Semiclassical quantization of superstrings: AdS5 × S5 and beyond

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Abstract

We discuss semiclassical quantization of closed superstrings in AdS5 × S5. We consider two basic examples: point-like string boosted along large circle of S5 and folded string rotating in AdS5. In the first case we clarify the general structure of the sigma model perturbation theory for the energy of string states beyond the 1-loop order (related to the plane-wave limit). In the second case we argue that the large spin limit of the expression for the ground-state energy (i.e. for the dimension of the corresponding minimal twist gauge theory operator) has the form $S + f(\lambda) \ln S$ to all orders in the $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ expansion, in agreement with the AdS/CFT duality. We also suggest the extension of the semiclassical approach to near-conformal (near-AdS) cases on the example of the fractional D3-brane on conifold background.

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1. Introduction

The AdS/CFT duality \([1]\) relates the string theory in \(AdS_5 \times S^5\) to \(\mathcal{N} = 4\) \(SU(N)\) SYM theory in 4-d flat space with parameters \(\frac{R^4}{\alpha'^2} = \lambda \equiv g_{YM}^2 N\), \(g_s = \frac{\lambda}{N}\). The “classical” (tree-level) string theory limit corresponds to the ‘t Hooft limit: \(g_s \to 0\), \(\frac{R^4}{\alpha'^2}\) = fixed. Full understanding of the tree-level string theory in \(AdS_5 \times S^5\) would allow one to compute various “interpolating functions” \(f_i(\lambda)\) that enter the observables like SYM entropy \(S = f_1(\lambda)N^2V_3T^3\), rectangular Wilson loop expectation value (“quark-antiquark” potential) \(\langle W(C) \rangle \sim \exp[-f_2(\lambda)\frac{T}{T}]\), anomalous dimensions of composite operators \(\Delta - \Delta_0 = f_3(\lambda)\) (related to masses of the corresponding string modes), etc. While exact computations of these functions is beyond the reach at present, finding the leading terms in them in small \(\alpha' \sim \lambda^{-1/2}\) (large \(\lambda\)) expansion is already of interest, as they provide information about strong-coupling behaviour of the SYM theory. In some cases certain structures in the \(\alpha'\) expansion of \(f_i(\lambda)\) may be given exactly by the first few (say, classical and 1-loop) terms in the semiclassical expansion. This would then allow the comparison with perturbative gauge theory results. This is indeed what happens in the large R-charge example considered in \([2,3]\) (see also \([4]\)).

Given that the \(AdS_5 \times S^5\) string action \([5]\) is highly non-linear, the semiclassical \(\alpha'\)-expansion near a particular string configuration is the simplest way to explore the string-theory side of the duality. Such expansion was developed earlier in the open-string (Wilson loop) sector \([6,7,8,9]\). Recently, similar semiclassical approach was suggested in the closed string sector \([3,10]\), where one can relate the energy of a particular string state to the dimension of the corresponding operator in dual gauge theory. While the interest to a particular large \(S^5\) angular momentum sector of closed string modes was drawn by the observation \([11]\) that GS string in the R-R plane-wave background \([12]\) is exactly solvable, one does not actually need to use the results of \([12,11,13]\) to arrive at the conclusions of \([2]\): as explained in \([3,10]\), all one needs to do is to expand the original \(AdS_5 \times S^5\) action \([7]\) near a particular point-like string classical solution and compute the 1-loop correction to the energy.

The semiclassical expansion approach is more general than the one based on starting directly with the plane-wave background: (i) it clarifies the place of the large R-charge sector in the context of the standard AdS/CFT correspondence: one concentrates on specific string states with large \(S^5\) angular momentum (avoiding possible ambiguities in how one is to take the Penrose limit or identify \(p^+\) and \(p^-\) with \(J\) and \(\Delta - J\), cf. \([2,14]\)); the plane-wave
(Penrose) limit is recognized as being simply the sigma model 1-loop approximation.\(^1\) It makes possible to systematically extend the computation of the energy/dimension and thus the check of the AdS/CFT correspondence beyond the leading (1-loop) order \([10]\). (iii) it allows one to investigate (by expanding, in much the same way, near other classical solutions) other interesting subsectors of string states with large quantum numbers which can be again related to particular operators in gauge theory.

One may start with any stable classical string solution\(^2\) carrying linear or angular momentum, or simply having non-zero energy proportional to the string tension \(\sim \sqrt{\lambda}\) times an “oscillator number” (a “non-topological soliton”), which is thus large in the \(\alpha' \to 0\) or \(\lambda \gg 1\) limit. For the simplest point-like string solution, one has two “irreducible” choices: (i) massless geodesic running parallel to the boundary of AdS\(_5\), and (ii) massless geodesic running along big circle of S\(_5\) (carrying angular momentum \(J\)). In the latter case the 1-loop approximation is equivalent to exact string quantization in the plane-wave (Penrose) limit of the AdS\(_5 \times S^5\) geometry. The basic extended string configuration is the string rotating near the center of AdS\(_5\) (and carrying spin \(S\)) \([15,3]\). One may also consider the “mixed” \((J,S)\) case \([10]\). Solutions with oscillations were considered in \([16,3]\) and \([17,18]\) (see also \([19]\) for other related examples in the context of the semiclassical approach). Computing quantum string corrections to the energies of the string states allows one to determine the strong-coupling expansions of the anomalous dimensions of the corresponding composite operators on the SYM side.

The study of special sectors of string states with large quantum numbers is equivalent to semiclassical expansion since the energy and conserved charges then scale as string tension and thus are large in the large \(\lambda\) limit. From general perspective, given the non-linearity of the AdS\(_5 \times S^5\) string action, one may try to do semiclassical expansions near different points in the classical solution space and then try to patch the resulting expansions together. Interpolating between the expressions for the string spectra obtained near different expansion points may lead to a progress in understanding the structure of the

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\(^1\) This suggests, in particular, that it may be somewhat misleading to try to find some special holography in this case that would extend to the full interaction level.

\(^2\) The classical string solutions will depend only on the string-frame metric and (in the extended string case) on the NS-NS \(B_{mn}\) field. However, the form of the sigma model quantum corrections (and thus a possibility of interpolation to weakly-coupled gauge theory results) will be sensitive to detailed structure of other background fields (dilaton and R-R field strengths).
string spectrum in $AdS_5 \times S^5$. An example of such interpolation was described in [10] and will be reviewed below.

It is useful to recall how similar correspondence between expansions near different classical solutions is achieved in flat space. The standard option is to expand near massless point-like string solution $x_0 = x_9 = p\tau$. Gauging away fluctuations of $x_0 + x_9$, this is equivalent to light-cone gauge quantization where excited string modes are identified with small fluctuations near point-like vacuum ("supergraviton") state. One finds that the one-loop approximation here is exact and $P^+ = E + P_9 = \frac{2p}{\alpha'}$, $P^- = E - P_9 = \frac{1}{\alpha'}(P^2 + \frac{2}{\alpha'} \sum_{n=-\infty}^{\infty} |n|N_n)$. Alternatively, one may expand near the stationary classical solution describing folded closed string rotating near its center of mass:

$$x_0 = \kappa \tau, \quad x_1 = r(\sigma) \cos w\tau, \quad x_2 = r(\sigma) \sin w\tau, \quad r(\sigma) = \frac{\kappa}{w} \sin w\sigma,$$

where the periodicity in $\sigma$ implies quantization condition $w = n = 1, 2, ...$. The solution with $n = 1$ has the classical energy and spin related by $E = \sqrt{\frac{2}{\alpha'} S}$, i.e. it corresponds to a closed-string state on the leading Regge trajectory in the standard oscillator vacuum, $(a_1^\dagger a_{-1}^\dagger)^2 |0\rangle$. Expanding near this solution one finds a tower of oscillator string states with given angular momentum. The ground state in this sector (representing the unexcited classical solution) may be described as a coherent state of oscillator string states, $|O\rangle_S \sim e^{\sqrt{S}a_1^\dagger e^{\sqrt{S}a_{-1}^\dagger}} |0\rangle$. Similar expansions may be developed near other oscillating string solution. Given that the flat-space string action is essentially quadratic in the conformal gauge so that one knows the general classical string solution one is able to establish correspondence between expansions near different classical starting points. Detailed relations between different expansion points in the case of strings in $AdS_5 \times S^5$ should be of course much more intricate.

Below we shall first review the form of the GS action in $AdS_5 \times S^5$ in the "non-conformal" light-cone gauge (with the light-cone directions parallel to the boundary of $AdS_5$) and discuss point-like string solutions in Poincare and global coordinates (section 2). We shall then develop semiclassical expansion near rotating string solutions, by considering in turn the $S^5$-boost $J \neq 0$, $AdS_5$ rotation $S \neq 0$ and the mixed $J, S \neq 0$ cases (section 3). In section 4 we shall suggest an extension of the semiclassical approach to more "realistic" near-AdS (near-conformal) cases.

### 2. Light-cone gauge GS action and point-like classical solutions

Let us start with recalling the form of the quadratic fermionic term in the type IIB GS action in $AdS_5 \times S^5$

$$L_F = i(\sqrt{-g}g^{ab}\delta^{IJ} - e^{ab}s^{IJ})\bar{\theta}^I \tilde{q}_a D_b \theta^J + O(\theta^4),$$

(2.1)
where $I, J = 1, 2$, $s^{IJ} = \text{diag}(1,-1)$, $\varrho_a \equiv \Gamma_A E^A_M \partial_a X^M$, and $E^A_M$ is the vielbein of the 10-d target space $AdS_5 \times S^5$ metric (see [33] for details). In the conformal gauge $\sqrt{-g g^{ab}} = \eta^{ab} = \text{diag}(-1,1)$. The covariant derivative $D_a = \partial_a X^M D_M$ is the projection of the 10-d derivative $D^I_M = (\partial_M + \frac{i}{4} \omega^A_M \Gamma_{AB}) \delta^I_J - \frac{1}{8} \Gamma_{A1...A5} \Gamma_M \epsilon^{IJ}$. Since $\theta$ are 10-d MW spinors of the same chirality and since $F^5 = R^{-1}(\epsilon_5 + *\epsilon_5)$ for the $AdS_5 \times S^5$ background, $D_a$ can be put into the following form

$$D_a \theta^I = (\delta^I_J D_a - \frac{i}{2} R^{-1} \epsilon^I_J \Gamma_{*} \varrho_a) \theta^J , \quad \Gamma_{*} \equiv i \Gamma_{01234} , \quad \Gamma^2_{*} = 1 , \quad (2.2)$$

where $D_a = \partial_a + \frac{i}{4} \partial_a X^M \omega^A_M \Gamma_{AB}$. Thus the action contains a fermionic “mass term” originating from the R-R coupling [5].

As usual, the GS fermionic kinetic term contains a potentially problematic factor $\partial X$. However, a perturbative expansion near a particular $X \neq \text{const}$ string configuration is well-defined, producing (after fixing an appropriate $\kappa$-symmetry gauge) a non-degenerate fermionic propagator.

An alternative is to use, as in flat space, the light-cone gauge [20]. The fermionic $\kappa$-symmetry gauge $\Gamma^+ \theta^I = 0$ can be supplemented by $x^+ = p^+ \tau$ (this can be justified using, e.g., the phase-space approach [20]). Here $x^+$ is a light-cone direction parallel to the boundary in the Poincare coordinates, i.e. corresponding to a light-cone direction in the gauge theory. A short-cut way to arrive at the resulting light-cone gauge action is to use, instead of the conformal gauge, the “diagonal gauge” [21]

$$\sqrt{-g g^{ab}} = \text{diag}(-z^2, z^{-2}) , \quad z = z(\tau, \sigma) , \quad (2.3)$$

where $z$ is the “radial” coordinate of $AdS_5$ in the Poincare parametrization, i.e. $(m = 0,1,2,3, \ p = 1,...,6)$

$$(ds^2)_{AdS_5 \times S^5} = \frac{R^2}{z^2} (dx_m dx_m + dz_p dz_p) , \quad z^2 = z_p z_p . \quad (2.4)$$

In what follows we shall often set $R = 1$. The bosonic part of the string action is then $(\xi^a = (\tau, \sigma))$

$$I = \frac{1}{2} T \int d^2 \xi \left[ \dot{x}_m \dot{x}_m + \dot{z}_p \dot{z}_p - \frac{1}{z^3} (x'_m x'_m + z'_p z'_p) \right] , \quad T \equiv \frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi \alpha'} . \quad (2.5)$$

One can then impose $x^+ = p^+ \tau$ (we shall use decomposition $x^m = (x^+, x^-, x^s)$, $s = 1, 2$) and derive the corresponding light-cone Hamiltonian that will coincide with the one in
Splitting the fermionic components into 4+4 complex Grassmann variables \( \theta_i, \eta_i \) transforming in fundamental representation of \( SU(4) \), the light-cone gauge Lagrangian can be put into the form \([20,22]\) (rescaling \( \sigma \) to absorb \( p^+ \) and overall effective string tension \( T \) factors)

\[
L = \frac{1}{2} \left[ \dot{x}_s \dot{x}_s + (\dot{z}^p - i \eta_i \rho^{pqj} \eta^j z_q z^{-2})^2 + i (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i - h.c.) - z^{-2} (\eta^2)^2 - T^2 z^{-4} (x'_s x'_s + z'_p z'_p) \right] \\
- T \left[ z^{-3} \eta^i \rho^p_{ij} z_p (\theta'^j - i \sqrt{2} z^{-1} \eta^j x') + h.c. \right]. \tag{2.6}
\]

Here \( \rho \)-matrices are blocks of \( \Gamma \)-matrices and \( x = x_1 + i x_2 \). Like many light-cone gauge actions in curved space this action is not manifestly 2-d Lorentz invariant, but has a well-defined kinetic term. Expanding the 2-d fields in Fourier modes in \( \sigma \) one gets a non-linear quantum-mechanical system for the infinite number of modes. The theory \( (2.6) \) has two obvious limits: (i) the “particle theory limit” \( T \to \infty \) (i.e. \( \lambda \to \infty \)) in which all fields become independent of \( \sigma \) and the spectrum of the corresponding Hamiltonian is the same as the spectrum of the supergravity modes in \( AdS_5 \times S^5 \) background \([23]\); (ii) the “tensionless string limit” \( T \to 0 \) (i.e. \( \lambda \to 0 \)) in which all the \( \sigma \)-derivative terms in \( (2.6) \) are to be omitted but the fields still depend on \( \sigma \) \([20,22]\); in this case, the presence of the non-linear interaction terms involving \( z_p \) and \( \eta \)-fermions in \( (2.6) \) implies that the string does not split into an infinite collection of decoupled oscillators as that happens in flat space \([22]\). Detailed study of this limit and the spectrum of the light-cone theory \( (2.6) \) remains an important open problem.

The non-linearity of \( (2.6) \) in \( z \) (and the singularity of the interaction terms near the boundary of \( AdS_5 \) where \( z \to 0 \)) suggests that to be able to learn more about the quantum properties of this theory (e.g., using standard perturbation theory) one needs to expand near particular string configurations with non-zero background values of \( z \).

The simplest classical string configurations are point-like null geodesics in \( AdS_5 \times S^5 \) space. They are easy to find from the action \( (2.5) \) in the diagonal gauge. In general, for the special class of string backgrounds for which \( z(\tau, \sigma) = f(\tau) h(\sigma) \) the solutions in the standard conformal gauge and in the diagonal gauge \( (2.3) \) are related by the 2-d coordinate transformation:

\[
\tau \to F(\tau), \quad \sigma \to H(\sigma), \quad \dot{F}(\tau) = f^2(\tau), \quad H'(\sigma) = h^2(\sigma), \quad z(\tau, \sigma) = f(\tau) h(\sigma). \tag{2.7}
\]
The $\sigma$-independent (point-like) solutions of equations of motion plus constraints corresponding to (2.5), (2.3) are the same straight null lines as in flat space ($a_m, p_m, u_p, v_p = \text{const}$)

\[ x_m = a_m + p_m \tau, \quad z_p = u_p + v_p \tau, \quad p_m p_m + v_p v_p = 0. \]  

(2.8)

Transforming this solution to the conformal gauge using (2.7) one finds the general form of point-like string solution in $AdS_5 \times S^5$ which was previously given in [7]

\[ x_m = a_m + p_m q(\tau), \quad z_p = u_p + v_p q(\tau), \]  

(2.9)

\[ q(\tau) = \frac{\nu}{v^2} \tan(\nu \tau) - \frac{u \cdot v}{v^2}, \quad \nu^2 \equiv u^2 v^2 - (u \cdot v)^2. \]  

(2.10)

It is easy to indentify the two non-trivial cases:

(I) Null geodesic parallel to the boundary of $AdS_5$: If $p_m p_m \equiv -p_0^2 + p_i^2 = -v_m v_m = 0$, then $v_m = 0$ and the solution is the null geodesic parallel to the boundary located at a distance $z$ from the boundary ($z_p = u_p = \text{const}$). We can choose $p_m$ and $u_p$ so that the solution is given by

\[ x_0 = x_3 = p \tau, \quad z_1 = u = \text{const}, \quad x_1, x_2, z_2, ..., z_6 = 0. \]  

(2.11)

The expansion near this classical solution (with fluctuations of $x^+ = x_0 + x_3$ gauged away) is equivalent to the expansion of the light-cone action (2.6) near the constant $z_p$ point. Here the energy is $E = P_0 = P_3 = \sqrt{\lambda} p \gg 1$. It is easy to see from (2.6) that here (in contrast to the case of the $S^5$-geodesic discussed below) the resulting quadratic fluctuation action is trivial, i.e. is the same as in flat space, giving the 1-loop correction to the energy $E_1 = \frac{1}{2p} \sum |n| N_n$. However, the 2-loop corrections to the energies of strings states will be non-trivial ($E_2 = \frac{1}{\sqrt{\lambda}} F(p, n)$) and would be interesting to compute explicitly.

(II) Null geodesic transverse to the boundary of $AdS_5$ (along the big circle of $S^5$). If $p_m p_m = -v_m v_m$ is non-zero, then the translational invariance in $x_m$ and $SO(1, 3) \times SO(6)$ rotational symmetry allows us to bring the corresponding solution in the diagonal gauge (2.8) to the form

\[ x_0 = p \tau, \quad z_1 = u = \text{const}, \quad z_2 = p \tau, \]  

(2.12)

where we also used the translational invariance in $\tau$ to set $u_2 = 0$. This is a straightline geodesic in the $(z_1, z_2)$ plane parallel to the $z_2$-axis. Note that the expansion near this solution can be again described by the light-cone action (2.6) (the light-cone gauge

\footnote{3 It thus has the same form in the conformal and diagonal gauges.}
choice means only that the "boundary" light-cone coordinate $x^+$ does not contain quantum fluctuations).

This second solution (2.12) corresponding to a straight motion of a particle in the plane $(z_1, z_2)$ is carrying a non-zero angular momentum since $z_1 \neq 0$. It can also be viewed as a null geodesic in $AdS_5 \times S^5$ running along the big circle of $S^5$. To show this let us first transform (2.12) to the conformal gauge, i.e. write down the solution (2.9) with $u_p = (u, 0, 0, 0, 0), \; v_p = (0, p, 0, 0, 0), \; u \cdot v = 0, \; a_0 = 0, \; p_0 = p$:

$$x_0 = p \tan \nu \tau \; , \; \; \; z_1 = u \; , \; \; \; z_2 = p \tan(\nu \tau) \; , \; \; \; \nu = u p \; .$$ \hspace{1cm} (2.13)

For simplicity let us also rescale $x_m$ and $z_p$ by $1/p$ and choose $u = p$. Then (2.13) can be written as

$$x_0 = \tan t \; , \; \; \; \; z_1 = 1 = z \cos \varphi \; , \; \; \; z_2 = \tan t = z \sin \varphi \; .$$ \hspace{1cm} (2.14)

where

$$t = \nu \tau \; , \; \; \; \varphi = \nu \tau \; , \; \; \; \; z = \frac{1}{\cos \nu \tau} \; .$$ \hspace{1cm} (2.15)

Here $\varphi$ is, in fact, the angle of large circle of $S^5$ and $t$ is the global time coordinate of $AdS_5$. Indeed, in global coodinates

$$ds^2_{AdS_5} = - \cosh^2 \rho \; dt^2 + d\rho^2 + \sinh^2 \rho \; d\Omega_3 \; ,$$ \hspace{1cm} (2.16)

$$d\Omega_3 = d\beta_1^2 + \cos^2 \beta_1 (d\beta_2^2 + \cos^2 \beta_2 \; d\beta_3^2) \; ,$$

while the angle $\varphi$ of $S^5$ related to $z_p$ coordinates by $dz_p dz_p = dz^2 + z^2 d\varphi^2 + dz_n dz_n$.

In general, the transformation between the Poincare and the global coordinates of $AdS_5$ can be done as follows (we set the radius of $AdS_5$ to be 1 and use the Minkowski signature):

$$X_0 = \frac{x_0}{z} = \cosh \rho \; \sin t \; , \; \; \; X_5 = \frac{1}{2z} \left( 1 + z^2 - x_0^2 + x_i^2 \right) = \cosh \rho \; \cos t \; ,$$ \hspace{1cm} (2.17)

$$X_i = \frac{x_i}{z} = n_i \sinh \rho \; , \; \; \; X_4 = \frac{1}{2z} \left( -1 + z^2 - x_0^2 + x_i^2 \right) = n_4 \sinh \rho \; , \; \; \; n_i^2 + n_4^2 = 1 \; .$$ \hspace{1cm} (2.18)

$$\tan t = \frac{2x_0}{1 + z^2 - x_0^2 + x_i^2} \; , \; \; \; \; \; z^{-1} = \cosh \rho \; \cos t - n_4 \sinh \rho \; .$$ \hspace{1cm} (2.19)

Here $X_0, X_i, X_4 \; (i = 1, 2, 3)$ are the coordinates of $R^{2,4}$: the $AdS_5$ metric is induced from the flat $R^{2,4}$ one by the embedding (see, e.g., [1]) $X_0^2 + X_5^2 - X_i^2 - X_4^2 = 1$. The unit vector $n_k \; (k = 1, 2, 3, 4), \; n_i^2 + n_4^4 = 1$, parametrizes the 3-sphere: $dn_k \; dn_k = d\Omega_3$. The obvious
point-like solution in global coordinates (2.16), i.e. $t = \nu \tau$, $\rho = 0$, $\varphi = \nu \tau$ (with all other angles being trivial), then becomes equivalent to (2.15).

Expanding the covariant $AdS_5 \times S^5$ string action (in the conformal gauge or in diagonal gauge (2.5)) near the above solution one reproduces (see [10] and below), in the 1-loop approximation, the same spectrum as found in the case of the plane-wave background [11,13].

3. Semiclassical expansion near rotating string solutions

3.1. Classical energy and classical solutions

Let us first supplement the transformation between the Poincare and global coordinates of $AdS_5$ (2.17)–(2.19) by the relation between the corresponding energies (corresponding to $t$ and $x_0$ time coordinates). The energy in global coordinates expressed in terms of the Poincare coordinates $x_m, z_p$ in (2.4) as functions of both $\tau$ and $\sigma$ is

$$E = \sqrt{\lambda} \mathcal{E} = \sqrt{\lambda} \int \frac{d\sigma}{2\pi} \mathcal{E}_d = \sqrt{\lambda} \int \frac{d\sigma}{2\pi} \cosh^2 \rho \ i$$

$$= \frac{1}{2} \sqrt{\lambda} \int \frac{d\sigma}{2\pi} \left[ (1 + z^2 + x^2) \mathcal{P}_0 - 2x_0 \mathcal{D} \right], \quad (3.1)$$

$$\mathcal{P}_0 = \frac{1}{z^2} \dot{x}_0, \quad \mathcal{D} = \frac{1}{2z^2} \frac{\partial}{\partial \tau} (z^2 + x^2), \quad x^2 = -x_0^2 + x_i^2. \quad (3.2)$$

Here $\mathcal{P}_0$ is the energy density corresponding to translations in $x_0$ and $\mathcal{D}$ is the dilatation ($z \to kz$, $x_m \to kx_m$) charge density. One can show that $E$ is related to the superconformal generators in the Poincare patch as follows

$$E = \frac{1}{2} (\mathcal{P}_0 + \mathcal{K}_0) = \frac{1}{2} \sqrt{\lambda} \int \frac{d\sigma}{2\pi} (\mathcal{P}_0 + \mathcal{K}_0), \quad \mathcal{K}_0 = (z^2 + x^2) \mathcal{P}_0 - 2x_0 \mathcal{D}. \quad (3.3)$$

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4 The 1-loop approximation (i.e. taking $\alpha' \to 0$ and rescaling the coordinates) is related to the strict Penrose limit [24].

5 Note that in general $E$ may not be conserved if solutions do not decay fast enough in $\sigma$: the translation invariance in $t$ implies that $\partial_a E^a = 0$, $E^a = \frac{\partial L}{\partial \partial_a t}$, so that one needs $\int d\sigma \frac{\partial L}{\partial \partial_a t} = 0$ for the energy to be conserved.

6 At the boundary of $AdS_5$ (i.e. $z = 0$) the standard superconformal generators are $\mathcal{D} = -ix^m \partial_m$, $\mathcal{P}_m = -i \partial_m$, $\mathcal{K}_m = -2x_m \mathcal{D} + x^2 \mathcal{P}_m = i(2x_m x_n - x^2 \eta_{mn}) \partial_n$, $x^2 = x_m x_m = -x_0^2 + x_i^2$, and so $[\mathcal{D}, \mathcal{P}_0] = i \mathcal{P}_0$, $[\mathcal{D}, \mathcal{K}_0] = -i \mathcal{K}_0$, $[\mathcal{P}_0, \mathcal{K}_0] = -2i \mathcal{D}$, etc. The above generators acting on the $AdS_5$ coordinates are extensions of these to the bulk.
We can also express the energy density in (3.1) as
\[ E_d = \frac{1}{2} \frac{(1 + z^2 + x^2)^2}{z^2} \frac{\partial}{\partial \tau} \left( \frac{x_0}{1 + z^2 + x^2} \right). \] (3.4)

For the solutions discussed below one has the following relation
\[ z^2 - x_0^2 + x_i^2 = 1. \] (3.5)

Thus \( D = 0 \) (see (3.2)) and the energy \( E \) in global coordinates (3.4) coincides with the energy \( P_0 \) in the Poincare coordinates:
\[ E = P_0 = K_0 = \sqrt{\lambda} \int \frac{d\sigma}{2\pi} \frac{1}{z^2} \dot{x}_0, \quad D = 0. \] (3.6)

Next, let us summarize the (conformal-gauge) form of the simplest classical string solutions in \( AdS_5 \times S^5 \) written in global (2.16) and Poincare (2.4) coordinates of \( AdS_5 \) (\( G \) will stand for the global and \( P \) for the Poincare coordinate form). As was already discussed above, for the point-like string rotating in \( R^6 \) (or, equivalently, boosted along a big circle in \( S^5 \))
\[ G: \quad t = \nu \tau, \quad \varphi = \nu \tau, \] (3.7)
\[ P: \quad x_0 = \tan t, \quad z = \frac{1}{\cos t}, \quad \varphi = t = \nu \tau. \] (3.8)
The spinning string in \( AdS_5 \) is described by (the angle \( \phi \) is \( \beta_3 \) in (2.16)) \([15,3]\)
\[ G: \quad t = \kappa \tau, \quad \phi = \omega \tau, \quad \rho = \rho(\sigma), \quad \rho^2 = \kappa^2 \cosh^2 \rho - w^2 \sinh^2 \rho, \] (3.9)
or, equivalently, by
\[ P: \quad x_0 = \tan t, \quad z = \frac{1}{\cos t \cosh \rho}, \quad x_1 = r \cos \phi, \quad x_2 = r \sin \phi, \quad r \equiv \frac{\tanh \rho}{\cos t} \] (3.10)
where \( t = t(\tau), \quad \phi = \phi(\tau), \) and \( \rho = \rho(\sigma) \) are given by (3.3). Here \( \rho \) changes from 0 to its maximal value \( \rho_0 = \text{Arctanh} \frac{\kappa}{w} \) and the parameter \( w \) is a function of \( \kappa \). In Poincare coordinates the string moves towards the horizon (center of AdS), rotating and stretching. More general solution can be obtained by combining the rotation in \( AdS_5 \) with the boost in \( S^5 \) \([10]\) (see section 3.4 below).

One may also consider string spinning in \( S^5 \) \([3]\), for which \( t = \kappa \tau, \quad \varphi = \nu \tau, \quad \psi = \psi(\sigma) \) (\( \psi \) is an angle of \( S^5 \) along which the string is stretched, \( ds^2_{S^5} = d\psi^2 + \sin^2 \psi \, d\varphi^2 + ... \)), as
well as a more general solution interpolating between this and the above two, i.e. string spinning in AdS$_5$ as well as in S$^5$ [7].

Let us mention also some of the oscillating string solutions [16,3,18]. For the string oscillating in AdS$_5$

\[
G: \quad t = t(\tau), \quad \phi = w\sigma, \quad \rho = \rho(\tau), \quad \dot{t} = \frac{\kappa}{\cosh^2 \rho}, \quad \dot{\rho}^2 = \frac{\kappa^2}{\cosh^2 \rho} - w^2 \sinh^2 \rho. \quad (3.11)
\]

The form of this solution in Poincare coordinates is similar to the one in the rotation case (3.10)

\[
P: \quad x_0 = x_0(\tau), \quad x_1 = r \cos \phi, \quad x_2 = r \sin \phi, \quad 
\frac{r}{z} = \sinh \rho, \quad \frac{x_0}{z} = \cosh \rho \sin \phi, \quad z = \frac{1}{\cosh \rho} \cos \phi, \quad x_0 = \tan \phi. \quad (3.12)
\]

For the string oscillating in S$^5$ one has $t = \kappa \tau$, $\phi = w\sigma$, $\psi = \psi(\tau)$. There are also more general solutions incorporating the above as special cases [25].

The form of a classical solution cannot depend on the value of the string tension, i.e. on $\sqrt{\lambda}$, which appears as a factor in front the string action $I = \frac{\sqrt{\lambda}}{4\pi} \int d^2 \xi \, G_{MN}(x) \partial_a x^M \partial^a x^N$. Thus the classical energy can be written as $E = \sqrt{\lambda} \mathcal{E}(N)$, where $N$ stands for all constant parameters (like $\nu, w, \kappa$ in the above expressions (3.7)–(3.12)) that enter the classical solution. These parameters should be fixed in the standard sigma model loop ($\frac{1}{\sqrt{\lambda}}$) expansion. However, some of them may be quantized in the full quantum theory, i.e. $\sqrt{\lambda}N = N=\text{integer}$ (being related to canonical momenta the quantized parameters should contain a factor of string tension). For example, the global charges like the S$^5$ and AdS$_5$ angular momentum components $J = \sqrt{\lambda} \nu$, $S = \sqrt{\lambda} \mathcal{S}(\kappa)$, etc., will take integer values. In addition, for the oscillating solutions there will be an integer “oscillation number” parameter $N_{\text{osc}} = \sqrt{\lambda}N_{\text{osc}} \gg 1$: as usual for stationary “non-topological” soliton-type solutions there will be a semiclassical path integral phase quantization condition (see [26,10]). Expressed in terms of these quantized parameters the classical energy will look like

\[
E = \sqrt{\lambda} \mathcal{E}(\frac{J}{\sqrt{\lambda}}, \frac{S}{\sqrt{\lambda}}, \frac{N_{\text{osc}}}{\sqrt{\lambda}}). \quad (3.13)
\]

According to the AdS/CFT duality [3], the energy in global coordinates should give the expression for the strong-coupling limit of the (canonical+anomalous) dimension of the corresponding SYM operator. The string sigma model corrections to (3.13) discussed below will represent subleading strong-coupling expansion terms in the anomalous dimension.
As was observed in [16], in the limit of large values of $N$ (and thus of the quantum numbers $N \gg 1$) the classical energy of a string solution in any $AdS_p$ space goes as linear function of $N$, i.e. $E \sim N$. Here $N$ can be an angular momentum or an oscillation number or a combination of the two. This is to be compared to $E \sim \sqrt{N}$ in the flat space case. This remarkable linear behaviour (seen explicitly on specific examples of solutions in [15,3,10,17,18]) is a consequence of the curvature of the $AdS_p$ and is perfectly consistent with the AdS/CFT duality: the large $N$ expression for $E$ (i.e. of the full dimension) should start with canonical dimension of the corresponding gauge-theory operator.

Below we shall discuss the classical relation (3.13) and quantum string sigma model corrections to it on specific examples. Our aim will be to try to draw some general conclusions about the expression for the quantum energy $E$ on the string side of the AdS/CFT duality.

3.2. Quantum corrections to point-like string rotating in $S^5$

Let us now consider the quantum corrections to the energy of the point-like $S^5$-rotating solution (3.7). Its classical energy and angular momentum $E = \sqrt{\lambda} \nu$ and $J = \sqrt{\lambda} \nu$ are proportional to the parameter $\nu$ which should be fixed in the semiclassical expansion in powers of the inverse string tension $\frac{1}{\sqrt{\lambda}}$. Then $J \gg 1$ in the large $\sqrt{\lambda}$ limit.

The general strategy is as follows. One is supposed to expand the string action near a given classical solution and then compute corrections in sigma model and string perturbation theory. If one is interested in quantum corrections to the energies of string states in global coordinates in $AdS_5$ one should consider string theory on the cylinder\footnote{For an alternative description of the semiclassical approximation for the 2-point function of vertex operators in Poincare coordinates see [27].} (or cylinder with handles attached if one goes beyond tree level in string coupling) and compute the expectation value of the energy operator $\langle \Psi | \hat{E} | \Psi \rangle$ in the sector of states with quantized angular momentum $[10]$:

$$\langle \Psi | \hat{J} | \Psi \rangle = J = \sqrt{\lambda} \nu = \text{integer}.$$

The parameters of the string perturbation theory are $\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}} \ll 1$, $\nu = \frac{J}{\sqrt{\lambda}} = \text{fixed}$ and $g_s \ll 1$. They can be expressed in terms of the parameters $\lambda'$ and $g_2$ [2,28,29] that enter the SYM perturbation theory for the anomalous dimensions of operators with large R-charge:

$$\lambda' \equiv \frac{\lambda}{J^2} = \frac{1}{\nu^2}, \quad \text{i.e.} \quad \frac{1}{\sqrt{\lambda}} = \frac{1}{J} \frac{1}{\sqrt{N}},$$

$$g_2 \equiv \frac{J^2}{N}, \quad \text{i.e.} \quad g_s = g_{YM}^2 = \frac{\lambda}{N} = \lambda' \frac{J^2}{N} = \lambda' g_2.$$

$$\lambda' \equiv \frac{\lambda}{J^2} = \frac{1}{\nu^2}, \quad \text{i.e.} \quad \frac{1}{\sqrt{\lambda}} = \frac{1}{J} \frac{1}{\sqrt{N}}, \quad (3.14)$$

$$g_2 \equiv \frac{J^2}{N}, \quad \text{i.e.} \quad g_s = g_{YM}^2 = \frac{\lambda}{N} = \lambda' \frac{J^2}{N} = \lambda' g_2. \quad (3.15)$$
Then the semiclassical string limit $\nu=\text{fixed}, \lambda \to \infty, g_s \to 0$ (i.e. the sigma model 1-loop approximation in string theory on the cylinder) is equivalent to $\lambda'=\text{fixed}, J \to \infty, g_2 \to 0$. Choosing $\nu$ to be large corresponds to $\lambda'<1$. In general, the expression for the energy of string states computed in both sigma model and string loop perturbation theory will be

$$E = E\left(\frac{1}{\sqrt{\lambda}}, g_s\right) = E\left(\frac{1}{\sqrt{\lambda}}, \lambda' g_2\right). \quad (3.16)$$

Again, sending $g_2$ to zero and then $J$ to infinity for fixed $\lambda'$ on the SYM side is thus equivalent to the sigma model 1-loop approximation in tree-level string theory. The energy of a string state with some quantum number $n = \{n_i\}$ (i.e. the expectation value of the quantum space-time energy operator $E = \langle \Psi_n | \hat{E} | \Psi_n \rangle$) may be written as

$$E = \sum_{l=0}^{\infty} E_l + O(g_s^2), \quad E_l = \frac{1}{(\sqrt{\lambda})^{l-1}} E_l(\nu; n), \quad (3.17)$$

where $l = 0, 1, 2, ...$ is the sigma model loop order. Residual supersymmetry preserved by the classical solution implies that $E_l$ with $l > 0$ should vanish for the ground state (and all other supergravity modes in the spectrum).

Expanding the bosonic part of the $AdS_5 \times S^5$ string action near the classical solution (3.7) or (3.8) one finds the following quadratic term in the Lagrangian for the fluctuation fields [3,10]

$$L_{2B} = -(\partial \tilde{t})^2 + (\partial \tilde{\varphi})^2 + (\partial \tilde{\xi}_k)^2 + (\partial \tilde{\psi}_k)^2 + \nu^2 (\tilde{\xi}_k^2 + \tilde{\psi}_k^2). \quad (3.18)$$

Here $\xi_k$ and $\psi_k$ ($k = 1, 2, 3, 4$) are 4+4 fluctuations in other directions of $AdS_5$ and $S^5$ rescaled by $(\sqrt{\lambda})^{-1/2}$. The mass terms originate from the curvature of the two spaces. The next term in the expansion of the sigma model Lagrangian is a quartic interaction term which has the following symbolic structure [10]

$$L_{4B} \sim \frac{1}{\sqrt{\lambda}} \left[ -\xi^2 (\partial \tilde{\xi})^2 + \tilde{\psi}^2 (\partial \tilde{\varphi})^2 + \tilde{\xi}^2 (\partial \tilde{\xi})^2 + \tilde{\psi}^2 (\partial \tilde{\psi})^2 + \nu^2 (\tilde{\xi}^4 + \tilde{\psi}^4) \right], \quad (3.19)$$

so that the parameter $\nu$ of the classical solution determines the masses as well as the potential terms for the 8 bosonic fluctuation fields. More generally, the expansion of the bosonic part of the action has the structure (here $\xi$ stands for all 8 dynamical fluctuation fields and in contrast to (3.18), (3.19) we do not rescale $\xi$ by $(\sqrt{\lambda})^{-1/2}$)

$$L_B \sim \sqrt{\lambda} \left[ (\partial \xi)^2 + \nu^2 \xi^2 + \sum_{m=1}^{\infty} \left[ c_m \xi^{2m} (\partial \xi)^2 + b_m \nu^2 \xi^{2m+2} \right] \right]. \quad (3.20)$$


The form of the quadratic fermionic term is easily found from the general expression (2.1), (2.2). Choosing the natural \( \kappa \)-symmetry gauge (which is essentially “imposed” on us by the choice of the background) \( \Gamma^+ \theta^I = 0 \), \( \Gamma^\pm = \mp \Gamma_0 + \Gamma_\phi \), we get [10]

\[
L_{2F} = -i\nu(\bar{\theta}^I \Gamma_-^I \theta^I + \theta^I \Gamma_+^I \bar{\theta}^I - 2\nu \theta^I \Gamma_-^I \Pi \theta^I),
\]

where \( \Pi \equiv i\Gamma^\ast \Gamma_0 = \Gamma_{1234} \), \( \Pi^2 = I \), and the mass term originated from the R-R coupling.

Redefining the spinors, we end up with an action that can be interpreted as describing 4+4 massive 2-d fermions (with Dirac-type mass terms, i.e. \( S_R \partial_+ S_R + S_L \partial_+ S_L \pm \nu S_L S_R \)).

Higher-order interacting fermionic terms that accompany (3.19) have the following structure

\[
L_F \sim \frac{1}{(\sqrt{\lambda})^{1/2}} [\bar{\theta} \theta((\partial \bar{\xi} + \partial \bar{\psi}^\dagger) + \nu \theta((\partial \bar{\xi} + \partial \bar{\psi}^\dagger) + \nu \theta)] + \cdots
\]

(3.22)

The quadratic part of the expanded \( AdS_5 \times S^5 \) action (3.18), (3.21) is the same as the full GS action for the plane-wave R-R background of [12] that was found in [11]. This is not surprising being a consequence of the fact that the plane-wave background is the Penrose limit of the \( AdS_5 \times S^5 \) near the same null geodesic [12,2]. The parameter \( \nu \) can be identified with the product of \( \alpha' p^+ \) in the light-cone gauge condition \( x^+ \equiv t + \phi = \alpha' p^+ \tau \) in [11,13] and a scale of the plane-wave background.

To find the quantum correction to the energy one may use the constraints to eliminate \( \tilde{x}^- = \tilde{t} - \tilde{\phi} \) or, equivalently, impose the “light-cone” gauge \( \tilde{x}^+ = 0 \). Since for classical trajectory \( t = \nu \tau \), the 10-d energy and the 2-d energy are directly related (as in the familiar case of the static gauge), with the proportionality factor \( \frac{1}{\nu} \). The quantum correction to the energy is then determined by the expectation value of the 2-d Hamiltonian [10]

\[
E - J = \frac{1}{\nu} \int_0^{2\pi} \frac{d\sigma}{2\pi} \langle \Psi | H_{2d}(\bar{\xi}, \bar{\psi}, \theta) | \Psi \rangle.
\]

(3.23)

Here \( H_{2d} \) corresponds to the above action (3.20), (3.21), (3.22) describing 4+4 massive bosonic and 4+4 massive fermionic “transverse” fluctuation modes and their interactions as prescribed by the full \( AdS_5 \times S^5 \) GS action expanded near the classical trajectory.

Ignoring the interactions, i.e. omitting the 2-loop \( O(\frac{1}{\sqrt{\lambda}}) \) and higher corrections to \( E \), one finds the same expression as in [11,2,13]

\[
E = \sqrt{\lambda} \nu + \frac{1}{\nu} \sum_{n=-\infty}^{\infty} \sqrt{n^2 + \nu^2} N_n = \sqrt{\lambda} \nu + \sum_{n=-\infty}^{\infty} \sqrt{1 + \frac{1}{\nu^2} n^2} N_n.
\]

(3.24)

---

8 Here \( \rho_0 = \Gamma_M \partial_0 X^M = \nu(\Gamma_0 + \Gamma_\phi) \), \( D^{IJ}_a = \delta^{IJ} \partial_a - \frac{i}{2} \epsilon^{IJ} \Gamma_a \rho_a \), \( \Gamma_\ast = i\Gamma_{1234} \).
Expressed in terms of $J$ and $\lambda$, the classical plus 1-loop correction to the energy is thus

$$E = J + \sum_{n=-\infty}^{\infty} \sqrt{1 + \frac{\lambda}{J^2} n^2} N_n \equiv J + \sum_{n=-\infty}^{\infty} \sqrt{1 + \lambda' n^2} N_n . \quad (3.25)$$

The analyticity of the latter square root expression in $\lambda'$ suggests the possibility of direct comparison with the SYM perturbation theory expression for the corresponding anomalous dimensions obtained in the limit $J \to \infty$, $\lambda' =$fixed \[2,29\]; indeed, one finds the exact agreement with the square root correction in (3.25) \[30\].

Why this happens is clear from the structure of the string perturbative expansion: higher-loop sigma model corrections are suppressed by powers of $\frac{1}{\sqrt{\lambda}}$ and thus by powers of $\frac{1}{J}$ (for fixed $\lambda'$). Still, there is a small miracle in that the string expression (3.25) has regular expansion in power series in $\lambda'$—otherwise, one would need to do a non-perturbative computation on the SYM side to be able to compare to the above string result. One may wonder if the same analyticity property holds also at higher orders in sigma-model loop expansion, i.e. at subleading orders in $\frac{1}{J}$ expansion, thus allowing one to hope extend the comparison with simply perturbative SYM theory beyond $J = \infty$ limit.

Indeed, it is possible to argue that all higher-order terms in (3.17) should have the structure

$$E_l = \frac{1}{\nu} (E_{2d})_l = \frac{1}{(\sqrt{\lambda} \nu)^{l-1}} F_l(\frac{\lambda}{\nu^2}; n), \quad F_l = c_l(n) + \frac{1}{\nu^2} d_l(n) + O\left(\frac{1}{\nu^4}\right) . \quad (3.26)$$

The expanded GS action defines an UV finite massive 2-d QFT on a cylinder, with interaction vertices containing only two derivatives or two powers of $\nu$ (cf. (3.20),(3.22)). Thus the energy of a particular oscillator string state with quantum numbers $n$ should have a regular inverse-mass ($\frac{1}{\nu}$) expansion, i.e. each $l > 1$ correction $E_l$ in (3.17) should vanish in the limit of $\nu \to \infty$ (for fixed $\lambda$). This implies (3.26)\[3 which may be written also as

$$E_l = \frac{1}{J^{l-1}} F_l(\frac{\lambda}{J^2}; n) = \frac{1}{J^{l-1}} F_l(\lambda'; n) , \quad (3.27)$$

with $F_l$ having a regular power series expansion for small $\lambda'$. As a result, in the large $J$ sector, one should be able to re-interpret the string $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ expansion as an expansion in positive powers of $\lambda' = \frac{\lambda}{J^2} = \frac{1}{\nu^2}$.

This implies that the comparison between the string theory and the SYM theory can be extended to subleading orders in $\frac{1}{J}$ without need to go beyond perturbation theory in

\[9 It may be possible also to show that in general $c_l = 0.$
\( \lambda' \) on the SYM side. The leading 2-loop correction to the energy of excited string states (like \( a_1^\dagger a_{-1}^\dagger |0> \)) has the following structure \( [10,31,32] \)

\[
E_2 = \frac{1}{\sqrt{\lambda} \nu} F_2(\frac{1}{\nu^2}; n) = \frac{1}{f} F_2(\frac{\lambda}{f^2}; n) = \frac{\lambda}{f^3} d_2(n) + \ldots .
\]

(3.28)

As follows from (3.27), the \( \frac{1}{f} \) term in the energy (anomalous dimension) may appear only from the 2-loop string correction (i.e. there is no non-trivial function of \( \sqrt{\lambda} \) multiplying \( \frac{1}{f} \) term). Therefore that the coefficient \( d_2 \) in (3.28) should be reproduced by the leading perturbative (single YM interaction vertex insertion) computation on the SYM side (see [32]). It would be interesting to find the complete string-theory expression for the 2-loop function \( F_2 \) in (3.28) \( [14] \) as well as to extend the gauge-theory computation of [30] to subleading order in \( \frac{1}{f} \). A complication in checking the AdS/CFT duality at subleading orders in \( \frac{1}{f} \) is that one needs also to modify [32] the definition [2] of the composite SYM operators corresponding to the string modes.

3.3. Quantum corrections to string rotating in \( AdS_5 \)

Let us now consider the spinning string solution (3.9) \( [15,3] \). Here the classical energy (which is the same in global or Poincare coordinates, see (3.6)) \( E = E_0 = \sqrt{\lambda} \mathcal{E}(\kappa, w) \) and the spin \( S = \sqrt{\lambda} S(\kappa, w) \) depend on the classical parameters \( \kappa, w \). The latter are related by the \( \sigma \)-periodicity condition, \( w = w(\kappa) \), so that one can express the energy in terms of spin, \( \mathcal{E} = \mathcal{E}(S) \) or \( E = E(S, \sqrt{\lambda}) \). One may view \( S \) as the basic parameter of the classical solution which is fixed in the semiclassical expansion in \( \frac{1}{\sqrt{\lambda}} \).

In the “short string” (\( \mathcal{S} \ll 1 \)) limit one finds that \( \mathcal{E} \approx \sqrt{2S} \), i.e. \( E \approx \sqrt{2\sqrt{\lambda} S} \); this is the same linear Regge trajectory relation as in flat space (not surprising, since the short string is located at the center of \( AdS_5 \) which is approximately flat). For “long string” (\( \mathcal{S} \gg 1 \)) one finds that \( \mathcal{E} \approx \mathcal{S} \), i.e. \( E \approx S \) \( [15] \), in agreement with general behavior of \( E \) in AdS space discussed in section 3.1. Since \( S \) should take integer values at the quantum level, this suggests [3], by analogy with the large R-charge case [4], that this string state should correspond to a CFT operator with large canonical dimension equal to

\[ ^{10} \text{For that one needs to expand the Hamiltonian (see (3.23), (3.22)) } \mathcal{H}_{2d} = \mathcal{H}_0 + \frac{1}{(\sqrt{\lambda})^{1/2}} \mathcal{H}_1 + \frac{1}{(\sqrt{\lambda})^{1/2}} \mathcal{H}_2 + \ldots \text{ and the quantum state } |\Psi\rangle = |\Psi_0\rangle + \frac{1}{(\sqrt{\lambda})^{1/2}} |\Psi_1\rangle + \ldots \text{, and to take into account that both } \langle \Psi_0 | \mathcal{H}_0 | \Psi_0 \rangle \text{ and } \langle \Psi_0 | \mathcal{H}_1 | \Psi_1 \rangle \text{ may contribute to the order } \frac{1}{\sqrt{\lambda}} \text{ correction.} \]
A remarkable observation made in [3] is that the relation \( E \approx S + \ldots \) contains also a
subleading logarithmic term,

\[
\mathcal{E}_{S \gg 1} \approx S + \frac{1}{\pi} \ln S + \ldots, \quad \text{i.e.} \quad E_0 \approx S + a_0 \sqrt{\lambda \ln \frac{S}{\sqrt{\lambda}}} + \ldots, \quad a_0 = \frac{1}{\pi}.
\] (3.29)

This is an example of the general relation (3.13). As another example, let us mention that
for the string oscillating in \( AdS_5 \) (3.11) one finds [16,18] \((N \gg 1)\):
\[ E \approx N + c_0 \sqrt{w N} + \ldots, \]

i.e.
\[ E \approx N + c_1 \lambda w^2 + \ldots. \]

Eq. (3.29) may be compared [3] to the known similar behaviour [33] of the anomalous
dimension of the dual gauge theory operator, e.g. \( \text{Tr}(\Phi^* D_{i_1} \ldots D_{i_S} \Phi) \), in an asymptotically
free theory (i.e. near a UV fixed point). Since the rotating string state is not BPS, the
relation (3.29) is expected to be renormalized by quantum corrections on both string-theory
and gauge-theory sides. On the gauge-theory side, it is believed (see [33] and discussion in
[3]) that the anomalous dimension should not contain higher powers of \( \ln S \) (even though
they appear in individual graphs). Fortunately, one is able to argue (see [10] and below)
that the same is true also on the string-theory side: there is a simple scaling argument
that implies that all sigma model \( \alpha' \sim \frac{1}{\sqrt{\lambda}} \) corrections to the energy of the rotating string
solution (3.29) do not produce higher than first powers of \( \ln S \) in \( E \), i.e. they modify (3.29)
o only by an “interpolating function” of the string tension (i.e. of the ‘t Hooft coupling)

\[
E_{\sqrt{\lambda} \gg 1} \approx S + f(\lambda) \ln S + \ldots, \quad f(\lambda \gg 1) = a_0 \sqrt{\lambda} + a_1 + \frac{a_2}{\sqrt{\lambda}} + \ldots.
\] (3.30)

This is the same kind of modification due to quantum corrections that happens on the
gauge-theory side, where \( f(\lambda \ll 1) = b_1 \lambda + b_2 \lambda^2 + \ldots \).

The starting point of the semiclassical quantization is again the GS action in \( AdS_5 \times
S^5 \) on a 2-d cylinder expanded near the solution (3.9). This is a conformal 2-d theory [1], i.e.
should be no 2-d UV divergences in expansion near any classical string solution. One needs
to impose [10] the angular momentum quantization condition \( \langle \Psi | \hat{S} | \Psi \rangle = S = \sqrt{\lambda} S = \text{integer} \)
and compute \( \langle \Psi | \hat{E} | \Psi \rangle = E(S) \). The semiclassical string spectrum contains the ground
state \( |\Psi_0\rangle \) representing the rotating string solution, plus a tower of excited string modes
(corresponding to small oscillations on top of the macroscopic rotation). In contrast to
the point-like \( S^5 \)-rotating solution discussed above, here the space-time (and effective 2-d)
supersymmetry is broken, and thus there should be a non-zero quantum correction to the
ground-state energy, \( \langle \Psi_0 | \hat{E} | \Psi_0 \rangle - E_0 \neq 0 \).
The bosonic part of the quadratic fluctuation Lagrangian can be put into the form

$$L_{2B} = - (\partial \tilde{t})^2 - \mu_i^2 \tilde{t}^2 + (\partial \tilde{\phi})^2 + \mu_\phi^2 \tilde{\phi}^2 + 4\tilde{\rho}(\kappa \sinh \rho \partial_0 \tilde{t} - w \cosh \rho \partial_0 \tilde{\phi})$$

$$+ (\partial \tilde{\rho})^2 + \mu_\rho^2 \tilde{\rho}^2 + (\partial \tilde{\beta}_i)^2 + \mu_\beta^2 \tilde{\beta}_i^2 + (\partial \tilde{\psi}_p)^2 , \quad (3.31)$$

where $\beta_i (i = 1, 2)$ and $\phi \equiv \beta_3$ are the $S^3$ angles of $AdS_5$ (see (2.11)), and $\psi_p$ are fluctuations of 5 angles of $S^5$. Two out of 10 fluctuations may be eliminated using the conformal-gauge constraints. This action describes a mixed system of 2-d fields with non-constant ($\sigma$-dependent) masses:

$$\mu_i^2 = m_i^2(\sigma) - \kappa^2 , \quad \mu_\phi^2 = m_\phi^2(\sigma) - w^2 , \quad \mu_\rho^2 = m_\rho^2(\sigma) - \kappa^2 - w^2 , \quad \mu_\beta^2 = m_\beta^2(\sigma) , \quad (3.32)$$

$$m^2 \equiv 2\rho'^2 - 2\kappa^2 \cosh^2 \rho - 2w^2 \sinh^2 \rho .$$

The quadratic fermionic action can be found from (2.1),(2.2). After applying (as in [1]) a particular $\sigma$-dependent fermionic rotation $\theta^I = U(\sigma)\theta^I$ and imposing the $\kappa$-symmetry gauge $\theta^1 = \theta^2$ one gets an action describing 4+4 set of 2-d fermions with a $\sigma$-dependent mass as in (3.32) [11]:

$$L_{2F} = i\bar{\theta} \tau^a \partial_a \theta + \bar{\theta} M \theta , \quad M = i\Gamma_{234} m_F(\sigma) , \quad m_F = \rho'(\sigma) . \quad (3.33)$$

“Diagonalizing” the bosonic fluctuations, it is possible to show the validity of the sum rule\[11\] $\sum m_B^2 - \sum m_F^2 = 0$, which explicitly checks the UV finiteness of the 2-d theory. The quantum correction to the ground-state energy is given (as in (3.23)) by the expectation value of the 2-d Hamiltonian corresponding to (3.31),(3.33)

$$E = E_0 + E_1 + ... = \sum + \frac{1}{\kappa} E_{2d} , \quad E_{2d} = \frac{1}{\kappa} \langle 0 | \hat{H}_{2d} | 0 \rangle . \quad (3.34)$$

To find the quantum correction in the “long-string” limit one notes that for $S \gg 1$ the masses of the 2-d fluctuation fields are approximately constant, $\rho' \approx \kappa$, and $\kappa (S \gg 1) \approx$ 

\[11\] Its existence reflects the fact that the effective 2-d supersymmetry is broken spontaneously by the classical solution.
\( \frac{1}{\pi} \ln S \gg 1 \) Then the 1-loop correction \( E_1 \) to the ground-state energy becomes

\[
E_1 \approx \frac{1}{2\kappa} \sum_{n=1}^{\infty} \left[ \sqrt{n^2 + 4\kappa^2} + 2\sqrt{n^2 + 2\kappa^2} + 5\sqrt{n^2} - 8\sqrt{n^2 + \kappa^2} \right].
\]  

(3.35)

The large \( \kappa \) (large mass) asymptotics is then \( E_1 = \frac{1}{2\pi} [-\frac{3}{2} \ln 2 \kappa^2 + O(\kappa)] \), i.e.

\[
E_1(S \gg 1) \approx a_1 \ln S , \quad a_1 = -\frac{3\ln 2}{4\pi} .
\]  

(3.36)

This is consistent with (3.30), i.e. there are no 1-loop quantum corrections that grow faster than \( \ln S \) for large \( S \).

Let us now argue \( [10] \) that the same should be true also at higher orders in inverse string tension expansion, i.e. that for large \( S \) \( (\kappa \approx \frac{1}{\pi} \ln S, \; S \gg 1) \)

\[
E_l = \frac{1}{\kappa}(E_{2d})_l \approx \frac{1}{\kappa(\sqrt{\lambda})^{l-1}}[b_1 k^2 + O(\kappa)] \approx \frac{1}{(\sqrt{\lambda})^{l-1}} a_l \ln S + O(1) .
\]  

(3.37)

The 2-d quantum field theory in question contains a collection of massive fields with approximately constant (for large \( S \)) masses proportional to \( \kappa \) and with interaction terms containing two derivatives or two powers of \( \kappa \). It may be modelled by a Lagrangian (cf. (3.20))

\[
L \sim \sqrt{\lambda} \left[ (1 + n_1 \xi^2 + n_2 \xi^4 + ...)(\partial \xi)^2 + m^2 (\xi^2 + k_1 \xi^4 + ...) \right] , \quad m \sim \kappa ,
\]  

(3.38)

supplemented by massive fermionic terms which should lead to cancellation of all 2-d UV divergences. The theory is defined on the cylinder \( 0 < \tau < T, \; 0 \leq \sigma < 2\pi L \), where \( T \to \infty \) and \( L \) is fixed (in the above discussion we put \( L = 1 \)). Since this theory must be UV finite,

\[\text{12}\] Though the masses do change near the end-points, one is able to argue that since the change occurs only during a very short interval of \( \sigma \), this does not, for large \( S \), significantly influence the spectrum of the Laplace operators.

\[\text{13}\] For constant masses the expression for the quantum correction to the energy becomes essentially the same as in section 6.5 of \( [9] \). Here we consider the theory on a flat cylinder \((\tau, \sigma)\), so that the individual field contributions to the 2-d vacuum energy are the same as in the theory with the kinetic term \(-\partial^2 + m^2\). This is an analog of the 2-d vacuum energy \((D - 2) \sum_{n=1}^{\infty} n = (D - 2) \zeta(-1) = -\frac{D-2}{12}\) in the closed bosonic string case. It vanishes in the standard flat-space GS string case when all masses are set to zero.
dimensional considerations imply that the $n$-loop contribution to the corresponding 2-d effective action (and thus to the vacuum energy) should scale as

$$\Gamma_l = T(E_{2d})_l = \frac{1}{(\sqrt{\lambda})^{l-1}} V_2 m^2 Q_l(mL), \quad V_2 = TL,$$  \hspace{1cm} (3.39)

where $Q_l$ is a finite function of dimensionless ratio of the two IR scales $1/m$ and $L$. Since in the infinite volume limit $L \to \infty$ the function $Q_l$ should approach a finite constant $c_l$, the same should be true also for fixed $L$ but for large $m$, i.e. $Q_l(mL \gg 1) \to c_l$. As a result, for large $m \sim \kappa$ we get (setting $L = 1$)

$$(E_{2d})_l \approx \frac{1}{(\sqrt{\lambda})^{l-1}} c_l m^2 \approx \frac{1}{(\sqrt{\lambda})^{l-1}} b_l \kappa^2,$$ \hspace{1cm} (3.40)

thus confirming (3.37). This implies that string corrections to the energy of the rotating string solution will produce a non-trivial function $f(\lambda)$ in (3.30) but will not lead to faster-growing $\ln^k S$ terms. Combined with the expected absence of the higher-order $\ln^k S$ terms in the anomalous dimension of the minimal twist operators on the gauge theory side [33,3] this represents a new non-trivial check of the gauge theory – string theory correspondence.

3.4. String rotating in $AdS_5$ and boosted in $S^5$

Let us now consider the classical solution that generalizes the above two, i.e. describes a folded closed string rotating in $AdS_5$ with its center of mass moving along the big circle of $S^5$ [10] (cf. (3.7), (3.9))

$$G: \quad t = \kappa \tau, \quad \phi = w \tau, \quad \varphi = \nu \tau, \quad \rho = \rho(\sigma), \quad \rho'' = \kappa^2 \cosh^2 \rho - w^2 \sinh^2 \rho - \nu^2.$$ \hspace{1cm} (3.41)

The corresponding string modes will carry two global charges, $J = \sqrt{\lambda} \nu$ and $S = \sqrt{\lambda} S$ with the energy being function of them $E = E(J, S)$. They will be counterparts of gauge-theory operators which have large spin and large R-charge at the same time. The expression for the classical energy $E = E(J, S) = \sqrt{\lambda} \mathcal{E}(\nu, S)$ will contain both $E = J$ and $E = S + a_0 \sqrt{\lambda} \ln S + ...$ cases as special limits.

---

14 In our case of several fields with different masses the functions $Q_l$ will also contain finite ratios of the masses $\sim \kappa$, but these will stay constant in the limit of large $\kappa$.

15 The classical parameters are again related by the $\sigma$-periodicity condition [10]: $w = w(S, \nu)$, $\kappa = \kappa(S, \nu)$, etc.
One interesting aspect of this solution is that it illustrates the interpolation between expansions near different semiclassical points. For example, in the limit of large $\nu$ and small $S$ the dependence of the classical energy $E = E(J,S)$ on $S$ will be the same as in the quantum oscillator part of (3.24) corresponding to point-like $J \neq 0$ solution: as in flat space, the classical spin can be built out of quantum-oscillator spins. The agreement between the classical and quantum expressions here is non-trivial since it depends on the curvature of $AdS_5$.

In the “short-string” limit one has $(2S)^2 \ll 1 + \nu^2$, and finds different expressions for $E$ depending on values of $\nu$. If $\nu \ll 1$ then $S \ll 1$ and we get the same relation as in flat space:

$$E \approx \sqrt{J^2 + 2\sqrt{\lambda} S}.$$  (3.42)

Indeed, $J$ has the meaning of a linear momentum along the $\varphi$-direction of $S^5$ and $S$ is the spin of a state on the leading Regge trajectory. If $\nu \gg 1$ (so that $\nu \gg 2S$) then

$$E \approx J + S + \frac{\lambda}{2J^2} S + ... .$$  (3.43)

This is to be compared with the quantum-corrected energy of the point-like solution with large $J$ (3.24),(3.25). For the oscillator states on the leading Regge trajectory $(a_1^* a_{-1}^*)^{\frac{S}{2}} |0\rangle$ one has $n = \pm 1,$ $N_1 = N_{-1} = \frac{S}{2}$ and thus (3.25) takes the form $E \approx J + (1 + \frac{\lambda}{2J^2} + ...)S + ...$ which is in perfect agreement with (3.43). This provides a non-trivial check of consistency of the semiclassical expansion. For $J \gg S$ the corresponding gauge-theory operators [2] should be [17]:

$$\sum_{i_1,...,i_S=1}^{J} \frac{1}{\sqrt{J^N J^2}} \ Tr(...ZD_{i_1}Z...ZD_{i_S}Z...) e^{\frac{2\pi i}{J^2}(l_1+...+l_S)},$$ i.e. they should contain $J$ factors of $Z$-scalar and $S$ covariant derivatives (at positions $l_k$) so that their canonical dimension is $\Delta_0 = J + S$. As was noted in [17], the $O(\frac{1}{J^2})$ correction to the anomalous dimension of such operators should indeed appear from a perturbative computation on the SYM side, in agreement with string-theory result (3.43).

In the “long-string” limit the spin parameters is always large, $S \gg 1$. If $\nu \ll \ln S$ we get the expression similar to the one in the $J = 0$ case (3.24)

$$E \approx S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \frac{\pi J^2}{2\sqrt{\lambda} \ln \frac{S}{\sqrt{\lambda}}} + ... ,$$  (3.44)

---

16 Notice again that the presence of the $S$-term in both (3.43) and the expression following from (3.25) is due to the curvature of $AdS_5$. 

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while if $\nu \gg \ln S$

\[ E \approx S + J + \frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J} + \ldots , \tag{3.45} \]

i.e. for large enough $J$ (or R-charge) we no longer find the $\ln S$ term \[10\].

As in the previous special $S = 0$ and $J = 0$ cases, one can again develop the semiclassical quantization of this solution \[10\], finding the quantum correction to the ground state energy (cf. \((3.23),(3.34)\)) as well the energies of the tower of string oscillator states that have the same $S$ and $J$ quantum numbers but different oscillator occupation numbers.

Other similar “non-topological soliton”-type solutions for strings in $AdS_5 \times S^5$ were discussed in \[16,3,17,18\]. Different classical solutions “probe” different types of string motions, i.e. different sectors of the complete string spectrum, and their further study should be useful.

4. Near-conformal (near-AdS) cases

It is of obvious interest to try to generalize the semiclassical approach to string – gauge theory duality to non-conformal (and less supersymmetric) cases. For example, the $S + f(\lambda) \ln S$ form of the dimension of minimal twist operators should be universal, i.e. it should be found \[3\] in all (e.g., non-supersymmetric, asymptotically free) gauge theories near a (UV) fixed point. It is then natural to try to reproduce this behavior on the string theory side for non-conformal examples of duality like the one in \[35\]. The notion of (anomalous) dimension is defined only near a conformal fixed point, so the semiclassical approach as described in \[3,10\] and above should directly apply only in the “near-AdS” region. The UV region of the corresponding background is described by the type IIB supergravity solution of \[35\] representing fractional D3-branes on a conifold ($N$ D3’s plus $M$ D5’s wrapped on a 2-cycle): its UV (large distance) asymptotics is approximately $AdS_5 \times T^{1,1}$.

In trying to repeat the analysis done above for the $AdS_5 \times S^5$ case in the near-conformal case one faces two problems. The first is to find the form of the corresponding classical string solutions. The non-conformal backgrounds are naturally written in the Poincare-type coordinates (more precisely, their $AdS$ limit gives the $AdS$ metric in Poincare coordinates), while the rotating string solutions in $AdS_5 \times S^5$ are simplest in the global coordinates (cf. \((3.9),(3.10)\)). Here it will be useful to be guided by the form of the $AdS_5 \times S^5$ solutions in the Poincare coordinates \((3.8),(3.10)\).
The second (related) problem is how to define the string-theory analog of conformal dimension, given that the background is only asymptotically \( AdS \). In particular, the global-coordinate energy \( E \) will no longer be conserved. On the gauge theory side, this may be viewed as a reflection of running of gauge couplings (which enter the expression for anomalous dimension) once one moves away from the conformal point.\(^{17}\) Below we shall use the same definition of \( E \) as in the \( AdS \) case expressed in terms of the Poincare-coordinate fields \((3.1)\). The definition of dimension or \( E \) away from the conformal point is of course ambiguous, but viewing the geometry of \((35)\) as a perturbation of \( AdS_5 \times T^{1,1} \), this ambiguity should not matter to leading order in deviation from the conformal point.

Our aim will be to find the correction to the string energy \( E \) (and thus to the corresponding dimension) due to non-conformality of the background. We shall consider both the case of the point-like string moving along a circle in \( T^{1,1} \) and a folded string rotating parallel to the boundary.

### 4.1. Perturbed solution for point-like string rotating in \( T^{1,1} \)

Our starting point will be the following generalization of the \( AdS_5 \times S^5 \) metric

\[
ds^2 = h^{-1/2}(y)dx_m dx_m + h^{1/2}(y)(dy^2 + y^2 ds_5^2),
\]

\( h = \frac{Q(y)}{y^4} \), \( Q = 1 + \epsilon q(y) \). \( (4.1) \)

Here \( \epsilon = 0 \) corresponds to the conformal case, e.g., \( AdS_5 \times T^{1,1} \) if \( ds_5^2 = ds_{T^{1,1}}^2 \). An example of \((4.1)\) is the solution of \((35)\) where (after appropriate rescaling)\(^{18}\)

\[
q = \ln y, \quad \epsilon = \frac{3g_s M^2}{2\pi N}.
\]

Assuming \( \epsilon \ll 1 \) and the range of \( y \) such that \( \epsilon \ln y \) is small, this metric is close to \( AdS_5 \times T^{1,1} \) and can be written as (expanding in powers of \( \epsilon \))\(^{19}\)

\[
ds^2 = \frac{1}{z^2}[1 - \frac{1}{2}\epsilon q(z)]dx_m dx_m + \[1 + \frac{1}{2}\epsilon q(z)]\frac{dz^2}{z^2} + [1 + \frac{1}{2}\epsilon q(z)]ds_{T^{1,1}}^2 + O(\epsilon^2)
\]

\(^{17}\) It may be possible to relate \( \frac{\partial E}{\partial \tau} \) to an analog of RG equation on the gauge theory side.

\(^{18}\) This metric is supported by certain p-form fluxes, which will not be important at the classical level for the solutions we discuss below.

\(^{19}\) In terms of global coordinates of \( AdS_5 \) this metric can be written as

\[
ds^2 = Q^{-1/2}(y)[-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3] + Q^{1/2}(y)ds_{T^{1,1}}^2 + \frac{dy^2}{y^2}[Q^{1/2}(y) - Q^{-1/2}(y)],
\]

where \( y = y(\rho, t, \beta_i) \). It is clear that for \( Q \neq 1 \) the correction to the metric starts depending on \( t \).
\( \phi \) is an angle of \( 1 - E \)
cancels out, on \( \text{AdS} \)
The string action on this background may thus be interpreted as perturbed string action
on \( \text{AdS}_5 \times T^{1,1} \).

Let us now find the generalization of the solution (3.8) to the leading order in \( \epsilon \) (now \( \phi \) is an angle of \( T^{1,1} \)).

Assuming the ansatz \( x_0 = x_0(\tau), z = z(\tau), \varphi = \varphi(\tau) \) with all other coordinates being zero or constant the perturbed solution corresponding to (4.4) should have the form (we set \( \nu = 1 \))

\[
x_0 = \tan \tau + \epsilon T(\tau), \quad z = \frac{1}{\cos \tau} [1 - \epsilon Z(\tau)], \quad \varphi = \tau + \epsilon \Phi(\tau),
\]

\[
\cos^2 \tau \dot{T} = -2Z + \frac{1}{2} F, \quad \dot{\Phi} = -\frac{1}{2} F, \quad \tan \tau \dot{Z} = \frac{1}{\cos^2 \tau} Z - \frac{1}{2} F, \quad F(\tau) \equiv q(z(\tau)).
\]

The above relations are valid for arbitrary \( q(z) \). Computing the value of the energy density (defined as in the \( \text{AdS} \) case by (3.4)) on the deformed solution of the form (4.6) we get

\[
\mathcal{E}_d = 1 + \frac{\epsilon}{\cos^2 \tau} (Z + \sin \tau \cos \tau \dot{Z} + \cos^2 \tau \dot{T}) + O(\epsilon^2).
\]

It then follows from the form of the solution in (4.7) that the leading \( O(\epsilon) \) term in \( E \) cancels out, \( \mathcal{E}_d = 1 + O(\epsilon^2) \), i.e.

\[
E = J + \epsilon \gamma_1(\tau) + \epsilon^2 \gamma_2(\tau) + ... , \quad \gamma_1 = 0.
\]

Note that \( \gamma_1 = 0 \) for any \( F(\tau) \), but higher-order corrections certainly should not vanish.

Thus there is no leading \( O(\frac{A^2}{N}) \) correction to the anomalous dimension of the corresponding “ground-state” large R-charge operator \( \text{Tr}(A_2B_2)^J \) due to running of the gauge couplings in the dual \( SU(N) \times SU(N+M) \) gauge theory. This is reminiscent of the

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20 In the “plane-wave” context this null geodesic in \( \text{AdS}_5 \times T^{1,1} \) was considered in [36,37,38].

21 The equations for \( x_0 \) and \( \varphi \) are directly integrated while the equation for \( z \) is given by the constraint

\[
-\frac{1}{z^2} [1 - \frac{1}{2} \epsilon q(z)] \dot{x}_0^2 + \left[ 1 + \frac{1}{3} \epsilon q(z) \right] \frac{\dot{z}^2}{z^4} + \left[ 1 + \frac{1}{3} \epsilon q(z) \right] \dot{\varphi}^2 = 0, \quad \text{i.e.} \quad [1 + \frac{1}{2} \epsilon q(z)](-z^2 + \frac{\dot{z}^2}{z^4}) + [1 - \frac{1}{2} \epsilon q(z)] + O(\epsilon^2) = 0.
\]

The explicit solution for \( Z \) is \( q = \ln \cos \tau \): \( Z(\tau) = -\frac{1}{2} \tan \tau \int \tau \, \frac{dt'}{\csc t'} \ln \cos t', \) i.e. it can be expressed in terms of a polylog function. For general \( Q(y) \) the equations for the corresponding null geodesic are \( y^2 \dot{x}_0 = \kappa Q^{1/2}(y), \ \phi = w Q^{-1/2}(y), \ y^2 + w^2 y^2 Q^{-1}(y) - \kappa^2 = 0 \) (see also [38]).
vanishing of the leading correction to the anomalous dimension of the operator $\text{Tr}(A_iB_j)$ observed in [39] – the correction to the dimension starts at a rather high order in $M/N$: 
\[ \Delta = -\frac{1}{2} + c_1 \frac{\zeta(3)}{\sqrt{\lambda}} (\frac{M}{N})^4 + \ldots \] It would be interesting to find a field-theory explanation of why the dimension of the chiral operator $\text{Tr}(A_2B_2)$ of the conformal $SU(N) \times SU(N)$ theory [40] does not shift if we make an $O(M^2)$ marginal perturbation away from the conformal point.\(^\text{22}\)

4.2. Perturbed solution for rotating string

Let us now carry out a similar computation for the spinning string solution (3.10). The aim is to see how the $E(S)$ relation (3.29) found in the $AdS_5$ case is modified when the $AdS_5$ the metric is replaced by its deformation in (4.7). The internal $T^{1,1}$ part of the metric will not be important as the corresponding angles will be kept fixed. Here we expect a non-trivial correction to $E$ due to the deformation,

\[ E = S + [a_0 \sqrt{\lambda} + \epsilon \eta_1(\tau) + O(\epsilon^2)] \ln S + \ldots, \] (4.10)

where $\tau$-dependence of the function $\eta_1$ should be reflecting broken conformal invariance (i.e. changing of anomalous dimension with scale due to running of gauge couplings).\(^\text{22}\)

It is useful to change the coordinate $\sigma$ to $s = \tanh \rho$, where $\rho(\sigma)$ is a solution of the condition in (3.9) (we set $\kappa = 1$):

\[ \left( \frac{ds}{d\sigma} \right)^2 = (1 - s^2)(1 - w^2 s^2) \equiv G^2(s). \] (4.11)

The relevant part of the string action in conformal gauge is then

\[ I \sim \int d\tau ds \left[ z^{-2}(1 + \frac{1}{2}\epsilon \ln z)G^{-1}(-\dot{x}_0^2 + \dot{r}^2 + r^2 \dot{\phi}^2) \right. \]

\[ \left. -G(-\dot{x}_0^2 + r'^2 + r^2 \phi'^2) + z^{-2}(1 - \frac{1}{2}\epsilon \ln z)(G^{-1} \dot{z}^2 - Gz'^2) \right], \] (4.12)

\(^\text{22}\) The same may be true also for the LS [41] perturbations of $\mathcal{N}=4$ SYM theory. The supergravity dual of large R-charge limit of LS conformal point was considered in [42,43]. Let us note also that a discussion of a plane wave limit of a solution describing flow between two conformal points was given in [44,43].

\(^\text{23}\) The dependence on $\tau$ may be traded for the dependence on $z$ that plays the role of an energy scale.
where prime now stands for the derivative over $s$. The $AdS$ rotating solution (3.10) is then generalized, to leading order in $\epsilon$, as follows

$$x_0 = \tan \tau + \epsilon T(\tau, s), \quad z = \frac{\sqrt{1 - s^2}}{\cos \tau} [1 - \epsilon Z(\tau, s)],$$

$$\phi = w\tau + \epsilon P(\tau, s), \quad r = \frac{s}{\cos \tau} [1 + \epsilon R(\tau, s)].$$

(4.13)

The aim is to find the perturbed solution, and then to compute the leading correction to the energy density (3.4) (where $x^2 = -x_0^2 + r^2$). In general, we get (cf. (4.8))

$$\mathcal{E}_d = \frac{1}{1 - s^2} \left( 1 + \frac{\epsilon}{\cos^2 \tau} \left[ (1 + s^2 \cos 2\tau)Z + s^2 \cos 2\tau R + \cos^2 \tau \dot{T} - s^2 \sin \tau \cos \tau \dot{R} + (1 - s^2) \sin \tau \cos \tau \dot{Z} \right] \right) + O(\epsilon^2).$$

(4.14)

Note that in (3.10), i.e. at the zeroth order in $\epsilon$, $\partial_\tau (\dot{z}) = 0$. Assuming that this will be true also at non-zero order, i.e. that $\dot{Z} + \dot{R} = 0$, one is able to separate $\tau$ and $s$ coordinates in the solution, $T = T(\tau), \quad P = P(\tau), \quad Z = A(\tau) + B(s), \quad R = -A(\tau) + C(s)$. One is then to solve the resulting string equations, $w(\tan \tau + 8\dot{A}) + 2\dot{P} = 0, \quad \tan \tau + 4\dot{A} - 4 \sin \tau \cos \tau \dot{T} + 2 \cos^2 \tau \dot{T} = 0$, etc., and the conformal-gauge constraints. Finally, one is to compute the energy (3.1),(4.14) and the spin (the conserved charge corresponding to translations in $\phi$ in (4.12)) and to compute $\eta_1(\tau)$ in (4.10). We leave detailed discussion of this computation for future.

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References

[1] J. M. Maldacena, “The large $N$ limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [hep-th/9711200]. O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [hep-th/9905111].

[2] D. Berenstein, J. Maldacena and H. Nastase, “Strings in flat space and pp waves from $N = 4$ super Yang Mills,” JHEP 0204, 013 (2002) [hep-th/0202021].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B 636, 99 (2002) [hep-th/0204051].

[4] A. M. Polyakov, “Gauge fields and space-time,” hep-th/0110196.

[5] R. R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in $AdS_5 \times S^5$ background,” Nucl. Phys. B 533, 109 (1998) [hep-th/9805028].

[6] J. Maldacena, “Wilson loops in large $N$ field theories,” Phys. Rev. Lett. 80, 4859 (1998) [hep-th/9803021]. S. J. Rey and J. Yee, “Macroscopic strings as heavy quarks in large $N$ gauge theory and anti-de Sitter supergravity,” Eur. Phys. J. C 22, 379 (2001) [hep-th/9803001].

[7] R. Kallosh and A. A. Tseytlin, “Simplifying superstring action on $AdS_5 \times S^5$,” JHEP 9810, 016 (1998) [hep-th/9808088].

[8] S. Forste, D. Ghoshal and S. Theisen, “Stringy corrections to the Wilson loop in $N = 4$ super Yang-Mills theory,” JHEP 9908, 013 (1999) [hep-th/9903042].

[9] N. Drukker, D. J. Gross and A. A. Tseytlin, “Green-Schwarz string in $AdS_5 \times S^5$ : Semi-classical partition function,” JHEP 0004, 021 (2000) [hep-th/0001204]. A. A. Tseytlin, “‘Long’ quantum superstrings in $AdS_5 \times S^5$ ,” hep-th/0008107.

[10] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$ ,” JHEP 0206, 007 (2002) [hep-th/0204226].

[11] R. R. Metsaev, “Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background,” Nucl. Phys. B 625, 70 (2002) [hep-th/0112044].

[12] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “A new maximally supersymmetric background of IIB superstring theory,” JHEP 0201, 047 (2002) [hep-th/0110242]. “Penrose limits and maximal supersymmetry,” hep-th/0201081.

[13] R. R. Metsaev and A. A. Tseytlin, “Exactly solvable model of superstring in plane wave Ramond-Ramond background,” Phys. Rev. D 65, 126004 (2002) [hep-th/0202109].

[14] D. Berenstein and H. Nastase, “On lightcone string field theory from super Yang-Mills and holography,” hep-th/0205048.

[15] H. J. de Vega and I. L. Egusquiza, “Planetoid String Solutions in $3 + 1$ Axisymmetric Spacetimes,” Phys. Rev. D 54, 7513 (1996) [hep-th/9607056].
[16] H. J. de Vega, A. L. Larsen and N. Sanchez, “Semiclassical quantization of circular strings in de Sitter and anti-de Sitter space-times,” Phys. Rev. D 51, 6917 (1995) [hep-th/9410213].

[17] J. G. Russo, “Anomalous dimensions in gauge theories from rotating strings in $AdS_5 \times S^5$,” JHEP 0206, 038 (2002) [hep-th/0205244].

[18] J. A. Minahan, “Circular Semiclassical String Solutions on $AdS_5 \times S^5$,” [hep-th/0209047].

[19] A. Armoni, J. L. Barbon and A. C. Petkou, “Orbiting strings in AdS black holes and $N = 4$ SYM at finite temperature,” JHEP 0206, 058 (2002) [hep-th/0205280].

[20] R. R. Metsaev and A. A. Tseytlin, “Superstring action in $AdS_5 \times S^5$: kappa-symmetry light cone gauge,” Phys. Rev. D 63, 046002 (2001) [hep-th/0007030]. R. R. Metsaev, C. B. Thorn and A. A. Tseytlin, “Light-cone superstring in AdS space-time,” Nucl. Phys. B 596, 151 (2001) [hep-th/0009171].

[21] A. Polyakov, unpublished (private communication, Jan. 2000).

[22] A. A. Tseytlin, “On limits of superstring in $AdS_5 \times S^5$,” [hep-th/0201112].

[23] R. R. Metsaev, “Light cone gauge formulation of IIB supergravity in AdS(5) x S(5) background and AdS/CFT correspondence,” Phys. Lett. B 468, 65 (1999) [hep-th/9908114].

[24] K. Sfetsos and A. A. Tseytlin, “Four-dimensional plane wave string solutions with coset CFT description,” Nucl. Phys. B 427, 245 (1994) [hep-th/9404063].

[25] J. Russo, unpublished.

[26] R. F. Dashen, B. Hasslacher and A. Neveu, “The Particle Spectrum In Model Field Theories From Semiclassical Functional Integral Techniques,” Phys. Rev. D 11, 3424 (1975).

[27] A. Polyakov, talk at Strings 2002, www.damtp.cam.ac.uk/strings02/avt/polyakov/

[28] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, “A new double-scaling limit of $N = 4$ super Yang-Mills theory and PP-wave strings,” [hep-th/0205033].

[29] D. J. Gross, A. Mikhailov and R. Roiban, “Operators with large R charge in $N = 4$ Yang-Mills theory,” [hep-th/0205066].

[30] A. Santambrogio and D. Zanon, “Exact anomalous dimensions of $N = 4$ Yang-Mills operators with large R charge,” [hep-th/0206079].

[31] A. Tseytlin, talk at KIAS Workshop on Strings and Branes, Seoul, May 20-31, 2002 [http://m.kias.re.kr/program/program.html]; talk at Strings 2002, Cambridge, July 15-20, 2002, [http://www.damtp.cam.ac.uk/strings02/avt/tseytlin/].
[32] A. Parnachev and A. V. Ryzhov, “Strings in the near plane wave background and AdS/CFT,” [hep-th/0208010].
[33] D. J. Gross and F. Wilczek, “Asymptotically Free Gauge Theories. 2,” Phys. Rev. D 9, 980 (1974). A. Gonzalez-Arroyo and C. Lopez, “Second Order Contributions To The Structure Functions In Deep Inelastic Scattering. 3. The Singlet Case,” Nucl. Phys. B 166, 429 (1980). G. P. Korchemsky and G. Marchesini, “Structure function for large x and renormalization of Wilson loop,” Nucl. Phys. B 406, 225 (1993) [hep-ph/9210281].
[34] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities,” JHEP 0008, 052 (2000) [hep-th/0007191].
[35] I. R. Klebanov and A. A. Tseytlin, “Gravity duals of supersymmetric SU(N) x SU(N+M) gauge theories,” Nucl. Phys. B 578, 123 (2000) [hep-th/0002159].
[36] N. Itzhaki, I. R. Klebanov and S. Mukhi, “PP wave limit and enhanced supersymmetry in gauge theories,” JHEP 0203, 048 (2002) [hep-th/0202153].
[37] J. Gomis and H. Ooguri, “Penrose limit of N = 1 gauge theories,” [hep-th/0202157].
[38] L. A. Pando Zayas and J. Sonnenschein, “On Penrose limits and gauge theories,” JHEP 0205, 010 (2002) [hep-th/0202186].
[39] S. Frolov, I. R. Klebanov and A. A. Tseytlin, “String corrections to the holographic RG flow of supersymmetric SU(N) x SU(N+M) gauge theory,” Nucl. Phys. B 620, 84 (2002) [hep-th/0108100].
[40] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B 536, 199 (1998) [hep-th/9807080].
[41] R. G. Leigh and M. J. Strassler, “Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory,” Nucl. Phys. B 447, 95 (1995) [hep-th/9503121].
[42] R. Corrado, N. Halmagyi, K. D. Kennaway and N. P. Warner, “Penrose limits of RG fixed points and pp-waves with background fluxes,” [hep-th/0205314].
[43] D. Brecher, C. V. Johnson, K. J. Lovis and R. C. Myers, “Penrose limits, deformed pp-waves and the string duals of N = 1 large N gauge theory,” [hep-th/0206045].
[44] E. G. Gimon, L. A. Pando Zayas and J. Sonnenschein, “Penrose limits and RG flows,” [hep-th/0206033].