Explanatory relations in arbitrary logics based on satisfaction systems, cutting and retraction

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Abstract

The aim of this paper is to introduce a new framework for defining abductive reasoning operators based on a notion of retraction in arbitrary logics defined as satisfaction systems. We show how this framework leads to the design of explanatory relations satisfying properties of abductive reasoning, and discuss its application to several logics. This extends previous work on propositional logics where retraction was defined as a morphological erosion. Here weaker properties are required for retraction, leading to a larger set of suitable operators for abduction for different logics.

Keywords: Explanatory relation, Retraction, Cutting, Satisfaction systems.

1 Introduction

Since its introduction by Charles Peirce in \cite{Peirce1932}, abduction has motivated a large body of research in several scientific fields, e.g. philosophy of science, logics, law, artificial intelligence, to mention a few. Abduction, whatever the adopted view on its treatment, involves a background theory (\(T\)), an observation also called explanandum (\(\phi\)), and an explanation (\(\psi\)). The observation may be seen as a surprising phenomenon that is inconsistent with the background theory. It may also be consistent with the background theory but not directly entailed by this theory, which is the case considered in this paper. Several constraints can be imposed on the explanations and on the process of their
production. One can allow changing the background theory, or not, consider as non relevant explanations those that entail the observation on their own without engaging the background knowledge. Hence, several forms of abduction can be defined depending on the chosen criteria. Despite their divergence, most of these models agree to define abduction as an explanatory reasoning allowing us to infer the best explanation of an observation. This contributes to the field of explainable artificial intelligence. Explanatory relations, trying to model common sense and everyday reasoning, find applications in many domains, such as diagnosis [16, 22], forensics [30], argumentation [11, 12], language understanding [31], image understanding [4, 40], etc. (it is out of the scope of this paper to describe applications exhaustively). Then, as a form of inference, several works have studied rationality postulates that are more appropriate to govern the process of selecting the best explanations, e.g. [24, 35]. From a computational point of view, a very large number of papers has tackled the definition of abductive procedures, mainly in propositional logics. An attractive approach, governed by what is called the AKM model, is based on semantic tableaux tailored for particular logics (e.g. propositional logics [3], first order and modal logics [14, 15], which was the basis for several extensions (e.g. [3, 13, 22, 28]). In these works, the explanatory reasoning process is split into two stages: (i) generating a set of hypotheses from the formulas that allow closing the open branches in the tableau constructed from \( T \cup \{ \neg \varphi \} \), and (ii) selecting the preferred solutions from this plain set by considering some of the criteria discussed above.

Our aim in this paper is to introduce a new framework for defining abductive reasoning operators in arbitrary logics in the framework of satisfaction systems. To this end, we propose on a new notion of cutting, from which operators of retraction are derived. We show that this framework leads to the design of explanatory relations satisfying the rationality postulates of abductive reasoning introduced in [35] and adapted here to the proposed more general framework, and present applications in several logics. This extends previous work on abduction in propositional logics where retraction was defined as a morphological erosion [8, 9, 10], as well as abduction in description logics for image understanding [4]. Here weaker properties are required for retraction, that allow defining a larger set of suitable operators for abduction for different logics. This approach is similar to the one proposed for revision in [1], where revision operators were defined from relaxation in satisfaction systems, and then instantiated in various logics. An important feature of the proposed explanations based on retraction is that generation and selection steps are merged, or at least the set of generated hypotheses is reduced, thus facilitating the selection step.

The paper is organized as follows. In Section 2, we recall the useful definitions and properties of satisfaction systems, and provide examples in propositional logic, Horn logic, first order logic, modal propositional logic and description logic. In Section 3 we introduce our first contribution, by defining a notion of cutting, from which explanations are then defined. In Section 4, we propose to define particular cuttings, based on retraction of formulas. Then in Section 5, we instantiate the proposed general framework in various logics.
2 Satisfaction systems

We recall here the basic notions of satisfaction systems needed in this paper. The presentation follows the one in [1], where we give a more complete presentation of satisfaction systems, including the properties and their proofs, that are omitted here.

2.1 Definition and examples

Definition 1 (Satisfaction system). A satisfaction system $\mathcal{R} = (\text{Sen}, \text{Mod}, \models)$ consists of

- a set $\text{Sen}$ of sentences,
- a class $\text{Mod}$ of models, and
- a satisfaction relation $\models \subseteq \text{Mod} \times \text{Sen}$.

Let us note that the non-logical vocabulary, so-called signature, over which sentences and models are built, is not specified in Definition 1. Actually, it is left implicit. Hence, as we will see in the examples developed in the paper, a satisfaction system always depends on a signature.

Example 1. The following examples of satisfaction systems are of particular importance in computer science and in the remainder of this paper.

Propositional Logic (PL) Given a set of propositional variables $\Sigma$, we can define the satisfaction system $\mathcal{R}_\Sigma = (\text{Sen}, \text{Mod}, \models)$ where $\text{Sen}$ is the least set of sentences finitely built over propositional variables in $\Sigma$, the symbols $\top$ and $\bot$ (denoting tautologies and antilogies, respectively), and Boolean connectives in $\{\neg, \lor, \land, \Rightarrow\}$, $\text{Mod}$ contains all the mappings $\nu : \Sigma \to \{0, 1\}$ ($0$ and $1$ are the usual truth values), and the satisfaction relation $\models$ is the usual propositional satisfaction.

Horn Logic (HCL) A Horn clause is a sentence of the form $\Gamma \Rightarrow \alpha$ where $\Gamma$ is a finite (possibly empty) conjunction of propositional variables and $\alpha$ is a propositional variable. The satisfaction system of Horn clause logic is then defined as for PL except that sentences are restricted to be conjunctions of Horn clauses.

Modal Propositional Logic (MPL) Given a set of propositional variables $\Sigma$, we can define the satisfaction system $\mathcal{R}_\Sigma = (\text{Sen}, \text{Mod}, \models)$ where

- $\text{Sen}$ is the least set of sentences finitely built over propositional variables in $\Sigma$, the symbols $\top$ and $\bot$, Boolean connectives in $\{\neg, \lor, \land, \Rightarrow\}$, and modalities in $\{\Box, \Diamond\}$;
- $\text{Mod}$ contains all the Kripke models $(I, W, R)$ where $I$ is an index set, $W = (W^i)_{i \in I}$ is a family of functions from $\Sigma$ to $\{0, 1\}$, and $R \subseteq I \times I$ is an accessibility relation;

\footnote{The set of logical symbols is defined in each particular logic and does not depend on a theory.}
• the satisfaction of sentences by Kripke models, \((I, W, R) \models \varphi\), is defined by 
  \((I, W, R) \models_i \varphi\) for every \(i \in I\) where \(\models_i\) is defined by structural induction 
  on sentences as follows:
  - \((I, W, R) \models_i p\) iff \(p \in W^i\) for every \(p \in \Sigma\),
  - Boolean connectives are handled as usual,
  - \((I, W, R) \models_i \Box \varphi\) iff \((I, W, R) \models_j \varphi\) for every \(j \in I\) such that \((i, j) \in R\), and
  - \(\Diamond \varphi\) is the same as \(\neg \Box \neg \varphi\).

First Order Logic (FOL) and Many-sorted First Order Logic We detail here only 
the many-sorted variant of FOL, FOL being a particular case. Signatures are 
triplets \((S, F, P)\) where \(S\) is a set of sorts, and \(F\) and \(P\) are sets of function 
and predicate names respectively, both with arities in \(S^* \times S\) and \(S^+\) respectively 
\((S^+\) is the set of all non-empty sequences of elements in \(S\) and \(S^* = S^+ \cup \{\epsilon\}\) 
where \(\epsilon\) denotes the empty sequence). In the following, to indicate that a function 
name \(f \in F\) (respectively a predicate name \(p \in P\)) has for arity \((s_1 \ldots s_n, s)\) 
(respectively \(s_1 \ldots s_n\)), we will note \(f : s_1 \times \ldots \times s_n \rightarrow s\) (respectively \(p : s_1 \times \ldots \times s_n\)).

Given a signature \(\Sigma = (S, F, P)\), we can define the satisfaction system 
\(\mathcal{R}_\Sigma = (\text{Sen}, \text{Mod}, \models)\) where:

- \(\text{Sen}\) is the least set of sentences built over atoms of the form \(p(t_1, \ldots, t_n)\) 
  where \(p : s_1 \times \ldots \times s_n \in P\) and \(t_i \in T_F(X)_s\) for every \(i, 1 \leq i \leq n\) 
  \((T_F(X)_s)\) is the term algebra of sort \(s\) built over \(F\) with sorted variables 
in a given set \(X\) by finitely applying Boolean connectives in \(\{\neg, \vee, \wedge, \Rightarrow\}\) 
  and quantifiers in \(\{\forall, \exists\}\).

- \(\text{Mod}\) is the class of models \(\mathcal{M}\) defined by a family \((M_s)_{s \in S}\) of non-empty 
  sets (one for every \(s \in S\)), each one equipped with a function \(f^M : M_{s_1} \times \ldots \times M_{s_n} \rightarrow M_s\) for every 
  \(f : s_1 \times \ldots \times s_n \rightarrow s \in F\) and with an \(n\)-ary 
  relation \(p^M \subseteq M_{s_1} \times \ldots \times M_{s_n}\) for every \(p : s_1 \times \ldots \times s_n \in P\).

- Finally, the satisfaction relation \(\models\) is the usual first-order satisfaction.

As for PL, we can consider the logic \(\text{FHCL}\) of first-order Horn Logic whose 
models are those of FOL and sentences are restricted to be conjunctions of 
universally quantified Horn sentences (i.e. sentences of the form \(\Gamma \Rightarrow \alpha\) where \(\Gamma\) 
is a finite conjunction of atoms and \(\alpha\) is an atom).

Description logic (DL) Signatures are triplets \((N_C, N_R, I)\) where \(N_C\), \(N_R\) and \(I\) are 
nonempty pairwise disjoint sets where elements in \(N_C\), \(N_R\) and \(I\) are called 
concept names, role names and individuals, respectively.

Given a signature \(\Sigma = (N_C, N_R, I)\), we can define the satisfaction system 
\(\mathcal{R}_\Sigma = (\text{Sen}, \text{Mod}, \models)\) where:

- \(\text{Sen}\) contains all the sentences of the form \(C \subseteq D\), \(x : C\) and \((x, y) : r\) 
  where \(x, y \in I\), \(r \in N_R\) and \(C\) is a concept inductively defined from \(N_C \cup \)

\(^2\)The description logic defined here is better known under the acronym \(\text{ALC}\).
\{\top\} and binary and unary operators in \{\_\cap, \_\cup\} and in \{\neg, \forall, \exists\}, respectively.

- \text{Mod} is the class of models \mathcal{I} defined by a set \Delta^\mathcal{I} equipped for every concept name \(A \in N_C\) with a set \(A^\mathcal{I} \subseteq \Delta^\mathcal{I}\), for every relation name \(r \in N_R\) with a binary relation \(r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}\), and for every individual \(x \in I\) with a value \(x^\mathcal{I} \in \Delta^\mathcal{I}\).

- The satisfaction relation \(\models\) is then defined as:
  - \(\mathcal{I} \models C \subseteq D\) iff \(C^\mathcal{I} \subseteq D^\mathcal{I}\),
  - \(\mathcal{I} \models x : C\) iff \(x^\mathcal{I} \in C^\mathcal{I}\),
  - \(\mathcal{I} \models (x, y) : r\) iff \((x^\mathcal{I}, y^\mathcal{I}) \in r^\mathcal{I}\),
where \(C^\mathcal{I}\) is the evaluation of \(C\) in \(\mathcal{I}\) inductively defined on the structure of \(C\) as follows:
  - if \(C = A\) with \(A \in N_C\), then \(C^\mathcal{I} = A^\mathcal{I}\);
  - if \(C = \top\) then \(C^\mathcal{I} = \Delta^\mathcal{I}\);
  - if \(C = C' \cup D'\) (resp. \(C = C' \cap D'\)), then \(C^\mathcal{I} = C'^\mathcal{I} \cup D'^\mathcal{I}\) (resp. \(C^\mathcal{I} = C'^\mathcal{I} \cap D'^\mathcal{I}\));
  - if \(C = \neg C'\), then \(C^\mathcal{I} = \Delta^\mathcal{I} \setminus C'^\mathcal{I}\);
  - if \(C = \forall r.C'\), then \(C^\mathcal{I} = \{x \in \Delta^\mathcal{I} | \forall y \in \Delta^\mathcal{I}, (x, y) \in r^\mathcal{I}\}\) implies \(y \in C'^\mathcal{I}\);
  - if \(C = \exists r.C'\), then \(C^\mathcal{I} = \{x \in \Delta^\mathcal{I} | \exists y \in \Delta^\mathcal{I}, (x, y) \in r^\mathcal{I}\}\) and \(y \in C'^\mathcal{I}\).

### 2.2 Knowledge bases and theories

Let us now consider a fixed but arbitrary satisfaction system \(\mathcal{R} = (\text{Sen}, \text{Mod}, \models)\) (since the signature \(\Sigma\) is supposed fixed, the subscript \(\Sigma\) will be omitted from now on).

**Notation 1.** Let \(T \subseteq \text{Sen}\) be a set of sentences.

- \(\text{Mod}(T)\) is the sub-class of \(\text{Mod}\) whose elements are models of \(T\), i.e. for every \(\mathcal{M} \in \text{Mod}(T)\) and every \(\varphi \in T, \mathcal{M} \models \varphi\). When \(T\) is restricted to a formula \(\varphi\) (i.e. \(T = \{\varphi\}\)), we will denote the class of model of \(\{\varphi\}\) by \(\text{Mod}(\varphi)\), rather than \(\text{Mod}(\{\varphi\})\).
- \(Cn(T) = \{\varphi \in \text{Sen} | \forall \mathcal{M} \in \text{Mod}(T), \mathcal{M} \models \varphi\}\) is the set of semantic consequences of \(T\). In the following, we will also denote \(T \models \varphi\) to mean that \(\varphi \in Cn(T)\).
- \(\varphi \equiv_T \psi\) iff \(\text{Mod}(T \cup \{\varphi\}) = \text{Mod}(T \cup \{\psi\})\).
- Let \(\mathcal{M} \subseteq \text{Mod}\). Let us note \(\mathcal{M}^* = \{\varphi \in \text{Sen} | \forall \mathcal{M} \in \mathcal{M}, \mathcal{M} \models \varphi\}\). When \(\mathcal{M}\) is restricted to one model \(\mathcal{M}\), \(\mathcal{M}^*\) will be equivalently noted \(\mathcal{M}^*\).
Let us note $\text{Triv} = \{ M \in \text{Mod} \mid M^* = \text{Sen} \}$, i.e. the set of models in which all formulas are satisfied. In PL, MPL and FOL, Triv is empty because the negation is considered. Similarly, the negation is involved in the DL ALC, hence Triv is empty. In HCL, Triv only contains the unique model where all propositional variables have a truth value equal to 1. In FHCL, Triv contains all models $M$ where for every predicate name $p : s_1 \times \ldots \times s_n \in P$, $p^M = M_{s_1} \times \ldots \times M_{s_n}$.

**Definition 2** (Knowledge base and theory). A knowledge base (KB) $T$ is a finite set of sentences (i.e. $T \subseteq \text{Sen}$ and the cardinality of $T$ belongs to $\mathbb{N}$). A set of sentences $T$ is said to be a theory if and only if $T = \text{Cn}(T)$.

A theory $T$ is **finitely representable** if there exists a KB $T' \subseteq \text{Sen}$ such that $T = \text{Cn}(T')$.

A class of models $M \subseteq \text{Mod}$ is **finitely axiomatizable** if there exists a finite KB $T$ such that $\text{Mod}(T) = M$. A satisfaction system $R$ is **finitely axiomatizable** if each of its classes of models $M \subseteq \text{Mod}$ is finitely axiomatizable.

Note that in DL, a knowledge base consists classically of a set of axioms (in the form $C \sqsubseteq D$), called TBox, and a set of assertions (in the form $x : C$ or $(x, y) : r$), called ABox.

Classically, the consistency of a theory $T$ is defined as $\text{Mod}(T) \neq \emptyset$. The problem of such a definition of consistency is that its significance depends on the considered logic. Hence, this consistency is significant for FOL, while in FHCL it is a trivial property since each set of sentences is consistent because $\text{Mod}(T)$ always contains $\text{Triv}$ which is non-empty. Here, for the notion of consistency to be more appropriate for our purpose of defining abduction for the largest family of logics, we propose a more general definition of consistency, the meaning of which is that given a theory $T$, $\text{Mod}(T)$ is not restricted to trivial models.

**Definition 3** (Consistency). $T \subseteq \text{Sen}$ is consistent if $\text{Cn}(T) \neq \text{Sen}$.

**Proposition 1** ([1]). For every $T \subseteq \text{Sen}$, $T$ is consistent if and only if $\text{Mod}(T) \setminus \text{Triv} \neq \emptyset$.

Hence, for every $T \subseteq \text{Sen}$, $T$ is inconsistent is equivalent to $\text{Mod}(T) = \text{Triv}$.

### 2.3 Internal logic

Following [17, 27], the satisfaction system-independent definition of Boolean connectives is straightforward. This will be useful when we give general results of preserving explanatory relation along Boolean connectives. Let $R$ be a satisfaction system. A sentence $\varphi'$ is a

- **semantic negation** of $\varphi$ when $\text{Mod}(\varphi') = \text{Mod} \setminus \text{Mod}(\varphi)$;
- **semantic conjunction** of $\varphi_1$ and $\varphi_2$ when $\text{Mod}(\varphi') = \text{Mod}(\varphi_1) \cap \text{Mod}(\varphi_2)$;
• **semantic disjunction** of $\varphi_1$ and $\varphi_2$ when $\text{Mod}(\varphi') = \text{Mod}(\varphi_1) \cup \text{Mod}(\varphi_2)$;

• **semantic implication** of $\varphi_1$ and $\varphi_2$ when $\text{Mod}(\varphi') = (\text{Mod} \setminus \text{Mod}(\varphi_1)) \cup \text{Mod}(\varphi_2)$.

$\mathcal{R}$ has (semantic) negation when each sentence has a negation. It has (semantic) conjunction (respectively disjunction and implication) when any two sentences have conjunction (respectively disjunction and implication). As usual, we note negation, conjunction, disjunction and implication by $\neg$, $\land$, $\lor$ and $\Rightarrow$.

**Example 2.** PL has all semantic Boolean connectives. FOL has all semantic Boolean connectives when sentences are restricted to closed formulas, otherwise (i.e. sentences can be open formulas) it only has semantic conjunction. Finally, MPL has only semantic conjunction.

### 3 Explanation in satisfaction systems

The process of inferring the best explanation of an observation is usually known as **abduction**. In a logic-based approach, the background of abduction is given by a knowledge base (KB) $T$ and a formula $\varphi$ (the observation) such that $T \cup \{\varphi\}$ is consistent. Besides this fact, which can be expressed equivalently as $T \not\models \neg \varphi$, some works further require that $T \not\models \varphi$. We do not impose this last requirement here.

Let us start by introducing the notion of explanation of $\varphi$ with respect to $T$.

**Definition 4 (Set of explanations).** Let $T$ be a KB. Let $\varphi \in \text{Sen}$ be a formula consistent with $T$. The **set of explanations** of $\varphi$ over $T$ is the set $\text{Expla}_T(\varphi)$ defined as:

$$\text{Expla}_T(\varphi) = \{ \psi \in \text{Sen} \mid \text{Mod}(T \cup \{\psi\}) \neq \text{Triv} \text{ and } T \cup \{\psi\} \models \varphi \}$$

Note that this definition does not impose that $\psi \not\models \varphi$. In some cases a preferred explanation of $\varphi$ with respect to the background knowledge base $T$ could be a formula $\psi$ such that $\psi \models \varphi$.

Since abduction aims to infer the best explanations, the notion of explanation given in Definition 4 only captures candidate explanations of $\varphi$ with respect to $T$. Some additional properties are needed to define the key notion of “preferred explanations”. Following the works in [3, 24, 25, 26, 35, 36], we will study some preference criteria and give their logical properties when abduction is regarded as a form of inference.

**Definition 5 (Explanatory relation).** Let $T$ be a KB. An **explanatory relation for $T$** is a binary relation $\triangleright \subseteq \text{Sen} \times \text{Sen}$ such that:

$$\forall \varphi, \psi \in \text{Sen}, \varphi \triangleright \psi \implies \psi \in \text{Expla}_T(\varphi)$$

Now, we define an (abstract) explanatory relation, the behavior of which will consist in cutting in $\text{Mod}(T \cup \{\varphi\})$ as much as possible but still under the constraint that it remains consistent (i.e. it is not equal to $\text{Triv}$). A cutting will then generate a sequence
of subsets of \( \mathcal{P}(\text{Mod}(T \cup \{\varphi\})) \) that we can order by inclusion. Moreover, this sequence cannot be extended by inverse inclusion. This gives rise to the notion of a cutting for a KB \( T \) and a formula \( \varphi \).

**Definition 6** (Cutting). Let \( T \) be a KB and let \( \varphi \) be a formula. A cutting for \( T \) and \( \varphi \) is any \( C \subseteq \mathcal{P}(\text{Mod}(T \cup \{\varphi\})) \) such that for every \( M \in C \), \( \text{Triv} \subseteq M \), \( C \) is closed under set-theoretical union and contains \( \text{Mod}(T \cup \{\varphi\}) \), and the poset \((C, \subseteq)\) is well-founded. \(^3\)

Let us denote \( \text{Min}(C) \) the set of minimal elements for \( \subseteq \) in \( C \).

In the following, given a KB \( T \) and a formula \( \varphi \), a cutting for \( T \) and \( \varphi \) will be denoted \( C_\varphi \). \(^4\)

Note that in Definition 6 we do not impose that \( T \cup \{\varphi\} \) is consistent ("who can do more, can do less"). However, the case where it is not would not be very interesting since \( \text{Mod}(T \cup \{\varphi\}) \setminus \text{Triv} \) would then be empty.

**Remark 1.** If \( \text{Mod}(T \cup \{\varphi\}) \neq \text{Triv} \) then there exists a trivial cutting for \( \varphi \), namely \( C_\varphi = \{\text{Mod}(T \cup \{\varphi\})\} \).

As \((C_\varphi, \subseteq)\) is closed under set-theoretical union and then it is inductive, by the Hausdorff maximal principle, every chain is contained in any maximal chain (and then maximal chains exist). Moreover, as \((C_\varphi, \subseteq)\) is well-founded, every maximal chain has a least element which belongs to \( \text{Min}(C_\varphi) \).

**Definition 7** (Explanatory relation based on cuttings). Let \( T \) be a KB, and let us define a set of cuttings \( C \) by choosing a cutting \( C_\varphi \) for every \( \varphi \) in \( \text{Sen} \): \( C = \{C_\varphi \mid \varphi \in \text{Sen}\} \).

Let us define the binary relation \( \triangleright_C \subseteq \text{Sen} \times \text{Sen} \) as follows:

\[
\varphi \triangleright_C \psi \iff \begin{cases} 
\text{Mod}(T \cup \{\psi\}) \neq \text{Triv}, \text{ and} \\
\exists M \in \text{Min}(C_\varphi), \text{Mod}(T \cup \{\psi\}) \subseteq M
\end{cases}
\]

By Remark 1 \( C \) is well defined. Obviously, \( \triangleright C \) is an explanatory relation. We will later add some stability properties to \( C \) to ensure good properties of this explanatory relation.

**Remark 2.** If \( C_\varphi \) is a cutting for \( T \) and \( \varphi \), then we can define a relation \( \triangleright_C \) based on cuttings such that \( \varphi \triangleright_C \psi \) satisfies the equivalence of Definition 7 (i.e. \( C_\varphi \) is precisely the cutting chosen for \( \varphi \) in the set \( C \)).

The next example shows how our general definition via cuttings can capture some explanatory relations defined in the literature.

\(^3\)Let us recall that a poset \((X, \subseteq)\) is well-founded if every non-empty subset \( S \subseteq X \) has a minimal element with respect to \( \subseteq \), or equivalently there does not exist any infinite descending chain.

\(^4\)To simplify the notations, \( T \) does not index cuttings because as we will see, \( T \) will be often constant.

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Example 3. Abduction via semantic tableau \[14\] and resolution \[39\] generates a cutting, and then an explanatory relation. We illustrate this fact for abduction via semantic tableau in the framework of the propositional logic \[1\].

Semantic tableaux are used as refutation systems. Let \( S \) be a set of propositional formulas. The tableau expansion rules are as follows:

\[\begin{align*}
\neg - \text{rules:} & \quad \frac{\neg \varphi \land \varphi}{\bot} \quad \frac{\neg \varphi}{\bot} \\
\alpha - \text{rules:} & \quad \frac{\varphi \land \varphi}{\varphi} \quad \frac{(\neg \varphi \Rightarrow \varphi)}{\neg \varphi} \quad \frac{(\neg \varphi \lor \varphi)}{\bot} \\
\beta - \text{rules:} & \quad \frac{\varphi \lor \varphi}{\varphi} \quad \frac{\varphi \Rightarrow \varphi}{\neg \varphi} \quad \frac{\varphi \Rightarrow \neg \varphi}{\bot}
\end{align*}\]

A tableau \( T \) is then a sequence of sets of sets of formulas \( (\Gamma_1, \ldots, \Gamma_n, \ldots) \) such that, for every \( i \), \( \Gamma_{i+1} \) is obtained from \( \Gamma_i \) by the application of a tableau expansion rule on a formula of a set \( S \) in \( \Gamma_i \). At each step \( i \), every set \( S \) in \( \Gamma_i \) which contains both \( p \) and \( \neg p \) for some propositional variable \( p \) is removed from \( \Gamma_i \).

A formula \( \varphi \) is a theorem of a KB \( T \) if there exists a finite sequence \( (\Gamma_1, \ldots, \Gamma_n) \) such that \( \Gamma_1 = \{ T \} \cup \{ \neg \varphi \} \) and \( \Gamma_n = \emptyset \). As an example, let us show that \( a \) is a theorem of \( \{ a \land c, a \Rightarrow b \} \). The tableau method provides the finite sequence \( \Gamma_1 = \{ \{ a \land c, a \Rightarrow b, \neg a \} \}, \Gamma_2 = \{ \{ a, c, a \Rightarrow b, \neg a \} \}, \) using \( \alpha \)-rules. The set \( \Gamma_2 \) contains a unique set, with both \( a \) and \( \neg a \), which is then removed, and \( \Gamma_2 \) becomes empty.

Let us observe that the tableau expansion rules break propositional formulas on their main Boolean connectives. Hence, tableaux are necessarily finite, and then two cases can occur:

1. the last set \( \Gamma_n \) of the sequence is empty, and then we have that \( T \models \varphi \); or
2. every \( S \) in \( \Gamma_n \) only contains literals but no literal has its negation in \( S \).

Following \[14\], if \( g \) is any consistent choice function for the elements of \( \Gamma_n \), i.e., for \( \Gamma_n = \{ S_1, \ldots, S_{m_n} \} \), \( g(S_n) \in S_n \), then if \( \psi = \neg g(S_{i_1}) \land \ldots \land \neg g(S_{i_m}) \) is consistent with \( T \), then \( \psi \) is an explanation of \( \varphi \) for \( T \) (\( \psi \) is even the minimal one according to the definition of minimality given in \[14\]).

We now show that the way the tableau \( T \) is generated in \[14\] defines a cutting \( C_T \). Before defining the cutting \( C_T \), let us introduce some useful notions. Let \( T = (\Gamma_1, \ldots, \Gamma_n) \) be a tableau for \( T \cup \{ \neg \varphi \} \) such that \( \Gamma_i = \{ S_1, \ldots, S_{m_i} \} \). For every \( j \), \( 1 \leq j \leq m_i \), let us denote \( \psi_{ij} \), the disjunction of the negation of all the literals \( l \in S_{ij} \), i.e., \( \psi_{ij} = \lor \{ \neg l \mid l \text{ literal and } l \in S_{ij} \} \). Then, let us set \( \psi_i = \land_{1 \leq j \leq m_i} \psi_{ij} \). We can define the cutting \( C_T \) as follows:

\[ C_T = \{ \text{Mod}(T \cup \{ \varphi \}) \} \cup \{ \bigcup_{1 \leq i \leq n} \{ \text{Mod}(\psi_i) \} \} \]

Obviously we have \( \text{Mod}(T \cup \{ \varphi \}) \subseteq C_T \) and \( \text{Triv} \notin C_T \). Moreover, for any \( i \), \( \text{Mod}(\psi_i) \subseteq \text{Mod}(T \cup \{ \varphi \}) \), hence \( C_T \in \mathcal{P}(\text{Mod}(T \cup \{ \varphi \})) \). It is not difficult to show

\[\text{Note that semantic tableau methods have been extended to modal logic} \ [6][15], \text{first-order logic} \ [33][38], \text{DL} \ [39], \text{etc.}, \text{and in the same way we would be able to generate a cutting from them.}\]
that for every \( i, 1 \leq i \leq n \), \( \text{Mod}(\psi_{i+1}) \subseteq \text{Mod}(\psi_i) \subseteq \text{Mod}(T \cup \{\varphi\}) \). Moreover, the tableau \( T \) is finite, which completes the proof that \( C_T \) is a cutting.

Let us illustrate this construction on an example. Let \( T = \{ f \Rightarrow m, t \vee s, r \Rightarrow m \} \) be the KB and let \( \varphi = m \) be the observation. The tableau method applied to \( T \cup \{\neg \varphi\} \) generates four sets \( \Gamma_1, \ldots, \Gamma_4 \) where:

- \( \Gamma_1 = \{ \{ f \Rightarrow m, t \vee s, r \Rightarrow m, -m \} \} \);
- \( \Gamma_2 = \{ \{ \neg f, t \vee s, r \Rightarrow m, -m \} \} \);
- \( \Gamma_3 = \{ \{ \neg f, t, r \Rightarrow m, m, s \Rightarrow m, -m \} \} \);
- \( \Gamma_4 = \{ \{ \neg f, t, -r, -m \} \} \).

This leads to the following formulas \( \psi_1, \ldots, \psi_4 \):

- \( \psi_1 = m \);
- \( \psi_2 = f \vee m \);
- \( \psi_3 = (f \vee m \vee t) \land (f \vee m \vee s) \);
- \( \psi_4 = (f \vee m \vee -t \vee r) \land (f \vee m \vee -s \vee r) \).

A consistent choice satisfying minimality is for instance \( f \vee r \).

It is interesting to note that there is an alternative way of looking at \( \triangleright C \). The descending chains to obtain the minimal element \( M \) provide a method to order the models of \( \text{Mod}(T \cup \{\varphi\}) \).

**Definition 8** (Relation on models). Let \( T \) be a KB and let \( \varphi \) be a formula such that \( \text{Mod}(T \cup \{\varphi\}) \neq \text{Triv} \). Let \( C_\varphi \) be a cutting for \( T \) and \( \varphi \). Let us define \( \preceq \subseteq \text{Mod} \times \text{Mod} \) as follows:

\[
\mathcal{M} \preceq_\varphi \mathcal{M}' \iff \exists C \subseteq C_\varphi, \text{s.t.} C \text{ is a maximal chain} \quad \forall \mathcal{M} \in C, \mathcal{M}' \in \mathcal{M} \Rightarrow \mathcal{M} \in \mathcal{M}'
\]

(1)

Let \( M \subseteq \text{Mod} \) and \( \preceq \) be a binary relation over \( M \). We define \( \preceq \) as \( \mathcal{M} \prec \mathcal{M}' \) if and only if \( \mathcal{M} \preceq \mathcal{M}' \) and \( \mathcal{M}' \not\prec \mathcal{M} \). We also define \( \text{Min}(\mathcal{M}, \preceq) = \{ \mathcal{M} \in \mathcal{M} \mid \forall \mathcal{M}' \in \mathcal{M}, \mathcal{M}' \not\prec \mathcal{M} \} \). Note that the relation \( \preceq_\varphi \) is reflexive, but not necessarily transitive (hence it is not a pre-order).

**Theorem 1.** Let \( C_\varphi \) be the cutting for a KB \( T \) and a formula \( \varphi \) used in the definition of \( \triangleright C \). For any \( \psi \in \text{Sen} \), the following equivalence holds:

\[
\varphi \triangleright_\varphi \psi \iff \begin{cases}
\text{Mod}(\text{Triv}, T \cup \{\psi\}) \neq \emptyset, \text{ and}
\text{Mod}(\text{Triv}, T \cup \{\psi\}) \subseteq \text{Min}(\text{Mod}(T \cup \{\psi\}) \setminus \text{Triv}, \preceq_\varphi)
\end{cases}
\]

Proof. (\( \Rightarrow \)) By definition of \( \triangleright C \), we have \( \text{Mod}(T \cup \{\psi\}) \setminus \text{Triv} \neq \emptyset \). Let us suppose \( \mathcal{M} \in \text{Mod}(T \cup \{\psi\}) \setminus \text{Triv} \). By the definition of \( \triangleright C \), the statement \( \varphi \triangleright_\varphi \psi \) means that there exists \( \mathcal{M} \in \text{Min}(\mathcal{C}_\varphi) \) such that \( \text{Mod}(T \cup \{\psi\}) \subseteq \mathcal{M} \). As \( (C_\varphi, \preceq) \) satisfies the Hausdorff maximal principle, there exists a maximal chain \( C \), the least element of which is \( M \). Hence, by the definition of \( \preceq_\varphi \), for every \( \mathcal{M}' \in C \), we have that:
Theorem 2. Let \( R \) be a satisfaction system, \( T \) a KB, \( C \) a set of cuttings and \( \triangleright_C \) the explanatory relation based on cuttings of Definition 7. The following properties are satisfied, for every \( \varphi, \varphi', \psi, \psi' \):

1. for every \( M' \in M \), \( M \preceq_{C_\varphi} M' \) and \( M' \preceq_{C_\varphi} M \), and
2. for every \( M' \in M' \setminus M \), \( M \prec_{C_\varphi} M' \).

This proves that \( M \in \text{Min}(\text{Mod}(T \cup \{\varphi\}) \setminus \text{Triv}, \preceq_{C_\varphi}) \).

\( (\Leftarrow) \) Let us suppose that \( \varphi \triangleright_C \psi \). This means that either \( \text{Mod}(T \cup \{\psi\}) = \text{Triv} \) and in this case the conclusion is obvious, or there does not exist a minimal element \( M \in \text{Min}(C_\varphi) \) such that \( \text{Mod}(T \cup \{\psi\}) \subseteq M \). Let \( M \) be the least element (for inclusion) of \( C_\varphi \) such that \( \text{Mod}(T \cup \{\psi\}) \subseteq M \). This least element \( M \) exists because \( C_\varphi \) contains \( \text{Mod}(T \cup \{\varphi\}) \) and \( (C_\varphi, \subseteq) \) is well-founded. As \( (C_\varphi, \subseteq) \) satisfies the Hausdorff maximal principle, there exists a maximal chain \( C \) which contains \( M \), and then \( M \) cannot be the least element of \( C \). Therefore, there exist some models \( M' \) which belong to some elements \( M' \in C \) such that \( \text{Mod}(T \cup \{\psi\}) \nsubseteq M' \) whence we can deduce that for some models \( M \in \text{Mod}(T \cup \{\psi\}) \setminus \text{Triv} \), we have \( M' \prec_{C_\varphi} M \).  

The explanatory relation \( \triangleright_C \) satisfies a number of logical properties. Most of these properties are (rationality) postulates defined in [35] up to some adaptations. Let us recall them, adapted to the satisfaction system context, for any KB \( T \), explanatory relation \( \triangleright \) for \( T \) and formulas \( \varphi, \varphi', \psi, \in \text{Sen} \):

- **LLE**
  \[ \varphi \equiv \varphi' \quad \frac{\varphi \triangleright \psi}{\varphi' \triangleright \psi} \]
- **RLE**
  \[ \psi \equiv \psi' \quad \frac{\varphi \triangleright \psi}{\varphi \triangleright \psi'} \]
- **E-CM**
  \[ \frac{\varphi \triangleright \psi}{\varphi \land \phi^\prime \triangleright \psi^\prime} \]
- **E-Cut**
  \[ \frac{\varphi \land \phi^\prime \triangleright \psi}{\exists \psi' (\varphi \land \phi^\prime \triangleright \psi') \land \psi'} \]
- **E-R-Cut**
  \[ \frac{\varphi \land \phi^\prime \triangleright \psi}{\exists \psi' (\varphi \land \phi^\prime \triangleright \psi') \land \psi'} \]
- **E-DR**
  \[ \frac{\varphi \triangleright \psi \lor \varphi \triangleright \psi'}{\varphi \triangleright (\psi \lor \psi')} \]
- **ROR**
  \[ \frac{\varphi \triangleright \psi \lor \varphi \triangleright \psi'}{\varphi \triangleright \psi'} \]
- **RS**
  \[ \frac{\varphi \triangleright \psi \land \psi'}{\varphi \triangleright \psi} \]
- **E-Reflexivity**
  \[ \frac{\varphi \triangleright \psi}{\varphi \equiv \psi} \]
- **E-Con**
  \[ Mod(T \cup \{\varphi\}) \neq \text{Triv} \iff \exists \psi, \varphi \triangleright \psi \]

Now, we will show that, with an appropriate structure on the set of cuttings \( C \), adding a limited set of rather intuitive stability and monotony requirements, we can get strong results on the explanatory relation \( \triangleright_C \), according to the above postulates. Recall that \( C \) is defined by choosing a cutting \( C_\varphi \) for each \( \varphi \in \text{Sen} \). A first requirement is that for every \( \varphi, \varphi' \) we have:

\[ \text{If } \varphi \equiv_T \varphi', \text{ then } C_\varphi = C_{\varphi'} \quad (2) \]

This will be directly used in Property (1) of the following Theorem.
1. Assume that $C$ satisfies Equation $2^1$ If $\varphi \equiv_T \varphi'$, $\psi \equiv_T \psi'$ and $\varphi \triangleright_C \psi$, then $\varphi' \triangleright_C \psi'$.

2. If $\varphi \triangleright_C \psi$ and $T \cup \{\psi'\} \models \psi$ with $\text{Mod}(T \cup \{\psi'\}) \neq \text{Triv}$, then $\varphi \triangleright_C \psi'$.

3. $\psi \in \text{Expl}_T(\varphi)$ if there exists a relation $\triangleright_C$ based on cuttings such that $\varphi \triangleright_C \psi$.

4. If $\mathcal{R}$ is finitely axiomatizable for every $\mathcal{M} \subseteq \text{Mod}$ and has conjunction, then for every cutting $C_\varphi$, we have that $\text{Expl}_T(\varphi) \neq \emptyset$ and $\exists \psi \in \text{Sen}, \varphi \triangleright_C \psi$, where $\triangleright_C$ is a relation based on cuttings such that the cutting associated with $\varphi$ is $C_\varphi$.

Proof. 1. The first property is obviously satisfied because $\varphi \equiv_T \varphi'$ and $\psi \equiv_T \psi'$ mean that $\text{Mod}(T \cup \{\varphi\}) = \text{Mod}(T \cup \{\varphi'\})$ and $\text{Mod}(T \cup \{\psi\}) = \text{Mod}(T \cup \{\psi'\})$ and, by assumption, $C_\varphi = C_{\varphi'}$.

2. $\varphi \triangleright_C \psi$ means that there exists $\mathcal{M} \in \text{Min}(C_\varphi)$ such that $\text{Mod}(T \cup \{\psi\}) \subseteq \mathcal{M}$, and $\text{Mod}(T \cup \{\varphi'\}) \neq \text{Triv}$. As $\text{Mod}(T \cup \{\psi'\}) \subseteq \text{Mod}(T \cup \{\psi\})$ (hence $\text{Mod}(T \cup \{\psi'\}) \subseteq C_\varphi$), and $\text{Mod}(T \cup \{\psi'\}) \neq \text{Triv}$, we can deduce that $\varphi \triangleright_C \psi'$.

3. The “if part” is obvious. To prove the “only if part”, let us notice that for every $\psi \in \text{Expl}_T(\varphi)$, we can build in $P(\text{Mod}(T \cup \{\varphi\}))$ a saturated chain $C_{\varphi'}$ starting at $\text{Mod}(T \cup \{\psi\})$. As $\psi \in \text{Expl}_T(\varphi)$, this saturated chain satisfies all the conditions of Definition $6^1$ and then it is a cutting for $T$ and $\varphi$. By Remark $2^1$ we can define the relation $\triangleright_C$ based on cuttings such that the cutting corresponding to $\varphi$ is precisely $C_{\varphi'}$. By construction of $C_{\varphi'}$, it is clear that $\varphi \triangleright_C \psi$.

4. Again by Remark $2^1$ it makes sense to consider $\triangleright_C$ as the explanatory relation defined by a family of cuttings in which the one associated with $\varphi$ is $C_\varphi$. Let $\mathcal{M} \in \text{Min}(C_\varphi)$. As $\mathcal{R}$ is finitely axiomatizable, there exists a finite KB $T'$ such that $\text{Mod}(T') = \mathcal{M}$. Let us set $\psi = \bigwedge_{\varphi' \in T'} \varphi'$. We obviously have that $T \cup T'$ is consistent, hence $\text{Mod}(T \cup \{\psi\}) \neq \text{Triv}$, and then we can conclude that $\varphi \triangleright_C \psi$.

It is interesting to note that property (1) generalizes to satisfaction systems the properties LLE and RLE of $[35]$. Similarly, Property (2) corresponds to RS, and Properties (3) and (4) to E-Con.

If $\mathcal{R}$ also has Boolean connectives in $\{\wedge, \vee, \Rightarrow\}$, the explanatory relation $\triangleright_C$ satisfies additional logical properties.

**Lemma 1.** If $C_{\varphi'}$ is a cutting for $T$ and $\varphi$, $\text{Mod}(\varphi') \subseteq \text{Mod}(\varphi)$ and $\text{Mod}(\varphi') \neq \text{Triv}$, then $C_{\varphi'} = \{M \cap \text{Mod}(\varphi') \mid M \in C_{\varphi}\} \setminus \text{Triv}$ is a cutting for $T$ and $\varphi'$.

**Proof.** For every $M \in C_{\varphi'}$, we have that $M \cap \text{Mod}(\varphi') \subseteq \text{Mod}(T \cup \{\varphi'\})$. It is clear that if $C_{\varphi'}$ is not well-founded so it is $C_{\varphi}$. Moreover, $C_{\varphi'}$ is closed by union of sets. Thus all the conditions for defining a cutting are satisfied.

**Lemma 2.** If $C_{\varphi'}$ is a cutting for $T$ and $\varphi$ and $\text{Mod}(\varphi') \cap \text{Mod}(T \cup \{\varphi\}) \neq \text{Triv}$, then $C_{\varphi'} = \{M \cap \text{Mod}(\varphi') \mid M \in C_{\varphi}\} \setminus \text{Triv}$ is a cutting for $T$ and $\varphi'$.

**Proof.** Similar to the one of the previous lemma.
Theorem 3. Let $\mathcal{R}$ be a satisfaction system with conjunction, disjunction and implication. Let $T$ be a KB, and $C$ a set of cuttings satisfying Equation 5, 6. The following properties are satisfied, for every $\varphi, \varphi', \psi, \psi'$:

5. If $\varphi \triangleright_C \psi$ and $\text{Mod}(T \cup \{\psi \land \psi'\}) \neq \text{Triv}$, then $\varphi \triangleright_C \psi \land \psi'$.

6. If $\varphi \triangleright_C \psi$ and $T \cup \{\psi\} \models \varphi'$, then $\varphi \land \varphi' \triangleright_C \psi$.

7. If $\varphi \triangleright_C \psi \land \psi'$ and $T \cup \{\psi\} \models \psi'$, then $\varphi \triangleright_C \psi$.

8. If $\varphi \land \varphi' \triangleright_C \psi$ (and $\text{Mod}(\varphi') \cap \text{Mod}(T \cup \{\varphi\}) \neq \text{Triv}$), then $\varphi \triangleright_C \psi$.

9. If $\varphi \triangleright_C \psi$, then $\varphi \lor \varphi' \triangleright_C \psi$.

10. If $\varphi \lor \varphi' \triangleright_C \psi$ and $T \cup \{\psi\} \models \varphi$, then $\varphi \triangleright_C \psi$.

11. If $\varphi \triangleright_C \psi \lor \psi'$ and $\text{Mod}(T \cup \{\psi\}) \neq \text{Triv}$, then $\varphi \triangleright_C \psi$.

12. For every cutting $C_{\varphi}$ such that $\subseteq_{C_{\varphi}}$ is total, if $\varphi \triangleright_C \psi$ and $\varphi \triangleright_C \psi'$, then $\varphi \triangleright_C \psi \lor \psi'$.

13. If $(\varphi \rightarrow \varphi') \triangleright_C \psi$ and $T \cup \{\psi\} \models \varphi$, then $\varphi' \triangleright_C \psi$.

14. If $\varphi \triangleright_C \psi$, then $\psi \triangleright_C \psi$, for $C_{\varphi} = \{M \cap \text{Mod}(\psi) | M \in C_{\varphi}\}$.

Proof. 5. By hypothesis, there exists $M \in \text{Min}(C_{\varphi})$ such that $\text{Mod}(T \cup \{\psi\}) \subseteq M$. Obviously, we have that $\text{Mod}(T \cup \{\psi \land \psi'\}) \subseteq \text{Mod}(T \cup \{\psi\})$, and then as $T \cup \{\psi \land \psi'\}$ is consistent, we can deduce that $\varphi \triangleright_C \psi \land \psi'$.

6. By hypothesis, there exists $M \in \text{Min}(C_{\varphi})$ such that $\text{Mod}(T \cup \{\psi\}) \subseteq M$. As we have further $\text{Mod}(T \cup \{\psi\}) \subseteq \text{Mod}(\varphi')$, we have that $\text{Mod}(T \cup \{\psi\}) \subseteq M \cap \text{Mod}(\varphi')$. By Lemma 2, $C_{\varphi \land \varphi'}$ is a cutting, and then we obviously have that $\text{Min}(C_{\varphi \land \varphi'}) = \{M \cap \text{Mod}(\varphi') | M \in \text{Min}(C_{\varphi})\}$. We can then conclude that $\varphi \land \varphi' \triangleright_C \psi$.

7. By hypothesis, there exists $M \in \text{Min}(C_{\varphi})$ such that $\text{Mod}(T \cup \{\psi \land \psi'\}) \subseteq M$. As $T \cup \{\psi\} \models \psi'$, we have that $\text{Mod}(T \cup \{\psi \land \psi'\}) = \text{Mod}(T \cup \{\psi\})$, whence we can deduce that $\varphi \triangleright_C \psi$. 

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8. Let us notice that if $C_\varphi$ is a cutting and $\text{Mod}(\varphi') \cap \text{Mod}(T \cup \{\varphi\}) \neq \text{Triv}$ then so is $C_{\varphi \land \varphi'}$ by Lemma 2 and $\text{Min}(C_{\varphi \land \varphi'}) = \{M' \cap \text{Mod}(\varphi') \mid M' \in \text{Min}(C_\varphi)\}$. By hypothesis, there exists $M \in \text{Min}(C_{\varphi \land \varphi'})$ such that $\text{Mod}(T \cup \{\psi\}) \subseteq M$. Hence, there exists $M' \in C_\varphi$ such that $M = M' \cap \text{Mod}(\varphi')$, and then $\text{Mod}(T \cup \{\psi\}) \subseteq M'$, whence we can deduce that $\varphi \triangleright_C \psi$.

9. If $C_{\varphi \lor \varphi'}$ is a cutting, then so is $C_\varphi$ by Lemma 1. By hypothesis, there exists $M \in \text{Min}(C_\varphi)$ such that $\text{Mod}(T \cup \{\psi\}) \subseteq M$. Hence, there exists $M' \in \text{Min}(C_{\varphi \lor \varphi'})$ such that $M = M' \cap \text{Mod}(\varphi)$, and then $\text{Mod}(T \cup \{\psi\}) \subseteq M'$, whence we can conclude that $\varphi \lor \varphi' \triangleright_C \psi$.

Properties (10), (13) and (14) can be proved similarly to Property (6), and Property (11) is a direct consequence of the fact that $\text{Mod}(\psi) \subseteq \text{Mod}(\psi \lor \psi')$.

Let us finish by the proof of Property (12). By hypothesis that $\leq_{C_\varphi}$ is total, the poset $(C_\varphi, \subseteq)$ contains a unique maximal chain $C$. Hence, both $\text{Mod}(T \cup \{\psi\})$ and $\text{Mod}(T \cup \{\psi'\})$ are included in the unique minimal element $M$ of $C$. Obviously, we have that $\text{Mod}(T \cup \{\psi \lor \psi'\}) \subseteq M$, whence we can conclude that $\varphi \lor \varphi' \triangleright_C \psi$. □

Properties (6), (12) and (14) are extensions of the postulates E-CM, ROR and E-Reflexivity defined in [35]. Properties (8) and (9) are revisited forms of the postulates E-C-Cut and LOR, adapted to satisfaction systems and explanations based on cuttings.

The following two results are easy to prove, and therefore we omit the proof.

Lemma 3. Let $C_\varphi$ and $C_{\varphi'}$ be cuttings for $T$ and $\varphi$ (respectively $\varphi'$). Then the set $C_\varphi \oplus C_{\varphi'} = \{A \cup B \mid A \in C_\varphi$ and $B \in C_{\varphi'}\}$ is a cutting for $T$ and $\varphi \lor \varphi'$.

Corollary 1. Suppose that $\triangleright_C$ is an explanatory relation defined on cuttings such that $C_\varphi$ and $C_{\varphi'}$ are the cuttings for $T$ and $\varphi$ and for $T$ and $\varphi'$ respectively. Furthermore, suppose that the cutting for $\varphi \lor \varphi'$ is precisely $C_\varphi \oplus C_{\varphi'}$. If $\varphi \triangleright_C \psi$ or $\varphi' \triangleright_C \psi$, then $\varphi \lor \varphi' \triangleright_C \psi$.

This result is a stronger version of E-DR.

Concerning E-R-Cut we have to note that property (8) is a stronger version of E-R-Cut.

4 Cutting based on retraction

In this section, we introduce some more constraints on cuttings, to be able to propose concrete examples in various logics in Section 5. The idea is to define particular cuttings from “retractions”, that consist in transforming any KB $T$ into a new consistent one $T'$ such that $\text{Mod}(T') \subseteq \text{Mod}(T)$. This notion of retraction draws inspiration from Bloch & al.’s works in [8, 9, 10] on Morpho-Logics where some retractions have been defined based on erosions from mathematical morphology [12] (see Section 5). Here, we propose to generalize this notion in the framework of satisfaction systems, and a retraction is defined on $\text{Sen}$ as follows.
Definition 9 (Retraction). A retraction is a mapping $\kappa : \text{Sen} \to \text{Sen}$ satisfying, for every $\varphi \in \text{Sen}$ such that $\text{Mod}(\varphi) \neq \text{Triv}$, the two following properties:

- **Anti-extensivity:** $\text{Mod}(\kappa(\varphi)) \subseteq \text{Mod}(\varphi)$.
- **Vacuum:** $\exists k \in \mathbb{N}, \text{Mod}(\kappa^k(\varphi)) = \text{Triv}$ where $\kappa^0$ is the identity mapping, and for all $k > 0$, $\kappa^k(\varphi) = \kappa(\kappa^{k-1}(\varphi))$.

The condition for $\varphi$ not to be a tautology in Definition 9 allows us to eliminate the trivial case where $(\bigwedge T) \land \varphi$ is a tautology for a KB $T$ and an observation $\varphi$, and then in this case $\varphi$ would not deserve any explanation from $T$.

Example 4. Many examples of retractions can be defined in PL. Here, we propose to define retractions from the tableau expansion rules given in Example 3. In Section 5.1, we will study another retraction for PL but based on erosions from mathematical morphology.

Let us recall that the tableau expansion rules break propositional formulas on their main Boolean connectives, and only the $\beta$-rules require a choice. Hence, given a choice such as for instance choosing at each time the left element of the formula (i.e. the formula $\varphi_1$ in $\beta$-rules), we start by inductively defining a mapping $h : \text{Sen} \to \text{Sen}$ as follows:

1. $h(p) = p$ for every $p \in \Sigma$;
2. $h(\neg \varphi) = h(\varphi)$;
3. $h(\neg \top) = \top$;
4. $h(\varphi_1 \land \varphi_2) = h(\varphi_1) \land h(\varphi_2)$;
5. $h(\neg(\varphi_1 \Rightarrow \varphi_2)) = h(\varphi_1) \land h(\neg \varphi_2)$;
6. $h(\neg(\varphi_1 \lor \varphi_2)) = h(\neg \varphi_1) \lor h(\neg \varphi_2)$;
7. $h(\varphi_1 \lor \varphi_2) = h(\varphi_1)$;
8. $h(\varphi_1 \Rightarrow \varphi_2) = h(\neg \varphi_1)$;
9. $h(\neg(\varphi_1 \land \varphi_2)) = h(\neg \varphi_1)$.

We could have just as easily defined the mapping $h$ as follows: the first six cases are identical, and

- $h(\varphi_1 \lor \varphi_2) = h(\varphi_2)$ or $h(\varphi_1 \lor \varphi_2) = h(\varphi_1) \land h(\varphi_2)$;
- $h(\varphi_1 \Rightarrow \varphi_2) = h(\varphi_2)$ or $h(\varphi_1 \Rightarrow \varphi_2) = h(\varphi_1) \land h(\varphi_2)$;
- $h(\neg(\varphi_1 \land \varphi_2)) = h(\neg \varphi_2)$ or $h(\neg(\varphi_1 \land \varphi_2)) = h(\neg \varphi_1) \land h(\neg \varphi_2)$. 

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Hence, each application of the mapping $h$ corresponds to a step of a path in the tableau. By structural induction on $\varphi$, it is easy to show that $\text{Mod}(h(\varphi)) \subseteq \text{Mod}(\varphi)$, hence $h$ verifies the anti-extensivity property. Moreover, it is also obvious to show that $h(h(\varphi)) = h(\varphi)$, and then except if $\varphi$ is an antilogy, we cannot have $\text{Mod}(h^k(\varphi)) = \text{Triv}$ for some $k$. This means that $h$ does not satisfy the vacuum property. Hence, $h$ is not a retraction. Now it is quite obvious to define a retraction from $h$. Indeed, given a mapping $h$ defined as previously, let us define the retraction $\kappa_h : \text{Sen} \to \text{Sen}$ as follows:

$$\kappa_h(\varphi) = \begin{cases} \bot & \text{if } h(\varphi) = \varphi \\ h(\varphi) & \text{otherwise} \end{cases}$$

It is not difficult to show that $\kappa_h$ is a retraction.

Here, we introduce two cuttings based on retraction: $C_{\text{clcr}}$ (last consistent retraction) and $C_{\text{lnr}}$ (last non-trivial retraction). In $C_{\text{clcr}}$ (respectively in $C_{\text{lnr}}$), we define a unique sequence of ordered models (cf. Proposition 3) which approximates the most central part of $T$ (respectively of $T \cup \{\varphi\}$).

**Definition 10** (Cuttings based on retraction). Let $\kappa$ be a retraction. Let $T \subseteq \text{Sen}$ be a KB and $\varphi$ be a sentence such that $T \cup \{\varphi\}$ is consistent. Let us define the two subsets of $P(\text{Mod}(T \cup \{\varphi\}))$ as follows:

$$C_{\text{clcr}} = \{\text{Mod}(\kappa^k(\bigwedge T) \land \varphi) \mid k \in \mathbb{N}, \text{Mod}(\kappa^k(\bigwedge T) \land \varphi) \neq \text{Triv}\}$$  \hspace{1cm} (7)

$$C_{\text{lnr}} = \{\text{Mod}(\kappa^k(\bigwedge T) \land \varphi) \mid k \in \mathbb{N}, \text{Mod}(\kappa^k(\bigwedge T) \land \varphi) \neq \text{Triv}\}$$  \hspace{1cm} (8)

where $\bigwedge T = \varphi_1 \land \ldots \land \varphi_n$ if $T = \{\varphi_1 \land \ldots \land \varphi_n\}$.

**Proposition 2.** $C_{\text{clcr}}$ and $C_{\text{lnr}}$ as defined in Equations 7 and 8 are cuttings for $T$ and $\varphi$.

**Proof.** Let us observe that in $C_{\text{clcr}}$ (respectively in $C_{\text{lnr}}$) we have a unique maximal chain of finite size the least element of which is $\text{Mod}(\kappa^n(\bigwedge T) \land \varphi)$ (respectively $\text{Mod}(\kappa^n(\bigwedge T \land \varphi))$) where $n = \sup\{k \in \mathbb{N} \mid \text{Mod}(\kappa^k(\bigwedge T) \land \varphi) \neq \text{Triv}\}$ (respectively $\text{Mod}(\kappa^n(\bigwedge T \land \varphi)) \neq \text{Triv}$) (by the vacuum property such a $n$ exists). Obviously, both sets are closed under set-theoretical inclusion and are well-founded. \(\square\)

**Proposition 3.** Both $\preccurlyeq_{C_{\text{clcr}}}$ and $\preccurlyeq_{C_{\text{lnr}}}$ derived from the cuttings defined in Equations 7 and 8 as in Equation 1 are total pre-orders.

**Proof.** This is a direct consequence of the fact that both $C_{\text{clcr}}$ and $C_{\text{lnr}}$ have a unique maximal chain. \(\square\)

Following Definition 7 these two cuttings give rise to two explanatory relations defined as follows:

$$\varphi \triangleright_{C_{\text{clcr}}} \psi \iff \begin{cases} \text{Mod}(T \cup \{\psi\}) \neq \text{Triv}, \quad \text{and} \\ \text{Mod}(T \cup \{\psi\}) \subseteq \text{Mod}(\kappa^n(T) \cup \{\varphi\}) \end{cases}$$
ϕ \triangleright_{c_{lr}} \psi \iff \begin{cases} \text{Mod}(T \cup \{ \psi \}) \neq \text{Triv}, \text{ and} \\ \text{Mod}(T \cup \{ \psi \}) \subseteq \text{Mod}(\kappa_n(T \cup \{ \varphi \})) \end{cases}

where \( n = \sup \{ k \in \mathbb{N} \mid \text{Mod}(\kappa^k(T \cup \{ \varphi \})) \neq \text{Triv} \} \) (respectively \( n = \sup \{ k \in \mathbb{N} \mid \text{Mod}(\kappa^k(T \cup \{ \varphi \})) \neq \text{Triv} \} \).

**Corollary 2.** Both explanation relations \( \triangleright_{c_{lr}} \) and \( \triangleright_{c_{lnr}} \) satisfy all the logical properties of Theorems 2 and 3.

**Proof.** Again this is derived from the fact that both \( C_{lr} \) and \( C_{lnr} \) have a unique maximal chain. Only the property (12) of Theorem 3 requires for \( \leq_c \) to be total. This has been proved for \( \leq_{c_{lr}} \) and \( \leq_{c_{lnr}} \) in Proposition 3.

An example showing that these two definitions may provide different explanations of the same \( \varphi \) is given in the case of \( \text{PL} \) in Section 5.1.

## 5 Applications

In this section, we illustrate our general approach by defining abduction based on retractions for the logics \( \text{PL} \), \( \text{HCL} \), \( \text{FOL} \), \( \text{MPL} \) and the \( \text{DL} \) \( \text{ALC} \).

### 5.1 Explanatory relations based on retraction in \( \text{PL} \)

Here, drawing inspiration from Bloch & al.’s works in [8, 9, 10] on Morpho-Logics, we define retractions based on erosions from mathematical morphology [7]. To define retractions in \( \text{PL} \), we will apply set-theoretic morphological operations. First, let us recall basic definitions of erosion in mathematical morphology [7]. In complete lattices, an algebraic erosion is an operator that commutes with the infimum of the lattices. Concrete definitions of erosions often involve the notion of structuring element. Let us first consider the case of a lattice defined as the power set of some set (e.g. \( \mathbb{R}^n \)), with the inclusion relation. Let \( X \) and \( B \) be two subsets of \( \mathbb{R}^n \). The erosion of \( X \) by the structuring element \( B \), denoted by \( E_B(X) \), is defined as follows:

\[
E_B(X) = \{ x \in \mathbb{R}^n \mid B_x \subseteq X \}
\]

where \( B_x \) denotes the translation of \( B \) at \( x \). More generally, erosions in any space can be defined in a similar way by considering the structuring element as a binary relationship between elements of this space.

In \( \text{PL} \), knowing that we can identify any propositional formula \( \varphi \) with its set of interpretations \( \text{Mod}(\varphi) \), this leads to the following erosion of a formula \( \varphi \):

\[
\text{Mod}(E_B(\varphi)) = \{ \nu \in \text{Mod} \mid B_\nu \subseteq \text{Mod}(\varphi) \}
\]

where \( B_\nu \) contains all the models that satisfy some relationship with \( \nu \). The relationship standardly used is based on a discrete distance \( \delta \) between models, and the most commonly used is the Hamming distance \( d_H \) where \( d_H(\nu, \nu') \) for two propositional models...
over a same signature $\Sigma$ is the number of propositional symbols that are instantiated differently in $\nu$ and $\nu'$. In this case, we can rewrite the erosion of a formula as follows:

$$\text{Mod}(E_B(\varphi)) = \{\nu \in \text{Mod} | \forall \nu' \in \text{Mod}, \delta(\nu, \nu') \leq 1 \Rightarrow \nu' \in \text{Mod}(\varphi)\}$$

This consists in using the distance ball of radius 1 as structuring element. To ensure the non-consistency condition to our retraction based on erosion, we need to add a condition on distances, the differentiation property.

**Definition 11** (Differentiation property). Let $\delta$ be a discrete distance over a set $S$. Let us note $\Gamma_x$ for every $x \in S$, the set $\Gamma_x = \{y \in S | \delta(x, y) \leq 1\}$. The distance $\delta$ has the differentiation property if for every $x, y \in S$, $\Gamma_x \neq \Gamma_y$.

The Hamming distance trivially satisfies the differentiation property.

**Proposition 4.** $E_B$ is a retraction for finite signatures $\Sigma$, and when it is based on a distance between models that satisfies the differentiation property.

**Proof.** It is anti-extensive since any erosion defined from a reflexive relationship is anti-extensive. Since $\delta$ is a distance, $\delta(\nu, \nu) = 0$ for any $\nu$ and $\nu \in B_\nu$, and thus for every $\varphi$ and for every model $\nu \in \text{Mod}(E_B(\varphi))$, we have that $\nu \in \text{Mod}(\varphi)$. Let $\varphi$ be a propositional formula such that $\text{Mod}(\varphi) \neq \text{Mod}(\Sigma)$. As $\delta$ satisfies the differentiation property, there necessarily exists a model $\nu \in \text{Mod}(\varphi)$ and a model $\nu' \in \text{Mod}(\Sigma) \setminus \text{Mod}(\varphi)$ such that $\delta(\nu, \nu') \leq 1$ and $\nu' \notin \text{Mod}(\varphi)$. Hence, each application of $E_B$ removes at least one model. As $\Sigma$ is a finite signature, $\text{Mod}(\varphi)$ is finite, and then there is $k \in \mathbb{N}$ such that $\text{Mod}(E_B^k(\varphi)) = \emptyset$.

Let us first illustrate the instantiation of the two proposed definitions of explanation, when the retraction is an erosion using a ball of the Hamming distance as structuring element. Let us consider three propositional variables $a, b, c$, a KB $T = \{a \lor b \lor c\}$, and the observation to be explained $\varphi = (a \land \neg b \land c) \lor (a \land b \land \neg c) \lor (a \land \neg b \land \neg c)$. Models can be graphically represented as the vertices of a cube, as shown in Figure 1 (for instance the bottom left vertex is $\neg a \land \neg b \land \neg c$ while the top right one is $a \land b \land c$).

![Figure 1: An example of last consistent erosion.](image)

---

6As the negation is considered in PL, the set $\text{Triv}$ is empty, and then the consistency of a formula $\varphi$ can be defined by the fact that $\text{Mod}(\varphi) = \emptyset$. 

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It is easy to show that $\varepsilon(T)$ is consistent with $\varphi$, but $\varepsilon^2(T)$ is not [9]. Hence, for $\bowtie_{clcr}$, explanations $\psi$ are such that $\text{Mod}(\psi) \subseteq \{(a \land \neg b \land c) \lor (a \land b \land \neg c)\}$. Similarly, it is easy to see that $\varepsilon(T \land \varphi) = \bot$, hence the explanations $\psi$ for $\bowtie_{clcr}$ are such that $\text{Mod}(\psi) \subseteq \text{Mod}(T \land \varphi)$. In particular $\psi = (a \land \neg b \land \neg c)$ is a potential explanation of $\varphi$ for $\bowtie_{clcr}$ but not for $\bowtie_{clcr}$. This is an example where the two proposed explanatory relations introduced in Definition [10] provide different results.

Let us now show that the choice of the structuring element used in the erosions can impact the obtained explanations. This example is adapted from [9]. Let us consider the explanatory relation $\bowtie_{clcr}$, the KB $T = \{a \Rightarrow c, b \Rightarrow c, a \lor b\}$, and the observation $c$.

1. With the standard ball $B_\omega = \{\omega' \in \text{Mod} \mid d_H(\omega, \omega') \leq 1\}$, where $d_H$ denotes the Hamming distance, we get $\varepsilon^1(T) = \bot$. Thus, we have in particular,

   $c \bowtie_{clcr} (a \land b)$.

2. Now we use $B^{ab}_\omega = \{\omega' \in B_\omega \mid \omega(x) = \omega'(x) \text{ for all } x \notin \{a, b\}\}$, i.e. $B^{ab}_\omega$ contains the valuations in $B_\omega$ that agree with $\omega$ outside $\{a, b\}$. Then $\varepsilon^1(T) = a \land b \land c$ and $\varepsilon^2(T) = \bot$. Thus

   $c \bowtie_{clcr} (a \land b)$.

Notice that $c \not\bowtie_{clcr} (a \lor b)$.

3. Finally, let us consider the following structuring element

   $B^{ab}_\omega = \{\omega \cup \{\omega' \in \text{Mod} \mid d_H(\omega, \omega') = 2 \text{ and } \omega(x) = \omega'(x) \text{ for all } x \notin \{a, b\}\}$

Then $\varepsilon^1(T) = \varepsilon^2(T) = (\neg a \land b \land c) \lor (a \land \neg b \land c)$. Thus,

   $c \bowtie_{clcr} (a \land \neg b) \lor (\neg a \land b)$.

Notice that $c \not\bowtie_{clcr} (a \land b)$.

These different results can be interesting in situations where different explanations may be expected. This is illustrated by the following examples, that are further discussed in [9]:

1. $a = \text{rained\_last\_night}$
   $b = \text{sprinkle\_was\_on}$
   $c = \text{grass\_is\_wet}$

   The “common sense cautious explanation” of $c$ is $a \lor b$.

2. $a = \text{low\_taxes}$
   $b = \text{investment\_increases}$
   $c = \text{economy\_grows}$

An explanation that enhances the chances of achieving the goal of making the economy to grow is $a \land b$. 

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3. 

\[ a = \text{book\_was\_left\_somewhere\_else} \]
\[ b = \text{somebody\_took\_the\_book} \]
\[ c = \text{book\_is\_not\_in\_the\_shelf} \]

An explanation based on the principle of the “Ockham’s razor” will select either
\[ a \text{ or } b \] but not both, that is to say, \((a \land \neg b) \lor (\neg a \land b)\).

5.2 Explanatory relations based on retraction in HCL

Following our work in [1] on the definition of revision operators based on relaxations
for HCL, we propose here to extend retractions that we have defined in the framework
PL to deal with the Horn fragment of propositional formulas. First, let us recall some
useful notions.

**Definition 12 (Model intersection).** Given a propositional signature \( \Sigma \) and two \( \Sigma \)-
models \( \nu, \nu' : \Sigma \rightarrow \{0, 1\} \), we note \( \nu \cap \nu' : \Sigma \rightarrow \{0, 1\} \) the \( \Sigma \)-model defined by:

\[
\nu \cap \nu' (p) = \begin{cases}
1 & \text{if } \nu (p) = \nu' (p) = 1 \\
0 & \text{otherwise}
\end{cases}
\]

Given a set of \( \Sigma \)-models \( S \), we note

\[
\text{cl}_\cap (S) = S \cup \{ \nu \cap \nu' \mid \nu, \nu' \in S \}
\]

which is the closure of \( S \) under intersection of positive atoms.

For any set \( S \) closed under intersection of positive atoms, there exists a Horn
sentence \( \varphi \) that defines \( S \) (i.e. \( \text{Mod}(\varphi) = S \)). Given a distance \( \delta \) between models,
we then define a retraction \( \kappa \) as follows: for every Horn formula \( \varphi \), \( \kappa(\varphi) \) is any Horn
formula \( \varphi' \) such that \( \text{Mod}(\varphi') = \text{cl}_\cap (\text{Mod}(\text{EB}(\varphi))) \) (by the previous property, we
know that such a formula \( \varphi' \) exists).

**Proposition 5.** With the same conditions as in Proposition 4, the mapping \( \rho \) is a
retraction.

Again both explanatory relations \( \triangleright_{tcr} \) and \( \triangleright_{lnr} \) can be defined from \( \kappa \) using Definition 10.

5.3 Explanatory relations based on retraction in FOL

A trivial way to define a retraction in FOL is to map any formula to an antilogy. A
less trivial and more interesting retraction consists in replacing existential quantifiers
involved in the formula to be retracted by universal ones. A dual approach has been
adopted in [1] for defining revision operators using dilations in FOL. In the following
we suppose that, given a signature, every formula \( \varphi \) in \( \text{Sen} \) is a conjunction of formulas
in prenex form (i.e. \( \varphi \) is of the form \( \bigvee_j Q_j x_1^j \ldots Q_{n_j} x_{n_j}^j \psi_j \) where each \( Q_i^j \) is in \( \{\forall, \exists\} \)). Let us define the retraction \( \kappa \) as follows, for an antilogy \( \tau \):
Hence, we have many possible explanations such as the trivial one following normal form because in this case we have that is always reflexive) where the formula modal logics such as such that for every or formula and defined for by replacing which is as well as the prenex form of anti-extensive, and satisfies the vacuum property because in a finite number of steps, we always reach the antilogy .

**Proposition 6.** \( \kappa \) is a retraction.

**Proof.** \( \kappa \) is obviously anti-extensive, and satisfies the vacuum property because in a finite number of steps, we always reach the antilogy \( \tau \). □

**Example 5.** To illustrate our approach, let us consider the example taken from [14] and defined by the KB which only contains the formula \( \forall x.\forall y.\forall z.(p(x, y) \land p(y, z) \Rightarrow p(x, z)) \) and the observation \( \varphi = \exists w.p(w, w) \). According to the explanatory relation we consider (i.e. either \( \triangleright_{\text{ler}} \) or \( \triangleright_{\text{lnr}} \), the retracted formula will be different. For \( \triangleright_{\text{ler}} \), only the formula \( \forall x.\forall y.\forall z.(p(x, y) \land p(y, z) \Rightarrow p(x, z)) \) is retracted. But in this case, to preserve consistency, the maximum number of retraction steps to apply is 0. Hence, we have many possible explanations such as the trivial one \( \exists w.p(w, w) \) (i.e. \( \varphi \triangleright_{\text{ler}} \varphi \)). The minimal explanation \( \exists x.\exists y.p(x, y) \land p(y, x) \) given in [14] also satisfies \( \varphi \triangleright_{\text{lnr}} \exists x.\exists y.p(x, y) \land p(y, x) \).

For \( \triangleright_{\text{lnr}} \), we can directly retract the formula \( \forall x.\forall y.\forall z.(p(x, y) \land p(y, z) \Rightarrow p(x, z)) \land \exists w.p(w, w) \), but in this case to preserve consistency, the maximum number of retraction steps is 0, and we come up with the previous case. Now, we can also consider the prenex form of \( \forall x.\forall y.\forall z.(p(x, y) \land p(y, z) \Rightarrow p(x, z)) \land \exists w.p(w, w) \) which is \( \forall x.\forall y.\forall z.\exists w.(p(x, y) \land p(y, z) \Rightarrow p(x, z)) \land p(w, w) \). Here, to preserve consistency, the maximum number of retraction steps to apply is 1. We then obtain the formula \( \forall x.\forall y.\forall z.\exists w.(p(x, y) \land p(y, z) \Rightarrow p(x, z)) \land p(w, w) \), and then a possible explanation here is \( \varphi \triangleright_{\text{lnr}} \forall w.p(w, w) \). In contrast, we now have that \( \varphi \not\triangleright_{\text{lnr}} \exists x.\exists y.p(x, y) \land p(y, x) \).

### 5.4 Explanatory relations based on retraction in MPL

By the classical first-order correspondence of MPL, we can easily adapt the retraction defined for FOL by replacing \( \diamond \) by \( \Box \). Now, we can go further when dealing with formulas of the form \( \Box \ldots \Box \varphi \). Indeed, in MPL, we have that \( \text{Mod}(\varphi) \subseteq \text{Mod}(\Box \varphi) \). Hence, we can remove in formulas the most external \( \Box \). Of course, when dealing with modal logics such as T, S4, B and S5 (i.e. the accessibility relation of Kripke models is always reflexive) where the formula \( \Box \varphi \Rightarrow \varphi \) is a tautology, this is of no interest because in this case we have that \( \varphi \equiv \Box \varphi \). This gives rise to the following retraction \( \kappa \): here also we suppose that every formula \( \varphi \) in \( \text{Sen} \) is a conjunction of formulas \( \varphi \) in the following normal form \( \varphi = M_1 \ldots M_n, \psi \) where each \( M_i \) is in \{\( \Box, \Diamond \}\}).

- \( \kappa(\tau) = \tau \) if \( \tau \) is any antilogy;
• $\kappa(\varphi) = \tau$ if $\varphi$ is modality free;

• $\kappa(\square \varphi) = \varphi$;

• Let $\varphi = M_1 \ldots M_n \psi$ be a formula such that the set $E_\varphi = \{i, 1 \leq i \leq n \mid M_i = \Diamond \} \neq \emptyset$. Then, $\kappa(M_1 \ldots M_n \varphi) = \bigvee_{i \in E_\varphi} \varphi_i$ where $\varphi_i = M'_1 \ldots M'_n \psi_i$ such that for every $j \neq i, 1 \leq j \leq n, M'_j = M_j$ and $M'_i = \square$;

• $\kappa(\bigwedge_{j} M^j_1 \ldots M^j_n \psi) = \bigwedge_{j} \kappa(M^j_1 \ldots M^j_n \psi)$.

**Proposition 7.** $\kappa$ is a retraction.

**Proof.** Similarly to Proposition 6 and the fact that $\text{Mod}(\varphi) \subseteq \text{Mod}(\square \varphi)$, $\kappa$ is anti-extensive and satisfies the vacuum property. \(\square\)

### 5.5 Explanatory relations based on retraction in DL

Abduction in DL can take different forms: concept abduction, TBox abduction, ABox abduction and knowledge base abduction (see e.g. [23, 29, 32]).

**Definition 13 (Abduction types in DL).** Let $\mathcal{L}, \mathcal{L}'$ be two arbitrary description logics, $\mathcal{K} = (T, A)$ a knowledge base in $\mathcal{L}$ with $T$ the TBox and $A$ the Abox, $C, D$ two concepts in $\mathcal{L}$ satisfiable with respect to $\mathcal{K}$ (i.e. that admit non empty interpretations). Abduction forms in DL are as follows:

- **Concept abduction:** given an observation concept $O$ in $\mathcal{L}$ satisfiable w.r.t. $\mathcal{K}$, the set of explanations introduced in Definition 4 writes as:

$$\text{Expla}_{\mathcal{K}}(O) = \{ H \mid H \text{ satisfiable w.r.t. } \mathcal{K} \text{ and } \mathcal{K} \models H \sqsubseteq O \}.$$

The set of concepts in $\text{Expla}_{\mathcal{K}}(O)$ may possibly be expressed in another description logic $\mathcal{L}'$.

- **TBox abduction:** let $C \sqsubseteq D$ be satisfiable w.r.t. $\mathcal{K}$, the set of explanations is made of axioms defined as:

$$\text{Expla}_{\mathcal{K}}(C \sqsubseteq D) = \{ E \sqsubseteq F \mid E \sqsubseteq F \text{ satisfiable w.r.t. } \mathcal{K} \text{ and } \mathcal{K} \cup \{ E \sqsubseteq F \} \models C \sqsubseteq D \}.$$

- **ABox abduction:** let $S_o$ be a set of assertions representing the observation, the set of explanations is the set $S_o$ of ABox assertions such that $S_o$ satisfiable w.r.t. $\mathcal{K}$ and $\mathcal{K} \cup S_o \models S_o$.

- **KB abduction:** let $\{ \varphi \}$ be a consistent set of ABox or TBox assertions w.r.t. $\mathcal{K}$. A solution of knowledge base abduction, considered as a combination of TBox abduction and ABox abduction, is any finite set $S = \{ \psi_i, i = 1 \ldots n \}$ in $\mathcal{L}'$ satisfiable w.r.t. $\mathcal{K}$ and such that $\mathcal{K} \cup S \models \{ \varphi \}$.

As in any other logic, additional constraints can be used to find the preferred explanations in $\text{Expla}_{\mathcal{K}}$ (minimality, etc. (see e.g. [5]).

Let us illustrate these notions on an example inspired from an image interpretation task.
Example 6. Suppose we have an image, in which we have identified three regions a, b and c (Figure 2). Region b has been identified as Hat and has a color attribute Red, while region c has been identified as Beard. There is a spatial relation hasOnTop that links a to b, and a spatial relation hasPart linking a to c. Furthermore, the background knowledge tells us that smurf leaders are smurfs that wear red hats (i.e. have them on top) and have beards. In this example, a good approach should be able to come up with the explanation that region a might be a smurf leader.

The background knowledge is encoded as a TBox:

\[ T = \{ \text{SmurfLeader} \sqsubseteq \exists \text{hasPart}.\text{Beard} \sqcap \exists \text{hasOnTop}.\text{RedHat}, \]
\[ \text{RedHat} \equiv \text{Hat} \sqcap \exists \text{hasColor}.\text{Red} \} \]

The observation is encoded as an ABox

\[ A_o = \{(a, b) : \text{hasOnTop}, \]
\[ (a, c) : \text{hasPart}, \]
\[ b : \text{Hat}, \]
\[ b : \exists \text{hasColor}.\text{Red}, \]
\[ c : \text{Beard} \} \]

A preferred explanation for this example is \( A_e = \{ a : \text{SmurfLeader} \} \).

In Concept abduction, we need to say which region we want to explain, here region a. Then the most specific concept for this region is:

\[ O = \text{msc}(a) = \exists \text{hasPart}.\text{Beard} \sqcap \exists \text{hasOnTop}.(\text{Hat} \sqcap \exists \text{hasColor}.\text{Red}). \]

\[ \]

---

The example results from a discussion with Felix Distel during his visit at LTCI, Télécom ParisTech (summer 2013).
Concept abduction now looks for a concept description $C$ such that $T \models C \subseteq O$. A solution that is both length minimal\(^8\) and $\sqsubseteq$-maximal would be $C = \text{SmurfLeader}$ which is one of the expected solutions.

In this paper, we consider the general form of abduction in DL, i.e. KB abduction. The other forms can be seen as particular cases. Explanatory relations of a KB in DL can be defined in two ways:

- When the logic is equipped with the disjunction and full negation constructors, as it is the case of the logic $\mathcal{ALC}$ and its extensions, the theory is transformed into an internalized concept on which the retraction operators act. The internalized concept is defined as follows: $C_T := \prod_{(C \subseteq D) \in T}(\neg C \cup D)$. When Abox assertions are considered one can also internalize the Abox to an equivalent concept provided that nominals are part of the syntax. Nominals are concept descriptions having as semantics: $(\{o\})^I = \{o^I\}$, where $(\mathcal{I}, \Delta^I)$ is an interpretation. Then the Abox assertions are transformed into concept inclusions as follows: $a : C$ corresponds to $\{a\} \sqsubseteq C$, $(a, b) : r$ corresponds to $\{a\} \sqsubseteq r, \{b\}$, etc.

- When the logic does not allow for full negation, a possible workaround consists in retracting all the formulas at the same time. Hence, one needs to define concrete retraction operators both on concepts and on formulas. $\Sigma$-formula retraction can be defined in two ways (other definitions may also exist). For sentences of the form $C \subseteq D$, a first possible approach consists in retracting the set of models of $D$ while the second one amounts to “relax” the set of models of $C$ (see e.g. [1] for definitions of relaxations in satisfaction systems, with several examples in DL).

Note also that retracting a concept (or formula) amounts to “relaxing” its negated form, and can then be seen as its dual operator. The notion of concept relaxation has been first introduced in description logics to define dissimilarity measures between concepts in [18, 19], and has been extended to define revision operators in arbitrary logics in [1]. It is extensive and exhaustive, i.e. $\exists k \in \mathbb{N}$ such that $\rho^k(C) \equiv \top$, where $\rho^k$ denotes $k$ iterations of a relaxation $\rho$. Relaxation of formulas have been defined from retraction of concepts to come up with revision operators, in particular within the context of description logics. In [1], some retraction operators of DL-concept descriptions have been introduced. These operators, designed for the purpose of revision are too strong, since their aim is to remove formulas in the background knowledge that are inconsistent with the new acquired one. The philosophy behind abduction is slightly different. One should add new knowledge to the set of consequences of the background theory. Hence, the retraction operator should act on the entire KB rather than just seeking a subpart that is more appropriate to revise. Concretely, this means that instead of relaxing the formulas, potentially each one to a different extent, as done for revision, for abduction, all formulas have to be retracted in the same way.

Hence, we need to introduce new retraction operators suited to the purpose of abduction. In what follows, we restrict ourselves to the context of the logic $\mathcal{ALC}$, as defined in Section 2 possibly enriched with nominals.

---

\(^8\)The length of a concept is defined as the number of atomic concepts appearing in it.
Definition 14 (Concept Retraction). Let $C(\Sigma)$ be the set of concept descriptions defined over a signature $\Sigma$. A (concept) retraction is an operator $\kappa : C(\Sigma) \rightarrow C(\Sigma)$ that satisfies the following two properties for all $C \in C(\Sigma)$ such that $C$ is not equivalent to $\top$:

1. $\kappa$ is anti-extensive, i.e. $\kappa(C) \sqsubseteq C$, and
2. $\kappa$ satisfies the vacuum property, i.e. $\exists k \in \mathbb{N}, \kappa^k(C) = \bot$ where $\kappa^0$ is the identity mapping, and for all $k > 0, \kappa^k(C) = \kappa(\kappa^{k-1}(C))$.

This definition is a direct instantiation to DL of Definition 9. Now we propose, as an example, the following operator to define a particular retraction in $\mathcal{ALC}$.

Definition 15. Given an $\mathcal{ALC}$-concept description $C$ we define an operator $\kappa_f$ recursively as follows.

- For $C = A \in N_C$ (i.e. an atomic concept), $\kappa_f(C) = \bot$.
- For $C = \neg A$, $\kappa_f(C) = \bot$.
- For $C = \bot$, $\kappa_f(C) = \bot$.
- For $C = \top$, $\kappa_f(C) = \top$.
- For $C = C_1 \sqcup C_2$, $\kappa_f(C_1 \sqcup C_2) = (\kappa_f(C_1) \sqcup C_2) \sqcap (C_1 \sqcup \kappa_f(C_2))$.
- For $C = C_1 \sqcap C_2$, $\kappa_f(C_1 \sqcap C_2) = \kappa_f(C_1) \sqcap \kappa_f(C_2)$.
- For $C = \forall r.D$, with $r \in N_R$, $\kappa_f(C) = \forall r.\kappa_f(D)$.
- For $C = \exists r.D$, $\kappa_f(C) = (\forall r.D) \sqcup (\exists r.\kappa_f(D))$.

Note that this definition assumes that any concept is rewritten using standard De Morgan rules so that negations apply only on atomic concepts.

Proposition 8. The operator $\kappa_f$ is a retraction.

Proof. The proof is straightforward, by induction on the structure of $C$. \qed

Example 7. Let us illustrate this retraction operator on Example 6 using the explanatory relations introduced in Section 7. To ease the reading, we will note the concepts by capital letters and roles by small letters, e.g. $B=$Beard, $S=$SmurfLeader, $H=$Hat, $R=$Red, $t=$hasOnTop, $p=$hasPart, $c=$hasColor. The unfolded TBox writes as:

$\mathcal{T} = \{ S \sqsubseteq \exists p.B \sqcap \exists t.(H \sqcap \exists c.R) \}$

and the observation writes as:

$\varphi = \exists p.B \sqcap \exists t.(H \sqcap \exists c.R)$

Note that $\mathcal{T} = \{ S \subseteq \varphi \}$, and $C_\mathcal{T} = \neg S \sqcup \varphi$, where $C_\mathcal{T}$ is the internalized concept of the TBox.
First, note that we have:

\[ \kappa_f(\varphi) = \kappa_f(\exists p. B \land \exists t. (H \land \exists c. R)) = \forall p. B \land \forall t. (H \land \exists c. R) \]  

(9)

and

\[ \kappa_f^2(\varphi) = \kappa_f(\kappa_f(\varphi)) = \kappa_f(\forall p. B \land \forall t. (H \land \exists c. R)) = \bot \]  

(10)

Let us now consider the two explanatory relations \( \triangleright_{C_{\text{clcr}}} \) and \( \triangleright_{C_{\text{clnr}}} \) defined in Section 4, applied here in ALC and with \( \kappa_f \).

Case 1 (\( \triangleright_{C_{\text{clcr}}} \)). The \( \triangleright_{C_{\text{clcr}}} \) relation amounts to take the last retraction of \( C_T \) that is still consistent with \( \varphi \), i.e.:

\[ \varphi \triangleright_{C_{\text{clcr}}} \psi \iff \psi \subseteq \kappa_f^2(\neg S \cup \varphi) \cap \varphi \]

We have:

\begin{align*}
\kappa_f^1(\neg S \cup \varphi) &= \varphi \cap (\neg S \cup \kappa_f^1(\varphi)) \\
\kappa_f^2(\neg S \cup \varphi) &= \kappa_f^1(\varphi) \cap (\kappa_f^1(\varphi) \cap (\kappa_f^2(\varphi) \cup \neg S)) \\
&= \kappa_f^1(\varphi) \cap (\kappa_f^2(\varphi) \cup \neg S)) \\
&= \kappa_f^1(\varphi) \cap \neg S, \text{ since } \kappa_f^2(\varphi) = \bot \\
\kappa_f^3(\neg S \cup \varphi) &= \bot
\end{align*}

Then \( n = 2 \) and

\[ \psi \subseteq (\kappa_f^1(\varphi) \cap \neg S) \cap \varphi \]

with \( \kappa_f^1(\varphi) = \forall p. B \land \forall t. (H \land \exists c. R) \). Since \( \kappa_f \) is anti-extensive, \( \kappa_f^1(\varphi) \cap \varphi = \kappa_f^1(\varphi) \), and

\[ \psi \subseteq (\forall p. B \land \forall t. (H \land \exists c. R) \cap \neg S) \]

Case 2 (\( \triangleright_{C_{\text{clnr}}} \)). The \( \triangleright_{C_{\text{clnr}}} \) relation amounts to take the last non empty retraction of \( C_T \cap \varphi \), i.e.:

\[ \varphi \triangleright_{C_{\text{clnr}}} \psi \iff \psi \subseteq \kappa_f^n((\neg S \cup \varphi) \cap \varphi) \]

with the largest possible value of \( n \) such that the retraction is not empty. Since we have \( (\neg S \cup \varphi) \cap \varphi = \varphi \), according to Equations 9 and 10, \( n = 1 \), and then

\[ \psi \subseteq \forall p. B \land \forall t. (H \land \exists c. R) \cup (S \cap \neg \varphi) \]

A possible solution is then \( S \) according to subset minimality, which well fits the intuition.

Several other retractions could be proposed. In particular, several relaxations proposed in [1] for revision could be modified to become retractions. For instance, relaxing \( C \subseteq D \) can be performed either by retracting \( C \) or by relaxing \( D \). Similarly, retracting \( C \subseteq D \) could be performed either by relaxing \( C \) or by retracting \( D \).
6 Conclusion

In this paper, we proposed a new framework for abduction in satisfaction systems, by introducing the notion of cutting, which provides a structure on the set of models among which an explanation can be found. Inspired by previous work in propositional logic where abduction was defined from morphological erosions, we proposed to define cuttings from the more general notion of retraction, and prove a set of rationality postulates for the derived explanatory relations. The generic feature of the proposed approach is illustrated by providing concrete examples of retractions, cuttings and explanatory relations in various logics.

Future work will aim at further analyzing the structure of the set of cuttings $C$ for a theory $T$, and the properties of the derived relation $\preceq_C$. The examples in DL could also be further investigated, by considering other types of retractions as well as various fragments of $\mathcal{ALC}$, as done for revision in [1]. Links between the proposed approach with other abduction methods could also deserve to be investigated, such as with sequent calculus, prime implicants [37] or equational logic [20, 21]. To address the question of uncertainty in the observations, or in the theory, the proposed approach could be extended to the case of fuzzy logic, based on our previous work on mathematical morphology in the framework of institutions [2]. Finally applications will be further developed, in particular for image understanding and spatial reasoning.

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