1. Introduction

Given a primitive sublattice $P$ of signature $(1, \rho - 1)$ of the K3 lattice $\Lambda_{K3} := E_8 \perp E_8 \perp U \perp U \perp U$, let $T = T_P := P^\perp$ be the orthogonal lattice and $\Gamma = \Gamma_P$ be the subgroup of $O(\Lambda_{K3})$ consisting of elements $g$ satisfying $g|_P = \text{id}_P$. By abuse of notation, we write the image of $\Gamma$ under the injection $g \mapsto g|_T \in O(T)$ by the same symbol. A $P$-polarized K3 surface in the sense of Nikulin [Nik79a] is a pair $(Y, j)$ of a K3 surface $Y$ and a primitive lattice embedding $j : P \hookrightarrow \text{Pic} Y$. As explained in [Dol96, Section 3], the global Torelli theorem [PSŞ71, BR75] and the surjectivity of the period map [Tod80] shows that the period map gives an isomorphism from the coarse moduli scheme of pseudo-ample $P$-polarized K3 surfaces to the quotient $M := D/\Gamma$ of the bounded Hermitian domain

$$\mathcal{D} := \{ [\Omega] \in \textbf{P}(T \otimes \mathbb{C}) \mid (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0 \}$$

of type IV.

Let

$$\tilde{\mathcal{D}} := \{ \Omega \in T \otimes \mathbb{C} \mid (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0 \}$$

be the total space of a principal $\mathbb{C}^\times$-bundle over $\mathcal{D}$. A modular form of weight $k \in \mathbb{Z}$ and character $\chi \in \text{Char}(\Gamma) := \text{Hom}(\Gamma, \mathbb{C}^\times)$ is a holomorphic function $f : \tilde{\mathcal{D}} \rightarrow \mathbb{C}$ satisfying

(i) $f(\alpha z) = \alpha^{-k} f(z)$ for any $\alpha \in \mathbb{C}^\times$, and

(ii) $f(\gamma z) = \chi(\gamma) f(z)$ for any $\gamma \in \Gamma$.

The vector spaces $A_k(\Gamma, \chi)$ of modular forms constitute the ring

$$\tilde{A}(\Gamma) := \bigoplus_{k=0}^{\infty} \bigoplus_{\chi \in \text{Char}(\Gamma)} A_k(\Gamma, \chi)$$

of modular forms. We also write the subring of modular forms without characters as

$$A(\Gamma) := \bigoplus_{k=0}^{\infty} A_k(\Gamma).$$

The main result of this paper is the following:

**Theorem 1.1.** The graded ring of modular forms with characters of $O(2, 4; \mathbb{Z})$ is generated by modular forms of weights 4, 4, 6, 8, 10, 12, and 30 with three relations of weights 8, 20, and 60:

$$\tilde{A}(O(2, 4; \mathbb{Z})) \cong \mathbb{C}[t_4, t_6, t_8, t_{10}, t_{12}, s_4, s_{10}, s_{30}]/(s_4^2 - \Delta_8(t), s_{10}^2 - \Delta_{20}(t), s_{30}^2 - \Delta_{60}(t)),$$

where the polynomials $\Delta_8(t)$, $\Delta_{20}(t)$, and $\Delta_{60}(t)$ are given in (3.4), (2.12), and (3.6) respectively.

This paper is organized as follows: In Section 2, we prove that the coarse moduli scheme of $U \perp E_7 \perp E_7$-polarized K3 surfaces is the double cover of the weighted projective space $\textbf{P}(4, 6, 8, 10, 12)$ branched along the divisor defined by $\Delta_{20}(t)$. We give a proof of Theorem 1.1 in Section 3. In Section 4, we discuss the relation with the configuration space of six lines on $\textbf{P}^2$ following [MSY92].

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Theorem 2.1. The graded ring $A(\Gamma_1)$ of automorphic forms with respect to $\Gamma_1$ is given by
\begin{equation}
A(\Gamma_1) = \mathbb{C}[t_4, t_6, t_8, t_{10}, s_{10}, t_{12}]/(s_{10}^2 - \Delta_20(t)),
\end{equation}
where the lower indices indicate the weights, and $\Delta_20(t)$ is defined in (2.12).

Proof. Giving a $P_1$-polarized K3 surface is equivalent to giving an elliptic K3 surface with a section and two singular fibers containing an $E_7$-configuration (i.e., of Kodaira type $II^*$ or $III^*$), which we may assume to lie above 0 and $\infty$ on the base $\mathbb{P}^1$. An elliptic K3 surface with a section admits a Weierstrass model of the form
\begin{equation}
z^2 = y^3 + g_2(x, w)y + g_3(x, w)
\end{equation}
in $\mathbb{P}(1, 4, 6, 1)$ (cf. e.g. [SS10, Section 4]). Recall that the elliptic surface (2.2) has a singular fiber of type either $II^*$ or $III^*$ at $a \in \mathbb{P}^1$ only if $\text{ord}_a g_2(x, w) \geq 3$ and $\text{ord}_a g_3(x, w) \geq 5$ (cf. e.g. [Mir89, Table IV.3.1]). This requires
\begin{align}
g_2(x, w) &= u_{5,3}x^5w^3 + u_{4,4}x^4w^4 + u_{3,5}x^3w^5, \\
g_3(x, w) &= u_{7,5}x^7w^5 + u_{6,6}x^6w^6 + u_{5,7}x^5w^7.
\end{align}

It has a singularity worse than rational double points on the fiber at $a \in \mathbb{P}^1$ if and only if $\text{ord}_a(g_2) \geq 4$ and $\text{ord}_a(g_3) \geq 6$ (cf. e.g. [Mir89, Proposition III.3.2]). This is the case if and only if either $u_{3,5} = u_{5,7} = 0$ (for $a = 0$) or $u_{5,3} = u_{7,5} = 0$ (for $a = \infty$). The parameter
\begin{equation}
u = (u_{5,3}, u_{4,4}, u_{3,5}, u_{7,5}, u_{6,6}, u_{5,7}) \in U := \mathbb{C}^6 \setminus \{u_{3,5} = u_{5,7} = 0 \text{ or } u_{5,3} = u_{7,5} = 0\}
\end{equation}
appearing in the Weierstrass model (2.2) is unique up to the action of $(\mathbb{C}^\times)^2$ given by
\begin{align}
\mathbb{C}^\times &\ni \lambda: ((x, y, z, w), (u_{i,j})_{i,j}) \mapsto (x, \lambda^2 y, \lambda^3 z, w), (\lambda^{(i+j)/2}u_{i,j})_{i,j}) \\
\mathbb{C}^\times &\ni \mu: ((x, y, z, w), (u_{i,j})_{i,j}) \mapsto ((\mu^{-1}x, y, z, \mu w), (\mu^{-1}j u_{i,j})_{i,j}).
\end{align}

Note that the former $\mathbb{C}^\times$-action rescales the holomorphic volume form
\begin{equation}
\Omega = \text{Res} \ \frac{w dx \wedge d y \wedge d z}{z^2 - y^3 - g_2(x, w)y - g_3(x, w; u)}
\end{equation}
as
\begin{equation}
\Omega_{\lambda u} = \text{Res} \ \frac{w dx \wedge d(\lambda^2 y) \wedge d(\lambda^3 z)}{(\lambda^3 z)^2 - (\lambda^2 y)^3 - g_2(x, w; \lambda u)(\lambda^2 y) - g_3(x, w; \lambda u)} = \lambda^{-1}\Omega_u,
\end{equation}
whereas the latter (which comes from the automorphism of the base $\mathbb{P}^1$ fixing 0 and $\infty$) keeps it invariant. The categorical quotient $T := U/C^\times_u$ is the coarse moduli scheme of pairs $(Y, \Omega)$ consisting of a $P_1$-polarized K3 surface $Y$ and a holomorphic volume form $\Omega \in H^0(\omega_Y)$ on $Y$. The coordinate ring $\mathbb{C}[T]$ of $T$ is generated by six elements
\begin{align}
t_4 &:= u_{4,4}, \\
t_6 &:= u_{6,6}, \\
t_8 &:= u_{5,3}u_{3,5}, \\
t_{10} &:= u_{5,3}u_{5,7} + u_{3,5}u_{7,5}, \\
ts_{10} &:= u_{5,3}u_{5,7} - u_{3,5}u_{7,5}, \\
t_{12} &:= u_{7,5}u_{5,7},
\end{align}
with one relation
\begin{equation}
s_{10}^2 = \Delta_20(t) := t_{10}^2 - 4t_8t_{12}.
\end{equation}
The boundary of the affinization $\overline{T} := \text{Spec} \mathbb{C}[T]$ is given by
\begin{equation}
\{ t_8 = t_{10} = s_{10} = t_{12} = 0 \} \cong \mathbb{C}_{t_4} \times \mathbb{C}_{t_6}.
\end{equation}
The period map induces an isomorphism of the graded ring of modular forms and the coordinate ring $\mathbb{C}[T]$. The weight of the modular form is identified with the weight of the $\mathbb{C}_x^\times$-action, and Theorem 1.1 is proved.

3. MODULAR FORMS OF $O(2,4;\mathbb{Z})$ WITH CHARACTERS

The lattice $T_1$ has a unique extension $T_1 \subset T_2 \subset T_1 \otimes \mathbb{Q}$ of index 2 to an odd unimodular lattice $T_2 \cong U \uplus \langle -1 \rangle \uplus \langle 1 \rangle \uplus \langle -1 \rangle \uplus \langle 1 \rangle$, and $T_1$ is the sublattice of $T_2$ consisting of even elements;
\begin{equation}
T_1 \cong \{ v \in T_2 \mid \langle v, v \rangle \in 2\mathbb{Z} \}.
\end{equation}

It follows that $\Gamma_2 := O(2,4;\mathbb{Z}) := O(T_2)$ can naturally be identified with $O(T_1)$, so that
\begin{equation}
A(O(T_2)) \cong A(O(T_1)) = A(\Gamma_1)^{\langle \sigma_1 \rangle}.
\end{equation}
Since $\sigma_1$ acts on $U$ by sending $u_{i,j}$ to $u_{j,i}$, one obtains a proof of the following:

\textbf{Theorem 3.1} ([Vin10, Theorem 1]). The graded ring $A(O(2,4;\mathbb{Z}))$ of automorphic forms is given by
\begin{equation}
A(O(2,4;\mathbb{Z})) = \mathbb{C}[t_4, t_6, t_8, t_{10}, t_{12}]
\end{equation}
where the lower indices indicate the weights.

In fact, this proof of Theorem 3.1 already appears in [CMS19]. In particular, (2.2) is identical to [CMS19, (4.13)] up to an obvious change of coordinates. Note that ‘isomorphisms of $H \uplus E_7(\mathbb{Z})$ lattice polarized K3 surfaces’ in [CMS19, Proposition 4.3,(b)] come from the action of $\sigma_1$, and as such are not isomorphisms of lattice polarized K3 surfaces in the sense of [Dol96].

The coarse moduli space $M := D / O(T_1)$ of $P_1$-polarized K3 surfaces up to the action of $\sigma_1$ is an open subvariety of its Satake–Baily–Borel compactification Proj $A(O(T_1)) \cong \mathbb{P}(4,6,8,10,12)$. Although $M$ and the the orbifold quotient $\mathbb{M} := [D / O(T_1)]$ are closely related, the canonical morphism $\mathbb{M} \rightarrow M$ is not an isomorphism even in codimension 1. In order to obtain an orbifold which is isomorphic to $\mathbb{M}$ in codimension 1 (so that the total coordinate rings are isomorphic), we first consider the stacky weighted projective space $\mathbb{P} := \mathbb{P}(4,6,8,10,12)$, defined as the quotient of $\mathbb{C}^5 \setminus \mathbf{0}$ by a $\mathbb{C}^\times$-action with this weight. The morphism $\mathbb{M} \rightarrow M$ lifts to a morphism $\mathbb{M} \rightarrow \mathbb{P}$, which is an isomorphism in codimension 0, since the generic stabilizers are $\{ \pm \text{id} \}$ on both sides.

Stabilizers of $\mathbb{M}$ along divisors come from reflections, and besides $\sigma_1$ appearing above, there are two more reflections that one can easily find in $O(T_1)$. The first one, which we call $\sigma_2$, is the reflection along the $(−2)$-vector whose reflection hyperplane is defined by
\begin{equation}
\Delta_8(t) := t_8.
\end{equation}
In terms of $P_1$-polarized K3 surfaces, this divisor corresponds to the locus where the Picard lattice contains $U \uplus E_7 \uplus E_8$. The second one, which we call $\sigma_3$, is the reflection along the $(−2)$-vector whose reflection hyperplane corresponds to the locus where the Picard lattice contains $U \uplus E_7 \uplus E_7 \uplus A_1$. In order to describe this locus, first consider the discriminant $4g_2(x,w,t)^3 + 27g_3(x,w,t)^2$ of $y_9^3 + g_2(x,w,t)y + g_3(x,w,t)$ as a polynomial of $y$, which is the product of $x^9w^9$ and a homogeneous polynomial $h(x,w,t)$, of degree 6 in $(x,w)$ and degree 12 in $t$. Note that the discriminant of the polynomial $\sum_{i=0}^{n} a_i x^i w^{n-i}$ with respect to $(x,w)$ is homogeneous of degree $2(n-1)$ in $\mathbb{Z}[a_0,\ldots,a_n]$ if $\deg a_0 = \cdots = \deg a_n = 1$. It follows that the discriminant $k_{120}(t)$ of $h(x,w,t)$ with respect to $(x,w)$ is a homogeneous polynomial of degree $2 \cdot 5 \cdot 12 = 120$ in $t$. A general point on the divisor of $\mathbb{P}(4,6,8,10,12)$ defined by $k_{120}(t)$ corresponds to the locus where two fibers of Kodaira type $I_1$ collapse into one fiber. This divisor has two components; a general point on one corresponds to the case when there exists a point $p = [x:w]$ on $\mathbb{P}^1$ such that neither $g_2$ nor $g_3$ vanishes at $p$, and a general point on the other component corresponds to the case when both $g_2$ and $g_3$ vanishes at $p$. In the former case, the resulting singular fiber is of Kodaira type $I_1$, and the surface acquires an $A_1$-singularity. In the latter case, the resulting singular fiber is of Kodaira type $I_2$, and the surface
does not acquire any new singularity. The defining equation of the latter component is the resultant of \(g_2\) and \(g_3\). It is given as the determinant

\[
(3.5) \quad r_{20}(t) = \begin{vmatrix} u_{5,3} & u_{4,4} & u_{3,5} \\ u_{5,3} & u_{4,4} & u_{3,5} \\ u_{7,5} & u_{6,6} & u_{5,7} \end{vmatrix}
\]

of the Sylvester matrix, which is homogeneous of degree 20. As shown in [HU, Lemma 6.1], the polynomial \(k_{120}(t)\) is divisible by \(r_{20}(t)^3\), and a direct calculation using a computer algebra system shows that the quotient

\[
(3.6) \quad \Delta_{60}(t) := k_{120}(t)/r_{20}(t)^3
\]

is irreducible.

Recall from [AGV08, Cad07] that the root construction is an operation which adds a stabilizer along a divisor. Let \(T\) be the stack obtained from \(P\) by the root construction of order 2 along the divisor on \(P\) defined by \(\Delta_{88}(t) := \Delta_8(t)\Delta_{20}(t)\Delta_{60}(t)\), which is the quotient of the double cover of \(P\) branched along \(\Delta_{88}(t)\) by the group \(G\) of deck transformations. The Picard group of \(T\) (or the \(G\)-equivariant Picard group of \(P\)) is generated by the pull-back \(O_T(1) := p^*O_P(1)\) of the generator \(O_P(1)\) of the Picard group of \(P\) by the structure morphism \(p: T \to P\) and three line bundles \(O_T(D_i)\) for \(i = 4, 10, 30\) such that the space \(H^0(O_T(D_i))\) is generated by an element \(s_i\) satisfying \(s_i^2 = \Delta_i \in H^0(O_T(i)) \cong H^0(O_P(i))\).

The ramification formula for the canonical bundle gives

\[
(3.7) \quad \omega_T \cong p^*(\omega_P \otimes O_T(D_4 + D_{10} + D_{30}))
\]

\[
(3.8) \quad \cong O_T(-40) \otimes O_T(D_4 + D_{10} + D_{30})
\]

\[
(3.9) \quad \cong O_T(4) \otimes O_T(-44 + D_4 + D_{10} + D_{30}).
\]

Note that \(O_T(-44 + D_4 + D_{10} + D_{30})\) is an element of order two in \(\text{Pic}\, T\). By comparing (3.9) with

\[
(3.10) \quad \omega_{M_0} \cong O_M(4) \otimes \det
\]

which follows from (the proof of) [HU, Proposition 5.1], one concludes that \(M\) has no further stabilizer along a divisor, so that the lift \(M \to T\) of \(M \to P\) is an isomorphism in codimension 1. It follows that \(\text{Pic}\, M \cong \text{Pic}\, T\) is isomorphic to \(\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3\), and the total coordinate ring (also known as the Cox ring) of \(M\) is given by

\[
(3.11) \quad \bigoplus_{L \in \text{Pic}\, M} H^0(L) \cong \mathbb{C}[t_4, t_6, t_8, t_{10}, t_{12}, s_4, s_{10}, s_{30}]/(s_4^2 - \Delta_8(t), s_{10}^2 - \Delta_{20}(t), s_{30}^2 - \Delta_{60}(t)).
\]

A character of \(O(T_1)\) gives a line bundle on \(M\), so that the ring \(\tilde{A}(O(T_1))\) of modular forms with characters is a subring of the total coordinate ring, which in fact is the whole of it since any line bundle on \(M\) comes from a character in the case at hand. This can be seen by noting that the universal cover \(\tilde{D} \to \tilde{M}\) factors through a \((\mathbb{Z}/2\mathbb{Z})^3\)-cover defined by the equation appearing on the right hand side of (3.11), so that the line bundles \(O\tilde{D}(-i + D_i)\) for \(i = 4, 10, 30\) come from characters of the orbifold fundamental group of \(\tilde{M}\), which is isomorphic to \(O(T_1)\) by definition.

Note that (4.5) below produces two distinct elements of order two in \(\text{Char}(O(T_1))\); one comes from the sign representation of \(\mathfrak{S}_6\) and the other comes from the unique non-trivial representation of \(\langle \tau \rangle\). Yet another element of order two comes from the determinant representation \(\text{det}: O(T_1) \to \{\pm 1\}\), which does not belong to the subgroup of \(\text{Char}(O(T_1))\) generated by the above two characters. Since \(\text{det}(g) = -1\) for any reflection \(g \in O(T_1)\), the character of the modular form \(s_4s_{10}s_{30}\) is \(\text{det}\). As we explain in Section 4, the characters of the modular forms \(s_{10}\) and \(s_{30}\) come from the non-trivial representation of \(\langle \tau \rangle\) and the sign representation of \(\mathfrak{S}_6\) respectively.

4. Configuration of six lines on the plane

Let \(L_1, \ldots, L_6\) be six lines on \(P^2\) in very general position, and \(Y\) be the K3 surface obtained as the resolution of 15 ordinary double points on the double cover of \(P^2\) branched along the union of these six lines. Fix an isometry \(H^2(Y; \mathbb{Z}) \cong \Lambda_{K3}\) and regard the Picard lattice \(P_3\) of \(Y\) as a sublattice of \(\Lambda_{K3}\). The lattice \(P_3\) is an even lattice of signature \((1, 15)\) generated by the classes of strict transforms
of the lines and 15 exceptional divisors. The primitive embedding of $P_3$ into $\Lambda_{K3}$ is unique up to the action of $O(\Lambda_{K3})$ by [Nik79b, Theorem 1.14.4]. The orthogonal lattice $T_3$ of $P_3$ inside $\Lambda_{K3}$ is isometric to $T_{2}(2) \cong U(2) \perp U(2) \perp A_1 \perp A_1$, so that one has
\begin{equation}
O(T_3) \cong O(T_2).
\end{equation}
Set $\Gamma_3 := \Gamma_{P_3}$ and $\Gamma(2) := \{g \in O(T_3) \mid g \equiv \text{id}_{T_3} \mod 2\}$.

**Lemma 4.1.** One has $\Gamma_3 = \Gamma(2)$.

**Proof.** It follows from [Nik79b, Corollary 1.5.2] that $\Gamma_3$ is the kernel of the natural homomorphism from $O(T_3)$ to the group of automorphisms of the discriminant group of $T_3$ (i.e., the quotient of the dual lattice $T^\vee := \text{Hom}(T_3, \mathbb{Z})$ by the natural injection $T_3 \hookrightarrow T^\vee$).

It is shown in [MSY92, Proposition 2.7.3 and Corollary 2.7.4] that $\Gamma(2)$ is a reflection group generated by reflections along 20 $(-2)$-vectors given on [MSY92, p. 103]. One can easily check that every reflection acts on $T^\vee_2/T_3$ as the identity, so that $\Gamma(2) \subset \Gamma_3$.

On the other hand, it is shown in [MSY92, Proposition 2.8.2] that the quotient group $O(T_3)/\Gamma(2)$ is the finite group given by
\begin{equation}
\left\{ \begin{pmatrix} Y_4 & 0 \\ 0 & \eta_2 \end{pmatrix} \mid Y_4 \in \text{Sp}(2, \mathbb{Z}/2\mathbb{Z}), \eta_2 \in \{I_2, U\} \right\},
\end{equation}
where
\begin{equation}
\text{Sp}(2, \mathbb{Z}/2\mathbb{Z}) := \{Y_4 \in \text{GL}(4, \mathbb{Z}/2\mathbb{Z}) \mid Y_4^T(U \oplus U)Y_4 = U \oplus U\}.
\end{equation}
Note that $\text{Sp}(2, \mathbb{Z}/2\mathbb{Z})$ is generated by
\begin{equation}
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ I_2 \end{pmatrix}, \begin{pmatrix} U & I_2 \\ I_2 & U \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} U \\ 1 \\ 0 \end{pmatrix}, \text{and} \begin{pmatrix} I_2 \\ 1 \\ 1 \end{pmatrix},
\end{equation}
and isomorphic to the symmetric group $\mathfrak{S}_6$ of degree 6. One can easily see that if $h \in O(T_3)/\Gamma(2)$ is not the identity, then $h$ induces a non-trivial transformation on $T^\vee_2/T_3$, so that $\Gamma(2) = \Gamma_3$. \hfill \square

We write the element of order 2 in $O(T_3)/\Gamma_3$ represented by $I_4 \oplus U$ as $\tau$, so that
\begin{equation}
O(T_3)/\Gamma_3 \cong \mathfrak{S}_6 \times \langle \tau \rangle.
\end{equation}

The *Igusa quartic* is the Siegel modular variety of genus 2, which can naturally be identified with the moduli spaces of
- principally polarized abelian surfaces,
- genus two curves, and
- hyperelliptic curves of genus two, i.e., configurations of six points on $\mathbb{P}^1$.

It can be described as a quartic hypersurface in $\mathbb{P}^4$, defined by the equations
\begin{equation}
\sum_{i=0}^5 x_i = 0,
\end{equation}
\begin{equation}
\left(\sum_{i=0}^5 x_i^2\right)^2 = 4 \sum_{i=0}^5 x_i^4
\end{equation}
in $\mathbb{P}^5$. The GIT quotient $\overline{X}(3, 6)$ of $(\mathbb{P}^2)^6$ by the action of $\text{PGL}_3 \cong \text{Aut} \mathbb{P}^2$ with respect to the democratic weight is known to be the double cover of $\mathbb{P}^4$ branched along the Igusa quartic by [DO88]. The period map from the configuration space $X(3, 6)$ of six lines on $\mathbb{P}^2$ in general position to the modular variety $M_3 := D/\Gamma_3$ extends to an isomorphism from $\overline{X}(3, 6)$ to the Satake–Baily–Borel compactification $\overline{M}_3$ of $M_3$ by [MSY92]. As explained in [MSY92, Section A.2], the action of $\tau$ on $M_3$ gives an involution on $M_3$ whose fixed locus is the moduli space of six points on a conic, which can naturally be identified with the Igusa quartic; the natural projection $M_3 \to M_3/\langle \tau \rangle \cong \mathbb{P}^4$ is a double cover of $\mathbb{P}^4$ branched along the Igusa quartic. The residual action of $\mathfrak{S}_6$ on $M_3/\langle \tau \rangle$
is the projectivization of the natural action of $\mathcal{S}_6$ on $\{(x_1, \ldots, x_6) \in \mathbb{A}^6 \mid x_1 + \cdots + x_6 = 0\}$ by permutation of coordinates. The quotient

$$\mathcal{D}/O(T_3) \cong M_3/\mathcal{S}_6 \times \langle \tau \rangle \cong \mathbb{P}^1/\mathcal{S}_6 \cong \text{Spec } A(O(T_3))$$

is the weighted projective space $\mathbb{P}(2, 3, 4, 5, 6) = \text{Proj } \mathbb{C}[t_4, t_6, t_8, t_{10}, t_{12}]$, where $t_{2i}$ are symmetric functions of degree $i$ in $x_1, \ldots, x_6$. The projection

$$M_1 := \mathcal{D}/\Gamma_1 \to \mathcal{D}/O(T_1) \cong \mathbb{P}(2, 3, 4, 5, 6)$$

is the double cover branched along the hypersurface defined by $\Delta_{20}(t)$. It follows that the character of $s_{10}$ is the composite of $O(2, 4; \mathbb{Z}) \cong O(T_3) \to O(T_3)/\Gamma_3 \cong \mathcal{S}_6 \times \langle \tau \rangle \to \langle \tau \rangle$ and the non-trivial representation of $\langle \tau \rangle$.

Since the branch locus of the double cover of $\mathbb{P}(2, 3, 4, 5, 6)$ associated with the sign representation of $\mathcal{S}_6$ is the discriminant $\Delta_{60}(t) = \prod_{1 \leq i < j \leq 6}(x_i - x_j)^3$, the character of the modular form $s_{30}$ is the composite of the surjection $O(2, 4; \mathbb{Z}) \to \mathcal{S}_6$ and the sign representation of $\mathcal{S}_6$.

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