Dynamics of (Pseudo) Automorphisms of 3-space:
Periodicity versus positive entropy

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§0. Introduction. We consider the family of birational maps of 3-space which may be written in affine coordinates as

\[ f_{\alpha, \beta} : (x_1, x_2, x_3) \mapsto \left( x_2, x_3, \frac{\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3} \right). \] (0.1)

The algebraic iterates \( f_{\alpha, \beta}^n := f_{\alpha, \beta} \circ \cdots \circ f_{\alpha, \beta} \) are rational maps for all \( n \in \mathbb{Z} \). Here we study the dynamics of \( f = f_{\alpha, \beta} \), by which we mean the behavior of \( f^n \) as \( n \to \pm \infty \). We have invertible dynamics since \( f \) has a rational inverse, but it does not behave like a diffeomorphism (or even a homeomorphism). There are two difficulties if we want to regard \( f \) as a mapping of points. First, there is the set of indeterminacy \( \mathcal{I}(f) \), which have no analogue in dimension 2. Diller and Favre [DiF] showed that there is a new manifold \( \pi \) because the identity \( (\pi^*)^n = (f^n)^* \) may not be valid. Given a birational map \( f \) in dimension 2, Diller and Favre [DiF] showed that there is a new manifold \( \pi : Y \to X \) such that the iterates of the induced map \( f_Y \) behave naturally on cohomology, in the sense that \( (f_Y^n)^* = (f^n)^* \). In dimension greater than 2, however, no such theorem is known.

Given a rational map \( f : X \to X \) there is a well-defined pullback map on cohomology, \( f^* : H^*(X) \to H^*(X) \). Passage to cohomology, however, may not be compatible with iteration because the identity \( (f^*)^n = (f^n)^* \) may not be valid.

Within the family (0.1) we find the first known examples of pseudo-automorphisms of positive entropy on blowups of \( \mathbf{P}^3 \):

**Theorem 1.** Suppose that \( \alpha = (a, 0, \omega, 1) \) and \( \beta = (0, 1, 0, 0) \) where \( a \in \mathbb{C} \setminus \{0\} \) and \( \omega \) is a non-real cube root of the unity. Then there is a modification \( \pi : Z \to \mathbf{P}^3 \) such that \( f_Z \) is a pseudo-automorphism. The dynamical degrees \( \delta_1(f) = \delta_2(f) \approx 1.28064 > 1 \) are equal and are given by the largest root of \( t^8 - t^5 - t^4 - t^3 + 1 \). The entropy of \( f_Z \) is the logarithm of the dynamical degree and is thus positive.

**Theorem 2.** For the mappings in Theorem 1, there is a 1-parameter family of surfaces \( S_c \subset Z, c \in \mathbb{C} \) which have the invariance \( f_S_c = S_{\omega c} \). For generic \( c \), \( S_c \) is K3, and the restriction \( f^3\mid_{S_c} \)

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is an automorphism. For generic $c$ and $c'$, the surfaces $S_c$ and $S_{c'}$ are biholomorphically inequivalent, and the automorphisms $f^3|_{S_c}$ and $f^3|_{S_{c'}}$ are not smoothly conjugate.

The surface $S_0$ is invariant, and the restriction $f_{S_0}$ is an automorphism which has the same entropy as $f$. This is smaller than the entropy of the automorphism constructed in [M2, Theorem 1.2] and is thus the smallest known entropy for a projective $K3$ surface automorphism.

Closely related to the dynamics of $f_z$ is the $(1,1)$-current $T^+$ which is expanded by $f_z^n$, and a current $T^-$ for $f_z^{-1}$. This is obtained in §7, as well as the invariant $(2,2)$-current $T^+ \wedge T^−$. The slices of $T^\pm$ and $T^+ \wedge T^−$ on the surfaces $S_c$ give the expanded/contracted currents, as well as the unique invariant measure, for the automorphism $f|_{S_c}$.

The following mappings have quadratic degree growth and complete integrability:

**Theorem 3.** Suppose that $\beta = (0,1,0,0)$ and either $\alpha = (0,0,\omega,1)$ or $\alpha = (a,0,1,1)$ where $a \in \mathbb{C} \setminus \{1\}$, $\omega \neq 1$, and $\omega^3 = 1$. Then the degree of $f^n$ grows quadratically in $n$. Further, there is a modification $\pi : Z \to \mathbb{P}^3$ such that $f_Z$ is a pseudo-automorphism. There is a two-parameter family of surfaces $S_c$, $c = (c_1,c_2) \in \mathbb{C}^2$ which are invariant under $f^3$. For generic $c$ and $c'$, $S_c$ is a smooth $K3$ surface, and $S_c \cap S_{c'}$ is a smooth elliptic curve.

For the mappings in Theorems 1 and 3, $f$ is reversible on the level of cohomology: $f_Z^* f_Z^{(−1)*} = (f_Z^*)^{-1}$. The identity $\delta_1(f) = \delta_2(f)$ for such maps is a consequence of the duality between $H^{1,1}$ and $H^{2,2}$, so they are not cohomologically hyperbolic, in the terminology of [G2]. For each of these maps, the family of invariant $K3$ surfaces becomes singular at an invariant 8-cycle $\mathcal{R}$ of rational surfaces (see (7.2)). We show that the restriction $f|_\mathcal{R}$ is not birationally conjugate to a surface automorphism: see Appendix C for the maps in Theorem 1 and Proposition 8.2 for the maps in Theorem 3. By Corollary 1.6, then, we have:

**Theorem 4.** Let $f$ be a map from Theorems 1 and 3. If $a \neq 1$, then $f$ is not birationally conjugate to an automorphism.

We note that for birational surface maps, the degree growth of the iterates determines whether the map is birationally conjugate to an automorphism: This occurs if and only if either (i) the degrees are bounded or degree growth is quadratic (see [DiF]), or (ii) if the dynamical degree is a Salem number (see [BC]). Theorem 4 shows that this result does not hold in dimension 3.

We will also determine which mappings $f_{\alpha,\beta}$ are periodic, or finite order, in the sense that $f^n = id$ for some $p > 0$. In contrast to Theorem 4, it was shown by de Fernex and Ein [dFE] that if $f$ is a rational map of finite order, then there is a modification $f_X$ as above, which is an automorphism of $X$. If $f_X$ is periodic, then $f_X^n$ will also be periodic.

In (4.1) and (4.2) we identify conditions which are necessary for $f$ to be periodic and are sufficient for the existence of a space $Z = Z_{\alpha,\beta}$ such that $f_Z$ is a pseudo-automorphism. We show that for a map in (0.1), if $f_Z$ is periodic, then $f$ also turns out to be periodic. The birational map (0.1) may also be considered as a 3-step linear fractional recurrence: given $z_0$, $z_1$, $z_2$, we define a sequence $\{z_n\}$ by

$$z_{n+3} = \frac{\alpha_0 + \alpha_1 z_n + \alpha_2 z_{n+1} + \alpha_3 z_{n+2}}{\beta_0 + \beta_1 z_n + \beta_2 z_{n+1} + \beta_3 z_{n+2}}$$  \hspace{1cm} (0.2)$$

The recurrence (0.2) is said to be periodic if the sequence $\{z_n\}$ is periodic for all choices of initial terms $z_0$, $z_1$ and $z_2$. Equivalently, $f_{\alpha,\beta}^p = id$ for some $p$. For all $r > 0$ there are $r$-step recurrences of the form (0.2). In [BK2] we determined the possible periods for 2-step linear
fractional recurrences. McMullen [M1] has explained the periods that arise by showing that
the corresponding (2-dimensional) $f_{\alpha,\beta}$ represent certain Coxeter elements.

Here we determine all possible periods for 3-step recurrences (0.2). To rule out trivial
cases, we assume that the coefficients satisfy (2.3), and we have:

**Theorem 5.** The only nontrivial periods for (0.2) are 8 and 12. Each periodic recurrence is
equivalent to one of the following:

$$z_{n+3} = \frac{1 + z_{n+1} + z_{n+2}}{z_n} \quad z_{n+3} = \frac{-1 - z_{n+1} + z_{n+2}}{z_n} \quad (\text{period 8})$$

$$z_{n+3} = \frac{\eta/(1 - \eta) + \eta z_{n+1} + z_{n+2}}{\eta^2 + z_n} \quad \eta^3 = -1 \quad (\text{period 12})$$

In the notation of (0.1), the first case corresponds to $\beta = (0, 1, 0, 0), \alpha = (\pm 1, 0, \pm 1, 1)$, and
the second case to $\beta = (\eta^2, 1, 0, 0), \alpha = (\eta/(1 - \eta), 0, \eta, 1)$.

Each of these mappings has a different structure; these structures are described in Theorems 6.10 and 6.11. The first period 8 recurrence above was found by Lyness [L], and the second one was found by Cs"oronyi and Laczkovic [CL] (see also [CGM1]). We note that the
der period 12 recurrences are the case $k = 3$ of a general phenomenon exhibited in [BK4]: For each
$k$, there are $k$-step linear fractional recurrences with period $4k$. There is a literature dealing
with $r$ step recurrences of the form (0.2). We refer to the books [KoL], [KuL], [GL], [CaL] and
the extensive bibliographies they contain. That direction of research is largely concerned with
the case where the structural parameters $\alpha, \beta$, as well as the dynamical points, are real and
positive. This avoids the difficulty that the denominator in (0.2) might vanish, causing the
expression to be undefined; but the restriction to positive numbers leads to a subdivision into
a large number of distinct cases to be treated separately.

In working with the family $f_{\alpha,\beta}$, we work with the pointwise iterates as much as possible,
but this runs into difficulties if the orbit enters the indeterminacy locus. We can often deal with
this by blowing up certain subsets. In this way we convert these subsets into hypersurfaces,
and we then deal with the hypersurfaces by passing to $f^*$ on Pic. This allows us to convert
many difficulties with indeterminate orbits into more tractable problems of Linear Algebra.

This paper is organized as follows. §1 assembles some general information about rational
maps and the geometry of blowing up. §2 gives the specific behaviors of the maps (0.1). It
is evident, then, that there are two possibilities, defined by (3.1), which we call “critical”
and “non-critical,” and in §3 we show that any periodic map must be critical. We study the
structure of general critical maps in §4. In Theorem 5.1 we show that if $f$ is a critical map
satisfying (5.1), then $f_Z$ is a pseudo-automorphism. Pseudo-automorphisms are discussed in
§5, together with the possibilities for the induced map $f^*_Z$ on cohomology. In §6 we determine
the periodic mappings and give the proof of Theorem 5. In §7 we give the proof of Theorems 1
and 2. At the end of §7 we present a different pseudo-automorphism with positive entropy; it
has properties similar to those given in Theorems 1 and 2, but we do not discuss it in detail.
The proof of Theorem 3 is given in §8.

§1. Rational Maps A rational map $f : P^d \to P^d$ is given by a $(d+1)$-tuple of homogeneous
polynomials, all of the same degree: $f = [f_0 : \cdots : f_d]$. We may divide $f$ by g.c.d.$(f_0, \ldots, f_d)$
so that $f_j$’s have no common polynomial factor. We define the degree of $f$, $\deg(f)$, to be the (common) degree of the $f_j$’s. The indeterminacy locus of $f$ is defined by

$$\mathcal{I}(f) = \{x \in \mathbb{P}^d : f_0(x) = \cdots = f_d(x) = 0\}$$

and is a subvariety of codimension at least 2, and $f$ defines a holomorphic mapping $f : \mathbb{P}^d \setminus \mathcal{I}(f) \to \mathbb{P}^d$. If $S$ is an irreducible subvariety of $\mathbb{P}^d$, and $S \not\subset \mathcal{I}(f)$, we define the strict transform, written simply as $f(S)$, to be the closure of $f(S - \mathcal{I}(f))$. We say that an irreducible variety $V$ is exceptional for a rational mapping $f$ if $V \not\subset \mathcal{I}(f)$, and if the dimension of $f(V - \mathcal{I}(f))$ is strictly less than the dimension of $V$. Following [DO, p. 64], we say that $f : X \to Y$ is a pseudo-isomorphism if $f$ is birational, and if neither $f$ nor $f^{-1}$ has an exceptional hypersurface. It follows that if $f$ is a pseudo-isomorphism, then $f : X \setminus \mathcal{I}(f) \to Y \setminus \mathcal{I}(f^{-1})$ is biholomorphic. If $X = Y$, we say that $f$ is a pseudo-automorphism.

**Theorem 1.1.** If $f : X \to Y$ is a pseudo-isomorphism between 3-dimensional manifolds, then the indeterminacy locus has no isolated points.

**Proof.** Suppose that there is an isolated point $p \in \mathcal{I}(f)$. Since $f^{-1}$ has no exceptional hypersurfaces, $f$ must blow $p$ up to a curve $C' \subset Y$. Now we consider the behavior of $f^{-1}$ on $C'$. We must have $C' \subset \mathcal{I}(f^{-1})$, for if $f^{-1}$ is regular at a point $q \in C'$, then $f^{-1}$ must map an open subset of $C'$ to $p$. Thus the jacobian of $f^{-1}$ must vanish at $q$. Since the jacobian vanishes on a hypersurface, $f^{-1}$ would have an exceptional hypersurface containing $q$. Thus $q$ must be indeterminate. Since the total transform of $q$ under $f^{-1}$ is given by $\bigcap_{\epsilon > 0} (f^{-1}(B(q, \epsilon) - \mathcal{I}(f^{-1})))$, it must be connected, and it must be a curve $C$ containing $p$. But since $p$ was an isolated point of $\mathcal{I}(f)$, there are nearby points $p' \in C - \mathcal{I}(f)$. Since $f$ is regular at these points, it must map them to $q$, and thus $f$ must have an exceptional hypersurface. By this contradiction, we see that $\mathcal{I}(f)$ has no isolated points.

For a rational map $f : X \to X$, we consider the iterates $f^j = f \circ \cdots \circ f$, $j > 0$. If $\Sigma$ is an irreducible hypersurface, then $\Sigma \not\subset \mathcal{I}(f^j)$ for reasons of dimension, so we may consider the sequence of varieties $V_j := f^j(\Sigma)$, for $j > 0$. Since we will be interested in knowing to what extent the iterates of $f$ behave like a pointwise-defined dynamical system, we note: If $S \not\subset \mathcal{I}(g)$ is irreducible and if $g(S) \not\subset \mathcal{I}(f)$, then $S \not\subset \mathcal{I}(f \circ g)$, and $f(\gamma_f) = (f \circ g)(\gamma_f)$. We may also define $f$ at points of indeterminacy. Let $\gamma_f = \{(x, y) \in (\mathbb{P}^d - \mathcal{I}) \times \mathbb{P}^d : y = f(x)\}$ denote the graph of $f$ at its regular points, and we let $\Gamma$ denote the closure of $\gamma_f$ inside $\mathbb{P}^d \times \mathbb{P}^d$. It follows that $\Gamma$ is an irreducible variety of dimension $d$, and there are holomorphic projections $\pi_j : \Gamma \to \mathbb{P}^d$, $j = 1, 2$, onto the first and second factors, respectively, and we have $f = \pi_2 \circ \pi_1^{-1}$ on $\mathbb{P}^d - \mathcal{I}$. For a point $p \in \mathbb{P}^d$, we define the total transform to be $f_* p := \pi_2(\pi_1^{-1} p)$, and then we define $f_*(g(S)) := \bigcup_{p \in S} f_* p$. It is easily seen that we have: If $\Sigma$ is an irreducible hypersurface, then $f_*(\gamma_f(\Sigma)) \supset (f \circ g)(\Sigma)$.

**Proposition 1.2.** Suppose that $f : X \to X$ is rational, and suppose that for each exceptional hypersurface $E$ and for $m > 0$, we have $f^m(E - \mathcal{I}) \not\subset \mathcal{I}$. If follows that $(f^*)^n = (f^n)^*$ on $H^{1,1}(X)$.

**Proof.** It is sufficient to show that $(f^*)^2 = (f^2)^*$ on $Pic(X)$. If $D$ is a divisor, then $f^* D$ is the divisor on $X$ which is the same as $f^{-1} D$ on $X - \mathcal{I}$. Since $\mathcal{I}$ has codimension at least 2, we also have $(f^2)^* D = f^*(f^* D)$ on $X - \mathcal{I} - f^{-1}(\mathcal{I})$. By our hypothesis $f^{-1}(\mathcal{I})$ has codimension at least 2. Thus we have $(f^2)^* D = (f^*)^2 D$ on $X$. 

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In a similar way, we may define \( f^* : H^{p,q}(X) \to H^{p,q}(X) \). That is, if \( \beta \) is a \((p,q)\) form on X, then the pullback \( \pi_2^* \beta \) is a smooth form on \( \Gamma \). We may let \([\pi_2^* \beta]\) denote the reinterpretation of the form as a current, and we may push it forward to obtain a current \( f^* \beta = \pi_1_*[\pi_2^* \beta] \) on X. This pulls smooth forms back to currents and is well defined at the level of cohomology classes. If \( \alpha \in H^{p',q'} \) is an element of the dual cohomology group, then we have \( \langle \alpha, f^* \beta \rangle = \langle \pi_1^* \alpha, \pi_2^* \beta \rangle \).

Now if \( f \) is birational and \( g = f^{-1} \), then

\[
\langle g^* \alpha, \beta \rangle = \langle \pi_1^* \alpha, \pi_2^* \beta \rangle = \langle \alpha, f^* \beta \rangle
\]  

(1.1)

If we have \((f^n)^* = (f^*)^n\) on \( H^{p,q} \) for \( n \geq 0 \), then this gives us \((g^n)^* = (g^*)^n\) on \( H^{p',q'} \).

The following is proved along standard lines:

**Proposition 1.3.** If \( f \) is a pseudo-automorphism, then we have \((f^n)^* = (f^*)^n\) on \( H^{1,1} \) for all \( n \in \mathbb{Z} \). In particular, \((f^{-1})^* = (f^*)^{-1}\).

From this we get the following:

**Proposition 1.4.** If \( f : X \to X \) is a pseudo-automorphism on a \( d \)-dimensional manifold, then for all \( n \in \mathbb{Z} \) we have \((f^*)^n = (f^n)^*\) on both \( H^{1,1} \) and \( H^{d-1,d-1} \). Further, the characteristic polynomials of \( f^* \) on \( H^{1,1} \) and \( H^{d-1,d-1} \) are the same and therefore the first and the \( d - 1 \)st dynamical degree are the same.

**Proof.** From Proposition 1.3, we have \( f^*(f^{-1})^* = id \) and so we have that if \( \alpha \in H^{1,1} \) and \( \beta \in H^{d-1,d-1} \), then \( f^* \alpha \cdot f^* \beta = ((f^{-1})^* f^*) \alpha \cdot \beta = \alpha \cdot \beta \). Further, we conclude that \((f^*)^n = (f^n)^*\) on \( H^{d-1,d-1} \) for all \( n \in \mathbb{Z} \).

Now let us define some specific blowup situations. This will serve to define the constructions we will use in the sequel, and it allows us to exhibit the models of indeterminate behavior that we will encounter.

**Blowing up a point and a line which contains it.** We use \((x_0, x_1, x_2) \mapsto [x_0 : x_1 : x_2 : 1]\) as local coordinates in a neighborhood of \( e_3 := [0 : 0 : 0 : 1] \in \mathbb{P}^3 \). Let \( X_1 \) be the space obtained by blowing up a point \( e_3 \) and we let \( E_3 \) denote the fiber over \( e_3 \). We may use

\[
\pi_1 : X_1 \ni (s_0, s_1, \xi_2)_1 \mapsto [\xi_2 s_0 : \xi_2 s_1 : \xi_2 : 1] \in \mathbb{P}^3
\]

(1.2)
as a local coordinate system for a neighborhood of \( E_3 \cap \{ x_0 = x_1 = 0 \} \) in \( X_1 \). It follows that the exceptional fiber \( E_3 = \{ \xi_2 = 0 \} \) in this coordinate system.

Let \( \Sigma_{01} = \{ x_0 = x_1 = 0 \} \subset \mathbb{P}^3 \) denote the \( x_2 \)-axis. The strict transform of \( \Sigma_{01} \) inside \( X_1 \) may be written as \( \Sigma_{01} = \{ s_0 = s_1 = 0 \} \). Thus \( \Sigma_{01} \cap E_3 = \{ s_0 = s_1 = \xi_2 = 0 \} \). Let \( X_2 \) be a complex manifold obtained by blowing up \( \Sigma_{01} \) in \( X_1 \). We can define a local coordinate system of \( X_2 \) via \( \pi_2 : X_2 \ni (t_0, \eta_1, \xi_2)_2 \mapsto (t_0 \eta_1, \eta_1, \xi_2)_1 \in X_1 \). Thus \( \pi_2 \circ \pi_1 : X_2 \to \mathbb{P}^3 \) is given, in this coordinate neighborhood, by

\[
\pi_1 \circ \pi_2 : X_2 \ni (t_0, \eta_1, \xi_2)_2 \mapsto [t_0 \eta_1 \xi_2 : \eta_1 \xi_2 : \xi_2 : 1] \in \mathbb{P}^3.
\]

(1.3)
The inverse of \( \pi_1 \) (resp. \( \pi_2 \)) gives a model of indeterminate behavior that blows up the point \((0, 0, 0)\) (resp. the line \( \{ x_1 = x_2 = 0 \} \)) to a hyperplane:

\[
\pi_1^{-1} : (x_1, x_2, x_3) \mapsto (x_1/x_3, x_2/x_3, x_3), \quad \pi_2^{-1} : (x_1, x_2, x_3) \mapsto (x_1/x_2, x_2, x_3).
\]

(1.4)
Blowing up two intersecting lines. Let \( \pi_1 : Z_1 \to \mathbb{P}^3 \) be the blowup of the \( x_1 \)-axis \( \Sigma_{02} = \{x_0 = x_2 = 0\} \subset \mathbb{P}^3 \). We use local coordinate system in \( Z_1 \)

\[
\pi_1 : Z_1 \ni (\xi, x, s)_{Z_1} \mapsto [s\xi : x : s : 1] \in \mathbb{P}^3
\]

Let us denote the blowup fiber over the point \( o = \Sigma_{01} \cap \Sigma_{02} = [0 : 0 : 0 : 1] \in \mathbb{P}^3 \) as \( \mathcal{F}_o^1 \) then in this coordinate system we have \( \mathcal{F}_o^1 = \{s = x = 0\} \). The strict transform of the \( x_2 \)-axis in \( Z_1 \) is given by \( \ell_2 = \{\xi = x = 0\} \) and \( \mathcal{F}_o^1 \cap \ell_2 = (0, 0, 0)_{Z_1} \). Now let \( Z_2 \) be the blow up of \( \ell_2 \) with a local coordinate system

\[
\pi := \pi_1 \circ \pi_2 : (t, \eta, s)_{Z_2} \in Z_2 \mapsto (t, t\eta, s)_{Z_1} \in Z_1
\]

\[
\mapsto [ts : t\eta : s : 1] \in \mathbb{P}^3
\]

We denote the second (new) fiber over \( o \) as \( \mathcal{F}_o^2 \), so \( \mathcal{F}_o^2 = (0, \eta, 0)_{Z_2} \). Let us also use \( \mathcal{F}_o^1 \) for its strict transform in \( Z_2 \), so \( \mathcal{F}_o^1 \cup \mathcal{F}_o^2 = \pi_2^{-1} \circ \pi_1^{-1} \{o\} \) and \( \mathcal{F}_o^1 = (t, 0, 0)_{Z_2} \).

Let \( \tau[x_0 : x_1 : x_2 : x_3] = [x_0 : x_2 : x_1 : x_3] \) be the involution that interchanges the \( x_1 \)- and \( x_2 \)-axes. It follows that \( \tau \) induces the involution \( \tilde{\tau} = \pi^{-1} \circ \tau \circ \pi \) on \( Z_2 \). In coordinates, we have

\[
\tilde{\tau} : (t, \eta, s) \mapsto (s/\eta, \eta, t\eta),
\]

which will serve as our third model of indeterminate behavior. We note that \( \tilde{\tau} \) is regular on \( \mathcal{F}_o^2 - \mathcal{F}_o^1 \), while each point of \( \mathcal{F}_o^1 \) blows up to the variety \( \mathcal{F}_o^1 \).

Similarly we can blow up the \( x_2 \)-axis first and then the strict transform of \( x_1 \)-axis. Performing similar computations, we obtain a blowup space \( \hat{\pi} : Y_2 \to \mathbb{C}^3 \). The identity map \( \hat{\pi} \) on \( \mathbb{P}^3 \) lifts to a map \( \hat{\iota} : Z_2 \to Y_2 \), which in local coordinates is similar to \( \tilde{\tau} \).

Remark. Suppose that \( \gamma' \) and \( \gamma'' \) are curves in \( \mathbb{P}^3 \) which intersect transversally at points \( \{p_1, \ldots, p_N\} \). We have local coordinate systems for \( 1 \leq j \leq N \) so that \( p_j \) is the origin, and \( \gamma' \) (resp. \( \gamma'' \)) coincides with the \( x \)-axis (resp. the \( y \)-axis) in a neighborhood of \( p_j \). Since the operation of blowing up the axes is local near \( p_j \), we may construct a blowup space \( \pi : W \to \mathbb{P}^3 \) in which \( \gamma' \) and \( \gamma'' \) are both blown up, and over each \( p_j \) we are free to choose whether \( \gamma' \) or \( \gamma'' \) was blown up first, independently of the choices over \( p_k \) for \( k \neq j \).

Theorem 1.5. Let \( f \) be a birational map of \( X \). Let \( X_0 \subset X \) be a hypersurface such that the strict transform is \( f(X_0) = X_0 \). Let \( \varphi : X \to Y \) is a birational map which conjugates \( (f, X) \) to an automorphism \( (g, Y) \). Then there is a birational map \( \hat{\varphi} : X \to \hat{Y} \) such that the strict transform \( \hat{Y}_0 := \hat{\varphi}(X_0) \) is a nonsingular hypersurface, and the induced map \( \hat{g} := \hat{\varphi} \circ f \circ \hat{\varphi}^{-1} \) gives an automorphism of \( \hat{Y} \).

Proof. We may assume that \( X_0 \) is irreducible. Since \( X_0 \) is a hypersurface, we may take its strict transform \( \varphi(X_0) \). If \( \varphi(X_0) \) is a point in \( Y \), then it is fixed by \( g \). If \( \pi_1 : Y_1 \to Y \) be the blowup of the point \( \varphi(X_0) \), then \( g \) lifts to an automorphism of \( Y_1 \). Let \( \phi_1 := \pi_1^{-1} \circ \varphi \). If \( \varphi_1(X_0) \) is again a point, we can repeat this blowing-up process until \( \varphi_1(X_0) \) has dimension \( > 0 \), which we may assume to be 1. If the singular locus of \( \varphi_1(X_0) \) is nonempty, it is finite and invariant under \( f_1 \). Now we can blow up the singular set of \( \varphi_1(X_0) \) finitely many times and have a new blow up space \( \pi_2 : Y_2 \to Y_1 \). Since we were blowing up invariant sets, the induced birational map \( g_2 \) of \( Y_2 \) is again an automorphism. Now the image \( \varphi_2(X_0) \) must be a nonsingular curve, which must be invariant. We can blow up this curve, and repeat the process finitely many times so that \( \varphi_3(X_0) \) has dimension \( > 1 \). We continue this process until \( \varphi_N(X_0) \) is a nonsingular hypersurface in \( Y_N \), and now we set \( \hat{Y} = Y_N \).
Corollary 1.6. Let $f$ be a birational map of $X$. Let $X_0 \subset X$ be a hypersurface for which the strict transform is $f(X_0) = X_0$. Let $\varphi : X \to Y$ is a birational map which conjugates $(f, X)$ to an automorphism $(g, Y)$. Then the restriction $(f_{X_0}, X_0)$ is birationally conjugate to an automorphism.

Proof of Theorem 4. Let $f$ be as in Theorem 4. In §C we study the restriction of $f^S$ to the plane $\Sigma_3 = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 : x_3 = 0\}$. There we show that this restricted mapping is not birationally equivalent to an automorphism of $\Sigma_3$. Thus Theorem 4 is a consequence of Corollary 1.6. \hfill \Box

§2. Linear fractional recurrences. The maps (2.2) are among the Cremona transformations of 3-space which are discussed in Chapter 10 of [H]. We discuss general properties of these transformations, and for the generic parameters (2.10) we construct a new space $\pi : X \to \mathbb{P}^3$, such that passing to the induced map $f_X$ effectively eliminates one of the exceptional components.

For $\{i_1, \ldots, i_k\} \subset \{0, 1, 2, 3\}$, we use the notation

$$\Sigma_{i_1, \ldots, i_k} = \{x \in \mathbb{P}^3 : x_{i_j} = 0, 1 \leq j \leq k\}, \quad (2.1)$$

and for a vector $A = (A_0, \ldots, A_3)$ we will write $A \cdot x = A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3$. In homogeneous coordinates the maps (0.1) take the form

$$f[x_0 : x_1 : x_2 : x_3] = [x_0\beta \cdot x : x_2\beta \cdot x : x_3\beta \cdot x : x_0\alpha \cdot x] \quad (2.2)$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)$. In the sequel, we will assume

$$\alpha \neq \lambda \beta, \quad \beta \neq (\beta_0, 0, 0, 0), \quad (\alpha_1, \beta_1) \neq (0, 0). \quad (2.3)$$

Note that if one of the first two conditions does not hold, then $f$ is linear, and if the third condition does not hold, then $f$ is independent of $x_1$ and thus $f$ is actually a 2-step recurrence. If we set $\gamma = \beta_1\alpha - \alpha_1\beta$, then we have

$$\mathcal{I} = \Sigma_{\beta\gamma} \cup \Sigma_{\alpha\beta} \cup \{e_1\}$$

where $\Sigma_\beta = \{\beta \cdot x = 0\}$, $\Sigma_\gamma = \{\gamma \cdot x = 0\}$, $\Sigma_0 = \{x_0 = 0\}$, $\Sigma_{\beta\gamma} = \Sigma_\beta \cap \Sigma_\gamma$, $\Sigma_{\alpha\beta} = \Sigma_0 \cap \Sigma_\beta$, and $e_1 = [0 : 1 : 0 : 0] = \Sigma_{023}$.

The Jacobian determinant of $f$ is given by $2x_0(\gamma \cdot x)(\beta \cdot x)^2$. Thus we see that the exceptional hypersurfaces are $\mathcal{E} = \{\Sigma_0, \Sigma_\beta, \Sigma_\gamma\}$. The action of $f$ on the exceptional varieties is given as follows: for $\lambda_2, \lambda_3 \in \mathbb{C}, (\lambda_2, \lambda_3) \neq (0, 0),$

$$\Sigma_\beta \mapsto e_3,$$

$$f : \Sigma_0 \cap \{\lambda_2x_2 = \lambda_3x_3\} \mapsto [0 : \lambda_3 : \lambda_2 : 0],$$

$$\Sigma_\gamma \cap \{\lambda_2x_2 = \lambda_3x_3\} \mapsto \Sigma_{BC} \cap \{\lambda_2x_1 = \lambda_3x_2\}$$

where we set $\breve{\alpha} = (\alpha_0, \alpha_2, \alpha_3, 0)$, $\breve{\beta} = (\beta_0, \beta_2, \beta_3, 0)$, and

$$B = (-\alpha_1, 0, 0, 0), \quad C = \beta_1\breve{\alpha} - \alpha_1\breve{\beta}.$$
Thus \(\Sigma_\beta\) is blown down to a point. The pencil of lines in \(\Sigma_\gamma\) passing through \(e_1 \in \Sigma_0 \cap \Sigma_\gamma\) are all mapped to points in \(\Sigma_{BC}\). The pencil of lines in \(\Sigma_0\) passing through \(e_1\) are all mapped to points on the line \(\Sigma_{03}\), which is again one of the exceptional lines. We have strict transforms:

\[
f : \Sigma_0 \mapsto \Sigma_{03} \mapsto e_1
\]

The inverse is given by

\[
f^{-1}[x_0 : x_1 : x_2 : x_3] = [x_0 B \cdot x : x_0 \alpha \cdot x - x_3 \beta \cdot x : x_1 B \cdot x : x_2 B \cdot x],
\]

and the indeterminacy locus is \(I(f^{-1}) = \Sigma_{0B} \cup \Sigma_{BC} \cup \{e_3\}\). The Jacobian of \(f^{-1}\) is \(2 x_0 (C \cdot x)(B \cdot x)^2\), so the exceptional hypersurfaces are \(E(f^{-1}) = \{\Sigma_0, \Sigma_B, \Sigma_C\}\) and for \(\mu_1, \mu_2 \in C, (\mu_1, \mu_2) \neq (0, 0)\),

\[
\Sigma_B \mapsto e_1,
\]

\[
f^{-1} : \Sigma_0 \cap \{\mu_1 x_1 = \mu_2 x_2\} \mapsto \Sigma_{0\beta} \cap \{\mu_1 x_2 = \mu_2 x_3\},
\]

\[
\Sigma_C \cap \{\mu_1 x_1 = \mu_2 x_2\} \mapsto \Sigma_{\beta \gamma} \cap \{\mu_1 x_2 = \mu_2 x_3\}.
\]

Now let us construct the space \(\pi_1 : X_1 \to \mathbb{P}^3\) by blowing up a point \(e_1\), and then the space \(\pi_2 : X \to X_1\) obtained by blowing up a line \(\Sigma_{03}\). We set

\[
\pi = \pi_1 \circ \pi_2 : X \to \mathbb{P}^3.
\]

Let \(S_{03}\) denote the blowup fiber over the strict transformation of \(\Sigma_{03}\) in \(X_1\) and \(E_1\) for the strict transformation of \(\pi_1^{-1} e_1\) in \(X_1\). For the induced map on \(X\), the orbit of \(\Sigma_0\) becomes

\[
f_X : \Sigma_0 \to S_{03} \to E_1 \to \Sigma_B
\]

If \(X\) and \(Y\) are irreducible, we will say that a rational map \(f : X \dashrightarrow Y\) is dominant if the rank of \(df\) is equal to the dimension of \(Y\) on a dense open set. Let us define a generic condition:

\[
\beta_1 \neq 0, \quad \beta_1 \alpha_2 \neq \alpha_1 \beta_2, \quad \text{and} \quad \beta_1 \alpha_3 \neq \alpha_1 \beta_3.
\]

For simplicity we use the same notation for both a variety and its strict transform, if there is no possibility of confusion.

**Proposition 2.1.** If (2.10) holds, then all the maps in (2.9) are dominant, so \(E(f_X) = \{\Sigma_\beta, \Sigma_\gamma\}\). Further, \(I(f_X) = \Sigma_{0\beta} \cup \Sigma_{\beta \gamma}\).

**Proof.** Let us first consider the restriction to \(S_{03}\). We may use the local coordinates for a neighborhood of \(S_{03}\), \(\{(s_0, x_2, \xi_3)_{S_{03}} \mapsto [s_0 : 1 : x_2 : s_0 \xi_3]\}\). For the neighborhood of the exceptional fiber \(E_1\) over \(e_1\), we use \((t_0, \xi_3, \xi_3)_{E_1} \mapsto [t_0 : 1 : t_0 \xi_2 : t_0 \xi_3]\). It follows that \(S_{03} = \{(0, x_2, \xi_3)_{S_{03}}\}\) and \(E_1 = \{(0, \xi_2, \xi_3)_{E_1}\}\). Using these local coordinates we have

\[
f_X|_{S_{03}} : (0, x_2, \xi_3)_{S_{03}} \mapsto \left(0, \xi_3, \frac{\alpha_1 + \alpha_2 x_2}{\beta_1 + \beta_2 x_2} \right)_{E_1}.
\]

To have a dominant map, it is required that \(\beta_1 \alpha_2 \neq \alpha_1 \beta_2\). For the restrictions of the induced birational map to \(S_{03}\) and \(E_1\) are given by linear maps:

\[
f_X : \Sigma_0 \ni [0 : x_1 : x_2 : x_3] \mapsto \left(0, \frac{x_3}{x_2}, \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3} \right)_{S_{03}}
\]

\[
f_X : E_1 \ni (0, \xi_2, \xi_3)_{E_1} \mapsto [\beta_1 : \beta_1 \xi_2 : \beta_1 \xi_3 : \alpha_1] \in \Sigma_B.
\]

We see that \(f_X|_{E_1}\) is dominant because \(\beta_1 \neq 0\). And since \(\beta_1 \alpha_2 \neq \alpha_1 \beta_2\) and \(\beta_1 \alpha_3 \neq \alpha_1 \beta_3\), we see that \(f_X|_{\Sigma_0}\) is dominant. \(\square\)
Thus in passing to $f_X$, we have removed one exceptional hypersurface and one point of indeterminacy. There is a group of linear conjugacies acting on the family (0.2). For $(\lambda, c, \mu) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}$, we set

\[
(\alpha, \beta) \mapsto (\lambda \alpha, \lambda \beta)
\]

(2.11a)

\[
(\alpha, \beta) \mapsto (\alpha_0, c\alpha_1, c^2\beta_1, c^2 \beta_2, c^2 \beta_3)
\]

(2.11b)

\[
(\alpha, \beta) \mapsto (\alpha_0', \alpha_1', \alpha_2', \beta_0', \beta_1, \beta_2, \beta_3)
\]

(2.11c)

\[
\alpha_0' = \alpha_0 + \mu(\alpha_1 + \alpha_2 + \alpha_3) + \mu(\beta_0 + \mu \beta_1 + \mu \beta_2 + \mu \beta_3), \quad \alpha_1' = \alpha_1 - \mu \beta_1,
\]

\[
\alpha_2' = \alpha_2 - \mu \beta_2, \quad \alpha_3' = \alpha_3 - \mu \beta_3, \quad \beta_0' = \beta_0 + \mu(\beta_1 + \beta_2 + \beta_3).
\]

The first action corresponds to the homogeneity of $f$. The action (2.11b) corresponds to a scaling of $(x_1, x_2, x_3)$ in affine coordinates, and (2.11c) comes from translation by the vector $(\mu, \mu, \mu)$. Note that these actions preserve the form of the recurrence relation.

§3. Non-critical maps. A map $f$ of the form (2.2) is critical if (3.1) holds:

\[
\beta_2 = \beta_3 = 0, \quad \text{and} \quad \beta_1 \alpha_2 \alpha_3 \neq 0.
\]

(3.1)

In this section we establish the following:

**Theorem 3.1.** If $f$ is not critical, then $f$ is not periodic.

We will use the following criterion:

**Proposition 3.2.** Suppose that $f : X \to X$ is periodic, i.e., $f^p_X$ is the identity for some $p > 1$. If $E \subset X$ is an exceptional hypersurface, then $f^j E \subset \mathcal{I}(f_X)$ for some $1 \leq j < p$.

**Proof.** Since $E$ is exceptional, then $\text{codim}(f(E)) \geq 2$. Let us consider the sequence of varieties $V_j := f_X^j(\Sigma)$. If $V_j \not\subset \mathcal{I}(f_X)$ for all $j$, then applying the strict transform of $f$ repeatedly, we have $f_X^{j+1}(E) = f_X(V_j)$ for all $j$, so $\text{codim}(f(V_j)) \geq 2$ for all $j$. On the other hand, we must have $f_X^p(E) = E = V_p$. $lacksquare$

The proof of Theorem 3.1 will involve several cases, so we start with some Lemmas.

**Lemma 3.3.** Let $\pi : X \to \mathbb{P}^3$ be the complex manifold defined in (2.8). If $\beta_1 = 0$, then there is a hypersurface $V \subset X$ such that $f_X^n V$ is a point of $X - \mathcal{I}$ either for all $n \geq 1$ or for all $n \leq -1$.

**Proof.** We use the local coordinates for a neighborhood of $S_{03}$ and a neighborhood of $E_1$ defined in §2. Since $\beta \neq (\beta_0, 0, 0, 0)$, we first assume that $\beta_3 \neq 0$ (and thus we may suppose $\beta_3 = 1$) and to consider various cases.

(i) Case $\beta_2 \neq 0$: In this case, the orbit of $\Sigma_\beta$ is:

\[
f_X : \Sigma_\beta \mapsto e_3 \mapsto (0, 0, \alpha_3)_{S_{03}} \mapsto (0, \alpha_3, \alpha_2/\beta_2)_{E_1} \mapsto e_3 \in \Sigma_0 \setminus \mathcal{I}
\]

Thus the orbit is periodic and remains a regular point of $f_X$.

(ii) Case $\beta_2 = 0, \alpha_2 = 0$. If $\beta_0 + \alpha_3 = 0$, then we have $f_X^2(e_3) = e_3$. If $\beta_0 + \alpha_3 \neq 0$, then $f_X^3(e_3) = e_3$. In both cases, the orbit of $\Sigma_\beta$ is pre-periodic.

(iii) Case $\beta_2 = 0, \alpha_2 \neq 0$. With this parameters, we have a two-cycle between $\Sigma_{02}$ and the fiber over $e_2$, $\{(0, 0, \xi_3)_{S_{03}} : \xi_3 \in \mathbb{C}\}$.

\[
f_X : \begin{cases}
[0 : x_1 : 0 : x_3] \in \Sigma_{02} \mapsto (0, 0, (\alpha_1 x_1 + \alpha_3 x_3)/x_3)_{S_{03}} \\
(0, 0, \xi_3)_{S_{03}} \mapsto [0 : \beta_0 + \xi_3 : 0 : \alpha_2] \in \Sigma_{02}.
\end{cases}
\]
Since $f_X^2(e_3) = [0 : \beta_0 + \alpha_3 : 0 : \alpha_2]$, and both $\Sigma_{02}$ and the fiber over $e_2$ in $S_{03}$ are disjoint from $\mathcal{I}$, the forward orbit of $\Sigma_{\beta}$ is disjoint from $\mathcal{I}$.

(iv) Case $\beta_3 = 0$ and $\beta_2 \neq 0$. Under the backward mapping, $\Sigma_0$ is exceptional. To see the backward iteration, let us use a different local coordinate system in a neighborhood of $E_1$:

$$(\zeta_0, \zeta_2, t_3)_{E_1} \mapsto [\zeta_0 t_3 : 1 : \zeta_2 t_3 : t_3] \in \mathbb{P}^3.$$

Using this local coordinates we see

$$\Sigma_0 \ni [0 : x_1 : x_2 : x_3] \mapsto (0, x_1/x_2, 0)_{E_1} \in E_1 \cap \Sigma_0$$

$$f_X^{-1} : E_1 \ni (\zeta_0, \zeta_2, 0)_{E_1} \mapsto (0, \alpha_1 \zeta_0 / (\beta_2 - \alpha_2 \zeta_0), \zeta_2 / \zeta_0)_{S_{03}} \in S_{03}$$

$$S_{03} \ni (0, x_2, \xi_3)_{S_{03}} \mapsto [0 : -\alpha_2 - \alpha_3 x_2 + \beta_2 \xi_3 : \alpha_1 : \alpha_1 x_2] \in \Sigma_0$$

Let $p = \Sigma_0 \cap E_1 \cap S_{03}$. It follows that $f_X^{-1} p = p$, that is, $p$ is a fixed point for the inverse mapping.

Now let us suppose that $\beta_1 \neq 0$. Using actions (2.11a–c), we may assume that $\beta_1 = 1$ and $\alpha_1 = 0$.

**Lemma 3.4.** Suppose that $\beta_1 = 1, \alpha_1 = 0$. If either $\beta_2 \neq 0$ or $\beta_3 \neq 0$, then $\Sigma_0$ is exceptional and preperiodic for $f^{-1}$.

**Proof.** Let us first assume that $\beta_3 \neq 0$. Then

$$f^{-1} : \Sigma_0 \ni [0 : x_1 : x_2 : x_3] \mapsto [0 : -(\beta_2 x_1 + \beta_3 x_2) : x_1 : x_2] \in \Sigma_{03}$$

and $\Sigma_{03}$ is invariant under $f^{-1}$:

$$f^{-1} : [0 : -(\beta_2 x_2 + \beta_3 x_3) : x_2 : x_3] \mapsto [0 : \beta_2 (\beta_2 x_2 + \beta_3 x_3) - \beta_3 x_2 : -(\beta_2 x_2 + \beta_3 x_3) : x_2].$$

Now suppose $\beta_3 = 0$ and $\beta_2 \neq 0$. In this case we have

$$f^{-1} : \Sigma_0 \ni [0 : x_1 : x_2 : x_3] \mapsto [0 : -\beta_2 x_1 : x_1 : x_2] \mapsto [0 : \beta_2^2 : -\beta_2 : 1] \in \Sigma_{03} \setminus \mathcal{I}(f^{-1})$$

and this last point is fixed under $f^{-1}$.

Let $\pi : Z \to \mathbb{P}^3$ be the complex manifold obtained by blowing up $e_2$ and $\Sigma_{02}$ and let $E_2$ and $S_{02}$ be the corresponding blow-up divisors. In the following Lemma, we use the local coordinates $(s_0, x_1, \xi_2)_{S_{02}} \mapsto [s_0 : x_1 : s_0 \xi_2 : 1]$ in a neighborhood of $S_{02} = \{s_0 = 0\}$ and $(u_0, \eta_1, \eta_3)_{E_2} \mapsto [u_0 : \eta_1 u_0 : 1 : \eta_3 u_0]$ in a neighborhood of $E_2 = \{u_0 = 0\}$.

**Lemma 3.5.** Suppose that $\beta_1 = 1, \alpha_1 = \beta_2 = \beta_3 = 0$. If either $\alpha_2 = 0$ or $\alpha_3 = 0$ then $S_{02}$ is exceptional and preperiodic for $f_Z$ or $f_Z^{-1}$.

**Proof.** If $\alpha_2 = \alpha_3 = 0$ then the mapping is basically linear.

(i) Case $\alpha_3 = 0$ and $\alpha_2 \neq 0$:

$$f_Z : (0, t, \xi_2)_{S_{02}} \mapsto (0, \xi_2, 0)_{E_2} \mapsto (0, \beta_0 + \xi_2, 0)_{S_{02}} \mapsto (0, 0, 0)_{E_2} \mapsto (0, \frac{\beta_0}{\alpha_2}, 0)_{S_{02}}.$$

(ii) Case $\alpha_2 = 0$ and $\alpha_3 \neq 0$:

$$f_Z^{-1} : (0, t, \xi_2)_{S_{02}} \mapsto (0, -\beta_0, \xi_2)_{E_2} \mapsto (0, -\alpha_3, -\beta_0)_{S_{02}} \mapsto (0, -\beta_0, -\beta_0)_{E_2}.\]
Theorem 3.6. If $f$ is not critical, then there exists a complex manifold $X$ such that either there is an exceptional hypersurface $E \subset X$ for an induced birational map $f_X$ such that $f_X^n E \not\in \mathcal{I}(f)$ for $n = 1, 2, \ldots$, or the analogous statement holds for $f_X^{-1}$.

Proof. Let $X$ denote either the space $X$ or $Z$ in the Lemmas above. This Theorem follows from the Lemmas 3.3–5.

Proof of Theorem 3.1. If $f$ is not critical, then Theorem 3.6 says that in each case that there is an exceptional hypersurface that does not map into $\mathcal{I}(f_X)$. By Proposition 3.2, then, $f$ is not periodic.

§4. Critical Maps. Here we study critical maps in general. Let us recall the condition for $f$ being critical: $\beta_2 = \beta_3 = 0$ and $\beta_1 \alpha_2 \alpha_3 \neq 0$. Using the action (2.11a–c) we may assume that a critical map satisfies:

$$\beta_1 = 1, \beta_2 = \beta_3 = 0, \alpha_1 = 0, \alpha_2 \neq 0, \alpha_3 = 1.$$ (4.1)

In this section, we show (Lemma 4.2) that for every critical map there is a blowup space $π : Y \to X$ such that the induced map $f_Y$ has only one exceptional hypersurface, which is $Σ_γ$.

We determine the indeterminacy locus of $f_Y$ (Corollary 4.6) and the dynamical degree for the generic case (Theorem 4.8).

Proposition 4.1. If $f$ is critical then $f^{-1}$ is conjugate to a critical map.

Proof. Let $β = (β_0, 1, 0, 0)$ and $α = (α_0, 0, α_2, 1)$ be parameters of a critical map $f$. We consider a linear map $ϕ : [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_3 : x_2 : x_1]$. It follows that we have

$$ϕ^{-1} \circ f^{-1} \circ ϕ : [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 x_1 : x_2 x_1 : x_3 x_1 : x_0 (α_0 x_0 + β_0 x_1 + 2 x_2 + α_2 x_3)].$$

Thus $f^{-1}$ is conjugate to a critical map of the form (2.2) with parameter values $β' = (0, 1, 0, 0)$ and $α' = (α_0, -β_0, 1, α_2)$ which satisfy the condition (3.1).

Remark. By Proposition 4.1, each result for $f$ corresponds to a result for $f^{-1}$. The translation between $f$ and $f^{-1}$ is guided by notation: $β \leftrightarrow B, γ \leftrightarrow C, 1 \leftrightarrow 3$: thus $f^{-1}Σ_C = Σ_βγ$, etc.

If (4.1) holds, it follows that $e_3 = Σ_0 \cap Σ_β \cap \{x_2 = 0\} \in \mathcal{I}$, and

$$f : Σ_β \dashrightarrow e_3 \dashrightarrow Σ_0 \dashrightarrow Σ_03 \dashrightarrow e_1 \dashrightarrow Σ_B$$ (4.2)

We define a new complex manifold $π_Y : Y \to \mathbb{P}^3$ by blowing up $e_1$ and $e_3$, then the strict transform of $Σ_01$, followed by the strict transform of $Σ_03$. (Equivalently, we start with $X$ and blow up the strict transform of $e_1$ and $Σ_03$.) For $j = 1, 3$, we denote the exceptional divisor over $e_j$ by $E_j$ and the exceptional divisor over $Σ_{0j}$ by $S_{0j}$ for $j = 1, 3$. The induced birational map $f_Y : Y \to Y$ maps

$$f_Y : Σ_β \to E_3 \to S_01 \to Σ_0 \to S_{03} \to E_1 \to Σ_B$$ (4.3)

Lemma 4.2. The maps in (4.3) are dominant; $Σ_γ$ is the unique exceptional hypersurface for $f_Y$, and $Σ_C$ is the unique exceptional hypersurface for $f_Y^{-1}$.

Proof. Using the local coordinates defined in §2, we have

$$Σ_β \setminus Σ_βγ \ni [x_0 : -β_0 x_0 : x_2 : x_3] \mapsto (0, \frac{x_2}{x_0}, \frac{x_3}{x_0}) E_3 \in E_3$$

$$f_Y : E_3 \ni (0, ξ_1, ξ_2) E_3 \mapsto (0, ξ_2, β_0 + ξ_1) S_{01} \in S_{01}$$

$$S_{01} \setminus Σ_βγ \ni (0, η_1, x_2) S_{01} \mapsto [0 : x_2 (β_0 + η_1) : (β_0 + η_1) : 1 + α_2 x_2] \in Σ_0.$$
In Proposition 2.1 we showed that the maps $\Sigma_0 \to S_{03} \to E_1 \to \Sigma_B$ are dominant. It follows that $\Sigma_1$ is the only exceptional hypersurface for $f_Y$, and $\Sigma_C$ is the only one for $f_Y^{-1}$. \qed

For $p \in \mathbf{P}^3$, we will say that a point of $\pi_Y^{-1}p$ is at level 1 if it could have been obtained by a blowup a point or curve in $\mathbf{P}^3$. Thus the points of all fibers are of level 1, unless they lie over $e_1$, $e_3$, or $e_2 = \Sigma_{01} \cap \Sigma_{03}$. The fibers $E_1 \cap S_{03}$ and $E_3 \cap S_{01}$ represent the points of $E_1$ and $E_3$ which are at level 2. Over $e_2$, we define $\mathcal{F}_{e_2}^1 := S_{01} \cap \pi_Y^{-1}e_2$ and $\mathcal{F}_{e_2}^2 := S_{03} \cap \pi_Y^{-1}e_2$. We see that $\mathcal{F}_{e_2}^i$, for $i = 1, 2$ is at level $i$.

We see that the three curves on level 2 are not indeterminate:

**Lemma 4.3.** If $f$ is critical, then the indeterminacy loci $\mathcal{I}(f_Y)$ and $\mathcal{I}(f_Y^{-1})$ do not contain $E_1 \cap S_{03}$, $E_3 \cap S_{01}$, or $\mathcal{F}_{e_2}^2$.

**Proof.** Let us first consider the blowup fiber over $E_3 \cap \Sigma_{01}$. For this fiber let us use a local coordinate $(\xi_0, \xi_1, t_2)_{E_3} \mapsto [t_2\xi_0 : t_2\xi_1 : t_2 : 1] \in \mathbf{P}^3$. It follows that the strict transformation of $\Sigma_{01} = \{(0, 0, t_2)_{E_3}\}$ and $E_3 \cap \Sigma_{01} = (0, 0, 0)_{E_3}$. The local coordinates in a neighborhood of the second blowup fiber over $E_3 \cap S_{01}$ is given by $(\eta_0, u_1, t_2)_{E_0^3} \mapsto (\eta_0u_1, u_1, t_2)_{E_3} \mapsto [\eta_0u_1t_2 : u_1t_2 : t_2 : 1] \in \mathbf{P}^3$ and we have the second blowup fiber $E_3 \cap S_{01} = \{(\eta_0, 0, 0)_{E_0^3}\}$. With these coordinates, we see that

$$f_Y : (\eta_0, 0, 0)_{E_0^3} \mapsto (0, 0, \eta)_{S_{01}} = S_{01} \cap \Sigma_0$$

where $(\xi, t, x_3)_{S_{01}} \mapsto [\xi t : t : 1 : x_3]$ gives a local coordinates near $S_{01}$. It follows that the second blowup fiber $E_3 \cap S_{01}$ is not indeterminate for $f_Y$. The computations for $f_Y^{-1}$ and for $E_1 \cap S_{03}$ are essentially the same, and we see that $E_1 \cap S_{03}$ and $E_3 \cap S_{01}$ are not indeterminate for $f_Y$ or to $f_Y^{-1}$.

To consider the second blowup fiber $\mathcal{F}_{e_2}^2$, we use local coordinates $(\xi, s, x_3)_{S_{01}} \mapsto [\xi s : s : 1 : x_3]$. In this coordinates we see that $S_{01} = \{s = 0\}$ and the strict transform of $\Sigma_{03} = \{\xi = 0, x_3 = 0\}$. Thus the local coordinates near the blowup of $\Sigma_{03}$ is given by $(\eta, s, t)_{S_{03}} \mapsto (\eta s, s, t)_{S_{01}} \mapsto [\eta s : s : 1 : t]$ and we have $\mathcal{F}_{e_2}^2 = \{(\eta, 0, 0)_{S_{03}}\}$. With this coordinates, we have

$$f_Y : \mathcal{F}_{e_2}^2 \ni (\eta_0, 0, 0)_{S_{03}} \mapsto (0, 0, \alpha_2\eta)_{E_1} = E_1 \cap \Sigma_0$$

where $(\xi_0, t_2, \xi_3)_{E_1} \mapsto [\xi_0t_2 : 1 : t_2 : \xi_3t_2]$ is local coordinates near $E_1$. Similarly we see that $f_Y^{-1} \mathcal{F}_{e_2}^2 = E_3 \cap \Sigma_0$ and the mapping is dominant. \qed

Recall from §2 that in $\mathbf{P}^3$ each point on $\Sigma_{\beta\gamma}$ blows up to a line in $\Sigma_C$. Note that $[0 : 0 : 1 : -\alpha_2] = \Sigma_{\beta\gamma} \cap \Sigma_{01}$, and let $\mathcal{F}_{0\beta\gamma} := \pi_Y^{-1}(\Sigma_{\beta\gamma} \cap \Sigma_{01})$. Note that the base point is the intersection of $\Sigma_{01}$ and $\Sigma_{\beta\gamma}$, two indeterminate lines. Similarly, we write $\mathcal{F}_{0BC} := \pi_Y^{-1}(\Sigma_{BC} \cap \Sigma_{03})$.

**Lemma 4.4.** If $f$ is critical, the fiber $\mathcal{F}_{0\beta\gamma}$ is a component of $\mathcal{I}(f_Y)$, and we have $f_Y : \mathcal{F}_{0\beta\gamma} \setminus \Sigma_{\beta\gamma} \rightarrow \mathcal{F}_{0BC}$.

**Proof.** Let us consider a local coordinates in a neighborhood of the fiber $\mathcal{F}_{0\beta\gamma}$ and a local coordinates in a neighborhood of $\mathcal{F}_{0BC}$:

$$(s_0, \eta_1, x)_{S_{01}} \sim [s_0 : \eta_1s_0 : 1 : x] \in \mathbf{P}^3 \quad \text{and} \quad \mathcal{F}_{0\beta\gamma} = \{s_0 = 0, x = -\alpha_2\}$$

$$(s_0, x, \eta_3)_{S_{03}} \sim [s_0 : 1 : x : \eta_3s_0] \in \mathbf{P}^3 \quad \text{and} \quad \mathcal{F}_{0BC} = \{s_0 = 0, x = -\alpha_2\}.$$
we see that for each \( \eta_1 \) we have
\[
f_Y(0, \eta_1, -\alpha_2)_{s_{01}} = \lim_{s_0 \to 0, x \to -\alpha_2} \left( s_0, x, \frac{1}{\beta_0 + \eta_1} \frac{\alpha_0 s_0 + \alpha_2 + x}{s_0} \right)_{s_{03}}
\]
\[= \{(0, -\alpha_2, \eta)_{s_{03}} : \eta \in \mathbb{C} \} = \mathcal{F}_{0BC}
\]
\[\square\]

Let us define a set \( S \subset Y \) to be \emph{totally invariant} if it is completely invariant for the total transform, or if for all \( p \in S \), we have \( (f_Y)_\ast p \subset S \) and for all \( p \notin S \) we have \( (f_Y)_\ast p \cap S = \emptyset \).

**Lemma 4.5.** If \( f \) is critical, then \( \Sigma_{02} \) is indeterminate for \( f_Y \). Each point of \( \Sigma_{02} \) blows up to \( \mathcal{F}_{e_1} \), and \( \mathcal{F}_{e_1} \) is mapped smoothly to \( \Sigma_{02} \). The set \( \Sigma_{02} \cup \mathcal{F}_{e_1} \) is totally invariant.

**Proof.** Recall that \( f_{\Sigma_{02}} = e_2 \), and the point \( e_2 \) was blown up. We consider points \([s : 1 : s \xi : x]\) which are close to \( \Sigma_{02} \) when \( s \) is small. We see that
\[
f_Y : [s : 1 : s \xi : x] \mapsto \left[ \frac{s \xi}{x} : 1 : s \frac{\alpha_0 s + \alpha_2 s \xi + x}{x(\beta_0 s + 1)} \right] = \left( \frac{s \xi}{x}, s \frac{\alpha_0 s + \alpha_2 s \xi + x}{x(\beta_0 s + 1)} \right)_{01}.
\]
Letting \( s \to 0 \) we see that \( f_Y[0 : 1 : 0 : x] \sim \{(\eta, 0, 0)_{01}\} \). Using the same local coordinates we also see that
\[
f_Y : \mathcal{F}_{e_1} \ni (\eta, 0, 0)_{01} \to [0 : 1 : 0 : \frac{\alpha_2 \eta}{\beta_0 \eta + 1}] \in \Sigma_{02}.
\]
For the second statement, we first notice that from (4.3) \( f_Y((\Sigma_{02} \cup \mathcal{F}_{e_1})^c - \mathcal{I}(f_Y)) \) is disjoint from the set \( \Sigma_{02} \cup \mathcal{F}_{e_1} \). Since \( \mathcal{I}(f_Y) = \Sigma_{\beta \gamma} \cup \mathcal{F}_{0\beta \gamma} \cup \Sigma_{02} \) and \( \Sigma_{\beta \gamma} \cap \Sigma_{02} = \emptyset \), we see that the set \( \Sigma_{02} \cup \mathcal{F}_{e_1} \) and \( \Sigma_{\beta \gamma} \cup \mathcal{F}_{0\beta \gamma} \) are disjoint. It follows that the set \( \Sigma_{02} \cup \mathcal{F}_{e_1} \) is totally invariant. \(\square\)

**Corollary 4.6.** If \( f \) is critical, then \( \mathcal{I}(f_Y) = \Sigma_{\beta \gamma} \cup \mathcal{F}_{0\beta \gamma} \cup \Sigma_{02} \) has pure dimension 1.

The behavior of \( f_Y \) at \( \Sigma_{02} \) is, in suitable coordinates, given by the third model (1.5). The behavior of \( f_Y \) at \( \mathcal{F}_{0\beta \gamma} \), as seen in Lemma 4.4, is different from the model (1.5). Further, we note that by Proposition 4.1 and the remark following it, the analogues of Lemmas 4.2–5 all hold for \( f_Y \). For instance, \( \Sigma_C \) is the unique exceptional hypersurface for \( f_Y^{-1} \), \( \mathcal{I}(f_Y^{-1}) = \mathcal{F}_{e_2} \cup \mathcal{F}_{0BC} \cup \Sigma_{BC} \), and each point of \( \mathcal{F}_{0BC} \) blows up under \( f_Y^{-1} \) to \( \mathcal{F}_{0\beta \gamma} \).

**Corollary 4.7.** If \( f \) is critical, then \( f_Y^j \Sigma_{\gamma} \cap (\Sigma_{02} \cup \mathcal{F}_{e_1}) = \emptyset \) for all \( j \geq 0 \).

**Proof.** By Lemma 4.5, it suffices to consider the case \( j = 0 \). By (4.1), \( e_2 \notin \Sigma_{\gamma} \in \mathbb{P}^3 \), so the fiber over \( e_2 \) remains disjoint from \( \Sigma_{\gamma} \) inside \( Y \). Now \( e_1 = \Sigma_{02} \cap \Sigma_{\gamma} \) in \( \mathbb{P}^3 \) and we see that \( \Sigma_{02} \) and \( \Sigma_{\gamma} \) are separated when we blow up \( e_1 \) to make \( Y \). \(\square\)

Recall that the degree complexity is \( \delta(f) = \lim_{n \to \infty}(\deg(f^n))^{1/n} \). If \( \delta(f) > 1 \), then the degrees of the iterates \( f^n \) grow exponentially in \( n \). In particular, \( f \) cannot be periodic if \( \delta(f) > 1 \).

**Theorem 4.8.** If \( f \) is critical, and if \( f_Y^m \Sigma_{\gamma} \notin \Sigma_{\beta \gamma} \cup \mathcal{F}_{0\beta \gamma} \), then the first dynamical degree is \( \delta(f) \sim 1.32472 \), the largest root of \( x^3 - x - 1 \).

**Proof.** Using Corollary 4.7 we see that \( f_Y^m \Sigma_{\gamma} \cap \mathcal{I}(f_Y) = \emptyset \) for all \( m \geq 1 \). Thus by Proposition 1.2 we have \( (f_Y^*)^m = (f_Y^m)^* \) for all \( m \). Thus \( \delta(f) \) is the spectral radius of \( f_Y^* \). Inside the Picard group Pic\( (Y) \), we let \( H_Y \) be the class of a generic hyperplane in \( Y \), and we have
\[
f_Y^* : H_Y \to 2H_Y - E_1 - E_3 - S_{01}, \quad S_{01} \to E_3 \to \Sigma_{\beta} = H_Y - E_3 - S_{01},
\]
\[
E_1 \to S_{03} \to \Sigma_0 = H_Y - E_1 - E_3 - S_{01} - S_{03}
\]
(4.4)
The characteristic polynomial of this transformation is \((x^2 + 1)(x^3 - x - 1)\), so \(\delta(f)\) is as claimed.

Now we give the existence of Green currents, which are invariant currents with the equidistribution properties given in the following:

**Theorem 4.9.** If \(f\) is as in Theorem 4.8, then there is a positive closed current \(T_Y^+\) in the class of \(\alpha_Y^+\) with the properties: \(f_Y^* T_Y^+ = T_Y^+\), and if \(\Xi^+\) is a smooth form which represents \(\alpha_Y^+\), then \(\lim_{n \to \infty} \delta_1(f)^{-n} f_Y^n \Xi_Y^+ = T_Y^+\) in the weak sense of currents on \(Y\).

**Proof.** Recall from Corollary 4.6 that \(\mathcal{I}(f_Y) = \Sigma_{02} \cup \Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma}\). The total forward image of this set is \(\pi_2 \pi_1^{-1} \mathcal{I}(f_Y) = \mathcal{F}_{\epsilon_2} \cup \mathcal{F}'_{0BC} \cup \Sigma_C\). We will show that if \(\sigma \subset \pi_2 \pi_1^{-1} \mathcal{I}(f_Y)\) is any curve, then \(\alpha_Y^+ \cdot \sigma \geq 0\). The Theorem will then be a consequence of Theorem 1.3 of [Ba].

Up to a scalar multiple, we may write \(\alpha = H_Y - c_1 E_1 - c_3 E_3 - c_{01} S_{01} - c_{03} S_{03}\). Then since \(f_Y^*\) is given by (4.4) we have \(1 > c_1 > c_3 > c_{01} = c_{03} > 0\), and \(c_1 + c_3 = 1\). Let us start with \(\mathcal{F}_{0\beta\gamma} \subset \mathcal{I}(f_Y)\). Points of this curve are blown up to \(\mathcal{F}_{0BC}\). The curve \(\sigma = \mathcal{F}_{0BC}\) is the exceptional fiber inside \(S_{03}\) over the point \(\Sigma_{03} \cap \Sigma_{03} \in \mathbb{P}^3\). Thus \(\sigma \cdot S_{03} = -1\), so \(\alpha \cdot \sigma = c_{03} > 0\). Points of the indeterminate curve \(\Sigma_{02}\) blow up to \(\sigma = \mathcal{F}_{\epsilon_2}\). In this case, we have that \(\sigma \cdot S_{01}\) and \(\sigma \cdot S_{03}\), are \(\pm 1\), with opposite signs, depending on the order of blow-up of \(S_{01}\) and \(S_{03}\). Thus \(\sigma \cdot \alpha_Y^+ \geq 0\).

The other possibility is that \(\sigma \subset \Sigma_C\). In this case, we have \(\sigma \cdot H = \deg(\sigma) = \sigma \cdot S_{01}\). Further, if we let \(m_3\) denote the multiplicity of \(\sigma\) at \(e_3\), then \(m_3\) is bounded above by \(\deg(\sigma)\). If \(\sigma\) is a curve in \(\Sigma_C\), then it is represented by \(L + m_1 \mathcal{F}_{01}^1 + m_3 e_3\), where \(\mathcal{F}_{01}^1\) represents a fiber of \(S_{01}\), and \(e_3 = E_3 \cap \Sigma_C\). The multiplicities \(m_1\) and \(m_3\) are bounded below by \(-\deg(\sigma)\). Since \(\mathcal{F}_{01}^1 \cdot S_{01} = -1 = E_3 \cdot e_3\), we have \(\sigma \cdot \alpha_Y^+ \geq \deg(\sigma)(1 - c_{01} - c_3) > 0\).

**Lemma 4.10.** Suppose that \(\Sigma_\gamma\) is not preperiodic. If either \(f_Y^i \Sigma_\gamma = \mathcal{F}_{0\beta\gamma}\) or \(f_Y^i \Sigma_\gamma\) is a point of \(\mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}\), then \(f_Y^{i+1} |_{\Sigma_\gamma}\) has rank 1, and \(f_Y^{i+1} \Sigma_\gamma = \mathcal{F}_{0BC}\).

**Proof.** Let us describe the indeterminate behavior of \(f_Y\) at \(\Sigma_{\beta\gamma}\) and \(\mathcal{F}_{0\beta\gamma}\). Up to coordinate changes in domain and range, we may assume that the indeterminate curve is \(\{\xi_2 = \xi_3 = 0\}\), and the maps behave like

\[
(s_1, \xi_2, \xi_3) \mapsto (s_1, \xi_2, \xi_3) \quad \text{near } \Sigma_{\beta\gamma}
\]

\[
(s_1, \xi_2, \xi_3) \mapsto (s_1, \xi_2, \xi_3) \quad \text{near } \mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}.
\]

The behavior near \(\Sigma_{\beta\gamma}\) is given in (2.7), and \(\mathcal{F}_{0\beta\gamma}\) is given in Lemma 4.4.

We will track the forward orbit \(f_Y^i \Sigma_\gamma\). Without loss of generality, we may assume that \(f_Y^i \Sigma_\gamma\) is not \(\Sigma_{\beta\gamma}\), for \(1 \leq i \leq j\). Let us choose coordinates \((u_1, u_2, v)\) such that \(\Sigma_\gamma = \{v = 0\}\), and the exceptional fibers of \(\Sigma_\gamma\) are \(\{v = 0, u_1 = \text{const}\}\). Thus \(f\) has the form \((u_1, u_2, v) \mapsto (u_1, u_2 v, v)\). By making coordinate changes in the range, we may represent the iterated map near \(\Sigma_\gamma\) as \((u_1, u_2 v, v)\) as long as \(f_Y^i \Sigma_\gamma\) is not an exceptional curve in \(\Sigma_\gamma\), and not a component of \(\mathcal{I}(f_Y^i)\).

Now suppose that \(f_Y^i \Sigma_\gamma = f_Y^{-1} \Sigma_{BC}\) coincides with \(\mathcal{F}_{0\beta\gamma}\), which we will write as \(\{\xi_2 = \xi_3 = 0\}\), as above. Thus we may assume that \((s, \xi_2, \xi_3) = (u_1, u_2 v, v)\), so \(f^{i+1}\) has the form \((u_1, u_2, v) \mapsto (u_1 u_2, u_2 v, v)\), which is a map of rank 1.
As we continue to iterate $f_Y$, the other possibility is that $f_Y^k \Sigma_\gamma \subset \Sigma_\gamma$ is an exceptional curve. The coordinate of the map which varies when $\nu = 0$ must be inside $\{u_1 = \text{const}\} \subset \Sigma_\gamma$, which means that the map must be like $\Sigma_\gamma \ni (u_2v, u_1u_2, v) \mapsto (u_2v, u_1u_2v, v)$, which belongs to $\Sigma_{BC}$. Now we continue to iterate this point forward. It cannot re-enter $\Sigma_\gamma$, because otherwise the orbit would re-enter $\Sigma_{BC}$ and become pre-periodic. Thus the only possibility is to enter the indeterminacy locus. Let $N$ denote the first positive integer for which $f_N^j \Sigma_\gamma \subset \Sigma_\gamma$ was an exceptional curve. Then $\mathcal{I}(f_Y) \cap \bigcup_{i=0}^{N-1} \Sigma_{BC}$ is finite. Thus the forward orbit of the point can intersect the indeterminacy locus only finitely many times. If the point enters $\Sigma_{\beta_\gamma} = \{\xi_2 = \xi_3 = 0\}$, then the next image is $(s, \xi_2, \xi_3) = (u_2v, u_1u_2v, v) \mapsto (u_2v, u_1u_2v)$, which has rank 1 again, and we continue as before.

\section{Pseudo-automorphisms.}

In this section we assume that $f$ is critical and we consider the condition

\[
 f_Y^N \Sigma_\gamma = \Sigma_{\beta_\gamma} \quad \text{for some } N. \tag{5.1}
\]

We give conditions for $f_Y$ to be birationally equivalent to a pseudo-automorphism (Theorem 5.1).

Suppose that (5.1) holds. For $1 \leq j \leq N - 1$, we consider four possibilities:

(i) $f_Y^j \Sigma_\gamma$ is a one of the exceptional fibers in $\Sigma_\gamma$, that is $f_Y^j \Sigma_\gamma = \Sigma_\gamma \cap \{\lambda_2x_2 = \lambda_3x_3\}$ for some $\lambda_2, \lambda_3 \in \mathbb{C}$ and $f_Y^{j+1} \Sigma_\gamma$ is a point in $\Sigma_{BC}$.

(ii) $f_Y^j \Sigma_\gamma$ is a point of indeterminacy in $\Sigma_{\beta_\gamma}$, and this point blows up to one of fibers of $\Sigma_C$.

(iii) $f_Y^j \Sigma_\gamma \subset \mathcal{F}_{0\beta_\gamma}$, and $f_Y^{j+1} \Sigma_\gamma = \mathcal{F}_{0BC}$.

(iv) None of the above.

\begin{theorem}
 Suppose that a critical map $f$ satisfies (5.1), and that whenever case (iii) occurs above, then $f_Y^j \Sigma_\gamma = \mathcal{F}_{0\beta_\gamma}$. Then there is a blowup space $\pi : Z \to Y$ such that $f_Z$ is a pseudo-automorphism.
\end{theorem}

\begin{proof}
 Let us define two sets of subvarieties: $\Lambda_1 = \{f_Y^j \Sigma_\gamma \mid \dim f_Y^j \Sigma_\gamma = 0, 1 \leq j \leq N\}$ and $\Lambda_2 = \{f_Y^j \Sigma_\gamma \mid \dim f_Y^j \Sigma_\gamma = 1, 1 \leq j \leq N\}$. We construct a blowup space $\pi_Z : Z \to Y$ obtained by first blowing up points in $\Lambda_1$ and then blowing up curves in $\Lambda_2$. We denote by $\mathcal{F}_1, \ldots, \mathcal{F}_N$ the blowup fibers over $f_Y \Sigma_\gamma = \Sigma_{BC}$, $f_Y^2 \Sigma_\gamma, \ldots, f_Y^N \Sigma_\gamma = \Sigma_{\beta_\gamma}$, respectively.

We claim that the induced map $f_Z$ on the orbit of $\Sigma_\gamma$ are dominant. To show this we need to check the map $f_Z$ on $S_j$ where $f_Y^j \Sigma_\gamma \subset \mathcal{E}(f_Y)$ or $f_Y^j \Sigma_\gamma \subset \mathcal{I}(f_Y)$. For $\mathcal{F}_1$ we use local coordinates

\[
 (x_1, \xi, s)_1 \mapsto [1 : x_1 : -\alpha_0 - \alpha_2x_1 + s\xi : s] \in \mathbb{P}^3, \quad \text{and } \{s = 0\} = \mathcal{F}_1.
\]

Using this we see that

\[
 f_Z : \Sigma_\gamma \ni [x_0 : x_1 : x_2 : -\alpha_0x_0 - \alpha_2x_2] \mapsto (\frac{x_2}{x_0}, \beta_0 + \frac{x_1}{x_0}, 0) \in \mathcal{F}_1. \tag{5.2}
\]

Suppose the possibility (i) occurs, that is $f_Y^j \Sigma_\gamma = \Sigma_\gamma \cap \{x_2 = \lambda x_3\}$ for some $\lambda \in \mathbb{C}$. Local coordinates near $\mathcal{F}_j$ and $\mathcal{F}_{j+1}$ are given by

\[
 (s, x, \xi)_j \mapsto [-(\alpha_2\lambda + 1)/\alpha_0 + s : x : \lambda + s\xi : 1] \in \mathbb{P}^3, \quad \text{and } \{s = 0\} = \mathcal{F}_j.
\]

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\[(s, \xi_1, \xi_2)_{j+1} \mapsto \left[-(\alpha_2 \lambda + 1)/\alpha_0 + s : \lambda + s\xi_1 : 1 : s\xi_2\right] \in \mathbb{P}^3, \quad \text{and } \{s = 0\} = \mathcal{F}_{j+1}.\]

It follows that
\[
f_Z : \mathcal{F}_j \ni (0, x, \xi) \mapsto (0, \xi, (1 + \alpha_2 \lambda)(\alpha_0 + \alpha_2 \xi) / \beta_0 - \alpha_0 x + \alpha_2 \beta_0 \lambda)_{j+1} \in \mathcal{F}_{j+1}. \quad (5.3)
\]

In case the second possibility (ii), i.e. $f_Y^j \Sigma_\gamma$ is a point in $\Sigma_{\beta\gamma}$ occurs is essentially identical to the first possibility (i). For the third possibility (iii), due to Lemma 6.3 we only need to check the induced map on the blowup fiber over $f_Y^j \Sigma_\gamma = \mathcal{F}_{0\beta\gamma}$. We use local coordinates near $\mathcal{F}_j$, the blowup fiber over $\mathcal{F}_{0\beta\gamma}$, and $\mathcal{F}_{j+1}$, the blowup fiber over $\mathcal{F}_{0BC}$:
\[
(s, \eta_1, \eta_2)_{j+1} \mapsto (s, -\alpha_2, \eta_1 + s\eta_2)_{s,0,3} \mapsto [s : -\alpha_2 : s\eta_1 + s^2\eta_2] \in \mathbb{P}^3.
\]

Using these local coordinates we see that
\[
f_Z : \mathcal{F}_j \ni (0, \eta_1, \eta_2) \mapsto \left(0, s\eta_1 + s^2\eta_2 : 1 / \alpha_0 \right)_{j+1} \in \mathcal{F}_{j+1}. \quad (5.4)
\]

Now for the case when $f_Y^j \Sigma_\gamma = E_3 \cap \Sigma_0$, $f_Y^{j+1} \Sigma_\gamma = E_2 \cap \Sigma_0$, $f_Y^{j+2} \Sigma_\gamma = E_1 \cap \Sigma_0$ we use local coordinates
\[
(\xi_0, \xi_1, t)_{j} \mapsto (t\xi_0, \xi_1, t)_{E_3} \mapsto [t^2 \xi_0 : t\xi_1 : t : 1] \in \mathbb{P}^3
\]
\[
(\xi_0, \xi_1, t)_{j} \mapsto (t\xi_0, \xi_1, t)_{E_2} \mapsto [t^2 \xi_0 : t\xi_1 : 1 : t] \in \mathbb{P}^3
\]
\[
(\xi_0, \xi_1, t)_{j} \mapsto (t\xi_0, \xi_1, t)_{E_1} \mapsto [t^2 \xi_0 : 1 : t\xi_1 : t] \in \mathbb{P}^3.
\]

We have
\[
f_Z : \mathcal{F}_j \ni (\xi_0, \xi_1, 0) \mapsto \left(\frac{\xi_1^2}{\xi_0^2}, \frac{\xi_1}{\xi_0}, 0\right)_{j+1} \in \mathcal{F}_{j+1} \quad (5.5)
\]
\[
f_Z : \mathcal{F}_{j+1} \ni (\xi_0, \xi_1, 0) \mapsto \left(\frac{\xi_1^2}{\alpha_2 \xi_0^2}, \frac{\xi_1}{\alpha_2 \xi_0}, 0\right)_{j+1} \in \mathcal{F}_{j+1}. \quad (5.6)
\]

The last part we have to check is $f_Z$ on $\mathcal{F}_N$, the blowup fiber of the line of indeterminacy $\Sigma_{\beta\gamma}$. The local coordinates we use near $\mathcal{F}_N$ is given by
\[
(\xi_0, \xi_1, \xi_2) \mapsto [1 : -\beta_0 + s : x_2 : -\alpha_0 - \alpha_2 x_2 + s\xi_3] \in \mathbb{P}^3
\]

and we get
\[
f_Z : \mathcal{F}_N \ni (0, x_2, \xi_3) \mapsto [1 : x_2 : -\alpha_0 - \alpha_2 x_2 : \xi_3] \in \Sigma_C. \quad (5.7)
\]

From (5.2–7) we see that the induced mapping $f_Z$ is dominant on the orbit of $\Sigma_\gamma$ and therefore $f_Z$ has no exceptional hypersurface. By Lemma 4.1, it follows that $f_Z^{-1}$ also has no exceptional hypersurface, so $f_Z$ is a pseudo-automorphism. \[\square\]

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Lemma 5.2. Suppose that a critical map \( f \) satisfies (5.1), and that whenever case (iii) occurs above, then the possibility (iii) can occur at most once.

**Proof.** Suppose that there are \( 0 < j_1 < j_2 < N \) such that \( f_Y^{j_1} \Sigma_\gamma, f_Y^{j_2} \Sigma_\gamma \in \mathcal{F}_{0\beta\gamma} \). It follows that \( f_Y^{j_1+1} \Sigma_\gamma = f_Y^{j_2+1} \Sigma_\gamma = \mathcal{F}_{0\beta} \) and thus \( \mathcal{F}_{0\beta} \) is fixed under \( f_Y^{j_2-j_1} \). Since \( 0 < j_1 < j_2 < N \), we have that the dimension \( f_Y^{j_1} \Sigma_\gamma \leq 1 \) for all \( j = j_1 + 1, \ldots, j_2 \). Using the fact that \( \mathcal{F}_{0\beta} \) is fixed under \( f_Y^{j_2-j_1} \), we conclude that the dimension \( f_Y^{j_1} \Sigma_\gamma \leq 1 \) for all \( j \geq 1 \) which contradicts to the assumption that \( f \) satisfies (5.1).

\[
\begin{array}{cccccccccccc}
\Sigma_\gamma & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & \cdots & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 8 & \rightarrow & 9 & \rightarrow & \cdots & \rightarrow & 10 & \rightarrow & 11 \\
\Sigma_{bc} & \rightarrow & \Sigma_{vt}
\end{array}
\]

Figure 5.1. A hypothetical orbit: \( d_1 = 2, u_1 = 4, d_2 = 7, u_2 = 9, m_d = m_u = 2, N = 11 \).

Let \( N \) be the smallest positive integer such that \( f_Y^N \Sigma_\gamma = \Sigma_{\beta\gamma} \). Let \( m_d \) positive integers \( d_1 < d_2 < \cdots < d_{m_d} \) denote the number of iteration to have the possibility (i), that is for each \( j = 1, \ldots, m_d \), \( f_Y^{d_j} \Sigma_\gamma \) is \( \Sigma_\gamma \cap \{ \lambda_2 x_2 = \lambda_3 x_3 \} \) for some \( \lambda_2, \lambda_3 \in \mathbb{C} \). Let \( u_1 < u_2 < \cdots < u_{m_u} \) be \( m_u \) positive integers such that \( f_Y^{u_j} \Sigma_\beta \) is of possibility (ii) for \( j = 1, \ldots, m_u \). We also set \( m_s \) be a positive integer such that \( f_Y^{m_s+2} \Sigma_\gamma \in \mathcal{F}_{0\beta\gamma} \) if such possibility occurs. If there is no such case we set \( m_s = \infty \). To illustrate this numbering scheme, a hypothetical orbit of \( \Sigma_\gamma \) is given in Figure 5.1. Here we have assumed that we are in the simpler case \( m_s = \infty \), which means that the orbit never enters \( \mathcal{F}_{0\beta\gamma} \), so the case (iii) does not occur. Thus in Figure 5.1 the dimension can increase from 0 to 1 only via case (ii).

Let us use the numbers \( m_s, m_u, m_d, u_j, d_j \) and \( N \) to define four Laurent polynomials:

\[
Q_1 := -1 - \sum_{j=1}^{m_d} \frac{1}{d_j} + \frac{1}{m_s+1},
\]

\[
Q_2 := \frac{1}{tm_s} \left( \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \frac{1}{t^4} \right),
\]

\[
Q_3 := -1 - \sum_{j=1}^{m_d} \frac{1}{d_j} + \frac{1}{m_s} \left( 1 + \frac{1}{t} - \frac{1}{t^4} \right),
\]

\[
Q_4 := -t - t \sum_{j=1}^{m_d} \frac{1}{d_j} - t \sum_{j=1}^{m_u} \frac{1}{u_j} - \frac{1}{t^{N-1}} - \frac{1}{t^{m_s}} \left( \frac{1}{t} + \frac{1}{t^3} \right).
\]

**Theorem 5.3.** If \( f \) is pseudo-automorphism and if the possibility (iii) can occur at most once, then the dynamical degree of \( f \) is given by the largest root of the polynomial

\[
\chi_f(t) := t^{N-1} \left[ (Q_1 - Q_4)t^3 + (2Q_1 - Q_2 - Q_3 - Q_4)t^2 + (Q_1 - Q_3)t + Q_4 \right].
\]

**Proof.** By Corollary 6.2 and Lemma 6.3, we see that \( f_Y \) satisfies the hypotheses of Theorem 5.1. Let \( f_Z \) be the corresponding pseudo-automorphism. The dynamical degree will be the modulus of the largest root of the characteristic polynomial of \( f_Z^* \). In the Appendix we show that the characteristic polynomials given by \( \chi_f \).
§6. Periodic maps. In this section, we determine all possible periodic 3-step recurrences. By §3, we may assume (4.1). The question of periodicitities for maps (4.1) with $\beta_0 = 0$ has been answered by Csörnyei and Laczkovic [CL]: they have shown that the only periodicitities in this case are the two period 8 maps given in the Theorem stated in the Introduction. We will consider the general case where $\beta_0$ is possibly nonzero. We start by giving a necessary condition for a map to be periodic.

Proposition 6.1. If $f$ is pseudo-automorphism and if $E$ is an exceptional hypersurface then there is an exceptional hypersurface $E'$ for $f^{-1}$ such that $f^nE = E'$ for some $n > 0$ and the co-dimension of $f^jE$ is $\geq 2$ for all $j = 1, \ldots, n - 1$.

Proof. Suppose $f$ has period $p$. Since $f^pE = E$ and codim $fE \geq 2$, it follows that there exists $0 < n \leq p$ such that codim $f^{n-1}E \geq 2$ and codim $f^nE = 1$. Thus $f^nE$ is an exceptional for $f^{-1}$.

Since $f$ is critical, $\dim f^j\Sigma_\beta < 2$ for $j = 1, 2$, and $f^3\Sigma_\beta = \Sigma_0$; further, $\dim f^j\Sigma_0 < 2$ for $j = 1, 2$, and $f^3\Sigma_0 = \Sigma_\beta$. By Lemma 4.2 the only exceptional hypersurface for $f_Y$ is $\Sigma_\gamma$, and the only exceptional hypersurface for $f_Y^{-1}$ is $\Sigma_C$. This gives us the following necessary condition for $f$ to be periodic.

Corollary 6.2. If $f$ is periodic, then $f$ is critical and there is some $n > 0$ such that $f^n_Y \Sigma_\gamma = \Sigma_{\beta_\gamma}$ and $f^{-n}_Y \Sigma_C = \Sigma_{BC}$.

Proof. If $f$ is pseudo-automorphism then so is $f_Y$. Since both $f_Y$ and $f_Y^{-1}$ have the unique exceptional hypersurface, there exists $n \geq 0$ such that $f^n_Y \Sigma_\gamma = \Sigma_{\beta_\gamma}$ which blows up to a hypersurface $\Sigma_C$. If $f$ is periodic then so is $f^{-1}$ and thus $f_{Y}^{-n} \Sigma_C = \Sigma_{BC}$.

Lemma 6.3. Suppose $f$ is periodic. If $f^n_Y \Sigma_\gamma \subset \mathcal{F}_{0\beta_\gamma}$ (possibility (iii) in §4), then $f^{i+1}_Y \Sigma_\gamma = \mathcal{F}_{0\beta_\gamma}$. Thus $f_Z$ is a pseudo-automorphism.

Proof. Suppose $f$ is a periodic map with period $p$. For each $i = 1, \ldots, p$, let us set $V_i = f_Y^i \Sigma_\gamma$. It follows that $(f_Y^{-1})^i \Sigma_\gamma = V_{p-i}$. If $V_j \in \mathcal{F}_{0\beta_\gamma}$ then $V_{j+1} = \mathcal{F}_{0\beta_\gamma}$. Applying Lemma 4.10 to $f_Y^{-1}$ we see that $f_Y^{-1}V_{j+1} = V_j = \mathcal{F}_{0\beta_\gamma}$. It follows that $f$ satisfies every condition in Theorem 5.1 and therefore $f_Z$ is a pseudo-automorphism.

Lemma 6.4. If $f$ is periodic then $\chi_f(t)$ is self-reciprocal, and $\chi_f = \chi_{f^{-1}}$.

Proof. A polynomial $p(z) = \sum_{i=0}^{k} a_i z^i$, $a_i \in \mathbb{C}$ is self-reciprocal if $p(z) = \pm z^k p(1/z)$. If $f$ is periodic then the characteristic polynomial of $f_Z^*$, $\chi(t)$ is a product of cyclotomic factors and thus $\chi(t)$ is self-reciprocal. Furthermore by Lemma 6.3 $f_Z$ is a pseudo-automorphism and therefore $(f_Z^*)^{-1} = (f_Z^{-1})^*$. It follows that $\chi_f$ and $\chi_{f^{-1}}$ are integer polynomials with the same roots.

Lemma 6.5. If $f$ is periodic, then there is a non-negative integer $m$ such that

(1) $m = m_u = m_d < N$, $1 < d_1 < u_1 < \cdots < d_m < u_m < N$ and
(2) $N - u_j = d_{m+1-j}$, $N - d_j = u_{m+1-j}$ for $j = 1, \ldots, m$.

Proof. From (5.8) we see that the characteristic polynomial $\chi = \chi_f(t)$ is given by $\chi(t) =$
\[ t^{N-1}(t^2 + 1)\varphi(t), \]

where

\[ \varphi(t) = \frac{1}{t^{m_1}}(t-1)(t+1) + \frac{1}{t^{N-1}}(t^N(t^3-t-1) + (t^3 + t^2 - 1)) + (t^3 + t^2) \left( t \sum_{j=1}^{m_u} \frac{1}{t^u_j} + (t-1) \sum_{j=1}^{m_d} \frac{1}{t^d_j} \right) - t \sum_{j=1}^{m_u} \frac{1}{t^u_j} - t \sum_{j=1}^{m_d} \frac{1}{t^d_j}. \]  

(6.1)

By Lemma 6.4, \( \chi(t) \) should be self-reciprocal. Since the first part of \( \chi \) and the first line of (6.1) are self-reciprocal, it suffices to consider the case \( m_u = \infty \) and \( m_u m_d \neq 0 \). In this case \( \dim f_Z^j \Sigma_\gamma = \dim f_Z^{j+1} \Sigma_\gamma \) if and only if \( j \not\in \{u_i, i = 1, \ldots, m_u\} \cup \{d_i, i = 1, \ldots, m_d\} \). Thus it is clear that we have \( m = m_u = m_d < N \) and \( 1 < d_1 < u_1 < \cdots < d_m < u_m < N \) for some positive integer \( m \). Thus we have

\[ f_Z : \Sigma_\gamma \rightarrow \Sigma_{BC} \rightarrow \cdots \rightarrow f_Z^{d_1} \Sigma_\gamma \rightarrow p_1 \in \Sigma_{BC} \rightarrow \cdots \rightarrow q_1 \in \Sigma_{\beta \gamma} \]

or

\[ f_Z^{d_1} \Sigma_\gamma \rightarrow \cdots \rightarrow f_Z^{d_2} \Sigma_\gamma \rightarrow \cdots \rightarrow f_Z^{d_1} \Sigma_\gamma = \Sigma_{\beta \gamma} \Rightarrow \Sigma_C \]

By interchanging the roles of \( \Sigma_{\beta \gamma}, \Sigma_\gamma \) and \( \Sigma_{BC}, \Sigma_C \), we see that the characteristic polynomial for \( f^{-1} \) is given by \( \chi_{f^{-1}}(t) = t^{N-1}(t^2 + 1)\varphi(t) \) where

\[ \varphi(t) = \frac{1}{t^{N-1}}(t^N(t^3-t-1) + (t^3 + t^2 - 1)) + (t^3 + t^2) \left( t \sum_{j=1}^{m} \frac{1}{t^{N-d_j}} + (t-1) \sum_{j=1}^{m} \frac{1}{t^{N-u_j}} \right) - t \sum_{j=1}^{m} \frac{1}{t^{N-d_j}} - t \sum_{j=1}^{m} \frac{1}{t^{N-u_j}}. \]  

(6.2)

Since both \( f \) and \( f^{-1} \) have the same characteristic polynomial, we obtain the second statement of the Lemma by comparing \( \chi_f \) and \( \chi_{f^{-1}} \). \( \square \)

**Lemma 6.6.** Suppose \( f \) is periodic.

(a) If \( m \) is even, then for all \( j = 1, \ldots, m, 2 \leq u_j - d_j \leq d_1 \).

(b) If \( m \) is odd, then \( 1 \leq u_{(m+1)/2} - d_{(m+1)/2} \leq d_1 \) and for all \( j \neq (m+1)/2, 2 \leq u_j - d_j \leq d_1 \).

**Proof.** Suppose \( j_* \) is the smallest positive integer such that \( u_{j_*} - d_{j_*} > d_1 \). Then we have

(1) \( f_Y^{d_{j_*}} \Sigma_\gamma \) is a fibration in \( \Sigma_\gamma \), (2) \( f_Y^{d_{j_*}+1} \Sigma_\gamma \) is a point in \( f_Y^i \Sigma_\gamma \) for \( i = 1, \ldots, d_1 \), and (3) \( f_Y^{d_{j_*}+1} \Sigma_\gamma \in f_Y^{d_{j_*}+1} \Sigma_\gamma \) which is a point in \( \Sigma_{BC} \). It follows that the exceptional hypersurface \( \Sigma_\gamma \) is pre-periodic which contradicts to the hypothesis \( f \) is periodic. If \( u_j - d_j = 1 \) then \( f_Y^{d_{j_*}+1} \Sigma_\gamma = \Sigma_{BC} \cap \Sigma_{\beta \gamma} = f_Y^{d_{j_*}} \Sigma_\gamma \). Thus the situation \( u_j - d_j = 1 \) can happens at most once and by Lemma 6.5 we see that \( (N - d_j) - (N - u_j) = u_{j+1} - d_{j+1} = 1 \). It follows that \( j = m -j+1 \) and thus \( j = (m+1)/2 \). \( \square \)

**Lemma 6.7.** Suppose \( f \) is critical and \( m \geq 1 \) then

(a) \( d_1 \neq 1, 3, 4 \)

(b) If \( d_1 = 2 \) then \( m = 1 \), and either (i) \( \alpha_0 = \alpha_2 = 1 \) and \( \beta_0 = 0 \) or (ii) \( \alpha_0 = \eta^2 \alpha_2 = \eta, \beta = \eta^2 \) where \( \eta^2 - \eta + 1 = 0 \).

(c) If \( m \geq 2 \) is odd then for \( j = 1, \ldots, m - 1, d_{j+1} - u_j \geq 5. \)
(d) If \( m \geq 2 \) is even then for \( 1 \leq j \leq m-1 \) and \( j \neq m/2 \), \( d_{j+1} - u_j \geq 5 \) and \( d_{m/2+1} - u_{m/2} \geq 4 \).

**Proof.** (a) \( d_1 = 1 \) means \( \Sigma_{BC} \) is a line through \( e_1 \) in \( \Sigma_\gamma \). Since \( \Sigma_{BC} = \{ x_3 = 0, \alpha_0 x_0 + \alpha_2 x_1 + x_2 = 0 \} \) and \( \alpha_2 \neq 0 \), it follows that \( e_1 \notin \Sigma_{BC} \) and thus \( d_1 \neq 1 \). Since \( \Sigma_{BC} \subset \Sigma_3 \), we have \( f_3^3 \Sigma_\gamma = f_3^2 \Sigma_{BC} \subset \Sigma_3 \) which doesn’t contain \( e_1 \). Furthermore \( f_3 \Sigma_1 = \{ \beta_0 x_0 : \beta_0 x_2 : \beta_0 x_3 : \alpha_0 x_0 + \alpha_2 x_2 + x_3 \} \) if \( \beta_0 \neq 0 \) and \( f_3 \Sigma_1 \) is a line in the blowup fiber \( E_3 \) if \( \beta_0 = 0 \). It follows that \( f_3 \Sigma_1 \) does not contain \( e_1 \). Therefore \( d_1 \neq 3 \) or 4. The statement for (b) can be confirmed by direct computation. For each \( j \leq m-1 \), \( f_{u_j}^{u_j+1} \Sigma_\gamma \) is a line through \( e_3 \) in \( \Sigma_C \) which can be parametrized as \( t \mapsto \{ [1 : \mu : -\alpha_0 - \alpha_2 \mu : t] \} \) for some fixed \( \mu \in \mathbb{C} \cup \{ \infty \} \). By computing the forward iteration of \([1 : \mu : -\alpha_0 - \alpha_2 \mu : t]\) we see that \( d_{j+1} - u_j \neq 1, 2, \) or 3. Furthermore \( d_{j+1} - u_j = 4 \) if and only if \( f_{u_j}^{u_j+1} \Sigma_\gamma = \Sigma_C \cap \{ (1 + \alpha_2 \beta_0) x_0 + \alpha_2 x_1 = 0 \} \). It follows that \( d_{j+1} - u_j = 4 \) occurs only once. Suppose \( d_{j+1} - u_j = 4 \) for some \( 1 \leq j \leq m - 1 \). By Lemma 6.5 we see that \( (N - u_j) - (N - d_{j+1}) = d_{m-j+1} - u_{m-j} = 4 \). It follows that \( j + 1 = m - j + 1 \) and thus \( j = m/2 \). The statement (c) and (d) follows.

Direct computation shows the following properties:

**Lemma 6.8.** Suppose \( f \) is critical, then \( \dot{\varphi}(t) \) defined in (6.2) satisfies

(a) \( \dot{\varphi}(1) = 0 \), and

(b) \( \dot{\varphi}'(1) = 7(m + 1) - (N + \sum_{j=1}^{m} (u_j - d_j)) \).

**Lemma 6.9.** Suppose that \( m \geq 2 \) and that \( f \) is critical satisfying (5.1). Then

\[
N + \sum_{j=1}^{m} (u_j - d_j) \geq 9m + 3 \quad \text{if } \ m \text{ is odd}
\]

\[
N + \sum_{j=1}^{m} (u_j - d_j) \geq 9m + 4 \quad \text{if } \ m \text{ is even}
\]

Thus \( \varphi' \) is critical, so \( \varphi \) has a root greater than 1.

**Proof.** Suppose \( m \) is even. By Lemma 6.5 we see that

\[
N + \sum_{j=1}^{m} (u_j - d_j) = 2d_1 + 4(u_1 - d_1) + 2(d_2 - u_1) + \cdots + 2(d_m - u_{m-1}) + 4(u_m - d_m) + (d_m - u_m)
\]

By Lemma 6.7, (a) and (b), we have \( d_1 \geq 5 \). Applying Lemma 6.6 (a) and Lemma 6.7 (d) we have

\[
N + \sum_{j=1}^{m} (u_j - d_j) \geq 2 \cdot 5 + 4 \cdot 2 + 2 \cdot 5 + \cdots + 4 \cdot 2 + 4 = 9m + 4.
\]

Similarly when \( m \) is odd

\[
N + \sum_{j=1}^{m} (u_j - d_j) = 2d_1 + 4(u_1 - d_1) + 2(d_2 - u_1) + \cdots + 2(d_{m+1})/2 + 2(d_{m+1})/2.
\]

Again applying Lemma 6.6 (b) and Lemma 6.7(c) we have

\[
N + \sum_{j=1}^{m} (u_j - d_j) \geq 2 \cdot 5 + 4 \cdot 2 + 2 \cdot 5 + \cdots + 2 \cdot 5 + 2 = 9m + 3.
\]
Theorem 6.10. If $f$ is periodic with $m = 0$ and $m_s = \infty$ then $f$ is one of the following:

- $\alpha = (-1, 0, -1, 1)$, $\beta = (0, 1, 0, 0)$: $f_{\alpha \beta}$ has period 8 and there is a conic $Q$ such that
  $$f_Y : \Sigma_\gamma \to \Sigma_{BC} \to Q \to \Sigma_{\beta \gamma} \sim \Sigma_C.$$ 

- $\alpha = (-1/2, 0, -1, 1)$, $\beta = (1, 1, 0, 0)$: $f_{\alpha \beta}$ has period 12, and
  $$f_Y : \Sigma_\gamma \to \Sigma_{BC} \to L_1 \to L_2 \to \Sigma_{\beta \gamma} \sim \Sigma_C$$
  where we set $L_1 = \Sigma_2 \cap \{x_0 + x_3 = 0\}$ and $L_2 = \Sigma_1 \cap \{x_0 + x_2 = 0\}$.

Proof. The polynomial defined in (5.8) is also given by

$$\chi(t) = (t^2 + 1) \left( t^N (t^3 - t - 1) + t^3 + t^2 - 1 \right).$$

It follows that $\chi(t)$ has a root bigger than 1 if and only if $N \geq 8$ and in case $N = 7$ the matrix representation of $f_Z^7$ has $3 \times 3$ Jordan block with eigenvalue 1. Thus we need to check the situation $f^{n+1} \Sigma_\gamma = \Sigma_{\beta \gamma}$ only for $n \leq 5$. For this, let us parametrize $\Sigma_{BC} = \{(1 : t : -\alpha_0 - \alpha_2 t : 0)\}$ and let $[f_0^{(n)} : f_1^{(n)} : f_2^{(n)} : f_3^{(n)}]$ denote the $n$-th iteration of $\Sigma_{BC}$. If $f^{n+1} \Sigma_\gamma = \Sigma_{\beta \gamma}$ then for all $t$ we have

$$\beta_0 f_0^{(n)} + f_1^{(n)} = 0, \quad \alpha_0 f_0^{(n)} + \alpha_2 f_2^{(n)} + f_3^{(n)} = 0 \quad \text{(6.3)}$$

Since equations in (6.3) are polynomials in $t$ whose coefficients are integer polynomials in the variables $\beta_0, \alpha_0, \alpha_2$, we may use the computer show that for $0 \leq n \leq 5$, the only two possibilities are those listed above.

Theorem 6.11. If $f$ is periodic with $m = 1$ and $m_s = \infty$, then $f$ is one of the following:

- $\alpha = (1, 0, 1, 1)$, $\beta = (0, 1, 0, 0)$: $f$ has period 8, $\Sigma_{BC} \cap \Sigma_{\beta \gamma} \neq \emptyset$, and
  $$f_Y : \Sigma_\gamma \to \Sigma_{BC} \to \Sigma_\gamma \cap \Sigma_2 \to \Sigma_{BC} \cap \Sigma_{\beta \gamma} \sim \Sigma_C \cap \Sigma_2 \to \Sigma_{\beta \gamma} \sim \Sigma_C.$$ 

- $\alpha = (\eta/(1 - \eta), 0, \eta, 1)$, $\beta = (\eta^2, 1, 0, 0)$ and $\eta^3 = -1$, $\eta \neq -1$: $f_{\alpha \beta}$ has period 12, and
  $$f_Y : \Sigma_\gamma \to \Sigma_{BC} \to \Sigma_\gamma \cap \Sigma_2 \to p_1 \in \Sigma_{BC} \to p_2 \in \Sigma_{\beta \gamma} \sim \Sigma_C \cap \Sigma_1 \to \Sigma_{\beta \gamma} \sim \Sigma_C,$$
  where $p_1 = [1 : 0 : -\eta^2 : 0] \in \Sigma_{BC}$ and $p_2 = [1 : -\eta^2 : 0 : -\eta^2] \in \Sigma_{\beta \gamma}$.

Proof. From (5.8) the characteristic polynomial of $f_Z^7$ is given by

$$\chi(t) = t^{N-(u_1+d_1)}(t^2 + 1)(t^{d_1+1}) \left( t^{u_1}(t^3 - t - 1) + t^3 + t^2 - 1 \right).$$

It follows that $\chi(t)$ has a root bigger than 1 if and only if $u_1 \geq 8$. If $u_1 = 7$, the $f_Z^7$ has a $3 \times 3$ Jordan block. Thus if $f_Z^7$ is periodic then $d_1 \leq 5 < u_1$. By direct computation of $f^n \Sigma_\gamma = f^{n-1} \Sigma_{BC}$ for $n = 1, \ldots, 5$, we can easily check the two conditions (i) $f^n \Sigma_{BC} \subset \Sigma_\gamma$,
(ii) $f^n \Sigma_{BC} \subset \{x_3 = \lambda x_2\}$ for some $\lambda \in \mathbb{C}$ and thus we see that there are only two possibilities listed in this Theorem.
Theorem 6.12. If \( m \geq 2, m_s = \infty \) then \( f \) has exponential degree growth (and is not periodic).

Proof. By Lemmas 6.8 and 6.9 we see that \( \chi_N(1) = 0 \) and \( \chi'_N(1) = 2(3 + \varphi'(1)) = 2(-2m+1) \). It follows that if \( m \geq 1 \) then \( f \) has positive entropy. Now suppose \( m = 0 \), we have \( \chi_N(t) = (t^2 + 1)(t^N(t^3 - t - 1) + t^{N-m_s-2}(t-1)(t^2 + t + 1) + t^3 + t^2 - 1) \). If \( f \) is periodic then the characteristic polynomial for \( f^*_Z \) should be self-reciprocal. It follows that \( m_s = (N - 4)/2 \). It follows that \( N \) should be even and

\[
\chi_N(t) = (t^2 + 1) \left( t^N(t^3 - t - 1) + t^{N/2}(t-1)(t^2 + t + 1) + t^3 + t^2 - 1 \right).
\]

By inspection we see that \( m_s \geq 3 \) and it follows that \( N \geq 10 \). We can also check that \( \chi_N(1) = 0 \) and \( \chi'_N(t) = 10 - N \). Therefore if \( m_s \neq 3 \) then \( f \) is not periodic. In case \( m_s = 3 \), the matrix representation for \( f^*_Z \) has \( 3 \times 3 \) Jordan block with eigenvalue 1 and all other eigenvalues have modulus 1.

Proof of Theorem 5: The statement of the Theorem 5 in the Introduction follows from Theorems 6.10–13.

We remark that in the proof of Theorem 6.13, we see that if \( m_s = 3 \) and \( m = 0 \), then the degree of \( f^n \) is quadratic in \( n \). This case occurs for \( \alpha = (a, 0, 1, 1) \) and \( \beta = (0, 1, 0, 0) \), which is the so-called Lyness process and will be discussed in §8.

§7. Pseudo-automorphisms with positive entropy In this section we consider the case

\[
\beta = (0, 1, 0, 0) \quad \text{and} \quad \alpha = (a, 0, \omega, 1) \tag{7.1}
\]

where \( \omega^2 + \omega + 1 = 0 \) and \( a \in \mathbb{C} \setminus \{0\} \). With this choice of parameters, we see that \( f \) is critical and that \( \Sigma_B = \Sigma_3 \) and \( \Sigma_\beta = \Sigma_1 \). Since the maps \( f : \Sigma_3 \to \Sigma_2 \to \Sigma_1 \) are dominant, (4.3) gives an 8-cycle of dominant maps

\[
f_Y : \Sigma_1 \to E_3 \to S_{01} \to \Sigma_0 \to S_{03} \to E_1 \to \Sigma_3 \to \Sigma_2 \to \Sigma_1 \tag{7.2}
\]

Since this 8-cycle is fundamental to our understanding of \( f \) in this case, we will refer to the union of these 8 hypersurfaces as the rotor and denote it as \( R \). Clearly, \( f^8_Y \) fixes each component of the rotor; in addition, it has a relatively simple expression. On \( \Sigma_3 \), or example, we have:

\[
f^8_Y : \Sigma_3 \ni [x_0 : x_1 : x_2 : 0] \\
\quad \mapsto [x_0(ax_0 + \omega x_2)(ax_0 + ax_1 + \omega x_2) \\
\quad \quad \quad : x_1(x_1x_2 + a \omega x_2^2 + a \omega x_0 x_1 + a \omega x_0 x_2 + \omega x_0 x_2 + \omega^2 x_0 x_2 + \omega^2 x_2^2) \\
\quad \quad \quad : \omega x_2(ax_0 + \omega x_2)(x_1 + a \omega x_0 + \omega^2 x_2) : 0] \in \Sigma_3.
\tag{7.3}
\]

The restriction of \( f^8_Y \) to the rotor is studied in §C.

Note that by (7.1), \( \Sigma_{BC} = \Sigma_3 \cap \Sigma_C \) and \( \Sigma_{\beta \gamma} = \Sigma_1 \cap \Sigma_\gamma \). Using (7.2) we may verify that \( f_Y \) satisfies condition (5.1), which in this case is

\[
f_Y^1 : \Sigma_\gamma \not\subset F_{0 \beta \gamma}, \quad \text{for all} \ 1 \leq j \leq 10, \quad \text{and} \quad f_Y^{11} = \Sigma_{\beta \gamma} \tag{7.4}
\]

We define the space \( \pi_Z : Z \to Y \) by successively blowing up the 11 curves \( \gamma_j := f_Y^j \Sigma_\gamma, \ 1 \leq j \leq 11 \). The dynamical degree, being a birational invariant, is independent of the order in which the \( \gamma_j \)'s are blown up.

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Theorem 7.1. The induced map $f_Z$ is a pseudo-automorphism, and the dynamical degree of $f$ is greater than 1.

Proof. From (7.4) we see that $f_Y$ satisfies the condition in Theorem 5.1, so $f_Z$ is a pseudo-automorphism. By Theorem 5.3, the characteristic polynomial of $f_Z'$ is $t^{11}(t^3 - t - 1) + t^3 + t^2 - 1 = (-1 + t)(1 + t)(1 + t^2)(1 - t - t^2 + t^4 + t^5)$. Thus $\delta(f)$ is the largest root of this polynomial, which is approximately 1.28064.

The space $Z$ has been defined earlier, but now let us be more precise: we define $Z$ as the space obtained by blowing up first $11 \subset Y$, then we blow up the strict transform of $\gamma_9$ in the resulting space, followed by blowing up the strict transform of $\gamma_9$, and continuing this way until we blow up the strict transform of $\gamma_1$. We will use the notation $\Gamma_j$ to denote the exceptional divisor of the blowup of $\gamma_j$. There are no points where three distinct $\gamma_j$'s intersect. If $p = \gamma_j \cap \gamma_k$, with $j > k$, then we blow up $\gamma_j$ first, and we refer to the fiber in $\Gamma_j$ over $p$ as the first fiber over $p$, and write it as $\mathcal{F}^1_p$. We then blow up the strict transform of $\gamma_j$, and the blowup fiber over the point $\gamma_k \cap \gamma_j$ is equal to $\Gamma_j \cap \Gamma_k$.

Let us describe some of the intersections of the $\gamma_j$'s. $\delta$ is constant on each line in $\Sigma_\gamma$ passing through $e_1$. Further, $\gamma_1 \subset \Sigma_3$, $\gamma_2 \subset \Sigma_2$, and $\gamma_6 \subset \Sigma_0$, and $e_1 = \Sigma_0 \setminus \Sigma_2 \setminus \Sigma_3$. We set $\ell_2 = \Sigma_\gamma \setminus \Sigma_3$, $\ell_3 = \Sigma_\gamma \setminus \Sigma_2$, and $\ell_7 = \Sigma_\gamma \setminus \Sigma_0$. Thus we have $f(\ell_j) = \gamma_j \cap \gamma_j$ for $j = 2, 3, 7$. The curve $\gamma_9 \subset \Sigma_1$ is a conic, and $\gamma_9 \cap \gamma_1$ consists of two points. We let $\ell_9'$ and $\ell_9''$ denote the two lines in $\Sigma_\gamma$ for which $f(\ell_9' \cup \ell_9'') = \gamma_1 \cap \gamma_9$. This accounts for all the curves $\gamma_j$ which intersect $\gamma_1$. As a consequence of the order of blowup, the first fiber $\mathcal{F}^1_{f(\ell_j)} = \mathcal{F}^1_{\gamma_1 \cap \gamma_j}$, $j = 2, 3, 7$ is contained in $\Gamma_j$ and similarly for $\ell_9', \ell_9''$.

There is a similar situation for the $\gamma_j$'s which intersect $\gamma_{11}$. The curves $\gamma_5$, $\gamma_9$ and $\gamma_{10}$ each intersect $\gamma_{11}$ in a single point, and $\gamma_{3}$, which intersects $\gamma_{11}$ in 2 points, and this accounts for all the intersection points between $\gamma_{11}$ and the other $\gamma_j$'s.

Let us use the notation $\pi_1 : Z_1 \rightarrow Y$ for the manifold obtained by blowing up the curve $\gamma_{11} \subset Y$. This is the first blowup performed in the construction of $Z$. Let $f_{Z_1} : Z_1 \rightarrow Z_1$ be the induced map. Since $\mathcal{I}(f_Y) = \gamma_{11} \cup \Sigma_{02} \cup \mathcal{F}^1_{03\gamma}$, it follows that $\mathcal{I}(f_{Z_1}) \subset \Sigma_{02} \cup \mathcal{F}^1_{03\gamma} \cup \gamma_{10} \cup \Gamma_{11}$.

Lemma 7.2. $\mathcal{I}(f_{Z_1}) = \mathcal{F}^1_{03\gamma} \cup \Sigma_{02} \cup \gamma_{10}$.

Proof. We have seen already that the indeterminacy locus is contained in $\Sigma_{02} \cup \mathcal{F}^1_{03\gamma} \cup \gamma_{10} \cup \Gamma_{11}$, so it suffices to show that $\mathcal{I}(f_{Z_1}) \cap \Gamma_{11}$ consists of the two points $\gamma_{10} \cap \Gamma_{11}$ and $\mathcal{F}^1_{03\gamma} \cap \gamma_{11}$. Thus we look at $f_{Z_1}$ in coordinate charts that cover $\Gamma_{11}$. We will look first at $\Gamma_{11} \cap \pi^{-1}_1(\gamma_{11} - \gamma_{10})$.

In the local coordinates $(s, \zeta, x_3)_{S_{01}} \mapsto [s : s\zeta : 1 : x_3] \in \mathbb{P}^3$ in the neighborhood of $S_{01} - E_1 = \{s = 0, \zeta \neq \infty\}$, we have $\gamma_{11} = \{\zeta = 0, as + \omega + x_3 = 0\}$ and $\mathcal{F}^1_{03\gamma} \cap \gamma_{11} = (0, 0, -\omega)_{S_{01}}$. We use the local coordinate maps $(s, t, \eta)' : U' \mapsto U'$ and $(s, \eta, t)'' : U'' \mapsto U''$ so that $\pi_1$ is given by

\[
\pi' : U' \ni (s, t, \eta)' \mapsto (s, -as - \omega + t\eta)_{S_{01}}
\]

\[
\pi'' : U'' \ni (s, \eta, t)'' \mapsto (s, \eta, -as - \omega + t)_{S_{01}}
\]

It is evident that $\Gamma_{11} \ni \{t = 0\}$ in both coordinate charts, and $U' \cup U'' \ni \pi^{-1}_1(\gamma_{11} - \gamma_{10})$. The induced map $f \circ \pi_1 : U' \cup U'' \rightarrow \mathbb{P}^3$ is given by

\[
U' \ni (s, t, \eta)' \mapsto [s : 1 : -as + t\eta - \omega]
\]

\[
U'' \ni (s, \eta, t)'' \mapsto [s\eta : \eta : (as + t - \omega) : 1].
\] (7.4)

so we see that $\{t = 0\}$ is mapped to $\Sigma_C$. 23
From (7.4) we see that the map \( f \circ \pi_1 : U' \cup U'' \to \mathbb{P}^3 \) is everywhere regular. The only points of \( \Sigma_C \) which is blown up in the construction of \( Y \) are \( e_3 \) and \([0 : 1 : -\omega : 0]\) which is point the base point of \( F_{0BC}^1 \). By (7.4), the preimage of \( e_3 \) is \((0, 0, 0)'' \in U''\), and the preimage \([0 : 1 : -\omega : 0]\) is \((0, 0, 0)' \in U'\). Working in local coordinates in \( Y \) over \( e_3 \), we find that \( f \circ \pi_1 : U'' \to Y \) is everywhere regular. Thus we conclude that \( f_Y \circ \pi_1 : Z_1 \to Y \) is regular on \((U'' - (0, 0, 0)') \cup U''\). Now in order to pass to \( f_{Z_1} \) we need to consider the point \( \gamma_{11} \cap \Sigma_C \) which is blown up. However, this is the image point of \( \gamma_{10} \cap \gamma_{11} \), which is not in our coordinate chart. We note that \((0, 0, 0)'\) is the point \( F_{0\beta\gamma}^1 \cap \Sigma_{11} \), so we conclude that \( f_{Z_1} \) is regular at all points of \( \Gamma_{11} - (\pi_1^{-1}(\gamma_{11} \cap \gamma_{10}) \cup F_{0\beta\gamma}^1) \).

Now we consider \( \Gamma_{11} \cap \pi^{-1}(\gamma_{11} - F_{0\beta\gamma}^1) \), which does not lie over any of the centers of blowup in the construction of \( Y \). As in (5.7), we may use the local coordinates \((s, x_2, \zeta) \mapsto [1 : s : x_2 : -a - \omega x_2 + s\zeta] \) in a neighborhood of \([s = 0, \zeta \neq \infty] \subset \Gamma_{11} - \pi_1^{-1}(F_{0\beta\gamma}^1) \), and we get

\[
f_{Z_1} : \Gamma_{11} \ni (0, x_2, \zeta) \mapsto [1 : x_2 : -a - \omega x_2 : \zeta] \in \Sigma_C \quad \text{if } (0, x_2, \zeta) \neq (0, 0, a\omega - a). \quad (7.5)
\]

Similarly using the local coordinates \((\zeta, x_2, s) \mapsto [1 : s\zeta : x_2 : -a - \omega x_2 + s] \in \Gamma_{11} \), we have

\[
f_{Z_1} : \Gamma_{11} \ni (\zeta, x_2, 0) \mapsto [\zeta : x_2\zeta : \zeta(-a - \omega x_2) : 1] \in \Sigma_C \quad \text{if } (\zeta, x_2, 0) \neq \left(\frac{1}{a\omega - a}, 0, 0\right). \quad (7.6)
\]

Since both \((0, 0, a\omega - a)\) in (7.5) and \((1/(a\omega - a), 0, 0)\) in (7.6) correspond to the point \( \gamma_{11} \cap \gamma_{10} \), combining with the previous conversation about \( \Gamma_{11} - \pi_1^{-1}(\gamma_{11} \cap \gamma_{10}) \), we conclude that \( f_{Z_1} \) is regular at all points of \( \Gamma_{11} - (\gamma_{10} \cup F_{0\beta\gamma}^1) \).

\[\square\]

**Lemma 7.3.** The three curves \( \gamma_5, \gamma_{11}, F_{0\beta\gamma}^1 \) intersect transversally inside \( Y \), and \( \gamma_1, \gamma_7, F_{0BC}^1 \) intersect transversally inside \( Y \). Thus, inside \( Z_1 \), the strict transform of \( F_{0\beta\gamma}^1 \) is disjoint from the strict transforms of \( \gamma_j \), \( 1 \leq j \leq 10 \).

**Proof.** It suffices to prove the first statement. We may write \( \gamma_{11} \subset \mathbb{P}^3 \) as \( s \mapsto [s : 0 : 1 : -as - \omega] \). This intersects \( \Sigma_01 \) in the point \([0 : 0 : 1 : -\omega] \). We use the coordinate system \( \pi : (u, \eta, x_3) \mapsto [u : u\eta : 1 : x_3] \in \mathbb{P}^3 \). Thus \( F_{0\beta\gamma}^1 \) is \([u = 0, x_3 = -\omega] \). In this coordinate system, \( \gamma_{11} \) becomes \( s \mapsto (s, 0, -as - \omega) \), so \( \gamma_{11} \) crosses \( S_01 \) when \( s = 0 \), at the point \((0, 0, -\omega) \). On the other hand, if we map \( \gamma_{11} \) backward under \( f_Y^{-6} \), we find an expression for \( \gamma_5 \). The base point is given by \([0 : 0 : 1 : s] \), and the fiber coordinate is given by \( \eta = (1 + as)(1 + as + \omega(1 + a - s))/ \omega(-1 + s + as - \omega(1 + a - as)) \). Thus when the base point is \([0 : 0 : 1 : -\omega] \), we have \( \eta = 0 \). Thus all three curves meet at \((u, \eta, x_3) = (0, 0, -\omega) \). The curve \( \gamma_{11} \) is transverse to \( S_01 \), but \( \gamma_5 \) and \( F_{0\beta\gamma}^1 \) are tangential to \( S_01 \), so \( \gamma_{11} \) is transverse to the other two, and \( \gamma_5 \) is transverse to \( \{x_3 = -\omega\} \), while \( F_{0\beta\gamma}^1 \) is tangential to this set.

\[\square\]

For \( 2 \leq j \leq 11 \), let \( \pi_j : Z_j \to Z_{j-1} \) be the blowup of the strict transform of \( \gamma_{12-j} \) inside \( Z_{j-1} \) and \( \pi : Z \to Y = \pi_{11} \circ \pi_{10} \circ \cdots \circ \pi_1 \), that is, we blowup \( \gamma_{11} \) first, then \( \gamma_{10} \), then \( \gamma_9 \), etc. Let \( f_{Z_j} : Z_j \dashrightarrow Z_j, f_Z : Z \dashrightarrow Z \) denote the induced map.

**Lemma 7.4.** For \( 1 \leq j \leq 10 \), \( T(f_{Z_j}) = F_{0\beta\gamma}^1 \cup \Sigma_{02} \cup \gamma_{11-j} \).

**Proof.** Suppose \( p \) is a point of \( \gamma_j \cap \gamma_k, 1 \leq j < k \leq 10 \). Because of the order of blowup, \( \gamma_k \) is blown up before \( \gamma_j \) and \( \gamma_{k+1} \) is blown up before \( \gamma_{j+1} \). Since \( f_Y \) is regular at \( p \) and the order of blowups at \( p \) is consistent with the order of blowups at \( f_Y(p) \), the induced map \( f_{Z_j} \) is a local biholomorphism in a neighborhood of the exceptional divisor over \( p \) for \( 12 - j \leq i \leq 11 \).

\[24\]
Notice that for all $1 \leq j \leq 11$ the strict transformation of $\gamma_j$ does not intersect $\Sigma_{02}$ in $Y$. Suppose $\gamma_j$ intersects $F_{10\beta\gamma}$ at a point $q$. Using the local coordinates in the neighborhood $(s, \zeta, x_3)_{S_{01}}$, we may assume that $q = (0, \zeta, -\omega)_{S_{01}}$ and $\gamma_j(s) = (Q_1(s), Q_2(s) + \zeta, Q_3(s) - \omega)_{S_{01}}$, where $\gamma_j = \{\gamma_j(s), s \in \mathbb{C}\}$, and $\gamma_j(0) = q$. Consider two local coordinate charts covering the exceptional divisor over the point $q$:

\[(s, t, \eta) \mapsto (Q_1(s), Q_2(s) + \zeta + t, Q_3(s) - \omega + t\eta)_{S_{01}}\]

\[(s, \eta, t) \mapsto (Q_1(s), Q_2(s) + \zeta + t\eta, Q_3(s) - \omega + t)_{S_{01}}\]

With a computation similar to Lemma 7.2, we see that the induced map is regular everywhere on the exceptional divisor over $q$, $F(q)$, except the point of intersection $F(q) \cap F_{10\beta\gamma}$. Now since the curve $\gamma_{11-j}$ is the pre-image of $\gamma_{12-j}$, we have $\mathcal{I}(f_{Z_j}) = F_{10\beta\gamma} \cup \Sigma_{02} \cup \gamma_{11-j}$. \qed

From the previous Lemma we have $\mathcal{I}(f_{Z_{10}}) = F_{0\beta\gamma} \cup \Sigma_{02} \cup \gamma_1$. Since $\Sigma_\gamma$ is the pre-image of $\gamma_1$, we have $\mathcal{I}(f_Z) \subset F_{1\beta\gamma} \cup \Sigma_{02} \cup \Sigma_\gamma$. From (5.2) we see that for all most every line $\ell \subset \Sigma_\gamma$, through $e_1$ in $\Sigma_\gamma$, $f$ maps $\ell$ regularly to a point $q \in \gamma_1$. In our construction, we blew up $\gamma_{11}, \ldots, \gamma_2$ before $\gamma_1$. Thus the map $f_Z$ will map $\ell$ regularly to the fiber of $\Gamma_1$ over $q$ unless $q$ is an intersection point of $\gamma_1 \cap \gamma_j$ for some $2 \leq j \leq 11$.

**Lemma 7.5.** Suppose $q \in \gamma_1 \cap \gamma_j$ for some $j = 2, \ldots, 11$ and $\ell_j \subset \Sigma_\gamma$ be the line which mapped to $q$ by $f_Y$. The line $\ell_j \subset \mathcal{I}(f_Z)$ and every point in $\ell_j$ blows up to the first blowup fiber $F_{1\gamma}$. \[\]

**Proof.** Let us parametrize $\gamma_1 = \{\gamma_1(t) = [-\frac{1}{a}(1 + \omega t) : t : 1 : 0], t \in \mathbb{C}\}$. Let us set $q = \gamma_1(t_*)$ for some $t_* \in \mathbb{C}$ and $\gamma_j = \{\gamma_j(s) = [Q_0(s) - \frac{1}{a}(1 + \omega t_*) : Q_1(s) + t_* : Q_2(s) + 1 : Q_3(s)]\}$. The line $\ell_j$ is given by the strict transform in $Y$ of the line connecting $e_1$ and $q = [-\frac{1}{a}(1 + \omega t_*) : 0 : t_* + 1]$ in $\mathbb{P}^3$. To see the image of the line $\ell_j$, we consider the set $U = \{-\frac{1}{a}(1 + \omega t_*) + s\zeta : u : t_* + s : 1\}$ which has the property that $U \cap \{s = 0\} = \ell_j - \{e_1\}$. Since the point $q$ is blown up twice, let us consider a local coordinate charts for $\pi_{12-j}(\gamma_j)$:

\[(v, \xi, s)_{\gamma_j} \mapsto \left[\frac{Q_0(s) - \frac{1}{a}(1 + \omega t_*)}{Q_2(s) + 1} + v : \frac{Q_1(s) + t_*}{Q_2(s) + 1} + v\xi : 1 : \frac{Q_3(s)}{Q_2(s) + 1}\right].\]

Using the induced map $f_Z$, we see that

\[f_Z : \ell_j \ni [-\frac{1}{a}(1 + \omega t_*) : u : t_* + 1] \sim \{(0, \xi, 0), \xi \in \mathbb{C}\} \subset \Gamma_j,\]

that is, each point in $\ell_j$ blows up to a whole first blowup fiber over $q$. \qed

Before Lemma 7.2, we enumerated the possibilities for lines $\ell$ and points $q$ as in the hypotheses of Lemma 7.5. Thus we may combine Lemmas 7.2–5 to have the following Theorem:

**Theorem 7.6.** The indeterminacy locus $\mathcal{I}(f_Z) = \Sigma_{02} \cup F_{1\beta\gamma} \cup \ell_2 \cup \ell_3 \cup \ell_7 \cup \ell_9 \cup \ell_9'$. If $\zeta$ is a point of one of the lines $\ell$, then $f_Z$ blows up $\zeta$ to the first fiber $F_{f(\ell)}$. \[\]

Now we give the existence of Green currents for the invariant class $\alpha = \alpha_{Z}^{+} \in H^{1,1}(Z)$. 25
Theorem 7.7. There is a positive closed current $T^+_Z$ in the class of $\alpha^+_Z$ with the property: if $\Xi^+$ is a smooth form which represents $\alpha^+_Z$, then $\lim_{n \to \infty} \delta_1(f)^{-n} f^*_Z \Xi^+_Z = T^+_Z$ in the weak sense of currents on $Z$.

Proof. The map $f^*_Z$ is given in Appendix A, where we are in Case (II). Working directly with the matrix (A.1), we see that the invariant class is given by:

$$
\alpha = H_Z - c_1 E_1 - c_3 E_3 - c_{01} \bar{S}_{01} - c_{03} \bar{S}_{03} - \sum_{j=1}^{11} c_j' F_j
$$

where $c_1, c_3 > 0$, $c_1 + c_3 = 1$, $c'_1 > c'_3 > \cdots > c'_1 > 0$, and $c_{01} = c_{03} = c'_1$. As in Theorem 4.9, we will show that $\alpha^+_Z \cdot \sigma$ for each curve $\sigma$ inside the forward image of $\mathcal{I}(f_Z)$. The result will follow from Theorem 1.3 of [Ba].

Let us start with $f_{0\beta\gamma} \subset \mathcal{I}(f_Z)$. Points of this curve are blown up to $f_{0\beta\gamma}$. The curve $\sigma = f_{0\beta\gamma}$ is the exceptional fiber inside $S_{03}$ over the point $\Sigma_{03} \cap \Sigma_{03} \in \mathbb{P}^3$. Thus $\sigma \cdot S_{03} = -1$.

In the construction of $Z$, $\gamma_7$ will be blown up to create the exceptional divisor $\Gamma_7$. At this stage, by Lemma 7.3, $\sigma$ and $\gamma_1$ become separated. Thus $\sigma \cdot \Gamma_1 = 0$, and $\sigma \cdot \Gamma_7 = 1$, so, so $\alpha \cdot \sigma = c_{03} - c_7 > 0$.

Points of the indeterminate curve $\Sigma_{02}$ blow up to $\sigma = f_{e_2}$. In this case, we have that $\sigma \cdot S_{01}$ and $\sigma \cdot S_{03}$, are $\pm 1$, with opposite signs, so $\sigma \cdot \alpha^+_Z = \pm c_{01} \mp c_{03} = 0$ as was seen in the proof of Theorem 4.9.

The other possibility is $\ell \subset \mathcal{I}(f_Z)$, for one of the indeterminate lines in $\Sigma_{\gamma}$. This blows up to one of the first fibers $\sigma = f_{\ell_1}$. In this case, $\sigma$ crosses $\Gamma_1$ transversally, so $\sigma \cdot \Gamma_1 = 1$. On the other hand, $\sigma \subset \Gamma_j$ for some $j > 1$, so we have $\sigma \cdot \Gamma_j = -1$. Thus $\sigma \cdot \alpha^+_Z = c'_j - c'_1 > 0$.

Remark. Considering the symmetry between $f$ and $f^{-1}$, we find that $\mathcal{I}(f_Z^{-1}) = \Sigma_{02} \cup f_{0\beta\gamma} \cup \bigcup_{\eta} f^{1}_{\eta}$, where the $\eta$'s are the intersection points of $\gamma_1$ with the curves $\gamma_2$, $\gamma_3$, $\gamma_7$, and $\gamma_9$.

If we instead blow up the $\gamma_j$'s in the order $\gamma_1$, $\gamma_2$, $\ldots$, and call the resulting space $\hat{Z}$. Then we have $\mathcal{I}(f_{\hat{Z}}) = \Sigma_{02} \cup f_{0\beta\gamma} \cup \bigcup_{\eta} f^{1}_{\eta}$, where the $\eta \in \gamma_11$ are the points of intersection with $\gamma_3$, $\gamma_5$, $\gamma_9$, and $\gamma_{10}$. Each of these points $\eta$ is blown up by $f_{\hat{Z}}$ to a line of the pencil in $\Sigma_{\gamma}$ passing through $e_3$.

Thus we can apply a similar argument to $\alpha^-_Z$ to obtain the Green current for $f_Z^{-1}$.

Corollary 7.8. There is a positive closed current $T^-_Z$ in the class of $\alpha^-_Z$ with the property: if $\Xi^-$ is a smooth form which represents $\alpha^-_Z$, then $\lim_{n \to \infty} \delta_1(f)^{-n} f^{-n}_{Z} \Xi^-_Z = T^-_Z$ in the weak sense of currents on $Z$.

Next we show what happens to the invariant fibration when we lift it to $Z$. Let us set $P_0 = x_0 x_1 x_2 x_3$, and let $P_1$ be a homogeneous quartic polynomial defined in Appendix B. For $c \in \mathbb{C}$, let us set $S_c = \{cP_0 + P_1 = 0\}$, so the rotor $\mathcal{R}$ corresponds to $c = \infty$. Since we have $f(S_c) = S_{\infty}$, the surface $S_0$ is invariant.

Proposition 7.9. The variety $S_0 := \{P_1 = 0\} \subset \mathbb{P}^3$ has singular points at $e_1$, $e_3$ and the fixed points $p_{\pm}$. If $p_{\pm}$ are blown up (in additional to the $e_1$ and $e_3$ which were blown up to construct $Y$), then the strict transform of $S_0$ is a nonsingular $K3$ surface.

Proof. Using the computer, we find that the critical points of $P_1$ occur exactly at $e_1$, $e_3$ and $p_{\pm} = (x_{\pm}, x_{\pm}, x_{\pm}) \in \mathbb{C}^3$ where $x_{\pm}$ are the roots of $x^2 = a + (1 + \omega)x$. (Mathematica, for
instance, can do this.) Further, $p_{\pm}$ are singular points of type $A_1$. The singular points $e_1$ and $e_3$ are type $A_1$ unless $a = (1 + 2\omega)/(1 - \omega)$, in which case they are type $A_2$. In either case, it follows (see, for instance, [EJ, Lemma 3.1 and Remark 3.2]) that $S_0$ is $K3$.

**Corollary 7.10.** For all but finitely many values of $c \in \mathbb{C}$, the strict transform of $S_c$ in $Z$ is a nonsingular $K3$ surface.

Let $\mathcal{P} \subset Y$ denote the (finite) set of all intersection points of distinct curves $\gamma_j \cap \gamma_k$. Since the $\gamma_j$ lie in the rotor, we have $\mathcal{P} \subset \mathcal{R}$. The rotor is the union of 8 smooth hypersurfaces which intersect transversally, so the singular locus of $\mathcal{R}$ is the set where two (or more) of these surfaces intersect. We will write $\mathcal{P}_s$ (resp. $\mathcal{P}_r$) for the points of $\mathcal{P}$ which are contained in the singular (resp. regular) locus of $\mathcal{R}$.

While $Z$ itself depends on the order in which the curves $\gamma_j$ are blown up, the following Propositions are valid for any ordering of the blowups.

**Proposition 7.11.** For $p \in \mathcal{P}_r$, there is a unique $c_p \in \mathbb{C}$ such that $S_{c_p} \subset Y$ is singular at $p$. This is a conical singularity, and the strict transform $S_{c_p} \subset Z$ contains the first fiber $\mathcal{F}_{c_p}$.

**Proof.** Without loss of generality, we may choose coordinates $(x, y, z)$ so that $p = 0, L = z$ near $p$, and $\mathcal{R} = \{z = 0\}$. Let us suppose that $p \in f_Y^1 \Sigma_{BC} \cap f_Y^k \Sigma_{BC}$. Since the curves $f_Y^1 \Sigma_{BC}$ are contained in $\mathcal{R}$ and intersect transversally, we may suppose that near $p$ the curves $f_Y^1 \Sigma_{BC}$ and $f_Y^k \Sigma_{BC}$ coincide with the $x$- and $y$-axes. Thus the tangent to $\{M = 0\}$ at $p$ is given by $z = 0$, so we may suppose that $M = \lambda z + xy + \cdots$. The surfaces are then $S_c = \{M + cL = 0\} = \{\lambda z + xy + cz + \cdots = 0\}$. The surface $S_c$ is singular if $c = -\lambda$. We blow up the $x$-axis by the coordinate change $(x, s, \eta) \mapsto (x, s, s\eta)$. The first fiber is $\mathcal{F}_p^1 = \{x = s = 0\}$. The strict transforms of the surfaces are $S_c = \{\lambda z + cy + x = 0\}$. The strict transform of the $y$-axis is now the $s$-axis, which is contained in each $S_c$. Otherwise, the $S_c$'s are disjoint. The strict transform of $S_{-\lambda}$ contains $\mathcal{F}_p^1$. After we blow up the $s$-axis, the surfaces are all disjoint and smooth.

**Proposition 7.12.** For $p \in \mathcal{P}_s$, $S_c$ is smooth at $p$ for all $c \in \mathbb{C}$. The first fiber is contained in the rotor: $\mathcal{F}_{c}^1 \subset \mathcal{R} \subset Z$.

**Proof.** We may assume that $p$ is a normal crossing of two of the hypersurfaces of $\mathcal{R}$. Thus we may choose coordinates $(x, y, z)$ such that $p = 0$, and $L = xy$ near $p$. We may assume that $f_Y^1 \Sigma_{BC}$ is the $x$-axis, and $f_Y^k \Sigma_{BC}$ is the $y$-axis. Since $M$ contains both axes, we may assume that $M = z + \varphi$, where $\varphi$ is divisible by $xy$. Thus $S_c = \{M + cL = z + \varphi + cy + x\}$ is smooth for all $c \in \mathbb{C}$. When we blow up the $x$-axis, we use coordinates $(x, s, \eta) \mapsto (x, s, s\eta)$. The strict transforms are then $S_c = \{s\eta + \tilde{\varphi} + csx = 0\}$, where $\tilde{\varphi}$ is divisible by $xs$. Dividing this equation by $s$, we have $S_c = \{s\eta + \tilde{\varphi}(x, s, \eta) + cx = 0\}$, where $\tilde{\varphi}(0, s, 0) = 0$, since $S_c$ contains the $s$-axis (the strict transform of the $y$-axis). We have $\mathcal{F}_p^1 = \{x = s = 0\}$. Now we blow up the $s$-axis via the coordinates $(\xi, s, t) \mapsto (\xi, t, s) = (x, s, \eta)$. This gives the new strict transforms $S_c = \{1 + \tilde{\psi}(\xi, s, t) + c\xi = 0\}$, where $\tilde{\psi}(\xi, s, t) = t^{-1}\psi(\xi, t, s)$ is regular. The strict transform of $\mathcal{F}_p^1$ is now $\{\xi = s = 0\}$, which is disjoint from the $S_c$'s.

If $p' \in \gamma_1 \cap \gamma_9$, then there is a unique $c' \in \mathbb{C}$ be such that $S_{c'}$ is singular at $p'$. Let $\ell_9'$ denote the line for which $f(\ell_9') = p'$. By Theorem 7.6 and Proposition 7.11, it follows that $f_Z$ maps $\ell_9'$ to the strict transform of $S_{c'}$ inside $Z$. Thus the total transform of $\ell_9'$ under $f_Z^2$ is contained in $S_{\omega,c'}$. Let $p''$ denote the other point of $\gamma_1 \cap \gamma_9$, and let $c'' \in \mathbb{C}$ denote the corresponding parameter. Let $\tilde{S} = S_{c'} \cup S_{\omega,c'} \cup S_{\omega^2,c'} \cup S_{c'',c'} \cup S_{\omega,c''} \cup S_{\omega^2,c''}$ we see that $\tilde{S}$ is a
$f_Z$-invariant set which contains $\ell'_0 \cup \ell''_0$. Let $\mathcal{R}$ denote the strict transform of the rotor in $Z$. The sets $\mathcal{R}$, $\hat{S}$, and $\Sigma_{02} \cup \mathcal{F}^1_{e_2}$ are totally invariant, and we break the indeterminacy locus into three sets:

$$\mathcal{I}(f_Z) = (\Sigma_{02} \cup \mathcal{F}^1_{e_2}) \cup (\mathcal{I}(f_Z) \cap \mathcal{R}) \cup (\mathcal{I}(f_Z) \cap \hat{S})$$

with $\mathcal{I}(f_Z) \cap \mathcal{R} = \ell_2 \cup \ell_3 \cup \ell_7 \cup \mathcal{F}^1_{03Y}$, and $\mathcal{I}(f_Z) \cap \hat{S} = \ell'_9 \cup \ell''_9$. We set

$$\Omega = Z - (\hat{S} \cup \mathcal{R} \cup \Sigma_{02} \cup \mathcal{F}^1_{e_2})$$

By Propositions 7.11 and 7.12, $\mathcal{R}$ is disjoint from the strict transform of each $S_c$. Thus $f_Z$ is regular on $\Omega$, and $\Omega$ is invariant under $f_Z$.

**Proposition 7.13.** For every $S_c$ in $\Omega$ the dynamical degree of the restriction is $\delta(f^3_Z) = \delta_1(f)^3$.

*Proof.* Let us denote $\Gamma$ a hypersurface in $Z$ whose cohomology class in $H^{1,1}(Z)$ is $H_Z$. It follows that the degree of $f^{-3n}_Z \Gamma$ grows like $\delta_1(f)^{3n}$. On the other hand $S_c \subset \Omega$ does not contain an irreducible component of the indeterminacy locus for $f_Z$. It follows that we have $S_c \cap (f^3_Z)^{-n} \Gamma = (f^3_Z)^{-n}(S_c \cap \Gamma)$. Because $S_c$ is non-singular and $f^3_Z$ is pseudo-automorphism, the degree of $(f^3_Z)^{-n}(S_c \cap \Gamma) = (f^3_c)^{-n}(S_c \cap \Gamma)$ is $4 \delta_1(f)^{3n}$. Thus the dynamical degree of $f^3_c$ is $\delta_1(f)^3$.

Using the fact that $f_Z$ is regular on the large invariant set $\Omega$, we avoid the difficulties that can occur in defining the entropy of a map (see [G1]).

**Theorem 7.14.** The entropy of $f$ is $\log \delta_1(f)$.

*Proof.* Since $f$ is equivalent to a pseudo automorphism, and $f^*_Z$ is conjugate to $(f^*_Z)^{-1}$, both the first and the second dynamical degrees are equal. Combining the result in [DS] and the fact that $h_{top}(f) \geq h_{top}(f_0)$, we have the inequality

$$\log \delta_1(f) \geq h_{top}(f) \geq h_{top}(f_0) = \log \delta_1(f)$$

which gives the result.

Since $\Xi^\pm$ and $f_Z$ are regular on $\Omega$, the potential of $g^\pm$ is continuous on $\Omega$. Thus may define the wedge product $T_2 := T^+ \wedge T^-$ as a positive, closed $(2,2)$-current on $\Omega$, and we have:

**Proposition 7.15.** $\lim_{n \to \infty} \delta_1(f)^{-2n} f^*_Z f^{-n} \Xi^+ \wedge f^*_Z f^{-n} \Xi^- = T_2$ exists as a $(2,2)$-current on $\Omega$.

We have seen that the restrictions $f^3|S_c$ are automorphisms, and there are invariant currents $\mu_c^\pm$ on $S_c$, as well as invariant measures $\mu_c := \mu^+_c \wedge \mu^-_c$ (see [C]). The following property leads us to consider $T^\pm$ and $T_2$ as the “bifurcation currents” for the family $\{f^3|S_c\}$ (see [DuF]).

**Theorem 7.16.** For $S_c \subset \Omega$ the slices by $S_c$ are well-defined and give the corresponding dynamical objects: $T^\pm|S_c = \mu^\pm_c$, and $T_2|S_c = \mu_c$.

*Proof.* If we set $h = f^3$, then the class $[S_c]$ is invariant under $h^*$. Thus $\alpha^+: [S_c] \in H^{1,1}(S_c)$ is a class that is expanded by a factor of $\delta_1(f)$. It follows that the restriction $\Xi^+|S_c$ gives the expanded class, and this converges to $\mu^+_c$. Similarly, the normalized pullbacks/push-forwards of $\Xi^+ \wedge \Xi^-$ on $S_c$ will converge to $\mu_c$. 

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Theorem 7.17. For generic \( c', c'' \), the maps \( f^3|_{S_{c'}} \) and \( f^3|_{S_{c''}} \) are not smoothly conjugate, and the surfaces \( S_{c'} \) and \( S_{c''} \) are not isomorphic.

Proof. There is an invariant 6-cycle of curves, \( \Gamma_j \), \( j = 0, \ldots, 5 \) for \( f \). For generic \( c \), \( \Gamma_j \cap S_c \) is a saddle 2-cycle for \( f^3|_{S_c} \). The multipliers of this saddle cycle are not constant in \( c \), so the maps \( f^3|_{S_c} \) are not smoothly conjugate. Since the automorphism group of \( S_c \) is disconnected, we see that the family \( \{ S_c \} \) cannot consist of surfaces which are all isomorphic to each other. \( \square \)

Remark. If \( a_2 \neq 1 \) is a primitive 5th root of unity and \( a_0 = b_0 = 0 \), then we may repeat most of the arguments in this section for this map. In particular, we have:

Theorem 7.18. If \( a_2 \) is a primitive 5th root of unity, and \( a_0 = b_0 = 0 \), then \( f \) is equivalent to a pseudo-automorphism, and the dynamical degrees \( \delta_1(f) = \delta_2(f) \approx 1.3211018 > 1 \) are the largest root of \( t^{19}(t^3 - t - 1) + t^3 + t^2 - 1 \). The entropy of \( f \) is \( \log \delta_1(f) > 0 \). Furthermore there are two quartic polynomials which are invariant in the sense of (B.1). This gives a family of K3 surfaces which are invariant under \( f^5 \).

§8. Pseudo-automorphisms which are completely integrable. Let us consider two cases for maps of the form (2.2):

\[
\alpha = (a, 0, 1, 1), \ a \neq 1, \ \text{and} \ \beta = (0, 1, 0, 0) \quad (8.1a)
\]

\[
\alpha = (0, 0, \omega, 1), \ \omega^3 = 1, \ \omega \neq 1, \ \text{and} \ \beta = (0, 1, 0, 0) \quad (8.1b)
\]

The map (8.1a) has been extensively studied under the name Lyness process. The maps (8.1a) and (8.1b) exhibit similarities to the maps in the previous section: they are critical maps, and the iterates of the critical image \( \Sigma_{BC} \) go “once around” the rotor and land on \( \Sigma_{\beta\gamma} \). The difference with §7 is that \( f_Y^4\Sigma_{BC} = \mathcal{F}_{0\beta\gamma} \) is an indeterminate curve, and by Lemma 4.4 this fiber is mapped to \( \mathcal{F}_{0BC} \), that is, \( f_Y^4\Sigma_{BC} = \mathcal{F}_{0\beta\gamma} \subset S_{01} \) and \( f_Y^5\Sigma_{BC} = \mathcal{F}_{0BC} \subset S_{03} \). Thus \( \Sigma_{BC} \) arrives at \( \Sigma_{\beta\gamma} \) one step faster than was the case in §7.

Let \( \pi : Z \rightarrow Y \) denote the space obtained by blowing up the orbit \( f^j\Sigma_{BC}, f^{-j}\Sigma_{\beta\gamma}, 0 \leq j \leq 4 \) (one curve less than the construction in §7).

Theorem 8.1. The induced map \( f_Z \) is a pseudo-automorphism, and the iterates of \( f \) have quadratic degree growth.

Proof. Since \( f_Y^0\Sigma_{BC} = \Sigma_{\beta\gamma} \) and \( f_Y^4\Sigma_{BC} = \mathcal{F}_{0\beta\gamma}, f_Y^5\Sigma_{BC} = \mathcal{F}_{0BC} \), we see that \( f_Y \) satisfies the condition in Theorem 5.1. This theorem then follows from Theorems 5.1 and 5.3. \( \square \)

Proposition 8.2. In cases (8.1a) and (8.1b), the induced rotor map \( f^8|_{\Sigma_3} \) has linear degree growth. This map is not birationally conjugate to a surface automorphism.

Proof. In the case (8.1b), the restriction of \( f_Y^8 \) to \( \Sigma_3 \) is given by setting \( a_0 = 0 \) in (7.3), so we find the degree 2 birational map:

\[
f_Y^8|_{\Sigma_3} : [x_0 : x_1 : x_2 : 0] \mapsto [x_0\omega^2x_2 : x_1(x_1 + \omega^2x_0 + \omega^2x_2) : \omega^2x_2(x_1 + \omega^2x_2) : 0]
\]

This map has three distinct exceptional lines. Two of exceptional lines are mapped to fixed points \([1 : 0 : -1 : 0]\) and \([0 : 1 : -a : 0]\). The remaining exceptional line is mapped to a point of indeterminacy \( e = [1 : -1 : 0 : 0] \). We let \( W \) be the blowup space obtained by blowing up \( \Sigma_3 \).
at e. The induced map has only two exceptional lines which are mapped to fixed points and therefore the induced map is algebraically stable. The action on Pic is given by the matrix \[
\begin{pmatrix}
2 & 1 \\
-1 & 0
\end{pmatrix}
\]
which has an eigenvalue 1 with 2 \times 2 jordan block. It follows that the degree of restriction map grows linearly.

The analysis in the case (8.1a) is essentially the same. The induced rotor map is now:

\[
f_Y^8|_{\Sigma_3}: [x_0 : x_1 : x_2 : 0] \mapsto [x_0(ax_0 + ax_1 + x_2) : x_1(x_0 + x_1 + x_2) : x_2(ax_0 + x_1 + x_2) : 0]
\]

This map has three exceptional lines. Two of them are mapped to fixed points [1 : 0 : −1] and [0 : 1 : −a]. The third exceptional line is mapped to [1 : −1 : 0], which is indeterminate. After we blow up the point [1 : −1 : 0], the induced map is algebraically stable and the action on Pic has an eigenvalue 1 with 2 \times 2 jordan block.

Finally, since the restriction of \( f_w \) to the rotor has linear degree growth. It follows from [DiF] that this restriction is not an automorphism.

We consider first the Lyness map, i.e., case (8.1a). This is known to be integrable, and the invariant polynomials are given in [CGM1] and [KoL]. These invariant polynomials, which satisfy (B.1) with \( t = 1 \), are:

\[
\begin{align*}
Q_0 & = x_0x_1x_2x_3 \\
Q_1 & = (ax_0 + x_1 + x_2 + x_3)(x_0 + x_1)(x_0 + x_2)(x_0 + x_3) \\
Q_2 & = (x_0(ax_0 + x_1 + x_2 + x_3 + x_1x_3)(x_0 + x_1 + x_2)(x_0 + x_2 + x_3).
\end{align*}
\]

The set \( \{ Q_0 = 0 \} \) gives an invariant 8-cycle of rational surfaces, which is the rotor \( R \subset Y \). (Although \( Q_0 = 0 \) consists of 4 irreducible components in \( \mathbb{P}^3 \), it yields an 8-cycle inside \( Y \) because these components map through the indeterminacy locus, which is blown up to yield an additional 4 divisors.) The set \( \{ Q_1 = 0 \} \) gives an invariant 4-cycle, and \( \{ Q_2 = 0 \} \) gives an invariant 3-cycle; the components of the 8-, 4-, and 3-cycles are rational surfaces. As we observed in §4, \( f_Y \) induces dominant maps on each of these cycles. And as in Proposition 8.2, we may show that the restriction of \( f^4 \) to the 4-cycle, and the restriction of \( f^3 \) to the 3-cycle both have linear degree growth.

Let us define the surfaces \( S_c = \{ Q_c = 0 \} \) with \( Q_c := c_0Q_0 + c_1Q_1 + c_2Q_2 \). If we also write \( S_c \) for its strict transform inside \( Z \), we have \( fS_c = S_c \).

**Theorem 8.3.** For generic \( c \), the surface \( S_c \) is an irreducible K3 surface.

**Proof.** For generic \( c \), we find that \( S_c \) has 16 singular points: two of them are \( e_1, e_3 \), which are type \( A_2 \), and there are 14 more which are of type \( A_1 \). In the construction of \( Z \), we blew up \( e_1 \) and \( e_3 \). Then we blew up \( f^j\Sigma_{BC} \), \( 0 \leq j \leq 10 \), and the other 14 singular points are contained in these curves. It follows that the strict transform of \( S_c \) inside \( Z \) is smooth and thus K3.

**Theorem 8.4.** For generic \( c \) and \( c' \), the intersection \( S_c \cap S_{c'} \) is an elliptic curve. The restriction of \( f^3 \) to \( S_c \) has quadratic degree growth.

**Proof.** Since \( S_c \) is a K3 surface, it has trivial canonical bundle. Thus the birational map \( f^3 \) of \( S_c \) must be an automorphism. For generic \( c \) and \( c' \neq c \), the intersections \( S_c \cap S_{c'} \) give an invariant fibration of \( S_c \). Since \( f^3|_{S_c} \) is an automorphism, then by [DiF] the intersection \( S_c \cap S_{c'} \) is an elliptic curve and the restriction of \( f \) to the family of K3 surfaces has quadratic degree growth.
The map (8.1b) is similar. In this case the solutions to (B.1) take the form:

\[
\begin{align*}
R_0 &= x_0 x_1 x_2 x_3 \\
R_1 &= (x_0 + \omega x_1)(x_0 + \omega x_2)(x_0 + \omega x_3)(x_1 + \omega^2 x_2 + \omega x_3) \\
R_2 &= \omega x_1 x_3 (x_0 + \omega x_1)(x_0 + \omega x_3) \\
&\quad + \omega^2 x_0 x_2 (x_0 x_1 + \omega x_3) + x_2(\omega x_1 + x_3) + \omega^2 x_0 x_2
\end{align*}
\]

(8.3)

where \( t_{R_0} = 1 \), \( t_{R_1} = \omega^2 \), and \( t_{R_2} = \omega^2 \). As before, we see that \( f_Z \) will have an invariant 8-cycle given by the rotor \( R \subset Z \). And \( \{ R_1 = 0 \} \) will give a 4-cycle of rational surfaces. For generic \( c \), the singularities of the surface \( S_c = \{ \sum c_j R_j = 0 \} \) are \( e_1, e_3 \) (type A2) and \( e_2 \) (type A1). As in Theorems 8.3 and 8.4, we have:

**Theorem 8.5.** In case (8.1b): for generic \( c \), \( S_c \) is a K3 surface, \( f^3 \) is an automorphism of \( S_c \) with quadratic growth, and the intersections \( S_c \cap S_{c'} \) are elliptic curves.

§A. Appendix: Computing the Characteristic Polynomial for \( f_Z^3 \). Let us consider a critical map \( f \) satisfying condition (5.1), and let \( m_u, m_d, m_s, d_j, u_j \) and \( N \) be the numbers defined in §5. We define the \((N + 5) \times (N + 5)\) matrix

\[
\begin{pmatrix}
2 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\
-1 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & -1 & 1 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & -1 \\
* & * & & & & & & & \\
\vdots & \vdots & & & & & & & \\
* & * & & & & & & &
\end{pmatrix}
\]  

(A.1)

where the *’s indicate that the 7th through the \( N + 5 \)th rows remain to be specified. We will define the \( j \)th row \( r_j \) in terms of the elements \( e_k \), which are vectors of length \( N + 5 \) in which the \( k \)th entry is 1, and all other entries are 0:

(a) if \( j = N - d_i \) for some \( i = 1, \ldots, m_d \), then \( r_{j+6} = e_{j+5} - e_{N+5} \)

(b) if \( j = N - u_i \) for some \( i = 1, \ldots, m_u \), then \( r_{j+6} = -e_1 - e_5 + e_{j+5} - e_{N+5} \)

(c) if \( j = N - m_s \),

\[
\begin{align*}
r_{j+2} &= -e_1 - e_3 + e_{j+1} - e_{N+5}, \quad r_{j+3} = -e_3 + e_{j+2} \\
r_{j+4} &= -e_1 - e_3 - e_5 + e_{j+3} - e_{N+5}, \quad r_{j+5} = -e_1 - e_3 - e_5 + e_{j+4} \\
r_{j+6} &= -e_5 + e_{j+5}
\end{align*}
\]

(d) Otherwise, \( r_{j+6} = e_{j+5} \).

Let \( \pi : Z \to P^3 \) be the space constructed in Theorems 5.1 and 5.3.

**Proposition A.1.** The matrix (A.1) represents \( f_Z^3 \).

**Proof.** There are three cases to consider. Although we define a different basis in each case, the matrix (A.1) representing \( f_Z^3 \) is the same.
Case (I) : There is no \( 1 \leq j \leq N \) such that \( f^j_Y \Sigma_\gamma \subset \mathcal{F}_{0\beta \gamma} \cup \Sigma_{\beta \gamma} \).
In this case for all \( j \neq d_i, u_k, m_s, m_s+1, \ldots, m_s+4 \), \( 1 \leq i \leq m_d, 1 \leq k \leq m_u \) we have \( \pi \mathcal{F}_j \nsubseteq \Sigma_0 \cup \Sigma_\beta \cup \Sigma_{\gamma} \) and therefore

\[
f^j_Z H_Z = 2H_Z - E_1 - S_{01} - E_3 - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2} - \mathcal{F}_{m_s+4}
\]

\[
\{ \Sigma_0 \} = H_Z - E_1 - S_{03} - S_{01} - E_3 - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2} - \mathcal{F}_{m_s+3} - \mathcal{F}_{m_s+4}
\]

\[
\{ \Sigma_\beta \} = H_Z - S_{01} - E_3 - \mathcal{F}_N - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2}
\]

\[
\{ \Sigma_{\gamma} \} = H_Z - E_1 - \mathcal{F}_N - \mathcal{F}_{m_s+1} - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \sum_{i=1}^{m_d} \mathcal{F}_{d_i}.
\]

Since we have \( f^j_Z : E_1 \mapsto S_{03} \mapsto \{ \Sigma_0 \}, S_{01} \mapsto E_3 \mapsto \{ \Sigma_\beta \}, \mathcal{F}_j \mapsto \mathcal{F}_{j-1} \) for all \( j = 2, \ldots, N \), and \( \mathcal{F}_1 \mapsto \{ \Sigma_\gamma \} \) using the ordered basis \( \{ H_Z, E_1, S_{03}, S_{01}, E_3, \mathcal{F}_N, \mathcal{F}_{N-1}, \ldots, \mathcal{F}_2, \mathcal{F}_1 \} \) for \( \text{Pic}(Z) \) we see that (A.1) is the matrix representation for \( f^j_Z \).

Case (II) : There are \( \kappa \) positive integers \( 1 < s_1 < \cdots < s_\kappa < N \) such that for \( j = 1, \ldots, \kappa \), \( f^j_Y \Sigma_\gamma \subset \Sigma_{\beta} \setminus \ell_\beta \cup \Sigma_{\beta \gamma} \cup \ell'_\beta \) where \( f^3_Y \ell'_\beta = E_3 \cap \Sigma_0 \).

For this case let us use the ordered basis \( \tilde{\mathcal{B}} = \{ H_Z, \hat{E}_1, \tilde{S}_{03}, \tilde{S}_{01}, \hat{E}_3, \mathcal{F}_N, \mathcal{F}_{N-1}, \ldots, \mathcal{F}_2, \mathcal{F}_1 \} \) for \( \text{Pic}(Z) \) where \( \hat{E}_3 = E_3 + \sum_{i=1}^{3} \mathcal{F}_{s_i+1}, \tilde{S}_{01} = S_{01} + \sum_{i=1}^{\kappa} \mathcal{F}_{s_i+2}, \tilde{S}_{03} = S_{03} + \sum_{i=1}^{\kappa} \mathcal{F}_{s_i+4}, \) and \( \hat{E}_1 = E_1 + \sum_{i=1}^{3} \mathcal{F}_{s_i+5} \). Using this new ordered basis we can see that

\[
f^*_Z : \hat{E}_1 \mapsto \tilde{S}_{03} \mapsto \{ \Sigma_0 \} + \sum_{i=1}^{\kappa} \mathcal{F}_{s_i+3}
\]

\[
= H_Z - \hat{E}_1 - \tilde{S}_{03} - \tilde{S}_{01} - \hat{E}_3 - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2} - \mathcal{F}_{m_s+3} - \mathcal{F}_{m_s+4}
\]

In a similar way we may compute \( f^*_Z \) of \( H_Z, \tilde{S}_{01}, \hat{E}_3 \) and \( \mathcal{F}_N \) and see that the matrix representation with \( \tilde{\mathcal{B}} \) is given by (A.1).

Case (III) : There are \( \tau \) positive integers \( 1 < q_1 < \cdots < q_\tau < N \) such that \( f^q_Y \Sigma_\gamma \subset \ell'_\beta \) for \( j = 1, \ldots, \tau \).

Let us consider the ordered basis \( \tilde{\mathcal{B}} = \{ H_Z, \hat{E}_1, \tilde{S}_{03}, \tilde{S}_{01}, \hat{E}_3, \mathcal{F}_N, \mathcal{F}_{N-1}, \ldots, \mathcal{F}_2, \mathcal{F}_1 \} \) for \( \text{Pic}(Z) \) where \( \hat{E}_3 = E_3 + \sum_{i=1}^{\tau} (F_{q_{i+1}} + F_{q_{i+3}}), \tilde{S}_{01} = S_{01} + \sum_{i=1}^{\tau} (F_{q_{i+2}} + F_{q_{i+4}}), \tilde{S}_{03} = S_{03} + \sum_{i=1}^{\tau} (F_{q_{i+4}} + F_{q_{i+6}}), \) and \( \hat{E}_1 = E_1 + \sum_{i=1}^{\tau} (F_{q_{i+5}} + F_{q_{i+7}}) \). Since \( f^q_Y \ell'_\beta = \Sigma_\beta \cap S_{01} \), we have

\[
\{ \Sigma_\beta \} = H_Z - \tilde{S}_{01} - \hat{E}_3 - \sum_{i=1}^{\tau} (F_{q_i} + F_{q_{i+1}} + 2F_{q_{i+2}} + F_{q_{i+3}} + F_{q_{i+4}})
\]

\[
- \mathcal{F}_N - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2}
\]

\[
= H_Z - \tilde{S}_{01} - \hat{E}_3 - \sum_{i=1}^{\tau} (F_{q_i} + F_{q_{i+2}}) - \mathcal{F}_N - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2}.
\]
It follows that we have

\[ f_Z^* : \hat{S}_01 \mapsto \hat{E}_3 \mapsto \{ \Sigma_{\beta} \} + \sum_{i=1}^{\tau} (F_{q_i} + F_{q_i+2}) \]

\[ = \mathcal{H}_Z - \hat{S}_01 - \hat{E}_3 - F_N - \sum_{i=1}^{m_s} F_{u_i} - F_{m_s} - F_{m_s+1} - F_{m_s+2} \]

For the other basis elements, computations are essentially identical and thus we see that (A.1) represents \( f_Z^* \) with respect to the ordered basis \( \mathcal{B} \).

According to the previous Proposition, we see that the characteristic polynomial of \( f_Z^* \) only depends on \( m_u, m_d, m_s, d_j, u_j \) and \( N \).

**Lemma A.2.** The characteristic polynomial of \( f_Z^* \) is given by

\[ \pm t^{N-1} (t^2 + 1) \left[ (Q_1 - Q_4)t^4 + (2Q_1 - Q_2 - Q_3 - Q_4)t^2 + (Q_1 - Q_3)t + Q_4 \right]. \]

**Proof.** We subtract \( tI \) from the matrix (A.1) and perform a sequence of row operations on it. Step (i): we add or subtract the 6th row to the rows whose last entry is 1 or \(-1\) and then (ii) for \( j = 1, \ldots, N - 1 \), we subtract \( 1/t^j \) times the \( N + 4 - j \)th row from \( 6 \)th row. This gives

\[ \det(f_Z^* - tI) = \det \begin{pmatrix} A & 0 \\ \ast & B \end{pmatrix} \]

where

\[ A = \begin{pmatrix}
1 - t & 0 & 1 & 0 & 0 & -t \\
1 - t & -t & 0 & 0 & 1 & 0 \\
0 & 1 & -1 - t & 0 & 0 & 0 \\
-1 & 0 & -1 & -t & -1 & 0 \\
-1 & 0 & -1 & 1 & -1 - t & 0 \\
Q_1 & 0 & Q_2 & 0 & Q_3 & Q_4
\end{pmatrix}, \quad B = \begin{pmatrix}
-t & 0 & 0 & \cdots & 0 & 0 \\
1 & -t & 0 & \cdots & 0 & 0 \\
0 & 1 & -t & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & -t & 0 \\
0 & 0 & 1 & -t & \end{pmatrix} \]

with \( Q_1, Q_2, Q_3, Q_4 \) as in §5. We have \( \det(f_Z^* - t \text{Id}) = (-1)^{N-1} t^{N-1} \det(A) \), and we evaluate \( \det(A) \) to obtain the polynomial given above. \( \square \)

**§B. Appendix: Invariant Polynomials.** We will look for polynomials \( P(x) = \sum a_I x^I \) which are invariant in the sense that

\[ P \circ f = t \cdot j_f \cdot P \]  \hspace{1cm} (B.1)

where \( t \neq 0 \) is constant, and \( j_f = 2x_0(\gamma \cdot x)(\beta \cdot x)^2 \) is the Jacobian determinant. If \( P \) and \( Q \) are solutions to (B.1) with multipliers \( t_P \) and \( t_Q \), then \( \varphi = P/Q \) is a rational function with the invariance property: \( \varphi \circ f = (t_P t_Q^{-1}) \varphi \). If \( P \) is a solution to (B.1), then \( P \) defines a meromorphic 3-form \( \Omega_P \): on the set \( x_0 \neq 0 \), it is given by \( P(1, x_1, x_2, x_3)^{-1} dx_1 \wedge dx_2 \wedge dx_3 \). This is invariant in the sense that \( f^* \Omega_P = t_P^{-1} \Omega_P \). It follows that \( \{ P = 0 \} \) is an \( f \)-invariant surface
which represents the canonical class in $\mathbb{P}^3$ and its strict transforms are invariant surfaces which represent the canonical classes in $Y$ and $Z$.

The equation (B.1) can be rewritten as a system of linear equations for the coefficients of the monomials in $P$. This system can be solved directly for all the maps in §7 and §8. For instance, in §7 $\omega$ is a non-real root of unity and $a_0 = a \neq 0$, and we find a solution for $t = \omega^2$:

\[
P_1 = (1 - \omega) \left( a^2 x_0^4 + (1 + a)x_0 x_1 x_2 + x_1^2 x_2^2 + ax_1 x_2 x_3^2 \right) - (2 + \omega) \left( x_0 x_2^3 + (1 + a)x_0 x_1 x_3 + ax_1 x_2 x_3 + ax_0^2 x_3 \right) + (1 + 2\omega) \left( ax_0^2 x_1^2 + ax_0 x_1^2 x_2 + ax_1^2 x_2 x_3 + ax_0 x_2 x_3^2 + ax_0^2 x_1 (1 + a + 2\omega - a\omega) + (1 - 2a + 2\omega - a\omega) \left( x_0^2 x_1 x_3 + x_0 x_2^2 x_3 \right) + x_0^2 x_2 (1 - a + 2\omega + a\omega) - (2 - a + \omega + a\omega) \left( x_0^3 x_1 + x_0 x_1 x_3^2 + ax_0^3 x_3 (1 - 2a - \omega - a\omega) \right) + (1 + a)x_0^2 x_2 x_3 (1 + a - \omega + 2a\omega) + ax_0^3 x_2 (2 + a + \omega + 2a\omega) \right).
\]

**§C. Appendix: The Rotor Map.** Let $g := f^8_{\Sigma_3}$ denote the rotor map restricted to $\Sigma_3$, which is written in coordinates in (7.3). By factoring the jacobian determinant, we see that there are four exceptional curves.

\[
C_1 = \{ ax_0 + \omega x_2 = 0 \}
\]
\[
C_2 = \{ ax_0 + ax_1 + \omega x_2 = 0 \}
\]
\[
C_3 = \{ a\omega x_0 + x_1 + \omega^2 x_2 = 0 \}
\]
\[
C_4 = \{ a\omega x_0^2 + a\omega x_0 x_1 + a\omega x_0 x_2 + \omega^2 x_0 x_2 + x_1 x_2 + \omega^2 x_2 = 0 \}
\]

**Lemma C.1.** If $a \neq \omega^j$ and $a^j \neq \omega^{j+2}$ for all $j \geq 2$ then $g$ is not birationally conjugate to an automorphism.

**Proof.** The exceptional curves $C_2$ and $C_4$ mapped to a three cycle: $g : C_2 \mapsto [0 : 1 : -a\omega] \mapsto [0 : 1 : -a] \mapsto [0 : 1 : -a\omega^2] \mapsto [0 : 1 : -a\omega]$ and $g : C_4 \mapsto [1 : 0 - \omega^2] \mapsto [1 : 0 : -\omega] \mapsto [1 : 0 : -\omega^2]$. For $C_3$ we see that $g^3 C_3 = [1 : -\omega^2 (\omega/a)^{j-1} : 0]$ for all $j \geq 1$. It follows that these three curves have orbits that do not encounter the indeterminacy locus of $g$. The remaining exceptional curve $C_1$ mapped to $e_1 = [0 : 1 : 0]$, which is indeterminate. We let $W$ be the space obtained by blowing up $\Sigma_3$ at $e_1$, and we let $E_1$ be the corresponding exceptional divisor. Under the induced map $g_W$ we have $g_W(E_1) = E_1$ and the orbit of the strict transform of $C_1$ remains in $E_1$ and does not encounter the indeterminacy locus of $g_W$.

Now if $H$ denote the class of a generic line in $W$, then $\langle H, E_1 \rangle$ is an ordered basis for $\text{Pic}(W)$. The action on $\text{Pic}$ is given by the matrix

\[
g_W^* = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}
\]

The largest eigenvalue is $\lambda = (3 + \sqrt{5})/2$ and invariant class is given by $\theta = \lambda H - E_1$. Since $\theta^2 = \lambda^2 - 1 \neq 0$, it follows from [DF, Theorem 5.4] that $g$ is not birationally conjugate to an automorphism.

**Lemma C.2.** If $a^j = \omega^{j-2}$ for some $j \geq 2$ then $g$ is not birationally conjugate to an automorphism.

**Proof.** In case $a^j = \omega^{j-2}$ for some $j \geq 2$, the orbits of three exceptional curves $C_2, C_3$, and $C_4$ are the same as the previous Lemma. After we blowup $e_1$ on $\Sigma_3$, the strict transformation of $C_1$
mapped to a point of indeterminacy after \( j \)-th iteration of \( g_W \). We let \( W_2 \) be the space obtained by blowing up \( W \) at \( g^k_C_1 \) for \( k = 1, \ldots, j \) and we let \( F_k, 1 \leq k \leq j \) be the corresponding exceptional divisors. Under the induced map \( g_{W_2} \), the exceptional line \( C_1 \) is removed and the orbits of remaining three exceptional curves do not encounter the indeterminacy locus of \( g_{W_2} \).

Let \( \langle H, F_j, F_{j-1}, \ldots, F_1, E_1 \rangle \) be the ordered basis for \( \text{Pic}(W_2) \). The characteristic polynomial of the action on \( \text{Pic} \) is given by \( t^j + 2 - 4t^{j+1} + 3t^j + t^2 - 2t + 1 \). It follows that the dynamical degree is not a Salem number. Thus by [DiF], \( g \) is not birationally conjugate to an automorphism.

\( \square \)

**Lemma C.3.** If \( a_j = \omega^{j+2} \) for some \( j \geq 2 \) then \( g \) is not birationally conjugate to an automorphism.

**Proof.** When \( a_j = \omega^{j+2} \), the orbit of \( C_3 \) is different from Lemma C.1, that is \( g^{j+1}C_3 = \[ 1 : -1 : 0 \] \), which is indeterminate. We let \( W_3 \) be the space obtained by blowing up \( \Sigma_3 \) at \( e_1 \) and \( g^kC_3, 1 \leq k \leq j + 1 \), and we let \( E_1 \) and \( F_k, 1 \leq k \leq j + 1 \) be the corresponding exceptional divisors. Using the ordered basis \( \langle H, F_{j+1}, F_j, \ldots, F_1, E_1 \rangle \) for \( \text{Pic}(W_3) \), we see that the characteristic polynomial of the action on \( \text{Pic} \) is given by \( t^j + 3 - 3t^{j+2} + t^{j+1} + t \). Similarly as in Lemma C.2, the dynamical degree is not a Salem number and therefore \( g \) is not birationally conjugate to an automorphism.

\( \square \)

**Lemma C.4.** If \( a = \omega \) then \( g \) is not birationally conjugate to an automorphism.

**Proof.** In this case we see that \( C_2 \) is mapped to a point of indeterminacy under 2 iterations and \( C_4 \) is also mapped to a point of indeterminacy under 3 iterations. After we blowup \( e_1 \), we can check that the orbits of other two remaining exceptional lines does not encounter the indeterminacy locus. After we blow up the orbit of \( C_2 \) and the orbit of \( C_4 \), we see that the dynamical degree of \( g \) is given by the largest root of the polynomial \( t^3 - t^2 - 2t - 1 \). Again since this number is not a Salem number we have our result.

\( \square \)

**Lemma C.5.** If \( a = \omega^2 \) then \( g \) is not birationally conjugate to an automorphism.

**Proof.** If \( a = \omega^2 \) the each component of \( g \) has the same factor \( x_0 + x_1 + \omega^2x_2 \). It follows that the restriction of \( f_Y^3 \) to \( \Sigma_3 \) is a degree 2 birational map. There are two exceptional lines and both exceptional lines are mapped to points of indeterminacy. After we blowup the points on the orbits of three exceptional lines, we see that the induced map has one exceptional line which is mapped to a point of indeterminacy. Once we blow up this point of indeterminacy, we see that the induced map has no exceptional lines and therefore the induced map is algebraically stable. Furthermore the characteristic polynomial of the action on \( \text{Pic} \) is \( t(1+t)(t-1)^3 \) and the action on \( \text{Pic} \) has \( 2 \times 2 \) Jordan block. It follows that the degree of \( g \) grows linearly. According to [DiF], we have that \( g \) is not birationally conjugate to an automorphism.

\( \square \)

**Lemma C.6.** If \( a = 1 \) then the degree \( g \) grows quadratically.

**Proof.** For this case all four exceptional curves are mapped to points of indeterminacy: \( g : C_1 \mapsto e_1, C_2 \mapsto [0 : 1 : -\omega], C_3 \mapsto [1 : -\omega^2 : 0] \mapsto [1 : -1 : 0] \) and \( g : C_4 \mapsto [1 : 0 : -\omega^2] \). We let \( Z \) be the space obtained by blowing up \( \Sigma_3 \) at all five points in the orbit of exceptional curves and we let \( E_1, Q_2, Q_3, Q_4, \) and \( Q_5 \) be the corresponding exceptional divisors. Under the induced map \( g_Z \), there is a unique exceptional line which is the strict transformation of \( C_1 \). We see that \( g_ZC_1 \) is a point of indeterminacy of \( g_Z \). By blowing up one more point on \( E_1 \) we make the induced map an algebraically stable. Let us denote \( Q_1 \), the exceptional divisors corresponding to the point blow ups on \( E_1 \). Let us use \( \langle H, F_1, F_2, F_3, F_4, F_5, E_1 \rangle \).
as the ordered basis of $\text{Pic}$. The characteristic polynomial of the action on $\text{Pic}$ is given by $(t - 1)^4(t + 1)(t^2 + t + 1)$ and the matrix representation of the action on $\text{Pic}$ has $3 \times 3$ Jordan block. It follows that the degree of $g$ grows quadratically.

References

[Ba] T. Bayraktar, Green currents for meromorphic maps of compact Kähler manifolds. J. of Geometric Analysis, arXiv:1107.3063v2

[BK1] E. Bedford and KH Kim, On the degree growth of birational mappings in higher dimension, J. Geom. Anal. 14 (2004), 567-596.

[BK2] E. Bedford and KH Kim, Periodicities in linear fractional recurrences: Degree growth of birational surface maps, Michigan Math. J. 54 (2006), 647-670.

[BK4] E. Bedford and KH Kim, Linear fractional recurrences: periodicities and integrability, arXiv:0910.4409

[BC] J. Blanc and S. Cantat, personal communication.

[CaL] E. Camouzis and G. Ladas, Dynamics of Third Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/CRC Press, 2008.

[C] S. Cantat, Dynamique des automorphismes des surfaces $K3$, Acta Math. 187 (1): 1–57, 2001.

[CGM1] A. Cima, A. Gasull and F. Mañosas, On periodic rational difference equations of order $k$. J. Difference Equ. Appl. 10 (2004), no. 6, 549–559.

[CGM2] A. Cima, A. Gasull and V. Mañosa, Dynamics of the third order Lyness’ difference equation, J. Difference Equ. Appl. 13 (2007), no. 10, 855–884. arXiv:math.DS/0612407v1

[CL] M. Csörnyei and M. Laczkovich, Some periodic and non-periodic recursions, Monatshefte für Mathematik 132 (2001), 215-236.

[dFE] T. de Fernex and L. Ein, Resolution of indeterminacy of pairs, Algebraic geometry, 165–177, de Gruyter, Berlin, 2002.

[DiF] J. Diller and C. Favre, Dynamics of bimeromorphic maps of surfaces, Amer. J. of Math., 123 (2001), 1135–1169.

[DS] T.-C. Dinh and N. Sibony, Une borne supérieure pour l’entropie topologique d’une application rationnelle. Ann. of Math. (2) 161 (2005), no. 3, 1637–1644.

[DOL] I. Dolgachev and D. Ortland, Point Sets in Projective Spaces and Theta Functions, Astérisque, Vol. 165, 1988.

[DuF] R. Dujardin and C. Favre, Distribution of rational maps with a preperiodic critical point. Amer. J. Math. 130 (2002), no. 4, 979–1032.

[EJ] A-S Elsenhans and J. Jahnel, Determinantal quartics and the computation of the Picard group, arXiv:1010.1923v1

[GL] E.A. Grove and G. Ladas, Periodicities in Nonlinear Difference Equations, Kluwer Academic Publishers, 2005.

[G1] V. Guedj, Entropie topologique des applications méromorphes. Ergodic Theory Dynam. Systems 25 (2005), no. 6, 18471855.

[G2] V. Guedj, Théorie ergodique des transformations rationnelles. arXiv:math/0611302

[H] H.P. Hudson, Cremona Transformations in Plane and Space, Cambridge U. Press, 1927.

[KoL] V.I. Kocic and G. Ladas, Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers 1993.
[KuL] M. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations*, CRC Press, 2002.

[L] R.C. Lyness, Notes 1581, 1847, and 2952, Math. Gazette 26 (1942), 62, 29 (1945), 231, and 45 (1961), 201.

[M1] C.T. McMullen, Dynamics on blowups of the projective plane. Publ. Math. I.H.E.S. No. 105 (2007), 49–89.

[M2] C.T. McMullen, K3 surfaces, entropy and glue.  
http://www.math.harvard.edu/~ctm/papers/home/text/papers/glue/glue.pdf

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