ZQ} n]'[\text{PR} n]'[\text{QR} n].
\]

(Property (P)) follows from the fact that \( D_y(U_n) = [\text{ZP} n]'[\text{ZQ} n]'[\text{PR} n]'[\text{QR} n] \) and Lemma \( 3.3 \).

Now take \( n > 10 \), \( X = \{1u_n x, 2u_n z_1, 1u_n q_n, 2u_n x, 1u_n y, 2u_n p_1, 1u_n t_n, 2u_n y\} \) and \( \mathcal{X} = \{x, y\} \). Evidently, set \( X \) is left-unstable in \( U_n \approx V_n \).

Let us check the second condition of Lemma \( 44 \). Let \( U \in [U_n]_S \) be a word so that \( \text{occ}_U(x) = \text{occ}_U(y) = 2 \) and set \( X \) is left-stable in \( U_n \approx U \), i.e. \((1u x) <_U (2u z_1) <_U (1u q_n) <_U (2u x) <_U (1u y) <_U (2u p_1) <_U (1u t_n) <_U (2u y)\).

Let \( u \) be a word in less than \( n/2 \) variables so that \( \Theta(u) = U \) for some substitution \( \Theta : A \to A^+ \). Since the word \( u \) has less than \( n/2 \) variables, for some \( c \in \text{OccSet}(u) \) and \( 2 < i < n - 1 \) both \( 2u z_i \) and \( 2u z_{i+1} \) are contained in \( \Theta_u(c) \). Then property (P) implies that \( c \) must be the only occurrence of a linear variable \( t_i \) in \( u \). Similarly, the word \( u \) contains a linear letter \( t \) so that \( \Theta_u(t) \) contains both \( 2u r_i \) and \( 2u t_{i+1} \).

Since \( \Theta_u^{-1} \) is a homomorphism from \((\text{OccSet}(U),<_U)\) to \((\text{OccSet}(u),<_u)\), we have that \( \Theta_u^{-1}(1u x) \leq_u \Theta_u^{-1}(2u z_1) \leq_u (u t_1) \leq_u \Theta_u^{-1}(1u q_n) \leq_u \Theta_u^{-1}(2u x) \leq_u \Theta_u^{-1}(1u y) \leq_u \Theta_u^{-1}(2u p_1) \leq_u (u t_2) \leq_u \Theta_u^{-1}(1u t_n) \leq_u \Theta_u^{-1}(2u y)\).

Claim 3. The word \( u(\Theta^{-1}(x) \cup \Theta^{-1}(y) \cup \{t_1, t_2\}) \) is an isoterm for \( S \).

Proof. In view of Fact \( \text{14} \), there are only two possibilities for set \( \Theta^{-1}(x) \) and two possibilities for set \( \Theta^{-1}(y) \). If set \( \Theta^{-1}(x) \) contains variable \( x \) with \( \text{occ}_u(x) = 2 \) and
set $\Theta^{-1}(y)$ contains variable $y$ with $\text{occ}_u(x) = 2$ then $u(x, y, t_1, t_2) = xtyt_2y$ is an isoterm for $S$. If either set $\Theta^{-1}(x)$ or set $\Theta^{-1}(y)$ contains a variable that is linear in $u$ then we have $xtyt_2y \preceq \{xxt, txx, ttx\} \preceq u(\Theta^{-1}(x) \cup \Theta^{-1}(y) \cup \{t_1, t_2\})$. □

Now let $u \approx v$ be an arbitrary identity of $S$ and $V = \Theta(v)$. Claim 4 implies that set of variables $\Theta^{-1}(\{x, y\}) = \Theta^{-1}(x) \cup \Theta^{-1}(y)$ is stable in $u \approx v$. Consequently, set $\{x, y\}$ is stable in $U \approx V$ by Lemma 4.3. In particular, this means that $\text{occ}_V(x) = \text{occ}_V(y) = 2$ i.e. each variable in $X = \{x, y\}$ is stable in $U \approx V$.

**Claim 4.** Set

$$\{\Theta_u^{-1}(1ux), \Theta_u^{-1}(2uz_1), \Theta_u^{-1}(1uq_n), \Theta_u^{-1}(2ux), \Theta_u^{-1}(1uy), \Theta_u^{-1}(2up_1), \Theta_u^{-1}(1ur_n), \Theta_u^{-1}(2uy)\}$$

is left-stable in $u \approx v$.

*Proof.* In view of Claim 4 and Lemma 4.8, it is only left to prove that the following sets are left-stable in $u \approx v$: $$\{\Theta_u^{-1}(1ux), \Theta_u^{-1}(2uz_1), \Theta_u^{-1}(1uq_n), \Theta_u^{-1}(2ux), \Theta_u^{-1}(1uy), \Theta_u^{-1}(2up_1), \Theta_u^{-1}(1ur_n), \Theta_u^{-1}(2uy)\}.$$ We only show that set $\{\Theta_u^{-1}(1ux), \Theta_u^{-1}(2uz_1)\}$ is left-stable in $u \approx v$. (Proofs for the other three sets are similar.) Indeed, $\Theta_u^{-1}(1ux)$ and $\Theta_u^{-1}(2uz_1)$ are occurrences of some variables $x$ and $z$ in $u$. By Fact 4.4, variables $x$ and $z$ occur at most twice in $u$ and if each of them occurs twice in $u$ then $1ux = \Theta_u^{-1}(1ux), 2ux = \Theta_u^{-1}(2ux), 1uz = \Theta_u^{-1}(1uz)$ and $2az = \Theta_u^{-1}(2uz_1)$. If either $x$ or $z$ is linear in $u$ then set $\{1ux, 1uz\}$ is left-stable in $u \approx v$ because $xtyt_2y \preceq \{xxt, txx, ttx\}$. If both $x$ and $z$ occur twice in $u$ then the set $\{1ux, 2az\}$ is left-stable in $u \approx v$ because $u(x, z, t) = ztxzt_1x$ is an isoterm for $S$. □

Claim 4 and Lemma 4.7 imply that set

$$U_n u(X) = \{1ux, 2uz_1, 1uq_n, 2ux, 1uy, 2up_1, 1ur_n, 2uy\}$$

is left-stable in $U \approx V$. Therefore, monoid $S$ is non-finitely based by Lemma 4.4.

**Row 4.** Here $W = \{xxxy, xtytx\}$ and $U_n = xtyz_1^2 z_2^2 \ldots z_n^2 \simeq xtyz_1^2 z_2^2 \ldots z_n^2 x = V_n$. Let $S$ be a monoid so that each word in $W$ is an isoterm for $S$ and $S$ satisfies the identity $U_n \approx V_n$ for each $n > 1$.

**Claim 5.** If $U \in [U_n]_S$ then $U$ satisfies the following properties:

$(P1)$ $\text{occ}_U(x) = 2$;

$(P2)$ $D_x(U) = ytyz_1^2 z_2^2 \ldots z_n^2$;

$(P3)$ $1ux \preceq_U u t$;

$(P4)$ $1ux \preceq_U 2ux$.

*Proof.* Property (P1) follows from the fact that $xx$ is an isoterm for $S$. Property (P2) follows from the fact that $D_x(U_n) = ytyz_1^2 z_2^2 \ldots z_n^2 \simeq xxxy$. Property (P3) follows from the fact that $xtx$ is an isoterm for $S$. If $2ux \preceq_U 1ux$ then monoid $S$ would have satisfied the identity $U(x, t, z_n) = xtxz_n z_n x \simeq xt_n z_n x = U_n(x, t, z_n)$. Since this identity is false in $S$, Property (P4) is verified. □

Now take $n > 10$ and $X = \{(1u_n x, 1u_n y)\}$. Evidently, set $X = \{(1u_n x, 1u_n y)\}$ is left-unstable in $U_n \approx V_n$. 15
Let us check the second condition of Lemma 2.2. Let $U \in [U_n]_S$ be a word so that set $X = \{1u_nx, 1uNy\}$ is left-stable in $U_n \cong U$, i.e. $1u_nx <_U 1uNyt$.

Let $u$ be a word in less than $n/2$ variables so that $\Theta(u) = U$ for some substitution $\Theta : \mathfrak{A} \to \mathfrak{A}^+$. Since the word $u$ has less than $n/2$ variables, for some $c \in \text{Occ}(u)$ and $2 < i < n - 1$ both $1u_iz_i$ and $1u_iz_{i+1}$ are contained in $\Theta_u(c)$. Then property (P2) implies that $c$ must be the only occurrence of a linear variable $t_1$ in $u$.

Since $\Theta_u^{-1}$ is a homomorphism from $(\text{Occ}(U), <_U)$ to $(\text{Occ}(u), <_u)$, Properties (P1)-(P4) imply that $\Theta_u^{-1}(1u_x) \leq_u \Theta_u^{-1}(1u_y) \leq_u \Theta_u^{-1}(2u_y) \leq_u (ut) \leq_u \Theta_u^{-1}(2u_x)$.

Now $\Theta_u^{-1}(1u_x)$ and $\Theta_u^{-1}(1u_y)$ are occurrences of some variables $x$ and $y$ in $u$. By Fact 4.4, variables $x$ and $y$ occur at most twice in $u$ and if each of them occurs twice in $u$ then $1u_x = \Theta_u^{-1}(1u_x)$, $2u_x = \Theta_u^{-1}(2u_x)$, $1u_y = \Theta_u^{-1}(1u_y)$ and $2u_y = \Theta_u^{-1}(2u_y)$.

Now let $u \equiv v$ be an arbitrary identity of $S$ and $V = \Theta(v)$. If either $x$ or $y$ is linear in $u$ then set $\{1u_x, 1u_y\}$ is left-stable in $u \equiv v$ because $x = xyty \leq\{xty, ttx, xt\}$. If both $x$ and $y$ occur twice in $u$ then the set $\{1u_x, 1u_y\}$ is left-stable in $u \equiv v$ because $u(x, y, t, t_1) = xytyt_1x$ is an isomer for $S$. Since the set $\{1u_x, 1u_y\}$ is left-stable in $u \equiv v$, we have $(1v_x) \leq_v (1v_y)$. Then Lemma 4.5 implies that $(1v_y) <_v (1v_x)$. This means that the set $l_{u_n, u}(X) = \{1u_n, 1u_y\}$ is left-stable in $U \equiv V$. Therefore, monoid $S$ is non-finitely based by Lemma 2.2.

Row 5. Here $W = \{xtytxy, xtyxy, xtytxy\}$ and $U_n = xy[Zn]yty[nZ] \approx yx[Zn]xyt[ZA] = V_n$. Let $S$ be a monoid so that each word in set $W$ is an isomer for $S$ and $S$ satisfies the identity $U_n \equiv V_n$ for each $n > 1$. If $U \in [U_n]_S$ then $U$ satisfies the following property:

(P) $D_{x,y}(U) = [Zn]t[nZ] = z_1z_2 \ldots z_nz_{n+1}$.

(Property (P) follows from the fact $D_{x,y}(U_n) = [Zn]t[nZ] \succeq xtytxy$.)

Now take $n > 10$, $X = \{1u_n, x, 1u_nz_1, 1u_nz_n, 1u_yn, 2u_n, y, 2u_nx\}$ and $X = \{x, y\}$. Evidently, set $X$ is left-unstable in $U_n \equiv V_n$.

Let us check the second condition of Lemma 4.1. Let $U \in [U_n]_S$ be a word so that occ$(u) = occ(u) = 2$ and set $X$ is left-stable in $U_n \equiv U$, i.e. $(1uNyt) <_U (1uNyt) <_U (1uNyt) <_U (2uNyt) <_U (2uNyt)$.

Let $u$ be a word in less than $n/2$ variables so that $\Theta(u) = U$ for some substitution $\Theta : \mathfrak{A} \to \mathfrak{A}^+$. Since the word $u$ has less than $n/2$ variables, for some $c \in \text{Occ}(u)$ and $2 < i < n - 1$ both $2uz_i$ and $2uz_{i+1}$ are contained in $\Theta_u(c)$. Then property (P2) implies that $c$ must be the only occurrence of a linear variable $t$ in $u$.

Since $\Theta_u^{-1}$ is a homomorphism from $(\text{Occ}(U), <_U)$ to $(\text{Occ}(u), <_u)$, we have that $\Theta_u^{-1}(1u_x) \leq_u \Theta_u^{-1}(1u_y) \leq_u \Theta_u^{-1}(1u_z) \leq_u (ut) \leq_u \Theta_u^{-1}(1u_nz_1) \leq_u \Theta_u^{-1}(1u_nz_n) \leq_u \Theta_u^{-1}(2u_nyt) \leq_u \Theta_u^{-1}(2u_nx)$.

Claim 6. The word $u(\Theta^{-1}(x) \cup \Theta^{-1}(y) \cup \{t\})$ is an isomer for $S$.

Proof. In view of Fact 4.4 there are only two possibilities for set $\Theta^{-1}(x)$ and two possibilities for set $\Theta^{-1}(y)$. If set $\Theta^{-1}(x)$ contains variable $x$ with occ$(x) = 2$ and set $\Theta^{-1}(y)$ contains variable $y$ with occ$(y) = 2$ then $u(x, y, t) = xtytxy$ is an isomer for $S$. If either set $\Theta^{-1}(x)$ or set $\Theta^{-1}(y)$ contains a variable that is linear in $u$ then we have $xtytxy \leq u(\Theta^{-1}(x) \cup \Theta^{-1}(y) \cup \{t\})$. □
Now let \( u \approx v \) be an arbitrary identity of \( S \) and \( V = \Theta(v) \). Claim \( 4 \) implies that set of variables \( \Theta^{-1}(\{x, y\}) = \Theta^{-1}(x) \cup \Theta^{-1}(y) \) is stable in \( u \approx v \). Consequently, set \( \{x, y\} \) is stable in \( U \approx V \) by Lemma \( 1.3 \). In particular, this means that \( \text{occ}_V(x) = \text{occ}_V(y) = 2 \). i.e. each variable in \( X = \{x, y\} \) is stable in \( U \approx V \).

**Claim 7.** Set \( \{\Theta_u^{-1}(11u), \Theta_u^{-1}(1iy), \Theta_u^{-1}(1iz), \Theta_u^{-1}(1u z_n), \Theta_u^{-1}(2uy), \Theta_u^{-1}(2ux)\} \)

is left-stable in \( u \approx v \).

**Proof.** In view of Claim \( 6 \) and the fact that \( xytyx \preceq [Zn][t][nZ] \), it is only left to prove that the following sets are left-stable in \( u \approx v \): \( \{\Theta_u^{-1}(1iy), \Theta_u^{-1}(1iz)\} \), \( \{\Theta_u^{-1}(1u z_n), \Theta_u^{-1}(2uy)\} \).

Indeed, \( \Theta_u^{-1}(1iy) \) and \( \Theta_u^{-1}(1iz) \) are occurrences of some variables \( y \) and \( z \) in \( u \). By Fact \( 4.4 \), variables \( y \) and \( z \) occur at most twice in \( u \) and if each of them occurs twice in \( u \) then \( 1uy = \Theta_u^{-1}(1iy) \), \( 2uy = \Theta_u^{-1}(2uy) \), \( 1uz = \Theta_u^{-1}(1uz) \) and \( 2uz = \Theta_u^{-1}(2uz) \). If either \( y \) or \( z \) is linear in \( u \) then set \( \{1uy, 1uz\} \) is left-stable in \( u \approx v \) because \( xytyx \preceq xtx \). If both \( y \) and \( z \) occur twice in \( u \) then the set \( \{1uy, 1uz\} \) is left-stable in \( u \approx v \) because \( u(y, z, t) = yztzy \).

Similarly, \( \Theta_u^{-1}(2uy) \) and \( \Theta_u^{-1}(1uz) \) are occurrences of some variables \( y \) and \( z \) in \( u \). By Fact \( 4.4 \), variables \( y \) and \( z \) occur at most twice in \( u \) and if each of them occurs twice in \( u \) then \( 1uy = \Theta_u^{-1}(1iy) \), \( 2uy = \Theta_u^{-1}(2uy) \), \( 1uz = \Theta_u^{-1}(1uz) \) and \( 2uz = \Theta_u^{-1}(2uz) \). If either \( y \) or \( z \) is linear in \( u \) then set \( \{2uy, 1uz\} \) is left-stable in \( u \approx v \) because \( xytyx \preceq xtx \). If both \( y \) and \( z \) occur twice in \( u \) then the set \( \{2uy, 1uz\} \) is left-stable in \( u \approx v \) because \( u(y, z, t, t') = yztzyz \). (Here \( ut' = \Theta_u^{-1}(1u) \).) \( \square \)

Claim \( 4 \) and Lemma \( 2.5 \) imply that set \( I_{1u,n}(X) = \{11u, 1uy, 1uz, 1u z_n, 2uy, 2ux\} \) is left-stable in \( U \approx V \). Therefore, monoid \( S \) is non-finitely based by Lemma \( 4.1 \)

**Row 6.** Similar to Row 5.

**Row 7.** Here \( W = \{xtytx, xytx\} \) and \( U_n = [Xn][nX][Y n][nY] \approx [Y n][nY][Xn][nX] = V_n \). Let \( S \) be a monoid so that each word in set \( W \) is an isoterm for \( S \) and \( S \) satisfies the identity \( U_n \approx V_n \) for each \( n > 1 \). If \( U \in [U_n]_S \) then \( U \) satisfies the following properties:

- (P1) \( D_{\{x_1, \ldots, x_n\}}(U) = [Xn][nX] = y_1y_2 \ldots y_n \).
- (P2) \( D_{\{y_1, \ldots, y_n\}}(U) = [Xn][nX] = x_1x_2 \ldots x_n \).
- (Properties (P1) and (P2) follow from the fact that \( D_{\{y_1, \ldots, y_n\}}(U_n) = [Xn][nX] = x_1x_2 \ldots x_n \).

Now take \( n > 10 \) and \( X = \{2ux, 1uy, 1y_1\} \). Evidently, set \( X = \{2ux, 1uy, 1y_1\} \) is left-unstable in \( U_n \approx V_n \).

Let us check the second condition of Lemma \( 2.2 \). Let \( U \in [U_n]_S \) be a word so that set \( X = \{2ux, 1uy, 1y_1\} \) is left-stable in \( U_n \approx U \), i.e. \( 2ux < u 1uy \).

Let \( u \) be a word in less than \( n/2 \) variables so that \( \Theta(u) = U \) for some substitution \( \Theta : \mathcal{A} \to \mathcal{A}^+ \). Since the word \( u \) has less than \( n/2 \) variables, for some \( c \in \text{OccSet}(u) \) and \( 2 < i < n-1 \) both \( 2ux_i \) and \( 2ux_{i+1} \) are contained in \( \Theta(u)(c) \). Then property (P2) implies that \( c \) must be the only occurrence of a linear variable \( t_i \) in \( u \). Similarly, the word \( u \) contains a linear letter \( t_2 \) so that \( \Theta(u(t_2)) \) contains both \( 2uy_i \) and \( 2uy_{i+1} \).
Since $\Theta_u^{-1}$ is a homomorphism from $(\text{OccSet}(U), <_U)$ to $(\text{OccSet}(u), <_u)$, we have that $\Theta_u^{-1}(1u_1) \leq u(1u_1) \leq u \Theta_u^{-1}(2u_1) \leq u \Theta_u^{-1}(1u_1) \leq u \Theta_u^{-1}(2u_1)$.

Now $\Theta_u^{-1}(2u_1)$ and $\Theta_u^{-1}(1u_1)$ are occurrences of some variables $x$ and $y$ in $u$. By Fact 4.4, variables $x$ and $y$ occur at most twice in $u$ and if each of them occurs twice in $u$ then $1u_1x = \Theta_u^{-1}(1u_1)x_1$, $2u_1x = \Theta_u^{-1}(2u_1)x_1$, $1u_1y = \Theta_u^{-1}(1u_1)y_1$ and $2u_1y = \Theta_u^{-1}(2u_1)y_1$.

Now let $u \approx v$ be an arbitrary identity of $S$ and $V = \Theta(v)$. If either $x$ or $y$ is linear in $u$ then set $\{1u_1x, 1u_1y\}$ is left-stable in $u \approx v$ because $xtxyty \leq xtx$. If both $x$ and $y$ occur twice in $u$ then the set $\{2u_1x, 1u_1y\}$ is left-stable in $u \approx v$ because $u(x, y, t) = xtxyty$ is an isomorphism for $S$. Since the set $\{2u_1x, 1u_1y\}$ is left-stable in $u \approx v$, we have $(2u_1x) \leq (1u_1y)$. Then Lemma 4.1 implies that $(2u_1x) \leq (1u_1y)$. This means that set $\{u_1v_1, 1u_1y\}$ is left-stable in $U \approx V$. Therefore, monoid $S$ is non-finitely based by Lemma 2.2.

**Row 8.** Here $W = \{xxyy\} \cup \{ytpyztxx, x^m-dtx^dtx^dty\vert 0 < d < m\}$ for some $m > 2$ and $U_n = ytpyztxx = ytpyztxx \approx ytpyztxx = V_n$. Let $S$ be a monoid so that each word in set $W$ is an isomorphism for $S$ and $S$ satisfies the identity $U_n \approx V_n$ for each $n > 1$. If $U \in [U_n]_S$ then $U$ satisfies the following property:

(P) $D_x(U_n) = ytpyztxx$.

(Property (P) follows from the fact $D_x(U_n) = ytpyztxx \leq xxyy$)

Now take $n > 10$, $X = \{u_1v_1, 1u_1x, 2u_1y, 1u_1z, mUx, u_1t_2\}$ and $U = \{x\}$. Evidently, set $X$ is left-unsupported in $U \approx V_n$.

Let us check the second condition of Lemma 4.1. Let $U \in [U_n]_S$ be a word so that $\text{occ}_U(x) = m$ and set $X$ is left-stable in $U_n \approx U$, i.e. $(u_1v_1) < u (1u_1x) < u (2u_1y) < u (1u_1z) < u (mUx) < u (u_1t_2)$.

Let $u$ be a word in less than $n/2$ variables so that $\Theta(u) = U$ for some substitution $\Theta : A \to A^+$. Since the word $u$ has less than $n/2$ variables, for some $c \in \text{OccSet}(u)$ and $2 < i < n - 1$ both $1up_i$ and $1up_{i+1}$ are contained in $\Theta(c)$. Then property (P2) implies that $c$ must be the only occurrence of a linear variable $t$.

Since $\Theta_u^{-1}$ is a homomorphism from $(\text{OccSet}(U), <_U)$ to $(\text{OccSet}(u), <_u)$, we have that $\Theta_u^{-1}(u_1t_1) \leq u \Theta_u^{-1}(1u_1x) \leq u \Theta_u^{-1}(2u_1y) \leq u (u_1t) \leq u \Theta_u^{-1}(u_1z) \leq u \Theta_u^{-1}(mUx) \leq u \Theta_u^{-1}(u_1t_2)$.

**Claim 8.** Each variable in $\Theta^{-1}(x)$ is stable in $u$ with respect to $S$.

**Proof.** If variable $x \in \Theta^{-1}(x)$ occurs in $u$ exactly $m$ times, then $1u_1x = \Theta_u^{-1}(1u_1x)$ and $mUx = \Theta_u^{-1}(mUx)$. Since $(1u_1x) < u (u_1t) < u (mUx)$ we have that $u(x, t) = x^dtx^m-d$ for some $0 < d < m$. Since this word is an isomorphism for $S$, variable $x$ is stable in $u$ with respect to $S$. If variable $y \in \Theta^{-1}(x)$ occurs in $u$ less than $m$ times, then $y$ is stable in $u$ with respect to $S$ because the word $y^{m-1}$ is an isomorphism for $S$.

Now let $u \approx v$ be an arbitrary identity of $S$ and $V = \Theta(v)$. Then by Claim 8 each variable in $\Theta^{-1}(x)$ is stable in $u \approx v$. Therefore, variable $x$ is stable in $U \approx V$, i.e. $\text{occ}_U(x) = \text{occ}_V(x) = m$.

**Claim 9.** Set $I_{U_n}(X) = \{u_1v_1, 1u_1x, 2u_1y, 1u_1z, mUx, u_1t_2\}$ is left-stable in $U \approx V$. 18
Proof. In view of the fact that \(yt_1yp_2\ldots p_nz_2z \geq xxyy\), it is enough to prove that sets \(\{ut_1, 1u^{x_2}, 2u^{y_2}\}\) and \(\{1u^{z_m}, m^{u_2}, u^{t_2}\}\) are left-stable in \(U \approx V\). We only show that set \(\{ut_1, 1u^{x_2}, 2u^{y_2}\}\) is left-stable in \(U \approx V\). (Proof for the other set is symmetric.)

Since variable \(t_1\) is linear in \(U\), we may assume that \(t_1\) is linear in \(u\) and \(u^{t_1} = \Theta_u^{-1}(ut_1)\). Since \(y\) occurs twice in \(U\), we may assume that either \(1u^{y_1} = \Theta_u^{-1}(1u^{y_1})\) and \(2u^{y_1} = \Theta_u^{-1}(2u^{y_1})\) or \(u^{t_3} = \Theta_u^{-1}(1u^{y_1})\) and \(u_y = \Theta_u^{-1}(2u^{y_1})\). In view of Fact \(2.3\) we may assume that \(\Theta_u^{-1}(1u^{x_2}) = 1u^{x_2}\).

If variable \(x\) occurs \(m\) times in \(u\), then \(\Theta_u^{-1}(x) = \{x\}\), \(occ_u(x) = m\) and \(\Theta_u^{-1}(m^{u_2}) = m^{u_2}\). Since \(1u^{x_2} \leq_u (ut) \leq u^{mu_2}\), we have that \(u(x, y, t_1, t) = yt_1x^dyy^c\) for some \(d, p > 0\) and \(d + c + p = m\). Since for each \(0 < d < m\) the word \(yt_1x^dyy^ctx^p\) is an isoterm for \(S\), we have that \((v^{t_1}) \leq_v (1v^{x_2}) \leq v(-2v^{y_1})\) (or \((v^{t_1}) \leq_v (1v^{x_2}) \leq v(v^{y_1})\)). Since \(\Theta_u^{-1}(x) = \{x\}\), we have that \(\Theta_u^{-1}(1v^{x_2}) = 1v^{x_2}\). Since \(\Theta_u^{-1}\) is a homomorphism from \((\text{OccSet}(V), \leq_V)\) to \((\text{OccSet}(V), <_V)\) and in view of Lemma \(4.5\) we conclude that \((v^{t_1}) <_V (1v^{x_2}) < v(2v^{y_1})\).

If variable \(x\) occurs less than \(m\) times in \(u\), then \(u(x, y, t_1) = yt_1x^dyy^c\) (or \(u(x, y, t_1) = t_1x^dyy^c\)) so that \(d > 0\) and \(occ_u(x) = d + c < m\). Since the words \(yt_1x^dyy^c\) and \(x^dyy^c\) are isoterm for \(S\), we have that \((v^{t_1}) <_V (1v^{x_2}) < v(2v^{y_1})\) (or \((v^{t_1}) <_V (1v^{x_2}) < v(v^{y_1})\)). In view of Fact \(2.3\) for some variable \(z\) we have \(\Theta_u^{-1}(1v^{x_2}) = 1v^{x_2}\). If \(z = x\) then Lemma \(4.5\) implies that \((v^{t_1}) <_V (1v^{x_2}) < v(2v^{y_1})\).

If \(z \neq x\) then we have \((1v^{z}) <_V (1v^{x_2}) \leq v(2v^{y_1})\) (or \((1v^{z}) <_V (1v^{x_2}) \leq v(v^{y_1})\)) and \((1u^{x_2}) < u(1u^{z})\). Let us assume that \((1v^{z}) \leq v(v^{t_1})\) and obtain a contradiction. Since by Claim \(8\) we have \(occ_u(z) = occ_v(z) < m\), the word \(v(z, t_1)\) is an isoterm for \(S\). Consequently, \((1u^{z}) \leq_u (u^{t_1})\). But on the other hand \((u^{t_1}) \leq u(1v^{x_2}) \leq u(1u^{z})\). To avoid a contradiction, we must assume that \((v^{t_1}) \leq (1v^{z})\).

Since \(\Theta_u^{-1}\) is a homomorphism from \((\text{OccSet}(V), \leq_V)\) to \((\text{OccSet}(V), <_V)\) and \((v^{t_1}) \leq_v (1v^{x_2}) \leq v(2v^{y_1})\) (or \((v^{t_1}) \leq_v (1v^{x_2}) \leq v(v^{y_1})\)) we have that \((v^{t_1}) <_V (1v^{x_2}) < v(2v^{y_1})\).

Therefore, monoid \(S\) is non-finitely based by Lemma \(4.1\)

It is easy to verify (see articles \([16, 17]\)) that if \(W\) is one of the eight sets of words in the left column of Table \(4\) then the finite monoid \(S(W)\) satisfies the corresponding identities in the right column, and consequently, is non-finitely based by Theorem \(4.3\).

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References

[1] J. Isbell, *Two examples in varieties of monoids*, Mathematical Proceedings of the Cambridge Philosophical Society, 68, No.2, September 1970, 265–266.

[2] M. Jackson, *On the finite basis problem for finite Rees quotients of free monoids*, Acta. Sci. Math. (Szeged) 67, (2001), 121–159.

[3] M. Jackson, *Finiteness properties of varieties and the restriction to finite algebras*, Semigroup Forum, 70, (2005), 159–187.

[4] M. Jackson, O. Sapir, *Finitely based, finite sets of words*, Internat. J. Algebra Comput., 10, No.6 (2000), 683–708.

[5] E. W. H. Lee, *Finitely generated limit varieties of aperiodic monoids with central idempotents*, J. Algebra Appl. 8 (2009), No. 6, 779–796.

[6] E. W. H. Lee, *Maximal Specht varieties of monoids*, Mosc. Math. J. 12 (2012), 787-802.

[7] E. W. H. Lee, *A sufficient condition for the non-finite basis property of semigroups*, Monatsh. Math. 168 (2012), 461-472.

[8] E. W. H. Lee, *Finitely based monoids obtained from non-finitely based semigroups*, accepted by Universitatis Iagellonicae Acta Mathematica.

[9] J. R. Li, W. T. Zhang, Y. F. Luo, *On the finite basis problem for certain 2-limited words*, Acta Math. Sin. (Engl. Ser.) 29 (2013), 571-590.

[10] R. McKenzie, *Tarski’s finite basis problem is undecidable*, Internat. J. Algebra Comput., 6, (1996), 49–104.

[11] P. Perkins, *Bases for equational theories of semigroups*, J. Algebra 11 (1969), 298–314.

[12] M. Sapir, *Problems of Burnside type and the finite basis property in varieties of semigroups*, Math. USSR Izvestiya, 30, No.2 (1988), 295–314.

[13] M. Sapir, *Inherently nonfinitely based finite semigroups*, Math. USSR Sbornik, 61, No.1 (1988), 155–166.

[14] O. Sapir, *Finitely based words*, Internat. J. Algebra Comput., 10, No.4 (2000), 457–480.

[15] O. Sapir, *Finitely based monoids*, preprint.

[16] O. Sapir, *Finitely based words with at most two non-linear variables*, preprint.

[17] O. Sapir, *Finitely based sets of block-2-simple words*, preprint.
[18] O. Sapir, *Finitely generated permutative varieties*, Semigroup Forum, **78**, (2009), 427–449.

[19] L. Shevrin, M. Volkov, *Identities of semigroups*, Russian Math (Iz. VUZ), **29**, No.11, (1985), 1–64.

[20] L. Shneerson, *On the axiomatic rank of varieties generated by a semigroup or monoid with one defining relation*, Semigroup Forum, **39**, (1989), 17–38.

[21] M. Volkov, *The finite basis problem for finite semigroups*, Sci. Math. Jpn., **53**, (2001), 171–199.

[22] W. T. Zhang, Luo, Y. F, *A new example of a minimal nonfinitely based semigroup*, Bull. Aust. Math. Soc. **84**, (2011), 484–491.

[23] W. T. Zhang, *Existence of a new limit variety of aperiodic monoids*, Semigroup Forum, **86**, (2013), 212–220.