Power law deformation of Wishart–Laguerre ensembles of random matrices

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Abstract. We introduce a one-parameter deformation of the Wishart–Laguerre or chiral ensembles of positive definite random matrices with Dyson index $\beta = 1, 2$ and 4. Our generalized model has a fat-tailed distribution while preserving the invariance under orthogonal, unitary or symplectic transformations. The spectral properties are derived analytically for finite matrix size $N \times M$ for all three values of $\beta$, in terms of the orthogonal polynomials of the standard Wishart–Laguerre ensembles. For large $N$ in a certain double-scaling limit we obtain a generalized Marčenko–Pastur distribution on the macroscopic scale, and a generalized Bessel law at the hard edge which is shown to be universal. Both macroscopic and microscopic correlations exhibit power law tails, where the microscopic limit depends on $\beta$ and the difference $M - N$. In the limit where our parameter governing the power law goes to infinity we recover the correlations of the Wishart–Laguerre ensembles. To illustrate these findings, the generalized Marčenko–Pastur distribution is shown to be in very good agreement with empirical data from financial covariance matrices.

Keywords: matrix models, models of financial markets

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Dedicated to Oriol Bohigas on the occasion of his 70th birthday
1. Introduction

Ensembles of matrices with random entries have been introduced in the pioneering works by Wigner and Dyson in the early 1950s, in an attempt to provide a statistical description of energy levels of heavy nuclei based on few and minimal symmetry requirements. Today’s applications cover all areas in physics, and we refer the reader to [1] for a review. However, many years before these works of Wigner and Dyson, John Wishart had already introduced a random matrix ensemble in his studies on samples from a multivariate population. This ensemble of random matrices can be obtained as follows. Take a rectangular matrix $X$ of size $M \times N$ ($M > N$), whose entries are independent Gaussian variables with mean zero and variance one in the real, complex or quaternion domain, labelled with the Dyson index: $\beta = 1, 2$ and $4$, respectively. Then form the (non-normalized) covariance matrix of size $N \times N$, $W = X^\dagger X$, which is positive definite. By comparison to such matrices one can establish which part of the spectrum of empirical covariance matrices carries genuine information and which should be discarded as contaminated by pure noise.

The ensemble of such matrices $W$ is known in random matrix theory (RMT) as the Wishart, Laguerre or chiral ensemble and will be denoted by WL in the following. It has since appeared in many different contexts, such as multivariate statistical data analysis [2], analysis of the capacity of channels with multiple antennae and receivers [3], low energy quantum chromodynamics and other gauge theories [4,5], quantum gravity [6,7], knowledge networks [8], finance [9] and also in statistical physics problems, such as a class of $(1+1)$-dimensional directed polymer problems [10]. Very recent analytical results
include statistics of large deviations for the maximum eigenvalue [11] and distributions related to entangled random pure states [12].

Because of their invariance, the WL ensembles can be written in terms of eigenvalues and completely solved, for any finite $N$ and $M$, as well as in the large $N$ limit. We only give a brief account of what is known; more details follow in the main part later.

The WL ensemble can be exactly solved through the standard orthogonal polynomial technique [13]. For a Gaussian weight functions all spectral correlation functions can be expressed at finite $N$ in terms of orthogonal ($\beta = 2$) or skew orthogonal ($\beta = 1, 4$) Laguerre polynomials; see [13] for a review. In the large $N$ limit one has to distinguish between smooth macroscopic and oscillatory microscopic correlations on the level of the mean spacing between eigenvalues. The most well known macroscopic result is the spectral density of Marčenko and Pastur (MP) [14], generalizing the semicircle in the Wigner–Dyson class (WD). For the microscopic domain much effort has been spent on the so-called hard edge at the origin for the positive definite eigenvalues, which is absent in the WD class. It is governed by the Bessel law [15, 4], [16]–[18] labelled with $\beta$ and $M - N$ being finite. The robustness or universality under polynomial deformations of the Gaussian weight function has also been proven [19, 20]. This perturbation destroys the independence of the uncorrelated degrees of freedom of $X$, without changing the microscopic correlation functions. Connected macroscopic density correlation functions also remain universal under such deformations [6].

What is the motivation to further generalize WL ensembles? In the finance and risk management domain, the empirical covariance of a set of $N$ assets over a temporal window of size $M$ has been under scrutiny for some time [9], and its eigenvalues were shown to be distributed in reasonably good agreement with the MP law, as if they were originated from a completely uncorrelated data series. However, the same analyses repeated by several groups [21]–[23] on different data sets have shown that either a part of the spectrum corresponding to extremely low eigenvalues—the most interesting for portfolio selections—or the fat tails are not reproduced by this crude approach. This has led to the appearance of more sophisticated models [23]–[25], e.g. the multivariate Student distribution where the variance of each matrix entry becomes a random variable. Another deformation of the WL ensemble has been introduced in [26] where sparse matrices $X$ were considered. This setting has many applications in communication theory and in complex networks (namely in the study of spectral properties of adjacency matrices). All of these generalizations lead to a deformation of the MP law, and thus lie outside the WL class. However, the lack of invariance in such models generically spoils the complete solvability for all correlation functions.

Conversely, a model with power law tails which is exactly solvable in principle appeared in [25]. However, the analysis in [25] was restricted to the macroscopic spectral density, whereas many more interesting results and ultimately a complete solution of the model can be obtained. This goal is achieved by exploiting techniques and results introduced in previous works, where the WD class was generalized using non-standard entropy maximization [27, 28] and super-statistical approaches [29, 30].

We will follow these lines and provide a complete solution for all three $\beta$ values of a generalized WL model with rotational invariance, with an emphasis on the issues of universality and complete integrability for all spectral correlations, both macroscopic and microscopic. After a proper rescaling and normalization, our $N$-independent correlations...
depend only on a single parameter $\alpha$, given by the power law. In the limit of a large parameter $\alpha \gg 1$ we recover all standard WL correlations. For completeness, we also mention another generalization of WL which exploits a different direction. In [31] the unitary WL were generalized to display critical statistics.

Our paper is organized as follows. In section 2 we define our generalized WL ensembles as a one-parameter deformation, including a general non-Gaussian potential $V$ for all three $\beta = 1, 2$ and 4. The general solution for finite $N$ is given, applying the method of (skew) orthogonal polynomials to an integral transform of the standard WL ensembles.

In section 3 we take the macroscopic large $N$ limit for the special case of a Gaussian potential. Here macroscopic refers to the smooth part of the spectrum, considering correlations on a distance large compared to the mean level spacing. We re-derive a generalization of the semicircle and the MP spectral density; see [27]–[29] and [25] respectively. As a new result we compute the average position of an eigenvalue and the position of a pseudo-edge. The former leads to the correct scaling with $N$ and an $N$-independent generalized spectral density displaying a power law decay.

Section 4 is devoted to the microscopic large $N$ limit, where correlations of the order of the mean level spacing are computed. Here we can use the known universal WL results as an input to our model. In the case when the difference of the matrix dimension $M - N$ is kept finite, the spectrum has a hard edge at the origin. In two subsections we compute its microscopic density there, generalizing the universal Bessel law as well as the corresponding smallest eigenvalue distribution for all three $\beta$ values.

In section 5 the nearest neighbour spacing distribution in the bulk of the spectrum is computed using a Wigner surmise at $N = 2$ for our generalized model.

Our conclusions and outlook are presented in section 6, including a comparison of the generalized MP density to data from financial correlation matrices. In two appendices technical details are collected.

2. Definition of the model and the finite $N$ solution for a general potential

The joint probability density of our generalized Wishart–Laguerre ensembles is defined as follows in terms of matrix elements:

$$P_\gamma[X] \, dX \propto \left(1 + \frac{n\beta}{\gamma} \text{Tr} V(X^\dagger X)\right)^{-\gamma} \, dX,$$

where $X$ is a matrix of size $M \times N$ with real, complex and quaternion real elements for the values $\beta = 1, 2$ and 4, respectively. We define $M = N + \nu$ for later convenience, where $\nu \geq 0$ may be either finite or of order $O(N)$ in the large $N$ limit. The integration measure $dX$ is defined by integrating over all independent matrix elements of $X$ with a flat measure. Expectation values of an operator $O$ (denoted by $\langle O \rangle_\gamma$) are defined with respect to $P_\gamma$ in the usual sense.

The real positive parameter $\gamma$ is to be specified below, and we keep an additional variance-like parameter $n > 0$. The so-called potential $V$ is taken to be a polynomial of finite degree $d$, although some of the universal results that we inherit from the WL ensembles are known to hold for a much larger class of functions.
The measure (2.1) is well defined and integrable only if the following condition holds:

$$\gamma d > \frac{\beta}{2} N(N + \nu).$$

This condition, which is derived in appendix A, can be seen when changing to radial coordinates for the matrix $X^\dagger X$; see e.g. [32,28]. In particular, the large $N$ behaviour of any spectral property cannot be taken for fixed $\gamma$, and a prescription as regards the way both quantities should approach infinity needs to be given, respecting the inequality equation (2.2).

A similar model was introduced earlier generalizing the (non-chiral) Wigner–Dyson ensembles for a Gaussian potential [27,29,28], and a similar interplay between the deformation parameter $\gamma$ and the matrix size $N$ was observed.

In the limit of an infinite deformation parameter

$$\lim_{\gamma \to \infty} \mathcal{P}_\gamma[X] = \exp\left[-n\beta \text{Tr} V(X^\dagger X)\right] \equiv \mathcal{P}[X]$$

we recover the standard Wishart–Laguerre or chiral ensembles denoted by $\mathcal{P}[X]$. Typically, in these ensembles one chooses $n \sim N$ when taking the large $N$ limit to obtain an $N$-independent macroscopic density. In our case, this limit will be more involved, and we will keep $n$ general for the time being.

The generalized ensembles can be related to the standard WL through the following integral representation:

$$(1 + z)^{-\gamma} = \int_0^{\infty} d\xi \ e^{-\xi} \xi^{-1} \ e^{-\xi z}.$$ 

Inserting this into the definition equation (2.1) we obtain

$$\mathcal{P}_\gamma[X] = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} d\xi \ e^{-\xi} \xi^{-1} \exp\left[-\xi \frac{n\beta}{\gamma} \text{Tr} V(X^\dagger X)\right].$$

This relation is the key starting point for solving our generalized model both for finite and for large $N$. The same trick was used for the generalization of the Gaussian Wigner–Dyson ensembles introduced previously in [27]–[29]. In fact, a similar technique was employed much earlier in [32] when solving the fixed and restricted trace ensembles by writing them as integral transforms of the Wigner–Dyson ensembles.

An advantage of our model over some other generalizations of WL [25,23] is its invariance under orthogonal, unitary or symplectic transformations. In particular, for $\beta = 1$ Burda et al [25] considered a very general family of probability distributions of the form $\mathcal{P}_f[X] \sim f(\text{Tr} X^\dagger C^{-1} X A^{-1})$ where $C$ and $A$ represent the true correlation and autocorrelation matrices respectively, and $f$ is a non-negative and normalized weight function. Only in the special case $C = A = 1$ is invariance recovered. This approach has been modified in [23] to allow for a time-dependent random volatility.

From equation (2.5) (or equation (2.1)) we can immediately go to an eigenvalue basis of the positive definite matrix $X^\dagger X$, to obtain the following joint probability distribution.
cases, for which we refer the reader to [13] for more details.

It is expressed through the jpdf of the standard WL:

\[ P(\lambda_1, \ldots, \lambda_N; \xi) \equiv \prod_{i=1}^{N} \lambda_{i}^{(1/2)(\nu+1)-1} \prod_{j>k}^{N} |\lambda_j - \lambda_k|^\beta, \quad (2.7) \]

depending on \( \xi \) through its weight \( -\xi (n\beta/\gamma)V(\lambda_i) \). In both jpdf’s we have suppressed the constant from the integration over the angular degrees of freedom.

For completeness we also define the corresponding partition function

\[ Z(\xi) \equiv \int_0^\infty \prod_{i=1}^{N} d\lambda_i \left( 1 + \frac{n\beta}{\gamma} \sum_{i=1}^{N} V(\lambda_i) \right)^{-\gamma} \prod_{i=1}^{N} \lambda_i^{(1/2)(\nu+1)-1} \prod_{j>k}^{N} |\lambda_j - \lambda_k|^\beta \]

which is again an integral over the standard, \( \xi \)-dependent WL partition function

\[ Z(\xi) \equiv \int_0^\infty \prod_{i=1}^{N} d\lambda_i \lambda_i^{(3/2)(\nu+1)-1} \exp \left[-\xi \frac{n\beta}{\gamma} V(\lambda_i) \right] \prod_{j>k}^{N} |\lambda_j - \lambda_k|^\beta. \quad (2.9) \]

Because of this linear relation between the generalized and standard ensembles we can immediately express all \( k \)-point eigenvalue density correlation functions, denoted by \( R \) for finite \( N \), in terms of each other. They are defined in the usual way [13]:

\[ R_\gamma(\lambda_1, \ldots, \lambda_k) \equiv \frac{N!}{(N-k)!} \zeta_\gamma \int_0^\infty d\lambda_{k+1} \cdots d\lambda_N P_\gamma(\lambda_1, \ldots, \lambda_N) \]

\[ = \int_0^\infty d\xi \ e^{-\xi} \xi^{\gamma-1} \frac{Z(\xi)}{\Gamma(\gamma)} R(\lambda_1, \ldots, \lambda_k; \xi), \quad (2.11) \]

where the \( k \)-point correlation functions of the standard ensembles depend on \( \xi \) through the exponent in the measure

\[ R(\lambda_1, \ldots, \lambda_k; \xi) \equiv \frac{(N-k)!}{k!} \zeta(\xi) \int_0^\infty d\lambda_{k+1} \cdots d\lambda_N P(\lambda_1, \ldots, \lambda_N; \xi). \quad (2.12) \]

The latter can be solved using the method of (skew) orthogonal polynomials [13], expressing them through the determinant of the kernel of the orthogonal polynomials for \( \beta = 2 \), or the Pfaffian of the matrix kernel of skew orthogonal polynomials for \( \beta = 1 \) and 4. We only detail the simpler \( \beta = 2 \) case here and briefly outline the \( \beta = 1 \) and 4 cases, for which we refer the reader to [13] for more details.
Let us define monic orthogonal polynomials and their norms for $\beta = 2$ as follows:
\[
\int_0^\infty d\lambda \lambda^n \exp \left[ -\xi \frac{2n}{\gamma} V(\lambda) \right] P_k(\lambda)P_l(\lambda) = h_k\delta_{kl}.
\] (2.13)

Introducing their kernel
\[
K_N(\lambda, \mu) \equiv (\lambda \mu)^{\nu/2} e^{-\xi(\eta/\gamma)(V(\lambda)+V(\mu))} \sum_{k=0}^{N-1} h_k^{-1} P_k(\lambda)P_k(\mu),
\] (2.14)

and applying the Christoffel–Darboux identity for $\lambda \neq \mu$
\[
\sum_{k=0}^{N-1} h_k^{-1} P_k(\lambda)P_k(\mu) = h_{N-1}^{-1} \frac{P_N(\lambda)P_{N-1}(\mu) - P_N(\mu)P_{N-1}(\lambda)}{\lambda - \mu},
\] (2.15)

we can express all eigenvalue correlations of the standard ensemble through this kernel [13],
\[
R(\lambda_1, \ldots, \lambda_k; \xi) = \det_{1 \leq i,j \leq k} [K_N(\lambda_i, \lambda_j)].
\] (2.16)

We thus arrive at
\[
R_\gamma(\lambda_1, \ldots, \lambda_k) = \int_0^\infty d\xi \ e^{-\xi} \xi^{\gamma-1} \frac{Z(\xi)}{\Gamma(\gamma)Z_\gamma} \det_{1 \leq i,j \leq k} [K_N(\lambda_i, \lambda_j)],
\] (2.17)

which is the main result of this section. The simplest example is the spectral density
\[
R_\gamma(\lambda) \text{ given by the integral over the single kernel } K_N(\lambda, \lambda). \text{ Notice that it is normalized to } N = \int_0^\infty d\lambda R_\gamma(\lambda).
\]

In order to take $N$ large we only need to know the asymptotic of the polynomials
\[
P_N(\lambda), \text{ take a finite determinant of size } k \text{ of the asymptotic kernel and integrate once with respect to } \xi.
\]

As pointed out already, the same result, equation (2.17), holds for $\beta = 1$ and 4 when replacing the determinant by a Pfaffian, $\text{Pf}[K_N(\lambda_i, \lambda_j)]$, where $\kappa_N$ is a $2 \times 2$ matrix kernel. For a Gaussian weight its skew orthogonal polynomials are explicitly known as well in terms of Laguerre polynomials [17,18].

For completeness, we also give the partition function occurring inside the integrand in equation (2.17) in terms of the norms $h_i$ of orthogonal polynomials:
\[
Z(\xi) = N! \prod_{i=0}^{N-1} h_i = N! h_0^N \prod_{i=0}^{N-1} r_i^{N-i},
\] (2.18)

or their ratios $r_i \equiv h_{i+1}/h_i$. An identical result holds for $\beta = 1$ and 4 in terms of the skew orthogonal norms [13].

In the Gaussian case the ratio of partition functions $Z(\xi)/\Gamma(\gamma) Z_\gamma$ in equation (2.17) can be obtained most explicitly at finite $N$ for all three values of $\beta$ by changing to radial coordinates; see appendix A for a derivation.

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3. The macroscopic large $N$ limit for a Gaussian potential

In this section, we will restrict consideration to the Gaussian potential $V(\lambda) = \lambda$, deriving a generalization of the MP spectral density of WL for all three $\beta$. It exhibits a non-compact support and power law behaviour for large arguments, and we will distinguish two cases. In the first subsection, we will deal with matrices that become asymptotically quadratic with $M - N = \nu = \mathcal{O}(1)$. In the standard WL ensembles, at large $N$ the spectral support is an interval on the positive semi-axis, where the origin represents a hard edge. In the second subsection, we take the limit $\lim_{M,N \to \infty} N/M \equiv c$, (3.1)

with $c < 1$, corresponding to the case $M - N = \nu = \mathcal{O}(N)$. In the WL ensembles, the resulting MP macroscopic spectral density takes support on a positive interval. Conversely, in both cases $c = 1$ and $c < 1$ our generalized macroscopic density will have support on the full positive real semi-axis.

3.1. The generalized semicircle for $c = 1$

The finite $N$ density for the WL ensembles $R(\lambda; \xi)$ is well known in terms of Laguerre polynomials; see equation (B.7) for $\beta = 2$ in the appendix B. This results in an explicit integral representation for the spectral density of our generalized ensembles; see equation (B.10). Despite this result, it is rather difficult to extract information about the macroscopic large $N$ limit from those analytical formulae, both for the standard and for the generalized WL ensembles.

Hence, we follow here an alternative route, already exploited in [27]–[29]: we directly insert the large $N$ result into equation (2.11). The $N \gg 1$ asymptotic of the WL density is known and is given by the Marčenko–Pastur law, which is in fact the semicircle law in squared variables at $c = 1$:

$$
\lim_{N \gg 1} R(\lambda) = \frac{n}{\pi} \sqrt{\frac{2N}{n\lambda} - 1} \quad \text{with } \lambda \in (0, 2N/n].
$$

(3.2)

It is given here for the Gaussian weight $\exp[-n\beta \lambda]$ for all three $\beta$, and can be derived easily using the Coulomb gas approach and saddle point method (see also [15]).

In order to obtain an $N$-independent macroscopic density we have to rescale the argument of the density by the mean eigenvalue position, $\lambda \to \langle \lambda \rangle x$, and divide by $N$ to normalize the density to unity. The mean position of an eigenvalue or first moment, $\langle \lambda \rangle$, can be computed in the generalized model for both finite $N$ and $\nu$ (see appendix A),

$$
\langle \lambda \rangle \equiv \frac{\int_0^\infty d\lambda \lambda R_\gamma(\lambda)}{\int_0^\infty d\lambda R_\gamma(\lambda)} = \frac{1}{N} \langle \text{Tr}(X^\dagger X) \rangle_\gamma = \gamma(N + \nu) \quad \text{with } \gamma = \frac{2n(\gamma - (\beta/2)N(N + \nu) - 1)}{(N + \nu)}. \tag{3.3}
$$

(3.3)

We will comment on the existence of this first moment later, and we will also need this equation again when we consider $c < 1$. It correctly reproduces the known result for WL in the limit $\lim_{\gamma \to \infty} \langle \lambda \rangle \gamma = \langle \lambda \rangle = (N + \nu)/2n$. For the WL case we thus obtain the
following known $N$- and $\beta$-independent macroscopic density from equation (3.2):

$$
\rho(x) \equiv \lim_{N \to \infty} \frac{1}{N} \langle \lambda \rangle R(x|\lambda) = \frac{1}{2\pi} \sqrt{\frac{1}{x} - 1} \quad \text{with } x \in (0, 4].
$$

It is normalized to unity and has mean $\langle x \rangle = 1$.

We can now repeat the same steps for our generalized model, where we will need equation (3.2) for the weight $\exp[-\xi n \beta \lambda/\gamma]$ (see equation (2.7)). Because of the rescaling with respect to $\langle \lambda \rangle_\gamma$ we now have to specify the $N$ dependence of $\gamma$. We keep the following combination fixed:

$$
\alpha \equiv \lim_{N, \gamma \to \infty} \left[ \gamma - \frac{\beta}{2} N(N + \nu) - 1 \right], \quad \text{with } \alpha > 0 \text{ finite.}
$$

This satisfies the constraint (2.2). We thus obtain for the generalized macroscopic density

$$
\rho_\alpha(x) \equiv \lim_{N, \gamma \to \infty} \frac{1}{N} \langle \lambda \rangle_\gamma R_\gamma(x|\lambda_\gamma) = \lim_{N, \gamma \to \infty} \frac{1}{N} \langle \lambda \rangle_\gamma \int I \, d\xi \, e^{-\xi} \xi^{\gamma - 1} \frac{Z(\xi)}{\Gamma(\gamma) Z_\gamma \gamma \pi} \sqrt{\frac{2N\gamma}{n\xi x(\lambda)_\gamma}} - 1,
$$

where the integration is restricted to the interval $I = (0, (2N\gamma/n\langle \lambda \rangle_\gamma)] = (0, 4\alpha/x]$. In order to compute the integral we still need the following quantity inside the integrand:

$$
\frac{Z(\xi)}{\Gamma(\gamma) Z_\gamma} = \frac{\xi^{-(\beta/2)N(N+\nu)}}{\Gamma(\gamma - (\beta/2)N(N + \nu))},
$$

the ratio of partition functions given here for finite $N$. This result is derived in appendix A; see equation (A.5). Inserting all ingredients into equation (3.6) and taking limits with the definition (3.5) we arrive at the following result after changing variables:

$$
\rho_\alpha(x) = \frac{1}{2\alpha \pi \Gamma(\alpha + 1)} \left( \frac{4\alpha}{x} \right)^{\alpha + 2} \int_0^1 dt \, \exp \left[ \frac{-4\alpha}{x} t \right] t^{\alpha + 1} \sqrt{\frac{1}{t} - 1} = \frac{\Gamma(\alpha + 3/2)}{4\alpha \sqrt{\pi} \Gamma(\alpha + 1) \Gamma(\alpha + 3)} \left( \frac{4\alpha}{x} \right)^{\alpha + 2} 1_{\text{F}_1} \left( \alpha + \frac{3}{2}, \alpha + 3; -\frac{4\alpha}{x} \right). \quad (3.8)
$$

Here we have introduced the confluent or Kummer hypergeometric function

$$
1_{\text{F}_1}(a; b; z) = \frac{\Gamma(b)}{\Gamma(b - a) \Gamma(a)} \int_0^1 dt \, e^{zt} t^{a - 1} (1 - t)^{b - a - 1}. \quad (3.9)
$$

Equation (3.8) is the main result of this subsection, the macroscopic spectral density of our generalized WL. It has an unbounded support $(0, \infty)$, and the density as well as its first moment are normalized to unity:

$$
\int_0^\infty dx \rho_\alpha(x) = 1 = \int_0^\infty dx \rho_\alpha(x). \quad (3.10)
$$

Note that due to this normalization, the parameter $n$ has completely dropped out. We are left with a one-parameter class of densities depicted in figure 1, which approach the WL density for $\alpha \to \infty$ as discussed below.
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Figure 1. The macroscopic generalized density of equation (3.8), $\rho_\alpha(x)$, shown on the positive real line $\mathbb{R}_+$ for $\alpha = -0.5, 0.1, 0.5$ and $14$ in light blue, blue, red and green, respectively (left), and its map $\vartheta_\alpha(y) = |y|\rho_\alpha(y^2)$ to the full real line $\mathbb{R}$ (right). Note that the MP or semicircle density given in black for comparison has compact support on $(0, 4]$ and $[-2, 2]$ respectively.

From the expansion for small arguments $\text{I}_F(a, b; z) = 1 + (a/b)z + \cdots$ we can immediately read off the power law decay of our new density equation (3.8):

$$\lim_{x \to \infty} \rho_\alpha(x) = x^{-(\alpha+2)} \frac{\Gamma(\alpha + 3/2)}{4\alpha \sqrt{\pi} \Gamma(\alpha + 1) \Gamma(\alpha + 3)} (4\alpha)^{\alpha+2} (1 + \mathcal{O}(1/x)). \quad (3.11)$$

Because of $\alpha > 0$ the decay is always faster than quadratic. However, if we drop the requirement for the existence of the first moment we can allow for $\alpha$ to take values $-1 < \alpha < 0$ while satisfying the constraint equation (2.2). Keeping the same formal rescaling in equation (3.6) we arrive at the same result, equation (3.8), now valid for $-1 < \alpha$ and $\alpha \neq 0$. Our density can thus describe power laws in between linear and quadratic decay as well; see figure 1. The same feature could be incorporated in the generalized Gaussian Wigner–Dyson model [27,29,28].

As a check we can take the limit $\alpha \to \infty$ on our final result equation (3.8). This amounts to decoupling the dependences on $\gamma$ and $N$, and thus we expect to recover the MP density at $\gamma = \infty$. By taking a saddle point approximation, we find that this is indeed the case, $\lim_{\alpha \to \infty} \rho_\alpha(x) = \rho(x)$. Hence, the density $\rho_\alpha(x)$ is a well-behaved deformation of the MP density for $c = 1$.

We can also derive the behaviour of the density $\rho_\alpha(x)$ close to the origin. Using the large argument asymptotic for the confluent hypergeometric function at negative argument, $\lim_{z \to \infty} \text{I}_F(a, b; -z) = z^{-a} \Gamma(b)/\Gamma(b - a)(1 + \mathcal{O}(1/z))$, we find that

$$\lim_{x \to 0} \rho_\alpha(x) = x^{-1/2} \frac{\Gamma(\alpha + 3/2)}{\pi^{1/2} \Gamma(\alpha + 1) \sqrt{\alpha}} (1 + \mathcal{O}(x)). \quad (3.12)$$

Therefore all our generalized densities have a square root singularity at the origin, just like the MP density.

To illustrate our findings we first plot equation (3.8) in figure 1 (left) for different values of $\alpha$, and compare it to the semicircle density equation (3.4). In order to visualize

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1 We use the obvious notation $\alpha^{\alpha+1} = \exp((\alpha + 1) \ln |\alpha|)$. 

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the square root singularity for all $\alpha$ we map the density from the positive to the full real axis by defining

$$\vartheta_\alpha(y) \equiv |y|\rho_\alpha(y^2), \quad (3.13)$$

that is to a normalized density on $\mathbb{R}$, $\int_{-\infty}^{\infty} dy \vartheta_\alpha(y) = 1$. In this form it equals the deformed semicircle law derived from generalizing the Gaussian Wigner–Dyson ensembles in [27, 29, 28], where we have eliminated all irrelevant parameters.

The same map, equation (3.13), takes the MP density, equation (3.4), to the semicircle, as mentioned already several times:

$$\vartheta(y) = \frac{1}{2\pi} \sqrt{4 - y^2}. \quad (3.14)$$

Both densities are shown in figure 1 (right). For $\alpha = 14$ it already approximates the semicircle very well, whereas for $\alpha \to 0$ the generalized density rapidly approximates a $\delta$ function. For negative $-1 < \alpha < 0$ the height of the maximum in figure 1 left goes down again; for $\alpha = -0.5$ the curve is even below the semicircle.

As a final point of this subsection let us discuss the issue of macroscopic universality of our generalized density, equation (3.8). This is a direct consequence of the (non-)universality of the MP or semicircular density, due to the linear relationship equation (2.11).

The semicircle possesses a certain weak universality in the sense that it is the same for all three Gaussian ensembles at $\beta = 1, 2, 4$, as well as for independently distributed random variables, as was shown already by Wigner. Consequently our generalized density is universal in this weak sense, too.

On the other hand the invariant deformation of the Gaussian potential by a polynomial in the definition of our model is clearly non-universal, as the semicircle becomes a polynomial times one or several square root cuts, and we refer the reader to [6] for details at $\beta = 2$. Remarkably, it was found in [6] for $\beta = 2$ that all macroscopic connected density correlation functions with two or more $k$-points are universal under such perturbations $V$, depending only on a finite number of parameters for any degree $d$. Does this universality persist in our model? The answer is found to be no, simply by looking at the definition of the connected two-point density:

$$R_\gamma^{\text{conn}}(\lambda, \mu) \equiv R_\gamma(\lambda, \mu) - R_\gamma(\lambda)R_\gamma(\mu). \quad (3.15)$$

We use the same definition for the standard WL density. It no longer relates linearly to the corresponding connected WL two-point density, as we would subtract an integral of the product of two one-point densities, instead of the product of two integrated one-point densities:

$$R_\gamma^{\text{conn}}(\lambda, \mu) = \int_0^\infty d\xi \ e^{-\xi} \xi^{\gamma-1} \frac{Z(\xi)}{\Gamma(\gamma)} R(\lambda, \mu; \xi) \nonumber$$

$$- \int_0^\infty d\xi \ e^{-\xi} \xi^{\gamma-1} \frac{Z(\xi)}{\Gamma(\gamma)} R(\lambda; \xi) \int_0^\infty d\xi \ e^{-\xi} \xi^{\gamma-1} \frac{Z(\xi)}{\Gamma(\gamma)} R(\mu; \xi) \nonumber$$

$$\neq \int_0^\infty d\xi \ e^{-\xi} \xi^{\gamma-1} \frac{Z(\xi)}{\Gamma(\gamma)} R^{\text{conn}}(\lambda, \mu; \xi). \quad (3.16)$$
The universal macroscopic connected two-point function obtained in the large $N$ limit $\rho^{\text{conn}}(x, y; \xi)$ in [6] will thus mix with the non-universal density $\rho(x; \xi)$, and the same feature persists for higher $k$-point connected correlators. This should not come as a surprise as the same situation was encountered in the fixed or restricted trace ensembles [32,33], being an integral transformation of the classical Wigner–Dyson ensembles. As observed there [33], our microscopic correlations will remain universal; see section 4.

In section 3.2 we will study the limit $N/M \to c < 1$. The corresponding standard WL ensemble can be mapped to the so-called generalized Penner model [7] with positive definite matrices. The extra determinant from the Jacobian of this change of variables can be written as an extra logarithm in the potential $V \to V + N \ln |\lambda|$. For the same reason as given above the universal findings of [7] for the unitary ensemble $\beta = 2$ do not translate to the macroscopic limit in section 4 either.

### 3.2. The generalized Marčenko–Pastur law for $c < 1$

In this subsection we deal with the limit in which the matrix $X$ remains rectangular, that is both $M$ and $N = cM$ become large such that $\lim_{N,M \to \infty} N/M = c < 1$. This limit is particularly relevant for applications to real data series.

We will follow the same steps as in section 3.1. It is known that in the standard WL ensembles with weight $\exp[-n\beta \lambda]$ the corresponding density is given by the following MP law:

$$
\lim_{N \gg 1} R(\lambda) = \frac{n}{\pi \lambda} \sqrt{\left( \frac{\lambda - N}{2n} X_+ \right) \left( \frac{N}{2n} X_- - \lambda \right)}, \quad \text{with } \lambda \in \left[ \frac{N}{2n} X_-, \frac{N}{2n} X_+ \right].
$$

For later convenience we have defined the bounds

$$
X_\pm \equiv (c^{-1/2} \pm 1)^2 \quad \text{with } 0 < c < 1.
$$

In the limit $c \to 1$ we recover from equation (3.17) the semicircle equation (3.2) from the last section.

The $N$-independent density is again obtained after rescaling by the mean eigenvalue position:

$$
\langle \lambda \rangle_\gamma = \frac{\gamma N}{2nc(\gamma - (\beta/2c)N^2 - 1)},
$$

where we have used $M = N + \nu = N/c$. For large $\gamma$ we obtain the quantity $\langle \lambda \rangle = N/2nc$, which is the average position for the standard WL in our limit $c < 1$. We thus obtain for the rescaled density MP density

$$
\rho(x) = \lim_{N \to \infty} \frac{1}{N} \langle \lambda \rangle R(x; \langle \lambda \rangle)
\quad = \frac{1}{2\pi cx} \sqrt{(x - cX_-(cX_+ - x))} \quad \text{with } x \in [cX_-, cX_+].
$$

It is normalized to unity with mean $\langle x \rangle = 1$.

For the generalized model we have to insert equation (3.17), now with weight $\exp[-\xi n\beta/\gamma \lambda]$, into equation (2.11) and rescale with respect to equation (3.19).
As previously, we keep fixed
\[ \alpha \equiv \lim_{N, \gamma \to \infty} \left[ \gamma - \frac{\beta}{2c} N^2 - 1 \right], \quad \text{with} \quad \alpha > 0, \tag{3.21} \]
as in equation (3.5). The rescaled generalized density is thus given by
\[ \rho_\alpha(x) \equiv \lim_{N, \gamma \to \infty} \frac{1}{N} \langle \lambda \rangle, \quad R_\gamma(x(\lambda) \gamma) \]
\[ = \lim_{N, \gamma \to \infty} \frac{1}{N} \langle \lambda \rangle \int_{\mathcal{I}} d\xi \, e^{-\xi - 1} \frac{Z(\xi)}{\Gamma(\gamma) \pi^2\gamma^2 x(\lambda) \gamma} \]
\[ \times \left( x(\lambda) \gamma - \frac{N \gamma}{2 n \xi} \right) \left( \frac{N \gamma}{2 n \xi} X_+ - x(\lambda) \gamma \right), \tag{3.22} \]
where \( \mathcal{I} \equiv [(c\alpha/x)X_-, (c\alpha/x)X_+] \). Filling in all definitions and changing variables we finally arrive at the following main result of this subsection:
\[ \rho_\alpha(x) = \frac{1}{2 \pi c \alpha \Gamma(\alpha + 1)} \left( \frac{\alpha}{\alpha} \right)^{\alpha + 2} \int_{X_-}^{X_+} dt \exp \left[ -\frac{\alpha}{\alpha} t \right] t^\alpha \sqrt{(t - X_-)(X_+ - t)}. \tag{3.23} \]
Our density is normalized to unity and has first moment \( \langle x \rangle = 1 \). This result was derived previously (modulo different notation) for \( \beta = 1 \) in [25], using different methods. The integral in equation (3.23) can be computed in principle in terms of a confluent hypergeometric series in two variables (see [34], formulae 3.385 and 9.261(1)), but the integral form is more convenient for numerical evaluations and an asymptotic analysis.

As a first check we recover equation (3.8) in the limit \( c \to 1 \). Furthermore, one can show that the following limit holds: \( \lim_{c \to 1} \rho_\alpha(x) = \rho(x) \), recovering the MP density equation (3.20). Thus the large \( N \) limit and the large \( \gamma \) limit are again well behaved. We have also checked in appendix B that the convergence with \( N \) towards the density of equation (3.8) is very fast; see figure B.1. In fact it is faster than for WL. Our remark from the previous subsection allowing \( -1 < \alpha < 0 \) applies here too.

It is easy to see how the density decays for large \( x \gg 1 \):
\[ \rho_\alpha(x) \sim x^{-(\alpha + 2)} \left( \frac{\alpha}{\alpha} \right)^{\alpha + 2} \frac{1}{2 \pi c \alpha \Gamma(\alpha + 1)} C(1 + \mathcal{O}(1/x)), \tag{3.24} \]
with the same power law as for \( c = 1 \). The constant \( C \) is given by
\[ C \equiv \frac{1}{8} (X_+ - X_-)^2 X_+^\alpha \pi F_1 \left( \begin{matrix} 3/2, -\alpha; 3; -\frac{X_+ - X_-}{X_-} \end{matrix} \right). \tag{3.25} \]
The asymptotic for small values of \( x \) is less obvious to obtain, and we find
\[ \rho_\alpha(x) \sim x^{-\alpha - 1/2} \exp \left[ -\frac{\alpha}{\alpha} X_- \right] \mathcal{D}, \tag{3.26} \]
where the constant \( \mathcal{D} = X_+^\alpha (X_+ - X_-)^{1/2} (c\alpha)^{-1/2} / 16 \Gamma(\alpha + 1) \).

While the MP density of the standard WL ensemble, equation (3.20), has compact support in \( [cX_, cX_+] \), our generalized density is non-vanishing on the entire real positive axis, even for \( c < 1 \). To the left of the edge in MP, \( x < cX_- \), our density decreases but remains non-zero. Below a certain point that we will call a pseudo-edge, \( X_- \), our

\small
\[ \text{doi:10.1088/1742-5468/2008/09/P09002} \]

\normalsize
density becomes exponentially suppressed. From the asymptotic (3.26) it is possible to give an estimate for $X_-$ below which the density becomes negligible. The reasoning goes as follows: writing the asymptotic (3.26) as

$$\rho_\alpha(x) \approx \exp \left[ - \left( (\alpha + 1/2) \log x + \frac{c\alpha}{x} X_- \right) \right]$$

(3.27)

the exponential damping conventionally begins at the point $X_-$ where

$$(\alpha + 1/2) \log X_- + \frac{c\alpha}{X_-} X_- \approx 1.$$ (3.28)

For the case depicted in figure 2 ($c = 0.3$ and $\alpha = 3$), the above estimate reads $X_- \approx 0.055\ldots$, in reasonable agreement with the inset.

4. The universal microscopic large $N$ limit for a general potential $V$

In this section we consider a different large $N$ limit, the microscopic limit, which takes us to the scale of the mean level spacing and thus to the distribution of individual eigenvalues. Our findings will be universal for a general non-Gaussian polynomial potential $V(\lambda)$ for all three $\beta = 1, 2, 4$, inheriting the corresponding universality from the standard ensembles.

We will only consider the case $c = 1$ and the so-called hard edge here, deriving a generalized Bessel law for the microscopic densities and their first-eigenvalue distributions.
For $c < 1$ the local distribution at the inner (and outer) soft edge of the standard WL ensembles follows the Tracy–Widom law. Although it would be very interesting to derive the corresponding generalization, we have not managed so far, and leave this task for future investigation.

In the first subsection the microscopic densities are derived while the second subsection is devoted to the first-eigenvalue distributions. The matching of the two is illustrated in many figures throughout this section, being an important consistency check.

### 4.1. The generalized universal Bessel law

Let us first recall the definition of the microscopic limit in the standard WL ensembles, resulting in the universal Bessel law. We will discuss in detail the case $\beta = 2$ and then only quote the results for $\beta = 1, 4$.

For simplicity consider the Gaussian case first. Starting from the weight $\exp[-n\beta \lambda]$ we first scale out the mean eigenvalue $\lambda \rightarrow x \langle \lambda \rangle$, just as in equation (3.4) for the macroscopic limit. On top of that we make a further rescaling by the mean level spacing $4N^2 x = y$, keeping $y$ fixed. We therefore define the microscopic limit as

$$
\rho^{(\beta)}_\nu(y) = \lim_{N \rightarrow \infty} \frac{1}{2N^2} \langle \lambda \rangle R \left( \frac{y}{4N^2} \langle \lambda \rangle \right),
$$

where $R(\lambda)$ is the spectral density for finite $N$ in one of the three WL ensembles. The result now depends on $\beta$ and $\nu$ as indicated through the indices, in contrast to the semicircle law. We can then apply the following asymptotic:

$$
\lim_{k \rightarrow \infty} k^{-\nu} L_k \left( \frac{z^2}{4k} \right) = \left( \frac{z}{2} \right)^{-\nu} J_\nu(z)
$$

to the orthogonal Laguerre polynomials in the finite $N$ density; see equation (B.7) for $\beta = 2$. We obtain

$$
\rho^{(2)}_\nu(y) = \lim_{N \rightarrow \infty} \frac{2n}{2N^2 \langle \lambda \rangle} \left( \frac{y \langle \lambda \rangle 2n}{4N^2} \right)^\nu e^{-\left(2ny/4N^2\langle \lambda \rangle\right)} \sum_{k=0}^{N-1} \frac{k!}{(k+\nu)!} L_k^{\nu} \left( \frac{2ny k}{4N^2 \langle \lambda \rangle} \right)^2
$$

$$
= \frac{1}{2} \int_0^1 dt J_\nu(\sqrt{ty})^2 = \frac{1}{2} (J_\nu(\sqrt{y})^2 - J_{\nu-1}(\sqrt{y}) J_{\nu+1}(\sqrt{y})),
$$

after replacing the sum by an integral\(^2\) with variable $t = k/N$. The Bessel density is plotted for different values of $\nu$ in figure 4 together with the first eigenvalue from the next subsection, after changing to the conventional squared variables $y \rightarrow y^2$ (see equation (3.13)):

$$
\vartheta^{(2)}_\nu(y) = \frac{|y|}{2} (J_\nu(y)^2 - J_{\nu-1}(y) J_{\nu+1}(y)).
$$

This result is universal [19], being valid for any potential $V$ with spectral support including the origin. We only have to rescale in equation (4.1) by the macroscopic density in terms of

\(^2\) Instead of equation (B.7) we could have applied the Christoffel–Darboux identity, equation (B.9), leading to the same result.

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the squared variables $\pi\vartheta(0)$ for a general potential $V$, instead of the Gaussian macroscopic density of equation (3.14) where $\pi\vartheta(0) = 1$. In other words all orthogonal polynomials, equation (2.13), tend to Bessel $J$ functions in the microscopic limit.

We can now repeat the above analysis for our generalized microscopic density. The scaling of $\gamma$ with $N$ keeping $\alpha$ fixed is kept throughout this entire section. In section 3 we found that the density diverges exactly like the standard density as an inverse square root; see equation (3.12). For that reason the microscopic rescaling is the same, without changing powers of $N$. However, the constant in front of the macroscopic density at the origin is not the standard Gaussian result, $1/\pi = \vartheta(0)$, but given by equation (3.12),

$$\vartheta_\alpha(0) = b/\pi, \quad \text{with} \qquad b = \frac{\Gamma(\alpha + 3/2)}{\Gamma(\alpha + 1)\sqrt{\alpha}}. \quad (4.5)$$

We therefore define the microscopic limit as

$$\rho_{\alpha,\nu}^{(\beta)}(y) \equiv \lim_{N,\gamma \to \infty} \frac{1}{2N^2b^2} \langle \lambda \rangle_\gamma R_\gamma \left( \frac{y}{4N^2b^2} \langle \lambda \rangle_\gamma \right). \quad (4.6)$$

Because of the universality we just stated we can restrict ourselves to doing the computation for the orthogonal polynomials with Gaussian weight $\exp[-2n\xi^2\lambda]/\gamma$ in equation (2.13). Taking the microscopic limit, equation (4.6), and inserting equation (4.3), we obtain

$$\vartheta_{\alpha,\nu}^{(2)}(y) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty d\xi \ e^{-\xi} \left( \frac{\xi^2}{2\alpha b^2} \right)^2 \left(J_{\nu} \left( \frac{y}{b} \sqrt{\xi/\alpha} \right) \right)^2 - J_{\nu-1} \left( \frac{y}{b} \sqrt{\xi/\alpha} \right) J_{\nu+1} \left( \frac{y}{b} \sqrt{\xi/\alpha} \right). \quad (4.7)$$

This is the first main result of this subsection given here in terms of squared values. The integral could be expressed in terms of generalized hypergeometric functions, but for plots this representation is preferable. Note that in our calculation we have inserted the ratio of partition functions, equation (3.7), for the Gaussian models. This quantity is again universal as in the large $N$ limit: not only the polynomials themselves, but also their norms become universal. Starting from equation (2.18) the standard WL partition function is given as follows:

$$\lim_{N \to \infty} (\log[Z(\xi)] - \log[N!h_0^N]) = \lim_{N \to \infty} N \sum_{i=0}^{N-1} \left( 1 - \frac{i}{N} \right) \log[r_i] = \int_0^1 dt (1 - t) \log[r(t)]. \quad (4.8)$$

The $\xi$-dependent ratios of the norms determined from the so-called string or recursion equation at finite $N$ have a universal limit $r(t)$ [19]. Re-exponentiating and inserting this universal result, equation (4.8), into equation (2.8), the generalized partition function $Z_\gamma$ also becomes universal in the large $N$ limit, and thus the ratio equation (3.7) does as well. The constant factor that we have subtracted on the left-hand side of equation (4.8) cancels out when taking the ratio.
The new microscopic density, equation (4.7), generalizing the Bessel law, equation (4.4), is shown in figures 3 and 4 for various values of $\alpha$ and $\nu$. The plots include the corresponding first eigenvalues to be derived later. As a check we can take the limit $\alpha \to \infty$ to analytically re-obtain equation (4.4) from equation (4.7). This is illustrated in figure 3 where we observe that the convergence is rather slow. We have checked that for $\alpha \sim O(150)$ the first three maxima become indistinguishable.

The procedure for $\beta = 1$ and 4 is the same as above and so we can be more concise. The corresponding microscopic densities of the WL ensembles are again universal [20]. Computed initially in [17,18] the expressions obtained can be simplified. They can be expressed through the $\beta = 2$ density, equation (4.4), plus extra terms as shown for $\beta = 1$ [35] and $\beta = 4$ [36]:

$$
\vartheta^{(1)}_{\nu}(y) = \vartheta^{(2)}_{\nu}(y) + \frac{1}{2} J_{\nu}(|y|) \left( 1 - \int_0^{|y|} dt J_{\nu}(t) \right),
$$

(4.9)

$$
\vartheta^{(4)}_{\nu}(y) = \vartheta^{(2)}_{2\nu}(2y) - \frac{1}{2} J_{2\nu}(2y) \int_0^{2|y|} dt J_{2\nu}(t).
$$

(4.10)
Figure 5. Varying $\nu$ for $\beta = 1$: the generalized microscopic density $\vartheta^{(1)}_{\alpha,\nu}(y)$, equation (4.11) (blue), with its first eigenvalue (green) at $\alpha = 0.1$ versus the corresponding WL Bessel density $\vartheta^{(1)}_{\nu}(y)$, equation (4.9) (black), and its first eigenvalue (red) at $\nu = 0$ (left), $\nu = 1$ (middle), and $\nu = 3$ (right).

Figure 6. Varying $\nu$ for $\beta = 4$: the generalized microscopic density $\vartheta^{(4)}_{\alpha,\nu}(y)$, equation (4.12) (blue), at $\alpha = 0.1$ versus the corresponding WL Bessel density $\vartheta^{(4)}_{\nu}(y)$, equation (4.10) (black), at $\nu = 0$ (left), $\nu = 1$ (middle), and $\nu = 2$ (right). For $\nu = 0$ we also display the respective first eigenvalue for the generalized (green) and WL ensembles (red).

The generalized densities immediately follow. Because of the linear relationship they are also expressed through the generalized $\beta = 2$ density equation (4.7):

\[
\vartheta^{(1)}_{\alpha,\nu}(y) = \vartheta^{(2)}_{\alpha,\nu}(y) + \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \xi \, \varepsilon^{-\xi} \xi^\alpha \sqrt{\frac{\xi}{ab^2}} \frac{1}{2} J_{\nu}\left(\frac{y}{b} \sqrt{\xi/\alpha}\right)
\times \left( 1 - \int_0^{\sqrt{\xi/ab^2}} \frac{\varepsilon^{-\xi}}{\xi} \, dt J_{\nu}(t) \right), \tag{4.11}
\]

\[
\vartheta^{(4)}_{\alpha,\nu}(y) = \vartheta^{(2)}_{\alpha,2\nu}(2y) - \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \xi \, \varepsilon^{-\xi} \xi^\alpha \sqrt{\frac{\xi}{ab^2}} \frac{1}{2} J_{2\nu}\left(\frac{2y}{b} \sqrt{\xi/\alpha}\right)
\times \int_0^{\sqrt{\xi/ab^2}} \frac{\varepsilon^{-\xi}}{\xi} \, dt J_{2\nu}(t). \tag{4.12}
\]

For a given $\nu$ the inner integral over the single Bessel $J$ function can be performed analytically. It is given in terms of Bessel $J$ functions for odd values of $\nu$, e.g. $\int_0^{\infty} dt J_{1}(t) = J_0(\nu)$, and additional Struve functions for even $\nu$.

The generalized microscopic densities $\vartheta^{(\beta)}_{\alpha,\nu}(y)$ are compared to the standard ones in figure 5 for $\beta = 1$, and in figure 6 for $\beta = 4$.

Higher order correlation functions can be computed along the same lines by inserting the asymptotic Bessel kernels into equation (2.17), and we only quote the simplest final
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result for $\beta = 2$:

$$\vartheta^{(2)}_{\alpha,\nu}(y_1, \ldots, y_k) = \prod_{i=1}^k \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty d\xi \, e^{-\xi} \xi^{\alpha} \prod_{j=1}^k \xi_j \sqrt{\frac{\xi}{\alpha b}}$$

$$\times \prod_{1 \leq i, j \leq k} \left[ J_\nu \left( \frac{y_i/b}{\sqrt{\xi/\alpha}} \right) J_{\nu+1} \left( \frac{y_j/b}{\sqrt{\xi/\alpha}} \right) - (i \leftrightarrow j) \right].$$  \hspace{1cm} (4.13)

The corresponding results for $\beta = 1, 4$ are given in terms of a Pfaffian of a matrix kernel [18], and for a discussion of a relation between the three universal kernels we refer the reader to [20].

A feature that we observe for all three $\beta$ is that for $\alpha \leq O(1)$ the oscillations of the Bessel density are completely smoothed out, apart from the first peak. A similar feature was observed in a generalization of the unitary WL ensemble for critical statistics [31]. However, no power law tails seem to be present in such a model, where only the generalized microscopic density and number variance were computed.

It is known that for standard WL the maxima of the Bessel density correspond to the locations of individual eigenvalues [37], as we will see in the next subsection. On the other hand the microscopic density of the WL ensembles in the bulk is completely flat, equalling $1/\pi$ in our normalization. We may thus suspect that in the generalized model the bulk is approached much faster than in the standard WL, where localized maxima persist to $y \gg 10$. We therefore focus mainly on the first-eigenvalue distribution in the generalized model which is the subject of section 4.2.

4.2. The generalized universal first-eigenvalue distribution at the hard edge

The probability that the interval $(0, s]$ is empty of eigenvalues is defined as follows:

$$E_\gamma(s) \equiv \frac{1}{Z_\gamma} \int_s^\infty d\lambda_1 \cdots d\lambda_N \mathcal{P}_\gamma(\lambda_1, \ldots, \lambda_N)$$

$$= \int_0^\infty d\xi \, e^{-\xi} \xi^{\gamma-1} \frac{Z(\xi)}{\Gamma(\gamma) Z_\gamma} E(s; \xi),$$  \hspace{1cm} (4.14)

where the gap probability of the WL ensembles is defined as

$$E(s; \xi) \equiv \frac{1}{Z(\xi)} \int_s^\infty d\lambda_1 \cdots d\lambda_N \mathcal{P}(\lambda_1, \ldots, \lambda_N; \xi).$$  \hspace{1cm} (4.15)

Both quantities are normalized to unity at $s = 0$ and vanish at $s = \infty$. The distribution of the first eigenvalue $p(s)$ simply follows by differentiation:

$$p_\gamma(s) \equiv -\frac{\partial}{\partial s} E_\gamma(s),$$  \hspace{1cm} (4.16)

and likewise for WL. In WL the gap probability $E(s)$ and the first-eigenvalue distribution $p(s)$ are explicitly known and universal in the microscopic large $N$ limit for all $\nu$ at $\beta = 2$, for odd values of $\nu$ and 0 at $\beta = 1$, and for $\nu = 0$ at $\beta = 4$. This has been shown by various authors independently [16, 38, 39, 37]. In some cases only finite $N$ results are

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known in terms of a hypergeometric function of a matrix valued argument [40, 41], from which limits are difficult to extract.

Although $p(s)$ follows from $E(s)$, the most compact universal formulae are known directly for $p(s)$ for all three $\beta$ [37]. There, also the distributions of second and higher eigenvalues are given, which we will not consider here.

For pedagogical reasons we start once more with an explicit calculation for $\beta=2$ and the Gaussian ensemble. At $\nu = 0$ the pre-exponential factor is absent and we have for WL with weight $\exp[-2n\lambda]$

$$E(s) = \frac{1}{Z} \int ds \prod_{i=1}^{N} \lambda_i \exp \left[ -2n \sum_{i=1}^{N} \lambda_i \right] \prod_{j<k} |\lambda_j - \lambda_k|^2 = \exp[-2nN s], \quad (4.18)$$

shifting all integration variables by $s$ and using the invariance of the Vandermonde determinant. This result is exact for any $N$ and identical to the properly rescaled large $N$ result when keeping $Ns$ fixed.

The generalized ensemble follows easily, by inserting this result into equation (4.15):

$$E_\gamma(s) = \int_0^\infty d\xi \frac{\exp[-\xi^\gamma - 2n\xi Ns]}{\Gamma(\gamma) \xi^{\gamma-1}} \left( 1 + \frac{2nN s}{\gamma} \right)^{-\gamma N^2}, \quad (4.19)$$

which is also exact for finite and infinite $N$. This very fact implies that we have full control of the large $N$ limit. Because of the fat tails in our distributions it was not a priori clear if the scales at the hard edge and in the bulk would mix. This implicit assumption of a separation of scales in section 4.1 is thus fully justified.

The microscopic limit can be taken following equation (4.1) for WL:

$$\mathcal{E}^{(\beta)}(y) = \lim_{N,\gamma \to \infty} E \left( \frac{y}{4N^2 b^2} \right), \quad (4.20)$$

where we explicitly indicate the dependence on $\beta$ and $\nu$, as in section 4.1. As a result we obtain for $\beta=2$ and $\nu=0$

$$\mathcal{E}^{(2)}_{\nu=0}(y) = \exp \left[ -\frac{1}{4} y^2 \right], \quad (4.21)$$

and for the corresponding generalized gap probability

$$\mathcal{E}^{(2)}_{\alpha, \nu=0}(y) = \lim_{N,\gamma \to \infty} E_\gamma \left( \frac{y}{4N^2 b^2} \right) = \left( 1 + \frac{y^2}{4\alpha b^2} \right)^{-(\alpha+1)}. \quad (4.22)$$

The first-eigenvalue distribution can be compared to the microscopic densities $\vartheta_\alpha(y)$ in squared variables:

$$\vartheta^{(2)}_{\nu=0}(y) = \frac{\partial}{\partial y} \mathcal{E}^{(2)}_{\nu=0}(y^2) = \frac{1}{2} |y| \exp \left[ -\frac{1}{4} y^2 \right], \quad (4.23)$$

for WL, and for the generalized ensemble:

$$\vartheta^{(2)}_{\alpha, \nu=0}(y) = \frac{\partial}{\partial y} \mathcal{E}^{(2)}_{\alpha, \nu=0}(y^2) = |y| \frac{(\alpha + 1)}{2\alpha b^2} \left( 1 + \frac{y^2}{4\alpha b^2} \right)^{-(\alpha+2)}. \quad (4.24)$$

These distributions are all normalized to unity,

$$\int_0^\infty dy \vartheta^{(\beta)}_{\alpha, \nu}(y) = 1. \quad (4.25)$$

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The restriction to a Gaussian potential in the discussion above can be lifted, as the first-eigenvalue distributions for all \( \nu \) are universal, including the ratio in partition functions that we have inserted again. Equations (4.23) and \((4.24)\) are compared to the corresponding densities, equations \((4.4)\) and \((4.7)\), in figure \(3\) for various values of \( \alpha \).

Next we give the first-eigenvalue distribution for general \( \nu \). Here we directly use the most compact universal expression \([37]\) for \( \wp^{(2)}(y) \) in WL, without making the detour over \( \mathcal{E}^{(2)}(y) \) \([39]\):

\[
\wp^{(2)}_{\alpha}(y) = \frac{1}{2} \left| y \right| \exp \left[ -\frac{1}{4} y^2 \right] \text{det}_{1 \leq i,j \leq \nu} \left[ I_{i-j+2}(\left| y \right|) \right].
\] (4.26)

In addition to the exponential in equation (4.23) it contains a determinant of finite size \( \nu \times \nu \) over the modified Bessel \( I \) function, which is absent at \( \nu = 0 \). Knowing that the properly rescaled microscopic gap probability is a function of the form \( \mathcal{E}^{(2)}(y; \xi) = \mathcal{E}^{(2)}(\sqrt{\alpha b^2 y}) \) (see equation \((4.7)\)), we obtain for the generalized first-eigenvalue distribution in terms of squared eigenvalues

\[
\wp^{(2)}_{\alpha,\nu}(y) = \frac{1}{\Gamma(\alpha + 1)} \int_0^{\infty} d\xi \, e^{-\xi} \xi^{\alpha} (-\partial_{\xi}) \mathcal{E}^{(2)}(y \sqrt{\alpha b^2})
\]

\[
= \frac{1}{\Gamma(\alpha + 1)} \int_0^{\infty} d\xi \, e^{-\xi} \xi^{\alpha} \frac{\xi y^2}{2 \alpha b^2} \exp \left[ -\frac{\xi y^2}{4 \alpha b^2} \right] \text{det}_{1 \leq i,j \leq \nu} \left[ I_{i-j+2}(\left| y \right/b \sqrt{\xi/\alpha}) \right].
\] (4.27)

For \( \nu = 1 \) containing only one Bessel \( I \) the integral can be performed and is given in terms of a hypergeometric function:

\[
\wp^{(2)}_{\alpha,1}(y) = \frac{|y|}{2 \alpha b^2 \Gamma(\alpha + 1)} \int_0^{\infty} d\xi \, e^{-\xi} \xi^{\alpha+1} \exp \left[ -\frac{\xi y^2}{4 \alpha b^2} \right] \text{I}_2 \left( \frac{y}{b \sqrt{\xi/\alpha}} \right)
\]

\[
= \frac{\Gamma(\alpha + 3)}{|y| \Gamma(\alpha + 1)} \left( 1 + \frac{y^2}{4 \alpha} \right)^{-(\alpha+3)} \text{I}_1 \left( \alpha + 3, 3; 1 + \frac{4 \alpha}{y^2} \right)^{-1}.
\] (4.28)

It is shown in figure \(4\) (middle), together with \( \nu = 2 \) (right). For increasing \( \nu \) however, the integral representation equation \((4.27)\) is more convenient.

We now turn to \( \beta = 1 \). Here the first-eigenvalue distribution of the WL ensembles is only known explicitly for odd values of \( \nu \) in the large \( N \) limit \([37]\), given by a Pfaffian with indices running over half-integers:

\[
\wp^{(1)}_{\nu}(y) = \text{const.} \left| y \right|^{(3-\nu)/2} \exp \left[ -\frac{1}{8} y^2 \right] \text{Pf}_{(\nu/2)+1 \leq i,j \leq (\nu/2)-1} \left[ (i-j) I_{i+j+3}(\left| y \right|) \right].
\] (4.29)

The constant in front is determined by the normalization to unity and can be computed case by case. An exception is for \( \nu = 0 \) where the distribution was calculated in \([16]\):

\[
\wp^{(1)}_{0}(y) = \frac{1}{4} (2 + \left| y \right|) \exp \left[ -\frac{\left| y \right|}{2} - \frac{y^2}{8} \right].
\] (4.30)

For \( \nu = 1 \) and \( 3 \) we have from equation \((4.29)\)

\[
\wp^{(1)}_{1}(y) = \frac{1}{4} \left| y \right| \exp \left[ -\frac{1}{8} y^2 \right],
\] (4.31)

\[
\wp^{(1)}_{3}(y) = \frac{1}{2} \exp \left[ -\frac{1}{8} y^2 \right] I_3(\left| y \right|).
\] (4.32)
The corresponding generalized formula for general odd $\nu$ thus reads

$$\varphi_{\alpha, \nu}^{(1)}(y) = \text{const.} \cdot \frac{1}{\Gamma(\alpha + 1)b} \int_0^\infty d\xi \ e^{-\xi} \left( \xi^\alpha \sqrt{\frac{\xi}{\alpha}} \right)^{(3-\nu)/2} \exp \left[ -\frac{1}{8\alpha b^2} \xi y^2 \right] \times \text{Pf}_{-(\nu/2)+1 \leq i, j \leq (\nu/2)-1} \left[ (i - j) I_{i+j+3} \left( \frac{|y|}{b} \sqrt{\xi/\alpha} \right) \right],$$

(4.33)

up to the normalization constant. For the simplest examples $\nu = 0, 1$ and 3 displayed in figure 5 we have

$$\varphi_{\alpha, 0}^{(1)}(y) = \frac{1}{4\sqrt{\alpha \Gamma(\alpha + 1)b}} \int_0^\infty d\xi \ e^{-\xi} \xi^{\alpha+1/2} \left( 2 + \frac{|y|}{b} \sqrt{\xi/\alpha} \right) \exp \left[ -\frac{|y|}{2b} \sqrt{\xi/\alpha} - \frac{\xi y^2}{8\alpha b^2} \right],$$

(4.34)

$$\varphi_{\alpha, 1}^{(1)}(y) = \frac{(\alpha + 1)}{4\alpha b^2} |y| \left( 1 + \frac{y^2}{8\alpha b^2} \right)^{-(\alpha+2)},$$

(4.35)

$$\varphi_{\alpha, 3}^{(1)}(y) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty d\xi \ e^{-\xi} \xi^\alpha \sqrt{\frac{\xi}{\alpha}} b \exp \left[ -\frac{\xi y^2}{8\alpha b^2} \right] I_3 \left( \frac{|y|}{b} \sqrt{\xi/\alpha} \right).$$

(4.36)

While the density is modified only rather mildly compared to WL, the first eigenvalue changes considerably. We have checked that the curves converge to WL for large $\alpha$, where the convergence to the density is much faster than for $\beta = 2$. The case $\nu = 0$ in figure 5 left is the only example where the microscopic densities do not vanish at $x = 0$. The fact that they have the same limit, $\frac{1}{2}$, can also be seen analytically, exploiting that $J_\nu(0) = \delta_{\nu,0}$ (see equations (4.9) and (4.11)). As mentioned above, for $\nu = 2k$ with $k \in \mathbb{N}_+$ the first eigenvalue is not available to date.

Finally we turn to $\beta = 4$. In principle the result is known for the WL ensemble:

$$\varphi^{(4)}_0(y) = \text{const.} \cdot |y|^{\nu+(3/2)} \exp \left[ -\frac{1}{2} y^2 \right] Z_{3/2}(\{|y|\}_{2w+1}).$$

(4.37)

Here $Z_{3/2}(\{|y|\}_{2w+1})$ is the large $N$ matrix model partition function at topological charge $3/2$ with $2\nu + 1$ degenerate masses at value $y$, and we refer the reader to [37] for a more detailed discussion of these objects. This partition function is generally known explicitly only for an even number of masses, except at $\nu = 0$. There we have

$$\varphi^{(4)}_0(y) = \frac{1}{2} \sqrt{2\pi} \ |y|^{3/2} \exp \left[ -\frac{1}{2} y^2 \right] I_{3/2}(y) = |y| \left( \cosh(y) - \frac{1}{y} \sinh(y) \right) \exp \left[ -\frac{1}{2} y^2 \right].$$

(4.38)

Thus the generalized distribution depicted in figure 6 is given by

$$\varphi^{(4)}_{\alpha}(y) = \frac{1}{\alpha \Gamma(\alpha + 1)b^2} \int_0^\infty d\xi \ e^{-\xi} \xi^{\alpha+1} \left( |y| \cosh \left( \frac{y}{b} \sqrt{\xi/\alpha} \right) - b \sqrt{\frac{\alpha}{\xi}} \sinh \left( \frac{y}{b} \sqrt{\xi/\alpha} \right) \right) \exp \left[ -\frac{\xi y^2}{2\alpha b^2} \right].$$

(4.39)

We note that the oscillations get smoothed out as already observed at $\beta = 2$. The convergence in $\alpha$ to the WL quantity is again slow, as for $\beta = 2$. 

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5. The Wigner surmise in the bulk

In this section we study the distribution of spacing between eigenvalues in the bulk in our generalized model. In contrast to the previous sections, this one does not take $N$ to be large, but uses the $N=2$ results, following the original idea of Wigner.

In the Wigner–Dyson ensembles with a Gaussian potential, the spacing distribution has the simple form

\[ \mathcal{P}_{\text{WD}}^{(\beta)}(s) = \alpha s^\beta e^{-\beta s^2}, \] (5.1)

known as Wigner’s surmise. The known constants $A, B$ follow from normalization and can be found e.g. in [1]. For the WL case the corresponding expression can be computed from the jpdf where we have introduced the notation $\bar{\nu} \equiv \frac{\beta}{2}(\nu + 1) - 1$:

\[ \mathcal{P}^{(\beta)}(s) = C s^{\beta + \bar{\nu} + 1/2} K_{\bar{\nu} + 1/2}(n\beta s), \] (5.2)

where $K_{\mu}(x)$ is a modified Bessel function and the constant $C$ is given by

\[ C = \left(2^{-1/2+\beta+\bar{\nu}}(\nu\beta)^{-3/2-\beta-\bar{\nu}}\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(1+\nu+\frac{\beta}{2}\right)\right)^{-1}. \] (5.3)

Only for $\bar{\nu} = 0$ one recovers Wigner’s surmise, using $K_{-1/2}(x) = \sqrt{\pi/2x} e^{-x}$ (apart from squared variables in the exponent there). For general $\bar{\nu}$ one easily gets an exponential decay for large spacing $\sim s^{\beta+\bar{\nu}} e^{-n\beta s}$, from the asymptotic expression of the Bessel function (which equals $\lim_{x \to \infty} K_{\mu}(x) \sim \sqrt{\pi/2x} e^{-x}$). However, we expect that only the $\bar{\nu} = 0$ expression will lead to a good approximation of the infinite $N$ case.

Conversely, for the generalized model the spacing distribution has a different expression:

\[ \mathcal{P}_{\gamma}^{(\beta)}(s) = \gamma s^{\beta+\bar{\nu}} \left(\frac{\gamma}{2n\beta} + \frac{s}{2}\right)^{\bar{\nu}+1-\gamma} \binom{2F_1\left(-\bar{\nu}, -\nu + 1; -\bar{\nu} + \gamma; \frac{1}{2} - \frac{\gamma}{2n\beta s}\right)}{1}, \] (5.4)

where $2F_1(a, b; c; z)$ is a hypergeometric function. The constant $C_{\gamma}$ can be computed as

\[ C_{\gamma} = \frac{2^{1-\gamma} B(\bar{\nu} + 1, -2\bar{\nu} - 1 + \gamma) \Gamma(\gamma) \Gamma(1 + \beta/2)}{\Gamma(\bar{\nu} + 1) \Gamma(1 + \beta) \Gamma(\bar{\nu} + 1 + \beta/2) \Gamma(\gamma - 2 - \beta - 2\bar{\nu})} \left(\frac{n\beta}{\gamma}\right)^{2+2\nu+\beta-\gamma}, \] (5.5)

and $B(\alpha, \beta)$ is a Beta function. The large $s$ decay is now a pure power law $\sim s^{-(\varpi+1)}$ where we have defined

\[ \varpi \equiv \gamma - \beta - 2\bar{\nu} - 2 > 0, \] (5.6)

required to be positive for convergence\(^3\).

For $\bar{\nu} = 0$, the spacing distribution takes a much simpler form:

\[ \mathcal{P}_{\gamma}^{(\beta)}(s) = \frac{n(n\beta/\gamma)^{\beta} \Gamma(\gamma - 1)}{\gamma \Gamma(\beta) \Gamma(\gamma - \beta - 2)} s^{\beta} \left(1 + \frac{sn\beta}{\gamma}\right)^{1-\gamma}. \] (5.7)

\(^3\) This condition is equal to equation (2.2) derived for the partition function at $N=2$. 

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Figure 7. Comparison between $\hat{P}_\gamma^{(\beta)}(x)$ (blue, green) and $\hat{P}^{(\beta)}(x)$ (red), for $\beta = 1, 2, 4$ (from left to right). The $\gamma$ value for the blue curves is chosen in such a way that the combination $\varpi$ is kept constant to the value 2. The green curves have $\gamma = 12, 12, 25$ from left to right, and correctly approach the limiting WL curve.

It agrees with the corresponding quantity in the generalized WD ensembles found in [29]. In this particular case, we can compute the mean level spacing explicitly:

\[
\langle \langle s \rangle \rangle_\gamma = \int_0^{\infty} s P^{(\beta)}(s) \, ds = \left( \frac{\gamma}{n\beta} \right) \frac{1 + \beta}{\gamma - \beta - 3},
\]

(5.8)

which converges to the WL Wigner surmise value for $\gamma \to \infty$:

\[
\langle \langle s \rangle \rangle = \int_0^{\infty} s P^{(\beta)}(s) \, ds = \frac{1 + \beta}{n\beta}.
\]

(5.9)

After defining the rescaled quantities having mean spacing 1,

\[
\hat{P}_\gamma^{(\beta)}(x) = \langle \langle s \rangle \rangle_\gamma P^{(\beta)}(\langle \langle s \rangle \rangle_\gamma x),
\]

(5.10)

\[
\hat{P}^{(\beta)}(x) = \langle \langle s \rangle \rangle P^{(\beta)}(\langle \langle s \rangle \rangle x),
\]

(5.11)

we can compare the curves for all three $\beta$ at $\bar{\nu} = 0$ (and $n = 1$), in figure 7. The power law tail modification compared to the standard WL spacing distribution is evident in the plots. Because in the large $N$ limit $\gamma$ scales with $N$, we keep the combination $\varpi$ in equation (5.6) fixed in order to be able to compare to spacing distributions at large $N$.

Both in this and in the previous section the power law tail of the macroscopic density is seen to persist on the microscopic level of the mean level spacing. It would be very interesting to confirm this on real data sets.

6. Conclusions and outlook

We have introduced a generalization of all three ensembles of random matrices called Wishart–Laguerre or chiral ensembles, replacing the exponential of a non-Gaussian potential by a fat-tailed distribution with parameter $\gamma$. In the limit $\gamma \to \infty$ we can recover the exponential weight and thus the standard ensembles. This modification leads to the appearance of correlations with a power law, governed by a single parameter. Such a behaviour is found in many systems in Nature, e.g. in the wide area of complex networks. WL ensembles are often used in comparing to eigenvalues from covariance matrices of real data sets. To illustrate the potential of our generalization we show a
Figure 8. Comparison between the rescaled eigenvalue distribution from financial data [42] and the macroscopic density $\rho_\alpha(x)$ for the generalized model, equation (3.23), red dots and solid blue respectively. The best fit gives a value of $\alpha \approx 0.95$, which corresponds to a power law decay as $\rho_\alpha(x) \sim x^{-2.95}$.

comparison to financial data in figure 8. The eigenvalues of the covariance matrix from time series of stock data clearly show a power law behaviour. These are well described by the generalized Marčenko–Pastur density, refining previous comparisons to the standard Marčenko–Pastur law.

The solution of our generalized models relied heavily on the possibility of writing them as an integral transform of the standard WL ensembles. This generalization is thus in the spirit of super-statistics where other models have been constructed already. The virtue of our model is its invariance for all three symmetry classes, allowing us to go to an eigenvalues basis and to study universality. We were able to show that the generalized macroscopic density which was known in the Gaussian case is only weakly universal.

In contrast, all microscopic densities are universal under any invariant deformations by polynomial potentials. This macroscopic/microscopic dichotomy should not come as a surprise, being observed previously for the restricted trace ensembles.

We exploited the linear relation to standard WL to solve our model exactly at finite $N$ for any polynomial potential, using the formalism of orthogonal polynomials therein. In the subsequent large $N$ double-scaling limit, where $\gamma$ is scaled with $N$, we derived all density correlations in the macroscopic limit for quadratic and rectangular matrices, and in the microscopic limit at the hard edge for all three values of $\beta$. Here we have mainly focused on the spectral density itself and the first-eigenvalue distribution. The general formalism for higher density or higher individual eigenvalue correlation

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functions was provided and is straightforward to use if such quantities were to be needed.

While the hard edge was solved exhaustively, persisting in our model for asymptotically quadratic matrices, we only provided a Wigner surmise in the bulk. Here, more detailed properties of correlation functions could be investigated, including a possible generalization of the Tracy–Widom distribution at the soft edges. This is left for future investigations.

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Appendix A. The partition function and first moment of the Gaussian models

The purpose of this appendix is threefold. First, we derive the condition under which our generalized model defined in equation (2.1) is convergent. Second, we compute both the generalized and standard Gaussian partition functions in order to determine their $\xi$-dependent ratio needed in the computation of all eigenvalue correlation functions. Third, we compute the first moment as a function of $N$, $\nu$, $\beta$ and $\gamma$ which is needed for the rescaling of the eigenvalues in the large $N$ limit.

All three steps will be performed by changing variables from independent matrix elements or eigenvalues to radial coordinates, following [32].

The generalized partition function reads in terms of eigenvalues

$$Z_\gamma = \int_0^\infty \mathcal{L}_N \prod_{i=1}^N \lambda_i \prod_{i=1}^N \lambda_i^{(1/2)\beta(\nu+1)-1} \prod_{j>k}^{N} |\lambda_j - \lambda_k|^{\beta} \frac{1}{(1 + (n\beta/\gamma) \sum_{i=1}^N V(\lambda_i))^\gamma} \times \prod_{j>k}^{N} |\lambda_j/r - \lambda_k/r|^{\beta} \frac{1}{(1 + (n\beta/\gamma) \sum_{i=1}^N V(r\lambda_i/r))^\gamma}. \tag{A.1}$$

Here we have changed to radial coordinates of the $N$-component vector of the eigenvalues $(\lambda_1, \ldots, \lambda_N)$, and $d\Omega_N$ denotes the angular integration over the $N$-dimensional unit sphere $\Omega(N)$. The eigenvectors $e_i = \lambda_i/r$ of norm unity span $\Omega(N)$ and no longer depend on the radius. Collecting all powers of $r$ in the numerator and comparing to the leading power of the denominator $d\gamma$ at large $r \gg 1$, the integral only converges if the following inequality

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\[ \frac{\beta}{2} N(N + \nu) - 1 - \gamma d < 1, \quad (A.2) \]

which is exactly equation (2.2). Next we compute the partition function \( Z(\xi) \), where for the rest of this appendix we restrict our consideration to the Gaussian potential \( V(\lambda) = \lambda \). The same steps can be taken for a purely monic potential \( V(\lambda) = \lambda^d \) as well.

In principle we could repeat the same calculation in terms of eigenvalues as above, but it will be more instructive to start directly from the matrix elements:

\[
Z(\xi) = \int dX \exp \left[ -\xi n \frac{\beta}{\gamma} \text{Tr} X^\dagger X \right]
= \int_{\Omega(\beta N(N + \nu))} da_{\beta N(N + \nu)} \int_0^\infty dr r^{\beta N(N + \nu) - 1} \exp \left[ -\xi n \frac{\beta}{\gamma} r^2 \right]
= \frac{1}{2} \left( \frac{\gamma}{\xi n \beta} \right)^{(\beta/2) N(N + \nu)} \Gamma \left( \frac{\beta}{2} N(N + \nu) \right) \int_{\Omega(\beta N(N + \nu))} da_{\beta N(N + \nu)}. \quad (A.3)
\]

Here we have used radial coordinates for the \( \beta N(N + \nu) \) component vector of all independent matrix elements \( X_{ij} \), with squared norm \( r^2 = \text{Tr} X^\dagger X \). We do not need to compute the angular integral explicitly as it cancels out below.

If we insert the result, equation (A.3), into the relation (2.8) we immediately obtain

\[
Z(\xi) = \frac{1}{\Gamma(\gamma)} \int_0^\infty d\xi \ e^{-\xi} \xi^{-\gamma} Z(\xi)
= \frac{1}{2\Gamma(\gamma)} \left( \frac{\gamma}{\beta n} \right)^{(\beta/2) N(N + \nu)} \Gamma \left( \gamma - \frac{\beta}{2} N(N + \nu) \right) \Gamma \left( \frac{\beta}{2} N(N + \nu) \right)
\times \int_{\Omega(\beta N(N + \nu))} da_{\beta N(N + \nu)}. \quad (A.4)
\]

Combining the last two equations we arrive at

\[
\frac{Z(\xi)}{\Gamma(\gamma) Z_\gamma} = \xi^{-(\beta/2) N(N + \nu)} \Gamma(\gamma - (\beta/2) N(N + \nu)), \quad (A.5)
\]

the ratio of the two Gaussian partition functions at finite values of \( N, \nu \) and \( \gamma \) valid for all three \( \beta \).

In the last step of this appendix we compute the mean eigenvalue position in the Gaussian model. It can be defined either through the spectral density (see equation (3.3)) or in terms of the first moment, where we start with the standard WL ensembles:

\[
\langle \lambda(\xi) \rangle = \frac{1}{Z(\xi)} \int dX \frac{1}{N} \text{Tr} \left( X^\dagger X \right) \exp \left[ -\xi n \frac{\beta}{\gamma} \text{Tr} X^\dagger X \right]
= \frac{1}{N Z(\xi)} \int_{\Omega(\beta N(N + \nu))} da_{\beta N(N + \nu)} \int_0^\infty dr r^{\beta N(N + \nu) - 1} \exp \left[ -\xi n \frac{\beta}{\gamma} r^2 \right]
= \frac{\gamma}{2n \xi} (N + \nu). \quad (A.6)
\]

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Note that the $\beta$ dependence has cancelled out. We can immediately use this result to compute the same quantity for the generalized Gaussian model:

$$\langle \lambda \rangle_\gamma = \frac{1}{Z_\gamma} \int d\mathbf{X} \frac{(1/N) \text{Tr} (\mathbf{X}^\dagger \mathbf{X})}{(1 + (n \beta/\gamma) \text{Tr} \mathbf{X}^\dagger \mathbf{X})^\gamma}$$

$$= \frac{1}{NZ_\gamma \Gamma(\gamma)} \int_0^\infty d\xi \ e^{-\xi} \xi^{\gamma-1} Z(\xi) \langle \lambda(\xi) \rangle$$

$$= \frac{\gamma(N+\nu)}{2n(\gamma - (\beta/2)N(N+\nu) - 1)}.$$

(A.7)

This result is used in the rescaling for both large $N$ limits. As a check, it reduces to equation (A.6) in the limit $\gamma \to \infty$, with a weight $\exp[-n\beta r^2]$.

Appendix B. The explicit $\beta = 2$ solution for all $k$-point densities at finite $N$ and $\gamma$

In this appendix we present all details for the solution of the generalized WL ensemble with unitary invariance $\beta = 2$ and Gaussian potential $V(\lambda) = \lambda$. In this case the orthogonal polynomials of the WL ensemble are known to be Laguerre, allowing for an explicit solution at finite $N$ and finite $\gamma$.

The orthogonal polynomials in equation (2.13) read for the weight function $\exp[-\xi(2n/\gamma)\lambda]$

$$P_k(\lambda) = (-)^k k! \left( \frac{\gamma}{2n\xi} \right)^k L^\nu_k \left( \frac{2n\xi}{\gamma} \lambda \right),$$

with norms

$$h_k = k! (k + \nu)! \left( \frac{\gamma}{2n\xi} \right)^{2k+\nu+1}.$$

(B.1)

Here the Laguerre polynomials are defined as usual:

$$L^\nu_k(z) = \sum_{j=0}^{k} (-1)^j \left( \frac{k + \nu}{k - j} \right) z^j j!,$$

with $L^\nu_k(z)' = -L^{\nu+1}_k(z)$.

Using equation (2.18) we can immediately read off the WL partition function from the norms:

$$Z(\xi) = \left( \frac{\gamma}{2n\xi} \right)^{N\nu+N^2} \prod_{k=0}^{N-1} (k + 1)! (k + \nu)!.$$

(B.4)

As the next step we can compute the partition function, equation (2.8), given by

$$Z_\gamma = \frac{1}{\Gamma(\gamma)} \int_0^\infty d\xi \ e^{-\xi} \xi^{\gamma-1} \left( \frac{\gamma}{2n\xi} \right)^{N\nu+N^2} \prod_{k=0}^{N-1} (k + 1)! (k + \nu)!$$

$$= \left( \frac{\gamma}{2n} \right)^{N\nu+N^2 } \frac{\Gamma(\gamma - N(N+\nu))}{\Gamma(\gamma)} \prod_{k=0}^{N-1} (k + 1)! (k + \nu)!.$$

(B.5)

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This leads to the following ratio needed for example inside the relation (2.11):

$$
\frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} = \xi^{-N(N+\nu)} \frac{1}{\Gamma(\gamma - N(N + \nu))}.
$$

(B.6)

It confirms independently part of the result from the previous appendix, equation (A.5) for \( \beta = 2 \).

The spectral density for finite \( N \) follows on inserting this ratio as well as the standard Laguerre density at finite \( N \),

$$
R(\lambda; \xi) = \lambda^{\nu} e^{-\xi 2n/\gamma \lambda} \sum_{k=0}^{N-1} \frac{k!}{(k + \nu)!} \left( \frac{2n \xi}{\gamma} \right)^{\nu+1} L_k^{\nu} \left( \frac{2n \xi}{\gamma} \lambda \right)^2,
$$

(B.7)

into (2.11):

$$
R_\gamma(\lambda) = \frac{1}{\Gamma(\gamma - N(N + \nu))} \int_0^\infty d\xi \ e^{-\xi \gamma^{-1-N(N+\nu)} R(\lambda; \xi)}.
$$

(B.8)

With the help of equation (B.3) we can derive and simplify the Christoffel–Darboux identity for Laguerre polynomials of equal arguments:

$$
\sum_{k=0}^{N-1} \frac{k!}{\Gamma(k + 1 + \nu)} L_k^{\nu}(y)^2 = \frac{N!}{\Gamma(N + \nu)} \left[ L_N^{\nu}(y) L_{N-1}^{\nu+1}(y) - L_{N-1}^{\nu}(y) L_{N-2}^{\nu+1}(y) \right].
$$

(B.9)

We thus arrive at our final result for the generalized density at finite \( N \) and \( \gamma \):

$$
R_\gamma(\lambda) = \frac{N!}{\Gamma(\gamma - N^2 - N\nu)\Gamma(N + \nu)} \left( \frac{2n \xi}{\gamma} \right)^{\nu+1} \lambda^{\nu} \int_0^\infty d\xi \ e^{-\xi (1+2n/\gamma \lambda)} \xi^{\gamma N^2 + N\nu + \nu} \times \left[ L_N^{\nu} \left( \frac{2n \xi}{\gamma \lambda} \right) L_{N-1}^{\nu+1} \left( \frac{2n \xi}{\gamma \lambda} \right) - L_{N-1}^{\nu} \left( \frac{2n \xi}{\gamma \lambda} \right) L_{N-2}^{\nu+1} \left( \frac{2n \xi}{\gamma \lambda} \right) \right].
$$

(B.10)

This single integration over an exponential times polynomials can be performed explicitly, at the expense of a double summation. Since this equivalent result is not very illuminating or useful for the asymptotic we do not display it here.

Proceeding along the same lines as above we can write down the general result for the \( k \)-point density correlations functions as they follow from equation (2.17):

$$
R_\gamma(\lambda_1, \ldots, \lambda_k) = \frac{N!^k (n/\gamma)^{k \nu}}{\Gamma(\gamma - N^2 - N\nu)\Gamma(N + \nu)^k} \times \prod_{j=1}^k \lambda_j^{\nu} \int_0^\infty d\xi \ e^{-\xi (1+2n/\gamma \sum_{j=1}^k \lambda_j)} \xi^{\gamma_{1-N^2-(N-k)\nu}} \times \det_{1 \leq i, j \leq k} \left[ \frac{L_N^{\nu} \left( \frac{2n \xi}{\gamma \lambda_i} \right) L_{N-1}^{\nu} \left( \frac{2n \xi}{\gamma \lambda_j} \right) - L_{N-1}^{\nu} \left( \frac{2n \xi}{\gamma \lambda_j} \right) L_{N-2}^{\nu} \left( \frac{2n \xi}{\gamma \lambda_i} \right)}{\lambda_j - \lambda_i} \right].
$$

(B.11)

In order to compare the finite \( N \) result (B.10) with the macroscopic \( \gamma \)- and \( N \)-independent density \( \vartheta_\alpha(x) \), equation (3.13), we adopt the following procedure:
Figure B.1. The macroscopic generalized semicircle density $\vartheta_\alpha(x)$, equation (3.13), for $\alpha = 1.02$ and 14 (green and dashed red), compared with the finite $N = 4$ result $\hat{\vartheta}_\alpha(x)$ (blue and dash–dotted orange).

(1) We rescale $R_\gamma(\lambda)$ with mean value $\langle \lambda \rangle_\gamma$ equation (3.3) and normalize to 1: $\hat{\rho}_\gamma(x) \equiv N^{-1}\langle \lambda \rangle_\gamma R_\gamma(\langle \lambda \rangle_\gamma x)$.

(2) Next, we express $\gamma$ as a function of $\alpha$ and $N$, $\gamma = \alpha + N(N + \nu) + 1$, and pass to squared variables: $\hat{\vartheta}_\alpha(x) \equiv |x|\hat{\rho}_\gamma(x^2)$.

(3) Then, we compare $\hat{\vartheta}_\alpha(x)$ and $\vartheta_\alpha(x)$ for $\nu = 0$ in figure B.1. The agreement is already very good for $N = 4$, apart from for the region close to the origin.

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