On Generalized Stone’s Theorem

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Abstract. It is known that the generator of a strictly continuous one parameter unitary group in the multiplier algebra of a $C^*$-algebra is affiliated to that $C^*$-algebra. We show that under natural non degeneracy conditions, this self adjoint unbounded operator lies indeed in the (unbounded) multiplier algebra of the Pedersen’s ideal of the $C^*$-algebra.

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1. Introduction

One of the main objectives of the Heisenberg formulation of the Quantum Mechanics is to give appropriate models for the commutation relations, the most famous one of which being

$$[P, Q] = i\hbar I$$

where $\hbar$ is the Planck constant and $P, Q$ are the quantum position and quantum momentum. It was known from the beginning that bounded linear operators cannot satisfy such a relation (convince yourself by checking this for matrices where you have a trace for free!). In particular this does not happen in a $C^*$-algebra. Although projective limit of $C^*$-algebras can include unbounded operators, this cannot happen in a projective limit also. (Just recall that some of their quotients are $C^*$-algebras [Ph88b]). One classical trick is to replace this type of relation by a stronger commutation property, which in this case is

$$U_tQU_{-t} = e^{i\hbar t}Q$$

where $U_t = e^{itH}$, for some closed operator $H$. It has been shown in [HQV] that any strictly continuous one parameter group of unitaries $(U_t)$ in the multiplier

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algebra of a \(C^\ast\)-algebra \(A_0\) has a generator \(H\) (a version of Stone’s theorem) which could be chosen to be in \(A_0^n\) (the set of densely defined linear operators on \(A_0\) which are affiliated with \(A_0\) [Wr91]). Here we show that indeed \(H\) could be chosen more specifically. Let \(A_{00} = K(A_0)\) be the Pedersen’s ideal of \(A_0\) and \(\Gamma(A_{00})\) be the topological algebra of (unbounded) multiplier on \(A_{00}\) [LT]. We show that \(H\) could be chosen to be in \(A = \Gamma(A_{00})\).

2. Stone’s theorem

We start with some lemmas from [Wr91] in which we replace \(A_0^n\) with \(A = \Gamma(A_{00})\). We keep the notations of the above paragraph all over the paper. We also freely use the notations and terminology of [Wr91]. All morphisms are supposed to be non degenerate, and following [Wr 91] we use \(\text{Mor}\) to denote the set of morphisms.

The following lemma has been already proved in [Am], but we bring the proof here for the sake of completeness.

**Lemma 2.1.** Let \(A_0\) and \(B_0\) be \(C^\ast\)-algebras. Let \(\phi_0 \in \text{Mor}(A_0, B_0)\) be strictly non degenerate (that is \(B_{00} \subseteq \phi_0(A_{00})B_0\)). Then \(\phi_0\) extends uniquely to a morphism \(\phi \in \text{Mor}(A, B)\) such that \(\phi(A_{00})B_0\) is a core for \(\phi(T)\) and \(\phi(T)(\phi_0(a)b) = \phi_0(Ta)b\) for all \(a \in A_{00}, b \in B_0\), and \(T \in A = \Gamma(A_{00})\).

**Proof** Since \(\phi_0\) preserves the spectral theory, \(\phi_0(A_{00}) \subseteq B_{00}\). On the other hand, \(\phi_0(A_{00})\) is clearly a dense ideal of \(\phi_0(A_0)\). Therefore \(B_0\phi_0(A_{00})B_0\) is an ideal of \(B_0\) which is dense in \(B_0\phi_0(A_0)B_0\). Now if \(\phi_0\) is non degenerate then \(\phi_0(A_0)B_0\) is dense in \(B_0\), so \(B_0\phi_0(A_0)B_0\) is dense in \(B_0B_0 = B_0\), i.e. \(B_0\phi_0(A_{00})B_0\) is a dense ideal of \(B_0\) and so contains \(B_{00}\). But \(B_0\phi_0(A_{00})B_0 \subseteq B_0B_{00}B_0 \subseteq B_{00}\), hence the equality holds. If \(\phi_0\) is strictly non degenerate then \(\phi_0(A_{00})B_0 \supseteq B_{00}\). The converse inclusion follows from the fact that \(\phi_0(A_{00}) \subseteq B_{00}\). Hence \(\phi_0(A_{00})B_0 = B_{00}\). Now the right hand side is self adjoint and the adjoint of the left hand side is \(B_0\phi_0(A_{00})\), hence \(B_0\phi_0(A_{00}) = B_{00}\), \(\square\)

Now each \(T \in A = \Gamma(A_{00})\) could be considered as an element of \(A_0^n\) and \(\phi_0\) also extends to \(\tilde{\phi} : A_0^n \to B_0^n\) by [Wr91, 1.2]. But there is no ambiguity as we have

**Lemma 2.2.** With the above notations, \(\tilde{\phi}(T) = \phi(T)\). In particular we have \(\phi(z_T) = z_{\phi(T)}\).

**Proof** The first statement follows from above lemma and the facts that by assumption \(\phi_0(A_{00})B_0 \supseteq B_{00}\) and \(\tilde{\phi}(T)(\phi_0(a)b) = \phi_0(Ta)b\) \((a \in D(T) \supseteq A_{00}, b \in B_0)\) [Wr91, 1.2]. The second statement is proved for \(\tilde{\phi}\) in [Wr91,1.2], and follows from the first for \(\phi\). \(\square\)

Next we prove a technical result about \(z\)-transform [Wr91], where \(z\) is a bounded continuous function on \(\mathbb{C}\). Here we take \(z(\lambda) = \lambda(1 + \lambda\bar{\lambda})^{\frac{1}{2}}\) (see [Am2] for the properties of \(z\)-transform on unbounded multiplier algebra).
Proposition 2.1. Let $A_0$ be a $C^*$-algebra and $T \in A$ be self adjoint. Let $z_T \in \mathcal{M}(A_0)$ be the $z$-transform of $T$. Then $\sigma(z) \subseteq [-1, 1]$. Assume that
\[ \text{span}\{f(z_T) a : f \in C_{00}(-1, 1), a \in A_0\} \supseteq A_{00}. \]
Then there is a unique $\phi = \phi_T \in \text{Mor}(C(\mathbb{R}), A)$ such that $\phi_T(id) = T$. Moreover $\phi(C_{00}(\mathbb{R}))A_0 \supseteq A_{00}$ and $\phi_T(z) = z_T$, for $z(t) = t(1 + t^2)^{\frac{1}{2}}$, $t \in \mathbb{R}$.

Proof $z_T \in \mathcal{M}(A_0)$ is self adjoint and $\|z_T\| \leq 1$, so its spectrum $\sigma(z_T)$ is contained in $[-1, 1]$. The same is true for $z \in C_0(\mathbb{R})$. We use the continuous functional calculus to show the uniqueness. If $\phi(id) = T$ then $\phi(z) = z_T$ and $\phi(f \circ z) = f(z_T)$, for each $f \in C_b(-1, 1) = C([-1, 1])$. But each element of $C_0(\mathbb{R})$ is of the form $f \circ z$, where $f \in C_0(-1, 1)$ and the uniqueness follows.

For the existence, let’s define $\phi_0 : C_0(\mathbb{R}) \to M(A_0)$ by $\phi_0(f \circ z) = f(z_T)$. Then $\phi_0$ is clearly a *-homomorphism. Also by assumption, $\phi_0(C_{00}(\mathbb{R}))A_0 \supseteq A_{00}$ which means that $\phi_0$ is strictly non degenerate. By [Am, Theorem 3.4], $\phi_0$ extends (uniquely) to some $\phi \in \text{Mor}(C(\mathbb{R}), \Gamma(A_{00}))$. Now $\phi(f \circ z) = f(z_T)$, for each $f \in C_0(-1, 1)$. In particular, for $f(t) = t(1 - t^2)^{\frac{1}{2}}$, we get $f \circ z = id$ and so $\phi(id) = f(z_T) = T$.

Let $\phi_0$ and $\phi$ be as in the proof of the above proposition. Then $\phi$ is not injective in general. But we can do the classical trick to make it injective. Consider $\ker(\phi_0) = \{f \in C_0(\mathbb{R}) : f = 0 \text{ on } \sigma(T)\}$. Then $C_0(\mathbb{R})/\ker(\phi_0) = C_0(\sigma(T))$ and the corresponding quotient map identifies with the restriction map $\pi : C_0(\mathbb{R}) \to C_0(\sigma(T))$. Let $id_{\sigma(T)} = \pi(id)$, then we have

Proposition 2.2. Let $A_0$ be a $C^*$-algebra and $T \in \Gamma(A_{00})$, then there is a unique embedding $\psi = \psi_T \in \text{Mor}(C(\sigma(T)), \Gamma(A_{00}))$ such that $\psi_T(id_{\sigma(T)}) = T$.

Proof Since $\pi$ is onto, there is a function $\psi_0$ such that $\psi_0\pi = \phi_0$. Then $\psi_0$ is a *-homomorphism. To see that it is strictly non degenerate it is enough to observe that $\psi_0(C_0(\sigma(T)))A_0 = \phi_0(C_0(\mathbb{R}))A_0 \supseteq A_{00}$. Therefore it extends to a morphism $\psi \in \text{Mor}(C(\sigma(T)), \Gamma(A_{00}))$.

Next let us show that $\psi$ is one-one. It is clear that $\psi_0$ is one-one (since $\psi_0 = \phi_0\pi$ where $\pi(ker\phi_0) = \{0\}$). Take any $F \in C(\mathbb{R})$ such that $\psi(F) = 0$, then $0 = \psi(F)\psi_0(f)a = \psi_0(Ff)a$, for each $f \in C_0(\sigma(T))$ and $a \in A_0$. This means that $\psi_0(Ff) \in M(A_0)$ multiplies $A_0$ into 0, i.e. $\psi_0(Ff) = 0$. Hence $Ff = 0$ for each $f \in C_0(\sigma(T))$, and so $F = 0$.

Now we are prepared to prove the generalization of Stone’s theorem. Let $A_0$ be a $C^*$-algebra, $A_{00} = K(A_0)$ its Pedersen’s ideal, and $A = \Gamma(A_{00})$ be the algebra of (unbounded) multipliers of $A_{00}$ [LT]. For each $t \in \mathbb{R}$ consider the function $e_t \in C(\mathbb{R})$ defined by $e_t(s) = \exp(is) \quad (s \in \mathbb{R})$. Let $h \in A = \Gamma(A_{00})$ and $U_t = \phi_h(e_t) = \exp(iht)$. Then $(U_t)_{t \in \mathbb{R}}$ is a one parameter strictly continuous unitary group in $M(A_0)$ (the strict continuity follows from the fact that $\phi_h$ is non degenerate). Moreover, if $h \in b(\Gamma(A_{00})) = M(A_0)$, then this is also norm continuous. Conversely each strictly continuous unitary group in $M(A_0)$ is of this form, for some $h \eta A_0$ [HQV]. Here we want $h$ to be actually
in $\Gamma(A_{00})$. Clearly for this to happen, we would need to put some condition on the unitary group. This is the content of the following result. The proof is quite similar to [HQV, 2.1]. Here we only sketch those parts of the proof which have to be modified. But first a definition.

**Definition 2.1.** Let $A_0$ be a $C^*$-algebra and $(U_t)_{t \in \mathbb{R}}$ be a one parameter strictly continuous unitary group in $M(A_0)$. Let’s define $\alpha : L^1(\mathbb{R}) \to M(A_0)$ by $\alpha(f) = \int_{\mathbb{R}} f(t)U_t dt$, where

$$\left( \int f(t)U_t dt \right)x = \int f(t)U_t xdt \quad (x \in A_0)$$

is in the Bochner sense. This extends to a $*$-homomorphism $\alpha \in Mor(C^*(\mathbb{R}), A_0)$ [HQV]. We say that $(U_t)$ is (strictly) non degenerate, if the morphism $\alpha$ is (strictly) non degenerate (cf. [Am]).

**Theorem 2.1. (Generalized Stone’s Theorem)** Let $A_0$ be a $C^*$-algebra and $(U_t)_{t \in \mathbb{R}}$ be a one parameter strictly continuous strictly non degenerate unitary group in $M(A_0)$. Then there exists a self adjoint $h \in \Gamma(K(A_0)))$ such that $U_t = \exp(ith)$ for $t \in \mathbb{R}$. Moreover, if $(U_t)_{t \in \mathbb{R}}$ is norm continuous, then $h \in M(A_0)$.

**Proof** Define $\alpha \in Mor(C^*(\mathbb{R}), A_0)$ as above. Then $\alpha$ is strictly non degenerate, and so it extends to a morphism of the corresponding pro-$C^*$-algebras, which we still denote it by $\alpha \in Mor(\Gamma(K(C^*(\mathbb{R})))$, $\Gamma(K(A_0)))$ (see the discussion before Theorem 3.2 in [Am]). Also it is well known that the Fourier transform $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ extends to an isomorphism $\mathcal{F} \in Mor(C^*(\mathbb{R}), C_0(\mathbb{R}))$ with $\mathcal{F}(\iota_t) = \lambda_t$, where $\lambda$ is the left regular representation of $\mathbb{R}$. Since $\mathcal{F}$ is surjective we can extend it to an isomorphism $\mathcal{F} \in Mor(\Gamma(K(C^*(\mathbb{R})))$, $C(\mathbb{R}))$ [LT]. Also it is clear that $\alpha(\iota_t) = U_t \ (t \in \mathbb{R})$. Define $h = (\alpha \circ \mathcal{F}^{-1})(id) \in \Gamma(A_{00})$. This is self adjoint and by the uniqueness part of Proposition 1.2 we have $\phi_h = \alpha \circ \mathcal{F}^{-1}$. Now $U_t = \alpha(\iota_t) = \alpha(\mathcal{F}^{-1}(\iota_t)) = \phi_h(\iota_t) = \exp(ith)$, which finishes the proof of the first part of the theorem.

Now, for each $t \in \mathbb{R}$ we get

$$\|U_t - 1\| = \|\phi_h(\iota_t) - 1\| = \|\phi_h(\iota_t)\pi(\iota_t - 1)\| = \|\pi(\iota_t - 1)\|$$

$$= \sup\{|e^{it\lambda} - 1| : \lambda \in \sigma(h)\}.$$

So, if the unitary group is norm continuous, then $\sigma(h)$ is bounded. Hence $h \in b(\Gamma(K(A_0)))$ [Ph88b]. But $b(\Gamma(K(A_0))) = M(A_0)$ [Ph88a], so $h \in M(A_0)$ and the proof is complete. □

Next, following [HQV], we show that $h$ can be found by differentiating the unitary group $(U_t)_{t \in \mathbb{R}}$.

**Proposition 2.3.** Let $A_0$ be a $C^*$-algebra, $(U_t)_{t \in \mathbb{R}}$ a one parameter strictly continuous unitary group in $M(A_0)$, and $h$ a self adjoint element of $\Gamma(K(A_0)))$
such that $U_t = \exp(it h)$ for $t \in \mathbb{R}$. Define the (unbounded) operator $H : D(H) \subseteq A_0 \rightarrow A_0$ by

(1) \[ D(H) = \{ a \in A_0 : t \mapsto U_t a \text{ is } C^1 \} \]

(2) \[ Ha = \left. \frac{d}{dt} \right|_{t=0} U_t a = \lim_{t \to 0} (U_t a - a)/t \quad (a \in D(H)). \]

Then $h \subseteq iH = i \frac{d}{dt} |_{t=0} U_t$.

**Proof** Using minimality of $A_{00}$ we have $A_{00} \subseteq \sqrt{1-z_h A_0}$. Now this last set is contained in $D(H) = D(iH)$ [HQV, 2.2]. The fact that $h(x) = iH(x)$, for each $x \in A_{00}$, follows from the calculations of [HQV, 2.2]. \qed

This in particular shows the uniqueness of the element $h \in \Gamma(A_{00})$ for which $U_t = \exp(it h)$ \quad ($t \in \mathbb{R}$). We call this element the *infinitesimal generator* of the one parameter group $(U_t)_{t \in \mathbb{R}}$.

3. **ELEMENTS AFFILIATED WITH GROUP $C^*$-ALGEBRA**

The group $C^*$-algebras are important objects in the theory of quantum groups. One reason is that they are *dual objects* to continuous functions. When the underlying group is not discrete (non compact case), one would expect some unbounded elements to come into the play. These can not belong to the group $C^*$-algebra, but they are usually affiliated with it. The problem of finding all elements affiliated with $C^*(G)$ is open in general. Some attempts are done to find them in the case that $G$ is a Lie group [Wr91]. In this case we know that the elements of the corresponding Lie algebra, considered as differential operators, are affiliated with the group $C^*$-algebra. The question of whether all elements of the universal enveloping algebra of $G$ are affiliated with $C^*(G)$ was left open. In this section we give an affirmative answer to this question.

Let $G$ be a Lie group, and $\pi : G \rightarrow B(\mathcal{X})$ be a (strongly continuous) representation of $G$ on a Banach space $\mathcal{X}$. We say that $x \in \mathcal{X}$ is a $C^\infty$-vector ($analytic$ vector, respectively) of $\pi$ if the map $g \mapsto \pi(g)x$ from $G$ to $\mathcal{X}$ is a $C^\infty$ (analytic, respectively) function. We denote the set of all such elements $x \in \mathcal{X}$ by $D^\infty = D^\infty(\pi)$ ($D^\omega(\pi)$, respectively). Then this is a dense subset of $\mathcal{X}$. Indeed L. Gårding showed that if $\phi \in C^\infty_0(G)$ and $\pi(\phi)x = \int_G \pi(t)x \phi(t)dt \quad (x \in \mathcal{X})$, then $D^\infty_0 = \pi(C^\infty_0(G))\mathcal{X} \subseteq \mathcal{X}$ is dense. This is called the *Gårding domain* of $\pi$. Let $\mathcal{G} = L(G)$ be the Lie algebra of $G$, i.e. the set of all left invariant vector fields on $G$ (at identity). For each $X \in \mathcal{G}$, define

$$\pi(X)x = \lim_{h \to 0} \frac{\pi(\exp hX)x - x}{h} \quad (x \in D^\infty)$$

Then $\pi(X) : D^\infty \subseteq \mathcal{X} \rightarrow D^\infty \subseteq \mathcal{X}$ is a densely defined unbounded operator on $\mathcal{X}$, and it is *skew symmetric* if $\pi$ is unitary (and $\mathcal{X}$ a Hilbert space).

Harish-Chandra noticed that $D^\infty$ could have a subspace $D$ such that $\pi(X)D \subseteq D$ but $\pi(g)D \not\subseteq D$. Therefore he suggested replacing $D^\infty$ with $D^\omega$. He
showed that \( D^\omega \) is dense in \( \mathcal{X} \) for certain representations of a semisimple Lie group [HC]. P. Cartier and J. Dixmier gave a proof for all unitary representations [CD], and E. Nelson used a generalization of the fundamental solution of the heat equation on Lie groups to prove this for an arbitrary representation. He also used analytic vectors to give a sufficient condition for a representation of a Lie algebra to be induced by a unitary representation of the Lie group [Nel]. Later I. Segal showed that this is equivalent to the \textit{complete positivity} of the representation with respect to an appropriate cone [Seg].

We are mainly interested in the case of the universal representation. For any locally compact group \( G \), the universal representation \( u : G \to M(C^*(G)) \) is determined by the following universal property: For each \( C^* \)-algebra \( A \) and each representation \( \pi : G \to A \), there is a unique \( \tau \in \text{Mor}(C^*(G), A) \) such that \( \pi = \tau u \). Then \( u \) is continuous and open with respect to the strict topology. Now let \( G \) be a Lie group and let \( \mathcal{G} \) be its Lie algebra of dimension \( N \) with a basis \( X_1, X_2, \ldots, X_N \). Then the \textit{universal enveloping algebra} \( U = U(G) \) is an \( * \)-algebra under \( X^* = -X \), \( X \in \mathcal{G} \). Elements of \( U \) are differential operators on \( G \) commuting with the right translations: Take \( D^\infty = D^\infty(u) \), then each \( X \in U \) defines \( du(X) : D^\infty \subseteq C^*(G) \to \mathcal{B}(H_u) \)

\[
du(X)a = Xu(g)a|_{g=e} = \lim_{t \to 0} \frac{1}{t}(u(\exp tX)a - a)
\]

which is a closable operator whose closure is simply denoted by \( X : D^\infty \subseteq C^*(G) \to C^*(G) \). For the reasons explained in the beginning of this section, we are interested in elements affiliated with \( C^*(G) \). We know that for \( X \in U \), if the differential equation \( X^*Xf = -f \) has only trivial bounded \( C^\infty \)-solution, then \( X, X^* \eta C^*(G) \) [Wr95,2.1]. It is known that all the elements of \( \mathcal{G} \) and also the \textit{elliptic operator} \( \Delta = -\sum_1^N X_i^2 \), where \( X_i \)'s form a basis of \( \mathcal{G} \), are affiliated with \( C^*(G) \) [Wr95]. The \( X_i \)'s are skew self-adjoint, where as \( \Delta \) is self-adjoint and positive (yes positive!).

Now let \( A_0 \) be a \( C^* \)-algebra and \( G \) be a Lie group acting on \( A_0 \) through a strictly continuous representation \( u : G \to M(A_0) \). Then

\[
C^\infty(u) = \{x \in A_0 : g \mapsto u(g)x \text{ is } C^\infty\}
\]

contains the dense subspace of \( A_0 \) spanned by all elements \( u(f)x \), with \( f \in C^\infty_0(G) \) and \( x \in A_0 \).

**Proposition 3.1.** With the above notation, to each \( X \in \mathcal{G} \) there corresponds an element \( \hat{X} \in \Gamma(A_0) \) such that

\[
\exp \hat{X} = u(e^{tX}) \quad (t \in \mathbb{R}).
\]

Moreover \( \hat{X} \) leaves \( C^\infty(u) \) invariant and \([X,Y]^\hat{\cdot} = [\hat{X},\hat{Y}]\), when restricted to \( C^\infty(u) \), for all \( X,Y \in \mathcal{G} \).
Proof Take $X \in \mathcal{G}$ and put $U_t = u(e^{tX})$. Then Theorem 2.1 applies and gives $\hat{X} \in \Gamma(A_{00})$ with $U_t = e^{t\hat{X}}$ for each $t \in \mathbb{R}$ (here $\hat{X}$ is skew self adjoint). The rest is proved as in [HQV, 2.4]. □

Corollary 3.1. With the above notation, $\mathcal{U}(\mathcal{G}) \subseteq \Gamma(K(C^*(G)))$.

Proof Just observe that $G$ acts strictly continuously on $C^*(G)$ through the universal representation and apply above proposition. □

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