Input-to-State Stability Analysis of Subhomogeneous Cooperative Systems

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This work was supported in part by the National Natural Science Foundation of China under Grant 11671227 and Grant 61374074, and in part by the Science and Technology Project of the University of Jinan under Grant XKY1701.

ABSTRACT This paper analyzes the input-to-state stability (ISS) properties of subhomogeneous cooperative systems for the first time. A new definition of finite-time ISS (FISS) for nonlinear positive systems is given. For stability analysis, the max-separable Lyapunov function is used, and the min-separable Lyapunov-like function is presented for the first time in this paper. Using these functions, we obtain our main results on solution estimates for subhomogeneous positive systems. According to the proposed method of analysis, the ISS properties of these systems can be easily judged only on the basis of their differential and algebraic characteristics. Simulation examples verify the validity of our results.

INDEX TERMS Subhomogeneous cooperative system, positive system, finite-time input-to-state stability, max-separable Lyapunov function, min-separable Lyapunov-like function.

I. INTRODUCTION

Positive systems are dynamical systems whose input, output and state variables are constrained to be nonnegative at all times whenever the initial conditions are nonnegative [1]. Systems of this kind can be seen in many real-world processes in areas such as economics, biology, ecology and communications. Due to their importance and wide applicability, the analysis and control of positive systems have attracted considerable attention from the control community (see, e.g., [2]–[9] and references therein).

Since the performance of a real control system is affected to some extent by uncertainties such as unmodeled dynamics, parameter perturbations, exogenous disturbances, and measurement errors, research on the robustness of control systems always plays a vital role in the development of control theory and technology. Aiming at robustness analyses of nonlinear control systems, Sontag developed the concept of input-to-state stability (ISS) and various extensions thereof, such as integral ISS (iISS) and finite-time ISS (FISS) ([10]–[13]). This concept has been applied to continuous-time dynamical systems [14], discrete-time systems [15], switched systems [16], [17], stochastic systems [17], [18], Markovian jump systems [18], time-delayed systems [19], sampled-data nonlinear systems [20], networked control systems [21], etc.

In [22], the ISS properties of a positive cooperative system were used as a basis for comparison to judge the ISS properties of general nonlinear systems, in which the ISS definition used for positive systems was Sontag’s ISS. In [23], for better consistency with the positivity property of a positive system, new ISS definitions and corresponding criteria were provided for homogeneous cooperative systems. Note that subhomogeneous cooperative systems [24] are also a class of positive systems that has attracted increasing attention from scholars. Such a system can include terms such as $\frac{a_1}{a_2+a_3}$ for $a > 0$ and $\tau > 0$, which arise frequently in models of biochemical reaction networks [25]. However, for subhomogeneous cooperative systems, research has focused mainly on stability [24], [26], [27], such as the robust stability of delay [28]. Subhomogeneous cooperative systems that are disturbed by exogenous input have not been studied in the framework of ISS theory.

In this paper, the ISS properties of two classes of subhomogeneous cooperative systems, which have a unique equilibrium at the origin or a unique strictly nonzero equilibrium, will be analyzed for the first time. First, new definitions of ISS and FISS for nonlinear positive systems are introduced. As useful tools for stability analysis, the max-separable Lyapunov function ([29], [30]) is used to estimate the upper bound on the solutions, and the min-separable Lyapunov-like function is newly introduced to estimate the lower bound on the solutions. Using these functions, we obtain our main
results on the solution estimates for these two classes of subhomogeneous positive systems. According to our results, the ISS properties of these systems can be judged on the basis of only their differential and algebraic characteristics. Simulation examples verify the validity of our results.

The remainder of this paper is organized as follows: Section 2 provides some notations and introduces some preliminary results on the systems of interest, relevant definitions, and some useful tools. Section 3 gives the solution estimates for the two classes of subhomogeneous cooperative systems classified by their equilibrium positions. In section 4, some simulated examples are presented to illustrate our results. Section 5 offers some concluding remarks.

Notation: Throughout this paper, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of all nonnegative real numbers, the natural numbers and the natural numbers including zero, respectively. $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ denote $n$-dimensional real space and $n \times m$-dimensional real matrix space, respectively. In particular, a matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a *Metzler* matrix if and only if its off-diagonal entries $a_{ij}$, $\forall \, i \neq j$, are nonnegative [1]. The transpose of a vector or matrix is denoted by a superscript $T$. For matrices $A, B \in \mathbb{R}^{n \times m}$, we write that $A \geq B$ ($A \leq B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} \leq b_{ij}$) for $i \in \mathbb{T}_n = \{1, 2, \ldots, n\}$ and $j \in \mathbb{T}_m$; $A > B$ ($A < B$) if $A \geq B$ ($A \leq B$) and $A \neq B$; and $A > B$ ($A < B$) if $a_{ij} > b_{ij}$ ($a_{ij} < b_{ij}$) for $i \in \mathbb{T}_n$ and $j \in \mathbb{T}_m$.

The positive orthant $\mathbb{R}_{+}^n$ in $\mathbb{R}^n$ is the set $\{x | x_i > 0, \forall \, i \}$ of positive $n$-dimensional vectors; $\mathbb{R}_{+}^{n \times m}$ denotes a vector whose components are all one. The $p$-norm on $\mathbb{R}^n$ is denoted by $\| \cdot \|_p$, where $p$ is usually omitted in the case that $p = 2$. The max-norm is denoted by $\| \cdot \|_{\infty}$. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. Given a vector $v \in \mathbb{R}^n$, $v > 0$, the weighted $L_\infty$-norm is defined as $\|x\|_{L_\infty} = \max_{1 \leq i \leq n} |x_i| / v_i$. All vectors are column vectors unless otherwise specified. $C^1$ denotes all $i$-times continuously differentiable functions; $C^{1,k}$ denotes all functions with an $i$-times continuously differentiable first component and a $k$-times continuously differentiable second component. Finally, we denote the composition of two functions $\psi : A \rightarrow B$ and $\psi : B \rightarrow C$ by $\psi \circ \phi : A \rightarrow C$.

The function classes $\mathcal{K}$ and $\mathcal{K}_\infty$ considered for comparison are the sets of all continuous functions $\{\gamma : \mathbb{R}_+^n, \gamma(0) = 0, \gamma$ is strictly increasing $\}$ and $\{\gamma \in \mathcal{K} : \gamma$ is unbounded $\}$, respectively. The class of continuous positive-definite functions $\alpha : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is denoted by $\mathcal{PD}$. A function $\beta : \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is of class $\mathcal{K}\mathcal{L}$ if for fixed $t \geq 0$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for fixed $s \geq 0$, the function $\beta(s, \cdot)$ is nonincreasing with $\lim_{t \rightarrow \infty} \beta(s, t) = 0$.

A function $h : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is said to belong to the generalized $\mathcal{K}$ ($G\mathcal{K}$) class if it is continuous with $h(0) = 0$ and satisfies

$$
\begin{cases}
    h(r_1) > h(r_2), & \text{if } h(r_1) \neq 0; \\
    h(r_1) = h(r_2) = 0, & \text{if } h(r_1) = 0; \\
    \forall \, r_1 > r_2.
\end{cases}
$$

A function $\beta : \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ belongs to the generalized $\mathcal{K}\mathcal{L}$ function ($G\mathcal{K}\mathcal{L}$ function) class if for each fixed $t \geq 0$, the function $\beta(s, t)$ is a generalized $\mathcal{K}$ function, and for each fixed $s \geq 0$, it decreases to zero as $t \rightarrow T$ for some $T \leq \infty$.

**II. PRELIMINARY RESULTS**

Consider the following continuous-time $n$-dimensional nonlinear system:

$$
\dot{x} = f(x, u), \quad t \geq t_0, \tag{1}
$$

where $x \in \mathbb{R}_+^n$ and $u \in \mathcal{L}_\infty^m$ are the system state and input, respectively; $\mathcal{L}_\infty^m$ denotes the set of all measurable and locally essentially bounded inputs $u \in \mathbb{R}_+^m$ on $[t_0, \infty)$ with norm

$$
\|u\| = \text{ess sup}\{\|u(t)\|_{\infty}\}; \tag{2}
$$

$f : \mathbb{R}_+^n \times \mathcal{L}_\infty^m \rightarrow \mathbb{R}^n$ is continuously differentiable on $(x, u)$ and satisfies $f(0, 0) \equiv 0$; and the initial data are $x_0 = (x_{01}, \ldots, x_{0n})^T \in \mathbb{R}_+^n$.

We use $\mathcal{U}_\infty$ to denote the set of all inputs. Accordingly, given two $u_1, u_2 \in \mathcal{U}_\infty$, we write $u_1 \geq u_2$ if $u_1(t) \geq u_2(t)$ for all $t \geq t_0$. (To be more precise, this and other definitions should be interpreted in an "almost everywhere" sense, since the inputs are Lebesgue-measurable functions.) We interpret $x(t_0, x_0, u)$ as the state at time $t$ with the initial data $x_0$ and the external input $u(\cdot)$. Sometimes, for cases that are independent of context, we write $x(t)$ instead of $x(t, t_0, x_0, u)$.

For systems of the form given in (1), we introduce the following definitions and propositions.

**Definition 1** [32]: System (1) is called *monotone* if the implication

$$
u_1 \geq u_2, x_1 \geq x_2 \Rightarrow x(t, t_0, x_1, u_1) \geq x(t, t_0, x_2, u_2)
$$

holds for all $t \geq t_0$.

This definition states that trajectories of monotone systems starting from ordered initial conditions preserve the same ordering for all inputs $u(\cdot) \in \mathcal{U}_\infty$ and $t \geq t_0$. By choosing $u_2 = 0$ and $x_2 = 0$, since $x(t, t_0, 0, 0) = 0$ for all $t \geq t_0$, it is easy to see that

$$
x_1 \in \mathbb{R}_+^n, u_1 \geq 0 \Rightarrow x(t, t_0, x_1, u_1) \in \mathbb{R}_+^n, \forall \, t \geq t_0.
$$

This shows that the positive orthant $\mathbb{R}_+^n$ is an invariant set for a monotone system of the form given in (1). Thus, monotone systems with an equilibrium point at the origin are defined to be *positive systems*.

**Proposition 2**: [32] System (1) is monotone if and only if, for all $\xi_1, \xi_2 \in V := \text{int} \mathbb{R}_+^n$, the interior of $\mathbb{R}_+^n$, $u_1, u_2 \in \mathcal{W} := \text{int} \mathcal{U}$:

$$
\xi_1 \geq \xi_2, u_1 \geq u_2 \Rightarrow f(\xi_1, u_1) - f(\xi_2, u_2) \geq 0. \tag{3}
$$

Note that if $f$ is continuously differentiable on $\mathbb{R}_+^n \times \mathcal{L}_\infty^m$, then condition (3) is equivalent to the requirement that, for all $i, j \in \mathbb{T}_n$,

$$
\frac{\partial f_i}{\partial x_j}(x, u) \geq 0, \quad \forall \, x \in \mathcal{V}, \forall \, u \in \mathcal{W}, \forall \, i \neq j. \tag{4}
$$
\[
\frac{\partial f_i}{\partial u_j}(x, u) \geq 0, \quad \forall x \in \mathbb{R}^n_+, \forall u \in \mathcal{V}.
\] (5)

If (4) and (5) hold, system (1) is called cooperative.

We also introduce the following new definition of FISS and corresponding research tools (the max-separable Lyapunov function and the min-separable Lyapunov-like function), which are needed for ISS analysis.

**Definition 3:** A nonlinear positive system of the form given in (1) is said to possess finite-time input-to-state stability (FISS) if there exist functions \( \beta \in \mathcal{KL} \) and \( \gamma \in \mathcal{K} \) such that

\[
x(t) \leq \beta(\max_{i \in I_n} x_{yi}), t - t_0, t \leq \gamma(\|u\|)l_n
\] (6)

and the settling time ([31]) satisfies \( T = T(t_0, x_0) < +\infty \).

**Remark 4:** If the first function satisfies \( \beta \in \mathcal{KL} \) in the above definition, then the nonlinear positive system (1) is said to possess input-to-state stability (ISS). Under the preconditions of a nonlinear positive system and initial data \( x_0 \geq 0 \), (6) is equivalent to any of the following three inequalities:

\[
\|x(t)\|_\infty \leq \beta(\|x_0\|_\infty, t - t_0) + \gamma(\|u\|),
\]
\[
\|x(t)\|_1 \leq \eta\beta(\|x_0\|_1, t - t_0) + n\gamma(\|u\|),
\]
\[
\|x(t)\|_2 \leq \sqrt{n}\beta(\|x_0\|_2, t - t_0) + \sqrt{n}\gamma(\|u\|).
\]

From the properties of \( \mathcal{KL}(\mathcal{KL}) \) functions [33], the nonlinear positive system (1) also possesses ISS (FISS) in the senses of the \( l_\infty \), \( l_1 \) and \( l_2 \)-norms. Therefore, our new ISS (FISS) definition is coincident with the usual ISS (FISS) definition for general systems.

**Remark 5:** With input \( u \equiv 0 \), by the state estimate given in (6), the zero equilibrium of system (1) is finite-time asymptotically stable. In fact, due to the positiveness of the system, the solution to system (1) will only asymptotically tend toward zero in finite time in \( \mathbb{R}^n_+ \). Therefore, we can also say that it possesses "upper" finite-time stability because the solution is above the zero equilibrium. Conversely, if the solution is below the zero equilibrium, we can say that the system possesses "lower" finite-time stability. In the study of positive systems, this terminology is especially useful for systems with nonzero equilibria. Based on this idea, the terminology of upper ISS (FISS) and lower finite-time stability will be used directly without corresponding definitions being explicitly given.

**Definition 6** [29]: For system (1), the following function \( V(x) \) is called the max-separable Lyapunov function:

\[
V(x) = \max_{i \in I_n} V_i(x_i),
\] (7)

with scalar functions \( V_i : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \).

Since the Lyapunov function in (7) is not necessarily continuously differentiable, we consider its upper-right Dini derivative along the solutions of system (1), as follows:

\[
D^+ V(x) = \lim_{h \to 0^+} \frac{V(x + hf(x, u)) - V(x)}{h}.
\] (8)

The following result shows that if the functions \( V_i \) in (7) are continuously differentiable, then (8) admits an explicit expression.

**Proposition 7** [34]: Consider \( V : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \) in (7), and let \( V_i : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \) be continuously differentiable for all \( i \in I_n \). Then, the **upper-right Dini derivative** (8) is given by

\[
D^+ V(x) = \max_{j \in \mathcal{J}(x)} \left\{ \frac{\partial V_j}{\partial x_j}(x)f_j(x, u) \right\},
\] (9)

where \( \mathcal{J}(x) \) is the set of indices for which the maximum in (7) is attained, i.e.,

\[
\mathcal{J}(x) = \{ j \in I_n | V_j(x_j) = V(x) \}.
\]

Similarly, if we define the min-separable Lyapunov-like function as

\[
V(x) = \min_{i \in I_n} V_i(x_i),
\] (10)

its lower-right Dini derivative along the solutions of system (1) can be defined as

\[
D^- V(x) = \lim_{h \to 0^+} \frac{V(x + hf(x, u)) - V(x)}{h}.
\]

If the functions \( V_i \) in (10) are continuously differentiable, then

\[
D^- V(x) = \min_{j \in \mathcal{J}(x)} \left\{ \frac{\partial V_j}{\partial x_j}(x)f_j(x, u) \right\},
\]

where \( \mathcal{J}(x) \) is the set of indices for which the minimum in (10) is attained, i.e.,

\[
\mathcal{J}(x) = \{ j \in I_n | V_j(x_j) = V(x) \}.
\]

**III. Stabilité Analysis of Subhomogeneous Positive Systems with Positive Disturbances**

Consider subhomogeneous systems with positive disturbances of the following form:

\[
\dot{x} = F(x, w) = f(x) + g(x)w,
\] (11)

where \( x \in \mathbb{R}_+^n \) is the system state and \( w : [0, +\infty) \rightarrow \mathbb{R}_+^m \) is the disturbance, which belongs to \( \mathcal{W} \).

Our aim in this paper is to analyze the ISS properties of a subhomogeneous cooperative system of the form given in (11) that is in affine form. To this end, the following definitions and proposition are needed to describe the system functions \( f \) and \( g \).

**Definition 8** [35]: A continuous vector field \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n \), which is \( C^1 \) on \( \mathbb{R}_+^n \setminus \{0\} \), is said to be cooperative if the Jacobian matrix \( \partial f / \partial x(a) \) is Metzler for all \( a \in \mathbb{R}_+^n \setminus \{0\} \). Cooperative vector fields exhibit the following property.

**Proposition 9** [36]: Let the vector field \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n \) be cooperative. For any two vectors \( x, y \in \mathbb{R}_+^n \), with \( x_i \geq y_i \), and \( x \geq y \), we have \( f_i(x) \geq f_i(y) \).

Given an n-tuple \( r = (r_1, \ldots, r_n) \) of positive real numbers and \( \lambda > 0 \), the dilation map \( \delta_{r, \lambda}^\lambda(x) : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n \) is given by

\[
\delta_{r, \lambda}^\lambda(x) = (\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n).
\]

**Definition 10** [3]: For an \( \alpha > 0 \), the vector field \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n \) is said to be homogeneous of degree \( \alpha \) with respect
to $\delta^r_\lambda(x)$ if for all $x \in \mathbb{R}^n$ and all real $\lambda \geq 0$, $f(\delta^r_\lambda(x)) = \lambda^r \delta^r_\lambda(f(x))$.

If $r = (1, \cdots, 1)$, then $\delta^r_\lambda(x)$ is the standard dilation map. When $\alpha = 1$, $f$ is said to be homogeneous of degree one.

Definition 11 [24]: $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be subhomogeneous of degree $\tau > 0$ if $f(\lambda x) = \lambda^\tau f(x)$, for all $\lambda \in \mathbb{R}_+$, where $\lambda \in \mathbb{R}$ with $\lambda \geq 1$.

Definition 12 [5]: $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ is order-preserving on $\mathbb{R}^n_+$ if $g(x) \geq g(y)$ for any $x, y \in \mathbb{R}^n_+$ such that $x \succeq y$.

Concretely, the ISS properties will be considered in this section for two cases: system (11) without a positive disturbance and with a unique equilibrium at the origin or a unique equilibrium at $p > 0$ in int $\mathbb{R}^n_+$.

A. EQUILIBRIUM AT THE ORIGIN

First, we will show the positiveness of system (11).

Theorem 13: If $f$ is cooperative on $\mathbb{R}^n \setminus \{0\}$ with $f(0) = 0$ and $g$ is nonnegative and order-preserving on $\mathbb{R}^n_+$, then system (11) is positive.

Proof: Due to the nonnegative and order-preserving properties of $g$, for all $\xi_1, \xi_2 \in \mathcal{V}$ and $u_1, u_2 \in \mathcal{W}$,

$$
\xi_1 \succeq \xi_2, \quad u_1 \succeq u_2 \Rightarrow g(\xi_1)u_1 - g(\xi_2)u_2 \geq 0.
$$

From the above inequality and the cooperative property of $f$, for all $\xi_1, \xi_2 \in \mathcal{V}$ and $u_1, u_2 \in \mathcal{W}$,

$$
\xi_1 \succeq \xi_2, \quad u_1 \succeq u_2 \Rightarrow F(\xi_1, u_1) - F(\xi_2, u_2) \geq 0.
$$

From Proposition 2, system (11) is monotone. Because $F(0, 0) = 0$, system (11) is positive.

Next, based on the following assumption, the stability analysis of a continuous-time subhomogeneous system of the form given in (11) with a positive disturbance input will be considered.

Assumption 1: i) $f$ is continuous and subhomogeneous of degree $\tau > 0$ on $\mathbb{R}^n$ and continuously differentiable and cooperative on $\mathbb{R}^n \setminus \{0\}$, and $g$ is nonnegative and order-preserving on $\mathbb{R}^n_+$ and bounded by $M > 0$. ii) $f(0) = 0$. iii) There exists a vector $v = (v_1, \cdots, v_n)^T > 0$ such that $f(v) < 0$.

Based on Assumption 1, we give our first main result as follows.

Theorem 14: If Assumption 1 holds, then the continuous-time subhomogeneous system (11) possesses FISS if its degree is $\tau \in (0, 1)$. If its degree is $\tau \geq 1$ and $\|w\| \geq \frac{2M}{\min_{i \in \mathcal{A}_n} |[f(v_i)]|}$, then system (11) possesses ISS. If its degree is $\tau \geq 1$ and $\|w\| \geq \frac{\min_{i \in \mathcal{A}_n} |[f(v_i)]|}{2M}$, then system (11) possesses FISS.

Proof: (1) $\tau \in (0, 1)$: Let us define a max-separable Lyapunov function of the form

$$
V(x) = \max_{i \in \mathcal{A}_n} \frac{x_i}{v_i},
$$

where $v$ is a vector satisfying Assumption 1. Let $\sigma \in [0, \infty)$ denote the time at which the trajectory enters the set $\mathcal{B} = \{x \in \mathbb{R}^n_+ | V(x) < \chi(\|w\|)\}$ for the first time, where $\chi(\cdot) = \frac{2M}{\min_{i \in \mathcal{A}_n} |[f(v_i)]|} \times \sigma$. We will complete the proof by considering the following two cases: $x_0 \in B^c$ and $x_0 \in B \setminus \{0\}$.

Case 1. $x_0 \in B^c$. In this case, for any $t \in [t_0, \sigma]$, $V(x) \geq \chi(\|w\|)$. Let the subscript $m$ denote the index such that $V_m = V$. From (11) and the monotonicity of system (11), by $x \prec \|x\|_\infty v$,

$$
\frac{\dot{x}_m(t)}{v_m} \leq \frac{1}{v_m}(f_m(x(t)) + M\|w\|) \leq \frac{1}{v_m}(f_m(\|x\|_\infty v) + M\|w\|) \leq \frac{f_m(v)}{2v_m} \|x\|_\infty^2.
$$

Based on the relationship between $V(x)$ and 1, we divide the discussion into two further cases. If $V(x) \geq 1$, then by the subhomogeneous nature of the function $f$ and the definition of the set $B$, we have

$$
D^+ V(x) \leq \frac{f_m(v)}{2v_m} V(x).
$$

Due to the nonnegative property of $V(x)$, we take

$$
\tilde{\beta}_1(r, s) = \begin{cases} 0, & r = 0, s = 0; \\ 0, & r \neq 0, s \geq \frac{\tau^{-1} - \tau}{(\tau - 1)^2} \frac{f_m(v)}{2v_m}; \\ \tau^{-1} + (1 - \tau) \frac{f_m(v)}{2v_m}, & r \neq 0, s < \frac{\tau^{-1} - \tau}{(\tau - 1)^2} \frac{f_m(v)}{2v_m}; \\ \tau^{-1} \frac{f_m(v)}{2v_m}, & r < (1 - \tau) \frac{f_m(v)}{2v_m} \\
\end{cases}
$$

where $f_m(v) < 0$ and $\tilde{\beta}_1 \in \mathcal{GKL}$. From (13),

$$
V(t) \leq \tilde{\beta}_1(V_0, t - t_0).
$$

By taking $\tilde{\beta}_1(r, s) = \nu_{\max}^{\frac{\tau^{-1}}{\min_{i \in \mathcal{A}_n} v_i}} (1 - \min_{i \in \mathcal{A}_n} v_i)$, $\nu_{\max} = \max \{v_1, \cdots, v_n\}$ and $\nu_{\min} = \min \{v_1, \cdots, v_n\}$, from Lemma 4.2 in [33], it can be found that $\tilde{\beta}_1 \in \mathcal{GKL}$ and

$$
x(t) \leq \tilde{\beta}_1(\max_{i \in \mathcal{A}_n} [x_i], t - t_0) I_n.
$$

According to the definition of $\tilde{\beta}_1(r, s)$, there exists a settling time $T_1 \leq \max_{i \in \mathcal{A}_n} \left\{\frac{(\max_{i \in \mathcal{A}_n} v_i)^{-1} - 1}{\tau^{-1} \frac{f_m(v)}{2v_m}}\right\} \leq \infty$ such that $V(x)$ converges to 1 from $V_0$ when $t \rightarrow t_0 + T_1$.

If $V(x) < 1$, then from (12), due to the cooperative nature of $f$ and the definition of $B$, we have $f_m(\|x\|_\infty v) \leq f_m(v)$ and

$$
\frac{\dot{x}_m(t)}{v_m} \leq \frac{f_m(v)}{2v_m},
$$

i.e.,

$$
D^+ V(x) \leq \frac{f_m(v)}{2v_m}.
$$

(14)
Due to the nonnegative property of $V(x)$, we take

$$
\bar{\beta}_2(r, s) = \begin{cases} 
0, & r = 0, s = 0; \\
0, & r \neq 0, s \geq \frac{r}{f_m(v)}; \\
\frac{f_m(v)}{2v_m}r, & r \neq 0, s < \frac{r}{f_m(v)}. 
\end{cases}
$$

It is easy to see that $\bar{\beta}_2 \in GKL$. From (14),

$$V(x) \leq \beta_2(V_0, t - t_0).$$

Taking $\beta_2(r, s) = v_{\text{max}} \bar{\beta}_2(\frac{r}{f_m(v)}, s)$, by Lemma 4.2 in [33], we can obtain $\bar{\beta}_2 \in GKL$ and

$$x(t) \leq \beta_2(\max_{x \in \mathcal{I}_a}(x), t - t_0) \mathbf{1}_n. \quad (15)$$

According to the definition of $\bar{\beta}_2(r, s)$, there exists a settling time $T_2 \leq \max_{x \in \mathcal{I}_a}(\frac{2v_m}{\beta_2}) \leq \frac{1}{\beta_2} < \infty$ such that $V(x)$ goes to 0 from either $V_0$ or 1 when $t \to t_0 + T_1$. From the above, we can write

$$\beta = \begin{cases} 
\beta_1(\max_{x \in \mathcal{I}_a}(x), t - t_0), & t_0 \leq t \leq t_0 + T_1; \\
\beta_2(1, t - t_0 - T_1), & t > t_0 + T_1;
\end{cases}$$

when $V_0 > 1$. Similarly, we can write $\beta = \beta_2(\max_{x \in \mathcal{I}_a}(x), t - t_0)$ when $V_0 \leq 1$, such that for any $t \in [t_0, \sigma]$,

$$x(t) \leq \beta(\max_{x \in \mathcal{I}_a}(x), t - t_0) \mathbf{1}_n.$$

Let us now turn our attention to the interval $t \in (\sigma, \infty)$. Since $\frac{\text{d}x(t)}{\text{d}t} \leq Mv_{\text{max}} \|x(t)\|$, we have

$$x(t) \leq 2Mv_{\text{max}} \min_{x \in \mathcal{I}_a}(\|x(t)\|) \mathbf{1}_n, \quad (16)$$

Therefore, for any $t \in [t_0, \infty)$,

$$x(t) \leq \beta(\max_{x \in \mathcal{I}_a}(x), t - t_0) \mathbf{1}_n + 2Mv_{\text{max}} \min_{x \in \mathcal{I}_a}(\|f(x(t))\|) \mathbf{1}_n. \quad (17)$$

Case 2: $x_0 \in \mathcal{B}$. Following the proof of Case 1,

$$x(t) \leq 2Mv_{\text{max}} \min_{x \in \mathcal{I}_a}(\|f(x(t))\|) \mathbf{1}_n, \quad (18)$$

When $t = t_0$, by the definition of the set $\mathcal{B}$, we obtain

Then, by (16) and (17), for any $t \in [t_0, \infty)$,

$$x(t) \leq \beta(\max_{x \in \mathcal{I}_a}(x), t - t_0) \mathbf{1}_n + 2Mv_{\text{max}} \min_{x \in \mathcal{I}_a}(\|f(x(t))\|) \mathbf{1}_n. \quad (19)$$

Therefore, system (11) possesses FISS.

(2) $\tau \geq 1$ : When $x_0 \in \mathcal{B}$, (12) also holds. Based on (12), if $V(x) \geq 1$, then (13) also holds, and furthermore,

$$D^+ V(x) \leq \frac{f_m(v)}{2v_m} V'(x) \leq \frac{f_m(v)}{2v_m} V(x).$$

From this result, through simple calculation, we can obtain

$$V(x(t)) \leq V_0 e^{\frac{f_m(v)}{2v_m}(t-t_0)}$$

and

$$x(t) \leq \beta(\max_{x \in \mathcal{I}_a}(x), t - t_0) \mathbf{1}_n.$$

By incorporating the proof procedure for (1) Case 2, we can obtain

$$x(t) \leq \beta(\max_{x \in \mathcal{I}_a}(x), t - t_0) \mathbf{1}_n + \frac{2Mv_{\text{max}}}{\min_{x \in \mathcal{I}_a}(\|f(x(t))\|)} \mathbf{1}_n. \quad (19)$$

From the selected $\beta$ functions ($\beta_1 \in KKL$ and $\beta_2 \in GKL$), we can obtain that if $\|x(t)\| \geq 1$, i.e., $\|x(t)\| \geq \min_{x \in \mathcal{I}_a}(\|f(x(t))\|)$, then system (11) possesses ISS; if $\|x(t)\| < \frac{\min_{x \in \mathcal{I}_a}(\|f(x(t))\|)}{2M}$, then system (11) possesses FISS.

B. EQUILIBRIUM AT $p > 0$

We next derive state bounding estimates for subhomogeneous cooperative systems with positive disturbances. A subhomogeneous cooperative system without disturbance is assumed to have a unique equilibrium at $p > 0$ in $\text{int} \mathbb{R}_+^n$. To prove the main result of this section, for $p \in \text{int} \mathbb{R}_+^n$, we adopt the notations $R_1(p) = \{x : x > p\}$ and $R_2(p) = \{x \in \text{int} \mathbb{R}_+^n : x < p\}$. Assumption 2: i) $f$ is continuous and subhomogeneous of degree $\tau > 0$ on $\mathbb{R}^n$ and continuously differentiable and cooperative on $\mathbb{R}^n \setminus \{0\}$, and $g$ is nonnegative and order-preserving on $\mathbb{R}_+^n$ and bounded by $M$. ii) $\bar{x} = f(x)$ has a unique equilibrium at $p > 0$. iii) There exist a vector $v_1 + p \in R_1(p)$ such that $f(v_1 + p) < 0$ and a vector $v_2 \in R_2(p)$ such that $f(v_2) > 0$. Theorem 15: Under Assumption 2, the following two statements hold.

(1) The equilibrium $p$ of the continuous-time subhomogeneous system (11) possesses upper FISS if the degree is $\tau \in (0, 1)$; if the degree is $\tau \geq 1$ and $\|x(t)\| \geq \frac{\min_{x \in \mathcal{I}_a}(\|f(x(t))\|)}{2M}$, then system (11) possesses FISS.
then the equilibrium $p$ of system (11) possesses upper ISS; and if the degree is $\tau \geq 1$ and $\parallel w \parallel < \frac{\min_{x \in \mathbb{R}_+} \{ f(x) \}}{2M}$, then the equilibrium $p$ of system (11) possesses upper FISS.

(2) For all $\tau > 0$, the equilibrium $p$ of system (11) is lower finite-time stable.

Proof: (1) For a system of the form given in (11) with a unique equilibrium at $p$, we can obtain the system

$$\dot{y} = f(y) + g(y)w$$

with a unique zero equilibrium. From Assumption 2 and the cooperativeness of $f, f(v_1) < f(v_1 + p) < 0$. From Theorem 14 and its proof, if we define the max-separable Lyapunov function

$$V_1(x) = \max \{ \frac{x_i}{v_1}, i \in I_n \},$$

we can find that (18) and (19) still hold for the state $y$ and the initial state $y_0$ with $v_1$ substituted for the vector $v$. Therefore, for system (11) with a unique equilibrium at $p > 0$ in $\mathbb{R}_+^n$, if $\tau \in (0, 1)$, then there exists a $\beta \in GKL$ such that

$$x(t) \leq p + \beta(\max_{i \in I_n} \{ x_i \} - p, t - t_0)1_n$$

i.e., the equilibrium $p$ possesses upper FISS; similarly, if the degree is $\tau \geq 1$ and $\parallel w \parallel < \frac{\min_{x \in \mathbb{R}_+} \{ f(x) \}}{2M}$, then the equilibrium $p$ of system (11) possesses upper ISS; and if the degree is $\tau \geq 1$ and $\parallel w \parallel < \frac{\min_{x \in \mathbb{R}_+} \{ f(x) \}}{2M}$, then the equilibrium $p$ of system (11) possesses upper FISS.

(2) Let us define a Lyapunov function of the form

$$V_2(x) = \min_{i \in I_n} \{ \frac{x_i}{v_2} \},$$

where $v_2$ is a positive vector as in Assumption 2. Let the subscript $m$ denote the index such that $V_{2m} = V_2$. We consider the problem of solution estimation in two cases: $x_0 \in \mathcal{A} := \{ x \mid V_2(x) \leq 1 \}$ and $x_0 \in \mathcal{A}^c := \{ x \mid V_2(x) > 1 \}$. Let $\eta \in [t_0, \infty)$ denote the time at which the trajectory enters the set $\mathcal{A}$ for the first time.

Case 1. $x_0 \in \mathcal{A}$. In this case, for any $t \in [t_0, \eta],$

$$\frac{\dot{V}_2(x)}{v_{2m}} \geq \frac{f_m(x(t)) + (g(x)w)_m}{v_{2m}} \geq \frac{1}{v_{2m}} f_m(V_2(x)v_2).$$

Furthermore, by the subhomogeneity property of the function $f$ and $V_2(x) \leq 1$, we have

$$\frac{\dot{x}_m(t)}{v_{2m}} \geq \frac{f_m(V_2(x))}{v_{2m}} V_2(x), \quad t \in [t_0, \eta],$$

i.e.,

$$D^- V_2(x) \geq \frac{f_m(V_2(x))}{v_{2m}} V_2(x), \quad t \in [t_0, \eta].$$

If $\tau \in (0, 1)$, we can obtain

$$V_2^1(x) \geq V_{2,0}^1 + (1 - \tau) \frac{f_m(V_2)}{v_{2m}} (t - t_0), \quad t \in [t_0, \eta],$$

where $V_{2,0} = V_2(x(t_0))$. Therefore, for any $t \in [t_0, \eta],$

$$x(t) \geq v_{2min}^{\tau} \frac{\min_{i \in I_n} \{ x_i \}^{1 - \tau}}{v_{2max}^{\tau}} \frac{1}{n},$$

where $v_{2min} = \min\{v_{21}, \ldots, v_{2n}\}$ and $v_{2max} = \max\{v_{21}, \ldots, v_{2n}\}$.

If $\tau = 1$, then

$$x(t) \geq v_{2min} \frac{\min_{i \in I_n} \{ x_i \} e^{\frac{f_m(v_2)(t - t_0)}{v_{2max}}}}{v_{2max}} 1_n, \quad t \in [t_0, \eta].$$

If $\tau > 1$, then for any $t \in [t_0, \eta],$

$$x(t) \geq v_{2min} \frac{\min_{i \in I_n} \{ x_i \} e^{\frac{f_m(v_2)(t - t_0)}{v_{2max}}}}{v_{2max}} 1_n.$$

(22)

From (21), (22) and (23), it can be found that, for all cases of $\tau > 0$, if the initial data satisfy $0 < x_0 < v_2$, then the state $x$ will increase to $v_2$ in finite time.

When $t \in (\eta, \infty)$ and $V_2(x) > 1$, $x \geq v_2$. Due to the cooperativeness of $f$ and the nonnegativity of $g,

$$\frac{\dot{x}_m(t)}{v_{2m}} = \frac{1}{v_{2m}} (f_m(x(t)) + (g(x)w)_m) \geq \frac{1}{v_{2m}} f_m(v_2).$$

It can be found that, for any $t \in (\eta, \infty),$

$$x(t) \geq v_{2min} \min_{i \in I_n} \{ x_i \} + f_m(v_2)(t - t_0)1_n.$$

(24)

Therefore, the solution will increase to $p$ in finite time.

Case 2. $x_0 \not\in \mathcal{A}$. Following the proof for Case 1,

$$x(t) \geq v_{2min} \min_{i \in I_n} \{ x_i \} + f_m(v_2)(t - t_0)1_n, \quad t \in (t_0, \infty).$$

When $t = t_0$, by the definition of the set $\mathcal{A}$, we obtain

$$x(t_0) \geq v_{2min} \min_{i \in I_n} \{ x_i \} 1_n.$$

Then, by the above two inequalities, for any $t \in [t_0, \infty),$

$$x(t) \geq v_{2min} \min_{i \in I_n} \{ x_i \} + f_m(v_2)(t - t_0)1_n.$$

(25)

From (21), (22), (23), and (24) combined with (25), if the initial data satisfy $0 < x_0 < p$, then the solution $x(t, t_0, x_0)$ will increase to the equilibrium $p$ in finite time. Therefore, for all $\tau > 0$, the equilibrium $p$ of system (11) is lower finite-time stable.

Remark 16: In Theorem 15, for any solution of system (11), the corresponding upper and lower bounds should be satisfied simultaneously. Generally, if the system state satisfies $0 < x < p$, then the state will increase to the equilibrium $p$ in finite time; once $x > p$, the states will remain ultimately
bounded by an upper bound as given in (20) or similar. When the lower bound is considered, due to the choice of the min-separable Lyapunov-like function, the initial state \( x_0 \) is required to belong to \( int \mathbb{R}_+^n \), which excludes the case of \( V_2(x) = 0 \). From the solution estimates in the proof of Theorem 15, the system states will increase from zero and reach the equilibrium \( p \) in finite time. Overall, we can consider the equilibrium \( p \) of system (11) to be lower finite-time stable. For \( \tau \in (0, 1) \), the equilibrium \( p \) of system (11) is said to possess upper FISS. If \( \tau \geq 1 \) and \( \|w\| \geq \frac{\min \left\{ \|f(y(t))\| \right\}}{2M} \), then the equilibrium \( p \) of system (11) possesses upper ISS. If the degree is \( \tau \geq 1 \) and \( \|w\| < \frac{\min \left\{ \|f(y(t))\| \right\}}{2M} \), then the GKL function \( \beta_2 \) in Theorem 14 will be enabled; thus, the equilibrium \( p \) of system (11) possesses upper FISS.

IV. SIMULATION EXAMPLES

In this section, three examples are presented to demonstrate the effectiveness and usefulness of our main results.

**Example 1:** Consider the following system, which originates from biochemical reaction networks:

\[
\dot{x} = f(x) + g(x)w = \begin{pmatrix} -x_1 + \frac{x_2}{a + x_1} \\ -x_2 + \frac{a + x_2}{b + x_1} \end{pmatrix} + w,
\]

where \( a > 1, b > 1 \) and \( g(x) \equiv E \) is the identity matrix. It can be easily checked that \( f \) is \( C^1 \) on \( \mathbb{R}_+^n \), cooperative and subhomogeneous of degree 1. Hence, system (26) with \( w \equiv 0 \) is positive and monotone. Additionally, \( f(x) = 0 \) has two solutions; one is \( x = 0 \), and the other is

\[
x = \begin{pmatrix} 1 - ab \\ 1 + a \\ 1 - ab \\ 1 + b \end{pmatrix}.
\]

Since \( a, b > 1 \), the second solution is outside the positive orthant. Hence, the positive system (26) with \( w \equiv 0 \) has a unique equilibrium in \( \mathbb{R}_+^n \).

If we take \( a = 2 \) and \( b = 3 \), then there exists a vector \( v = (1, 1)^T \) such that \( f(v) < 0 \). If the bounded nonnegative disturbance is taken to be \( w = \begin{pmatrix} |\sin(t)| \\ |\cos(t)| \end{pmatrix} \), then by Theorem 14, system (26) possesses ISS. The simulated curves of \((x_1, x_2)^T\) with initial values \((2, 3)^T\) are shown in Fig. 1. From Fig. 1, it can be seen that the states converge to a bounded region.

For the case of \( w \equiv 0 \), the simulated curves of \((x_1, x_2)^T\) with initial values \((2, 3)^T\) are shown in Fig. 2. From Fig. 2, it can be seen that the states will converge to zero.

**Example 2:** Consider the nonlinear dynamical system given by (26) with \( 0 < a < 1 \) and \( 0 < b < 1 \). \( w \) is the disturbance input bounded by 1. Similar to Example 1, \( f \) is \( C^1 \) on \( \mathbb{R}_+^n \), cooperative and subhomogeneous of degree 1. The subhomogeneous nonlinear system (26) with \( w \equiv 0 \) has two equilibria: one is \( x = 0 \), and the other, given by (27), is in the positive orthant. If we take \( a = 0.5 \) and \( b = 0.5 \), then the two state equilibria are \( 0 \) and \( p = (0.5, 0.5)^T \). There exist vectors \( v_1 = (0.5, 0.5)^T \) and \( v_2 = (0.25, 0.25)^T < p \) such that \( f(v_1 + p) < 0 \) and \( f(v_2) > 0 \), respectively. If we take the bounded disturbance to be \( w = \begin{pmatrix} |\sin(t)| \\ |\cos(t)| \end{pmatrix} \), then by Theorem 15, the equilibrium \( p \) possesses upper ISS and is lower finite-time stable. The simulated curves of \((x_1, x_2)^T\) with initial values \((2, 3)^T\) and \((0.25, 0.5)^T\) are shown in Fig. 3. From Fig. 3, it can be seen that the states converge to a bounded region.

For the case of \( w \equiv 0 \), the simulated curves of \((x_1, x_2)^T\) with initial values \((2, 3)^T\) and \((0.25, 0)^T\) are shown in Fig. 4. From Fig. 4, it can be seen that all the states will converge to the nonzero equilibrium \((0.5, 0.5)^T\).

**Example 3:** Consider the system

\[
\dot{x} = f(x) + g(x)w = \begin{pmatrix} (x_1^2 + x_2^2)^{\frac{1}{4}} - (6x_1^2 + x_2^2)^{\frac{1}{4}} - 1 \\ (x_1^2 + x_2^2)^{\frac{1}{4}} - (x_1^2 + 5x_2^2)^{\frac{1}{4}} - 1 \end{pmatrix} + w.
\]
It can be easily checked that \( f \) is \( C^1 \) on \( \mathbb{R}^n_+ \), cooperative and subhomogeneous of degree \( \frac{3}{2} \). Hence, (28) with \( w \equiv 0 \) is positive and monotone. Additionally, \( f(x) = 0 \) has one equilibrium at \((2.0235, 2.2624)^T\). Hence, the positive system (28) with \( w \equiv 0 \) has a unique equilibrium in \( \mathbb{R}^n_+ \).

There exist vectors \( v_1 = (3, 3)^T \) and \( v_2 = (1, 1)^T \) such that \( f(v_1) < 0 \) and \( f(v_2) > 0 \). If the bounded disturbance is taken to be \( w = \left( \frac{1}{2} \sin(t) \right)^T \), then by Theorem 15, the equilibrium of system (28) possesses FISS. The simulated curves of \((x_1, x_2)^T\) with initial values of \((4, 5)^T\) and \((0.5, 1)^T\) are shown in Fig. 5. From Fig. 5, it can be seen that the states converge to a bounded region.

For the case of \( w \equiv 0 \), the simulated curves of \((x_1, x_2)^T\) with initial values of \((4, 5)^T\) and \((0.5, 1)^T\) are shown in Fig. 6. From Fig. 6, it can be seen that the states will converge to \((2.0235, 2.2624)^T\) in finite time.

V. CONCLUSION
In this paper, an ISS analysis is performed for the first time for two classes of subhomogeneous cooperative systems classified by their equilibrium position. By means of the max-separable Lyapunov function and the min-separable Lyapunov-like function methods which are presented for the first time in this paper, the upper and lower bounds on the solutions to these systems can be estimated. From these estimates, the systems’ ISS properties can be identified and can be judged only on the basis of the differential and algebraic characteristics of the systems. The presented method is easy to apply. Simulated examples verify the validity of our results.

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