Automorphisms of $sl(2)$ and dynamical $r$-matrices.

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Abstract

Two outer automorphisms of infinite-dimensional representations of $sl(2)$ algebra are considered. The similar constructions for the loop algebras and yangians are presented. The corresponding linear and quadratic $R$-brackets include the dynamical $r$-matrices.

1 Introduction

We are concerned with the representation theory of loop algebras and yangians and with the method of separation of variables in framework of the inverse scattering method. Our aim is to apply the representation theory of $sl(2)$ and $sl(N)$ algebras into the $r$-matrix formalism.

Let us start with the following Levi-Civita theorem [6]: if the Hamilton-Jacobi equation

$$H = g^{ij}(x_1, \ldots, x_n) p_i p_j + h^j(x_1, \ldots, x_n) p_j + U(x_1, \ldots, x_n), \quad \{p_i, x_j\} = \delta_{ij},$$

(1.1)
can be integrated by separation of variables then the equation is integrated with $U = 0$ (1.1), i.e. in the absence of a force, in the same system of coordinates.

This note is devoted to the similar problem. Consider a classical hamiltonian system completely integrable on a $2n$-dimensional symplectic manifold $D$. It means that system possesses $n$ independent integrals $I_1, \ldots, I_n$ in the involution

$$\{I_i(x, p), I_j(x, p)\} = 0, \quad i, j = 1, \ldots, n,$$

where $\{x_j, p_j\}_{j=1}^n$ is some coordinate system in $D$.

Introduce the following mapping

$$I_j \to I'_j = \sum_{i=1}^n a_{ij}(x, p) \cdot [I_i + v_i(x, p)],$$

(1.2)

here $a_{ij}$ and $v_j$ are certain functions on $D$.

Under the suitable conditions for functions $a_{ij}$ and $v_j$, the mapping (1.2) could define a new integrable system on $D$ with independent integrals $I'_1, \ldots, I'_n$ in the involution

$$\{I'_i(x, p), I'_j(x, p)\} = 0, \quad i, j = 1, \ldots, n,$$

and could preserve the property of separability in the same coordinate system.

Remind, that systems discovered by Stäckel (see review [4]) have the similar properties. Their integrals are

$$I_j = \sum_{i=1}^n a_{ij}(x_1, \ldots, x_n)[p_i^2 + v_i(x_i)],$$

(1.3)
with the special functions $a_{ij}(x_1,\ldots,x_n)$ and the arbitrary functions $v_i(x_i)$.

We try to make a first step to understand of the algebraic roots of constraints on the functions $a_{ij}$ and $v_j$, which guarantee the properties of integrability and separability for mapping (1.2) in the framework of the inverse scattering method.

## 2 Outer automorphisms of representations of $sl(2)$

Let $W$ be an infinite-dimensional representation of the Lie algebra $sl(2)$ in linear space $V$ defined in the Cartan-Weil basis $\{s_3, s_\pm\} \in \text{End}(V)$ equipped with the natural bracket

$$[s_3, s_\pm] = \pm s_\pm, \quad [s_+, s_-] = 2s_3$$

and the single Casimir operator

$$\Delta = s_3^2 + \frac{1}{2}(s_+ s_- + s_- s_+).$$

(2.4)

If operator $s_+$ is invertible in $\text{End}(V)$ then the mapping

$$s_3 \rightarrow s'_3 = s_3, \quad s_+ \rightarrow s'_+ = s_+, \quad s_- \rightarrow s'_- = s_- + fs_+^{-1}, \quad f \in \mathbb{C}$$

(2.6)

is an outer automorphism of the space of infinite-dimensional representations of $sl(2)$ in $V$. The mapping (2.4) shifts the spectrum of $\Delta$ on the parameter $f$

$$\Delta \rightarrow \Delta' = \Delta + f.$$  

(2.7)

Let us call the mapping (2.6) an additive automorphism.

Assuming in addition the value of Casimir operator $\Delta$ is equal to zero $\Delta = 0$ in $W$, then the mapping

$$s_j \rightarrow s'_j = s_j \cdot (1 - fs_+^{-1}), \quad j = \pm, 3, \quad f \in \mathbb{C}$$

$$\Delta \rightarrow \Delta' = \Delta \cdot (1 - fs_+^{-1})^2, \quad \Delta = \Delta' = 0,$$

(2.8)

is another outer automorphism of representation $W$. Let us call the mapping (2.8) a multiplicative automorphism.

These special infinite-dimensional representations $W$ could be obtained by using the previous additive automorphism (2.4). Notice, that the similar representations with $\Delta = 0$ are well known in the quantum conformal field theory [4].

We can suppose that additive (2.6) and multiplicative (2.8) automorphisms define the one-parametric realizations $W(f)$ of $sl(2)$.

For instance, realization of $sl(2)$ with one free parameter $f$ in the classical mechanics is given by

$$s_3 = \frac{xp}{2}, \quad s_+ = \frac{x^2}{2}, \quad s_- = -\frac{p^2}{2} + \frac{f}{x^2}, \quad \Delta' = f,$$

(2.9)

where $(x,p)$ is a pair of canonical coordinate and momenta with the classical Poisson bracket $\{p, x\} = 1$. Similar realization of generators $s_j$ as the differential operators in quantum case is

$$s_3 = x\partial_x - l, \quad s_+ = x, \quad s_- = x\partial_x^2 - 2l\partial_x + \frac{f}{x}, \quad \Delta' = l(l+1) + f.$$  

(2.10)
For the quantum Calogero system the more sophisticated representation of $sl(2)$ is equal to

$$s_3 = \frac{1}{4} \sum_{k=1}^{N} (x_k D_k + D_k x_k), \quad s_+ = 2 \sum_{k=1}^{N} x_k^2, \quad s_- = -2 \sum_{k=1}^{N} D_k^2,$$

where $x_k$ are coordinates and $D_k$ are the corresponding Dunkl operators.

Using the inner automorphisms of $sl(2)$ the one parametric mappings (2.6) and (2.8) can be generalized. Let $W$ be an infinite-dimensional representation of $sl(2)$ in $V$ defined by operators $s_j \in \text{End}(V)$ with the standard bracket and the single quadratic Casimir operator $\Delta$

$$[s_i, s_j] = \varepsilon_{ijk} s_k, \quad \Delta = s_1^2 + s_2^2 + s_3^2. \quad \tag{2.11}$$

Introduce the general linear operator $b \in \text{End}(V)$ as

$$b = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3, \quad \alpha_j \in \mathbb{C}, \quad \tag{2.12}$$

where $\alpha_j$ are three arbitrary parameters. If $b$ is invertible operator, then the mapping

$$s_j \to s_j' = s_j + \alpha_j b^{-1} \quad \tag{2.13}$$

is an outer automorphism of the space of infinite-dimensional representations of $sl(2)$ in $V$, if the parameters $\alpha_j$ lie on the cone

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0. \quad \tag{2.14}$$

For this mapping the spectrum of the Casimir operator $\Delta$ (2.11) is additively shifted on the parameter $f = \sum \alpha_j$

$$\Delta \to \Delta' = \Delta + f = \Delta + \sum_{j=1}^{3} \alpha_j. \quad \tag{2.15}$$

If the value of Casimir operator $\Delta$ is equal to zero $\Delta = 0$ and the general linear operator $b \in \text{End}(V)$ (2.12) is invertible, then the mapping

$$s_j \to s_j' = s_j(1 - b^{-1}), \quad \Delta \to \Delta' = \Delta \cdot (1 - b^{-1})^2 \quad \tag{2.16}$$

is an outer multiplicative automorphism of representation $W$.

Automorphisms (2.13) and (2.16) define the general two and three-parametric realizations $W(\alpha_j)$ of $sl(2)$.

For the $sl(N)$ algebra the similar additive automorphism shifted the highest central element of $N$-th order can be taken in the form

$$s_{ij} \to s'_{ij} = s_{ij} + A_{ij}^{(m-1)} B^{(m)}, \quad s_{ij} \in \text{End}(V), \quad m = N(N-1)/2. \quad \tag{2.17}$$

In operators $s_{ij}$ functions $B^{(m)}$ and $A_{ij}^{(m-1)}$ are certain polynomials of degrees $m$ and $m - 1$. Here we shall not go into details of this problem, one example related to $sl(3)$ will be presented in Section 4.

Motivated by realization (2.3) we present one application of automorphism (2.6) in the theory of integrable systems. Consider a classical Hamiltonian system completely integrable on phase space $D = \mathbb{R}^{2n}$ with the natural hamiltonian

$$H = \sum_{j=1}^{n} p_j^2 + V(x_1, \ldots, x_n).$$
Let the phase space be identified completely or partially with the \( m \) coadjoint orbits in \( sl(2)^* \) as (2.9). Then the mapping
\[
H \rightarrow H' = H + \sum_{j=1}^{m} f_j x_j^2, \quad f_j \in \mathbb{R}
\] (2.18)
preserves the properties of integrability and separability. The list of such systems can be found in [7].

The main our aim is in developing similar constructions for the loop algebras and yangians. Some examples of the integrable systems related with this approach have been considered in [3, 10, 11].

3 Dynamical \( r \)-matrices associated to \( \tilde{sl}(2) \)

All details of the general \( r \)-matrix scheme can be found in review [8] and in references therein.

We recall briefly necessary elements of \( r \)-matrix scheme for loop algebra \( g = \tilde{sl}(2) \) to assume the standard identification of the dual spaces.

The loop algebra \( g = \tilde{sl}(2) \) consists of Laurent polynomials \( s_j(\lambda) = \sum k s_{jk}^j \lambda^k, \quad j = 1, 2, 3 \) of spectral parameter \( \lambda \) with coefficients in \( a = sl(2) \) and commutator \( [s_i^j, s_j^m] = \varepsilon_{ijk}s_k^\lambda \lambda^l+m \). The standard \( R \)-bracket associated with loop algebra \( g \) is defined by the following decomposition of \( g \) into a linear sum of two subalgebras
\[
g = g_+ \oplus g_-, \quad g_+ = \oplus_{i\geq 0}a^i \lambda^j, \quad g_- = \oplus_{i<0}a^i \lambda^j, \quad R = P_+ - P_-. \] (3.1)

Here \( R \) is a standard \( r \)-matrix and \( P_\pm \) means the projection operators onto \( g_\pm \) parallel \( g_\mp \).

The Lax equation may be presented in the form
\[
\frac{dL(\lambda)}{dt} = -\text{ad}^*_g A \cdot L, \quad A = \frac{1}{2} R(dP(L)), \] (3.2)
where Lax matrix \( L(\lambda) \) belongs to \( g^* \), \( P \) is an \( \text{ad}^* \)-invariant polynomial on \( a^* \). For algebra \( \tilde{sl}(2) \) polynomial \( P(L) \) is a function of the unique invariant polynomial \( \Delta(\lambda) = \sum_{j=1}^{3} s_j^2(\lambda) \). Let \( P(L) = \phi(\lambda)\Delta(\lambda) \), where \( \phi \) is a functions of \( \lambda \). The integrals of motion \( I_k \) related to flow (3.2) are
\[
I_k(L) = \text{Res}_{\lambda=0}(\phi_k(\lambda)\Delta(\lambda)), \] (3.3)
where \( \phi_k(\lambda) \) are various functions of spectral parameter defining a complete set of integrals of motion [8].

The Lax matrix \( L(\lambda) \) (3.2) is defined on the whole infinite-dimensional phase space \( g^* \). It is well known that the standard \( R \)-bracket associated with (3.1) has also a large collection of finite-dimensional Poisson (\( \text{ad}^*_R \)-invariant) subspaces [8]
\[
\mathcal{L}_{M,N} = \bigoplus_{j=-M}^{N} a^* \lambda^j, \quad \text{provided} \ M \geq 0; \ N \geq 1. \] (3.4)

and, as a rule, the concrete physical systems are related to the restrictions of the flow (3.2) to certain low-dimensional Poisson submanifolds \( \mathcal{L}_{M,N} \) (3.4).

The \( r \)-matrix scheme is extended easily to the twisted subalgebras of loop algebra \( g \) and corresponding \( r \)-matrices have rational, trigonometric and elliptic dependence on spectral parameter. We shall work with a tensor form of the Lax equations and \( R \)-bracket on \( \tilde{sl}(2) \), that allows us to consider all the \( r \)-matrices simultaneously. In addition, for brevity, we shall consider the \( sl(2, \mathbb{R}) \) only.
The Lax matrix $L(\lambda)$ \((3.2)\) associated to the hamiltonian \((3.3)\) is given by

$$L(\lambda) = \sum_{k=1}^{3} s_k(\lambda) \sigma_k,$$

$$\Delta(\lambda) = \frac{1}{2} \text{tr} L^2(\lambda) = -\det L(\lambda), \quad (3.5)$$

where $\sigma_j$ are Pauli matrices. The $R$-bracket related to decomposition \((3.1)\) take the following form

$$\{ L(\lambda), L(\mu) \} = [r_{12}(\lambda, \mu), L(\lambda)] - [r_{21}(\lambda, \mu), L(\mu)], \quad (3.6)$$

where the standard notations are introduced:

$$L(\lambda) = L(\lambda) \otimes I,$$

$$r_{12}(\lambda, \mu) = \sum_{k=1}^{3} w_k(\lambda, \mu) \cdot \sigma_k \otimes \sigma_k, \quad r_{21}(\lambda, \mu) = \Pi r_{12}(\mu, \lambda) \Pi, \quad (3.7)$$

Here $\Pi$ is the permutation operator of auxiliary spaces and $w_k(\lambda, \mu)$ are certain functions of spectral parameters only. Their explicit dependence on $\lambda, \mu$ is not important for the moment.

Returning now to the additive automorphism of representations of $sl(2)$ \((2.13)\) we introduce the related mappings on the loop algebras $\tilde{g} = sl(2, \mathbb{R})$ or $\tilde{g} = \oplus^n sl(2, \mathbb{R})$.

The entries of the Lax matrix are Laurent polynomials with coefficients in $sl(2)$. Let us consider the mapping transforming these coefficients. If parameters $a_j$ lie on the cone \((2.14)\), then the first mapping similar to the automorphism \((2.13)\)

$$s_j(\lambda) = \sum_k s_j \lambda^k \rightarrow s_j'(\lambda) = \sum_k [s_j + \alpha_j b^{-1}] \lambda^k, \quad b = \sum_{j=1}^{3} \alpha_j s_j, \quad (3.8)$$

is a canonical change of variables on $sl(2)$. This mapping is defined for the infinite-dimensional representations of $sl(2)$ and mapping \((3.8)\) is a Poisson map with respect to the first natural Lie-Poisson bracket and to the second linear Lie-Poisson bracket associated with the standard $R$-matrix \((3.1)\) on $sl(2)$. For the certain Lax representation $L(\lambda)$ mapping \((3.8)\) transforms the integrals of motion $I_k$ similar to \((2.18)\).

Let us introduce the general linear operator $b(\lambda) \in \tilde{sl}(2)$

$$b(\lambda) = \sum_{j=1}^{3} \alpha_j(\lambda) s_j(\lambda), \quad (3.9)$$

where $\alpha_j(\lambda)$ are functions of spectral parameter and of central elements on $\tilde{sl}(2)$. If functions $\alpha_j(\lambda)$ lie on the cone

$$\alpha_1^2(\lambda) + \alpha_2^2(\lambda) + \alpha_3^2(\lambda) = 0, \quad (3.10)$$

then the second mapping similar to the automorphism \((2.13)\)

$$s_j(\lambda) \rightarrow s_j'(\lambda) = s_j(\lambda) + \alpha_j(\lambda) b^{-1}(\lambda), \quad (3.11)$$

is a Poisson map with respect to the first natural Lie-Poisson bracket. For the mapping \((3.11)\) the Lax matrix \((3.7)\), the invariant polynomial $\Delta(\lambda)$ and integrals of motion $I_k$ \((3.3)\) are
additively shifted
\[
L(\lambda) \to L'(\lambda) = L(\lambda) + \sum_{k=1}^{3} \alpha_k \sigma_k \cdot b^{-1}(\lambda),
\] (3.12)
\[
\Delta(\lambda) \to \Delta'(\lambda) = \Delta(\lambda) + V(\lambda) = \Delta(\lambda) + \sum_{k=1}^{3} \alpha_k(\lambda),
\] (3.13)
\[
I_k \to I_k' = I_k + U_k, \quad U_k = \text{Res}_{\lambda=0} (\phi_k(\lambda)V(\lambda)) \in \mathbb{C}.
\] (3.14)

Restriction (3.10) guarantees that initial integrals of motion \( I_k \) (3.3) transform by some constants \( U_k \in \mathbb{C} \) (3.12). Hence, two Lax matrices \( L(\lambda) \) and \( L'(\lambda) \) (original and the image of mapping (3.11)) correspond to the same integrable system.

The entries of the initial matrix \( L(\lambda) \) belong to \( \mathfrak{g}^* \). The entries of the matrix \( L'(\lambda) \) are the Laurent polynomials of spectral parameter \( \lambda \) with coefficients from the universal enveloping algebra of \( \mathfrak{a}^* \). Nevertheless, the second Poisson bracket \( \{L'(\lambda), L'(\mu)\} \) can be directly calculated, because of all the necessary Poisson brackets between \( s_j(\lambda) \) and \( b(\lambda) \) are preassigned by (3.6).

Further we consider a special class of the linear dynamical \( R \)-bracket related to \( \widetilde{sl}(2) \) and defined by the following second restriction on coefficients \( \alpha_j(\lambda) \)
\[
\{b(\lambda), b(\mu)\} = g(\lambda, \mu)b(\lambda) - g(\mu, \lambda)b(\mu), \quad \lambda, \mu \in \mathbb{C},
\] (3.15)

or, that is equivalent,
\[
w_j(\lambda, \mu)\alpha_i(\mu)\alpha_j(\lambda) - w_i(\lambda, \mu)\alpha_i(\mu)\alpha_j(\lambda) = g(\lambda, \mu) \cdot \alpha_k(\lambda),
\] (3.16)
where \((j, i, k)\) are cyclic permutations of indices \((1, 2, 3)\) and the scale function \( g(\lambda, \mu) \) depends of the spectral parameters only. This restriction is closely related with the separation of variables method. It guarantees that all zeroes of \( b(\lambda) \) are mutually commuting. Below we assume that both conditions (3.10) and (3.14) are always fulfilled for the mapping (3.11).

In this case we get the Lax matrix \( L'(\lambda) \) (3.12) obeys the linear \( R \)-bracket (3.14), where constant \( r_{ij} \)-matrices substituted by \( r'_{ij} \)-matrices depending on dynamical variables
\[
r_{12}(\lambda, \mu) \to r'_{12} = r_{12} + \sum_{i,j=1}^{3} \alpha_{ij}(\lambda, \mu) \sigma_i \otimes \sigma_j,
\] (3.17)
with coefficients \( \alpha_{ij} \) being
\[
\alpha_{ij}(\lambda, \mu) = g(\lambda, \mu) \frac{\alpha_j(\mu) \alpha_i(\mu) w_i(\lambda, \mu)}{b^2(\mu)} - g(\mu, \lambda) \frac{\alpha_i(\lambda) \alpha_j(\lambda) w_j(\lambda, \mu)}{b^2(\lambda)}.
\] (3.18)

The proof see in [5].

Dynamical matrices \( r'_{jk}(\lambda, \mu) \) (3.17) obey the classical dynamical Yang-Baxter equation
\[
[r'_{12}(\lambda, \mu), r'_{13}(\lambda, \nu)] + [r'_{12}(\lambda, \mu), r'_{23}(\mu, \nu)] + [r'_{32}(\nu, \mu), r'_{13}(\lambda, \nu)] + \]
\[
+ [L'_2(\mu), r'_{13}(\lambda, \nu)] - [L'_3(\nu), r'_{12}(\lambda, \mu)] + [X_{123}(\lambda, \mu, \nu), L'_2(\mu) - L'_3(\nu)] = 0,
\] (3.19)
where an explicit expression of the tensor \( X(\lambda, \mu, \nu) \) is not important for the moment (see [3, 3, 12]). This equation is the image of a standard 2-cocycle in the new object, which consists of
Laurent polynomials of spectral parameter $\lambda$ with coefficients from the corresponding universal enveloping algebra.

We can suppose that mapping (3.11) defines family of Lax matrices $L'(\lambda)$ and dynamical $r$-matrices (3.17) with a fixed set of parameters $\alpha_j(\lambda)$ for certain integrable system. Thus, the Lax matrix $L(\lambda)$ (3.2), the constant $r$-matrices (3.7) and the standard classical Yang-Baxter equation on $\mathfrak{sl}(2)$ are the limits of the dynamical ones at $\alpha_j(\lambda) = 0$.

The mapping (3.11) allows us to construct an infinite set of the Lax matrices associated to loop algebra $\tilde{\mathfrak{sl}}(2)$, which correspond to different integrable systems. According to [5] we introduce an infinite set of mappings

$$s_j(\lambda) \rightarrow s'_j(\lambda) = s_j(\lambda) + \left[ \alpha_j(\lambda)b^{-1}(\lambda) \right]_{M_jN_j},$$

(3.20)

where $[z]_{MN}$ means restriction of $z$ onto the certain Poisson subspace $\mathcal{L}_{MN}$ (3.4) of the standard $R$-bracket (3.1).

As an example, we can use the linear combinations of the following Laurent and Fourier projections

$$[z]_{MN} = \left[ \sum_{k=-\infty}^{+\infty} z_k \lambda^k \right]_{MN} \equiv \sum_{k=-M}^{N} z_k \lambda^k,$$

(3.21)

and

$$[z]_{MN} = \left[ \sum_{k=-\infty}^{+\infty} z_k \exp(k \cdot \lambda) \right]_{MN} \equiv \sum_{k=-M}^{N} z_k \exp(k \cdot \lambda),$$

if the corresponding $r$-matrices have rational or trigonometric dependence on spectral parameter $\lambda$.

Generally speaking, we can not describe now all the possible restrictions of the mapping (3.11) to certain low-dimensional submanifolds, which allow us to define the second Poisson bracket on $\mathfrak{sl}(2)$. However, for a fairly large class of properly defined projections (3.21) the mappings (3.20) are the Poisson maps with respect to the second $R$-bracket and the corresponding dynamical $r$-matrices obey the dynamical Yang-Baxter equation (3.19).

For instance, an application of projections (3.21) yields dynamical $r$-matrices

$$r_{12}(\lambda, \mu) \rightarrow r'_{12} = r_{12} + \sum_{i,j=1}^{3} \alpha_{ij}(\lambda, \mu) \sigma_i \otimes \sigma_j,$$

(3.22)

with the following coefficients

$$\alpha_{ij}(\lambda, \mu) = g(\lambda, \mu)w_i(\lambda, \mu) \left[ \frac{\alpha_{ji}(\mu)\alpha_i(\mu)}{b^2(\mu)} \right]_{MN} - g(\mu, \lambda)w_j(\lambda, \mu) \left[ \frac{\alpha_{ji}(\lambda)\alpha_j(\lambda)}{b^2(\lambda)} \right]_{MN}.$$

(3.23)

The essential feature of the restricting mappings (3.20) is, in comparison with the mapping (3.11), that the invariant polynomial $\Delta(\lambda)$ and all integrals of motion $I_k$ are shifted now on the items depending on dynamical variables

$$\Delta(\lambda) \rightarrow \Delta_{MN}(\lambda) = \Delta(\lambda) + V_{MN}(s_j, \alpha_j, \lambda),$$

(3.24)

$$I_k \rightarrow I'_k = I_k + \text{Res}_{\lambda=0}(\phi_k(\lambda)V_{MN}(s_j, \alpha_j, \lambda)).$$
where $V_{MN}$ being

$$V_{MN}(s_j, \alpha_j, \lambda) = \sum_{k=1}^{3} \left( 2s_k(\lambda) + [a_j(\lambda)b^{-1}(\lambda)]_{MN} \right) [a_j(\lambda)b^{-1}(\lambda)]_{MN}. \quad (3.25)$$

Hence, images of integrals of motion $I'_k$ of the mapping $\phi$ are functionally different from the original ones $I_k$.

The second feature of the mapping $\phi$ is that it can be also applied to the finite-dimensional representations of $sl(2)$. Consider the multipole Lax matrices related to the rational r-matrix $\mathfrak{g}$. Let $\mathfrak{g} = \oplus^n sl(2, \mathbb{R})$ and elements of mapping $\phi$ are

$$\alpha_1 = 1, \quad \alpha_2 = i, \quad \alpha_3 = 0, \quad b(\lambda) = \sum_{k=1}^{n} \frac{s_{+}^{(k)}}{\lambda - e_k}, \quad e_j \neq e_k \in \mathbb{R}. \quad (3.26)$$

The Taylor projection of $b^{-1}(\lambda)$ is determined by the following recurrence relations

$$\left[ b^{-1}(\lambda) \right]_{MN}^{M=0} = \sum_{k=0}^{N} \mathcal{V}_k \lambda_{N-k}, \quad \mathcal{V}_k = \sum_{i=1}^{n} \left( s_{+}^{(i)} \sum_{j=0}^{k-1} \mathcal{V}_{k-1-j} e_j \right), \quad \mathcal{V}_0 = 1. \quad (3.27)$$

The Taylor projection $\left[ b^{-1}(\lambda) \right]_{MN}^{M=0}$ is the well-defined polynomial of the nilpotent operators $s_{+}^{(k)}$ without the negative powers, which can be also used for finite-dimensional representations of $sl(2)$.

Once again, the mappings $\phi$ are not just isomorphisms of the Lax matrices $L(\lambda)$ and $L'(\lambda)$, in contrast with mapping $\phi$, but they also preserve the second Poisson bracket together with all the commuting integrals of motion $I'_k$, i.e. preserve the properties of integrability and separability.

Thus, the mappings $\phi$ play the role of a dressing procedure allowing to construct the Lax matrices $L'_M(\lambda)$ for an infinite set of new integrable systems starting from the single known Lax matrix $L(\lambda)$ associated to one integrable model.

Now we shall briefly discuss the second multiplicative automorphism $\Delta'$. Let us start with some Lax matrix $L(\lambda) \in sl(2)^*$. This matrix relates to the integrable system with the Lax representation $\Delta$ and with the following integrals of motion $\Delta_k$:

$$I_k = \text{Res}_{\lambda=0} (\phi_k(\lambda) \Delta(\lambda)).$$

Recall, that the multiplicative automorphism $\Delta'$ of $sl(2)$ transforms the Casimir operator by the rule

$$\Delta \rightarrow \Delta' = \Delta \cdot \varphi(b) = \Delta \cdot (1 - b^{-1})^2.$$

Motivated by this transformation we consider the new set of integrals $I'_k$ (see 1.2) defined by

$$I'_k = \text{Res}_{\lambda=0} (\varphi_k(b, \lambda) \cdot \Delta(\lambda)), \quad (3.28)$$

where $\varphi_k(b, \lambda)$ are certain functions of $b(\lambda)$ and of the spectral parameter $\lambda$. The involution conditions

$$\{I'_i, I'_j\} = 0, \quad i, j = 1, \ldots, n,$$

yield the set of equations in $\varphi_k(b, \lambda)$

$$\text{Res}_{\lambda=0} \left\{ \Delta(\lambda), b(\mu) \right\} \varphi_j(\lambda)\varphi_k(\mu) \left( \frac{\partial \ln \varphi_j(b(\lambda)) \Delta(\lambda)}{\partial b} - \frac{\partial \ln \varphi_k(b(\mu)) \Delta(\mu)}{\partial b} \right) = 0.$$


New integrals of motion $I'_j$ could be functionally different from the original ones $I_j$.

In the quantum case the Poisson brackets should be replaced by the standard commutator relations on $sl(2)$ and the non-dynamical linear $R$-matrix bracket (3.6) becomes the following commutator relations

$$\left[ L(\lambda), L(\mu) \right] = [\mathbf{v}_{12}(\lambda, \mu), \mathbf{L}(\lambda)] - [\mathbf{v}_{21}(\lambda, \mu), \mathbf{L}(\mu)], \quad [\mathbf{v}(\lambda, \mu) = -i\hbar r(\lambda, \mu).$$

(3.29)

Theory of the general quantum linear $R$-bracket with dynamical $r$-matrices is not well developed yet, but a nice feature of the presented Poisson mappings (3.11) and (3.20) is that it admits a natural quantization. All the necessary commutator relations between operators $s_j(\lambda)$ and linear invertible operator $b(\lambda)$ (3.3) are preassigned by (3.29) and, therefore, the introduction of the quantum dynamical $r$-matrices (3.17) is a straightforward calculation.

A similar calculation yields the direct quantum counterpart of the dynamical Yang-Baxter equation.

### 4 Dynamical $r$-matrices and separation of variables

The presented algebraic construction are intimately connected with the method of separation of variables in classical mechanics. We shall use technique developed by Sklyanin in framework of $r$-matrix formalism and based on the application of the Baker-Akhiezer function [9]. Recall, that the Baker-Akhiezer function $\Phi(\lambda)$ is the eigenvector of the Lax matrix

$$L(\lambda)\Phi(\lambda) = z(\lambda)\Phi(\lambda), \quad \{H, \Phi\} = \Phi_t(\lambda) = A\Phi(\lambda),$$

(4.1)

corresponding to the eigenvalue $z(\lambda)$, which has certain analyticity properties. Here $A$ is a second matrix in the Lax representation (3.3). Since an eigenvector is defined up to a scalar factor, to exclude the ambiguity in the definition of $\Phi(\lambda)$ one has to fix a normalization of $\Phi(\lambda)$ imposing a linear constraint

$$\sum_{j=1}^{N} \beta_j(\lambda)\Phi_j(\lambda) = 1.$$

The main purpose of this Section is to find a correspondence between the outer automorphisms of infinite-dimensional representations of $sl(2)$ and the normalization of the Baker-Akhiezer function.

According to the Sklyanin recipe the variables of separation $(x_j, p_j)$ are defined by the poles of the properly normalized Baker-Akhiezer function $\Phi(\lambda)$ and the corresponding eigenvalues $z(\lambda)$ of the Lax matrix [9]. For the Lax matrix $L(\lambda) \in sl(2)^*$ the poles $x_j$ of $\Phi(\lambda)$ are zeroes of the following function

$$B(\lambda, \beta_1, \beta_2) = \beta_1^2(\lambda)s_+(\lambda) - \beta_2^2(\lambda)s_-(\lambda) - 2\beta_1(\lambda)\beta_2(\lambda)s_0(\lambda).$$

(4.2)

Introduce functions $\alpha_j(\lambda)$ as

$$\alpha_1(\lambda) = \lambda [\lambda(\lambda - 1)/2] f(\lambda), \quad \alpha_2(\lambda) = i(\lambda^2 + \lambda + 1)f(\lambda), \quad \alpha_3 = -2\lambda f(\lambda),$$

(4.3)

where $f(\lambda)$ is a certain common function of a spectral parameter only. The coefficients $\alpha_j(\lambda)$ lie on the cone (3.10) and define the linear operator $b(\lambda, \alpha_j) = B(\lambda, \beta_1, \beta_2)$ (3.9) for the mappings (3.11) and (3.20). The second restriction (3.15) on $\alpha_j(\lambda)$ guarantees that all zeroes $x_j$ of $b(\lambda)$ are mutually commuting $\{x_j, x_k\} = 0$. 


For the linear $R$-bracket the eigenvalues $z(\lambda)$ of the Lax matrix $L(\lambda)$ corresponding to zeroes $x_j$ are canonically conjugated momenta $p_j$. The pairs of variables $(x_j, p_j)$ lie on the spectral curve

$$W(p_j, x_j) = 0, \quad W(z, \lambda) = \det(z - L(\lambda)) = z^2 - \Delta(\lambda),$$

(4.4)

which fits exactly to separated equations [4].

The equations (4.1) and divisor of poles $\Psi(\lambda)$ are covariant with respect to the mappings (3.11-3.20)

$$L \to L' = L + \Delta L_{MN}, \quad H \to H' = H + V_{MN}. $$

These mappings could be considered as an analogous of a standard Darboux transformation.

The mappings (3.20) change the separated equations (4.4) as

$$p_j^2 - \Delta(x_j, I_k) = 0 \to p_j^2 - \Delta'(x_j, I'_k) = 0,$$

(4.5)

where $\Delta'(\lambda)$ and new integrals of motion $I'_k$ are given by (3.24). Thus, by taking the single Lax matrix $L(\lambda)$ associated to some integrable system, which can be integrated by separation of variables in coordinates $\{x_j, p_j\}$, we get an infinite set of completely integrable systems determined by the mappings (3.21), which are separable in the same variables $\{x_j, p_j\}$.

The linear operator $b(\lambda)$ (3.9) is a symmetric function of its zeroes $\{x_j\}_{j=1}^n$ and dynamical $r$-matrices (3.17) depends of the spectral parameters and only half of the dynamical variables $\{x_j\}_{j=1}^n$. Moreover, if normalization $\beta(\lambda)$ of $\Psi(\lambda)$ is given by arbitrary constant numeric vector, then three functions $\alpha_j(\lambda)$ (4.3) differ by certain numeric constants $\beta_j$

$$\alpha_j(\lambda) = \beta_j f(\lambda), \quad \beta_j \in \mathbb{C}, \quad \sum \beta_j^2 = 0.$$

(4.6)

Then by (4.6) the invariant polynomial $\Delta(\lambda)$ and all the integrals of motion $I_k$ (3) are shifted on items said to be potentials depending of separated coordinates $x_j$ (4.2):

$$\Delta' = \Delta + V_{MN}(x_1, \ldots, x_n) = \Delta + 2b(x_j, \lambda) \left[ f(\lambda) \right]_{MN},$$

$$I'_k = I_k + U_k(x_1, \ldots, x_n) = I_k + \text{Res}_{\lambda=0} \left( \phi_k(\lambda)V_{MN}(x_1, \ldots, x_n) \right).$$

(4.7)

Thus, all integrable systems related to the mappings (3.20) with the constant normalization of the corresponding Baker-Akhiezer function obey to the Levi-Civita theorem (1.1). Integers of motion $I_j'$ (3.28) associated to the multiplicative automorphism (2.16) could be differ from original ones on certain functions $a_{ij}(x_1, \ldots, x_n)$ of only half of the dynamical variables $\{x_j\}_{j=1}^n$ (see (1.2) and the Stäckel systems (1.3)). Here functions $a_{ij}$ are defined by (3.28) for any concrete representation.

Method of separation of variables for the Lax matrices $L(\lambda) \in sl(N)^*$ was studied [3]. The separated coordinates are obtained as zeroes of the certain polynomial $B(\lambda)$ of degree $N(N - 1)/2$ in components of $L(\lambda)$. The spectral curve $W(z, \lambda) = \det(z - L(\lambda))$ is a nonhyperelliptic algebraic curve for $N > 2$ and the Levi-Civita theorem cannot be applied directly to this case. Still the polynomial $B(\lambda)$ depending of separated coordinates can be used to construct a counterpart of the mappings (1.11) and (3.20) for $sl(N)$.

Consider, for instance, the loop algebra $sl(3)$. Let the Lax matrix $L(\lambda) \in sl(3)^*$ obeys the standard linear $R$-bracket (3.6) with rational $r$-matrix $r_{12} = (\lambda - \mu)^{-1}\Pi$, where $\Pi$ is the permutation operator: $\Pi x \otimes y = y \otimes x, \ x, y \in \mathbb{C}^3$. The entries $s_{ij}(\lambda)$ of the Lax matrix $L(\lambda)$
are constructed in variables \( s_{ij} \) \((i, j = 1, 2, 3. \sum s_{ii} = 0)\) with the following standard Poisson brackets

\[
\{s_{ij}, s_{km}\} = s_{im} \delta_{jk} - s_{kj} \delta_{im},
\]

which define the natural Lie-Poisson bracket on \( \widetilde{sl(3)}^* \). According to \cite{9}, the simplest choice of normalization \( \beta(\lambda) \) of \( \Psi(\lambda) \) is

\[
\beta_1 = \beta_2 = 0, \quad \beta_3 = 1,
\]

then the polynomial \( B(\lambda) \) is given by

\[
B(\lambda) = s_{32}(\lambda)u_{13}(\lambda) - s_{31}(\lambda)u_{23}(\lambda),
\]

where \( u_{ij}(\lambda) \) is \((ij)\)-cofactor of the determinant of \( L(\lambda) \). The polynomial \( B(\lambda) \) \cite{4.9} allows to introduce the following mapping

\[
\begin{align*}
\bar{s}_{ij} &\rightarrow \bar{s}'_{ij} = s_{ij}, \quad (ij) \neq (13), (23), \\
s_{13} &\rightarrow \bar{s}'_{13} = s_{13} + s_{32}f(\lambda)B^{-1}(\lambda), \\
s_{23} &\rightarrow \bar{s}'_{23} = s_{23} - s_{31}f(\lambda)B^{-1}(\lambda),
\end{align*}
\]

as an analog of the mapping \cite{4.11} with the normalization \cite{4.3}. This mapping leaves fixed the spectral invariants \( \tau_1 \) and \( \tau_2 \) and shifts the third invariant polynomial \( \tau_3 \)

\[
\begin{align*}
\tau_1(\lambda) &= \text{tr}L(\lambda) = 0 \rightarrow \tau'_1(\lambda) = \tau_1(\lambda) = 0, \\
\tau_2(\lambda) &= \text{tr}L^2(\lambda) \rightarrow \tau'_2(\lambda) = \tau_2(\lambda), \\
\tau_3(\lambda) &= \det L(\lambda) \rightarrow \tau'_3(\lambda) = \tau_3(\lambda) + f(\lambda),
\end{align*}
\]

The general Lax matrix \( L'(\lambda) \) defined by \cite{4.10} obeys the linear \( R \)-bracket with the dynamical \( r \)-matrices. As an example, the Lax representations for the Henon-Heiles system and a system with quartic potential \cite{1} can be embedded into the proposed scheme by using the more sophisticated normalization.

So, we can apply the known polynomials \( B(\lambda) \) and method of separation of variables to construct the analogs of mappings \cite{4.11} and \cite{4.20} for the loop algebra \( sl(N) \). It would be interesting to find the general counterparts of outer automorphisms \cite{2.13} and \cite{2.16} for infinite-dimensional representations of \( sl(N) \) and to consider the inverse problem of choosing the correct normalization of Baker-Akhiezer function by using these automorphisms. A similar, but more difficult problem arises for the simple Lie algebras other than \( sl(N) \).

5 Dynamical \( r \)-matrices and quadratic \( R \)-bracket

Now we consider analogs of the additive and multiplicative automorphisms of \( sl(2) \) \cite{2.6}-\cite{2.8} for the quadratic \( R \)-bracket in classical mechanics.

Introduce the formal particular mapping

\[
T(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} 
\]

\( (u) \rightarrow T'(u) = \begin{pmatrix} A + fD^{-1} & B \\ C & D \end{pmatrix} 
\]

\( (u) \).

If the initial matrix \( T(u) \) obeys the standard quadratic \( R \)-bracket

\[
\frac{1}{2} \{ T(u), T(v) \} = [r(u - v), T(u)T(v)], \quad \text{with} \quad r = (u - v)^{-1}\Pi,
\]

\( \frac{1}{2} \)
then the image $T'(u)$ of mapping (5.1) obeys the dynamical quadratic $R$-bracket

$$\langle 1T'(u), 2T'(v) \rangle = \langle r(u-v), T'(u)T'(v) \rangle + T'(v)s_{21}T'(u) - T'(u)s_{12}T'(v).$$

(5.3)

Here dynamical matrices $s_{jk}$ are given by

$$s_{12}(u, v) = \frac{1}{u-v} \left[ \frac{f(u)}{D^2(u)} - \frac{f(v)}{D^2(v)} \right] \cdot \sigma_- \otimes \sigma_+,$$

$$s_{21}(u, v) = \Pi s_{12}(v, u) \Pi.$$

(5.4)

In contrast to the linear case, the mapping (5.1) related to additive automorphism (2.6) changes the form of $R$-bracket. However, if the functions $D(u)$ and $f(u)$ (5.1) are independent on spectral parameter $u$, then the dynamical matrices $s_{jk}$ in (5.3) go to zero [10]. Moreover, in this case, we can use the more general mapping (see (5.1))

$$A(u) \rightarrow A'(u) = A(u) + g(D),$$

where $g(D)$ is an arbitrary function on entry $D$. This mapping changes the standard $R$-bracket (5.2) to dynamical bracket (5.3), but it preserves the property of integrability

$$\{ \text{tr} T'(u), \text{tr} T'(v) \} = 0.$$

We have to emphasize that additive and multiplicative automorphisms of $sl(2)$ (2.6)-(2.8) give rise to the integrable systems associated to the two root systems $BC_n$ and $D_n$ [10], respectively.

Consider the special solutions

$$T_A(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & -A(-u) \end{pmatrix},$$

(5.5)

of the classical reflection equation

$$\langle 1T(u), 2T(v) \rangle = \left[ r(u-v), \frac{1}{T(u)} \frac{2}{T(v)} \right] +$$

$$+ \frac{1}{T(u)r(u+v)} \frac{2}{T(v)r(u+v)} \frac{1}{T(u)}.$$

(5.6)

with the rational $r = (u-v)^{-1} \Pi r$-matrix. An application of the additive and multiplicative automorphisms of $sl(2)$ (2.6)-(2.8) is more tricky in the quadratic case.

If the entry $B(u)$ is independent of spectral parameter $u$, i.e. $\partial B/\partial u = 0$, we can construct the new solution of the reflection equation (5.6)

$$T_C = T_A + \begin{pmatrix} 0 & 0 \\ \gamma B^{-1} & 0 \end{pmatrix}, \quad \gamma \in \mathbb{R}.$$

(5.7)

Assuming in addition that entry $A(u)$ is a linear function of $u$, i.e. $A(u) = ua_1 + a_2$, let us introduce the second boundary matrix

$$T_{BC} = T_C + \begin{pmatrix} \frac{\alpha}{u} + \beta & 0 \\ \alpha(A(u) - A(-u)) + \beta(A(u) + A(-u)) & B^{-1} \frac{\alpha}{u} - \beta \end{pmatrix},$$

(5.8)

$$\alpha, \beta, \in \mathbb{R}$$

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which is new solution of reflection equation (5.6). These solution $T_C$ (5.7) and $T_{BC}$ (5.8) could be associated with the additive automorphism (2.6).

If the central element $\Delta = \det T_A(u) = 0$ is equal to zero and $\partial B/\partial u = 0$ the third boundary matrix

$$T_D = \begin{pmatrix} A(u) - A(-u)B^{-1} & B \cdot (1 - B^{-1})^2 \\ C(u) & -A(-u) + A(u)B^{-1} \end{pmatrix},$$

is a solution of reflection equation (5.6). This solution could be associated with the multiplicative automorphism (2.8).

As an example, we consider the Toda lattices. Let the initial boundary matrix $T_A(u)$ is given by

$$T_A(u) = \begin{pmatrix} (u-p) \exp(q) & \exp(2q) \\ u^2 - p^2 & (u+p) \exp(q) \end{pmatrix},$$

where $(q,p)$ is a pair of canonically conjugate variables. According to [10] matrices $T_A$, $T_{BC}$ and $T_D$ correspond to the Toda lattices associated with the Lie algebras of $A_n$, $B_n (\beta = \gamma = 0)$, $C_n (\alpha = \beta = 0)$ and $D_n$ series, respectively. Among the hamiltonians, in comparison with (1.2), there are

$$H_A = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} \exp(x_{j+1} - x_j),$$

$$H_{BC} = H_A + \gamma \exp(-2x_1) + (2\alpha + 2\beta p_1) \exp(-x_1),$$

$$H_D = H_A + \exp(-x_1 - x_2).$$

In classical and quantum mechanics the boundary matrices (5.8) and (5.9) have been used for the relativistic Toda lattices and the Neumann top, Kowalewski top and Toda lattice associated to the Lie algebra $G_2$.

6 Conclusions

The outer automorphisms of infinite-dimensional representation of $sl(2)$ give possibility to construct new Lax matrices. The corresponding linear and quadratic $R$-brackets include the dynamical $r$-matrices, which obey the dynamical Yang-Baxter equations.

For the loop algebras as a second step we applied the certain projection of general Lax matrix onto the low-dimensional subspaces, which preserve the $R$-bracket. Thus, the set of the Lax matrices associated to the different integrable system can be obtained.

The similar dynamical deformations of quadratic $R$-bracket has been applied for the construction of Lax matrix for integrable systems associated to the root systems $BC_n$ and $D_n$.

Among the known and possible examples there are: a wide class of the Stäckel systems; the integrable extensions of the classical tops - Euler top, Manakov and Steklov tops, Lagrange and Goryachev-Chaplygin top; generalizations of the Heisenberg and Gaudin magnets; the Toda and the Calogero-Moser systems.

The following problems, however, remain open. We have not an exhaustive description of outer automorphisms of infinite-dimensional representations of simple Lie algebras and of the all admissible low-dimensional submanifolds, which allow a well-defined restriction to the corresponding linear classical and quantum $R$-brackets.
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