TYPE-BASED TERMINATION FOR FUTURES

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Abstract. In sequential functional languages, sized types enable termination checking of programs with complex patterns of recursion in the presence of mixed inductive-coinductive types. In this paper, we adapt sized types and their metatheory to the concurrent setting. We extend the semi-axiomatic sequent calculus, a subsuming paradigm for futures-based functional concurrency, and its underlying operational semantics with recursion and arithmetic refinements. The latter enables a new and highly general sized type scheme we call sized type refinements. As a widely applicable technical device, we type recursive programs with infinitely deep typing derivations that unfold all recursive calls. Then, we observe that certain such derivations can be made infinitely wide but finitely deep. The resulting trees serve as the induction target of our termination result, which we develop via a novel logical relations argument.

1. Introduction

Note: this is an extended version of an eponymous paper that appeared in FSCD 2022 that includes further examples (Examples 2.4, 2.5, and 2.7), a more straightforward presentation of the metatheory (Section 4) based on Kripke logical relations [Plo73], and a representative set of the corresponding proofs (Sections 3 and 4).

Adding (co)inductive types and terminating recursion (including productive corecursive definitions) to any programming language is a non-trivial task, since only certain recursive programs constitute valid applications of (co)induction principles. Briefly, inductive calls must occur on data smaller than the input and, dually, coinductive calls must be guarded by further codata output. In either case, we are concerned with the decrease of (co)data size—height of data and observable depth of codata—in a sequence of recursive calls. Since inferring this exactly is intractable, languages like Agda (before version 2.4) [AP16] and Coq [The21] resort to conservative syntactic criteria like the guardedness check.

One solution that avoids syntactic checks is to track the flow of (co)data size at the type level with sized types, as pioneered by Hughes et al. [HPS96] and further developed by others.

Key words and phrases: type-based termination, sized types, futures, concurrency, infinite proofs.

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Inductive and coinductive types are indexed by the height and observable depth of their data and codata, respectively. Consider the equirecursive type definitions in Example 1.1 adorned with our novel sized type refinements: \( \text{nat}[i] \) describes unary natural numbers less than or equal to \( i \) and \( \text{stream}_A[i] \) describes infinite \( A \)-streams that allow the first \( i+1 \) elements to be observed before reaching potentially undefined or divergent behavior. \( \oplus \) and \( \& \) are respectively analogous to eager variant record and lazy record types in the functional setting.

**Example 1.1** (Recursive types).

\[
\text{nat}[i] = \oplus \{ \text{zero} : 1, \text{succ} : i > 0 \land \text{nat}[i-1] \}
\]

\[
\text{stream}_A[i] = \& \{ \text{head} : A, \text{tail} : i > 0 \Rightarrow \text{stream}_A[i-1] \}
\]

Note that \( \text{stream}_A[i] \) is not polymorphic, but is parametric in the choice of \( A \) for demonstrative purposes.

The phrases \( \phi \land \ldots \) and \( \phi \Rightarrow \ldots \) are constrained types, so that the succ branch of \( \text{nat}[i] \) produces a nat at height \( i-1 \) when \( i > 0 \) whereas the tail branch of \( \text{stream}_A[i] \) can produce the remainder of the stream at depth \( i-1 \) assuming \( i > 0 \). Starting from \( \text{nat}[i] \), recursing on, for example, \( \text{nat}[i-1] \) (\( i > 0 \) is assumed during elimination so that \( i-1 \) is well-defined) produces the size sequence \( i > i-1 > i-2 > \ldots \) that eventually terminates at 0, agreeing with the (strong) induction principle for natural numbers. Dually, starting from \( \text{stream}_A[i] \), recursing into \( \text{stream}_A[i-1] \) (again, \( i > 0 \) is assumed during introduction so that \( i-1 \) is well-defined) produces the same well-founded sequence of sizes, agreeing with the coinduction principle for streams. In either case, a recursive program terminates if its call graph generates a well-founded sequence of sizes in each code path. Most importantly, the behavior of constraint conjunction and implication during elimination and introduction encodes induction and coinduction, respectively. To see how sizes are utilized in the definition of recursive programs, consider the type signatures below. We will define the code of these programs in Example 2.3.

**Example 1.2** (Evens and odds I). Postponing the details of our typing judgment for the moment, the signature below describes definitions that project the even- and odd-indexed substreams (referred to by \( y \)) of some input stream (referred to by \( x \)) at half of the original depth. Note that indexing begins at zero.

\[
\begin{align*}
\text{evens} & : \text{stream}_A[2i] \vdash y \leftarrow \text{evens} \; i \; x :: (y : \text{stream}_A[i]) \\
\text{odds} & : \text{stream}_A[2i+1] \vdash y \leftarrow \text{odds} \; i \; x :: (y : \text{stream}_A[i])
\end{align*}
\]

An alternate typing scheme that hides the exact size change is shown below—given a stream of arbitrary depth, we may project its even- and odd-indexed substreams of arbitrary depth, too. We provide implementations for both versions in Example 2.3.

\[
\begin{align*}
\forall j. \text{stream}_A[j] \vdash y \leftarrow \text{evens} \; i \; x :: (y : \text{stream}_A[i]) \\
\forall j. \text{stream}_A[j] \vdash y \leftarrow \text{odds} \; i \; x :: (y : \text{stream}_A[i])
\end{align*}
\]

\( \exists j. X[j] \) and \( \forall j. X[j] \) denote full inductive and coinductive types, respectively, classifying (co)data of arbitrary size. In general, less specific type signatures are necessary when the exact size change is difficult to express at the type level [XP99]. For example, in relation to an input list of height \( i \), the height \( j \) of the output list from a list filtering function may be constrained as \( j \leq i \).
Sized types are *compositional*: since termination checking is reduced to an instance of typechecking, we avoid the brittleness of syntactic termination checking. However, we find that *ad hoc* features for implementing size arithmetic in the prior work can be subsumed by more general *arithmetic refinements* [DP20b, XP99], giving rise to our notion of sized type refinements that combine the “good parts” of modern sized type systems. First, the instances of constraint conjunction and implication to encode inductive and coinductive types, respectively, in our system are similar to the bounded quantifiers in MiniAgda [Abe12], which gave an elegant foundation for mixed inductive-coinductive functional programming, avoiding continuity checking [Abe08]. Unlike the prior work, however, we are able to modulate the specificity of type signatures: (slight variations of) those in Example 1.2 are given in CIC$_\ell$ [Sac14] and MiniAgda [Abe12, Abe]. Furthermore, we avoid transfinite indices in favor of permitting some unbounded quantification (following Vezzosi [Vez15]), achieving the effect of somewhat complicated infinite sizes without leaving finite arithmetic.

Moreover, some prior work, which is based on sequential functional languages, encodes recursion via various fixed point combinators that make both mixed inductive-coinductive programming [Bas18] and substructural typing difficult, the latter requiring the use of the ! modality [Wad12]. Thus, like $F^{\omega}_{\cop}$ [AP16], we consider a signature of parametric recursive definitions. However, we make typing derivations for recursive programs infinitely deep by unfolding recursive calls *ad infinitum* [Bro05, LR19], which is not only more elegant than finitary typing, but also simplifies our termination argument. To prove termination of program reduction, we observe that *arithmetically closed* typing derivations, which have no free arithmetic variables or constraint assumptions, can be translated to infinitely wide but finitely deep trees of a different judgment. The resulting derivations are then the induction target for our proof, leaving the option of making the original typing judgment arbitrarily rich. Thus, although our proposed language is not substructural, this result extends to programs that use their data substructurally. In short, our contributions are as follows:

1. A general system of sized types based on arithmetic refinements subsuming features of prior systems, such as the mixed inductive-coinductive types of MiniAgda [Abe12] as well as the linear size arithmetic of CIC$_\ell$ [Sac14]. Moreover, we do not depend on transfinite arithmetic.
2. The first language for mixed inductive-coinductive programming that is a subsuming paradigm [Lev04] for futures-based functional concurrency.
3. A method for proving termination in the presence of infinitely deep typing derivations by translation to infinitely wide but finitely deep trees.

We define SAX$^\infty$, which extends the semi-axiomatic sequent calculus (SAX) [DPP20] with arithmetic refinements, recursion, and infinitely deep typing derivations (Section 2). Then, we define an auxiliary type system called SAX$^\omega$ which has infinitely wide but finitely deep derivations to which we translate the derivations of SAX$^\infty$ (Section 3). Then, we show that all SAX$^\omega$-typed programs terminate by a novel logical relations argument over *configurations* of processes that capture the state of a concurrent computation (Section 4).

2. SAX$^\infty$

In this section, we extend SAX [DPP20] with recursion and arithmetic refinements in the style of Das and Pfenning [DP20b]. SAX is a logic-based formalism and subsuming paradigm [Lev04] for concurrent functional programming that conceives call-by-need and call-by-value strategies as particular concurrent schedules [PP20]. Concurrency and parallelism devices
like fork/join, futures [Hal85], and SILL-style [TCP13] monadic concurrency can all be encoded and used side-by-side in SAX [PP20].

To review SAX, let us make observations about proof-theoretic polarity. In the sequent calculus, inference rules are either invertible—can be applied at any point in the proof search process, like the right rule for implication—or noninvertible, which can only be applied when the sequent “contains enough information,” like the right rules for disjunction. Connectives that have noninvertible right rules are positive and those that have noninvertible left rules are negative. The key innovation of SAX is to replace the noninvertible rules with their axiomatic counterparts in a Hilbert-style system. Consider the following right rule for implication as well as the original left rule in the middle that is replaced with its axiomatic counterpart on the right.

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \quad \frac{\Gamma, A \rightarrow B \vdash A, B \vdash C}{\Gamma, A \rightarrow B \vdash B + C} \quad \frac{\Gamma, A \rightarrow B, A \vdash B}{\Gamma, A \rightarrow B, A \vdash B}$$

Since the axiomatic rules drop the premises of their sequent calculus counterparts, cut elimination corresponds to asynchronous communication just as the standard sequent calculus models synchronous communication [CP10]. In particular, SAX has a shared memory interpretation, mirroring the memory-based semantics of futures [Hal85]. A future $x$ of type $A$ either contains an object of type $A$ or is not yet populated. A process reading from $x$ either succeeds immediately or blocks if $x$ is not yet populated. As a result, the sequent becomes the typing judgment (extended with arithmetic refinements in the style of [DP20b]):

$$\overline{\cdot} \Gamma, x : A, y : B, \ldots \vdash P :: (z : C)$$

where the arithmetic variables in $\mathcal{V}$ are free in the constraints (arithmetic formulas) in $\mathcal{C}$, the types in $\Gamma$, the process $P$, and type $C$; moreover, the address variables in $\Gamma$, which are free in $P$, stand for addresses of memory cells representing futures. In particular, $P$ reads from $x, y, \ldots$ (sources) and writes to $z$ (a destination) according to the protocols specified by $A, B, \ldots$ and $C$, respectively. $z$ is written to exactly once corresponding to the population of a future [Hal85]. Lastly, the vector (indicated by the overline) of arithmetic expressions $\overline{\cdot}$ will be used to track the sizes encountered at each recursive call as mentioned in the introduction. Now, let us examine the definitions of types and processes. For our purposes, detailed syntaxes for expressions $e$ and formulas $\phi$ are unnecessary.

**Definition 2.1 (Type).** Types are defined by the following grammar, presupposing some mutually recursive type definitions of the form $X[\overline{\cdot}] = A_X[\overline{\cdot}]$. Positive types (in the left column) and negative types (in the right column) are colored red and black, respectively. Recursive type names are colored blue since they take on the polarity of their definitia.

$$A, B := \begin{array}{ll}
  1 & \text{unit} \\
  A \otimes B & \text{eager pair} \\
  \oplus \{ \ell : A_{\ell} \}_{\ell \in S} & \text{eager variant record} \\
  \phi \land A & \text{constraint conjunction} \\
  \exists i. A(i) & \text{arithmetic dependent pair} \\
  X[\overline{\cdot}] & \text{equirecursive type} \\
  A \rightarrow B & \text{function} \\
  & \text{function} \\
  & \text{function} \\
  & \text{function} \\
  & \text{function} \\
  & \text{function} \end{array}$$

There are eight kinds of processes: two for the structural rules (identity and cut), one for each combination of type polarity (positive or negative) and rule type (left or right), one for definition calls, and one for unreachable code.
Definition 2.2 (Process). Processes are defined by the following grammar. The superscripts $R$ and $W$ indicating reading from or writing to a cell.

\[ P, Q := y^W \leftarrow x^R \quad \text{copy contents of } x \text{ to } y \]

\[ x \leftarrow P(x); Q(x) \quad \text{allocate } x, \text{ spawn } P \text{ to write to } x \text{ and concurrently proceed as } Q, \text{ which may read from } x \]

\[ x^W.V \quad \text{write value } V \text{ to } x \]

\[ \text{case } x^R K \quad \text{read value stored in } x \text{ and pass it to continuation } K \]

\[ \text{case } x^W K \quad \text{write continuation } K \text{ to } x \]

\[ x^R.V \quad \text{read continuation stored in } x \text{ then pass value } V \text{ to it} \]

\[ y \leftarrow f \bar{\tau} \bar{\pi} \quad \text{expands to } P_f(\bar{\tau}, \bar{\pi}, y) \text{ from a signature of mutually recursive definitions of the form } y \leftarrow f \bar{\tau} \bar{\pi} = P_f(\bar{\tau}, \bar{\pi}, y) \]

| impossible | unreachable code due to inconsistent arithmetic context |

The first two kinds of processes correspond to the identity and cut rules. Values $V$ and continuations $K$ are specified on a per-type-and-rule basis in the following two tables. Note the address variable $x$ distinguished by each rule.

| polarity | right rule | left rule | type(s) | value | continuation $K$ |
|----------|-----------|-----------|---------|-------|------------------|
| positive | $x^W.V$   | case $x^R K$ | 1       | $\langle \rangle$ | $\langle \rangle \Rightarrow P$ |
|          |           |           | $\otimes, \rightarrow$ | $\langle y, z \rangle$ | $\langle y, z \rangle \Rightarrow P(y, z)$ |
|          |           |           | $\&$, $\oplus$ | $\ell y$ | $\{ \ell y \Rightarrow P(y) \}_{\ell \in S}$ |
| negative | case $x^W K$ | $x^R.V$ | $\wedge, \Rightarrow$ | $\langle *, y \rangle$ | $\langle *, y \rangle \Rightarrow P(y)$ |
|          |           |           | $\forall$, $\exists$ | $\langle e, y \rangle$ | $\langle i, y \rangle \Rightarrow P(i, y)$ |

To borrow terminology from linear logic, the “multiplicative” group $(1, \otimes, \rightarrow)$ is concerned with writing addresses, whereas the “additive” group $(\oplus, \&)$ is concerned with writing labels and their case analysis. Constrained types read and write a placeholder $\ast$ indicating that a constraint is asserted or assumed. However, we will suppress instances of $\ast$ in the example code given, since assumptions and assertions are inferrable in the absence of consecutively alternating constraints (e.g., $\phi \wedge (\psi \Rightarrow A)$). On the other hand, the arithmetic data communicated by quantifiers are visible since inference is difficult in general [DP20c].

Now that we are acquainted with the process syntax, let us complete Example 1.2.

**Example 2.3 (Evens and odds II).** Recall that we are implementing the following signature and stream $\sigma_A[i] = \&\{ \text{head} : A, \text{tail} : i > 0 \Rightarrow \sigma_A[i - 1] \}$.

\[
i : \vdash x : \sigma_A[2i] \vdash^i y \leftarrow \text{evens } i \ x :: (y : \sigma_A[i])
\]

\[
i : \vdash x : \sigma_A[2i + 1] \vdash^i y \leftarrow \text{odds } i \ x :: (y : \sigma_A[i])
\]

The even-indexed substream retains the head of the input, but its tail is the odd-indexed substream of the input’s tail. The odd-indexed substream, on the other hand, is simply the even-indexed substream of the input’s tail. Operationally, the heads and tails of both substreams are computed on demand similar to a lazy record. Unlike their sequential
counterparts, however, the recursive calls proceed concurrently due to the nature of cut. Since our examples will keep constraints implicit, we indicate when constraints are assumed or asserted inline for clarity.

\[
y \leftarrow \text{evens} \; i \; x = \text{case} \; y^W \{ \text{head} \; h \Rightarrow x^R, \text{head} \; h, \\
\text{tail} \; y_t \Rightarrow x_t \leftarrow x^R, \text{tail} \; x_t; y_t \leftarrow \text{odds} \; (i - 1) \; x_t \}
\]

\[
y \leftarrow \text{odds} \; i \; x = x_t \leftarrow x^R, \text{tail} \; x_t; y \leftarrow \text{evens} \; i \; x_t
\]

By inlining the definition of odds in evens and vice versa, both programs terminate according to our criterion from the introduction even though odds calls evens with argument \(i\). However, we sketch an alternate termination argument for similar such definitions at the end of Section 3. On the other hand, consider the alternate signature we gave.

\[
i;::x; \forall j. \text{stream}_{A[j]} \vdash y \leftarrow \text{evens} \; i \; x :: (y : \text{stream}_{A[i]}) \\
i;::x; \forall j. \text{stream}_{A[j]} \vdash y \leftarrow \text{odds} \; i \; x :: (y : \text{stream}_{A[i]})
\]

First, we define head and tail observations on streams of arbitrary depth. Since they are not recursive, we do not bother tracking the size superscript of the typing judgment, since they can be inlined. Moreover, we take the liberty to nest values (boxed and highlighted yellow), which can be expanded into SAX [PP20].

\[
y \leftarrow \text{head} \; x = x^R \begin{cases} (0, \text{head} \; y) \leftrightharpoons j+1>0 \text{ asserted} \end{cases}
\]

The implementation of odds and evens follows almost exactly as before with the above observations in place. Note that we use the abbreviation \(y \leftarrow f \in \Gamma; V; C \vdash e; Q = y \leftarrow (y' \leftarrow f \in \Gamma; V; C); Q\) for convenience.

\[
y \leftarrow \text{evens} \; i \; x = \text{case} \; y^W \{ \text{head} \; h \Rightarrow y \leftarrow \text{head} \; x, \\
\text{tail} \; y_t \Rightarrow x_t \leftarrow \text{tail} \; x_t; y_t \leftarrow \text{odds} \; (i - 1) \; x_t \}
\]

\[
y \leftarrow \text{odds} \; i \; x = x_t \leftarrow \text{tail} \; x_t; y \leftarrow \text{evens} \; i \; x_t
\]

Refer to Figure 1 for the full process typing judgment—we will comment on specific rules when necessary, but section 5 of [DPP20] discusses the propositional rules more closely. In particular, the arithmetic typing rules make use of a well-formedness judgment \(V; C \vdash e\) and entailment \(V; C \vdash \phi\). Most importantly, there are two rules for recursive calls; let us reproduce them below.

\[
\frac{\vdash \Gamma; C \vdash P :: (y : A)}{\vdash \Gamma; C \vdash \infty P :: (y : A)}
\]
Our process typing judgment is itself mixed inductive-coinductive [DA09]—we introduce the auxiliary judgment \( \forall; C; \Gamma \vdash \varpi P :: (y : A) \) that is coinductively generated by the \( \infty \) rule (indicated by the double line). Since the premise of the call rule refers to \( \forall; C; \Gamma \vdash \varpi P :: (y : A) \),
all valid typing derivations are trees whose infinite branches have a call-\(\infty\) pair occurring infinitely often, representing the unfolding of a recursive process. At each unfolding, we check that the arithmetic arguments have decreased (from \(\tau\) to \(\bar{\tau}\)) lexicographically\(^1\) for termination.

We conjecture that finite-time typechecking only requires restricting our type system (including equality of equirecursive types \([DP20a]\)) to circular derivations, which can be represented as finite trees with loops, provided decidable arithmetic (e.g., Presburger). Such a restricted system may be put in correspondence with a finitary system that detects said loops \([SD03, Bro05, Dag21, PP20]\) where arithmetic assertions can be discharged mechanically \([DP20a]\). In Example 2.4 below, we show a hypothetical instance of typechecking.

**Example 2.4** (Typechecking). The process definition below, whose type signature is \(i; :: x : \text{nat}[i] \vdash i \ y \leftarrow \text{eat} \ i \ x :: (y : 1)\), traverses a unary natural number by induction to produce a unit. Recall \(\text{nat}[i] = \oplus\{\text{zero} : 1, \text{succ} : i > 0 \land \text{nat}[i-1]\}\).

\[y \leftarrow \text{eat} i \ x = \text{case} x^R \{ \text{zero} z \Rightarrow y^W \leftarrow z^R, \text{succ} z \Rightarrow y \leftarrow \text{eat} (i - 1) z\}\]

Now, let us construct a typing derivation of its body below.

\[
\begin{align*}
\vdash i; :: x : \text{nat}[i] \vdash i - 1 < i & \quad \text{call} \\
\vdash i; :: \text{nat}[i - 1] \vdash (y : 1) & \\
\vdash \text{nat}[i - 1] \vdash (y : 1) & \\
\vdash \text{nat}[i - 1] \vdash (y : 1) & \\
\end{align*}
\]

For space, we omit the process terms. Of importance is the instance of the call rule for the recursive call to eat: the check \(i - 1 < i\) verifies that the process terminates and the loop \([i - 1]/i][z/x]D\) “ties the knot” on the typechecking process. Mutually recursive programs, then, are checked by circular typing derivations that are mutually recursive in the metatheory.

Note that in subsequent examples, in lieu of writing unwieldy typing derivations, we will demarcate when arithmetic constraints are assumed and asserted, as well as how a recursive call is checked. Now, to illustrate compositionality of typechecking, i.e., termination checking without full source code availability, we show below how to develop a terminating program against a library thereof.

**Example 2.5** (Naïve Quicksort). A naïve “functional quicksort” can be implemented against the following signature of definitions assuming \(\text{nat} = \exists i. \text{nat}[i]\) and \(\text{list}[m] = \oplus\{\text{nil} : 1, \text{cons} : \text{nat} \otimes (n > 0 \land \text{list}[n - 1])\}\) — the latter classifying natural number lists of length at most \(n\).

\[
m, n; :: x : \text{list}[m], y : \text{list}[n] \vdash z \leftarrow \text{append} (m, n) (x, y) :: (z : \text{list}[m + n])
\]

\[
k; :: p : \text{nat}, x : \text{list}[k] \vdash y \leftarrow \text{partition} k (p, x) :: (y : \exists m, n. k = m + n \land \text{list}[m] \otimes \text{list}[n])
\]

That is, we assume that we have definitions that (1) append two lists together and (2) partitions one by a pivot. Then, at a high level, quicksort is a size-preserving definition.

\(^1\)If two vectors have different lengths, then zeroes are appended to the shorter one.
with the input list length as its termination measure. For brevity, we nest patterns (boxed and highlighted yellow), which can be expanded into nested matches [PP20].

\[
\begin{align*}
    k; \cdot x : \text{list}[k] & \vdash y \leftarrow \text{qsort } k \ x :: (y : \text{list}[k]) \\
y & \leftarrow \text{qsort } k \ x = \\
\text{case } x^R \{ \text{nil } x' \Rightarrow y^W \leftarrow x^R, \\
    \text{cons } (h, t) \Rightarrow z \leftarrow \text{partition } (k - 1) (h, t); \\
\text{case } z^R \{ \langle m, n, l, r \rangle \Rightarrow l' \leftarrow \text{qsort } m \ l; r' \leftarrow \text{qsort } n \ r;
    \text{case } z^R \{ \langle m, n, l, r \rangle \Rightarrow l' \leftarrow \text{qsort } m \ l; r' \leftarrow \text{qsort } n \ r;
    \text{case } z^R \{ \langle m, n, l, r \rangle \Rightarrow l' \leftarrow \text{qsort } m \ l; r' \leftarrow \text{qsort } n \ r;
    \text{case } z^R \{ \langle m, n, l, r \rangle \Rightarrow l' \leftarrow \text{qsort } m \ l; r' \leftarrow \text{qsort } n \ r;
\end{align*}
\]

Note that despite not being structurally recursive, quicksort still passes the termination check since size information flows from partition, to recursive calls to quicksort, then finally to append.

Now, consider the following two examples that demonstrate a use case of mixed induction-coinduction in concurrency.

**Example 2.6 (Left-fair streams).** Let us define the mixed inductive-coinductive type \( \text{lfair}_{A,B}[i,j] \) of left-fair streams [Bas18]: infinite \( A \)-streams where each element is separated by finitely many elements in \( B \). Once again, these types are not polymorphic, but are parametric in the choice of \( A \) and \( B \) for demonstration.

\[
\begin{align*}
\text{lfair}_{A,B}[i,j] &= \oplus \{ \text{now} : \& \{ \text{head} : A, \text{tail} : \text{lfair}_{A,B}[i,j], \text{later} : B \otimes \text{lfair}_{A,B}[i,j] \} \\
\text{lfair}_{A,B}[i,j] &= i > 0 \Rightarrow \exists j'. \ \text{lfair}_{A,B}[i - 1, j'] \\
\text{lfair}_{A,B}[i,j] &= j > 0 \land \text{lfair}_{A,B}[i,j - 1]
\end{align*}
\]

In particular, \( i \) bounds the observation depth of the \( A \)-stream whereas \( j \) bounds the height of the \( B \)-list in between consecutive \( A \) elements. Thus, this type is defined by lexicographic induction on \( (i,j) \). First, the provider may offer an element of \( A \), in which case the observation depth of the stream decreases from \( i \) to \( i - 1 \) (in the coinductive part, \( \text{lfair}_{A,B}[i,j] \)). As a result, \( j \) may be “reset” as an arbitrary \( j' \). On the other hand, if an element of “padding” in \( B \) is offered, then the depth \( i \) does not change. Rather, the height of the \( B \)-list decreases from \( j \) to \( j - 1 \) (in the inductive part, \( \text{lfair}_{A,B}[i,j] \)). By using left-fair streams, we can model processes that permit some timeout behavior but are eventually productive, since consecutive elements of type \( A \) are interspersed with only finitely many timeout acknowledgements of type \( B \). Armed with this type, we can define a projection operation [Bas18] that removes all of a left-fair stream’s timeout acknowledgements...
concurrently, returning an $A$-stream. As before, nested patterns are boxed and highlighted.

\[ \begin{align*}
i, j; \vdash x : \text{ifair}_{A,B}[i,j] \vdash (i,j) y & \leftarrow \text{proj} (i,j) x :: (y : \text{stream}_A[i]) \\
y & \leftarrow \text{proj} (i,j) x = \\
\text{case } x^R (\text{now } s \Rightarrow \text{case } y^W (\text{head } h \Rightarrow s^R, \text{head } h, \\
\text{tail } t) \Rightarrow u \leftarrow s^R. \text{tail } u; \\
i > 0 \text{ assumed} \\
\text{case } u^R ((j', x') \Rightarrow t \leftarrow \text{proj} (i-1, j') (x')), \\
i > 0 \text{ asserted} i, j, j' > 0 \Rightarrow (i-1, j') < (i, j) \text{ checked} \\
j > 0 \text{ assumed} \langle h, x' \rangle \Rightarrow y \leftarrow \text{proj} (i, j-1) (x') \\
i, j > 0 \Rightarrow (i, j-1) < (i, j) \text{ checked} \\
\end{align*} \]

**Example 2.7** (Stream processors). The mixed inductive-coinductive type $\text{sp}_{A,B}[i,j]$ of stream processors of input depth $i$ and output depth $j$ represents *continuous* (in the sense of [GHP09]) functions from $\text{stream}_A[i]$ to $\text{stream}_B[j]$; we define it below. Ghani et al. [GHP09] define it as the nested greatest-then-least fixed point $\nu X. \mu Y. (A \times Y) + (B \times X)$, but we adapt the version by Abel [Abe12] to finite size arithmetic.

\[ \begin{align*}
\text{sp}_{A,B}[i,j] & = \oplus \{ \text{get} : A \rightarrow \text{sp}_{A,B}[i,j], \text{put} : \& \{ \text{now} : B, \text{rest} : \text{sp}_{A,B}[i,j] \} \} \\
\text{sp}_{A,B}[i,j] & = i > 0 \land \forall j'. \text{sp}_{A,B}[i-1, j'] \\
\text{sp}_{A,B}[i,j] & = j > 0 \Rightarrow \text{sp}_{A,B}[i, j-1] \\
\end{align*} \]

Such functions may consume finitely many elements of type $A$ from the input stream (the inductive part $\text{sp}_{A,B}^I[i]$) before outputting arbitrarily many elements of type $B$ onto the output stream (the coinductive part $\text{sp}_{A,B}^C[i]$). This requires a lexicographic induction on $(i, j)$—in the inductive part, the input depth decreases to $i-1$, so the new output depth $j'$ may be arbitrary. In the coinductive part, $i$ stays the same, so $j$ must decrease (to $j-1$).

Let us interpret stream processors as functions on streams via a concurrent ”run” function (as opposed to the sequential version from prior work [Abe12, AP16]). Below, we give the type signature and code, with nested values boxed and highlighted yellow as usual.

\[ \begin{align*}
i, j, p : \text{sp}_{A,B}[i,j], x : \text{stream}_A[i] \vdash y \leftarrow \text{run} (i, j) (p, x) :: (y : \text{stream}_B[j]) \\
y & \leftarrow \text{run} (i, j) (p, x) = \\
\text{case } p^R \{ \text{get } f \Rightarrow h \leftarrow x^R. \text{head } h; p' \leftarrow f^R \{ h, (j, p') \} \} \\
\text{later} \langle h, (j, p') \rangle \Rightarrow y \leftarrow \text{proj} (i, j-1) (x') \\
i > 0 \text{ assumed} i, j, j' > 0 \Rightarrow (i, j-1) < (i, j) \text{ checked} \\
j > 0 \text{ assumed} \langle j, x' \rangle \Rightarrow p' \leftarrow o^R. \text{rest } p'; t \leftarrow \text{run} (i, j-1) (p', x) \} \\
j > 0 \text{ asserted} i, j, j' > 0 \Rightarrow (i, j-1) < (i, j) \text{ checked} \\
\end{align*} \]
If the processor issues a “get,” then the head of the input stream is consumed, recursing on its tail. Otherwise, the output stream is constructed recursively, first issuing the element received from the processor. It is clear that the program terminates by lexicographic induction on \((i, j)\).

3. SAX\(^\omega\)

Proving termination of program reduction in the presence of infinitely deep typing derivations typically requires techniques that deviate from standard (inductive) logical relations arguments \([\text{Bro}05, \text{DP}19]\). As a result, we give a purely inductive process typing called SAX\(^\omega\) with the judgment \(\Gamma \vdash \omega P :: (x : A)\) (Figure 2). By dropping the arithmetic and constraint contexts, the rules \(\exists L\omega\) and \(\forall R\omega\) have one premise per natural number \(n\) instead of introducing a new arithmetic variable (like \(\omega\)-rules in arithmetic \([\text{Sch}77]\)). Moreover, the premises of \(\wedge L\omega\) and \(\Rightarrow R\omega\) assume the closed constraint \(\phi\) (which has no free arithmetic variables) holds at the meta level instead of adding it to a constraint context.

Most importantly, the call rule does not refer to a coinductively-defined auxiliary judgment, because in the absence of free arithmetic variables, the tracked size arguments decrease from some \(\bar{n}\) to \(\bar{n}'\) to etc. Since the lexicographic order on fixed-length natural number vectors is well-founded, this sequence necessarily terminates. To rephrase: the exact number of recursive calls is known. While this system is impractical for type checking, we can translate arithmetically closed SAX\(^\infty\) derivations to SAX\(^\omega\) derivations. In fact, any SAX\(^\infty\) derivation can be made arithmetically closed by substituting each of its free arithmetic variables for numbers that validate (and therefore discharge) its constraints. By trading infinitely deep derivations for infinitely wide but finitely deep ones, we may complete a logical relations argument by induction over a SAX\(^\omega\) derivation. Thus, let us examine the translation theorem.

\[
D \vdash \omega P :: (x : A)
\]

**Theorem 3.1 (Translation).** If \(\vdash \vdash \Gamma \vdash \omega P :: (x : A)\), then \(\Gamma \vdash \omega P :: (x : A)\).

**Proof.** By lexicographic induction on \((\bar{n}, D)\), we cover the important cases. We show the proof as a transformation \(\rightsquigarrow\) on derivations with \(IH\) as the induction hypothesis.

(1) When SAX\(^\infty\) derivation \(D\) ends in the identity or an axiomatic rule, we are done by the corresponding SAX\(^\omega\) rule.

\[
D = \vdash ; \vdash ; \Gamma, x : A \vdash ^\pi y^W \leftarrow x^R :: (y : A) \overset{id}\rightsquigarrow \Gamma, x : A \vdash ^\omega y^W \leftarrow x^R :: (y : A) \overset{id^\omega}\]

\[
\Gamma, x : A \vdash \omega \quad y^W \leftarrow x^R :: (y : A) \quad \text{id}^\omega
\]
\[
\Gamma, x : A \vdash \omega \quad P(x) :: (x : A) \quad \Gamma, x : A \vdash \omega \quad Q(x) :: (z : C) \quad \text{cut}^\omega
\]
\[
\Gamma \vdash \omega \quad x \leftarrow P(x) ; Q(x) :: (z : C) \quad 1R^\omega
\]
\[
\Gamma, x : 1 \vdash \omega \quad P :: (z : C) \quad \text{1L}^\omega
\]
\[
\Gamma, x : A \vdash \omega \quad y : A, z : B \vdash \omega \quad P(y, z) :: (w : C) \quad \otimes \text{L}^\omega
\]
\[
\Gamma, y : A \vdash \omega \quad P(y, z) :: (z : B) \quad \rightarrow \text{R}^\omega
\]
\[
\Gamma, x : A \rightarrow B, y : A \vdash \omega \quad x^R, y, z :: (z : B) \quad \rightarrow \text{L}^\omega
\]
\[
\Gamma, y : A_k \vdash \omega \quad x^W, k y :: (x : \oplus \{ \ell : A_l \}_{l \in S}) \quad \oplus \text{R}^\omega
\]
\[
\{ \Gamma \vdash \omega \quad P(y) :: (x : A_l) \}_{l \in S} \quad \& \text{R}^\omega
\]
\[
\Gamma, y : A(n) \vdash \omega \quad x^W, (n, y) :: (x : \exists i. A(n)) \quad \exists \text{R}^\omega
\]
\[
\Gamma \vdash \omega \quad P(n, y) :: (y : A(n)) \quad \text{for all } n \in \mathbb{N} \quad \forall \text{R}^\omega
\]
\[
\Gamma \vdash \omega \quad x^W, (i, y) :: (x : \forall i. A(i)) \quad \forall \text{L}^\omega
\]
\[
\Gamma, x : \forall i. A(i) \vdash \omega \quad x^R, (n, y) :: (y : A(n)) \quad \forall \text{L}^\omega
\]
\[
\Gamma \vdash \omega \quad x^R, (\ast, y) :: (x : \phi \land A) \quad \wedge \text{R}^\omega
\]
\[
\Gamma, y : A \vdash \omega \quad x^W, (\ast, y) :: (x : \phi \land A) \quad \Rightarrow \text{R}^\omega
\]
\[
\Gamma, x : \phi \land A \vdash \omega \quad P(y) :: (z : C) \quad \text{if } \vdash \phi \quad \wedge \text{L}^\omega
\]
\[
\Gamma \vdash \omega \quad P(y) :: (y : A) \quad \text{if } \vdash \phi \quad \Rightarrow \text{L}^\omega
\]
\[
\vdash \phi \quad \Rightarrow \text{L}^\omega
\]
\[
\begin{align*}
\text{(no rule for impossible)}
\end{align*}
\]
\[
\Gamma, x : A \vdash \omega \quad y \leftarrow f \overline{\pi} \quad \text{call}^\omega
\]

Figure 2: SAX\(^\omega\) Typing Rules
(2) When SAX∞ derivation $D$ ends in an invertible propositional rule with subderivation $D'$, we proceed by induction on $(\pi, D')$.

$$
\begin{align*}
D' \\
\vdots; \pi, y : A \vdash P(y, z) :: (z : B) \\
\Gamma, y : A \vdash P(y, z) :: (x : A \to B) \quad \rightarrow R
\end{align*}
$$

$$
\begin{align*}
\Gamma, y : A \vdash P(y, z) :: (z : B) \\
\rightarrow R
\end{align*}
$$

(3) When SAX∞ derivation $D$ ends in $\exists L$ or $\forall R$, its subderivation $D'$ introduces a fresh arithmetic variable $i$. The $m^{th}$ premise of the corresponding SAXω rules $\exists L^\omega$ and $\forall R^\omega$ are fulfilled by induction on $(\pi, [m/i]D')$.

$$
\begin{align*}
D' \\
i; \pi, y : A \vdash P(i, y) :: (y : A(i)) \\
\Gamma, y : A \vdash P(y) :: (x : \forall i. A(i)) \quad \forall R
\end{align*}
$$

$$
\begin{align*}
\Gamma \vdash P(m, y) :: (y : A(m)) \text{ for all } m \in \mathbb{N} \\
\rightarrow R^\omega
\end{align*}
$$

(4) Analogously, when SAX∞ derivation $D$ ends in $\land L$ or $\Rightarrow R$, its subderivation $D'$ assumes the closed constraint $\phi$. The premises of the corresponding SAXω rules $\land L^\omega$ and $\Rightarrow R^\omega$

$$
\begin{align*}
D' \\
\vdots; \pi, y : A \vdash P(y) :: (y : A) \\
\Gamma \vdash P(y) :: (x : \phi \Rightarrow A) \quad \Rightarrow R
\end{align*}
$$

(5) Finally, assume SAX∞ derivation $D$ ends in the call rule with subderivation $D'$. By inversion, $D'$ ends in the $\infty$ rule with subderivation $D''$. Although $D''$ may be larger than $D$, we have some new arithmetic arguments $n'' < \pi$. Thus, we are done by induction
on \((\bar{m}', D'')\) then the \(\text{SAX}\omega\) call rule.

\[
\begin{align*}
\text{IH}(\bar{n}', D'') \\
\vdots \
\cdot, x: A \vdash n' \varphi(n', x, y) :: (y : A)
\end{align*}
\]

\[
\begin{align*}
D' &= \vdots \cdot, x: A \vdash n' < n \\
\text{call} &\Rightarrow \mathcal{IH}(\bar{n}', D'') \\
\vdots \
\cdot, \Gamma, x: A \vdash \varphi(n', \bar{x}) :: (y : A)
\end{align*}
\]

As we mentioned in the introduction, we can make the \(\text{SAX}\infty\) judgment arbitrarily rich to support more complex patterns of recursion. As long as derivations in that system can be translated to \(\text{SAX}\omega\), the logical relations argument over \(\text{SAX}\omega\) typing that we detail in Section 4 does not change. For example, consider the following additions.

1. **Multiple blocks**: To support multiple blocks of definitions, we may simply impose the requirement that mutual recursion may not occur across blocks. In other words, the call graph across blocks is directed acyclic, imposing a well-founded order on definition names: \(g < f\) iff \(f\) calls \(g\). As a result, translation of the definition \(f\) may proceed by lexicographic induction on \((f, \bar{n}, D)\). For example, let \(f\) call \(g\). If \(g\) is defined in a different block than \(f\), then the arithmetic arguments it applies \((\bar{n})\) may increase. Otherwise, \(n\) must decrease, since \(g\) is “equal” to \(f\) (in this order).

2. **Mutual recursion with priorities**: Definitions in a block can be ordered by priority: if \(g < f\), then \(f\) can call \(g\) with arguments of the same size. In Example 2.3, odds calls evens with arguments of the same size but evens calls odds with arguments of lesser size. As a result, evens \(<\) odds. If \(<\) is well-founded (like in this example), then translation of \(f\) may proceed by lexicographic induction on \((\bar{n}, f, D)\).

### 4. Operational Semantics and Termination

In this section, we will give an operational semantics for configurations of processes. Then, we will show that all \(\text{SAX}\omega\)-typed processes terminate, which accounts for \(\text{SAX}\infty\) by way of Theorem 3.1. Program execution based on processes alone is impractical, because cut elimination only facilitates communication between two processes at a time. Thus, DeYounge et al. [DPP20] define programs in \(\text{SAX}\) as configurations of simultaneously executing processes and the memory cells with which they communicate. Relatedly, the metatheory of the \(\pi\)-calculus must be defined up-to structural congruence to achieve a similar effect [PCPT14].

**Definition 4.1 (Configuration)**. Let \(a, b, c, \ldots\) be cell addresses and \(W := V | K\). A configuration \(C\) is defined by the following grammar.
\[ C := \cdot \quad \text{empty configuration} \]
\[ \mid \text{proc} a P \quad \text{process } P \text{ writing to cell addressed by } a \]
\[ \mid \text{!cell } a W \quad \text{persistent (marked with !) cell addressed by } a \text{ with contents } W \]
\[ \mid C, C \quad \text{join of two configurations} \]

\( C \) denotes a multiset of objects (processes and cells), so the join and empty rules form a commutative monoid. However, we also require that an address refers to at most one object in \( C \). Lastly, a configuration \( F \) is final iff it only contains (persistent) cells.

Now, let \( \Gamma \) and \( \Delta \) be contexts that associate cell addresses to types. The configuration typing judgment given in Figure 3, \( \Gamma \vdash C :: \Delta \), means that the objects in \( C \) are well-typed with sources in \( \Gamma \) and destinations in \( \Delta \) (note that we are allowing the process typing judgment to use addresses in place of address variables). Notice that the typing rules preserve the invariant \( \Gamma \subseteq \Delta \) thanks to the persistence of memory cells.

![Configuration Typing](image)

Figure 3: Configuration Typing

Configuration reduction \( \rightarrow \) is given as multiset rewriting rules [CS09] in Figure 4, which replace any subset of a configuration matching the left-hand side with the right-hand side. However, \( ! \) indicates objects that persist across reductions. Principal cuts encountered in a configuration are resolved by passing a value to a continuation also given in Figure 4 as the relation \( V \bowtie K = P \).

![Operational Semantics](image)

Figure 4: Operational Semantics

The first rule for \( \rightarrow \) corresponds to the identity rule and copies the contents of one cell into another. The second rule, which is for cut, models computing with futures [Hal85]: it allocates a new cell to be populated by the newly spawned \( P \). Concurrently, \( Q \) may read from said new cell, which blocks if it is not yet populated. The third and fourth rules resolve
principal cuts by passing a value to a continuation, whereas the fifth one resolves definition calls. Lastly, the final two rules perform the action of writing to a cell.

Now, we are ready to prove termination. Relatedly, refer to Das and Pfening [DP20a] for a proof of type safety for a session type system with arithmetic refinements. In contrast to the termination proof for base SAX [DP20], we explicitly construct a model of SAX in sets of terminating configurations, also known as semantic typing [App01, HLKB21]. This leaves open several possibilities—for example, we could reason about programs that fail to syntactically typecheck [JJKD17, DTK+19] or analyze fixed points of semantic type constructors. Our approach mirrors that for natural deduction:

1. We define semantic types: Kripke relations [Plo73] over cell contents $W$ where the worlds are the (terminating) configurations to which they belong.
2. We show compatibility lemmas that reflect the syntactic typing rules of processes, objects, and configurations in semantic typing.
3. This culminates in a fundamental theorem of the logical relation, from which termination is immediate.

Now, let us begin with the definition of semantic type.

**Definition 4.2** (Semantic type). A semantic type $\mathcal{A}, \mathcal{B}, \ldots$ is a set of pairs of cell contents and final configurations, writing $F \in \llbracket W : \mathcal{A} \rrbracket$ for $(W, F) \in \mathcal{A}$, such that:

- **Kripke monotonicity**: if $F \in \llbracket W : \mathcal{A} \rrbracket$, then $F, F' \in \llbracket W : \mathcal{A} \rrbracket$ for all $F'$.

Then, let $F \in \llbracket a : \mathcal{A} \rrbracket \triangleq \text{!cell } a W \in F$ for some $W$ where $F \in \llbracket W : \mathcal{A} \rrbracket$. We extend membership in semantic types to terminating configurations inductively in the usual way:

$$
\frac{F \in \llbracket a : \mathcal{A} \rrbracket}{C \in \llbracket a : \mathcal{A} \rrbracket} \quad \frac{C' \in \llbracket a : \mathcal{A} \rrbracket}{C \in \llbracket a : \mathcal{A} \rrbracket}
$$

Monotonicity is necessary, primarily, to prove the compatibility lemmas. In the next definition, we quickly define each semantic type in **boldface** based on its syntactic counterpart such that monotonicity is immediate.

**Definition 4.3** (Semantic types).

1. $F \in \llbracket () : 1 \rrbracket$ only.
2. $F \in \llbracket (a, b) : \mathcal{A} \otimes \mathcal{B} \rrbracket \triangleq F \in \llbracket a : \mathcal{A} \rrbracket$ and $F \in \llbracket b : \mathcal{B} \rrbracket$.
3. $F \in \llbracket \langle x, y \rangle \Rightarrow P(x, y) : \mathcal{A} \rightarrow \mathcal{B} \rrbracket \triangleq$ for all $F' \supseteq F$, if $F' \in \llbracket a : \mathcal{A} \rrbracket$, then $F', \text{proc} b (P(a, b)) \in \llbracket b : \mathcal{B} \rrbracket$ for $a$ and $b$ fresh in $F$ and $F'$, respectively.
4. $F \in \llbracket \{ k : \mathcal{A}_k \}_{k \in S} \rrbracket \triangleq k \in S$ and $F \in \llbracket a : \mathcal{A}_k \rrbracket$.
5. $F \in \llbracket \{ \ell x \Rightarrow P(x) \}_{\ell \in S} : \& \{ \ell : \mathcal{A}_\ell \}_{\ell \in S} \rrbracket \triangleq F, \text{proc} a (P_k(a)) \in \llbracket a : \mathcal{A}_k \rrbracket$ for $k \in S$ and for $a$ fresh in $F$.

Assume $\mathcal{F}$ is an $N$-indexed semantic type and that $\phi$ is a closed constraint.

1. $F \in \llbracket (n, a) : \mathcal{F} \rrbracket \triangleq F \in \llbracket a : \mathcal{F}(n) \rrbracket$.
2. $F \in \llbracket (i, x) \Rightarrow P(i, x) : \forall \mathcal{F} \rrbracket \triangleq F, \text{proc} a (P(n, a)) \in \llbracket a : \mathcal{F}(n) \rrbracket$ for all $n \in N$ and $a$ fresh in $F$.
3. $F \in \llbracket (\ast, a) : \phi \land \mathcal{A} \rrbracket \triangleq \vdots \vdash \phi$ and $F \in \llbracket a : \mathcal{A} \rrbracket$.
4. $F \in \llbracket (\ast, x) \Rightarrow P(x) : \phi \Rightarrow \mathcal{A} \rrbracket \triangleq$ if $\vdots \vdash \phi$, then $F, \text{proc} a (P(a)) \in \llbracket a : \mathcal{A} \rrbracket$ for $a$ fresh in $F$.

Positive semantic types are defined by *intension*—the contents of a particular cell—whereas negative semantic types are defined by *extension*—how interacting with a continuation produces the desired result. Analogously for the $\lambda$-calculus, the semantic positive
product is defined as containing pairs of terminating terms, whereas the semantic function space contains all terms that terminate under application [GTL89, AP16]. Now, to state the compatibility lemmas, we need to define the semantic typing judgment.

**Definition 4.4** (Semantic typing judgment). Let $\Gamma$ and $\Delta$ be contexts associating cell addresses to semantic types.

1. $F \in [\Gamma] \triangleq F \in [a : \mathcal{A}]$ for all $a : \mathcal{A} \in \Gamma$.
2. $C \in [\Gamma] \triangleq C \in [a : \mathcal{A}]$ for all $a : \mathcal{A} \in \Gamma$.
3. $\Gamma \vdash C :: \Delta \triangleq$ for all $F \in [\Gamma]$, we have $F, C \in [\Delta]$.

In natural deduction, the equivalent judgment $\Gamma \vdash e : \mathcal{A}$ is defined by quantifying over all closing value substitutions $\sigma$ with domain $\Gamma$, then stating $\sigma(e) \in \mathcal{A}$. Similarly, we ask whether the configuration $C$ terminates at the desired semantic type(s) when “closed” by a final configuration $F$ providing all the sources from which $C$ reads. Immediately, we reproduce the standard backwards closure result.

**Lemma 4.5** (Backward closure). If for all reducts $C'$ of $C$, $\Gamma \vdash C' :: \Delta$, then $\Gamma \vdash C :: \Delta$.

We are finally ready to prove a representative sample of compatibility lemmas, all of which are in Figure 5 (the dashed lines indicate that they are lemmas). Afterwards, we can tackle objects and configurations.

**Lemma 4.6** (id). $\Gamma, a : \mathcal{A} \vdash \text{proc}\; (b^W \leftarrow a^R) :: (b : \mathcal{A})$

**Proof.** Assuming $F \in [\Gamma, a : \mathcal{A}]$, we want to show $F, \text{proc}\; (b^W \leftarrow a^R) \in [b : \mathcal{A}]$. By assumption, there is some $\text{!cell}\; W \in F$, and so because $F, \text{proc}\; (b^W \leftarrow a^R) \rightarrow F, \text{!cell}\; b\; W$ is the only possible reduction, it suffices to show $F, \text{!cell}\; b\; W \in [b : \mathcal{A}]$ by Lemma 4.5, which is immediate from monotonicity.

The reader may have noticed that each semantic type’s definition encodes its own noninvertible rule, which makes the admissibility of rules like $\otimes R$ immediate. Invertible rules require more effort; consider $\otimes L$ and $\rightarrow R$ below.

**Lemma 4.7** ($\otimes L$). If $\Gamma, c : \mathcal{A} \otimes \mathcal{B}, a : \mathcal{A}, b : \mathcal{B} \vdash \text{proc}\; d (P(a, b)) :: (d : \mathcal{C})$, then $\Gamma, c : \mathcal{A} \otimes \mathcal{B} \vdash \text{proc}\; d (\text{case}\; c^R ((x, y) \Rightarrow P(x, y))) :: (d : \mathcal{C})$.

**Proof.** Assuming $F \in [\Gamma, c : \mathcal{A} \otimes \mathcal{B}]$, we want to show that $F, \text{proc}\; d (\text{case}\; c^R ((x, y) \Rightarrow P(x, y))) \in [d : \mathcal{C}]$. Since $F \in [c : \mathcal{A} \otimes \mathcal{B}]$, we have $\text{!cell}\; c\; (a, b) \in F$ where $F \in [a : \mathcal{A}]$ and $F \in [b : \mathcal{B}]$. In sum, $F \in [\Gamma, c : \mathcal{A} \otimes \mathcal{B}, a : \mathcal{A}, b : \mathcal{B}]$, so by the premise, $F, \text{proc}\; d (P(a, b)) \in [d : \mathcal{C}]$. Since $F, \text{proc}\; d (\text{case}\; c^R ((x, y) \Rightarrow P(x, y))) \rightarrow F, \text{proc}\; d (P(a, b))$ is the only reduction, we are done by Lemma 4.5.

**Lemma 4.8** ($\rightarrow R$). If $\Gamma, a : \mathcal{A} \vdash \text{proc}\; b (P(a, b)) :: (b : \mathcal{B})$, then $\Gamma \vdash \text{!cell}\; c (\langle x, y \rangle \Rightarrow P(x, y)) :: (c : \mathcal{A} \rightarrow \mathcal{B})$.

**Proof.** Assuming $F \in [\Gamma]$, we want to show that $F, \text{!cell}\; c (\langle x, y \rangle \Rightarrow P(x, y)) \in [c : \mathcal{A} \rightarrow \mathcal{B}]$, i.e., for all $F' \subseteq F$ where $F' \in [a : \mathcal{A}]$, we have $F', \text{proc}\; b (P(a, b)) \in [b : \mathcal{B}]$. By monotonicity, $F' \in [\Gamma, a : \mathcal{A}]$, so we are done by the premise.

With these compatibility lemmas in hand, we are almost ready to construct a correspondence between the syntactic typing of processes and configuration objects with the semantic typing thereof. First, we need a semantic interpretation of (syntactic) types.
\[ \Gamma, a : \mathcal{A} \vdash \text{proc } b (b^W \leftarrow a^R) :: (b : \mathcal{A}) \quad \text{id} \]
\[ \Gamma \vdash \text{proc } a \; P :: (a : \mathcal{A}) \quad \Gamma, a : A \vdash \text{proc } c (Q(a)) :: (c : \mathcal{C}) \quad \text{cut} \]
\[ \Gamma, a : \mathcal{A} \vdash \text{proc } b \; P :: (b : \mathcal{B}) \quad \Gamma \vdash \text{proc } c (x \leftarrow P; Q(x)) :: (c : \mathcal{C}) \]
\[ \Gamma, a : 1 \vdash \text{proc } b \; P :: (b : \mathcal{C}) \quad \Gamma, a : 1 \vdash \text{proc } b (\text{case } a^R (\langle \phi \rangle \Rightarrow P)) :: (b : \mathcal{C}) \quad \text{1L} \]
\[ \Gamma, a : \mathcal{A}, b : \mathcal{B} \vdash \text{cell } c (a, b) :: (c : \mathcal{A} \otimes \mathcal{B}) \quad \otimes R \]
\[ \Gamma, c : \mathcal{A} \otimes \mathcal{B}, a : \mathcal{A}, b : \mathcal{B} \vdash \text{proc } d (P(a, b)) :: (d : \mathcal{D}) \quad \otimes L \]
\[ \Gamma, a : \mathcal{A} \vdash \text{proc } b \; P :: (b : \mathcal{B}) \]
\[ \Gamma \vdash \text{cell } c ((x, y) \Rightarrow P(x, y)) :: (c : \mathcal{A} \rightarrow \mathcal{B}) \]
\[ \Gamma, a : \mathcal{A} \rightarrow \mathcal{B}, a : \mathcal{A} \equiv \text{proc } b (c^R, a, b) :: (b : \mathcal{B}) \quad \rightarrow L \]
\[ \{\Gamma, b : \mathcal{E} \otimes \{\ell : \mathcal{E}_{\ell} \}_{\ell \in \mathcal{S}}, a : \mathcal{A} \vdash \text{proc } c (P(k(a))) :: (c : \mathcal{C})\}_{k \in \mathcal{S}} \quad \oplus L \]
\[ \Gamma, a : \mathcal{A} \vdash \text{cell } b (k a) :: (b : \mathcal{A} \oplus \{\ell : \mathcal{A}_{\ell} \}_{\ell \in \mathcal{S}}) \quad \oplus R \]
\[ \Gamma, a : \mathcal{A} \rightarrow \mathcal{B}, a : \mathcal{A} \vdash \text{proc } b (c^R, a, b) :: (b : \mathcal{B}) \quad \& L \]
\[ \{\Gamma, b : \exists \mathcal{F}, a : \mathcal{F}(n) \vdash \text{proc } c (P(n, a)) :: (c : \mathcal{C})\}_{n \in \mathcal{N}} \quad \exists L \]
\[ \Gamma, a : \mathcal{A} \rightarrow \mathcal{B}, a : \mathcal{A} \vdash \text{proc } b (c^R, a, b) :: (b : \mathcal{B}) \quad \forall L \]
\[ \{\Gamma, b : \forall \mathcal{F} \vdash \text{proc } a (b^R, n, a) :: (a : \mathcal{F}(n))\}_{n \in \mathcal{N}} \quad \forall L \]
\[ \Gamma, a : \mathcal{A} \rightarrow \mathcal{B}, a : \mathcal{A} \vdash \text{proc } b (P(b)) :: (b : \mathcal{A}) \quad \text{id} \]
\[ \Gamma, a : \mathcal{A} \rightarrow \mathcal{B}, a : \mathcal{A} \vdash \text{proc } a (\text{case } a^R (\langle * , y \rangle \Rightarrow P(y))) :: (c : \mathcal{C}) \quad \text{AR} \]
\[ \Gamma, a : \mathcal{A} \rightarrow \mathcal{B}, a : \mathcal{A} \vdash \text{proc } b (a^R, \ast, b) :: (b : \mathcal{A}) \quad \text{L} \]
\[ \Gamma, b : \mathcal{A} \vdash \text{proc } a (f \leftarrow f \bar{\mathcal{A}}) :: (a : \mathcal{A}) \quad \text{call} \]
\[ \text{(no rule for } \boxed{\text{impossible}} \text{) } \]

Figure 5: Compatibility Lemmas

**Definition 4.9** (Semantic interpretation). We define \(\downarrow[n](A)\) by lexicographic induction on \((n, A)\) that sends arithmetically closed types to semantic ones. At a recursive type, \(n\) is stepped down to allow \(A\) to potentially grow larger. Note that \(\lambda\) marks a meta-level anonymous function and that \(\phi\) is closed.
For example, consider when Part 1 follows by case analysis on SAX then the part number, yielding induction hypotheses

\[ (\exists i. A(i))_n \equiv \exists \lambda m. (A(m))_n \]

Now, let

\[ \{ A \} = F \in \{ A \}_n \]

for some \( n \)—intuitively, all of the (syntactic) types we have considered so far are defined by a lexicographic induction on their arithmetic indices, so \( \{ A \} \) classifies configurations \( C \) for which there is some number of recursive unfoldings of \( A \) that successfully establish \( C \) as a member of (the denotation of) \( A \) due to the aforementioned induction. \( \{ \} \) is then extended to contexts \( \Gamma \) and \( \Delta \) in the obvious way.

**Lemma 4.10** (Semantic object typing).

\[
\begin{align*}
\langle 1 \rangle_n & \equiv 1 \\
\langle a \otimes B \rangle_n & \equiv \langle a \rangle_n \otimes \langle B \rangle_n \\
\langle \exists \{ \ell : A \} \rangle & \equiv \exists \{ \ell : \langle A \rangle \}_n \in S \\
\langle \forall i. A(i) \rangle_n & \equiv \forall \lambda m. (A(m))_n \\
\langle \phi \land A \rangle_n & \equiv \phi \land \langle A \rangle_n \\
\langle X [\mathcal{M}] \rangle_{0} & \equiv \emptyset \\
\langle \Xi [\mathcal{M}] \rangle_{n} & \equiv \langle A \rangle_n \rightarrow \langle B \rangle_n \\
\langle \phi \Rightarrow A \rangle_n & \equiv \phi \Rightarrow \langle A \rangle_n \\
\langle X [\mathcal{M}] \rangle_{n+1} & \equiv \langle A \rangle_n \rightarrow \langle B \rangle_n 
\end{align*}
\]

Proof. Part 1 follows by case analysis on SAX\( ^\omega \) derivation \( D \) applying the relevant compatibility lemmas. For example, consider when \( D \) ends in \( \otimes \mathcal{R} \) (once again, representing the theorem as a translation \( \leadsto \)).

\[
\begin{align*}
D' & \equiv \Gamma, a : A, b : B \vdash \text{case} c^W. \langle y, z \rangle \Rightarrow P(y, z) \vdash (c : A \rightarrow B) \otimes \mathcal{R} \leadsto \\
\& \langle \Xi [\mathcal{M}] \rangle_{n} & \equiv \langle A \rangle_n \rightarrow \langle B \rangle_n \\
\& \langle X [\mathcal{M}] \rangle_{n+1} & \equiv \langle A \rangle_n \rightarrow \langle B \rangle_n 
\end{align*}
\]

We prove parts 2 and 3 simultaneously by lexicographic induction on SAX\( ^\omega \) derivation \( D \) then the part number, yielding induction hypotheses \( \text{IH}_2(\text{derivation}) \) and \( \text{IH}_3(\text{derivation}) \) indexed by part number. First, part 2 refers to part 3 on the typing subderivation for the process contained in \( K \). For example, consider when \( D \) ends in \( \rightarrow \mathcal{R} \)—upon induction, the size of the derivation decreases, allowing the part number to increase. Below, we use ellipses to indicate applications of the inductive hypothesis, even though we are not constructing syntactic typing derivations.

\[
\begin{align*}
D' & \equiv \Gamma, a : A, b : B \vdash (b : B) \rightarrow \mathcal{R} \leadsto \\
\& \langle \Xi [\mathcal{M}] \rangle_{n} & \equiv \langle A \rangle_n \rightarrow \langle B \rangle_n \\
\& \langle X [\mathcal{M}] \rangle_{n+1} & \equiv \langle A \rangle_n \rightarrow \langle B \rangle_n 
\end{align*}
\]

\[
\begin{align*}
\Gamma, a : A, b : B \vdash \text{case} c^W. \langle (y, z) \Rightarrow P(y, z) \rangle \vdash (c : A \rightarrow B) \rightarrow \mathcal{R} \leadsto \\
\& \langle \Xi [\mathcal{M}] \rangle_{n} & \equiv \langle A \rangle_n \rightarrow \langle B \rangle_n \\
\& \langle X [\mathcal{M}] \rangle_{n+1} & \equiv \langle A \rangle_n \rightarrow \langle B \rangle_n 
\end{align*}
\]
In part 3, if \( P \) reads a cell with a synchronous rule, then we invoke the relevant compatibility lemma. For example, consider \( \rightarrow \L^\omega \).

\[
\Gamma, c : A \rightarrow B, a : A \vdash \omega c^R, (a, b) :: (b : B) \quad \rightarrow \L^\omega
\]

\[
\Gamma, a : (a, b) \vdash c^R, (a, b) :: (b : B)
\]

However, if \( P \) reads a cell with an asynchronous rule, then we invoke part 3 on the typing subderivation. For example, consider when \( D \) ends in \( \otimes \L^\omega \)—upon induction, the size of the derivation decreases, but the part number stays the same.

\[
D' \vdash \Gamma, c : A \otimes B, a : A, b : B \vdash \omega P(a, b) :: (d : C)
\]

\[
D = \Gamma, c : A \otimes B \vdash \omega \text{case} e^R ((x, y) \Rightarrow P(y, z)) :: (d : C) \quad \otimes \L^\omega
\]

\[
\langle \Gamma \rangle, c : (A \otimes B), a : (A), b : (B) \vdash \text{proc } d (P(a, b)) :: (d : (C))
\]

\[
\langle \Gamma \rangle, c : (A) \otimes (B) \vdash \text{proc } d (\text{case } c^R ((x, y) \Rightarrow P(x, y))) :: (d : (C))
\]

If \( P \) writes a continuation \( K \), then \( \text{proc } a (\text{case } a^W K) \rightarrow \text{!cell } a K \), so we invoke part 2 on \( \SAX^\omega \) derivation \( D \) and conclude by Lemma 4.5. For example, we consider again the case when \( D \) ends in \( \rightarrow \R^\omega \)—upon induction, the size of the derivation stays the same and the part number decreases.

\[
\Gamma, a : A \vdash \omega P(a, b) :: (b : B)
\]

\[
D = \Gamma, c : A \vdash \omega \text{case} c^W ((y, z) \Rightarrow P(y, z)) :: (c : A \rightarrow B) \quad \rightarrow \R^\omega
\]

\[
\langle \Gamma \rangle, c : (A) \vdash \text{!cell } c ((y, z) \Rightarrow P(y, z)) :: (c : (A) \rightarrow (B))
\]

\[
\langle \Gamma \rangle, c : (A) \otimes (B) \vdash \text{proc } c (\text{case } c^W ((y, z) \Rightarrow P(y, z))) :: (c : (A) \rightarrow (B))
\]

Writing a value follows symmetrically since \( \text{proc } a (a^W V) \rightarrow \text{!cell } a V \), invoking part 1.

\[
D = \Gamma, a : A, b : B \vdash \omega c^W, (a, b) :: (c : A \otimes B) \quad \otimes \R^\omega
\]

\[
\langle \Gamma \rangle, c : (A), b : (B) \vdash \text{!cell } c (a, b) :: (c : (A) \otimes (B))
\]

\[
\langle \Gamma \rangle, a : (A), b : (B) \vdash \text{proc } c (c^W, (a, b)) :: (c : (A) \otimes (B))
\]

\[
\text{Part 1 on } \langle \Gamma \rangle, a : (A) \vdash \text{proc } b (b^W \leftarrow a^R) :: (b : (A))
\]

Lastly, identity directly invokes its semantic counterpart...

\[
D = \Gamma, a : A \vdash \omega b^W \leftarrow a^R :: (b : A) \quad \text{id}^\omega
\]

\[
\langle \Gamma \rangle, a : (A) \vdash \text{proc } b (b^W \leftarrow a^R) :: (b : (A)) \quad \text{id}
\]
... whereas cut invokes the induction hypothesis (the size of the derivation decreases and the part number stays the same).

\[
D = \frac{D_1 \in \Gamma \vdash P(a) :: (a : A) \quad D_2 \in \Gamma, a : A \vdash Q(a) :: (c : C)}{\Gamma \vdash x \leftarrow P(x); Q(x) :: (c : C)} \quad \text{cut}^{\omega} \sim\sim
\]

\[
\text{IH}_3(D_1) \quad \text{IH}_3(D_2)
\]

\[
(\Gamma) \models \text{proc } P :: (a : \{A\}) \quad (\Gamma), a : \{A\} \models \text{proc } c(Q(a)) :: (c : \{C\})
\]

\[
(\Gamma) \models \text{proc } c(x \leftarrow P; Q(x)) :: (c : \{C\}) \quad \text{cut}
\]

Now that processes and objects have been resolved, it remains to derive the semantic configuration typing rules.

**Lemma 4.11** (Semantic configuration typing).

1. **Empty:** \( \Gamma \models \cdot :: \Gamma \)
2. **Join:** If \( \Gamma \models C :: \Gamma' \) and \( \Gamma' \models C' :: \Delta \), then \( \Gamma \models C, C' :: \Delta \).

**Proof.**

1. Immediate.
2. Assuming \( F \in [\Gamma] \), we want to show \( F, C, C' \in [\Delta] \). By induction on the first premise, \( F, C \in [\Gamma'] \), we proceed by cases. If \( F, C \) is final, then we are done by the second premise. Otherwise, if \( F, C \rightarrow F, C_1 \) for some \( C_1 \), then we are done by induction hypothesis.

The previous lemmas establish the fundamental theorem, i.e., compatibility of the syntax with semantics.

\[
D
\]

**Theorem 4.12** (Fundamental theorem). If \( \Gamma \models C :: \Delta \), then \( (\Gamma) \models C :: (\{\Delta\} \).

**Proof.** By induction on the configuration typing derivation \( D \), the empty and join cases are discharged by Lemma 4.11. The object typing cases are covered by Lemma 4.10, noting that \( (\Gamma) \) persists across the semantic sequent due to memory cell persistence and monotonicity.

The fundamental theorem of the logical relation entails termination of closed well-typed configurations, as desired.

**Corollary 4.13** (Termination). If \( \cdot \models C :: \Delta \), then \( C \) terminates, i.e., either \( C \) is final or, inductively, \( C' \) terminates for all reducts \( C' \) of \( C \).

## 5. Related Work

Our system is closely related to the sequential functional language of Lepigre and Raffalli [LR19], which utilizes circular typing derivations for a sized type system with mixed inductive-coinductive types, also avoiding continuity checking. In particular, their well-foundedness criterion on circular proofs seems to correspond to our checking that sizes decrease between recursive calls. However, they encode recursion using a fixed point combinator and use transfinite size arithmetic, both of which we avoid as we explained in the introduction. Moreover, our metatheory, which handles infinite typing derivations (via
mixed induction-coinduction at the meta level), seems to be both simpler and more general since it does not have to explicitly rule out non-circular derivations. Nevertheless, we are interested in how their innovations in polymorphism and Curry-style subtyping can be integrated into our system, especially the ability to handle programs not annotated with sizes.

**Sized types.** Sized types are a type-oriented formulation of size-change termination [LJBA01] for rewrite systems [TG03, BR09]. Sized (co)inductive types [BFG+04, Bla04, Abe08, AP16] gave way to sized mixed inductive-coinductive types [Abe12, AP16]. In parallel, linear size arithmetic for sized inductive types [CK01, Xi01, BR06] was generalized to support coinductive types as well [Sac14]. We present, to our knowledge, the first sized type system for a concurrent programming language as well as the first system to combine both features from above. As we mentioned in the introduction, we use unbounded quantification [Vez15] in lieu of transfinite sizes to represent (co)data of arbitrary height and depth. However, the state of the art [Abe12, AP16, CLB23] supports polymorphic, higher-kindred, and dependent types, which we aim to incorporate in future work.

**Size inference.** Our system keeps constraints implicit but arithmetic data explicit at the process level in agreement with observations made about constraint and arithmetic term reconstruction in a session-typed calculus [DP20c]. On the other hand, systems like CICℓ [Sac14] and CIC∗ [CLB23] have comprehensive size inference, which translates recursive programs with non-sized (co)inductive types to their sized counterparts when they are well-defined. Since our view is that sized types are a mode of use of more general arithmetic refinements, we do not consider size inference at the moment.

**Infinite and circular proofs.** Validity conditions of infinite proofs have been developed to keep cut elimination productive, which correspond to criteria like the guardedness check [BDS16, BT17, DP19, DP20c]. Although we use infinite typing derivations, we explicitly avoid syntactic termination checking for its non-compositionality. Nevertheless, we are interested in implementing such validity conditions as uses of sized types as future work. Relatedly, cyclic termination proofs for separation logic programs can be automated [BBC08, TB20], although it is unclear how they could generalize to concurrent programs (in the setting of concurrent separation logic) as well as codata.

**Session types.** Session types are inextricably linked with SAX, as it also has an asynchronous message passing interpretation [PP21]. Severi et al. [SPTDC16] give a mixed functional and concurrent programming language where corecursive definitions are typed with Nakano’s later modality [Nak00]. Since Vezzosi [Vez15] gives an embedding of the later modality and its dual into sized types, we believe that a similar arrangement can be achieved in our setting. In any case, we support recursion schemes more complex than structural (co)recursion [LM16].

**π-calculi.** Certain type systems for π-calculi [Kob06, Pad14, GKL14] guarantee the eventual success of communication only if or regardless of whether processes diverge [DP22]. Considering a configuration C such that \( \Gamma ⊢ C :: (\Gamma, a : X[n]) \) where \( X[i] \) is a positive coinductive type, we conjecture that \( |C| \), which has all constraint and arithmetic data erased, is similarly “productive” even if it may not terminate. Intuitively, \( C \) writes a number of cells as a function of \( n \) then terminates, so \( |C| \) represents \( C \) in the limit since \( X[i] \) is positive...
coinductive. However, this behavior is more desirable in a message passing setting rather than in our shared memory setting.

On the other hand, there are type systems that themselves guarantee termination—some assign numeric levels to each channel name and restrict communication such that a measure induced by said levels decreases consistently [DS06, DHS10, CH16]. While message passing is a different setting than ours, we are interested in the relationship between sizes and levels, if any. Other such type systems constrain the type and/or term structure; the language $\mathcal{P}$ [San06] requires grammatical restrictions on both types and terms, the latter of which we are trying to avoid. On the other hand, the combination of linearity and a certain acyclicity condition [YBH04] on graph types [Yos96] is also sufficient. Our system is able to guarantee termination despite utilizing non-linear types, but it remains open how type refinements compare to graph types.

6. CONCLUSION AND FUTURE WORK

We have presented a highly general concurrent language that conceives mixed inductive-coinductive programming as a mode of use of arithmetic refinements. Moreover, we prove termination via a novel logical relations argument in the presence of infinitely deep typing derivations that is mediated through infinitely wide but finitely deep (inductive) typing. There are three main points of interest for future work.

(1) **Richer types**: to mix linear [Pfe20], affine linear, non-linear, etc. references to memory as well as persistent and ephemeral memory, we conjecture that moving to a type system based on adjoint logic [PP21] is appropriate. In that case, sizes could be related to the grades of the adjoint modalities [Som21]. Furthermore, we are interested in generalizing to substructural, polymorphic, higher-kindred [DDMP21], and dependent types [CP96, KPB15].

(2) **Implementation**: we are interested in developing a convenient surface language (perhaps a functional one [PP20]) for SAX and implementing our type system, following Rast [DP20a], an implementation of resource-aware session types that includes arithmetic refinements. Perhaps various validity conditions of infinite proofs can be implemented as implicit uses of sized type refinements.

(3) **Message passing**: we would like to transport our results to the asynchronous message passing interpretation of SAX [PP21], avoiding a technically difficult detour through asynchronous typed $\pi$-calculi [DCPT12].

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