ON HYPERCYCLIC CONVOLUTION OPERATORS AND CONVOLUTION EQUATIONS

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ABSTRACT. We introduce a new technique that generates new examples of hypercyclic convolution operators in situations where the known techniques are not effective. We also apply this technique to prove new existence and approximation theorems for convolution equations.

1. Introduction

If $X$ is a topological space, a map $f : X \to X$ is hypercyclic if the set $\{ x, f(x), f^2(x), \ldots \}$ is dense in $X$ for some $x \in X$. In this case, $x$ is said to be a hypercyclic vector for $f$. Hypercyclic translation and differentiation operators on spaces of entire functions of one complex variable were first investigated by Birkhoff [8] and MacLane [34]. Godefroy and Shapiro [27] pushed these results quite further by proving that every convolution operator on spaces of entire functions of several complex variables which is not a scalar multiple of the identity is hypercyclic. For the theory of hypercyclic operators and its ramifications we refer to [3, 4, 28] and the references therein. Results on the hypercyclicity of convolution operators on spaces of entire functions of infinitely many complex variables appeared later (see, e.g., [1, 5, 6, 7, 9, 10, 11, 20, 28, 50]). In 2007, Carando, Dimant and Muro [10] proved some general results, including a solution to a problem posed in [2], that encompass as particular cases several of the above mentioned results. In [5], using the theory of holomorphy types, Bertoloto, Botelho, Fávaro and Jatobá generalized the results of [10] to a more general setting. For instance, the following theorem from [5], when restricted to $E = C^n$ and $\mathcal{P}_\Theta(mE) = \mathcal{P}(mC^n)$ recovers the famous result of Godefroy and Shapiro [27] on the hypercyclicity of convolution operators on $\mathcal{H}(C^n)$:

**Theorem 1.1.** [5, Theorem 2.7] Let $E'$ be separable and $(\mathcal{P}_\Theta(mE))_{m=0}^\infty$ be a $\pi_1$-holomorphy type from $E$ to $C$. Then every convolution operator on $\mathcal{H}_\Theta(E)$ which is not a scalar multiple of the identity is hypercyclic.

However, the spaces $\mathcal{P}_\Theta(mE)$ need to be Banach spaces and thus $\mathcal{H}_\Theta(E)$ becomes a Fréchet space. When the spaces $\mathcal{P}_\Theta(mE)$ are quasi-Banach, the respective space $\mathcal{H}_\Theta(E)$ is not Fréchet and then the arguments used to prove the result above, for instance the Hypercyclicity Criterion obtained independently by Kitai [32] and Gethner and Shapiro [26], do not work.

Also, several spaces of holomorphic mappings, which have arisen with the development of the theory of polynomial and operator ideals (see, for instance [15, 47, 48]), are not Fréchet spaces and thus the investigation of this more general setting seems to be relevant.

The same problem happens in the investigation of existence and approximation results for convolution equations. This line of investigation was initiated by Malgrange [34] and developed by several authors (see, for example [12, 13, 14, 20, 21, 22, 23, 24, 29, 30, 37, 38, 39, 41, 42, 45]). In this context, a result of [23] (refined in [5]) gives a general method to prove existence and approximation results for convolution equations defined on certain spaces of entire functions of bounded type. The general process

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to prove existence and approximation results for convolution equations on $\mathcal{H}_{gb}(E)$ involves three main steps:

(i) To establish an isomorphism between the topological dual of $\mathcal{H}_{gb}(E)$ and a certain space $\mathcal{E}$ of exponential-type holomorphic functions via Borel transform.

(ii) To prove a division theorem for holomorphic functions on $\mathcal{E}$, that is, if $fg = h$, $g \neq 0$, $g, h \in \mathcal{E}$, $f \in \mathcal{H}(E')$, then it is possible to show that $f \in \mathcal{E}$.

(iii) To handle the results of (i) and (ii) and use Hahn–Banach type theorems and the Dieudonné–Schwartz theorem that appears in [18].

The absence of Hahn–Banach and Dieudonné–Schwartz theorems for more general settings is a crucial obstacle for the development of a general theory. For some classes of polynomials $P_\alpha(mE)$ there are duality results via Borel transform but the step (iii) can not be accomplished.

In this paper we develop general approach that, together with the recent results of [5], allow us to deal with these problematic cases of hypercyclicity and existence and approximation results for convolution equations. More precisely, in some cases in which there are duality results via Borel transform but the Hypercyclicity Criterion and the handling of step (iii) fail, we create a slightly different domain (with a locally convex topology) where this criterion and steps (i)-(iii) are applicable. It is worth mentioning that the proof of division theorems (step (ii)) for holomorphic functions is always a hard work (see e.g., [13, 14, 22, 31, 33, 36, 40]).

The paper is organized as follows:

In Section 2 we develop a general and technical results that will be used to provide new examples of hypercyclic convolution operators and new existence and approximation results for convolution equations, when $P_\alpha(mE)$ are quasi-Banach spaces.

In Section 3 we present background results that will be needed in the next section.

In Section 4 we obtain hypercyclic results for convolution operators and existence and approximation results for convolution equations in a case that was not possible before. We use as a prototype of model the class of Lorentz nuclear polynomials.

Throughout the paper $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$. The letters $E$ and $F$ will always denote complex Banach spaces and $E'$ represents the topological dual of $E$ and $E''$ its bidual. The Banach space of all continuous $m$-homogeneous polynomials from $E$ into $F$ endowed with its usual sup norm is denoted by $P(mE; F)$. The subspace of $P(mE; F)$ of all polynomials of finite type is represented by $P_f(mE; F)$. The linear space of all entire mappings from $E$ into $F$ is denoted by $\mathcal{H}(E; F)$. When $F = \mathbb{C}$ we write $P(mE)$, $P_f(mE)$ and $\mathcal{H}(E)$ instead of $P(mE; \mathbb{C})$, $P_f(mE; \mathbb{C})$ and $\mathcal{H}(E; \mathbb{C})$, respectively. For the general theory of homogeneous polynomials and holomorphic functions we refer to Dineen [17] and Mujica [24]. If $G$ and $H$ are vector spaces and $(\cdot, \cdot)$ is a bilinear form on $G \times H$, we denote by $(G, H, (\cdot, \cdot))$ (or $(G, H)$ for short) the dual system. We denote by $\sigma(G, H)$ the weak topology with respect to the dual system $(G, H)$, that is, the coarsest topology on $G$ for which the linear forms $x \to \langle x, y \rangle$, $y \in H$ are continuous.

2. Main results

Let $n \in \mathbb{N}$ and suppose that $(\mathcal{P}_\Delta(nE), \| \cdot \|_\Delta)$ is a quasi-normed space of $n$-homogeneous polynomials defined on $E$ such that the inclusion $\mathcal{P}_\Delta(nE) \ni P \mapsto \|P\| \in \mathcal{P}_\Delta(nE)$ is continuous and $P_f(nE) \subset \mathcal{P}_\Delta(nE)$. Let $C_{\Delta_n} > 0$ be such that $\|P\| \leq C_{\Delta_n} \|P\|_\Delta$, for all $P \in \mathcal{P}_\Delta(nE)$. Suppose that the normed space $(\mathcal{P}_\Delta(nE'), \| \cdot \|_\Delta') \subset \mathcal{P}_\Delta(nE')$ is such that the Borel transform

$$B : (\mathcal{P}_\Delta(nE'), \| \cdot \|) \to (\mathcal{P}_\Delta(nE'), \| \cdot \|_\Delta')$$

given by $B(T)(\varphi) = T(\varphi^n)$, for all $\varphi \in E'$ and $T \in \mathcal{P}_\Delta(nE')$, is a topological isomorphism.

We will show that the pair

$$(\mathcal{P}_\Delta(nE), \mathcal{P}_\Delta(nE'))$$

is a dual system. More precisely, we will prove that there exists a bilinear form $(\cdot, \cdot)$ on

$$(\mathcal{P}_\Delta(nE) \times \mathcal{P}_\Delta(nE'))$$

such that

$$(\varphi, T) = \langle \varphi^n, T \rangle$$

for all $\varphi \in E'$ and $T \in \mathcal{P}_\Delta(nE')$. 

This bilinear form $(\cdot, \cdot)$ is continuous with respect to the weak topology $(\mathcal{P}_\Delta(nE), \| \cdot \|_\Delta)$.
such that the following conditions hold:

(S1) \( \langle P; Q \rangle = 0 \) for all \( Q \in \mathcal{P}_\Delta(\sigma E') \) implies \( P = 0 \).
(S2) \( \langle P; Q \rangle = 0 \) for all \( P \in \mathcal{P}_\Delta(\sigma E) \) implies \( Q = 0 \).

Let

\[ \langle \cdot, \cdot \rangle : \mathcal{P}_\Delta(\sigma E) \times \mathcal{P}_\Delta(\sigma E') \to \mathbb{K} \]

be defined by

\[ \langle P; Q \rangle = B^{-1}(Q)(P). \]

It is clear that \( \langle \cdot, \cdot \rangle \) is bilinear.

For \( 0 \neq Q \in \mathcal{P}_\Delta(\sigma E') \), we have \( B^{-1}(Q) \neq 0 \) (because \( B \) is an isomorphism). Hence

\[ \langle P; Q \rangle = B^{-1}(Q)(P) \neq 0 \]

for some \( P \in \mathcal{P}_\Delta(\sigma E) \), and so (S2) holds.

Now, if \( 0 \neq P \in \mathcal{P}_\Delta(\sigma E) \), then there is \( x \in E \) such that \( P(x) \neq 0 \). We consider \( A_x \in \mathcal{E}'' \) defined by

\[ A_x(\varphi) = \varphi(x), \]

for all \( \varphi \in E' \) and define

\[ T : \mathcal{P}_\Delta(\sigma E) \to \mathbb{K} \]
\[ T(P) = P(x). \]

Obviously \( T \) is linear. Moreover, \( T \) is continuous and \( \| T \| \leq C_{\Delta_n} \| x \|^n \). In fact, for every \( P \in \mathcal{P}_\Delta(\sigma E) \)

\[ |T(P)| = |P(x)| \leq \| P \| \| x \|^n \leq C_{\Delta_n} \| x \|^n \| P \| \]

and so \( \| T \| \leq C_{\Delta_n} \| x \|^n \). Note that

\[ B(T)(\varphi) = T(\varphi^n) = \varphi(x)^n = (A_x(\varphi))^n \]

for all \( \varphi \in E' \). We conclude that the polynomial

\[ (A_x)^n : E' \to \mathbb{K} \]
\[ (A_x)^n(\varphi) = \varphi(x)^n \]

belongs to \( \mathcal{P}_\Delta(\sigma E') \) and

\[ <P, (A_x)^n > = B^{-1}((A_x)^n)(P) = T(P) = P(x) \neq 0. \]

Thus (S1) is proved and hence the pair \( (\mathcal{P}_\Delta(\sigma E), \mathcal{P}_\Delta(\sigma E')) \) is a dual system.

Now, let

\[ U = \{ P \in \mathcal{P}_\Delta(\sigma E) ; \| P \| \leq 1 \}. \]

Since the bipolar of \( U \), denoted by \( U^{\circ\circ} \), is absorbing we can consider the corresponding gauge

\[ p_U(\delta) = \inf \{ \delta > 0 ; P \in \delta U^{\circ\circ} \}, \]

defined for all \( P \) in \( \mathcal{P}_\Delta(\sigma E) \). Recall that the polar of \( U \) is defined by

\[ U^\circ = \{ Q \in \mathcal{P}_\Delta(\sigma E') ; |<P, Q>| \leq 1 \text{ for all } P \in U \}. \]

Hence

\[ U^\circ = \{ Q \in \mathcal{P}_\Delta(\sigma E') ; |B^{-1}(Q)(P)| \leq 1 \text{ for all } P \in \mathcal{P}_\Delta(\sigma E) , \| P \| \leq 1 \} \]
\[ = \{ Q \in \mathcal{P}_\Delta(\sigma E') ; \| B^{-1}(Q) \| \leq 1 \}. \]

Moreover,

\[ U^{\circ\circ} = \{ P \in \mathcal{P}_\Delta(\sigma E) ; |<P, Q>| \leq 1 \text{ for all } Q \in U^\circ \}
\[ = \{ P \in \mathcal{P}_\Delta(\sigma E) ; |B^{-1}(Q)(P)| \leq 1 , \text{ for all } Q \in \mathcal{P}_\Delta(\sigma E') \text{ with } \| B^{-1}(Q) \| \leq 1 \} \]
and hence
\[ p_{U^{\infty}} (P) = \inf \{ \delta > 0; \| B^{-1} (Q) (P) \| \leq \delta, \text{ for all } Q \in \mathcal{P}_{\Delta'} (\alpha E') \text{ with } \| B^{-1} (Q) \| \leq 1 \} . \]

Since
\[ \| B^{-1} (Q) (P) \| \leq \| B^{-1} (Q) \| \| P \|_{\Delta} \leq \| P \|_{\Delta} , \]
for all \( Q \in \mathcal{P}_{\Delta'} (\alpha E') \) with \( \| B^{-1} (Q) \| \leq 1 \), it follows that
\[ (2.1) \quad p_{U^{\infty}} (P) \leq \| P \|_{\Delta} , \]
for all \( P \in \mathcal{P}_{\Delta} (\alpha E) \).

Note that \( p_{U^{\infty}} \) is a norm on \( \mathcal{P}_{\Delta} (\alpha E) \). In fact, we only have to prove that \( p_{U^{\infty}} (P) = 0 \) implies \( P = 0 \). If \( p_{U^{\infty}} (P) = 0 \), then
\[ \| B^{-1} (Q) (P) \| = 0 \]
for all \( Q \in \mathcal{P}_{\Delta'} (\alpha E') \) with \( \| B^{-1} (Q) \| \leq 1 \). So, we conclude that
\[ < P, Q > = \| B^{-1} (Q) (P) \| = 0 \]
for all \( Q \in \mathcal{P}_{\Delta'} (\alpha E') \). Hence, from (S1) it follows that \( P = 0 \).

From now on we will often use the Bipolar Theorem, which asserts that the bipolar of \( U \) coincides with the \( \sigma (\mathcal{P}_{\Delta} (\alpha E), \mathcal{P}_{\Delta'} (\alpha E')) \)-closure of the absolutely convex hull \( \Gamma (U) \) of \( U \).

**Proposition 2.1.** If \( P \in \mathcal{P}_{\Delta} (\alpha E) \) then
\[ \| P \| \leq C_{\Delta} p_{U^{\infty}} (P) . \]

**Proof.** We know that
\[ \| P \| \leq C_{\Delta} \| P \|_{\Delta} , \]
for all \( P \in \mathcal{P}_{\Delta} (\alpha E) \). If \( P \) belongs to the absolutely convex hull \( \Gamma (U) \) of \( U \), then
\[ P = \sum_{j=1}^{m} \lambda_j P_j , \]
where \( P_j \in U, \lambda_j \in K, j = 1, \ldots, m, \) for some \( m \in \mathbb{N}, \) and
\[ \sum_{j=1}^{m} |\lambda_j| \leq 1 . \]

Since \( P_j \in U \), we have
\[ \| P_j \| \leq C_{\Delta} \| P_j \|_{\Delta} \leq C_{\Delta} . \]
So,
\[ (2.2) \quad \| P \| = \left\| \sum_{j=1}^{m} \lambda_j P_j \right\| \leq \sum_{j=1}^{m} |\lambda_j| \| P_j \| \leq C_{\Delta} \sum_{j=1}^{m} |\lambda_j| \leq C_{\Delta} \]
for every \( P \in \Gamma (U) \). Now if \( P \in U^\infty \), which is the \( \sigma ((\mathcal{P}_{\Delta} (\alpha E), \mathcal{P}_{\Delta'} (\alpha E')))-\)closure of \( \Gamma (U) \), let \( (P_i)_{i \in I} \) be a net in \( \Gamma (U) \) such that
\[ \lim_{i \in I} |< P_i, Q >| = |< P, Q >| \]
for all \( Q \in \mathcal{P}_{\Delta'} (\alpha E') \). So we have
\[ \lim_{i \in I} |B^{-1} (Q) (P_i)| = |B^{-1} (Q) (P)| \]
for all \( Q \in \mathcal{P}_{\Delta'} (\alpha E') \). In particular, for \( x \in B_E \), we have
\[ |P (x)| = |B^{-1} ((A_x)\alpha) (P)| = \lim_{i \in I} |P_i (x)| \leq C_{\Delta} \| x \| . \]
Hence
\[(2.3)\] \[\|P\| \leq C_{\Delta_n}\]
for every \(P \in U^{\infty}\). Finally, for \(0 \neq P \in \mathcal{P}_\Delta (n E)\), let
\[R = (p_{U^{\infty}} (P))^{-1} P.\]

We thus have
\[p_{U^{\infty}} (R) = 1,\]
and this implies that
\[\|B^{-1} (Q) (R)\| \leq 1\]
for all \(Q \in \mathcal{P}_{\Delta'} (n E')\) with \(\|B^{-1} (Q)\| \leq 1\). Thus \(R \in U^{\infty}\) and consequently, from \(\text{2.8}\) we conclude that \(\|R\| \leq C_{\Delta_n}\), i.e.,
\[\|(p_{U^{\infty}} (P))^{-1} P\| \leq C_{\Delta_n}\]
and the result follows. \(\square\)

We denote the completion of the space \((\mathcal{P}_\Delta (n E), p_{U^{\infty}})\) by \((\mathcal{P}_{\Delta} (n E), \|\cdot\|_{\Delta})\). So the restriction of \(\|\cdot\|_{\Delta}\) to \(\mathcal{P}_\Delta (n E)\) is \(p_{U^{\infty}}\) and Proposition \(\text{2.1}\) implies that
\[\mathcal{P}_\Delta (n E) \subset \mathcal{P} (n E)\]
and
\[(2.4)\] \[\|P\| \leq C_{\Delta_n} \|P\|_{\Delta},\]
for all \(P\) in \(\mathcal{P}_{\Delta} (n E)\).

**Definition 2.2.** The elements of \(\mathcal{P}_{\Delta} (n E)\) are called quasi- \(\Delta\) \(n\)-homogeneous polynomials.

**Remark 2.3.** When \(\mathcal{P}_\Delta (n E)\) is a Banach space then we have \(\|\cdot\|_{\Delta} = \|\cdot\|_{\Delta}\) and \(\mathcal{P}_{\Delta} (n E) = \mathcal{P}_\Delta (n E)\).

In fact, in this case, \(U\) is the closed unit ball in \(\mathcal{P}_\Delta (n E)\), hence balanced and convex and \(\Gamma (U) = U\).

By using the Bipolar Theorem we have
\[U^{\infty} = \overline{\Gamma (U)}^{(\mathcal{P}_\Delta (n E), p_{\Delta} (n E'))} = \overline{\Gamma (\mathcal{P}_\Delta (n E), p_{\Delta} (n E'))} = U,\]
and the last equality follows from Banach-Mazur Theorem. Hence
\[p_{U^{\infty}} (P) = \inf \{\delta > 0; P \in \delta U^{\infty}\} = \inf \{\delta > 0; P \in \delta U\} = \inf \{\delta > 0; \|P\|_{\Delta} \leq \delta\} = \|P\|_{\Delta} .\]

The next theorem plays a fundamental role for the applications of the next sections. It assures that the dual of \(\mathcal{P}_{\Delta} (n E)\) and \(\mathcal{P}_\Delta (n E)\), by the Borel transform, are the same.

**Theorem 2.4.** The linear mapping
\[\overline{\mathcal{B}}: (\mathcal{P}_\Delta (n E)', \|\cdot\|) \longrightarrow (\mathcal{P}_{\Delta'} (n E'), \|\cdot\|_{\Delta'})\]
\[\overline{\mathcal{B}} (T) (\varphi) = T (\varphi'^n)\]
is a topological isomorphism.

**Proof.** We know that \((\mathcal{P}_\Delta (n E), p_{U^{\infty}})\) is dense in \((\mathcal{P}_{\Delta} (n E), \|\cdot\|_{\Delta})\). Thus the topological duals of the both spaces are isometrically isomorphic. So we only need to prove that \(\mathcal{P}_\Delta (n E)\) has the same topological dual for the norm \(p_{U^{\infty}}\) and for the quasi-norm \(\|\cdot\|_{\Delta}\). By \(\text{2.1}\), for each \(T \in (\mathcal{P}_\Delta (n E), p_{U^{\infty}})'\) we have
\[\sup_{P \in U} \|T (P)\| \leq \sup_{p_{U^{\infty}} (P) \leq 1} |T (P)|\]
and this implies that the inclusion
\[(\mathcal{P}_\Delta (n E), p_{U^{\infty}})' \hookrightarrow (\mathcal{P}_\Delta (n E), \|\cdot\|_{\Delta})',\]
is continuous. Now, let \( T \in (\mathcal{P}_\Delta (^n E), \| \cdot \|_\Delta)' \). If \( P \in \Gamma (U) \), then 

\[
P = \sum_{j=1}^{m} \lambda_j P_j, 
\]

where \( P_j \in U, \lambda_j \in \mathbb{K}, j = 1, \ldots, m \), for some \( m \in \mathbb{N} \), and 

\[
\sum_{j=1}^{m} |\lambda_j| \leq 1. 
\]

Thus

\[
(2.5) \quad |T (P)| \leq \sum_{j=1}^{m} |\lambda_j| \|T (P_j)\| \leq \sup_{Q \in U} |T (Q)| \sum_{j=1}^{m} |\lambda_j| \leq \sup_{Q \in U} |T (Q)| < +\infty. 
\]

If \( P \in U^\infty \), then there exists a net \( (P_i)_{i \in I} \subset \Gamma (U) \) such that

\[
|T (P) | = \lim_{i \in I} |T (P_i) | \leq \sup_{Q \in U} |T (Q) | < +\infty. 
\]

Hence \( T \) is bounded over \( U^\infty \) and so continuous for \( p_{U^\infty} \), as we wanted to show. \( \square \)

The next result will be necessary in Section 4. It is clear that since \( \mathcal{P}_f (^n E) \) is contained in \( \mathcal{P}_\Delta (^n E) \), then \( \mathcal{P}_f (^n E) \) is contained in \( \mathcal{P}_\Delta (^n E) \).

**Proposition 2.5.** Let \( n \in \mathbb{N} \).

(a) If there exists \( K > 0 \) such that \( \| \varphi^n \|_\Delta \leq K \| \varphi \|_\Delta^n \), for all \( \varphi \in E' \), then

\[
\| \varphi^n \|_\Delta \leq K \| \varphi \|_\Delta^n \leq KC_{\Delta}^n \| \varphi \|_\Delta^n 
\]

for all \( \varphi \in E' \).

(b) If \( \mathcal{P}_f (^n E) \) is dense in \( (\mathcal{P}_\Delta (^n E), \| \cdot \|_\Delta) \), then \( \mathcal{P}_f (^n E) \) is dense in \( (\mathcal{P}_\Delta (^n E), \| \cdot \|_\Delta) \).

**Proof.** (a) By inequality (2.4) we have \( \|P\|_\Delta \leq \|P\|_\Delta^n \), for every \( P \in \mathcal{P}_\Delta (^n E) \). In particular, for every \( \varphi \in E' \),

\[
\| \varphi^n \|_\Delta \leq \| \varphi^n \|_\Delta^n \leq K \| \varphi \|_\Delta^n . 
\]

Besides, (2.4) assures that

\[
\| \varphi \|_\Delta \leq C_{\Delta} \| \varphi \|_\Delta 
\]

for every \( \varphi \in E' \). Now the result follows from the last two inequalities.

(b) We know that \( p_{U^\infty} (\cdot) \leq \| \cdot \|_\Delta (\text{see } (2.1)) \) and \( p_{U^\infty} (P) = \|P\|_\Delta, \) for all \( P \in \mathcal{P}_\Delta (^n E) \). Using this fact and the density of \( \mathcal{P}_f (^n E) \) in \( (\mathcal{P}_\Delta (^n E), \| \cdot \|_\Delta) \), the result follows. \( \square \)

Now we are interested to connect the previous construction with the concept of holomorphy type that we recall below. The notation for the derivatives of polynomials that we use are the same introduced by L. Nachbin in [16].

**Definition 2.6.** A **holomorphy type** \( \Theta \) from \( E \) to \( F \) is a sequence of Banach spaces \( (\mathcal{P}_\Theta (^n E; F))^\infty_{n=0} \), the norm on each of them being denoted by \( \| \cdot \|_\Theta \), such that the following conditions hold true:

1. Each \( \mathcal{P}_\Theta (^n E; F) \) is a linear subspace of \( \mathcal{P}(^n E; F) \).
2. \( \mathcal{P}_\Theta (^n E; F) \) coincides with \( \mathcal{P}(^n E; F) \) as a normed vector space.
3. There is a real number \( \sigma \geq 1 \) for which the following is true: given any \( k \in \mathbb{N}_0, n \in \mathbb{N}_0, k \leq n, a \in E \) and \( P \in \mathcal{P}_\Theta (^n E; F) \), we have

\[
\frac{\hat{d}^{k} P(a)}{k!} \in \mathcal{P}_\Theta (^n E; F) \text{ and } \| \frac{\hat{d}^{k} P(a)}{k!} \|_\Theta \leq \sigma^n \| P \|_\Theta |a|^{n-k}. 
\]
It is plain that each inclusion $\mathcal{P}_\Theta(nE; F) \subseteq \mathcal{P}(nE; F)$ is continuous and that $\|P\| \leq \sigma^n\|P\|_\Theta$ for every $P \in \mathcal{P}_\Theta(nE; F)$.

The definition of holomorphy type motivates the next definition for quasi-normed spaces of homogeneous polynomials.

**Definition 2.7.** For each $n \in \mathbb{N}_0$, let $(\mathcal{P}_\Delta(nE), \|\cdot\|_\Delta)$ be a quasi-normed space, where $\mathcal{P}_\Delta(\{0\}E) = \mathbb{C}$. The sequence $(\mathcal{P}_\Delta(nE))_{n=0}^\infty$ is stable for derivatives if

1. $\hat{d}^k P(x) \in \mathcal{P}_\Delta(kE)$ for each $n \in \mathbb{N}_0$, $P \in \mathcal{P}_\Delta(nE)$, $k = 0, 1, \ldots, n$ and $x \in E$.
2. For each $n \in \mathbb{N}_0$, $k = 0, 1, \ldots, n$, there is a constant $C_{n, k} \geq 0$ such that
   $$\left\|\hat{d}^k P(x)\right\|_\Delta \leq C_{n, k} \|P\|_\Delta \|x\|^{n-k},$$
   for all $x \in E$.

**Theorem 2.8.** Let $(\mathcal{P}_\Delta(nE))_{n=0}^\infty$ be a sequence stable for derivatives. If $P \in \mathcal{P}_\Delta(nE)$, then

$$\hat{d}^k P(x) \in \mathcal{P}_\Delta(kE)$$

and

$$\left\|\hat{d}^k P(x)\right\|_\Delta \leq C_{n, k} \|P\|_\Delta \|x\|^{n-k},$$

for every $k = 0, 1, \ldots, n$ and $x \in E$, where $C_{n, k}$ is the constant of Definition 2.7.

**Proof.** By hypothesis we have

$$\left\|\hat{d}^k P(x)\right\|_\Delta \leq C_{n, k} \|P\|_\Delta \|x\|^{n-k},$$

for all $P \in \mathcal{P}_\Delta(nE)$, $k = 0, 1, \ldots, n$ and $x \in E$. Let $U_k = \{Q \in \mathcal{P}_\Delta(kE) : \|Q\|_\Delta \leq 1\}$ and let $V_k$ be the absolutely convex hull of $U_k$. Let $p_{V_k}$ be the gauge of $V_k$ (note that $p_{V_k}$ is a norm, since $V_k$ is a bounded, balanced and convex neighborhood of zero). Consider

$$\psi : \mathcal{P}_\Delta(nE) \rightarrow \mathcal{P}_\Delta(kE)$$

$$\psi(P) = \hat{d}^k P(x)$$

We know that

$$p_{V_k}(Q) \leq \|Q\|_\Delta$$

for every $Q \in \mathcal{P}_\Delta(kE)$. In fact, since $V_k$ is convex, balanced and absorbing, we have

$$V_k \subset \{Q \in \mathcal{P}_\Delta(kE) : p_{V_k}(Q) \leq 1\}$$

and, for $Q \in \mathcal{P}_\Delta(kE)$, $Q \neq 0$, we have

$$\left\|\frac{Q}{\|Q\|_\Delta}\right\|_\Delta = 1.$$

So,

$$\frac{Q}{\|Q\|_\Delta} \in U_k \subset V_k$$

and

$$p_{V_k}\left(\frac{Q}{\|Q\|_\Delta}\right) \leq 1,$$

which shows (2.7). From (2.7) we get

$$p_{V_k}(\hat{d}^k P(x)) \leq \left\|\hat{d}^k P(x)\right\|_\Delta \leq C_{n, k} \|P\|_\Delta \|x\|^{n-k}.$$
for every \( P \in \mathcal{P}_\Delta (\mathbb{V}) \). Now let \( Q \in V_n \). Then
\[
Q = \sum_{j=1}^{m} \lambda_j P_j
\]
with \( P_j \in U_n, j = 1, \ldots, m \) and \(|\lambda_1| + \cdots + |\lambda_m| = 1\). Hence
\[
(2.9) \quad p_{V_n}(d^k Q(x)) \leq \sum_{j=1}^{m} |\lambda_j| p_{V_k} \left( d^k P_j(x) \right)
\]
\[
\leq C_{n,k} \| P_j \|_{\Delta} \| x \|^{n-k}
\]
\[
\leq C_{n,k} \| x \|^{n-k}.
\]
for every \( Q \in V_n \).

If \( P \in \mathcal{P}_\Delta (\mathbb{V}), P \neq 0 \), then for every \( \varepsilon > 0 \) we have
\[
p_{V_n} \left( \frac{P}{p_{V_n}(P) + \varepsilon} \right) < 1
\]
and it follows from definition of \( p_{V_n} \) that
\[
\frac{P}{p_{V_n}(P) + \varepsilon} \in 1V_n = V_n.
\]
Hence, from (2.9) we have
\[
p_{V_k} \left( \frac{d^k P(x)}{p_{V_n}(P) + \varepsilon} \right) \leq C_{n,k} \| x \|^{n-k}
\]
for every \( \varepsilon > 0 \). Since \( \varepsilon > 0 \) is arbitrary, we obtain
\[
(2.10) \quad p_{V_k}(d^k P(x)) \leq C_{n,k} \| x \|^{n-k} p_{V_n}(P).
\]
From the Bipolar Theorem we know that \( U_n^{oo} \) is the weak closure of \( V_n \). It is clear that (2.8) holds for \( n \) in the place of \( k \), so we also have
\[
V_n \subset \{ Q \in \mathcal{P}_\Delta (\mathbb{V}) : p_{V_n}(Q) \leq 1 \}.
\]
Note that \( (\mathcal{P}_\Delta (\mathbb{V}), p_{V_n}) \) is consistent with the dual system \( (\mathcal{P}_\Delta (\mathbb{V}), \mathcal{P}_\Delta (\mathbb{V}')) \), and this means that the dual of \( \mathcal{P}_\Delta (\mathbb{V}) \) with the topologies \( p_{V_n} \) and \( \| \cdot \|_\Delta \) is the same. In fact,
\[
p_{U_n^{oo}} \leq p_{V_n} \leq \| \cdot \|_\Delta
\]
and \( (\mathcal{P}_\Delta (\mathbb{V}), p_{U_n^{oo}}) \) is dense in \( (\mathcal{P}_\Delta (\mathbb{V}), \| \cdot \|_\Delta) \). Hence
\[
(\mathcal{P}_\Delta (\mathbb{V}), p_{U_n^{oo}})' = (\mathcal{P}_\Delta (\mathbb{V}), \| \cdot \|_\Delta)'
\]
\[
= (\mathcal{P}_\Delta (\mathbb{V}'), \| \cdot \|_{\Delta'})
\]
\[
= (\mathcal{P}_\Delta (\mathbb{V}), \| \cdot \|_\Delta)'.
\]
From [49, 3.1 p. 130] we know that the closure of a convex set is the same no matter how we choose the topology (consistent with the dual system). Since \( V_n \) is convex, we have
\[
\{ Q \in \mathcal{P}_\Delta (\mathbb{V}) : p_{V_n}(Q) \leq 1 \} = \{ Q \in \mathcal{P}_\Delta (\mathbb{V}) : p_{V_n}(Q) \leq 1 \}^\psi_{V_n}
\]
\[
= \{ Q \in \mathcal{P}_\Delta (\mathbb{V}) : p_{V_n}(Q) \leq 1 \}^{(\mathcal{P}_\Delta (\mathbb{V}), \mathcal{P}_\Delta (\mathbb{V}'))}.
\]
Hence
\[
(2.11) \quad U_n^{oo} = \overline{\{ Q \in \mathcal{P}_\Delta (\mathbb{V}) : p_{V_n}(Q) \leq 1 \}^{(\mathcal{P}_\Delta (\mathbb{V}), \mathcal{P}_\Delta (\mathbb{V}'))}} = \{ Q \in \mathcal{P}_\Delta (\mathbb{V}) : p_{V_n}(Q) \leq 1 \}.
\]
Thus we have
\[
(2.12) \quad p_{U_n^{oo}}(\psi(P)) \leq p_{V_k}(\psi(P))
\]
for every \( P \in U_n^\infty \).

Now, let \( P \in \mathcal{P}_\Delta (^{m}E), \ P \neq 0 \). From the argument below \( \text{(2.9)} \) we have
\[
P \in \mathcal{U}_n^\infty
\]
and hence
\[
p_{U_n^\infty}(P) \leq C_{n,k} \|x\|^{n-k} p_{U_n^\infty}(P).
\]
This implies that \( \psi \) is a continuous linear mapping from \( (\mathcal{P}_\Delta (^{m}E), p_{U_n^\infty}) \) into \( (\mathcal{P}_\Delta (^{k}E), p_{U_n^\infty}) \). We can now extend \( \psi \) to the completions \( (\mathcal{P}_\Delta (^{m}E), \|\cdot\|_\Delta) \) and \( (\mathcal{P}_\Delta (^{k}E), \|\cdot\|_\Delta) \) and the proof is done. \( \square \)

**Corollary 2.9.** If \( (\mathcal{P}_\Delta (^{m}E))_{n=0}^{\infty} \) is stable for derivatives for \( C_{n,k} \leq \frac{n!}{(n-k)!} \), then \( (\mathcal{P}_\Delta (^{m}E))_{n=0}^{\infty} \) is a holomorphy type.

**Proof.** Since conditions (1) and (2) of Definition \( \text{(2.4)} \) are clear, we only have to prove (3). We will show that for \( \sigma = 2 \), we obtain (3). Let \( P \in \mathcal{P}_\Delta (^{m}E), \ k \in \mathbb{N}_0, \ k \leq n \) and \( x \in E \). By Theorem \( \text{(2.8)} \) we have
\[
\left\| \hat{d}^k P(x) \right\|_\Delta \leq \frac{n!}{(n-k)!} \|P\|_\Delta \|x\|^{n-k}.
\]
Hence
\[
\left\| \frac{1}{k!} \hat{d}^k P(x) \right\|_\Delta \leq \frac{n!}{k!(n-k)!} \|P\|_\Delta \|x\|^{n-k} \leq 2^n \|P\|_\Delta \|x\|^{n-k},
\]
as we wanted to show. \( \square \)

Corollary \( \text{(2.9)} \) tells us how to use the results of this section to obtain a holomorphy type \( (\mathcal{P}_\Delta (^{m}E))_{n=0}^{\infty} \) from \( (\mathcal{P}_\Delta (^{m}E))_{n=0}^{\infty} \). We use this result in Section 11 to give a concrete example of how to obtain new hypercyclic results for convolution operators and new existence and approximation results for convolution equations.

### 3. Prerequisites for the applications

The following notions will be very important in the next section.

**Definition 3.1.** Let \( (\mathcal{P}_\Theta (^{m}E; F))_{m=0}^{\infty} \) be a holomorphy type from \( E \) to \( F \). A given \( f \in \mathcal{H}(E; F) \) is said to be of \( \Theta \)-holomorphy type of bounded type if
\[
(i) \ \hat{d}^m f(0) \in \mathcal{P}_\Theta (^{m}E; F), \ \text{for all} \ m \in \mathbb{N}_0,
(ii) \ \lim_{m \to \infty} \left( \frac{1}{m!} \|\hat{d}^m f(0)\|_\Theta \right) = 0.
\]

The vector subspace of \( \mathcal{H}(E; F) \) of all such \( f \) is denoted by \( \mathcal{H}_{\Theta}(E; F) \) and becomes a Fréchet space with the topology \( \tau_\Theta \) generated by the family of seminorms
\[
f \in \mathcal{H}_{\Theta}(E; F) \mapsto \|f\|_{\Theta, \rho} = \sum_{m=0}^{\infty} \frac{\rho^m}{m!} \|\hat{d}^m f(0)\|_\Theta,
\]
for all \( \rho > 0 \) (see \( \text{[23] Proposition 2.3} \)).

When \( F = \mathbb{C} \) we represent \( \mathcal{H}_{\Theta}(E; \mathbb{C}) : = \mathcal{H}_{\Theta}(E) \)

The next two definitions are slight variations of the concepts of \( \pi_1 \) and \( \pi_2 \) holomorphy types (originally introduced in \( \text{[23]} \)) and they can be found in \( \text{[5]} \).
Definition 3.2. A holomorphy type \( (P_\Theta (mE; F))_{m=0}^\infty \) from \( E \) to \( F \) is said to be a \( \pi_1 \)-holomorphy type if the following conditions hold:

(i) Polynomials of finite type belong to \( (P_\Theta (mE; F))_{m=0}^\infty \) and there exists \( K > 0 \) such that
\[
\|\phi^m \cdot b\|_\Theta \leq K^m \|\phi\| \cdot \|b\|
\]
for all \( \phi \in E', \, b \in F \) and \( m \in \mathbb{N} \);

(ii) For each \( m \in \mathbb{N}_0 \), \( P_f (mE; F) \) is dense in \( (P_\Theta (mE; F), \| \cdot \|_\Theta) \).

Definition 3.3. A holomorphy type \( (P_\Theta (mE))_{m=0}^\infty \) from \( E \) to \( F \) is said to be a \( \pi_2 \)-holomorphy type if for each \( T \in [H_\Theta (E)]' \), \( m \in \mathbb{N}_0 \) and \( k \in \mathbb{N}_0 \), \( k \leq m \), the following conditions hold:

(i) If \( P \in P_\Theta (mE) \) and \( A: E^m \rightarrow \mathbb{C} \) is the unique continuous symmetric \( m \)-linear mapping such that \( P = A \), then the \((m - k)\)-homogeneous polynomial
\[
T \left( A(\cdot)^k \right): E \rightarrow \mathbb{C}
\]
\[
y \mapsto T \left( A(\cdot)^k y^{m-k} \right)
\]
belongs to \( P_\Theta (m-kE) \);

(ii) For constants \( C, \rho > 0 \) such that
\[
|T(f)| \leq C \|f\|_{\Theta, \rho} \quad \text{for every } f \in H_\Theta (E)
\]
(which exist since \( T \in [H_\Theta (E)]' \)), there is a constant \( K > 0 \) such that
\[
\|T(A(\cdot)^k)||_\Theta \leq C \cdot K^m \rho^k \|P\|_\Theta \quad \text{for every } P \in P_\Theta (mE).
\]

When \( \Theta \) is a \( \pi_1 \)-holomorphy type from \( E \) to \( F \), it is clear that the Borel transform
\[
B_\Theta: [P_\Theta (mE; F)]' \rightarrow P(mE'; F'), \quad B_\Theta T(\phi)(y) = T(\phi^m y),
\]
for \( T \in [P_\Theta (mE; F)]' \), \( \phi \in E' \) and \( y \in F \), is well-defined and linear. Moreover, \( B_\Theta \) is continuous and injective from conditions (i) and (ii) of Definition 3.2. So, denoting the range of \( B_\Theta \) in \( P(mE'; F') \) by \( P_\Theta (mE'; F') \), the correspondence
\[
B_\Theta T \in P_\Theta (mE'; F') \mapsto \|B_\Theta T\|_{\Theta'} := \|T\|
\]
defines a norm on \( P_\Theta (mE'; F') \).

In this fashion the spaces \( ([P_\Theta (mE; F)]', \| \cdot \|) \) and \( (P_\Theta (mE'; F'), \| \cdot \|_{\Theta'}) \) are isometrically isomorphic. For more details on this isomorphism we refer [5] or [23].

Definition 3.4. [5] Definition 2.6] Let \( \Theta \) be a holomorphy type from \( E \) to \( \mathbb{C} \).

(a) For \( a \in E \) and \( f \in H_\Theta (E) \), the translation of \( f \) by \( a \) is defined by
\[
\tau_0 f: E \rightarrow \mathbb{C}, \quad \tau_0 f(x) = f(x - a).
\]

By [23 Proposition 2.2] we have \( \tau_0 f \in H_\Theta (E) \).

(b) A continuous linear operator \( L: H_\Theta (E) \rightarrow H_\Theta (E) \) is called a convolution operator on \( H_\Theta (E) \) if it is translation invariant, that is,
\[
L(\tau_0 f) = \tau_0 (L(f))
\]
for all \( a \in E \) and \( f \in H_\Theta (E) \).

(c) For each functional \( T \in [H_\Theta (E)]' \), the operator \( \Gamma_\Theta (T) \) is defined by
\[
\Gamma_\Theta (T): H_\Theta (E) \rightarrow H_\Theta (E) \quad \Gamma_\Theta (T)(f) = T * f,
\]
where the convolution product \( T * f \) is defined by
\[
(T * f)(x) = T(\tau_{-x} f) \quad \text{for every } x \in E.
\]

(d) \( \delta_0 \in [H_\Theta (E)]' \) is the linear functional defined by
\[
\delta_0: H_\Theta (E) \rightarrow \mathbb{C}, \quad \delta_0(f) = f(0).
\]
3.1. Hypercyclic results. Using the techniques developed in Section 2 in the final section we shall provide new nontrivial applications of the following hypercyclicity results:

**Theorem 3.5.** [2] Theorem 2.7 Let $E'$ be separable and $(P_\Theta(mE))_{m=0}^\infty$ be a $\pi_1$-holomorphy type from $E$ to $\mathbb{C}$. Then every convolution operator on $\mathcal{H}_{\Theta h}(E)$ which is not a scalar multiple of the identity is hypercyclic.

**Theorem 3.6.** [2] Theorem 2.8 Let $E'$ be separable, $(P_\Theta(mE))_{m=0}^\infty$ be a $\pi_1-\pi_2$-holomorphy type and $T \in [\mathcal{H}_{\Theta h}(E)]'$ be a linear functional which is not a scalar multiple of $\delta_0$. Then $\Gamma_\Theta(T)$ is a convolution operator that is not a scalar multiple of the identity, hence hypercyclic.

3.2. Existence and approximation results.

**Definition 3.7.** Let $(P_\Theta(mE))_{m=0}^\infty$ be a $\pi_1$-holomorphy type from $E$ to $\mathbb{C}$. An entire function $f \in \mathcal{H}(E')$ is said to be of $\Theta'$-exponential type if

(i) $\tilde{d}^m f(0) \in P_\Theta(mE')$ for every $m \in \mathbb{N}_0$;

(ii) There are constants $C \geq 0$ and $c > 0$ such that

$$\|\tilde{d}^m f(0)\|_{\Theta'} \leq C c^m,$$

for all $m \in \mathbb{N}_0$.

The vector space of all such functions is denoted by $\text{Expo'}(E')$.

**Definition 3.8.** [23] Definition 4.4] Let $U$ be an open subset of $E$ and $\mathcal{F}(U)$ a collection of holomorphic functions from $U$ into $\mathbb{C}$. We say that $\mathcal{F}(U)$ is closed under division if, for each $f$ and $g$ in $\mathcal{F}(U)$, with $g \neq 0$ and $h = f/g$ a holomorphic function on $U$, we have $h \in \mathcal{F}(U)$.

The quotient notation $h = f/g$ means that $f(x) = h(x) \cdot g(x)$, for all $x \in U$.

Now we are able to enunciate two results for convolution equations defined on $\mathcal{H}_{\Theta h}(E)$ that we will use, together with the techniques of Section 2 to obtain new existence and approximation results for convolution equations:

**Theorem 3.9.** [23] Theorem 4.2 If $(P_\Theta(mE))_{m=0}^\infty$ is a $\pi_1-\pi_2$-holomorphy type, $\text{Expo'}(E')$ is closed under division and $L: \mathcal{H}_{\Theta h}(E) \longrightarrow \mathcal{H}_{\Theta h}(E)$ is a convolution operator, then the vector subspace of $\mathcal{H}_{\Theta h}(E)$ generated by the exponential polynomial solutions of the homogeneous equation $L = 0$, is dense in the closed subspace of all solutions of the homogeneous equation, that is, the vector subspace of $\mathcal{H}_{\Theta h}(E)$ generated by

$$\mathcal{L} = \{ P \exp \varphi; P \in P_\Theta(mE), m \in \mathbb{N}_0, \varphi \in E', L(P \exp \varphi) = 0 \}$$

is dense in

$$\ker L = \{ f \in \mathcal{H}_{\Theta h}(E); Lf = 0 \}.$$

**Theorem 3.10.** [23] Theorem 4.4 If $(P_\Theta(mE))_{m=0}^\infty$ is a $\pi_1-\pi_2$-holomorphy type, $\text{Expo'}(E')$ is closed under division and $L: \mathcal{H}_{\Theta h}(E) \longrightarrow \mathcal{H}_{\Theta h}(E)$ is a non zero convolution operator, then $L(\mathcal{H}_{\Theta h}(E))$ is equal to $\mathcal{H}_{\Theta h}(E)$.

4. Applications

4.1. Lorentz nuclear and summing polynomials: the basics. For the sake of completeness we will recall the concepts of Lorentz summing polynomials introduced in [23] and Lorentz nuclear polynomials introduced in [25] and related results. We start introducing some notations.

We denote by $c_0(E)$ the Banach space (with the sup norm $\| \cdot \|_{\infty}$) composed by the sequences $(x_j)_{j=1}^\infty$ in the Banach space $E$ so that $\lim_{n \to \infty} x_n = 0$ and $c_00(E)$ is the subspace of $c_0(E)$ formed by the sequences $(x_j)_{j=1}^\infty$ for which there is a $N_0$ such that $x_n = 0$ for all $n \geq N_0$. When $E = \mathbb{K} := \mathbb{R}$ or $\mathbb{C}$ we write $c_0$ and $c_00$ instead of $c_0(\mathbb{K})$ and $c_00(\mathbb{K})$, respectively. If $u = (u_j) \in c_00(E)$, the symbol $\text{card}(u)$ denotes the cardinality of the set $\{ j; u_j \neq 0 \}$.

As usual $\ell_0(E)$ represents the Banach space of bounded sequences in the Banach space $E$, with the sup norm and $\ell_\infty := \ell_\infty(\mathbb{K})$. If $m \in \mathbb{N}$, $(x_j)_{j=1}^m$ denotes $(x_1, \ldots, x_m, 0, 0, \ldots)$, and when $(x_j)_{j=1}^\infty$ is a
sequence of positive real numbers, we say that \((x_j)_{j=1}^\infty\) admits a non-increasing rearrangement if there is an injection \(\pi : \mathbb{N} \to \mathbb{N}\) such that \(x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots\) and \(x_j = x_{\pi(i)}\) for some \(i\) whenever \(x_j \neq 0\). If \(p \geq 1\), then \(p'\) denotes the conjugate of \(p\), i.e., \(\frac{1}{p} + \frac{1}{p'} = 1\).

**Definition 4.1.** Let \(E\) be a Banach space and \(0 < r, q < +\infty\).

(a) For \(x = (x_j)_{j=1}^\infty \in \ell_\infty(E)\) we define
\[
a_{E,n}(x) := \inf \{\|x - u\|_\infty : u \in c_00(E) \text{ and } \text{card}(u) < n\}.\]

(b) The Lorentz sequence space \(\ell_{(r,q)}(E)\) consists of all sequences \(x = (x_j)_{j=1}^\infty \in \ell_\infty(E)\) such that
\[
\left( n^{\frac{r}{q'} - \frac{1}{q}} a_{E,n}(x) \right)_{n=1}^\infty \in \ell_q.
\]
For \(x \in \ell_{(r,q)}(E)\) we define the quasi-norm
\[
\|x\|_{(r,q)} = \left\| \left( n^{\frac{r}{q'} - \frac{1}{q}} a_{E,n}(x) \right)_{n=1}^\infty \right\|_q.
\]

(c) \(\ell^w_{(r,q)}(E)\) is the space of all sequences \((x_j)_{j=1}^\infty\) in \(E\) such that \(\|\varphi(x_n)\|_{n=1}^\infty < \infty\) for every \(\varphi \in E'\). For \(x \in \ell^w_{(r,q)}(E)\) we define the quasi-norm
\[
\|(x_n)_{n=1}^\infty\|_{w,(s,q)} := \sup_{\|\varphi\| \leq 1} \|\varphi(x_n)\|_{n=1}^\infty\|_{(r,q)}.
\]

With the respective quasi-norms, \(\ell_{(r,q)}(E)\) and \(\ell^w_{(r,q)}(E)\) become complete spaces.

It is well known that \(\ell_{(r,q)}(E) \subset c_0(E)\) and if \(x = (x_j)_{j=1}^\infty \in c_0(E)\), then the sequence \((\|x_j\|_{j=1}^\infty)\) admits a non-increasing rearrangement.

**Definition 4.2.** (Definition 4.1) If \(0 < p, q, r, s < \infty\), an \(n\)-homogeneous polynomial \(P \in \mathcal{P}(nE; F)\) is Lorentz \(((s,p);(r,q))\)-summing if \((P(x_j))_{j=1}^\infty \in \ell_{(s,p)}(F)\) for each \((x_j)_{j=1}^\infty \in \ell^w_{(r,q)}(E)\).

The vector space composed by the Lorentz \(((s,p);(r,q))\)-summing \(n\)-homogeneous polynomials from \(E\) to \(F\) is denoted by \(\mathcal{P}_{\text{as}}((s,p);(r,q))(nE; F)\). When \(n = 1\) we write \(\mathcal{L}_{\text{as}}((s,p);(r,q))(E; F)\).

When \(s = p\), we write \(\mathcal{L}_{\text{as}}((s,r);(r,q))\) instead of \(\mathcal{L}_{\text{as}}((s,s);(r,q))\); when \(r = q\), we denote \(\mathcal{L}_{\text{as}}((s,p);(q,q))\) instead of \(\mathcal{L}_{\text{as}}((s,p);(q,q))\).

Note that when \(n = 1, s = p\) and \(r = q\) we have the usual concept of absolutely \((p; q)\)-summing operator. The space of absolutely \((p; q)\)-summing operators from \(E\) to \(F\) is represented by \(\mathcal{L}_{\text{as}}((p; q))(E; F)\). When \(p = q\), we simply write \(\mathcal{L}_{\text{as}, p}\) instead of \(\mathcal{L}_{\text{as}}((p; q))\). For the theory of absolutely summing linear operators we refer to \([10]\).

**Theorem 4.3.** (Theorem 4.2) For \(P \in \mathcal{P}(nE; F)\), the following conditions are equivalent:

1. \(P\) is Lorentz \(((s,p);(r,q))\)-summing.
2. There is \(C \geq 0\) such that
\[
\|\|(P(x_j))_{j=1}^m\|_{(s,p)} \leq C \|\|(x_j)_{j=1}^m\|_{w,(r,q)}\]
for all \(m \in \mathbb{N}\) and \(x_1, \ldots, x_m \in E\).
3. There is \(C \geq 0\) such that
\[
\|\|(P(x_j))_{j=1}^\infty\|_{(s,p)} \leq C \|\|(x_j)_{j=1}^\infty\|_{w,(r,q)}\]
for all \((x_j)_{j=1}^\infty \in \ell^w_{(r,q)}(E)\).

The infimum of the constants \(C\) for which the above inequalities hold is a quasi-norm (denoted by \(\|\cdot\|_{\text{as}}((s,p);(r,q))\)) for \(\mathcal{P}_{\text{as}}((s,p);(r,q))(nE; F)\) and under this quasi-norm, \(\mathcal{P}_{\text{as}}((s,p);(r,q))(nE; F)\) is complete.
Definition 4.4. Let $E$ and $F$ be Banach spaces, $n \in \mathbb{N}$ and $r, q, s, p \in [1, \infty]$ such that $r \leq q$, $s' \leq p'$ and
\[
1 \leq \frac{1}{q} + \frac{n}{p'}.
\]
An $n$-homogeneous polynomial $P : E \to F$ is Lorentz $((r, q); (s, p))$-nuclear if
\[
P(x) = \sum_{j=1}^{\infty} \lambda_j (\varphi_j(x))^{tr} y_j,
\]
with $(\lambda_j)_{j=1}^{\infty} \in \ell_{(r, q)}$, $(\varphi_j)_{j=1}^{\infty} \in \ell_{(s', p')}^{w}(E')$ and $(y_j)_{j=1}^{\infty} \in \ell_{\infty}(F)$.

We denote by $P_{N,((r, q);(s, p))}(n; E; F)$ the subset of $P(n; E; F)$ composed by the $n$-homogeneous polynomials which are Lorentz $(r, q); (s, p))$-nuclear. We define
\[
\|P\|_{N,((r, q);(s, p))} = \inf \left\{ \|\lambda_j\|_{\ell_{(r, q)}} \left\|\left(\varphi_j\right)_{j=1}^{\infty}\right\|_{\ell_{(s', p')}^{w}(E')} \left\|y_j\right\|_{\ell_{\infty}(F)} \right\},
\]
where the infimum is considered for all representations of $P \in P_{N,((r, q);(s, p))}(n; E; F)$ of the form (4.1).

Note that
\[
\|P\| \leq \|P\|_{N,((r, q);(s, p))}.
\]
From now on, unless stated otherwise, $r, q \in [1, \infty]$ and $s, p \in [1, \infty]$, with $r \leq q$ and $s' \leq p'$.

Proposition 4.5. The space $\left(\mathcal{P}_{N,((r, q);(s, p))}(n; E; F), \|\cdot\|_{N,((r, q);(s, p))}\right)$ is a complete quasi-normed space. Besides, for $t_n$ given by
\[
1 = \frac{1}{q} + \frac{n}{p'},
\]
there is a $M \geq 0$ so that
\[
\|P + Q\|_{N,((r, q);(s, p))}^{t_n} \leq M \left(\|P\|_{N,((r, q);(s, p))}^{t_n} + \|Q\|_{N,((r, q);(s, p))}^{t_n}\right).
\]
For this reason we call this quasi-norm by \textit{"{q}uasi-$t_n$-norm"}.

Theorem 4.6. If $E'$ has the bounded approximation property, then the linear mapping
\[
\Psi : \mathcal{P}_{N,((r, q);(s, p))}(n; E; F)' \to \mathcal{P}_{as ((r', q'); (s', p'))}(n; E'; F')
\]
given by $\Psi(T) = P_{T}$ is a topological isomorphism, where the map $P_{T} : E' \to F'$ is given by
\[
P_{T}(\varphi)(y) = T(\varphi^n y).
\]

4.2. New hypercyclic, existence and approximation results. Now, suppose that $E'$ has the bounded approximation property. The three steps below are common steps to obtain hypercyclic results (Theorems 3.9 and 4.10) and existence and approximation results (Theorems 3.9 and 5.10) for convolution operators:

1. To obtain the spaces $\mathcal{P}_{N,((r, q);(s, p))}(n; E)$, for all $n \in \mathbb{N}$, according to Definition 2.2.
2. To prove that $\left(\mathcal{P}_{N,((r, q);(s, p))}(n; E)\right)_{n=0}^{\infty}$ is a holomorphic type.
3. To prove that $\left(\mathcal{P}_{N,((r, q);(s, p))}(n; E)\right)_{n=0}^{\infty}$ is a $\pi_{1}-\pi_{2}$-holomorphic type.

A further step to obtain Theorems 3.9 and 5.10 is:

4. To prove that $Exp_{as ((r', q'); (s', p'))}(E')$ (see Definition 3.7 (b)) is closed under division.

Step (1) is satisfied due to Proposition 4.5 and Theorem 4.6. In fact, Theorem 4.6 assures that the Borel transform is an isomorphism between $\mathcal{P}_{N,((r, q);(s, p))}(n; E; F)'$ and $\mathcal{P}_{as ((r', q'); (s', p'))}(n; E'; F')$. Thus, we can consider, for each $n \in \mathbb{N}$, the space $\mathcal{P}_{N,((r, q);(s, p))}(n; E)$, of all Lorentz $((r, q); (s, p))$-quasi-nuclear $n$-homogeneous polynomials from $E$ to $\mathbb{C}$, according to Definition 2.2.
Now, let us prove Step 2, i.e., \((P_{N,(r,q);(s,p)})^{(n)}E\) is a holomorphy type. We only have to prove that \((P_{N,(r,q);(s,p)})^{\infty}E\) is stable for derivatives with constant \(C_{n,k} = \frac{n!}{(n-k)!}\) (see Proposition 4.7 below) and the result follows from Corollary 2.9.

**Proposition 4.7.** If \(P \in P_{N,(r,q);(s,p)}^{(n)}E\), \(k = 1, \ldots, n\) and \(x \in E\), then \(d^k P (x) \in P_{N,(r,q);(s,p)}^{(k)}E\) and

\[
\left\| d^k P (x) \right\|_{N,(r,q);(s,p)} \leq \frac{n!}{(n-k)!} \left\| P \right\|_{N,(r,q);(s,p)} \left\| x \right\|^{n-k}.
\]

**Proof.** Let

\[
P (x) = \sum_{j=1}^{\infty} \lambda_j \varphi_j (x)^n y_j,
\]

with \((\lambda_j)_{j=1}^{\infty} \in \ell_{(r,q)}, (\varphi_j)_{j=1}^{\infty} \in \ell_{w,(s',p')}^{w} (E')\) and \((y_j)_{j=1}^{\infty} \in \ell_{\infty} (F)\). Then

\[
d^k P (x) = \frac{n!}{(n-k)!} \sum_{j=1}^{\infty} \lambda_j \varphi_j (x)^{n-k} \varphi_j^k y_j,
\]

for \(k = 1, \ldots, n\). Let \(y = \varphi_j (x)\), and note that

\[
\left\| \left( \lambda_j (\varphi_j (y))^{n-k} \right)^{\infty}_{j=1} \right\|_{(r,q)} \leq \left\| (\lambda_j)^{\infty}_{j=1} \right\|_{(r,q)} \sup_{j \in \mathbb{N}} \left\| \varphi_j (y)^{n-k} \right\|.
\]

Now we obtain

\[
\frac{n!}{(n-k)!} \left\| x \right\|^{n-k} \left\| \left( \lambda_j (\varphi_j (y))^{n-k} \right)^{\infty}_{j=1} \right\|_{(r,q)} \leq \frac{n!}{(n-k)!} \left\| x \right\|^{n-k} \left( \lambda_j \right)^{\infty}_{j=1} \left( \varphi_j \right)^{\infty}_{j=1} \left( y_j \right)^{\infty}_{j=1} \sup_{j \in \mathbb{N}} \left\| \varphi_j (y)^{n-k} \right\| \leq \frac{n!}{(n-k)!} \left\| x \right\|^{n-k} \left( \lambda_j \right)^{\infty}_{j=1} \left( \varphi_j \right)^{\infty}_{j=1} \left( y_j \right)^{\infty}_{j=1} \sup_{j \in \mathbb{N}} \left\| \varphi_j (y)^{n-k} \right\| \leq \frac{n!}{(n-k)!} \left\| x \right\|^{n-k} \left( \lambda_j \right)^{\infty}_{j=1} \left( \varphi_j \right)^{\infty}_{j=1} \left( y_j \right)^{\infty}_{j=1} \sup_{j \in \mathbb{N}} \left\| \varphi_j (y)^{n-k} \right\| \leq +\infty.
\]

Thus we have that \((4.2)\) is a valid Lorentz \(((r,q);(s,p))\)-nuclear representation of \(d^k P (x)\) and in view of the last inequalities we can write

\[
\left\| d^k P (x) \right\|_{N,(r,q);(s,p)} \leq \frac{n!}{(n-k)!} \left\| x \right\|^{n-k} \left\| (\lambda_j)^{\infty}_{j=1} \right\|_{(r,q)} \left\| (\varphi_j)^{\infty}_{j=1} \right\|_{w,(s',p')} \left\| (y_j)^{\infty}_{j=1} \right\|_{\infty}.
\]

Hence

\[
\left\| d^k P (x) \right\|_{N,(r,q);(s,p)} \leq \frac{n!}{(n-k)!} \left\| P \right\|_{N,(r,q);(s,p)} \left\| x \right\|^{n-k}
\]

as we wanted to show. 

Now we are able to define the space of Lorentz \(((r,q);(s,p))\)-quasi-nuclear entire mappings of bounded type, according to Definition 3.1.

**Definition 4.8.** An entire mapping \(f : E \rightarrow \mathbb{C}\) is said to be Lorentz \(((r,q);(s,p))\)-quasi-nuclear of bounded type if

1. \(\hat{d}^{n} f (0) \in P_{N,(r,q);(s,p)}^{(n)}E\), for all \(n \in \mathbb{N}_{0}\),
2. \(\lim_{n \rightarrow \infty} \left( \frac{1}{n!} \left\| \hat{d}^{n} f (0) \right\|_{N,(r,q);(s,p)} \right)^{\frac{1}{n}} = 0\).

The space of all entire mappings \(f : E \rightarrow \mathbb{C}\) that are Lorentz \(((r,q);(s,p))\)-quasi-nuclear of bounded
type is denoted by $\mathcal{H}_{\tilde{N},((r,q);(s,p))}(E)$ and it is a Fréchet space with the topology generated by the family of seminorms:

\begin{equation}
(4.3) \quad f \in \mathcal{H}_{\tilde{N},((r,q);(s,p))}(E) \mapsto \|f\|_{\mathcal{H}_{\tilde{N},((r,q);(s,p))},\rho} = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \|\hat{f}^{(m)}(0)\|_{\tilde{N},((r,q);(s,p))},
\end{equation}

for all $\rho > 0$.

Now we have to prove that $\left(\mathcal{P}_{\tilde{N},((r,q);(s,p))}(nE)\right)_{n=0}^{\infty}$ is a $\pi_1$-$\pi_2$-holomorphy type.

**Proposition 4.9.** $\left(\mathcal{P}_{\tilde{N},((r,q);(s,p))}(nE)\right)_{n=0}^{\infty}$ is a $\pi_1$-holomorphy type.

**Proof.** In [25, Example 4.5] it was proved that $\mathcal{P}_f (nE)$ is contained in $\mathcal{P}_{\tilde{N},((r,q);(s,p))}(nE)$ and

$$\|\phi^n\|_{\mathcal{N},((r,q);(s,p))} = \|\phi\|^n$$

for all $\phi \in E'$ and $n \in \mathbb{N}$. Besides, from [25, Lemma 4.10] we know that the space of finite type polynomials $\mathcal{P}_f (nE)$ is dense in $\left(\mathcal{P}_{\mathcal{N},((r,q);(s,p))}(nE),\|\|_{\mathcal{N},((r,q);(s,p))}\right)$. Thus, it follows from Proposition 4.5 that $\mathcal{P}_f (nE)$ is dense in $\left(\mathcal{P}_{\tilde{N},((r,q);(s,p))}(nE),\|\|_{\tilde{N},((r,q);(s,p))}\right)$ and

\begin{equation}
(4.4) \quad \|\phi^n\|_{\tilde{N},((r,q);(s,p))} = \|\phi\|^n,
\end{equation}

for all $\phi \in E'$ and $n \in \mathbb{N}$, since the constants $K$ and $C_{\Delta_n}$, in this case, may be taken equal to 1. \hfill \square

**Proposition 4.10.** $\left(\mathcal{P}_{\tilde{N},((r,q);(s,p))}(nE)\right)_{n=0}^{\infty}$ is a $\pi_2$-holomorphy type.

**Proof.** Let $T \in \left[\mathcal{H}_{\tilde{N},((r,q);(s,p))}(E)\right]'$, $n, k \in \mathbb{N}_0$, $k \leq n$ and $C, \rho > 0$ be constants such that

\begin{equation}
(4.5) \quad |T(f)| \leq C \|f\|_{\tilde{N},((r,q);(s,p))}, \quad \text{for every } f \in \mathcal{H}_{\tilde{N},((r,q);(s,p))}(E).
\end{equation}

For $P \in \mathcal{P}_{\tilde{N},((r,q);(s,p))}(nE)$, we will show that the $(n-k)$-homogeneous polynomial

$$T \left(\hat{A}(\cdot)^k\right) : E \rightarrow \mathbb{C}
\quad y \mapsto T \left(\hat{A}(\cdot)^k y^{n-k}\right)$$

where $A : E^n \rightarrow \mathbb{C}$ is the unique continuous symmetric $n$-linear mapping such that $P = A$, belongs to $\mathcal{P}_{\tilde{N},((r,q);(s,p))}(n-kE)$ and

$$\|T(\hat{A}(\cdot)^k)\|_{\tilde{N},((r,q);(s,p))} \leq C \cdot \rho^k \|P\|_{\tilde{N},((r,q);(s,p))}.$$

First, suppose that $P \in \mathcal{P}_{\tilde{N},((r,q);(s,p))}(nE)$. Then

$$P = \sum_{j=1}^{\infty} \lambda_j \varphi_j^n,$$

with $(\lambda_j) \in \ell_{(r,q)}$ and $(\varphi_j) \in \ell_{(s',p')}(E')$, and for every $y \in E$ we have

$$T \left(\hat{A}(\cdot)^k\right) (y) = T(A \cdot k y^{n-k}) = T \left(\sum_{j=1}^{\infty} \lambda_j \varphi_j^k \varphi_j(y)^{n-k}\right)
= \sum_{j=1}^{\infty} \lambda_j T \left(\varphi_j^k\right) \varphi_j(y)^{n-k}.$$
Now, to prove that $T \left( \hat{A}(\cdot)^k \right) \in \mathcal{P}_{N,((r,q);(s,p))}(n-k,E)$, we only have to show that $(\lambda_j T (\varphi_j^k)) \in \ell_{(r,q)}$ since we already have $(\varphi_j) \in \ell^{w}_{(s',p')}(E')$. First, note that

$$
\left\| (T (\varphi_j^k))_{j=1}^{\infty} \right\|_{(r,q)} = \sup_j \left\| T (\varphi_j^k) \right\|_{\tilde{N}((r,q);(s,p)),p} = C \rho^k \sup_j \| \varphi_j^k \|_{\tilde{N}((r,q);(s,p))} = C \rho^k \sup_j \| \varphi_j \|^k
$$

and so $\lambda_j T (\varphi_j^k) \in \mathcal{P}_{N,((r,q);(s,p))}(n-k,E)$. Moreover

$$
\left\| T \left( \hat{A}(\cdot)^k \right) \right\|_{N,((r,q);(s,p))} \leq \left\| \lambda_j T (\varphi_j^k) \right\|_{(r,q)} \left\| (\varphi_j^k)_{j=1}^{\infty} \right\|_{w,(s',p')} = C \rho^k \left\| (\varphi_j^k)_{j=1}^{\infty} \right\|_{w,(s',p')} \leq C \rho^k \left\| (\varphi_j^k)_{j=1}^{\infty} \right\|_{w,(s',p')} \leq C \rho^k \left\| (\varphi_j^k)_{j=1}^{\infty} \right\|_{w,(s',p')} \leq C \rho^k \left\| P \right\|_{N,((r,q);(s,p))} \text{.}
$$

and proceeding in a similar way as in the proof of Theorem 4.12, the result follows. \qed

Now we can state the hypercyclicity results in this new framework:

**Theorem 4.11.** Let $E'$ be separable. Then every convolution operator on $\mathcal{H}_{N,((r,q);(s,p))}(E)$ which is not a scalar multiple of the identity is hypercyclic.

**Theorem 4.12.** Let $E'$ be separable and $T \in \mathcal{H}_{N,((r,q);(s,p))}'(E)$ be a linear functional which is not a scalar multiple of $\delta_0$. Then $\hat{T}_{N,((r,q);(s,p))}(T)$ is a convolution operator that is not a scalar multiple of the identity, hence hypercyclic.

Now, according to Definition 4.13(b) we introduce the Lorentz-summing functions of exponential type:

**Definition 4.13.** An entire mapping $f : E \rightarrow \mathbb{C}$ is said to be of Lorentz $((s,p);(r,q))$-summing exponential type if $\hat{f}^n f(0) \in \mathcal{P}_{as((s,p);(r,q))}(E)$, for all $n \in \mathbb{N}_0$, and there are $C \geq 0$ and $c > 0$ such that

$$
\left\| \hat{f}^n f(0) \right\|_{as((s,p);(r,q))} \leq C e^n,
$$

for all $n \in \mathbb{N}_0$.

The vector space of all these mappings is denoted by $Exp_{as((s,p);(r,q))}(E)$. 

This definition was motivated by the definition of mappings of exponential type. We recall the concept below (see [29]):

**Definition 4.14.** An entire mapping \( f: E \rightarrow F \) is said to be of exponential type if one of the following equivalent conditions hold:

(i) There are \( C \geq 0 \) and \( c > 0 \) such that \( \|f(x)\| \leq C \exp(c|x|) \), for all \( x \in E \).

(ii) There are \( D \geq 0 \) and \( d > 0 \) such that \( \|d^m f(0)\| \leq Cc^m \), for all \( m \in \mathbb{N} \).

(iii) \( \limsup_{m \to \infty} \|d^m f(0)\|^\frac{1}{m} < +\infty \).

We denote by \( \text{Exp}(E; F) \) the vector space of all entire mappings of exponential type from \( E \) into \( F \). When \( F \) is the scalar field \( \mathbb{C} \) we denote \( \text{Exp}(E) \) instead of \( \text{Exp}(E; \mathbb{C}) \).

The next two lemmata will be necessary for the proof of the division theorem (Theorem 4.17). The first is a division result due to Gupta (see [29] [30]).

**Lemma 4.15.** If \( f, g \) and \( h \) are entire mappings on \( E \) with values in \( \mathbb{C} \), \( f \neq 0 \), \( h(x) = f(x)g(x) \) for all \( x \in E \), with \( f \) and \( h \) of exponential type on \( E \), then \( g \) is of exponential type on \( E \).

**Lemma 4.16.** Let \( r, q, s, p \in [1, +\infty] \) with \( r \leq q \) and \( F \) be a Banach space. If \( f \in \text{Exp}(F) \) and \( g \in \text{Exp}_s((r', q'); (s', p'))(F) \) then \( fg \) is in \( \text{Exp}_s((r', q'); (s', p'))(F) \).

**Proof.** For each \( k \in \mathbb{N} \), it follows from the uniqueness of the power series of a holomorphic function around a point of its domain that

\[
d^k f(0) = \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \frac{1}{l!} \frac{d^lf(0)(x)}{d^l g(0)(x)}
\]

for all \( x \in F \). Since \( f \in \text{Exp}(F) \), there are \( C \geq 0 \) and \( c > 0 \) such that

\[
\|d^m f(0)\| \leq Cc^m,
\]

for every \( n \in \mathbb{N} \). For \( \|(x_j)_{j=1}^{\infty}\|_{w,(s', p')} \leq 1 \), we have \( \|x_j\| \leq 1 \) for every \( j \in \mathbb{N} \) and so

\[
\left|d^n f(0)(x_j)\right| \leq Cc^n,
\]

for every \( n \in \mathbb{N} \). Thus

\[
\left|d^k (fg)(0)(x_j)\right| \leq C \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \left|\frac{1}{l!} \frac{d^lf(0)(x_j)}{d^l g(0)(x_j)}\right|.
\]

for all \( j \in \mathbb{N} \). Let \( \pi: \mathbb{N} \rightarrow \mathbb{N} \) be an injection. Since \( r' \geq q' \) we have (using Lemma 3.1 of [43])

\[
\sum_{j=1}^{\infty} \left( \frac{k}{l!} \frac{d^k}{d^l g(0)(x_{\pi(j)})} \right)^q \leq C \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \left|\frac{1}{l!} \frac{d^lf(0)(x_{\pi(j)})}{d^l g(0)(x_{\pi(j)})}\right|^q.
\]

By Theorem 4.13 we have

\[
\left\|\left(\frac{d^k g(0)(x_j)}{d^l g(0)(x_j)}\right)^q\right\| \leq \left\|\frac{d^k g(0)}{d^l g(0)}\right\|_{s((r', q'), (s', p'))} \left\|\frac{d^k g(0)}{d^l g(0)}\right\|_{s((r', q'), (s', p'))},
\]

for all \( l = 1, \ldots, k \). Thus

\[
\sum_{j=1}^{\infty} \left( \frac{k}{l!} \frac{d^k (fg)(0)(x_{\pi(j)})}{d^l (fg)(0)(x_{\pi(j)})} \right)^q \leq C \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \left|\frac{1}{l!} \frac{d^lf(0)(x_{\pi(j)})}{d^l g(0)(x_{\pi(j)})}\right|^q \leq \left\|\frac{d^k (fg)(0)}{d^l (fg)(0)}\right\|_{s((r', q'), (s', p'))},
\]
and we conclude that \( \left( \left| \hat{d}^k (fg)(0)(x_{\sigma(j)}) \right| \right)_{j=1}^{\infty} \in \ell_{(r',q')} \) and thus \( \left( \hat{d}^k (fg)(0)(x_{\sigma(j)}) \right)_{j=1}^{\infty} \in c_0 \).

Since \( \pi \) is arbitrary we have
\[
\left\| \left( \hat{d}^k (fg)(0)(x_{\sigma(j)}) \right)_{j=1}^{\infty} \right\|_{(r',q')} \leq C \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} c^l \left\| \hat{d}^{k-l} g(0) \right\|_{a_s((r',q'),(s',p'))},
\]
for some injection \( \sigma : \mathbb{N} \to \mathbb{N} \). Hence
\[
\left\| \left( \hat{d}^k (fg)(0)(x_j) \right)_{j=1}^{\infty} \right\|_{(r',q')} \leq C \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} c^l \left\| \hat{d}^{k-l} g(0) \right\|_{a_s((r',q'),(s',p'))} \left\| (x_j)_{j=1}^{\infty} \right\|_{w,(s',p')},
\]
for all \( j \in \mathbb{N} \), we have
\[
\left\| (y_j)_{j=1}^{\infty} \right\|_{w,(s',p')} = 1
\]
and using the previous estimates we obtain
\[
\left\| \left( \hat{d}^k (fg)(0)(x_j) \right)_{j=1}^{\infty} \right\|_{(r',q')} \leq C \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} c^l \left\| \hat{d}^{k-l} g(0) \right\|_{a_s((r',q'),(s',p'))} \left\| (x_j)_{j=1}^{\infty} \right\|_{w,(s',p')}.
\]
Using again Theorem 4.15 it follows that \( \hat{d}^k g(0) \in P_{a_s((r',q');(s',p'))}(k F) \) and
\[
\left\| \hat{d}^k (fg)(0) \right\|_{a_s((r',q'),(s',p'))} \leq C \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} c^l \left\| \hat{d}^{k-l} g(0) \right\|_{a_s((r',q'),(s',p'))}.
\]
Since \( g \in Exp_{a_s((r',q');(s',p'))}(F) \) there are \( D \geq 0 \) and \( d > 0 \) such that
\[
\left\| \hat{d}^n g(0) \right\|_{a_s((r',q'),(s',p'))} \leq Dd^n,
\]
for all \( n \in \mathbb{N} \). Hence
\[
\left\| \hat{d}^k (fg)(0) \right\|_{a_s((r',q'),(s',p'))} \leq CD \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} c^l d^{k-l} = CD(c+d)^k
\]
and \( fg \in Exp_{a_s((r',q');(s',p'))}(F) \).

\[ \square \]

**Theorem 4.17 (Division Theorem).** Let \( r, q, s, p \in [1, +\infty] \) with \( r \leq q \) and \( F \) be a Banach space. Then \( Exp_{a_s((r',q');(s',p'))}(F) \) is closed under division, that is, if \( f, g \) and \( h \) are entire mappings on \( F \) with values in \( C \), \( f \neq 0 \), and \( h = fg \) with \( f \) and \( h \in Exp_{a_s((r',q');(s',p'))}(F) \), then \( g \) is also in \( Exp_{a_s((r',q');(s',p'))}(F) \).

**Proof.** For each \( k \in \mathbb{N} \) we have
\[
\hat{d}^k h(0)(x) = f(0)\hat{d}^k g(0)(x) + \sum_{l=1}^{k} \frac{k!}{l!(k-l)!} \hat{d} f(0)(x) \hat{d}^{k-l} g(0)(x)
\]
and so
\[
f(0)\hat{d}^k g(0)(x) = \hat{d}^k h(0)(x) - \sum_{l=1}^{k} \frac{k!}{l!(k-l)!} \hat{d} f(0)(x) \hat{d}^{k-l} g(0)(x)
\]
for all \( x \in F \). Let us suppose \( f(0) \neq 0 \). For \( \left\| (x_j)_{j=1}^{\infty} \right\|_{w,(s',p')} \leq 1 \), we have \( \left\| x_j \right\| \leq 1 \) for every \( j \in \mathbb{N} \). Since \( \left\| \cdot \right\| \leq \left\| \cdot \right\|_{a_s((r',q'),(s',p'))} \), it follows from Definition 4.14 ii) that \( g \) is of exponential type. So there are \( C \geq 0 \) and \( c > 0 \) such that
\[
\left\| \hat{d}^k g(0) \right\| \leq Cc^k.
\]
Therefore

\[
\left| \hat{d}^k g(0)(x_j) \right| \leq \frac{1}{|f(0)|} \left| \hat{d}^k h(0)(x_j) \right| + \frac{C}{|f(0)|} \sum_{l=1}^{k} \frac{k!}{l!(k-l)!} c^{k-l} \left| \hat{d}^l f(0)(x_j) \right|
\]

for every \( j \in \mathbb{N} \). Let \( \pi : \mathbb{N} \rightarrow \mathbb{N} \) be an injection. Since \( r' \geq q' \), we have (using Lemma 3.1 of \[43\])

\[
\left[ \sum_{j=1}^{\infty} \left( j^\frac{1}{q'} - \frac{1}{q'} \left| \hat{d}^k g(0)(x_{\pi(j)}) \right| \right)^q \right]^\frac{1}{q'}
\]

\[
\leq \frac{1}{|f(0)|} \left[ \sum_{j=1}^{\infty} \left( j^\frac{1}{q'} - \frac{1}{q'} \left| \hat{d}^k h(0)(x_{\pi(j)}) \right| \right)^q \right]^\frac{1}{q'}
\]

\[
+ \frac{C}{|f(0)|} \sum_{l=1}^{k} \frac{k!}{l!(k-l)!} c^{k-l} \left[ \sum_{j=1}^{\infty} \left( j^\frac{1}{q'} - \frac{1}{q'} \left| \hat{d}^l f(0)(x_{\pi(j)}) \right| \right)^q \right]^\frac{1}{q'}
\]

\[
\leq \frac{1}{|f(0)|} \left( \left| \hat{d}^k h(0)(x_j) \right|_2 \right)^\infty_{j=1} \left( \left| \hat{d}^l f(0)(x_j) \right|_2 \right)^\infty_{j=1} + \frac{C}{|f(0)|} \sum_{l=1}^{k} \frac{k!}{l!(k-l)!} c^{k-l} \left( \left| \hat{d}^l f(0)(x_j) \right|_2 \right)^\infty_{j=1}.
\]

By Theorem \[43\] we have

\[
\left( \left| \hat{d}^k h(0)(x_j) \right|_2 \right)^\infty_{j=1} \leq \left( \left| \hat{d}^k h(0) \right|_{as(r', q'), (s', p')} \right) \left( \left| x_j \right|_2 \right)^\infty_{j=1} \leq 1,
\]

\[
\left( \left| \hat{d}^l f(0)(x_j) \right|_2 \right)^\infty_{j=1} \leq \left( \left| \hat{d}^l f(0) \right|_{as(r', q'), (s', p')} \right) \left( \left| x_j \right|_2 \right)^\infty_{j=1} \leq 1,
\]

for all \( l = 1, \ldots, k \). Thus

\[
\left( \sum_{j=1}^{\infty} \left( j^\frac{1}{q'} - \frac{1}{q'} \left| \hat{d}^k g(0)(x_{\pi(j)}) \right| \right)^q \right)^\frac{1}{q'}
\]

\[
\leq \frac{1}{|f(0)|} \left( \left| \hat{d}^k h(0) \right|_{as(r', q'), (s', p')} \right) \left( \left| x_j \right|_2 \right)^\infty_{j=1} \left( \left| \hat{d}^l f(0) \right|_{as(r', q'), (s', p')} \right) \left( \left| x_j \right|_2 \right)^\infty_{j=1}.
\]

We conclude that \( \left( \hat{d}^k g(0)(x_{\pi(j)}) \right)^\infty_{j=1} \in \ell_{(r', q')} \) and thus \( \left( \hat{d}^k g(0)(x_{\pi(j)}) \right)^\infty_{j=1} \in \ell_0 \). Since \( \pi \) is arbitrary we have

\[
\left( \sum_{j=1}^{\infty} \left( j^\frac{1}{q'} - \frac{1}{q'} \left| \hat{d}^k g(0)(x_{\pi(j)}) \right| \right)^q \right)^\frac{1}{q'} = \left( \hat{d}^k g(0)(x_j) \right)^\infty_{j=1} \left( x_j \right)^\infty_{j=1},
\]

for some injection \( \sigma : \mathbb{N} \rightarrow \mathbb{N} \).

Now, let \( 0 \neq (x_j)^\infty_{j=1} \in \ell_{(s', p')}(F) \). Defining

\[
y_j = \frac{x_j}{\left( \left( x_j \right)^\infty_{j=1} \right)_{w,(s', p')}},
\]

for all \( j \in \mathbb{N} \), we have \( \left( y_j \right)^\infty_{j=1} \left( x_j \right)^\infty_{j=1} = 1 \) and using the previous estimates we obtain

\[
\left( \hat{d}^k g(0)(x_j) \right)^\infty_{j=1} \left( x_j \right)^\infty_{j=1}
\]

\[
\leq \left( \frac{1}{|f(0)|} \left| \hat{d}^k h(0) \right|_{as((r', q'), (s', p'))} \right) \left( \left| x_j \right|_2 \right)^\infty_{j=1} \left( \left| \hat{d}^l f(0) \right|_{as((r', q'), (s', p'))} \right) \left( \left| x_j \right|_2 \right)^\infty_{j=1}.
\]
Using again Theorem 4.13, it follows that \( \hat{d}^k g(0) \in \mathcal{P}_{as((r',q');(s',p'))} (kF) \). Since \( f, h \in \text{Exp}_{as((r',q');(s',p'))} (F) \), there are \( A, B \geq 0 \) and \( a, b > 0 \) such that

\[
\left\| \hat{d}^k h(0) \right\|_{as((r',q'),(s',p'))} \leq A a^k
\]

and

\[
\left\| \hat{d}^k f(0) \right\|_{as((r',q'),(s',p'))} \leq B b^l,
\]

for \( l = 1, \ldots, k \). Hence

\[
\left\| \left( \hat{d}^k g(0) / (x_j) \right)_{j=1}^\infty \right\|_{as((r',q'))} \leq \left( \frac{A a^k}{|f(0)|} + \frac{C B}{|f(0)|} \sum_{l=1}^k \frac{1}{l! (k-l)!} \right) \left\| (x_j)_{j=1}^\infty \right\|_{w,(s',p')},
\]

for all \( (x_j)_{j=1}^\infty \in \ell^w_{(s',p')}(F) \) and by definition of \( \left\| \cdot \right\|_{as((r',q'),(s',p'))} \) (see Theorem 4.13) we have

\[
\left\| \hat{d}^k g(0) \right\|_{as((r',q'),(s',p'))} \leq D d^k,
\]

with \( D = \frac{A+BC}{|f(0)|} \geq 0 \) and \( d = a + b + c \geq 0 \).

Now, suppose that \( f(0) = 0 \) and define

\[
f_0(x) = f(x) + \psi(x)
\]

and

\[
h_0(x) = h(x) + \psi(x) g(x)
\]

for all \( x \in F \), where \( \psi \in \text{Exp}_{as((r',q');(s',p'))} (F) \), \( \psi(0) \neq 0 \) and \( \psi \) is non constant (for example, let \( \psi(x) = 1 + P(x) \), with \( P \in \mathcal{P}_{as(s,p);(r,q)} (nF) \) for some \( n \neq 0 \)). Thus \( h_0 = f_0 g_0, \ f_0(0) \neq 0, \ f_0 \in \text{Exp}_{as((r',q');(s',p'))} (F) \) and by Lemma 4.14, \( h_0 \in \text{Exp}_{as((r',q');(s',p'))} (F) \). Applying the result we just have proved, it follows that \( g \in \text{Exp}_{as((r',q');(s',p'))} (F) \). \( \square \)

For \( F = E' \), we have that \( \text{Exp}_{as((r',q');(s',p'))} (E') \) is closed under division, and so it follows from Theorems 3.9 and 3.10 that we have existence and approximation results for convolution equations defined on \( \mathcal{H}_{\tilde{N}b,(r,q);(s,p)} (E) \) as enunciated below.

**Theorem 4.18.** The vector subspace of \( \mathcal{H}_{\tilde{N}b,(r,q);(s,p)} (E) \) generated by the exponential polynomial solutions of the homogeneous equation \( L = 0 \), is dense in the closed subspace of all solutions of the homogeneous equation, that is, the vector subspace of \( \mathcal{H}_{\tilde{N}b,(r,q);(s,p)} (E) \) generated by

\[
\mathcal{L} = \left\{ P \exp \varphi; P \in \mathcal{P}_{N,((r,q);(s,p))} (mE), m \in \mathbb{N}_0, \varphi \in E', L (P \exp \varphi) = 0 \right\}
\]

is dense in

\[
\ker L = \left\{ f \in \mathcal{H}_{\tilde{N}b,(r,q);(s,p)} (E); L f = 0 \right\}.
\]

**Theorem 4.19.** If \( L \) is a non zero convolution operator, then

\[
L \left( \mathcal{H}_{\tilde{N}b,(r,q);(s,p)} (E) \right) = \mathcal{H}_{\tilde{N}b,(r,q);(s,p)} (E).
\]

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