Dynamic indifference pricing via the $G$-expectation

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Abstract

We study the dynamic indifference pricing with ambiguity preferences. For this, we introduce the dynamic expected utility with ambiguity via the nonlinear expectation–$G$-expectation, introduced by Peng (2007). We also study the risk aversion and certainty equivalent for the agents with ambiguity. We obtain the dynamic consistency of indifference pricing with ambiguity preferences. Finally, we obtain comparative statics.

Keywords: Dynamic indifference pricing, model uncertainty, $G$-expectation, ambiguity

1 Introduction

Hodges and Neuberger (1989) introduce the concept of the indifference pricing to study the optimal replication of contingent claims under transaction costs. In Hodges and Neuberger (1989), the utility is defined by the von Neumann-Morgenstern expected utility. Since then, there are several works studying the indifference pricing. The interested reader can refer to Henderson and Hobson (2009) and the references therein. Recently, Giammarino and Barrieu (2013) study indifference pricing with uncertainty averse preferences of Cerreia Vioglio et al. (2011). However, Giammarino and Barrieu (2013) study the indifference pricing in a static framework. The objective of this paper is to investigate the indifference pricing in a dynamic framework. In contrast to the static setup, we obtain the time consistency of the indifference pricing for the agents with ambiguity.

This paper investigates dynamic indifference pricing in the presence of model uncertainty, in which no reference probability measure is fixed, and our priors can be singular with each other. We will study such problem via the $G$-expectation.

Motivated by risk measures and volatility uncertainty problems, Peng (2007) introduces a new theory—the $G$-expectation. Since $G$-expectation is associated with a set of nonequivalent priors, which are not mutually absolutely continuous, $G$-expectation is a nice tool to study model uncertainty.

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problems in finance and economics. The G-expectation has many applications in finance and economics. For example, Epstein and Ji (2013, 2014) study the asset pricing with ambiguity preferences. Beissner (2013) studies the equilibrium theory with ambiguous volatility. Vorbrink (2014) studies the arbitrage theory in the financial market with volatility uncertainty. Riedel and Bessiner (2014) study the Arrow-Debreu equilibria by continuous trading under volatility uncertainty.

We first introduce the dynamic expected utility with ambiguity using the theory of the G-expectation, and study the related properties, e.g., Jensen inequality. By virtue of the dynamic expected utility with ambiguity, we then study the risk aversion and certainty equivalent for the agents with ambiguity. Finally, we investigate the dynamic indifference pricing with ambiguity and study comparative statics. We obtain the dynamic consistency of the ask price and bid price for the agents with ambiguity.

This paper is organized as follows. In Section 2, we introduce our framework, and recall the theory of the G-expectation. In Section 3, we introduce the dynamic expected utility with ambiguity, and study its properties. The risk aversion for the agents with ambiguity is also studied in this section. Section 4 investigates the certainty equivalent for the agents under model uncertainty. In Section 5, we study the dynamic indifference pricing. Finally, in Section 6, we give the comparative statics.

2 Framework

Following Peng (2007, 2008, 2010), we recall the notions of the sublinear expectation, the G-expectation and the related properties, which we will use in what follows.

2.1 Sublinear expectations

Let \( \Omega \) be a given nonempty set and \( \mathcal{H} \) be a linear space of real functions defined on \( \Omega \) such that if \( x_1, \ldots, x_n \in \mathcal{H} \), then \( \varphi(x_1, \ldots, x_n) \in \mathcal{H} \), for each \( \varphi \in C_{lip}(\mathbb{R}^m) \). Here \( C_{lip}(\mathbb{R}^m) \) denotes the linear space of functions \( \varphi \) satisfying

\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^n + |y|^n)|x - y|,
\]

for all \( x, y \in \mathbb{R}^m \), for some \( C > 0 \) and \( n \in \mathbb{N} \), both depending on \( \varphi \). The space \( \mathcal{H} \) is considered as a set of random variables.

**Definition 2.1** A functional \( \hat{E} : \mathcal{H} \mapsto \mathbb{R} \) is called a sublinear expectation if it satisfies the following properties: for all \( X, Y \in \mathcal{H} \),

(i) monotonicity: if \( X \geq Y \), then \( \hat{E}[X] \geq \hat{E}[Y] \);

(ii) preservation of constants: \( \hat{E}[c] = c \), for all \( c \in \mathbb{R} \);

(iii) subadditivity: \( \hat{E}[X] + \hat{E}[Y] \leq \hat{E}[X + Y] \);

(iv) positive homogeneity: \( \hat{E}[\lambda X] = \lambda \hat{E}[X] \), for all \( \lambda \geq 0 \).
The triple \((\Omega, \mathcal{H}, \hat{E})\) is called a sublinear expectation space.

**Remark 2.2** The sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is a generalization of the classical probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with the linear expectation associated with \(\mathbb{P}\).

### 2.2 \(G\)-expectations

Let \(\Omega = C_0([0, T]; \mathbb{R})\) be the space of all real valued continuous functions \((\omega_t)_{t \in [0,T]}\) with \(\omega_0 = 0\), equipped with the distance

\[
\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \left[ \left( \max_{t \in [0,i]} |\omega^1_t - \omega^2_t| \right) \lor 1 \right], \quad \omega^1, \omega^2 \in \Omega.
\]

We denote by \(\Theta\) a fixed non-empty and closed subset of \(\mathbb{R}^+\). For each \(\varphi \in C_{l,\text{lip}}(\mathbb{R})\), let \(u_\varphi\) be the unique viscosity solution of the following parabolic partial differential equation:

\[
\begin{cases}
\frac{\partial u}{\partial t} = G(\frac{\partial^2 u}{\partial x^2}), & (t, x) \in (0, T) \times \mathbb{R}, \\
u(0, x) = \varphi(x),
\end{cases}
\]

where

\[
G(a) = \frac{1}{2} \sup_{\gamma \in \Theta} (\gamma a).
\]

Let \(B\) be the canonical process of \(\Omega\). We consider the following space on \(\Omega\):

\[
L^0(\mathcal{F}_t) := \{ \varphi(B_{t_1}, \cdots, B_{t_m}) \mid t_1, \cdots, t_m \in [0,t], \text{ for all } \varphi \in C_{l,\text{lip}}(\mathbb{R}^m), \ m \geq 1 \}.
\]

We can construct a sublinear expectation \(\hat{E}\) defined on \(L^0(\mathcal{F}_T)\) as follows:

(i) For \(0 \leq s < t \leq T\) and \(\varphi \in C_{l,\text{lip}}(\mathbb{R})\),

\[
\hat{E}[\varphi(B_t - B_s)] = \hat{E}[\varphi(B_{t-s})] = u_\varphi(t-s,0).
\]

(ii) For \(m \geq 1, \varphi \in C_{l,\text{lip}}(\mathbb{R}^m)\) and \(0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq T\), we set

\[
\hat{E}[\varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_m})] = \hat{E}[\phi(B_{t_1}, B_{t_2}, \cdots, B_{t_{m-1}})],
\]

where

\[
\phi(x_1, x_2, \cdots, x_{m-1}) = \hat{E}[\phi(x_1, x_2, \cdots, x_{m-1}, B_{t_m} - B_{t_{m-1}} + x_{m-1})].
\]

For \(m \geq 1, \varphi \in C_{l,\text{lip}}(\mathbb{R}^m)\) and \(0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq T\), the related conditional expectation of \(\varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_m})\) under \(L^0(\mathcal{F}_j), (j = 1, \cdots, m)\) as follows:

\[
\hat{E}_{t_j}[\varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_m})] = \psi(B_{t_1}, B_{t_2}, \cdots, B_{t_j}),
\]
where
\[ \psi(x_1, x_2, \ldots, x_j) = \hat{E}[\psi(x_1, x_2, \ldots, x_j, B_{t_{j+1}} - B_{t_j} + x_j, \ldots, B_{t_m} - B_{t_{m-1}} + x_{m-1})]. \]

Let \( L(F_t) \) be the completion of \( L^0(F_t) \) under the norm \( \hat{E}[|\cdot|] \). Then the operator \( \hat{E}[] \) (resp., \( \hat{E}_t[] \)) can be continuously extended to \( L(F_T) \) (resp., \( L(F_t) \)). The sublinear expectation \( \hat{E}[] \) is called the \( G \)-expectation, and \( \hat{E}_t[] \) is called the conditional \( G \)-expectation. The canonical process \( B \) is called \( G \)-Brownian motion.

**Proposition 2.3** For all \( 0 \leq s \leq t \leq T \), the following holds for all \( X,Y \in L(F_T) \):

(i) If \( X \geq Y \), then \( \hat{E}_t[X] \geq \hat{E}_t[Y] \).

(ii) \( \hat{E}_t[\lambda X] = \lambda \hat{E}_t[X] \), for all \( \lambda \geq 0 \).

(iii) \( \hat{E}_t[\eta] = \eta \), for all \( \eta \in L(F_t) \).

(iv) \( \hat{E}_t[X] + \hat{E}_t[Y] \geq \hat{E}_t[X + Y] \).

(v) \( \hat{E}_s[\hat{E}_t[X]] = \hat{E}_s[X] \).

After the above basic definition we now introduce the notion of \( G \)-normal distribution.

**Definition 2.4** \((G\text{-normal distribution})\) A random variable \( X \) in a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called \( G \)-normal distributed, denoted by \( X \sim \mathcal{N}(0, \Theta) \), if for each \( \varphi \in C_{l,\text{lip}}(\mathbb{R}) \), the following function defined by
\[ u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)] \text{, } (t, x) \in [0, \infty) \times \mathbb{R}, \]
is the unique viscosity solution of equation (2.1).

**Remark 2.5** If we take \( \Theta = [\sigma^2, \sigma^2] \), where \( \sigma^2 \leq \sigma^2 \), and \( X \sim \mathcal{N}(0, [\sigma^2, \sigma^2]) \), then \( \hat{E}[-X] = -\hat{E}[X] = 0 \), and \( \hat{E}[X^2] = \sigma^2 \geq \hat{E}[-X^2] = \sigma^2 \). Which means that \( X \) has volatility uncertainty. The interval \([\sigma^2, \sigma^2] \) characterizes the volatility uncertainty of \( X \). If \( \sigma^2 = \sigma^2 \), then \( G \)-expectation is just the classical linear expectation.

### 2.3 \( G \)-expectations and multiple priors

Let \( W \) be a standard Brownian motion under a probability \( P \) on \( \Omega \), and \( F^W = (F^W_t)_{t \geq 0} \) be the filtration generated by \( W \):
\[ F^W_t := \sigma\{W_s, s \in [0, T]\} \vee \mathcal{N}, \]
where \( \mathcal{N} \) is the collection of all \( P \)-null sets. We denote by \( A^\Theta_{0,T} \) all the \( \Theta \)-valued \( F^W \)-adapted processes on an interval \([0, T]\). For \( \theta \in A^\Theta_{0,T} \), let \( P_\theta \) be the law of \( \left\{ \int_0^t \theta_s dW_s, t \in [0, T]\right\} \). The following proposition is the characterization of the \( G \)-expectation.
Proposition 2.6 For $X \in L(F_T)$, we have

$$\hat{E}[X] = \sup_{\theta \in A^\Theta_{0,T}} E_{P_\theta}[X].$$

Remark 2.7 The priors $\left\{ P_\theta, \theta \in A^\Theta_{0,T} \right\}$ are a set of nonequivalent probability measures. For more details, see Peng (2007, 2010) and Epstein and Ji (2013, 2014).

For $\theta \in A^\Theta_{0,T}$, we let

$$A^\Theta(t, \theta) := \left\{ \theta' \in A^\Theta_{0,T}, \theta' = \theta \text{ on } [0, t] \right\}.$$

We now give the characterization of the conditional $G$-expectation.

Proposition 2.8 For each $\theta \in A^\Theta_{0,T}$ and $X \in L(F_T)$, then for all $t \in [0, T]$

$$\hat{E}_t[X] = \text{esssup}_{\theta' \in A^\Theta(t, \theta)} E_{P_{\theta'}}[X|F^W_t], P_\theta\text{-a.s.}$$

3 Dynamic expected utility with ambiguity

In this section, we introduce the dynamic expected utility with ambiguity and investigate the related properties. We also study the risk aversion in this framework. For this, we first introduce the superlinear expectation and study the related properties, which will be also used in the following sections.

3.1 Superlinear expectations

For $t \in [0, T]$ and $X \in L(F_T)$, we introduce the superlinear expectation:

$$E_t[X] = -\hat{E}_t[-X].$$

We denote by $E[X] := E_0[X]$. The superlinear expectation $E_t[\cdot]$ has the following properties.

Proposition 3.1 For all $0 \leq s \leq t \leq T$, the following holds for all $X, Y \in L(F_T)$:

(i) If $X \geq Y$, then $E_t[X] \geq E_t[Y].$

(ii) $E_t[\lambda X] = \lambda E_t[X]$, for all $\lambda \geq 0.$

(iii) $E_t[X + \eta] = E_t[X] + \eta$, for all $\eta \in L(F_t).$

(iv) $E_t[X] + E_t[Y] \leq E_t[X + Y].$

(v) $E_s[E_t[X]] = E_s[X].$

Proof: From Proposition 2.3 we can check the proposition holds. We only give the proof of (iv). From (iv) in Proposition 2.8 we have

$$\hat{E}_t[-X - Y] \leq \hat{E}_t[-X] + \hat{E}_t[-Y].$$

Therefore,

$$E_t[X + Y] = -\hat{E}_t[-X - Y] \geq -\hat{E}_t[-X] - \hat{E}_t[-Y] = E_t[X] + E_t[Y].$$
3.2 Jensen inequality

As we know, the classical Jensen inequality plays an important role in economics and finance. The following proposition shows that the Jensen inequality also holds for the model with uncertainty averse preferences.

**Proposition 3.2** Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a continuous, concave and increasing function. Then, for \( t \in [0, T] \) and \( X \in L(\mathcal{F}_T) \), the following holds:

\[
\mathbb{E}_t[\varphi(X)] \leq \varphi(\mathbb{E}_t[X]). \tag{3.1}
\]

**Proof:** Let

\[
\land_\varphi := \{(a, b) \in \mathbb{R}^2 | \varphi(x) \leq ax + b, \text{ for all } x \in \mathbb{R}\}.
\]

Then

\[
\varphi(x) = \inf_{(a, b) \in \land_\varphi} (ax + b).
\]

Since \( \varphi \) is increasing, we have \( a \geq 0 \). From Proposition 3.1 it follows that

\[
\mathbb{E}_t[\varphi(X)] \leq a\mathbb{E}_t[X] + b, \text{ for all } (a, b) \in \land_\varphi.
\]

Therefore,

\[
\mathbb{E}_t[\varphi(X)] \leq \inf_{(a, b) \in \land_\varphi} (a\mathbb{E}_t[X] + b) = \varphi(\mathbb{E}_t[X]).
\]

The proof is complete. \( \Box \)

If \( \varphi \in C^2(\mathbb{R}) \), then we have the following proposition, which we will use later.

**Proposition 3.3** If \( \varphi \in C^2(\mathbb{R}) \), then the following are equivalent:

(i) The function \( \varphi \) is concave.

(ii) For all \( t \in [0, T] \) and \( X \in L(\mathcal{F}_T) \), the following holds:

\[
\mathbb{E}_t[\varphi(X)] \leq \varphi(\mathbb{E}_t[X]).
\]

**Proof:** (i) \( \implies \) (ii) Using a similar argument of Proposition 5.4.6 in Peng (2007), we can prove that (i) implies (ii). We omit the proof here.

(ii) \( \implies \) (i) For \( a \in \mathbb{R} \), we denote by

\[
\underline{G}(a) = \frac{1}{2} \inf_{\gamma \in \Theta} (\gamma^2a). \tag{3.2}
\]

From (ii) we know that, for all \( t \in [0, T] \) and \( X \in L(\mathcal{F}_T) \),

\[
\mathbb{E}_t[\varphi(X)] \leq \varphi(\mathbb{E}_t[X]).
\]

Using a similar argument of Proposition 5.4.6 in Peng (2007), we have, for all \( (x, y, z) \in \mathbb{R}^3 \),

\[
\underline{G}(\varphi'(y)x + \varphi''(y)z^2) - \varphi'(y)\underline{G}(x) \leq 0.
\]
In particular, for fixed \((x_0, y_0) \in \mathbb{R}^2\) and each \(z \in \mathbb{R}\), we have
\[
G(\varphi'(y_0)x_0 + \varphi''(y_0)z^2) - \varphi'(y_0)G(x_0) \leq 0. \quad (3.3)
\]
Since \(G(\cdot)\) is a positive homogenous and superadditive function, then
\[
G(\varphi'(y_0)x_0 + \varphi''(y_0)z^2) - \varphi'(y_0)G(x_0) \\
\geq \frac{G(\varphi''(y_0))}{C}(\varphi''(y_0))z^2 + G(\varphi'(y_0)x_0) - \varphi'(y_0)G(x_0) \\
= \frac{G(\varphi''(y_0))}{C}(\varphi''(y_0))z^2 + C,
\]
where \(C = G(\varphi'(y_0)x_0) - \varphi'(y_0)G(x_0)\), which is independent of the choice \(z\).

If \(\varphi'(y_0) \geq 0\), by the definition of \(G\) in (3.2), we have
\[
G(\varphi'(y_0)x_0) = \frac{1}{2} \inf_{\gamma \in \Theta} (\gamma^2 \varphi'(y_0)x_0) = \varphi'(y_0)\frac{1}{2} \inf_{\gamma \in \Theta} (\gamma^2 x_0) = \varphi'(y_0)G(x_0).
\]
If \(\varphi'(y_0) \leq 0\), using (3.2) again, we have
\[
G(\varphi'(y_0)x_0) = \frac{1}{2} \inf_{\gamma \in \Theta} (\gamma^2 \varphi'(y_0)x_0) = -\frac{1}{2} \varphi'(y_0) \inf_{\gamma \in \Theta} (-\gamma^2 x_0) \\
= \frac{1}{2} \varphi'(y_0) \sup_{\gamma \in \Theta} (\gamma^2 x_0) \leq \frac{1}{2} \varphi'(y_0) \inf_{\gamma \in \Theta} (\gamma^2 x_0) = \varphi'(y_0)G(x_0).
\]
Therefore, \(C \leq 0\).

From (3.3) it follows that
\[
G(\varphi''(y_0))z^2 \leq -C,
\]
for all \(z \in \mathbb{R}\), from which we get
\[
G(\varphi''(y_0)) \leq 0,
\]
Therefore,
\[
\varphi''(y_0) \leq 0, \text{ for } y_0 \in \mathbb{R},
\]
From the above we can prove that
\[
\varphi''(y) \leq 0, \text{ for all } y \in \mathbb{R},
\]
i.e., \(\varphi\) is concave. The proof is complete. \(\square\)

### 3.3 Expected utility with ambiguity

**Definition 3.4** Let \(u : \mathbb{R} \to \mathbb{R}\) be continuous, concave and increasing. Then, \(U\) is called a dynamic expected utility, if for all \(t \in [0, T]\), \(U_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)\) is defined by
\[
U_t(X) = \mathbb{E}_t[u(X)], \text{ for } X \in L(\mathcal{F}_T).
\]

**Remark 3.5** From Proposition 2.2 we know that, for each \(\theta \in \mathcal{A}_0^\Theta\) and \(X \in L(\mathcal{F}_T)\), then for all \(t \in [0, T]\),
\[
U_t(X) = \underset{x \in \mathcal{A}_0^\Theta(t, \theta)}{\text{ess inf}} \ S_{\mathcal{P}_W}[u(X)|\mathcal{F}_t^W], \text{ } \mathcal{P}_0\text{-a.s.}
\]

7
The dynamic expected utility $U$ has the following properties.

**Proposition 3.6** For all $0 \leq t \leq T$, the following holds for all $X,Y \in L(\mathcal{F}_T)$:

(i) If $X \geq Y$, then $U_t(X) \geq U_t(Y)$.

(ii) $U_t(X) = u(X)$, for all $X \in L(\mathcal{F}_t)$.

(iii) For $\lambda \in (0, 1)$, $\lambda U_t(X) + (1 - \lambda)U_t(Y) \leq U_t(\lambda X + (1 - \lambda)Y)$.

(iv) For $s \in [0, t]$, $U_s(X) = U_s(U_t(X))$.

(v) For $s \in [0, t]$, if $U_t(X) \geq U_t(Y)$, then $U_s(X) \geq U_s(Y)$.

**Proof:**

(i) Since $X \geq Y$ and $u$ is increasing, then from (i) in Proposition 3.1 it follows that

$$U_t(X) = \mathbb{E}_t[u(X)] \geq \mathbb{E}_t[u(Y)] = U_t(Y).$$

(ii) From (iii) in Proposition 3.1 we have

$$U_t(X) = \mathbb{E}_t[u(X)] = u_t(X).$$

(iii) From the concavity of $u$, (i), (ii) and (iv) in Proposition 3.1 we have

$$U_t(\lambda X + (1 - \lambda)Y)$$

$$= \mathbb{E}_t[u(\lambda X + (1 - \lambda)Y)]$$

$$\geq \mathbb{E}_t[\lambda u(X) + (1 - \lambda)u(Y)]$$

$$\geq \lambda \mathbb{E}_t[u(X)] + (1 - \lambda)\mathbb{E}_t[u(Y)]$$

$$= \lambda U_t(X) + (1 - \lambda)U_t(Y).$$

(iv) From (v) in Proposition 3.1 it follows that

$$U_s(X) = \mathbb{E}_s[u(X)] = \mathbb{E}_s[\mathbb{E}_t[u(X)]] = U_s(U_t(X)).$$

(v) Since $U_t(X) \geq U_t(Y)$ and $u$ is increasing, then from (i) and (iv) it follows that

$$U_s(X) = U_s(U_t(X)) \geq U_s(U_t(Y)) = U_s(Y).$$

The proof is complete. $\square$

### 3.4 Risk Aversion

In comparison with the von Neumann-Morgenstern expected utility under linear expectation, we study the risk aversion for the expected utility with ambiguity.

**Definition 3.7** A decision maker is risk averse if for all $t \in [0, T]$,

$$U_t(X) \leq u(\mathbb{E}_t[X]), \quad \text{for all } X \in L(\mathcal{F}_t).$$
Using Proposition 3.3, we have the following proposition.

**Proposition 3.8** If \( u \in C^2(\mathbb{R}) \), then a decision maker is risk averse if and only if \( u \) is concave.

**Definition 3.9** For \( t \in [0,T] \), let \( u_1, u_2 \) be two continuous functions. \( u_1 \) is more risk averse than \( u_2 \) at time \( t \) if for all \( X \in L(F_T) \)

\[
\mathbb{E}_t[u_1(X)] \leq u_1(\mathbb{E}_t[X]) \implies \mathbb{E}_t[u_2(X)] \leq u_2(\mathbb{E}_t[X]).
\]

**Proposition 3.10** If \( u_2(u_1^{-1}(x)) \) is a continuous, concave and increasing function of \( x \), then \( u_1 \) is more risk averse than \( u_2 \).

**Proof:** Since \( u_2(u_1^{-1}(x)) \) is a concave and increasing function of \( x \), using Proposition 3.2, for \( X \in L(F_T) \) and \( t \in [0,T] \), we have

\[
\mathbb{E}_t[u_2(X)] = \mathbb{E}_t[u_2(u_1^{-1}(u_1(X)))] \leq u_2(u_1^{-1}(\mathbb{E}_t[u_1(X)])).
\]

Since \( \mathbb{E}_t[u_1(X)] \leq u_1(\mathbb{E}_t[X]) \) and \( u_2(u_1^{-1}(x)) \) is increasing in \( x \), then we have

\[
\mathbb{E}_t[u_2(X)] \leq u_2(\mathbb{E}_t[X]).
\]

The proof is complete. \( \square \)

4 Certainty Equivalent

Pratt (1964) introduces the classical notion of the static certainty equivalent. We first study the static certainty equivalent, then we study the dynamic case.

4.1 Static Certainty Equivalent

In this subsection, we study the static certainty equivalent in the presence of ambiguous volatility. Let us consider a decision maker with ambiguity, who has an asset \( x \) and utility function \( u \) (\( u \) is concave and \( u' > 0, u'' < 0 \)). The ambiguity premium \( \pi \) is such that he would be indifferent between receiving an uncertainty payoff \( X \) and receiving a deterministic amount \( \mathbb{E}[X] - \pi \). The ambiguity premium \( \pi \) depends on \( x \) and \( X \), and we denote it by \( \pi(x,X) \).

Using expected utility with ambiguity in Section 3, we have

\[
u(x + \mathbb{E}[X] - \pi(x,X)) = \mathbb{E}[u(x + X)]. \tag{4.1}\]

For given two reals \( \underline{\sigma}, \bar{\sigma} \) with \( 0 \leq \underline{\sigma} \leq \bar{\sigma} \), we suppose that \( X \) is \( G \)-normal distributed, i.e., \( X \sim \mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2]) \). From Section 2 and Section 3 we know that \( X \) does not have mean uncertainty, i.e., \( \mathbb{E}[-X] = -\mathbb{E}[X] = 0 \), but \( X \) has volatility uncertainty, i.e., \( \mathbb{E}[X^2] = \bar{\sigma}^2 \leq -\mathbb{E}[-X^2] = \underline{\sigma}^2 \). Using Taylor expansion on both sides of (4.1) around \( x \) we have

\[
u(x - \pi(x,X)) \approx \nu(x) - u'(x)\pi(x,X), \tag{4.2}\]

9
and
\[ E[u(X + x)] \approx u(x) + \mathbb{E}[u'(x)X + \frac{1}{2} u''(x)X^2] \]
\[ = u(x) + u'(x)\mathbb{E}[X + \frac{1}{2} u''(x)X^2]. \]

Since \( E[-X] = -E[X] = 0 \) and \( -E[-X^2] = \sigma^2 \), we have
\[ E[u(X + x)] \approx u(x) + u'(x)\mathbb{E}[-\frac{1}{2} r(x)X^2] \]
\[ = u(x) + \frac{1}{2} r(x)u'(x)\mathbb{E}[-X^2] \]
\[ = u(x) - \frac{1}{2} r(x)u'(x)\sigma^2, \quad (4.3) \]

where \( r(x) = -\frac{u''(x)}{u'(x)} \) is the absolute risk aversion of Pratt (1964).

From (4.1), (4.2) and (4.3) we have
\[ \pi(x, X) \approx \frac{1}{2} r(x)\sigma^2. \]

The ambiguity premium of a decision maker with ambiguity preferences depends on \( \sigma^2 \), but does not depend on \( \sigma^2 \). This is different from the classical case in Pratt (1964). If \( \sigma^2 = \sigma^2 \), then this is the classical case.

### 4.2 Dynamic certainty equivalent

The objective of this subsection is to study the dynamic certainty equivalent. From the dynamic point of view, a key notion is that of the time consistency, which means that for any \( 0 < r < s < t \leq T \), the certainty equivalent at \( r \) of \( X \) defined at \( t \), can be indifferently evaluated directly or using an intermediate time \( s \).

**Definition 4.1** Let \( u \) be a continuous, strictly increasing and concave function. Then for \( X \in L(F_T) \), we define the dynamic certainty equivalent \( C_t(X) \) of \( X \) as follows:
\[ C_t : L(F_T) \to L(F_t), \]
satisfies
\[ u(C_t(X)) = \mathbb{E}_t[u(X)]. \]

**Remark 4.2** Since \( u \) is strictly increasing, the above definition is well defined. Clearly, we can write it as follows:
\[ C_t(X) = u^{-1}(\mathbb{E}_t[u(X)]). \]

The dynamic certainty equivalent has the following properties.

**Proposition 4.3** For \( 0 \leq s \leq t < T \), and \( X, Y \in L(F_T) \), the following properties hold.
(i) If \( X \in L(\mathcal{F}_t) \), then \( C_t(X) = X \).

(ii) \( C_s(X) = C_s(C_t(X)) \).

(iii) If \( C_t(X) \leq C_t(Y) \), then \( C_s(X) \leq C_s(Y) \). In particular, if \( X \leq Y \), then \( C_s(X) \leq C_s(Y) \).

(iv) \( C_s(X) \leq E_s[C_t(X)] \). In particular, \( C_s(X) \leq E_s[X] \).

**Proof:** (i). By (iii) in Proposition 3.1 and the definition of the dynamic certainty equivalent, we have

\[
C_t(X) = u^{-1}(\mathbb{E}_t[u(X)]) = u^{-1}(u(X)) = X.
\]

(ii). By (v) in Proposition 3.1 and the definition of the dynamic certainty equivalent, we have

\[
C_s(X) = u^{-1}(\mathbb{E}_s[u(X)]) = u^{-1}(\mathbb{E}_s[\mathbb{E}_t[u(X)]])) = u^{-1}(\mathbb{E}_s[u(C_t(X))]) = u^{-1}(u(C_s(C_t(X)))) = C_s(C_t(X)).
\]

(iii). From (ii) it follows that

\[
C_s(X) = C_s(C_t(X)) \leq C_s(C_t(Y)) = C_s(Y).
\]

In particular, we take \( t = T \) and from (i) it follows that, if \( X \leq Y \), then \( C_s(X) \leq C_s(Y) \).

(iv). From (ii) we have

\[
C_s(X) = C_s(C_t(X)) = u^{-1}(\mathbb{E}_s[u(C_t(X))]).
\]

Therefore, by Jensen inequality in Proposition 3.2 we get

\[
\mathbb{E}_s[u(C_t(X))] \leq u(\mathbb{E}_s[C_t(X)]).
\]

From the above inequalities it follows that

\[
C_s(X) \leq \mathbb{E}_s[C_t(X)]
\]

In particular, taking \( t = T \) and using (i) we get \( C_s(X) \leq \mathbb{E}_s[X] \). \( \square \)

**Remark 4.4** The dynamic certainty equivalent for \( X \) at time \( s \) can be obtained in two ways. We can get the dynamic certainty equivalent for \( X \) at time \( s \) directly. Also, we can first get the dynamic certainty equivalent for \( X \) at time \( t > s \), then get the dynamic certainty equivalent for \( C_t(X) \) at time \( s \).

**Remark 4.5** For \( 0 < s < t \leq T \), if \( C_t(X) = C_t(Y) \), then from (iii) we know that \( C_s(X) = C_s(Y) \), which means that the indifference of certainty equivalent between \( X \) and \( Y \) at time \( s \) can carry over to any earlier time \( s < t \), that is, when less information is available.
5 Dynamic indifference pricing

Giammarino and Barrieu (2013) study indifference pricing with uncertainty averse preferences in a static framework. In their framework, it is impossible to study indifference pricing in a dynamic framework. This section will investigate the indifference pricing in a dynamic framework.

In this section, we give the definition of the dynamic indifference pricing via the expected utility with ambiguity, which is studied in Section 3. We also investigate the properties of the dynamic indifference pricing. In particular, we obtain the time consistency, which is a crucial property for the indifference bid price and indifference ask price.

5.1 Indifference bid price

Definition 5.1 Let $u \in C(\mathbb{R})$ be a strictly increasing and concave function. For $X \in L(F_T)$, $b_t : L(F_T) \to L(F_t)$ is called the indifference bid price of $X$ with respect to the utility $U_t$ at the time $t$ if it satisfies

$$u(Y - b_t(X)) = U_t(Y - X),$$

for $Y \in L(F_t)$.

At time $t$, a decision maker is endowed with a monetary payoff $Y \in L(F_t)$ and a contingent claim $X \in L(F_T)$ (a short position), which happens at time $T$. The decision maker contemplates a transaction which allows him to transfer the contingent claim $X$ in exchange for paying $b_t(X) \in L(F_t)$ at time $t$. The decision maker is in the position of the buyer of a policy, which enables him not to have the contingent claim.

Proposition 5.2 For all $0 \leq t \leq T$, the following holds for all $X_1, X_2 \in L(F_T)$:

(i) If $X_1 \geq X_2$, then $b_t(X_1) \geq b_t(X_2)$.

(ii) $b_t(X) = X$, for all $X \in L(F_t)$.

(iii) For $s \in [0, t]$, if $b_t(X_1) \leq b_t(X_2)$, then $b_s(X_1) \leq b_s(X_2)$.

(iv) For $s \in [0, t]$, $b_s(X) = b_s(b_t(X))$.

Proof: (i) For $Y \in L(F_t)$, since $X_1 \geq X_2$, then from (i) in Proposition 3.6 it follows that

$$u(Y - b_t(X_1)) = U_t(Y - X_1) \leq U_t(Y - X_2) = u(Y - b_t(X_2)).$$

Since $u$ is strictly increasing, we have

$$b_t(X_1) \geq b_t(X_2).$$

(ii) From (ii) in Proposition 3.6 we have, for $Y \in L(F_t)$,

$$u(Y - b_t(X)) = U_t(Y - X) = u(Y - X).$$
Since $u$ is strictly increasing, we have

$$b_t(X) = X.$$  

(iii) For $Y \in L(F_s)$, since $b_t(X_1) \leq b_t(X_2)$, then from (iv) in Proposition 3.6 it follows that

$$u(Y - b_s(X_1)) = U_s(Y - X_1) = U_s(U_t(Y - X_1))$$

$$= U_s(u(Y - b_t(X_1))) \geq U_s(u(Y - b_t(X_2)))$$

$$= U_s(U_t(Y - X_2)) = U_s(Y - X_2)$$

$$= u(Y - b_s(X_2)).$$

Since $u$ is strictly increasing, we have

$$b_s(X_1) \leq b_s(X_2).$$

(iv) From Definition 5.1 and expected utility with ambiguity in Section 3 we have, for $Y \in L(F_s)$,

$$u(Y - b_s(b_t(X))) = U_s(Y - b_t(X))$$

$$= \mathbb{E}_s[u(Y - b_t(X))]$$

$$= \mathbb{E}_s[\mathbb{E}_t[u(Y - b_t(X))]]$$

$$= \mathbb{E}_s[U_t(Y - b_t(X))]$$

$$= \mathbb{E}_s[u(Y - X)]$$

$$= U_s(Y - X)$$

$$= u(Y - b_s(X)).$$

Since $u$ is strictly increasing, we have

$$b_s(X) = b_s(b_t(X)).$$

The proof is complete. \hfill \Box

**Remark 5.3** The property (iii) is called the time consistency of the bid price. For $0 < s < t \leq T$, and $X_1, X_2 \in L(F_T)$, if $b_t(X_1) = b_t(X_2)$, then from (iii) we know that $b_s(X_1) = b_s(X_2)$, which means that the bid prices of $X_1$ and $X_2$ are equal at time $t$, then they should be equal at any previous time $s < t$.

**5.2 Indifference ask price**

**Definition 5.4** Let $u \in C(\mathbb{R})$ be a strictly increasing and concave function. For $X \in L(F_T)$, $a_t : L(F_T) \rightarrow L(F_t)$ is called the indifference ask price of $X$ with respect to the utility $U_t$ at the time $t$ if it satisfies

$$u(Y + a_t(X)) = U_t(Y + X),$$

for $Y \in L(F_t)$.
At time $t$, a decision maker is endowed with a monetary payoff $Y \in L(F_t)$ and a contingent claim $X \in L(F_T)$ (a long position), which will be known at time $T$. The ask price $a_t(X) \in L(F_t)$ is the smallest amount that the decision maker would willingly sell $X \in L(F_T)$.

**Remark 5.5** By Definition 5.1 we know that, for $X \in L(F_T)$ and $Y \in L(F_t)$,

$$u(Y - b_t(-X)) = U_t(Y + X).$$

Then we have

$$u(Y - b_t(-X)) = u(Y + a_t(X)).$$

Since $u$ is strictly increasing we have

$$a_t(X) = -b_t(-X).$$

From Proposition 5.2 we have the following proposition.

**Proposition 5.6** For all $0 \leq t \leq T$, the following holds for all $X_1, X_2 \in L(F_T)$:

(i) If $X_1 \geq X_2$, then $a_t(X_1) \geq a_t(X_2)$.

(ii) $a_t(X) = X$, for all $X \in L(F_t)$.

(iii) For $s \in [0, t]$, if $a_t(X_1) \leq a_t(X_2)$, then $a_s(X_1) \leq a_s(X_2)$.

(iv) For $s \in [0, t]$, $b_s(X) = b_s(b_t(X))$.

**Remark 5.7** The property (iii) is called the time consistency of the ask price. For $0 < s < t \leq T$, and $X_1, X_2 \in L(F_T)$, if $a_t(X_1) = a_t(X_2)$, then from (iii) we know that $a_s(X_1) = a_s(X_2)$, which means that the ask prices of $X_1$ and $X_2$ are equal at time $t$, then they should be equal at any previous time $s < t$.

**6 Comparative statics**

In this section, we give the comparative statics. In order to study the characterization of uncertainty aversion, we use the notation $U_t^u, \Theta$, where $t \in [0, T]$, $u$ is the utility, and $\Theta$ is the parameter which generates the $G$-expectation $\widehat{\mathbb{E}}$ via equation (2.1). Moreover, the related ask and bid prices are denoted by $a_t^u, \Theta$ and $b_t^u, \Theta$, respectively.

The following notion of comparative uncertainty aversion in a dynamic framework is similar to the definition of comparative uncertainty aversion in a static framework in Ghirardato and Marinacci (2002), and Giammarino and Barrieu (2013), and the definition of comparative risk aversion in Yaari (1969).
\textbf{Definition 6.1} For \( t \in [0, T] \), let \( u_1, u_2 \in C(\mathbb{R}) \) be two strictly increasing and concave functions. A decision maker \( U_t^{u_2, \Theta_2} : L(\mathcal{F}_T) \rightarrow L(\mathcal{F}_t) \) is said to be more uncertainty averse than \( U_t^{u_1, \Theta_1} : L(\mathcal{F}_T) \rightarrow L(\mathcal{F}_t) \) if

\[ u_1(Y + \bar{Y}) \geq U_t^{u_1, \Theta_1}(Y + X) \implies u_2(Y + \bar{Y}) \geq U_t^{u_2, \Theta_2}(Y + X), \]

for all \( Y, \bar{Y} \in L(\mathcal{F}_t) \) and \( X \in L(\mathcal{F}_T) \).

\textbf{Proposition 6.2} For all \( 0 \leq t \leq T \), the following statements are equivalent:

(i) \( U_t^{u_2, \Theta_2} \) is said to be more uncertainty averse than \( U_t^{u_1, \Theta_1} \).

(ii) \( b_t^{u_1, \Theta_1}(X) \leq b_t^{u_2, \Theta_2}(X) \), for all \( X \in L(\mathcal{F}_T) \).

(iii) \( a_t^{u_1, \Theta_1}(X) \geq a_t^{u_2, \Theta_2}(X) \), for all \( X \in L(\mathcal{F}_T) \).

\textbf{Proof:} (i) \( \iff \) (ii) From Definition 5.1 and Definition 6.1 we know that, for all \( Y, \bar{Y} \in L(\mathcal{F}_t) \) and \( X \in L(\mathcal{F}_T) \), \( U_t^{u_2, \Theta_2} \) is more uncertainty averse than \( U_t^{u_1, \Theta_1} \) if and only if

\[ u_1(Y + \bar{Y}) \geq u_1(Y - b_t^{u_1, \Theta_1}(-X)) \implies u_2(Y + \bar{Y}) \geq u_2(Y - b_t^{u_2, \Theta_2}(-X)), \]

that is

\[ -b_t^{u_1, \Theta_1}(-X) \leq \bar{Y} \iff -b_t^{u_2, \Theta_2}(-X) \leq \bar{Y}, \]  

(6.1)

where we use the fact that \( u_1 \) and \( u_2 \) are strictly increasing. Since (6.1) holds for all \( X \in L(\mathcal{F}_T) \), then (6.1) is equivalent to the following,

\[ b_t^{u_1, \Theta_1}(X) \geq -\bar{Y} \iff b_t^{u_2, \Theta_2}(X) \geq -\bar{Y}. \]

Therefore, \( U_t^{u_1, \Theta_1} \) is more uncertainty averse than \( U_t^{u_2, \Theta_2} \) if and only if

\[ b_t^{u_1, \Theta_1}(X) \leq b_t^{u_2, \Theta_2}(X). \]

(ii) \( \iff \) (iii) Using the fact that

\[ a_t^{u_1, \Theta_1}(X) = -b_t^{u_1, \Theta_1}(-X), \quad a_t^{u_2, \Theta_2}(X) = -b_t^{u_2, \Theta_2}(-X), \]

we can easily get the equivalence of (ii) and (iii). The proof is complete. \( \square \)

\textbf{Proposition 6.3} For all \( 0 \leq t \leq T \) and \( X \in L(\mathcal{F}_T) \), \( u \in C(\mathbb{R}) \) is strictly increasing, if \( \Theta_1 \subset \Theta_2 \), then

(i) \( b_t^{u, \Theta_1}(X) \leq b_t^{u, \Theta_2}(X) \),

(ii) \( a_t^{u, \Theta_1}(X) \geq a_t^{u, \Theta_2}(X) \).
**Proof:** Since $\Theta_1 \subset \Theta_2$, then we have $A^{\Theta_1}(t, \theta) \subset A^{\Theta_2}(t, \theta)$. From Remark 5.3 it follows that, for each $\theta \in A^{\Theta}_{0,T}$, $X \in L(F_T)$, and for all $t \in [0, T]$

$$U_t(X) = \operatorname{essinf}_{\theta' \in A^{\Theta}(t, \theta)} E_{P_{\theta'}}[u(X)|F_t^W], \ P_\theta\text{-a.s.}$$

Therefore,

$$u(Y - b_t^{u,\Theta_1}(X)) = U_t(Y - X) = \operatorname{essinf}_{\theta' \in A^{\Theta_1}(t, \theta)} E_{P_{\theta'}}[u(Y - X)|F_t^W] \geq \operatorname{essinf}_{\theta' \in A^{\Theta_2}(t, \theta)} E_{P_{\theta'}}[u(Y - X)|F_t^W] = u(Y - b_t^{u,\Theta_2}(X)).$$

Since $u$ is strictly increasing, then we have

$$b_t^{u,\Theta_1}(X) \leq b_t^{u,\Theta_2}(X).$$

Since

$$a_t^{u,\Theta_1}(X) = -b_t^{u,\Theta_1}(-X), \ a_t^{u,\Theta_2}(X) = -b_t^{u,\Theta_2}(-X),$$

we can easily get

$$a_t^{u,\Theta_1}(X) \geq a_t^{u,\Theta_2}(X).$$

The proof is complete. $\square$

**Proposition 6.4** For $t \in [0, T]$, let $u_1, u_2 \in C(\mathbb{R})$ be two strictly increasing and concave functions. If $u_1(u^{-1}_2(x))$ is a concave and increasing function of $x$, then for $X \in L(F_T)$,

(i) $b_t^{u_1,\Theta}(X) \geq b_t^{u_2,\Theta}(X)$,

(ii) $a_t^{u_1,\Theta}(X) \geq a_t^{u_2,\Theta}(X)$.

**Proof:** For $X \in L(F_T)$ and $t \in [0, T]$, $U_t(X) = E_t[u(X)]$. Therefore,

$$u_1(Y - b_t^{u_1,\Theta}(X)) = E_t[u_1(Y - X)].$$

Since $u_1$ is strictly increasing, then we have

$$b_t^{u_1,\Theta}(X) = Y - u_1^{-1}(E_t[u_1(Y - X)]).$$

Similarly,

$$b_t^{u_2,\Theta}(X) = Y - u_2^{-1}(E_t[u_2(Y - X)]).$$

Thus,

$$b_t^{u_1,\Theta}(X) - b_t^{u_2,\Theta}(X) = u_2^{-1}(E_t[u_2(Y - X)]) - u_1^{-1}(E_t[u_1(Y - X)]). \quad (6.2)$$
Since $u_1(u_2^{-1}(x))$ is a concave and increasing function of $x$, by using Proposition 3.2 we have
\[
E_t[u_1(Y - X)] = E_t[u_1(u_2^{-1}(u_2(Y - X)))] 
\leq u_1(u_2^{-1}(E_t[u_2(Y - X)])).
\]

Since $u_1$ is strictly increasing, then we have
\[
u_1^{-1}(E_t[u_1(Y - X)]) \leq u_2^{-1}(E_t[u_2(Y - X)]).
\]

Therefore, from (6.2) we have
\[
\frac{b_t^{u_1,\Theta}(X)}{b_t^{u_2,\Theta}(X)} \geq 1.
\]

Since
\[
\frac{a_t^{u_1,\Theta}(X)}{a_t^{u_1,\Theta}(-X)}, \frac{a_t^{u_2,\Theta}(X)}{a_t^{u_2,\Theta}(-X)},
\]
we can easily get
\[
\frac{a_t^{u_1,\Theta}(X)}{a_t^{u_1,\Theta}(X)} \geq 1.
\]

The proof is complete. □

References

Beissner, P., 2013. Radner equilibria under ambiguous volatility. Working paper.

Cerreia Vioglio, S., Maccheroni, F., Marinacci, M., Montrucchio. L. 2011. Uncertainty averse preferences. Journal of Economic Theory, 146, 1275-1330.

Epstein, L., Ji, S. 2013. Ambiguous volatility and asset pricing in continuous time. Review of Financial Studies, 26, 1740-1786.

Epstein, L., Ji, S. 2014. Ambiguous volatility, possibility and utility in continuous time. Journal of Mathematical Economics, 50, 269-282.

Giammarino, F., Barrieu, P., 2013. Indifference pricing with uncertainty averse preferences. Journal of Mathematical Economics, 49, 22-27.

Ghirardato, P., Marinacci, M., 2002. Ambiguity made precise: a comparative foundation. Journal of Economic Theory, 102, 251-289.

Henderson, V., Hobson, D., 2009. Utility indifference pricing—an overview. In Volume on Indifference Pricing: Theory and Applications edited by René Carmona, Princeton University Press.

Hodges, S., Neuberger, A., 1989. Optimal replication of contingent claims under transaction costs. Review of Futures Markets, 8, 222-239.
Peng, S., 2007. *G*-Brownian motion and dynamic risk measure under volatility uncertainty. [arXiv:0711.2834](https://arxiv.org/abs/0711.2834).

Peng, S., 2007. *G*-expectation, *G*-Brownian motion and related stochastic calculus of Ito type, in: Stochastic Analysis and Applications, in: Abel Symp., vol. 2, Springer, Berlin, pp. 541-567.

Peng, S., 2008. Multi-dimensional *G*-Brownian motion and related stochastic calculus under *G*-expectation. Stochastic Process. Appl. 118 (12), 2223-2253.

Peng, S., 2010. Nonlinear expectations and stochastic calculus under uncertainty. [arXiv:1002.4546v1](https://arxiv.org/abs/1002.4546v1).

Pratt, J., 1964. Risk aversion in the small and in the large. Econometrica, 32, 122–136.

Riedel, F., Bessiner, P., 2014. Non-Implementability of Arrow-Debreu equilibria by continuous trading under volatility uncertainty. Working paper, Bielefeld University.

Vorbrink, J., 2014. Financial markets with volatility uncertainty. Journal of Mathematical Economics 53, 64–78.

Yaari, M., 1969. Some remarks on measures of risk aversion and on their uses. Journal of Economic Theory 1, 315-329.