Closed-String Tachyon Condensation and the On-Shell Effective Action of Open-String Tachyons

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We study how the effect of closed-string tachyon condensation can enter into the on-shell effective action of open-string tachyons in the bosonic case. We also consider open-string one-loop quantum corrections to the on-shell action. We use a sigma-model approach with boundary terms, and we utilize some results of boundary string field theory (BSFT) to define the on-shell effective action. We regard D-instanton-like objects with appropriate weight as closed-string tachyon tadpoles, and we insert them into worldsheets to analyze the effect of closed-string tachyons.

§1. Introduction

One of the important subjects in string theory is the construction of an off-shell formulation. Recently, studies of this subject have proceeded in the context of open-string tachyon condensation, using string field theories. \(^1\) - \(^3\) String field theories provide tools to analyze the spacetime action of open-string tachyons, and they have been utilized \(^4\) - \(^11\) to check Sen’s conjecture. \(^12\) - \(^15\) In particular, Sen’s conjecture has been confirmed \(^16\) - \(^18\) in the framework of boundary string field theory (BSFT). \(^2\), \(^3\), \(^19\) - \(^21\) BSFT gives the exact spacetime action of open-string tachyons at the tree level, \(^16\) - \(^18\) and we can derive the descent relation of such spacetime actions on D-branes of various dimensions. \(^22\) Currently, we know that an unstable D-brane decays completely when the open-string tachyons living on it fully condense. Therefore, it is thought that the open-string sector disappears after the open-string tachyon condensation, and only the closed-string sector is left.

Although studies of open-string tachyons have progressed, it is still difficult to treat closed-string tachyon condensation. No one knows precisely what happens after closed-string tachyon condensation. Therefore, determining the fate of both the open-string sector and the closed-string sector is very interesting, and studies of closed-string tachyons represent an important approach to making such a determination.

In this paper, we attempt to treat closed-string tachyon condensation in bosonic strings. However, we do not consider off-shell closed-string dynamics. Our aim here is to calculate physical quantities on the background in which closed-string tachyons have already condensed. Specifically, we consider the on-shell effective action of bosonic open-string tachyons on the background in which closed-string tachyons have already condensed. We also consider open-string one-loop quantum corrections

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to such an on-shell effective action. In this article, we do not attempt to obtain off-shell effective action of open-string tachyons on the background in which closed-string tachyons are condensed. It is quite difficult to construct open-string field theories on such a background, although those on the usual flat background without closed-string tachyon condensation have been constructed. With these points in mind, this paper is intended as an investigation of a method of obtaining a consistent on-shell effective action of open-string tachyons on the background in which closed-string tachyons have condensed. We propose a basic procedure to obtain a consistent on-shell effective action. Basically, we use a sigma-model approach.\textsuperscript{23, 24} We also utilize the results of BSFT to define the on-shell effective action of open-string tachyons for each vacuum. We treat oriented strings for simplicity.

The organization of this article is as follows. In §2, we present a basic strategy to treat closed-string tachyon condensation. The method we use here is based on the model that was originally proposed by Green.\textsuperscript{25-29} We regard D-instanton-like objects with appropriate weights as closed-string tachyon tadpoles, and we insert them into worldsheets to analyze the effect of closed-string tachyons. In §3, we propose a sigma-model approach with boundary terms to analyze the corrections from closed-string tachyons and open-string loops. We point out that we can utilize the results of BSFT to define the on-shell effective action of open-string tachyons. Although it is very difficult to estimate the effect of closed-string tachyon condensation precisely, we calculate the modified on-shell effective action of open-string tachyons for some special cases in §4 and in 5. In general, the on-shell effective action of open strings possesses an instability due to closed-string tachyons if loop corrections are included. We propose a basic procedure to obtain a consistent on-shell effective action that has no instability due to closed-string tachyons in §5. Although it is still difficult to fix the weight of the closed-string tachyon tadpole, the condition that the instability due to closed-string tachyons vanishes provides a constraint on the weight. Using the procedure we propose there, we can obtain a finite correction to the on-shell effective action, at least at the one-loop level of open strings, in principle. In the final section, we give a conclusion and make some remarks. A problem involving the equation of motion in BSFT is pointed out in the Appendix. We also propose a method of avoiding this problem in order to utilize the results of BSFT.

\textbf{§2. Tachyon condensation and tadpoles on the worldsheet}

In this section, we recall a previous attempt to describe strings on the background in which tachyons are condensed. First, let us recall the formulation of field condensation in usual quantum field theories. In point-particle field theory with field condensation (that is, in the theory with a nonzero expectation value of the field), we can calculate correct quantities if we know the correct expectation value of the field, even with a perturbation around an incorrect vacuum. For example, we can calculate the exact propagator with a perturbation around a tachyonic vacuum by attaching tadpole diagrams to the tree propagator.\textsuperscript{2)} Even though the mass squared

\textsuperscript{2)} See for example Ref. 30).
is negative in a description around such a vacuum, tadpoles with appropriate weight create an additional shift of the mass squared, and cause the total mass squared to become positive (or zero). The weight attached to the tadpole corresponds to the expectation value of the field. In such a case, although the tachyonic mode exists in a perturbative theory around a tachyonic vacuum, the theory is not incorrect, and only the “vacuum” is incorrect. In field theories, the true vacuum or exact expectation value of the field can be obtained using the Schwinger-Dyson equation. In this way we can get the correct weight of the tadpoles, we can calculate the correct propagator, and so on.

The important point here is that we do not need the exact Feynman rules around the true vacuum. We can reproduce them with the Feynman rules around the tachyonic vacuum if we attach tadpoles to the graphs with appropriate weight. Therefore, this method is suitable even for first quantized theories.

We now draw an analogy between field theories and string theories and introduce tachyon “tadpoles” into the framework of string theory.\footnote{Attempts to describe tachyon condensation using the emission of massless scalar particles from string diagrams are found in Refs. 31–34}. Green proposed many years ago that an off-shell extension of the string amplitude is realized by insertion of Dirichlet boundaries into a worldsheet.\footnote{Most of the following arguments already appear in Ref. 36), in which an attempt is made to construct consistent noncritical strings.} We naturally assume that the inserted Dirichlet boundaries correspond to the tadpoles in string theories. We also assume that macroscopic holes with Dirichlet boundary conditions on the worldsheet represent closed-string tachyon tadpoles and Dirichlet boundaries inserted into the boundary of the worldsheet represent open-string tachyon tadpoles. The Dirichlet boundary conditions we use are “D-instanton-like” boundary conditions, namely,

\[ X^\mu(\xi_i) = x^{\mu}_{0(i)}, \]

where the subscript \( i \) distinguishes the tadpoles, and \( \xi_i \) denotes the worldsheet coordinate where the tadpole is inserted. \( x^{\mu}_{0(i)} \) is a constant that corresponds to the position of the tadpole in spacetime. Therefore, each Dirichlet boundary is mapped to a single point \( x^{\mu}_{0(i)} \) in spacetime. The contribution of the D-instanton-like tadpole is integrated over \( x^{\mu}_{0(i)} \) with appropriate weight. Therefore, if the weight does not depend on the spacetime coordinates, translational invariance is restored. It is known that these point-like energy densities on strings greatly alter the behavior of the strings.\footnote{Some argument for the macroscopic hole as a tachyonic state is also given in Ref. 37).}

The reasons we regard the above D-instanton-like Dirichlet boundaries as tachyon tadpoles are as follows.\footnote{\textsuperscript{\textcopyright} }\footnote{\textsuperscript{\textasteriskcentered}}

- Tachyon tadpoles are off-shell in general. For example, if the expectation value of a tachyon does not depend on the spacetime coordinates, the weight should be constant and tadpoles do not carry momentum. In such a case, tachyon tadpoles are off-shell because tachyons have non-zero mass squared. We also know that
off-shell states in string theories do not correspond to local emission vertices. Thus we naturally assume that a tachyon tadpole is a non-local macroscopic boundary inserted into the worldsheet. Namely, the tadpole for a closed-string tachyon should be a macroscopic hole in the worldsheet, and the open-string tachyon tadpole should be a macroscopic line inserted into the boundary of the worldsheet.

- We assume that we have a stable bosonic-string vacuum, which might not yet be known. We also assume that we have some dynamics that lead us there from a 26-dimensional flat tachyonic vacuum. Our goal here is to obtain the on-shell quantities of strings around such a stable vacuum, in the case that it exists. More precisely, we attempt to represent physics just on the stable vacuum using well-known string Feynman rules obtained around the tachyonic vacuum. Therefore, we must preserve Weyl invariance on the worldsheet even if we include tachyon tadpoles.\(^\dagger\) It is natural to impose Dirichlet boundary conditions on the macroscopic boundaries as tadpoles, which maintain Weyl invariance. This is because boundaries with Neumann boundary conditions can emit on-shell open strings, and this does not seem to be a natural property for tadpoles that are coupled to an external (tachyon) field. In this sense, Neumann boundaries would correspond to the tadpoles which represent vacuum polarization when the emitted strings form loops.

- It has been reported that the insertion of D-instanton-like boundaries into worldsheets alters the vacuum state of string theories radically.\(^\ddagger\) In addition, the naive estimation in Ref. 36) suggests that they can shift the mass squared of tachyons.

Although we do not have rigorous proof that the macroscopic boundaries mentioned above represent tachyon tadpoles in string theory, we call these D-instanton-like Dirichlet boundaries “tachyon tadpoles” in this paper; a macroscopic hole with the above Dirichlet boundary condition is a closed-string tachyon tadpole, and an inserted Dirichlet boundary on the edge of the worldsheet is an open-string tachyon tadpole. We emphasize that they are not D-instantons as solitonic solutions of string theories. D-instantons are physical objects with definite tension (although they might be unstable), while tachyon tadpoles are parts of Feynman diagrams with the weight attached to them corresponding to the expectation value of the tachyon field.

We insert closed-string tachyon tadpoles into the worldsheet in order to reproduce the background in which closed-string tachyons are condensed. However, we do not use open-string tachyon tadpoles in this paper for simplicity, because we can utilize the results of BSFT to describe open-string tachyon condensation.

Of course, we must consider the proper weights of the string wave functions on the tadpoles. Unfortunately, we do not have a Schwinger-Dyson equation for string

\(^\dagger\) If we wished to describe off-shell dynamics of strings, we should insert macroscopic boundaries as tadpoles that break Weyl invariance.
theory, and we do not know how to obtain the correct weight. Thus, we cannot give a rigorous treatment of closed-string tachyon condensation. However, we derive a constraint for the weight in §5.

§3. Sigma-model approach and closed-string tachyon condensation

In this section, we present a basic strategy to analyze the on-shell effective action of open-string tachyons on the background in which closed-string tachyons are condensed. We basically use a sigma-model approach. We also utilize the results of BSFT to define the partition function of open strings and to identify the vacuum of open-string tachyons. Therefore, we begin with a brief review of the sigma-model approach.

3.1. Sigma-model approach on the background in which closed-string tachyons are condensed

The basic idea of the sigma-model approach is that the spacetime action \( S \) for string fields is provided by the renormalized partition function \( Z \) in the background fields. That is,

\[
S = Z. \tag{3.1}
\]

For example, the tree-level spacetime action for open strings is given as the partition function derived on the disk worldsheet with background fields on the boundary. However, the relation (3.1) is not correct in the off-shell region, where we do not have conformal symmetry on the worldsheet, for bosonic strings in general. Therefore, we use (3.1) only for the case in which we calculate the on-shell spacetime action.

Next, let us extend the idea of the sigma-model approach to the case that we consider open strings on the background in which closed-string tachyons are condensed. As discussed in §2, the worldsheet on the stable vacuum can be reconstructed with the worldsheets on the tachyonic vacuum. More precisely, the effect of closed-string tachyon condensation is represented by the worldsheets to which closed-string tachyon tadpoles are attached. The partition functions obtained on such worldsheets would correspond to the correction terms to the spacetime action. In this article, we assume that the closed-string coupling \( g_c \) is sufficiently small, and we calculate only annulus diagrams as the correction terms. In this case, we should also consider the annulus worldsheet that represents an ordinary open-string one-loop quantum correction, since it has the same topology.\(^\ast\) The difference between the two types of annuli lies in the boundary conditions imposed on the inner circle. The worldsheets that represent the quantum corrections have the same boundary conditions on the inner circle as those on the outer circle, while the diagrams that include closed-string tachyon tadpoles have inner circles on which D-instanton-like boundary conditions are imposed. We regard the outer circle as an ordinary boundary of an open-string worldsheet that is coupled to a general D-brane.

Therefore, if we expand the total partition function with respect to the closed-

\(^\ast\) We are not sure that this treatment is completely valid. We discuss a related problem in §6.
string coupling constant, we obtain

\[
Z_{\text{total}} = Z_0 + g_c \int_0^\infty \frac{dt}{2t} \prod_i \int \frac{dx_0^\mu}{\sqrt{2\pi\alpha'}} w(x_0^\mu) Z_{\text{tad}}^1(t, x_0^\mu) + g_c \int_0^\infty \frac{dt}{2t} Z_{\text{loop}}^1(t) + O(g_c^2),
\]

(3.2)

where \(x_0^\mu\) denotes the position of D-instanton-like tadpoles in the spacetime, \(w(x_0^\mu)\) is the weight attached to the tadpoles, which depends on \(x_0^\mu\) in general, and \(t\) is the moduli of the annulus, which is defined in §4.

The leading order term \(Z_0\) is the partition function on the disk with no tadpole and no loop. The terms of order \(g_c\) consist of two parts; one of them contains the partition function \(Z_{\text{tad}}^1(t, x_0^\mu)\), which has one closed-string tachyon tadpole, and the other is composed of the partition function \(Z_{\text{loop}}^1(t)\), which has one open-string loop.

In this paper, as an example of a concrete calculation of the effect of closed-string tachyon condensation, we attempt to calculate the on-shell effective action of open-string tachyons on the background in which closed-string tachyons are condensed. We write this effective action as \(S_{\text{cond}}\). Basically, we assume the relation \(S_{\text{cond}} = Z_{\text{total}}\) in the framework of the sigma-model approach. In this sense, the sigma-model approach provides a sufficiently powerful tool for us. However, as we find bellow, the boundary term in the worldsheet action used in BSFT is useful to define the on-shell value of \(Z_{\text{total}}\) even in sigma-model approach. For this reason, we make a brief review of BSFT in the next subsection.

3.2. A short review of BSFT for the bosonic case

In the framework of the sigma-model approach, the dependence on the renormalization scheme of the partition function is considered to be equivalent to the field redefinition of \(S\). This equivalence holds as long as the divergence of the partition function is logarithmic. However, it was pointed out that a power-law divergence breaks this equivalence, and there results an ambiguity in the manner of obtaining \(S\) from the partition function in such a case.\(^{38,39}\) String theories that include tachyons have this problem.\(^{24}\)

BSFT provides us a solution to this problem. It gives us a natural extension of the relation (3.1) in the sigma-model approach.\(^*\) The relation between the spacetime action \(S\) and the partition function \(Z\) in BSFT is given as

\[
S(\lambda^i) = Z(\lambda^i) \left(1 + \sum_i \beta^i(\lambda^i) \frac{\partial}{\partial \lambda^i} \log Z(\lambda^i)\right),
\]

(3.3)

where \(Z(\lambda^i)\) is the partition function given by

\[
Z(\lambda^i) = \int \mathcal{D}X \exp\{-S_w(X, \lambda^i)\},
\]

(3.4)

\(^*\) Therefore, what we learn that BSFT can be applied to studies of the sigma-model approach. Recent studies of this kind are also found in Refs. 40–42.)
and the $\beta^i(\lambda^i)$ are $\beta$-functions obtained with the worldsheet action $S_w$. The worldsheet action for the construction of BSFT for open-string tachyons is

$$S_w = \frac{1}{4\pi \alpha'} \int \sigma \tau \partial_a X^\mu \partial^n X_\mu + \frac{1}{4\pi \alpha'} \int \theta \left\{ 2\alpha' a + \sum_{i=1}^{26} u_i (X(\theta)^i)^2 \right\}. \quad (3.5)$$

The quantities $\lambda_i$ in (3.3) and (3.4) are coupling constants included in the boundary term of $S_w$; they are $a$ and the $u_i$ in this case. They control the configuration of the open-string tachyon $T_{\text{open}}$ through the relation

$$T_{\text{open}} = a + \sum_{i=1}^{26} \frac{u_i X_i^2}{2\alpha'}. \quad (3.6)$$

The worldsheet $\Sigma$ is chosen to be a disk, on which we have rigid rotational symmetry. We do not need conformal symmetry in this case, and the boundary term in (3.5) breaks conformal invariance, in general.

Then the BSFT action for open-string tachyons is given by

$$S(a, u_i) = Z(a, u_i) \left( 1 + a + \sum_i u_i - \sum_i \frac{\partial}{\partial u_i} \ln Z(a, u_i) \right), \quad (3.7)$$

where $Z(a, u_i)$ is the partition function defined by (3.4) and (3.5). It is given exactly by

$$Z(a, u_i) = Ae^{-a} \left( \prod_{i=1}^{26} \sqrt{u_i e^{\gamma u_i} \Gamma(u_i)} \right) \equiv Z_0(a, u_i). \quad (3.8)$$

Here $A$ is a normalization factor, which is determined so that $S(0,0) = Z(0,0) = V_{25} T_{25}$, where $V_{25}$ is the volume of D25-brane and $T_{25}$ is the tension of D25-brane. We find that $A$ should be fixed as $A = T_{25}(2\pi \alpha')^{-13}$ in the Appendix. The precise definitions of $S(0,0)$ and $Z(0,0)$ are given bellow.

If we do not consider a closed-string sector, the tree level spacetime action of open-string tachyons is given by (3.7). The action (3.7) has several classical solutions which correspond to unstable D-branes of various dimensions. If we substitute the corresponding classical solution $(a_*, u_{*i})$ into the action, $S(a_*, u_{*i})$ gives the energy density times the volume of the corresponding D-brane. Explicitly, we obtain the tension of the corresponding D-brane if we divide the on-shell value of $S(a, u_i)$ by the volume of the spacetime where open strings exist. The value of $a_*$ and the $u_{*i}$ are 0 or $\infty$. We obtain the classical solution $(a_*, u_{*i})$ by taking the limit discussed bellow.

Note that (3.3) becomes the equivalent to (3.1) if we have conformal symmetry on the worldsheet. Conformal symmetry is recovered if we take the limit

$$a \rightarrow \begin{cases} 0 \\ \infty \end{cases},$$

$$u_i \rightarrow \begin{cases} 0 \\ \infty \end{cases}. \quad (3.9)$$
In general, we take a limit in which \( n_0 \) of the quantities \( u_i \) go to zero and \( n_\infty \) of the \( u_i \) go to \( \infty \), where \( n_0 + n_\infty = 26 \). If we take the above limit while imposing the stationary condition for the variable \( a \), the variables \( a \) and \( u_i \) converge to a classical solution given by \( a_s \) and \( u_{s,i} \). The stationary condition we use here is obtained in the Appendix as

\[
a = \sum_{i=1}^{26} \left( -u_i + u_i \frac{\partial}{\partial u_i} \ln Z_0(u_i) \right) + \sum_s \left( \frac{1}{2} - \frac{u_s x_s^2}{2\alpha'} \right),
\]

where the values \( u_s \) in the last term are those taken to zero in the conformal limit, and therefore the constant term coming from the sum in the last term is \( \frac{n_0}{2} \). The values \( x^s \) are the spacetime coordinates parallel to the D-brane, and they correspond to the zero modes of the \( X^s \). We can check easily that \( S(a, u_i) = Z_0(a, u_i) \) if (3.10) is satisfied. Then, the on-shell action is obtained as

\[
S(a_s, u_{s,i}) = \lim_{(a, u_i) \to (a_s, u_{s,i})} S(a, u_i) = \lim_{(a, u_i) \to (a_s, u_{s,i})} Z(a, u_i),
\]

where the limits are taken along (3.10). The limit \( u_i \to \infty \) yields Dirichlet boundary conditions for \( X^i \), and therefore \( n_\infty \) corresponds to the number of directions that are perpendicular to the corresponding D-brane. Here, the action (3.7) becomes the tension times the volume of D(\( n_0 - 1 \))-brane after we take the above limit. Actually, there are some delicate problems concerning the method of taking the conformal limits. We discuss the relevant details in the Appendix.

In the original spirit of BSFT, (3.7) is the spacetime action that describes open-string tachyons even in the off-shell region. BSFT might not be necessary for us, since we wish to obtain the on-shell action, which can also be obtained from the sigma-model approach. However, the boundary term that includes the variables \( a \) and \( u_i \) also provides us a good definition of the on-shell partition function through the above conformal limit, even within the sigma-model approach; the \( u_i \) play the role of a regulator, and the combination of \( a \) and the \( u_i \) acts as a navigator, which leads us to each vacuum through the above conformal limit.

The partition function on a disk without boundary terms has an IR divergence. The quantities \( u_i \) act as regulators, which regularize the IR divergence through integration over the zero-modes of \( X^\mu \), which correspond to the spacetime coordinates. Positive \( u_i \) prevent such divergence. Furthermore, we calculate the partition function with the boundary term in (3.5) and take the conformal limit (3.9) while imposing the relation (3.10). Then we obtain the on-shell effective action at each vacuum for open strings. In this sense, the variables \( a \) and \( u_i \) play the role of a navigator that leads us to each vacuum of the open-string tachyons; \( n_0 \) determines the dimension of the D-brane where the open strings exist, and (3.9) with (3.10) provides us the correct normalization of the partition function at each vacuum.

It is hoped that we can construct an extended BSFT that also describes off-shell dynamics of open-string tachyons on the background in which closed-string tachyons are condensed, using the partition function \( Z^{\text{total}} \) instead of \( Z_0 \). We might also be able to include quantum corrections in BSFT.\(^{\text{4)}}\) However, there are some subtleties

\(^{\text{4)}}\) Some attempts to obtain a loop-corrected BSFT action are found in Refs. 43–47.
involved in the construction of such an extended BSFT, as we discuss in the next subsection. Therefore, we do not attempt to extend BSFT here, and we carry out our analysis only for the on-shell properties of strings on the basis of the sigma-model approach, including \( O(g_c) \) corrections. Thus, we do not regard the quantities \( a \) and \( u_i \) as dynamical variables, as those in BSFT. Instead, we use the results of BSFT only for the definition of the on-shell effective action of open-string tachyons in the framework of the sigma-model approach.

To restate our position, we are not concerned with off-shell actions in this paper. In this case, we are not able to obtain the profile of the open-string tachyon potential. However, we can obtain the absolute value of the effective action of open-string tachyons at each vacuum on the background in which closed-string tachyons are condensed.

3.3. Subtleties in the construction of BSFT including \( O(g_c) \) corrections

In order to represent the dynamics of open-string tachyons on the background in which closed-string tachyons are condensed, it may be possible to construct BSFT using a partition function that includes \( O(g_c) \) terms. However, there are some non-trivial problems involved in the naive extension of BSFT. These subtleties are as follows.

First, in the definition of BSFT,\(^{2,3}\) we consider a disk worldsheet with rotational symmetry. Rigid rotational symmetry is required to obtain the BSFT action. However, it seems that we still have various choices of how to insert macroscopic holes into the rotationally invariant disk when we construct worldsheets that have an annulus topology. Most of the worldsheets we obtain after the insertion of a hole do not have rotational symmetry. In the on-shell case, we have Weyl invariance, and we can map all of such worldsheets to an annulus that is rotationally invariant. However, the boundary term that breaks Weyl invariance does not allow us to naively transform the worldsheet to an annulus. Furthermore, we lose rotational symmetry if we insert more than one hole, even in the case that we have Weyl symmetry.

Second, even if we have a good idea of how to construct BSFT with worldsheets of arbitrary topology, it is difficult to calculate contributions of all the worldsheets of different configurations without conformal invariance.

§4. Correction terms at order \( g_c \)

4.1. Setting up the analysis

We now attempt to calculate the on-shell spacetime action of open-string tachyons on the background in which closed-string tachyons are condensed. Although the exact calculation of the effect of the insertion of multiple tadpoles and open-string loops is very difficult, we can calculate the partition function at order \( g_c \). According to the argument in the previous sections, we must calculate \( Z_{\text{tad}}^1 \) and \( Z_{\text{loop}}^1 \).

To begin with, let us consider only a partition function with one closed-string tadpole, in order to simplify the analysis. An ordinary open-string one-loop partition function can be obtained easily, and we mention it later.
First, we consider the annulus $M$. The inner circle of the annulus represents the tadpole, and we impose D-instanton-like Dirichlet boundary conditions on it. The outer circle of the annulus is the usual boundary of the worldsheet, which couples to the open-string tachyons. We include the boundary term of the worldsheet action only for the outer circle. We set the radius of the outer circle to 1 and that of the inner circle to $r$. This annulus becomes the disk that is used to define BSFT if we remove the inner boundary. The worldsheet action on the annulus $M$ is given by

$$
S_{\text{worldsheet}} = \frac{1}{4\pi\alpha'} \int_M d\sigma d\tau \partial_\alpha X^\mu \partial^\alpha X_\mu + \frac{1}{4\pi\alpha'} \int_{\partial M_{\text{out}}} d\theta \left\{ 2\alpha' a + \sum_{i=1}^{26} u_i X_i(\theta)^2 \right\},
$$

(4.1)

and the boundary conditions for the $X^\mu$ are

$$
n^\alpha \partial_\alpha X^\mu + u_\mu X^\mu|_{\partial M_{\text{out}}} = 0,
$$

(4.2)

$$
X^\mu|_{\partial M_{\text{in}}} = x^\mu_0,
$$

(4.3)

where $\partial M_{\text{in}}$ and $\partial M_{\text{out}}$ denote the inner boundary and the outer boundary, respectively. Here $n^\alpha$ is the unit normal vector. Note that we do not sum over $\mu$ in the second term of the left-hand side of (4.2).

We also must integrate the partition function over the moduli. We have two types of moduli here. One of them is $r$, the radius of the inner circle of the annulus. The other is $x_0^\mu$, the position of the D-instanton-like tadpole in spacetime. After the integration over $x_0^\mu$, we take the conformal limit to justify our calculation. Then we fix the gauge, and integrate over $r$. Although the weight $w(x_0^\mu)$ can depend on $x_0^\mu$ nontrivially in general, we have no technique to determine it rigorously as yet. Therefore, we regard $w$ as a constant in this section. An argument regarding the $w(x_0^\mu)$ dependence of $w$ is made in §5.

4.2. Calculation of $Z_{1}^{\text{tad}}$

Now we begin the calculation of $Z_{1}^{\text{tad}}$. We have four types of parameters that control the partition function. They are $a$, $u_i$, $x_0^\mu$ and $r$. We can divide the calculating procedure into several parts as follows.

4.2.1. The $u_i$ dependent part and the $a$ dependent part

First, we calculate the $u_i$ dependent part of the partition function with the action (4.1) employing the method in Refs. 3) and 48). Let us first consider the case in which only one of $u_i$ is non-zero and the others are zero. We omit the subscript $i$ in this case.

First, we set $z = \sigma + i\tau$. The components of the metric on the annulus $M$ are

$$
g_{zz} = g_{zz} = \frac{1}{2}, \ g_{zz} = g_{zz} = 0. \ The \ Green's \ function \ on \ the \ annulus \ should \ obey
$$

$$
-\frac{1}{\pi\alpha'} \partial_z \partial_{\bar{z}} G(z, w) = \delta^2(z, w),
$$

(4.4)
and the boundary conditions are
\[ ((z\partial_z + \bar{z}\partial_{\bar{z}})G(z, w) + u)|_{z=1} = 0, \]
\[ G(z, w)|_{z=r} = x_0^\mu. \]

To begin with, we calculate the partition function for \( x_0^\mu = 0 \), and then we determine the \( x_0^\mu \) dependent factor. If \( x_0^\mu = 0 \), the above stated requirement determines the Green’s function to be
\[ G(z, w) = -\frac{\alpha'}{2} \ln |z - w|^2 - \frac{\alpha'}{2} \ln |1 - zw|^2 - \frac{\alpha'}{4} \frac{u}{1 - u \ln r} \ln |z|^2 \ln |w|^2 \]
\[ + \frac{\alpha'}{2} \frac{1}{1 - u \ln r} \left( \ln |z|^2 + \ln |w|^2 \right) - \alpha' \frac{\ln r}{1 - u \ln r} \]
\[ - \frac{\alpha'}{2} \sum_{k=1}^{\infty} \left( \frac{1}{k} (k - u)r^{2k} \right) \left\{ \left( \frac{z}{w} \right)^k + \left( \frac{\bar{z}}{\bar{w}} \right)^k + \left( \frac{\bar{z}}{w} \right)^k + \left( \frac{z}{\bar{w}} \right)^k \right\} \]
\[ - \frac{\alpha'}{2} \sum_{k=1}^{\infty} \left( \frac{2u + (k - u)r^{2k}}{k} (k - u)r^{2k} \right) \left\{ (zw)^k + (\bar{z}w)^k \right\} \]
\[ - \frac{\alpha'}{2} \sum_{k=1}^{\infty} \left( \frac{k + u}{k} (k - u)r^{2k} \right) \left\{ (zw)^{-k} + (\bar{z}w)^{-k} \right\}. \]

Therefore, we obtain
\[ \langle X(\theta)^2 \rangle = -\alpha' \frac{\ln r}{1 - u \ln r} - 2\alpha' \left( \frac{1}{k} (k + u + (k - u)r^{2k}) \right), \]

after an appropriate subtraction of the divergent terms. Then using the relation
\[ \frac{d}{du} \ln Z_1^{\text{tad}} = -\frac{1}{4\pi \alpha'} \int_0^{2\pi} d\theta \langle X(\theta)^2 \rangle, \]

we obtain
\[ Z_1^{\text{tad}} = \left\{ \frac{ue^{\gamma u} \Gamma(u)}{\sqrt{1 - u \ln r}} \prod_{k=1}^{\infty} \left( \frac{k + u}{k} (k + u + (k - u)r^{2k}) \right) \right\} \times ( \text{factors including } a, x_0, r), \]

where \( \gamma \) is Euler’s constant. In general, we obtain the partition function with non-zero \( u_i \) as
\[ Z_1^{\text{tad}} = \prod_{i=1}^{26} \left\{ \frac{u_i e^{\gamma u_i} \Gamma(u_i)}{\sqrt{1 - u_i \ln r}} \prod_{k=1}^{\infty} \left( \frac{k + u_i}{k} (k + u_i + (k - u_i)r^{2k}) \right) \right\} \times ( \text{factors including } a, x_0, r). \]

We can easily incorporate the \( a \) dependent factor as
\[ Z_1^{\text{tad}} = e^{-a} \prod_{i=1}^{26} \left\{ \frac{u_i e^{\gamma u_i} \Gamma(u_i)}{\sqrt{1 - u_i \ln r}} \prod_{k=1}^{\infty} \left( \frac{k + u_i}{k + u_i + (k - u_i)r^{2k}} \right) \right\} \]
\[
\times (\text{factors including } x_0, r)
= Z_0(a, u_i) \prod_{i=1}^{26} \left\{ \sqrt{\frac{u_i}{1 - u_i \ln r}} \prod_{k=1}^{\infty} \left( \frac{k + u_i}{k + u_i + (k - u_i)r^{2k}} \right) \right\} \\
\times (\text{factors including } x_0, r),
\tag{4.12}
\]
where \( Z_0(a, u_i) = e^{-a} \prod_{i=1}^{26} \sqrt{u_i} \exp(\gamma u_i \Gamma(u_i)) \). Here we have absorbed the factor \( \prod_{i=1}^{26} \prod_{k=1}^{\infty} \frac{1}{k} \) into \( e^{-a} \) and renormalized, as in Ref. 3).

4.2.2. The \( x_0^\mu \) dependent part

Next, we fix the factor including \( x_0^\mu \). We determine the \( x_0^\mu \) dependent factor as follows. First, we map the annulus \( M \) to a cylinder \( M' \) with the diffeomorphism
\[
z = e^\rho t,
\tag{4.13}
\]
where \( \rho \equiv \xi^1 + i\xi^2 \) represents the complex coordinates on the cylinder. The length of the cylinder is \( \pi \) and its periodicity is \( 2\pi t \). The inner circle of the annulus is mapped to one side of the cylinder, where \( \xi^1 = -\pi \), and the outer circle is mapped to the other side, where \( \xi^1 = 0 \). \( t \) is related to \( r \) as
\[
r = e^{-\frac{\pi}{t}}.
\tag{4.14}
\]
Now \( t \) is the moduli of the cylinder and the annulus, which has already appeared in (3.2). Of course, the partition function does not change under this transformation, while the metric on the cylinder is now
\[
g_{\rho\bar{\rho}} = g_{\rho\rho} = \frac{|\exp(\frac{\rho}{t})|^2}{2t^2}.
\tag{4.15}
\]
Next, we carry out a Weyl transformation as
\[
g_{\rho\bar{\rho}} \to \frac{1}{|\exp(\frac{\rho}{t})|^2} g_{\rho\bar{\rho}} = \frac{1}{2t^2}.
\tag{4.16}
\]
Note that this rescaling does not change the metric where \( \xi^1 = 0 \), and therefore the boundary action is not affected. Furthermore, the bulk action possesses Weyl symmetry. Thus the partition function does not change under the above two successive transformations, even though the boundary action breaks conformal invariance.

Now we have the worldsheet action on the cylinder \( M' \) given by
\[
S = \frac{1}{4\pi\alpha'} \int_{M'} d\xi^1 d\xi^2 \partial_\alpha X^\mu \partial^\alpha X_\mu
+ \frac{1}{4\pi\alpha'} \int_{\partial M'_\text{out}} d\theta' \frac{1}{t} \left\{ 2\alpha' a + \sum_{i=1}^{26} u_i X_i(\theta')^2 \right\}.
\tag{4.17}
\]
Next, we use the Minkowski signature on the cylinder and calculate the Hamiltonian. The mode expansion of \( X^\mu \) on the cylinder is
\[
X^\mu(\xi^0, \xi^1) = x_0^\mu \frac{t - u_\mu \xi^1}{t + u_\mu \pi} + (\text{oscillating modes}),
\tag{4.18}
\]
where we do not sum over $\mu$ on the right-hand side. The worldsheet action contains only terms quadratic in the $X^\mu$ and a constant term. Thus the zero-mode part of the Hamiltonian decouples from the oscillating-mode part. The oscillating modes give the partition function that is obtained in the case that $x_0^\mu = 0$. Therefore, we only have to calculate the zero-mode part of the Hamiltonian, which yields the $x_0^\mu$ dependent factor. We easily obtain

$$H = \frac{1}{4\pi\alpha'} \sum_{\mu=1}^{26} \frac{u_\mu}{t + u_\mu \pi} (x_0^\mu)^2 + \frac{1}{4\pi\alpha'} \frac{2\alpha' a}{t} + \text{(contribution of oscillating modes)}. \quad (4.19)$$

Therefore, the $x_0^\mu$ dependent factor of the partition function can be obtained from

$$Z_1^{\text{tad}}(x_0^\mu) = \text{Tr} e^{-H 2\pi i} = e^{-\frac{\alpha'}{2\pi} \sum_{\mu=1}^{26} \frac{u_\mu}{1 + u_\mu \pi} (x_0^\mu)^2} Z_1^{\text{tad}}(x_0^\mu = 0). \quad (4.20)$$

Note that this process also reproduces the $a$ dependent factor explicitly.

Next, we integrate the $x_0^\mu$ dependent factor over $x_0^\mu$. We regard the weight $w$ as a constant for simplicity, and we do not include it in the integrand. This integration can be carried out easily, and we have

$$\int d^{26}x_0^\mu \exp \left( -\frac{t}{2\alpha'} \sum_{\mu=1}^{26} \frac{u_\mu}{t + u_\mu \pi} (x_0^\mu)^2 \right) = \prod_{\mu=1}^{26} \left( \frac{2\alpha' t + u_\mu \pi}{t - u_\mu} \right). \quad (4.21)$$

We find that the inverse of the right-hand side of (4.21) is included in the right-hand side of (4.12), using the relation (4.14). Therefore, we obtain

$$\frac{1}{(2\pi\alpha')^{26/2}} \int d^{26}x_0^\mu Z_1^{\text{tad}} = Z_0 \left\{ \prod_{i=1}^{26} \prod_{k=1}^{\infty} \frac{k + u_i}{k + u_i + (k - u_i)e^{-2\pi k/t}} \right\} \times \text{(factors including $t$)}. \quad (4.22)$$

We note that finite values of the $u_i$ lead to the reduction

$$\sqrt{\frac{t + u_i \pi}{t u_i}} \sqrt{\frac{u_i}{1 - u_i \ln r}} = 1 \quad (4.23)$$

in the calculation of (4.22). This is one merit of using the variables $u_i$ as regulators.

4.2.3. The $t$ dependent part

The $t$ dependent factor is obtained as follows. First we set $x_0^\mu = 0$, and we treat only one of the $u_i$ as non-zero, again. (Again we omit the subscript $i$.) First, we calculate the energy-momentum tensor on the cylinder $M'$. The energy-momentum tensor on the annulus $M$ is obtained as

$$\langle T_{zz} \rangle = -\frac{1}{\alpha'} \lim_{w \to z} \left\{ \partial_z \partial_w G(z, w) + \frac{\alpha'}{2} \left( \frac{1}{(z - w)^2} \right) \right\}$$

$$= \frac{1}{z^2} \left\{ \frac{u}{4 - u \ln r} - \sum_{k=1}^{\infty} \frac{k(k - u) r^{2k}}{k + u + (k + u) r^{2k}} \right\}. \quad (4.24)$$
The anti-holomorphic part is obtained similarly. Then, the energy-momentum tensor on the cylinder $M'$ is

$$\langle T_{\rho\rho} \rangle = \left( \frac{dz}{d\rho} \right)^2 \langle T_{zz} \rangle + \frac{26}{12} \{ z, \rho \}$$

$$= \frac{1}{t^2} \left\{ \frac{1}{4} \frac{u}{1 - u \ln r} - \sum_{k=1}^{\infty} \frac{k(k - u)r^{2k}}{k + u + (k + u)r^{2k}} - \frac{26}{24} \right\}. \quad (4.25)$$

We use the following relation for the calculation of the partition function;

$$\delta \ln Z_{\text{tad}}^1 = - \frac{1}{4\pi \alpha'} \int d^2 \rho \sqrt{g} \left\{ \delta g_{\rho\rho} \langle T_{\rho\rho} \rangle + \delta g_{\bar{\rho}\bar{\rho}} \langle T_{\bar{\rho}\bar{\rho}} \rangle \right\}. \quad (4.26)$$

Suppose that we change the periodicity of the cylinder as $2\pi t \to 2\pi (t + \delta t)$, while keeping the metric unchanged. We can realize the same shift of the partition function without changing the periodicity if we shift the metric by the amount

$$\delta g_{\rho\rho} = \delta g_{\bar{\rho}\bar{\rho}} = -\frac{1}{2t \ell^2} \delta t, \quad (4.27)$$

instead. Recall that the metric on the cylinder is given in (4.16) as

$$g_{\rho\rho} = g_{\bar{\rho}\bar{\rho}} = \frac{1}{2t^2}, \quad g_{\rho\bar{\rho}} = g_{\bar{\rho}\rho} = 0. \quad (4.28)$$

Using the above relations, we obtain the differential equation

$$\frac{d}{dt} \ln Z_{\text{tad}}^1 = 2\pi \langle T_{\rho\rho} \rangle$$

$$= \frac{2\pi}{t^2} \left\{ \frac{1}{4} \frac{u}{1 - u \ln r} - \sum_{k=1}^{\infty} \frac{k(k - u)r^{2k}}{k + u + (k + u)r^{2k}} - \frac{26}{24} \right\}. \quad (4.29)$$

Therefore, we obtain

$$Z_{\text{tad}}^1 = e^{\frac{26\pi}{t^2 \ell^2}} \sqrt{\frac{t}{t + u \pi}} \prod_{k=1}^{\infty} \frac{1}{k + u + (k - u)e^{-2\pi k/t}} \times \text{(factors including } a, u, x_0^\mu) \quad (4.30)$$

Note that the above procedure also exactly reproduces the factors that include both $u$ and $t$.

4.2.4. Conformal limit and the integration over the moduli $t$

We obtain

$$ (2\pi \alpha')^{-26/2} \int d^2 x_0^\mu Z_1(a, u_i, x_0^\mu, t) = Z_0(a, u_i) e^{\frac{26\pi}{t^2 \ell^2}} \prod_{i=1}^{26} \prod_{k=1}^{\infty} \frac{k + u_i}{k + u_i + (k - u_i)e^{-2\pi k/t}} \times \text{constant}, \quad (4.31) $$
after we incorporate the results of the previous subsections. We absorb the ambiguity of the overall factor into the weight $w$. Therefore, we set the constant factor to 1.

As the final step, we integrate (4.31) over the moduli $t$. Of course, the gauge fixing procedure is justified only in the conformal limit. Therefore we take this limit and perform the integration over $t$, including the Faddeev-Popov determinant.

We take several types of conformal limits, as mentioned in §3. Using the result of BSFT, the disk partition function $Z_0(a, u_i)$ becomes

$$Z_0(a, u_i) \longrightarrow V_{n_0-1}T_{n_0-1} \quad (4.32)$$

in the conformal limit in which we take $n_0$ of the quantities $u_i$ to be zero and the other $(26 - n_0)$ quantities $u_i$ to be $\infty$. Here $V_{n_0-1}$ is the volume of the D$(n_0 - 1)$-brane, and $T_{n_0-1}$ is the tension of the D$(n_0 - 1)$-brane at the tree level.

The Faddeev-Popov determinant on the annulus is given in the usual way, since we do not have ghosts in the boundary action. It is obtained as

$$\eta(it)^2 = \frac{1}{t} e^{-\frac{2\pi}{it}} \prod_{k=1}^{\infty} (1 - e^{-\frac{2\pi k}{t}})^2. \quad (4.33)$$

Then, we obtain the correction term, which includes the contribution of one closed-string tadpole in the conformal limit, as

$$g_c \int_0^\infty \frac{dt}{2t} \prod_\mu \int \frac{dx_0^\mu}{\sqrt{2\pi\alpha'}} w(x_0^\mu) Z_1^{\text{tad}}(t, x_0^\mu, n_0) = g_c V_{n_0-1}T_{n_0-1} I^{\text{tad}}(n_0), \quad (4.34)$$

where

$$I^{\text{tad}}(n_0) \equiv \int_0^\infty \frac{dt}{2t} \prod_\mu \int \frac{dx_0^\mu}{\sqrt{2\pi\alpha'}} w(x_0^\mu) \frac{Z_1^{\text{tad}}(t, x_0^\mu, n_0)}{Z_0(n_0)}$$

$$= w \int_0^\infty \frac{dt}{2t^2} e^{2\pi/t} \prod_{k=1}^{\infty} \left[ (1 - e^{-\frac{2\pi k}{t}})^{n_0-24} (1 + e^{-\frac{2\pi k}{t}})^{-n_0} \right]. \quad (4.35)$$

Note that $w$ is treated as a constant here.

4.3. One-loop open-string partition function on a D$(n_0 - 1)$-brane

The ordinary one-loop partition functions for open-strings are easily obtained, as described in string textbooks. We consider an annulus that is the same as the annulus $M$ in §4.1, except for the boundary conditions on the inner circle. The boundary conditions on the inner circle are the same as those on the outer circle in this case. The one-loop open-string partition function on the D$(n_0 - 1)$-brane is given as

$$g_c \int_0^\infty \frac{dt}{2t} Z_1^{\text{loop}}(t) = g_c V_{n_0-1}T_{n_0-1} I^{\text{loop}}(n_0), \quad (4.36)$$

where

$$I^{\text{loop}}(n_0) \equiv h \int_0^\infty \frac{dt}{2t^2} e^{2\pi/t} \prod_{k=1}^{\infty} (1 - e^{-\frac{2\pi k}{t}})^{-24}. \quad (4.37)$$

*) For example, see Ref. 49) .
Here $h$ is a constant that determines the ratio of the one-loop correction term to the leading term $Z_0$. It is fixed by the stipulation that the theory be unitary, as is usually the case in calculations of amplitudes.

\section{Calculation of the on-shell effective action}

\subsection{Conformal limit and the on-shell effective action}

We are now ready to discuss the on-shell effective action of open-string tachyons on the background in which closed-string tachyons are condensed. We identify the vacua for open-string tachyons as in the framework of BSFT as follows.\textsuperscript{*}) Closed-string tachyons are assumed to already be condensed here.

- **perturbative vacuum**: $(a, u_i) \to (0, 0)$
  This corresponds to the situation in which open-string tachyons are on a D25-brane.

- **non-perturbative vacuum**: $(a, u_i) \to (\infty, 0)$
  In this case, open-string tachyons are condensed completely. We find that $Z_0 = 0$, and therefore $Z_{\text{total}} = 0$.

- **intermediate vacuum**: $a \to \infty$, $n_0$ of the $u_i \to 0$, and $n_\infty$ of the $u_i \to \infty$
  We suppose that the open-string tachyons are on a D$(n_0 - 1)$-brane at this vacuum. Open-string tachyons are partially condensed here.

The on-shell effective action of open-string tachyons at each vacuum is given in (3.2). Using the results of the previous section, we obtain

\begin{equation}
Z_{\text{total}} = V_{n_0 - 1} T_{n_0 - 1} \left\{ 1 + g_c \left( I_{\text{tad}}(n_0) + I_{\text{loop}}(n_0) \right) + O(g_c^2) \right\}. \tag{5.1}
\end{equation}

We use an Euclidean signature in the spacetime metric in this paper, and thus $Z_{\text{total}}/V_{n_0 - 1}$ is the energy density of open-string tachyons. If Sen’s conjecture is still valid in the case that we consider $O(g_c)$ corrections, it corresponds to the modified D$(n_0 - 1)$-brane tension.

\subsection{Elimination of the divergence in the effective action}

Next, we estimate the value of $Z_{\text{total}}$ at each vacuum. In this step, there is the difficulty that divergent terms exist. The divergence of $I_{\text{loop}}$ and $I_{\text{tad}}$ in the small $t$ region corresponds to the IR divergence in the closed-string channel, which occurs when the inner circle of the annulus becomes infinitesimally small. The divergence in the large $t$ region is an IR divergence in the open-string channel, due to the propagation of light open-string modes circulating the annulus.

We now consider the divergence in the closed-string channel. $I_{\text{loop}}(n_0)$ can be rewritten as

\begin{equation}
I_{\text{loop}}(n_0) = (2\pi)^{(26-n_0)/2} h \int_0^{\infty} \frac{ds}{4\pi} e^{s (n_0-26)/2} \prod_{k=1}^{\infty} (1 - e^{-ks})^{-24}
\end{equation}

\textsuperscript{*)} We take the conformal limit while imposing the stationary condition (3.10). In §6 we discuss a problem that can arise when we include $O(g_c)$ corrections.
\[
= (2\pi)^{(26-n_0)/2} \hbar \int_0^\infty \frac{ds}{4\pi} s^{(n_0-26)/2} [e^s + 24 + O(e^{-s})], \quad (5.2)
\]
and \(I_{\text{tad}}(n_0)\) can be written as
\[
I_{\text{tad}}(n_0) = w \int_0^\infty \frac{ds}{4\pi} e^s \prod_{k=1}^\infty \left[(1 - e^{-ks})^{n_0-24}(1 + e^{-ks})^{-n_0}\right]
= w \int_0^\infty \frac{ds}{4\pi} [e^s + (24 - 2n_0) + O(e^{-s})], \quad (5.3)
\]
where \(s = 2\pi/t\). The origin of the divergence in the closed-string channel is light closed-string fields, namely closed-string tachyons and dilatons.

5.2.1. The divergence from tachyons

The divergence coming from the first terms in the last lines of (5.2) and (5.3) is due to zero-momentum closed-string tachyons emitted as tadpoles into the vacuum. This type of divergence is spurious. We can obtain finite contributions from these terms by analytic continuation. In this case, the regularized partition function can be a complex number after the analytic continuation, in general. Its imaginary part corresponds to the decay rate of the unstable vacuum through the relation

\[
\tau_{\text{decay}} = -2\text{Im}\{E\}, \quad (5.4)
\]
where \(\tau_{\text{decay}}\) is the decay rate of the unstable vacuum per unit volume, and \(E\) is the regularized energy density of the unstable vacuum. We assume \(E = Z_{\text{total}}/V_{n_0-1}\) here. The imaginary part from closed-string tachyons has to be removed, since we wish to represent the effective action, which does not have an instability due to closed-string tachyons. We propose a method to cancel the imaginary part in §5.3.

5.2.2. The divergence from massless string fields

A fatal divergence comes from the second terms in the last lines of (5.2) and (5.3). This divergence is due to on-shell dilatons emitted into the vacuum. It causes a conformal anomaly. We have several methods to eliminate this divergence for oriented strings.

- Fischler-Susskind mechanism
  If we put appropriate vertex operators onto the worldsheet, the background fields are shifted and the conformal anomaly is absorbed. This is called the Fischler-Susskind mechanism.\(^{51,52}\) The vertex operators to be inserted are those of massless string fields, and the weights attached to them are chosen so that the fatal divergence vanishes. This mechanism causes the contribution of zero-momentum tachyons to be non-zero in general. This property is desirable in the treatment of closed-string tachyon condensation.

- Cancellation between loops and tadpoles
  We have another method to cancel the anomaly, which is presented in Ref. 28). Let us consider the case of \(n_0 = 26\), for example. If we choose the weight \(w\)
appropriately as
\[(24 - 2n_0)w = -(2\pi)^{(26-n_0)/2}24h, \tag{5.5}\]

the dilaton tadpole coming from \(I^\text{tad}(26)\) and that from \(I^\text{loop}(26)\) cancel. However, we do not use this method in this paper for the reason given in §5.3.

5.2.3. The divergence in the open-string channel

We also have divergence from the open-string channel in \(I^\text{loop}\) and \(I^\text{tad}\), in general. This occurs when the radius of the inner circle becomes equal to that of the outer circle of the annulus. We also have two types of divergence here, one due to open-string tachyons and the other to massless open-strings.

The divergence caused by open-string tachyons can also be regularized using analytic continuation. In general, an imaginary part appears after the analytic continuation. This reflects the instability of the vacuum due to open-string tachyons.\(^\ast\) The imaginary part is eliminated by open-string tachyon condensation. There is also a fatal divergence from massless open-string fields here. This divergence can also be removed using the Fischler-Susskind mechanism.\(^\ast\ast\)

It might desirable to treat open-string tachyon condensation using “open-string tachyon tadpoles”, as we do in the context of closed-string tachyon condensation. However, our main purpose in this paper is investigation of the effect of closed-string tachyon condensation. For this reason, we do not consider the details of the mechanism of open-string tachyon condensation here. The effect of closed-string tachyon condensation on open-string tachyon condensation is discussed in §5.4.

5.3. Construction of a consistent on-shell effective action

Our goal in this paper is to describe physical quantities on the stable vacuum of closed-strings. Therefore \(\tau_{\text{decay}}\) coming from closed-string tachyons should vanish, and the worldsheet should be conformally invariant. Thus, natural criteria which on-shell effective actions should obey are as follows.

- The conformal anomaly must be removed.
- The imaginary part of the spacetime action due to closed-string tachyons must vanish after analytic continuation.

These criteria lead us to the following procedure to obtain a consistent on-shell effective action. The procedure we propose here consists of three steps.

1. Analytic continuation to regularize tachyonic divergence:

First, we make the spurious IR divergence from tachyons finite using analytic continuation.

\(^\ast\) This treatment of the divergence from open-string tachyons, considering one-loop corrections to BSFT, is also found in Refs. 46) and 47).

\(^\ast\ast\) See, for example, Refs. 58)–62).
Closed-String Tachyon Condensation and the On-Shell Effective Action

2. Cancellation of the imaginary part:
   Next, we set the weight $w$ so that the imaginary part in $Z^{\text{total}}$ due to closed-string tachyons vanishes. If we choose $w$ appropriately, the imaginary part in open-string loops and that in closed-string tachyon tadpoles cancel.

3. Elimination of the conformal anomaly:
   In general, after the above two steps, we still have the fatal divergence from massless string modes. In the last step, we use the Fischler-Susskind mechanism to cancel the remaining divergence and make the worldsheet conformally invariant. In this step, we do not use cancellation of the fatal divergences of loops and tadpoles mentioned near (5.5) in §5.2.2, since we have already fixed the value of $w$ in the previous step. The condition for $w$ to cancel the fatal divergence is not compatible with the elimination of the imaginary part, in general.

   In the second step, we remove the imaginary part from closed-string tachyons. That is, the appropriate choice of $w$ eliminates the instability of the vacuum due to closed-string tachyons. After the last step, strings exist in a curved spacetime, in general. The insertion of closed-string tachyon tadpoles and massless string-field vertex operators into worldsheets effectively creates such a background. The above procedure is the most natural method to obtain an on-shell spacetime action satisfying the above criteria. The strategy proposed here also acts as a constraint for possible $w$.

   It is important to note that there is still an ambiguity in the determination of the weight $w$. To understand the above procedure, let us estimate the imaginary part in the closed-string channel in $I^{\text{loop}}$. We obtain

$$
\text{Im}\{I^{\text{loop}}(n_0)\} = \text{Im}\left\{\int_0^{\infty} \frac{ds}{4\pi} s^{(n_0-26)/2} e^{(1+i\epsilon)s}\right\} = \frac{1}{4\pi} \frac{\pi}{\Gamma\left(\frac{26-n_0}{2}\right)} (5.6)
$$

by using the relation

$$
\text{Im}\left\{\int_0^{\infty} \frac{ds}{s} s^{-\alpha} e^{(\gamma+i\epsilon)s}\right\} = \frac{\pi}{\Gamma(1+\alpha)} \gamma^\alpha. \quad \text{(for } \gamma > 0) \quad (5.7)
$$

Therefore, we have an imaginary part in $I^{\text{loop}}(n_0)$ in the case $n_0 \leq 25$. Contrastingly, $I^{\text{tad}}(n_0)$ given by (5.3) does not have an imaginary part from closed-string tachyons. This is due to our treatment of $w$. Note that we have regarded $w$ as a constant in the calculation of $I^{\text{tad}}(n_0)$ in §4. However, $w$ can depend on $x_0^\mu$ in general. If $w$ has a nontrivial dependence on $x_0^\mu$, $I^{\text{tad}}(n_0)$ also can be a complex number after the analytic continuation. For example, let us assume that $w$ is given by

$$
w(x_0) = w_{n_0} (2\pi \alpha')^{(26-n_0)/2} \delta^{(26-n_0)}(x_0^2 - 0), \quad (5.8)
$$
where \( x_0^i \) is the spacetime coordinate perpendicular to the D\((n_0 - 1)\)-brane. Then \( I_{\text{tad}}(n_0) \) is obtained as

\[
I_{\text{tad}}(n_0) = \int_0^\infty \frac{dt}{2t} \prod_{\mu} \int \frac{dx_0^\mu}{\sqrt{2\pi \alpha'}} w(x_0^\mu) Z_1^{\text{tad}}(t, x_0^\mu, n_0) \frac{Z_0(n_0)}{Z_0(n_0)}
\]

\[
= w_{n_0} \int_0^\infty \frac{dt}{2t^2} e^{2\pi/t} \left( \frac{t}{\pi} \right)^{26-n_0} \frac{1}{2\pi} \prod_{k=1}^{\infty} \left[ (1 - e^{-2\pi k/t})^{n_0 - 24} (1 + e^{-2\pi k/t})^{-n_0} \right]
\]

\[
= 2^{(26-n_0)/2} w_{n_0} \int_0^\infty \frac{ds}{4\pi} e^{s(n_0-26)/2} \prod_{k=1}^{\infty} \left[ (1 - e^{-ks})^{n_0 - 24} (1 + e^{-ks})^{-n_0} \right]
\]

\[
= 2^{(26-n_0)/2} w_{n_0} \int_0^\infty \frac{ds}{4\pi} e^{s(n_0-26)/2} \left[ e^s + (24 - 2n_0) + O(e^{-s}) \right]. \quad (5.9)
\]

In this case, if we set \( w_{n_0} \) as

\[
w_{n_0} = -\pi^{(26-n_0)/2} h, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
The divergence in the open-string channel occurs in the large $t$ region, in general. However, if $n_0 = 0$, $I_{\text{loop}}(n_0) + I_{\text{tad}}(n_0)$ vanishes completely. Thus, the “D($-1$)-brane” has no instability due to open-string tachyons at order $g_c$ in this case. Of course, we cannot conclude immediately that bosonic D($-1$)-branes are stable on the nontrivial background, which the procedure in §5.3 with the choice of the weight as (5.8) and (5.10) creates. However, the above considerations seem to imply the possibility of the existence of some nontrivial stable vacuum for open bosonic strings. Further study is needed to obtain any definite conclusion in this regard. The any case, we can calculate the modified energy density $Z_{\text{total}}/V_{n_0-1}$ of open-string tachyons as (5.1) up to $O(g_c^2)$, using the above procedure. If Sen’s conjecture still holds for the case in which we include $O(g_c)$ corrections, the energy density $Z_{\text{total}}/V_{n_0-1}$ corresponds to the corrected tension of the D($n_0-1$)-brane.

§6. Conclusion and discussion

We have proposed a basic strategy to obtain a modified on-shell effective action for open-string tachyons, which includes loop corrections and the effect of closed-string tachyon condensation. The basic tool we used is the sigma-model approach. We included a boundary term in the worldsheet action to utilize the results of BSFT in defining the on-shell partition function. To incorporate the effect of closed-string tachyon condensation, we inserted D-instanton-like macroscopic holes into the world-sheets, and we regarded them as closed-string tachyon tadpoles. We presented a natural procedure to obtain a consistent effective action, using the tadpoles. The procedure we proposed provides a constraint for the weight that is attached to the tadpole, although we cannot determine the weight rigorously at this stage. The instability due to closed-string tachyons vanishes through this procedure. We obtain a finite energy density of open-string tachyons with this method, in principle. The most important problem is determining how to derive the weight $w$ rigorously. At present, this involves an ambiguity. The rigorous determination of $w$ is left as an important future project.

Now we give some remarks. First, the sigma-model approach using the results of BSFT presented in this paper can be used to estimate the energy density of open-string tachyons even if we do not consider the effect of closed-string tachyons, in principle, as $Z_{\text{total}}$ with $w = 0$ represents the on-shell effective action of open-string tachyons with one-loop quantum corrections.\(^{\ast}\) Therefore, the method proposed here should be useful to study the quantum corrections to the D-brane tension if we regard $Z_{\text{total}}/V_{n_0-1}$ as the tension of a D($n_0 - 1$)-brane.\(^{\ast\ast}\) However, there is an unresolved point. We took the conformal limit while imposing the stationary condition (3.10) in the derivation of the on-shell effective action of open-string tachyons. The stationary condition (3.10) is the equation of motion for the variable $a$ in the framework of BSFT, as mentioned in the Appendix. In this sense, when we include

\(^{\ast}\) In this case, we should consider, for example, the Fischler-Susskind mechanism or $SO(2^{13})$ unoriented strings to eliminate the conformal anomaly from loop diagrams.

\(^{\ast\ast}\) A calculation of quantum corrections to the tension of D-branes is found in Ref. 63).
$O(g_c)$ corrections in the effective action, the condition (3-10) can possess a term
\[
a = \sum_{i=1}^{25} \left( -u_i + u_i \frac{\partial}{\partial u_i} \ln Z_0(u_i) \right) + \sum_s \left( \frac{1}{2} \frac{u_s x_s^2}{2 \alpha'} \right) + g_c f(u_i),
\]
as a correction, where $f(u_i)$ is a function of the $u_i$ that we cannot determine rigorously at this stage. If $f(u_i)$ does not vanish in the conformal limit, we have an extra factor $e^{-g_c f}$ in the on-shell effective action, and then we have an additional $O(g_c)$ correction term $-g_c f Z_0$ in the effective action. So far, we have ignored the possibility of the existence of the above correction. We cannot make a rigorous argument to treat this problem, since we do not have an extended BSFT including $O(g_c)$ corrections, from which we should derive the explicit form of $f(u_i)$. Thus there is room for further investigation of this problem. In this article, we simply assume that $f(u_i)$ is not divergent and that it does not affect the procedure for obtaining a consistent on-shell effective action presented in §5, even if $f(u_i)$ is non-zero and finite in the conformal limit.

Second, the most difficult problem regarding the topic of this paper is the rigorous determination of the weight $w(x_0^\mu)$. It would be interesting to attempt utilizing string dualities to obtain some useful information concerning the weight.

Third, the choice of the weight in (5.8) and (5.10) means that the D-instanton-like tadpoles are localized on the D($n_0 - 1$)-brane. Inhomogeneous distributions of closed-string tadpoles like this also suggest the modification of the spacetime structure from flat spacetime. It is pointed out that the spacetime positions where we can fix ends of strings using Dirichlet boundary conditions are restricted in the subspace of the total target space if dilatons have a nontrivial dependence on the spacetime coordinates.\(^{36}\) We have to check the consistency of the Dirichlet boundary conditions we used with the configuration of the background dilatons created by the Fischler-Susskind mechanism. We leave this problem for further investigation. To study the relation between closed-string tachyon condensation and the dynamical nature of spacetime structure is a very interesting and an important project.

Fourth, there is also a fundamental question concerning the value of $w$. We implicitly assumed that $g_c w$ is sufficiently small, so that the expansion with respect to the topology of worldsheets gives us a good approximation. However, the expectation value of fields in point-particle field theories can be of the order of $(\text{coupling})^{-1}$ in general. Actually, all the diagrams with tadpoles without loops are at the tree level, for any number of the attached tadpoles, in point-particle field theories. In this sense, $g_c w$ also can be of order 1, and we might have to consider all the worldsheets that include multiple tadpoles, as well as the annulus $M$ we consider in this paper. This problem will become clear if we obtain the correct value of $w$. In any case, the basic strategy we presented in this paper to obtain a consistent effective action is still natural, though the cancellation of the imaginary parts only within the annulus diagrams, done in §5, is not correct in this case.

Finally, we treated oriented strings in this paper for simplicity. If we consider unoriented strings, we have another method to eliminate the fatal divergence; we can cancel this divergence if we include cross-cap diagrams and make an appropriate
choice of the gauge group. In this case, the situation seems to be more complicated. We may be able to make the effective action finite and remove the instability due to closed-string tachyons using a combination of several methods, the Fischler-Susskind mechanism, the inclusion of cross caps with the appropriate choice of gauge group, and a suitable choice of the weight of closed-string tachyon tadpoles. The required gauge group could be different from the ordinary $SO(2^{13})$ group, in general. Consideration of the spacetime gauge anomaly is also important to make the model consistent. Construction of consistent (anomaly free) string theories using D-instanton-like objects with appropriate weights, as first pointed out by Green in Ref. 28), may be interesting, even without considering the problem of tachyons.

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Appendix A

A Comment on the Stationary Conditions in BSFT

There are some delicate problems in taking the conformal limit in BSFT. We discuss them in detail here. Let us consider the stationary condition for the variable $a$. The condition $\frac{\partial}{\partial a}S(a, u_i) = 0$ for the action given by (3.7) and (3.8) yields this condition as

$$a = \sum_{i=1}^{26} \left( -u_i + u_i \frac{\partial}{\partial u_i} \ln Z_0(u_i) \right).$$  \hfill (A.1)

However, we find that (A.1) gives

$$\lim_{u_i \to 0} a = \frac{-26}{2} \neq 0$$ \hfill (A.2)

if we take the limit $u_i \to 0$. This implies that the equation of motion for $a$, $\frac{\partial}{\partial a}S(a, u_i) = 0$, is not compatible with the conformally invariant solution $(a_*, u_{i*}) = (0, 0)$. This is not a desirable situation.

We find the same problem in another place. For small values of $u_i$ we obtain

$$S(a, u_i) = A(a + 1 + \frac{26}{2})e^{-a} \prod_{i=1}^{26} \frac{1}{\sqrt{u_i}}$$

$$+ Ae^{-a} \left( \sum_{i=1}^{26} u_i \right) \prod_{i=1}^{26} \frac{1}{\sqrt{u_i}}$$

$$+ \cdots,$$ \hfill (A.3)

where the first term corresponds to the potential term and the second term corresponds to the kinetic term in the derivative truncated action $S(T_{\text{open}})$ for open-string
tachyons $T_{\text{open}}$.\(^{16),17)} \) $S(T_{\text{open}})$ is given by \(^{16),17)}$

$$
S(T_{\text{open}}) = A \int \frac{d^{26}x}{\sqrt{2\pi\alpha^\prime}^{26}} \left\{ \alpha^\prime e^{-T_{\text{open}}(\partial T_{\text{open}})^2} + e^{-T_{\text{open}}(1 + T_{\text{open}})} \right\}, \quad (A.4)
$$

where we have defined $T_{\text{open}}$ as $T_{\text{open}} = a + \sum_{i=1}^{26} u_i x_i^2$, with $x_i$ the zero mode of $X^i$. Here we find from (A.4) that the factor $A$ should be fixed as $A = T_{25}^2 (2\pi\alpha^\prime)^{13}$. The potential term in (A.4) becomes

$$
A \int \frac{d^{26}x}{\sqrt{2\pi\alpha^\prime}^{26}} e^{-a(1 + a)} \quad (A.5)
$$
in the small $u_i$ region, while (A.3) implies

$$
\text{potential} \propto e^{-a\left(a + 1 + \frac{26}{2}\right)}. \quad (A.6)
$$

Here we again find the additional term $\frac{26}{2}$ in (A.6).

The origin of the additional term is found in the following calculation. If we calculate the potential term in (A.4) explicitly, we obtain

$$
A \int \frac{d^{26}x}{\sqrt{2\pi\alpha^\prime}^{26}} e^{-T_{\text{open}}(1 + T_{\text{open}})} = A e^{-a(a + 1)} \int \frac{d^{26}x}{\sqrt{2\pi\alpha^\prime}^{26}} \exp \left\{ - \sum_{i=1}^{26} u_i x_i^2 \right\}
$$

$$
+ A e^{-a} \int \frac{d^{26}x}{\sqrt{2\pi\alpha^\prime}^{26}} \frac{u_i x_i^2}{2\alpha^\prime} \exp \left\{ - \sum_{i=1}^{26} \frac{u_i x_i^2}{2\alpha^\prime} \right\}. \quad (A.7)
$$

The first term on the right-hand side gives (A.5). We emphasize that the second term needs delicate treatment when we take the conformal limit $u_i \to 0$. If we take the limit before we perform the integration over $x^i$, the second term in (A.7) vanishes, while if we take the limit after the integration, it does not vanish, and it yields an additional term.

We have learned from the above discussion that we should take the conformal limit $u_i \to 0$ before we perform the integration over $x^i$. However, the BSFT action $S(a, u_i)$ does not possess the integral explicitly; this integration has already been performed as an integration over the zero modes of the $X^i$ when we perform the path integral in the derivation of the partition function $Z_0$. Thus, we apply the following trick. First, we insert the identity

$$
1 = \int \frac{d^{26}y}{\sqrt{\pi}^{26}} \exp \left\{ - \sum_{i=1}^{26} u_i y_i^2 \right\}
$$

$$
= \int \frac{d^{26}x}{\sqrt{b\pi\alpha^\prime}^{26}} \exp \left\{ - \sum_{i=1}^{26} \frac{u_i x_i^2}{b\alpha^\prime} \right\}, \quad (\text{for } u_i > 0) \quad (A.8)
$$

into the partition function $Z_0$, where $y^i$ is a dimensionless parameter. We have also introduced the parameter $b$ as

$$
y^i = \frac{x^i}{\sqrt{b\alpha^\prime}}, \quad (A.9)
$$
Now, we wish to regard $x^i$ in (A.9) as the zero mode of $X^i$, which corresponds to the spacetime coordinates. In the partition function $Z_0$, we have the integral over the zero mode as

$$
\int d^{26}x \exp\left\{-a - \sum_{i=1}^{26} \frac{u_ix_i^2}{2\alpha'}\right\},
$$

(A.10)

which is extracted from (3.4) and the boundary term in (3.5). Therefore we naturally set $b = 2$, and we insert

$$
1 = \int \frac{d^{26}x}{\sqrt{2\pi\alpha'}} \exp\left\{-\sum_{i=1}^{26} \frac{u_ix_i^2}{2\alpha'}\right\}
$$

(A.11)

into $Z_0$. Then the action $S(a, u_i)$ is rewritten as

$$
S(a, u_i) = \int \frac{d^{26}x}{\sqrt{2\pi\alpha'}} \mathcal{L}(a, u_i, x^i),
$$

(A.12)

where

$$
\mathcal{L}(a, u_i, x^i) = Z_0^x \left(1 + a + \sum_i u_i - \sum_i u_i \frac{\partial}{\partial u_i} \ln Z_0^x\right),
$$

$$
Z_0^x(a, u_i, x^i) = A e^{-a} \prod_{i=1}^{26} u_i e^{\gamma u_i} \Gamma(u_i) e^{-\frac{u_ix_i^2}{2\alpha'}}.
$$

(A.13)

Taking the conformal limit $u_i \to 0$ before the integration over the spacetime coordinates means that we should take the limit while enforcing the condition $\frac{\partial}{\partial a} \mathcal{L} = 0$, not the condition $\frac{\partial}{\partial a} S = 0$. Then we obtain the condition

$$
a = \sum_{i=1}^{26} \left(-u_i + u_i \frac{\partial}{\partial u_i} \ln Z_0^x\right)
$$

$$
= \sum_{i=1}^{26} \left(-u_i + u_i \frac{\partial}{\partial u_i} \ln Z_0 + \frac{1}{2} - \frac{u_ix_i^2}{2\alpha'}\right),
$$

(A.14)

instead of (A.1), from $\frac{\partial}{\partial a} \mathcal{L} = 0$. The condition (A.14) gives $a \to 0$ when we take $u_i \to 0$. We also find

$$
\lim_{u_i \to 0} \mathcal{L} = (1 + a)e^{-a}
$$

(A.15)

if we take the limit subject to (A.14). Therefore, our problem is solved.

However, we encounter the opposite situation when we take the limit $u_i \to \infty$. In this case we find that we should take the limit $u_i \to \infty$ after the integration over the spacetime coordinates. For example, if we take $u_1 \to \infty$ while maintaining the condition (A.14), we find that $\mathcal{L}$ diverges as $\mathcal{L} \to O(\sqrt{\alpha'})$. On the other hand, we obtain the correct result

$$
\lim_{u_1 \to \infty} S(a, u_i) = \frac{2\pi\sqrt{\alpha'}}{\text{Vol}} S(0, 0),
$$

(A.16)
where Vol is the spacetime volume in the $x^1$ direction, if we take the limit $u_1 \to \infty$ after the integration over the spacetime coordinate $x^1$. Thus, we obtain the correct ratio of tensions of D-branes if we take the limit while enforcing the condition

$$a = -u_1 + u_1 \frac{\partial}{\partial u_1} \ln Z_0,$$

(A.17)

which corresponds to (A.1). In this case, we take $u_i \to 0$ subject to (A.14) for $i \neq 1$. The condition (A.17) is used to derive the ratio of tensions of D-branes in Ref. 17).

We thus find that the most suitable stationary condition is

$$a = \sum_{i=1}^{26} \left( -u_i + u_i \frac{\partial}{\partial u_i} \ln Z_0 \right) + \sum_s \left( \frac{1}{2} - \frac{u_s x_s^2}{2\alpha'} \right),$$

(A.18)

where the $u_s$ are the variables that are taken to zero in the conformal limit. That is, we have Neumann boundary conditions in the $x^s$ direction after we take the conformal limit and the number of coordinates $x^s$ is $n_0$. This is the condition given in (3.10). In this case, $S(a, u_i)$ should be written

$$S(a, u_i) = \left( \prod_s \int \frac{dx^s}{\sqrt{2\pi\alpha'}} \right) Z_0 \left( 1 + a + \sum_i u_i - \sum_i u_i \frac{\partial}{\partial u_i} \ln Z_0 \right),$$

$$Z_0(a, u_i, x^s) = A e^{-a} \left( \prod_{i=1}^{26} \sqrt{u_i} e^{\gamma u_i} \Gamma(u_i) \right) \prod_s \sqrt{u_s} e^{-\frac{u_s x_s^2}{2\alpha'}}.$$  

(A.19)

Note that only the integration over $x^s$ has to be restored. We can easily check that $S(a, u_i) = Z(a, u_i)$ if the stationary condition (A.18) is satisfied. Of course, the variable $a$ should be inside the integral over $x^s$ when we use the condition (A.18). Thus we should take the conformal limit while maintaining (A.18), which is also given in (3.10), and we have to rewrite $S(a, u_i) = Z(a, u_i)$ in the form of (A.19). The zero-mode integrals in the $x^s$ directions should be performed after we take the limit $u_s \to 0$. We can obtain the correct value for the on-shell effective action of open-string tachyons if we use the stationary condition (A.18).

We now give a final comment. The second term in (A.7) always vanishes if we consider BSFT in a finite volume spacetime. In this case, we can use the condition (A.1). Therefore, considering a finite volume spacetime as an IR regularization and using the condition (A.1) represents another solution.

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