Abstract

We study Gaussian mechanism in the shuffle model of differential privacy (DP). Particularly, we characterize the mechanism’s Rényi differential privacy (RDP), showing that it is of the form:

$$\epsilon(\lambda) \leq \frac{1}{\lambda - 1} \log \left( \frac{e^{-\lambda/2\sigma^2}}{n^\lambda} \sum_{k_1 + \ldots + k_n = \lambda; k_1, \ldots, k_n \geq 0} \binom{\lambda}{k_1, \ldots, k_n} e^{\sum_{i=1}^{n} k_i^2 / 2\sigma^2} \right)$$

We further prove that the RDP is strictly upper-bounded by the Gaussian RDP without shuffling. The shuffle Gaussian RDP is advantageous in composing multiple DP mechanisms, where we demonstrate its improvement over the state-of-the-art approximate DP composition theorems in privacy guarantees of the shuffle model. Moreover, we extend our study to the subsampled shuffle mechanism and the recently proposed shuffled check-in mechanism, which are protocols geared towards distributed/federated learning. Finally, an empirical study of these mechanisms is given to demonstrate the efficacy of employing shuffle Gaussian mechanism under the distributed learning framework to guarantee rigorous user privacy.

1 Introduction

The shuffle/shuffled model [14, 22] has attracted attention recently as an intermediate model of trust in differential privacy (DP) [16, 17]. Within this setup, each user sends a locally differentially private (LDP) [33] report to a trusted shuffler, where the collected reports are anonymized/shuffled, before being forwarded to the untrusted analyzer/server. When viewed in the central model, the central DP parameter, $\epsilon$, can be smaller than the local one, $\epsilon_0$ [22], a phenomenon also known as privacy amplification. This yields better utility-privacy trade-offs than the local model of DP in general, without relying on a highly trusted server as in the central model of DP. The shuffle model can be realized in practice through a Trusted Execution Environment (TEE) [8], mix-nets [12], or peer-to-peer protocols [40]. Various aspects of the shuffle model have been investigated in the literature [4, 5, 13, 14, 23, 24, 26, 27, 29, 36].

A prominent use case of the shuffle model is in differentially private distributed learning, or federated learning (FL) [31, 41]. The non-private version of FL proceeds roughly as follows. The orchestrating server initiates the learning by sending a model to users. Then, each user calculates the gradient/model update using her own data, and sends the gradient to the server while keeping her data local.

In order to incorporate DP to distributed learning, it is convenient to first consider the probably most popular (and state-of-the-art) approach of learning with DP under the centralized setting, i.e., differentially private stochastic gradient descent (DP-SGD) [16, 45]. Here, the Gaussian mechanism is employed, where noise in the form of Gaussian/normal distribution is applied to a batch of clipped gradients to attain DP model updates. As learning often requires repeated interaction, privacy

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2 The source code of our implementation is available at [https://github.com/spliew/shuffgauss](https://github.com/spliew/shuffgauss)

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composition is vital when working under a pre-determined privacy budget. In DP-SGD, privacy is conventionally accounted for via the moments accountant \[ 1 \], which is essentially an accounting method based on the more general Rényi differential privacy (RDP) \[ 43 \], a notion of DP that provides generally much tighter DP composition, leading to a significant saving of privacy budget compared to strong composition theorems \[ 20, 32 \]. The subsampling effect \[ 46 \] is typically leveraged as well to achieve acceptable levels of privacy. Moreover, the central-DP version of DP-SGD has been adapted for federated learning \[ 42 \].

However, perhaps due to the difficulty of handling approximate DP mechanisms (particularly in privacy accounting), such as the Gaussian mechanism, the shuffle Gaussian mechanism, and its applications in distributed learning have not been explored in depth \[ 36, 39 \]. Distributed learning in the local or shuffle model is often conducted by utilizing variants of the LDP-SGD algorithm \[ 15, 21 \] in the literature, which is arguably more sophisticated than the Gaussian mechanism. \[ 4 \]

In this paper, we provide a characterization of the RDP of the shuffle Gaussian mechanism. Our principal result is showing that the Rényi divergence of the shuffle Gaussian mechanism is

\[
D_\lambda(\mathcal{M}(D)||\mathcal{M}(D')) = \frac{1}{\lambda - 1} \log \left( \frac{e^{-\lambda/2\sigma^2}}{n^{\lambda}} \sum_{k_1,\ldots,k_n=\lambda; k_1,\ldots,k_n \geq 0} \left( \sum_{i=1}^n k_i^2 / 2\sigma^2 \right) \right)
\]  

To calculate the above expression numerically, we reduce it to a partition problem in number theory. This enables us to perform tight privacy composition of any Gaussian-noise randomized mechanism coupled with a shuffler. Furthermore, the shuffle Gaussian RDP characterization allows us to compute the RDP of subsampled shuffle Gaussian and shuffled check-in Gaussian mechanisms, which are mechanisms tailored to distributed learning. To gauge the utility-privacy trade-offs of the variants of shuffle Gaussian mechanism in distributed learning, we also perform machine learning tasks experimentally in later sections.

**Related work.** Early studies on the shuffle model put focus on \( \epsilon_0 \)-local randomizer \[ 4, 14, 22 \]. In \[ 5, 23 \], extension to approximate DP, i.e., \( (\epsilon_0, \delta_0) \)-local randomizer has been worked on. While methods for composing optimally approximate DP mechanisms are known \[ 32, 44 \], they are unable to compose optimally mechanisms that satisfy multiple values of \( (\epsilon, \delta) \) simultaneously, such as the Gaussian mechanism.

A preliminary analysis of the shuffle Gaussian mechanism using the Fourier/numerical accountant \[ 28, 37, 49 \] has been performed in \[ 36 \]. This approach is known to give tight composition in general, but faces the curse of dimensionality; only \( n \ll 10 \) (\( n \) being the database size) can be evaluated within reasonable accuracy and amount of computation, not suitable for evaluating tasks like distributed learning.

Distributed learning with differential privacy is traditionally studied under the central model setting \[ 9, 30, 42 \]. Most studies of distributed learning under the local or shuffle model have investigated only the \( \epsilon_0 \)-local randomizer \[ 7, 21, 24, 26 \]. In \[ 39 \], the Gaussian mechanism is used in distributed learning, but the accounting of privacy is performed based on approximate DP, less optimal compared to our RDP approach.

**Paper organization.** In the next section, we provide preliminaries of this paper, containing various related results of DP, as well as problem formulation. Then, we prove our main results characterizing the RDP of shuffle Gaussian. In Section 4, the shuffle Gaussian mechanism is extended for applications in distributed learning. Before closing, we give the numerical results in Section 5.

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3The LDP-SGD algorithm proceeds roughly as follows. The client-side algorithm first clips user gradient, and flips its sign with a certain probability. Then, a unit vector is sampled randomly to make an inner product with the processed gradient to yield the inner product’s sign. The sign is again flipped with a certain probability, before sending it along with the unit vector to the server. The server-side algorithm includes normalizing the aggregated reports, and applying a \( l_2 \) projection when updating the model. See Algorithms 4 and 5 in \[ 21 \] for details.
2 Preliminaries

Problem formulation. Let there be \( n \) users. Let \( D = (x_1, \ldots, x_n) \) be a database of size \( n \), where each of \( i, i \in [n] \) is a data instance held by user \( i \). \( x_i \) is a \( d \)-dimensional vector, \( x_i \in \mathbb{R}^d \) and w.l.o.g., normalized to be \( \|x_i\|_2 \in [0,1] \).

Each user applies Gaussian mechanism \( G \) with variance \( \sigma \) on \( x_i \) and sends it, \( \tilde{x}_i \) to the shuffler. Let \( S_n : \mathcal{Y}^n \rightarrow \mathcal{Y}^n \) be the shuffling operation which takes \( \tilde{x}_i \)'s as input and outputs a uniformly and randomly permuted \( \tilde{x}_i \)'s. The randomization mechanism \( M \) of interest is therefore
\[
M(D) = S_n(G(x_1), \ldots, G(x_n)).
\]

Our purpose is to characterize and quantify the DP of \( M \) through RDP.

Let us next give definitions related to differential privacy and other known results.

Definition 1 (Central Differential Privacy [18]). A randomization mechanism, \( M : \mathcal{D}^n \rightarrow S \) with domain \( \mathcal{D}^n \) and range \( S \) satisfies central \((\epsilon, \delta)\)-differential privacy (DP), where \( \epsilon \geq 0, \delta \in [0,1] \), if for any two adjacent databases \( D, D' \in \mathcal{D}^n \) with \( n \) data instances and for any subset of outputs \( S \subseteq S \), the following holds:
\[
\Pr[M(D) \in S] \leq e^{\epsilon} \cdot \Pr[M(D') \in S] + \delta.
\] (2)

In this paper, adjacent databases are referred to as databases that have one data instance replaced by another; also known as “replacement” DP in the literature. The \((\epsilon, \delta)\)-DP mechanism is also said to satisfy approximate DP, or is \((\epsilon, \delta)\)-indistinguishable. Furthermore, we simply refer to \((\epsilon, \delta)\)-DP as “DP” when it is unambiguous. We also abbreviate \( \mathcal{D}^n \) as \( \mathcal{D} \) when the context is clear.

The randomization mechanism is known as local randomizer when the mechanism is applied to each data instance instead of the whole database. Formally, this form of mechanism is said to satisfy local DP:

Definition 2 (Local Differential Privacy (LDP) [33]). A randomization mechanism \( A : \mathcal{D} \rightarrow S \) satisfies local \((\epsilon, \delta)\)-DP if for all pairs \( x, x' \in \mathcal{D} \), \( A(x) \) and \( A(x') \) are \((\epsilon, \delta)\)-indistinguishable.

We often use \( \epsilon_0 \) instead of \( \epsilon \) when referring to LDP. A mechanism satisfying \((\epsilon_0, 0)\)-LDP is also called an \( \epsilon_0 \)-LDP randomizer.

Next, we define Rényi differential privacy, the main privacy notion used throughout this work (see also [10] [19]).

Definition 3 (Rényi Differential Privacy (RDP) [43]). A randomization mechanism \( M : \mathcal{D} \rightarrow S \) is said to satisfy \((\lambda, \epsilon)\)-RDP, for \( \lambda \in (1, \infty) \), if for any adjacent databases \( D, D' \in \mathcal{D} \), the Rényi divergence of order \( \lambda \) between \( M(D) \) and \( M(D') \), as defined below, is upper-bounded by \( \epsilon \):
\[
D_{\lambda}(M(D)||M(D')) := \frac{1}{\lambda - 1} \log \left( \mathbb{E}_{y \sim M(D')} \left[ \left( \frac{M(D)(y)}{M(D')(y)} \right)^{\lambda} \right] \right) \leq \epsilon,
\] (3)

where \( M(D)(y) \) denotes \( M \) taking \( D \) as input to output \( y \) with certain probability.

We take a functional view of \( \epsilon \), writing it as \( \epsilon(\lambda) \). We occasionally call \( \epsilon(\lambda) \) the RDP, and also refer to \( D_{\lambda}(M(D)||M(D')) \) as the RDP when the context is clear.

The RDP is most useful at composing DP mechanisms, where it has cleaner composition than approximate DP. The formal description is given below.

Lemma 1 (Adaptive RDP composition [43]). Let mechanisms \( M_1, M_2 \) taking \( D \in \mathcal{D} \) as input be \((\lambda_1, \epsilon_1),(\lambda_2, \epsilon_2)\)-RDP respectively, the composed mechanism, \( M_1 \times M_2 \) satisfies \((\lambda_1 + \epsilon_1, \epsilon_2)\)-RDP.

After accounting for the privacy with RDP, the RDP notion is often converted to the conventional and interpretable approximate DP notion. The conversion is given by the following lemma.

Lemma 2 (RDP-to-DP conversion [3][11]). A mechanism \( M \) satisfying \((\lambda, \epsilon(\lambda))\)-RDP also satisfies \((\epsilon, \delta)\)-DP, where \( 1 < \delta < 0 \) is arbitrary and \( \epsilon \) is given by
\[
\epsilon = \min_{\lambda} \left( \epsilon(\lambda) + \frac{\log(1/\delta) + (\lambda - 1) \log(1 - 1/\lambda) - \log(\lambda)}{\lambda - 1} \right).
\] (4)
In the above definitions, we have not specified what the underlying randomization mechanisms are. We next give the definition of the core mechanism used in this work, the Gaussian mechanism.

**Definition 4 (Gaussian mechanism).** Given \( x \in \mathbb{R}^d \), the Gaussian mechanism applied to \( x \) is a mechanism with parameter \( \sigma \) that adds zero mean isotropic Gaussian perturbation of variance \( \sigma^2 \) to \( x \), outputting \( \mathcal{M}(x) = x + \mathcal{N}(0, \sigma^2 I_d) \).

### 3 The RDP of shuffle Gaussian

We first derive the Rényi divergence (of order \( \lambda \)) of the shuffle Gaussian mechanism in this section.

Subsequently, an upper bound on the notion is given, followed by a description of the numerical techniques for evaluating the RDP.

The main result of this paper is given by the following theorem.

**Theorem 1 (Shuffle Gaussian RDP).** The Rényi divergence (of order \( \lambda \)) of the shuffle Gaussian mechanism with variance \( \sigma^2 \) for neighboring datasets with \( n \) instances is given by

\[
D_\lambda(\mathcal{M}(D) \| \mathcal{M}(D')) = \frac{1}{\lambda - 1} \log \left( \frac{e^{-\lambda/2\sigma^2}}{n^{\lambda}} \sum_{k_1, \ldots, k_n = \lambda; k_1, \ldots, k_n \geq 0} \left( \frac{\lambda}{k_1, \ldots, k_n} \right) e^{\sum_{i=1}^n k_i^2/2\sigma^2} \right)
\]

**Proof.** We consider w.l.o.g. adjacent databases \( D, D' \in \mathbb{R}^n \) with one-dimensional data instances \( D = (0, \ldots, 0), D' = (1, 0, \ldots, 0) \).\(^4\) Note that the databases can also be represented in the form of \( n \)-dimensional vector: where \( D \) is simply 0, an \( n \)-dimensional vector with all elements equal to zero, and \( D' = e_1 \), where \( e_i \) is a unit vector with the elements in all dimensions except the \( n \)-th (\( i \in [n] \)) one equal to zero.

The shuffle Gaussian mechanism first applies Gaussian noise (with variance \( \sigma^2 \)) to each data instance, and subsequently shuffle the noisy instances. The shuffled output of \( D \) distributes as \( \mathcal{M}(D) \sim (\mathcal{N}(0, \sigma^2), \ldots, \mathcal{N}(0, \sigma^2)) \), an \( n \)-tuple of \( \mathcal{N}(0, \sigma^2) \). Intuitively, the adversary sees an anonymized set of randomized data of size \( n \). Due to the homogeneity of \( D \), we can write \( \mathcal{M}(D) \sim \mathcal{N}(0, \sigma^2 I_n) \) using the vector notation mentioned above.\(^5\)

On the other hand, since all data instances except one is 0 in \( D' \), the mechanism can be written as a mixture distribution: \( \mathcal{M}(D') \sim \frac{1}{n} (\mathcal{N}(e_1, \sigma^2 I_n) + \ldots + \mathcal{N}(e_n, \sigma^2 I_n)) \). Intuitively, each output has probability \( 1/n \) carrying the non-zero element, and hence the mechanism as a whole is a uniform mixture of \( n \) distributions. In summary, the neighbouring databases of interest are

\[
\mathcal{M}(D) \sim \mathcal{N}(0, \sigma^2 I_n), \quad \text{(5)}
\]

\[
\mathcal{M}(D') \sim \frac{1}{n} (\mathcal{N}(e_1, \sigma^2 I_n) + \ldots + \mathcal{N}(e_n, \sigma^2 I_n)) \quad \text{(6)}
\]

Note that \( \mathbb{P}(\mathcal{N}(0, \sigma^2)) = \exp(-x^2/2\sigma^2)/\sqrt{2\pi\sigma^2} \), and

\[
\frac{n \cdot (2\pi\sigma^2)^{n/2}}{\sqrt{2\pi\sigma^2}} \cdot \mathbb{P}(\mathcal{M}(D')(x)) = \exp \left[ -(x_1 - 1)^2/2\sigma^2 - \sum_{i=2}^n x_i^2/2\sigma^2 \right] + \ldots
\]

\[
+ \exp \left[ -(x_n - 1)^2/2\sigma^2 - \sum_{i=1}^{n-1} x_i^2/2\sigma^2 \right]. \quad \text{(7)}
\]

That is, for \( j \in [n] \), the \( j \)-th component of the mixture distribution contains a term proportional to \( \exp \left[ -(x_j - 1)^2/2\sigma^2 - \sum_{i \neq j} x_i^2/2\sigma^2 \right] \).

\(^4\)This can be done w.l.o.g. by normalizing and performing unitary transform on the data instances. See also Appendix B.1 of [37].

\(^5\)Note that the vector space defined here is over the dataset dimension instead of input dimension as in Definition 4.
We are concerned with calculating the following quantity, $E_{x \sim \mathcal{M}(D)} \left[ \left( \frac{\mathcal{M}(D')(x)}{\mathcal{M}(D)(x)} \right)^\lambda \right]$ for RDP:

$$
E_{x \sim \mathcal{M}(D)} \left[ \left( \frac{\mathcal{M}(D')(x)}{\mathcal{M}(D)(x)} \right)^\lambda \right] = \int \left( \frac{\exp \left[ -(x_1 - 1)^2/2\sigma^2 - \sum_{i=2}^{n} x_i^2/2\sigma^2 \right] + \ldots }{n \exp \left[ -\sum_{i=1}^{n} x_i^2/2\sigma^2 \right]} \right)^\lambda \exp\left[ -\sum_{i=1}^{n} x_i^2/2\sigma^2 \right] \frac{d^n x}{(2\pi\sigma^2)^{n/2}} \tag{8}
$$

$$
= \int \left( \frac{\exp \left[ -(x_1 - 1)^2/2\sigma^2 + x_2^2/2\sigma^2 + \ldots \right]}{n} \right)^\lambda \exp\left[ -\sum_{i=1}^{n} x_i^2/2\sigma^2 \right] \frac{d^n x}{(2\pi\sigma^2)^{n/2}} \tag{9}
$$

$$
= \int \left( \exp \left[ (2x_1 - 1)/2\sigma^2 \right] + \ldots \right)^\lambda \exp\left[ -\sum_{i=1}^{n} x_i^2/2\sigma^2 \right] \frac{d^n x}{n^\lambda (2\pi\sigma^2)^{n/2}} \tag{10}
$$

$$
= \int \left( \sum_{i=1}^{n} \exp \left[ (2x_i - 1)/2\sigma^2 \right] \right)^\lambda \exp\left[ -\sum_{i=1}^{n} x_i^2/2\sigma^2 \right] \frac{d^n x}{n^\lambda (2\pi\sigma^2)^{n/2}} \tag{11}
$$

Let us explain the above calculation in detail. We first notice that $E_{x \sim \mathcal{M}(D)} \left[ \left( \frac{\mathcal{M}(D')(x)}{\mathcal{M}(D)(x)} \right)^\lambda \right]$ is an $n$-th dimensional integral of $x_i$ ($i \in [n]$), as shown in Equation 8. For each term $j \in [n]$ in the nominator of $(\ldots)^\lambda$, we divide it by the denominator $\exp\left[ -\sum_{i=1}^{n} x_i^2/2\sigma^2 \right]$, yielding $\exp\left[ -(x_j - 1)^2/2\sigma^2 - \sum_{i \neq j} x_i^2/2\sigma^2 + \sum_{i=1}^{n} x_i^2/2\sigma^2 \right] = \exp\left[ -(x_j - 1)^2/2\sigma^2 + x_j^2/2\sigma^2 \right]$, with all $x_i$ terms in $i \in [n]$ except $j$ canceled out (Equation 9). The term can be further simplified to $\exp\left[ (2x_j - 1)/2\sigma^2 \right]$ as in Equation 11.

Then, we expand the expression $(\ldots)^\lambda$ using the multinomial theorem:

$$
\left( \sum_{i=1}^{n} \exp \left[ (2x_i - 1)/2\sigma^2 \right] \right)^\lambda = \sum_{k_1 + \ldots + k_n = \lambda; k_1, \ldots, k_n \geq 0} \left( k_1, \ldots, k_n \right) \prod_{i=1}^{n} \exp \left[ k_i(2x_i - 1)/2\sigma^2 \right], \tag{12}
$$

where $\left( k_1, \ldots, k_n \right) = \binom{\lambda}{k_1, \ldots, k_n}$ is the multinomial coefficient, and $k_i \in \mathbb{Z}^+$ for $i \in [n]$.

Before proceeding, we make a detour to calculate the following integral (with $k \in \mathbb{Z}^+$):

$$
\int \exp[k(2x - 1)/2\sigma^2] \exp[-x^2/2\sigma^2] \frac{dx}{\sqrt{2\pi\sigma^2}} = \int \frac{dx}{\sqrt{2\pi\sigma^2}} \exp[-(k^2 - k^2)/2\sigma^2] = \exp[k^2/2\sigma^2].
$$

Using the above expression and Equation 12, we can write Equation 11 as

$$
\int \sum_{k_1 + \ldots + k_n = \lambda; k_1, \ldots, k_n \geq 0} \left( k_1, \ldots, k_n \right) \prod_{i=1}^{n} \exp \left[ k_i(2x_i - 1)/2\sigma^2 \right] \frac{d^n x}{n^\lambda (2\pi\sigma^2)^{n/2}} = \frac{1}{n^\lambda} \sum_{k_1 + \ldots + k_n = \lambda; k_1, \ldots, k_n \geq 0} \left( k_1, \ldots, k_n \right) \prod_{i=1}^{n} \exp \left[ k_i^2/2\sigma^2 - k_i/2\sigma^2 \right], \tag{13}
$$

where we have moved the $1/n^\lambda$ factor to the front. Noticing that $\prod_{i=1}^{n} \exp[-k_i] = \exp[-\sum_{i=1}^{n} k_i]$ and that $\sum_{i=1}^{n} k_i = \lambda$ under the multinomial constraint, we have

$$
\frac{1}{n^\lambda} \sum_{k_1 + \ldots + k_n = \lambda; k_1, \ldots, k_n \geq 0} \left( k_1, \ldots, k_n \right) \prod_{i=1}^{n} \exp \left[ k_i^2/2\sigma^2 - k_i/2\sigma^2 \right]. \tag{14}
$$
\[ e^{-\lambda/2\sigma^2} \sum_{k_1 + \ldots + k_n = \lambda; \ k_1, \ldots, k_n \geq 0} \left( \begin{array}{c} \lambda \\ k_1, \ldots, k_n \end{array} \right) e^{\sum_{i=1}^n k_i^2/2\sigma^2}. \]  

Combining the above expression with Equation 3, we obtain the expression given in Equation 1 as desired.

Using the above results, we next give an upper bound on the shuffle Gaussian RDP.

**Corollary 1** (Upper bound of shuffle Gaussian RDP). The shuffle Gaussian RDP \( \epsilon(\lambda) \) is upper-bounded by \( \lambda/2\sigma^2 \).

**Proof.** From the Cauchy-Schwartz inequality, \( \sum_{i=1}^n k_i^2 \leq (\sum_{i=1}^n k_i)^2 = \lambda^2 \). Then, from Equation 14

\[ e^{-\lambda/2\sigma^2} \sum_{k_1 + \ldots + k_n = \lambda; \ k_1, \ldots, k_n \geq 0} \left( \begin{array}{c} \lambda \\ k_1, \ldots, k_n \end{array} \right) e^{\sum_{i=1}^n k_i^2/2\sigma^2} \leq e^{-\lambda/2\sigma^2} \sum_{k_1 + \ldots + k_n = \lambda; \ k_1, \ldots, k_n \geq 0} \left( \begin{array}{c} \lambda \\ k_1, \ldots, k_n \end{array} \right) e^{\lambda^2/2\sigma^2} \]

\[ = e^{(\lambda^2-\lambda)/2\sigma^2}, \]

using in the last line, \( \sum_{k_1 + \ldots + k_n = \lambda; \ (k_1, \ldots, k_n)} = n^\lambda \). Substituting this to Equation 3, we get the desired expression.

This corollary states that the upper bound of the shuffle Gaussian RDP is equal to the Gaussian RDP without shuffling 44. Let us emphasize that this is a strictly tight upper bound compared with previous studies. In early analyses of shuffling 522, the resulting \( \epsilon \) after shuffling can be larger than the local one, \( \epsilon_0 \), especially at the large \( \epsilon_0 \) region. In 23, the shuffle \( \epsilon \) is shown, only numerically, to saturate at \( \epsilon_0 \). We have instead proved in the above corollary, at least for the Gaussian mechanism, that the upper bound is analytically and strictly the same as the one without shuffling. This also means that there is no such phenomenon as “privacy degradation” due to shuffling in Gaussian mechanism.

### 3.1 Evaluating the shuffle Gaussian RDP numerically

Evaluating the shuffle Gaussian RDP numerically is non-trivial, as \( n \) can be as large as \( 10^5 \) in distributed learning scenarios and \( \lambda \) can be large as well. Here, we describe our numerical recipes for calculating the RDP.

We exploit the permutation invariance inherent in Equation 1 to perform more efficient computation. Equation 1 contains a sum of multinomial coefficients multiplied by an expression with integer \( k_i \) (\( i \in [n] \)) constrained by \( k_1 + \ldots + k_n = \lambda \). We first obtain all possible \( k_i \)'s with the above constraint, without counting same summands differing only in the order. This is integer partition, a subset sum problem known to be NP 35. It is also related to partition in number theory 2. Nevertheless, the calculation can be performed efficiently (e.g., 34) and one can cache the partitions to save computational time.

Note that there is a one-to-one mapping between the term \( e^{\sum_{i=1}^n k_i^2/2\sigma^2} \) in Equation 14 and the partition obtained above. We can therefore calculate the number of permutation corresponding to each partition, multiplied by the multinomial coefficient, and \( e^{\sum_{i=1}^n k_i^2/2\sigma^2} \), and sum over all the partitions to obtain the numerical expression of Equation 14. For a partition with the number of repetition of exponents with the same degrees (or the counts of unique \( k_1, \ldots, k_n \) being \( \kappa_1, \ldots, \kappa_l \) (\( l \leq n \)), the number of permutation can be calculated to be \( n!/(\kappa_1! \ldots \kappa_l!) \). See Appendix B for details. The algorithm of calculating the RDP can also be found in Algorithm 1.

Since Equation 1 is of the form of a logarithm, evaluation with the log-sum-exp technique leads to more stable results. Moreover, the factorials appearing in the multinomial coefficients can be calculated efficiently with the log gamma function (implemented in scipy).
Algorithm 1 Numerical evaluation of Equation 1

1: Inputs: Database size $n$, RDP order $\lambda$, variance of Gaussian mechanism $\sigma^2$.
2: Output: RDP parameter $\epsilon$.
3: Initialize: $\Gamma : \{\emptyset\}$, Sum = 0.
4: $\Gamma \leftarrow \text{GetPartition}(\lambda)$
5: for $\rho \in \Gamma$ do
6: \{$\kappa_i$\} $\leftarrow \text{GetUniqueCount}(\rho)$
7: Sum $\leftarrow$ Sum $+ e^{-\lambda/2\sigma^2} \prod_{\kappa_i \in \rho} \frac{\lambda n! \prod_{j=1}^{\kappa_i} e^{\sum_{j=1}^{\kappa_i} k_j^2}}{k_1! \ldots k_\kappa_i!}$ where $\{k_i\} \in \rho$
8: Return: $\epsilon = \text{Sum}$

4 Extensions of the Shuffle Gaussian

In this section, we extend the study of the shuffle Gaussian RDP to mechanisms more attuned to distributed learning. Specifically, we consider mechanisms where only a subset of the dataset participates in training, typical for large-scale distributed learning. Throughout our study, we consider protecting the privacy of users each holding a single data instance.

4.1 Subsampled shuffle Gaussian mechanism

We first consider the mechanism where a fixed number of users, $m$, is sampled randomly from $n$ available users to participate in training. The sampled users randomize their data with Gaussian noises and pass them to a shuffler. We call this overall procedure the subsampled (without replacement) shuffle Gaussian mechanism.

The RDP of the subsampled shuffle Gaussian mechanism is given by the following theorem.

**Theorem 2** (Subsampled Shuffle Gaussian RDP). Let $n$ be the total number of users, $m$ be the number of subsampled users in each round of training, and $\gamma$ be the subsampling rate, $\gamma = m/n$. Also let $\epsilon_{m}^{SG}(\lambda)$ be the shuffle Gaussian RDP given in Theorem 1. The subsampled shuffle Gaussian mechanism satisfies $(\lambda, \epsilon_{SSG}^{\gamma,m}(\lambda))$-RDP, where

$$
\epsilon_{SSG}^{\gamma,m}(\lambda) \leq \frac{1}{\lambda - 1} \log \left( 1 + \gamma^2 \frac{\lambda}{2} \min \left\{ 4(e^{\epsilon_{m}^{SG}(2)} - 1), 2e^{\epsilon_{m}^{SG}(2)} \right\} + \sum_{j=3}^{\lambda} 2\gamma^j \left( \frac{\lambda}{j} \right) e^{(j-1)e^{\epsilon_{SG}^{\gamma,j}(2)}} \right)
$$

(16)

*Proof sketch.* The proof is a direct application of Theorem 1 and the subsampled RDP results of [47]. The full proof can be found in Appendix.

4.2 Shuffled Check-in Gaussian mechanism

In practice, it is difficult to achieve uniform subsampling with a fixed number of users via the orchestrating server [5, 26]. It is more natural to let the user decide (with her own randomness) whether to participate in training. This is known as the shuffled check-in mechanism [39]. The mechanism may also be considered as the shuffle version of Poisson subsampling in the literature.

Let $n$ be the total number of users, and $\gamma$ be the check-in rate (the probability of a user participating in training) The shuffled check-in’s RDP is given as follows [39].

$$
\epsilon_{SCI}(\lambda) \leq \frac{1}{\lambda - 1} \log \left( \sum_{k=1}^{n} \binom{n}{k} \gamma^k (1 - \gamma)^{n-k} e^{(\lambda - 1)e^{\epsilon_{SG}^{\gamma,k}(\lambda)}} \right).
$$

(17)

Here, $\epsilon_{SG}^{\gamma,k}(\lambda)$ is the subsampled shuffle Gaussian RDP defined in Equation 16 with $k$ the number of shuffled instances, $\gamma$ the subsampling rate.

The direct evaluation of the above equation is inefficient numerically, as it involves a sum over $n$. We opt for a more efficient computation, which requires the use of the following conjecture.

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See also [25], where a similar protocol except that an $\epsilon_0$-local randomizer is utilized, has been considered.
Table 1: the value of shuffle $\epsilon$ under composition. The clones method \cite{23} is compared with our RDP-based method.

| No. of composition | 1     | 2     | 3     | 4     | 5     | 6     | 7     |
|--------------------|-------|-------|-------|-------|-------|-------|-------|
| Clones             | 0.18623 | 0.38461 | 0.59516 | 0.79355 | 1.02241 | 1.22689 | 1.43138 |
| Ours               | 0.22820 | 0.22820 | 0.22821 | 0.22821 | 0.22821 | 0.22822 | 0.22822 |

Conjecture 1 (Monotonicity). The expression in Equation 14, that is,

$$e^{-\lambda/2\sigma^2} \sum_{k_1+\ldots+k_n=\lambda; k_1,\ldots,k_n \geq 0} \left( \frac{\lambda}{\sum_{i=1}^{n} k_i} \right)^{\sum_{i=1}^{n} k_i^2/2\sigma^2},$$

is a monotonically decreasing function with respect to $n$.

Although we are not able to prove rigorously the above-mentioned conjecture yet, our empirical scan over $n$ shows that it is indeed monotonically decreasing (at least within our scope of investigation). With this conjecture, we present the following theorem for calculating the shuffled check-in Gaussian RDP with efficiency.

Theorem 3 (Shuffled Check-in Gaussian RDP). Let $n$ be the total number of users, and $\gamma$ be the check-in rate. Also let $e^{SSG}_{\gamma,m}(\lambda)$ be the subsampled shuffle Gaussian RDP defined in Equation 16, with $k$ the number of shuffled instances, $\gamma$ the subsampling rate. If Conjecture 1 holds, the shuffled check-in Gaussian mechanism satisfies $(\lambda, e^{SCI}(\lambda))$-RDP, where

$$e^{SCI}(\lambda) \leq \frac{1}{\lambda - 1} \log \left( e^{(\lambda-1)e^{SSG}(\lambda)-\Delta^2 n\gamma/\sigma^2} + e^{(\lambda-1)e^{SSG}(1-\Delta)n\gamma+1}(\lambda) \right).$$ (18)

Here, $\Delta \in [0, 1]$ is arbitrary.

Proof sketch. The proof follows closely to the techniques used in \cite{39}, where monotonicity and the Chernoff bound are utilized to prove the theorem. The full proof can be found in Appendix A. ■

The main advantage of using Equation 18 is that the calculation of the RDP with this theorem is $O(n)$ times more efficient computationally compared to Equation 17.

5 Numerical Results

In this section, we present the numerical evaluation results of the shuffle Gaussian and its extensions. We also use the latter mechanisms to perform machine learning tasks under the distributed setting.

5.1 Numerical results of the shuffle Gaussian

An upper bound on the shuffle model with an $(\epsilon, \delta)$-LDP randomizer is given by the clones method \cite{23}, which is our target of comparison. According to \cite{23}, given $n$ instances undergone $(\epsilon_0, \delta_0)$-LDP randomization and further shuffled such that $\epsilon_0 \leq \log(\frac{n}{\delta_0 \log(23)})$ for any $\delta \in [0, 1]$, the overall DP is $(\epsilon, \delta + (s^2 + 1)(1 + e^{-\epsilon_0/2})n\delta_0)$-DP. Here, $\epsilon = O(\epsilon_0 \sqrt{\epsilon_0 \log(1/\delta) \sqrt{n}})$ when $\epsilon_0 > 1$.

The Gaussian mechanism is however governed by the noise variance, $\sigma$, and there is an infinite pair of corresponding $(\epsilon, \delta)$. To use the clones method, we fix $\sigma$, $\delta_0 = \delta$ and scan over $\delta_0$ to find parameters satisfying $\delta_{\text{final}} \leq 1/n$, which is the overall DP $\delta$ parameter. Finally, mechanism composition is performed with the strong composition theorem \cite{32}.

In Table 1 we compare privacy accounting using the clones method with ours (with RDP), fixing $n = 60,000$, $\sigma = 9.48$ (corresponding to $(1, 1/n)$-LDP) and global sensitivity 1. We set the maximum $\lambda$ to be evaluated as 30. While the clones method gives tighter result without composition, the RDP accounting shows tighter results by a large margin with composition.

Let us plot the $\epsilon$ dependence at larger number of composition. In Figure 1a, we show the RDP and upper bound, again demonstrating significant amplification due to shuffling.
5.2 Distributed learning

We perform an experiment under the shuffled check-in protocol with a setup similar to the one given in [39]. We use the MNIST handwritten digit dataset [38], with each user holding one data instance from it. A convolutional neural network is used as the target model \( \theta \in \mathbb{R}^d \) under the distributed learning setting. In each round, each participating user calculates the gradient \( g \in \mathbb{R}^d \) using the received model. The user clips the gradient, \( g \leftarrow g \cdot \min \{1, \frac{C}{\|g\|} \} \) with a clipping factor \( C \), and add Gaussian noise to it, \( \tilde{g} \rightarrow g + \mathcal{N}(0, C^2 \sigma^2 I_d) \) before sending it to the server.

The total number of user is 60,000, i.e., the size of the MNIST train dataset. The check-in rate is set to be \( \gamma = 0.1 \), \( C = 0.01 \), \( \sigma = 5 \). The test dataset is used to evaluate the classification accuracy.

We compare our approach with two methods introduced in [39]. The first method performs conservatively the privacy accounting of approximate DP of subsampling and shuffling (Baseline). The second method converts approximate DP of (subsampled) shuffling to RDP, and perform the privacy accounting under the shuffled check-in framework (generic bound).

We note that the experiments of interest can be simulated conveniently by making simple modifications to existing libraries that implement DP-SGD (e.g., Opacus [48]). This is done by noticing that in DP-SGD, given a batch of clipped gradients \( \tilde{g} \), Gaussian noise is applied to the sum of the gradients, i.e.,

\[
\sum_{i=1}^{B} \tilde{g}_i \leftarrow \sum_{i=1}^{B} \tilde{g}_i + \mathcal{N}(0, \sigma^2)
\]

Here, \( B \) is the number of samples. In shuffle Gaussian or local Gaussian, the server sums over the received gradients randomized with Gaussian noises,

\[
\sum_{i=1}^{B} \{ \tilde{g}_i + \mathcal{N}(0, \sigma^2) \} = \sum_{i=1}^{B} \tilde{g}_i + \mathcal{N}(0, B\sigma^2)
\]

Therefore, we can simply multiply \( \sigma \) by a factor of \( \sqrt{B} \) in existing DP-SGD libraries to simulate the decentralized setting of shuffle/local Gaussian.

Figure 1b shows the accuracy-versus-\( \epsilon \) plot of our experiment. Here, a total number of 5,540 rounds of training has been run. Our accounting requires less \( \epsilon \) to reach optimal accuracy (80%). Note also that the generic accounting method in [39] is rather sophisticated (further requiring an inner round of optimization) and requires \( O(n) \) times more computation compared to ours.

6 Conclusion

In this paper, we have analyzed the RDP of shuffle Gaussian mechanism, and applied it to distributed learning. There are still some open questions. Calculating the RDP at large \( \lambda \) is computationally
infeasible. It has deep relation with number theory and looking for hints from there to calculate these values is a possible future direction. A rigorous proof of Conjecture \[ \text{[1]} \] is also needed.

As various settings of the Gaussian mechanism under the centralized learning setting, i.e., DP-SGD, are relatively well studied, we hope that our privacy accountant of shuffle Gaussian opens an avenue that inspires more in-depth studies on various datasets and settings of distributed learning under the shuffle model, using the relatively simple Gaussian mechanism, similar to DP-SGD.

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A Additional Proofs

A.1 Proof of Theorem 2

**Proof.** Theorem 9 of [47] states that for subsampling rate $\gamma$, a mechanism $\mathcal{M}$ satisfying $(\lambda, \epsilon(\lambda))$-RDP applied to a subsampled set of data, satisfy $(\lambda, \epsilon'(\lambda))$, where

\[
\epsilon'(\lambda) \leq \frac{1}{\lambda - 1} \log[1 + \gamma^2 \binom{\lambda}{2}] \min \left\{ 4(e^m(2) - 1), e^m(2) \min\{2, (e^{\epsilon(\infty)} - 1)^2\} \right\} \\
+ \sum_{j=3}^{\lambda} \gamma^j \binom{\lambda}{j} e^{(j-1)\epsilon_m(j)} \min\{2, (e^{\epsilon(\infty)} - 1)^j\}
\]

Since $\epsilon(\infty)$ is unbounded, we can simplify it to the expression given in Theorem 2 in a straightforward way. ■

A.2 Proof of Theorem 3

**Proof.** The Chernoff bound states that $\mathbb{P}[X \leq (1 - \Delta)\mu] \leq e^{-\Delta^2 \mu/2}$ for all $0 < \Delta < 1$.

We split the summation over $k$ of

\[
\sum_{k=1}^{n} \binom{n}{k} \gamma^k (1 - \gamma)^{n-k} e^{(\lambda-1)\epsilon_{SG}^{SSG}}
\]

into two parts, those equal or less than $(1 - \Delta)n\gamma$, and those equal or larger than $(1 - \Delta)n\gamma + 1$, assuming that $(1 - \Delta)n\gamma$ is an integer. As Conjecture 1 tells us that the expression in Equation 11 is monotonically decreasing, the largest value of $\epsilon_{SG}^{SSG}(\lambda)$ for $k \in [1, (1 - \Delta)n\gamma]$ is $\epsilon_{SG}^{SSG}(\lambda)$. The first part of the summation can then be bounded by

\[
e^{-\Delta^2 n\gamma/2} \epsilon_{SG}^{SSG}(\lambda),
\]

using the Chernoff bound.

Similarly, the largest value of $\epsilon_{SG}^{SSG}(\lambda)$ for $k \in [(1 - \Delta)n\gamma + 1, n]$ is $\epsilon_{SG}^{SSG}(\lambda)$. The second part of the summation is then bounded by

\[
\epsilon_{SG}^{SSG}(n - (1 - \Delta)n\gamma - 1) \lambda
\]

Combining the above two terms gives the desired theorem. ■

B Algorithms

B.1 GetUniqueCount: Note on the permutation invariance of Equation 14

It is best to describe the counting factor in our numerical evaluation of Equation 14 via an example. Consider again $(x_1 + x_2 + x_3)^3$. Expanding the multinomials, the terms with exponents of the form $x_1^3 x_2$ are

\[
3(x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2).
\]

Here, the factor 3 comes from the multinomial coefficient $\binom{3}{2,1,0}$. These terms $x_1^2 x_2, x_1^2 x_3, x_2^2 x_1, x_2^2 x_3, x_3^2 x_1, x_3^2 x_2$ belong to the same subset of the form $x_i^2 x_j$, which do not have repetition of exponent with same degrees. It has $3! = 6$ elements in total. This subset contributes effectively a factor of $3 \times 6 = 18$ to the expansion in Equation 14

\[
18 e^{(2^2 + 1)^2/2\sigma^2},
\]

ignoring factor unrelated to the multinomial coefficients.
Consider the same expansion, but the terms with exponents of the form $x_i x_j x_k$:

$$6 x_1 x_2 x_3.$$ 

Here, the multinomial coefficient is $\binom{3}{1,1,1} = 6$. Since the number of repetition of exponent with the same degree is 3 ($x_i, x_j, x_k$ has the same degree 1), the subset has $3!/(3!) = 1$ element. This subset contributes effectively a factor of $6 \times 1 = 6$ to the expansion in Equation 14,

$$6 e^{(1^2+1^2)/2\sigma^2},$$

ignoring factor unrelated to the multinomial coefficients.

This procedure of calculating the contributing coefficients is called `GetUniqueCount` in Algorithm 1.

### B.2 Algorithm 1 description

We describe our algorithm for evaluating Equation 1.

Given $\lambda$, we find all partition of integers satisfying $k_1, \ldots, k_n = \lambda$. We denote the operation by `GetPartition`. For each of the partition, we obtain the number of counts for each unique $k_i$; let them be $\kappa_1, \ldots, \kappa_i$ ($i \leq n$), and denote the operation by `GetUniqueCount`. Subsequently, we calculate the value:

$$e^{-\lambda/2\sigma^2} \frac{\lambda}{n^\lambda} \binom{\lambda}{k_1 \ldots k_n} \frac{n!}{\kappa_1! \ldots \kappa_i!} e^{\sum_{j=1}^n k_j^2},$$

and make summation over all the partitions. The algorithm is shown in Algorithm 1.