An Optimal Choice Dictionary

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Abstract. A choice dictionary is a data structure that can be initialized with a parameter $n \in \mathbb{N} = \{1, 2, \ldots\}$ and subsequently maintains an initially empty subset $S$ of $\{1, \ldots, n\}$ under insertion, deletion, membership queries and an operation choice that returns an arbitrary element of $S$. The choice dictionary is fundamental in space-efficient computing and has numerous applications. The best previous choice dictionary can be initialized with a universe size $n$ and a second parameter $t \in \mathbb{N}$ in constant time and subsequently executes all operations in $O(t)$ time and occupies $n + O(n(t/w)^{\ell} + \log n)$ bits on a word RAM with a word length of $w = O(\log n)$ bits. We describe a new choice dictionary that, following a constant-time initialization, executes all operations in constant time and, in addition to the space needed to store $n$, occupies only $n + 1$ bits, which is shown to be optimal if $w = o(n)$. Allowed $\lceil \log_2(n+1) \rceil$ bits of additional space, the new data structure also supports iteration over the set $S$ in constant time per element.

Keywords. Data structures, space efficiency, choice dictionaries, bounded universes.

1 Introduction

Following similar earlier definitions [35] and concurrently with that of [2], the choice-dictionary data type was introduced by Hagerup and Kammer [8] as a basic tool in space-efficient computing and is known to have numerous applications [2,5,8,10,11]. Its precise characterization is as follows:

Definition 1.1. A choice dictionary is a data structure that can be initialized with an arbitrary integer $n \in \mathbb{N} = \{1, 2, \ldots\}$, subsequently maintains an initially empty subset $S$ of $U = \{1, \ldots, n\}$ and supports the following operations, whose preconditions are stated in parentheses:

- $\text{insert}(\ell)$ ($\ell \in U$): Replaces $S$ by $S \cup \{\ell\}$.
- $\text{delete}(\ell)$ ($\ell \in U$): Replaces $S$ by $S \setminus \{\ell\}$.
- $\text{contains}(\ell)$ ($\ell \in U$): Returns 1 if $\ell \in S$, 0 otherwise.
- $\text{choice}$: Returns an (arbitrary) element of $S$ if $S \neq \emptyset$, 0 otherwise.

As is common and convenient, we use the term “choice dictionary” also to denote data structures that implement the choice-dictionary data type. Following the initialization of a choice dictionary $D$ with an integer $n$, we call (the constant) $n$ the universe size of $D$ and (the variable) $S$ its client set. If a choice dictionary $D$ can operate only if given access to $n$ (stored outside of $D$), we say that $D$ is externally sized. Otherwise, for emphasis, we may call $D$ self-contained.

Our model of computation is a word RAM [117] with a word length of $w \in \mathbb{N}$ bits, where we assume that $w$ is large enough to allow all memory words in use to be addressed. As part of ensuring this, in the discussion of a choice dictionary with universe size $n$ we always assume that $w > \log_2 n$. The word RAM has constant-time operations for addition, subtraction and multiplication modulo $2^w$, division with truncation ($x/y \mapsto \lfloor x/y \rfloor$ for $y > 0$), left shift modulo $2^w$ ($x \ll y \mapsto x \cdot 2^y$, where $x \ll y = x \cdot 2^y$), right shift ($x \gg y \mapsto x \gg y = \lfloor x/2^y \rfloor$), and bitwise Boolean operations (AND, OR and XOR (exclusive or)). We also assume a constant-time operation to load an integer that deviates from $\sqrt{w}$ by at most a constant factor—this enables the proof of Lemma [2,13].

The best previous choice dictionary [8, Theorem 7.6] can be initialized with a universe size $n$ and a second parameter $t \in \mathbb{N}$ in constant time and subsequently executes all operations in $O(t)$ time and occupies $n + O(n(t/w)^{\ell} + \log n)$ bits. Let us call a choice dictionary atomic if it executes all operations including the initialization in constant time. Then, for every constant
$t \in \mathbb{N}$, the result just cited implies the existence of an atomic choice dictionary that occupies $n + O(n/w^t + \log n)$ bits when initialized for universe size $n$. Here we describe an externally sized atomic choice dictionary that needs just $n + 1$ bits, which is optimal if $w = o(n)$. The optimality of the bound of $n + 1$ bits follows from a simple argument of [9,12]: Because the client set $S$ of a choice dictionary with universe size $n$ can be in $2^n$ different states, any two of which can be distinguished via calls of contains, if the choice dictionary uses only $n$ bits it must represent each possible state of $S$ through a unique bit pattern. Since $S$ is in one particular state following the initialization, the latter must force each of $n$ bits to a specific value, which needs $\Omega(n/w)$ time.

In addition to being more space-efficient than all earlier choice dictionaries, the new data structure is also significantly simpler than its best predecessors, and in an actual implementation its operations are likely to be faster by a constant factor. The new choice dictionary owes much to a recent data structure of Katoh and Goto [12] that implements so-called initializable arrays. A connection between choice dictionaries and initializable arrays was first noted by Hagerup and Kammer [9], who observed that the light-path technique, invented in [8] in the context of choice dictionaries, also yields initializable arrays better than those known at the time. Here we show that an ingenious and elegant data representation devised by Katoh and Goto, slightly modified and used together with additional operations, yields a choice dictionary that leaves little to be desired. Our main result, formulated in Theorem 1.2 below, can be expressed informally as follows: At the price of one additional bit, a bit vector can be augmented with the operations “clear all” and “locate a 1”.

**Theorem 1.2.** There is an externally sized atomic choice dictionary that, when initialized for universe size $n$, occupies $n + 1$ bits in each quiescent state (i.e., between operations) and needs $O(w)$ additional bits of transient space during the execution of each operation.

## 2 A Simple Reduction

For $n \in \mathbb{N}$, the *bit-vector representation* over $U = \{1, \ldots, n\}$ of a subset $S$ of $U$ is the sequence $(d_1, \ldots, d_n)$ of $n$ bits with $d_\ell = 1 \Leftrightarrow \ell \in S$, for $\ell = 1, \ldots, n$, or its obvious layout in $n$ consecutive bits in memory. If we represent the client set of a choice dictionary with universe size $n$ via its bit-vector representation $B$, the choice-dictionary operations translate into the reading and writing of individual bits in $B$ and the operation choice, which now returns the position of a nonzero bit in $B$ (0 if all bits in $B$ are 0). It is trivial to carry out all operations other than initialization and choice in constant time. In the special case $n = O(w)$, the latter operations can also be supported in constant time. This is a consequence of the following lemma.

**Lemma 2.1 ([8]).** Given a nonzero integer $\sum_{j=0}^{w-1} 2^j d_j$, where $d_j \in \{0, 1\}$ for $j = 0, \ldots, w-1$, constant time suffices to compute $\max J$ and $\min J$, where $J = \{j \in \{0, \ldots, w-1\} \mid d_j = 1\}$.

We use the externally sized atomic choice dictionary for universe sizes of $O(w)$ implied by these considerations to handle the few bits left over when we divide a bit-vector representation of $n$ bits into pieces of a fixed size. The details are as follows:

Let $b$ be a positive integer that can be computed from $w$ and $n$ in constant time using $O(w)$ bits (and therefore need not be stored) and that satisfies $b \geq \log_2 n$, but $b = O(w)$. In order to realize an externally sized choice dictionary $D$ with universe size $n$ and client set $S$, partition the bit-vector representation $B$ of $S$ into $N = \lfloor n/(2b) \rfloor$ segments $B_1, \ldots, B_N$ of exactly $2b$ bits each, with $n' = n \bmod (2b)$ bits left over. If $n' \neq 0$, maintain (the set corresponding to) the last $n'$ bits of $B$ in an externally sized atomic choice dictionary $D_2$ realized as discussed above. Assume without loss of generality that $N \geq 1$. The following lemma is proved in the remainder of the paper:

**Lemma 2.2.** There is a data structure that, if given access to $b$ and $N$, can be initialized in constant time and subsequently occupies $2bN + 1$ bits and maintains a sequence $(a_1, \ldots, a_N) \in \{0, \ldots, 2^{2b} - 1\}^N$, initially $(0, \ldots, 0)$, under the following operations, all of which execute in
constant time: \textit{read}(i) (i \in \{1, \ldots, N\}), which returns \(a_i\); \textit{write}(i, x) (i \in \{1, \ldots, N\} and \(x \in \{0, \ldots, 2^b - 1\}\)), which sets \(a_i\) to \(x\); and \textit{nonzero}, which returns an \(i \in \{1, \ldots, N\}\) with \(a_i \neq 0\) if there is such an \(i\), and 0 otherwise.

For \(i = 1, \ldots, N\), view \(B_i\) as the binary representation of an integer and maintain that integer as \(a_i\) in an instance of the data structure of Lemma 2.2. This yields an externally sized atomic choice dictionary \(D_1\) for the first \(2bN\) bits of \(B\): To carry out \textit{insert}, \textit{delete} or \textit{contains}, update or inspect the relevant bit in one of \(a_1, \ldots, a_N\), and to execute \textit{choice}, call \textit{nonzero} and, if the return value \(i\) is positive, apply an algorithm of Lemma 2.1 to \(a_i\). It is obvious how to realize the full choice dictionary \(D\) through a combination of \(D_1\) and \(D_2\). The only nontrivial case is that of the operation \textit{choice}. To execute \textit{choice} in \(D\), first call \textit{choice} in \(D_1\) (say). If the return value \(i\) is positive, it is a suitable return value for the parent call. Otherwise call \textit{choice} in \(D_2\), increase the return value by \(2bN\) if it is positive, and return the resulting number. \(D\) is atomic because \(D_1\) and \(D_2\) are, and the total number of bits used by \(D\) is \(2bN + 1 + n' = n + 1\). Theorem 1.2 follows.

3 The Main Construction

In this section we prove Lemma 2.2 except that we relax the space bound by allowing \(2bN + w\) bits instead of \(2bN + 1\) bits.

3.1 The Storage Scheme

This subsection describes how the sequence \((a_1, \ldots, a_N)\) is represented in memory in \(2bN + w\) bits. Most of the available memory stores an array \(A\) of \(N\) cells \(A[1], \ldots, A[N]\) of 2b bits each. In addition, a \(w\)-bit word is used to hold an integer \(k \in \{0, \ldots, N\}\) best thought of as a “barrier” that divides \(V = \{1, \ldots, N\}\) into a part to the left of the barrier, \(\{1, \ldots, k\}\), and a part to its right, \(\{k + 1, \ldots, N\}\). We often consider a \((2b)\)-bit quantity \(x\) to consist of a \textit{lower half}, denoted by \(z\) and composed of the \(b\) least significant bits of \(x\) (i.e., \(z = x \text{ and } (2^b - 1)\)) and an \textit{upper half}, \(\bar{x} = x \gg b\), and we may write \(x = (z, \bar{x})\). A central idea is that the upper halves of \(A[1], \ldots, A[N]\) are used to implement a matching on \(V\) according to the following convention: Elements \(i\) and \(j\) of \(V\) are matched exactly if \(\overline{A[i]} = j\), \(\overline{A[j]} = i\), and precisely one of \(i\) and \(j\) lies to the left of the barrier, i.e., \(i \leq k < j\) or \(j \leq k < i\). In this case we call \(j\) the \textit{mate} of \(i\) and vice versa. The assumption \(b \geq \log_2 n\) ensures that the upper half of each cell in \(A\) can hold an arbitrary element of \(V\). A function that inputs an element \(i\) of \(V\) and returns the mate of \(i\) if \(i\) is matched and \(i\) itself if not is easily coded as follows:

\[
\text{mate}(i):
\begin{align*}
&i' := \overline{A[i]}; \\
&\text{if } (1 \leq i \leq k < i' \leq N \text{ or } 1 \leq i' \leq k < i \leq N) \text{ and } \overline{A[i']} = i \text{ then return } i'; \\
&\text{return } i;
\end{align*}
\]

For all \(i \in V\), call \(i\) \textit{strong} if \(i\) is matched and \(i \leq k\) or \(i\) is unmatched and \(i > k\), and call \(i\) \textit{weak} if it is not strong. The integers \(A[1], \ldots, A[N]\) and \(k\) represent the sequence \((a_1, \ldots, a_N)\) according to the following storage invariant: For all \(i \in V\),

- \(a_i = 0\) exactly if \(i\) is weak;
- if \(i\) is strong and \(i > k\), then \(a_i = A[i]\);
- if \(i\) is strong and \(i \leq k\), then \(a_i = (A[i], A[\text{mate}(i)])\).

The storage invariant is illustrated in Fig. 1. The following drawing conventions are used here and in subsequent figures: The barrier is shown as a thick vertical line segment with a triangular base. Each pair of mates is connected with a double arrow, and a cell \(A[i]\) of \(A\) is shown in a darker hue if \(i\) is strong. A question mark indicates an entry that can be completely arbitrary, except that it may not give rise to a matching edge, and the upper and lower halves of some cells of \(A\) are shown separated by a dashed line segment.
3.2 The Easy Operations

The data structure is initialized by setting $k = N$, i.e., by placing the barrier at the right end. Then the matching is empty, and all elements of $V$ are to the left of the barrier and weak. Thus the initial value of $(a_1, \ldots, a_N)$ is $(0, \ldots, 0)$, as required. The implementation of $read$ closely reflects the storage invariant:

$$read(i):$$

- if $mate(i) \leq k$ then return $0$; (* $i$ is weak exactly if $mate(i) \leq k$ *)
- if $i > k$ then return $A[i]$; else return $(A[i], A[mate(i)])$;

The code for $nonzero$ is short but a little tricky:

$$nonzero:$$

- if $k = N$ then return $0$;
- return $mate(N)$;

The implementation of $write(i, x)$ is easy if $i$ is weak and $x = 0$ (then nothing needs to be done) or $i$ is strong and $x \neq 0$. In the latter case the procedure $simple\_write$ shown below can be used. The only point worth noting is that writing to $A[i]$ when $i$ is strong and $i > k$ may create a spurious matching edge that must be eliminated.

$$simple\_write(i, x):$$

- if $i \leq k$ then $(A[i], A[mate(i)]) := (x, \top)$;
- else $A[i] := x$;
- $i' := mate(i)$;
- if $i' \neq i$ then $A[i'] := i'$; (* eliminate a spurious matching edge *)

3.3 Insertion and Deletion

The remaining, more complicated, operations of the form $write(i, x)$ are those in which $a_i$ is changed from zero to a nonzero value—call such an operation an $insertion$—or vice versa—a $deletion$. When the data structure under development is used to realize a choice dictionary, insertions and deletions are triggered by (certain) insertions and deletions, respectively, executed on the choice dictionary. Insertions and deletions are the operations that change the barrier and usually the matching. In fact, $k$ is decreased by 1 in every insertion and increased by 1 in every deletion, so $k$ is always the number of $i \in V$ with $a_i = 0$.

The various different forms that an insertion may take are illustrated in Figs. 2 and 3. The situation before the insertion is always shown above the situation after the insertion. A “1” outside of the “stripes” indicates the position of an insertion and symbolizes the nonzero value to be written, while a “1” inside the stripes symbolizes that value after it has been written. The various forms of a deletion are illustrated in Figs. 4 and 5. Here a “0” indicates the position of a deletion, while a “1” symbolizes the nonzero value that is to be replaced by zero.

There are many somewhat different cases, but for each it is easy to see that the storage invariant is preserved and that the sequence $(a_1, \ldots, a_N)$ changes as required. It is also easy to turn the figures into a $write$ procedure that branches into as many cases. Here we propose the following realization of $write$ that is terser, but needs a more careful justification.
Fig. 2. Insertion to the left of the barrier.

Fig. 3. Insertion to the right of the barrier.

Fig. 4. Deletion to the left of the barrier.

Fig. 5. Deletion to the right of the barrier.
\textbf{write}(i, x):
\begin{itemize}
\item $x_0 := \text{read}(i)$; (* the value to be replaced by $x$ *)
\item $i' := \text{mate}(i)$;
\item if $x \neq 0$ then
  \begin{itemize}
  \item if $x_0 = 0$ then (* an insertion *)
    \begin{itemize}
    \item $k' := \text{mate}(k)$; (* $k = k$ will cross the barrier *)
    \item $u := \text{read}(k)$; (* save $a_k$ *)
    \item $k := k - 1$; (* move the barrier left *)
    \item \text{simple_write}(k + 1, u); (* reestablish the value of $a_k$ *)
    \item if $i \neq k'$ then \{ $A[i'] := k'$; $A[k'] := i'$; \}
    \item \text{simple_write}(i, x); (* $i$ was or has been made strong *)
  \end{itemize}
  \item else (* $x = 0$ *)
    \begin{itemize}
    \item if $x_0 \neq 0$ then (* a deletion *)
      \begin{itemize}
      \item $k' := \text{mate}(k + 1)$; (* $k = k + 1$ will cross the barrier *)
      \item $u := \text{read}(k')$; (* save $a_{k'}$ *)
      \item $k := k + 1$; (* move the barrier right *)
      \item $A[i'] := k'$; $A[k'] := i'$; (* match $i'$ and $k'$ *)
      \item if $k' \neq i$ then \text{simple_write}(k', v); (* reestablish the value of $a_{k'}$ *)
    \end{itemize}
    \end{itemize}
\end{itemize}
\end{itemize}

To see the correctness of the procedure \text{write} shown above, consider a call \text{write}(i, x) and assume that it gives rise to an insertion or a deletion, since in the remaining cases the procedure is easily seen to perform correctly. Let $k_0$ be the value of $k$ (immediately) before the call. Because the call changes the value of $k$ by 1, a single element $\tilde{k}$ of $V$ crosses the barrier, i.e., is to the left of the barrier before or after the call, but not both. In the case of an insertion, $\tilde{k} = k_0$; in that of a deletion, $\tilde{k} = k_0 + 1$.

Assume that $i$ does not cross the barrier, i.e., that $i \neq \tilde{k}$. Because the call changes $a_i$ from zero to a nonzero value or vice versa, $i$ must change its matching status, i.e., be matched before or after the call, but not both. In detail, if $i$ is matched before the call, its mate at that time, if different from $\tilde{k}$, must find a new mate, which automatically leaves $i$ unmatched. If $i$ is unmatched before the call, $i$ itself must find a mate. We can unify the two cases by saying that if $i' = \text{mate}(i)$ (evaluated before the call under consideration has changed $k$ and $A$) is not $\tilde{k}$, then $i'$ must find a (new) mate. If $i' \neq \tilde{k}$, moreover, $i'$ is to the left of the barrier in the case of an insertion and to the right of it in the case of a deletion.

Assume now that the call does not change $a_{\tilde{k}}$, i.e., that $\tilde{k} \neq i$. Then, because $\tilde{k}$ crosses the barrier, it must also change its matching status: If $\tilde{k}$ is matched before the call, its mate at that time, if different from $i$, must find a new mate, and otherwise $\tilde{k}$ itself must find a mate. As above, this can be expressed by saying that if $k' = \text{mate}(\tilde{k})$ (evaluated before the call has changed $k$ and $A$) is not $i$, then $k'$ must find a new mate. Moreover, after the call $k'$ is to the right of the barrier in the case of an insertion and to the left of it in the case of a deletion.

Exclude the special cases identified above by assuming that \{ $i, i'$ \} \cap \{ $k, k'$ \} = \emptyset. Then it can be seen that all required changes to the matching can be effectuated by matching $i'$ and $k'$, which is what the procedure \text{write} does. In the case of an insertion, this makes $i$ strong, which implies that $a_i$ can be set to $x$ simply by executing \text{simple_write}(i, x) at the very end.

In addition, with $\ell = \min\{i', k'\}$, it must be ensured that the call does not change $a_{\ell}$ except if $\ell = i$. In the case of an insertion, $\ell = i'$, and if $i' \neq i$, the mate of $i'$ switches from being $i$ to being $k'$, so that it suffices to execute $A[k'] := A[i]$, which happens in the procedure. The same assignment is executed if $i' = i$, in which case it is useless but harmless, given that it takes place before the call \text{simple_write}(i, x). In the case of a deletion, $\ell = k'$. Here the procedure plays it safe by remembering the value of $a_{k'}$ before the call in a variable $v$ and restoring $a_{k'}$ to that value at the end, unless $k' = i$, via the call \text{simple_write}(k', v). This is convenient because $a_{k'}$ is not stored in a unique way before the call.

At this point, $i'$ and $k'$ have been “taken care of”, but $\tilde{k}$ still needs attention. In the case of a deletion, either $\tilde{k} = k'$ or $\tilde{k}$ is weak, so nothing more needs to be done. In an insertion, the procedure saves the original value of $a_{\tilde{k}}$ in $u$ and restores it afterwards through the statement
**simple_write**(\(k + 1, u\)). This is necessary and meaningful only if \(\tilde{k}\) is strong. If \(\tilde{k}\) is weak, however, the effect of the statement—except for the harmless possible elimination of a spurious matching edge—is canceled through the subsequent assignment to \(\overline{A[k]}\) and \(\overline{A[k']}\).

We still need to consider the special cases that were ignored above, namely calls with \(\{i, k\} \subseteq \{\tilde{k}, k'\} \neq \emptyset\). These form part (b) of Figs. 2–5. In fact, the number of special cases is quite limited. If \(\tilde{k}\) is weak before an insertion or strong before a deletion, it is unmatched. Thus if \(i = \tilde{k}\), we have \(i = k'\), and \(i' = \tilde{k}\) implies \(\tilde{i} = \tilde{k}\). On the other hand, each of the statements \(i = k'\) and \(i' = \tilde{k}\) implies the other one. Thus there are two cases to consider: (1) \(i = i' = \tilde{k} = k'\) and (2) \(i = k' \neq \tilde{k} = i'\).

In case (1), all writing to \(A[i]\) happens to \(A[i]\). For insertion, the execution of \(\text{simple_write}(i, x)\) at the very end ensures the correctness of the call. For deletion, the execution of \(\overline{A[k]} := i'\) at the end ensures that \(i\) is unmatched, which is all that is required. In case (2), after an insertion, \(i\) and \(\tilde{k}\) are both to the right of the barrier, \(a_i\) and \(a_{\tilde{k}}\) are both nonzero, and the execution of \(\text{simple_write}(i, x)\) and \(\text{simple_write}(k + 1, u)\) ensures that \(A[i]\) and \(\overline{A[k]}\) have the correct values after the call. After a deletion, \(i\) and \(\tilde{k}\) are both to the left of the barrier and \(a_i = a_{\tilde{k}} = 0\), and the execution of \(\overline{A[k]} := k'\) and \(\overline{A[k']} := i'\) in fact ensures that \(i\) and \(\tilde{k}\) are both unmatched, which is all that is required.

Since all operations of the data structure have been formulated as pieces of code without loops and \(b = O(w)\), it is clear that the operations execute in constant time.

### 4 The Last Bits and Pieces

#### 4.1 Reducing the Space Requirements

The space requirements of the data structure of the previous section can be reduced from \(2bN + w\) bits to \(2bN + 1\) bits, as promised in Lemma 2.2, by a method of 9,12. First, \(b\) is chosen to satisfy not only \(b \geq \log_2 n\), but \(b \geq 2 \log_2 n\), which is clearly still compatible with \(b = O(w)\). As a result, for each \(i \in V\) to the left of the barrier, \(A[i]\) has at least \(2 \log_2 n - \lceil \log_2 N \rceil \geq \lceil \log_2 N \rceil\) unused bits. If \(k \geq 1\), we store \(k\) in the unused bits of \(A[1]\) (the unused bits of \(A[2]\), . . . , \(A[k]\) continue to be unused). When \(k = 0\), even \(A[1]\) is to the right of the barrier and there are no unused bits in \(A\), so we use a single bit outside of \(A\) to indicate whether \(k\) is nonzero. The resulting data structure occupies exactly \(2bN + 1\) bits.

#### 4.2 The Choice of \(b\)

A practical choice dictionary based on the ideas of this paper would probably content itself with the main construction of Section 3 and refrain from applying the construction of Subsection 4.1 to squeeze out the last few bits. Then there is no reason to choose \(b\) larger than \(w\), and \(b = w\) seems the best choice. This yields a self-contained atomic choice dictionary that occupies \(n + 2w\) bits when used with universe size \(n\).

If \(w\) is even and \(w \geq 2 \log_2 n\), another plausible choice is \(b = w/2\), which allows an entry in the array \(A\) to be manipulated with a single instruction and simplifies the access to cells of \(A\). It seems likely, however, that the gains in certain scenarios from choosing \(b = w/2\) instead of \(b = w\) are small and can be reduced still further through an optimization of the case \(b = w\) that omits superfluous operations on upper or lower halves of cells in \(A\).

If the space needed for an externally sized choice dictionary is to be reduced all the way to \(n + 1\) bits for universe size \(n\), \(b = 2w\) seems the best choice.

#### 4.3 A Self-Contained Atomic Choice Dictionary

In order to convert the externally sized atomic choice dictionary of Theorem 1.2 to a self-contained one, we must augment the data structure with an indication of the universe size \(n\). This can clearly always be done with \(w\) additional bits. If a space bound is desired that depends
only on \( n \), \( n \) must be stored as a so-called self-delimiting numeric value. Assume first that the most significant bits in a word are considered to be its “first” bits, i.e., the ones to be occupied by a data structure of fewer than \( w \) bits (the “big-endian” convention). Then one possibility is to use the code \( \gamma' \) of Elias [3]: With \( \text{bin}(n) \) denoting the usual binary representation of \( n \in \mathbb{N} \) (e.g., \( \text{bin}(10) = 1010 \)), store \( n \) in the form of the string \( 0^{\text{bin}(n)}-1 \text{bin}(n) \), which can be decoded in constant time with an algorithm of Lemma 2.1. Since \( |\text{bin}(n)| = \lfloor \log_2(n + 1) \rfloor \), this yields a space bound for the self-contained choice dictionary of \( n + 2\lfloor \log_2(n + 1) \rfloor \) bits. If instead the least significant bits of a word are considered to be its first bits (the “little-endian” convention), the scheme needs to be changed slightly: The string \( 0^{\text{bin}(n)}-1 \text{bin}(n) \) is replaced by \( \tilde{\text{bin}}(n)0^{\lfloor \text{bin}(n) \rfloor - 1} \), where \( \tilde{\text{bin}}(n) \) is the same as \( \text{bin}(n) \), except that the leading 1 is moved to the end.

Incidentally, if an application can guarantee that \( k \) never becomes zero, the method of Subsection 4.1 can be used to “hide” \( n \) as well as \( k \) in the array \( A \) if we choose \( b \geq 4\lfloor \log_2(n + 1) \rfloor \). This yields a restricted self-contained atomic choice dictionary that occupies \( n + 1 \) bits. The restriction is satisfied, e.g., if the universe \( \{1, \ldots, n\} \) always contains \( 4b - 1 \) consecutive elements that do not belong to the client set.

### 4.4 Iteration

We say that a choice dictionary with client set \( S \) supports iteration in constant time per element if there are constant-time operations to reset the iteration, to return (“enumerate”) some “next” element of \( S \), and to test whether the iteration is complete, i.e., whether all elements of \( S \) have been enumerated. A more precise description can be found in [8].

Using Lemma 2.1 it is easy to enumerate the positions of the nonzero bits in a bit vector of length \( O(w) \) in constant time per position output—two shifts can be used to clear the bits whose positions were already enumerated. This reduces the problem of supporting constant-time iteration for the choice dictionary of Theorem 1.2 to that of supporting constant-time iteration over \( \{i \in \{1, \ldots, N\} | a_i \neq 0\} \) for the data structure of Lemma 2.2. And the latter is easy: For \( i = k + 1, \ldots, N \), enumerate \( \text{mate}(i) \). The iteration needs to remember \( \lfloor \log_2(n + 1) \rfloor \) bits of state information between calls.

### 4.5 Additional Operations and Features

Compared to the best previous choice dictionary [8], the choice dictionary of the present paper, while doing better on the “core business” of a choice dictionary, lacks many important additional operations and features. If an application needs any of these additional capabilities, it must still resort to the choice dictionary of [8].

First, the new choice dictionary supports iteration over the client set \( S \), but it is not robust in the sense of [8]: If \( S \) is modified during an iteration through insertions and deletions, the data structure will not suffer a run-time error and will not enumerate elements outside of \( S \), but an element of \( S \) may be enumerated more than once, and it may not be enumerated at all even though it belongs to \( S \) throughout the iteration.

Second, there is no obvious way to extend the new dictionary to several colors, i.e., to the maintenance of several pairwise disjoint subsets of \( U = \{1, \ldots, n\} \), where \( n \) is the universe size, or even to support the operation \text{choice}, which returns an element in the complement of the client set with respect to \( U \).

Operations that could be supported in constant time are batched insertion and deletion of all elements of a subset \( I \) of \( U \) with \( \max I - \min I = O(w) \) presented via a bit-vector representation in a constant number of words, as well as batched inspection, in the sense of \text{contains}, of \( O(w) \) consecutive elements of \( U \). However, this is a characteristic shared with earlier choice dictionaries.

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