Distributed Submodular Maximization with Parallel Execution

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Abstract—The submodular maximization problem is widely applicable in many engineering problems where objectives exhibit diminishing returns. While this problem is known to be NP-hard for certain subclasses of objective functions, there is a greedy algorithm which guarantees approximation at least 1/2 of the optimal solution. This greedy algorithm can be implemented with a set of agents, each making a decision sequentially based on the choices of all prior agents. In this paper, we consider a generalization of the greedy algorithm in which agents can make decisions in parallel, rather than strictly in sequence. In particular, we are interested in partitioning the agents, where a set of agents all make a decision simultaneously based on the choices of prior agents, so that the algorithm terminates in limited iterations. We provide bounds on the performance of this parallelized version of the greedy algorithm and show that dividing the agents evenly among the sets in the partition yields an optimal structure. We additionally show that this optimal structure is still near-optimal when the objective function exhibits a certain monotone property. Lastly, we show that the same performance guarantees can be achieved in the parallelized greedy algorithm even when agents can only observe the decisions of a subset of prior agents.

I. INTRODUCTION

Submodular maximization is an important topic with relevance to many fields and applications, including sensor placement [1], outbreak detection in networks [2], maximizing and inferring influence in a social network [3], [4], document summarization [5], clustering [6], assigning satellites to targets [7], path planning for multiple robots [8], and leader selection and resource allocation in multiagent systems [9], [10]. An important similarity among these applications is the presence of an objective function which exhibits a “diminishing returns” property. For instance, a group of leaders can impose some influence on a social network, but the marginal gain in influence achieved by adding a new leader to the group decreases as the size of the group increases. Objective functions (such as influence) satisfying this property are submodular.

The submodular maximization problem is to choose a set of elements (such as leaders) which maximize the submodular objective function, according to some constraints. This problem is known to be NP-hard for certain subclasses of submodular functions [11]. Therefore, much research has focused on how to approximate the optimal solution [12], [13], [14], [15]. The overall message of this research is that simple algorithms can perform well by providing solutions which are guaranteed to be within some factor of optimal.

One such algorithm is the greedy algorithm, first proposed in [16]. It was shown in this seminal work that for certain classes of constraints the solution provided by the greedy algorithm is guaranteed to be within $1 - 1/e$ of the optimal, and within $1/2$ of the optimal for the more general case [17]. Since then, more sophisticated algorithms have been developed to show that there are many instances of the submodular maximization problem which can be solved efficiently within the $1 - 1/e$ guarantee [12], [18]. It has also been shown that progress beyond this level of optimality is not possible using a polynomial-time algorithm [19].

More recent research has focused on distributed algorithms, since in many cases having a centralized agent with access to all the relevant data is untenable [20], [6], [21]. In this case, the greedy algorithm can be generalized using a set of $n$ agents, each with its own decision set. The combined set of decisions by the agents is evaluated by the submodular objective function, which they seek to maximize. Each agent chooses sequentially, maximizing its marginal contribution compared to the actions of the prior agents. In this setting, the greedy algorithm has been shown to provide a solution within $1/2$ of the optimal.

Observe that with this distributed greedy algorithm, each agent must have access to all prior agents’ decisions, meaning that the algorithm requires $n$ iterations to terminate. For large $n$, this may be unacceptable. Therefore, researchers have explored how to parallelize the greedy algorithm [5], [22], [23]. While much has been done, many of these parallelization techniques still require some entity to have access to all decisions – in essence, part of the parallelization technique is how to assign decision sets to agents.

A natural extension is to consider the case where no such centralized authority is present, i.e., the agents’ decision sets are determined a priori and cannot be modified. Terminating the greedy algorithm in $q < n$ iterations thus requires a partitioning of the agents into sets $1, \ldots, q$, where now each agent in set $j$ simultaneously chooses an action which maximizes its marginal contribution relative to the actions chosen by all the agents in sets $1, \ldots, j - 1$. In this setting, one can ask the following questions:

1) What is the best way to partition the agents? Should the agents be spread evenly across the sets in the partition, or should set 1 or set $q$ be larger than the others?

2) If some additional structure is known about the sub-
modular objective function, how does that affect the partitioning strategy?

In the non-parallelized setting, recent research has explored how the performance of the greedy algorithm degrades as we relax the information sharing constraint that each agent must have access to the decisions of all prior agents [24, 25]. Therefore, it is a natural extension to ask in the parallelized setting:

3) If we relax the requirement that an agent in partition set \( j \) must have access to the decisions of all agents in sets \( 1, \ldots, j - 1 \), can the same level of performance be maintained by the greedy algorithm?

In response to the questions listed above, the contributions of this paper are the following:

1) Theorem 1 shows that partitioning the agents into equally-sized sets yields the highest performance guarantee of the greedy algorithm, given a time constraint \( q \). The performance guarantee is also shown, in terms of \( n \) and \( q \).

2) Theorem 2 shows that if we know some additional structure about the submodular function, i.e., a lower bound on each agent’s relative marginal contribution, then a near-optimal partitioning strategy is the same as that of Theorem 1. Increased performance guarantees are also shown.

3) Theorems 3 and 4 prove that the above performance guarantees on the greedy algorithm can be preserved even with less information sharing among the agents.

II. Model

Consider a base set \( S \) and a set function \( f: 2^S \to \mathbb{R}_{\geq 0} \). Given \( A, B \subseteq S \), we define the marginal contribution of \( A \) with respect to \( B \) as

\[
f(A \mid B) = f(A \cup B) - f(B)
\]

In this paper, we are interested in distributed algorithms for maximization of submodular functions, where there is a collection of \( n \) decision-making agents, named as \( N = \{1, \ldots, n\} \), and a set of decisions \( S \). And each decision-making agent is associated with a set of permissible decisions \( X_i \subseteq S \) so that \( X_1, \ldots, X_n \) form a partition of \( S \). For rest of the paper, we refer to decision-making agents as simply agents. Then the collection of decisions by all agents \( x = \{x_1, \ldots, x_n\} \) is called an action, and we denote the set of all actions as the action profile \( X = X_1 \times \cdots \times X_n \). An action \( x \in X \) is evaluated by an objective function \( f: 2^S \to \mathbb{R}_{\geq 0} \) if \( f(x) = f(\cup_i \{x_i\}) \). Furthermore, we restrict our attention to \( f \)'s that are submodular functions, which satisfy the following properties:

- **Normalized** \( f(\emptyset) = 0 \).
- **Monotonic** \( f(\{e\} \mid A) \geq 0 \) for all \( e \in S \) and \( A \subseteq S \).
- **Submodular** \( f(\{e\} \mid A) \geq f(\{e\} \mid B) \) for all \( A \subseteq B \subseteq S \) and \( e \in S \setminus B \).

For simplicity, we will refer to the set of submodular functions as \( \mathcal{F} \). The goal of the submodular maximization problem is to find:

\[
x^\text{opt} \in \arg \max_{x \in X} f(x)
\]  \hspace{1cm} (2)

For convenience, we overload \( f \) to accept elements: \( f(e) = f(\{e\}) \) for \( e \in S \), and allow multiple inputs: \( f(A, B) = f(A \cup B) \) for \( A, B \subseteq S \). We also denote \( \{1, \ldots, k\} = [k] \) for any nonnegative integer \( k \) and \( \{i, i+1, \ldots, j\} = : j \) for any nonnegative integers \( i \leq j \). For any \( x \in X \) and \( M \subseteq N \), we will write \( f(x_M) \) with the understanding that we mean \( f(x_M) = f(\cup_{i \in M} \{x_i\}) \). For instance \( f(x_{1:10}) = f(x_1, \ldots, x_{10}) \).

A distributed greedy algorithm can be used to approximate the problem as stated in (2). In the greedy algorithm, the agents make decision in sequential order according to their names and each agent attempts to maximize its marginal contribution with respect to the choices of prior agents:

\[
x_{i}^\text{sol} \in \arg \max_{x_i \in X_i} f(x_i \mid x_1:i-1)
\]  \hspace{1cm} (3)

Note that the greedy solution \( x^\text{sol} \) may not be unique. In the context of this paper, we assume that \( x^\text{sol} \) is the worst solution among all possible greedy solutions.

Let \( \gamma(f, X) \) be \( f(x^\text{sol})/f(x^\text{opt}) \) from the greedy algorithm subject to objective function \( f \) and action profile \( X \). Then, to measure the quality of the greedy algorithm, we consider its competitive ratio:

\[
\gamma = \inf_{f \in \mathcal{F}, X} \gamma(f, X)
\]  \hspace{1cm} (4)

The well known result from [17] states that \( \gamma = 1/2 \), which means that (3) guarantees that the performance of the greedy algorithm is always within 1/2 of optimal. Furthermore, only one agent can make a decision at any given time and thus a solution will be found in precisely \( n \) iterations. The central focus of this work is on characterizing the achievable competitive ratio for situations where a system-design does not have the luxury of having \( n \) iterations to construct a decision. To this end, we introduce the parallelized greedy algorithm to allow multiple agents make decisions at the same time. More formally, we consider a situation where the system is given limited number of iterations \( q \leq n \) and needs to come up with an iteration assignment \( P: [n] \to [q] \) so that each agent makes decision only based on the choices made by agents from earlier iterations:

\[
x_i^\text{sol} \in \arg \max_{x_i \in X_i} f(x_i \mid x_{N_i})
\]  \hspace{1cm} (5)

where \( N_i = \{j \mid P(j) < P(i)\} \subseteq \{1, \ldots, i - 1\} \). Note, that to maintain consistency with (5), \( P \) must preserve the implicit ordering of the greedy algorithm: \( P(i) \leq P(j) \) whenever \( i < j \).

We are interested in the performance of (5) given a fixed number of agents and number of iterations. We define the competitive ratio of a time step assignment \( P \) as:

\[
\gamma(P) = \inf_{f \in \mathcal{F}, X} \gamma(f, X, P)
\]  \hspace{1cm} (6)

where \( \gamma(f, X, P) \) is \( f(x^\text{sol})/f(x^\text{opt}) \) from the parallelized greedy algorithm subject to objective function \( f \), action
The target set as possible. An instance of this problem is the weighted set cover problem \[18\]. Given a target set \(Y\) and \(n\) elements, the goal is to find the smallest possible weighted set cover. The optimal solution is to assign each element to a separate subset, which achieves \(\beta\)-strict monotonicity.

In this section, we present the best possible competitive ratio for the parallelized greedy algorithm and optimal iteration assignment which achieves such a bound. We perform this analysis for both general submodular objective functions and submodular objective functions with an additional property which we call \(\beta\)-strict monotonicity.

Note that according to \[5\], a pair of agents do not utilize each other’s decisions (in either direction) when they are in the same iteration. This intuitively represents some “blindspots” in the parallelized greedy algorithm compared to the original greedy algorithm. One way to reduce those “blindspots” is to divide the agents evenly among the available iterations, in other words, we want the number of agents deciding in parallel to be as close to the concurrency as possible. Theorems \[1\] and \[3\] illustrate that this idea yields the best possible iteration assignment.

**Theorem 1:** Given a parallelized submodular maximization problem with \(n\) agents and \(q\) iterations, the competitive ratio is:

\[
\rho(n, q) = \begin{cases} 
\frac{1}{r} & \text{if } n \equiv 1 \pmod{q} \\
\frac{1}{r} + 1 & \text{otherwise}
\end{cases}
\]

In particular, when \(n \equiv 1 \pmod{q}\),

\[
P^*_{n,q} = \left\lfloor \frac{i}{r} \right\rfloor \quad \text{if } i < n
\]

\[
P^*_{n,q} = \frac{i}{r} \quad \text{if } i = n
\]

Otherwise,

\[
P^*_{n,q} = \left\lfloor \frac{i}{r} \right\rfloor
\]

The above optimal iteration assignment is illustrated by Figure 2.

In this section, we present the best possible competitive ratio for the parallelized greedy algorithm and optimal iteration assignment which achieves such a bound. We perform this analysis for both general submodular objective functions and submodular objective functions with an additional property which we call \(\beta\)-strict monotonicity.

**III. Optimal Parallel Structures**

An example of the submodular maximization problem is the weighted set cover problem \[18\]. Given a target set \(Y\) and a mapping \(T: S \to 2^Y\) from the decisions to subsets of \(Y\). The value of an action profile \(x \in X\) is determined by a weight function \(w: Y \to \mathbb{R}_{\geq 0}\):

\[
f(x) = \sum_{y \in \cup_i T(x_i)} w(y)
\]

Intuitively, this problem aims to “cover” as much of the target set as possible. An instance of this problem is illustrated by Figure 1.
Lower Bound on Competitive Ratio

Fig. 2: Examples of optimal iteration assignments as given by Theorem 1. Each agent, represented by a node, is named as 1, ..., n. Each column, labeled by a Roman numeral, contains all agents executed at a given iteration. Also shown are some examples of iteration assignments that are not optimal. According to Theorem 1, the competitive ratio in Fig. 2a is 1/3 and the competitive ratio in Fig. 2c is also 1/3. On the other hand, in Section IV-C, Lemma 1 shows that the competitive ratio in Fig. 2b and Fig. 2d are both 1/4. Also, according to Theorem 2, when the objective function is β-strictly monotone, Fig. 2a and Fig. 2c are nearly-optimal, but Fig. 2b and 2d are not.

Note that as β → 0, (16) converges to (11), and as β → 1, ρ(n,q) converges to 1. This confirms the intuition that for higher β, actions have less “overlap” and as a result the greedy algorithm can perform closer to the optimal. For more detail, Figure 3 illustrates how the lower bound changes with respect to β at different r. Also, as r → ∞ both the lower and upper bounds converge to β. So the lower and upper bounds are close to each other when concurrency is high.

We will present a more general version of Theorem 2 and its proof in Section IV-D.

IV. Parallelization as Information Exchange

A. Preliminaries

In this section, we will employ several concepts from graph theory. Throughout this section, we will assume that we have an undirected graph G = (V, E):

Definition 2: Nodes K ⊆ V form a clique if for all distinct i, j ∈ K, (i,j) ∈ E. A clique cover of G is a partition of V so that each set in the partition forms a clique. And the clique cover number θ(G) is the least number of cliques necessary to form a clique cover.

Definition 3: Nodes I ⊆ V form an independent set if for all distinct i, j ∈ I, (i,j) /∈ E. The maximum independent set is an independent set with the largest possible number of nodes. And the independence number α(G) is the size of the maximum independent set.

B. Parallization and Information

In both [3] and [5], each agent uses the choices of prior agents to make its own decision. Alternatively, we can view this process as an agent communicating its decision to other agents who depend on this piece of information. We can thus model this information exchange as an undirected graph G = (V,E), where V = N represents the agents and each edge (i,j) ∈ E represents information being exchanged between two agents. Since the greedy algorithm has an implicit ordering induced by the agents’ names, for every pair i < j so that (i,j) ∈ E, agent j requires knowing
A non-optimal assignment for $n = 5, q = 2$.

A non-optimal assignment for $n = 5, q = 3$.

The induced information graph of iteration assignments from Figure 2 as discussed in Section V. In the information graphs, an edge $(i, j)$ where $i < j$ represents that the iteration of agent $i$ is before the iteration of agent $j$ and hence agent $j$ requires knowing the choice made by agent $i$ before making its own decision. Each edge can be thought of as the flow of information exchanges between the agents in which a prior agent share its choice so later agents can decide.

From the definitions, it is clear that $P_{n,q} \subseteq G_{n,q}$. Hence we can adopt competitive ratios for information graphs so that (6), (8) and (9) become:

$$\gamma(G) = \inf_{f,X,G} \gamma(f, X, G) \quad \text{(20)}$$

and (14) and (15) become:

$$\gamma'(G) = \inf_{f \in F, X} \gamma(f, X, G) \quad \text{(23)}$$

where $\gamma(f, X, G)$ is $f(x^{\text{sol}})/f(x^{\text{opt}})$ from the generalized greedy algorithm subject to objective function $f$, action profile $X$ and information graph $G$.

It is natural to ask whether the above generalization of parallelized greedy algorithm yields higher competitive ratio under the same number of agents and iterations. Also, in some applications, the information exchange may incur some costs, and therefore having a information graph with many edges is undesirable. So we wish to answer a second question, whether we can achieve the same competitive ratio in (11) and (16) with information graphs that have fewer edges than that of (12) and (13). Theorems 3 and 4 as presented below, show that generalizing to information graphs does not yield higher competitive ratio but allows us to achieve the same optimal competitive ratio with fewer edges.

### C. Extension of Theorem 1 to Information Graphs

We will present and prove a version of Theorem 1 generalized to information graphs, and Theorem 1 follows from the same logic.

**Theorem 3:** Given a parallelized submodular maximization problem with $n$ agents and $q$ iterations, the competitive ratio is:

$$\eta(n, q) = \begin{cases} \frac{1}{q} & \text{if } n \equiv 1 \pmod{q} \\ \frac{1}{r+1} & \text{otherwise} \end{cases} \quad \text{(25)}$$
Furthermore, (26) has approximately information graph induced by (12) and (13), respectively. This is true because every agent (except agent 1) and no information need to be exchanged among threads. "delightfully parallel" so that we can partition the agents one agent from each prior time step. Additionally, (27) is achievable.

Additionally, if there exists \(\gamma\) such that graph achieves the equality \(\gamma(G) = 1/r\). Now let \(G\) be the graph described in (26) and we can show that \(\gamma(G) \geq 1/r\). First note that for any \(i < n\), \(\overline{P}(G;i) = \lfloor i/(r-1) \rfloor\) and if \(i = n\), \(\overline{P}(G;i) = q\). Hence this graph is in \(G_{n,q}\). Let \(T = \{i \mid i \leq (r-1)(q-1)\}\), and note that for all \(i < n\), \(N_i \subseteq T\). Then we have:

\[
\begin{align*}
f(x^{opt}) &\leq f(x^{opt}, x^*_{sol}) \\
&= f(x^*_{sol}) + \sum_{i=1}^{n} f(x^*_{opt} \mid x^*_{opt,i}, x^*_{sol}) \\
&\leq f(x^*_{sol}) + \sum_{i=1}^{n} f(x^*_{opt} \mid x^*_i) \\
&= f(x^*_{sol}) + \sum_{j=0}^{r-1} f(x^*_{opt}(n+q) \mid x^*_j) \\
&\leq f(x^*_{sol}) + \sum_{j=1}^{r-1} f(x^*_{opt}(n+q) \mid x^*_j) \\
&\leq r f(x^*_{sol})
\end{align*}
\]

where (30a) and (30g) follow from monotonicity, (30b) and (30f) follow from telescoping sums, (30c) follows from submodularity, (30d) follows from the graph structure as defined in (26), and (30e) follows from the definition of the greedy algorithm as stated in (17). From here we can conclude that \(q(n,q) = 1/r\) when \(n \equiv 1 \pmod{q}\).

Now we consider the case where \(n \equiv 1 \pmod{c}\). If \(\alpha(G) > r\), then by Lemma 1 \(\gamma(G) \geq 1/(r+1)\). If \(\alpha(G) = r\), then \(|D_k| \leq r\) and apply the pigeonhole principle on \(D_{1,q-1}\), there exists some \(k < q\), so that \(D_k = r\). The above is true because the condition \(n \equiv 1 \pmod{c}\) ensures that \(n-r > (q-1)(r-1)\). Therefore there must exist some \(w \leq n, d \in D_k\) so that \(d \in N_w\). Then, Lemma 1 implies that \(\gamma(G) \leq 1/(r+1)\). Now let \(G\) be the graph described by (27). For any \(i \in n\), \(\overline{P}(G;i) = \lfloor i/r \rfloor\), hence this graph is in \(G_{n,q}\). Also note that it consists of \(r\) disjoint cliques, therefore \(r = \alpha(G) = \theta(G)\). By Lemma 1, we have \(\gamma(G) \geq 1/(r+1)\), hence the equality is achieved.

D. Extension of Theorem 2 to Information Graphs

We will prove a version of Theorem 2 generalized to information graphs, and the original theorem follows from the same logic.

\[\text{Theorem 4: Given a parallelized submodular and } \beta\text{-strictly monotone maximization problem with } n \text{ agents and}\]

\[\begin{align*}
\text{Lemma 1: Given any information graph } G = (V,E), \quad & \frac{1}{\alpha(G)} \geq \gamma(G) \geq \frac{1}{\theta(G) + 1} \\
\text{Lemma 2: For any } G \in G_{n,q}, \quad & \alpha(G) \geq r.
\end{align*}\]
We want decisions are distinct. Suppose not and for some $i$

$$\frac{(r - 1)\beta + 1}{r} \geq \eta'(n, q) \geq \frac{(r - 1)\beta + 1}{r - \beta + 1} \quad (31)$$

Furthermore, the graph $G$ as defined in (27) achieves the lower bound $\gamma'(G) \geq \frac{(r - 1)\beta + 1}{r - \beta + 1}$.

To prove Theorem 4, we present a lower and an upper bound on the competitive ratio subject to $\beta$-strict monotonicity in a similar fashion as Lemma 1. The following lemma is proven in the Appendix.

**Lemma 3:** Given any information graph $G$,

$$\frac{\alpha(G) - 1\beta + 1}{\alpha(G)} \geq \gamma'(G) \geq \frac{(\theta(G) - 1\beta + 1)}{\theta(G) - \beta + 1} \quad (32)$$

Now we come back to the proof of Theorem 3. The upper bound in (31) follows from combining Lemma 2 and the lower bound in Lemma 3. Consider the graph $G$ described by (27); as we already argued previously in Section IV-C, $r = \alpha(G) = \theta(G)$ and $G \in \mathcal{G}_{n,q}$. Hence, from the lower bound in Lemma 3, we conclude that $G$ achieves the lower bound in (31), so we are done.

**V. Conclusion**

In this paper, we derived bounds on the competitive ratio of the parallelized greedy algorithm for both submodular objective functions and those with an additional property of $\beta$-strictly monotone. We also provided the optimal design which achieves such bound and showed that a graph theoretic approach yields more effective parallelization that still achieves the same competitive ratio.

There are several directions of future research. One possibility is to consider whether employing other structural properties on the objective functions can also improve the competitive ratio of greedy algorithm. In particular, we are interested in properties that consider fixed number actions at once because $\beta$-strict monotonicity considers arbitrarily many actions at once. Another possible direction is to consider applying something other than the marginal contribution in making the greedy decisions.

**APPENDIX PROOF OF LEMMA 3**

We first show the lower bound on $\gamma'(G)$. In this proof, we consider some minimal clique covering $\{K_1, K_2, \ldots, K_0\}$ of $G$. Let $K(i)$ denotes the clique containing the $i$ node, and $\sigma_j = f(x_{i}^{opt})$. We will upper bound $f(x_{i}^{opt})$ in terms of $\sigma$'s. Then we express $\sigma_j / f(x_{i}^{opt})$ as a concave function in $\sigma_j$ and using convexity to derive the final lower bound.

First, we need to have an appropriate objective function $f$ and action profile $X$. Given $(f \in F', X)$, we can assume that $X_j = \{x_{i}^{opt}\}$ without affecting the competitive ratio. We want $x_{i}^{opt} \neq x_{i}^{sol}$ for all $i \in N$, which ensures that all decisions are distinct. Suppose not and for some $i \in N$, $x_{i}^{opt} = x_{i}^{sol} = u$, then we transform $(f, X)$ to $(\tilde{f}, \tilde{X})$ such that $x_{j}^{sol} = x_{j}^{opt}$, $\tilde{x}_{j}^{opt} = x_{i}^{opt}$ for any $j \neq i$, $\tilde{x}_{i}^{opt} = u$ and $\tilde{x}_{i}^{opt} = v$. Also define $\tilde{f}$ as:

$$\begin{align*}
\tilde{f}(u, v, A) &= f(u, A) + \beta f(u) \\
\tilde{f}(u, A) &= \tilde{f}(v, u, A) = f(u, A) \\
\tilde{f}(A) &= f(A)
\end{align*}$$

where $A \subseteq S \setminus X_i$. Note that the above transformation does not affect the competitive ratio because $\tilde{f}(x_{i}^{opt}) = f(x_{i}^{opt})$ and $\tilde{f}(x_{i}^{opt}) = f(x_{i}^{opt})$. Also, from some simple algebra:

$$\begin{align*}
\tilde{f}(u, v, A) &= f(u, A) + \beta f(u) \quad (34a) \\
\tilde{f}(u, A, v) &= f(y, u, v) = f(y | u, A) \quad (34b)
\end{align*}$$

where $A \subseteq S \setminus X_i, v \in S \setminus X_i$. From (34a), it is easy to verify that $(\tilde{f}, \tilde{X})$ satisfies all properties of submodular functions and $\beta$-strict monotonicity; but for brevity, we will not explicitly show them here. Hence, for the rest of the proof, we can safely assume that $x_{i}^{opt} \neq x_{i}^{sol}$ for all $i \in N$.

We bound $f(x_{i}^{opt})$ through strict monotonicity:

$$\begin{align*}
\beta \sum_{j=1}^{\theta} \sigma_j &\leq f(x_{i}^{opt}) + \sum_{j=1}^{\theta} \sum_{i \in K_j} \beta f(x_{i}^{opt}) \\
&= f(x_{i}^{opt}) + \sum_{i=1}^{n} \beta f(x_{i}^{opt}) \quad (35a) \\
&\leq f(x_{i}^{opt}) + \sum_{i=1}^{n} f(x_{i}^{opt} | x_{i:1}^{opt}, x_{i}^{opt} \setminus K(i)) \quad (35b) \\
&= f(x_{i}^{opt}) + \sum_{i=1}^{n} \beta f(x_{i}^{opt} | x_{i}^{opt} \setminus K(i)) \quad (35c)
\end{align*}$$

where (35a) follows from submodularity, (35b) follows from the fact that $K'$s are disjoint, (35c) follows from $\beta$-strict monotonicity, and (35d) is a telescoping sum.

Then we bound $f(x_{i}^{opt}, x_{i}^{opt})$ using properties of submodular functions and the greedy algorithm.

$$\begin{align*}
f(x_{i}^{opt}, x_{i}^{opt}) &= f(x_{i}^{opt}) + \sum_{i=1}^{n} f(x_{i:1}^{opt} | x_{i:1}^{opt}, x_{i}^{opt}) \quad (36a) \\
&\leq f(x_{i}^{opt}) + \sum_{i=1}^{n} f(x_{i}^{opt} | x_{i}^{opt} \setminus K(i)) \quad (36b) \\
&\leq f(x_{i}^{opt}) + \sum_{i=1}^{n} f(x_{i}^{opt} | x_{i:1}^{opt} \setminus K(i)) \quad (36c) \\
&\leq f(x_{i}^{opt}) + \sum_{i=1}^{n} f(x_{i}^{opt} | x_{i}^{opt} \setminus K(i)) \quad (36d) \\
&= f(x_{i}^{opt}) + \sum_{j=1}^{\theta} \sum_{i \in K_j} f(x_{i}^{opt} | x_{i}^{opt} \setminus K_j) \quad (36e) \\
&= f(x_{i}^{opt}) + \sum_{j=1}^{\theta} \sigma_j \quad (36f)
\end{align*}$$

where (36a) and (36f) are telescoping sums, (36b) and (36d) follow from submodularity, (36c) follows from the fact that $K''$s are disjoint, and (36e) follows from the greedy algorithm as defined in (17).
Combining (35) and (36), we have that
\[ \frac{f(x^{\text{opt}})}{f(x^{\text{sol}})} \leq 1 + (1 - \beta) \sum_{j=1}^{\theta} \frac{\sigma_j}{f(x^{\text{sol}})} \] (37)

And we can upper bound \( \sigma_j \)’s using strict monotonicity in a manner same as that of (35).
\[ \sigma_j + \sum_{i \notin j} \beta \sigma_i \leq \sigma_j + \sum_{i \notin K_j} \beta f(x^{\text{sol}}) \] (38a)
\[ \leq \sigma_j + \sum_{i \notin K_j} f(x^{\text{sol}} | x^{\text{sol}}_{1:i-1} \cup x^{\text{sol}}_{j}) \] (38b)
\[ = f(x^{\text{sol}}) \] (38c)

Hence for any \( 1 \leq j \leq \theta \),
\[ \frac{\sigma_j}{f(x^{\text{sol}})} \leq \frac{1}{\beta} \frac{\sigma_j}{\sigma_j + (1 - \beta) \sigma_j} = \frac{\sigma_j}{\beta \sigma_j + (1 - \beta) \sigma_j} \] (39)

where we WLOG impose the normalization that \( \sum_{i=1}^{\theta} \sigma_i = \theta \). Note that the RHS is concave with respect to \( \sigma_j \). Recall Jensen’s inequality states that given a convex function \( g \) and numbers \( y_1, \ldots, y_m \), then \( \sum_{i=1}^{m} g(y_i) \) is minimized when \( y_1 = \cdots = y_m \). Therefore, substituting \( \sigma_1 = \cdots = \sigma_\theta = 1 \) into (37) yields:
\[ \frac{1}{\gamma'(G)} = \frac{f(x^{\text{opt}})}{f(x^{\text{sol}})} \leq 1 + \frac{(1 - \beta) \theta}{\beta (\theta - 1) + 1} = \frac{\theta - 1 + \beta}{\beta (\theta - 1) + 1} \] (40)

Hence we derived the lower bound.

Now we show the upper bound on \( \gamma'(G) \). Let \( I \) be a maximum independent set of \( G \) and \( \alpha = \left| I \right| \). Consider \( X = \{u_1, \ldots, u_\alpha, v_1, \ldots, v_\alpha\} \) and action profile determined by:
\[ X_i = \begin{cases} \{u_i, v_i\} & \text{if } i \in I \\ \emptyset & \text{otherwise} \end{cases} \] (41)

Define the objective function \( f \) so that for any \( x \in X \),
\[ f(x) = \min \left( 1 - \beta, \sum_{u_i \in x} 1 - \beta \right) + \sum_{v_i \in x} \beta + \sum_{v_i \in x} 1 \] (42)

Note that through this construction, \( f \) is a submodular function and is also \( \beta \)-strictly monotonic. \( f \) also has the following properties for any \( i \in I \):
1) \( f(u_i) = f(v_i) = 1 \).
2) \( f(u_i | x_{\setminus i}) = f(u_i) \) for any \( x \in X \) because by definition, we have \( X_i \cap I = \emptyset \).
3) \( f(v_i | B) = f(v_i) \) for any \( B \subseteq S \setminus \{v_i\} \).

From these properties, the agents in \( I \) are equally incentivized to pick either option. In the greedy solution, the actions are the \( u \)'s. And in the optimal solution, the actions are the \( v \)'s. This results in \( f(x^{\text{sol}}) \leq (1 - \beta) + \alpha \beta \) and \( f(x^{\text{opt}}) = \alpha \), hence \( \gamma'(G) \leq ((\alpha - 1) \beta + 1) / \alpha \).

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