AN ISOPERIMETRIC INEQUALITY FOR THE FIRST STEKLOV-DIRICHLET LAPLACIAN EIGENVALUE OF CONVEX SETS WITH A SPHERICAL HOLE

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Abstract. In this paper we prove the existence of a maximum for the first Steklov-Dirichlet eigenvalue in the class of convex sets with a fixed spherical hole under volume constraint. More precisely, if \( \Omega = \Omega_0 \setminus B_{R_1} \), where \( B_{R_1} \) is the ball centered at the origin with radius \( R_1 > 0 \) and \( \Omega_0 \subset \mathbb{R}^n \), \( n \geq 2 \), is an open bounded and convex set such that \( B_{R_1} \subset \Omega_0 \), then the first Steklov-Dirichlet eigenvalue \( \sigma_1(\Omega) \) has a maximum when \( R_1 \) and the measure of \( \Omega \) are fixed. Moreover, if \( \Omega_0 \) is contained in a suitable ball, we prove that the spherical shell is the maximum.

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1. INTRODUCTION AND MAIN RESULTS

Let \( \Omega_0 \subset \mathbb{R}^n \), \( n \geq 2 \), be an open, bounded, connected set with Lipschitz boundary such that \( B_{R_1} \subset \Omega_0 \), where \( B_{R_1} \) is the open ball of radius \( R_1 > 0 \) centered at the origin such that its closure is strictly contained in \( \Omega_0 \) and let us set \( \Omega := \Omega_0 \setminus B_{R_1} \). The first Steklov-Dirichlet eigenvalue of \( \Omega \) is defined by

\[
\sigma_1(\Omega) = \min_{v \in H^1_{\partial B_{R_1}}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla v|^2 \, dx}{\int_{\partial \Omega_0} v^2 \, d\mathcal{H}^{n-1}},
\]

where \( H^1_{\partial B_{R_1}}(\Omega) \) is the set of Sobolev functions on \( \Omega \) vanishing on \( \partial B_{R_1} \) (see Section 2 for the precise definition).

Denoting by \( \nu \) the outer unit normal to \( \partial \Omega_0 \), any minimizer of (1.1) satisfies the following problem

\[
\begin{aligned}
\Delta u &= 0 & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= \sigma(\Omega) u & \text{on } \partial \Omega_0 \\
u &= 0 & \text{on } \partial B_{R_1}.
\end{aligned}
\]

(1.2)
with \( \sigma(\Omega) = \sigma_1(\Omega) \). For more details on \( \sigma_1(\Omega) \) and the problem (1.2) we refer the reader to Section 2.2.

When \( R_1 = 0 \), (1.2) is the classical Steklov-Laplacian eigenvalue problem. In this case, Weinstock in [W1, W2] proved an isoperimetric inequality for the first non-trivial Steklov eigenvalue in two dimensions. More precisely, he showed that among all simply connected sets of the plane with prescribed perimeter, the disk maximizes the first non-trivial Steklov-Laplacian eigenvalue. In [BFNT] the authors proved that Weinstock inequality holds true in any dimension, provided they restrict to the class of convex sets with fixed perimeter. In [B], it is proved that the ball is still a maximizer for the first non-trivial Steklov eigenvalue among all bounded open sets with Lipschitz boundary of \( \mathbb{R}^n \), \( n \geq 2 \), with fixed volume. Stability and instability results are also studied (for instance we refer to [BDPR, BN, GLMPT]).

When we consider a spherical hole with homogeneous Dirichlet boundary condition, that is \( R_1 > 0 \), the Steklov-Dirichlet eigenvalue problem (1.2) is substantially different. The study of an eigenvalue problem on sets with a spherical hole is actually a topic of interest (see [BKPS, K1, PPT, PW] and the references therein). In particular, problem (1.2) has been recently considered by several authors (see for instance [D, F, HLS, HP, GP, PPS, VS]). In [PPS] it is proved that the first eigenvalue \( \sigma_1(\Omega) \), as defined in (1.1), is bounded from above when both the volume of \( \Omega \) and the radius of the inner ball are fixed among the class of nearly spherical sets of \( \mathbb{R}^n \). Moreover the authors prove that the spherical shell is a local maximizer.

The aim of this paper is twofold. First, we prove the existence of a maximum for \( \sigma_1(\Omega) \) in the class of sets \( \Omega = \Omega_0 \setminus \overline{B}_{R_1} \), where \( \Omega_0 \subset \mathbb{R}^n \), \( n \geq 2 \), is an open bounded and convex set containing \( B_{R_1} \), keeping \( R_1 \) and the measure of \( \Omega \) fixed. Actually, we prove this existence result also when the hole is not spherical, but it is an open, convex set \( K \subset \Omega_0 \) with non-empty interior.

Our second aim is to find the shape of the maximum when the hole is spherical. We stress that, when \( \Omega_0 = B_{R_1}(x_0) \) is a ball centered at \( x_0 \) with radius \( R_2 > R_1 \), in [F, VS] it is proved that \( \sigma_1(\Omega) \) achieves the maximum when \( \Omega \) is the spherical shell, that is when the two balls are concentric.

Our goal is to prove that this is also true for a suitable class of annular sets. More precisely our main result is the following.

**Theorem 1.1.** Let \( R_1 > 0 \), \( \Omega_0 \subset \mathbb{R}^n \) be an open, bounded and convex set, \( n \geq 2 \), such that \( B_{R_1} \subset \Omega_0 \subset B_R \), where \( B_R \) is the ball centered at the origin with radius \( R \) given by

\[
R = \begin{cases} 
R_1 e^{\sqrt{n}} & \text{if } n = 2 \\
R_1 \left[ \frac{(n-1)+(n-2)\sqrt{2(n-1)}}{n-1} \right]^{\frac{1}{n-1}} & \text{if } n \geq 3.
\end{cases}
\]  

(1.3)

Then, denoting by \( \Omega = \Omega_0 \setminus \overline{B}_{R_1} \), the following inequality holds

\[
\sigma_1(\Omega) \leq \sigma_1(A_{R_1,R_2}).
\]  

(1.4)
where \( A_{R_1,R_2} \) is the spherical shell of radii \( R_1 < R_2 \) having the same volume as \( \Omega \).

We observe that the convexity assumption is not just technical but it is natural when dealing with Steklov-Dirichlet eigenvalues (see [FS]).

The outline of the paper is the following. In the next Section we set the notation and collect some basic results about the Steklov-Dirichlet eigenvalue problem that will be needed in the sequel. In Section 3, firstly we prove some estimates for \( \sigma_1(\Omega) \) in terms of suitable geometrical quantities related to \( \Omega \) and then we state and prove the existence result. In Section 4, by using a suitable weighted isoperimetric inequality, we prove that the spherical shell is a maximum for \( \sigma_1(\Omega) \) when both the volume of \( \Omega \) and the radius of the inner ball are fixed. Eventually, in Section 5, we discuss on a maximization problem for \( \sigma_1(\Omega) \) under perimeter constraint.

2. Preliminary results

2.1. Notations and basic facts. Throughout this paper, we denote by \( B_R(x_0) \) the ball centered at \( x_0 \in \mathbb{R}^n \) with radius \( R > 0 \), by \( B_R \) the ball centered at the origin with radius \( R \) and by \( B, S^{n-1} \) and \( \omega_n \) respectively the unit ball of \( \mathbb{R}^n \) centered at the origin, its boundary and its volume. Let \( R_1, R_2 \) be such that \( 0 < R_1 < R_2 \), the spherical shell will be denoted as follows:

\[
A_{R_1,R_2} = \{ x \in \mathbb{R}^n : R_1 < |x| < R_2 \}.
\]

Moreover, the \((n - 1)\)-dimensional Hausdorff measure in \( \mathbb{R}^n \) will be denoted by \( \mathcal{H}^{n-1} \). The Euclidean scalar product in \( \mathbb{R}^n \) is denoted by \((\cdot,\cdot)\).

Let \( D \subseteq \mathbb{R}^n \) be an open bounded set and let \( E \subseteq \mathbb{R}^n \) be a measurable set. For the sake of completeness, we recall here the definition of the perimeter of \( E \) in \( D \) (see for instance [AFP, M]), that is

\[
P(E;D) = \sup \left\{ \int_E \text{div} \varphi \, dx : \varphi \in C_c^\infty(D;\mathbb{R}^n), ||\varphi||_\infty \leq 1 \right\}.
\]

The perimeter of \( E \) in \( \mathbb{R}^n \) will be denoted by \( P(E) \) and, if \( P(E) < \infty \), we say that \( E \) is a set of finite perimeter. Moreover, if \( E \) has Lipschitz boundary, it holds

\[
P(E) = \mathcal{H}^{n-1}(\partial E).
\]

The Lebesgue measure of a measurable set \( E \subset \mathbb{R}^n \) will be denoted by \( V(E) \). Moreover, we define the inradius of \( E \subset \mathbb{R}^n \) as

\[
\rho(E) = \sup_{x \in E} \inf_{y \not\in E} |x - y|, \tag{2.1}
\]

while the diameter of \( E \) is

\[
diam(E) = \sup_{x,y \in E} |x - y|.
\]
If $E$ is an open, bounded and convex set of $\mathbb{R}^n$ with non-empty interior, we have (see for instance [DPBG, Sc])

$$\rho(E) \leq \frac{nV(E)}{P(E)}, \quad (2.2)$$

and the following (see [EFT, GWW, Sc]):

$$P(E)^{n-1} > \omega_{n-1} n^{n-2} \operatorname{diam}(E) V(E)^{n-2}. \quad (2.3)$$

Finally, we recall the definition of Hausdorff distance between two non-empty compact sets $E, F \subset \mathbb{R}^n$, that is (see for instance [Sc])

$$\delta_H(E, F) = \inf \{ \varepsilon > 0 : E \subset F + B_\varepsilon, \ F \subset E + B_\varepsilon \}.$$  

Note that, if $E, F$ are both convex sets, then $\delta_H(E, F) = \delta_H(\partial E, \partial F)$.

Let $\{E_k\}_{k \in \mathbb{N}}$ be a sequence of non-empty compact subsets of $\mathbb{R}^n$, we say that $E_k$ converges to $E$ in the Hausdorff sense and we denote

$$E_k \xrightarrow{\mathcal{H}} E$$

if and only if $\delta_H(E_k, E) \to 0$ as $k \to \infty$. Moreover, we say that $\{E_k\}_{k \in \mathbb{N}}$ converges in measure to $E$, and we write $E_k \to E$, if $\chi_{E_k} \to \chi_E$ in $L^1(\mathbb{R}^n)$, where $\chi_E$ and $\chi_{E_k}$ are the characteristic functions of $E$ and $E_k$ respectively. In what follows we recall some properties of the convex bodies, i.e. compact convex sets without empty interior. We refer to [Sc] for further properties and the details.

Let $K \subset \mathbb{R}^n$ be a bounded convex body. The support function $h_K$ of $K$ is defined as follows

$$h_K : \mathbb{S}^{n-1} \to \mathbb{R}, \quad h_K(x) = \sup_{y \in K} \langle x, y \rangle.$$  

If the origin belongs to $K$ then $h_K$ is non-negative and $h_K(x) \leq \operatorname{diam}(K)$ for every $x \in \mathbb{S}^{n-1}$. Let $K \subset \mathbb{R}^n$ be a bounded convex body such that the origin is an interior point of $K$. The radial function of $K$ is defined as follows

$$\rho_K(x) = \sup \{ \lambda \geq 0 : \lambda x \in K \}, \quad x \in \mathbb{S}^{n-1}, \quad (2.4)$$

and it is a Lipschitz function. The radial map is

$$r_K : \mathbb{S}^{n-1} \to \partial K, \quad r_K(x) = x \rho_K(x). \quad (2.5)$$

Then

$$\partial K = \{ x \rho_K(x), x \in \mathbb{S}^{n-1} \}.$$  

Let us define the minimum and the maximum distance of $\partial K$ from the origin as follows

$$R_m = \min_{\mathbb{S}^{n-1}} \rho_K(x), \quad R_M = \max_{\mathbb{S}^{n-1}} \rho_K(x). \quad (2.7)$$

Moreover if $f : \partial K \to \mathbb{R}$ is $\mathcal{H}^{n-1}$–integrable the following formula for the change of variable given by the radial map holds:

$$\int_{\partial K} f \, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} f(r_K(x)) \frac{\rho_K(x)}{h_K(\nu_K(r_K(x)))} \, d\mathcal{H}^{n-1}, \quad (2.8)$$
where $\nu_K(r_K(x))$ is the outer unit normal to $\partial K$ at the point $r_K(x) = x \rho_K(x)$. We have (see for example [Sc])

$$\nu_K(r_K(x)) = \frac{x \rho_K(x) - \n \rho_K(x)}{\sqrt{(\rho_K(x))^2 + (\n \rho_K(x))^2}},$$

where by $\n \rho_K$ we denote the the component of $\n \rho_{K}$ tangential to $S_{n-1}$. So, we observe that (2.8) is equivalent to

$$\int_{\partial K} f \, dH_{n-1} = \int_{S_{n-1}} f(r_K(x))(\rho_K(x))^{n-1}\left(1 + \left(\frac{|\n \rho_K(x)|}{\rho_K(x)}\right)^2\right) \, dH_{n-1}.$$  

The following result holds (see for instance [Ch], [HLYZ], [Sc]).

**Lemma 2.1.** Let $K_n$ and $K$ be bounded convex bodies containing the origin for any $n \in \mathbb{N}$ and such that $K_n \to K$ in the Hausdorff sense. For any $n \in \{0, 1, 2, \ldots\}$, let $h_{K_n}, \rho_{K_n}$ be the support function and the radial function $K_n$, respectively. Then the following statements hold

(i) Let $h_K$ be the support function of $K$ then

$$\sup_{x \in S_{n-1}} |h_{K_n}(x) - h_K(x)| \to 0.$$

(ii) Let $\rho_K$ the radial function of $K$ then

$$\sup_{x \in S_{n-1}} |\rho_{K_n}(x) - \rho_K(x)| \to 0.$$

(iii) Let $x \in \partial K$ and $x_n \in \partial K_n$, $n \in \mathbb{N}$, points where $\nu_K(x)$ and $\nu_{K_n}(x_n)$ are well defined and such that

$$\lim_{n \to +\infty} x_n = x.$$

Then

$$\lim_{n \to +\infty} \nu_{K_n}(x_n) = \nu_K(x).$$

By (2.8), Lemma 2.1 and the Lebesgue’s convergence Theorem we immediately get

**Theorem 2.2.** Let $K_n$ and $K$ be bounded convex bodies containing the origin for any $n \in \mathbb{N}$ and such that $K_n \to K$ in the Hausdorff sense. Let

$$f_n : \partial K_n \to \mathbb{R}, \quad f : \partial K \to \mathbb{R}$$

be $H_{n-1}$ measurable functions such that

(i) there exists $C > 0$ such that

$$\|f\|_{L^\infty(\partial K)} \leqslant C, \quad \|f_n\|_{L^\infty(\partial K_n)} \leqslant C, \quad \forall n \in +\mathbb{N},$$

(ii) if $x_n \in \partial K_n$ is such that $\lim_{n \to +\infty} x_n = x \in \partial K$, $f_n$ is defined in $x_n$ and

$$\lim_{n \to +\infty} f_n(x_n) = f(x).$$
Then
\[
\lim_{n \to +\infty} \int_{\partial K_n} f_n(x_n) \, d\mathcal{H}^{n-1} = \int_{\partial K} f(x) \, d\mathcal{H}^{n-1}.
\]

2.2. The Steklov-Dirichlet eigenvalue problem. Let \( R_1 > 0 \) and \( \Omega_0 \subset \mathbb{R}^n \) be an open bounded connected set with Lipschitz boundary and such that \( B_{R_1} \subset \Omega_0 \), that means \( \overline{B}_{R_1} \subset \Omega_0 \). Let us set \( \Omega := \Omega_0 \setminus \overline{B}_{R_1} \).

We denote the set of Sobolev functions on \( \Omega \) vanishing on \( \partial B_{R_1} \) by \( H^1_{\partial B_{R_1}}(\Omega) \), that is (see [ET]) the closure in \( H^1(\Omega) \) of the following set
\[
C^\infty_{\partial B_{R_1}}(\Omega) := \{ u | u \in C^\infty_0(\mathbb{R}^n), \text{spt}(u) \cap \partial B_{R_1} = \emptyset \}.
\]

Let us consider the following Steklov-Dirichlet eigenvalue problem in \( \Omega \)
\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= \sigma(\Omega)u \quad \text{on } \partial \Omega_0 \\
u &= 0 \quad \text{on } \partial B_{R_1},
\end{aligned}
\tag{2.9}
\]

where \( \nu \) is the outer unit normal to \( \partial \Omega_0 \). The spectrum of (2.9) is discrete and the eigenvalue can be ordered as follows
\[
0 < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \ldots
\]

In [PPS] the authors study the first eigenvalue of (2.9), which has the following variational characterization
\[
\sigma_1(\Omega) = \min_{\substack{v \in H^1_{\partial B_{R_1}}(\Omega) \\
v \not\equiv 0}} \frac{\int_{\Omega} |\nabla v|^2 \, dx}{\int_{\partial \Omega_0} v^2 \, d\mathcal{H}^{n-1}},
\tag{2.10}
\]

and they also prove the following result.

**Proposition 2.3.** Let \( R_1 > 0 \) and \( \Omega_0 \subset \mathbb{R}^n \) be an open bounded connected set with Lipschitz boundary and such that \( B_{R_1} \subset \Omega_0 \) and let \( \Omega := \Omega_0 \setminus \overline{B}_{R_1} \). There exists a function \( u \in H^1_{\partial B_{R_1}}(\Omega) \) which achieves the minimum in (2.10) and it is a weak solution to the problem (2.9). Moreover \( \sigma_1(\Omega) \) is simple and the first eigenfunctions have constant sign in \( \Omega \).

In particular they prove the following upper bound
\[
\sigma_1(\Omega) \leq C(n, R_1, V(\Omega)) V^{\frac{1}{n}}(\Omega),
\tag{2.11}
\]

where
\[
C(n, R_1, V(\Omega)) = \frac{2}{n \omega_n^\frac{4}{n} \left( \frac{V(\Omega)}{2 \omega_n} + R_1^\frac{n}{n-1} \right)^\frac{1}{n} - R_1}.
\]
As a consequence, the first Steklov-Dirichlet eigenvalue remains bounded from above, when the volume of $\Omega$ and $R_1$ are fixed.

Obviously $\sigma_1(\Omega)$ is bounded also when we fix the perimeter of $\Omega$, that is equivalent to fix the perimeter of $\Omega_0$, instead of the volume. Indeed by (2.11) and the isoperimetric inequality, we can deduce the following upper bound

$$\sigma_1(\Omega) \leq \frac{2V_{n/2}(\Omega)}{n\omega_n^{1/2} \left( \frac{V(\Omega)}{2\omega_n} + R_1^n \right)^{1/2} - R_1} \leq C(n) \frac{P^{1/(n-1)}(\Omega_0)}{R_1^2},$$

(2.12)

where $C(n)$ is a positive constant that depends only on the dimension $n$.

In particular the following scaling property for $\sigma_1(\Omega)$ holds:

$$\sigma_1(t\Omega) = \frac{1}{t} \sigma_1(\Omega), \quad \forall t > 0.$$

Now we recall some known results about the first Steklov-Dirichlet eigenvalue when $\Omega$ is a spherical shell. In this case, in [VS] the authors find the explicit expression of the first eigenfunction.

**Proposition 2.4.** Let $A_{R_1,R_2}$ be the spherical shell with radii $R_2 > R_1 > 0$. The first eigenfunction associated to $\sigma_1(A_{R_1,R_2})$, is a radially symmetric, positive, strictly increasing function and it is given by

$$w(r) = \begin{cases} 
\ln r - \ln R_1 & \text{for } n = 2 \\
\frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} & \text{for } n \geq 3,
\end{cases}$$

(2.13)

with $r = |x|$. The corresponding first Steklov-Dirichlet eigenvalue can be computed and it is the following

$$\sigma_1(A_{R_1,R_2}) = \begin{cases} 
\frac{1}{R_2 \log \left( \frac{R_2}{R_1} \right)} & \text{for } n = 2 \\
\frac{R_2 \left( \frac{R_2}{R_1} \right)^{n-2} - 1}{n-2} & \text{for } n \geq 3.
\end{cases}$$

(2.14)

**Remark 2.5.** We point out that by (2.14), we have that $\sigma_1(A_{R_1,R_2})$ is increasing with respect to the radius of the inner ball, $R_1$, that is

$$\sigma_1(A_{R_1,R_2}) < \sigma_1(A_{r_1,R_2}), \quad \text{if } r_1 > R_1.$$ 

Moreover it holds

$$\lim_{R_1 \to 0} \sigma_1(A_{R_1,R_2}) = 0,$$

(2.15)

that is $\sigma_1(A_{R_1,R_2})$ tends to the first trivial Steklov eigenvalue of the Laplacian for $R_1$ which goes to zero. Finally we stress that an easy computation gives that $\sigma_1(A_{R_1,R_2})$ is decreasing with respect to the external radius $R_2$, that is

$$\sigma_1(A_{R_1,R_2}) < \sigma_1(A_{r_1,R}), \quad \text{if } \tilde{R} < R_2.$$
3. UPPER AND LOWER BOUNDS FOR $\sigma_1(\Omega)$ AND EXISTENCE RESULT

In this Section we prove an upper and lower bound for $\sigma_1(\Omega)$ in terms of $R_m$ and $R_M$, that are the minimal and maximal distance from the origin of the outer boundary as defined in (2.7). Then, we prove an existence results for a maximizer among convex sets with fixed inner ball and fixed volume and we also generalize it in the case of a suitable not spherical hole.

3.1. Estimates in terms of $R_m$ and $R_M$. The proof follows an idea used in [KS] for the planar case and in [GM, V] for any dimension to obtain a lower bounds for the first Steklov Laplacian eigenvalue.

**Theorem 3.1.** Let $R_1 > 0$ and $\Omega_0 \subset \mathbb{R}^n$ be an open bounded connected set with Lipschitz boundary such that $B_{R_1} \subset \Omega_0$ and let $\Omega = \Omega_0 \setminus \overline{B}_{R_1}$. Then, it holds

$$\frac{1}{\max_{\mathbb{S}^{n-1}} \left( \sqrt{1 + \frac{\|\nabla \tau \rho_0\|^2}{\rho_0^2}} \right)} \left( \frac{R_m}{R_M} \right)^{n-1} \sigma_1(A_{R_1,R_m}) \leq \sigma_1(\Omega) \leq \left( \frac{R_M}{R_m} \right)^{n-1} \sigma_1(A_{R_1,R_m}),$$

where $R_m$ and $R_M$ are defined in (2.7), $\rho_0$ is the radial function of $\Omega_0$ defined in (2.4) and $A_{R_1,R_m}$ is the spherical shell with radii $R_1$ and $R_m$.

Moreover, the equality case holds if and only if $\Omega$ is a ball $B_R$ centered at the origin of radius $R > 0$.

**Proof.** Let $u \in H^1_{\partial B_{R_1}}(\Omega)$ be a positive eigenfunction for $\sigma_1(\Omega)$, then

$$\sigma_1(\Omega) = \frac{\int_{\Omega} \|\nabla u\|^2 \, dx}{\int_{\partial \Omega_0} u^2 \, d\mathcal{H}^{n-1}}.$$  

(3.2)

By using spherical coordinates and the notation introduced in Section 2:

$$\partial \Omega_0 = \{ x \rho_0(x), x \in \mathbb{S}^{n-1} \},$$

the denominator in (3.2) becomes

$$\int_{\partial \Omega_0} u^2 \, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} u^2 \sqrt{1 + \left( \frac{\|\nabla \tau \rho_0\|}{\rho_0} \right)^2} (\rho_0)^{n-1} \, d\mathcal{H}^{n-1}.$$  

Then, we have

$$(R_m)^{n-1} \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1} \leq \int_{\partial \Omega_0} u^2 \, d\mathcal{H}^{n-1} \leq (R_M)^{n-1} \max_{\mathbb{S}^{n-1}} \left( \sqrt{1 + \frac{\|\nabla \tau \rho_0\|^2}{\rho_0^2}} \right) \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1}.$$  

(3.3)

Let us now take into account the numerator in (3.2). Since

$$\Omega = \{ s \in \mathbb{R}^n : s = x r, x \in \mathbb{S}^{n-1}, R_1 \leq r \leq \rho_0(x) \},$$

...
Remark 3.2. We observe that the lower bound in (3.1) gives that

\[ \int_{\Omega} |\nabla u|^2 \, ds = \int_{U} \int_{R_{1}} \left\{ \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_{r} u|^2 \right\} r^{n-1} \sqrt{g} \, dr \, dy, \quad (3.4) \]

where \( \sqrt{g} \) is the determinant of the matrix \( \tilde{g}_{ij} \), that is the standard metric on \( S^{n-1} \) and \( \nabla_{r} u \) is the component of \( \nabla u \) tangential to \( S^{n-1} \). Then, we have

\[ \int_{U} \int_{R_{1}} \left\{ \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_{r} u|^2 \right\} r^{n-1} \sqrt{g} \, dr \, dy, \leq \int_{\Omega} |\nabla u|^2 \, ds \leq \]

\[ \int_{U} \int_{R_{1}} \left\{ \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_{r} u|^2 \right\} r^{n-1} \sqrt{g} \, dr \, dy. \quad (3.5) \]

Combining (3.3) and (3.5) and recalling (3.2), we get

\[ \frac{1}{\max_{S^{n-1}} \left( \sqrt{1 + \frac{|\nabla_{r} \rho_{0}|^2}{\rho_{0}^2}} \right)} \left( \frac{R_{m}}{R_{M}} \right)^{n-1} \sigma_{1}(A_{R_{1},R_{m}}) = \]

\[ \int_{U} \int_{R_{1}} \left\{ \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_{r} u|^2 \right\} r^{n-1} \sqrt{g} \, dr \, dy, \leq \sigma_{1}(\Omega) \leq \]

\[ (R_{M})^{n-1} \max_{S^{n-1}} \left( \sqrt{1 + \frac{|\nabla_{r} \rho_{0}|^2}{\rho_{0}^2}} \right) \int_{S^{n-1}} u^2 \, dH^{n-1} \]

\[ \frac{\int_{U} \int_{R_{1}} \left\{ \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_{r} u|^2 \right\} r^{n-1} \sqrt{g} \, dr \, dy,}{(R_{m})^{n-1} \int_{S^{n-1}} u^2 \, dH^{n-1}} = \left( \frac{R_{M}}{R_{m}} \right)^{n-1} \sigma_{1}(A_{R_{1},R_{m}}). \quad (3.6) \]

Finally, we stress that the equality case implies that all the inequalities become equalities. So, we have that \( \nabla_{r} \rho_{0} = 0 \) and \( \rho_{0}(x) = R \), with \( R > R_{1} \) constant. \( \square \)

Remark 3.2. We observe that the lower bound in (3.1) gives that \( \sigma_{1}(\Omega) > 0 \) being \( R_{1} > 0 \) fixed. Moreover, (3.1) also implies a continuity results: \( \sigma_{1}(\Omega) \to 0 \) as \( R_{1} \to 0 \).

3.2. An upper bound for \( \sigma_{1}(\Omega) \) for not spherical hole. In this subsection we prove an upper bound in the case of a not spherical hole. Let \( K \subset \mathbb{R}^{n} \) be a convex body such that \( K \subseteq \Omega_{0} \) and let \( \Omega_{K} = \Omega_{0} \backslash \overline{K} \). In this case, according to [ET], the natural space of functions that we have to consider to define \( \sigma_{1}(\Omega_{K}) \) are \( C^{\infty}_{\partial K}(\Omega_{K}) \) and \( H^{1}_{\partial K}(\Omega_{K}) \). In particular the classical arguments of Calculus of Variations apply, as showed in [PPS], and \( \sigma_{1}(\Omega_{K}) \) is well defined.
Let us now assume that the volume $|\Omega| = \omega$ and the inradius $\rho(K) = \tilde{r}$ of $K$ are fixed. Let us consider $A_{\tilde{r}, \tilde{R}}$ the spherical shell with radii $\tilde{r}$ and $\tilde{R}$, where $\tilde{R}$ is such that $|A_{\tilde{r}, \tilde{R}}| = |\Omega|/2$. So, we have

$$\tilde{R} = \left(\frac{|\Omega|}{2\omega_n} + \tilde{r}^n\right)^{1/n}. \quad (3.7)$$

We also consider the following test function $\varphi : \mathbb{R}^n \setminus K \to [0, \infty)$:

$$\varphi(x) = \begin{cases} d_K(x) & \text{if } 0 \leq d_K(x) \leq \tilde{R} \\ \tilde{R} & \text{if } d_K(x) \geq \tilde{R} \end{cases}, \quad (3.8)$$

where

$$d_K(x) := \inf_{y \in \partial K} ||x - y||.$$ 

and we denote by $K_t$ the set

$$K_t = \{ x \in \mathbb{R}^n \setminus K : d_K(x) < t \}. \quad (3.9)$$

We have now to distinguish two cases. If $K_{\tilde{R}} \subseteq \Omega$, then, using the test function (3.8) in the variational characterization, we have

$$\sigma_1(\Omega) \leq \int_{K_{\tilde{R}}} |\nabla d_K|^2 \, dx \leq \frac{|K_{\tilde{R}}|}{\tilde{R}^2 P(\Omega_0)} \leq \frac{|\Omega|}{\tilde{R}^2 n\omega_n^{1/n}|\Omega|^{1-1/n}}$$

$$= \frac{|\Omega|^{1/n}}{n\omega_n^{1/n} \left(\frac{|\Omega|}{2\omega_n} + \tilde{r}^n\right)^{2/n}} = C(n, \tilde{r}, |\Omega|),$$

where we have used the fact that $|\nabla d_K(x)| = 1$ a.e., the classical isoperimetric inequality and (3.7).

Finally, let us consider the case when $K_{\tilde{R}} \not\subseteq \Omega$. We will use the following notations: $\partial^c \Omega_0 = \partial \Omega_0 \cap K_{\tilde{R}}$ and $\partial^c \Omega_0 = \partial \Omega_0 \setminus \partial^c \Omega_0$. Using as before the test function (3.8), we have

$$\sigma_1(\Omega) \leq \frac{\int_{K_{\tilde{R}} \cap \Omega} |\nabla d_K|^2 \, dx}{\int_{\partial^c \Omega_0} d_K^2 d\mathcal{H}^{n-1} + \int_{\partial^c \Omega_0} \tilde{R}^2 d\mathcal{H}^{n-1}} \leq \frac{|K_{\tilde{R}} \cap \Omega|}{R^2 |\partial^c \Omega_0|} \leq \frac{2|\Omega|}{R^2 n\omega_n^{1/n}|\Omega|^{1-1/n}} = 2C(n, \tilde{r}, |\Omega|), \quad (3.10)$$

where we have used the relative isoperimetric inequality (see [PPS, Proposition 2.4] and the references therein).
3.3. The existence result. Inequality (2.11) ensures that the Steklov-Dirichlet eigenvalue \( \sigma_1(\Omega) \), defined in (1.1), is bounded from above if the volume of \( \Omega \) is fixed. In this section we prove the existence of a maximizer among convex sets with fixed internal ball and fixed volume. Let \( \omega > 0 \) and \( R_1 > 0 \) be fixed, then by \( A_{R_1}(\omega) \) we will denote the class of convex sets having measure \( \omega \) and containing the ball \( B_{R_1} \), that is
\[
A_{R_1}(\omega) := \{D = K \setminus B_{R_1}, K \subseteq \mathbb{R}^n \text{ open, bounded, convex}: B_{R_1} \subseteq K, V(D) = \omega\}.
\]

The main theorem of this section is the following existence result.

**Theorem 3.3.** Let \( \omega > 0 \) and \( R_1 > 0 \) be fixed. There exists a set \( E \in A_{R_1}(\omega) \), such that
\[
\max_{D \in A_{R_1}(\omega)} \sigma_1(D) = \sigma_1(E).
\]

**Proof.** The upper bound (2.11) implies that there exists \( M > 0 \) such that
\[
\sup_{D \in A_{R_1}(\omega)} \sigma_1(D) = M < +\infty.
\]

Hence, there exists a sequence \( \{E_k\}_{k \in \mathbb{N}} \subseteq A_{R_1}(\omega) \) such that
\[
\lim_{k \to \infty} \sigma_1(E_k) = M.
\]

In order to show the desired result, we need to prove the existence of a set \( E \in A_{R_1}(\omega) \) such that \( E_k \overset{h}{\to} E \) with \( \sigma_1(E) = M \).

Firstly we prove that, up to a subsequence, \( \{E_k\}_{k \in \mathbb{N}} \) converges to a certain \( E \in A_{R_1}(\omega) \) in the Hausdorff metric.

Being \( \{E_k\}_{k \in \mathbb{N}} \subseteq A_{R_1}(\omega) \) then, for every \( k \in \mathbb{N} \) there exists a convex set \( E_{0,k} \), such that \( B_{R_1} \subseteq E_{0,k} \),
\[
E_k = E_{0,k} \setminus B_{R_1}
\]
and
\[
\omega_0 := V(E_{0,k}) = \omega + \omega_n R_1^n.
\]

By the Blaschke selection Theorem and the continuity of the volume functional with respect to the Hausdorff measure (see Sc as a reference), it is enough to show that \( \{E_{0,k}\}_{k \in \mathbb{N}} \) is equibounded.

We proceed by contradiction assuming that
\[
\lim_{k \to +\infty} \text{diam}(E_{0,k}) = +\infty. \tag{3.11}
\]

Inequality (2.2) gives
\[
\rho(E_{0,k}) \leq \frac{nV(E_{0,k})}{P(E_{0,k})}, \tag{3.12}
\]
where \( \rho(E_{0,k}) \) is the inradius of \( E_{0,k} \) defined in (2.1).
The assumption (3.11) and the inequality (2.3) imply that the right-hand side in (3.12) tends to 0 as $k \to +\infty$, being $V(E_{0,k})$ fixed. Therefore, by (3.12), we have
\[
\lim_{k \to +\infty} \rho(E_{0,k}) = 0,
\]
which is in contradiction with
\[
0 < R_1 < \rho(E_{0,k}).
\]
Hence, the equiboundeness is proved and then $\{E_k\}_{k \in \mathbb{N}}$ converges up to a subsequence to a set $E \in A_{R_1}(\omega)$ in the Hausdorff metric. Hence, by the definition of $A_{R_1}(\omega)$, there exists an open bounded convex set $E_0$ such that $E = E_0 \setminus \overline{B}_{R_1}$.

In order to complete the proof, we will prove that
\[
M = \lim_{k} \sigma_1(E_k) \leq \sigma_1(E).
\]
Let $u \in H^1_{\chi_B R_1}(E)$ be the first positive eigenfunction associated to $\sigma_1(E)$, such that
\[
\int_{\partial E_0} u^2 \, d\mathcal{H}^{n-1} = 1.
\]
Hence, we have
\[
\sigma_1(E) = \int_E |\nabla u|^2 \, dx.
\]
By the extension theorem (see for instance [C, S] for Lipschitz domains), we can extend $u$ in $\mathbb{R}^n$ obtaining a function $\tilde{u} \in H^1_{\chi_B R_1}(\mathbb{R}^n)$ such that $\tilde{u} = u$, a.e. in $E$, and
\[
\|\tilde{u}\|_{H^1_{\chi_B R_1}(\mathbb{R}^n)} \leq c(n)\|u\|_{H^1_{\chi_B R_1}(E)},
\]
for some positive constant $c = c(n)$. For every $k \in \mathbb{N}$ we define $u_k$ as the restriction of $\tilde{u}$ in $E_k$. Using $u_k$ as a test function for $\sigma_1(E_k)$, we have
\[
\sigma_1(E_k) \leq \frac{\int_{E_k} |\nabla \tilde{u}|^2 \, dx}{\int_{\partial E_{0,k}} \tilde{u}^2 \, d\mathcal{H}^{n-1}}.
\]
In order to get (3.13), we prove that the right-hand side in (3.14) converges to $\sigma_1(E)$. We observe that
\[
\int_{E_k} |\nabla \tilde{u}|^2 \, dx - \int_E |\nabla \tilde{u}|^2 \, dx = \int_{\mathbb{R}^n} (\chi_{E_k} - \chi_E) |\nabla \tilde{u}|^2 \, dx \to 0,
\]
since $E_k \to E$ in the Hausdorff metric and by the dominated convergence theorem.

In order to conclude the proof we have to prove
\[
\int_{\partial E_{0,k}} \tilde{u}^2 \, d\mathcal{H}^{n-1} \to \int_{\partial E_0} u^2 \, d\mathcal{H}^{n-1} = 1.
\]
The equiboundedness of the sequence \( \{ E_{0,k} \}_{k \in \mathbb{N}} \) guarantees the existence of a ball \( B_R \) centered at the origin with radius \( R > 0 \) such that \( E_{0,k} \subset B_R \), for every \( k \in \mathbb{N} \). Extending \( \tilde{u} \) to zero in \( B_{R_1} \) and by using an approximation argument, we can suppose that \( \tilde{u} \in C^\infty(B_R) \). Then (3.16) follows by Theorem 2.2.

Finally, passing to the limit in (3.14), by (3.15) and (3.16), we get (3.13), that is

\[
M \leq \sigma_1(E)
\]

and, consequently, we can conclude that

\[
\sigma_1(E) = M,
\]

obtaining the desired claim. \( \square \)

**Remark 3.4.** We observe that the above existence result holds even when we consider \( \Omega_K = \Omega_0 \setminus K \), where \( K \) is a convex body strictly contained in \( \Omega_0 \). Indeed, by using the upper bound (3.10), the proof can be done following line by line the one just discussed in the case of a spherical hole.

### 4. PROOF OF THE MAIN RESULT

In this section we give the proof of the main result. The idea is to take as test function in the quotient (2.10) the eigenfunction of the spherical shell with the same measure as \( \Omega \). Before giving the proof, we need a preliminary result.

**Lemma 4.1.** Let \( R_1 > 0 \) and let \( f \) be the function defined in \( \mathbb{R}^+ \) as

\[
f(t) = \begin{cases} 
\log^2 \left( R_1^2 / t \right) \sqrt{t} & n = 2 \\
\left( \frac{1}{R_1^2} - \frac{1}{t} \right)^2 \frac{t^{n-1}}{n} & n \geq 3.
\end{cases}
\]

Then, \( f \) is convex for every \( \alpha_- (n) R_1^n \leq t \leq \alpha_+ (n) R_1^n \), where

\[
\alpha_\pm (n) = \begin{cases} 
e^{\pm \sqrt{2}} & n = 2 \\
\left( \frac{(n-1) \pm (n-2) \sqrt{2(n-1)}}{n-1} \right)^{n} & n \geq 3.
\end{cases}
\]

**Proof.** Let us begin with the bidimensional case. After an easy computation one can see that

\[
f''(t) = \frac{2 - \log^2 \left( \sqrt{t} / R_1 \right)}{4t \sqrt{t}},
\]

which gives immediately the conclusion.

Now let us consider \( n \geq 3 \). After some computations the second derivative of the function is the following

\[
f''(t) = t^{\frac{3}{n} - 3} \left[ \frac{R_1^{4-2n}}{n} \left( \frac{1}{n} - 1 \right) t^{2 - \frac{4}{n}} + \frac{2R_1^{2-n}}{n} \left( 1 - \frac{1}{n} \right) t^{1 - \frac{2}{n}} + \left( \frac{3}{n} - 2 \right) \left( \frac{3}{n} - 1 \right) \right].
\]
If we call \( y = t^{1 - \frac{2}{n}} \), the previous function is non-negative if and only if
\[
g(y) = \frac{R_1^{4 - 2n}}{n} \left( \frac{1}{n} - 1 \right) y^2 + \frac{2R_1^{2 - n}}{n} \left( 1 - \frac{1}{n} \right) y + \left( \frac{3}{n} - 2 \right) \left( \frac{3}{n} - 1 \right) \geq 0.
\]
It is not difficult to check that the zeros of \( g(y) \) are
\[
y_{\pm} = R_1^{n - 2} \left( n - 1 \pm (n - 2)\sqrt{2(n - 1)} \right).
\]
Being \( y_{-} = 0 \) for \( n = 3 \) and \( y_{-} < 0 \) for every \( n \geq 4 \) it must be \( y_{-} \leq y \leq y_{+} \), which concludes the proof. \( \square \)

Now we can prove the main result.

**Proof of the Theorem 1.1.** Let us consider the fundamental solution \( w \), given in (2.13), as a test function in (2.10). Then,
\[
\sigma_1(\Omega) \leq \frac{\int_{\Omega} |\nabla w|^2 \, dx}{\int_{\partial \Omega} w^2 \, d\mathcal{H}^{n-1}}
\]
In order to prove the result we will show that
\[
\int_{\Omega} |\nabla w|^2 \, dx \leq \int_{A_{R_1, n_2}} |\nabla w|^2 \, dx = \sigma_1(A_{R_1, n_2}). \tag{4.1}
\]
Since \( |\nabla w|^2 \) is a non-negative radially symmetric decreasing function for any \( n \geq 2 \), it coincides with its Schwarz symmetrization. Hence by the Hardy-Littlewood inequality \([K, \text{Th. 1.2.2}]\), we have
\[
\int_{\Omega} |\nabla w|^2 \, dx = \int_{\Omega_0} |\nabla w|^2 \, dx - \int_{B_{R_1}} |\nabla w|^2 \, dx \leq \int_{B_{R_2}} |\nabla w|^2 \, dx - \int_{B_{R_1}} |\nabla w|^2 \, dx = \int_{A_{R_1, n_2}} |\nabla w|^2 \, dx. \tag{4.2}
\]
Hence, it remains to prove the following inequality
\[
\int_{\partial \Omega_0} w^2 \, d\mathcal{H}^{n-1} \geq \int_{\partial B_{R_2}} w^2 \, d\mathcal{H}^{n-1}. \tag{4.3}
\]
Let \( \rho_0 \) be the radial function of \( \Omega_0 \) defined in (2.4). By (2.6), \( \partial \Omega_0 \) can be represented as follows
\[
\partial \Omega_0 = \{ x \rho_0(x), x \in \mathbb{S}^{n-1} \},
\]
with $R_1 < \rho_0(\theta) \leq \tilde{R}$ and $\tilde{R}$ defined in (1.3).

Firstly, let us consider the case $n = 2$. If we denote by $z(\theta) = R^2(\theta) = \rho_0^2(x(\theta))$, being $V(\Omega_0) = V(B_{R_2})$, it holds

$$R_2 = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} z(\theta) d\theta}. \quad (4.4)$$

Moreover, we get

$$\int_{\partial \Omega_0} w^2 \, ds = \int_{\partial \Omega_0} (\log(|x|) - \log R_1)^2 \, ds = \int_0^{2\pi} \log^2 \left( \frac{R(\theta)}{R_1} \right) R(\theta) \sqrt{1 + \left( \frac{R'(\theta)}{R(\theta)} \right)^2} \, d\theta \geq \int_0^{2\pi} \log^2 \left( \frac{R(\theta)}{R_1} \right) R(\theta) \, d\theta = \int_0^{2\pi} \log^2 \left( \frac{\sqrt{z(\theta)}}{R_1} \right) z(\theta) \, d\theta$$

$$\geq 2\pi \log^2 \left( \frac{R_2}{R_1} \right) \sqrt{\int_0^{2\pi} \frac{z(\theta) \, d\theta}{2\pi}} = 2\pi R_2 \log^2 \left( \frac{R_2}{R_1} \right) = \int_{\partial B_{R_2}} w^2 \, ds,$$  

(4.5)

where, since $\rho_0(x) \leq \tilde{R}$, the last inequality follows by Lemma 4.1 and by Jensen’s inequality. This conclude the proof of (4.3) in the bidimensional case.

Now, let us consider the case $n \geq 3$ and we proceed in a similar way. Moreover since

$$V(\Omega_0) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_0^n(x) \, d\mathcal{H}^{n-1}$$

and being $V(\Omega_0) = V(B_{R_2})$, it holds

$$R_2 = \left( \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} z(x) \, d\mathcal{H}^{n-1} \right)^{\frac{1}{n}}, \quad (4.6)$$
where \( z(x) = \rho_0^0(x) \). Then, we have
\[
\int_{\partial \Omega_0} w^2 \, d\mathcal{H}^{n-1} = \int_{\partial \Omega_0} \left( \frac{1}{R_1^{n-2}} - \frac{1}{|x|^{n-2}} \right)^2 \, d\mathcal{H}^{n-1}
\]
\[
= \int_{\mathbb{S}^{n-1}} \left( \frac{1}{R_1^{n-2}} - \frac{1}{(\rho_0(x))^{n-2}} \right)^2 \left( \rho_0(x) \right)^{n-1} \sqrt{1 + \left( \frac{\nabla \rho_0(x)}{\rho_0(x)} \right)^2} \, d\mathcal{H}^{n-1}
\]
\[
\geq \int_{\mathbb{S}^{n-1}} \left( \frac{1}{R_1^{n-2}} - \frac{1}{(z(x))^{n-2}} \right)^2 \left( z(x) \right)^{n-1} \, d\mathcal{H}^{n-1}
\]
\[
\geq n \omega_n \left[ \frac{1}{R_1^{n-2}} - \frac{n \omega_n}{\left( \int_{\mathbb{S}^{n-1}} z(x) \, d\mathcal{H}^{n-1} \right)^{\frac{n-2}{n}}} \right]^2 \left( \int_{\mathbb{S}^{n-1}} z(x) \, d\mathcal{H}^{n-1} \right)^{\frac{n-1}{n}}
\]
\[
= n \omega_n \left( \frac{1}{R_1^{n-2}} - \frac{1}{R_2^{n-2}} \right)^2 R_2^{n-1} = \int_{\partial B_{R_2}} w^2 \, d\mathcal{H}^{n-1}.
\]

where last inequality follows by Lemma 4.1 and by Jensen’s inequality, being \( \rho_0(x) \leq R \). This gives (4.3) for \( n \geq 3 \) and concludes the proof. \( \square \)

5. Some remarks about the perimeter constraint

The estimate (2.12) states that the first Steklov-Dirichlet eigenvalue is bounded from above also when we keep the outer perimeter and the radius of the inner ball fixed. So, it is natural to investigate if there exists a set which maximizes \( \sigma_1(\Omega) \) in the following class

\[ B_{R_1}(\kappa) := \{ D = K \setminus \overline{B_{R_1}}, \, K \subset \mathbb{R}^n, \text{ open, convex : } B_{R_1} \subseteq K, \, P(K) = \kappa \}, \]

where \( R_1 > 0 \) and \( \kappa > n \omega_n R_1^{-n-1} \). Arguing as Theorem 3.3, we obtain the following existence result under a perimeter constraint.

**Theorem 5.1.** Let \( \kappa > n \omega_n R_1^{-n-1} \) be fixed. There exists a set \( \Omega \in B_{R_1}(\kappa) \) such that

\[ \sup_{D \in B_{R_1}(\kappa)} \sigma_1(D) = \sigma_1(\Omega). \]

**Remark 5.2.** We stress that inequality (1.2) continues to hold true even if we fix the perimeter of \( \Omega_0 \). Indeed the isoperimetric inequality ensures that the ball \( B_{R_2} \) centered at the origin and having the same measure than \( \Omega_0 \) is contained in the ball centered at the origin and having the same perimeter than \( \Omega_0 \).

On the other hand we cannot prove, instead, the inequality (4.3) under the perimeter constraint in order to obtain that the spherical shell is still a maximum for \( \sigma_1(\Omega) \). Indeed, if we proceed as in the proof of Theorem 4.1, for instance in the planar case, equation (4.1) has to be replaced by the following inequality:

\[ 2 \pi R_2 = P(B_{R_2}) = P(\Omega_0) = \int_0^{2\pi} R(\theta) \sqrt{1 + \left( \frac{R'(\theta)}{R(\theta)} \right)^2} \, d\theta \geq \int_0^{2\pi} R(\theta) \, d\theta, \quad (5.1) \]
where $R(\theta) = \rho_0(x(\theta))$ Then, in the last step of (1.5), after using Jensen’s inequality, we do not obtain the first Steklov-Dirichlet eigenvalue of the spherical shell, since (5.1) is not an equality.

In support of this fact, we give the following numerical counterexample obtained by using Maxima. We consider $R_1 = 10^{-5}$ and $\Omega_0$ an ellipse with the same perimeter as $A_{R_1,1}$. Let $a$ and $b$ the semi-axes of the ellipse. In order to compute the integral over the ellipse, we used the formula $P(\Omega_0) = 2\pi \sqrt{a^2 + b^2}$, which is an approximation by excess for the perimeter of the ellipse. Here we have chosen $b = 1.1$. We obtain

$$D(A_{R_1,1}) \approx 832,820208 > 828,919156 \approx D(\Omega_0),$$

where $D(\Omega_0) = \int_{\partial \Omega_0} w^2 \, ds$ and $w$ is the fundamental solution defined in (2.13).

This means that we cannot study separately the numerator and denominator terms to obtain inequality (1.4) under perimeter constraint.

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