TORIC VECTOR BUNDLES OVER A DISCRETE VALUATION RING  
AND BRUHAT-TITS BUILDINGS  

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Abstract. We give a classification of rank $r$ torus equivariant vector bundles $E$ on a  
toric scheme $X$ over a discrete valuation ring, in terms of piecewise affine maps $\Phi$ from  
the polyhedral complex of $X$ to the extended Bruhat-Tits building of $\text{GL}(r)$. This is  
an extension of Klyachko’s classification of torus equivariant vector bundles on a toric  
variety over a field. We also give a simple criterion for equivariant splitting of $E$ into a  
sum of toric line bundles in terms of its piecewise affine map.  

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Comments are welcome!  

Introduction  

Let $T$ be an $n$-dimensional split algebraic torus over a field $K$ with cocharacter lattice  
$N \cong \mathbb{Z}^n$. Let $X = X_\Sigma$ be a $T$-toric variety associated to a fan $\Sigma$ in $N_\mathbb{R} = N \otimes \mathbb{R} \cong \mathbb{R}^n$. A  
torus equivariant vector bundle (or toric vector bundle for short) on $X$ is a vector bundle  
$E$ on $X$ together with a linear action of $T$ that lifts that of $X$. Toric vector bundles have  
been classified by Kaneyama [Kan75] (in terms of certain data of cocycles) and by Klyachko  
[Kly90] (in terms of certain data of compatible filtrations, see Section 1.1). In this paper  

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we extend Klyachko’s classification to toric schemes over a discrete valuation ring $O$. Our classification, on the other hand, extends the known classification of toric line bundles on toric schemes (see [KKMS73, §IV.3(e)], [BGPS14, Section 3.6]). We introduce the notion of a piecewise affine map from the polyhedral complex, defining the toric scheme, to the extended Bruhat-Tits building of a general linear group. Our main result states that the isomorphism classes of toric vector bundles over a toric scheme are in one-to-one correspondence with such piecewise affine maps.

The present paper can be considered as a continuation of ideas in [KM22] where classifying torus equivariant principal bundles (or toric principal bundles for short) on toric varieties over a field is connected with the theory of Tits buildings of algebraic groups. The main result in [KM22] states that for a reductive group $G$, toric principal $G$-bundles on $X_\Sigma$ are classified by piecewise linear maps from the fan $\Sigma$ to the cone over the Tits building of $G$.

We begin with recalling some of the main concepts involved. Throughout $K$ denotes a discretely valued field with valuation $\text{val} : K \rightarrow \mathbb{Z}$. We denote by $O$ the corresponding valuation ring $\{x \mid \text{val}(x) \geq 0\}$. Also $\varpi$ denotes a uniformizer, that is, a generator of the unique maximal ideal $m = \{x \mid \text{val}(x) > 0\}$. The scheme $\text{Spec}(O)$ has two points: the generic point $\eta$ corresponding to the prime ideal $\{0\}$ and the special point $o$ corresponding to the maximal ideal $m$. For a scheme $X$ over $\text{Spec}(O)$, we call $X_\eta$ (the fiber over $\eta$) the generic fiber, and $X_o$ (the fiber over $o$) the special fiber.

Let $T$ be a split torus over $\text{Spec}(O)$. A toric scheme over $\text{Spec}(O)$ is a normal integral separated scheme of finite type $X$ over $\text{Spec}(O)$ equipped with a dense open embedding $T_K = T_o \hookrightarrow X_0$ and an action of $T$ on $X$ over $\text{Spec}(O)$ which extends the translation action of $T_K$ on itself. While toric varieties (over a field) are classified by fans in $N_\mathbb{R} \cong \mathbb{R}^n$, toric schemes (over $\text{Spec}(O)$) are classified by fans $\Sigma$ in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ (see Section 1.2). For a fan in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ we denote the corresponding toric scheme by $X_\Sigma$. By intersecting cones in $\Sigma$ with $N_\mathbb{R} \times \{1\}$ one obtains a polyhedral complex $\Sigma_1$ in the affine space $N_\mathbb{R} \times \{1\}$. The generic fiber $X_\eta$ is a usual toric variety over $K$ with corresponding fan $\Sigma_0$ obtained by intersecting the cones in $\Sigma$ with $N_\mathbb{R} \times \{0\}$. For the rest of the paper, we denote the identity element in $T_\eta \hookrightarrow X$ by $x_0$.

The insight from [KM22] is that the right gadgets for classification of toric principal bundles are buildings of algebraic groups. Buildings are special kinds of simplicial complexes arising in the classification theory of algebraic groups over arbitrary fields. Abstractly speaking, a building is an (infinite) simplicial complex together with certain distinguished subcomplexes called apartments that satisfy a list of axioms (see Definition 5.1).

There are two important kinds of buildings: spherical buildings and affine buildings. In a spherical building each apartment is a triangulation of a sphere while in an affine building each apartment is a triangulation of an affine (or Euclidean) space. To a reductive algebraic group over a field one associates its Tits building which is a typical example of a spherical building. Moreover, to a reductive algebraic group over a discretely valued field $K$ one associates its Bruhat-Tits building which is a typical example of an affine building.

Let $E \cong K^r$ be an $r$-dimensional $K$-vector space. To state the main results of the paper we need to briefly review the construction of the Bruhat-Tits building $\mathcal{B}_{\text{aff}}(E)$ as well as the extended Bruhat-Tits building $\mathcal{B}_{\text{aff}}(E)$ of the group $\text{GL}(E)$. Roughly speaking, the Bruhat-Tits building $\mathcal{B}_{\text{aff}}(E)$ is obtained by gluing infinitely many $(r-1)$-dimensional real affine spaces along some simplices. We have one affine space for each direct sum decomposition of $E$ into 1-dimensional subspaces. The extended building $\mathcal{B}_{\text{aff}}(E)$ is just an extension of $\mathcal{B}_{\text{aff}}(E)$ by a direct sum with $\mathbb{R}$. Below we describe the Bruhat-Tits building of the group $\text{GL}(E)$ in two ways (see Section 2.1 for details):
(i) as a simplicial complex which we denote by $\Delta_{\text{aff}}(E)$,

(ii) as a topological space which we denote by $\mathfrak{B}_{\text{aff}}(E)$.

The topological space $\mathfrak{B}_{\text{aff}}(E)$ is a geometric realization of the abstract simplicial complex $\Delta_{\text{aff}}(E)$, that is, each simplex in $\Delta_{\text{aff}}(E)$ can be identified with a subset of $\mathfrak{B}_{\text{aff}}(E)$ homeomorphic to a standard simplex and these simplices intersect along their common sub-simplices. By abuse of terminology we refer to both as the Bruhat-Tits building of $\text{GL}(E)$.

We recall that a lattice $\Lambda$ (also called an $O$-lattice) in $E$ is a full rank $O$-submodule. In other words, a lattice is a finitely generated submodule of $E$ that spans the latter as a vector space. Two lattices $\Lambda$ and $\Lambda'$ are said to be equivalent if there is nonzero $c \in K^\times$ such that $c\Lambda = \Lambda'$. The vertices in $\Delta_{\text{aff}}(E)$ correspond to equivalence classes $[\Lambda]$ of lattices $\Lambda \subset E$. Two equivalence classes $[\Lambda], [\Lambda']$ are connected to each other if there are representatives $\Lambda, \Lambda'$ such that $\varpi\Lambda \subsetneq \Lambda' \subsetneq \Lambda$. The $(k - 1)$-dimensional simplices in $\Delta_{\text{aff}}(E)$ then correspond to $k$ mutually connected vertices $[\Lambda_1, \ldots, [\Lambda_k]$, in other words, chains of lattices such that $\varpi\Lambda_k \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_k$. A $K$-vector space basis $B = \{b_1, \ldots, b_r\}$ for $E$ defines a distinguished subcomplex, namely an apartment, in $\Delta_{\text{aff}}(E)$. The vertices in the apartment of $B$ consist of the equivalence classes of lattices $\Lambda$ of the form $\Lambda = \sum_{i=1}^r \varpi^n_i \mathcal{O}b_i$, for any choice of $a_i \in \mathbb{Z}$. Hence the vertices in the apartment can be identified with the points in $\mathbb{Z}^r / \langle (1,\ldots,1) \rangle \cong \mathbb{Z}^{r-1}$.

Alternatively, the set of lattices in $E$ can be identified with the set of integer-valued additive norms (see Section 2.1). An additive norm on $E$ is a function $v : E \to \mathbb{R} = \mathbb{R} \cup \{\infty\}$ satisfying the following axioms:

1. $v(\lambda e) = \text{val}(\lambda) + v(e)$, for all $e \in E$ and $\lambda \in K$.
2. (Non-Archimedean property) $v(e_1 + e_2) \geq \min\{v(e_1), v(e_2)\}$, for all $e_1, e_2 \in E$.
3. $v(e) = \infty$ if and only if $e = 0$.

Two additive norms are said to be equivalent if their difference is a constant. Given a basis $B = \{b_1, \ldots, b_r\}$, an additive norm $v$ is said to be adapted to $B$ if the following holds. For any $e = \sum_i c_ib_i \in E$ we have $v(e) = \min\{\text{val}(c_1) + v(b_1) \mid i = 1, \ldots, r\}$.

The set of equivalence classes of additive norms on $E$ is the (geometric realization of the) Bruhat-Tits building of $\text{GL}(E)$ which we denote by $\mathfrak{B}_{\text{aff}}(E)$. For a basis $B$, the set of equivalence classes of additive norms adapted to it is the (geometric realization of the) apartment associated to $B$ which we denote by $A_{\text{aff}}(B)$.

Furthermore, we denote the set of all additive norms on $E$ by $\mathfrak{B}_{\text{aff}}(E)$ and call it the extended Bruhat-Tits building of $E$. For a basis $B$, we denote the set of all additive norms adapted to it by $A_{\text{aff}}(B)$ and call it the extended apartment of $B$.

**Definition 1** (Piecewise affine map to an affine building). Let $\Sigma_1$ be a polyhedral complex in $N_E \times \{1\}$. We say that a map $\Phi : [\Sigma_1] \to \mathfrak{B}_{\text{aff}}(E)$ is a piecewise affine map if the following holds:

(a) For every polyhedron $\Delta \in \Sigma_1$, there is an apartment $\tilde{A}_{\text{aff}}(\Delta)$ in $\mathfrak{B}_{\text{aff}}(E)$ such that $\Phi(\Delta)$ lands in $\tilde{A}_{\text{aff}}(\Delta)$.

(b) The map $\Phi|_{\Delta} : \Delta \to \tilde{A}_{\text{aff}}(\Delta)$ is the restriction of an affine map from $N_E \times \{1\}$ to the affine space $\tilde{A}_{\text{aff}}(\Delta)$.

We say that $\Phi$ is integral if, for every $\Delta \in \Sigma_1$, $\Phi|_{\Delta}$ is the restriction of an integral affine map from $N_E \times \{1\}$ to $\tilde{A}_{\text{aff}}(\Delta)$.

Recall that given a piecewise affine function $\phi : [\Sigma_1] \to \mathbb{R}$, one can define $\phi_0 : [\Sigma_0] \to \mathbb{R}$, the linear part of $\phi_0$. Here $\Sigma_0$ is the collection of recession cones of the polyhedra in $\Sigma_1$.  

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More generally, for a piecewise affine map $\Phi$, we can define $\Phi_0 : |\Sigma_0| \to \overline{B}_{\text{sph}}(E)$, the linear part of $\Phi$. Here $\overline{B}_{\text{sph}}(E)$ is the cone over the spherical building of $GL(E)$ which can be identified with the collection of prevaluations on $E$ (see Section 2).

There is a natural notion of morphism of piecewise affine maps (Definition 3.7), and the collection of piecewise affine maps to extended Bruhat-Tits buildings of finite dimensional vector spaces $E$ form a category.

Let $X_\Sigma$ be a toric scheme over $\text{Spec}(O)$ associated to a fan $\Sigma$ in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ with corresponding polyhedral complex $\Sigma_1$ in $N_\mathbb{R} \times \{1\}$. Throughout, we will assume that $\Sigma_1$ determines $\Sigma$. That is, all the cones in $\Sigma$ are cones over the polyhedra in $\Sigma_1$ or faces of such cones. For example, this is the case if the support of $\Sigma$ is the whole $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$, or equivalently, if the support of $\Sigma_1$ is the whole $N_\mathbb{R} \times \{1\}$ (this means that the toric scheme $X_\Sigma$ is proper over $\text{Spec}(O)$, see also [GKP14, Section 3.5]). Given a toric vector bundle $E$ over $X_\Sigma$, we denote the fiber $E_{x_0}$, over the distinguished point $x_0$, by $E$. Our main result is the following.

**Theorem 2** (Main theorem, first part). With notation as above, there is a bijection between the isomorphism classes of toric vector bundles on $X_\Sigma$ and the piecewise affine maps from $|\Sigma_1|$ to $\overline{B}_{\text{aff}}(E)$. Moreover, this bijection extends to an equivalence of the corresponding categories.

The moral of Theorem 2 is that the simplicial complex $\overline{B}_{\text{aff}}(E)$ can be considered as an analogue of the classifying space of $GL(E)$ for rank $r$ toric vector bundles over toric schemes.

**Theorem 3** (Main theorem, second part). Let $E$ be a toric vector bundle over a toric scheme $X_\Sigma$ with the corresponding piecewise affine map $\Phi : |\Sigma_1| \to \overline{B}_{\text{aff}}(E)$. Then the generic fiber $E_{\eta}$, which is a (usual) toric vector bundle over the (usual) toric variety $X_\eta = X_{\Sigma_0}$, corresponds to $\Phi_0 : |\Sigma_0| \to \overline{B}_{\text{sph}}(E)$, the linear part of $\Phi_0$, which is a piecewise linear map.

Theorems 2 and 3 extend the well-known results on classification of torus equivariant line bundles on a toric scheme ([BGP14, Section 3.3]).

**Remark.** We made the assumption that the fan $\Sigma$ (in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$) is determined by its corresponding polyhedral complex $\Sigma_1$ (in $N_\mathbb{R} \times \{1\}$). One can modify the statement of Theorem 2 so that this assumption is not necessary. To this end, one uses the notion of a piecewise linear map from the fan $|\Sigma|$ to the “completed” extended building $\overline{B}_{\text{aff}}(E) \cup \overline{B}_{\text{sph}}(E)$.

**Remark.** We expect that Theorem 2 can be used to give a classification of equivariant vector bundles on normal complexity-one $T$-varieties, in terms of piecewise affine maps, extending the classification of equivariant line bundles on these varieties. In this regard the results in [IS15] might be relevant.

We say that a toric vector bundle $E$ over a toric scheme splits equivariantly if it is equivariantly isomorphic to a direct sum of toric line bundles. We give the following criterion for equivariant splitting (Theorem 4.2).

**Theorem 4** (Criterion for equivariant splitting). A toric vector bundle $E$ over a toric scheme $X_\Sigma$ splits equivariantly if and only if the image of the corresponding piecewise affine map $\Phi : |\Sigma_1| \to \overline{B}_{\text{aff}}(E)$ lands in one extended apartment.

We use the above criterion to give an example of a toric vector bundle over $\mathbb{P}^1_K$ which does not split equivariantly. This is in contrast with the case of toric vector bundles over a field $K$: every toric vector bundle over $\mathbb{P}^1_K$ splits equivariantly (see [KM22, p. 5, A little application]).
We end this introduction with a question and a conjecture. An important key step in the proof of Theorem 2 is the local equivariant triviality of toric vector bundles over affine toric schemes (Proposition 3.3). We ask whether this result extends to toric principal bundles over toric schemes. More precisely, let $G$ be a reductive algebraic group over $\text{Spec}(O)$. A toric principal $G$-bundle over a toric scheme $\mathfrak{X}$ is a principal $G$-bundle on $\mathfrak{X}$ together with an action of $T$ lifting that of $\mathfrak{X}$ such that the $T$-action and the $G$-action commute.

**Question 5.** Is a toric principal bundle $\mathcal{P}$ over an affine toric scheme $\mathfrak{X}$ equivariantly trivial? That is, is there a $(G \times T)$-equivariant isomorphism between $\mathcal{P}$ and $\mathfrak{X} \times G \rightarrow \mathfrak{X}$, where $G$ acts on $\mathfrak{X} \times G$ by multiplication on the second factor and $T$ acts on $\mathfrak{X} \times G$ diagonally via a homomorphism $T \rightarrow G$?

Provided that the answer to the above question is positive, it is reasonable to make the following conjecture. This is an analogue of the main result in [KM22].

**Conjecture 6.** The toric principal bundles over a toric scheme $\mathfrak{X}_\Sigma$ are classified by piecewise affine maps from $\Sigma$ to the (extended) Bruhat-Tits building of $G$.

We end the introduction with a question/conjecture motivated by equivariant splitting of toric vector bundles.

**Question 7.** Does Helly’s theorem for the Bruhat-Tits building of $\text{GL}(2, K)$ (see Example 4.4 and [Kly90] Theorem 6.1.2) hold for other buildings?

The structure of the paper is as follows: Section 1 reviews preliminary material about Klyachko classification of toric vector bundles over toric varieties, as well as a review of toric schemes. Section 2 recalls Tits and Bruhat-Tits buildings associated to the general linear group. Section 3 contains the statements and proofs of our main results. Finally Section 4 gives a simple criterion for equivariant splitting of toric vector bundles.

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**Notation:**

- $K$, a discretely valued field with valuation $\text{val}: K \rightarrow \mathbb{Z}$
- $\mathcal{O}$, the valuation ring of $(K, \text{val})$
- $m$, the maximal ideal in $\mathcal{O}$
- $\varpi$, a uniformizer for $\mathcal{O}$ i.e. a generator of $m$
- $k = \mathcal{O}/m$, the residue field of $\mathcal{O}$
- $\text{Spec}(\mathcal{O})$, the affine scheme associated to $\mathcal{O}$
- $\eta$, the generic point of Spec, i.e. the point associated to the prime ideal $\{0\}$
- $o$, the special point of Spec, i.e. the point associated to the maximal ideal $m$
- For a scheme $\mathfrak{X}$ over $\text{Spec}$, we denote the fibers of $\mathfrak{X}$ over $\eta$ and $o$ by $\mathfrak{X}_\eta$ and $\mathfrak{X}_o$ respectively.
- $T \cong \mathbb{G}_m^n$, a split torus over $\mathcal{O}$. We have $T_\eta \cong (K^\times)^n$ and $T_o \cong (k^\times)^n$.
- $N$ and $M$, cocharacter and character lattices of $T$, with $N_R = N \otimes \mathbb{R}$ and $M_R = M \otimes \mathbb{R}$
- $\bar{N} = N \times \mathbb{Z}$ and $\bar{M} = M \times \mathbb{Z}$
- For a cone $\sigma \subset N_R \times \mathbb{R}_{\geq 0}$, we let $\Delta = \sigma \cap (N_R \times \{1\})$. Conversely, for a polyhedron $\Delta \subset N_R \times \{1\}$ we let $\sigma \subset N_R \times \mathbb{R}_{\geq 0}$ be the cone over $\Delta$.
- $\Sigma$, a fan in $N_R \times \mathbb{R}_{\geq 0}$
• $\Sigma_1$, a polyhedral complex in $N_\mathbb{R} \times \{1\}$ obtained by intersecting the cones in $\Sigma$ with $N_\mathbb{R} \times \{1\}$
• $\mathcal{X} = \mathcal{X}_\Sigma$, toric scheme associated to a fan $\Sigma$.
• $\mathcal{B}_{\text{aff}}(E)$, the collection of all additive norms on a $K$-vector space $E$ modulo homotheties. It can be considered as (the geometric realization of) the Bruhat-Tits building of $GL(E)$ (as well as $SL(E)$ or $PGL(E)$).
• $\mathcal{B}_{\text{aff}}(E)$, the collection of all additive norms on a $K$-vector space $E$. It is called the extended Bruhat-Tits building of $GL(E)$.

1. Preliminaries on toric varieties and toric schemes

1.1. Klyachko’s classification of toric vector bundles over a field. In this section we review Klyachko’s classification of toric vector bundles [Kly90]. We mainly follow the exposition in [Pay08 Section 2]. The first classification of toric vector bundles goes back to [Kan75]. We refer the reader to [Pay08 Section 2.4] for a nice brief history of the subject.

Let $T \cong G_m^\mathfrak{n}$ denote an $n$-dimensional algebraic torus over a field $\mathbb{k}$. We let $M$ and $N$ denote its character and cocharacter lattices respectively. We also denote by $M_\mathbb{R}$ and $N_\mathbb{R}$ the $\mathbb{R}$-vector spaces spanned by $M$ and $N$. For cone $\sigma \in N_\mathbb{R}$ let $M_\sigma$ be the quotient lattice:

$$M_\sigma = M/(\sigma^+ \cap M).$$

Let $\Sigma$ be a (finite rational polyhedral) fan in $N_\mathbb{R}$ and let $X_\Sigma$ be the corresponding toric variety. Also $X_\sigma$ denotes the invariant affine open subset in $X_\Sigma$ corresponding to a cone $\sigma \in \Sigma$. We denote the support of $\Sigma$, that is the union of all the cones in $\Sigma$, by $|\Sigma|$. For each $i$, $\Sigma(i)$ denotes the subset of $i$-dimensional cones in $\Sigma$. In particular, $\Sigma(1)$ is the set of rays in $\Sigma$. For each ray $\rho \in \Sigma(1)$ we let $v_\rho$ be the primitive vector along $\rho$, i.e. $v_\rho$ is the unique vector on $\rho$ whose integral length is equal to 1.

We say that $\mathcal{E}$ is a toric vector bundle on $X_\Sigma$ if $\mathcal{E}$ is a vector bundle on $X_\Sigma$ equipped with a $T$-linearization. This means that there is an action of $T$ on $\mathcal{E}$ which lifts the $T$-action on $X_\Sigma$ such that the action map $\mathcal{E}_x \to \mathcal{E}_{tx}$ for any $t \in T$, $x \in X_\Sigma$ is linear.

We fix a point $x_0 \in X_0 \subset X_\Sigma$ in the dense orbit $X_0$. We often identify $X_0$ with $T$ and think of $x_0$ as the identity element in $T$. We let $E = E_{x_0}$ denote the fiber of $\mathcal{E}$ over $x_0$. It is an $r$-dimensional vector space where $r = \text{rank}(\mathcal{E})$.

For each cone $\sigma \in \Sigma$ we have an invariant open subset $X_\sigma \subset X_\Sigma$. The space of sections $H^0(X_\sigma, \mathcal{E})$ is a $T$-module. We let $H^0(X_\sigma, \mathcal{E})_u \subseteq H^0(X_\sigma, \mathcal{E})$ be the weight space corresponding to a weight $u \in M$; these spaces define the weight decomposition:

$$H^0(X_\sigma, \mathcal{E}) = \bigoplus_{u \in M} H^0(X_\sigma, \mathcal{E})_u.$$ 

Every section in $H^0(X_\sigma, \mathcal{E})_u$ is determined by its value at $x_0$. Thus, by restricting sections to $E = \mathcal{E}_{x_0}$, we get an embedding $H^0(X_\sigma, \mathcal{E})_u \hookrightarrow E$. Let us denote the image of $H^0(X_\sigma, \mathcal{E})_u$ in $E$ by $E_u^\sigma$. Note that if $u' \in \sigma^+ \cap M$ then multiplication by the character $\chi_u^{u'}$ gives an injection $H^0(X_\sigma, \mathcal{E})_u \hookrightarrow H^0(X_\sigma, \mathcal{E})_{u-u'}$. Moreover, the multiplication map by $\chi_u^{u'}$ commutes with the evaluation at $x_0$ and hence induces an inclusion $E_u^\sigma \subseteq E_{u-u'}^\sigma$. If $u' \in \sigma^+$ then these maps are isomorphisms and thus $E_u^\sigma$ depends only on the class $[u] \in M_\sigma = M/(\sigma \cap M)$. For a ray $\rho \in \Sigma(1)$ we write

$$E_j^\rho = E_{u_j}^\rho,$$
for any \( u \in M \) with \( \langle u, v_\rho \rangle = j \) (all such \( u \) define the same class in \( M_\rho \)). Equivalently, one can define \( E_\rho^u \) as follows (see [Kly90 §0.1]). Pick a point \( x_\rho \) in the orbit \( O_\rho \) and let:
\[
E_\rho^u = \{ e \in E \mid \lim_{t \to x_\rho} \chi^u(t)^{-1}(t \cdot e) \text{ exists in } \mathcal{E} \},
\]
where \( t \) varies in \( T \) in such a way that \( t \cdot x_\rho \) approaches \( x_\rho \). One checks that \( E_\rho^u \) does not depend on the choice of \( x_\rho \) and only depends on \( j = \langle u, v_\rho \rangle \).

We thus have a decreasing filtration of \( E \):
\[
\cdots \supset E_{j-1}^\rho \supset E_j^\rho \supset E_{j+1}^\rho \supset \cdots
\]

An important step in the classification of toric vector bundles is that a toric vector bundle over an affine toric variety is equivariantly trivial. That is, it decomposes \( T \)-equivariantly as a sum of trivial line bundles. Let \( \sigma \) be a strictly convex rational polyhedral cone with corresponding affine toric variety \( X_\sigma \). Given \( u \in M \), let \( \mathcal{L}_u \) be the trivial line bundle \( X_\sigma \times \mathbb{A}^1 \) on \( X_\sigma \) where \( T \) acts on \( \mathbb{A}^1 \) via the character \( -u \). One observes that the toric line bundle \( \mathcal{L}_u \) in fact only depends on the class \( [u] \in M_\rho \). Hence we also denote this line bundle by \( \mathcal{L}_{[u]} \).

One has the following ([Kly90 Proposition 2.1.1]):

**Proposition 1.1.** Let \( \mathcal{E} \) be a toric vector bundle of rank \( r \) on an affine toric variety \( X_\sigma \). Then \( \mathcal{E} \) splits equivariantly into a sum of line bundles \( \mathcal{L}_u \):
\[
\mathcal{E} = \bigoplus_{i=1}^r \mathcal{L}_{[u_i]}
\]
where \( [u_i] \in M_\rho \).

We denote the multiset \( \{[u_1], \ldots, [u_r]\} \subset M_\rho \) by \( u(\sigma) \). The above shows that the filtrations \((E_\rho^u)_{i \in \mathbb{Z}} \mid \rho \in \Sigma(1)\) satisfy the following compatibility condition: There is a decomposition \( E = \bigoplus_{j=1}^r L_j \) of \( E \) into a direct sum of 1-dimensional subspaces \( L_j \) and a multiset \( u(\sigma) = \{[u_1], \ldots, [u_r]\} \subset M_\rho \) such that for any rated \( \rho \in \sigma(1) \) we have:
\[
E_\rho^u = \sum_{(u_j,v_\rho) \ge i} L_j.
\]

We call a collection of decreasing \( \mathbb{Z} \)-filtrations \( \{(E_\rho^u) \mid \rho \in \Sigma(1)\} \) a compatible collection of filtrations if for any \( \sigma \in \Sigma \) there is a direct sum decomposition \( E = \bigoplus_{j=1}^r L_j \) of \( E \) into 1-dimensional subspaces and a multiset \( \{[u_1], \ldots, [u_r]\} \subset M_\rho \) such that (2) holds. We also need the notion of a morphism between compatible filtrations. Let \( E, E' \) be finite dimensional vector spaces with compatible collections of filtrations \( \{(E_\rho^u) \mid \rho \in \Sigma(1)\} \) and \( \{(E'_\rho^u) \mid \rho \in \Sigma(1)\} \) respectively. A morphism between these compatible collections is a linear map \( F : E \to E' \) such that \( F(E_\rho^u) \subset E'_\rho^u \), for all \( i \in \mathbb{Z} \) and \( \rho \in \Sigma(1) \).

The following is Klyachko’s result on classification of toric vector bundles (see [Kly90 Theorem 2.2.1]).

**Theorem 1.2 (Klyachko).** The category of toric vector bundles \( \mathcal{E} \) on \( X_\Sigma \) is naturally equivalent to the category of compatible collections of filtrations on finite dimensional vector spaces \( E \).

### 1.2. Toric schemes over a discrete valuation ring.

Let \( \mathcal{O} \) be a discrete valuation ring with field of fractions \( K \). We let \( \varpi \) denote a uniformizer for \( \mathcal{O} \), namely an ideal generator for \( \mathfrak{m} \). We let \( k = \mathcal{O}/\mathfrak{m} \) be the residue field of \( \mathcal{O} \). Also \( o \) and \( \eta \) denote the special and generic points in \( \text{Spec}(\mathcal{O}) \) corresponding to the maximal ideal \( \mathfrak{m} \) and the prime ideal \( \{0\} \) respectively.
For a scheme $\mathcal{X}$ over $\text{Spec}(O)$ we let $\mathcal{X}_\eta = \mathcal{X} \times_{\text{Spec}(O)} \text{Spec}(K)$ and $\mathcal{X}_o = \mathcal{X} \times_{\text{Spec}(O)} \text{Spec}(k)$ be the generic fiber and special fiber of $\mathcal{X}$ respectively.

We let $N$ and $M$ be dual free abelian groups of rank $n$, $\tilde{N} = N \times \mathbb{Z}$ and $\tilde{M} = M \times \mathbb{Z}$. We also let $T$ denote the split torus over $\text{Spec}(O)$ with cocharacter lattice $N$. For $n \in M$ we denote the corresponding character of $T$ by $\chi^n$. The generic fiber $T_\eta$ is the split torus over $K$ isomorphic to $(K^\times)^n$. The coordinate ring $K[T_\eta]$ is isomorphic to the Laurent polynomial algebra with coefficients in $K$ in $n$ indeterminates.

A toric scheme $\mathcal{X}$ is an irreducible normal scheme of finite type over $\text{Spec}(O)$ with a $T$-action with a choice of an open embedding of $T_\eta$ in $\mathcal{X}$ such that the action of $T$ on $\mathcal{X}$ extends the translation action of $T_\eta$ on itself. We denote the image of the identity point in $T_\eta$ in $\mathcal{X}$ with $x_0$.

For a rational polyhedral cone $\sigma \subset N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ we let $\Delta$ be the polyhedron obtained by intersecting $\sigma$ and the plane $N_\mathbb{R} \times \{1\}$ (it may be empty). Conversely, for a rational polyhedron $\Delta \subset N_\mathbb{R} \times \{1\}$. We let $\sigma \subset N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ denote the cone over $\Delta$. Then $\sigma \leftrightarrow \Delta$ gives a one-to-one correspondence between rational polyhedra in $N_\mathbb{R} \times \{1\}$ and rational polyhedral cones in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ that intersect $N_\mathbb{R} \times \{1\}$.

**Definition 1.3.** Let $\Sigma$ be a fan in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$. We denote by $\Sigma_1$ the polyhedral complex in $N_\mathbb{R} \times \{1\}$ obtained by intersecting all the cones in $\Sigma$ by $N_\mathbb{R} \times \{1\}$. Conversely, for a polyhedral complex $\Sigma_1$ in $N_\mathbb{R} \times \{1\}$ we let $\Sigma$ be the fan consisting of the cones over the polyhedra in $\Sigma$ as well as the faces of these cones.

We recall that the recession cone $\text{recc}(\Delta)$ of a polyhedron $\Delta \subset N_\mathbb{R}$ is defined as:

$$\text{recc}(\Delta) = \{x \in N_\mathbb{R} \mid x + \Delta \subset \Delta\}.$$
(d) *In particular, the irreducible components of the special fiber $(X^o)$ are in bijection with the vertices of $\Sigma$. Let $\rho$ be a vertex in $\Sigma$ and let $(v_\rho, \ell_\rho)$ be the primitive vector on the ray generated by $\rho$. Then $\ell_\rho$ is the order of vanishing of the uniformizer $\varpi$ along the irreducible component of $X^o$ corresponding to $\rho$.*

**Example 1.6.** Let $n = 1$ and $K = \mathbb{C}((t))$ be the field of formal Laurent series in one indeterminate $t$. Consider the polyhedral complex $\Sigma_1$ in $\mathbb{R} \times \{1\}$ consisting of the polyhedra $(-\infty, 0], [0, 1], [1, \infty)$ (Figure 1). The corresponding fan $\Sigma$ is depicted in Figure 1. Intuitively, the toric scheme $X_{\Sigma}$ is the family over $S$ whose generic fiber is a projective line $\mathbb{P}^1$ and the special fiber is a union of two copies of $\mathbb{P}^1$ glued at a fixed point (Figure 2).

\[\Sigma_0\]

\[\Sigma\]

\[\mathbb{R}_{\geq 0}\]

\[\mathbb{N}_{\mathbb{R}}\]

**Figure 1.** A fan $\Sigma$ defining a toric scheme with $\mathbb{P}^1$ as general fiber

**Figure 2.** The polyhedral complex $\Sigma$ associated to the fan $\Sigma$

**Figure 3.** The general and special fibers of the toric scheme associated to the fan $\Sigma
Definition 1.7. A toric line bundle on $X$ is a line bundle $L$ over $X$ together with a linear action of $T$ on $L$.

We also define a framed toric line bundle on $X$ to be a toric line bundle together with a choice of a nonzero vector $e$ in the fiber $(L_{x_0})_{x_0}$. The choice of $e$ is equivalent to the choice of a $K$-linear isomorphism between $(L_{x_0})_{x_0}$ and $K$.

Remark 1.8. Our notion of a framed toric line bundle is the same as the notion of a toric line bundle with a toric section in [BGPS14, Definition 3.6.4]. We point out that we use the term toric line bundle to mean a torus equivariant line bundle.

Recall that a function $\Phi : N_R \to \mathbb{R}$ is an affine function if $\Phi$ is a linear function followed by a translation. In other words, there is a dual vector $u \in M_R$ and $b \in \mathbb{R}$ such that for all $x \in N_R$ we have:

$$\Phi(x) = \langle u, x \rangle + b.$$  

We say that $\Phi$ is an integral affine function if $\Phi(N) \subset \mathbb{Z}$. This means that $u \in M$ and $b \in \mathbb{Z}$.

Let $\Sigma$ be a polyhedral complex. We recall that $|\Sigma|$ denotes the support of $\Sigma$, the union of all the polyhedra in $\Sigma$.

Definition 1.9 (Piecewise affine function). We say that a function $\phi : |\Sigma| \to \mathbb{R}$ is piecewise affine, if for every polyhedron $\sigma \in \Sigma$, the function $\phi|_\sigma$ is the restriction of an affine function on $N_R$ to $\sigma$. We say that $\phi$ is an integral piecewise affine function, if for every $\sigma \in \Sigma$, the linear $\phi|_\sigma$ is the restriction of an integral affine function on $N_R$ to $\sigma$, that is, it maps $N$ to $Z$.

One has the following (it is essentially the same as [KKMSD73, Chap IV, Section 3, item (h)]).

Theorem 1.10. There is a bijection between the set of (framed) toric line bundles $L$ over $X_\Sigma$ and integral piecewise affine functions $\phi : \Sigma \to \mathbb{R}$.

Our Theorem 3.8 extends the above (Theorem 1.10) to toric vector bundles over $S$.

2. Preliminaries on buildings

A building is an (abstract) simplicial complex together with a collection of distinguished subcomplexes called apartments that satisfy certain axioms (see Definition 5.1). One can think of the notion of building as a “discretization” of the notion of symmetric space from Lie theory and differential geometry.

There are two important kinds of buildings: spherical buildings and affine buildings.

- In a spherical building, every apartment is a triangulation of a sphere. To an algebraic group $G$ over a field there corresponds a spherical building which is usually referred to as the Tits building of $G$. As a simplicial complex, it encodes the relative position of parabolic subgroups of $G$.
- In an affine building every apartment is a triangulation of an affine (Euclidean) space. To an algebraic group $G$ over a discretely valued field there corresponds an affine building which is usually referred to as the Bruhat-Tits building of $G$. As a simplicial complex, it encodes the relative position of parahoric subgroups of $G$.

The main results of the paper are concerned with the Bruhat-Tits building of a general linear group $\text{GL}(r, K)$ over a discretely valued field $K$. The Tits building of $\text{GL}(r, K)$ also makes an appearance in our main results as the boundary at infinity of the Bruhat-Tits building. We discuss the usual descriptions of these buildings in Sections 2.1 and 2.2.
Section 2.3 we explain how the Tits building can be regarded as the boundary at infinity of the Bruhat-Tits building.

The notion of a building was invented by Jacques Tits (see the introduction in [AB08] and references therein) in order to describe semisimple algebraic groups over an arbitrary field. Given a linear algebraic group \( G \), Tits showed how to associate a simplicial complex \( \Delta(G) \) together with an action of \( G \) (the Tits building of \( G \)). The action of \( G \) imposes very strong combinatorial regularity conditions on the complex \( \Delta \). Taking these conditions as axioms for a class of simplicial complexes, Tits gave his first definition of a building. A part of the data defining a building \( \Delta \) is a Coxeter group \( W \). As a simplicial complex, each apartment in a building is the Coxeter complex of \( W \). The building \( \Delta \) is glued together from multiple copies of the Coxeter complex. When \( W \) is a finite Coxeter group, the action of this group divides the Euclidean space into a finite number of cones and the Coxeter complex is the triangulation of sphere obtained by intersecting these chambers with the unit sphere. Hence, for a finite Coxeter group, (the geometric realization of) its Coxeter complex is a topological sphere, and the corresponding buildings are affine buildings. Iwahori–Matsumoto, Borel-Tits and Bruhat-Tits showed that in analogy with Tits’s construction of Tits buildings, affine buildings can also be constructed from algebraic groups \( G \) over a discretely valued field (local non-Archimedean field). This is the Bruhat-Tits buildings of \( G \).

As mentioned before, in this paper, we are only concerned with the Bruhat-Tits building of a general linear group \( \text{GL}(r, K) \) and we do not discuss the technical construction of the Bruhat-Tits building of an arbitrary linear algebraic group. For the general construction we refer the reader to [BT72] or or [RTW15, Section 3]. In the appendix (Section 5) we give a brief review of buildings of algebraic groups. For the general theory of buildings we refer the reader to the monographs [AB08] and [Gar97].

2.1. The Bruhat-Tits building of \( \text{GL}(r, K) \). In this section we review the descriptions of the Bruhat-Tits building of the general group in terms of lattices and additive norms. The reader can also consult [AB08, Section 6.9] and [RTW15, Section 1.2.1].

Let \( E \cong K^r \) be an \( r \)-dimensional vector space over \( K \). We first describe the Bruhat-Tits building of \( \text{GL}(E) \) as an abstract simplicial complex which we denote by \( \Delta_{\text{aff}}(E) \). This description is in terms of of lattices in \( E \). Then we describe the geometric realization of the simplicial complex \( \Delta_{\text{aff}}(E) \) as a topological space which we denote by \( \mathfrak{B}_{\text{aff}}(E) \). This geometric realization is in terms of additive norms on \( E \). Every simplex in \( \Delta_{\text{aff}}(E) \) corresponds to a subset of \( \mathfrak{B}_{\text{aff}}(E) \) homeomorphic to a standard simplex. By abuse of terminology we refer to both \( \Delta_{\text{aff}}(E) \) and \( \mathfrak{B}_{\text{aff}}(E) \) as the Bruhat-Tits building of \( \text{GL}(E) \).

2.1.1. Lattices. A lattice \( \Lambda \) in \( E \) is a rank \( r \) free \( \mathcal{O} \)-submodule in \( E \). Two lattices \( \Lambda_1, \Lambda_2 \) are said to be equivalent if for some nonzero \( c \in K^\times \) we have \( c\Lambda_1 = \Lambda_2 \). We denote the equivalence class of a lattice \( \Lambda \) by \([\Lambda]\).

The vertices in the Bruhat-Tits building \( \Delta_{\text{aff}}(E) \) correspond to the equivalence classes of lattices in \( E \). Two vertices \([\Lambda_1]\) and \([\Lambda_2]\) are connected if for some representatives \( \Lambda_1, \Lambda_2 \) we have \( \varpi \Lambda_1 \not\subsetneq \Lambda_2 \subsetneq \Lambda_1 \). A \((k-1)\)-simplex in \( \Delta_{\text{aff}}(E) \) corresponds to \( k \) vertices \([\Lambda_1], \ldots, [\Lambda_k]\) such that:

\[
\varpi \Lambda_k \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_k.
\]

We call a chain of lattices as in (3) a periodic lattice chain.
A full periodic lattice chain is a chain of lattices

\[ \mathcal{O} \Lambda_r \subseteq \Lambda_1 \subseteq \ldots \subseteq \Lambda_r. \]  

Then each subsequent quotient \( \Lambda_{i+1}/\Lambda_i \) is a locally free \( \mathcal{O} \)-module of rank 1. The full periodic lattice chains correspond to simplices of maximum dimension (also called chambers).

Let \( \{b_1, \ldots, b_r\} \) be a \( K \)-vector space basis for \( E \) and let \( \Lambda_i \subset E \) be the free \( \mathcal{O} \)-module (lattice) generated by \( \{b_1, \ldots, b_i, \varpi b_{i+1}, \ldots, \varpi b_r\} \). The chain of lattices

\[ \bigoplus_{i=1}^r \mathcal{O} b_i = \mathcal{O} \Lambda_r \subseteq \Lambda_1 \subseteq \ldots \subseteq \Lambda_r = \bigoplus_{i=1}^r \mathcal{O} b_i \]

is a full periodic lattice chain. Moreover, every full periodic lattice chain has the above form \( \mathcal{O} \) for some choice of a basis \( B \).

A frame \( L = \{L_1, \ldots, L_r\} \) is a direct sum decomposition \( E = \bigoplus_{i=1}^r L_i \) of \( E \) into 1-dimensional subspaces \( L_i \). One thinks of frames as equivalence classes of vector space bases up to multiplication of basis elements by nonzero scalars. To a frame \( L \) there corresponds a subcomplex of \( \Delta_{\text{aff}}(E) \) called the apartment associated to \( L \) as follows. Let \( B = \{b_1, \ldots, b_r\} \) be a basis of \( E \) with \( b_i \in L_i \). The vertices in the apartment are equivalence classes \([\Lambda]\) where \( \Lambda \) is of the form

\[ \Lambda = \bigoplus_{i=1}^r \mathcal{O} \varpi^{a_i} b_i, \]

for any choice of \( a_i \in \mathbb{Z} \). Note that a different choice of a basis \( B \) in the same frame \( L \) gives rise to the same collection of vertices but possibly with different coordinates \((a_1, \ldots, a_r)\). More precisely, let \( \{b_1', \ldots, b_r'\} \) be another basis with \( b'_i = \lambda_i b_i \), \( 0 \neq \lambda_i \in K \). Then \( \Lambda \) from \( \mathcal{O} \) can be written as

\[ \Lambda = \bigoplus_{i=1}^r \mathcal{O} \varpi^{a_i - \text{val}(\lambda_i)} b'_i. \]

Clearly, if \((a_1, \ldots, a_r)\) runs over all the points in \( \mathbb{Z}^r \) then \((a_1 - \text{val}(\lambda_1), \ldots, a_r - \text{val}(\lambda_r))\) also runs over all the points in \( \mathbb{Z}^r \). The set of vertices in the apartment is a copy of \( \mathbb{Z}^r \) but without a specified point as the origin. A choice of a basis \( B \), with the frame \( L \), distinguishes a point in the apartment as the origin.

**Remark 2.1** (Iwahori and parahoric subgroups). The \( \text{GL}(E) \)-stabilizer of a full periodic lattice chain is called an Iwahori subgroup. Let us fix a basis \( B \) and identify \( E \) with \( K^r \). Then the Iwahori subgroup that is the stabilizer of a full lattice chain as in \( \mathcal{O} \) is equal to the preimage of a Borel subgroup \( B \subset \text{GL}(r, k) \) under the evaluation map \( p : \text{GL}(r, \mathcal{O}) \to \text{GL}(r, k) \) sending \( \varpi \) to 0. The stabilizer of a periodic lattice chain is called a parahoric subgroup. It is the preimage, under \( p \), of a parabolic subgroup in \( \text{GL}(r, k) \).

Each frame \( L \) in \( E \) determines a split maximal torus \( H_L \subset \text{GL}(E) \). The simplices in the apartment of \( L \) correspond to the parahoric subgroups in \( \text{GL}(E) \) that contain the maximal torus \( H_L \).

2.1.2. *Additive norms*. Next we describe the geometric realization \( \mathfrak{B}_{\text{aff}}(E) \) of the simplicial complex \( \Delta_{\text{aff}}(E) \) in terms of *additive norms* on \( E \). This realization is due to Goldman and Iwahori and precedes the work of Bruhat and Tits on general theory of Bruhat-Tits buildings of algebraic groups (see [GI63]).
Definition 2.2 (Additive norm). We call a function $v : E \to \mathbb{R} = \mathbb{R} \cup \{\infty\}$ an additive norm if the following hold:

1. For all $e \in E$ and $\lambda \in K$ we have $v(\lambda e) = \text{val}(\lambda) + v(e)$.
2. For all $e_1, e_2 \in E$, the non-Archimedean inequality $v(e_1 + e_2) \geq \min\{v(e_1), v(e_2)\}$ holds.
3. $v(e) = \infty$ if and only if $e = 0$.

An additive norm is integral if it attains values in $\mathbb{Z} = \mathbb{Z} \cup \{\infty\}$.

To an integral additive norm $v$ one assigns the lattice $\Lambda_v$ given by:

$$\Lambda_v = \{e \in E \mid v(e) \geq 0\}.$$  

Conversely, a lattice $\Lambda$ gives rise to an additive norm $v_\Lambda : E \to \bar{\mathbb{Z}}$ by:

$$v_\Lambda(e) = \max\{k \in \mathbb{Z} \mid \omega^{-k} e \in \Lambda\}.$$  

One verifies that the maps $v \mapsto \Lambda_v$ and $\Lambda \mapsto v_\Lambda$ are mutually inverse and provide a one-to-one correspondence between integral additive norms and lattices.

We say that an additive norm $v : E \to \mathbb{R}$ is adapted to a frame $L = \{L_1, \ldots, L_r\}$ for $E$ if the following holds. For any $e = \sum_i e_i$, $e_i \in L_i$, we have:

$$v(e) = \min\{v(e_i) \mid i = 1, \ldots, r\}.$$  

In other words, if $B = \{b_1, \ldots, b_r\}$ is a basis with $b_i \in L_i$, then for any $e = \sum_i \lambda_i b_i$ we have:

$$v(e) = \min\{\text{val}(\lambda_i) + v(b_i) \mid i = 1, \ldots, r\}.$$  

That is, the additive norm $v$ is determined by its values on the basis $B$. In particular, the additive norm $v_\Lambda$ associated to a lattice $\Lambda = \bigoplus_{i=1}^{r} \mathcal{O} \omega^{a_i} b_i$, adapted to $B$, is given by

$$v_\Lambda(e) = \min\{\text{val}(\lambda_i) + v(b_i) \mid i = 1, \ldots, r\}.$$  

Note that if $v$ is an additive norm then for any $a \in \mathbb{R}$, $v_a$ defined by $v_a(e) = v(e) + a$ is also an additive norm. Two additive norms $v$, $v'$ are said to be equivalent if their difference is a constant.

Definition 2.3 (Geometric realization of Bruhat-Tits building). We define $\mathfrak{B}_{\text{aff}}(E)$, the (geometric realization of) the Bruhat-Tits building of $\text{GL}(E)$ to be the set of equivalence classes of additive norms on $E$. For a frame $L$, the (geometric realization of) the apartment associated to $L$, denoted by $\Lambda_{\text{aff}}(L)$, is the set of equivalence classes of all additive norms adapted to $L$.

Definition 2.4 (Extended Bruhat-Tits building). We let $\tilde{\mathfrak{B}}_{\text{aff}}(E)$ to be the set of all additive norms on $E$ (without taking the equivalence classes). It is sometimes called the extended Bruhat-Tits building of $\text{GL}(E)$ (see [RTW15, Remark 1.23]). For a frame $L$, we denote the set of all additive norms adapted to $L$ by $\Lambda_{\text{aff}}(L)$ and call it the extended apartment associated to $L$.

Remark 2.5. The lattices in $E = K^r$ correspond to closed points of the affine Grassmannian $\text{GL}(r, K)/\text{GL}(r, O)$.

2.1.3. Bruhat-Tits building of $\text{GL}(2, K)$. Let $B = \{b_1, b_2\}$ be a basis for $E = K^2$. Consider the lattice

$$\Lambda = \mathcal{O} b_1 \oplus \mathcal{O} b_2.$$

The equivalence class $[\Lambda]$ is comprised of the lattices $\mathcal{O} \omega^a b_1 \oplus \mathcal{O} \omega^a b_2$, $a \in \mathbb{Z}$. The vertices in the apartment $\Lambda_{\text{aff}}(B)$ consist of all the equivalence classes of lattices of the form:

$$\Lambda_{a_1, a_2} = \mathcal{O} \omega^{a_1} b_1 \oplus \mathcal{O} \omega^{a_2} b_2.$$
Thus the set of vertices in $A_{\text{aff}}(B)$ is $\{[\Lambda_a, 0] \mid a \in \mathbb{Z}\}$ and $A_{\text{aff}}(B)$ is a 1-dimensional affine space.

A vertex $[\Lambda']$ is connected to the vertex $[\Lambda]$ if for some representative $\Lambda'$ we have:

$\varpi \Lambda \subsetneq \Lambda' \subsetneq \Lambda$.

Let $\overline{\Lambda'}$ be the image of $\Lambda'$ in the $k$-vector space $\Lambda/\varpi \Lambda \cong k^2$. One sees that $\Lambda' \mapsto \overline{\Lambda'}$ gives a one-to-one correspondence between the lattices $\Lambda'$, $\varpi \Lambda \subsetneq \Lambda' \subsetneq \Lambda$, and the $k$-vector subspaces $L$, $\{0\} \subsetneq L \subsetneq \Lambda/\varpi \Lambda$. It follows that the set of vertices $[\Lambda']$ connected to $[\Lambda]$ is in one-to-one correspondence with the points on the projective line $\mathbb{P}(\Lambda/\varpi \Lambda) \cong \mathbb{P}^1$. In particular, if the residue field $k$ is a finite field with $q$ elements, then every vertex is connected to $q + 1$ other vertices. It can also be shown that the graph obtained by connecting the vertices as above is a tree, that is, there are no cycles. Moreover, if the base field $K$ is complete then any one-sided infinite path lies in an apartment (see [Ser03, Chap. II, §1.1] and [Cas, Section I.2]). An example is $K = \mathbb{Q}_p$, the field of $p$-adic numbers (Figure 6).
Let us determine for what other bases \( B' = \{ b_1', b_2' \} \) the vertex \([\Lambda]\) lies in the corresponding apartment \( A_{B'}\). If \([\Lambda]\) lies in \( A_{B'}\) then we can find \( \gamma_1, \gamma_2 \in \mathbb{Z}\) such that:

\[
\mathcal{O}b_1 \oplus \mathcal{O}b_2 = \Lambda = \mathcal{O}(\varpi^{\gamma_1} b_1' + \mathcal{O}(\varpi^{\gamma_2} b_2').
\]

Let \( C = [c_{ij}], 1 \leq i, j \leq 2, \) be the change of coordinates matrix from \( B \) to \( B'\). That is, \( b_i' = \sum_{j=1}^{2} c_{ij} b_j \). The equation (8) is then equivalent to

\[
CD \in \text{GL}(2, \mathcal{O}),
\]

where \( D = \text{diag}(\varpi^{\gamma_1}, \varpi^{\gamma_2}). \) For instance, we can take \( \gamma_1 = \gamma_2 = 0 \) and \( C = \begin{bmatrix} 1 & \varpi^\delta \\ 0 & 1 \end{bmatrix} \) with \( \delta \geq 0 \). Then \( CD = C \in \text{GL}(2, \mathcal{O}) \) and thus

\[
\mathcal{O}b_1 \oplus \mathcal{O}b_2 = \Lambda = \mathcal{O}(\varpi^\delta b_1 + b_2).
\]

Hence \( \Lambda \) belongs to the apartment \( A_{B'}\) where \( B' = \{ b_1, \varpi^\delta b_1 + b_2 \} \).

To demonstrate how apartments can overlap we consider the apartments associated to the bases \( B = \{ b_1, b_2 \} \) and \( B' = \{ b_1 + b_2, b_1 \} \). The vertices in the apartments \( A_B\) and \( A_{B'}\) correspond respectively to the chain of lattices:

\[
\cdots \subset \mathcal{O}b_1 \oplus \mathcal{O}\varpi b_2 \subset \mathcal{O}b_1 \oplus \mathcal{O}b_2 \subset \mathcal{O}b_1 \oplus \mathcal{O}\varpi^{-1} b_2 \subset \cdots
\]

and

\[
\cdots \subset \mathcal{O}(b_1 + b_2) \oplus \mathcal{O}\varpi b_2 \subset \mathcal{O}(b_1 + b_2) \oplus \mathcal{O}b_2 \subset \mathcal{O}(b_1 + b_2) \oplus \mathcal{O}\varpi^{-1} b_2 \subset \cdots.
\]

One verifies that for any \( k \geq 0, \mathcal{O}(b_1 + b_2) \oplus \mathcal{O}\varpi^k b_2 = \mathcal{O}(b_1 + b_2) \oplus \mathcal{O}\varpi^{-k} b_2\). Thus the two chains of lattices overlap on the half-infinite path with vertices \([\mathcal{O}b_1 \oplus \mathcal{O}\varpi^{-k} b_2] = [\mathcal{O}(b_1 + b_2) \oplus \mathcal{O}\varpi^{-k} b_2], k \geq 0\), but become separate afterwards (Figure 7).

2.1.4. Floor, ceiling and pull-back of an additive norm. In this section we discuss some basic operations on additive norms, namely the floor and ceiling functions and pull-back, and prove some facts about them for future use in Section 3.3.

One has a natural partial order \( \preceq \) on the set \( \mathcal{F}(E) \) of all additive norms. Let \( v, v' \) be additive norms on \( E \). We say that \( v \preceq v' \) if \( v(e) \leq v'(e) \), for all \( e \in E \).

Recall that for \( r \in \mathbb{R}\), the floor \( \lfloor r \rfloor \) (respectively the ceiling \( \lceil r \rceil \)) is the largest integer less than or equal to \( r \) (respectively the smallest integer greater than or equal to \( r \)).

**Definition 2.6** (Floor and ceiling of an additive norm). For an additive norm \( v : E \to \mathbb{R} \) we define its floor \( \lfloor v \rfloor \) and ceiling \( \lceil v \rceil : E \to \mathbb{Z} \) to be functions from \( E \) to \( \mathbb{Z} \) given by:

\[
\lfloor v \rfloor (e) = \lfloor v(e) \rfloor,
\]

\[
\lceil v \rceil (e) = \lceil v(e) \rceil, \quad \forall e \in E.
\]

**Proposition 2.7.** \( \lfloor v \rfloor \) and \( \lceil v \rceil \) are additive norms.

**Proof.** We write the proof for the ceiling, the proof for the floor is similar. We know that there exists a basis \( B = \{ b_1, \ldots, b_r \} \) such that \( v \) is adapted to \( B \). Thus:

\[
v \left( \sum_{i=1}^{r} \lambda_i b_i \right) = \min \{ \text{val}(\lambda_i) + v(b_i) \mid i = 1, \ldots, r \}.
\]
Now
\[ [v] \left( \sum_{i} \lambda_i b_i \right) = [v(\sum_{i} \lambda_i b_i)] \]
\[ = \left[ \min \{ \text{val}(\lambda_i) + v(b_i) \mid i = 1, \ldots, r \} \right] \]
\[ = \min \{ \text{val}(\lambda_i) + [v(b_i)] \mid i = 1, \ldots, r \}. \]

But any expression \( \min \{ \text{val}(\lambda_i) + a_i \mid i = 1, \ldots, r \} \), \( a_i \in \mathbb{R} \), defines an additive norm. This proves that \([v]\) is indeed an additive norm. \( \square \)

The floor (respectively ceiling) of an additive norm \( v \) is the greatest integral additive norm that is smaller than or equal to \( v \) (respectively smallest integral additive norm that is greater than or equal to \( v \)).

**Definition 2.8** (Pull-back of an additive norm). Let \( F : E \rightarrow E' \) be a linear map between \( K \)-vector spaces \( E \) and \( E' \). Given an additive norm \( v' : E \rightarrow \mathbb{R} \) one can consider its pull-back \( F^*(v') \) given by
\[ F^*(v')(e) = v'(F(e)), \quad \forall e \in E. \]

We note that \( F^*(v') \) may not be an additive norm on \( E \) for the following reason: \( F \) may have nonzero kernel in which case \( F^*(v') \) assigns value \( \infty \) to nonzero elements in the kernel. We will need the following in Section 3.3.
**Proposition 2.9.** Let $F : E \to E'$ be a linear map between $K$-vector spaces $E$ and $E'$. Let $v$ and $v'$ be integral additive norms on $E$ and $E'$ respectively. Then, for all $e \in E$,

$$v(e) \leq F^*(v')(e)$$

if and only if

$$F(\Lambda_v) \subset \Lambda_{v'}.$$

Recall that $\Lambda_v, \Lambda_{v'}$ denote the lattices corresponding to the integral additive norms $v, v'$ respectively (see [7]).

**Proof.** Suppose $v(e) \leq F^*(v')(e)$, for all $e \in E$, but $F(\Lambda_v) \not\subset \Lambda_{v'}$. Then there exists $e \in \Lambda_v$ with $F(e) \not\in \Lambda_{v'}$. Since $e \in \Lambda_v$ we have $v(e) \geq 0$ and since $F(e) \not\in \Lambda_{v'}$ we have $v'(F(e)) < 0$, the contradiction proves the claim. Conversely, suppose $F(\Lambda_v) \subset \Lambda_{v'}$ then for any $e \in E$ we have:

$$v(e) = \max\{ k \mid \varpi^{-k}e \in \Lambda_v \},$$

$$\leq \max\{ k \mid \varpi^{-k}F(e) \in F(\Lambda_v) \},$$

$$\leq \max\{ k \mid \varpi^{-k}F(e) \in \Lambda_{v'} \},$$

$$= v'(F(e)).$$

□

### 2.2. The Tits building of $GL(r, K)$

We now turn to the discussion of Tits building of $GL(E)$, which is a spherical building associated to the group $GL(E)$. As in the case of Burhat-Tits building, we describe the Tits building of $GL(E)$ in two ways:

(i) as a simplicial complex $\Delta_{sph}(E)$ and,

(ii) its geometric realization $\mathfrak{B}_{sph}(E)$ as a topological space.

The abstract simplicial complex, $\Delta_{sph}(E)$, can be described as follows. Its vertices are the proper nonzero vector subspaces of $E$. Two subspaces $F_1$ and $F_2$ are connected if one of them is a subset of the other. The $(k-1)$-simplices of $\Delta_{sph}(E)$ are formed by sets of $k$ mutually connected vertices, hence $(k-1)$-simplices correspond to partial flags

$$F_k = (\{0\} \subset F_1 \subset \cdots \subset F_k \subsetneq E).$$

Maximal simplices, i.e. chambers, correspond to complete flags. The apartments in $\Delta_{sph}(E)$, correspond to frames in $E$. Recall that a frame in $E$ is a direct sum decomposition $E = \bigoplus_{i=1}^r L_i$ into one-dimensional subspaces $L_i$. A flag $F_*$ is said to be adapted to a frame $L = \{L_1, \ldots, L_r\}$ if all the subspaces in the flag are spanned by a subset of the $L_i$. The apartment corresponding to a frame $L$ is the collection of all flags adapted to $L$.

A convenient way to describe the geometric realization of the spherical building of $GL(E)$ is as the space of prevaluations on $E$. (see [KM22 Section 1.4]). This is an analogue of the Goldman-Iwahori realization of the Bruhat-Tits building of $GL(E)$ as the space of equivalence classes of additive norms on $E$ (see Section 2.1). This description of Tits building of the general linear group appears in [KM22] but the authors are not aware of another reference for this.

**Definition 2.10 (Prevaluation).** We call a function $v : E \to \overline{\mathbb{R}}$ a prevaluation if the following hold:

1. For all $e \in E$ and $0 \neq \lambda \in K$ we have $v(\lambda e) = v(e)$.
(2) For all \( e_1, e_2 \in E \), the non-Archimedean inequality \( v(e_1 + e_2) \geq \min\{v(e_1), v(e_2)\} \) holds.

(3) \( v(e) = \infty \) if and only if \( e = 0 \).

We note that the only difference with the definition of an additive norm (Definition 2.2) is item (1). We call a prevaluation \( v \) integral if it attains values in \( \mathbb{Z} \). In short, a prevaluations is a valuation on \( E \) extending the trivial valuation on \( K \), and an additive norm is a valuation on \( E \) extending the valuation \( \text{val} \) on \( K \).

**Remark 2.11.** The term prevaluation is taken from the paper [KK12, Section 2.1]. Prevaluation is a standard commutative algebra notion although in most of the literature the term valuation on a vector space is used. We use the term prevaluation to distinguish prevaluations from valuations on rings.

For a prevaluation \( v \), consider the decreasing filtration \( \{E_{v \geq a}\}_{a \in \mathbb{R}} \), where \( E_{v \geq a} = \{e \in E \mid v(e) \geq a\} \). Let

\[
\{0\} \subset F_1 \subset \cdots \subset F_k \subset E,
\]

be all the distinct subspaces among the \( E_{v \geq a} \). They determine the flag associated to the simplex that contains \( v \).

An integral prevaluation \( v \) on \( E \) gives a \( \mathbb{Z} \)-filtration \( \{E_{v \geq a}\}_{a \in \mathbb{Z}} \). One verifies that this gives a one-to-one correspondence between integral prevaluations and \( \mathbb{Z} \)-filtrations \( \{E_a\}_{a \in \mathbb{Z}} \) such that \( \bigcap_{a \in \mathbb{Z}} E_a = \{0\} \) and \( \bigcup_{a \in \mathbb{Z}} E_a = E \).

We say that a prevaluation \( v \) is adapted to a frame \( L = \{L_1, \ldots, L_r\} \) if the values of \( v \) can be obtained by taking a minimum of the values \( v(L_i) \). More precisely, for any \( e \in E \) let us write \( e = \sum_i e_i, e_i \in L_i \). Then:

\[
v(e) = \min\{v(e_i) \mid i = 1, \ldots, r\}.
\]

This is equivalent to requiring that, for any \( a \in \mathbb{R} \), the subspace

\[
F_a = \{e \in E \mid v(e) \geq a\}
\]

is spanned by some of the \( L_i \).

Two prevaluations \( v_1, v_2 \) are said to be equivalent if there is \( \lambda > 0 \) such that \( \lambda v_1 = v_2 \). The (geometric realization of the) Tits building \( \mathcal{B}_{\text{sph}}(E) \) can be identified with the set of all equivalence classes of prevaluations on \( E \). The collection of all equivalence classes of prevaluations that are adapted to a given frame \( L \) is the (geometric realization of the) apartment associated to \( L \). Each such prevaluation \( v \) is uniquely determined by the values \( \{v(L_1), \ldots, v(L_r)\} \). Hence this space can be identified with the \((r-1)\)-dimensional sphere \( \mathbb{R}^r/\mathbb{R}_{>0} \cong S^{r-1} \) (since we are taking equivalence classes of prevaluations).

We denote the set of all prevaluations on \( E \) by \( \mathfrak{B}_{\text{sph}}(E) \) and call it the cone over the Tits building of \( \text{GL}(E) \). For a frame \( L \), we denote the set of all prevaluations on \( E \) that are adapted to \( L \) by \( \mathfrak{A}_{\text{sph}}(L) \) and call it the cone over the apartment associated to \( L \).

### 2.3. Building at infinity

As usual let \( E \cong K^r \) be a finite dimensional vector space over a discretely valued field \( K \). We now explain that \( \mathfrak{B}_{\text{sph}}(E) \), the space of prevaluations on \( E \), can be regarded as the 'boundary at infinity' of \( \mathfrak{B}_{\text{aff}}(E) \), the space of additive norms on \( E \). This is related to a general construction for affine buildings. Given an affine building \( \mathfrak{B}_{\text{aff}}(E) \) one can consider the parallelism equivalence classes of rays in the buildings (that is, each equivalence class consists of all the rays parallel to each other). This set of equivalence classes naturally has a structure of a spherical building (see Section 5).
Take a basis $B = \{b_1, \ldots, b_r\}$ for $E$. Recall that the extended apartment $\tilde{A}_{\text{aff}}(B)$ can be identified with the set of all additive norms $v : E \to \mathbb{R}$ that are adapted to the basis $B$, namely, for any $e = \sum_i \lambda_i b_i$ we have:

\[
v(e) = \min\{\operatorname{val}(\lambda_i) + v(b_i) \mid i = 1, \ldots, r\}.
\]

In $\tilde{A}_{\text{aff}}(B)$, a ray is given by $\rho : t \mapsto v_t$ where $v_t$ is the additive norm adapted to $B$ given by

\[
v_t(e) = \min\{\operatorname{val}(\lambda_i) + a_i + tv_i \mid i = 1, \ldots, r\},
\]

where $a = (a_1, \ldots, a_r)$, $v = (v_1, \ldots, v_r) \in \mathbb{R}^r$. Then an equivalence class of parallel rays in the apartment $\tilde{A}_{\text{aff}}(B)$ is obtained by fixing $v$ and varying $a$. Let us consider $v_\infty : E \to \mathbb{R}$ defined by:

\[
v_\infty(e) = \lim_{t \to \infty} \frac{v_t(e)}{t}.
\]

We have:

\[
v_\infty(e) = \lim_{t \to \infty} \min\{\operatorname{val}(\lambda_i)/t + a_i/t + v_i \mid i = 1, \ldots, r\}
= \min\{v_i \mid \lambda_i \neq 0\}.
\]

Thus $v_\infty : E \to \mathbb{R}$ is in fact a prevaluation, adapted to the basis $B$, which represents the parallelism equivalence class of a ray $\rho$ in $\mathcal{B}_{\text{aff}}(E)$. We recall that the cone over the spherical
building $\mathcal{B}_{\text{sph}}(E)$ can be viewed as the space of all prevaluations on $E$. This explains that $\mathcal{B}_{\text{sph}}(E)$ can be viewed as the boundary at infinity of $\mathcal{B}_{\text{aff}}(E)$.

Recall that, for a basis $B$, we denote by $\mathcal{A}_{\text{sph}}(B)$ the set of all prevaluations adapted to the basis $B$, that is, the cone over the apartment associated to $B$. The above shows that $\mathcal{A}_{\text{sph}}(B)$ can be thought of as the boundary at infinity of the extended affine apartment $\mathcal{A}_{\text{aff}}(B)$.

3. Toric vector bundles over a discrete valuation ring

As before let $\mathfrak{X} = \mathfrak{X}_\Sigma$ be a toric scheme over $\text{Spec}(\mathcal{O})$. Here $\Sigma$ is a fan in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ with the corresponding polyhedral complex $\Sigma_1$ in $N_\mathbb{R} \times \{1\}$. We recall that $\mathfrak{X}$ comes with a distinguished point $x_0$ in the open chart $\mathfrak{X}_0$ giving an identification of this open set with the torus $T_\eta = T_K$.

**Definition 3.1.** A toric vector bundle over $\mathfrak{X}$ is a $T$-linearized vector bundle $\mathcal{E}$ over $\mathfrak{X}$. A toric vector bundle is the same as a $T$-linearized locally free sheaf of constant rank $\mathcal{O}_{\mathfrak{X}}$-modules on $\mathfrak{X}$. By a morphism of toric vector bundles over $\mathfrak{X}_\Sigma$, we mean a torus equivariant morphism of vector bundles. Toric vector bundles over $\mathfrak{X}_\Sigma$ and their morphisms form a category.

We refer the reader to [Kly90, Definition 1.6] for the definition of a $T$-linearization of a vector bundle on a $T$-scheme (the definition in the above mentioned reference is for line bundles but the same works for vector bundles as well). In short, a $T$-linearization of $\mathcal{E}$ is a lifting of action of $T$ on $\mathfrak{X}$ to an action of $T$ on $\mathcal{E}$ by morphisms of vector bundles.

**Assumption 3.2.** We assume that the fan $\Sigma$ is determined by the polyhedral complex $\Sigma_1$. That is, each cone in $\Sigma$ is a cone over a polyhedron in $\Sigma_1$ or a face of such cone. Geometrically, the assumption says that every $T_K$-orbit in the generic fiber $\mathfrak{X}_\eta$ extends to the special fiber $\mathfrak{X}_\sigma$, that is, its closure intersects $\mathfrak{X}_\sigma$. In particular, if $\Sigma$ is a complete polyhedral complex this condition is satisfied.

Under the above assumption (Assumption 3.2) in this section we give a classification of toric vector bundles over $\mathfrak{X}$ in terms of piecewise affine maps from $\Sigma_1$ to $\mathcal{B}_{\text{aff}}(E)$, the extended Bruhat-Tits buildings of $\text{GL}(E)$, for finite dimensional $K$-vector spaces $E$.

3.1. Local equivariant triviality. Similar to the case of toric vector bundles over a field, the following is a key step in the classification. The proof is similar to the one for toric vector bundles over a field (see [Kly90, Proposition 2.1.1]).

**Proposition 3.3.** Let $\mathcal{E}$ be a toric vector bundle on an affine toric scheme $\mathfrak{X}_\sigma$. Then $\mathcal{E}$ is equivariantly trivial. That is, there exist characters $u_1, \ldots, u_r \in M$ such that $H^0(\mathfrak{X}_\sigma, \mathcal{E}) = \bigoplus_{i=1}^r S_{\sigma, i}$ where $T$ acts on $S_{\sigma, i} \cong S_{\sigma}$ via character $u_i$.

**Proof.** It suffices to show there are $T$-weight sections $s_1, \ldots, s_r \in H^0(\mathfrak{X}_\sigma, \mathcal{E})$ that freely generate $H^0(\mathfrak{X}_\sigma, \mathcal{E})$ as an $S_{\sigma}$-module. If the cone $\sigma$ lies in $N_\mathbb{R} \times \{0\}$, there is no special fiber $\mathfrak{X}_{\sigma, o}$ and we are in the case of a toric vector bundle over an affine toric variety $\mathfrak{X}_{\sigma, o}$. Thus, we can assume that $\sigma$ does not lie in $N_\mathbb{R} \times \{0\}$. In this case, $\sigma$ corresponds to the unique closed $T_\sigma$-orbit $O_\sigma$ in the special fiber $\mathfrak{X}_\sigma$. Recall that $T_\sigma = T_k$ is the torus over the residue field $k = \mathcal{O}/m$. Since $O_\sigma$ is an orbit, the vector bundle $\mathcal{E}_{|O_\sigma}$ is equivariantly trivial. That is, we can find a basis of $T$-weight sections $s'_1, \ldots, s'_r \in H^0(O_\sigma, \mathcal{E}_{|O_\sigma})$. By general properties of coherent sheaves on affine schemes, one knows that the $S_{\sigma}$-module morphism:

$$j^* : H^0(\mathfrak{X}_\sigma, \mathcal{E}) \to H^0(O_\sigma, \mathcal{E}_{|O_\sigma}),$$
induced by the inclusion morphism \( j : O_\sigma \to X_\sigma \) is surjective. Note that the \( S_\sigma \)-module structure on \( H^0(O_\sigma, E_{O_\sigma}) \) comes from the homomorphism \( S_\sigma \to k[O_\sigma] \) determined by the embedding \( O_\sigma \to X_\sigma \). Moreover, \( j^* \) is \( T \)-equivariant. In other words, \( j^* \) is \( M \)-grading preserving. Thus we can find \( T \)-weight sections \( s_1, \ldots, s_r \in H^0(X, E) \) such that \( j^*(s_i) = s'_i \), for all \( i \). We claim that these \( s_i \) freely generate \( H^0(X, E) \) as an \( S_\sigma \)-module. First we claim that for any \( x \in X \), the images of the \( s_i \) in the fiber \( E_x \) are linearly independent. This is because the subscheme of \( X_\sigma \) where these images are linearly dependent is a closed \( T \)-invariant subscheme and hence, if non-empty, should contain \( O_\sigma \). But since the \( s'_i \) are a basis for \( H^0(O_\sigma, E_{O_\sigma}) \), this subscheme does not contain \( O_\sigma \) as a subscheme, and thus is empty which proves the claim. Now from the assumption that \( E \) is locally free we conclude that for all \( x \in X \) the images of \( s_i \) in \( E_x \) form a basis. It follows that the \( s_i \) are an \( S_\sigma \)-basis for \( H^0(X_\sigma, E) \) as required.

\[ \Phi(x) = \Phi_0(x) + b. \]

The linear map \( \Phi_0 \) is called the linear part of \( \Phi \). We say that \( \Phi \) is an integral affine function if \( \Phi(N) \subset \mathbb{Z}^r \). This means that the entries of the matrix of \( \Phi_0 \), with respect to a \( \mathbb{Z} \)-basis for \( N \) and the standard basis for \( \mathbb{Z}^r \), are integers, and moreover, \( b \in \mathbb{Z}^r \).

As usual, let \( \Sigma \) be a fan in \( N_\mathbb{R} \times R_{\geq 0} \) and \( \Sigma_1 \) the polyhedral complex obtained by intersecting the cones in \( \Sigma \) with the affine hyperplane \( N_\mathbb{R} \times \{1\} \). Similarly, \( \Sigma_0 \) is the fan obtained by taking the cones in \( \Sigma \) that lie in the hyperplane \( N_\mathbb{R} \times \{0\} \).

**Definition 3.4** (Piecewise affine map to \( \mathcal{B}(E) \)). Let \( \Sigma_1 \) be a polyhedral complex in the affine space \( N_\mathbb{R} \times \{1\} \). A map \( \Phi : |\Sigma_1| \to \mathcal{B}(E) \) is a piecewise affine map if the following holds:

(a) For any polyhedron \( \Delta \in \Sigma_1 \), there is an extended apartment \( A_{a}(\Delta) \) (not necessarily unique) such that the restriction \( \Phi_\Delta := \Phi|_\Delta \) maps \( \Delta \) into \( A_{a}(\Delta) \).

(b) For any \( \Delta \in \Sigma_1 \), \( \Phi_\Delta \) is the restriction of an affine map from \( N_\mathbb{R} \times \{1\} \) to the affine space \( A_{a}(\Delta) \).

We say that \( \Phi \) is an integral piecewise affine map if for each \( \Delta \in \Sigma_1 \), \( \Phi_\Delta \) is the restriction of an integral affine map from \( N_\mathbb{R} \times \{1\} \) to \( A_{a}(\Delta) \).

Let \( \Phi : |\Sigma_1| \to \mathcal{B}(E) \) be an integral piecewise affine map. For any polyhedron \( \Delta \in \Sigma_1 \), the requirement that \( \Phi_\Delta := \Phi|_\Delta : \Delta \to A_{a}(\Delta) \) is an integral affine map means that there exists a \( K \)-basis \( B_\Delta = \{b_\Delta,1, \ldots, b_\Delta,r\} \) for \( E \) and \( \{u_\Delta,1, \ldots, u_\Delta,r\} \subset M \) such that for any \( x \in \sigma \), \( \Phi_\Delta(x) \), as an additive norm on \( E \), is given by

\[ \Phi_\Delta(x) \left( \sum_i \lambda_i b_\Delta,i \right) = \min\{\text{val}(\lambda_i) + \langle u_\Delta,i, x \rangle \mid i = 1, \ldots, r\}. \]
Definition 3.5 (Linear part of a piecewise affine map). Let $\Phi : |\Sigma_1| \to \mathcal{B}_{\text{aff}}(E)$ be a piecewise affine map. There is a well-defined piecewise linear map $\Phi_0 : |\Sigma_0| \to \mathcal{B}_{\text{aff}}(E)$ such that for each polyhedron $\Delta \in \Sigma_1$, $\Phi_0$ is the linear part of the affine map $\Phi|_{\Delta} : \Delta \to \mathcal{A}_{\text{aff}}(\Delta)_0$. The linear part of $\Phi|_{\Delta}$ is a linear map from the recession cone $\text{recc}(\Delta)$ to $\mathcal{A}_{\text{aff}}(\Delta)_0$, the space of prevaluations adapted to $B$. As explained in Section 2.3, $\mathcal{A}_{\text{aff}}(\Delta)_0$ can be thought of as the boundary at infinity of the extended affine apartment $\mathcal{A}_{\text{aff}}(\Delta)$.

Remark 3.6. The above definitions extend to arbitrary affine buildings.

Definition 3.7 (Morphism of piecewise affine maps). Let $F : E \to E'$ be a linear transformation between finite dimensional $K$-vector spaces $E$, $E'$ and let $\Phi : |\Sigma_1| \to \mathcal{B}(E)$ and $\Phi' : |\Sigma_1| \to \mathcal{B}(E')$ be two piecewise affine maps. We say that $F$ gives a morphism of piecewise affine maps $\Phi \to \Phi'$ if for any $x \in |\Sigma_1|$ we have:

$$ \Phi(x) \leq F'(\Phi(x)),$$

(see Definition 2.8 and cf. Proposition 2.9).

3.3. Main theorems. Recall that $\Sigma$ denotes a fan in $N\mathbb{R} \times \mathbb{R}_{\geq 0}$ with $\Sigma_1$ the corresponding polyhedral complex in $N\mathbb{R} = N\mathbb{R} \times \{1\}$ and $X_\Sigma$ the corresponding toric scheme over $\text{Spec}(O)$.

We break our classification of toric vector bundles on $X_\Sigma$ into the following theorems (Theorems 3.8, 3.10, 3.11). We present the proofs later in this section. The proofs involve the notion of ceiling of an additive norm (Definition 2.9).

Recall that we fix a point $x_0$ in the open orbit in the generic fiber $X_\eta$.

Theorem 3.8. There is a one-to-one correspondence between the isomorphism classes of toric vector bundles over a toric scheme $\mathfrak{X} = \mathfrak{X}_\Sigma$ and the integral piecewise affine maps from $|\Sigma_1|$ to $\mathcal{B}_{\text{aff}}(E)$, the extended Bruhat-Tits buildings of general linear groups of finite dimensional $K$-vector spaces $E$.

Remark 3.9. Let $\Sigma$ be a fan in $N\mathbb{R}$ and $X = X_\Sigma$ the corresponding toric variety over a field $K$. The result [KM22, Theorem 1] applied to $GL(r, K)$ shows that toric vector bundles over a toric variety $X_\Sigma$ are classified by piecewise linear maps $\Phi : |\Sigma| \to \mathcal{B}_{\text{aff}}(E)$. This is a reformulation of the Klyachko classification of toric vector bundles (see also [KM22, Example 2.8]).

If $E$ is a toric vector bundle over a toric scheme $\mathfrak{X}$ then the generic fiber $\mathfrak{X}_\eta$ is a usual toric variety over the field $K$ and $\mathcal{E}|_{\mathfrak{X}_\eta}$ is a usual toric vector bundle on it. The next theorem describes the piecewise linear map (equivalently, the Klyachko filtrations) associated to the toric vector bundle $\mathcal{E}|_{\mathfrak{X}_\eta}$.

Theorem 3.10. Given a toric vector bundle $E$ with associated integral piecewise affine map $\Phi : |\Sigma_1| \to \mathcal{B}_{\text{aff}}(E)$, the toric vector bundle $\mathcal{E}|_{\mathfrak{X}_\eta}$ over the generic fiber $\mathfrak{X}_\eta$, which is a toric vector bundle over the field $K$, corresponds to the piecewise linear map $\Phi_0 : |\Sigma_0| \to \mathcal{B}_{\text{aff}}(E)$, which is the linear part of $\Phi$ (see [KM22, Example 2.8]).

Theorem 3.11. The correspondence $E \mapsto \Phi_E$ gives an equivalence of categories between the category of toric vector bundles over the toric scheme $\mathfrak{X} = \mathfrak{X}_\Sigma$ and the category of integral piecewise affine maps from $|\Sigma_1|$ to $\mathcal{B}_{\text{aff}}(E)$, for finite dimensional $K$-vector spaces $E$.

Remark 3.12. One can modify the statements of Theorems 3.8 and 3.10 so that Assumption 3.2 is not necessary. To this end, one should use the notion of a piecewise linear map from the fan $|\Sigma|$ to the “completed” extended building $\mathcal{B}_{\text{aff}}(E) \cup \mathcal{B}_{\text{aff}}(E)$.
Proofs of Theorems 3.8 and 3.10. Recall that $\Sigma$ is the fan in $\mathbb{N}_r \times \mathbb{R}_{\geq 0}$ consisting of the cones over the polyhedra in $\Sigma_1$ as well as faces of such cones. For each $\sigma \in \Sigma$, we have the affine toric scheme $X_\sigma = \text{Spec}(S_\sigma)$ where

$$S_\sigma = \mathcal{O}[\chi^u \omega^k \mid (u, k) \in \sigma^\vee \cap \tilde{M}] \subset K[T_\eta].$$

In particular, $S_0 = K[T_\eta]$ is the coordinate ring of the torus $T_\eta \cong (K^\times)^r$, isomorphic to the Laurent polynomial ring over $K$ in $r$ indeterminates. The affine toric schemes $X_\sigma$, $\sigma \in \Sigma$, glue together to form the toric scheme $X = X_\Sigma$. The scheme $X_0 = \text{Spec}(K[T_\eta])$ comes with a distinguished point $x_0 = 1 \in T_\eta$.

Let $E$ be a toric vector bundle on $X_\Sigma$ of rank $r$. First we see how to associate to $E$ a piecewise affine map $\Phi = \Phi_E : [\Sigma_1] \to \tilde{\mathcal{E}}_{\text{aff}}(E)$. The toric vector bundle $E$ is equivalent to the data of a $T$-linearized sheaf $\mathcal{F}$ of locally free $\mathcal{O}_X$-modules of rank $r$ on $X$. Let $E = E_{x_0}$ which is an $r$-dimensional $K$-vector space. For each cone $\sigma \in \Sigma$, consider the $M$-graded $S_\sigma$-module $\mathcal{F}_\sigma := H^0(\mathcal{F}_\sigma, \mathcal{E}|_{X_\sigma})$. Since $E$ is trivial over the open orbit $X_0 = T_\eta$, we can identify $E|_{X_0}$ with $T_\eta \times E$ and hence $\mathcal{F}_0$ with $K[T_\eta] \otimes_K E$. Then, for each $\sigma \in \Sigma$, the restriction of $\mathcal{F}_\sigma$ to $X_0$ gives us an embedding

$$\mathcal{F}_\sigma \hookrightarrow \mathcal{F}_0 = K[T_\eta] \otimes_K E.$$  

Consider the decomposition of $\mathcal{F}_\sigma$ into $M$-homogeneous components:

$$\mathcal{F}_\sigma = \bigoplus_{u \in M} \mathcal{F}_{\sigma,u},$$

where $\mathcal{F}_{\sigma,u} \subset \chi^u E \subset K[T_\eta] \otimes_K E$ is the $u$-homogeneous component of $\mathcal{F}_\sigma$. Note that the $u$-homogeneous component is the $(-u)$-weight space in $\mathcal{F}_{\sigma}$. Since the $u$-homogeneous component in $K[T_\eta] \otimes_K E$ is $\chi^u E$, we can write:

$$\mathcal{F}_{\sigma,u} = \chi^u \Lambda_{\sigma,u},$$

for some $\mathcal{O}$-submodule $\Lambda_{\sigma,u} \subset E$.

On the other hand, by the local equivariant triviality (Proposition 3.3) the $M$-graded $S_\sigma$-module $\mathcal{F}_\sigma$ is free of rank $r$ and we can write:

$$\mathcal{F}_\sigma = \bigoplus_{i=1}^r S_\sigma \cdot s_{\sigma,i},$$

where $s_{\sigma,i}$ is an $M$-homogeneous section whose $M$-degree we denote by $u_{\sigma,i}$, that is, the image of $s_{\sigma,i}$ in $K[T_\eta] \otimes_K E$ lands in $\chi^{u_{\sigma,i}} E$. Let us write:

$$s_{\sigma,i} = \chi^{u_{\sigma,i}} b_{\sigma,i},$$

for some $b_{\sigma,i} \in E$. Since (12) is a direct sum, the set of vectors $B_\sigma = \{b_{\sigma,1}, \ldots, b_{\sigma,r}\}$ forms a $K$-vector space basis for $E$. We thus have:

$$\mathcal{F}_\sigma = \bigoplus_{i=1}^r \mathcal{F}_{\sigma,u} \otimes \mathcal{O} \cdot \chi^{u'} \omega^{k'} \chi^{u_{\sigma,i}} b_{\sigma,i}.$$  

To compute the $u$-homogeneous component $\mathcal{F}_{\sigma,u}$ in (13) we should sum over all $(u', k')$ such that $u' = u - u_{\sigma,i}$ and $(u', k') \in \sigma^\vee \cap \tilde{M}$. This means that

$$\langle (u - u_{\sigma,i}, k'), y \rangle \geq 0, \ \forall y \in \sigma.$$
Then:

\[ \Lambda_{\sigma,u} = \sum_{i=1}^{r} \sum_{k' : (u-u_{\sigma,i},k') \in \sigma^\vee \cap \widetilde{M}} \mathcal{O} \varpi^{k'} b_{\sigma,i}. \]  

We note that the condition \((u-u_{\sigma,i},k') \in \sigma^\vee \) is equivalent to \((u-u_{\sigma,i},k') \in \rho^\vee \), for all \(\rho \in \sigma(1)\). where \(\sigma(1)\) denotes the set of rays in \(\sigma\). Thus, from (14), we have:

\[ \Lambda_{\sigma,u} = \bigcap_{\rho \in \sigma(1)} \Lambda_{\rho,u}, \]

Hence we can conclude that the sheaf \(F\) is completely determined by the \(\mathcal{O}\)-modules \(\Lambda_{\rho,u}\), for \(\rho \in \Sigma(1)\) and \(u \in M\).

Let \(\rho\) be a ray in \(\Sigma\). We consider two cases:

**Case 1:** The ray \(\rho\) does not lie in \(N_\mathbb{R} \times \{0\}\). Let \((v,1)\) be the unique point of intersection of \(\rho\) and \(N_\mathbb{R} \times \{1\}\). Note that \(v\) is a rational point but in general may not be a lattice point. Then:

\[ (u-u_{\sigma,i},k') \in \rho^\vee \Longleftrightarrow \langle u-u_{\sigma,i}, v \rangle + k' \geq 0 \]

\[ \Longleftrightarrow k' \geq \langle u_{\sigma,i} - u, v \rangle \]

\[ \Longleftrightarrow k' \geq \lceil \langle u_{\sigma,i} - u, v \rangle \rceil. \]

Here \([r]\) denotes the ceiling of a real number \(r\), that is, the smallest integer greater than or equal to \(r\). From (17) we then get:

\[ \Lambda_{\rho,u} = \bigoplus_{i=1}^{r} \mathcal{O} \varpi^{\lceil \langle u_{\sigma,i} - u, v \rangle \rceil} b_{\sigma,i}; \]

Since \(B_{\sigma} = \{b_{\sigma,1}, \ldots, b_{\sigma,r}\}\) is a \(K\)-basis for \(E\) we see that, for any \(u \in M\), the \(\mathcal{O}\)-module \(\Lambda_{\rho,u} \subset E\) is of full rank, i.e. it is a lattice.

**Case 2:** The ray \(\rho\) lies in \(N_\mathbb{R} \times \{0\}\). Let \((v,0)\) be a primitive vector in \(\rho\). Then:

\[ (u-u_{\sigma,i},k') \in \rho^\vee \Longleftrightarrow \langle u-u_{\sigma,i}, v \rangle \geq 0, \forall k' \in \mathbb{Z}. \]

Noting that \(K = \bigcup_{k' \in \mathbb{Z}} \mathcal{O} \varpi^{k'}\) we get:

\[ \Lambda_{\rho,u} = \bigoplus_{i: \langle u,v \rangle \geq \langle u_{\sigma,i}, v \rangle} K \cdot b_{\sigma,i}, \]

In other words, \(\Lambda_{\rho,u}\) depends only on the integer \(\langle u,v \rangle\), that is, on the class of \([u]\) of \(u\) in \(M_\rho := M/(\rho^\perp \cap M) \cong \mathbb{Z}\). Let us write:

\[ E^\rho_j = \Lambda_{\rho,u} \]

where \(j = \langle -u,v \rangle\) (the reason for \(-u\) instead of \(u\) is to make the \(\{E^\rho_j\}_{j \in \mathbb{Z}}\) a decreasing filtration and hence match with the convention in [Kly90]). We observe that for rays \(\rho\) that lie in \(\sigma \cap (N_\mathbb{R} \times \{0\})\), the equation (20) for the filtrations \(\{E^\rho_j\}_{j \in \mathbb{Z}}\) is the Klyachko compatibility condition. It follows, exactly as in the classification of toric vector bundles over a field (Theorem 1.2) that \(F\) restricted to the generic fiber \(X_\eta\) is determined, up to an equivariant isomorphism, by the decreasing \(\mathbb{Z}\)-filtrations \(\{E^\rho_j\}_{j \in \mathbb{Z}}\), \(\rho \in \Sigma(1)\).

Now we are ready to construct the piecewise affine map \(\Phi\) associated to the toric vector bundle \(E\). Let \(\Delta \in \Sigma_1\) be a polyhedron with \(\sigma \in \Sigma\) the corresponding cone over it. In this case, we denote the basis \(B_{\sigma} = \{b_{\sigma,i}\}\) by \(B_\Delta = \{b_{\sigma,i}\}\). Also, if \(v\) is a vertex in \(\Delta\) with corresponding ray \(\rho \in \sigma(1)\), we denote the \(\mathcal{O}\)-module \(\Lambda_{\rho,u}\) by \(\Lambda_{v,u}\).
Let \( \tilde{A}_{\text{aff}}(\Delta) \) be the extended apartment in the extended building \( \tilde{B}_{\text{aff}}(E) \) corresponding to the basis \( B_\Delta \). Recall that it consists of additive norms that are adapted to the basis \( B_\Delta \) (see Section 2.1). Note that choice of a basis \( B_\Delta \) gives an identification of the extended apartment \( \tilde{A}_{\text{aff}}(\Delta) \) with \( \mathbb{R}^r \). We define the affine map \( \Phi_\Delta : \Delta \to \tilde{A}_{\text{aff}}(\Delta) \) to be the linear map given by \( \Phi_\Delta(x) = (\langle u_{\sigma,1}, x \rangle, \ldots, \langle u_{\sigma,r}, x \rangle) \). In terms of additive norms, \( \Phi_\Delta(x) : E \to \mathbb{R} \) is the additive norm given by

\[
\Phi_\Delta(x) \left( \sum_i \lambda_i b_{\Delta,i} \right) = \min \{ \text{val}(\lambda_i) + \langle u_{\sigma,i}, x \rangle \mid i = 1, \ldots, r \}.
\]

For a vertex \( v \) of the polyhedron \( \Delta \) and \( u \in M \), the lattice \( \Lambda_{v,u} \) can be recovered from \( \Phi_\Delta \) as

\[
\Lambda_{v,u} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{C}}[\Phi_\Delta(v)(b_{\Delta,i} - (u,v))] b_{\Delta,i}.
\]

We also need to check that the affine map \( \Phi_\Delta \) contains the data of filtrations \( \{E_j\}_{j \in \mathbb{Z}} \), where the \( \rho \in N_\mathbb{R} \times \{0\} \) are horizontal rays in the cone \( \sigma \) (see (20) and (21)). We see that the filtrations \( \{E_j\}_{j \in \mathbb{Z}} \) can be recovered from the linear part \( \Phi_{\Delta,0} \) of \( \Phi_\Delta \). As in Section 2.3 for any \((x,0)\) in the linear span of the recession cone of \( \Delta \), we can compute the value of the prevaluation \( \Phi_{\Delta,0}(x,0) \) on a vector \( e = \sum_i \lambda_i b_{\Delta,i} \) by:

\[
\Phi_{\Delta,0}(x,0) \left( \sum_i \lambda_i b_{\Delta,i} \right) = \lim_{t \to \infty} \min \{ \text{val}(\lambda_i) + \langle u_{\Delta,i}, tx \rangle \mid i = 1, \ldots, r \},
\]

\[
= \min \{ \langle u_{\Delta,i}, x \rangle \mid i = 1, \ldots, r \}.
\]

Let \((v,0)\) be the primitive vector for a ray \( \rho \) in \( \sigma \), the cone over \( \Delta \). In particular, if we let \( x = -v \), the prevaluation \( \Phi_{\Delta,0}(-v,0) \) is given by:

\[
\Phi_{\Delta,0}(-v,0) \left( \sum_i \lambda_i b_{\Delta,i} \right) = \min \{ \langle u_{\Delta,i}, -v \rangle \mid i = 1, \ldots, r \},
\]

and its corresponding filtration \( \{E_j\}_{j \in \mathbb{Z}} \) is:

\[
E_j = \text{span}_K \{ b_{\Delta,i} \mid j \leq \langle u_{\Delta,i}, -v \rangle \}.
\]

But this is the same as (20), the filtration associated to the ray \( \rho \in \Sigma_0 \), as required.

It remains to show that the \( \Phi_\Delta : \Delta \in \Sigma_1 \), glue together to give a piecewise affine map \( \Phi : [\Sigma_1] \to \tilde{B}_{\text{aff}}(E) \). To this end, we need to show that for all \( \Delta, \Delta' \in \Sigma_1 \), \( \Phi_\Delta \) and \( \Phi_{\Delta'} \) coincide on the intersection \( \Delta \cap \Delta' \). This is the content of the next lemma. If the vertices of \( \Delta \cap \Delta' \) are lattice points, the lemma is easy to show because we can remove the ceiling function in the expression for \( \Lambda_{\sigma,v} \) in (23). In general, the presence of the ceiling function adds extra details to the argument.

**Lemma 3.13.** Let \( v \) be a vertex of the convex polyhedron \( \Delta \cap \Delta' \). Then the additive norms \( \Phi_\Delta(v) \) and \( \Phi_{\Delta'}(v) \) coincide.

**Proof.** Let \( B_\Delta = \{ b_{\Delta,i} \} \) and \( B_{\Delta'} = \{ b_{\Delta',i} \} \) be the bases corresponding to \( \Delta \) and \( \Delta' \) respectively. We note that, by (23), the additive norm corresponding to \( \Lambda_{v,u} \) is:

\[
[\Phi_\Delta(v)(\cdot) - \langle u,v \rangle] = [\Phi_{\Delta'}(v)(\cdot) - \langle u,v \rangle].
\]
By contradiction, suppose $\Phi_\Delta(v) \neq \Phi_{\Delta'}(v)$. Then for some $e \in E$, $\Phi_\Delta(v)(e) \neq \Phi_{\Delta'}(v)(e)$. Let $e = \sum \lambda_i b_{\Delta,i} = \sum \lambda'_i b_{\Delta',i}$. From (23) we see that, after a reordering of basis elements if necessary, there is an index $j$ such that:

$$\Phi_\Delta(e) = \text{val}(\lambda_j) + \langle u_{\sigma,j}, v \rangle \neq \text{val}(\lambda'_j) + \langle u'_{\sigma,j}, v \rangle = \Phi_{\Delta'}(e).$$

On the other hand, in (24), if we let $u = u_{\Delta,j}$ we get:

$$\lfloor \langle u_{\Delta',j}, v \rangle - \langle u_{\Delta,j}, v \rangle \rfloor = \text{val}(\lambda_j) - \text{val}(\lambda'_j).$$

Also, in (24), if we let $u = u_{\Delta',j}$, in the same way we obtain:

$$\lfloor \langle u_{\Delta,j}, v \rangle - \langle u'_{\Delta,j}, v \rangle \rfloor = \text{val}(\lambda'_j) - \text{val}(\lambda_j).$$

It follows that $\langle u_{\Delta,j}, v \rangle - \langle u'_{\Delta,j}, v \rangle$ is an integer. Thus, $\Phi_\Delta(v)(e) - \Phi_{\Delta'}(v)(e)$ is an integer. This together with (24) implies that $\Phi_\Delta(v)(e) = \Phi_{\Delta'}(v)(e)$. The contradiction proves the claim. □

We have shown that $\Phi_\Delta$ and $\Phi_{\Delta'}$ coincide on the vertices of the polyhedron $\Delta \cap \Delta'$. We need to show that the linear parts of $\Phi_\Delta$ and $\Phi_{\Delta'}$ also coincide on the recession cone of $\Delta \cap \Delta'$. Let $\rho \in \Sigma_0$ be a ray in the recession cone of the polyhedron $\Delta \cap \Delta'$ with primitive vector $(v,0)$. As we saw above, both prevalevaluations $\Phi_{\Delta,0}(v,0)$ and $\Phi_{\Delta',0}(v,0)$ on $E$ correspond to the filtration $\{E^p\}$ on $E$ and hence coincide.

Now $\Phi_\Delta$ and $\Phi_{\Delta'}$ are two affine maps on $\Delta \cap \Delta'$ that coincide on the vertices of $\Delta \cap \Delta'$ and also have the same linear parts on $\Delta \cap \Delta'$. This implies that they coincide everywhere on $\Delta \cap \Delta'$ as required. Hence the $\Phi_{\Delta'}$, $\Delta \in \Sigma_1$, glue together to give a piecewise affine map from $|\Sigma_1|$ to the extended building $\mc{B}_{\text{aff}}(E)$.

Note that the above arguments also show that $\Phi$ is uniquely determined by the vector bundle $\mc{E}$. This is because, as shown above, the $\Phi_\Delta$ (and hence $\Phi$) are determined by the lattices $\Lambda_{\rho,u}, u \in M_\rho$ (for rays $\rho$ in $\Sigma$ that intersects $N_\mathbb{R} \times \{1\}$), and the filtrations $\Lambda_{\rho}$ (for rays $\rho$ that lie in $N_\mathbb{R} \times \{0\}$).

In conclusion we have a piecewise affine map $\Phi : |\Sigma_1| \to \mc{B}_{\text{aff}}(E)$ that determines the sheaf $\mc{F}$, and its linear part $\Phi_0$ gives the Klyachko’s data for the restriction of the sheaf $\mc{F}$ to the generic orbit $X_{\eta}$.

Conversely, let $\Phi : |\Sigma_1| \to \mc{B}_{\text{aff}}(E)$ be a piecewise affine map. One verifies that reversing the construction gives a toric vector bundle $\mc{E}_\Phi$ over the toric scheme $X_\Sigma$.

Next we prove the claim about the equivalence of categories. Again, if all the vertices in the polyhedral complex $\Sigma_1$ are lattice points, then arguments are easier. In general, we need extra details involving the ceiling function.

Proof of Theorem 3.11. A $T$-equivariant morphism of vector bundles is the same as a $T$-equivariant morphism of the corresponding locally free sheaves of modules. Let $\mc{E}$, $\mc{E}'$ be toric vector bundles on $X = X_\Sigma$ with the corresponding sheaves of $O_X$-modules $\mc{F}$ and $\mc{F}'$ respectively. Let $F : \mc{F} \to \mc{F}'$ be a $T$-equivariant morphism. Since $T_\eta$ is open in $X$, $F$ is determined by its restriction to $T_\eta$. On the other hand, $F$ is $T$-equivariant and hence $F|_{T_\eta}$ is determined by its restriction to $E = \mc{E}_{T_\eta}$. We note that $F : \mc{E}_{x_0} \to \mc{E}'_{x_0}$ is a $K$-linear map. We would like to characterize which $K$-linear maps $F : \mc{E}_{x_0} \to \mc{E}'_{x_0}$ give rise to a $T$-equivariant morphisms $F : \mc{F} \to \mc{F}'$. In fact, one verifies that a linear map $F$ gives rise to a $T$-equivariant morphism if and only if for any $\sigma \in \Sigma$ and $u \in M$ we have:

$$F(\Lambda_{\sigma,u}) \subset \Lambda'_{\sigma,u}.\tag{25}$$
We note that since $\Lambda_{\sigma,u} = \bigcap_{\rho \in \Sigma(1)} \Lambda_{\rho,u}$, the above is equivalent to

$$F(\Lambda_{\rho,u}) \subset \Lambda'_{\rho,u}, \quad \forall \rho \in \Sigma(1), \forall u \in M.$$  

The next lemma shows that the condition in (26) is equivalent to $\Phi$ being dominated by $F^*(\Phi')$. This finishes the proof of Theorem 3.11. \hfill $\square$

**Lemma 3.14.** The following are equivalent:

(a) For all $\rho \in \Sigma$ and $u \in M$, 

$$F(\Lambda_{\rho,u}) \subset \Lambda'_{\rho,u},$$

where $\Lambda_{\rho,u}$ is given by (18) if $\rho$ intersects $N_R \times \{1\}$ and is given by (20) otherwise.

(b) For all $x \in |\Sigma_1|$, $u \in M$ and $e \in E$:

$$\left[ \Phi(x)(e) - \langle u, v \rangle \right] \leq \left[ \Phi'(x)(F(e)) - \langle u, x \rangle \right].$$

(c) For all $x \in |\Sigma_1|$ and $e \in E$:

$$\Phi(x)(e) \leq F^*(\Phi'(x))(e) := \Phi'(x)(F(e)).$$

**Proof.** (a) $\Rightarrow$ (b): Define $\Psi_u(x)(e) = [\Phi(x)(e) - \langle u, v \rangle]$ and $\Psi'_u(x)(e) = [\Phi'(x)(F(e)) - \langle u, x \rangle]$. Then $\Psi_u$ and $\Psi'_u$ are piecewise affine functions on $|\Sigma_1|$. The condition in (a) implies that, for any vertex $v$ in $\Sigma_1$, $\Psi_u(v) \leq \Psi'_u(v)$, and moreover the linear part of $\Psi_u$ is less than or equal to the linear part of $\Psi'_u$. These put together imply that $\Psi_u(x) \preceq \Psi'_u(x)$, for all $x \in |\Sigma_1|$. 

(b) $\Rightarrow$ (c): Suppose by contradiction that there is $x$ and $e$ such that $\Phi(x)(e) > \Phi'(x)(F(e))$. By the piecewise affineness of $\Phi'$ we know that, for any polyhedron $\Delta \in \Sigma_1$ containing $x$, there is a basis $B'_\Delta = \{b'_\Delta, \ldots, b'_{\Delta,r}\}$ for $E'$ and characters $\{u'_\Delta, \ldots, u'_{\Delta,r}\} \subset M$ such that for all $e' = \sum_i \lambda_i b'_i \in E$ we have:

$$\Phi'(x)(e') = \min\{\text{val}(\lambda_i) + \langle u'_\Delta, x \rangle \mid i = 1, \ldots, r\}.$$ 

Let $j$ be the index where the minimum above is attained for $e' = F(e)$, thus

$$\Phi'(x)(F(e)) = \text{val}(\lambda_j) + \langle u'_{\Delta,j}, x \rangle.$$ 

Letting $u = u'_{\Delta,j}$, we see from (24) that $\Phi'(x)(F(e)) - \langle u, x \rangle \in \mathbb{Z}$. But

$$\Phi(x)(e) - \langle u, x \rangle > \Phi'(x)(F(e)) - \langle u, x \rangle.$$ 

Since the right-hand side is an integer we see that

$$\left[ \Phi(x)(e) - \langle u, x \rangle \right] > \left[ \Phi'(x)(F(e)) - \langle u, x \rangle \right],$$

which contradicts the assumption in (b).

(c) $\Rightarrow$ (a): First suppose for all $x \in |\Sigma|$ and $e \in E$ we have $\Phi(x)(e) \leq F^*(\Phi'(x))(e) := \Phi'(x)(F(e))$. Then, for all $u \in M$, $x \in |\Sigma|$, $e \in E$ one has

$$\Phi(x)(e) - \langle x, u \rangle \leq \Phi'(x)(F(e)) - \langle x, u \rangle,$$

$$\Phi(x)(e) - \langle x, u \rangle \leq \Phi'(x)(F(e)) - \langle x, u \rangle.$$ 

From Proposition (24) we now conclude that $F(\Lambda_{\rho,u}) \subset \Lambda'_{\rho,u}$ as required. \hfill $\square$

**Example 3.15.** Let the base field be the 2-adic field $\mathbb{Q}_2$. Recall that the Bruhat-Tits building $\mathfrak{B}_\text{aff}(\mathbb{Q}_2)$ of $\text{GL}(2, \mathbb{Q}_2)$ is the 3-regular infinite tree depicted in Figure 8. Each apartment in the Bruhat-Tits building is a two-sided infinite path in the tree. Each extended apartment is the Cartesian product of a two-sided infinite path and $\mathbb{R}$.  

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Consider the canonical model of $\mathbb{P}^1$ over $\mathbb{Q}_2$. Its fan $\Sigma$ is the Cartesian product of fan of $\mathbb{P}^1$ and $\mathbb{R}_{\geq 0}$. The polyhedral complex $\Sigma_1$ in $N_\mathbb{R} \times \{1\}$ consists of three polyhedra $(-\infty, 0]$, $\{0\}$ and $[0, \infty)$. The generic fiber is $\mathbb{P}^1$ over $\mathbb{Q}_2$ and the special fiber is $\mathbb{P}^1$ over the residue field $\mathbb{F}_2$. In Figure 10, we have marked three one-sided infinite paths $I_1$, $I_2$ and $I_1'$. We let $\tilde{I}_1$, $\tilde{I}_2$ and $\tilde{I}_1'$ denote the Cartesian products of the $I_1$, $I_2$ and $I_1'$ with $\mathbb{R}$ respectively.

Consider piecewise affine maps $\Phi_i : \Sigma \rightarrow \mathfrak{B}_{\text{aff}}(E)$, $i = 1, 2$, where $\Phi_1$ is given by some affine maps $\Phi_1|_{(-\infty, 0]} : (-\infty, 0] \rightarrow \tilde{I}_1$ and $\Phi_1|[0, \infty) : [0, \infty) \rightarrow \tilde{I}_2$. Similarly, $\Phi_2$ is given by some affine maps $\Phi_2|_{(-\infty, 0]} : (-\infty, 0] \rightarrow \tilde{I}_1$ and $\Phi_2|[0, \infty) : [0, \infty) \rightarrow \tilde{I}_2$. The piecewise affine maps $\Phi_i$, $i = 1, 2$, give rise to toric vector bundles $E_{\Phi_i}$, $i = 1, 2$, of rank 2 on the canonical model of $\mathbb{P}^1$. In Example 4.3 we will show that $E_{\Phi_1}$ splits equivariantly into sum of line bundles while $E_{\Phi_2}$ does not.

4. Equivariant splitting of toric vector bundles on toric schemes

In this section we give a simple criterion for splitting of a toric vector bundle into a sum of toric vector bundles in terms of its associated piecewise affine map. Similar criterion exists for toric vector bundles over field (see [KM22, p. 5, A little application]).

**Definition 4.1.** We say that a toric vector bundle $E$ splits equivariantly if it is equivariantly isomorphic to a direct sum of toric line bundles.

**Theorem 4.2** (Criterion for equivariant splitting). A toric vector bundle $E$ over $\mathfrak{X}_\Sigma$ splits equivariantly if and only if the image of the corresponding piecewise affine map $\Phi_E$ lands in one extended apartment.

**Proof.** Let $E = \bigoplus_{i=1}^r L_i$ where the $L_i$ are toric line bundles on $\mathfrak{X}_\Sigma$. As before, the local equivariant triviality (Proposition 3.3) applied to each toric line bundle $L_i$ implies that
there exists a basis $B = \{b_1, \ldots, b_r\}$ such that for any $\sigma \in \Sigma$, the space of sections $\mathcal{F}_\sigma = H^0(X_\sigma, \mathcal{E}|_{X_\sigma})$ is given by (cf. (13)):

$$\mathcal{F}_\sigma = \bigoplus_{i=1}^{r} \bigoplus_{(u',k') \in \sigma^\vee \cap \mathcal{M}} \mathcal{O} \cdot \chi_{u'} \omega^k \chi_{\sigma,i} b_i \subset \mathcal{E} \otimes K[T]$$. 

for some set of weights $\{u_{\sigma,i} \mid i = 1, \ldots, r\}$. The corresponding piecewise affine map $\Phi_\mathcal{E}$ is then given by (cf. (22)):

$$\Phi_\mathcal{E}(x) \left( \sum_{i} \lambda_i b_i \right) = \min \{ \operatorname{val}(\lambda_i) + \langle u_{\sigma,i}, x \rangle \mid i = 1, \ldots, r \}$$

for any $x \in \Delta = \sigma \cap (N_\mathbb{R} \times \{1\})$. This shows that all the additive norms $\Phi_\mathcal{E}(x), x \in |\Sigma|$, are adapted to the basis $B$ and hence the image of $\Phi_\mathcal{E}$ lands in the extended apartment $A_{\operatorname{aff}}(B)$.

Conversely, if there is a basis $B = \{b_i\}$ such that all the additive norms $\Phi_{W} (x)$ are adapted to $B$ then, as above, one sees that $\mathcal{E} = \bigoplus L_i$ where $L_i$ is the toric line bundle whose sheaf of sections $\mathcal{F}_i$ is given as follows: for any $\sigma \in \Sigma$,

$$\mathcal{F}_i(X_\sigma) = \bigoplus_{(u',k') \in \sigma^\vee \cap \mathcal{M}} \mathcal{O} \cdot \chi_{u'} \omega^k \chi_{\sigma,i} b_i$$

Example 4.3. In Example 3.15 the image of $\Phi_1$ lands in an apartment, that is, an infinite two-sided path, while the image of $\Phi_2$ does not. It follows from Theorem 4.2 that the toric vector bundle $\mathcal{E}_{\Phi_1}$, associated to the piecewise affine map $\Phi_1$, is equivariantly split while $\mathcal{E}_{\Phi_2}$, associated to $\Phi_2$, is not. This is in contrast to the field case: over a field, any toric vector bundle over $\mathbb{P}^1$ splits equivariantly (see [Kly90, Theorem 6.1.2] as well as [KM22, p. 5, A little application]). The example of $\mathcal{E}_{\Phi_2}$ shows that this is not the case over a discrete valuation ring.

Example 4.4 (A Helly’s theorem for the tree of $\text{GL}(2, \mathbb{Q}_p)$). We observe that a version of Kyachko’s analogue of Helly’s theorem ([Kly90, Theorem 6.1.2]) holds for the infinite $(p+1)$-regular tree $T_p$. Let $A$ be a finite collection of vertices in the tree $T_p$ such that any 3 vertices in $A$ lie on the same path. Then all of the vertices in $A$ lie on the same path. One can give a direct combinatorial proof of the above fact as follows: By induction on $|A|$, it suffices to show that any $k-1$ vertices in $A$ lie on a path then any $k$ vertices also lie on a path. Take vertices $v_i \in A, i = 1, \ldots, k$. Let $P$ be a path containing the vertices $v_i \in A, i = 1, \ldots, k-1$. We can assume that $v_1$ and $v_{k-1}$ are the first and last vertices in $A$ that appear in $P$. We recall that, in a tree, there is a unique path joining any two given vertices. Now by assumption, $v_1, v_{k-1}, v_k$ lie on the same path $Q$. The uniqueness of path joining $v_1$ and $v_{k-1}$ then implies that all the $v_i, i = 1, \ldots, k$, lie on $Q$ as well. This proves the claim.

5. Appendix: Buildings associated to a reductive algebraic group

In this section we provide a quick overview of the theory of buildings associated to linear algebraic groups. This generality is not needed in the rest of the paper, but hopefully is helpful to put the results of the paper in a larger framework and give the reader some perspective.

We start by giving the definition of a building as an abstract simplicial complex.
A cone over the (geometric realization of) the apartment we denote by $\Delta_{\text{sph}}$. For parabolic subgroups $P_1, P_2$, the simplex corresponding to $P_1$ is a face of that of $P_2$ if $P_2 \subset P_1$. The apartments $\Delta_{\text{sph}}(G)$ correspond to maximal tori in $G$. For a parabolic subgroup $P$ and a maximal torus $H \subset G$, the simplex of $P$ lies in the apartment of $H$ if $H$ is a maximal torus in $P$. For given maximal torus $H$, let $\Lambda^\vee(H)$ denote its cocharacter lattice and put $\Lambda^\vee_k(H) = \Lambda^\vee(H) \otimes \mathbb{Z}^\mathbb{R}$. The apartment corresponding to $H$ is the Coxeter complex of $(G, H)$, i.e., the triangulation of the unit sphere in $\Lambda^\vee_k(H)$, with respect to some inner product, obtained by intersecting the Weyl chambers with the sphere (see Figures 8 and 9). It is well-known that there is a one-to-one correspondence between parabolic subgroups that contain the maximal torus $H$ (standard parabolic subgroups) and the faces of Weyl chambers in $\Lambda^\vee_k(H)$ (equivalently the simplices in the Coxeter complex of $(G, H)$). We define the space $B_{\text{sph}}(G)$ to be the infinite union of the unit spheres in $\Lambda^\vee(H)$ corresponding to the maximal tori $H \subset G$ where we glue two spheres along their common simplices. That is, suppose a parabolic subgroup $P$ contains two maximal tori $H$ and $H'$. Then we glue the simplices corresponding to $P$ in the unit spheres in $\Lambda^\vee_k(H)$ and $\Lambda^\vee_k(H')$ together. We refer to $B_{\text{sph}}(G)$ as the geometric realization of the Tits building of $G$ (see [KM22, Section 1]). By abuse of terminology we may refer to $B_{\text{sph}}(G)$ as the Tits building of $G$ as well.

Similarly, we can take the infinite union of real vector spaces $\Lambda^\vee(H)$ for all maximal tori $H \subset G$, and glue two vector spaces along the faces of Weyl chambers that correspond to the same parabolic subgroup. We denote the resulting space by $\mathcal{B}_{\text{sph}}(G)$ and call it the cone over the (geometric realization) of the Tits building of $G$ (see [KM22, Section 1]).

For a maximal torus $H \subset G$ we denote the unit sphere in $\Lambda^\vee_k(H)$ by $A_H$. More canonically, $A_H$ can be defined as the quotient of $\Lambda^\vee_k(H)$ by $\mathbb{R}_{>0}$. We refer to $A_H$ as the geometric realization of the apartment associated to $H$. Similarly, we call the $\mathbb{R}$-vector $\Lambda^\vee_k(H)$, the cone over the (geometric realization of) the apartment $A_H$ and sometimes denote it by $\tilde{A}_H$.

Let $K$ be a field together with a discrete valuation $\text{val} : K \to \mathbb{Z}$. As usual, $\mathcal{O}$ and $\mathfrak{m}$ denote the valuation ring of $(K, \text{val})$ and its unique maximal ideal respectively. We denote a uniformizer for $\text{val}$ by $\varpi$.

To a reductive group $G$ defined over $K$ one can associate its Bruhat-Tits building which is an example of an affine building. Each apartment in the Bruhat-Tits building is a triangulation of an affine space, namely the affine Coxeter complex of $G$ corresponding to its affine Weyl group.
The Bruhat-Tits building $\mathfrak{B}_{aff}(G)$ is obtained by taking the infinite union of affine spaces associated to maximal split tori in $G$ and gluing them along certain subsets corresponding to parahoric subgroups in $G$.

**Remark 5.2.** As abstract simplicial complexes, the Bruhat-Tits building of a reductive algebraic group $G$ is the same as that of its semisimple quotient $G/Z(G)$, where $Z(G)$ is the center of $G$.

To any affine building $\mathfrak{B}_{aff}$ one can associate its *spherical building at infinity* $\mathfrak{B}_{\infty}$. One can compactify an affine building by adding its spherical building at infinity to it. This is based on compactifying the Euclidean space $\mathbb{R}^r$ to a ball by attaching a sphere $S^{r-1}$. We will use this in Theorem 3.10 to relate the combinatorial data of a toric vector bundle on a toric scheme and that of its general fiber.

Firstly, we remark that (the geometric realization of) a building is a metric space. For a spherical building of each apartment is isometric to a unit sphere, while for an affine building each apartment is isometric to a Euclidean space.

Let $\mathfrak{B}_{aff}$ be an $r$-dimensional affine building. We denote the distance between two points $x, y$ by $d(x, y)$. A ray in $\mathfrak{B}_{aff}$ is a subset that is isometric to the half-line $[0, \infty)$. Let $\rho : [0, \infty) \to \mathfrak{B}_{aff}$ be an isometry into its image representing a ray. By abuse of notation we refer to the ray represented by $\rho$, namely the image of $\rho$, also by $\rho$. One shows that any ray necessarily lies in an apartment and hence, after identifying the apartment with $\mathfrak{B}_{aff}$, a ray $\rho$ can be written as $\rho(t) = x_0 + tv$, $t \geq 0$, for some $x_0, v \in \mathbb{R}^r$.

Two rays $\rho, \rho'$ are said to be *parallel* if there is a bound $b \geq 0$ such that for any $x \in \rho$ there exists $y \in \rho'$ with $d(x, y) \leq b$, and conversely for any $y \in \rho'$ there is $x \in \rho$ such that $d(x, y) \leq b$. This parallel equivalence relation satisfies an analogue of *parallel postulate* in Euclidean geometry. Namely, for any ray $\rho$ and any point $x$ in the building there is a unique ray $\rho'$ passing through $x$ and parallel to $\rho$ (see [Gar97, Section 16.8]). Note that in general $\rho$ and $\rho'$ may not land in the same apartment.

The spherical building at infinity $\mathfrak{B}_{\infty}$ of $\mathfrak{B}_{aff}$ is defined to be the space whose points are equivalence classes of parallel rays. For any apartment $A$ (which is isometric to $\mathbb{R}^r$) in $\mathfrak{B}_{aff}$, the corresponding apartment at infinity $A_{\infty}$ consists of equivalence classes of parallel rays in $A$. One observes that $A_{\infty}$ is isometric to the sphere $S^{r-1}$. It can be shown that $\mathfrak{B}_{\infty}$ has the structure of a spherical building ([Gar97, Section 16.9]).

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