ON THE INNER PRODUCTS OF SOME DELIGNE–LUSZTIG TYPE REPRESENTATIONS

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Abstract. In this paper we introduce a family of Deligne–Lusztig type varieties attached to connected reductive groups over quotients of discrete valuation rings, naturally generalising the higher Deligne–Lusztig varieties and some constructions related to the algebraisation problem raised by Lusztig. We establish the inner product formula between the representations associated to these varieties and the higher Deligne–Lusztig representations.

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1. Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring with a uniformiser $\pi$ and with a finite residue field $\mathcal{O}/\pi = \mathbb{F}_q$. Since $\mathcal{O}$ is a profinite ring, the study of smooth representations of reductive groups over $\mathcal{O}$ leads to the study of representations of the groups over $\mathcal{O}/\pi^r$, for all $r \in \mathbb{Z}_{>0}$; in this paper, we will be concerned with these latter objects.

Let $G$ be a connected reductive group scheme over $\mathcal{O}_r := \mathcal{O}/\pi^r$. When $r = 1$, $G(\mathcal{O}_1) = G(\mathbb{F}_q)$ is a finite group of Lie type; Deligne and Lusztig [DL76] founded a geometric approach to its representations. In the Corvallis paper [Lus79], Lusztig proposed a generalisation of this geometric theory for $G(\mathcal{O}_r)$, for any positive integer $r \geq 1$. The proofs in the positive characteristic case was presented by Lusztig himself in [Lus04], which was later generalised by Stasinski for the general case in [Sta09], by the use of the Greenberg functor. When the involved parameters satisfying some regularity conditions, these representations are irreducible; meanwhile, for $r \geq 2$, following Shintani [Shi68], Gérardin [Gér75] found a purely algebraic method to construct some irreducible representations of these groups.

It is a very interesting question that whether the geometrically constructed representations of Lusztig coincides with the algebraically constructed representations of Gérardin. At even levels (i.e. $r$ is even), this problem was investigated recently, and was proved to be true, as expected by Lusztig; for $G = \text{GL}_n$ this is in [Che17] and for a general $G$ this is in [CS17].
In this paper, we introduce a family of varieties \( L^{-1}(FU^{r-b,b}) \), labelled by \( b \in [0, r] \cap \mathbb{Z} \), such that the alternating sum (see Definition 2.2)

\[
R^\theta_{T,U,b} := \sum_{i \in \mathbb{Z}} (-1)^iH^i_c(L^{-1}(FU^{r-b,b}), \mathbb{Q}_\ell)_\theta
\]

is a virtual representation of \( G(\mathcal{O}_r) \), where the parameter \( \theta \) is a character of certain finite abelian group \( T^F \) (here \( \ell \) is a prime not equal to \( \text{char}(\mathbb{F}_q) \)). When \( b = r \), this coincides with the higher Deligne–Lusztig representation \( R^\theta_{T,U} \) introduced in [Lus79]; when \( r \) is even and \( b = r/2 \), this coincides with Gérardin’s representation introduced in [Gér75].

Our aim is to compute the inner product of the higher Deligne–Lusztig representation \( R^\theta_{T,U} \) and the general \( R^\theta_{T,U,b} \): In particular, we will show that

\[
\langle R^\theta_{T,U,b}, R^\theta_{T,U} \rangle_{G(\mathcal{O}_r)} = 1,
\]

when \( \theta \) is regular and in general position (see Theorem 2.4).

Computing inner products of this type is one of the core steps in many situations in Deligne–Lusztig theory. Here the general principle is that one should first transfer the inner product of this type into an (equivariant) Euler characteristic of certain variety (see Lemma 3.1), then partition the variety into small pieces, which reduces the problem of computing the Euler characteristic of the variety to computing that of each small pieces; however, to make this argument practically work, one usually needs more sophisticated constructions to deal with different specific situations.

Note that, in the case \( b = r \), (2) implies the irreducibility of the higher Deligne–Lusztig representations, and was proved in [Lus04] with \( \text{char}(\mathcal{O}) > 0 \) and in [Sta09] in general.

Also note that, in the case \( r \) is even and \( b = r/2 \), (2) is the main step in the algebraisation of higher Deligne–Lusztig representations at even levels, and was proved in [Che17] with \( G = \text{GL}_n \) and in [CS17] in general.

However, the methods in the \( b = r \) case and the \( b = r/2 \) case faced some obstructions in the general case; the reasons are roughly that the variety \( U^{r-b,b} \) is in general not the Greenberg functor image of the unipotent radical of a Borel subgroup (while \( U^{0,r} = U \) is so), and in general not stable under the Frobenius action or the Weyl group conjugation (while \( U^{r/2,r/2} \) is so). In this paper we will overcome these obstructions, thus complete (2) for any \( r \) and \( b \). In the special case that \( r \) is odd and \( b = (r + 1)/2 \), this result is expected to be useful in the algebraisation of higher Deligne–Lusztig representations at odd levels.

For \( x, y \) in an algebraic group, we will use the conjugation notation \( x^y = y^{-1}xy = y^{-1}x \). For the alternating sum of \( \ell \)-adic cohomology groups \( \sum_i(-1)^iH^i_c(-, \mathbb{Q}_\ell) \), we will write \( H^*_c(-, \mathbb{Q}_\ell) \) for short. If an involved Frobenius \( F \) is applied to some object \( * \), then we occasionally drop the “()” in \( F(*) \) and write \( F* \).

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2. Deligne–Lusztig Varieties at Various Pages

Fix an arbitrary \( r \in \mathbb{Z}_{>0} \). Let \( \mathcal{O}^{ur} \) be the ring of integers in the maximal unramified extension of the fraction field of \( \mathcal{O} \), and put \( \mathcal{O}^{ur} := \mathcal{O}^{ur}/\pi^r \). For a smooth affine group scheme \( H \) over \( \mathcal{O}^{ur} \), according to the Greenberg functor \( \mathcal{F} \) introduced in [Gre61] and [Gre63], there is an associated affine algebraic group \( H = \mathcal{F}H \) over \( k := \overline{\mathbb{F}}_q \) satisfying several nice properties. In [Sta09], this Greenberg functor technique was used to generalise the constructions and results in [Lus04], from the positive characteristic case to the general case. Detailed modern treatments of the Greenberg functors can be found in [Sta12] and [BDA16].

Let \( \mathbb{G} \) be a connected reductive group over \( \mathcal{O} \) (i.e. a smooth affine group scheme over \( \text{Spec}(\mathcal{O}_r) \)), whose geometric fibres are connected reductive algebraic groups in the usual sense; this is the definition used in [DG70, XIX 2.7]), and let \( \mathbb{G} \) be the base change of \( \mathbb{G} \) to \( \mathcal{O}^{ur} \). In general, a surjective algebraic group endomorphism on \( G = G_r = \mathcal{F}G \) with finitely many fixed points is called a Frobenius endomorphism. In this paper we only focus on the following typical situation: The Frobenius element in \( \text{Gal}(k/\mathbb{F}_q) \) extends to an automorphism of \( \mathcal{O}^{ur} \), then by the Greenberg functor it gives a rational structure of \( G \) over \( \mathbb{F}_q \), such that \( G(\mathcal{O}^{ur}) \cong G(k) \) and \( G(\mathcal{O}_r) \cong GF \) as abstract groups, where \( F \) denotes the associated geometric Frobenius endomorphism (see the terminology in [DM91, Chapter 3]); this allows us to use the geometry of \( G \) to study the representations of \( \mathbb{G}(\mathcal{O}_r) \). Let \( L \colon g \mapsto g^{-1}F(g) \) be the Lang endomorphism on \( G \).

To define our fundamental objects (1), we recall some notation used in [Sta09] and [CS17]: For any \( i \in [1, r] \cap \mathbb{Z} \), the modulo \( \pi^i \) reduction map gives a surjective algebraic group endomorphism \( G_i \to G_r \). We denote the kernel subgroup by \( G^i = G_r^i \), and put \( G^0 := G \) (do not mix it with the identity component \( G^0 \)); similar notation applies to the closed subgroups of \( G \). Given a closed subgroup \( H \) of \( G \), we call it \( F \)-stable (or \( F \)-rational, or simply rational) if \( F(H) \subseteq H \). Let \( T \) be a maximal torus of \( G \) such that \( T = \mathcal{F}T \) is \( F \)-stable, let \( B \) be a Borel subgroup containing \( T \), and let \( U \) (resp. \( U^- \)) be the unipotent radical of \( B \) (resp. the opposite of \( B \)). Denote the associated algebraic groups by \( B = \mathcal{F}B \), \( U = \mathcal{F}U \), and \( U^- = \mathcal{F}U^- \), respectively.

For any \( b \in [0, r] \cap \mathbb{Z} \), consider the unipotent algebraic group \( U^{r-b} := U^{r-b}(U^-)^b \). Note that, when \( r \) is even and \( b = r/2 \), this is a commutative group, and in this case it is denoted by \( U^\pm \) in [CS17].

**Definition 2.1.** For \( b \in [0, r] \cap \mathbb{Z} \), we call \( L^{-1}(FU^{r-b}) \subseteq G \) a Deligne–Lusztig variety at page \( b \).

Note that \( GF \) acts on \( L^{-1}(FU^{r-b}) \) by the left multiplication, and \( TF \) acts on \( L^{-1}(FU^{r-b}) \) by the right multiplication, so, after fixing an arbitrary rational prime \( \ell \nmid q \), we obtain the following construction.

**Definition 2.2.** For \( b \in [0, r] \cap \mathbb{Z} \) and \( \theta \in \hat{T}F := \text{Hom}(T^F, \overline{\mathbb{Q}}_\ell) \), we call the virtual \( GF \)-representation

\[
R_{T,U,b}^\theta := H^r_*(L^{-1}(FU^{r-b}), \overline{\mathbb{Q}}_\ell)_\theta
\]

a Deligne–Lusztig representation at page \( b \).

This generalises the constructions studied previously in [Lus79], [Lus04], [Sta09], [Che17], and [CS17]. In those works one central theme is the inner product formula; in this paper we
continue this theme by completing the formula between the higher Deligne–Lusztig representation \( R_{T,U}^\theta = R_{T,U,0}^\theta \) and the general \( R_{T,U,b}^\theta \), for any \( b \). To state our main result, we need to recall the notion of regularity:

Let \( \Phi = \Phi(G, T) \) be the set of roots of \( T \), let \( \Phi^+ \subseteq \Phi \) be the subset of positive roots with respect to \( B \), and let \( \Phi^- := \Phi \setminus \Phi^+ \) be the corresponding subset of negative roots. Given a root \( \alpha \in \Phi \), we denote by \( T^\alpha \) the image of the coroot \( \check{\alpha} \), and write \( T^\alpha := FT^\alpha \); similarly, we write \( U_\alpha \subseteq U \) for the root subgroup of \( \alpha \), and write \( U_\alpha \) for the Greenberg functor image. Put \( T^\alpha := (T^\alpha)^{r-1} \).

**Definition 2.3.** Let \( a \) be a positive integer such that \( T^\alpha \) is \( F^a \)-stable for all \( \alpha \in \Phi \). Consider the norm map \( N^{F^a}_F(t) := t \cdot F(t) \cdots F^{a-1}(t) \) on \( T^{F^a} \). A character \( \theta \in \hat{T}^F \) is called regular, if it is non-trivial on \( N^{F^a}_F(T^\alpha) \) for every \( \alpha \in \Phi \).

The notion of regularity is indeed independent of the choice of the integer \( a \); see [Lus04, 1.5] and [Sta09, 2.8].

Since \( O_r^w \) is a strictly henselian local ring, by [Sta09, 2.1] and [DG70, XXII 3.4] we see that \( W(T) := N_G(T)/T \cong W(T_1) := N_{G_1}(T_1)/T_1 \). Now we can state the formula:

**Theorem 2.4.** Let \( b \in [0, r] \cap \mathbb{Z} \). If \( \theta \in \hat{T}^F \) is regular, then

\[
\langle R_{T,U,b}^0, R_{T,U}^0 \rangle_{G(o)} = \# \text{Stab}_{W(T)^F}(\theta).
\]

In particular, if \( \theta \) is moreover in general position, then \( \langle R_{T,U,b}^0, R_{T,U}^0 \rangle_{G(o)} = 1 \).

The remaining part of this paper devotes to its proof.

### 3. Proof of Theorem 2.4

We start with the following general lemma which allows us to transform the inner product into an Euler characteristic.

**Lemma 3.1.** Suppose that we are given a connected algebraic group over \( \mathbb{F}_q \); denote by \( \mathcal{G} \) its base change to \( \mathbb{F}_q \). Let \( \sigma \) be the associated geometric Frobenius endomorphism, and let \( L_\alpha \) be the associated Lang morphism. If \( X \) and \( X' \) are two closed subvarieties of \( \mathcal{G} \), then the morphism

\[
\kappa: \mathcal{G}^\sigma \setminus (L^{-1}(X) \times L^{-1}(X')) \rightarrow \{(x, x', y) \in X \times X' \times \mathcal{G} \mid x\sigma(y) = yx'\},
\]

given by \( (g, g') \mapsto (L_\alpha(g), L_\alpha(g'), g^{-1}g') \), is an isomorphism. (Indeed, in the proof of our main result, only the bijectivity part of this morphism is needed.)

This can be proved in the same way as the special case in [Car93, Page 222]:

**Proof.** Consider the variety \( S := \{(x, x', y) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G} \mid x\sigma(y) = yx'\} \), which contains \( \{(x, x', y) \in X \times X' \times \mathcal{G} \mid x\sigma(y) = yx'\} \) as a closed subvariety. Note that, in the defining equation of \( S \), the component \( x' \) is determined by \( x \) and \( y \); this fact, together with the surjectivity of the Lang map, imply that the following morphism is surjective:

\[
\iota: \mathcal{G} \times \mathcal{G} \rightarrow S, \quad (g, g') \mapsto (L_\alpha(g), L_\alpha(g'), g^{-1}g').
\]

Note that the fibres of \( \iota \) are \( \mathcal{G}^\sigma \)-orbits, so it suffices to show that the induced bijective morphism \( \bar{\iota}: \mathcal{G}^\sigma \setminus (\mathcal{G} \times \mathcal{G}) \rightarrow S \), extending \( \kappa \), is an isomorphism. By [Bor91, 6.6] it suffices to show that \( S \) is smooth and \( \iota \) is separable.
The smoothness of $S$ follows from the fact that $S \cong G \times G$ via $(x, x', y) \mapsto (x, y)$.

Meanwhile, consider $\iota' : S \to G \times G$ given by $(x, x', y) \mapsto (x, x')$. Then $\iota' \circ \iota$ is a Lang morphism, which is known to be separable, thus $\iota$ is also separable.

Now we turn to the proof of the theorem itself.

First, when $\theta$ is regular, it is known that $R_{T, U}^\theta$ is independent of $U$, so we only need to deal with the case $b \in [r/2, r] \cap \mathbb{Z}$. Moreover, the case $b = r$ and the case $b = r/2$ (for $r$ even) are also known. (See [Lus04, Corollary 2.4], [Sta09, Corollary 3.4], [Che17, Theorem 4.3.2], and [CS17, Theorem 4.1].) Thus, in order to simplify some boundary cases discussions, we will assume $b \in (r/2, r) \cap \mathbb{Z}$ till the end of this paper (note that, this is a non-empty condition only when $r \geq 3$).

By the Künneth formula and Lemma 3.1, we have

$$\langle R_{T, U, b, U}^\theta, R_{T, U}^\theta \rangle_{G(O_x)} = \dim \sum_i (-1)^i H^i_c(\Sigma, \mathbb{Q}_L)_{\theta^{-1}, \theta},$$

where

$$\Sigma := \{(x, x', y) \in F(U^{r-b, b}) \times F(U) \times G \mid xF(y) = yx'\},$$

on which $T^F \times T^F$ acts by $(t, t') : (x, x', y) \mapsto (x^t, (x')^{t'}, t^{-1}yt')$.

We want to compute the cohomology following a general principle of Lusztig, namely, we will first decompose $\Sigma$ into locally closed pieces according to the Bruhat decomposition, and then compute the cohomology of each piece.

Consider the Bruhat decomposition $G_1 = \coprod_{w \in W(T)} B_1 \hat{v} B_1$ of $G_1 = G(k)$ (here $\hat{v}$ denotes a lift of $v$ in $G_1$), which lifts to a decomposition $G = \coprod_{w \in W(T)} G_v$, where

$$G_v := (U \cap \hat{v}U \hat{v}^{-1})(\hat{v}(U)^{-1}\hat{v}^{-1})\hat{v}TU$$

(here we again use $\hat{v}$ to denote a lift of $v$ in $G$); see e.g. the proof of [Sta09, Lemma 2.3]. This results a finite partition of $\Sigma$ into locally closed subvarieties

$$\Sigma = \coprod_{w \in W(T)} \Sigma_v,$$

where

$$\Sigma_v := \{(x, x', y) \in F(U^{r-b, b}) \times FU \times G_v \mid xF(y) = yx'\}.$$

For each $v$, consider the variety

$$Z_v := (U \cap \hat{v}U \hat{v}^{-1}) \times \hat{v}(U)^{-1}\hat{v}^{-1}.$$  

Taking the multiplication morphism $Z_v \to \hat{v}U \hat{v}^{-1}$, we obtain a locally trivial fibration $\hat{\Sigma}_v \to \Sigma_v$ by an affine space, where

$$\hat{\Sigma}_v := \{(x, x', u', u^-, \tau, u) \in F(U^{r-b, b}) \times FU \times Z_v \times T \times U \mid xF(u'u^- \hat{v} \tau u) = u'u^- \hat{v} \tau ux'\},$$

on which $T^F \times T^F$ acts by

$$(t, t') : (x, x', u', u^-, \tau, u) \mapsto (x^t, (x')^{t'}, (u')^t, (u^-)^t, (t\hat{v})^{-1} \tau t', u'^t).$$

Taking the change of variable $x'F(u)^{-1} \mapsto x'$, we can rewrite $\hat{\Sigma}_v$ as

$$\hat{\Sigma}_v = \{(x, x', u', u^-, \tau, u) \in F(U^{r-b, b}) \times FU \times Z_v \times T \times U \mid xF(u'u^- \hat{v} \tau) = u'u^- \hat{v} \tau ux'\},$$
on which the $T^F \times T^F$-action does not change; for our purpose it suffices to compute the dimension of the $\theta^{-1} \times \theta$-isotypical part of $H^*_c(\hat{\Sigma}_v)$, for every $v$.

We want to stratify each $\hat{\Sigma}_v$, by stratifying each $\mathcal{Z}_v$. Let us first make the following notation convention. For $\beta \in \Phi^-$, let $F(\beta) \in \Phi$ be the root defined by $F(U)_{F(\beta)} = F(U_\beta)$. Similarly, we can define $F$ on $\Phi^+$, thus obtain a bijection on $\Phi = F(\Phi^-) \sqcup F(\Phi^+)$ and hence also on $\{U_\beta\}_{\beta \in \Phi}$. Consider the length function on the roots with respect to $(T,FB)$; we denote by $\text{ht}(\cdot)$ the absolute value of the length function; we fix an arbitrary total order on $F(\Phi^-)$ refining the order given by $\text{ht}(\cdot)$. For $z \in U^-$ and $\beta' \in F(\Phi^-)$, let $x_{\beta'}^F(z) \in (FU)_\beta$ be defined by the decomposition

$$F(z) = \prod_{\beta \in F(\Phi^-)} x_{\beta}^F(z),$$

where the product is taken with respect to the following order: If $\beta_1 < \beta_2$, then $x_{\beta_1}^F(z)$ is left to $x_{\beta_2}^F(z)$.

For $\beta \in \Phi^-$, let $\mathcal{Z}_v^\beta \subseteq \mathcal{Z}_v$ be the subvariety consisting of the elements $(u',u^-)$ satisfying that, for $z := (u'u^-)^\beta \in U^-$, one has $x_{\beta'}^F(z) = 1$ whenever $\beta' < F(\beta)$, and $x_{F(\beta)}^F(z) \neq 1$. This gives a finite stratification into locally closed subvarieties

$$\mathcal{Z}_v = (\bigcup_{\beta \in \Phi^-} \mathcal{Z}_v^\beta) \sqcup \mathcal{Z}_v^c,$$

where $\mathcal{Z}_v^c := \mathcal{Z}_v \setminus (\bigcup_{\beta \in \Phi^-} \mathcal{Z}_v^\beta)$. Furthermore, for $i = 0, \ldots, r$, let $\mathcal{Z}_v^\beta(i)$ be the pre-image of $\hat{\delta}(U^-) \cap G^0$ (recall that $G^0 := G$) along the multiplication morphism $\mathcal{Z}_v^\beta \to \hat{\delta}(U^-)$; for $i = 0, \ldots, r-1$, put $\mathcal{Z}_v^\beta(i)^* := \mathcal{Z}_v^\beta(i) \setminus \mathcal{Z}_v^\beta(i+1)$. The above stratification can then be refined to be

$$\mathcal{Z}_v = (\bigcup_{\beta \in \Phi^-} \sqcup_{i=0}^{i=r-1} \mathcal{Z}_v^\beta(i)^*) \sqcup \mathcal{Z}_v^c.$$

This naturally corresponds to a stratification of $\hat{\Sigma}_v$:

$$\hat{\Sigma}_v = (\sqcup_{\beta,i} \Sigma_{v,\beta,i}) \sqcup \Sigma_{v,c},$$

where

$$\Sigma_{v,\beta,i} := \{(x',x',u'^-,\tau,u) \in F(U'_{r-b,b}) \times FU \times Z_v^\beta(i)^* \times T \times U \mid xF(u'u^-\hat{\tau} x) = u'u^-\hat{\tau} x u'\}$$

and

$$\Sigma_{v,c} := \{(x',x',u'^-,\tau,u) \in F(U'_{r-b,b}) \times FU \times Z_v^c \times T \times U \mid xF(u'u^-\hat{\tau} x) = u'u^-\hat{\tau} x u'\}.$$
Lemma 3.2. One has \( \dim H_{c}^{*}(\Sigma_{v,c})_{\theta^{-1},\theta} = \begin{cases} 1, & \text{if } v \in \text{Stab}_{W(T)^F}(\theta) \\ 0, & \text{otherwise.} \end{cases} \)

Lemma 3.3. One has \( \dim H_{c}^{*}(\Sigma_{v,\beta,i}^{\prime\prime})_{\theta^{-1},\theta} = 0, \) where \( i = 0, \ldots, r-1 \) and \( \beta \in \Phi^{-}. \)

Lemma 3.4. One has \( \dim H_{c}^{*}(\Sigma_{v,\beta,i}^{\prime\prime})_{\theta^{-1},\theta} = 0, \) where \( i = 0, \ldots, r-1 \) and \( \beta \in \Phi^{-}. \)

(As we will see in the below, the first two lemmas are true for any \( \theta \), no matter whether \( \theta \) is regular; however, our proof of the third lemma relies on the regularity of \( \theta \).

Within these results we can deduce that \[
\dim H_{c}^{*}(\Sigma)_{\theta^{-1},\theta} = \sum_{v \in \text{Stab}_{W(T)^F}(\theta)} 1 = \# \text{Stab}_{W(T)^F}(\theta),
\]
which concludes the theorem. The remaining part of this section devotes to the proofs of these lemmas.

We start with the first two, which are much easier and can be proved simultaneously:

Proof of Lemma 3.2 and Lemma 3.3. First, by taking the change of variables \( xF(u'u^{-}) \mapsto x \),
we can rewrite \( \Sigma^{\prime\prime}_{v,\beta,i} \) as
\[
\tilde{\Sigma}_{v,\beta,i}^{\prime\prime} := \{(x, x', u', u^{-}, \tau, u) \in F(U^{r-hb}) \times FU \times Z^{b}(i)^{\prime\prime} \times T \times U \mid xF(\hat{\tau}u) = u'u^{-}\hat{\tau}ux'\},
\]
on which the \( TF \times TF \)-action does not change.

Consider \( H = \{(t, t') \in T_{1} \times T_{1} \mid tF(t^{-1}) = F(\hat{\tau})t'F(t'^{-1})F(\hat{\tau}^{-1})\}; \) this is an algebraic group
and it acts on both \( \Sigma_{v,c} \) and \( \Sigma_{v,\beta,i}^{\prime\prime} \) by naturally extending the \( T_{1}F \times T_{1}F \)-action (note
that \( T_{1} \) is a subgroup of \( T \).) The identity component \( H^{o} \) is a torus, thus by basic properties
of \( \ell \)-adic cohomology (see e.g. [DM91, 10.15]) we have
\[
\dim H_{c}^{*}(\Sigma_{v,c})_{\theta^{-1},\theta} = \dim H_{c}^{*}(\Sigma_{v,c}^{H_{o}})_{\theta^{-1},\theta}
\]
and
\[
\dim H_{c}^{*}(\Sigma_{v,\beta,i}^{\prime\prime})_{\theta^{-1},\theta} = \dim H_{c}^{*}(\tilde{\Sigma}_{v,\beta,i}^{\prime\prime})^{H_{o}}_{\theta^{-1},\theta}.
\]
The Lang–Steinberg theorem implies that both the first and the second projections of \( H^{o} \) to
\( T_{1} \) are surjective, thus
\[
(\Sigma_{v,c})^{H_{o}} = \{(1, 1, 1, 1, \tau, 1) \mid F(\hat{\tau}u) = \hat{\tau}\}^{H_{o}} \quad \text{and} \quad (\tilde{\Sigma}_{v,\beta,i}^{\prime\prime})^{H_{o}} = \emptyset.
\]
So it remains to deal with \( (\Sigma_{v,c})^{H_{o}} \).

Note that \( \hat{\tau}T \) is non-empty only if \( v \in W(T)^F; \) we only need to deal with this non-
empty case. As \( \{(1, 1, 1, 1, \tau, 1) \mid F(\hat{\tau}u) = \hat{\tau}\} \) is a finite set, it admits only the trivial action
of the connected group \( H^{o} \), thus
\[
(\Sigma_{v,c})^{H_{o}} = (\hat{\tau}T)^F,
\]
hence \( H_{c}^{*}(\Sigma_{v,c}) = \overline{\mathbb{Q}}_{l}[(\hat{\tau}T)^F], \) on which \( TF \times TF \) acts by \( (t, t') : \hat{\tau} \mapsto \hat{\tau}(t\hat{\tau}^{-1})t' \) (note
that this is the regular representation of both the left \( TF \) and the right \( TF \) in \( TF \times TF \)). In
particular, the irreducible subrepresentations of \( H_{c}^{*}(\Sigma_{v,c}) \) are of the form \( (\phi^{\hat{\tau}})^{-1} \times \phi \),
where \( \phi \) runs over \( TF \). Therefore, \( H_{c}^{*}(\Sigma_{v,c})_{\theta^{-1},\theta} \) is non-zero (in which case it is of dimension 1) if
and only if \( v \in W(T)^F \) and \( \theta^{\hat{\tau}} = \theta \), that is, if and only if \( v \in \text{Stab}_{W(T)^F}(\theta). \)
Now we turn to Lemma 3.4; its proof is more difficult than those of Lemma 3.2 and Lemma 3.3. We divide it into three cases, namely

(I) \( i \geq b \);

(II) \( i < r - b \);

(III) \( r - b \leq i < b \).

We will treat them separately.

Proof of Lemma 3.4 (case (I) \( i \geq b \)). In this case, \( \Sigma_{v, \beta, i} = \emptyset \) by our construction. \( \square \)

In order to deal with the other two cases, we need the following slight generalisation of the technical result [CS17, Lemma 4.6]:

Lemma 3.5. For \( i \in \{0, \ldots, b - 1\} \), \((u', u^-) \in \mathcal{Z}_v^\beta(i)'\), \( z := (u'u^-)^\delta \), and \( \xi \in \U_{-F(\beta)}^{r-i-1} \), one has

\[
[\xi, F(z)] := \xi F(z) \xi^{-1} F(z)^{-1} = \tau_{\xi, F(z)} \omega_{\xi, F(z)},
\]

where \( \tau_{\xi, F(z)} \in \mathcal{T}_{-F(\beta)}' \) and \( \omega_{\xi, F(z)} \in F(U')^{-1} \) are uniquely determined. Moreover,

\[
U_{-F(\beta)}^{r-i-1} \rightarrow \mathcal{T}_{-F(\beta)}', \quad \xi \mapsto \tau_{\xi, F(z)}
\]

is a surjective morphism admitting a section \( \Psi_{F(z)}^{-F(\beta)} \) such that \( \Psi_{F(z)}^{-F(\beta)}(1) = 1 \) and such that the map

\[
\mathcal{Z}_v^\beta(i)' \times \mathcal{T}_{-F(\beta)}' \rightarrow U_{-F(\beta)}^{r-i-1}, \quad ((u', u^-), \tau) \mapsto \Psi_{F(z)}^{-F(\beta)}(\tau)
\]

is a morphism.

Proof. A similar argument of [CS17, Lemma 4.6] works here (actually, it works for any \( i \in \{0, \ldots, r - 2\} \)); we record it for the completeness and for that we will use part of the argument later. Write \( F(z) = x_{F(\beta)}^{F(z)} F(z') \) (see the notation in (3)), then

\begin{equation}
[\xi, F(z)] = \xi \cdot x_{F(\beta)}^{F(z)} \cdot F(z') \cdot \xi^{-1} \cdot (F(z'))^{-1} \cdot (x_{F(\beta)}^{F(z)})^{-1} = [\xi, x_{F(\beta)}^{F(z)}] \cdot x_{F(\beta)}^{F(z)} [\xi, F(z')].
\end{equation}

We need to determine \([\xi, x_{F(\beta)}^{F(z)}] \) and \( x_{F(\beta)}^{F(z)} [\xi, F(z')] \).

Following the notation in [DG70, XX], we write \( p_\alpha : (\mathcal{O}_m)^{\mathcal{O}_w} \cong U_\alpha \) for every \( \alpha \in \Phi \) (and we use the same notation \( p_\alpha \) for the isomorphism \( \mathcal{F}(\mathcal{O}_m)^{\mathcal{O}_w} \cong U_\alpha \) induced by the Greenberg functor). Then there exists \( \alpha \in \mathbb{G}_m(\mathcal{O}_w) \) such that, for all \( x, y \in \mathcal{O}_m(\mathcal{O}_w) \), we have

\begin{equation}
p_\alpha(y)p_\alpha(x) = p_\alpha\left(\frac{x}{1 + axy}\right)\dot{\alpha}(1 + axy)^{-1} p_\alpha\left(\frac{y}{1 + axy}\right).
\end{equation}

(See [DG70, XX 2.2].) Let \( x \) and \( y \) be such that \( p_{-F(\beta)}(x) = \xi \) and \( p_{F(\beta)}(y) = x_{F(\beta)}^{F(z)} \) (note that in our case \((xy)^2 = 0\), so \((1 + axy)^{-1} = 1 - axy\)). By applying (5) to the commutator

\[
[p_\alpha(x), p_\alpha(y)] = p_\alpha(x)p_{-\alpha}(y)p_{-\alpha}(-x)p_{-\alpha}(-y)
\]

with \( \alpha := -F(\beta) \), we see that

\begin{equation}
[\xi, x_{F(\beta)}^{F(z)}] = p_\alpha(x)p_{-\alpha}(y)p_{-\alpha}(-x)p_{-\alpha}(-y)
\end{equation}

(6)

\[
= p_\alpha(x)p_{-\alpha}\left(\frac{-x}{1 - axy}\right)\dot{\alpha}(1 + axy)^{-1} p_{-\alpha}\left(\frac{y}{1 - axy}\right)p_{-\alpha}(y)
\]

\[
= \dot{\alpha}(1 + axy)p_{-\alpha}(axy^2).
\]
Note that, since $\xi \in G^{r-i-1}$ and $x^{F(z)}_{F,b} \in G^i$ (in other words, $\pi^{r-i-1} | x$ and $\pi^i | y$), we have $p_{F,b}(axy^2) \in U^{r-1}_{F,b}$ (note that $p_{F,b}(axy^2) = 1$ unless $i = 0$). In the below one shall see that $\bar{\alpha}(1 + axy)$ is the desired $\tau_{\xi,z}$.

Now turn to $[\xi, F(z')]$; we want to show that $[\xi, F(z')] \in F(U)^{r-1}$. Let us do this by induction on $\# \{ \beta' \in F \Phi^- \mid x^{F,\beta'}_{\beta} \neq 1 \}$ (a lightly different argument is applied to the corresponding result in the published version of [CS17]).

If $\# \{ \beta' \in F \Phi^- \mid x^{F,\beta'}_{\beta} \neq 1 \} = 1$, then we have $F(z') = p_{\beta'}(y_0)$ for some $\beta' \in F \Phi^-$ and $y_0 \in G_u(\mathcal{O}^w)$, so by the Chevalley commutator formula (see [Dem65, 3.3.4.1]) we get

$$[\xi, F^z] = \prod_{j, j' \geq 1, j \beta' + j'(-F(\beta)) \in \Phi} p_{j \beta' + j'(-F(\beta))}(a_{j, j'} y_0^j x^{\beta'}_{\beta}) \in \prod_{j, j' \geq 1, j \beta' + j'(-F(\beta)) \in \Phi} (U_{j \beta' + j'(-F(\beta))})^{r-1}$$

for some $a_{j, j'} \in \mathcal{O}^w$. In this formula, if $j \beta' + j'(-F(\beta)) \in \Phi^+$ for some $j, j'$ and if $y_0^j x^{\beta'}_{\beta}$ is non-zero, then the non-zero coefficients of the simple roots in $\beta' + (-F(\beta))$ are all greater than zero and there exists at least one non-zero coefficient (recall that $\xi \in G^{r-i-1}$ and $z' \in G^i$); this implies that $ht(\pi) > ht(\beta')$, which contradicts our assumption on $z$, so $[\xi, F(z')] = F(U)^{r-1}$ in this case.

Now, suppose that $[\xi, F^z] \in F(U)^{r-1}$ for all $\# \{ \beta' \in F \Phi^- \mid x^{F,\beta'}_{\beta} \neq 1 \} \leq N$. Then for $\# \{ \beta' \in F \Phi^- \mid x^{F,\beta'}_{\beta} \neq 1 \} = N + 1$, take a decomposition $F^z = \prod_{\beta' \in F \Phi^-} x^{F,\beta'}_{\beta} = z_1 z_2$ in a way such that both $[\xi, z_1]$ and $[\xi, z_2]$ are in $F(U)^{r-1}$; note that $[\xi, z_1] = [\xi, z_2] = \xi$. As $z_1, z_2 \in F(U)^{r-1}$, thus $[\xi, F^z] \in F(U)^{r-1}$ also for $\# \{ \beta' \in F \Phi^- \mid x^{F,\beta'}_{\beta} \neq 1 \} = N + 1$. So by the induction principle we always have $[\xi, F^z] \in F(U)^{r-1}$.

By (4) and (6) we get

$$[\xi, F^z] = [\xi, F^{x^{F,\beta}_{\beta}}] \cdot x^{F,\beta}_{\beta} [\xi, F^{z_1}] = \bar{\alpha}(1 + axy) \cdot p_{F,\beta}(axy^2) \cdot x^{F,\beta}_{\beta} [\xi, F^z].$$

Now put

$$\tau_{\xi, F^z} = \bar{\alpha}(1 + axy)$$

and

$$\omega_{\xi, F^z} = p_{F,\beta}(axy^2) \cdot x^{F,\beta}_{\beta} [\xi, F^z].$$

From the above we see that $\tau_{\xi, F^z} \in T_{-F(\beta)}$ and $\omega_{\xi, F^z} \in F(U)^{r-1}$ (as $[\xi, F^z] \in F(U)^{r-1}$). The elements $\tau_{\xi, F^z}$ and $\omega_{\xi, F^z}$ are uniquely determined because of the Iwahori decomposition.

Now, as $\tau_{\xi, F(z)}$ is defined to be $\bar{\alpha}(1 + ap^{-1}_{\beta}(\xi)p_{F,\beta}(x^{F,\beta}_{\beta}))$, the map $\xi \mapsto \tau_{\xi, F^z}$, whose target is a connected 1-dimensional algebraic group, is a surjective algebraic group morphism (note that $F^z \mapsto x^{F,\beta}_{\beta}$ is a projection, hence a morphism). We define $\Psi_{F^z}^{F,\beta}$ in the following way: The isomorphism of additive groups

$$(\pi^i) \cong \mathcal{O}^w_{r-i}, \quad \pi^i a + (\pi^r) \mapsto a + (\pi^{r-i})$$

induces an isomorphism of affine spaces (by the Greenberg functor)

$$\mu_i : (\mathcal{F}(\mathbb{G}_a)\mathcal{O}^w)^i \rightarrow (\mathcal{F}(\mathbb{G}_a)\mathcal{O}^w)_{r-i}.$$
Note that this isomorphism depends on the choice of $\tau$. Meanwhile, let
\[ \mu^i : (F(G_a)_{O^w})_{\tau - i} \cong F(G_a)_{O^w}/(F(G_a)_{O^w})^{-i} \rightarrow F(G_a)_{O^w} \]
be a section morphism to the quotient morphism such that $\mu^i(0) = 0$ ($\mu^i$ exists because $F(G_a)_{O^w}$ is an affine space). For $\tau \in T^{-F^\beta}$ we put
\[ \Psi_{F^\beta}^{-1}(\tau) := p_{-F^\beta} \left( (\alpha - 1) \cdot \mu_i \left( (\alpha - 1)(\tau) - 1 \right) \cdot \mu_i \left( p_{-F^\beta}(x_{F^\beta}) \right)^{-1} \right) \].

Here $\alpha^{-1}$ is defined on $T^{-F^\beta} = (F_T^{-F^\beta})^{-1} \cong (F(G_m)_{O^w})^{-1}$ as the inverse to $\alpha$, and we view $\alpha^{-1}(\tau)$ as an element in $F(G_a)_{O^w}$ by the natural open immersion $(G_m)_{O^w} \rightarrow (G_a)_{O^w}$, so the minus operation $\alpha^{-1}(\tau) - 1$ is well-defined. On the other hand, by our assumption on $z$, $\mu_i \left( p_{-F^\beta}(x_{F^\beta}) \right)$ is an element in $F(G_m)_{O^w}$, so its multiplicative inverse exists. Moreover, the product operation is by viewing $(G_m)_{O^w}$ as $F(G_a)_{O^w}$ as a ring scheme (resp. $k$-ring variety). Thus $\Psi_{F^\beta}^{-1}(\tau)$ is well-defined; we need to check that it is a section morphism.

By the definition of $\mu_i$ and $\mu^i$, for $\tau \in T^{-F^\beta}(k)$ we have
\[ \tau_{\Psi_{F^\beta}^{-1}(\tau)} \cdot F^\beta = \alpha \left( 1 + \alpha \cdot p_{-F^\beta}(\Psi_{F^\beta}^{-1}(\tau)) p_{-F^\beta}(x_{F^\beta}) \right) = \alpha \left( 1 + \mu_i \left( (\alpha - 1)(\tau) - 1 \right) \cdot \mu_i \left( p_{-F^\beta}(x_{F^\beta}) \right)^{-1} \right) \cdot p_{-F^\beta}(x_{F^\beta}) = \alpha \left( 1 + \mu_i \cdot (\alpha - 1)(\tau) - 1 \right) = \tau. \] (For the third line, note that $\alpha^{-1}(\tau)$ is of the form $1 + s\pi^{-1}$ for some $s \in O^w$, as an element in $G_m(O^w)$.) Thus $\tau \mapsto \Psi_{F^\beta}^{-1}(\tau) \mapsto \tau_{\Psi_{F^\beta}^{-1}(\tau)} \cdot F^\beta$ is the identity map on the $k$-points $T^{-F^\beta}(k)$ of the 1-dimensional affine space $T^{-F^\beta} \cong A^1_k$, hence it is the identity morphism, so $\Psi_{F^\beta}^{-1}$ is a section to $\xi \mapsto \tau_{\xi, F(z)}$; the other assertions follow from the definition of $\Psi_{F^\beta}^{-1}$.

Taking the changes of variables $\hat{\nu} \tau \hat{\nu}^{-1} \tau^{-1} \rightarrow \tau^{-1} u^{-1} \tau \rightarrow u^{-1}$, and then $\tau^{-1} u^{-1} \tau \rightarrow \tau^{-1} u^{-1} \tau \rightarrow u^{-1}$ (in this order), we can rewrite $\Sigma_{v, \beta, i}$ as
\[ \Sigma_{v, \beta, i} := \{(x, x', u', u^-, \tau, u) \in FU^{-b} \times FU \times Z_v(i') T \times U \mid xF(\tau u^{-1} u^{-1}) = \tau u^{-1} u^{-1} \hat{\nu} x' \}, \]
on which $(t, t') \in T^F \times T^F$ acts by sending $(x, x', u', u^-, \tau, u)$ to
\[ (t^{-1} x, t^{-1} x', (\hat{\nu} t')^{-1} x' - (\hat{\nu} t')^{-1} u^{-1} - (\hat{\nu} t')^{-1} u^{-1} (\hat{\nu} t')^{-1} u^{-1} - (\hat{\nu} t')^{-1} u^{-1} (\hat{\nu} t'), t^{-1} \tau (\hat{\nu} t'), t^{-1} u'). \]
To show $\text{dim} H^c_{\tau}(\Sigma_{v, \beta, i} \theta^{-1})_F = 0$ for $i < b$, it suffices to show
\[ \text{dim} H^c_{\tau}(\Sigma_{v, \beta, i} \theta^{-1})_F = 0 \]
for $i < b$, where the subscript $T^{(r-1)F}$ denotes the subgroup $T^{(r-1)F} \times 1 \subseteq T^F \times T^F$. Note that $T^{(r-1)F}$ acts on $\Sigma_{v, \beta, i}$ by
\[ t : (x, x', u', u^-, \tau, u) \mapsto (x, x', u', u^-, t^{-1} \tau, u). \]

**Proof of Lemma 3.4 (case (II) $i < r - b$)**. In this case, note that $Z_{v}(i') \subset Z_v$ is just the closed subvariety consisting of the pairs $(u', u^-)$ satisfying that:

1. $z := (u'u^-)^\beta \in (U^-)^i \backslash (U^-)^{i+1};$
2. $\bar{x}_{F(z)}^{F_{\beta}} \neq 1;$
3. $\bar{x}_{F(z)}^{F_{\beta}} = 1$ for all $\beta' < F(\beta).$
Consider
\[ H_\beta := \{ t \in T^{r-1} \mid F(\hat{v})^{-1} F(t) t^{-1} F(\hat{v}) \in T^{-F(\beta)} \} ; \]
this is a closed subgroup of \( T^{r-1} \), and it contains \((T^{r-1})^F\).

For any \( t \in H_\beta \), we have a morphism \( g_t: FU \to FU \) given by
\[ g_t: x' \mapsto x' \cdot \Psi^{-F(\beta)} F(\hat{v})^{-1} F(t^{-1}) t F(\hat{v})^{-1} , \]
where the parameter is \( z := \hat{v}^{-1} u'u^{-v} \) with \((u', u^-) \in \mathcal{Z}_\nu(i)'\) (here \( \Psi^{-F(\beta)} \) is the morphism in Lemma 3.5). By Lemma 3.5, if \( F(t) = t \), then \( g_t = \text{Id} \).

Meanwhile, for any \( t \in H_\beta \), we have a morphism \( f_t: FU^{r-b_b} \to FU^{r-b_b} \) given by
\[ f_t: x \mapsto x \cdot F(\tau) (t^{-1} F(\hat{v})^{-1} x^{-1} g_t(x') F(t)) , \]
where the parameters are \( x' \in FU \), \( \tau \in T \), and \( z := \hat{v}^{-1} u'u^{-v} \) with \((u', u^-) \in \mathcal{Z}_\nu(i)'\). We need to check that this is well-defined, i.e. to check that the right hand side is in \( FU^{r-b_b} \): By Lemma 3.5 we have
\[ F(z)x^{-1} g_t(x') F(z^{-1}) = \Psi^{-F(\beta)} F(\hat{v})^{-1} F(t^{-1}) F(t) F(\hat{v})^{-1} . \]
for some \( \omega \in U^{r-1}(U^{-1})^{r-1} \), so
\[ (x^{-1} f_t(x))^{F(\tau)} = (F(\hat{v}) \Psi)^t . (F(\hat{v}) \omega)^{F(t)} \in \prod_{a \in \Phi} U_{a}^{-i-1} \subseteq \prod_{a \in \Phi} (U_{a})^b \subseteq FU^{r-b_b} , \]
where \( \Psi := \Psi^{-F(\beta)} F(\hat{v})^{-1} F(t^{-1}) F(t) F(\hat{v})^{-1} \). Thus \( x^{-1} f_t(x) \in FU^{r-b_b} \), hence \( f_t \) is well-defined. Like \( g_t \), if \( F(t) = t \), then \( f_t = \text{Id} \).

Now for any \( t \in H_\beta \), by combining the above constructions we get the following automorphism of \( \tilde{\Sigma}_{\nu,\beta,i} \):
\[ h_t: (x, x', u', u^-, \tau, u) \mapsto (f_t(x), g_t(x'), u', u^-, t^{-1} \tau, u) , \]
where the parameter is \( z := \hat{v}^{-1} u'u^{-v} \). We need to check that this is well-defined, i.e. to check that the image satisfies the defining equation of \( \tilde{\Sigma}_{\nu,\beta,i} \), that is, satisfies
\[ f_t(x) F(t^{-1} \tau u' u^- \hat{v}) = t^{-1} \tau u' u^- \hat{v} g_t(x'). \]
Indeed, by expanding the definition of \( f_t \) we see that:
\[ f_t(x) F(t^{-1} \tau u' u^- \hat{v}) = x \cdot F(\tau) (t^{-1} F(\hat{v}) (x^{-1} g_t(x')) F(t)) \cdot F(t^{-1} \tau u' u^- \hat{v}) \]
\[ = t^{-1} x F(\tau u' u^- \hat{v}) x^{-1} g_t(x') \]
\[ = t^{-1} \tau u' u^- \hat{v} g_t(x'). \]
(For the second line, note that \( t \in T^{r-1} \) commutes with the elements in \( G^{r-1} \); for the third line, use the property \( x F(\tau u' u^- \hat{v}) = \tau u' u^- \hat{v} x \cdot \hat{v} \)). So \( h_t \) is well-defined. Clearly, if \( F(t) = t \), then \( h_t \) coincides with the \( (T^{r-1})^F \)-action, so (see the discussions in [DL76, p. 136] or [Che17, Lemma 4.3.4]) the induced endomorphism of \( h_t \) on \( H^{*}_c(\tilde{\Sigma}_{\nu,\beta,i}) \) is the identity map for any \( t \) in the identity component \((H_\beta)^0 \) of \( H_\beta \).

Let \( a \in \mathbb{Z}_{\geq 0} \) be such that \( F^a(F(\hat{v}) T^{-F(\beta)} F(\hat{v})^{-1}) = F(\hat{v}) T^{-F(\beta)} F(\hat{v})^{-1} \). By continuity, the images of the norm map \( N^a_F(t) = t \cdot F(t) \cdots F^{a-1}(t) \) on \( F(\hat{v}) T^{-F(\beta)} F(\hat{v})^{-1} \) form a connected
subgroup of $H_\beta$, hence are contained in $(H_\beta)^\circ$. Therefore $N_{F}^{F_\alpha}((F(\tilde{v})T^{-F(\beta)}F(\tilde{v})^{-1})^{F_\alpha}) \subseteq (T^{r-1})^F \cap (H_\beta)^\circ$. Thus, the regularity of $\theta$ implies

$$H_c^*(\Sigma_{v,\beta,i})_{\theta^{-1}}|_{N_{F}^{F_\alpha}((F(\tilde{v})T^{-F(\beta)}F(\tilde{v})^{-1})^{F_\alpha})} = 0.$$ 

In particular, $H_c^*(\Sigma_{v,\beta,i})_{\theta^{-1}|_{T^{r-1}F}} = 0. \quad \square$

We use a modification of the argument of the case (II) to deal with the case (III).

**Proof of Lemma 3.4 (case (III) $r - b \leq i < b$).** In this case, by our construction, $\Sigma_{v,\beta,i}$ (or more precisely, $Z_{v}^\beta(i')$) is non-empty only if there is a $\beta_1 \in \Phi^-$ such that $v(\beta_1) \in \Phi^-$; we only need to deal with this non-empty case. Denote by $B_v$ the set consisting of the elements $\beta_1 \in \Phi^-$ satisfying: $F(\beta_1) \geq F(\beta)$ and $v(\beta_1) \in \Phi^-$. Then by the assumption that $r - b \leq i < b$ we have a stratification by locally closed subvarieties

$$Z_{v}^\beta(i') = \bigcup_{\beta_1 \in B_v} Z_{v}^{\beta,\beta_1}(i),$$

where $Z_{v}^{\beta,\beta_1}(i) \subseteq Z_{v}^{\beta}(i')$ is the subvariety consisting of the pairs $(u', u^-) \in Z_{v}^{\beta}(i')$ satisfying the following properties: (write $z := (u'u^-)^b$ for $(u', u^-) \in Z_{v}^{\beta}(i')$)

1. $x_{F(v(\beta_1))}^F(z) \notin F(U^-)^b$;
2. $x_{F(v(\beta_2))}^F(z) \in F(U^-)^b$ for all $\beta_2 \in B_v$ satisfying that $F(\beta_2) < F(\beta_1)$.

This partition of $Z_{v}^{\beta}(i')$ naturally induces a partition of $\Sigma_{v,\beta,i}$:

$$\Sigma_{v,\beta,i} = \bigcup_{\beta_1 \in B_v} \Sigma_{v,\beta,\beta_1,i},$$

where

$$\Sigma_{v,\beta,\beta_1,i} = \{(x, x', u', u^-, \tau, u) \in FU^{r-b} \times FU \times Z_{v}^{\beta,\beta_1}(i) \times T \times U \mid x F(\tau u' u^- \hat{v}) = \tau u' u^- \hat{v} u x'\};$$

clearly each $\Sigma_{v,\beta,\beta_1,i}$ inherits the $(T^{r-1})^F$-action.

Consider

$$H_{\beta_1} := \{t \in T^{r-1} \mid F(\tilde{v})^{-1} F(t)t^{-1} F(\hat{v}) \in T^{-F(\beta_1)}\},$$

which is a closed subgroup of $T^{r-1}$ containing $(T^{r-1})^F$. In the following, for $(u', u^-) \in Z_{v}^{\beta,\beta_1}(i)$, we put $z := (u'u^-)^b$ and we write $F(z) = F(z_0) \cdot F(z_1)$, where

$$F(z_0) := \prod_{\alpha \in F(\Phi^-), \beta' < F(\beta_1)} x^{F(z)}_{\beta'}$$

and

$$F(z_1) := \prod_{\alpha \in F(\Phi^-), \beta' > F(\beta_1)} x^{F(z)}_{\beta'};$$

here the products are taken in the order as in (3).

For $t \in H_{\beta_1}$, we have a morphism $g_t: FU \to FU$ defined by

$$g_t: x' \mapsto x' \cdot \Psi_{F(z_1)}^{-F(\beta_1)} \left(F(\hat{v})^{-1} F(t)^{-1} F(\tilde{v})^{-1}\right),$$

where the parameter is $z := \hat{v}^{-1} u'u^- \hat{v}$ with $(u', u^-) \in Z_{v}^{\beta,\beta_1}(i)$; this $\Psi_{F(z_1)}^{-F(\beta_1)}$ is the morphism in Lemma 3.5. Note that, if $F(t) = t$, then $g_t = \text{Id.}$. 

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Meanwhile, for $t \in H_{\beta_1}$, we have a morphism $f_t : FU^{r-b} \to FU^{r-b}$ defined by

$$f_t : x \mapsto x \cdot F(\tau)^{-1} \cdot F(\tilde{\nu}) \left( x' \left( x' \cdot g_t(x') \right) \cdot F(t) \right) ,$$

where the parameters are $x' \in FU$, $\tau \in T$, and $z := \tilde{\nu}^{-1} u'u^{-}\tilde{\nu}$ with $(u', u^-) \in \mathbb{Z}_v^\beta \beta_1(i)$. To see this is well-defined, we need to check that the right hand side is in $FU^{r-b}$. By Lemma 3.5 we have

$$F(z) x'^{-1} g_t(x') F(z)^{-1} = F(\tilde{\nu}) \left( \Psi_{F(z)}^{-1} (F(\tilde{\nu})^{-1} F(\tilde{\nu}) F(t)^{-1}) \cdot F(\tilde{\nu})^{-1} F(\tilde{\nu}) F(t)^{-1} \cdot \omega \right)$$

for some $\omega \in U^{r-1}(U^-)^{r-1}$, so (recall that $i > 0$)

$$(x^{-1} f_t(x)) F(\tau) = (F(\tilde{\nu}) \Psi)^{t} \cdot (F(\tilde{\nu}) \omega) F(t)$$

where $\Psi := \Psi_{F(z)}^{-1} (F(\tilde{\nu})^{-1} F(\tilde{\nu}) F(t)^{-1})^{-1}$; it remains to show that $F(\tilde{\nu}) \Psi \in FU^{r-b}$. By the Chevalley commutator formula (see [Dem65, 3.3.4.1] or the relevant part in the proof of Lemma 3.5) we have $F(\tilde{\nu}) \Psi = \Psi \cdot \omega_1$ for some $\omega_1 \in U^{r-1}(U^-)^{r-1}$. Now, as $b > 0$, it suffices to show that $F(\tilde{\nu}) \Psi \in FU^{r-b}$, which follows immediately from our assumption that $\beta_1 \in B_v$. Thus $f_t$ is well-defined. And like $g_t$, if $F(t) = t$, then $f_t = \text{Id}$.

Now for any $t \in H_{\beta_1}$, by combining the above constructions we get the following automorphism of $\Sigma_{v, \beta, \beta_1, i}$:

$$h_t : (x, x', u', u^-, \tau, u) \mapsto (f_t(x), g_t(x'), u', u^-, t^{-1} \tau, u),$$

where the parameter is $z := \tilde{\nu}^{-1} u'u^\tilde{\nu}$. As in the case (II), we can check that this is well-defined by expanding the definition of $f_t$:

$$f_t(x) F(t^{-1} \tau u'u^\tilde{\nu}) = x \cdot F(\tau) \left( t^{-1} \cdot F(\tilde{\nu}) \left( x' \cdot g_t(x') \right) \cdot F(t) \right) \cdot F(t^{-1} \tau u'u^\tilde{\nu})$$

$$= t^{-1} x F(\tau u'u^\tilde{\nu}) x'^{-1} g_t(x')$$

$$= t^{-1} \tau u'u^\tilde{\nu} \cdot \tau u'u^\tilde{\nu} g_t(x').$$

So $h_t$ is well-defined. Similarly, if $F(t) = t$, then $h_t$ coincides with the $(T^{-1})F$-action, so the induced endomorphism of $h_t$ on $H^*_{c}(\Sigma_{v, \beta, \beta_1, i})$ is the identity map for any $t$ in the identity component $(H_{\beta_1})^0$ of $H_{\beta_1}$.

Let $a \in \mathbb{Z}_{>0}$ be such that $F^a(F(\tilde{\nu}) T^{-F(\tilde{\nu})} F(\tilde{\nu})^{-1}) = F(\tilde{\nu}) T^{F(\tilde{\nu})} F(\tilde{\nu})^{-1}$. Again, the images of the norm map $N^a_F(t) = t \cdot F(t) \cdots F^{a-1}(t)$ on $F(\tilde{\nu}) T^{-F(\tilde{\nu})} F(\tilde{\nu})^{-1}$ form a connected subgroup of $H_{\beta_1}$, hence are contained in $(H_{\beta_1})^0$. Thus $N_{F}^{a}((F(\tilde{\nu}) T^{F(\tilde{\nu})} F(\tilde{\nu})^{-1}) F^a) \subseteq (T^{-1})F \cap (H_{\beta_1})^0$. Finally, the regularity of $\theta$ implies that

$$H^c_{c}(\Sigma_{v, \beta, \beta_1, i})_{\theta^{-1}} \bigg|_{N^a_F((F(\tilde{\nu}) T^{F(\tilde{\nu})} F(\tilde{\nu})^{-1}) F^a)} = 0.$$ 

Therefore $H^c_{c}(\Sigma_{v, \beta, \beta_1, i})_{\theta^{-1}} (\phi_{r-1}) = 0$ for all $\beta_1 \in B_v$, so $H^c_{c}(\Sigma_{v, \beta, i})_{\theta^{-1}} (\phi_{r-1}) = 0$. 

This completes the proof of the theorem.

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