ON THE ASYMPTOTIC BEHAVIOR
OF WEAKLY LACUNARY SERIES

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ABSTRACT. Let $f$ be a measurable function satisfying
\[ f(x + 1) = f(x), \quad \int_0^1 f(x) \, dx = 0, \quad \text{Var}_{[0, 1]} f < +\infty, \]
and let $(n_k)_{k \geq 1}$ be a sequence of integers satisfying $n_{k+1}/n_k \geq q > 1$ ($k = 1, 2, \ldots$). By the classical theory of lacunary series, under suitable Diophantine conditions on $n_k$, $(f(n_k x))_{k \geq 1}$ satisfies the central limit theorem and the law of the iterated logarithm. These results extend for a class of subexponentially growing sequences $(n_k)_{k \geq 1}$ as well, but as Fukuyama showed, the behavior of $f(n_k x)$ is generally not permutation-invariant; e.g. a rearrangement of the sequence can ruin the CLT and LIL. In this paper we construct an infinite order Diophantine condition implying the permutation-invariant CLT and LIL without any growth conditions on $(n_k)_{k \geq 1}$ and show that the known finite order Diophantine conditions in the theory do not imply permutation-invariance even if $f(x) = \sin 2\pi x$ and $(n_k)_{k \geq 1}$ grows almost exponentially. Finally, we prove that in a suitable statistical sense, for almost all sequences $(n_k)_{k \geq 1}$ growing faster than polynomially, $(f(n_k x))_{k \geq 1}$ has permutation-invariant behavior.

1. INTRODUCTION

Let $f$ be a measurable function satisfying
\[ f(x + 1) = f(x), \quad \int_0^1 f(x) \, dx = 0, \quad \text{Var}_{[0, 1]} f < +\infty \]
and let $(n_k)_{k \geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition
\[ n_{k+1}/n_k \geq q > 1 \quad (k = 1, 2, \ldots). \]
In the case $n_k = 2^k$, Kac [14] proved that $f(n_k x)$ satisfies the central limit theorem
\[ N^{-1/2} \sum_{k=1}^N f(n_k x) \xrightarrow{D} N(0, \sigma^2) \]
with respect to the probability space [0, 1] equipped with the Lebesgue measure, where
\[ \sigma^2 = \int_0^1 f^2(x) \, dx + 2 \sum_{k=1}^{\infty} \int_0^1 f(x)f(2^k x) \, dx. \]

Gaposhkin \cite{11} extended (3) to the case when the fractions \( n_{k+1}/n_k \) are all integers or if \( n_{k+1}/n_k \to \alpha \), where \( \alpha \) is irrational for \( r = 1, 2, \ldots. \) On the other hand, an example of Erdős and Fortet (see \cite{15}, p. 646) shows that the CLT (3) fails if \( n_k = 2^k - 1. \) Gaposhkin also showed (see \cite{12}) that the asymptotic behavior of \( \sum_{k=1}^N f(n_k x) \) is intimately connected with the number of solutions of the Diophantine equation
\[ an_k + bn_l = c, \quad 1 \leq k, l \leq N. \]

Improving these results, Aistleitner and Berkes \cite{1} gave a necessary and sufficient condition for the CLT (3). For related laws of the iterated logarithm, see \cite{5}, \cite{11}, \cite{13}, and \cite{17}.

The previous results show that for arithmetically “nice” sequences \( (n_k)_{k \geq 1} \), the system \( f(n_k x) \) behaves like a sequence of independent random variables. However, as an example of Fukuyama \cite{9} shows, this result is not permutation-invariant: a rearrangement of \( (n_k)_{k \geq 1} \) can change the variance of the limiting Gaussian law or ruin the CLT altogether. A complete characterization of the permutation-invariant CLT and LIL for \( f(n_k x) \) under the Hadamard gap condition (1) is given in our forthcoming paper \cite{3}. In particular, it is shown there that in the harmonic case \( f(x) = \cos 2\pi x, f(x) = \sin 2\pi x \) the CLT and LIL for \( f(n_k x) \) hold after any permutation of \( (n_k)_{k \geq 1}. \)

For subexponentially growing \( (n_k)_{k \geq 1} \) the situation changes radically. Note that in the case \( f(x) = \cos 2\pi x, f(x) = \sin 2\pi x, \) the unpermuted CLT and LIL remain valid under the weaker gap condition
\[ n_{k+1}/n_k \geq 1 + ck^{-\alpha}, \quad 0 < \alpha < 1/2; \]
see Erdős \cite{8}, Takahashi \cite{18}, \cite{19}. However, as the following theorem shows, the slightest weakening of the Hadamard gap condition (2) can ruin the permutation-invariant CLT and LIL.

**Theorem 1.** For any positive sequence \( (\varepsilon_k)_{k \geq 1} \) tending to 0, there exists a sequence \( (n_k)_{k \geq 1} \) of positive integers satisfying
\[ n_{k+1}/n_k \geq 1 + \varepsilon_k, \quad k \geq k_0 \]
and a permutation \( \sigma : \mathbb{N} \to \mathbb{N} \) of the positive integers such that
\[ N^{-1/2} \sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x - b_N \overset{D}{\to} G, \]
where \( G \) is a non-Gaussian distribution with characteristic function given by (14) \cite{10} and \( (b_N)_{N \geq 1} \) is a numerical sequence with \( b_N = O(1) \). Moreover, there exists a permutation \( \sigma : \mathbb{N} \to \mathbb{N} \) of the positive integers such that
\[ \limsup_{N \to \infty} \frac{\sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x}{\sqrt{2N \log \log N}} = +\infty \quad \text{a.e.} \]
As we will see, with a slight change of the norming sequence \( N^{-1/2} \) in (5) the limit distribution \( G \) can also be chosen as the Cauchy distribution with density \( \pi^{-1}(1+x^2)^{-1} \). The proof of Theorem 1 will also show that (6) can be improved to

\[
\limsup_{N \to \infty} \frac{\sum_{k=1}^{N} \cos 2\pi \sigma(k)x}{\sqrt{N \log N}} > 0 \quad \text{a.e.}
\]

It should be noted that the subexponential gap condition (4) does not imply the permutation-invariant behavior of \( f(n_k x) \) even for arithmetically “nice” sequences \((n_k)_{k \geq 1}\). Indeed, the sequence \((n_k)_{k \geq 1}\) in Theorem 1 can be chosen so that it satisfies conditions B, C, G in our paper [7] implying very strong independence properties of \( \cos 2\pi n_k x, \sin 2\pi n_k x \), including the CLT and LIL. In fact, it is not easy to construct subexponential sequences \((n_k)_{k \geq 1}\) satisfying the permutation-invariant CLT and LIL: the only known example (see [2]) is the Hardy-Littlewood-Pólya sequence, i.e. the sequence generated by finitely many primes and arranged in increasing order; the proof uses deep number-theoretic tools. The purpose of this paper is to introduce a new, infinite order Diophantine condition \( A_\omega \) which implies the permutation-invariant CLT and LIL for \( f(n_k x) \) and then to show that, in a suitable statistical sense, almost all sequences \((n_k)_{k \geq 1}\) growing faster than polynomially satisfy \( A_\omega \). Thus, despite the difficulties to construct explicit examples, the permutation-invariant CLT and LIL are rather the rule than the exception.

Given a nondecreasing sequence \( \omega = (\omega_1, \omega_2, \ldots) \) of positive numbers tending to \(+\infty\), let us say that a sequence \((n_k)_{k \geq 1}\) of different positive integers satisfies

**Condition \( A_\omega \)** if for any \( N \geq N_0 \) the Diophantine equation

\[
a_1 n_{k_1} + \ldots + a_r n_{k_r} = 0, \quad 2 \leq r \leq \omega_N, \quad 0 < |a_1|, \ldots, |a_r| \leq N^{\omega_N}
\]

with different indices \( k_j \) and nonzero integer coefficients \( a_j \) has only such solutions, where all \( n_{k_j} \) belong to the smallest \( N \) elements of the sequence \((n_k)_{k \geq 1}\).

Clearly, this property is permutation-invariant and it implies that for any fixed nonzero integer coefficients \( a_j \) the number of solutions of (8) with different indices \( k_j \) is at most \( N^r \).

**Theorem 2.** Let \( \omega = (\omega_1, \omega_2, \ldots) \) be a nondecreasing sequence tending to \(+\infty\) and let \((n_k)_{k \geq 1}\) be a sequence of different positive integers satisfying condition \( A_\omega \). Then for any \( f \) satisfying (11) we have

\[
N^{-1/2} \sum_{k=1}^{N} f(n_k x) \overset{P}{\to} \mathcal{N}(0, \|f\|^2),
\]

where \( \|f\| \) denotes the \( L_2(0,1) \) norm of \( f \). If \( \omega_k \geq (\log k)^\alpha \) for some \( \alpha > 0 \) and \( k \geq k_0 \), then we also have

\[
\limsup_{N \to \infty} \frac{\sum_{k=1}^{N} f(n_k x)}{(2N \log \log N)^{1/2}} = \|f\| \quad \text{a.e.}
\]

Condition \( A_\omega \) is different from the usual Diophantine conditions in lacunarity theory, which typically involve 4 or fewer terms. In contrast, \( A_\omega \) is an “infinite order” condition; namely, it involves equations with arbitrary large order. As noted, the usual Diophantine conditions do not suffice in Theorem 2. Given any \( \omega_k \uparrow \infty \), it is not hard to see that any sufficiently rapidly growing sequence \((n_k)_{k \geq 1}\) satisfies \( A_\omega \); on the other hand, we do not have any “concrete” subexponential examples for
Theorem 3. Let \( g \) growing faster than polynomially satisfy condition \( A_\omega \) for some appropriate \( \omega \).

However, we will show that, in a suitable statistical sense, almost all sequences growing faster than polynomially satisfy condition \( A_\omega \) for some appropriate \( \omega \).

To make this precise requires defining a probability measure over the set of such sequences, or, equivalently, a natural random procedure to generate such sequences. A simple procedure is to choose \( n_k \) independently and uniformly from the integers in the interval

\[
I_k = [a(k - 1)^{\omega_k - 1}, ak^{\omega_k}), \quad k = 1, 2, \ldots
\]

Note that the length of \( I_k \) is at least \( a_\omega (k - 1)^{\omega_k - 1} \geq a_\omega \) for \( k = 2, 3, \ldots \) and equals \( a \) for \( k = 1 \), and thus choosing \( a \) large enough, each \( I_k \) contains at least one integer. Let \( \mu_\omega \) be the distribution of the random sequence \((n_k)_{k \geq 1}\) in the product space \( I_1 \times I_2 \times \ldots \).

**Theorem 3.** Let \( \omega_k \uparrow \infty \) and let \( f \) be a function satisfying (11). Then with probability one with respect to \( \mu_\omega \), the sequence \((f(n_k))_{k \geq 1}\) satisfies the CLT (10) after any permutation of its terms, and if \( \omega_k \geq (\log k)^{\alpha} \) for some \( \alpha > 0 \) and \( k \geq k_0 \), \((f(n_k))_{k \geq 1}\) also satisfies the LIL (11) after any permutation of its terms.

The sequences \((n_k)_{k \geq 1}\) provided by \( \mu_\omega \) satisfy \( n_k = O(k^{\omega_k}) \); for slowly increasing \( \omega_k \), the so-obtained sequences grow much slower than exponentially, in fact they grow barely faster than polynomial speed. If \( \omega_k \) grows so slowly that \( \omega_k - \omega_{k-1} = o((\log k)^{-1}) \), then the so-obtained sequence \((n_k)_{k \geq 1}\) has the precise speed \( n_k \sim k^{\omega_k} \).

We do not know if there exist polynomially growing sequences \((n_k)_{k \geq 1}\) satisfying the permutation-invariant CLT or LIL. The proof of Theorem 3 will also show that with probability 1, the sequences provided by \( \mu_\omega \) satisfy \( A_{\omega^*} \) with \( \omega^* = (\omega_1^{1/2}, \omega_2^{1/2}, \ldots) \).

2. Proofs

2.1. Proof of Theorem 1. We begin with the CLT part. Let \((\varepsilon_k)_{k \geq 1}\) be a positive sequence tending to 0. Let \( m_1 < m_2 < \ldots \) be positive integers such that \( m_{k+1}/m_k \geq 2^k \), \( k = 1, 2, \ldots \) and all the \( m_k \) are powers of 2; let \( r_1 \leq r_2 \leq \ldots \) be positive integers satisfying \( 1 \leq r_k \leq k^2 \). Put \( I_k = \{m_k, 2m_k, \ldots, r_km_k\} \); clearly the sets \( I_k \), \( k = 1, 2, \ldots \) are disjoint. Define the sequence \((n_k)_{k \geq 1}\) by

\[
(n_k)_{k \geq 1} = \bigcup_{j=1}^{\infty} I_j.
\]

Clearly, if \( n_k, n_{k+1} \in I_j \), then \( n_{k+1}/n_k \geq 1 + 1/r_j \) and thus if \( r_j \) grows sufficiently slowly, the sequence \((n_k)_{k \geq 1}\) satisfies the gap condition (4). Also, if \( r_j \) grows sufficiently slowly, there exists a subsequence \((n_{k_i})_{i \geq 1}\) of \((n_k)_{k \geq 1}\) which has exactly the same structure as the sequence in (12), just with \( r_k \sim k \). By the proof of Theorem 1 in [4], \((\cos 2\pi n_{k_i}x)_{i \geq 1}\) satisfies

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \cos 2\pi n_{k_i}x - b_N \overset{D}{\rightarrow} G,
\]
As in [6, Lemma 2.1], the random variables 

\[ \sigma \] 

and thus 

\[ \text{satisfying (4) that contains a subsequence } (\sigma) \] 

where \((\sigma)\) in Berkes and Philipp [6]. Similarly as above, we construct a sequence \((\sigma)\) of Theorem 1. 

Then 

\[ \exp \left\{ \int_{R \setminus \{0\}} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) dL(x) \right\} , \] 

where 

\[ L(x) = \begin{cases} 
-\frac{1}{\pi} \int_{-1}^{0} F(t) dt & \text{if } 0 < x \leq 1, \\
\frac{1}{\pi} \int_{-1}^{0} G(t) dt & \text{if } -1 \leq x < 0, \\
0 & \text{if } |x| > 1 
\end{cases} \] 

and 

\[ F(t) = \lambda\{x > 0 : \sin x/x \geq t\}, \quad G(t) = \lambda\{x > 0 : \sin x/x \leq -t\} \quad (t > 0), \] 

where \(\lambda\) is the Lebesgue measure. Define a permutation \(\sigma\) in the following way: 

- for \(k \notin \{1, 2, 4, \ldots, 2^m, \ldots\}\), \(\sigma(k)\) takes the values of the set \(\{k_1, k_2, \ldots\}\) in consecutive order; 
- for \(k \in \{1, 2, 4, \ldots, 2^m, \ldots\}\), \(\sigma(k)\) takes the values of the set \(N \setminus \{k_1, k_2, \ldots\}\) in consecutive order. 

Then \(\sigma\) is a permutation of \(N\) and the sums 

\[ \sum_{k=1}^{N} \cos 2\pi n_{\sigma(k)} x \quad \text{and} \quad \sum_{i=1}^{N} \cos 2\pi n_{ki} x \] 

differ at most in \(2 \log_2 N\) terms. Therefore, \([1.3]\) implies \([1.5]\), proving the first part of Theorem 1. 

The proof of the LIL part of Theorem 1 is modeled after the proof of Theorem 1 in Berkes and Philipp [6]. Similarly as above, we construct a sequence \((n_k)_{k \geq 1}\) satisfying \([4]\) that contains a subsequence \((\mu_k)_{k \geq 1}\) of the form \([12]\) with 

\[ I_k = \{m_k, 2m_k, \ldots, r_km_k\}, \] 

where \(r_k \sim k \log k\) and \((m_k)_{k \geq 1}\) is growing fast; specifically we choose \(m_k\) in such a way that it is a power of 2 and \(m_{k+1} \geq r_k 2^{2k} m_k\). Let \(\mathcal{F}_i\) denote the \(\sigma\)-field generated of the dyadic intervals 

\[ \nu 2^{-(\log_2 m_i - i)}, (\nu + 1)2^{-(\log_2 m_i - i)}, \quad 0 \leq \nu < 2^{(\log_2 m_i) + i}. \] 

Write 

\[ X_i = \cos 2\pi m_i x + \cdots + \cos 2\pi r_i m_i x \] 

and 

\[ Z_i = \mathbb{E}(X_i | \mathcal{F}_i). \] 

Then for all \(x \in (0, 1), \) 

\[ |X_i(x) - Z_i(x)| \ll r_i^2 m_i 2^{-(\log_2 m_i) - i} \] 

and thus 

\[ \sum_{i \geq 1} |X_i(x) - Z_i(x)| < \infty \quad \text{for all } x \in (0, 1). \] 

As in [6, Lemma 2.1], the random variables \(Z_1, Z_2, \ldots\) are independent, and as in [6, Lemma 2.2], for almost every \(x \in (0, 1)\) we have 

\[ \limsup_{i \to \infty} X_i/r_i \geq 2/\pi. \]
Assume that for a fixed $x$ and some $i \geq 1$ we have $X_i/r_i \geq 1/\pi$. Then either

$$|X_1 + \cdots + X_{i-1}| \geq r_i/2\pi$$

or

$$|X_1 + \cdots + X_i| \geq r_i/2\pi.$$

Since the total number of summands in $X_1, \ldots, X_i$ is $\ll i^2 \log i$, and since $r_i \gg (i^2 \log i)^{1/2}(\log(i^2 \log i))^{1/2}$, we have

$$\limsup_{N \to \infty} \frac{\left| \sum_{k=1}^{N} \cos 2\pi \mu_k x \right|}{\sqrt{N \log N}} > 0 \ a.e.,$$

and, in particular,

$$\limsup_{N \to \infty} \frac{\left| \sum_{k=1}^{N} \cos 2\pi \mu_k x \right|}{\sqrt{2N \log \log N}} = +\infty \ a.e.$$

Thus we constructed a subsequence of $(n_k)_{k \geq 1}$ failing the LIL and similarly as above, we can construct a permutation $(n_{\sigma(k)})_{k \geq 1}$ of $(n_k)$ failing the LIL as well.

Note that the just completed proof of Theorem 1 provides the stronger relation (7) instead of (6). Also, if relation $r_k \sim k$ just preceding relation (13) is replaced by $r_k \sim k \log \log k$, then the proof of Theorem 2 in [4] shows that (5) will be replaced by

$$a_N^{-1} \sum_{k=1}^{N} \cos 2\pi n_{\sigma(k)} x = b_N \to G,$$

where $G$ is the Cauchy distribution with density $\pi^{-1}(1 + x^2)^{-1}$ and $a_N \sim c\sqrt{N/\log \log N}$ with some $c > 0$ and $b_N = O(1)$.

2.2. Proof of Theorem 2

Lemma 1. Let $\omega_k \uparrow \infty$ and let $(n_k)_{k \geq 1}$ be a sequence of different positive integers satisfying condition $A_{\omega}$. Let $f$ satisfy (1) and put $S_N = \sum_{k=1}^{N} f(n_k x)$, $\sigma_N = (\text{ES}_N^2)^{1/2}$. Then for any $p \geq 3$ we have

$$\mathbb{E} S_N^p = \left\{ \begin{array}{ll} \frac{\mu^p}{(p/2)!} 2^{-p/2} \sigma_N^p + O(T_N) & \text{if } p \text{ is even,} \\ O(T_N) & \text{if } p \text{ is odd,} \end{array} \right.$$}

where

$$T_N = \exp(p^2)N^{(p-1)/2}(\log N)^p$$

and the constants implied by the $O$ are absolute.

Proof. Fix $p \geq 2$ and choose the integer $N$ so large that $\omega_{\lceil N^{1/4} \rceil} \geq 8p$. Without loss of generality we may assume that $f$ is an even function and that $\|f\|_{\infty} \leq 1$, $\text{Var}_{[0,1]} f \leq 1$; the proof in the general case is similar. Let

$$f \sim \sum_{j=1}^{\infty} a_j \cos 2\pi jx$$

be the Fourier series of $f$. $\text{Var}_{[0,1]} f \leq 1$ implies that

$$|a_j| \leq j^{-1}$$
(see Zygmund [20, p. 48]), and writing
\[ g(x) = \sum_{j=1}^{N_p} a_j \cos 2\pi j x, \quad r(x) = f(x) - g(x), \]
we have
\[ \|g\|_\infty \leq \text{Var}_{[0,1]} f + \|f\|_\infty \leq 2, \quad \|r\|_\infty \leq \|f\|_\infty + \|g\|_\infty \leq 3 \]
by (4.12) of Chapter II and (1.25) and (3.5) of Chapter III of Zygmund [20]. Letting \( \| \cdot \| \) and \( \| \cdot \|_p \) denote the \( L_2(0,1) \), resp. \( L_p(0,1) \), norms, (18) yields for any positive integer \( n \),
\[ \|r(nx)\|^2 = \|r(x)\|^2 = \frac{1}{2} \sum_{j=N_p+1}^\infty a_j^2 \leq N^{-p}. \]
By Minkowski’s inequality,
\[ \|S_N\|_p \leq \left\| \sum_{k=1}^N g(n_k x) \right\|_p + \left\| \sum_{k=1}^N r(n_k x) \right\|_p \]
and
\[ \sum_{k=1}^N \|r(n_k x)\|_p \leq 3 \sum_{k=1}^N \|r(n_k x)\|_p \leq 3 \sum_{k=1}^N \|r(n_k x)\|^2/p \leq 3 \sum_{k=1}^N N^{-1} \leq 3. \]
Similarly,
\[ \left\| S_N - \sum_{k=1}^N g(n_k x) \right\| \leq \left\| \sum_{k=1}^N r(n_k x) \right\| \leq N^{-\frac{p}{2}+1}, \]
and therefore
\[ \left\| S_N \right\|^p - \left\| \sum_{k=1}^N g(n_k x) \right\|_p \leq p \max \left( \left\| S_N \right\|^{p-1}, \left\| \sum_{k=1}^N g(n_k x) \right\|^{p-1} \right) \left\| S_N \right\| - \left\| \sum_{k=1}^N g(n_k x) \right\| \leq p \left( N(\log \log N)^2 \right)^{\frac{p-1}{2}} N^{-\frac{p}{2}+1} \ll p (\log \log N)^{p-1} N^{1/2} \]
since by a result of Gál [10] and Koksma [16],
\[ \|S_N\|^2 \ll N(\log \log N)^2 \quad \text{and} \quad \sum_{k=1}^N \|g(n_k x)\|^2 \ll N(\log \log N)^2, \]
where the implied constants are absolute.

By expanding and using elementary properties of the trigonometric functions we get
\[ E \left( \sum_{k=1}^N g(n_k x) \right)^p \]
\[ = 2^{-p} \sum_{1 \leq j_1, \ldots, j_p \leq N^p} a_{j_1} \cdots a_{j_p} \sum_{1 \leq k_1, \ldots, k_p \leq N} \mathbb{I}\{\pm j_1 n_{k_1} \pm \ldots \pm j_p n_{k_p} = 0\}, \]

with all possibilities of the signs ± within the indicator function. Assume that $j_1, \ldots, j_p$ and the signs ± are fixed, and consider a solution of $\pm j_1 n_{k_1} \pm \cdots \pm j_p n_{k_p} = 0$. Then the set $\{1, 2, \ldots, p\}$ can be split into disjoint sets $A_1, \ldots, A_l$ such that for each such set $A$ we have $\sum_{i \in A} \pm j_i n_{k_i} = 0$ and no further subsums of these sums are equal to 0. Group the terms of $\sum_{i \in A} \pm j_i n_{k_i}$ with equal $k_i$. If after grouping there are at least two terms, then by the restriction on subsums, the sum of the coefficients $j_i$ in each group will be different from 0 and will not exceed

$$pN^p \leq \omega_{[N^{1/4}]} N^{\frac{1}{2} \omega_{[N^{1/4}]}} \leq 2^{\frac{1}{2} \omega_{[N^{1/4}]}} N^{\frac{1}{2} \omega_{[N^{1/4}]}} \leq N^{\frac{1}{2} \omega_{[N^{1/4}]}} \quad (N \geq N_0).$$

Also the number of terms after grouping will be at most $p \leq \omega_{[N^{1/4}]}$ and thus applying condition $A_\omega$ with the index $[N^{1/4}]$ shows that within a block $A$ the $n_{k_i}$ belong to the smallest $[N^{1/4}]$ terms of the sequence. Thus letting $|A| = m$, the number of solutions of $\sum_{i \in A} \pm j_i n_{k_i} = 0$ is at most $N^{m/4}$. If after grouping there is only one term, then all the $k_i$ are equal and thus the number of solutions of $\sum_{i \in A} \pm j_i n_{k_i} = 0$ is at most $N$. Thus if $m \geq 3$, then the number of solutions of $\sum_{i \in A} \pm j_i n_{k_i} = 0$ in a block is at most $N^{m/3}$. If $m = 2$, then the number of solutions is clearly at most $N$. Thus if $s_i = |A_i| \quad (1 \leq i \leq l)$ denotes the cardinality of $A_i$, the number of solutions of $\pm j_1 n_{k_1} \pm \cdots \pm j_p n_{k_p} = 0$ admitting such a decomposition with fixed $A_1, \ldots, A_l$ is at most

$$\prod_{\{i: s_i \geq 3\}} N^{s_i/3} \prod_{\{i: s_i = 2\}} N = N^{\frac{1}{3} \sum_{\{i: s_i \geq 3\}} s_i + \frac{1}{3} \sum_{\{i: s_i = 2\}} s_i} = N^{\frac{1}{3} \sum_{\{i: s_i \geq 3\}} s_i + \frac{1}{3} \sum_{\{i: s_i \geq 3\}} (p - \sum_{\{i: s_i \geq 3\}} s_i)} = N^{\frac{1}{3} - \frac{1}{3} \sum_{\{i: s_i \geq 3\}} s_i}.$$

If there is at least one $i$ with $s_i \geq 3$, then the last exponent is at most $(p - 1)/2$ and since the number of partitions of the set $\{1, \ldots, p\}$ into disjoint subsets is at most $p! 2^p$, we see that the number of solutions of $\pm j_1 n_{k_1} \pm \cdots \pm j_p n_{k_p} = 0$, where at least one of the sets $A_i$ has cardinality $\geq 3$, is at most $p! 2^p N^{(p-1)/2}$. If $p$ is odd, there are no other solutions and thus using (18) the inner sum in (23) is at most $p! 2^p N^{(p-1)/2}$ and consequently, taking into account the $2^p$ choices for the signs ±1,

$$\left| \mathbb{E} \left( \sum_{k \leq N} g(n_k x) \right)^p \right| \leq p! 2^p N^{(p-1)/2} \sum_{1 \leq j_1, \ldots, j_p \leq N^p} |a_{j_1} \cdots a_{j_p}| \ll \exp(p^2) N^{(p-1)/2} (\log N)^p.$$
If \( p \) is even, there are also solutions where each \( A \) has cardinality 2. Clearly, the
correction of the terms in (23), where \( A_1 = \{1, 2\}, A_2 = \{3, 4\}, \ldots, \) is
\[
\left( \frac{1}{4} \sum_{1 \leq i, j \leq N^{2p}} \sum_{1 \leq k, \ell \leq N} a_i a_j \mathbb{I} \{ \pm in_k \pm jn_\ell = 0 \} \right)^{p/2} = \left( \mathbb{E} \left( \sum_{k \leq N} g(n_k x) \right)^2 \right)^{p/2}
\]
\[
= \left\| \sum_{k \leq N} g(n_k x) \right\|^p
\]
\[
= \| S_N \|^p + O \left( p(\log \log N)^{p-1} N^{1/2} \right)
\]
by (21).

Since the splitting of \( \{1, 2, \ldots, p\} \) into pairs can be done in
\( \frac{p!}{(p/2)!^2} \) different ways, we proved that
\[
(24) \quad \mathbb{E} \left( \sum_{k \leq N} g(n_k x) \right)^p = \begin{cases} 
\frac{p!}{(p/2)!^2} 2^{-\frac{p}{2}} \sigma_N^p + O(T_N), & \text{if } p \text{ is even} \\
O(T_N), & \text{if } p \text{ is odd}
\end{cases}
\]
according as \( p \) is even or odd; here
\[
T_N = \exp(p^2) N^{(p-1)/2} (\log N)^p.
\]
Now, letting \( G_N = \sum_{k \leq N} g(n_k x) \) we get, using (20), (22) and (24),
\[
| \mathbb{E} S_N^p - \mathbb{E} G_N^p | \leq p \max \left( \| S_N \|^p - 1, \| G_N \|^p - 1 \right) \cdot \| S_N \|_p - \| G_N \|_p \\
\ll p \left( \frac{p!}{(p/2)!^2} 2^{-\frac{p}{2}} \sigma_N^p \right)^{p-1} \\
\ll T_N,
\]
where in the last step we used the fact that \( \sigma_N \ll \sqrt{\log \log N} \) by (22).
This completes the proof of Lemma 1. \( \Box \)

**Lemma 2.** Let \( \omega \uparrow \infty \) and let \( (n_k)_{k \geq 1} \) be a sequence of different positive integers
satisfying condition \( A_\omega \). Then for any \( f \) satisfying (1) we have
\[
(25) \quad \int_0^1 \left( \sum_{k=1}^N f(n_k x) \right)^2 \, dx \sim \| f \|^2 N \quad \text{as } N \to \infty.
\]

**Proof.** Clearly, \( \omega_{[N^{1/4}]} \geq 4 \) for sufficiently large \( N \) and thus applying Condition \( A_\omega \)
for the index \( [N^{1/4}] \) it follows that for \( N \geq N_0 \) the Diophantine equation
\[
(26) \quad j_1 n_{i_1} + j_2 n_{i_2} = 0, \quad i_1 \neq i_2, \quad 0 < |j_1|, |j_2| \leq N
\]
has only such solutions, where \( n_{i_1}, n_{i_2} \) belong to the set \( J_N \) of \( [N^{1/4}] \) smallest
elements of the sequence \( (n_k)_{k \geq 1} \). Write \( p_N(x) \) for the \( N \)-th partial sum of the
Fourier series of \( f \), and \( r_N \) for the \( N \)-th remainder term. Then we have for any \( f \)
satisfying (1),
\begin{equation}
\left\| \sum_{k=1}^{N} f(n_k x) \right\| \geq \left\| \sum_{k \in [1, N] \setminus J_N} p_N(n_k) \right\| - \left\| \sum_{k \in J_N} f(n_k x) \right\| - \left\| \sum_{k \in [1, N] \setminus J_N} r_N(n_k) \right\|.
\end{equation}

Using the previous remark on the number of solutions of (26) we get, as in (23),
\begin{equation}
\left\| \sum_{k \in [1, N] \setminus J_N} p_N(n_k) \right\| = (N - [N^{1/4}])^{1/2} \|p_N\| \sim N^{1/2} \|f\|,
\end{equation}

since \( \|p_N\| \to \|f\| \). Further, \( \|r_N\| \ll N^{-1/2} \) by (18) and thus using Minkowski’s inequality and the results of Gál and Koksma mentioned in (22), we get
\begin{equation}
\left\| \sum_{k \in J_N} f(n_k x) \right\| \ll N^{1/4}, \quad \left\| \sum_{k \in [1, N] \setminus J_N} r_N(n_k) \right\| \ll \sqrt{N} \log \log N \|r_N\| \ll \log \log N.
\end{equation}

These estimates, together with (27), prove Lemma 2.

Lemma 1 and Lemma 2 imply that for any fixed \( p \geq 2 \), the \( p \)-th moment of \( S_N/\sigma_N \) converges to \( \frac{p!}{(p/2)!} \left[ \log \log N \right]^{2-p/2} \) if \( p \) is even and to 0 if \( p \) is odd; in other words, the moments of \( S_N/\sigma_N \) converge to the moments of the standard normal distribution. By \( \sigma_N \sim \|f\| \sqrt{N} \) and a well-known result in probability theory, this proves the CLT part of Theorem 2. The proof of the LIL part of Theorem 2 is more involved, and we will give just a sketch of the proof. The details can be modeled after the proof of Theorem 1. The crucial ingredient is Lemma 3 below, which yields the LIL part of Theorem 2 just as Theorem 1 follows from Lemma 3.

Let \( \theta > 1 \) and define \( \Delta_M' = \{ k \in \mathbb{N} : \theta^{M} < k \leq \theta^{M+1} \} \) and \( T_M' = \sum_{k \in \Delta_M'} f(n_k x) \). By the standard method of proof of the LIL, we need precise bounds for the tails of \( T_M' \) and also, a near independence relation for the \( T_M' \) for the application of the Borel-Cantelli lemma in the lower half of the LIL. From the set \( \Delta_M' \) we remove its \( \theta^{M/2} \) elements with the smallest value of \( n_k \) (recall that the sequence \( (n_k)_{k \geq 1} \) is not assumed to be increasing) and denote the remaining set by \( \Delta_M \). Since the number of removed elements is \( \ll |\Delta_M'|^{1/4} \), this operation does not influence the partial sum asymptotics of \( T_M' \). As in the proof of Lemma 1 we assume that we have a representation of \( f \) in the form (17) and that (15) holds. Define
\begin{equation}
g_M(x) = \sum_{j=1}^{\lfloor \theta^{M/2} \rfloor} a_j \cos 2\pi j x, \quad \sigma_M^2 = \int_0^1 \left( \sum_{k \in \Delta_M} g_M(n_k x) \right)^2 dx
\end{equation}

and
\begin{equation}
T_M = \sum_{k \in \Delta_M} g_M(n_k x), \quad Z_M = T_M/\sigma_M.
\end{equation}

From Lemma 2 it follows easily that
\begin{equation}
\sigma_M \gg |\Delta_M|^{1/2}.
\end{equation}

Assume that \( (n_k)_{k \geq 1} \) satisfies Condition \( A_\omega \) for a sequence \( (\omega_k)_{k \geq 1} \) with \( \omega_k \geq (\log k)^\alpha \) for some \( \alpha > 0, k \geq k_0 \). Without loss of generality we may assume
0 < \alpha < 1/2. Choose \delta > 0 so small that for sufficiently large \( r \),

\[
(29) \quad \left( \log \theta^{\sqrt{r}/4} \right)^{\alpha} > 4 \left( \log \theta \right)^{\delta}.
\]

**Lemma 3.** For sufficiently large \( M, N \) satisfying \( N^{1-\alpha/2} \leq M \leq N \), and for positive integers \( p, q \) satisfying \( p + q \leq (\log N)^{\delta} \) we have

\[
EZ_{M}^{p} Z_{N}^{q} = \begin{cases} 
\frac{p!}{(p/2)!2^{p/2} (q/2)!2^{q/2}} + O(R_{M,N}) & \text{if } p, q \text{ are even}, \\
O(R_{M,N}) & \text{otherwise},
\end{cases}
\]

where

\[
R_{M,N} = 2^{p+q}(p + q)! (\log M)^{p+q} |\Delta M|^{-1/2}.
\]

**Proof.** Note that \( N \leq M^{1+3\alpha/4} \) and thus, setting \( L = [\theta^{M/4}] \), relation \( p + q \leq (\log N)^{\delta} \) and \( (29) \) imply

\[
p + q \leq \frac{1}{4} (\log \theta^{\sqrt{N}/4})^{\alpha} \leq \frac{1}{4} (\log \theta^{M/4})^{\alpha} \leq \omega L
\]

and by a simple calculation,

\[
(p + q) \left( \theta^{N} \right)^{2} \leq [\theta^{M/4}]^{(\log (\theta^{M/4}))^{\alpha}} \leq L^{\omega L}
\]

provided \( N \) is large enough. Applying condition \( A_{\omega} \) we get that for all solutions of the equation

\[
(30) \quad \pm j_{1} n_{k_{1}} \pm \cdots \pm j_{p} n_{k_{p}} \pm j_{p+1} n_{k_{p+1}} \pm \cdots \pm j_{p+q} n_{k_{p+q}} = 0
\]

with different indices \( k_{1}, \ldots, k_{p+q} \), where

\[
1 \leq j_{i} \leq (p + q) \left( \theta^{N} \right)^{2},
\]

the \( n_{k_{i}} \) belong to the \( \left[ \theta^{M/4} \right] \) smallest elements of \( (n_{k})_{k \geq 1} \). By construction not a single one of these elements is contained in \( \Delta_{M} \) or \( \Delta_{N} \). Thus the equation \( (30) \) subject to \( (31) \) has no solution \( (k_{1}, \ldots, k_{p+q}) \), where \( k_{1}, \ldots, k_{p+q} \) are different and satisfy

\[
k_{1}, \ldots, k_{p} \in \Delta_{M}, \quad k_{p+1}, \ldots, k_{p+q} \in \Delta_{N}.
\]

Now

\[
EZ_{M}^{p} Z_{N}^{q} = \frac{2^{-p-q}}{\sigma_{M}^{-p} \sigma_{N}^{-q}} \sum_{1 \leq j_{1}, \ldots, j_{p} \leq \theta^{M/2}, \, 1 \leq j_{p+1}, \ldots, j_{p+q} \leq \theta^{N/2}} \sum_{k_{1}, \ldots, k_{p} \in \Delta_{M}, \, k_{p+1}, \ldots, k_{p+q} \in \Delta_{N}} a_{j_{1}} \cdots a_{j_{p+q}} 1 \{ \pm j_{1} n_{k_{1}} \pm \cdots \pm j_{p+q} n_{k_{p+q}} = 0 \}.
\]

If for some \( k_{1}, \ldots, k_{p+q} \) we have (note that in \( (32) \) these indices need not be different)

\[
(33) \quad \pm j_{1} n_{k_{1}} \pm \cdots \pm j_{p+q} n_{k_{p+q}} = 0,
\]

then grouping the terms of the equation according to identical indices, we get a new equation of the form

\[
\sum_{i=1}^{s} j_{i}^{l_{i}} m_{i} = 0, \quad l_{1} < \ldots < l_{s}, \quad s \leq p + q, \quad j_{i}^{l_{i}} \leq (p + q) (\theta^{N})^{2}
\]

and using the above observation, all the coefficients \( j_{i}^{l_{i}} \) must be equal to 0. In other words, in any solution of \( (33) \) the terms can be divided into groups such that in each group the \( n_{k_{i}} \) are equal and the sum of the coefficients is 0. Consider
solutions in (32) is since, as we noted, \( p \) and \( q \) are even, and similarly to the proof of Lemma 1 the contribution of such solutions in (32) is

\[
\frac{p!}{(p/2)!2^{p/2}} \frac{q!}{(q/2)!2^{q/2}}.
\]

Consider now the solutions of (33), where at least one group has cardinality \( 2 \). This can happen only if both \( p \) and \( q \) are even, and similarly to the proof of Lemma 1, the contribution of such solutions cannot exceed

\[
|\Delta_M|^R |\Delta_N|^S \leq |\Delta_M|^{p/2} |\Delta_N|^{q/2} |\Delta_M|^{-1/2} \ll \sigma_M^p \sigma_N^q |\Delta_M|^{-1/2},
\]

where we used (28) and the fact that \( |\Delta_M| \leq |\Delta_N| \). Since the number of partitions of the set \( \{1, 2, \ldots, p + q\} \) into disjoint subsets is at most \((p + q)!2^{p+q}\) and since the number of choices for the signs \pm in (33) is at most \( 2^{p+q}\), we see, after summing over all possible values of \( j_1, \ldots, j_{p+q} \), that the contribution of the solutions containing at least one group with cardinality \( \geq 3 \) in (32) is at most \( 2^{p+q}\) \((p + q)!|\Delta_M|^{-1/2} (\log|\Delta_N|)^{p+q}\). This completes the proof of Lemma 3.

The rest of the proof of the LIL part of Theorem 2 can be modeled following the lines of Lemma 4, Lemma 5, Lemma 6 and the proof of Theorem 1 in [2].

2.3. Proof of Theorem 3 Let \( \omega_k \uparrow \infty \) and set \( \eta_k = \frac{1}{2} \omega_k^{1/2}, \eta = (\eta_1, \eta_2, \ldots) \).

Clearly

\[
(2k)^{\eta_k^2 + 2\eta_k} \leq (2k)^{\omega_k/2} \leq k^{-2} |I_k| \quad \text{for } k \geq k_0
\]

since, as we noted, \(|I_k| \geq \omega_k(k - 1)^{\omega_k - 1} \geq (k/2)^{\omega_k - 1}\) for large \( k \). We choose \( n_k, k = 1, 2, \ldots \), independently and uniformly from the integers of the intervals \( I_k \) in (31). We claim that, with probability 1, the sequence \( (n_k)_{k \geq 1} \) is increasing and satisfies condition \( A_\eta \). To see this, let \( k \geq 1 \) and consider the numbers of the form

\[
(a_1 n_{i_1} + \ldots + a_s n_{i_s})/d,
\]

where \( 1 \leq s \leq \eta_k, 1 \leq i_1, \ldots, i_s \leq k - 1, a_1, \ldots, a_s, d \) are nonzero integers with \(|a_1|, \ldots, |a_s|, |d| \leq k^{\eta_k} \). Since the number of values in (35) is at most \((2k)^{\eta_k^2 + 2\eta_k}\), and by (34), the probability that \( n_k \) equals any of these numbers is at most \( k^{-2} \). Thus by the Borel-Cantelli lemma, with probability 1 for \( k \geq k_1, n_k \) will be different from all the numbers in (35) and thus the equation

\[
a_1 n_{i_1} + \ldots + a_s n_{i_s} + a_{s+1} n_k = 0
\]

has no solution with \( 1 \leq s \leq \eta_k, 1 \leq i_1 < \ldots < i_s \leq k - 1, 0 < |a_1|, \ldots, |a_{s+1}| \leq k^{\eta_k} \). By monotonicity, the equation

\[
a_1 n_{i_1} + \ldots + a_s n_{i_s} = 0
\]

has no solutions provided the indices \( i_\nu \) are all different, the maximal index is at least \( k \), the number of terms is at most \( \eta_k \), and \( 0 < |a_1|, \ldots, |a_s| \leq k^{\eta_k} \). In other words, \( (n_k) \) satisfies condition \( A_\eta \). Now using Theorem 2 we get Theorem 3.
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