Meanfield Approximation For Field Theories On The Worldsheet Revisited ¹

Korkut Bardakci²

Department of Physics
University of California at Berkeley
and
Theoretical Physics Group
Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

Abstract

This work is the continuation of the earlier efforts to apply the mean field approximation to the continuum world sheet formulation of planar $\phi^3$ theory. The previous attempts were either simple but without solid foundation or better founded but excessively complicated. In this article, we present an approach both simple, and also systematic and well founded. We are able to carry through the leading order mean field calculation analytically, and with a suitable tuning of the coupling constant, we find string formation.

¹This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, of the U.S. Department of Energy under Contract DE-AC03-76SF00098, and in part by the National Science Foundation Grant PHY-0098840
²e-mail: kbardakci@lbl.gov
1. Introduction

Over the last couple of years, the present author and Charles Thorn have been pursuing a program of summation of planar graphs in field theory \[1,2,3,4\]. Because of its simplicity, the field theory that has been most intensively investigated so far is the $\phi^3$ theory, although Thorn and collaborators have made considerable progress in extending the program to more physical theories \[5,6,7\]. The basic idea, which goes back to 't Hooft \[8\], is to represent Feynman graphs on the world sheet by a suitable choice of the light cone coordinates. 't Hooft’s representation, which was non-local, was later reformulated as a local field theory on the world sheet by introducing additional non-dynamical fields \[1\]. This reformulation has no new physical content; it merely reproduces perturbation theory. However, it provides a setup well suited for the study of string formation in field theory. This is an old problem that has attracted recent renewed interest \[9,10,11,12\] following the discovery of AdS/CFT correspondence \[13,14\].

Our approach to the problem of string formation starts with the world sheet description of the $\phi^3$ field theory mentioned above, and we look for the phenomenon of “condensation” of Feynman graphs. This phenomenon will be defined more precisely later in the paper, but roughly it means that the lines that form Feynman graphs on the world sheet become dense, and graphs of asymptotically infinite order dominate the perturbative expansion. Furthermore, the original non-dynamical world sheet fields become dynamical and string formation takes place. Whether the scenario described above really happens is of course a question of dynamics. So far, the only tool used to investigate this problem in the present context has been the mean field or the self consistent field approximation \[2,3,4\]. The accuracy of the mean field approximation is questionable; however, one can hope that at least a qualitative understanding of the relevant dynamical issues would emerge.

The main virtue of the mean field method is its simplicity. There is, however, no unique way to do the mean field calculation, and it all depends on the choice of the order parameter. In reference \[2\], the simplest choice was made for the order parameter by taking it to be the expectation value of a local field. This makes the subsequent calculation quite tractable. However, this early attempt, at least in its continuum version, relied on a number of questionable steps and approximations that are hard to justify. In order to overcome these difficulties, in references \[3\] and \[4\] the order parameter was taken to be the expectation value of two fields at different points (two
point function). This also has the advantage of providing a better probe of the problem; but the disadvantage is that the calculation becomes too complicated to carry out analytically.

In this article, we revisit the earlier calculation [2] based on a simple order parameter to see whether the difficulties associated with it can be overcome without sacrificing its basic simplicity. We will mainly focus on the treatment based on the continuum world sheet given in this reference. Reexamining this treatment, we identify the following problems:

a) The boundary conditions on the world sheet were imposed only approximately through the so-called $\beta$ trick. The exact boundary conditions could only be recovered in the problematic $\beta \to \infty$ limit.

b) If we try to impose the boundary conditions exactly by means of Lagrange multipliers, as was done in [1] and as we shall do here, we avoid the problem discussed in a), but instead we encounter another difficulty. The action has then an unfixed gauge invariance, which can lead to ill-defined results.

c) There are two kinds of fields in the problem: The matter and the ghost fields. The contribution of each sector is quadratically divergent, but there is a subtle partial cancellation between them. Unless great care is exercised, the result can depend on the regulation scheme used.

d) The use of light cone coordinates obscures the covariance of the theory. An approximation scheme, such as the mean field approximation, could easily violate Lorentz invariance.

In this article, we propose the following improvements to overcome the problems listed above:

a) The boundary conditions will be imposed exactly by means of Lagrange multipliers.

b) To avoid the resulting gauge invariance problem, we introduce a gauge fixing term. This is probably the most important new idea introduced in the present work. As is well known, working with a gauge unfixed action can lead to lots of confusion.

c) To keep track of the cancellation between the matter and ghost fields, we impose supersymmetry on the world sheet. This idea was already in the air in [1], what we have done here is to formulate it explicitly. Also, in treating divergent quantities, we first combine the contributions of the matter and ghost sectors, and then we regulate the sum in a way that does not spoil the cancellation between them. The answer obtained in this fashion is unambiguous.

d) There is a particular subgroup of the Lorentz group under which the
light cone variables transform linearly. In particular, under boosts along the 1 direction, the variables $x^\pm$ and $p^\pm$ scale (see section 2 for the definition of these variables). All of this is familiar from the light cone treatment of the bosonic string, where, among other things, the importance of invariance under this special boost was recognized [15]. In the present context, this was discussed in reference [4]. In this paper, following the treatment given in [4], we will try to preserve invariance under this special subgroup at each step. We will not, however, try to investigate invariance under the full Lorentz group. In string theory, full Lorentz invariance in the light cone framework is realized only for a fixed critical value of $D$, where $D$ is the dimension of the transverse space [15]. If the same is true here, this problem is clearly beyond the scope of a leading large $D$ approximation.

When these improvements are incorporated in the mean field approximation of reference [2], the result is a systematic approximation scheme without any of the ambiguities encountered in [2]. It is also simple enough so that we are able to carry out the leading order calculation analytically. With a suitable tuning of the coupling constant, we find string formation, and in addition, we discover that a new string mode has been dynamically generated.

In organizing this paper, the goal was to present a complete and self-contained treatment which should be intelligible even to a reader unfamiliar with the previous work on the subject. As a result, there a good deal of overlap with reference [2], and some overlap with references [3] and [4]. When we preview the rest of the paper below, we will try to make clear what is new and what is a review or elaboration of the material covered in the earlier work cited above.

In section 2, we briefly review both the rules for Feynman graphs in light cone variables [8] and the local field theory on the world sheet from which these graphs follow [1]. We also discuss the transformation properties of the fields under the special boost mentioned earlier, which manifests itself as a scale transformation on the world sheet. This is an abridged version of a more complete discussion given in [4].

In section 3, world sheet supersymmetry is introduced and a supersymmetric free action $S_0$ is constructed. This is a new idea in the present context. $S_0$ differs from the corresponding action introduced in [1] and used in all the subsequent work in the structure of the terms involving ghosts, and also in the presence of a relatively insignificant bosonic field $r$ required by SUSY. The cancellation of quantum corrections between matter and ghost fields, which was the reason for the introduction of ghosts in the first place, is now
guaranteed by supersymmetry.

The boundary conditions accompanying \( S_0 \) are enforced by a term in the action, \( S_1 \), given in section 4. Both the boundary conditions and \( S_1 \) are substantially the same as those in [1]. To express the new term in the action fully in the language of field theory, one needs world sheet fermions which were introduced earlier [1,4]. We have found it slightly more convenient to work with a somewhat different set of fermions, although it is easy to show by means of a Bogoliubov transformation that the two are completely equivalent.

In section 5, we show that the action constructed so far is invariant under a simple gauge transformation. It is therefore important to fix this gauge, and we show how to do it. We also note that it is possible to introduce some arbitrary constants in the boundary and gauge fixing terms in the action. Although the exact theory is not sensitive to these constants, we keep them around to see how the affect an approximate calculation. The material covered in this section is completely new.

The mean field approximation which is at the basis of the present work is discussed in section 6 from the point of view of the large \( D \) limit. All of this is standard material, well known from the solution of the vector models in in the large \( N \) (in this case, large \( D \)) limit [16]. The only thing new here compared to [2] and [3] is the manner in which the singular determinants resulting from integration over the matter and ghost fields are regulated: The two determinants are combined into an expression less singular than each one separately, and regulating the combined expression, we get an unambiguous and scale invariant answer.

In section 7, an effective action is constructed and evaluated by the saddle point method in the large \( D \) limit. This effective action is pretty close to but still different in detail from the one derived in [2]. A standard bosonic string action with positive slope emerges from this calculation. The important question is then whether this slope is finite. We find that, by suitably tuning the coupling constant of the \( \phi^3 \) theory, the slope can be made finite. This tuning can be regarded as renormalization: A cutoff dependent coupling constant is traded for the finite slope parameter.

It seems somewhat surprising that starting with an unstable field theory, no sign of instability has so far appeared in the string picture. One possibility is that we have not gone far enough. The calculation of the intercept, which we have not undertaken here, may show that, as in the bosonic string, some lowest lying states are tachyonic. Or, it may be that the instability is not
visible in the leading order of the large $D$ limit. These possibilities are discussed in at the end of section 7.

So far, all the calculations were carried out in the leading order of the large $D$ limit. In section 8, we compute a non-leading correction in this limit by expanding the composite field $\rho$ (see section 4 for its definition) around its mean value $\rho_0$. In addition, $\rho$ is assumed to be slowly varying, and an expansion up to second order in powers of derivatives of this field is carried out. This is essentially a repetition of the computation done in the Appendix B of [2] from the standpoint of the present approach. We find that, from the world sheet perspective, $\rho$ becomes a dynamical, propagating field, and from the string perspective, the string acquires an additional mode, with the same slope as all the other modes. Finally, in the Appendix, we show that, the mean field computation presented here is completely equivalent to the standard large $N (D)$ treatment of vector models.

2. A Brief Review

The Feynman graphs of masses $\phi^3$ have a particularly simple form in the mixed light cone representation of 't Hooft [8]. In this representation, the evolution parameter is $x^+$ also denoted by $\tau$, and the conjugate Hamiltonian is $p^-$, and the Minkowski evolution operator is given by

$$T = \exp(-ix^+p^-).$$

(1)

In this notation, a Minkowski vector $v^\mu$ has the light cone components $(v^+, v^-, v)$, where $v^\pm = (v^0 \pm v^1)/\sqrt{2}$, and the boldface letters label the transverse directions. A propagator that carries momentum $p$ is pictured as a horizontal strip of width $p^+$ and length $\tau = x^+$ on the world sheet, bounded by two solid lines (Fig.1). The lines forming the boundary carry transverse momenta $q_1$ and $q_2$, where

$$p = q_1 - q_2,$$

and the corresponding propagator is given by

$$\Delta(p) = \frac{\theta(\tau)}{2p^+} \exp \left( -i \frac{\tau}{2p^+} p^2 \right).$$

(2)

More complicated graphs consist of several horizontal solid line segments (Fig.2). The beginning and the end of each segment is where the $\phi^3$ interaction takes place, and a factor of $g$ is associated with each such point, where
$g$ is the coupling constant. Finally, one has to integrate over the position of the interaction vertices, as well as the momenta carried by the solid lines. We note that momentum conservation is automatic in this formulation. A typical light cone graph is pictured in Fig. 2.

It was shown in [1] that the light cone Feynman rules described above can be reproduced by a local field theory on the world sheet. The world sheet is parametrized by coordinates $\sigma$ along the $p^+$ direction and $\tau$ along the $x^+$ direction, and the transverse momentum $\mathbf{q}$ is promoted to a bosonic field $\mathbf{q}(\sigma, \tau)$ defined everywhere on the world sheet. In addition, two fermionic fields (ghosts) $b(\sigma, \tau)$ and $c(\sigma, \tau)$ are needed. In contrast to $\mathbf{q}$, which has $D$ components, $b$ and $c$ each have $D/2$ components, where $D$ is the dimension of the transverse space, assumed to be even. The action on the world sheet, with the Minkowski signature (+,-), is given by

$$S_0 = \int_0^{p^+} d\sigma \int_{\tau_i}^{\tau_f} d\tau \left( \mathbf{b}' \cdot \mathbf{c}' - \frac{1}{2} \mathbf{q}'^2 \right), \quad (3)$$
where the prime denotes derivative with respect to $\sigma$. This action is supplemented by the Dirichlet conditions

$$\dot{q} = 0, \ b = c = 0,$$  \hspace{1cm} (4)

on the solid lines (boundaries), where the dot denotes derivative with respect to $\tau$. Imposing Dirichlet conditions on $q$ is equivalent to fixing them to be $\tau$ independent on the solid lines and then integrating over them. If we fix $q$ to be $q_1$ and $q_2$ on two adjacent solid lines and solve the equations of motion

$$q'' = 0$$

subject to these boundary conditions, we find that the resulting classical action reproduces the exponential factor in eq.(2). However, there is also an unwanted quantum contribution, given by

$$-\frac{1}{2} D \det(\partial^2_\sigma)$$

which is exactly cancelled by the corresponding determinant resulting from integrating over $b$ and $c$.  

\hspace{1cm} 7
The action formulation described above, which was extensively used in the previous work [2,3,4]. Although it is basically correct, it has some unsatisfactory features. For one thing, although we have nothing new to say about this problem in this paper, the factor of $1/(2p^+)$ in front of the exponential in eq.(2) is missing. Also, the splitting of the ghost fields into two components $b$ and $c$ is somewhat unnatural; it leads to the artificial condition that $D$ is even, and it does not look rotationally invariant, although there is of course no real violation of rotation symmetry in the transverse space. In the next section, we will show that the introduction of supersymmetry on the world sheet leads to a more symmetric ghost sector and the condition that $D$ be even is no longer needed. We also think that it results in a more natural and elegant approach.

Finally, we would like to discuss briefly the question of Lorentz invariance. This is a non-trivial problem, since the use of the light cone coordinates obscures the Lorentz transformation properties of the dynamical variables. There is, however, a special subgroup of the Lorentz group under which the light cone coordinates have simple linear transformation properties $^3$. If $L_{i,j}$ are the angular momenta and $K_i$ are the boosts, the generators of this subgroup are

$$L_{i,j}, \quad M_{+,\tau} = K_1, \quad M_{+,i} = K_i + L_{1,i}, \quad (5)$$

where indices $i$ and $j$ run from 2 to $D+2$. It turns out that invariance under all the generators except for one is rather trivially satisfied by the propagator (2) or the field theories (3). The non-trivial transformation, generated by $K_1$, corresponds to scaling of $x^+$ and $p^+$ by a constant $u$:

$$x^+ \rightarrow x^+/u, \quad p^+ \rightarrow p^+/u, \quad (6)$$

and the tranverse momenta $\mathbf{q}$ are unchanged. The reason this transformation is critical is that although it is easy to construct classically scale invariant theories, this symmetry is in general broken by quantum corrections. This is familiar from the study of conformal invariance in field theory.

Now, let us examine the scale invariance of the action (3)[4]. Since the coordinates $\sigma$ and $\tau$ must transform like $p^+$ and $x^+$, the transformation laws of the fields are

$$\mathbf{q}(\sigma, \tau) \rightarrow \mathbf{q}(u\sigma, u\tau), \quad b, c(\sigma, \tau) \rightarrow b, c(u\sigma, u\tau), \quad (7)$$

$^3$In this article, we will not investigate invariance under the remaining Lorentz generators. If the string analogy is to be trusted, this is where the critical dimension becomes important [15], and a large $D$ approximation is clearly inadequate for treating this problem.
and the classical action is invariant under this transformation. On the other hand, the quantum corrections are singular and need a cutoff for their definition. This cutoff would break scale invariance, were it not for the cancellation between the ghost and matter fields. So what we have here is a potential violation of scale invariance by an anomaly, which is eliminated by the cancellation between the matter and ghost sectors. This is nothing but the cancellation of the determinants discussed earlier. We would like to stress that the quantum contribution in question is a world sheet effect; it potentially present even in the case of a free propagator and it has nothing to do with the target space ultraviolet divergences.

In addition to the scaling of the bulk, we have to consider the scaling behaviour of the boundary conditions given by eq.(4). These are scale invariant as they stand, but the integration over the position of the boundaries is not invariant, since the $\sigma$ coordinate scales. The factor of $1/(2p^+)$ provides the measure needed to make this integration scale invariant. Although we will not present here a general recipe for the inclusion of this factor, we will be careful to preserve the scale invariance of various integrals that occur in the course of the mean field calculation. It can be shown that, in any case, this factor does not contribute to the leading order of the mean field calculation [2,3].

3. SUSY On The World Sheet

We generalize the momentum $q$ to form a SUSY multiplet:

$$Q = q + \theta_1 b + \theta_2 c + \theta_1 \theta_2 r,$$

(8)

where $\theta_1, 2$ are the usual anticommuting coordinates and all the fields represented by boldface letters are vectors in the D dimensional transverse space. The two SUSY generators are ($i = 1, 2$)

$$Q_i = \frac{\partial}{\partial \theta_i^*} + \theta_i \partial_\sigma,$$

(9)

and they satisfy

$$Q_i^2 = Q_2^2 = \partial_\sigma, \ [Q_1, Q_2] = 0.$$

(10)

The infinitesimal transformations are given by

$$Q \rightarrow Q + \sum_i \epsilon_i Q_i, Q]$$

(11)
where $\epsilon_{1,2}$ are anticommuting infinitesimal parameters. We also define two covariant derivatives

$$D_i = \frac{\partial}{\partial \theta_i} - \theta_i \partial_\sigma,$$

which anticommute with the SUSY generators and satisfy the same equations as (10), apart from the change of the sign of $\partial_\sigma$.

Having introduced the framework of supersymmetry, we write down the supersymmetric analogue of the action (3):

$$S_0 \to \int_0^{p^+} d\sigma \int d\tau \int d^2\theta \left( -\frac{1}{2} D_1 Q \cdot D_2 Q \right)$$

$$= \int_0^{p^+} d\sigma \int d\tau \left( -\frac{1}{2} (q'^2 + b \cdot b' + c \cdot c' + r^2) \right),$$

supplemented by the boundary conditions

$$\dot{q} = 0, \quad b = 0, \quad c = 0, \quad r = 0,$$

on the solid lines. We now have twice as many ghost fields as before, but they appear with only first order derivative in $\sigma$, and as a result, there is again complete cancellation between the determinants (quantum corrections) coming from the integration of the matter and ghost fields. It is clear that this cancellation is a consequence of supersymmetry. In fact, the main motivation for introducing SUSY was to obtain this cancellation as a systematic consequence of a symmetry, and not as some accident. In the rest of this article, we will exclusively use this supersymmetric form of the action.

Although there is supersymmetry in the bulk of the worldsheet, it is broken by the boundary conditions, since the condition on $q$ differs from those on $b$ and $c$. This breakdown of SUSY is essential, since supersymmetric boundary conditions would lead to a complete cancellation between the matter and the ghost sectors, resulting in a trivial propagator. This has no effect on the cancellation of the quantum contributions, since the cancellation occurs in the bulk and it is insensitive to the boundary conditions.

Since we now have a new expression for $S_0$, we have to reinvestigate the invariance under the scaling transformations (6,7). The action given by eq.(13) is invariant if the fields transform as

$$q(\sigma, \tau) \to q(u \sigma, u \tau), \quad b(\sigma, \tau) \to \sqrt{u} b(u \sigma, u \tau),$$

$$c(\sigma, \tau) \to \sqrt{u} c(u \sigma, u \tau), \quad r(\sigma, \tau) \to u r(u \sigma, u \tau).$$
Again, a potential quantum anomaly that could violate scale invariance is cancelled as a consequence of supersymmetry.

4. The World Sheet Action

In this section, we review the construction of an action that incorporates the boundary conditions (14), which will be implemented by introducing a bosonic Lagrange multipliers $y(\sigma, \tau)$ and $z(\sigma, \tau)$, and the fermionic Lagrange multipliers $b(\sigma, \tau)$ and $c(\sigma, \tau)$ [1]. The corresponding term in the action is

$$S_1 = \int_0^{p^+} d\sigma \int d\tau \rho \left( y \cdot \dot{q} + \bar{b} \cdot \dot{b} + \bar{c} \cdot \dot{c} + z \cdot r \right),$$

(16)

where $\rho(\sigma, \tau)$ is equal to 1 on the solid lines and it is equal to 0 elsewhere. This definition is singular on the continuum world sheet; we will give a more precise definition on the discretized world sheet below. The factor $\rho$ ensures that the boundary conditions are imposed only on the solid lines; elsewhere, $\rho$ vanishes and there are no constraints on the fields.

In the earlier work [1,2,4], it was shown how to construct $\rho$ in terms of a fermionic field. We will present here a slightly different version of this construction. In order to avoid singular expressions, it is best to start by discretizing the $\sigma$ coordinate into segments of length $\Delta \sigma = a$, with $p^+ = Na$. The parameter $a$ plays the role of a cutoff, which will in any case be needed later on. The specific form of this cutoff is not important; for example, a cutoff in the Fourier modes conjugate to $\sigma$ would serve just as well. Fig.3 shows $N$ equally spaced lines parallel to the $\tau$ direction. For the convenience of exposition, the absence of a solid line in a given position is pictured by the presence of a dotted line in the same position. In other words, the solid lines mark the boundaries and the dotted lines fill the bulk of the world sheet. On each line, we introduce two fermionic variables $\psi_i(\sigma_n, \tau)$ and their conjugate $\bar{\psi}_i(\sigma_n, \tau)$, where $i = 1, 2$, $\sigma_n = na$, and $n$ is an integer in the range $0 \leq n \leq N$. The free fermionic action is given by

$$S_2 = \sum_n \int d\tau \ i \bar{\psi}(\sigma_n, \tau) \dot{\psi}(\sigma_n, \tau),$$

(17)

and the fermions satisfy the usual anticommutation relations:

$$[\bar{\psi}_i(\sigma_n, \tau), \psi_j(\sigma_n, \tau)]_+ = \delta_{i,j} \delta_{m,n}.$$
The function of the fermions is to keep track of the solid and dotted lines. The vacuum state, defined by

$$\psi_i |0\rangle = 0,$$

corresponds to the trivial situation where all the lines in the graph are dotted. The $\tau$ independent state corresponding to a single solid line at $\sigma = \sigma_n$ is

$$|\sigma_n\rangle = \bar{\psi}_1(\sigma_n, \tau)\bar{\psi}_2(\sigma_n, \tau)|0\rangle.$$  \hspace{1cm} (18)

This is an eternal solid line, extending indefinitely for both positive and negative $\tau$, since the operators $\bar{\psi}_i$ and hence the above state, are independent of $\tau$ by the equation of motion following from (17). Several solid lines are represented by the state

$$\prod_n \bar{\psi}_1(\sigma_n)\bar{\psi}_2(\sigma_n)|0\rangle.$$

We note that, a state with a double solid line, which has no graphical interpretation, vanishes by fermi statistics. Having set up our fermionic system,
we can express $\rho$ in eq.(16) in terms of fermions:

$$
\rho(\sigma_n, \tau) = \frac{1}{2} \sum_{i=1,2} \bar{\psi}_i(\sigma_n, \tau) \psi_i(\sigma_n, \tau),
$$

(19)

and it is easy to check that $\rho$ is 1 if there is a solid line located at $\sigma = \sigma_n$, and it is zero if the line is dotted. The ground state expectation value of this composite field, $\rho_0$, will play an important role in what follows. In any finite order of perturbation theory, where the density of solid lines is essentially zero, $\rho_0$ is zero. On the other hand, a non-zero $\rho_0$ means that a portion of the world sheet with finite area is occupied by the solid lines, which we interpret as condensation of solid lines. It is also clear that this can only happen if large (infinite) order graphs dominate the perturbation series. It will be shown later that, at least in the mean field approximation, the condensation of solid lines leads to string formation. One can think of $\rho_0$ as an order parameter that distinguishes between the stringy phase and the perturbative phase of the same theory.

So far, all the lines, whether solid or dotted, are eternal. We need an interaction term in the action which will convert dotted lines into solid lines and vice versa. Remembering that the transition between solid and dotted lines is accompanied by a factor of the coupling constant $g$, we set

$$
S_3 = g \sum_n \int d\tau \left( \bar{\psi}_1(\sigma_n, \tau) \bar{\psi}_2(\sigma_n, \tau) + \psi_2(\sigma_n, \tau) \psi_1(\sigma_n, \tau) \right),
$$

(20)

and it is easy to show that this term in the action does the required job.

As we mentioned earlier, the fermions introduced in this paper are somewhat different from those used in the earlier work [1,2,4]. However, it is not difficult to show that the two are connected by the Bogoliubov transformation

$$
\bar{\psi}_1 \leftrightarrow \psi_1,
$$

and, as a result, they are physically equivalent.

Finally, we will consider the continuum limit, with $a \to 0$. The dictionary for taking this limit is

$$
\sum_n \to \frac{1}{a} \int d\sigma, \quad \frac{1}{\sqrt{a}} \psi_i(\sigma_n, \tau) \to \psi_i(\sigma, \tau),
$$

$$
\frac{1}{a} \rho(\sigma_n, \tau) \to \rho(\sigma, \tau) = \frac{1}{2} \sum_i \bar{\psi}_i(\sigma, \tau) \psi_i(\sigma, \tau).
$$

(21)
For the continuum fermions, we use the same notation as the discrete ones, but with $\sigma_n$ replaced by $\sigma$. They satisfy anticommutation relations similar to the discretized version, with the Kroenecker delta replaced by the Dirac delta function. The expressions for $S_2$ and $S_3$ (eqs. (17) and (20)) remain unchanged if one replaces the sum over $\sigma_n$ by integration over $\sigma$. In particular, there is no explicit dependence on $a$. The state corresponding to a solid line at $\sigma = \sigma'$ is now represented by

$$|\sigma'\rangle = \bar{\psi}_1(\sigma', \tau) \psi_2(\sigma', \tau) |0\rangle,$$

and

$$\rho(\sigma, \tau)|\sigma'\rangle = \delta(\sigma - \sigma')|\sigma'\rangle. \quad (22)$$

In what follows, although we will mostly work with the continuum fermions, from time to time we will also need the world sheet with discretized $\sigma$ to have well defined expressions.

The total action is the sum of various pieces given by eqs. (13), (17) and (20):

$$S = S_0 + S_1 + S_2 + S_3. \quad (23)$$

As it stands, this action suffers from a serious problem: It is not well defined. We will discuss this problem and present its solution in the next section.

We close with a discussion of the scaling properties of the fermions. Invariance of the canonical algebra, or of the free action for the continuum fermions requires the transformation law

$$\psi(\sigma, \tau) \rightarrow \sqrt{u} \psi(u\sigma, u\tau), \quad \bar{\psi}(\sigma, \tau) \rightarrow \sqrt{u} \bar{\psi}(u\sigma, u\tau), \quad (24)$$

and as result

$$\rho(\sigma, \tau) \rightarrow u \rho(u\sigma, u\tau), \quad \bar{\rho}(\sigma, \tau) \rightarrow u \bar{\rho}(u\sigma, u\tau). \quad (25)$$

We also note that, from its definition $Na = p^+$, it follows that the cutoff parameter $a$ must scale as $p^+$:

$$a \rightarrow a/u. \quad (26)$$

It is now easy to check that, with the possible exception of the interaction term $S_3$, all the terms of the fermionic action are scaling invariant. The interaction term violates scaling, unless the coupling constant $g$ is allowed to transform. Of course, the original field theory coupling constant is a
Lorentz scalar and cannot transform. However, the coupling constant on the world sheet is closely related to but not identical to the field theory coupling constant, and it need not be a Lorentz scalar. In fact, Lorentz invariance requires that

\[ g \to u \cdot g \]

under scaling. The simplest way to secure this is to use \( p^+ \), the only other physical parameter at our disposal and make the replacement

\[ g \to \frac{g}{p^+}. \quad (27) \]

\( p^+ \) takes care of the scaling (see eq.(6)), and the newly defined \( g \) is a true scalar.

5. The Gauge Fixed Worldsheet Action

Consider eq.(16) for \( S_1 \): Since \( \rho \) vanishes everywhere except on the solid lines (the boundary), the integrand is independent of the Lagrange multipliers \( y, b, \bar{c} \) and \( z \) in the bulk of the world sheet. As a result, the functional integration over the bosonic variables is divergent\(^4\) and the corresponding integration over the fermionic variables vanishes. The result is an ill defined expression of the type infinity times zero. This is similar to what happens in gauge theories before gauge fixing. In fact, \( S_1 \) is invariant under the gauge transformation

\[
\begin{align*}
    y &\to y + \bar{\rho} y_0, \quad b \to b + \bar{\rho} b_0, \\
z &\to z + \bar{\rho} z_0, \quad \bar{c} \to \bar{c} + \bar{\rho} \bar{c}_0,
\end{align*}
\]

where \( y_0, z_0, \bar{b}_0 \) and \( \bar{c}_0 \) are arbitrary functions and \( \bar{\rho} \) is defined by

\[ \bar{\rho} = 1 - \rho, \]

for discretized \( \sigma \) and

\[ \bar{\rho} = \frac{1}{a} - \rho = \frac{1}{2} \sum_i \psi_i \bar{\psi}_i \quad (29) \]

for continuous \( \sigma \), where \( \rho \) is given by (19) in the first case and by (21) in the second case. In either case, since

\[ \rho \bar{\rho} = 0, \quad (30) \]

\(^4\)I thank Charles Thorn for stressing this point, although at the time I did not think it was important.
invariance of $S_1$ under the transformations (28) follows. It is this gauge invariance that is responsible for the singular behaviour of the functional integrals over the Lagrange multipliers mentioned above.

The cure for this problem is to gauge fix the action, but only in the bulk of the world sheet (on the dotted lines), where $\rho = 0$. We will also demand the gauge fixing term to be supersymmetric, since our philosophy is to keep intact the SUSY in the bulk and violate it only on the boundaries. We therefore promote the Lagrange multipliers into a supersymmetric multiplet

$$Y = z + \theta_1 \bar{c} + \theta_2 \bar{b} + \theta_1 \theta_2 y, \quad (31)$$

and write the gauge fixing part of the action as

$$S_4 = \int_0^{p^+} d\sigma \int d\tau \int d^2 \theta \frac{1}{2} \alpha \bar{\rho} Y \cdot Q = \int_0^{p^+} d\sigma \int d\tau \alpha \bar{\rho} \left( y \cdot z + \bar{b} \cdot \bar{c} \right), \quad (32)$$

where $\alpha$ is a gauge fixing parameter. This is the simplest gauge fixing term which is supersymmetric and which vanishes on the boundaries (solid lines) where $\bar{\rho} = 0$ and no gauge fixing is needed. The previously singular functional integral over the bulk of the world sheet where $\rho = 0$ is now equal to unity:

$$\int_{\rho=0} Dz \ D\bar{z} \ Db \ D\bar{c} \ exp(iS_4) = 1. \quad (33)$$

We note that the $\alpha$ dependent integration measures of the bosonic and fermionic functional integrals cancel. We believe that the absence of gauge fixing on the continuum world sheet caused some of the problems encountered in the earlier work.

There is one more technical issue which we have to address here. There is some arbitrariness in the equation for (16) for $S_1$; it could be replaced by the following more general expression:

$$S_1 \rightarrow \int_0^{p^+} d\sigma \int d\tau \rho \left( \beta_1 y \cdot \dot{q} + \beta_2 \bar{b} \cdot b + \beta_3 \bar{c} \cdot c + \beta_4 z \cdot r \right). \quad (34)$$

Here, $\beta_{1,2,3,4}$ are arbitrary constants. They can be eliminated by absorbing them into the definition of the Lagrange multipliers. So long as one is dealing with the exact expression for the action, the introduction of the $\beta$’s changes nothing, and one could just as well set them all equal to one, as in eq.(16).
However, if an approximation scheme is used, the results may well depend on these constants. These remarks also apply to the gauge fixing parameter $\alpha$: The exact theory is independent of this constant but an approximate calculation may introduce some dependence. In the next section, when we carry out a mean field calculation, we will be able to see to what extent our results are sensitive to the choice of these constants.

Finally, we have to make sure that the gauge fixing term $S_4$ and also $S_1$ are scale invariant. This is indeed the case for $S_4$ if under scaling

\[
\begin{align*}
    y(\sigma, \tau) &\rightarrow y(u\sigma, u\tau), & \vec{b}(\sigma, \tau) &\rightarrow \sqrt{u}\vec{b}(u\sigma, u\tau), \\
    \vec{c}(\sigma, \tau) &\rightarrow \sqrt{u}\vec{c}(u\sigma, u\tau), & z(\sigma, \tau) &\rightarrow uz(u\sigma, u\tau),
\end{align*}
\]

and eq.(25) is taken into consideration. Now that we know the scaling properties of all the fields, we can check $S_1$ (eq.(16)). The first three terms are indeed invariant as they stand, but in the last term, we have to let $\beta_4 \rightarrow p^+ \beta_4$.

6. The Mean Field Approximation

In the last section, the final form of the action which is supersymmetric, gauge fixed and scale invariant was worked out. Since the mean field approximation will be based on it, we start by writing it down in full:

\[
S = \sum_{n=0}^{n=4} S_n
\]

\[
= \int_0^{p^+} d\sigma \int d\tau \left( -\frac{1}{2} (q^2 + b \cdot b' + c \cdot c' + r^2) + \rho (y \cdot \dot{q} + \beta_2 \vec{b} \cdot \vec{b}) + \beta_3 \vec{c} \cdot \vec{c} + \beta_4 p^+ (z \cdot r) + \alpha \bar{\rho} (y \cdot z + \vec{b} \cdot \vec{c}) + i \bar{\psi} \psi + \frac{g}{p^+} (\bar{\psi}_1 \psi_2 + \psi_2 \psi_1) \right),
\]

where $\rho$ and $\bar{\rho}$ are given by (21) and (29).

The mean field approximation was developed and applied to the world sheet action in the earlier work [2,3,4]. Here, we need the simplest version of it used in [2]. We notice that eq.(36) represents a vector model, which can be solved in the large $D$ limit [16]. The standard approach is to replace scalar products of various vector fields, such as the bilinear terms in the expression

\[
S = \sum_{n=0}^{n=4} S_n
\]

\[
= \int_0^{p^+} d\sigma \int d\tau \left( -\frac{1}{2} (q^2 + b \cdot b' + c \cdot c' + r^2) + \rho (y \cdot \dot{q} + \beta_2 \vec{b} \cdot \vec{b}) + \beta_3 \vec{c} \cdot \vec{c} + \beta_4 p^+ (z \cdot r) + \alpha \bar{\rho} (y \cdot z + \vec{b} \cdot \vec{c}) + i \bar{\psi} \psi + \frac{g}{p^+} (\bar{\psi}_1 \psi_2 + \psi_2 \psi_1) \right),
\]

where $\rho$ and $\bar{\rho}$ are given by (21) and (29).

The mean field approximation was developed and applied to the world sheet action in the earlier work [2,3,4]. Here, we need the simplest version of it used in [2]. We notice that eq.(36) represents a vector model, which can be solved in the large $D$ limit [16]. The standard approach is to replace scalar products of various vector fields, such as the bilinear terms in the expression
for $S_0$ (eq.(13)), by their vacuum expectation values, which are then treated as classical c-number fields. The functional integral over the remaining fields is carried out exactly, and the resulting effective action is minimized with respect to the classical fields. This method is known to be equivalent to a large $N$ (in this case, $D$ replaces $N$) saddle point calculation [16]. Instead of the approach sketched above, we find it much simpler to replace the bilinears in the fermions, $\rho$ and $\bar{\rho}$ by their classical expectation values. In the Appendix, we show that this is completely equivalent to replacing the bilinears in vector fields by their expectation values.

The problem is considerably simplified by starting with the setup where the total transverse momentum $p$ carried by the whole graph is zero:

$$p = \int_0^{p^+} d\sigma \dot{q} = 0. \quad (37)$$

This configuration can always be reached by a suitable Lorentz transformation. It allows us to impose the periodic boundary conditions

$$q(\sigma = 0, \tau) = q(\sigma = p^+, \tau). \quad (38)$$

This setup is translationally invariant in both the $\sigma$ and the $\tau$ directions, and we shall see that translation invariance will play an important role in simplifying the mean field calculation.

We start by explicitly introducing the composite field $\rho$ by adding a new term $\Delta S$ to the action:

$$S \to S + \Delta S,$$

$$\Delta S = \int_0^{p^+} d\sigma \int d\tau \kappa \left( \frac{1}{2} \sum_i \bar{\psi}_i \psi_i - \rho \right), \quad (39)$$

where $\kappa$ is a Lagrange multiplier, and $\bar{\rho}$ is given in terms of $\rho$ through eq.(29). In the large $D$ limit, we can treat $\kappa$ and $\rho$ as classical fields (See the Appendix). In other words, we make the replacement

$$\kappa \to \kappa_0 = \langle \kappa \rangle, \quad \rho \to \rho_0 = \langle \rho \rangle, \quad \bar{\rho} \to \bar{\rho}_0 = \langle \bar{\rho} \rangle, \quad (40)$$

where $\langle \rangle$ denotes the expectation value in the ground state of the field in question. Translation invariance on the world sheet means that both $\kappa_0$ and $\rho_0$ are constants independent of the coordinates $\sigma$ and $\rho$. With this simplification, it is possible to carry out explicitly all the functional integration
over the fields. We first consider the integration over the fermions; that part of the action involving the fermions is given by

$$S_f = \int_0^{p^+} d\sigma \int d\tau \left( i\bar{\psi}\dot{\psi} + \frac{g}{p^+}(\bar{\psi}_1\bar{\psi}_2 + \psi_2\psi_1) + \frac{\kappa_0}{2} \sum_i \bar{\psi}_i\psi_i \right). \quad (41)$$

Instead of working with the action, we find it more convenient to diagonalize the corresponding Hamiltonian. In order to avoid singular expressions, we first discretize the $\sigma$ coordinate as in eqs.(17) and (20). There is a complete decoupling of the dynamics in the $\sigma$ direction; as a result, the total Hamiltonian can be written as a sum of $N$ mutually commuting Hamiltonians:

$$H = \sum_n H_n, \quad (42)$$

where,

$$H_n = - \left( g'(\bar{\psi}_1\bar{\psi}_2 + \psi_2\psi_1) + \frac{\kappa_0}{2} \sum_i \bar{\psi}_i\psi_i \right)_{\sigma = \sigma_n}, \quad (43)$$

and

$$g' = g/p^+. \quad (44)$$

We observe that $H_n$ acts on the two states

$$|0\rangle$$

corresponding to a dotted line at $\sigma = \sigma_n$ and

$$|n\rangle = \bar{\psi}_1(\sigma_n)\bar{\psi}_2(\sigma_n)|0\rangle$$

corresponding to a solid line at the same position as a two by two matrix:

$$H_n|0\rangle = -g'|n\rangle, \quad H_n|n\rangle = -g'|0\rangle - \kappa_0|n\rangle. \quad (44)$$

The eigenvalues are

$$E_{f}^{\pm} = -\frac{1}{2}\kappa_0 \pm \frac{1}{2}\sqrt{\kappa_0^2 + 4g'^2}, \quad (45)$$

where the square root is defined to be positive. Since we are interested in the ground state, we have to pick the lower energy $E_{f}^{-}$ at each $\sigma = \sigma_n$ and add up to get the total fermionic energy:

$$E_f = NE_f^{-} = \frac{p^+}{a} E_f^{-}. \quad (46)$$
Next, we will carry out the functional integrations over the vector fields in eq.(36), setting
\[ \rho \rightarrow \rho_0, \quad \bar{\rho} \rightarrow \bar{\rho}_0 = \frac{1}{a} - \rho_0. \]

We first split the action into \( S_m \), the matter action, and \( S_g \), the ghost action:

\[ S_m = \int_{0}^{p^+} d\sigma \int d\tau \left( -\frac{1}{2} (q'^2 + r^2) + \rho_0 (\beta_1 y \cdot \dot{q} + \beta_4 p^+ z \cdot \dot{r}) + \alpha \bar{\rho}_0 y \cdot \dot{z} \right), \]
\[ S_g = \int_{0}^{p^+} d\sigma \int d\tau \left( -\frac{1}{2} (\bar{b} \cdot b' + \bar{c} \cdot c') + \rho_0 (\beta_2 \bar{b} \cdot \dot{b} + \beta_3 \bar{c} \cdot \dot{c}) + \alpha \bar{\rho}_0 \bar{b} \cdot \dot{\bar{c}} \right). \]

(47)

The integrations over \( y, r, \bar{b} \) and \( \bar{c} \) are Gaussian and they can be done trivially. The (singular) Jacobians coming from integrations over the matter and the ghost fields cancel, with the result,

\[ S_m \rightarrow \int_{0}^{p^+} d\sigma \int d\tau \left( -\frac{1}{2} q'^2 + \frac{1}{2} \left( \frac{\beta_1 \beta_4 p^+ \bar{\rho}_0^2}{\alpha \bar{\rho}_0} \right)^2 q'^2 \right), \]
\[ S_g \rightarrow \int_{0}^{p^+} d\sigma \int d\tau \left( -\frac{1}{2} b \cdot b' - \frac{1}{2} c \cdot c' + \frac{\beta_2 \beta_3 \bar{\rho}_0^2}{\alpha \bar{\rho}_0} c \cdot b \right). \]

(48)

We note that \( S_m \) is the (Minkowski) world sheet action for a string, with the slope parameter \( \alpha' \), where

\[ 4\alpha'^2 = \left( \frac{\beta_1 \beta_4 p^+ \bar{\rho}_0^2}{\alpha \bar{\rho}_0} \right)^2. \]

(49)

It may seem like string formation is almost automatic; however, the string picture breaks down if the slope is zero, which happens for \( \rho_0 = 0 \). The parameter \( \rho_0 \) is therefore the order parameter that distinguishes between the stringy and the perturbative phases of the same field theory. Roughly speaking, since \( \rho_0 \) measures the fraction of the world sheet area occupied by the solid lines, each graph in perturbation theory corresponds to \( \rho_0 = 0 \). This is because any finite number of solid lines, being one dimensional, have vanishing area, and as to be expected, perturbative field theory is then the zero slope limit of the string theory. A non-zero slope requires \( \rho_0 \neq 0 \), which means that the solid lines condense to occupy a finite fraction of the
area of the world sheet. Therefore, $\rho_0$ serves as an order parameter which distinguishes between two phases: $\rho_0 = 0$ in the perturbative field theory phase and $\rho_0 \neq 0$ in the stringy phase. In the next section, we will find that $\rho_0 \neq 0$ in the mean field approximation.

Since $S_{m,g}$ have quadratic dependence on the fields, the functional integrals can be carried out. Defining

$$\int \exp(iS_{m,g}) = \exp(is_{m,g}),$$

we have, after Euclidean rotation in $\tau$,

$$S^e_m = -\frac{1}{2} D Tr \ln \left(-\partial^2_\sigma - A^2 \partial^2_\tau\right),$$

$$S^e_g = D Tr \ln \left(-\partial^2_\sigma - B^2 \partial^2_\tau\right),$$

(50)

where,

$$A = \frac{\beta_1 \beta_4 p^+ \rho_0^2}{\alpha \rho_0}, \quad B = -i \frac{\beta_2 \beta_3 \rho_0^2}{\alpha \rho_0}.$$

Rewriting the $Tr \ln$’s in terms of momenta $k_0$ and $k_1$ conjugate to $\tau$ and $\sigma$ gives

$$S^e = S^e_m + S^e_g = D \frac{\tau_f - \tau_i}{4\pi} \int dk_0 \sum_n \ln \left(\frac{(k_1^n)^2 + B^2}{(k_1^n)^2 + A^2 k_0^2}\right),$$

(51)

with

$$k_1^n = \frac{2\pi n}{p^+}.$$

This is only a formal result, since the integration over $k_0$ and summation over $n$ lead to a quadratic divergence, and before one can make sense of it, it must be regulated. In fact, we are precisely interested in the coefficient of this quadratic divergence, which, after it is regulated by a cutoff, will be the dominant term in the answer. Since the answer is somewhat sensitive to the cutoff procedure, we will try to explain the motivation for the regulator we use. First, we make use of a simplification: Only the leading cutoff dependent part of the answer is of interest and this dependence is sensitive only to the large $k_1^n$ and large $k_0$ behaviour of the integrand in the above equation. Therefore, we can safely replace the summation over $n$ by the integration over the continuous variable $k_1$. By the same token, we can set $B = 0$ in the integrand:

$$S^e \to (\tau_f - \tau_i) \frac{D p^+}{8\pi^2} \int dk_0 \int dk_1 \ln \left(\frac{k_1^2}{k_1^2 + A^2 k_0^2}\right).$$

(52)
In eq.(51), the contributions of the matter and ghost sectors were combined into a single log. This was intentional: We are going to regulate the combined contributions of the two sectors, rather then regulate them separately. To see why, we observe that

a) The combined term is less singular then each term treated separately. In fact, for a fixed $k_0$, the integral over $k_1$ converges, so that we need to regulate only the integral over $k_0$. This is no accident: it can be traced back to the cancellation of the singular determinants between the matter and the ghost sectors. Regulating each term seperately could spoil this cancellation.

b) We saw in section 2 that scale invariance on the world sheet is necessary for Lorentz invariance in the target space. Therefore, the regulated expresion for $S^e$ should be scale invariant. Under scaling, $k_0$ and $k_1$ transform as

$$k_0 \to u k_0, \quad k_1 \to u k_1,$$

and from eq.(50), it is easy to check that $A$ is scale invariant. It then follows that the integrand in eq.(52) is scale invariant. This is of course related to the cancellation discussed above, if we recall from section 3 that the cancellation of determinants and scale invariance are intimately connected. Again, it was essential to combine the matter and the ghost terms to arrive at a scale invariant integrand.

The scale invariance of the integrand in eq.(52) is necessary but not sufficient for the scale invariance of $S^e$; one also needs a regulator that respects scaling. We have seen that only the $k_0$ integration has to be regulated, which we regulate by introducing a second cutoff (in addition to $a$) in the $\tau$ direction. Again the precise form of the regulator is not important, so long as it respects scaling. As a simple example, we will consider a sharp cutoff in $k_0$:

$$S^e \to (\tau_f - \tau_i) \frac{D p^+}{8\pi^2} \int_{-\lambda/p^+}^{\lambda/p^+} dk_0 \int dk_1 \ln \left( \frac{k^2_1}{k^2_0 + A^2 k^2_0} \right), \quad (53)$$

where $\lambda$ is the cutoff parameter. The factor of $p^+$ is inserted so that the limits of $k_0$ integration are invariant under scaling. By a change of variables, we can evaluate this integral as

$$S^e = -\frac{D}{8\pi^2} \frac{\tau_f - \tau_i}{p^+} A \Lambda, \quad (54)$$

where,

$$\Lambda = \int_{-\lambda}^{\lambda} dk_0 \int dk_1 \ln \left( \frac{k^2_1 + k^2_0}{k^2_1} \right). \quad (55)$$

22
We note that

a) $\Lambda$ is a positive cutoff dependent constant. Since this is the only cutoff dependent parameter in the result, we may as well replace the original cutoff parameter $\lambda$ by $\Lambda$.

b) The simple linear dependence of $S^e$ on $A$ is going to be important in the following development. This dependence is a fairly robust result: It follows from the change of variable

$$k_1 \rightarrow A k_1,$$

independent of the details of how the integral over $k_0$ is regulated.

c) The dependence on $p^+$ is required by scale invariance, again independent of the form of the regulator.

The preceding discussion leads to the conclusion that $S^e$ has the unique form given by eq. (54), provided that we combine the matter and ghost determinants before regularizing and we use a regulator that respects scale invariance.

Finally, we would like to comment on the factor of $i$ in the definition of $B$ in eq. (54). If the product $\beta_2 \beta_3$ is real, $B^2$ will be negative, and from eq. (), $S^e$ will be complex. This is, of course, a signal for instability. This does not concern us here: Since we are interested only in the leading cutoff dependence, we use eq. (51), where the $B^2$ term is absent. However, even for the non-leading part of $S_e$, it is possible to avoid this problem by taking the product $\beta_2 \beta_3$ to be pure imaginary. We recall that the constants $\beta_2$ and $\beta_3$ appeared in front of the Lagrange multipliers that set the fields $\bar{b}$ and $\bar{c}$ equal to zero. If these fields were bosonic, complex values for these constants would not be permissible, but for fermionic fields, complex coefficients are allowed. Nevertheless, this may still show up as an instability in some non-leading order in the large $D$ limit.

7. String Formation

We can now put together various terms derived in the last section and write down the full effective action $S^{eff}$:

$$S^{eff} = S^e - (\tau_f - \tau_i)(E_f + p^+ \kappa_0 \rho_0) = \frac{\tau_f - \tau_i}{p^+} \tilde{S},$$

$$\tilde{S} = -\frac{D}{8\pi^2} \frac{\beta_1 \beta_2 p^+ \rho_0^2}{\alpha \rho_0} \Lambda + \frac{(p^+)^2}{2a} \left( \kappa_0 + \sqrt{\kappa_0^2 + 4g^2} \right) - (p^+)^2 \kappa_0 \rho_0.$$  

(56)
Now let us go back to the question posed in Section 5: How does the effective action depend on the arbitrary constants $\beta_{1,2,3,4}$ that appear in eq.(36) and the gauge fixing parameter $\alpha$? The answer is that, at least within the present approximation, it does not depend on $\beta_2$ and $\beta_3$ at all. Also, the dependence on $\alpha$ and $\beta_1$ and $\beta_4$ is rather trivial. Since the cutoff parameter $\Lambda$ is arbitrary to begin with, by redefining it through

$$\Lambda \rightarrow \frac{\beta_1\beta_4}{\alpha} \Lambda,$$

one can eliminate all the reference to these parameters.

Next we will search for the saddle point of the effective action in the variables $\rho_0$ and $\kappa_0$, or equivalently in the variables $x$ and $y$, and see whether this corresponds to the minimum value of the ground state energy. To start with, the expression for $\tilde{S}$ can be simplified considerably by a series of redefinitions:

$$\tilde{\Lambda} = \frac{\beta_1\beta_4}{8\pi^2 \alpha} \Lambda, \quad \rho_0 = \frac{x}{a},$$

$$\bar{\rho}_0 = \frac{1-x}{a}, \quad \kappa_0 = 2\frac{D\tilde{\Lambda}}{p^+} y, \quad g' = \frac{g}{p^+} = \frac{D\tilde{\Lambda}}{p^+} \tilde{g}.$$  \hspace{1cm} (57)\hspace{1cm}

It is also convenient to define a scale invariant cutoff parameter $a'$ by

$$a' = a/p^+.$$ \hspace{1cm}

In terms of these new variables, $\tilde{S}$ is given by

$$\tilde{S} = D \frac{\tilde{\Lambda}}{a'} F(x, y),$$  \hspace{1cm} (58)\hspace{1cm}

where

$$F(x, y) = -\frac{x^2}{1-x} - 2xy + y + \sqrt{y^2 + \tilde{g}^2},$$  \hspace{1cm} (59)\hspace{1cm}

and $y$ ranges from $-\infty$ to $+\infty$, whereas $x$ is limited to the interval $0 \leq x \leq 1$.

A number of features of this expression are worth noting:

a) There is a factor of $D$ multiplying the whole expression. Therefore, in the large $D$ limit, the dominant contribution comes from the saddle point.

b) Every variable in this expression is scale invariant.

c) The factor of $D$ appearing in the definition of $\tilde{g}$ is the standard “large N” factor [8] needed to have the correct limit.
d) Although we have so far introduced two independent cutoff parameters $\Lambda$ and $a$, or equivalently, $\tilde{\Lambda}$ and $a'$, only the combination $\tilde{\Lambda}/a'$ appears in the expression for $\tilde{S}$.

The integral to be evaluated in the large $D$ limit is

$$
Z = \int_{-\infty}^{+\infty} dy \int_{0}^{1} dx \exp (i S^{e f f}),
$$

$$
S^{e f f} = \gamma F(x, y),
$$

and the constant $\gamma$ is defined by

$$
\gamma = D \frac{\tilde{\Lambda}}{a'} \frac{\tau f - \tau i}{p^+}.
$$

First, let us consider the integration over $y$ for a fixed value of $x$ in the interval $(0,1)$. This integral can be evaluated exactly in terms of a Bessel function, but since we need only the large $D$ result, we will instead use the saddle point approximation. The saddle point $y_s$ is at

$$
\partial_y F(x, y) = 0 \rightarrow y_s = \frac{2x - 1}{2\sqrt{x - x^2}} \tilde{g}.
$$

In this equation, both $\tilde{g}$ and the square root are defined to be positive. The contour of integration in the complex $y$ plane can be distorted into the curve whose equation is

$$
\text{Re} \left( -2xy + y + \sqrt{y^2 + \tilde{g}^2} \right) = 2\tilde{g}\sqrt{x - x^2}.
$$

This curve passes through the saddle point, and with the above choice of the branch of the square root, as

$$
\text{Im}(y) \rightarrow \pm \infty,
$$
on the curve,

$$
\text{Im}(F(x, y)) \rightarrow +\infty,
$$
and therefore the integral

$$
\int_{y=\text{curve}} dy \exp (i\gamma F(x, y))
$$
is exponentially convergent. Using this contour of integration justifies the evaluation of the integral over $y$ by setting $y = y_s$ in the integrand. In Fig.4, the curve defined by eq.(62) in the complex $y$ plane is pictured for $\tilde{g} = 1$ and $x = 1/2$, when the saddle point is at $y_s = 0$. The branch cuts of the square root run from $i\tilde{g}$ to $+i\infty$ and from $-i\tilde{g}$ to $-i\infty$ and the contour asymptotes the vertical lines $\text{Re}(y) = \pm \tilde{g}$.

After the saddle point evaluation of the integral over $y$, we are left with the integral over $x$:

$$Z \to \int_0^1 dx \exp (-i\gamma f(x)),$$

$$\text{(63)}$$
where

\[
f(x) = \frac{x^2}{1 - x} - 2\tilde{g}\sqrt{x - x^2}.
\]  \hspace{1cm} (64)

The function \(f(x)\) is pictured in Fig. 5 for \(\tilde{g} = 20\). It has a single minimum in the interval \(0 \leq x \leq 1\) at \(x = x_m\), which satisfies

\[
f'(x_m) = \frac{2x_m - x_m^2}{(1 - x_m)^2} - \tilde{g} \frac{1 - 2x_m}{\sqrt{x_m - x_m^2}} = 0. \hspace{1cm} (65)
\]

At the minimum, \(f(x)\) is negative, corresponding to a negative minimum energy

\[
E_0 = D\frac{\Lambda}{a'p^+}f(x_m),
\]

which is the energy of the ground state in this approximation. The situation is similar for other positive values of \(\tilde{g}\): There is only a single minimum in the interval \((0,1)\), corresponding to a negative ground state energy.

We have just seen that the ground state corresponds to a value of \(x\) between 0 and 1. Let us remember that

\[
x = 0 \rightarrow \rho_0 = 0
\]
corresponds to the trivial case of a world sheet with all dotted lines. The opposite limit of
\[ x = 1 \to \bar{\rho}_0 = 0 \]
corresponds to a world sheet with only solid lines. A value of \( x \) in between these two extremes implies an intermediate world sheet texture: The solid and dotted lines each occupy a finite fraction of the area of the world sheet. Recalling our earlier discussion following eq.(49), we see that indeed a condensate of the solid lines has formed.

Let us now see whether a sensible string picture emerges. In particular, the slope parameter (eq.(49))
\[
\alpha' = \frac{\beta_1 \beta_4 \rho_0^2}{2\alpha \bar{\rho}_0} = \frac{\beta_1 \beta_4 x_m^2}{2\alpha(1 - x_m)a'},
\]
is the only physical parameter to emerge from the mean field calculation. The theory should be renormalized by requiring it to be a finite number independent of any cutoff. To see how this happens, we first get rid of the irrelevant constants \( \beta_{1,4} \) and \( \alpha \) by suitably redefining the cutoff parameter \( a' \):
\[
\alpha' \to \frac{x_m^2}{2(1 - x_m)a'}.
\]
In order to have a finite slope in the limit \( a' \to 0 \), \( x_m \), and therefore, \( \tilde{g} \) should also go to zero in the same limit. Solving (65) in the small \( \tilde{g} \) limit, we have
\[
x_m \approx \left( \frac{\tilde{g}}{2} \right)^{2/3},
\]
and
\[
\alpha' \approx \frac{\tilde{g}^{1/3}}{2^{7/3} a'}.
\]
Therefore, in the limit of the cutoff \( a' \) tending to zero, the coupling constant \( \tilde{g} \) should be fine tuned so that the ratio
\[
\frac{\tilde{g}^{1/3}}{a'}
\]
stays constant. The theory is renormalized by trading the cutoff dependent coupling constant \( \tilde{g} \) for the cutoff independent physical parameter \( \alpha' \) through eq.(69).
There is one more comparison one can make between the parameters of the $\phi^3$ theory and the string theory: One could try to relate the string intercept to the field theory mass and the coupling constant. If we identify the mass square of the lowest lying state on the string trajectory with the ground state energy corresponding to $\tilde{S}$ (eq.58), the meanfield approximation gives a negative cutoff dependent answer. In this article, we will not consider the question of renormalization of the ground state energy, which is the same as the renormalization of the intercept. There is, however, the following simple possibility. Instead of starting with a $\phi^3$ theory with zero bare mass, we could have started with a non-zero cutoff dependent bare mass. Fine tuning this bare mass, it may be possible to end up with a finite renormalized ground state energy. To carry out this program, however, our formalism has to be extended to include a non-zero bare mass. This we leave to future research.

It may seem surprising that, starting with an unstable field theory, so far we have not encountered any sign of instability on the string side. Of course, as mentioned above, we have not calculated the renormalized intercept. In the end, upon calculating this intercept, just as in the case of the bosonic string, some of the lowest lying states may turn out to be tachyonic. Another possibility is that, in the leading order of the large $D$ approximation, the instability may not be visible. For example, it is easy to construct a simple quantum mechanics problem with $D$ degrees of freedom, where there is a metastable state which decays by tunneling. It is usually the case that tunneling is suppressed in the leading large $D$ limit, and the metastable state becomes stable. One has to go beyond the leading order to see signs of instability. It is possible that this is what happens in the model we are studying here.

8. An Additional String Mode

In this section, we are going to compute a particular correction to the leading mean field or large $D$ result. We recall that, in this limit, the composite field $\rho$ can be replaced by its ground state expectation value $\rho_0$, but to go beyond the leading term, one has to expand in powers of the fluctuations around the mean. Of course, an exact computation of the full expansion is
impractical; however, as in section 6, we look for the dominant contribution in the limit of large cutoff. In this case, this is a logarithmically divergent term which dominates the rest of the terms, which are finite. To isolate this contribution, we split the composite field $\rho$ into the constant mean value $\rho_0$ (see eq.(70)), and a fluctuating part $\chi$. Along with a power series expansion in $\chi$, we treat $\chi$ as a slowly varying function, and we expand it around a fixed point in powers of the coordinates $\sigma$ and $\tau$. This latter expansion, which is sometimes called the derivative expansion, is very useful in isolating divergent terms in the perturbation series and it is widely used in literature. The point is that increasing order in this expansion goes with increasing number of derivatives on $\chi$, and by dimensional arguments, this results in greater convergence. The divergent terms therefore appear only in the lowest orders of the derivative expansion and they are easy to identify. We shall see that the leading divergence is quadratic, but this is already included in the calculation done in section 6 with a constant $\rho_0$. The next leading divergence is logarithmic, which is the contribution we are going to calculate. The rest of the terms in the expansion are finite. The logarithmic term has a special physical significance: it provides a kinetic energy term for $\chi$ in the action and so it promotes $\chi$ into a new propagating degree of freedom. This phenomenon should be familiar from other two dimensional models, such as the $CP(N)$ model [17] or the Gross-Neveu model [18]. In contrast, the finite terms in the expansion are non-local and they do not seem to have any special physical significance.

Our starting point is eq.(48) for $S_m$, but with $\rho_0$ replaced by $\rho$, since we are considering fluctuations of $\rho$ around the mean value $\rho_0$. We define

$$\left(\frac{\beta_1 \beta_4 p^+ \rho^2}{\alpha \rho}\right)^2 = A^2(1 + \chi),$$

(70)

where $A$ is defined by eq.(50) and,

$$\chi = \left(\frac{\rho_0 \rho^2}{\bar{\rho} \rho_0}\right)^2 - 1,$$

(71)

is the fluctuating field. Doing the functional integral over $q$ gives the following contribution to the action:

$$S' = \frac{i}{2} DT r \ln \left(\partial_\sigma^2 - A^2 \partial_\tau (1 + \chi) \partial_\tau\right).$$

(72)
We are going to examine in detail only terms up to second order in $\chi$; it will then be easy to figure out the contribution of the higher order terms. We therefore expand $S'$ up to second order:

$$S' = S'_0 + S'_1 + S'_2 + \ldots,$$

$$S'_0 = \frac{i}{2} D Tr \ln \left( \partial^2_{\tau} - A^2 \partial^2_{\sigma} \right),$$

$$S'_1 = -\frac{i}{2} D A^2 Tr \left( (\partial^2_{\sigma} - A^2 \partial^2_{\tau})^{-1} \partial_{\tau} \chi \partial_{\tau} \right),$$

$$S'_2 = \frac{i}{4} D A^4 Tr \left( (\partial^2_{\sigma} - A^2 \partial^2_{\tau})^{-1} \partial_{\tau} \chi \partial_{\tau} (\partial^2_{\sigma} - A^2 \partial^2_{\tau})^{-1} \partial_{\tau} \chi \partial_{\tau} \right). \quad (73)$$

The zeroth order term $S'_0$ was called $S'_m$ in section 6 and it was already calculated there (eq.(50)). The first order term $S'_1$ is represented by the graph in Fig.6, where the external line in this graph carries zero momentum. Its contribution is given by

$$S'_1 = c_1 \int d\sigma \int d\tau \chi(\sigma, \tau),$$

where $c_1$ is a quadratically divergent constant. In addition, there other contributions from higher order terms, represented by graphs of the form given in Fig.7, which generate the following series in $S'$:

$$S' \approx \int d\sigma \int d\tau \sum_{n=1}^{\infty} c_n (\chi(\sigma, \tau))^n. \quad (74)$$
This can be thought of as a potential for $\chi$, which, when minimized, determines the expectation value of $\chi$. However, all this does is to shift the value of $\rho_0$, which we have already calculated in the previous section. This ambiguity is due to the intrinsic arbitrariness in the split made in eq.(70): Only the particular combination of $\rho_0$ and $\chi$ that appears on the right hand side of (70) is well defined: Shifting $\rho_0$ and the expectation value of $\chi$ while keeping the right hand side of (70) constant will not change anything. We can resolve this ambiguity by setting the expectation value of $\chi$ equal to zero. In that case, there is no shift in the value of $\rho_0$, and we can drop the terms given in eq.(74). We shall do so in what follows.

Let us now focus on $S'_2$, the term quadratic in $\chi$, which is represented by the graph in Fig.8. In momentum space, we can set

$$S'_2 = -\frac{i}{4}DA^1P^+ \int d^2k' I(k'_0, k'_1)\bar{\chi}(k'_0, k'_1)\bar{\chi}(-k'_0, -k'_1),$$  

(75)

where $\bar{\chi}$ is the Fourier transform of $\chi$, and

$$I(k_0, k_1) = \int d^2k \frac{(4k_0^2 - (k'_0)^2)^2}{((2k_1 + k'_1)^2 - A^2(2k_0 + k'_0)^2)((2k_1 - k'_1)^2 - A^2(2k_0 - k'_0)^2)}.$$

(76)

So far, this expression is exact, but now, we are going to carry out the derivative expansion explained earlier, which coincides with the expansion in powers of the momentum $k'$. The zeroth order term in $k'$ contributes to the potential in $\chi$, and we have explained above that if one starts with the
correct value of the expectation value of $\rho$, this term is already taken care of. The first order term in $k'$ vanishes, and the second order term has a logarithmic divergence. Setting

$$I = I_0 + I_1 + I_2 + \ldots,$$

$I_1$ is zero and $I_2$ has the form

$$I_2 = I_{2,0} (k'_0)^2 + I_{2,1} (k'_1)^2,$$

and, after Euclidean rotation in $k_0$,

$$I_{2,0} = \frac{i}{2} \int d^2k \frac{k_0^2(3A^2k_0^2k_1^2 + k_1^4)}{(k_1^2 + A^2k_0^2)^4},$$

$$I_{2,1} = \frac{i}{2} \int d^2k \frac{k_0^4(k_1^2 - A^2k_0^2)}{(k_1^2 + A^2k_0^2)^4}. \quad (78)$$

The integrals are elementary. After the change of variables by

$$k_1 = r \sin(\theta), \quad A k_0 = r \cos(\theta),$$

the $\theta$ integrals are easily done, leaving a logarithmically divergent $r$ integral:

$$I_{2,0} = \frac{i\pi}{2A^3} \int_{\epsilon}^{\lambda} \frac{dr}{r}, \quad I_{2,1} = -\frac{i\pi}{2A^5} \int_{\epsilon}^{\lambda} \frac{dr}{r}. \quad (79)$$

We have introduced both an infrared cutoff $\epsilon$ and an ultraviolet cutoff $\lambda$ to regulate the divergent $r$ integral. Putting this back into the equation for $S'_2$ (eq.(i)), and rewriting it in the position space, we have

$$S'_2 \simeq \frac{D R}{32\pi A} \int d\sigma \int d\tau \left( A^2(\partial_r \chi)^2 - (\partial_\sigma \chi)^2 \right), \quad (80)$$

Figure 8: One Loop Contribution to the Propagator
where we have defined

\[ R = \int_{\epsilon}^{\lambda} \frac{dr}{r}. \]

We close this section with a couple of comments on this result:
a) The contribution of the \( \chi \) mode to the action is exactly the same as the contribution of \( q \) to \( S_m \) (eq.(48)), apart from an overall cutoff dependent multiplicative constant. This constant can be eliminated by rescaling \( \chi \) (wave function renormalization). The important point is that it is positive; otherwise, this term would have the wrong (ghostlike) signature.
b) There is therefore an additional string mode represented by \( \chi \), with the same slope

\[ \alpha' = A/2 \]
as the other modes. This changes the dimension of the target space from \( D \) to \( D + 1 \), and therefore it can regarded as a non-leading contribution in the large \( D \) limit.
c) If we continued the derivative expansion of \( S_2' \) beyond second order, we would get terms with higher derivatives of \( \chi \) with respect to \( \tau \) and \( \sigma \). By simple power counting, these terms would be finite and therefore they would not be of interest to us.
d) Let us now consider terms higher then second order in \( \chi \) in the expansion of \( S' \). The logarithmically divergent contribution still comes from second order in the derivative expansion. Consider the graph of Fig.9, where two external lines carry finite momenta and the rest of the external carry zero momentum. A simple generalization of the calculation given above in the case of the second order term in \( \chi \) shows that this term must be of the form

\[ S' \simeq \int d\sigma \int d\tau h(\chi) \left( (\partial_\tau \chi)^2 - A^2(\partial_\sigma \chi)^2 \right), \]  

and \( h(\chi) \) is a function whose power series expansion in \( \chi \) starts with the positive constant that appears in eq.(80). Redefining the field \( \chi \) by

\[ \chi \rightarrow g(\chi), \]

and choosing the function \( g \) so that it satisfies

\[ h(f(\chi))(g'(\chi))^2 = 1, \]

one can get replace \( h \) by one. This shows that a simple field redefinition gets us back to the second order result given by eq.(80).
To complete our discussion, we should also consider the contribution coming from $S_g$ in eq.(50). This contribution has a quite different structure compared to the one coming from the matter sector. It has a linear cutoff dependence, and it depends only on $\sigma$ derivatives of $\chi$ and not $\tau$ derivatives. We remind the reader of the motivation behind integrating over the matter fields in $S_m$: The integration produced a kinetic energy term for the originally non-propagating field $\chi$. Since integrating over the ghost fields produces nothing of comparable interest, it is probably best to leave $S_g$ as it is.

9. Conclusions

This article is a direct follow up of reference [2]. The goal is to give a systematic treatment which resolves various problems encountered in [2], while still staying within the framework of the simple meanfield approximation developed in that reference. The crucial improvement over the treatment given in [2] is the fixing of an accidental gauge invariance. In addition, there are various other technical improvements: Supersymmetry is introduced on the world sheet to keep better track of matter-ghost cancellation, and a better treatment of the singular determinants is presented. Also, following reference [4], we show how to impose partial Lorentz invariance.

The meanfield method used here and in [2] is sufficiently simple to allow the carrying out of a fairly complete treatment. In the leading order of this approximation, a condensation of Feynman graphs takes place, which means that large order Feynman graphs dominate the perturbation series. As a consequence, with a suitable tuning of the coupling constant, a string with finite slope is formed. We also show that a new dynamical degree of freedom emerges, extending the transverse dimension of the string from $D$ to $D + 1$. Although the string appears to be stable in the leading order, we argue that the fundamental instability of the underlying field theory probably shows up in the non-leading orders of the approximation.

The question whether string formation takes place in a given field theory is fundamentally important but difficult to answer in practice. The meanfield approximation used in this article, free from its earlier shortcomings, is simple yet powerful enough to answer this question in the case of the $\phi^3$ theory. The possibility of applying this method to more realistic theories appears more appears within reach. Another valuable line of future research would be to try to improve over the meanfield approximation.
Acknowledgements

This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, of the US Department of Energy under Contract DE-AC03-76SF00098, and in part by the National Science Foundation under Grant PHY-0098840.

Appendix

In this appendix, we will discuss an alternative, and more conventional way of doing the mean field approximation, and we will show that it is completely equivalent to the approach used in this article. Instead of $\rho$ (eq.(39)), we introduce a different set of composite fields:

$$
\Delta S = \int d\sigma \int d\tau \left( \kappa_1 (y \cdot \dot{q} - \lambda_1) + \kappa_2 (b \cdot b - \lambda_2) + \kappa_3 (\dot{c} \cdot c - \lambda_3) + \kappa_4 (z \cdot r - \lambda_4) + \kappa_5 (y \cdot z - \lambda_5) + \kappa_6 (\bar{b} \cdot \bar{c} - \lambda_6) \right). \tag{82}
$$

In the large $D$ limit, the fields $\kappa_i$ and $\lambda_i$ become classical and they can be replaced by their ground state expectation values:

$$
\kappa_i \to \langle \kappa_i \rangle, \quad \lambda_i \to \langle \lambda_i \rangle.
$$

The justification for this is well known [16]: The composite fields $\lambda_i$ are each sum of $D$ terms, and consequently, they grow like $D$ as $D$ becomes large. On the other hand, compared to this, the quantum fluctuations are suppressed by a factor of $1/\sqrt{D}$. As a result, in leading order large $D$ these fields become classical. In contrast, there is no comparable direct argument for why $\rho$ should become classical in this limit. Therefore, from the perspective of large $N$ (large $D$) physics, the approach sketched in this appendix is better motivated than the approach developed in the main body of the article. The disadvantage is that, at least initially, many more auxiliary fields have to be introduced. We will show presently that the expectation values of these extra fields can easily be expressed in terms $\kappa_0$ and $\rho_0$ introduced earlier. To do this, we first write $S_f$, the fermionic part of the action, in terms of the fields introduced in eq.(82):

$$
S_f = \int d\sigma \int d\tau \left( i \bar{\psi} \dot{\psi} + \frac{g}{p^+} (\bar{\psi}_1 \psi_2 + \psi_2 \bar{\psi}_1) \right)
+ \frac{1}{2} \left( \lambda_1 + \lambda_2 + \lambda_3 + p^+ \lambda_4 - \alpha \lambda_5 - \alpha \lambda_6 \right) \sum_i \bar{\psi}_i \psi_i 
+ \frac{\alpha}{a} (\lambda_5 + \lambda_6), \tag{83}
$$
where, to keep things simple, we have set all the \( \beta \)'s equal to one. Comparing this with the \( S_f \) given by eq.(41) leads to the identification

\[
\kappa_0 = \lambda_1 + \lambda_2 + \lambda_3 + p^+\lambda_4 - \alpha\lambda_5 - \alpha\lambda_6, \tag{84}
\]

so that the two expressions for \( S_f \), apart from an additive term, agree. We now consider all the \( \lambda \) dependent terms in

\[ \Delta S + S_f \]

and write down the equations of motion by varying the \( \lambda \)'s, subject, however, to the constraint that \( \kappa_0 \) (eq.(84)) is held fixed. These equations give

\[
\kappa_1 = \kappa_2 = \kappa_3 = \frac{\kappa_4}{p^+}, \quad \kappa_5 = \kappa_6 = \alpha\left(\frac{1}{a} - \kappa_1\right). \tag{85}
\]

Consequently, there is only one independent \( \kappa \), say \( \kappa_1 \). With the further identification

\[
\kappa_1 = \rho_0, \quad \frac{1}{a} - \kappa_1 = \bar{\rho}_0, \tag{86}
\]

we recover the action of eq.(36) with \( \beta \)'s set equal to one, establishing the equivalence of the two approaches.

References

1. K.Bardakci and C.B.Thorn, Nucl.Phys. B 626 (2002) 287, hep-th/0110301
2. K.Bardakci and C.B.Thorn, Nucl.Phys. B 652 (2003) 196, hep-th/0206205
3. K.Bardakci and C.B.Thorn, Nucl.Phys. B 661 (2003) 235, hep-th/0212254
4. K.Bardakci, Nucl.Phys. B 667 (2004) 354, hep-th/0308197
5. C.B.Thorn, Nucl.Phys. B 637 (2002) 272, hep-th/0203167
6. S.Gudmundsson, C.B.Thorn and T.A.Tran, Nucl.Phys. B 649 (2003) 3, hep-th/0209102
7. C.B.Thorn and T.A.Tran, Nucl.Phys. B 677 (2004) 289, hep-th/0307203
8. G.'tHooft, Nucl.Phys. B 72 (1974) 461.
9. A.Clark, A.Karsch, P.Kovtun and D.Yamada, Phys.Rev. **D 68** (2003) 066011.

10. R.M.Koch, A.Jevicki and J.P.Rodrigues, Phys.Rev. **D 68** (2003) 065012, hep-th/0305042.

11. O.Aharony, J.Marsano, S.Minwalla, K.Papadodimas and M.V Van Raamsdonk, hep-th/0310285.

12. R.Gopakumar, hep-th/0402063.

13. J.M.Maldacena, Adv.Theor.Math.Phys. **2** (1998) 281, hep-th/9711200.

14. O.Aharony, S.S.Gubser, J.Maldacena, H.Ooguri and Y.Oz, Phys.Rep. **323** (2000) 183, hep-th/9905111.

15. P.Goddard, J.Goldstone, C.Rebbi and C.B.Thorn, Nucl.Phys. **B 56** (1973) 109.

16. For a recent review of the large N method, see M.Moshe and J.Zinn-Justin, Phys.Rept. **385** (2003) 69, hep-th/0306133.

17. A.D’Adda, M.Luscher and P.Di Vecchia, Nucl.Phys. **B 146** (1978) 63.

18. D.Gross and A.Neveu, Phys.Rev. **D 10** (1974) 3235.