Affine Weyl Group Approach to Painlevé Equations

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Abstract

An overview is given on recent developments in the affine Weyl group approach to Painlevé equations and discrete Painlevé equations, based on the joint work with Y. Yamada and K. Kajiwara.

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1. Introduction

The purpose of this paper is to give a survey on recent developments in the affine Weyl group approach to Painlevé equations and discrete Painlevé equations.

It is known that each of the Painlevé equations from $P_{II}$ through $P_{VI}$ admits the action of an affine Weyl group as a group of Bäcklund transformations (see a series of works [16] by K. Okamoto, for instance). Furthermore, the Bäcklund transformations (or the Schlesinger transformations) for the Painlevé equations can already be thought of as discrete Painlevé equations with respect to the parameters. The main idea of the affine Weyl group approach to (discrete) Painlevé systems is to extend this class of Weyl group actions to general root systems, and to make use of them as the common underlying structure that unifies various types of discrete system ([10]). In this paper, we discuss several aspects of affine Weyl group symmetry in nonlinear systems, based on a series of joint works with Y. Yamada and K. Kajiwara.

Before starting the discussion of (discrete) Painlevé equations, we recall some definitions, following the notation of [4]. A (generalized) Cartan matrix is an integer matrix $A = (a_{ij})_{i,j \in I}$ (with a finite indexing set) satisfying the conditions

$$a_{ii} = 2; \quad a_{ij} \leq 0 \quad (i \neq j); \quad a_{ij} = 0 \iff a_{ji} = 0.$$  

(1.1)

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The Weyl group $W(A)$ associated with $A$ is defined by the generators $s_i$ ($i \in I$), called the simple reflections, and the fundamental relations

$$s_i^2 = 1, \quad (s_i s_j)^{m_{ij}} = 1 \quad (i \neq j), \quad (1.2)$$

where $m_{ij} = 2, 3, 4, 6$ or $\infty$, according as $a_{ij}a_{ji} = 0, 1, 2, 3$ or $\geq 4$. When the Cartan matrix $A = (a_{ij})_{i,j=0}^l$ is of affine type (of type $A_l^{(1)}, B_l^{(1)}, \ldots, D_4^{(3)}$), the corresponding Weyl group is called an affine Weyl group.

We fix some notation for the case of type $A_l^{(1)}$ that will be used throughout this paper. The Cartan matrix $A = (a_{ij})_{i,j=0}^l$ of type $A_l^{(1)}$ is defined by

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad (l = 1), \quad A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad (l \geq 2). \quad (1.3)$$

The affine Weyl group $W(A_l^{(1)}) = \langle s_0, s_1, \ldots, s_l \rangle$ is defined by the following fundamental relations:

$$(l = 1) : \quad s_0^2 = s_1^2 = 1,$$

$$(l \geq 2) : \quad s_i^2 = 1, \quad s_i s_j = s_j s_i \quad (j \neq i, i \pm 1), \quad (s_i s_j)^3 = 1 \quad (j = i \pm 1), \quad (1.4)$$

where we have identified the indexing set with $\mathbb{Z}/(l + 1)\mathbb{Z}$. We also define an extension $\tilde{W}(A_l^{(1)}) = \langle s_0, \ldots, s_l, \pi \rangle$ of $W(A_l^{(1)})$ by adjoining a generator $\pi$ (rotation of indices) such that $\pi s_i = s_{i+1} \pi$ for all $i = 0, 1, \ldots, l$; we do not impose the relation $\pi^{l+1} = 1$.

2. Variations on the theme of $P_{1IV}$

In this section, we present several examples of affine Weyl group action of type $A_2^{(1)}$ to illustrate the role of affine Weyl group symmetry in (discrete) Painlevé equations and related integrable systems.

2.1. Symmetric form of $P_{1IV}$

Consider the following system of nonlinear differential equations for three unknown functions $\varphi_j = \varphi_j(t)$ ($j = 0, 1, 2$):

$$(N_{1IV}) \quad \begin{cases} \varphi'_0 = \varphi_0(\varphi_1 - \varphi_2) + \alpha_0, \\ \varphi'_1 = \varphi_1(\varphi_2 - \varphi_0) + \alpha_1, \\ \varphi'_2 = \varphi_2(\varphi_0 - \varphi_1) + \alpha_2, \end{cases} \quad (2.1)$$

where $' = d/dt$ denotes the derivative with respect to the independent variable $t$, and $\alpha_j = 0$ ($j = 0, 1, 2$) are parameters. When $\alpha_0 + \alpha_1 + \alpha_2 = 0$, this system
provides an integrable deformation of the Lotka-Volterra competition model for three species. When $\alpha_0 + \alpha_1 + \alpha_2 = k \neq 0$, it is essentially the fourth Painlevé equation

$$(P_{IV}) \quad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + (t^2 - \alpha)y + \frac{\beta}{y}. \quad (2.2)$$

In fact, from $(\varphi_0 + \varphi_1 + \varphi_2)' = k$, we have $\varphi_0 + \varphi_1 + \varphi_2 = kt + c$. Under the renormalization $k = 1, c = 0$, system (2.1) can be written as a second order equation for $y = \varphi_0$; it is transformed into $P_{IV}$ with $\alpha = \alpha_2 - \alpha_1, \beta = -2\alpha_0^2$ by the change of variables $t \to \sqrt{2}t, y \to -y/\sqrt{2}$. In view of this fact, we call (2.1) the symmetric form of the fourth Painlevé equation ($N_{IV}$). This type of representation for $P_{IV}$ was introduced by [19], [1] in the context of nonlinear dressing chains, and by [12] in the study of rational solutions of $P_{IV}$.

The symmetric form $N_{IV}$ provides a convenient framework for describing the discrete symmetry of $P_{IV}$. Let $K = C(\alpha, \varphi)$ be the field of rational functions in the variables $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $\varphi = (\varphi_0, \varphi_1, \varphi_2)$. We define the derivation $': K \to K$ by using formulas (2.1) together with $\alpha'_j = 0\ (j = 0, 1, 2)$; we regard the differential field $(K, ')$ as representing the differential system $N_{IV}$. In this setting, we say that an automorphism of $K$ is a Bäcklund transformation for $N_{IV}$ if it commutes with the derivation $'$. (A Bäcklund transformation as defined above means a birational transformation of the phase space that commutes with the flow defined by the nonlinear differential system.) As we will see below, $N_{IV}$ has four fundamental Bäcklund transformations that generate the extended affine Weyl group $\widehat{W} = \langle s_0, s_1, s_2, \pi \rangle$ of type $A_2^{(1)}$. Identifying the indexing set $\{0, 1, 2\}$ with $Z/3Z$, we define the automorphisms $s_i\ (i = 0, 1, 2)$ and $\pi$ of $K$ by

$$s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \quad s_i(\varphi_j) = \varphi_j + \frac{\alpha_i}{\varphi_i} u_{ij} \quad (i, j = 0, 1, 2),$$

$$\pi(\alpha_j) = \alpha_{j+1}, \quad \pi(\varphi_j) = \varphi_{j+1} \quad (j = 0, 1, 2). \quad (2.3)$$

Here $A = (a_{ij})^2_{i,j=0}$ stands for the Cartan matrix of type $A_2^{(1)}$, and $U = (u_{ij})^2_{i,j=0}$ for the orientation matrix of the Dynkin diagram (triangle) in the positive direction:

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \quad (2.4)$$

These automorphisms $s_i$ and $\pi$ commute with the derivation $'$, and satisfy the fundamental relations

$$s_i^2 = 1, \quad (s_is_{i+1})^3 = 1, \quad \pi s_i = s_{i+1} \pi \quad (i = 0, 1, 2) \quad (2.5)$$

for the generators of $\widehat{W}(A_2^{(1)})$. Hence we obtain a realization of the extended affine Weyl group $\widehat{W}(A_2^{(1)})$ as a group of Bäcklund transformations for $N_{IV}$. Notice that the action of the affine Weyl group $W = \langle s_0, s_1, s_2 \rangle$ on the $\alpha$-variables is identical to its canonical action on the simple roots.
We remark that the affine Weyl group symmetry is deeply related to the structure of special solutions of \( P_{1\mathbb{V}} \) (with the parameters \( \alpha_j \) as in \( N_{1\mathbb{V}} \)). Along each reflection hyperplane \( \alpha_j = n \) \((j = 0, 1, 2; n \in \mathbb{Z})\) in the parameter space, \( P_{1\mathbb{V}} \) has a one-parameter family of classical solutions expressed in terms of Toeplitz determinants of Hermite-Weber functions; each solution of this class is obtained by Bäcklund transformations from a seed solution at \( \alpha_j = 0 \) which satisfies a Riccati equation. Also, at each point of the \( W \)-orbit of the barycenter \((a_0, a_1, a_2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) of the fundamental alcove, it has a rational solution expressed in terms of Jacobi-Trudi determinants of Hermite polynomials.

### 2.2. \( q \)-Difference analogue of \( P_{1\mathbb{V}} \)

We now introduce a multiplicative analogue of the birational realization (2.3) of the extended affine Weyl group \( \tilde{W} = \langle s_0, s_1, s_2, \pi \rangle \) ([5]). Taking the field of rational functions \( \mathcal{L} = \mathbb{C}(a, f) \) in the variables \( a = (a_0, a_1, a_2) \) and \( f = (f_0, f_1, f_2) \), we define the automorphisms \( s_0, s_1, s_2, \pi \) of \( \mathcal{L} \) as follows:

\[
\begin{align*}
\pi(a_j) &= a_{j+1}, \\
\pi(f_j) &= f_{j+1} (j = 0, 1, 2),
\end{align*}
\]

where \( a_j \) are the multiplicative parameters corresponding to the simple roots \( \alpha_j \). These automorphisms again satisfy the fundamental relations for the generators of \( \tilde{W} \). In the following, the \( \tilde{W} \)-invariant \( a_0 a_1 a_2 = q \) plays the role of the base for \( q \)-difference equations. If one parameterizes \( a_j \) and \( f_j \) as

\[
a_j = e^{-\epsilon^2 \alpha_j/2}, \quad f_j = -e^{-\epsilon \varphi_j} \quad (j = 0, 1, 2)
\]

with a small parameter \( \epsilon \), one can recover the original formulas (2.3) from (2.6) by taking the limit \( \epsilon \to 0 \).

A \( q \)-difference analogue of (the symmetric form of) \( P_{1\mathbb{V}} \) is given by

\[
(q P_{1\mathbb{V}}) \left\{ \begin{array}{l}
T(f_0) = a_0 a_1 f_1 \frac{1 + a_2 f_2 + a_2 a_0 f_2 f_0}{1 + a_0 f_0 + a_0 a_1 f_0 f_1} \\
T(f_1) = a_1 a_2 f_2 \frac{1 + a_0 f_0 + a_0 a_1 f_0 f_1}{1 + a_1 f_1 + a_1 a_2 f_1 f_2} \\
T(f_2) = a_2 a_0 f_0 \frac{1 + a_1 f_1 + a_2 a_1 f_2 f_1}{1 + a_2 f_2 + a_2 a_0 f_2 f_0} \\
T(a_j) = a_j \quad (j = 0, 1, 2),
\end{array} \right.
\]

where \( T \) stands for the discrete time evolution ([5]). Notice that (2.8) implies \( T(f_0 f_1 f_2) = (a_0 a_1 a_2)^2 f_0 f_1 f_2 = q^2 f_0 f_1 f_2 \); hence one can consistently introduce a time variable \( t \) such that \( f_0 f_1 f_2 = t^2 \). If we consider \( f_j \) as functions of \( t \), the discrete time evolution \( T \) is identified with the \( q \)-shift operator \( t \to q t \), so that \( T f_j(t) = f_j(q t) \). In this sense, formula (2.8) defines a system of nonlinear \( q \)-difference equations, which we call the fourth \( q \)-Painlevé equation (\( q P_{1\mathbb{V}} \)).
The time evolution \( T \), regarded as an automorphism of \( L \), commutes with the action of \( \tilde{W} \) that we already described above. Namely, the \( q \)-difference system \( qP_{IV} \) admits the action of the extended affine Weyl group \( W \) as a group of Bäcklund transformations. Again, by taking the limit as \( \varepsilon \to 0 \) under the parametrization (2.7), one can show that the \( q \)-difference system \( qP_{IV} \), as well as its affine Weyl group symmetry, reproduces the differential system \( N_{IV} \). It is known that \( qP_{IV} \) defined above shares many characteristic properties with the original \( P_{IV} \). For example, it has classical solutions expressed by continuous \( q \)-Hermite-Weber functions, and rational solutions expressed by of continuous \( q \)-Hermite polynomials, analogously to the case of \( P_{IV} \) ([5], [7]). We also remark that, when \( a_0a_1a_2 = 1 \), one can regard \( qP_{IV} \) as a discrete integrable system which generalizes a discrete version of the Lotka-Volterra equation.

### 2.3. Ultra-discretization of \( P_{IV} \)

It should be noticed that the discrete time evolution of \( qP_{IV} \) is defined in terms of a subtraction-free birational transformation; we say that a rational function is subtraction-free if it can be expressed as a ratio of two polynomials with real positive coefficients. Recall that there is a standard procedure, called the ultra-discretization, of passing from subtraction-free rational functions to piecewise linear functions ([18], [2], see also [15]). Roughly, it is the procedure of replacing the operations

\[
a \cdot b \rightarrow A + B, \quad a/b \rightarrow A - B, \quad a + b \rightarrow \max(A, B).
\]  

(2.9)

Introducing the variables \( A_j, F_j \) \((j = 0, 1, 2)\), from \( qP_{IV} \) we obtain the following system of piecewise linear difference equations by ultra-discretization:

\[
(uP_{IV}) \left\{ \begin{array}{l}
T(F_0) = A_0 + A_1 + F_1 + \max(0, A_2 + F_2, A_2 + A_0 + F_2 + F_0) \\
- \max(0, A_0 + F_0, A_0 + A_1 + F_0 + F_1), \\
T(F_1) = A_1 + A_2 + F_2 + \max(0, A_0 + F_0, A_0 + A_1 + F_0 + F_1) \\
- \max(0, A_1 + F_1, A_1 + A_2 + F_1 + F_2), \\
T(F_2) = A_2 + A_0 + F_0 + \max(0, A_1 + F_1, A_1 + A_2 + F_1 + F_2) \\
- \max(0, A_2 + F_2, A_2 + A_0 + F_2 + F_0), \\
T(A_j) = A_j \quad (j = 0, 1, 2),
\end{array} \right.
\]  

(2.10)

which we call the fourth ultra-discrete Painlevé equation \( (uP_{IV}) \). Simultaneously, the affine Weyl group symmetry of \( qP_{IV} \) is ultra-discretized as follows:

\[
s_i(A_j) = A_j - A_ia_{ij}, \quad s_i(F_j) = F_j + u_{ij} \left( \max(A_i, F_i) - \max(0, A_i + F_i) \right), \\
\pi(A_j) = A_{j+1}, \quad \pi(F_j) = F_{j+1} \quad (i, j = 0, 1, 2).
\]  

(2.11)

This time, the extended affine Weyl group \( \tilde{W} \) is realized as a group of piecewise linear transformations on the affine space with coordinates \((A, F)\). We also remark that, when \( A_0 + A_1 + A_2 = Q = 0 \), \( uP_{IV} \) gives rise to an ultra-discrete integrable system. It would be an interesting problem to analyze special solutions of the ultra-discrete system \( uP_{IV} \).
3. Discrete symmetry of Painlevé equations

In this section, we propose a uniform description of discrete symmetry of the Painlevé equations $P_J$ for $J = \text{II, IV, V, VI}$. We also give some remarks on a generalization of this class of birational Weyl group action to arbitrary root systems.

3.1. Hamiltonian system $H_J$

It is known that each Painlevé equation $P_J$ ($J = \text{II, III,.., VI}$) is equivalently expressed as a Hamiltonian system

$$ (H_J) : \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} $$

(3.1)

with a polynomial Hamiltonian $H = H(q, p, t, \alpha) \in \mathbb{C}(t)[q, p, \alpha]$ depending on parameters $\alpha = (\alpha_1, \ldots, \alpha_l)$ (see [3], for instance). Setting $\mathcal{K} = \mathbb{C}(q, p, t, \alpha)$, we define the Poisson bracket $\{\cdot, \cdot\} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and the Hamiltonian vector field $\delta : \mathcal{K} \to \mathcal{K}$ by

$$ \{\varphi, \psi\} = \frac{\partial \varphi}{\partial p} \frac{\partial \psi}{\partial q} - \frac{\partial \varphi}{\partial q} \frac{\partial \psi}{\partial p}, \quad \delta(\varphi) = \{H, \varphi\} + \frac{\partial \varphi}{\partial t} \quad (\varphi, \psi \in \mathcal{K}). $$

(3.2)

In this setting, a Bäcklund transformation for $H_J$ is understood as an automorphism $w : \mathcal{K} \to \mathcal{K}$ that commutes with $\delta$. We also say that $w$ is canonical if it preserves the Poisson bracket: $w(\{\varphi, \psi\}) = \{w(\varphi), w(\psi)\}$ for any $\varphi, \psi \in \mathcal{K}$.

For each $J = \text{II, III,.., VI}$, it is known that the parameter space for $H_J$ is identified with the Cartan subalgebra of a semisimple Lie algebra, and that an extension of the corresponding affine Weyl group acts on $\mathcal{K}$ as a group of Bäcklund transformations ([16]). A table of fundamental Bäcklund transformations for $H_J$ can be found in [9].

If the Hamiltonian $H$ is chosen appropriately, the affine Weyl group symmetry of $H_J$ for $J = \text{II, IV, V, VI}$ can be described in a universal way in terms of root systems. With the notation of [4], the type of the affine root system is specified as follows:\footnote{In the case of $H_{\text{III}}$, one can use an extension of the affine Weyl group, either of type $C_2^{(1)}$ or of $2A_1^{(1)}$, for describing the same group of Bäcklund transformations. It seems natural to expect that the same principle to be discussed below should apply to $H_{\text{III}}$ as well, but we have not completely understood the case of $H_{\text{III}}$ yet.}

$$ \begin{array}{c|ccccc} H_J & H_{\text{II}} & H_{\text{IV}} & H_{\text{V}} & H_{\text{VI}} \\ \hline X_1^{(1)} & A_1^{(1)} & A_2^{(1)} & A_3^{(1)} & D_4^{(1)} \end{array} $$

(3.3)
The corresponding Cartan matrix $A = (a_{ij})_{i,j=0}$ is given by

$$
A_1^{(1)}: \quad A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad A_2^{(1)}: \quad A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}
$$

$$
A_3^{(1)}: \quad A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad D_4^{(1)}: \quad A = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}
$$

respectively. For the description of affine Weyl group symmetry, we make use of the following Hamiltonian $H = H(q, p, t, \alpha)$:

$$
H_{\Pi} : \quad H = \frac{1}{2}p(p - 2q^2 + t) + \alpha_1 q,
$$

$$
H_{\Pi} : \quad H = qp(2p - q - 2t) - 2\alpha_1 p - \alpha_2 q,
$$

$$
H_V : \quad tH = q(q - 1)(p + t) - (\alpha_1 + \alpha_3)qp + \alpha_3 p + \alpha_2 t q,
$$

$$
H_{VI} : \quad (t - 1)H = q(q - 1)(q - t)p^2 - \{(\alpha_0 - 1)q(q - 1) + \alpha_4(q - 1)(q - t)
$$

$$
+ \alpha_3 q(q - t)\} p + \alpha_1 (\alpha_1 + \alpha_2)(q - t).
$$

The parameter $\alpha_0$ is defined so that $\alpha_0 + \alpha_1 + \cdots + \alpha_l = 1$ for $J = \Pi, \Pi, V, \Pi, V$, and $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$ for $J = VI$. (Conventionally, the null root is normalized to be the constant 1.)

### 3.2. Discrete symmetry of $H_J$

Our main observation concerning the discrete symmetry of $H_J$ can be summarized as follows.

**Theorem 1** Choose the polynomial Hamiltonian $H \in \mathbb{C}[t][q,p,\alpha]$ for $H_J$ ($J = \Pi, \Pi, V, \Pi, V, VI$) as in (3.5). Then there exists a set $\{\varphi_0, \varphi_1, \ldots, \varphi_l\}$ of nonzero elements in the Poisson algebra $\mathcal{R} = \mathbb{C}[q,p,t]$ with the following properties:

1. The elements $\varphi_i$ ($i = 0, 1, \ldots, l$) satisfy the Serre relations of type $X_i^{(1)}$

$$
ad_{\{\varphi_i\}}(\varphi_j)^{-a_{ij}} \varphi_j^+ = 0 \quad (i \neq j),
$$

where $ad_{\{\varphi\}} = \{\varphi, \cdot\}$ stands for the adjoint action of $\varphi$ by the Poisson bracket.

2. For each $i = 0, 1, \ldots, l$, define $s_i$ to be the unique automorphism of $\mathcal{K} = \mathbb{C}(q,p,t,\alpha)$ such that

$$
s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij} \quad (j = 0, 1, \ldots, l),
$$

$$
s_i(\psi) = \exp \left( \frac{\alpha_i}{\varphi_i} \right) \varphi_i(\varphi_i(\psi)) \quad (\psi \in \mathcal{R} = \mathbb{C}[q,p,t]).
$$

Then these $s_i$ are canonical Bäcklund transformations for $H_J$. Furthermore, the subgroup $W = \langle s_0, s_1, \ldots, s_l \rangle$ of Aut($\mathcal{K}$) is isomorphic to the affine Weyl group $W(X_l^{(1)})$. 


Note that, for each \( \psi \in \mathcal{R} \), \( s_i(\psi) \) is determined as a finite sum
\[
s_i(\psi) = \psi + \frac{\alpha_i}{\varphi_i}(\varphi_i, \psi) + \frac{1}{2!}\left(\frac{\alpha_i}{\varphi_i}\right)^2\{\varphi_i, \{\varphi_i, \psi\}\} + \cdots, \tag{3.8}
\]
since the action of \( \text{ad}_{\{\varphi_i\}} \) on \( \mathcal{R} \) is locally nilpotent. A choice of the generators \( \varphi_i \) \( (i = 0, 1, 2, \ldots, l) \) with the properties of Theorem 1 is given as follows:

\[
\begin{align*}
H_{II} : & \quad \varphi_0 = -p + 2q^2 + t, \quad \varphi_1 = p, \\
H_{IV} : & \quad \varphi_0 = -p + \frac{q}{2} + t, \quad \varphi_1 = \frac{-q}{2}, \quad \varphi_2 = p, \\
H_{V} : & \quad \varphi_0 = p + t, \quad \varphi_1 = t(q), \quad \varphi_2 = -p, \quad \varphi_3 = t(1-q), \\
H_{VI} : & \quad \varphi_0 = q - t, \quad \varphi_1 = 1, \quad \varphi_2 = -p, \quad \varphi_3 = q - 1, \quad \varphi_4 = q.
\end{align*}
\tag{3.9}
\]

We remark that, in the case of \( H_J \) \( (J = II, IV, V) \) of type \( A_l^{(1)} \) \( (l = 1, 2, 3) \), we also have the Bäcklund transformation \( \pi \) corresponding to the diagram rotation; its action is given simply by \( \pi(\alpha_j) = \alpha_{j+1}, \pi(\varphi_j) = \varphi_{j+1} \).

If we use the polynomials \( \varphi_j \) as dependent variables, the Hamiltonian system \( H_{IV} \), for example, is rewritten as
\[
\frac{d\varphi_j}{dt} = 2\varphi_j(\varphi_{j+1} - \varphi_{j+2}) + \alpha_j \quad (j = 0, 1, 2) \tag{3.10}
\]
with the convention \( \varphi_{j+3} = \varphi_j \), from which we obtain the symmetric form \( N_{IV} \) by a simple rescaling of the variables. We remark that the polynomials \( \varphi_j \) are the factors of the “leading term” of the Hamiltonian \( H \). Also, in the context of irreducibility of Painlevé equations, the polynomials \( \varphi_j \) are the fundamental invariant divisors along the reflection hyperplanes \( \alpha_j = 0 \) (see [8], [17], for instance). When \( \alpha_j = 0 \), the specialization of \( H_J \) by \( \varphi_j = 0 \) gives rise to a Riccati equation that reduces to a linear equation of hypergeometric type; for \( J = II, IV, V \) and \( VI \), the differential equations of Airy, Hermite-Weber, Kummer and Gauss appear in this way, respectively.

Apart from differential equations, this class of birational realization of Weyl groups as in Theorem 1 can be formulated for an arbitrary Cartan matrix by means of Poisson algebras (see [13], for the details). In this sense, Bäcklund transformations for Painlevé equations \( P_J \) \( (J = II, IV, V, VI) \) have a universal nature with respect to root systems. In the case where \( A \) is of affine type, such a birational realization of the affine Weyl group appears as the symmetry of systems of nonlinear partial differential equations of Painlevé type, obtained by similarity reduction from the principal Drinfeld-Sokolov hierarchy (of modified type). The case of type \( A_l^{(1)} \) will be mentioned in the next section. As for the original Painlevé equations, \( P_{II}, P_{IV} \) and \( P_V \) are in fact obtained by similarity reduction from the \((l+1)\)-reduced modified KP hierarchy for \( l = 1, 2, 3 \), respectively. For \( P_{VI} \), an \( 8 \times 8 \) Lax pair is constructed in [14] in the framework of the affine Lie algebra \( \hat{\mathfrak{so}}(8) \). This Lax pair is compatible with the affine Weyl group symmetry of Theorem 1. It is not clear, however, how this construction should be understood in relation to the Drinfeld-Sokolov hierarchy of type \( D_4^{(1)} \).
4. Painlevé systems with $W(A_l^{(1)})$ symmetry

In this section, we introduce Painlevé systems and $q$-Painlevé systems with affine Weyl group symmetry of type $A_l$; this part can be regarded as a generalization, to higher rank cases, of the variations of $P_{IV}$ discussed in Section 2. In the following, we fix two positive integers $M, N$, and consider a Painlevé system, as well as its $q$-version, attached to $(M, N)$.

4.1. Painlevé system of type $(M, N)$

We investigate the compatibility condition for a system of linear differential equations

\[ N\partial_z \vec{\psi} = A\vec{\psi}, \quad \partial_{t_m} \vec{\psi} = B_m \vec{\psi} \quad (m = 1, \ldots, M), \tag{4.1} \]

where $\vec{\psi} = (\psi_1, \ldots, \psi_N)^t$ is the column vector of unknown functions, and $A, B_m$ are $N \times N$ matrices, both depending on $(z, t) = (z, t_1, \ldots, t_M)$. We assume that

\[ B_m = \sum_{k=0}^{m-1} \text{diag}(b^{(m,k)}) \Lambda^k + \Lambda^m, \quad (m = 1, \ldots, M), \tag{4.2} \]

where $\Lambda = \sum_{i=1}^{N-1} E_{i,i+1} + zE_{N,1}$ denotes the cyclic matrix, $E_{ij} = (\delta_{a,i}\delta_{b,j})_{a,b}$ being the matrix units, and $b^{(m,k)} = (b^{(m,k)}_1, \ldots, b^{(m,k)}_N)$ are $N$-vectors depending only on $t$. Note that the compatibility condition

\[ \partial_{t_n}(B_m) - \partial_{t_m}(B_n) + [B_m, B_n] = 0 \quad (m, n = 1, \ldots, M) \tag{4.3} \]

is the Zakharov-Shabat equation of the $N$-reduced modified KP hierarchy (restricted to the first $M$ time variables). As for the matrix $A$, we set

\[ A = -\text{diag}(\rho) + \sum_{k=1}^{M} kt_k B_k, \quad \rho = (N-1, N-2, \ldots, 0). \tag{4.4} \]

Then the compatibility condition

\[ \partial_{t_m}(A) - N\partial_z(B_m) + [A, B_m] = 0 \quad (m = 1, \ldots, M), \tag{4.5} \]

reduces to the homogeneity condition

\[ \sum_{n=1}^{M} nt_n \partial_{t_n}(b^{(m,k)}) = (k - m)b^{(m,k)} \quad (1 \leq k \leq m \leq M) \tag{4.6} \]

for the coefficients of the $B$ matrices. We define the Painlevé system of type $(M, N)$ to be the system of nonlinear partial differential equations (4.3) with the similarity constraint (4.6).

We remark that, when $(M, N) = (3, 2), (2, 3), (2, 4)$, this system reduces essentially to the Painlevé equations $P_{III}, P_{IV}, P_{V}$, respectively. When $(M, N) = (2, N)$
We consider this condition as the Zakharov-Shabat equation for the KP hierarchy; in this formulation, all the time variables $t_1, \ldots, t_M$ are treated equally. Note that, as to the Euler operator $T = T_1 \cdots T_M$, we have

$$T\psi = B_T\psi, \quad B_T = T_2 \cdots T_M(B_1)T_3 \cdots T_M(B_2) \cdots T_M(B_{M-1})B_M.$$ (4.13)
In the linear $q$-difference system (4.10), we choose the following matrix for $A$:

$$A = \text{diag}(\kappa)^{-1} B_T, \quad \kappa = (q^{N-1}, q^{N-2}, \ldots, 1).$$  \hspace{1cm} (4.14)

Then the compatibility condition

$$T_m(A)B_m = T_{q^n, z}(B_m)A \quad (m = 1, \ldots, M)$$  \hspace{1cm} (4.15)

is equivalent to the homogeneity condition

$$T_1 \cdots T_M(u_i^{(m)}) = u_i^{(m)} \quad (i = 1, \ldots, N; m = 1, \ldots, M).$$  \hspace{1cm} (4.16)

We define the $q$-Painlevé system of type $(M, N)$ to be the system of nonlinear $q$-

difference equations (4.12) for $M \times N$ unknown functions $u_i^{(m)} (m = 1, \ldots, M; i = 1, \ldots, N)$ with the similarity constraint (4.16). This system can be written in
the form

$$T_m(u_i^{(n)}) = F_i^{(m,n)}(t, u) \quad (m, n = 1, \ldots, M; i = 1, \ldots, N);$$  \hspace{1cm} (4.17)

in general, these $F_i^{(m,n)}(t, u)$ are complicated rational functions. It turns out, however, that by introducing new variables

$$x_j^i = \frac{1}{t_i} T_{i+1} T_{i+2} \cdots T_M(u_j^{(i)}) \quad (i = 1, \ldots, M; j = 1, \ldots, N),$$  \hspace{1cm} (4.18)

the time evolution of the $q$-Painlevé system can be described explicitly by means of
a birational affine Weyl group action on the $x$-variables.

4.3. A birational Weyl group action on the matrix space

For describing the time evolution $T_i$ of the $q$-Painlevé system, we introduce
a birational action of the direct product $\tilde{W}(A_{M-1}^{(1)}) \times \tilde{W}(A_{N-1}^{(1)})$ of two extended
affine Weyl groups. In the following, we use the notation

$$\tilde{W}^M = \langle r_{0}r_{1} \ldots r_{M-1}, \omega \rangle, \quad \tilde{W}^N = \langle s_{0}s_{1} \ldots s_{N-1}, \pi \rangle$$  \hspace{1cm} (4.19)

for the two extend affine Weyl groups. Introducing two parameters $q, p$, we take
$\mathbb{K} = \mathbb{C}(q, p)$ as the ground field. Let $\mathcal{K} = \mathbb{K}(x)$ be the field of rational functions in the $MN$ variables $x_j^i \ (1 \leq i \leq M; 1 \leq j \leq N)$; we regard the $x$-variables as
the canonical coordinates of the affine space of $M \times N$ matrices. For convenience, we extend the indices $i, j$ of $x_j^i$ to $\mathbb{Z}$ by setting $x_j^{i+M} = qx_j^i, x_{j+N}^i = px_j^i$.

We define the automorphisms $r_k \ (k \in \mathbb{Z}/M\mathbb{Z}), \ \omega, \ s_l \ (l \in \mathbb{Z}/N\mathbb{Z}), \ \pi$ of $\mathcal{K}$ as follows:

$$r_k(x_j^i) = px_j^{i+1} \frac{P_j^{i}}{P_j^{i+1}}, \quad r_k(x_j^{i+1}) = p^{-1} x_j^i \frac{P_j^i}{P_j^{i+1}} \quad (i = k \mod M);$$

$$r_k(x_j^i) = x_j^i \quad (i \neq k \mod M); \quad \omega(x_j^i) = x_j^{i+1};$$

$$s_l(x_j^i) = qx_j^{i+1} \frac{Q_j^{-i}}{Q_j^{-i+1}}, \quad s_l(x_j^{i+1}) = q^{-1} x_j^i \frac{Q_j^i}{Q_j^{i+1}} \quad (j = l \mod N);$$

$$s_l(x_j^i) = x_j^i \quad (j \neq l \mod N); \quad \pi(x_j^i) = x_{j+1}^i.$$

(4.20)
where

\[ P_j^i = \sum_{k=1}^{N} \prod_{a=0}^{k-1} x_{j+a}^i \prod_{a=k+1}^{N} x_{j+a}^{i+1}, \quad Q_j^i = \sum_{k=1}^{M} \prod_{a=0}^{k-1} x_{j+a}^i \prod_{a=k+1}^{M} x_{j+a}^{i+1}. \]  

(4.21)

Note that all these automorphisms represent subtraction-free birational transformations on the affine space of \( M \times N \) matrices.

**Theorem 2** The automorphisms \( r_0, \ldots, r_{M-1}, \omega \) and \( s_0, \ldots, s_{M-1}, \pi \) of \( K \) defined as above give a realization of the product \( \tilde{W}^M \times \tilde{W}_N \) of extended affine Weyl groups.

By using this birational action of affine Weyl group \( \tilde{W}^M = \langle r_0, \ldots, r_{M-1}, \omega \rangle \) we define \( \gamma_1, \ldots, \gamma_M \) by

\[ \gamma_k = r_{k-1} \cdots r_1 \omega r_M \cdots r_k \quad (k = 1, \ldots, M). \]  

(4.22)

We remark that these elements \( \gamma_k \) generate a free abelian subgroup \( L \simeq \mathbb{Z}^M \), and that the extended affine Weyl group \( W^M \) decomposes into the semidirect product \( L \rtimes S_M \) of \( L \) and the symmetric group of degree \( M \) that acts on \( L \) by permuting the indices for \( \gamma_k \).

**Theorem 3** In terms of the variables \( x_j^i \) defined by (4.18), the time evolution of the \( q \)-Painlevé system of type \( (M, N) \) is described by

\[ T_k(x_j^i) = \gamma_k^{-1}(x_j^i) \quad (1 \leq i \leq M; 1 \leq j \leq N) \]  

(4.23)

for all \( k = 1, \ldots, M \), where \( \gamma_k \) is defined by (4.22) through the birational action of \( \tilde{W}^M \) with \( p = 1 \).

This theorem means that the discrete time evolutions \( T_k \) (\( k = 1, \ldots, M \)) of the \( q \)-Painlevé system of type \( (M, N) \) coincides with the commuting discrete flows \( \gamma_k^{-1} \) (\( k = 1, \ldots, M \)) arising from the affine Weyl group action of \( \tilde{W}^M \) with \( p = 1 \). Furthermore the \( q \)-Painlevé system admits the action of extended affine Weyl group \( \tilde{W}_N \) of type \( A_{N-1}^{(1)} \) as a group of Backlund transformations. One can show that the fourth \( q \)-Painlevé equation \( qP_{11} \) discussed in Section 2 arises from the \( q \)-Painlevé system of type \( (M, N) = (2, 3) \), consistently with the differential case.

Finally we give some remarks on the ultra-discretization. From the birational action of \( \tilde{W}^M \times \tilde{W}_N \) with two parameters \( q, p \), we obtain a piecewise linear action of the same group on the space of \( M \times N \) matrices, with two parameters \( Q, P \) corresponding to \( q, p \). When \( P = 0 \), the commuting piecewise linear flows \( \gamma_k \in \tilde{W}^M \) may be called the ultra-discrete Painlevé system of type \( (M, N) \). When \( P = Q = 0 \), it specializes to an integrable ultra-discrete system; it gives rise to an \( M \)-periodic version of the box-ball system.

This class of piecewise linear action is tightly related to the combinatorics of crystal bases. The coordinates of the \( M \times N \) matrix space can be identified with the coordinates for the tensor product \( B^\otimes M \) of \( M \) copies of the crystal basis \( B \) for the symmetric tensor representation of \( GL_N \). Under this identification, it turns out that the piecewise linear transformations \( r_k \) and \( s_l \) with \( P = Q = 0 \) represent the action of the combinatorial \( R \) matrices and the Kashiwara’s Weyl group action on \( B^\otimes M \), respectively (see [15]).
References

[1] V.E. Adler: Nonlinear chains and Painlevé equations, Physica D 73 (1994), 335–351.
[2] A. Berenstein, S. Fomin and A. Zelevinsky: Parametrization of canonical bases and totally positive matrices, Adv. in Math. 122 (1996), 49–149.
[3] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida: From Gauss to Painlevé—A Modern Theory of Special Functions, Vieweg, 1991.
[4] V.G. Kac: Infinite dimensional Lie algebras, 3rd Edition, Cambridge University Press, 1990.
[5] K. Kajiwara, M. Noumi and Y. Yamada: A study on the fourth q-Painlevé equation, J. Phys. A: Math. Gen. 34 (2001), 8563–8581.
[6] K. Kajiwara, M. Noumi and Y. Yamada: Discrete integrable systems with $W(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ symmetry, to appear in Lett. Math. Phys. (nlin.SI/0106029).
[7] K. Kajiwara, M. Noumi and Y. Yamada: q-Painlevé systems arising from q-KP hierarchy, preprint (nlin.SI/0112045).
[8] M. Noumi and K. Okamoto: Irreducibility of the second and the fourth Painlevé equations, Funkcial. Ekvac. 40 (1997), 139–163.
[9] M. Noumi, K. Takano and Y. Yamada: Bäcklund transformations and the manifolds of Painlevé systems, to appear in Funkcial. Ekvac.
[10] M. Noumi and Y. Yamada: Affine Weyl groups, discrete dynamical systems and Painlevé equations, Commun. Math. Phys. 199 (1998), 281–295.
[11] M. Noumi and Y. Yamada: Higher order Painlevé equations of type $A_l^{(1)}$, Funkcial. Ekvac. 41 (1998), 483–503.
[12] M. Noumi and Y. Yamada: Symmetries in the fourth Painlevé equations and Okamoto polynomials, Nagoya Math. J. 153 (1999), 53–86.
[13] M. Noumi and Y. Yamada: Birational Weyl group action arising from a nilpotent Poisson algebra, in Physics and Combinatorics 1999, Proceedings of the Nagoya 1999 International Workshop (Eds. A.N. Kirillov, A. Tsuchiya and H. Umemura), 287–319, World Scientific, 2001.
[14] M. Noumi and Y. Yamada: A new Lax pair for the sixth Painlevé equation associated with $\hat{so}(8)$, to appear in Microlocal Analysis and Complex Fourier Analysis, World Scientific (math-ph/0203030).
[15] M. Noumi and Y. Yamada: Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions, preprint (math-ph/0203030).
[16] K. Okamoto: Studies of the Painlevé equations I, Ann. Math. Pura Appl. 146 (1987), 337–381; II, Japan. J. Math. 13 (1987), 47–76; III, Math. Ann. 275 (1986), 221–255; IV, Funkcial. Ekvac. 30 (1987), 305–332.
[17] H. Umemura and H. Watanabe: Solutions of the second and fourth Painlevé equations, I, Nagoya Math. J. 148 (1997), 151–198.
[18] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma: From soliton equations to integrable cellular automata through a limiting procedure, Phys. Rev. Lett. 76 (1996), 3247–3250.
[19] A.P. Veselov and A.B. Shabat: A dressing chain and the spectral theory of Schrödinger operator, Funct. Anal. Appl. 27 (1993), 81–96.