COHOMOLOGY OF LIE ALGEBRAS OF POLYNOMIAL VECTOR FIELDS
ON THE LINE OVER FIELDS OF CHARACTERISTIC 2

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Abstract. For a field $F$, let $L_k(F)$ be the Lie algebra of derivations $f(t)\frac{d}{dt}$ of the polynomial ring $F[t]$, where $f(t)$ is a polynomial of degree $\geq k$. For any $k \geq -1$, we build a basis of the space of continuous cohomology of the Lie algebra $L_k(F)$ with coefficients in the trivial module $F$ for the case where $\text{char}(F) = 2$. The main result obtained is an analog of the famous Goncharova’s Theorem for the case $\text{char}(F) = 0$ and $k \geq 1$.

Introduction

Let $W(F)$ be the vector space of polynomials in the indeterminate $t$ over a field $F$; consider $W(F)$ as the Lie algebra with commutator

$$[f_1(t), f_2(t)] = f_1(t)f_2'(t) - f_2(t)f_1'(t),$$

where $f'(t)$ denotes the derivative of the polynomial $f(t)$. This algebra is called the Lie algebra of polynomial vector fields on the line over $F$. The vectors $c_i = t^{i+1}$, where $i \geq -1$, form a basis of $W(F)$, and

$$[c_a, c_b] = (b-a)c_{a+b}.$$

The Lie algebra $W(F)$ contains a decreasing sequence of the Lie subalgebras

$$W(F) = L_{-1}(F) \supset L_0(F) \supset L_1(F) \supset L_2(F) \supset \cdots,$$

where $L_k(F) \subset W(F)$ is spanned by the vectors $c_i$ with $i \geq k$. Let

$$H^\ast(L_k(F)) = \bigoplus_{q=0}^{\infty} H^q(L_k(F))$$

be the graded algebra of continuous cohomology[^1] of $L_k(F)$ with coefficients in the trivial module $F$.

When $\text{char}(F) = 0$, the dimensions of the spaces $H^q(L_k(F))$ are finite. It is easy to find them for $k = -1, 0$. For $k \geq 1$, these dimensions were computed by L. V. Goncharova in the article [2].

In article [4] (see also [5]) for any $k \geq 1$, a special “filtering” basis of the Chevalley-Eilenberg complex of the Lie algebra $L_k(\mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers, was offered. In this basis, the coboundary operator acts simply enough and allows, in particular, to obtain a transparent proof of Goncharova’s result.

When the characteristic of $F$ is positive, the spaces $H^q(L_k(F))$ have infinite dimensions. But since the algebra $L_k(F)$ is graded by integers $\geq k$ (degrees), its Chevalley-Eilenberg complex decomposes into a direct sum of finite-dimensional complexes, enumerated by integers $n \geq k$ (degrees of the cochains). Therefore

$$H^q(L_k(F)) \simeq \bigoplus_{n \geq k} H^q_{(n)}(L_k(F))$$

is a direct sum of finite-dimensional spaces, where $H^q_{(n)}(L_k(F))$ is the space of $q$-dimensional cohomology of the mentioned $n$th degree subcomplex. Thus, the algebra $H^\ast(L_k(F))$ is naturally bigraded as a sum of finite-dimensional spaces uniquely defined by the numbers $q$ and $n$.

Probably, one can build an analog of the “filtering” basis (at least in small dimensions of cochains) for algebras $L_k(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the prime field with $p$ elements and $k \geq 1$. But attempts to use the

[^1] “Continuous” means that in the Chevalley-Eilenberg complex, which we use to define cohomology, we consider as cochains only linear functionals on $\Lambda^\ast(L_k)$ with the finite-dimensional supports; see [4]
information gathered in this direction to compute the spaces $H^*(L_k(\mathbb{Z}_p))$ when $p > 2$ leads to troubles, although it allows one to obtain some partial results.

For $p = 2$, the approach to compute the spaces $H^*(L_k(\mathbb{F}))$ with the help of a “filtering” basis is successful for any $k \geq 1$. In \[4\] I formulate a theorem that describes a basis of the spaces $H^*(L_k(\mathbb{Z}_2))$ and the numbers $\dim H^q_\nu(L_k(\mathbb{Z}_2))$. A detailed proof is given only in the most interesting case $k = 1$. For $k > 1$, one can prove the theorem from \[4\] by induction on $k$, similarly to what was done in the article \[4\] in the case $F = \mathbb{Q}$.

The construction of “filtering” basis for $p = 2$ follows the same schema as in the case of $F = \mathbb{Q}$; first, I build a special set of vectors, which spans the space $C^*(L_1(\mathbb{F}))$. Their linear independence follows from a combinatorial formula. When $F = \mathbb{Q}$ this formula is equivalent to the Sylvester identity in the partitions theory (see \[1\] or \[6\]). The corresponding formula in case of $F = \mathbb{Z}_2$ leads to another interesting identity on partitions, which is established in \[6\] (see formula \[5\] below).

This article is organized as follows. In \[4\] I define a complex, for which the cohomology will be computed.

In \[4\] I consider only the case $k = 1$. The definitions on integer partitions necessary to formulate the theorem on a “filtering” basis in the cochain complex in this case are collected in \[4\].

The theorem itself (Theorem 3.4) is formulated and proved in \[4\].

In \[5\] I compute the multiplication in $H^*(L_k(\mathbb{Z}_2))$. In particular, I show that, as algebra, it is generated by 1- and 2-dimensional cohomological classes. In addition I formulate a conjecture that completely describes the multiplicative structure of $H^*(L_1(\mathbb{Z}_2))$.

In \[4\] I formulate a theorem describing a basis of the space $H^*(L_k(\mathbb{Z}_2))$ for any $k \geq 1$, thus generalizing the result obtained for $k = 1$.

In \[7\] I use the results of \[4\] to compute the cohomology of the Lie algebras $L_0(\mathbb{Z}_2)$ and $L_{-1}(\mathbb{Z}_2)$. As an application, I offer an explicit description of the set of equivalence classes of central extensions of Lie algebra $W(\mathbb{Z}_2)$ with one-dimensional kernel, i.e. the set of exact sequences of the Lie algebras

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \tilde{W}(\mathbb{Z}_2) \rightarrow W(\mathbb{Z}_2) \rightarrow 0$$

considered up to natural isomorphisms between such sequences (see \[3\], Sec. 7.6).

**Notation:**

$|M|$ — the cardinality of the finite set $M$.

$M_1 \sqcup M_2$ — the disjoint union of the sets $M_1$ and $M_2$.

For a finite set of integers $I$, set $I^- := \text{min}(I)$.

For an integer $a$, let $\beta(a) := \begin{cases} 0 & \text{if } a \equiv 0 \mod 2, \\ 1 & \text{if } a \equiv 1 \mod 2. \end{cases}$

Symbol $L_k$ is used as a synonym of $L_k(\mathbb{Z}_2)$.

In what follows all vector spaces are defined over field $\mathbb{Z}_2$.

1. **The complex $C^*(L_k)$**

The homology of Lie algebra $L_k$ with coefficients in the trivial module $\mathbb{Z}_2$ is the homology of complex

$$0 \leftarrow C_0(L_k) \leftarrow \cdots \leftarrow C_i(L_k) \leftarrow \cdots \leftarrow C_{-1}(L_k) \leftarrow C_0(L_k),$$

where $C_0(L_k) = \mathbb{Z}_2$, the space $C_q(L_k) := \bigwedge^q L_k$ of $q$-dimensional chains for $q > 0$ is the $q$th exterior power of the space $L_k$, and the action of the boundary operator $d$ on the chain $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q}$, where $i_1 < i_2 < \cdots < i_q$, is defined by the formula

$$d(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q}) = \sum_{1 \leq a < b \leq q} (i_b + i_a)e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \hat{e}_{i_a} \wedge \cdots \wedge \hat{e}_{i_b} \wedge \cdots \wedge e_{i_q},$$

where

$$\hat{e}_{i_j} := \begin{cases} 0 & \text{if } i_j \equiv 0 \mod 2, \\ e_{i_j} & \text{if } i_j \equiv 1 \mod 2. \end{cases}$$

$\bigwedge$ means the disjoint union of the sets $M_1$ and $M_2$. For $\mathbb{F} = \mathbb{Q}$ and $k = 1, 2$, the product in the algebra $H^*(L_k(\mathbb{Q}))$ is zero, see [5], Remark 2.7. The general formula for multiplication in $H^*(L_k(\mathbb{Q}))$ when $k > 0$ remains unknown.
Set $C_*(L_k) = \bigoplus_{q=0}^{\infty} C_q(L_k)$. The space $C^{(n)}_q(L_k) \subset C_*(L_k)$, spanned by the chains $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q}$, where $i_1 + i_2 + \cdots + i_q = n$, is a subcomplex and $C_*(L_k) = \bigoplus_{n \geq 0} C^{(n)}_q(L_k)$ is a direct sum of complexes.

(By definition, $C^0_0(\emptyset) = \{0\}$ for any integer $n$.)

Define the space of $q$-dimensional cochains of algebra $L_k$ as

$$C^q(L_k) := \bigoplus_{n \geq 0} C^q_n(L_k), \quad \text{where} \quad C^q_n(L_k) := \text{Hom}_F(C^{(n)}_q(L_k), F).$$

Let us introduce on $C_*(L_k)$ an Euclidean metric such that the set of chains $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q}$ consists an orthonormal basis. This metric defines for any $q$ and any appropriate $n$ a unique isomorphism $C^{(n)}_q(L_k) \cong C^q_n(L_k)$ which allow us to treat the chains of the algebra $L_k$ as its cochains.

Then the action of the coboundary operator $\delta_k$, i.e. the one dual to $d$, is expressed by the formulas:

$$\delta_k(e_{i_1}) = \sum_{a+b=i, k \leq a < b} (a + b) e_a \wedge e_b,$$

$$\delta_k(e_{i_1} \wedge \cdots \wedge e_{i_q}) = \sum_{1 \leq a \leq q} e_{i_1} \wedge \cdots \wedge \delta_k(e_{i_a}) \wedge \cdots \wedge e_{i_q}. \quad (1)$$

**Definition 1.1.** For any integer $n \geq 0$, the cohomology space of the complex

$$0 \longrightarrow C^1_n(L_k) \overset{\delta_k}{\longrightarrow} C^2_n(L_k) \overset{\delta_k}{\longrightarrow} \cdots \overset{\delta_k}{\longrightarrow} C^{q-1}_n(L_k) \overset{\delta_k}{\longrightarrow} C^q_n(L_k) \overset{\delta_k}{\longrightarrow} \cdots$$

is denoted by $H^*(L_k)$ and called the cohomology space of degree $n$ of algebra $L_k$.

The space $H^*(L_k) := \bigoplus_{n \geq 0} H^*_n(L_k)$ is called the space of cohomology of the Lie algebra $L_k$.

Let $I \subset L_{-1}$ be an ideal. The adjoint action of $L_{-1}$ on $I$ induces an action $L_{-1}$ on $C^*(I)$, which commutes with the coboundary operator. Therefore, it turns $H^*(I)$ into an $L_{-1}$-module. In our treatment of the cochains the action of $e_r \in L_{-1}$ on $C^*(I)$ is uniquely defined by the formula

$$e_r(e_{i_1} \wedge \cdots \wedge e_{i_q}) = \sum_{a=1}^q i_a e_{i_1} \wedge \cdots \wedge e_{i_{a-r}} \wedge \cdots \wedge e_{i_q}. \quad (3)$$

**2. Partitions**

This section gathers some material pertaining to partitions and used throughout the article.

**Definition 2.1.** A partition is a finite ordered set of natural numbers $\langle i_1, i_2, \ldots, i_q \rangle$, referred to the parts of the partition such that $i_1 \leq i_2 \leq \cdots \leq i_q$. A partition is said to be strict if $i_1 < i_2 < \cdots < i_q$.

The numbers $\|I\| := i_1 + i_2 + \cdots + i_q$ and $|I| = q$ are called the degree and length of $I$, respectively.

A subpartition of $I$ is an ordered subset $J \subset I$. The union $I_1 \sqcup I_2$ of partitions $I_1$ and $I_2$ is the partition whose set of parts is the disjoint union of the sets $I_1$ and $I_2$.

**Definition 2.2.** Let $I$ and $I'$ be distinct partitions such that $\|I\| = \|I'\|$. We write $I' \subset I$ if either $|I'| < |I|$, or $|I'| = |I| = q$ and

$$i'_1 + \cdots + i'_r \leq i_1 + \cdots + i_r \quad \text{for any} \ r \in [1, q].$$

**Definition 2.3.** A marked partition is a pair $\langle I; J \rangle$, where $I$ is a partition and $J \subset I$. The elements of $J$ are called the marked parts. We identify each partition $I$ with the marked partition $\langle I; \emptyset \rangle$. The numbers

$$\|\langle I; J \rangle\| := \|I\|, \quad |\langle I; J \rangle| := |I| + |J|, \quad \text{and} \quad |I|$$

are called the degree, the length, and the reduced length of $\langle I; J \rangle$, respectively.

A marked partition $\langle I; J \rangle$ is called strict if $I$ is strict.

Define $\langle I_1; J_1 \rangle \sqcup \langle I_2; J_2 \rangle := \langle I_1 \sqcup I_2; J_1 \sqcup J_2 \rangle$.

Instead of explicitly indicating the set of marked parts, we often underline these parts in $I$. For example, $\langle 1, 4, 6, 7 \rangle :\langle 4, 6, 7 \rangle = \langle 1, 4, 6, 7 \rangle$.

**Definition 2.4.** Let $\langle I; J \rangle$ and $\langle I'; J' \rangle$ be distinct marked partitions. We write $\langle I'; J' \rangle \subset \langle I; J \rangle$ if either $I' \subset I$, or $I' = I$ and $J' < J$, where $<$ stands for the lexicographical order.
For example, \( \langle 5 \rangle < (2, 3) \), \( \langle 3, 2 \rangle < \langle 5, 7 \rangle \) and \( \langle 2, 6 \rangle < \langle 3, 5 \rangle \).

One can readily show that the \( < \) is a partial order on the set of marked partitions. Below, we use the following

**Lemma 2.5 (E).** If \( \langle I'; J' \rangle \not\subseteq \langle I; J \rangle \) and \( \langle I'_1; J'_1 \rangle < \langle I_1; J_1 \rangle \), then \( \langle I'; J' \rangle \cup \langle I'_1; J'_1 \rangle < \langle I; J \cup \langle I_1; J_1 \rangle \).

**Definition 2.6.** A partition \( I = \langle i_1, i_2, \ldots, i_q \rangle \) is called *regular* if \( i_{m+1} - i_m \geq 2 \) for any \( m \in [1, q-1] \).

A *dense* partition is a partition of the form \( \langle a, a+2, a+4, \ldots, a+2(q-1) \rangle \).

A *special* partition is a dense partition of the form \( \langle 1, 3, \ldots, 2q-1 \rangle \).

An *even* or *odd* partition is a partition where all parts are either even or odd, respectively.

**Definition 2.7.** For any regular partition \( I \), there is a unique decomposition \( I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_s \), where \( I_1, I_2, \ldots, I_s \) are the dense subpartitions of the maximal possible length. This decomposition is called the *canonical decomposition of \( I \).* The dense partitions \( I_1, I_2, \ldots, I_s \) are called the *simple components* of \( I \).

The minimal parts of the odd non-special simple components are called the *leading parts of \( I \).* The quantity of leading parts is denoted by \( \text{ind}(I) \).

**Definition 2.8.** A marked partition \( \langle I; J \rangle \) is called *regular* if \( I \) is regular and \( J \) is a subset of the set of leading parts of \( I \). Otherwise it called *singular*.

For a regular marked partition \( \langle I; J \rangle \), the decomposition
\[
\langle I; J \rangle = \langle I_1; J_1 \rangle \sqcup \langle I_2; J_2 \rangle \sqcup \cdots \sqcup \langle I_s; J_s \rangle,
\]
where \( I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_s \) is the canonical decomposition, is called the *canonical decomposition of \( \langle I; J \rangle \).* The marked \( k \)-partitions \( \langle I_1; J_1 \rangle, \langle I_2; J_2 \rangle, \ldots, \langle I_s; J_s \rangle \) are called its *simple components.*

### 3. A basis of \( e \)-monomials of the complex \( C^*(L_1) \)

**Definition 3.1.** A *\( k \)-monomial* is a chain of the form \( e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q} \in C_q(L_k) \), where \( I = \langle i_1, i_2, \ldots, i_q \rangle \) is a strict \( k \)-partition. The set of \( k \)-monomials \( e_I \) with \( |I| = q \) forms a basis of \( C_q(L_k) \).

**Definition 3.2.** Given \( \langle I; J \rangle \), where \( I = \langle i_1, i_2, \ldots, i_q \rangle \), define
\[
e_{\langle I; J \rangle} = \tilde{e}_{i_1} \wedge \tilde{e}_{i_2} \wedge \cdots \wedge \tilde{e}_{i_q} \in C^*(L_1), \quad \text{where} \quad \tilde{e}_{i_a} = \begin{cases} e_{i_a} & \text{if } i_a \not\in J, \\ \delta_1(e_{i_a}) & \text{if } i_a \in J. \end{cases}
\]

A nonzero cochain of the form \( e_{\langle I; J \rangle} \in C^*(L_k) \) is called an *\( e \)-monomial.*

For example, \( e_{\langle I; J \rangle} \) is an \( e \)-monomial for any regular marked partition \( \langle I; J \rangle \).

Keeping in mind the correspondence between marked partitions and \( e \)-monomials, we apply the notions related to such partitions (degree, length, regularity, order \( \prec \), etc.), to \( e \)-monomials. Lemma 2.5 implies the following Lemma:

**Lemma 3.3.** If \( e_{\langle I; J \rangle} \not\subseteq e_{\langle I'; J' \rangle} \), \( e_{\langle I_1; J_1 \rangle} \not\prec e_{\langle I'_1; J'_1 \rangle} \), and \( e_{\langle I'; J' \rangle} \wedge e_{\langle I'_1; J'_1 \rangle} \neq 0 \), then
\[
e_{\langle I; J \rangle} \wedge e_{\langle I_1; J_1 \rangle} \prec e_{\langle I'; J' \rangle} \wedge e_{\langle I'_1; J'_1 \rangle}.
\]

The next claim is the main result of this section:

**Theorem 3.4.** The set of regular \( e \)-monomials forms a basis of the space \( C^*(L_1) \). The decomposition of any singular \( e \)-monomial \( e_{\langle I; J \rangle} \) in this basis is such that
\[
e_{\langle I; J \rangle} = \sum_{\langle I'; J' \rangle \not\subseteq \langle I; J \rangle} e_{\langle I'; J' \rangle}.
\] (4)

**Proof.** Let \( D^q(n) \) be the set of partitions of degree \( n \) and length \( q \), and let \( M^q(n) \) be the set of regular marked partitions of degree \( n \) and length \( q \). Since \( \dim C^q(L_1) = |D^q(n)| \), and assuming the existence of decomposition (4), linear independence of the regular \( e \)-monomials follows from the combinatorial identity
\[
|D^q(n)| = |M^q(n)|.
\] (5)

This identity is proved in [8] (Theorem 5.6 for \( k = 1 \) and \( \lambda = 2 \)).
Therefore, it is sufficient to establish only the existence of the decomposition (4). The singular e-monomials of reduced length 2 are exhausted by the following e-monomials
\[ e_i \land e_{i+1}, \quad e_2 \land e_1, \quad e_{2i} \land e_1, \quad e_{2i} \land e_{2i+1}, \quad e_{2i+1} \land e_{2i+1}, \quad e_{2i} \land e_{2i+1} \land e_{2i+1}, \]
where \( i \geq 1 \) is an appropriate integer. For this set of e-monomials of degree \( n \), the decomposition (4) directly follows from the easily checked identities
\[ \sum_{a+b=n, 1 \leq a \leq b} e_a \land e_b = \delta_1(e_n) \quad \text{if } n \equiv 1 \mod 2, \]  
\[ \sum_{a+b=n, a \geq b} e_a \land e_1 = 0, \]  
\[ \sum_{a+b=n, 1 \leq a \leq b} \delta_1(e_a) \land e_b = 0. \]

Let us replace any submonomial of reduced length \( i \) with a linear combination of the regular ones. As a result, we present the monomial \( \langle I; J \rangle \) as a linear combination of e-monomials \( e_{\langle I'; J' \rangle} \) such that \( e_{\langle I'; J' \rangle} < e_{\langle I; J \rangle} \). This follows from Lemma 3.3.

Let us now apply the same procedure to each singular e-monomial \( e_{\langle I'; J' \rangle} \) of the obtained linear combination. Since the set of e-monomials with a fixed degree is finite, a finite number of such iterations leads to the decomposition (4).

\[ \square \]

4. Computing the space \( H^*(L_1) \)

Let \( M_0 \) be the set of marked regular partitions. For \( r \geq 1 \), define
\[ M_r := M_{r-1} \setminus \{ \text{the set of maximal partitions in } M_{r-1} \text{ with respect to order } < \}. \]

In view of Theorem 3.4, the sequence of partially ordered sets \( M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \) induces a filtration of the vector spaces
\[ C^*(L_1) = C^*(M_0) \supseteq C^*(M_1) \supseteq C^*(M_2) \supseteq \cdots, \]  
where \( C^*(M_r) \subseteq C^*(L_1) \) is the space spanned by regular e-monomials \( e_{\langle I; J \rangle} \) such that \( \langle I; J \rangle \in M_r \).

For the canonical decomposition \( \langle I; J \rangle = \langle I_1; J_1 \rangle \cup \cdots \cup \langle I_s; J_s \rangle \), formula (2) implies that
\[ \delta_1(e_{\langle I; J \rangle}) = \sum_{1 \leq a \leq s} e_{\langle I_1; J_1 \rangle} \land \cdots \land \delta_1(e_{\langle I_s; J_s \rangle}) \land \cdots \land e_{\langle I_r; J_r \rangle}. \]  

This expression, Theorem 3.4 and Lemma 3.3 show that (9) is a filtration of the complex \( C^*(L_1) \).

**Definition 4.1.** For \( x, y \in C^*(M_r) \), we write \( x \approx y \) if \( x - y \in C^*(M_{r+1}) \).

**Lemma 4.2.** Let \( I \) be a dense partition. Then
\[ \delta_1(e_I) \approx 0 \]  
if \( I \) is special or even,
\[ \beta(|I|) e_{\langle I; I \rangle} \]  
otherwise.

**Proof.** For any special or even partition \( I \), this is clear since \( \delta_1(e_I) = 0 \) for such a partition.

Formula (7) implies that \( e_{a+2r} \land \delta_1(e_{a+2r+1}) \approx \delta_1(e_{a+2r}) \land e_{a+2(r+1)} \) for any \( r \geq 0 \). Therefore, for the remaining partitions, Lemma 4.2 follows from formula (2) and Lemma 3.3. \( \square \)

**Lemma 4.3.** Let \( I \) be a dense odd non-special partition of even length. Then there exist cocycles \( \varepsilon_I \) and \( \varepsilon_{\langle I; I \rangle} \) of the complex \( C^*(L_1) \) such that
\[ \varepsilon_I \approx e_I \quad \text{and} \quad \varepsilon_{\langle I; I \rangle} \approx e_{\langle I; I \rangle}. \]

**Proof.** Formulas (7) and (9) imply that, for \( a \geq 3 \) odd, the cochains
\[ \varepsilon_{\langle a; a \rangle} := \sum_{r=0}^{a-1} e_{a-2r} \land e_{a+2r+2}, \quad \varepsilon_{\langle a; a+2 \rangle} := \sum_{r=0}^{a-1} \delta_1(e_{a-2r}) \land e_{a+2r+2}. \]  
(11)
are cocycles of the complex $C^*(L_1)$. Therefore, for $I = \langle a, a + 2, \ldots, a + 2(q - 1) \rangle$, $q$ even, and $a \geq 3$ odd, formula (10) implies that the following cocycles are cocycles as well:

$$
\varepsilon_I := \varepsilon_{\langle a, a + 2 \rangle} \wedge \varepsilon_{\langle a + 4, a + 6 \rangle} \wedge \cdots \wedge \varepsilon_{\langle a + 2(q - 2), a + 2(q - 1) \rangle},
$$

$$
\varepsilon_{\langle I, I^- \rangle} := \varepsilon_{\langle a, a + 2 \rangle} \wedge \varepsilon_{\langle a + 4, a + 6 \rangle} \wedge \cdots \wedge \varepsilon_{\langle a + 2(q - 2), a + 2(q - 1) \rangle}.
$$

The formulas $\varepsilon_I \approx \varepsilon_1$ and $\varepsilon_{\langle I, I^- \rangle} \approx \varepsilon_{\langle I, I^- \rangle}$ follow from Theorem 3.4 and Lemma 3.3.

For dense partitions $I$ not mentioned in Lemma 4.3, define

$$
\varepsilon_I := \varepsilon_1, \quad \varepsilon_{\langle I, I^- \rangle} := \delta_1(\varepsilon_I) \quad \text{if } I \text{ is odd, non-special, and } \beta(|I|) = 1.
$$

These formulas, together with the formulas from Lemma 4.3, define cocycles $\varepsilon_I$ and $\varepsilon_{\langle I, I^- \rangle}$ for any dense regular partition $I$. Cocycles of such a form are said to be the simple $\varepsilon$-monomials.

**Definition 4.4.** Let $\langle I; J \rangle = \langle I_1; J_1 \rangle \sqcup \cdots \sqcup \langle I_s; J_s \rangle$ be the canonical decomposition of a regular partition. Define $\varepsilon_{\langle I; J \rangle} := \varepsilon_{\langle I_1; J_1 \rangle} \wedge \cdots \wedge \varepsilon_{\langle I_s; J_s \rangle}$. The cochain $\varepsilon_{\langle I; J \rangle}$ is called the $\varepsilon$-monomial corresponding to $\langle I; J \rangle$. Its simple $\lambda$-factors are called the simple components of the $\varepsilon$-monomial $\varepsilon_{\langle I; J \rangle}$.

Since the matrix of passage from the set of $\varepsilon$-monomials to the basis of regular $\varepsilon$-monomials is a square lower triangle matrix with units on the main diagonal, Theorem 3.4 and Lemmas 3.3 and 4.2 imply the following result:

**Theorem 4.5.** The set of $\varepsilon$-monomials is a basis of complex $C^*(L_1)$. For a simple $\varepsilon$-monomial $\varepsilon_{\langle I; J \rangle}$, we have:

$$
\delta_1 \left( \varepsilon_{\langle I; J \rangle} \right) = \begin{cases} 
\varepsilon_{\langle I, I^- \rangle} & \text{if } I \text{ is an odd non-special partition, } J = \emptyset, \text{ and } \beta(|I|) = 1, \\
0 & \text{otherwise.}
\end{cases}
$$

From this theorem, formula (10), and the definition of the tensor product of complexes, we obtain

**Corollary 4.6.** For a regular partition $I$, let $T^*(I) \subset C^*(L_1)$ be the linear span of the set of $\varepsilon$-monomials $\varepsilon_{\langle I, J \rangle}$. Then $T^*(I)$ is a subcomplex in $C^*(L_1)$, and

$$
C^*(L_1) = \bigoplus_I T^*(I),
$$

where $I$ runs over the set of regular partitions. For the canonical decomposition $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_s$, we have an isomorphism of complexes

$$
T^*(I) \cong T^*(I_1) \otimes T^*(I_2) \otimes \cdots \otimes T^*(I_s).
$$

**Theorem 4.7.** Let $R(n)$ be the set of the regular partitions of degree $n$, for which each simple component is either special, or has an even degree.

Then any $\varepsilon$-monomial $\varepsilon_{\langle I; J \rangle}$, where $I \in R(n)$, is a nonzero cocycle of the complex $C^*(L_1)$. Cohomological classes of these cocycles form a basis of the space $H^q_{\langle n \rangle}(L_1)$. Moreover, the classes with $|I| + |J| = q$ form a basis of the space $H^q_{\langle n \rangle}(L_1)$. In particular,

$$
\sum_{q=1}^{\infty} \dim H^q_{\langle n \rangle}(L_1) t^q = \sum_{I \in R(n)} (1 + t)^{\text{ind}(I)} t^{|I|}.
$$

**Proof.** Theorem 4.5 implies that, for a dense partition $I$, we have

$$
H^*(T^*(I)) = \begin{cases} 
0 & \text{if } I \text{ is odd non-special partition and } \beta(|I|) = 1, \\
T^*(I) & \text{otherwise.}
\end{cases}
$$

Therefore Theorem 4.7 follows from Corollary 4.6 and the Künneth isomorphism

$$
H^*(T^*(I)) \cong H^*(T^*(I_1)) \otimes H^*(T^*(I_2)) \otimes \cdots \otimes H^*(T^*(I_s)).
$$

□
For example, $R(12) = \{\langle 12 \rangle, \langle 2, 10 \rangle, \langle 4, 8 \rangle, \langle 5, 7 \rangle, \langle 1, 3, 8 \rangle, \langle 2, 4, 6 \rangle \}$. If $I \in R(12)$ and $I \neq \langle 5, 7 \rangle$, then $\text{ind}(I) = 0$. Since $\text{ind}(\langle 5, 7 \rangle) = 1$, we have
\[
\sum_{q=1}^{\infty} \dim H^q_{\langle 12 \rangle}(L_1) t^q = t + 3t^2 + 3t^3.
\]

5. On the multiplicative structure of the algebra $H^*(L_1)$

In the space $H^*(L_1)$, the exterior product of cochains in the complex $C^*(L_1)$ induces a multiplication that turns $H^*(L_1)$ into an algebra. Theorem 4.7 and formulas (11) imply that $\varepsilon$-monomials $e_1$ and
\[
x(i) := e_{2i}, \quad y(i) := \sum_{r=0}^{i-1} e_{2i-2r-1} \wedge e_{2i+2r+1}, \quad \text{where } i \geq 1,
\]
\[
z(i) := \sum_{r=0}^{i-2} \delta_1(e_{2i-2r-1} \wedge e_{2i+2r+1}), \quad \text{where } i \geq 2
\]
are cocycles which represent nonzero cohomological classes of $L_1$. Let us denote them by $e, x_i, y_i,$ and $z_i,$ respectively. Theorem 4.7 implies that these classes generate the algebra $H^*(L_1)$.

Lemma 5.1. In the algebra $H^*(L_1)$, we have $e^2 = 0, x_i^2 = y_i^2 = 0$ for all $i \geq 1$ and
\[
e \cdot x_1 = e \cdot y_1 = 0,
\]
\[
z_i = \sum_{a=1}^{i-1} x_{2a} \cdot y_{i-a}, \quad \text{where } i \geq 2,
\]
\[
\sum_{a=0}^{i-1} x_{2a+1} \cdot y_{i-a} = 0, \quad \sum_{a=0}^{i-1} y_{i-a} \cdot y_{i+a+1} = 0, \quad \text{where } i \geq 1.
\]
In particular, the algebra $H^*(L_1)$ is generated by the classes $e, x_i, y_i,$ where $i \geq 1$.

Proof. Formulas $e \wedge x_1 = e \wedge y_1 = 0$ are obvious. The remaining formulas follow from the directly checked relations
\[
z(i) = \sum_{a=1}^{i-1} x(2a) \wedge y(i-a) + \delta_1 \left( \sum_{m=0}^{i-2} e_{2i-4m-3} \wedge e_{2i+4m+3} \right), \quad \text{where } i \geq 2,
\]
\[
\sum_{a=0}^{i-1} x(2a+1) \wedge y(i-a) = \delta_1 \left( \sum_{m=1}^{i-1} e_{4m-1-2\beta(i)} \wedge e_{4(i-m)+3+2\beta(i)} \right), \quad \text{where } i \geq 1,
\]
\[
\sum_{a=0}^{i-1} y(i-a) \wedge y(i+a+1) = 0, \quad \text{where } i \geq 1.
\]

In addition, Lemma 4.3 implies that
\[
\begin{align*}
e_1 \wedge e_3 \wedge \ldots \wedge e_{2q-1} & \approx \begin{cases} y(1) \wedge y(3) \wedge \ldots \wedge y(q-1) & \text{if } \beta(q) = 0, \\
e_1 \wedge y(2) \wedge y(4) \wedge \ldots \wedge y(q-1) & \text{if } \beta(q) = 1. \end{cases}
\end{align*}
\]

To finish the proof, it remains to apply theorems 4.7 and 5.4.

Let $\mathbb{P}[E, X, Y]$ be the exterior algebra generated over $\mathbb{Z}_2$ by $E, X_i,$ and $Y_i,$ where $i \geq 1$. Consider $\mathbb{P}[E, X, Y]$ as a bigraded algebra having defined the bigrading as follows:
\[
\deg(E) = (1, 1), \quad \deg(X_i) = (1, 2i), \quad \deg(Y_i) = (2, 4i).
\]
Conjecture 5.2. Let $A$ be a homogeneous ideal in $\mathbb{P}[E, X, Y]$ with generators

$$E \wedge X_1, \quad E \wedge Y_1, \quad \sum_{a=0}^{i-1} X_{2a+1} \wedge Y_{i-a}, \quad \sum_{a=0}^{i-1} Y_{i-a} \wedge Y_{i+a+1}, \quad \text{where } i \geq 1.$$ 

Then we have the exact sequence of bigraded algebras

$$0 \rightarrow A \rightarrow \mathbb{P}[E, X, Y] \xrightarrow{\pi} H^*(L_1) \rightarrow 0,$$

where $\pi(E) = e$, $\pi(X_i) = x_i$, $\pi(Y_i) = y_i$.

Remark 5.3. Thanks to Theorem 4.7 one can reduce this conjecture to a combinatorial question about integer partitions. Namely, let $P^q(n)$ be the set of pairs $(K, L)$ of partitions such that

1. $K = \langle k_1, k_2, \ldots, k_q \rangle$ is a strict partition and $L = \langle l_1, l_2, \ldots, l_b \rangle$ is a regular partition.
2. $a + 2b = q$ and $2k_1 + 2k_2 + \cdots + 2k_q + 4l_1 + 4l_2 + \cdots + 4l_b = n$.
3. $|k_i - l_j| \geq 2$ for all $i \in [1, a]$ such that $\beta(k_i) = 1$, and for all $j \in [1, b]$.

On the other hand, let $M^q_k(n)$ be the set of marked regular partitions $(I; J)$ of degree $n$, length $q$, and such that the degree of any simple component of $I$ is even. It is easy to see that conjecture 5.2 follows from the conjectural identity $|P^q(n)| = |M^q_k(n)|$.

6. A basis of the space $H^*(L_k)$ for $k \geq 1$

Definition 6.1. A $k$-partition is a pair $(k; I)$, where $I$ is a partition and $I^- \supseteq k$.

We write $k$-partitions as usual partitions, emphasizing that we only consider $k$-partitions. For example, one may consider $(2, 7)$ as either a 1-partition or a 2-partition; these objects are different.

Definition 6.2. A regular $k$-partition $I = \langle i_1, i_2, \ldots, i_q \rangle$ is called special if $i_q < 2(k + q - 1)$.

It is easy to check that the quantity of the special $k$-partitions of length $q$ is equal to \( \binom{q+k-1}{k-1} \).

Definition 6.3. A regular $k$-partition is simple if either it is special, or dense.

Definition 6.4. For any regular $k$-partition $I$, there exists a unique decomposition $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_s$, where $I_1, I_2, \ldots, I_s$ are simple $k$-partitions of the maximal possible length. This decomposition is called the canonical decomposition of $I$. The partitions $I_1, I_2, \ldots, I_s$ are called the simple components of $I$.

The minimal parts of the odd non-special simple components of $I$ are called the leading parts of $I$. The quantity of them is denoted by $\text{ind}_k(I)$.

For example, for $I = \langle 3, 5, 9, 13, 15, 18 \rangle$ and $k = 1, 2, 3$, the canonical decompositions are

$$I = \begin{cases} &\langle 3, 5 \rangle \sqcup \langle 9 \rangle \sqcup \langle 13, 15 \rangle \sqcup \langle 18 \rangle \quad \text{if } k = 1, \\ &\langle 3, 5, 9 \rangle \sqcup \langle 13, 15 \rangle \sqcup \langle 18 \rangle \quad \text{if } k = 2, \\ &\langle 3, 5, 9 \rangle \sqcup \langle 13, 15 \rangle \sqcup \langle 18 \rangle \quad \text{if } k = 3. \end{cases}$$

Therefore, $\text{ind}_1(I) = 3$, $\text{ind}_2(I) = 2$, and $\text{ind}_3(I) = 1$ respectively.

Definition 6.5. We say that $(I; J)$ is a $k$-partition if $I$ is a $k$-partition. A marked $k$-partition $(I; J)$ is called regular if $I$ is regular and $J$ is a subset of the set of the leading parts of $I$.

Theorem 6.6. Let $R_k(n)$ be the set of $k$-regular partitions of degree $n$, for each simple component of which is either special, or has an even degree.

For any regular marked $k$-partition $(I; J)$, where $I \in R_k(n)$, one can uniquely define a nonzero cocycle $\varepsilon_{(I; J)} \in C^*_n(L_k)$. Cohomological classes of these cocycles form a basis of the space $H^*_n(L_k)$. Moreover, the classes with $|I| + |J| = q$ form a basis of the space $H^q_n(L_k)$. In particular,

$$\sum_{q=1}^{\infty} \dim H^q_n(L_k) t^q = \sum_{I \in R_k(n)} (1 + t)^{\text{ind}_k(I)} t^{|I|}.$$
7. Computing the spaces $H^*(L_0)$ and $H^*(L_{-1})$

In this section $k = 0$ or $k = -1$. Since in both the cases $L_1 \subset L_k$ is an ideal, to compute $H^*(L_k)$ one could use the corresponding Hochshild-Serre spectral sequence. But we prefer a direct computation, which agrees with the computations from \[\text{[4]}\]

Let $A^q_{(n)}(k)$ be the vector subspace of $C^q_{(n)}(L_k)$ spanned by the cochains $c = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_q}$ such that $e_0 \wedge c \neq 0$. Obviously,

$$A^q_{(n)}(0) = C^q_{(n)}(L_1), \quad A^q_{(n)}(-1) = \begin{cases} \mathbb{Z}_2 e_n & \text{if } q = 1 \text{ and } n \neq 0, \\ e_0 \wedge A^{q-1}_{(n+1)}(k) \oplus C^q_{(n)}(L_1) & \text{if } q > 1 \text{ and } n \geq 0, \end{cases}$$

Then

$$C^q_{(n)}(L_k) = \begin{cases} \mathbb{Z}_2 e_n & \text{if } q = 1, \\ e_0 \wedge A^{q-1}_{(n+1)}(k) \oplus A^q_{(n)}(k) & \text{if } q > 1. \end{cases}$$

Since $e_0$ is a cocycle of the complex $C^*(L_k)$, the space $e_0 \wedge A^q_{(n)}(k)$ is a subcomplex in $C^q_{(n)}(L_k)$. Using the natural projection, let us identify the space $A^q_{(n)}(k)$ with the space of the factor-complex $C^q_{(n)}(L_k)/e_0 \wedge A^q_{(n)}(k)$ and transfer its differential, denoted by $\delta$, to $A^q_{(n)}(k)$. Then

$$H^q(e_0 \wedge A^q_{(n)}(k)) \cong e_0 \wedge H^{q-1}(A^q_{(n)}(k)).$$

The exact sequence of complexes

$$0 \longrightarrow e_0 \wedge A^q_{(n)}(k) \longrightarrow C^q_{(n)}(L_k) \longrightarrow A^q_{(n)}(k) \longrightarrow 0,$$

induces the exact cohomology sequence

$$\cdots \longrightarrow H^{q-1}_{(n)}(A^q_{(n)}(k)) \longrightarrow e_0 \wedge H^{q-1}_{(n)}(A^q_{(n)}(k)) \longrightarrow H^q_{(n)}(L_k) \longrightarrow H^q_{(n)}(A^q_{(n)}(k)) \longrightarrow e_0 \wedge H^q_{(n)}(A^q_{(n)}(k)) \longrightarrow \cdots,$$

where $b$ is the Bockstein homomorphism. Its definition implies that for any $h \in H^q_{(n)}(A^q_{(n)}(k))$ we have

$$b(h) = ne_0 \wedge h.$$

Therefore, if $n$ odd then $H^q_{(n)}(L_k) = \{0\}$. Otherwise

$$H^q_{(n)}(L_k) \cong \begin{cases} \mathbb{Z}_2 e_n & \text{if } q = 1, \\ e_0 \wedge H^{q-1}_{(n)}(A^q_{(n)}(k)) \oplus H^q(A^q_{(n)}(k)) & \text{if } q > 1. \end{cases} \quad (12)$$

For $k = 0$ this implies the following result:

**Theorem 7.1.** If $n$ is odd then $H^q_{(n)}(L_0) = \{0\}$. Otherwise

$$H^q_{(n)}(L_0) \cong \begin{cases} \mathbb{Z}_2 e_n & \text{if } q = 1, \\ e_0 \wedge H^{q-1}_{(n)}(L_1) \oplus H^q_{(n)}(L_1) & \text{if } q > 1. \end{cases}$$

Let now $k = -1$. Since $\delta(e_{-1}) = 0$, we see that $e_{-1} \wedge C^q_{(n+1)}(L_1) \subset A^q_{(n)}(-1)$ is a subcomplex and $H^q(e_{-1} \wedge C^q_{(n+1)}(L_1)) \cong e_{-1} \wedge H^q_{(n+1)}(L_1)$. The exact sequence of complexes

$$0 \longrightarrow e_{-1} \wedge C^q_{(n+1)}(L_1) \longrightarrow A^q_{(n)}(-1) \longrightarrow C^q_{(n)}(L_1) \longrightarrow 0,$$

induces the exact cohomology sequence

$$\cdots \longrightarrow H^{q-1}_{(n)}(L_1) \longrightarrow e_{-1} \wedge H^{q-1}_{(n+1)}(L_1) \longrightarrow H^q(A^q_{(n)}(-1)) \longrightarrow H^q_{(n)}(L_1) \longrightarrow e_{-1} \wedge H^q_{(n+1)}(L_1) \longrightarrow \cdots.$$

The definition of $b$ implies that for any $h \in H^q_{(n)}(L_1)$ we have

$$b(h) = e_{-1} \wedge e_{-1}(h),$$

where $e_{-1}(h)$ denotes the action of $e_{-1}$ on $h \in H^q_{(n)}(L_1)$ (see formula \[\text{[3]}\]). It is directly checked that for $a \geq 1$ odd we have

$$e_{-1}(e_{a,a+2}) = \delta_1(e_{2a+3}) \quad \text{and} \quad e_{-1}(e_{2,a+2}) = \delta_1(e_2 \wedge e_{a+3}).$$

Since in addition $e_{-1}(e_i) = 0$ for any $i \geq 2$ even, from Theorem \[\text{[17]}\] it follows that $b(h) = 0$. Thus,

$$H^q(A^q_{(n)}(-1)) \cong e_{-1} \wedge H^{q-1}_{(n+1)}(L_1) \oplus H^q_{(n)}(L_1).$$
Therefore, from isomorphism \((12)\) we obtain the following result:

**Theorem 7.2.** If \(n\) is odd then \(H^q_{(n)}(L_{-1}) = \{0\}\). Otherwise

\[
H^q_{(n)}(L_{-1}) \cong \begin{cases} 
\mathbb{Z}_2 e_n & \text{if } q = 1, \\
e_{-1} \wedge e_0 \wedge H^{q-2}_{(n+1)}(L_1) \bigoplus e_0 \wedge H^{q-1}_{(n)}(L_1) \bigoplus e_{-1} \wedge H^{q-1}_{(n+1)}(L_1) \bigoplus H^q_{(n)}(L_1) & \text{if } q > 1.
\end{cases}
\]

**Corollary 7.3.** The cohomological classes of the following set of cocycles in \(C^*_u(n)(L_{-1})\):

\[
u_{a,b}(n) := e_{2a} \wedge e_{2b}, \quad \text{where } 2a + 2b = n \text{ for integers } 0 \leq a < b,
\]

\[
u(n) := \sum_{r=0}^{\frac{n}{2}} e_{\frac{n}{2} - 2r - 1} \wedge e_{\frac{n}{2} + 2r + 1} \quad \text{if } n \equiv 0 \mod 4
\]

constitute a basis of the space \(H^2_{(n)}(L_{-1})\). In particular,

\[
\dim H^2_{(n)}(L_{-1}) = \left\lfloor \frac{n}{4} \right\rfloor + 1.
\]

Thanks to a well known interpretation of the 2-dimensional cohomology of Lie algebras with trivial coefficients (see [3], Sec.7.6), the cocycles formulas from this claim explicitly describe the basis of the space of equivalence classes of the central extensions with one-dimensional kernel of Lie algebra \(W(\mathbb{Z}_2) = L_{-1}\).

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