ON ALGEBRAS GENERATED BY POSITIVE OPERATORS

ROMAN DRNOVŠEK

Abstract. We study algebras generated by positive matrices, i.e., matrices with non-negative entries. Some of our results hold in more general setting of vector lattices. We reprove and extend some theorems that have been recently shown by Kandič and Šivic. In particular, we give a more transparent proof of their result that the unital algebra generated by positive idempotent matrices $E$ and $F$ such that $EF \geq FE$ is equal to the linear span of the set \{I, E, F, EF, EFE, FEE, (EF)^2, (FE)^2\}, and so its dimension is at most 9. We give examples of two positive idempotent matrices that generate unital algebra of dimension $2n$ if $n$ is even, and of dimension $(2n-1)$ if $n$ is odd.

We also prove that the algebra generated by positive matrices $B_1, B_2, \ldots, B_k$ is triangularizable if $AB_i \geq B_iA$ ($i=1,2,\ldots,k$) for some positive matrix $A$ with distinct eigenvalues.

Key words: positive matrices, positive idempotents, vector lattices, commutativity, triangularizability
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1. Introduction

Recently, Kandič and Šivic [6] have studied order analogs of Gerstenhaber’s theorem stating that the dimension of the unital algebra generated by two commuting $n \times n$ matrices is at most $n$. They showed that the dimension of the unital algebra generated by two positive $n \times n$ matrices $A$ and $B$ is at most $n(n+1)/2$ provided its commutator $[A, B] = AB - BA$ is also positive (see Theorem 3.1). Here positivity of a matrix means that it has nonnegative entries. We prove an extension of this result in the case when the matrix $A$ has distinct eigenvalues (see Theorem 3.2). Under the same assumption on $A$ we then consider the unital algebra generated by the super left-commutant of $A$, that is the collection of all positive matrices $B$ such that $[A, B] \geq 0$. We prove that the dimension of this algebra is at most $n(n+1)/2$ (see Corollary 3.3). If $A$ is a positive diagonal matrix with distinct diagonal entries, then this upper bound is attained (see Theorem 3.3).

It has been also shown in [6] that 9 is the largest dimension of the unital algebra generated by two positive idempotent matrices $E$ and $F$ satisfying $EF \geq FE$ (see Corollary 3.4).
Moreover, the paper \cite{6} provides a nontrivial example showing that this upper bound can be attained. In our paper this result is proved in more transparent way that also gives some insight in constructing the just-mentioned example. Extensions to the vector lattice setting are also considered.

In \cite{5}, it is shown that a unital algebra generated by two $n \times n$ matrices with quadratic minimal polynomials is at most $2n$-dimensional if $n$ is even, and at most $(2n - 1)$-dimensional if $n$ is odd. Examples in \cite{5} show that the bounds on dimensions are sharp even in the case of idempotents. We give such examples in which the two idempotents are also positive matrices.

2. Preliminaries

Since some of our results hold in general setting of vector lattices, we recall some basic definitions and properties of vector lattices and operators on them. For the terminology and details not explained here we refer the reader to \cite{1} or \cite{2}.

Let $L$ be a vector lattice with the positive cone $L^+$. The band

$$S^d := \{x \in L : |x| \wedge |y| = 0 \text{ for all } y \in S\}$$

is called the disjoint complement of a set $S$ of $L$. A band $B$ of $L$ is said to be a projection band if $L = B \oplus B^d$. If every band of $L$ is a projection band, we say that the vector lattice $L$ has the projection property.

Let $A$ be a positive (linear) operator on a vector lattice $L$. The null ideal $\mathcal{N}(A)$ is the ideal in $L$ defined by

$$\mathcal{N}(A) = \{x \in L : A|x| = 0\}.$$ 

When $\mathcal{N}(A) = \{0\}$, we say that the operator $A$ is strictly positive. The range ideal $\mathcal{R}(A)$ of $A$ is the ideal generated by the range of $A$, that is,

$$\mathcal{R}(A) = \{y \in L : \exists x \in L^+ \text{ such that } |y| \leq Ax\}.$$ 

An operator $A$ on $L$ is called order continuous if every net $\{x_\alpha\}$ order converging to zero is mapped to the net $\{Ax_\alpha\}$ order converging to zero as well. It is easy to verify that the null ideal of an order continuous positive operator is always a band of $L$. On the other hand, the range ideal of an order continuous positive operator is not necessarily a band, even if the operator is idempotent.
Example 2.1. Let \( E : l^2 \to l^2 \) be the order continuous positive operator defined by \( Ex = \langle x, u \rangle u \), where \( u = (2^{-1/2}, 2^{-2/2}, 2^{-3/2}, 2^{-4/2}, 2^{-5/2}, \ldots) \in l^2 \). Since \( \|u\|_2 = 1 \), we have \( E^2 = E \). Clearly, \( \mathcal{R}(E) \) is the ideal generated by \( u \), and it is not equal to \( l^2 \), as \( (2^{-1/2}, 2 \cdot 2^{-2/2}, 3 \cdot 2^{-3/2}, 4 \cdot 2^{-4/2}, 5 \cdot 2^{-5/2}, \ldots) \notin \mathcal{R}(E) \). On the other hand, we have \( \mathcal{R}(E)^d = \{0\} \).

We will make use of the following simple lemma.

Lemma 2.2. Let \( L \) be an Archimedean vector lattice. Let \( A \) be a positive operator on \( L \) such that \( \mathcal{R}(A)^d = \{0\} \), and let \( B \) be an order continuous positive operator on \( L \) such that \( BA = 0 \). Then \( B = 0 \).

Proof. Assume first that \( 0 \leq y \in \mathcal{R}(A) \). Then there is a positive vector \( x \in L^+ \) such that \( y \leq Ax \). It follows that \( 0 \leq By \leq B Ax = 0 \), and so \( By = 0 \).

Assume now that \( y \in L \). Since \( \mathcal{R}(A)^dd = L \), there exists a net \( \{y_\alpha\} \subset \mathcal{R}(A) \) order converging to \( y \). As \( By_\alpha = 0 \) and \( B \) is order continuous, we obtain that \( By = 0 \). \( \square \)

A family \( \mathcal{F} \) of operators on a \( n \)-dimensional vector space \( X \) is reducible if there exists a nontrivial subspace of \( X \) that is invariant under every operator from \( \mathcal{F} \). Otherwise, \( \mathcal{F} \) is irreducible. A family \( \mathcal{F} \) is said to be triangularizable if there is a basis of \( X \) such that all operators in \( \mathcal{F} \) have upper triangular representation with respect to that basis. Clearly, triangularizability is equivalent to the existence of a chain of invariant subspaces

\[ \{0\} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = X \]

with the dimension of \( M_j \) equal to \( j \) for each \( j = 0, 1, 2, \ldots, n \). Any such chain is called a triangularizing chain of \( \mathcal{F} \). Order analogs of these concepts are defined as follows.

A family \( \mathcal{F} \) of operators on an \( n \)-dimensional vector lattice \( L \) is said to be ideal-reducible if there exists a nontrivial ideal of \( L \) that is invariant under every operator from \( \mathcal{F} \). Otherwise, we say that \( \mathcal{F} \) is ideal-irreducible. A family \( \mathcal{F} \) is said to be ideal-triangularizable if it is triangularizable and at least one of (possibly many) triangularizing chains of \( \mathcal{F} \) consists of ideals of \( L \). More information on triangularizability can be found in [9].

For a complex \( n \times n \) matrix \( A \), the commutant \( \{A\}' \) is the algebra of all matrices \( B \) such that \( AB = BA \). For a family \( \mathcal{F} \) of complex \( n \times n \) matrices, let \( \text{lin} (\mathcal{F}) \) and \( \text{alg} (\mathcal{F}) \) denote the subspace and the algebra generated by the family \( \mathcal{F} \), respectively. By \( J_n \) we denote the nilpotent \( n \times n \) Jordan block. For \( i, j \in \{1, 2, \ldots, n\} \), let \( E_{ij} \) denote the \( n \times n \) matrix whose entries are all 0 except in the \((i, j)\) cell, where it is 1.
Let $A$ be a positive $n \times n$ matrix. The super left-commutant $\langle A \rangle$ is the collection of all positive matrices $B$ such that $[A, B] \geq 0$. Similarly, the super right-commutant $[A]$ is the collection of all positive matrices $B$ such that $[A, B] \leq 0$. Since $B \in \langle A \rangle$ if and only if $B^T \in [A^T]$, we will consider super left-commutants only. It is easy to verify that $\langle A \rangle$ is an additive and multiplicative semigroup of positive matrices. It follows easily that $\text{lin} (\langle A \rangle) = \text{alg} (\langle A \rangle)$. More about super commutants can be found in [1].

3. Positive matrices

Searching for order analogs of Gerstenhaber’s theorem, Kandić and Šivic [6] have recently proved the following theorem [6, Theorem 3.2]. In fact, this theorem also follows from [7, Proposition 4.3].

**Theorem 3.1.** Let $A$ and $B$ be positive $n \times n$ matrices such that $[A, B] \geq 0$. Then the unital algebra $\mathcal{A}$ generated by $A$ and $B$ is triangularizable, and so its dimension is at most $n(n+1)/2$.

This result raises a question under which conditions the whole super left-commutant $\langle A \rangle$ is triangularizable. A possible answer is given by the following theorem and its corollary.

**Theorem 3.2.** Let $A$ be a positive $n \times n$ matrix with distinct eigenvalues. Let $B_1, B_2, \ldots, B_k$ be positive matrices such that $[A, B_i] \geq 0$ for all $i = 1, \ldots, k$. Then the unital algebra $\mathcal{A}$ generated by $A, B_1, B_2, \ldots, B_k$ is triangularizable, and so its dimension is at most $n(n+1)/2$.

**Proof.** Let $S = A + B_1 + \ldots + B_k$. Then, up to similarity with a permutation matrix, we may assume that

$$S = \begin{pmatrix}
S_{11} & S_{12} & S_{13} & \ldots & S_{1m} \\
0 & S_{22} & S_{23} & \ldots & S_{2m} \\
0 & 0 & S_{33} & \ldots & S_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & S_{mm}
\end{pmatrix}$$

and

$$A = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & \ldots & A_{1m} \\
0 & A_{22} & A_{23} & \ldots & A_{2m} \\
0 & 0 & A_{33} & \ldots & A_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A_{mm}
\end{pmatrix},$$

where $S_{11}, S_{22}, \ldots, S_{mm}$ are ideal-irreducible matrices. Since $0 \leq A \leq S$ and $0 \leq B_i \leq S$ for each $i$, every product of length $l$ formed from the matrices $A, B_1, \ldots, B_k$ is dominated by $S^l$. It follows that every member of $\mathcal{A}$ has the same block form as $S$, except that the diagonal blocks are not necessarily ideal-irreducible. Since $[A, S] = \sum_{i=1}^k [A, B_i] \geq 0$, we have $[A_{jj}, S_{jj}] \geq 0$ for all $j$. 
Fix \( j \in \{1, 2, \ldots, m\} \). By [3, Theorem 2.1], we obtain that \([A_{jj}, S_{jj}] = 0\), as \( S_{jj} \) is ideal-irreducible. Let \( B_{jj}^{(i)} \) denotes the \((j,j)\)-block of the matrix \( B_i \). Since \( 0 = [A_{jj}, S_{jj}] = \sum_{i=1}^{k} [A_{jj}, B_{jj}^{(i)}] \) and \([A_{jj}, B_{jj}^{(i)}] \geq 0\), we conclude that \([A_{jj}, B_{jj}^{(i)}] = 0\) for all \( i = 1, \ldots, k\). Since \( A \) has distinct eigenvalues, the same holds for \( A_{jj} \), and so its commutant \( \{A_{jj}\}' \) is diagonalizable. If \( A_{jj} \) denotes the algebra of all compressions to the \((j,j)\)-block of the members of \( A \), then \( A_{jj} \subseteq \{A_{jj}\}' \), and so the algebra \( A_{jj} \) is diagonalizable as well. It follows that the whole algebra \( A \) is triangularizable. \( \square \)

**Corollary 3.3.** Let \( A \) be a positive \( n \times n \) matrix with distinct eigenvalues. Then the algebra \( \text{alg} (\langle A \rangle) = \text{lin} (\langle A \rangle) \) is triangularizable, and so its dimension is at most \( n(n+1)/2 \).

**Proof.** Because of finite-dimensionality there exist positive matrices \( B_1, B_2, \ldots, B_k \) in \( \langle A \rangle \) such that \( \text{alg} (\langle A \rangle) = \text{lin} \{B_i : i = 1, \ldots, k\} \). We now apply Theorem 3.2. \( \square \)

If the matrix \( A \) in Corollary 3.3 is diagonal, more can be said. Consider first the special case.

**Proposition 3.4.** Let \( A \) be a positive diagonal \( n \times n \) matrix with strictly decreasing diagonal entries. Then the algebra generated by \( \langle A \rangle \) is equal to the algebra generated by \( A \) and \( J_n \), and it coincides with the algebra of all upper triangular matrices.

**Proof.** Let \( A \) be the algebra generated by \( A \) and \( J_n \), let \( S \) be the algebra generated by the super left-commutant \( \langle A \rangle \), and let \( UT \) be the algebra of all upper triangular matrices. Since \([A, J_n] \geq 0\), we have \( J_n \in \langle A \rangle \), and so \( A \subseteq S \). As \( A \) has distinct diagonal entries, there exists a polynomial \( p_i \) such that \( p_i(A) = E_{ii} \), so that \( E_{ii} \in A \) for all \( i = 1, 2, \ldots, n \). For any \( 1 \leq i < j \leq n \) we have \( E_{ii} J_n^{j-i} = E_{ij} \), and so \( UT \subseteq A \subseteq S \). By Corollary 3.3 the dimension of the algebra \( S \) is at most \( n(n+1)/2 \), so that we finally conclude that \( S = UT = A \). \( \square \)

It is well-known that the commutant of a complex diagonal \( n \times n \) matrix with distinct diagonal entries is equal to the algebra of all diagonal matrices, and so it has dimension \( n \). The following theorem says that the super left-commutant of a positive diagonal matrix with distinct diagonal entries spans maximal triangularizable algebra.

**Theorem 3.5.** Let \( A \) be a positive diagonal \( n \times n \) matrix with distinct diagonal entries. Then the algebra generated by \( \langle A \rangle \) is permutation similar to the algebra of all upper triangular matrices, and so its dimension is \( n(n+1)/2 \).
Proof. There exists a permutation matrix $P$ such that the positive diagonal matrix $P^T A P$ has strictly decreasing diagonal entries. Now, we apply Proposition \ref{prop:decreasing_diagonal} \hfill \Box

Examples show that in Corollary \ref{cor:decreasing_diagonal} we cannot omit the assumption on the eigenvalues of $A$. As a trivial example, we can take $A$ to be the identity matrix. More interesting examples can be obtained if we want, in addition, that the matrix $A$ is ideal-irreducible.

Example 3.6. Let $A = ee^T$ be an ideal-irreducible $n \times n$ matrix, where $e = (1, 1, \ldots, 1)^T$. Then $\langle A \rangle$ consists of all positive multiples of doubly stochastic matrices. Recall that a positive $n \times n$ matrix $S$ is doubly stochastic if $S e = e$ and $S^T e = e$, that is, each of its rows and columns sums to 1. Clearly, the super left-commutant $\langle A \rangle$ is not triangularizable provided $n \geq 3$. The dimension of the algebra generated by $\langle A \rangle$ is $(n-1)^2+1 = n^2-2n+2$ that is greater than $n(n+1)/2$ when $n \geq 5$.

4. Positive idempotents with positive commutators

Let us begin with a supplement of \cite[Theorem 6.3]{6}.

Theorem 4.1. Let $E$ a positive idempotent operator on a Archimedean vector lattice $L$, and let $A$ be an operator on $L$ such that either $AE \geq EA$ or $AE \leq EA$.

(a) If $N(E) = \{0\}$, then $AE = EAE$ and $(AE - EA)^2 = 0$.
(b) If $R(E) = L$, then $EA = EAE$ and $(AE - EA)^2 = 0$.
(c) If $N(E) = \{0\}$ and $R(E) = L$, then $AE = EA$.

Suppose, in addition, that the operators $A$ and $E$ are order continuous.

(d) If $R(E)^d = \{0\}$, then $EA = EAE$ and $(AE - EA)^2 = 0$.
(e) If $N(E) = \{0\}$ and $R(E)^d = \{0\}$, then $AE = EA$.

Proof. We consider only the case that $AE \geq EA$, as the other case can be treated similarly.

(a) It follows from $E(AE - EAE) = 0$ that $AE - EAE = 0$, since $AE - EAE = (AE - EA)E \geq 0$ and $N(E) = \{0\}$. Now, we have

$$E(AE - EA)^2 = EAEAE - EAEA - EA^2E + EAEAE = EA(EAE - AE) = 0.$$ 

Since $(AE - EA)^2 \geq 0$ and $N(E) = \{0\}$, we conclude that $(AE - EA)^2 = 0$.

(b) Since $EAE - EA = E(AE - EA) \geq 0$ and $R(E) = L$, the equality $(EAE - EAE)E = 0$ implies that $EA - EAE = 0$. We now compute

$$(AE - EA)^2 E = AEAE - AEAE - EA^2E + EAEAE = (EA - EAE)AE = 0.$$
Since \((AE - EA)^2 \geq 0\) and \(\mathcal{R}(E) = L\), we conclude that \((AE - EA)^2 = 0\).

(c) This is a direct consequence of (a) and (b).

(d) Using Lemma 2.2 the proof goes similar lines as the proof of (b).

(e) This follows from (a) and (d).

\(\square\)

The following result is a slight improvement of \([6, \text{Theorem 6.4}]\).

**Theorem 4.2.** Let \(L\) be a vector lattice, and let \(E\) and \(F\) be positive idempotent operators on a vector lattice \(L\) such that either \(EF \geq FE\) or \(EF \leq FE\). Let \(A\) be the unital algebra generated by \(E\) and \(F\).

(i) If either \(\mathcal{N}(E) = \{0\}\) or \(\mathcal{R}(E) = L\), then
\[
A = \text{lin} \{I, E, F, EF, FE, FEF\}.
\]

(ii) If \(\mathcal{N}(E) = \{0\}\) and \(\mathcal{R}(E) = L\), then
\[
A = \text{lin} \{I, E, F, EF\}.
\]

If the operators \(E\) and \(F\) are order continuous, the assumption \(\mathcal{R}(E) = L\) can be replaced by a (weaker) condition \(\mathcal{R}(E)^d = \{0\}\).

**Proof.** If \(\mathcal{N}(E) = \{0\}\), then Theorem 4.1(a) gives that \(EFE = FE\). If \(\mathcal{R}(E) = L\), then Theorem 4.1(b) implies that \(EFE = EF\). In both cases, we conclude that \(A = \text{lin} \{I, E, F, EF, FE, FEF\}\), proving the assertion (i). The assertion (ii) is a direct consequence of Theorem 4.1(c). The last assertion holds because of Theorem 4.1(d) and (e). \(\square\)

The proof of the main result of this section (Theorem 4.4) is based on the following key result.

**Theorem 4.3.** Let \(L\) be a vector lattice with the projection property. Let \(E\) be an order continuous positive idempotent operator on \(L\). Let \(A\) be an order continuous positive operator on \(L\) such that either \(AE \geq EA\) or \(AE \leq EA\). Then
\[
(AE - EA)^2E = E(AE - EA)^2 = 0,
\]
or equivalently
\[
(FA)^2E = EA^2E.
\]

**Proof.** Since \(E\) is order continuous, its null ideal \(\mathcal{N}(E)\) is a band of \(L\). Let us define the bands \(L_1, L_2, L_3\) and \(L_4\) by \(L_1 = \mathcal{N}(E) \cap \mathcal{R}(E)^d\), \(L_2 = \mathcal{N}(E) \cap \mathcal{R}(E)^dd\), \(L_3 = \mathcal{R}(E) \cap \mathcal{R}(E)^d\), \(L_4 = \mathcal{R}(E) \cap \mathcal{R}(E)^dd\).
\( \mathcal{N}(E)^d \cap \mathcal{R}(E)^{dd} \) and \( L_4 = \mathcal{N}(E)^d \cap \mathcal{R}(E)^d \). With respect to the band decomposition \( L = L_1 \oplus L_2 \oplus L_3 \oplus L_4 \), the idempotent \( E \) has the form

\[
E = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & X & Z \\
0 & 0 & G & Y \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

where \( G, X, Y \) and \( Z \) are positive operators on the appropriate bands. It follows from \( E^2 = E \) that \( G^2 = G, XG = X, GY = Y \) and \( Z = XY \). Therefore, we have

\[
E = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & XG & XGY \\
0 & 0 & G & GY \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 \\
X \\
I \\
0
\end{pmatrix} G \begin{pmatrix}
0 & 0 & I & Y
\end{pmatrix},
\]

\( \mathcal{N}(G) = \{0\} \) and \( \mathcal{R}(G)^d = \{0\} \). Writing \( A = [A_{ij}]_{i,j=1}^4 \) with respect to the same decomposition, we compute

\[
AE = \begin{pmatrix}
0 & 0 & (A_{12}X + A_{13})G & (A_{12}X + A_{13})GY \\
0 & 0 & (A_{22}X + A_{23})G & (A_{22}X + A_{23})GY \\
0 & 0 & (A_{32}X + A_{33})G & (A_{32}X + A_{33})GY \\
0 & 0 & (A_{42}X + A_{43})G & (A_{42}X + A_{43})GY
\end{pmatrix},
\]

and

\[
EA = \begin{pmatrix}
0 & XG(A_{31} + YA_{41}) & 0 & 0 \\
G(A_{31} + YA_{41}) & XG(A_{32} + YA_{42}) & XG(A_{33} + YA_{43}) & XG(A_{34} + YA_{44}) \\
0 & G(A_{32} + YA_{42}) & G(A_{33} + YA_{43}) & G(A_{34} + YA_{44}) \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

We now consider two cases.

Case 1: \( AE \geq EA \). Comparing the \((3,1)\)-block we conclude that \( G(A_{31} + YA_{41}) = 0 \). Since \( A_{31} + YA_{41} \geq 0 \) and \( \mathcal{N}(G) = \{0\} \), we have \( A_{31} + YA_{41} = 0 \), and so \( A_{31} = 0 \) and \( YA_{41} = 0 \). As \( \mathcal{N}(Y) = \{0\} \) we obtain that \( A_{41} = 0 \). Similarly, the equality \( G(A_{32} + YA_{42}) = 0 \) implies that \( A_{32} = 0 \) and \( A_{42} = 0 \). Comparing the \((3,3)\)-block we have \( A_{33}G \geq G(A_{33} + YA_{43}) \). If we apply the idempotent \( G \) on both sides, we obtain that \( GYA_{43}G = 0 \). Since \( \mathcal{N}(G) = \{0\}, \mathcal{R}(G)^d = \{0\} \) and \( \mathcal{N}(Y) = \{0\} \), we use Lemma 2.2 to conclude that \( A_{43} = 0 \). Consequently, we have the inequality \( A_{33}G \geq GA_{33} \), and so \( A_{33}G = GA_{33} \), by Theorem 4.1(c). This shows that the commutator of \( A \) and \( E \) has the form

\[
AE - EA = \begin{pmatrix}
0 & 0 & + & + \\
0 & 0 & + & + \\
0 & 0 & 0 & + \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
where each + denotes a positive block (that can be also 0). It follows that

$$(AE - EA)^2 = \begin{pmatrix} 0 & 0 & 0 & + \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so

$$E(AE - EA)^2 = 0 = (AE - EA)^2 E.$$  

Case 2: $AE \leq EA$. Comparing the $(1,3)$-block we obtain that $(A_{12}X + A_{13})G = 0$. Since $A_{12}X + A_{13}$ is a positive order continuous operator and $\mathcal{R}(G)^d = \{0\}$, we have $A_{12}X + A_{13} = 0$ by Lemma 2.2 and so $A_{12}X = 0$ and $A_{13} = 0$. Because of $\mathcal{R}(X)^d = \{0\}$ we finally obtain that $A_{12} = 0$. Similarly, the equality $(A_{42}X + A_{43})G = 0$ implies that $A_{42} = 0$ and $A_{43} = 0$. Comparing the $(3,3)$-block we have $(A_{32}X + A_{33})G \leq GA_{33}$. Applying the idempotent $G$ on both sides, we obtain that $GA_{32}XG = 0$. Since $\mathcal{N}(G) = \{0\}$, $\mathcal{R}(G)^d = \{0\}$ and $\mathcal{R}(X)^d = \{0\}$, we use Lemma 2.2 to conclude that $A_{32} = 0$. Consequently, we have the inequality $A_{33}G \leq GA_{33}$, and so $A_{33}G = GA_{33}$, by Theorem 4.1 (c). This shows that the commutator of $A$ and $E$ has the form

$$EA - AE = \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & + & + \\ + & 0 & 0 & + \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

It follows that

$$(AE - EA)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & 0 & + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so

$$E(AE - EA)^2 = 0 = (AE - EA)^2 E.$$  

The main result of this section now easily follows. With a different proof it was already shown in [6, Theorem 6.6] under slightly weaker assumption, as the order continuity of the idempotent $E$ was not needed.

**Theorem 4.4.** Let $L$ be a vector lattice with the projection property. Let $E$ and $F$ be order continuous positive idempotent operators on $L$ such that $EF \geq FE$. Then $(EF)^2E = EFE$ and $(FE)^2F = FEF$, and so the unital algebra $\mathcal{A}$ generated by $E$ and
$F$ is equal to the linear span of the set
$$\{I, E, F, EF, FE, EFE, FEF, (EF)^2, (FE)^2\}.$$ In particular, the dimension of $A$ is at most 9.

**Proof.** Applying Theorem 4.3 twice we obtain that $(EF)^2E = EFE$ and $(FE)^2F = FEF$. The remaining conclusions of the theorem are then clear. \qed

Theorem 4.4 can be slightly generalized using an extension theorem for positive order continuous operators [2, Theorem 1.65].

**Theorem 4.5.** Let $L$ be an Archimedean vector lattice. Let $E$ and $F$ be order continuous positive idempotent operators on $L$ such that $EF \geq FE$. Then the unital algebra generated by $E$ and $F$ is equal to the linear span of the set
$$\{I, E, F, EF, FE, EFE, FEF, (EF)^2, (FE)^2\}.$$ 

**Proof.** Let $L^\delta$ be the Dedekind completion of $L$. Since $L$ is order dense in $L^\delta$, the operator $E : L \to L^\delta$ is order continuous. By [2, Theorem 1.65], there is a unique order continuous linear idempotent extension $E_0 : L^\delta \to L^\delta$. If $F_0$ is an order continuous linear idempotent extension of $F$, then $E_0F_0 \geq F_0E_0$, and so we can apply Theorem 4.4 to complete the proof. \qed

Theorem 4.4 can be also slightly extended in the case when the order dual $L^\sim$ separates points of a vector lattice $L$. This condition is satisfied for normed lattices.

**Theorem 4.6.** Let $L$ be a vector lattice whose order dual $L^\sim$ separates points of $L$. Let $E$ and $F$ be positive idempotent operators on $L$ such that $EF \geq FE$. Then the unital algebra $A$ generated by $E$ and $F$ is equal to the linear span of the set
$$\{I, E, F, EF, FE, EFE, FEF, (EF)^2, (FE)^2\}.$$ 

**Proof.** By [2, Theorem 1.73], the order adjoint $T^\sim$ of a positive operator $T$ on $L$ is necessarily order continuous. Therefore, $E^\sim$ and $F^\sim$ are order continuous positive idempotent operators on $L^\sim$ such that $E^\sim F^\sim = (FE)^\sim \leq (EF)^\sim = F^\sim E^\sim$. By Theorem 4.4 the unital algebra $A^\sim = \{A^\sim : A \in A\}$ that is generated by $E^\sim$ and $F^\sim$ is equal to the linear span of the set
$$\{I, E^\sim, F^\sim, E^\sim F^\sim, F^\sim E^\sim, E^\sim F^\sim E^\sim, F^\sim E^\sim F^\sim, (E^\sim F^\sim)^2, (F^\sim E^\sim)^2\}.$$ Since the order dual $L^\sim$ separates points of $L$, the conclusion of the theorem follows. \qed
In the special case of matrices, Theorem 4.4 gives the following result.

**Corollary 4.7.** Let $E$ and $F$ be positive idempotent $n \times n$ matrices such that $EF \geq FE$. Then the unital algebra generated by $E$ and $F$ is equal to the linear span of the set \{I, E, F, EF, FE, EFE, FEF, (EF)^2, (FE)^2\}, and so its dimension is at most 9.

The upper bound of Theorem 4.4 (and Corollary 4.7) can be attained, as [6, Example 6.8] shows. Let us rewrite this example in such a way that the idempotent $F$ has the block form appeared in the proof of Theorem 4.4 and the unital algebra generated by $E$ and $F$ is ideal-triangularizable.

**Example 4.8.** Define the ideal-triangularizable idempotent positive matrices $E$ and $F$ by

$$E = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ 

Direct verifications prove that $EF \geq FE$ and that the elements of the set \{I, E, F, EF, FE, EFE, FEF, (EF)^2, (FE)^2\} are linearly independent matrices.

Several papers have been published about semigroups of idempotents, called bands in the abstract semigroup theory (see e.g. [4]). So, the following corollary of Theorem 4.4 is perhaps interesting.

**Corollary 4.9.** Let $L$ be a vector lattice with the projection property. Let $E$ and $F$ be order continuous positive idempotent operators on $L$ such that $EF \geq FE$. Suppose that the semigroup $S$ generated by $E$ and $F$ consists of positive idempotents. Then

$$S = \{E, F, EF, FE, EFE, FEF\},$$

and so the unital algebra generated by $E$ and $F$ is at most 7-dimensional.

The bound 7 in the last theorem cannot be improved. An example showing this can be obtained from Example 4.8 by deleting the last row and the last column of the matrices $E$ and $F$. 
If in Corollary 4.7 positivity of the commutator $[E, F]$ is removed, the dimension of the unital algebra generated by $E$ and $F$ is at most $2n$, as we have the following theorem (proved in [5]).

**Theorem 5.1.** A unital algebra generated by two $n \times n$ matrices with quadratic minimal polynomials is at most $2n$-dimensional if $n$ is even, and at most $(2n - 1)$-dimensional if $n$ is odd.

Examples in [5] show that the bounds on dimensions are sharp even in the case of idempotents. Now, we give such examples in which the two idempotents are also positive matrices.

**Example 5.2.** Let $n \in \mathbb{N}$ be an even integer, so that $n = 2k$ for some $k \in \mathbb{N}$. If $k = 1$, then define positive idempotents $E$ and $F$ by

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

Then the algebra $\mathcal{A}$ generated by $E$ and $F$ is equal to the algebra of all $2 \times 2$ matrices. In the proof of this conclusion one can use the fact that $2E_{11} = EF \in \mathcal{A}$. So, the dimension of $\mathcal{A}$ is 4 when $n = 2$.

Assume now that $k \geq 2$. Define positive idempotents $E$ and $F$ by

$$E = \begin{pmatrix} I & 2I \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} I & 0 \\ P & 0 \end{pmatrix},$$

where $I$ is the $k \times k$ identity matrix and $P$ is the $k \times k$ permutation matrix corresponding to the largest cycle:

$$P = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$ 

Define

$$C = E - F = \begin{pmatrix} 0 & 2I \\ -P & 0 \end{pmatrix}.$$ 

Then we claim that the following matrices from the unital algebra $\mathcal{A}$ generated by $E$ and $F$ are linearly independent:

$$C^{2j} = \begin{pmatrix} (-2P)^j & 0 \\ 0 & (-2P)^j \end{pmatrix}, \quad \text{and} \quad C^{2j+1} = \begin{pmatrix} 0 & 2(-2P)^j \\ -P(-2P)^j & 0 \end{pmatrix},$$ 

and...
\[ C^{2j}E = \begin{pmatrix} (-2P)^j & 2(-2P)^j \\ 0 & 0 \end{pmatrix} \] and \[ C^{2j+1}E = \begin{pmatrix} 0 & 0 \\ -P(-2P)^j & (-2P)^{j+1} \end{pmatrix}, j = 1, 2, \ldots, k. \]

To verify this claim, assume that
\[
\sum_{j=1}^{k} (a_j C^{2j} + b_j C^{2j+1} + c_j C^{2j} E + d_j C^{2j+1} E) = 0
\]
for some numbers \(a_j, b_j, c_j, d_j, j = 1, 2, \ldots, k\). It follows that
\[
\sum_{j=1}^{k} (a_j + c_j)(-2P)^j = 0,
\]
\[
\sum_{j=1}^{k} (b_j + c_j)2(-2P)^j = 0,
\]
\[
\sum_{j=1}^{k} (b_j + d_j)P(-2P)^j = 0,
\]
\[
\sum_{j=1}^{k} (a_j(-2P)^j + d_j(-2P)^{j+1}) = 0.
\]

From the first three equations we obtain that \(a_j = b_j = -c_j = -d_j\). Inserting this in the last equation, we get that
\[
a_1(-2P) + \sum_{j=2}^{k} (a_j - a_{j-1})(-2P)^j - a_k(-2)^k(-2P) = 0.
\]

This yields \(a_1 = a_2 = \ldots = a_k\) and \(a_1 = (-2)^k a_k\), and so \(a_j = 0\) for all \(j\). This complete the proof of the claim.

Since the dimension of \(\mathcal{A}\) is at most \(2n = 4k\) by Theorem 5.1, this dimension must be equal to \(2n\).

**Example 5.3.** Let \(n = 2k + 1\) for some \(k \in \mathbb{N}\). Take \(E, F\) and \(C = E - F\) as in Example 5.2 and let
\[
\tilde{E} = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} \quad \text{and} \quad \tilde{F} = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}.
\]

Then \(\tilde{E}\) and \(\tilde{F}\) are positive idempotent matrices, and the algebra \(\mathcal{A}\) generated by them has dimension \(4k + 1 = 2n - 1\). In the proof of the last claim (for \(k \geq 2\)) one can use the fact that
\[
\begin{pmatrix} 1 & 0 \\ 0 & (-2)^k I \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ C^{2k} \end{pmatrix} = (\tilde{E} - \tilde{F})^{2k} \in \mathcal{A} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 4^k I \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (-2)^k I \end{pmatrix}^2 \in \mathcal{A}
\]

imply that
\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}.
\]


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Roman Drnovšek
Department of Mathematics
Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 19
SI-1000 Ljubljana, Slovenia
e-mail : roman.drnovsek@fmf.uni-lj.si