Production of Fock mixtures in trapped ions for motional metrology

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Abstract

We present a protocol to produce a class of non-thermal Fock state mixtures in trapped ions. This class of states features a clear metrological advantage with respect to the ground state, thus overcoming the standard quantum limit without the need for full sideband cooling and Fock-state preparation on a narrow electronic transition. The protocol consists in the cyclic repetition of red-sideband (RSB), measurement and preparation laser pulses. By means of the Kraus map representation of the protocol, it is possible to relate the length of the RSB pulses to the specific class of states that can be generated. With the help of numerical simulations, we analyse the parametric regime where these states can be reliably reproduced.

1. Introduction

Trapped ions are a platform of reference for the implementation and testing of quantum information protocols\cite{1,2}, with several recent achievements in the quantum computing race\cite{3}. Beyond quantum computation, quantum logic spectroscopy\cite{4} has opened up a useful avenue in quantum metrology\cite{5}, both in the context of optical clocks\cite{6} and force sensing\cite{7–10}. Force sensing may find applications of varied nature involving, for instance, precision spin-resonance imaging\cite{11}, enhanced atomic force microscopy\cite{12}, mass spectrometry\cite{13,14} and tests of fundamental physical phenomena\cite{15}. Nevertheless, non-vanishing fluctuations of the motional ground state set a fundamental limit (standard quantum limit, SQL) in the precision of many quantum sensors.

An ongoing effort to improve the sensing capabilities of trapped ions is underway, with promising strategies arising in recent times. One of these consists in the use of non-thermal motional states. Squeezed states were one of the first workarounds to SQL that were proposed and implemented to great success\cite{16} in the photonic context, and has been proposed\cite{17,18} and implemented\cite{19–21} in the context of trapped ions. Cat states have also featured prominently as alternatives to squeezed states\cite{22}. In order to avoid the accurate control of the phase of squeezed states and cat states with respect to the measured force, excited Fock states ($n = 1, 2, \ldots$) have recently been proposed\cite{23}, and even superpositions of Fock states\cite{24}, as a means to achieve metrological advantages with respect to the ground state. Their production involves ground state cooling\cite{25}, often via a transition that resolves motional Fock states and therefore severely delays the cooling process. Although higher cooling rates can be obtained by applying electromagnetically induced transparency\cite{26}, they cannot be arbitrarily increased\cite{27,28} unless more elaborate implementations are considered\cite{29}.

Here we analyse a novel and simple protocol for the production of Fock state mixtures that circumvents the need for ground-state laser cooling and repeated blue-sideband (BSB) cycles to prepare a specific Fock state. These mixtures, that we term \textit{trapped states}, retain a significant metrological advantage with respect to the ground state. They feature a reduced entropy with respect to the states they were created from and, in addition, their parity can be well defined. In this paper we explore in detail the production process as well as
the properties of trapped states, which can be generated with red-sideband (RSB) excitations although also BSB excitations can be used.

The paper is structured as follows. In section 2 we present the protocol formally, we define the concept of trapped states, and also analyse their form in the special case of an initially thermal ion. Section 3 introduces the Kraus map representation of the protocol. In section 4 we explore the applicability of the trapped state in the context of quantum metrology and displacement sensitivity. In section 5 we extend the idea to the use of BSB excitations. Finally, in section 7 we simulate the results numerically under realistic conditions by employing a Lindblad master equation and provide the final conclusions of the work.

2. Implementation of the protocol

We consider implementation of our protocol in a trapped-ion architecture. Let us model the electronic degrees of freedom of a trapped ion with a two-level system: an electronic ground state $|g\rangle$ and an excited state $|e\rangle$, with transition frequency $\omega$. Its motional degrees of freedom are approximated by a quantum harmonic oscillator of natural frequency $\nu$. Control is exerted by means of a laser field of frequency $\omega_\ell$ that induces Rabi oscillations of frequency $\Omega$ and is characterized by a Lamb–Dicke parameter $\eta$ [30]. The Hamiltonian in the rotating wave approximation (RWA) with respect to $\omega_\ell$ is

$$ H = \frac{\Delta}{2} \sigma_z + \nu a^\dagger a + \frac{\Omega}{2} \left( \sigma^+ D(i\eta) + \sigma^- D^\dagger(i\eta) \right), \tag{1} $$

where $\Delta = \omega - \omega_\ell$ is the detuning of the laser with respect to the transition, $a$ is the annihilation operator of the harmonic oscillator, $D(\alpha) = \exp\left(\alpha a + \alpha^* a^\dagger\right)$ is the displacement operator, $\sigma^+ = |e\rangle \langle g|$, $\sigma^- = (\sigma^+)^\dagger$ are the spin raising and lowering operators respectively and $\sigma_z = 1 - 2\sigma^+ \sigma^-$. The Lamb–Dicke regime is quantitatively expressed by $\eta \sqrt{\langle n_{\text{ph}}\rangle} \ll 1$, where $\langle n_{\text{ph}}\rangle$ is the average phonon number. In this regime, transitions that modify the motional state by more than a single phonon are strongly suppressed. In this limit, setting the detuning $\Delta = \nu$, and additionally performing a RWA with respect to $\nu (a^\dagger a + \sigma_z/2)$, yields the RSB Hamiltonian

$$ H_{\text{RSB}} = \frac{\hbar \Omega}{2} (a \sigma^+ - a^\dagger \sigma^-), \tag{2} $$

also known as the Jaynes–Cummings Hamiltonian [31]. It generates Rabi oscillations between states $|g, n\rangle \leftrightarrow |e, n-1\rangle$ at a frequency $\eta \Omega \sqrt{n}$.

The protocol, that we call selective population trapping (SPT) protocol, simply consists in the periodic alternation of RSB laser pulses with measurement and preparation steps. In particular, we consider three steps:

(i) A RSB pulse is applied for a time $\tau = 2\pi/(\eta \Omega \sqrt{n_0})$, where $n_0 > 0$ is an integer of our choice that determines the form of the trapped state.

(ii) A measurement of the electronic state of the ion is performed at time $\tau$.

(iii) The ion is projected back into its electronic ground state $|g\rangle$.

The effect of this process is illustrated in figure 1(a). Population of most Fock states cascades down analogously to a sideband-cooling scheme. Nevertheless, the length of the sideband pulse $\tau$ matches the RSB period of motional Fock states $|n_0 m^2\rangle$, where $m$ is any natural number. All states for which the sideband pulse represents a full Rabi oscillation will remain trapped. In particular, the first state for which this occurs is $|n_0\rangle$ (corresponding to $m = 1$), which clarifies the meaning of the parameter $n_0$ determining the time of the RSB pulse $\tau$. Also the ground state (corresponding to $m = 0$) retains population, although in this case because it has no RSB excitation. Eventually, motional Fock mixtures are generated that have the form

$$ \mu_\tau = \sum_{m=0}^{\infty} p_\tau (m) \langle n_0 m^2 \rangle \langle n_0 m^2 \rangle, \tag{3} $$

where the initial population of all states below trap $|n_0 (m+1)^2\rangle$ has been deposited in trap $|n_0 m^2\rangle$, so that, for an initial motional state $\mu_0$, we have

$$ p_\tau (m) = \sum_{k=n_0 m^2}^{n_0 (m+1)^2 - 1} \langle k | \mu_0 | k \rangle. \tag{4} $$




Figure 1. (a) Production process and energy levels of a trapped state for \( n_0 = 1 \). (b) Trapped (blue bars) and thermal (red dots) probability distributions for various values of \( \langle n_{\text{ph}} \rangle \) and \( n_0 = 1 \) (associated with the trap series 0, 1, 4, 9, 16, ...). (c) Thermal (red-dashed line) and trapped state (blue-solid line) von Neumann entropy for \( n_0 = 1 \) as a function of \( \langle n_{\text{ph}} \rangle \). (d) Experimental realisation through a third energy level with a fast decay rate back to the electronic ground state.

For the particular case of an initially thermal distribution \( \mu_0^{\text{th}} \) of inverse temperature \( \beta \), we have \( \langle k | \mu_0^{\text{th}} | k \rangle = (1 - e^{-\beta \nu}) e^{-\beta \nu k} \), and the final trapped-state distribution becomes \( p_{\text{tr}}^{\text{th}} (m) = e^{-\beta \nu n_0 m^2} - e^{-\beta \nu (m+1)^2} \). Some examples are presented in figure 1(b).

The state described by equation (3) is a non-thermal probability distribution. The function \( p_{\text{tr}} (m) \) may even be non-monotonic: the position of its maximum depends only on the initial state \( \mu_0 \) and the time \( \tau \), as it can be clearly seen in figure 1(b). Since the protocol concentrates population in a few trapping levels, it is expected to reduce the entropy of the state, as shown in figure 1(c). A proof that this is always the case for an initially diagonal state in the Fock basis can be found in the appendix.

Periodic electronic state measurement and preparation, corresponding to steps (ii) and (iii) of the protocol, may be implemented by using electronic shelving techniques [32]: an additional laser resonantly couples \( |e\rangle \) to a higher excited level of the ion \( |f\rangle \), which has a fast decay rate back to the electronic ground state of the system, see figure 1(d). This technique is also commonplace in implementations of standard sideband cooling in order to increase cooling rates.

3. Kraus maps analysis and steady state

As a way to analyse the dynamics of the motional degrees of freedom, we employ the Kraus sum representation of quantum processes [33]. The effect of the sideband, measurement and preparation pulses are summarized by Kraus maps \( K_e \) or \( K_g \), depending on the outcome of the electronic state measurement. Disregarding the measurement outcome, the unconditional evolution of the density matrix \( \mu (\tau) \) of the motional degrees of freedom is
\[
\mu(\tau) = \sum_{i=\epsilon,\delta} K_i \mu_0 K_i^\dagger.
\]

Kraus maps satisfy the condition \( \sum_{i=\epsilon,\delta} K_i^\dagger K_i = 1 \). Under the described protocol, they are computed as \( K_i = \langle \tilde{i} | U(\tau) \rangle |g\rangle \), with \( U(\tau) = e^{-iH_{\text{ion}} \tau} \) being the unitary evolution operator associated with the RSB pulse. The final expressions read

\[
K_\epsilon = \sum_{n=0}^{\infty} \cos \left( \sqrt{n + \frac{1}{2}} \eta \Omega \tau \right) |n\rangle \langle n|,
\]

\[
K_\delta = -\sum_{n=0}^{\infty} \sin \left( \sqrt{n + \frac{1}{2}} \eta \Omega \tau \right) |n+1\rangle \langle n|.
\]

Since by step (iii) of the protocol the electronic state is prepared back into \( |g\rangle \), the same set of Kraus maps can be used to describe repeated iterations of the SPT-Protocol. This is a useful property in order to extract the steady state of the Fock state populations.

From the structure of the Kraus maps it can be seen that the evolution of populations and coherences is decoupled. In particular, we may describe the stroboscopic evolution of the vector of populations \( \mathbf{p}(t) = (p_0, p_1, \ldots, p_n, p_{n+1}, \ldots)^T \) [with \( p_n = \langle n | \mu(t) | n \rangle \) and \( t = k\tau \) any integer multiple of \( \tau \)] by means of the dynamical map \( \mathcal{E}(\tau) \). It is a matrix whose components are related to the Kraus maps

\[
\mathcal{E}_{mn}(\tau) = \sum_{i=\epsilon,\delta} \langle m | K_i(\tau) | n \rangle \langle n | K_i^\dagger(\tau) | m \rangle.
\]

The steady state populations \( \mathbf{p}^s \) satisfy the equation \( \mathbf{p}^s = \mathcal{E}(\tau) \mathbf{p}^s \), which implies

\[
\sin^2 \left( \sqrt{n + \frac{1}{2}} \eta \Omega \tau \right) p_n^s = \sin^2 \left( \sqrt{n + \frac{1}{2}} \eta \Omega \tau \right) p_{n+1}^s.
\]

Beyond the trivial solution (which corresponds to \( p_0^s = 1 \) and \( p_n^s = 0 \) for any \( n > 0 \) and describes thus a pulsed implementation of sideband cooling), this equation illustrates the reason for the choice \( \tau = 2\pi/(\eta \Omega \sqrt{m_0}) \), since it is only for this case that additional solutions exist, corresponding to the trapped states.

### 4. Displacement sensitivity and quantum metrology

We now analyse the metrological advantage trapped states can have with respect to the ground state in the field of displacement sensitivity. In the spirit of [23], a phase-space displacement \( \alpha \) is implemented by letting the ion interact with an external electric field. This corresponds to the state transformation \( \mu_\alpha \rightarrow D(\alpha) \mu_\alpha D(\alpha) \) with \( D(\alpha) = \exp(\alpha \sigma^\downarrow - \sigma^\uparrow \alpha^* \sigma) \). In order to gain information of the displacement \( \alpha \), a measurement of the motional state \( |n\rangle \) of the ion is implemented. This measurement carries some information about \( \alpha \). In particular, with the help of the overlap function \( \xi(\alpha) = \text{tr} \{ |n\rangle \langle n | D(\alpha) \mu_\alpha D(\alpha) \} \) between the initial and the displaced state one can express the Fisher information of the measurement with

\[
\mathcal{F}(\alpha) = \frac{1}{\xi(\alpha) [1 - \xi(\alpha)]} \left[ \frac{d\xi(\alpha)}{d\alpha} \right]^2.
\]

The Fisher information is directly linked to the achievable measurement sensitivity \( \Delta \alpha \) by means of the Cramer–Rao bound \( \Delta \alpha^{\text{CR}} \), given by

\[
\Delta \alpha \geq \Delta \alpha^{\text{CR}} = \frac{1}{\sqrt{NF(\alpha)}}.
\]

It can be used to quantify the metrological gain in comparison to the SQL as

\[
g = \frac{\mathcal{F}(\alpha)}{\mathcal{F}_{\text{SQL}}},
\]

where \( \mathcal{F}_{\text{SQL}} = 4 \) is the quantum Fisher information (QFI) of the motional ground state \( |0\rangle \). The QFI is the maximum Fisher information that can be obtained from a state assuming any positive operator-valued measurements (POVMs) can be performed. It is therefore an upper bound to the Fisher information associated with a specific experiment. More details on the calculations of state overlap and the associated Fisher information are included in the Appendices.
4.1. Thermal state
The QFI of a thermal state characterized by an average population \( \langle n_{ph} \rangle \) is \( 4/(2 \langle n_{ph} \rangle + 1) \), upper bounded by the SQL and therefore unsuitable for metrological purposes. As illustrated in figure 2, the maximum value for the Fisher information that can be obtained by means of a measurement of the ground state of a displaced thermal state decreases with an increasing \( \langle n_{ph} \rangle \) and it appears at larger values of the displacement amplitude \( |\alpha| \). This justifies the use of sideband cooling to achieve higher sensitivities, although trapped states can overcome the SQL without it, as we will now show.

4.2. Trapped state
The QFI of a trapped state with \( n_0 > 1 \) is \( 8 \langle n \rangle + 4 \), where

\[
\langle n \rangle = n_0 \sum_{m=0}^{\infty} m^2 p_{tr}(m)
\]

is its average phonon number. Therefore, it is similar to the QFI of the \( n \)th Fock state, which reads \( 8n + 4 \). For the case \( n_0 = 1 \), the QFI is slightly smaller and has the form \( 8 \langle n \rangle + 4 - 16p_{tr}(0)p_{tr}(1)/[p_{tr}(0) + p_{tr}(1)] \).

Nevertheless, as can be seen in figure 3, the choice \( n_0 = 1 \) may provide reasonable metrological advantage taking into account the non-trivial dependency of \( \langle n \rangle \) with respect to the initial thermal occupation \( \langle n_{ph} \rangle \). Furthermore, \( \langle n \rangle \) increases with the initial temperature \( \langle n_{ph} \rangle \), which indicates that the trapped-state approach may be suitable in experiments with poor Doppler cooling. In particular, QFI of trapped states with \( n_0 = 1 \) beats all other choices for sufficiently large \( \langle n_{ph} \rangle \).

Figure 4 shows the Fisher information of a trapped state as a function of the displacement amplitude \( |\alpha| \). We consider trapped states created from thermal states with different values of temperature represented by their initial average phonon number \( \langle n_{ph} \rangle \). For each value, two curves are presented, each one corresponding to a measurement for a different Fock state. The green curve indicates a measurement of the most likely Fock state of the given trapped mixture \( \mu_{tr} \), i.e. the Fock state \( |n_0m^2\rangle \) with highest \( p_{tr}(m) \). The blue curve corresponds instead to a measurement of the next Fock state \( |n_0m^2 + 1\rangle \), which is not a member of the trapping series. Blue curves do not feature higher peaks in the Fisher information, they do however result in higher Fisher information values for very small displacement amplitudes. By comparison with figure 2, it becomes apparent that trapped states produce significantly higher Fisher information than thermal states. Importantly, in all cases there exists a range of \( \alpha \) for which Fisher information exceeds the SQL. This can be seen in figure 4 in all subplots, with the effect becoming especially noticeable for values above \( \langle n_{ph} \rangle = 5 \). As opposed to thermal states, achievable values of Fisher information by trapped states increase with larger initial temperatures. This proves that trapped states can be useful for metrological purposes, since they can provide higher displacement sensitivities than the motional ground state. Additionally, the difference between the maximum Fisher informations of the cases \( n_0 = 1 \) and \( n_0 = 2 \) also increases with temperature, resulting in a bigger metrological advantage for \( n_0 = 2 \). The projective measurement that we model for the
Quantum Fisher information of trapped states of different $n_0$ as a function of the average occupation number $\langle n_{ph} \rangle$ of the thermal states they were created from. The SQL ($F_Q = 4$) is indicated as a guide to the eye, as well as the function $8\langle n_{ph} \rangle + 4$. The $n_0 = 1$ trapped state requires an initial occupation number of 0.7 to overcome the SQL. For large initial occupation $\langle n_{ph} \rangle$, states of smaller $n_0$ perform better. All lines follow the same tendency except $n_0 = 1$ which is gradually better on increasing $\langle n_{ph} \rangle$.

Trapped-state Fisher information as a function of the displacement amplitude $|\alpha|$ for different values of the average phonon number of the initial thermal state $\langle n_{ph} \rangle$ and measured Fock state $n$. The SQL ($F_Q = 4$) is indicated as a guide to the eye.

State readout is not the optimal POVM for reaching the QFI, and even higher Fisher information are expected under optimization of the measurement scheme. An investigation of this aspect deserves a separate study.

In particular, we can expect the following metrological gains as calculated with the expression $g_{SQL}^{\alpha} = \max_\alpha F_{tr}^{\alpha} / F_{SQL}$:
additional RSB pulses to excite the ground state. A difference in comparison to the RSB case is that now the first member of the trap series can be arbitrarily chosen and is not necessarily the motional ground state. For a BSB-pulse time $\tau = 2\pi/(\eta \Omega \sqrt{m_0})$, the final mixture now has the form

$$p_{n} = \sin^2 \left( \sqrt{n+1} \frac{\eta \Omega}{2} \tau \right) p_{n-1}^{\mathrm{ts}}.$$  \hspace{1cm} (17)

Solving for the steady state of these Kraus maps, the following equations arise:

$$\sin^2 \left( \sqrt{n+1} \frac{\eta \Omega}{2} \tau \right) p_n^{\mathrm{ts}} = \sin^2 \left( \sqrt{n} \frac{\eta \Omega}{2} \tau \right) p_{n-1}^{\mathrm{ts}}.$$  \hspace{1cm} (16)

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The corresponding Kraus maps have the form

$$K_n = \sum_{n=0}^{\infty} \cos \left( \sqrt{n+1} \frac{\eta \Omega}{2} \tau \right) |n\rangle \langle n|,$$

$$K_n = \sum_{n=0}^{\infty} \sin \left( \sqrt{n+1} \frac{\eta \Omega}{2} \tau \right) |n+1\rangle \langle n|.$$  \hspace{1cm} (14)

$$H_{\text{BSB}} = -\frac{i n \Omega}{2} (a^+ \sigma^+ - a \sigma^-).$$

A difference in comparison to the RSB case is that now the first member of the trap series can be arbitrarily chosen and is not necessarily the motional ground state $|0\rangle$.

A very interesting feature of the trapped state produced by a BSB excitation is that one can reduce the entropy of the motional degree of freedom (the proof of this is formally equivalent to the one presented in the appendix) by increasing its energy. Additionally, with the BSB version of the protocol it is possible to create mixtures of definite odd Fock state parity, whereas RSB mixtures can be created with definite even parity. A possible application of BSB trapping could be the production of almost perfect excited Fock states $|n_F\rangle$, by choosing an initial temperature such that $p_0 + p_1 + \ldots + p_{m_0} \approx 1$, avoiding therefore RSB cooling along with additional RSB pulses to excite the ground state.

5. BSB trapping

Here we extend the idea of SPT for the case of a BSB excitation. The SPT protocol stays as defined in section 2, with the only exception being that now a BSB transition is implemented in the first step. Since all the analytical derivations for this section are formally identical to the ones detailed in the RSB case, we are going to simply present the results in a compact form and only focus on the differences between the two processes. The Hamiltonian in this case is also known as anti-Jaynes–Cummings Hamiltonian and is given in the RWA by

$$\langle n_{\phi} \rangle = 3 :$$

$$g_{n=1}^{\mathrm{SQL}} = \frac{F_{\text{tr}} (\alpha \approx 0.3)}{F_{\text{SQL}}} \approx 1.3 \text{dB},$$

$$g_{n=2}^{\mathrm{SQL}} = \frac{F_{\text{tr}} (\alpha \approx 0.5)}{F_{\text{SQL}}} \approx 1.3 \text{dB},$$

$$\langle n_{\phi} \rangle = 10 :$$

$$g_{n=1}^{\mathrm{SQL}} = \frac{F_{\text{tr}} (\alpha \approx 0.2)}{F_{\text{SQL}}} \approx 3 \text{dB},$$

$$g_{n=2}^{\mathrm{SQL}} = \frac{F_{\text{tr}} (\alpha \approx 0.2)}{F_{\text{SQL}}} \approx 3.5 \text{dB},$$

$$\langle n_{\phi} \rangle = 15 :$$

$$g_{n=1}^{\mathrm{SQL}} = \frac{F_{\text{tr}} (\alpha \approx 0.2)}{F_{\text{SQL}}} \approx 3.6 \text{dB},$$

$$g_{n=2}^{\mathrm{SQL}} = \frac{F_{\text{tr}} (\alpha \approx 0.2)}{F_{\text{SQL}}} \approx 4.9 \text{dB}.$$  \hspace{1cm} (18)

It is worth stressing that trapped states have finite entropy (see figure 1(c)) and still are expected to provide a metrological gain with respect to the pure ground state (of zero entropy).

6. Rate

We now derive an estimate for the production rate of trapped states and compare it to that of sideband cooling. We show that, for a large parametric region, the rate of production of trapped states beats the rate of continuous sideband cooling into the ground state. The former is linearly proportional to the Lamb–Dicke parameter and thus superior to the quadratic proportionality of sideband cooling (note that $\eta \ll 1$). As we have shown, trapped states are metrologically superior to the ground state. Therefore, an additional
transformation of the ground state into Fock or squeezed states would still be required, further delaying their production as opposed to trapped states. This is therefore a crucial advantage of trapped states and a central result of this article.

The vector of populations under sideband cooling is governed by a differential rate equation of the form [34]

\[
\frac{dp_n}{dt} = -n R p_n + (n + 1) R p_{n+1},
\]

where we have neglected BSB effects and

\[
R < \frac{\eta^2 \Omega^2}{2 \Gamma^*}.
\]

This rate \( R \) is an upper bound to the speed of the overall cooling process. Sideband cooling works in the sideband-resolved regime \( \Gamma < \nu \) and requires weak driving \( \Omega < \nu \) to reach the ground state. It reaches optimal values in the regime \( \Omega \simeq \Gamma \), so that optimal cooling rates can be achieved of the order \( R_{SC} \simeq \eta^2 \Omega / 2 \).

The evolution of the vector of populations for trapped-state generation is governed by the dynamical map equation (8) such that \( \mathbf{p}(t + \tau) = \mathbf{E}(\tau) \mathbf{p}(t) \). From this, a difference rate equation may be derived of the form

\[
p_n(t + \tau) - p_n(t) \approx -\frac{\sin^2 \left( \sqrt{\frac{\eta^2 \Omega^2}{\Gamma^*}} \right)}{\frac{\eta^2 \Omega^2}{\Gamma^*}} p_n(t) + \frac{\sin^2 \left( \sqrt{n + 1} \frac{\eta^2 \Omega}{\Gamma^*} \right)}{\frac{\eta^2 \Omega^2}{\Gamma^*}} p_{n+1}(t),
\]

from which, by inspecting equation (19), a mean rate may be obtained

\[
\bar{R}_w = \frac{\sin^2 \left( \sqrt{n + 1} \frac{\eta^2 \Omega^2}{\Gamma^*} \right)}{\frac{\eta^2 \Omega^2}{\Gamma^*}} = \sqrt{n_0} \sin^2 \left( \sqrt{\frac{\eta^2 \Omega}{\Gamma^*}} \right),
\]

where \( \tau = 2\pi / (\eta \Omega \sqrt{n_0}) \) has been substituted in the second equality. The most relevant aspect of this result is its linear dependence with respect to the Lamb–Dicke parameter \( \eta \) as opposed to the quadratic dependence of equation (20). Noting the Lamb Dicke regime \( \eta \ll 1 \) is a prerequisite in most trapped-ion experiments, this makes the rate of trapped states far superior than sideband cooling. Beyond this important consideration, it can be observed that, unlike equation (20), a dependence on the level \( n \) remains, so that the rate decreases with increasing \( n \) and, in particular, vanishes as expected for the trapping levels \( n = n_0 m^2 \). In practice, larger initial occupation numbers \( \langle n_0 \rangle \) will cool more slowly but will achieve larger final \( \langle n \rangle \) and associated QFI. A choice of larger \( n_0 \) increases the overall rates, but decreases the final \( \langle n \rangle \) as shown in figure 3. Taking this interplay into account, the trapped-state protocol allows reaching metrologically relevant states at a rate substantially higher than the ground state can be reached with sideband cooling. We further investigate this end at hand of numerical simulations below.

7. Numerical simulation

So far, the SPT protocol has been presented analytically, assuming a Jaynes–Cummings Hamiltonian (equation (2)) or an anti-Jaynes–Cummings Hamiltonian (equation (14)). Additionally, both the measurement and the spontaneous decay of the ion back to the electronic ground state have been assumed to be instantaneous. In this section, we test the predictions in a more realistic scenario by performing numerical simulations that include both finite ground state preparation time and the full Hamiltonian (equation (1)).

7.1. Lindblad master equation and evolution in two steps

In this numerical simulation we model the SPT-Protocol as a two-step process: (1) the unitary evolution of the closed electronic-motional system for time \( \tau \) and (2) the measurement followed by the spontaneous decay of the ion due to photon emission taking an extra time \( \tau_{\rightarrow g} \). This is done by means of a master equation in Lindblad form [35] as given in [36]. It acts on the total density matrix \( \rho \), involving both electronic and motional degrees of freedom, and is expressed by the following equation

\[
\frac{d}{dt} \rho = -i [H, \rho] + \frac{\Gamma}{2} \left( 2 \sigma^- \tilde{\rho} \sigma^+ - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^- \right)
\equiv \mathcal{L}(\Omega, \Gamma) \rho,
\]
where $\Gamma$ is the decay rate between the two qubit states, $\mathcal{L}$ stands for the Liouvillian of the system and $\tilde{\rho}$ is a displaced density matrix containing the recoil effect of spontaneous emission

$$\tilde{\rho} = \frac{1}{2} \int_{-1}^{1} W(x) e^{i\eta'(a+\dagger)} \rho e^{-i\eta'(a+\dagger)} dx.$$  \hfill (24)

The angular distribution of spontaneous emission in a dipolar transition is expressed by $W(x) = 3(1+x^2)/4$ and we take $\eta' \ll \eta$. By using the full Hamiltonian from equation (1), we include the effects of off-resonant carrier and BSB excitations, as well as all higher-order Lamb–Dicke terms.

The dynamical map describing the evolution of $\rho$ is the result of the product $\mathcal{E}_{\text{sim}}(\tau + \tau_{e-ag}) = \mathcal{E}_a(\tau_{e-ag}) \cdot \mathcal{E}_u(\tau)$, where $\mathcal{E}_a(t) = \exp[\mathcal{L}(\Omega t)]\rho(0)$ represents the unitary part of the evolution (with $\Gamma = 0$) and $\mathcal{E}_d(t) = \exp[\mathcal{L}(\Omega, \Gamma t)]$ the dissipative part of the evolution (with $\Omega = 0$). This is illustrated in figure 5.

### 7.2. Results and discussion

We compare the analytical prediction for the population vector $\mathbf{p}_R$ after $R$ repetitions of the SPT protocol as provided by the ideal dynamical map $\mathcal{E}(\tau)$ from equation (8), $\mathbf{p}_R = \mathcal{E}^R(\tau)\mathbf{p}_0$, and as computed from the numerical simulation, $\rho[R\tau + R\tau_{e-ag}] = \mathcal{E}_{\text{sim}}(\tau + \tau_{e-ag})\rho(0)$. This facilitates identification and analysis of the three parameter requirements ($R \gg 1$, $\eta \ll 1$ and $\Omega \ll \nu$) necessary for SPT to work. For simplicity, throughout this section we use $n_0 = 1$. Simulations were performed with a cutoff $\dim_m = 14$ for the motional Hilbert space (maximum Fock state number), which was found to produce sufficient convergence.

#### 7.2.1. Number of repetitions

First, varying the number of repetitions $R$ and comparing the results leads to an understanding of how many applications of the SPT-Protocol are necessary in order to observe the effect of population trapping expected in the steady state. For the parameter regime in which trapped states are well approximated ($\eta \ll 1$ and $\Omega \ll \nu$), we find that $R = 30$ is sufficient to reach a reasonable approximation to the steady state, since higher values do not appreciably modify the distribution.

#### 7.2.2. Lamb–Dicke parameter

A small $\eta$ leads to population distributions significantly closer to the analytical predictions, as can be seen in figure 6. This is due to the Lamb–Dicke approximation $\eta \sqrt{\langle \eta \rho \rangle} \ll 1$, which loses its validity both as $n$ or $\eta$ increase. For values of $\eta \leq 0.02$, the simulated results completely match the analytical ones after about 20 repetitions of the protocol. Values in the range $0.02 < \eta < 0.06$ only approximate the predictions within a margin of about $10\%$, while for higher values the two predictions do not match properly. For larger $\eta$, the Lamb–Dicke approximation is no longer valid and one needs to account for higher order terms of the form $a^{(m+n)}b^n$, which couple states with quantum numbers that vary by $n$, i.e. $|g, m \rangle \to |e, m + n \rangle$. The role of $n$ in the loss of validity of the Lamb–Dicke approximation is also clear from the simulation: as $\eta$ increases, lower values of $n$ are affected. As shown in the top right subplot of figure 6, a small increase in $\eta$ primarily affects trapping state $n = 9$, while further increase of $\eta$ (bottom right) unstabilizes the $n = 4$ trapping state as well. Meanwhile the $n = 1$ trapping state remains stable and accumulates the population that was not properly accumulated by the higher traps.

An interesting effect that can be observed in the bottom right subplot of figure 6, is the formation of a trap at $n = 10$ for $\eta = 0.1$, which cannot be explained within the Jaynes–Cummings type model employed in the previous sections. The trapping at that state is very persistent, and is present even after 1000 repetitions of the protocol. We suspect this behaviour to be due to the Debye–Waller reduction factor of the Lamb–Dicke parameter due to high-order effects [37] and will be investigated elsewhere. At any rate, it is an indication...
that the concept of SPT could be extended to models that account for all Lamb–Dicke orders. Even in regimes where the production of trapped states is not ideal, just a few repetitions of the protocol produces metrologically relevant states for a large range of the Lamb–Dicke parameter, as is shown in figure 7 in terms of the QFI. Although the behaviour is highly non-trivial, just a few applications suffice to overcome the SQL.

7.2.3. Rabi frequency
We now focus on how different Rabi frequencies influence the results, as shown in figure 8. Increased $\Omega$ leads to some population escaping the traps due to off-resonant carrier excitation, as the corresponding term in the Hamiltonian of equation (1) becomes more important. In order to remain within the regime of validity of the RWA underlying $H_{\text{RSB}}$, we observe that Rabi and trap frequencies require a separation of approximately three orders of magnitude. Values of $\Omega$ that are only two orders of magnitude larger than $\nu$ quickly destabilize the traps. In contrast to an increased Lamb–Dicke parameter, this affects the lower lying traps first. It has also been observed that, for higher Rabi frequency values that are within the range of about $(0.5 - 1)\%$ of the trap frequency, the analytical result is better reproduced for about twenty repetitions of the protocol, instead of the thirty shown in the figures.
Figure 8. Comparison between the analytical predictions $p_{\text{tr}}$ (blue) and simulated results $p_{\text{sim}}$ (green) for the probability distribution of Fock populations for $R =$ 30 repetitions of the protocol applied on an initially thermal state $p_{\text{th}}$ (red) for various values of the Rabi frequency. For all the subplots, the following parameter values have been used: $n_0 = 1$, $\eta = 0.02$, $\beta = 0.01\nu^{-1}$, $\delta = \nu$ and $\Gamma_{e\rightarrow g} = 1000\nu$.

Figure 9. Quantum Fisher information as a function of time for the trapped-state protocol for different values of $n_0$ and comparison to sideband cooling. For reference, the shown data correspond to a total number of repetitions 20, 27, 34 and 39 respectively for $n_0 = 1, 2, 3$ and 4, whereas sideband cooling corresponds to the continuous application of the cooling pulse with the same parameters. Initial occupation number $\langle n_{\text{ph}} \rangle = 4$ in all cases. Other parameters are $\eta = 0.02$, $\Omega = 0.01\nu$, $\delta = \nu$ and $\Gamma_{e\rightarrow g} = 1000\nu$ for trapped states and $\Gamma_{e\rightarrow g} = \Omega$ for sideband cooling, which maximizes its cooling rate.

In conclusion, population trapping as described in section 2 is experimentally applicable in the regimes given by $\eta \leq 0.02$ and $\Omega \leq 0.005\nu$, which directly results from the approximations made in the derivation of the Hamiltonian in equation (2).

7.2.4. Rate
We now analyse the increase of QFI as a function of time for the trapped-state protocol and compare it to sideband cooling. As shown in figure 9, trapped states are able to amply surpass the SQL in a time where sideband cooling has hardly had any impact in the QFI of the initial thermal state. This corresponds to just a few repetitions of the trapped-state protocol.

The reason behind its superior performance is twofold. On the one hand, it directly cools into a metrologically valuable state. On the other hand, as reasoned above, it cools faster than the sideband cooling protocol by a factor of $\eta$ thanks to the separation of coherent and dissipative processes in two separate pulses. This second reason is illustrated by figure 10(left), which compares the time evolution of the phonon...
occupation number $\langle n \rangle$ for trapped states and sideband cooling. Regardless of the value of $n_0$, all forms of the protocol cool faster than sideband cooling under the same conditions of laser intensity (Rabi frequency) and Lamb–Dicke parameter. An exponential fit of the curves allows us to extract an effective rate for each $n_0$ and for different initial occupation numbers $\langle n_{ph} \rangle$ (figure 10, right).

As opposed to sideband cooling, the effective rate depends on the initial temperature and additionally also on the value $n_0$ of the protocol. In line with the predictions of equation (22), the rate generally improves with increasing $n_0$ and decreasing $\langle n_{ph} \rangle$. Apart from this trend, the effective rate of trapped states is consistently larger than that of sideband cooling, due to the respectively linear vs quadratic dependence on the Lamb–Dicke parameter in the decay rates of each Fock level. For this particular parameter regime, this translates into an improvement of over an order of magnitude for $n_0 = 4$ and $\langle n_{ph} \rangle = 2$, which can still be enhanced as $n_0$ increases and $\langle n_{ph} \rangle$ decreases. It is important stressing that the sideband cooling still requires additional manipulation pulses to reach a metrologically useful state.

8. Conclusions

We have presented a protocol for the creation of a special class of motional states in trapped ions. The protocol is simple and involves the alternate concatenation of red or BSB pulses with measurement and preparation pulses. The form of the generated state depends on the duration of the sideband pulses and populates only Fock states proportional to perfect square numbers. The presence in the mixture of excited Fock states makes them especially suitable for motional metrology, improving on the SQL without the need for long sideband cooling pulses. The protocol works best with small values of the Lamb–Dicke parameter in order to avoid deleterious high-order effects, and small Rabi frequencies to suppress carrier effects. Such a protocol might be applied in a Penning trap to measure the motional frequencies of a single ion and even extend it to the case of an unbalanced two-ion crystal where the motional modes of the crystal have to be probed [13, 14].

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Trapped and thermal state entropy

The Von Neumann entropy \([38]\) of the system described by a density matrix \(\mu\) is given by

\[ S = -k_B \text{tr} (\mu \ln \mu), \quad (A.1) \]

which for a diagonal density matrix simplifies to

\[ S = -k_B \sum_{m=0}^{\infty} p_m \ln (p_m), \quad (A.2) \]

and essentially quantifies the degree of mixedness for a given state. Since in a trapped state there are significantly fewer states populated, one would expect a lower entropy compared to the thermal state that was used to produce it. The following table shows the first few probabilities involved in the sum of equation (A.2) for both a diagonal and a trapped state, assuming \(n_0 = 1\), in order to get an idea on how they relate to each other.

| Thermal | \(p_0\) | \(p_1\) | \(p_2\) | \(p_3\) | \(p_4\) |
|---------|--------|--------|--------|--------|--------|
| Trapped | \(p_0\) | \(\sum_{m=1}^{3} p_m\) | 0 | 0 | \(\sum_{m=4}^{8} p_m\) |

It is apparent that the contribution to entropy generated by three states with probabilities \(p_1\), \(p_2\) and \(p_3\) respectively is replaced in the trapped-state case by the contribution of a single state of total probability \(p_1 + p_2 + p_3\).

In general, we have trapped-state probabilities

\[ p_{tr} (m) = \sum_{k=A}^{B} p_k, \quad (A.3) \]

where \(A \equiv n_0 \cdot m^2\) and \(B \equiv n_0 (m + 1)^2 - 1\). The contribution to the entropy corresponding to a single trap and the states until the next trap is

\[ p_{tr} (m) \ln [p_{tr} (m)] = \ln [p_{tr} (m)] \sum_{k=A}^{B} p_k. \quad (A.4) \]

If we compare it with the contribution to the entropy of the same states in the original distribution, we may write

\[ \sum_{k=A}^{B} p_k \ln (p_k) = p_A \ln (p_A) + p_{A+1} \ln (p_{A+1}) + \ldots + p_B \ln (p_B). \quad (A.5) \]

Considering that \(p_{tr} (m) > p_k\) with \(k\) between \(A \equiv n_0 \cdot m^2\) and \(B \equiv n_0 (m + 1)^2 - 1\) leads to \(\ln [p_{tr} (m)] > \ln (p_k)\). In combination with equations (A.4) and (A.5) we get

\[ p_{tr} (m) \ln [p_{tr} (m)] > \sum_{k=A}^{B} p_k \ln (p_k). \quad (A.6) \]

Summing over the remaining traps and multiplying with \(-k_B\) yields the trapped entropy \(S_{tr}\) and original entropy \(S_0\) the predicted result

\[ S_{tr} < S_0. \quad (A.7) \]
Appendix B. State overlap

For a prepared state of the form \( \rho_0 = \sum_{m=0}^{\infty} p_m |m\rangle \langle m| \) the state overlap between the pure state and displaced prepared state can be calculated by

\[
\xi(\alpha) = \text{tr}\left\{ |n\rangle \langle n| D(\alpha) \sum_{m=0}^{\infty} p_m |m\rangle \langle m| D^\dagger(\alpha) \right\} = \sum_{m=0}^{\infty} p_m |\langle n| D(\alpha) |m\rangle|^2.
\]

The general formula for the scalar product between two displaced Fock states as given in [39] is

\[
\langle n| D^\dagger(\beta) D(\alpha) |m\rangle = \sqrt{\frac{m!}{n!}} (\alpha - \beta)^{n-m} |\beta| \alpha \mathcal{L}^{n-m}_m \{ (\alpha - \beta)(\alpha^* - \beta^*) \}
\]

with \( \langle \beta| \alpha \rangle = \exp \{ \alpha \beta^* - \frac{1}{2} (\alpha^* \alpha + \beta^* \beta) \}. \)

Inserting \( \beta = 0 \) gives

\[
\xi(\alpha) = e^{-|\alpha|^2} \sum_{m=0}^{\infty} p_m m! |\alpha|^2 (n-m) \mathcal{L}^{n-m}_m \{ |\alpha|^2 \}^2,
\]

with \( \mathcal{L}^n_m(x) \) being the generalized Laguerre polynomials as defined in [40].

Inserting the thermal and trapped probability distribution in equation (B.2) gives the respective overlap expressions.

Appendix C. Fisher information

The Fisher information is a measure of how quickly a probability distribution \( P(x|\theta) \) changes with respect to the parameter \( \theta \). In order to derive an analytical expression for the Fisher information we follow the method from [23].

The precision of an estimation is limited by the Cramer–Rao bound as

\[
\Delta \theta_{\text{est}} \geq \Delta \theta_{\text{CR}} = \frac{1}{\sqrt{N F(\theta)}},
\]

where \( \Delta \theta_{\text{est}} \) is an arbitrary estimator for \( \theta \), \( N \) is the number of repeated measurements, and

\[
F(\theta) = \sum_x \frac{1}{P(x|\theta)} \left[ \frac{\partial P(x|\theta)}{\partial \theta} \right]^2
\]

is the classical Fisher Information. In metrological applications, one is interested in maximizing the precision estimation of the parameter \( \theta \). For this purpose, a minimum \( \Delta \theta_{\text{CR}} \), or equivalently a maximum \( F(\theta) \) is required.

The probability distribution \( P(x|\theta) = \text{tr}\{ \Pi_x \mu(\theta) \} \) depends on the quantum state \( \mu(\theta) \) and the choice of the performed measurement, described by the projectors \( \{ \Pi_x \} \). For processes where the parameter \( \theta \) is imprinted by a unitary process, i.e. \( \mu(\theta) = U(\theta) \mu_0 U^\dagger(\theta) \), the Fisher information has a lower bound given by

\[
F(\theta) \geq \frac{1}{\langle \Delta M \rangle_{\mu(\theta)}^2} \left[ \frac{\partial \langle M \rangle_{\mu(\theta)}}{\partial \theta} \right]^2,
\]

where \( \langle M \rangle_{\mu(\theta)} = \text{tr}\{ M \mu(\theta) \} \) is the mean value and \( \langle \Delta M \rangle_{\mu(\theta)}^2 = \langle M^2 \rangle_{\mu(\theta)} - \langle M \rangle_{\mu(\theta)}^2 \) the variance of the measured observable \( M = \sum_x x \Pi_x \).

If there exist only two possible outcomes from a measurement, \( x = 0,1 \), this bound is tight. Considering that the probabilities must add up to one, \( P(0|\theta) = 1 - P(1|\theta) \) and the variance becomes

\[
(\Delta M)_{\mu(\theta)}^2 = P(1|\theta) [1 - P(1|\theta)].
\]

Let us now consider the metrological protocol described in the manuscript. Starting with a state \( \rho_0 \), we are interested in how a displacement \( D(\alpha) \) affects the probability distribution of the state. In other words, we
investigate how sensitive the ion is to that displacement. The higher the sensitivity (quantified by the Fisher information), the more precise the parameter estimation is for the parameter \( \theta = \alpha \). The displacement operator is unitary and transforms the density matrix as

\[
\mu (\alpha) = D (\alpha) \mu_0 D^\dagger (\alpha). \tag{C.5}
\]

A projective measurement for a pure Fock state \( |n\rangle \langle n| \) has only two possible outcomes, and therefore the bound (C.3) is tight and the measured observable \( M \) takes the form

\[
M = \sum_{n=0,1} x \Pi_k = |n\rangle \langle n|. \tag{C.6}
\]

This results in a mean value of

\[
\langle M \rangle_{\mu(\alpha)} = \text{tr} \{ M \mu(\alpha) \} = \xi (\alpha). \tag{C.7}
\]

Combining this with \( P(1|\alpha) = \text{tr} \{ \Pi_k \mu(\alpha) \} = \xi (\alpha) \) and the tightness of the bound (C.3) results in

\[
\mathcal{F}(\alpha) = \frac{1}{\xi(\alpha)|1-\xi(\alpha)|} \left[ \frac{d\xi(\alpha)}{d\alpha} \right]^2. \tag{C.8}
\]

Appendix D. QFI

The calculation of the QFI for a mixed state \( \rho \) with respect to an operator \( A \) may be performed by means of the expression

\[
F_Q [\rho, A] = 2 \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |\langle k| A |l \rangle|^2, \tag{D.1}
\]

where \( \lambda_k \) and \( |k\rangle \) are the eigenvalues and eigenvectors of the density matrix \( \rho \), respectively, and the summation goes over all \( k \) and \( l \) such that \( \lambda_k + \lambda_l > 0 \).

We move on to calculate the QFI for the case of the position operator \( X = a + a^\dagger \) and mixed states of a harmonic oscillator that may be diagonalized in terms of the basis of Fock states \( |k\rangle \) with probabilities \( p_k \). In that case we may rewrite the above formula in terms of a single sum as

\[
F_Q [\rho, X] = 4 \sum_{l>0} \frac{(p_{l-1} - p_l)^2}{p_{l-1} + p_l} l, \tag{D.2}
\]

since \( |\langle k| X |l \rangle|^2 = |\sqrt{l} \delta_{k,l-1} + \sqrt{l} \delta_{k,l+1}|^2 \).

D.1. Thermal state

For a thermal state with density matrix

\[
\mu_{\phi}^{th} = (1 - e^{-\beta \nu}) \sum_{n=0}^{\infty} e^{-\beta \nu n} |n\rangle \langle n|, \tag{D.3}
\]

we may operate

\[
F_Q [\mu_{\phi}^{th}, X] = 4 (1 - e^{-\beta \nu}) \sum_{l>0} \frac{e^{-\beta \nu (l-1)} - e^{-\beta \nu l}}{e^{-\beta \nu (l-1)} + e^{-\beta \nu l}} l, \tag{D.4}
\]

\[
F_Q [\mu_{\phi}^{th}, X] = 4 \frac{(1 - e^{-\beta \nu})^3}{1 + e^{-\beta \nu}} e^{\beta \nu} \sum_{l>0} e^{-\beta \nu l} l, \tag{D.5}
\]

\[
F_Q [\mu_{\phi}^{th}, X] = 4 \frac{(1 - e^{-\beta \nu})^2}{1 + e^{-\beta \nu}} e^{\beta \nu} \langle n_{ph} \rangle. \tag{D.6}
\]

By making use of the relations \( Z = (1 - e^{-\beta \nu})^{-1}, \langle n_{ph} \rangle = e^{-\beta \nu} Z \) and \( \langle n_{ph} \rangle + 1 = Z \),
\[
F_Q [\mu^\text{th}_{\text{tr}}, X] = 4 \frac{1 - e^{-\beta \nu}}{1 + e^{-\beta \nu}},
\]
(D.7)

and finally
\[
F_Q [\mu^\text{th}_{\text{tr}}, X] = \frac{4}{2 \langle n_{\text{ph}} \rangle + 1},
\]
(D.8)

D.2. Trapped state

Let us consider a thermal trapped state
\[
\mu^\text{th}_{\text{tr}} = \sum_{m=0}^{\infty} P^\text{th}(m) |n_0 \cdot m^2\rangle \langle n_0 \cdot m^2|, \{n_0 \} \in \mathbb{N},
\]

with
\[
P^\text{th}(m) = e^{-\beta \hbar \omega_0 m^2} - e^{-\beta \hbar \omega_0 (m+1)^2}.
\]

Let us recall \(p_k \neq 0\) only if \(k = n_0 m^2\). We distinguish now the cases \(n_0 > 1\) and \(n_0 = 1\).

- **If** \(n_0 > 1\),

\[
F_Q^{n_0 > 1} [\mu^\text{th}_{\text{tr}}, X] = 4 \sum_{m} P^\text{th}(m) (2n_0 m^2 + 1).
\]

Noticing \(\langle n \rangle = n_0 \sum_m m^2 P^\text{th}(m)\) we get the final form,
\[
F_Q^{n_0 > 1} [\mu^\text{th}_{\text{tr}}, X] = 8 \langle n \rangle + 4.
\]

- **If** \(n_0 = 1\) a pair of neighbouring states with non-zero population exists, \(\{|0\rangle, |1\rangle\}\) and it is necessary to distinguish the first two terms of the sum in equation (D.2). Instead of contributing with \(4p_0 + 12p_1\), they contribute with \(4(p_0 - p_1)^2/(p_0 + p_1) + 8p_1\). By adding and subtracting \(4p_0 + 4p_1\), we may arrive at the expression

\[
F_Q^{n_0 = 1} [\mu^\text{th}_{\text{tr}}, X] = 8 \langle n \rangle + 4 - 16p_0^\text{th}(0)p_1^\text{th}(1) + p_0^\text{th}(0) + p_1^\text{th}(1).
\]

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