MALLIAVIN CALCULUS FOR NON-COLLIDING PARTICLE SYSTEMS

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Abstract. In this paper, we use Malliavin calculus to show the existence and continuity of density functions of $d$-dimensional non-colliding particle systems such as hyperbolic particle systems and Dyson Brownian motion with smooth drift. For this purpose, we apply results proved by Florit and Nualart (1995) and Naganuma (2013) on locally non-degenerate Wiener functionals.

1. Introduction

1.1. Background. In the theory of stochastic differential equations (SDEs), the existence and regularity/smoothness of density functions with respect to the Lebesgue measure of solutions of SDEs is a major research topic for which there are many results and methodologies of study. Let us comment on the approach from parabolic partial differential equations (PDEs) and the approach from stochastic analysis.

Regarding the approach from parabolic PDEs, a fundamental solution of a PDE is known to exist if its coefficients are bounded and Hölder continuous and if its diffusion coefficient is uniformly elliptic (see Friedman [Fri64]). The fundamental solution is a density function of a solution to the corresponding SDE by the Feynman–Kac formula. The idea of the proof is based on Levi’s parametrix method (perturbation of the drift), which has been extended to a solution of an SDE with an $L^p$-valued drift coefficient (Portenko [Por90]) and a path-dependent drift coefficient (Kusuoka [Kus17] and Makhlof [Mak16]). The parametrix method leads to the differentiability (resp. Hölder continuity) of the density function with respect to the initial variable (resp. terminal variable).

On the other hand, as an approach from stochastic analysis, Malliavin calculus is a powerful tool, and it is well-known that, under the Hörmander condition, a solution of an SDE with infinitely differentiable coefficients has a smooth density function. Because there exists a criterion that non-degenerate Wiener functionals in the Malliavin sense admit smooth density functions with respect to the Lebesgue measure, we obtain the result by showing that the solution is non-degenerate. For general theory of Malliavin calculus and applications for solutions of SDEs, see [MT17, Shi04, Nua06, IW89]. However, the criterion cannot be applied to solutions of SDEs with singular coefficients. A squared Bessel process is a typical example.
of such SDEs; its diffusion coefficient is singular at the origin although the coefficient is locally smooth. Naganuma [Nag13] proposed an approach to access the squared Bessel process. He refined the notion of the local non-degeneracy of Wiener functionals introduced by Florit and Nualart [FN95] and showed that solutions of squared-Bessel-type SDEs (and therefore of Bessel-type SDEs) admit continuous density functions (see [Nag13, Theorem 2.2]). Note that inverse moments of the processes play a crucial role in the argument. As another approach, De Marco [DM11] showed the local existence of smooth density functions of solutions of SDEs if their coefficients are locally smooth.

In this paper, we consider non-colliding particle systems of Dyson type and show that it admits a continuous density function. A typical example of such a system is the $\beta$-Dyson Brownian motion, which describes the dynamics of non-colliding Brownian particles. More precisely, for $d \geq 2$ and $T > 0$, the $d$-dimensional Dyson Brownian motion $X = \{X(t) = (X_1(t), \ldots, X_d(t))^\top \}_{0 \leq t \leq T}$ with a parameter $\beta \geq 1$ is defined by a unique solution of an SDE

\[
\begin{aligned}
\begin{cases}
    dX_i(t) = \frac{\beta}{2} \sum_{k,k \neq i} \frac{dt}{X_i(t) - X_k(t)} + dW_i(t), & i = 1, \ldots, d, \\
    X(0) = \bar{x},
\end{cases}
\end{aligned}
\]

where $\bar{x}$ is a deterministic initial condition belonging to $\Delta_d = \{(x_1, \ldots, x_d)^\top \in \mathbb{R}^d; x_1 < \cdots < x_d\}$ and $W = \{W(t) = (W_1(t), \ldots, W_d(t))^\top \}_{0 \leq t \leq T}$ is a $d$-dimensional standard Brownian motion on the canonical probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}(t)_{0 \leq t \leq T}$ satisfying the usual conditions. The parameter $\beta$ is called the inverse temperature. It is known that the process $X$ with $\beta = 1, 2, 4$ is obtained as an eigenvalue process of some matrix-valued Brownian motion ([Dys62], [AGZ10], [Kat15], [Meh04]). Further, $X$ with $\beta = 2$ is obtained as a standard Brownian motion with the non-colliding condition or Doob’s $h$-transform of an absorbing Brownian motion in $\Delta_d$ ([Gra99], [Bia95]). Therefore, this process is studied using various methods.

The existence of density functions of Dyson Brownian motion has been studied in various ways. Because the Dyson Brownian motion with parameter $\beta = 1, 2, 4$ is obtained as an eigenvalue process of a matrix-valued Brownian motion, we see the existence of density functions and explicit forms (see [AGZ10, Theorem 2.5.2], [Meh04, Theorem 3.3.1]). For $\beta = 2$, we can also derive the density function in the context of Karlin-McGregor formula, Brownian motion with the non-colliding condition and Doob’s $h$-transform (see [Kat15, Theorem 3.3 and Equation (3.32)]). Meanwhile, as a general framework for an analytical approach to Dyson Brownian motion, a (radial) Dunkl process has been studied. The Dunkl process is a càdlàg Markov process with martingale property and its infinitesimal generator is the Dunkl Laplacian, which is a differential operator with a root system in $\mathbb{R}^d$ (for more details, see [GRY08]). Moreover, the semigroup density of the Dunkl process exists and can be express by the normalized spherical Bessel functions (see [Rös98]). The radial part of Dunkl process is a solution of some stochastic differential equation (see [GRY08, Corollary 6.6]) valued in the fundamental Weyl chamber of associated
root system and its semigroup density also exists and have some representation (and therefore a Dyson Brownian motion has a density for all $\beta \geq 1$ because it is a radial Dunkl process with the root system type $A$). It is worth noting that in the theory of Dunkl process, the diffusion coefficient $\sigma$ must be the identity matrix.

The aim of the present paper is to apply Malliavin calculus to non-colliding particle systems such as hyperbolic particle systems (see [CL01]) and Dyson Brownian motion with smooth drift. We prove that under some moment condition (see Assumption 1.1 below), solutions of non-colliding particle systems admit continuous density functions with respect to the Lebesgue measure. We use the result by Florit and Nualart [FN95] and Naganuma [Nag13] to deal with the singularity of the diffusion coefficient $(x_i - x_k)^{-1}$. As with Bessel-type processes, the inverse moment of $X_i(t) - X_k(t)$ plays an important role in the proof. We show the integrability by using the Girsanov transformation for non-colliding particle systems, which was inspired by Yor [Yor80] (this approach can also be found in [Chy06]).

1.2. Main Result. We treat an extension of (1.1) and consider the existence and continuity of the density function of its solution at time $t > 0$.

We consider a constant diffusion coefficient $\sigma = (\sigma_{ij})_{1 \leq i,j \leq d}$ that is an invertible matrix. Next, we introduce a drift coefficient $f = (f_1, \ldots, f_d)^T : \Delta_d \to \mathbb{R}^d$ consisting of a singular part $a$ and a smooth part $b$ as follows. Let $a = (\alpha_{ik})_{1 \leq i,j \leq d}$ be a symmetric matrix with non-negative components. For $i > j$ (resp., $i < j$), we set $I_{ij} = (0, \infty)$ (resp., $I_{ij} = (-\infty, 0)$). We define $\phi_{i,j} : I_{ij} \to I_{ij}$ by $\phi_{i,j}(\xi) = \alpha_{ij}/\xi$ for every $i \neq j$ and $a = (a_1, \ldots, a_d)^T : \Delta_d \to \mathbb{R}^d$ by

$$a_i(x) = \sum_{k,k \neq i} \phi_{ik}(x_i - x_k) = \sum_{k,k \neq i} \frac{\alpha_{ik}}{x_i - x_k}.$$ 

Let $b = (b_1, \ldots, b_d)^T : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth function which has bounded derivatives of all orders ($b$ itself need not be bounded). We set $f_i = a_i + b_i$.

For such coefficients $f$ and $\sigma$, we consider a solution $X = \{X(t)\}_{0 \leq t \leq T}$ to an SDE

$$dX_i(t) = f_i(X(t)) \, dt + \sum_{k=1}^d \sigma_{ik} \, dW_k(t), \quad i = 1, \ldots, d,$$

$$X(0) = \bar{x} \in \Delta_d$$

as an extension of (1.1).

Assumption 1.1. Let $0 < T < \infty$ be fixed.

1. Existence and uniqueness of a strong solution $X$ to SDE (1.3) such that $P(X(t) \in \Delta_d$ for all $0 \leq t \leq T) = 1$.

2. For some $q > d$, it holds that

$$\max_{1 \leq i,j \leq d} \sup_{0 \leq t \leq T} E[|X_i(t) - X_j(t)|^{-6d}] < \infty.$$

The following is our main theorem.

Theorem 1.2. Let $0 < t \leq T$. Under Assumption 1.1, the solution $X(t)$ admits a continuous density function with respect to the Lebesgue measure.
We prove this theorem by showing that $X(t)$ is locally non-degenerate in the sense of [FN95] and [Nag13]. For details, see Section 3. Note that we do not assume that the diffusion matrix $\sigma$ is identity matrix. For examples of such SDEs, see Corollary 1.5 below.

By similar arguments, we might show the smoothness under stronger assumption for $q$ in Assumption 1.1 (2). It is known that a Dyson Brownian motion can be considered with $X(0) = \bar{x} \in \Delta_d$. However, for proving Theorem 1.2, we need the inverse moment condition (1.4) thus the initial value $X(0) = \bar{x}$ is not in the boundary of $\Delta_d$.

Next, we propose and prove a criterion of Assumption 1.1. For this purpose, we need additional assumptions on $f = a + b$ and $\sigma$ as follows.

Assumption 1.3. (1) (a) $\sigma = I$, where $I$ is the identity matrix.
(b) $\alpha_{ik} = \alpha$ in (1.2).
(c) $b = \mu + c$, where $\mu = (\mu_1, \ldots, \mu_d)^\top$ and $c = (c_1, \ldots, c_d)^\top : \mathbb{R}^d \to \mathbb{R}^d$ are smooth functions such that

$\bullet$ $\mu_i$ depends only on the $i$th argument, that is, $\mu_i(x) = \tilde{\mu}_i(x_i)$ for some one-variable function $\tilde{\mu}_i$.

$\bullet$ all derivatives of $\mu_i$ are bounded ($\mu_i$ itself need not be bounded).

$\bullet$ $\mu_k(x) \geq \mu_l(x)$ for any $x = (x_1, \ldots, x_d) \in \Delta_d$ and $k > l$.

$\bullet$ $c_i$ is bounded together with all its derivatives.

(2) $\alpha > 6d + 1/2$.

For example, if $\mu_i(x) = \tilde{\mu}_i x_i$ for a constant $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_d)^\top$ with $\tilde{\mu}_k \geq \tilde{\mu}_l$ for $k > l$, then $\mu$ satisfies all the conditions. Then, we obtain the following.

Theorem 1.4. Let $0 < t \leq T$. Under Assumption 1.3, the solution $X$ exists and $X(t)$ admits a continuous density function with respect to the Lebesgue measure.

We comment on Theorem 1.4. We show in two steps that Assumption 1.3 implies Assumption 1.1 (1). First, we consider the case $c = 0$ and $\alpha \geq 1/2$, in which the results of [RS93, Lemma 1], [CL97, Theorem 3.1], and [GM14, Theorem 2.2, Corollary 6.2] ensure that (1.3) satisfies Assumption 1.1 (1). For general $c$ and $\alpha \geq 1/2$, see Proposition 4.1. Note that we can show Proposition 4.1 using only the existence and uniqueness of solutions of (1.3) with $c = 0$ and $\alpha = 1/2$. Next, we consider Assumption 1.1 (2). To ensure the condition holds, we need Assumption 1.3 (2); see Proposition 4.2. The proofs of Propositions 4.1 and 4.2 are based on the Girsanov transformation.

We obtain the following for non-identity diffusion matrix $\sigma$, which is not considered in the theory of Dunkl process.

Corollary 1.5. Let $0 < t \leq T$ and define $\sigma^2 \alpha := \max_{i=1, \ldots, d} \sum_{k=1}^d \sigma_{ik}^2$. We assume $\alpha_{ik} = \alpha > (6d + 1)d\sigma^2\alpha/3$ in (1.2) and $b = \mu$ satisfies Assumption 1.3 (c). Then the solution $X$ exists and $X(t)$ admits a continuous density function with respect to the Lebesgue measure.
Proof. The existence and strong uniqueness follows from [NT17, Theorem 3.6]. The assumption on $\alpha$ implies that there exists $q > d$ such that $6q < \frac{3\alpha}{\sqrt{d}} - 1$. Thus [NT17, Lemma 3.4] ensures Assumption 1.1 (2). \hfill \Box

Finally, we note that our framework covers hyperbolic particle systems. We set $\mu = 0$ and define $c = (c_1, \ldots, c_n)^\top : \mathbb{R}^d \to \mathbb{R}^d$ by

$$c_i(x) = \sum_{k, k \neq i} \alpha_{ik} \psi(x_i - x_k), \quad \text{where} \quad \psi(\xi) = \begin{cases} 0, & \xi = 0, \\ \coth \xi - \frac{1}{\xi}, & \xi \neq 0. \end{cases}$$

Because $\psi$ is smooth and all its derivatives are bounded, $c$ is smooth and all its derivatives are bounded. In addition, we obtain

$$f_i(x) = \sum_{k, k \neq i} \alpha_{ik} \left( \frac{1}{x_i - x_k} + \psi(x_i - x_k) \right) = \sum_{k, k \neq i} \alpha_{ik} \coth(x_i - x_k).$$

### 1.3. Notation and Structure.

In the present paper, we use the following notation. For $x, y \in \mathbb{R}^n$, $|x|$ and $(x, y)$ denote the Euclidian norm and the Euclidian inner product, respectively. Let $\text{Mat}_n(\mathbb{R})$ be the set of all real square matrices of size $n$ and $\text{Sym}_n(\mathbb{R})$ be the set of all real symmetric matrices of size $n$. We set $|A| = (\sum_{i,j=1}^n A_{ij}^2)^{1/2}$ for $A = (A_{ij})_{1 \leq i, j \leq n} \in \text{Mat}_n(\mathbb{R})$, which is a norm $\cdot$ on $\text{Mat}_n(\mathbb{R})$. Note that the norm $\cdot$ is sub-multiplicative; that is, $|AB| \leq |A||B|$. For $A \in \text{Mat}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$, $A^\top$ and $x^\top$ stand for the transposes of $A$ and $x$, respectively. The set of $\mathbb{R}^n$-valued continuous functions defined on $[0, T]$ is denoted by $C([0, T]; \mathbb{R}^n)$. For $1 < p < \infty$, $L^p([0, T]; \mathbb{R}^n)$ stands for the set of $\mathbb{R}^n$-valued power-$p$ integrable functions on $[0, T]$. For a smooth $\mathbb{R}^n$-valued function $g = (g_1, \ldots, g_n)^\top$ defined on $\mathbb{R}^n$, we write $g'_j = \partial g_j/\partial x_j$ and $g''_{ijk} = \partial^2 g_{ij}/\partial x_j \partial x_k$. We also regard $g' = (g'_j)_{1 \leq i, j \leq n}$ as a $\text{Mat}_n(\mathbb{R})$-valued function. The Kronecker delta is denoted by $\delta_{jk}$. An indicator function of a set $F$ is denoted by $1_F$.

This paper is structured as follows. In Section 2, we make some remarks on the drift coefficient $f$ and introduce approximating SDEs of (1.3). In Section 3, we apply Malliavin calculus to non-colliding particle systems by using the approximating SDEs introduced in Section 2. The proof of Theorem 1.2 is deferred to Section 3. In Section 4, we show Theorem 1.4. In Appendix A, we study some properties of matrix-valued ordinary differential equations (ODEs).

### 2. Preliminaries

#### 2.1. Remarks on the Drift Coefficient.

We show some properties of the singular part $a$ of the drift coefficient $f$ from (1.3).

**Lemma 2.1.** We have the following:

1. For all $n \in \mathbb{N} \cup \{0\}$ and $\xi \in I_{ij}$, we have $\frac{d^n}{d\xi^n}(\phi_{ij})(\xi) = (-1)^{n+1} \frac{d^n}{d\xi^n}(\phi_{ij})(-\xi)$.

2. For all $x \in \Delta_d$, $a'(x)$ is symmetric and given by

$$a'_{ij}(x) = \begin{cases} \sum_{l \neq i} \phi''_{il}(x_i - x_l), & i = j, \\ -\phi''_{jj}(x_i - x_j), & i \neq j. \end{cases}$$
Proof. We show Assertion (1) by induction. The case for \( n = 0 \) follows from the definition. Assume that the assertion holds for some \( n \). Then, we have

\[
\frac{1}{h} \left( \frac{d^n \phi_{ij}}{d \xi^n}(\xi + h) - \frac{d^n \phi_{ij}}{d \xi^n}(\xi) \right) = \frac{(-1)^{n+1}}{n!} \left( \frac{d^n \phi_{ij}}{d \xi^n}(-\xi - h) - \frac{d^n \phi_{ij}}{d \xi^n}(-\xi) \right).
\]

By letting \( h \to 0 \), we obtain the assertion for \( n + 1 \). The proof is complete. \( \square \)

Lemma 2.2. For any \( x \in \Delta_d \) and \( y, z \in \mathbb{R}^d \), we have

(2.1) \[
\sum_{i=1}^{d} x_i a_i(x) = \sum_{k,l,k>l} (x_k - x_l) \phi_{kl}(x_k - x_l) = \sum_{k,l,k>l} \alpha_{kl},
\]

(2.2) \[
\langle y, a'(x) y \rangle = \sum_{k,l,k>l} \phi'_{kl}(x_k - x_l)(y_k - y_l)^2 \leq 0,
\]

(2.3) \[
\sum_{j,k=1}^{d} a'_{ijk}(x) y_j z_k = \sum_{l, \ell \neq i} \phi''_{ll}(x_i - x_l)(y_i - y_l)(z_i - z_l).
\]

We show (2.1) and (2.2) by using the following identity:

(2.4) \[
\sum_{i=1}^{d} \xi_i \sum_{l \neq i} \eta_{li} = \sum_{k,l,k>l} \{ \xi_k \eta_{kl} + \xi_l \eta_{lk} \}
\]

for any \( \{ \xi_i \}_{1 \leq i \leq d} \) and \( \{ \eta_{lj} \}_{1 \leq i, j \leq d, i \neq j} \).

Proof of Lemma 2.2. We show (2.1). From (2.4) and Lemma 2.1 (1), we have

\[
\sum_{i=1}^{d} x_i a_i(x) = \sum_{i=1}^{d} x_i \sum_{l \neq i} \phi_{li}(x_i - x_l)
\]

\[
= \sum_{k,l,k>l} \{ x_k \phi_{kl}(x_k - x_l) + x_l \phi_{lk}(x_l - x_k) \}
\]

\[
= \sum_{k,l,k>l} (x_k - x_l) \phi_{kl}(x_k - x_l).
\]

We show (2.2). Lemma 2.1 (2) implies

\[
\langle a'(x)y \rangle_i = \sum_{j=1}^{d} a'_{ij}(x) y_j = \left( \sum_{l \neq i} \phi'_{li}(x_i - x_l) \right) y_i + \sum_{j \neq i} (-\phi'_{ij}(x_i - x_j)) y_j
\]

\[
= \sum_{l \neq i} \phi''_{li}(x_i - x_l)(y_i - y_l).
\]

This expression and (2.4) yield

\[
\langle y, a'(x)y \rangle = \sum_{i=1}^{d} y_i \langle a'(x)y \rangle_i = \sum_{i=1}^{d} y_i \left( \sum_{l \neq i} \phi''_{li}(x_i - x_l)(y_i - y_l) \right)
\]

\[
= \sum_{k,l,k>l} \{ y_k \phi_{kl}(x_k - x_l)(y_k - y_l) + y_l \phi_{lk}(x_l - x_k)(y_l - y_k) \}
\]

\[
= \sum_{k,l,k>l} \phi''_{kl}(x_k - x_l)(y_k - y_l)^2.
\]

Direct computation yields (2.3). The proof is complete. \( \square \)
2.2. Approximating SDEs. To apply Malliavin calculus to a solution of (1.3), we must consider how to approximate SDEs. For this purpose, we define for the drift coefficient $f$ a family of approximations $\{f^{(\epsilon)}\}_{0<\epsilon<1}$ on $\mathbb{R}^d$.

First, we define a family of functions $\{\rho_{\epsilon}\}_{0<\epsilon<1}$ that approximates the function $ho: \mathbb{R} \to \mathbb{R}$ defined by $ho(\xi) = \xi 1_{[0,\infty)}(\xi)$ as follows. We set

$$
\bar{\lambda}(\xi) = \begin{cases} e^{-1/\xi}, & \xi > 0, \\
0, & \xi \leq 0,
\end{cases}
$$

and define

$$
\rho_{\epsilon}(\xi) = \epsilon + \int_{\epsilon}^{\xi} \lambda_{\epsilon}(\eta) \, d\eta.
$$

Next, for $i \neq j$, we introduce $\{\phi^{(\epsilon)}_{ij}\}_{0<\epsilon<1}$ that approximates $\phi_{ij}$ by

$$
\phi^{(\epsilon)}_{ij}(\xi) = \begin{cases} \phi_{ij}(\epsilon) + \int_{\epsilon}^{\xi} \phi'_{ij}(\rho_{\epsilon}(\eta)) \, d\eta, & i > j, \\
\phi_{ij}(-\epsilon) + \int_{-\epsilon}^{\xi} \phi'_{ij}(\rho_{\epsilon}(\eta)) \, d\eta, & i < j.
\end{cases}
$$

Finally, we define $a^{(\epsilon)}(x) = (a^{(\epsilon)}_1, \ldots, a^{(\epsilon)}_d)^{\top}$ and $f^{(\epsilon)} = (f^{(\epsilon)}_1, \ldots, f^{(\epsilon)}_d)^{\top}: \mathbb{R}^d \to \mathbb{R}^d$ by

$$
a^{(\epsilon)}_i(x) = \sum_{k,k \neq i} \phi^{(\epsilon)}_{ik}(x_i - x_k),
$$

$$
f^{(\epsilon)}_i = a^{(\epsilon)}_i + b_i.
$$

To approximate the solution of (1.3), we consider a solution $X^{(\epsilon)}$ to an SDE

$$
(2.5) \quad \begin{cases} dX^{(\epsilon)}(t) = f^{(\epsilon)}(X^{(\epsilon)}(t)) \, dt + \sum_{k=1}^{d} \sigma_{ik} \, dW_k(t), & i = 1, \ldots, d, \\
X^{(\epsilon)}(0) = \bar{x}.
\end{cases}
$$

Note that this SDE admits a unique strong solution because $f^{(\epsilon)}$ is smooth and all its derivatives are bounded for all $0 < \epsilon < 1$.

Remark 2.3. The functions $\lambda_{\epsilon}$ and $\rho_{\epsilon}$ have the following properties.

- $0 \leq \lambda_{\epsilon}(\xi) \leq 1$ for $\xi \in \mathbb{R}$ and $\lambda_{\epsilon}(\xi) = 0$ (resp. $1$) for $\xi \leq 0$ (resp. $\epsilon \leq \xi$).
- $\rho_{\epsilon}$ is smooth and non-decreasing. We have

$$
\rho_{\epsilon}(\xi) = \begin{cases} \epsilon + \int_{\epsilon}^{\xi} \lambda_{\epsilon}(\eta) \, d\eta, & \xi \leq 0, \\
\xi, & \epsilon \leq \xi.
\end{cases}
$$

In addition, we obtain that $\xi \leq \rho_{\epsilon}(\xi)$ for $0 < \xi < \epsilon$.

Remark 2.4. Note that $\{\phi^{(\epsilon)}_{ij}\}_{0<\epsilon<1}$ and $\{a^{(\epsilon)}\}_{0<\epsilon<1}$ are good approximations of $\phi_{ij}$ and $a$, respectively. For example, $\phi^{(\epsilon)}_{ij}$ is smooth and non-increasing on $\mathbb{R}$. For $i > j$, $\phi^{(\epsilon)}_{ij}$ is expressed as

$$
(2.6) \quad \phi^{(\epsilon)}_{ij}(\xi) = \begin{cases} \phi_{ij}(\epsilon) + \int_{\epsilon}^{\xi} \phi'_{ij}(\rho_{\epsilon}(\eta)) \, d\eta + \phi'_{ij}(\rho_{\epsilon}(0))\xi, & \xi \leq 0, \\
\phi_{ij}(\xi), & \epsilon \leq \xi.
\end{cases}
$$
In particular, $\phi_{ij}^{(n)}(\xi) \geq 0$ for $\xi \leq 0$. In addition, we obtain that
\[
\phi_{ij}^{(c)}(\xi) = \phi_{ij}(\epsilon) + \int_{\epsilon}^{\xi} \phi'_{ij}(\rho_\epsilon(\eta))\,d\eta \leq \phi_{ij}(\epsilon) + \int_{\epsilon}^{\xi} \phi_{ij}(\eta)\,d\eta = \phi_{ij}(\xi)
\]
for $0 < \xi < \epsilon$. In the estimate, we used the fact that $\xi \leq \rho_\epsilon(\xi)$ for $\xi > 0$.

We obtain the assertions of Lemmas 2.1 and 2.2 in which $\phi_{ij}$ and $a$ are replaced by $\phi_{ij}^{(c)}$ and $a^{(c)}$, respectively. Indeed, we obtain the following.

Lemma 2.5. We have the following.

1. For all $n \in \mathbb{N} \cup \{0\}$ and $\xi \in I_{ij}$, we have $\frac{d^n \phi_{ij}^{(c)}}{d\xi^n}(\xi) = (-1)^{n+1}\frac{d^n \phi_{ij}^{(c)}}{d\xi^n}(-\xi)$.
2. For all $x \in \Delta_d$, $a^{(c),j}(x)$ is symmetric and given by
\[
a^{(c),j}_{ij}(x) = \begin{cases}
\sum_{i \neq j} \phi_{kl}^{(c),j}(x_i - x_l), & i = j, \\
-\phi_{ij}^{(c),j}(x_i - x_j), & i \neq j.
\end{cases}
\]

Lemma 2.6. For any $x \in \Delta_d$ and $y, z \in \mathbb{R}^d$, we have
\[
\sum_{i=1}^{d} x_i a^{(c),i}_{ij}(x) = \sum_{k,l,k,l \neq i} (x_k - x_l)\phi_{kl}^{(c)}(x_k - x_l) \leq \sum_{k,l,k \neq l} \alpha_{kl},
\]
\[
\langle y, a^{(c),j}(x) y \rangle = \sum_{k,l,k,l \neq i} \phi_{kl}^{(c),j}(x_k - x_l) (y_k - y_l)^2 \leq 0,
\]
\[
\sum_{j=1}^{d} a^{(c),i}_{ij}(x) y_j z_k = \sum_{l \neq i} \phi_{kl}^{(c),i}(x_i - x_l) (y_i - y_l)(z_i - z_l).
\]

Proof of Lemma 2.5. Assertion (1) is shown by induction. Indeed, we have
\[
\phi_{ij}^{(c)}(\xi) = -\phi_{ij}(-\epsilon) + \int_{\epsilon}^{\xi} \phi'_{ij}(-\rho_\epsilon(\eta))\,d\eta
\]
\[
= -\phi_{ij}(-\epsilon) - \int_{-\epsilon}^{-\xi} \phi'_{ij}(-\rho_\epsilon(-\eta))\,d\eta
\]
\[
= -\phi_{ij}^{(c)}(-\xi).
\]

This is Assertion (1) for $n = 0$. We can prove the assertion for $n \geq 1$ in the same way as for Lemma 2.1 (1). Assertion (2) is a consequence of Assertion (1). \qed

Proof of Lemma 2.6. From (2.6) and (2.7), we see that $\xi \phi_{ij}^{(c)}(\xi) \leq \alpha_{ij}$ for $i > j$. We can prove the same inequality for $i < j$ in the same way. These inequalities imply the inequality part of (2.8). The other assertions are easily proved. \qed

3. Malliavin Calculus for Non-colliding Particle Systems

3.1. Basics of Malliavin Calculus. We collect basic results on Malliavin calculus; see [MT17, Shi04, Nua06, IW89] for more details.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the canonical probability space; that is, $\Omega = C([0,T]; \mathbb{R}^d)$, $\mathcal{F}$ is the Borel $\sigma$-field on $\Omega$, and $\mathbf{P}$ is the Wiener measure. Set $W(\omega) = \omega$ for $\omega \in \Omega$. Under the probability measure $\mathbf{P}$, $W$ is a $d$-dimensional standard Brownian motion. The Cameron–Martin space associated with $d$-dimensional standard Brownian motion is denoted by $\mathcal{H}$; that is, $\mathcal{H}$ consists of all elements $h \in \Omega$ that
have a Radon–Nikodym derivative \( \dot{h} \) with respect to the Lebesgue measure and \( h \in L^2([0,T];\mathbb{R}^d) \).

Let \( K \) be a real separable Hilbert space, \( k \in \mathbb{N} \cup \{0\} \), and \( 1 < p < \infty \). We denote by \( \mathcal{D}_{k,p}(K) \) the Sobolev space of \( K \)-valued Wiener functionals defined on \((\Omega,\mathcal{F},\mathbb{P})\) in the Malliavin sense with differentiability index \( p \) and integrability index \( p \). If there is no risk of confusion, we write simply \( \mathcal{D}_{k,p}(\mathbb{R}) \). We set \( L^p(K) = \mathcal{D}_{0,p}(K) \). For \( F \in \mathcal{D}_{k,p}(K) \) and \( 0 \leq l \leq k \), \( D^lF \) denotes the \( l \)th derivative of \( F \).

We use the following sufficient condition to ensure the existence and continuity of a density function of a Wiener functional.

**Proposition 3.1** ([FN95, Theorem 2.1], [Nag13, Theorem 2.2]). Let \( 1 < p, q < \infty \) satisfy \( 1/p + 1/q \leq 1 \). Suppose \( q > d \). A functional \( F \in \mathcal{D}_{1,p}(\mathbb{R}^d) \) admits a continuous density function on \( \mathbb{R}^d \) if there exists an \( \mathcal{S}^d \)-valued Wiener functional \( U = (U_1,\ldots,U_d) \in \mathcal{D}_{1,q}(\mathcal{S}^d) \) such that \( (DF_k,U_k)_{\mathcal{S}^d} = \delta_{jk} \) a.s. for any \( 1 \leq j,k \leq d \).

### 3.2. Malliavin Differentiability

Let \( X \) be a solution of (1.3). In this subsection, we show Malliavin differentiability of \( X(t) \) for every \( 0 \leq t \leq T \) (Proposition 3.6). Throughout of this subsection, we suppose that Assumption 1.1 (1) holds. Before starting our discussion, we comment on the drift coefficients \( f = a + b \) and \( f^{(\epsilon)} = a^{(\epsilon)} + b \). Combining (2.1) and the linear growth condition of \( b \), we obtain that there exists a positive constant \( K \) such that

\[
2 \sum_{i=1}^d x_i f_i(x) + |\sigma|^2 = 2 \sum_{i=1}^d x_i a_i(x) + 2 \sum_{i=1}^d x_i b_i(x) + |\sigma|^2 \leq K(1 + |x|^2)
\]

for any \( x \in \Delta_d \). In particular, \( 2 \sum_{k,l;k > l} \alpha_{kl} + |\sigma|^2 \leq K \) holds. From (2.8) and the linear growth condition of \( b \), the same inequality holds for \( f^{(\epsilon)} \), and the constant \( K \) is independent of \( \epsilon \). Since \( b' \) is bounded, the constant

\[
M = \sup_{x \in \mathbb{R}^d} |b'(x)|
\]

is finite.

To express the derivative \( DX(t) \), we introduce processes \( Y \) and \( Z \) as solutions to the following \( \text{Mat}_n(\mathbb{R}) \)-valued stochastic ODEs:

\[
\begin{align*}
dY(t) &= +f'(X(t))Y(t) \, dt, \quad Y_0 = I, \\
dZ(t) &= -Z(t)f'(X(t)) \, dt, \quad Z_0 = I.
\end{align*}
\]

For auxiliary consideration, we introduce solutions \( Y^{(\epsilon)} \) and \( Z^{(\epsilon)} \) to \( \text{Mat}_n(\mathbb{R}) \)-valued ODEs

\[
\begin{align*}
dY^{(\epsilon)}(t) &= +f^{(\epsilon),i}(X^{(\epsilon)}(t))Y^{(\epsilon)}(t) \, dt, \quad Y_0^{(\epsilon)} = I, \\
dZ^{(\epsilon)}(t) &= -Z^{(\epsilon)}(t)f^{(\epsilon),i}(X^{(\epsilon)}(t)) \, dt, \quad Z_0^{(\epsilon)} = I.
\end{align*}
\]

Here, \( X^{(\epsilon)} \) is a solution to (2.5). Note that these four ordinary differential equations admit unique solutions with probability one because, for a fixed \( \omega \in \Omega \), \( f'(X(\bullet))(\omega) \)
and \( f(x') \cdot (X^{(c)}(\bullet))(\omega) \) are continuous on \([0, T]\) and thus bounded. A simple calculation implies that \( Z \) and \( Z^{(c)} \) are the inverse matrices of \( Y \) and \( Y^{(c)} \), respectively. The next lemma shows the relationship between \( X, X^{(c)} \), and so on.

**Lemma 3.2.** Under Assumption 1.1 (1), there exists a random variable \( 0 < \epsilon_0(T) < 1 \) such that \( X^{(c)}(t) = X(t), Y^{(c)}(t) = Y(t), \) and \( Z^{(c)}(t) = Z(t) \) hold for any \( 0 \leq t \leq T \) and \( 0 < \epsilon < \epsilon_0(T) \).

**Proof.** For every \( 0 < \epsilon < \min_{2 \leq i \leq d}(\bar{x}_i - \bar{x}_{i-1}) \wedge 1 \), we define a stopping time \( \tau^{(c)} \) by

\[
\tau^{(c)} = \inf\{0 \leq t \leq T; X(t) \in \Delta^{(c)}_d \} \wedge \inf\{0 \leq t \leq T; X^{(c)}(t) \in \Delta^{(c)}_d \},
\]

where \( \Delta^{(c)}_d = \{(x_1, \ldots, x_d)^\top \in \Delta_d; \min_{2 \leq i \leq d}(x_i - x_{i-1}) > \epsilon \} \). Because \( f = f^{(c)} \) on \( \Delta^{(c)}_d \), we see that \( X(t) = X^{(c)}(t) \) holds for any \( 0 \leq t \leq \tau^{(c)} \). Assumption 1.1 (1) ensures that a random variable \( \epsilon_0(T) := \sup\{\epsilon \in (0, 1); X_t = X_t^{(c)}; \forall t \in [0, T]\} \) exists and it holds that \( \tau^{(c)} = T \) for all \( 0 < \epsilon < \epsilon_0(T) \). Hence, \( X(t) = X^{(c)}(t) \) holds for all \( 0 \leq t \leq T \) and \( 0 < \epsilon < \epsilon_0(T) \). This implies that \( Y(t) = Y^{(c)}(t) \) and \( Z(t) = Z^{(c)}(t) \) hold for any \( 0 \leq t \leq T \) and \( 0 < \epsilon < \epsilon_0(T) \).

We give estimates of \( Y(t)Z(s) \) and \( Y^{(c)}(t)Z^{(c)}(s) \) for \( 0 \leq s \leq t \leq T \).

**Lemma 3.3.** For any \( 0 \leq s \leq t \leq T \) and \( v \in \mathbb{R}^d \), we have

\[
\begin{align*}
|Y(t)Z(s)| &\leq e^{M(t-s)}\sqrt{d}, & |Y(t)Z(s)v| &\leq e^{M(t-s)}|v|, \\
|Y^{(c)}(t)Z^{(c)}(s)| &\leq e^{M(t-s)}\sqrt{d}, & |Y^{(c)}(t)Z^{(c)}(s)v| &\leq e^{M(t-s)}|v|.
\end{align*}
\]

In particular, the absolute values of the eigenvalues of \( Y(t)Z(s) \) and \( Y^{(c)}(t)Z^{(c)}(s) \) are less than or equal to \( e^{M(t-s)} \). Here, \( M \) is a non-negative constant defined by (3.2).

**Proof.** It follows from (2.2) and (2.9) that the eigenvalues of \( a'(x) \) and \( a^{(c)}(x) \) are less than or equal to zero. Recall the boundedness of \( b' \) and the definition of \( M \). Using Proposition A.1, we obtain the assertions. \( \square \)

**Lemma 3.4.** Let \( 4 \leq p < \infty \). We have

\[
E \left[ \sup_{0 \leq t \leq T} |X(t)|^p \right] = C(1 + |\bar{x}|^p), \quad \sup_{0 \leq t \leq T} E \left[ \sup_{0 \leq s \leq T} |X^{(c)}(t)|^p \right] \leq C(1 + |\bar{x}|^p),
\]

where \( C \) is a positive constant that depends only on \( p, T, \) and \( K \).

**Proof.** Because we can give estimates of \( E[\sup_{0 \leq t \leq T} |X(t)|^p] \) and \( E[\sup_{0 \leq t \leq T} |X^{(c)}(t)|^p] \) in the same way, we consider \( E[\sup_{0 \leq t \leq T} |X^{(c)}(t)|^p] \) only. In this proof, \( C' \) and \( C'' \) are positive constants that depend only on \( p, T, \) and \( K \).

Applying Itô's formula to (2.5), we have \( |X^{(c)}(t)|^2 = |\bar{x}|^2 + A^{(c)}(t) + M^{(c)}(t) \), where

\[
A^{(c)}(t) = \int_0^t \left\{ 2 \sum_{i=1}^d X^{(c)}(s) f^{(c)}(X^{(c)}(s)) + |\sigma|^2 \right\} ds,
\]

\[
M^{(c)}(t) = 2 \sum_{i=1}^d \sum_{k=1}^d \int_0^t X^{(c)}(s) \sigma_{ik} dW_k(s).
\]
Recalling (3.1) and setting $\overline{A}(t) = K \int_0^t (1 + |X(s)|^2)^2 \, ds$, we see that $A(t) \leq \overline{A}(t)$. Hence, $|X(t)|^2 \leq |\overline{A}(t) + M(t)|$, which implies $|X(t)|^p \leq 3p^{2-1}(|\overline{A}(t)|^p + |M(t)|)^{p/2}$.

We estimate the expectations of $\sup_{0 \leq s \leq t} |\overline{A}(t)|^p$ and $\sup_{0 \leq s \leq T} |M(t)|^p$.

From the Jensen inequality, we have

$$E \left[ \sup_{0 \leq u \leq t} |\overline{A}(u)|^p \right] \leq C' \left( 1 + \int_0^t E \left[ \sup_{0 \leq u \leq s} |X(u)|^p \right] \, ds \right).$$

Note that $\langle M(t) \rangle = 4 \int_0^t |(X(s))|^2 \, ds \leq 4 \int_0^t |X(s)|^2 |\sigma|^2 \, ds$ holds and that $\xi^2 \eta^2 \leq \xi^4 + \eta^4$ for any $\xi, \eta \in \mathbb{R}$. Using these inequalities, the Burkholder–Davis–Gundy inequality, and the Jensen inequality, we have

$$E \left[ \sup_{0 \leq u \leq t} |M(u)|^p \right] \leq C_p E[\langle M(t) \rangle^{p/4}] \leq C'' \left( 1 + \int_0^t E \left[ \sup_{0 \leq u \leq s} |X(u)|^p \right] \, ds \right).$$

Here, $C_p$ is a constant that appears in the Burkholder–Davis–Gundy inequality and depends only on $p$.

Combining the above, we obtain

$$E \left[ \sup_{0 \leq u \leq t} |X(u)|^p \right] \leq 3p^{2-1} \left( |\overline{A}|^p + |M|^p \right) \left( 1 + \int_0^t E \left[ \sup_{0 \leq u \leq s} |X(u)|^p \right] \, ds \right).$$

This and Gronwall’s inequality imply the assertion.

From Lemmas 3.3 and 3.4, we obtain the next result on the differentiability of $X(t)$.

**Lemma 3.5.** Let $0 \leq t \leq T$ and $1 < p < \infty$. We have $X(t) \in D^{1,p}(\mathbb{R}^d)$ and

$$(DX_t^{(c)}(t))_n = \int_0^1 1_{[0,t]}(s) (Y(t)Z(s)\sigma)_{in} \, ds.$$  

Furthermore, it holds that

$$\sup_{0<\epsilon<1} \|X(t)\|_{D^{1,p}(\mathbb{R}^d)} \leq C,$$

where $C$ is a positive constant that depends only on $|\overline{x}|$, $K$, $M$, $T$, and $p$.

**Proof.** We have $X(t) \in D^{1,p}(\mathbb{R}^d)$ and (3.3) from the standard theory of Malliavin calculus; see [MT17, Theorem 5.5.1], [Nua06, Theorem 2.2.1].

We estimate $E[\|DX_t^{(c)}(t)\|_{D^{1,p}}^p]$. Because the boundedness of $Y(t)Z(s)\sigma$ is deduced from Lemma 3.3, we obtain

$$\|DX_t^{(c)}(t)\|_{D^{1,p}}^p = \int_0^t |Y(t)Z(s)\sigma|^2 \, ds \leq \int_0^t e^{2M(t-s)} |\sigma|^2 \, ds = \frac{e^{2Mt} - 1}{2M} |\sigma|^2,$$

where $(e^{2Mt} - 1)/(2M) = t$ for $M = 0$. This implies

$$E[\|DX_t^{(c)}(t)\|_{D^{1,p}}^p] \leq \left( \frac{e^{2Mt} - 1}{2M} |\sigma|^2 \right)^{p/2}.$$ 

Combining this estimate and Lemma 3.4, we obtain (3.4).
Proposition 3.6. Let $0 \leq t \leq T$ and $1 < p < \infty$. Under Assumption 1.1 (1), we have $X(t) \in D^{1,p}({\mathbb R}^d)$ and

$$(3.5) \quad (DX_1(t)) = \int_0^1 1_{[0,t]}(s)(Y(t)Z(s)) \sigma ds.$$  

Proof. Lemmas 3.2 and 3.5 imply

$$\lim_{\epsilon \downarrow 0} X^{(c)}(t) = X(t) \quad \text{a.s.,} \quad \lim_{\epsilon \downarrow 0} DX^{(c)}(t) = \text{RHS of (3.5)} \quad \text{a.s.}$$

Lemma 3.4 implies the uniform integrability of $\{X^{(c)}(t)\}_{0 < c < 1}$ and $\{DX^{(c)}(t)\}_{0 < c < 1}$. Hence,

$$\lim_{\epsilon \downarrow 0} X^{(c)}(t) = X(t) \quad \text{in } L^p({\mathbb R}^d), \quad \lim_{\epsilon \downarrow 0} DX^{(c)}(t) = \text{RHS of (3.5)} \quad \text{in } L^p(\mathcal{F}^d).$$

Combining these $L^p$-convergences with the closability of $D$ ([Nua06, Proposition 1.2.1], [Shi04, Corollary 4.14]), we obtain the assertion. \hfill \Box

3.3. Non-degeneracy. For every $0 < t \leq T$, we show the existence and continuity of the density of a solution $X(t)$ to (1.3). To prove this assertion, we find an $\mathcal{F}^d$-valued Wiener functional $U$ that satisfies the assumption of Proposition 3.1. Throughout of this subsection, we suppose that Assumption 1.1 holds.

We set

$$\gamma = I - e^{-(M+1)t}Y(t), \quad u_k = (u_{1k}, \ldots, u_{dk})^T, \quad k = 1, \ldots, d,$$

where $M$ is a non-negative constant defined by (3.2) and

$$(3.6) \quad u_{nk} = \int_0^1 1_{[0,t]}(s) \left( \sigma^{-1}\{(M + 1)t - f'(X(s))\} \right)_{nk} e^{-(M+1)(t-s)} ds, \quad n = 1, \ldots, d.$$  

Here, $\gamma$ depends on $t$; however, we suppress $t$ for notational simplicity. In what follows, we show that an $\mathcal{F}^d$-valued Wiener functional $U = (U_1, \ldots, U_d)$ defined by

$$U_j = \sum_{k=1}^d u_k(\gamma^{-1})_{kj}, \quad j = 1, \ldots, d,$$

satisfies the assumption of Proposition 3.1.

First, we show that $\gamma$ is invertible.

Lemma 3.7. Under Assumption 1.1, $|\det \gamma| \geq (1 - e^{-t})^d$.

Proof. We denote the eigenvalues of $Y(t)$, which may be complex number, by $\lambda_1, \ldots, \lambda_d$. We then have $|\lambda_i| \leq e^{Mt}$ for all $1 \leq i \leq d$ from Lemma 3.3 with $s = 0$. Hence, $|1 - e^{-(M+1)t}\lambda_i| \geq 1 - e^{-t}$. Noting that $\det \gamma = e^{-d(M+1)t} \det(e^{(M+1)t}I - Y(t))$ and putting $\xi = e^{(M+1)t}$ into the polynomial $\det(\xi I - Y(t)) = \prod_{i=1}^d (\xi - \lambda_i)$, we obtain

$$\det \gamma = e^{-d(M+1)t} \prod_{i=1}^d (e^{(M+1)t} - \lambda_i) = \prod_{i=1}^d (1 - e^{-(M+1)t}\lambda_i).$$

Combining the above, we obtain the assertion. \hfill \Box

Next, we study the differentiability of $Y(t)$ in the Malliavin sense.
**Lemma 3.8.** Under Assumption 1.1, \(Y(t) \in \mathcal{D}^{1,2q}(\mathbb{R}^d)\).

**Proof.** Let us consider a solution \(\mathcal{Y}^{(\epsilon)}\) of

\[
d\mathcal{Y}^{(\epsilon)}(t) = + f^{(\epsilon),r}(X(t))\mathcal{Y}^{(\epsilon)}(t) \, dt, \quad \mathcal{Y}^{(\epsilon)}_0 = I.
\]

From [Nua06, Lemma 2.2.2] and the boundedness of \(f^{(\epsilon),r}\), we have \(\mathcal{Y}^{(\epsilon)}(t) \in \mathcal{D}^{1,2q}(\mathbb{R}^d)\) and see that \(D\mathcal{Y}^{(\epsilon)}_{im}(t)\) is identified with the process \((\Xi^{(\epsilon)}_{im}(r,t))_{1 \leq n \leq d}\) satisfying

\[
\Xi^{(\epsilon)}_{im}(r,t) = \sum_{j,k=1}^d \int_r^t f^{(\epsilon),r}_{ijk}(X(s))\tilde{Y}_{km}(r,s)\Xi^{(\epsilon)}_{jm}(r,s) \, ds + \sum_{j=1}^d \int_r^t f^{(\epsilon),r}_{ij}(X(s))\Xi^{(\epsilon)}_{jm}(r,s) \, ds,
\]

where \(\tilde{Y}_{km}(r,s) = 1_{(0,a)}(r)(Y(s)Z(r)\sigma)_{kn}\). To show the assertion by a similar argument to that used for Proposition 3.6, we give uniform estimates of \(E[|\mathcal{Y}^{(\epsilon)}_{im}(t)|^{2q}]\) and \(E[||D\mathcal{Y}^{(\epsilon)}_{im}(t)||^{2q}]\) in \(\epsilon\).

It follows from a similar argument to that in Lemma 3.3 that \(|\mathcal{Y}^{(\epsilon)}(t)| \leq e^{Mt}\sqrt{d}\).

In the rest of this proof, we give a uniform estimate of \(E[||D\mathcal{Y}^{(\epsilon)}_{im}(t)||^{2q}]\) in \(\epsilon\). To this end, we write \(\Xi^{(\epsilon)}_{imn}(r,t) = (\Xi^{(\epsilon)}_{1mn}(r,t), \ldots, \Xi^{(\epsilon)}_{dmn}(r,t))^\top\) and \(||\Xi^{(\epsilon)}_{imn}(r,t)||^2 = \sum_{i=1}^d \Xi^{(\epsilon)}_{imn}(r,t)^2\). Note

\[
||D\mathcal{Y}^{(\epsilon)}_{im}(t)||^2_{\mathcal{B}} \leq \sum_{i=1}^d ||D\mathcal{Y}^{(\epsilon)}_{im}(t)||^2_{\mathcal{B}} = \int_0^t \sum_{n=1}^d ||\Xi^{(\epsilon)}_{imn}(r,t)||^2 \, dr.
\]

In the rest of this proof, we estimate \(||\Xi^{(\epsilon)}_{imn}(r,t)||^2\) uniformly in \(r\). The fundamental theorem of calculus and the fact that \(\Xi^{(\epsilon)}_{imn}(r,r)^2 = 0\) yield

\[
||\Xi^{(\epsilon)}_{imn}(r,t)||^2 = 2 \sum_{i=1}^d \int_r^t \Xi^{(\epsilon)}_{imn}(r,s) \, d\Xi^{(\epsilon)}_{imn}(r,s) = 2 \int_r^t g^{(\epsilon)}_{mn}(r,s) \, ds + 2 \int_r^t h^{(\epsilon)}_{mn}(r,s) \, ds,
\]

where

\[
g^{(\epsilon)}_{mn}(r,s) = \sum_{i=1}^d \Xi^{(\epsilon)}_{imn}(r,s) \sum_{j,k=1}^d f^{(\epsilon),r}_{ijk}(X(s))\tilde{Y}_{km}(r,s)\Xi^{(\epsilon)}_{jm}(s),
\]

\[
h^{(\epsilon)}_{mn}(r,s) = \sum_{i=1}^d \Xi^{(\epsilon)}_{imn}(r,s) \sum_{j=1}^d f^{(\epsilon),r}_{ij}(X(s))\Xi^{(\epsilon)}_{jm}(r,s).
\]

We define functions \(g^{(\epsilon),a}_{mn}(r,s)\) and \(g^{(\epsilon),b}_{mn}(r,s)\) by replacing \(f^{(\epsilon)}\) by \(a^{(\epsilon)}\) and \(b\), respectively, in (3.7). We use similar symbols for \(h^{(\epsilon),a}_{mn}(r,s)\) and \(h^{(\epsilon),b}_{mn}(r,s)\). Because \(f^{(\epsilon)} = a^{(\epsilon)} + b\), we have \(g^{(\epsilon)}_{mn}(r,s) = g^{(\epsilon),a}_{mn}(r,s) + g^{(\epsilon),b}_{mn}(r,s)\) and \(h^{(\epsilon)}_{mn}(r,s) = h^{(\epsilon),a}_{mn}(r,s) + h^{(\epsilon),b}_{mn}(r,s)\).
We first estimate $g_{mn}^{(e),a}(r, s)$. From (2.10) and (2.4), we have
\[
g_{mn}^{(e),a}(r, s) = \sum_{i=1}^{d} \Xi_{mn}^{(e),i}(r, s) \sum_{l \neq i} \phi_{kl}^{(e),i}(X_i(s) - X_l(s)) \times \{\tilde{Y}_{im}(r, s) - \tilde{Y}_{lm}(r, s)\} \{Y_{im}^{(e)}(s) - Y_{lm}^{(e)}(s)\}
\]
\[
= \sum_{k, l, k > l} \{\Xi_{kmn}^{(e),i}(r, s) - \Xi_{lmn}^{(e),i}(r, s)\} \phi_{kl}^{(e),i}(X_k(s) - X_l(s)) \times \{\tilde{Y}_{kn}(r, s) - \tilde{Y}_{ln}(r, s)\} \{Y_{km}^{(e)}(s) - Y_{lm}^{(e)}(s)\}.
\]
By combining this expression, the inequality $\phi_{kl}^{(e),i}(\xi) \leq 2\alpha_{kl}/\xi^3$ for $\xi > 0$, and Lemma 3.3, we have
\[
|g_{mn}^{(e),a}(r, s)| \leq C_1 \sum_{k, l, k > l} \frac{|\Xi_{kmn}^{(e),i}(r, s) - \Xi_{lmn}^{(e),i}(r, s)|}{|X_k^{(e)}(s) - X_l^{(e)}(s)|^3},
\]
where $C_1$ is a positive constant such that
\[
|2\alpha_{kl}\{\tilde{Y}_{kn}(r, s) - \tilde{Y}_{ln}(r, s)\}\{Y_{km}^{(e)}(s) - Y_{lm}^{(e)}(s)\}| \leq C_1
\]
for any $k, l, m$, and $n$, and for $r < s$. Hence, Young’s inequality implies
\[
|g_{mn}^{(e),a}(r, s)| \leq C_1 \sum_{k, l, k > l} \frac{1}{2} \left\{\left(\Xi_{kmn}^{(e),i}(r, s) - \Xi_{lmn}^{(e),i}(r, s)\right)^2 + \frac{1}{(X_k^{(e)}(s) - X_l^{(e)}(s))^6}\right\} \leq 2(d - 1)C_1|\Xi_{mn}^{(e)}(r, s)|^2 + \frac{C_1}{2} \sum_{k, l, k > l} \frac{1}{(X_k^{(e)}(s) - X_l^{(e)}(s))^6}.
\]
Next, we estimate $g_{mn}^{(e),b}(r, s)$. Noting the boundedness of $b''$ and Lemma 3.3, we see that there exists a positive constant $C_2$ that satisfies
\[
\sum_{i=1}^{d} \left(\sum_{j, k=1}^{d} b''_{ijk}(X(s))\tilde{Y}_{kn}(r, s)\tilde{Y}_{jn}(r, s)\right)^2 \leq C_2^2
\]
for any $r \leq s$. H"older’s inequality and Young’s inequality imply
\[
|g_{mn}^{(e),b}(r, s)| \leq |\Xi_{mn}^{(e)}(r, s)|C_2 \leq \frac{C_2}{2} \left[1 + |\Xi_{mn}^{(e)}(r, s)|^2\right].
\]
Finally, we estimate $h_{mn}^{(e),a}(r, s)$ and $h_{mn}^{(e),b}(r, t)$. Note that
\[
h_{mn}^{(e),a}(r, s) = (\Xi_{mn}^{(e)}(r, s), a^{(e),i}(X(s))\Xi_{mn}^{(e)}(r, s)),
\]
\[
h_{mn}^{(e),b}(r, s) = (\Xi_{mn}^{(e)}(r, s), b'(X(s))\Xi_{mn}^{(e)}(r, s)).
\]
Hence, by using (2.9) and noting the boundedness of $b'$, we have
\[
h_{mn}^{(e),a}(r, t) \leq 0, \quad |h_{mn}^{(e),b}(r, t)| \leq M|\Xi_{mn}^{(e)}(r, s)|^2.
\]
Combining the above, we obtain
\[
|\Xi_{mn}^{(e)}(r, t)|^2 \leq C_3(t - r) + C_4 \sum_{k, l, k > l} \int_t^s \frac{ds}{(X_k(s) - X_l(s))^6} + C_5 \int_r^t |\Xi_{mn}^{(e)}(r, s)|^2 ds.
\]
Therefore, Gronwall’s inequality implies
\[
|\Xi(t)(r,t)|^2 \leq \left\{ C_3 (t-r) + C_4 \sum_{k,l,k>l} \int_r^t \frac{ds}{(X_k(s)-X_l(s))^6} \right\} e^{C_5(t-r)}
\]
\[
\leq \left\{ C_3 t + C_4 \sum_{k,l,k>l} \int_0^t \frac{ds}{(X_k(s)-X_l(s))^6} \right\} e^{C_5 t}.
\]
This and Assumption 1.1 (2) imply that $E[\|DY_{\Omega}^{m}[\eta]\|_\beta]$ is finite. 

**Lemma 3.9.** Under Assumption 1.1, $\gamma^{-1} \in D^{1.2q}(R^d)$.

**Proof.** We use [Nua06, Proposition 1.2.3] to prove this assertion. Recall that $\gamma^{-1} = \Gamma^{-T}/\det \gamma$, where $\Gamma$ is the cofactor matrix of $\gamma$. Because all elements of $\gamma$ are bounded and belong to $D^{1.2q}$ (see Lemmas 3.3 and 3.8), we have $\Gamma \in D^{1.2q}(R^d)$ and $\det \gamma \in D^{1.2q}$. Lemma 3.7 yields $1/\det \gamma \in D^{1.2q}$. Hence, the assertion holds.

**Lemma 3.10.** Under Assumption 1.1, $u_k \in D^{1.2q}(S)$ for every $1 \leq k \leq d$.

**Proof.** Let $A_n(s)$ be the integrand in (3.6); that is, $u_n = \int_0^s A_n(s) ds$. By [Shi04, pp.125–126], the assertion holds if we have

1. $\Psi(s) \in D^{1.2q}(R)$ for all $0 \leq s \leq T$,
2. $E \left( \int_0^T |\Psi(s)|^2 ds \right)^q < \infty$,
3. $E \left( \int_0^T \|D\Psi(s)\|^2_\beta ds \right)^q < \infty$

for $\Psi(s) = A_n(s)$.

First, we set $\Psi(s) = (X_i(s)-X_j(s))^{-2}$ for $i \neq j$ and show that Assertions (1), (2), and (3) hold. Assumption 1.1 (2) and Proposition 3.6 yield Assertion (1). Jensen’s inequality and Assumption 1.1 (2) imply Assertion (2). Because $D\Psi(s)$ is identified with
\[
(-2)(X_i(s)-X_j(s))^{-3}(\tilde{Y}_i(s) - \tilde{Y}_j(s))_{1 \leq n \leq d},
\]
where $\tilde{Y}_i(u, s) = 1_{[0,s]}(u)(Y(s)Z(u)\sigma)_i$, we see that $\|D\Psi(s)\|^2_\beta$ is bounded above by
\[
\int_0^T |(-2)(X_i(s)-X_j(s))^{-3}(\tilde{Y}_i(u, s) - \tilde{Y}_j(u, s))_{1 \leq n \leq d}|^2 du
\]
\[
= \int_0^T 4(X_i(s)-X_j(s))^{-6}|(\tilde{Y}_i(u, s) - \tilde{Y}_j(u, s))_{1 \leq n \leq d}|^2 du
\]
\[
\leq 16d^2|\sigma|^2\epsilon^{2MT} (X_i(s)-X_j(s))^{-6}.
\]
This estimate and Assumption 1.1 (2) ensure Assertion (3).

We conclude this proof by showing Assertions (1), (2), and (3) for $\Psi(s) = A_n(s)$. From the above discussion for $\Psi(s) = (X_i(s)-X_j(s))^{-2}$, the assertions are valid for $\Psi(s) = \alpha_{ij}^*(X(s))$. Because $X(s) \in D^{1.2q}(S)$ and $b_{ij}$ has bounded derivatives, the assertions hold for $\Psi(s) = b_{ij}^*(X(s))$. Because $f = a + b$, Assertions (1), (2), and (3) hold for $\Psi(s) = A_n(s)$. The proof is complete.

We are now in a position to prove our main theorem.
Proof of Theorem 1.2. We have \( X(t) \in D^{1,p} \) for \( 1 < p < \infty \) with \( 1/p + 1/q \leq 1 \) from Proposition 3.6. From Lemmas 3.9 and 3.10, we have \( \gamma^{-1} \in D^{1,2q}(R^d) \) and \( u \in D^{1,q}(\delta^d) \), which implies \( U \in D^{1,q}(\delta^d) \). In the rest of this proof, we show \( \langle DX_i, U_j \rangle_{\delta} = \delta_{ij} \). Proposition 3.6 and the definition (3.6) imply
\[
\frac{d}{ds} DX_i(s) = 1_{[0,t]}(s) \sum_{k,l=1}^d Y_{ik}(t) Z_{kl}(s) \sigma_{ln},
\]
respectively. By summing the product of these terms over \( n \), we obtain
\[
\langle DX_i(t), u_j \rangle_{\delta} = \int_0^t \sum_{k,l=1}^d Y_{ik}(t) Z_{kl}(s) ((M + 1) \delta_{ij} - f'_{ij}(X(s))) e^{-(M+1)(t-s)} ds.
\]
The integration by parts formula implies
\[
\langle DX_i(t), u_j \rangle_{\delta} = \sum_{k=1}^d Y_{ik}(t) e^{-(M+1)t} \int_0^t \sum_{l=1}^d Z_{kl}(s) ((M + 1) \delta_{lj} - f'_{lj}(X(s))) e^{M+1s} ds
\]
\[
= \sum_{k=1}^d Y_{ik}(t) e^{-(M+1)t} \int_0^t \left\{ Z_{kj}(s) \frac{de^{M+1s}}{ds} + \frac{dZ_{kj}}{ds}(s) e^{M+1s} \right\} ds.
\]
From this, we have
\[
\langle DX_i(t), U_j \rangle_{\delta} = \sum_{k=1}^d \langle DX_i(t), u_k \rangle_{\delta} \gamma^{-1}_{kj} = \sum_{k=1}^d \gamma_{ik} \gamma^{-1}_{kj} = \delta_{ij}.
\]
The proof is complete. \( \square \)

4. Proof of Theorem 1.4

In this section, we show that Assumption 1.3 implies Assumption 1.1. As a result we obtain Theorem 1.4. We denote by \( X^{\alpha,\mu+c} \) a solution to (1.3) with \( \alpha_{ik} = \alpha \), \( b = \mu + c \), and \( \sigma = 1 \) and call it a Dyson Brownian motion with a parameter \( \alpha \) and a smooth drift \( \mu + c \). The goal of this section is to show the following propositions.

Proposition 4.1. Let Assumption 1.3 (1) and \( \alpha \geq 1/2 \) be satisfied. Then, there exists a unique strong solution \( X^{\alpha,\mu+c} \) to (1.3) such that \( P(X^{\alpha,\mu+c}(t) \in \Delta_d \) for all \( 0 \leq t \leq T) = 1 \).

Proposition 4.2. Let Assumption 1.3 (1) and \( \alpha > 1/2 \) be satisfied. For any \( 0 \leq q < \alpha - 1/2 \), we have
\[
\max_{1 \leq i \leq d} \sup_{0 \leq t \leq T} E[|X_{i}^{\alpha,\mu+c}(t) - X_{k}^{\alpha,\mu+c}(t)|^{-q}] < \infty.
\]
Theorem 1.4 is a direct consequence of these propositions as follows.

**Proof of Theorem 1.4.** Proposition 4.1 implies Assumption 1.1 (1). From Proposition 4.2, we have Assumption 1.1 (2) under the conditions that $c$ is bounded together with the first derivatives and $\alpha > 6d + 1/2$. This completes the proof. □

4.1. **Existence and Uniqueness.** This subsection is devoted to proving Proposition 4.1. First, we show the pathwise uniqueness for (1.3).

**Lemma 4.3.** The pathwise uniqueness of solutions of (1.3) holds.

**Proof.** Let $(X, W)$ and $(\tilde{X}, W)$ be two solutions of (1.3). Itô’s formula yields

$$|X(t) - \tilde{X}(t)|^2 = 2 \int_0^t A(X(s), \tilde{X}(s)) \, ds + 2 \int_0^t B(X(s), \tilde{X}(s)) \, ds,$$

where

$$A(x, \tilde{x}) = \alpha \sum_{i=1}^d (x_i - \tilde{x}_i)(a_i(x) - a_i(\tilde{x})), \quad B(x, \tilde{x}) = \sum_{i=1}^d (x_i - \tilde{x}_i)(b_i(x) - b_i(\tilde{x})).$$

From (2.4) and $(\xi - \eta)(\xi^{-1} - \eta^{-1}) \leq 0$ for $\xi, \eta > 0$, we have

$$A(x, \tilde{x}) = \alpha \sum_{i=1}^d \{ (x_i - \tilde{x}_i) - (x_i - \tilde{x}_i) \} \left( \frac{1}{x_i - \tilde{x}_i} - \frac{1}{\tilde{x}_i - \tilde{x}_i} \right) \leq 0.$$

The Lipschitz continuity of $b$ implies $|B(x, \tilde{x})| \leq K|x - \tilde{x}|^2$ for some $K$ that is independent of $x$ and $\tilde{x}$. Hence, $|X(t) - \tilde{X}(t)|^2 \leq 2K \int_0^t |X(s) - \tilde{X}(s)|^2 \, ds$. Therefore, from Gronwall’s inequality, we conclude the pathwise uniqueness. □

Next, we show the existence of solutions of (1.3). Recall that there exists a unique strong solution of (1.3) for $\alpha \geq 1/2$ and $c = 0$ from [RS93, Lemma 1], [CL97, Theorem 3.1], and [GM14, Theorem 2.2, Corollary 6.2]. We use this result to show weak existence and uniqueness in law. The proof is based on the method of [Yor80, Lemma 4.5] (see also [PY81, Proposition 2.1] and [RY99, Chapter XI, Exercise 1.22]), which is the Girsanov transformation for Bessel processes to restrict its parameter $\alpha$ to $1/2$. Note that in this proof, we use only the result on the unique existence of a strong solution of (1.3) with $\alpha = 1/2$ and $c = 0$.

Before starting our discussion, we fix the notation. Let $\alpha \geq 1/2$. Set $\nu = \alpha - 1/2$ and $h(x) = \prod_{k,l,k \neq l}(x_k - x_l)$ for $x \in \Delta_d$. Let $X^{1/2, \mu}$ be a strong solution of (1.3) with $\alpha = 1/2$ and $c = 0$. We define processes $M = \{M(t)\}_{0 \leq t \leq T}$ and $Z = \{Z(t)\}_{0 \leq t \leq T}$ used in the Girsanov transformation by

$$M(t) = \sum_{i=1}^d \sum_{k,l,k \neq l} \int_0^t \frac{dW_i(s)}{X^{1/2, \mu}_i(s) - X^{1/2, \mu}_k(s)}, \quad Z(t) = \exp \left( \nu M(t) - \frac{\nu^2}{2} (M(t)) \right).$$

Because $X^{1/2, \mu}$ satisfies $P(X^{1/2, \mu}(t) \in \Delta_d$ for all $0 \leq t \leq T) = 1$, the process $M$ is well-defined and a local martingale. Although we see that $Z$ is a local martingale because it is a solution of an SDE

$$Z(t) = 1 + \nu \int_0^t Z(s) \, dM(s),$$

we can show that $Z$ is a martingale as follows.
Lemma 4.4. Let Assumption 1.3 (1) and \( \alpha \geq 1/2 \) be satisfied. Then, the process \( Z \) is expressed as

\[
Z(t) = \frac{h(X^{1/2,\mu}(t))^\nu}{h(x)^\nu} \exp \left( -\nu \sum_{k,l;k>l} \int_0^t \frac{\mu_k(X^{1/2,\mu}(s)) - \mu_l(X^{1/2,\mu}(s))}{X_k^{1/2,\mu}(s) - X_l^{1/2,\mu}(s)} \, ds \right)
\]

and is a martingale.

Proof. We set \( F(x) = \log h(x) = \sum_{k,l;k>l} \log(x_k - x_l) \) for \( x \in \Delta_d \). Then the derivatives of \( F \) are given by

\[
\frac{\partial F}{\partial x_i}(x) = \sum_{k,l;k\neq i} \frac{1}{x_i - x_k} =: u_{i1}(x), \quad \frac{\partial^2 F}{\partial x_i^2}(x) = -\sum_{k,l;k\neq i} \frac{1}{(x_i - x_k)^2} =: -u_{i2}(x),
\]

for all \( i = 1, \ldots, d \). From [AGZ10, p.252], we note that

\[
\sum_{i=1}^d u_{i1}(x)^2 = \sum_{i=1}^d u_{i2}(x).
\]

Applying Itô’s formula and using (4.3), we have \( F(X^{1/2,\mu}(t)) = F(\bar{x}) + A(t) + M(t) \), where

\[
A(t) = \sum_{i=1}^d \int_0^t u_{i1}(X^{1/2,\mu}(s)) \mu_i(X^{1/2,\mu}(s)) \, ds.
\]

Hence,

\[
Z(t) = \exp \left( \nu\{F(X^{1/2,\mu}(t)) - F(\bar{x}) - A(t)\} - \frac{\nu^2}{2} \langle M \rangle(t) \right).
\]

Next, we calculate \( Z(t) \). The definition of \( F \) yields

\[
\exp(\nu\{F(X^{1/2,\mu}(t)) - F(\bar{x})\}) = \frac{h(X^{1/2,\mu}(t))^\nu}{h(x)^\nu}.
\]

From (2.4), we have

\[
\sum_{i=1}^d u_{i1}(x)\mu_i(x) = \sum_{i=1}^d \mu_i(x) \sum_{k,l;k\neq i} \frac{1}{x_i - x_k} = \sum_{k,l;k>l} \frac{\mu_k(x) - \mu_l(x)}{x_k - x_l},
\]

which implies

\[
A(t) = \sum_{k,l;k>l} \int_0^t \frac{\mu_k(X^{1/2,\mu}(s)) - \mu_l(X^{1/2,\mu}(s))}{X_k^{1/2,\mu}(s) - X_l^{1/2,\mu}(s)} \, ds.
\]

From (4.3), we obtain

\[
\langle M \rangle(t) = \sum_{i=1}^d \int_0^t u_{i1}(X^{1/2,\mu}(s))^2 \, ds = \sum_{i=1}^d \int_0^t u_{i2}(X^{1/2,\mu}(s)) \, ds.
\]

Combining the above, we obtain the expression for \( Z(t) \).
Next, we show that $Z$ is a martingale. Since $\mu_k(x) \geq \mu_l(x)$ for $k > l$, from the expression of $Z$, we have for any $p > 1$ and stopping time $\tau \leq T$,

$$|Z(\tau)|^p \leq \sup_{0 \leq t \leq T} \frac{h(X^{1/2,\mu}(t))^{\mu_p}}{h(x)^{\mu_p}}.$$  

Lemma 3.4 therefore yields the family of random variables $Z(\tau)$, is uniformly integrable. Hence from [RY99, Proposition 1.7 in chapter IV], $Z$ is a martingale. □

**Lemma 4.5.** Let Assumption 1.3 (1) and $\alpha \geq 1/2$ be satisfied. If we assume that $c = 0$, then we have the following:

1. A weak solution of (1.3) on $\Delta_d$ for all $0 \leq t \leq T$ exists and uniqueness in law holds.
2. For any measurable function $g: C([0,T]; \mathbb{R}^d) \to \mathbb{R}$, we have

$$E[g(X^{\alpha,\mu})] = E[g(X^{1/2,\mu})Z(T)]$$

provided that all the above expectations exist.

**Proof.** We define a new measure $P_T(F) = E[Z(T)1_F]$ for $F \in \mathcal{F}(T)$. Then, because $Z$ is a martingale, $P_T$ is a probability measure. From the Girsanov theorem, the process $B = \{(B_1(t), \ldots, B_d(t))\}_{0 \leq t \leq T}$ defined by

$$B_i(t) = W_i(t) - \langle W_i, \nu \rangle(t) = W_i(t) - \nu \sum_{k,k \neq i} \int_0^t \frac{ds}{X^{1/2,\mu}_i(s) - X^{1/2,\mu}_k(s)}$$

is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}(T), P_T)$. Moreover, we observe that

$$X^{1/2,\mu}_i(t) = \bar{x}_i + \int_0^t \left( \sum_{k,k \neq i} \frac{\alpha}{X^{1/2,\mu}_i(s) - X^{1/2,\mu}_k(s)} + \mu_i(X^{1/2,\mu}(s)) \right) ds + W_i(t)$$

$$= \bar{x}_i + \int_0^t \left( \sum_{k,k \neq i} \frac{\alpha}{X^{1/2,\mu}_i(s) - X^{1/2,\mu}_k(s)} + \mu_i(X^{1/2,\mu}(s)) \right) ds + B_i(t),$$

and thus $(X^{1/2,\mu}, B)$ is a weak solution of (1.3) with $\alpha \geq 1/2$ and $c = 0$ on the probability space $(\Omega, \mathcal{F}(T), P_T)$. The uniqueness follows from the uniqueness of the case $\alpha = 1/2$ and $c = 0$. This concludes the proof of the statement. □

We generalize Lemma 4.5 as follows.

**Lemma 4.6.** Let Assumption 1.3 (1) and $\alpha \geq 1/2$ be satisfied. Then we have the following:

1. A weak solution of (1.3) on $\Delta_d$ for all $0 \leq t \leq T$ exists and uniqueness in law holds.
2. For any measurable function $g: C([0,T]; \mathbb{R}^d) \to \mathbb{R}$, we have

$$E[g(X^{\alpha,\mu+c})] = E \left[ g(X^{\alpha,\mu}) \exp \left( \sum_{i=1}^d \int_0^T c_i(X^{\alpha,\mu}(s)) dW_i(s) \right)^{-1/2} \right]$$
provided that all the above expectations exist.

(3) Let \( p > 1 \). For any measurable function \( g : C([0, T]; \mathbb{R}^d) \to \mathbb{R} \), we have

\[
E[|g(X_{\alpha,\mu}^c)|] \leq C E[|g(X_{\alpha,\mu})|^{1/p}]^{1/p},
\]

where \( C \) is a positive constant that is independent of \( g \).

Proof. Let \( (X_{\alpha,\mu}, W) \) be a weak solution of (1.3) with \( \alpha \geq 1/2 \) and \( c = 0 \). We set \( \tilde{M}(t) = \sum_{i=1}^{d} \int_{0}^{t} c_i(X_{\alpha,\mu}(s)) \, dW_i(s) \). Because \( \langle \tilde{M} \rangle(t) = \int_{0}^{t} |c(X_{\alpha,\mu}(s))|^2 \, ds \) and \( c \) is bounded, the process \( \tilde{M} \) satisfies the Novikov condition. Hence, for every \( q \geq 1 \), \( \{ \tilde{Z}_q(t) = \exp(q\tilde{M}(t) - \frac{q^2}{2} \langle \tilde{M} \rangle(t)) \}_{0 \leq t \leq T} \) is a martingale starting at 1.

Next, we prove Assertion (1). By using the Girsanov transformation and the weak existence and uniqueness in law of solutions of (1.3) with \( c = 0 \) (Lemma 4.5), we see weak existence and uniqueness in law of solutions of (1.3) with any function \( c \). Note that \( E[g(X_{\alpha,\mu}^c)] = E[g(X_{\alpha,\mu})\tilde{Z}_1(T)] \) holds. The proof of Assertion (2) is complete.

Next, we prove Assertion (3). For any \( p, q > 1 \) with \( 1/p + 1/q = 1 \), Hölder’s inequality yields \( E[|g(X_{\alpha,\mu}^c)|] \leq E[|g(X_{\alpha,\mu})|^{1/p}]^{1/p} E[|\tilde{Z}_1(T)|^{1/q}]^{1/q} \). Therefore, we need to prove that \( E[|\tilde{Z}_1(T)|^{q}] \) is finite. Because \( \tilde{Z}_1(T)^q = e^{q(\bar{a}-1)\langle \tilde{M} \rangle(T)/2}\tilde{Z}_q(T) \), \( c \) is bounded and \( \tilde{Z}_q \) is a martingale starting at 1, we have \( E[|\tilde{Z}_1(T)|^{q}] \leq e^{q(\bar{a}-1)R^2/2} \), where \( R \) is a positive constant such that \( |c(x)| \leq R \) for any \( x \in \mathbb{R}^d \). This proves Assertion (3).

\[ \square \]

Proof of Proposition 4.1. Lemma 4.3 and Lemma 4.6 (1) imply the assertion. \( \square \)

4.2. Inverse Moments. Next, we prove Proposition 4.2.

Lemma 4.7. Let Assumption 1.3 (1) and \( \alpha \geq 1/2 \) be satisfied. Assume that \( c = 0 \). For any \( 0 \leq q \leq \alpha - 1/2 \), we have (4.1).

Proof. Applying Lemma 4.5 (2) with \( g(w) = |w_1(t) - w_k(t)|^{-q} \) for \( w \in C([0, T]; \mathbb{R}^d) \), we have

\[
E[|X_{\alpha,\mu}^c(t) - X_{\alpha,\mu}^c(t)|^{-q}] \leq \frac{1}{h(x)^{\alpha-1/2}} \frac{h(X_{\alpha,\mu}(t))^{\alpha-1/2}}{|X_{\alpha,\mu}^c(t) - X_{\alpha,\mu}^c(t)|^q} \leq C' h(x)^{\alpha-1/2},
\]

where \( C' \) is a positive constant that depends only on \( \alpha, q, \tilde{x}, T, K, \) and \( d \). In the last estimate, we used Lemma 3.4 because the integrand is reducible with respect to \( |X_{\alpha,\mu}^c(t) - X_{\alpha,\mu}^c(t)|^2 \).

\[ \square \]

Proof of Proposition 4.2. We use Lemma 4.6 (3) with \( p = (\alpha - 1/2)/q > 1 \) and Lemma 4.7. Then

\[
E[|X_{\alpha,\mu}^c(t) - X_{\alpha,\mu}^c(t)|^{-q}] \leq C E[|X_{\alpha,\mu}^c(t) - X_{\alpha,\mu}^c(t)|^{-(\alpha-1/2)/q}] < \infty,
\]

which implies the conclusion.

\[ \square \]
Appendix A. Estimate of Solution of Matrix-valued ODE

We consider a continuous Sym$_n$(R)-valued function $a$ defined on $[0, \infty)$ and denote the eigenvalues of $a(t)$ by $\lambda_1(t), \ldots, \lambda_n(t)$. We assume that a constant $L$ exists such that $\lambda_i(t) \leq L$ holds for any $1 \leq i \leq n$ and $t \geq 0$. The assumption $a(t) \in$ Sym$_n$(R) implies that $a(t)$ is diagonalizable; more precisely, there exists an orthogonal matrix $q(t)$ such that $a(t) = q(t)^\top \Lambda(t) q(t)$, where $\Lambda(t) = \text{diag}\{\lambda_1(t), \ldots, \lambda_n(t)\}$. We consider a continuous Mat$_n$(R)-valued function $b$ defined on $[0, \infty)$ and assume that a positive constant $M$ exists such that $|b(t)| \leq M$ for any $t$. Set $f(t) = a(t) + b(t)$.

For a function $f$ satisfying the conditions above, we consider Mat$_n$(R)-valued ODEs

\[
\frac{dy}{dt}(t) = +f(t)y(t), \quad y(0) = I, \\
\frac{dz}{dt}(t) = -z(t)f(t), \quad z(0) = I.
\]

From $\frac{d(vw)}{dt}(t) = 0$, we have $y(t)z(t) = z(t)y(t) = I$. For every $0 \leq s \leq t$, we set $\tilde{y}(s, t) = y(t)z(s)$. Then, we have

\[
\frac{d\tilde{y}}{dt}(s, t) = +f(t)\tilde{y}(s, t), \quad \tilde{y}(s, s) = I.
\]

**Proposition A.1.** For any $0 \leq s \leq t < \infty$ and $v \in \mathbb{R}^n$, we have

\[
|\tilde{y}(s, t)| \leq e^{(L+M)(t-s)}|I|, \quad |\tilde{y}(s, t)v| \leq e^{(L+M)(t-s)}|v|.
\]

**Proof.** We have the assertion because we can prove that $|\tilde{y}(s, t)|$ and $|\tilde{y}(s, t)v|^2$ satisfy the assumption of Gronwall’s inequality; that is,

\[
\frac{1}{2} \frac{d}{dt}|\tilde{y}(s, t)|^2 \leq (L + M)|\tilde{y}(s, t)|^2, \quad \frac{1}{2} \frac{d}{dt}|\tilde{y}(s, t)v|^2 \leq (L + M)|\tilde{y}(s, t)v|^2.
\]

Because we can prove these two inequalities in a similar way, we prove the first inequality only. Note that

\[
\frac{1}{2} \frac{d}{dt}|\tilde{y}(s, t)|^2 = \langle a(t)\tilde{y}(s, t), \tilde{y}(s, t) \rangle + \langle b(t)\tilde{y}(s, t), \tilde{y}(s, t) \rangle.
\]

We write $\tilde{y} = \tilde{y}(s, t)$ for notational simplicity. Because $a(t)$ is diagonalizable by the orthogonal matrix $q(t)$ and $\Lambda(t) \leq LI$, we have

\[
\langle a(t)\tilde{y}, \tilde{y} \rangle = \langle q(t)^\top \Lambda(t) q(t)\tilde{y}, \tilde{y} \rangle = \langle \Lambda(t) q(t)\tilde{y}, q(t)\tilde{y} \rangle \leq L|q(t)\tilde{y}|^2 = L|\tilde{y}|^2.
\]

The boundedness of $b(t)$ implies $|\langle b(t)\tilde{y}, \tilde{y} \rangle| \leq |b(t)| |\tilde{y}| |\tilde{y}| = M|\tilde{y}|^2$. Combining them, we obtain the assertion and complete the proof. \qed

From Proposition A.1, we see that the absolute values of the eigenvalues of $\tilde{y}(s, t)$ are less than or equal to $e^{(L+M)(t-s)}$ as follows. Let $\lambda$ be an eigenvalue of $\tilde{y}(s, t)$ and $v$ be an eigenvector corresponding to $\lambda$ with $|v| = 1$. Then $|\lambda| = |\langle \lambda v, v \rangle| = |\langle \tilde{y}(s, t)v, v \rangle| \leq e^{(L+M)(t-s)}$. 

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