Fundamental Limits of Byzantine Agreement

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Abstract

Byzantine agreement (BA) is a distributed consensus problem where $n$ processors want to reach agreement on an $\ell$-bit message or value, but up to $t$ of the processors are dishonest or faulty. The challenge of this BA problem lies in achieving agreement despite the presence of dishonest processors who may arbitrarily deviate from the designed protocol. The quality of a BA protocol is measured primarily by using the following three parameters: the number of processors $n$ as a function of $t$ allowed (resilience); the number of rounds (round complexity); and the total number of communication bits (communication complexity). For any error-free BA protocol, the known lower bounds on those three parameters are $3t + 1$, $t + 1$ and $\Omega(\max\{n\ell, nt\})$, respectively, where a protocol that is guaranteed to be correct in all executions is said to be error free.

In this work, by using coding theory we design a coded BA protocol (termed as COOL) that achieves consensus on an $\ell$-bit message with optimal resilience, asymptotically optimal round complexity, and asymptotically optimal communication complexity when $\ell \geq t \log n$, simultaneously. The proposed COOL is an error-free and deterministic BA protocol that does not rely on cryptographic technique such as signatures, hashing, authentication and secret sharing (signature free). It is secure against computationally unbounded adversary who takes full control over the dishonest processors (information-theoretic secure). The main idea of the proposed COOL is to use a carefully-crafted error correction code that provides an efficient way of exchanging “compressed” information among distributed nodes, while keeping the ability of detecting errors, masking errors, and making a consistent and validated agreement at honest distributed nodes. We show that our results can also be extended to the setting of Byzantine broadcast, aka Byzantine generals problem, where the honest processors want to agree on the message sent by a leader who is potentially dishonest. This work reveals that coding is an effective approach for achieving the fundamental limits of Byzantine agreement and its variants.

Index Terms

Multi-valued Byzantine agreement, Byzantine broadcast, information-theoretic security, signature-free protocol, error-free protocol, error correction codes.

I. INTRODUCTION

Byzantine agreement (BA), as originally proposed by Pease, Shostak and Lamport in 1980, is a distributed consensus problem where $n$ processors want to reach agreement on some message (or value), but up to $t$ of the processors are dishonest or faulty [1]. The challenge of this BA problem lies in achieving agreement despite the presence of dishonest processors who may arbitrarily deviate from the designed protocol. One variant of the problem is Byzantine broadcast, aka Byzantine generals problem, where the honest processors want to agree on the message sent by a leader who is potentially dishonest [2]. Byzantine agreement and its variants are considered to be the fundamental building blocks for distributed systems and cryptography including Byzantine-fault-tolerant (BFT) distributed computing, distributed storage, blockchain protocols, state machine replication and voting, just to name a few [1]–[15].

To solve the Byzantine agreement problem, a designed protocol needs to satisfy the following conditions: every honest processor eventually outputs a message and terminates (termination); all honest processors output the same message (consistency); and if all honest processors hold the same initial message then they output this initial message (validity). A protocol that satisfies the above three conditions in all executions is said to be error free. The quality of a BA protocol is measured primarily by using three parameters:

- Resilience: the number of processors $n$ as a function of $t$ allowed.
- Round complexity: the number of rounds of exchanging information.
- Communication complexity: the total number of communication bits.

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TABLE I
COMPARISON OF PROPOSED AND SOME OTHER ERROR-FREE SYNCHRONOUS BA PROTOCOLS.

| Protocols | resilience | communication complexity | round complexity | error free | signature free |
|-----------|------------|--------------------------|------------------|------------|---------------|
| ℓ-1-bit   | n ≥ 3ℓ + 1 | Ω(n^2ℓ)                  | Ω(ℓt)            | yes        | yes           |
| [4]       | n ≥ 3ℓ + 1 | O(nℓ + n^4\sqrt{ℓ} + n^6) | Ω(\sqrt{ℓ} + n^2) | yes        | yes           |
| Proposed  | n ≥ 3ℓ + 1 | O(max{nl, nt log n})      | O(t)             | yes        | yes           |

For any error-free BA protocol, the known lower bounds on those three parameters are respectively

$$3t + 1 \quad \text{ (cf. [1], [2]),} \quad t + 1 \quad \text{ (cf. [16], [17]),} \quad \Omega(\max\{n\ell, nt\}) \quad \text{ (cf. [3], [18])}$$

where ℓ denotes the length of message being agreed upon. It is worth mentioning that, in practice the consensus is often required for multi-valued message rather than just single-bit message [3]–[5], [19]. For example, in BFT consensus protocols of Libra and Hyperledger Fabric Blockchains proposed by Facebook and IBM respectively, the message being agreed upon could be a transaction or transaction block with size scaled from 1KB to 1MB [20]–[26]. Also, in practice the consensus is often expected to be 100% secure and error free in mission-critical applications such as online banking and smart contracts [27]–[31].

Achieving an error-free multi-valued consensus with optimal resilience, optimal round complexity and optimal communication complexity simultaneously is widely believed to be the “holy-grail” in the BA problem. The multi-valued BA problem of achieving consensus on an ℓ-bit message could be solved by invoking ℓ instances of 1-bit consensus in sequence, which is termed as ℓ-1-bit scheme. However, this scheme will result in communication complexity of Ω(n^2ℓ) bits, because Ω(n^2) is the lower bound on communication complexity of 1-bit consensus given n ≥ 3t + 1 [18], [32], [33]. In 2006, Fitzi and Hirt provided a probabilistically correct multi-valued BA protocol by using a hashing technique, which results in communication complexity of O(nℓ + n^3(n + κ)) bits for some κ serving as a security parameter [3]. This improves the communication complexity to O(nℓ) when ℓ ≥ n^3, but at the cost of non-zero error probability. In 2011, Liang and Vaidya provided an error-free BA protocol with communication complexity O(nℓ + n^4\sqrt{ℓ} + n^6) bits, which is optimal when ℓ ≥ n^6 [4]. However, in the practical regime of ℓ < n^6, this communication complexity is sub-optimal. Furthermore, the round complexity of the BA protocol in [4] is at least Ω(\sqrt{ℓ} + n^2), which is sub-optimal. In some previous work, randomized algorithms were proposed to reduce communication and round complexities but the termination cannot be 100% guaranteed [5], [34]–[38]. In some other work, the protocols were designed with cryptographic technique such as signatures, hashing, authentication and secret sharing [9], [17], [38], [39]. However, the protocols with such cryptographic technique are vulnerable to attacks from the adversary with very high computation power, e.g., using supercomputer or quantum computer possibly available in the future, and hence not error free. A protocol that doesn’t rely on cryptographic technique mentioned above is said to be signature free. A protocol is said to be information-theoretic secure if it is secure against computationally unbounded adversary who takes full control over the dishonest processors.

In this work we show that coding is a promising approach for solving the long-standing open problem in BA. Specifically, by using coding theory we are able to design an error-free signature-free information-theoretic-secure multi-valued BA protocol with optimal resilience, asymptotically optimal round complexity, and asymptotically optimal communication complexity when ℓ ≥ t \log n, simultaneously (see Table I), focusing on the BA setting with synchronous communication network. In a nutshell, carefully-crafted error correction codes provide an efficient way of exchanging “compressed” information among distributed nodes, while keeping the ability of detecting errors, masking errors, and making a consistent agreement at honest distributed nodes. In this work the proposed protocol is developed first for the Byzantine agreement problem. We show that the proposed protocol can also be generalized to the Byzantine broadcast problem with the same performance as in the Byzantine agreement problem.
The remainder of this paper is organized as follows. Section II describes the system models. Section III provides the main results of this work. Section IV discusses some advantages, issues, and challenges of coding for BA. The proposed COOL developed from coding theory is described in Section V for the BA setting. In Section VII the proposed COOL is shown to be extended to the setting of Byzantine broadcast. The work is concluded in Section VIII. Throughout this paper, \(|\bullet|\) denotes the magnitude of a scalar or the cardinality of a set. \([m_1 : m_2]\) denotes the set of integers from \(m_1\) to \(m_2\), for some nonnegative integers \(m_1 \leq m_2\). If \(m_1 > m_2\), \([m_1 : m_2]\) denotes an empty set. Logarithms are in base 2. \([m]\) denotes the least integer that is no less than \(m\), and \(|m|\) denotes the greatest integer that is no larger than \(m\). \(f(x) = O(g(x))\) implies that \(\limsup_{x \to \infty} |f(x)|/g(x) < \infty\). \(f(x) = \Omega(g(x))\) implies that \(\liminf_{x \to \infty} f(x)/g(x) > 0\). \(f(x) = \Theta(g(x))\) implies that \(f(x) = O(g(x))\) and \(f(x) = \Omega(g(x))\).

II. SYSTEM MODELS

In the BA problem, \(n\) processors want to reach agreement on an \(\ell\)-bit message (or value), but up to \(t\) of the processors are dishonest (or faulty). Processor \(i\) holds an \(\ell\)-bit initial message \(u_i\), \(\forall i \in [1 : n]\). To solve this BA problem, a designed protocol needs to satisfy the termination, consistency, and validity conditions mentioned in the previous section.

We consider the synchronous BA, in which every two processors are connected via a reliable and private communication channel, and the messages sent on a channel are guaranteed to reach to the destination on time. The index of each processor is known to all other processors. We assume that a Byzantine adversary takes full control over the dishonest processors and has complete knowledge of the state of the other processors, including the \(\ell\)-bit initial messages.

As mentioned, a protocol that satisfies the termination, consistency, and validity conditions in all executions is said to be error free. A protocol that doesn’t rely on the cryptographic technique such as signatures, hashing, authentication and secret sharing is said to be signature free. A protocol that is secure (satisfying the termination, consistency, and validity conditions) against computationally unbounded adversary is said to be information-theoretic secure.

We also consider the BB problem, where the honest processors want to agree on the message sent by a leader who is potentially dishonest. To solve the BB problem, a designed protocol needs to satisfy the termination, consistency, and validity conditions. In the BB problem, termination and consistency conditions remain the same as that in the BA problem, while the validity condition is slightly different from that in the BA problem. Specifically, the validity condition of the BB problem requires that, if the leader is honest then all honest processors should agree on the message sent by the leader. Other definitions follow similarly from that of the BA problem.

III. MAIN RESULTS

In this work, by using coding theory we design a coded BA protocol, termed as COOL, that achieves consensus on an \(\ell\)-bit message with promising performance in resilience, round complexity, and communication complexity. This proposed COOL is also extended to the Byzantine broadcast setting. The main results of this work are summarized in the following theorems. The proofs are provided in Sections V-VII.

**Theorem 1** (BA problem). The proposed COOL is an error-free signature-free information-theoretic-secure multi-valued BA protocol that achieves the consensus on an \(\ell\)-bit message with optimal resilience, asymptotically optimal round complexity, and asymptotically optimal communication complexity when \(\ell \geq t \log n\), simultaneously.

The proposed COOL achieves the consensus on an \(\ell\)-bit message with resilience of \(n \geq 3t + 1\) (optimal), round complexity of \(O(t)\) rounds (asymptotically optimal), and communication complexity of \(O(\max\{n\ell, nt \log n\})\) bits (asymptotically optimal when \(\ell \geq t \log n\)), simultaneously. The description of COOL is provided in Section V. Note that, for any error-free BA protocol, the known lower bounds
on resilience, round complexity and communication complexity are $3t + 1$ (cf. [1], [2]), $t + 1$ (cf. [16], [17]), and $\Omega(\max\{n\ell, nt\})$ (cf. [3], [18]), respectively.

**Theorem 2** (BB problem). *The proposed adapted COOL is an error-free signature-free information-theoretic-secure multi-valued BB protocol that achieves the consensus on an $\ell$-bit message with optimal resilience, asymptotically optimal round complexity, and asymptotically optimal communication complexity when $\ell \geq t \log n$, simultaneously.*

The proposed COOL designed for the BA setting can be adapted into the BB setting, which achieves the consensus on an $\ell$-bit message with the same performance of resilience, round complexity and communication complexity as in the BA setting. Note that the for any error-free BB protocol, the known lower bounds on resilience, round complexity and communication complexity remain the same as for the error-free BA protocol. The description of adapted COOL for the BB setting is provided in Section VII. Before describing the proposed COOL, let us first describe the advantages, issues, and challenges of coding for BA in the following section.

### IV. CODING FOR BA: ADVANTAGES, ISSUES, AND CHALLENGES

Coding and information theory provide very elegant and powerful ways of detecting or correcting errors in possibly corrupted data via carefully constructed codes with certain properties, which have revolutionized the digital era. This work seeks to use these concepts and techniques to bring long-lasting benefits in a new field: Byzantine agreement.

#### A. Advantages of using coding

For the multi-valued Byzantine agreement problem, error correction codes can be utilized to achieve consensus for the entire string of message but not bit by bit. In order to see the advantages of coding for Byzantine agreement, let us discuss the following three schemes.

- **Scheme with full data transmission**: For the $\ell$-1-bit scheme with bit-by-bit consensus in sequence, since distributed processors need to exchange the whole message data, it is not surprising that this scheme results in a sub-optimal communication complexity (see Table I).
- **Non-coding scheme with reduced data transmission**: To reduce the communication complexity, one possible solution is to let distributed processors exchange only one piece of data, instead of the whole data. However, if the data is uncoded, this reduction in data exchange could possibly lead to a failure, i.e., not satisfying termination, consistency or validity conditions. This is because honest processors might not get enough information from others and hence this protocol is more vulnerable to the attacks from dishonest processors.
- **Coding scheme with reduced data transmission**: If the data is coded with error correction code, the distributed processors could possibly be able to detect and correct errors, even with reduced data transmission. Hence, the protocol using carefully-crafted coding scheme with reduced data transmission could be robust against attacks from dishonest processors and could be error free. Table II provides some comparison between the three schemes.

| Schemes                  | communication complexity | error free |
|--------------------------|--------------------------|------------|
| scheme with full DT      | high                     | yes        |
| non-coding scheme with reduced DT | low                 | no         |
| coding scheme with reduced DT | low                 | yes        |
B. Error correction codes

The $(n, k)$ Reed-Solomon error correction code encodes $k$ data symbols from Galois Field $GF(2^c)$ into a codeword that is consisting of $n$ symbols from $GF(2^c)$, for

$$n \leq 2^c - 1$$

(cf. [40]). We can use $c$ bits to represent each symbol from $GF(2^c)$, which implies that a vector consisting of $k$ symbols from $GF(2^c)$ can be represented using $kc$ bits of data. The error correction code can be constructed by Lagrange polynomial interpolation [41]. This constructed code is a variant of Reed-Solomon code with minimum distance $d = n - k + 1$, which is optimal according to the Singleton bound. For an input vector $x \triangleq [x_1, x_2, \ldots, x_k]^\top$ and an output vector $z \triangleq [z_1, z_2, \ldots, z_n]^\top$ with data symbols from $GF(2^c)$, the encoding is described below

$$z_i = h_i^\top x \quad i \in [1 : n]$$  \hspace{1cm} (1)

where $h_i \triangleq [h_{i,1}, h_{i,2}, \ldots, h_{i,k}]^\top$ and

$$h_{i,j} \triangleq \prod_{p=1 \atop p \neq j}^{k} \frac{i - p}{j - p}, \quad i \in [1 : n], \quad j \in [1 : k].$$

Note that when $k = 1$, it becomes a non-coding scheme, with coefficients set as $h_i = h_{i,1} = 1$ for $i \in [1 : n]$. An $(n, k)$ error correction code can correct up to $\lfloor \frac{n-k}{2} \rfloor$ Byzantine errors, that is, up to $\lfloor \frac{n-k}{2} \rfloor$ Byzantine errors in $n$ symbol observations (see [1]) can be corrected by applying some efficient decoding algorithms for Reed-Solomon code, such as, Berlekamp-Welch algorithm and Euclid’s algorithm [40], [42], [43]. More generally, an $(n, k)$ error correction code can correct up to $f_c$ Byzantine errors and simultaneously detect up to $f_d$ Byzantine errors in $n'$ symbol observations if and only if the conditions of $2f_c + f_d \leq n' - k$ and $n' \leq n$ are satisfied.

C. Coding scheme with reduced DT: what could possibly go wrong?

In the BA problem, if distributed processors exchange information with reduced data transmission, the honest processors might not be able to get enough information from others and might be vulnerable to attacks from dishonest processors. In this sub-section, we will describe some possible issues when using coding scheme with reduced data transmission. In the next section, we will show how to handle the issues by carefully designing the protocol.

1) Attack example: violating validity condition: Let us consider an example where all honest processors hold the same initial message $\bar{w}$, for some non-empty $\ell$-bit value of $\bar{w}$. By using coding scheme, each processor could encode its initial message $\bar{w}$ into a symbol with reduced size, as in (1). In this way, the processors could exchange information with reduced data transmission. For example, Processor $i$, $\forall i \in [1 : n]$, could encode its initial message $\bar{w}$ into a symbol as $y_i = h_i^\top \bar{w}$, where $h_i$ is defined below (1). The size of $y_i$ is $c = \ell/k$ bits, which is relatively small compared to the original message size, when $k$ is sufficiently large. By exchanging the $\ell/k$-bit information $y_i$, instead of the $\ell$-bit message $\bar{w}$, for $i \in [1 : n]$, the communication complexity could be significantly reduced.

However, if the protocol is not well designed, the honest processors might not be able to make a correct consensus output that is supposed to be $\bar{w}$ in this case under the validity condition. For example, if $k$ is designed to be too large, then the honest processors might not be able to correct the errors injected by the dishonest processors, and hence make a wrong consensus output, e.g., as a default value $\phi$, violating the validity condition (see Fig. [1]).

\footnote{For the extended Reed-Solomon codes, the constraint can be relaxed to $n \leq 2^c + 1$, and to $n \leq 2^c + 2$ in some cases.}
Fig. 1. An attack example violating the validity condition with \((t = 4, n = 13)\). The light-cyan square nodes refer to honest processors, while the rest nodes refer to dishonest processors.

2) Attack example: violating consistency condition: Let us consider another example with three disjoint groups in \(n\)-processor network\(^2\) Group \(F\), Group \(\bar{A}_1\) and Group \(\bar{A}_2\). Group \(F\) is a set of \(t\) dishonest processors. Group \(\bar{A}_j\) is a set of honest processors holding the same initial message \(\bar{w}_j\), for \(j = 1, 2, |\bar{A}_1| = t + 1, |\bar{A}_2| = t\), and \(\bar{w}_1 \neq \bar{w}_2\) (see an example in Fig. 2). By using coding scheme, Processor \(i\), \(i \in \bar{A}_j\) could encode its initial \(\ell\)-bit message \(\bar{w}_j\) into an \(\ell/k\)-bit symbol as \(y_i = h_i^T \bar{w}_j\). By sending the “compressed” information \(y_i\) to other processors, the communication complexity could be significantly reduced. However, if the protocol is not well designed, the honest processors might make inconsistent consensus outputs.

To attack the protocol, dishonest processors could send out inconsistent information, e.g., sending \(y_j' = h_j^T \bar{w}_1\) to Group \(\bar{A}_1\) and \(y_j'' = h_j^T \bar{w}_2\) to Group \(\bar{A}_2\) respectively, \(\forall j \in F\). In this scenario, each processor in Group \(\bar{A}_j\) might make a consensus output as \(\bar{w}_1\). This is because, for each processor in Group \(\bar{A}_1\), at least \(2t + 1\) observations received from Group \(\bar{A}_1\) (including itself) and Group \(F\) are encoded from message \(\bar{w}_1\). Hence, each processor in Group \(\bar{A}_1\) has an illusion that at least \(2t + 1\) processors hold the same initial message \(\bar{w}_1\). At the same time, each processor in Group \(\bar{A}_2\) might make a consensus output as \(\bar{w}_2\). This is because, \(2t\) observations received from Group \(\bar{A}_2\) and Group \(\bar{F}\) are encoded from message \(\bar{w}_2\). Furthermore, some observations received from Group \(\bar{A}_1\) could possibly be expressed as

\[
h_i^T \bar{w}_1 = h_i^T \bar{w}_2 \quad \text{for some } i \in \bar{A}_1
\]

when \((\bar{w}_1 - \bar{w}_2)\) is in the null space of \(h_i\). With the condition in (2), each processor in Group \(\bar{A}_2\) has an illusion that at least \(2t + 1\) processors hold the same initial message \(\bar{w}_2\). At the end, honest processors might make inconsistent consensus outputs (see Fig. 2).

3) Attack example: violating termination condition: For the coding scheme with reduced data transmission, if the protocol is not designed well, it might not be able to terminate. For example, one possible termination condition at a processor is that it needs to hear at least \(2t + 1\) “ready” signals from other processors, where the “ready” signal could be sent when a processor is able to correct up to \(t\) errors

\(^2\)The use of the notation \(\bar{A}_i\) is to differentiate it from the other notation \(A_i\) that will be shown later. Note that \(\bar{A}_i\) does not mean the “complement” of \(A_i\). The “complement” notation is not used throughout this paper.
from the received observations. However, due to the inconsistent information injected from dishonest processors, there might be some events that the termination condition will never be satisfied and hence the protocol won’t terminate.

D. Coding scheme with reduced DT: what are the challenges?

For the coding scheme with reduced date transmission, since processors exchange coded information with reduced size (cf. [1]), it might create some illusion at some processors as described in Section IV-C.2. As shown in (2), when Processor \( i \) sends a coded information \( \bar{h}^i_w \) to other processors, this coded information could possibly be expressed as \( \bar{h}^i_w = \bar{h}^i_{w_2} \), for some different \( w_1 \) and \( w_2 \), which creates an illusion at other processors that Processor \( i \) might initially hold the message \( w_2 \), but not \( w_1 \). Removing such illusion is one of the challenges in designing the coding scheme with reduced date transmission. This is because, sending the coded information with reduced dimension, e.g., \( \bar{h}^i_w \), could not reveal enough information about the initial message \( w_1 \). To make it efficient and secure, i.e., satisfying termination, consistency and validity conditions, the coding scheme needs to be carefully crafted.

Another challenge lies in a constraint of Reed-Solomon error correction code used in the coding scheme with reduced date transmission. The \((n, k)\) Reed-Solomon error correction code encodes \( k \) data symbols from Galois Field \( GF(2^c) \) into a codeword that is consisting of \( n \) symbols from \( GF(2^c) \), but under the constraint of \( n \leq 2^c - 1 \), or equivalently \( \log(n + 1) \leq c \) (see Section IV-B). This constraint implies that each symbol from \( GF(2^c) \) is represented by at least \( \log(n + 1) \) bits. If every processor sends one such symbol to all other processors, it will result in a communication complexity of at least \( \Omega(n^2 \log n) \) bits. As it will be shown later on, this is the main reason that there is still a multiplicative gap (within \( \log n \) between the communication complexity of proposed protocol and the known lower bound, when \( \ell < t \log n \). When \( \ell \geq t \log n \), the proposed protocol achieves the asymptotically optimal communication complexity of \( \Theta(n \ell) \) bits. To close the gap in the regime of \( \ell < t \log n \), one might need to overcome the challenge incurred by the constraint \( n \leq 2^c - 1 \) of Reed-Solomon error correction code.

V. COOL: coded Byzantine agreement protocol

This section will describe the proposed COOL: coded Byzantine agreement protocol. The design of COOL is based on the coding scheme with reduced data transmission. In a nutshell, carefully-crafted error correction codes provide an efficient way of exchanging “compressed” information among distributed nodes, while keeping the ability of detecting errors, masking errors, and making a consistent and validated agreement at honest distributed nodes. To this end, it will be shown that the proposed COOL is an error-free signature-free information-theoretic-secure multi-valued BA protocol that achieves the consensus on an \( \ell \)-bit message with resilience of \( n \geq 3t + 1 \) (optimal), round complexity of \( O(t) \) rounds (asymptotically optimal), and communication complexity of \( O(\max\{n\ell, nt \log n\}) \) bits (asymptotically optimal when \( \ell \geq t \log n \)), simultaneously. This result will serve as the achievability proof of Theorem 1.

For this BA problem, Processor \( i \), \( \forall i \in [1 : n] \), initially has the \( \ell \)-bit input message \( w_i \). At first the parameters \( k \) and \( c \) are designed as

\[
k \triangleq \left\lfloor \frac{t}{2} \right\rfloor + 1, \quad c \triangleq \left\lceil \max\{\ell, (t/2 + 1) \log(n + 1)\} \right\rceil.
\]

With the above values of \( k \) and \( c \), it holds true that the condition of the Reed-Solomon error correction code, i.e., \( n \leq 2^c - 1 \), is satisfied (see Section IV-B). The \((n, k)\) Reed-Solomon error correction code will be used to encode the \( \ell \)-bit initial message \( w_i \), \( \forall i \in [1 : n] \). When \( \ell \) is less than \( kc \) bits, the \( \ell \)-bit message \( w_i \) will be first extended to a \( kc \)-bit data by adding \( (kc - \ell) \) bits of redundant zeros (zero padding). In the description of the proposed COOL, \( t \) is considered such that \( t \leq (n - 1)/3 \) and \( t = \Omega(n) \). Later on we will discuss the case when \( t \) is relatively small compared to \( n \).

The proposed COOL consists of at most four phases (see Fig. 3), which are described in the following sub-sections. The proposed COOL is also described in Algorithm 1. Two examples of COOL are provided in Fig. 4 and Fig. 5, which address the issues described in Section IV-C.2 and Section IV-C.1, respectively. Some notations of COOL are summarized in Table III.
Phase 1: exchange compressed information and decode message

Phase 1 has two steps. The idea is to exchange “compressed” information and learn it.

1) Exchange compressed information: Processor $i$, $\forall i \in [1 : n]$, sends a coded value

$$y_i \triangleq h_i^T \mathbf{w}_i$$

(4)

to all other processors, where $h_i$ is defined below (1).

2) Decode message: Processor $i$, $\forall i \in [1 : n]$, decodes the message with the observations $\{y_1, y_2, \cdots, y_n\}$ obtained from $n$ processors respectively. Note that different processors might have different views on $\{y_1, y_2, \cdots, y_n\}$, due to the inconsistent information sent from dishonest processors. Let $\mathbf{w}^{(i)}$ denote the decoded message at Processor $i$. At Processor $i$, an observation $y_j$ is marked as an observation error if

$$y_j \neq h_j^T \mathbf{w}^{(i)}$$

(5)

for some $j$. The decoding at Processor $i$ in this step is said to be successful if there exist one unique non-empty decoded message $\mathbf{w}^{(i)}$ such that both of the following two conditions are satisfied: (a) the number of observation errors is no more than $t$; (b) the decoded message matches the original message, i.e., $\mathbf{w}^{(i)} = \mathbf{w}_i$. If the decoding is not successful, then Processor $i$ sets $\mathbf{w}^{(i)} = \phi$ as a default value. If the decoding is successful, then Processor $i$ sets a binary success indicator, denoted by $s_i$, as

$$s_i = 1$$

(6)

else sets $s_i = 0$, for $i \in [1 : n]$.

Remark 1. Since $y_i$ defined in (4) has only $c$ bits, the total communication complexity among the network for the first step of Phase 1, denoted by $b_1$, is $b_1 = cn(n - 1)$ bits. If Processor $i$, $\forall i \in [1 : n]$, sends the whole message $\mathbf{w}_i$ to other processors, then the total communication complexity would be $\ell n(n - 1)$ bits. Compared to the whole message $\mathbf{w}_i$, the value of $y_i$ can be considered as a compressed information. By exchanging compressed information, instead of whole messages, the communication complexity is significantly reduced in this step.

Remark 2. In Phase 1, exchanging “compressed” information reduces the communication complexity, however, it also creates some potential issues due to the lack of full original information. As shown in Fig. 4, the dishonest processors could send inconsistent information to two different groups of honest processors, that is, $y'_j \triangleq h_j^T \mathbf{w}_1$ to Group $A_1$ and $y''_j \triangleq h_j^T \mathbf{w}_2$ to Group $A_2$ respectively, $\forall j \in F$. Due to this inconsistent information, together with the condition of $h_1^T \mathbf{w}_1 = h_2^T \mathbf{w}_2$, the honest processors from different groups consequently decode the messages differently, which might lead to inconsistent consensus outputs as described in Section IV-C2. In the next phases the effort is to address the potential issues by detecting errors, masking errors, and broadcasting “trusted” information.
B. Phase 2: detect errors

Phase 2 consists of three steps. The goal is to detect errors from the honest processors.

1) Exchange decoded information: Processor $i$, $\forall i \in [1 : n]$, sends

$$y_j^{(i)} \triangleq h_j^T w^{(i)}$$

(7)

to Processor $j$, $\forall j \in [1 : n], j \neq i$.

2) Detect errors: Processor $i$, $\forall i \in [1 : n]$, uses a binary error indicator $e_i$ to indicate whether if the observation $y_i$ sent from Processor $i$ is causing a certain number of observation errors at other processors or not. Specifically, Processor $i$ sets

$$e_i = 1$$

(8)

if at least one of the following two conditions is satisfied: (a) $s_i = 0$; (b) at least $t + 1$ values of $y_j^{(i)}$ obtained from $n$ processors are different from $y_i$. If none of the above two conditions is satisfied, then Processor $i$ sets $e_i = 0$. Processor $i$ sets $w^{(i)} = \phi$ if $e_i = 1$, else keeps the original value of $w^{(i)}$.

3) Exchange error indicators: Processor $i$, $\forall i \in [1 : n]$, sends the value of $e_i$ to other processors. Let

$$\mathcal{E} \triangleq \{j : e_j = 1, j \in [1 : n]\}.$$  

(9)

Note that different processors might have different views on $\mathcal{E}$, due to the inconsistent information possibly sent from dishonest processors.

Remark 3. Since $y_j^{(i)}$ defined in (7) has only $c$ bits, the total communication complexity for the first step of Phase 2, denoted by $b_2$, is $b_2 = cn(n - 1)$ bits. Since $e_i$ defined in (8) has only 1 bit, the total communication complexity for the third step of Phase 2, denoted by $b_3$, is $b_3 = n(n - 1)$ bits.

Remark 4. For the example in Fig. 4 during Phase 2 the dishonest processors could send inconsistent information to Group $\mathring{A}_1$ and Group $\mathring{A}_2$, that is, $h_j^T \bar{w}_1$ to Processor $j$ for $j \in \mathring{A}_1$ and $h_j^T \bar{w}_2$ to Processor $j'$ for $j' \in \mathring{A}_2$. In this phase Processor $i$, $i \in \{6, 7, 8\} \subset \mathring{A}_2$, identifies that its initial message doesn’t match the majority of other processors’ initial messages, and then updates the decoded message as $w^{(i)} = \phi$, for $i \in \{6, 7, 8\}$. Note that in Phase 2, Processor 9 still thinks that its initial message does match the majority
of other processors’ initial messages because of the condition $h_7^0 \bar{w}_1 = h_7^0 \bar{w}_2$. This condition implies that $y_9 = y_9^{(j)}$ for $j \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ from the view of Processor 9, and hence results in an output of $e_9 = 0$ at Processor 9.

**Remark 5.** The goal in Phase 2 is to detect as much as possible the errors, i.e., the mismatch of the initial messages among the honest processors. The idea here is to let each honest processor tell if its initial message is matched with others’ or not, from the majority point of view. Based on our protocol design, once an honest processor, say, Processor $i$ for some $i$, tells the error indicator $e_i = 1$ to other processors, it is ensured that its initial message is not matched with other honest processors’ initial messages, from the majority point of view. Note that, in Phase 2, not all of the errors from the honest processors could be detected out. The goal in the next phase is to handle the rest of the errors from the honest processors.

**C. Phase 3: mask errors and re-decode messages**

Phase 3 has five steps. The goal is to mask the rest of errors from the honest processors.

1) **Mask errors identified in Phase 2:** Processor $i$, $\forall i \notin \mathcal{E}$, sets

$$y_j = \phi, \ \forall j \in \mathcal{E}. \quad (10)$$

2) **Re-decode messages:** Processor $i$, $\forall i \notin \mathcal{E}$, decodes the message again as in Phase 1 using updated observations $\{y_1, \cdots, y_n\}$ and then updates $u_i^{(t)}$ and $s_i$.

3) **Exchange success indicators:** Processor $i$, $\forall i \in [1:n]$, sends success indicator $s_i$ to others.

4) **Vote:** Processor $i$, $\forall i \in [1:n]$, sets a binary vote as

$$v_i = 1 \quad (11)$$

if it receives no less than $2t + 1$ number of ones from $n$ success indicators $\{s_1, s_2, \cdots, s_n\}$, else sets $v_i = 0$. The indicator $v_i$ can be considered as a vote from Processor $i$ for going to next phase or stopping in this phase.

5) **One-bit consensus on the $n$ votes:** In this step the system runs one-bit consensus on the $n$ votes $\{v_1, v_2, \cdots, v_n\}$ from all processors. By using the one-bit consensus from [32], [33], the correct consensus can be made by using $O(nt)$ bits of communication complexity, and $O(t)$ rounds of round complexity, for $t < n/3$. If the consensus of the votes $\{v_1, v_2, \cdots, v_n\}$ is 1, then every honest processor goes to next phase, else every honest processor sets $u_i^{(t)} = \phi$ and considers it as a final consensus and stops here.

**Remark 6.** Since $s_i$ has only 1 bit, the total communication complexity for the third step of Phase 3, denoted by $b_4$, is $b_4 = n(n - 1)$ bits.

**Remark 7.** The one-bit consensus from [32], [33] ensures that: (a) every honest processor eventually outputs a consensus value; (b) all honest processors should have the same consensus output; (c) if all honest processors have the same vote then the consensus should be the same as the vote of the honest processors. Since we run the one-bit consensus from [32], [33], the total communication complexity for fifth step of Phase 3, denoted by $b_5$, is $b_5 = O(nt)$ bits, while the round complexity of this step is $O(t)$ rounds, which dominates the total round complexity of the proposed protocol.

**Remark 8.** The idea of Phase 3 is to mask out the errors from honest processors whose initial messages don’t match the majority of other processors’ initial messages, but could not be detected out in Phase 2 due to the condition as in (2). For the example in Fig. 4 at the second step of Phase 3, Processor 9 re-decodes the message and outputs $w_9^{(t)} = \phi$. This is because the number of observation errors is at least 7, corresponding to the updated observations $\{y_j\}_{j=2}^{9}$ originally received from Processors 2-8 respectively, from the view of Processor 9.
D. Phase 4: broadcast correct coded messages and make consensus

This phase is taken place only when one-bit consensus of the votes \( \{v_1, v_2, \ldots, v_n\} \) is 1. This phase has five steps. For notational convenience, let

\[
S_1 \triangleq \{i : s_i = 1, i \in [1 : n]\}, \quad S_0 \triangleq \{i : s_i = 0, i \in [1 : n]\}. \tag{12}
\]

1) **Broadcast correct coded messages**: Processor \( i, \forall i \in S_1 \) sends the value of \( y_j^{(i)} = h_j^t w^{(i)} \) to Processor \( j, \forall j \in S_0 \).

2) **Update information with majority rule**: Processor \( i, \forall i \in S_0 \) updates the value of \( y_i \) as

\[
y_i \leftarrow \text{Majority}((y_i^{(j)} : j \in S_1)) \tag{13}
\]

where \( \{y_i^{(j)} : j \in S_1\} \) were received in the previous step. Majority(\( \bullet \)) is a function that returns the most frequent value in the list, based on majority rule. For example, Majority(1, 2, 2) = 2.

3) **Broadcast updated information**: Processor \( i, \forall i \in [1 : n] \), sends the value of \( y_i \) to Processor \( j, \forall j \in S_0 \).

4) **Decode the message again**: Processor \( i, \forall i \in S_0 \), decodes the message again using the new observations \( \{y_1, y_2, \ldots, y_n\} \).

5) **Stop**: Processor \( i, \forall i \in [1 : n] \), outputs consensus as decoded message \( w^{(i)} \) and stops.

**Remark 9.** Since \( y_i^{(i)} \) has only \( c \) bits, the total communication complexity for the first step of Phase 4, denoted by \( b_6 \), is upper bounded by \( b_6 \leq cn(n - 1) \) bits. Since \( y_i \) has only \( c \) bits, the total communication complexity for the third step of Phase 4, denoted by \( b_7 \), is upper bounded by \( b_7 \leq cn(n - 1) \) bits.

**Remark 10.** The idea of Phase 4 is to broadcast “trusted” information from the set of honest processors whose initial messages match the majority of other processors’ initial messages. In this way, the set of honest processors whose initial messages don’t match the majority of other processors’ initial messages could calibrate and update their information. In some scenarios, the protocol could terminate at Phase 3, depending on the information sent from dishonest processors. In any scenario, all honest processors eventually make the same consensus output. For the example in Fig. 4 since the size of \( A_1 \) is bigger
than the size of $F$, it guarantees that Group $A_2$ could calibrate and update their information successfully in the second step of Phase 4, eventually making the same consensus output as Group $A_1$.

**Remark 11.** Fig. 3 depicts another example of COOL, addressing the issue described in Section IV-C1. In this example, all honest processors have the same initial message as $\bar{w}_1$. In each phase all honest processors output the same decoded message as $\bar{w}_1$, no matter what information sent by the dishonest processors. All honest processors eventually make the consistent and validated consensus outputs.

**E. Provable performance of COOL**

For the proposed COOL, the provable performance is summarized in the following lemmas.

**Lemma 1.** COOL achieves the consensus on an $\ell$-bit message with the resilience of $n \geq 3t + 1$, the round complexity of $O(t)$ rounds, and the communication complexity of $O(\max\{n\ell, nt \log n\})$ bits.

**Proof.** The proposed COOL achieves the consensus on an $\ell$-bit message as long as $n \geq 3t + 1$ (see Lemma 2). For the proposed COOL, the total communication complexity is computed as

$$b_{\text{total}} = \sum_{i=1}^{7} b_i = O(cn(n - 1) + n^2) = O(\max\{\ell/t, \log n\}n(n - 1) + n^2) = O(\max\{\ell n, n^2 \log n\}) \text{ bits}$$

where $b_1, b_2, \ldots , b_7$ are expressed in Remarks 1, 3, 6, 7 and 9. Recall that $c = \left\lceil \frac{\max\{\ell, (t+1)\log n\}}{k} \right\rceil$ and $k = \left\lceil \frac{t}{2} \right\rceil + 1$ (see (3)). In the description of the proposed protocol, $t$ is considered such that $t \leq (n - 1)/3$ and $t = \Omega(n)$. In this case, the total communication complexity becomes

$$b_{\text{total}} = O(\max\{\ell n, nt \log n\}) \text{ bits}.$$ 

If $t$ is relatively small, we can simply randomly select $n' = 3t + 1$ processors for running the consensus as described in the proposed protocol (just replacing $n$ with $n' = 3t + 1$). After the consensus those selected $n' = 3t + 1$ processors will send the agreed message using coding to the rest of the processors, in a way that each of the selected $n' = 3t + 1$ processors sends a coded message with $c$ bits (like in Phase 1). The proposed protocol guarantees that the consensus among the selected $n' = 3t + 1$ processors satisfies termination, consistency and validity conditions. After the consensus, the $n' = 3t + 1$ processors uses $(n', t)$ error correction code as described in Phase 1 to send the agreed message to each of the rest of the processors. This communication, each selected processor just sends $c$ bits of information coded from the agreed message, that is, $h_i^T \bar{w}$, where $i$ belongs to the set of indices of the selected $n' = 3t + 1$ processors, and $\bar{w}$ denotes the agreed message. The communication complexity in this step is $(3t + 1)(n - 3t - 1)c$ bits. In this scenario, the total communication complexity is

$$b_{\text{total}} = \sum_{i=1}^{7} b_i = O(ct^2 + t^2 + t(n-t)c) = O(cn(t^2 + t) = O(\max\{\ell/t, \log n\}nt + t^2) = O(\max\{\ell n, nt \log n\}) \text{ bits.}$$

Thus, by combining the above two cases, it is concluded that the total communication complexity of COOL is

$$b_{\text{total}} = O(\max\{\ell n, nt \log n\}) \text{ bits.}$$

For the proposed protocol, the round complexity is dominated by the round complexity of the one-bit consensus in Phase 3, which is $O(t)$ rounds. Therefore, the round complexity of the proposed protocol is $O(t)$ rounds. □

**Lemma 2.** For $n \geq 3t + 1$, the proposed COOL is an error-free BA protocol, i.e., it satisfies the termination, consistency and validity conditions in all executions.

**Proof.** The proof of Lemma 2 borrows tools from coding theory and linear algebra, with key steps given below.
• Given $n \geq 3t + 1$, and given the designed $k$, after Phase 1 there exists at most 2 groups of honest processors, where the honest processors within the same group successfully decode the same non-empty message and the messages decoded in different groups are different (like $\bar{A}_1$ and $\bar{A}_2$ in Fig. 4).

• Given $n \geq 3t + 1$, and given the designed $k$, after Phase 3 there exists at most 1 group of honest processors, where the honest processors within this group successfully decode the same non-empty message (like $\bar{A}_1$ in Fig. 4), and the honest processors outside this group could not successfully decode the messages (like $\bar{A}_2$ in Fig. 4).

• Given $n \geq 3t+1$, when the protocol stops at Phase 3, it is happened only when the honest processors have inconsistent initial messages, and eventually they have the same consensus output $\phi$.

• Given $n \geq 3t + 1$, when the protocol goes to Phase 4, it is ensured that there exists exactly 1 group of honest processors successfully decoding the same non-empty message (like $\bar{A}_1$ in Fig. 4), and the size of this group is at least $t + 1$. This makes sure that the information of the rest honest processors can be calibrated, in the presence of inconsistent information sent from dishonest processors. In this way, all honest processors eventually have the same consensus output.

• Given $n \geq 3t + 1$, if all honest processors have the same initial message, it is guaranteed that they will go to Phase 4 and have the validated consensus output.

Specifically, the proof of Lemma 2 will use the following lemmas whose proofs are provided in Section VI.

**Lemma 3.** Given $n \geq 3t + 1$, at the end of Phase 1 of COOL there exists at most 2 groups of honest processors, where the honest processors within the same group successfully decode the same non-empty message and the messages decoded in different groups are different.

*Proof.* See Section VI-B.

**Lemma 4.** Given $n \geq 3t + 1$, at the end of Phase 3 of COOL there exists at most 1 group of honest processors, where honest processors within the same group successfully decode the same non-empty message, and honest processors outside this group could not successfully decode the messages.

*Proof.* See Section VI-C.

**Lemma 5.** Given $n \geq 3t + 1$, all honest processors reach the same agreement in COOL.

*Proof.* See Section VI-D.

**Lemma 6.** Given $n \geq 3t + 1$, if all honest processors have the same initial message, then at the end of COOL all honest processors agree on this initial message.

*Proof.* See Section VI-E.

**Lemma 7.** Given $n \geq 3t + 1$, all honest processors eventually output messages and terminate in COOL.

*Proof.* See Section VI-F.

From Lemmas 5-7 it reveals that, given $n \geq 3t + 1$, the consistency, validity and termination conditions are all satisfied in COOL, which completes the proof of Lemma 2. The above Lemma 3 and Lemma 4 are used for the proofs of Lemmas 5-7.

Note that Lemma 2 holds true without using any assumptions on the cryptographic technique such as signatures, hashing, authentication and secret sharing, which implies that COOL is a signature-free protocol. Lemma 2 also holds true even when the adversary, who takes full control over the dishonest processors, has unbounded computational power, which implies that COOL is an information-theoretically-secure protocol. The results of Lemma 1 and Lemma 2 serve as the achievability proof of Theorem 1.

From the proposed COOL, it reveals that coding is an effective approach for solving the BA problem. In a nutshell, carefully-crafted error correction codes provide an efficient way of exchanging “compressed” information among distributed nodes, while keeping the ability of detecting errors, masking errors, and making a consistent and validated agreement at honest distributed nodes.
Algorithm 1: COOL protocol, code for Processor $i$, $i \in [1 : n]$

**Phase 1**
1: Processor $i$ sends $y_i = h_i^T w_i$ to all other processors.  \hspace{1cm} // Exchange compressed information
2: Processor $i$ decodes the message with received observations $\{y_1, y_2, \cdots, y_n\}$.  \hspace{1cm} // Decode message
3: if (the number of observation errors $\leq t$) \&\& $(w^{(i)} = w_i)$ then
4: \hspace{1cm} Processor $i$ sets $w^{(i)} = w_i$ and $s_i = 1$.
5: else
6: \hspace{1cm} Processor $i$ sets $w^{(i)} = \phi$ and $s_i = 0$.

**Phase 2**
7: Processor $i$ sends $y^{(i)}_j = h^T_j w^{(i)}$ to Processor $j$, $\forall j \in [1 : n], j \neq i$.  \hspace{1cm} // Exchange decoded information
8: if ($s_i == 0$) \hspace{1cm} \hspace{1cm} // Detect errors
9: \hspace{1cm} \hspace{1cm} Processor $i$ sets $e_i = 1$ and $w^{(i)} = \phi$.
10: else
11: Processor $i$ sets $e_i = 0$.
12: Processor $i$ sends the value of $e_i$ to other processors.  \hspace{1cm} // Exchange error indicators

**Phase 3**
13: Processor $i$ sets $y_j = \phi$, $\forall j \in E \triangleq \{p : e_p = 1, p \in [1 : n]\}$.  \hspace{1cm} // Mask errors identified in Phase 2
14: if ($e_i == 0$) then  \hspace{1cm} // Re-decode message
15: \hspace{1cm} Processor $i$ decodes the message again using updated observations $\{y_1, \cdots, y_n\}$.
16: if (the number of observation errors $> t$) \hspace{1cm} // Vote
17: \hspace{1cm} \hspace{1cm} Processor $i$ sets $w^{(i)} = \phi$ and $s_i = 0$.
18: \hspace{1cm} \hspace{1cm} Processor $i$ sends success indicator $s_i$ to all other processors.  \hspace{1cm} // Exchange success indicators
19: \hspace{1cm} \hspace{1cm} if ($\sum_{j=1}^n s_j >= 2t + 1$) then
20: \hspace{1cm} \hspace{1cm} \hspace{1cm} // Vote
21: \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} Processor $i$ sets the binary vote as $v_i = 1$.
22: \hspace{1cm} \hspace{1cm} else
23: \hspace{1cm} \hspace{1cm} \hspace{1cm} Processor $i$ sets the binary vote as $v_i = 0$.
24: \hspace{1cm} \hspace{1cm} \hspace{1cm} Processor $i$ runs the one-bit consensus with all other processors on the $n$ votes $\{v_1, v_2, \cdots, v_n\}$, by using the one-bit consensus protocol from [32], [33].  \hspace{1cm} // One-bit consensus on the $n$ votes
25: \hspace{1cm} \hspace{1cm} if (the consensus of the votes $\{v_1, v_2, \cdots, v_n\}$ is 1) then
26: \hspace{1cm} \hspace{1cm} \hspace{1cm} // Vote
27: \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} Processor $i$ goes to next phase.
28: \hspace{1cm} \hspace{1cm} else
29: \hspace{1cm} \hspace{1cm} \hspace{1cm} Processor $i$ sets $w^{(i)} = \phi$ and considers it as a final consensus and stops here.

**Phase 4**
28: if ($s_i == 1$) then  \hspace{1cm} // Broadcast correct coded messages
29: \hspace{1cm} Processor $i$ sends $y^{(i)}_j = h^T_j w^{(i)}$ to Processor $j$, $\forall j \in S_0 \triangleq \{p : s_p = 0, p \in [1 : n]\}$.
30: if ($s_i == 0$) then  \hspace{1cm} // Update information with majority rule
31: \hspace{1cm} Processor $i$ updates the value of $y_i$ as $y_i \leftarrow \text{Majority}(\{y^{(j)}_i : j \notin S_0, j \in [1 : n]\})$.
32: Processor $i$ sends the value of $y_i$ to Processor $j$, $\forall j \in S_0$.  \hspace{1cm} // Broadcast updated information
33: if ($s_i == 0$) then  \hspace{1cm} // Decode the message again
34: \hspace{1cm} Processor $i$ decodes the message again using new observations $\{y_1, y_2, \cdots, y_n\}$ and updates $w^{(i)}$.  \hspace{1cm} // Stop
35: Processor $i$ outputs consensus as the decoded message $w^{(i)}$ and stops.
TABLE III
SUMMARY OF SOME NOTATIONS FOR THE PROPOSED COOL.

| notation | interpretation |
|----------|----------------|
| \( k \)  | error correction code parameter (see (3)) |
| \( c \)  | error correction code parameter (see (3)) |
| \( w_i \) | initial message at Processor \( i \) |
| \( w^{(i)} \) | decoded message at Processor \( i \) |
| \( h_i \) | an encoding vector defined below (1) |
| \( y_i \) | a value encoded with \( h_i \) and initial message \( w_i \) of Processor \( i \) (see (4)) |
| \( y_j^{(i)} \) | a value encoded with \( h_j \) and decoded message \( w^{(i)} \) of Processor \( i \) (see (7)) |
| \( s_i \) | a success indicator for indicating whether the decoding at Processor \( i \) is successful or not (see (6)) |
| \( S_1 \) | a set of indices of success indicators \( \{s_1, s_2, \ldots, s_n\} \) whose values are ones (see (12)) |
| \( S_0 \) | a set of indices of success indicators \( \{s_1, s_2, \ldots, s_n\} \) whose values are zeros (see (12)) |
| \( e_i \) | an error indicator for indicating whether the observation \( y_i \) sent from Processor \( i \) is causing a certain number of observation errors at other nodes or not (see (8)) |
| \( E \) | a set of indices of error indicators whose values are ones (see (9)) |
| \( v_i \) | a vote from Processor \( i \) for going to next phase or stopping in the current phase (see (11)) |

VI. ANALYSIS ON TERMINATION, CONSISTENCY AND VALIDITY PROPERTIES OF COOL

In this section we will provide analysis on the termination, consistency and validity properties of the proposed COOL. Specifically, we will prove Lemmas 3-7 by borrowing tools from coding theory and linear algebra. In order to prove the lemmas, at first we will classify \( n \) processors into different groups as in the following sub-section.

A. Groups in network

We will classify \( n \) processors in the network into the following disjoint groups: Group \( F \), Group \( \bar{A}_l \), \( l \in [1 : \eta] \), and Group \( B \), for some non-negative integer \( \eta \). The group classification is based on the values of the processors’ decoded messages \( \{w^{(i)}\}_{i=1}^n \) obtained in Phase 1.

- **Group \( F \):** Group \( F \) includes the indices of all of the dishonest processors. The number of dishonest processors is denoted by \( t \), that is,

  \[ |F| = t. \]  

- **Group \( \bar{A}_l \), \( l \in [1 : \eta] \):** Consider \( w^{(i)} \) as the value of decoded message obtained in Phase 1 at Processor \( i \), \( i \in [1 : n] \). Group \( \bar{A}_l \) is a subset of honest processors whose indices are defined as

  \[ \bar{A}_l \triangleq \{i : w^{(i)} = \bar{w}_l, \ i \notin F\}, \ l \in [1 : \eta] \]  

for some non-empty \( \ell \)-bit values \( \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_\eta \) such that \( \bar{w}_l \neq \phi, \forall j \in [1 : \eta] \) and

\[ \bar{w}_l \neq \bar{w}_j, \ \forall l, j \in [1 : \eta], \ j \neq l \]

for some non-negative integer \( \eta \). This definition implies that, at the end of Phase 1 of COOL, all of the processors in the Group \( \bar{A}_l \) decode the same non-empty message \( \bar{w}_l \), for \( l \in [1 : \eta] \). The messages decoded in different groups are different.
• Group $B$: Group $B$ is a subset of honest processors whose indices are defined as

$$B \triangleq \{ i : w^{(i)} = \phi, \ i \notin \mathcal{F} \}$$  \hspace{1cm} (16)

where $w^{(i)}$ is the value of decoded message obtained in Phase 1. From our definition, Group $B$ includes the honest processors who could not decode their messages successfully in Phase 1.

Since the above disjoint groups cover all of the $n$ processors, it holds true that

$$\sum_{l=1}^\eta |\bar{A}_l| + |B| + |\mathcal{F}| = n.$$  \hspace{1cm} (17)

For Processor $i$, $i \in B$, its original message $w_i$ might satisfy the equality of $h_i^T w_i = h_i^T \bar{w}_l$ for some $l$, $l \in [1 : \eta]$. With this motivation, Group $B$ can be further divided into the following disjoint sub-groups:

$$B_l \triangleq \{ i : \ i \in B, \ h_i^T w_i = h_i^T \bar{w}_l, \ h_i^T w_i \neq h_i^T \bar{w}_j, \ \forall j \in [1 : l - 1] \}, \ \ l \in [1 : \eta]$$  \hspace{1cm} (18)

$$B_0 \triangleq B \setminus \{ \cup_{j=1}^\eta B_j \}.$$  \hspace{1cm} (19)

By combining $\bar{A}_l$ and $B_l$ (see (15) and (18)) together, it gives a bigger group $A_l$ defined by

$$A_l \triangleq \bar{A}_l \cup B_l, \ \ l \in [1 : \eta].$$  \hspace{1cm} (20)

At this point, (17) can be rewritten as

$$\sum_{l=1}^\eta |A_l| + |B_0| + |\mathcal{F}| = n.$$  \hspace{1cm} (21)

Based on our definition of $A_l$, as well as $B_l$ and $\bar{A}_l$, it holds true that $h_i^T w_i = h_i^T \bar{w}_l$, $\forall i \in A_l$. In other words, for any Processor $i$, $i \in A_l$, the coded value

$$y_i = h_i^T w_i$$

(see (4)) sent from this processor to all other processors can be represented as

$$y_i = h_i^T w_i = h_i^T \bar{w}_l, \ \ \forall i \in A_l$$  \hspace{1cm} (22)

for $l \in [1 : \eta]$.

For some $i \in A_l$, the equality of $h_i^T \bar{w}_l = h_i^T \bar{w}_j$ might be satisfied for some $j$, $j \neq l$, $l, j \in [1 : \eta]$. With this motivation, Group $A_{l,j}$, $l \in [1 : \eta]$, can be further divided into some (possibly overlapping) sub-groups defined as follows:

$$A_{l,j} \triangleq \{ i : \ i \in A_l, \ h_i^T \bar{w}_l = h_i^T \bar{w}_j \}, \ \ j \neq l, \ l, j \in [1 : \eta]$$  \hspace{1cm} (23)

$$A_{l,l} \triangleq A_l \setminus \{ \cup_{j=1,j \neq l}^\eta A_{l,j} \}, \ \ l \in [1 : \eta].$$  \hspace{1cm} (24)

Note that $A_{l,j}$ could be overlapping with $A_{l,j'}$, because the conditions of $h_i^T \bar{w}_l = h_i^T \bar{w}_j$ and $h_i^T \bar{w}_l = h_i^T \bar{w}_{j'}$ could be satisfied at the same time, for some $l, j, j' \in [1 : \eta], \ j \neq j', l \neq j, l \neq j'$. However, $A_{l,l}$ is not overlapping with $A_{l,j}$, for any $l, j \in [1 : \eta], j \neq l$. Similarly, Group $\bar{A}_l$ defined in (15) can be further divided into some (possibly overlapping) sub-groups as:

$$\bar{A}_{l,j} \triangleq \{ i : \ i \in \bar{A}_l, \ h_i^T \bar{w}_l = h_i^T \bar{w}_j \}, \ \ j \neq l, \ l, j \in [1 : \eta]$$  \hspace{1cm} (25)

$$\bar{A}_{l,l} \triangleq \bar{A}_l \setminus \{ \cup_{j=1,j \neq l}^\eta \bar{A}_{l,j} \}, \ \ l \in [1 : \eta].$$  \hspace{1cm} (26)
B. Proof of Lemma 3

We will prove Lemma 3 in this sub-section. Specifically, we will prove that, given \( n \geq 3t + 1 \), at the end of Phase 1 of COOL there exists at most 2 groups of honest processors, where the honest processors within the same group successfully decode the same non-empty message and the messages decoded in different groups are different. In other words, we will prove that

\[
\eta \leq 2
\]

where \( \eta \) denotes the number of the groups of \( \bar{A}_1, \bar{A}_1, \ldots, \bar{A}_\eta \), based on the group classification in the previous sub-section.

Our proof for Lemma 3 is based on proof by contradiction. Proof by contradiction is one type of proof that establishes the truth or the validity of a claim. The approach is to show that assuming the claim to be false leads to a contradiction. Let us first assume that the claim in Lemma 3 is false. Specifically, let us assume that

\[
\eta \geq 3. \tag{27}
\]

Later on we will prove that this assumption leads to a contradiction.

For Processor \( i \), for \( i \in \bar{A}_l \) and \( l \in [1 : \eta] \), one of the requirements for successfully decoding the message in Phase 1 is that the number of observation errors is no more than \( t \). Recall that, given the decoded message \( w^{(i)} \) at Processor \( i \), an observation \( y_j \) is marked as an observation error if

\[
y_j \neq h_j^T w^{(i)}
\]

(see (5)) for some \( j \). From this requirement, it holds true that in Phase 1 Processor \( i \), \( i \in \bar{A}_l \), needs to receive at least

\[
n - t
\]

observations that can be expressed as

\[
y_j = h_j^T \bar{w}_l
\]

for some \( j \). In other words, in Phase 1 Processor \( i \), \( i \in \bar{A}_l \), needs to receive at least

\[
n - t - t \tag{28}
\]

observations from the honest processors (including itself) that can be expressed as \( y_j = h_j^T \bar{w}_l \) for \( j \in \bar{A}_l \cup \{ \cup_{p=1,p \neq l}^{\eta} A_{p,l} \} \). Note that, for a dishonest Processor \( j \), the observation \( y_j \) sent from this processor to Processor \( i \) in Phase 1 can be expressed in the form of \( y_j = h_j^T \bar{w}_l \) for \( j \in F \) and \( |F| = t \). Also note that, for an honest Processor \( j \), the observation \( y_j \) sent from this processor to Processor \( i \) in Phase 1 can be expressed in the form of \( y_j = h_j^T \bar{w}_l \) only if \( j \in \bar{A}_l \cup \{ \cup_{p=1,p \neq l}^{\eta} A_{p,l} \} \), based on the definitions in (15), (18), (20) and (23). From the requirement in (28), it implies that

\[
|A_l| + \sum_{p=1,p \neq l}^{\eta} |A_{p,l}| \geq n - t - t, \quad \text{for} \quad l \in [1 : \eta]. \tag{29}
\]

From (29) we have

\[
\sum_{l=1}^{\eta} |A_l| + \sum_{l=1}^{\eta} \sum_{p=1,p \neq l}^{\eta} |A_{p,l}| \geq \eta(n - t - t). \tag{30}
\]

Note that the second term in the left hand side of (30) can be written as \( \sum_{l=1}^{\eta} \sum_{p=1,p \neq l}^{\eta} |A_{p,l}| = \sum_{l=1}^{\eta} \sum_{p=1,p \neq l}^{\eta} |A_{l,p}| \). From Lemma (9) described below, it implies that this term is upper bounded as

\[
\sum_{l=1}^{\eta} \sum_{p=1,p \neq l}^{\eta} |A_{p,l}| < (\eta - 1)k. \tag{31}
\]
By combining (30) and (31), it gives the following bound

\[ \sum_{l=1}^{\eta} |A_l| \geq \eta(n - t - t) - \sum_{l=1}^{\eta} \sum_{p=1, p \neq l}^{\eta} |A_{p,l}| \geq \eta(n - t - t) - (\eta - 1)k. \]  

(32)

On the other hand, from (21) it implies that

\[ \sum_{l=1}^{\eta} |A_l| = n - |F| - |B_0| = n - t - |B_0| \leq n - t. \]  

(33)

Finally, under the assumption of \( \eta \geq 3 \) (see (27)), the result of Lemma 8 described below implies that the following inequality holds true

\[ n - t \leq \eta(n - t - t) - (\eta - 1)k \]  

(34)

which leads to a contradiction, that is, the conclusions in (32) and (33) contradict with each other under the assumption of \( \eta \geq 3 \). Therefore, the assumption of \( \eta \geq 3 \) is not true and \( \eta \) should satisfy

\[ \eta \leq 2 \]

which proves Lemma 3. The two lemmas used in the proof are provided below.

**Lemma 8.** Under the assumption of \( \eta \geq 3 \) (see (27)), and given \( n \geq 3t + 1 \) and \( k = \lfloor \frac{t}{2} \rfloor + 1 \) (see (3)), the following inequality holds true

\[ n - t \leq \eta(n - t - t) - (\eta - 1)k. \]

**Proof.** First note that, under the assumption of \( \eta \geq 3 \), it holds true that \( \frac{1}{\eta - 1} \leq \frac{1}{2} \). Beginning with the value of \( k \), we have

\[ k = \lfloor \frac{t}{2} \rfloor + 1 \]  

(35)

\[ \leq \frac{t}{2} + 1 \]

\[ = t + 1 - \frac{t}{2} \]

\[ \leq t + 1 - \frac{t}{\eta - 1} \]  

(36)

\[ \leq n - 2t - \frac{t}{\eta - 1} \]  

(37)

\[ = \frac{(\eta - 1)n - (2\eta - 1)t}{\eta - 1} \]  

(38)

where (35) follows from the value of \( k = \lfloor \frac{t}{2} \rfloor + 1 \) (see (3)); (36) results from the derivation that \( \frac{1}{\eta - 1} \leq \frac{1}{2} \) under the assumption of \( \eta \geq 3 \); and (37) uses the condition of \( n \geq 3t + 1 \). By multiplying \( \eta - 1 \) at each side of (38), it gives \( (\eta - 1)k \leq (\eta - 1)n - (2\eta - 1)t \) and \( n - t \leq \eta(n - t - t) - (\eta - 1)k \). At this point we complete the proof of this Lemma 8.

**Lemma 9.** When \( \eta \geq 2 \), it holds true that

\[ \sum_{l=1}^{\eta} \sum_{j=1, j \neq l}^{\eta} |A_{l,j}| < (\eta - 1)k \]  

(39)

where \( A_{l,j} = \{ i : i \in A_l, \ h_i^l \bar{w}_l = h_i^j \bar{w}_j \} \) and \( \bar{w}_l \neq \bar{w}_j \) for \( j \neq l, l, j \in [1 : \eta] \) (see (23)).
Proof. The proof of Lemma 9 borrows tool from linear algebra. This lemma considers the case of \( \eta \geq 2 \). The proof will use the fact that
\[
\bar{w}_l - \bar{w}_j \neq 0, \quad \forall j \neq l, \ j, l \in [1 : \eta] \quad (40)
\]
as well as the fact that
\[
h_i^T(\bar{w}_l - \bar{w}_j) = 0 \quad \forall i \in A_{l,j}, \ j \neq l, \ l, j \in [1 : \eta] \quad (41)
\]
which follows from the definition of \( A_{l,j} = \{i : i \in A_l, \ h_i^T \bar{w}_l = h_i^T \bar{w}_j\} \) for \( j \neq l, l, j \in [1 : \eta] \). When \( l > j \), then \( \bar{w}_l - \bar{w}_j \) can be represented as
\[
\bar{w}_l - \bar{w}_j = \sum_{p=j}^{l-1}(\bar{w}_{p+1} - \bar{w}_p). \quad (42)
\]
Let us define \( H_{l,j} \) as an \( |A_{l,j}| \times k \) matrix such that
\[
H_{l,j} = \begin{bmatrix} h_{i_1}^T & h_{i_2}^T & \cdots & h_{i_{|A_{l,j}|}}^T \end{bmatrix} \quad \text{for} \quad i_1, i_2, \ldots, i_{|A_{l,j}|} \in A_{l,j}, \ i_1 < i_2 < \cdots < i_{|A_{l,j}|} \quad (43)
\]
for \( j \neq l, l, j \in [1 : \eta] \). Note that \( H_{l,j} \) is full rank and its rows are linearly independent, based on the definition of \( h_i \) below (1).

From (41) and (42), it implies that the following equality holds true that
\[
Hx = 0 \quad (44)
\]
where
\[
x = \begin{bmatrix} \bar{w}_2 - \bar{w}_1 \\ \bar{w}_3 - \bar{w}_2 \\ \vdots \\ \bar{w}_\eta - \bar{w}_{\eta-1} \end{bmatrix} \quad (45)
\]
and
\[
H_l \triangleq \begin{bmatrix} H_1 \\ H_2 \\ H_{\eta} \end{bmatrix}, \quad H'_l \triangleq \begin{bmatrix} H'_1 & 0 \\ 0 & H''_l \end{bmatrix} \quad l \in [1 : \eta] \quad (46)
\]
where
\[
H'_l \triangleq \begin{bmatrix} H_{l,1} & H_{l,1} & H_{l,1} & \cdots & H_{l,1} \\ 0 & H_{l,2} & H_{l,2} & \cdots & H_{l,2} \\ 0 & 0 & H_{l,3} & \cdots & H_{l,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H_{l,l-1} \end{bmatrix}, \quad H''_l \triangleq \begin{bmatrix} H_{l,l+1} & 0 & 0 & \cdots & 0 \\ H_{l,l+2} & H_{l,l+2} & 0 & \cdots & 0 \\ H_{l,l+3} & H_{l,l+3} & H_{l,l+3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{l,\eta} & H_{l,\eta} & \cdots & H_{l,\eta} \end{bmatrix} \quad (47)
\]
for \( l \in [1 : \eta] \). Note that \( H'_3 \triangleq \begin{bmatrix} H_{3,1} & H_{3,1} \\ 0 & H_{3,2} \end{bmatrix}, \ H'_2 = H_{2,1}, \ H'_1 = \phi, \ H''_\eta = \phi, \text{ and } H''_{\eta-1} = H_{\eta-1,\eta}. \)

Since \( H'_l \) is a block upper triangular matrix (see (47)), and given that \( H_{l,j} \) is full rank \( \forall j \in [1 : \eta], j \neq l \) (see (43)), then it holds true that \( H'_l \) is full rank, \( \forall l \in [1 : \eta] \). Similarly, since \( H''_l \) is a block lower triangular matrix, it also holds true that \( H''_l \) is full rank, \( \forall l \in [1 : \eta] \). With the full rank property of both \( H'_l \) and \( H''_l \), and given the block diagonal structure of \( H_l \) (see (46)), it implies that \( H_l \) is a full rank matrix.
In our setting, it holds true that $h_i$ and $h_j$ are linearly independent, $\forall i \in A_{l_1}, \forall j \in A_{l_2}$, given $l_1 \neq l_2, l_1, l_2 \in [1: \eta]$. Then, it implies that the rows of $H_{l_1}$ are linearly independent of the rows of $H_{l_2}$ for any $l_1 \neq l_2, l_1, l_2 \in [1: \eta]$. Finally, with full rank property of the matrices $H_1, H_2, \cdots, H_\eta$, and given that the rows of $H_{l_1}$ are linearly independent of the rows of $H_{l_2}$ for any $l_1 \neq l_2, l_1, l_2 \in [1: \eta]$, it can be concluded that $H$ (see (46)) is a full rank matrix.

Based on the definition in (46), the dimension of the matrix $H$ is $m_1 \times m_2$ where

$$m_1 = \sum_{l=1}^{\eta} \sum_{j=1,j \neq l}^{\eta} |A_{l,j}|, \quad m_2 = (\eta - 1)k. \quad (48)$$

From (40), the vector $x$ defined in (45) is required to be

$$x \neq 0 \quad (49)$$

because $\bar{w}_l - \bar{w}_l \neq 0, \forall j,l \in [1: \eta], j \neq l$. At this point, let us now go back to the equality of $Hx = 0$ described in (44). By combining the fact that $x \neq 0$ (see (49)) and the fact that the $m_1 \times m_2$ matrix $H$ is full rank, we can conclude that the following inequality must hold true

$$m_1 < m_2 \quad (50)$$

(otherwise $x = 0$, which contradicts with the fact that $x \neq 0$). Then, from (48) and (50) we finally conclude that

$$\sum_{l=1}^{\eta} \sum_{j=1,j \neq l}^{\eta} |A_{l,j}| < (\eta - 1)k$$

which completes the proof of this Lemma 9.

C. Proof of Lemma 4

At the end of Phase 3 of the proposed COOL, there exists a number of groups of honest processors, where the honest processors within the same group successfully decode the same non-empty message and the messages decoded in different groups are different. Let $\bar{\eta}$ denote the number of this kind of groups of honest processors, classified by the values of the processors’ decoded messages $\{\bar{w}(i)\}_{i=1}^{n}$ obtained in Phase 3, for some non-negative integer $\bar{\eta}$. To prove Lemma 4 with $n \geq 3t + 1$, we will specifically prove that

$$\bar{\eta} \leq 1.$$

From Lemma 3, we note that, at the end of Phase 1 of the proposed COOL, there exists at most 2 groups of honest processors, i.e., $\eta \leq 2$, where the honest processors within the same group successfully decode the same non-empty message and the messages decoded in different groups are different. Intuitively, $\bar{\eta}$ should not be more than $\eta$, due to the error detection, error masking, and message re-decoding in Phase 2 and Phase 3. If $\eta < 1$, it then automatically implies that $\bar{\eta} \leq 1$. In what follows we will prove $\bar{\eta} \leq 1$ just under the case of $\eta = 2$. 


Given \( \eta = 2 \), the definitions in (15)-(20) and (23)-(26) imply that

\[
\begin{align*}
B_1 & = \{ i : i \in B, \ h_i^T w_1 = h_i^T \bar{w}_1 \} \tag{51} \\
B_2 & = \{ i : i \in B, \ h_i^T w_1 = h_i^T \bar{w}_2, \ h_i^T w_i \neq h_i^T \bar{w}_1 \} \tag{52} \\
B_0 & = B \setminus \{B_1 \cup B_2\} \tag{53} \\
A_1 & = \bar{A}_1 \cup B_1 \tag{54} \\
A_2 & = \bar{A}_2 \cup B_2 \tag{55} \\
A_{1,2} & = \{ i : i \in A_1, \ h_i^T \bar{w}_1 = h_i^T \bar{w}_2 \} \tag{56} \\
A_{2,1} & = \{ i : i \in A_2, \ h_i^T \bar{w}_2 = h_i^T \bar{w}_1 \} \tag{57} \\
A_{1,1} & = A_1 \setminus A_{1,2} \tag{58} \\
A_{2,2} & = A_2 \setminus A_{2,1} \tag{59} \\
\bar{A}_{1,2} & = \{ i : i \in \bar{A}_1, \ h_i^T \bar{w}_1 = h_i^T \bar{w}_2 \} \tag{60} \\
\bar{A}_{2,1} & = \{ i : i \in \bar{A}_2, \ h_i^T \bar{w}_2 = h_i^T \bar{w}_1 \} \tag{61} \\
\bar{A}_{1,1} & = \bar{A}_1 \setminus \bar{A}_{1,2} \tag{62} \\
\bar{A}_{2,2} & = \bar{A}_2 \setminus \bar{A}_{2,1} \tag{63}
\end{align*}
\]

for some non-empty messages \( w_1 \) and \( \bar{w}_2 \) such that \( w_1 \neq \phi, \bar{w}_2 \neq \phi \) and \( \bar{w}_1 \neq \bar{w}_2 \). Given \( \eta = 2 \), from (29) it implies that the following two conditions need to be satisfied for the decodability in Phase 1 for Group \( \bar{A}_1 \) and Group \( \bar{A}_2 \) respectively:

\[
\begin{align*}
|A_1| + |A_{2,1}| & \geq n - t - t \tag{64} \\
|A_2| + |A_{1,2}| & \geq n - t - t. \tag{65}
\end{align*}
\]

In the following we will complete the proof by focusing on the following three cases

- **Case 1:** \(|A_1| + |B_0| \geq t + 1\) \tag{66}
  \(|A_2| + |B_0| < t + 1\) \tag{67}

- **Case 2:** \(|A_1| + |B_0| < t + 1\) \tag{68}
  \(|A_2| + |B_0| \geq t + 1\) \tag{69}

- **Case 3:** \(|A_1| + |B_0| \geq t + 1\) \tag{70}
  \(|A_2| + |B_0| \geq t + 1\) \tag{71}

Note that the following case

**Case 4:** \(|A_1| + |B_0| < t + 1\) \tag{72}
\(|A_2| + |B_0| < t + 1\) \tag{73}

does not exist. The argument is given as follows. Given the assumption in (72) (which is equivalent to \(|A_1| + |B_0| \leq t\)), it then implies that

\[
\begin{align*}
|A_2| & = n - |F| - |A_1| - |B_0| \tag{74} \\
& \geq n - |F| - t \tag{75} \\
& \geq t + 1 \tag{76}
\end{align*}
\]

where (74) follows from (21); (75) uses the assumption of \(|A_1| + |B_0| \leq t\) as in (72); (76) results from the condition of \(n \geq 3t + 1\) and \(|F| = t\). The conclusion in (76) contradicts with (73), which suggests that the case described in (72) and (73) does not exist.
1) Analysis for Case 1: Let us consider Case 1 described in (66) and (67). Recall that in the error detection step of Phase 2, Processor \(i, \forall i \in [1 : n]\), sends the value of

\[ y_j^{(i)} = h_j^i w^{(i)} \]

to Processor \(j, \forall j \in [1 : n], j \neq i\) (see (7)). From (51)-(63) it reveals that

\[ y_j^{(i)} \neq y_j, \forall i \in A_1 \cup B_0, \forall j \in \bar{A}_2 \]

(77) where \(y_j = h_j^j w_j\) (see (4)). With the condition in (66), i.e., \(|A_1| + |B_0| \geq t + 1\), and with (77), it implies that in the error detection step of Phase 2, Processor \(j\) sets

\[ e_j = 1, \forall j \in \bar{A}_2 \] (78)

(see (8)), meaning that the observation \(y_j\) sent from Processor \(j\) is causing more than \(t\) number of observation errors at other processors. The outcome in (78) implies that Processor \(j\) sets

\[ w^{(j)} = \phi, \forall j \in \bar{A}_2 \] (79)

in the error detection step of Phase 2. Since the processors within \(B\) could not decode their messages successfully, it gives the following outcomes

\[ w^{(j)} = \phi \text{ and } e_j = 1, \forall j \in B \] (80)

in the error detection step of Phase 2. With (78)-(80), it is true that Processor \(j\) sends out

\[ e_j = 1, \forall j \in \bar{A}_2 \cup B \] (81)

to all other processors in the step of exchanging error indicators of Phase 2. Then in the error masking step of Phase 3, Processor \(i, \forall i \in A_1 \cup \bar{A}_2,1\) sets

\[ y_j = \phi, \forall j \in \bar{A}_2 \cup B \] (82)

from its view (see (10)).

With the outcome in (82), Processor \(i, \forall i \in \bar{A}_2,1\) could not successfully re-decode the message in Phase 3, since the number of observation errors is more than \(t\). Specifically, with the outcome in (82), and with the definition of \(A_{1,1}\) in (60) and (62), i.e., \(A_{1,1} = \{i : i \in A_1, h_1^i w_1 \neq h_1^i w_2\}\), then from the view of Processor \(i, \forall i \in \bar{A}_2,1\), it holds true that

\[ y_j \neq h_j^i w^{(i)}, j \in \bar{A}_{1,1} \cup \bar{A}_{2,2} \cup B, \forall i \in \bar{A}_2,1 \] (83)

(see (5)) and that

\[ |\bar{A}_{1,1} \cup \bar{A}_{2,2} \cup B| = |\bar{A}_{1,1}| + |\bar{A}_{2,2}| + |B| \]

\[ = n - |F| - |\bar{A}_{1,2}| - |\bar{A}_{2,1}| \]

\[ > n - |F| - k \]

\[ \geq 2t + 1 - k \]

\[ \geq t + 1 \]

(84)-(88)

where (84) uses the disjoint property between \(\bar{A}_{1,1}, \bar{A}_{2,2}\) and \(B\); (85) is from (17) and the disjoint property between \(A_{1,1}, A_{2,2}, \bar{A}_{2,1}, \bar{A}_{2,2}\) and \(B\); (86) follows from Lemma 9 which implies that \(|A_{1,2}| + |A_{2,1}| < k\) and consequently that \(|A_{1,2}| + |A_{2,1}| \leq |A_{1,2}| + |A_{2,1}| < k\) for this case with \(\eta = 2\); (87) uses the condition that \(n \geq 3t + 1\) and \(|F| = t\); (88) results from the fact that \(t \geq k\) based on our design of \(k\) (see (3)).

From (83) and (88), it suggests that Processor \(i, \forall i \in \bar{A}_2,1\) could not successfully re-decode the message in Phase 3, since the number of observation errors is more than \(t\). Then Processor \(i\) sets

\[ w^{(i)} = \phi, \forall i \in \bar{A}_2,1 \] (89)
in the message re-decoding step of Phase 3. By combining the outcomes in (79) and (89), at this point we can conclude that Processor $i$, $\forall i \in \bar{A}_2$ could not successfully re-decode the message and $w^{(i)}$ is set as

$$w^{(i)} = \phi, \quad \forall i \in \bar{A}_2$$

(90)

at the end of Phase 3 of the proposed COOL. In other words, at the end of Phase 3, there exists at most 1 group of honest processors, where honest processors within the same group successfully decode the same non-empty message, and honest processors outside this group could not successfully decode the messages, for Case 1.

2) Analysis for Case 2: Due to the symmetry between Case 1 and Case 2, one can follows from the proof steps for Case 1 and show for Case 2 that Processor $i$, $\forall i \in \bar{A}_1$, could not successfully re-decode the message and $w^{(i)}$ is set as

$$w^{(i)} = \phi, \quad \forall i \in \bar{A}_1$$

(91)

at the end of Phase 3 of the proposed COOL. In other words, for Case 2, at the end of Phase 3 there exists at most 1 group of honest processors, where honest processors within the same group successfully decode the same non-empty message, and honest processors outside this group could not successfully decode the messages. Note that for Case 2 the condition we use for deriving (91) is (69), while for Case 1 the condition we use for deriving (90) is (66).

3) Analysis for Case 3: Note that the condition in (70) for Case 3 is the same as the condition in (66) for Case 1. By following the proof steps for Case 1 one can show for Case 3 that Processor $i$, $\forall i \in \bar{A}_2$ could not successfully re-decode the message and $w^{(i)}$ is set as

$$w^{(i)} = \phi, \quad \forall i \in \bar{A}_2$$

(92)

at the end of Phase 3 of the proposed COOL. Also note that the condition in (71) for Case 3 is the same as the condition in (69) for Case 3. Thus, one can also show for Case 3 that Processor $i$, $\forall i \in \bar{A}_1$, could not successfully re-decode the message and $w^{(i)}$ is set as

$$w^{(i)} = \phi, \quad \forall i \in \bar{A}_1$$

(93)

at the end of Phase 3 of the proposed COOL. In other words, for Case 3, none of the honest processors could successfully re-decode the non-empty message at the end of Phase 3.

By combining the analysis for the above three cases, at this point we complete the proof of Lemma 4.

D. Proof of Lemma 5

We will prove Lemma 5 in this sub-section. Specifically, given $n \geq 3t + 1$, we will prove that all of the honest processors reach the same agreement in COOL.

Recall that in the last step of Phase 3 in COOL, the system runs the one-bit consensus on the $n$ votes $\{v_1, v_2, \ldots, v_n\}$ from all processors, where $v_i = 1$ can be considered as a vote from Processor $i$ for going to Phase 4, while $v_i = 0$ can be considered as a vote from Processor $i$ for stopping in Phase 3, for $i \in [1 : n]$ (see (11)). The one-bit consensus from [32], [33] ensures that: every honest processor eventually outputs a consensus value; all honest processors should have the same consensus output; and if all honest processors have the same vote then the consensus should be the same as the vote of the honest processors. We will prove this lemma by considering each of the following two cases.

• Case (a): In Phase 3, the consensus of the votes $\{v_1, v_2, \ldots, v_n\}$ is 0.

• Case (b): In Phase 3, the consensus of the votes $\{v_1, v_2, \ldots, v_n\}$ is 1.

Analysis for Case (a): In Phase 3 of COOL, if the consensus of the votes $\{v_1, v_2, \ldots, v_n\}$ is 0, then all of the honest processors will set $w^{(i)} = \phi$ and consider $\phi$ as a final consensus and stop here. In this case, all of the honest processors agree on the same message.
Analysis for Case (b): In Phase 3 of COOL, if the consensus of the votes \( \{v_1, v_2, \ldots, v_n\} \) is 1, then all of the honest processors will go to Phase 4. In this case, at least one of the honest processors votes \( v_i = 1 \), for some \( i \notin F \).

Otherwise, all of the honest processors vote the same value such that \( v_i = 0 \), \( \forall i \notin F \) and the consensus of the votes \( \{v_1, v_2, \ldots, v_n\} \) should be 0, contradicting the condition of this case. In COOL, the condition of voting \( v_i = 1 \) (see (11)) is that Processor \( i \) receives no less than \( 2t + 1 \) number of ones from \( n \) success indicators \( \{s_1, s_2, \ldots, s_n\} \), i.e.,

\[
\sum_{j=1}^{n} s_j \geq 2t + 1 \tag{94}
\]

where the success indicator \( s_j = 1 \) sent from Processor \( j \) in Phase 3 means that Processor \( j \) successfully re-decodes the non-empty message in Phase 3, while \( s_j = 0 \) means that Processor \( j \) could not successfully re-decode its message. Since \( t \) dishonest processors might send the success indicators as ones, the outcome in (94) implies that at least \( t + 1 \) honest processors send out success indicators as ones in Phase 3.

Lemma 4 reveals that at the end of Phase 3 there exists at most 1 group of honest processors, where the honest processors within this group successfully decode the same non-empty message and the honest processors outside this group could not successfully decode the messages. Then, for this Case (b), with the outcome in (94) and the conclusion in Lemma 4, it implies that at the end of Phase 3 there exists exactly 1 group of honest processors with group size satisfying

\[
\text{group size} \geq t + 1 \tag{95}
\]

where the honest processors within this group successfully decode the same non-empty message and the honest processors outside this group could not successfully decode the messages. Let us use \( A \) to denote this group of honest processors and use \( \bar{w} \) to denote the non-empty message successfully re-decoded within this group in Phase 3. From our definitions and from (95) it holds true that

\[
w^{(i)} = \bar{w}, \quad \forall i \in A \tag{96}
\]

\[
|A| \geq t + 1. \tag{97}
\]

based on the values of the decoded messages in Phase 3.

Recall that in the first step of Phase 4, Processor \( i, \forall i \in S_1 \) sends the value of

\[
y_j^{(i)} = h_j^i \bar{w} \tag{98}
\]

to Processor \( j, \forall j \in S_0 \), where \( S_1 \) and \( S_0 \) are defined as \( S_1 = \{i : s_i = 1, i \in [1 : n]\} \) and \( S_0 = \{i : s_i = 0, i \in [1 : n]\} \) (see (12)). In this step, honest Processor \( i \) sends honest Processor \( j \) the value of

\[
y_j^{(i)} = h_j^i \bar{w}, \quad \forall i \in A, \quad \forall j \in \{p : p \in [1 : n], p \notin F, p \notin A\}. \tag{98}
\]

In the first step of Phase 4, even \( t \) dishonest processors send fake information to honest Processor \( j \) for \( j \in \{p : p \in [1 : n], p \notin F, p \notin A\} \), Processor \( j \) could still use the majority rule to update the value of \( y_j \) as

\[
y_j \leftarrow \text{Majority} \{(y_j^{(i)} : i \in S_1)\} = h_j^T \bar{w} \tag{99}
\]

(see (13) in the second step of Phase 4) due to the facts that \( |A| \geq t + 1 \) and that \( |F| = t \) (see (97)-(98)).

Then in the third step of Phase 4, Processor \( i, \forall i \in [1 : n], \) sends the value of \( y_i \) to Processor \( j, \forall j \in S_0 \). After that, Processor \( j, \forall j \in S_0 \), decodes the message \( w^{(j)} \) again using the new observations \( \{y_1, y_2, \ldots, y_n\} \). At this point, given the outcome in (99), and with the fact in (96), it holds true that Processor \( j, \forall j \in \{p : p \in [1 : n], p \notin F, p \notin A\} \) can successfully re-decode the message as

\[
w^{(j)} = \bar{w}, \quad \forall j \in \{p : p \in [1 : n], p \notin F, p \notin A\}. \tag{100}
\]

Finally, by combining the outcomes in (96) and (100), it reveals that all of the honest processors successfully decode the same message, i.e., \( w^{(j)} = \bar{w}, \forall j \in \{p : p \in [1 : n], p \notin F\} \), and all of the honest processors finally agree on this message. With this we complete the proof of Lemma 5.
E. Proof of Lemma 6

Now we will prove that, given \( n \geq 3t + 1 \), if all honest processors have the same initial message, then at the end of COOL all honest processors agree on this initial message. One example is depicted in Fig. 5.

Assume that all honest processors have the same initial message as \( \bar{w}_1 \), for some non-empty value \( \bar{w}_1 \) (see Fig. 5 as an example). For this scenario, in Phase 1 all honest processors will successfully decode the messages that are exactly the same as the initial message \( \bar{w}_1 \), because every honest processor will have no more than \( t \) number of observation errors (see (5)). Then, in Phase 2, none of the honest processors will be in the list of \( \mathcal{E} \) defined in (9), because every honest processor’s decoded message is matched with all other honest processors’ decoded messages. In Phase 2, all honest processors will keep the decoded messages as \( \bar{w}_1 \). In Phase 3, all honest processors will again successfully decode the messages that are exactly the same as the initial message \( \bar{w}_1 \), because every honest processor will have no more than \( t \) number of observation errors. In this scenario, the consensus of the votes \( \{v_1, v_2, \ldots, v_n\} \) in Phase 3 should be 1, i.e., all honest processors will go to Phase 4, as described in Case (b) in the previous sub-section. Given the same decoded message \( \bar{w}_1 \) as input at all of the honest processors, in Phase 4 all honest processors will again successfully decode the messages that are exactly the same as the initial message \( \bar{w}_1 \). Thus, for this scenario, all honest processors eventually agree on the initial message \( \bar{w}_1 \). At this point we complete the proof of Lemma 6.

F. Proof of Lemma 7

In the proposed COOL, given \( n \geq 3t + 1 \), it is guaranteed that all honest processors eventually terminate together at the last step of Phase 3, or terminate together at the last step of Phase 4. It is also guaranteed that every honest processor eventually outputs a message when it terminates, which completes the proof of Lemma 7.

VII. The Extension of COOL to the Byzantine Broadcast Problem

The proposed COOL designed for the BA setting can be adapted into the BB setting. Note that for any error-free BB protocol, the known lower bounds on resilience, round complexity and communication complexity remain the same as for the error-free BA protocol.

For the BB setting, we just add an additional step into the proposed COOL described in Section V. Specifically, for the BB setting, at first the leader is designed to send each processor an \( \ell \)-bit message that can be considered as the initial message in the BA setting. Then, COOL is applied into this BB setting from this step to achieve the consistent and validated consensus. We call the proposed COOL with the additional step mentioned above as an adapted COOL for this BB setting. When \( n \geq 3t + 1 \), this adapted COOL is guaranteed to satisfy the termination, consistency and validity conditions in all executions in this BB setting. Note that for the added step, i.e., sending initial messages from the leader to all processors, the additional communication complexity is just \( O(n\ell) \) bits. Therefore, the total communication complexity of the adapted COOL for this BB setting is

\[
O(\max\{n\ell, nt \log n\} + n\ell) = O(\max\{n\ell, nt \log n\}) \text{ bits.}
\]

Furthermore, the round complexity of the adapted COOL for this BB setting is

\[
O(t + 1) = O(t) \text{ rounds.}
\]

Hence, the adapted COOL is an error-free signature-free information-theoretic-secure BB protocol that achieves the consensus on an \( \ell \)-bit message with optimal resilience, asymptotically optimal round complexity, and asymptotically optimal communication complexity when \( \ell \geq t \log n \). This result serves as the achievability proof of Theorem 2.
VIII. Conclusion

In this work, we proposed COOL, a deterministic BA protocol designed from coding theory, which achieves the BA consensus on an $\ell$-bit message with optimal resilience, asymptotically optimal round complexity, and asymptotically optimal communication complexity when $\ell \geq t \log n$, simultaneously. Our protocol is guaranteed to be correct in all executions (error-free) and it does not rely on cryptographic technique such as signatures, hashing, authentication and secret sharing (signature-free). Furthermore, our protocol is robust even when the adversary has unbounded computational power (information-theoretic secure). The adapted COOL also achieves the BB consensus on an $\ell$-bit message with optimal resilience, asymptotically optimal round complexity, and asymptotically optimal communication complexity when $\ell \geq t \log n$, simultaneously. This work reveals that coding is an effective approach for achieving the fundamental limits in resilience, round complexity and communication complexity of Byzantine agreement and its variants.

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