Extremal conjugated unicyclic and bicyclic graphs with respect to total-eccentricity index

Mehar Ali Malik and Rashid Farooq

School of Natural Sciences,
National University of Sciences and Technology, Sector H-12, Islamabad, Pakistan

Abstract

Let $G$ be a molecular graph. The total-eccentricity index of graph $G$ is defined as the sum of eccentricities of all vertices of $G$. The extremal trees, unicyclic and bicyclic graphs, and extremal conjugated trees with respect to total-eccentricity index are known. In this paper, we extend these results and study the extremal conjugated unicyclic and bicyclic graphs with respect to total-eccentricity index.

Keywords: Topological indices, total-eccentricity index, conjugated graphs.

AMS Classification: 05C05, 05C35

1 Introduction

Let $G$ be an $n$-vertex molecular graph with vertex set $V(G)$ and edge set $E(G)$. The vertices and edges of $G$ respectively correspond to atoms and chemical bonds between atoms. A topological index $T$ is a numerical quantity associated with the chemical structure of a molecule. The aim of such association is to correlate these indices with various physico-chemical properties of a chemical compound. An edge between two vertices $u, v \in V(G)$ is denoted by $uv$. The order and size of $G$ are respectively the cardinalities $|V(G)|$ and $|E(G)|$. The neighbourhood $N_G(v)$ of a vertex $v$ in $G$ is the set of vertices adjacent to $v$. A simple graph is a graph without loops and multiple edges. All graphs considered in this paper are simple. The degree $d_G(v)$ of a vertex $v$ in $G$ is the cardinality $|N_G(v)|$. A graph $G$ is called a $k$-regular graph if $d_G(v) = k$ for all $v \in V(G)$. A vertex of degree 1 is called a pendant vertex. Let $P_n, C_n$ and $K_n$ respectively denote an $n$-vertex path, cycle and a complete graph. An $n$-vertex complete bipartite graph and $n$-vertex star is respectively denoted by $K_{a,b}$ and $K_{1,n-1}$ (or simply $S_n$), where $a + b = n$. A $(v_1, v_n)$-path with vertex set $\{v_1, v_2, \ldots, v_n\}$ is denoted by $v_1v_2\ldots v_n$. A graph $G$ is said to be connected if there exists a path between every pair of vertices in

*Corresponding author.

Email addresses: alies.camp@gmail.com, mehar.ali@sns.nust.edu.pk (M. A. Malik), farook.ra@gmail.com (R. Farooq).
A maximal connected subgraph of a graph is called a component. A vertex \( v \in V(G) \) is called a cut-vertex if deletion of \( v \), along with the edges incident on it, increases the number of components of \( G \). A maximal connected subgraph of a graph without any cut-vertex is called a block. A tree is a connected graph containing no cycle. Thus, an \( n \)-vertex tree is a connected graph with exactly \( n - 1 \) edges. An \( n \)-vertex unicyclic graph is a simple connected graph which contains \( n \) edges. Similarly, an \( n \)-vertex bicyclic graph is a simple connected graph which contains \( n + 1 \) edges.

A matching \( M \) in a graph \( G \) is a subset of edges of \( G \) such that no two edges in \( M \) share a common vertex. A vertex \( u \) in \( G \) is said to be \( M \)-saturated if an edge of \( M \) is incident with \( u \). A matching \( M \) is said to be perfect if every vertex in \( G \) is \( M \)-saturated. A conjugated graph is a graph that contains a perfect matching. In graphs representing organic compounds, perfect matchings correspond to Kekulé structures, playing an important role in analysis of the resonance energy and stability of hydrocarbons [6].

The distance \( d_G(u, v) \) between two vertices \( u, v \in V(G) \) is defined as the length of a shortest path between \( u \) and \( v \) in \( G \). If there is no path between vertices \( u \) and \( v \) then \( d_G(u, v) \) is defined to be \( \infty \).

The eccentricity \( e_G(v) \) of a vertex \( v \in V(G) \) is defined as the largest distance from \( v \) to any other vertex in \( G \). The diameter \( \text{diam}(G) \) and radius \( \text{rad}(G) \) of a graph \( G \) are respectively defined as:

\[
\text{rad}(G) = \min_{v \in V(G)} e_G(v),
\]
\[
\text{diam}(G) = \max_{v \in V(G)} e_G(v).
\]

A vertex \( v \in V(G) \) is said to be central if \( e_G(v) = \text{rad}(G) \). The graph induced by all central vertices of \( G \) is called the center of \( G \), denoted as \( C(G) \). A vertex \( w \) is called an eccentric vertex of a vertex \( v \) in \( G \) if \( d_G(v, w) = e_G(v) \). The set of all eccentric vertices of \( v \) in a graph \( G \) is denoted by \( E_G(v) \).

The first topological index was introduced by Wiener [18] in 1947, to calculate the boiling points of paraffins. In 1971, Hosoya [8] defined the notion of Wiener index for any graph as the half sum of distances between all pairs of vertices. The average-eccentricity of an \( n \)-vertex graph \( G \) was defined in 1988 by Skorobogatov and Dobrynin [16] as:

\[
\text{avec}(G) = \frac{1}{n} \sum_{u \in V(G)} e_G(u).
\]

In the recent literature, a minor modification of average-eccentricity index \( \text{avec}(G) \) is used and referred as total-eccentricity index \( \tau(G) \). It is defined as:

\[
\tau(G) = \sum_{u \in V(G)} e_G(u).
\]

The eccentric-connectivity index and the Randić index of a graph \( G \) are defined respectively by \( \xi(G) = \sum_{u \in V(G)} d_G(v)e_G(v) \) and \( R(G) = \sum_{uv \in E(G)} (d_G(u)d_G(v))^{-\frac{1}{2}} \). Liang and Liu [11] proved a conjecture on the relation between the average-eccentricity and Randić index. Dankelmann and Mukwembi [3] obtained upper bounds on the average-eccentricity in terms of several graph parameters. Smith et al. [17] studied the extremal values of total-eccentricity index in trees. Ilic [9] studied some extremal graphs with respect to average-eccentricity. Farooq et al. [4] studied the extremal unicyclic and bicyclic graphs and extremal conjugated
trees with respect to total-eccentricity index. For more details on topological indices of graphs and networks, the author is referred to [1, 2]. In this paper, we extend the results of [4] to conjugated unicyclic and bicyclic graphs.

For some special families of graphs of order \( n \geq 4 \), the total-eccentricity index is given as follows:

1. For a \( k \)-regular graph \( G \), we have \( \tau(G) = \frac{\xi(G)}{k} \).
2. \( \tau(K_n) = n \).
3. \( \tau(K_{m,n}) = 2(m + n), m, n \geq 2 \).
4. The total-eccentricity index of a star \( S_n \), a cycle \( C_n \) and a path \( P_n \) is given by

\[
\tau(S_n) = 2n - 1, \quad (1.5)
\]
\[
\tau(C_n) = \begin{cases} 
\frac{3n^2}{4} - \frac{n}{4} & \text{if } n \equiv 0 \pmod{2} \\
\frac{n^2}{2} - \frac{n}{4} - \frac{1}{4} & \text{if } n \equiv 1 \pmod{2},
\end{cases} \quad (1.6)
\]
\[
\tau(P_n) = \begin{cases} 
\frac{3n^2}{4} - \frac{n}{4} & \text{if } n \equiv 0 \pmod{2} \\
\frac{n^2}{2} - \frac{n}{4} - \frac{1}{4} & \text{if } n \equiv 1 \pmod{2}.
\end{cases} \quad (1.7)
\]

Let \( \{v_1, v_2, \ldots, v_n\} \) be the vertices of a path \( P_n \). Let \( U_2 \) be a unicyclic graph obtained from \( P_n \) by joining \( v_1 \) and \( v_3 \) by an edge. Similarly, let \( B_2 \) be a bicyclic graph obtained from \( P_n \) by joining \( v_1 \) with two vertices \( v_3 \) and \( v_4 \). Note that when \( n \equiv 0 \pmod{2} \), the graphs \( U_2 \) and \( B_2 \) are conjugated and are denoted by \( \overline{U}_2 \) and \( \overline{B}_2 \), respectively. In Figure 1, we give two 7-vertex bicyclic graphs \( U_2 \) and \( B_2 \). For \( n \equiv 0 \pmod{2} \), let \( S_n^* \) be an \( n \)-vertex conjugated tree obtained by identifying one vertex each from \( \frac{n-2}{2} \) copies of \( P_3 \) and deleting a single pendent vertex. Let \( v \) be the unique central vertex of \( S_n^* \). Let \( U_1 \) be a conjugated unicyclic graph obtained from \( S_n^* \) by adding an edge between \( v \) and any vertex not adjacent to \( v \). In a similar fashion, let \( B_1 \) be a conjugated bicyclic graph obtained from \( S_n^* \) by adding two edges between \( v \) and any two vertices not adjacent to \( v \) (see Figure 2).

Figure 1: The 7-vertex unicyclic and bicyclic graphs \( U_2 \) and \( B_2 \).

Figure 2: The 8-vertex conjugated unicyclic and bicyclic graphs \( \overline{U}_1, \overline{U}_2, \overline{B}_1 \) and \( \overline{B}_2 \).
Now we give some previously known results on the center of a graph from [7] and some results on extremal graphs with respect to total-eccentricity index from [4]. In next theorem, we give a result dealing with the location of center in a connected graph.

**Theorem 1.1** (Harary and Norman [7]). *The center of a connected graph $G$ is contained in a block of $G$.*

The only possible blocks in a unicyclic graph are $K_1, K_2$ or a cycle $C_k$. Thus the following corollary gives the center of an $n$-vertex conjugated unicyclic graph $U$.

**Corollary 1.2.** *If $U$ is an $n$-vertex conjugated unicyclic graph with a unique cycle $C_k$, then $C(U) = K_1$ or $K_2$, or $C(U) \subseteq C_k$.*

The following results give the extremal unicyclic and bicyclic graphs with respect to total-eccentricity index.

**Theorem 1.3** (Farooq et al. [4]). *Among all $n$-vertex unicyclic graphs, $n \geq 4$, the graph $U_2$ shown in Figure 1 has maximal total-eccentricity index.*

**Theorem 1.4** (Farooq et al. [4]). *Among all $n$-vertex bicyclic graphs, $n \geq 5$, the graph $B_2$ shown in Figure 1 has the maximal total-eccentricity index.*

In Section 2, we find extremal conjugated unicyclic and bicyclic graphs with respect to total-eccentricity index.

## 2 Conjugated unicyclic and bicyclic graphs

In this section, we find extremal conjugated unicyclic and bicyclic graphs with respect to total-eccentricity index. In (2.1), we give the total-eccentricity index of the conjugated graphs $\overline{U}_1$, $\overline{U}_2$, $\overline{B}_1$ and $\overline{B}_2$ which can easily be computed.

$$
\tau(\overline{U}_1) = \frac{7}{2}n - 3, \quad \tau(\overline{U}_2) = \frac{3n^2}{4} - n - \frac{3}{4},
\tau(\overline{B}_1) = \frac{7}{2}n - 4, \quad \tau(\overline{B}_2) = \frac{3}{4}n^2 - n - 2. \tag{2.1}
$$

Using Theorem 1.1 and Corollary 1.2, we prove the following result.

**Remark 2.1.** *When $n = 4$, the graph shown in Figure 3(a) has the smallest total-eccentricity index among all 4-vertex conjugated unicyclic graphs. When $n = 6$, the graphs shown in Figure 3(b) and Figure 3(c) have smallest total-eccentricity index among 6-vertex conjugated unicyclic graphs. When $n = 8$, the graph shown in Figure 3(d) has smallest total-eccentricity index among 8-vertex conjugated unicyclic graphs.*

**Theorem 2.1.** *Let $n \equiv 0 \pmod{2}$ and $n \geq 10$. Then among all $n$-vertex conjugated unicyclic graphs, the graph $\overline{U}_1$ shown in Figure 2 has the minimal total-eccentricity index.*

**Proof.** Let $\overline{U}_1$ be the $n$-vertex conjugated unicyclic graph shown in Figure 2. Let $\overline{U}$ be an arbitrary $n$-vertex conjugated unicyclic graph with a unique cycle $C_k$. We show that
\(\tau(U) \geq \tau(U_1)\). Let \(n_i\) denote the number of vertices with eccentricity \(i\) in \(U\). If \(k \geq 8\) or \(\text{rad}(U) \geq 4\) then
\[
\tau(U) \geq 4n > \frac{7n}{2} - 3 = \tau(U_1).
\]
In the rest of the proof, we assume that \(k \in \{3,4,5,6,7\}\) and \(\text{rad}(U) \in \{2,3\}\). Let \(k \in \{6,7\}\).

If \(x\) is a vertex of \(U\) such that \(x\) is not on \(C_k\), then \(e_U(x) \geq 4\). Also, it is easily seen that there are at most five vertices on \(C_k\) with eccentricity 3. Thus
\[
\tau(U) \geq 3(5) + 4(n - 5) = 4n - 5 > \frac{7n}{2} - 3 = \tau(U_1).
\]

We complete the proof by considering the following cases.

**Case 1.** When \(\text{rad}(U) = 3\) and \(k \in \{3,4,5\}\). By Corollary 1.2, \(C(U) = K_1\) or \(C(U) = K_2\) or \(C(U) \subseteq C_k\). This shows that \(U\) has at most five vertices with eccentricity 3. Thus the inequality (2.2) holds in this case.

**Case 2.** When \(\text{rad}(U) = 2\) and \(k \in \{4,5\}\). Then \(\text{diam}(U) \leq 2\text{rad}(U) = 4\) and there will be exactly one vertex in \(C(U)\). That is, \(n_2 = 1\). Let \(v\) be the vertex with \(e_U(v) = 2\). Then \(v \in V(C_k)\). Considering several possibilities for longest possible paths (of length 2) starting from \(v\) and that \(U\) is conjugated, one can see that \(U\) is isomorphic to one of the graphs shown in Figure 4. Moreover, observe that \(n_4 \geq \frac{n}{2} - 1\).

Since \(n_2 + n_3 + n_4 = n\) and \(n_2 = 1\), we can write
\[
\begin{align*}
\tau(U) &= 2n_2 + 3n_3 + 4n_4 \\
&= 2 + 3(n_3 + n_4) + n_4 \\
&= 3n - 1 + n_4 \\
&\geq 3n - 1 + \frac{n}{2} - 1 = \frac{7n}{2} - 2 > \tau(U_1).
\end{align*}
\]
Case 3. When \( \text{rad}(\overline{U}) = 2 \) and \( k = 3 \). Then \( n_2 = 1 \) (see Figure 5). Let \( v \) be the unique central vertex of \( \overline{U} \). Then either \( v \) is a vertex of \( C_3 \) or \( v \) is adjacent to a vertex of \( C_3 \). When \( v \in V(C_3) \), then \( \overline{U} \) is isomorphic to one of the graphs shown in Figure 5(a), 5(b) or 5(c). In this case, all vertices with eccentricity 4 are pendent. This gives \( n_4 \geq \frac{n}{2} - 2 \). Therefore

\[
\tau(\overline{U}) = 2(1) + 3(n_3 + n_4) + n_4 \\
\geq 3n - 1 + \frac{n}{2} - 2 \\
= \frac{7n}{2} - 3 = \tau(U_1).
\]

Similarly, if the central vertex \( v \) is not on \( C_3 \), then \( U \) is isomorphic to one of the graphs shown in Figure 5(d) or 5(e). Note that \( n_4 \geq \frac{n}{2} \). Thus \( \tau(\overline{U}) \geq 2(1) + 3(n - 1) + \frac{n}{2} = \frac{7n}{2} - 1 > \tau(U_1) \).

Combining all the cases, we see that \( U_1 \) is the minimal graph with respect to total-eccentricity index. This completes the proof.

![Figure 5: The \( n \)-vertex conjugated unicyclic graphs discussed in Case 3.](image)

The following theorem gives the maximal conjugated unicyclic graphs with respect to total-eccentricity index.

**Theorem 2.2.** Let \( n \equiv 0(\text{mod}2) \). Then the \( n \)-vertex conjugated unicyclic graph corresponding to the maximal total-eccentricity index is the graph \( \overline{U}_2 \) shown in Figure 2.

**Proof.** Note that the class of all \( n \)-vertex conjugated unicyclic graphs forms a subclass of the class of all \( n \)-vertex unicyclic graphs. From Theorem 1.3 we see that among all \( n \)-vertex unicyclic graphs, the graph \( U_2 \) (see Figure 1) has the largest total-eccentricity index. Since \( U_2 \) admits a a perfect matching when \( n \equiv 0(\text{mod}2) \), the result follows.

**Corollary 2.3.** For an \( n \)-vertex conjugated unicyclic graph \( \overline{U} \), we have \( \frac{7n}{2} - 3 \leq \tau(\overline{U}) \leq \frac{3n^2}{4} - n - \frac{3}{4} \).

**Proof.** Using Theorem 2.1, Theorem 2.2 and equation (2.1), we obtain the required result.

The next theorem gives the minimal conjugated bicyclic graphs with respect to total-eccentricity index.

**Remark 2.2.** Let \( n = 4 \). Then among all 4-vertex conjugated bicyclic graphs, one can easily see that the graph shown in Figure 6(a) has the minimal total-eccentricity index. Similarly, when \( n = 6 \) and 8, then the graphs respectively shown in Figure 6(b) and Figure 6(c) have the minimal total-eccentricity index among all 6-vertex and 8-vertex conjugated bicyclic graphs.
Figure 6: The $n$-vertex conjugated bicyclic graphs with minimal total-eccentricity index when $n = 4, 6$ and 8.

**Theorem 2.4.** Let $n \equiv 0 \pmod{2}$ and $n \geq 10$. Then among the $n$-vertex conjugated bicyclic graphs, the graph $\overline{B}_1$ shown in Figure 2 has the minimal total-eccentricity index.

**Proof.** Let $\overline{B}_1$ be the $n$-vertex conjugated bicyclic graph shown in Figure 2. Let $\overline{B}$ be an arbitrary $n$-vertex conjugated bicyclic graph and $\overline{B} \not\equiv \overline{B}_1$. Let $C(\overline{B})$ denote the center of $\overline{B}$ and $n_i$ denote the number of vertices with eccentricity $i$. The proof is divided into two cases depending upon the number of cycles in $\overline{B}$.

**Case 1.** When $\overline{B}$ contains two edge-disjoint cycles $C_{k_1}$ and $C_{k_2}$ of lengths $k_1$ and $k_2$, respectively. Without loss of generality, assume that $k_1 \leq k_2$. If $\tau(\overline{B}) \geq 4$ or $k_2 \geq 8$, then

$$\tau(\overline{B}) \geq 4n > \frac{7n}{2} - 4 = \tau(\overline{B}_1).$$

Thus, we assume that $k_2 \in \{3, 4, 5, 6, 7\}$ and $\tau(\overline{B}) \in \{2, 3\}$. If $k_2 \in \{6, 7\}$, then for any vertex $x \notin V(C_{k_2})$, $e_\overline{B}(x) \geq 4$. Moreover, as $k_2 \leq 7$, the number of vertices with eccentricity 3 are at most 7. Thus

$$\tau(\overline{B}) \geq 3(7) + 4(n - 7) = 4n - 7 > \frac{7n}{2} - 4 = \tau(\overline{B}_1). \quad (2.3)$$

We consider the following three subcases.

**(a)** Let $\text{rad}(\overline{B}) = 3$ and $k_2 \in \{3, 4, 5\}$. By Theorem 1.1, we have $|V(C(\overline{B}))| \leq 7$. Thus $\tau(\overline{B})$ satisfies equation (2.3).

**(b)** Let $\text{rad}(\overline{B}) = 2$ and $k_2 \in \{4, 5\}$. Take $v \in V(C(\overline{B}))$. We observe that the center $C(\overline{B})$ is contained in $C_{k_2}$ and $n_2 = 1$. Assume $v$ to be the unique central vertex of $\overline{B}$. Then for several possible choices for possible pendent vertices in the conjugated graph $\overline{B}$, one can observe that $\overline{B}$ is one of the graphs shown in Figure 7. Moreover $n_4 \geq \frac{n}{2} - 2$. Then $\tau(\overline{B}) \geq 2n_2 + 3n_3 + 4n_4 = 2n_2 + 3(n_3 + n_4) \geq 2(1) + 3(n - 1) + \frac{n}{2} - 2 = \frac{7n}{2} - 3 > \tau(\overline{B}_1)$.

**(c)** Let $\text{rad}(\overline{B}) = 2$ and $k_2 = 3$. If $C(\overline{B}) \subseteq C_{k_2}$ then $C(\overline{B}) = K_1$, otherwise $n \leq 8$ which is not true. Similarly, if $C(\overline{B}) \not\subseteq C_{k_2}$, then $C(\overline{B}) = K_2$ or $K_1$. Since $n_2 \not\geq 2$, we have $n_2 = 1$. Let $c$ be the unique central vertex. Then $\overline{B}$ is isomorphic to one of the graphs shown in Figure 8. When $c$ is not a vertex of $C_{k_1}$ or $C_{k_2}$, then $n_4 \geq \frac{n}{2} + 1$ and $\tau(\overline{B}) \geq 2(1) + 3(n - 1) + \frac{n}{2} + 1 = \frac{7n}{2} > \tau(\overline{B}_1)$. If $c$ is a vertex of $C_{k_1}$ or $C_{k_2}$, then $n_4 \geq \frac{n}{2} - 3$. Thus $\tau(\overline{B}) \geq 2(1) + 3(n - 1) + \frac{n}{2} - 3 = \frac{7n}{2} - 4 = \tau(\overline{B}_1)$.

**Case 2.** When cycles of $\overline{B}$ share some edges. Then there are cycles $C_{k_1}$, $C_{k_2}$ and $C_{k_3}$ in $\overline{B}$ of lengths $k_1$, $k_2$ and $k_3$, respectively. Without loss of generality, assume that $k_1 \leq k_2 \leq k_3$. Then $k_1, k_2 \geq 3$ and $k_3 \geq 4$. Let $Q$ be the subgraph of $\overline{B}$ induced by the vertices of $C_{k_1}$, $C_{k_2}$ and $C_{k_3}$. Clearly, $Q$ contains the cycle $C_{k_3}$. Assume that $V(C_{k_3}) = \{v_1, v_2, \ldots, v_{k_3}\}$. Then
Figure 7: The $n$-vertex conjugated bicyclic graphs discussed in Case 1-(b).

Figure 8: The $n$-vertex conjugated bicyclic graphs studied in Case 1-(c).

$\overline{B}$ is minimal with respect to total-eccentricity index if $Q$ can be obtained from the cycle $C_{k_3}$ by adding the edge $v_1v_{\left\lfloor \frac{k_3}{2} \right\rfloor + 1}$ or $v_1v_{\left\lfloor \frac{k_3}{2} \right\rfloor + 2}$.

When $k_3 \geq 13$, then $\epsilon_{\overline{B}}(w) \geq 4$ for all $w \in V(\overline{B})$. This gives us $\tau(\overline{B}) \geq 4n > \tau(B_1)$. Assume that $\text{rad}(\overline{B}) \leq 3$ and $k_3 \leq 12$. We consider the following two subcases:

(a) When $k_3 \in \{9, 10, 11, 12\}$. Then $\text{rad}(\overline{B}) = 3$. There exists a vertex $c$ with degree 3 in $Q$ such that $c \in C(\overline{B})$ and $c$ has at most $\left\lfloor \frac{n}{2} \right\rfloor + 1$ neighbours not in $Q$. Clearly, all of these vertices will have eccentricity 4. Then $\frac{n}{2} - 5$ such vertices can have unique pendant vertices with eccentricity 5. Moreover, there will be at least one vertex with eccentricity 5 in $Q$, otherwise $\text{rad}(\overline{B}) \neq 3$. Thus $n_5 \geq \frac{n}{2} - 4$. Moreover, at most 3 vertices in $Q$ can have eccentricity 3. This gives $1 \leq n_3 \leq 3$. When $k_3 = 9$, such a graph $\overline{B}$ is shown in Figure 9(a). The vertex $v \in V(Q)$ such that $\epsilon_{\overline{B}}(v) = 5$ is also shown in the figure. Using the facts that $n = n_3 + n_4 + n_5$ and $n_4 + n_5 = n - n_3 \geq n - 3$. We get

\[
\tau(\overline{B}) = 3n_3 + 4n_4 + 5n_5 \\
= 3n_3 + 4(n_4 + n_5) + n_5 \\
\geq 3(1) + 4(n - 3) + \frac{n}{2} - 4 \\
= \frac{9n}{2} - 13 \geq \tau(B_1).
\]

(b) When $k_3 \in \{4, 5, 6, 7, 8\}$. We first assume that $\text{rad}(\overline{B}) = 3$. Then by Theorem 1.1, we have $1 \leq n_3 \leq 8$. Thus $\tau(\overline{B}) \geq 3(8) + 4(n - 8) = 4n - 8 \geq \tau(B_1)$. On the other hand, when $\text{rad}(\overline{B}) = 2$. Then $n_2 = 1$. When $C(\overline{B}) \not\subseteq Q$, then the minimal graph $\overline{B}$ is isomorphic to the graph shown in Figure 9(b). Clearly, $n_4 \geq \frac{n}{2}$. Thus $\tau(\overline{B}) = 2n_2 + 3(n_3 + n_4) + n_4 \geq
2(1) + 3(n - 1) + \frac{n}{2} = \frac{7n}{2} - 1 > \tau(\overline{B}_1). \quad \text{When } C(\overline{B}) \subseteq Q, \text{ then } \overline{B} \text{ is isomorphic to one of the graphs shown in Figures 9(c)–9(h). It can be seen that } n_4 \geq \frac{n}{2} - 2. \text{ Thus we can write } \\
\tau(\overline{B}) = 2n_2 + 3(n_3 + n_4) + n_4 \geq 2(1) + 3(n - 1) + \frac{n}{2} - 2 = \frac{7n}{2} - 3 = \tau(\overline{B}_1).

Combining the results of Case 1 and Case 2, we see that among all conjugated bicyclic graphs, \( \overline{B}_1 \) has the minimal total-eccentricity index. The proof is complete.

**Theorem 2.5.** Let \( n \equiv 0 \pmod{2} \). Then among the \( n \)-vertex conjugated bicyclic graphs, the graph \( \overline{B}_2 \) shown in Figure 2 has the maximal total-eccentricity index.

**Proof.** For \( n \equiv 0 \pmod{2} \), the proof can be derived from the proof of Theorem 1.4.

**Corollary 2.6.** For any conjugated bicyclic graph \( \overline{B} \), we have \( \frac{7n}{2} - 4 \leq \tau(\overline{B}) \leq \frac{3n^2}{4} - n - 2. \)

**Proof.** The result follows by using Theorem 2.4, Theorem 2.5 and equation (2.3).

### 3 Conclusion

In this paper, we extended the results of Farooq et al. [4] and studied the extremal conjugated unicyclic and bicyclic graphs with respect to total-eccentricity index.

**Acknowledgements**

The authors are thankful to the Higher Education Commission of Pakistan for supporting this research under the grant 20-3067/NRPU/R&D/HEC/12/831.

**References**

1. S. Akhter, R. Farooq, Computing the eccentric connectivity index and eccentric adjacency index of conjugated trees, Util. Math. (To appear)

2. T. Doslić, M. Salehi, Eccentric connectivity index of composite graphs, Utilitas Mathematica, 95, 3-22 (2014).
[3] P. Dankelmann, S. Mukwembi, Upper bounds on the average eccentricity, Discrete Appl. Math. 167, 72-79 (2014).

[4] R. Farooq, M.A. Malik, J. Rada, Extremal graphs with respect to total-eccentricity index, submitted.

[5] R. Farooq, S. Akhter, J. Rada, Total-eccentricity index of trees with fixed pendent vertices, submitted.

[6] I. Gutman, O. Polansky, Mathematical Concepts in Organic Chemistry, Springer-Verlag, Berlin (1986).

[7] F. Harary, R.Z. Norman, The dissimilarity characteristic of Husimi trees, Ann. Math. 58(1), 134-141 (1953).

[8] H. Hosoya, Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, Bull. Chem. Soc. Jpn. 4, 2332-2339 (1971).

[9] A. Ilić, On the extremal properties of the average eccentricity, Comput. Math. Appl. 64, 2877-2885 (2012).

[10] C. Jordan, Sur les assemblages de lignes, J. Reine Agnew. Math. 70, 185-190 (1869).

[11] M. Liang and J. Liu, Proofs of conjectures on the Randić index and average eccentricity, Discrete Appl. Math. 202, 188–193 (2016).

[12] M.J. Morgan, S. Mukwembi, H.C. Swart, On the eccentric connectivity index of a graph, Discrete Math. 311, 1229-1234 (2011).

[13] J. Ma, Y. Shi, Z. Wang, J. Yue, On Wiener polarity index of bicyclic networks, Sci. Rep. 6, 19066 (2016).

[14] M.J. Morgan, S. Mukwembi, H.C. Swart, A lower bound on the eccentric connectivity index of a graph, Discrete Appl. Math. 160, 248-258 (2012).

[15] V. Sharma, R. Goswami, A.K. Madan, Eccentric connectivity index: a novel highly discriminating topological descriptor for structure-property and structure-activity studies, J. Chem. Inf. Comput. Sci. 37, 273-282 (1997).

[16] V. A. Skorobogatov, A. A. Dobrynin, Metric analysis of graphs, MATCH Commun. Math. Comp. Chem. 23, 105-151 (1988).

[17] H. Smith, L. Szekely, H. Wang, Eccentricity sums in trees, Discrete Appl. Math. 65, 41-51 (2004).

[18] H. Wiener, Structural determination of the paraffin boiling points, J. Am. Chem. Soc. 69, 17-20 (1947).

[19] B. Zhou, Z. Du, On eccentric connectivity index, MATCH Commun. Math. Comput. Chem. 63, 181-198 (2010).