The phase space exchange pseudopotential for fermions and bosons

A S Larkin¹,² and V S Filinov²

¹ Moscow Institute of Physics and Technology, Institutskiy Pereulok 9, Dolgoprudny, Moscow Region 141700, Russia
² Joint Institute for High Temperatures of the Russian Academy of Sciences, Izhorskaya 13 Bldg 2, Moscow 125412, Russia
E-mail: alexanderlarkin@rambler.ru

Abstract. Effects of the Bose–Einstein statistics and the Fermi–Dirac one in the phase space quantum thermodynamics are considered. The Wigner function of ideal gases of fermions and bosons is calculated explicitly in pair approximation. It is shown that exchange effects can be taken into account using pseudopotential in the phase space, attractive for bosons and repulsive for fermions. This result is generalized to non-ideal systems.

1. Introduction

The approach to equilibrium quantum thermodynamics, based on the Wigner function, is very promising for simulations of strongly coupled many-particle systems. Using the phase space, it allows one to calculate various distribution and correlation functions and to average physical quantities directly [1]. Moreover, the Wigner approach opens the way to studying of kinetic properties of matter in terms of classical dynamics [2, 3].

Although the Wigner function of strongly coupled system cannot be calculated analytically, it can be represented in form of path integral [4, 5] and be calculated numerically with path-integral Monte-Carlo methods (PIMC) [6]. Two PIMC methods for calculation of Wigner function were developed: SMPIMC (single momentum PIMC) [7], LAPIMC (linear approximation PIMC) and HAPIMC (harmonic approximation PIMC) [8, 9]. SMPIMC method allows performing calculations for strongly coupled fermionic systems with moderate degeneracy, but it has two fundamental limitations. Firstly, it uses the single momentum Wigner function instead of full Wigner function. Secondly, it cannot deal with degenerate bosonic systems. On the other hand, LAPIMC and HAPIMC methods use the full Wigner function and, in principle, are able to deal with fermionic and bosonic system. However, effects of the Bose–Einstein or Fermi–Dirac statistics have not been taken into account adequately.

In this paper, we take into account effects of Bose–Einstein and Fermi–Dirac in LAPIMC and HAPIMC formalism. We derive the exchange pseudopotential in pair approximation ab initio, instead of semi-empirical pseudopotential from [8, 10]. The fermionic pseudopotential reflects the Pauli principle prohibiting to any fermions to be at the same quantum state at once. The bosonic pseudopotential reflects a tendency of bosons to be at the same quantum state at once.
2. Wigner function of canonical ensemble

We consider a general \( N \)-particle system with the Bose–Einstein statistics or the Fermi–Dirac one. The Hamiltonian of this system is

\[
\hat{H} = \sum_{a=1}^{N} \frac{\hat{p}_a^2}{2m} + U(\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_N).
\]

The operator of potential energy \( U(\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_N) \) defines interparticle interaction or interaction between particles and external field.

The Wigner function of the canonical \((N, V, T)\) ensemble is defined as a Fourier transform of the density matrix:

\[
W(p, q) = \frac{Z(N, V, T)^{-1}}{\pi \hbar} \int d\xi e^{i(p\xi)/\hbar} \langle q - \xi/2 | e^{-\beta \hat{H}} | q + \xi/2 \rangle,
\]

(2)

where \( Z(N, V, T) \) is the partition function of canonical ensemble, \( \beta = 1/(kT) \), the coordinate states are symmetrized or antisymmetrized over particle permutations. \( N \)-particle states with prescribed coordinates and momenta can be expressed with single-particle states \(|q\rangle, |p\rangle\):

\[
|q\rangle = \frac{1}{N!} \sum_P (\pm 1)^P |q_P1\rangle |q_P2\rangle \ldots |q_PN\rangle,
\]

\[
|p\rangle = \frac{1}{N!} \sum_P (\pm 1)^P |p_P1\rangle |p_P2\rangle \ldots |p_PN\rangle.
\]

(3)

The symbol \( P \) denotes permutations of particles with the same spin projection, and the sum is taken over all permutations. The symbol \( Pa \) denotes the number of the particle rearranged with particle \( a \). The sign “+” corresponds to the Bose–Einstein statistics, the sign “−” corresponds to the Fermi–Dirac statistics.

The Wigner function (2) cannot be calculated analytically in general case because of the non-commutativity of coordinate and momentum operators. However it can be represented in form of path integral [6]:

\[
W(p, q) = Z(N, V, T)^{-1} \lambda^{-S} \sum_P (\pm 1)^P \int d\xi \int_{z(0)=z(1)=0}^{1} Dz(\tau) \exp \left\{ \frac{i}{\hbar} (p\xi) \right. \\
- \frac{\pi}{\lambda^2} [P(q + \xi/2) - (q - \xi/2)]^2 - \int_0^1 d\tau \pi \dot{z}^2(\tau) - \int_0^1 d\tau \beta U(q(\tau)) \left\}
\]

(4)

where functional integral \( Dz(\tau) \) is taken over closed trajectories \( z(\tau) \). To obtain more explicit expression for Wigner function, one has to perform Fourier transform of (4). This can be done in linear or harmonic approximation [9], when the potential energy is expanded into a Taylor series up to the first or the second order correspondingly:

\[
U(\lambda z(\tau) + q + \xi(\tau - 1/2)) \approx U(\lambda z(\tau) + q) + (\tau - 1/2)\xi_{a,i} \frac{\partial U(\lambda z(\tau) + q)}{\partial q_{a,i}} + \frac{1}{2}(\tau - 1/2)^2 \xi_{a,i} \xi_{b,j} \frac{\partial^2 U(\lambda z(\tau) + q)}{\partial q_{a,i} \partial q_{b,j}},
\]

(5)

with particle indices \( a, b = 1, 2, \ldots, N \) and spatial indices \( i, j = 1, 2, 3 \). In the papers [6, 7] we derived expressions for the Wigner function in linear and harmonic approximation for identical
Since these states are eigenstates of the Hamiltonian \( \hat{W} \), i.e., without effects of the Bose–Einstein or Fermi–Dirac statistics:

\[
W_{\text{LA}}(p, q) = Z(\beta, V)^{-1} \int_{z(0)=z(1)=0} \text{D}z(\tau) \exp \left\{ -\frac{1}{4\pi} \rho_{a,i} \rho_{a,i} \right\} \cos \left\{ \rho_{a,i} J_{a,i}(q, z(\tau)) \right\} \\
\times \exp \left\{ -\pi K[\tau] - \pi P[q, z(\tau)] \right\},
\]

\[
W_{\text{HA}}(p, q) = Z(\beta, V)^{-1} \int_{z(0)=z(1)=0} \text{D}z(\tau) \det |\chi_{a,i:b,j}(q, z(\tau))|^{1/2} \\
\times \exp \left\{ -\frac{1}{4\pi} \rho_{a,i} \chi_{a,i:b,j}^{-1}[q, z(\tau)] \rho_{b,j} + \pi J_{a,i}[q, z(\tau)] \chi_{a,i:b,j}^{-1}[q, z(\tau)] J_{b,j}[q, z(\tau)] \right\} \\
\times \cos \left\{ \rho_{a,i} \chi_{a,i:b,j}^{-1}[q, z(\tau)] J_{b,j}[q, z(\tau)] \right\} \times \exp \left\{ -\pi K[\tau] - \pi P[q, z(\tau)] \right\},
\]

where functionals \( K \) and \( P \) are scalar, functional \( J \) is \( 3N \)-vector and functional \( \chi \) is \( (3N \times 3N) \)-matrix depending on coordinates \( q \) and trajectories \( z(\tau) \):

\[
K[\tau] = \int_0^1 d\tau \dot{z}_{a,i}(\tau) z_{a,i}(\tau),
\]

\[
P[q, z(\tau)] = \frac{\beta}{\pi} \int_0^1 d\tau U(q + \lambda z(\tau)),
\]

\[
J_{a,i}[q, z(\tau)] = \frac{\beta \lambda}{2\pi} \int_0^1 d\tau (\tau - 1/2) \frac{\partial U(q + \lambda z(\tau))}{\partial q_{a,i}},
\]

\[
\chi_{a,i:b,j}[q, z(\tau)] = \delta_{a,b} \delta_{i,j} + \frac{\beta \lambda^2}{2\pi} \int_0^\beta d\tau (\tau - 1/2) \frac{\partial^2 U(q + \lambda z(\tau))}{\partial q_{a,i} \partial q_{b,j}}.
\]

In the present paper, we generalize linear and harmonic approximations to the systems with the Bose–Einstein and Fermi–Dirac statistics by considering contributions of pair, triple and other permutations.

3. Ideal gas

Let us consider an ideal gas, i.e., system consisting of \( N \) non-interacting identical particles. The Hamiltonian is

\[
\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + \cdots + \frac{\hat{p}_N^2}{2m},
\]

and the particles can be either bosons with an integer spin or fermions with a half-integer spin. To be short, we will omit the spin indices \( \sigma \) and summation over them. We are going to calculate the Wigner function of the system (8) directly from the definition (2). To do this, we use the complete system of \( N \)-particle states with prescribed momenta:

\[
1 = \int \frac{dp}{(2\pi \hbar)^N} |p\rangle |p\rangle.
\]

Since these states are eigenstates of the Hamiltonian \( \hat{H} |p\rangle = H(p) |p\rangle \), the statistical operator can be calculated explicitly:

\[
W_0(p, q) = \int \frac{d^{3N} \rho d^{3N} \chi}{(2\pi \hbar)^{3N}} \exp \left\{ \frac{i}{\hbar} (p \xi - \frac{\beta}{2m} (p)^2 \right\} |q - \xi/2 |p\rangle |q + \xi/2\rangle.
\]
Since the summation is performed over all particles, we can write \( \bar{\rho}(\vec{q}) \) instead of \( \bar{\rho}(\vec{q}, P \xi) \), where \( P^{-1}a \) is reverse permutation of the particle \( a \), i.e., number of the particle rearranged with \( a \). The integral over \( \xi \) gives us a delta function, and the expression for the Wigner function:

\[
W_0(p, q) = \sum_P (\pm 1)^P \int \frac{d^3p d^3q}{(2\pi\hbar)^{3N}} \exp\left\{ \frac{i}{\hbar} (p \xi) - \frac{i}{\hbar} (\vec{P} \xi/2) - \frac{i}{\hbar} (\vec{P} - \vec{P} \xi/2) \right\}
\]

(11)

Since the summation is performed over all particles, we can write \( \bar{\rho}(P \xi) \) instead of \( \bar{\rho}(P^{-1} \xi) \), where \( P^{-1}a \) is reverse permutation of the particle \( a \), i.e., number of the particle rearranged with \( a \). The integral over \( \xi \) gives us a delta function, and the expression for the Wigner function:

\[
W_0(p, q) = \sum_P (\pm 1)^P \int d^3\tilde{p}_1 \ldots d^3\tilde{p}_N \left[ \prod_{a=1}^N \delta(3) \left( \vec{p}_a - \frac{\vec{P}_a + \vec{P}_{P^{-1}a}}{2} \right) \right]
\times \exp\left\{ -\frac{\beta}{2m} \left( \vec{p}_a + \sum_{a=1}^N \vec{p}_a (q_a - q_{P_a}) \right) \right\}.
\]

(12)

Here all permutations are considered: identical, pair, triple and other. Let us consider contributions from different permutations.

Identical permutation \( Pa = a \) leads to Maxwellian function:

\[
w_{P1}(\vec{p}_a, q_a) = + \exp\left\{ -\frac{\beta}{2m} \vec{p}_a^2 \right\}.
\]

(13)

This expression does not depend on coordinate, because there is no interaction of particles.

In case of pair permutation \( Pa = b, Pb = a \) one can change variables: \( \tilde{p}_{ab} = (\vec{p}_a + \vec{p}_b)/2, \)
\( \tilde{p}_{ab} = \vec{p}_a - \vec{p}_b, \) so the integral in (12) can be rewritten:

\[
w_{P2}(\vec{p}_a, \vec{p}_b, q_a, q_b) = (\pm 1) \delta(3) (\vec{p}_a - \vec{p}_b) \int d^3\tilde{p}_{ab} \exp\left\{ -\frac{\beta}{2m} (\vec{p}_a^2 + \vec{p}_b^2) \right\}
\]

(14)

The integral over \( \tilde{p}_{ab} \) can be taken, so the contribution of the pair permutation:

\[
w_{P2}(\vec{p}_a, \vec{p}_b, q_a, q_b) = (\pm 1) \delta(3) \left( \frac{\vec{p}_a - \vec{p}_b}{2\pi\hbar} \right)^{\lambda'} \exp\left\{ -\frac{\beta}{2m} (\vec{p}_a^2 + \vec{p}_b^2) - \frac{\pi}{\lambda'^2} (q_a - q_b)^2 \right\},
\]

(15)

where \( \lambda' = \lambda/\sqrt{2} \) is de Broglie wavelength of the “doubled” particle.

In case of triple permutation, for example \( Pa = b, Pb = c, Pc = a \), the reverse permutation is \( P^{-1}a = c, P^{-1}b = a, P^{-1}c = b \). Let us change the variables:

\[
\tilde{p}_{ab} = \frac{\vec{p}_a + \vec{p}_b}{2}, \quad \tilde{p}_{bc} = \frac{\vec{p}_b + \vec{p}_c}{2}, \quad \tilde{p}_{ca} = \frac{\vec{p}_c + \vec{p}_a}{2}.
\]

(16)

Jacobian is \( J(\tilde{p}_{ab}, \tilde{p}_{bc}, \tilde{p}_{ca}) = 4 \). As result the delta-functions in (12) separates, and the contribution of the triple permutation is

\[
w_{P3}(\vec{p}_a, \vec{p}_b, \vec{p}_c, q_a, q_b, q_c) = 4 \exp\left\{ -\frac{\beta}{2m} (3\vec{p}_a^2 + 3\vec{p}_b^2 + 3\vec{p}_c^2 - 2\vec{p}_a \vec{p}_b - 2\vec{p}_b \vec{p}_c - 2\vec{p}_c \vec{p}_a) \right\}
\]

(17)

Permutations of the higher order have the similar structure.

As it follows from (15), a contribution of the pair permutation is non-zero only if momenta of the particles \( a \) and \( b \) are equal, and it depends on coordinates through Gaussian exponent. In case
of the Fermi–Dirac statistics this fact is manifestation of the Pauli principle: two fermions cannot be at the same quantum state. Formally, the expression (15) is infinite due to $\delta^{(3)}(\mathbf{p}_a - \mathbf{p}_b)$, and does not have physical sense. However the phase space should be considered up to elementary cells with $6N$-volume $\Delta W = (2\pi \hbar)^{3N}$. In this case, the infinite delta function must be replaced by the Kronecker symbol, which is equal to 1 when $\mathbf{p}_a$ and $\mathbf{p}_b$ are in the same cell, and zero in the other case. In order to find the exact correspondence, one should calculate the two-particle momentum distribution function for the particles $a$ and $b$ integrating the contributions (13) and (15) over coordinates $\mathbf{q}_a$, $\mathbf{q}_b$:

$$ f(\mathbf{p}_a, \mathbf{p}_b) = V^2 \exp\left\{ -\frac{\beta}{2m} \left( \mathbf{p}_a^2 + \mathbf{p}_b^2 \right) \right\} \left[ 1 \pm \frac{(2\pi \hbar)^3}{V} \delta^{(3)}(\mathbf{p}_a - \mathbf{p}_b) \right].$$

An infinite delta function is countervailed by volume $V$, which is infinite in the thermodynamic limit. One can easily derive analogous formula for discrete phase space and compare it with (18). This leads to the following correspondence between the delta-function and the Kronecker symbol:

$$ (2\pi \hbar)^3 \delta^{(3)}(\mathbf{p}_a - \mathbf{p}_b) = V \delta^{(3)}_{\mathbf{p}_a, \mathbf{p}_b}. $$

The Kronecker symbol can be approximated by the Gaussian exponent:

$$ \delta^{(3)}_{\mathbf{p}_a, \mathbf{p}_b} = \exp\left\{ -\frac{(\mathbf{p}_a - \mathbf{p}_b)^2}{4\pi \hbar^2 \alpha^2} \right\}, $$

where parameter $\alpha$ should be tuned for dispersion being about less then the phase cell: $\alpha \leq 0.5$. On the other hand, the integral over momenta of (15) must lead to correct coordinate distribution in the pair approximation:

$$ g_P^2(\mathbf{q}_a, \mathbf{q}_b) \approx \frac{1}{2^{3/2} \lambda^6 \lambda^3} \left\{ \frac{V}{\lambda^3} \right\} \left( \frac{\alpha}{\lambda} \right)^3 \exp\left\{ -\frac{\pi}{\lambda^2} (\mathbf{q}_a - \mathbf{q}_b)^2 \right\}, $$

so $\alpha \approx \lambda V^{-1/3}$. Thus, the contribution of identical and pair permutation into Wigner function of ideal gas approximately is

$$ w_{P1+2}(\mathbf{p}_a, \mathbf{p}_b, \mathbf{q}_a, \mathbf{q}_b) \approx \exp\left\{ -\frac{\beta}{2m} \left( \mathbf{p}_a^2 + \mathbf{p}_b^2 \right) \right\} \left[ 1 \pm \exp\left\{ -\frac{(\mathbf{p}_a - \mathbf{p}_b)^2(\lambda')^2}{4\pi \hbar^2 \alpha^2} \right\} \right], $$

$$ -\frac{\pi}{\lambda^2} (\mathbf{q}_a - \mathbf{q}_b)^2 + \ln\left\{ \frac{V}{\lambda^3} \right\}. $$

If the distance between $a$ and $b$ is quite large or their momenta differs significantly from each other, i.e.,

$$ \frac{\pi}{\lambda^2} (\mathbf{q}_a - \mathbf{q}_b)^2 + \frac{(\mathbf{p}_a - \mathbf{p}_b)^2(\lambda')^2}{4\pi \hbar^2 \alpha^2} \geq \ln\left\{ \frac{V}{\lambda^3} \right\}, $$

then the expression in (22) is positive even in case of the Fermi–Dirac statistics. Therefore the contribution of identical and pair permutation into the Wigner function can be rewritten:

$$ w_{P1+2}(\mathbf{p}_a, \mathbf{p}_b, \mathbf{q}_a, \mathbf{q}_b) \approx \exp\left\{ -\frac{\beta}{2m} \left( \mathbf{p}_a^2 + \mathbf{p}_b^2 \right) - \beta v_{ab} \right\}, $$

$$ v_{ab} = -\frac{1}{\beta} \ln\left\{ 1 \pm \exp\left\{ -\frac{(\mathbf{p}_a - \mathbf{p}_b)^2(\lambda')^2}{4\pi \hbar^2 \alpha^2} - \frac{\pi}{\lambda^2} \left( \mathbf{q}_a - \mathbf{q}_b \right)^2 + \ln\left\{ \frac{V}{\lambda^3} \right\} \right\} \right]. $$

where $v_{ab}$ has a meaning of exchange pseudopotential between $a$ and $b$ in the phase space. Note that in case of the Bose–Einstein statistics the condition (23) is always satisfied so pseudopotential exist in any case.

Contribution of the triple permutation (17) completely differs from (15) at first sight. Firstly, it does not contain delta functions explicitly. Secondly, the dependence on momenta is non-Maxwellian. Thirdly, the dependence on coordinates is described by an exponent from imaginary
In case of the Fermi–Dirac statistics, we define the required delta functions and Maxwellian distribution arise:

\[ f_{P3}(p_a, p_b, p_c) \sim \frac{1}{2} \exp \left\{ -\frac{\beta}{2m} (p_a^2 + p_b^2 + p_c^2) \right\} \]
\[ \times \delta^{(3)} \left( \frac{(p_a - p_b) \lambda}{2\pi \hbar} \right) \delta^{(3)} \left( \frac{(p_b - p_c) \lambda}{2\pi \hbar} \right) \delta^{(3)} \left( \frac{(p_c - p_a) \lambda}{2\pi \hbar} \right), \tag{25} \]

so triple permutations has the same structure as pair permutations in calculations of average values and momentum distributions. At the same time, if one integrates (17) over all momenta, then the coordinate distribution function \( g_{P3}(q_a, q_b, q_c) \) will have the form of a Gaussian exponent of coordinates, similar to (18). With permutations of higher orders the situation is the same. Thus, the contribution of all permutations to the Wigner function of an ideal gas can be taken into account in the pair approximation in the following way:

\[ W_0(p, q) \approx \exp \left\{ -\frac{\beta}{2m} (p_1^2 + \cdots + p_N^2) \right\} \prod_{a < b} \left[ 1 + \delta^{(3)} \left( \frac{(p_a - p_b) \lambda}{2\pi \hbar} \right) \right] \]
\[ \times \exp \left\{ -\frac{\pi}{\lambda^2} (q_a - q_b)^2 \right\}, \tag{26} \]

where product is taken over all pairs of particles \( a, b \) with equal spin projection.

Finally we replace the delta functions in (26) with the Kronecker symbols according to (20) with the parameter \( \alpha \approx \lambda V^{-1/3} \). Introducing pair exchange pseudopotential \( v_{ab} \) from (24), we obtain the following expression for Wigner function of ideal gas in the pair approximation:

\[ W_0(p, q) \approx \exp \left\{ -\sum_{a=1}^{N} \frac{\beta}{2m} p_a^2 - \sum_{a>b} v_{ab} \right\}. \tag{27} \]

This expression is meaningful only if the argument of the logarithm is positive. In case of the Bose–Einstein statistics this condition is always satisfied, and bosonic exchange pseudopotential is

\[ v_{BE,ab} = -\frac{1}{\beta} \ln \left[ 1 + \exp \left\{ -\frac{(p_a - p_b)^2 \lambda^2}{4\pi \hbar^2 \alpha^2} - \frac{\pi}{\lambda^2} (q_a - q_b)^2 + 3 \ln \frac{1}{\alpha} \right\} \right]. \tag{28} \]

In case of the Fermi–Dirac statistics, we define \( v_{ab} \) in the following way:

\[ v_{FD,ab} = -\frac{1}{\beta} \ln \left[ 1 - \exp \left\{ -\frac{(p_a - p_b)^2 \lambda^2}{4\pi \hbar^2 \alpha^2} - \frac{\pi}{\lambda^2} (q_a - q_b)^2 + 3 \ln \frac{1}{\alpha} \right\} \right], \]
\[ v_{FD,ab} = -\frac{\pi}{\lambda^2} (q_a - q_b)^2 + \frac{(p_a - p_b)^2 \lambda^2}{4\pi \hbar^2 \alpha^2} \geq 3 \ln \frac{1}{\alpha}, \]
\[ v_{FD,ab} = -\frac{\pi}{\lambda^2} (q_a - q_b)^2 + \frac{(p_a - p_b)^2 \lambda^2}{4\pi \hbar^2 \alpha^2} \leq 3 \ln \frac{1}{\alpha}. \tag{29} \]

Fermionic pseudopotential takes into account permutations of fermions with momenta and coordinates satisfying this condition. Therefore, it can be used in calculations when the main integral contribution is given by this region of the phase space.

Let us consider a quasiclassical limit of pseudopotentials \( v_{BE,ab} \) and \( v_{FD,ab} \). When the characteristic lengths of the system are much greater than De Broglie wavelength \( \lambda \), the exponents in (28) and (29) tends to zero. In this case the exchange interaction is not manifested and the system has the Boltzmann statistics.

Typical fermionic exchange pseudopotential (29) is shown in figure 1. It reflects the Pauli principle, which prohibits two fermions from being at the same quantum state. Exchange
Figure 1. Fermionic exchange pseudopotential $v_{ab}$ in one-dimensional case when the parameter $\alpha^2$ is equal to 0.04 (a) and 0.16 (b).

Figure 2. Bosonic exchange pseudopotential $v_{ab}$ in one-dimensional case when the parameter $\alpha^2$ is equal to 0.04 (a) and 0.16 (b).

pseudopotential $v_{ab}$ becomes infinite if two particles are in the same cell of the phase space, and tends to zero if momenta and coordinates of the particles are different.

Typical bosonic exchange pseudopotential (28) is shown in figure 2. In contrast to the fermionic case, it is attractive and finite in all points of the phase space. This pseudopotential takes into account tendency of bosons to gather at the same quantum state. Pseudopotential tends to zero if momenta and coordinates of the particles are different.

4. Non-ideal gas
Let us generalize results (12), (28) and (29) to the systems with interaction between particles in linear and harmonic approximation for Wigner function. One can write contributions of different permutations to the Wigner function as well as in the ideal case (12).
In exactly the same way as in the ideal case, one can write the contributions to the Wigner function of various permutations, similarly to the expression (12). The only difference is that Fourier variable $\xi$ is also presented in the expansion of the potential energy. Therefore delta function in linear approximation depends on the combination of momenta and derivatives of the potential energy through vector-functional $J$:

$$\mathbf{p}_a \rightarrow \widetilde{p}_a = \mathbf{p}_a + \frac{2\pi \hbar}{\lambda} J[q, z(\tau)].$$

(30)

In the linear approximation all contributions of identical (13), pair (15) and other permutations are quite similar, with the only difference in the Maxwellian and delta functions by the functionals $K$, $P$, $J$. Thus our path to the expressions (27), (28), (29) in the ideal case is held, and the Wigner function in the linear approximation is

$$W_{LA}(p, q) = Z(\beta, V)^{-1} \int_{z(0)=z(1)=0} Dz(\tau) \exp \left\{-\frac{1}{2m} \sum_{a=1}^{N} \beta \mathbf{p}_a^2 - \beta \sum_{a>b} \tilde{v}_{ab} \right\} \times \cos \left\{ \rho_{a,i} J_{a,i}[q, z(\tau)] \right\} \exp \left\{ -\pi K[z(\tau)] - \pi P[q, z(\tau)] \right\},$$

(31)

with exchange potential depending on the derivative of potential energy:

$$\tilde{v}_{BE,ab} = -\frac{1}{\beta} \ln \left[ 1 + \exp \left\{ -\frac{(-\mathbf{p}_a + \mathbf{p}_b)^2 (\lambda')^2}{4\pi \hbar^2 \alpha^2} - \frac{\pi}{\lambda^2} (q_a - q_b)^2 + 3 \ln \frac{1}{\alpha} \right\} \right],$$

$$\tilde{v}_{FD,ab} = -\frac{1}{\beta} \ln \left[ 1 - \exp \left\{ -\frac{(-\mathbf{p}_a + \mathbf{p}_b)^2 (\lambda')^2}{4\pi \hbar^2 \alpha^2} - \frac{\pi}{\lambda^2} (q_a - q_b)^2 + 3 \ln \frac{1}{\alpha} \right\} \right],$$

$$\tilde{v}_{FD,ab} = \infty,$$

$$\tilde{v}_{FD,ab} = -\frac{1}{\beta} \ln \left[ 1 + \exp \left\{ -\frac{(-\mathbf{p}_a + \mathbf{p}_b)^2 (\lambda')^2}{4\pi \hbar^2 \alpha^2} - \frac{\pi}{\lambda^2} (q_a - q_b)^2 + 3 \ln \frac{1}{\alpha} \right\} \right].$$

(32)

In the case of harmonic approximation, there is an additional quadratic term $\xi^T (\chi - E) \xi$, where $E$ is unit matrix. Therefore the integral over $\xi$ leads to Gaussian exponent instead of delta function:

$$\delta^{(3N)} \left( p_a - \frac{\mathbf{\bar{p}}_a + P^{-1} \mathbf{\bar{p}}}{2} \right) \rightarrow \exp \left\{ -\frac{1}{4} \left( p - \frac{\mathbf{\bar{p}} + P^{-1} \mathbf{\bar{p}}}{2} \right)_a \right\} \chi_{a,i}^{-1} (\chi - E)_{a,i;b,j}^{-1} \left( p - \frac{\mathbf{\bar{p}} + P^{-1} \mathbf{\bar{p}}}{2} \right)_b \right\}.$$  

(33)

However, within the framework of the harmonic approximation, the inverse matrix of the second derivatives $(\chi - E)^{-1}$ is numerically large so difference between delta function and Gaussian exponent is negligible and all reasoning for linear approximation remains true. As a result, harmonic approximation for Wigner function with pair exchange:

$$W_{HA}(p, q) = Z(\beta, V)^{-1} \int_{z(0)=z(1)=0} Dz(\tau) \det \left| \chi^{-1}_{a,i,b,j}[q, z(\tau)] \right|^{1/2} \times \exp \left\{ -\frac{1}{4\pi} \rho_{a,i} \chi^{-1}_{a,i,b,j}[q, z(\tau)] \rho_{b,j} - \beta \sum_{a>b} \tilde{v}_{ab} + \pi J_{a,i}[q, z(\tau)] \chi^{-1}_{a,i,b,j}[q, z(\tau)] J_{b,j}[q, z(\tau)] \right\} \times \cos \left\{ \rho_{a,i} \chi^{-1}_{a,i,b,j}[q, z(\tau)] J_{b,j}[q, z(\tau)] \right\} \times \exp \left\{ -\pi K[z(\tau)] - \pi P[q, z(\tau)] \right\},$$

(34)

with exchange pseudopotential (32).
5. Conclusion
In the paper, general many-particle quantum system in thermodynamic equilibrium is considered using Wigner formalism. We have taken into account exchange effects in the Wigner function in case of the Bose–Einstein statistics and the Fermi–Dirac one. We have calculated the Wigner function of an ideal gas explicitly. We have shown that exchange effects in this case can be taken into account using pseudopotential in the phase space. In pair approximation the bosonic pseudopotential is quite exact, and reflects the tendency of bosons to be at the same quantum state. The fermionic pseudopotential reflects the Pauli principle prohibiting to any fermions to be at the same quantum state at once. Linear and harmonic approximations for non-ideal system were generalized to fermions and bosons, and the exchange pseudopotential was generalized on interacting systems.

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