Abstract

The algebras $Q_n$ describe the relationship between the roots and coefficients of a non-commutative polynomial. I. Gelfand, S. Gelfand, and V. Retakh have defined quotients of these algebras corresponding to graphs. In this work we find the Hilbert series of the class of algebras corresponding to the graph $K_3$. We also show this algebra is Koszul.

1 Koszul Algebras

There are a number of equivalent definitions of Koszul algebras including this lattice definition from Ufnarovskij [4].

Definition 1. A quadratic algebra $A = \{V, R\}$ (where $V$ is the span of the generators and $R$ the span of the generating relations in $V \otimes V$) is Koszul if the collection of $n-1$ subspaces $\{V^{\otimes n-1} \otimes R \otimes V^{\otimes n-1}\}$, generates a distributive lattice in $V^{\otimes n}$ for any $n$.

In [4] the following criterion is given for distributivity of a modular lattice:

Theorem 2. Suppose $\{x_1, \ldots, x_n\}$ generates the modular lattice $\Omega$. If any proper subset of $\{x_1, \ldots, x_n\}$ generates a distributive sublattice then $\Omega$ is distributive iff for any $2 \leq k \leq n-1$ the triple $x_1 \vee \cdots \vee x_k \wedge \cdots \wedge x_n$ is distributive.

Applying theorem 2 to definition 1 we get the following corollary we will use in various chapters throughout this work:

Corollary 1.1. The quadratic algebra $A = \{V, R\}$ (where $V$ is the span of the generators and $R$ the span of the generating relations in $V \otimes V$) is Koszul if $RV^{n-2} \cap VRV^{n-2} \cap \cdots \cap V^{n-2}RV^{n-2}a$, $V^{n-2}RV^{n-2}$, $V^n RV^{n-a-2} + \cdots + V^{n-2}R$ is a distributive triple in $V^n$ for any $a$ and $n$ with $2 \leq a \leq n-2$.

We will also need the following theorem from [4].

Theorem 2. A quadratic algebra $A$ is Koszul iff its dual algebra $A^*$ is Koszul. In the situation where they are both Koszul the Hilbert series of $A$ is given by $\frac{1}{\prod_{i \in A} (1 - x^i)}$ where $h(x)$ is the Hilbert series of $A$.

2 $Q_n$ and $Q_n(G)$

Let $P(x) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - \cdots + (-1)^n a_0$ be a polynomial over a division algebra. I. Gelfand and V. Retakh [2] studied relationships between the coefficients $a_i$ and a generic set $\{x_1, \ldots, x_n\}$ of solutions of $P(x) = 0$. For any ordering $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ one can construct pseudoroots $y_{i_k}$, $k = 1, \ldots, n$, (certain rational functions in $x_{i_1}, \ldots, x_{i_n}$) that give a decomposition $P(t) = (t - y_{i_1}) \cdots (t - y_2) (t - y_1)$ where $t$ is a central variable.

In [3] I. Gelfand, V. Retakh, and R. Wilson introduced the algebra $Q_n$ of all pseudo-roots of a generic noncommutative polynomial, determined a basis for this algebra and studied its structure. The algebras $Q_n$ have a presentation given by generators $u(A), \emptyset \neq A \subseteq [n]$ and relations

$$\sum_{C, D \subseteq A} [u(C \cup i), u(D \cup j)] = \left( \sum_{E \subseteq A} u(E \cup i \cup j) \right) \sum_{F \subseteq A} (u(F \cup i) - u(F \cup j))$$

for all $A \subseteq [n], i, j \in [n] \setminus A, i \neq j$.

In [4] I. Gelfand, S. Gelfand, and V. Retakh introduced a class of quotient algebras of $Q_n$ corresponding to graphs on $n$ nodes. Let $G$ be a graph with vertex set $[n] = \{1, 2, \ldots, n\}$ and edge set $E$ composed of elements of $P([n])$ with cardinality two (hence $G$ has no loops of multiple edges). We can then consider the quotient algebra $Q_n(G)$ we get by adding the additional relations $u([i,j]) = 0$ if $\{i, j\} \notin E$ to $Q_n$.

The following theorem gives a nice presentation of the algebra $Q_n(G)$.

Theorem 3. Let $G$ be a graph on $n$ nodes with edge set $E$. Then the algebra $Q_n(G)$ is generated by the elements $u(i)$ for $i \in [n]$ and $u(i,j)$ for $\{i, j\} \in E$ with the following relations (assume $u(i,j) = 0$ if $\{i, j\} \notin E$):

(i) $u(i), u(j) = u(i, j)(u(i) - u(j))$ if $i \neq j$, $i, j \in [n]
(ii) u(i, k), u(j, k) + [u(i, k), u(j)] + [u(i), u(j, k)] = u(i, j)(u(i, k) - u(j, k))$ for distinct $i, j, k \in [n]
(iii) $u(i, j), u(k, l) = 0$ for distinct $i, j, k, l \in [n]$. 




3 The Algebra $K_3$

Here we consider the algebra that is generated by the graph $K_3$. By theorem $\Box$ this algebra has generators $u(1), u(2), u(3), u(12), u(13), u(23)$ together with the following relations in $V \otimes V$ which we will refer to as $r_1$ through $r_5$:

- $r_1 = [u(1), u(2)] + u(12)(u(2) - u(1)) = 0$
- $r_2 = [u(2), u(3)] + u(23)(u(3) - u(2)) = 0$
- $r_3 = [u(3), u(1)] + u(13)(u(1) - u(3)) = 0$
- $r_4 = [u(12), u(23)] + [u(12), u(3)] + [u(1), u(23)] - u(13)(u(12) - u(23)) = 0$
- $r_5 = [u(12), u(13)] + [u(12), u(3)] + [u(2), u(13)] - u(23)(u(12) - u(13)) = 0$

We have only five relations because all other possible combinations in i) and ii) are linear combinations of these five. We also have no relations of type iii) because $n = 3$ and we do not have four distinct integers to work with.

We define an increasing filtration on $K_3$ by defining $F_n$ to be the span of all monomials $u(A_1)u(A_2)\cdots u(A_k)$ such that $\sum_{i=1}^k |A_i| \leq n$. It is clear that our $F_i$ are subspaces with the properties $F_iF_j \subseteq F_{i+j}$. We set the define $F_0$ to be the span of 1.

Now we form $gr(K_3)$ in the usual way. Take $G_i = F_i/F_{i-1}$ and set $gr(K_3) = \bigoplus_i G_i$ and then define multiplication in $gr(K_3)$ so for all $a \in F_i, b \in F_j, (a + F_{i-1})(b + F_{j-1}) = ab + F_{i+j-1}$. Note that there is a non-linear map $gr : K_3 \to grK_3$ that sends $a \in F_i, a \notin F_{i-1}$ to $a + F_{i-1}$ in $gr(K_3)$ and sends 0 to 0.

4 A New Presentation of $gr(K_3)$

For ease of notation, let us temporarily set $a = u(1), b = u(2), c = u(3), d = u(12), e = u(23),$ and $f = u(13)$. Notice our relations in $K_3$ then become:

- $r_1 = db - da + ab - ba$
- $r_2 = ec - cb + bc - cb$
- $r_3 = fa - fc + ca - ac$
- $r_4 = de - ed - fd + fe + dc - cd + ac - ea$
- $r_5 = df - fd - cd + ef + dc - cd + bf - fb$

Therefore in $gr(K_3)$ (if we allow each generator to represent itself under the image $gr$) we know the following relations hold:

- $d \cdot b = da$
- $c \cdot c = cb$
- $f \cdot c = fc$
- $f \cdot e = fd - de + cd$
- $f \cdot d = df + ef - ed$

Now this list of relations might not be enough for a presentation of $gr(K_3)$, for other relations may hold true and be needed as well. Call the algebra generated by the five truncated relations above “chopped” $K_3$ (or $ch(K_3)$). We know that since $gr(K_3)$ is a quotient of $ch(K_3)$ (it has possibly more relations) we can verify that the two are equal by showing they have the same Hilbert series. Since $gr(K_3)$ has the same Hilbert series as $K_3$ we can compare the series for $ch(K_3)$ to the one for $K_3$ instead.

First let us use the diamond lemma to compute the Hilbert series of $K_3$.

**Theorem 4.** A basis for $K_3$ is given by the set of monomials in $T(V)$ containing none of the following substrings: $ce, cd, cb, ca, bf, ba$.

**Proof.** Order the set of monomials first by monomial length, and then lexicographically with the ordering $c > b > e > f > a > d$ (which is actually $u(3) > u(2) > u(23) > u(13) > u(1) > u(12)$). We then have the following reductions (after replacing $r_5$ with $r_5 - r_4$):

- $cd \rightarrow de - ed - fd + fe + dc + ae - ea$
- $bf \rightarrow fb + ae - ea + fe - ef + de - df$
- $ba \rightarrow ab + db - da$
- $cb \rightarrow bc + ec - cb$
- $ca \rightarrow ac + fc - fa$

This gives us $cba$ and $cbf$ as two ambiguities that need to be resolved. It is not hard to check that if we compute $c(ba) - (cb)a$ and reduce with the above five relations we get zero. However in order to resolve $cbf$ it turns out we must add the relation: $ce = cfb + efa + ace + ccc - fae - cea + dce + de^2 + ae^2 + fe^2 - fde - def - aef - eaf - fef + fdf - bcf - ecf + efb + ef - e^2f - e^2a - edf$
However, adding this relation creates no new ambiguities. So we have found a basis in the set of monomials not containing the strings ce, cf, cd, ch, cd, ce, cf, cf, cd.

**Theorem 5.** The Hilbert series of $K_3$ is $H(x) = \frac{1}{(1-x)^3}$. 

**Proof.** We must count the number of monomials of length $n$ in $T(V)$ not containing any of the strings listed in theorem 4. Call such monomials the valid monomials of length $n$.

Let $T_n$ be the number of valid monomials of length $n$.

Let $J_n$ be the number of valid monomials of length $n$ that begin with $b$.

Let $K_n$ be the number of valid monomials of length $n$ that begin with $c$.

We can note right away that $T_{n+1} = 4T_n + J_{n+1} + K_{n+1}$. Since words beginning with $b$ can not be followed by $a$ or $f$, we get $J_{n+1} = 2T_{n-1} + J_n + K_n$. Words beginning with $c$ can be followed by $c, f,$ or $e$, but in the case they can not next be followed by $f$. This gives us $K_{n+1} = T_{n-1} + K_n + 3T_{n-2} + J_{n-1} + K_{n-1}$. By counting the valid words up to length three, we also get initial conditions. Thus we obtain the following system of recurrences:

- $T_{n+1} = 4T_n + J_{n+1} + K_{n+1}$
- $J_{n+1} = 2T_{n-1} + J_n + K_n$
- $K_{n+1} = T_{n-1} + K_n + 3T_{n-2} + J_{n-1} + K_{n-1}$
- $T_1 = 6, T_2 = 31, T_3 = 157$

To solve this system notice first that in the second equation $J_n + K_n = J_{n+1} - 2T_{n-1}$. Plugging this into our first equation gives $J_{n+2} = T_{n+1} - 2T_n$. We can use this to get rid of the $J$'s in the first and third equations to get the system:

- $T_{n+1} = K_{n+1} + 5T_n - 2T_{n-1}$
- $K_{n+1} = K_n + K_{n-1} + T_{n-1} + 4T_{n-2} - 2T_{n-3}$

Solving the first for $K_{n+1}$ and substituting into the second gives us $T_{n+1} = 6T_n - 5T_{n-1} + T_{n-2}$. Using generating functions and our initial conditions we can quickly find that the Hilbert series for $K_3$ is $H(x) = \frac{1}{(1-x)^3}$.

Now we must find the Hilbert series of $ch(K_3)$. Consider an ordering first by monomial length and then lexicographically with $f > e > d > c > b > a$. We get the following reductions in $ch(K_3)$.

- $fe \rightarrow ef + df - de$
- $fd \rightarrow df + ef - ed$
- $db \rightarrow da$
- $ec \rightarrow eb$
- $fe \rightarrow fa$

We have two ambiguities to resolve this time: $fec$ and $fdb$. An attempt to resolve the $fec$ ambiguity shows it necessary to add the relation $efb = efa - dfb + dfa$. With this new relation, we can resolve the $fdb$ ambiguity. However, we created a new ambiguity by adding our $efb$ relation. In order to resolve $febf$ we must toss in the relation $efbf = efa + dfb - dfb + \frac{1}{3}efb - \frac{1}{3}efb - \frac{1}{3}dfb + \frac{13}{3}dfe$. We now have to worry about the ambiguity $febf$. We can deal with all these ambiguities at once with the following lemma.

**Lemma 1.** Suppose we need to resolve an ambiguity of the form $ef^n b = ev_n + dw_n - \alpha_n df^n b$ where $v_n$ and $w_n$ are linear combinations of monomials of length $n + 1$. Suppose also that the terms in $v_n$ and $w_n$ are all less than or equal then $f^n b$ and that $\alpha_n$ is a positive real number. Then the ambiguity $ef^n b$ can be resolved by adding a relation of the form $ef^{n+1} b = ev_{n+1} + dw_{n+1} - \alpha_{n+1} df^{n+1} b$ where $v_{n+1}$ and $w_{n+1}$ are linear combinations of monomials of length $n + 2$, the terms in $v_{n+1}$ and $w_{n+1}$ are all less than $f^{n+1} b$, and $\alpha_{n+1}$ is a positive real number.

**Proof.** We have $(fe) f b = f (ef^n b) = ef^{n+1} b + df^{n+1} b - (d + f)(ev_n + dw_n - \alpha_n df^n b) = ef^{n+1} b + df^{n+1} b - dev_n - dfw_n - \alpha_n df^n b - dev_n - dfw_n - \alpha_n df^n b + \alpha_n ef^n b - \alpha_n df^n b$.

Thus $ef^{n+1} b + \alpha_n ef^n b = d(-f^{n+1} b + dw_n + \alpha_n df^n b + f v_n + f w_n - \alpha_n f^{n+1} b + e(f v_n + f w_n - dw_n + \alpha_n df^n b + d v_n - d w_n + \alpha_n df^n b)$ and dividing by $1 + \alpha_n$ gives us:

- $ef^{n+1} b = \frac{1}{1 + \alpha_n} (f v_n + f w_n - dw_n + \alpha_n df^n b) + \frac{1}{1 + \alpha_n} (d(-f^{n+1} b + dw_n + \alpha_n df^n b + f v_n + f w_n) - \frac{1}{1 + \alpha_n} ef^n b)$

Setting $v_n = \frac{1}{1 + \alpha_n} (f v_n + f w_n - dw_n + \alpha_n df^n b), w_n = \frac{1}{1 + \alpha_n} (-f^{n+1} b + dw_n + \alpha_n df^n b + f v_n + f w_n)$ and taking $\alpha_n$ to the positive real constant $\frac{3}{1 + \alpha_n}$ completes the proof.
With this lemma, we see that the only bad words are ones containing strings of the following forms:  
$fe, fd, db, ec, fc, \text{or } ef^nb$ for $n \geq 1$. We can now use this to find the Hilbert series of $ch(K_3)$.

**Proposition 4.1.** The Hilbert series of $ch(K_3)$ is equal to the Hilbert series of $K_3$. 

**Proof.** We wish to count the strings not containing $fe, fd, db, ec, fc, \text{or } ef^nb$ for $n \geq 1$ as a substring. To count all such strings let $T_n$ be the total number of valid monomials of length $n$. Let $K_n$ be the number of valid monomials of length $n$ beginning with $d$. Let $L_n$ and $M_n$ be the corresponding numbers for $f$ and $e$ respectively.

We can immediately see that $T_n = K_n + L_n + M_n + 3T_{n-1}$. We know that if a word begins with $d$, it can be followed by any smaller valid word not beginning with $b$. This gives us $K_n = 2T_{n-2} + K_{n-1} + L_{n-1} + M_{n-1}$. The words beginning with $f$ can be followed by $f, a$ or $b$ giving us $L_n = 2T_{n-2} + L_{n-1}$.

Finally we must count the words beginning with $e$. If $e$ is followed by any valid word not beginning with $f$ or $c$ then we are okay. There will be $2T_{n-2} + K_{n-1} + M_{n-1}$ of these. If the second letter does happen to be $f$ then the next letter can only be $a$ or $f$. If it is $a$ then we can follow up with any valid word (which adds $T_{n-3}$ to the equation), but if it is $f$ we are once again in an $a$ or $f$ situation. This time the $a$ case ends up adding $T_{n-4}$. We can repeat this down the line to add $T_{n-5} + T_{n-6} + \cdots + T_1$ and finally we add 2 (or $2T_0$) for the $ef \ldots fa$, and $ef \ldots ff$ cases. This gives us $M_n = 2T_{n-2} + K_{n-1} + M_{n-1} + \sum_{k=3}^{n} T_{n-k} + T_0$.

We must now solve the following system of recurrences:

\[
T_n = K_n + L_n + M_n + 3T_{n-1}
\]

\[
K_n = 2T_{n-2} + K_{n-1} + L_{n-1} + M_{n-1}
\]

\[
L_n = 2T_{n-2} + L_{n-1}
\]

\[
M_n = 2T_{n-2} + K_{n-1} + M_{n-1} + \sum_{k=3}^{n} T_{n-k} + T_0
\]

To do this, first we must get rid of the summation for $M_n$. Set $R_n = M_n - M_{n-1}$ which equals $2T_{n-2} + K_{n-1} + M_{n-1} + \sum_{k=3}^{n} T_{n-k} + T_0 - 2T_{n-3} - K_{n-2} - M_{n-2}$. Substituting for $K_n$ and $L_n$ we have

\[
R_n = 2T_{n-2} - T_{n-3} + K_{n-1} - K_{n-2} + T_0.
\]

Now we can get rid of our $M_n$ terms altogether by replacing $T_n$ and $K_n$ with $T_n - K_n = 3T_{n-1} - 2T_{n-2} + K_{n-1} + L_{n-1} + R_n$ and $T_n - K_n + 1 = 3T_{n-1} + K_{n-1} + L_{n-1} + M_{n-2} - T_{n-2} - K_{n-1} - L_{n-1} = T_{n-1}$. By plugging $L_n - L_{n-1} = 2T_{n-2}$ into our $T_n$ equation we are left with the following system:

\[
T_n = 3T_{n-1} + 2K_n - K_{n-1} + R_n
\]

\[
R_n = 2T_{n-2} - T_{n-3} + R_{n-1} + K_{n-1} - K_{n-2}
\]

\[
K_n = T_{n-1} - T_{n-2}
\]

Substituting for $K_n$ leads to the system:

\[
T_n = 5T_{n-1} - 3T_{n-2} + T_{n-3} + R_n
\]

\[
R_n = 3T_{n-2} - T_{n-3} + R_{n-1} + R_{n-1}
\]

We can now solve for $R_n$ in the first equation and substitute into the second equation to get $T_n = 6T_{n-1} - 5T_{n-2} + T_{n-3}$. This is the same recurrence we used to generate the Hilbert series of $K_3$. Checking the length one, two, and three cases gives the same initial conditions as well. Therefore the two algebras must have the same Hilbert series.

**Corollary 4.1.** $ch(K_3) \cong gr(K_3)$. 

**Proof.** We know $ch(K_3)$ has the same graded dimension as $K_3$ which has the same graded dimension as $gr(K_3)$. Since $ch(K_3)$ is a quotient of $gr(K_3)$ with the same graded dimension, the two must be isomorphic.

From now on we will list the chopped relations for a presentation of $gr(K_3)$.

## 5 Reduction to $gr(K_3)$

The following corollary from [2] is useful here.

**Corollary 5.1.** Let $0 = F_0W \subset F_1W \subset \cdots \subset F_{l-1}W \subset F_lW$ be a filtered vector space and $X_1, \ldots, X_n$ a collection of subspaces; then the following two conditions are equivalent:

(a) the whole set of subspaces $F_0W, \ldots, F_lW, X_1, \ldots, X_n \subset W$ is distributive.

(b) the associated graded collection $grX_1, \ldots, grX_n$ in the associated graded vector space $grW$ is distributive are for any $1 \leq i < j \leq n$ either of the two equivalent conditions holds:
\( gr(X_i + X_j) = grX_i + grX_j \) or \( gr(X_i \cap X_j) = grX_i \cap grX_j \).

Hence, if the set \( \{ grX_i \} \) generates a distributive lattice in \( grV \), and \( gr(X_i \cap X_j) = gr(X_i) \cap gr(X_j) \) for all \( i \) and \( j \), then the set \( \{ X_i \} \) generates a distributive lattice in \( V \). We wish to check that \( gr(RV \cap VR) = gr(RV) \cap gr(VR) \). By looking at \( gr \) as a function and only using the basic rules for set maps and intersections we see that \( gr(RV \cap VR) \subseteq gr(RV) \cap gr(VR) \). It will be enough for us to show that the dimension of \( gr(RV) \cap gr(VR) \) is one and that \( gr(RV \cap VR) \) is not the zero subspace.

For the second part, notice that the vector \( r_1c + r_2d + r_3b + r_4(c-b) + r_5(a-c) \) is equal to the vector \( ar_2 + br_3 + cr_1 + d(r_2 + r_3) + e(r_1 + r_3) + f(r_1 + r_2) \) where \( r_1, \ldots, r_5 \) are the relations we defined earlier for \( K_3 \). This shows that \( RV \cap VR \) is not zero, and thus \( gr(RV \cap VR) \) is not the zero subspace.

**Proposition 5.1.** \( \dim gr(RV) \cap gr(VR) = 1 \)

**Proof.** In the last section we showed that \( gr(R) \) is the span of \( \{ fe - ef - df + de, fd - df - ef + ed, d(b-a), e(c-b), f(a-c) \} \). Since these are the relations we will be working with throughout the rest of this paper, we officially set:

\[
\begin{align*}
r_1 &= db - da \\
r_2 &= ec - eb \\
r_3 &= fa - fc \\
r_4 &= de - ed - fd + fe \\
r_5 &= df - fd - ed + ef
\end{align*}
\]

Call the span of these five relations \( S \) (so \( S = gr(R) \)). Our goal is to show \( SV \cap VS \) is of dimension one.

Suppose \( x \in SV \cap VS \) and \( x \neq 0 \). Since all the monomials in \( VS \) contain no \( a, b, \) or \( c \) in the middle spot, we can replace \( SV \) with \( sp\{r_4, r_5\} \otimes V \). As all the monomials in \( SV \) contain no \( a, b, \) or \( c \) in the first slot we can replace \( VS \) with \( sp\{d, e, f\} \otimes S \).

Suppose \( x \in sp\{r_4, r_5\} \otimes V \cap sp\{d, e, f\} \otimes S \). Then we can write \( x = r_1v_1 + r_5v_2 \) for some \( v_1, v_2 \in V \). Now if the coefficient of \( c \) in \( v_1 \) was nonzero we would have an \( f e e \) term with nothing else that could cancel it out. Hence we would have an \( f e e \) appearing in \( VS \) which is not possible. If the coefficient of \( f \) in \( v_1 \) was nonzero then we would have a \( d f f \) which could only happen in \( VS \) is the coefficient of \( dr_5 \) was nonzero. But this would give us a nonzero \( d e e \) term with nothing that could cancel it out, which is also not possible. If the coefficient of \( d \) was nonzero we would have an \( f d d \) appearing which implies that \( a r_4 + \beta f r_5 \) appears in \( VS \) with \( \alpha \neq -\beta \). As \( f f e \) can not appear, \( \alpha \) must be zero, so \( \beta \neq 0 \). But then we would have an \( -\beta f f d \) appearing with nothing to cancel it out with. This shows that \( v_1 \in sp\{a, b, c\} \). A similar argument shows that \( v_2 \) is in this same span. We know now that \( x \in SV \cap VS \) implies \( x \in sp\{r_4, r_5\} \cap sp\{a, b, c\} \cap sp\{d, e, f\} \otimes S \). We can replace this last \( S \) with \( sp\{r_1, r_2, r_3\} \) after noticing that only \( a, b, \) and \( c \) can now appear in the last slot. So far we know \( SV \cap VS = sp\{r_4, r_5\} \cap sp\{a, b, c\} \cap sp\{d, e, f\} \otimes sp\{r_1, r_2, r_3\} \)

Now suppose \( x = r_1v_1 + r_5v_2 \) where \( v_1, v_2 \in sp\{a, b, c\} \) and \( x \neq 0 \). Suppose also that the coefficient of \( b \) in \( v_1 \) is 0. Then no \( d e b \) or \( f e b \) can appear in \( x \) (with a non-zero coefficient). Looking in \( VS \) we see this means \( dr_2 \) and \( fr_2 \) must have coefficients of zero. This means no \( d e c \) or \( f e c \) can appear in \( x \) either. Hence \( v_1 \) is a constant multiple of \( a \). However this constant must be 0, otherwise the term \( d e a \) would appear, and there is no way to achieve that in \( VS \). So in this case \( v_1 = 0 \) and \( x = r_5v_2 \). Then \( v_2 \) would have to be a multiple of \( a \) since no \( d f b \) and \( f d c \) can occur. So \( x \) is a constant multiple of \( r_5a \). But \( r_5a \) is not in \( VS \) so we have reached a contradiction.

We now know that if \( x = r_1v_1 + r_5v_2 \in SV \cap VS \) then the coefficient of \( b \) in \( v_1 \) is non-zero, so we can scale \( x \) to make this coefficient 1. Hence \( d e b + f e b \) appears in \( x \). Looking in \( VS \) we see this implies \(-d e c - f e c \) appears as well. This implies the coefficient of \( c \) in \( v_1 \) is \(-1 \). As \( d f c \) can not appear, this term must cancel, meaning the coefficient of \( c \) in \( v_2 \) is 1. This means \( d f c + e f c \) appears in \( x \). Looking in \( SV \) we see \(-d f a - e f a \) appears, implying the coefficient of \( a \) in \( v_2 \) is \(-1 \). Since the coefficients of \( a \) in \( v_1 \) and \( b \) in \( v_2 \) must be 0 (look at \( SV \) to see this) we get that \( x = r_4(b-c) + r_5(c-a) \). It is easy to see this \( x \) is in \( VS \) because it is equal to \((c+f)r_1 - (d+f)r_2 - (d+e)r_3 \).

\( \square \)

6 A Reduction to Two Cases

Recall from corollary \( \Box \) that \( K_3 \) will be Koszul if we show that \( RV^{a-2} \cap VR^{a-2} \cap \cdots \cap V^{a-2}RV^{a-2} \cap V^{a-1}RV^{a-1}RV^{a-2} \cap V^{a-2}R \) is a distributive triple in \( V^n \) for any \( n \) and \( 2 \leq a \leq n - 2 \).

We will require the following lemma of Serconek and Wilson (lemma 1.1 from \( \Box \)):
Lemma 2. If

1) $V = \sum_{i=1}^{n} V_i$ is graded as a vector space

2) $X_j$ is a collection of subspaces of $V$

3) Each $X_j = \sum_{i=1}^{n} (X_j \cap V_i)$

then $\{X_j\}_j$ is distributive if and only if for all $i \in I$, $\{X_j \cap V_i\}_j$ is distributive in $V_i$.

We now choose a particular $\{1,2\}^n$ grading of $V$ to apply this lemma to. Set $V^n_{(1,2,\ldots,n)}$ to be the span of all monomials $u(A_1)u(A_2)\ldots u(A_n)$ so that $|A_k| = i_k$. For example, in the $n = 2$ case $r_1$, $r_2$, and $r_3$ are in the $V^2_{(2,1)}$, space and $r_4$ and $r_5$ are in the $V^2_{(2,2)}$ space. Hence $R = (R \cap V^2_{(2,1)}) + (R \cap V^2_{(2,2)})$.

This shows that property three of lemma 2 will apply to our set $\{V^n_{(1,2,\ldots,n)}\}$.

We have to show that $\{RV^{n_2}_{(2,1)} \cap (V^n)_{\alpha}, \ldots, RV^{n_2}_{(2,1)} \cap (V^n)_{\alpha}\}$ generates a distributive lattice in $(V^n)_{\alpha}$ for $\alpha \in \{1,2\}^n$. Notice that we need only check the $\alpha = (1,1,\ldots,1)$ such that $1_i \geq 1_{i+1} \geq \cdots \geq 1_n$. This is because $R \in V^2_{(2,1)} + V^2_{(2,2)}$ so if a $(1,2)$ appears somewhere in the string $\alpha$ then one of our $RV^{n_2}_{(2,1)} \cap (V^n)_{\alpha}$ will be zero. Since we can show proper subsets of $\{RV^{n_2}_{(2,1)} \cap (V^n)_{\alpha}, \ldots, RV^{n_2}_{(2,2)} \cap (V^n)_{\alpha}\}$ are distributive, we are done for such $\alpha$.

Next notice that the subspace $RV \cap VR$ we found earlier is contained in $V^3_{(2,2,1)}$. Combining corollary 1 with lemma 2 we have to check that $RV^{n_2}_{(2,1)} \cap \cdots \cap RV^{n_3}_{(2,1)} \cap (V^n)_{\alpha}, RV^{n_2}_{(2,2)} \cap (V^n)_{\alpha}$ is a distributive triple for any decreasing $\alpha$ and $2 \leq \alpha \leq n - 2$. However if the last two digits of $\alpha$ are $(1,1)$ then the last term of the third element $(RV^{n_2}_{(2,1)} \cap (V^n)_{\alpha}) = 0$ and we will be done because proper subsets are distributive. We are done with all cases except when $\alpha$ contains a two in the second to last spot. This leaves $\alpha = (2,2,\ldots,2,1)$ and $\alpha = (2,2,\ldots,2)$.

Next suppose $a > 2$. Notice that our first term in our triple is $RV^{n_2}_{(2,1)} \cap VR^{n_3}_{(2,1)} \cap \cdots \cap RV^{n_2}_{(2,1)} \cap (V^n)_{\alpha}$ which is contained in $(RV \cap VR)V^{n-3}_{(2,1)} \cap (V^n)_{\alpha}$. Since $\alpha$ cannot not start with $(2,2,1)$ and $RV \cap VR$ is contained in this graded space, this term must be zero. Hence we need only check the case where $\alpha = 2$.

From here on we can check a set $\{X_1, \ldots, X_k\}$ is distributive in the $\alpha$ case we mean that the set $\{X_1 \cap (V^n)_{\alpha}, \ldots, X_k \cap (V^n)_{\alpha}\}$ is distributive. Thus we must show $RV^{n_2}_{(2,2,1)} \cap VR^{n_3}_{(2,2,1)} \cap \cdots \cap RV^{n_2}_{(2,2,1)}$ is a distributive triple in the $(2,2,\ldots,2)$ cases. This amounts to showing $RV^{n_2}_{(2,2,1)} \cap (V^n)_{\alpha}$ is distributive for any decreasing $\alpha$ and $2 \leq \alpha \leq n - 2$. However if the last two digits of $\alpha$ are $(1,1)$ then the last term of the third element $(V^n)_{\alpha} = 0$ and we will be done because proper subsets are distributive. We are done with all cases except when $\alpha$ contains a two in the second to last spot. This leaves $\alpha = (2,2,\ldots,2,1)$ and $\alpha = (2,2,\ldots,2)$.

Introducing a little more terminology makes this statement simpler. Define $W_k = RV^{k-2}_{(2,2,1)} \cap VR^{k-3}_{(2,2,1)} \cap \cdots \cap VR^{k-2}_{(2,2,1)} \cap (V^n)_{\alpha}$ for $k \geq 2$ and set $W_k$ to the zero subspace otherwise. Then we must show that $X \in RV^{n_2}_{(2,2,1)} \cap WV_{n-1} \implies x \in V^{k-2}_{(2,2,1)} \cap WV_{n-2}$ for our two cases. Notice also that this statement is trivial for $n < 4$.

Next we look for a more natural spanning set for $R$. Let $\sigma$ be the permutation $(1,2,3)$. Notice that the map $T$ sending $u(A)$ to $u(\sigma(A))$ sends $sp(r_1, r_2, r_3)$ and $sp(r_4, r_5)$ both back to themselves. Hence $R$ is $T$ invariant. Letting $\omega$ be a primitive cube root of one, the following elements form a basis for $R$ consisting only of eigenvectors. Set:

$u_1 = u(12) + u(23) + u(13)$

$u_\omega = u(12) + u(23) + \omega u(13)$

$u_{\omega^2} = u(12) + \omega u(23) + \omega u(13)$

$v_1 = u(1) + u(2) + u(3)$

$v_2 = u(1) + u(2) + \omega u(3)$

$v_3 = u(1) + \omega^2 u(2) + \omega u(3)$

then our relations become

$r_1 = (u_1 + u_\omega + u_{\omega^2})(\omega - 1)v_1 + (\omega^2 - 1)v_2$

$r_2 = (\omega u_1 + u_\omega + \omega u_{\omega^2})(\omega^2 - \omega)v_1 + (\omega - \omega^2)v_2$

$r_3 = (\omega u_1 + u_\omega + \omega^2 u_{\omega^2})(1 - \omega^2)v_1 + (1 - \omega)v_2$

$r_4 = u_1^2 - 2 u_1 u_\omega + u_1 u_{\omega^2}$

$r_5 = u_{\omega^2}^2 - 2 u_1 u_{\omega^2} + u_2 u_1$

If we set $c = v_1, b = v_2, d = u_1, e = u_\omega, f = u_{\omega^2}$ then we get the simpler looking set:

$r_1 = (e + f + d)((\omega - 1)a + (\omega^2 - 1)b)$

$r_2 = (e + \omega f + \omega^2 d)((\omega^2 - \omega)a + (\omega - \omega^2)b)$

$r_3 = (e + \omega^2 f + \omega d)((1 - \omega^2)a + (1 - \omega)b)$

$r_4 = f^2 - 2 de - cd$

$r_5 = e^2 - 2 df + fd$
Keep in mind that \( r_4 \) and \( r_5 \) still sit in \( V_{(2,2)}^2 \) and \( r_1, r_2, \) and \( r_3 \) still sit in \( V_{(2,1)}^2 \) since \( d, e, \) and \( f \) are multiples of \( u(12), u(23), \) and \( u(12) \) and \( a, b, \) and \( c \) are multiples of \( u(1), u(2), \) and \( u(3) \). This will be our spanning set for \( R \) as we move on to the two last cases.

## 7 The Two Last Cases

Our main goal is to prove the following lemma in the \((2,2,\cdots ,2)\) and \((2,2,\cdots ,1)\) cases.

### Lemma 3.

Suppose \( n \geq 2 \) then

a) If \( z_1, z_2 \in V^n, dz_1 + ez_2 \in W_{n+1} \) then \( z_1, z_2 \in W_n \)

b) If \( y_1, y_2 \in V^n, ey_1 + fy_2 \in W_{n+1} \) then \( y_1, y_2 \in W_n \)

c) If \( x_1, x_2 \in V^n, dx_1 + fx_2 \in W_{n+1} \) then \( x_1, x_2 \in W_n \)

**Proof.** Suppose we knew the lemma was true for either \( n = 1 \) or \( n = 2 \). Then we can assume by induction that the lemma holds for \( n-1 \). Suppose we are in the a) case and set \( z = dz_1 + ez_2 \in W_{n+1} \). Then we can write \( z = r_4 h_1 + r_5 h_2 + V W_n = f( fh_1 + dh_2 ) + e( eh_2 + dh_1 ) - 2d( eh_1 + fh_2 ) + V W_n \). This means \( f h_1 + dh_2 \in W_n \). Since \( z \in dV^n + eV^n \) we know that \( f( fh_1 + dh_2 ) \in V W_n \) so \( f h_1 + dh_2 \in W_n \). By our inductive hypothesis \( h_1, h_2 \in W_{n-1} \). So \( z \in r_4 W_{n-1} + r_5 W_{n-1} + V W_n \subset W_n \), and hence \( z_1, z_2 \in W_n \).

The b) and c) cases are similar.

We have left to find basis cases for lemma. In the \((2,2,\cdots ,2)\) situation we can find one when \( n = 1 \).

Assume \( dz_1 + ez_2 \in W_2 = R = \alpha_4 + \beta_5 \). If \( \alpha \neq 0 \) we would have an \( f^2 \) appearing which we could not cancel, and hence a contradiction. Similarly \( \beta \) must be 0 as well and hence \( z_1, z_2 \) are both in \( W_1 = 0 \).

Now that we know that the lemma is true in both cases we can prove the following proposition thus completing our proof that \( K_3 \) is Koszul.

### Proposition 7.1.

If \( n \geq 4, x \in RV_{n-2} \cap V_{n-1} \) then \( x \in V^2 W_{n-2} \)

**Proof.** Since \( x \in RV_{n-2} \) we can write \( x = r_4 h_1 + r_5 h_2 = ( f^2 - 2de - cd ) h_1 + ( e^2 - 2df + fd ) h_2 \). This means \( f h_1 + dh_2 \in W_n \). Since \( x \in V W_{n-1} \) we know that \( f h_1 + dh_2 , eh_2 + dh_1 , \) and \( eh_1 + fh_2 \) are all in \( W_{n-1} \). Since \( n - 1 \geq 3 \), lemma \( \Box \) applies and thus \( h_1, h_2 \in W_{n-2} \). Thus \( x \in r_4 W_{n-2} + r_5 W_{n-2} \subset V^2 W_{n-2} \) and we are done.

### Theorem 6.

\( K_3 \) is Koszul.

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