TOPOLOGICAL INVARIANTS FROM QUANTUM GROUP $\mathcal{U}_\xi\mathfrak{sl}(2\mid 1)$ AT ROOTS OF UNITY

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Abstract. In this article we construct link invariants and 3-manifold invariants from the quantum group associated with Lie superalgebra $\mathfrak{sl}(2\mid 1)$. This construction based on nilpotent irreducible finite dimensional representations of quantum group $\mathcal{U}_\xi\mathfrak{sl}(2\mid 1)$ where $\xi$ is a root of unity of odd order $1$. These constructions use the notion of modified trace [3] and relative $G$-modular category [2].

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1. Introduction

The vanishing of the dimension of an object $V$ in a ribbon category $C$ is an obstruction when one studies the Reshetikhin - Turaev link invariant. If the dimension of a simple object $V$ of $C$ is zero, then the quantum invariants of all (framed oriented) links with components labelled by $V$ are equal to zero, i.e. they are trivial. To overcome this difficulty, the authors N. Geer, B. Patureau-Mirand and V. Turaev introduced the notion of a modified dimension (see [7]). The modified dimension may be non-zero when $\text{dim}_C(V) = 0$. Using the modified dimension, for example on the class of projective simple objects, they defined an isotopy invariant $F'(L)$ (the renormalized Reshetikhin-Turaev link invariant) for any link $L$ whose components are labelled with objects of $C$ under the only assumption that at least one of the labels belongs to the set of projective ambidextrous objects. Here $F'(L)$ is a nontrivial link invariant (see [7]). This modified dimension is used to construct the quantum invariants in [2], [5].

The existence of a modified dimension relates strictly with the definition of modified traces (see [3]). In the article [4], the authors showed that a necessary and sufficient condition for the existence of a modified trace on an ideal generated by a simple object $J$ is that $J$ is an ambidextrous object.

The Lie superalgebras (see [10]) are the generalizations of Lie algebras used by physicists to describe supersymmetry. Deformations of these superalgebras and their representations are partially known. For
Lie superalgebra $\mathfrak{sl}(2|1)$, the irreducible representations of its deformations $\mathcal{U}_\xi \mathfrak{sl}(2|1)$ at roots of unity are described in [1]. Using these representations and developing the idea of modified traces open up the method for constructing a quantum invariant of framed links with components labelled by irreducible representations.

The aim of this article is to construct a link invariant and a 3-manifold invariant from quantum group $\mathcal{U}_\xi \mathfrak{sl}(2|1)$ at the root of unity. Note that the Lie superalgebra $\mathfrak{sl}(2|1)$ having superdimension zero, $\mathfrak{sl}(2|1)$-weight functions are trivial. Hence combining them with the Kontsevich integral or the LMO invariant also give trivial link and 3-manifold invariants. The paper contains five sections. In section 2, we recall the monoidal category, pivotal category, braided category and, ribbon category definitions. In section 3, we describe the quantum superalgebra $\mathcal{U}_\xi \mathfrak{sl}(2|1)$ where $\xi$ is a root of unity of the odd order and by adding two elements $h_1, h_2$ to $\mathcal{U}_\xi \mathfrak{sl}(2|1)$, we have the Hopf superalgebra $\mathcal{U}_H^H \mathfrak{sl}(2|1)$. This complement helped us to construct the non semi-simple ribbon category $\mathcal{C}^H$ of the nilpotent simple finite dimensional representations of $\mathcal{U}_H^H \mathfrak{sl}(2|1)$. In section 4 we prove that a typical module over $\mathcal{U}_H^H \mathfrak{sl}(2|1)$ is an ambidextrous module and that a modified trace exists on the ideal of projective modules $\text{Proj}$. This modified trace will be used to construct a link invariant. In section 5, we prove that the category $\mathcal{C}^H$ is $G$-modular relative ([2]) and we construct a 3-manifold invariant using this property.

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2. Preliminaries

2.1. Monoidal category.

**Definition 2.1.** A monoidal category $\mathcal{C}$ is a category enhanced with a bifunctor called tensor product $\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unity object $I$ such that

$$I \otimes \cdot = \cdot \otimes I = \text{Id}_\mathcal{C} \quad \text{and} \quad (\cdot \otimes \cdot) \otimes \cdot = \cdot \otimes (\cdot \otimes \cdot).$$

We write $V \in \mathcal{C}$ to denote an object $V$ in the category $\mathcal{C}$ and call $\text{Hom}_\mathcal{C}(V, W)$ the morphisms in $\mathcal{C}$ from $V \in \mathcal{C}$ to $W \in \mathcal{C}$ and $\text{End}_\mathcal{C}(V) = \text{Hom}_\mathcal{C}(V, V)$.

We say that $\mathcal{C}$ is a monoidal $K$-linear category if for all $V, W \in \mathcal{C}$, the morphisms $\text{Hom}_\mathcal{C}(V, W)$ form a $K$-module and the composition and the tensor product are bilinear and $\text{End}_\mathcal{C}(I) \cong K$. An object $V \in \mathcal{C}$ is said to be simple if $\text{End}_\mathcal{C}(V) \cong K$ as a unitary $K$-algebra. An object $W \in \mathcal{C}$ is a direct sum of $V_1, ..., V_n \in \mathcal{C}$ if there is for $i = 1, ..., n, f_i \in$
Hom\(_C(V, W)\), for \(i \neq j\) and \(\sum_{i=1}^n f_i \circ g_i = \text{Id}_W\). An object \(W \in \mathcal{C}\) is semi-simple if it is a direct sum of simple objects. The category \(\mathcal{C}\) is semi-simple if all objects are semi-simple and Hom\(_C(V, W) = \{0\}\) for any pair of non-isomorphic simple objects in \(\mathcal{C}\).

### 2.2. Pivotal category.

**Definition 2.2.** Let \(\mathcal{C}\) be a monoidal category and \(A, B \in \mathcal{C}\). A duality between \(A\) and \(B\) is given by a pair of morphisms (\(\alpha \in \text{Hom}_\mathcal{C}(\mathbb{I}, B \otimes A), \beta \in \text{Hom}_\mathcal{C}(A \otimes B, \mathbb{I})\)) such that

\[
(\beta \otimes \text{Id}_A) \circ (\text{Id}_A \otimes \alpha) = \text{Id}_A \quad \text{and} \quad (\text{Id}_B \otimes \beta) \circ (\alpha \otimes \text{Id}_B) = \text{Id}_B.
\]

A pivotal category (or sovereign) is a strict monoidal category \(\mathcal{C}\), with a unity object \(\mathbb{I}\), equipped with the data for each object \(V \in \mathcal{C}\) of its dual object \(V^* \in \mathcal{C}\) and of four morphisms

\[
\begin{align*}
\overline{\text{ev}}_V : V^* \otimes V \to \mathbb{I}, & \quad \overline{\text{coev}}_V : \mathbb{I} \to V \otimes V^*, \\
\overline{\text{ev}}_V : V \otimes V^* \to \mathbb{I}, & \quad \overline{\text{coev}}_V : \mathbb{I} \to V^* \otimes V
\end{align*}
\]

such that \((\overline{\text{ev}}_V, \overline{\text{coev}}_V)\) and \((\overline{\text{ev}}_V, \overline{\text{coev}}_V)\) are dualities which induce the same functor duality and the same natural isomorphism \((V \otimes W)^* \simeq W^* \otimes V^*\). Thus, the right and left dual coincide in \(\mathcal{C}\): for every morphism \(h : V \to W\), we have

\[
h^* = (\overline{\text{ev}}_W \otimes \text{Id}_{V^*}) \circ (\text{Id}_{V^*} \otimes h \otimes \text{Id}_{V^*}) \circ (\text{Id}_{V^*} \otimes \overline{\text{coev}}_V)
\]

and for \(V, W \in \mathcal{C}\), the isomorphisms \(\gamma_{V,W} : W^* \otimes V^* \to (V \otimes W)^*\) are given by

\[
\gamma_{V,W} = (\overline{\text{ev}}_W \otimes \text{Id}_{(V \otimes W)^*}) \circ (\text{Id}_{V^*} \otimes \overline{\text{ev}}_V \otimes \text{Id}_{(V \otimes W)^*}) \circ (\text{Id}_{V^*} \otimes \overline{\text{coev}}_{V \otimes W})
\]

The family of isomorphisms

\[
\Phi = \{\Phi_V = (\overline{\text{ev}}_V \otimes \text{Id}_{V^*}) \circ (\text{Id}_{V^*} \otimes \overline{\text{coev}}_{V^*}) : V \to V^{**}\}_{V \in \mathcal{C}}
\]

is a monoidal natural isomorphism called the pivotal structure.

### 2.3. Ribbon category.

A braided category is a tensor category \(\mathcal{C}\) provided with a braiding \(c : \) for all objects \(V\) and \(W\) of \(\mathcal{C}\), we have an isomorphism

\[
c_{V,W} : V \otimes W \to W \otimes V.
\]

These isomorphisms are natural and for all objects \(U, V\) and \(W\) of \(\mathcal{C}\), we have

\[
c_{U;V,W} = (\text{Id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \text{Id}_W) \quad \text{and} \quad c_{U;V,W} = (c_{U;W,V} \otimes \text{Id}_V) \circ (\text{Id}_U \otimes c_{V,W}).
\]
If the category $\mathcal{C}$ is pivotal and braided, we can define a family of natural isomorphisms

$$\theta_V = \text{ptr}_R(c_{V,V}) = (\text{Id}_V \otimes \overrightarrow{\text{ev}}_V) \circ (c_{V,V} \otimes \text{Id}_V) \circ (\text{Id}_V \otimes \overrightarrow{\text{coev}}_V) : V \to V.$$ 

We say that $\theta$ is a twist if it is compatible with the dual in the following sense

$$\forall V \in \mathcal{C}, \theta_V^* = (\theta_V)^*$$

which is equivalent to $\theta_V = \text{ptr}_L(c_{V,V}) = (\overleftarrow{\text{ev}}_V \otimes \text{Id}_V) \circ (\text{Id}_V^* \otimes c_{V,V}) \circ (\overleftarrow{\text{coev}}_V \otimes \text{Id}_V) : V \to V.$

A ribbon category is a braided pivotal category in which the family of isomorphisms $\theta$ is a twist.

3. Quantum superalgebra $U_{\xi}\mathfrak{sl}(2|1)$

3.1. Hopf superalgebra $U_{\xi}\mathfrak{sl}(2|1)$.

**Definition 3.1.** Let $\ell \geq 3$ be an odd integer and $\xi = \exp(\frac{2\pi i}{\ell})$. The superalgebra $U_{\xi}\mathfrak{sl}(2|1)$ is an associative superalgebra on $\mathbb{C}$ generated by the elements $k_1, k_2, k_1^-, k_2^-, e_1, e_2, f_1, f_2$ and the relations

- $k_1 k_2 = k_2 k_1,$
- $k_i k_i^- = 1, \ i = 1, 2,$
- $k_i e_j k_i^- = \xi^{a_{ij}} e_j, k_i f_j k_i^- = \xi^{-a_{ij}} f_j \ i, j = 1, 2,$
- $e_1 f_1 - f_1 e_1 = \frac{k_1 - k_1^-}{\xi - \xi^{-1}}, e_2 f_2 + f_2 e_2 = \frac{k_2 - k_2^-}{\xi - \xi^{-1}},$
- $[e_1, f_2] = 0, [e_2, f_1] = 0,$
- $e_2^2 = f_2^2 = 0,$
- $e_1^2 e_2 - (\xi + \xi^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0,$
- $f_1^2 f_2 - (\xi + \xi^{-1}) f_1 f_2 f_1 + f_2 f_1^2 = 0.$

The last two relations are called the Serre relations. The matrix $(a_{ij})$ is given by $a_{11} = 2, a_{12} = a_{21} = -1, a_{22} = 0$. The odd generators are $e_2, f_2$.

We define $\xi^x := \exp(\frac{2\pi i x}{\ell})$, afterwards we will use the concepts

$$\{x\} = \xi^x - \xi^{-x}, [x] = \frac{\xi^x - \xi^{-x}}{\xi - \xi^{-1}}.$$ 

Set $e_3 = e_1 e_2 - \xi^{-1} e_2 e_1, f_3 = f_2 f_1 - \xi f_1 f_2$. The Serre relations become

$$e_1 e_3 = \xi e_3 e_1, f_3 f_1 = \xi^{-1} f_1 f_3.$$
Furthermore

\[ e_2 e_3 = -\xi e_3 e_2, \quad f_3 f_2 = -\xi^{-1} f_2 f_3, \]
\[ e_3 f_3 + f_3 e_3 = \frac{k_1 k_2 - k_1^{-1} k_2^{-1}}{q - q^{-1}}, \]
\[ e_3^2 = f_3^2 = 0. \]

According to [11], \( \mathcal{U}_q \mathfrak{sl}(2|1) \) is a Hopf superalgebra with the coproduct, counit and antipode as below

\[
\Delta(e_i) = e_i \otimes 1 + k_i^{-1} \otimes e_i \quad i = 1, 2,
\]
\[
\Delta(f_i) = f_i \otimes k_i + 1 \otimes f_i \quad i = 1, 2,
\]
\[
\Delta(k_i) = k_i \otimes k_i \quad i = 1, 2,
\]
\[
S(e_i) = -k_i e_i, \quad S(f_i) = -f_i k_i^{-1}, \quad S(k_i) = k_i^{-1} \quad i = 1, 2,
\]
\[
\epsilon(k_i) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0 \quad i = 1, 2.
\]

The center and representations of \( \mathcal{U}_q \mathfrak{sl}(2|1) \) were studied by B. Abdelzalim, D. Arnaudon and M. Bauer [1]. We focus on the case of nilpotent representations of type \( \mathfrak{B} \) with the condition \( \ell \) odd.

**Remark 3.2.**
1. Because \((e_i \otimes 1)(k_1^{-1} \otimes e_1) = \xi^2(k_1^{-1} \otimes e_1)(e_1 \otimes 1)\) and \((\ell)_\xi \equiv \frac{1 - \xi^\ell}{1 - \xi} = 0\) then \(\Delta(e_i^\ell) = \sum_{m=0}^\ell \binom{\ell}{m} (e_1 \otimes 1)^m (k_1^{-1} \otimes e_1)^{\ell-m} = e_1^\ell \otimes 1 + k_1^{-\ell} \otimes e_1^\ell.\) We have \(\Delta^{op}(e_i^\ell) = 1 \otimes e_1^\ell + e_1^\ell \otimes k_1^{-\ell}\) at the same time. It is known that \(e_1^\ell, f_1^\ell, k_1^\ell \in \mathbb{Z}\) where \(\mathbb{Z}\) is the center of \(\mathcal{U}_q \mathfrak{sl}(2|1)\), so \(\Delta(e_i^\ell) \in \mathbb{Z} \otimes \mathbb{Z}.\) It follows that there exists no element \(R \in \mathcal{U}_q \mathfrak{sl}(2|1) \otimes \mathcal{U}_q \mathfrak{sl}(2|1)\) such that \(\Delta^{op}(x) = R \Delta(x) R^{-1} \forall x \in \mathcal{U}_q \mathfrak{sl}(2|1),\) i.e. the superalgebra \(\mathcal{U}_q \mathfrak{sl}(2|1)\) is not quasitriangular.
2. We think that the quotient superalgebra \(\mathcal{U} / (e_1^\ell, f_1^\ell)\) is not quasitriangular but \(\mathcal{U} / (e_1^\ell, f_1^\ell, k_1^\ell - 1, k_2^\ell - 1)\) should be, a proof of this might be found along the lines of [12] where the author uses a version of quantum group with divided power. This is not the quotient that interests us in this article.
3. The unrolled version \(U^*_{\mathfrak{B}} \mathfrak{sl}(2|1)\) seems to be quasitriangular only in a topological sense (see [5]). However, we will show in Theorem 3.7 and Proposition 3.8 that some representations (the weight modules) form a ribbon category.

The superalgebra \(\mathcal{U}_q \mathfrak{sl}(2|1) / (e_1^\ell, f_1^\ell)\) has a Poincaré–Birkhoff-Witt basis \(\{e_0^p e_0^q e_1^p e_0^q k_1 k_2 f_2^p f_2^q, f_3^p f_3^q, \rho, \sigma, \rho', \sigma' \in \{0, 1\}, p, p' \in \{0, 1, ..., \ell - 1\}, s, t \in \mathbb{Z}\}\), its Borel part is a superalgebra \(\mathcal{U}(\mathfrak{n}_+)\) which has a vector space basis \(\{e_0^p e_0^q e_1^p \rho, \sigma \in \{0, 1\}, p \in \{0, 1, ..., \ell - 1\}\}\). It is well known that \(\mathcal{U}(\mathfrak{n}_+)\) is a Nichols algebra of diagonal type associated with the generalized Dynkin diagram \(\Sigma_3^{2} - \xi^{2} - 1\) (see [9]). We now explain this point of view. We consider the group algebra \(B = \mathbb{C}G\) in which \(G\) is an abelian group generated by \(k_1, k_2\), a vector space \(V \otimes \mathbb{C}\) generated by \(e_1, e_2.\)
Here $B$ is a Hopf algebra and $(V, \cdot, \delta)$ is a Yetter-Drinfeld module on $B$ [9], where the action $\cdot : B \otimes V \to V$ of $B$ on $V$ is determined by

$$k_1 \cdot e_1 = \xi^2 e_1, \ k_1 \cdot e_2 = \xi^{-1} e_2, \ k_2 \cdot e_1 = \xi^{-1} e_1, \ k_2 \cdot e_2 = -e_2,$$

the matrix determining the bicharacter is $(g_{ij})_{2 \times 2}, g_{ij} = (-1)^{|i||j|} \xi^{a_{ij}}$ where $|1| = 0, |2| = 1$ and the coaction $\delta : V \to B \otimes V$ of $B$ on $V$ is given by

$$\delta(e_i) = k_i \otimes e_i \ i = 1, 2.$$

It is clear that $\delta(b \cdot v) = b(1)v(-1)S(b(3)) \otimes b(2) \cdot v(0)$ for all $b \in B, v \in V$. Here we use the Sweedler notation and write $(\Delta \otimes \Id)\Delta(b) = b(1) \otimes b(2) \otimes b(3)$, $\delta(v) = v(-1) \otimes v(0)$ for $b \in B, v \in V$.

Using Hopf algebra $B$ and Yetter-Drinfeld module $V$ we can determine the Nichols algebra $B(V) = T(V)/\mathcal{J}(V)$ where $T(V) = \bigoplus_{n=0}^{\infty} V^\otimes n$ is the tensor algebra of $V$ with the braided coproduct $\Delta(v) = 1 \otimes v + v \otimes 1$ and counit $\epsilon(v) = 0$ for $v \in V$, $\mathcal{J}(V)$ is the maximal coideal of $T(V)$.

We now check that $\epsilon_2^2$ and the Serre relation $w = e_1 e_3 - \xi e_3 e_1$ are in $\mathcal{J}(V)$. We have $\tilde{\Delta}(\epsilon_2^2) = \tilde{\Delta}(\epsilon_2)\tilde{\Delta}(\epsilon_2) = (1 \otimes e_2 + e_2 \otimes 1)(1 \otimes e_2 + e_2 \otimes 1) = 1 \otimes e_2^2 + (k_2 \cdot e_2) \otimes e_2 + e_2 \otimes e_2 + e_2^2 \otimes 1 = 1 \otimes e_2^2 + e_2^2 \otimes 1$, so $e_2^2 \in \mathcal{J}(V)$.

We calculate

$$\tilde{\Delta}(e_3) = \tilde{\Delta}(e_1)\tilde{\Delta}(e_2) - \xi^{-1}\tilde{\Delta}(e_2)\tilde{\Delta}(e_1)$$

$$= (1 \otimes e_1 + e_1 \otimes 1)(1 \otimes e_2 + e_2 \otimes 1) - \xi^{-1}(1 \otimes e_2 + e_2 \otimes 1)(1 \otimes e_1 + e_1 \otimes 1)$$

$$= 1 \otimes e_1 e_2 + (k_1 \cdot e_2) \otimes e_1 + e_1 \otimes e_2 + e_1 e_2 \otimes 1$$

$$- \xi^{-1}(1 \otimes e_2 e_1 + (k_2 \cdot e_1) \otimes e_2 + e_2 \otimes e_1 + e_2 e_1 \otimes 1)$$

$$= 1 \otimes e_3 + e_3 \otimes 1 + (1 - \xi^{-2})e_1 \otimes e_2.$$

And a similar calculation gives us

$$\tilde{\Delta}(e_1)\tilde{\Delta}(e_3) = 1 \otimes e_1 e_3 + \xi e_3 \otimes e_1 + (1 - \xi^{-2})\xi^2 e_1 \otimes e_1 e_2$$

$$+ e_1 \otimes e_3 + e_1 e_3 \otimes 1 + (1 - \xi^{-2})e_1^2 \otimes e_2,$$

and

$$\tilde{\Delta}(e_3)\tilde{\Delta}(e_1) = 1 \otimes e_3 e_1 + \xi e_1 \otimes e_3 + e_3 \otimes e_1 + e_3 e_1 \otimes 1$$

$$+ (1 - \xi^{-2})e_1 \otimes e_2 e_1 + (1 - \xi^{-2})\xi^{-1}e_1^2 \otimes e_2.$$

Thus we have

$$\tilde{\Delta}(w) = \tilde{\Delta}(e_1)\tilde{\Delta}(e_3) - \xi\tilde{\Delta}(e_3)\tilde{\Delta}(e_1)$$

$$= 1 \otimes w + w \otimes 1 + (\xi^2 - 1)e_1 \otimes e_1 e_2 + e_1 \otimes e_3$$

$$- \xi^2 e_1 \otimes e_3 - (\xi - \xi^{-1})e_1 \otimes e_2 e_1$$

$$= 1 \otimes w + w \otimes 1.$$
This implies that \( w \in \mathcal{J}(V) \). The bosonization of \( \mathcal{B}(V) \) is then isomorphic to a Hopf subalgebra of the bosonization of the Hopf superalgebra \( \mathcal{U}\mathfrak{sl}(2|1) \).

3.2. Pivotal Hopf superalgebra \( \mathcal{U}\mathfrak{sl}(2|1) \).

**Proposition 3.3.** Given \( \phi_0 = k_1^{-\ell}k_2^{-\ell} \), so \( \forall u \in \mathcal{U}\mathfrak{sl}(2|1), S^2(u) = \phi_0u\phi_0^{-1} \).

**Proof.** This can be verified for generator elements \( k_i, e_i, f_i, i = 1, 2 \). □

It follows that the Hopf superalgebra \( \mathcal{U}\mathfrak{sl}(2|1) \) provided with the pivotal element \( \phi_0 = k_1^{-\ell}k_2^{-\ell} \) is pivotal superalgebra (see [13]).

Given \( \mathcal{C} \) the even category of representations of \( \mathcal{U}\mathfrak{sl}(2|1) \) in \( \mathbb{C} \)-vector spaces of finite dimension, the category \( \mathcal{C} \) is pivotal. If \( V \) is an object of \( \mathcal{C} \), its dual is a \( \mathbb{C} \)-vector space \( V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) \) provided with the action of \( u \) given by \( (u, \varphi) \mapsto (-1)^{\deg u \deg \varphi} \circ \rho_V(S(u)) \) where \( \rho_V : \mathcal{U}\mathfrak{sl}(2|1) \rightarrow \text{End}_{\mathbb{C}}(V) \) is the representation of \( \mathcal{U}\mathfrak{sl}(2|1) \).

The unity element of category \( \mathcal{C} \) is the module \( \mathbb{C} \) provided with the representation \( \epsilon : \mathcal{U}\mathfrak{sl}(2|1) \rightarrow \mathbb{C} \cong \text{End}_{\mathbb{C}}(\mathbb{C}) \). If one has a basis \( (e_i)_i \) of \( V \) with dual basis \( (e_i^*)_i \), it can be described by dual morphisms

\[
\overline{e}_V : e_i^* \otimes e_j \mapsto e_i^*(e_j) = \delta_i^j, \quad \overline{\text{coev}}_V : 1 \mapsto \sum_i e_i \otimes e_i^*, \quad \overline{\epsilon}_V : e_i \mapsto (-1)^{\deg e_i} e_i^*(\phi_0.e_i), \quad \overline{\text{coev}}_V : 1 \mapsto \sum_i e_i^* \otimes (-1)^{\deg e_i} (\phi_0^{-1}.e_i).
\]

3.3. Category of nilpotent weight modules.

3.3.1. **Typical module.** We consider the even category \( \mathcal{C} \) of the nilpotent finite dimensional representations over \( \mathcal{U}\mathfrak{sl}(2|1) \), its objects are finite dimensional representations of \( \mathcal{U}\mathfrak{sl}(2|1) \) on which \( e_i^* = f_i^* = 0 \) and \( k_1, k_2 \) are diagonalizable. If \( V, V' \in \mathcal{C} \), \( \text{Hom}_{\mathcal{C}}(V, V') \) is formed by the even morphisms between these two modules (see [6]). Each nilpotent simple module (called ”of type \( \mathfrak{B} \)” in section 5.2 [1]) is determined by the highest weight \( \mu = (\mu_1, \mu_2) \in \mathbb{C}^2 \) and is denoted \( V_{\mu_1, \mu_2} \) or \( V_{\mu} \). Its highest weight vector \( w_{0,0,0} \) satisfies

\[
e_1w_{0,0,0} = 0, \quad e_2w_{0,0,0} = 0, \quad k_1w_{0,0,0} = \lambda_1w_{0,0,0}, \quad k_2w_{0,0,0} = \lambda_2w_{0,0,0}
\]

where \( \lambda_i = \xi^\mu_i \) with \( i = 1, 2 \).

For \( \mu = (\mu_1, \mu_2) \in \mathbb{C}^2 \) we say that \( \mathcal{U}\mathfrak{sl}(2|1) \)-module \( V_{\mu} \) is typical if it is a simple module of dimension \( 4\ell \). Other simple modules are said to be atypical.

The basis of a typical module is formed by vectors \( w_{\rho,\sigma,p} = f_2^p f_3^\rho f_1^\sigma w_{0,0,0} \) where \( \rho, \sigma \in \{0, 1\}, 0 \leq p < \ell \). The odd elements are \( w_{0,1,p} \) and \( w_{1,0,p} \).
others are even. The representation of typical $\mathcal{U}_q\mathfrak{sl}(2|1)$-module $V_{\mu_1,\mu_2}$

is determined by

$$
\begin{align*}
k_1 w_{\rho,\sigma,p} &= \lambda_1 \xi^{\rho-\sigma-2p} w_{\rho,\sigma,p}, \\
k_2 w_{\rho,\sigma,p} &= \lambda_2 \xi^{\rho+p} w_{\rho,\sigma,p}, \\
f_1 w_{\rho,\sigma,p} &= \xi^{\rho-p} w_{\rho,\sigma,p+1} - \rho(1 - \sigma) \xi^{-\sigma} w_{\rho-1,\sigma+1,p}, \\
f_2 w_{\rho,\sigma,p} &= (1 - \rho) w_{\rho+1,\sigma,p}, \\
e_1 w_{\rho,\sigma,p} &= -\sigma(1 - \rho) \lambda_1 \xi^{-2p+1} w_{\rho+1,\sigma-1,p} + [p][\mu_1 - p + 1] w_{\rho,\sigma,p-1}, \\
e_2 w_{\rho,\sigma,p} &= \rho[\mu_2 + p + \sigma] w_{\rho-1,\sigma,p} + (\sigma(1 - \rho) \lambda_2^{-1} \xi^{-p} w_{\rho,\sigma-1,p+1}.
\end{align*}
$$

where $\rho, \sigma \in \{0, 1\}$ and $p \in \{0, 1, \ldots, \ell - 1\}$.

We also have $V_\mu \simeq V_{\mu+\vartheta} \iff \vartheta \in (\ell \mathbb{Z})^2$.

**Remark 3.4.** The module $V_\mu$ is typical if $[\mu_1 - p + 1] \neq 0 \forall p \in \{1, \ldots, \ell - 1\}$ ($\mu_1 \neq p - 1 + \frac{i}{\ell} \mathbb{Z}$) and $[\mu_2][\mu_1 + \mu_2 + 1] \neq 0 (\mu_2 \neq \frac{i}{\ell} \mathbb{Z})$ (see (1)).

We define a $\mathbb{C}$-superalgebra $\mathcal{U}_q^H\mathfrak{sl}(2|1)$ (note $\mathcal{U}^H$) as $\mathcal{U}_q^H\mathfrak{sl}(2|1) = \langle \mathcal{U}_q\mathfrak{sl}(2|1), h_i, i = 1, 2 \rangle$ with the relations in $\mathcal{U}_q\mathfrak{sl}(2|1)$ and $[h_i, e_j] = \frac{\delta_{ij}}{2} [h_i, f_j] = -a_{ij} e_j, [h_i, f_j] = 0, [h_i, k_j] = 0$. The superalgebra $\mathcal{U}_q^H$ is a Hopf superalgebra where $\Delta, S$ and $\epsilon$ are determined as in $\mathcal{U}_q\mathfrak{sl}(2|1)$ and by

$$
\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, S(h_i) = -h_i, \epsilon(h_i) = 0 \quad i = 1, 2.
$$

We consider the even category $\mathcal{C}^H$ of nilpotent finite dimensional $\mathcal{U}_q^H$-modules (precise that $e_1 = f_1 = 0$) for which $\xi h_i = k_i$ as diagonalizable operators. The category $\mathcal{C}^H$ is pivotal similar to $\mathcal{C}$ (see Section 3.2). We define the actions of $h_i, i = 1, 2$ on the basis of $V_{\mu_1,\mu_2}$ by

$$
h_1 w_{\rho,\sigma,p} = (\mu_1 + \rho - \sigma - 2p) w_{\rho,\sigma,p}, h_2 w_{\rho,\sigma,p} = (\mu_2 + \sigma + p) w_{\rho,\sigma,p}.
$$

Thus $V_{\mu_1,\mu_2}$ is a weight module of $\mathcal{C}^H$. A module in $\mathcal{C}^H$ is said to be typical if, seen as a $\mathcal{U}_q\mathfrak{sl}(2|1)$-module, it is typical. For each module $V$ we note $\overline{V}$ the same module with the opposite parity. We set $G = \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}$ and for each $\overline{p} \in G$ we define $\mathcal{C}_{\overline{p}}^H$ as the sub-category of weight modules which have their weights in the coset $\overline{p}$ (modulo $\mathbb{Z} \times \mathbb{Z}$). So $\{\mathcal{C}_{\overline{p}}^H\}_{\overline{p} \in G}$ is a $G$-gradation (where $G$ is an additive group): let $V \in \mathcal{C}_{\overline{p}}^H, V' \in \mathcal{C}_{\overline{p}'}^H$, then the weights of $V \otimes V'$ are congruent to $\overline{p} + \overline{p}'$ (modulo $\mathbb{Z} \times \mathbb{Z}$). Furthermore, if $\overline{p} \neq \overline{p}'$ then $\text{Hom}_{\mathcal{C}^H}(V, V') = 0$ because a morphism preserves weights.

We also define

$$
G_s = \{\overline{g} \in G \text{ such that } \exists V \in \mathcal{C}_{\overline{g}}^H \text{ simple of } \mathcal{C}_{\overline{g}}^H \text{ and atypical}\}.
$$
It follows from [1] that
\[ G_s = \left\{ \frac{0}{2}, \frac{T}{2} \right\} \times \mathbb{C}/\mathbb{Z} \cup \mathbb{C} \times \left\{ \frac{0}{2}, \frac{T}{2} \right\} \cup \left\{ (\overline{1} \mu_1, \overline{1} \mu_2) : \overline{1} \mu_1 + \overline{1} \mu_2 \in \left\{ \frac{0}{2}, \frac{T}{2} \right\} \right\}. \]

3.3.2. Character of representations of \( \mathcal{U}^H \mathfrak{sl}(2|1) \).

**Definition 3.5.** The character of a weight module \( V \) is
\[ \chi_V = \sum_{\mu} \dim(E_{\mu}(V))X_{\mu_1}X_{\mu_2} \]
where \( E_{\mu}(V) \) is the proper subspace of the proper value \( \mu = (\mu_1, \mu_2) \) of \((h_1, h_2)\).

Note that we do not use the concept of a super-character defined as above by replacing the dimension by the super-dimension.

A finite dimensional representation of \( \mathcal{U}^H \mathfrak{gl}(2) \), subalgebra generated by \( e_1, f_1, k_1 \) is defined by \( V = \text{Vect}(v_0, ..., v_{\ell-1}) \) \[1\]
\[ k_1v_p = \lambda_1 \xi^{-2p}v_p \text{ with } p \in \{0, 1, ..., \ell - 1\}, \]
\[ f_1v_p = v_{p+1} \text{ with } p \in \{0, 1, ..., \ell - 2\} \text{ and } f_1v_{\ell-1} = 0, \]
\[ e_1v_p = [p][\mu_1 - p + 1]v_{p-1}, \quad \xi^{\mu_1} \equiv \lambda_1, \]
\[ k_2v_p = \lambda_2 \xi^p v_p \text{ with } p \in \{0, 1, ..., \ell - 1\}. \]

It extends to the generators \( h_1, h_2 \) by
\[ h_1v_p = (\mu_1 - 2p)v_p \text{ with } p \in \{0, 1, ..., \ell - 1\}, \]
\[ h_2v_p = (\mu_2 + p)v_p \text{ with } p \in \{0, 1, ..., \ell - 1\}. \]

so that \( \xi^{h_i} = k_i, \ i = 1, 2 \) on \( V \). We have the character of representation of \( \mathcal{U}^H \mathfrak{gl}(2) \)
\[ \chi_{\mathcal{U}^H \mathfrak{gl}(2)} = X_{\mu_1}X_{\mu_2} \frac{1 - x^{\ell}}{1 - x} \text{ where } x = X_1^{-2}X_2. \]

In the case of a typical representation, the nilpotent representation \( V_{\mu_1, \mu_2} \) of \( \mathcal{U}^H \mathfrak{gl}(2|1) \) with highest weight \( (\mu_1, \mu_2) \) is determined by
\[ k_1w_{\rho, \sigma, p} = \lambda_1 \xi^{\rho - \sigma - 2p}w_{\rho, \sigma, p}, \]
\[ k_2w_{\rho, \sigma, p} = \lambda_2 \xi^{\rho + p}w_{\rho, \sigma, p} \]
with \( h_1w_{\rho, \sigma, p} = (\mu_1 + \rho - \sigma - 2p)w_{\rho, \sigma, p} \) and \( h_2w_{\rho, \sigma, p} = (\mu_2 + \sigma + p)w_{\rho, \sigma, p} \).

So the nilpotent representation \( V_{\mu_1, \mu_2} \) has the following character
\[ (1) \]
\[ \chi_{\mathcal{U}^H \mathfrak{sl}(2|1)} = X_{\mathcal{U}^H \mathfrak{gl}(2)} + X_{\mathcal{U}^H \mathfrak{sl}(2)} + X_{\mathcal{U}^H \mathfrak{gl}(2)} + X_{\mathcal{U}^H \mathfrak{gl}(2)} = X_{\mu_1}X_{\mu_2} \frac{1 - x^{\ell}}{1 - x}(1 + X_1)(1 + X_1x). \]
3.3.3. Braided category $\mathcal{C}^H$. The $\mathbb{C}$-superalgebra $U_q\mathfrak{sl}(2|1)$ can be seen as the specialisation at $q = \xi$ of the $\mathbb{C}(q)$-superalgebra $U_q\mathfrak{sl}(2|1)$ of the $h$-adic quantized enveloping superalgebra of $\mathfrak{sl}(2|1)$ where $q = e^h \in \mathbb{C}[[h]]$. In articles [11,13] the authors showed that: $\mathcal{R}^q = \mathcal{R}^q\mathcal{K}_q$ where
\[
\mathcal{R}^q = \sum_{i=0}^{\infty} \frac{1}{(i)_q!} e_i \otimes f_i = \sum_{j=0}^{\infty} \frac{(-1)^j e_j \otimes f_j}{(j)_q!} = \sum_{k=0}^{\infty} \frac{(-1)^k e_k \otimes f_k}{(k)_q!},
\]
$(0)_q! = 1, (n)_q! := (1)_q(2)_q \cdots (n)_q, (k)_q = \frac{1-q^k}{1-q}$ and $\mathcal{K}_q = q^{-h_1 \otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2}$ is a universal $R$-matrix element of superalgebra $U_q\mathfrak{sl}(2|1)$. That is, we have the following relations
\[
(\Delta \otimes \text{Id})(\mathcal{R}^q) = \mathcal{R}^q_{13} \mathcal{R}^q_{23}, \quad (\text{Id} \otimes \Delta)(\mathcal{R}^q) = \mathcal{R}^q_{13} \mathcal{R}^q_{12}, \quad \Delta^{op}(x)\mathcal{R}^q = \mathcal{R}^q \Delta(x)
\]
for all $x \in U_q\mathfrak{sl}(2|1)$. The superalgebra $U_q\mathfrak{sl}(2|1)$ has a Poincaré-Birkhoff-Witt basis $\{e_i^p e_j^q h_1^{s_1} h_2^{s_2} f_1^p f_2^q f_3^p p, p, p' \in \mathbb{N}, \rho, \sigma, \rho', \sigma' \in \{0, 1\}, s_1, s_2 \in \mathbb{N}\}$. Using this basis we can write $U_q\mathfrak{sl}(2|1)$ as a direct sum $U_q\mathfrak{sl}(2|1) = U^< \oplus I$ where $U^<$ is a $\mathbb{C}(q)$-module generated by the elements $e_i^p e_j^q h_1^{s_1} h_2^{s_2} f_1^p f_2^q f_3^p p$ for $0 \leq p, p' < \ell; \rho, \sigma, \rho', \sigma' \in \{0, 1\}, s_1, s_2 \in \mathbb{N}$ and $I$ is generated by the other monomials. Set $p : U_q\mathfrak{sl}(2|1) \rightarrow U^<$ the projection with kernel $I$. We define
\[
\mathcal{R}^< := p \otimes p(\mathcal{R}^q) = p \otimes \text{Id}(\mathcal{R}^q) = \text{Id} \otimes p(\mathcal{R}^q).
\]
The proposition below shows that the "truncated $R$-matrix" $\mathcal{R}^<$ satisfies the properties of an $R$-matrix "modulo truncation".

**Proposition 3.6.** $\mathcal{R}^<$ satisfies:

1. $(p \otimes p \otimes p)(\Delta \otimes \text{Id}(\mathcal{R}^<)) = (p \otimes p \otimes p)(\mathcal{R}^q_{13} \mathcal{R}^q_{23})$,
2. $(p \otimes p \otimes p)(\text{Id} \otimes \Delta(\mathcal{R}^<)) = (p \otimes p \otimes p)(\mathcal{R}^q_{13} \mathcal{R}^q_{12})$,
3. $(p \otimes p)(\mathcal{R}^< \Delta^{op}(x)) = (p \otimes p)(\Delta(x)\mathcal{R}^<)$ for all $x \in U_q\mathfrak{sl}(2|1)$.

Proof. The above relations and $p \circ p = p$ give us $(p \otimes p \otimes p)(\Delta \otimes \text{Id}(\mathcal{R}^q)) = (p \otimes p \otimes p)(\Delta \otimes \text{Id}(\mathcal{R}^q)) = (p \otimes p \otimes p)(\Delta \otimes \text{Id}(\mathcal{R}^<))$. At the same time $(p \otimes p \otimes p)(\mathcal{R}^q_{13} \mathcal{R}^q_{23}) = (p \otimes p \otimes p)(\mathcal{R}^q_{13} \mathcal{R}^q_{12})$. So
\[
(2) \quad (p \otimes p \otimes p)(\Delta \otimes \text{Id}(\mathcal{R}^<)) = (p \otimes p \otimes p)(\mathcal{R}^q_{13} \mathcal{R}^q_{23}).
\]
Similarly we also have
\[
(3) \quad (p \otimes p \otimes p)(\text{Id} \otimes \Delta(\mathcal{R}^<)) = (p \otimes p \otimes p)(\mathcal{R}^q_{13} \mathcal{R}^q_{12}).
\]
For the third equality, it is enough to check on the generator elements. It is true when $x = h_i$ because $\Delta(h_i)$ is symmetric and $\Delta(h_i)(e_j \otimes f_j) = e_j \otimes h_i f_j + h_i e_j \otimes f_j = e_j \otimes f_j(h_i - a_{ij}) + e_j(h_i + a_{ij}) \otimes f_j = e_j \otimes f_j(1 \otimes (h_i - a_{ij}) + (h_i + a_{ij}) \otimes 1) = (e_j \otimes f_j)\Delta(h_i)$.

For $x = e_i$ we have $(p \otimes p)(\Delta^{op}(e_i)\mathcal{R}^q) = (p \otimes p)(1 \otimes e_i + e_i \otimes k_i^{-1})\mathcal{R}^q = (p \otimes p)((1 \otimes e_i)\mathcal{R}^q) + (p \otimes p)((e_i \otimes k_i^{-1})\mathcal{R}^q) = (p \otimes p)((1 \otimes e_i)\mathcal{R}^<) + (p \otimes p)((e_i \otimes k_i^{-1})\mathcal{R}^<) = (p \otimes p)(\Delta^{op}(e_i)\mathcal{R}^<)$. On the other side
\begin{align*}
(p \otimes p)(\mathcal{R}^{\mathcal{Q}} \Delta(e_i)) & = (p \otimes p)(\mathcal{R}^{\mathcal{Q}} \Delta(e_i)). \quad \text{So we have } (p \otimes p)(\Delta^{\mathcal{Q}p}(e_i)\mathcal{R}^{\mathcal{Q}}) = (p \otimes p)(\mathcal{R}^{\mathcal{Q}} \Delta(e_i)).

\text{For } x = f_i \text{ we proceed analogously. So we deduce that }
(p \otimes p)(\Delta^{\mathcal{Q}p}(x)\mathcal{R}^{\mathcal{Q}}) = (p \otimes p)(\mathcal{R}^{\mathcal{Q}} \Delta(x)) \forall x \in \mathcal{U}_q \mathfrak{sl}(2|1).
\end{align*}

Let $\mathcal{K}$ be the operator in $\mathcal{C}^\mathcal{H} \otimes \mathcal{C}^\mathcal{H}$ defined by

$$
\mathcal{K} = \xi^{-h_1 \otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2}
$$

that is $\forall V, W \in \mathcal{C}^\mathcal{H}, \mathcal{K}_{V \otimes W} = \exp \left( \rho_{V \otimes W} \left( \frac{2\pi i}{h} (-h_1 \otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2) \right) \right)$ is a linear map on the finite dimensional vector space $V \otimes W$. For example, if $w_{\rho, \sigma, p} \otimes w_{\rho', \sigma', q'} \in V_\mu \otimes V_{\mu'}$ one has

$$
\mathcal{K}_{V \otimes W}(w_{\rho, \sigma, p} \otimes w_{\rho', \sigma', q'}) = \xi^{-(\mu_1 + \rho - \sigma - 2p)(\mu_1' + \rho' - \sigma' - 2q')} \cdot \exp \left( \frac{2\pi i}{h} (-h_1 \otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2) \right) w_{\rho, \sigma, p} \otimes w_{\rho', \sigma', q'}.
$$

We have

\begin{equation}
(\Delta \otimes \text{Id}(\mathcal{K})) = \mathcal{K}_{13} \mathcal{K}_{23}, (\text{Id} \otimes \Delta)(\mathcal{K}) = \mathcal{K}_{13} \mathcal{K}_{23}.
\end{equation}

Let $\mathcal{R}^{\mathcal{Q}}$ be the universal truncated quasi $R$-matrix of $\mathcal{U}_q \mathfrak{sl}(2|1), q = e^h \in \mathbb{C}[[h]]$ given by $\mathcal{R}^{\mathcal{Q}} = p \otimes p(\mathcal{R}^{\mathcal{Q}}) = \text{Id} \otimes \text{Id}(\mathcal{R}^{\mathcal{Q}}) = p \otimes \text{Id}(\mathcal{R}^{\mathcal{Q}})$, i.e:

$$
\mathcal{R}^{\mathcal{Q}} = \sum_{i=0}^{\ell-1} \frac{(1)^i e_1^i \otimes f_1^i}{(i)_q!} \sum_{j=0}^{1} \frac{(-1)^i e_3^j \otimes f_3^j}{(j)_q!} \sum_{k=0}^{1} \frac{(-1)^k e_2^k \otimes f_2^k}{(k)_q!}
$$

Set $\mathcal{R} = \mathcal{R}^{\mathcal{Q}}|_{q=\xi}, i.e:

$$
\mathcal{R} = \sum_{i=0}^{\ell} \frac{(1)^i e_1^i \otimes f_1^i}{(i)_{\xi}!} \sum_{j=0}^{1} \frac{(-1)^i e_3^j \otimes f_3^j}{(j)_{\xi}!} \sum_{k=0}^{1} \frac{(-1)^k e_2^k \otimes f_2^k}{(k)_{\xi}!} \in \mathcal{U}^\mathcal{H} \otimes \mathcal{U}^\mathcal{H}.
$$

**Theorem 3.7.** The operator $\mathcal{R} = \mathcal{R} \mathcal{K}$ led to a braiding $\{ \mathcal{C}_{V,W} \}$ in the category $\mathcal{C}^\mathcal{H}$ where $\mathcal{C}_{V,W} : V \otimes W \rightarrow W \otimes V$ is determined by $v \otimes w \mapsto \tau(\mathcal{R}(v \otimes w))$. Here $\tau : V \otimes W \rightarrow W \otimes V, v \otimes w \mapsto (-1)^{\text{deg} v \cdot \text{deg} w} w \otimes v$.

**Proof.** It is sufficient to prove that the operator $\mathcal{R}$ satisfies

\begin{equation}
(\Delta \otimes \text{Id})(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}, (\text{Id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}, \mathcal{R} \Delta^{\mathcal{Q}p}(x) = \Delta(x) \mathcal{R}
\end{equation}

for all $x \in \mathcal{U}^\mathcal{H}$.

Let $\chi_q : \mathcal{U}_q \mathfrak{sl}(2|1) \otimes \mathcal{U}_q \mathfrak{sl}(2|1) \rightarrow \mathcal{U}_q \mathfrak{sl}(2|1) \otimes \mathcal{U}_q \mathfrak{sl}(2|1)$ be the automorphism determined by $x \otimes y \mapsto K_q(x \otimes y) K_q^{-1}$, this one induces an automorphism $\chi_{\xi} : \mathcal{U}^\mathcal{H} \otimes \mathcal{U}^\mathcal{H} \rightarrow \mathcal{U}^\mathcal{H} \otimes \mathcal{U}^\mathcal{H}$. We consider the element $\mathcal{R}^{\mathcal{Q}}$ of $\mathcal{U}_q \mathfrak{sl}(2|1) \otimes \mathcal{U}_q \mathfrak{sl}(2|1)$, Proposition 3.6 implies the relations

\begin{align*}
(\Delta \otimes \text{Id})(\mathcal{R}) & = \mathcal{R}_{13} (\chi_q)_{13} (\mathcal{R}_{23}), \\
(\text{Id} \otimes \Delta)(\mathcal{R}) & = \mathcal{R}_{13} (\chi_q)_{13} (\mathcal{R}_{12}), \\
(\mathcal{R} (\chi_q))(\Delta^{\mathcal{Q}p}(x)) & = \Delta(x) \mathcal{R} \text{ for all } x \in \mathcal{U}^\mathcal{H}.
\end{align*}

We will prove the equality \((6)\), and that the other two are similar. From the first equality of the Proposition 3.6 we deduce that $$(\Delta \otimes \text{Id})(\mathcal{R}) = \mathcal{R}_{13} (\chi_q)_{13} (\mathcal{R}_{23})$$.
Id)(\mathcal{R}^g\mathcal{K}_q) = \mathcal{R}^g_{13}(\mathcal{K}_q)_{13}\mathcal{R}^g_{23}(\mathcal{K}_q)_{23}. The term in the left of this equality is equal to \((\Delta \otimes \text{Id})(\mathcal{R}^g)(\Delta \otimes \text{Id})(\mathcal{K})\) = \Delta \otimes \text{Id}(\mathcal{R}^g)(\mathcal{K})_{13}\mathcal{K}_{23}. The right one is equal to \(\mathcal{R}^g_{13}(\mathcal{K}_q)_{13}\mathcal{R}^g_{23}(\mathcal{K}_q)_{23} = \mathcal{R}^g_{13} (\chi_q)_{13} (\mathcal{R}^g_{23}(\mathcal{K}_q)_{23}).\) Now because \(\mathcal{K}_q\) is invertible, the result is \(\Delta \otimes \text{Id}(\mathcal{R}^g) = \mathcal{R}^g_{13} (\chi_q)_{13}(\mathcal{R}^g_{23}).\)

The element \(\mathcal{R}^g\) has no pole when \(q\) is a root of unity of order \(\ell\). Hence we can specialize this relation at \(q = \xi\) and \(\Delta \otimes \text{Id}(\mathcal{R}) = \mathcal{R}^g_{13} (\chi_q)_{13}(\mathcal{R}^g_{23}).\) Finally, as operators on \(V_1 \otimes V_2 \otimes V_3\) in which \(V_1, V_2, V_3 \in \mathcal{C}^H\), equation \([4]\) implies that

\[
\Delta \otimes \text{Id}(\mathcal{R}) = (\Delta \otimes \text{Id})(\mathcal{R})(\Delta \otimes \text{Id})(\mathcal{K})
= \mathcal{R}^g_{13} (\chi_q)_{13}(\mathcal{R}^g_{23})\mathcal{K}_{13}\mathcal{K}_{23}
= \mathcal{R}^g_{13}\mathcal{K}_{13}\mathcal{R}^g_{23}\mathcal{K}_{13}\mathcal{K}_{23}
= \mathcal{R}^g_{13}\mathcal{K}_{13}\mathcal{R}^g_{23}\mathcal{K}_{23}
= \mathcal{R}^g_{13}\mathcal{R}^g_{23}\mathcal{K}_{23}.
\]

Thus the relations of equation \([5]\) hold.

The category \(\mathcal{C}^H\) is pivotal and braided with the braiding \(c_{V,W} : V \otimes W \rightarrow W \otimes V, v \otimes w \mapsto \tau \circ \mathcal{R}(v \otimes w)\) where \(V, W \in \mathcal{C}^H\).

3.3.4. Ribbon category \(\mathcal{C}^H\). To prove the next proposition we will use the semi-simplicity of \(\mathcal{C}_g\) \((g \in G \setminus G_s)\) which is proven later in Theorem 3.17.

**Proposition 3.8.** The family of isomorphisms \(\theta_V : V \rightarrow V\) determined by \(\theta_V = (\text{Id}_V \otimes \overline{\text{ev}}_V)(c_{V,V} \otimes \text{Id}_V)\cdot(\text{Id}_V \otimes \overline{\text{coev}}_V), V \in \mathcal{C}^H\) is a twist. That is \(\theta_V = \theta'_V \forall V \in \mathcal{C}^H\) where \(\theta'_V = (\overline{\text{ev}}_V \otimes \text{Id}_V)(\text{Id}_V \otimes c_{V,V})(\overline{\text{coev}}_V \otimes \text{Id}_V)\).

**Proof.** Firstly, if \(V\) is a typical module of highest weight \(\mu = (\mu_1, \mu_2), V \in \mathcal{C}_g^H, g \in G \setminus G_s\), we have \(\theta'_V = (\overline{\text{ev}}_V \otimes \text{Id}_V)(\text{Id}_V \otimes c_{V,V})(\overline{\text{coev}}_V \otimes \text{Id}_V) = X_1X_2X_3.\)

We use the vector of lowest weight \((\mu_1 - 2\ell + 2, \mu_2 + \ell)\) of \(V, w_{1,1,\ell-1} := w_\infty\), to calculate.

\[
X_3(w_\infty) = \sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}w_{\rho,\sigma,p}^* \otimes \phi_0^{-1}w_{\rho,\sigma,p} \otimes w_\infty
= \sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}\xi^{\ell \mu_1+2\mu_2+2\sigma+2p}w_{\rho,\sigma,p}^* \otimes w_{\rho,\sigma,p} \otimes w_\infty.
\]

\[
X_2X_3(w_\infty) = \sum_{\rho,\sigma,p}(-1)^{\rho+\sigma}\xi^{\ell \mu_1+2\mu_2+2\sigma+2p}w_{\rho,\sigma,p}^* \otimes (\tau \circ \mathcal{R})(w_{\rho,\sigma,p} \otimes w_\infty).
\]

\[
\mathcal{K}(w_{\rho,\sigma,p} \otimes w_\infty) = \xi^{-h_1 \otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2}w_{\rho,\sigma,p} \otimes w_\infty
= \xi^{-\mu_1(\mu_2+\sigma+p+\ell)-\mu_2(\mu_1+2\mu_2+\sigma+p+2)-2(\sigma+p)}w_{\rho,\sigma,p} \otimes w_\infty.
\]
\[ X_2 X_3(w_{\infty}) = \sum_{\rho, \sigma, p} (-1)^{\rho + \sigma} \xi^{\ell_{\mu_1} + 2\mu_2} \xi^{-\mu_1(\mu_2 + \sigma + p + \ell) - \mu_2(\mu_1 + 2\mu_2 + \sigma + p + 2)} \rho_{\rho_{\sigma}, \rho_{p}} \otimes w_{\infty} \otimes w_{\rho_{\sigma}, \rho_{p}} \]
\[ = \sum_{\rho, \sigma, p} (-1)^{\rho + \sigma} \xi^{-\mu_1(\mu_2 + \sigma + p) - \mu_2(\mu_1 + 2\mu_2 + \sigma + p)} w_{\rho_{\sigma}, \rho_{p}} \otimes w_{\infty} \otimes w_{\rho_{\sigma}, \rho_{p}}. \]

So
\[ X_1 X_2 X_3(w_{\infty}) = \sum_{\rho, \sigma, p} (-1)^{\rho + \sigma} \xi^{-\mu_1(\mu_2 + \sigma + p) - \mu_2(\mu_1 + 2\mu_2 + \sigma + p)} w_{\rho_{\sigma}, \rho_{p}}(w_{\infty}) \otimes w_{\rho_{\sigma}, \rho_{p}} \]
\[ = \xi^{-\mu_1(\mu_2 + \ell) - \mu_2(\mu_1 + 2\mu_2 + 2)} w_{\infty}. \]

Secondly, we have
\[ \theta_V = (\text{Id}_V \otimes \text{coev}_V)(c_{\nu V} \otimes \text{Id}_V^*) (\text{Id}_V \otimes \text{coev}_V) = Y_1 Y_2 Y_3. \]
\[ Y_3(w_{0,0,0}) = \sum_{\rho, \sigma, p} w_{0,0,0} \otimes w_{0,0,0} \otimes w_{\rho_{\sigma}, \rho_{p}} w_{\rho_{\sigma}, \rho_{p}} \]
\[ Y_2 Y_3(w_{0,0,0}) = \sum_{\rho, \sigma, p} (\tau \circ \mathcal{R})(w_{0,0,0} \otimes w_{\rho_{\sigma}, \rho_{p}}) \otimes w_{\rho_{\sigma}, \rho_{p}} \]
where
\[ \mathcal{K}(w_{0,0,0} \otimes w_{\rho_{\sigma}, \rho_{p}}) = \xi^{-\mu_1(\mu_2 + \sigma + p) - \mu_2(\mu_1 + \rho - \sigma - 2p) - 2\mu_2(\mu_1 + \sigma + p)} w_{0,0,0} \otimes w_{\rho_{\sigma}, \rho_{p}}. \]
\[ \mathcal{R}(w_{0,0,0} \otimes w_{\rho_{\sigma}, \rho_{p}}) = \xi^{-\mu_1(\mu_2 + \sigma + p) - \mu_2(\mu_1 + \rho - \sigma - 2p) - 2\mu_2(\mu_1 + \sigma + p)} w_{0,0,0} \otimes w_{\rho_{\sigma}, \rho_{p}}. \]
\[ Y_2 Y_3(w_{0,0,0}) = \sum_{\rho, \sigma, p} \xi^{-\mu_1(\mu_2 + \sigma + p) - \mu_2(\mu_1 + \rho - \sigma - 2p) - 2\mu_2(\mu_1 + \sigma + p)} w_{\rho_{\sigma}, \rho_{p}} \otimes w_{\rho_{\sigma}, \rho_{p}} \]
\[ = \xi^{-\mu_1(\mu_2 + \ell) - \mu_2(\mu_1 + 2\mu_2 + 2)} w_{0,0,0}. \]

We can deduce that \( \theta_V = \theta_V \) for every typical module \( V \) with highest weight \( \mu = (\mu_1, \mu_2), V \in \mathcal{C}_g^H, g \in G \setminus G_s \). Note that the calculation does not change if we reverse the parity of vectors. So we have the affirmation for a semi-simple module in degree \( g \in G \setminus G_s \). Let a module \( W \in \mathcal{C}_g^H, g \in G \). By Theorem 3.17 it exists \( h \in G \) such that \( \mathcal{C}_h^H, \mathcal{C}_{g+h}^H \) are semi-simple. For a module \( V \in \mathcal{C}_h^H \) we have \( W \otimes V \in \mathcal{C}_g^H \) is semi-simple.

Because \( \theta_{W \otimes V} = (\theta_W \otimes \theta_V)(c_{WV} \otimes c_{WV}^*) = (\theta_W' \otimes \theta_V')(c_{WV} \otimes c_{WV}) \) and \( \theta_V = \theta_V' \), we deduce that \( \theta_W = \theta_W' \forall W \in \mathcal{C}_h^H \). i.e. the family \( \theta_V \) is a twist.

Lemma 3.9. Let \( \mu = (\mu_1, \mu_2) \in \mathbb{C} \times \mathbb{C}, \) then the value of the twist \( \theta_{V_{\mu}} \) on a simple module \( V_{\mu} \) with highest weight \( \mu \) is \( \xi^{-\ell_{\mu_1} - 2\mu_2(1 + \mu_1 + \mu_2)} \text{Id}_{V_{\mu}}. \)
That is,
\[ \theta_{V_{\mu}} = \xi^{-\ell_{\mu_1} - 2\mu_2(1 + \mu_1 + \mu_2)} \text{Id}_{V_{\mu}} = -\xi^{-2(\alpha_2^2 + \alpha_1 \alpha_2)} \text{Id}_{V_{\mu}}. \]
where $\alpha = (\alpha_1, \alpha_2) = (\mu_1 - \ell + 1, \mu_2 + \frac{\xi}{2})$.

\textbf{Proof.} By the proof of Proposition 3.8, $\theta_{V_{\mu_1, \mu_2}} = \xi^{-\ell \mu_1 - 2\mu_2 (1 + \mu_1 + \mu_2)} \text{Id}_{V_{\mu}}$. \hfill $\square$

The category $\mathcal{C}^H$ is a ribbon category. Let $\mathcal{C}$ be the ribbon category of $\mathcal{C}$-colored oriented ribbon graphs in the sense of Turaev [14]. The set of morphisms $T((V_1, \pm), \ldots, (V_n, \pm))$, $((W_1, \pm), \ldots, (W_n, \pm)))$ is a space of linear combinations of $\mathcal{C}$-colored ribbon graphs. The ribbon Reshetikhin-Turaev functor $F : \mathcal{T} \to \mathcal{C}^H$ is defined by the Penrose graphical calculus.

\textbf{Definition 3.10.} If $T \in \mathcal{T}((V_{\mu}, +), (V_{\mu'}, +))$ where $V_{\mu}$ is a simple weight module of $\mathcal{U}^H_q sl(2|1)$, then $F(T) = x. \text{Id}_{V_{\mu}} \in \text{End}_{\mathcal{C}(\mu, \mu')} (V_{\mu})$ for $x \in \mathbb{C}$. We define the bracket of $T$ by $\langle T \rangle = x$. For example, if $V_{\mu}, V_{\mu'} \in \mathcal{C}^H$, we define $S'(V_{\mu}, V_{\mu'}) = \langle \begin{array}{c} \mu' \\ V_{\mu} \end{array} \rangle$.

We write $S'(\mu, \mu')$ for $S'(V_{\mu}, V_{\mu'})$.

Another example is the bracket of the twist $\langle \begin{array}{c} \mu' \\ V_{\mu} \end{array} \rangle = -\xi^{-(\alpha_1^2 + \alpha_2 \alpha_2)}$, $(\alpha_1, \alpha_2) = (\mu_1 - \ell + 1, \mu_2 + \frac{\xi}{2})$.

\textbf{Proposition 3.11.} Let $V = V_{\mu}$ be a typical module, $V' = V_{\mu'}$ be a simple module, then

$$S'(\mu, \mu') = \xi^{-4\alpha_1 \alpha_2 - 2(\alpha_2 \alpha_1 + \alpha_1 \alpha_2)} \frac{\ell \alpha_1 \alpha_1 \alpha_2 \alpha_2}{\alpha_1 \alpha_1}$$

where $\alpha = (\alpha_1, \alpha_2) = (\mu_1 - \ell + 1, \mu_2 + \frac{\xi}{2})$, $\alpha' = (\alpha_1', \alpha_2') = (\mu_1' - \ell + 1, \mu_2' + \frac{\xi}{2})$.

\textbf{Proof.} Let $S = S(\mu, \mu') \in \text{End}_{\mathcal{C}}(V_{\mu'}, V_{\mu'})$ be the endomorphism determined by the diagram $\begin{array}{c} \mu' \\ V_{\mu} \end{array}$. We have

$$S'(\mu, \mu') \text{Id}_{V_{\mu'}} = (\text{Id}_{V'} \otimes \text{ev}_{V})(c_{V,V'} \otimes \text{Id}_{V'})(c_{V',V} \otimes \text{Id}_{V'})(\text{Id}_{V'} \otimes \text{coev}_{V})$$

$$= X_1 X_2 X_3 X_4.$$
Lemma 3.13. mined up to an isomorphism and parity by its character.

Furthermore, the element \((\mathcal{R} - 1)(w_{\rho,\sigma,p} \otimes w_{0,0,0}^\prime, \mu)\) is a sum of vectors of the form \(v \otimes w^\prime\) where \(v^\prime\) is a weight vector of \(V_{\mu,\mu^\prime}\) and \(v^\prime\) is a weight vector of \(V_{\mu,\mu^\prime}\) which has a higher weight than \(w_{\rho,\sigma,p}\).

\[X_2X_3X_4(w_{0,0,0}^\prime) = \sum_{\rho,\sigma,p} \xi^{-\mu_1'(\mu_2+\sigma+p)-\mu_2'(\mu_1+\rho-\sigma-2p)-2\mu_2'(\mu_2+\sigma+p)}(w_{\rho,\sigma,p} \otimes w_{0,0,0}) \otimes w_{\rho,\sigma,p}^*\]

By the definition \(S(\mu, \mu')(w_{0,0,0}^\prime) = S'(\mu, \mu')w_{0,0,0}^\prime\), we deduce the proposition. \(\square\)

Definition 3.12. If \(\mu = (\mu_1, \mu_2) \in (\mathbb{C} \backslash \frac{1}{2}\mathbb{Z} \cup (-1 + \frac{1}{2}\mathbb{Z})) \times \mathbb{C} \backslash \frac{1}{2}\mathbb{Z}\) and \(\mu_2 + \mu_1 + 1 \in \mathbb{C} \backslash \frac{1}{2}\mathbb{Z}\), we define

\[d(\mu) = \frac{\{\mu_1 + 1\}}{\ell(\mu_1)}\{\mu_2\}{\mu_2 + \mu_1 + 1}\]

so there is a symmetry

\[d(\mu')S'(\mu, \mu') = d(\mu)S'(\mu', \mu)\]

3.3.5. Semi-simplicity of category \(\mathcal{C}^H\). Remember that \(G = \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z}\) and \(G_s = \{g \in G\) such that \(\exists V \in \mathcal{C}^H\) simple of \(\mathcal{C}^H\) and atypical\}.

Lemma 3.13. If \(\mathcal{C}^H_{\mathfrak{p}}\) is semi-simple, then a module \(V \in \mathcal{C}^H_{\mathfrak{p}}\) is determined up to an isomorphism and parity by its character.

The above lemma and the character of representation \(V_{\mu_1,\mu_2} \otimes V_{\mu_1',\mu_2'}\) gives us the following theorem.

Theorem 3.14. Let \(V_{\mu}, V_{\mu'}\) be two typical modules. If \(\mu + \mu' \not\in G_s\) then \(V_{\mu_1,\mu_2} \otimes V_{\mu_1',\mu_2'} = \bigoplus_{k=0}^{\ell-1}(V_{\mu_1+\mu_1'-2k,\mu_2+\mu_2'+k} \oplus V_{\mu_1+\mu_1'-2k+1,\mu_2+\mu_2'+k})\)
\[ \nabla_{\mu_1+\mu_1'-2k,\mu_2+\mu_2'+k+1} \otimes V_{\mu_1+\mu_1'-2k-1,\mu_2+\mu_2'+k+1} \] where \( \nabla \) is the module \( V \) with opposite parity.

**Proof.** According to the formula \([1]\), we have

\[
\chi_{\mu_1,\mu_2} \otimes \chi_{\mu_1',\mu_2'} = \chi_{\mu_1,\mu_2} \chi_{\mu_1',\mu_2'}
\]

\[
= X_1^{\mu_1+\mu_1'} X_2^{\mu_2+\mu_2'} \frac{1 - x^\ell}{1 - x} (1 + X_1)(1 + X_1 x) \sum_{k=0}^{\ell-1} (X_1^{-2} X_2)^k (1 + X_1 + X_2 + X_1^{-1} X_2)
\]

\[
= \frac{1 - x^\ell}{1 - x} (1 + X_1)(1 + X_1 x) \sum_{k=0}^{\ell-1} X_1^{\mu_1+\mu_1'-2k} X_2^{\mu_2+\mu_2'+k} + X_1^{\mu_1+\mu_1'-2k+1} X_2^{\mu_2+\mu_2'+k + 1}
\]

\[
+ X_1^{\mu_1+\mu_1'-2k} X_2^{\mu_2+\mu_2'+k + 1} + X_1^{\mu_1+\mu_1'-2k-1} X_2^{\mu_2+\mu_2'+k + 1}.
\]

The analysis of parity of highest weight vectors can be used to conclude. \( \square \)

**Remark 3.15.** Not all terms in the decomposition of the above theorem are distinct.

We defined a graduation \( \mathcal{C}^H = \bigoplus_{\pi \in G} \mathcal{C}^H_\pi \). Let \( \text{Proj} \) be the subcategory of \( \mathcal{C}^H \) containing projective modules, \( \text{Proj} \) is an ideal (see \([2]\)), i.e. \( \text{Proj} \) is closed under retracts and absorbent to the tensor product. We have the following proposition.

**Proposition 3.16.** For \( \pi \in G \), the three conditions below are equivalent

1. All the simple \( \mathcal{U}_\pi \text{sl}(2|1) \)-modules of \( \mathcal{C}_\pi \) are projective.
2. The category \( \mathcal{C}_\pi \) is semi-simple.
3. The \( \mathbb{C} \)-superalgebra of finite dimension \( \mathcal{U}/(k_1^\ell - \xi^{\ell \pi}, k_2^\ell - \xi^{\ell \pi}) \) is semi-simple where \( \mathcal{U} = \mathcal{U}_\pi \text{sl}(2|1)/(e_1^\ell, f_1^\ell) \).

**Proof.** The equivalence is classic knowing that \( \mathcal{C}_\pi \) is also a category of the \( \mathcal{U}/(k_1^\ell - \xi^{\ell \pi}, k_2^\ell - \xi^{\ell \pi}) \)-modules. \( \square \)

**Theorem 3.17.**

1. If \( \pi \in G \setminus G_s \) then \( \mathcal{C}^H_\pi \) is semi-simple.
2. A typical \( \mathcal{U}_\pi^H \text{sl}(2|1) \)-module is projective.

We select and fix a \( \pi \in G \setminus G_s \), note \( \mu_i = (\mu_1 + i_1, \mu_2 + i_2) \in \pi \), \( i_1, i_2 = 0, 1, ..., \ell - 1 \), that is \( \mu_i \in \{(\mu_1 + i_1, \mu_2 + i_2) : \ i_1, i_2 = 0, 1, ..., \ell - 1\} \). We have the two following lemmas.

**Lemma 3.18.** For all \( \mu_i, \mu_j \in \pi : \mu_i \neq \mu_j \) there exists \( z_{ij} \in \mathcal{Z} \) such that \( \chi_{\mu_i}(z_{ij}) \neq \chi_{\mu_j}(z_{ij}) \) where \( \chi_{\mu_i}(z_{ij}) \in \mathbb{C} \) is defined by \( \rho_{\mu_i}(z_{ij}) = \chi_{\mu_i}(z_{ij}) \text{Id}_{V_{\mu_i}} \).

**Proof.** We consider \( \mu = (\mu_1, \mu_2), \mu' = (\mu_1 + k, \mu_2 + m) \) \( k, m = 0, 1, ..., \ell - 1 \). We suppose that \( \forall \ z \in \mathcal{Z} : \chi_{\mu}(z) = \chi_{\mu'}(z) \). Consider the central
elements $C_p$ where $p \in \mathbb{Z}$ (see [1]). We have
\[
\chi_{\mu}(C_p) = (\xi - \xi^{-1})^2 \xi^{(2p-1)(\mu_1+2\mu_2)}[\mu_2 + \mu_1 + 1],
\]
\[
\chi'_{\mu}(C_p) = (\xi - \xi^{-1})^2 \xi^{(2p-1)(\mu_1+2\mu_2+k+2m)}[\mu_2 + \mu_1 + k + m + 1].
\]
Because $\chi_{\mu}(C_p) = \chi'_{\mu}(C_p)$ and $[\mu_2][\mu_2 + \mu_1 + 1] \neq 0$, we deduce that
\[
\begin{cases}
\frac{\chi_n(C_{p+1})}{\chi'_n(C_{p+1})} = \frac{\chi_n(C_p)}{\chi'_n(C_p)} \\
\chi_{\mu}(C_p) = \chi'_{\mu}(C_p).
\end{cases}
\]
This is equivalent to
\[
\begin{align*}
\xi^{2(\mu_1+2\mu_2)} &= \xi^{2(\mu_1+2\mu_2+k+2m)} \\
\xi^{(2p-1)(\mu_1+2\mu_2)}[\mu_2 + \mu_1 + 1] &= \xi^{(2p-1)(\mu_1+2\mu_2+k+2m)}[\mu_2 + \mu_1 + k + m + 1],
\end{align*}
\]
which implies
\[
2(k + 2m) = 0 \pmod{\ell \mathbb{Z}}
\]
and
\[
[\mu_2][\mu_2 + \mu_1 + 1] = \xi^{k+2m}[\mu_2 + m][\mu_2 + \mu_1 + k + m + 1].
\]
Because $\ell$ odd, (9) implies $k + 2m = 0 \pmod{\ell \mathbb{Z}} \iff k + m = -m \pmod{\ell \mathbb{Z}}$. On the other hand, (10) is equivalent to $[a][b] = [a + m][b - m] \iff -[a - b + m][m] = 0 \iff [-\mu_1 - 1 + m][m] = 0 \Rightarrow m = 0$ where $a = \mu_2, b = \mu_1 + \mu_2 + 1$. Because $m = 0$, we have $k = 0 \pmod{\ell \mathbb{Z}} \Rightarrow k = 0$.

**Lemma 3.19.** Let $\mathcal{V}$ be a vector space over $\mathbb{C}$, $I$ be a finite set and consider a family of $\mathbb{C}$-linear functions $a_i : \mathcal{V} \rightarrow \mathbb{C}, i \in I$. If for all $i \neq j \exists u_{ij} \in \mathcal{V}$ such that $a_i(u_{ij}) \neq a_j(u_{ij})$, then it exists $u_0 \in \mathcal{V}$ such that $\forall i \neq j \ a_i(u_0) \neq a_j(u_0)$.

**Proof.** We set $u = \sum_{i \neq j} x_{ij} u_{ij} \in \mathcal{V}$ with $x_{ij} \in \mathbb{C}, i, j \in I$. We note $x = (x_{ij}) \in \mathbb{C}^N$. We consider the set $X = \{ x \in \mathbb{C}^N : \sum_{i \neq j} (a_i(u_{ij}) - a_j(u_{ij}))x_{ij} = 0 \}$, this is a finite reunion of hyperplanes of $\mathbb{C}^N$. This proves that $\exists x \notin X$ and this $x$ does not have the above property. That is, it exists $u_0 \in \mathcal{V}$ such that $a_i(u_0) \neq a_j(u_0)$ for all $i \neq j$.

Now we change the basis of module $V_{\mu}$ as follows. We set
\[
w'_{\rho,\sigma,p} = \begin{cases}
w_{\rho,\sigma,p} & \text{if } \rho = \sigma = 0, 1 \\
f_1^{(\rho)}w_{1,0,0} & \text{if } \rho = 1, \sigma = 0 \\
e_1^{(\rho-1)}w_{0,1,r-1} & \text{if } \rho = 0, \sigma = 1
\end{cases}
\]
where $p = 0, \ldots, \ell - 1$. For the basis $\{w'_{\rho,\sigma,p}\}$ we have the actions

\[
\begin{align*}
k_1 w'_p &= \xi^{\mu_1+\rho-\sigma-2p} w'_{\rho,\sigma,p}, \\
k_2 w'_p &= \xi^{\mu_2+\sigma+p} w'_{\rho,\sigma,p}, \\
e_1 w'_{1,0,p} &= [p][\mu_1 + 2 - p] w'_{1,0,p-1}, \\
f_1 w'_{1,0,p} &= w'_{1,0,p+1}, \\
e_1 w'_{0,1,p} &= w'_{0,1,p-1}, \\
f_1 w'_{0,1,p} &= [p + 1][\mu_1 - (p + 1)] w'_{0,1,p+1}.
\end{align*}
\]

**Proof of the theorem** 3.17 We begin to show that $\mathfrak{g}_\Pi$ is semi-simple. We set $\mathcal{A} = \mathcal{U}(k_1 - \xi^{\mu_1}, k_2 - \xi^{\mu_2})$. The density theorem implies that the application $\rho : \mathcal{A} \to \prod_{\mu_1} \text{End}(V_{\mu_1}) \cong \prod_{i=1}^{\ell^2} \mathcal{M}_{4\ell}(\mathbb{C})$ is surjective. We give here an elementary proof.

By Lemma 3.18 and 3.19, it exists an element $z \in \mathcal{Z}$ such that $\forall \mu_1 \neq \mu_j \chi_{\mu_1}(z) \neq \chi_{\mu_j}(z)$ and we set $z_i = \chi_{\mu_i}(z)$ $i = 1, \ldots, \ell^2$ and we introduce the ideal $J = \prod_{\ell^2}(z - z_i)\mathcal{A}$.

Firstly, we consider the representation $\rho : \mathcal{A}/(z - z_i) \to \text{End}_C(V_{\mu_i})$. We will prove that $\rho$ is a surjection. We have $\text{End}_C(V_{\mu_i}) \cong \mathcal{M}_{4\ell}(\mathbb{C})$.

We consider the elements $\Omega = \frac{k_1\xi^{+1} + k_1\xi^{-1}}{1!} + f_1 e_1 = \frac{k_1\xi^{+1} + k_1\xi^{-1}}{1!} + e_1 f_1, c = k_1 k_2^2, k_1$ in $\mathfrak{g}_\mathfrak{s}(2|1)$. The actions of these elements on the basis $w'_{\rho,\sigma,p}$ are defined by

\[
\begin{align*}
\Omega w'_{0,0,p} &= (\xi^{\mu_1+1} + \xi^{-\mu_1-1}) w'_{0,0,p}, \\
\Omega w'_{1,1,p} &= (\xi^{\mu_1+1} + \xi^{-\mu_1-1}) w'_{1,1,p}, \\
\Omega w'_{0,1,p} &= \xi^{\mu_1} + \xi^{-\mu_1} \{1\}^2 w'_{0,1,p}, \\
\Omega w'_{1,0,p} &= \xi^{\mu_1+2} w'_{0,1,p}, \\
c w'_{\rho,\sigma,p} &= \xi^{\mu_1+2\mu_2+\rho+\sigma} w'_{\rho,\sigma,p}, \\
k_1 w'_{\rho,\sigma,p} &= \xi^{\mu_1+\rho-\sigma-2p} w'_{\rho,\sigma,p}.
\end{align*}
\]

We now check that for all $w'_{\rho,\sigma,m} \neq w'_{\rho',\sigma',j}$ $\exists u \in \{\Omega, c, k_1\}$ such that $\chi_{\mu_1}(u) \neq \chi_{\mu_1}(j)$ where $\rho(u) w'_{\rho,\sigma,m} = \chi_{\rho,\sigma,m}(u) w'_{\rho,\sigma,m}$. Indeed, if $\rho + \sigma \neq \rho' + \sigma'$ then we select $u = c$ and we have $cw'_{\rho,\sigma,m} \neq cw'_{\rho',\sigma',j}$. If $\rho + \sigma = \rho' + \sigma'$ then we consider two cases: if $(\rho, \sigma) \neq (\rho', \sigma')$ we select $u = \Omega$ and $\Omega w'_{\rho,\sigma,m} \neq \Omega w'_{\rho',\sigma',j}$; if $(\rho, \sigma) = (\rho', \sigma')$ we select $u = k_1$ and we have $k_1 w'_{\rho,\sigma,m} \neq k_1 w'_{\rho',\sigma',j}$ because $m \neq j$.

By Lemma 3.19, it exists a vector $u_0 \in \mathcal{C}(\Omega, c, k_1)$-space generated by the elements $\Omega, c, k_1$ such that $\chi_{\mu_1}(u_0) \neq \chi_{\mu_1}(j)(u_0)$ for all $w'_{\rho,\sigma,m} \neq w'_{\rho',\sigma',j}$.

\[
\begin{align*}
k_1 w'_{\rho,\sigma,p} &= \xi^{\mu_1+\rho-\sigma-2p} w'_{\rho,\sigma,p}, \\
k_2 w'_{\rho,\sigma,p} &= \xi^{\mu_2+\sigma+p} w'_{\rho,\sigma,p}, \\
e_1 w'_{1,0,p} &= [p][\mu_1 + 2 - p] w'_{1,0,p-1}, \\
f_1 w'_{1,0,p} &= w'_{1,0,p+1}, \\
e_1 w'_{0,1,p} &= w'_{0,1,p-1}, \\
f_1 w'_{0,1,p} &= [p + 1][\mu_1 - (p + 1)] w'_{0,1,p+1}.
\end{align*}
\]
This demonstrates that \( A \) is the identity. Thus, the application \( A \) is surjective.

For \( i \in \{1, ..., \ell^2\}, j \in \{1, ..., 4\ell\} \) we have \( \rho(A/(z - z_i))(v_j) \subset V_{\mu} \) (here we note \( v_j \) the \( j \)-th vector of the basis) and \( V_{\mu} \) is simple. Thus we deduce \( \rho(A/(z - z_i))(v_j) = V_{\mu} \). This proves that there exists \( a_0 \in A/\{z - z_i\} \) such that \( \rho(a_0)(v_j) = v_n \) \( \forall n \in \{1, ..., 4\ell\} \).

The endomorphism \( \rho(a_0) \) determines the matrix \( (\rho(a_0)) \) where \( \rho(a_0)_{jn} = 1 \). The matrix \( E_{jn} \) is equal to \( E_{jj} = \rho(a_0)_{jn}E_{nn} \), i.e. the matrix \( E_{jn} \) is the image of an element in \( A/\{z - z_i\} \). So the application \( \rho \) is a surjection. This implies that the application \( \prod_{i=1}^{\ell^2} A/(z - z_i) \rightarrow \prod_{i=1}^{\ell^2} \mathcal{M}_{4\ell}(\mathbb{C}) \) is surjective.

Secondly, the composition \( \prod_{i=1}^{\ell^2} A/(z - z_i) \rightarrow A/J \rightarrow \prod_{i=1}^{\ell^2} A/(z - z_i) \) is the identity. Thus, the application \( A/J \rightarrow \prod_{i=1}^{\ell^2} A/(z - z_i) \) is surjective. We deduce a series of surjections \( A \rightarrow A/J \rightarrow \prod_{i=1}^{\ell^2} A/(z - z_i) \rightarrow \prod_{i=1}^{\ell^2} \mathcal{M}_{4\ell}(\mathbb{C}) \), this sequence determines the surjection \( A \rightarrow \prod_{i=1}^{\ell^2} \mathcal{M}_{4\ell}(\mathbb{C}) \).

Furthermore, the two algebras \( A \) and \( \prod_{i=1}^{\ell^2} \mathcal{M}_{4\ell}(\mathbb{C}) \) have the same dimension \( 16\ell^4 \). This implies that this surjection is an isomorphism. This demonstrates that \( A \) is semi-simple. The category \( \mathcal{C}_\pi \) is also semi-simple.

Now we prove that \( \mathcal{C}_\pi^H \) is semi-simple. Let \( V^H \) be a module in \( \mathcal{C}_\pi^H \). Set \( W = \text{Ker} e_1 \cap \text{Ker} e_2 \cap \text{Ker} e_3 \), it is a vector space of the highest weight vectors (the weights for \( (h_1, h_2) \)). We call \( \{v_j\}_{j=1}^n \) a basis of weight vectors of \( W \), we have \( h_i v_j = \mu_j v_j, i = 1, 2 \). So each \( v_j \) generates a \( \mathcal{U}_\pi^H \mathfrak{sl}(2|1) \)-module \( V_j \),

\[
V_j = \mathcal{U}_\pi^H \mathfrak{sl}(2|1).v_j = \mathcal{U}_\pi \mathfrak{sl}(2|1).v_j = \mathcal{U}_-.v_j
\]

where \( \mathcal{U}_- = \text{Alg}(f_1, f_2, f_3) \subset \mathcal{U}_\pi \mathfrak{sl}(2|1) \) and \( \text{dim}(\mathcal{U}_-) = 4\ell \). Thus \( \text{dim}(V_j) \leq 4\ell \) and \( V_j \) is simple (because there is no module in \( \mathcal{C}_\pi^H \) of dimension strictly between 0 and \( 4\ell \)).

Set \( V' = \sum_{i=1}^{n} V_i \subset V^H \). We can write \( V^H = V' \oplus V'' \) as a \( \mathcal{U}_\pi \mathfrak{sl}(2|1) \)-module. However \( W \subset V' \) which implies \( V'' = 0 \) (because there is no highest weight vector in \( V'' \) and \( V^H = V'_0 = \sum_{i=1}^{n} V_i \). Because the \( V_i \) are simple, so \( V^H = \bigoplus_{i \in I} V_i \) where \( I \subset \{1, ..., n\} \). Thus \( V^H \) is semi-simple.

For the second assertion (2), if \( V \in \mathcal{C}_\pi^H \) and \( \mathcal{C}_\pi^H \) is semi-simple, then \( V \) is projective. If not, (2) follows from \( S'(V_\mu, V) \neq 0 \) where \( V_\mu \) is any projective typical module which implies that \( V \) is a direct factor of \( V_\mu \otimes V \otimes V'' \in \text{Proj} \). This implies that \( V \) is a projective module. \( \square \)
4. Modified traces on the projective modules

4.1. Ambidextrous module. For each object $V$ of the category $\mathcal{C}$ and any endomorphism $f$ of $V \otimes V$ set

$$\ptr_R(f) = (\Id_V \otimes \overline{\text{ev}}_V) \circ (f \otimes \Id_{V^*}) \circ (\Id_{V^*} \otimes \overline{\text{coev}}_V) \in \text{End}(V),$$
$$\ptr_L(f) = (\overline{\text{ev}}_V \otimes \Id_{V^*}) \circ (\Id_{V^*} \otimes f) \circ (\overline{\text{coev}}_V \otimes \Id_{V^*}) \in \text{End}(V).$$

In the ribbon category $\mathcal{C}^H$ of nilpotent weight $U_q^H \mathfrak{sl}(2|1)$-modules, we say that a module $V$ is ambidextrous if $V$ simple and $\ptr_L(f) = \ptr_R(f)$ for all $f \in \text{End}(V \otimes V)$ (see [7]).

**Theorem 4.1.** Each typical module $V_\mu$ of category $\mathcal{C}^H$ is an ambidextrous module.

**Proof.** We will prove this theorem in two steps:

Step 1. Proving the existence of two nonzero $U_q^H \mathfrak{sl}(2|1)$-invariant vectors $x_+ w_+$ and $x_+ w_-$.

Step 2. Applying Theorem 3.1.3 [3] gives us the affirmation that $V_\mu$ is ambidextrous.

Call $v_+, v'_+$ the highest weight vectors of $V_\mu$, $V_{\mu}^*$ and $v_-, v'_-$ the lowest weight vectors of $V_\mu, V_{\mu}^*$. Set $x_- = f_2 f_3 f_1^{-\ell}, x_+ = e_2 e_3 e_1^{-\ell}, w_+ = v_+ \otimes v'_+, w_- = v_- \otimes v'_-$. We will prove that the two vectors $x_- w_+$ and $x_+ w_-$ are $U_q^H \mathfrak{sl}(2|1)$-invariant.

We consider the actions of generator elements $e_i, h_i, f_i$ on $x_- w_+$. The highest weight vector (resp. lowest) of $V_\mu$ is $v_+ = w_{1,1,\ell-1}^+$ (resp. $v_- = w_{0,0,0}^+$). The highest weight vector (resp. lowest) of $V_{\mu}^*$ is $v'_+ = w_{1,1,\ell-1}^+ \otimes v'_+$ (resp. $v'_- = w_{0,0,0}^+$).

The weight of vector $w_+ = v_+ \otimes v'_+$ is equal to the sum of the weights of $v_+$ and $v'_+$. That is weight($w_+ = (\mu_1, \mu_2) + (\mu_1 + 2\ell - 2, -\mu_2 - \ell) = (2\ell - 2, -\ell)$. Furthermore, weight($x_- w_+ = weight(f_2 f_3 f_1^{-\ell} w_+ = -\ell weight(e_1) - 2 weight(e_2) + weight(w_+) = -\ell(2, -1) - 2(-1, 0) + (2\ell - 2, -\ell) = (0, 0)$. It implies that $h_i x_- w_+ = 0$.

We also have the relations below between the generator elements in $U_q^H \mathfrak{sl}(2|1)$ (see (B1) [3]):

$$f_1 f_2^p f_3^p f_1^p = \xi^p \sigma f_2 f_3 f_1 f_1^{p+1} - \rho (1 - \sigma) \xi^{-p} f_2 f_3 f_1^{p+1} f_1^{p+1} f_1^p,$$
$$f_2 f_3 f_1^p f_1^p = (1 - \rho) f_2 f_3 f_1^{p+1}$$

$$[e_1, f_2 f_3 f_1^{p+1}] = \sigma (1 - \rho) (1 - \sigma) f_2 f_3 f_1^{p+1} f_3^{p+1} f_1^{p+1} + [p] f_2 f_3 f_1^{p+1} [h_1 - p + 1],$$
$$e_2 f_2 f_3 f_1^{p+1} f_1^p = (-1)^{p+\sigma} f_2 f_3 f_1^{p+1} f_2 f_3 f_1^{p+1} f_1^p h_2 + p + \sigma + \sigma (-1)^{p+\sigma} f_2 f_3 f_1^{p+1} f_1^p \xi^{-p} h_2 - p$$

where $(p, \rho, \sigma) \in \mathbb{N} \times \{0, 1\} \times \{0, 1\}$. With the above relations, it is easy to check $f_1 x_- w_+ = 0$.

The fourth relation above gives us $e_2 f_2 f_3 f_1^{p+1} f_2 f_3 f_1^{p+1} e_2 = f_3 f_1^{p+1} [h_2 + \ell]$. Because $e_2 (v_+ \otimes v'_+) = 0$ and $[h_2 + \ell](v_+ \otimes v'_+) = 0$, we deduce $e_2 x_- w_+ = 0$.
The third relation gives \([e_1, f_2 f_3 f_1^{-1}] = [\ell - 1] f_2 f_3 f_1^{-2}[h_1 - \ell + 2]\). Because \(e_1(v_+ \otimes v'_+) = 0\) and \([h_1 - \ell + 2](v_+ \otimes v'_+) = 0\), we deduce \(e_1 x_- w_+ = 0\).

Consequently, we conclude that \(x_- w_+\) is an \(U\mathfrak{l}(2|1)\)-invariant vector. The demonstration that the vector \(x_+ w_-\) is \(U\mathfrak{l}(2|1)\)-invariant is analogous using the relations obtained by applying the automorphism \(\omega\) of superalgebra \(U\mathfrak{l}(2|1)\) where \(\omega(e_i) = (-1)^{\deg e_i} f_i\), \(\omega(f_i) = (-1)^{\deg f_i} e_i\), \(\omega(h_i) = k_i^{-1}\), \(\omega(h_i) = -h_i\), \(i = 1, 2\).

Furthermore \(\Delta x_- = x_- \otimes 1 + \) a sum of tensor products of two elements of \(U\mathfrak{l}(2|1)\) with negative weight. Thus \(\Delta x_- w_+\) contains the nonzero vector \(x_- v_+ \otimes v'_+ = f_2 f_3 f_1^{-1} v_+ \otimes v'_+ = w_1, v_{1, \ell - 1} \otimes v'_+\). We conclude that the vector \(x_- w_+\) is nonzero. Similarly, the vector \(x_+ w_-\) is nonzero.

For step 2, we use the following results:

The decomposition of the tensor product \(V \otimes V^*\) is a direct sum of indecomposable modules

\[ V \otimes V^* = P_1 \oplus ... \oplus P_m. \]

The set of invariant vectors \(w \in V \otimes V^*\) is in bijection with \(\text{coev}_V (\mathbb{C})\) because \(\text{Hom}_\mathfrak{g}(\mathbb{C}, V \otimes V^*) \cong \text{Hom}_\mathfrak{g}(V, V) \cong \mathbb{C}\).

The vector \(w_+\) (resp. \(w_-\)) is the highest weight vector (resp. lowest weight vector) of \(V \otimes V^*\). Then there exists a unique integer \(k\) (resp. \(l\)) such that \(w_+ \in P_k\) (resp. \(w_- \in P_l\)). The weight of \(w_+\) (resp. \(w_-\)) is \(\lambda_+ = (2\ell - 2, -\ell)\) (resp. \(\lambda_- = (-2\ell + 2, \ell)\)). Because \(\lambda_- = -\lambda_+\) and \((V \otimes V^*)^* \cong (V \otimes V^*)\), this implies \(P_+^\ast \cong P_l\).

In addition, \(\text{coev}_V (1) \in P_1\), \(\text{coev}_V (1) \in P_k\) because \(x_+ P_l \subset P_l, x_- P_k \subset P_k\), then \(P_k = P_l\). That is \(P_k = P_k^\ast\). By Theorem 3.1.3 [4], it gives us the affirmation that \(V_\mu\) ambidextrous.

\[ \square \]

**Remark 4.2.** All typical modules are projective and ambidextrous.

### 4.2. Modified traces on the projective modules.

**Definition 4.3.** Let \(\mathcal{I}\) be an ideal of \(\mathcal{C}\) (see [3]). The family of linear applications \(t = (t_V : \text{End}_\mathfrak{g}(V) \rightarrow k)_{V \in \mathcal{I}}\) is a trace (modified trace) on \(\mathcal{I}\) if it satisfies:

\(\forall U, V \in \mathcal{I}, \forall W \in \mathcal{C},\)

\[ \forall f \in \text{Hom}_\mathfrak{g}(U, V), \forall g \in \text{Hom}_\mathfrak{g}(V, U), t_V(f \circ g) = t_U(g \circ f) \]

\[ \forall f \in \text{End}_\mathfrak{g}(V \otimes W), t_V \otimes W(f) = t_V(\text{ptr}_R(f)). \]

We also have

\[ \forall f \in \text{End}_\mathfrak{g}(W \otimes V), t_W \otimes V(f) = t_V(\text{ptr}_L(f)). \]

Given \(V\) as a typical module. The module \(V\) is ambidextrous and projective. This implies that the ideal generated by this module is \(\mathcal{I}_V = \text{Proj}\) (see [3]). So we have the following theorem.
Theorem 4.4. There exists a unique modified trace \( t = \{ t_P \}_{P \in \text{Proj}} \) on the ideal \( \text{Proj} \) of projective modules of \( \mathcal{C}^H \),

\[
    t_P : \text{End}(P) \to \mathbb{C}, \; P \in \text{Proj}.
\]

If \( P = V_\mu \) is a typical module, then \( t_{V_\mu}(f) = \langle f \rangle d(\mu), f \in \text{End}(V_\mu), \)

\[ d(\mu) = t_{V_\mu}(\text{Id}_{V_\mu}) \]

is determined by the Definition 3.12.

4.3. Invariants of embedded graphs. Recall that \( \mathcal{C}^H \) is the \( \mathbb{C} \)-linear ribbon category of the finite dimensional representations of \( \mathcal{U}^H \mathfrak{sl}(2 | 1) \), \( \text{Proj} \) is the ideal of projective modules and \( t \) is a trace on \( \text{Proj} \).

We call \( G \) the set of \( \mathcal{C}^H \)-colored closed ribbon graphs, that are the \( \mathcal{C}^H \)-colored ribbon graphs in \( S^3 \). We have \( G \cong \text{End}_\mathcal{T}(\emptyset) \).

We use the concept of a cutting presentation of \( \mathcal{C}^H \)-colored closed ribbon graph: If a diagram \( T \) represents a \( \mathcal{C}^H \)-colored ribbon graph which is an endomorphism of \( \mathcal{T} \), its lower and upper parts are formed by the same sequences of \( k \) vertical colored strands. It is then possible, as for a braid of \( k \) strands, to consider the closure \( \hat{T} \) obtained by joining its \( k \) top vertices to its \( k \) bottom vertices by \( k \) parallel strands. This construction is actually the categorical trace in \( \mathcal{T} \): we have \( \hat{T} = \text{tr}_\mathcal{T}(T) \in \text{End}_\mathcal{T}(\emptyset) \). We say that \( T \) is a cutting presentation with \( k \) strands of the closed graph \( \hat{T} \) and that \( \hat{T} \) is the closure of \( T \) (see [13]).

A closed graph \( T \) of \( \mathcal{T} \) is said to be \( \mathcal{C}^H \)-colored admissible if there is at least one strand of \( T \) colored by \( P \in \text{Proj} \). Let \( \mathcal{G}_a \) be the set of isotopy classes of \( \mathcal{C}^H \)-colored admissible ribbon graphs.

From the trace \( t \) on \( \text{Proj} \) we have the theorem below.

Theorem 4.5. The application

\[
    F' : \mathcal{G}_a \to \mathbb{C} \\
    \hat{T} \mapsto t_P(F(T))
\]

is well defined. Here, \( P \in \text{Proj}, T \in \text{End}_\mathcal{T}((P, +)) \) is a cutting presentation with one strand of \( \hat{T} \). That is to say the complex number \( t_P(F(T)) \) does not depend on the choice of \( T \) but only of the isotopy class of the \( \mathcal{C}^H \)-colored graph \( \hat{T} \).

Proof. First, we select an edge of \( \hat{T} \) and cut, we have the graph \( T \). Then, we select and cut a second edge of \( \hat{T} \), we have the graph \( T' \). By cutting \( \hat{T} \) in both these places, one obtains a graph \( T_2 \in \text{End}_\mathcal{G}_a((P, +), (P', +)) \) which is a presentation with two strands of \( \hat{T} \) and such that \( T = \)
Finally we use the properties of the compatibility of trace $t$:

$$t_{P}(F(T)) = t_{P}(\text{ptr}_R(F(T_2))) = t_{P\otimes P'}(F(T_2)) = t_{P'}(\text{ptr}_L(F(T_2))) = t_{P'}(F(T')).$$ □

Remark 4.6. In the case $P = V_{\mu}$ typical, we have

$$F'(\begin{array}{c}
  \\ T' \\
\end{array}) = d(\mu) \begin{array}{c}
  \\ T \\
\end{array}.$$ 

The affirmation of the above theorem gives us a link invariant in the following corollary.

Corollary 4.7. The application $F': \{\text{links } \mathbb{C}^2 - \text{colored}\} \rightarrow \mathbb{C}$ enables to associate with each link with $n$ ordered components a meromorphic function on $\mathbb{C}^{2n}$.

5. Invariant of 3-manifolds

In the article [2] the authors constructed $\mathcal{C}$-decorated 3-manifold invariants where $\mathcal{C}$ is a ribbon category. In the previous section, it was proven that $\mathcal{C}^H$ is a ribbon category, this suggests we construct an invariant of $\mathcal{C}^H$-decorated 3-manifolds. We recall some concepts, definitions and results from [2].

5.1. Relative $G$-modular categories. Let $\mathcal{C}$ be a $k$-linear ribbon category where $k$ is a field. A set of objects of $\mathcal{C}$ is said to be commutative if for any pair $\{V, W\}$ of these objects, we have $c_{V,W} \circ c_{W,V} = \text{Id}_{W \otimes V}$ and $\theta_V = \text{Id}_V$. Let $(Z, +)$ be a commutative group. A realization of $Z$ in $\mathcal{C}$ is a commutative set of objects $\{\varepsilon_t\}_{t \in Z}$ such that $\varepsilon_0 = I$, $\text{qdim}(\varepsilon_t) = 1$ and $\varepsilon_t \otimes \varepsilon_t' = \varepsilon_{t+t'}$ for all $t, t' \in Z$.

A realization of $Z$ in $\mathcal{C}$ induces an action of $Z$ on isomorphism classes of objects of $\mathcal{C}$ by $(t, V) \mapsto \varepsilon_t \otimes V$. We say that $\{\varepsilon_t\}_{t \in Z}$ is a free realization of $Z$ in $\mathcal{C}$ if this action is free. This means that $\forall t \in Z \setminus \{0\}$ and for any simple object $V \in \mathcal{C}$, $V \otimes \varepsilon_t \not\simeq V$. We call simple $Z$-orbit the reunion of isomorphism classes of an orbit for this action.

Definition 5.1 ([2]). Let $(G, \times)$ and $(Z, +)$ be two commutative groups. A $k$-linear ribbon category $\mathcal{C}$ is $G$-modular relative to $X$ with modified dimension $d$ and periodicity group $Z$ if

1. the category $\mathcal{C}$ has a $G$-grading $\{\mathcal{C}_g\}_{g \in G}$,
2. the group $Z$ has a free realization $\{\varepsilon^t\}_{t \in Z}$ in $\mathcal{C}_1$ (where $1 \in G$ is the unit),
3. there is a $Z$-bilinear application $G \times Z \rightarrow k^\times, (g, t) \mapsto g^t$ such that $\forall V \in \mathcal{C}_g, \forall t \in Z, c_{V,\varepsilon^t} \circ c_{\varepsilon^t, V} = g^t \text{Id}_{\varepsilon^t \otimes V}$,
Indeed, we have \( Z = (5) \) is satisfied. Theorem 4.4 implies that condition (6) is true.

Definition 5.1 are satisfied.

Because \( w \) is bilinear application \( G \circ R(\rho, \sigma, p) \), we consider a typical module \( \mathcal{V} \) that the scalar \( \Delta_+ \) defined in Figure 1 is nonzero, similarly, there exists an element \( g \in G \setminus \mathcal{X} \) and an object \( V \in \mathcal{C}_g \) such that the scalar \( \Delta_- \) defined in Figure 7 is nonzero, (8) the morphism \( S(U, V) = F(H(U, V)) \neq 0 \in \text{End}_\mathcal{E}(V) \), for all simple objects \( U, V \in \text{Proj} \), where

\[
H(U, V) = \begin{cases} \mathcal{V} & \text{in } \text{End}_\mathcal{E}((V, +)) \end{cases}.
\]

The category \( \mathcal{C}_g^H \) of \( U^H_\mathcal{E}(\mathfrak{s}(2)[1]) \)-modules is \( G \)-modular relative to \( \mathcal{X} \). Indeed, we have \( \mathcal{C}_g^H \) being \( G \)-graded by \( G = \mathbb{C}/\mathbb{Z} \times \mathbb{C}/\mathbb{Z} \). We set \( Z = \mathbb{Z} \times \mathbb{Z} \) and \( \{e^n\}_{n \in \mathbb{Z}} \) the set of simple highest weight modules \( n = (n_1, n_2, \ell) \), i.e. \( e^n \) is a \( U^H_\mathcal{E}(\mathfrak{s}(2)[1]) \)-module of dimension 1 (with the basis \( \{w\} \) determined by \( h_1 w = n_1 \ell w, h_2 w = n_2 \ell w, e_i w = f_i w = 0 \). Because \( c_{e_m, e^n} = \tau \) and \( \theta_{e^n} = \text{Id} \), the two conditions (1) and (2) of the Definition 5.1 are satisfied.

We consider a typical module \( V_\mu \). We have \( c_{e^n, V_\mu}(w \otimes w_{\rho, \sigma, p}) = \tau \circ R(w \otimes w_{\rho, \sigma, p}) = \xi^{-n_1 \ell \mu_2 - n_2 \ell \mu_1 - 2n_2 \ell \mu_2} w_{\rho, \sigma, p} \otimes w \). Next \( c_{V_\mu, e^n} \circ c_{e^n, V_\mu}(w \otimes w_{\rho, \sigma, p}) = c_{V_\mu, e^n}(\xi^{-n_1 \ell \mu_2 - n_2 \ell \mu_1 - 2n_2 \ell \mu_2} w_{\rho, \sigma, p} \otimes w) = \xi^{-2 \ell (\mu_2 n_1 + (\mu_1 + 2 \mu_2)n_2)} w \otimes w_{\rho, \sigma, p} \). So we can determine the \( Z \)-bilinear application \( G \times Z \rightarrow \mathbb{C}^\times, (\mu, n) \mapsto \xi^{-2 \ell (\mu_2 n_1 + (\mu_1 + 2 \mu_2)n_2)} \text{Id}_{e^n} \otimes V_\mu(w \otimes w_{\rho, \sigma, p}) \). This means that we have condition (3) of the definition. Condition (4) is also satisfied with \( \mathcal{X} = G_s = \{\frac{1}{2}, 0, \frac{1}{2}\} \times \mathbb{C}/\mathbb{Z} \cup \mathbb{C}/\mathbb{Z} \times \{\frac{1}{2}, 0, \frac{1}{2}\} \cup \{(\mu_1, \mu_2): \mu_1 + \mu_2 \in \{0, \frac{1}{2}\}\} \). It was proven that \( \mathcal{C}_g^H \) is semi-simple for \( g \in G \setminus G_s \) (Theorem 3.17) and \( V_\mu \otimes e^n \simeq V_{\mu + \ell n} \), i.e. the condition (5) is satisfied. Theorem 4.4 implies that condition (6) is true.
To compute $\Delta_-$, we first use the graphical calculus

$$
F\left(\begin{array}{c}
\Omega_{\mu} \\
\mu
\end{array}\right) = \sum_{s,t=0}^{\ell-1} d(\mu_{st}) F\left(\begin{array}{c}
\theta_{\mu}^{-1} \\
\mu
\end{array}\right)
$$

$$
= \sum_{s,t=0}^{\ell-1} d(\mu_{st}) \left(\theta_{\mu}^{-1}\right) \left(\theta_{\mu_{st}}^{-1}\right) F\left(\begin{array}{c}
\mu
\end{array}\right)
$$

$$
= \sum_{s,t=0}^{\ell-1} d(\mu_{st}) \left(\theta_{\mu}^{-1}\right) \left(\theta_{\mu_{st}}^{-1}\right) S'(\mu_{st}, \mu) \text{Id}_{\mu}.
$$

We have

$$
\left\langle \theta_{\mu}^{-1} \right\rangle = -\xi^{2(\alpha_2^2 + \alpha_1 \alpha_2)}, \quad \left\langle \theta_{\mu_{st}}^{-1} \right\rangle = -\xi^{2((\alpha_2 + t)^2 + (\alpha_1 + s) + \alpha_2 + t)}
$$

and $S'(\mu_{st}, \mu) = \xi^{-4\alpha_2(\alpha_2 + t) - 2(\alpha_2(\alpha_1 + s) + \alpha_1(\alpha_2 + t))} \frac{1}{\ell d(\mu)}$.

Thus

$$
F\left(\begin{array}{c}
\Omega_{\mu} \\
\mu
\end{array}\right) = \sum_{s,t=0}^{\ell-1} d(\mu_{st}) \frac{1}{\ell d(\mu)} \frac{1}{\ell \alpha_1} \sum_{s,t=0}^{\ell-1} \frac{\{\alpha_1 + s\}}{\{\alpha_2 + t\}\{\alpha_1 + \alpha_2 + s + t\}} \xi^{2(t^2 + st)} \text{Id}_{\mu}
$$

$$
= \frac{1}{\ell d(\mu)} \frac{1}{\ell \alpha_1} \sum_{s,t=0}^{\ell-1} \left(\xi^{-(\alpha_2 + t)} \frac{\{\alpha_1 + s\}}{\{\alpha_2 + t\}} - \frac{\xi^{-(\alpha_1 + \alpha_2 + s + t)}}{\{\alpha_1 + \alpha_2 + s + t\}}\right) \xi^{2(t^2 + st)} \text{Id}_{\mu}.
$$

Because

$$
\sum_{s,t=0}^{\ell-1} \frac{\xi^{-(\alpha_2 + t)} \xi^{2(t^2 + st)}}{\{\alpha_2 + t\}} = \sum_{t=0}^{\ell-1} \xi^{2t^2} \frac{\xi^{-(\alpha_2 + t)}}{\{\alpha_2 + t\}} \sum_{s=0}^{\ell-1} \xi^{2st}
$$

$$
= \sum_{t=0}^{\ell-1} \xi^{2t^2} \frac{\xi^{-(\alpha_2 + t)}}{\{\alpha_2 + t\}} \ell \delta_{t}^{0}
$$

$$
= \ell \xi^{-\alpha_2} \frac{\{\alpha_2\}}{\{\alpha_2\}},
$$
\[
\sum_{s,t=0}^{\ell-1} \xi^{-\{\alpha_1+\alpha_2+s+t\}} \xi^{2(t^2+st)} \frac{1}{\xi^{2(\alpha_1+\alpha_2+s+t)}} = - \sum_{s,t=0}^{\ell-1} \xi^{2(t^2+st)} \frac{1}{1 - \xi^{2(\alpha_1+\alpha_2+s+t)}} \\
= - \sum_{s,t=0}^{\ell-1} \xi^{2(t^2+st)} \sum_{k=0}^{\infty} \xi^{2k(\alpha_1+\alpha_2+s+t)} \\
= - \sum_{k=0}^{\infty} \sum_{t=0}^{\ell-1} \xi^{2(t^2+ka_1+ka_2+kt)} \sum_{s=0}^{\ell-1} \xi^{2(s+k)t} \\
= - \sum_{k=0}^{\infty} \sum_{t=0}^{\ell-1} \xi^{2(t^2+ka_1+ka_2+kt)) \delta_{t+k}^{(0) \mod \ell N} \\
= -\ell \left(1 + \sum_{t=0}^{\ell-1} \xi^{2t} \sum_{j=1}^{\infty} \xi^{2(j-t)(\alpha_1+\alpha_2+t)}\right) \\
= -\ell \left(1 + \sum_{t=0}^{\ell-1} \xi^{-2t(\alpha_1+\alpha_2)} \xi^{2t(\alpha_1+\alpha_2)} \frac{\xi^{2(\alpha_1+\alpha_2)}}{1 - \xi^{2(\alpha_1+\alpha_2)}}\right) \\
= -\ell + \frac{\xi^{\alpha_1+\alpha_2}}{\{\alpha_1 + \alpha_2\}} \\
\]

then

\[
F \left( \delta^\infty_{\mu} \right) = \frac{1}{d(\mu)\{\ell\alpha_1\}} \left( \frac{1}{\xi^{\alpha_2}\{\alpha_2\}} - \frac{\xi^{\alpha_1+\alpha_2}}{\{\alpha_1 + \alpha_2\}} + 1 \right) \text{Id}_{V_\mu} \\
= \frac{1}{\{\alpha_1\}} \left( \{\alpha_1 + \alpha_2\} \xi^{-\alpha_2} - \{\alpha_2\} \xi^{\alpha_1+\alpha_2} + \{\alpha_2\} \{\alpha_1 + \alpha_2\} \right) \text{Id}_{V_\mu} \\
= \frac{1}{\{\alpha_1\}} \{\alpha_1\} = \text{Id}_{V_\mu}.
\]

This means that \(\Delta_- = 1\).

By using the automorphism \(\omega\) of superalgebra \(\mathcal{U}\mathfrak{sl}(2|1)\) where \(\omega(e_i) = (-1)^{\deg e_i} f_i, \omega(f_i) = (-1)^{\deg e_i} e_i, \omega(h_i) = k_i^{-1}, \omega(h_i) = -h_i, i = 1, 2\) and computing we also have \(\Delta_+ = 1\). Condition (8) is obviously true.

Hence category \(\mathcal{C}^H\) is relatively \(G\)-modular.

### 5.2. Invariants of 3-manifolds.

**Definition 5.2.** Let \((M,T,\omega)\) be a triple where \(M\) is a compact connected oriented 3-manifold, \(T \subset M\) is a \(\mathcal{C}^H\)-colored ribbon graph (possibly empty) and \(\omega \in H^1(M \setminus T, G)\).

1. The triple \((M,T,\omega)\) is compatible if each edge \(e\) of \(T\) is colored by an element of \(\mathcal{C}_{\omega(m_e)}\) where \(m_e\) is an oriented meridian of the edge \(e\).

2. Let \(L \cup T \subset S^3\) where \(L\) is an oriented link in \(S^3 \setminus T\) which gives a presentation of \((M,T)\) by surgery. The presentation \(L \cup T\) is
computable if for each component $L_i$ of $L$ whose meridian is denoted $m_i$, we have $\omega(m_i) \notin \mathcal{X}$.

We suppose that $(M, T, \omega)$ is a compatible triple.

**Definition 5.3.** The formal linear combination $\Omega_\pi = \sum_{\nu, \in \pi} d(V_\nu) V_\nu$, is a Kirby color of degree $\bar{\pi} \in G \setminus G_s$ if $\{V_\nu\}$ is a set of representatives of simple $\mathbb{Z}$-orbits of $G_\pi$.

**Theorem 5.4.** Let $(M, T, \omega)$ a compatible triple admitting a computable presentation $L \cup T \subset S^3$ then

$$N(M, T, \omega) = F'(L_\omega \cup T)$$

is a well defined topological invariant, i.e. depends only on the diffeomorphism class of the triple $(M, T, \omega)$ where $L_\omega$ is obtained as the link $L$ in which we have colored the $i$-th component $L_i$ by a Kirby color of degree $\omega(m_i)$ where $m_i$ is a meridian of $L_i$.

5.3. **Example.** We consider an example in the case $\ell = 3$. Let $M$ be the lens space $L(5, 2)$ which is given by surgery presentation on the Hopf link $L$ (Figure 2). It has two oriented components $L_i$, $i = 1, 2$ with framings $3, 2$ and let $m_i$ be an oriented meridian of $L_i$. The linking matrix of $L$ with respect to the components $L_i$ is

$$\text{lk} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$ 

Let $\omega \in H^1(M \setminus T, G)$ and suppose that the triple $(M, \emptyset, \omega)$ is computable. We compute the values $\omega = (\omega^1, \omega^2)$ where $\bar{\pi} = \omega^1 = \omega(m_1), \overline{\pi} = \omega^2 = \omega(m_2)$ from the equations $3\overline{\pi} + \bar{\pi} = 0$ and $\overline{\pi} + 2\bar{\pi} = 0$ (in $\mathbb{Z}/\mathbb{Z} \times \mathbb{Z}/\mathbb{Z}$). Hence $\bar{\pi} = \left(\frac{k}{5}, \frac{4k}{5}\right), \overline{\pi} = \left(\frac{2k}{5}, \frac{4k}{5}\right), k = 1, \ldots, 4$. Here we set $\omega_k = (\omega_k^1, \omega_k^2), \omega_k^1 = \left(\frac{k}{5}, \frac{2k}{5}\right), \omega_k^2 = \left(\frac{2k}{5}, \frac{4k}{5}\right), k = 1, \ldots, 4$. We have $\omega_4 = -\omega_1, \omega_3 = -\omega_2$. Using variables as in Lemma 3.9 we have $(\alpha_1, \alpha_2) = \bar{\pi} + (-\ell + 1, \frac{\ell}{2}) = (\frac{k}{5} - 2, \frac{2k}{5} + \frac{3}{2}), (\alpha'_1, \alpha'_2) = \overline{\pi} + (-\ell + 1, \frac{\ell}{2}) = (\frac{2k}{5} - 2, \frac{4k}{5} + \frac{3}{2})$.

We color the $i$-th component $L_i$ by a Kirby color of degree $\omega(m_i)$, i.e. $\Omega_{\omega(m_1)} = \Omega_\pi = \sum_{s,t=0}^2 d(\alpha_{st}) V_{\alpha_{st}}$ and $\Omega_{\omega(m_2)} = \Omega_\overline{\pi} = \sum_{i,j=0}^2 d(\alpha'_{ij}) V_{\alpha'_{ij}}$ where $\alpha_{st} = (\alpha_1 + s, \alpha_2 + t), \alpha'_{ij} = (\alpha'_1 + i, \alpha'_2 + j)$. By Lemma 3.9...
Proposition 3.11 we have
\[ N(M, \emptyset, \omega) = \sum_{s,t} \sum_{i,j} d(\alpha_{st})d(\alpha'_{ij}) \left( \theta_{V_{\alpha_{st}}}, \theta_{V'_{\alpha'_{ij}}} \right)^2 d(\alpha_{st})S'(\alpha'_{ij}, \alpha_{st}) \]
in which
\[ d(\alpha_{st}) = \frac{\{\alpha_1 + s\}}{\ell(\ell(\alpha_1 + s))\{\alpha_2 + t\}\{\alpha_1 + \alpha_2 + s + t\}}, \]
\[ \left( \theta_{V_{\alpha_{st}}} \right) = -\xi^{-2((\alpha_2 + t)^2 + (\alpha_1 + s)(\alpha_2 + t))}, \]
\[ \left( \theta_{V'_{\alpha'_{ij}}} \right) = -\xi^{-2((\alpha'_2 + j)^2 + (\alpha'_1 + i)(\alpha'_2 + j))}, \]
\[ S'(\alpha'_{ij}, \alpha_{st}) = \frac{1}{\ell d(\alpha_{st})} \xi^{-4(\alpha'_2 + j)(\alpha_2 + t) - 2((\alpha'_2 + j)(\alpha_1 + s) + (\alpha'_1 + i)(\alpha_2 + t))}. \]

Using computer algebra software Sagemath, we have \((\xi^{\frac{1}{10}}\text{ has degree 8 over } \mathbb{Q})\)
\[ N(M, \emptyset, \pm\omega_1) = \frac{1}{15} \left( -2\xi^\frac{7}{10} - 2\xi^\frac{3}{5} - 2\xi^\frac{2}{5} + 5\xi^\frac{1}{3} + 2\xi^\frac{1}{6} \right), \]
\[ N(M, \emptyset, \pm\omega_2) = \frac{1}{15} \left( -7\xi^\frac{7}{10} - 2\xi^\frac{3}{5} + 4\xi^\frac{2}{5} + 4\xi^\frac{1}{3} + 5\xi^\frac{1}{6} - 4 \right). \]

In this case, the result \(N(M, \emptyset, \omega) = N(M, \emptyset, \bar{\omega})\) is consistent with \((M, \emptyset, \omega) \simeq (M, \emptyset, \bar{\omega})\).

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