Upper Bounds For Families Without Weak Delta-Systems

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Abstract
For $k \geq 3$, a collection of $k$ sets is said to form a weak $\Delta$-system if the intersection of any two sets from the collection has the same size. Erdős and Szemerédi asked about the size of the largest family $\mathcal{F}$ of subsets of $\{1, \ldots, n\}$ that does not contain a weak $\Delta$-system. In this note we improve upon the best upper bound due to the author and Sawin, and show that

$$|\mathcal{F}| \leq \left(\frac{2}{3} \Theta(C) + o(1)\right)^n$$

where $\Theta(C)$ is the capset capacity. In particular, this shows that

$$|\mathcal{F}| \leq (1.8367 \ldots + o(1))^n.$$ 

1 Introduction

A collection of $k$ sets for $k \geq 3$ is said to form a $k$-sunflower, or a $\Delta$-system, if the intersection of any two sets from the collection is the same. This notion was introduced in 1960 by Erdős and Rado [1], and they famously conjectured that for $k \geq 3$, there exists $C_k > 0$ depending on $k$, such that any $k$-sunflower-free family $\mathcal{F}$ of sets of size $m$ satisfies

$$|\mathcal{F}| \leq C_k^m.$$ 

This was one of Erdős’s favorite problems, for which he offered $1000$ [2, Problem 90], and while still out of reach, progress was made recently by Alweiss, Lovett, Wu and Zhang [3] (see also [4–7]).
In 1974, Erdős, E. Milner, and Rado [8] introduced the related notion of a weak \( \Delta \)-system (see [9] for a survey). For \( k \geq 3 \), a collection of \( k \) sets is said to form a weak \( \Delta \)-system of size \( k \) if the intersection of any two sets from the collection has the same size. Let \( G_k(n) \) denote the size of the largest family of subsets of \( \{1, \ldots, n\} \) that does not contain a weak \( \Delta \)-system of size \( k \). Erdős and Szemerédi asked about the growth rate of \( G_k(n) \) [10] (see also [2, Problem 94]), and in their paper they gave the lower bound

\[
G_k(n) \geq n^{\frac{\log n}{4 \log \log n}}.
\]

The current best lower bound, due to Kostochka and Rödl [11] (which improved upon [12]), is

\[
G_k(n) \geq k^{c(n \log n)^{\frac{1}{3}}}.
\]

Frankl and Rödl [13] resolved a conjecture of Erdős and Szemerédi, and proved that for every \( k \), there exists \( \varepsilon_k > 0 \) such that \( G_k(n) < (2 - \varepsilon_k)^n \). In the case \( k = 3 \), the stronger upper bound

\[
G_3(n) \leq \left( \frac{3}{2^{2/3}} + o(1) \right)^n = (1.889881 \ldots + o(1))^n
\]

is a consequence of the upper bound for 3-sunflower-free families due to the author and Sawin [14]. That result was proven using the slice-rank method, which was introduced by Tao [15] following the polynomial method breakthrough of Croot, Lev, and Pach [16], and Ellenberg and Gijswijt [17]. Throughout, we refer to a 3-sunflower-free family simply as sunflower-free since it contains no \( k \)-sunflower for any \( k \). The bound for sunflower-free sets achieved in [14] cannot be improved without a substantial change in approach, since the result also applies to multicolored sunflower-free sets (see [18] for a definition). The multicolored lower bounds from the work of Kleinberg, Sawin, and Speyer, [19], and Pebody [20], imply that for multicolored sunflower-free sets, the bound in equation (1.1) is optimal up to sub-exponential factors. Note that if a family does not contain a weak \( \Delta \)-system for \( k = 3 \), then it does not contain a weak \( \Delta \)-system for any \( k \).

In this paper we relate the size of the largest family that does not contain a weak \( \Delta \)-system in \( \{0, 1\}^n \) to the size of the largest capset in \( \mathbb{F}_3^n \), and improve upon the upper bound for \( G_3(n) \). A set \( A \subseteq \mathbb{F}_3^n \) is called a capset if there is no triple \( x, y, z \in A \), not all equal, such that \( x + y + z = 0 \) (mod 3). Equivalently, \( A \) is a capset if there does not exist a triple \( x, y, z \), not all equal, such that for every coordinate \( i \),

\[
\{x_i, y_i, z_i\} \in \{\{0, 1, 2\}, \{0, 0, 0\}, \{1, 1, 1\}, \{2, 2, 2\}\}.
\]
Let $C_n$ denote the size of largest capset in $\mathbb{F}_3^n$, and define the capset capacity, $\Theta(C)$, to be

$$\Theta(C) = \limsup_{n \to \infty} (C_n)^{1/n}.$$ 

In particular, Ellenberg and Gijswijt proved that

$$\Theta(C) \leq \min_{0 < t < 1} t^{-2}(1 + t + t^2) = \frac{3}{8}\sqrt{207 + 33\sqrt{33}} = 2.7551046 \ldots.$$ 

The notation $\Theta(C)$ is used since this quantity is precisely the Shannon Capacity of the hypergraph with three elements and one edge, see [21] for details on this notation.

Our main result is:

**Theorem 1** Let $X$ be a set of size $|X| = n$, and let $\mathcal{F}$ be a collection of subsets of $X$, and suppose that $\mathcal{F}$ does not contain a weak $\Delta$-system. Then

$$|\mathcal{F}| \leq \left(\frac{2}{3}\Theta(C) + o(1)\right)^n,$$

where $\Theta(C)$ is the capset capacity. In particular due to (1.2) we have that

$$|\mathcal{F}| \leq \left(\frac{1}{4}\sqrt{207 + 33\sqrt{33}} + o(1)\right)^n = (1.8367 \ldots + o(1))^n.$$

Equivalently, Theorem 1 states that

$$G_3(n) \leq \left(\frac{2}{3}\Theta(C) + o(1)\right)^n.$$

To prove this, we examine sets without non-trivial equilateral triangles in $\{0, 1\}^n$, where an equilateral triangle is a triple $x, y, z \in \mathbb{R}^n$ such that $\|x - y\| = \|y - z\| = \|z - x\|$, and it is said to be trivial if $x = y = z$. In the next section, we prove the following upper bound:

**Theorem 2** Let $A \subset \{0, 1\}^n$ that does not contain a non-trivial equilateral triangle. Then $|A| \leq \left(\frac{2}{3}\Theta(C) + o(1)\right)^n$ where $\Theta(C)$ is the capset capacity.

Since we are working in $\{0, 1\}^n$, coordinate-wise distances are either 0 or 1, and so the above result holds for any $L^p$ norm. Let us begin by deducing Theorem 1 from Theorem 2.

**Proof of Theorem 1 assuming Theorem 2** Let $\mathcal{F}$ be a family of subsets of $\{1, \ldots, n\}$, and suppose that $\mathcal{F}$ does not contain a weak $\Delta$-system. Every subset $A \subset \{1, 2, \ldots, n\}$ corresponds to a vector $x \in \{0, 1\}^n$ where $x_i = 1$ if and only if $i \in A$, and so our family $\mathcal{F}$ corresponds to a set $A \subset \{0, 1\}^n$. In this setting, three vectors $x, y, z \in \{0, 1\}^n$ form a weak $\Delta$-system if and only if $\langle x, y \rangle = \langle y, z \rangle = \langle z, x \rangle$. For $x \in \{0, 1\}^n$, the weight
of $x$ is defined to be the number of non-zero entries. If $x$, $y$, $z$ all have the same weight $w$, then $\|x\|_2^2 = \|y\|_2^2 = \|z\|_2^2$, and so $x$, $y$, $z$ form a weak $\Delta$-system if and only if $\|x - y\|_2^2 = \|y - z\|_2^2 = \|z - x\|^2$, that is, if and only if $x$, $y$, $z$ form an equilateral triangle. Let $A_w$ denote the elements of $A$ of weight $w$. Since $\sum_{w=0}^n |A_w| = |A|$, there must exist $w$ such that $|A_w| \geq |A|/n+1$. Then $A_w$ is a set that does not contain an equilateral triangle, and so by Theorem 2, $|A_w| \leq (2^{1/3} \Theta(C))^n$, and the proof is complete.

\[ \square \]

2 Subsets of $\{0, 1\}^n$ Avoiding Equilateral Triangles

To prove Theorem 2, we first prove a lemma that allows us to upper bound the density of the largest set without equilateral triangles in $\{0, 1\}^n$ by the the relative density of the largest set without equilateral triangles inside any subset of $\{0, 1\}^n$. Then we define a mapping that allows us to use the Ellenberg–Gijswijt capset bound to upper bound the density of the largest set without equilateral triangles among the elements of weight $w$.

For any $x \in \{0, 1\}^n$ define the map $f_x : \{0, 1\}^n \to \{0, 1\}^n$ by

\[ f_x(y) = x + y \pmod{2}. \]

**Lemma 1** For any $x \in \{0, 1\}^n$, $f_x$ is an isometry. That is, for any $y, z \in \{0, 1\}^n$ we have that $\|y - z\| = \|f_x(y) - f_x(z)\|$.

**Proof** Given $x, y, z$, examine coordinate by coordinate. If $y_i = z_i$, then $x_i + y_i = x_i + z_i \pmod{2}$, and so the distance is still 0. If $y_i \neq z_i$, then $x_i + y_i \neq x_i + z_i \pmod{2}$, and so once again the distance is preserved. $\square$

For any set $B \subset \{0, 1\}^n$, let $w_\Delta(B)$ denote the size of the largest subset of $B$ that does not contain an equilateral triangle, and define

\[ \delta(B) \Delta(B) = \frac{w_\Delta(B)}{|B|}. \]

Note that if $B$ does not contain any equilateral triangles, then $\delta(B) = 1$.

**Lemma 2** Let $A \subset \{0, 1\}^n$ that does not contain an equilateral triangle. Then

\[ |A| \leq 2^n \min_{B \subset \{0, 1\}^n} \delta(B). \]

**Proof** Let $A, B \subset \{0, 1\}^n$ be given, and suppose that $A$ does not contain an equilateral triangle. For every element $x \in \{0, 1\}^n$ consider $f_x(A) \cap B$. For each pair of elements, $a \in A, b \in B$, there is one and only one element $x \in \{0, 1\}^n$ such that $f_x(a) = b$, which implies that

\[ \sum_{x \in \{0, 1\}^n} |f_x(A) \cap B| = |A||B|. \]
and hence there exists \( x \in \{0, 1\}^n \) such that
\[
\frac{|A|}{2^n} \leq \frac{|f_x(A) \cap B|}{|B|}.
\]
Since \( A \) does not contain an equilateral triangle, by Lemma 1, neither does \( f_x(A) \). Hence
\[
\frac{|f_x(A) \cap B|}{|B|} \leq \delta_\Delta(B),
\]
by definition of \( \delta_\Delta \), and the lemma follows.

We say that a subset of \( \{0, 1\}^n \) is sunflower-free if it does not contain three elements \( x, y, z \), not all equal, such that \( \{x_i, y_i, z_i\} \subseteq \{0, 0, 0\}, \{1, 1, 1\}, \{0, 0, 1\} \) for every \( i \). Note that in this definition of sunflower-free, we do allow triples where two of the three are equal, and so for example \( A = \{(0, 1), (1, 1)\} \) is not sunflower-free since \( (0, 1), (0, 1), (1, 1) \) form a sunflower. For a set of vectors of a fixed weight, this definition of sunflower-free is the same if the three vectors are required to be distinct.

Consider the map \( F : \mathbb{F}_3^n \to \{0, 1\}^n \) defined coordinate-wise by \( F_i(0) = 0 \), \( F_i(1) = 1 \) and \( F_i(2) = 0 \) for each \( i \).

**Lemma 3** Let \( A \subset \{0, 1\}^n \) be a sunflower-free set. Then \( F^{-1}(A) \) is a capset.

**Proof** We will show that if \( x, y, z \in \mathbb{F}_3^n \) are not all equal, and if \( F(x), F(y), F(z) \) is not a sunflower, then \( x + y + z \neq 0 \) (mod 3). If \( F(x) = F(y) = F(z) \), then we cannot have \( x + y + z = 0 \) unless \( x = y = z \) since \( \{0, 2\}^n \) is a capset in \( \mathbb{F}_3^n \). If \( F(x), F(y), F(z) \) are not all equal, and do not form a sunflower, then there exists a coordinate \( i \) such that \( \{F(x_i), F(y_i), F(z_i)\} = \{0, 1, 1\} \). This implies that \( \{x_i, y_i, z_i\} = \{\ast, 1, 1\} \) where \( \ast \) is either a 0 or a 2, and in either case this guarantees that \( x + y + z \neq 0 \).

Let \( B_k \subset \{0, 1\}^n \) denote the set of vectors of weight \( k \).

**Theorem 3** If \( A \subset B_k \) does not contain a sunflower, we have that
\[
|A| \leq \frac{\Theta(C)^n}{2^{n-k}}
\]
where \( \Theta(C) \) is the capset capacity.

**Proof** Let \( A \subset B_k \) be a sunflower-free set. Each element in \( A \) has \( n - k \) zeros, and so
\[
|F^{-1}(A)| = |A| \cdot 2^{n-k}.
\]
The result follows since Lemma 3 implies that \( |F^{-1}(A)| \leq \Theta(C)^n \) since \( \Theta(C)^n \) bounds from above the size of the largest capset of size \( n \).

Putting this all together, we prove Theorem 2.

**Proof of Theorem 2** A set without equilateral triangles in \( \mathbb{R}^n \) gives rise to such a set in \( \mathbb{R}^{n+m} \) by appending \( m \) 0’s to each vector, so we may suppose that \( 3|n \) which can only
impact the bound by at most a factor of 4. Among the vectors of weight \( n/3 \), every sunflower is an equilateral triangle. Since

\[
|B_{n/3}| = \binom{n}{n/3} = \left( \frac{3}{2^{2/3}} + o(1) \right)^n,
\]
due to Stirling’s approximation, Theorem 3 implies that for \( B_{n/3} \),

\[
\delta_\Delta(B_{n/3}) \leq \left( \frac{\Theta(C)}{2^{2/3}} \right)^n \cdot \left( \frac{3}{2^{2/3}} + o(1) \right)^{-n} = \left( \frac{\Theta(C)}{3} + o(1) \right)^n.
\]

Lemma 2 implies that for any \( A \subset \{0, 1\}^n \) that does not contain an equilateral triangle, we have

\[
|A| \leq 2^n \delta_\Delta(B_{n/3}) \leq \left( \frac{2\Theta(C)}{3} + o(1) \right)^n,
\]
and the result follows.

\[ \square \]

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