ANALYTIC DEFINITION OF CURVES AND SURFACES
BY PARABOLIC BLENDING

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SUMMARY

A procedure for interpolating between specified points of a curve or surface is described. The method guarantees slope continuity at all junctions. A surface panel divided into $p \times q$ contiguous patches is completely specified by the coordinates of $(p+1) \times (q+1)$ points. Each individual patch, however, depends parametrically on the coordinates of 16 points, allowing shape flexibility and global conformity.

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I. Interpolation Between points on a Space Curve

We assume that a space curve is approximately defined by a sequence of points \( \{\vec{A}, \vec{B}, \vec{C}, \ldots\} \), each of which are vectors, i.e., \( \vec{A} \equiv A_x, A_y, A_z \) in a Cartesian reference frame. We propose an interpolation scheme that defines the curve between each pair of adjacent points so that the curve is as smooth as possible. Consider four adjacent points: D,E,F,G.

\[ \text{Figure 1} \]

The distance \( t \) along the chord between E and F defines a scalar variable that we use to parametrize a curve \( \vec{c}(t) \) between E and F. \( \vec{c}(t) \) will be a blend of two parabolas.

The three points \( \vec{D}, \vec{E}, \vec{F} \), define a parabola \( \vec{p}(r) \) as follows. Let \( r \) be the distance along the chord between \( \vec{D} \) and \( \vec{F} \). Let \( u \) be the distance perpendicular to \( r \) in the plane defined by \( \vec{D}, \vec{E}, \vec{F} \). The parabola,

\[
   u = \alpha r(d - r),
\]

has an axis perpendicular to the line along \( r \). \( d = \text{distance between } \vec{D} \text{ and } \vec{F} \). We choose \( \alpha \) so that this parabola passes through \( \vec{E} \). Then \( \vec{p}(r) \) can be taken to be the parabola, Eq.(1).
Similarly, the three points $\vec{E}, \vec{F}, \vec{G}$, define a parabola $q(s)$ as follows. Let $s$ be the distance along the chord between $\vec{E}$ and $\vec{G}$. Let $v$ be the distance perpendicular to $s$ in the plane defined by $\vec{E}, \vec{F}, \vec{G}$. The parabola,

$$v = \beta s(e - s),$$

(2)

has an axis perpendicular to the line along $s$. $e \equiv$ distance between $\vec{E}$ and $\vec{G}$. We choose $\beta$ so that this parabola passes through $\vec{F}$. Then $q(s)$ can be taken to be the parabola, Eq.(2). If $\vec{E}, \vec{F}$, and $\vec{G}$ happen to be collinear, Eq.(2) will be $v \equiv 0$, a straight line.

We now define $c(t)$ as a blend of $p(r)$ and $q(s)$ as follows:

$$\vec{c}(t) = [1 - (t/t_o)]\vec{p}(r) + (t/t_o)\vec{q}(s).$$

(3)

$t_o \equiv$ distance between $\vec{E}$ and $\vec{F}$. Consequently, the two blending functions [the coefficients of $\vec{p}(r)$ and $\vec{q}(s)$] vary linearly between 0 and 1. Eq.(3) is not completely specified until we define a relation between $t$ and $r$ and between $t$ and $s$. This can be done only by dropping perpendiculars to the lines along $r$ and $s$ respectively; so $r = r(t)$ and $s = s(t)$ are defined by this geometric operation.

In like manner we construct a curve $\vec{c}_i(t_i)$ between each adjacent pair of points. It is easy to prove that where two $\vec{c}_i(t_i)$ connect, their slopes are equal; so the entire curve will be continuous and smooth. To see this, consider Eq.(3) rewritten as follows.

$$\vec{c}(t) = \vec{p}(t) + (t/t_o)[\vec{q}(t) - \vec{p}(t)].$$

(4)

The slope at point $\vec{E}$ is

$$(d\vec{c}/dt)_E = (d\vec{p}/dt)_E + (t/t_o)[d\vec{q}/dt - d\vec{p}/dt]_E + (1/t_o)(\vec{q} - \vec{p})_E.$$
The second term on the right hand side is zero because \( t = 0 \) at \( E \); and the third term on the right hand side is zero because \( \vec{q} = \vec{p} \) at \( E \). Consequently,

\[
\left( \frac{d\vec{c}}{dt} \right)_E = \left( \frac{d\vec{p}}{dt} \right)_E. \tag{6}
\]

The slope of \( \vec{c}(t) \) at \( E \) equals the slope of the parabola \( \vec{p}(r) \), which passes through \( D, E, \) and \( F \). This same parabola determines the slope of \( \vec{c} \) at \( E \) for the curve between \( D \) and \( E \) by an identical argument. Consequently slope continuity is assured.

Blending of two parabolas is possible only if the interval is an interior one. If the curve starts at point \( A \), then interpolation between \( A \) and \( B \) should be by the single parabola, defined as above, through \( A, B, \) and \( C \). Accordingly, in order to specify carefully the shape of the curve near its endpoint, the distance between \( A \) and \( B \) should be smaller than that for interior intervals. There is no requirement that the points be equally, or nearly equally spaced. Obviously point density should be higher where the curvature is higher.

If an interpolated segment of the curve does not behave according to some a priori intent, an extra point can be inserted to provide the required constraint. Aside from endpoints discussed above, it should not be necessary to
specify for special treatment interior points at inflections or cusps, provided
the latter are not intended to be infinitely sharp. Precision is reproducing
such features can be assured merely by specifying extra points sufficiently
near to these characteristic features.

The curve \( \vec{c}(t) \), Eq.(3) or (4), is of cubic order in the Cartesian coordi-
nates. Consequently it can adequately represent an inflection that should
occur within an interval. It should be apparent that the cubic function ap-
propriate to the interval EF does not pass through points D and G. (This is
an important difference in comparison to simple fit by cubic polynomials.)
The manner of construction (as a blend of two parabolas) guarantees that
spurious wiggles will not be introduced, as frequently happens when simple
cubes are forced to pass through four points of a curve.

II. Interpolation Between Points on a Space Surface

We assume that a space surface is defined by a net of points \( \vec{A}_{ij} \), such as
intersections of lines in the figure below.

\[\text{Figure 3}\]
Connecting lines between adjacent points of the net can be constructed by the blending algorithm described in Sec. I. Clearly a good surface interpolation for the patch EFGH must take into account the global shape implied by the adjacent points, \( L, M, N, \ldots V, W \). Consider the patch EFGH. A set of coordinates to which points on the patch are related can be defined, say \( x \) and \( y \). \( x = \text{constant} \) (between 0 and 1) defines a space curve through the four points a,b,c,d which, in the notation of Sec. I, are points \( t = xt_o \) of curves MN, EF, GH, VU respectively. The line \( \ell \), between b and c, is then defined by the blending algorithm. Similarly, a line \( m \) is defined by \( y = \text{constant} \) (between 0 and 1). Surface point \( \vec{z}(x, y) \) can be taken to be

\[
\vec{z}(x, y) = \vec{\ell}(y), \quad (7)
\]

\[
\vec{z}(x, y) = \vec{m}(x). \quad (8)
\]

These two surfaces will differ slightly since \( \ell \) and \( m \) will not in general intersect. They do coincide, however, on all network lines. If computational labor is not a factor, one may take the average of (7) and (8):

\[
\vec{z}(x, y) = \frac{1}{2} \vec{\ell}(y) + \frac{1}{2} \vec{m}(x). \quad (9)
\]

An advantage of these surface interpolation schemes is that all surface slopes will be continuous at boundaries between patches. The surface shape of each patch will depend, as it should, on the behavior of the surface surrounding it.

Parabolic blending along \( \ell \), \( m \) lines is possible only for interior patches. Interpolation for patches at an edge will employ a single parabola for \( \ell \) or \( m \), like the end interval of Sec. I. Corner patches will necessitate single parabolas for both \( \ell \) and \( m \) lines. Accordingly good edge definition requires a narrow
spacing for the network lines adjacent to edges of the surface, as shown in the figure.

An entire surface panel consisting of \( pq \) patches is specified by the coordinates of \((p + 1)(q + 1)\) points, slightly more than one point per patch on the average. The shape of a single (interior) patch, however, depends on 16 points. This signifies both flexibility and global relationship.

The optimum arrangement of network points will naturally depend on the shape of the panel that is to be represented. If the panel has a ridge, for example, the network should be arranged so that a network line passes along the crest. Closely spaced network lines (approximately parallel to the crest) should also be included to provide adequate definition.

III. Analysis

In order to carry out the procedures described in the preceding sections it is necessary to have explicit formulas for the parabolic functions and coordinates. We need to this only for one case.

\[ \vec{J} \text{ is the point along DF obtained by dropping a perpendicular from } \vec{E}. \text{ If } \vec{J} \text{ is } \vec{D} + x(\vec{F} - \vec{D}), \text{ then:} \]
\[ \{ \vec{E} - [\vec{D} + x(\vec{F} - \vec{D})] \} \cdot (\vec{F} - \vec{D}) = 0. \]  

(10)
It follows that
\[ x = (\vec{E} - \vec{D}) \cdot (\vec{F} - \vec{D}) / d^2 \]  \hspace{1cm} (11)

where \( d \equiv |\vec{F} - \vec{D}| \). The equation of a point \( \vec{p} \) on the parabola \( \vec{p}(r) \) is
\[ \vec{p}(r) = \vec{D} + (r/d)(\vec{F} - \vec{D}) + \alpha r(d - r)(\vec{E} - \vec{J}). \]  \hspace{1cm} (12)

The coefficient \( \alpha \) is determined by requiring \( \vec{p}(xd) = \vec{E} \). The last term of (12) must satisfy,
\[ \alpha xd(d - xd)(\vec{E} - \vec{J}) = \vec{E} - \vec{J}. \]  \hspace{1cm} (13)

So,
\[ \alpha = 1/[d^2 x(1 - x)]. \]  \hspace{1cm} (14)

The equation of the parabola is now completely specified: Eq.(12) together with values for \( \alpha \) and \( x \), Eqs.(14) and (11).

The only problem that remains is to find the relation between \( t \) and \( r \). From the figure it follows that
\[ r = xd + t \cos \theta. \]  \hspace{1cm} (15)

We have,
\[ \cos \theta = (\vec{F} - \vec{E}) \cdot (\vec{F} - \vec{D}) / dt_0 \]  \hspace{1cm} (16)

where \( t_0 = |\vec{F} - \vec{E}| \). Consequently, Eq.(15) is specified completely, so Eq.(12) can be written as a function of \( t \).

In the alternative event that \( t \) is along the chord from \( \vec{D} \) to \( \vec{E} \), so that \( t_0 = |\vec{E} - \vec{D}| \), we have instead of (15),
\[ r = t(\vec{E} - \vec{D}) \cdot (\vec{F} - \vec{D}) / dt_0. \]  \hspace{1cm} (17)
Figure 1
Figure 2
