Generating the texture of symmetric fermion mass matrices and anomalous hierarchy patterns for the neutrinos from an extra abelian symmetry.

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Abstract
Assuming that a horizontal abelian (gauge) symmetry is at the origin of texture zeros in the fermion mass matrices we show how realistic mass patterns can be generated stepwise with a small number of effective operators from GUT’s and string theory. It is interesting to note that, in the minimal scenario, if the up quark and the down quark mass matrices are generated simultaneously their texture is form-equivalent and contains only two zeros. Textures with more zeros (and thus more predictability) emerge if the up and down sectors are generated stepwise. This last possibility leads to rather anomalous neutrino patterns with mass degenerate neutrinos and large $\nu_\mu - \nu_\tau$ or $\nu_e - \nu_\tau$ mixing even in the absence of hierarchy in $M_R$, the mass matrix of the righthanded neutrinos, and as required by the LSND results if they were to be confirmed by future experiments. In contrast if the quark mass matrices have two zero entries (or less) the neutrino mass-mixing spectrum is independent of the texture structure of $M_R$ -if the latter does not contain any hierarchy of scales- and obeys the quadratic seesaw.
1 Introduction.

In the hope to find the underlying symmetry principle for the long list of mass and mixing parameters which characterize the fermion spectrum and enter in the Standard Model (SM) as free parameters there have been many attempts since the late sixties to describe it with a minimum of observable quantities which can be related to a more fundamental theory [1]. It is for example known that the observed mass and mixing hierarchies of the fermion spectrum can be successfully described in terms of the Wolfenstein parameter \( \lambda \approx 0.22 \) which, to a good approximation, gives the Cabbibo mixing \(|V_{us}|\) [2].

Taking into account the present experimental uncertainties [3] one finds the following mass ratios for the up and down quarks:

\[
m_u : m_c : m_{t/M_X} = \left( \xi_1(M_X)\lambda^8 : \xi_2(M_X)\lambda^4 : 1 \right) \times m_t(M_X) \tag{1}
\]

\[
m_d : m_s : m_{b/M_X} = \left( \xi'_1(M_X)\lambda^4 : \xi'_2(M_X)\lambda^2 : 1 \right) \times m_b(M_X), \tag{2}
\]

with \( \xi_{1,2}, \xi'_{1,2} \leq \mathcal{O}(\lambda) \). The scale dependent constants \( \xi_1, \xi_2 \) and \( \xi'_1, \xi'_2 \) contain the radiative corrections from the running of the Yukawa couplings (since the gauge couplings are “family blind”) from the electroweak scale \( M_Z \) to a scale \( M_X \) which can be as high as the grand unification scale \( M_G = 10^{16} \) GeV or even the Planck scale \( M_P = 10^{19} \) GeV. For the SM as for the two-Higgs doublet models these corrections are still within the margin of the experimental uncertainties and are to leading order equal for the two ratios \( m_u/m_t \) and \( m_c/m_t \) (and for \( m_d/m_b \) and \( m_s/m_b \)) due to the large mass of the top quark, or more generally speaking due to the predominance of the third-generation Yukawa couplings [4]. The same applies also to the mixing elements of the CKM matrix, written in the Wolfenstein parametrisation as:

\[
V_{\text{CKM}/M_X} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \frac{\lambda}{1 - \frac{\lambda^2}{2}} & A(M_X)\lambda^3(\rho + i\eta) \\
-\lambda & \frac{\lambda}{1 - \frac{\lambda^2}{2}} & A(M_X)\lambda^2 \\
A(M_X)\lambda^3(1 - \rho + i\eta) & -A(M_X)\lambda^2 & 1
\end{pmatrix}, \tag{3}
\]

where \( A \) is of order one. The running of the Cabbibo angle is negligible, while the running of the small mixing elements is incorporated in \( A(M_X) \) and is practically the same for \(|V_{13}|\) and \(|V_{23}|\). In the SM \( A(M_X) \) is increasing with the energy whereas in the minimal supersymmetric standard model (MSSM) and in the general two-Higgs doublet case it is decreasing [4] while remaining
within the uncertainties of the present day experimental data. Because of this it is not necessary to work within a particular model to understand how the different powers of lambda arise, nor to fix the scale at which the fermion mass and mixing patterns are generated. We shall therefore set up a framework which could be implemented in different models, though, it may appear more natural from within the MSSM with unification of gauge couplings and partial unification of Yukawa couplings at $M_G$. In the latter the regularity of the spectrum in terms of $\lambda$ is improved at the ultraviolet, where one also notices that ratios of up quark masses and down quark masses are related through a $\lambda^2 \leftrightarrow \lambda$ transformation. Compared to the almost equal spacing between neighbouring quark mass levels the hierarchy in the charged lepton sector is rather anomalous:

$$m_e : m_\mu : m_{\tau/M_X} = \left( \eta_1(M_X) \lambda^5 : \eta_2(M_X) \lambda : 1 \right) \times m_e(M_X), \quad (4)$$

where again the constants $\eta_1, \eta_2$ contain the radiative corrections. Therefore, in models with unification of the Yukawa couplings of the third generation there is no such unification for the first two generations. Instead the following approximate relations:

$$m_\tau \simeq m_b \quad m_e \cdot m_\mu \simeq m_d \cdot m_s \quad . \quad (5)$$

hold at the unification scale for various supersymmetric grand unified theories (SUSY GUT’s) [5].

Given the fact that at low energies one has only 13 observables (six quark and three lepton masses, the three mixing angles and the CP violating phase of the CKM matrix) one cannot fix the entries of the quark and lepton mass matrices $M_u, M_d$ and $M_e$ at $M_X$, even by assuming that the latter are hermitian. This has led to different Ansätze [1,6] in which some of the entries are zero while the others are given in powers of $\lambda$. In the quark sector the maximum number of zeros that one can have is five (counting together those in $M_u$ and $M_d$, but without counting symmetric entries twice) and there are five different pairs of $M_u - M_d$ textures at $M_G$ which lead to masses and mixings which are compatible with the present-day experimental values [6].

1Zeros are in fact small entries which can be neglected to leading order.

2A texture in this context exhibits the power behaviour of fermion mass matrices in terms of some scale, and, is normalized with respect to the largest entry.
In fact, the zeros in the mass matrices can be thought off as “relics” of a new symmetry which is not “family-blind”, while the small non-zero entries could well be correction terms generated after symmetry breaking. This “old” idea [7] has been recently reinvestigated [8-10] in the light of a new way of obtaining also the successful $\sin^2\theta_W = 3/8$ result of the canonical gauge coupling unification which consists in extending the gauge group of the standard model by a horizontal $U(1)_X$ factor whose anomalies can be cancelled by the Green-Schwarz mechanism [11].

Previous investigations have shown that if the effective theory contains higher dimension operators with a few extra scalar fields one can generate symmetric textures with two zeros [9,10] or asymmetric textures [12] which can successfully reproduce the $O(\lambda^n)$ behaviour of realistic quark mass matrices. Here we shall show that within this minimal scheme one can also generate symmetric textures with three zeros which lead to more predictive Ansatz for the quark mass matrices and discuss the consequences for the neutrino spectrum in the two cases.

2 Generating the textures of the quark mass matrices.

2.1 Generating symmetric textures with two zeros.

Let us assume the existence of a family-dependent $U(1)_X$ symmetry at $M_P$, with respect to which the quarks and leptons carry charges $\alpha_i$ and $a_i$ respectively, where $i = 1, 2, 3$ is the generation index. We first consider the up quark mass matrix. Given the predominant role played by the top quark we start with a rank-one matrix and make a choice for the charges such that only the $(3,3)$ renormalizable coupling $t^c t h_1$ is allowed. This fixes the charge of the light Higgs $h_1$ to $-2\alpha$ ($\alpha \equiv \alpha_3$). We expect the other entries to be generated by higher-dimension operators which may occur at the string compactification level and contain combinations of scalar fields some of which acquire vacuum expectation values (vev’s) leading to spontaneous symmetry breaking and large masses for the non-observable part of the spectrum. In the most minimal scenario one will have just one singlet field or a pair of such fields $\sigma_{\pm}$ developing equal (vev’s) along a “D-flat” direction and carrying opposite charges $\pm 1$. They can give rise to higher-order couplings
$q_i^c h_1 (\frac{\sigma_+}{M})^{2\alpha - \alpha_i - \alpha_j} |q_j |$. Notice that when the exponent is positive (negative) only the field $\sigma_+ (\sigma_-)$ can contribute. The new scale $E = \frac{\sigma_+}{M}$ which enters in the quark mass matrix is the ratio of the symmetry breaking scale to the scale which governs these higher-dimension operators, and could be the string unification scale $M_S \simeq 10^{18}$ GeV or $M_P$ or in fact any intermediate scale. The power with which this scale appears in the different Yukawa entries is such as to compensate the charge of $q_i^c h_1 q_j$. If $E$ is a small number one finds two universal hierarchy patterns in the texture which is generated:

$$M_x \sim \begin{pmatrix} E^{x_1} & E^{x_1+x_2} & E^{x_1} \\ E^{x_1+x_2} & E^{x_2} & E^{x_2} \\ E^{x_1} & E^{x_2} & 1 \end{pmatrix} |x_{1,2}| = |\alpha - \alpha_{1,2}|,$$

namely $m_{11} \sim m_{13}^2$ and $m_{22} \sim m_{23}^2$, where by $m_{ij}$ we denote the value of the entry $ij$. The choices $|x_2| = 1$ and $|x_1| = 4$ or $|x_2| = 1$ and $|x_1| = 2$ lead to:

$$M_x^{(1)} \sim \begin{pmatrix} E^8 & E^3 & E^4 \\ E^3 & E^2 & E \\ E^4 & E & 1 \end{pmatrix} \quad \text{or} \quad M_x^{(2)} \sim \begin{pmatrix} E^4 & E^3 & E^2 \\ E^3 & E^2 & E \\ E^2 & E & 1 \end{pmatrix}.$$

If $E$ is of order $\lambda^2$ the two textures above correspond to the phenomenologically acceptable Ansätze à la Fritzsch or à la Giudice for the up quark mass matrix. Notice that for generating $M_x^{(2)}$ only one singlet is needed while for generating $M_x^{(1)}$ a pair of singlets with opposite charges are needed. Approximating the $(1,1)$ entry in the textures of eq.(7) by a zero leads to the well known relation: $V_{us} \simeq \sqrt{m_d/m_s}$ [1]. Setting the $(1,3)$ entry in $M_x^{(1)}$ or the $(1,2)$ entry in $M_x^{(2)}$ to zero leads to further mass mixing relations [1,6]. The generation of other phenomenologically acceptable textures having a zero also in the $(2,2)$ or the $(2,3)$ entry (but not in both entries simultaneously) necessitates a more complicated mechanism involving extra singlets and mixing with heavy Higgses, a case which will be discussed in the next section.

Given the up-quark textures of eq.(7) we shall try to construct realistic textures for the down quark sector. The assumption of symmetric mass matrices and the $SU(2)_L$ symmetry require the equality between the charges of the up and down quarks. Assuming again that only the $(3,3)$ renormalizable coupling is allowed leads to the other light Higgs $h_2$ carrying the same charge.

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3The original Ansatz had a zero in the $(2,2)$ entry.
as \( h_1 \). This means that this \( U(1)_X \) is anomalous and needs a cancellation mechanism. Notice that the choice of a particular texture for \( M_u \) has already fixed the texture of \( M_d \) in terms of some new scale \( \mathcal{E}' \) which has to be of order \( \lambda \) to give the correct mass spectrum of eq.(2). The origin of this difference in scale \( \mathcal{E}' \approx \mathcal{E}^{1/2} \) is yet unknown. Since the Ansatz \( M_u^{(2)}, M_d^{(2)} \) is phenomenologically not acceptable, the only sucessfull Ansatz which can be generated within this minimal scenario for the up and down quark mass matrices is the one which resembles the Fritzsch Ansatz and contains four zeros in total:

\[
M_u^i \sim \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix}, \quad M_d^i \sim \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & \lambda^2 & \lambda \\ 0 & \lambda & 1 \end{pmatrix},
\]

where we have approximated by zeros the suppressed entries.

### 2.2 Generating symmetric textures with three zeros.

In order to generate three-zero textures additional scalar fields are needed. In SUSY GUT’s as well as in many string compactification schemes these are commonplace. There are Higgs multiplets \( H_i \) which are responsible for the breaking of the GUT symmetry as well as heavy singlets \( \tilde{\sigma}_i \) [13] and thus higher dimensional couplings \( q_i^j H (\frac{\tilde{\sigma}_i}{\mathcal{M}})^{2\beta - \alpha_i - \alpha_j} q_j \), where by \(-2\beta\) we denote the charge of \( H \), which give rise to the following texture:

\[
M_z \sim \begin{pmatrix}
\tilde{\mathcal{E}}^{2|z_1|} & \tilde{\mathcal{E}}^{1|z_1+z_2|} & \tilde{\mathcal{E}}^{1|z_1+z|} \\
\tilde{\mathcal{E}}^{1|z_1+z_2|} & \tilde{\mathcal{E}}^{1|z_2|} & \tilde{\mathcal{E}}^{1|z_2+z_2|} \\
\tilde{\mathcal{E}}^{1|z_1+z|} & \tilde{\mathcal{E}}^{1|z_2|} & 1 + \tilde{\mathcal{E}}^{2|z|} 
\end{pmatrix},
\]

whith \(|z_1| = |\beta - \alpha_i|\), and \(|z| = |\beta - \alpha|\). When the difference between the light- and heavy-Higgs charges is larger than between the charges of the heavy Higgs and the quarks, \( i.e. \ z \gg z_{1,2} \), this automatically gives suppressed (1,3) and (2,3) mass entries. For the particular choice \( z_2 = 1 \) and \( z_1 = 2 \) (or \( z_2 = 1 \) and \( z_1 = 4 \)) one obtains to leading order the following texture:

\[
M_z^{(1)} \sim \begin{pmatrix} 0 & \tilde{\mathcal{E}}^3 & 0 \\ \tilde{\mathcal{E}}^3 & \tilde{\mathcal{E}}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

\(^4\text{This choice requires a pair of singlets with opposite charges.}\)
Another interesting texture structure arises when $|z_2 + z| = 1$ and $|z_1 + z_2| = 3$, while $|z|, |z_{1,2}| \gg 0$:

$$M_z^{(2)} \sim \begin{pmatrix} 0 & \tilde{E}^3 & 0 \\ \tilde{E}^3 & 0 & \tilde{E} \\ 0 & \tilde{E} & 1 \end{pmatrix}.$$  \hspace{1cm} (11)

The texture of eq. (11) is the original Ansatz of Fritzsch having an extra zero in the (2,2) entry, while the texture of eq. (10) has an extra zero in the (2,3) entry, as compared to the two-zero texture $M_z^{(1)}$ of eq. (7). As there are no solutions with six zeros (counting the zeros of the up quark and down quark sector together) the most predictive Ansatz require a mixed scenario which will first generate $M_z^{(1)}$ or $M_z^{(2)}$ and subsequently generate the missing entries needed for obtaining $M_x^{(1)}$ or $M_x^{(2)}$. This means that either the up quark and down quark sectors are generated independently of each other or progressively one from another. There are two possible scenarios where this can be accomplished.

### 2.3 The mixed cases

In the first scenario the up-quark mass texture is generated first through mixing of a set of singlet fields with heavy Higgs fields: $u_i^c H (\frac{<\bar{\sigma}>}{M})^{2\beta-\alpha_i-\alpha_j} u_j$ with $\tilde{E} = \frac{<\bar{\sigma}>}{M} \sim \lambda^2$. The down-quark mass texture is generated when another set of singlet fields mixes with the light Higgses: $d_i^c h_1 (\frac{<\sigma>}{M})^{2\alpha-\alpha_i-\alpha_j} d_j$ with $E = \frac{<\sigma>}{M} \sim \lambda$. In this way one obtains the following two set of phenomenologically acceptable solutions:

$$M_u^I \sim \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \hspace{1cm} M_d^I \sim \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & \lambda^2 & \lambda \\ 0 & \lambda & 1 \end{pmatrix}$$  \hspace{1cm} (12)

and

$$M_u^{II} \sim \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & 0 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix} \hspace{1cm} M_d^{II} \sim \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & \lambda^2 & \lambda \\ 0 & \lambda & 1 \end{pmatrix}.$$  \hspace{1cm} (13)

In the other scenario the up- and down-quark sectors are generated independently of each other from the couplings: $u_i^c h_1 (\frac{<\bar{\sigma}>}{M})^{2\alpha-\alpha_i-\alpha_j} u_j$ and
\[ d_i^c H(\frac{<\tilde{\sigma}>}{M})^{2\beta - \alpha_1 - \alpha_j} d_j \] respectively, giving rise to the following two solutions:

\[
M_{u}^{IV} \sim \begin{pmatrix}
0 & \lambda^6 & 0 \\
\lambda^6 & \lambda^4 & \lambda^2 \\
0 & \lambda^2 & 1
\end{pmatrix}, \quad M_{d}^{IV} \sim \begin{pmatrix}
0 & \lambda^3 & 0 \\
\lambda^3 & \lambda^2 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (14)
\]

and

\[
M_{u}^{V} \sim \begin{pmatrix}
0 & 0 & \lambda^4 \\
0 & \lambda^4 & \lambda^2 \\
\lambda^4 & \lambda^2 & 1
\end{pmatrix}, \quad M_{d}^{V} \sim \begin{pmatrix}
0 & \lambda^3 & 0 \\
\lambda^3 & \lambda^2 & 0 \\
0 & 0 & 1
\end{pmatrix}. \quad (15)
\]

One should notice that the set of textures \((I, II, IV, V)\) which have emerged from a particular choice of the charges \(\alpha_1, \alpha_2, \alpha\) and \(\beta\) correspond to the approximate 5-zero texture solutions of ref.[6]. Notice however that the parametrization of the entries of the down quark mass matrix in terms of powers of lambda is somewhat ambiguous because in the MSSM the factors in front are of \(O(\lambda)\) or of \(O(\lambda^{1/2})\). In ref.[...] it was shown that alternative textures which are apriori possible due to this ambiguity cannot be generated within this minimal scenario.

3 Generating the texture of the lepton mass matrices

3.1 The charged lepton mass texture

Assuming simply the gauge symmetries of the SM the \(U(1)_X\) charges \(a_i\) of the leptons are not related to those of the quarks. Allowing however the coupling \(\tau^c \tau h_2\) leads to \(a_3 = \alpha\). Another constraint comes from the second mass relation of eq.(5) which implies that \(a_1 + a_2 = \alpha_1 + \alpha_2\). The simplest way to satisfy this relation is to have \(a_1 = \alpha_1\) and \(a_2 = \alpha_2\). Since the early days of grand unification it is known that in order to obtain also for the first two generations acceptable mass relations between the charged leptons and down quarks the (2,2) entry of \(M_d\) should be multiplied by a factor \(\kappa = -3\), the other entries of \(M_e\) and \(M_d\) been equal [14]. In this way, though there
can be no explanation of the factor minus three in this approach

\[ M_e = M_d \quad M_e^{(i,II)} \sim \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & \kappa \lambda^2 & \lambda \\ 0 & \lambda & 1 \end{pmatrix} \quad M_e^{(IV,V)} \sim \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & \kappa \lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

The alternative is a texture:

\[ M_e^* \sim \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

which can be generated from the texture \( M_z \) in eq.(9) replacing \( |z_i| \to |\beta - a_i| \) and setting \( |z_2| = 1/2 \) and \( |z_1| = 5/2 \) or \( 7/2 \) when \( |z| \gg |z_{1,2}| \). This choice is compatible with the 5-zero solutions of eqs.(12-15) and obviously with the 4-zero Ansatz of eq.(8) but at the expense of introducing an extra scale in addition to the two scales needed for generating the quark textures.

### 3.2 The Dirac neutrino mass texture

We turn now to the neutrino sector. Again as a consequence of the SU(2)\(_L\) symmetry and our symmetric Ansatz the lefthanded and righthanded neutrinos, \( \nu_i \) and \( N_i \), become charged under the \( U(1)_X \) with the same charges \( a_i \) as the charged leptons. Obviously the presence of the \( N_i \)'s implies a larger symmetry than the minimal extension of the SM by an extra \( U(1) \) factor, but also the assumption of symmetric mass matrices can find its justification only in the context of a left-right symmetric theory. Let us first discuss the generation mechanism for Dirac neutrino mass terms: \( M_D^i N_i^c \nu_j \). Since these are of the same type as the mass terms in the quark and charged lepton sector it is natural to adopt the same approach. Then, because the charges \( a_i \) have been fixed through the charged lepton Ansatz,

\[ M_D^c = M_e. \]

Furthermore for the choice \( a_1 = \alpha_1 \) and \( a_2 = \alpha_2 \), which led to eq.(16), one obtains the well known GUT relations:

\[ M_D^c = M_u \quad \text{or} \quad M_D^c = M_d. \]

9
The left equation implies four distinct textures for $M^D$:

$$M_u^{D(i)} \sim \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_u^{D(II)} \sim \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & 0 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix}$$

$$M_u^{D(i,IV)} \sim \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix} \quad M_u^{D(V)} \sim \begin{pmatrix} 0 & 0 & \lambda^4 \\ 0 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{pmatrix},$$

while the equation on the right leads to two possible textures:

$$M_u^{D(i,I,II)} \sim \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & \lambda^2 & \lambda \\ 0 & \lambda & 1 \end{pmatrix} \quad M_u^{D(IV,V)} \sim \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & \lambda^2 & \lambda \\ 0 & \lambda & 1 \end{pmatrix}.$$  \hspace{1cm} (22)

Finally the more general case of eq.(18) implies two textures which are identical to the previous ones, $M_e^{D(i,I,II)} = M_d^{D(i,I,II)}$ and $M_e^{D(IV,V)} = M_d^{D(IV,V)}$ plus an extra case:

$$M_e^{D(\ast)} \sim \begin{pmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \hspace{1cm} (23)$$

### 3.3 The Majorana neutrino mass texture

On the other hand, Majorana mass terms $M_R N_i^c N_j$ need not be generated the same way the other mass terms have been generated so far. In compactified string models, due to the absence of large Higgs representations, righthanded neutrinos donot get tree-level masses, so all entries in $M_R$ are due to nonrenormalizable operators, and nothing is a priori known concerning the particular texture of $M_R$ or the existence of a possible hierarchy in this sector. The only constraints come from the requirement that the seesaw-suppressed masses of the ordinary neutrinos should be below the experimental upper limits. For this, $M_R$ has to be a nonsingular matrix and its scale should be well above the electroweak scale, or else one has to consider the case of unstable neutrinos and the related problems from astrophysical and cosmological bounds [...]. Therefore in addition to the operators that generated the textures of the up and down quarks and the charged leptons...
one will need at least an extra piece to set the Majorana mass scale. Common examples are operators containing the heavy Higgses which have been used for generating the texture in eq.(9), namely \( N_c^i H H N_j \), whose scale is of \( \mathcal{O}(M^2_2/M_S) \approx 10^{14} \text{ GeV} \) multiplied for some orbifold suppression factor \( C = 1 - 10^{-3} \):

\[
R = C \frac{<H><H>}{M_S} \approx 10^{11} - 10^{14} \text{GeV}.
\]  

(24)

In this case \( M_R^{ij} \neq 0 \) implies \( \alpha_i + \alpha_j = 4\beta \).

Starting with the 4-zero Ansatz of eq.(8) for which only the charges \( \alpha_1 \) and \( \alpha_2 \) have been specified one can fix the charges of the light and heavy Higgs bosons relative to each other such that there is one entry which is different from zero. In order to generate more non-zero entries and thus a nonsingular \( M_R \) there are two alternatives paths. Either some of the Majorana entries must be generated perturbatively in a similar way as the entries in the other mass matrices (this implies a hierarchy of righthanded neutrino scales) [...], or, one should need more higher-dimension operators: \( N_c^i H_k H_l N_j \), \((k, l = 1, 2)\), containing two heavy Higgs multiplets \( H_1 \) and \( H_2 \). Following this second approach one can generate three alternative Majorana mass textures containing the maximum number of (four) zero entries [...]

\[
M^{(b)}_R \sim \begin{pmatrix} 0 & 0 & R \\ 0 & R & 0 \\ R & 0 & 0 \end{pmatrix}, \quad M^{(c)}_R \sim \begin{pmatrix} 0 & R & 0 \\ R & 0 & 0 \\ R & 0 & R \end{pmatrix}, \quad M^{(d)}_R \sim \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}. \]

(25)

For the same reason, starting from the 5-zero texture solutions of eqs.(12-15) two extra heavy Higgs fields - in addition to the one which led to eq.(9) - are needed in order to generate the Majorana textures of above. There may be factors which multiply the non-zero entries due to the possibility that the two heavy Higgses do not acquire the same vacuum expectation value, or due to another cause.

\[\text{\footnotesize{5Notice that with respect to ref.[...]} we have assumed that the entries in } M_R \text{ are all equal or of the same order of magnitude } R.\]
4 The neutrino mass spectrum

Given the texture structure of the quark and lepton mass matrices one can calculate the neutrino masses and the lepton mixing and trace back the different mass-mixing patterns to the positioning of texture zeros in the mass matrices and thus to the symmetries of the latter.

When $M_R$ is a nonsingular matrix, the seesaw mechanism leads to three light neutrinos, which are obtained upon diagonalisation of the reduced $3 \times 3$ mass matrix:

$$M^{eff}_\nu \simeq M^{D^\dagger} M^{-1}_R M^D.$$  \hspace{1cm} (26)

Furthermore if one makes the Ansatz:

$$M_R \simeq 1 \times R \quad \text{or} \quad M_R \simeq M^D,$$  \hspace{1cm} (27)

as this has been common place in the literature, this obviously leads to the quadratic seesaw relation:

$$m_{\nu_1} : m_{\nu_2} : m_{\nu_3} = (z^6 : z^4 : 1)m_0,$$  \hspace{1cm} (28)

where the neutrino masses scale either as the up quark masses squared

$$M_D \simeq M_u \quad z = \lambda^2 \quad m_0 = \frac{m_t^2}{R},$$  \hspace{1cm} (29)

or they scale as the down quark masses squared

$$M_D \simeq M_d \quad z = \lambda \quad m_0 = \frac{m_b^2}{R}.$$  \hspace{1cm} (30)

In refs.[...], it was pointed out that the quadratic seesaw spectrum of eq.(28) is obtained if the Majorana sector has no symmetries on its own, which means that it has the same symmetries as the quark sector or it has no symmetries at all.

In order to study this in more detail let us write the up and down quark textures of eqs.(8,12-15) in a common form, by means of a set of parameters $\alpha, \beta, \gamma, \delta$ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$, which for simplicity can be one or zero:

$$M_u = \begin{pmatrix} 0 & \alpha \lambda^6 & \delta \lambda^4 \\ \alpha \lambda^6 & \beta \lambda^4 & \gamma \lambda^2 \\ \delta \lambda^4 & \gamma \lambda^2 & 1 \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & \tilde{\alpha} \lambda^4 & 0 \\ \tilde{\alpha} \lambda^4 & \tilde{\beta} \lambda^2 & \tilde{\gamma} \lambda \\ 0 & \tilde{\gamma} \lambda & 1 \end{pmatrix}. \hspace{1cm} (31)$$
Setting for example $\delta = 0$ or $\alpha = 0$ one obtains the *Ansatz à la Fritzsch* or *à la Giudice*. Then the effective light-neutrino mass matrix assumes a universal form in terms of the scale $z$ which characterizes the Dirac neutrino texture $M^D$, and as function of the minors of the matrix $M_R$ $r_{i=1,...,6}$ which are obtained by omitting the row and column containing the corresponding $R_i$ entry, e.g., $r_3 = R_1 R_2 - R_4^2$:

$$\frac{M_{\nu}^{\text{eff}}}{m_{\nu}^2/\Delta} = r_3 \begin{pmatrix} \delta z^4 & \gamma \delta z^3 & \delta z^2 \\ \gamma \delta z^3 & \gamma^2 z^2 & \gamma z \\ \delta z^2 & \gamma z & 1 \end{pmatrix}$$

$$+ r_6 z \begin{pmatrix} (\beta \delta + \alpha \gamma) z^3 & (\alpha \gamma + \beta \delta) z^2 \\ (\alpha \gamma + \beta \delta) z^2 & (\gamma^2 + \beta) z \\ (\gamma^2 + \beta) z & 2\gamma \end{pmatrix}$$

$$+ r_2 z^2 \begin{pmatrix} \alpha^2 z^4 & \alpha \beta z^3 & \alpha \gamma z^2 \\ \alpha \beta z^3 & \beta^2 z^2 & \beta \gamma z \\ \alpha \gamma z^2 & \beta \gamma z & \beta^2 \end{pmatrix}$$

$$+ r_5 z^2 \begin{pmatrix} 2\alpha \gamma z^2 & \delta^2 z^2 \\ \delta^2 z^2 & (\alpha + \gamma \delta) z \\ (\alpha + \gamma \delta) z & 2\delta \end{pmatrix}$$

$$+ r_4 z^3 \begin{pmatrix} \alpha^2 z^3 & 2\alpha \gamma z^2 & (\alpha \gamma + \beta \delta) z \\ (\alpha \gamma + \beta \delta) z & (\alpha \gamma + \beta \delta) z & 2\gamma \delta \\ (\alpha \gamma + \beta \delta) z & 2\gamma \delta & \delta^2 \end{pmatrix}$$

If $r_3 \neq 0$ and the entries of the Majorana mass matrix are all of the same order of magnitude the texture of $M_{\nu}^{\text{eff}}$ is, as in the quark sector, either of the Fritzsch type with nearest neighbour mixing or of the Giudice type leading to the spectrum of eq.(12). In contrast when $r_3 = 0$ as a result of a symmetry between $N_1$ and $N_2$ the neutrino spectrum can be drastically distorted. For example for the quark textures of solution .... a texture zero appears in the $(3,3)$ entry of $M_{\nu}^{\text{eff}}$ leading to ..... Notice that in general ‘anomalous’ neutrino mass-mixing patterns can be obtained only for the five-zero texture solutions of Table 1. The four-zero solutions require very particular conditions of strong hierarchy in $M_R$ to give mass degenerate neutrinos.