Stretched Exponential Decay of a Quasiparticle in a Quantum Dot

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The decay of a quasiparticle in an isolated quantum dot is considered. At relatively small time the probability to find the system in the initial state decays exponentially: \( P(t) \sim \exp(-\Gamma t) \), in accordance with the golden rule. However, the contributions to \( P(t) \) accounting for the discreteness of final three-particle states, five-particle states, etc. decay much slower being \( \sim (\Delta \Gamma)^n \exp(-\Gamma t/(2n+1)) \) for \( 2n+1 \) final particles. Here \( \Delta \sim \Gamma \) is the level spacing for three-particle states available via the direct decay. These corrections are dominant at large enough time and slow down the decay to become \( \ln(P(t)) \sim -\sqrt{t} \) asymptotically. \( P(t) \) fluctuates strongly in this regime and the analytical formula for the distribution \( W(P) \) is found.

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The problem of the decay of a quasiparticle in closed mesoscopic systems has attracted a permanent interest last years [1,11]. The very existence of the (irreversible) decay becomes non-trivial in the case of a discrete spectrum. In a many particle system one should in addition understand the role of the whole hierarchy of the discreteness, ranging from the single-particle levels (level spacing \( \Delta \)) to the exponentially dense spectrum of complicated many-particle excitations. It was first realized that the decay into two particles and one hole, relevant for an infinite geometry, does not apply for the low excitation energies in the finite system. Namely, for a diffusive quantum dot, the coupling to the three-particle states describes the broadening of the quasiparticle only for excitation energy \( \epsilon \gg \sqrt{g} \Delta \), where \( g \gg 1 \) is the dimensionless conductance [2,4]. Utilization of the ideas of delocalization in the Fock space for the quasiparticle decay in Ref. [4] has stimulated further active research in this field [5,6]. However, the main interest was concentrated around the limited range of excitation energies \( \sqrt{g/\ln(g)} < \epsilon/\Delta < \sqrt{g} \). In spite of the beauty of the applied theoretical approaches experimentally it may be difficult to look for the effects sensitive to the logarithm of large parameter (see however [11]). Therefore the goal of this paper is to consider the effect of the discreteness of the spectrum for much higher energy, where the usual three-particle decay is expected to determine the quasiparticle lifetime. As we will see, even for \( \epsilon \gg \sqrt{g} \Delta \) the decay in the confined geometry at large enough time is strongly modified and slowed down.

The simple fermionic Hamiltonian models the effect of electron interaction in closed quantum dot [1] (in the nuclear physics an analogous Hamiltonian was introduced already a long time ago [11,12])

\[
H = \sum \epsilon_i c_i^+ c_i + \sum V_{ijkl} c_i^+ c_j^+ c_k c_l .
\]  

The single-particle energies \( \epsilon_i \) are randomly distributed around \( \epsilon_F = 0 \) with mean level spacing \( \Delta \). The depth of the Fermi sea is always much larger than the excitation energy. The Gaussian random two-particle interaction has zero mean and variance given by \( g \)

\[
\Gamma = 2 V^2 = \Delta^2/g^2 .
\]  

The conductance \( g \gg 1 \) measured in units of \( c^2/h \) is a large parameter in our problem. We are interested in the stability of simple single-particle excitations above the ground state of \( H \). Due to the weak interaction the low lying excited states are almost unperturbed. However, the level spacing for three-particle states (two particles and one hole) accessible for the direct decay decreases while increasing the excitation energy \( \epsilon \)

\[
\Delta_3 = 4 \Delta^3/\epsilon^2
\]  

and at \( \epsilon > \sqrt{g}\Delta \) the matrix element of the interaction \( g \) exceeds \( \Delta_3 \). Now the Fermi golden rule is applied to find the width of the quasiparticle

\[
\Gamma = 2\pi |V|^2 = \frac{\pi \epsilon^2}{2g^2 \Delta} .
\]  

The natural quantity characterising the time evolution of the single-particle state is the return probability \( P(t) \), which is the probability to find a system in the initial state \( |0\rangle \) after time \( t \)

\[
P = |\langle 0 |e^{-iHt}|0\rangle|^2 = \int e^{i(\epsilon'-\epsilon)t} G_{00}(\epsilon) G^*_{00}(\epsilon') \frac{d\epsilon'd\epsilon}{4\pi^2} .
\]  

Here we have introduced \( G_{00}(\epsilon) \), the diagonal matrix element of the single-particle Greens function of the Hamiltonian \( H \).

The authors of Ref. [4] have used the Cayley tree model in order to describe the quasi-particle disintegration into many-particle states. One site of the lattice (tree) is associated with the single-particle state which is connected with many three-particle states (sites). Each of the three-particle states is connected with a number of five-particle
states, and so on. On the Cayley tree the exact Greens function is given by the simple formula:

$$G_{00}(\varepsilon) = \frac{1}{\varepsilon_\lambda - \varepsilon_0 - \sum_n \frac{V_n^2}{(\varepsilon_\lambda - E_n)} - \sum_m \frac{V_{nm}^2}{(\varepsilon_\lambda - E_m)} - \ldots}$$

(6)

where $\varepsilon_\lambda = \varepsilon + i\lambda$ with some small $\lambda$. Each new denominator of the continued fraction corresponds to a mixing with the next generation (a set of states in the Fock space having one more excited particle and one more hole). $E_n = \varepsilon_n + \varepsilon_n + \varepsilon_n, E_m = \varepsilon_m + \varepsilon_m + \varepsilon_m + \varepsilon_m + \varepsilon_m, and$ so on. $V_n$ is the matrix element connecting the single-particle and three particle states and $V_{nm}$ is the matrix element connecting the $n$-th three-particle and $m$-th five-particle states. All matrix elements satisfy Eq. (6). Not all effects of the two-particle interaction are taken into account by the Cayley tree model. Nevertheless, one may associate any term of the perturbative expansion (in $V$) of Eq. (6) with a certain class of true diagrams of many-body perturbation theory. The diagrams not present in Eq. (6) should be taken into account separately, but they will not change the results of this paper. In particular, the interaction of the initial particle with the three-particle states is completely taken into account by the Cayley tree. Replacing the first sum in the denominator in Eq. (6) by the integral over averages, one gets

$$\overline{G_{00}(\varepsilon)} = G_0(\varepsilon) = (\varepsilon - \varepsilon_0 + i\Gamma/2)^{-1},$$

(7)

where the width

$$\Gamma = 2 \int \frac{V^2}{\varepsilon_\lambda - \varepsilon - i\Gamma/2} \, dx = 2\pi \frac{V^2}{\Delta_3}$$

(8)

effectively does not depend on the secondary width $\Gamma_n$. The pole of $G_0$ leads to the exponential decay of the return probability $P(t) \sim \exp(-\Gamma t)$ and it is hard to change this result while considering separately $G(\varepsilon)$ and $G'(\varepsilon')$. The slow tail of $P(t)$ at large $t$ may only come from terms in the correlation function $\overline{G_{00}(\varepsilon)}G_{00}(\varepsilon')$ that are non-analytic in $\varepsilon - \varepsilon'$. Let at first stage forget about the decay of secondary states in Eq. (6) (i.e. we put $V_{nm} = 0$). Now the expansion of the Greens function (6) around the averaged value (Eq. (7)) gives

$$G_{00} = G_0 + G_0^2 \left\{ \sum_n \frac{V_n^2}{\varepsilon_\lambda - E_n} - \int \frac{V^2}{\varepsilon_\lambda - x} \, dx \right\},$$

(9)

Again $\varepsilon_\lambda = \varepsilon + i\lambda$. Only the correlation of the second term {...} in $G_{00}(\varepsilon)$ with its analog in $G_{00}(\varepsilon')$ may lead to the correction singular in $\varepsilon - \varepsilon'$. It is convenient to replace the first sum in Eq. (8) also by an integral via

$$\sum F(E_n) \to \int \sum \delta(x - E_n)F(x) \, dx.$$ 

Now the only non-trivial part in the calculation of the correlation function is the averaging

$$\left( \sum_n V_n^2 \delta(x - E_n) \right) - \frac{V^2}{\Delta_3} \left( \sum_n V_{nm}^2 \delta(y - E_m) - \frac{V^2}{\Delta_3} \right) = \frac{V^2}{\Delta_3} \delta(x - y).$$

(10)

We suppose that $\varepsilon_n$ are independent random variables (Poisson statistics). In fact the problem of the decay of a simple state into the quasi-continuum of discrete states (just like our decay into three particles) has already been considered by several authors [14–16]. In particular, the case of constant interaction and Wigner-Dyson statistics for secondary states is considered in [14].

With the use of Eqs. (8,11) one finds ($G_0' \equiv G_0(\varepsilon')$)

$$G_{00}(\varepsilon)G_{00}(\varepsilon') = G_0G_0' + (G_0G_0')^2 \frac{2\pi i V^4}{\varepsilon - \varepsilon - 2i\lambda \Delta_3},$$

(11)

which after substitution into Eq. (3) gives (for Gaussian random interaction $V^2 = 3\Delta^2/2g^2$)

$$P(t) = e^{-\Gamma t} + \frac{3\Delta_3}{2\pi\Gamma} \left( \frac{\Gamma}{\Gamma - 2\lambda} \right)^3 e^{-2\lambda t}.$$

(12)

Here $2\lambda$ works effectively as the constant intrinsic width of the three-particle excited states. After the initial state spreads over the $\sim \Gamma/\Delta_3$ directly connected states, the usual Breit-Wigner decay saturates. It is important for this result that the interaction matrix elements and/or the three-particle interlevel spacings essentially fluctuate. Otherwise the coherent decay may proceed up to much larger times ($\Delta_3^{-1}$) and even may become reversible.

The many particle origin of the final states is in fact ignored in Eq. (12). In the real problem the energy and width of the “intermediate” final state are given by the sum of energies and widths of the constituent particles $E_n = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \approx \varepsilon$ and $\Gamma_n = \sum \Gamma_i \sim \sum \varepsilon_i^2$ [4]. In order to take into account this energy dependence of the secondary width one should return to Eq. (3) and consider the three-particle density of states as the sum over individual particle energies. This means that in Eq. (12)

$$\frac{e^{-2\lambda t}}{\Delta_3} \to \int \exp \left\{ - \sum_{i=1}^{3} \Delta_i t \right\} \delta(\varepsilon - \sum \varepsilon_i) \prod \frac{d\varepsilon_i}{\Delta}$$

(13)

The return probability now takes the form

$$P = e^{-\Gamma t} + \frac{27\sqrt{3}}{4} \frac{\Delta_3}{\Gamma t} e^{-\Gamma t/3} + \ldots.$$

(14)
Here the states from the second generation are allowed to decay and their widths are distributed within $\Gamma/3 < \Gamma_n < \Gamma$. However, the fluctuations of the widths, and even more important, correlations of the fluctuations, are still not taken into account. The second term in the r.h.s. of Eq. (14) is not small necessarily and for $\Gamma > (3/2)\Gamma$ even becomes dominant. Here we introduced a special notation for the large logarithm $L = \ln(\Gamma/\Delta_3)$. However even at these times the prefactor $1/\Gamma t$ in the return probability is only logarithmically small. We will neglect the contributions of this type in the calculation of further corrections to the return probability.

In the way similar to the derivation of Eq. (3) we may extract from the Greens function the term responsible for fluctuations in the coupling of three-particle to five-particle states

$$G_{00} = G_0 + \ldots + G^2_{00} \sum_n G^2_n V^2_n \tag{15}$$

$$\times \int \left( \sum_m V^2_{nm} \delta(x - E_m) - \frac{V^2}{\Delta_n} \right) \frac{dx}{\varepsilon_\lambda - x} + \ldots .$$

Here $\Delta_n \sim \Delta_3$ is the interval between the five-particle states into which the given three-particle state decays. The connected average of the five-particle term in Eq. (15) with the same contribution from $G_{00}(\varepsilon'_i)$ leads to a pole at $\varepsilon' - \varepsilon = i\Gamma/5$ in complete analogy with Eq. (13). Here again $\Gamma/5$ is the smallest total width for the five particles with given total energy $\sum \varepsilon_i = \varepsilon$. The straightforward estimation of the five-particle contribution to the return probability gives

$$P_5 \sim (\Delta_3/\Gamma)^2 e^{-\Gamma t/5} \tag{16}$$

Neither the overall numerical constant nor the “weak” $\sim 1/\Gamma t$ prefactor are shown here. Non-Cayley-tree contributions to $P_5$ coming from the corrections to $G_{00}$ not included in the continued fraction Eq. (3) are $\Gamma/\Delta$ times smaller and decay also like $e^{-\Gamma t/5}$ (or faster) at large time $t$.

Generalisation of Eq. (14) for arbitrary number of final particles gives

$$P_{2n+1} \sim (\Delta_3/\Gamma)^n e^{-\Gamma t/(2n+1)} . \tag{17}$$

Adding one more particle-hole pair in the final state costs a small factor $\Delta_3/\Gamma$, but the time evolution of this contribution is governed by the smallest joint width for $2n+1$ particles with fixed total energy, which is $\Gamma/(2n+1)$. The total return probability is given by the sum over all many particle contributions (17) (starting from $n = 0$). However, at any given moment of time one of the $P_{2n+1}$ dominates and all others may be neglected. Figure 1 shows the function $\ln P(t)$ and illustrates how it is formed by different contributions from Eq. (17). The two consecutive values $P_{2n+1}$ coincides and the strength of decay is changed at $\Gamma t = (2n - 1)(2n + 1)L/2$. For large $\Gamma t$ one may also replace the piecewise linear $\ln P$ by single smooth function (see Figure) such that

$$P_{anym} \sim e^{-\sqrt{2\Delta t}} = \exp \left\{ -\sqrt{\ln \left( \frac{\varepsilon}{\sqrt{g\Delta}} \right)} \frac{4\pi \varepsilon^2}{g^2 \Delta} \right\} \tag{18}$$

This formula should be compared with the usual decay $\exp(-\Gamma t) = \exp(-\pi \varepsilon^2 t/2g^2 \Delta)$. Eq. (18) is the main result of this paper. Since the true many particle density of states is $\delta^{-1} \sim \exp(-2\pi \varepsilon/6\Delta)$ this decay may formally proceed until time $t \sim (g^2/\varepsilon)1/\ln(\varepsilon/\sqrt{g\Delta})$.

Within the interval $3L/2 < \Gamma t < 15L/2$ the second term in Eq. (13) dominates. Consider the distribution of $P(t)$ in this region. The direct generalisation of the calculation described by Eqs. (13)(12) gives

$$P(t)^n = n!P(t)^n \tag{19}$$

Here $n!$ simply accounts for the number of ways how $n$ Greens functions $G_{00}(\varepsilon_i)$ may be contracted with $n$ $G_{00}(\varepsilon'_i)$. The distribution function corresponding to the eq. (19) is evidently

$$W(P) = 1/P \exp \{ -\ln P \} . \tag{20}$$

Moreover, at any time only one contribution (17), corresponding to fixed number of final particles, dominates in $P(t)$. Therefore, Eq. (19) and consequently Eq. (20) are also valid for any large time $\Gamma t > 3L/2$. The strong fluctuations of $P(t)$ in the asymptotics (20) are in contrast with the usual decay of the return probability at $\Gamma t < 3L/2$, where one has $\var P(t) \ll P(t)^2$.

It is seen from Eq. (20) that the stretched asymptotic decay of a quasiparticle in a finite system described by Eq. (13) does not require any special realization of the quantum dot. The interesting asymptotic effects due to rare fluctuations of disorder were considered more than a decade ago in Ref. [18]. (See also the recent discussion of the decay in case of strong interaction [19].)

All the above results only correspond to the case of relatively high excitation energy. Below $\varepsilon \sim \sqrt{g\Delta}$ the width expected from the golden rule become smaller then the level spacing for available three-particle states $\Gamma < \Delta_3$ [1]. Still at $\sqrt{g} \ln g \ll \varepsilon/\Delta \ll \sqrt{g}$ quasiparticles are unstable [1]. Not much is known about the time dependence of the return probability $P(t)$ in this region. The decay now is not exponential at any time. One may convert the analysis of Ref. [1] in order to get the time dependent series

$$P_{1<\Delta_3} = 1 - \frac{\varepsilon^2}{g\Delta^2} \sum_{n=0} c_n \left( \frac{\varepsilon^2}{g\Delta^2} \ln \left( \frac{\Delta}{g} \right) \right)^n , \tag{21}$$

where the $c_n$ are some (unknown) numerical coefficients. Unfortunately, Eq. (21) is valid only for $1 - P \ll 1$. This
result may be used for an estimation of the time after which the decay of the return probability starts, but it does not teach us about the functional form of $P(t)$.

To conclude, in this paper we have considered the decay of a single-particle excitation in a finite fermionic system with random two-particle interaction. Our results may be interpreted as follows. At short time $t$ the decay into two-particles and one hole proceeds with the Breit-Wigner width $\Gamma$, in accordance with the golden rule. At sufficiently large time this piece-wise linear exponential decay may be described by the smoothed formula (18) with $\ln(\Gamma L)$. When the decay of a single-particle excitation in a finite fermionic system with random two-particle interaction. Our results may be interpreted as follows. At short time $t$ the decay into two-particles and one hole proceeds with the Breit-Wigner width $\Gamma$, in accordance with the golden rule. At sufficiently large time this piece-wise linear exponential decay may be described by the smoothed formula (18) with $\ln(\Gamma L)$. When the decay of a single-particle excitation in a finite fermionic system with random two-particle interaction. Our results may be interpreted as follows. At short time $t$ the decay into two-particles and one hole proceeds with the Breit-Wigner width $\Gamma$, in accordance with the golden rule. At sufficiently large time this piece-wise linear exponential decay may be described by the smoothed formula (18) with $\ln(\Gamma L)$.

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FIG. 1. The logarithm of the return probability $P$ as a function of $\Gamma t$, both measured in units of large logarithm $L = \ln(\Gamma L)$ (thick polygonal line). The thin dashed lines are the individual contributions due to the three-particle decay ($-\Gamma L$) and due to the correlations in the $2n + 1$-particle final state ($\sim -\Gamma L/(2n + 1)$). The thick dashed line is the smoothed asymptotics $P \sim \exp(-2\Gamma L)$. 

To perform the averaging it is convenient to write each multiplier here as a sum of two terms

$$ \sum (V_n^2 - V_n^2) \delta(x - \varepsilon_n) + \gamma L \sum \delta(x - \varepsilon_n) - \Delta^3 \gamma^{-1} ,$$

one responsible for the fluctuations of the potential $V_n$ and another for the positions of the levels $\varepsilon_n$. 
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