Comment on: "Quantum dynamics of a general time-dependent coupled oscillator".

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\begin{abstract}
By using dynamical invariants theory, Hassoul et al. \cite{1,2} investigate the quantum dynamics of two (2D) and three (3D) dimensional time-dependent coupled oscillators. They claim that, in the 2D case, introducing two pairs of annihilation and creation operators uncouples the original invariant operator so that it becomes the one that describes two independent subsystems. For the 3D case, the authors pretend that they have obtained a diagonalized invariant which is exactly the sum of three simple harmonic oscillators.

We show that their investigations suffer from basic errors and therefore the found results are not valid.
\end{abstract}

1 Introduction

The harmonic oscillator is one of the most important models in quantum mechanics, and one of the few ones that has an exact analytical solution what made it applicable in the study of the dynamical properties of different physical systems. Recently, Hassoul et al \cite{1,2} have investigated quantum dynamical properties of a two (2D) and three (3D) dimensional time-dependent coupled oscillator. Their studies are based on the theory of two-dimensional (2D) and three-dimensional (3D) dynamical invariants.

Oscillators are the core of both natural systems and technological devices which made the study of dynamical properties of coupled oscillatory systems extremely demanded though their complex motion especially when the parameters are time-dependent and/or the dimension of the system is more than two.

The time-dependent coupled oscillator that has been a point of interest in the research field for a few years now \cite{3-8}, model various physical systems \cite{9-13} and helped in explaining

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numerous physical interacting systems including trapped atoms [20], nano-optomechanical resonances [21, 22], electromagnetically induced transparency [23], stimulated Raman effects [24], time-dependent Josephson phenomena [25], and systems of three isotropically coupled spins 1/2 [26]. Coupled oscillator are fundamental for quantum technologies such as quantum computing and quantum cryptography [27, 28, 29]

In what follows, we highlight many basic and mathematical flaws made by Hassoul et al. in their recent papers [1, 2] while investigating the quantum dynamical properties of a general time-dependent coupled oscillator using two and three-dimensional dynamical invariants. From there, we proceeded to reevaluate the entire study and approach the whole subject in a more scientifically coherent manner. In section 2, we recapitulate the basic errors made in [1] where a 2D quantum invariant operator is introduced for the study of a 2D time-dependent coupled oscillator system, then we show that the linear canonical transformation defined in [1] can not be adapted to the considered problem without a constraint on the mass and to solve the problem in question, we introduce more adapting canonical transformations. In Section 3, we consider the 3D case investigated in [2] and highlight with some simple mathematical calculation all the flawed results found in [2].

2 Two-dimensional time-dependent harmonic oscillators

The Hamiltonian of the time-dependent coupled oscillator that Hassoul et al. [1] consider is

\[ H(t) = \frac{1}{2} \sum_{i=1}^{2} \left[ \frac{P_i^2}{m_i(t)} + c_i(t)X_i^2 \right] + \frac{1}{2}c_3(t)X_1X_2, \]

where \( m_i(t) \), \( c_i(t) \) and \( c_3(t) \) are arbitrary functions of time. They assume a quantum invariant operator of the system of the form

\[ I(t) = \frac{1}{2} \sum_{i=1}^{2} \left[ \alpha_i(t)P_i^2 + \beta_i(t)(X_iP_i + P_iX_i) + \gamma_i(t)X_i^2 \right] + \frac{1}{2}\eta(t)X_1X_2, \]

where the parameters \( \alpha_i(t) \), \( \beta_i(t) \), \( \gamma_i(t) \) (\( i = 1, 2 \)) and \( \eta(t) \) are real and differentiable functions of time. The substitution of (1) and (2) into the invariance condition

\[ \frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H] = 0, \]

implies the auxiliary equations given in [1] as Eqs. (8-14) and an additional condition which was not mentioned

\[ \alpha_1(t)m_1(t) = \alpha_2(t)m_2(t), \]

we note that the commutator in relation (3) is replaced with Poisson brackets in the classical case.

The authors of Ref. [1] pretend to obtain the solution of the equations (8-13), omitting to mention that the so-called solution \( m_i(t) = 1/\alpha_i(t) \) must obey the following constraint equation

\[ \ddot{m}_i(t) - \frac{1}{2} \frac{\dot{m}_i^2(t)}{m_i(t)} + 2(\delta_i m_i(t) - c_i(t)) = 0, \]
which is difficult to be solved. This last equation (6) is obtained from the auxiliary equations (8-13) by noting that

\[ \alpha_i(t)\gamma_i(t) - \beta_i^2(t) = \delta_i, \]

with \( \delta_i \) being a real constant. Condition (6) is not mentioned in [1]. Note that putting \( \frac{1}{m_i} = m_i(t) \) and changing \( c_i(t) \) by \( m_i(t)c_i(t) \) gives the famous auxiliary equation of \( \rho_i \) [30, 31, 32, 33].

We emphasize that the invariant operator (23) in [1] can not be decoupled into (35) simply because the canonical transformations (41-42) can be adapted to the considered problem if and only if \( m_1(t) = m_2(t) \) and \( \omega_1^2(t) = \omega_2^2(t) \). However, in order to solve the problem in question, we introduce the following more adapting canonical transformations that should be used as

\[
\begin{pmatrix}
  X_1 \\
  X_2
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{\sqrt{m_1(t)}} \cos \left( \frac{\theta}{2} \right) & -\frac{1}{\sqrt{m_1(t)}} \sin \left( \frac{\theta}{2} \right) \\
  \frac{1}{\sqrt{m_2(t)}} \sin \left( \frac{\theta}{2} \right) & \frac{1}{\sqrt{m_2(t)}} \cos \left( \frac{\theta}{2} \right)
\end{pmatrix} \begin{pmatrix}
  Q_1 \\
  Q_2
\end{pmatrix},
\]

and set

\[
\frac{1}{\sqrt{m_i(t)}} P_i + \sqrt{m_i(t)} \beta_i X_i = P_i,
\]

to write the invariant operator (2) in the form

\[
I(t) = \frac{1}{2} \sum_{i=1}^{2} (P_i^2 + \omega_i^2 Q_i^2) + \Gamma(t) Q_1 Q_2,
\]

this invariant describes a coupled harmonic oscillator where

\[
\omega_1^2(t) = \left( \int_0^t \frac{[c_1 \dot{m}_1/m_1] dt}{m_1} - \frac{\dot{m}_1^2}{4m_1^2} \right) \cos^2 \left( \frac{\theta}{2} \right) + \left( \int_0^t \frac{[c_2 \dot{m}_2/m_2] dt}{m_2} - \frac{\dot{m}_2^2}{4m_2^2} \right) \sin^2 \left( \frac{\theta}{2} \right)
\]

\[
+ \left( \int_0^t c_3 [\dot{m}_1/2m_1 + \dot{m}_2/2m_2] dt \right) \sin (\theta),
\]

\[
\omega_2^2(t) = \left( \int_0^t \frac{[c_1 \dot{m}_1/m_1] dt}{m_1} - \frac{\dot{m}_1^2}{4m_1^2} \right) \sin^2 \left( \frac{\theta}{2} \right) + \left( \int_0^t \frac{[c_2 \dot{m}_2/m_2] dt}{m_2} - \frac{\dot{m}_2^2}{4m_2^2} \right) \cos^2 \left( \frac{\theta}{2} \right)
\]

\[
- \left( \int_0^t c_3 [\dot{m}_1/2m_1 + \dot{m}_2/2m_2] dt \right) \sin (\theta),
\]

\[
\Gamma(t) = - \left[ \left( \int_0^t \frac{[c_1 \dot{m}_1/m_1] dt}{m_1} - \frac{\dot{m}_1^2}{4m_1^2} \right) - \left( \int_0^t \frac{[c_2 \dot{m}_2/m_2] dt}{m_2} - \frac{\dot{m}_2^2}{4m_2^2} \right) \right] \sin (\theta)
\]

\[
+ \frac{\int_0^t c_3 [\dot{m}_1/2m_1 + \dot{m}_2/2m_2] dt}{\sqrt{m_1m_2}} \cos (\theta),
\]

\[\]
the separation of variables in eq. (9) requires the coefficient $\Gamma(t)$ to be equal to zero allowing us to determine the angle $\theta$ as

$$
\tan(\theta) = \frac{1}{\sqrt{m_1 m_2}} \int_0^t c_3 [\dot{m}_1/2m_1 + \dot{m}_2/2m_2] \, dt
$$

$$
\times \left[ \left( \frac{1}{m_1} \int_0^t [c_1 \dot{m}_1/m_1] \, dt - \frac{\dot{m}_1^2}{4m_1^2} \right) - \left( \frac{1}{m_2} \int_0^t [c_2 \dot{m}_2/m_2] \, dt - \frac{\dot{m}_2^2}{4m_2^2} \right) \right]^{-1},
$$

(13)

this last equation is the same as (32) in [1] but without clarification in terms of how it is deduced.

It is useful that we reexpress the invariant operator (9) in terms of the annihilation and creation operators $a_i$ and $a_i^\dagger$, respectively, as

$$
I(t) = \sum_{i=1}^2 \hbar \omega_i \left( a_i^\dagger a_i + \frac{1}{2} \right),
$$

(14)

where

$$
a_1 = \frac{1}{\sqrt{2\hbar \omega_1}} \left[ \omega_1 \left( \sqrt{m_1} \cos \left( \frac{\theta(t)}{2} \right) X_1 + \sqrt{m_2} \sin \left( \frac{\theta(t)}{2} \right) X_2 \right) 
+ i \left( \frac{\mathcal{P}_1}{\sqrt{m_1}} \cos \left( \frac{\theta(t)}{2} \right) + \frac{\mathcal{P}_2}{\sqrt{m_2}} \sin \left( \frac{\theta(t)}{2} \right) \right) \right],
$$

(15)

$$
a_1^\dagger = \frac{1}{\sqrt{2\hbar \omega_1}} \left[ \omega_1 \left( \sqrt{m_1} \cos \left( \frac{\theta(t)}{2} \right) X_1 + \sqrt{m_2} \sin \left( \frac{\theta(t)}{2} \right) X_2 \right) 
- i \left( \frac{\mathcal{P}_1}{\sqrt{m_1}} \cos \left( \frac{\theta(t)}{2} \right) + \frac{\mathcal{P}_2}{\sqrt{m_2}} \sin \left( \frac{\theta(t)}{2} \right) \right) \right],
$$

(16)

$$
a_2 = \frac{1}{\sqrt{2\hbar \omega_2}} \left[ \omega_2 \left( -\sqrt{m_1} \sin \left( \frac{\theta(t)}{2} \right) X_1 + \sqrt{m_2} \cos \left( \frac{\theta(t)}{2} \right) X_2 \right) 
+ i \left( -\frac{\mathcal{P}_1}{\sqrt{m_1}} \sin \left( \frac{\theta(t)}{2} \right) + \frac{\mathcal{P}_2}{\sqrt{m_2}} \cos \left( \frac{\theta(t)}{2} \right) \right) \right],
$$

(17)

$$
a_2^\dagger = \frac{1}{\sqrt{2\hbar \omega_2}} \left[ \omega_2 \left( -\sqrt{m_1} \sin \left( \frac{\theta(t)}{2} \right) X_1 + \sqrt{m_2} \cos \left( \frac{\theta(t)}{2} \right) X_2 \right) 
- i \left( -\frac{\mathcal{P}_1}{\sqrt{m_1}} \sin \left( \frac{\theta(t)}{2} \right) + \frac{\mathcal{P}_2}{\sqrt{m_2}} \cos \left( \frac{\theta(t)}{2} \right) \right) \right].
$$

(18)

Moreover, to our knowledge, the invariant operator in the Lewis and Riesenfeld theory [30] has time-independent eigenvalues whereas the frequencies $\omega_i^2$ are time-dependent which does not imply time-independent eigenvalues of the invariant as claimed in [1] and consequently it is not easy to obtain the phases.

4
Three-dimensional time-dependent harmonic oscillators

A generalization of the 2D coupled oscillator to a 3D one has been incorrectly made by Hassoul et al. in [2] where the authors study general time-dependent three coupled nano-optomechanical oscillators. The Hamiltonian operator of the 3D system reads

$$H(t) = \frac{1}{2} \sum_{i=1}^{3} \left[ \frac{P_i^2}{m_i(t)} + m_i(t) \omega_i^2(t) X_i^2 \right] + \frac{1}{2} \left[ k_{12}(t) X_1 X_2 + k_{13}(t) X_1 X_3 + k_{23}(t) X_2 X_3 \right], \quad (19)$$

where $X_i$ and $P_i$ are the canonical coordinates and momenta, $k_{12}(t)$, $k_{13}(t)$ and $k_{23}(t)$ are the coupling parameters. They choose, in this case, an invariant operator of the form

$$O(t) = \frac{1}{2} \sum_{i=1}^{3} \left[ A_i(t) P_i^2 + B_i(t) (X_i P_i + X_i P_i) + C_i(t) X_i^2 \right] + \frac{1}{2} \left[ D_{12}(t) X_1 X_2 + D_{13}(t) X_1 X_3 + D_{23}(t) X_2 X_3 \right], \quad (20)$$

where the parameters $A_i(t)$, $B_i(t)$, $C_i(t)$, $D_{12}(t)$, $D_{13}(t)$ and $D_{23}(t)$ are time-dependent real and differentiable functions.

They substitute the Hamiltonian and the invariant formulae in the invariance condition (3) to derive the formulae of the parameters in (19) where they give six differential equations (equations (8)-(13) in [2]) with their possible solutions (equations (14)-(19). In fact, they failed again in mentioning other conditions just like before, since the substitution of (19) and (20) in (3) implies nine equations given as

$$\dot{A}_i(t) = -\frac{2B_i(t)}{m_i(t)}, \quad (21)$$

$$\dot{B}_i(t) = -\frac{C_i(t)}{m_i(t)} + m_i(t) \omega_i^2(t) A_i(t), \quad (22)$$

$$\dot{C}_i(t) = 2m_i(t) \omega_i^2(t) B_i(t), \quad (23)$$

$$\dot{D}_{12}(t) = \frac{k_{12}(t)}{2} [B_1(t) + B_2(t)], \quad (24)$$

$$\dot{D}_{13}(t) = \frac{k_{13}(t)}{2} [B_1(t) + B_3(t)], \quad (25)$$

$$\dot{D}_{23}(t) = \frac{k_{23}(t)}{2} [B_2(t) + B_3(t)], \quad (26)$$

$$\frac{D_{13}(t)}{D_{12}(t)} = \frac{k_{13}(t)}{k_{12}(t)}, \quad (27)$$

$$\frac{D_{12}(t)}{D_{23}(t)} = \frac{k_{12}(t)}{k_{23}(t)}, \quad (28)$$

$$\frac{D_{23}(t)}{D_{13}(t)} = \frac{k_{23}(t)}{k_{13}(t)}. \quad (29)$$
Similarly to the 2D case, the authors of Ref. [2] pretend to obtain the solution of equations (21)-(26), omitting to mention an important detail: the following constraint equation that the mass must obey when considering \( m_i(t) = 1/\alpha_i(t) \)

\[
\ddot{m}_i(t) - \frac{1}{2} \dot{m}_i^2(t) + 2 \left( \delta_i - \omega_i^2(t) \right) m_i(t) = 0, \tag{30}
\]

which is difficult to be solved. This last equation is obtained from the auxiliary equations (21)-(26) by noting that

\[
A_i(t)C_i(t) - B_i^2(t) = \delta_i, \tag{31}
\]

with \( \delta_i \) being a real constant. Condition (31) is not mentioned in [2]. Note that putting \( \frac{1}{\rho_i} = m_i(t) \) gives the famous auxiliary equation of \( \rho_i \) \[30, 31, 32, 33\].

Thus the given solutions (equations (14)-(19) in [2]) impose a constraint on the system that the authors did not pay attention to. The system cannot be resolved for any given mass.

The authors proceed, with the aim to have a diagonalized invariant operator \( \mathcal{O}(t) \), to diagonalize the matrix \( k \) (formula (30) in [2]) using the invertible matrix \( R \) (formula (46) in [2]) as

\[
R^{-1} k R = \begin{pmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{pmatrix}.	ag{32}
\]

Note that substituting the expressions (49-51) of \( w_i \) [2] in the diagonalized invariant operator \( \mathcal{O}(t) \) does not lead to expression (28).

A straightforward evaluation of the product \( R^{-1} k R \) leads to the following elements

\[
M_{11} = \frac{1}{2} \left( \omega_1^2 + \omega_2^2 \right) + \frac{1}{4} \left( K_{13} + K_{23} \right) + \frac{1}{2} K_{12}, \tag{33}
\]

\[
M_{12} = \frac{\sqrt{3}}{2} \left[ \frac{\lambda_+ \left( \omega_1^2 - \omega_2^2 \right)}{2} \left( K_{12} - K_{23} - 2 \Omega^2 \right) + \frac{\lambda_+}{4} \left( K_{23}^2 - K_{13}^2 \right) \right], \tag{34}
\]

\[
M_{13} = \frac{\sqrt{3}}{2} \left[ \frac{\lambda_- \left( \omega_1^2 - \omega_2^2 \right)}{2} \left( K_{12} - K_{23} + 2 \Omega^2 \right) + \frac{\lambda_-}{4} \left( K_{23}^2 - K_{13}^2 \right) \right], \tag{35}
\]

\[
M_{21} = \frac{1}{4 \sqrt{3} \lambda_+ \Omega^2} \left( \frac{K_{23} - K_{12} - 2 \Omega^2}{K_{23} - K_{13}} \right) \left[ \omega_1^2 + \omega_2^2 - 2 \omega_3^2 + K_{12} - \frac{K_{13} + K_{23}}{2} \right]
+ \frac{1}{4 \sqrt{3} \lambda_+ \Omega^2} \left[ \omega_2^2 - \omega_1^2 + \frac{K_{23} - K_{13}}{2} \right], \tag{36}
\]

\[
M_{22} = \frac{1}{4 \Omega^2} \left[ \frac{(K_{23} - K_{12})^2 - 4 \Omega^4}{K_{23} - K_{13}} \right] \left( \frac{\omega_2^2 - \omega_1^2}{2} \right)
+ \frac{1}{4 \Omega^2} \left[ \frac{(\omega_1^2 + \omega_2^2)}{2} \right] \frac{K_{23} - K_{12}}{2} \frac{K_{23} - K_{12} + 2 \Omega^2}{4} + \frac{K_{23} - K_{13}}{4}
- \frac{(K_{23} - K_{12} - 2 \Omega^2)}{4 \Omega^2} \left[ \frac{K_{23} - K_{12} + 2 \Omega^2}{2} + \frac{K_{23} + K_{13}}{4} \right], \tag{37}
\]

\[
\frac{1}{4 \Omega^2} \]

\[
\frac{1}{4 \Omega^2} \left[ \frac{(K_{23} - K_{12})^2 - 4 \Omega^4}{K_{23} - K_{13}} \right] \left( \frac{\omega_2^2 - \omega_1^2}{2} \right)
+ \frac{1}{4 \Omega^2} \left[ \frac{(\omega_1^2 + \omega_2^2)}{2} \right] \frac{K_{23} - K_{12}}{2} \frac{K_{23} - K_{12} + 2 \Omega^2}{4} + \frac{K_{23} - K_{13}}{4}
- \frac{(K_{23} - K_{12} - 2 \Omega^2)}{4 \Omega^2} \left[ \frac{K_{23} - K_{12} + 2 \Omega^2}{2} + \frac{K_{23} + K_{13}}{4} \right], \tag{37}
\]

\[
\frac{1}{4 \Omega^2} \]

\[
\frac{1}{4 \Omega^2} \left[ \frac{(K_{23} - K_{12})^2 - 4 \Omega^4}{K_{23} - K_{13}} \right] \left( \frac{\omega_2^2 - \omega_1^2}{2} \right)
+ \frac{1}{4 \Omega^2} \left[ \frac{(\omega_1^2 + \omega_2^2)}{2} \right] \frac{K_{23} - K_{12}}{2} \frac{K_{23} - K_{12} + 2 \Omega^2}{4} + \frac{K_{23} - K_{13}}{4}
- \frac{(K_{23} - K_{12} - 2 \Omega^2)}{4 \Omega^2} \left[ \frac{K_{23} - K_{12} + 2 \Omega^2}{2} + \frac{K_{23} + K_{13}}{4} \right], \tag{37}
\]

\[
\frac{1}{4 \Omega^2} \]

\[
\frac{1}{4 \Omega^2} \left[ \frac{(K_{23} - K_{12})^2 - 4 \Omega^4}{K_{23} - K_{13}} \right] \left( \frac{\omega_2^2 - \omega_1^2}{2} \right)
+ \frac{1}{4 \Omega^2} \left[ \frac{(\omega_1^2 + \omega_2^2)}{2} \right] \frac{K_{23} - K_{12}}{2} \frac{K_{23} - K_{12} + 2 \Omega^2}{4} + \frac{K_{23} - K_{13}}{4}
- \frac{(K_{23} - K_{12} - 2 \Omega^2)}{4 \Omega^2} \left[ \frac{K_{23} - K_{12} + 2 \Omega^2}{2} + \frac{K_{23} + K_{13}}{4} \right], \tag{37}
\]

\[
\frac{1}{4 \Omega^2} \]
\[ M_{23} = \frac{\lambda_-}{4\lambda_+\Omega^2} \left[ \frac{(K_{23} - K_{12} - 2\Omega^2)^2}{K_{23} - K_{13}} \right] \left( \frac{\omega_2^2 - \omega_1^2}{2} \right) \]
\[ + \frac{\lambda_-}{4\lambda_+\Omega^2} \left[ \frac{(\omega_1^2 + \omega_2^2 - K_{12})}{2} \right] (K_{23} - K_{12} - 2\Omega^2) + \frac{(K_{23} - K_{13})^2}{4} \]
\[ - \frac{\lambda_- (K_{23} - K_{12} - 2\Omega^2)}{4\lambda_+\Omega^2} \left[ \frac{K_{23} - K_{12} - 2\Omega^2}{2} + \omega_3^2 - \frac{K_{23} + K_{13}}{4} \right], \] (38)

\[ M_{31} = \frac{1}{4\sqrt{3}\lambda_+\Omega^2} \left[ \frac{(K_{12} - K_{23} - 2\Omega^2)^2}{K_{23} - K_{13}} \right] \left( \frac{\omega_1^2 + \omega_2^2 - 2\omega_3^2 + K_{12} - \frac{K_{13} + K_{23}}{2}}{2} \right) \]
\[ + \frac{1}{4\sqrt{3}\lambda_+\Omega^2} \left[ \frac{\omega_1^2 - \omega_2^2 + \frac{K_{13} - K_{23}}{2}}{2} \right], \] (39)

\[ M_{32} = \frac{\lambda_+}{4\lambda_-\Omega^2} \left[ \frac{(K_{12} - K_{23} - 2\Omega^2)^2}{K_{23} - K_{13}} \right] \left( \frac{\omega_1^2 - \omega_2^2}{2} \right) \]
\[ + \frac{\lambda_+}{4\lambda_-\Omega^2} \left[ \frac{(\omega_1^2 + \omega_2^2 - K_{12})}{2} \right] (K_{12} - K_{23} - 2\Omega^2) - \frac{(K_{23} - K_{13})^2}{4} \]
\[ - \frac{\lambda_+ (K_{12} - K_{23} - 2\Omega^2)}{4\Omega^2\lambda_-} \left[ \frac{K_{23} - K_{12} + 2\Omega^2}{2} + \omega_3^2 - \frac{K_{13} + K_{23}}{4} \right], \] (40)

\[ M_{33} = \frac{\lambda_+}{4\lambda_-\Omega^2} \left[ \frac{(K_{12} - K_{23})^2 - 4\Omega^4}{K_{23} - K_{13}} \right] \left( \frac{\omega_1^2 - \omega_2^2}{2} \right) \]
\[ + \frac{\lambda_+}{4\lambda_-\Omega^2} \left[ \frac{(\omega_1^2 + \omega_2^2) - K_{12}}{2} \right] (K_{12} - K_{23} + 2\Omega^2) - \frac{(K_{23} - K_{13})^2}{4} \]
\[ - \frac{\lambda_+ (K_{12} - K_{23} - 2\Omega^2)}{4\lambda_-\Omega^2} \left[ \frac{K_{23} - K_{12} - 2\Omega^2}{2} + \omega_3^2 - \frac{K_{23} + K_{13}}{4} \right], \] (41)

the formula (35) of [2] cannot be obtained unless the parameters of k obey
\[ K_{12} = K_{13} = K_{23} \quad \text{and} \quad \omega_1^2 = \omega_2^2 = \omega_3^2, \] (42)

which implies that \( \Omega^2 = 0 \) and the eigenvalues \( \Omega_i^2 \) read
\[ \Omega_1^2 = \omega_1^2 + K_{12}, \] (43)
\[ \Omega_2^2 = \omega_1^2 - \frac{K_{12}}{2}, \] (44)
\[ \Omega_3^2 = \omega_1^2 - \frac{K_{12}}{2}. \] (45)
Furthermore, as mentioned in Section 2, it is crucial in the Lewis and Riesenfeld theory [30] for the invariant operator to have time-independent eigenvalues whereas in [2] the eigenvalues $\Omega_i^2$ are time-dependent. To see this, we calculate the time derivative of $\Omega_i^2$ as

$$\frac{d\Omega_i^2}{dt} = \frac{d}{dt} \left( \frac{D_{12}}{\sqrt{m_1m_2}} \right) + \frac{d}{dt} \left( \frac{D_{13}}{\sqrt{m_1m_3}} \right),$$

(46)

$$\frac{d\Omega_2^2}{dt} = -\frac{d}{dt} \left( \frac{D_{23}}{\sqrt{m_2m_3}} \right) + \frac{d\Omega_2^2}{dt},$$

(47)

$$\frac{d\Omega_3^2}{dt} = -\frac{d}{dt} \left( \frac{D_{23}}{\sqrt{m_2m_3}} \right) - \frac{d\Omega_2^2}{dt},$$

(48)

one can see that even if $\Omega^2 = 0$, the parameters $D_{12}, D_{13}, D_{23}$ and the masses $m_i$ are defined as time-dependent. Therefore the eigenvalues $\Omega_i^2$ are not time-independent as they are supposed to be. This is once again a fundamental error that contradicts with the Lewis Riesenfeld theory.

Finally, we suspect that the authors have calculated the phases (56) in [1] ((76) in [2]) by taking the invariant operator instead of the Hamiltonian operator in which the term $X_1X_2$ has been omitted. Knowing that the dynamics of the system is ruled by the Hamiltonian operator and not by the invariant operator. It seems that the authors take the results of [30, 31, 32, 33] and simply set $\dot{\rho} = m(t)$ as if the invariant operator is the generator of the dynamics. Apparently, they present a study of coupled systems from a quantum point of view (this has an analog in the classical theory) and claim to prove that the solution to the time-dependent Schrödinger equation with the mixed term $X_1X_2$ in the Hamiltonian can be reduced to the solution of a time-independent Schrödinger equation involving the quantum invariant. We believe that is incorrect because the coupled terms $X_1X_2$ in the Hamiltonian have a contribution and cannot be omitted.

This paper is an opportunity to draw the reader’s attention to the references [14-15] cited in [1] which contain errors in dealing with time-dependent systems without taking into account the generating function of the canonical transformation [34, 35].

It is clear from the analysis above that Hassoul et al.’s analytical expressions (23), (26)-(32), (41)-(42) and (56) in [1] and expressions (28), (38)-(45) and (76)-(80) in [2] cannot be correct. Consequently, all the physical conclusions derived from such equations, are based on wrong analytical layout.

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