Revelation Gap for Pricing from Samples

Yiding Feng  
yidingfeng2021@u.northwestern.edu  
Northwestern University  
Evanston, IL, USA

Jason D. Hartline  
hartline@northwestern.edu  
Northwestern University  
Evanston, IL, USA

Yingkai Li  
yingkai.li@u.northwestern.edu  
Northwestern University  
Evanston, IL, USA

ABSTRACT
This paper considers prior-independent mechanism design, in which a single mechanism is designed to achieve approximately optimal performance on every prior distribution from a given class. Most results in mechanism design focus on mechanisms with truth-telling equilibria, a.k.a., truthful mechanisms. To quantify the loss of the restriction to truthful mechanisms, the revelation gap is introduced. Feng and Hartline [FOCS 2018] introduce the revelation gap to quantify the loss of the restriction to truthful mechanisms. We solve a main open question left in Feng and Hartline [FOCS 2018]: namely, we identify a non-trivial revelation gap for revenue maximization.

Our analysis focuses on the canonical problem of selling a single item to a single agent with only access to a single sample from the agent’s valuation distribution. We identify the sample-bid mechanism (a simple non-truthful mechanism) and upper-bound its prior-independent approximation ratio by 1.835 (resp. 1.296) for regular (resp. MHR) distributions. We further prove that no truthful mechanism can achieve prior-independent approximation ratio better than 1.957 (resp. 1.543) for regular (resp. MHR) distributions. Thus, a non-trivial revelation gap is shown as the sample-bid mechanism outperforms the optimal prior-independent truthful mechanism. On the hardness side, we prove that no (possibly non-truthful) mechanism can achieve prior-independent approximation ratio better than 1.073 even for uniform distributions.

1 INTRODUCTION
One important research direction in modern computer science focuses on multi-party computation. Two fundamental concerns in this area are (i) who should be doing what part of the computation; and (ii) what are their incentives to do it correctly. The second concern has been studied extensively in the economics field of mechanism design. For the first concern, however, the system design field and the mechanism design field have different high-level guidelines. The end-to-end argument [cf. 51] – a long-standing principle in system design – suggests that the computation should be done where the data is, i.e., in a decentralized fashion. On the other hand, due to revelation principle (see next paragraph), the mechanism design literature favors systems where the entire computation is done by a center with other participants truthfully reporting their portion of the input data to the optimization. Addressing this discrepancy, in this paper, we argue that such decentralization idea from the system design field is beneficial even in purely economic terms when robust mechanisms are desired.

Revelation principle, a seminal observation in mechanism design suggests that if there is a mechanism with good equilibrium outcome, there is a mechanism which achieves the same outcome in a truth-telling equilibrium. This constructed mechanism asks agents to report true preferences, simulates the agent strategies in the original mechanism, and outputs the outcome of the simulation. Due to this guiding principle, a vast number of studies in mechanism design focus on truthful mechanisms (i.e., ones where revealing preferences truthfully forms an equilibrium). However, successful applications – e.g., first-price auction, generalized second-price auction for advertisers in sponsored search – suggest a great practical impact for non-truthful mechanisms. From the view of multi-party computation, the mechanism itself as well as the participating agents can be thought as different parties in the system, where agents have their private preference as their input data. Truthful mechanisms correspond to systems where the optimization is done by the center (i.e. mechanism) and other parties (i.e. agents) only truthfully report their preference. Non-truthful mechanisms correspond to systems recommended by the end-to-end argument [51], where agents are also perform some of the computation (i.e. computing their strategies).

To provide a theoretical understanding of the potential inadequacy of revelation principle and advantages of non-truthful mechanisms, we consider questions from prior-independent mechanism design, in which a mechanism is designed for agents with preferences drawn from an unknown distributions (a.k.a. prior). The goal is to
identify robust mechanisms – ones with good (multiplicative) prior-independent approximation to the optimal mechanism that is tailored to the distribution of preferences. In prior-independent mechanism design, it is not generally without loss to restrict to truthful mechanisms – the equilibrium strategies for Bayesian agents in non-truthful mechanisms are a function of their prior and thus the construction of truthful mechanism via revelation principle is no longer prior-independent. Nonetheless, similar to other lines of research in mechanism design, most results in prior-independent mechanism design focus, with loss of generality, on truthful mechanisms. To understand the loss of the restriction to truthful mechanisms, Feng and Hartline [29] introduce revelation gap, a quantification of optimal prior-independent approximation ratio among all truthful mechanisms vs. the optimal prior-independent approximation ratio among all (possibly non-truthful) mechanisms. They identify a non-trivial revelation gap in any canonical model for revenue maximization, which is another important and presumably technically more challenging objective in mechanism design.

1.1 Main Results

In this paper, we focus on revenue maximization in a canonical single-item environment for a single agent with a single sample access, i.e., the agent’s value is drawn from an unknown distribution but the mechanism can access a single sample (independent to agent’s value) from that distribution [cf. 4, 26]. The agent knows her private valuation and the distribution for valuation, but she does not know the sample of the mechanism. Our main theorem identifies a non-trivial revelation gap for revenue maximization in this model. This theorem follows from three results. First, we introduce the (non-truthful) sample-bid mechanism and obtain an upper bound of its prior-independent approximation ratio. Second, we obtain a lower bound of the optimal prior-independent approximation ratio among all possible mechanisms. Third, we show that any truthful mechanism is equivalent to a sampled-based pricing mechanism introduced by Alloah and Besbes [4] where the authors lower-bound and upper-bound the optimal prior-independent approximation ratio among all sample-based pricing mechanisms. See Table 1 for a summary of all three results. Since the prior-independent approximation ratio of the sample-bid mechanism is strictly better than the optimal prior-independent approximation ratio among all truthful mechanisms, we immediately get our non-trivial revelation gap for revenue maximization.

In the model of a single agent with single-sample access, the class of non-truthful mechanisms is rich, which includes fairly complicated mechanisms. For example, mechanisms can ask agents to report both her value and prior; or include multiple rounds of communication between seller and agent who sequentially reveal their private information. Nonetheless, our upper bound of the optimal prior-independent approximation ratio is attained by a simple non-truthful mechanism – sample-bid mechanism defined as follow.

- **Sample-bid mechanism:** Given parameter $\alpha$ and sample $s$, the sample-bid mechanism solicits a non-negative bid $b \geq 0$, charges the agent $\alpha \cdot \min\{b, s\}$, and allocates the item to the agent if $b \geq s$.

From the agent’s perspective, she reports a bid to compete for the item against a random sample realized from the same valuation distribution; and regardless of whether she wins or loses, she will always be charged $\alpha \cdot \min\{b, s\}$. In fact, the agent’s optimal bidding strategy could be overbidding or underbidding, depending on the value as well as the distribution. The sample-bid mechanism has the similar format as the Becker–DeGroot–Marschak method [9] which has been studied and implemented in experimental economics for understanding agents’ perception of the random event.

In order to beat the optimal prior-independent approximation ratio among all truthful mechanisms, we need to show the approximation for the sample-bid mechanism is strictly better than 1.957 < 2 for regular distributions, and 1.543 < $\alpha$ for MHR distributions. However, most approximation techniques and results for non-truthful mechanisms in the literature only provide similar or larger constants – for instance, smoothness property, permeability, and revenue covering property in price of anarchy [cf. 28, 39, 48, see more discussion in related work]. One the other hand, analyzing the approximation of truthful mechanisms is relatively easier. In revenue maximization, one analysis approach used extensively for truthful mechanisms is the revenue curve reduction (see next paragraph). This approach has lead to tight or nearly tight results in both prior-independent approximation [3, 4, 37] and Bayesian approximation [2, 43, 44].

Revenue curves [cf. 11] give an equivalent representation of agent’s valuation distribution and enable clean characterizations of the revenue of any mechanism [see e.g. 1, 11, 47]. The high-level goal of

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1 We impose a technical assumption (i.e. scale-invariant) to the class of truthful mechanisms, which is common in prior-independent mechanism design [3, 4, 37].
2 Recall that the agent knows the distribution of the sample but does not know its realization.

|                | Class of truthful mechanisms | Class of all mechanisms |
|----------------|------------------------------|-------------------------|
| Regular dists. | 1.957$^\circ$                | 1.296$^\circ$           |
| MHR dists.     | 1.543$^\circ$                | 1.075$^\circ$           |

$^\circ$ [4] and Lemma 7.2; $^\circ$ Theorem 4.1; $^\circ$ Theorem 5.1; $^\circ$ Theorem 6.1.
revenue curve reduction is to identify a subclass of revenue curves that has closed form and over which the worst approximation guarantee is attained. The main argument is to design a (problem or mechanism) specific modification to the revenue curve (converting an arbitrary revenue curve into a revenue curve from the subclass) and analyze the impact of revenue from the modification on the given mechanism. Note that revenue is the expected payment of the agents when they bid optimally. For truthful mechanisms, after the modification has been designed, it is sufficient to study how payment changes for every bid in the modification, since agents are bidding truthfully (i.e., bids equal values). However, for non-truthful mechanisms, converting a revenue curve to another one will lead to changes in both the payment for each bid and the optimal bidding strategy of each agent. This makes the revenue curve reduction approach more difficult for non-truthful mechanisms, and thus, results of non-truthful mechanisms in the literature rarely uses this technique. In this paper, due to the simplicity of our model and the sample-bid mechanism, we are able to apply this technique by carefully (but relatively loosely) disentangling these two impacts and then analyzing them separately.

Our final result for the single-agent pricing from samples model provides a lower bound on the optimal prior-independent approximation ratio among the class of all mechanisms. This result contrasts with multi-agent models where there there exists complicated and arguably impractical non-truthful mechanism whose prior-independent approximation is arbitrarily close to 1. The crucial observation for proving this lower bound is that for pointmass distributions, the agent perfectly knows the seller’s sample. Thus, she can strategically imitate the behavior of the values in other distributions. This restricts the seller’s ability to extract revenue from the agent, which leads to a prior-independent approximation ratio at least 1.073 even on the restricted subclass of MHR distributions (in fact, even on uniform distributions). Our lower bound also suggests that it will be non-trivial to identify the non-truthful mechanism which attains the optimal prior-independent approximation ratio.

It should be noted that our better-performing non-truthful prior-independent mechanisms do not come without drawbacks relative to truthful prior-independent mechanisms. Elegantly, truthful prior-independent mechanisms do not require prior knowledge by any party. In contrast, non-truthful prior-independent mechanisms generally require some knowledge of the prior on the part of the agents. From this perspective, our results show that a seller is able to extract strictly higher revenue from the agent by taking advantage of information that the agent possesses and is able to strategize with respect to.

1.2 Important Directions

Despite the practical importance of non-truthful mechanisms, the literature on mechanism design almost exclusively considers the design of truthful mechanisms. Thus, the most general direction from this paper is to systematically build a theory for the design of non-truthful mechanisms with good performance guarantee.

Some recent works on this topic are equilibrium analysis of i.i.d. rank-based mechanism [14], robust analysis of welfare and revenue for classic mechanisms in practice (i.e. price of anarchy, see discussion in related work), estimating revenue and welfare in a mechanism from equilibrium bids in another mechanism [15, 16], and the sample complexity of non-truthful mechanisms in asymmetric environments [38].

Though Feng and Hartline [29] and this paper demonstrate non-trivial revelation gap for both welfare-maximization and revenue maximization, both gaps are constant. Thus, one interesting open question left is to identify a superconstant revelation gap in a canonical model where simple non-truthful mechanisms are sufficient to beat the optimal prior-independent truthful mechanisms, and we conjecture that the single-agent with single-sample access model without any regularity assumption on distributions might be a good candidate to answer this question.

Prior-independent mechanism design for a single item with symmetric agents is an extensively studied model [10, 25, 32]. The fundamental difficulty is to pin down the optimal prior-independent approximation ratio even for the two-agent setting. Recently, Allouah and Besbes [3] obtain the tight bounds of optimal prior-independent truthful mechanism for MHR distributions, and Hartline, Johnsen, and Li [37] generalize it to regular distributions. In both works, the main technique is the revenue curve reduction. An open question here is to identify simple non-truthful mechanism which outperforms the optimal prior-independent truthful mechanism in this canonical single-item two-agents model.

In this work, we apply the revenue curve reduction approach—a powerful technique of approximation analysis for truthful mechanisms—to a non-truthful mechanism. Our argument is not as general as ones for truthful mechanisms and thus there are gaps between the lower bound and upper bound. Besides sharpening these bounds as an open question, an important open question is to design general analysis framework on revenue curves for non-truthful mechanisms.

1.3 Related Work

Prior-independent mechanism design, as a standard framework for understanding the robustness of mechanisms, has been applied to single-dimensional mechanism design [3, 26, 29, 32, 37, 49], multi-dimensional mechanism design [24, 33, 50], makespan minimization [17], mechanism design for risk-averse agents [31], and mechanism design for agents with interdependent values [13]. Except Fu, Hartline, and Hoy [31] and Feng and Hartline [29], all other results focus on truthful mechanisms.

There is a significant area of research studying mechanism design with sample access from the distribution of agents’ preference, which has two regimes—small number of samples, and large number of samples. In the former regime, literature studies the approximation of mechanisms with a single-sample access [4, 6, 20, 21, 26, 27, 30], and mechanisms with two-sample access [7, 23]. In the latter regime, the goal is to minimize the sample complexity, i.e., number of sample to achieve (1−ε)-approximation.
We consider the prior-independent mechanism design with a single agent. We assume that distribution \( F \) supported on \([v_{\min}, v_{\max}]\) is the valuation distribution (a.k.a. prior) for the agent. This paper focuses on the single-item revenue-maximization model. The authors upper-bound the mal prior-independent approximation without any restriction on mechanisms and give an analysis framework based on this property. With this smoothness framework, the authors later tighten the welfare-approximation of the first-price auction by \( \epsilon \) and the welfare-approximation of the all-pay auction by \( 2 \). These two results are later tightened by Christodoulou, Sgouritsa, and Tang [18] for the all-pay auction and Hoy, Taggart, and Wang [40] for the first-price auction using some mechanism-specific arguments. Hartline, Hoy, and Taggart [36] introduce a geometric framework for analyzing the price of anarchy for both welfare and revenue. Hartline, Hoy, and Taggart [36] introduce a geometric framework for analyzing the price of anarchy for both welfare and revenue. In general, there is no incentive compatible mechanism which outputs the welfare-maximizing allocation, but allows mechanisms where agents cross-report their beliefs on valuations.

\[ \text{Definition 2.2:} \text{For any valuation distribution } F \text{, the mechanism } M = (\hat{x}, \hat{p}) \text{ is incentive compatible (IC) if the agent's utility truthfully is her best response, i.e., } b(v, F) = v \text{ for all } v \text{ and } F. \]

\[ \text{Definition 2.3:} \text{The mechanism } M = (\hat{x}, \hat{p}) \text{ is individually rational (IR) if the agent's utility under her best response is non-negative, i.e., } \max_{b} v \cdot \hat{x}(b, F) - \hat{p}(b, F) \geq 0 \text{ for all } v \text{ and } F. \]

Revenue Curve. For any distribution \( F \), let \( q(v, F) = 1 - F(v) \) be the quantile for the distribution, and \( v(q, F) = \text{the value } v \text{ such that } q = 1 - F(v) \). Here we introduce the revenue curve in quantile space [cf. 11], which is a useful tool in the revenue analysis.

\[ \text{Definition 2.2:} \text{For any valuation distribution } F \text{, the revenue curve } R(q, F) \text{ of the mechanism } M \text{ is a mapping from any } q \in [0, 1] \text{ to the optimal revenue from an agent with value drawn from } F \text{ subject to the constraint that the item is allocated with } \text{ex ante probability } q. \]

We consider the prior-independent mechanism design with a single sample access. Namely, the seller does not know the valuation distribution \( F \) but has a single sample \( s \) drawn from \( F \). The agent knows the valuation distribution \( F \) but does not observe the sample, and the value \( v \) of the agent is independent of the sample \( s \). A mechanism \( M = (\hat{x}, \hat{p}) \) includes an allocation rule \( \hat{x} : R \times R \to [0, 1] \) mapping from the agent’s bid \( b \) and the sample \( s \) to the allocation probability of the item; and a payment rule \( \hat{p} : R \times R \to \mathbb{R} \) mapping from the agent’s bid \( b \) and the sample \( s \) to the payment charged from the agent. Let \( \hat{x}(b, F) = E_{x \sim F}[\hat{x}(x, s)] \) and \( \hat{p}(b, F) = E_{s \sim F}[\hat{p}(x, s)] \) be the expected allocation and payment over the randomness of the sample \( s \) drawn from distribution \( F \). The seller first announce the mechanism \( M = (\hat{x}, \hat{p}) \) to the buyer, and then the sample \( s \) and value \( v \) are realized from distribution \( F \). The agent report a bid \( b \) based on her private value \( v \), and the seller implements the mechanism \( M \) with input \( b \) and sample \( s \). We assume that the seller has full commitment power on implementing the mechanism.

Given a mechanism \( (\hat{x}, \hat{p}) \) and distribution \( F \), the best response of the agent is \( b(_\ast, F) : \mathbb{R} \to \mathbb{R} \) which maximizes her expected utility, i.e., for every value \( v \), \( b(v, F) \in \arg\max_{b \in \mathbb{R}} v \cdot \hat{x}(b, F) - \hat{p}(b, F) \). A mechanism \( (\hat{x}, \hat{p}) \) is incentive compatible (IC) if reporting the agent’s value truthfully is her best response, i.e., \( b(v, F) = v \) for all \( v \) and \( F \). A mechanism \( (\hat{x}, \hat{p}) \) is individually rational (IR) if the agent’s utility under her best response is non-negative, i.e., \( \max_{b} v \cdot \hat{x}(b, F) - \hat{p}(b, F) \geq 0 \) for all \( v \) and \( F \).
In the later analysis in the paper, when \( F \) is clear from the context, we omit it in the notation and only use \( R(q) \) to represent the revenue curve and \( q(v) \) to represent the quantile of value \( v \). Let \( \phi(v) = v - \frac{1 - F(v)}{f(v)} \) be the virtual value of the agent.

**Definition 2.3.** An valuation distribution \( F \) is regular if the virtual value of the agent is weakly increasing.

**Theorem 2.1 (47).** A distribution \( F \) is regular if and only if the corresponding revenue curve \( R(q, F) \) is concave.

**Theorem 2.2 (47).** For any distribution \( F \) and any mechanism with interim allocation and payment rule \( x(v), p(v) \), the expected revenue of the seller equals the expected virtual value of the agent plus the payment of the lowest value \( v \), i.e., \( E_{v \sim F} [p(v)] = E_{v \sim F} [x(v)\phi(v)] + p(\phi(v)) \).

Finally, we define the monopoly reserve and monopoly quantile of the agent given the revenue curve \( R \).

**Definition 2.4.** The monopoly quantile of the agent is \( q_m = \arg \max_q R(q) \), and the monopoly reserve of the agent is \( v_m = \frac{R(q_m)}{q_m} \).

### 3 THE SAMPLE-BID MECHANISM

In this section, we introduce the main mechanism considered in this paper, the sample-bid mechanism.

**Definition 3.1** (sample-bid mechanism). Given parameter \( \alpha \) and sample \( s \), the sample-bid mechanism solicits a non-negative bid \( b \geq 0 \), charges the agent \( \alpha \cdot \min\{b, s\} \), and allocates the item to the agent if \( b \geq s \).

In the sample-bid mechanism, the agent reports her bid without knowing the realization of the sample. From her perspective, the utility \( u(v, b, F) \) for her who has value \( v \), reports bid \( b \), and competes with sample \( s \) is

\[
u(v, b, F) = v \cdot \frac{F(b) - ab \cdot (1 - F(b)) - \alpha \int_b^{\max\{b,v\}} tdF(t) }{Pr_{v \sim F}[s \leq b]} \text{ payment when } s \geq b
\]

\[
u(v, b, F) = v \cdot \frac{F(b) - ab \cdot (1 - F(b)) - \alpha \int_b^{\max\{b,v\}} tdF(t) }{Pr_{v \sim F}[s > b]} \text{ payment when } s \leq b
\]

Note that reporting bid equal to zero, the utility of agent is zero. Thus, sample-bid mechanism is individually rational.

**Lemma 3.1.** The sample-bid mechanism is individually rational.

On the other hand, reporting bid equal to agent’s value is not the best response in general. We provide a characterization of agent’s optimal bid as follows.

**Lemma 3.2.** In the sample-bid mechanism, given any parameter \( \alpha \) and distribution \( F \), the optimal bid \( b(v, F) \) for the agent with value \( v \) satisfies the constraint that

\[
v = \alpha \cdot \frac{1 - F(b(v, F))}{f(b(v, F))},
\]

or \( b(v, F) \in \{0, \infty\} \). Ties are broken according to the utility of the agent.

**Proof.** The agent’s utility from reporting bid \( b \) is

\[
u(v, b, F) = v \cdot F(b) - ab(1 - F(b)) - \alpha \int_b^{\max\{b,v\}} tdF(t)
\]

Consider the first order condition with respect to bid \( b \), if the optimal bid is obtained in the interior, we have

\[
f(b) \left(v - \alpha \cdot \frac{1 - F(b)}{f(b)} \right) = 0
\]

as a necessary condition for the optimality of the bid \( b \). Otherwise, the optimal bid is obtained on the boundary, where \( b(v, F) \in \{0, \infty\} \).

Note that there might exist multiple bids \( b \) that satisfies the constraint (1) in Lemma 3.2. In that case, the agent chooses the bid which satisfies (1) and maximizes her utility. Another observation (Lemma 3.3) of the sample-bid mechanism is that the expected revenue of the seller scales linearly with the valuation distribution. Since the optimal revenue scales linearly with the valuation distribution as well, to analyze the prior-independent approximation ratio of the sample-bid mechanism, we can focus on the valuation distributions such that the optimal revenue is normalized to 1.

**Lemma 3.3.** Denote by \( r \) the revenue of the sample-bid mechanism with any parameter \( \alpha \) and any valuation distribution \( F \). For any \( p > 0 \) and distribution \( F^\rho \) such that \( F^\rho \) is \( F^\rho \) scaled by \( p \), i.e., \( F^\rho(v) = F^\rho(pv) \) for all \( v \), the revenue of the sample-bid mechanism with parameter \( \alpha \) and distribution \( F^\rho \) is \( pr \).

**Proof.** First we show that for any value \( v \), the bid of value \( v \) given distribution \( F^\rho \) is equivalent to the bid of value \( pv \) given distribution \( F^\rho \) scaled by \( p \). The reason is that \( F^\rho(v) = F^\rho(pv) \) and \( F^\rho(v) = pF^\rho(pv) \). Therefore, by Lemma 3.2, the first order condition implies that the optimal bid satisfies \( b(pv, F^\rho) = \rho \cdot b(v, F^\rho) \). Moreover, the payment satisfies

\[
\tilde{p}(\rho b, F^\rho) = \alpha \rho (1 - F^\rho(b)) + \alpha \int_0^{\rho b} t dF^\rho(t)
\]

By taking expectation over the valuation, the expected revenue is scaled by \( p \) as well.

We finish this section by providing two simple monotonicity properties of the sample-bid mechanism and defer other more complicated characterizations required in our analysis to the later sections.

**Lemma 3.4.** In the sample-bid mechanism, given any parameter \( \alpha \) and distribution \( F \), the expected payment for bid \( b \) is monotonically non-decreasing in \( b \).
Proof. By definition, the expected payment \( \hat{p}(b, F) \) of bid \( b \) over the randomness of the sample \( s \sim F \) is
\[
\hat{p}(b, F) = ab \cdot (1 - F(b)) + \int_b^{\max\{b, \xi\}} t \, dF(t)
\]
Taking the derivative with respect to bid \( b \), we have
\[
\frac{\partial \hat{p}(b, F)}{\partial b} = a(1 - F(b)) - af(b) + aF(b) = a(1 - F(b)) \geq 0,
\]
which finishes the proof.

\( \square \)

Lemma 3.5. In the sample-bid mechanism, given any parameter \( \alpha \) and distribution \( F \), the optimal bid \( b(v, F) \) is monotonically non-decreasing in value \( v \).

Proof. By Myerson [47], the equilibrium allocation of the agent is non-decreasing in value \( v \). Moreover, given the auction format, the equilibrium allocation of the agent is increasing in the bid, and thus the optimal bid \( b(v, F) \) is non-decreasing in the value \( v \). \( \square \)

4 THE SAMPLE-BID MECHANISM FOR MHR DISTRIBUTIONS

In this section, we analyze the prior-independent approximation ratio of the sample-bid mechanism over the class of MHR distributions.

Definition 4.1. A distribution \( F \) is MHR if the hazard rate \( \frac{f(v)}{1 - F(v)} \) is monotonically non-decreasing in \( v \).

Theorem 4.1. For the sample-bid mechanism with \( \alpha = 0.824 \), the prior-independent approximation ratio over the class of MHR distributions is between [1.295, 1.296].

The lower bound in Theorem 4.1 is shown in the following example.

Example 4.2. For the sample-bid mechanism with \( \alpha = 0.824 \), let \( F \) be the valuation distribution such that \( F(v) = 1 - e^{-v^2} \) for \( v \in [0, 0.43) \) and \( F(v) = 1 \) for \( v \in [0.43, \infty) \). It is easy to verify that \( F \) is MHR. Moreover, the optimal revenue is 0.2797 while the expected revenue of the sample-bid mechanism, which equals the expected revenue of posting a price equal to 0.824 fraction of the expected welfare, is 0.2519. Thus, the prior-independent approximation ratio of the sample-bid mechanism with \( \alpha = 0.824 \) is at least 1.295.

Before the proof of the upper bound in Theorem 4.1, we first introduce a characterization of the agent’s optimal bid when the sample distribution \( F \) is MHR; and a technical property for MHR distributions.

Lemma 4.2. In the sample-bid mechanism, given any parameter \( \alpha \) and MHR distribution \( F \), the optimal bid \( b(v, F) \) for the agent with value \( v \) is
\[
b(v, F) = \begin{cases} 0 & \text{if } v < \alpha E_{\hat{E}_{\alpha}}[s], \\ \infty & \text{otherwise}. \end{cases}
\]

Proof. By the proof of Lemma 3.2, the derivative of the utility given the bid \( b \) is
\[
f(b) \left( v - \alpha \cdot \frac{1 - F(b)}{f(b)} \right),
\]
where the sign of the above expression flips from negative to positive only once when the bid \( b \) increases from 0 to infinity since \( F \) is MHR. Thus the utility is a quasi-convex function of the bid, which implies that the maximum utility is attained at extreme points, i.e., bid 0 or \( \infty \). Note that the utility for bidding 0 is always 0, while the utility for bidding \( \infty \) is \( u(v, \infty, F) = v - \alpha E_{\hat{E}_{\alpha}}[s] \). Hence, the agent bid \( \infty \) if and only her value \( v \) is at least \( \alpha E_{\hat{E}_{\alpha}}[s] \).

\( \square \)

Lemma 4.3 (4). For any MHR distribution with any pair of quantile and values \((v_1, q_1), (v_2, q_2)\) such that \( q_1 = q(v_1) \leq q_2 = q(v_2) \) and \( v_1 \geq v_2 \). Then for any \( v \geq v_2 \), we have \( q(v) \geq q_2 \cdot e^{\frac{\ln(\frac{v}{v_1})}{q_1}} \).

Lemma 4.4. The expected value of any MHR distribution with monopoly quantile \( q_m \) is \( v \geq q_m \cdot m \ln q_m \).

Proof. The expected value of the agent is
\[
\int_0^\infty q(v) \, dv = \int_0^{q_m} e^{-q_m - q_m \ln q_m} \, dv = \frac{1}{q_m \cdot m \ln q_m} \left( e^{-q_m - q_m \ln q_m} - e^{-q_m} \right) = \frac{q_m - 1}{q_m \cdot m \ln q_m},
\]
where the inequality holds by applying Lemma 4.3 with \( q_1 = q_m, v_1 = \frac{1}{q_m} \) and \( q_2 = 1, v_2 = 0 \).

\( \square \)

Now, we are ready to show Theorem 4.1.

Proof of the upper bound in Theorem 4.1. Fix any MHR distribution \( F \). Let \( w \hat{=} E_{\hat{E}_{\alpha}}[s] \). Note that by Lemma 4.2, our mechanism is equivalent to posting price \( \alpha w \) to the agent. Next we analyze the approximation ratio by considering the cases \( \alpha w \geq v_m \) and \( \alpha w < v_m \) and optimize the parameter \( \alpha \) such that the approximation ratio of both cases coincide. Recall that it is without loss of generality to normalize the expected revenue of the optimal mechanism to 1, i.e., \( q_m \cdot v_m = 1 \).

First we consider the case when \( \alpha w < v_m \). By Lemma 4.4, we have \( w \geq \frac{q_m - 1}{q_m \ln q_m} \) and by combining Lemma 4.3 with \( (v_1, q_1) = (v_m, q_m) \) and \( (v_2, q_2) = (0, 1) \), we have \( q(\alpha w) \geq e^{\alpha(q_m - 1)} \). Thus, the expected revenue in this case is
\[
\alpha w \cdot q(\alpha w) \geq \frac{\alpha(q_m - 1)}{q_m \ln q_m} \cdot e^{\alpha(q_m - 1)}.
\]

Then we consider the case when \( \alpha w \geq v_m \). In this case, combining Lemma 4.3 with \( (v_1, q_1) = (w, q_w) \) and \( (v_2, q_2) = (v_m, q_m) \), where \( q_w \geq 1/\alpha \) is the quantile of the welfare [see 8], for any value \( v \geq v_m \), we have \( q(\alpha w) \geq q_m \cdot e^{\frac{\ln(q_w)}{q_m}} \). Thus the expected revenue is
\[
\alpha w \cdot q(\alpha w) \geq \alpha w \cdot q_m \cdot e^{\frac{\ln(q_w)}{q_m}} \geq \alpha w \cdot q_m \cdot e^{\frac{1}{q_m} \ln \left( \frac{q_w}{q_m} \right)}.
\]
5 THE SAMPLE-BID MECHANISM FOR REGULAR DISTRIBUTIONS

In this section, we analyze the prior-independent approximation of the sample-bid mechanism over the class of regular distributions.

**Theorem 5.1.** For the sample-bid mechanism with $\alpha = 0.7$, the prior-independent approximation ratio over the class of regular distributions is between $[1.628, 1.835]$.

The lower bound in Theorem 5.1 is shown in the following example.

**Example 5.1.** For the sample-bid mechanism with $\alpha = 0.7$, let $F$ be the valuation distribution such that $F(v) = \frac{0.365}{v-0.735}$ for $v \in [1, \infty)$. It is easy to verify that $F$ is regular. Moreover, the optimal revenue is $1$ while the expected revenue of the sample-bid mechanism is $0.614$. Thus, the prior-independent approximation ratio of the sample-bid mechanism with $\alpha = 0.7$ is at least $1.628$.

In Section 5.1, we introduce some technical characterizations of the sample-bid mechanism which will be used in the subsequent analysis. In Sections 5.2 and 5.3, we study the prior-independent approximation ratio of the sample-bid mechanism over the class of regular distributions with monopoly quantile $q_m \geq 0.62$ and $q_m \leq 0.62$ respectively. By Lemma 3.3, without loss of generality, we restrict our attention to the class of regular valuation distributions where the optimal revenue for the distributions is exactly $1$ (i.e., $v_m = q_m = 1$), and then lower-bound the expected revenue of the sample-bid mechanism with $\alpha = 0.7$.

Here we sketch the high-level approach to lower-bound the expected revenue of the sample-bid mechanism in both regimes (Sections 5.2 and 5.3). Given a regular distribution $F$, we define a value threshold $v^*(F)$ as the smallest value whose optimal bid is at least monopoly reserve $v_m(F)$, i.e.,

$$v^*(F) \triangleq \inf \{v : b(v, F) \geq v_m(F)\}.$$

Denote $q(v^*(F), F)$ by $q^*(F)$. By Lemma 3.4 and Lemma 3.5, the expected revenue $\text{Rev}_F(SB)$ of the sample-bid mechanism SB for valuation $F$ can be lower-bounded as follows,

$$\text{Rev}_F(SB) = \int_0^1 p(v(q, F), F) \, dq \geq p(v^*(F), F) \cdot q^*(F) + \int_{q^*(F)}^1 p(v(q, F), F) \, dq.$$

where $p(v, F)$ is the expected payment of the agent, with value $v$ and valuation distribution $F$, in the sample-bid mechanism. We then analyze $p(v^*(F), F), q^*(F),$ and $p(v(q, F), F)$ for $q \geq q^*(F)$. By providing lower bounds as the functions of $q_m(F)$ and other some parameters of $F$. Finally, by numerically evaluating the value of lower bounds for all possible parameters, we conclude that the expected revenue in the sample-bid mechanism for all regular distribution (with monopoly revenue $1$) is at least $0.545$, which implies the prior-independent approximation ratio $\frac{\text{Rev}_F(SB)}{\text{Rev}_{\text{OPT}}(F)} \approx 1.835$ of the sample-bid mechanism in Theorem 5.1. The details for discretizations and numerical evaluations can be found in the online appendix. Note that the bounds for the approximation ratio of the sample-based pricing mechanisms in Alloah and Besbes [4] are also obtained by numerical analysis, which requires solving a relatively more complicated dynamic program. In contrast, our numerical analysis only requires brute force enumeration of a few parameters.

As we discussed in Section 2, every valuation distribution $F$ can be represented by its induced revenue curve $R$ where $R(q) \triangleq q \cdot F^{-1}(1-q)$ for all $q \in [0,1]$. In the remaining of the section, all statements, notations and analysis (except Lemma 5.3) will be presented in the language of revenue curves instead of valuation distributions.

5.1 Technical Properties of the Sample-Bid Mechanism

In this subsection, we introduce some technical characterizations of the sample-bid mechanism which will be used in the later analysis.

To establish a lower bound on the expected revenue of of a truthful mechanism, a classic approach - revenue curve reduction - [e.g. 2, 3] is as follows: (i) start with an arbitrary revenue curve $R_1$, (ii) convert it to another revenue $R_2$ with closed-form formula while the optimal revenue remains the same, (iii) argue that the expected revenue for $R_2$ is at most the expected revenue for $R_1$ while the optimal revenue remains the same, and finally (iv) evaluate the expected revenue for $R_2$ for all possible parameters. In this section, we want to apply a similar approach to the sample-bid mechanism because it is a non-truthful mechanism. A new technical difficulty arises in step (iii). When comparing $R_1$ and $R_2$, for truthful mechanisms, it is sufficient to study the change in the expected payment (i.e. $p(b, R_1)$ and $p(b, R_2)$) for each bid $b$. However, for non-truthful mechanisms (e.g. sample-bid mechanism), the optimal bid of the agent changes when the revenue curve $R_1$ is replaced by $R_2$. In Lemma 5.2, we provide a characterization of optimal bid when we switch from $R_1$ to $R_2$ in a specific way (illustrated in Figure 1). We use it as a building block repeatedly in Section 5.2 and Section 5.3. Intuitively, the following lemma characterizes the phenomenon that increasing the revenue curve for high values does not affect the agent’s preference for low bids.

**Lemma 5.2.** In the sample-bid mechanism, consider any quantile $q^1 \in [0,1]$ and any pair of revenue curves $R_1, R_2$ such that $R_1(q) \leq R_2(q)$ for any quantile $q \leq q^1$ and $R_1(q^1) = R_2(q^1)$. Letting $b^\dagger = R_1(q^1)$. For any value $v$ and any bid $b^\dagger \geq b^\dagger$, if an agent with value $v$ and revenue curve $R_1$ prefers bid $b^\dagger$ than $b^\dagger$, i.e., $u(v, b^\dagger, R_1) \geq u(v, b^\dagger, R_2)$, then an agent with value $v$ and revenue curve $R_2$ also prefers bid $b^\dagger$ than $b^\dagger$, i.e., $u(v, b^\dagger, R_2) \geq u(v, b^\dagger, R_2)$.

---

9Let $R$ be the revenue curve induced by valuation distribution $F$. In Section 5.2, we lower-bound the expected revenue as a function of $q_m(F)$ and $R(0)$. In Section 5.2, we lower-bound the expected revenue as a function of $q_m(F)$, $q(v^*(F), F)$ and $R(0)$.
Thus, we compute the difference between $u(v, \frac{v}{m}, F)$ and $u(v, b, F)$ for any value $v \geq v_m$ and bid $b \in [v_m, v/\alpha]$ as follows,

$$u(v, \frac{v}{m}, F) - u(v, b, F) = \int_{b}^{v/\alpha} \alpha f(t) \left( \frac{v}{\alpha} - \frac{1 - F(t)}{f(t)} \right) dt \geq \int_{b}^{v/\alpha} \alpha f(t) \left( t - \frac{1 - F(t)}{f(t)} \right) dt \geq 0$$

where the last inequality uses the fact that $t - \frac{1 - F(t)}{f(t)} \geq 0$ for all $t \geq v_m$ if $F$ is regular. 

\[ \square \]

### 5.2 Regular Distributions with Monopoly Quantile $q_m \geq 0.62$

In this subsection, we analyze the approximation ratio of the sample-bid mechanism over the class of regular distributions with monopoly quantile $q_m \geq 0.62$.

**Lemma 5.4.** For the sample-bid mechanism with $\alpha = 0.7$, the approximation ratio over the class of regular distributions with monopoly quantile $q_m \geq 0.62$ is at most 1.835.

Fix an arbitrary revenue curve $R$, let $v^*(R) \triangleq \inf \{ v : b(v, R) \geq v_m(R) \}$ be the smallest value whose optimal bid $b(v, R)$ for revenue curve $R$ is at least the monopoly reserve $v_m(R)$. Since Lemma 3.5 guarantees that $b(v, R)$ is weakly non-decreasing in $v$, $v^*(R)$ is well-defined, $b(v, R) \geq v_m(R)$ for all $v \geq v^*(R)$, and $b(v, R) < v_m(R)$ for all $v < v^*(R)$. Denote $q(v^*(R), R)$ by $q^*(R)$. We decompose the proof of Lemma 5.4 by considering the following two subregimes - Lemma 5.5 for revenue curve $R$ with $v^*(R) \leq v_m(R)$; and Lemma 5.7 for revenue curve $R$ with $v^*(R) \geq v_m(R)$.

**Lemma 5.5.** Given any concave revenue curve $R$ such that $q_m(R) \geq 0.62$ and $v^*(R) \leq v_m(R)$, the revenue of the sample-bid mechanism with $\alpha = 0.7$ is a 1.835-approximation of the optimal revenue.

**Proof.** Fix an arbitrary concave revenue curve $R$ satisfying the requirement in the lemma statement, i.e., $q_m(R) \geq 0.62$ and $v^*(R) \leq v_m(R)$. Consider an arbitrary value $v \geq v^*(R)$. By Lemma 3.5, the optimal bid of an agent with value $v$ is at least $v_m(R)$. Thus, together with Lemma 3.4, her expected payment in sample-bid mechanism is at least the expected payment $\hat{p}(v_m(R), R)$ of bidding $v_m(R)$, and

$$\hat{p}(v_m(R), R) = 0.7 v_m(R) q_m(R) + 0.7 \int_{q_m(R)}^{v_m(R)} \frac{R(q)}{q} dq = 0.7 + 0.7 \int_{q_m(R)}^{1} \frac{1-q}{q} dq$$

$$\geq 0.7 + 0.7 \int_{q_m(R)}^{1} \frac{1-q}{q} dq = \frac{0.7 \log(q_m(R))}{1 - q_m(R)}.$$
which is at least 0.545 for all \( q \) where the inequality uses the fact that (1) \( R \) is concave, which implies that \( R(q) \geq \frac{1-q}{1-q_{m}(R)} \) for all \( q \geq q_{m}(R) \); and (2) \( v_{m}(R)q_{m}(R) \) is normalized to 1 for the revenue curve \( R \). Since \( v^{\ast}(R) \leq v_{m}(R) \), each value with quantile smaller than \( q_{m}(R) \) has \( p(v_{m}(R), R) \) as a lower bound of its payment in the sample-bid mechanism. Thus, a lower bound of the expected revenue \( \text{Rev}_{R}(SB) \) for revenue curve \( R \) in the sample-bid mechanism is

\[
\text{Rev}_{R}(SB) = \int_{0}^{1} p(v(q,F), F) dq \geq p(v^{\ast}(R), R) \cdot q^{\ast}(R)
\]

\[
\geq \hat{p}(v_{m}(R), R) \cdot q_{m}(R) \geq \frac{0.7 \log(q_{m}(R))q_{m}(R)}{1-q_{m}(R)}
\]

which is at least 0.545 for all \( q_{m}(R) \geq 0.62 \). This finishes the proof, since we (without loss of generality) consider revenue curve \( R \) with optimal revenue equal to 1, i.e., \( v_{m}(R) \cdot q_{m}(R) = 1 \). \( \square \)

Before diving into the subregime where \( v^{\ast}(R) \geq v_{m}(R) \), we provide a characterization (Lemma 5.6) of the optimal bid for concave revenue curves with monopoly quantile greater than 0.62. Specifically, Lemma 5.6 guarantees that \( b(v,R) = 0 \) for all value \( v < v^{\ast}(R) \). The proof of Lemma 5.6 is provided in the online appendix.

**Lemma 5.6.** In the sample-bid mechanism with parameter \( \alpha = 0.7 \), given any value \( v \) and any concave revenue curve \( R \) with \( q_{m}(R) \geq 0.62 \), the optimal bid \( b(v,R) \) for an agent with value \( v \) and revenue curve \( R \) satisfies \( b(v,R) \in \{0\} \cup \{v_{m}(R), \infty\} \).

Now, we provide the approximation guarantee for revenue curve \( R \) with \( v^{\ast}(R) \geq v_{m}(R) \).

**Lemma 5.7.** Given any concave revenue curve \( R \) such that \( q_{m}(R) \geq 0.62 \) and \( v^{\ast}(R) \geq v_{m}(R) \), the revenue of the sample-bid mechanism with \( \alpha = 0.7 \) is a 1.835-approximation of the optimal revenue.

**Proof.** The proof is done in four major steps:

\[
\text{Rev}_{R}(SB) = \int_{0}^{1} p(v(q,F), F) dq \geq p(v^{\ast}(R), R) \cdot q^{\ast}(R)
\]

\[
\geq \hat{p}(v_{m}(R), R) \cdot q_{m}(R) \geq \frac{0.7 \log(q_{m}(R))q_{m}(R)}{1-q_{m}(R)}
\]

which is at least 0.545 for all \( q_{m}(R) \geq 0.62 \). This finishes the proof, since we (without loss of generality) consider revenue curve \( R \) with optimal revenue equal to 1, i.e., \( v_{m}(R) \cdot q_{m}(R) = 1 \). \( \square \)
with value \( v \) well. On the other side, every quantile \( \alpha \) with (Lemma 5.6) and gains zero virtual surplus while their virtual value by construction. We claim that the allocation for each \( b \) with value \( v \) is the right-hand derivative of \( R \). See Figure 4 for a graphical illustration. Invoking Lemma 5.3 and Lemma 5.6, with the same argument for values with positive virtual values in step 1, we can conclude that \( q^*(R_2) \leq q^*(R_1) \), every quantile \( q \leq q^*(R_2) \) weakly decreases since the virtual value is non-negative while the allocation decreases. Note that in sample-bid mechanism, the payment for lowest type is always 0, i.e., \( p(0) = 0 \). By Theorem 2.2, the expected revenue (a.k.a. virtual surplus) for \( R_2 \) is at most the expected revenue (a.k.a. virtual surplus) for \( R_1 \).

**Step 2:** flattening the revenue curve for all quantiles \( q \geq q^* \).

In this step, we start with revenue curve \( R_2 \) constructed in step 1, and consider a sequence of revenue curves \( R_2^{(0)}, R_2^{(1)}, \ldots \) where \( R_2^{(0)} = R_2 \) and \( R_2^{(i+1)} \) is recursively defined on \( R_2^{(i)} \) as follows,

\[
R_2^{(i)}(q) \begin{cases} 
R_2(q) & q \in [0, q^*(R_2^{(i)})) \\
1 & q \in \left[ 1 - \frac{R_2(q) - R_2(q^*(R_2^{(i)}))}{R_2(q) - R_2(q^*(R_2^{(i)}))}, q^*(R_2^{(i)}), 1 \right], \\
R_2(q^*(R_2^{(i)})) \cdot (q - R_2(q^*(R_2^{(i)}))) + q^*(R_2^{(i)}) & \text{o.w.}
\end{cases}
\]

where \( R_2^{(i)}(q^*(R_2^{(i)})) \) is the right-hand derivative of \( R_2^{(i)}(q) \) at \( q = q^*(R_2^{(i)}) \). See Figure 4 for a graphical illustration. Invoking Lemma 5.3 and Lemma 5.6, with the same argument for values with positive virtual values in step 1, we can conclude that \( q^*(R_2^{(i)}) \) and the expected revenue for \( R_2^{(i)} \) in the sample-bid mechanism is weakly decreasing in \( i \).

Note that by construction, the sequence of revenue curves \( R_2^{(0)}, R_2^{(1)}, \ldots \) converges to a revenue curve \( R_2 \) whose expected revenue in the sample-bid mechanism is at most the revenue for \( R_2 \), and satisfying the following characterization,

\[
R_3(q) \begin{cases} 
R_3(q) & q \in [0, q^*(R_3)) \\
1 & q \in \left[ 1 - \frac{R_3(q) - R_3(q^*(R_3))}{R_3(q) - R_3(q^*(R_3))}, q^*(R_3), 1 \right], \\
R_3(q^*(R_3)) \cdot (q - R_3(q^*(R_3))) + R_3(q^*(R_3)) & \text{o.w.}
\end{cases}
\]

We claim that the expected revenue of the sample-bid mechanism with \( \alpha = 0.7 \) for revenue curve \( R_2 \) is at most that of revenue curve \( R_1 \). To see this, consider the virtual surplus for both revenue curves. By our assumption that \( v^*(R_1) \geq v_m(R_1) \), every quantile \( q > v_m(R_1) \) has negative virtual value \( R_2'(q) \) in \( R_1 \), bids zero (Lemma 5.6) and gains zero virtual surplus while their virtual value \( R_2'(q) \) becomes zero in \( R_2 \) and thus gains zero virtual surplus as well. On the other side, every quantile \( q \leq v_m(R_1) \) has identical virtual value by construction. We claim that the allocation for each of these quantiles weakly decreases. To see this, note that the allocation of bidding any bid \( b \geq v_m(R_1) = v_m(R_2) \) is the same for both revenue curves \( R_1 \) and \( R_2 \), and the expected payment increases by a constant when the revenue curve \( R_1 \) is replace by \( R_2 \). Thus the agent’s preference among all bids \( b \geq v_m(R_1) \) is the same in both revenue curves \( R_1 \) and \( R_2 \). However, the utility of bidding \( b \geq v_m(R_2) \) is lower when the revenue curve is \( R_2 \), which implies that there may exist value \( v \) such that the agent may prefer bidding 0 to bidding above the monopoly reserve in \( R_2 \), while strictly prefer bidding above the monopoly reserve in \( R_1 \). By Lemma 5.6, the optimal bid for any value \( v \) is not in \((0, v_m(R_2)) \). Thus, we conclude that \( q^*(R_2) \leq q^*(R_1) \) and (1) the optimal bid (as well as the allocation) for every quantile \( q \leq q^*(R_1) \) in both \( R_1 \) and \( R_2 \) remains the same; and (2) for every quantile \( q \in [q^*(R_2), q^*(R_1)) \), the optimal bid quantile \( q = 0 \) when the revenue curve is \( R_2 \). This guarantees that the virtual surplus for every quantile \( q \leq q_m(R_1) \) weakly decreases since the virtual value is non-negative while the allocation decreases.

**Step 3:** flattening the revenue curve for all quantiles \( q \leq q_m(R_1) \).

For any revenue curve \( R \), let \( p(v^*(R), R) \) be the expected payment in the sample-bid mechanism of an agent with value \( v^*(R) \) and revenue curve \( R \). Due to Lemma 3.4 and Lemma 3.5, \( p(v^*(R), R) \cdot q^*(R) \) is a valid lower bound of the expected revenue in the sample-bid mechanism for an agent with revenue curve \( R \). In this step,
instead of analyzing the expected revenue, we argue that we can convert any revenue curve $R_3$ (constructed in step 2) into another revenue curve $R_4$, such that (i) $v^*(R_4) = v^*(R_3)$; (ii) $q^*(R_4) \leq q^*(R_3)$; and (iii) $p(v^*, R_4) \leq p(v^*, R_3)$. Finally, by showing that $p(v^*(R_4), R_4) \cdot q^*(R_4) \geq 0.545$, we finish the proof of the lemma.

Given the revenue curve $R_3$ constructed in step 2, for any $r_0 \in [0, 1]$, we define a revenue curve $R_4^{(v)}$ as follows,

$$R_4^{(v)}(r_0) \triangleq \begin{cases} \begin{array}{l} r_0 + (1 - r_0) \frac{q}{q_m(R_3)} & q \in [0, q_m(R_3)] , \\ 1 & q \in [q_m(R_3), 1] . \end{array} \end{cases}$$

See the black curves in Figure 6 as an example. We claim that there exists $r_0^* \in [0, 1]$ s.t. $R_4^{(v)}(r_0^*)$ satisfies properties (i) (ii) (iii) mentioned above. To see this, consider the argument as follows.

By construction, for every value $v$, every bid $b$, the utility $u(b, v, R_4^{(v)})$ is decreasing continuously in $r_0$. Thus, $v^*(R_4^{(v)})$ is decreasing continuously in $r_0$. Let $b^*$ be the optimal bid of an agent with value $v^*(R_3)$ and revenue curve $R_3$. Denote $q(b^*, R_3) b^2$. Consider revenue curve $R_4^{(v)}$ where $R_4 \triangleq 1 - \frac{q_m(R_3)}{q_m(R_3) - q}(1 - R_3(q^*))$.

By construction, $R_4^{(v)}(q) \geq R_3(q)$ for all $q \leq q^*$, and $R_4^{(v)}(q) \leq R_3(q)$ for all $q \geq q^*$. See Figure 6 for a graphical illustration. Note that by construction,

$$u(v^*(R_3), b^*, R_4^{(v)}(q)) = v^*(R_3) \cdot (1 - q^*) - ab^* \cdot q^* - \alpha \int_{q^*}^{1} \frac{R_4^{(v)}(q)}{q} dq$$

$$\geq v^*(R_3) \cdot (1 - q^*) - ab^* \cdot q^* - \alpha \int_{q^*}^{1} R_3(q) dq$$

$$= u(v^*(R_3), b^*, R_3) = 0$$

Thus, $v^*(R_4^{(v)}) \leq v^*(R_3)$. Next, consider revenue curve $R_4^{(v)}$, where $R_4 \triangleq 1 - \frac{q_m(R_3)}{q_m(R_3) - q}(1 - R_3(q^*))$. By construction, $R_4^{(v)}(q) \geq R_3(q)$ for all $q \in [0, 1]$. See Figure 6 for a graphical illustration. Thus, $v^*(R_4^{(v)}) \geq v^*(R_3)$ with the same argument for $R_4^{(v)}$. Therefore, we know that there exists $r_0^* \in [0, 1]$ such that $v^*(R_4^{(r_0^*)}) = v^*(R_3)$.

Denote $R_4^{(v)}$ by $R_4$ and show that $R_4$ satisfies properties (ii) $q^*(R_4) \leq q^*(R_3)$ and (iii) $p(v^*, R_4) \leq p(v^*, R_3)$ with the argument below.

Lemma 5.3 implies that $b^* > v^*(R_3)$. Combining with the fact that $r_0^* \geq 0$, we know that property (ii) ($q^*(R_4) \leq q^*(R_3)$) is satisfied. See Figure 7 for a graphical illustration.

Combining the first order condition in Lemma 3.2 and construction of $R_4$, it is guaranteed that the optimal bid $b^*$ of value $v^*$ for revenue curve $R_4$ is at most $b^2$. Furthermore, $q(b^*, R_4) \geq q(b^*, R_3)$ by construction. By the definition, the optimal utility of value $v^*(R)$ for any revenue curve $R$ is zero. Thus, $p(v^*, R_4) = v^* \cdot (1 - q^*) = v^* \cdot (1 - q^*(b^*, R_3)) = p(v^*, R_3)$.

Step 4: lower-bounding the expected revenue on $R_4$. So far, we have shown that for an arbitrary revenue curve satisfying the assumptions in lemma statement, its expected revenue in the sample-bid mechanism is lower-bounded by $p(v^*(R_4), R_4) \cdot q^*(R_4)$ for $R_4$ pinned down by some $(r_0, q_m)$ as follows,

$$R_4 \triangleq \begin{cases} \begin{array}{l} r_0 + (1 - r_0) \frac{q}{q_m(R_3)} & q \in [0, q_m(R_3)] , \\ 1 & q \in [q_m(R_3), 1] . \end{array} \end{cases}$$

By numerically verifying $p(v^*(R_4), R_4) \cdot q^*(R_4) \geq 0.545$ for all $(r_0, q_m) \in [0, 1]^2$, we finish the proof. The details of this numerical evaluation is elaborated on in the online appendix.

5.3 Regular Distributions with Monopoly Quantile $q_m \leq 0.62$

In this subsection, we analyze the prior-independent approximation ratio of the sample-bid mechanism over the class of regular distributions with monopoly quantile $q_m \leq 0.62$.

Lemma 5.8. For the sample-bid mechanism with $\alpha = 0.7$, the prior-independent approximation ratio over the class of regular distributions with monopoly quantile $q_m \leq 0.62$ is at most 1.835.

Fix an arbitrary revenue curve $R$, let $v^*(R) = \inf \{ v : b(v, R) \geq v_m(R) \}$ be the smallest value whose optimal bid $b(v, R)$ for revenue curve $R$ is at least $v_m(R)$. Since Lemma 3.5 guarantees that $b(v, R)$ is weakly non-decreasing in $v$, $v^*(R)$ is well-defined, $b(v, R) \geq v_m(R)$ for all $v \geq v^*(R)$. Furthermore, by Lemma 5.3, we know that $b(v, R) \geq v_m(R)$ for all $v \geq \max\{v^*(R), v_m(R)\}$. Denote $g(v^*(R), R)$ by $q^*(R)$.

By Lemma 3.4 and Lemma 3.5, the expected revenue $Rev_{(SB)}$ of the sample-bid mechanism for revenue curve $R$ can be lower-bounded as follows,

$$Rev_{(SB)} = \int_{0}^{1} p(v(q), R, R) dq$$

$$= \int_{0}^{\min\{q^*(R), q_m(R)\}} p(v(q), R, R) dq$$

$$+ \int_{\min\{q^*(R), q_m(R)\}}^{q^*(R)} p(v(q), R, R) dq + \int_{q^*(R)}^{1} p(v(q), R, R) dq$$

$$\geq p(v_m(R), R) \cdot \min\{q^*(R), q_m(R)\}$$

$$+ p(v_m(R), R) \cdot \max\{0, q^*(R) - q_m(R)\} + \int_{q^*(R)}^{1} p(v(q), R, R) dq.$$ .

Denote $q(v_m(R), R)$ by $q^*(R)$, and $\int_{0}^{\min\{q_m(R), q(q)\}} p(v(q), R, R) dq$ by $w(R)$. In the online appendix, we will provide parametric lower-bound for the critical value $v^*(R)$, the quantile $q^*(R)$, and the expected payment $Rev_{(SB)}$. As the functions of $q_m(R)$, $q^*(R)$, $w(R)$, and $v^*(R)$, putting all pieces together, we show Lemma 5.8 by providing a lower bound of expected revenue in the sample-bid mechanism as a function of $q_m(R)$, $q^*(R)$, $w(R)$, and $v^*(R)$, numerically evaluating its value for all possible parameters. The details of the numerical evaluations in this section are similar to those of Lemma 5.7, which are elaborated on in the online appendix.

6 PRIOR-INDEPENDENT APPROXIMATION LOWER BOUND

In this section, we show that no mechanism can achieve prior-independent approximation better than 1.07 even when the class
of distributions are uniform distributions. Note that point mass distributions are special cases of the uniform distributions. The lower bound we will prove in this section holds for more general families of mechanisms than the single-round mechanisms that we introduced in Section 2. Here we will show that even when the agent and the seller have multiple rounds of communication in general messages spaces, no mechanism can achieve prior-independent approximation better than 1.07. However, since our analysis does not hinge on the exact format of the mechanism, we will not formally introduce the model for multi-rounds of communication.

**Theorem 6.1.** For a single item, a uniformly distributed agent, and a single valuation sample, the prior-independent approximation ratio for revenue maximization is at least 1.07.

The main idea for proving Theorem 6.1 is as follows. Consider two scenarios where the valuation distribution for the agent is actually a pointmass

\[ v \in [1, 2] \]  

Note that the optimal mechanism for an agent with value from the uniform distribution \( U[1, 2] \) is to allocate the item with expected payment 1. Thus if the mechanism is optimal for this setting, when the valuation distribution for the agent is actually a pointmass with some value \( v \in [1, 2] \), the agent can always imitate the type in a uniform distribution \( U[1, 2] \) to win the item and pay at most 1 in expectation. This indicates that the optimal prior-independent approximation ratio is strictly above 1. By leveraging the approximation ratio in those two cases, we show that the optimal ratio is at least 1.07.

Before the proof of Theorem 6.1, we first introduce several notations and present several properties for non-truthful mechanisms \( M \) with prior-independent approximation ratio \( \beta \).

**Lemma 6.2.** For single item, single agent, any distribution \( F \) with support \( [v, \bar{v}] \), for non-truthful mechanism with prior-independent approximation ratio \( \beta \), the interim allocation for agent with highest value \( \bar{v} \) is \( x(\bar{v}, F) \geq \frac{1}{\beta} \).

Proof. Suppose the interim allocation for agent with value \( \bar{v} \) is \( x(\bar{v}, F) < \frac{1}{\beta} \). Since the interim allocation is monotone, the maximum expected virtual welfare for mechanism under distribution \( F \) is less than \( \frac{1}{\beta} \) of the optimal expected virtual welfare, which implies the revenue is less than \( \frac{1}{\beta} \) of the optimal revenue and the approximation ratio for distribution \( F \) is higher than \( \beta \), a contradiction.

**Lemma 6.3.** For single item, single agent, and any uniform distribution \( F \) with support \( [v, \bar{v}] \) such that \( 2\bar{v} \geq \bar{v} \), for a non-truthful mechanism with prior-independent approximation ratio \( \beta \), the interim utility for agent with highest value \( \bar{v} \) is \( u(\bar{v}, F) \geq \frac{1}{2} \left( \bar{v} - \sqrt{\bar{v}^2 - \frac{4\bar{v}}{\beta} (\bar{v} - \bar{v})} \right) \).

Proof. For uniform distribution \( F \) with support \( [v, \bar{v}] \) such that \( 2\bar{v} \geq \bar{v} \), the optimal mechanism \( \text{OPT}_F \) is to post price \( \bar{v} \) with expected revenue \( \bar{v} \). Suppose the utility for agent with value \( \bar{v} \) is \( u(\bar{v}, F) < \frac{1}{2} \left( \bar{v} - \sqrt{\bar{v}^2 - \frac{4\bar{v}}{\beta} (\bar{v} - \bar{v})} \right) \), the optimal mechanism subject to this constraint is to post price \( \bar{v} - u(\bar{v}, F) \), with expected revenue \( \frac{u(\bar{v}, F)}{\bar{v}} \cdot (\bar{v} - u(\bar{v}, F)) < \frac{1}{\beta} \), a contradiction.

**Lemma 6.4.** For single item, single agent, any point mass distribution \( F \) with support \( \bar{v} \), for non-truthful mechanism with prior-independent approximation ratio \( \beta \), the interim utility for agent with value \( \bar{v} \) is \( u(\bar{v}, F) \leq \bar{v}(1 - 1/\beta) \).

Proof. Suppose the interim utility in this case is \( u(\bar{v}, F) > \bar{v}(1 - 1/\beta) \), the expected revenue is at most the social welfare minus the expected utility, which is at most \( \bar{v} - u(\bar{v}, F) < \frac{\bar{v}}{\beta} \), contradicting the fact that mechanism \( M \) achieves prior-independent approximation ratio \( \beta \).

Proof of Theorem 6.1. Suppose mechanism \( M \) inducing interim allocation and payment rule \( x \) and \( p \) achieves prior-independent approximation ratio \( \beta \). Consider uniform distribution \( F \) with support \( [1, 2] \). By Lemma 6.2 and 6.3, we have \( x(2, F) \geq \frac{1}{\beta} \) and \( u(2, F) \geq 1 - \sqrt{1 - 1/\beta} \). For any sample \( s \in [1, 2] \), the expected allocation and payment of agent with value 2 given the sample \( s \) satisfies the constraint that

\[ s \cdot x(2, F, s) - p(2, F, s) \leq s \left( 1 - \frac{1}{\beta} \right) \]  

otherwise for distribution \( F_s \) with point mass on \( s \), an agent with value \( s \) can imitate the behavior of an agent with value 2 in uniform distribution to achieve utility strictly higher than \( s (1 - 1/\beta) \), and by Lemma 6.4, this contradicts to the assumption that mechanism \( M \) achieves prior-independent approximation ratio \( \beta \). Taking expectation over sample \( s \) for the left hand side of equation (2), we have

\[ E_s \left[ s \cdot x(2, F, s) - p(2, F, s) \right] \geq E_s \left[ s \cdot x(2, F, s) \right] - (2 - u(2, F)) \]  

\[ \geq \int_1^{1 + 1/\beta} s ds - (2 - u(2, F)) \]  

where the last inequality holds because \( x(2, F) \geq \frac{1}{\beta} \) and the worst case happens when \( x(2, F, s) = 0 \) for any sample \( s \geq 1 + 1/\beta \). Taking expectation over sample \( s \) for the right hand side of equation (2), we have

\[ E_s \left[ s \left( 1 - \frac{1}{\beta} \right) \right] = \frac{3}{2} \left( 1 - \frac{1}{\beta} \right) \]  

Combining the inequalities, we have

\[ \frac{1}{2} \left( 1 + \frac{1}{\beta} \right)^2 - \frac{1}{2} - (1 + \sqrt{1 - 1/\beta}) \leq \frac{3}{2} \left( 1 - \frac{1}{\beta} \right) \]  

By solving the inequality, we have \( \beta \geq 1.0737 \).

**7 REVELATION GAP**

Feng and Hartline [29] proposed the revelation gap to quantify the difference between the worst case performance of the optimal truthful mechanism and the optimal non-truthful mechanism in prior-independent mechanism design. They showed that a non-trivial revelation gap exists for the welfare maximization problem
for agents with budgets. In this section, we show that a revelation gap also exists for the revenue maximization problem when considering the single-item single-agent setting with single-sample access.

Let $\mathcal{M}$ be the family of truthful mechanisms, the family of mechanisms such that the agent maximizes her utility by truthfully revealing her valuation to the seller, i.e., $b^*(v, F) = v$ for all value $v$ and valuation distributions $F$. Let $\mathcal{M}$ be the family of all mechanisms. We define

$$\beta_{\text{MECHS}, \text{DISTS}}(\mathcal{M}, \text{DISTS}) \triangleq \min_{\mathcal{M} \in \text{MECHS}} \Gamma(\mathcal{M}, \text{DISTS})$$

as the optimal prior-independent approximation ratio among the family of mechanisms MECHS. The revelation gap for a family of distributions DISTS is then defined as the ratio

$$\beta_{\text{MECHS}, \text{DISTS}}(\mathcal{M}, \text{DISTS})$$

Definition 7.1. A mechanism is scale-invariant if the interim allocation is invariant of the scale, i.e., $x(\alpha v, \alpha F) = x(v, F)$ for any distribution $F$, valuation $v$ and any $\alpha > 0$.

Allouah and Besbes [4] characterized the prior-independent approximation ratio of the truthful mechanisms under the assumption of scale-invariance for sample-based pricing mechanisms. Note that in contrast, our lower bound result shown in Theorem 6.1 does not require the assumption on scale-invariance. Here is the formal definition of sample-based pricing mechanisms.

Definition 7.2. Given function $\alpha : \mathbb{R} \rightarrow \Delta(\mathbb{R})$ mapping from the sample to the randomized price, for sample $s$, the sample-based pricing mechanism solicits a non-negative bid $b \geq 0$, allocates the item to the agent if $b \geq \alpha(s)$, and charges the agent $\alpha(s) \cdot 1(b \geq \alpha(s))$.

It can be observed that the bid allocation rules of both sample-bid mechanism and sample-based pricing are similar (i.e. competing against the sample), and the difference is the payment semantics.

**Theorem 7.1 (4).** Under the assumption of scale-invariance, for single-item setting with regular valuation distribution, when seller has access to a single sample, the prior-independent approximation ratio of the optimal sample-based pricing mechanism is bounded in $(1.957, 1.996)$. Moreover, when the valuation distribution is MHR, the prior-independent approximation ratio is bounded in $(1.543, 1.575)$.

Given an arbitrary valuation distribution and any mechanism that is incentive compatible only for the given valuation distribution, the mechanism may not be equivalent to any sample-based pricing mechanism. The is because the agent only maximizes her utility by taking expectation over the sample. However, we can show that if the mechanism is incentive compatible for all possible prior distributions, then it is equivalent to consider posting a randomized price to the agent based on the realization of the sample, i.e., a sample-based pricing mechanism.

**Lemma 7.2.** For any mechanism with allocation $\tilde{x}$ and payment $\tilde{p}$ that is incentive compatible and individual rational for all valuation distributions, there exists a sample-based pricing mechanism that generates the same expected allocation and payment pointwise for any valuation of the agent and any realization of the sample.

**Proof.** First we claim that, for any truthful mechanism with allocation $\tilde{x}$ and payment $\tilde{p}$, the induced allocation rule $\tilde{x}(\cdot, s)$ and payment rule $\tilde{p}(\cdot, s)$ are incentive compatible and individual rational given any realization of the sample $s$.

First we prove the incentive compatibility. Suppose by contradiction, there exists constant $\epsilon > 0$, sample $s$ and value $v, v'$ such that

$$u(\tilde{x}(v'), s) - \tilde{x}(v, s) \geq u(\tilde{x}(v), s) - \tilde{x}(v, s) + \epsilon.$$  

Let $F$ be an arbitrary distribution with positive density everywhere on the support $[0, \infty)$. Define $H$ as $u(v, v', F) - u(v, v', F')$ as the utility loss for value $v$ to misreport $v'$ when the distribution is $F$. Given constant $\delta > 0$, let $F'$ be the distribution such that with probability $1 - \delta$, the value of the agent is $s$ and with probability $\delta$, the value is drawn from distribution $F$. It is easy to verify that both $v$ and $v'$ are in the support of distribution $F'$. Moreover, the utility loss for misreporting $v'$ is

$$u(v, v, F) - u(v, v', F') \geq (1 - \delta)\epsilon + \delta H$$

where $(1 - \delta)\epsilon + \delta H > 0$ for sufficiently small $\delta$. This implies that the mechanism is not incentive compatible for distribution $F'$, a contradiction.

Similarly, for individual rationality, if there exists constant $\epsilon > 0$, sample $s$ and value $v, v'$ such that

$$u(\tilde{x}(v), s) - \tilde{p}(\tilde{x}, v, s) \leq -\epsilon,$$

there exists a distribution $F'$ supported on $[0, \infty)$ such that agent with value $v$ is not individual rational given distribution $F'$.

Finally, since for any sample $s$, the induced mechanism is incentive compatible, the allocation $\tilde{x}(v, s)$ is monotone in $v$ for any sample $s$. Moreover, individual rationality implies that the payment of the agent is 0 if she does not win the item. Thus the mechanism can be implemented as sample-based pricing mechanism for any realized sample. \qed

Lemma 7.2 suggests that under the assumption of scale invariance, the bounds on prior-independent approximation ratio of sample-based pricing in Theorem 7.1 carry over to truthful mechanisms. Then combining it with Theorem 5.1 and 6.1, we have the following corollary.

**Corollary 7.3.** Under the assumption of scale-invariance, for single-item setting with regular valuation distribution, when seller has access to a single sample, the revelation gap is bounded in $(1.066, 1.859)$. Moreover, when the valuation distribution is MHR, the revelation gap is bounded in $(1.190, 1.467)$.

**ACKNOWLEDGMENTS**

This work is funded by NSF 1618502. The full version of the paper is available at [https://arxiv.org/abs/2102.13496](https://arxiv.org/abs/2102.13496).

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