THE GENEALOGY OF CONVEX SOLIDS

GÁBOR DOMOKOS, ZSOLT LÁNGI AND TÍMEA SZABÓ

Abstract. The shape of homogeneous, smooth convex bodies as described by the Euclidean distance from the center of gravity represents a rather restricted class $\mathcal{M}_C$ of Morse-Smale functions on $\mathbb{S}^2$. Here we show that even $\mathcal{M}_C$ exhibits the complexity known for general Morse-Smale functions on $\mathbb{S}^2$ by exhausting all combinatorial possibilities: every 2-colored quadrangulation of the sphere is isomorphic to a suitably represented Morse-Smale complex associated with a function in $\mathcal{M}_C$ (and vice versa). We prove our claim by an inductive algorithm, starting from the path graph $P_2$ and generating convex bodies corresponding to quadrangulations with increasing number of vertices by performing each combinatorially possible vertex splitting by a convexity-preserving local manipulation of the surface. Since convex bodies carrying Morse-Smale complexes isomorphic to $P_2$ exist, this algorithm not only proves our claim but also defines a hierarchical order among convex solids and generalizes the known classification scheme in [36], based on the number of equilibria. Our expansion algorithm is essentially the dual procedure to the algorithm presented by Edelsbrunner et al. in [20], producing a hierarchy of increasingly coarse Morse-Smale complexes. We point out applications to pebble shapes.

1. Introduction

In this paper we prove that the arrangement of equilibrium points on the surfaces of convex solids exhausts all combinatorial possibilities and also defines a hierarchical order among such objects.

The shape of homogeneous, smooth convex bodies $K \in \mathcal{K}$ as described by the Euclidean distance from the center of gravity represents a rather restricted class $\mathcal{M}_C$ of Morse-Smale functions on $\mathbb{S}^2$. We assume that the critical points of these functions are generic singularities; namely, that the determinant of the Hessian of the function at such a point is not zero. These points are the vertices of the Morse-Smale complex (cf. [31]) associated with $K$, the edges of the complex are special integral curves of the gradient: heteroclinic orbits connecting singularities. In physical terms, these singularities are identical to static equilibrium points of $K$ when balanced on a horizontal plane under uniform gravity. The study of static equilibria was initiated by Archimedes [26] and attracts even current interest (cf. [11], [12], [13], [27] or [37]). Since the numbers of stable, unstable and saddle points ($S, U, H$, respectively) are related by the Poincaré-Hopf formula as

\begin{equation}
S + U - H = 2,
\end{equation}
the so-called primary classification system in \[36\] could be set up by omitting saddles, based on the integers \(\{S, U\}\) and it was shown that this system is complete, i.e. there are no empty classes. Our current motivation is to go beyond this result and establish the completeness according to a more refined (secondary) classification based on the topological arrangement of generic equilibria of convex bodies, i.e., to show that every combinatorially possible arrangement physically also exists.

The class of general Morse-Smale functions \(M\) on \(S^2\) is known to exhaust all combinatorial possibilities. Similarly to the primary equilibrium classification, saddles can again be omitted from the Morse-Smale complex to obtain the so-called quasi-dual representation \[19\]: every Morse-Smale complex associated with a function in \(M\) is isomorphic to a \(2\)-colored \((S, U)\) quadrangulation in class \(Q\) (\[19\], \[20\] and \[38\], for more detail see Section 2 and Figure 1) and the inverse statement is also relatively easy to see by direct construction. We introduce the shorthand notation \(Q(K)\) for the quasi-dual representation of the Morse-Smale complex associated with the convex body \(K\). Our main goal is to show that even \(M_C\) exhibits the same complexity as \(M\):

**Theorem 1.** For every \(Q^* \in Q\) there is a homogeneous convex body \(K^*\), with a \(C^\infty\)-class boundary, such that \(Q(K^*) = Q^*\).

We introduce \(Q_n \subset Q\) for the \(2\)-colored quadrangulations of the sphere with \(S + U = n\) vertices and the class of convex bodies with exactly \(S + U = n\) extremal points will be denoted by \(K_n \subset K\). It is relatively easy to see (as we show in Section 2 based on results from \[4\], \[9\] and \[33\]) that by subsequent applications of an operation called *face contraction* (where two diagonal vertices in a quadrangular face are merged and the resulting double edges are also merged) every quadrangulation \(Q^* \in Q_n\) can be collapsed onto the stem-graph \(P_2\) (the path graph with 2 vertices \[24\]) via a sequence of \((n - 1)\) graphs. If we list the latter in reverse order, we obtain what we call *combinatorial expansion sequences*:

\[
Q_2 \equiv P_2, Q_3, Q_4, \ldots, Q_{n-1}, Q_n \equiv Q^* \quad \text{such that} \quad Q_i \in Q_i.
\]

Subsequent elements of \((\text{2})\) are connected via *vertex splittings* (cf.\[4\], \[9\]), the inverse operation of face contraction. Both face contraction and vertex splitting can be carried out on any \(Q \in Q_n, n > 2\), and thus the sequences \((\text{2})\) can be generated both forward (vertex splittings) and backward (face contractions). Neither the forward nor the backward sequence is unique; the number of combinatorial possibilities as well as alternative representations in other graph classes are described in Section 2. To prove Theorem 1 in Section 3 we show that each combinatorially possible vertex splitting can be realized on convex bodies by a convexity-preserving, local manipulation of the surface (consisting of two consecutive, local truncations) which we call *equilibrium splitting*. By this operation we can create *geometric expansion sequences*

\[
K_2, K_3, K_4, \ldots, K_{n-1}, K_n \equiv K^* \quad \text{such that} \quad K_i \in K_i,
\]

which are generated in such a way that they are linked to an arbitrarily, a priori given combinatorial expansion sequence \((\text{2})\) via

\[
Q(K_i) = Q_i.
\]

Combinatorial sequences belonging to an arbitrary graph \(Q^* \in Q_n\) can be generated by running \((\text{2})\) in reverse order, arriving at \(P_2\). Since convex bodies in class \(K_2\)
carrying Morse-Smale complexes homeomorphic to $P_2$ exist (cf. [36]), subsequently we can run (3) forward to obtain the desired convex body $K^*$ with $Q(K^*) = Q^*$.

This algorithm not only proves Theorem 1 but also defines a hierarchical order among convex solids and generalizes the primary scheme of [36], which uses only the numbers $S, U$ of stable and unstable equilibria to classify convex shapes. Recently, in [29] it was shown that an inductive algorithm based on [36] is incapable of generating all combinatorial possibilities. We also remark that unlike combinatorial sequences (2) which can be run both forward and backward, our geometrical expansion sequences (3) can be run only forward from an arbitrary initial convex body $K$. Nevertheless, in a different setting, the geometric analogy of face contraction was considered in [8].

The local truncations associated with each geometric expansion step are extremely delicate. To have the ability of performing any combinatorially possible splitting (i.e. to obtain a convex body with the desired Morse-Smale complex) one has to render the vicinity of the given critical point arbitrarily sensitive before the actual splitting is achieved with a planar truncation. We achieve arbitrary sensitivity in Lemma 3 by constructing a preliminary truncation with a sphere the radius of which is sufficiently close to the distance of the equilibrium point to be split from the centre of gravity. If only the critical point is specified, however, the combinatorial structure of the splitting is arbitrary then the geometric task is substantially simpler [36].

Our expansion algorithm is essentially the dual procedure to the one presented by Edelsbrunner et al. [20] (cf. also [14]), producing a hierarchy of increasingly coarse Morse-Smale complexes (i.e. running combinatorial expansion sequences in reverse order compared to (2)); a topic attracting current interest in computational topology (cf. [5] or [25]). Beyond illustrating that $\mathcal{M}_C$ exhausts all possible combinatorial possibilities and offering a modest link between Morse theory and convex geometry, geometric expansion sequences defined in equation (3) appear to be the natural building blocks for the description of pebble shape evolution under collisional abrasion.

We describe graph expansions based on earlier results in Section 2; the main proof of our statement is contained in Section 3 where we show the existence of equilibrium splittings associated with arbitrary vertex splittings. We illustrate our results and discuss some related issues (including pebble abrasion) in Section 4.

### 2. Graph expansions

In this section, based on earlier results, we summarize the combinatorial properties of Morse-Smale complexes associated with general Morse-Smale functions in $\mathcal{M}$. In Subsection 2.1 we describe three alternative graph representations, one of which is 2-colored quadrangulations in class $\mathcal{Q}$. Subsection 2.2 describes the inductive algorithm which generates $\mathcal{Q}$ from a single stem-graph. By isomorphism we mean embedding-preserving isomorphism (cf. [10]) and by quadrangulation we mean a finite graph embedded on the 2-sphere such that it may have multiple edges and each face is bounded by a closed walk of length 4 (cf. [11] or [9]). Also, we follow Archdeacon et al. [1] and regard the path graph (cf. [24]) $P_3$ with 3
vertices as a quadrangulation. For clarity we now list the four different classes of quadrangulations appearing in the paper:

- $Q^{0,s}$ is the class of simple quadrangulations without coloring.
- $Q^0$ is the class of quadrangulations without coloring.
- $Q$ is the 2-colored version of the latter. Note that every graph in $Q^0$ is bipartite; that is, 2-colorable (cf. [1] and [32]).
- $Q^{3\ast}$ is the class of special 3-colored $(S, U, H)$ quadrangulations with $\deg(v) = 4$ for any $v \in H$ and $S + U - H = 2$ (cf. [1]).

In addition, we also use the class $T^{3\ast}$ of special 3-colored $(S, U, H)$ triangulations with $\deg(v) = 4$ for any $v \in H$. In this case (1) follows from the previous properties.

2.1. Graph representations of Morse-Smale complexes in $\mathcal{M}$. We start with a summary of the concepts of Morse theory that we use in the paper.

Let $f : S^2 \to \mathbb{R}$ be a smooth ($C^\infty$-class differentiable) function defined on the 2-sphere. The points with $(\text{grad } f)(p) = 0$ are called the critical points of $f$. If $f$ has finitely many nondegenerate critical points (i.e. the determinant of the Hessian of $f$ at each of them is not zero), we say that $f$ is a Morse function defined on $S^2$. In this case we distinguish maximum, minimum and saddle points, depending on the number of the negative eigenvalues of the Hessian.

It is known that for any Morse function $f$ on $S^2$, the integral curves of the vector field $\text{grad } f$ start and end at (different) critical points of $f$, are pairwise disjoint and, apart from the critical points, they cover $S^2$. For any critical point $p$, the union of the integral curves starting/ending at $p$ is called the ascending/descending manifold of $p$, respectively. Both the ascending and the descending manifolds form a CW-decomposition of $S^2$. The complex of the descending manifolds of $S^2$ is called the Morse complex of $f$.

Assume that any descending and ascending manifold of $f$ intersect transversally. Then $f$ is called a Morse-Smale function, and the complex, obtained as the intersection of the Morse complexes of $f$ and $-f$, is called the Morse-Smale complex of $f$. We note that, even though in the literature $f$ is usually assumed to be of class $C^\infty$ (or at least $C^2$), these notions can be interpreted in the same way under the assumption that $f$ is of class $C^1$, and each critical point of $f$ has a $C^2$-class neighborhood.

Dong et al. [19] introduced three different ways for representing a Morse-Smale complex associated with a function in $\mathcal{M}$. It was shown in [20] and [38] that such a Morse-Smale complex can be uniquely represented by a 3-colored quadrangulation in $Q^{3\ast}$, where colors correspond to the 3 types of non-degenerate critical points (maxima, minima and saddles), each quadrangle is bounded by a closed walk consisting of a maximum, a saddle, a minimum and a saddle in this order around the face and the degree of every saddle is 4. We follow Dong et al. [19] and call this representation of the Morse-Smale complex the primal Morse-Smale graph. These graphs are special 3-colored quadrangulations, since vertices corresponding to one color (saddles) are always of degree 4. These special vertices can be removed from the primal Morse-Smale graph without losing information (cf. [19]): first we connect maxima and minima in the quadrangles, resulting in a triangulated Morse-Smale graph in class $T^{3\ast}$, then we cancel saddle points and edges incident
to them (see Figure 1). Since the degree of every saddle is 4, the resulting graph is a 2-colored quadrangulation in class $Q$. We call this representation the quasi-dual Morse-Smale graph (cf. [19]). In this section we rely on the latter, however, in Section 3 the primal graph representation is preferred. We remark that all three representations (primal, triangulated and quasi-dual) are equivalent in the sense that they are mutually uniquely identified.

Due to a well-known bijection [22] between planar maps and 2-colored quadrangulations of the sphere, Morse-Smale complexes can be also represented by the even more compact class of planar maps as well. (A planar map is the equivalence class of topologically equivalent drawings of a planar graph on the sphere). The transformation between the two classes is shown on Figure 2: starting with a graph in class $Q$, first we connect with an edge the two (not necessarily distinct) stable vertices of a face, then we cancel unstable points and edges incident to them. The resulting graph has $S$ vertices, $U$ faces and $H$ edges, thus, in this representation, the Poincare-Hopf formula (1) is equivalent with Euler’s formula. Although compactness is a great advantage of this planar map representation, it is also a disadvantage: connection between combinatorial and geometric expansions would be difficult to interpret, therefore we rely on the more expressive quasi-dual representation in this section.

2.2. Inductive generation of $Q$ by vertex splittings. Let us regard a simple quadrangulation $Q^{0,s} \in Q^{0,s}$ and let $F = (v_1, v_2, v_3, v_4)$ be a face of $Q^{0,s}$ with
boundary walk $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$, where $v_1 = v_5$ and edges are $e_i = \{v_i, v_{i+1}\}$, $i = 1, 2, 3, 4$ (cf. Figure 3 a), left). On multigraphs $Q^0 \in Q^0$ the named vertices and edges may not be distinct, however, the definition of quasi-dual representation (i.e. that these graphs have been generated by removing the saddle points of degree 4 from the triangular representation) admits only two kinds of coincidences: two diagonal vertices $v_2$ and $v_4$ may coincide, and in this case the edges $e_2$ and $e_3$ may coincide. These two cases are illustrated in Figure 3 b) and c), left. Note that in Figure 3 b) the internal domain bordered by the edges $e_2$ and $e_3$ is not a quadrangular face but necessarily contains additional vertices.

Vertex splitting can be defined as the inverse of a combinatorial operation called face contraction (cf. [1], [9] or [33]). Let us apply a face contraction on a quadrangulation of $F$ at vertices $\{v_1, v_3\}$ produces a new quadrangulation $Q_2^{0} \in Q_2^{0}$ for $F$ by identifying the vertices $v_1$ and $v_3$, the edges $e_1$ and $e_2$, and the edges $e_3$ and $e_4$. Since the faces of $Q_2^{0}$ are identical to the faces of $Q_0^{0}$ except $F$, $Q_2^{0}$ is also a quadrangulation, if $n > 3$. The exceptional case when $Q_3^{0} = P_2$ and $Q_2^{0} = P_2$ will also be treated as a special face contraction, when $e_1 = e_4$, $e_2 = e_3$ and $v_2 = v_4$ (cf. Figure 3 d)), although $P_2$ is clearly not a quadrangulation. Thus, any $Q^0 \in Q^0$ can be contracted to $P_2$ through a sequence of face contractions. Therefore, $Q^0$ can be generated inductively from $P_2$ via a combinatorial expansion sequence of type (2), consisting of vertex splittings, the inverse transformation of face contraction. The above argument is a slight generalization of the results of Bagatelli [11], and also Negami and Nakamoto [33], who defined face contraction inside the class $Q^0$s of simple quadrangulations and showed that $Q^0$s can be generated inductively from $P_2$.

**Figure 3.** Face contraction on a graph in class $Q$. Like in [33], triangles incident to some vertices indicate that one or more edges may occur at that position around the vertex. Here, $v_1$ and $v_3$ are minimum points; their counterparts, which remove a maximum point from the graph, can be performed by switching the colors.
The quasi-dual representations $Q \in Q$ of Morse-Smale complexes associated with convex bodies are two-colored $(S, U)$ quadrangulations, however, extension of the expansion scheme to colored multigraphs is straightforward. On each face diagonal vertices have identical color, so after generating the color-less graph $Q^0 \in Q^0$, the 2-colored version $Q \in Q$ can be generated by applying two alternative coloring schemes, so if $Q^0$ can be generated from $P_2$ by vertex splittings, so can be $Q$. Thus we showed that the combinatorial expansion sequences (2) described in the Introduction are sufficient to generate the quasi-dual representation of any Morse-Smale function associated with a convex body $K \in K$. Conversely, given any such a body, the associated sequences can be generated by running (2) backward via face contractions.

Since a face contraction can be applied to any face of a quadrangulation, the sequence of graphs from $Q_n \in Q_n$ to $Q_2 = P_2$ is, in general, not unique. In special applications additional criteria may be applied to single out one sequence among the combinatorially possible ones. Face contraction on multigraphs was already used in [20] and [14] to simplify Morse-Smale complexes. Edelsbrunner et al. used the primal representation of the complex in which a face contraction (defined in the quasi-dual representation) emerges as a double edge contraction [15]. Their goal was to produce a hierarchy of increasingly coarse Morse-Smale complexes, therefore they applied an extra metric criteria (growing height differences on the edges of the graph) which resulted in an unambiguous sequence of graphs. This algorithm was also used in [17] to identify macroscopically perceptible static equilibrium points of 3D scanned pebbles.

As here we do not use any metric criteria while applying face contractions on a quadrangulation, the combinatorial expansion sequence (2) is not unique. Thus, in Section 3 we have to show that all possible vertex splittings, leading from a graph $Q_k$ to any $Q_{k+1}$ can be geometrically performed on a convex body as an equilibrium splitting in a way that convexity is not affected. At an arbitrary vertex $v_i$ of $Q_k$, vertex splitting increases the degree of two incident vertices of $v_i$ (on Figure 3, the split vertex is $v_1 = v_3$, the incident vertices are $v_2$ and $v_4$). If $v_i$ has degree $d(v_i)$, we have $P_i = \binom{d(v_i) + 1}{2}$ possibilities to choose the two (not necessarily distinct) neighbors. The possibilities for splitting a vertex with degree 4 are shown on Figure 4 a). Since $Q_k$ has $k$ vertices, we have altogether $\sum_{i=1}^{k} P_i$ possibilities to generate a graph $Q_{k+1}$ from $Q_k$. In Section 3 we prove Theorem 1 by showing that all these possibilities can be geometrically constructed; that is, a convex body can be modified in the vicinity of an arbitrary stable or unstable point in such a way that it remains convex and its Morse-Smale complex is modified according to an arbitrary splitting of that vertex.

3. Geometric realization of graph expansions

Throughout this section, by the center of a convex body $K$ we mean its center of gravity. Furthermore, the Morse-Smale complex of $K$ is meant to be the Morse-Smale complex defined on bd $K$ by the Euclidean distance function from the center of $K$. If we measure distance from a different point $w$, then we write about the Morse-Smale complex of $K$ with respect to $w$. To represent these complexes, in this section we use the primal representation, introduced in the previous section.
Our algorithm is based on the repeated applications of two truncations $T_1$ and $T_2$ which are described in Subsections 3.2 and 3.3 respectively. Inside these steps, we also use a smoothing subroutine $S$ which is described in Subsection 3.1. Starting from the mono-monostatic body $K_2 \in \mathcal{K}_2$ (associated with the path-graph $P_2 \equiv Q_2$, cf. [30]) these steps are applied in 2-cycles $T_1, T_2$ to obtain the element $K_{i+1} \in \mathcal{K}_{i+1}$ from $K_i \in \mathcal{K}_i$ in the geometric expansion series ([8]), with prescribed Morse-Smale representation $Q_{i+1} \in \mathcal{Q}_{i+1}$.

**Figure 4.** Vertex splittings at vertex $v$. Here, $v$ is a minimum point (of course, $v$ could also be a maximum). a) Vertex splitting in the quasi-dual representation in class $\mathcal{Q}$ ($d(v) = 4$) b) Geometric interpretation of the previous vertex splittings as splitting equilibrium point $p$ in the triangulated representation in class $\mathcal{T}^3\ast(d(p) = 8)$. The truncated piece of the body is illustrated by the white domain.
Throughout these steps, we assume that $K$ is a convex body such that

- the Euclidean distance function of $K$ is Morse-Smale (cf. Subsection 2.1);
- $\text{bd } K$ has nonzero principal curvatures at any critical point of this function.

For simplicity, we call this property (*).

Below we briefly introduce the steps $T_1$, $T_2$ and subroutine $S$ before giving a detailed description.

**Subroutine S: Smoothing:** Here we start with a convex body $K$ satisfying (*) except that it has a $C^1$-class boundary and construct a $C^\infty$-class small perturbation $K'$ of $K$, such that $K'$ satisfies (*), with a Morse-Smale complex homeomorphic to that of $K$. This perturbation is carried out in such a way that if $\text{bd } K$ is a spherical cap in a neighborhood of a critical point, then the same holds for a neighborhood of the corresponding critical point of $K'$.

**Step $T_1$: Truncation with a ‘sphere’:** In this step, for any $K$ satisfying (*), we construct a convex body $K'$ satisfying (*), with a Morse-Smale complex homeomorphic to that of $K$, such that the critical point of $K'$ corresponding to $p$ has a spherical cap neighborhood in $\text{bd } K'$.

**Step $T_2$: Vertex splitting:** In this step, for any convex body $K$ satisfying (*), and any stable or unstable point $p$ of $K$ that has a spherical cap neighborhood in $\text{bd } K$, and also any combinatorially possible splitting of $p$, we construct a convex body $K'$ satisfying (*), such that its Morse-Smale complex is homeomorphic to the ‘split’ complex of $K$, using the chosen splitting.

Throughout this section, we use the following notations and assumptions. For a point $q \in \mathbb{R}^3$ and $r > 0$, the open (respectively closed) ball with center $q$ and radius $r$ is denoted by $B_r(q)$ (respectively, $\bar{B}_r(q)$). For simplicity, we set $B = B_1(o)$ and $\bar{B} = \bar{B}_1(o)$. We consider a convex body $K \subset \mathbb{R}^3$ with a stable or unstable point $p \in \text{bd } K$. Without loss of generality, we may assume that the center of $K$ is the origin, and $p = (0, 0, 1)$. We denote the integral curves of $K$ that connect $p$ to a saddle point of $K$, in cyclic order around $p$, by $\Gamma_1, \Gamma_2, \ldots, \Gamma_k = \Gamma_0$. Furthermore, we let $t_i$ be the unit tangent vector of $\Gamma_i$ at $p$.

### 3.1. Smoothing subroutine $S$

Here we show that if $K$ satisfies (*) except that it has a $C^1$-class boundary, then it can be modified to a convex body $K'$ satisfying (*), with a Morse-Smale complex homeomorphic to that of $K$.

Observe that, as $o$ is the center of $K$, we have $o \in \text{int } K$. Let $f$ denote the distance function of $K$; that is, $f$ is defined by $f(u) = \min\{\lambda : \lambda \geq 0, u \in \lambda K\}$ (cf. [6]). Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a nonnegative, $C^\infty$-class function such that $\text{supp } h \subseteq B$ and $\int_{\mathbb{R}^3} h(x) \, dx = 1$. Such a function is called by various names in the literature: **mollifier** (cf. [21]), or **bump function** (cf. [28]) or **probability distribution**. Clearly, we may choose $h$ in a way that its symmetry group is $O^3$. Observe that by setting $h_t(x) = \frac{h(x/t)}{t^3}$, we obtain a family of $C^\infty$-class functions with $\text{supp } h_t \subseteq tB$ and $\int_{\mathbb{R}^3} h_t(x) \, dx = 1$. 
Let us define the function \( g_t : \mathbb{R}^3 \to \mathbb{R} \) as the convolution
\[
g_t(x) = \int_{\mathbb{R}^3} f(x-y)h(t) \, dy.
\]

Clearly, for every \( t \), \( g_t \) is \( C^\infty \)-class, and as the integral average of convex functions is convex, it is convex. In particular, it follows that the set \( K_t = g_t^{-1}([0,1]) \) is compact and convex, and hence it is a convex body for sufficiently small values of \( t \). Note that \( \text{bd} K_t \) is a \( C^\infty \)-class submanifold of \( \mathbb{R}^n \).

Furthermore, it is known (cf. [23] or [28]), that if \( U \) is a compact set and \( f \) is \( C^n \)-class on \( U \) for some \( 2 \leq n < \infty \), then on \( U \) \( g_t \) converges uniformly to \( f \), together with its derivatives up to order \( n \), as \( t \to 0 \). Thus, for sufficiently small values of \( t \), that is, for any \( t \in [0,\varepsilon] \) for some \( \varepsilon > 0 \), the Morse-Smale complex of \( K_t \) with respect to \( o \) is homeomorphic to that of \( K \); and \( \text{bd} K_t \) has nonzero principal curvatures at the critical points of \( K \). This implies that \( K_t \) satisfies (*)

We are left to show that, for small values of \( t \), the Morse-Smale complex of \( K_t \) with respect to its center is homeomorphic to that with respect to \( o \). To do this, we observe that for any \( t \in [0,\varepsilon] \) there is some \( \rho > 0 \) such that the Morse-Smale complex of \( K_t \) with respect to \( u \) is homeomorphic to that with respect to \( o \), for any \( u \in \text{int} B_\rho \). Let the radius of the largest such ball be \( \rho(t) \). Clearly, \( \rho(t) \) is a continuous function of \( t \), and thus, by compactness, there is a value of \( \rho \) such that for any \( t \in [0,\varepsilon] \), the Morse-Smale complex of \( K_t \) with respect to any point of \( \text{int} B_\rho \) is homeomorphic to that with respect to \( o \). Note that the center \( o_t \) of \( K_t \) tends to \( o \) as \( t \to 0 \). Thus, if \( t \) is sufficiently small, \( |o_t| < \rho \). Choosing \( t \) with this property and setting \( K' = K_t \), the assertion readily follows.

**Remark 1.** Note that if the symmetry group of \( f \) and \( h \) is \( O^3 \), then the same holds for \( g_t \) for every value of \( t \). Thus, if \( p \in \text{bd} K \) has a spherical cap neighborhood, then so does the corresponding point of \( K' \).

### 3.2. Step T3: Truncation with a sphere.

We show that if \( K \) satisfies (*), and has a stable or unstable point at \( p \), then there is a local modification \( K' \) of \( K \) such that \( K' \) satisfies (*), the Morse-Smale complex of \( K' \) is homeomorphic to that of \( K \), and the stable/unstable point of \( K' \) corresponding to \( p \) has a spherical cap neighborhood in \( \text{bd} K' \).

Recall that we may assume that \( p = (0,0,1) \) and that the center of \( K \) is \( o \). Then the condition that \( p \) is nondegenerate is equivalent to saying that \( ||p||\kappa_1 \neq 1 \neq ||p||\kappa_2 \), where \( \kappa_1 \) and \( \kappa_2 \) are the two principal curvatures of \( \text{bd} K \) at \( p \).

Now we define a two-parameter family of truncations of \( K \), denoted by \( K_R(\varepsilon) \).

Let \( z = f(x,y) \) be a function the graph of which is a neighborhood of \( p \) in \( \text{bd} K \).

Furthermore, for any \( \phi \in [0,\pi] \), let \( f_L(t) = f(t \cos \phi, t \sin \phi) \) be the restriction of \( f \) to the line \( L \) containing \( (\cos \phi, \sin \phi) \) and the origin. Consider a value of \( R \) such that \( \frac{1}{R} < \frac{1}{\delta} \) is strictly smaller than any of the two principal curvatures of \( \text{bd} K \) at \( p \). Then \( \frac{1}{R} < |f''_L(0)| \) for any line \( L \). Since \( \text{bd} K \) is \( C^\infty \)-class in a neighborhood of \( p \), so is \( f \) in a neighborhood of \( (0,0) \). Thus, for some \( \delta > 0 \), \( \frac{1}{R} < f''(t) \) for any \( |t| \leq \delta \).

We choose \( \delta \) satisfying this condition, and consider the part \( S \), with points for the coordinates of which \( \sqrt{x^2 + y^2} \leq \delta \) and \( z > 0 \), of the sphere of radius \( R \) that touches the plane \( \{z = 1\} \) at \( p \) from the side containing the origin. Clearly, \( S \) is
Lemma 1. Let $K$ be smaller than any of the two principal curvatures of $\text{bd} K$ at $p$. Then there are constants $\lambda_1, \lambda_2, \lambda_3 > 0$ depending only on $R$ and $K$ such that for every sufficiently small $\varepsilon > 0$ the following hold.

1. no point of $(\text{bd} K) \cap B_{\lambda_3 \sqrt{c}}(p)$ belongs to $K_R(\varepsilon)$.
2. $(\text{bd} K) \setminus B_{\lambda_2 \sqrt{c}}(p) \subseteq K_R(\varepsilon)$.
3. If $D$ is a convex body, with center $o'$ and satisfying $K_R(\varepsilon) \subseteq D \subseteq K$, then $|o'| \leq \lambda_3 \varepsilon^2$.

Proof. First, observe that $K_R(\varepsilon)$ is convex, as it is the intersection of convex bodies.

Recall the two-variable function $f$ from the definition of $K_R(\varepsilon)$. Like there, let $f|_L : \mathbb{R} \to \mathbb{R}$ denote its restriction $f|_L(t) = f(t \cos \phi, t \sin \phi)$, where $\phi \in [0, \pi]$, on a line $L$ passing through the origin and $(\cos \phi, \sin \phi)$. Let $h$ denote the one-variable function, defined on $[-R, R]$, the graph of which is a semicircle, of radius $R$, and with maximum $h(0) = 1 - \varepsilon$. By the conditions in Lemma 1, we have $0 > f|_L''(0) > h''(0)$. Hence, from the second-degree Taylor polynomials of $f|_L$ and $h$, we obtain that

$$\varepsilon - \lambda_1^* t^2 < f|_L(t) - h(t) < \varepsilon - \lambda_2^* t^2$$

for some positive constants $\lambda_1^*, \lambda_2^*$. Thus, (1) and (2) clearly follow with $\lambda_1 = \frac{1}{\lambda_1^*}$ and $\lambda_2 = \frac{1}{\lambda_2^*}$.

Now we show (3). Let $o' = (x', y', z')$. Recall that

$$o' = \frac{\int_{q \in D} q \, dV}{\int_{q \in D} 1 \, dV}.$$ 

First, we estimate $x'$. Note that the part of $K$ outside $D$ can be covered by an axis-parallel brick of side-lengths $2\lambda_2 \sqrt{c}, 2\lambda_2 \sqrt{c}$ and $c \varepsilon > 0$ for some constant $c$ independent of $\varepsilon$. Thus,

$$\left| \int_{q \in D} 1 \, dV - \int_{q \in K} 1 \, dV \right| \leq 4\lambda_2^2 c \varepsilon^2,$$

and

$$\left| \int_{q \in D} x \, dV - \int_{q \in K} x \, dV \right| \leq 4\lambda_2^3 c \varepsilon^{5/2},$$

from which $|x'| \leq \lambda_3 \varepsilon^2$ follows. In the same way, we may obtain similar bounds for $y'$ and $z'$, which readily yields the assertion. $\square$

We note that Lemma 1 can be applied for the degenerate case $R = \infty$ as well. The next lemma guaranties that the truncated body $K_R(\varepsilon)$ has the same numbers and types of static equilibrium points.
Lemma 2. Let $p$ be a stable point of $K$, or let $p$ be an unstable point and $R < \| p \|$. If $R$ satisfies the conditions in Lemma [1] then there is an $\varepsilon > 0$ such that for every $t \in [0, \varepsilon]$ the following holds: $K_R(t)$ and $K = K_R(0)$ have the same number of stable/unstable and saddle points. Furthermore, the coordinates of these points are continuous functions of $t$.

Proof. Note that as the Euclidean distance function from a fixed point is $C^\infty$-class at every point but the origin, all the partials of any order at any point of $\partial K$ change continuously when translating $K$. Hence, applying the Poincaré-Hopf Theorem [2] to a compact neighborhood of any equilibrium point shows that the numbers of the stable/unstable/saddle points of $K$ do not change under a translation by a small vector, and their coordinates are continuous functions of the translation vector. Similarly, the gradient vector field changes continuously under a translation of $K$. Thus, $o$ has a neighborhood $U$ such that for any $u \in U$, the Morse-Smale complex of $K$ is homeomorphic to that of $K + u$, or, in other words, the Morse-Smale complex of $K$ with respect to $u$ is homeomorphic to that with respect to $o$.

If $t$ is sufficiently small, then the center $o_t$ of $K_R(t)$ is contained in $U$. Hence, all the stable/unstable and saddle points of $K$ but $p$ change continuously as functions of $t$. Note that by (1) and (3) of Lemma [1] the stable/unstable point of $K$, with respect to $o_t$, that corresponds to $p$ is contained in the part truncated by $C_R(t)$. Thus, this point does not belong to $K_R(t)$. On the other hand, by (2) of the same lemma, there is exactly one equilibrium point of $K_R(t)$ on the part belonging to $C_R(t)$.

We leave it to the reader to show that no plane, perpendicular to $[u, q]$ for some $u \in V$, supports $K_R(t)$ at a point $q \in (\partial K) \cap \partial C_R(t)$, and thus, there are no more equilibrium points of $K_R(t)$. □

Now we show that Step $T_1$ can be carried out using a sphere of radius $R$ for some value of $R$. To do this, consider some $K_R(\varepsilon)$, where $\varepsilon > 0$ and $R > 0$ satisfy the conditions of Lemma [2]. By this lemma, we may assume that $K_R(\varepsilon)$ has the same numbers of stable/unstable and saddle points as $K$ does. Let $o_R$ denote the center of $K_R(\varepsilon)$, and $p_R$ be the critical point of $K_R(\varepsilon)$ corresponding to $p$. Consider a closed $C^1$-class curve $g \subset \partial K_R(\varepsilon)$, separating $p_R$ from all the other critical points of $K_R(\varepsilon)$, which is disjoint from the truncated, spherical part, and is not tangent to any integral curve of $\partial K$, with respect to $o_R$, that intersects $g$.

Now we show that even though $\partial K_R(\varepsilon)$ is not $C^1$-class, we may still apply the smoothing algorithm described in Subroutine $S$. More specifically, we show that if we smooth $K_R(\varepsilon)$ using the function $h_t$ described in the subroutine, then the Morse-Smale complex of the resulting body $K'$ is homeomorphic to that of $K$. Let $g' \subset \partial K'$ be the curve corresponding to $g$. Observe that as the gradient vector field changes continuously as a function of $t$, for sufficiently small $t$ $g'$ is not tangent to any integral curve of $K'$ intersecting it. Furthermore, any such integral curve ends at a critical point in the region inside $g'$ and for small values of $\varepsilon$ and $t$ there is a unique critical point in this region, which we denote by $p'$. Thus, we may choose values of $\varepsilon$ and $t$ such that the Morse-Smale complex of $K'$ is homeomorphic to that of $K$. By Remark [4], $p'$ has a spherical neighborhood, and the radius of this sphere is arbitrarily close to $R$. 


As a result of our consideration, we may assume that the examined critical point \( p \) has a spherical cap neighborhood in \( \text{bd} \, K \) of radius \( R \) for some value of \( R \). In the next lemma, we show that this truncation can be carried out using a sphere of any radius \( r \), where \( 0 < r < ||p|| \) if \( p \) is an unstable point, and \( r > ||p|| \) if \( p \) is a stable point. To formulate it, recall that the integral curves connecting \( p \) to a saddle point of \( K \) are denoted by \( \Gamma_1, \Gamma_2, \ldots, \Gamma_k = \Gamma_0 \) in cyclic order around \( p \).

**Lemma 3.** Let \( K \) satisfy (*), and assume that \( p \) has a spherical cap neighborhood in \( \text{bd} \, K \), of radius \( R \). Let \( 0 < r < ||p|| \) if \( p \) is an unstable point, and \( r > ||p|| \) if \( p \) is a stable point, and let \( \delta > 0 \) arbitrary. Then there is a convex body \( K' \subseteq K \) satisfying (*) and the following.

- The Morse-Smale complex of \( K' \) is homeomorphic to that of \( K \).
- Denoting the critical point of \( K' \) corresponding to \( p \) by \( p' \), \( p' \) has a spherical cap neighborhood in \( \text{bd} \, K' \), of radius arbitrarily close to \( r \).
- Denoting the integral curve of \( K' \) corresponding to \( \Gamma_i \) by \( \Gamma_i' \) for every \( i \), and by \( t_i' \) the unit tangent vector of \( \Gamma_i' \) at \( p' \), we have that \( ||t_i' - t_i|| < \delta \).

**Proof.** First, let \( r < R \). Then we set \( p' = p \). Let \( U \) be a (sufficiently small) neighborhood of \( o \) such that for every \( u \in U \), the Morse-Smale complex of \( K \) with respect to \( u \) is homeomorphic to that with respect to \( o \). By (3) of Lemma [1] there is an \( \varepsilon > 0 \) such that for any convex body \( K' \), satisfying \( (K \setminus B_\varepsilon(p)) \subseteq K' \subseteq K \), the center of \( K' \) is contained in \( U \).

Clearly, we may replace the part of \( \text{bd} \, K \) in \( B_\varepsilon(p) \) with a surface of rotation \( S' \), with the line containing \( [o, p] \) as its axis of symmetry, in such a way that:

- \( p \in S' \subseteq K \) and \( (\text{bd} \, K \setminus B_\varepsilon(p)) \cup S' \) is the boundary of a convex body \( K' \),
- the modified body \( K' \) has a \( C^1 \)-class boundary,
- a neighborhood of \( p \) in \( S' \) belongs to a sphere of radius \( r \),
- there is no equilibrium point on \( S' \) but \( p \), with respect to any point of \( U \) on the line containing \( [o, p] \).

Note that the center of \( K' \) lies on the line containing \( [o, p] \), and thus all the integral curves emanating from \( p \) in \( B_\varepsilon(p) \) are the meridians of \( S' \). Let these curves be denoted by \( \Gamma'_1, \ldots, \Gamma'_k = \Gamma'_0 \) in cyclic order, with \( t'_i \) denoting the unit tangent vector of \( \Gamma'_i \) at \( p \). Thus, since the integral curves of \( K \) with respect to \( u \in U \), connecting \( p \) and the saddle points of \( K \) are deformed continuously when modifying \( u \), the assertion readily follows for sufficiently small \( U \). To obtain a convex body with a \( C^\infty \)-class boundary, we may finally apply Subroutine S.

If \( r > R \), we may use a truncation by a sphere of radius \( r \), like in the previous part of Step \( T_1 \). \qed

**Remark 2.** Observe that the two critical points of the convex body \( K \) with Morse-Smale complex \( P_2 \) are the ‘poles’ of the parametrization (cf. [35]), and thus, may not have \( C^\infty \)-class neighborhoods. Nevertheless, the curves \( P \cap \text{bd} \, K \), where \( P \) is any closed half plane containing a critical point of \( K \) and its center in its relative boundary, are \( C^\infty \)-class curves with nonzero second derivatives at the critical points. Thus, a straightforward modification of our arguments can be applied in this case.

### 3.3. Step \( T_2 \): Vertex splittings

In this subsection, given a convex body \( K \) satisfying (*), with a critical point \( p \) that has a spherical cap neighborhood in \( \text{bd} \, K \),
we describe the equilibrium splitting corresponding to any given combinatorially possible splitting defined in Section 2 such that the resulting convex body $K'$ satisfies (*). An example for a complete list of combinatorially possible vertex splittings is illustrated in Figure 4(a), the corresponding approximate geometric truncations are shown in Figure 4(b). First, we consider the case that $p$ is a stable point of $K$. The truncation is illustrated in Figure 5.

Recall that the center of $K$ is the origin, and $p = (0, 0, 1)$. Furthermore, the integral curves of $K$, connecting $p$ and the saddle points of $K$, in cyclic order around $p$, are $\Gamma_i$, where $i = 1, 2, \ldots, k$, and the unit tangent vector of $\Gamma_i$ at $p$ is $t_i$. Let $R$ and $q$ denote the radius and the center of the sphere, containing a spherical cap neighborhood of $p$. We denote this neighborhood by $S$. Then, according to Lemma 3 in Step $T_1$, we may assume that $R$ is arbitrarily close to any given number greater than $|p|$. Thus, the arc of any integral curve of $K$ within $S$, including the $\Gamma_i$s, are great circle arcs.

Let $G_1$ and $G_2$ be two open great circle arcs in $S$, starting at $p$, that are, in the interiors of the two regions, bounded by the $\Gamma_i$s, that we want to split. These different regions are shaded in Figure 4(b). We may choose $G_1$ and $G_2$ in a way that they do not belong to the same great circle. We intend to truncate $K$ near $p$ with a plane in such a way that there is a new stable point on the plane, a new saddle point on its boundary, and the saddle point is connected to the stable point corresponding to $p$, to the stable point on the plane and to the two unstable points in the boundaries of the regions containing $G_1$ and $G_2$. To do this, we use a planar section that touches $G_1$ and $G_2$. To examine the properties of these sections, we first prove a technical lemma that we are going to use in the construction.

**Lemma 4.** Let $C$ be a circle in the plane $\mathbb{R}^2$ with the origin as its center and with radius 1. Let $p = (0, t)$ with $t > 0$. Let $q_1 = (u_1, v_1)$ and $q_2 = (u_2, v_2)$ be two points of $C$ such that $v_1 > 0$.

1. If $[q_1, q_2]$ is perpendicular to $[p, q_1]$, then $\lim_{u_1 \to 0} \frac{u_2}{u_1} = \frac{2t}{1-t}$.
2. If the angle of $[q_1, q_2]$ and $[p, q_1]$ is $\frac{\pi}{2} - cu_1$ for some constant $c$ independent of $u_1$, then $\lim_{u_1 \to 0} \frac{u_2}{u_1} = \frac{2t}{1-t} + 2c$.

**Proof.** We note that $u_2 = \frac{(1-u^2)u_1}{1-2\sqrt{1-u^2}t+u^2}$, from which (1) immediately follows. The second part can be proven with a similar elementary computation. Note that in
the second case the angle between \([p, q_1]\) and the segment connecting \(p\) and its orthogonal projection on \([q_1, q_2]\) is \(c u_1\).

Now, consider a spherical circle on \(S\) that touches \(G_1\) and \(G_2\), and let \(u_1\) and \(u_2\) denote, respectively, the distance of its closest and its farthest point from the \(z\)-axis. Let \(L\) denote the limit of \(u_2/u_1\) as \(u_1\) tends to zero. Note that this limit exists, is positive and does not depend on the radius of \(S\), only on the angle of \(G_1\) and \(G_2\); this follows from the observation that spherical space ‘locally’ is Euclidean.

By Lemma \(3\) we may slightly deform \(K\) in a way that \(G_1\) and \(G_2\) are still in the interiors of the two chosen regions, and that \(2(R-1) < L\). On the other hand, by (1) of Lemma \(4\) if \(2(R-1) < L\), then there is a sufficiently small circle \(C\) on \(S\) touching \(G_1\) and \(G_2\) such that the truncation of \(K\) by the plane containing \(C\) has a stable point, with respect to \(o\), on the truncated, disk-shaped part. Furthermore, by (2) of the same Lemma, we may choose \(0 < \tau < 1\) in a way that if \(C\) is sufficiently small, then the distance of this point from the boundary of \(C\) is at least \(\tau\) times the radius of the circle.

Let \(H_\varepsilon\) denote the plane perpendicular to and intersecting \([o, p]\), at the distance \(\varepsilon\) from \(p\). Let \(K_\varepsilon\) denote the truncation of \(K\) by this plane. Let \(U_\varepsilon\) be the set of the centers of all the convex bodies \(D\) satisfying \(K_{2\varepsilon} \subseteq D \subseteq K\). Clearly, we may choose a sufficiently small \(\varepsilon\) such that the Morse-Smale complex of \(K\) with respect to any point of \(U_\varepsilon\) is homeomorphic to that with respect to \(o\). Furthermore, using (3) of Lemma \(4\) we may choose \(\varepsilon\) in a way that both the diameter of \(U_\varepsilon\), and the diameter of the projection \(U_\varepsilon^S\) on \(S\) of \(U_\varepsilon\) from \(q\) is at most \(c \varepsilon^2\) for some \(c > 0\) independent of \(\varepsilon\). Note that for small values of \(\varepsilon\), these projections belong to \(S\). Furthermore, for any fixed \(\delta > 0\), we may also choose \(\varepsilon\) in a way that the intersection points, with the circle \(K \cap H_\varepsilon\), of any integral curve connecting the projection of a point \(u\) from \(q\) on \(S\) and a saddle point of \(K\) with respect to \(u\) moves on an arc of angle not greater than \(\delta\), where \(u\) runs over \(U_\varepsilon\). We denote these integral curves by \(\Gamma^u_1, \Gamma^u_2, \ldots, \Gamma^u_k = \Gamma^u_0\).

Let \(C_\varepsilon\) be the circle that touches \(G_1\), \(G_2\) and \(H_\varepsilon\) on the side of \(H_\varepsilon\) not containing \(o\), and \(K_\varepsilon\) be the truncation of \(K\) by the plane containing \(C_\varepsilon\). Note that the radius of \(C_\varepsilon\) is at least \(\bar{c} \sqrt{\varepsilon}\) for some positive constant \(\bar{c}\). Thus, for sufficiently small \(\varepsilon\), the following are satisfied.

- for any \(u \in U_\varepsilon\), \(C_\varepsilon\) intersects exactly those curves from amongst the \(\Gamma^u_i\)'s that we want to intersect with the planar section.
- No such intersection point is on \(G_1\) and \(G_2\).
- The orthogonal projection of \(U_\varepsilon\) on the plane of \(C_\varepsilon\) is contained in the interior of \(C_\varepsilon\).
- \(U^S_\varepsilon\) is contained in \(K_\varepsilon\), and is disjoint from \(C_\varepsilon\).

Clearly, under the conditions described in the previous paragraph, \(K_\varepsilon\) has three equilibrium points in the closed half plane bounded by \(H_\varepsilon\) and containing \(p\): a stable point \(p'\) ‘near’ \(p\) (that is, in \(U^S_\varepsilon\)), another stable point \(s\) on the planar disk bounded by \(C_\varepsilon\), and a saddle point \(h\) on \(C_\varepsilon\). This saddle point is connected to both \(p'\) and \(s\): the integral curve connecting it to \(p'\) is a great circle arc, and the other one is a straight line segment. Furthermore, for every \(\Gamma_i\) on the truncated side of \(G_1 \cup G_2\), there is a (piecewise differentiable) integral curve of \(K_\varepsilon\) connecting \(s\) to
the corresponding saddle point of $K_{\varepsilon}$, and for every $\Gamma_i$ on the other side of $G_1 \cup G_2$, there is a similar curve ending at $p'$. This implies, by exclusion, that $h$ is connected to the two unstable points in the two chosen regions containing $G_1$ and $G_2$.

Finally, observe that by replacing the truncating plane containing $C_{\varepsilon}$ by a ball of sufficiently large radius results in a homeomorphic Morse-Smale complex. Furthermore, ‘to smooth out’ $\partial K_{\varepsilon}$, we may replace $C_{\varepsilon}$ by a sufficiently small toroidal surface, which results in a convex body $K'$ with a $C^1$-class boundary, such that its Morse-Smale complex is homeomorphic to the one split in the required way. We remark that $K_2 \subseteq K'$, and thus, the center of $K'$ is still contained in $U_{\varepsilon}$. Finally, we may apply Subroutine $S$ to obtain a convex body with a $C^\infty$-class boundary, which finishes the proof for a stable point.

One may wonder whether we could have proven our statement using only one local truncation by a plane. However, as we already indicated in the Introduction, this appears to be impossible: The phenomenon described in Lemma 1 shows that to carry out an arbitrary splitting of an arbitrary (given) vertex we need one additional free parameter and this is the radius of the truncating sphere. In terms of local quantities, this implies that even in the special case when the chosen equilibrium is an umbilical point (i.e. its vicinity is spherical), the required splitting cannot be realized by a planar truncation unless the principal curvature at this point is contained in a given interval determined by the directions of the edges starting at this vertex. As a consequence, we need to adjust the principal curvature before truncating with a plane. This step is achieved in Lemma 3.

For an unstable point we may apply a similar consideration. In this case, instead of a truncation with a ball of large radius, we expand the original ball with a circular cone the axis of which contains the ball center $q$.

4. Summary and related issues

The central idea of our paper is to associate vertex splittings with localized geometric transformations. The latter are defined in such a way that we can control all combinatorial possibilities. Next we show that vertex splittings arise in a spontaneous way in various geometric settings where they may or may not exhaust the full combinatorial catalog. From this point of view, our construction creates a framework to study these processes.

4.1. A road map for the generic bifurcations of one-parameter vector fields. If we consider generic, 1-parameter families of gradient vector fields on the 2-sphere then it is not true that every element of such a family is Morse-Smale. Rather, these families produce two types of singularities at which this property may fail: saddle-node bifurcations and saddle-saddle connections. The former is a local bifurcation while the latter is a non-local bifurcation. A saddle-node bifurcation corresponds to a vertex splitting or a face contraction on the quasi-dual graph representation of the Morse-Smale complex, while a saddle-saddle connection corresponds to a transformation called diagonal slide. Each gradient vector field can be uniquely associated with the quasi-dual graph representation of its Morse-Smale complex, so the evolution of one-parameter families can be studied on a metagraph $\mathcal{G}$ the vertices of which are graphs $Q \in \mathcal{Q}$, representing the Morse-Smale complexes and the edges of $\mathcal{G}$ correspond to generic bifurcations in one-parameter
families. Any such family will appear as a path on $\mathcal{G}$. Convex bodies associated with some selected graphs (selected vertices of $\mathcal{G}$) are illustrated in Figure 6/a3. A small portion of $\mathcal{G}$ is illustrated in Figure 6/a1. Vertices are classified based on the number of stable $(S)$ and unstable $(U)$ equilibrium points. Solid edges represent vertex splittings, dashed edges represent saddle-saddle connections, Figure 6/a2 shows the latter inside the primary equilibrium class $\{S, U\} = \{2, 3\}$. Nakamoto [33] showed that any two 2-colored quadrangulations with the same number of vertices from the two different colors can be transformed into each other by diagonal slides, thus, saddle-saddle connections inside any primary equilibrium class connect any two graphs. However, it is not known whether there exists a geometric correspondence for this combinatorial connection inside the primary equilibrium classes.

Observe that $\mathcal{G}$ is not oriented, new critical points may emerge or disappear at generic bifurcations. Theorem 4 states that the oriented subgraph $\mathcal{G}_v \subset \mathcal{G}$ (illustrated in Figure 6/b1) containing only vertex splittings, exists among gradient fields associated with convex bodies. Combinatorial expansion sequences (2) associated with an $n$-vertex graph $Q_n \in Q_n$ appear on this oriented metagraph as an oriented path of length $n - 2$, starting at the root ($P_2$). Observe that in the $k$th step a vertex in the box-diagonal $S + U = k$ is selected. Two such sequences are illustrated in Figure 6/b2. Beyond theoretical interest, these metagraphs admit the study of interesting physical phenomena some of which we briefly discuss below.

4.2. Inhomogeneous bodies. Our first example are inhomogeneous bodies. So far, throughout the paper we assumed convexity and material homogeneity. Relaxing the latter constraint is equivalent to keep the convex surface $\partial d K$ as the boundary of the body but let the mass be concentrated at the center of gravity $C$. As the location of $C$ is varied in time as a curve $r_C(t)$, it generates a one-parameter family of gradient vector fields on $\partial d K$. A classical result in catastrophe theory states that the number of critical points of the gradient changes if and only if $r_C(t)$ transversely passes through one of the two caustics of the body [34]. Caustics (also known as focal surfaces) are the two surfaces formed by the curvature centers corresponding to the principal curvatures of $\partial d K$. Figure 7 (a)-c) shows the two caustics of an ellipsoid, a) corresponding to curvature minimum, b) corresponding to curvature maximum, c) intersection (superposition) of both caustics. When $r_C(t)$ transversely crosses the caustic defined by the minimal principal curvature, a saddle and an unstable point meet at a saddle-node; when $r_C(t)$ transversely crosses the other caustic, a saddle and a stable point collide. Every saddle-node bifurcation corresponds to a vertex splitting (or face contraction, depending on the direction) on the quasi-dual Morse-Smale graph, so at each such event the corresponding path on $\mathcal{G}$ will move from one box-diagonal $S + U = k$ to one of its neighbor diagonals. Figure 7 (d) shows the different Morse-Smale graphs in the different domains determined by the intersections of the two caustics. It is easy to see that if $C$ is located far enough from the center of gravity of the homogeneous body then the corresponding Morse-Smale complex is represented by the path graph $P_2$.

4.3. Collisonal abrasion: chipping of rocks. Our second example is pebble abrasion via collisions. This process is most often described by averaged geometric PDEs, the most general such model is given by Bloore [7] as
Figure 6. a1) Metagraph $\mathcal{G}$ corresponding to generic bifurcations in one-parameter families of vector fields on the sphere a2) Saddle-saddle connections inside the primary equilibrium class $\{S, U\} = \{2, 3\}$ a3) Examples for convex bodies with some selected graphs b1) Oriented subgraph $\mathcal{G}_v \subset \mathcal{G}$ corresponding to vertex splittings b2) Two selected combinatorial expansion sequences b3) Morse-Smale complexes associated with real pebbles, derived with expansion sequences from the ellipsoid
\[
v = 1 + 2bH + cK
\]

where \(v\) is the attrition speed along the inward surface normal, \(H, K\) are the Mean and Gaussian curvatures, \(b, c\) are constants. Solutions of these PDEs correspond to (inward propagating) wave fronts. The actual physical process is somewhat different: it is based on discrete collisions where small amounts of material are being removed in a strongly localized area. Simple but natural interpretations of the discrete, physical abrasion process are chipping algorithms [18], [35] and [30] where in each step a small amount of material is chipped off at point \(p\) by intersecting the body with a plane resulting as a small parallel translation of the tangent at \(p\). We call such an operation a chipping event and their sequence a chipping sequence.

In Section 3 we showed that any vertex splitting can be achieved by a suitably chosen convexity-preserving local truncation. It is not very difficult to show a related, though converse statement: any sufficiently small chipping event will either leave the Morse-Smale complex invariant or result in one or two consecutive vertex splittings. If we regard the material abraded in a chipping event as a random variable \(\chi\) with very small, but finite expected value \(E(\chi)\) and very small variance (as it is often done in chipping algorithms) then most chipping events will be sufficiently small to form finite chipping sequences the subsequences of which are geometric expansion sequences of type (3).

Chipping sequences do not represent a rigorous, algorithmic discretization of the PDEs, rather, they can be regarded as an alternative, discrete approximation of the physical process. As it was pointed out in [16] and [17], pebble surfaces display equilibria on two, well-separated scales. While the PDE description accounts for the evolution of global equilibria, local equilibria, corresponding to the fine structure of the surface are only captured by the chipping model. Figure 8 shows the high-accuracy scan of a real pebble with equilibria and the primal representation of the Morse-Smale complex. We can observe flocks of local equilibria accumulated around global equilibrium points. In fact, one motivation behind chipping algorithms is to better understand the interplay between the two scales. Chipping models appear to be successful in explaining laboratory experiments (cf. [30] and [35]) as well as geological field observations [35]. The connection between chipping models and our geometric expansion algorithms suggests that the number of local equilibrium points may increase in abrasion processes for finite time intervals. Whether and how this process interacts with the evolution of global equilibria is an open question not addressed in the current paper.

Figure 6/b3 illustrates two, rather short geometrical expansion sequences leading to Morse-Smale complexes associated with real pebbles. While the abrasion of these pebbles was not monitored, given their simple Morse-Smale complex and nearly-ellipsoidal shape it is realistic to assume that these geometric expansion sequences are subsequences of the actual physical abrasion process (modeled by chipping sequences) which produced these shapes. Needless to say, many more experiments are needed to verify this theory.

4.4. Concluding remarks. In this paper we showed that Morse-Smale functions \(M_C\) associated with convex bodies exhaust all combinatorial possibilities, i.e., to any two-colored quadrangulation \(Q^* \in Q\) of the sphere there exists a convex body
$K^*$ such that $Q^*$ is the quasi-dual representation of the Morse-Smale complex associated with $K^*$. We proved our claim by showing that to each possible combinatorial expansion sequence $Q_i$ generated by vertex splittings, there exists a coupled geometric expansion sequence $K_i$ such that $Q_i$ is the quasi-dual representation of the Morse-Smale complex associated with $K_i$. Geometric expansion sequences were created by local, convexity preserving truncations of the convex body. Beyond proving the existence of all combinatorially possible convex shapes they also define a hierarchical order among them. We also showed that these sequences appear to be part of the natural geometrical description of collisional abrasion.

5. Acknowledgement

This research was supported by OTKA grant T104601. The authors are indebted to E. Makai Jun., G. Etési and Sz. Szabó for their valuable comments on smooth approximations of continuous functions. Z. Lángi also acknowledges the support of the Fields Institute for Research in Mathematical Sciences, University of Toronto, Toronto ON, Canada, and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

References

[1] Archdeacon, D., Hutchinson, J., Nakamoto, A., Negami, S. and Ota, K., Chromatic numbers of quadrangulations on closed surfaces, J. Graph Theory 37 (2001), 100-114.
[2] Arnold, V.I., Ordinary differential equations, 10th printing, MIT Press, Cambridge, 1998.
[3] Arnold, V.I. (Ed.), Dynamical Systems V: Bifurcation Theory and Catastrophe Theory, Springer-Verlag, Berlin, 1994.
[4] Bagatelj, V., An inductive definition of the class of 3-connected quadrangulations of the plane, Discrete Math. 78 (1989), 45-53.
[5] Bauer, U., Lange, C. and Wardetzky, M., Optimal topological simplification of discrete functions on surfaces, Discrete Comput. Geom. 47 (2012), 347-377.
[6] Bonnesen, T., Fenchel, W., Theory of Convex Bodies, Moscow, Idaho: L. Boron, C. Christenson and B. Smith, BCS Associates, 1987.
[7] Bloore, F.J., The Shape of Pebbles, Math. Geol. 9 (1977) 113-122.
[8] Bremer, P.T., Edelsbrunner, H., Hamann, B. and Pascucci, V., A multi-resolution data structure for two-dimensional Morse-Smale functions, Proceeding VIS ’03 (2003), 139-146.
[9] Brinkmann, G., Greenberg, S., Greenhill, C., McKay, B.D., Thomas, R. and Wollan, P., Generation of simple quadrangulations of the sphere, Discrete Math. 305 (2005), 33-54.
[10] Brinkmann, G. and McKay, B.D., Fast generation of planar graphs, MATCH Commun. Math. Comput. Chem 58 (2007), 323-357.
[11] Conway, J.H. and Guy, R., Stability of polyhedra, SIAM Rev. 11 (1969), 78-82.
[12] Dawson, R., Monostatic Simplexes, Amer. Math. Monthly 92 (1985), 541-546.
[13] Dawson, R. and Finbow, W., What shape is a loaded die?, Math. Intelligencer 22 (1999), 32-37.
[14] Dey, T.K., Li, K., Luo, C., Ranjan, P., Safa, I. and Wang, Y., Persistent Heat Signature for Pose-oblivious Matching of Incomplete Models, Computer Graphics Forum 29 (2010), 1545-1554.
[15] Diestel, R., Graph Theory, 3rd edition, Springer-Verlag, Heidelberg, 2005.
[16] Domokos G., Lángi Z. and Szabó, T., On the equilibria of finely discretized curves and surfaces, Monatsh. Math. 168 (2012), 321-345.
[17] Domokos G., Sipos A.Á. and Szabó, T., The mechanics of rocking stones: equilibria on separated scales, Math. Geosci. 44 (2012), 71-89.
[18] Domokos G., Sipos A.Á. and Várkonyi P. Continuous and discrete models for abrasion processes, Per. Pol. Arch. 40 (2009) 3-8.
[19] Dong, S., Bremer, P.-T., Garland, M., Pascucci, V. and Hart, J.C., Spectral surface quadrangulation, ACM T. Graphic 25 (2006), 1057-1066.
[20] Edelsbrunner, H., Harer, J. and Zomorodian, A., Hierarchical Morse-Smale complexes for piecewise linear 2-manifolds, Discrete Comput. Geom. 30 (2003), 87-107.

[21] Evans, L., Partial differential equations, Graduate Texts in Mathematics 19, American Mathematical Society, Providence RI, 1998.

[22] Fuss, E., Counting unrooted maps using tree-decomposition, Seminaire Lotharingien de Combinatoire 54A (2007), Article B54Al

[23] Ghomi, M., The problem of optimal smoothing for convex functions, Proc. Amer. Math. Soc. 130 (2002), 2255-2259.

[24] Gross, J. T. and Yellen, J., Graph Theory and Its Applications, 2nd ed., Boca Raton, FL: CRC Press, 2006.

[25] Gyulassy, A., Natarajan, V., Pascucci and V., Hanam, B., Efficient computation of Morse-Smale complexes for three-dimensional scalar functions, IEEE Trans. Vis. Comput. Graph. 13 (2007), 1440-1447.

[26] Heath, T. I. (Ed.), The Works of Archimedes, Cambridge University Press, 1897.

[27] Hesp, A., A double-tipping tetrahedron, SIAM Rev. 9 (1967), 599-600.

[28] Hirsch, M., Differential topology, Graduate Texts in Mathematics 33, Springer-Verlag, New York-Heidelberg, 1986.

[29] Kápolnai, R. and Domokos, G., Inductive generation of convex bodies, in: The 7th Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications, 2011, pp. 170-178.

[30] Krapivsky, P.L. and Redner S., Smoothing a rock by chipping, Phys. Rev. E 9 (2007), 75(3 Pt 1):031119.

[31] Milnor, J., Morse Theory, Princeton Univ. Press, New Jersey, 1963.

[32] Nakamoto, A., Generating quadrangulations of surfaces with minimum degree at least 3, J. Graph Theory 30 (1999), 229-234.

[33] Negami, S, and Nakamoto, A., Diagonal transformations of graphs on closed surfaces, Sci. Rep. Yokohama Nat. Univ., Sec. I 40 (1993), 71-97.

[34] Poston, T. and Stewart, J., Catastrophe theory and its applications, Pitman, London, 1978.

[35] Sipos A.Á., Domokos G., Wilson A. and Hovius N. A Discrete Random Model Describing Bedrock Erosion, Math. Geosci. 43 (2011) 583-591.

[36] Várkonyi, P.L. and Domokos, G., Static equilibria of rigid bodies: dice, pebbles and the Poincaré-Hopf Theorem, J. Nonlinear Sci. 16 (2006), 255-281.

[37] Zamfirescu, T., How do convex bodies sit?, Mathematica 42 (1995), 179-181.

[38] Zomorodian, A., Topology for computing, Cambridge University Press, 2005.

GÁBOR DOMOKOS, DEPT. OF MECHANICS, MATERIALS AND STRUCTURES, BUDAPEST UNIVERSITY OF TECHNOLOGY, MÜEGYETEM RAKPART 1-3., BUDAPEST, HUNGARY, 1111

E-mail address: domokos@iit.bme.hu

ZSOLT LÁNGI, DEPT. OF GEOMETRY, BUDAPEST UNIVERSITY OF TECHNOLOGY, EGYR JÓZSEF U. 1., BUDAPEST, HUNGARY, 1111

E-mail address: zlangi@math.bme.hu

TÍMEA SZABÓ, DEPT. OF MECHANICS, MATERIALS AND STRUCTURES, BUDAPEST UNIVERSITY OF TECHNOLOGY, MÜEGYETEM RAKPART 1-3., BUDAPEST, HUNGARY, 1111

E-mail address: tszabo@szt.bme.hu
Figure 7. Caustics of an ellipsoid. a) Caustic defined by the minimal principal curvature b) Caustic defined by the maximal principal curvature c) Superposition of the two caustics d) Quasi-dual Morse-Smale graphs (in class $Q$) in the different domains, shown on the plane section through axes $x$ and $z$. 

Figure 8. The Morse-Smale complex of a real pebble: primal Morse-Smale graph in class $Q^3$ a) drawn on the pebble’s surface b) drawn on the plane