Symmetric cumulants and event-plane correlations in Pb+Pb collisions

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The ALICE Collaboration has recently measured the correlations between amplitudes of anisotropic flow in different Fourier harmonics, referred to as symmetric cumulants. We derive approximate relations between symmetric cumulants involving \(v_1\) and \(v_5\) and the event-plane correlations measured by ATLAS. The validity of these relations is tested using event-by-event hydrodynamic calculations. The corresponding results are in better agreement with ALICE data than existing hydrodynamic predictions. We make quantitative predictions for three symmetric cumulants which are not yet measured.

Anisotropic flow is the key observable showing that the matter produced in an ultrarelativistic nucleus-nucleus collision behaves collectively as a fluid [1]. Following the discovery of flow fluctuations [2] and triangular flow [3], a “flow paradigm” has emerged, which states that particles are emitted independently (up to short-range correlations) but with a momentum distribution that fluctuates event to event [4]. The azimuthal (\(\varphi\)) distribution in a given event is written as a Fourier series:

\[
P(\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} V_n e^{-in\varphi},
\]

where \(V_n = v_n \exp(in\Psi_n)\) is the (complex) anisotropic flow coefficient in the \(n\)th harmonic, and \(V_{-n} = V^*_n\). Both the magnitude [2] and phase [2, 6] of \(V_n\) fluctuate event to event. In the last five years or so, an extremely rich phenomenology has emerged from this simple paradigm. RMS values of \(v_n\) have been measured up to \(n = 6\) [7–10], and more recently, the full probability distribution of \(v_2\) [11]. An even wider variety of new observables can be constructed by combining different Fourier harmonics [12–14]. This new direction was pioneered by the ALICE collaboration which measured the angular correlation between \(V_2\) and \(V_3\) [8, 15], and then explored systematically by the ATLAS collaboration which analyzed fourteen mixed correlations involving relative phases between Fourier harmonics, dubbed event-plane correlations [16].

Recently, the ALICE collaboration has taken a new step in this direction [17] by measuring the correlation between the magnitudes of different Fourier harmonics using a cumulant analysis [18]. We define the normalized symmetric cumulant \(sc(n, m)\) with \(n \neq m\) by

\[
sc(n, m) \equiv \frac{\langle v_n^2 v_m^2 \rangle - \langle v_n^2 \rangle \langle v_m^2 \rangle}{\langle v_n^2 \rangle \langle v_m^2 \rangle}. \tag{2}
\]

ALICE has measured \(sc(3, 2)\) and \(sc(4, 2)\) as a function of centrality. While these two quantities are formally similar, the hydrodynamic mechanisms giving rise to these correlations differ. Elliptic flow, \(v_2\), and triangular flow, \(v_3\), are both determined to a good approximation by linear response to the anisotropies of the initial density profile in the corresponding harmonics [19, 20]. Therefore, \(sc(3, 2)\) directly reflects correlations present in the initial spatial density profile, which are preserved by the hydrodynamic evolution as the spatial anisotropy is converted into a momentum anisotropy. Standard models for the initial density indeed reproduce the negative sign and overall (small) magnitude of the measured \(sc(3, 2)\) for all centralities [17]. By contrast, \(V_4\) gets a significant nonlinear contribution proportional to \(V_2^2\) generated by the hydrodynamic evolution [21, 22] in addition to the linear contribution from the initial anisotropy in the fourth harmonic [24, 25]. The nonlinear response explains [24] the large event-plane correlation between \(V_2\) and \(V_4\). It also explains qualitatively why \(sc(4, 2)\) is positive.

In this paper, we derive a proportionality relation between \(sc(4, 2)\) and the corresponding event-plane correlation, where the proportionality constant involves the fluctuations of \(v_2\). Using this, we are able to relate recent ALICE measurements with previously measured quantities, which circumvents the most typical limitation of hydrodynamic predictions that depend on initial conditions or medium properties [27, 33]. The sole assumption underlying our derivation is that the linear and nonlinear contributions to \(V_4\) are independent. The validity of this assumption is tested using hydrodynamic calculations. The value of \(sc(4, 2)\) derived using our relation and previous ATLAS measurements is compared with the recent direct measurement by ALICE. We make predictions along the same lines for \(sc(5, 2)\), \(sc(5, 3)\) and \(sc(4, 3)\), which are not yet measured.

We decompose \(V_4\) and \(V_5\) into linear and non-linear parts [22]

\[
V_4 = V_{4L} + \chi_4 V_2^2 \\
V_5 = V_{5L} + \chi_5 V_2 V_3. \tag{3}
\]

We define \(\chi_4\) and \(\chi_5\) in such a way that the linear correlations between linear and nonlinear parts vanish, that is, \(\langle V_{4L}(V_2)^2 \rangle = \langle V_{5L} V_2 V_3 \rangle = 0\). We now introduce a measure of the relative magnitude of the linear and nonlinear parts via the Pearson correlation coefficients

1 Note the ALICE collaboration uses the same notation for the numerator only.
These equations are exact and simply follow from the definition of \( \chi \). They are depicted in Fig. 1.

We now assume that the linear parts \( V_{4L} \) and \( V_{5L} \) are statistically independent of \( V_2 \) and \( V_3 \). This is a stronger statement than just assuming that the linear correlation vanishes. As will be shown below, it is a reasonable approximation in hydrodynamics. Then, only the nonlinear response contributes to the correlation between \( v_4 \) and \( v_2 \), and Eq. (6) gives:

\[
\langle v_4^2 v_2^2 \rangle - \langle v_4^2 \rangle \langle v_2^2 \rangle = \chi_4^2 \left( \langle v_2^4 \rangle - \langle v_4^2 \rangle \langle v_2^2 \rangle \right). \tag{7}
\]

Two We only consider the event-plane correlations measured using the scalar-product method, which are denoted by the subscript “w” in the ATLAS paper and have a clear interpretation in terms of \( V_n \), in contrast to the results obtained using the event-plane method \([34]\).

FIG. 1. (Color online) Schematic picture of the relation between the event-plane angle \( \Phi_{24} \) in Eq. (4) and the decomposition Eq. (3). The legs of the triangle correspond to the rms values of the linear and nonlinear parts, and the hypotenuse is the rms \( v_4 \). A similar figure can be drawn for \( V_5 \).

between \( V_4 \), or \( V_5 \), and their nonlinear parts:

\[
\cos \Phi_{24} = \frac{\text{Re}(V_4(V_2^*)^2)}{\sqrt{\langle v_2^2 \rangle} \langle v_2^2 \rangle},
\]

\[
\cos \Phi_{235} = \frac{\text{Re}(V_4V_2V_3^*)}{\sqrt{\langle v_3^2 \rangle} \langle v_3^2 \rangle}, \tag{4}
\]

where \( \Phi_{24} \) and \( \Phi_{235} \) lie between 0 and \( \pi \). The first angle \( \Phi_{24} \) corresponds precisely to the event-plane correlation measured by ATLAS \([10]\) and denoted by \( \langle \cos(4(\Phi_2 - \Phi_4) \rangle_w \). The second angle \( \Phi_{235} \) almost corresponds to the quantity denoted by \( \langle \cos(2\Phi_2 + 3\Phi_3 - 5\Phi_5) \rangle_w \). The only difference is that the latter has \( \langle v_2^2 \rangle \langle v_3^2 \rangle \) in the denominator, instead of \( \langle v_2^2 v_3^2 \rangle \). Therefore the precise relation is

\[
\cos \Phi_{235} = \frac{(\cos(2\Phi_2 + 3\Phi_4 - 5\Phi_5) \rangle_w}{\sqrt{1 + sc(3, 2)}}, \tag{5}
\]

where \( sc(3, 2) \) is defined in Eq. (2).

Inserting Eq. (3) into Eq. (4), one obtains

\[
\chi_4^2 \langle v_2^2 \rangle = \langle v_4^2 \rangle \cos^2 \Phi_{24},
\]

\[
\chi_5^2 \langle v_3^2 v_5^2 \rangle = \langle v_5^2 \rangle \cos^2 \Phi_{235}. \tag{6}
\]

These equations are exact and simply follow from the definition of \( \chi_4 \) and \( \chi_5 \). They are depicted in Fig. 1.

We now assume that the linear parts \( V_{4L} \) and \( V_{5L} \) are statistically independent of \( V_2 \) and \( V_3 \). This is a stronger statement than just assuming that the linear correlation vanishes. As will be shown below, it is a reasonable approximation in hydrodynamics. Then, only the nonlinear response contributes to the correlation between \( v_4 \) and \( v_2 \), and Eq. (6) gives:

\[
\langle v_4^2 v_2^2 \rangle - \langle v_4^2 \rangle \langle v_2^2 \rangle = \chi_4^2 \left( \langle v_2^4 \rangle - \langle v_4^2 \rangle \langle v_2^2 \rangle \right). \tag{7}
\]

Similar relations can be written for the correlations between \( v_4^2 \) and \( v_3^2 \), \( v_5^2 \) and \( v_5^2 \) or \( v_3^2 \) or \( v_5^2 \). Substituting in \( \chi_4 \) and \( \chi_5 \) extracted from Eqs. (6), one obtains

\[
sc(4, 2) = \left( \frac{\langle v_2^4 \rangle}{\langle v_2^2 \rangle^2} - 1 \right) \cos^2 \Phi_{24}
\]

FIG. 2. (Color online) Test of Eqs. (8) using hydro calculations. Symbols correspond to the left-hand sides of Eqs. (8), dark shaded bands to the right-hand sides. Light-shaded bands correspond to Eqs. (9) and (12). Errors are statistical and estimated via jackknife resampling.
The equation for $sc(4, 2)$ can also be tested against existing data. The left-hand side has been measured by ALICE [17] while the quantities entering the right-hand side (moments of the $v_2$ distribution, event-plane correlation) have been measured by ATLAS, ALICE and ATLAS have different acceptances, both in transverse momentum ($p_t$) and pseudorapidity ($\eta$), so that the comparison is not quite apples to apples. However, we expect that the quantities entering Eq. (8) (ratios of moments, event-plane correlations) depend weakly on the $p_t$ range. The effect of the acceptance in $\eta$ will be discussed below. The moments of $v_2$ are not measured directly but can be expressed [22] as a function of the cumulants $v_2(2)$, $v_2(4)$ and $v_2(6)$, which are measured [46]. Note that the ATLAS $v_2(2)$ is biased by nonflow correlations since no rapidity gap is implemented ($v_2(4)$ and $v_2(6)$ are expected to be free of nonflow correlations whether or not there is a gap). However, we have compared the ratios $v_2(4)/v_2(2)$ from ALICE [8] (where $v_2(2)$ has a gap) and ATLAS (where $v_2(2)$ has no gap) and found that they are compatible, which suggests that the nonflow contribution to the integrated $v_2(2)$ measured by ATLAS is small (nonflow effects are known to be large at high $p_t$).

Figure 3 displays the comparison between the left-hand side of Eq. (8) measured by ALICE [17] and the right-hand side using ATLAS data. Agreement is reasonable for all centralities. In particular, our data-driven approach gives a better result for $sc(4, 2)$ than existing hydrodynamic predictions [17] [30]. Based on the hydrodynamic calculation of Fig. 2 one would expect that the right-hand side of Eq. (8) is larger than the left-hand side. However, it is the other way around above 30% centrality. One reason may be that the event-plane correlation for ATLAS uses a much larger pseudorapidity window ($|\eta| < 4.8$) than ALICE ($|\eta| < 0.8$). Now, the phase of $V_n$ depends slightly on rapidity [17] [39], which induces a decoherence of azimuthal correlations for larger $\Delta \eta$. Due to these longitudinal flow fluctuations, the event-plane correlation measured by ATLAS is smaller than what ALICE would measure in a more central rapidity window. Ideally, the comparison between the two sides of Eq. (8) should be done in the exact same rapidity window.

We now make predictions for $sc(4, 3)$, $sc(5, 2)$ and
Taking the ratio of Eqs. (11) and inserting into Eq. (8), one obtains
\[
sc(4,3) \approx \frac{\langle v_2^2 \rangle \left( \langle v_2^2 \rangle - \langle v_2 \rangle \langle v_2^2 \rangle \right)}{\langle v_2^4 \rangle - \langle v_2^2 \rangle^2} \text{sc}(3,2) \cos^2 \Phi_{24}.
\] (12)

The right-hand side of this equation is shown as a light-shaded band in Fig. 2 (b). It is very close to the dark-shaded banded for all centralities, thus showing that the decomposition in Eq. (10) appropriately takes into account the correlation between \(v_2\) and \(v_3\).

Figure 4 displays our predictions for \(sc(5,3)\), \(sc(5,2)\) and \(sc(4,3)\) using Eqs. (9) and (12), where we use ATLAS data for the quantities in the right-hand side. For \(sc(4,3)\), we use ALICE data for \(sc(3,2)\), and the other quantities in the right-hand side of Eq. (12) (moments of \(v_2\) and \(v_3\) measured in the near future). These new observables will allow to test hydrodynamic behavior directly, provided that one also measures higher-order correlations between \(v_2\) and \(v_3\) such as \(\langle v_2^4 \rangle / \langle v_2^2 \rangle^2\).

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