Towards an exact description of Gravitational Waves with a Positive Cosmological Constant: Basic Framework

Adrian Boîtier and Shubhanshu Tiwari

November 1, 2022

Abstract

Gravitational waves (GW) are a natural consequence of general relativity (GR), first derived by Einstein in 1918, but their existence was debated as the derivation was only available in the linearized version of the theory. Only in 1960 the existence of GW in full GR was established. The non-trivial task of finding a workable definition for gravitational waves in general relativity was achieved and their existence as a solution to the Einstein equations for vanishing cosmological constant has been proven. But although much work has been done to extend this proof the case of a positive cosmological constant (which is a best fit model of cosmology with current observations) a description in the full non-linear theory is still missing. We take inspiration from the Bondi-Sachs formalism and approach the problem in a novel way, by describing a geometry via geodesics and thus removing the degeneracy between coordinates and the metric.

Our formalism uses infinitesimal spherical triangles as geometry generating elements, to relate the geodesic flow bundle to Gaussian curvature. This ultimately allows us to calculate all geodesics from a curvature field up to second order. In this paper we describe the first step of the space-like part of the solution, in which we reduce an n-dimensional problem to a 2-dimensional triangulation problem.

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1 Introduction

The question, whether general relativity predicts gravitational waves is an old one and is rigorously answered for the case of the cosmological constant being 0. However, the current observations suggest that the universe is in an accelerated expansion which is best described by the standard model of cosmology (Λ-CDM) where the cosmological constant (Λ) is positive [1]. Moreover, we now have multiple observations of gravitational wave (GW) events [2] experimentally proving their existence. These observational results motivate strongly the theoretical exploration of gravitational waves with a positive Λ. In this work we lay the basic framework which might be helpful in proving the existence of GW with a positive Λ.

Before such a proof could be attempted one should find an invariant definition of a gravitational wave, since coordinate artifacts can be (and were) confused with GW as the discussion in [3] shows. An invariant definition was achieved by [4]. The Bondi-Sachs formalism [5, 6] uses outgoing null-rays to create a coordinate system on an asymptotically flat space-time. Using this formalism Robinson and Trautman [7] managed to prove the existence of gravitational waves in GR for Λ = 0. Unfortunately, this proof is not straight forwardly generalizable to arbitrary backgrounds, since there the asymptotic symmetry group is Diff(Σ) which makes it difficult to find an invariant notion of energy-momentum carried by gravitational waves (GW). Ashtekar and collaborators in a series of papers pointed out the difficulties of extending the Bondi-Sachs formalism to Λ ≠ 0 and achieved a description of GW on a de-Sitter background in the weak field limit [8–12]. A description of GW on arbitrary backgrounds in full GR, however, is not yet achieved.

The Bondi-Sachs formalism works with a $\frac{1}{t}$-expansion. This had to be replaced with a late-time expansion in the case of the background de-Sitter metric and it is reasonable to assume that it has to be specifically adapted to whatever background metric one is working on, which for a generic background cannot be done.

We took inspiration from the Bondi-Sachs formalism in another way: There, a special chart is created using outgoing null-rays which achieves that we have some understanding of what these coordinates represent and are thus not confused about whether the metric describes a wave or not.

We tried to go a step further and use the geodesics themselves to describe a geometry instead of using some of them to set up a coordinate system and then use the metric in that chart for this purpose. Consequently, we relate the geodesics directly to curvature i.e., we construct the analogue relation to:

\[
R^\mu_{\nu\rho\sigma} = -\Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\eta_{\nu\sigma,\rho} \Gamma^\mu_{\eta\rho} + \Gamma^\nu_{\nu\sigma} \Gamma^\mu_{\eta\rho}, \quad \Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} (g_{\nu\rho,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu}),
\]

where the metric is related to curvature via second order coupled partial differential equations.

If one manages to find a metric to a given curvature tensor one in general faces the problem we pointed out above. Namely that we have to understand the coordinates in which the metric is expressed to understand if it describes a GW or not. One way to achieve this is to solve the geodesic equations

\[
\frac{D}{dt} \gamma^\mu = \ddot{\gamma}^\mu + \Gamma^\mu_{\rho\sigma} (\gamma) \dot{\gamma}^\rho \dot{\gamma}^\sigma = 0, \quad \gamma : I \subset \mathbb{R} \rightarrow \mathcal{M}
\]

and then use these geodesics to construct a new coordinate system as it is done in the Bondi-Sachs formalism or for the Edington-Finkelstein coordinates.

Proper time distances between two spacetime events can then be calculated by integrating the tangent vector field along such a geodesic, since distances on a differential manifold are defined as the infimum of the arclengths of connecting curves.

\[
d(x, y) = \inf \{l[\gamma] : [a, b] \rightarrow \mathcal{M}, \gamma(a) = x, \gamma(b) = y\}, \quad l[\gamma] = \int_a^b \sqrt{g_{\gamma(t)} (\dot{\gamma}(t), \dot{\gamma}(t))} \, dt
\]

In our formalism the quantity describing geometry is related to the one describing curvature by a transcendental equation.

In this work we restrict ourselves to the Riemannian case which describes the spatial sub-sheets of a space-time. Thus there is no time evolution in the problem, but we are currently working on an extension of the formalism to the pseudo-Riemannian case, for which the Riemannian case serves as a foundation.
1.1 Outline and resulting Procedure

Since we want to describe a geometry using only geodesics, we start by defining and rederiving many fundamental concepts.

A geometry is essentially a collection of distances between all points in a space and angles between all direction at any point. Distances in a manifold are inherently related to geodesics. If the infimum in Eq. (3) exists, there is a connecting curve with minimal arclength, which is by definition called a geodesic. We thus use arclength parametrized geodesics \( \gamma \) together with a map to the \((n-1)\)-sphere \( S^{n-1} \), representing the directions at any given point, to get as close to (fundamental) geometrical properties as possible. Interestingly, this ended up excluding the null-geodesics and we thus work on the complement of geodesics at the origin to the ones used in the Bondi-Sachs formalism. We bundle the geodesics of flows from all points together, analogue to the tangent bundle and relate this geodesic flow bundle to curvature via the sine- and cosine-laws on any geodesic triangle.

This way we manage to skip the step from the Riemann-tensor (1) and the Einstein equations to the geodesic equations (red arrows in Fig. 1) and reduce the coupled non-linear differential equations to an analytic operation (green-gray arrow in Fig. 1). We achieve this by design: if the metric is the main quantity. One relates the metric to curvature and then first solves for the metric before one can calculate any other quantity one might want from there. So, if we make geodesics to the main quantity, we calculate them directly from curvature. For the same reason we restrict the parametrization of the geodesics to arclength to get the distances directly ingrained in our main quantity. We develop a spherical triangulation method in the follow-up paper, which allows us to calculate the geodesic flow bundle from a curvature field. The distances and angles can then be then read out from the arguments of the geodesic flow bundle and thus the integral in Eq. (3) reduces to a transcendental equation.

The following diagram Fig. 1 summarizes, how the geodesic flow bundle is related to the metric one. The arrows show, from which quantities we can calculate others. The analogue to the Riemann curvature tensor in our formalism is the source curvature \( K \) (Sec. 4). We are not certain yet, whether an analogue of the energy-momentum tensor is needed.

Figure 1: Our ultimate goal is represented by the blue dashed arrow. The black arrows stand for calculations which are straight forward, whereas the red ones mark in general unsolvable equations. Green stands for the steps which are derived in this work and the gray arrow is covered in the follow-up paper. The gray dotted relations are yet unknown or not properly worked out (Parallel Transport).

The presentation is not strictly structured according to the resulting methodology, since we would otherwise have to use concepts which were not introduced jet. We define the geodesic flow bundle (GFB) in Sec. 2 and investigate its properties. Then we construct a special case of normal charts, which we call faithful normal chart in Sec. 3. Ultimately, we want to use this formalism on the Einstein equations. The first step in that direction, is to connect the geodesic flow bundle to curvature, which we do in Sec. 4. We find that it is more appropriate in this formalism to treat curvature as a source of geometry rather then an effect of it and thus introduce the notion of a source curvature field. This field can be related to the Riemann curvature tensor using parallel transport. We discuss how a natural notion of parallel transport arises from the GFB in Appendix C but discussing its relation to Riemann curvature in full detail is postponed to a later publication.
We relate our formalism to the metric in Sec. 5, to lay a ground for comparison and check, whether our results so far are consistent, before we start calculating the geodesic flow bundle from a source curvature field, which we will present in the follow-up paper.

We provide some examples in the supplement to visualize the geodesic flow bundle and the faithful normal charts. We also demonstrate, how one can calculate the metric from the GFB on the example of the 2-sphere, 3-sphere, flat n-dimensional space and give an example of a geometry, which cannot be described using a metric.

Finally, a method emerges from this discussion, through which we can calculate the geodesic flow bundle up to second order in the arclength parameter from a source curvature field. The resulting procedure works as follows:

0. We treat curvature as a source field for geometry and consider it our input from which we calculate the geodesics.

1. Then we systematically embed two-dimensional hypersurfaces in an n-dimensional manifold which splits the problem into a one of tilt angles and a two-dimensional one. Since we restrict ourselves to Riemannian plane GFBs this angular problem is relatively simple and consists of solving transcendental equations obtained from rotations.

2. The remaining two-dimensional problem contains the largest part of the complexity and amounts to an extension of the sine- and cosine-laws to varying curvature fields. The follow-up paper is dedicated to solving it up to second order via the spherical triangulation of a geodesic triangle.

The same procedure can be used in the case of a pseudo-Riemannian space-time. We develop and test it on the simplest example in this and the follow up paper, before we use it on more generic cases.

2 Definition of the Geodesic Flow Bundle

Usually, one defines a metric which is a scalar product field, to impose a geometry on a manifold. The geometric quantities i.e. distances and angles can then be calculated from the metric. But not all notions of distances can be expressed as scalar products, like for example the Manhattan metric

\[ d(x, y) = \sum_{i=1}^{n} |x_i - y_i| \]

In the context of differential manifold we interpret this as imposing the Manhattan metric consistently on an \( \varepsilon \)-Ball around each point in the manifold. Constructing such a metric using a scalar product field would require us to represent the metric via a scalar product on every tangent space. This notion of distance on a vector space is however not representable by a matrix.

Despite the fact, that we can define a scalar product coordinate independently we still need to choose coordinates when we actually want to use it. And we are trying to use a linear structure on an otherwise in general nonlinear entity. We might lose or obscure some of the non-linear structure when we do this.

If we are given a metric expressed in an arbitrary chart, it can be challenging to discern the underlying geometry of the manifold since the components are all coordinate dependent and thus cannot be interpreted as long as we do not understand what coordinates we are working with. To do so we need geodesics in a parametrization affine to their arclengths, since we then understand the parametrization and we know that curves are 1-dimensional submanifolds and thus coordinate independent. Additionally, we know that geodesics trace out the shortest paths between two sufficiently close points. To find these geodesics however, we would have to solve the geodesic equations and then integrate to compute a distance. We propose to instead describe the geometry on a manifold by using the geodesics themselves instead of a scalar product.

2.1 Geodesic flow from a point

Instead of choosing a scalar product field or imposing a metric which is a function which maps two points to a positive real number. We choose a family of curves \( \{ \gamma_\Omega \}_{\Omega \in D_o} \) starting from a origin point \( o \in M \) on the manifold \( M^n \) and heading out in all directions \( \Omega \in D_o \). We consider this family of curves as geodesics by definition and define the distance from \( o \) to any other point \( p \) to be the parameter value of the curve at that point. We thereby
consider the parameter \( \lambda \) to be the arclength by definition as well. Let \( M^n \) be an \( n \) dimensional differential manifold and \( x \in M \).

\[
\gamma_{o,\Omega} : (-\varepsilon, \varepsilon) \to M \quad \lambda \mapsto \gamma_{o,\Omega}(\lambda) \quad \varepsilon > 0 \quad \land \quad \gamma_{o,\Omega}(0) = o;
\]

\[
d(o, x) := \lambda, \quad \text{with } \lambda \in (0, \varepsilon) \text{ and } \Omega \in D_o \text{ s.t. } \gamma_{o,\Omega}(\lambda) = x \quad (5)
\]

As a definition for the set of directions at a point \( p \in M \) we can use the following equivalence class of curves through \( p \).

\[
D_p := \{[\gamma_p] \sim | \gamma_p : (-\varepsilon, \varepsilon) \to M, \text{ smooth }, \gamma_p(0) = p \}, \quad \gamma_p \sim \gamma_p' \Leftrightarrow \dot{\gamma}_p(0) \propto \dot{\gamma}_{p'}(0). \quad (6)
\]

This way we do not need to impose any additional structure, other than the manifold \( M \) itself. Directions should already be a property of a manifold and thus it should not be necessary to introduce additional structure to describe them. The family of curves \( \{\gamma_{o,\Omega}\}_{\Omega \in D_o} \) defines a flow in the neighbourhood around \( o \) without the origin point \( o \) itself.

**Definition 1** (Geodesic Flow from a Point). The geodesic flow from \( x \in M \) is the map:

\[
\phi : D_x \to \mathcal{C}^\infty(I; M), \quad I = (0, \varepsilon), \quad \text{s.t. } (7) \quad \gamma_{\Omega}(\lambda) \text{ are injective (8)}
\]

\[
\Omega \mapsto \gamma_{\Omega}(I), \quad I = (0, \varepsilon), \quad \text{s.t. } (7) \quad \gamma_{\Omega} \text{ is smooth (9)}
\]

\[
\Rightarrow X(\gamma_{\Omega}(\lambda)) = \dot{\gamma}_{\Omega}(\lambda) \in \Gamma (TM\setminus\{x\}) \quad (10)
\]

As concluded from condition (9) the tangent vectors of the flow from \( x \) form a smooth vector field \( X \) on \( M \setminus \{x\} \), which by definition is a central field of unit vectors. We briefly recap the standard notion of the flow of a vector field in differential geometry and write the geodesic flow from a point in a more conventional way:

A local flow of \( X \in \Gamma(TM) \) is a family \( \{\phi^t\}_{t \in I} \), with \( \phi^t : U \subset M \to \phi^t(U) \subset M \), where \( 0 \in I \subset \mathbb{R} \) is an interval. The function \( \phi^t(x) := \phi(x, t) = \gamma_x(t) \) is given by the integral curves i.e. they satisfy:

\[
\frac{\partial \phi(x, t)}{\partial t} = X(\phi(x, t)) \quad \land \quad \phi(x, 0) = x, \quad \forall x \in U \subset M, \forall t \in I 
\]

With a bit of tweaking on the arguments the conditions are satisfied trivially since we start from the integral curve.

\[
\phi^t(x) = \phi(x, t) = \gamma_{\Omega}(\lambda + t) = \phi(\Omega, \lambda + t), \quad \text{with } \Omega \text{ and } \lambda \text{ s.t. } \gamma_{\Omega}(\lambda) = x \quad (12)
\]

### 2.1.1 Angular structure at a point

With the geodesic flow from a point \( p \in M \) we have so far defined the distances between \( p \) and any other point in \( M \) using the map \( d(p, \cdot) \) from 5. We now study the choice of angles at a point.

An angle is measured by the circle arc \( L \), which two directions enclose, divided by the radius of the circle \( r \) or more simply by the unit circle arc, that two directions enclose. Thereby a special case of a metric is imposed / defined on the set of directions \( D_p \) at a point \( p \in M \).

\[
\angle : D_p \times D_p \to [0, 2\pi) \quad (13)
\]

The unit sphere \( S^{(n-1)} \) is the union of all unit circles around a point \( p \in M^n \). It thus provides a convenient way of defining the metric on \( D_p \).

**Proposition 1.** The set of directions \( D_p \) at a point \( p \in M^n \) forms a manifold, which is diffeomorphic to the sphere \( S^{n-1} \).

\[
\Phi_D : D_p \to S^{n-1}, \quad \Omega = [\gamma_p] \sim \mapsto \hat{\Omega} \quad (14)
\]

**Proof.** The maps

\[
\Phi_D : \gamma \in \Omega, \quad \gamma \to \hat{\Omega} = \frac{\dot{\gamma}}{\|\dot{\gamma}\|} \in S^{n-1} \quad \Phi^{-1}_D : \hat{\Omega} \mapsto \{\gamma \in C^\infty((-\varepsilon, \varepsilon); M)|\gamma(0) = p, \gamma \propto \hat{\Omega}\} = \Omega \quad (15)
\]

are independent of the representant and smooth inverse of each other. \( \square \)
The unit sphere is defined as a subset of $\mathbb{R}^n$ endowed with the standard scalar product, which induces the embedding $i$.

$$S^{n-1} := \{ x \in \mathbb{R}^n \| x \| = 1 \}, \quad \Rightarrow \quad i : S^{n-1} \hookrightarrow \mathbb{R}^n$$

This embedding together with the standard scalar product $\cdot$ on the ambient space $\mathbb{R}^n$ defines an angular metric on $S^{(n-1)}$. Combined with a choice of diffeomorphism $\Phi_D$ this induces a metric on $D_p$.

$$\mathcal{L} : D_p \times D_p \xrightarrow{\Phi_D} S^{n-1} \times S^{n-1} \rightarrow [0, 2\pi)$$

$$d(\hat{\Omega}, \hat{\Omega}') := \inf_\gamma \{ l(\gamma) : \gamma : [0, 1] \rightarrow S^{n-1}, \gamma(0) = \hat{\Omega}, \gamma(1) = \hat{\Omega}' \}, \quad l[\gamma] = \int_0^1 \| \dot{\gamma}(t) \| \, dt$$

In other words, angles are the distances on the unit sphere between two directions.

In the case of a space-time this concept generalizes to generalized angles being distances on the unit hyperboloid under the Minkowski metric. Upon further investigation we found that some direction pairs are related to angles and others to arclengths of the unit hyperbola which are known as rapidities.

In the case of a space-time this concept generalizes to some direction pairs being related by arclengths of unit hyperbolas rather than great-circle arcs.

**Definition 2 (Direction Sphere at a Point).** The direction sphere is the sphere $S^{n-1}$ endowed with the metric $\mathcal{L}$, representing the directions $D_p$ at a point $p \in \mathcal{M}$, using the map $\Phi_D$.

**Remark.** The degree of freedom (d.o.f.) of the arclength is thereby replaced by the auxiliary d.o.f. and thus the ambient space is well suited to model the direction sphere at a point.

**Proposition 2.** The construction of the direction sphere allows us to calculate the angle between two directions $\Omega, \Omega' \in D_p$ via the embedding $i$:

$$\mathcal{L}(\Omega, \Omega') = \cos^{-1} i \circ \Phi_D(\Omega) \cdot i \circ \Phi_D(\Omega').$$

**Proof.** Due to the rotation symmetry of the sphere, we can without loss of generality assume that the two directions are mapped to the 1, 2-plane and specifically $i \circ \Phi_D(\Omega) = \hat{\Omega} = \hat{e}_1$. The direction vectors on $S^{n-1}$ and the connecting geodesic are then given by:

$$\hat{\Omega} = i \circ \Phi_D(\Omega) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\Omega}' = i \circ \Phi_D(\Omega') = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad \gamma(t) = \begin{pmatrix} \cos(\theta t) \\ \sin(\theta t) \\ 0 \end{pmatrix} \in S^{n-1} \subset \mathbb{R}^n$$

We see, that the arclength of the geodesic agrees with the scalar product of the direction vectors:

$$l[\gamma] = \int_0^1 \sqrt{(-\theta \sin(\theta t))^2 + (\theta \cos(\theta t))^2} \, dt = \theta = \cos^{-1} \cos \theta = \cos^{-1} \hat{\Omega} \cdot \hat{\Omega}'$$

To use the geodesic flow in concrete examples we need to systematically parametrize its curves. Since the curves are labelled by the direction in which they head out from a point $o$ we can use a parametrization of the unit sphere. We still want to have as much geometrical information i.e. maintain the angular structure in our geodesic flow and we want to note what the arbitrary choices of such a parametrization are.

In two dimensions we can pick a direction and label it with 0 i.e. define it as our origin or initial direction. We will refer to points lying on the geodesic heading out in this direction as, with $q(c) = \gamma_o(c)$. If we now demand that the angular parameter represents actual angles i.e.

$$\Delta \varphi := \mathcal{L}(\gamma_o, \varphi, \gamma_o, \varphi') = |\varphi - \varphi'|$$

then we only have two choices left: in which direction do we want to increase the parameter and what domain do we want to use. For example $(-\pi, \pi]$ or $[0, 2\pi)$ as domains and then we could also invert the angular parameter $\varphi \mapsto -\varphi$.

The system is constrained because the angles are inherently 2-dimensional and thus to a large degree dictate what
coordinates we need to pick if they are supposed to reflect the angular structure.

In this work we will use the following parametrization map:

$$\hat{\Omega}^{(1)} : [0, 2\pi) \rightarrow S^1, \quad \varphi \mapsto \hat{\Omega}^{(1)}(\varphi) = \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$$

(23)

It should be noted that we used the arclength parameter of the geodesic flow from a point we name 0 on $S^1$. Because angles are 2-dimensional concepts, the angular structure becomes much more complex and we have much more choices for our angular parameters, when we introduce another dimension:

We begin in the same way as before by picking a random direction and calling it original direction $\Omega_0$ and mapping it to $\Phi_D(\Omega_0) = (1, 0, 0) = \hat{0}$ on $S^2$. We then consider a geodesic $\gamma_{q,\hat{\beta}_1}$ heading out form a point $q(c)$ on $\gamma_{0,\hat{\beta}_1}$ in a direction $\hat{\beta}_1$. We now have two angular variables for every geodesic and we start labelling the ones heading out from 0, which cross $\gamma_{q,\hat{\beta}_1}$ with $\varphi$. We choose to increase the first angular parameter $\varphi$ in direction of $\gamma_{q,\hat{\beta}_1}$. The directions of these geodesics from 0 may in general be mapped to a generic curve on $S^2$ from $\hat{0}$ i.e. the second angular parameter of these geodesics could vary: $\vartheta = \vartheta(\varphi)$. We can however choose the second angular variable to be $\vartheta = 0$ for all these geodesics, which would map the directions to the great arc $\gamma_{0,\hat{0}}$ on the 2-sphere.

$$\hat{\Omega}^{(2)}(\varphi, \vartheta) \quad \text{s.t.} \quad \gamma_{q,\hat{\Omega}^{(2)}(\varphi,0)}(l) = \gamma_{q,\hat{\beta}_1}(l) \Rightarrow \bigcup_{\varphi} [\gamma_{q,\hat{\Omega}^{(2)}(\varphi,0)}] \sim \Phi_D \mapsto \gamma_{0,\hat{0}}^{S^2}$$

(24)

Now we see that we can again take the arclength of the $S^1$-flow as the $\varphi$ parameter, embedding the 2-dimensional case in the 3-dimensional one:

$$\hat{\Omega}^{(1)}(\varphi) \hookrightarrow \hat{\Omega}^{(2)}(\varphi, 0) = \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix}$$

(25)

We continue this procedure, by considering a geodesic $\gamma_{q,\hat{\Omega}_\vartheta}$ from $q$, which does not lie in the initial plane. As before, we can choose to label the geodesics $\gamma_{q,\hat{\Omega}^{(2)}(\varphi,\vartheta)}$ which cross $\gamma_{q,\hat{\Omega}_\vartheta}$ in a way such that their directions are mapped to a great arc $\gamma_{0,\hat{0}}^{S^2}$ on $S^2$.

$$\bigcup_{\varphi} [\gamma_{q,\hat{\Omega}^{(2)}(\varphi,\vartheta)}] \sim \Phi_D \mapsto \gamma_{0,\hat{0}}^{S^2}$$

(26)

We end up with the parametrization:

$$\hat{\Omega}^{(2)} : [0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow S^2, \quad (\varphi, \vartheta) \mapsto \hat{\Omega}^{(2)}(\varphi, \vartheta) = \begin{pmatrix} \cos \varphi \\ \cos \vartheta \sin \varphi \\ \sin \vartheta \sin \varphi \end{pmatrix}$$

(27)

By comparing (27) with the Sec. S2 in the supplement we see, that the angular structure at 0 is just the flow from $o_{S^2} = 0$ on the 2-sphere. The first angular parameter $\varphi$ coincides with the arclength $\lambda_{S^2}$ and the second angular
parameter \( \vartheta \) is the same as the first and only angular parameter \( \varphi \) of the geodesic flow on \( S^2 \): \( \varphi = \lambda \), \( \vartheta = \varphi \).

So, we connected the geodesic flow on a 3-dimensional manifold, which has 2 angular degrees of freedom, with the one on the 2-dimensional sphere, which has 1 angular degree of freedom.

Furthermore, we can parametrize the flow from \( o \) on the 3-sphere, using the flow from \( o \) on the 2-sphere and then precede to parametrizing the flow on the 4-sphere with the one on the 3-sphere and so on, see Fig. 3.

\[
\rho_{\alpha,\Omega}(\lambda) \xrightarrow{\lambda \mapsto \varphi} \rho_{\alpha,\Omega(\varphi)}(\lambda) \quad \varphi \mapsto \vartheta \quad \vartheta \mapsto \vartheta_2 \quad \vartheta_2 \mapsto \vartheta_3 \quad \ldots \quad (28)
\]

That way we can construct a meaningful parametrization in the sense, that all parameters correspond to angles i.e. arc-lengths on the unit sphere and not some unknown function of them.

\[
\hat{\Omega}^{(n)} : [0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^{n-2} \rightarrow S^{n-1}, \quad \hat{\Omega}^{(n-1)}(\varphi, \vartheta_1, \ldots, \vartheta_{n-2}) = \begin{pmatrix}
\cos \varphi \\
\sin \vartheta_1 \sin \varphi \\
\sin \vartheta_2 \cos \vartheta_1 \sin \varphi \\
\vdots \\
\sin \vartheta_{n-2} \cos \vartheta_{n-3} \cos \vartheta_1 \sin \varphi
\end{pmatrix}
\]

Using the map \( \Phi_D \) we can parametrize the directions:

\[
[0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^{n-2} \rightarrow S^{n-1}, \quad \hat{\Omega}^{(n-1)}(\varphi, \vartheta_1, \ldots, \vartheta_{n-2}) \xrightarrow{\Phi_D^{-1}} \hat{D}_p, \quad (\varphi, \vartheta_1, \ldots, \vartheta_{n-1}) \mapsto \hat{\Omega} \mapsto \Omega \quad (30)
\]

and the angle between two parametrized directions is calculated via:

\[
\angle(\hat{\Omega}, \hat{\Omega}') = \cos^{-1} \hat{\Omega} \cdot \hat{\Omega}', \quad \hat{\Omega} = \iota \circ \Phi_D(\Omega) = \iota \circ \Phi_D \circ \Phi_D^{-1} \circ \hat{\Omega}^{(n-1)}(\varphi, \vartheta_1, \ldots, \vartheta_{n-1}) = \iota \circ \hat{\Omega}^{(n-1)}(\varphi, \vartheta_1, \ldots, \vartheta_{n-1}) \quad (31)
\]

We will from now on identify the directions with the positions on the sphere and the images of the map \( \iota \circ \hat{\Omega}^{n-1} \) and abbreviate it with \( \hat{\Omega} \).
The angular labelling of the flow from a point $o$ may seem very similar to the exponential map. We explain the difference between the two concepts in Appendix A.

### 2.2 Geodesic Flow Bundle

So far, we have defined the distances from a point $o$ to any other point $p$ in $U \subset \mathcal{M}$ and the angles at $o$. To complete the geometric information, we need a geodesic flow at every point. Just as we organize the tangent spaces at all points into a bundle, we now organize all geodesic flows into the geodesic flow bundle.

**Definition 3** (Geodesic Flow Bundle). We define the geodesic flow bundle by:

$$
\Phi \mathcal{M} = \bigcup_{x \in \mathcal{M}} \{x\} \times o_x = \bigcup_{x \in \mathcal{M}} \{x\} \times \{\hat{\gamma}_{x,\Omega}\}_{\Omega \in S^{n-1}}
$$

(32)

It is often handy, to collect the geodesic flows in a function. Thus, we introduce the flow bundle function:

$$
\Phi: \mathcal{M} \times S^{n-1} \times I \to \mathcal{M}, \quad (x, \hat{\Omega}, \lambda) \mapsto \Phi(x; \hat{\Omega}, \lambda)
$$

(33)

which we will for most cases consider to be smooth and has to satisfy the following consistency conditions:

- $\gamma_{x,\Omega}(\lambda) = y \Rightarrow \gamma_{y,\Omega}(\tau) = \gamma_{x,\Omega}(\lambda + \tau), \quad \tau \in (0, 2\pi) \land \gamma_{y,\Omega}(\tau) = \gamma_{x,\Omega}(\lambda - \tau), \quad \tau \in (0, \lambda)$

(34)

- $\forall x, y, z \in \mathcal{M}$, with $\gamma_{x,\Omega}(\lambda) = z$, $\gamma_{x,\Omega}(\tau) = y$ and $\gamma_{y,\omega}(t) = z$ : $\lambda \leq \tau + t$ (triangle inequality)

(35)

- $\lambda = \tau + t \Rightarrow \Omega = \Omega', \quad \land \gamma_{y,\omega}(t) = \gamma_{x,\Omega}(\tau + t)$

(36)

**Remark.** Here $\Omega$ denotes the respective directions at the point where the geodesic heads out. More explicitly: when the first index is the point $x$, then the second index $\Omega$ is a direction at $x$.

$$
\gamma_{x,\Omega} \Rightarrow \Omega = \Omega(x) = [\gamma_{x,\Omega}]_{\sim} \in \mathcal{D}_x
$$

(37)

The directions are supposed to be interpreted in the following way:

$$
\Omega(y) = [\tau \mapsto \gamma_{x,\Omega}(\lambda + \tau)]_{\sim} \in \mathcal{D}_y, \quad -\Omega(y) = [\tau \mapsto \gamma_{x,\Omega}(\lambda - \tau)]_{\sim} \in \mathcal{D}_y
$$

(38)

Note however, that we will see in the next subsection, that in a curved manifold it is not possible to label all geodesics in a way that the direction labels remain constant along the geodesics. We can do this for the geodesics from one point, we chose the ones from the origin $o$ but will have changing direction labels for geodesics which do not cross $o$.

Intuitively we think of a geodesic as a path which locally always goes straight on i.e. never changes it’s direction. The consistency condition (34) incorporates that, by demanding, that if we head out from a point $y$ on a geodesic in the direction or opposite direction of that geodesic, then that geodesic from $y$ overlaps with the previous one. The third condition (36) asserts, that in the case where all three points are aligned on the same geodesic the triangle inequality (35) reduces to the consistency condition (34).

The distances are now defined by:

$$
d(x, y) = \lambda, \quad \gamma_{x,\Omega}(\lambda) = y \land \gamma_{y,\Omega}(\lambda) = x
$$

(39)

The definition of the geodesic flow bundle is valid for arbitrary cases including the case of a space-time. What requires adapting is the angular structure, where the Riemannian case forms a part of the pseudo-Riemannian angular structure. The other parts work analogously with hyperbolas replacing the circles.

#### 2.2.1 Angular structure on the Geodesic Flow Bundle

Since we are dealing with the directions at every point $p \in \mathcal{M}$ we need to promote our parametrization function to a field on $\mathcal{M}$:

$$
\hat{\Omega}_p^{(n-1)}: \mathcal{M} \to \text{Diff}(P, S^{n-1}), \quad p \mapsto \hat{\Omega}_p^{(n-1)}, \quad P = [0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^{n-2}
$$

(40)

The vector space $P$ (or subset of a vector space) in general denotes any viable angular parameter space but we stick to our choice above to assure consistency throughout this work.
We continue our angular labelling scheme from $o$ to any other point $p \in M$, while satisfying the consistency condition (34) and restricting ourselves to angles as parameters. We can incorporate (34) more directly into the angular labelling of the flow from $o$ by keeping the angular label constant i.e. assigning the same label to $\Omega(p)$ and $\Omega(o)$.

We start with the 2-dimensional case, where we had the parametrization function $\hat{\Omega}^{(1)}_o(\varphi)$ at $o$ and thus the geodesics from $o$ where labelled with $\gamma_o,\varphi$. Let $p \in M$ be the point, which can be reached by heading out in direction $\alpha$ from $o$ and covering the distance $l$, then the label of two directions at $p$ are already determined by (34):

$$p = \gamma_o,\alpha(l); \quad \gamma_o,\alpha(l + \lambda) \overset{!}{=} \gamma_p,\alpha(\lambda) \wedge \gamma_o,\alpha(l - \lambda) \overset{!}{=} \gamma_p,\alpha + \pi(\lambda)$$

(41)

$$\Rightarrow \quad \alpha \mapsto \Omega_\alpha := [\gamma_o,\alpha(l + \lambda)]_\sim \wedge \alpha + \pi \mapsto \Omega_{\alpha + \pi} := [\gamma_o,\alpha(l - \lambda)]_\sim, \quad \Omega_\alpha, \Omega_{\alpha + \pi} \in D_p,$$

(42)

If we now demand, that the angular parametrization is smooth, especially at $o$ and that $P$ is the same and corresponds to angles everywhere, then the parametrization field $\hat{\Omega}^{(1)}_p$ is uniquely determined on $M$.

$$\lim_{l \to 0} \hat{\Omega}^{(1)}_p = \hat{\Omega}^{(1)}_o \Rightarrow \quad \angle_p([\gamma_o,\alpha(l + \lambda)]_\sim, [\gamma_p,0]_\sim) \overset{!}{=} \alpha \wedge \gamma_p,0 = \gamma_p,\alpha - \alpha$$

(43)

Consequently the geodesic with angular label 0 at $p$ is the one which encloses the angle $\alpha$ with $\gamma_o,\alpha$ at $p$ and lies in direction of decreasing $\varphi$. It is as expected uniquely determined but we may be a bit surprised when we look at the example of $S^2$ plotted in Fig. 4, since $\gamma_p,0$ intuitively seems to point in a different direction than $\gamma_o,0$. If we move $p$ closer to the original geodesic $\gamma_o,0$ the angle $\alpha$ tends to 0 and in that limit the two geodesics coincide.

$$\lim_{\alpha \to 0} \gamma_{p,\alpha} = \gamma_{p,0} \wedge \lim_{\alpha \to 0} \gamma_{o,\alpha} = \gamma_{o,0} \Rightarrow \lim_{\alpha \to 0} \gamma_{p,\alpha}(\lambda) = \gamma_{o,0}(l + \lambda)$$

(44)

We also note that geodesics from $q$ do not enclose the same angle with the 0-geodesic from another point $p$ and thus their angular label changes although we continue in the same direction.

Figure 4: We plot a collection of curves from the geodesic flow bundle on $S^2$ embedded in its ambient space $\mathbb{R}^3$ on the left and in the faithful normal chart at $o$, introduced in Sec. 3, on the right. The green thick line is the original geodesic $\gamma_{o,0}$ in this example and is continued by the blue thick one $\gamma_{q,0}$ which is thus also labelled with 0. The angle between the original geodesic and the yellow thick line at $o$ is $\alpha = \frac{\pi}{6}$ and thus the yellow thick line $\gamma_{o,\frac{\pi}{6}}$ and its continuation, the red thick line $\gamma_{o,\frac{\pi}{2}}$ have that angular label. The black thick line $\gamma_{p,0}$ is the one with the label 0 and thus the angle between it and the red thick one is $\frac{\pi}{6}$ as well.
We are now ready to investigate the 3-dimensional case. We again consider a point \( p \in M \) in an arbitrary direction \( \hat{\Omega}^{\alpha} := (\hat{\omega}^{\alpha}, \hat{\Omega}^{\beta}, \hat{\Omega}^{\gamma}) \) at a distance \( t \) from \( o \). Then again, the labels of the forward and backwards direction from \( o \) are already determined.

\[
p = \gamma_{o,\hat{\Omega}^{\alpha}}(t); \quad \gamma_{o,\hat{\Omega}^{\alpha}}(l + \lambda) = \gamma_{p,\hat{\Omega}^{\alpha}}(\lambda) \quad \hat{\Omega}^{\beta}, \hat{\Omega}^{\gamma}(l - \lambda) = \gamma_{p,\hat{\Omega}^{\alpha}}(\lambda)
\]

\[
\Rightarrow \quad (\alpha_1, \alpha_2) \xrightarrow{\hat{\Omega}^{(2)}} \hat{\Omega}^{\alpha} = [\gamma_{o,\hat{\Omega}^{\alpha}}(l + \lambda)], \quad (\alpha_1 + \pi, \alpha_2) \xrightarrow{\hat{\Omega}^{(1)}} -\hat{\Omega}^{\alpha} = [\gamma_{o,\hat{\Omega}^{\alpha}}(l - \lambda)]. \quad \hat{\Omega}^{\alpha}, -\hat{\Omega}^{\alpha} \in D_p,
\]

**Definition 4** (Geodesic Surfaces). Under the condition that geodesics form consistent surfaces a geodesic surface is defined as the image set of a one-parameter group of geodesics:

\[
U_{\vartheta}^{\beta} := \phi_{\vartheta}^{\beta}([0, 2\pi]), \quad \phi_{\vartheta}^{\beta}: [0, 2\pi) \rightarrow \Phi M \quad \varphi \mapsto \gamma_{x,\hat{\Omega}^{(n-1)}(\varphi, \vartheta)}
\]

for \( x \) and \( \vartheta = (\vartheta_1, \ldots, \vartheta_{n-2}) \) fixed.

**Remark.** These surfaces are naturally parametrized by \( \lambda \) and \( \varphi \) and are sub-flows of the flow from \( x \).

In 3-dimensions the geodesic surfaces can be labelled with the \( \alpha_2 \)-parameter, which represents the angle between the two geodesic surfaces \( U_{\alpha_2} \) and \( U_0 \) at every point \( p \in M \), and can be written as sets:

\[
U_{\alpha_2} := \bigcup_{\varphi} \gamma_{o,\hat{\Omega}^{(2)}}(\varphi, \alpha_2)
\]

**Definition 5** (Plane Geodesic Flow Bundles). When all geodesics in a geodesic surface \( U_{\vartheta}^{\beta} \) can be labelled consistently with \( \vartheta \), we call it a plane geometry and we say that it has a plane geodesic flow bundle.

**Remark.** In a faithful normal chart (see Sec. 3) geodesic surfaces appear as planes.

We see on the example of the 3-sphere in Sec. S3 that such geometries exists and we thus restrict ourselves to plane geometries in this work. We use this example to visualize the labelling scheme we describe in the following in Fig. 5.

The first angular parameter of a geodesic heading out from a point \( q = \gamma_{o,\hat{\Omega}^{\alpha}}(c) \) on the original geodesic will change if we look at the coinciding geodesic from a point \( p = \gamma_{q,\hat{\Omega}^{(2)}}(\beta_1, \beta_2)(a) \) further up i.e.:

\[
\gamma_{q,\hat{\Omega}^{(2)}}(\beta_1, \beta_2)(a + \lambda) = \gamma_{p,\hat{\Omega}^{(2)}}(\beta_1', \beta_2')(a) \quad \text{where in general:} \quad \beta_1 \neq \beta_1',
\]

as we can see from the discussion above of the 2-dimensional case. The second angular parameter does not change however by our assumption (plane GFB). We will see in Sec. 5 that this assumption assures that the metric can be diagonalized.

To complete the labelling scheme let us investigate the 3-dimensional analogue of the geodesic in original direction at \( p \) and the consistency with the limiting cases:

Finding the geodesic with label \( \hat{\Omega} \) at \( p \) is less straight forward in this case. In continuation of \( \hat{\Omega}^{(2)} \) from \( o \) to other points \( p \) inherit the angular structure of the two dimensional case in the surfaces \( U_{\alpha_2} \). So, we know from the previous case, that we just have to rotate \( \hat{\Omega}_o \) back by \( \alpha_1 \) in that surface \( U_{\alpha_2} \) at \( p \), to arrive at \( \hat{\Omega}_p \). In other words the direction labelled with \( \hat{\Omega}_p^{(2)}(\alpha_1, \alpha_2) \) means, that it encloses the angle \( \alpha_1 \) with \( \hat{\Omega}_p = \hat{\Omega}^{(2)}(0, 0) \) and is tangential to the surface \( U_{\alpha_2} \). We interpret the angular labels at \( p \) in the same way as we did at \( o \), which guaranties consistency in the limiting case \( t \rightarrow 0 \).

The original surface \( U_0 \) at \( p \) is the one, whose intersection with \( U_{\alpha_2} \) is \( \gamma_{p,\hat{\Omega}} \) and encloses the angle \( \alpha_2 \) with \( U_{\alpha_2} \). In the plot in Fig. 5, which is the 3-dimensional analogue to Fig. 4, the tangent plane of \( U_0 \) at \( p \) is not parallel to the ones at \( o \) and \( q \) but will agree with those in the limits of \( a \rightarrow 0 \) or \( t \rightarrow 0 \).

We incorporate the consistency condition (34) into the angular labelling of the flow from \( o \) and that we only use angles i.e. arclengths of the unit sphere to parametrize directions determines the angular parametrization of the entire geodesic flow bundle uniquely, given the angular parametrization at \( o \).

The arguments given for the 3-dimensional case can be generalized to \( n \)-dimensions since it is the lowest dimensional generic case. Surfaces then have multiple parameters \( \alpha_2, \ldots, \alpha_{n-1} \), which are specific choices of the parameters \( \vartheta_1, \ldots, \vartheta_{n-1} \). And we know now that the angular parametrization at \( o \) is carried along the geodesics from \( o \) in a unique fashion.
Figure 5: We sketch, how we rotate by \(-\phi_p\) in the plane \(U_{\alpha_2}\) to find the geodesic \(\gamma_{p,\hat{0}}\) in direction \(\hat{0}\) from \(p\) and then tilt the plane \(U_{\alpha_2}\) by \(-\alpha_2\), using \(\hat{0}(p)\) as axis.

3 Faithful Normal Chart

The geodesic flow bundle induces a special case of normal coordinates at every point \(o \in \mathcal{M}^n\). We can construct a chart on a neighborhood \(U\) around \(o\) by labelling every point \(p \in U\) with the arclength parameter \(\lambda\) and the angular parameters \(\varphi, \vartheta_2, \ldots, \vartheta_{n-2}\) of the geodesic flow from \(o\) to \(p\).

**Definition 6 (Faithful Normal Chart).** The faithful normal chart (FNC) at a point \(o \in \mathcal{M}\) is the tuple \((U, \mathcal{N}_o)\), where the map is defined as:

\[
\mathcal{N}_o : U \subset \mathcal{M} \to \mathbb{R}^n, \quad p \mapsto (l, \varphi, \vartheta_1, \ldots, \vartheta_{n-2}), \quad \hat{\Omega} \in S^{n-1}, \quad l \in \mathbb{R}_+ \quad s.t. \quad p = \gamma_{o,\hat{\Omega}^n}(\varphi, \vartheta_1, \ldots, \vartheta_{n-2})(l). \tag{50}
\]

We get a special normal chart, which correctly represents the angles at \(o\) and the distances from \(o\) to any other point \(p\), which can be reached via geodesic from \(o\).

The chart represents a path description of an observer at point \(o\) which is that we are supposed to head out in the direction \(\hat{\Omega}(\varphi, \vartheta_1, \ldots, \vartheta_{n-2})\) and walk straight on for a distance \(l\) to arrive at the point \(p\). So, we are essentially completing the chart on \(S^{n-1}\) given by the inverse of the parametrization function \(\hat{\Omega}^{-1} \circ \Phi_D : D_o \to S^{n-1} \to P\), with the geodesic parametrization by arclength.

From here we can get a Cartesian version of the faithful normal coordinates by using the embedding \(\iota\) of \(S^{n-1}\) at \(o\) into \(\mathbb{R}^n\), which we specified in (29):

\[
\mathcal{C}_o : U \to \mathbb{R}^n, \quad p \mapsto \vec{x} = (x_1, \ldots, x_n), \quad \vec{x} = l (\iota \circ \hat{\Omega}^n)(\varphi, \vartheta_1, \ldots, \vartheta_{n-2}) \tag{51}
\]
We can construct this embedding via group actions on \( o \). An arbitrary point \( p \) can be reached, by a translation along \( \gamma_{0,\hat{0}} \) by \( c \), a rotation in the \( x_1, x_2 \)-plane with angle \( \varphi \) at \( o \) and subsequent rotations in the \( x_2, x_i \)-planes at \( o \) with angles \( \vartheta_i \), with \( i \in \{3, \ldots, n\} \).

\[
\hat{\Omega}^{(n-1)}(\varphi, \vartheta_1, \ldots, \vartheta_{n-2}) = R_{x_2, x_n}(\vartheta_{n-2}) \circ \ldots \circ R_{x_2, x_3}(\vartheta_1) \circ R_{x_1, x_2}(\varphi) \cdot \hat{0}
\]  

(52)

where \( \hat{0} = (1, 0, \ldots, 0) \).

It is more natural to work with the polar form, since this does not require an additional construction, but we use this mapping to generate plots.

In this chart the geodesic surfaces \( U_{\vartheta_1, \ldots, \vartheta_{n-1}} \) are planes and thus all geodesics lie in these planes through \( o \). The parameters \( \vartheta_1, \ldots, \vartheta_{n-1} \) describe the angles by which these planes are tilted from the original plane \( U_{\vartheta_1, \ldots, \vartheta_n} \). So, we basically accomplished to embed the 2-dimensional case in the \( n \)-dimensional one and we dealt with the relations between these planes. This will allows us to reduce an \( n \)-dimensional problem into a 2-dimensional one.

We will see in Sec. 4 that this plane structure is exactly what we need, to describe curvature in our formalism and relate it to the geodesic flow bundle.

The faithful normal coordinates would precisely represent how an observer at \( o \) would perceive a Riemannian manifold. The straight lines are paths which the observer would follow if it would decide to walk straight on. If the observer would after a distance \( c \) decide to change direction and continue walking straight on, then that path would also appear as a straight line from \( o \), since it lies in the same geodesic plane.

The geodesic flow bundle \( \Phi_M \) does not require charts but provides an atlas \( A_{FNC} := \bigcup_{p \in M} N_p \) which is tightly related to it and carries it’s entire geometric information. In other words, we can choose charts, but we don’t have to, if we don’t want to.

We demonstrate the geodesic flow bundle in faithful normal charts on the Manhattan metric and explain the calculations to the plot in Figure 4 in detail in the supplement.

From the view point of the metric formalism we would expect that we would now introduce a notion of connection and parallel transport, to define the Riemann curvature tensor. We find in Appendix C that a natural notion of parallel transport is already ingrained in the GFB formalism. We do however not need it to relate the GFB to curvature as we will see in the following section. The parallel transport will become useful, when one wants to relate the GFB formalism to the Riemann tensor and can provide a consistency check of our results, which we outline in the follow-up paper.

## 4 Curvature

In this section we define a notion of curvature, which is suited to our approach. We work from the assumption that curvature is a property of any point in the manifold which may be described by some set of numbers. A property which can in a sense be seen as the local source or generator of the manifold’s global geometry.

\[
K : M \to A, \quad p \mapsto K(p), \quad A \text{ a set.} \tag{53}
\]

Since we view the curvature as the source of the geometry, we are looking for a way to calculate the geodesic flow bundle from the values of the field.

In the metric formalism we work from the geometry and define a quantity which “measures” an effect of the non-flatness of that geometry. Thus the defined curvature quantities describe an effect of some source and not the source itself.

### 4.1 Relating geodesics to curvature

We note that in faithful normal coordinates all geodesics in plane GFB’s lie in planes through \( o \) and all geodesics from \( o \) are straight lines. Thus the entire geometric information lies in the relation between two flows. More specifically in the flow from a point \( p \) expressed in terms of the flow from another point \( o \). This is exactly describing the flow from \( p = \gamma_{o,\hat{0}}(c) \), with \( \Omega_p = \hat{\Omega}^{(n-1)}(\varphi_p, \vartheta_{1,p}, \ldots, \vartheta_{n-2,p}) \) in faithful normal coordinates at \( o \): \( N_p(\gamma_{p,\hat{0}}(\lambda)) \).

But we already know, that \( N_p(\gamma_{p,\hat{0}}(\lambda)) \) lies in a plane through \( o \) which turns this into a 2-dimensional problem as
sketched in Fig. 6. We can without loss of generality consider the geodesic triangle spanned by \( o \) and \( \gamma_{q,\hat{\beta}}(\lambda) \), with \( \hat{\beta} = \hat{\Omega}^{(n-1)}(\beta, 0, \ldots, 0) \), lying on the original geodesic \( q = \gamma_{\alpha,\hat{\lambda}}(c) \), since a generic triangle spanned by a geodesic \( \gamma_{p,\hat{\Omega}} \), with \( \hat{\Omega} = \hat{\Omega}^{(n-1)}(\beta_1, \ldots, \beta_{n-1}) \) from a point \( p = \gamma_{o,\hat{\Omega}}(l) \) is tilted by the angles \( \beta_2, \ldots, \beta_{n-2} \), then rotated in the \( x_1, x_2 \)-plane with angle \( \varphi_p \) and then again everything is tilted with the angles \( \vartheta_{1,p}, \ldots, \vartheta_{2,p} \).

![Figure 6: We sketch, how we tilt and rotate the geodesic triangle from the original plane to the point \( p \) with tilt \( \beta_2 \) with respect to the rotated original plane (red).](image)

In other words our generic problem is expressing the geodesic \( \gamma_{q,\hat{\beta}}(\lambda) \) from \( q = \gamma_{o,0}(c) \) in faithful normal coordinates in 2 dimensions at \( o \). This automatically leads us to a triangle since we need to find the geodesic \( \gamma_{o,\alpha}(l) \) connecting \( o \) with \( p = \gamma_{q,\hat{\beta}}(\lambda) \). More precisely we need to find the arclength of the top-line \( b \) and the opening angle \( \alpha \), given the distance to \( q \) i.e. arclength of the base line \( c \), direction angle \( \beta \), arclength \( \lambda \) and a curvature field \( K \).

\[
N_o(\gamma_{q,\hat{\beta}}(\lambda))_{U_o} = \begin{pmatrix} b(\lambda) \\ \alpha(\lambda) \end{pmatrix}, \quad b(K; \beta, c, \lambda), \quad \alpha(K; \beta, c, \lambda), \quad \hat{\beta} = \hat{\Omega}^{(n-1)}(\beta, 0, \ldots, 0) \tag{54}
\]

We call \( b \) and \( \alpha \) the fundamental solution to the curved triangle problem and we describe in a follow up paper how we calculate these two quantities from a curvature field on the geodesic triangle.

### 4.2 Infinitesimal triangles as geometry generators

In this section we investigate the curvature at a single point \( o \) and since the entire geometric information lies in the relation between two flows (the flow from the origin point \( o \) in FNC looks the same for every geometry), we consider the flow from a neighbouring point \( q = \gamma_{o,0}(\varepsilon) \) at a fixed but arbitrarily small distance \( \varepsilon \) from \( o \). We then trace out the first infinitesimal part of the flow from \( q \) i.e. only until an arbitrarily small parameter value \( \delta \lambda > 0 \). The important question is now, whether a single value determines the infinitesimal start of all the geodesics of the flow from \( q \) i.e. the distribution \( \delta l(\beta) \) only has one degree of freedom or whether one can do more to it. We aim to answer the following question Is it possible to choose \( \delta l(\beta) \) differently for every \( \beta \) or are there restrictions to that?

**Lemma 3.** We find that the flat space limit restricts \( \delta l(\beta) \) to one degree of freedom. Thus, there is a single curvature value per infinitesimal triangle.

**Proof.** If we assume, that an infinitesimal triangle in an arbitrary manifold can be sufficiently described by a spherical, flat, or pseudo-spherical one, then the first infinitesimal part of the geodesic is uniquely defined by the
two infinitesimals $\delta\lambda$, $\delta l$ and the angle $\beta$, since with these a point can uniquely be determined. The distance coordinate can be calculated from a Gaussian curvature value $K$, using the cosine law for constant curvature

$$\delta l(K, \beta) = \cos_K^{-1} \left( \cos_K \varepsilon \cos_K \delta\lambda + K \sin_K \varepsilon \sin_K \delta\lambda \cos(\pi - \beta) \right), \quad \cos_K x := \cos(\sqrt{K} x), \quad \sin_K x := \frac{\sin(\sqrt{K} x)}{\sqrt{K}}$$

and from there we can calculate the angle coordinate, using the sine law for constant curvature and thus obtain the first infinitesimal segment of a geodesic from $q$ in faithful normal coordinates:

$$\varphi(K, \beta) = \sin^{-1} \left( \frac{\sin_K \delta\lambda}{\sin_K \delta l(K)} \sin(\pi - \beta) \right), \quad (56)$$

$$\mathcal{N}_o(\gamma_{q, \beta}(\delta\lambda)) = \begin{pmatrix} \delta l \\ \varphi \end{pmatrix} (K, \beta) \quad (57)$$

$$\forall \varepsilon, \delta\lambda > 0, \quad \forall \beta \in [0, 2\pi) \quad \exists K \in \mathbb{R}$$

to generate any consistent distribution $\delta l(\beta)$

Figure 7: A sketch of the infinitesimal triangle.

The question is though, whether this allows us to generate all consistent distributions (geodesic stomps) $\delta l(\beta)$. Every Riemannian manifold is locally flat, which means that in the limit of $\varepsilon, \delta\lambda \to 0$ the relation between the 4 quantities has to converge to the cosine law in a plane:

$$\delta l^2(\beta) \xrightarrow{\varepsilon, \delta\lambda \to 0} \varepsilon^2 + \delta\lambda^2 - \varepsilon \delta\lambda \cos(\pi - \beta)$$

(58)

We cannot modify the relation for a flat space, by adding a term of second order or lower, since we would violate the limit above (58). On the other hand, only the leading order matters, since we are considering an infinitesimal triangle. Thus, the only change we can make whilst still satisfying (58) is multiplying both sides with a constant factor $K$ which is exactly what the constant curvature cosine law does in this limit.

$$\delta l^2 = \varepsilon^2 + \delta\lambda^2 - \varepsilon \delta\lambda \cos(\pi - \beta) \xrightarrow{K \in \mathbb{R}} \cos_K \delta l = \cos_K \varepsilon \cos_K \delta\lambda + K \sin_K \varepsilon \sin_K \delta\lambda \cos(\pi - \beta)$$

(59)

$$\varepsilon, \delta\lambda \to 0 \quad K \delta l^2 \sim K \varepsilon^2 + K \delta\lambda^2 - K \varepsilon \delta\lambda \cos(\pi - \beta)$$

(60)

So, in general the geodesics from $q$ are generated by a sequence of triangles spanned by the side lengts $\delta\lambda$ with curvature values $K_1, K_2, \ldots$ as sketched in Fig. 8. In the limit of $\delta\lambda \to 0$ we take infinite curve segments and a direction dependent curvature field $K(\varphi)$ at $o$ and we get a flow with finite arclength.

Figure 8: A sketch that visualizes, how infinitesimal triangles generate a geodesic.
We conclude that any geodesic flow bundle and thus any geometry can be generated by infinitesimal triangles with constant curvature. More precisely, if we divide a manifold into an increasingly fine sampling of constant curvature triangles, we can describe any geometry.

This means that the curvature degrees of freedom of a 2-dimensional manifold are one curvature value per infinitesimal triangle which in terms of this construction here means, that every point can at most have a direction dependent curvature field \( K(\phi) \), because we can make the triangles as slim as we like. So, measuring the curvature at a point \( o \) could be achieved by pointing an arbitrarily slim triangle attached to \( o \) in different directions and thus scanning the curvature at that point like a radar.

The same argument can be made for a space-time using the cosine- and sine-laws for triangles on a pseudo-Riemannian hyperboloid.

### 4.3 Curvature in 2 dimensions

**Lemma 4.** The curvature at any point in a 2-dimensional real Riemannian manifold can be sufficiently described by a single real number \( K \in \mathbb{R} \).

**Proof.** To show that the curvature at a point of a 2-dimensional real Riemannian manifold can be sufficiently described by a single real number we consider two isosceles triangles with legs of length \( \varepsilon > 0 \) and vertex angles \( \delta \). Motivated by the findings in the previous section we consider these generator triangles to have a priori different curvatures \( K_1 \) and \( K_2 \) and thus side lines with different lengths \( \delta \lambda_1 \) and \( \delta \lambda_2 \). This means that the geodesics which form the sides of the two triangles are completely determined by the curvature values in these triangles: \( \delta \lambda_1(K_1) \) and \( \delta \lambda_2(K_2) \).

But the isosceles triangle with vertex angle \( 2\delta \) is also a generator triangle and thus its side length is completely determined by it’s curvature value \( \delta \lambda(K) \). And we know from the previous section, that it only has one. In the limit of \( \varepsilon, \delta \to 0 \) the union of the two smaller triangles coincide with the larger one and \( \delta \lambda_1(K_1) + \delta \lambda_2(K_2) = d\lambda(K) \). Since we can only have one curvature value in a generator triangle, we conclude that the three otherwise conflicting values must agree \( K_1 = K_2 = K \).

\[
\exists K_1, K_2 \in \mathbb{R} : \quad \cos_{K_1} \delta \lambda_1 = \cos_{K_2}^2 \varepsilon + \sin_{K_1}^2 \varepsilon \cos \delta \\
\cos_{K_2} \delta \lambda_2 = \cos_{K_2}^2 \varepsilon + \sin_{K_2}^2 \varepsilon \cos \delta
\]

but also we have:

\[
\delta \lambda_1 + \delta \lambda_2 \xrightarrow{\delta \to 0} \delta \lambda
\]

\[
\exists K \in \mathbb{R} : \quad \cos_K \delta \lambda = \cos_K^2 \varepsilon + \sin_K^2 \varepsilon \cos 2\delta
\]

\[
\Rightarrow K_1 = K_2 = K
\]

Figure 9: Two joined infinitesimal triangles form another larger, but still infinitesimal and thus generating triangle, when the side line is adjusted to be a geodesic.

So, we find that the curvature in a 2-dimensional manifold can be described by a scalar field. This means, that sectional curvature describes all degrees of freedom to create a geometry on a manifold.

### 4.4 The \( n \)-dimensional case

**Definition 7** (Great circle function). We call a function on the great circles of the \((n-1)\)-sphere a great circle function:

\[
o : O^n \to \mathbb{R}, \quad O^n := \{S^1 \subset S^{n-1} \mid S^1 \text{ a geodesic on } S^{n-1}\}, \quad n \geq 2.
\]

**Remark.** A smooth great arc function on the directions \( \mathcal{D}_p \) around \( p \in \mathcal{M} \) is an element in \( C^\infty(\Phi_{D}^{-1}(O^n); \mathbb{R}) \).

**Theorem 5.** Using the mapping \( \Phi_{D} \) of directions \( \mathcal{D}_p \) at \( p \) to the sphere \( S^{n-1} \) we can describe all degrees of freedom of curvature at this point by a great circle function.

**Proof.** If the isosceles triangles from the previous argument are tilted to one-another in a higher dimensional manifold i.e., do not lie in the same geodesic surface. Then that argument does not work anymore since we cannot form a covering triangle by taking the union. Thus, we retain the degrees of freedom of curvature in different planes at a point.
Figure 10: If the two infinitesimal geodesic triangles do not lie in the same geodesic surface, then they cannot be joined to a larger triangle.

If we imagine an arbitrarily small sphere around the origin point \( o \) in FNC (faithful normal coordinates), then the geodesic planes on which we can have one curvature value intersect with that sphere on a great circle. So, we do have a sort of direction dependent curvature at \( o \), where all directions which are mapped to the same great circle on the sphere must have the same curvature value. Thus, the curvature at \( o \) can be described by a smooth function on the great circles of a sphere.

**Definition 8 (Source Curvature).** We define a smooth source curvature field as a great arc field that determines the geometry over application of the cosine law to infinitesimal triangles:

\[
K : \mathcal{M} \to C^\infty(\Phi_D^{-1}(O^n); \mathbb{R})
\]

\[
K_0 = K(o)(\mathcal{C}); \quad \lim_{\varepsilon \to 0} \cos K_0(\varepsilon e) = \lim_{\varepsilon \to 0} \cos K_0(\varepsilon a) \cos K_0(\varepsilon b) - K_0 \sin K_0(\varepsilon a) \sin K_0(\varepsilon b) \cos \theta,
\]

\( o, q, p \in \mathcal{M}, \) with \( q = \gamma_{o,\Omega}(\varepsilon e), \) \( p = \gamma_{q,\Omega'}(\varepsilon b) = \gamma_{o,\Omega'}(\varepsilon b), \) \( \Omega, \Omega' \in \mathcal{C} = \Phi_D^{-1}(C) \subset D_o, \) \( C \in \mathbb{O}^n, \) \( \tilde{\Omega} \in D_q, \)
\( \theta = \angle_p([\gamma_{q,\Omega}(\varepsilon a + \lambda)], [\gamma_{o,\Omega'}(\varepsilon b + \lambda)]).\)

We can use the geodesic flow bundle on the sphere, to parameterize the great circles on the sphere and thus a great circle function at a point. Looking back at Eq. (28) and Fig. 3 in Sec. 2.1.1 we remind ourselves, that the flow bundle parameters \( \lambda_{S^2}, \varphi_{S^2} \) on the sphere \( S^2 \) correspond to the angular parameters \( \varphi, \vartheta \) of the flow from \( o \) in the manifold \( \mathcal{M}^3 \). So, the curvature at \( o \) can be parametrized with the direction parametrization function \( \hat{\Omega} \) restricted to the parameter domain \( P' = [0, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \). We will mark the angular parameters of curvature with a subscript \( K \) to differentiate them from the direction parameters of a geodesic:

\[
\lambda_{S^2} = \varphi_K \land \varphi_{S^2} = \vartheta_K \implies K_o : [0, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}, \quad (\varphi_K, \vartheta_K) \mapsto K_o(\varphi_K, \vartheta_K)
\]

Using the sequence from Eq. (28) again we can straightforwardly extend the curvature parametrization to \( n \)-dimensions:

\[
\lambda_{S^{n-1}} = \varphi, \quad \varphi_{S^{n-1}} = \vartheta_1, \quad \vartheta_{S^{n-1}} = \vartheta_2, \ldots \quad \vartheta_{n-3} = \vartheta_{n-2}
\]

\[
\Rightarrow K_o : P' = [0, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})^{n-2} \to \mathbb{R}, \quad (\varphi, \vartheta_1, \ldots, \vartheta_{n-2}) \mapsto K_{\hat{\Omega}(n-1)}|_{\varrho'}(o)
\]

And to parametrize the curvature on the entire manifold \( \mathcal{M}^n \) we promote it to a field

\[
K : \mathcal{M} \to C^\infty(P'; \mathbb{R}), \quad p \mapsto K_{\hat{\Omega}(n-1)}|_{\varrho'}(p)
\]
4.5 Parametrization of a 3-dimensional Gaussian curvature field

We note that the flow from the initial geodesic $\gamma^{S^2}_{\hat{q}(\varphi),\vartheta}$ parametrizes all great circles on the 2-sphere. Thus, we can provide a cleaner definition of the set of great circles and great circle functions on the set of directions $D_p \simeq S^2$ at a point $p \in M^3$:

$$o : O^3 = \left\{ \gamma^{S^2}_{\hat{q}(\varphi_K),\vartheta_K} | \varphi_K \in [0, \pi), \vartheta_K \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right\} \rightarrow \mathbb{R} \quad (68)$$

We generated the geodesics of a flow from an arbitrary point $p \in M$ from the curvature value of an infinitesimal generator triangle. We want to know now how to read out the correct value from the curvature function $K_p(\varphi_K, \vartheta_K)$ at $p \in M^3$ to such a geodesic heading out from $p$ in direction $\hat{\Omega}_p(\beta_1, \beta_2)$.

Since geodesic surfaces become planes in faithful normal coordinates and these planes are directly related to the great circles, we need the relation between a geodesic $\gamma_{p,\hat{\Omega}_p}$ from a point $p$ and the geodesic $\gamma^{S^2}_{\hat{q}(\varphi_K),\vartheta_K}$ on the 2-sphere $S^2 \simeq D_o$ at $o$, which corresponds to the plane $U$, in which $\gamma_{p,\hat{\Omega}_p}$ lies (see Fig. 11). The plane $U$ is obtained by rotating the original plane $U_0$ in the following way:

$$U = R_z(\vartheta_p) \circ R_x(\varphi_p) \circ R_x(\alpha_2 - \vartheta_p) \cdot U_0 \quad (69)$$

We want to have the parameters $\varphi_K$ and $\vartheta_K$ we used in (68). This amounts to translating to the simpler rotation:

$$U = R_z(\varphi_K) \circ R_x(\vartheta_K) \cdot U_0 \quad (70)$$

![Figure 11](image)

Figure 11: The black great arc is the intersection of the original plane with the unit sphere in yellow, which represents the directions of the point $o$ in the centre of the sphere. The great arc in blue is associated to the plane in which the geodesic in question $\gamma_{p,\hat{\Omega}_p}$ lies. The tilt parameter $\alpha_2 = \beta_2$ in red is the angle between that great arc and the geodesic in black with angular label 0 on the sphere at that point $\hat{\Omega}_p \in S^2 \simeq D_o$.

We use the sine and cosine laws on $S^2$ (i.e. $K = 1$) to relate the great circle function parameters $\varphi_K, \vartheta_K \in P'$ to the direction parameters $\varphi_p, \vartheta_p$ of the point $p = \gamma_{o,\hat{\Omega}_p}(l)$ and the tilt $\alpha_2 = \beta_2$ of the plane, in which the geodesic in question $\gamma_{p,\hat{\Omega}_p}$ lies. In other words we solve the equation:

$$\gamma^{S^2}_{\hat{\Omega}_{p,\alpha_2}(\tau + \lambda) = \gamma^{S^2}_{\hat{q}(\varphi_K),\vartheta_K}(\lambda).}$$

We are describing a geodesic from an arbitrary point $\hat{\Omega}_p$ on $S^2$ in terms of one originating from a point $\hat{\Omega}(\varphi_K,0) = \gamma^{S^2}_{0,0}(\varphi_K)$ on the initial geodesic $\gamma^{S^2}_{0,0}$. This is essentially applying a special case of the consistency condition (34).
The problem splits into different cases which we discuss more in Appendix B. For simplicity we restrict ourselves to one case.

\[
\varphi_K = \tan^{-1}\left(\frac{\sin \varphi_p}{|\cot(\alpha_2 - \vartheta_p)| \sin \vartheta_p + \cos \varphi_p \cos \vartheta_p}\right), \quad \vartheta_K = \sin^{-1}\left(\frac{\sin \varphi_p \sin(\alpha_2 - \vartheta_p)}{\sin \varphi_K(\varphi_p, \vartheta_p, \alpha_2)}\right)
\] (71)

Extending it to more cases is possible but a computationally hectic task which requires one to adjust the branches of the inverse trigonometric functions.

Ultimately, we parametrize the curvature field \(K\) with the location \(p \in \mathcal{M}\) in faithful normal coordinates \(\mathcal{N}(p) = (l, \varphi_p, \vartheta_p)\) and the orientation parameters \(\varphi_K, \vartheta_K\) of the geodesic plane, in which the geodesic in question lies: \(K_p(\varphi_K, \vartheta_K)\).

### 4.6 Restrictions to the curvature field

We parametrized the curvature field and found, that the degrees of freedom at every point correspond to the great-circles on the \((n-1)\)-sphere at this point. But there might be dependencies which reduce the degrees of freedom. To investigate, whether this is the case we start with choosing a scalar curvature field on every plane through \(o\) in a smooth manner: \(\hat{\Omega}_{\mathcal{N}}(l, \varphi)\). Now all geodesics on these planes in FNC are determined, including the ones, which form the curved surface depicted in Figure 12.

But we know that all geodesics lie on geodesic planes through \(o\). So, the entire geometry of a manifold is determined by the curvature field on the geodesic surfaces through a single point (w.l.o.g. \(o\)).

It should be noted that the remaining degrees of freedom are much more than the Riemann tensor can incorporate. We describe an example of a geometry in the supplement S4, which cannot be expressed in the metric formalism.

### 5 Relation to the metric

We constructed a collection of charts in Sec. 3, which are tightly related to the geodesic flow bundle. These charts allow us to relate the geodesic flow bundle to the metric. In this section we derive the procedure to calculate the components of the metric in a faithful normal chart.

For this we need the tangent vector fields of the coordinate lines of \(l\) and \(\Omega\). With coordinate lines we mean the curves traced out by increasing only one coordinate of the faithful normal chart \(\mathcal{N}\). This creates the flows:

\[
\phi_{\mathcal{N}}^{l}(p) = (l_p + l, \bar{\vartheta}_p), \quad \mathcal{N}(p) = (l_p, \bar{\vartheta}_p), \quad \phi_{\mathcal{N}}^{\varphi}, \quad \phi_{\mathcal{N}}^{\vartheta_1}, \quad \ldots \quad \phi_{\mathcal{N}}^{\vartheta_{n-2}},
\] (72)

we collect the angular parameters in the parameter vectors \(\vec{\vartheta} = (\varphi, \vartheta_1, \ldots, \vartheta_{n-2})\), \(\bar{\vartheta}_p = \vartheta_i^p \in \mathcal{P}\).
In the case of the coordinate lines coincide with the geodesic flow from \( o \) and thus the tangent vector field is just its parameter derivative:

\[ \phi^j_{\lambda} = \gamma_{o, \tilde{\Omega}_p} \Rightarrow \partial_t(p(\tilde{\Omega}_p, l_p)) = \dot{\gamma}_{o, \tilde{\Omega}_p}(l_p) = \frac{d}{d\lambda} \gamma_{o, \tilde{\Omega}_p}(\lambda) \bigg|_{\lambda=l_p} \]  

(73)

The geodesic \( \gamma_{o, \tilde{\Omega}_p} \) is by definition parametrized by arclength \( \lambda \) and thus: \( \| \gamma_{o, \tilde{\Omega}_p}(\lambda) \| = 1 \ \forall \lambda \in I \).

\[ g_{\lambda}(p) = g_p(\partial_t(p), \partial_t(p)) - g_p(\dot{\gamma}_{o, \tilde{\Omega}_p}(l_p), \dot{\gamma}_{o, \tilde{\Omega}_p}(l_p)) = \| \gamma_{o, \tilde{\Omega}_p}(\lambda) \|^2 = 1 \ \forall p \in U \]  

(74)

The coordinate lines of the angular parameters \( \vartheta^i \) lie on \((n-1)\)-spheres and their tangent vector fields are thus orthogonal to the \( l \)-line rays.

\[ \partial_{\vartheta^i}(p) = \dot{\phi}^i_{\lambda} \bigg|_{\vartheta^i = \vartheta^i_p} \perp \dot{\phi}^i_{\lambda}(p) = \gamma_{o, \tilde{\Omega}_p}(l_p) \Leftrightarrow \beta = \mathcal{L}_p([\dot{\phi}^i_{\lambda}(p)]\), [\gamma_{o, \tilde{\Omega}_p}] = \frac{\pi}{2} \]  

(75)

\[ g_{\vartheta^i \vartheta^j}(p) = g_p(\dot{\phi}^i_{\lambda}(p), \dot{\gamma}_{o, \tilde{\Omega}_p}(l_p)) = \| \dot{\phi}^i_{\lambda}(p) \| \| \dot{\gamma}_{o, \tilde{\Omega}_p}(l_p) \| \cos \beta = 0 \ \forall p \in U \]  

(76)

The vector fields of our angular parameters form an orthogonal basis on the \((n-1)\)-sphere, wherever it is not degenerate. We can calculate the vector fields of the angular coordinate lines, by taking derivatives of the angular parametrization function \( \tilde{\Omega} \), since the \( l \)-coordinate is held constant anyways. Since the angular parametrization we use is generated by subsequently acting rotations in orthogonal planes\(^2\) considering the embedding \( i \) of \( S^{n-1} \) in \( \mathbb{R}^n \) on the initial direction \( \hat{0} \), the orbits of each group action are automatically orthogonal to the subspace in which the orbits of the previous actions lie. And thus, all angular coordinate lines are orthogonal to each other.

\[ \dot{\phi}^i_{\lambda}(p) = l_p \tilde{\Omega}_{\vartheta^i}(\hat{0}_p) \Rightarrow \tilde{\Omega}(\hat{0}) = R_{\vartheta_{n-2}} \circ \cdots \circ R_{\vartheta_{2} \vartheta_{1}}(\vartheta_1) \circ R_{\vartheta_{1} \vartheta_{0}}(\vartheta_0) \Rightarrow \dot{\phi}^i_{\lambda} \perp \dot{\phi}^j_{\lambda}, i \neq j \]  

(77)

Thus, the metric in faithful normal coordinates is always diagonalized:

\[ g_{\vartheta^i \vartheta^j}(p) = 0, \ i \neq j, \ \forall p \in U \]  

(78)

The coordinate lines \( \dot{\phi}^i_{\lambda}(p) \) are in general not geodesics (consider for example the constant \( \theta \) lines \( \dot{\phi}^\theta_{\lambda}(p) \) on the 2-sphere) and thus their tangent vectors are in general not unit vectors. To find the diagonal entry of \( \dot{\vartheta}^i \) in the metric we use the tangent vector of a geodesic heading out in direction of increasing \( \vartheta^i \) from a point \( p \) and the already determined components of the metric.

\[ \hat{\vartheta}^i := N \frac{d\tilde{\Omega}}{d\vartheta^i}(p) \in S^{n-1}, \ \dot{\gamma}_{\hat{\vartheta}^i}(0) = \frac{d}{d\lambda} \mathcal{N}_o(\gamma_{\hat{\vartheta}^i}(\lambda)) \bigg|_{\lambda=0} \Rightarrow \partial_j(p) = \dot{\vartheta}^j(p, \hat{\vartheta}^i; 0) \partial_j(p) + \dot{\vartheta}^j(p, \hat{\vartheta}^i; 0) \partial_{\vartheta^i}(p) \]  

(79)

where \( N \) is a normalization constant to make it a unit vector.

Since this is a geodesic (and in general not a coordinate line) it has length 1. Writing this in components gives us an equation with which we can determine the missing metric component:

\[ g_{\vartheta^i \vartheta^j}(p) = 1 - \frac{\dot{\vartheta}^i(p, \hat{\vartheta}^i; 0)^2}{\dot{\vartheta}^j(p, \hat{\vartheta}^i; 0)^2} \]  

(79)

We saw in Sec. 4.1 that calculating the geodesic flow bundle from the curvature field boils down to calculating the length of the top-line \( b \) and opening angle \( \alpha \) of a triangle from a curvature field \( K \) and is thus essentially a

\(^2\)The concept of rotations around an axis only works in the special case of 3-dimensions. In higher dimensions the axes would generalize to \( n-2 \)-dimensional submanifolds. In the 2-dimensional case we can see however, that rotations are an action in a surface. There is no additional direction in which we could have an axis.

\(^3\)We are not talking about the angular structure at a point \( p \) here, since this is in general not correctly represented in the faithful normal chart. We are using the angular structure at \( o \), which is projected outwards, using geodesics.
2-dimensional problem. So, we want to express this tangent vector in terms of the solution to this problem (54) (fundamental solution), which we will derive in the following sections.

The geodesics \( \gamma_{q,\hat{\Omega}_\alpha} \) from a point \( q = \gamma_{o,\hat{\Omega}}(c) \) on the original geodesic can be directly written in terms of the fundamental solution \( b(K) \), \( \alpha(K) \) and the tilts \( \beta_2, \ldots, \beta_{n-1} \) at \( q \) of the plane in which the triangle lies. In a plane GFB the tilt parameters \( \alpha_2, \ldots, \alpha_{n-1} \) at \( o \) are the same as the ones at \( q \) and we have:

\[
\gamma_{q,\hat{\Omega}_\beta}(\lambda) = l \hat{\Omega}(\tilde{\alpha}) = \gamma_{o,\hat{\Omega}_\alpha}(l) \quad \Rightarrow \quad N_o(\gamma_{q,\hat{\Omega}_\beta}) = \left( b(K; c, \beta_1; \lambda), \alpha(K; c, \beta_1; \lambda), \beta_2, \ldots, \beta_{n-1} \right)^T
\]

To calculate the flow from a different point \( p \) using the fundamental solution \( (b, \alpha) \) we just rotate the tilted triangle to \( p \) according to our parametrization function \( \hat{\Omega} \), see Fig. 13\(^4\). Then we have to solve Eq. (81) again to obtain the components \((l, \varphi, \vartheta_2, \ldots, \vartheta_{n-2})\) in FNC:

\[
\gamma_{p,\hat{\Omega}_\beta}(\lambda) = R_{x_1, x_2}(\vartheta_{n-2}, p) \circ \cdots \circ R_{x_3, x_2}(\vartheta_{1, p}) \circ R_{x_1, x_2}(\varphi_p) \cdot \gamma_{q,\hat{\Omega}_\beta}(l)
\]

\[
= l(K) R_{x_1, x_2}(\vartheta_{n-2}, p) \circ \cdots \circ R_{x_3, x_2}(\vartheta_{1, p}) \circ R_{x_1, x_2}(\varphi_p) \cdot \hat{\Omega}(\varphi(K), \beta_2, \ldots, \beta_{n-1}) = l \hat{\Omega}(\tilde{\vartheta}) = \gamma_{o,\hat{\Omega}}(l)
\]

Note that \( l \) drops out and this equation is only about the angular part which is the same for every geometry. One gets a solution of the type: \( \vartheta^j(\hat{\Omega}_p, \hat{\Omega}_o) \).

The 2-dimensional case is trivial and we present the solutions for the 3-dimensional case in Sec. S3, but in general this is a nontrivial transcendental equation and attempting to derive the general \( n \)-dimensional solution is not within the scope of this work.

Now we are ready to do this with the geodesics \( \gamma_{p,\hat{\theta}^i} \), which we need to calculate the metric components. We calculated the direction vectors \( \hat{\theta}^i \) at an arbitrary point \( p \). Since we solve the problem at a point \( q \) on the original geodesic we have to rotate these direction vectors back to \( q \) to find the parameters \( \hat{\beta}^i \):

\[
\hat{\Omega}_\beta = \hat{\Omega}(\tilde{\beta}) = \hat{\Omega}_p^{-1}(\varphi_p) \circ \hat{\Omega}_p^{-1}(\vartheta_{1, p}) \circ \cdots \circ \hat{\Omega}_p^{-1}(\vartheta_{n-2, p}) \cdot \hat{\theta}^i(p) \quad \Rightarrow \quad \hat{\beta}^i
\]

Finally, the FNC component functions of the geodesics \( \gamma_{p,\hat{\theta}^i} \) to the coordinate direction vectors \( \hat{\theta}^i \) are determined by the direction vectors \( \hat{\Omega}_p \) of \( p \) (containing the information to rotate the triangle from \( q \) to \( p \)) and \( \hat{\Omega}_o \), describing the triangle as it would be at \( q \). The triangle vector \( \hat{\Omega}_q \) is composed of the opening angle \( \alpha_1 \), which is the angular component of the fundamental solution \( \alpha(K) \), and the tilts \( \hat{\beta}^i \).

\[
\vartheta^i(p, \tilde{\beta}^i; \lambda) = \vartheta^i \left( \hat{\Omega}_p, \hat{\Omega} \left( \alpha(K; c, \beta_1^i; \lambda), \beta_2^i, \ldots, \beta_{n-1}^i \right) \right) = N_o(\gamma_{p,\hat{\theta}^i}(\lambda))^{i+1}
\]

The +1 in the index comes from \( l \) being the first coordinate which is just the distance component of the fundamental solution evaluated for the correct angular coordinate:

\[
l(p, \tilde{\beta}^i; \lambda) = b(K; c, \beta_1^i; \lambda)
\]

We demonstrate this procedure on the examples in the supplement for which we can calculate the functions \( b(K) \) and \( \alpha(K) \) with the constant curvature cosine- and sine-laws. The counter example in S4 demonstrates, that the metric cannot pick up all the curvature degrees of freedom, that a GFB can describe.

\(^4\)Note that we are using \( \beta_2 \) as a coordinate in Fig. 13 rather than the angle between the red plane and the blue triangle as it is used in Eq. (81). The angle between the red plane \( U_{\beta_2} \) and the blue triangle \( U_{\beta_3} \) in the plot is given by \( \beta_2 - \vartheta_p \). So, the two conventions are related by: \( \beta_2 \leftrightarrow \beta_2 - \vartheta_p \). The \( \beta_i \) are the angular coordinates at \( q \), but not anymore, when we rotate them to \( p \). So, they are the angular labels the geodesic \( \gamma_{p,\hat{\Omega}_\beta} \) would have, if it were heading out from \( q \).

\(^5\)The triangle vector \( \hat{\Omega}_o \) can be interpreted as a \( n \)-dimensional generalization of the opening angle, additionally describing the orientation of the opening angle via the tilt parameters.
Figure 13: The figure shows, how a geodesic triangle is embedded in an $n$-dimensional manifold and how we rotate and tilt the flow from $q$ to get the flow from $p$.

6 Conclusions

We introduced the geodesic flow bundle in an attempt to describe a geometry in terms of geometric quantities i.e., distances and angles. We found, that the structure of the directions at a point can be described via the geodesic flow bundle on the unit-sphere, and that there is a natural way to define the angular labels, by subsequently embedding lower dimensional spheres into the next higher dimensional ones until we reach $S^{n-1}$ where $n$ is the dimension of the manifold.

We introduced the special case of normal chart, which preserves distances from the centre and angles at the centre and the coordinates consists purely of geometrically meaningful quantities.

When trying to understand the relation of the geodesic flow bundle to the curvature of a geometry we found that infinitesimal spherical triangles can serve as generators of a geometry. From this we concluded that the sectional curvature, a value on an infinitesimal disc around a point, properly characterizes the degrees of freedom a curvature distribution has. This led us to introduce the great circle functions (i.e., functions on the great circles of the $n$-sphere) to get to a parametrization of a generic source curvature field, which is compatible to our formalism and includes all degrees of freedom. We have shown that the geodesic flow bundle contains more information, than the metric i.e., it includes all geodesics of the manifold in arclength parametrized form, we can calculate the metric from the geodesic flow bundle. Doing the converse is only possible, if one manages to solve the geodesic equations for a generic case. We calculated the metric from the GFB for the example in the supplement S4, to compare our parametrization of curvature fields to the Riemann tensor. From this example we conclude that there are geometries, which the metric cannot describe and thus curvature degrees of freedom, which are not included in the Riemann tensor.
With this, we laid the foundation for the pseudo-Riemannian case and solved a part of it. We reduced the problem of calculating the GFB from a source curvature field on an \( n \)-dimensional manifold to a two-dimensional problem. The next step is to solve this two-dimensional problem, which we do up to second order in a follow-up paper. After that we extend the formalism to the pseudo-Riemannian case i.e., we discuss the different cases and how the Riemannian one is embedded in the pseudo-Riemannian flow bundle.

By establishing a coordinate independent formalism, we got rid of the degeneracy (confusion) between gravitational waves and coordinates. Gravitational waves can however also blend into the background, since it is the background itself which forms a wave. We can use the invariant definition of gravitational waves proposed in [4] on the geodesic flow bundle. It states, that A) the Riemann tensor characterizes the presence of radiation which translates to the source curvature taking this role and that B) in empty space-time gravitational radiation propagates with the fundamental velocity. For the point B) we need to extend our formalism to the pseudo-Riemannian case. These two conditions are met for source curvature field with a disturbance propagating along the null rays i.e. constant phase along these rays.

Acknowledgements

A.B. is supported by the Forschungskredit of the University of Zurich Grant No. FK-21-083 S.T. is supported by Swiss National Science Foundation Grant No. 200020 182047. Symbolic manipulation as well as some of the plots have been done using Mathematica [13].

A Comparison of the geodesic flow from a point with the exponential map

Our construction is similar to the exponential map, whose definition we give here briefly for comparison. We use the definition 2.51 given in the lecture script by M. Burger and S. Tornier [14] in Sec. 2.3 for this purpose.

Let \((\mathcal{M}, g)\) be a Riemannian manifold, with connection \(\nabla\). Then

\[
\forall x \in \mathcal{M} \exists \text{ a neighbourhood } U \subset \mathcal{M} \land \varepsilon, \delta > 0 : \quad C : \quad \{(p, v) \in TU|\|v\| < \varepsilon\} \times (-\delta, \delta) \to \mathcal{M} \quad ((p, v), t) \mapsto c_{(p,v)}(t) \quad (86)
\]

is a well defined smooth map, where \(c_{(p,v)}\) is the unique geodesic satisfying \(c_{(p,v)}(0) = p\) and \(\dot{c}_{(p,v)}(0) = v\).

When \(\|v\|\) is chosen small enough, then \(\delta\) can be bigger than 1, so one can define a subset of the tangent bundle,

\[
\Omega = \{(p, v) \in T\mathcal{M}|c_{(p,v)} \text{ is defined on } (-\delta, \delta), \delta > 1\} \quad (87)
\]

on which the exponential map is defined:

\[
\exp : \quad \Omega \to \mathcal{M}, \exp((p, v)) = c_{(p,v)}(1) \quad (88)
\]

We do not want a map involving tangent spaces, since it is not necessary to introduce them in our formalism. Doing so, would mix up the directions with the distances, just so we would have to separate them again. Also, there are in general not many restrictions on the parametrization of the geodesics, when defining the exponential map. We need it to be parametrized by arclength however.

The intention here is to construct a local isomorphism between the tangent bundle and the manifold constructed by geodesics. We, however, want to collect a family of curves, which we consider as geodesics by definition, to give our manifold a geometric structure instead of using the metric.
There are three cases of triangles which all yield equivalent equations:

\[ \begin{align*}
\alpha_2 \geq \vartheta_p & \implies \alpha_2 \geq 0 : \\
\alpha_2 < \vartheta_p, \ \alpha_2 < 0 : & \\
\alpha_2 < \vartheta_p, \ \alpha_2 > 0 : \\
\end{align*} \]

The signs in the first equations all cancel, only the second equation in (90) has a different sign. We square this equation, and thus get rid of the sign change, and add it to the squared cosine law, to get an equation from which we can get \( \varphi_K \).

\[ 1 = \sin^2 \tau + \cos^2 \tau = \frac{\sin^2 \vartheta_p \sin^2 \varphi_K}{\sin^2(\alpha_2 - \vartheta_p)} + \left( \cos \varphi_p \cos \varphi_K + \sin \varphi_p \sin \varphi_K \cos \vartheta_p \right)^2 \]

\[ \iff A \sin^2 \varphi_K + B \sin(2\varphi_K) = C \]

\[ A = \frac{\sin^2 \varphi_p}{\sin^2(\alpha_2 - \vartheta_p)} + \sin^2 \varphi_p \left( \cos^2 \vartheta_p + 1 \right) - 1, \quad B = \frac{1}{2} \sin(2\varphi_p) \cos \vartheta_p, \quad C = \sin^2 \varphi_p \]

This equation has two solutions which are both defined piece-wise.

\[ \varphi_K = \begin{cases} 
\tan^{-1} \left( \frac{C}{B + \sqrt{B^2 + (A-C)C}} \right), & B \geq 0 \\
\tan^{-1} \left( \frac{C}{B - \sqrt{B^2 + (A-C)C}} \right) + \pi, & B < 0 
\end{cases} \]

\[ \varphi_K = \begin{cases} 
\tan^{-1} \left( \frac{C}{B - \sqrt{B^2 + (A-C)C}} \right), & B \geq 0 \\
\tan^{-1} \left( \frac{C}{B + \sqrt{B^2 + (A-C)C}} \right), & B < 0 
\end{cases} \]

The one on the left represents our choice of domain \( \varphi_K \in [0, \pi) \), the one on the left returns negative angles. Since \( \vartheta_p \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) we can reduce the condition on \( B \) into one on \( \varphi_p \):

\[ B \geq 0 \iff \varphi_p \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, \pi\right) \quad \land \quad B \leq 0 \iff \varphi_p \in \left[\frac{\pi}{2}, \pi\right) \cup \left[\frac{3\pi}{2}, 2\pi\right) \]

From the plots in Fig. 14 we see, that the denominator \( B + \sqrt{B^2 + (A-C)C} \) of the solution is positive, wherever \( B \geq 0 \) and \( B - \sqrt{B^2 + (A-C)C} \) is negative wherever \( B \leq 0 \). We plot the denominators on the domains of the angular position coordinates \( \varphi_p \) and \( \vartheta_p \), for a sample of tilt parameters \( \alpha_2 \). Since the denominator in the case of \( B \geq 0 \) is positive we use the principle branch of \( \arctan \) and the side branch \( \tan^{-1}(x) + \pi \) for the case of \( B \leq 0 \), where the denominator is negative. This results in a map:

\[ \varphi_K : [0, 2\pi) \times (-\pi, \pi] \rightarrow [0, \pi) \times (-\pi, \pi] \rightarrow \varphi_K = \begin{cases} 
\varphi_p, & B \geq 0 \\
\varphi_p - \pi, & B < 0 
\end{cases} \]

Where the \( \varphi_p \) domain is folded:

\[ \varphi_K^+ : [0, \frac{\pi}{2}) \cup [\frac{3\pi}{2}, \pi) \rightarrow [0, \pi), \quad \varphi_K^- : [\frac{\pi}{2}, \pi) \cup [\frac{3\pi}{2}, 2\pi) \rightarrow [\frac{\pi}{2}, \pi) \]

This is consistent with the fact, that a plane rotated by \( \pi \) coincides with itself.

The other curvature plane parameter \( \vartheta_K \) can now be calculated from the sine-law and \( \varphi_K \):

\[ \vartheta_K = \sin^{-1} \left( \frac{\sin \varphi_p \sin(\alpha_2 - \vartheta_p)}{\sin \varphi_K(\varphi_p, \vartheta_p, \alpha_2)} \right) \]

The principal branch of the arcsine nicely coincides with the domain of \( \vartheta_K \) in this case.

So, we ultimately parametrize the curvature field \( K \) with the location \( p \in M \) in faithful normal coordinates \( N(p) = (l, \varphi_p, \vartheta_p) \) and the orientation parameters \( \varphi_K, \vartheta_K \) of the geodesic plane, in which the geodesic in question lies.

\[ K_p(\varphi_K, \vartheta_K) \]
C Parallel transport

It seems to be a priori not clear, how we would compare directions at two different points on a manifold. Especially if we use tangent spaces to represent them. To deal with this we would usually introduce the notion of covariant derivative or equivalently a parallel transport. This introduces an additional structure and thus additional degrees of freedom. If we work with the geodesic flow bundle however we see that there is no choice to be made. The parallel transport is already built into the flow bundle structure.

In classical Euclidean geometry we understand parallel as two straight lines which have the same distance between each other everywhere. Parallely transporting a straight-line along another straight-line would mean, that we do not change the angle between the two as we do the transport. When we use straight lines as representants of directions we can conclude that we transport a direction parallely along a straight-line, if and only if we leave the angle between the direction and the straight-line invariant.

We translate this concept into curved geometries, by replacing the straight lines with geodesics, which are locally straight lines. To make use of our angular structure we start w.l.o.g. form the origin $\text{o}$ and preserve the direction $\hat{\Omega}(\text{o})$ along the geodesic to $p = \gamma_{\text{o},\hat{\Omega}}(l)$, by keeping the angular labels constant.

$$P_{o-p,\gamma_{o,\hat{\Omega}}}(\widehat{\Omega}_\beta) = \hat{\Omega}_\beta \quad (101)$$

For example in Fig. 4 the black line is the geodesic from $p$ in direction $0$, which is the direction of the original geodesic from $o$ transported along $\gamma_{o,\alpha}$ and thus encloses the angle $\alpha$ with this geodesic at $o$ as well as at $p$.

If we would try to define a parallel transport along non-geodesics in this formalism, we would need another flow bundle, since we need a curve between any two points on the manifold to parallely transport between this pair of points. This would mean however that we just introduced two conflicting flow bundles and in that case the way
we would identify directions at different points would be inconsistent with the geometry we imposed on the manifold.

To parallelly transport a direction along a non-geodesic we would just transport along geodesics between increasingly many points on the curve i.e., approximate the curve with geodesics.

\[ \{ x_i \}_{i=0}^N \subset c(I) \in C^\infty(I; M), \quad I \subset \mathbb{R}, \quad x_0 = a, \quad x_N = b \]  

At each of these points we get a kink, and we pick up a change in the angle, since we continue with a geodesic that does not continue in the same direction. If we take the limit to infinitely many sampling points, forming an infinitely dense sampling of the curve everywhere, these changes become infinitely small, but we have infinitely many of them. The kinks vanish, and the approximation coincides with the smooth non-geodesic.

\[ \lim_{N \to \infty} \bigcup_{i=0}^N \gamma_{x_i, \Omega_i}(\lambda) = c(\lambda), \quad \gamma_{x_i, \Omega_i}(\lambda_i) = x_{i+1} \quad \forall i < N \land \gamma_{x_N, \Omega_N}(\lambda_N) = b \]  

Figure 15: A sketch of how we connect sample points \( x_i \) (red) of a in general non-geodesic curve \( c(\lambda) \) (blue) with geodesics \( \gamma_{x_i, \Omega_i} \) (purple). We transport a direction (black) piecewise along the geodesics and observe, that the angles \( \varphi_i \) (orange) change at the sample points.

We can use the inverse of the exponential map to connect the parallel transport to the classical formalism using tangent spaces:

\[ \forall X \in T_x M \quad \exists \Omega \in S^{n-1} \land \alpha \in \mathbb{R} : X = d_\lambda \gamma_{x, \Omega}(\alpha \lambda)|_{\lambda=0} \]  

\[ P_{c,a,b}(X) = P_{c,a,b}(d_\lambda \gamma_{x, \Omega}(\alpha \lambda)|_{\lambda=0}) = \lim_{N \to \infty} \left( \bigcup_{i=0}^N P_{\gamma_{x_i, \Omega_i}, a, \lambda_i} \right) \left( d_\lambda \gamma_{x, \Omega}(\alpha \lambda)|_{\lambda=0} \right) \]  

Extending the angle \( \Omega \) and scale factor \( \alpha \) to fields, the covariant derivative can then be defined as:

\[ \frac{DX}{d\lambda}(\lambda) = \lim_{\varepsilon \to 0} \frac{P_{c,a,\lambda+\varepsilon}(X(c(\lambda)))) - X(c(\lambda))}{\varepsilon}, \quad \forall X \in \Gamma(TM), \quad \Omega(x) \land \alpha(x) : X(x) = d_\lambda \gamma_{x, \Omega(x)}(\alpha(x)\lambda)|_{\lambda=0} \]  

Then we can define the connection over the integral curves:

\[ \nabla_Y X := \frac{DX}{d\lambda}, \quad \text{with } c \in C^\infty(I; M) \text{ s.t. } \dot{c}(\lambda) = Y(c(\lambda)) \quad \forall X,Y \in \Gamma(TM) \]  

Since the Levi-Civita connection can be directly calculated from the metric and is thus entirely determined by the geometry, it is reasonable to assume, that it coincides with our notion of parallel transport. Under the assumption, that this is the case, we conclude that all other connections introduce additional structure on to the tangent spaces, which is unrelated to geometry. Also note that in our formalism the tangent spaces are not necessarily present. We can introduce them, but we do not need them.

We note a difference to the metric formalism in that here we already have a natural notion of parallel transport from the geodesic flow bundle, and we then have to derive the corresponding covariant derivative from that. In the metric formalism it is natural, to start with the connection since we can calculate the connection coefficients of the Levi-Civita connection (Christoffel symbols) directly from the metric. And then we derive the parallel transport
From Sec. 4 we know, that the connection is not needed, to relate the geodesic flow bundle to curvature and is thus of limited importance in this formalism. Though, it can be used to relate the notion of curvature in the GFB formalism to the Riemann tensor. However, calculating the connection from the parallel transport in the flow bundle formalism is a non-trivial task and goes beyond the scope of this work.

A more practical way, to compare the flow bundle formalism to the metric one, is to calculate the metric from the geodesic flow bundle in FNC, which we do in Sec. 5. One can then calculate the connection coefficients in FNC. We present a plan in the follow-up paper for an analytic consistency check of our main result with the geodesic equations in FNC, using the Levi-Civita connection. Carrying this out in full detail however is a cumbersome and difficult task and is not presented here.

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Contents

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The following examples demonstrate the concept of the geodesic flow bundle. We use the faithful normal chart to visualize the geodesics. Since we can plot the geodesic flow bundle onto the 2-sphere embedded in $\mathbb{R}^3$ and it is a non-trivial example, it is the most helpful example to understand how this structure works.

1 The Geodesic Flow Bundle of the Manhattan metric

The geodesic flow bundle on the Euclidean plane $E \simeq \mathbb{R}^2$, in Cartesian coordinates is given by:

$$\Phi_E(q; \alpha, \lambda) = \begin{pmatrix} c \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix},$$

restricting to the $x$-axis $q = (c, 0)$ for simplicity.

Now we can construct a flow bundle on $\mathbb{R}^2$ which induces the Manhattan / maximum metric:

$$d(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

We leave the angles invariant $\varphi \mapsto \varphi$ but change the distances $r = \sqrt{x^2 + y^2} \mapsto l = \max\{|x|, |y|\}$.

$$\Phi_M((c, 0); \beta, \lambda') = l \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \quad l(c, \beta, \lambda') = \max\{|c + \lambda' \cos \beta|, |\lambda' \sin \beta|\},$$

$$\varphi(c, \beta, \lambda') = \sin^{-1} \frac{ac \sin \beta}{\sqrt{(\lambda')^2 + c^2 + 2 \lambda' c \cos \beta}}, \quad \lambda' = \lambda \max\{|\cos \beta|, |\sin \beta|\}$$

We need to transform the parameter to make sure, that if $\Phi_E(q; \beta, \lambda) = p$ then $\Phi_M(q; \beta, \lambda') = p$ ends at the same point in the new geometry.

$$\Phi_M(o, \alpha, 0) = o \land \Phi_M(o, \alpha, b) = p \implies d(o, p) = l = \max\{|c + \lambda' \cos \beta|, |\lambda' \sin \beta|\} \quad \checkmark$$
$$\Phi_M(q, \beta, 0) = q \land \Phi_M(q; \beta, \lambda') = p \implies d(q, p) = \lambda' = \max\{|\lambda \cos \beta|, |\lambda \sin \beta|\} \quad \checkmark$$
2 The Geodesic Flow Bundle and Faithful Normal Coordinates of the 2-sphere $S^2$

We choose polar coordinates on the sphere to compare the flow bundle of $S^2$ embedded in $\mathbb{R}^3$ with the faithful normal coordinates.

$$p(\theta, \phi) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ \sin \theta \end{pmatrix}, \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \quad \phi \in [0, 2\pi) \quad (7)$$

The sphere $S^{n-1}$ is the subset $S^{n-1} \subset \mathbb{R}^n$, which is invariant under the group action of $O(n)$ on $\mathbb{R}^n$ i.e. every point $p$ in $S^{n-1}$ is mapped to another point on the sphere under the action of the rotation group.

$$\cdot : \quad O(n) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad \mapsto \quad R \cdot x \in S^{n-1} \quad \forall R \in O(n) \forall x \in S^{n-1}, \quad \text{i.e.} \quad O(n) \cdot S^{n-1} = S^{n-1} \quad (8)$$

We can generate a geodesic by acting a rotation $R \in O(n)$ onto an origin point $o$. We choose our origin at $o = (1,0,0)$ and pick rotations around the $z$-axis to trace out our first geodesic. This geodesic coincides with the half equator (the green line continued by the blue one in Fig. 2).

$$R_z(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \in O(n); \quad \gamma_{o,0}(\lambda) = R_z(\lambda) \cdot o = \begin{pmatrix} \cos \lambda \\ \sin \lambda \\ 0 \end{pmatrix} = p(0,\lambda), \quad \lambda \in (0, \pi) \quad (9)$$

We then use the $O(n)$-action again to generate the flow from $o$ (green gradient lines in Fig. 2), acting rotations around the $x$-axis on the original geodesic.

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix} \in O(n); \quad \gamma_{o,\varphi}(\lambda) = R_x(\varphi) \cdot \gamma_{o,0}(\lambda) = \begin{pmatrix} \cos \lambda \\ \cos \varphi \sin \lambda \\ \sin \varphi \sin \lambda \end{pmatrix}, \quad (10)$$

with $\lambda \in (0, \pi)$ and $\alpha \in [0, 2\pi)$.

The geodesic flow from $o$ defines the faithful normal coordinates at $o$. A point $p \in M$, which can be reached by a geodesic from $o$, is labelled with the direction and arclength of that geodesic: $p = \gamma_{o,\varphi}(l) \rightarrow p(\varphi,l)$.

$$\mathcal{N} : \quad S^2 \subset \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad \mapsto \quad (l, \varphi)(p), \quad l \in \mathbb{R}_+, \quad \varphi \in [0, 2\pi) : \quad \gamma_{o,\varphi}(l) = p \quad (11)$$
The geodesic flow from any point appears as straight lines which display distances correctly in the faithful normal coordinates at that point.

Next we rotate the flow from $o$ around the $z$-axis to generate the flow from the original geodesic (blue gradient lines in Fig. 2).

$$\gamma_{q(c),\beta}(\lambda) = R_z(c) \cdot \gamma_{o,\beta}(\lambda) = \begin{pmatrix} \cos c \cos \lambda - \cos \beta \sin c \sin \lambda \\ \cos \beta \cos c \sin \lambda + \sin c \cos \lambda \\ \sin \beta \sin \lambda \end{pmatrix}, \quad q(c) = \gamma_{o,0}(c)$$  

(12)

To find the expression of $\gamma_{q,\beta}$ in the $o$-chart $N_o$ (faithful normal coordinates at $o$) we need to solve the equation:

$$\gamma_{q(c),\beta}(\lambda) = p(l, \varphi) = \gamma_{o,\varphi}(l),$$

(13)

for $\varphi$ and $l$ in terms of $c, \beta$ and $\lambda$.

$$\left( \frac{\cos c \cos \lambda - \cos \beta \sin c \sin \lambda}{\sin \beta \sin \lambda} \right) = \left( \frac{\cos \lambda}{\sin \varphi \sin \lambda} \right) \Rightarrow N_o(\gamma_{q,\beta}(\lambda)) = \begin{pmatrix} l(c, \beta, \lambda) \\ \varphi(c, \beta, \lambda) \end{pmatrix},$$

(14)

$$l(c, \beta, \lambda) = \cos^{-1} \left( \frac{\cos \lambda \cos c + \sin \lambda \sin c \cos(\pi - \beta)}{\sqrt{1 - (\cos \lambda \cos c + \sin \lambda \sin c \cos(\pi - \beta))^2}} \right),$$

(15)

$$\varphi(c, \beta, \lambda) = \sin^{-1} \left( \frac{\sin \lambda \sin \beta}{\sqrt{1 - (\cos \lambda \cos c + \sin \lambda \sin c \cos(\pi - \beta))^2}} \right),$$

(16)

Figure 2: We plot a collection of curves from the geodesic flow bundle on $S^2$ embedded in its ambient space $\mathbb{R}^3$ on the left and in the faithful normal chart at $o$ on the right. The green thick line is the original geodesic $\gamma_{o,0}$ in this example and is continued by the blue thick one $\gamma_{q,0}$ which is thus also labelled with 0. The angle between the original geodesic and the yellow thick line at $o$ is $\alpha = \frac{\pi}{6}$ and thus the yellow thick line $\gamma_{o,\frac{\pi}{6}}$ and its continuation, the red thick line $\gamma_{p,0}$ have that angular label. The black thick line $\gamma_{p,0}$ is the one with the label 0 and thus the angle between it and the red thick one is $\frac{\pi}{6}$ as well.

Due to the symmetry of the sphere, it is straight forward to generalize the flow from a point $q$ on the original geodesic to the one from a generic point $p \in \mathcal{M}$, represented by the red gradient lines in Fig. 2:

$$p = \gamma_{o,\varphi_p}(l_p); \quad \mathcal{N}_o(\gamma_{p,\beta}(\lambda)) = \begin{pmatrix} l(p, \beta, \lambda) \\ \varphi(p, \beta, \lambda) \end{pmatrix} = \begin{pmatrix} b(l_p, \beta, \lambda) \\ \varphi_p + \alpha(l_p, \beta, \lambda) \end{pmatrix},$$

(17)
The functions $b$ and $\alpha$ coincide with $l$ and $\varphi$ in Eq. (14). We see, that in 2 dimensions the only change in the coordinate functions $l$ and $\varphi$ when we move to a generic point is the constant shift $\varphi_p$, which rotates the flow from $q$ to $p$.

We can then calculate the metric from the derivatives of $l$ and $\varphi$ with respect to arclength $\lambda$ at $\lambda = 0$ of the geodesic heading out in increasing $\varphi$-direction at $p$, i.e. $\beta = \frac{\pi}{2}$.

\[
\dot{l}(l_p, \frac{\pi}{2}, 0) = 0, \quad \varphi(l_p, \frac{\pi}{2}, 0) = \frac{1}{\sin b} \quad \Rightarrow \quad g_{\varphi \varphi} = \frac{1 - l^2(l_p, \frac{\pi}{2}, 0)}{\sin^2(l_p, \frac{\pi}{2}, 0)} = \sin^2 l_p
\]

\[
\Rightarrow \quad g(p) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 l_p \end{pmatrix}
\]

**3 Example on the 3-sphere**

To get the flow from $q$ in faithful normal coordinates we use the 2-dimensional case and tilt the triangle as described in the main text.

\[
\gamma_{q,\Omega_\beta}(\lambda) = (l_p, \beta_1; l) \dot{\Omega}(\varphi(l_p, \beta_1; l), \vartheta) = \gamma_{o,\Omega}(l), \quad \vartheta = \beta_2, \quad \text{for } K(p) = 1, \quad N_o(\gamma_{q,\Omega_\beta}(\lambda)) = \left(\frac{b(l_p, \beta_1; l)}{\beta_2}\right)
\]

We then rotate the entire triangle to another point $p = (l_p, \varphi_p, \vartheta_p)_\mathcal{N}$ in a way, which is consistent with the angular coordinates. To find the normal coordinates of $\gamma_{p,\Omega_\beta}(\lambda)$ we need to solve the following equation.

\[
\gamma_{p,\Omega_\beta}(\lambda) = R_{x_2,x_3}(\vartheta_p) \circ R_{x_1,x_2}(\varphi_p) \circ \gamma_{q,\Omega_\beta}(\lambda) = \dot{\Omega}(\varphi, \vartheta) = \gamma_{o,\Omega}(l)
\]

We see that this is only a problem of angles at $o$ and thus will be generally valid, independent of the curvature involved. The dependence on the curvature field lies in the functions of the fundamental solution $b$ and $\alpha$. The solution to the angular problem is:

\[
\begin{align*}
\varphi(\hat{\Omega}_p, \hat{\Omega}_\alpha) &= \tan^{-1}\left(\sqrt{(1 - \cos \alpha_2)(1 + \cos \alpha_2)}\right), \quad \vartheta(\hat{\Omega}_p, \hat{\Omega}_\alpha) = \tan^{-1}\left(\frac{\sin \varphi_p \cos \alpha_2 + \cos \vartheta_p \sin \alpha_1 \sin \alpha_2}{\cos \vartheta_p \cos \alpha_2 - \sin \vartheta_p \sin \alpha_1 \sin \alpha_2}\right), \\
\hat{\Omega}_p &= \hat{\Omega}(\varphi_p, \vartheta_p), \quad \hat{\Omega}_\alpha = \hat{\Omega}(\alpha_1, \alpha_2), \quad \alpha_1 = \alpha(c, \beta_1, \lambda), \quad \alpha_2 = \beta_2 \\
\cos \alpha_2 &= \cos \alpha_1 \cos \varphi_p - \sin \alpha_1 \sin \varphi_p \cos \alpha_2, \quad \tilde{\cos \alpha_2} = \cos \alpha_1 \sin \varphi_p + \sin \alpha_1 \cos \varphi_p \cos \alpha_2
\end{align*}
\]

To calculate the metric we need the geodesics $\gamma_{p,\varphi}(\lambda)$ and $\gamma_{p,\beta}(\lambda)$, which head out in $\varphi$- and $\vartheta$-direction respectively from an arbitrary point $p \in \mathcal{M}$. To keep the map from directions to $S^2$ consistent across the manifold i.e. satisfy the consistency condition, we need to rotate the directions at $p$ back to $q$, to determine the parameters $\beta_1$ and $\beta_2$:

\[
\hat{\Omega}(\beta_1, \beta_2) = R_{x_2,x_3}^{-1}(\varphi_p) \circ R_{x_1,x_2}^{-1}(\vartheta_p) \circ \hat{\Omega}_\beta
\]

The parameter $\beta_1$ is directly related to the angle of the triangle at $q$ via $\beta = \pi - \beta_1$ and $\beta_2$ describes the tilt of the triangle with respect to the original plane, rotated from $q$ to $p$.

To calculate the metric we need geodesics, which head out in $\varphi$- and $\vartheta$-direction from any point $p \in \mathcal{M}$. Increasing one angular variable at a point in the faithful normal chart creates a flow of coordinate lines:

\[
N_o(p) = N(\gamma_o,\Omega(l)) = N(l \dot{\Omega}(\varphi, \vartheta)) = (l, \varphi, \vartheta)_\mathcal{N}
\]

\[
\phi_N^\varphi(p) = l_p \dot{\Omega}(\varphi + \varphi_p, \vartheta_p), \quad \phi_N^\vartheta(p) = l_p \dot{\Omega}(\varphi_p, \vartheta + \vartheta)
\]
We calculate the tangent vector field of coordinate line flows:

\[
\partial_{\varphi}(p) = \dot{\varphi}(p) = l_p \frac{\partial \hat{\Omega}}{\partial \varphi}(\varphi_p, \vartheta_p) = l_p \begin{pmatrix} -\sin \varphi_p \\ \cos \vartheta_p \cos \varphi_p \\ \sin \vartheta_p \cos \varphi_p \end{pmatrix}.
\]

\[
\partial_{\theta}(p) = \dot{\theta}(p) = l_p \frac{\partial \hat{\Omega}}{\partial \theta}(\varphi_p, \vartheta_p) = l_p \begin{pmatrix} 0 \\ -\sin \vartheta_p \sin \varphi_p \\ \cos \vartheta_p \sin \varphi_p \end{pmatrix}.
\]

The two tangent vector fields are orthogonal to each other at every point \(\varphi(p) \cdot \dot{\vartheta}(p) = 0, \forall p \in \mathcal{M}\). The vector field \(\dot{\varphi}\) is a unit vector field, but \(\dot{\vartheta}\) is not. To obtain an element of \(\mathbb{S}^2\), which we use to parametrize the geodesic flow with, we need to normalize \(\dot{\vartheta}\). Especially because it also vanishes at \(q\) where we apply our solution.

\[
\dot{\vartheta}(p) = \frac{1}{\sin \varphi_p} \vartheta(p)
\]

We calculate the \(\beta_1\) and \(\beta_2\) parameters of \(\varphi\) by solving:

\[
\hat{\Omega}(\beta_1, \beta_2) \overset{!}{=} R_{x_1, x_2}^{-1}(\varphi_p) \circ R_{x_2, x_3}^{-1}(\vartheta_p) \circ \hat{\varphi}(\varphi_p, \vartheta_p) \Rightarrow \beta_1 = \frac{\pi}{2}, \beta_2 = 0
\]

Then the components of \(\gamma_{p, \varphi}(\lambda)\) are given by:

\[
\varphi \left( \hat{\Omega}_p, \hat{\Omega} \left( \alpha \left( c, \frac{\pi}{2}; \lambda \right), 0 \right) \right) = \tan^{-1} \left( \frac{\sqrt{1 - \cos^2 c \cos^2 \lambda}}{\sqrt{1 + \sin^2 c \cos \lambda}} \sqrt{\cos^2 \varphi_0 + \sin c \cot \lambda (\sin(2 \varphi_p) + \sin c \sin^2 \varphi_p \cot \lambda)} \right)
\]

\[
\vartheta \left( \hat{\Omega}_p, \hat{\Omega} \left( \alpha \left( c, \frac{\pi}{2}; \lambda \right), 0 \right) \right) = \vartheta_p, \quad l \left( c, \frac{\pi}{2}; \lambda \right) = \tan^{-1} \left( \frac{\sqrt{1 - \cos^2 c \cos^2 \lambda}}{\cos \cos \lambda} \right)
\]

To calculate the metric component, we need the tangent vector at \(p\) in faithful normal coordinates, i.e. the derivatives of the components. It is trivial to confirm, that this tangent vector indeed has a vanishing \(\theta\)-component, since it is constant.

\[
l(p, \varphi; 0) = d_{\lambda} l \left( c, \frac{\pi}{2}; \lambda \right) \bigg|_{\lambda = 0} = 0, \quad \varphi(p, \varphi; \lambda) = d_{\lambda} \varphi \left( \hat{\Omega}_p, \hat{\Omega} \left( \alpha \left( c, \frac{\pi}{2}; \lambda \right), 0 \right) \right) \bigg|_{\lambda = 0} = \frac{1}{\sin c}
\]

And thus using our formula for the metric components, which we derived in the main text we get:

\[
g_{\varphi\varphi}(p) = \frac{1 - l^2(p, \varphi; 0)}{\vartheta^2(p, \varphi; 0)} = \sin^2 c
\]

Repeating the process for \(\gamma_{p, \vartheta}(\lambda)\) we get:

\[
\hat{\Omega}(\beta_1, \beta_2) \overset{!}{=} R_{x_1, x_2}^{-1}(\varphi_p) \circ R_{x_2, x_3}^{-1}(\vartheta(p)) \circ \hat{\vartheta} \Rightarrow \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{2}
\]

\[
\varphi \left( \hat{\Omega}_p, \hat{\Omega} \left( \alpha \left( c, \frac{\pi}{2}; \lambda \right), \frac{\pi}{2} \right) \right) = \tan^{-1} \left( \frac{\sqrt{1 - \cos^2 c \cos^2 \lambda}}{\sin c \cos \varphi_p \cos \lambda} \left( \frac{1 + \sin^2 c \sin^2 \varphi_p \cot^2 \lambda}{1 + \sin^2 c \cos^2 \lambda} \right) \right)
\]

\[
\vartheta \left( \hat{\Omega}_p, \hat{\Omega} \left( \alpha \left( c, \frac{\pi}{2}; \lambda \right), \frac{\pi}{2} \right) \right) = \tan^{-1} \left( \frac{\sin \vartheta_p \sin c \sin \varphi_p \cot \lambda + \cos \vartheta_p}{\cos \vartheta_p \sin c \sin \varphi_p \cot \lambda - \sin \vartheta_p} \right)
\]

And finally the components of the tangent vector \(\dot{\gamma}_{p, \vartheta}(0)\) at \(p\) are:

\[
\dot{l}(p, \vartheta; 0) = 0, \quad \varphi(p, \vartheta; 0) = 0, \quad \dot{\vartheta}(p, \vartheta; 0) = \frac{1}{\sin c \sin \varphi_p}
\]
which leads to the metric component:

\[
g_{\vartheta\vartheta}(p) = 1 - \frac{\dot{l}^2(p, \hat{\vartheta}; 0)}{\dot{\vartheta}^2(p, \hat{\vartheta}; 0)} = \sin^2 \varphi \sin^2 \varphi_p
\]

Now, that we have all components we can write down the metric in faithful normal coordinates and calculate the curvature tensors, to check whether our result for this example makes sense.

\[
g(c, \varphi, \vartheta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sin^2 c & 0 \\
0 & 0 & \sin^2 c \sin^2 \varphi_p
\end{pmatrix}
\]

The non-vanishing Christoffel symbols are:

\[
\Gamma^i_{kl} = \frac{1}{2} g^{ij} (g_{jk,l} + g_{jl,k} - g_{kl,j}); \quad \Gamma^c_{\varphi\varphi} = -\cos c \sin c, \quad \Gamma^c_{\varphi\vartheta} = -\cos c \sin c \sin^2 \varphi, \quad \Gamma^c_{\vartheta\vartheta} = -\cos \varphi \sin \varphi
\]

Using these we can calculate the components of the Riemann tensor. Again, we list the non-vanishing ones:

\[
R^c_{\varphi\vartheta\vartheta} = -R^c_{\vartheta\varphi\vartheta} = \sin^2 c, \quad R^c_{\varphi\varphi\varphi} = -R^c_{\vartheta\vartheta\varphi} = \sin^2 c \sin^2 \varphi
\]

We contract the Riemann tensor to obtain the Ricci tensor and then contract again to get the Riemann curvature scalar:

\[
R_{jl} = R^i_{jil} = 2 \begin{pmatrix}
1 & 0 & 0 \\
0 & \sin^2 c & 0 \\
0 & 0 & \sin^2 c \sin^2 \varphi
\end{pmatrix}, \quad R = R^i_i = g^{ij} R_{ij} = 6
\]

And we get the expected result for the unit 3-sphere. For comparison the general formula for the \(n\)-sphere with radius \(r\) is:

\[
R \left( \mathcal{S}^n(r) \right) = \frac{n(n - 1)}{r^2}
\]

Figure 3: We show a sample of the basis vectors we use to calculate the metric on the left and visualize how we rotate the vectors at \(p\) to \(q\), where the angular parameters \(\beta_1\) and \(\beta_2\) are defined. The \(\varphi\)-coordinate line at \(p\) is drawn in orange and the \(\vartheta\) one in blue. The geodesics heading out in the same direction as these two are drawn in orange and blue gradients respectively.
4 Constant curvature on each central plane

From the main text we know, that if we want to pick a curvature field we can pick an arbitrary point w.l.o.g. \( o \) and then choose a curvature field on every geodesic surface through \( o \) in a smooth way. We again restrict ourselves to plane geometries and choose the following source curvature field, where the curvature on each such geodesic plane is constant:

\[
K_p(\varphi_K, \vartheta_K) = \cos^2 \varphi_K \cos^2(2\vartheta_K), \quad \forall p \in \mathcal{M} \tag{46}
\]

Although these geodesic surfaces, labelled by \( \varphi_K \) and \( \vartheta_K \), appear as planes in the faithful normal coordinates at \( o \), they are 2-spheres of different radii. The field varies in different directions in the same way at each point and is thus not position but purely direction dependent.

To compare these curvature degrees of freedom with the Riemann tensor we try to calculate the metric. Due to the position independence it suffices to look at the point \( q(c) = \gamma_{o,0}(c) \). In the main text we saw that we need the geodesics heading out in the directions of increasing \( \varphi \) and \( \vartheta \) at each point. We already determined in Sec. 3 that the direction labels for \( \hat{\varphi} \) are \( \beta_1 = \frac{\pi}{2}, \beta_2 = 0 \) and the ones for \( \hat{\vartheta} \) are \( \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{2} \). But the geodesics heading out in these directions agree with the unit sphere since the curvature is given by:

\[
\varphi_p = \vartheta_p = 0, \quad \alpha_2 = \beta_2 \tag{47}
\]

\[
\Rightarrow \varphi_K(0,0,0) = 0, \quad \vartheta_K(0,0,0) = 0, \quad K_p(0,0) = 1, \quad \text{in } \hat{\varphi}\text{-direction}, \tag{48}
\]

\[
\Rightarrow \varphi_K \left(0,0,\frac{\pi}{2}\right) = 0, \quad \vartheta_K \left(0,0,\frac{\pi}{2}\right) = \frac{\pi}{2}, \quad K_p \left(0,\frac{\pi}{2}\right) = 1, \quad \text{in } \hat{\vartheta}\text{-direction.} \tag{49}
\]

using the functions \( \varphi_K(\varphi_p, \vartheta_p, \alpha_2) \) and \( \vartheta_K(\varphi_p, \vartheta_p, \alpha_2) \) we derived in the main text.

Thus we find, that the geodesic planes labelled by \( (0,0) \) and \( (0, \frac{\pi}{2}) \) have constant curvature 1, thus the geodesics in these surfaces are the ones of the unit 2-sphere and when we use their derivatives to calculate the metric components we get the metric of the unit sphere.

But if we rotate the basis by \( \frac{\pi}{4} \) we instead get:

\[
\varphi_K \left(0,0,\frac{\pi}{4}\right) = 0, \quad \vartheta_K \left(0,0,\frac{\pi}{4}\right) = \frac{\pi}{4}, \quad K_p \left(0,\frac{\pi}{4}\right) = 0, \quad \text{in } \hat{\varphi}\text{-direction,} \tag{50}
\]

\[
\varphi_K \left(0,0,-\frac{\pi}{4}\right) = 0, \quad \vartheta_K \left(0,0,-\frac{\pi}{4}\right) = -\frac{\pi}{4}, \quad K_p \left(0,-\frac{\pi}{4}\right) = 0, \quad \text{in } \hat{\vartheta}\text{-direction,} \tag{51}
\]

keeping in mind, that the domain of \( \alpha_2 \) is between \(-\frac{\pi}{2}\) and \( \frac{\pi}{2} \) and thus we have \( \alpha_2 = -\frac{\pi}{4} \) for the \( \hat{\vartheta} \)-direction instead of \( \frac{3\pi}{4} \), which both label the same plane.

With a choice of basis that lies in these two planes we get the flat metric, because these planes are indeed the only real planes through \( o \) in this geometry. We conclude that the metric and thus also the Riemann tensor is not able to pick up this degree of freedom.

In Fig. 4 we see that the flow of this geometry agrees with the sphere in the directions \( 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \) but in the other directions the blue geodesics do not cover the grey ones of the sphere since the curvature in those directions is lower and thus the geodesics are bent less.

Figure 4: A 3D plot of the geodesic flow from the origin point in green and from the point \( q = \left(\frac{\pi}{2}, 0, 0\right) \) in blue, with the flow of the 3-sphere in gray for comparison. A side view from the right-hand side is provided on the right.
5 \textit{n-dimensional flat space}

To deal with the \( n \)-dimensional case we first need the tangent vector fields \( \hat{\theta}_i \), which we gain by taking derivatives of \( \hat{\Omega}^{(n-1)} \):

\[
\hat{\Omega}^{(n-1)} = \begin{pmatrix}
\cos \varphi \\
\cos \vartheta_{n-3} \cos \vartheta_{n-3} \ldots \cos \vartheta_{1} \sin \varphi \\
\sin \vartheta_{1} \sin \varphi \\
\sin \vartheta_{2} \cos \vartheta_{1} \sin \varphi \\
\sin \vartheta_{3} \cos \vartheta_{2} \cos \vartheta_{1} \sin \varphi \\
\vdots \\
\sin \vartheta_{n-2} \cos \vartheta_{n-3} \ldots \cos \vartheta_{1} \sin \varphi
\end{pmatrix}, \quad \hat{\varphi}(p) = \hat{\Omega}^{(n-1)} = \begin{pmatrix}
-\sin \varphi \\
-\sin \vartheta_{n-3} \cos \vartheta_{n-3} \ldots \cos \vartheta_{1} \cos \varphi \\
\sin \vartheta_{1} \cos \varphi \\
\sin \vartheta_{2} \cos \vartheta_{1} \cos \varphi \\
\sin \vartheta_{3} \cos \vartheta_{2} \cos \vartheta_{1} \cos \varphi \\
\vdots \\
\sin \vartheta_{n-2} \cos \vartheta_{n-3} \ldots \cos \vartheta_{1} \cos \varphi
\end{pmatrix}
\]

\[
\hat{\vartheta}_{i}(p) = \hat{\Omega}^{(n-1)} = \begin{pmatrix}
0 \\
-\cos \vartheta_{n-2} \cos \vartheta_{n-3} \ldots \cos \vartheta_{1} \sin \varphi \\
\cos \vartheta_{1} \sin \varphi \\
-\sin \vartheta_{2} \sin \vartheta_{1} \sin \varphi \\
-\sin \vartheta_{3} \cos \vartheta_{2} \sin \vartheta_{1} \sin \varphi \\
\vdots \\
-\sin \vartheta_{n-2} \cos \vartheta_{n-3} \ldots \cos \vartheta_{1} \sin \varphi
\end{pmatrix}, \quad \hat{\vartheta}_{i}(p) = \hat{\Omega}^{(n-1)} = \begin{pmatrix}
0 \\
-\cos \vartheta_{n-2} \ldots \sin \vartheta_{i} \ldots \cos \vartheta_{1} \sin \varphi \\
\sin \vartheta_{1} \sin \varphi \\
\vdots \\
\cos \vartheta_{i} \ldots \cos \vartheta_{1} \sin \varphi \\
-\sin \vartheta_{n-2} \ldots \sin \vartheta_{i} \ldots \cos \vartheta_{1} \sin \varphi
\end{pmatrix}
\]

and then normalize them:

\[
\hat{\varphi} \cdot \hat{\varphi} = \sin^{2} \varphi + \cos^{2} \vartheta_{n-2} \cos^{2} \vartheta_{n-3} \ldots \cos^{2} \vartheta_{1} \sin^{2} \varphi + \sin^{2} \vartheta_{1} \cos^{2} \varphi + \sin^{2} \vartheta_{2} \cos^{2} \vartheta_{1} \cos \varphi \\
+ \sin^{2} \vartheta_{3} \cos^{2} \vartheta_{2} \cos^{2} \vartheta_{1} \cos^{2} \varphi + \ldots + \sin^{2} \vartheta_{n-2} \cos \vartheta_{n-3} \ldots \cos \vartheta_{1} \cos^{2} \varphi \\
= \sin^{2} \varphi + (\cos^{2} \vartheta_{n-2} + \sin^{2} \vartheta_{n-2}) \cos^{2} \vartheta_{n-3} \ldots \cos^{2} \vartheta_{1} \sin^{2} \varphi + \sin^{2} \vartheta_{1} \cos^{2} \varphi + \sin^{2} \vartheta_{2} \cos^{2} \vartheta_{1} \cos \varphi \\
+ \sin^{2} \vartheta_{3} \cos^{2} \vartheta_{2} \cos^{2} \vartheta_{1} \cos^{2} \varphi + \ldots + \sin^{2} \vartheta_{n-3} \cos^{2} \vartheta_{n-4} \ldots \cos^{2} \vartheta_{1} \cos^{2} \varphi \\
= \sin^{2} \varphi + (\cos^{2} \vartheta_{n-3} + \sin^{2} \vartheta_{n-3}) \cos^{2} \vartheta_{n-4} \ldots \cos^{2} \vartheta_{1} \sin^{2} \varphi + \sin^{2} \vartheta_{1} \cos^{2} \varphi + \sin^{2} \vartheta_{2} \cos^{2} \vartheta_{1} \cos \varphi \\
+ \sin^{2} \vartheta_{3} \cos^{2} \vartheta_{2} \cos^{2} \vartheta_{1} \cos^{2} \varphi + \ldots + \sin^{2} \vartheta_{n-4} \cos^{2} \vartheta_{n-5} \ldots \cos^{2} \vartheta_{1} \cos^{2} \varphi \\
\vdots \\
= \sin^{2} \varphi + (\cos^{2} \vartheta_{3} + \sin^{2} \vartheta_{3}) \cos^{2} \vartheta_{2} \cos^{2} \vartheta_{1} \sin^{2} \varphi + \sin^{2} \vartheta_{1} \cos^{2} \varphi + \sin^{2} \vartheta_{2} \cos^{2} \vartheta_{1} \cos \varphi \\
= \sin^{2} \varphi + (\cos^{2} \vartheta_{2} + \sin^{2} \vartheta_{2}) \cos^{2} \vartheta_{1} \cos^{2} \varphi + \sin^{2} \vartheta_{1} \cos^{2} \varphi \\
= \sin^{2} \varphi + (\cos^{2} \vartheta_{1} + \sin^{2} \vartheta_{1}) \cos^{2} \varphi = \sin^{2} \varphi + \cos^{2} \varphi = 1,
\]

\[
\hat{\vartheta}_{i} \cdot \hat{\vartheta}_{1} = 0 + \cos^{2} \vartheta_{n-2} \cos^{2} \vartheta_{n-3} \ldots \cos^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi + \cos^{2} \vartheta_{1} \sin^{2} \varphi + \sin^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi \\
+ \sin^{2} \vartheta_{3} \cos^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi + \ldots + \sin^{2} \vartheta_{n-2} \cos^{2} \vartheta_{n-3} \ldots \cos^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi \\
= (\cos^{2} \vartheta_{n-2} + \sin^{2} \vartheta_{n-2}) \cos^{2} \vartheta_{n-3} \ldots \cos^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi + \cos^{2} \vartheta_{1} \sin^{2} \varphi + \sin^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi \\
+ \sin^{2} \vartheta_{3} \cos^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi + \ldots + \sin^{2} \vartheta_{n-3} \ldots \cos^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi \\
\vdots \\
= (\cos^{2} \vartheta_{3} + \sin^{2} \vartheta_{3}) \cos^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi + \cos^{2} \vartheta_{1} \sin^{2} \varphi + \sin^{2} \vartheta_{2} \sin^{2} \vartheta_{1} \sin^{2} \varphi \\
= (\cos^{2} \vartheta_{2} + \sin^{2} \vartheta_{2}) \sin^{2} \vartheta_{1} \sin^{2} \varphi + \cos^{2} \vartheta_{1} \sin^{2} \varphi = (\sin^{2} \vartheta_{1} + \cos^{2} \vartheta_{1}) \sin^{2} \varphi = \sin^{2} \varphi
\]

\[
\Rightarrow \hat{\vartheta}_{1} = \frac{1}{\sin \varphi} \hat{\vartheta}_{i},
\]

(56)
\[\vec{\theta}_i \cdot \vec{\varphi}_i = \cos^2 \vartheta_{n-2} \ldots \sin^2 \vartheta_i \cdots \cos^2 \vartheta_1 \sin^2 \varphi + \sin^2 \vartheta_1 \sin^2 \varphi + \ldots + \cos^2 \vartheta_1 \cdots \cos^2 \vartheta_1 \sin^2 \varphi + \ldots + \sin^2 \vartheta_1 \sin^2 \varphi + \sin^2 \vartheta_1 \sin^2 \varphi + \ldots + \cos^2 \vartheta_i \sin^2 \varphi + \ldots + \sin^2 \vartheta_i \sin^2 \varphi + \ldots + \cos^2 \vartheta_i \sin^2 \varphi + \ldots + \sin^2 \vartheta_{n-2} \sin^2 \varphi = 1\]

\[\dot{\vartheta}(p, \hat{\varphi}; \lambda) = d\lambda \vartheta(p, \hat{\varphi}; \lambda) \bigg|_{\lambda=0} = 1\]

The distance function for a flat space is given by:

\[d_{\lambda} \varphi \bigg|_{\lambda=0} = \frac{1}{c} \quad \dot{\varphi}(p, \hat{\varphi}; \lambda) = d\lambda \vartheta(p, \hat{\varphi}; \lambda) \bigg|_{\lambda=0} = \frac{1}{c \sin \varphi_p}\]
We get the metric:

\[
g = \begin{pmatrix}
1 & 0 & 0 \\
0 & c^2 & 0 \\
0 & 0 & c^2 \sin^2 \varphi
\end{pmatrix},
\]

(65)

the non-vanishing Christoffel symbols are:

\[\Gamma^c_{\varphi\varphi} = -c, \quad \Gamma^c_{\vartheta\vartheta} = -c \sin^2 \varphi, \quad \Gamma^\varphi_{\vartheta\vartheta} = \frac{1}{c}, \quad \Gamma^\vartheta_{\varphi\vartheta} = -\cos \varphi \sin \varphi, \quad \Gamma^\vartheta_{c\vartheta} = \frac{1}{c}, \quad \Gamma^d_{\varphi\vartheta} = \cot \varphi\]

(66)

The Riemann tensor vanishes, as expected for flat space.