MATRIX VALUED SPHERICAL FUNCTIONS ASSOCIATED TO THE COMPLEX PROJECTIVE PLANE

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TO E. H. ZARANTONELLO, teacher and friend.

ABSTRACT. The main purpose of this paper is to compute all irreducible spherical functions on $G = SU(3)$ of arbitrary type $\delta \in \hat{K}$, where $K = S(U(2) \times U(1)) \simeq U(2)$. This is accomplished by associating to a spherical function $\Phi$ on $G$ a matrix valued function $H$ on the complex projective plane $P_2(\mathbb{C}) = G/K$. It is well known that there is a fruitful connection between the hypergeometric function of Euler and Gauss and the spherical functions of trivial type associated to a rank one symmetric pair $(G, K)$. But the relation of spherical functions of types of dimension bigger than one with classical analysis, has not been worked out even in the case of an example of a rank one pair. The entries of $H$ are solutions of two systems of ordinary differential equations. There is no ready made approach to such a pair of systems, or even to a single system of this kind. In our case the situation is very favorable and the solution to this pair of systems can be exhibited explicitly in terms of a special class of generalized hypergeometric functions $p+1F_p$.

1. INTRODUCTION AND STATEMENT OF RESULTS

The complex projective plane can be realized as the homogeneous space $G/K$, where $G = SU(3)$ and $K = S(U(2) \times U(1))$. We are interested in determining, up to equivalence, all irreducible spherical functions, associated to the pair $(G, K)$. If $(V, \pi)$ is a finite dimensional irreducible representation of $K$ in the equivalence class $\delta \in \hat{K}$, a spherical function on $G$ of type $\delta$ is characterized by

i) $\Phi : G \rightarrow \text{End}(V)$ is analytic.

ii) $\Phi(k_1gk_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.

iii) $[\Delta_2\Phi](g) = \lambda_2\Phi(g)$, $[\Delta_3\Phi](g) = \lambda_3\Phi(g)$ for all $g \in G$ and for some $\lambda_2, \lambda_3 \in \mathbb{C}$.

Here $\Delta_2$ and $\Delta_3$ are two algebraically independent generators of the polynomial algebra $D(G)^G$ of all differential operators on $G$ which are invariant under left and right multiplication by elements in $G$. A particular choice of these operators is given in Proposition 5.1.

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The set $\hat{K}$ can be identified with the set $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$. If $k = \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right)$, with $A \in U(2)$ and $a = (\det A)^{-1}$, then
\[
\pi(k) = \pi_{n,\ell}(A) = (\det A)^n A^\ell,
\]
where $A^\ell$ denotes the $\ell$-symmetric power of $A$, defines an irreducible representation of $K$ in the class $(n, \ell) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$.

For $n \geq 0$, the representation $\pi_{n,\ell}$ of $U(2)$ extends to a unique holomorphic multiplicative map of $M(2, \mathbb{C})$ into $\text{End}(V_\pi)$, which we shall still denote by $\pi_{n,\ell}$. For any $g \in M(3, \mathbb{C})$, we shall denote by $A(g)$ the left upper $2 \times 2$ block of $g$, i.e.
\[
A(g) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.
\]

For any $\pi = \pi_{n,\ell}$ with $n \geq 0$ let $\Phi_\pi : G \longrightarrow \text{End}(V_\pi)$ be defined by
\[
\Phi_\pi(g) = \Phi_{n,\ell}(g) = \pi_{n,\ell}(A(g)).
\]

Then, in Theorem 2.10 we prove that $\Phi_\pi$ is a spherical function of type $(n, \ell)$. This is the main result of Section 2. These particular spherical functions will play a crucial role in the rest of the paper.

Observe that when $n < 0$ the function $\Phi_\pi$ above can be defined on the open set
\[
\mathcal{A} = \{ g \in G : \det A(g) \neq 0 \}.
\]
The group $G = \text{SU}(3)$ acts in a natural way in the complex projective plane $P_2(\mathbb{C})$. This action is transitive and $K$ is the isotropy subgroup of the point $(0, 0, 1) \in P_2(\mathbb{C})$. Therefore $P_2(\mathbb{C}) = G/K$. We shall identify the complex plane $\mathbb{C}^2$ with the affine plane $\{ (x, y, 1) \in P_2(\mathbb{C}) : (x, y) \in \mathbb{C}^2 \}$.

The canonical projection $p : G \longrightarrow P_2(\mathbb{C})$ maps the open dense subset $\mathcal{A}$ onto the affine plane $\mathbb{C}^2$. Observe that $\mathcal{A}$ is stable by left and right multiplication by elements in $K$.

To determine all spherical functions $\Phi : G \longrightarrow \text{End}(V_\pi)$ of type $\pi = \pi_{n,\ell}$ we use the function $\Phi_\pi$ in the following way: in the open set $\mathcal{A}$ we define a function $H$ by
\[
H(g) = \Phi(g) \Phi_\pi(g)^{-1},
\]
where $\Phi$ is supposed to be a spherical function of type $\pi$. Then $H$ satisfies
\begin{enumerate}
  \item $H(e) = I$.
  \item $H(gk) = H(g)$, for all $g \in \mathcal{A}, k \in K$.
  \item $H(kg) = \pi(k)H(g)\pi(k^{-1})$, for all $g \in \mathcal{A}, k \in K$.
\end{enumerate}
Property ii) says that $H$ may be considered as a function on $\mathbb{C}^2$.

The fact that $\Phi$ is an eigenfunction of $\Delta_2$ and $\Delta_3$, makes $H$ into an eigenfunction of certain differential operators $D$ and $E$ on $\mathbb{C}^2$. For completeness the explicit computation of these operators is carried out fully in Section 3.

In Section 3 we take full advantage of the $K$-orbit structure of $P_2(\mathbb{C})$ combined with property iii) of our functions $H$. The affine plane $\mathbb{C}^2$ is $K$-stable and the corresponding line at infinity $L$ is a $K$-orbit. Moreover the $K$-orbits in $\mathbb{C}^2$ are the spheres $S_r = \{ (x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = r^2 \}$. Thus
we can take the points \((r, 0) \in S_r\) and \((1, 0, 0) \in L\) as representatives of \(S_r\) and \(L\), respectively. Since \((M, 0, 1) = (1, 0, \frac{1}{M}) \to (1, 0, 0)\) when \(M \to \infty\), the closed interval \([0, \infty]\) parametrizes the set of \(K\)-orbits in \(P_2(C)\).

Thus there exist ordinary differential operators \(\tilde{D}\) and \(\tilde{E}\) on the open interval \((0, \infty)\) such that
\[
(DH)(r, 0) = (\tilde{D}\tilde{H})(r), \quad (EH)(r, 0) = (\tilde{E}\tilde{H})(r),
\]
where \(\tilde{H}(r) = H(r, 0), r \in (0, \infty)\). These operators \(\tilde{D}\) and \(\tilde{E}\) are explicitly given in Theorems 5.1 and 5.2. We need to compute a number of second order partial derivatives of the function \(H : C^2 \to \text{End}(V_\pi)\) at the point \((r, 0)\). This detailed computation is broken down in a number of lemmas included in the Appendix for the benefit of the reader.

Theorems 5.1 and 5.2 are given in terms of linear transformations. The functions \(\tilde{H}\) turn out to be diagonalizable (Proposition 5.10). Thus, in an appropriate basis of \(V_\pi\) we can write \(\tilde{H}(r) = (h_0(r), \ldots, h_\ell(r))\). Then we give in Corollaries 5.15 and 5.16 the corresponding statements of these theorems in terms of the scalar functions \(h_i\).

In Section 6 we take into account the behavior of the function \(\tilde{H}\) associated to a spherical function \(\Phi\) when \(r \to \infty\). The corresponding asymptotic behavior of \(\tilde{H}\) is given in the first part of Proposition 6.1. Our strategy to obtain all spherical functions of \((G, K)\) will be to find all eigenfunctions \(\tilde{H}\) of \(\tilde{D}\) and \(\tilde{E}\) which satisfy this behavior at infinity. This is the point of the second part of Proposition 6.1.

In Section 7 we consider the natural inner product among continuous maps from \(G\) to \(\text{End}(V_\pi)\), which makes \(\Delta_2\) and \(\Delta_3\) symmetric (Proposition 7.2). In Proposition 7.1 we give the explicit expression of this inner product when restricted to those functions which satisfy \(\Phi(k_1 g k_2) = \pi(k_1) \Phi(g) \pi(k_2)\).

We also introduce here the variable \(t = (1 + r^2)^{-1}\) which converts the operators \(\tilde{D}\) and \(\tilde{E}\) into new operators \(D\) and \(E\). Then we present a classical analysis motivation for the inner product given in Proposition 7.1.

In Section 8 we give a complete description of a method to obtain all \(C^\infty\) solutions of \(DH = \lambda H\). This construction rests on some remarkable factorizations of higher order differential operators with Gauss’ hypergeometric operator as one of the factors. This construction revolves around introducing an integer parameter \(0 \leq k \leq \ell\) as well as a parameter \(w\) involved in expressing \(\lambda\) appropriately.

The case \(\ell = 0\) is analyzed in detail and the solutions are given generically in terms of \(\mathbb{F}_1\). For higher \(\ell\) the solutions are obtained by acting on \(\mathbb{F}_1\) either by differential operators or by appropriate shift operators.

As we remark in Section 2 there exists a one to one correspondence between the set of all equivalence classes of finite dimensional irreducible representations of \(G\) which contain the representation \(\pi_{n, \ell}\) of \(K\), and the set of all equivalence classes of irreducible spherical functions of \((G, K)\) of type \((n, \ell)\). This is the starting point we take in Section 9 to get a parametrization of all these equivalence classes in terms of all tuples \((p, q, k_1, k_2) \in \mathbb{Z}^4\) such
that \( p + q \geq k_1 \geq q \geq k_2 \geq 0 \), with \( n = k_1 + 2k_2 - p - 2q \), \( \ell = k_1 - k_2 \). This is the content of Corollary 9.3. Also in Corollary 9.4 we give the eigenvalues \( \lambda \) and \( \mu \) which correspond to the differential operators \( D \) and \( E \) in terms of these parameters. Moreover, we introduce here two integer parameters \( w, k \) subject to the following four inequalities: \( 0 \leq w \), \( 0 \leq k \leq \ell \), \( 0 \leq w + n + k \), which give a very convenient parametrization of the irreducible spherical functions of type \( (n, \ell) \).

Section 10 starts with Proposition 10.1 where the local behaviour of the functions \( H = H(t) \) in the variable \( t = (1 + r^2)^{-1} \) at \( t = 0 \) associated to a spherical function \( \Phi \) of type \( (n, \ell) \) is established. In particular it is proved that \( H \) is analytic and that has a zero of order at least \(-n - \ell\), when \( n + \ell < 0 \). Thus we aim at getting joint solutions to \( DH = \lambda H \) and \( EH = \mu H \) in the form of formal power series \( H(t) = \sum_{j \geq 0} t^j H_j \). Since \( E \) commutes with \( D \) we look at the linear map defined by it on the vector space \( V(\lambda) \) of all formal solutions of \( DH = \lambda H \). This equation gives rise to a three term recursion relation for the coefficients \( H_j \).

When \( n \geq 0 \), one sees that the linear map \( \eta : H(t) \mapsto H(0) \) is an isomorphism of \( V(\lambda) \) onto \( \mathbb{C}^{\ell+1} \). Using \( \eta \), \( E \) becomes an \((\ell + 1) \times (\ell + 1)\) matrix \( L = L(\lambda) \). In Corollary 10.4, \( \eta \) is used to establish a one to one correspondence between joint solutions to \( DH = \lambda H \), \( EH = \mu H \) and the eigenvalues \( \mu \) of \( L(\lambda) \). Moreover there is no joint eigenfunction given by formal power series unless the pair \((\lambda, \mu)\) lies on the union of \((\ell + 1)\) straight lines. For any point on this curve, if \( H_0 \) is an eigenvector of \( L(\lambda) \) with eigenvalue \( \mu \), the recursion relation becomes a two term recursion and can be solved explicitly in terms of generalized hypergeometric functions. In particular they converge for \(|t| < 1\).

When \( n < 0 \) the isomorphism \( \eta \) must be redefined and the corresponding matrix \( L \) is not as simple as the one obtained when \( n \geq 0 \). This suggest that our choice of the function \( H \) associated to a spherical function is most appropriate only when \( n \geq 0 \).

In Section 11 we illustrate in full detail the results obtained above in the cases \( \ell = 0, 1, 2 \) and \( n \) an arbitrary integer. The main ingredient here is a particular class of generalized hypergeometric functions among those of the form \( {}_pF_p \). For each \( \ell \), we need to deal with several values of \( p \) in the range \( 1 \leq p \leq \ell + 1 \) and thus for \( \ell = 0 \) (and \( n = 0 \)) we obtain the known result involving only \( {}_2F_1 \).

Generalized hypergeometric functions have numerator and denominator parameters, as well as an independent variable \( t \). The functions that enter in our explicit formulas satisfy the condition that all but two of the numerator parameters exceed by one all but one of the denominator parameters. In the classical case, with \( \ell = 0 \), there is not enough room for this phenomenon to show up.

We state a fairly explicit conjecture on the dependence of these hypergeometric functions on its main parameters, namely the ones that are not
related by the shift described above. These two numerator parameters and
the denominator one exhaust the ones that appear in the case of $\mathbf{2F}_1$.

In view of the very important role played by all functions $p+1\mathbf{F}_p$, $p$ arbitrary,
in our description of the entries of the matrix valued spherical functions, it is worth to note that certain generalized hypergeometric functions,
more precisely the functions $3n-4\mathbf{F}_{3n-5}$, have appeared in the expression of
matrix entries of representations of the group $U(n)$. In this case one can
consult [16] as well as the very systematic treatment in [22]. This very nice
result extends to the case of arbitrary $n$ the well known results involving
Gauss’ function $\mathbf{2F}_1$ in the case of $U(2)$. It would be nice to see the functions identified here playing a useful role in other problems in geometry or
physics.

In Section 12 we observe that our spherical functions satisfy not only
a differential equation, but also an appropriate recursion relation in the
discrete variable $w$. We display the results in full in the case $\ell = 2$ and $n$
nonnegative.

This three term recursion relation is well known in the classical case, when
$\ell = 0$, and it gives the main way to compute in a numerically stable way
the so called classical orthogonal polynomials. Its importance is not only
a practical matter. It gives one of the earliest instances of the bispectral
property. By repeated applications of the Darboux process this innocent
looking property is tied up with all sorts of other issues of recent interest in
mathematical physics, including nonlinear integral evolution equations, $W$-
algebras, interesting isomonodromy deformations, the whole area of random
matrix models, etc. There are even $q$ versions of all of this. Moreover this
is not restricted to the case of one (either spatial or spectral) variable.

It is natural to wonder about the relation between the matrix valued
functions constructed in Section 12 and the relatively new theory of matrix
valued orthogonal polynomials. This issue has been addressed in [5]. It
suffices to state here that the functions considered in Section 12 do not
satisfy all the conditions in the theory, see for instance [5].

Finally we remark that we have chosen this example as one of the simplest
to analyze among the symmetric spaces of rank one leading to matrix valued
spherical functions. We hope to deal with other simple examples in the
near future. One important case of scalar valued spherical functions of non
trivial type has been considered in Part I, Chapter 5 of [13]. The referee
has wondered what is the perspective for doing analogous things in the case
of $SU(n)$, and E. Stein had also suggested to consider this more general
problem. At this point this appears to be a very interesting challenge.

One can only speculate that many of the connections that make Gauss’
function a vital part of mathematics at the end of the twenty century will
be shared by its matrix valued version discussed here.

It is a pleasure to thank the referee for a very thorough job. In partic-
ular the referee has redone independently a few of the computer algebra
checks that we had originally done with Maxima by using very nice Maple worksheets.

2. Spherical functions

Let $G$ be a locally compact unimodular group and let $K$ be a compact subgroup of $G$. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of $K$; for each $\delta \in \hat{K}$, let $\xi_\delta$ denote the character of $\delta$, $d(\delta)$ the degree of $\delta$, i.e. the dimension of any representation in the class $\delta$, and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure $dk$ on $K$ normalized by $\int_K dk = 1$.

We shall denote by $V$ a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and by $\text{End}(V)$ the space of all linear transformations of $V$ into $V$. Whenever we shall refer to a topology on such vector space we shall be talking about the unique Hausdorff linear topology on it.

By definition a zonal spherical function ([11]) $\varphi$ on $G$ is a continuous complex valued function which satisfies $\varphi(e) = 1$ and

$$\varphi(x)\varphi(y) = \int_K \varphi(xky) \, dk \quad x, y \in G.$$  

A fruitful generalization of the above concept is given in the following definition.

**Definition 2.1.** ([9], [1]) A spherical function $\Phi$ on $G$ of type $\delta \in \hat{K}$ is a continuous function on $G$ with values in $\text{End}(V)$ such that

i) $\Phi(e) = I$. ($I =$ identity transformation).

ii) $\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky) \, dk$, for all $x, y \in G$.

**Proposition 2.2.** ([9], [1]) If $\Phi : G \rightarrow \text{End}(V)$ is a spherical function of type $\delta$ then:

i) $\Phi(kgk') = \Phi(k)\Phi(g)\Phi(k')$, for all $k, k' \in K$, $g \in G$.

ii) $k \mapsto \Phi(k)$ is a representation of $K$ such that any irreducible subrepresentation belongs to $\delta$.

Concerning the definition let us point out that the spherical function $\Phi$ determines its type univocally (Proposition 2.2) and let us say that the number of times that $\delta$ occurs in the representation $k \mapsto \Phi(k)$ is called the **height** of $\Phi$.

Let $\varphi$ be a complex valued continuous solution of the equation (1). If $\varphi$ is not identically zero then $\varphi(e) = 1$. (cf. [11], p. 399). This result generalizes in the following way: we shall say that a function $\Phi : G \rightarrow \text{End}(V)$ is **irreducible** whenever $\Phi(g)$, $g \in G$, is an irreducible family of linear transformations of $V$ into $V$. Then we have

**Proposition 2.3.** ([9]) Let $\Phi$ be an $\text{End}(V)$-valued continuous solution of equation ii) in Definition 2.1. If $\Phi$ is irreducible then $\Phi(e) = I$. 

Spherical functions of type $\delta$ arise in a natural way upon considering representations of $G$. If $g \mapsto U(g)$ is a continuous representation of $G$, say on a complete, locally convex, Hausdorff topological vector space $E$, then

$$P(\delta) = \int_K \chi_\delta(k^{-1})U(k) \, dk$$

is a continuous projection of $E$ onto $P(\delta)E = E(\delta)$; $E(\delta)$ consists of those vectors in $E$, the linear span of whose $K$-orbit is finite dimensional and splits into irreducible $K$-subrepresentations of type $\delta$. Whenever $E(\delta)$ is finite dimensional, the function $\Phi : G \to \text{End}(E(\delta))$ defined by $\Phi(g)a = P(\delta)U(g)a$, $g \in G$, $a \in E(\delta)$ is a spherical function of type $\delta$. In fact, if $a \in E(\delta)$ we have

$$\Phi(x)\Phi(y)a = P(\delta)U(x)P(\delta)U(y)a = \int_K \chi_\delta(k^{-1})P(\delta)U(x)U(k)U(y)a \, dk = \left(\int_K \chi_\delta(k^{-1})\Phi(xky) \, dk\right) a.$$

If the representation $g \mapsto U(g)$ is topologically irreducible (i.e. $E$ admits no non-trivial closed $G$-invariant subspace) then the associated spherical function $\Phi$ is also irreducible.

If a spherical function $\Phi$ is associated to a Banach representation of $G$ then it is quasi-bounded, in the sense that there exists a semi-norm $\rho$ on $G$ and $M \in \mathbb{R}$ such that $\|\Phi(g)\| \leq M\rho(g)$ for all $g \in G$. Conversely, if $\Phi$ is an irreducible quasi-bounded spherical function on $G$, then it is associated to a topologically irreducible Banach representation of $G$ (Godement, see [24]). Thus if $G$ is compact any irreducible spherical function on $G$ is associated to a Banach representation of $G$, which is finite dimensional by Peter-Weyl theorem.

From now on we shall assume that $G$ is a connected Lie group. Then it is not difficult to prove that any spherical function $\Phi : G \to \text{End}(V)$ is differentiable ($C^\infty$), and moreover that it is analytic. Let $D(G)$ denote the algebra of all left invariant differential operators on $G$ and let $D(G)^K$ denote the subalgebra of all operators in $D(G)$ which are invariant under all right translation by elements in $K$.

In the following proposition $(V, \pi)$ will be a finite dimensional representation of $K$ such that any irreducible subrepresentation belongs to the same class $\delta \in \hat{K}$.

**Proposition 2.4.** ([20], [31]) A function $\Phi : G \to \text{End}(V)$ is a spherical function of type $\delta$ if and only if

i) $\Phi$ is analytic.

ii) $\Phi(k_1gk_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.

iii) $[D\Phi](g) = \Phi(g)[D\Phi](e)$, for all $D \in D(G)^K$, $g \in G$.

Let us observe that if $\Phi : G \to \text{End}(V)$ is a spherical function then $\Phi : D \mapsto [D\Phi](e)$ maps $D(G)^K$ into $\text{End}_K(V)$ ($\text{End}_K(V)$ denotes the space
of all linear maps of $V$ into $V$ which commute with $\pi(k)$ for all $k \in K$ defining a finite dimensional representation of the associative algebra $D(G)^K$. Moreover the spherical function is irreducible if and only if the representation $\Phi : D(G)^K \to \text{End}_K(V)$ is surjective. As a consequence of this we have:

**Proposition 2.5.** ([20], [3]) The following properties are equivalent:

i) $D(G)^K$ is commutative.

ii) Every irreducible spherical function of $(G,K)$ is of height one.

In this paper the pair $(G,K)$ is $(\text{SU}(3), \text{S(U(2) × U(1)))}$. Then it is known that $D(G)^K$ is abelian in this case; moreover $D(G)^K$ is isomorphic to $D(G)^G \otimes D(K)^K$ (cf. [3], [17]), where $D(G)^G$ (resp. $D(K)^K$) denotes the subalgebra of all operators in $D(G)$ (resp. $D(K)$) which are invariant under all right translations from $G$ (resp. $K$). Now a famous theorem of Harish-Chandra says that $D(G)^G$ (resp. $D(K)^K$) is a polynomial algebra in two algebraically independent generators $\Delta_2$ and $\Delta_3$ (resp. $Z$ and $\Delta_K$).

The first consequence of this is that all irreducible spherical functions of our pair $(G,K)$ are of height one.

The second consequence is that to find all spherical functions of type $\delta \in \hat{K}$ is equivalent to taking any irreducible representation $(V,\pi)$ of $K$ in the class $\delta$ and to determine all analytic functions $\Phi : G \to \text{End}(V)$ such that

1. $\Phi(k_1 g k_2) = \pi(k_1) \Phi(g) \pi(k_2)$, for all $k_1, k_2 \in K, g \in G$.
2. $[\Delta_j \Phi](g) = \Phi(g) [\Delta_j \Phi](e), \ j = 2, 3$.

In fact, because $Z$ and $\Delta_K$ are in $D(K)^K$ and $\Phi$ satisfies (1) we have

$$[Z \Phi](g) = \Phi(g) \hat{\pi}(Z) = \Phi(g) [Z \Phi](e)$$

and

$$[\Delta_K \Phi](g) = \Phi(g) \hat{\pi}(\Delta_K) = \Phi(g) [\Delta_K \Phi](e).$$

Here $\hat{\pi} : \mathfrak{k}_\mathbb{C} \to \text{End}(V_\pi)$ denotes the derivative of the representation $\pi$ of $K$. We also denote with $\hat{\pi}$ the representation of $D(K)$ in $\text{End}(V_\pi)$ induced by $\hat{\pi}$.

Therefore an analytic function $\Phi$ which satisfies (1) and (2) verifies conditions i), ii) and iii) of Proposition 2.4, and hence it is a spherical function.

The group $G = \text{SU}(3)$ consists of all $3 \times 3$ unitary matrices of determinant one. The subgroup $K = \text{S(U(2) × U(1))}$ consists of all unitary matrices of the form

$$k = \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}$$

where $A \in \text{U(2)}$ and $a = (\det A)^{-1}$.

Clearly the map $k \mapsto A$ defines a Lie isomorphism of $K$ onto $\text{U(2)}$. This isomorphism will be used freely in what follows. In particular $\hat{K}$ will be identified with $\text{U(2)}$. Let us recall that the identity representation $\pi_1$ of
U(2) in \( \mathbb{C}^2 \), as well as the \( \ell \)-symmetric power of it \( \pi_\ell : A \mapsto A^\ell \) of dimension \( \ell + 1 \) are irreducible. Moreover the representations \( \pi_{n,\ell} \) of \( U(2) \) defined by

\[
\pi_{n,\ell}(A) = (\det A)^n A^\ell \quad n \in \mathbb{Z}, \ell \in \mathbb{Z}_{\geq 0}
\]

is a complete set of representatives of elements in \( \hat{U}(2) \). Thus \( \hat{U}(2) \) can be identified with the set \( \mathbb{Z} \times \mathbb{Z}_{\geq 0} \).

Now let us observe that the representation \( \pi_{n,\ell} \) extends to a unique holomorphic representation of \( GL(2, \mathbb{C}) \) and that when \( n \geq 0 \) this extends in turn to a unique holomorphic multiplicative map of \( M(2, \mathbb{C}) \) into \( \text{End}(V_\ell) \) which we shall still denote by \( \pi_{n,\ell} \).

For any \( g \in M(3, \mathbb{C}) \), we shall denote by \( A(g) \) the left upper \( 2 \times 2 \) block of \( g \), i.e.

\[
A(g) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.
\]

Now we are in a position to introduce a particular spherical function on \( G \) of type \((n, \ell)\) for all \( n \geq 0 \). For any \( \pi = \pi_{n,\ell} \) with \( n \geq 0 \) we define the function \( \Phi_{\pi} : G \rightarrow \text{End}(V_\ell) \) in the following way:

\[
\Phi_{n,\ell}(g) = \pi_{n,\ell}(A(g)).
\]

Observe that when \( n < 0 \) the above formula defines a function \( \Phi_{\pi} \) on the set \( A = \{ g \in G : \det A(g) \neq 0 \} \). Although in this case \( \Phi_{\pi} \) is no longer spherical it will have a useful role to play.

To state the following lemma we need some notation. Let \( E_{i,j} \) denote the \( 3 \times 3 \) matrix with 1 in the \((i,j)\)-place and 0 elsewhere. We define the elements \( X_\beta \) and \( X_\gamma \) in the complexified Lie algebra \( \mathfrak{s}\mathfrak{l}(3, \mathbb{C}) \) of \( SU(3) \) by:

\[
X_\beta = E_{2,3} \quad \text{and} \quad X_\gamma = E_{1,3}.
\]

**Lemma 2.6.** The function \( \Phi_{\pi} \) has the following properties:

i) \( \Phi_{\pi}(k) = \pi(k) \), for all \( k \in K \).

ii) \( \Phi_{\pi}(k_1gk_2) = \Phi_{\pi}(k_1)\Phi_{\pi}(g)\Phi_{\pi}(k_2) \), for all \( k_1, k_2 \in K, g \in A \).

iii) \( X_\beta(\Phi_{\pi}(g)) = X_\gamma(\Phi_{\pi}(g)) = 0 \), for all \( g \in A \).

**Proof.** i) is obvious and ii) follows directly from the definition of \( \Phi_{\pi} \) upon observing that \( A(gk) = A(g)A(k) \) and \( A(kg) = A(k)A(g) \) for all \( g \in G, k \in K \).

iii) The function \( \Phi_{\pi} \) extends to a unique holomorphic function

\[
\Phi_{\pi} : SL(3, \mathbb{C}) \rightarrow \text{End}(V).
\]

Therefore for \( g \in G \) we have

\[
X_\beta \left( \Phi_{\pi} \right)(g) = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\pi}(g \exp t X_\beta) = 0,
\]

because \( A(g \exp t X_\beta) = A(g) \) for all \( t \). In a similar way it follows that \( X_\gamma(\Phi_{\pi}) = 0 \).

Before proving the next theorem we need some more preparation.
Lemma 2.7. Let $g \in \text{SU}(3)$ and let $g_{(ij)}$ be the $2 \times 2$ matrix obtained from $g$ deleting the $i$-th row and the $j$-th column. Then

$$g_{ij} = (-1)^{i+j} \det(g_{(ij)}).$$

The set $\mathcal{A} = \{ g \in G : \det A(g) \neq 0 \}$ is an open dense subset of $G$. Observe that by Lemma 2.7, $\mathcal{A}$ can also be described as the set of all $g \in G$ such that $g_{33} \neq 0$ and that $\mathcal{A}$ is stable by left and right multiplication by elements in $K$.

The group $G = \text{SU}(3)$ acts in a natural way in the complex projective plane $P_2(\mathbb{C})$. This action is transitive and $K$ is the isotropy subgroup of the point $(0,0,1) \in P_2(\mathbb{C})$. Therefore

$$P_2(\mathbb{C}) \simeq G/K.$$

Moreover the $G$-action on $P_2(\mathbb{C})$ corresponds to the action induced by left multiplication on $G/K$. We shall identify the complex plane $\mathbb{C}^2$ with the affine plane $\{ (x,y,1) \in P_2(\mathbb{C}) : (x,y) \in \mathbb{C}^2 \}$ under the map $(x,y) \mapsto (x,y,1)$.

The projection map $p : G \rightarrow P_2(\mathbb{C})$ defined by $p(g) = g \cdot (0,0,1)$ maps the open set $\mathcal{A}$ onto the affine plane $\mathbb{C}^2$.

Lemma 2.8. Given $H \in C^\infty(\mathbb{C}^2)$ denote also by $H \in C^\infty(\mathcal{A})$ the function defined by $H(g) = H(p(g)), g \in \mathcal{A}$. Then we have

$$(X_\beta H)(g) = \frac{\overline{g}_{11}}{g_{33}} \frac{\partial H}{\partial y} - \frac{\overline{g}_{21}}{g_{33}} \frac{\partial H}{\partial x}, \quad (X_\gamma H)(g) = -\frac{\overline{g}_{12}}{g_{33}} \frac{\partial H}{\partial y} + \frac{\overline{g}_{22}}{g_{33}} \frac{\partial H}{\partial x}.$$  

Corollary 2.9. A function $H \in C^\infty(\mathbb{C}^2)$ is antiholomorphic if and only if $(X_\beta H)(g) = (X_\gamma H)(g) = 0$ for all $g \in \mathcal{A}$.

Proof. The Cauchy Riemann equations say that $H$ is antiholomorphic precisely when $\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = 0$. Now the corollary follows from Lemma 2.8 upon observing that for $g \in \mathcal{A}$ the matrix

$$\frac{1}{g_{33}} \begin{pmatrix} \overline{g}_{11} & -\overline{g}_{21} \\ -\overline{g}_{12} & \overline{g}_{22} \end{pmatrix}$$

is non singular. \hfill $\square$

Theorem 2.10. For $n \geq 0$ and $\pi = \pi_n, \ell$, the function $\Phi_{\pi}$ is an irreducible spherical function of type $\pi$. 
Proof. We shall prove that $\Phi_\pi$ satisfies the integral equation
\[
\Phi_\pi(g)\Phi_\pi(a) = \int_K \chi_\pi(k^{-1})\Phi_\pi(gka)\,dk,
\]
for all $g, a \in G$.

We fix $g \in G$ and we consider the function $F : A \to \text{End}(V_\pi)$ defined by
\[
F(a) = \left(\int_K \chi_\pi(k^{-1})\Phi_\pi(gka)\,dk\right)\Phi_\pi(a)^{-1}.
\]
Then we have $F(ak) = F(a)$, for all $k \in K$. Therefore we may consider $F$ as a function defined on the affine plane $\mathbb{C}^2$ in $P_2(\mathbb{C})$. By Lemma 2.6 iii) we have $X_\beta F = X_\chi F = 0$. Thus $F : \mathbb{C}^2 \to \text{End}(V_\pi)$ is an antiholomorphic function (Corollary 2.3).

We fix $a \in A$ and let $p(a) = (x, y, 1)$. Now, let us consider the function $f(w) = F(\overline{w}x, \overline{w}y), w \in \mathbb{C}$. Then $f$ is a holomorphic function on $\mathbb{C}$ with values in $\text{End}(V_\pi)$.

Let $c(\vartheta) = \left(\begin{array}{ccc}
e^{-i\vartheta/3} & 0 & 0 \\
0 & e^{-i\vartheta/3} & 0 \\
0 & 0 & e^{2i\vartheta/3} \end{array}\right) \in K$. Then $p(c(\vartheta)) = (e^{-i\vartheta}x, e^{-i\vartheta}y, 1)$ and
\[
f(e^{i\vartheta}) = F(e^{-i\vartheta}x, e^{-i\vartheta}y) = F(c(\vartheta)a).
\]

We have
\[
F(c(\vartheta)a) = \left(\int_K \chi_\pi(k^{-1})\Phi_\pi(gkc(\vartheta)a)\,dk\right)\Phi_\pi(c(\vartheta)a)^{-1}
\]
\[
= \left(\int_K \chi_\pi(c(\vartheta)k^{-1})\Phi_\pi(gka)\,dk\right)\Phi_\pi(a)^{-1}\pi(c(\vartheta))^{-1}.
\]

Now we notice that $c(\vartheta)$ is in the center of $K$, thus $\pi(c(\vartheta))$ is a scalar (Schur’s lemma) and $\chi_\pi(c(\vartheta)k) = \pi(c(\vartheta))\chi_\pi(k)$. Therefore
\[
f(e^{i\vartheta}) = F(c(\vartheta)a) = F(a),
\]
\[
F(x, y) = f(1) = f(0) = F(0, 0),
\]
for all $(x, y) \in \mathbb{C}^2$. Therefore $F(a) = F(e)$ for all $a \in A$, hence
\[
\left(\int_K \chi_\pi(k^{-1})\Phi_\pi(gka)\,dk\right)\Phi_\pi(a)^{-1} = \int_K \chi_\pi(k^{-1})\Phi_\pi(gk)\,dk = \Phi_\pi(g)
\]
since $\int_K \chi_\pi(k^{-1})\pi(k)\,dk = I$ (orthogonality relations). So, we have proved that
\[
\Phi_\pi(g)\Phi_\pi(a) = \int_K \chi_\pi(k^{-1})\Phi_\pi(gka)\,dk,
\]
for any $g \in G$ and all $a \in A$. Since $A$ is dense in $G$ and $\Phi_\pi$ is continuous the theorem is proved.

\[\square\]

Proposition 2.11. Given any pair $(G, K)$ and a spherical function $\Phi$ on $G$ of type $\pi \in \hat{K}$ the function $\Phi^* : G \to \text{End} V^*$ defined by $\Phi^*(g) = \Phi(g^{-1})^T$ is spherical of type $\pi^*$, where $\pi^*$ denotes the contragradient representation of $\pi$.  

Proof. The assertion follows directly from Definition 2.1.

Corollary 2.12. If $\Phi$ is a spherical function of the pair $(SU(3), U(2))$ of type $(n, \ell)$, then $\Phi^*$ is spherical of type $(-n - \ell, \ell)$.

Proof. The only thing we have to observe is that $\pi^*_{n,\ell}$ is equivalent to $\pi_{-n-\ell,\ell}$.

Therefore to determine all spherical functions of type $(n, \ell)$ for all $n \in \mathbb{Z}$ it is enough to determine all spherical functions of type $(n, \ell)$ with $n \geq -\frac{\ell}{2}$. Observe that for $n \geq 0$ or $n \leq -\ell$ we have exhibited a particular spherical function of type $(n, \ell)$: precisely $\Phi_{n,\ell}$ or $\Phi_{-n-\ell,\ell}$, respectively.

It is useful to keep in mind the symmetry $n \mapsto -n - \ell$ and the partition of the integers in the form

\[ \mathbb{Z} = \{ n : n \geq 0 \} \cup \{ n : -\ell < n < 0 \} \cup \{ n : n \leq -\ell \}. \]

The spherical functions $\Phi_{n,\ell}$ correspond to taking an irreducible representation $U$ of $G$ and projecting to the subrepresentation of $K$ generated by a highest weight vector of $U$, precisely when $n \geq 0$. Moreover the spherical functions $\Phi^*_{n,\ell}$ correspond to taking an irreducible representation $U$ of $G$ and projecting to the subrepresentation of $K$ generated by a lowest weight vector of $U$, exactly when $n \leq -\ell$. In Section 10 this partition of $\mathbb{Z}$ will appear again. Notice that the exceptional interval $\{ n : -\ell < n < 0 \}$ occurs only when $\ell \geq 2$.

To exhibit explicitly a spherical function $\Psi_{n,\ell}$ on $G$ of type $(n, \ell)$ for all $n < 0$, we need to introduce the following notation. If $T$ and $S$ are linear operators on a finite dimensional vector space $V$ we shall denote by $T^{\ell-i} \cdot S^i$ the linear map induced by $\ell - i$ factors equal to $T$ and $i$ factors equal to $S$ acting on the space of symmetric tensors of rank $\ell$. Also for $g \in G$ we let $a(g) = g_{33}$ and

\[ B(g) = \begin{pmatrix} g_{13} g_{31} & g_{13} g_{32} \\ g_{23} g_{31} & g_{23} g_{32} \end{pmatrix}. \]

The proof of the following theorem will be based on Proposition 2.4 and will be given at the end of Section 4. This way of proving is needed because the function $\Psi_{n,\ell}$ does not satisfy a property like the one given in Lemma 2.6 iii) which allows one to establish Theorem 2.10.

**Theorem 2.13.** For $n \leq 0$

\[ \Psi_{n,\ell}(g) = \sum_{0 \leq i \leq \min\{-n, \ell\}} \binom{-n}{i} a(g)^{-n-i} A(g)^{\ell-i} \cdot B(g)^i \]

is an irreducible spherical function of type $(n, \ell)$.

Notice that $\Psi_{0,\ell} = \Phi_{0,\ell}$. 
3. Preliminaries

The Lie algebra of $G$ is $\mathfrak{g} = \{ X \in \mathfrak{gl}(3, \mathbb{C}) : X = -X^T, \, \text{tr} X = 0 \}$. Its complexification is $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(3, \mathbb{C})$. The Lie algebra $\mathfrak{k}$ of $K$ can be identified with $\mathfrak{u}(2)$ and its complexification $\mathfrak{k}_\mathbb{C}$ with $\mathfrak{gl}(2, \mathbb{C})$.

The following matrices form a basis of $\mathfrak{g}$.

$$ H_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2i \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, $$

$$ Y_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ of all diagonal matrices. The corresponding root space structure is given by

$$ X_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{-\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, $$

$$ X_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{-\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad H_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, $$

$$ X_\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X_{-\gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, $$

where

$$ \alpha(x_1E_{11} + x_2E_{22} + x_3E_{33}) = x_1 - x_2, $$

$$ \beta(x_1E_{11} + x_2E_{22} + x_3E_{33}) = x_2 - x_3, $$

$$ \gamma(x_1E_{11} + x_2E_{22} + x_3E_{33}) = x_1 - x_3. $$

We have

$$ X_\alpha = \frac{1}{2}(Y_1 - iY_2), \quad X_\beta = \frac{1}{2}(Y_5 - iY_6), \quad X_\gamma = \frac{1}{2}(Y_3 - iY_4), $$

$$ X_{-\alpha} = -\frac{1}{2}(Y_1 + iY_2), \quad X_{-\beta} = -\frac{1}{2}(Y_5 + iY_6), \quad X_{-\gamma} = -\frac{1}{2}(Y_3 + iY_4). $$

Let $Z = H_\alpha + 2H_\beta$, $\tilde{H}_1 = 2H_\alpha + H_\beta$ and $\tilde{H}_2 = H_\beta - H_\alpha$.

Now we shall prove the following lemma stated in Section 2.

**Lemma 2.8.** Given $H \in C^\infty(\mathbb{C}^2)$ denote also by $H \in C^\infty(A)$ the function defined by $H(g) = H(p(g))$, $g \in A$. Then we have

$$ (X_\beta H)(g) = \frac{\partial H}{\partial y} \frac{\partial H}{\partial y}, \quad (X_\gamma H)(g) = \frac{\partial H}{\partial y} + \frac{\partial H}{\partial x}. $$

**Proof.**

$$ 2X_\beta = Y_5 - iY_6 \quad \text{and} \quad 2X_\gamma = Y_3 - iY_4. \quad \text{We have} \quad p(g \exp tY_5) = \begin{pmatrix} g_{12} \tan t + g_{32} \tan t + g_{33} & g_{22} \tan t + g_{23} \\ g_{32} \tan t + g_{33} & g_{32} \tan t + g_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, $$

$$ = (u(t), v(t), 1) = (u_1(t) + iv_2(t), v_1(t) + iv_2(t), 1). $$
If we put \( \dot{u}_j = \left( \frac{d u_j}{dt} \right)_{t=0} \) and \( \dot{v}_j = \left( \frac{d v_j}{dt} \right)_{t=0} \) for \( j = 1, 2 \), then
\[
Y_5(H)(g) = \left( \frac{d}{dt} H(p(g \exp tY_5)) \right)_{t=0} = H_{x_1} \dot{u}_1 + H_{x_2} \dot{u}_2 + H_{y_1} \dot{v}_1 + H_{y_2} \dot{v}_2.
\]
Also by Lemma 2.7 we have,
\[
\dot{u} = -\frac{\mathcal{g}_{21}}{g_{33}}, \quad \dot{v} = \frac{\mathcal{g}_{11}}{g_{33}}.
\]
Similarly, for \( Y_6 \) let
\[
p(g \exp tY_6) = (\ddot{u}(t), \ddot{v}(t), 1) = (\ddot{u}_1(t) + i\ddot{u}_2(t), \ddot{v}_1(t) + i\ddot{v}_2(t), 1).
\]
Then
\[
Y_6(H)(g) = H_{x_1} \ddot{u}_1 + H_{x_2} \ddot{u}_2 + H_{y_1} \ddot{v}_1 + H_{y_2} \ddot{v}_2,
\]
where
\[
\ddot{u} = -\frac{i \mathcal{g}_{21}}{g_{33}}, \quad \ddot{v} = \frac{i \mathcal{g}_{11}}{g_{33}}.
\]
If \( u(t) = u_1(t) + iv_2(t) \) then \( \dot{u}_1 = \text{Re}(\ddot{u}) \) and \( \dot{u}_2 = \text{Im}(\ddot{u}) \). Therefore
\[
(3) \quad 2X_\beta(H)(g)
\]
\[
= H_{x_1} (\ddot{u}_1 - i\ddot{u}_2) + H_{x_2} (\ddot{u}_2 - i\ddot{u}_1) + H_{y_1} (\ddot{v}_1 - i\ddot{v}_2) + H_{y_2} (\ddot{v}_2 - i\ddot{v}_1)
\]
\[
= -H_{x_1} \frac{\mathcal{g}_{21}}{g_{33}} + iH_{x_2} \frac{\mathcal{g}_{21}}{g_{33}} + H_{y_1} \frac{\mathcal{g}_{11}}{g_{33}} - iH_{y_2} \frac{\mathcal{g}_{11}}{g_{33}}
\]
\[
= -2 \frac{\mathcal{g}_{21}}{g_{33}} \frac{\partial H}{\partial x} + 2 \frac{\mathcal{g}_{11}}{g_{33}} \frac{\partial H}{\partial y}.
\]
We proceed in the same way with \( (X_\gamma H)(g) \) and complete the proof of the lemma. \( \square \)

**Proposition 3.1.** \( D(G)^G \) as a polynomial algebra is generated by
\[
\Delta_2 = -H_\alpha^2 - \frac{1}{9} Z^2 - 2H_\alpha - 2Z - 4X_{-\alpha}X_\alpha - 4X_{-\beta}X_\beta - 4X_{-\gamma}X_\gamma
\]
and
\[
\Delta_3 = \frac{8}{9} H_\alpha^3 - \frac{8}{9} H_\beta^3 + \frac{4}{3} H_\alpha^2 H_\beta - \frac{4}{3} H_\alpha H_\beta^2 + 8H_\alpha^2 + 4H_\alpha H_\beta + 16H_\alpha + 8H_\beta
\]
\[
+ 4X_{-\alpha}X_\alpha H_\alpha + 8X_{-\alpha}X_\alpha H_\beta + 24X_{-\alpha}X_\alpha + 12 (X_{-\alpha}X_\beta + X_{-\gamma}X_\gamma)
\]
\[
- 4X_{-\beta}X_\beta \tilde{H}_1 - 4X_{-\gamma}X_\gamma \tilde{H}_2 + 12X_{-\beta}X_\gamma X_{-\alpha} + 12X_{-\gamma}X_\beta X_\alpha.
\]

**Proof.** Since \( G \) is a connected Lie group, to verify that \( \Delta_2 \) and \( \Delta_3 \) are in \( D(G)^G \) it is enough to check that they are of weight zero and that \( \text{ad}(X_\alpha)(\Delta_j) = \text{ad}(X_\beta)(\Delta_j) = 0 \) for \( j = 2, 3 \). This can be easily accomplished. To prove that \( 1, \Delta_2 \) and \( \Delta_3 \) are algebraically independent generators of \( D(G)^G \) one can use the well known Harish-Chandra isomorphism \( \xi : D(G)^G \rightarrow S(\mathfrak{h})^W \) where \( S(\mathfrak{h})^W \) denotes the Weyl group invariants in the symmetric algebra of \( \mathfrak{h} \). (See [23], Section 3.2.2.) We have
\[
\xi(\Delta_2) = -H_\alpha^2 - \frac{1}{9} Z^2 + 4, \quad \xi(\Delta_3) = -\frac{1}{9} Z^3 + H_\alpha^2 Z + Z^2 + 3H_\alpha - 36.
\]
If we put $X_1 = \frac{\sqrt{7}}{3} Z$ and $X_2 = H_{\alpha}$ then one can verify that $S(\mathfrak{h})^W$ is generated by the algebraically independent elements $1$, $X_1^2 + X_2^2$, and $X_1 (X_1^2 - 3X_2)$. (See Proposition 4 in [21]). This completes the proof of the proposition.

We write the operators $\Delta_2$ and $\Delta_3$ in the form

$$\Delta_2 = \Delta_{2,K} + \tilde{\Delta}_2, \quad \Delta_3 = \Delta_{3,K} + \tilde{\Delta}_3$$

where

$$\Delta_{2,K} = -H_{\alpha}^2 - \frac{1}{3} Z^2 - 2H_{\alpha} - 2Z - 4X_{-\alpha}X_{\alpha} \in D(K)^K,$$

$$\tilde{\Delta}_2 = -4(X_{-\beta}X_{\beta} + X_{-\gamma}X_{\gamma}) \in D(G)^K,$$

$$\Delta_{3,K} = \frac{8}{9} H_{\alpha}^3 - \frac{8}{3} H_{\beta}^3 \quad + \frac{4}{3} H_{\alpha}^2 H_{\beta} - \frac{4}{3} H_{\alpha} H_{\beta}^2 + 8H_{\alpha}^2 + 4H_{\alpha} H_{\beta} + 16H_{\alpha} + 8H_{\beta}$$

$$\quad + 4X_{-\alpha}X_{\alpha}H_{\alpha} + 8X_{-\alpha}X_{\alpha}H_{\beta} + 24X_{-\alpha}X_{\alpha} \in D(K)^K,$$

$$\tilde{\Delta}_3 = 12(X_{-\beta}X_{\beta} + X_{-\gamma}X_{\gamma}) - 4X_{-\beta}X_{\beta} \tilde{H}_1 - 4X_{-\gamma}X_{\gamma} \tilde{H}_2$$

$$\quad + 12X_{-\beta}X_{\gamma}X_{-\alpha} + 12X_{-\gamma}X_{\beta}X_{\alpha} \in D(G)^K.$$

4. Reduction to $P_2(\mathbb{C})$

We want to determine all spherical functions $\Phi : G \rightarrow \text{End}(V_{\pi})$ of type $\pi = \pi_{n,\ell}$. For this purpose we use the function $\Phi_\pi \in C^\infty(\mathcal{A}) \otimes \text{End}(V_{\pi})$ defined by $\Phi_\pi(g) = \pi(A(g))$, in the following way: in the open set $\mathcal{A}$ we define a function $H$ by

$$H(g) = \Phi(g) \Phi_\pi(g)^{-1},$$

where $\Phi$ is supposed to be a spherical function of type $\pi$. Then $H$ satisfies

i) $H(e) = I$.

ii) $H(gk) = H(g)$, for all $g \in \mathcal{A}, k \in K$.

iii) $H(kg) = \pi(k)H(g)\pi(k^{-1})$, for all $g \in \mathcal{A}, k \in K$.

The projection map $p : G \rightarrow P_2(\mathbb{C})$ defined before maps the open set $\mathcal{A}$ onto the affine plane $\mathbb{C}^2 = \{(x, y, 1) \in P_2(\mathbb{C}) : (x, y) \in \mathbb{C}^2\}$. Thus ii) says that $H$ may be considered as a function on $\mathbb{C}^2$.

The fact that $\Phi$ is an eigenfunction of $\Delta_2$ and $\Delta_3$, makes $H$ into an eigenfunction of certain differential operators on $\mathbb{C}^2$, to be determined now.

In the open set $\mathcal{A} \subset G$ let us consider the following differential operators. For $H \in C^\infty(\mathcal{A}) \otimes \text{End}(V_{\pi})$ let

$$D(H) = D_1(H) + D_2(H), \quad E(H) = E_1(H) + E_2(H),$$

where

$$D_1(H) = -4(X_{-\beta}X_{\beta} + X_{-\gamma}X_{\gamma})(H),$$

$$D_2(H) = -4 \left( X_{\beta}(H)X_{-\beta}(\Phi_{\pi})\Phi_{\pi}^{-1} + X_{\gamma}(H)X_{-\gamma}(\Phi_{\pi})\Phi_{\pi}^{-1} \right),$$

and

$$E_1(H) = \left( X_{\beta}(H)X_{-\beta}(\Phi_{\pi})\Phi_{\pi}^{-1} - X_{\gamma}(H)X_{-\gamma}(\Phi_{\pi})\Phi_{\pi}^{-1} \right).$$
Proof. In this way \( \Delta(X,Y) \) for \( H \in K \), then we shall also denote by \( \Phi(H) \) the function \( \Phi = H \Phi \). The function \( \Phi = H \Phi \) is an eigenfunction of \( \Delta_2 \) and \( \Delta_3 \) on \( A \) if and only if \( H \) is an eigenfunction of \( D \) and \( E \). Moreover, if \( \lambda, \mu, \lambda \) and \( \mu \) denote respectively the corresponding eigenvalues of \( \Delta_2, \Delta_3, D \) and \( E \) then

\[
\lambda = \tilde{\lambda} - \hat{\pi}(\Delta_2,K), \quad \mu = \tilde{\mu} + 3\lambda - \hat{\pi}(\Delta_3,K).
\]

Proposition 4.1. For \( H \in C^\infty(A) \otimes \text{End}(V) \) right invariant under \( K \), the function \( \Phi = H \Phi \) is an eigenfunction of \( \Delta_2 \) and \( \Delta_3 \) on \( A \) if and only if \( H \) is an eigenfunction of \( D \) and \( E \). Moreover, if \( \lambda, \mu, \lambda \) and \( \mu \) denote respectively the corresponding eigenvalues of \( \Delta_2, \Delta_3, D \) and \( E \) then

\[
\lambda = \tilde{\lambda} - \hat{\pi}(\Delta_2,K), \quad \mu = \tilde{\mu} + 3\lambda - \hat{\pi}(\Delta_3,K).
\]

Proof. If \( X \in \mathfrak{k} \) then \( X(H) = 0 \). So, \((XY)(H\Phi) = H(XY)(\Phi)\), for all \( X,Y \in \mathfrak{k} \). More generally \( \Delta(X\Phi) = H \Delta(\Phi) = H\Phi \hat{\pi}(\Delta) \), for all \( \Delta \in D(K) \).

Since \( X\beta(H\Phi) = X\gamma(H\Phi) = 0 \) we obtain

\[
(X_{-\beta}X_\beta + X_{-\gamma}X_\gamma)(H\Phi) = (X_{-\beta}X_\beta + X_{-\gamma}X_\gamma)(H)\Phi + X_\beta(H)X_{-\beta}(\Phi) + X_\gamma(H)X_{-\gamma}(\Phi).
\]

In this way

\[
\Delta_2(H\Phi) = (H\Phi)\hat{\pi}(\Delta_2,K) + D(H)\Phi.
\]

Similarly for \( \Delta_3 \) we have

\[
\Delta_3(H\Phi) = (H\Phi)\hat{\pi}(\Delta_3,K) - 4(X_{-\beta}X_\beta(H\Phi))\hat{\pi}(H_1)
\]

\[
-4(X_{-\gamma}X_\gamma(H\Phi))\hat{\pi}(H_2) + 12(X_{-\beta}X_\gamma(H\Phi))\hat{\pi}(X_{-\alpha})
\]

\[
+12(X_{-\gamma}X_\beta(H\Phi))\hat{\pi}(X_{-\alpha}) + 12(X_{-\beta}X_\beta + X_{-\gamma}X_\gamma)(H\Phi)
\]

\[
=(H\Phi)\hat{\pi}(\Delta_3,K) + E(H)\Phi + 3D(H)\Phi.
\]

By Schur’s lemma \( \hat{\pi}(\Delta_2,K) \) and \( \hat{\pi}(\Delta_3,K) \) are scalar because \( \Delta_2,K \) and \( \Delta_3,K \in D(K)^K \). Now it is clear that \( \Delta_2(H\Phi) = \tilde{\lambda}(H\Phi) \) and \( \Delta_3(H\Phi) = \tilde{\mu}(H\Phi) \) if and only if \( D(H) = \lambda H \) and \( E(H) = \mu H \) with

\[
\lambda = \tilde{\lambda} - \hat{\pi}(\Delta_2,K) \quad \text{and} \quad \mu = \tilde{\mu} - \hat{\pi}(\Delta_3,K) + 3\lambda.
\]

Given \( H \in C^\infty(C^2) \otimes \text{End}(V) \) we shall also denote by \( H \in C^\infty(A) \otimes \text{End}(V) \) the function defined by \( H(g) = H(p(g)) \), \( g \in A \). Moreover, if \( F \) is a linear endomorphism of \( C^\infty(A) \otimes \text{End}(V) \) which preserves the subspace \( C^\infty(A)^K \otimes \text{End}(V) \) of all functions which are right invariant by elements in \( K \), then we shall also denote by \( F \) the endomorphism of \( C^\infty(C^2) \otimes \text{End}(V) \) which satisfies \( \tilde{F}(H)(p(g)) = F(H)(g), g \in A \), \( H \in C^\infty(C^2) \otimes \text{End}(V) \).
Lemma 4.2. The differential operators $D_j$ and $E_j$ ($j = 1, 2$) introduced above, define differential operators $D_j$ and $E_j$ acting on $C^\infty(C^2) \otimes \text{End}(V_\pi)$.

Proof. The only thing we really need to proving is that $D_j$ and $E_j$ ($j = 1, 2$) preserve the subspace $C^\infty(A)^K \otimes \text{End}(V_\pi)$.

It is easy to see that $D_1 = -4(X_{-\beta}X_\beta + X_{-\gamma}X_\gamma) \in D(G)^K$. Then $D_1$ preserves $C^\infty(A)^K \otimes \text{End}(V_\pi)$. From (3) we get

$$D(H) = \Delta_2(D\Phi_\pi) \Phi^{-1} - \dot{\pi}(\Delta_2,K)H.$$ 

Since $\Delta_2(D\Phi_\pi) \Phi^{-1} \in C^\infty(A)^K \otimes \text{End}(V_\pi)$ it follows that $D$ and therefore $D_2$ preserve $C^\infty(A)^K \otimes \text{End}(V_\pi)$.

Let us now check that $E_2$ has the same property. Since $K$ is connected this is equivalent to verifying that for any $X \in \mathfrak{t}$ and all $H \in C^\infty(A)^K \otimes \text{End}(V_\pi)$ we have $XE_2(H) = 0$, which in turns amounts to prove that

$\begin{align*}
-XX_\beta(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(\tilde{H}_1) - X_\beta(H)XX_{-\beta}(\Phi_\pi)\dot{\pi}(\tilde{H}_1) \\
+ X_\beta(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(\tilde{H}_1)\dot{\pi}(\tilde{X}_1) - XX_\beta(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(\tilde{H}_2) \\
- X_\gamma(H)XX_{-\gamma}(\Phi_\pi)\dot{\pi}(\tilde{H}_2) + X_\gamma(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(\tilde{H}_2)\dot{\pi}(\tilde{X}_1) \\
+ 3XX_\beta(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_{-a}) + 3X_\gamma(H)XX_{-\beta}(\Phi_\pi)\dot{\pi}(X_{-a}) \\
- 3X_\gamma(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_{-a})\dot{\pi}(X) + 3XX_\beta(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(X_{ao}) \\
+ 3X_\beta(H)XX_{-\gamma}(\Phi_\pi)\dot{\pi}(X_{ ao}) - 3X_\beta(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(X_{ ao})\dot{\pi}(X) = 0.
\end{align*}$

Because $X(H) = 0$ and $X(\Phi_\pi) = \Phi_\pi\dot{\pi}(X)$, this is also equivalent to showing that

$\begin{align*}
- [X, X_\beta](H)X_{-\beta}(\Phi_\pi)\dot{\pi}(\tilde{H}_1) - X_\beta(H)[X, X_{-\beta}](\Phi_\pi)\dot{\pi}(\tilde{H}_1) \\
+ X_\beta(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(\tilde{H}_1)\dot{\pi}(\tilde{X}_1) - [X, X_\gamma](H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(\tilde{H}_2) \\
- X_\gamma(H)[X, X_{-\gamma}](\Phi_\pi)\dot{\pi}(\tilde{H}_2) + X_\gamma(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(\tilde{H}_2)\dot{\pi}(\tilde{X}_1) \\
+ 3[X, X_\beta](H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_{-a}) + 3X_\gamma(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_{-a}) \\
- 3X_\gamma(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_{-a})\dot{\pi}(X) + 3[X, X_\beta](H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(X_{ ao}) \\
+ 3X_\beta(H)[X, X_{-\gamma}](\Phi_\pi)\dot{\pi}(X_{ ao}) - 3X_\beta(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(X_{ ao})\dot{\pi}(X) = 0.
\end{align*}$

If we put $X = T \in \mathfrak{h}$ and use $\gamma(T) = \alpha(T) + \beta(T)$ we get

$\begin{align*}
- \beta(T)X_{-\beta}(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(\tilde{H}_1) + \beta(T)X_\beta(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(\tilde{H}_1) \\
- \gamma(T)X_\gamma(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(\tilde{H}_2) + \gamma(T)X_\gamma(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(\tilde{H}_2) \\
+ 3\gamma(T)X_\gamma(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_{-a}) - 3\beta(T)X_\gamma(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_{-a}) \\
- 3\alpha(T)X_\gamma(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_{-a}) + 3\beta(T)X_\beta(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(X_{ ao}) \\
- 3\gamma(T)X_\beta(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(X_{ ao}) + 3\alpha(T)X_\beta(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(X_{ ao}) = 0.
\end{align*}$

If we substitute $X = X_\alpha$ in (3) and use $[X_\alpha, X_\beta] = X_\gamma$, $[X_\alpha, X_{-\alpha}] = -X_{-\beta}$, $[X_\alpha, X_{-\gamma}] = H_{\alpha}$, $[X_\alpha, X_\gamma] = [X_\alpha, X_{-\beta}] = 0$, we obtain

$\begin{align*}
- X_\gamma(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(\tilde{H}_1) + \alpha(\tilde{H}_1)X_\beta(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_{ ao})
\end{align*}$
Proposition 4.3. For $H \in C^\infty(\mathbb{C}^2) \otimes \text{End}(V_\pi)$ we have

$$D_1(H)(x, y) = (1 + |x|^2 + |y|^2)(H_{x_1x_1} + H_{x_2x_2})(1 + |x|^2) + (H_{y_1y_1} + H_{y_2y_2})(1 + |y|^2) + 2(H_{y_1x_1} + H_{y_2x_2}) \text{Re}(x\overline{y}) + 2(H_{y_1x_2} - H_{y_2x_1}) \text{Im}(x\overline{y}).$$

Proposition 4.4. For $H \in C^\infty(\mathbb{C}^2) \otimes \text{End}(V_\pi)$ we have

$$D_2(H)(x, y) = -4 \frac{\partial H}{\partial x} \hat{\pi} \begin{pmatrix} x(1 + |x|^2) & x^2\overline{y} \\ y(1 + |x|^2) & x|y|^2 \end{pmatrix} - 4 \frac{\partial H}{\partial y} \hat{\pi} \begin{pmatrix} y|x|^2 & x(1 + |y|^2) \\ y^2\overline{x} & y(1 + |y|^2) \end{pmatrix}.$$
Proposition 4.6. For \( H \in C^\infty(\mathbb{C}^2) \otimes \text{End}(V_\pi) \) we have

\[
+E_2(H)(x, y) = 4 \frac{\partial H}{\partial x}\left( \hat{\pi}(0, x, y) \hat{\pi}(x, 0, y) \right) + \hat{\pi}\left( \begin{array}{c} 1 + |x|^2 \\ -2(1 + |x|^2) \end{array} \right) - 3(1 + |y|^2) \right)
\]

\[
\frac{\partial H}{\partial y}\left( \hat{\pi}(0, x, y) \right) + \hat{\pi}\left( \begin{array}{c} 1 + |x|^2 \\ -2(1 + |x|^2) \end{array} \right) - 3(1 + |y|^2) \right).
\]

5. Reduction to one variable

We are interested in considering the differential operators \( D \) and \( E \) applied to a function \( H \in C^\infty(\mathbb{C}^2) \otimes \text{End}(V_\pi) \) such that \( H(kp) = \pi(k)H(p)\pi(k)^{-1} \), for all \( k \in K \) and \( p \) in the affine complex plane \( \mathbb{C}^2 \). This property of \( H \) allows us to find ordinary differential operators \( \tilde{D} \) and \( \tilde{E} \) defined on the interval \((0, \infty)\) such that

\[
(D H)(r, 0) = (\tilde{D} \tilde{H})(r), \quad (E H)(r, 0) = (\tilde{E} \tilde{H})(r),
\]

where \( \tilde{H}(r) = H(r, 0) \). We also define differential operators \( \tilde{D}_1 \), \( \tilde{D}_2 \), \( \tilde{E}_1 \) and \( \tilde{E}_2 \) in the same way, that is \( (D_1 H)(r, 0) = (\tilde{D}_1 \tilde{H})(r), \quad (D_2 H)(r, 0) = (\tilde{D}_2 \tilde{H})(r), \quad (E_1 H)(r, 0) = (\tilde{E}_1 \tilde{H})(r), \quad (E_2 H)(r, 0) = (\tilde{E}_2 \tilde{H})(r) \).

The goal of this section is to prove the following theorems.

Theorem 5.1. We have

\[
\tilde{D}(\tilde{H})(r) = (1 + r^2)^2 \frac{d^2 \tilde{H}}{dr^2} + \left( 1 + \frac{r^2}{3} - 2r^2 \hat{\pi}(J) \right) \frac{d \tilde{H}}{dr}
\]

\[
+ \left( \frac{1 + \frac{r^2}{2}}{r^2} \right) \left( \hat{\pi}(J)^2 \tilde{H}(r) + \tilde{H}(r)\hat{\pi}(J)^2 - 2\hat{\pi}(J)\tilde{H}(r)\hat{\pi}(J) \right)
\]

\[
- \left( \frac{1 + \frac{r^2}{2}}{r^2} \right) \left( \hat{\pi}(T)^2 \tilde{H}(r) + \tilde{H}(r)\hat{\pi}(T)^2 - 2\hat{\pi}(T)\tilde{H}(r)\hat{\pi}(T) \right)
\]

\[
- 4\hat{\pi}(X_{-a})\tilde{H}(r)\hat{\pi}(X_a) + 4\tilde{H}(r)\hat{\pi}(X_{-a})\hat{\pi}(X_a),
\]

where \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).
Theorem 5.2. We have

\[
\hat{E}(\hat{H})(r)
= (1 + r^2)^2 \frac{d^2 \hat{H}}{dr^2} \hat{\pi}(\hat{H}_2) + \frac{(1 + r^2)^2}{r} \frac{d \hat{H}}{dr} \hat{\pi}(\hat{H}) + \frac{2(1 + r^2)}{r} \frac{d \hat{H}}{dr} \hat{\pi}(\hat{H}_1)
+ 2r(1 + r^2) \frac{d \hat{H}}{dr} \left(3 \hat{\pi}(X_\alpha)\hat{\pi}(X_{-\alpha}) - \hat{\pi}(H^*_\gamma)\hat{\pi}(\hat{H}_2)\right)
+ \frac{6(1 + r^2)^2}{r^2} \left(\hat{\pi}(X_\alpha) \frac{d \hat{H}}{dr} - \frac{d \hat{H}}{dr} \hat{\pi}(X_\alpha)\right) \hat{\pi}(X_{-\alpha})
- \frac{6(1 + r^2)^2}{r^2} \left(\hat{\pi}(X_{-\alpha}) \frac{d \hat{H}}{dr} - \frac{d \hat{H}}{dr} \hat{\pi}(X_{-\alpha})\right) \hat{\pi}(X_\alpha)
+ \frac{(1 + r^2)^2}{r^2} \left(\hat{\pi}(J^2) \hat{H}(r) + \hat{H}(r) \hat{\pi}(J)^2 - 2\hat{\pi}(J)\hat{H}(r) \hat{\pi}(J)\right) \hat{\pi}(\hat{H}_1)
- \frac{(1 + r^2)^2}{r^2} \left(\hat{\pi}(T^2) \hat{H}(r) + \hat{H}(r) \hat{\pi}(T)^2 - 2\hat{\pi}(T)\hat{H}(r) \hat{\pi}(T)\right) \hat{\pi}(\hat{H}_1)
+ 4 \left(\hat{\pi}(X_{-\alpha}) \hat{H}(r) - \hat{H}(r) \hat{\pi}(X_{-\alpha})\right) \left(-\hat{\pi}(X_\alpha)\hat{\pi}(\hat{H}_1) + 3\hat{\pi}(H^*_\gamma)\hat{\pi}(X_\alpha)\right).
\]

The proof of these theorems will be a direct consequence of Propositions 5.11, 5.12, 5.13 and 5.14. To reach this goal we need some preparatory material.

Since we are interested in considering functions \(H\) on \(P_2(\mathbb{C})\) such that \(H(kp) = \pi(k)H(p)\pi(k)^{-1}\), for all \(k \in K\) and \(p \in P_2(\mathbb{C})\) we need to see the \(K\)-orbit structure of \(P_2(\mathbb{C})\). The affine plane \(\mathbb{C}^2\) is \(K\)-stable and the corresponding line at infinity \(L = \{(x, y, 0) : x, y \in \mathbb{C}^2\}\) in \(P_2(\mathbb{C})\) is a \(K\)-orbit. Moreover the \(K\)-orbits in the affine plane correspond to the spheres

\[
S_r = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = r^2\}, \quad r \in \mathbb{R}_{\geq 0}
\]

in \(\mathbb{C}^2\). In each affine orbit we may choose as a representative the point \((r, 0) \in \mathbb{C}^2\), with \(r \geq 0\). In fact given \((x, y) \in \mathbb{C}^2\), there exists \(k \in K\) such that \(k(r, 0) = (x, y)\) where \(r^2 = |x|^2 + |y|^2\). Also we may choose \((1, 0, 0)\) as a representative of the line at infinity. Since \((M, 0, 1) = (1, 0, \frac{1}{M}) \rightarrow (1, 0, 0)\) when \(M \rightarrow \infty\), the closed interval \([0, \infty]\) parametrizes the set of \(K\)-orbits in \(P_2(\mathbb{C})\). This orbit structure is depicted in the figure below.
Turning to the proofs of Propositions 5.11, 5.12, 5.13 and 5.14 and referring to Propositions 4.3, 4.4, 4.5 and 4.6 we need to compute a number of first and second order partial derivatives of the function \( H \) at the point \( (r, 0) \in \mathbb{C}^2 \). These are given in Lemmas 5.3 through 5.9 whose proofs are included in the Appendix.

**Lemma 5.3.** At \( (r, 0) \in \mathbb{C}^2 \) we have
\[
H_{x_1}(r, 0) = \frac{d\tilde{H}}{dr}(r) \quad \text{and} \quad H_{x_1x_1}(r, 0) = \frac{d^2\tilde{H}}{dr^2}(r).
\]

**Lemma 5.4.** At \( (r, 0) \in \mathbb{C}^2 \) we have
\[
H_{y_1}(r, 0) = -\frac{1}{r} \left( \hat{\pi}(J)\tilde{H}(r) - \tilde{H}(r)\hat{\pi}(J) \right)
\]
and
\[
H_{y_1y_1}(r, 0) = -\frac{1}{r^2} \left( \hat{\pi}(J)^2\tilde{H}(r) + \tilde{H}(r)\hat{\pi}(J)^2 \right) + 2\frac{\hat{\pi}(J)\tilde{H}(r)\hat{\pi}(J)}{r^2},
\]
where \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

**Lemma 5.5.** At \( (r, 0) \in \mathbb{C}^2 \) we have
\[
H_{y_2}(r, 0) = \frac{i}{r} \left( \hat{\pi}(T)\tilde{H}(r) - \tilde{H}(r)\hat{\pi}(T) \right)
\]
and
\[
H_{y_2y_2}(r, 0) = \frac{1}{r} \left( \hat{\pi}(T)^2\tilde{H}(r) + \tilde{H}(r)\hat{\pi}(T)^2 \right) + 2\frac{\hat{\pi}(T)\tilde{H}(r)\hat{\pi}(T)}{r^2},
\]
where \( T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

**Lemma 5.6.** At \( (r, 0) \in \mathbb{C}^2 \) we have
\[
H_{x_2}(r, 0) = \frac{i}{r} \left( \hat{\pi}(H_\alpha)\tilde{H}(r) - \tilde{H}(r)\hat{\pi}(H_\alpha) \right)
\]
The function $\tilde{H}$ is diagonalizable.

**Proof.** Let $M = \{ m_0 : |a| = 1 \}$ where

$$m_a = \begin{pmatrix} a & 0 \\ 0 & -\frac{1}{a} \end{pmatrix}.$$ 

The subgroup $M$ fixes the points $(x, 0, 1)$ in the projective plane. Moreover $H(kg) = \tilde{\pi}(k)H(g)\tilde{\pi}(k^{-1})$, therefore $\tilde{H}(r) = \tilde{\pi}(m_0)\tilde{H}(r)\tilde{\pi}(m_0^{-1})$. Now since all finite dimensional irreducible $K$-modules are multiplicity free as $M$-modules, $\tilde{H}(r)$, $r \geq 0$, and $\tilde{\pi}(H_a)$ diagonalize simultaneously.

**Proposition 5.11.** We have

$$\tilde{D}_1(\tilde{H})(r) = (1 + r^2)^2 \frac{d^2 \tilde{H}}{dr^2} + (1 + r^2)(3 + r^2) \frac{1}{r} \frac{d \tilde{H}}{dr}$$ 

$$+ \left( \frac{1 + r^2}{r^2} \left[ \tilde{\pi}(J)^2 \tilde{H}(r) + \tilde{H}(r)\tilde{\pi}(J)^2 - 2\tilde{\pi}(J)\tilde{H}(r)\tilde{\pi}(J) \right] \right).$$
\[-\frac{(1+r^2)}{r^2} \left( \dot{\pi}(T)^2 \tilde{H}(r) + \ddot{\tilde{H}}(r)\dot{\pi}(T)^2 - 2\dot{\pi}(T)\ddot{\pi}(T) \right).\]

**Proof.** By Proposition 4.3 we have

\[D_1(H)(r, 0) = \left( H_{x_1x_1}(r, 0) + H_{x_2x_2}(r, 0) \right) (1 + r^2)^2 + \left( H_{y_1y_1}(r, 0) + H_{y_2y_2}(r, 0) \right) (1 + r^2).\]

Using the lemmas above and the fact that \(\tilde{H}(r)\) and \(\dot{\pi}(H_\alpha)\) commute because they are simultaneously diagonalizable, the proposition follows.

**Proposition 5.12.** We have

\[\dot{D}_2(\tilde{H})(r) = -2r(1 + r^2) \frac{d\tilde{H}}{dr} \dot{\pi}(H_\gamma) - 4\dot{\pi}(X_{-\alpha})\tilde{H}(r)\dot{\pi}(X_\alpha) + 4\ddot{\tilde{H}}(r)\dot{\pi}(X_{-\alpha})\dot{\pi}(X_\alpha).\]

**Proof.** By Proposition 4.4 we have

\[D_2(H)(r, 0) = -4 \left( \frac{\partial H}{\partial x}(r, 0) \dot{\pi} \left( \begin{array}{c} r(1 + r^2) \\ 0 \\ 0 \end{array} \right) + \frac{\partial H}{\partial y}(r, 0) \dot{\pi} \left( \begin{array}{c} 0 \\ r \\ 0 \end{array} \right) \right).\]

Now using the lemmas above we compute \(\frac{\partial H}{\partial x}(r, 0)\) and \(\frac{\partial H}{\partial y}(r, 0)\).

\[\frac{\partial H}{\partial x}(r, 0) = \frac{1}{2} \frac{\partial H}{\partial x_1}(r, 0) - \frac{i}{2} \frac{\partial H}{\partial x_2}(r, 0),\]

\[= \frac{1}{2} \frac{d\tilde{H}}{dr} + \frac{1}{2r} \dot{\pi}(H_\alpha) \tilde{H}(r) - \frac{1}{2r} \ddot{\tilde{H}}(r) \dot{\pi}(H_\alpha) = \frac{1}{2} \frac{d\tilde{H}}{dr},\]

\[\frac{\partial H}{\partial y}(r, 0) = \frac{1}{2} \frac{\partial H}{\partial y_1}(r, 0) - \frac{i}{2} \frac{\partial H}{\partial y_2}(r, 0),\]

\[= -\frac{1}{2r} \dot{\pi}(J) \tilde{H}(r) + \frac{1}{2r} \ddot{\tilde{H}}(r) \dot{\pi}(J) + \frac{1}{2r} \dot{\pi}(T) \tilde{H}(r) - \frac{1}{2r} \ddot{\tilde{H}}(r) \dot{\pi}(T)\]

\[= \frac{1}{r} \dot{\pi}(X_{-\alpha}) \tilde{H}(r) - \frac{1}{r} \ddot{\tilde{H}}(r) \dot{\pi}(X_{-\alpha}).\]

The proposition follows.
Proposition 5.13.

\[ \tilde{E}_1(\tilde{H})(r) \]
\[ = (1 + r^2)^2 \frac{d^2 \tilde{H}}{dr^2} \tilde{\pi}(\tilde{H}_2) + \frac{(1 + r^2)^2}{r} \frac{d \tilde{H}}{dr} \tilde{\pi}(\tilde{H}_2) + \frac{2(1 + r^2)}{r} \frac{d \tilde{H}}{dr} \tilde{\pi}(\tilde{H}_1) \]
\[ + \frac{6(1 + r^2)^2}{r} \left( \tilde{\pi}(X) \frac{d \tilde{H}}{dr} - \frac{d \tilde{H}}{dr} \tilde{\pi}(X) \right) \tilde{\pi}(X) \]
\[ - \frac{6(1 + r^2)}{r} \left( \tilde{\pi}(X) \frac{d \tilde{H}}{dr} - \frac{d \tilde{H}}{dr} \tilde{\pi}(X) \right) \tilde{\pi}(X) \]
\[ + \frac{(1 + r^2)}{r^2} \left( \tilde{\pi}(J)^2 \tilde{H}(r) + \tilde{H}(r) \tilde{\pi}(J)^2 - 2 \tilde{\pi}(J) \tilde{H}(r) \tilde{\pi}(J) \right) \tilde{\pi}(\tilde{H}_1) \]
\[ - \frac{(1 + r^2)}{r^2} \left( \tilde{\pi}(T)^2 \tilde{H}(r) + \tilde{H}(r) \tilde{\pi}(T)^2 - 2 \tilde{\pi}(T) \tilde{H}(r) \tilde{\pi}(T) \right) \tilde{\pi}(\tilde{H}_1). \]

Proof. By Proposition 4.5 we have

\[ E_1(H)(r, 0) = (1 + r^2) \left( H_{x_1 x_1}(r, 0) + H_{x_2 x_2}(r, 0) \right) \tilde{\pi}(\tilde{H}_2) \]
\[ + (1 + r^2) \left( H_{y_1 y_1}(r, 0) + H_{y_2 y_2}(r, 0) \right) \tilde{\pi}(\tilde{H}_1) \]
\[ - 3(1 + r^2) \left( H_{y_1 x_1}(r, 0) + H_{y_2 x_2}(r, 0) \right) \tilde{\pi}(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) \]
\[ + 3i(1 + r^2) \left( H_{x_2 y_1}(r, 0) - H_{x_1 y_2}(r, 0) \right) \tilde{\pi}(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}). \]

Since, as noticed before, \( \tilde{H}(r) \) and \( \tilde{\pi}(H) \) commute, from Lemmas 5.8 and 7.9 we get

\[ H_{x_2 y_1}(r, 0) = \frac{i}{2r^2} \left( \tilde{\pi}(J) \tilde{\pi}(H) - \tilde{\pi}(H) \tilde{\pi}(J) \right) \tilde{H}(r) \]
\[ - \frac{i}{2r^2} \tilde{H}(r) \left( \tilde{\pi}(J) \tilde{\pi}(H) - \tilde{\pi}(H) \tilde{\pi}(J) \right) \]
\[ = \frac{i}{r^2} \tilde{H}(r) \tilde{\pi}(T) - \frac{i}{r^2} \tilde{\pi}(T) \tilde{H}(r). \]

\[ H_{y_2 x_2}(r, 0) = \frac{1}{2r^2} \left( \tilde{\pi}(T) \tilde{\pi}(H) - \tilde{\pi}(H) \tilde{\pi}(T) \right) \tilde{H}(r) \]
\[ - \frac{1}{2r^2} \tilde{H}(r) \left( \tilde{\pi}(T) \tilde{\pi}(H) - \tilde{\pi}(H) \tilde{\pi}(T) \right) \]
\[ = \frac{1}{r^2} \tilde{H}(r) \tilde{\pi}(J) - \frac{1}{r^2} \tilde{\pi}(J) \tilde{H}(r). \]

In this way we obtain

\[ \tilde{E}_1(\tilde{H})(r) \]
\[ = (1 + r^2)^2 \frac{d^2 \tilde{H}}{dr^2} \tilde{\pi}(\tilde{H}_2) + \frac{(1 + r^2)^2}{r} \frac{d \tilde{H}}{dr} \tilde{\pi}(\tilde{H}_2) + \frac{2(1 + r^2)}{r} \frac{d \tilde{H}}{dr} \tilde{\pi}(\tilde{H}_1) \]
(10) \[ \frac{3(1 + r^2)}{r} \left( \hat{\pi}(J) \frac{d\hat{H}}{dr} \frac{d\hat{H}}{dr} \hat{\pi}(J) \right) \hat{\pi} \left( \begin{array}{c} 0 \\ 1 + r^2 \\ 0 \end{array} \right) \]

(11) \[ \frac{3(1 + r^2)}{r} \left( \hat{\pi}(T) \frac{d\hat{H}}{dr} \frac{d\hat{H}}{dr} \hat{\pi}(T) \right) \hat{\pi} \left( \begin{array}{c} 0 \\ 1 + r^2 \\ 0 \end{array} \right) \] 

\[ + \frac{1}{r^2} \left( \hat{\pi}(J)^2 \hat{H}(r) + \hat{H}(r) \hat{\pi}(J)^2 - 2\hat{\pi}(J)\hat{H}(r)\hat{\pi}(J) \right) \hat{\pi}(H_1) \]

\[ - \frac{1}{r^2} \left( \hat{\pi}(T)^2 \hat{H}(r) + \hat{H}(r) \hat{\pi}(T)^2 - 2\hat{\pi}(T)\hat{H}(r)\hat{\pi}(T) \right) \hat{\pi}(H_1). \]

We observe that combining lines (10) and (11), one gets

\[ (10) + (11) = \frac{3(1 + r^2)^2}{r} \hat{\pi}(J + T) \frac{d\hat{H}}{dr} \hat{\pi}(X_a) \]

\[ - \frac{3(1 + r^2)^2}{r} \hat{\pi}(J + T) \frac{d\hat{H}}{dr} \hat{\pi}(X_a) \]

By using that \( \hat{\pi}(T + J) = 2\hat{\pi}(X_a) \) and \( \hat{\pi}(T - J) = 2\hat{\pi}(X_{-a}) \) the proposition follows.

**Proposition 5.14.**

\[ E_2(\hat{H})(r) = 2r(1 + r^2) \frac{d\hat{H}}{dr} \left( 3\hat{\pi}(X_a)\hat{\pi}(X_{-a}) - \hat{\pi}(H)\hat{\pi}(\hat{H}_2) \right) \]

\[ + 4 \left( \hat{\pi}(X_{-a}) \hat{H}(r) - \hat{H}(r) \hat{\pi}(X_{-a}) \right) \left( -\hat{\pi}(X_a)\hat{\pi}(\hat{H}_1) + 3\hat{\pi}(H)\hat{\pi}(X_a) \right). \]

**Proof.** By Proposition 4.6 we have

\[ E_2(H)(r, 0) = \]

\[ 4 \frac{\partial H}{\partial x} (r, 0) \left( \hat{\pi} \left( \begin{array}{c} 0 \\ r \\ 0 \end{array} \right) \hat{\pi} \left( \begin{array}{c} 0 \\ 3 + r^2 \\ 0 \end{array} \right) \hat{\pi} \left( \begin{array}{c} 1 + r^2 \\ 0 \\ -2(1 + r^2) \end{array} \right) \right) \]

\[ + 4 \frac{\partial H}{\partial y} (r, 0) \left( \hat{\pi} \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \hat{\pi} \left( \begin{array}{c} -2 \\ 0 \\ 1 \end{array} \right) \hat{\pi} \left( \begin{array}{c} r \\ 0 \\ 0 \end{array} \right) \hat{\pi} \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \right). \]

By (8) and (9) we have

\[ \frac{\partial H}{\partial x} (r, 0) = \frac{1}{2} \frac{dH}{dr}, \quad \frac{\partial H}{\partial y} (r, 0) = \frac{1}{r} \left( \hat{\pi}(X_{-a})H(r) - H(r)\hat{\pi}(X_{-a}) \right). \]

Now the proposition follows easily.

The Theorems 5.1 and 5.2 are given in terms of linear transformations. Now we will give the corresponding statements in terms of matrices by choosing an appropriate basis.

If \( \pi = \pi_{n, \ell} \) it is well known (see [13], p.32) that there exists a basis \( \{v_i\}_{i=0}^\ell \) of \( V_\pi \) such that

\[ \hat{\pi}(H_{\alpha})v_i = (\ell - 2i)v_i, \]

\[ \hat{\pi}(X_{\alpha})v_i = (\ell - i + 1)v_{i-1}, \quad (v_{-1} = 0), \]

choosing an appropriate basis.
\[
\dot{\pi}(X_{-\alpha})v_i = (i + 1)v_{i+1}, \quad (v_{\ell+1} = 0).
\]

Since we are dealing with a representation of U(2) these relations have to be supplemented with
\[
\dot{\pi}(Z)v_i = (2n + \ell)v_i.
\]

This follows from
\[
\dot{\pi}(Z) = \left(\frac{d}{dt}\pi(e^t 0 0 e^t)\right)_{t=0} = \left(\frac{d}{dt}e^{2nt}\pi(e^t 0 0 e^t)\right)_{t=0} = (2n + \ell)I.
\]

We introduce the functions \(h_i(r)\) by means of the relations
\[
\bar{H}(r)v_i = h_i(r)v_i.
\]

**Corollary 5.15.** The function \(\bar{H}(r) = (h_0(r), \cdots, h_\ell(r))\), \((r > 0)\), satisfies \((\bar{D}\bar{H})(r) = \lambda\bar{H}(r)\) if and only if
\[
(1 + r^2)^2 h_i'' + \frac{(1 + r^2)}{r} (3 + r^2 - 2r^2(n + \ell - i)) h_i' - 4i(\ell - i + 1)(h_{i-1} - h_i) + \frac{4(1 + r^2)}{r} ((i + 1)(\ell - i)(h_{i+1} - h_i) + i(\ell - i + 1)(h_{i-1} - h_i)) = \lambda h_i
\]
for all \(i = 0, \cdots, \ell\).

**Proof.** We have \(J = X_\alpha - X_{-\alpha}, \quad T = X_\alpha + X_{-\alpha}\) and \(H_\gamma = \frac{1}{2}Z + \frac{1}{2}H_\alpha\). Then
\[
\dot{\pi}(J)^2\bar{H}(r)v_i = (\ell - i + 1)(\ell - i + 2)h_i v_{i-2}
\]
\[
- ((\ell - i + 1)i + (i + 1)(\ell - i)) h_i v_i + (i + 1)(i + 2)h_i v_{i+2},
\]
\[
\bar{H}(r)\dot{\pi}(J)^2v_i = (\ell - i + 1)(\ell - i + 2)h_i - 1 v_{i-2}
\]
\[
- ((\ell - i + 1)i + (i + 1)(\ell - i)) h_i v_i + (i + 1)(i + 2)h_i v_{i+2},
\]
\[
\dot{\pi}(J)\bar{H}(r)\dot{\pi}(J)v_i = (\ell - i + 1)(\ell - i + 2)h_{i-1} v_{i-2} + (i + 1)(i + 2)h_{i+1} v_{i+2}
\]
\[
- ((\ell - i + 1)i + (i + 1)(\ell - i)) h_i v_i + (i + 1)(i + 2)h_i v_{i+2},
\]
\[
\dot{\pi}(T)^2\bar{H}(r)v_i = (\ell - i + 1)(\ell - i + 2)h_i v_{i-2}
\]
\[
+ ((\ell - i + 1)i + (i + 1)(\ell - i)) h_i v_i + (i + 1)(i + 2)h_i v_{i+2},
\]
\[
\bar{H}(r)\dot{\pi}(T)^2v_i = (\ell - i + 1)(\ell - i + 2)h_i v_{i-2}
\]
\[
+ ((\ell - i + 1)i + (i + 1)(\ell - i)) h_i v_i + (i + 1)(i + 2)h_i v_{i+2},
\]
\[
\dot{\pi}(T)\bar{H}(r)\dot{\pi}(T)v_i = (\ell - i + 1)(\ell - i + 2)h_{i-1} v_{i-2} + (i + 1)(i + 2)h_{i+1} v_{i+2}
\]
\[
+ ((\ell - i + 1)i + (i + 1)(\ell - i)) h_i v_i + (i + 1)(i + 2)h_i v_{i+2},
\]
\[
\dot{\pi}(X_{-\alpha})\bar{H}(r)\dot{\pi}(X_\alpha)v_i = i(\ell - i + 1)h_{i-1} v_i,
\]
\[
\bar{H}(r)\dot{\pi}(X_{-\alpha})\dot{\pi}(X_\alpha)v_i = i(\ell - i + 1)h_i v_i.
\]

Therefore,
\[
(\bar{D}\bar{H})(r)v_i = \left((1 + r^2)^2 h_i''(r) + \frac{(1 + r^2)}{r} (3 + r^2 - 2r^2(n + \ell - i)) h_i'(r)
\right.
\]
\[
+ \frac{4(1 + r^2)}{r} ((i + 1)(\ell - i)h_{i+1}(r) + i(\ell - i + 1)h_{i-1}(r)
\left.)
\right)
We have

\[ \Psi_{n,\ell}(g) = \sum_{0 \leq i \leq \min(-n,\ell)} \binom{-n}{i} a(g)^{n-i} A(g)^{\ell-i} \cdot B(g)^i \]

The proof is finished. \( \square \)

**Corollary 5.16.** The function \( \tilde{H}(r) = (h_0(r), \cdots, h_\ell(r)), (r > 0), \) satisfies \( (\tilde{E}\tilde{H})(r) = \mu \tilde{H}(r) \) if and only if

\[
(n - \ell + 3i)(1 + r^2)^2 h_i'' + 6(i + 1)(\ell - i) \left( \frac{1+r^2}{r} \right) h_i' + 2(n + 2\ell - 3i) \left( \frac{1+r^2}{r} \right) h_i'
\]

\[ + 2r(1 + r^2)h_i' + (i + 1)(\ell - i) \left( \frac{1+r^2}{r} \right) h_i' + 4(n + 2\ell - 3i) \left( \frac{1+r^2}{r} \right) i \left( \frac{1+r^2}{r} \right) h_i' - 6i(\ell - i) + 6i(\ell - i) \left( \frac{1+r^2}{r} \right) h_i'
\]

\[ + 4(n + 2\ell - 3i) \left( \frac{1+r^2}{r} \right) i \left( \frac{1+r^2}{r} \right) h_i'
\]

\[ + 4i(\ell - i + 1)(2n + \ell + 3)(h_{i-1} - h_i) \]

for all \( i = 0, \cdots, \ell. \)

**Proof.** We have \( \tilde{H}_2 = \frac{1}{2}Z - \frac{3}{2}H_\alpha, \tilde{H}_1 = \frac{1}{2}Z + \frac{3}{2}H_\alpha, \tilde{H}_\alpha = \frac{1}{2}Z + \frac{1}{2}H_\alpha. \) By using the computations in the proof of Corollary 5.13 we obtain:

\[ (\tilde{E}\tilde{H})(r)v_i = \left( (n - \ell + 3i)(1 + r^2)^2 h_i'' + 6(i + 1)(\ell - i) \left( \frac{1+r^2}{r} \right) h_i' + 2(n + 2\ell - 3i) \left( \frac{1+r^2}{r} \right) h_i'ight.
\]

\[ + 2r(1 + r^2)h_i' + (i + 1)(\ell - i) \left( \frac{1+r^2}{r} \right) h_i' + 4(n + 2\ell - 3i) \left( \frac{1+r^2}{r} \right) i \left( \frac{1+r^2}{r} \right) h_i' - 6i(\ell - i) + 6i(\ell - i) \left( \frac{1+r^2}{r} \right) h_i'
\]

\[ + 4(n + 2\ell - 3i) \left( \frac{1+r^2}{r} \right) i \left( \frac{1+r^2}{r} \right) h_i' + 4i(\ell - i + 1)(2n + \ell + 3)(h_{i-1} - h_i) \]

\[ \left. v_i. \right) \]

\( \square \)

**Corollary 5.17.** If \( \tilde{H}(r) = (h_0(r), \cdots, h_\ell(r)), (r > 0), \) satisfies \( (\tilde{D}\tilde{H})(r) = \lambda \tilde{H}(r) \) and extends to a \( C^\infty \) function on \( r \geq 0, \) then \( h_0(0) = \cdots = h_\ell(0). \)

**Proof.** From Corollary 5.13, by multiplying by \( r^2 \) and evaluating at \( r = 0, \) we get

\[ (i + 1)(\ell - i)(h_{i+1}(0) - h_i(0)) + i(\ell - i + 1)(h_{i-1}(0) - h_i(0)) = 0 \]

for \( i = 0, 1, \cdots, \ell. \) Now the assertion follows by induction starting at \( i = 0 \) and ending at \( i = \ell - 1. \) \( \square \)

We take the opportunity to sketch a proof of Theorem 2.13 based on Corollaries 5.13 and 5.16.

**Theorem 2.13.** For \( n \leq 0 \)

\[ \Psi_{n,\ell}(g) = \sum_{0 \leq i \leq \min(-n,\ell)} \binom{-n}{i} a(g)^{n-i} A(g)^{\ell-i} \cdot B(g)^i \]
is an irreducible spherical function of type \((n, \ell)\).

**Proof.** Let \(\pi = \pi_{n, \ell}\) and let
\[
g(r) = \begin{pmatrix}
\frac{1}{(1+r^2)^{1/2}} & 0 & \frac{r}{(1+r^2)^{1/2}} \\
0 & 1 & 0 \\
\frac{-r}{(1+r^2)^{1/2}} & 0 & \frac{1}{(1+r^2)^{1/2}}
\end{pmatrix}.
\]

By Propositions 2.4 and 4.1 it is enough to check that the function \(\tilde{H}(r) = \Psi_{n, \ell}(g(r))\Phi_{\pi}(g(r))^{-1}\) is an eigenfunction of the differential operators \(\tilde{D}\) and \(\tilde{E}\) given respectively in Theorems 5.1 and 5.2.

Let \(\{e_1, e_2\}\) be the canonical basis of \(\mathbb{C}^2\) and let \(v_i = e_1^{\ell-i}e_2^i\) be the corresponding basis of the space of symmetric tensors of rank \(\ell\). Then we can look at the coordinate functions \((h_0, \ldots, h_\ell)\) of \(\tilde{H}\) associated to the basis \(\{v_i\}\). Then it is not difficult to check that
\[
h_i(r) = (1 + r^2)^n \sum_{0 \leq j \leq \min\{-n, \ell-i\}} (-1)^j \binom{-n}{j} \binom{\ell - i}{j} r^{2j}.
\]

Now it is straightforward but lengthy to verify that \((h_0, \ldots, h_\ell)\) is a simultaneous solution of the systems given in Corollaries 5.1 and 5.16 with \(\lambda = 4n(\ell + 2)\) and \(\mu = 4n(\ell + 2)(n - \ell)\).

6. Extension to \(G\)

Let us recall where we are: if \(\Phi\) is a spherical function on \(G\) of type \(\pi = \pi_{n, \ell}\) we have associated to it a function \(\tilde{H}(r) \neq \Psi_{n, \ell}(g(r))\Phi_{\pi}(g(r))^{-1}\) is an eigenfunction of the differential operators \(\tilde{D}\) and \(\tilde{E}\) (Theorems 5.1 and 5.2). Now we want to characterize the behavior of \(\tilde{H}\) when \(r \to \infty\).

The complement \(A^c\) of \(A\) in \(G\) is defined by the condition \(g_{33} = 0\). This set is clearly left and right invariant under \(K\).

For any \(t \in \mathbb{R}\) let
\[
a(t) = \begin{pmatrix}
\cos t & 0 & \sin t \\
0 & 1 & 0 \\
-\sin t & 0 & \cos t
\end{pmatrix}.
\]

Then for all \(-\pi/2 < t < \pi/2\), \(a(t) \in A\) and
\[
a(\pi/2) = \lim_{t \to \pi/2} a(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in A^c.
\]

Now if \(g \in A^c, p(g) = (g_{13}, g_{23}, 0) \in L\). Since the line at infinity \(L\) is a \(K\)-orbit, there exists \(k \in K\) such that \(p(g) = kp(a(\pi/2)) = p(ka(\pi/2))\). Therefore \(g = ka(\pi/2)k'\) for some \(k' \in K\). Thus \(A^c = K a(\pi/2)K\).

On \(A\) we have written \(\Phi = H\Phi_{\pi}\). Hence if we put \(A(t) = A(a(t))\), for \(-\pi/2 < t < \pi/2\) we have
\[
\Phi(a(t)) = (\cos t)^n H(a(t))A(t) = (\cos t)^n \tilde{H}(\tan t)A(t),
\]
since \( p(a(t)) = (\tan t, 0, 1) \).

If we make the change of variable \( r = \tan t \) and let \( t \to \pi/2 \) we obtain

\[
\Phi(a(\pi/2)) = \lim_{r \to \infty} (1 + r^2)^{-\frac{n}{2}} \tilde{H}(r) \left( \begin{array}{cc} (1 + r^2)^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{array} \right) ^\ell,
\]

where the exponent \( \ell \) denotes the \( \ell \)-th symmetric power of the matrix. If we use the basis \( \{v_i\}_{i=0}^\ell \) of \( V_\pi \) introduced before we have proved the existence of

\[
\lim_{r \to \infty} (1 + r^2)^{-\frac{(n+\ell-i)}{2}} h_i(r) = L_i
\]

for \( i = 0, \ldots, \ell \). This follows directly from \( \pi(H_\gamma)v_i = (n + \ell - i)v_i \) and \( \exp tH_\gamma = \left( \begin{array}{cc} e^t & 0 \\ 0 & 1 \end{array} \right) \).

Given \( x \in \mathbb{C} \) with \( |x| = 1 \) let us consider the element

\[
b(x) = \left( \begin{array}{ccc} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^{-1} \end{array} \right) \in K.
\]

Then \( b(x)a(\pi/2) = a(\pi/2)b(x^{-1}) \). Now

\[
\Phi(b(x)a(\pi))v_i = \pi(b(x))\Phi(a(\pi))v_i = x^{n+\ell-i}L_iv_i,
\]

\[
\Phi(a(\pi))b(x^{-1})v_i = \Phi(a(\pi))\pi(b(x^{-1}))v_i = L_ix^{-(n+\ell-i)}v_i.
\]

Therefore if \( i \neq n + \ell \), then \( L_i = 0 \). Thus we have proved the direct part of the following proposition.

**Proposition 6.1.** If \( \Phi \) is a spherical function on \( G \) of type \( \pi = \pi_{n,\ell} \) and \( \tilde{H} = (h_0, \ldots, h_\ell) \) is the associated function, then

i) \( \tilde{H} = \tilde{H}(r) \) is a \( C^\infty \)-function for \( 0 \leq r < \infty \) and \( \tilde{H}(0) = (1, \ldots, 1) \).

ii) If \( i \neq n + \ell \), \( \lim_{r \to \infty} (1 + r^2)^{-\frac{(n+\ell-i)}{2}} h_i(r) = 0 \).

If \( i = n + \ell \), \( \lim_{r \to \infty} h_i(r) = L_i \) exists.

Conversely if \( \tilde{H} \) is an eigenfunction of \( \tilde{D} \) which satisfies conditions i) and ii), then there exists a unique \( C^\infty \)-eigenfunction \( \Phi \) on \( G \) of \( \Delta_2 \) which satisfies condition ii) of Proposition 2A to which \( \tilde{H} \) is associated. Moreover if \( \tilde{H} \) is also an eigenfunction of \( \tilde{E} \) then \( \Phi \) is spherical of type \( (n, \ell) \).

This proposition will play a crucial role in Section 11. We will already come back to it in Section 14.

**Proof.** As we said before the second part is the only one which needs to be proved. First we want to extend \( \tilde{H} \) to the whole affine plane \( \mathbb{C}^2 \), in such a way that the extended function \( \tilde{H} \) satisfies \( \tilde{H}(kq) = \pi(kH(q)\pi(k^{-1})) \) for all \( k \in K \) and all \( q \in \mathbb{C}^2 \). We observe that for \( q = (x, y) \in \mathbb{C}^2 - \{(0, 0)\} \) the element

\[
k(q) = k(x, y) = \left( |x|^2 + |y|^2 \right)^{-\frac{1}{2}} \left( \begin{array}{c} x - y \\ y \\ x \end{array} \right)
\]
lies in \( K \) and \( q = k(q)(r,0) \) where \( r = (|x|^2 + |y|^2)^{\frac{1}{2}} \). Let
\[
H(q) = \pi(k(q)) \hat{H}(r) \pi(k(q)^{-1}).
\]
Then \( H \) is a \( C^\infty \) function on \( \mathbb{C}^2 - \{(0,0)\} \). Now we shall see that \( H \) extends to a continuous function on \( \mathbb{C}^2 \). In fact we can equip \( V_\pi \) with a scalar product such that \( \pi(k) \) becomes a continuous function on all \( k \in K \). If we denote with \( \| \cdot \| \) the corresponding operator norm on \( \text{End}(V_\pi) \) the we have
\[
\| H(q) - I \| = \| \pi(k(q))(\hat{H}(r) - I)\pi(k(q)^{-1}) \| = \| \hat{H}(r) - I \|,
\]
and our assertion follows.

Let us check now that \( H(uq) = \pi(u)H(q)\pi(u^{-1}) \) for all \( u \in K, q \in \mathbb{C}^2 \). We may assume that \( q = (x,y) \neq (0,0) \). Then \( k(uq)(r,0) = uq = uk(q)(r,0) \). Therefore there exists \( m \in M \) such that \( k(uq) = uk(q)m \). Thus
\[
H(uq) = \pi(k(uq)) \hat{H}(r) \pi(k(uq)^{-1})
= \pi(u) \pi(k(q)) \pi(m) \hat{H}(r) \pi(m^{-1}) \pi(k(q)^{-1}) \pi(u^{-1})
= \pi(u)H(q)\pi(u^{-1}),
\]
since \( \pi(m) \) and \( \hat{H}(r) \) commute (Proposition 5.10).

Now we lift the function \( H \) to a continuous function \( \tilde{H} = H \circ p \) on \( A \) with values in \( \text{End}(V_\pi) \) which is \( \mathcal{C}^\infty \) on \( A - K \). Then the function \( \Phi = \tilde{H}\Phi_\pi \) is a continuous function on \( A \) which is a \( \mathcal{C}^\infty \) eigenfunction of \( \Delta_2 \) on \( A - K \), satisfies \( \Phi(e) = I \) and \( \Phi(k_1gk_2) = \pi(k_1)\Phi(g)\pi(k_2) \) for \( k_1, k_2 \in K, g \in A \).

We shall next show that \( \Phi \) can be extended to a continuous function on \( G \) with property ii) of Proposition 2.4.

As was pointed out before \( A^c = Ka(\pi/2)K \) and \( a(\pi/2) = \lim_{t \to \pi/2} a(t) \). So we put
\[
\Phi(k_1a(\pi/2)k_2) = \pi(k_1) \lim_{t \to \pi/2} (\cos t)^n \hat{H}(\tan t)A(t)^{\ell}\pi(k_2).
\]
First of all we see that by hypothesis \( L = \lim_{t \to \pi/2} (\cos t)^n \hat{H}(\tan t)A(t)^{\ell} \) exists. But we still need to verify that if \( k_1a(\pi/2)k_2 = h_1a(\pi/2)h_2 \) then \( \pi(k_1)L\pi(k_2) = \pi(h_1)L\pi(h_2) \), to have a good definition of \( \Phi \). This is equivalent to proving that if \( ka(\pi/2) = a(\pi/2)h \) with \( k, h \in K \) then \( \pi(k)L = L\pi(h) \). It is easy to see that \( ka(\pi/2) = a(\pi/2)h \) happens if and only if \( k = b(x)m \) and \( h = b(x^{-1})m \), for some \( x \in \mathbb{C}, |x| = 1 \), and some \( m \in M \). Then
\[
\pi(k)L = \pi(b(x))\pi(m)L = \pi(b(x))L\pi(m) = L\pi(b(x^{-1}))\pi(m) = L\pi(h),
\]
because the hypothesis ii) of the proposition says precisely that \( \pi(b(x))L = L\pi(b(x^{-1})) \).

So we have a continuous function \( \Phi \) on \( G \) with values in \( \text{End}(V_\pi) \) which satisfies condition ii) of Proposition 2.4 and which is a \( \mathcal{C}^\infty \)-eigenfunction of \( \Delta_2 \) on the open set \( A - K \). Since the complement of this set \( A^c \cup K \) is of Haar measure zero in \( G \) and \( \Delta_2 \) is an elliptic differential operator we conclude
where the product runs over the eigenvalues $\pm$ with multiplicity 2 and $d g$ vector space $K / M$ normalized by an element of $G$ multiplicities, becomes

$$D(\text{Weyl's theorem, see for instance [14], p.96})$$

that $\Phi$ is a $C^\infty$-eigenfunction of $\Delta_2$. The proposition is proved. $\square$

7. The inner product

Given a finite dimensional irreducible representation $\pi = \pi_{i,\ell}$ of $K$ in the vector space $V_\pi$ let $(C(G) \otimes \text{End}(V_\pi))^{K \times K}$ be the space of all continuous functions $\Phi : G \to \text{End}(V_\pi)$ such that $\Phi(k_1 g k_2) = \pi(k_1) \Phi(g) \pi(k_2)$ for all $g \in G$, $k_1, k_2 \in K$. Let us equip $V_\pi$ with an inner product such that $\pi(k)$ becomes unitary for all $k \in K$. Then we introduce an inner product in the vector space $(C(G) \otimes \text{End}(V_\pi))^{K \times K}$ by defining

$$\langle \Phi, \Psi \rangle = \int_G \text{tr}(\Phi(g)\Psi(g)^*) \, dg,$$

where $dg$ denote the Haar measure on $G$ normalized by $\int_G dg = 1$, and where $\Psi(g)^*$ denotes the adjoint of $\Psi(g)$ with respect to the inner product in $V_\pi$.

Let us write $\Phi = H \Phi_\pi$, $\Psi = F \Phi_\pi$ on the open set $A$ of $G$ and put $H(\nu) = (h_1(\nu), \cdots, h_t(\nu)), \; F(\nu) = (f_1(\nu), \cdots, f_t(\nu))$ as we did in Section 3.

**Proposition 7.1.** If $\Phi, \Psi \in (C(G) \otimes \text{End}(V_\pi))^{K \times K}$ then

$$\langle \Phi, \Psi \rangle = 4 \sum_{i=0}^3 \int_0^\infty r^3 (1 + r^2)^{-(n+\ell+3-i)} h_i(r) \overline{f_i(r)} \, dr.$$

**Proof.** Let us consider the element $H = E_{13} - E_{31} \in \mathfrak{g}$. Then $H$ is conjugate by an element of $G$ to $i(H_0 + H_3)$, thus $\text{ad} \; H$ has 0 and $\pm i$ as eigenvalues with multiplicity 2 and $\pm 2i$ as eigenvalues with multiplicity 1.

Let $A = \exp \mathbb{R} H$ be the Lie subgroup of $G$ of all elements of the form

$$a(t) = \exp t H = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}, \quad t \in \mathbb{R}.$$

Then the function $D : A \to \mathbb{R}$ defined by

$$D(a(t)) = \prod_\nu \sin it\nu(H),$$

where the product runs over the eigenvalues $\nu = i, 2i$ of $\text{ad} \; H$ counted with multiplicities, becomes $D(a(t)) = -\sin^2 t \sin 2t$.

Now Corollary 5.12, p. 191 in [12] establishes that

$$\int_{G/K} f(gK) \, dg_K = 2 \int_{K/M} \left( \int_0^{\pi/2} |D(a(t))| f(\nu(a(t))K) \, dt \right) \, dk_M,$$

where $dg_K$ and $dk_M$ are respectively the invariant measures on $G/K$ and $K/M$ normalized by $\int_{G/K} dg_K = \int_{K/M} dk_M = 1$.

Since the function $g \mapsto \text{tr}(\Phi(g)\Psi(g)^*)$ is invariant under left and right multiplication by elements in $K$, we have

$$\langle \Phi, \Psi \rangle = 2 \int_0^{\pi/2} \sin^2 t \sin 2t \, \text{tr} \Phi(a(t)\Psi(a(t))^*) \, dt.$$
If we put \( r = \tan t \) for \( 0 < t < \pi/2 \) we have
\[
\text{tr} (\Phi(a(t)\Psi(a(t))^*) = \sum_{i=0}^{\ell} (1 + r^2)^{-(n+\ell+i)} h_i(r) f_i(r).
\]
(See Section 3). Then
\[
2 \int_0^{\pi/2} \sin^2 t \sin 2t \, \text{tr} (\Phi(a(t)\Psi(a(t))^*) \, dt ~ = \sum_{i=0}^{\ell} \int_0^\infty r^3 (1 + r^2)^{-(n+\ell+i)} h_i(r) f_i(r) \, dr.
\]
Thus the proposition follows.

\[\blacksquare\]

**Proposition 7.2.** If \( \Phi, \Psi \in (C^\infty(G) \otimes \text{End}(V_\pi))^K \times K \) then
\[
\langle \Delta_j \Phi, \Psi \rangle = \langle \Phi, \Delta_j \Psi \rangle \quad j = 1, 2.
\]

**Proof.** If we apply a left invariant vector field \( X \in \mathfrak{g} \) to the function \( g \mapsto \text{tr}(\Phi(g)\Psi(g)^*) \) on \( G \) and then we integrate over \( G \) we obtain
\[
0 = \int_G \text{tr} ((X\Phi)(g)\Psi(g)^*) \, dg + \int_G \text{tr} (\Phi(g)(X\Psi)(g)^*) \, dg.
\]
Therefore \( \langle X\Phi, \Psi \rangle = -\langle \Phi, X\Psi \rangle \). Now let \( \tau : \mathfrak{g}_C \to \mathfrak{g}_C \) be the conjugation of \( \mathfrak{g}_C \) with respect to the real linear form \( \mathfrak{g} \). Then \( -\tau \) extends to a unique antilinear involutive * operator on \( D(G) \) such that \( (D_1 D_2)^* = D_2^* D_1^* \) for all \( D_1, D_2 \in D(G) \). This follows easily from the fact that the universal enveloping algebra over \( \mathbb{C} \) of \( \mathfrak{g} \) is canonically isomorphic to \( D(G) \). Then it follows that \( \langle D\Phi, \Psi \rangle = \langle \Phi, D^* \Psi \rangle \).

Now \( H_\alpha^* = H_\alpha, H_\beta^* = H_\beta, X_\alpha^* = X_{-\alpha}, X_\beta^* = X_{-\beta}, X_\gamma^* = X_{-\gamma} \). From this it is easy to verify that \( \Delta_2^* = \Delta_2 \) and \( \Delta_3^* = \Delta_3 \). This completes the proof of the proposition.

We now describe an alternative way to motivate the choice of inner product made before Proposition 5.1.

Start from the system in Corollary 5.15. Introducing \( t = (1 + r^2)^{-1}, \) \( 0 < t < 1 \), we get
\[
t (1 - t) h''_i - ((n + \ell - i + 3)t - (n + \ell - i + 1)) h'_i \\
+ (\ell - i)(i + 1) \frac{h_{i+1} - h_i}{1 - t} - i(\ell - i + 1) t \frac{h_i - h_{i-1}}{1 - t} = \frac{\lambda}{4} h_i
\]
(12) with \( h'_i = \frac{dh_i}{dt} \). If \( D \) denotes the operator acting on the vector \( H(t) = (h_1(t), \ldots, h_\ell(t)) \) this can be put in the form
\[
DH(t) = D_1 H(t) + D_2 H(t) = \frac{\lambda}{4} H(t).
\]
Here
\[
(D_1 H) = \left( \frac{1}{(1 - t)^{1 + n + \ell - i}} \frac{d}{dt} \left( (1 - t)^{1 + n + \ell - i - 1} \frac{d}{dt} \right) h_i \right)
\]
and
\[(D_2H)(t) = (\ell - i)(i + 1) \frac{h_{i+1} - h_i}{1 - t} - i(\ell - i + 1) t \frac{h_i - h_{i-1}}{1 - t}.\]

The aim is to make \(D\) into a densely defined symmetric operator in some appropriate \(L^2\) space. A look at \(D_1\) suggests an obvious choice for an inner product among vector valued functions \(H(t)\) which are \(C^2\) and compactly supported inside the open set \((0, 1)\), namely
\[
\langle H, K \rangle = \sum_{i=0}^{\ell} \int_0^1 h_i(t) \overline{k_i(t)(1 - t)} t^{\ell + 1 - i} dt.
\]

If one were trying to make \(D_1\) symmetric, then each term in this sum could be multiplied by some arbitrary constant \(\mu_i > 0\). One check that the requirement
\[
\langle DH, K \rangle = \langle H, DK \rangle
\]
forces one to make the choice made above. A change of variables shows now that, up to a multiplicative constant, this is the inner product introduced before Proposition 7.1.

8. **The \(C^\infty\) eigenfunctions of \(D\)**

This section contains material that is crucial in determining explicit expressions for all the spherical functions associated to the complex projective plane. We display these results in Section 11 for some small values of \(\ell\).

In the process of obtaining these expressions we have uncovered what we think are interesting properties of eigenfunctions for the differential operator \(D\). In order to keep the length of this paper within reasonable bounds we face a problem in deciding what to include here, as well as the level of generality aimed for in the proof of some of these statements. We have opted for a rather mixed approach whereby several lines of attack are described while proofs are given only in some simple cases. We are confident that all the statements made below hold true, even if some of the proofs that we have at this time are lengthy and rather unilluminating. The statements made below are strong enough that they can be used in Section 11 to prove that we have exhibited all the spherical functions of the pair \((G, K)\).

Some of these different ways of getting at the solutions might be useful in potential applications of this material.

In subsection 5.1 we describe all solutions of \(DH = \lambda H\) in the case \(\ell = 0\) and \(n\) an arbitrary integer. This is just a review of the classical theory dealing with the indicial equation and related elementary material. The analysis is powerful enough to show that we have written down all spherical functions when \(\ell = 0\). In this case, and only in this case, we give a complete analysis including the non-analytic solutions of \(DH = \lambda H\) for all integer values of \(n\). We also give, for \(\ell = 0\), the “limit point, the limit circle” analysis of \(H\) of Weyl that would be needed to discuss several possible selfadjoint extensions of the symmetric operator \(D\).
Then we describe a way of constructing a $2(\ell + 1)$ dimensional space of solutions of $DH = \lambda H$ when $\ell$ is a non-negative integer. Here $\lambda$ is an arbitrary parameter.

This construction is based on reducing a second order system of $\ell + 1$ equations to a scalar equation of order $2(\ell + 1)$ and in observing, as the first in a string of miracles, that the corresponding differential operator has a nice factorization featuring the Gauss hypergeometric differential operator. This elimination process (from the bottom up) produces the last component of a (pair of) vector of solutions while the rest of the vector can then be trivially constructed by applying explicit differential operators to this last component. We then see how the entries in these column vectors can be used to give the first entries in a set of $2\ell$ other vector solutions of the same system $DH = \lambda H$. Once again the rest of these vectors are built by applying to their first component differential operators that can be read off from the equation. This construction has some of the flavor of an unrelated “miracle” uncovered by T. Koornwinder in [18]. See his comments at the end of Section 1.

The remarkable factorizations in Subsection 8.2 have not yet yielded to a nice and general proof for all values of $\ell$. They have been established by use of computer algebra for $\ell$ in the range $0, 1, \ldots, 10$.

An interesting feature of this construction is that it expresses the solution in terms of differential operators acting on the Gauss function $2F_1$. Alternatively one can express the solution as a linear combination of “shifted” versions of Gauss’ function.

We note that the elimination process mentioned above would fail if one were trying to apply it to determine solutions of $EH = \mu H$.

We start by recalling that the classical (scalar) equation of Gauss

\[(14) \quad \left( t(1-t) \left( \frac{d}{dt} \right)^2 + (c - (a + b + 1)t) \frac{d}{dt} - ab \right) h(t) = 0 \]

has a two-dimensional space of local solutions $\Phi(a, b, c; t)$ which for generic values of $a, b, c$ has a basis given by

$\varphi_1(t) = _2F_1 \left( \frac{a}{c}, \frac{b}{c}; t \right)$ and $\varphi_2(t) = t^{1-c} _2F_1 \left( \frac{a-c+1}{2-c}, \frac{b-c+1}{2-c}; t \right) \).$

Any equation of the form

\[ \left( t(1-t) \left( \frac{d}{dt} \right)^2 + (c - (p+1)t) \frac{d}{dt} - \mu \right) h(t) = 0 \]

can be solved easily in terms of these solutions. It is enough to introduce a parameter $w$ as one of the solutions of

$\mu = -w(w + p)$

and to put $a = -w$ and $b = w + p$. Notice that a different choice of a solution of the quadratic equation in $w$ amounts to exchanging $a$ and $b$ above and that they enter symmetrically in (14).
8.1. **All solutions of** $4D\mathcal{H} = \lambda \mathcal{H}$ **for** $\ell = 0$. With the strategy just outlined and the form of $D$ given in (12), we have with $\lambda = -4w(w + n + 2)$,

$$
\left( t(1-t) \left( \frac{d}{dt} \right)^2 + ((n+1)-(n+3)t) \frac{d}{dt} + w(w+n+2) \right) h_0(t) = 0.
$$

For generic values of the parameters the functions

$$
\varphi_w(t) = 2F_1 \left( -w, w + n + 2; \frac{1}{n+1}; t \right) \quad \text{and} \quad \psi_w(t) = t^{-n} 2F_1 \left( -w, w + n + 2; \frac{1}{1-n}; t \right)
$$

are linearly independent solutions. When $n+1$ or $1-n$ is a non-positive integer these expressions are of no use, and a more careful analysis is done below.

Since the replacement of $w$ by $-(w+n+2)$ does not change neither the equation nor the solutions $\varphi_w, \psi_w$, we will consider these values as equivalent.

We will show that for each integer $w$ satisfying $w \geq 0$ and $w + n \geq 0$ we have one solution of our equation that satisfies the conditions of Proposition 6.1, namely

$$
\lim_{t \to 1} h_0(t) = \text{finite and non-zero (15)}
$$

and

$$
\begin{align*}
\text{if } n = 0 & \text{ then } \lim_{t \to 0} h_0(t) \text{ exists,} \\
\text{if } n \neq 0 & \text{ then } \lim_{t \to 0} h_0(t) t^{n/2} = 0.
\end{align*}
$$

8.1.1. **Case** $n = 0$. The functions $\varphi_w(t)$ and $\psi_w(t)$ coincide and give us one solution. This is a polynomial for $w = 0, 1, 2, \ldots$. For other values of $w$ it behaves, close to $t = 1$, like a non-zero constant times $(1-t)^{-1}$.

For $w = -1$, $\varphi_w(t)$ is well defined and we get that

$$
\varphi_w(t) = \frac{1}{1-t} \quad \text{and} \quad (\log t) \varphi_w(t)
$$

are linearly independent solutions. Neither one of them nor any linear combination of them is bounded at both $t = 0$ and $t = 1$.

For $w \neq -1$, there is a unique solution of the form

$$(\log t) \varphi_w(t) + \sum_{0 \leq i} a_i t^i.
$$

This is linearly independent of $\varphi_w(t)$ and blows up at $t = 0$. Therefore when $n = 0$ we only have eigenfunctions meeting the asymptotic conditions (13) and (16) for $w = 0, 1, 2, \ldots$ (or the redundant set $w = -2, -3, \ldots$).

8.1.2. **Case** $n > 0$. Now $\varphi_w(t)$ is well defined for all $w$, and it is a polynomial if $w = 0, 1, 2, \ldots$. For other values of $w$ (except the “redundant” values $w = -n - 2, -n - 3, \ldots$) it behaves like a non-zero constant times $(1-t)^{-1}$ around $t = 1$. The function $\psi_w(t)$ is well defined and linearly independent of $\varphi_w(t)$ if $w = -n - 1, -n, \ldots, -2, -1$ and it is then a polynomial in $t$ times $t^n$. The product $t^{n/2} \psi_w(t)$ behaves like $t^{-n/2}$, and blows up, around $t = 0$. 

If \( w \neq -n - 1, -n, \ldots, -2, -1 \), then there is a unique solution of the form

\[
(\log t) \phi_w(t) + \sum_{-n \leq i, i \neq 0} a_i t^i.
\]

This solution is linearly independent of \( \psi_w(t) \) and its product with \( t^{n/2} \) is unbounded at \( t = 0 \). Thus the only values of \( w \) where there is an eigenfunction meeting the conditions (15) and (16) are \( w = 0, 1, 2, \ldots \).

8.1.3. Case \( n < 0 \). Now \( \psi_w(t) \) is always well defined, and it is a polynomial in \( t \) if \( w = -n, -n + 1, -n + 2, \ldots \) (and also if \( w = -2, -3, -4, \ldots \)). For other values of \( w \) it behaves like a non-zero constant times \( (1 - t)^{-1} \) around \( t = 1 \). The function \( \phi_w(t) \) is well defined if \( w = -1, 0, 1, \ldots, -n - 1 \), and it is in fact a polynomial in \( t \) which takes the value 1 at \( t = 0 \). It gives a solution that is linearly independent of \( \psi_w(t) \), and the product \( t^{n/2} \phi_w(t) \) is unbounded at \( t = 0 \).

If \( w \neq -1, 0, 1, \ldots, -n - 1 \), then there is a unique solution of the form

\[
(\log t) \psi_w(t) + \sum_{0 \leq i, i \neq -n} a_i t^i.
\]

This solution is linearly independent of \( \psi_w(t) \) and its product with \( t^{n/2} \) is unbounded at \( t = 0 \), since \( a_0 \neq 0 \).

By going over the results given above we see that for each integer \( w \) satisfying \( w \geq 0 \) and \( w + n \geq 0 \) we have one solution of our equation that satisfies the properties given in Proposition 6.1. For values of \( w \) that do not satisfy these conditions (once one takes into account the symmetry between \( -w \) and \( w + n + 2 \) mentioned earlier) there is no solution of the equation that satisfies the conditions in Proposition 6.1.

We stress that these conditions single out those solutions of our equation that correspond to spherical functions on \( G \). They should not be construed as “boundary conditions” in the traditional sense of specifying a self-adjoint extension of our densely defined symmetric operator.

In this regard we close the analysis of the case \( l = 0 \) with the observation that for \( n = 0 \) the end point \( t = 0 \) gives a limit circle case (in the terminology introduced by H. Weyl) while \( t = 1 \) gives a limit point case. For \( n \) non-zero both end points are of the limit point type. These statements can easily be proved from the detailed description of the solutions given above.

8.2. Solutions of \( 4D H = \lambda H \), \( \ell > 0 \). Consider now the system (12) given in Section 7. We fix the values of \( (\ell, n) \) and \( \lambda \). Exploiting the tridiagonal nature of our system of equations we can eliminate from the bottom up—all components of \( H(t) \) and obtain one equation in \( h_\ell(t) \) of order \( 2\ell + 2 \),

\[
P_{2\ell+2} \left( \frac{d}{dt} \right) h_\ell(t) = 0
\]
where the following factorization holds
\[ P_{2\ell+2} \left( \frac{d}{dt} \right) = A_{2\ell} \left( \frac{d}{dt} \right) \left( t(1-t) \left( \frac{d}{dt} \right)^2 + (n+1-(\ell+3)t) \frac{d}{dt} - \frac{4}{t} \right) \]
with \( A_{2\ell} \left( \frac{d}{dt} \right) \) a differential operator of order \( 2\ell \). For \( \lambda = -4w(w+n+\ell+2) \) and from the discussion in the introduction of the section, we can take for \( h_{\ell}(t) \) any local solution in the space
\[ \Phi(-w, w + n + \ell + 2, n + 1; t) \]
components \( h_i(t), i = 0, 1, \ldots, \ell - 1 \) of a vector solution \( H(t) \) by applying to \( h_{\ell}(t) \) differential operators of orders \( 2(\ell - i) \),
\[ h_i(t) = R_{2(\ell-i)} \left( \frac{d}{dt} \right) h_{\ell}(t). \]
These operators can be read off from the system \( 4DH = \lambda H \). In particular \( R_0 \left( \frac{d}{dt} \right) = I \).
We claim that from this vector solution (actually any vector in this two-dimensional space) we can readily obtain an extra set of \( \ell \) vector solutions (actually a \( 2\ell \) dimensional space of such).
Proceed as follows: Pick \( k = 1, \ldots, \ell \) and consider the system
\[ 4DH(t) = (\lambda - 4k(n + k + 1))H(t). \]
Eliminate (from the top down) all components of \( H(t) \) except \( h_0(t) \). At the last step one gets an equation of order \( 2\ell + 2 \) for \( h_0(t) \),
\[ M_{2\ell+2} \left( \frac{d}{dt} \right) h_0 = 0 \]
where another remarkable factorization holds, namely,
\[ M_{2\ell+2} \left( \frac{d}{dt} \right) \tilde{R}_{2(\ell-k)} \left( \frac{d}{dt} \right) = B_{4\ell-2k} \left( \frac{d}{dt} \right) \left( (1-t)t \left( \frac{d}{dt} \right)^2 + ((n+k+1)-(n+k+\ell+3)t) \frac{d}{dt} - \lambda \right). \]
Here \( B_{4\ell-2k} \left( \frac{d}{dt} \right) \) is a differential operator of order \( 4\ell - 2k \) and \( \tilde{R}_{2(\ell-k)} \left( \frac{d}{dt} \right) \) is the operator obtained from the \( R_{2(\ell-k)} \left( \frac{d}{dt} \right) \) introduced above by replacing \( n \) by \( n+k \). In consequence we can take for \( h_0(t) \) anything in the space
\[ \tilde{R}_{2(\ell-k)} \left( \frac{d}{dt} \right) \Phi(-w, w + n + k + \ell + 2, n + k + 1; t) \]
where \( w \) is now any root of
\[ \lambda = -4w(w+n+k+\ell+2). \]
Just as before, from $h_0(t)$ one gets the remaining components of a vector solution $H(t)$ by the recipe

$$h_k(t) = Q_{2k} \left( \frac{d}{dt} \right) h_0(t).$$

In conclusion we have $H(t)$ satisfying $DH = 4\left(-w(w + n + k + \ell + 2) - k(n + k + 1)\right)H$, and the relation between $w$ and the original $\lambda$ is given by

$$\lambda = -4w(w + n + k + \ell + 2) - 4k(n + k + 1).$$

Note that the first vector $H(t)$, constructed from $h_\ell(t)$, corresponds to $k = 0$. In a very explicit sense this $h_\ell(t)$ is the “seed” for the entire construction.

The discussion above is summarized as follows: For a fixed $(n, \ell)$ we associate to each integer $k$ satisfying $0 \leq k \leq \ell$ a two-dimensional space of solutions of $4DH = \lambda H$. The solutions can be constructed fairly explicitly as the result of applying certain differential operators to solutions of Gauss’s hypergeometric equation. The construction involves an auxiliary parameter $w$ related to $\lambda$ by

$$\lambda = -4w(w + n + k + \ell + 2) - 4k(n + k + 1).$$

The choice of one of the two roots of this quadratic equation does not affect the final result, i.e. the construction is invariant under the exchange of $-w$ into $w + n + k + \ell + 2$.

The parameters $k$ and $w$ will reappear in Section 11. In that section we will use the results of Section 9 giving a parametrization of those solutions of $4DH = \lambda H$ which correspond to spherical functions. The parameter $\lambda$ will be replaced by $\lambda/4$. For the convenience of the reader we anticipate here that these conditions will turn out to be $0 \leq k \leq \ell$, as above, and the extra requirement that $w$ or $-(w + n + k + \ell + 2)$ can be chosen to satisfy the conditions

$$0 \leq w, \quad 0 \leq w + n + k.$$

An instance of these conditions was seen in Subsection 8.1 in the special case of $\ell = 0$.

By using the well known differentiation formula

$$\frac{d}{dt} _2F_1\left( \begin{array}{cc} a, b \\ c \end{array} ; t \right) = \frac{ab}{c} _2F_1\left( \begin{array}{cc} a+1, b+1 \\ c+1 \end{array} ; t \right),$$

one can re-express the components of different vector solutions $H(t)$ of $4DH = \lambda H$ in terms of “shifted” version of $2F_1$, as seen below.

We give a fairly general formula for analytic solutions, at $t = 0$, when $n \geq 0$ and then illustrate it in a couple of cases.

For fixed $\ell$, and $n \geq 0$, and any integer $0 \leq k \leq \ell$ we have a vector solution $H^k(t)$ with components

$$H^k_j(t) = \sum_{p=-\min(k,j)}^{\ell-\max(k,j)} z(j, p) _2F_1\left( \begin{array}{cc} -w+p, w+n+\ell+2+k-\min(j,k) \\ n+1+\ell-j \end{array} ; t \right).$$
Here $\lambda = -4w(w+n+\ell+2+k) - 4k(n+k+1)$ and $z(j,p)$ are appropriate coefficients.

We include two examples. If $\ell = 1$ and $k = 1$ we have, up to a common scalar multiple and for $n \geq 0$

\[
H_0^1(t) = \frac{1}{2} F_1 \left( \frac{-w, w+n+4}{n+2} ; t \right),
\]
\[
H_1^1(t) = \frac{-(n+1)}{w+n+3} \left( (w+n+2) \frac{1}{2} F_1 \left( \frac{-w-1, w+n+3}{n+1} ; t \right) + \frac{1}{2} F_1 \left( \frac{-w, w+n+3}{n+1} ; t \right) \right).
\]

If $\ell = 2$ and $k = 2$ we have, up to a scalar multiple and for $n \geq 0$

\[
H_0^1(t) = \frac{1}{2} F_1 \left( \frac{-w, w+n+6}{n+3} ; t \right),
\]
\[
H_1^1(t) = \frac{-(n+2)}{w+n+5} \left( -(w+n+3) \frac{1}{2} F_1 \left( \frac{-w-1, w+n+5}{n+2} ; t \right) + \frac{1}{2} F_1 \left( \frac{-w, w+n+5}{n+2} ; t \right) \right),
\]
\[
H_2^1(t) = \frac{(n+1)(n+2)}{(w+n+5)} \left( \frac{w+n+3}{2} \frac{1}{2} F_1 \left( \frac{-w-2, w+n+4}{n+1} ; t \right) + \frac{1}{2} F_1 \left( \frac{-w, w+n+4}{n+1} ; t \right) \right).
\]

In Section 11 we will give more compact expressions for these solutions in terms of generalized hypergeometric functions. The relation between those expressions and the ones here can also be seen by using [6] and [7].

9. Parametrizations of spherical functions

As we pointed out in Section 2, there exists a one to one correspondence between the set of all equivalence classes of finite dimensional irreducible representations of $G$ which contain the representation $\pi_{n,\ell}$ of $K$ and the set of all equivalence classes of irreducible spherical functions of $(G,K)$ of type $(n,\ell)$.

We need to quote a restriction theorem of finite dimensional irreducible representations of $GL(n,\mathbb{C})$ to $GL(n-1,\mathbb{C})$ which we shall use. The equivalence classes of finite dimensional irreducible holomorphic representations of $GL(n,\mathbb{C})$ are parametrized by the $n$-tuples of integers $m_1 \geq m_2 \cdots \geq m_n$ in the following way: the diagonal subgroup of $GL(n,\mathbb{C})$ acts on the highest weight vector, with respect to the Borel subgroup of all upper triangular matrices, of the representation $\tau_{(m_1,\ldots,m_n)}$ by $t_1 E_{11} + \cdots + t_n E_{nn} \rightarrow t_1^{m_1} \cdots t_n^{m_n}$.

If $GL(n-1,\mathbb{C})$ is identified with the subgroup of all $(n-1) \times (n-1)$ matrices of $GL(n,\mathbb{C})$ in the following way

\[
GL(n-1,\mathbb{C}) \simeq \left( GL(n-1,\mathbb{C}) \begin{array}{c} 0 \\ 0 \end{array} \right),
\]

then we have (see [24], p.186):
Proposition 9.1. When we restrict the representation \( m_1 \geq m_2 \cdots \geq m_n \) of \( \text{GL}(n, \mathbb{C}) \) to \( \text{GL}(n-1, \mathbb{C}) \) it decomposes as the direct sum of the representations \( k_1 \geq \cdots \geq k_{n-1} \) of \( \text{GL}(n-1, \mathbb{C}) \) such that \( m_1 \geq k_1 \geq m_2 \geq k_2 \cdots \geq k_{n-1} \geq m_n, \) all of these with multiplicity one.

Any finite dimensional irreducible representation of \( G = \text{SU}(3) \) extends to a unique holomorphic irreducible representation of \( \text{SL}(3, \mathbb{C}) \), and all of these are restrictions of holomorphic irreducible representations of \( \text{GL}(3, \mathbb{C}) \). Moreover the highest weight of the restriction of \( \text{GL}(3, \mathbb{C}) \) is \( \lambda = (m_1 - m_2)\lambda_1 + (m_2 - m_3)\lambda_2 \) where \( \lambda_1, \lambda_2 \) are the fundamental weights of \( \text{GL}(3, \mathbb{C}) \) given by \( \lambda_1(H) = x_1 \) and \( \lambda_2(H) = -x_3 \), for \( H = x_1E_{11} + x_2E_{22} + x_3E_{33} \in \text{sl}(3, \mathbb{C}) \).

Now \( K = S(U(2) \times U(1)) \simeq U(2) \) and any finite dimensional irreducible representation of \( U(2) \) extends to a unique holomorphic irreducible representation of \( \text{GL}(2, \mathbb{C}) \). Thus a \( U(2) \)-irreducible subrepresentation of \( \tau_{m_1, m_2, m_3} \) corresponds to a \( K \)-irreducible subrepresentation parametrized by a pair \( (n, \ell) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \) which we want to determine.

The parameter \( \ell \) can be read off the action of the diagonal subgroup of \( \text{SU}(2) \times \{1\} \) on a highest weight vector of the subrepresentation \( (k_1, k_2) \) of \( U(2) \). Thus \( t^{k_1} (t^{-1})^{k_2} = t^{k_1-k_2} \) gives \( \ell = k_1 - k_2 \).

The parameter \( n \) can instead be read off the action of the center of \( K \) in the \( (k_1, k_2) \) subrepresentation of \( U(2) \). From

\[
\begin{pmatrix}
  t & 0 & 0 \\
  0 & t & 0 \\
  0 & 0 & t^{-2}
\end{pmatrix}
\begin{pmatrix}
  t^3 & 0 & 0 \\
  0 & t^3 & 0 \\
  0 & 0 & t^{-2}
\end{pmatrix}
\]

we get \( t^{2n} t^\ell = t^{3(k_1+k_2)} t^{-2(m_1+m_2+m_3)} \). Therefore \( 2n + k_1 - k_2 = 3(k_1+k_2) - 2(m_1+m_2+m_3) \) which gives \( n = k_1 + 2k_2 - m_1 - m_2 - m_3 \).

Now from the previous discussion we easily get

Proposition 9.2. When we restrict to \( K \) the finite dimensional irreducible representation of \( G \) with highest weight \( \lambda = p\lambda_1 + q\lambda_2 \) it decomposes as the direct sum of the representations \( \pi_{n, \ell} \) of \( K \) given by \( n = k_1 + 2k_2 - p - 2q, \ell = k_1 - k_2 \) with \( p + q \geq k_1 \geq q \geq k_2 \geq 0, \) all of these with multiplicity one.

Proof. The irreducible representation of \( G \) with highest weight \( \lambda = p\lambda_1 + q\lambda_2 \) can be realized as the restriction to \( G \) of the representation \( \tau_{m_1, m_2, m_3} \) of \( \text{GL}(3, \mathbb{C}) \) with \( m_1 = p + q, m_2 = q, m_3 = 0 \). Thus \( \tau_{m_1, m_2, m_3} \) restricted to \( U(2) \) is the direct sum of the representations \( (k_1, k_2) \) with \( p + q \geq k_1 \geq q \geq k_2 \geq 0, \) and a fortiori \( \tau_{m_1, m_2, m_3} \) restricted to \( K \) is the direct sum of \( \pi_{n, \ell} \) with \( n = k_1 + 2k_2 - p - 2q, \ell = k_1 - k_2 \). The proposition follows.

Corollary 9.3. The equivalence classes of irreducible spherical functions of \( (G, K) \) of type \( (n, \ell) \) can be parametrized by the set of all tuples \( (p, q, k_1, k_2) \in \mathbb{Z}^4 \) such that \( p + q \geq k_1 \geq q \geq k_2 \geq 0, \) with \( n = k_1 + 2k_2 - p - 2q \) and \( \ell = k_1 - k_2 \). Moreover if \( \Phi_{(p, q, k_1, k_2)} \) is a spherical function in the class \( (p, q, k_1, k_2) \) then

\[
\Delta_2 \Phi_{(p, q, k_1, k_2)} = -\frac{4}{9} (p^2 + q^2 + pq + 3p + 3q) \Phi_{(p, q, k_1, k_2)}
\]
\[ \Delta_3 \Phi_{(p,q,k_1,k_2)} = 4(\frac{2}{5}p^3 - \frac{2}{5}q^3 + \frac{1}{5}p^2q - \frac{1}{5}pq^2 + 2p^2 + pq + 4p + 2q) \Phi_{(p,q,k_1,k_2)}. \]

**Proof.** The only thing that remains to be proved are the last two formulas. Since \( \Delta_2 \) and \( \Delta_3 \) are in \( D(G)^G \) they act as scalars in any finite dimensional irreducible representation of \( G \). These scalars \( \lambda(\Delta_2) \) and \( \lambda(\Delta_3) \) can be computed by looking at the action of \( \Delta_2 \) and \( \Delta_3 \) on a highest weight vector.

From Proposition [12] it follows that

\[
\lambda(\Delta_2) = -\lambda(H_\alpha)^2 - \frac{1}{3}\lambda(Z)^2 - 2\lambda(H_\alpha) - 2\lambda(Z),
\]

\[
\lambda(\Delta_3) = \frac{8}{3}\lambda(H_\alpha)^3 - \frac{8}{5}\lambda(H_\beta)^3 + \frac{4}{3}\lambda(H_\alpha)^2\lambda(H_\beta) - \frac{4}{3}\lambda(H_\alpha)\lambda(H_\beta)^2
\]

\[+ 8\lambda(H_\alpha)^2 + 4\lambda(H_\alpha)\lambda(H_\beta) + 16\lambda(H_\alpha) + 8\lambda(H_\beta). \]

Replacing \( \lambda(H_\alpha) = p, \lambda(H_\beta) = q \) and \( \lambda(Z) = p + 2q \) in the above expressions the proof of the corollary is finished. \( \square \)

Going back to Proposition [4.1] we give in the following corollary the eigenvalues \( \lambda \) and \( \mu \) corresponding respectively to the differential operators \( D \) and \( E \) in terms of the parameters \( (n, l, p, q) \).

**Corollary 9.4.** We have

\[
\lambda = -4(\frac{2}{5}p^2 + q^2 + pq + 3p + 3q) + \frac{4}{3}(\ell^2 + n^2 + \ell n + 3\ell + 3n),
\]

\[
\mu = 4(\frac{2}{5}p^3 - \frac{2}{5}q^3 + \frac{1}{5}p^2q - \frac{1}{5}pq^2 + p^2 - q^2 + p - q)
\]

\[ - 4(\frac{2}{5}\ell^3 - \frac{2}{5}n^3 + \frac{1}{5}\ell^2n - \frac{1}{5}\ell n^2 + \ell^2 - n^2 + \ell - n). \]

For a given \((n, \ell)\) the four parameters \((p, q, k_1, k_2)\) are subject to two relations. One can introduce two integer free parameters \(w, k\) by means of the expressions

\[ k_1 = w + n + \ell + k, \]

\[ k_2 = w + n + k, \]

\[ p = w + \ell - k, \]

\[ q = w + n + 2k, \]

which are consistent with the relations in Proposition [12] and it is easy to see that the four inequalities in this proposition are equivalent to the following four conditions: \( 0 \leq w, 0 \leq k \leq \ell \) and \( 0 \leq w + n + k \). We recognize here the conditions on \( w, k \) mentioned in Section [8].

In terms of this new parameters we have

\[ \lambda = -4w(w + n + \ell + k + 2) - 4k(n + k + 1), \]

\[ \mu = \lambda(n + 3k - \ell) - 12k(\ell + 1 - k)(n + k + 1). \]

**10. The Analytic Eigenfunctions of D and E**

We take up the time honored method of formal power series expansions to look for simultaneous analytic eigenfunctions of the operators \( D \) and \( E \). The
system (18) given in Section 7 was obtained from the one given in Corollary 7.13 by making the change of variables \( t = (1 + r^2)^{-1} \) and we get

\[
\begin{align*}
  t (1 - t) h''_i + ((n + \ell - i + 1) - t(n + \ell - i + 3)) h'_i \\
  + (\ell - i)(i + 1) h_{i+1} - h_i - i(\ell - i + 1) t \frac{h_i - h_{i-1}}{1 - t} = \frac{\lambda}{t} h_i
\end{align*}
\]  

Similarly from Corollary 5.16 we get the system

\[
\begin{align*}
  (n - \ell + 3i) t (1 - t) h''_i - 3(i + 1)(\ell - i) h'_{i+1} \\
  + (n - \ell + 3i)(n + \ell - i + 1 - t(n + \ell - i + 3)) h'_i + 3i(\ell - i + 1) t h'_{i-1} \\
  + (n + 2\ell - 3i)(\ell - i)(i + 1) \frac{h_{i+1} - h_i}{1 - t} - (n + 2\ell - 3i)i(\ell - i + 1) \frac{h_i - h_{i-1}}{1 - t} \\
  - (2n + \ell + 3)i(\ell - i + 1)(h_i - h_{i-1}) = \frac{\mu}{t} h_i.
\end{align*}
\]

If we change \( \lambda/4 \) and \( \mu/4 \) respectively by \( \lambda \) and \( \mu \) and use matrix notation, both systems are equivalent to \( DH = \lambda H \) and \( EH = \mu H \) where the differential operators \( D \) and \( E \) are defined below.

\[
\begin{align*}
  DH &= t(1 - t)H'' + (A_0 - tA_1)H' + \frac{1}{1 - t}(B_0 - tB_1)H, \\
  EH &= t(1 - t)MH'' + (C_0 - tC_1)H' + \frac{1}{1 - t}(D_0 + tD_1)H.
\end{align*}
\]

Here \( H \) denotes the column vector \( H = (h_0, \ldots, h_\ell) \) and the coefficient matrices are given by:

\[
\begin{align*}
  A_0 &= \sum_{i=0}^\ell (n + \ell - i + 1)E_{ii}, \\
  A_1 &= \sum_{i=0}^\ell (n + \ell - i + 3)E_{ii}, \\
  B_0 &= \sum_{i=0}^\ell (i + 1)(\ell - i)(E_{i,i+1} - E_{i,i}), \\
  B_1 &= \sum_{i=0}^\ell i(\ell - i + 1)(E_{ii} - E_{i,i-1}), \\
  M &= \sum_{i=0}^\ell (n - \ell + 3i)E_{ii}, \\
  C_0 &= \sum_{i=0}^\ell (n - \ell + 3i)(n + \ell - i + 1)E_{ii} - 3 \sum_{i=0}^\ell (i + 1)(\ell - i)E_{i,i+1}, \\
  C_1 &= \sum_{i=0}^\ell (n - \ell + 3i)(n + \ell - i + 3)E_{ii} - 3 \sum_{i=0}^\ell i(\ell - i + 1)E_{i,i+1}, \\
  D_0 &= \sum_{i=0}^\ell (n + 2\ell - 3i)(i + 1)(\ell - i)(E_{i,i+1} - E_{ii}) \\
  &\quad - 3 \sum_{i=0}^\ell (n + \ell - i + 1)i(\ell - i + 1)(E_{ii} - E_{i,i-1}), \\
  D_1 &= \sum_{i=0}^\ell (2n + \ell + 3)i(\ell - i + 1)(E_{ii} - E_{i,i-1}).
\end{align*}
\]

**Proposition 10.1.** An irreducible spherical function \( \Phi \) on \( G \) of type \( (n, \ell) \) corresponds to a function \( H \) that is analytic at \( t = 0 \) and satisfies \( DH = \lambda H \) and \( EH = \mu H \) for some \( \lambda, \mu \in \mathbb{C} \). Moreover if \( n + \ell < 0 \) the function \( H(t) \) has a zero of order at least \(-n - \ell\) at \( t = 0 \).
Proof. In the previous sections we already established that \( H \) is an eigen-
function of \( D \) and \( E \).

that

\[
\tilde{H}(r) = (1 + r^2)^{n/2} \Phi \left( \frac{1}{(1 + r^2)^{1/2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -r \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^\ell.
\]

The first thing we get from here is that \( \tilde{H}(r) = \tilde{H}(-r) \). In fact we have

\[
\left( \begin{array}{ccc}
\frac{1}{2} & 0 & -r \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right)^{n/2}
\]

Moreover

\[
\pi \left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \quad \text{and} \quad \Phi \left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right)
\]

commute because they are diagonal matrices in the basis \( \{v_0, \ldots, v_\ell\} \). Hence

\( \tilde{H}(r) = \tilde{H}(-r) \).

If we make the change of variable \( s = 1/r \) then the function \( H = H(s) \) is
even and for \( s > 0 \) we have

\[
\tilde{H}(s) = \frac{(1 + s^2)^{n/2}}{s^n} \Phi \left( \frac{s}{(1 + s^2)^{1/2}} \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \right)^\ell.
\]

Thus \( h_i(s) \) is even and \( s^{n+\ell-i} h_i(s) \) is analytic at \( s = 0 \). Now in terms of
the variable \( t = (1 + r^2)^{-1} \) we have \( s^2 = t/(1 - t) \). Therefore \( h_i = h_i(t) \)
has a pole at \( t = 0 \) of order at most \( \left[ \frac{n+\ell-1}{2} \right] \) when \( n + \ell - i \geq 2 \). When
\( n + \ell - i \leq 1 \), \( h_i = h_i(t) \) is analytic at \( t = 0 \) and has also a zero or order at
least \( \left[ \frac{n+\ell-1}{2} \right] \), when \( n + \ell - i \leq -1 \).

If we put \( H(t) = \sum_{j \geq m} \mathcal{D}^j H_j \), and assume that \( H_m \neq 0 \), with column
vector coefficients \( H_j = (H_{0,j}, \ldots, H_{\ell,j}) \in \mathbb{C}^{\ell+1} \), then we have

(22) \[
H_{i,j} = 0 \quad \text{for all } j < - \left[ \frac{n + \ell - i}{2} \right].
\]

On the other hand from \( DH = \lambda H \) and (20) we get that the following three
term recursion relation holds for all \( j \)

(23)

\[
\begin{align*}
(j - 1)(j - 2) + (j - 1)A_1 - B_1 + \lambda]H_{j-1} \\
- [2j(j - 1) + j(A_0 + A_1) - B_0 + \lambda]H_j + (j + 1)(j + A_0)H_{j+1} = 0.
\end{align*}
\]
From here for \( j = m - 1 \) we obtain
\[
(24) \quad m(m + n + \ell - i)H_{i,m} = 0.
\]
Now assume that \( m < 0 \). From (24) we get \( H_{i,m} = 0 \) for all \( i \neq m + n + \ell \).
For \( i = m + n + \ell \) we have
\[
\left[ \frac{n + \ell - i}{2} \right] = \left[ \frac{-m}{2} \right] < -m.
\]
Then from (22) we obtain \( H_{m} = 0 \), which is a contradiction. Therefore \( m \geq 0 \) and \( H = H(t) \) is analytic at \( t = 0 \).

To prove the last assertion suppose that \( n + \ell < 0 \). First observe that \( m \geq -\left[ \frac{n + \ell}{2} \right] > 0. \) Now assume that \( m \leq -(n + \ell - 1) \). Then \( m + n + \ell - i \leq -1 - i < 0 \) for all \( 0 \leq i \leq \ell \). Hence (24) implies that \( H_{m} = 0 \), which completes the proof of the proposition.

Thus we want to consider now formal power series \( H(t) = \sum_{j=0}^{\infty} t^j H_j \) with column vector coefficients \( H_j \in \mathbb{C}^{\ell+1} \).

A formal solution of the system \( DH = \lambda H \) and \( EH = \mu H \) is then given by solving the pair of three term recursion relations given by (23) and
\[
(j - 1)(j - 2)M + (j - 1)C_1 + D_1 + \mu H_{j-1} + (j + 1)(jM + C_0)H_{j+1}
- 2j(j - 1)M + j(C_0 + C_1) - D_0 + \mu H_j = 0.
\]

Let \( V(\lambda) \) denote the vector space of all formal power series \( H(t) = \sum_{j=0}^{\infty} t^j H_j \) such that (23) holds. The differential operators \( D \) and \( E \), closely related to \( \Delta_2 \) and \( \Delta_3 \), commute. Therefore \( E \) restricts to a linear map of \( V(\lambda) \) into itself.

Now we shall first consider the case \( n \geq 0 \) where we have complete results and then we shall make some comments in the case \( n < 0 \).

Let \( \eta : V(\lambda) \to \mathbb{C}^{\ell+1} \) be the linear map defined by \( \eta : H(t) \mapsto H(0) \) and let \( L \) be the \((\ell + 1) \times (\ell + 1)\) matrix given by
\[
(26) \quad L = L(\lambda) = D_0 - C_0 A_0^{-1} (B_0 - \lambda).
\]
Notice that \( n \geq 0 \) insures that \( A_0 \) is invertible.

**Theorem 10.2.** If \( n \geq 0 \) then \( \eta : V(\lambda) \to \mathbb{C}^{\ell+1} \) is a surjective isomorphism. Moreover, the following is a commutative diagram
\[
\begin{array}{ccc}
V(\lambda) & \xrightarrow{E} & V(\lambda) \\
\downarrow\eta & & \downarrow\eta \\
\mathbb{C}^{\ell+1} & \xrightarrow{L} & \mathbb{C}^{\ell+1}
\end{array}
\]

**Proof.** The matrix \( j + A_0 \) is invertible for all \( j \geq 0 \) since
\[
\text{det}(j + A_0) = \prod_{i=0}^{\ell} (j + n + \ell - i + 1).
\]
Therefore the recurrence relation (23) shows that the coefficient vector \( H_0 = H(0) \) determines \( H(t) \). This proves the first assertion.
To prove the second statement let \( H(t) \in V(\lambda) \). Then from (21) and (23) we get
\[
\eta(EH(t)) = EH(0) = C_0 H'(0) + D_0 H(0) = C_0 H_1 + D_0 H_0
\]
\[
= (-C_0 A_0^{-1}(B_0 - \lambda) + D_0)H_0 = LH_0 = L(\eta(H(t))).
\]

In the next theorem we give the eigenvalues of the matrix \( L = L(\lambda) \) and their multiplicities, as well as their geometric multiplicities. We have only partial results on the eigenvectors of \( L \) and we do not display them here. In Section 11 we write them down in the cases \( \ell = 0, 1, 2 \).

**Theorem 10.3.** For any \((n, \ell) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}\) the characteristic polynomial of the matrix \( L(\lambda) \) is given by
\[
\det(\mu - L(\lambda)) = \prod_{k=0}^{\ell} (\mu - \mu_k(\lambda)),
\]
where \( \mu_k(\lambda) = \lambda(n - \ell + 3k) - 3k(\ell - k + 1)(n + k + 1) \). Moreover, all eigenvalues \( \mu_k(\lambda) \) of \( L(\lambda) \) have geometric multiplicity one. In other words all eigenspaces are one dimensional.

**Proof.** Let us consider the polynomial \( p \in \mathbb{C}[\lambda, \mu] \) defined by \( p(\lambda, \mu) = \det(\mu - L(\lambda)) \). For each integer \( k \) such that \( 0 \leq k \leq \ell \) let \( \lambda(w, k) = -w(w + n + \ell + k + 2) - k(n + k + 1) \) and let \( \mu_k(\lambda) = \lambda(n + 3k - \ell) - 3k(\ell - k + 1)(n + k + 1) \). Then from Theorem 10.2 and (17) we have
\[
p(\lambda(w, k), \mu_k(\lambda(w, k))) = 0,
\]
for all \( w \in \mathbb{Z} \) such that \( 0 \leq w \) and \( 0 \leq w + n + k \). Since there are infinitely many such \( w \), the polynomial function \( w \mapsto p(\lambda(w, k), \mu_k(\lambda(w, k))) \) is identically zero on \( \mathbb{C} \). Hence, for given \( k \) \((0 \leq k \leq \ell)\), we have \( p(\lambda, \mu_k(\lambda)) = 0 \) for all \( \lambda \in \mathbb{C} \).

Now \( \Lambda = \{ \lambda \in \mathbb{C} : \mu_k(\lambda) = \mu_{k'}(\lambda) \text{ for some } 0 \leq k < k' \leq \ell \} \) is a finite set, in fact \(| \Lambda | \leq \ell(\ell + 1)/2 \). Since for any \( \lambda \), \( p(\lambda, \mu) \) is a monic polynomial in \( \mu \) of degree \( \ell + 1 \), it follows that if \( \lambda \in \mathbb{C} - \lambda \) then
\[
p(\lambda, \mu) = \prod_{k=0}^{\ell} (\mu - \mu_k(\lambda))
\]
for all \( \mu \in \mathbb{C} \).

Now it is clear that (27) holds for all \( \lambda \) and all \( \mu \), which completes the proof of the first assertion.

To prove the second statement of the theorem we point out that the matrix \( L = L(\lambda) \) is a four diagonal matrix of the form
\[
L = \sum_{i=1}^{\ell} a_i E_{i,i-1} + \sum_{i=0}^{\ell} b_i E_{i,i} + \sum_{i=0}^{\ell-1} c_i E_{i,i+1} + \sum_{i=0}^{\ell-2} d_i E_{i,i+2},
\]
with
\[ a_i = 3i(\ell - i + 1)(n + \ell - i + 1), \]
\[ b_i = \lambda(n - \ell + 3i) - 3(i + 1)(\ell - i)(\ell - 2i) - 3i(\ell - i + 1)(n + \ell - i + 1), \]
\[ c_i = 3(i + 1)(\ell - i)(\ell - 2i - (n + \ell - i)^{-1}(i + 2)(\ell - i + 1 + \lambda)), \]
\[ d_i = 3(i + 1)(i + 2)(\ell - i - 1)(\ell - i)(n + \ell - i)^{-1}, \]
for all \( 1 \leq i \leq \ell. \)

If \( LH_\mu = \mu H_\mu \) with \( H_\mu = (h_0, \ldots, h_\ell) \in \mathbb{C}^{\ell + 1} \) then for all \( 0 \leq i \leq \ell \) we have
\[ a_i x_{i-1} + b_i x_i + c_i x_{i+1} + d_i x_{i+2} = \mu x_i, \]
where we must interpret \( a_0 = c_\ell = d_\ell = d_{\ell-1} = 0 \). From these equations we see that \( H_\mu \) is determined by \( x_\ell \), which proves that the geometric multiplicity of \( L \) is one.

\[ \square \]

**Corollary 10.4.** For \( n \geq 0 \) the formal solutions \( H(t) = \sum_{j=0}^{\infty} t^j H_j \) of our system \( DH = \lambda H, \quad EH = \mu H \) are parametrized, up to a nonzero multiplicative constant, by the eigenvalues \( \mu \) of the matrix \( L = L(\lambda) \). More precisely, if \( H_\mu \in \mathbb{C}^{\ell + 1} \) is a \( \mu \)-eigenvector of \( L \) then
\[ \eta^{-1}(H_\mu) = \sum_{j=0}^{\infty} t^j H_j \]
is the corresponding solution.

Notice that from the definition of \( \eta \) it follows that \( H_0 = H_\mu \).

This concludes the discussion of the case \( n \geq 0 \). In this case none of the components of \( H(t) \) vanishes at \( t = 0 \) unless \( H(t) \equiv 0 \).

When \( n < 0 \) things are not so simple. The structure of the order of the zeros of the components of \( H(t) \) at \( t = 0 \), leads us to consider the following cases: \( n \leq -\ell \) and \( -\ell < n < 0 \). A consequence of this structure is that the isomorphism \( \eta \) introduced in the case \( n \geq 0 \) must be redefined, as we do below.

The partition of the integers in the form
\[ (28) \quad \mathbb{Z} = \{ n : n \geq 0 \} \cup \{ n : -\ell < n < 0 \} \cup \{ n : n \leq -\ell \} \]
was already alluded to in the comments following Corollary 2.12.

When \( n < 0 \), if \( H \in \mathbb{V}(\lambda) \) then we have \( H(t) = \sum_{j=0}^{\infty} t^j H_j \) where \( a = \max \{ 0, -n - \ell \} \). See Proposition 10.1.

If \( n + \ell \leq 0 \) from (24), putting \( j = a - 1 \), we obtain \( a_i H_{i,a} = 0 \). Hence \( H_{i,a} = 0 \) for \( 1 \leq i \leq \ell. \) Similarly a closer look at (22) reveals that \( H_{i,a+k} = 0 \) for \( 0 \leq k \leq \ell - 1 \) and \( k + 1 \leq i \leq \ell \), and moreover that the map \( \eta : \mathbb{V}(\lambda) \to \mathbb{C}^{\ell + 1} \) defined by
\[ \eta(H) = (H_{0,a}, H_{1,a+1}, \ldots, H_{\ell,a+\ell}) \]
is a linear surjective isomorphism.

If \( 0 < n + \ell < \ell \) from (22) we get that \( H_{i,0} = 0 \) for all \( n + \ell + 1 \leq i \leq \ell \). But this does not follow from (23). So in this case we should consider the
vector space \( W(\lambda) = \{ H \in V(\lambda) : H_{i,0} = 0, \text{ for all } n + \ell + 1 \leq i \leq \ell \} \). Then it is easy to verify that the elements \( H \in W(\lambda) \) satisfy

\[
H_{i,k} = 0 \text{ for all } 0 \leq k \leq -n - 1 \text{ and } n + \ell + k + 1 \leq i \leq \ell,
\]

which is stronger than (22). In particular \( H_{i,1} = 0 \) for all \( n + \ell + 2 \leq i \leq \ell \). From this and (25) one can prove that the differential operator \( E \) restricts to a linear operator on \( W(\lambda) \). Moreover from (23) it follows that the map \( \eta : W(\lambda) \to \mathbb{C}^{\ell+1} \) defined by

\[
\eta(H) = (H_{0,0}, \ldots, H_{n+\ell,0}, H_{n+\ell+1,1}, H_{n+\ell+2,2}, \ldots, H_{\ell,-n})
\]

is a linear surjective isomorphism.

Notice that the structure of the order of zeros discussed above, insures that any analytic function \( H(t) \) in \( V(\lambda) \) or \( W(\lambda) \), depending on the value of \( n \), meets condition ii) of Proposition 6.1.

Unfortunately, the matrix \( L \) which completes the commutative diagram corresponding to the one given in Theorem 10.2, in the two cases going with \( n < 0 \), is not as simple as the one obtained when \( n \geq 0 \).

11. The spherical functions

The purpose of this section is to combine results given in previous sections to give a fairly explicit expression for the spherical functions associated to the complex projective plane in terms of a particular class of generalized hypergeometric functions. We have the following results which have been verified, by use of computer algebra, for values of \( \ell \) up to ten, but for which we do not have yet a general proof.

**Conjecture 11.1.** For a given \( \ell \geq 0 \), the spherical functions corresponding to the pair \((\ell, n)\) have components that are expressed in terms of generalized hypergeometric functions of the form \( p+2Fp+1 \), namely

\[
p+2Fp+1 \left( \begin{array}{c} a, b, a_1 + 1, \ldots, a_p + 1 \end{array}; \mathbf{t} \right) = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{j!(c)_j} (1 + d_1 j + \cdots + d_p j^p) t^j.
\]

In the case \( n \geq 0 \) we can give a more precise form of our results, namely

**Conjecture 11.2.** For \( \ell \geq 0 \) and \( n \geq 0 \) the spherical functions are given by diagonal matrices corresponding to \( 0 \leq k \leq \ell \). For a given \( k \) the entries along the diagonal make up a vector \( H_i^k(t) \), \( 0 \leq i \leq \ell \), with

\[
H_i^k(t) = H_i^k(0) \times \left( -w - \min(i, k), w + n + \ell + k + 2 - \min(i, k), (s_j + 1)_{n + \ell - i + 1, s_j} \right).
\]

The vector \( H_i^k(0) \) is an eigenvector of \( L(\lambda) \) defined in (23). The eigenvalues \( \lambda, \mu \) of \( D \) and \( E \) (see (20) and (21)) are given by \( \lambda = -w(w + n + \ell + k + 2) - k(n + k + 1) \) and \( \mu = \lambda(n - \ell + 3k) - 3k(\ell - k + 1) \). Finally \( s_j \) denotes a row vector of \( \ell - |i - k| \) denominator parameters and \( s_j + 1 \) the same coefficients shifted up by one.
Remark. By definition of the function \( p+2F_{p+1} \) on the left-hand side of Conjecture 11.1 we have
\[
\sum_{j=0}^{\infty} \frac{(a)_j(b)_j(s_1+1)_j \ldots (s_p+1)_j}{j!(c)_j \ldots (s_p)_j} t^j.
\]

Notice that the ratio of the \( p \) factors involving \( s_1, \ldots, s_p \) is given by
\[
\frac{(s_1+j)(s_2+j) \ldots (s_p+j)}{s_1s_2 \ldots s_p}
\]
and this serves to define the polynomial \( 1 + d_1j + \cdots + d_pj^p \) appearing in Conjecture 11.1.

The generalized hypergeometric functions \( p+1F_p \) have not, so far, played in representation theory and/or harmonic analysis a role comparable to that of the celebrated case of Gauss and Euler, namely \( 2F_1 \). An exception to this are the cases of \( p = 2, 3 \), and a few other low values of \( p \) as well documented for instance in [1]. Another very interesting exception to the statement above are the cases of the families \( 3n-4F_{3n-5} \) related to matrix entries for representations of \( U(n) \) discussed in [16] and [22]. For arbitrary \( p \) the monodromy of the corresponding equation has been analyzed in [2].

We observe that the special type of generalized hypergeometric functions that appear here do satisfy differential equations of lower order than the order \( p+2 \) that one should expect. This has been observed for instance in [19]. We are grateful to M. Ismail for this reference.

Next we give a guide that should allow the reader to see how the results in Sections 9 and 10, as well as Proposition 6.1 can be used to see that for \( \ell = 0,1,2, \ldots \) we are exhibiting all spherical functions of type \((n,\ell)\). We need to consider separately the cases:

a) \( n \geq 0 \),  
b) \( n \leq -\ell \),  
c) \( -\ell < n < 0 \),

as indicated in (28).

The results in Section 10 are complete for the case \( n \geq 0 \) and simpler than in the other cases, and we start with it.

arbitrary \( \lambda \in \mathbb{C} \) we should pick \( \mu \) as one of the eigenvalues \( \mu_k(\lambda) \), \( 0 \leq k \leq \ell \), of the matrix \( L(\lambda) \). Then we determine \( H_\mu \) as the properly normalized eigenvector. Starting with \( H_0 = H_\mu \) the three term recursion relation (28) reduces to a two term recursion, and this leads to our hypergeometric functions in terms of a parameter \( w \in \mathbb{C} \) related to \( \lambda \) by
\[
\lambda = -w(w+n+\ell+k+2) - k(n+k+1).
\]
Since these hypergeometric functions remain bounded at \( t = 1 \) only when they are polynomials, we are forced to take \( w = 0,1,2, \ldots \). Using the appropriate version of Gauss’ summation formula we see that we can normalize \( H_\mu \) in order to get \( H(1) = (1, \ldots, 1) \).

Recall that we have shown that the corresponding diagonal matrix is a scalar matrix, 5.14. Now Proposition 6.1 guarantees that \( H(t) \) in fact corresponds to a spherical function \( \Phi \) on \( G \) of type \((n,\ell)\). By refering to Section 9 one sees that we have a complete list.
It is remarkable that for the cases b) and c) the same machinery can be used, although we do not yet have the corresponding theoretical statements. The insistence on bounded eigenfunctions forces a different choice of \( w \), which again is consistent with the conditions \( 0 \leq w \) and \( 0 \leq w + n + k \) in Section 3.

11.0. The case \( \ell = 0 \). We see that the complete list of spherical functions is given (up to a scalar multiple) by:

a) Case \( n \geq 0 \). For \( w = 0, 1, 2, \ldots \) we have \( \lambda = -w(w+n+2), \mu = n\lambda \) and

\[
\varphi_w(t) = 2F_1\left(\frac{-w, w+n+2}{n+1}; t\right).
\]

b) Case \( n < 0 \). For \( w = -n, -n+1, \ldots \) we have \( \lambda = -w(w+n+2), \mu = n\lambda \) and

\[
\psi_w(t) = t^{-n} 2F_1\left(\frac{-w-n, w+n+2}{1-n}; t\right).
\]

These results were already given, and proved, in Section 3. In this case the operator \( E \) is proportional to \( D \).

11.1. The case \( \ell = 1 \). The complete list of matrix (or vector) valued spherical functions is given below.

a) Case \( n \geq 0 \). We have two families of spherical functions corresponding either to the choice \( \mu_0, \mu_1 \) indicated in Section 11 or the choice \( k = 0, 1 \) alluded to at the end of Section 10 and given in Section 10.

a.0) For \( k = 0 \), and \( w = 0, 1, 2, \ldots \) we have \( \lambda = -w(w+n+3), \mu = \lambda(n-1) \) and

\[
H_0 = \left(1 - \frac{\lambda}{n+1}\right),
\]

\[
H(t) = \left(\begin{array}{c}
\left(1 - \frac{\lambda}{n+1}\right) 3F_2\left(\frac{-w, w+n+3, \lambda-n}{n+2, \lambda-n-1}; t\right) \\
2F_1\left(\frac{-w, w+n+3}{n+1}; t\right)
\end{array}\right).
\]

a.1) For \( k = 1 \), and \( w = 0, 1, 2, \ldots \) we have \( \lambda = -w(w+n+4) - n - 2, \mu = (\lambda - 3)(n+2) \) and

\[
H_0 = \left(\begin{array}{c}
1 \\
-(n+1)
\end{array}\right),
\]

\[
H(t) = \left(\begin{array}{c}
2F_1\left(\frac{-w, w+n+4}{n+2}; t\right) \\
-(n+1) 3F_2\left(\frac{-w-1, w+n+3, \lambda}{n+1, \lambda-1}; t\right)
\end{array}\right).
\]

b) Case \( n < 0 \). Again we have two families corresponding to the choice \( k = 0 \) or 1 made in the case \( n \geq 0 \).
11.2. The case $\ell = 2$. The complete list follows.

\textbf{a)} Case $n \geq 0$.

\textbf{a.0)} For $k = 0$, and $w = 0, 1, 2, \ldots$ we have $\lambda = -w(w + n + 3)$, $\mu = \lambda(n - 1)$ and

$$H(t) = \frac{nt^{-n-1}}{3F_2} \left( \frac{w+2, -w-n-1, a+1}{-n, a}; t \right),$$

with $a = -w(w + n + 3) - 2n - 2$.

\textbf{b.1)} For $k = 1$, and $w = -n, -n + 1, \ldots$ we have $\lambda = -w(w + n + 4) - n - 2$, $\mu = (\lambda - 3)(n + 2)$ and

$$H(t) = \left( \frac{t}{n} \right)^{n-1} 3F_2 \left( \frac{w+3, -w-n-1}{1-n, b}; t \right),$$

with $b = -w(w + n + 4) - 2n - 3$.

\textbf{a.0)} For $k = 0$, and $w = 0, 1, 2, \ldots$ we have $\lambda = -w(w + n + 4)$, $\mu = \lambda(n - 2)$ and

$$H_0 = \left( \begin{array}{c} 1 + \frac{\lambda(\lambda-3(n+1))}{2(n+1)(n+2)} \\ 1 - \frac{\lambda}{n+1} \\ 1 \end{array} \right),$$

$$H(t) = \left( \begin{array}{c} 1 + \frac{\lambda(\lambda-3(n+1))}{2(n+1)(n+2)} \\ 1 - \frac{\lambda}{n+1} \\ 1 \end{array} \right) \cdot 3F_3 \left( \frac{-w, w+n+4, s_1+1, s_2+1}{n+3, s_1, s_2}; t \right),$$

with

$$s_1 s_2 = \frac{w(w+3)(w+n+1)(w+n+4)}{2} + (n+1)(n+2),$$

$$s_1 + s_2 = -w(w + n + 4) - n,$$

$$s_3 = -\frac{w(w+n+4)}{2} - \frac{n+1}{2}.$$

\textbf{a.1)} For $k = 1$, and $w = 0, 1, 2, \ldots$ we have $\lambda = -w(w + n + 5) - n - 2$, $\mu = \lambda(n + 1) - 6(n + 2)$ and

$$H_0 = \left( \begin{array}{c} \frac{\lambda}{(n+1)(n+2)} \\ -\frac{\lambda+2}{2(n+1)} \\ 1 \end{array} \right),$$

$$H(t) = \left( \begin{array}{c} \frac{\lambda}{(n+1)(n+2)} \\ -\frac{\lambda+2}{2(n+1)} \\ 1 \end{array} \right) \cdot 3F_2 \left( \frac{-w, w+n+5, s_4+1}{n+3, s_4}; t \right),$$

$$\ldots$$

$$3F_2 \left( \frac{-w-1, w+n+4, s_7+1}{n+1, s_7}; t \right).$$
with
\[s_4 = -\frac{w(w+n+5)}{2} - \frac{n+2}{2},\]
\[s_5 + s_6 = -\frac{w(w+n+5)}{2} - \frac{1}{2},\]
\[s_5s_6 = \frac{(w+1)(w+n+4)(w^2+nw+5w+n)}{8},\]
\[s_7 = -\frac{(w+1)(w+n+4)}{2}.

a.2) For \(k = 2, \) and \(w = 0, 1, 2, \ldots\) we have \(\lambda = -w(w + n + 6) - 2n - 6,\)
\(\mu = \lambda(n + 4) - 6(n + 3)\) and
\[H_0 = \begin{pmatrix} \frac{2}{(n+2)(n+1)} \\ -\frac{2}{n+1} \\ 1 \end{pmatrix},
\]
\[H(t) = \begin{pmatrix} \frac{2}{(n+2)(n+1)} \binom{2}{2} F_1 \left( -w, w+n+6; t; \frac{1}{n+3} \right) \\ -\frac{2}{n+1} \binom{3}{2} F_2 \left( -w-1, w+n+5, s_8+1; t; \frac{n+2}{n+1}, s_8 \right) \\ 4 F_3 \left( -w-2, w+n+4, s_9+1, s_{10}+1; t; \frac{n+1}{n+1}, s_9, s_{10} \right) \end{pmatrix},
\]
with
\[s_8 = -\frac{w(w+n+6)}{2} - \frac{n+5}{2},\]
\[s_9s_{10} = \frac{(w+1)(w+2)(w+n+4)(w+n+5)}{2},\]
\[s_9 + s_{10} = -w(w + n + 6) - (n + 6).
\]

b) Case \(n \leq -2.\) We have, as above, three families going with the choices \(k = 0, 1, 2.\)

b.0) For \(k = 0, \) and \(w = -n, -n + 1, \ldots\) we have \(\lambda = -w(w + n + 4),\)
\(\mu = \lambda(n - 2)\) and
\[H(t) = \begin{pmatrix} t^{-n-2} 4 F_3 \left( w+2, -w-n-2, s_1+1, s_2+1; t; \frac{-n-1}{n+1}, s_1, s_2 \right) \\ \frac{2}{n+1} t^{-n-1} 3 F_2 \left( w+3, -w-n-1, s_3+1; t; \frac{-n}{n+1}, s_3 \right) \\ \frac{2}{n(n+1)} t^{-n} 2 F_1 \left( w+4, -w-n; t; \frac{-n+1}{n+1} \right) \end{pmatrix},
\]
with
\[s_1s_2 = \frac{(w+2)(w+3)(w+n+1)(w+n+2)}{2},\]
\[s_1 + s_2 = -w(w + n + 4) - 3n - 4,\]
\[s_3 = -\frac{w(w+n+4)}{2} - \frac{3}{2}(n + 1).\]
b.1) For $k = 1$, and $w = -n - 1, n, -n + 1, \ldots$ we have $\lambda = -w(w + n + 5) - n - 2$, $\mu = \lambda(n + 1) - 6(n + 2)$ and

$$H(t) = \left( t^{-n-2} F_2 \left( \begin{array}{c} w+3, -w-n-2, s_1+1 \\ -n-1, s_1 \end{array} ; t \right) \right),$$

with $s_1, s_2, s_3, s_4$ given by the expressions

$$s_2 s_3 = \frac{(w+3)(w+n+2)(w^2+nw+5w+3n+2)}{8},$$

$$s_2 + s_3 = -\frac{w(w+n+5)}{2} - \frac{4n+5}{2},$$

$$s_1 = -\frac{w(w+n+5)}{2} - \frac{(3n+6)}{2},$$

$$s_4 = -\frac{w(w+n+5)}{2} - \frac{(3n+4)}{2}.$$

b.2) For $k = 2$, and $w = -n - 2, -n - 1, -n \ldots$ we have $\lambda = -w(w + n + 6) - 2n - 6$, $\mu = \lambda(n + 4) - 6(n + 3)$ and

$$H(t) = \left( t^{-n-2} F_1 \left( \begin{array}{c} w+4, -w-n-2 \\ -n-1 \end{array} ; t \right) \right),$$

with $s_1, s_2, s_3$ given by

$$s_1 = -\frac{w(w+n+6)}{2} - \frac{3n+7}{2},$$

$$s_2 s_3 = \frac{w(w+n+6)(w(w+n+6)+5n+13)}{2} + (3n^2 + 15n + 20),$$

$$s_2 + s_3 = -w(w + n + 6) - (3n + 6).$$

Finally, we come to the exceptional situation that arises when $-l < n < 0$.

c) Case $n = -1$. As in all the previous situations there are three families of spherical functions, corresponding to the choices $k = 0, 1, 2$. We observe that the expressions for $\mu_k(\lambda)$ in Section 14 are still valid when $n < 0$.

c.0) For $k = 0$, and $w = 1, 2, 3, \ldots$ we have $\lambda = -w(w + 3)$, $\mu = -3\lambda$ and

$$H(t) = \left( \begin{array}{c} 4F_3 \left( \begin{array}{c} -w, w+3, s_1+1, s_2+1 \\ 2, s_1, s_2 \end{array} ; t \right) \\ -w(w+3) \end{array} ; t \right),$$

and we have

$$s_1 s_2 = \frac{w^2(w+3)^2}{2},$$

$$s_1 + s_2 = -w^2 - 3w + 1,$$

$$s_3 = -\frac{w(w+3)}{2}. $$
c.1) For \( k = 1 \), and \( w = 0, 1, 2, \ldots \) we have \( \lambda = -w(w+4) - 1 \), \( \mu = -6 \) and
\[
H(t) = \begin{pmatrix}
3F_2\left(-w, w+4, \frac{s_1+1}{2}; t\right) \\
\frac{(-w^2+4w-1)}{2(w^2+4w+1)}\ 4F_3\left(-w-1, w+3, \frac{s_2+1}{1}, s_2, s_3; t\right) \\
t\ 3F_2\left(-w, w+4, \frac{s_1+1}{2}; t\right)
\end{pmatrix},
\]
with
\[
s_1 = -\frac{w^2+4w+1}{2},
\]
\[
s_2s_3 = \frac{(w+1)(w+3)(w^2+4w-1)}{8},
\]
\[
s_2 + s_3 = -\frac{w^2+4w+1}{2}.
\]

Remark. Note that in this case \( H_2(t) = tH_0(t) \).

c.2) For \( k = 2 \), and \( w = 0, 1, 2, \ldots \) we have \( \lambda = -w(w+5) - 4 \), \( \mu = 3\lambda - 12 \) and
\[
H(t) = \begin{pmatrix}
2F_1\left(-w, w+5; \frac{t}{2}\right) \\
-3F_2\left(-w-1, w+4, \frac{s_1+1}{1}, s_1; t\right) \\
-\frac{1}{2}(w+1)(w+4)t\ 4F_3\left(-w-1, w+4, \frac{s_2+1}{2}, s_2, s_3; t\right)
\end{pmatrix},
\]
with
\[
s_1 = -\frac{(w+1)(w+4)}{2},
\]
\[
s_2s_3 = \frac{(w+1)^2(w+4)^2}{2},
\]
\[
s_2 + s_3 = -(w^2 + 5w + 3).
\]

12. A matrix valued form of the bispectral property

In the case of \( \ell = 0 \) when our spherical functions are scalar valued and reduce to the well known case of Jacobi polynomials, one has a three-term recursion relation giving the product of \( t \) times \( \phi_{w}(t) \) as a linear combination, with coefficients independent of \( t \), of the functions \( \phi_{w-1}(t) \), \( \phi_{w}(t) \), and \( \phi_{w+1}(t) \).

The reasons behind this fact are well understood in terms of tensor products of representations. At a more elementary level this is just a consequence of dealing with a sequence of orthogonal polynomials.

From a different point of view this three-term recursion relation, along with the differential equation in \( t \) satisfied by \( \phi_{w}(t) \), can be seen as a basic instance of the “bispectral property” discussed in a more general context, first in [4] and subsequently in a variety of related contexts by several authors. For a good reference showing connections of this problem with lots of other areas see [10]. In this field the basic cases play a very crucial role, since they give rise to entire families of nontrivial bispectral situations by repeated applications of the Darboux process.
We finish this paper by displaying a rather intriguing matrix valued three term recursion relation that we have found for our spherical functions. We have not investigated the reasons behind it and we do not claim that this is the most natural such recursion in our context. We expect however to return to this problem in the future. In fact, since the initial version of this paper was completed we have made substantial progress in establishing the results illustrated below and the interested reader may want to see [3].

For given nonnegative integers \( n, \ell \) and \( w \) consider the matrix whose rows are given by the vectors \( H(t) \) corresponding to the values \( k = 0, 1, 2, \ldots, \ell \) discussed above. Denote the corresponding matrix by

\[
\Phi(w, t).
\]

12.1. The bispectral property. As a function of \( t \), \( \Phi(w, t) \) satisfies two differential equations

\[
D \Phi(w, t)^T = \Phi(w, t)^T \Lambda, \quad E \Phi(w, t)^T = \Phi(w, t)^T M.
\]

Here \( \Lambda \) and \( M \) are diagonal matrices with

\[
\Lambda(i, i) = -w(w + n + i + \ell + 1) - (i - 1)(n + i), \quad M(i, i) = \Lambda(i, i)(n - \ell + 3i - 3) - 3(i - 1)(\ell - i + 2)(n + i),
\]

\( 1 \leq i \leq \ell + 1; \) \( D \) and \( E \) are the differential operators introduced in (20) and (21). Moreover we have

**Theorem 12.1.** There exist matrices \( A_w, B_w, C_w \), independent of \( t \), such that

\[
A_w \Phi(w - 1, t) + B_w \Phi(w, t) + C_w \Phi(w + 1, t) = t \Phi(w, t).
\]

The matrices \( A_w \) and \( C_w \) consist of two diagonals each and \( B_w \) is tridiagonal. Assume, for convenience, that these vectors are normalized in such a way that for \( t = 1 \) the matrix \( \Phi(w, 1) \) consists of all ones.

We display below the formulas in the case \( \ell = 2, n \geq 0 \). We leave the \( \ell \) dependence quite explicit in these formulas to indicate how things go in the general case. If we are not in the case \( \ell = 2 \) the multiplicative factor 2 in the off-diagonal elements of the three matrices below has to be altered.

The nonzero entries of the matrix \( A_w \) are given as follows

\[
A_w(i, i) = w(w + \ell + 1)(w + n + i - 1)(w + n + \ell + i)(w + \ell - i + 2)^{-1}
\]
\[
\times (w + n + 2i - 1)^{-1}(2w + n + \ell + i - 1)^{-1}(2w + n + \ell + i + 1)^{-1},
\]

\[
A_w(i, i + 1) = 2w(w + \ell + 1)(w + \ell - i + 1)^{-1}(w + \ell - i + 2)^{-1}
\]
\[
\times (w + n + 2i - 1)^{-1}(2w + n + \ell + i + 1)^{-1}.
\]

Notice that if \( w = 0 \) we have \( A_0 = 0 \). This is useful since we have not defined \( \Phi(-1, t) \).

The nonzero entries of the matrix \( C_w \) are given as follows

\[
C_w(i, i) = (w + 1)(w + \ell + 2)(w + n + i)(w + n + \ell + i + 1)(w + \ell - i + 2)^{-1}
\]
\[
\times (w + n + 2i - 1)^{-1}(2w + n + \ell + i + 1)^{-1}(2w + n + \ell + i + 2)^{-1},
\]
For completeness we include here the proofs of those propositions and lemmas given, without them, in Sections 4 and 5.

**Proposition 13.** For $H \in C^\infty(\mathbb{C}^2) \otimes \text{End}(V_n)$ we have

$$D_1(H)(x, y) =$$

$$ (1 + |x|^2 + |y|^2) \left( (H_{x_1x_1} + H_{x_2x_2})(1 + |x|^2) + (H_{y_1y_1} + H_{y_2y_2})(1 + |y|^2) + 2(H_{y_1x_1} + H_{y_2x_2}) \Re(x \overline{y}) + 2(H_{y_1x_2} - H_{y_2x_1}) \Im(x \overline{y}) \right).$$

**Proof.** We have

$$D_1(H) = -4(X_{-\beta}X_\beta + X_{-\gamma}X_\gamma)(H) = (Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2)(H).$$
We begin by calculating \( Y_5^2(H)(g) \), for all \( g \in A \). We have
\[
Y_5^2(H)(g) = \left( \frac{d}{ds} \frac{d}{dt} H(p(g \exp(s + t)Y_5)) \right)_{s=t=0}.
\]

Let
\[
u(s, t) = \frac{g_{12}\tan(s + t) + g_{13}}{g_{32}\tan(s + t) + g_{33}} \quad \text{and} \quad v(s, t) = \frac{g_{22}\tan(s + t) + g_{23}}{g_{32}\tan(s + t) + g_{33}}.
\]

Then
\[
p(g \exp(s + t)Y_5) = (u(s, t), v(s, t), 1).
\]

Now using Lemma 2.7 we obtain
\[
\left( \frac{\partial u}{\partial s} \right)_{s=0} = \frac{g_{12}g_{33} - g_{13}g_{32}}{(g_{32}\sin t + g_{33}\cos t)^2} = -\frac{\mathcal{g}_{21}}{(g_{32}\sin t + g_{33}\cos t)^2}
\]
and
\[
\left( \frac{\partial v}{\partial s} \right)_{s=0} = \frac{g_{22}g_{33} - g_{23}g_{32}}{(g_{32}\sin t + g_{33}\cos t)^2} = \frac{\mathcal{g}_{11}}{(g_{32}\sin t + g_{33}\cos t)^2}.
\]

Therefore at \( s = t = 0 \) we have
\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} = -\frac{\mathcal{g}_{21}}{g_{33}}, \quad \frac{\partial v}{\partial s} = \frac{\partial v}{\partial t} = \frac{\mathcal{g}_{11}}{g_{33}}
\]
and
\[
\frac{\partial^2 u}{\partial s \partial t} = \frac{2g_{32}\mathcal{g}_{21}}{g_{33}^3}, \quad \frac{\partial^2 v}{\partial s \partial t} = \frac{-2g_{32}\mathcal{g}_{11}}{g_{33}^3}.
\]

Similarly let
\[
\tilde{u}(s, t) = \frac{ig_{12}\tan(s + t) + g_{13}}{ig_{32}\tan(s + t) + g_{33}} \quad \text{and} \quad \tilde{v}(s, t) = \frac{ig_{22}\tan(s + t) + g_{23}}{ig_{32}\tan(s + t) + g_{33}}.
\]

Then we obtain \( p(g \exp(s + t)Y_6) = (\tilde{u}(s, t), \tilde{v}(s, t), 1) \). So, at \( s = t = 0 \) we have
\[
\frac{\partial \tilde{u}}{\partial s} = \frac{\partial \tilde{u}}{\partial t} = \frac{i\mathcal{g}_{21}}{g_{33}}, \quad \frac{\partial \tilde{v}}{\partial s} = \frac{\partial \tilde{v}}{\partial t} = \frac{i\mathcal{g}_{11}}{g_{33}}
\]
and
\[
\frac{\partial^2 \tilde{u}}{\partial s \partial t} = \frac{-2g_{32}\mathcal{g}_{21}}{g_{33}^3}, \quad \frac{\partial^2 \tilde{v}}{\partial s \partial t} = \frac{2g_{32}\mathcal{g}_{11}}{g_{33}^3}.
\]

By using the chain rule we have
\[
Y_5^2(H)(g) = \sum_{i,j=1,2} H_{x_i x_j} \frac{\partial u_i}{\partial s} \frac{\partial u_j}{\partial t} + \sum_{i,j=1,2} H_{y_i y_j} \frac{\partial v_i}{\partial s} \frac{\partial v_j}{\partial t}
\]
\[
+ \sum_{i,j=1,2} H_{x_i y_j} \left( \frac{\partial^2 u_i}{\partial s \partial t} + \frac{\partial^2 v_j}{\partial s \partial t} \right) + \sum_{i=1,2} H_{x_i} \frac{\partial^2 u_i}{\partial t \partial s} + H_{y_i} \frac{\partial^2 v_i}{\partial t \partial s}
\]
and
\[
Y_6^2(H)(g) = \sum_{i,j=1,2} H_{x_i x_j} \frac{\partial \tilde{u}_i}{\partial s} \frac{\partial \tilde{u}_j}{\partial t} + \sum_{i,j=1,2} H_{y_i y_j} \frac{\partial \tilde{v}_i}{\partial s} \frac{\partial \tilde{v}_j}{\partial t}
\]
Upon considering the unique holomorphic representation of $\text{GL}(2, \mathbb{C})$

**Proof.**

We proceed in the same way with $Y_3^2 + Y_4^2$ and obtain

\[(Y_3^2 + Y_4^2)(H)(g) = (H_{x_1x_1} + H_{x_2x_2}) \left| \frac{g_{22}}{g_{33}} \right|^2 + (H_{y_1y_1} + H_{y_2y_2}) \left| \frac{g_{12}}{g_{33}} \right|^2 - \frac{2 \text{Re}(g_{12}g_{22})}{|g_{33}|^4} (H_{x_1x_1} + H_{x_2x_2}) - \frac{2 \text{Im}(g_{12}g_{22})}{|g_{33}|^4} (H_{x_1x_1} + H_{x_2x_2}) \cdot\]

Hence

\[\begin{aligned}
(D_1)(H)(g) &= (Y_3^2 + Y_4^2)(H)(g) + (Y_5^2 + Y_6^2)(H)(g) \\
&= (H_{x_1x_1} + H_{x_2x_2}) \left( \frac{1 - |g_{23}|^2}{|g_{33}|^4} \right) + (H_{y_1y_1} + H_{y_2y_2}) \left( \frac{1 - |g_{13}|^2}{|g_{33}|^4} \right) \\
&\quad + 2 (H_{x_1x_1} + H_{x_2x_2}) \text{Re}(g_{13}g_{23}) \left| \frac{g_{13}}{g_{33}} \right|^2 + 2 (H_{x_1y_1} + H_{x_2y_2}) \text{Im}(g_{13}g_{23}) \left| \frac{g_{13}}{g_{33}} \right|^2. \\
\end{aligned}\]

Now, using (3) the proposition follows.

Recall that the open dense subset $\mathcal{A}$ of $G$ was defined by the condition $\det(A(g)) \neq 0$ or equivalently by $g_{33} \neq 0$.

**Lemma 13.1.** For any $g \in \mathcal{A}$ let $B = \begin{pmatrix} 0 & g_{13} \\ 0 & g_{23} \end{pmatrix}$ and $C = \begin{pmatrix} g_{13} & 0 \\ g_{23} & 0 \end{pmatrix}$. Then

i) $(X_- \Phi_\pi)(g) = \pi(BA(g)^{-1}) \pi(A(g))$

ii) $(X_{-\gamma} \Phi_\pi)(g) = \pi(CA(g)^{-1}) \pi(A(g))$

**Proof.** Upon considering the unique holomorphic representation of $\text{GL}(2, \mathbb{C})$

which extends $\pi$ we may write

\[\Phi_\pi(g \exp tX_-) = \pi(A(g \exp tX_-)A(g)^{-1}) \pi(A(g))\]

for $|t|$ small. Moreover we have

\[\frac{d}{dt} \bigg|_{t=0} (X_- \Phi_\pi)(g) = \Phi_\pi(g \exp tX_-).\]
Therefore
\[(X_{-\beta} \Phi_\pi)(g) = \hat{\pi} \left( \left( \frac{d}{dt} A(g \exp tX_{-\beta}) \right)_{t=0} A(g)^{-1} \right) \pi(A(g)).\]

Since
\[\frac{d}{dt} \bigg|_{t=0} A(g \exp tX_{-\beta}) = B,\]
i) follows. In a similar way one establishes ii).

**Proposition 4.4.** For \(H \in C^\infty(\mathbb{C}^2) \otimes \text{End}(V_\pi)\) we have
\[D_2(H)(x, y) = -4 \frac{\partial H}{\partial x} \hat{\pi} \begin{pmatrix} x(1 + |x|^2) & x^2 y \\ y(1 + |x|^2) & x|y|^2 \end{pmatrix} - 4 \frac{\partial H}{\partial y} \hat{\pi} \begin{pmatrix} y|x|^2 & x(1 + |y|^2) \\ y^2 x & y(1 + |y|^2) \end{pmatrix}.\]

**Proof.** We have
\[D_2(H) = -4 \left( X_\beta(H) X_{-\beta}(\Phi_\pi) \Phi^{-1}_\pi + X_\gamma(H) X_{-\gamma}(\Phi_\pi) \Phi^{-1}_\pi \right).\]
By Lemma 2.3, for any \(g \in \mathcal{A}\) we have
\[(X_{-\beta} \Phi_\pi)(g) \Phi_\pi(g)^{-1} = \frac{1}{g_{33}} \hat{\pi} \begin{pmatrix} -g_{13}g_{21} & g_{13}g_{11} \\ -g_{23}g_{21} & g_{23}g_{11} \end{pmatrix}
\]
and
\[(X_{-\gamma} \Phi_\pi)(g) \Phi_\pi(g)^{-1} = \frac{1}{g_{33}} \hat{\pi} \begin{pmatrix} g_{13}g_{22} & -g_{13}g_{12} \\ g_{23}g_{22} & -g_{23}g_{12} \end{pmatrix}.\]
Also, by Lemma 2.3, we have
\[D_2(H)(g) = -4 \left( \frac{\bar{g}_{21}}{g_{33}} \frac{\partial H}{\partial x} + \frac{\bar{g}_{11}}{g_{33}} \frac{\partial H}{\partial y} \right), 
\[D_2(H)(g) = -4 \left( \frac{\bar{g}_{21}}{g_{33}} \frac{\partial H}{\partial x} - \frac{\bar{g}_{11}}{g_{33}} \frac{\partial H}{\partial y} \right) \frac{1}{g_{33}} \hat{\pi} \begin{pmatrix} g_{13}g_{21} & -g_{13}g_{11} \\ g_{23}g_{21} & -g_{23}g_{11} \end{pmatrix} \]
\[+ 4 \left( \frac{\bar{g}_{22}}{g_{33}} \frac{\partial H}{\partial x} - \frac{\bar{g}_{12}}{g_{33}} \frac{\partial H}{\partial y} \right) \frac{1}{g_{33}} \hat{\pi} \begin{pmatrix} -g_{13}g_{22} & g_{13}g_{12} \\ -g_{23}g_{22} & g_{23}g_{12} \end{pmatrix} \]
\[= -4 \left( \frac{\partial H}{\partial x} \hat{\pi}(P) - \frac{\partial H}{\partial y} \hat{\pi}(Q) \right),\]
where
\[P = \frac{\bar{g}_{21}}{g_{33}|g_{33}|^2} \begin{pmatrix} g_{13}g_{21} & -g_{13}g_{11} \\ g_{23}g_{21} & -g_{23}g_{11} \end{pmatrix} + \frac{\bar{g}_{22}}{g_{33}|g_{33}|^2} \begin{pmatrix} g_{13}g_{22} & -g_{13}g_{12} \\ g_{23}g_{22} & -g_{23}g_{12} \end{pmatrix} \]
\[= \frac{1}{g_{33}|g_{33}|^2} \begin{pmatrix} (|g_{21}|^2 + |g_{22}|^2)g_{13} + (g_{11}\bar{g}_{21} + g_{12}\bar{g}_{22})g_{13} \\ (|g_{21}|^2 + |g_{22}|^2)g_{23} - (g_{11}\bar{g}_{21} + g_{12}\bar{g}_{22})g_{23} \end{pmatrix}.\]
In (29) and (30) we have calculated

\[
\frac{1}{g_{33}|g_{33}|^2} \begin{pmatrix} (1 - |g_{23}|^2)g_{13} & g_{23}^2 \bar{g}_{23} \\ (1 - |g_{23}|^2)g_{23} & g_{13}|g_{23}|^2 \end{pmatrix} = \begin{pmatrix} x(1 + |x|^2) & x^2y \\ y(1 + |x|^2) & x|y|^2 \end{pmatrix}.
\]

In the last two steps we have used that \( g \in SU(3) \) and that \( x = g_{13}/g_{33}, y = g_{23}/g_{33} \). If we proceed in the same way with \( Q \) we complete the proof of the proposition. \( \square \)

**Proposition 4.5.** For \( H \in C^\infty(\mathbb{C}^2) \otimes \text{End}(V_\pi) \) we have

\[
E_1(H)(x, y) = (1 + |x|^2 + |y|^2) \begin{pmatrix} (H_{x_1x_1} + H_{x_2x_2}) & \frac{1}{2}(1 + |x|^2) \\ \frac{1}{2}(1 + |y|^2) \\ 0 \end{pmatrix} \begin{pmatrix} -(1 + |x|^2) & -3ixy \\ 0 & 2(1 + |x|^2) \end{pmatrix} \begin{pmatrix} 2(1 + |y|^2) \\ -3(1 + |y|^2) \end{pmatrix}.
\]

Proof. We have

(31)

\[
E_1(H) = -4 (X_{-\beta}X_{\beta}(H)) \Phi_\pi \hat{\pi}(\tilde{H}_1)\Phi^{-1}_\pi - 4 (X_{-\gamma}X_{\gamma}(H)) \Phi_\pi \hat{\pi}(\tilde{H}_2)\Phi^{-1}_\pi + 12 (X_{-\lambda}X_{\lambda}(H)) \Phi_\pi \hat{\pi}(X_{-\lambda})\Phi^{-1}_\pi + 12 (X_{-\lambda}X_{\lambda}(H)) \Phi_\pi \hat{\pi}(X_{\lambda})\Phi^{-1}_\pi.
\]

In (29) and (30) we have calculated \(-4 (X_{-\beta}X_{\beta}(H)) = (Y_5^2 + Y_6^2)(H) \) and \(-4 (X_{-\gamma}X_{\gamma}(H)) = (Y_3^2 + Y_4^2)(H) \). Now we shall compute

(32) \(-4X_{-\beta}X_{\beta}(H) = (Y_5Y_3 + Y_6Y_4)(H) + i (Y_6Y_3 - Y_5Y_4)(H) \).

For \( U, V \in \mathfrak{g} \) and \( g \in A \) we have

\[
UV(H)(g) = \left( \frac{d}{ds} \frac{d}{dt} H \left( p(g \exp sU \exp tV) \right) \right)_{s=t=0}.
\]

If \(|s|, |t|\) are small we can write

\[
p(g \exp sU \exp tV) = (x(s, t), y(s, t), 1).
\]
Also if we put \( x_1 = \text{Re}(x), x_2 = \text{Im}(x), y_1 = \text{Re}(y) \) and \( y_2 = \text{Im}(y) \), the chain rule gives

\[
UV(H)(g) = \sum_{i,j=1}^{2} H_{x_ix_j} \frac{\partial x_i}{\partial s} \frac{\partial x_j}{\partial t} + \sum_{i,j=1}^{2} H_{y_iy_j} \frac{\partial y_i}{\partial s} \frac{\partial y_j}{\partial t} + \sum_{i=1}^{2} H_{x_i} \frac{\partial^2 x_i}{\partial t \partial s} + H_{y_i} \frac{\partial^2 y_i}{\partial t \partial s}.
\]

For \( U = Y_5 \) and \( V = Y_3 \) we have

\[
x(s, t) = \frac{g_{11} \tan t + g_{12} \sin s + g_{13} \cos s}{g_{31} \tan t + g_{32} \sin s + g_{33} \cos s}
\]

and

\[
y(s, t) = \frac{g_{21} \tan t + g_{22} \sin s + g_{23} \cos s}{g_{31} \tan t + g_{32} \sin s + g_{33} \cos s}.
\]

Moreover at \( s = t = 0 \) we have

\[
\frac{\partial x}{\partial s} = - \frac{\overline{g}_{32}}{g_{33}}, \quad \frac{\partial x}{\partial t} = \frac{\overline{g}_{32}}{g_{33}}, \quad \frac{\partial y}{\partial s} = \frac{\overline{g}_{31}}{g_{33}}, \quad \frac{\partial y}{\partial t} = - \frac{\overline{g}_{12}}{g_{33}}.
\]

For \( U = Y_6 \) and \( V = Y_4 \) we have

\[
x(s, t) = \frac{i g_{11} \tan t + i g_{12} \sin s + g_{13} \cos s}{i g_{31} \tan t + i g_{32} \sin s + g_{33} \cos s}
\]

and

\[
y(s, t) = \frac{i g_{21} \tan t + i g_{22} \sin s + g_{23} \cos s}{i g_{31} \tan t + i g_{32} \sin s + g_{33} \cos s}.
\]

Moreover at \( s = t = 0 \) we have

\[
\frac{\partial x}{\partial s} = - \frac{i \overline{g}_{32}}{g_{33}}, \quad \frac{\partial x}{\partial t} = \frac{i \overline{g}_{32}}{g_{33}}, \quad \frac{\partial y}{\partial s} = \frac{i \overline{g}_{31}}{g_{33}}, \quad \frac{\partial y}{\partial t} = - \frac{i \overline{g}_{12}}{g_{33}}.
\]

For \( U = Y_6 \) and \( V = Y_3 \) we have

\[
x(s, t) = \frac{g_{11} \tan t + i g_{12} \sin s + g_{13} \cos s}{g_{31} \tan t + i g_{32} \sin s + g_{33} \cos s}
\]

and

\[
y(s, t) = \frac{g_{21} \tan t + i g_{22} \sin s + g_{23} \cos s}{g_{31} \tan t + i g_{32} \sin s + g_{33} \cos s}.
\]

Moreover at \( s = t = 0 \) we have

\[
\frac{\partial x}{\partial s} = - \frac{i \overline{g}_{32}}{g_{33}}, \quad \frac{\partial x}{\partial t} = \frac{\overline{g}_{32}}{g_{33}}, \quad \frac{\partial y}{\partial s} = \frac{\overline{g}_{31}}{g_{33}}, \quad \frac{\partial y}{\partial t} = - \frac{\overline{g}_{12}}{g_{33}}.
\]

For \( U = Y_5 \) and \( V = Y_4 \) we have

\[
x(s, t) = \frac{i g_{11} \tan t + g_{12} \sin s + g_{13} \cos s}{i g_{31} \tan t + g_{32} \sin s + g_{33} \cos s}
\]

and

\[
y(s, t) = \frac{i g_{21} \tan t + g_{22} \sin s + g_{23} \cos s}{i g_{31} \tan t + g_{32} \sin s + g_{33} \cos s}.
\]
Moreover at \( s = t = 0 \) we have
\[
\frac{\partial x}{\partial s} = \frac{g_{21}}{g_{33}^2}, \quad \frac{\partial x}{\partial t} = \frac{i g_{22}}{g_{33}^2}, \quad \frac{\partial y}{\partial s} = \frac{g_{11}}{g_{33}^2}, \quad \frac{\partial y}{\partial t} = -\frac{i g_{12}}{g_{33}^2}.
\]

Now if we add the four terms in the right hand side of (32) and observe that taking the real or imaginary part of a complex valued function commutes with taking a derivative, we obtain
\[
-4X_{-\gamma}X_\beta(H)(g) = -\frac{g_{21}g_{22}}{|g_{33}|^4} (H_{x_1x_1} + H_{x_2x_2}) - \frac{g_{11}g_{12}}{|g_{33}|^4} (H_{y_1y_1} + H_{y_2y_2})
\]
\[
+ \frac{(g_{11}g_{22} + g_{21}g_{12})}{|g_{33}|^4} (H_{x_1y_1} + H_{x_2y_2}) - \frac{i (g_{11}g_{22} - g_{21}g_{12})}{|g_{33}|^4} (H_{x_1y_1} - H_{x_2y_2}).
\]

In order to compute \( X_{-\gamma}X_\beta \) we observe that
\[
-4X_{-\gamma}X_\beta(H) = (Y_2Y_5 + Y_4Y_6)(H) + i (Y_4Y_5 - Y_2Y_6)(H)
\]
\[
= (Y_5Y_3 + Y_6Y_4)(H) - i (Y_6Y_3 - Y_5Y_4)(H),
\]
since if \( X \in \mathfrak{f} \) then \( X(H) = 0 \). Therefore it is easy to verify that
\[
-4X_{-\gamma}X_\beta(H)(g) = -\frac{g_{22}g_{21}}{|g_{33}|^4} (H_{x_1x_1} + H_{x_2x_2}) - \frac{g_{12}g_{11}}{|g_{33}|^4} (H_{y_1y_1} + H_{y_2y_2})
\]
\[
+ \frac{(g_{22}g_{11} + g_{12}g_{21})}{|g_{33}|^4} (H_{x_1y_1} + H_{x_2y_2}) - \frac{i (g_{22}g_{11} - g_{12}g_{21})}{|g_{33}|^4} (H_{x_1y_1} - H_{x_2y_2}).
\]

On the other hand we have
\[
\Phi_\pi(g)\hat{\pi} (\tilde{H}_1) \Phi_\pi(g)^{-1} = \frac{1}{g_{33}} \hat{\pi} \left( A(g) \tilde{H}_1 A(g)^{-1} \right)
\]
\[
= \frac{1}{g_{33}} \hat{\pi} \left( \begin{array}{cc}
2g_{11}g_{22} + g_{12}g_{21} & -3g_{11}g_{12} \\
g_{21}g_{22} & -g_{11}g_{22} - 2g_{12}g_{21}
\end{array} \right),
\]
\[
\Phi_\pi(g)\hat{\pi} (\tilde{H}_2) \Phi_\pi(g)^{-1} = \frac{1}{g_{33}} \hat{\pi} \left( \begin{array}{cc}
-g_{11}g_{22} - 2g_{12}g_{21} & 3g_{11}g_{12} \\
-3g_{21}g_{22} & 2g_{11}g_{22} + g_{12}g_{21}
\end{array} \right),
\]
\[
\Phi_\pi(g)\hat{\pi} (X_{-\alpha}) \Phi_\pi(g)^{-1} = \frac{1}{g_{33}} \hat{\pi} \left( \begin{array}{cc}
g_{12}g_{22} & -g_{12}^2 \\
g_{21}^2 & -g_{12}g_{22}
\end{array} \right),
\]
\[
\Phi_\pi(g)\hat{\pi} (X_{\alpha}) \Phi_\pi(g)^{-1} = \frac{1}{g_{33}} \hat{\pi} \left( \begin{array}{cc}
-g_{11}g_{22} & g_{11}^2 \\
g_{21}^2 & -g_{11}g_{22}
\end{array} \right).
\]
Now, by \((31)\), the function multiplying \(H_{x_1x_1} + H_{x_2x_2}\) is a matrix \(\hat{\pi}(A)\) where \(A\) is given by:

\[
A = \begin{pmatrix}
|g_{21}|^2 & |g_{22}|^2 \\
|g_{33}|^4 & |g_{33}|^4
\end{pmatrix}
\begin{pmatrix}
2g_{11}g_{22} + g_{12}g_{21} & -3g_{11}g_{12} \\
g_{21}g_{22} & -g_{11}g_{22} - 2g_{12}g_{12}
\end{pmatrix}
+ \begin{pmatrix}
|g_{22}|^2 \\
|g_{33}|^4
\end{pmatrix}
\begin{pmatrix}
2g_{11}g_{22} & 3g_{11}g_{12} \\
-3g_{21}g_{22} & 2g_{11}g_{22} + 2g_{12}g_{21}
\end{pmatrix}
+ \begin{pmatrix}
3g_{21}g_{22} \left( g_{12}g_{22} - g_{11}g_{12} \right) \\
|g_{33}|^4
\end{pmatrix}
\begin{pmatrix}
g_{21}g_{22} & -g_{11}g_{22} \left( g_{21}g_{22} - g_{11}g_{12} \right) \\
2g_{21}g_{22} & -g_{21}g_{22} \left( g_{21}g_{22} - g_{11}g_{12} \right)
\end{pmatrix}.
\]

If we recall that \(g \in SU(3)\), \(x = g_{13}/g_{33}\) and \(y = g_{23}/g_{33}\) then the coefficient \(A_{11}\) of \(A\) can be written in the following way,

\[
A_{11} = \frac{1}{|g_{33}|^4} \left( |g_{21}|^2(-g_{11}g_{22} + g_{12}g_{21}) + |g_{22}|^2(-g_{11}g_{22} + g_{12}g_{21}) \right)
= -\left( |g_{21}|^2 + |g_{22}|^2 \right) / |g_{33}|^4
= -(1 + |x|^2)(1 + |x|^2 + |y|^2).
\]

Similarly,

\[
A_{22} = \frac{1}{|g_{33}|^4} \left( |g_{21}|^2 + |g_{22}|^2 \right) (-2g_{21}g_{12} + 2g_{11}g_{22})
= 2(1 + |x|^2)(1 + |x|^2 + |y|^2).
\]

We also have \(A_{21} = 0\) and

\[
A_{12} = \frac{1}{|g_{33}|^4} \left( 3g_{11}g_{21}(-g_{11}g_{12} + g_{11}g_{22}) + 3g_{12}g_{22}(g_{11}g_{22} - g_{12}g_{12}) \right)
= \frac{3}{|g_{33}|^4} \left( g_{11}g_{21} + g_{12}g_{22} \right)
= -3x\overline{y}(1 + |x|^2 + |y|^2).
\]

Analogously we calculate the functions multiplying \(H_{y_1y_1} + H_{y_2y_2}\), \(H_{y_1x_1} + H_{y_2x_2}\), \(H_{x_1y_1} - H_{x_1y_2}\) and complete the proof of the proposition. \(\square\)

**Proposition 4.6.** For \(H \in C^\infty(\mathbb{C}^2) \otimes \text{End}(V_\pi)\) we have

\[
E_2(H)(x, y) = 4 \frac{\partial H}{\partial x} \left( \hat{\pi} \left( \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \right) \hat{\pi} \left( \begin{pmatrix} -2x\overline{y} & 0 \\ 3(1 + |x|^2) & x\overline{y} \end{pmatrix} + \hat{\pi} \left( \begin{pmatrix} 1 + |x|^2 & 3x\overline{y} \\ 0 & -2(1 + |x|^2) \end{pmatrix} \right) \right) \right)
+ 4 \frac{\partial H}{\partial y} \left( \hat{\pi} \left( \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \right) \hat{\pi} \left( \begin{pmatrix} -2(1 + |y|^2) & 0 \\ 3y\overline{x} & 1 + |y|^2 \end{pmatrix} + \hat{\pi} \left( \begin{pmatrix} y\overline{x} & 3(1 + |y|^2) \\ 0 & -2y\overline{x} \end{pmatrix} \right) \right) \right).
\]

**Proof.** We have

\[
E_2(H) = -4X_\beta(H)X_{-\beta}(\Phi_\pi) \hat{\pi}(\tilde{H}_1)\Phi_\pi^{-1} - 4X_\gamma(H)X_{-\gamma}(\Phi_\pi) \hat{\pi}(\tilde{H}_2)\Phi_\pi^{-1}
\]
\[ + 12X_\gamma(H)X_{-\beta}(\Phi_\pi)\dot{\pi}(X_\alpha)\Phi^{-1}_\pi + 12X_\gamma(H)X_{-\gamma}(\Phi_\pi)\dot{\pi}(X_\alpha)\Phi^{-1}_\pi. \]

For any \( g \in A \) we have, by Lemma \[2.3\]
\[ (X_\beta H)(g) = -\frac{\overline{g}_{21}}{g_{33}} \frac{\partial H}{\partial x} + \frac{\overline{g}_{11}}{g_{33}} \frac{\partial H}{\partial y}, \quad (X_\gamma H)(g) = -\frac{\overline{g}_{22}}{g_{33}} \frac{\partial H}{\partial x} - \frac{\overline{g}_{12}}{g_{33}} \frac{\partial H}{\partial y}, \]
and by Lemma \[2.1\]
\[ (X_{-\beta}\Phi_\pi)(g)\Phi_\pi(g)^{-1} = \frac{1}{g_{33}} \dot{\pi} \begin{pmatrix} -g_{13}g_{21} & g_{13}g_{11} \\ -g_{23}g_{21} & g_{23}g_{11} \end{pmatrix}, \]
\[ (X_{-\gamma}\Phi_\pi)(g)\Phi_\pi(g)^{-1} = \frac{1}{g_{33}} \dot{\pi} \begin{pmatrix} g_{13}g_{22} & -g_{13}g_{12} \\ g_{23}g_{22} & -g_{23}g_{12} \end{pmatrix}. \]

In \[33\] we computed \( \Phi_\pi(g)\dot{\pi}(A)\Phi_\pi(g)^{-1} \), for \( A = \dot{H}_1, \dot{H}_2, X_{-\alpha} \) and \( X_\alpha \). Therefore, \( E_2(H) = 4(\frac{\partial H}{\partial x} P + \frac{\partial H}{\partial y} Q) \) with
\[
P = -\frac{\overline{g}_{21}}{|g_{33}|^2} \dot{\pi} \begin{pmatrix} g_{13}g_{21} & -g_{13}g_{11} \\ g_{23}g_{21} & -g_{23}g_{11} \end{pmatrix} \dot{\pi} \begin{pmatrix} 2g_{11}g_{22} + g_{12}g_{21} & -3g_{11}g_{12} \\ 3g_{12}g_{11} & -g_{11}g_{22} - 2g_{12}g_{21} \end{pmatrix}
+ \frac{\overline{g}_{22}}{|g_{33}|^2} \dot{\pi} \begin{pmatrix} -g_{13}g_{22} & g_{13}g_{12} \\ -g_{23}g_{22} & g_{23}g_{12} \end{pmatrix} \dot{\pi} \begin{pmatrix} -g_{11}g_{22} & 2g_{12}g_{11} \\ -3g_{12}g_{11} & 2g_{11}g_{22} + g_{12}g_{21} \end{pmatrix}
- \frac{3\overline{g}_{22}}{|g_{33}|^2} \dot{\pi} \begin{pmatrix} g_{13}g_{21} & -g_{13}g_{11} \\ g_{23}g_{21} & -g_{23}g_{11} \end{pmatrix} \dot{\pi} \begin{pmatrix} g_{12}g_{11} & -g_{11}g_{12} \\ g_{21}g_{11} & g_{11}^2 \end{pmatrix}
+ \frac{3\overline{g}_{21}}{|g_{33}|^2} \dot{\pi} \begin{pmatrix} -g_{13}g_{22} & g_{13}g_{12} \\ -g_{23}g_{22} & g_{23}g_{12} \end{pmatrix} \dot{\pi} \begin{pmatrix} -g_{11}g_{22} & g_{11}^2 \\ -g_{21}g_{11} & -g_{11}g_{12} \end{pmatrix}.
\]

If we combine the first and third terms and the second and fourth terms, and use Lemma \[2.7\] several times we obtain
\[
P = \frac{1}{|g_{33}|^2} \dot{\pi} \begin{pmatrix} g_{13}g_{21} & -g_{13}g_{11} \\ g_{23}g_{21} & -g_{23}g_{11} \end{pmatrix}
\times \dot{\pi} \begin{pmatrix} 2g_{22}g_{13}g_{23} - g_{12}(1 - |g_{23}|^2) & -3g_{12}g_{13}g_{23} \\ -3g_{22}(1 - |g_{23}|^2) & 2g_{12}(1 - |g_{23}|^2) - g_{22}g_{13}g_{23} \end{pmatrix}
+ \frac{1}{|g_{33}|^2} \dot{\pi} \begin{pmatrix} -g_{13}g_{22} & g_{13}g_{12} \\ -g_{23}g_{22} & g_{23}g_{12} \end{pmatrix}
\times \dot{\pi} \begin{pmatrix} g_{12}g_{13}g_{23} - g_{11}(1 - |g_{23}|^2) & -3g_{11}g_{13}g_{23} \\ -3g_{21}(1 - |g_{23}|^2) & 2g_{11}(1 - |g_{23}|^2) - g_{21}g_{13}g_{23} \end{pmatrix}.
\]

Hence using \[\mathcal{H}\] we get
\[
P = \frac{1}{|g_{33}|^2} \dot{\pi} \begin{pmatrix} g_{13}g_{21} & -g_{13}g_{11} \\ g_{23}g_{21} & -g_{23}g_{11} \end{pmatrix}
\times \left( \dot{\pi} \begin{pmatrix} 2g_{22}g_{13}g_{23} & 0 \\ -3g_{22}(1 + |x|^2) & -g_{22}xg_{23} \end{pmatrix} + \dot{\pi} \begin{pmatrix} -g_{12}(1 + |x|^2) & -3g_{12}xg_{23} \\ 0 & 2g_{12}(1 + |x|^2) \end{pmatrix} \right)
+ \frac{1}{|g_{33}|^2} \dot{\pi} \begin{pmatrix} -g_{13}g_{22} & g_{13}g_{12} \\ -g_{23}g_{22} & g_{23}g_{12} \end{pmatrix}.
Thus,
\[ P = \frac{g_{22}}{|g_{33}|^2} \hat{\pi} \left( \begin{array}{cc} g_{13} - g_{11} & 0 \\ g_{23} - g_{21} & g_{23} g_{33} \end{array} \right) \hat{\pi} \left( \begin{array}{cc} 2x \bar{y} & 0 \\ -3(1 + |y|^2) - x \bar{y} & 0 \end{array} \right) \]
\[ + \frac{g_{21}}{|g_{33}|^2} \hat{\pi} \left( \begin{array}{cc} -g_{13} & g_{13} \bar{y} \\ -g_{23} & g_{23} \bar{y} \end{array} \right) \hat{\pi} \left( \begin{array}{cc} 2x \bar{y} & 0 \\ -3(1 + |y|^2) - x \bar{y} & 0 \end{array} \right) \]
\[ + \frac{g_{12}}{|g_{33}|^2} \hat{\pi} \left( \begin{array}{cc} g_{13} & -g_{13} \bar{y} \\ g_{23} & -g_{23} \bar{y} \end{array} \right) \hat{\pi} \left( \begin{array}{cc} 2x \bar{y} & 0 \\ -3(1 + |y|^2) - x \bar{y} & 0 \end{array} \right) \]
\[ + \frac{g_{11}}{|g_{33}|^2} \hat{\pi} \left( \begin{array}{cc} -g_{13} & g_{13} \bar{y} \\ -g_{23} & g_{23} \bar{y} \end{array} \right) \hat{\pi} \left( \begin{array}{cc} 2x \bar{y} & 0 \\ -3(1 + |y|^2) - x \bar{y} & 0 \end{array} \right) \].

Now, using Lemma 2.3 we obtain
\[ P = \frac{1}{|g_{33}|^2} \hat{\pi} \left( \begin{array}{cc} 0 & -g_{13} \bar{y} \\ 0 & -g_{23} \bar{y} \end{array} \right) \hat{\pi} \left( \begin{array}{cc} 2x \bar{y} & 0 \\ -3(1 + |y|^2) - x \bar{y} & 0 \end{array} \right) \]
\[ + \frac{1}{|g_{33}|^2} \hat{\pi} \left( \begin{array}{cc} -g_{13} \bar{y} & 0 \\ -g_{23} \bar{y} & 0 \end{array} \right) \hat{\pi} \left( \begin{array}{cc} 2x \bar{y} & 0 \\ -3(1 + |y|^2) - x \bar{y} & 0 \end{array} \right) \]
\[ = \hat{\pi} \left( \begin{array}{cc} 0 & x \\ 0 & y \end{array} \right) \hat{\pi} \left( \begin{array}{cc} -2x \bar{y} & 0 \\ 3(1 + |y|^2) - x \bar{y} & 0 \end{array} \right) + \hat{\pi} \left( \begin{array}{cc} x & 0 \\ y & 0 \end{array} \right) \hat{\pi} \left( \begin{array}{cc} 1 + |y|^2 & 3x \bar{y} \\ 0 & -2(1 + |y|^2) \end{array} \right) \].

If we proceed in the same way with \( Q \) we complete the proof of the proposition. \( \Box \)

Now we shall compute all first and second order partial derivatives that we used in Section \( \mathbf{3} \). We start with the following immediate results.

**Lemma 5.3.** At \((r,0) \in \mathbb{C}^2 \) we have
\[ H_{x_1}(r,0) = \frac{dH}{dr}(r) \quad \text{and} \quad H_{x_1 x_1}(r,0) = \frac{d^2H}{dr^2}(r). \]

As a preparation to other computations, given \((x,y) \in \mathbb{C}^2, x = x_1 + ix_2 \) and \( y = y_1 + iy_2 \) we need to choose an element in \( K \) which carries the point \((x,y)\) to the meridian \( \{ (r,0) : r > 0 \} \). We take
\[ A(x,y) = \frac{1}{s(x,y)} \begin{pmatrix} x - iy_2 \\ y \end{pmatrix} \in SU(2) \]where \( s(x,y) = \sqrt{|x|^2 + |y|^2}. \)

Then \((x,y) = A(x,y)(s(x,y),0) \) and
\[ H(x,y) = \pi(A(x,y))\bar{H}(s(x,y))\pi(A(x,y)^{-1}). \]

For simplicity we shall use the notation \((u_1,u_2,u_3,u_4) = (x_1,x_2,y_1,y_2).\)

As usual we denote \([A,B] = AB - BA.\)

**Proposition 13.2.** At \((r,0) \in \mathbb{C}^2 \) we have
\[ \frac{\partial H}{\partial u_j}(r,0) = \left[ \hat{\pi} \left( \begin{array}{c} \partial A \\ \partial u_j \end{array} \right) \bar{H}(r) \right] + \delta_{ij} \frac{d\bar{H}}{dr}(r). \]
and
\[
\frac{\partial^2 H}{\partial u_i \partial u_j}(r,0) = \frac{\partial^2 (\pi \circ A)}{\partial u_i \partial u_j} \tilde{H}(r) + \frac{\partial^2 (\pi \circ A^{-1})}{\partial u_i \partial u_j} + \delta_{ij} \frac{d^2 \tilde{H}}{dr^2} + \frac{1}{r} \delta_{ij}(1 - \delta_{ij}) \frac{d \tilde{H}}{dr} + \delta_{ij} \left[ \dot{\pi} \left( \frac{\partial A}{\partial u_i} \right), \frac{d \tilde{H}}{dr} \right] + \delta_{ij} \left[ \ddot{\pi} \left( \frac{\partial A}{\partial u_i} \right), \frac{d \tilde{H}}{dr} \right] - \ddot{\pi} \left( \frac{\partial A}{\partial u_i} \right) \tilde{H}(r) \dot{\pi} \left( \frac{\partial A}{\partial u_j} \right) - \dot{\pi} \left( \frac{\partial A}{\partial u_i} \right) \tilde{H}(r) \dot{\pi} \left( \frac{\partial A}{\partial u_j} \right).
\]

Proof. We have
\[
\tilde{H}(x,y) = \pi(A(x,y)) \tilde{H}(s(x,y)) \pi(A(x,y)^{-1}),
\]
then
\[
\frac{\partial \tilde{H}}{\partial u_j} = \frac{\partial (\pi \circ A)}{\partial u_j} (\tilde{H} \circ s)(\pi \circ A^{-1}) + (\pi \circ A) \frac{\partial (\tilde{H} \circ s)}{\partial u_j}(\pi \circ A^{-1}) + \frac{\partial (\pi \circ A^{-1})}{\partial u_j}.
\]

Moreover
\[
\frac{\partial (\tilde{H} \circ s)}{\partial u_j} = \left( \frac{d \tilde{H}}{dr} \circ s \right) \frac{\partial s}{\partial u_j}.
\]

We also note that \(A(r,0) = I\) and that at \((r,0)\) we have
\[
\frac{\partial s}{\partial u_j} = \delta_{1j}, \quad \frac{\partial (\pi \circ A)}{\partial u_j} = \dot{\pi} \left( \frac{\partial A}{\partial u_j} \right), \quad \frac{\partial (\pi \circ A^{-1})}{\partial u_j} = -\pi \left( \frac{\partial A}{\partial u_j} \right).
\]

Now replacing in (34) the first statement follows.

To compute the second order derivative we start observing that at \((r,0)\) we have
\[
\frac{\partial^2 (\tilde{H} \circ s)}{\partial u_i \partial u_j} = \frac{d^2 \tilde{H}}{dr^2} \frac{\partial s}{\partial u_i} \frac{\partial s}{\partial u_j} + \frac{d \tilde{H}}{dr} \frac{\partial^2 s}{\partial u_i \partial u_j} + \delta_{ij} \frac{d^2 \tilde{H}}{dr^2} + \frac{1}{r} \delta_{ij} (1 - \delta_{ij}) \frac{d \tilde{H}}{dr}.
\]

Now we differentiate the expression given in (34) with respect to \(u_i\) and evaluate it at \((r,0) \in \mathbb{C}^2\).

\[
\frac{\partial^2 H}{\partial u_i \partial u_j}(r,0) = \frac{\partial^2 (\pi \circ A)}{\partial u_i \partial u_j} \tilde{H}(r) + \frac{\partial^2 (\pi \circ A^{-1})}{\partial u_i \partial u_j} + \delta_{ij} \frac{d^2 \tilde{H}}{dr^2} + \frac{1}{r} \delta_{ij} (1 - \delta_{ij}) \frac{d \tilde{H}}{dr} + \delta_{ij} \left[ \dot{\pi} \left( \frac{\partial A}{\partial u_i} \right), \frac{d \tilde{H}}{dr} \right] + \delta_{ij} \left[ \ddot{\pi} \left( \frac{\partial A}{\partial u_i} \right), \frac{d \tilde{H}}{dr} \right] - \ddot{\pi} \left( \frac{\partial A}{\partial u_i} \right) \tilde{H}(r) \dot{\pi} \left( \frac{\partial A}{\partial u_j} \right) - \dot{\pi} \left( \frac{\partial A}{\partial u_i} \right) \tilde{H}(r) \dot{\pi} \left( \frac{\partial A}{\partial u_j} \right) - \delta_{ij} \frac{\partial s}{\partial u_j} \pi \left( \frac{\partial A}{\partial u_i} \right) + \tilde{H}(r) \frac{\partial^2 (\pi \circ A^{-1})}{\partial u_i \partial u_j}.
\]

Now rearranging the right hand side of the above expression the proof is finished. \(\square\)
Proposition 13.3. At \((r, 0) \in \mathbb{C}^2\) we have

\[
\frac{\partial^2 (\pi \circ A)}{\partial u_i \partial u_j} = \hat{\pi} \left( \frac{\partial^2 A}{\partial u_i \partial u_j} \right) - \frac{1}{2} \hat{\pi} \left( \frac{\partial A}{\partial u_i} \frac{\partial A}{\partial u_j} \right) - \frac{1}{2} \hat{\pi} \left( \frac{\partial A}{\partial u_j} \frac{\partial A}{\partial u_i} \right) + \frac{1}{2} \hat{\pi} \left( \frac{\partial A}{\partial u_i} \right) \hat{\pi} \left( \frac{\partial A}{\partial u_j} \right)
\]

(35)

and

\[
\frac{\partial^2 (\pi \circ A^{-1})}{\partial u_i \partial u_j} = - \hat{\pi} \left( \frac{\partial^2 A}{\partial u_i \partial u_j} \right) + \frac{1}{2} \hat{\pi} \left( \frac{\partial A}{\partial u_i} \frac{\partial A}{\partial u_j} \right) + \frac{1}{2} \hat{\pi} \left( \frac{\partial A}{\partial u_j} \frac{\partial A}{\partial u_i} \right) + \frac{1}{2} \hat{\pi} \left( \frac{\partial A}{\partial u_i} \right) \hat{\pi} \left( \frac{\partial A}{\partial u_j} \right).
\]

(36)

Proof. For \(|x|\) and \(|y|\) sufficiently small we consider

\[
X(x, y) = \log(A(x, y)) = B(x, y) - \frac{B(x, y)^2}{2} + \frac{B(x, y)^3}{3} - \cdots,
\]

where \(B(x, y) = A(x, y) - I\). Then

\[
\pi(A(x, y)) = \pi(\exp X(x, y)) = \exp \hat{\pi}(X(x, y)) = \sum_{j \geq 0} \frac{\hat{\pi}(X(x, y))^j}{j!}.
\]

Now we differentiate with respect to \(u_j\) to obtain

\[
\frac{\partial (\pi \circ A)}{\partial u_j} = \hat{\pi} \left( \frac{\partial X}{\partial u_j} \right) + \hat{\pi} \left( \frac{\partial X}{\partial u_j} \right) \hat{\pi}(X) + \frac{1}{2} \hat{\pi} \left( \frac{\partial X}{\partial u_j} \right) \hat{\pi}(X) \hat{\pi} \left( \frac{\partial X}{\partial u_j} \right) + \frac{1}{2} \hat{\pi} \left( \frac{\partial X}{\partial u_j} \right) \hat{\pi} \left( \frac{\partial X}{\partial u_j} \right) + \cdots.
\]

(38)

Since \(X(r, 0) = 0\), if we differentiate (38) and evaluate at \((r, 0)\) we obtain

\[
\frac{\partial^2 (\pi \circ A)}{\partial u_i \partial u_j} = \hat{\pi} \left( \frac{\partial^2 X}{\partial u_i \partial u_j} \right) + \frac{1}{2} \hat{\pi} \left( \frac{\partial X}{\partial u_i} \right) \hat{\pi} \left( \frac{\partial X}{\partial u_j} \right) + \frac{1}{2} \hat{\pi} \left( \frac{\partial X}{\partial u_j} \right) \hat{\pi} \left( \frac{\partial X}{\partial u_i} \right).
\]

To compute \(\frac{\partial X}{\partial u_j}\) and \(\frac{\partial^2 X}{\partial u_i \partial u_j}\) we differentiate (36) and we get

\[
\frac{\partial X}{\partial u_j} = \frac{\partial B}{\partial u_j} - \frac{1}{2} \left( \frac{\partial B}{\partial u_j} \right) B - \frac{1}{2} B \left( \frac{\partial B}{\partial u_j} \right) \frac{\partial B}{\partial u_j} + \frac{1}{3} B \left( \frac{\partial B}{\partial u_j} \right) B^2 + \frac{1}{3} B \left( \frac{\partial B}{\partial u_j} \right) B + \frac{1}{3} B^2 \left( \frac{\partial B}{\partial u_j} \right) + \cdots.
\]

Since \(B(r, 0) = 0\) we have

\[
\frac{\partial X}{\partial u_j}(r, 0) = \frac{\partial B}{\partial u_j}(r, 0) = \frac{\partial A}{\partial u_j}(r, 0).
\]
We also have at \((r, 0) \in \mathbb{C}^2\)
\[
\frac{\partial^2 X}{\partial u_i \partial u_j} = \frac{\partial^2 A}{\partial u_i \partial u_j} - \frac{1}{2} \frac{\partial A}{\partial u_j} \frac{\partial A}{\partial u_i} - \frac{1}{2} \frac{\partial A}{\partial u_i} \frac{\partial A}{\partial u_j}.
\]
Replacing in (39) the first statement in the lemma follows.

Now we compute \(\frac{\partial^2 (\pi \circ A^{-1})}{\partial u_i \partial u_j}\). We observe that
\[
\frac{\partial (\pi \circ A^{-1})}{\partial u_j} = - (\pi \circ A^{-1}) \frac{\partial (\pi \circ A)}{\partial u_j} (\pi \circ A^{-1}).
\]
Therefore at \((r, 0) \in \mathbb{C}^2\) we have,
\[
\frac{\partial^2 (\pi \circ A^{-1})}{\partial u_i \partial u_j} = - \frac{\partial (\pi \circ A^{-1})}{\partial u_i} \frac{\partial (\pi \circ A)}{\partial u_j} - \frac{\partial (\pi \circ A)}{\partial u_i} \frac{\partial (\pi \circ A^{-1})}{\partial u_j} - \frac{\partial (\pi \circ A)}{\partial u_i} \frac{\partial (\pi \circ A^{-1})}{\partial u_j}.
\]
Then the statement in (36) follows from (35) and
\[
\frac{\partial (\pi \circ A)}{\partial u_j} = \hat{\pi} \left( \frac{\partial A}{\partial u_j} \right), \quad \frac{\partial (\pi \circ A^{-1})}{\partial u_j} = - \hat{\pi} \left( \frac{\partial A}{\partial u_j} \right).
\]

Now we specialize the above propositions in the different cases we need.

**Lemma 5.4.** At \((r, 0) \in \mathbb{C}^2\) we have
\[
H_{y_1}(r, 0) = - \frac{1}{r} \left( \hat{\pi}(J) \hat{H}(r) - \hat{H}(r) \hat{\pi}(J) \right)
\]
and
\[
H_{y_1 y_1}(r, 0) = \frac{1}{r} \frac{d\hat{H}}{dr} + \frac{1}{r^2} \left( \hat{\pi}(J)^2 \hat{H}(r) + \hat{H}(r) \hat{\pi}(J)^2 \right) - \frac{2}{r^2} \hat{\pi}(J) \hat{H}(r) \hat{\pi}(J),
\]
where \(J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \).

**Proof.** To compute \(\partial A / \partial u_3\) at \((r, y_1) \in \mathbb{C}^2\) we consider
\[
A(x, y) = \frac{1}{\sqrt{r^2 + y_1^2}} \begin{pmatrix} r & -y_1 \\ y_1 & r \end{pmatrix}.
\]
Therefore
\[
\frac{\partial A}{\partial u_3}(r, 0) = - \frac{1}{r} J \quad \text{and} \quad \frac{\partial^2 A}{\partial u_3^2}(r, 0) = - \frac{1}{r^2} I.
\]
By Proposition [13.3] we have at \((r, 0) \in \mathbb{C}^2\)
\[
\frac{\partial^2 (\pi \circ A)}{\partial u_3^2} = \frac{\partial^2 (\pi \circ A^{-1})}{\partial u_3^2} = \frac{1}{r^2} \hat{\pi}(J)^2.
\]
Now from Proposition [13.2] we obtain
\[
\frac{\partial \hat{H}}{\partial y_1}(r, 0) = - \frac{1}{r} \left( \hat{\pi}(J) \hat{H}(r) - \hat{H}(r) \hat{\pi}(J) \right),
\]
\[
\frac{\partial^2 H}{\partial y^2_1}(r,0) = \frac{1}{r^2} \hat{\pi}(J)^2 \tilde{H}(r) + \frac{1}{r^2} \hat{H}(r) \hat{\pi}(J)^2 + \frac{1}{r} \frac{d \tilde{H}}{dr} - \frac{2}{r^2} \hat{\pi}(J) \tilde{H}(r) \hat{\pi}(J).
\]

This completes the proof of the lemma. \qed

**Lemma 5.5.** At \((r,0) \in \mathbb{C}^2\) we have

\[
H_{y_2}(r,0) = \frac{i}{r} \left( \hat{\pi}(T) \tilde{H}(r) - \tilde{H}(r) \hat{\pi}(T) \right)
\]

and

\[
H_{y_2y_2}(r,0) = \frac{1}{r} \frac{d \tilde{H}}{dr} - \frac{1}{r^2} \left( \hat{\pi}(T)^2 \tilde{H}(r) + \tilde{H}(r) \hat{\pi}(T)^2 \right) + \frac{2}{r^2} \hat{\pi}(T) \tilde{H}(r) \hat{\pi}(T),
\]

where \(T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\).

**Proof.** In this case we take

\[
A(x,y) = \frac{1}{\sqrt{r^2 + y^2}} \begin{pmatrix} r & iy_2 \\ iy_2 & r \end{pmatrix}.
\]

Therefore

\[
\frac{\partial A}{\partial u_4}(r,0) = \frac{i}{r} T \quad \text{and} \quad \frac{\partial^2 A}{\partial u_4^2}(r,0) = -\frac{1}{r^2} I.
\]

By Proposition 13.3, we have at \((r,0) \in \mathbb{C}^2\)

\[
\frac{\partial^2 (\pi \circ A)}{\partial u_4^2} = \frac{\partial^2 (\pi \circ A^{-1})}{\partial u_4^2} = -\frac{1}{r^2} \hat{\pi}(T)^2.
\]

Now from Proposition 13.2, we get

\[
\frac{\partial H}{\partial y_2}(r,0) = \frac{i}{r} \left( \hat{\pi}(T) \tilde{H}(r) - \tilde{H}(r) \hat{\pi}(T) \right),
\]

\[
\frac{\partial^2 H}{\partial y_2^2}(r,0) = -\frac{1}{r^2} \hat{\pi}(T)^2 \tilde{H}(r) - \frac{1}{r^2} \tilde{H}(r) \hat{\pi}(T)^2 + \frac{1}{r} \frac{d \tilde{H}}{dr} + \frac{2}{r^2} \hat{\pi}(T) \tilde{H}(r) \hat{\pi}(T).
\]

The proof of the lemma is finished. \qed

**Lemma 5.6.** At \((r,0) \in \mathbb{C}^2\) we have

\[
H_{x_2}(r,0) = \frac{i}{r} \left( \hat{\pi}(H_\alpha) \tilde{H}(r) - \tilde{H}(r) \hat{\pi}(H_\alpha) \right)
\]

and

\[
H_{x_2x_2}(r,0) = \frac{1}{r} \frac{d \tilde{H}}{dr} - \frac{1}{r^2} \left( \hat{\pi}(H_\alpha)^2 \tilde{H}(r) + \tilde{H}(r) \hat{\pi}(H_\alpha)^2 \right) + \frac{2}{r^2} \hat{\pi}(H_\alpha) \tilde{H}(r) \hat{\pi}(H_\alpha).
\]
Proof. In this case we have \( x = r + ix_2, \ y = 0, \) and
\[
A(x, y) = \frac{1}{\sqrt{r^2 + x^2}} \begin{pmatrix} r + ix & 0 \\ 0 & r - ix \end{pmatrix}.
\]
Therefore
\[
\frac{\partial A}{\partial u_2}(r, 0) = \frac{1}{r} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \frac{i}{r} H_\alpha \quad \text{and} \quad \frac{\partial^2 A}{\partial u_2^2}(r, 0) = -\frac{1}{r^2} I.
\]
By Proposition 13.3 we have at \((r, 0) \in \mathbb{C}^2\)
\[
\frac{\partial^2 (\pi \circ A)}{\partial u_2^2} = \frac{\partial^2 (\pi \circ A^{-1})}{\partial u_2^2} = -\frac{1}{r^2} \dot{\pi}(H_\alpha)^2.
\]
Now from Proposition 13.2 we have
\[
\frac{\partial H}{\partial x_2}(r, 0) = \frac{i}{r} \left( \dot{\pi}(H_\alpha) \tilde{H}(r) - \tilde{H}(r) \dot{\pi}(H_\alpha) \right)
\]
\[
\frac{\partial^2 H}{\partial x_2^2}(r, 0) = \frac{1}{r^2} \dot{\pi}(H_\alpha)^2 \tilde{H}(r) + \frac{1}{r^2} \tilde{H}(r) \dot{\pi}(H_\alpha)^2 + \frac{1}{r^2} \frac{d \tilde{H}}{dr}
\]
\[
- \frac{2}{r^2} \dot{\pi}(H_\alpha) \tilde{H}(r) \dot{\pi}(H_\alpha).
\]
This completes the proof of the lemma. \(\square\)

**Lemma 5.7**. At \((r, 0) \in \mathbb{C}^2\) we have
\[
H_{x_1y_1}(r, 0) = \frac{-1}{r} \left( \dot{\pi}(J) \frac{d \tilde{H}}{dr} - \frac{d \tilde{H}}{dr} \dot{\pi}(J) \right) + \frac{1}{r^2} \left( \dot{\pi}(J) \tilde{H}(r) - \tilde{H}(r) \dot{\pi}(J) \right).
\]

\[
H_{x_1y_2}(r, 0) = \frac{i}{r} \left( \dot{\pi}(T) \frac{d \tilde{H}}{dr} - \frac{d \tilde{H}}{dr} \dot{\pi}(T) \right) - \frac{i}{r^2} \left( \dot{\pi}(T) \tilde{H}(r) - \tilde{H}(r) \dot{\pi}(T) \right).
\]

**Proof.** It follows from Lemmas 5.4 and 5.5 by differentiating with respect to \(r\). \(\square\)

**Lemma 5.8**. At \((r, 0) \in \mathbb{C}^2\) we have
\[
H_{y_1x_2}(r, 0) = -\frac{i}{2r^2} \left( \dot{\pi}(H_\alpha) \dot{\pi}(J) + \dot{\pi}(J) \dot{\pi}(H_\alpha) \right) \tilde{H}(r)
\]
\[
- \frac{i}{2r^2} \tilde{H}(r) \left( \dot{\pi}(H_\alpha) \dot{\pi}(J) + \dot{\pi}(J) \dot{\pi}(H_\alpha) \right)
\]
\[
+ \frac{i}{r^2} \left( \dot{\pi}(H_\alpha) \tilde{H}(r) \dot{\pi}(J) + \dot{\pi}(J) \tilde{H}(r) \dot{\pi}(H_\alpha) \right).
\]

**Proof.** In this case we have \( x = r + ix_2, \ y = y_1, \) and
\[
A(x, y) = \frac{1}{\sqrt{r^2 + x^2 + y^2}} \begin{pmatrix} r + ix & -y_1 \\ y_1 & r - ix \end{pmatrix}.
\]
Therefore
\[
\frac{\partial A}{\partial u_2}(r, 0) = \frac{i}{r} H_\alpha, \quad \frac{\partial A}{\partial u_3}(r, 0) = -\frac{1}{r} J, \quad \frac{\partial^2 A}{\partial u_3 \partial u_2}(r, 0) = 0.
\]
By Proposition 13.3 we have at \((r, 0) \in \mathbb{C}^2\)
\[
\frac{\partial^2 (\pi \circ A)}{\partial u_3 \partial u_2} = \frac{\partial^2 (\pi \circ A^{-1})}{\partial u_3 \partial u_2} = -\frac{i}{2r^2} \left( \dot{\pi}(H_\alpha) \dot{\pi}(T) + \dot{\pi}(T) \dot{\pi}(H_\alpha) \right).
\]
Now the lemma follows from Proposition 13.2.

Lemma 5.9. At \((r, 0) \in \mathbb{C}^2\) we have
\[
H_{y_2 x_2}(r, 0) = -\frac{1}{2r^2} \left( \dot{\pi}(H_\alpha) \dot{\pi}(T) + \dot{\pi}(T) \dot{\pi}(H_\alpha) \right) \tilde{H}(r)
- \frac{1}{2r^2} \tilde{H}(r) \left( \dot{\pi}(H_\alpha) \dot{\pi}(T) + \dot{\pi}(T) \dot{\pi}(H_\alpha) \right)
+ \frac{1}{r^2} \left( \dot{\pi}(H_\alpha) \tilde{H}(r) \dot{\pi}(T) + \dot{\pi}(T) \tilde{H}(r) \dot{\pi}(H_\alpha) \right).
\]
Proof. In this case we have \(x = r + ix_2, y = y_2\), and
\[
A(x, y) = \frac{1}{\sqrt{r^2 + x^2 + y^2}} \begin{pmatrix} r + ix & iy_2 \\ y_2 & r - ix \end{pmatrix}.
\]
Therefore
\[
\frac{\partial A}{\partial u_2}(r, 0) = \frac{i}{r} H_\alpha, \quad \frac{\partial A}{\partial u_4}(r, 0) = \frac{i}{r} T, \quad \frac{\partial^2 A}{\partial u_4 \partial u_2}(r, 0) = 0.
\]
By Proposition 13.3 we have at \((r, 0) \in \mathbb{C}^2\)
\[
\frac{\partial^2 (\pi \circ A)}{\partial u_4 \partial u_2} = \frac{\partial^2 (\pi \circ A^{-1})}{\partial u_4 \partial u_2} = -\frac{1}{2r^2} \left( \dot{\pi}(H_\alpha) \dot{\pi}(T) + \dot{\pi}(T) \dot{\pi}(H_\alpha) \right).
\]
Now the lemma follows from Proposition 13.2.

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