SYMPLECTIC FORMS AND COHOMOLOGY
DECOMPOSITION OF ALMOST COMPLEX 4-MANIFOLDS

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1. Introduction

In this paper we continue to study differential forms on an almost complex 4–manifold \((M, J)\) following [18]. We are particularly interested in the subgroups \(H^+_J(M)\) and \(H^-_J(M)\) of the degree 2 real De Rham cohomology group \(H^2(M, \mathbb{R})\). These are the sets of cohomology classes which can be represented by \(J\)-invariant, respectively, \(J\)-anti-invariant real 2–forms. The goal pursued by defining these sub-groups is simple: understand the effects of the action of the almost complex structure on forms at the level of cohomology and introduce the idea of (real) cohomology type, via the almost complex structure. Certainly, the subgroups \(H^\pm_J(M)\) and their dimensions \(h^\pm_J\) are diffeomorphism invariants of the almost complex manifold \((M, J)\). We would like to show that these invariants appear to be interesting, particularly so in dimension 4. Here is the outline of our paper.

Our first main result, Theorem 2.3 in section 2, shows that on any compact almost complex 4-manifold the subgroups \(H^+_J(M)\) and \(H^-_J(M)\) will induce a direct sum decomposition of \(H^2(M, \mathbb{R})\). With the terminology introduced in [18], Theorem 2.3 says that any almost complex structure on a compact 4-dimensional manifold is \(C^\infty\)-pure and full. See section 2 for precise definitions. Theorem 2.3 appears to be specifically a 4-dimensional result, as a very recent preprint of Fino and Tomassini [10] shows the existence of a compact 6-dimensional nil-manifold with an almost complex structure which is not \(C^\infty\)-pure (the intersection of \(H^+_J(M)\) and \(H^-_J(M)\)) is non-empty).

Also in section 2, for a compact 4-manifold with an integrable \(J\), we show that subgroups \(H^+_J(M)\) and \(H^-_J(M)\) relate naturally with the (complex) Dolbeault cohomology groups. We also show that a complex type decomposition for cohomology does not hold for non-integrable almost complex structures (see Lemma 2.11 and Corollary 2.12).

In section 3 we compute the subgroups \(H^+_J(M)\) and \(H^-_J(M)\) and their dimensions \(h^+_J\) for almost complex structures related to integrable ones.

\footnote{We learned of the preprint [10] while putting together the final form of our paper. There are further interesting links between [10] and our paper (see further comments in section 2). The overlap is minimal though.}
The main focus is on metric related almost complex structures, i.e. almost complex structures which admit a common compatible metric.

In section 4 we focus on almost complex structures $J$ which admit compatible or tame symplectic forms and we give estimates for the dimensions $h^+_J$ in this case. If there are $J$-compatible symplectic forms, then the collection of cohomology classes of all such forms, the so-called $J$-compatible cone, $\mathcal{K}_J(M)$, is a (nonempty) open convex cone of $H^+_J(M)$. Thus it is important to determine the dimension $h^+_J$ of $H^+_J(M)$.

Our investigation of almost complex structures which are tamed by symplectic forms is also partly motivated by the following question of Donaldson (9):

**Question 1.1.** If $J$ is an almost complex structure on a compact 4–manifold $M$ which is tamed by a symplectic form $\omega$, is there a symplectic form compatible with $J$?

In [18] it was shown that the question has an affirmative answer when $J$ is integrable. For non-integrable $J$, positive answer is provided in a neighborhood of the integrable ones on $\mathbb{P}^2$ in [21] (see also [22]).

We observe in Theorem 4.3 that an estimate on $h^+_J$ which is immediate in the case of compatible $J$’s can be carried over to the case of tamed $J$’s as well. Moreover, using Taubes’ $\text{SW} = \text{GT}_{\omega}$, we show in Theorem 4.8 that the result of Theorem 4.3 can be often improved. Section 3 ends with an equivalent formulation of Donaldson’s Question 1.1.

Finally, in section 5 we discuss Donaldson’s approach to Question 1.1 via the symplectic version of the Calabi-Yau equation. We observe that his technique based on the Implicit Function Theorem can also be used to obtain a lower semi-continuity property of $h^+_J$ near a compatible $J$ (see Theorem 5.5). Via Theorem 2.3, we automatically get the upper semi-continuity property for $h^-_J$ near a compatible $J$.

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**Convention:** The groups indexed by $(p, q)$ arise from complex differential forms. The groups indexed by $\pm$ arise from real differential forms.

2. **Cohomology Decomposition of Almost Complex 4–Manifolds**

2.1. **The groups $H^+_J$.** Let $M$ be a compact $2n$-dimensional manifold and suppose $J$ is an almost complex structure on $M$. $J$ acts on the bundle of real 2-forms $\Lambda^2$ as an involution, by $\alpha(\cdot, \cdot) \to \alpha(J\cdot, J\cdot)$, thus we have the splitting,

$$\Lambda^2 = \Lambda^+_J \oplus \Lambda^-_J.$$  

We will denote by $\Omega^2$ the space of 2-forms on $M$ ($C^\infty$-sections of the bundle $\Lambda^2$), $\Omega^+_J$ the space of $J$-invariant 2-forms, etc. For any $\alpha \in \Omega^2$, the
J-invariant (resp. J-anti-invariant) component of \( \alpha \) with respect to the decomposition \([1]\) will be denoted \( \alpha' \) (resp. \( \alpha'' \)).

**Definition 2.1.** \([18]\) Let \( Z^2 \) denote the space of closed 2-forms on \( M \) and let \( Z^\pm = Z \cap \Omega^\pm_J \). Define
\[
H^\pm_J(M) = \{ a \in H^2(M; \mathbb{R}) | \exists \alpha \in Z^\pm_J such that [\alpha] = a \}.
\]

**2.2. The type decomposition of \( H^2(M; \mathbb{R}) \).** Obviously, \( H^+_J(M) + H^-_J(M) \subseteq H^2(M; \mathbb{R}) \), but if \( J \) is not integrable, it is not clear if equality holds and not even if the intersection of the two subspaces is trivial. The following definitions were also introduced in \([18]\):

**Definition 2.2.**

(i) \( J \) is said to be \( C^\infty \)-pure if \( H^+_J \cap H^-_J = 0 \);

(ii) \( J \) is said to be \( C^\infty \)-full if \( H^2(M; \mathbb{R}) = H^+_J(M) + H^-_J(M) \).

**Note:** The terms *pure* and *full* almost complex structures were also defined in \([18]\) in terms of currents. We will not use these in this paper, so we refer the reader to \([18]\) and \([10]\) for more on this. Note also that the preprint of Fino and Tomassini provides a number of interesting cases when the notions of pure and full almost complex structures are equivalent to the \( C^\infty \) counterparts (Theorem 3.7 and Corollary 4.1 in \([10]\)). See also Remark 2.6 below.

Our first result is

**Theorem 2.3.** If \( M \) is a compact 4-dimensional manifold then any almost complex structure \( J \) on \( M \) is \( C^\infty \)-pure and full. Thus, there is a direct sum cohomology decomposition
\[
H^2(M; \mathbb{R}) = H^+_J(M) \oplus H^-_J(M).
\]

Before the proof, we should set some more preliminaries and notations. The particularity of dimension 4 is that the Hodge operator \(*_g\) of a Riemannian metric \( g \) on \( M \) also acts as an involution on \( \Lambda^2 \). Thus, we have the well-known self-dual, anti-self-dual splitting of the bundle of 2-forms,
\[
\Lambda^2 = \Lambda^+_g \oplus \Lambda^-_g.
\]

We will denote by \( \Omega^\pm_g \) the space of sections of \( \Lambda^\pm_g \) and by \( \alpha^+, \alpha^- \) the self-dual, anti-self-dual components of a 2-form \( \alpha \). Since the Hodge-deRham Laplacian commutes with \(*_g\), the decomposition \([11]\) holds for the space of harmonic 2-forms \( \mathcal{H}_g \) as well. By Riemannian Hodge theory, we get the metric induced cohomology decomposition
\[
H^2(M; \mathbb{R}) = \mathcal{H}_g = \mathcal{H}^+_g \oplus \mathcal{H}^-_g
\]
As in Definition 2.1, one can define
\[
H^\pm_g(M) = \{ a \in H^2(M; \mathbb{R}) | \exists \alpha \in Z^\pm_g such that [\alpha] = a \}.
\]
Of course, \( Z^\pm := Z^2 \cap \Omega^\pm = \mathcal{H}^\pm_g \), so clearly \( H^+_g(M) = \mathcal{H}^+_g \), and \( \mathbb{H} \) can be written as
\[
H^2(M; \mathbb{R}) = H^+_g \oplus H^-_g.
\]
We will need the following special feature of the Hodge decomposition in dimension 4.

**Lemma 2.4.** If \( \alpha \in \Omega^+_g \) and \( \alpha = \alpha_h + d\theta + \delta\Psi \) is its Hodge decomposition, then \( (d\theta)_g^+ = (\delta\Psi)_g^+ \) and \( (d\theta)_g^- = -(\delta\Psi)_g^- \). In particular, the 2–form
\[
\alpha - 2(d\theta)_g^+ = \alpha_h
\]
is harmonic and the 2–form
\[
\alpha + 2(d\theta)_g^- = \alpha_h + 2d\theta
\]
is closed.

**Proof.** Since \( \ast \omega = \omega \), by the uniqueness of the Hodge decomposition, we have \( \ast (d\theta) = \delta\Psi, \ast (\delta\Psi) = d\theta \). The lemma follows. \( \square \)

Suppose now that \( J \) is an almost complex structure and \( g \) is a \( J \)-compatible Riemannian metric on the 4-manifold \( M \). The pair \((g, J)\) defines a \( J \)-invariant 2–form \( \omega \) by
\[
\omega(u, v) = g(Ju, v).
\]
Such a triple \((J, g, \omega)\) is called an almost Hermitian structure. Given \( J \), we can always choose a compatible \( g \). The relations between the decompositions (1) and (4) on a 4-dimensional almost Hermitian manifold are
\[
\Lambda^+_J = \mathbb{R}(\omega) \oplus \Lambda^-_g,
\]
\[
\Lambda^+_g = \mathbb{R}(\omega) \oplus \Lambda^-_J,
\]
\[
\Lambda^+_J \cap \Lambda^+_g = \mathbb{R}(\omega), \quad \Lambda^-_J \cap \Lambda^-_g = \{0\}.
\]
The following lemma is an immediate consequence of (8):

**Lemma 2.5.** Let \((M^4, g, J, \omega)\) be a 4-dimensional almost Hermitian manifold. Then \( Z^-_J \subset \mathcal{H}^+_g \) and the natural map \( Z^-_J \rightarrow H^-_J \) is bijective. More precisely, if \( \mathcal{H}^+_g \omega^\perp \) denotes the subspace of harmonic self-dual forms pointwise orthogonal to \( \omega \), we have
\[
H^-_J = Z^-_J = \mathcal{H}^+_g \omega^\perp.
\]
In particular, any closed, \( J \)-anti-invariant form \( \alpha (\alpha \neq 0) \) is non-degenerate on an open dense subset \( M' \subseteq M \).

**Proof.** Since \( \Lambda^-_J \subset \Lambda^+_g \), a closed \( J \)-anti-invariant 2–form is a self-dual harmonic form. In particular, there exists no non-trivial exact \( J \)-anti-invariant 2–form. Thus, the natural map \( Z^-_J \rightarrow H^-_J \) is bijective. The equality (identification) (10) is obvious. For the last statement, note that any self-dual form is non-degenerate on the complement of its nodal set \( M' = M \setminus \alpha^{-1}(0) \). On
the other hand, any harmonic form satisfies the unique continuation property, so if \( \alpha \neq 0 \), its nodal set \( \alpha^{-1}(0) \) has empty interior. In fact, from [6] it is known more: \( \alpha^{-1}(0) \) has Hausdorff dimension \( \leq 2 \).

We are now ready to give the proof of Theorem 2.3

**Proof of Theorem 2.3.** Let \( g \) be a \( J \)-compatible Riemannian metric and let \( \omega \) be the 2-form defined by \((g,J)\). We start by proving that \( J \) is \( C^\infty \)-pure.

If \( a \in H^+_J \cap H^-_J \), let \( \alpha' \in Z^+_J \), \( \alpha'' \in Z^-_J \), be representatives for \( a \). Then

\[
a \cup a = \int_M \alpha' \wedge \alpha'' = 0,
\]

but by Lemma 2.5, we also have

\[
a \cup a = \int_M \alpha'' \wedge \alpha'' = \int_M |\alpha''|^2_g \, d\mu_g.
\]

Thus \( \alpha'' = 0 \), so \( a = 0 \).

Next we prove that \( J \) is \( C^\infty \)-full. Suppose the contrary. Then there exists a class \( a \in H^2(M; \mathbb{R}) \) which is (cup product) orthogonal to \( H^+_J \oplus H^-_J \).

Since \( H^-_J \subset H^+_J \), we can assume \( a \in H^+_J \). Let \( \alpha \) be the harmonic, self-dual representative of \( a \) and denote \( f = \langle \alpha, \omega \rangle \). The function \( f \) is not identically zero, as otherwise it follows that \( a \in H^-_J \). Now we apply Lemma 2.4 to the self-dual form \( f\omega \). The closed form \((f\omega)_h + 2(f\omega)^{\text{exact}} \) is also \( J \)-invariant; indeed, it is equal to \( f\omega + 2((f\omega)^{\text{exact}})_g^- \). (Here and later, we shall denote \( \alpha^{\text{exact}} \) the exact part from the Hodge decomposition of a form \( \alpha \).) Thus \((f\omega)_h + 2(f\omega)^{\text{exact}} \) is a representative for a class \( b \in H^+_J \). But

\[
a \cup b = \int_M \alpha \wedge (f\omega)_h + 2(f\omega)^{\text{exact}} \geq 0 \quad dV
\]

\[
= \int_M f \omega + 2((f\omega)^{\text{exact}})_g^- \geq 0 \quad dV
\]

\[
= \int_M f^2 \quad dV \neq 0.
\]

This contradicts the assumption that \( a \) is orthogonal to \( H^+_J \oplus H^-_J \). \( \square \)

**Remark 2.6.** (i) Combining Theorem 2.3 with Theorem 3.7 from [10], it follows that any almost complex structure on a compact 4-dimensional manifold is not just pure and full for forms, but pure and full for currents as well.

(ii) Example 3.3 of [10] shows that Theorem 2.3 does not generalize to higher dimensions.

The following result was also proved in [10], Proposition 3.2.

**Proposition 2.7.** If \( J \) is an almost complex structure on a compact manifold \( M^{2n} \) and \( J \) admits a compatible symplectic structure, then \( J \) is \( C^\infty \)-pure.

**Proof.** On any almost Hermitian manifold \((M^{2n}, g, J, \omega)\), if \( \alpha \in \Omega^-_J \), then

\[
*_{g}(\alpha) = \alpha \wedge \omega^{n-2}.
\]
Thus, if \( \omega \) is symplectic and \( \alpha \) is closed, (11) implies that \( *_g(\alpha) \) is also closed. Hence, for any almost Kähler structure \((g,J,\omega)\), \( Z_J^+ \subset \mathcal{H}_g^2 \). It is straightforward now to generalize the first part of the proof of Theorem 2.3.

Let \( a \in H_J^2 \cap H_J^- \), and let \( \alpha' \in Z_J^+ \), \( \alpha'' \in Z_J^- \), be representatives for \( a \). Then
\[
a \cup a \cup [\omega]^{n-2} = \int_M \alpha' \wedge \alpha'' \wedge \omega^{n-2} = 0,
\]
but by (11) we also have
\[
a \cup a \cup [\omega]^{n-2} = \int_M \alpha'' \wedge \alpha'' \wedge \omega^{n-2} = \int_M |\alpha''|_g^2 d\mu_g.
\]
Thus \( \alpha'' = 0 \), so \( a = 0 \).

We end this subsection by noting that there may exist a duality between cohomology groups \( H_J^\pm \) and the corresponding homology groups defined in terms of currents, see [18].

2.3. The complexified \( H^2 \).

2.3.1. The groups \( H_J^{p,q} \). In all of the above, we referred to decompositions of real 2-forms. We present now the relation with the more familiar splitting of bi-graded complex 2-forms:

\[
\Lambda_C^2 = \Lambda_J^{2,0} \oplus \Lambda_J^{1,1} \oplus \Lambda_J^{0,2}
\]

The relation between the decompositions (1) and (12) is well known:

\[
\Lambda_J^+ = (\Lambda_J^{1,1})_R,
\]
\[
\Lambda_J^- = (\Lambda_J^{0,2} \oplus \Lambda_J^{2,0})_R.
\]

Note that the bundle \( \Lambda_J^- \) inherits an almost complex structure, still denoted \( J \), by
\[
\beta \in \Lambda_J^- \rightarrow J\beta \in \Lambda_J^-,
\]
where \( J\beta(X,Y) = -\beta(JX,Y) \).

**Definition 2.8.** Let \( H_J^{p,q} \) be the subspace of the complexified De Rham cohomology \( H^2(M;\mathbb{C}) \), consisting of classes which can be represented by a complex closed form of type \( (p,q) \).

**Lemma 2.9.** The groups \( H_J^{p,q} \) have the following properties:

\[
H_J^{p,q} = \overline{H_J^{p,q}},
\]
\[
H_J^{p,p} = (H_J^{p,p} \cap H^{2p}(M;\mathbb{R})) \otimes \mathbb{C},
\]
\[
(H_J^{p,q} + H_J^{q,p}) = ((H_J^{p,q} + H_J^{q,p}) \cap H^{p+q}(M;\mathbb{R})) \otimes \mathbb{C}.
\]
Proof. (14) follows from the fact that a complex form $\Psi$ is closed if and only if its conjugate $\overline{\Psi}$ is closed.

(15) follows from (14) and the following fact: Let $V$ be a real vector space and $W$ a complex subspace of $V \otimes_{\mathbb{R}} \mathbb{C}$, which as a subspace is invariant under conjugation. Then $W$ is the complexification of $W \cap V$ (see By Remark 2.5 on p. 139 in [7]).

We now investigate the relation between the groups $H^+_J$ and $H^{p,q}_J$. As we shall see in Lemma 2.11, when $J$ is not integrable, there is an important difference compared to what (13) would have predicted:

Lemma 2.10. For a compact almost complex manifold $(M,J)$ of any dimension,

$$(16) \quad H^+_J = H^{1,1}_J \cap H^2(M;\mathbb{R}),$$

and

$$H^{1,1}_J = H^+_J \otimes \mathbb{C}.$$  

Proof. The relation (17) is a consequence of (16) and (15) with $(p,p) = (1,1)$. So we just need to prove (16).

The inclusion $H^+_J \subseteq H^{1,1}_J \cap H^2(M;\mathbb{R})$ is clear, so we now prove the converse inclusion. An element in $H^{1,1}_J \cap H^2(M;\mathbb{R})$ can be represented by a complex $d$ closed $(1,1)$ form $\rho = \sigma + d\tau$, with $[\sigma] \in H^2(M;\mathbb{R})$. So it is also represented by the real $d$ closed $(1,1)$ form $\frac{1}{2}(\rho + \overline{\rho}) = \sigma + d(\tau + \overline{\tau})$.

When $J$ is integrable the same argument appears in the proof of Theorem 2.13 in [7].

The next lemma is a well known result (see e.g. [19]), recast in our terminology. It can be also be seen as a consequence and as a slight extension of Hitchin’s Lemma (13).

Lemma 2.11. Let $J$ be an almost complex structure on a compact 4-manifold.

$$(18) \quad (H^{2,0}_J + H^{0,2}_J) = \begin{cases} H^-_J \otimes \mathbb{C}, & \text{if } J \text{ is integrable,} \\ 0, & \text{if } J \text{ is not integrable.} \end{cases}$$

In particular, if $J$ is integrable, then

$$H^-_J = (H^{2,0}_J + H^{0,2}_J) \cap H^2(M;\mathbb{R}).$$

Proof. A (complex) form $\Phi \in \Omega^{2,0}_J$ is of the form

$\Phi = \beta + iJ\beta$, where $\beta \in \Omega^-_J$.

Assume $\beta \neq 0$. The point of the lemma is that $d\beta = 0$ and $d(J\beta) = 0$ occur simultaneously if and only if $J$ is integrable. To see this, let $Z_j = X_j - iJX_j$, $j = 1, 2, 3$ be arbitrary $(1,0)$ vector fields. Then

$$d\Phi(Z_1, \overline{Z}_2, \overline{Z}_3) = -\Phi(\overline{Z}_2, \overline{Z}_3^{1,0}, Z_1).$$
Assuming $d\beta = d(J\beta) = 0$, i.e. $d\Phi = 0$, the above relation implies $[Z_2,\overline{Z}_3]^{1,0} = 0$. This follows first on the set $M' = M \setminus \beta^{-1}(0)$, but then everywhere on $M$ by continuity, since $M'$ is dense in $M$ (see Lemma 2.5). This implies the integrability of $J$.

Conversely, assume that $J$ is integrable and we want to show that $d\beta = 0$ iff $d(J\beta) = 0$. Using $d = \partial + \overline{\partial}$, and $2\beta = \Phi = \overline{\Phi}$, we have

$$2d\beta = (\partial + \overline{\partial})(\Phi + \overline{\Phi}) = \overline{\partial}\Phi + \partial\overline{\Phi}.$$ 

(We used that $\partial\Phi = 0$ since it is a $(3,0)$ form on a complex surface.) Thus $d\beta = 0$ iff $\overline{\partial}\Phi = 0$. Similarly, $d(J\beta) = 0$ iff $\overline{\partial}(i\Phi) = 0$. But it is obvious that $\overline{\partial}\Phi = 0$ iff $\overline{\partial}(i\Phi) = 0$.

We will show in the examples of section 3 that it is possible for non-integrable $J$’s to have nonzero $H_J^{-}$. By the above two lemmas, we get:

**Corollary 2.12.** Suppose $J$ is an almost complex structure on a compact $4$–manifold. Then $J$ is always complex $\mathcal{C}^\infty$-pure in the sense $H_J^{1,1} \cap H_J^{2,0} \cap H_J^{0,2} = \{0\}$. Moreover, $J$ is also complex $\mathcal{C}^\infty$-full, i.e.,

$$H^2(M;\mathbb{C}) = H_J^{1,1} \oplus H_J^{2,0} \oplus H_J^{0,2};$$

if and only if $J$ is integrable or $h_J^{-} = 0$.

2.3.2. Dolbeault decomposition when $J$ is integrable. When $J$ is integrable, there is the Dolbeault decomposition which has long been discovered. We briefly recall this decomposition and relate it to the groups $H_J^{p,q}$ introduced in the previous subsection.

When $J$ is integrable, $\overline{\partial}^2 = 0$, so there is the $\overline{\partial}$ complex and the associated Dolbeault cohomology groups, which we denote by $H_{\overline{\partial}}^{p,q}(M)$. But our groups $H_{\overline{\partial}}^+(M)_{\mathbb{R}}$ are subgroups of the De Rham cohomology groups, and are generally different from the Dolbeault groups in arbitrary dimension. However, for a complex surface, they are closely related.

The Fröhlicher spectral sequence of the double complex

$$(\Omega(M) \otimes \mathbb{C} = \oplus \Omega^{p,q}, \partial, \overline{\partial})$$

reads (see P. 41-45, P. 140-141 in [7]):

$$E_1^{p,q} = H_{\overline{\partial}}^{p,q}(M) \Rightarrow H^{p+q}(M;\mathbb{C}).$$

The resulting Hodge filtration on $H^2(M;\mathbb{C})$ reads:

$$H^2(M;\mathbb{C}) = F^0(H^2) \supset F^1(H^2) \supset F^2(H^2) \supset 0,$$

where

$$H^2(M;\mathbb{C}) = \{[\alpha], \alpha \in \oplus_{p'>q'} \Omega^{p',q'} | d\alpha = 0\}.$$
Since

$$H^p_q(M) = E^p_q \to E_{\infty}^p_q = \frac{F^p(H^{p+1}(M; \mathbb{C}))}{F^{p+1}(H^{p+1}(M; \mathbb{C}))},$$

if the Fröhlicher spectral sequence degenerates at $E_1$, then

$$H^p_q(M) \cong \frac{F^p(H^{p+1}(M; \mathbb{C}))}{F^{p+1}(H^{p+1}(M; \mathbb{C}))}. \quad (21)$$

For $p + q = 2$ let

$$\mathcal{H}^p_q(M) = F^p(H^2) \cap \overline{F^q(H^2)}. \quad (22)$$

**Lemma 2.13.** $\mathcal{H}^p_q$ consists of De Rham classes which can be represented by a form of type $(p, q)$, i.e.

$$\mathcal{H}^p_q = H_j^{p,q}. \quad (23)$$

This should be known to experts; we record the argument here since it is useful to elucidate the relation between $H_j^+$ and $H_j^{1,1}.$

**Proof.** $F^2(H^2)$ consists of De Rham classes which can be represented by a form of type $(2, 0)$. Consequently, $\overline{F^2(H^2)}$ consists of classes of $(0, 2)$ forms.

It remains to show that $F^1(H^2) \cap \overline{F^1(H^2)}$ consists of De Rham classes which can be represented by a closed form of type $(1, 1)$. First of all, every such De Rham class lies in $F^1(H^2)$ and $\overline{F^1(H^2)}$. On the other hand, by definition, a class is in $F^1(H^2) \cap \overline{F^1(H^2)}$ if and only if it is represented by closed forms $\alpha_1 = \alpha_1^{1,1} + \alpha_1^{0,0}$ and $\alpha_2 = \alpha_2^{1,1} + \alpha_2^{0,0}$. Now $\alpha_1 - \alpha_2 = d\beta$, and it is easy to see that $\alpha_1 - d\beta^{1,0} = \alpha_2 + d\beta^{0,1}$ is a $d$-closed $(1, 1)$ form representing the same class. \qed

A weight 2 formal Hodge decomposition is a decomposition of the form

$$H^2(M; \mathbb{C}) = \oplus_{p+q=2} \mathcal{H}^p_q. \quad (24)$$

**Theorem 2.14.** ([7]) If $(M, J)$ is a Kähler manifold or a complex surface, then the Fröhlicher spectral sequence degenerates at $E_1$, and there is a weight 2 formal Hodge decomposition. Consequently,

$$
\begin{align*}
H_0^{2,0} &= E_\infty^{2,0} \cong F^2(H^2) & \cong \mathcal{H}^{2,0}, \\
H_0^{1,1} &= E_\infty^{1,1} \cong \frac{F^1(H^2)}{F^2(H^2)} & \cong F^1(H^2) \cap \overline{F^1(H^2)} & \cong \mathcal{H}^{1,1}, \\
H_0^{0,2} &= E_\infty^{0,2} \cong \frac{H^2(M; \mathbb{C})}{F^2(H^2)} & \cong F^2(H^2) & \cong \mathcal{H}^{0,2}.
\end{align*}
\quad (25)
$$

Together with (23), (16) and (19), we conclude

**Proposition 2.15.** If $J$ is integrable on a compact 4–manifold, then

$$H^{p,q}_j = H^{p,q}_\partial, \quad (26)$$

and

$$H^+_j = H^{1,1}_\partial \cap H^2(\mathbb{R}), \quad H^-_j = (H^{2,0}_\partial \oplus H^{0,2}_\partial) \cap H^2(\mathbb{R}). \quad (27)$$
3. Computations of $h^\pm_J$

Let us denote the dimension of $H^1_J$ by $h^\pm_J$. Notice that $h^\pm_J$ are diffeomorphism invariants of the almost complex manifold $(M, J)$. In this section we give several calculations of $h^\pm_J$.

3.1. When $J$ is integrable. If $J$ is integrable, it follows from Proposition 2.15 that

\[ h^+_J = h^{1,1}_0, \quad h^-_J = 2h^{2,0}_0. \]

Together with the signature theorem (Theorem 2.7 in [7]), we get

\[ h^+_J = \begin{cases} b^- + 1 & \text{if } b_1 \text{ even} \\ b^- & \text{if } b_1 \text{ odd} \end{cases}, \quad h^-_J = \begin{cases} b^+ - 1 & \text{if } b_1 \text{ even} \\ b^+ & \text{if } b_1 \text{ odd}. \end{cases} \]

It is a deep, but now well known fact that the cases $b_1$ even/odd correspond to whether the complex surface $(M, J)$ admits or not compatible Kähler structure.

Notice that when $J$ is integrable the dimensions $h^\pm_J$ are topological invariants. Such properties will not hold for general almost complex structures. However, we are still able to calculate the exact value of $h^\pm_J$ for almost complex structures which are metric related to integrable ones in 3.3. To achieve this we will first derive an estimate for general metric related almost complex structures.

3.2. Comparing metric related almost complex structures.

3.2.1. The space of $g$–compatible almost complex structures. We fix a Riemannian metric $g$. The space of almost complex structures on $M$ compatible with $g$ can be described as the space of $g$-self-dual 2-forms $\omega$ satisfying $|\omega|^2_g = 2$ point-wise on $M$.

Suppose we also fix a $g$–compatible pair $(J, \omega)$. Then any $g$–compatible almost complex structure corresponds to a 2–form

\[ \tilde{\omega} = f \omega + \gamma, \quad \text{with } \gamma \in \Omega^-_J, \ f \in C^\infty(M) \text{ so that } 2f^2 + |\gamma|^2 = 2. \]

For us, the following variation will be useful, extending an idea from [16]. Suppose further a section $\alpha \in \Omega^-_J$ is given. One can define new $g$-compatible almost complex structures as follows: pick smooth functions $f$ and $r$ on $M$ so that the form

\[ \tilde{\omega} = f \omega + r \alpha \]

satisfies $|\tilde{\omega}|^2_g = 2$, and let $\tilde{J}$ be the almost complex structure defined by $(g, \tilde{\omega})$. Equivalently, $f$ and $r$ should satisfy the point-wise condition

\[ 2f^2 + r^2|\alpha|^2_g = 2. \]

For any $\alpha \in \Omega^-_J$, one can find such functions $f$ and $r$. For instance, take $r$ to be small enough so that $r^2|\alpha|^2_g < 2$ everywhere and then $f$ is determined
up to sign by $f = \pm (1 - \frac{1}{2} r^2 |\alpha|_g^2)^{1/2}$. Junho Lee’s almost complex structures $J_\alpha$ (see [16]) are obtained for the specific choice 2

\begin{equation}
    r = -\frac{4}{2 + |\alpha|^2} \text{ and } f = \frac{2 - |\alpha|^2}{2 + |\alpha|^2}.
\end{equation}

Note that we actually get a pair of almost complex structures $J^\pm_\alpha$, as for the above choice of $r$, we have the sign freedom in choosing $f$. Junho Lee defines these almost complex structures on a Kähler surface $(M, J, g)$ and uses them as a tool for an easier computation of the Gromov-Witten invariants. Particularly important in his work are the almost complex structures $J_\alpha$ corresponding to closed $\alpha$’s, i.e $\alpha \in H^-_J$.

Another natural choice for $(r, f)$ is

\begin{equation}
    r = \pm f = \pm \frac{\sqrt{2}}{\sqrt{2 + |\alpha|^2}}.
\end{equation}

This corresponds to almost complex structures that arise from the forms $\pm \omega + \alpha$ conformally rescaled to satisfy the norm condition. We shall denote $\tilde{J}^\pm_\alpha$ the almost complex structures defined by $g$, (31) and (34).

Even more generally, given $\alpha$, we may choose $r$ so that $r^2 |\alpha|_g^2 \leq 2$, with equality at some points, but then at such points we have to require the smoothness of the function $(1 - \frac{1}{2} r^2 |\alpha|_g^2)^{1/2}$. Note also that if such points exists, then we no longer have an “up to sign choice” for $f$ overall.

3.2.2. Estimates for $g$–related almost complex structures. We again fix a Riemannian metric $g$.

**Definition 3.1.** Suppose $J$ and $\tilde{J}$ are two almost complex structures inducing the same orientation on a 4-manifold $M$. $J$ and $\tilde{J}$ are said to be $g$–related if they are both compatible with $g$.

We quickly point out the following simple facts:
1. If $g$ has this property, then so does any metric from its conformal class.  
2. If $J$ and $\tilde{J}$ are $g$–related then
   \begin{equation}
   \Lambda^-_J + \Lambda^-_{\tilde{J}} \subset \Lambda^+_g,
   \end{equation}
   and hence
   \begin{equation}
   H^-_J + H^-_{\tilde{J}} \subset \mathcal{H}^+_g.
   \end{equation}

The following observation is the key for the computations of $h^\pm_J$ we achieve in this section.

\footnote{There is a factor “2” difference in the convention for the norm of a two form between our paper and [16]. For us, if $(g, J, \omega)$ is a 4-dimensional almost Hermitian structure, $|\omega|_g^2 = 2$, whereas in [16], $|\omega|_g^2 = 1$. This explains the apparent difference between our $r$ and $f$ and those in Proposition 1.5 in [16].}
Proposition 3.2. Suppose $J$ and $\tilde{J}$ are $g$–related almost complex structures on a connected $4$-manifold $M$, with $\tilde{J}$ not identically equal to $\pm J$. Then \[ \dim (H^+_J \cap H^-_J) \leq 1. \]

Proof. Let $\omega$ and $\tilde{\omega}$ be the corresponding self-dual 2-forms. By assumption, the set \[ U = \{ p \in M \mid J(p) \neq \pm \tilde{J}(p) \} = \{ p \mid \dim (\text{Span}\{\omega(p), \tilde{\omega}(p)\}) = 2 \} \]
is a non-empty open set in $M$. Without loss of generality we can assume that $U$ is connected. Otherwise, we can make the reasoning below on a connected component of $U$.

Assume $H^+_J \cap H^-_J \neq \{0\}$ and let $\alpha_1, \alpha_2 \in Z^-_J \cap Z^-_J = H^-_J \cap H^-_J$, not identically zero. Let $U'$ be the open subset of $U$ where neither $\alpha_1$ or $\alpha_2$ vanishes. $U' \neq \emptyset$ because $\alpha_1$ and $\alpha_2$ are $g$-(self-dual)-harmonic forms, thus they satisfy the unique continuation property. Since on $U'$, $\text{Span}\{\omega, \tilde{\omega}\}$ is a 2-dimensional subspace of $\Lambda^+_2 M$ and $\alpha_1, \alpha_2$ are both orthogonal to this subspace, there exists $f \in C^\infty(U')$ such that $\alpha_2 = f \alpha_1$. Since $\alpha_1, \alpha_2$ are, by assumption, both closed, it follows that $0 = df \wedge \alpha_1$. But $\alpha_1$ is non-degenerate on $U'$ (it is self-dual, non-vanishing). Thus $df = 0$, so $f = \text{const}.$ on $U'$. It follows that $\alpha_2 = \text{const}. \alpha_1$ on $U'$, but, by unique continuation, this holds on the whole $M$. \[ \square \]

The next result shows that the estimate in Proposition 3.2 is sharp.

Proposition 3.3. Let $M$ be a connected 4-manifold, $J$ an almost complex structure on $M$ and assume that $\alpha \in Z^-_J = H^-_J$ is not identically zero. Suppose $g$ is a $J$-compatible metric. Then there exist infinitely many almost complex structures $\tilde{J}$ which are $g$–related to $J$ and $\neq \pm J$ such that $H^-_J \cap H^-_{\tilde{J}} = \text{Span}\{\alpha\}$.

Proof. Let $\omega$ be the 2-form associated to $(g, J)$. Consider the self-dual 2-form \[ (36) \quad \tilde{\omega} = f \omega + r J \alpha, \]
where $f$ and $r$ are $C^\infty$-functions so that \[ (37) \quad |\tilde{\omega}|_g^2 = 2f^2 + r^2 |\alpha|_g^2 = 2. \]
Observe that $\alpha$ is pointwise orthogonal to both $J \alpha$ and $\omega$, and hence it is pointwise orthogonal to $\tilde{\omega}$. Let $\tilde{J}$ be the almost complex structure induced from $\tilde{\omega}$ and $g$. Then $\alpha$ is $\tilde{J}$–anti-invariant by $g$ applied to $(\tilde{J}, g)$. Hence $\alpha \in H^-_J \cap H^-_{\tilde{J}}$ and our conclusion follows then from Proposition 3.2. \[ \square \]

Remark 3.4. In fact, any $g$-compatible almost complex structure $\tilde{J}$ such that $\alpha \in H^-_J \cap H^-_{\tilde{J}}$ will have a fundamental form $\tilde{\omega}$ given by (37) at least on the open dense set $M' = M \setminus \alpha^{-1}(0)$, with functions $f, r \in C^\infty(M')$ satisfying (36).
Now let us examine the special case $b^+ = 1$. It follows from (8) that $h^-_J \leq 1$ for any $\tilde{J}$ (even those not metric related to $J$). It only remains to analyze precisely when it is 1.

**Proposition 3.5.** Suppose $(M, J)$ is an almost complex 4–manifold with $b^+ = 1$. For any $J$–compatible metric $g$, let $\tau$ be the unique $g$–self-dual harmonic form (up to constant). Suppose $\tilde{J}$ is $g$–related to $J$ and $\omega$ is the fundamental form associated to $(g, \tilde{J})$. Then $h^-_J \leq 1$, and $h^-_J = 1$ if and only if $\omega$ is pointwise orthogonal to $\tau$.

This is easy to see since if a $g$–related $\tilde{J}$ has $h^-_J = 1$, then $\tau \in H^-_{\tilde{J}}$.

It is clear that $h^-_J = 0$ is the generic case. Notice also that if $(g, \tilde{J})$ is almost Kähler, then $h^-_J = 0$. See also the more general statement Corollary 4.4 and Remark 4.5.

Observe that compactness is not needed for Propositions 3.2 and 3.3.

As an immediate consequence of Proposition 3.2, we have

**Corollary 3.6.** In the space of almost complex structures compatible to a given metric $g$ on a compact 4–manifold, there is at most one $J$ such that

\[
(38) \quad h^-_J \geq \begin{cases} 
\frac{b^+ + 3}{2} & \text{if } b^+ \text{ is odd} \\
\frac{b^+ + 2}{2} & \text{if } b^+ \text{ is even.}
\end{cases}
\]

3.2.3. Metric related almost complex structures.

**Definition 3.7.** $J$ and $\tilde{J}$ are said to be metric related if they are $g$–related for some $g$.

It is not hard to see that $J$ and $\tilde{J}$ are metric related if and only if there exists a 3-dimensional sub-bundle $\Lambda^+ \subset \Lambda^2 M$, positive definite with respect to the wedge pairing, such that $\Lambda^-_J \subset \Lambda^+$, $\Lambda^-_{\tilde{J}} \subset \Lambda^+$. Note one important difference versus the “$g$-related” condition for a fixed $g$. The metric related condition is not transitive; that is, it is possible that $J_0$ is metric related to $J_1$ and $J_2$, but $J_1$ and $J_2$ are not metric related. Because of this, Corollary 3.6, for instance, is not automatically clear under just the metric related assumptions. However, Proposition 3.2 clearly extends to the metric related case. One immediate consequence is the following:

**Corollary 3.8.** Suppose $h^-_J = b^+$, then for any $\tilde{J}$ which is metric related to $J$ and not equal to $\pm J$, we have $h^-_{\tilde{J}} \leq 1$.

Suppose $h^-_J = b^+ - 1$, then for any $\tilde{J}$ which is metric related to $J$ and not equal to $\pm J$, we have $h^-_{\tilde{J}} \leq 2$. 
3.3. When \( \tilde{J} \) is metric related to a complex structure \( J \). In this subsection \( J \) is a complex structure on a compact 4-manifold \( M \) unless stated otherwise. Our goal is to compute \( \tilde{h}^2 \) for almost complex structures \( \tilde{J} \) metric related to \( J \). We start with the case which is immediate from Corollary 3.8.

3.3.1. Complex surfaces of non-Kähler type.

**Theorem 3.9.** Let \((M, J)\) be a compact complex surface with \( b_1 \) odd. If \( \tilde{J} \) is another almost complex structure on \( M \) metric related to \( J \), \( \tilde{J} \neq \pm J \), then

(i) \( \tilde{h}^2 = 0 \), or

(ii) \( \tilde{h}^2 = 1 \).

Case (i) is the generic situation.

Case (ii) occurs precisely when there exist \( \alpha \in H^2_{\tilde{J}} \) and functions \( f, r \) so that (36) and (37) hold on the set \( M' = M \setminus \alpha^{-1}(0) \) for some metric \( g \) compatible with both \( J \) and \( \tilde{J} \).

**Proof.** For a complex surface with \( b_1 \) odd we have by (28) and (29)

\[
\tilde{h}^2 = 2h^{2,0} = b^+.
\]

By Corollary 3.8 we get \( \tilde{h}^2 \leq 1 \). The case \( \tilde{h}^2 = 1 \) occurs as described in Remark 3.4. It remains to justify the statement that the case \( \tilde{h}^2 = 0 \) is the generic situation. If \( \tilde{h}^2 = b^+ = 0 \), then it follows from (8) that \( \tilde{h}^2 = 0 \) for any \( \tilde{J} \) on \( M \) (even not metric related to \( J \)). If \( \tilde{h}^2 \neq 0 \), we show in the following lemma that it is easy to produce examples with \( \tilde{h}^2 = 0 \).

**Lemma 3.10.** Let \((M, J)\) be a compact almost complex 4-manifold and assume that \( \tilde{h}^2 \geq 2 \) (this is the case for a complex surface with \( h^{2,0} \neq 0 \)). Then one can modify \( J \) on small open sets to obtain almost complex structures \( \tilde{J} \) on \( M \) with \( \tilde{h}^2 = 0 \).

**Proof.** By the assumption \( \tilde{h}^2 \geq 2 \), we can choose \( \alpha_1, \alpha_2 \in H^2_{\tilde{J}}(M) \) not scalar multiples of one another. Choose two points \( p_1, p_2 \in M_0 \), where \( M_0 \) denotes now \( M \setminus (\alpha_1^{-1}(0) \cup \alpha_2^{-1}(0)) \). Consider disjoint open sets \( U_1, U_2 \) in \( M_0 \) containing the points, respectively, and let \( f_1, f_2 \) be compactly-supported functions on \( U_1 \) and \( U_2 \) with \( f_1(p_1) = f_2(p_2) = 1 \). Let \( \beta = k(f_1\alpha_1 + f_2\alpha_2) \), where the constant \( k \) is chosen so that \( |\beta|^2 < 2 \) everywhere on \( M \). Let \( \tilde{J} \) be the almost complex structure associated to the form \( \tilde{\omega} = f\omega + \beta \), where \( f = (1 - |\beta|^2/2)^{1/2} \) (Of course, another almost complex structure is obtained by choosing \( f = -(1 - |\beta|^2/2)^{1/2} \) and the whole argument is the same.) We claim that \( \tilde{h}^2 = 0 \).

Indeed, if \( \tilde{h}^2 \neq 0 \), as in the proof of Proposition 3.3, there exists \( \alpha \in H^2_{\tilde{J}}(M) \) and a function \( f \), so that \( \beta = f\alpha \) on a dense open set \( M' \) in \( M \). But on \( U_1 \), \( \beta = f_1\alpha_1 \). By the argument in the proof of Proposition 3.2 it follows that \( \alpha \) and \( \alpha_1 \) are (non-zero) scalar multiples of one another on the
whole $M$. The same reasoning can be done with $\alpha_2$ instead of $\alpha_1$. It follows that $\alpha_1$ and $\alpha_2$ are scalar multiples of one another, which contradicts the assumption. □

This also concludes the proof of Theorem 3.9

Let us examine more closely the case of Kodaira surface. It is a surface with holomorphically trivial canonical bundle, and it has $h_J^+ = b^+ = 2$. Let $\beta$ the real part of a nowhere vanishing holomorphic $(2,0)$–form $\Phi$. Notice that $\beta$ is a closed form trivializing the canonical bundle. Moreover, $J\beta$ is also closed (in fact, the imaginary part of $\Phi$), and a base for $H^{-}_J$ is $\{\beta, J\beta\}$.

Suppose $g$ is a metric compatible with $J$ and $\omega$ is the non-degenerate form compatible with $(g, J)$. We can suppose the base of the rank 3 bundle $\Lambda^+_g$ is

$$\{\omega, \beta, J\beta\}.$$  

They are point-wise orthogonal to each other. We see that any almost complex structure compatible with $g$ corresponds to a form of the type in Equation (30). Now we further write the forms in Equation (30) as

$$f\omega + l\beta + sJ\beta,$$

where $f^2 = \frac{2 - |\beta|^2(l^2 + s^2)}{2}$. We denote the almost complex structure corresponding to the form by $J_{f,l,s}$.

Notice that

$$H^+_g = H^{-}_J.$$  

Thus the only possible self-dual harmonic forms are of type

$$a\beta + bJ\beta,$$

where $a$ and $b$ are constants.

So the only condition for this form lying in $H^{-}_{J_{f,l,s}}$ is $al + bs = 0$. Thus we have

$$h^-_{J_{f,l,s}} = \begin{cases} 2, & \{l, s\} \text{ has rank 0 (} f \equiv \pm 1, \tilde{J} = \pm J); \\ 1, & \{l, s\} \text{ has rank 1;} \\ 0, & \{l, s\} \text{ has rank 2.} \end{cases}$$

3.3.2. *Surfaces of Kähler type with topologically non-trivial canonical bundle.*

For surfaces of Kähler type we shall prove similar results to Theorem 3.9. However, in this case $h_J^+ = b^+ - 1$, so some extra work is needed beyond Proposition 3.2 and Corollary 3.8.

First we treat the case when the canonical bundle of $(M, J)$ is topologically non-trivial. In this case, essential is the fact that the canonical bundle is topologically non-trivial. Whether the surface is Kähler or not will not make a difference, so we will state the more general result.

**Theorem 3.11.** Let $(M, J)$ be a compact complex surface with non-trivial canonical bundle. If $\tilde{J}$ is another almost complex structure on $M$ metric related to $J$, $\tilde{J} \neq \pm J$, then we have the same conclusion as in Theorem 3.9.
The key tool in the proof will be the so called Gauduchon metrics (or standard Hermitian metrics) whose definition we recall below.

Given a Hermitian manifold \((M, g, J, \omega)\), the Lee form \(\theta\) is defined by \(d\omega = \theta \wedge \omega\), or, equivalently, by \(\theta = J \delta \omega\). It is known that \(d\theta\) is a conformal invariant. The case when \(\theta\) is closed (exact) corresponds to locally (globally) conformal Kähler metrics. Obviously, Hermitian metrics with \(\theta = 0\) are, in fact, Kähler metrics.

**Definition 3.12.** A Hermitian metric such that the Lie form is co-closed, i.e. \(\delta \theta = 0\), is called a Gauduchon metric.

Gauduchon proves in [11] the existence and uniqueness (up to homothety) of standard Hermitian metrics in each conformal class (in any dimension). One can say that the standard Hermitian metric is the “closest” to a Kähler metric in its conformal class. For us, the key property of a standard Hermitian metric in dimension 4 is the following:

**Proposition 3.13.** (Gauduchon) On a compact complex surface \(M\), endowed with a standard Hermitian metric \(g\), the trace of a harmonic, self-dual form is a constant.

For a proof of Proposition 3.13 see [12], Lemma II.3, or Proposition 3 in [4].

We are now ready for the

**Proof of Theorem 3.11.** Consider the conformal class of metrics compatible with both \(J\) and \(\tilde{J}\). This conformal class may not contain any Kähler metric. But as \(J\) is integrable, by the existence and uniqueness theorem of Gauduchon [11], we can choose the metric \(g\) which is a standard Hermitian metric with respect to \(J\).

Let \(\omega\) and \(\tilde{\omega}\) denote the fundamental two forms of \((g, J)\) and \((g, \tilde{J})\), respectively. Since \(\omega\) and \(\tilde{\omega}\) are both \(g\)-self-dual and of squared norm 2, they are related by (30).

Suppose \(h_{\tilde{J}} \neq 0\) and let \(\psi \in H_{\tilde{J}}\), not identically zero. We identify freely the cohomology classes in \(H_{\tilde{J}}\) with their (unique) representatives in \(Z_{\tilde{J}}\) (see Lemma 2.5). Since \(\psi\) is \(g\)-harmonic, from Proposition 3.13 it must be of the form \(\psi = a \omega + \alpha\), with \(a\) constant and \(\alpha \in \Omega_{\tilde{J}}\). Using (30), the point-wise condition \(\langle \psi, \tilde{\omega} \rangle = 0\) becomes \(2af + \langle \alpha, \beta \rangle = 0\) everywhere on \(M\). But \(\beta\) (and \(\alpha\)) must vanish somewhere on \(M\), otherwise the canonical bundle is topologically trivial (and even holomorphically trivial if \(\alpha\) is closed).

At a point \(p\) where \(\beta(p) = 0\), we have \(f^2(p) = 1 \neq 0\), thus it follows that \(a = 0\). Thus \(\psi = \alpha\), but since \(d\psi = 0\), it follows that \(\psi = \alpha \in H_{\tilde{J}}\). Thus we proved that \(H_{\tilde{J}} \subset H_{\tilde{J}}\). The fact that \(h_{\tilde{J}}\) equals 0 or 1 follows now from Proposition 3.2. The rest of the proof is now the same as in Theorem 3.9. 

\(\square\)
Remark 3.14. If a compact 4-manifold $M$ admits a pair of of integrable complex structure $(J_1, J_2)$ which are metric related then $M$ has a bi-Hermitian structure. The study of such structures has been very active recently, (see, for instance, [14] and the references therein), especially due to the link with generalized Kähler geometry ([13]). An easy consequence of our Theorem 3.11 is the observation that a compact 4-manifold $M$ with $b^+ = 2$, or $b^+ \geq 4$ does not admit a bi-Hermitian structure (compatible with the given orientation). This is not new, as it is easily seen from the classification results of [5] and [1], that manifolds admitting bi-Hermitian structures must have $b^+ \in \{0, 1, 3\}$.

Remark 3.15. If $\alpha \in H_J^-$, then the almost complex structures $\tilde{J}$ defined by (31) have $h^-_{\tilde{J}} = 1$, for any choice of $(r, f)$ satisfying (37). In particular this is true for Junho Lee’s almost complex structures $J_\alpha^\pm$ defined by (33) and the $\tilde{J}_\alpha^\pm$ defined by (34). Note that since $\alpha \in H_J^-$, $\alpha + iJ\alpha$ is a holomorphic $(2, 0)$ form on $M$, hence the zero set $\alpha^{-1}(0)$ is a canonical divisor on $(M, J)$.

3.3.3. Surfaces of Kähler type with topologically trivial but holomorphically non-trivial canonical bundle. Any such surface is a hyperelliptic surface. In this case $b^+ = 1$ and $h^-_{\tilde{J}} = 0$. Thus it is covered by Proposition 3.5.

3.3.4. Surfaces of Kähler type with holomorphically trivial canonical bundle. We first suppose that $(M, J)$ is a Kähler surface with holomorphically trivial canonical bundle. In this case $M$ is the K3 surface or the 4--torus and $b^+ = 3$.

In each conformal class of metrics compatible with $J$, let $g$ be the Gauduchon metric and $\omega$ the associated form. Then let us be in a similar set up as in the case of the Kodaira surface. In particular, every almost complex structure metric related to $J$ is of the type $J_{f, l, s}$ for some Gauduchon metric $g$.

The difference from the Kodaira surface case is that $b^+ = H_g^+ = 3$, rather than 2. As argued in Theorem 3.11 the $g$--harmonic form $\psi$ is of the form $c\omega + \alpha$, where $c$ is a constant and $\alpha \in \Omega_J$. Fix such a $\psi$ with $c = 1$ and $\alpha = u\beta + vJ\beta$, where $u, v$ are $C^\infty$ functions. Denote this $\psi$ by $\omega'$.

Thus the possible self-dual harmonic forms are of type

$$ c \cdot \omega' + a \cdot \beta + b \cdot J\beta, $$

where $a, b, c$ are constants. The only condition for this form lying in $H_{J_{f, l, s}}$ is then

$$ 2cf' + al' + bs' = 0, $$

where

$$ l' = l|\beta|^2, \quad s' = s|\beta|^2 \quad \text{and} \quad f' = 2f + ul' + vs' $$

are three functions. Therefore we have the following statement.
Proposition 3.16. Suppose \((M,J)\) is a Kähler surface with trivial canonical bundle. Let \(\beta\) be a closed form trivializing the canonical bundle, \(g\) a Gauduchon metric and \(\omega\) the associated form. Then
\[
h^*_J f',l',s' = 3 - \text{rank of span } \{f',l',s'\},
\]
with \(f',l',s'\) in \((39)\).

Notice that the case \(g\) is a hyperKähler metric corresponds to \(|\beta|^2 = 2\) point-wise. And the reader can compare the above theorem with Theorem 3.11 to see the expressions are essentially the same. The above theorem also tells us that \(h^-_J = 0\) is really a “generic” condition.

Example 3.17. We also consider Lee’s \(J_{\alpha}\). When \(g\) is not hyperKähler, we see that it has rank 2, so \(h^-_J = 1\) in our case. When \(g\) is hyperKähler, \(|\alpha|^2\) is constant, so we get \(h^-_J = 2\). Actually, in this case \(J_{\alpha}\) is also Kähler. Notice that \(J_{\alpha}\) is tamed. So the tameness is irrelevant to the dimension of \(h^-_J\).

Remark 3.18. Clearly any \(\tilde{J}\) in Proposition 3.16 with \(h^-_{\tilde{J}} \leq 1\) is not integrable. Moreover, it seems that most of \(\tilde{J}\) with \(h^-_{\tilde{J}} = 2\) in Proposition 3.16 is also not integrable. We would like to mention that Apostolov communicated to us that, via \([5]\), it is not hard to construct examples of non-integrable almost complex structures \(J'\) on K3 or \(T^4\) with \(h^-_{J'}\) exactly equal to 2, and he further speculated whether his examples exhaust all such almost complex structures. It is quite possible that his examples are related to our \(\tilde{J}\) and we hope to come back to this point in the future.

3.4. Well-balanced \(J\). We end this section with a result about \(h^-_J\) for almost complex structures \(J\) similar to integrable ones in a different sense.

Definition 3.19. We say that an almost Hermitian manifold \((M^4,g,J)\) is well-balanced if
\[
|\nabla (J\phi)|^2 = |\nabla \phi|^2, \text{ for any local section } \phi \in \Omega^1_J \text{ with } |\phi|^2 = 2.
\]
Note that it is equivalent to ask condition \((40)\) for any non-zero local section \(\phi \in \Omega^1_J\).

Lemma 3.20. Any 4-dimensional Hermitian or almost Kähler manifold is well-balanced.

Proof. If \((g,J,\omega)\) is any 4-dimensional almost Hermitian structure and \(\phi\) is a local section in \(\Omega^1_J\) with \(|\phi|^2 = 2\), we have
\[
\begin{align*}
\nabla \omega &= a \otimes \phi + b \otimes J\phi \\
\nabla \phi &= -a \otimes \omega + c \otimes J\phi \\
\n\nabla (J\phi) &= -b \otimes \omega - c \otimes \phi,
\end{align*}
\]
where \(a, b, c\) are local 1-forms. Thus
\[
|\nabla (J\phi)|^2 - |\nabla \phi|^2 = |b|^2 - |a|^2.
\]
It is well-known that for a Hermitian structure $b = Ja$, whereas for an almost Kähler one $b = -Ja$. In either case, $|b|^2 = |a|^2$.

**Definition 3.21.** We say that an almost Hermitian manifold $(M^4, g, J)$ has a Hermitian type Weyl tensor if

\[<W^+(J\beta), J\beta> = <W^+(\beta), \beta>, \text{ for any } \beta \in \Omega^-_J.\]

**Note:** It is known that if $J$ is integrable (i.e. $(M^4, g, J)$ is a Hermitian manifold), then $W$ commutes with $J$ when acting on $\Omega^-_J$, hence (42) holds in this case. Also, any almost Hermitian structure with an ASD metric trivially satisfies (42).

**Proposition 3.22.** Let $(M^4, J)$ be a compact almost complex 4-manifold which admits a compatible Riemannian metric $g$ so that $(g, J)$ is well-balanced and has Hermitian type Weyl tensor. Then $h^-_J = 0$ or $J$ is integrable.

**Proof.** The result follows immediately from Lemma 2.11 and the following

**Lemma 3.23.** Let $(M^4, g, J, \omega)$ be a compact, almost Hermitian 4-manifold which is well-balanced and has Hermitian type Weyl tensor. Then for any $\beta \in \Omega^-_J$, $d\beta = 0 \iff d(J\beta) = 0$.

**Proof of Lemma:** It’s enough to prove $d\beta = 0 \Rightarrow d(J\beta) = 0$. The well-known Weitzenböck formula for a 2-form $\psi$ is

\[\int_M (|d\psi|^2 + |\delta \psi|^2 - |\nabla \psi|^2) \, dV = \int_M \left(\frac{8}{3} |\psi|^2 - <W(\psi), \psi>\right) \, dV.\]

Applying this for $\beta$ and $J\beta$ and using the assumption on the Weyl tensor, we get

\[\int_M (|d\beta|^2 + |\delta \beta|^2 - |\nabla \beta|^2) \, dV = \int_M (|d(J\beta)|^2 + |\delta(J\beta)|^2 - |\nabla(J\beta)|^2) \, dV.\]

Now, by assumption $\beta \in \Omega^-_J$ and $d\beta = 0$, thus $\beta$ is harmonic, so it is non-vanishing on an open dense set in $M$. From the well-balanced assumption and continuity, we get that $|\nabla(J\beta)|^2 = |\nabla \beta|^2$ everywhere on $M$. Thus,

\[0 = \int_M (|d\beta|^2 + |\delta \beta|^2) \, dV = \int_M (|d(J\beta)|^2 + |\delta(J\beta)|^2) \, dV.\]

The lemma and the proposition are thus proved.

The following is an immediate consequence.

**Corollary 3.24.** A compact 4-dimensional almost Kähler structure $(g, J, \omega)$ with Hermitian type Weyl tensor and with $h^-_J \neq 0$ must be Kähler.

**Remark 3.25.** Under different additional conditions, some other integrability results have been obtained for compact, 4-dimensional almost Kähler manifolds $(g, J, \omega)$ with Hermitian type Weyl tensor (see [2], [3]).
Remark 3.26. The corollary implies that if we start with a Kähler surface 
\((M, g, J, \omega)\) and define the almost complex structures \(\tilde{J}_\alpha^\pm\) corresponding to (31) and (34) for an \(\alpha \in H^{-J}\), then \(\tilde{J}_\alpha^\pm\) cannot admit compatible almost Kähler structures with Hermitian-type Weyl tensor. Note, however, that they do admit compatible almost Kähler structures (since \(\pm \omega + \alpha\) is symplectic).

We end this section with the following question (See Remark 3.18).

Question 3.27. Are there (compact, 4-dimensional) examples of non-integrable almost complex structures \(J\) with \(h^{-J} \geq 3\)?

4. Estimates for \(h^+_{-J}\) when \(J\) is tamed by a symplectic form

From Theorem 2.3 on any compact 4-dimensional almost complex manifold \((M, J)\) we have

\[ h^+_{-J} + h^-_{-J} = b_2. \]

The decomposition (8) also leads to the following immediate estimates

\[ h^+_{-J} \geq b^-, \quad h^-_{-J} \leq b^+. \]

One reason for our interest in \(H^+_J\) stems from the following fact. If \(J\) admits compatible symplectic forms, then the set of all such forms, the \(J\)-compatible cone, \(K_J^c(M)\) is a (nonempty) open convex cone of \(H^+_J(M)\) [18]. Thus it is important to determine the dimension \(h^+_{-J}\) of \(H^+_J(M)\).

In light of the question of Donaldson mentioned in the introduction, it is also interesting to obtain information on the dimension \(h^+_{-J}\) in the case when \(J\) is just tamed by symplectic forms.

It was shown in [18] that an integrable \(J\) admits compatible Kähler structures if and only if it admits tamed symplectic forms. Thus we can state (29) in this context as follows:

\[ h^+_{-J} = \begin{cases} 
    b^-(M) + 1 & \text{if } J \text{ is tamed and integrable}, \\
    b^-(M) & \text{if } J \text{ is non-tamed and integrable}.
\end{cases} \]

4.1. A general estimate. When \(J\) admits a compatible symplectic form, we have the following easy improvement of (44):

**Proposition 4.1.** If \(J\) is almost Kähler, then

\[ h^+_{-J} \geq b^- + 1, \quad h^-_{-J} \leq b^+ - 1. \]

Actually, (46) can be obtained in a slightly more general setting:

**Lemma 4.2.** Suppose \((M, g, J, \omega)\) is a compact 4-dimensional almost Hermitian manifold. Assume that the harmonic part \(\omega_h\) of the Hodge decomposition of \(\omega\) is not identically zero. Then (46) holds.
Proof. Let \( \omega = \omega_h + d\theta + \delta \Psi \) be the Hodge decomposition of \( \omega \). From Lemma 2.4, \( \omega + 2(d\theta)^- = \omega_h + 2d\theta \) is a closed, \( J \)-invariant 2-form. By assumption, it represents a non-trivial cohomology class in \( H^+_g \cap H^+_J \) and the estimates follow.

Of course, if \((M, g, J, \omega)\) is almost Kähler, \( \omega = \omega_h \), so Proposition 4.1 is obvious. More interestingly, Lemma 4.2 implies that the estimates (46) hold for tamed \( J \)'s as well.

**Theorem 4.3.** Suppose \( J \) is tamed by a symplectic form \( \omega \). Then the estimates (46) still hold.

**Proof.** Write

\[
\omega = \omega' + \omega''
\]

with \( \omega' \in \Omega^+_J \) and \( \omega'' \in \Omega^-_J \). Explicitly,

\[
\omega'(v, w) = \frac{1}{2} \omega(v, w) + \frac{1}{2} \omega(Jv, Jw).
\]

Then \( \omega' \) is compatible with \( J \) and non-degenerate, thus it determines a Riemannian metric \( g \). From the pair \((\omega, J)\) we actually get a conformal class of metrics, these for which \( \Lambda^+_g = \text{Span}\{\omega, \Lambda^-_J\} \). The metric we fixed is singled out by imposing that \( |\omega'|^2 = 2 \).

We show that Lemma 4.2 can be applied to the almost Hermitian structure \((g, J, \omega')\). It is enough to show that the harmonic part \( \omega_h' \) is not identically zero. This is true because the following cup product is non-zero:

\[
[\omega_h'] \cup [\omega] = \int_M \omega_h' \wedge \omega = \int_M (\omega_h' + 2d\theta) \wedge \omega = \\
= \int_M (\omega' + 2(d\theta)^-) \wedge (\omega' + \omega'') = \int_M \omega' \wedge \omega' \neq 0.
\]

The following is an immediate consequence:

**Corollary 4.4.** If \( b^+ = 1 \) and \( J \) is tamed, then

\[
h_J^+ = 1 + b^- = b_2, \quad h_J^- = b^+ - 1 = 0.
\]

**Remark 4.5.** Lemma 4.2 can also be applied to show that if \( b^+ \geq 1 \), then the estimates (46) even hold for generic non-tamed almost complex structures \( J \) (but not all in view of (45)).

It has been shown [20] that non-tamed almost complex structures exist in any path-connected component of almost complex structures. This leads naturally to the following question (compare with Question 3.27): On a 4–manifold admitting symplectic structures, is there a non-tamed almost complex structure \( J \) in the path connected component of tamed almost complex structures, having \( h_J^+ = b^- \), or equivalently, \( h_J^- = b^+ \)?
Such a $J$ is necessarily non-integrable. In the case $b^+=1$, we have shown that there are non-integrable almost complex structures $\tilde{J}$ with $h^-\tilde{J}=1$ and are metric related a Kähler complex structure. But it is not clear that they are non-tamed.

4.2. SW estimate. Using Taubes’s SW=GT, we could obtain, in most cases, a stronger result than Theorem 4.3. For any almost complex structure, there is a canonical Spin$^c$ structure on $M$ whose characteristic class is $-K_J$. Sending this canonical Spin$^c$ structure to $0 \in H^2(M;\mathbb{Z})$ induces a bijective correspondence between the affine space of Spin$^c$ structures and the lattice $H^2(M;\mathbb{Z})$. Recall that given a metric $g$ on $M$, for each Spin$^c$ structure $c$, we can define the SW equations and the associated SW invariant $\text{SW}(M,g,c)$. It turns out as in the case of the dimension of the Harmonic forms, $\text{SW}(M,g,c)$ only depends on the pair $(M,c)$.

Definition 4.6. A cohomology class $e \in H^2(M;\mathbb{Z})$ is called an $J$–SW class if the SW invariant of the corresponding Spin$^c$ structure is nonzero. The $J$–SW subspace $V^J_{SW}$ is the subspace in $H^2(M;\mathbb{R})$ generated by $J$–SW basic classes.

It is natural to ask how the vector space $V^J_{SW}$ depends on $J$ or its canonical class $K_J$. Let us dispose this point straight away. If we use another $J'$ we will get the same vector space. The space $V^J_{SW}$ is actually a smooth invariant, uniquely determined by the SW affine space $\text{Aff}_{SW}$ generated by the SW basic classes $-K_J + 2e$ in the usual sense. This relies on the following general fact: If an affine space can be written as $A = v + L$ and $v' + L'$ for some vectors $v, v'$ and linear subspaces $L, L'$, then $L = L'$. The point is that the vector $v - v'$ is in $L$ and $L'$. Given any vector $w \in L$, $v + w \in A$ so $v + w = v' + w'$ for some $w' \in L'$. Thus $w = (v' - v) + w'$ is in $L'$ as well.

Therefore we shall denote $V^J_{SW}$ simply by $V_{SW}$ and its dimension by $r_{SW}$.

Definition 4.7. Choose any $g$ compatible with $J$ and consider the projection $P^+_g : H^2 = H^+_g \oplus H^-_g \to H^+_g$ as well as the subspace $V^g_{SW} = P^+_g(V_{SW})$ in $H^+_g$. Define $r^g_{SW}$ to be the dimension of $V^g_{SW}$, and $r^J_{SW}$ to be the maximum among all $J$–compatible $g$.

Theorem 4.8. Suppose $J$ is tamed by a symplectic form $\omega$. Then

$$V_{SW} \subseteq H^+_J.$$  

As a consequence, we have

\( h^+_J \geq b^- + r^J_{SW}, \quad h^-_J \leq b^+ - r^J_{SW}. \)

Proof. According to Taubes SW=GT$_\omega$, each class $e \in V_{SW}$ is represented by a $J$–holomorphic curve. Notice that for any $J$–holomorphic curve $C$ representing $e$ and any $J$–anti-invariant form $\alpha$, the integral $\int_C \alpha$ is well defined although $C$ might be singular. Moreover, the integral vanishes.
Therefore we conclude that the SW subspace $V_{SW}$ is orthogonal to $H^J_\perp$. From Theorem 2.3, it follows that $V_{SW} \subseteq H^+_J$. We can also conclude

$$H^-_g \oplus V^g_{SW} \subseteq H^+_J,$$

thus the estimates from the Theorem follow.

Now let us further estimate $r^J_{SW}$. Notice that for any $g$, each nonzero class in $H^-_g$ has negative square. Thus $P^g_+$ is injective on any semi-positive definite subspace of $V_{SW}$. It follows that $r^g_{SW}$ is at least the dimension of a maximal semi-positive definite subspace of $V_{SW}$. This dimension is a smooth invariant, which we denote by $r^0_{SW}$. In other words, $r^J_{SW} \geq r^g_{SW} \geq r^0_{SW}$.

One clean cut case is when $r^0_{SW} = r_{SW}$, i.e. $V^g_{SW}$ itself is semi-positive definite. Conjecturally this is the case for a minimal manifold admitting a symplectic structure, again due to $SW=GT_\omega$. Consequently, we conjecture that if $M$ is minimal, then for $J$ tamed by a symplectic form $\omega$,

$$h^-_J \leq b^+ - r_{SW}.$$

We remark we can construct minimal manifolds with arbitrarily high $r_{SW}$ by genus one fiber sums.

- If we vary the $J$–compatible metric $g$, is it possible to get a better bound of $r^J_{SW}$?
- Another possibility is $r^J_{SW} = r^g_{SW} = r^0_{SW}$. But this may need more than linear algebra.

For a minimal Kähler surface with $p_g \geq 0$ and nonzero real canonical class, $r_{SW} = 1$. The inequality is an equality for a Kähler $J$. But if the real canonical class is zero, then $r_{SW} = 0$.

For the one point blow up of a minimal Kähler surface, $e = K, E$ or $K-E$, where $E$ is the class of the exceptional curve. So $r_{SW} = 2$ and $r^0_{SW} = 1$. In this case for a Kähler $J$, it has to be true that $r^J_{SW} = 1$. But is it possible that for a non-integrable $J$, $r^J_{SW} = 2$?

4.3. A formulation of Donaldson’s question. We end this section by giving an equivalent formulation of Question 1.1. Suppose $\tilde{J}$ is an almost complex structure that is tamed by a symplectic form $\omega$ on a compact 4-manifold $M$. As noted in the proof of Theorem 4.3, the pair $(\tilde{J}, \omega)$ gives rise to a conformal class of Riemannian metrics $[g]$, so that $\Lambda^+_g = \text{Span}\{\omega, \Lambda^-_{\tilde{J}}\}$.

In the proof of Theorem 4.3, we chose in this conformal class the metric that will make $\omega'$, the $\tilde{J}$-invariant part of $\omega$, have point-wise norm $\sqrt{2}$.

For the comments below, we prefer to use another natural metric in this conformal class: we choose the metric $g$ so that $|\omega|^2_g = 2$ point-wise on $M$. Equivalently, $g$ is chosen so that $g$ and $\omega$ induce an almost Kähler structure $(g, J, \omega)$. Certainly, $\tilde{J}$ is also $g$-compatible, and let $\tilde{\omega}$ be the fundamental 2-form of $(g, \tilde{J})$. This can be written as in (30). Since $\tilde{J}$ is tamed by $\omega$, the
function $f$ is strictly positive on $M$. Thus, we can think that $\tilde{J}$ is induced by the metric $g$ and the 2-form $\omega + \frac{1}{f} \beta$, up to conformal rescaling by $f$.

Conversely, let $(M^4, g, J, \omega)$ be an almost Kähler manifold and let $\alpha \in \Omega^-_J$. Denote $\tilde{\omega}_\alpha = \omega + \alpha$. This is a non-degenerate, $g$ self-dual form, so (up to a conformal normalization) it induces another $g$-compatible almost complex structure which we denote $\tilde{J}_\alpha$. It is clear that $\tilde{J}_\alpha$ is tamed by $\omega$.

Donaldson’s Question [4.1] is equivalent to

**Question 4.9.** Is it true that for any almost Kähler manifold $(M^4, g, J, \omega)$ and any $\alpha \in \Omega^-_J$, the almost complex structure $\tilde{J}_\alpha$ is compatible with a symplectic form?

Using this set-up and Lemma 2.4, we obtain the following partial result.

**Proposition 4.10.** With the notations above, if the 2-form $\alpha$ satisfies the point-wise condition

\[(51) \quad 2 + |\alpha|^2 - 4 |(\alpha^\text{exact})_g^-|^2 > 0,
\]

then $\tilde{J}_\alpha$ is compatible with a symplectic form.

**Proof.** We just apply Lemma 2.4 to $\tilde{\omega}_\alpha = \omega + \alpha$. The form

$$\tilde{\omega}_\alpha + 2(\tilde{\omega}_\alpha^\text{exact})_g^- = \omega + \alpha + 2(\alpha^\text{exact})_g^-$$

is closed and $\tilde{J}_\alpha$-invariant. Condition (51) is equivalent to this form being point-wise positive definite. \hfill \square

**Remark 4.11.** When $\alpha$ is closed (hence harmonic), condition (51) is trivially satisfied. In this case, $\tilde{\omega}_\alpha$ is itself a symplectic form. Proposition 4.10 basically says that if $\alpha$ is not too far from being closed, then $\tilde{J}_\alpha$ is compatible with a symplectic form. The result can be seen in relation with the openness result of Donaldson [9] (see also next section).

If $\alpha$ does not satisfy (51), Lemma 2.4 still helps in the search for a symplectic form compatible with $\tilde{J}_\alpha$. Let $(M^4, g, J, \omega)$ be the fixed almost Kähler structure. Note that by (7) any $\tilde{J}_\alpha$-invariant form $\Omega_\alpha$ can be written as

$$\Omega_\alpha = f\tilde{\omega}_\alpha + \theta, \text{ with } f \in C^\infty(M) \text{ and } \theta \in \Omega^-_g.$$

Applying Lemma 2.4 to $f\tilde{\omega}_\alpha$, we get that $\Omega_\alpha$ is also closed if and only if $\theta - 2((f\tilde{\omega}_\alpha)^\text{exact})_g^-$ is closed, hence harmonic. Thus, renaming $\theta$, a potential symplectic form $\Omega_\alpha$ which is $\tilde{J}_\alpha$-compatible must be of the type

$$\Omega_\alpha = f\tilde{\omega}_\alpha + 2((f\tilde{\omega}_\alpha)^\text{exact})_g^- + \theta, \text{ with } f \in C^\infty(M) \text{ and } \theta \in \mathcal{H}^-_g.$$

Now the question becomes how should one choose $f \in C^\infty(M)$ and $\theta \in \mathcal{H}^-_g$ to satisfy $\Omega_\alpha^2 > 0$ everywhere on $M$. 

5. Symplectic Calabi-Yau equation and semi-continuity property of $h_J^+$

In this section, we use the beautiful ideas in [9] to study the variation of $h_J^+$ under small deformation of the almost complex structures.

5.1. Symplectic CY equation and openness. The classical Calabi-Yau theorem can be stated as follows: Let $(M, J, \omega)$ be a Kähler manifold. For any volume form $\sigma$ satisfying $\int_M \sigma = \int_M \omega^n$, there exists a unique Kähler form $\tilde{\omega}$ with $[\tilde{\omega}] = [\omega]$ s.t. $\tilde{\omega}^n = \sigma$.

Yau’s original proof of the existence ([24]) makes use of a continuity method between the prescribed volume form $\sigma$ and the natural volume form $\omega^n$. The proof of openness is by the implicit function theorem. The closedness part is obtained by a priori estimates.

5.1.1. Set up. Donaldson recently introduced the symplectic version of the Calabi-Yau equation in [9].

Let $(M, J)$ be a compact almost complex $2n$–manifold and assume that $\Omega$ is a symplectic form compatible with $J$. For any function $F$ with

$$\int_M e^F \Omega^n = \int_M \Omega^n$$

the symplectic CY equation is the following equation of a $J$–compatible symplectic form $\tilde{\omega}$,

$$\tilde{\omega}^n = e^F \Omega^n.$$ 

In [9], Donaldson further observed that solvability of the symplectic CY equation in dimension 4 may lead to some amazing results in four dimensional symplectic geometry. In particular, he points out the following link to Question 1.1.

First choose an almost complex structure $J_0$ compatible with $\omega$. Because the space of almost complex structures tamed by $\omega$ is path connected, we can then choose a path $\{J_t\}$, $t \in [0, 1]$, with $J_1 = J$.

Before going on to state the next step, let us suppose now that $M$ has dimension 4, and we fix a maximal positive space $H^2_+ \subset H^2(M; \mathbb{R})$ and a class $C \in H^2(M; \mathbb{R})$. Now comes the continuity method, which we state as the question $(D_t)$: Find a symplectic form $\omega_t$ compatible with $J_t$, satisfying

$$[\omega_t] \in C + H^2_+, \quad s.t. \quad \omega_t^2 = \omega^2.$$ 

Observe that $(D_0)$ is solved by taking $\omega_0 = \omega$ and $C = [\omega]$. Question 1.1 is nothing but to find an $\omega_1$ solving $(D_1)$. Let

$$T = \{ t | (D_t) \text{ has a solution} \} \subset [0, 1].$$

If we can prove that $T$ is open and closed, then we are done.
5.1.2. Openness. In Donaldson’s paper, he proves that $T$ is open by using the implicit function theorem. This only works for dimension 4. Donaldson actually works in the general setting of 2–forms on 4 manifolds.

Suppose $M$ is a 4–manifold with a volume form $\rho$ and a choice of almost-complex structure $J$. At any point $x$, $\rho$ and $J$ induce a volume form and a complex structure on the vector space $T_x(M)$. Denote by $P_x$ the set of positive $(1,1)$–forms whose square is the given volume form. Then $P_x$ is a three-dimensional submanifold in $\Lambda^2 T_x(M)$ (a sphere in a $(3,1)$–space).

We consider the 7–dimensional manifold $P$ fibred over $M$ with fiber $P_x$,

$$\mathcal{P}_\rho = \{ \omega^2 = \rho | \omega \text{ is compatible with } J \}.$$

It is a submanifold of the total space of the bundle $\Lambda^2$.

Now, we want to find a symplectic form $\omega$ which is compatible with $J$ and has fixed volume form with some cohomology conditions. That is, we are searching for $\omega$ satisfying the following conditions (we call this condition type $D$):

$$\begin{cases}
\omega & \subset \mathcal{P}_\rho, \\
d\omega & = 0, \\
[\omega] & \in C + H^2_+ \subset H^2(M; \mathbb{R}).
\end{cases}$$

(54)

Here $C$ is a fixed cohomology class and $H^2_+$ is a maximal positive subspace. Notice, we have three families of variables: $\rho$, $J$ and $C$. In particular, $C$ varies in a finite dimensional space.

We have the following result which is a slight variation of Proposition 1 in [9].

**Proposition 5.1.** Suppose $\omega$ is a solution of type $D$ constrain with given $\mathcal{P}$ and $C$. If we have a smooth family $\mathcal{P}^{(b)}$ of parameterized by a Banach space $B$, $\{\mathcal{P}^{(b)}\}$, with $\mathcal{P} = \mathcal{P}^{(0)}$ and $b$ varies in $B$, then we have a unique solution of the deformed constraint in a sufficiently small neighborhood of 0 in $B$. Further, this solution lies in a small neighborhood of $\omega$.

We will not use the uniqueness of the solution here, so we just indicate how to find a small neighborhood for which we have the existence.

For each point $x \in M$, the tangent space to $\mathcal{P}_x$ at $\omega(x)$ is a maximal negative space. Thus the solution $\omega$ determines a conformal structure on $M$. We fix a Riemannian metric $g$ in this conformal class. For small $\eta$, $\omega + \eta$ lying in $\mathcal{P}_\rho$ is expressed as

$$\eta^+ = Q(\eta),$$

where $Q$ is a smooth map with $Q(\eta) = O(\eta^2)$. After choosing 2–form representatives of $H^2_+$, closed forms $\omega + \eta$ satisfying our cohomological constraint can be expressed as $\omega + da + h$ where $h \in H^2_+$ and where $a$ is a 1–form satisfying the gauge fixing constraint $d^*a = 0$. Thus our constraints correspond
to the solutions of the PDE

\[ \begin{align*}
    d^* a &= 0 \\
    d^+ a &= Q(da + h) - h^+.
\end{align*} \]

(55)

Donaldson further observes that the linear operator

\[ d^* \oplus d^+ : \Omega^1/H^1 \longrightarrow \Omega^0/H^0 \oplus \Omega^2_+ / H^2_+ \]

is invertible. Thus we apply the following version of the implicit function theorem:

**Theorem 5.2.** Let \( X, Y, Z \) be Banach spaces. Let the map \( f : X \times Y \longrightarrow Z \) be Fréchet differentiable. If \( (x_0, y_0) \in X \times Y, f(x_0, y_0) = 0, \) and \( y \mapsto Df(x_0, y_0)(0, y) \) is a Banach space isomorphism from \( Y \) onto \( Z \). Then there exist neighborhoods \( U \) of \( x_0 \) and \( V \) of \( y_0 \) and a Fréchet differentiable function such that \( f(x, g(x)) = 0 \) and \( f(x, y) = 0 \) if and only if \( y = g(x) \), for all \( (x, y) \in X \times Y \).

To use this theorem, \( X \) is our parametrization space \( B, Y \) is \( \Omega^1/H^1, \) \( Z \) is \( \Omega^0/H^0 \oplus \Omega^2_+ / H^2_+ \). (For \( Y \) and \( Z \), we may first choose spaces with \( C^k \) norm, then after getting such a solution, by the uniqueness the solution should be smooth.) Then every condition is satisfied in our setting. We only need to remark that the condition a nearby form lies in a nearby \( \mathcal{P}^{(b)} \) reduces to a small perturbation of equation (55), which is still elliptic.

**Remark 5.3.** We can parameterize \( \rho \) by finite dimensional space. For example, \( \rho = a \cdot \omega^2 \) or the path used in Yau’s proof of Calabi conjecture.

We see that \( H^2_+ \) in the statement of the problem is crucial, or the linearization map is not invertible. We usually take \( C = [\omega] \), but sometimes we take \( C = 0 \) or other choices. Notice when \( b^+_2(M) = 1 \), if we choose \( C = 0 \) and \( H^2_+ = \mathbb{R} \cdot [\omega] \) we notice that the conditions \( \omega^2 = \omega^2 \) and \( [\omega] \in C + H^2_+ \) exactly tells us that \( [\omega] = [\omega] \).

5.2. **Semi-continuity property of \( h^\pm_J \).**

5.2.1. **Topology of the space of almost complex structures.** It is well known that the space of \( C^l \) almost complex structures has a natural separable Banach manifold structure. The tangent space \( T_J \mathcal{J}^l \) at \( J \) consists of \( C^l \) sections \( A \) of the bundle \( \text{End}(TM, J) \) such that \( AJ + JA = 0 \). This space can be denoted by \( \Omega^0_{J_l}(TM) \). It is a Banach space with \( C^l \) norm. Moreover, this gives rise to a local model for \( \mathcal{J}^l \) via \( Y \mapsto J\exp(-JY) \). Thus we can apply Proposition 5.1 to a Banach chart of \( J \) in the space of \( C^l \) almost complex structures \( \mathcal{J}^l \) endowed with \( C^l \) norm.

The corresponding space of \( C^\infty \) almost complex structures \( \mathcal{J} = \mathcal{J}^\infty \) is not a Banach space but a Fréchet space. In this case we can still apply Proposition 5.1 to a smooth path or a finite dimensional space (hence Banach) in \( \mathcal{J} \). That is to say, if an almost complex structure \( J \) has a solution of Calabi-Yau equation \( \tilde{\omega}^2 = \rho \) with a \( J \)–compatible form \( [\tilde{\omega}] \in C + H^2_+ \), then for any path through \( J \), there is a small interval near \( J \) such that the
Calabi-Yau type equation is solvable with conditions in \((D_t)\) in this interval. In the end we get a weak neighborhood—the union of all the intervals. Notice that this is not necessarily “a small ball” near \(J\), i.e. it may not have an interior point.

However, observe that the natural \(C^\infty\) topology is induced by the sequence of semi-norms \(C^0, C^1, \ldots, C^l, \ldots\). Locally, near a \(C^\infty\) almost complex structure \(J\), \(J\) is a subspace of \(J^l\) with finer topology. Thus we obtain,

**Corollary 5.4.** If we parameterize \(P(b)\) in Proposition 5.1 by a neighborhood \(N(J)\) of \(J\) in \(J\) by the \(C^\infty\) topology, i.e., then we can get a small neighborhood of \(J\) satisfying all the properties stated in Proposition 5.1 under the same topology.

5.2.2. Variations of \(h^\pm_J\).

**Theorem 5.5.** Suppose \(M\) is a 4-manifold with an almost complex structure \(J\) such that \(K^c_J(M)\) is non-empty. Then for any almost complex structure \(J'\) in a sufficiently small neighborhood of \(J\) as in Corollary 5.4, we have

\[
\begin{align*}
&\bullet K^c_{J'}(M) \neq \emptyset; \\
&h^+_J(M) \leq h^+_J'(M); \\
&h^-_J(M) \geq h^-_J'(M).
\end{align*}
\]

**Proof.** The first statement is a direct consequence of Corollary 5.4 and was already observed by Donaldson (see also [17]).

As \(K^c_{J'}(M)\) and \(K^c_J(M)\) are nonempty open sets in \(H^+_J(M)\) and \(H^+_J(M)\) respectively, to estimate \(h^+_J(M)\) and \(h^+_J(M)\), we only need to estimate the dimensions of \(K^c_{J'}(M)\) and \(K^c_J(M)\).

Let \(h = h^+_J(M)\). We choose \(h\) half lines which are “in general position”, i.e. the interior of their span is an open set of \(K^c_J(M)\). We suppose the \(h\) half lines are \(C\cdot[\omega_i]'s\) where \(\omega_i's\) are the \(J\)-compatible forms and \([\omega_i]'s\) have homology norm 1 with respect to some bases.

Then we use Proposition 5.1 for each \(i\) with fixed volume form \(\omega_i^2\). Then we have \(h\) neighborhoods \(N_i\) such that for \(J' \in N_i\), we have a \(J'\) compatible form \(\omega'_i\) which is a small perturbation of \(\omega_i\). Let \(N\) be the intersection of these \(h\) neighborhoods. Then for any \(J' \in N\), we have \(\omega'_i's\) which are still in the general position (because they are perturbed from a general position). And we see that the span of the \(h\) new half lines belongs to \(K^c_{J'}(M)\) because positive combinations of \(\omega'_i's\) are still \(J'\)-compatible forms. Hence we have \(h^+_J(M) \leq h^+_J'(M)\).

The last inequality is a consequence of the first and Theorem 2.3.

The first statement means that, on a 4-manifold, the space of almost Kähler complex structure \(J_{ak}\) is an open subset of \(J\).
Let us consider the stratification

\[ \mathcal{J} = \bigcup_{i=0}^{b^+} \mathcal{J}_i, \]

where \( j \in \mathcal{J}_i \) if \( h_j = i \). Then we have

**Corollary 5.6.** On a 4-manifold, \( \mathcal{J}_0 \cap \mathcal{J}_{ak} \) is open in \( \mathcal{J} \).

A natural question is whether \( \mathcal{J}_0 \) itself is open (and dense).

**Remark 5.7.** If Donaldson’s question can be answered positively, then we can replace the assumption \( K^c_j(M) \neq \emptyset \) by \( K^t_j(M) \neq \emptyset \). We also wonder whether this semi-continuity property holds for every \( J \).

**Remark 5.8.** If the almost complex deformation is replaced by complex deformation, the first item in the above theorem is essentially a theorem of Kodaira and Spencer. Their theorem is valid for any even dimension.

When \( J \) is integrable, \( h_J^+ \) is a topological invariant, in particular, it is invariant under complex deformation. The situation is different under an almost complex deformation, as we have seen. For instance, for Kodaira surface, when \( J \) is integrable \( h_J^+=b^- = 2 \), while if \( J \) is almost Kähler, or even tamed, then \( h_J^+ \geq b^- + 1 \).

**Remark 5.9.** Notice that there is also the \( \omega \)-harmonic homology studied by Koszul and Brylinski [8]. It could jump along a deformation of \( \omega \) [23].

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