Radiation Efficiency Cost of Resonance Tuning

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Abstract—Existing optimization methods are used to calculate the upper-bounds on radiation efficiency with and without the constraint on self-resonance. These bounds are used for the design and assessment of small electric-dipole-type antennas. We demonstrate that the assumption of lossless, lumped, external tuning skews the true nature of radiation efficiency bounds when practical material characteristics are used in the tuning network. A major result is that, when realistic (e.g., finite conductivity) materials are used, small antenna systems exhibit dissipation factors which scale as $(ka)^{-4}$, rather than $(ka)^{-2}$ as previously predicted under the assumption of lossless external tuning.

Index Terms—Radiation efficiency, antenna theory, optimization methods.

I. INTRODUCTION

RADIATION efficiency is a parameter of paramount importance for electrically small radiators since it significantly limits antenna performance [1]. Techniques for maximizing radiation efficiency [2], [3] or attempts to set physical bounds [4]–[10] on this parameter therefore naturally accompany developments in antenna technology.

In most cases, it is desirable to design a small antenna to be tuned (i.e., resonant) at a specified frequency. This can be accomplished either by designing the antenna itself to be self-resonant or through the use of an external tuning network. Although the high cost of resonance tuning in radiation efficiency was recognized long ago [3], many recent works assume that resonance tuning can be done in a lossless manner using external networks [4]–[7]. In fact, careful review reveals that this is a common assumption in many standard textbooks [1], [11], [12] as well. This assumption, however, leads to physical bounds which are unachievable by realistically tuned antenna designs [13]–[15].

Recently, the effect of resonance tuning has once more been taken into account by two different paradigms. The first approach [16], [17] used full-wave treatment of optimal currents on arbitrarily shaped lossy surfaces. The second employed spherical wave expansion [8] reaching analytic resonant bounds for spherical geometries.

The purpose of this paper is to show that the effect of resonant tuning on the radiation efficiency of electrically small antennas can be evaluated precisely for arbitrary surface current supports. Furthermore, we demonstrate that for electrically small antennas resonance tuning using realistically lossy materials leads to an unpleasant quartic frequency scaling of dissipation factor. This opposes the optimistic quadratic frequency scaling predicted by the bounds derived for systems externally tuned by lossless lumped circuits.

This paper is organized as follows. Sections II introduces definitions and restricting assumptions common to the entire manuscript. Section III introduces the radiation efficiency cost of resonance tuning on a canonical antenna example. Sections IV and V then show that the introductory observation is of general validity by presenting self-resonant radiation efficiency bounds. The bounds are compared to several realistic designs in Section VI. Paper is concluded in Section VII.

II. ASSUMPTIONS AND DEFINITIONS

Here we introduce several definitions and assumptions which help to obtain a mathematically tractable problem:

1) Time-harmonic steady state is assumed with angular frequency $\omega$.

2) An antenna is assumed to be tuned to resonance at a given frequency either by antenna current shaping (designed for self-resonance) or by a (potentially lossy) external lumped reactance connected to the antenna terminals.

3) An antenna and potential tuning element are made solely of a resistive sheet of given surface resistance $R_s$. No other material bodies are allowed.

4) When particular values of surface resistance $R_s$ are desired, the skin effect model $R_s = \sqrt{(\omega \mu) / (2\sigma)}$ is used with $\mu$ being a permeability and $\sigma$ being a conductivity. This model corresponds [18] to a metal sheet of thickness much higher than the penetration depth on which a current flows on one side only.

5) Within this paper, the radiation efficiency is defined as $\eta = 1 / (1 + \delta)$ with dissipation factor $\delta$ [19] being the ratio of cycle mean power lost by heat $P_{\text{loss}}$ to cycle mean power lost by radiation $P_{\text{rad}}$. The power $P_{\text{loss}}$ takes into account conduction losses in the antenna body as well as in the tuning network. By assumptions 3) and 4), conduction losses are the only thermal losses in the system. This definition of radiation efficiency is equivalent to that given in the IEEE Standard [20] for metallic antennas, when the matching network made of the same material is considered as a part of the antenna.

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system\(^1\).
6) When explicitly needed, the radiation efficiency and dissipation factor counting losses on the untuned antenna structure alone are denoted \(\eta_A\) and \(\delta_A\), respectively.
7) When an antenna with a well defined input port is tuned to resonance by a series lumped element with impedance \(Z_T = R_T + jX_T\), the dissipation factor of the entire system is given by [3]
\[
\delta = \delta_A \left( 1 + \frac{|X_A|}{Q_T R_{loss}} \right),
\]
where \(Z_A = R_{rad} + R_{loss} + jX_A\) is the impedance of the antenna, \(R_{rad}\) and \(R_{loss}\) distinguish between radiation and ohmic losses [22], and where \(Q_T = |X_T|/R_T\) is the Q-factor of the tuning element.
8) When applying (1), we assume in this paper that small losses inside tuning capacitors can be neglected \((Q_T \to \infty)\), while metallic conductance losses inside tuning inductors must be taken into account.

III. INTRODUCTORY EXAMPLE

As a motivating example, consider a practical HF band \((3 - 30\text{ MHz})\) scenario in which an electrically short dipole antenna is tuned to resonance by a series tuning coil. Assume the dipole antenna to be of total length \(\ell = 5\text{ m}\) and made of AWG 6 copper wire (2.055 mm radius) [23]. Calculations are carried out from 1 MHz \((\ell/\lambda = 0.0167, ka = 0.0524)\) to 28.9 MHz \((\ell/\lambda = 0.482, ka = 1.51)\), where \(k\) is the wavenumber, \(\lambda\) is the wavelength and \(a \approx \ell/2\) is the radius of the smallest sphere circumscribing the dipole. Note that the highest frequency is selected to be the self-resonance of the antenna, where no tuning inductance is required. The impedance \(Z_A\) and radiation efficiency \(\eta_A\) of the dipole antenna alone are calculated using NEC2++ [24] for the surface resistivity model shown in Section II with conductivity \(\sigma = 5.8 \times 10^7\text{ S m}^{-1}\). The results are shown in Fig. 1a and compared with an analytical prediction detailed in Appendix A.

Fig. 1. Simulated radiation efficiency \(\eta_A\) (top) and normalized dissipation factor \(\delta_A\) (bottom) of a 5 m dipole antenna made of copper wire of 2.055 mm radius.

When the dissipation factor corresponding to Fig. 1a is evaluated and normalized by the surface resistance \(R_s\), it follows the \((ka)^{-2}\) trend expected for electrically small dipole radiators [11], as can be seen from Fig. 1b. Normalization by vacuum impedance \(Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 120\pi\Omega\) maintains a unitless ordinate.

Up until this point, only the properties of the untuned antenna have been considered. To include the dissipation inside the tuning inductor, Q-factors \(Q_T\) for commercially available air-core inductors were obtained from [25]. The Q-factors values normalized by the frequency-dependent resistance of copper and frequency are shown in Fig. 2. These inductors are thus characterized with \(Q_T \approx 0.7 \times 10^{-9} \omega/R_s\) with little dependence on the value of inductance. This value of \(Q_T\) can thus (for this type of inductor) be substituted to (1) from which the radiation efficiency of the antenna plus the tuning element can be calculated. The result of this calculation is shown in Fig. 3a and Fig. 3b in absolute and normalized forms.

Comparison of Fig. 1b and Fig. 3b suggests that when losses in tuning elements are taken into account the following hypotheses are worthy of study:
- The radiation efficiency cost of resonance tuning is high, the most important contribution being the lossy tuning element.
- Properly normalized dissipation factor of a resonant antenna (self-resonant or tuned) follows a \((ka)^{-4}\) trend.
- Dissipation factor normalized as \((Z_0/R_s) (ka)^{\delta}\) depends\(^2\), in the electrically small regime, almost exclusively on the shape of an antenna and tuning inductor.

Following sections aim to show that the aforementioned

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\(^1\)The IEEE Standard Definition [20] of “antenna” as “That part of a transmitting or receiving system that is designed to radiate or to receive electromagnetic waves.” leaves room for interpretation in this regard, particularly for electrically small systems where a radiator and matching or loading elements are constructed of similar materials. Consider for example the equivalence between an inductively base-loaded short monopole and the same monopole tuned by an identical external coil [21].

\(^2\)Radiation efficiency cannot be easily normalized to remove the explicit dependence on size and material parameters. Exceptions are cases when \(\delta \gg 1\) or \(\delta \ll 1\).
IV. MATHEMATICAL TOOLS

The assumption of an antenna as an arbitrary surface \( S \) in \( \mathbb{R}^3 \) allows us to employ the electric field integral equation [26]

\[
-\hat{n} \times (\hat{n} \times (E_s(J) + E_i)) = R_s J \quad \text{on} \quad S,
\]

where \( E_s \) is the scattered electric field, \( E_i \) the incident electric field, \( J \) a surface current density, \( R_s \) the surface resistivity, and \( \hat{n} \) a unit normal to the surface \( S \). For computational purposes the fields and currents tangential to the surface can be modeled as a weighted sum of appropriate basis functions \( \{\psi_n\} \), i.e.,

\[
J(r) \approx \sum_n I_n \psi_n(r),
\]

in order to recast (2) into its matrix form [26]

\[
(Z + R_s \Psi) I = V,
\]

where \( I \) is a vector of expansion coefficients, \( V \) the excitation vector, \( Z = R + jX \) the impedance matrix [26] and \( \Psi \) the Gram matrix [16]. Throughout the remainder of this paper, we use Rao-Wilton-Glisson (RWG) basis functions [27] to represent current densities on simple surfaces, though alternative basis functions may be beneficial or necessary in accurately modeling certain physical or geometrical features (e.g., spherical shells, wires, or point contacts).

With the help of the aforementioned matrix formulation the complex power [28] can be written as

\[
P_{\text{rad}} + jP_{\text{react}} \approx \frac{1}{2} I^H (R + jX) I,
\]

and the cycle mean power lost as heat [28] can be written as

\[
P_{\text{loss}} \approx \frac{R_s}{2} I^H \Phi I.
\]

V. OPTIMAL CURRENT MAXIMIZING RADIATION EFFICIENCY

The classical procedure to find the current distribution on the surface \( S \) which maximizes radiation efficiency is to solve [19], [29]

\[
\Psi I_n = \delta_n R I_n,
\]

where matrices \( \Psi \) and \( R \) represent the antenna only, and take the current corresponding to its lowest eigenvalue. The resulting current distribution minimizes normalized dissipation factor \( \delta_A/R_s \) and thus maximizes radiation efficiency \( \eta_A \).

The solution to (7) is not necessarily self-resonant. If resonance is required, the dissipation factor obtained in (7) can only be achieved if there exists a lossless lumped element \( (Q_T \to \infty) \) that can tune the current to resonance without affecting dissipation [1], [4]–[7], [11], [12]. On planar regions this method generates constant current density which is an analytic solution to radiation efficiency maximization in the \( ka \to 0 \) limit [7]. Otherwise, the method generally tries to make the current distribution as uniform as possible. This solution neglects the effect of resonance tuning and is depicted for several canonical shapes in Fig. 4 as a function of electrical size \( ka \). Note that these curves scale as \((ka)^{-2}\) for \( ka < 1 \).

The additional constraint on self-resonance can be incorporated as [16], [17]

\[
\text{minimize} \quad I^H \Phi I,
\]

subject to

\[
I^H R I = 1,
\]

\[
I^H X I = 0.
\]

This optimization problem directly yields the normalized dissipation factor \( \delta_A/R_s \). As shown in Appendix B, its global optimum can be found in a deterministic way. Sample code in [30] shows a possible implementation of this procedure and Appendix C shows convergence of the results for increasing number of discretization elements.

The results generated by (8) are depicted by dashed lines in Fig. 4 for the same problems previously considered. The difference between solid and dashed curves in Fig. 4 at small electrical sizes shows the radiation efficiency cost of resonance tuning. The tuning cost is most easily described by a change from \((ka)^{-2}\) to \((ka)^{-4}\) scaling, which agrees well with the example shown in Section III and with the findings on a spherical shell [8], [9]. Results presented in Fig. 4 show that this phenomenon is of general nature for many electrically small objects.

The self-resonant current maximizing radiation efficiency on a rectangular support (Fig. 4, rectangular marks) is shown in Fig. 5a. This current shape is approximately optimal in the full frequency range of Fig. 4 and resembles a combination of electric-dipole-like and magnetic-dipole-like currents as suggested in [8], [9]. In fact, if the optimal current is evaluated on a spherical shell at small electrical sizes (Fig. 4, triangular marks), it precisely leads to a resonant combination of TM\(_{10}\) and TE\(_{10}\) spherical modes [8], [9].

Fig. 3. Calculated radiation efficiency (top) and normalized dissipation factor (bottom) of the dipole shown in Fig. 1 tuned by the non-ideal inductors presented in Fig. 2.
VI. COMPARISON OF BOUNDS WITH REALISTIC DESIGNS

A. Self-Resonant Antennas

The globally optimal current density depicted in Fig. 5a is difficult to realize as a driven antenna current in practice, especially when a design is restricted to have only one localized feed. The method described in [16], however, yields also all local optima of the problem in (8). The local optimum with the second lowest dissipation factor is depicted in Fig. 5b. The depicted current density suggests that structures from Fig. 6, which resemble a Julgalt pastry [31] and a Palmier pastry [32], could be good candidates for approaching the radiation efficiency bound. That this is the case is shown in Fig. 7 although it must be admitted that neither of the structures approach the bound closely (having dissipation factor at least six times higher than the bound).

Though the designs presented here are not necessarily the optimal antenna geometries for attaining maximum radiation efficiency, it has been shown in [14, design PMD2] that these designs have the highest radiation efficiency among planar meander designs. Despite of the potential suboptimality, both designs follow the \((ka)^{-4}\) trend predicted for self-resonant radiation efficiency bounds.

B. Antennas Externally Tuned by Realistic Components

Next, we will extend the analysis from Section III and show that antennas externally tuned by realistic components do not surpass the self-resonant bound and, in fact, stay well above the self-resonant spiral meanders when restricted to the same rectangular support. A fat dipole antenna, a bowtie antenna, and a rectangular loop antenna will be used as particular designs, see insets in Fig. 8. The first two examples are chosen for their space-filling properties, being inspired by the uniform current predicted by (7). The loop is chosen for possibility to use lumped capacitor, i.e., low-loss component, at small electrical sizes to achieve resonance.

Let us deal first with electric dipole antennas, a fat dipole and a bowtie, which will be tuned to resonance by realistic lossy inductors. To judge their radiation efficiency performance, imagine that one would desire the dissipation factor (1)
of an externally tuned antenna to be that of the self-resonant bound from Fig. 7 (circular marks). The relation in (1) can then be used to extract the Q-factor of the inductor $Q_T$ that would be necessary to achieve this goal. The resulting required inductor Q-factors $Q_T$ for the fat dipole and bowtie antennas are depicted in Fig. 8.

The question now stands if this required inductor Q-factor is achievable by realizable inductors. The negative answer is supported by an example of a planar spiral inductor and a helical inductor, which Q-factors are depicted in Fig. 9. In both cases, the inductors are made of the same material as the antenna, but due to the used normalization the particular choice of the material is of no relevance. To approximate the assumption of lumped tuning, let us suppose that the inductors are at least ten times smaller in electrical size, i.e., $k\alpha_T = k\alpha/10$. In this case the required tuning Q-factors from Fig. 8 are at least an order of magnitude higher than the realizable Q-factors from Fig. 9 and it is important to stress that this conclusion is just weakly dependent on electrical size and is independent on the inductor material. Note here that when for $k\alpha_T = k\alpha/N$, the values from Fig. 8 can directly be compared to values from Fig. 9 divided by $N$.

For comparison, the bowtie and fat dipole antennas tuned by planar spiral inductors are shown in Fig. 7. It can be seen that both externally tuned antennas have dissipation factors well above those of the self-resonant spiral meanders.

The case of externally tunned loop antenna differs from electric dipole antennas treated previously. Below the antiresonance of the loop the tuning can be done by low-loss capacitor (here we assume $Q_T \rightarrow \infty$) while above the antiresonance the tuning can be done in the same way as for the bowtie or fat-dipole antenna. The result of this tuning procedure is shown in Fig. 7. For the particular geometry used, the result is such that dissipation factor $\delta_A$ of the loop alone is already significantly higher than the self-resonant bound for rectangular support and there is thus no possibility to find a tuning network (not even lossless) with which the antenna will reach the radiation efficiency bound. The reason behind this result is that electrically small loop-like current exhibits dissipation factor that already scales with frequency as $(k\alpha)^{-4}$ and the tuning network can only worsen this behavior. In other words, it can be stated that a high cost of resonance tuning in all electrically small radiators is presented by the loop-like current that is used for resonance tuning.

One may ask whether using physically large inductors or superconducting materials in the matching network is a way around the dissipation factor scaling with $(k\alpha)^{-4}$. If matching components are made larger while still being considered approximately lumped (e.g., $k\alpha_T = k\alpha/8$ rather than $k\alpha_T = k\alpha/10$ in the preceding analysis), then above analysis holds, though the precise numerical results may slightly change. If, however, the tuning network is made similar in size to the antenna itself, then it becomes appropriate to include the spatial support of the tuning network into the derivation of the radiation efficiency bounds. Hence, the tuning network becomes part of the current optimization problem in (8) where it may even be leveraged as a source of radiation. In any case the dissipation factor of the system will scale as $(k\alpha)^{-4}$.

**VII. Conclusion**

It has been shown that radiation efficiency bounds assuming external tuning by lossless lumped elements are overly optimistic and that tighter self-resonant bounds can easily
be calculated. It was further demonstrated that selected self-resonant antennas can approach this bound and that their radiation efficiency surpasses that of the non-resonant antennas which one maximizes so-called dual function \( g(\lambda_1, \lambda_2) = \inf \left[ \mathcal{L}(I, \lambda_1, \lambda_2) \right] \), \( g(\lambda_1, \lambda_2) \) is the Lagrangian corresponding to (8) and inf denotes infimum. The supremum of dual function \( g(\lambda_1, \lambda_2) \) is the lower bound [34] to the original problem (8). In [35] it was however shown that for radiation problems of this kind there is no dual gap [34] present and the supremum of \( g(\lambda_1, \lambda_2) \) is the global optimum of (8). Since function \( g(\lambda_1, \lambda_2) \) is concave [34] it is assured that this global optimum can be approached to an arbitrary precision in a finite number of steps.

In the code available at [30], the supremum of \( g(\lambda_1, \lambda_2) \) is searched only among stationary points of the Lagrangian (15), which are guided by

\[
(\Psi - \lambda_1 X) I = \lambda_2 R I.
\]

This tremendously narrows the solution space and fixes the relation between Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \). The dual function (14) is therefore a function of single variable (\( \lambda_2 \)). Since function \( g(\lambda_1, \lambda_2) \) is concave [34] it is assured that this global optimum can be approached to an arbitrary precision in a finite number of steps.

This appendix briefly describes the method used to solve optimization problem (8). See [30] for an example of MATLAB implementation.

The solution is approached by a dual formulation [34] in which one maximizes so-called dual function

\[
g(\lambda_1, \lambda_2) = \inf \left[ \mathcal{L}(I, \lambda_1, \lambda_2) \right],
\]

where

\[
\mathcal{L}(I, \lambda_1, \lambda_2) = I^H \Psi I - \lambda_1 I^H X I - \lambda_2 (I^H R I - 1)
\]

is the Lagrangian corresponding to (8) and inf denotes infimum. The supremum of dual function \( g(\lambda_1, \lambda_2) \) is the lower bound [34] to the original problem (8). In [35] it was however shown that for radiation problems of this kind there is no dual gap [34] present and the supremum of \( g(\lambda_1, \lambda_2) \) is the global optimum of (8). Since function \( g(\lambda_1, \lambda_2) \) is concave [34] it is assured that this global optimum can be approached to an arbitrary precision in a finite number of steps.

The ratio of the loss resistance \( R_{\text{loss}} \) and the radiation resistance \( R_{\text{rad}} \) is the untuned dissipation factor. Expanding and taking the leading order term of the quotient of (11) and (10) yields

\[
\left( \frac{k \ell}{2} \right)^2 \frac{Z_0}{R_s} \delta_A \approx \frac{\ell}{r_w}.
\]

This agrees with the short dipole approximation of triangular current distribution [11] and demonstrates explicitly that, at small electrical size, the untuned dissipation factor scales as \((k \ell)^{-2}\), which is graphically presented in Fig. 1. Turning now to the tuned dissipation factor, formula (1) can be rewritten as

\[
\delta = \delta_A \left( 1 + \frac{k \ell |X_A|}{(k \ell)^2 \frac{Q_T R_s}{k \ell}} \frac{R_{\text{loss}}}{R_s} \right).
\]

At small electrical sizes \( ka \to 0 \) the input reactance of a wire dipole is capacitive \( X_A \propto k^{-1} \) and the term \( k \ell |X_A| \) is independent of frequency. The same holds for the terms \( Q_T R_s / (k \ell) \) and \( R_{\text{loss}} / R_s \) in (13) as can be seen from Fig. 2 and (11). This explicitly shows how the resonance tuning is responsible for the change from \((k \ell)^{-2}\) scaling to \((k \ell)^{-4}\) scaling when electrical size is small.

**APPENDIX B**

An analytical model used in Section III assumes a thin cylindrical dipole with length \( \ell \), radius \( r_w \) and current distribution

\[
I(\zeta) = I_0 \sin \left( \frac{k \ell}{2} - |\zeta| \right),
\]

where \( \zeta \in [-\ell/2, \ell/2] \). Radiation resistance \( R_{\text{rad}} \) of such a current reads [22]

\[
R_{\text{rad}} = \frac{Z_0}{\pi} \int_0^1 \frac{\left( \cos \left( \frac{k \ell}{2} \right) - \cos \left( \frac{k \ell}{2} \right) \zeta \right)^2}{\sin^2 \left( \frac{k \ell}{2} \right)(1 - \zeta^2)} d\zeta.
\]

Assuming the dipole made of a cylindrical surface (without end caps), which is covered by surface resistance \( R_s \), the loss resistance \( R_{\text{loss}} \) [22] can be calculated as

\[
\frac{R_{\text{loss}}}{R_s} = \left( \frac{\ell}{4 \pi r_w} \right) 1 - \frac{\sin (k \ell)}{k \ell}.
\]

This macrobasis is favorable in diagonalizing two of the three underlying operators. The unknown current vectors \( I \) and operators \( \Psi, X, R \) in the RWG basis are projected into this new macrobasis. This eases the computational burden since, to a high degree of precision, the optimal solution is composed of a few macrobasis functions [16].

This appendix briefly describes the method used to solve optimization problem (8). See [30] for an example of MATLAB implementation.

The solution is approached by a dual formulation [34] in which one maximizes so-called dual function

\[
g(\lambda_1, \lambda_2) = \inf \left[ \mathcal{L}(I, \lambda_1, \lambda_2) \right],
\]

where

\[
\mathcal{L}(I, \lambda_1, \lambda_2) = I^H \Psi I - \lambda_1 I^H X I - \lambda_2 (I^H R I - 1)
\]

The diameter of the helical inductor has been made slightly bigger than the inductor length, in order to achieve higher Q-factor [33]. The radius \( a_T \) describes a sphere circumscribing the inductor.
APPENDIX C

The lower bound on a self-resonant dissipation factor $\delta$ corresponding to a current flowing on a spherical shell is known analytically [9]. The spherical shell thus provides ideal grounds for testing the convergence of the optimization scheme outlined in Appendix B which was used to generate results in Fig. 4. To that point, several discretizations of a spherical shell have been used and the resulting self-tuned dissipation factors have been compared with the above-mentioned analytical result. The relative error $\varepsilon_{\text{err}}(T)$ of the numerical evaluation is shown in Fig. 10 for a particular choice of $T = \{72, 216, 600, 1176, 2400, 4056\}$ triangles.

![Graph showing relative error of self-resonant dissipation bound](image)

Fig. 10. Relative error of a self-resonant dissipation bound corresponding to a spherical shell of radius $a$ evaluated by the method detailed in Appendix B with respect to an analytical result presented in [9]. Numerical results were calculated in AToM [37].

It can be observed that the optimization scheme presented in Appendix B is numerically robust, achieving precision gain of approximately one digit per one order in number of triangles $T$. Although the analytical data for other shapes are not available, it can be expected that the precision for other canonical shapes presented in Fig. 4 will be similar, provided that their geometries and current paths are well represented by the chosen basis.

REFERENCES

[1] K. Fujimoto and H. Morishita, Modern Small Antennas. Cambridge University Press, 2013.
[2] G. S. Smith, “Radiation efficiency of electrically small multiturn loop antennas,” IEEE Trans. Antennas Propag., vol. 20, pp. 656–657, 1972.
[3] ——, “Efficiency of electrically small antennas combined with matching networks,” IEEE Trans. Antennas Propag., vol. 25, pp. 369–373, 1977.
[4] K. Fujita and H. Shirai, “A study of the antenna radiation efficiency for electrically small antennas,” in IEEE Antennas and Propagation Society International Symposium, 2013, pp. 1522–1523.
[5] A. Karlsson, “On the efficiency and gain of antennas,” Prog. Electromagn. Res., vol. 136, pp. 479–494, 2013.
[6] K. Fujita and H. Shirai, “Theoretical limitation of the radiation efficiency for homogenous electrically small antennas,” IEICE T. Electron., vol. E99C, pp. 2–7, 2015.
[7] M. Shalpari and D. V. Thiel, “Fundamental limitations for antenna radiation efficiency,” IEEE Trans. Antennas Propag., vol. 66, no. 8, pp. 3894–3901, 2018.
[8] C. Pfeiffer, “Fundamental efficiency limits for small metallic antennas,” IEEE Trans. Antennas Propag., vol. 65, pp. 1642–1650, 2017.
[9] V. Losenicky, L. Jelinek, M. Capek, and M. Gustafsson, “Dissipation factors of spherical current modes on multiple spherical layers,” IEEE Trans. Antennas Propag., 2018.
[10] H. L. Thal, “Radiation efficiency limits for elementary antenna shapes,” IEEE Trans. Antennas Propag., vol. PP, no. 99, pp. 1–1, 2018, early Access.
[11] W. L. Stutzman and G. A. Thiele, Antenna theory and design. John Wiley & Sons, 2012.
[12] J. L. Volakis, C. Chen, and K. Fujimoto, Small Antennas: Miniaturization Techniques & Applications. McGraw-Hill, 2010.
[13] S. R. Best and D. L. Hanna, “A performance comparison of fundamental small-antenna designs,” IEEE Antennas Propag. Mag., vol. 52, no. 1, pp. 47–70, Feb. 2010.
[14] S. R. Best and J. D. Morrow, “On the significance of current vector alignment in establishing the resonant frequency of small space-filling wire antennas,” IEEE Antennas Wireless Propag. Lett., vol. 2, pp. 201–204, 2003.
[15] S. R. Best, “Electrically small resonant planar antennas,” IEEE Antennas Propag. Mag., vol. 65, no. 1, pp. 329–341, Jan. 2017.
[16] L. Jelinek and M. Capek, “Optimal currents on arbitrarily shaped surfaces,” IEEE Trans. Antennas Propag., vol. 65, no. 3, pp. 38–47, June 2015.
[17] M. Gustafsson, M. Capek, and K. Schab, “Trade-off between antenna efficiency and Q-factor,” Electromagnetic Theory Department of Electrical and Information Technology Lund University Sweden, Tech. Rep., 2017. [Online]. Available: http://hup.lub.lu.se/record/48c12514-eba2-4493-b8a6-c8777051a002
[18] J. D. Jackson, Classical Electrodynamics, 3rd ed. Wiley, 1998.
[19] R. F. Harrington, “Antenna excitation for maximum gain,” IEEE Trans. Antennas Propag., vol. 13, no. 6, pp. 896–903, Nov. 1965.
[20] 145-2013 – IEEE Standard for Definitions of Terms for Antennas, IEEE Std., March 2014.
[21] C. Harrison, “Monopole with inductive loading,” IEEE Transactions on Antennas and Propagation, vol. 11, no. 4, pp. 394–400, July 1963.
[22] C. A. Balanis, Antenna Theory Analysis and Design, 3rd ed. Wiley, 2005.
[23] “Standard specification for standard nominal diameters and cross-sectional areas of AWG sizes of solid round wires used as electrical conductors,” ASTM International, West Conshohocken, PA, Standard, 2014.
[24] T. C. A. Molteno, “NEC2++: An NEC-2 compatible Numerical Electromagnetics Code”, Electronics Technical Reports, Tech. Rep., 2014.
[25] Square Air Core Inductors, Coilcraft, Oct. 2015, Document 720-2.
[26] R. F. Harrington, Field Computation by Moment Methods. Wiley – IEEE Press, 1993.
[27] S. M. Rao, D. R. Wilton, and A. W. Glisson, “Electromagnetic scattering by surfaces of arbitrary shape,” IEEE Trans. Antennas Propag., vol. 30, no. 3, pp. 409–418, May 1982.
[28] R. F. Harrington, Time-Harmonic Electromagnetic Fields, 2nd ed. Wiley –IEEE Press, 2001.
[29] M. Uzsoy and L. Solymár, “Theory of super-directive linear arrays,” Acta Physica Academiae Scientiarum Hungaricae, vol. 6, no. 2, pp. 185–205, December 1956.
[30] Optimal dissipation factor @ File Exchange. [Online]. Available: https://www.mathworks.com/matlabcentral/fileexchange/66412-optimal-dissipation-factor
[31] Julgall, Wikipedia, The Free Encyclopedia. [Online]. Available: https://sv.wikipedia.org/wiki/Lussekatt
[32] Palmier, Wikipedia, The Free Encyclopedia. [Online]. Available: https://en.wikipedia.org/wiki/Palmier
[33] H. Nagaoka, “The inductance coefficients of solenoids,” IEEE Trans. Antennas and Propagation, vol. 25, no. 4, pp. 329–341, May 1977.
[34] J. Nocedal and S. Wright, Numerical Optimization. New York, United States: Springer, 2006.
[35] M. Capek, M. Gustafsson, and K. Schab, “Minimization of antenna quality factor,” IEEE Trans. Antennas Propag., vol. 65, no. 8, pp. 4115–4123, Aug, 2017.
[36] Golden section search. Wikipedia, The Free Encyclopedia. [Online]. Available: https://en.wikipedia.org/wiki/Golden-section_search
[37] (2017) Antenna Toolbox for MATLAB (AToM). Czech Technical University in Prague. [Online]. Available: www.antennatoolbox.com