Neutrino mass sum-rule

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Abstract. Neutrino mass sum-rule is a very important research subject from theoretical side because neutrino oscillation experiment only gave us two squared-mass differences and three mixing angles. We review neutrino mass sum-rule in literature that have been reported by many authors and discuss its phenomenological implications.

1. Introduction
As we have already knew from the Standard Model of Particle Physics especially electroweak interaction model based on $SU(2)_L \times U(1)_Y$ gauge group, it is not possible to obtain a neutrino mass term in the Lagrangian of electroweak interaction when neutrino to be put as a Dirac particle. But, if neutrino is a Majorana particle, then we can have a mass term in the Lagrangian which is given by

$$L = \frac{1}{2} \nu_L C^{-1} M \nu_L + h, c.$$  \hspace{1cm} (1)

where $\nu_L$ contains a three left-handed neutrino filed, $C$ is the charge conjugation matrix, and $M$ is the Majorana mass matrix.

It was a very long time, before the neutrino oscillation phenomena was reported by the Superkamiokande collaboration in 1998 [1], neutrino mass is assumed to be zero or approximately zero. Unfortunately, the neutrino oscillations experiments only gave us the squared-mass difference between two neutrino flavors that undergo oscillations during its propagation in vacuum (two squared-mass differences), and three mixing angles that cannot be used to determined the absolute value of neutrino mass and its hierarchy. Another type of experiment that can be used to detect and determine the neutrino mass is neutrinoless double beta decay experiment. But, neutrinoless double beta decay experiment only give us upper bound of Majorana neutrino mass which is known as effective Majorana mass $\langle m_{ee} \rangle$. Thus, in order to determine neutrino masses by using the experimental data as an input, we should seek another way or relation as an additional parameter that can be used to determine neutrino masses. One of the relation that can be used to help us in determining the absolute value of neutrino mass is the relation that link all three neutrino masses which is know as neutrino mass sum-rule.

In this paper, we review neutrino mass sum-rule that have already reported by many authors and discuss its phenomenological implications on neutrino masses, mass hierarchy, and effective Majorana mass. The paper is organized as follow: in section 2 we review neutrino mass sum-rule that have already reported by many authors; in section 3 we discuss the penomenological
implications of the neutrino mass sum rule on neutrino mass and effective Majorana mass especially for normal hierarchy. Finally, the section 4 is devoted to conclusions.

2. Brief review of neutrino mass sum-rule

Neutrino mass sum rule is a relation among the neutrino masses $m_1, m_2, m_3$ which are known to be very small and it very useful for determining i. e. the hierarchy of neutrino mass whether it normal or inverted hierarchy, the absolute values of neutrino masses, and the effective neutrino mass $|m_{ee}|$ as measured in neutrinoless double beta decay when we use the experimental data of neutrino oscillation as input. The importance of neutrino mass sum rule relation has already been stressed as well in e. g. Refs [2, 3, 4]. The neutrino mass sum rule, which can be obtained from several flavor models based on non-Abelian discrete symmetries, can be classified into four neutrino mass sum rules as one can reads in Ref. [6]

$$\chi m_2 + \xi m_3 = m_1,$$  \hspace{1cm} (2)

$$\frac{\chi}{m_2} + \frac{\xi}{m_3} = \frac{1}{m_1},$$  \hspace{1cm} (3)

$$\chi \sqrt{m_2} + \xi \sqrt{m_3} = \sqrt{m_1},$$  \hspace{1cm} (4)

$$\frac{\chi}{\sqrt{m_2}} + \frac{\xi}{\sqrt{m_3}} = \frac{1}{\sqrt{m_1}},$$  \hspace{1cm} (5)

where $\chi$ and $\xi$ are model dependent complex constants. A sample of various neutrino mass sum rule and the groups generating them are summarized in [7] and a summary table of the present neutrino mass sum rule in literatures can be found in [4, 5]. As pointed out in [8] that the first three mass sum rule, a classification of all models predicting tribimaximal (TBM) mixing which generates mass relations similar to the first three sum rule, but the last case is a completely new case.

By referring to Ref. [4, 5], in literature, we have known that there are twelve neutrino mass sum-rule according to the general mass sum-rule that can be parameterized as follow

$$s(m_1, m_2, m_3, c_1, c_2, \phi_1, \phi_2, d, \Delta_{\chi 13}, \Delta_{\chi 23}) \equiv$$

$$c_1 (m_1 e^{-i\phi_1})^d e^{i\Delta_{\chi 13}} + c_2 (m_2 e^{-i\phi_2})^d e^{i\Delta_{\chi 23}} + m_3^d = 0,$$  \hspace{1cm} (6)

where $\phi_1$ and $\phi_2$ are Majorana phases, and the quantities $c_1, c_2, d, \Delta_{\chi 13}$, and $\Delta_{\chi 23}$ are the parameters that characterize the sum-rule.

If we put $\phi_1 = \phi_2 = 0$ and $\Delta_{\chi 13} = \Delta_{\chi 23} = 0$, then Eq. (6) reads

$$c_1 (m_1)^d + c_2 (m_2)^d + m_3^d = 0.$$  \hspace{1cm} (7)

It is apparent from Eq. (7) the neutrino mass sum-rule of Eq. (2) is easily obtained when we put $d = 1$, $c_1 = -\chi$, and $c_2 = -\xi$. To obtain Eq. (3) from Eq. (7) we should put $d = -1$, $c_1 = -\chi$, and $c_2 = -\xi$, and Eq. (4) is reproduced when we put $d = \frac{1}{2}$, $c_1 = -\chi$, and $c_2 = -\xi$. Finally, Eq. (5) will be obtained when we put $d = -\frac{1}{2}$, $c_1 = -\chi$, and $c_2 = -\xi$ into Eq. (7). It is an important task to explain why there are four possible values of parameter $d$ and to decide what are the feasible value of $d$ which is in agreement with the experimental data. Another question is why there are twelve values for $\chi$ and $\xi$ that make possible twelve type of neutrino mass sum-rule as one can find in literature (see Table 1). In this paper, we do not explain or answer the above questions, but we only evaluate and discuss the phenomenological implications of four types of neutrino mass sum-rule as shown in Eqs. (2)-(5).

According to the experimental result of neutrino oscillation that the experimen only measure the squared mass difference, not absolute value of neutrino masses i. e. $\Delta m^2_{21} > 0$ and $\Delta m^2_{31} > 0$
or $\Delta m_{31}^2 < 0$, then we can have two possible hierarchies of neutrino mass, i.e. normal hierarchy (NH) when $\Delta m_{21}^2 > 0$ and $\Delta m_{31}^2 > 0$ and inverted hierarchy (IH) when $\Delta m_{21}^2 > 0$ and $\Delta m_{31}^2 < 0$.

Meanwhile, neutrinoless double beta decay experiment only give us an upper bound of the effective Majorana mass $\langle m_{ee} \rangle$. The effective Majorana mass is given by

$$\langle m_{ee} \rangle = \left| \sum V_{ei}^2 m_i \right|,$$

(8)

where $V_{ei}$ is the $i$-th element of the first row of neutrino mixing matrix and $m_i$ is the $i$-th of the neutrino mass.

### Table 1. Possible neutrino mass sum-rule [7].

| $d$ | Type | Neutrino mass sum-rule | Group |
|-----|------|------------------------|-------|
| 1   | 1    | $m_1 + m_2 = m_3$      | $A_4, A_5, S_4, \Delta(54)$ |
| 2   | 1    | $m_1 + m_3 = 2m_2$   | $S_1$ |
| 3   | 2    | $2m_2 + m_3 = m_1$   | $A_4, S_4, T', T_7$ |
| 4   | 3    | $m_1 + m_2 = 2m_3$   | $S_4$ |
| 5   | 3    | $m_1 + \sqrt{3+1}m_3 = \sqrt{3-1}m_2$ | $A'_5$ |
| -1  | 6    | $m_1^{-1} + m_2^{-1} = m_3^{-1}$ | $A_4, S_4, A_5$ |
| 7   | 6    | $2m_2^{-1} + m_3^{-1} = m_1^{-1}$ | $A_4, T'$ |
| 8   | 6    | $m_1^{-3} + m_3^{-1} = 2m_2^{-1}$ | $A_4, T'$ |
| 9   | 6    | $m_3^{-1} \pm im_2^{-1} = m_1^{-1}$ | $A_4, \Delta(96)$ |
| -1/2| 10   | $\sqrt{m_1} - \sqrt{m_3} = 2\sqrt{m_2}$ | $A_1 \times Z_2$ |
| 11  | 10   | $\sqrt{m_1} + \sqrt{m_3} = 2\sqrt{m_2}$ | $A_4$ |
| -1/2| 12   | $m_1^{-1/2} + m_2^{-1/2} = 2m_3^{-1/2}$ | $S_4$ |

From Table 1, we can see that the $A_4$ symmetry is the most widely used to describe neutrino mass sum-rule. Since the $A_4$ symmetry is the most widely used as the underlying symmetry of the neutrino mass sum-rule, therefore we only evaluate the neutrino mass sum-rules that can be described by $A_4$ symmetry and other symmetries that can proceed the same mass sum-rule. Thus, for the next section we only evaluate and discuss neutrino mass sum-rules of type 1 and 3 in Table 1 because those types of neutrino mass sum-rule are the most general mass sum-rule according to the number of underlying symmetries that can be used to describe it.

### 3. Phenomenological implications of neutrino mass sum-rule

As stated in the previously, we only evaluate the most general neutrino masses in Table 1 (type 1 and 3) prediction on neutrino masses when we use the data of neutrino oscillations as input.

#### 3.1. Neutrino mass sum-rule of type 1

The neutrino mass sum-rule of type 1 reads

$$m_1 + m_2 = m_3.$$

(9)

From Eq. (9) we can have the following relation

$$m_2^2 + 2m_1m_2 - \Delta m_{31}^2 = 0,$$

(10)
where $\Delta m^2_{31} = m^2_3 - m^2_1$. After doing a little algebra, the Eq. (10) proceed

$$m_2 = -2m_1 + \sqrt{4m_1^2 + \Delta m^2_{31}}.$$  \hfill (11)

We have also another squared-mass difference

$$\Delta m^2_{21} = m^2_2 - m^2_1,$$  \hfill (12)

which can be measured in neutrino oscillation experiment. By inserting Eq. (11) into Eq. (12) and solving it to find $m_1$, then we have

$$m_1 = \sqrt{-105\Delta m^2_{21} - 15\Delta m^2_{31} + 60\sqrt{4\Delta m^2_{21} - \Delta m^2_{21}\Delta m^2_{31} + \Delta m^4_{31}}},$$  \hfill (13)

or

$$m_1 = \sqrt{-105\Delta m^2_{21} - 15\Delta m^2_{31} - 60\sqrt{4\Delta m^2_{21} - \Delta m^2_{21}\Delta m^2_{31} + \Delta m^4_{31}}},$$  \hfill (14)

By inserting the central values of squared-mass difference [9]

$$\Delta m^2_{21} = 7.59 \times 10^{-5} \text{ eV}^2,$$  \hfill (15)

$$\Delta m^2_{31} = 2.46 \times 10^{-3} \text{ eV}^2, \text{ for NH}$$  \hfill (16)

$$\Delta m^2_{31} = -2.36 \times 10^{-3} \text{ eV}^2, \text{ for IH}$$  \hfill (17)

into Eq. (13), and Eqs. (11) and (9), then we have

$$m_1 = 0.021158 \text{ eV}, \ m_2 = 0.022881 \text{ eV}, \ m_3 = 0.04404 \text{ eV},$$  \hfill (18)

for normal hierarchy (NH): $|m_1| < |m_2| < |m_3|$, and

$$m_1 = 0.027615 \text{ eV}, \ m_2 = -0.028956 \text{ eV}, \ m_3 = -0.001342 \text{ eV},$$  \hfill (19)

for inverted hierarchy (IH): $|m_3| < |m_1| < |m_2|$.

If we use Eq. (14) to determine $m_1$, $m_2$ from Eq. (11), and $m_3$ from Eq. (9)then we have the hierarchy of neutrino masses as follow

$$|m_1| < |m_3| < |m_2|,$$  \hfill (20)

$$|m_2| < |m_2| < |m_1|,$$  \hfill (21)

when $\Delta m^2_{31} > 0$, and

$$|m_2| < |m_1| < |m_3|,$$  \hfill (22)

$$|m_1| < |m_3| < |m_2|,$$  \hfill (23)

when $\Delta m^2_{31} < 0$. Thus, only neutrino mass of Eq. (13) with neutrino mass sum-rule of type 1 can predict the hierarchy of neutrino mass in agreement with the experimental data of neutrino oscillations. Plot of effective Majorana mass as function of the lightest neutrino mass for neutrino mass sum rule of type 1 (red line for NH and green line for IH) is displayed in Figure 1.
3.2. Neutrino mass sum-rule of Type 3

As shown in Table 1, the neutrino mass sum-rule of type 3 reads

$$2m_2 + m_3 = m_1.$$  \hspace{1cm} (24)

From neutrino mass sum-rule of Eq. (24) we can have

$$4m_2^2 + 4m_3m_2 + \Delta m_{31}^2 = 0,$$  \hspace{1cm} (25)

which then proceed

$$m_2 = -\frac{\Delta m_{31}^2 - 4\Delta m_{32}^2}{2\sqrt{-\Delta m_{31}^2 + 4\Delta m_{32}^2}},$$  \hspace{1cm} (26)

or

$$m_2 = \frac{\Delta m_{31}^2 - 4\Delta m_{32}^2}{2\sqrt{-\Delta m_{31}^2 + 4\Delta m_{32}^2}}.$$  \hspace{1cm} (27)

It is apparent from Eqs. (26) and (27) that neutrino mass $m_2$ is only as function of squared-mass differences $\Delta m_{31}^2$ and $\Delta m_{32}^2$. Since we need squared-mass difference $m_{32}^2$ to find out the value of $m_2$, then we can use the advantage of definition squared-mass differences $m_{31}^2$ and $m_{31}^2$ which then proceeds

$$\Delta m_{32}^2 = \Delta m_{31}^2 - \Delta m_{21}^2.$$  \hspace{1cm} (28)

By applying the same above procedure in determining the neutrino masses, from Eq. (26) we have neutrino masses for NH as follow

$$m_1 = 0.015869 \text{ eV}, \ m_2 = 0.018103 \text{ eV}, \ m_3 = 0.052075 \text{ eV},$$  \hspace{1cm} (29)
and when using Eq. (27) we have

\[ m_1 = 0.015869 \text{ eV}, \quad m_2 = 0.033972 \text{ eV}, \quad m_3 = -0.052075 \text{ eV}, \]  

(30)

which is consistent with normal hierarchy: \(|m_1| < |m_2| < |m_3|\).

Meanwhile, for IH, by using Eq. (27) we have neutrino masses

\[ m_1 = 0.018790i \text{ eV}, \quad m_2 = -0.016648i \text{ eV}, \quad m_3 = 0.052087i \text{ eV}, \]  

(31)

and when using Eq. (27) we have

\[ m_1 = 0.018790i \text{ eV}, \quad m_2 = 0.035439i \text{ eV}, \quad m_3 = -0.052087i \text{ eV}. \]  

(32)

Both the obtained neutrino masses in Eqs. (31) and (32) are inconsistent with the inverted hierarchy. Thus, we can only use the neutrino mass \( m_2 \) of Eq. (26) and neutrino mass sum-rule of type 3 to predict the correct hierarchy of neutrino mass. Plot of effective Majorana mass as function of the lightest neutrino mass for neutrino mass sum rule of type 3 (only allowed NH) is displayed in Figure 2.

![Figure 2. Plot of \( \langle m_{ee} \rangle \) as function of \( m_\nu \) for sum-rule of type 3](image)

4. Conclusions

We have briefly review neutrino mass sum-rule that can be read in literatures and we found there 12 type of neutrino mass sum-rule that can be derived from various symmetries. The most widely symmetry is \( A_4 \) symmetry. Based on the widely used symmetry that can be applied to derive the neutrino mass sum-rule, we choose two types neutrino mass sum-rule that have
already been reported in literature i.e. type 1 and type 3. When we evaluate the predictions of those both neutrino mass sum-rules on neutrino masses and its hierarchy by using the advantages of neutrino oscillations data as input, we find that the neutrino mass sum-rule of type 1 can predict neutrino mass hierarchy both in normal hierarchy and inverted hierarchy. Meanwhile, the neutrino mass sum-rule of type 3 can only predict neutrino mass hierarchy in normal hierarchy.

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