A NONUNIFORM MARKUS–YAMABE CONJECTURE: TRIANGULAR CASE VIA UNIFORMIZATION

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Abstract. We introduce a nonautonomous nonuniform Markus-Yamabe Conjecture, namely, a global problem on nonuniform asymptotic stability for nonautonomous differential systems, whose restriction to the autonomous case is related to the classical Markus–Yamabe Conjecture. Additionally, we prove that the conjecture is verified in dimension one and for a family of triangular systems of nonautonomous differential equations satisfying technical boundedness assumptions. An essential tool to carry out the proof in the triangular case is a necessary and sufficient condition ensuring the property of nonuniform exponential dichotomy for upper block triangular linear differential systems, which has been developed previously for the uniform case. We extend this result for a specific nonuniform exponential dichotomy and we obtain some byproducts having interest on itself, such as, the diagonal significance property.

1. Introduction

1.1. State of art. In the last decade, the problem of global stability for ordinary differential equations, also known as the Markus–Yamabe Conjecture, has been revisited from different approaches, namely, the case of continuous and discontinuous piecewise autonomous vector fields have been considered by J. Llibre & X. Zhang [24], L. Menezes [25], and Y. Zhang & X-S. Yang [33]. A infinite–dimensional perspective has been studied by H.M. Rodrigues et al. [30]. On a nonautonomous dynamical systems context, D. Cheban [7] worked in the framework of cocycles, while Á. Castañeda and G. Robledo [6] established a version of this problem of global stability in terms of the uniform exponential dichotomy.

Recall that the Markus-Yamabe Conjecture is a problem of global asymptotic stability for continuous autonomous dynamical systems on finite dimension, introduced in 1960 by L. Markus and H. Yamabe [27], which states that if the differential system \( \dot{x} = f(x) \), where \( f: \mathbb{R}^n \to \mathbb{R}^n \) of class \( C^1 \), \( f(0) = 0 \) and it is a Hurwitz vector field, that is, the eigenvalues of the Jacobian matrix of \( f \) have negative real part at any \( x \in \mathbb{R}^n \), or equivalently \( Jf(x) \) is a Hurwitz matrix for any \( x \), then the origin is globally uniformly asymptotically stable.

It is known that this global stability problem is true when \( n \leq 2 \). For details about the proof in the planar case, see R. Feßler in [14], A.A. Glutsyuk in [15] and C. Gutiérrez in [17]. When \( n \geq 3 \), the Markus–Yamabe Conjecture is false due to

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A. Cima et al. [8] found a polynomial vector field that satisfies the hypothesis of the problem, however the differential system associated to this vector field has a solution which escape to infinity.

In spite that the conjecture is now completely resolved in its autonomous classical version, many authors have dedicated to showing vector fields that satisfy both the hypothesis and its conclusion (see [5], [9], [10]). Remarkable examples of such setting is the case of triangular and gradient vector fields. Indeed, in a triangular context, L. Markus and H. Yamabe proved that the conjecture is true [27, Th.4] while P. Hartman [18, p. 539 Corollary 11.2] (see also [26]) showed that for the conjecture is also true gradient vector fields.

The main idea of this article is to settle a global nonuniform stability problem for a nonlinear nonautonomous system

\[ \dot{x} = f(t, x). \]

Notice that, in the linear case, the global nonuniform asymptotic stability is consequence of the nonuniform exponential stability, which can be described in terms of a particular nonuniform exponential dichotomy. We emphasize that this property of dichotomy is associated to a spectral theory, which will allow us to emulate the notion of Hurwitz vector fields to the nonuniform framework.

1.2. The Nonuniform conjecture. While the autonomous dynamical systems depend only on the time elapsed from the initial time \( t_0 \), the nonautonomous dynamical systems are also dependent on the initial time \( t_0 \) itself, which has several consequences to characterize limiting objects. As it was pointed out in [21, Ch.2] and [22, Sec.2.2], the dynamics arising from nonautonomous differential systems (1.1) can be formally described by two approaches where the above mentioned \( t_0 \) plays a key role: the skew product semiflows and the process formalism, also known as the two parameter \(-\) \((t, t_0)\) \(-\) semigroups.

Let us recall that the Markus–Yamabe Conjecture is stated in terms of the negativity of the real part of the eigenvalues of \( Jf(x) \) and the global attractiveness of the origin has a behavior described by the uniform asymptotic stability. Contrarily, it is known that the local stability of a linear nonautonomous system cannot be always determined by the eigenvalues, see e.g. [27, p.310]. However, we point out about the existence of several spectral theories based either on characteristic exponents (Lyapunov, Perron and Bohl exponents) or dichotomies [12], which have associated a wide range of asymptotic stabilities for nonautonomous linear systems, being the uniform asymptotic stability only a particular case.

In this article, we will work with the global nonuniform asymptotic stability to settle a nonautonomous and nonuniform Markus–Yamabe conjecture (NNMYC). We emulate the results obtained in [6] for a uniform version of the conjecture. Namely, we prove the NNMYC for \( n = 1 \) and give an example, in arbitrary dimension, of a nonautonomous quasilinear vector field satisfying the hypothesis of the nonuniform global stability problem and its conclusion. In [6] it was also proved that the uniform conjecture is also true for triangular systems. Nevertheless, we emphasize that, when we are working on the NNMYC, the triangular case is not an extension of the uniform case and we require to construct subtle and deeper methods to cope with this problem.

1.3. Triangular case setting. As we above stated, in this article we prove that this conjecture is also verified for a family of triangular vector fields. We point
out that the proof is completely different to the ones made in [27, Th.4] in an autonomous framework and [6, Cor.1] in a uniform nonautonomous case, respectively. In both previous works, as the underlying stability is the uniform one, we can use known results describing the stability of triangular differential systems in terms of its diagonal properties. A tool used in the proof of the triangular case in a nonautonomous uniform context is a result of F. Batelli and K.J. Palmer in [3] providing necessary and sufficient conditions ensuring that an upper triangular linear system has a uniform exponential dichotomy on the half line whenever its diagonal subsystems also have this property.

In order to obtain a similar tool to tackle the nonuniform context, we generalize the result of Batelli and Palmer by using the Lemma of Uniformization introduced by L. Zhou et al. in [35, Lemma 1] and its consequences. The generalization has a chain of byproducts, which allow us to conclude that NNMYC is verified for a family of triangular systems.

1.4. Structure. The section 2 gives a general setting: the subsection 2.1 describe the properties of nonuniform bounded growth, nonuniform contractions, nonuniform exponential dichotomies, nonuniform spectrum and explore some relations between them. The subsection 2.2 recalls the functional framework needed to introduce the Uniformization Lemma. The section 3 recalls the property of global nonuniform asymptotic stability for an equilibrium of (1.1) and introduces the NNMYC, additionally we prove that this conjecture is true in dimension one and for a family of quasilinear systems in arbitrary dimension. Our main results are immersed in the section 4, where we prove –Theorem 4– that the conjecture is verified for a family of triangular vector fields. The proof of this theorem is based in Theorem 3 and its consequences. The Theorem 3 establishes a relation between upper triangular linear systems and its diagonal subsystems associated with respect to the properties of nonuniform exponential dichotomy and nonuniform bounded growth. In particular, Theorem 3 establishes that if the diagonal subsystems have the half nonuniform bounded growth and nonuniform exponential dichotomy combined with the fact that the non diagonal block is bounded in terms a of a parametrized norm, then the nonuniform exponential dichotomy is preserved for the upper triangular system. Finally, in the Appendix, we give the a detailed proof of Theorem 3.

1.5. Notations. Throughout this paper, ||·|| and |·| will denote matrix and vector norms respectively. The set [0, +∞) is denoted by \(\mathbb{R}_+^n\) and the set of square \(n \times n\) matrices with real coefficients is denoted by \(M_n\), while \(I_n\) is the identity matrix. A continuous function \(G: \mathbb{R}_+^n \rightarrow [1, +\infty)\) will be called a growth rate.

2. Preliminary definitions and contextualization

2.1. Nonuniform bounded growth, nonuniform dichotomies and spectrum. We will recall basic properties about the nonautonomous linear differential system

\[
\dot{x} = A(t)x,
\]

where \(A: \mathbb{R}_+^n \rightarrow M_n(\mathbb{R})\) is a locally integrable matrix function. A basis of solutions of (2.1) is denoted by \(T(t)\), which satisfies \(\dot{T}(t) = A(t)T(t)\) and its corresponding evolution operator is \(T(t,s) = T(t)T^{-1}(s)\), then the solution of (2.1) with initial condition \(x_0\) at \(t_0\) defined by \(x(t,t_0,x_0) = T(t,t_0)x_0\).
Similarly as in the uniform case, there exist two definitions of nonuniform bounded growth in the literature:

**Definition 1.** The evolution operator $T(t,s)$ of (2.1) has a:

a) Full $(M(s),\nu)$–nonuniform bounded growth, namely, there exist a constant $\nu > 0$ and a growth rate $M : \mathbb{R}_0^+ \to [1, +\infty)$ such that
   \[ \|T(t,s)\| \leq M(s)e^{\nu(t-s)} \quad t, s \in \mathbb{R}_0^+, \]

b) Half $(M(s),\nu)$–nonuniform bounded growth, namely, there exist a constant $\nu > 0$ and a growth rate $M : \mathbb{R}_0^+ \to [1, +\infty)$ such that
   \[ \|T(t,s)\| \leq M(s)e^{\nu(t-s)} \quad t \geq s \geq 0. \]

We point out that there are no standard definition of bounded growth in the current literature and we are proposing the previous ones in order distinguished them and its consequences. Note that:

i) The property of half $(M(s),\nu)$–nonuniform bounded growth is considered in [35] p.686 under the name of nonuniform bounded growth.

ii) The property of the full $(M(s),\nu)$–nonuniform bounded growth is considered with the particular case of $M(s) = Me^{s\nu}$ where $\nu \geq 0$ and $M \geq 1$ by [11] p.547 and [34] p.1892, also under the name of nonuniform bounded growth.

iii) We propose to denote the particular case of $M(s) = M \geq 1$ as uniform bounded growth, which has been considered respectively by S. Siegmund [31] p.253 and W. Coppel [10] pp.8-9 in a full and half version respectively under the name of bounded growth.

From now on, in this paper we will work with the property of half $(Me^{\delta s},\nu)$–nonuniform bounded growth.

The property of exponential dichotomy plays an important role in the study of nonautonomous linear systems and –roughly speaking– refers that the solutions of (2.1) can be splitted as either exponential expansions or exponential contractions. A formal definition for the nonuniform framework is given by:

**Definition 2.** The linear system (2.1) has a $(K(s),\gamma)$–nonuniform exponential dichotomy on $\mathbb{R}_0^+$ if there exist a family of invariant projections $P(t) : \mathbb{R}^n \to \mathbb{R}^n$ for any $t \in \mathbb{R}_0^+$, a positive constant $\gamma$ and a growth rate $K : \mathbb{R}_0^+ \to [1, +\infty)$ such that:

(2.2) \[ T(t,s)P(s) = P(t)T(t,s) \quad \text{for } t \geq s \geq 0, \]

(2.3) \[
\begin{cases}
\|T(t,s)P(s)\| &\leq K(s)e^{-\gamma(t-s)} \quad \text{for } t \geq s \geq 0, \\
\|T(t,s)(I - P(s))\| &\leq K(s)e^{-\gamma(s-t)} \quad \text{for } 0 \leq t \leq s. 
\end{cases}
\]

The property (2.2) implies that the range of $P(\cdot)$ is invariant for any $t$, which motivates the name invariant projectors.

The inequalities (2.3) imply that $t \mapsto T(t,t_0)\xi$, the forward solution of (2.1) passing through $\xi \neq 0$ at $t = t_0$, can be splitted by $P(t_0)$ in $t \mapsto T(t,t_0)P(t_0)\xi$ and $t \mapsto T(t,t_0)(I - P(t_0))\xi$, whose behavior for any $t \geq t_0$ verifies that

\[ |T(t,t_0)P(t_0)\xi| \leq K(t_0)e^{-\gamma(t-t_0)}|P(t_0)\xi| \]

\[ (1/K(t_0))e^{\gamma(t-t_0)}|I - P(t_0)|\xi \leq |T(t,t_0)(I - P(t_0))\xi|, \]

\[ |T(t,t_0)(I - P(t_0))\xi| \leq |T(t,t_0)| |I - P(t_0)| \xi|, \]
that is, any solution is the sum of two solutions having a dichotomic exponential behavior: the \((K(s), \gamma)\)–nonuniform exponential contraction \(t \mapsto T(t, t_0)P(t_0)\xi\) and the \((K(s), \gamma)\)–nonuniform exponential expansion \(t \mapsto T(t, t_0)[I - P(t_0)]\). In this context, we will say that the system \((2.4)\) is a \((K(s), \gamma)\)–nonuniform exponential contraction if it has a \((K(s), \gamma)\)–nonuniform exponential dichotomy on \(\mathbb{R}_0^+\) with the projector \(P(t) = I\) for any \(t \geq 0\).

On the other hand, we emphasize that the growth rate \(K(\cdot)\) can take a wide range of possible behaviors. A distinguished case, which is older in the literature, is given by the constant function \(K(s) := K \geq 1\) and corresponds to the uniform exponential dichotomy on \(\mathbb{R}_0^+\) since the exponential contractions (expansions) have an exponential upper (lower) bound which are independent of \(t_0\). Having in mind this noteworthy case, we can see that the nonuniform exponential dichotomies have been residually defined as dichotomies where the growth rate \(K(\cdot)\) is not a constant function.

In this article, we will focus in the particular case of nonuniform exponential dichotomy having a growth rate defined by \(K(s) = K_0e^{\varepsilon s}\).

**Definition 3.** The linear system \((2.4)\) has a \((K_0e^{\varepsilon s}, \gamma)\)–nonuniform exponential dichotomy on \(\mathbb{R}_0^+\) if there exist a family of invariant projections \(P(t) : \mathbb{R}^n \to \mathbb{R}^n\) for any \(t \in \mathbb{R}_0^+\), a constant \(K \geq 1\), and a couple \((\gamma, \varepsilon)\) of constants such that \(\varepsilon \in [0, \gamma)\) and

\[
T(t, s)P(s) = P(t)T(t, s) \quad \text{for } t, s \geq 0,
\]

\[
\begin{cases}
|T(t, s)P(s)| & \leq K e^{-\gamma(t-s)}e^{\varepsilon s} \quad \text{for } t \geq s \geq 0, \\
|T(t, s)[I - P(s)]| & \leq K e^{-\gamma(s-t)}e^{\varepsilon s} \quad \text{for } 0 \leq t \leq s.
\end{cases}
\]

The above dichotomy has been considered in several works as [11][11][34] where is denoted as nonuniform exponential dichotomy and deserves some remarks:

- Note that if \(\varepsilon = 0\), we recover the uniform exponential dichotomy on \(\mathbb{R}_0^+\) and this prompts to denote the term \(e^{\varepsilon s}\) as the nonuniform part.
- In [11] Theorem 10.22, by using the Multiplicative Ergodic Theorem [11][10.27] and the Oseledelets-Pesin Reduction result [11] Theorem 10.28], it is shown that almost all variational equations obtained from a measure-preserving flow admit nonuniform exponential dichotomy and furthermore, the nonuniformity rate is arbitrarily small.

- There exists a spectral theory for the \((K_0e^{\varepsilon s}, \gamma)\)–nonuniform exponential dichotomy on \(\mathbb{R}\), which has been constructed in [11][11][34] and adapted to the half line by [19] in the continuous case and [2][2] in the discrete case.

**Definition 4.** ([11][34][37]) The \((K_0e^{\varepsilon s}, \gamma)\)–nonuniform exponential dichotomy spectrum of \((2.4)\) is the set \(\Sigma^+(A)\) of \(\lambda \in \mathbb{R}\) such that the system

\[
\dot{x} = [A(t) - \lambda I]x
\]

does not have a \((K_0e^{\varepsilon s}, \gamma)\)–nonuniform exponential dichotomy on \(\mathbb{R}_0^+\). The resolvent \(\rho(A)\) is defined as \(\mathbb{R} \setminus \Sigma^+(A)\), namely, the values of \(\lambda\) such that the system \((2.4)\) have a \((K_0e^{\varepsilon s}, \gamma)\)–nonuniform exponential dichotomy on \(\mathbb{R}_0^+\).

**Proposition 1.** ([11][31][34]) If the evolution operator \(T(t, s)\) of \((2.4)\) has a half \((M_0e^{\delta s}, \nu)\)–nonuniform bounded growth, then its nonuniform spectrum \(\Sigma^+(A)\)
is the union of \( m \) intervals where \( 0 < m \leq n \), that is,

\[
\Sigma^+(A) = \left\{ \begin{array}{l}
[a_1, b_1] \\
\text{or} \\
(-\infty, b_1)
\end{array} \right\} \cup [a_2, b_2] \cup \cdots \cup [a_{m-1}, b_{m-1}] \cup \left\{ \begin{array}{l}
[a_m, b_m] \\
\text{or} \\
[a_m, +\infty)
\end{array} \right\},
\]

with \(-\infty < a_1 \leq b_1 < \ldots < a_m \leq b_m < +\infty\).

The intervals \([a_i, b_i]\) are called spectral intervals for \( i = 2, \ldots, m - 1 \) while the intervals \( \rho_i(A) = (b_i, a_{i+1}) \) for \( i = 1, \ldots, m - 1 \) are called spectral gaps. Additionally, if the first spectral interval in (2.4) is \([a_1, b_1]\), we can define \( \rho_1(A) = (-\infty, a_1) \) and if the last spectral intervals in (2.5) is \([a_m, b_m]\), we can define \( \rho_{m+1}(A) = (b_m, +\infty) \). By the definition of \( \Sigma^+(A) \), it follows that for any \( \lambda \in \rho_j(A) \), the system (2.4) has a \((Ke^{\delta_\lambda}, \gamma)\)-nonuniform exponential dichotomy with \( P_j := P_j(\cdot) \). It can be proved, see e.g. [11], that:

a) If the first spectral interval is given by \([a_1, b_1]\), then \( P_1 = 0 \), \( P_{m+1} = I_n \) and 
\[
\dim \text{Range } P_i < \dim \text{Range } P_{i+1} \text{ for any } i = 1, \ldots, m.
\]

b) If the first spectral interval is given by \((-\infty, b_1]\), then \( P_{m+1} = I_n \) and 
\[
\dim \text{Range } P_i < \dim \text{Range } P_{i+1} \text{ for any } i = 2, \ldots, m.
\]

The nonuniform bounded growth definitions above stated allow to deduce boundedness properties for \( \Sigma^+(A) \). In fact, it is well known that if (2.1) has a full \((Me^{\delta_\lambda}, \nu)\)-nonuniform bounded growth, then \( \Sigma^+(A) \subset [-\nu, \nu] \) and we refer to Lemma 2.10 from [11] for a proof. In addition, it is easy to verify that if (2.1) has a half \((Me^{\delta_\lambda}, \nu)\)-nonuniform bounded growth, then \( \Sigma^+(A) \subset (-\infty, \nu] \).

The properties of \( \Sigma^+(A) \) and its spectral gaps provides an alternative characterization of the \((Ke^{\delta_\lambda}, \gamma)\)-nonuniform contractions.

**Lemma 1.** The system (2.1) has a \((Ke^{\delta_\lambda}, \gamma)\)-nonuniform exponential contraction if and only if \( \Sigma^+(A) \subset (-\infty, 0) \).

**Proof.** If \( \Sigma^+(A) \subset (-\infty, 0) \), it follows that \( 0 \in \rho_{m+1}(A) \) and (2.4) with \( \lambda = 0 \) coincides with (2.1), which has a nonuniform exponential dichotomy with projector \( P_{m+1} = I_n \). Then, from Definition 9 we obtain that (2.1) has a \((Ke^{\delta_\lambda}, \gamma)\)-nonuniform exponential contraction.

On the other hand, if (2.1) has a \((Ke^{\delta_\lambda}, \gamma)\)-nonuniform exponential contraction, we know that this equivalent to say that (2.1) has a \((Ke^{\delta_\lambda}, \gamma)\)-nonuniform exponential dichotomy with the identity as projector, which has full range, then we have that \( \lambda = 0 \in \rho_{m+1}(A) = (b_m, +\infty) \) and implies that \( \Sigma^+(A) \subset (-\infty, b_m) \) with \( b_m < 0 \).

2.2. **Uniformization Lemma.** Given a linear system (2.1) having a nonuniform exponential dichotomy and a half nonuniform bounded growth, the Uniformization Lemma provides a way to endow it with both an exponential dichotomy and uniform bounded growth. Nevertheless, the price to pay is to work in a functional framework described by parametrized vector and operator norms. This result was developed by L. Zhou, K. Lu and W. Zhang in [35, p.697]. As a previous step to its statement, we need to consider a family of norms in \( \mathbb{R}^n \) parametrized by \( \mathbb{R}_0^+ \) as \( \{ | \cdot |_t \}_{t \in \mathbb{R}_0^+} \), that is, \( | \cdot |_t \) is a norm of \( \mathbb{R}^n \) for any \( t \in \mathbb{R}_0^+ \).

Let \( \{ | \cdot |_t \}_{t \in \mathbb{R}_0^+} \) be a family of norms. By following [35], we summarize basic facts about this family:
a) By equivalence of norms, there exist two functions $L_i: \mathbb{R}_0^+ \to (0, +\infty)$ such that

$$L_1(t)|x| \leq |x|_t \leq L_2(t)|x| \quad \text{for any } t \in \mathbb{R}_0^+. \quad (2.6)$$

b) The family $\{| \cdot |_t\}_{t \in \mathbb{R}_0^+}$ is continuous if the mapping $t \mapsto |x|_t$ is continuous on $\mathbb{R}_0^+$ for any fixed $x \in \mathbb{R}^n$. In this case, it follows that the functions $L_i$ from (2.6) are continuous on $\mathbb{R}_0^+$, Prop.1.

c) A continuous family $\{| \cdot |_t\}_{t \in \mathbb{R}_0^+}$ is called uniformly lower bounded if $t \mapsto L_1(t)$ is uniformly bounded by a positive constant $L_1 > 0$ and (2.6) can be replaced by

$$L_1|x| \leq |x|_t \leq L_2(t)|x| \quad \text{for any } t \in \mathbb{R}_0^+. \quad (2.7)$$

In this article, we will assume that $\{| \cdot |_t\}_{t \in \mathbb{R}_0^+}$ is a uniformly lower bounded continuous family of norms. In consequence, we are implicitly assuming that $t \mapsto L_2(t)$ is a unbounded and continuous function.

The following technical lemma will be useful

Lemma 2. Given a couple of linear operators $U: (\mathbb{R}^n, | \cdot |_s) \to (\mathbb{R}^m, | \cdot |_t)$ and $U: (\mathbb{R}^n, | \cdot | ) \to (\mathbb{R}^m, | \cdot | )$ with norms defined by

$$||U||_{s,t} = \sup_{x \neq 0} \frac{|Ux|_t}{|x|_s} \quad \text{and} \quad ||U|| = \sup_{x \neq 0} \frac{|Ux|}{|x|},$$

it follows that

$$\frac{1}{\beta(t)} ||U||_{s,t} \leq ||U|| \leq \beta(s)||U||_{s,t} \quad \text{and} \quad \frac{1}{\beta(s)} ||U|| \leq ||U||_{s,t} \leq \beta(t)||U||, \quad (2.8)$$

where $\beta: \mathbb{R}_0^+ \to [1, \infty)$ is the continuous and upperly unbounded function:

$$\beta(t) = \frac{L_2(\tau)}{L_1}. \quad \text{Proof.}$$

By using (2.7) it can be proved that for any $x \in \mathbb{R}^n$ it follows that

$$\frac{|Ux|_t}{|x|_s} \geq \frac{L_1}{L_2(s)} \frac{|Ux|}{|x|} \quad \text{and} \quad \frac{|Ux|_t}{|x|_s} \leq \frac{L_2(t)}{L_1} \frac{|Ux|}{|x|},$$

and the Lemma follows. \qed

Remark 1. Given a linear operator $U: \mathbb{R}^p \to \mathbb{R}^q$ and a couple $(t, \tau)$ of positive real numbers. It will be useful to recall the estimations

$$||U\xi||_t \leq ||U||_{t,\tau}||\xi||_{\tau}, \quad (2.9)$$

$$||U\xi||_t \leq ||U||_{t,\tau}||\xi||_{\tau}, \quad (2.10)$$

$$||U\xi||_t \leq ||U||_{t,\tau}||\xi||_{\tau}. \quad (2.11)$$

Lemma 3 (Uniformization Lemma). The system (2.1) has both a $(K(s), \alpha)$–nonuniform exponential dichotomy and a half $(M(s), \nu)$–nonuniform bounded growth on $\mathbb{R}_0^+$ if and only if there exists a continuous family $\{| \cdot |_t\}_{t \in \mathbb{R}_0^+}$ of norms with a uniform lower bound such that Eq. (2.1) has a half uniform exponential dichotomy with respect to $\{| \cdot |_t\}_{t \in \mathbb{R}_0^+}, \ i.e., \ there \ are \ a \ projection \ P(t): \mathbb{R}^n \to \mathbb{R}^n$ and a couple
of constants \( \alpha > 0 \) and \( \kappa \geq 1 \) such that the invariant decomposition condition (the last two inequalities in (2.3)) can be replaced by
\[
\| T(t, s) P(s) \|_{s,t} \leq \kappa e^{-\alpha(t-s)} \quad \text{for } t \geq s \geq 0 \\
\| T(t, s)(I - P(s)) \|_{s,t} \leq \kappa e^{-\alpha(s-t)} \quad \text{for } s \geq t \geq 0
\]
and a half \((\mu, \nu)\)-uniform bounded growth with respect to \( \{| \cdot | \}_t \) \( t \in \mathbb{R}^+_0 \), namely,
\[
\| T(t, s)(I - P(s)) \|_{s,t} \leq \mu e^{\nu(t-s)} \quad t \geq s \in \mathbb{R}^+_0.
\]

Remark 2. A meticulous reading of the proof of the Uniformization Lemma [35, pp.697–700] shows that the family of norms \( \{ | \cdot | \}_t \) verifies
\[
| x |_t \leq L^2(t) | x |,
\]
where
\[
L^2(t) = M(t) + K(t).
\]

Remark 3. A version of the Uniformization Lemma, where the linear system (2.1) has the properties of \((Ke^{\varepsilon_s}, \gamma)\)-nonuniform exponential dichotomy and full \((Me^{2s}, \nu)\)-nonuniform bounded growth, has been used in [13]. Additionally, a linear system having the above properties is said to have a strong nonuniform exponential dichotomy.

3. Nonuniform Conjecture

In order to set the problem of nonuniform nonautonomous stability, we must to recall a formal definition of nonuniform asymptotic stability. For this purpose, let us consider the nonautonomous and nonlinear system of ordinary differential equations:
\[
(3.1) \quad \dot{x} = f(t, x)
\]
where \( f : \mathbb{R}^+_0 \times \mathbb{R}^n \to \mathbb{R}^n \) has properties ensuring the existence, uniqueness and unbounded forward continuation of solutions. The solution of (3.1) passing through \( x_0 \) at \( t_0 \) will be denoted as \( t \mapsto x(t, t_0, x_0) \). In addition, an equilibrium of (3.1) is a vector \( x^* \) such that \( f(t, x^*) = 0 \) for any \( t \in \mathbb{R}^+_0 \).

Definition 5. A unique equilibrium of (3.1) is globally nonuniformly asymptotically stable if, for any \( \eta > 0 \), there exists a \( \delta(t_0, \eta) > 0 \) such that
\[
|x_0 - x^*| < \delta(t_0, \eta) \Rightarrow |x(t, t_0, x_0) - x^*| < \eta \quad \forall t \geq t_0
\]
and for any \( x_0 \in \mathbb{R}^n \) it follows that \( \lim_{t \to +\infty} x(t, t_0, x_0) = x^* \).

There exists several alternative characterizations for the above defined stability, the most of them have made in terms of KL functions (see [20] for details). In the particular case where (3.1) is the linear system (2.1), it can be proved –with the help of KL functions– that if the linear system has a \((Ke^{\varepsilon_s}, \gamma)\)-nonuniform exponential contraction, then the origin is globally nonuniformly asymptotically stable.

Statement of the conjecture: As we have set forth the premises now we are able to state our nonuniform global stability problem.

Conjecture 1 (Nonuniform Nonautonomous Markus–Yamabe Conjecture (NNMYC)). Let us consider the nonlinear system
\[
(3.2) \quad \dot{x} = f(t, x)
\]
where \( f : \mathbb{R}^+_0 \times \mathbb{R}^n \to \mathbb{R}^n \). If \( f \) satisfies the following conditions
(G1) $f$ is continuous in $\mathbb{R}_0^+ \times \mathbb{R}^n$ and $C^1$ with respect to $x$. Moreover, $f$ is such that the forward solutions are defined in $[t_0, +\infty)$ for any $t_0 \geq 0$.

(G2) $f(t, x) = 0$ if $x = 0$ for all $t \geq 0$.

(G3) For any piecewise continuous function $t \mapsto \omega(t)$, the linear system

$$\dot{\vartheta} = Jf(t, \omega(t)) \vartheta,$$

where $Jf(t, \cdot)$ is the jacobian matrix of $f(t, \cdot)$, has a $(Ke^{\varepsilon s}, \gamma)$–nonuniform exponential dichotomy spectrum satisfying

$$\Sigma^+(Jf(t, \omega(t))) \subset (-\infty, 0).$$

Then the trivial solution of the nonlinear system (3.2) is globally nonuniformly asymptotically stable.

Let us recall that the autonomous version of the Markus–Yamabe conjecture is stated in terms of the eigenvalues spectrum of the jacobian matrix corresponding to the linearized vector field. In this context, the assumption (G3) mimics the above fact and since both spectra (eigenvalues and nonuniform spectrum) belong to $(-\infty, 0)$.

A uniform version of the above conjecture has been recently stated in [6], where the uniform conjecture was verified for scalar systems, triangular systems and a family of quasilinear systems where the nonlinearity has suitable properties.

In the rest of this section we will see that NNMYC is well posed. In fact, it can be verified for the scalar and the quasilinear cases.

**Theorem 1.** The NNMYC is verified for dimension $n = 1$.

**Proof.** The proof follows the lines of [6, Th.1] replacing the uniform context by the nonuniform one. □

**Theorem 2.** Let us consider the linear system (2.1) and its perturbation

$$\dot{x} = A(t)x + f(t, x),$$

where $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $\mathbb{R}_0^+ \times \mathbb{R}^n$ and $C^1$ with respect to $x$. Moreover, $f$ is such that the forward solutions are defined in $[t_0, +\infty)$ for any $t_0 \geq 0$.

If we assume that

i) The system (2.1) has a $(Ke^{\varepsilon s}, \alpha)$–nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projector $P(\cdot) = I$ and $\alpha > \varepsilon \geq 0$.

ii) The system (2.1) has a half nonuniform bounded growth on $\mathbb{R}_0^+$.

iii) $f(t, x) = 0$ if $x = 0$ for all $t \in \mathbb{R}_0^+$.

iv) There exists $\delta > 0$ sufficiently small such that for any piecewise continuous function $y : \mathbb{R}_0^+ \to \mathbb{R}^n$ it follows that

$$\|Jf(t, y(t))\| < \frac{\delta}{K^2} e^{-2\varepsilon t} \quad \text{for any } t \geq 0,$$

for some $\delta \in (0, \alpha K)$ sufficiently small.

Then the system (3.3) satisfies hypothesis of (NNMYC) and the trivial solution is globally nonuniformly asymptotically stable.

**Proof.** The proof follows the lines from [6, Th.2] and by using Theorem 2.5 from [23] combined with the property (3.4). □
Theorem 2 is related to the pioneer work of K.J. Palmer [28], which studies the topological conjugacy between the solutions of the linear system (2.1) and its perturbation (3.3). Roughly speaking, it shows that if (2.1) is a \((K e^{s_0}, \alpha)\)-nonuniform exponential contraction, then, under suitable conditions, the trivial solution of (3.3) is globally nonuniformly asymptotically stable, being an example verifying the nonuniform Markus-Yamabe conjecture of dimension \(n\).

We emphasize that the Palmer’s problem has been recently revisited due to lack of regularity (see [13] for details) of the topological conjugacy between the systems.

4. Triangular Vector Fields

The aim of this section is to prove NNMYC for triangular vector fields, that is, to show that the origin is a globally nonuniformly asymptotically stable for (3.2) when \(f(t, x)\) is triangular vector field.

As we say previously to Theorem 1, in [6] it was proved that the uniform conjecture is verified for triangular systems and we emphasize that can be seen as a consequence of the scalar case combined with a result of F. Batelli and K.J. Palmer [3] for upper triangular systems whose diagonal subsystems have the uniform exponential dichotomy. In consequence, if we intend to emulate the ideas of the proof for the uniform triangular case, it is necessary to generalize the Batelli–Palmer result to a nonuniform framework.

A first step to cope with this problem is to study the relation between the nonuniform exponential dichotomy properties of the upper block triangular systems

\[
\dot{z} = \begin{bmatrix} A(t) & C(t) \\ 0 & B(t) \end{bmatrix} z,
\]

with the nonuniform exponential dichotomy properties of the subsystems

\[
\dot{x} = A(t)x \quad \text{and} \quad \dot{y} = B(t)y,
\]

where \(x \in \mathbb{R}^n, y \in \mathbb{R}^m, z \in \mathbb{R}^{n+m}, A \in M_n(\mathbb{R}), B \in M_m(\mathbb{R})\) and \(C \in M_{nm}(\mathbb{R})\).

As we said, this problem has been addressed by F. Batelli and K.J. Palmer [3] in the context of the uniform exponential dichotomy on \(\mathbb{R}_0^+\) with an extension to the discrete case in [4].

In the case of the \((K(s), \gamma)\)-nonuniform exponential dichotomy on \(\mathbb{R}_0^+\), a first approach was carried out by L. Tien, L. Niehn and T. Chien in [32]. In particular, the first result of [32] is:

**Proposition 2 (Theorem 2.1 in [32]).** If the system (4.1) has a \((K(s), \gamma)\)-nonuniform exponential dichotomy on \(\mathbb{R}_0^+\) then the decoupled subsystems (4.2) also have a \((K(s), \gamma)\)-nonuniform exponential dichotomy on \(\mathbb{R}_0^+\).

As the \((K e^{s_0}, \gamma)\)-nonuniform exponential dichotomy is a particular case of the \((K(s), \gamma)\)-nonuniform exponential dichotomy, the following Corollary is immediate:

**Corollary 1.** If the system (4.1) has a \((K e^{s_0}, \gamma)\)-nonuniform exponential dichotomy on \(\mathbb{R}_0^+\) then the decoupled subsystems (4.2) also have a \((K e^{s_0}, \gamma)\)-nonuniform exponential dichotomy on \(\mathbb{R}_0^+\).

Our next result will show the converse of Proposition 2 which follows the lines of the converse result proved by Batelli and Palmer in [3, Th.1] but imposes boundedness conditions for \(C\) in terms of matrix norms \(\| \cdot \|_{t,t}\) described in the subsection 2.2 and also incorporates a creative use of the Uniformization Lemma.
Let us recall we are assuming that (2.1) has the property of half \((M e^{\delta s}, \nu)\)-nonuniform bounded growth and a \((K e^{\varepsilon s}, \gamma)\)-nonuniform exponential dichotomy on \(\mathbb{R}_0^+\). Then, by using the Uniformization Lemma and Remark 2, we have that the continuous norms \(\{ |\cdot|_t \}\) satisfy the inequalities
\[
(4.3) \quad L_1 |x| \leq |x|_t \leq (M + K) e^{\theta t} |x| \text{ for any } t \in \mathbb{R}_0^+,
\]
for some \(L_1 > 0\), where \(\theta := \max \{ \delta, \varepsilon \}\). By considering these specific norms, the inequalities (2.8) becomes
\[
(4.4) \quad \frac{1}{L} e^{-\theta t} ||U||_{s,t} \leq ||U|| \leq L e^{\theta t} ||U||_{s,t} \quad \text{and} \quad \frac{1}{L} e^{-\theta s} ||U|| \leq ||U||_{s,t} \leq L e^{\theta t} ||U||,
\]
where \(L = L_2 / L_1\). Now, the converse result of Corollary 1 is given by:

**Theorem 3.** Let us consider the upper block triangular system (4.1). If

i) The decoupled systems (4.2) have the property of half \((M e^{\delta s}, \omega)\)-nonuniform bounded growth and half \((\tilde{M} e^{\tilde{\delta} s}, \tilde{\omega})\)-nonuniform bounded growth respectively on \(\mathbb{R}_0^+\),

ii) The decoupled systems (4.2) have the properties of \((K e^{\varepsilon s}, \alpha)\)-nonuniform exponential dichotomy and \((\tilde{K} e^{\tilde{\varepsilon} s}, \tilde{\alpha})\)-nonuniform exponential dichotomy respectively on \(\mathbb{R}_0^+\),

iii) The non diagonal block verifies
\[
(4.5) \quad ||C||_{\tau, \infty} := \sup_{\tau \in \mathbb{R}_0^+} ||C(\tau)||_{\tau, \tau} < \infty,
\]

where
\[
||C(\tau)||_{\tau, \tau} = \sup_{x \neq 0} \frac{|C(\tau)x|_\tau}{|x|_\tau} \text{ with } |\cdot|_\tau \text{ verifying (4.3)},
\]

then (4.1) also has a \((\tilde{K} e^{\tilde{\varepsilon} s}, \tilde{\alpha})\)-nonuniform exponential dichotomy \(\mathbb{R}_0^+\), where \(\tilde{K}\) is a constant dependent of \(\alpha\) and \(\tilde{\alpha}\), \(\tilde{\varepsilon} \geq 0\) and \(\tilde{\alpha} = \min \{ \alpha, \tilde{\alpha} \}\), and a half \((M e^{\delta s}, \omega)\)-nonuniform bounded growth.

The proof of this result is plenty of bulky technicalities and the Uniformization Lemma combined with the functional setting from [35] play a key role. In order to give continuity to the reading of the article, the proof is written in the Appendix.

**Remark 4.** By using the left inequality (4.4), we have that \(||C(\tau)||_{\tau, \tau} \leq L e^{\theta \tau} ||C(\tau)||\) for any \(\tau \geq 0\). In consequence, a sufficient condition ensuring (4.5) is given by
\[
\sup_{\tau \in \mathbb{R}_0^+} L e^{\theta \tau} ||C(\tau)|| < \infty.
\]

**Corollary 2.** The upper block system
\[
(4.6) \quad \dot{x} = \begin{bmatrix}
A_1(t) & C_{12}(t) & \cdots & C_{1k}(t) \\
0 & A_2(t) & \cdots & C_{2k}(t) \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & A_k(t)
\end{bmatrix} x
\]

has a \((K e^{\varepsilon s}, \alpha)\)-nonuniform exponential dichotomy on \(\mathbb{R}_0^+\) if all the diagonal systems
\[
\dot{x}_i = A_i(t)x_i \quad \text{for } i = 1, \ldots, k
\]
have a half nonuniform bounded growth property and a \( (K_i e^{\xi_i \tau}, \alpha_i) \)-nonuniform exponential dichotomy on \( \mathbb{R}_0^+ \) (with \( i = 1, \ldots, k \)) provided that the upper diagonal blocks verifies

\[
\sup_{\tau \in \mathbb{R}_0^+} ||C_j(\tau)||_{\tau, \tau} < \infty \quad \text{for any } j \in \{2, \ldots, k\},
\]

where \( C_j(t) \) is defined by

\[
C_j(t) = \begin{bmatrix}
C_{1j}(t) \\
C_{2j}(t) \\
\vdots \\
C_{j-1,j}(t)
\end{bmatrix}.
\]

**Proof.** The proof will be carried out recursively.

If \( k = 2 \), the result is an immediate consequence of Theorem 3 since the matrix of the system (4.6) has similar structure to (4.1) with \( A_1(t) = A(t), A_2(t) = B(t) \) and \( C_{12}(t) = C(t) \).

If \( k = 3 \), the system (4.6) can be seen as having a similar structure that (4.1) with

\[
A(t) = \begin{bmatrix} A_1(t) & C_{12}(t) \\ 0 & A_2(t) \end{bmatrix}, \quad B(t) = A_3(t) \quad \text{and} \quad C(t) = C_3(t) = \begin{bmatrix} C_{13}(t) \\ C_{23}(t) \end{bmatrix}
\]

Note that the subsystems \( \dot{x} = A(t)x \) and \( \dot{y} = B(t)y \) have a nonuniform exponential dichotomy on \( \mathbb{R}_0^+ \). The first dichotomy property is a consequence of the case \( k = 2 \) while the second one is an hypothesis. As (4.7) is verified for \( j = 3 \), Theorem 3 implies that the system (4.6) with \( k = 3 \) has a nonuniform exponential dichotomy on \( \mathbb{R}_0^+ \) and the proof is achieved by a recursive way for \( k \geq 4 \).

A particular – but important – byproduct of the above Corollary is the following result when all the diagonal terms \( A_i(t) \) are scalar functions:

**Corollary 3.** The upper triangular system

\[
\dot{x} = \begin{bmatrix} a_1(t) & c_{12}(t) & \cdots & c_{1n}(t) \\ 0 & a_2(t) & \cdots & c_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_n(t) \end{bmatrix} x
\]

has a \( (Ke^{\xi \tau}, \alpha) \)-nonuniform exponential dichotomy on \( \mathbb{R}_0^+ \) if all the scalar differential equations

\[
\dot{x}_i = a_i(t)x_i \quad \text{for } i = 1, \ldots, k
\]

have a half \( (M_i e^{\eta_i \tau}, \omega_i) \)-nonuniform bounded growth property and a \( (K_i e^{\xi_i \tau}, \alpha_i) \)-nonuniform exponential dichotomy on \( \mathbb{R}_0^+ \) (with \( i = 1, \ldots, n \)) provided that

\[
\sup_{\tau \geq 0} |C_j(\tau)|_{\tau} = \sup_{\tau \geq 0} \left| \begin{bmatrix} c_{1j}(\tau) \\ c_{2j}(\tau) \\ \vdots \\ c_{j-1,j}(\tau) \end{bmatrix} \right|_{\tau} < \infty \quad \text{for any } j \in \{2, \ldots, n\}.
\]

**Remark 5.** Note that, when \( j = 2 \) in (4.10), we have

\[
\sup_{\tau \geq 0} |C_2(\tau)|_{\tau} = \sup_{\tau \geq 0} |c_{12}(\tau)|_{\tau} < \infty,
\]
which propels to consider $| \cdot |_\tau$ as a family of norms and $(\mathbb{R}, | \cdot |_\tau)$ as an $\mathbb{R}$–vector space of dimension one. It can be proved that there exists a continuous function $h : \mathbb{R}_0^+ \rightarrow (0, \infty)$ such that

$$|x|_\tau = h(\tau)|x|$$

for any $x \in \mathbb{R}$.

**Lemma 4.** Under the assumptions of Corollary 1 and Theorem 3, the $(Ke^x, \alpha)$–nonuniform exponential dichotomy spectrum of the upper block system (4.1) verifies

$$\Sigma^+(A) \cup \Sigma^+(B) = \Sigma^+(U)$$

with $U = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

**Proof.** Firstly, if $\lambda \in \rho(A) \cap \rho(B)$ we have that the systems (4.11)

$$\dot{x} = [A(t) - \lambda I]x \quad \text{and} \quad \dot{y} = [B(t) - \lambda I]y$$

have a nonuniform exponential dichotomy and Theorem 3 implies that $\lambda \in \rho(U)$ and consequently it follows that $\rho(A) \cap \rho(B) \subset \rho(U)$.

Secondly, if $\lambda \in \rho(U)$, the Corollary 1 implies that the subsystems (4.11) have a nonuniform exponential dichotomy on $\mathbb{R}^+_0$ which is equivalent to $\lambda \in \rho(A) \cap \rho(B)$, which implies that $\rho(U) \subset \rho(A) \cap \rho(B)$ and the result follows. □

Based on a recursive application of Lemma 4 we obtain the following descriptions for the nonuniform exponential dichotomy spectrum for an upper block system as in (4.6) and for an upper triangular system as in (4.8) respectively.

**Corollary 4.** Under the assumptions of Corollary 4, the nonuniform exponential dichotomy spectrum of the upper block system (4.6) is described by

$$\Sigma^+(A_1) \cup \Sigma^+(A_2) \cup \cdots \cup \Sigma^+(A_k).$$

**Corollary 5.** Under the assumptions of Corollary 3, the nonuniform exponential dichotomy spectrum of the upper triangular system (4.8) is described by

$$\Sigma^+(a_1) \cup \Sigma^+(a_2) \cup \cdots \cup \Sigma^+(a_n).$$

**Remark 6.** The previous Corollary says that the nonuniform exponential dichotomy spectrum of an upper triangular system coincides with the union of the spectra of the scalar equations (4.9). This property is known as diagonal significance and was introduced for the discrete uniform exponential dichotomy spectrum by C. Pötzsche in [29]. This fact is immediate in the autonomous case, while - counterintuitively - in the nonautonomous framework is not always verified.

Theorem 3 and its consequences provide the framework to state and prove our main result, namely, the nonautonomous nonuniform Markus–Yamabe conjecture is verified for triangular system of nonautonomous ordinary differential equations whose nondiagonal parts satisfy boundedness conditions described in terms of parametrized norms.

**Theorem 4.** Let us consider the triangular system

$$\begin{align*}
\dot{x}_1 &= f_1(t, x_1, x_2, \ldots, x_n) \\
\dot{x}_2 &= f_2(t, x_2, \ldots, x_n) \\
&\vdots \\
\dot{x}_n &= f_n(t, x_n),
\end{align*}$$

(4.12)

whose right part, namely $F(t, x)$, verifies (G1) and (G2). If for any piecewise continuous function $t \mapsto \theta(t)$ it is verified that
(a) There exist constants $k_i \geq 1$, $\alpha_i > 0$ and $\varepsilon_i \geq 0$, with $\varepsilon_i < \alpha_i$ such that
\[
\int_{s}^{t} \frac{\partial f_i}{\partial x_i}(\tau, \theta(\tau)) \, d\tau \leq \ln(k_i) - \alpha_i(t - s) + \varepsilon_i s \quad \text{for any } t \geq s \geq 0 \text{ and } i = 1, \ldots, n.
\]
(b) For any $j \in \{2, \ldots, n\}$ and any piecewise continuous function $t \mapsto \theta(t)$, the partial derivatives verify
\[
\sup_{\tau \geq 0} \left| \begin{pmatrix}
\frac{\partial f_1(\tau, \theta(\tau))}{\partial x_j} \\
\frac{\partial f_2(\tau, \theta(\tau))}{\partial x_j} \\
\vdots \\
\frac{\partial f_{n-1}(\tau, \theta(\tau))}{\partial x_j}
\end{pmatrix} \right| < \infty.
\]
then the trivial solution of (4.12) is globally nonuniformly asymptotically stable.

Proof. Firstly, we will prove that (G3) is verified. In fact, the statement (a) implies that, for any $i = 1, \ldots, n$, the nonuniform exponential dichotomy spectra of the differential equations
\[
\dot{x} = \frac{\partial f_i}{\partial x_i}(t, \theta(t))x
\]
verifies
\[
\Sigma^+ \left[ \frac{\partial f_i}{\partial x_i}(t, \theta(t)) \right] \subset (-\infty, 0).
\]
for any set of piecewise continuous functions $t \mapsto \theta(t) = (\theta_1(t), \ldots, \theta_n(t))$.

On the other hand, let us recall that the jacobian matrix $JF$ is upper triangular defined by
\[
JF(t, x_1, \ldots, x_n)_{ij} = \begin{cases} 
\frac{\partial f_i}{\partial x_j}(t, x_1, \ldots, x_n) & \text{if } i \leq j, \\
0 & \text{if } i > j.
\end{cases}
\]

Then, the statement (b) combined with (4.13) and Corollary 5 imply that
\[
\Sigma^+ [JF(t, \theta(t))] = \bigcup_{i=1}^{n} \Sigma^+ \left[ \frac{\partial f_i}{\partial x_i}(t, \theta(t)) \right] \subset (-\infty, 0)
\]
and (G3) follows.

Let $t \mapsto (x_1(t), x_2(t), \ldots, x_n(t))$ be a solution of (4.12) passing through $(x_1^0, x_2^0, \ldots, x_n^0)$ at $t = t_0$. Note that the scalar equation
\[
\dot{x}_n = f_n(t, x_n) \quad \text{with} \quad x_n(t_0) = x_n^0
\]
is a subsystem of (4.12) whose solution is denoted by $\phi_n(t) := x_n(t, t_0, x_n^0)$ and verifies $\lim_{t \to \infty} \phi_n(t) = 0$ globally and nonuniformly, as we can see by Theorem 1.

Now, we can see that the last two equations of (4.12) are
\[
\begin{align*}
\dot{x}_{n-1} &= f_{n-1}(t, x_{n-1}, x_n) \\
\dot{x}_n &= f_n(t, x_n)
\end{align*}
\]
with initial conditions $(x_{n-1}^0, x_n^0)$ at $t = t_0$. The solution of this system is denoted by $(\phi_{n-1}(t), \phi_n(t))$, where $\phi_n$ is defined above and $\phi_{n-1}$ is the solution of the scalar equation
\[
\dot{x}_{n-1} = f_n(t, x_{n-1}, \phi_n(t)) \quad \text{with} \quad x_{n-1}(t_0) = x_{n-1}^0
\]
and, as before, also verifies \( \lim_{t \to \infty} \phi_{n-1}(t) = 0 \) globally and nonuniformly, as we can see again by Theorem 1. The rest of the proof can be achieved in a recursive way. \( \square \)

**Appendix A. Proof of Theorem 3**

**A.1. Preliminaries.** By hypothesis, we know that the linear systems \( \dot{x} = A(t)x \) and \( \dot{y} = B(t)y \) have a \((Ke^{\varepsilon t}, \alpha) \) and \((Ke^{\tilde{\varepsilon} t}, \tilde{\alpha}) \)–nonuniform exponential dichotomy on \( \mathbb{R}^d_0 \) with projectors \( P^A(\cdot) \) and \( P^B(\cdot) \) respectively:

\[
\|X(t, s)P^A(s)\| \leq Ke^{-\alpha(t-s) + \varepsilon s} \quad \text{for } t \geq s \geq 0,
\]

\[
\|X(t, s)[I - P^A(s)]\| \leq Ke^{-\alpha(s-t) + \varepsilon s} \quad \text{for } s \geq t \geq 0,
\]

and

\[
\|Y(t, s)P^B(s)\| \leq \tilde{K}e^{-\tilde{\alpha}(t-s) + \tilde{\varepsilon} s} \quad \text{for } t \geq s \geq 0,
\]

\[
\|Y(t, s)[I - P^B(s)]\| \leq \tilde{K}e^{-\tilde{\alpha}(s-t) + \tilde{\varepsilon} s} \quad \text{for } s \geq t \geq 0,
\]

where \( X(t, s) \) and \( Y(t, s) \) are the evolution operators corresponding to the diagonal subsystems. Moreover, we also know that the above subsystems have the properties of half \((Me^{\ell s}, \omega) \) and half \((Me^{\tilde{\ell} s}, \tilde{\omega}) \)–nonuniform bounded growth on \( \mathbb{R}^d_0 \) respectively, that is

\[
\|X(t, s)\| \leq Me^{\omega(t-s) + \ell s} \quad \text{and} \quad \|Y(t, s)\| \leq \tilde{M}e^{\tilde{\omega}(t-s) + \tilde{\ell} s} \quad \text{for } t \geq s \geq 0,
\]

then, the Uniformization Lemma can be applied to both subsystems. On one hand, the dichotomy estimations become

\[
\|X(t, s)\|_{s,t} \leq Ke^{-\alpha(t-s)} \quad \text{for } t \geq s \geq 0,
\]

\[
\|X(t, s)[I - P^A(s)]\|_{s,t} \leq Ke^{-\alpha(s-t)} \quad \text{for } s \geq t \geq 0,
\]

and

\[
\|Y(t, s)\|_{s,t} \leq \tilde{K}e^{-\tilde{\alpha}(t-s)} \quad \text{for } t \geq s \geq 0,
\]

\[
\|Y(t, s)[I - P^B(s)]\|_{s,t} \leq \tilde{K}e^{-\tilde{\alpha}(s-t)} \quad \text{for } s \geq t \geq 0,
\]

while the half bounded growth properties become

\[
\|X(t, s)\|_{s,t} \leq Me^{\omega(t-s)} \quad \text{and} \quad \|Y(t, s)\|_{s,t} \leq \tilde{M}e^{\tilde{\omega}(t-s)} \quad t \geq s \geq 0.
\]

The Uniformization Lemma also ensures the existence of two family of norms: a family \( \{\|\cdot\|_t\} \) in \( \mathbb{R}^n \) and a family \( \{\|\cdot\|_t\} \) in \( \mathbb{R}^m \). Moreover, the inequalities (2.7) are verified with \( L^A_1, L^B_1, L^A_2(t) \) and \( L^B_2(t) \) respectively. By Remark 2 we know that

\[
L^A_2(t) = Me^{\ell t} + Ke^{\varepsilon t} \quad \text{and} \quad L^B_2(t) = \tilde{M}e^{\tilde{\ell} t} + \tilde{K}e^{\tilde{\varepsilon} t}
\]

and the inequalities (2.8) are verified with

\[
\beta^A(t) = \frac{L^A_2(t)}{L^A_1(t)} \leq Le^{\theta t} \quad \text{and} \quad \beta^B(t) = \frac{L^B_2(t)}{L^B_1(t)} \leq Le^{\theta t},
\]

where

\[
\theta = \max\{t, \ell, \varepsilon, \tilde{\ell}, \tilde{\varepsilon}\}.
\]

These constants \( L \) and \( \theta \) are useful to state the following result:
Lemma 5. For any $t, \tau, s \geq 0$, the evolution operators $X$ and $Y$ verify
\[
||X(t, \tau)Z(\tau)C(\tau)V(\tau)Y(\tau, s)|| \leq Le^{\theta t}||C||_{\tau, \infty}||X(t, \tau)Z(\tau)||_{\tau, t}||V(\tau)Y(\tau, s)||_{s, \tau},
\]
where $Z(\tau)$ is either $P^A(\tau)$ or $I_n - P^A(\tau)$ and $V(\tau)$ is either $P^B(\tau)$ or $I_m - P^B(\tau)$.

Proof. Let $\xi \in \mathbb{R}^n \setminus \{0\}$. By using (4.3) followed by (2.9) and recalling the dimensions of $C(\cdot)$, we have that
\[
|X(t, \tau)Z(\tau)C(\tau)V(\tau)Y(\tau, s)\xi| \leq \frac{1}{L_1} |X(t, \tau)Z(\tau)C(\tau)V(\tau)Y(\tau, s)\xi|_t
\]
\[
\leq \frac{1}{L_1} ||X(t, \tau)Z(\tau)||_{t, \tau}||C(\tau)||_{t, t}||V(\tau)V(\tau, s)||_{s, \tau}||\xi||_s.
\]
By using (2.10) followed by (2.11), (4.5), (4.3) combined with (A.5)–(A.6), it follows that
\[
|X(t, \tau)Z(\tau)C(\tau)V(\tau)Y(\tau, s)\xi| \leq \frac{1}{L_1} ||X(t, \tau)Z(\tau)||_{\tau, t}||C(\tau)||_{\tau, \tau}||V(\tau)V(\tau, s)||_{s, \tau}||\xi||_s
\]
\[
\leq Le^{\theta t}||C||_{\tau, \infty}||X(t, \tau)Z(\tau)||_{\tau, t}||V(\tau)V(\tau, s)||_{s, \tau}||\xi||_s
\]
and the Lemma follows. \qed

We will follow the lines of the work carried out by F. Batelli and K.J. Palmer in [3], which proved that evolution operator of the triangular system (4.1) is given by
\[
T(t, s) = \begin{bmatrix} X(t, s) & W(t, s) \\ 0 & Y(t, s) \end{bmatrix}
\]
where $W$ is a $n \times m$ matrix defined by
\[
W(t, s) := \int_s^t X(t, \tau)C(\tau)Y(\tau, s) \, d\tau.
\]
(A.7)

In addition, as in [3], let us consider:
\[
P(t) = T(t, 0) \begin{bmatrix} P^A(0) & LP^B(0) \\ 0 & P^B(0) \end{bmatrix} T(0, t) = \begin{bmatrix} P^A(t) & R(t) \\ 0 & P^B(t) \end{bmatrix},
\]
where $\mathcal{L} : \mathcal{R} P^B(0) \rightarrow (\mathcal{R} P^A(0))^\perp$ is the linking operator defined by
\[
\mathcal{L}\eta = -\int_0^\infty |I_m - P^A(0)|X(0, \tau)C(\tau)Y(\tau, 0)\eta \, d\tau,
\]
which plays an important role in the proof of Proposition 2. Moreover, $R(t)$ satisfies the matrix differential equation
\[
\dot{R} = A(t)R - RB(t) + C(t)P^B(t) - P^A(t)C(t) \quad \text{with} \quad R(0) = \mathcal{L}P^B(0),
\]
whose solution is defined by
\[
R(t) = X(t, 0)R(0)Y(0, t)
\]
(A.8)
\[
+ \int_0^t X(t, \tau)[C(\tau)P^B(\tau) - P^A(\tau)C(\tau)]Y(\tau, t) \, d\tau,
\]
or alternatively as follows

\[
R(t) = -\int_{t}^{+\infty} X(t, \tau)[I_n - P^A(\tau)]C(\tau)P^B(\tau)Y(\tau, t) \, d\tau = R_1(t)
\]

\[
- \int_{0}^{t} X(t, \tau)P^A(\tau)C(\tau)[I_m - P^B(\tau)]Y(\tau, t) \, d\tau = R_2(t)
\]

The Lemma 5 will be helpful to provide an estimation for \(R(t)\):

**Lemma 6.** For any \(t \geq 0\), the matrix \(R(t)\) verifies

\[
||R(t)|| \leq \frac{2\tilde{\kappa}}{\alpha + \tilde{\alpha}} Le^{\beta t}||C||_{\tau, \infty}.
\]

**Proof.** Let \(\xi \in \mathbb{R}^m \setminus \{0\}\). By using Lemma 6 we can deduce

\[
|R_1(t)\xi| \leq Le^{\beta t}||C||_{\tau, \infty} \int_{t}^{\infty} \|X(t, \tau)[I_n - P^A(\tau)][I_m - P^B(\tau)]Y(\tau, t)\|_{t, \tau} |\xi| \, d\tau.
\]

Now, we apply the Uniformization Lemma to (A.1) and (A.2) respectively, and we can deduce that

\[
|R_1(t)\xi| \leq \kappa \tilde{\kappa}Le^{\beta t}||C||_{\tau, \infty} |\xi| e^{(\alpha + \tilde{\alpha})t} \int_{t}^{\infty} e^{-(\alpha + \tilde{\alpha})\tau} \, d\tau
\]

\[
\leq \frac{\kappa \tilde{\kappa}}{\alpha + \tilde{\alpha}} Le^{\beta t}||C||_{\tau, \infty} |\xi|.
\]

Similarly, for the second term, by using Lemma 5 followed by the Uniformization Lemma, we can deduce that

\[
|R_2(t)\xi| \leq Le^{\beta t}||C||_{\tau, \infty} \int_{0}^{t} \|X(t, \tau)P^A(\tau)[I_m - P^B(\tau)]Y(\tau, t)\|_{t, \tau} |\xi| \, d\tau
\]

\[
\leq \kappa \tilde{\kappa}Le^{\beta t}||C||_{\tau, \infty} |\xi| e^{-(\alpha + \tilde{\alpha})t} \int_{0}^{t} e^{(\alpha + \tilde{\alpha})\tau} \, d\tau
\]

\[
\leq \frac{\kappa \tilde{\kappa}}{\alpha + \tilde{\alpha}} Le^{\beta t}||C||_{\tau, \infty} |\xi|,
\]

and the inequality (A.9) follows.

We will verify that \(P(\cdot)\) is an invariant projector. In fact, the property \(P^2(t) = P(t)\) for any \(t \geq 0\) is a consequence of its own definition, while the next result proves its invariance. This last property has not been proved in [3] and, in spite that can be deduced easily, we will prove it.

**Lemma 7.** The projector \(P(\cdot)\) is invariant, namely, it verifies the property

\[
T(t, s)P(s) = P(t)T(t, s)
\]

for any \(t, s \geq 0\).
\textbf{Proof.} Notice that
\[
T(t,s)P(s) = \begin{bmatrix}
X(t,s)P^A(s) & W(t,s)P^B(s) + X(t,s)R(s) \\
0 & Y(t,s)P^B(s)
\end{bmatrix}.
\]
As \(P^A(\cdot)\) and \(P^B(\cdot)\) are invariant projectors, we can see that the Lemma follows if and only if
\[
R(t)Y(t,s) + P^A(t)W(t,s) = W(t,s)P^B(s) + X(t,s)R(s) \quad \text{for any } t, s \geq 0.
\]
By defining \(R_{t,s}(0) := X(t,0)R(0,s)\), using (A.7) and (A.8) and considering \(t \geq s\), we can easily deduce that
\[
R(t)Y(t,s) + P^A(t)W(t,s) &= R_{t,s}(0) + \int_0^t X(t,\tau)[C(\tau)P^B(\tau) - P^A(\tau)C(\tau)]Y(\tau,s)\,d\tau \\
&+ \int_s^t X(t,\tau)P^A(\tau)C(\tau)Y(\tau,s)\,d\tau \\
&= R_{t,s}(0) + \int_0^s X(t,\tau)[C(\tau)P^B(\tau) - P^A(\tau)C(\tau)]Y(\tau,s)\,d\tau \\
&+ \int_s^t X(t,\tau)C(\tau)P^B(\tau)Y(\tau,s)\,d\tau \\
&= X(t,s)R(s) + W(t,s)P^B(s).
\]
A similar identity can be deduced considering \(t < s\) and the Lemma follows. \(\square\)

Gathering the above results, it can be proved that the triangular system (4.1) has a \((K^\varepsilon, \bar{\alpha})\)-nonuniform exponential dichotomy on \(\mathbb{R}_0^+\) with the above defined invariant projector \(P(t)\).

\textbf{Lemma 8.} There exist a constant \(K_3 \geq 1, \alpha_3 > 0\) and \(\varepsilon_3 \geq 0\), where \(\varepsilon_3 < \alpha_3\) such that
\[
||W(t,s)P^B(s) + X(t,s)R(s)|| \leq K_3e^{-\alpha_3(t-s) + \varepsilon_3 s} \quad \text{for } t \geq s \geq 0.
\]

\textbf{Proof.} In order to deduce this estimation, we will write
\[
W(t,s)P^B(s) + X(t,s)R(s) = \underbrace{\int_s^t X(t,\tau)P^A(\tau)C(\tau)P^B(\tau)Y(\tau,s)\,d\tau}_{:=D_1} \\
- \underbrace{\int_t^{+\infty} X(t,\tau)Q^A(\tau)C(\tau)P^B(\tau)Y(\tau,s)\,d\tau}_{:=D_2} \\
- \underbrace{\int_0^s X(t,\tau)P^A(\tau)C(\tau)Q^B(\tau)Y(\tau,s)\,d\tau}_{:=D_3},
\]
where \(I_n - P^A(\tau) = Q^A(\tau)\) and \(I_m - P^B(\tau) = Q^B(\tau)\).
By using again Lemma \[5\] followed by the Uniformization Lemma and recalling that \(s \leq t\), we have that:

\[
\|D_1\| \leq L e^{\theta s} |C|_{\tau, \infty} \int_s^t \|X(t, \tau)P^A(\tau)\|_{\tau, t} \|P^B(\tau)Y(\tau, s)\|_{s, \tau} d\tau \\
\leq L e^{\theta s} |C|_{\tau, \infty} K_1 \int_s^t e^{-\alpha(t-\tau)} e^{-\tilde{\alpha}(\tau-s)} d\tau,
\]

(A.11)

\[
\|D_2\| \leq L e^{\theta s} |C|_{\tau, \infty} \int_t^\infty \|X(t, \tau)Q^A(\tau)\|_{\tau, t} \|P^B(\tau)Y(\tau, s)\|_{s, \tau} d\tau \\
\leq L e^{\theta s} |C|_{\tau, \infty} K_1 \int_t^\infty e^{-\alpha(t-\tau)} e^{-\tilde{\alpha}(\tau-s)} d\tau,
\]

(A.12)

\[
\|D_3\| \leq L e^{\theta s} |C|_{\tau, \infty} \int_0^s \|X(t, \tau)P^A(\tau)\|_{\tau, t} \|Q^B(\tau)Y(\tau, s)\|_{s, \tau} ds \\
\leq L e^{\theta s} |C|_{\tau, \infty} K_1 \int_0^s e^{-\alpha(t-\tau)} e^{-\tilde{\alpha}(s-\tau)} d\tau.
\]

(A.13)

As a consequence of the estimates (A.11), (A.12), (A.13) and defining \(\kappa_f = \max\{\kappa, \tilde{k}\}\), we have:

\[
\|W(t, s)P^B(s) + X(t, s)R(s)\| \leq L e^{\theta s} |C|_{\tau, \infty} \kappa_f^2 \int_s^t e^{-\alpha(t-\tau)} e^{-\tilde{\alpha}(\tau-s)} d\tau
\]

\[+ L e^{\theta s} |C|_{\tau, \infty} \kappa_f^2 \int_t^{+\infty} e^{-\alpha(t-\tau)} e^{-\tilde{\alpha}(\tau-s)} d\tau + L e^{\theta s} |C|_{\tau, \infty} \kappa_f^2 \int_0^s e^{-\alpha(t-\tau)} e^{-\tilde{\alpha}(s-\tau)} d\tau \]

\[\leq L e^{\theta s} |C|_{\tau, \infty} \kappa_f^2 \cdot \left\{ \frac{e^{-\tilde{\alpha}(t-s)} - e^{-\alpha(t-s)}}{\alpha - \tilde{\alpha}} + \frac{e^{-\tilde{\alpha}(t-s)} - e^{-\alpha(t-s)}}{\alpha + \tilde{\alpha}} + \frac{e^{-\alpha(t-s)} - e^{-\alpha(t-s)}}{\alpha + \alpha} \right\} e^{-\alpha_1(t-s)} \]

\[\leq K_1 e^{-\alpha_1(t-s)+\theta s}, \]

and if \(\alpha \neq \tilde{\alpha}\), then (A.11) is verified with \(\alpha_1 = \min\{\alpha, \tilde{\alpha}\} > \theta\) and

\[K_1 = \max\left\{1, L |C|_{\tau, \infty} \kappa_f^2 \cdot \left[ \frac{1}{\alpha - \tilde{\alpha}} + \frac{1}{\alpha + \alpha} + \frac{1}{\alpha + \tilde{\alpha}} \right] \right\}.\]

Note that, similarly as done in [3], if \(\alpha = \tilde{\alpha}\), we can see that only the first term in the above brackets must be replaced by a new estimation of (A.11):

\[
\|D_1\| \leq L e^{\theta s} |C|_{\tau, \infty} K_1 \int_s^t e^{-\alpha(t-\tau)} e^{-\alpha(\tau-s)} d\tau \\
\leq L e^{\theta s} |C|_{\tau, \infty} K_1 \int_s^t e^{-\alpha(t-s)} d\tau \\
\leq L e^{\theta s} |C|_{\tau, \infty} K_1^2(t-s)e^{-\alpha(t-s)},
\]
and since the estimation \((t - s)e^{-\gamma(t-s)} \leq \frac{1}{\gamma e}\), for a positive \(\gamma\), if we have that \(\gamma < \alpha\) and \(\theta < \alpha - \gamma\), then the previous inequality becomes

\[
||D_1|| \leq Le^{\theta s}||C||_{\tau, \infty} K_1^2 (t - s) e^{-\alpha(t-s)} \leq Le^{\theta s}||C||_{\tau, \infty} K_1^2 \frac{1}{\gamma e} e^{-(\alpha - \gamma)(t-s)},
\]

thus we obtain that

\[
||W(t, s)P^B(s) + X(t, s)R(s)||
\]

\[
\leq Le^{\theta s}||C||_{\tau, \infty} K_1^2 \frac{1}{\gamma e} e^{-(\alpha - \gamma)(t-s)} + Le^{\theta s}||C||_{\tau, \infty} K_1^2 \frac{1}{\gamma e} e^{-(\alpha - \gamma)(t-s)}
\]

\[
\leq K_2 e^{-\alpha_2(t-s)+\theta s},
\]

where \(K_2 = \max \left\{ 1, 2L||C||_{\tau, \infty} K_1^2 \frac{1}{\gamma} \right\}\) and \(\alpha_2 = \alpha - \gamma > \theta\).

Furthermore, if we define \(K_3 := \max\{K_1, K_2\}\), \(\alpha_3 := \min\{\alpha_1, \alpha_2\}\) and \(\varepsilon_3 := \theta\), we can conclude the estimate (A.10).

**Lemma 9.** The evolution operator of (1.1) and the projector \(P(t)\) previously defined verify:

\[
||T(t, s)P(s)|| \leq \bar{K} e^{-(\tilde{\alpha} - \varepsilon s) + \varepsilon s} \text{ for any } t \geq s \geq 0,
\]

where \(\bar{K} \geq 1, \tilde{\alpha} > 0, \varepsilon \geq 0\), with \(\tilde{\alpha} > \varepsilon\).

**Proof.** Let us consider \((\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^m \setminus \{(0,0)\}\), then we have that

\[
\begin{vmatrix}
T(t, s)P(s)
\end{vmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix}
\]

\[
= \begin{bmatrix}
X(t, s)P^A(s) & W(t, s)P^B(s) + X(t, s)R(s) \\
0 & Y(t, s)P^B(s)
\end{bmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix},
\]

\[
\leq ||X(t, s)P^A(s)|| \cdot ||\xi_1|| + ||W(t, s)P^B(s) + X(t, s)R(s)|| \cdot ||\xi_2|| + ||Y(t, s)P^B(s)|| \cdot ||\xi_2||
\]

and due to the estimates (A.10) for the second summand deduced in the above Lemma, the fact that \(||\xi_i|| \leq ||(\xi_1, \xi_2)||\) and the estimations (A.1) and (A.2), the Lemma follows easily. □

The following two lemmas emulate the previous results considering the complementary projectors and \(t \leq s\). Allowing us to end the treatment and study of the dichotomy properties.

**Lemma 10.** There exist a constant \(K_3 \geq 1, \alpha_3 > 0\) and \(\varepsilon_3 \geq 0\), where \(\varepsilon_3 < \alpha_3\) such that

\[
||I_n - P^A(t)||W(t, s) - R(t)Y(t, s)|| \leq K_3 e^{-\alpha_3(s-t)+\varepsilon_3 s} \text{ for } s \geq t \geq 0.
\]

**Proof.** The proof is a charbon copy of the proof of the Lemma 5 and is left for the reader. □
Lemma 11. The evolution operator of (4.1) and the projector $P(t)$ previously defined verify:

$$\|T(t, s)[I - P(s)]\| \leq \tilde{K}e^{-\tilde{\alpha}(s-t)+\tilde{\varepsilon}s} \text{ for any } s \geq t \geq 0,$$

where $\tilde{K} \geq 1$, $\tilde{\alpha} > 0$, $\tilde{\varepsilon} \geq 0$, with $\tilde{\alpha} > \tilde{\varepsilon}$.

Proof. Let us consider $(\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^m \setminus \{(0, 0)\}$, then we have that

$$T(t, s)(I - P(s)) \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) = (I - P(t))T(t, s) \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right),$$

$$= \begin{bmatrix} Q^A(t)X(t, s) & Q^A(t)W(t, s) - R(t)Y(t, s) \\ 0 & Q^B(t)Y(t, s) \end{bmatrix} \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right),$$

$$\leq \|Q^A(t)X(t, s)\| \cdot |\xi_1| + \|Q^A(t)W(t, s) - R(t)Y(t, s)\| \cdot |\xi_2| + \|Q^B(t)Y(t, s)\| \cdot |\xi_2|$$

and the Lemma is a consequence of (A.13), which estimates the second summand, combined with the fact that $|\xi_i| \leq |(\xi_1, \xi_2)|$.

The next result shows that the nondiagonal submatrix $W(t, s)$ of $T(t, s)$ has a property reminiscent to the half nonuniform bounded growth.

Lemma 12. There exist a constant $M_1 \geq 1$, $\omega_3 > 0$ and $\theta \geq 0$ such that the operator $W(t, s)$, defined in (4.7), verifies

$$|W(t, s)| \leq M_1e^{\omega_3(t-s)+\theta s}, \quad t \geq s.$$

Proof. Let us recall that the systems $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have a half nonuniform bounded growth on $\mathbb{R}_0^+$ described in (A.3). By using Lemma 5 where $Z(\tau) = I_n$ and $V(\tau) = I_m$, combined with (A.4), which arises from the Uniformization Lemma, we can see that when $t \geq s \geq 0$:

$$|W(t, s)| \leq Le^{\theta s}|C||_{\tau, \infty} \int_s^t |X(t, \tau)||_{\tau, t}||Y(\tau, s)||_{s, \tau} \ d\tau$$

$$\leq Le^{\theta s}|C||_{\tau, \infty}\tilde{\mu}\int_s^t e^{\omega(t-\tau)}e^{\tilde{\omega}(t-s)} \ d\tau.$$
The second case is when $\omega = \tilde{\omega}$, then
\[
||W(t, s)|| \leq L e^{\theta s} ||C||_{\tau, \infty} \int_s^t e^{\omega(t-s)} \, d\tau \\
\leq L e^{\theta s} ||C||_{\tau, \infty} \mu^2 e^{\omega(t-s)} e^{(t-s)} \\
\leq M_2 e^{\omega_2(t-s) + \theta s},
\]
where
\[
M_2 = \max \{ 1, L ||C||_{\tau, \infty} \mu^2 \} \quad \text{and} \quad \omega_2 = \omega + 1.
\]
Based on the two cases analyzed, we can conclude that
(A.15) \[
||W(t, s)|| \leq M e^{\omega_3(t-s) + \theta s}, \quad t \geq s,
\]
where
\[
M_3 = \max \{ M_1, M_2 \} \quad \text{and} \quad \omega_3 = \max \{ \omega_1, \omega_2 \}.
\]

The last result shows that the evolution operator associated to the upper triangular system (2.1) has the property of half $$(Me^{\tilde{\theta}s}, \tilde{\omega})$$–nonuniform bounded growth.

Lemma 13. The evolution operator of (4.1) verify:
\[
||T(t, s)|| \leq \bar{M} e^{\bar{\omega}(t-s) + \bar{\theta}s} \quad \text{for any} \quad t \geq s, \quad \text{where} \quad \bar{M} \geq 1, \quad \bar{\omega} > 0, \quad \bar{\theta} \geq 0.
\]

Proof. If we consider $$(\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^m \setminus \{(0, 0)\}$$. Then, we have that
\[
||T(t, s)|| = \left|\begin{array}{c}
X(t, s) \\
Y(t, s)
\end{array}\right| \cdot ||(\xi_1, \xi_2)|| \\
= \left|\begin{array}{c}
X(t, s)\xi_1 + W(t, s)\xi_2 \\
Y(t, s)\xi_2
\end{array}\right| \\
\leq ||X(t, s)|| \cdot ||\xi_1|| + ||W(t, s)|| \cdot ||\xi_2|| + ||Y(t, s)|| \cdot ||\xi_2||
\]
and due to the estimation (A.15), the fact that $||\xi_1|| \leq ||(\xi_1, \xi_2)||$ and both estimations in (A.3), we can ensure that for $t \geq s$:
\[
||T(t, s)|| \leq \bar{M} e^{\bar{\omega}s} e^{\bar{\omega}(t-s)}.
\]

A.2. End of proof of Theorem 3. Firstly, the Lemmas 7, 9 and 11 imply that the triangular system (4.1) has a $$(Me^{\bar{\theta}s}, \tilde{\omega})$$–nonuniform exponential dichotomy in $\mathbb{R}^+$. Secondly, the Lemma 13 says that the system (4.1) has the property of half $$(Me^{\theta s}, \tilde{\omega})$$–nonuniform bounded growth and the Theorem follows.

Remark 7. A meticulous reading of this Appendix shows that the property of half nonuniform bounded growth is fundamental in several steps of the proof:
a) Is a necessary condition in order to use the Uniformization Lemma, which ensures the existence of a continuous family norms $\{\cdot|t|\}$ verifying the inequality $|x|_t \leq L_2(t)|x|$. The half nonuniform bounded growth property is a required tool to obtain explicit estimations for $L_2(\cdot)$ and (A.4).

b) The constants $\ell$ and $\tilde{\ell}$ are necessary to deduce (A.5) and (A.6), these identities are immersed in Lemma [2] which is the main key to deduce several estimations around the proof.

c) The previous facts, also shows that the boundedness properties of $||C(\tau)||_{\tau,\tau}$ involves estimations based in the half nonuniform bounded growth property.

We point out that in [32], the property of half nonuniform bounded growth is not considered neither in the statement of Uniformization Lemma (Lemma 2.2 in [32]) nor in the statement of Theorem 2.3.

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