THE FIBERED ISOMORPHISM CONJECTURE IN 
\(L\)-THEORY

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Abstract. This is the first of three articles on the Fibered Isomorphism Conjecture of Farrell and Jones for \(L\)-theory. We apply the general techniques developed in [15] and [16] to the \(L\)-theory case of the conjecture and prove several results.

Here we prove the conjecture, after inverting 2, for poly-free groups. In particular, it follows for braid groups. We also prove the conjecture for some classes of groups without inverting 2. In fact we consider a general class of groups satisfying certain conditions which includes the above groups and some other important classes of groups. We check that the properties we defined in [15] are satisfied in several instances of the conjecture.

1. Introduction

This is the first of three articles where we study the Fibered Isomorphism Conjecture of Farrell and Jones (\([4], 1.7\)) for \(L^{(-\infty)}\)-theory. The Isomorphism Conjecture was stated in \([4], 1.6\) for the pseudoisotopy theory, \(K\)-theory and the \(L^{(-\infty)}\)-theory. The conjecture says that these theories can be computed for a group if we can compute them for all its virtually cyclic subgroups. It is known that the conjecture is not true for the other \(L\)-theory functors \(L^h\) and \(L^s\) (see [5]). The Fibered Isomorphism Conjecture is stronger and is appropriate for induction arguments. This property is crucial for the method we use here. We developed some general techniques in [15] and [16] which applies to all the three cases of the conjecture and proved the Fibered Isomorphism Conjecture for the pseudoisotopy case for several classes of groups.

Here we see that the techniques can be used effectively to prove the Fibered Isomorphism Conjecture for \(L^{(-\infty)}\)-theory and for \(L^{(-\infty)}\)-theory for some well-known classes of groups, where \(L^{(-\infty)} = L^{(-\infty)} \otimes \mathbb{Z}[\frac{1}{2}]\). The advantage of taking \(L^{(-\infty)}\)-theory is that we can consider

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the ‘finite subgroups’ instead of the ‘virtually cyclic subgroups’. This will also help to prove the conjecture for $L^{(-\infty)}$-theory for a larger class of groups.

We also show that the two properties $(\mathcal{T}_c^H, \mathcal{P}_c^H)$ defined in [15] are satisfied in several instances of the conjecture for $L^{(-\infty)}$-theory.

Throughout the article by ‘group’ we mean ‘discrete countable group’ unless otherwise mentioned.

**Definition 1.1.** Let $\text{FICWF} (\text{FICWF})$ be the smallest class of groups satisfying the conditions 1 to 5 (i to iv) below.

1. $\text{FICWF}$ contains the cocompact discrete subgroups of linear Lie groups with finitely many components.
2. (Subgroup) If $H < G \in \text{FICWF}$ then $H \in \text{FICWF}$
3. (Free product) If $G_1, G_2 \in \text{FICWF}$ then $G_1 \ast G_2 \in \text{FICWF}$.
4. (Direct limit) If $\{G_i\}_{i \in I}$ is a directed sequence of groups with $G_i \in \text{FICWF}$. Then the limit $\lim_{i \in I} G_i \in \text{FICWF}$.
5. (Extension) For an exact sequence of groups $1 \to K \to G \to N \to 1$, if $K, N \in \text{FICWF}$ then $G \in \text{FICWF}$.
   i. $\text{FICWF}$ contains the cocompact discrete subgroups of Lie groups with finitely many components.
   ii. 2, 3 and 4 as above after replacing $\text{FICWF}$ by $\text{FICWF}$.
   iii. (Direct product) If $G_1, G_2 \in \text{FICWF}$ then $G_1 \times G_2 \in \text{FICWF}$.
   iv. (Polycyclic extension) For an exact sequence of groups $1 \to K \to G \to N \to 1$, if either $K$ is virtually cyclic and $N \in \text{FICWF}$ or $N$ is finite and $K \in \text{FICWF}$ then $G \in \text{FICWF}$.

For two groups $A$ and $B$ the wreath product $A \wr B$ is the semidirect product $A^B \rtimes B$ with respect to the regular action of $B$ on $A^B$. Here $A^B$ denotes the direct sum of copies of $A$ indexed by $B$. Let $\mathcal{VC}$ and $\mathcal{FIN}$ denote the class of virtually cyclic groups and the class of finite groups respectively.

We prove the following Theorem. For notations see Section 2.

**Theorem 1.1.** Let $\Gamma \in \text{FICWF}$ and $\Delta \in \text{FICWF}$. Let $G$ and $H$ be two groups with homomorphisms $\phi : G \to \Gamma \wr F$ and $\psi : H \to \Delta \wr K$ where $F$ and $K$ are finite groups. Then the following assembly maps are isomorphisms for all $n$.

$$\mathcal{H}_n^G(E_{\phi, FIN(V(1:F))}(G), L^{(-\infty)}) \to \mathcal{H}_n^G(pt, L^{(-\infty)}) \simeq L_n^{(-\infty)}(ZG).$$

$$\mathcal{H}_n^H(E_{\psi, \mathcal{VC}(\Delta(K))}(H), L^{(-\infty)}) \to \mathcal{H}_n^H(pt, L^{(-\infty)}) \simeq L_n^{(-\infty)}(ZH).$$

In other words the Fibered Isomorphism Conjecture of Farrell and Jones for the $L^{(-\infty)}$-theory ($L^{(-\infty)}$-theory) is true for the group $\Gamma \wr F$.
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$\langle \Delta \wr K \rangle$. Equivalently, the \( \text{FICwF}_L^H \langle \Gamma \rangle \) and the \( \text{FICwF}_V^H \langle \Delta \rangle \) are satisfied.

The notations in the above statement are described in the next section.

**Theorem 1.2.** Let \( \mathcal{C} (\mathcal{D}) \) be the class of groups which satisfy the \( \text{FICwF}_L^H \langle \Gamma \rangle \) (the \( \text{FICwF}_V^H \langle \Delta \rangle \)). Then \( \mathcal{C} (\mathcal{D}) \) has the properties 2 to 5 (ii to iv) after replacing \( \text{FICWF} \) (\( \text{FICWF} \)) by \( \mathcal{C} (\mathcal{D}) \) in Definition 1.1.

Our next goal is to show that \( \text{FICWF} \) and \( \text{FICWF} \) contains some well-known classes of groups.

**Theorem 1.3.** \( \text{FICWF} \) contains the following groups.
1. Virtually cyclic groups.
2. Free groups and abelian groups.
3. Poly-free groups. A poly-free group \( G \) admits a filtration by subgroups: \( 1 < G_0 < G_1 < \cdots < G_n = G \) so that \( G_i \) is normal in \( G_{i+1} \) and \( G_{i+1}/G_i \) is free. Here \( n \) is called the index of \( G \).
4. Strongly poly-free groups. See \([2] \), definition 1.1].
5. Full braid groups.
6. Cocompact discrete subgroups of Lie groups with finitely many components.
7. Groups whose some derived subgroup belong to \( \text{FICWF} \).
\( \text{FICWF} \) contains the following groups.
i. Virtually cyclic groups.
ii. Free groups and abelian groups.
iii. Groups appearing in 6 (by definition).
iv. Virtually polycyclic groups.

**Remark 1.1.** Here we should remark that the non-fibered version of the Isomorphism conjecture in \( L^{(-\infty)} \)-theory for a class of groups including poly-free groups and one-relator groups was proved in [[1], theorem 0.13].

In the next section we recall the statement of the conjecture in the more general context in equivariant homology theory from [1]. Also we recall some notations and definitions from [15]. For the statement of the original Isomorphism Conjectures see [[4], 1.6, 1.7].

When \( \Gamma \) and \( \Delta \) are torsion free, \( F = K = \{1\} \) and \( \phi \) and \( \psi \) are the identity maps, Theorem 1.1 reduces to the isomorphism of the classical assembly map in surgery theory. Therefore we have the following corollary. See 1.6.1 in [4] for details.
Corollary 1.1. Let $\Gamma \in \mathcal{FICWF}$ and $\Delta \in \mathcal{FICWF}$ and in addition assume that $\Gamma$ and $\Delta$ are torsion free. Then the following assembly maps are isomorphism for all $n$.

$$H_n(B\Gamma, L_{(-\infty)}^{(-\infty)}) \to L_n^{(-\infty)}(\mathbb{Z}\Gamma).$$

$$H_n(B\Delta, L_{(-\infty)}^{(-\infty)}) \to L_n^{(-\infty)}(\mathbb{Z}\Delta).$$

In other words the surgery groups $L_n^{(-\infty)}(\mathbb{Z}\Gamma)$ of $\Gamma$ and the surgery groups $L_n^{(-\infty)}(\mathbb{Z}\Delta)$ of $\Delta$ form generalized homology theories.

Since the surgery groups with different decoration defer by 2-torsion we also have the following. See [[6], section 5, para 1].

Lemma 1.1. $L_n^{(-\infty)}(\mathbb{Z}\Gamma) \simeq L_n^h(\mathbb{Z}\Gamma) \otimes \mathbb{Z}[\frac{1}{2}] \simeq L_n^s(\mathbb{Z}\Gamma) \otimes \mathbb{Z}[\frac{1}{2}]$ for any group $\Gamma$.

Therefore, Lemma 1.1 implies that the Theorem 1.1 is true for the functors $L_n^h \otimes \mathbb{Z}[\frac{1}{2}]$ and $L_n^s \otimes \mathbb{Z}[\frac{1}{2}]$ also. We have already mentioned in the Introduction that the Theorem 1.1 is not true for $L_n^h$ and $L_n^s$ if we do not tensor with $\mathbb{Z}[\frac{1}{2}]$.

Remark 1.2. The main ingredient behind the proof of Theorem 1.1 is [[4], theorem 2.1 and remark 2.1.3]. This was also used before to prove the $FIC_{\mathcal{L}}^{\mathcal{H}_i}$ and the $FIC_{\mathcal{V}}^{\mathcal{H}_i}$ in [6] for elementary amenable groups and for torsion free virtually solvable subgroups of $GL(n, \mathbb{C})$.

Remark 1.3. We also note here that the $IC_{\mathcal{V}}^{\mathcal{H}_i}$ is known for many classes of groups. See [[9], 5.3]. In [8] it was proved that the $IC_{\mathcal{V}}^{\mathcal{H}_i}$ is true for the fundamental groups of closed manifolds with $\mathbb{S}L \times \mathbb{E}^n$ structure for $n \geq 2$.

2. Statement of the Isomorphism Conjecture and some basic results in $L$-theory

Given a normal subgroup $H$ of a group $G$ by [[7], algebraic lemma] $G$ can be embedded as a subgroup in the wreath product $H \wr (G/H)$. We will always use this fact without mentioning.

Let $\mathcal{H}_i$ be an equivariant homology theory with values in $R$-modules for $R$ a commutative associative ring with unit. In this article we are considering the special case $R = \mathbb{Z}$.

In this section we always assume that a class of groups $\mathcal{C}$ is closed under isomorphisms, taking subgroups and taking quotients. We denote by $\mathcal{C}(G)$ the class of subgroups of a group $G$ which belong to $\mathcal{C}$.

Given a group homomorphism $\phi : G \to H$ and $\mathcal{C}$ a class of subgroups of $H$ define $\phi^*\mathcal{C}$ by the class of subgroups $\{K < G \mid \phi(K) \in \mathcal{C}\}$ of $G$. 



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For a class $\mathcal{C}$ of subgroups of a group $G$ there is a $G$-CW complex $E_\mathcal{C}(G)$ which is unique up to $G$-equivalence satisfying the property that for each $H \in \mathcal{C}$ the fixpoint set $E_\mathcal{C}(G)^H$ is contractible and $E_\mathcal{C}(G)^H = \emptyset$ for $H$ not in $\mathcal{C}$.

(Fibered) Isomorphism Conjecture: ([1], definition 1.1) Let $G$ be a group and $\mathcal{C}$ be a class of subgroups of $G$. Then the Isomorphism Conjecture for the pair $(G, \mathcal{C})$ states that the projection $p : E_\mathcal{C}(G) \to \text{pt}$ to the point $\text{pt}$ induces an isomorphism

$$H^n G (p) : H^n_\mathcal{C}(E_\mathcal{C}(G)) \simeq H^n_\mathcal{C}(\text{pt})$$

for $n \in \mathbb{Z}$.

And the Fibered Isomorphism Conjecture for the pair $(G, \mathcal{C})$ states that for any group homomorphism $\phi : K \to G$ the Isomorphism Conjecture is true for the pair $(K, \phi^* \mathcal{C})$.

Definition 2.1. ([15], definition 2.1) Let $\mathcal{C}$ be a class of groups. If the (Fibered) Isomorphism Conjecture is true for the pair $(G, \mathcal{C}(G))$ we say that the $\text{(F)IC}^H_\mathcal{C}$ is true for $G$ or simply say $\text{(F)IC}^H_\mathcal{C}(G)$ is satisfied. Also we say that the $\text{(F)ICwF}^H_\mathcal{C}(G)$ is satisfied if the $\text{(F)IC}^H_\mathcal{C}$ is true for $G \wr H$ for any finite group $H$.

Clearly, if $H \in \mathcal{C}$ then the $\text{(F)IC}^H_\mathcal{C}(H)$ is satisfied.

The following is easy to prove and is known as the hereditary property of the Fibered Isomorphism Conjecture.

Lemma 2.1. If the $\text{FIC}^H_\mathcal{C}(G)$ (the $\text{FICwF}^H_\mathcal{C}(G)$) is satisfied then the $\text{FIC}^H_\mathcal{C}(H)$ (the $\text{FICwF}^H_\mathcal{C}(H)$) is satisfied for any subgroup $H$ of $G$.

Let us denote by $\underline{L}^\mathcal{H}_*(\underline{L}^\mathcal{H}_*)$, the equivariant homology theory arises for the $\underline{L}^{(-\infty)}$-theory $(\underline{L}^{(-\infty)}$-theory). In the statement of Theorem 1.1 the notation $\mathcal{H}^G_n(-, \underline{L}^{(-\infty)}(-))$ stands for $\underline{L}^\mathcal{H}_n^G(-)$, where $\underline{L}^{(-\infty)}$ denotes the spectrum whose homotopy groups are the surgery groups $\underline{L}^{(-\infty)}_*(-, \underline{L}^{(-\infty)}(-))$. See [[9], section 6.2] for details.

Throughout the article a ‘graph’ is assumed to be connected and locally finite.

Definition 2.2. ([15], definition 2.2) We say that $\mathcal{T}_\mathcal{C}^\mathcal{H}_*$ ($\text{FIC}^\mathcal{H}_*$) is satisfied if for a graph of groups $\mathcal{G}$ with vertex groups (and hence edge groups) belonging to the class $\mathcal{C}$, the $\text{FIC}^\mathcal{H}_*$ (the $\text{FICwF}^\mathcal{H}_*$) for $\pi_1(\mathcal{G})$ is true.
We say that $\tau_T^H_\ast$ (or $\tau_T^H_\ast$) is satisfied if for a graph of groups $\mathcal{G}$ with trivial edge groups and the vertex groups belonging to the class $\mathcal{C}$, the $\text{FIC}_{\mathcal{C}}^H_\ast$ (the $\text{FICwF}_{\mathcal{C}}^H_\ast$) for $\pi_1(\mathcal{G})$ is true.

And we say that $\mathcal{P}_{\mathcal{C}}^H_\ast$ is satisfied if for $G_1, G_2 \in \mathcal{C}$ the product $G_1 \times G_2$ satisfies the $\text{FIC}_{\mathcal{C}}^H_\ast$.

We start with the following general lemma which easily follows from [[15], proposition 5.2] and [[16], lemma 3.4].

**Lemma 2.2.** Assume that $\mathcal{P}_{\mathcal{C}}^H_\ast$ is satisfied.

1. If the $\text{FIC}_{\mathcal{C}}^H_\ast$ (the $\text{FICwF}_{\mathcal{C}}^H_\ast$) is true for $G_1$ and $G_2$ then $G_1 \times G_2$ satisfies the $\text{FIC}_{\mathcal{C}}^H_\ast$ (the $\text{FICwF}_{\mathcal{C}}^H_\ast$).

2. Let $G$ be a finite index subgroup of a group $K$. If the group $G$ satisfies the $\text{FICwF}_{\mathcal{C}}^H_\ast$ then $K$ also satisfies the $\text{FICwF}_{\mathcal{C}}^H_\ast$.

3. Let $p : G \to Q$ be a group homomorphism. If the $\text{FICwF}_{\mathcal{C}}^H_\ast$ is true for $Q$ and for $p^{-1}(H)$ for all $H \in \mathcal{C}(Q)$ then the $\text{FICwF}_{\mathcal{C}}^H_\ast$ is true for $G$.

The following lemma is obvious.

**Lemma 2.3.** For a class of groups $\mathcal{C}$ the $\text{FIC}_{\mathcal{C}}^L_\ast$ implies the $\text{FIC}_{\mathcal{C}}^L_\ast$ and the $\text{FICwF}_{\mathcal{C}}^L_\ast$ implies the $\text{FICwF}_{\mathcal{C}}^L_\ast$.

If the Isomorphism Conjecture is true for a group with respect to a class $\mathcal{C}$ of subgroups then obviously it is true for the group with respect to a class of subgroups containing $\mathcal{C}$. The following Lemma shows that sometimes the converse is also true.

**Lemma 2.4.** If a group $G$ satisfies the $\text{FIC}_{\mathcal{C}}^L_\ast$ (the $\text{FICwF}_{\mathcal{C}}^L_\ast$) then it also satisfies the $\text{FIC}_{\mathcal{F}_{\mathcal{L}N}}^L_\ast$ (the $\text{FICwF}_{\mathcal{F}_{\mathcal{L}N}}^L_\ast$).

**Proof.** See [[6], lemma 5.1] or [[9], proposition 2.18].

**Lemma 2.5.** Let $\mathcal{C} = \mathcal{VC}$ or $\mathcal{FLN}$. Let $\{G_i\}_{i \in I}$ be a directed sequence of groups with limit $G$. If each $G_i$ satisfies $\#$ then $G$ also satisfies $\#$. Here $\#$ is one of the followings.

1. $\text{FIC}_{\mathcal{C}}^L_\ast$. 2. $\text{FIC}_{\mathcal{C}}^L_\ast$. 3. $\text{FIC}_{\mathcal{C}}^L_\ast$. 4. $\text{FICwF}_{\mathcal{C}}^L_\ast$.

**Proof.** For 1 and 2 it directly follows from [[6], theorem 7.1] and for 3 and 4 just note that for a finite group $F$, $G \wr F$ is a direct limit of $\{G_i \wr F\}_{i \in I}$ and then apply [[6], theorem 7.1].
Lemma 2.6. Let $\Gamma$ be a discrete cocompact subgroup of a Lie group with finitely many connected components. Then $\Gamma$ satisfies the $FIC^L_{\mathcal{H}_1^\mathcal{V}_C}$, $FIC^L_{\mathcal{H}_2^\mathcal{V}_C}$, $FIC^L_{\mathcal{F}_L\mathcal{N}^\mathcal{V}_C}$, $FICwF^L_{\mathcal{H}_1^\mathcal{V}_C}$, $FICwF^L_{\mathcal{H}_2^\mathcal{V}_C}$ and the $FICwF^L_{\mathcal{F}_L\mathcal{N}^\mathcal{V}_C}$.

Proof. By Lemma 2.3 $FIC^L_{\mathcal{H}_1^\mathcal{V}_C}$ implies $FIC^L_{\mathcal{H}_2^\mathcal{V}_C}$ and then apply Lemma 2.4 to get $FIC^L_{\mathcal{F}_L\mathcal{N}^\mathcal{V}_C}$. Similarly $FICwF^L_{\mathcal{H}_1^\mathcal{V}_C}$ implies $FICwF^L_{\mathcal{H}_2^\mathcal{V}_C}$ and then applying Lemma 2.4 we get $FICwF^L_{\mathcal{F}_L\mathcal{N}^\mathcal{V}_C}$. Therefore we only have to show that $\Gamma$ satisfies $FIC^L_{\mathcal{H}_1^\mathcal{V}_C}$ and $FICwF^L_{\mathcal{H}_2^\mathcal{V}_C}$. For $FIC^L_{\mathcal{H}_1^\mathcal{V}_C}$ it follows directly from [4], theorem 2.1 and remark 2.1.3.

For $FICwF^L_{\mathcal{H}_2^\mathcal{V}_C}$ we need the following lemma.

Lemma 2.7. Let $G$ be a Lie group with finitely many components and let $F$ be a finite group. Then the wreath product $G\wr F$ is again a Lie group with finitely many components with respect to the product topology on $G^F \times F$ where $F$ is given the discrete topology and $G^F$ denotes the $|F|$-times direct product of $G$.

Proof. Recall that an element of $G^F$ is of the form $(g_{f_1}, \ldots, g_{f_{|F|}})$ where $f_i \in F$. Now let $f \in F$. Then the regular action of $F$ on $G^F$ is by definition $f(g_{f_1}, \ldots, g_{f_{|F|}}) = (g_{f_1 f^{-1}}, \ldots, g_{f_{|F|} f^{-1}})$. It now follows from the definition of semi-direct product that the product and inverse operations on $G\wr F$ both are smooth. Therefore $G\wr F$ is a Lie group and clearly it has finitely many components.

Now if $\Gamma$ is a discrete cocompact subgroup of $G$ and $G$ has finitely many components then it is easy to verify that $G\wr F$ is a discrete cocompact subgroup of $G\wr F$ for any finite group $F$. Here the Lie group structure on $G\wr F$ is as described in Lemma 2.7.

Hence we can again use [[4], theorem 2.1 and remark 2.1.3] to see that the $FICwF^L_{\mathcal{H}_2^\mathcal{V}_C}$ is satisfied for $\Gamma$.

This completes the proof.

Lemma 2.8. $\mathcal{P}^L_{\mathcal{H}_1^\mathcal{V}_C}$, $\mathcal{P}^L_{\mathcal{H}_2^\mathcal{V}_C}$ and $\mathcal{P}^L_{\mathcal{F}_L\mathcal{N}^\mathcal{V}_C}$ are satisfied.

Proof. Recall that $\mathcal{P}^L_{\mathcal{H}_1^\mathcal{V}_C}$ states that the $FIC^L_{\mathcal{H}_1^\mathcal{V}_C}$ is true for $V_1 \times V_2$ for any two virtually cyclic groups $V_1$ and $V_2$. Let $V_1$ and $V_2$ be two such groups then $V_1 \times V_2$ contains a free abelian normal subgroup $H$ (say) (on at most 2 generators) of finite index. Hence $V_1 \times V_2$ is a subgroup of $H \wr ((V_1 \times V_2)/H)$. Therefore by Lemma 2.1 it is enough to prove the $FIC^L_{\mathcal{H}_1^\mathcal{V}_C}$ for $H \wr ((V_1 \times V_2)/H)$.

Now we need the following well known fact.
Lemma 2.9. Let $S$ be a closed orientable surface of genus $\geq 1$. Then $\pi_1(S)$ is a discrete cocompact subgroup of a Lie group with finitely many components.

Proof. If the genus of $S$ is 1 then $\pi_1(S)$ is a discrete cocompact subgroup of the Lie group of isometries of the flat Euclidean plane. And if the genus of $S$ is $\geq 2$ then the corresponding Lie group is the group of isometries of the hyperbolic plane. □

If $V_1 \times V_2$ is virtually cyclic then there is nothing to prove. If $H$ has rank 2 then applying Lemmas 2.9 and 2.6 we see that $\mathcal{P}_{\mathcal{H}^i_{V}}$ is satisfied. Next we apply Lemma 2.3 to see that $\mathcal{P}_{\mathcal{H}^i_{VC}}$ is also satisfied. And $\mathcal{P}_{\mathcal{H}^i_{FIN}}$ is obvious. □

Definition 2.3. Choose two classes of groups $C_1$ and $C_2$ so that $C_1 \subset C_2$. We say that $T_{\mathcal{H}^i_{C_1, C_2}}$ is satisfied if for a graph of groups $G$ with vertex groups belonging to the class $C_1$ the FIC$_{\mathcal{H}^i_{C_2(\pi_1(G))}}(\pi_1(G))$ is satisfied.

Lemma 2.10. $T_{\mathcal{H}^i_{FIN, VC}}$, $T_{\mathcal{H}^i_{FIN, VC}}$, $wT_{\mathcal{H}^i_{FIN}}$, $wT_{\mathcal{H}^i_{VC}}$ and $wT_{\mathcal{H}^i_{VC}}$ are satisfied.

Proof. At first we check $T_{\mathcal{H}^i_{FIN, VC}}$. So let $G$ be a group and $\mathcal{G}$ be a graph of finite groups with $\pi_1(\mathcal{G}) \simeq G$. If $\mathcal{G}$ is an infinite graph then we write $\mathcal{G}$ as an increasing union of finite subgraphs $\mathcal{G}_i$. Then $\pi_1(\mathcal{G}) \simeq \lim_{i \to \infty} \pi_1(\mathcal{G}_i)$. Hence using Lemma 2.5 we can assume that $\mathcal{G}$ is finite. It is now well known that $\pi_1(\mathcal{G})$ contains a finitely generated free subgroup of finite index. See [[15], lemma 3.2]. $T_{\mathcal{H}^i_{FIN, VC}}$ now follows from the following Main Lemma.

To check $wT_{\mathcal{H}^i_{FIN}}$ and $wT_{\mathcal{H}^i_{VC}}$ we only need to use Lemmas 2.3 and 2.4.

Next we prove $wT_{\mathcal{H}^i_{VC}}$ and $wT_{\mathcal{H}^i_{VC}}$.

Let $\mathcal{G}$ be a graph of groups with virtually cyclic vertex groups and trivial edge groups. As before we can assume that $\mathcal{G}$ is finite. Hence the group $\pi_1(\mathcal{G})$ is virtually free. This follows from the following two Lemmas.

Lemma 2.11. If a graph of groups $\mathcal{G}$ has trivial edge groups then $\pi_1(\mathcal{G})$ is isomorphic to the free product of a free group and the vertex groups of $\mathcal{G}$.

Proof. Apply [[15], lemma 6.2] and note that $\mathcal{G}$ is a direct limit of its finite subgraphs of groups. □
Lemma 2.12. Let $V_1$ and $V_2$ be two virtually free groups then $V_1 \ast V_2$ is virtually free.

Proof. We have a surjective homomorphism $p : V_1 \ast V_2 \to V_1 \times V_2$. Let $H_i$ be a free subgroup of $V_i$ of finite index for $i = 1, 2$. Hence $H = H_1 \times H_2$ has finite index in $V_1 \times V_2$. Note that $V_1 \ast V_2$ acts on a tree with trivial edge stabilizers and the vertex stabilizers are conjugate to $V_1$ or $V_2$. Hence $p^{-1}(H)$ also acts on the same tree. It follows that the edge stabilizers of this restricted action are again trivial and the vertex stabilizers are free. And hence $p^{-1}(H)$ is a free group by Lemma 2.11. This completes the proof. □

Therefore we can apply Lemmas 2.5 and the following Main Lemma to complete the proof of Lemma 2.10. □

Lemma 2.13. Assume that the $\#$ is true for two groups $G_1$ and $G_2$ then the $\#$ is true for the direct product $G_1 \times G_2$. Here $\#$ denotes one of the followings.

1. $\text{FIC}_{V_1C}^{H_1}$. 2. $\text{FIC}_{V_2C}^{H_2}$. 3. $\text{FICwF}_{V_1C}^{H_1}$. 4. $\text{FICwF}_{V_1\times C}^{H_1}$.

Proof. The proof is a combination of Lemma 2.8 and (1) of Lemma 2.2. □

Main Lemma. The $\text{FICwF}_{V_1C}^{\pi_1 H_1}$, $\text{FICwF}_{V_2C}^{\pi_1 H_2}$ and $\text{FICwF}_{\pi_1 H_1}$ are true for any virtually free group.

Proof. We only prove the lemma for the $\text{FICwF}_{V_1C}^{\pi_1 H_1}$. The other two conclusions will follow using Lemmas 2.3 and 2.4.

Let $\Gamma$ be a virtually free group and $G$ be a free normal subgroup of $\Gamma$ with $F$ the finite quotient group. Let $F'$ be another finite group and denote by $C$ the wreath product $F \wr F'$. Then we have the following inclusions.

$$\Gamma \wr F' < (G \wr F) \wr F' < G^{F \times F'} \wr C < (G \wr C) \times \cdots \times (G \wr C).$$

The inclusions are easy to check. (See [14], lemma 5.4 for the second inclusion). There are $|F \times F'|$ factors in the last term. Therefore using Lemmas 2.1 and 2.13 we see that it is enough to prove the $\text{FIC}_{V_1C}^{\pi_1 H_1}$ for $G \wr C$ for an arbitrary finite group $C$. Equivalently, we need to prove the $\text{FICwF}_{V_1C}^{\pi_1 H_1}$ for $G$. If $G$ is infinitely generated then let $G$ be the limit of a sequence of finitely generated subgroups of $G$. Hence by Lemma 2.5 we can assume that $G$ is finitely generated. Therefore $G$ is isomorphic to the fundamental group of an orientable 2-manifold $M$ with boundary. Consequently, $G$ is isomorphic to a subgroup of $\pi_1(M \cup \partial M)$, where $M \cup \partial M = S$ (say) denotes the double of $M$. Again using Lemma 2.1
it is enough to prove the $FICwF^i_{VC}$ for $\pi_1(S)$, where $S$ is a closed orientable surface. Without loss of generality we can assume that $S$ has genus $\geq 1$. Now applying Lemmas 2.9 and 2.6 we complete the proof of the Main Lemma.

Proposition 2.1. Assume that the $#$ is true for two groups $G_1$ and $G_2$ then the $#$ is true for the free product $G_1 \ast G_2$ also. Here $#$ is one of the followings:

1. $FIC^H^i_{VC}$. 2. $FIC^LH^i_{VC}$. 3. $FIC^LH^i_{FLN}$. 4. $FICwF^i_{VC}$. 5. $FICwF^L_{VC}$. 6. $FICwF^L_{FLN}$.

Proof. The proof follows from Lemmas 2.8, 2.10 and [[15], lemma 6.3].

Lemma 2.14. Let $1 \to K \to G \to N \to 1$ be an exact sequence of groups. Then the followings hold.

1. If the $FICwF^i_{FLN}(K)$ and the $FIC^H^i_{FLN}(N)$ are satisfied then the $FIC^H^i_{FLN}(G)$ is also satisfied.

2. If the $FICwF^i_{FLN}(K)$ and the $FICwF^L_{FLN}(N)$ are satisfied then the $FICwF^L_{FLN}(G)$ is also satisfied.

Proof. Apply (2) and (3) of Lemma 2.2 and note that $P^H^i_{FLN}$ is satisfied.

3. Braid groups

Let $\mathbb{C}^N$ be the $N$-dimensional complex space. A hyperplane arrangement in $\mathbb{C}^N$ is by definition a finite collection $\{V_1, V_2, \ldots, V_n\}$ of $(N-1)$-dimensional linear subspaces of $\mathbb{C}^N$.

Now we recall the definition of a fiber-type hyperplane arrangement from [[11], page 162]. Let us denote by $V_n$ the arrangement $\{V_1, V_2, \ldots, V_n\}$ in $\mathbb{C}^N$. $V_n$ is called strictly linearly fibered if after a suitable linear change of coordinates, the restriction of the projection of $\mathbb{C}^N - \bigcup_{i=1}^n V_i$ to the first $(N-1)$ coordinates is a fiber bundle projection whose base space is the complement of an arrangement $W_{n-1}$ in $\mathbb{C}^{N-1}$ and whose fiber is the complex plane minus finitely many points. By definition the arrangement 0 in $\mathbb{C}$ is fiber-type and $V_n$ is defined to be fiber-type if $V_n$ is strictly linearly fibered and $W_{n-1}$ is of fiber type. It follows by repeated application of homotopy exact sequence for fbration that the complement $\mathbb{C}^N - \bigcup_{i=1}^n V_i$ is aspherical and hence the fundamental group is torsion free.

Lemma 3.1. ([[7], theorem 5.3]) $\pi_1(\mathbb{C}^N - \bigcup_{i=1}^n V_i)$ is a strongly poly-free group.
Now recall that the pure braid group $PB_n$ on $n$ strings is by definition $\pi_1(\mathbb{C}^{n+1} - \cup_{i,j} V_{ij})$ where $V_{ij}$ is the hyperplane $x_i = x_j$ for $i < j$ and $x_i$'s being the coordinates in $\mathbb{C}^{n+1}$. One can show that $\{V_{ij}\}$ is a fiber-type arrangement and hence $PB_n$ is a strongly poly-free group. See [[2], theorem 2.1].

The full braid group $B_n$ is by definition $\pi_1((\mathbb{C}^{n+1} - \cup_{i,j} V_{ij})/S_{n+1})$ where the symmetric group $S_{n+1}$ on $(n + 1)$-symbols acts on $\mathbb{C}^{n+1} - \cup_{i,j} V_{ij}$ by permuting the coordinates. This action is free and therefore $PB_n$ is a normal subgroup of $B_n$ with quotient $S_{n+1}$.

Recall that in [2] we proved the following.

**Theorem 3.1.** (theorem 1.3 and corollary 1.4 in [2]) Let $\Gamma$ be the fundamental group of a fiber-type hyperplane arrangement complement or more generally a strongly poly-free group. Then $Wh(\Gamma) = K_0(\mathbb{Z}\Gamma) = K_i(\mathbb{Z}\Gamma) = 0$ for $i < 0$.

Theorem 3.1 and an application of the Rothenberg’s exact sequence show the following. See [[12], 17.2].

**Lemma 3.2.** Let $\Gamma$ be as in Theorem 3.1 then $L_n^{(-\infty)}(\mathbb{Z}\Gamma) \simeq L_n^h(\mathbb{Z}\Gamma) \simeq L_n^s(\mathbb{Z}\Gamma)$.

In the situation of $\Gamma$ as in Theorem 3.1, Lemma 3.2 shows that the 2-torsions which appear in the three surgery groups are also isomorphic.

In [17] we will use Corollary 1.1, Theorem 1.3 and Lemma 3.2 to calculate explicitly the surgery groups of the fundamental groups of fiber-type hyperplane arrangement complement in the complex space. In particular this applies to the pure braid groups.

4. **Proof of Theorem 1.3**

**Proof of Theorem 1.3.** 1 & i. Since $FICWF$ contains the discrete cocompact subgroups of (linear) Lie groups with finitely many components it follows that finite groups and the infinite cyclic group belong to $FICWF$. Next apply the ‘polycyclic extension’ condition to complete the proof of (1) (i)).

2 & ii. At first note that a countable infinitely generated group is a direct limit of finitely generated subgroups.

Now using (1) (i)) and the ‘free product’ condition we get that finitely generated free groups belong to $FICWF$ and since $FICWF$ has the property ‘direct limit’ the proof follows for infinitely generated free groups.

Using (1) (i)) and the ‘extension’ condition we see that finitely generated abelian groups belong to $FICWF$.
(FICWF). Therefore countable abelian groups belong to $\text{FICWF}$ (FICWF) by the ‘direct limit’ condition.

3. The proof is by induction on the index of the poly-free group. If $n = 1$ then $G$ is free and hence $G \in \text{FICWF}$ (apply (2)). So assume that poly-free groups of index $\leq n - 1$ belong to $\text{FICWF}$ and let $G$ has an index $n$ filtration. Note that $G_{n-1}$ is a poly-free group of index $n - 1$ and $G/G_{n-1}$ is a free group. Since $\text{FICWF}$ is closed under extensions we can now apply (2) and the induction hypothesis to show that $G \in \text{FICWF}$.

4 & 5. Recall from Section 3 that pure braid groups are strongly poly-free and strongly poly-free groups are poly-free. Also the full braid group $B_n$ contains the pure braid group $PB_n$ as a subgroup of finite index. Hence using 2 and since $\text{FICWF}$ is closed under extensions the proofs of 3 and 4 are complete.

6. Let $\Gamma$ be a cocompact discrete subgroup of a Lie group with finitely many components. Then following the steps in the proof of [[16], 2(a) of theorem 2.2] or of [[4], theorem 2.1] we have the following three exact sequences.

\[ 1 \to F \to \Gamma \to \Gamma' \to 1. \]

\[ 1 \to \Gamma_R \to \Gamma' \to \Gamma_S \to 1. \]

\[ 1 \to \Gamma_H \to \Gamma_S \to \Gamma_L \to 1. \]

Where $F$ is finite, $\Gamma_R$ is virtually poly-$\mathbb{Z}$, $\Gamma_H$ is virtually finitely generated abelian and $\Gamma_L$ is a cocompact discrete subgroup of a Linear Lie group with finitely many components. Now note that finitely generated free abelian groups and poly-$\mathbb{Z}$ groups (since they are also poly-free) belong to $\text{FICWF}$. Therefore, we can again apply the hypothesis that $\text{FICWF}$ is closed under extensions and use the above three exact sequences to complete the proof of 5.

7. Let $\Gamma$ be a group so that $\Gamma^{(n)} \in \text{FICWF}$ for some $n$. Using the extension condition it is enough to show that $\Gamma/\Gamma^{(n)} \in \text{FICWF}$, that is we need to show that $\text{FICWF}$ contains the solvable groups.

So let $\Gamma$ be a solvable group. Using the ‘direct limit’ condition in the definition of $\text{FICWF}$ we can assume that $\Gamma$ is finitely generated, for any countable infinitely generated group is a direct limit of finitely generated subgroups.
We say that $\Gamma$ is $n$-step solvable if $\Gamma^{(n+1)} = (1)$ and $\Gamma^{(n)} \neq (1)$. The proof is by induction on $n$. Since countable abelian groups belong to $\text{FICWF}$ (by (2)), the induction starts.

So assume that a finitely generated $k$-step solvable group for $k \leq n-1$ belongs to $\text{FICWF}$ and $\Gamma$ is $n$-step solvable.

We have the following exact sequence.

$$1 \to \Gamma^{(n)} \to \Gamma \to \Gamma/\Gamma^{(n)} \to 1.$$ 

Note that $\Gamma^{(n)}$ is abelian and $\Gamma/\Gamma^{(n)}$ is $(n-1)$-step solvable. Using the ‘extension’ condition and the induction hypothesis we complete the proof.

**iii.** This follows from the definition of $\text{FICWF}$.

**iv.** Using the ‘polycyclic extension’ condition and the following Lemma we complete the proof.

**Lemma 4.1.** Let $G$ be a virtually polycyclic group. Then $G$ contains a finite normal subgroup so that the quotient is a discrete cocompact subgroup of a Lie group with finitely many components.

**Proof.** See [[18], theorem 3, remark 4 in page 200].

**5. Proofs of Theorems 1.1 and 1.2**

**Proof of Theorem 1.1.** It is enough to prove the followings.

**A.** The $\text{FICwF}_{\mathcal{F}_L}^\mathcal{H}_c^*$ ($\text{FICwF}_{\mathcal{V}_c}^\mathcal{H}_c^*$) for the groups appearing in (1) ((i)) of Definition 1.1 is true.

**B.** The statement ‘The $\text{FICwF}_{\mathcal{F}_L}^\mathcal{H}_c^*$ ($\text{FICwF}_{\mathcal{V}_c}^\mathcal{H}_c^*$) is satisfied’ is closed under the operations described in 2 to 5 ((ii) to (iv)) of Definition 1.1.

**Proof of A.** The proof follows from Lemma 2.6.

**Proof of B.** The proof follows from Theorem 1.2.

This proves the Theorem 1.1.

**Proof of Theorem 1.2.** For **C**: 2, 3, 4 and 5 follows from Lemma 2.1, Proposition 2.1, Lemma 2.5 and Lemma 2.14 respectively.

For **D**: (ii) follows from Lemma 2.1, Proposition 2.1 and Lemma 2.5. (iii) follows from Lemma 2.13. (iv) follows using (3) of Lemma 2.2, Lemma 4.1 and Lemma 2.6. To apply (3) of Lemma 2.2 we will need the fact that if a group contains a finite normal subgroup with virtually cyclic quotient then the group is virtually cyclic. This follows from [[[15], lemma 6.1].
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