Towards a dynamical theory of observation

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We introduce a model of classical and quantum observation based on contextuality and dynamically evolving apparatus. Power sets of classical bits model the four classical states of elementary detectors, viz. the two normal yes/no signal states, the faulty or decommissioned state and the non-existence state. Operators over power-set registers are used to describe various physical scenarios such as the construction and decommissioning of physical devices in otherwise empty laboratories, the dynamics of signal states over those detectors, the extraction of information from those states and multiple observers. We apply our quantum formalism to the Elitzur–Vaidman bomb-tester experiment and the Hardy paradox experiment.

Keywords: observers; contextuality; quantum mechanics; qubits; registers

1. Introduction

The nineteenth and twentieth centuries may be summarized as the centuries of non-relativistic classical science and relativistic quantum science, respectively. Judging by the scale of activity and progress in the field, the twenty-first century may well turn out to be the century of neural science. The hard problem of consciousness and its relationship to brain function is being systematically worn down using all the quantum technologies and theories developed over the last 100 years.

Central to this programme is the concept of the observer, the enigmatic ‘I’ of I think therefore I am. The problem is that, despite the many triumphs of quantum mechanics, the physics of the observer and observation is still not well understood. An important problem is that we are not sure what the correct way to model observers mathematically is. The exophysical perspective, which assumes that observers stand outside the space–time arena in which SUOs (systems under observation) exist, remains the dominant paradigm in all the sciences.

Reality is different however. The empirical facts are that actual observers are part of the physical universe and can be observed by other observers. As Feynman (1982) wrote, ‘... we have an illusion that we can do any experiment that we want. We all, however, come from the same universe, have evolved with it, and don’t really have any real freedom. For we obey certain laws and have come from a certain past.’

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This raises a question central to the interpretation of quantum mechanics: are observers just more complicated versions of SUOs, a view of reality known as the endophysical perspective, or are they fundamentally different altogether? This question has been called the endo-versus-exo debate.

Another debate of central importance to the theory of observation involves the conflict between the classical world view and the quantum world view. The former postulates that SUOs and their properties exist independently of any observers or observation, whereas the latter cannot make sense without them.

Regardless of how observers are defined and whether classical or quantum principles are involved, physicists generally believe that classical information in some form is extracted from SUOs in actual physics experiments. In all branches of science, their language reflects this belief. Experimentalists talk of measuring an electron’s spin or the mass of a new particle, and so on. While this point of view is of immense practical value, it may be a fundamental conceptual error. To quote Heisenberg (1927): ‘the orbit (of the electron) comes into being only when we observe it’. It is difficult to think of any idea further from the classical world view than that one.

The conceptual issues in quantum mechanics such as wave–particle duality, quantum interference and non-locality gave the first indication that all might not be well with this perspective. We need only to look at the photon concept to appreciate some of the problems with the idea that quanta are particles (Paul 2004).

Developments in neuroscience are reinforcing the need to rethink the universality of the exophysical perspective. If large collections of neurons are seen to act in a coherent fashion more typical of observers than SUOs, then the boundaries between endo and exo physics will have been eroded. Just when does a brain become an observer rather than an SUO?

What seems to be missing is a dynamical theory of observation which regards observers and SUOs more on the same footing. In such a theory, observers would be subject to the same laws of physics as the SUOs that they were observing, just as said by Feynman (1982). Such a theory would ideally be capable of accounting for the creation and annihilation of observers and their apparatus, because in the real world, nothing lasts forever.

We are a long way from having such a theory of observation, but various authors have made some interesting comments on this topic (Anastopoulos 2006). This paper outlines our current thoughts on the subject to date. These are based firmly on standard principles of quantum mechanics extended to cover the processes of observation. The most important idea is to discuss observation in terms of signal states of apparatus, referred to as labstates, rather than states of SUOs.

2. Contextuality

When we look more closely at the meaning of observation, some concepts begin to emerge as more fundamental than others. Perhaps the most important of all is contextuality, the idea that every statement is true or physically meaningful only relative to its context. Outside of that context, nothing can or should be said about the truth or validity of that statement.
Contextuality makes sense of Heisenberg’s remark about electron trajectories, quoted above. We cannot say that electron trajectories exist or otherwise if we do not observe them. We should say nothing in that context. Contextuality also helps settle the particle–wave issue in quantum mechanics: there is no context in which an SUO can appear to be completely a particle and completely a wave.

A statement is *absolute* if it is true regardless of context. A central assumption of the classical world view is that there are mechanical absolutes. Newton (1687) recognized the need to clarify this point and took care to make specific statements about Absolute Space and Absolute Time in his monumental book *The Principia*. The problem is that quantum mechanics has no mechanical absolutes: wave-functions are contextual.

Contextuality leads to our first principle of quantum observation:

**Principle I.** *There are no absolutes in physics.*

With a veto on the absolute, contextuality enforces a better mental attitude towards important contextual concepts in physics such as entropy and probability. Contextuality makes us wary of formalisms which treat quantum wave-functions in absolute terms, as in Bohmian mechanics (Bohm 1952), the Multiverse paradigm (Deutsch 1997) and some versions of decoherence theory (Zurek 2002).

Contextuality gives an answer to the Einstein–Podolsky–Rosen (EPR) debate (Einstein *et al.* 1935), which is essentially about observation in quantum mechanics. We have to reject the concept of ‘element of reality’, slipped into the EPR paper as a seductive appeal to reason. If we subscribe to that idea then we are being manipulated into accepting the classical view that SUOs can have absolute properties. Such thinking is inconsistent with principle I stated above. Contextuality and the concept of an *equivalence class* leads to a useful definition of what is meant by an SUO. Recall that an equivalence class is a subset of a set, such that all elements of that subset have some property or properties in common. Those properties provide the context for the definition of that equivalence class. Spekkens (2005) gives a clear discussion of these concepts as they relate to measurements.

In our approach, SUOs are discussed in terms of equivalence classes defined by the physics and context of a given observation, an approach used by Ludwig (1983) and Kraus (1983) in their work on the quantum measurement problem. We define *relative external context* to be those properties of and that information about the state of an SUO which is washed out, or redundant, in the definition of an equivalence class, while *relative internal context* refers to the defining properties of an equivalence class associated with a given experiment. For example, in the measurement of electron spin in the Stern–Gerlach experiment (Gerlach & Stern 1922), the momentum and position of the electron plus whatever is happening in the rest of the universe represent the relative external context whereas whether the electron is in its spin up or down state is internal context, relative to that particular experiment.

The equivalence class approach leads to a second principle of quantum observation, implicit in all experimental science:

**Principle II.** *Relative external context can be ignored in any experiment.*
In standard quantum mechanics, this principle applies to state preparation as well as outcome detection. Peres (1993) made the point that a prepared quantum state of a system carries no memory of how it was prepared. A point in favour of the equivalence class perspective is that it accords with the spirit of Heisenberg’s remark about electron trajectories quoted above, since it is clearly the apparatus which defines the equivalence classes being observed.

3. Bits and qubits

*Classical bits* are central to our approach, being used to represent the process of observation in its simplest possible form. A classical bit $B \equiv \{0, 1\}$ is a set with two distinct elements, called classical bit states, plus a context which gives those states a meaning. Bit states are used to label equivalence classes of SUOs for those situations where only two alternatives exist as far as the observer is concerned at that time.

In recent years, the quantum analogue of a bit, known as a qubit, has found many uses, particularly in quantum computation (Nielsen & Chuang 2000). A qubit is altogether a more complicated mathematical object than a bit. One important difference is that whereas bits are not vector spaces, qubits are complex vector spaces, which means that elements of a qubit can be multiplied by complex numbers and can be added together. Another important difference is that a qubit contains a zero vector, which a bit does not. We will use the power set of a bit to get around that particular point.

It will be useful to us later to briefly review some basic aspects of a qubit now. Elements of a qubit $Q$ and its dual $Q^*$ are denoted by ket $|\psi\rangle$ and bra $\langle\psi|$ vectors, respectively. Given orthonormal bases $\{|0\rangle, |1\rangle\}$ and $\{|0\rangle, \langle 1|\}$ for a qubit and its dual, respectively, we define the projection operators $p^0 \equiv |0\rangle\langle 0|$, $p^1 \equiv |1\rangle\langle 1|$ and the transition operators $a \equiv |0\rangle\langle 1|$, $a^+ \equiv |1\rangle\langle 0|$. These operators satisfy the product rules given in table 1.

4. Elementary signal detectors

Our strategy is not to think of the SUOs as if they were ‘there’ but to discuss only that information which an observer can extract from elementary signal detectors (ESDs). Each ESD has only two possible normal states, known as the ground state and the signal state, respectively. Whenever an observer looks at a normally functioning ESD, they will find it only in one of these two possible states, denoted by 0 for the ground state and 1 for the signal state.

Our approach assumes that any observation can be described in terms of collections of ESDs. How many ESDs are needed in any particular experiment will depend on context. Some experiments, particularly many in quantum optics, can be described with a relatively small number of ESDs, while other experiments may require enormous numbers. An example of the latter type of experiment is the double-slit experiment, where we would require very many ESDs to model all the positions on the detecting screen where a photon could be detected.
Table 1. The product table for the four basic qubit operators, where 0 represents the zero operator. Entries in the main square represent products of left-most column elements with top row elements in that order.

|   | $p^0$ | $p^1$ | $a$ | $a^+$ |
|---|-------|-------|-----|-------|
| $p^0$ | $p^0$ | 0     | $a$ | 0     |
| $p^1$ | 0     | $p^1$ | 0   | $a^+$ |
| $a$   | 0     | $a$   | 0   | $p^0$ |
| $a^+$ | $a^+$ | 0     | $p^1$ | 0  |

ESDs are not restricted to the detection of position in space at a given time. An ESD is any process of observation which will return either a yes or a no answer. In principle, this could involve a great deal of spatially extended physical equipment operated over relatively long periods of time. Recent experiments in quantum optics such as the quantum eraser (Walborn et al. 2002) and delayed choice (Jacques et al. 2007) experiments have reinforced the message that the process of observation in quantum mechanics can appear not to follow classical patterns of causality or locality (Kim 2003).

A typical experiment will involve a time-dependent collection of ESDs. It would normally be assumed that prior to any run (or repetition) of the experiment, each ESD would have been set in its characteristic ground state. This has nothing to do with energy. The ground state of an ESD is simply whatever condition the observer regards the ESD as having in the absence of a response to any external stimulus.

If subsequently during the act of observation an ESD were found still in its ground state, that would be taken as indicating that nothing had happened at that ESD. If on the other hand, an ESD were found in its signal state, then something must have happened there, such as a particle impacting on a detector.

There are two important caveats to this interpretation which play a crucial role in the formalism: (i) an ESD might not exist, or (ii) it might exist but be faulty or decommissioned. These will be discussed in detail below.

## 5. Bit power sets

We will identify the two possible normal signal states of a functioning ESD as the two elements of a bit. As we have mentioned, however, bits are not vector spaces and there seems to be no meaning to the addition of bit state 0 to bit state 1, or even of the multiplication of a bit state by a real or complex number.

There is in fact a way of defining bit state addition, of a kind, in terms of set theory. We recall that the power set $\mathcal{P}(S)$ of a set is the set of all possible subsets of $S$ including the empty set $\emptyset$ and $S$ itself. The power set $\mathcal{P}(B)$ of a bit $B$ therefore has four distinct elements: $\mathcal{P}(B) = \{ |0\rangle, |1\rangle, |B\rangle, |\emptyset\rangle \}$, where we define $|0\rangle \equiv \{0\}$, $|1\rangle \equiv \{1\}$, $|B\rangle \equiv \{0, 1\}$ and $|\emptyset\rangle \equiv \{\emptyset\}$. In this scheme, $|\emptyset\rangle$ is a non-trivial element of $\mathcal{P}(B)$ and counts as one element of the power set.
We shall work in terms of the elements of $\mathcal{P}(B)$ rather than with the elements of $B$ itself, identifying elements $|0\rangle$ and $|1\rangle$ of $\mathcal{P}(B)$ as synonymous with bit states 0 and 1 of $B$. The value of using $\mathcal{P}(B)$ rather than $B$ itself is that the elements of the former are sets, so we can use the set properties of union and intersection to make some interesting constructions analogous to those found in qubit theory.

6. Interpretation

Before we proceed further, we need to resolve the following problem: the power set $\mathcal{P}(B)$ of a bit $B$ appears to have too many elements. Logic suggests that only the elements $|0\rangle$ and $|1\rangle$ of the power set are actually needed. What can the elements $|B\rangle$ and $|\emptyset\rangle$ represent?

We turn to the physics of observation to answer this question. An observation of an ESD can be regarded as the acquisition of an answer to a binary question, such a question being one with a yes or no answer. For example, we could ask the question $Q_1 \equiv$ is this ESD in its signal state? If we looked and found it was in that state, the answer would be yes and so the state of the ESD would be represented by the element 1 of the corresponding bit. Conversely, if the ESD was not found in its signal state, we would normally assume it was in its ground state and therefore we would represent that situation by the element 0.

The matter is not as straightforward as it seems, however. A subtle issue can arise concerning two-valued logic as it applies to physics. Given an ESD, there are two related binary questions. One is $Q_1$ and the other is $Q_2 \equiv$ is this ESD in its ground state? Logically, we would assume $Q_1$ and $Q_2$ were conjugate questions, but physically this need not be true. In an experiment, we could not always be certain that an answer no to $Q_1$ implies an answer yes to $Q_2$, because $Q_1$ and $Q_2$ might be questions asked at different physical locations, as in the Stern–Gerlach experiment. We must be careful not to rely on unwarranted counterfactuality when dealing with quantum physics. We should adhere as much as possible to the following principle advocated by Wheeler (1978) and Peres (1993):

**Principle III.** An experiment not actually done does not count.

This principle needs to be used carefully. We are allowed, in fact required, to superpose quantum amplitudes from signal detector sources whenever we do not know which source is the real one, i.e. when we have no which-way information. This leads to the fourth principle of observation:

**Principle IV.** Quantum superposition occurs in the absence of classical which-way information.

The most well known and potent demonstration of this principle is found in Feynman’s path integral formulation of standard quantum mechanics (Feynman & Hibbs 1965). The same principle applies also to our formalism whenever the observer chooses not to observe the signal status of particular ESDs (Jaroszkiewicz 2008a,b,c).
Another fundamental point concerns the existence of the ESDs themselves. Consider what happens in a real laboratory in the execution of a given run of an experiment. Before any observation of an ESD could be made for that run, the observer would have had to make a decision to perform a reading on it. Suppose the observer did make such a decision but was unaware that the ESD never actually existed. In such a case, even if the observer had decided to make an observation, no possible answer 0 or 1 could be found. This scenario will be interpreted as corresponding to the empty set element $|\emptyset\rangle$ of the power set $\mathcal{P}(B)$. In other words, $|\emptyset\rangle$ represents the answer yes to the binary question is it true that this ESD does not exist? We shall call the element $|\emptyset\rangle$ the empty state.

With this possibility and the two ‘normal’ possibilities of an ESD being in its ground state or its signal state, we have accounted for three of the four elements of the power set $\mathcal{P}(B)$. We account for the fourth element $|B\rangle$ as follows. Suppose that the ESD did exist and was accessible to the observer but had a technical problem and gave unreliable readings. Not all physical equipment works perfectly all the time. The element $|B\rangle \equiv |0\rangle \cup |1\rangle = \{0,1\}$ will be taken to represent such a scenario. Essentially, any answer that the observer obtained when the ESD was in state $|B\rangle$ would be known to the observer to be uncertain or ambiguous and therefore unreliable. We shall call the element $|B\rangle$ the faulty state. Another interpretation of the faulty state is that the ESD may have been decommissioned for one reason or another and is no longer functioning as a normal ESD capable of transmitting information to other ESDs.

7. Union and intersection

Now that we have an interpretation of all four elements of the power set $\mathcal{P}(B)$ we can explore the consequences of this line of thinking.

The elements of the power set $\mathcal{P}(B)$ are sets themselves and therefore union and intersection are defined for them. These generate the rules of a Boolean algebra, with $|\emptyset\rangle$ playing the role of the Boolean element $\mathbb{O}$ and $|B\rangle$ playing the role of the Boolean element $\mathbb{I}$. In this context, union $\cup$ and intersection $\cap$ play the roles of the idempotent, associative and commutative operations normally denoted by $\lor$ and $\land$, respectively, in the theory of Boolean algebras. Every element $t$ of a Boolean algebra has a complement $\bar{t}$, such that $t \lor \bar{t} = \mathbb{I}$ and $t \land \bar{t} = \mathbb{O}$. In our case, $|\bar{\emptyset}\rangle = |B\rangle$, $|\bar{B}\rangle = |\emptyset\rangle$, $|\bar{0}\rangle = |1\rangle$ and $|\bar{1}\rangle = |0\rangle$.

8. Questions and answers

The idea that observation is the process of getting answers to certain questions gives an insight into the essential difference between SUOs and observers. SUOs do not ask questions whereas observers do.

We now apply this idea to bit power sets. We introduced the elements of a bit as representing the answers to a binary question. Given $|0\rangle \equiv \{0\}$ represents no and $|1\rangle \equiv \{1\}$ represents yes, we denote the particular binary question involved by $(1|B)$ and write $(1|0) = 0$, $(1|1) = 1$. In other words, $(1|$ is the
compound question: does this ESD exist, and if so, is it working normally, and if so, is it in its signal state? The bit state $|1$ returns a simultaneous yes to all three component sub-questions.

We noted above that the Boolean algebra of the power set $\mathcal{P}(B)$ consists of elements each of which has a complement. Likewise, binary questions have their complements. The complement of the question $(1)$ will be denoted by $(0)$, which is the binary question does this ESD exist, and if so, is it working normally, and if so, is it in its signal state? Then we have the relations $(0)(0) = 1$, $(0)(1) = 0$.

By analogy we may introduce the questions $(B)$ and $(\emptyset)$, which have corresponding properties. We interpret $(B)$ as the question does this detector exist and if so, is it faulty? In the case of the empty state, it is more convenient to ask about non-existence rather than existence, so we define $(\emptyset)$ as the question is it true that the detector does not exist?

The situation has now become more complicated than expected, because now we have four binary questions asked of four power set states. An extension of notation is called for. We define $(2) \equiv (B)$ and $(3) \equiv (\emptyset)$, while $(2) \equiv (B)$ and $(3) \equiv (\emptyset)$. Then all 16 question and answer relations are given by the rule

$$(i|j) = \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker delta.

We should comment further on relations (8.1). We emphasize that a non-existent ESD cannot actually give a physical signal. The statement $(0|3) = 0$ should be interpreted as ‘if an ESD does not exist at a place then it is not true that, if we looked, we would find a detector in its ground state at that place’. Similarly, the statement $(3|3) = 1$ is equivalent to ‘if an ESD does not exist at a place then it is true that, if we looked, we would not find an ESD at that place’.

At this stage the four questions $(i)$, $i = 0, 1, 2, 3$, look like the basis elements of a dual vector space $V^*$ while the four answers $(j)$, $j = 0, 1, 2, 3$, look like the basis elements of a vector space $V$. For this reason, we shall call the elements $(i)$ the duals of the $(i)$). However, there are significant differences which we cannot expound on here, except to say that questions do not have the same status as answers: there is usually a temporal ordering relative to the observer, with questions being asked before answers can be obtained.

Another difference between bit questions and answers is that while $(0)$ and $(1)$ can be thought of as mutual complements on account of their physical interpretation, the same is not obviously the case as far as the physics of $(2)$ and $(3)$ are concerned, apart from the fact that an answer ‘yes’ to either tells us that the ESD can be ignored for information extraction purposes.

9. Bit operators

A bit operator is any mapping from the power set $\mathcal{P}(B)$ back into the power set. Given an element $(s)$ of $\mathcal{P}(B)$ and a bit operator $O$ then we denote the value of the operator’s action on $(s)$ by $O(s)$. There is a total of $4^4 = 256$ different bit operators and only a few will be of use to us.
A useful way of representing bit operators is via matrices. The elements $|i\rangle$ of the power set $\mathcal{P}(B)$ may be represented by column matrices $[i]$ given by

$$
|0\rangle \equiv |0\rangle \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad |1\rangle \equiv |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ldots
$$

(9.1)

We represent the action of bit operator $O$ on $|i\rangle$ by the action of a bit matrix $[O]$ on a column matrix $[i]$, such that

$$
O|i\rangle \equiv |Oi\rangle \equiv [O][i] \equiv [Oi].
$$

(9.2)

In this matrix representation the dual elements $(i)$ are represented by the row matrices $(0| \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, (1| \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ldots$ This is consistent with the question and answer relations $(8.1)$.

We can use the bit matrix representation to define the operational meaning of the dyadics $|i\rangle(j|$. These can then serve as formal basis elements in the expansion of bit operators. Given a bit operator defined by equation (9.2), we can write it as the formal (dyadic) expression $O = \sum_{i=0}^{3} |Oi\rangle(i|$. Products of bit operators are defined in the natural way: given bit operators $O_1$ and $O_2$, we define their ‘product’ $O_2O_1$ by their action on any element $|i\rangle$ of the power set $\mathcal{P}(B)$ according to the rule $O_2O_1|i\rangle \equiv O_2(O_1|i\rangle)$. This product rule is associative but not commutative. To see this we note that products of two bit matrices are also bit matrices, with the operator $O_2O_1$ being represented in the matrix representation by the matrix product rule $[O_2O_1] = [O_2][O_1]$. The result follows because matrix multiplication is associative but not commutative.

The following bit operators turn out to be useful:

(i) The identity $I$ maps every element back into itself, i.e. $I|s_i\rangle = |s_i\rangle$. Its matrix elements are given by the Kronecker delta, viz. $[I]_{ij} = \delta_{ij}$.

(ii) The annihilator $Z$ maps any element $|i\rangle$ of the power set $\mathcal{P}(B)$ into the empty state $|\emptyset\rangle$, viz., $Z|i\rangle = |\emptyset\rangle$, $i = 0, 1, 2, 3$, so its matrix representation is

$$
[Z] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}.
$$

(9.3)

The annihilator has a fundamental role in our theory: it represents the process of destroying and removing all traces of an already existing ESD.

(iii) The relative projection operators $P^0$ and $P^1$ have the action

$$
P^0|0\rangle = |0\rangle, \quad P^0|B\rangle = |\emptyset\rangle, \quad P^1|0\rangle = |\emptyset\rangle, \quad P^1|B\rangle = |\emptyset\rangle,$$

$$
P^0|1\rangle = |\emptyset\rangle, \quad P^0|\emptyset\rangle = |\emptyset\rangle, \quad P^1|1\rangle = |1\rangle \quad \text{and} \quad P^1|\emptyset\rangle = |\emptyset\rangle,
$$

(9.4)

so their matrix representations are

$$
[P^0] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix} \quad \text{and} \quad [P^1] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1
\end{bmatrix}.
$$

(9.5)
Table 2. The product table for the four basic bit operators, where $Z$ represents the annihilator. Entries in the main square represent products of left-most column elements with top row elements in that order.

|   | $P^0$ | $P^1$ | $A$ | $\tilde{A}$ |
|---|---|---|---|---|
| $P^0$ | $P^0$ | $Z$ | $A$ | $Z$ |
| $P^1$ | $Z$ | $P^1$ | $Z$ | $\tilde{A}$ |
| $A$ | $Z$ | $A$ | $Z$ | $P^0$ |
| $\tilde{A}$ | $\tilde{A}$ | $Z$ | $P^1$ | $Z$ |

These operators are idempotent, viz. $P^0 P^0 = P^0$, $P^1 P^1 = P^1$ and orthogonal, viz. $P^0 P^1 = P^1 P^0 = Z$. In this context, the annihilator $Z$ plays the role of a zero element.

(iv) The signal creation and signal annihilation operators $\tilde{A}$, $A$ are defined principally by their action on the normal states $|0\rangle$ and $|1\rangle$:

$$
\begin{align*}
A|0\rangle &= |\emptyset\rangle,
A|B\rangle &= |\emptyset\rangle,
\tilde{A}|0\rangle &= |1\rangle,
\tilde{A}|B\rangle &= |\emptyset\rangle,
A|1\rangle &= |0\rangle,
A|\emptyset\rangle &= |\emptyset\rangle,
\tilde{A}|1\rangle &= |\emptyset\rangle \quad \text{and} \quad \tilde{A}|\emptyset\rangle &= |\emptyset\rangle,
\end{align*}
$$

which gives the matrix representations

$$
[A] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad [\tilde{A}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.
$$

These operators are nilpotent, viz. $AA = \tilde{A}\tilde{A} = Z$, with $Z$ once again playing the role of a zero element.

The product rules for the operators $P^0$, $P^1$, $A$ and $\tilde{A}$ are given in table 2. Comparison with table 1 shows that these tables are isomorphic, provided the zero operator in table 1 is identified with the annihilator $Z$ in table 2.

(v) The construction operator $C$ acts on every element $|i\rangle$ of the power set and sets it to the ground state in readiness for observation, i.e. $C|i\rangle = |0\rangle$, $i = 0, 1, 2, 3$. There are two scenarios. If the bit is in its empty state then its ESD does not exist, so the action of the construction operator represents the physical construction of a standard ESD in its ground state in the laboratory, prior to any experiment. It is assumed that facilities exist in the laboratory for this. Alternatively, if the ESD already exists, then the construction operator resets it to its ground state if it is normal or repairs it and sets it to its ground state if it is faulty. This operator is represented by the matrix

$$
[C] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$
(vi) The **decommissioning operator** $D$ represents the action of decommissioning an already existing ESD, setting it into its faulty state $|2\rangle$. This operator does not reset states $|0\rangle$, $|1\rangle$ and $|2\rangle$ to the non-existence state $|3\rangle$ because in the real world, there will invariably be some remaining information in the form of *debris* which will inform the observer that apparatus has been decommissioned. This is an important feature of our discussion towards the end of this paper of the Elitzur–Vaidman bomb-tester experiment and Hardy’s paradox experiment.

The decommissioning operator is represented by the matrix

$$
[D] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

(9.9)

**10. Discrete time and space**

Classical mechanics generally assumes that SUOs move around continuous space continuously, but we take a different view. How observers interact with SUOs is always described in discrete terms, because real experiments always involve data extraction occurring at discrete times on finite numbers of ESDs. There are no truly continuous observations, either spatially or temporally. Some experiments do deal with continuous variables, such as temperature, but this is conceptually very different to what is being discussed here and will not be considered further in this article.

We do not refer to Hamiltonians or continuous unitary evolution either. Quantum dynamical evolution is discussed in terms of mappings from one quantum register to another and the unitary evolution operators of standard quantum mechanics are replaced by semi-unitary operators Jaroszkiewicz (2008a,b,c).

A typical experiment described by our formalism involves an observer interacting with a time-dependent number $r_n$ of ESDs at a countable number of times $t_n$, where the integer $n$ runs from some initial integer $M$ to some final integer $N > M$. There is no need to assume that $t_{n+1} - t_n$ always has the same value, or that $r_n$ is independent of $n$. The formalism allows for the creation and destruction of ESDs, something which happens in the real world.

**11. The laboratory and the universal register**

In our approach it is assumed that an observer exists in a physical environment referred to as the **laboratory**, $L$. This will have the facilities for the construction or introduction of apparatus consisting of a number of ESDs. At any given discrete time $n$, the observer will associate a state known as the **labstate** to the collection of ESDs at that time. This state could be a pure state or a mixed state. We shall restrict our attention in this paper to pure labstates for reasons of space.

A labstate carries information as to whether various ESDs exist in the first place and, if so, whether they are functioning normally and either in their ground or signal states, or whether they are faulty.
The power set approach to ESDs allows us to think of an absence of an ESD in \( \Lambda \) as an observable fact which can be representable mathematically. The state corresponding to an absent or non-functioning ESD is represented by the element \(|\emptyset\rangle\) of its associated power set. Therefore, we can represent a complete absence of any ESDs whatsoever by an infinite collection of such elements. This corresponds to an observer without any apparatus, i.e. an empty laboratory. We denote this labstate by the symbol \(|\Omega\rangle\) and call it the information void, or just the void. It represents a potential for existence, relative to a given observer.

If the observer’s laboratory \( \Lambda \) is in its void state \(|\Omega\rangle\), that does not mean that the laboratory \( \Lambda \) or the observer do not exist, or that there are no SUOs in \( \Lambda \). It means simply that the observer has no current means of acquiring any information. An empty laboratory devoid of any detectors is a physically meaningful concept, but one with no interesting empirical content.

The information void can be thought of as one element in an infinite set called the universal register \( \mathcal{O} \), the Cartesian product of an infinite number of bit power sets. We write

\[
|\Omega\rangle \equiv \prod_{i}^{\infty} |\emptyset_{i}\rangle \in \mathcal{O} \equiv \prod_{i}^{\infty} \mathcal{P}(\mathcal{B}_{i}),
\]

where the index \( i \) could in principle be discrete, continuous or a combination of both. The cardinality of the universal register as a measure of how many power sets belong to it may be assumed to be infinity, but precisely what sort of Cantorian cardinality it should be is not clear. If we think in terms of \( \Lambda \) sitting in continuous space, then we expect at least the cardinality, \( \mathfrak{c} \), of the continuum. However, that is a metaphysical statement, because there would not be enough energy in the universe to create a continuum of ESDs\(^1\). Principle II comes to our aid here. Real observers can only ever deal with finite numbers of ESDs in practice and by principle II we can generally ignore all potential ESDs.\(^2\)

The product notation in equation (11.1) is not essential but has been chosen to reflect the relationship between collections of power sets and the tensor products of qubit spaces that we encounter in the quantum version of this approach, discussed later. In our products, ordering is not significant, since labels keep track of the various terms. An arbitrary classical labstate \(|\Psi\rangle\) in the universal register \( \mathcal{O} \) will be of the form \( \prod_{i}^{\infty} |s_{i}\rangle \), where \(|s_{i}\rangle\) is one of the four elements of \( \mathcal{P}(\mathcal{B}_{i}) \).

Operators acting on universal register states will be denoted in blackboard bold font and act as follows. If \( O_{i} \) is a bit operator acting on elements of \( \mathcal{P}(\mathcal{B}_{i}) \), then \( \mathcal{O} \equiv \prod_{i}^{\infty} O_{i} \) acts on an arbitrary classical state \(|\Psi\rangle \equiv \prod_{i}^{\infty} |s_{i}\rangle\) according to the rule

\[
\mathcal{O}|\Psi\rangle \equiv \prod_{i}^{\infty} O_{i}|s_{i}\rangle.
\]  

For every classical register state \(|\Psi\rangle \equiv \prod_{i}^{\infty} |s_{i}\rangle \) there will be a corresponding dual register state \((\Psi| \equiv \prod_{i}^{\infty} (s_{i}|)\) where \((s_{i}|\) is dual to \(|s_{i}\rangle\). Classical register states

\(^1\)In other words, the universe cannot observe itself completely.

\(^2\)An absence of an ESD could be significant in some circumstances, however.
including the void satisfy the orthonormality condition

$$\la \Phi | \Psi \ra \equiv \prod_i^{\infty} (r_i) \prod_j^{\infty} |s_j\ra = \prod_i^{\infty} (r_i | s_i\ra) = \prod_i^{\infty} \delta_{r_i s_i}.$$

Classical register states $|\Phi\ra$ and $|\Psi\ra$ which differ in at least one bit power set element therefore satisfy the rule $(\Phi | \Psi) = (\Psi | \Phi) = 0$.

12. Contextual vacua

In conventional classical mechanics or Schrödinger–Dirac quantum mechanics, empty space is generally not represented by any specific mathematical object. In quantum field theory, however, empty space is represented by the vacuum, a normalized vector in an infinite dimensional Hilbert space. It has physical properties such as zero total momentum, zero total electric charge, etc., which, although bland, are physically significant attributes nevertheless.

In our approach we encounter an analogous concept. Starting with the void $|U\ra$, we represent the construction of a collection of ESDs in the laboratory $\Lambda$ by the application of a corresponding number of construction operators $C_i$ to their respective empty states $|\emptyset_i\ra$. For example, a labstate consisting of a single ESD $i$ in its ground state is given by $|\Psi\ra = C_i |\Omega\ra = (\prod_{j \neq i}^{\infty} |\emptyset_j\ra) \times |\emptyset_i\ra$, where $C_i$ is the register operator $C_i = (\prod_{j \neq i}^{\infty} I_j) \times C_i$. More generally, a state consisting of a number $r$ of ESDs each in its ground state is given by

$$|\Psi^r\ra = C_1 C_2 \ldots C_r |\Omega\ra,$$

where without loss of generality we label the ESDs involved from 1 to $r$. Such a state will be said to be a rank-$r$ ground state, or contextual vacuum state.

We can now draw an analogy between the vacuum of quantum field theory and the rank-$r$ ground states in our formalism. The physical three-dimensional space of conventional physics would correspond to a ground state of extremely large rank, if physical space were relevant to the experiment. This would be the case for discussions involving particle scattering or gravitation, for example. For many experiments however, such as the Stern–Gerlach experiment and quantum optics networks, physical space would be considered part of the relative external context and therefore could be ignored for the purposes of those experiments. It all depends on what the observer is trying to do.

In the real world there is more than one observer, so a theory of observation should take account of that fact. That is readily done in our theory. For example, the ground state for two or more distinct observers for which some commonality of time had been established would be represented by elements in $\mathcal{O}$ of the form

$$|\Psi^{1,2}\ra \equiv C_1^{1} C_2^{2} \ldots C_1^{r_1} C_2^{r_2} \ldots C_1^{r_1} C_2^{r_2} |\Omega\ra,$$

and so on, where superscripts refer to the different observers. If subsequent dynamics was such that the ESDs of observer 1 never sent signals to those of observer 2 and vice versa, then to all intents and purposes we could discuss each observer as if they were alone. If on the other hand some signals did pass between them, then that would be equivalent to having only one observer.
If no commonality of time or other context has been established between the observers, then there can be no physical meaning to equation (12.2). This is an important point in cosmology, where there are frequent discussions about multiple universe ‘bubbles’ beyond the limits of observation. The mere fact that astronomers have received light from extremely distant galaxies establishes a context between the signal preparation ESDs associated with those galaxies and the ESDs associated with the astronomers now and validates the use of General Relativity for those regions of spacetime. If no such signals have been received, then there is no such context. Therefore, relative to astronomers today, the universe beyond the horizon of observation can be meaningfully represented only by the information void, not the spatial vacuum. Something may be there, but we should not discuss it as if we had access to any form of information about it, such as its spacetime structure.

Much the same concern must be raised about the loss of information question in black hole physics. The answer to that question can come only from a careful understanding of the contextual relation between observers outside the critical radius and those that were assumed to be inside it.

13. Experiments

Long before any experiment can begin, the observer starts off with a laboratory $L$ in its void state $|\Omega\rangle$. Then at some time $t_{-1}$ before any runs can be taken, specific apparatus consisting of a finite number $r$ of ESDs has to be constructed in $L$. We will assume without loss of generality that these are all functioning normally and in their ground state, so the labstate $|\Psi, t_{-1}\rangle$ at that point is given by the right-hand side of equation (12.1). All of this is necessary before state preparation.

According to what we said earlier, external context involving ESDs in their empty state can be ignored. Therefore, we need only discuss those ESDs which subsequently are in states $|0\rangle$, $|1\rangle$ or $|\mathcal{B}\rangle$. A further simplification is that in real experiments, observers generally filter out observations from faulty ESDs (assuming these have been identified) by post-selecting only those labstates which contain the normal bit states $|0\rangle$ or $|1\rangle$. We shall confine our attention to such normal labstates until we deal with applications to quantum mechanics.

Given this condition, we can restrict our discussion at any given time $t_n$ to the physical register $\mathcal{R}_n$, a subset of the universal register $\mathcal{O}$ consisting of $2^{r_n}$ normal states, each of the form

$$|i_1 i_2 \ldots i_{r_n}\rangle \equiv |\tilde{i}_1 \rangle |\tilde{i}_2 \rangle \ldots |\tilde{i}_{r_n} \rangle,$$

(13.1)

where $i_j = 0$ or 1 for $j = 1, 2, \ldots, r_n$, such that $|\tilde{i}_j \rangle$ is in $\mathcal{P}(\mathcal{B}_n^j)$. The physical register $\mathcal{R}_n$ represents all those ESDs in the laboratory $L$ at time $n$ which exist and are not faulty.

Given the set of bits $\mathcal{B}_n^j \equiv \{|0_j\rangle, |1_j\rangle\}$ associated with $\mathcal{R}_n$, the label $j$ gives an ordering, so $\mathcal{R}_n$ can be regarded as the Cartesian product $\mathcal{B}_n^1 \times \mathcal{B}_n^2 \times \cdots \times \mathcal{B}_n^{r_n}$. When we come to discuss quantization towards the end of this paper,
every labstate (13.1) in $\mathcal{R}_n$ will be identified with a qubit tensor product state $|i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_n\rangle$, an element of the preferred basis for the associated qubit register.

The notation (13.1) will be referred to as the occupancy notation, as the integers $i_j$ can be interpreted as the answer to the question whether the $j$th ESD $\Delta_j$ contains a signal or is in its ground state. An occupancy value 0 means $\Delta_j$ is in its ground state while the occupancy value 1 means that $\Delta_j$ is in its signal state. These states satisfy the orthonormality conditions $(i_1 i_2 \ldots i_n |j_1 j_2 \ldots j_n) = \delta_{i_1 j_1} \delta_{i_2 j_2} \ldots \delta_{i_n j_n}$.

We define the signality of a given state in $\mathcal{R}_n$ to be the number of ones in the occupancy representation of that state. For example, the state $|00101101\rangle$ is a signality-four state in a rank-8 physical register. Signality allows us to partition the $2^{rn}$ states in $\mathcal{R}_n$ into a number of signal classes $S^0, S^1, \ldots, S^{rn}$. These are equivalence classes of states in $\mathcal{R}_n$ defined by the same signality.

Signality has physical significance. The zero-signal class $S^0$ consists of one state only, the ground state $|000\rangle$ of the physical register. States in the one-signal class $S^1$ correspond to what would normally be called a one-particle state, states in $S^2$ correspond to two-particle states, and so on.

There is a total of $r_n + 1$ distinct signal classes. The $d$th-signal class $S^d$ contains $C^r_d \equiv r_n! / d!(r_n - d)!$ distinct states. The $r_n$-signal class $S^{r_n}$ consists of only one state, the fully saturated state $|1_1 \rangle |1_2 \rangle \ldots |1_r \rangle$.

Given a rank-$r$ physical register $\mathcal{R}_r \equiv B^1 B^2 \ldots B^r$ we define the $r$ signal creation operators

$$\tilde{A}_i \equiv \left\{ \prod_{j \neq i} I_j \right\} \times A_i, \quad 1 \leq i \leq r,$$

and the $r$ signal annihilation operators

$$\tilde{A}_i \equiv \left\{ \prod_{j \neq i} I_j \right\} \times A_i, \quad 1 \leq i \leq r.$$

Then an application of the operator $A_i$ on the contextual rank-$r$ ground state $C_1 C_2 \ldots C_r |\Omega\rangle$ gives the rank-$(r-1)$ ground state

$$A_i C_1 C_2 \ldots C_r |\Omega\rangle = C_1 C_2 \ldots C_{i-1} C_{i+1} \ldots C_r |\Omega\rangle.$$

This is a non-zero labstate in $\tilde{\sigma}$, but is not an element of the original physical register $\mathcal{R}_r \equiv B^1 B^2 \ldots B^r$. What has happened is analogous to the convention qubit register result $a_i(|0_1 \rangle \otimes |0_2 \rangle \otimes \ldots \otimes |0_r \rangle) = 0$. In our case, we do not get zero, but the equivalent of it: the action of $A_i$ on the ground state $C_1 C_2 \ldots C_r |\Omega\rangle$ of $\mathcal{R}_r$ maps it into the ground state of a different physical register, one of rank $r - 1$, i.e. into a state orthogonal to every state in $\mathcal{R}_r$. The signal creation operators $\tilde{A}_i$ can be used to create the various signal classes discussed above, as follows. We start from the signality-zero class $S^0$, which consists of no application of any $\tilde{A}_i$ to the contextual ground state $|0\rangle \equiv C_1 C_2 \ldots C_r |\Omega\rangle$. 

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The signality-one class $S^1$ consists of states of the form $|2^{i-1}\rangle \equiv \bar{A}_i|0\rangle$ for $i = 1, 2, \ldots, r$, the signality-two class consists of all states of the form $|2^{i-1} + 2^{j-1}\rangle \equiv \bar{A}_i\bar{A}_j|0\rangle$ for $1 \leq i < j \leq r$, and so on. Finally, the signality-$r$ signal class consists of the single state $|2^r - 1\rangle \equiv \bar{A}_1\bar{A}_2\ldots\bar{A}_r|0\rangle$.

In the following discussion, we shall use the notation $|k\rangle$, $k = 0, 1, \ldots, 2^r - 1$ for the $2^r$ states in $\mathcal{R}^r$ and refer to it as the computational basis.

14. Classical particle signal mechanics

In this section, we shall restrict our attention to classical mechanics, to illustrate how our approach to observation can apply in that context. Quantum mechanics is discussed after that.

We consider now a physical register $\mathcal{R}^r$ of sufficiently large fixed rank $r$ such that it can serve as a model for a region of classical physical space over which particles can move. In this approach, particle motion is discussed in terms of the tracking of signals from a vast collection of ESDs over time. A particularly useful feature of this approach is that signal need not be conserved, which means that classical particle creation and annihilation is readily incorporated into the formalism.

There are several distinct forms of temporal evolution which could be discussed in such a scenario; the laws of mechanics for a given SUO could change with time or not, the SUO could be autonomous or interact with external forces, and signal could be conserved or not. We shall restrict our attention to autonomous SUOs with time-independent laws of dynamics, as these are generally of most interest. In principle, there should be no problem in dealing with other forms of dynamics, including those where the rank of the physical register changes with time. We could also deal with classical stochastic mechanics, which would incorporate Bayesian principles in a natural way.

In the following, all states are elements in the universal register $\mathcal{O}$ which also belong to $\mathcal{R}^r$, i.e. they represent the labstates of a fixed collection of normal ESDs, each of which can be found only in either its ground state or signal state.

We shall use the computational basis $\{|k\rangle : k = 0, 1, \ldots, 2^r - 1\}$ to represent the $2^r$ states in $\mathcal{R}^r$. Consider the temporal evolution of a system from state $|k\rangle$ at time $t$ to state $|Uk\rangle$ after one elementary time-step, where $k$ and $Uk$ are integers in the interval $[0, 2^r - 1]$. Denoting this transition as the action of some temporal evolution operator $U$ acting on the initial state $|k\rangle$, we write $U|k\rangle = |Uk\rangle$, $0 \leq k, Uk < 2^r$.

For a given $k$, there are in principle $2^r$ possible states $|Uk\rangle$ into which it could be mapped, and because there are $2^r$ values of $k$, we conclude that for a rank-$r$ classical register, there are $(2^r)^2^r$ distinct possible evolution operators in this form of mechanics. Even for very low rank physical registers, the number of possible operators is impressive. For example, a rank-2 register can have 256 different forms of autonomous, time-independent dynamics while a rank-3 register has $8^8 = 16,777,216$ different forms.

Most of the possible evolution operators over a physical register will not be useful. Many of them will correspond to irreversible and/or unphysical dynamical evolution and only a small subset will be of interest. We need to find some principles to guide us in our choice of evolution operator.
Recall that in standard classical mechanics, Hamilton’s equations of motion lead to Liouville’s theorem. This tells us that as we track a small volume element along a classical trajectory, this volume remains constant in magnitude though not necessarily constant in shape or orientation. This leads to the idea that a system of many non-interacting particles moving along classical trajectories in phase-space behaves like an incompressible fluid, such a phenomenon being referred to as a Hamiltonian flow.

An important characteristic of Hamiltonian flows is that flow lines never cross. We shall encode this idea into our approach to signal mechanics. There are two versions of this mechanics, one of which does not necessarily conserve signality while the other does. We consider the first one now.

\( (a) \) Permutation flows

The physical register \( \mathcal{R}^{r} \) contains \( 2^{r} \) labstates denoted by \( |k \rangle \), \( k = 0, 1, 2, \ldots, 2^{r} - 1 \). Consider a permutation \( P \) of the integers \( k \), such that under \( P \), \( k \rightarrow Pk \in [0,2^{r} - 1] \). Define the evolution of the labstate \( |k \rangle \) over one time step by \( |k \rangle \rightarrow U |k \rangle = |P(k) \rangle \). Such a process is reversible and will be referred to as a permutation flow.

There is a total of \( n! \) distinct permutations of \( n \) objects, so there are \((2^{r})!\) possible distinct permutation flow processes. For large \( r \), the number of permutation flows is a rapidly decreasing fraction of the number \((2^{r})^{2^{r}}\) all possible forms of register processes.

\( (b) \) Signal conserving flows

Most permutation flows will not conserve signality. We can readily identify the subset of the permutation flows which do conserve signality by using the occupancy notation. Consider a physical register state \( |\Psi_{n} \rangle \) at time \( t_{n} \) given by \( |\Psi_{n} \rangle = |i_{1}i_{2}\ldots i_{r} \rangle \) in the occupancy notation, where \( i_{j} = 0 \) or else 1 for \( 1 \leq j \leq r \).

Now let \( P^{*} \) be some permutation of the numbers 1, 2, \ldots, \( r \) and write \( P^{*}j \) to represent the number that \( j \) changes to under this permutation. Now suppose that \( |\Psi_{n} \rangle \) evolves into the labstate \( |\Psi_{n+1} \rangle \) at time \( t_{n+1} \) given by

\[
|\Psi_{n} \rangle \rightarrow |\Psi_{n+1} \rangle \equiv \mathbb{U}|\Psi \rangle = |i_{p_{*1}}i_{p_{*2}}\ldots i_{p_{*r}} \rangle. \tag{14.1}
\]

To determine the new occupancy of the \( j \)th bit, we just look at the occupancy of the \((P^{*}j)\)th bit. This may be summarized as the dynamical rule \( i_{j} \rightarrow i_{j}^{*} \equiv i_{P^{*}j} \).

We shall call this form of signal mechanics signal permutation dynamics.

In this form of dynamics, signality is automatically conserved. Another way of seeing this is to use the signal creation operators and note that if \( |\Psi_{n} \rangle \) has signality \( d \), then we can write \( |\Psi_{n} \rangle = \tilde{A}_{j_{1}}\tilde{A}_{j_{2}}\ldots\tilde{A}_{j_{d}}|0 \rangle \), where \( 1 \leq j_{1} < j_{2} < \cdots < j_{d} \leq r \). Then under the above permutation \( P^{*} \) of the integers 1, 2, \ldots, \( r \) the new state at time \( t_{n+1} \) takes the form

\[
|\Psi_{n+1} \rangle \equiv \mathbb{U}|\Psi_{n} \rangle = \tilde{A}_{P^{*}j_{1}}\tilde{A}_{P^{*}j_{2}}\ldots\tilde{A}_{P^{*}j_{d}}|0 \rangle. \tag{14.2}
\]

Then clearly signality is conserved.
The total number of distinct permutations of \( r \) objects is \( r! \), so there are that many distinct forms of signal permutation dynamics for a rank-\( r \) classical register. Since there are \((2^r)!\) distinct forms of permutation dynamics, the set of signal permutation dynamics forms a rapidly decreasing fraction of the set of all possible permutation dynamics.

Permutation flows have a number of features which have analogues in standard classical mechanics. First, permutation flows are reversible. Given a permutation \( P \), then its inverse \( P^{-1} \) always exists, because permutations form a group.

Another feature of permutation dynamics is the existence of orbits or cycles. A permutation of \( 2^r \) objects will in general contain cycles, which are subsets of the objects such that only elements within a given cycle replace each other under the permutation. This is relevant here because we have chosen to discuss time-independent autonomous systems, the evolution of which is given by repeated applications of the same permutation. Therefore, the structure of the cycles does not change and so each cycle consisting of \( p \) elements has a dynamical period \( p \). For example, the identity permutation gives a trivial form of mechanics where nothing changes. It has \( 2^r \) cycles each of period 1. At the other end of the spectrum, the permutation denoted by \((0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \rightarrow 2^r - 1 \rightarrow 0)\) has no cycles except itself and has period \( 2^r \). Therefore, any physical register evolving under time independent, autonomous permutation mechanics must return to its initial labstate no later than after \( 2^r \) time steps. This is the analogue of the Poincaré (1890) recurrence theorem.

15. Evolution and measurement

Any experiment consists of several distinct phases. We have discussed the creation of the apparatus and the evolution of the labstates. Now we turn to the process of measurement itself, which denotes the extraction of classical information from an SUO. Typically this information will be in the form of real numbers, and these can always be expressed in binary form, justifying our approach.

Context plays a vital role here. When for example an observer reports that a particle has been observed at position \( x = 1.5 \), what they mean is that positive signals have been detected at some normal ESD or ESDs associated with the number \( x = 1.5 \). This assignment is based on the context of the experiment: the observer will know on the basis of prior theoretical knowledge what those ESDs mean in terms of the physics of the SUO concerned, and therefore, what ‘values’ of some measurable quantity they represent.

So far, we have discussed the evolution on labstates. For each run or repetition of the experiment, this is modelled by the action of an evolution operator \( \mathbb{U}_N \) mapping initial labstates at time \( t_0 \) into final labstates at time \( t_N \). We need now to discuss how numbers are extracted at the end of an experiment consisting of a number of runs.

With reference to the position measurement discussed immediately above, we model the measurement process in terms of weighted relevant questions. What this means is this. Suppose the final physical register \( \mathcal{R}_N \) has rank \( r_N \). Assuming the experimentalist has established that each ESD is normal, then there will be a total of \( d_N \equiv 2^{r_N} \) possible normal labstates in this register. Therefore, the observer could ask a total of \( d_N \) normal questions. These questions are represented by the
dual labstates \{ \{ k \}: k = 0, 1, \ldots, d_N - 1 \}. Given a final labstate $|\Psi_N\rangle$, the answer yes or no to each question $|k\rangle \equiv \text{‘is it true that } |\Psi_N\rangle \text{ is } |k\rangle \text{?’}$ is represented by the number one or zero, respectively, and given by the bracket $(k|\Psi_N\rangle$.

Now the observer will generally have some theory as to what each answer $|k\rangle$ means physically. In many experiments, this will be some real number $X_k$. Therefore, the actual number $(X)_\Psi N$ obtained at time $t_N$ at end of a single run of the experiment can be written in the form:

$$
\langle X \rangle_\Psi N = (\Psi_N|X_N|\Psi_N),
$$

(15.1)

where $X_N \equiv \sum_{k=0}^{d_N-1} |k\rangle X_k(k)$ is an observable, a sum of dyadics representing a weighted relevant question.

Two comments are relevant. First, despite appearances, this is still a classical theory at this point. The final labstate $|\Psi_N\rangle$ is a single element in the final physical register, $R_N$, not a superposition of elements. Second, there is nothing in classical mechanics which rules out weighted sums of dyadics. For any element in $R_N$, all the possible answers $(k|\Psi_N\rangle$ are zero except for one of them, so equation (15.1) returns a physically sensible value for $(X)_\Psi N$.

A further refinement, anticipating the possibility of random variations in the starting with initial labstate $|\bar{\Psi}_0\rangle$. In such a case, a statistical approach can be taken.

We note that $|\Psi_N\rangle = \mathbb{U}_{N,0}|\Psi_0\rangle$ and $(\Psi_N| = (\Psi_N|\mathbb{U}_{N,0}$, where the evolution operator $\mathbb{U}_{N,0}$ maps elements of $R_0$ into elements of $R_N$ and similarly for the dual evolution operator $\mathbb{U}_{N,0}$. In general, it will be true that

$$
\mathbb{U}_{N,0}\mathbb{U}_{N,0} = \mathbb{I}_{R_0},
$$

(15.3)

the identity operator for $R_0$. However, because there is no requirement formally in this approach for the rank $r_N$ of the final physical register $R_N$ to equal the rank $r_0$ of the initial physical register $R_0$, it is possible that $\mathbb{U}_{N,0}\mathbb{U}_{N,0}$ does not equal $\mathbb{I}_{R_N}$. This corresponds to irreversible dynamics. In the analogous quantum formalism we have developed (Jaroszkiewicz 2008a, b, c), such evolution operators are referred to as semi-unitary.

Using equation (15.3) in equation (15.2), we may write

$$
\langle X \rangle_\Psi N = \text{Tr}(X_N\mathbb{U}_{N,0}\rho_0\mathbb{U}_{N,0}),
$$

where $\rho_0$ is the initial dyadic $|\Psi_0\rangle(|\Psi_0\rangle$.

(a) Random initial states

Real experiments normally consist of a large number of repetitions or runs of a basic process. However, it cannot always be guaranteed that the initial labstate is always the same. In principle, we could start with any one of $d_0 = 2^n$ initial labstates. In such a case, a statistical approach can be taken.

Consider a very large number $R$ of runs, such that there is a total of $R_k$ runs starting with initial labstate $|k\rangle$, for $k = 0, 1, \ldots, d_0 - 1$. Clearly, $\sum_{k=0}^{d_0-1} R_k = R$. Then in the limit of $R$ tending to infinity, we would assign a probability $\omega_k \equiv \lim_{R \to \infty} R_k/R$ for the initial labstate to be in state $|k\rangle$. 

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In such a scenario we define the initial density matrix $\rho_0 \equiv \sum_{k=0}^{d_0-1} \omega_k |k,0\rangle \langle k,0|$, where $|k,0\rangle$ is any one of the $d_0$ elements of the initial physical register $R_0$ and the $\omega_k$ are probabilities summing to unity. The formalism outlined above then gives the expectation values of operators.

16. Quantization

The formalism we have developed is readily extended to the quantum scenario, for which different principles hold concerning the interpretation and usage of the physical register. In this scenario, a rank $r_n$ physical register $R_n$ at time $n$ is identified with a preferred orthonormal basis $\{|k\rangle: k = 0, 1, \ldots, d_n - 1\}$ for a quantum register $Q_n \equiv Q^1 \otimes Q^2 \otimes \ldots \otimes Q^n$, where now the $Q^i$ are qubits. This register is a Hilbert space, the tensor product of $r_n$ qubits, each of which is identified with one ESD. Elements of this register are the labstates of interest and these can be multiplied by complex numbers and added together, unlike the classical scenario. Most of the formulae developed in the previous section can be taken wholesale into the quantum scenario. For example, equation (15.1) now corresponds to the expectation value of Hermitian operator $\hat{X} \equiv \sum_{k=0}^{d_N-1} |k\rangle X_k(k)$ relative to the normalized pure quantum state $|\Psi\rangle = \sum_{k=0}^{d_N-1} \psi_k|k\rangle$, where $\sum_{k=0}^{d_N-1} |\psi_k|^2 = 1$.

In applications to quantum optics, this approach has been extended to include SUO attributes such as internal spin (Jaroszkiewicz 2008b, c) in a generalization of the Ludwig–Kraus POVM formalism, which extends the work of von Neumann (1955) on quantum measurement. Properties such as spin and electric charge, conventionally interpreted as objective properties of SUO states, are encoded as contextual properties of ESDs. This is a realization of one of the aims of Feynman’s thesis, in which he wrote (Brown 2005): ‘and all of the apparent quantum properties of light and the existence of photons may be nothing more than the result of matter interacting with matter directly, and according to quantum mechanical laws’.

In the next two sections we show how our formalism describes experiments where the apparatus changes in one way or another during the experiment. In particular, the faulty state plays an important role in these experiments.

17. The Elitzur–Vaidman bomb-tester experiment

In this experiment, a stockpile of active (A) and dud (D) bombs is analysed, one by one, in order to find as many unexploded active bombs as possible. The approach follows that discussed in Elitzur & Vaidman (1993). In the schematic Mach-Zehnder circuit shown in figure 1, each numbered circle represents an ESD, or place where the observer could extract information in principle. The rectangles represent beam-splitters and the solid bars represent mirrors. Our convention is that a reflected beam undergoes a phase change of $\pi$ while a transmission results in a phase change of $\pi/2$. For each run of the experiment, a bomb is taken from the untested stockpile of bombs and is placed in contact with ESD 2, such that if the bomb is a dud (D) then ESD 2 acts normally and transmits onto ESD 5.
If on the other hand the bomb is active (A), then whenever the signal state is triggered at ESD2 the bomb explodes. In this case, ESD 2 becomes faulty and does not transmit onto ESD 5.

This experiment involves a randomly changing apparatus network, because the bomb being tested in a given run is immediately replaced by a new bomb from the unused stockpile. There are therefore two distinct networks to consider. We have to deal with each of these separately, for which a pure labstate description can be used. Then we use a density matrix approach to consider the combined experiment.

For each of the pure labstate calculations, the labstate, or current signal state of the network, is discussed in the Schrödinger picture. At any stage \( n \), the current labstate will be written in the form \( |J, n\rangle \). We define the contextual vacuum \( |0\rangle \equiv C_1 C_2 C_3 C_4 C_5 C_6 C_7 |\Omega\rangle \), where \( C_i \) is the construction operator for the \( i \)th ESD.

(a) Networks with a dud bomb

Starting at time \( t = 0 \), the initial labstate \( |D, 0\rangle \equiv \tilde{A}_1 |0\rangle \) changes by the following sequence of stages, or opportunities for information extraction from the apparatus:

\[
\begin{align*}
|D, 0\rangle &\rightarrow |D, 1\rangle = \frac{1}{\sqrt{2}} (i \tilde{A}_2 - \tilde{A}_3)|0\rangle \\
|D, 1\rangle &\rightarrow |D, 2\rangle = \frac{1}{\sqrt{2}} (-i \tilde{A}_5 + \tilde{A}_4)|0\rangle.
\end{align*}
\]

(17.1)

Using

\[
\begin{align*}
\tilde{A}_4 |0\rangle &\rightarrow \frac{1}{\sqrt{2}} (i \tilde{A}_6 - \tilde{A}_7)|0\rangle, \\
\tilde{A}_5 |0\rangle &\rightarrow \frac{1}{\sqrt{2}} (-\tilde{A}_6 + i \tilde{A}_7)|0\rangle
\end{align*}
\]

(17.2)

we find \( |D, 2\rangle \rightarrow |D, 3\rangle = \tilde{A}_6 |0\rangle \). Hence, no dud bomb ever coincides with a signal in ESD 7.

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(b) Networks with an active bomb

In this case, the initial labstate is given by $|A, 0\rangle \equiv \bar{A}_1 \mathbb{D}_2 |0\rangle$, where $\mathbb{D}_i$ is the decommissioning operator for ESD $i$. In this case we have

$$|A, 0\rangle \rightarrow |A, 1\rangle = \frac{1}{\sqrt{2}} (i \mathbb{D}_2 - \bar{A}_3) \mathbb{D}_2 |0\rangle,$$

$$|A, 1\rangle \rightarrow |A, 2\rangle = \frac{1}{\sqrt{2}} (i \mathbb{D}_2 + \bar{A}_4) \mathbb{D}_2 |0\rangle$$

and

$$|A, 2\rangle \rightarrow |A, 3\rangle = \frac{1}{2} (\sqrt{2} i \mathbb{D}_2 + i \bar{A}_6 - \bar{A}_7) \mathbb{D}_2 |0\rangle.$$ 

Using $\mathbb{D}_2 \mathbb{D}_2 = \mathbb{D}_2$, the state $\mathbb{D}_2 |0\rangle$ represents an explosion at ESD 2 coincident with no signal at either ESD 6 or 7. Hence this calculation gives the outcome probabilities $P(\text{Explode} | A) = 1/2$, $P(6 | A) = P(7 | A) = 1/4$.

(c) Random testing

Unfortunately, the observer does not know before each run whether a particular bomb is active or a dud. Consider a sequence of runs such that there is a (classical) probability $\omega_B$ of encountering an active bomb and a probability $\omega_D = 1 - \omega_A$ of a dud. In this case we take a density matrix approach. At the $n$th stage we define the density matrix

$$\rho(n) = \omega_A |A, n\rangle \langle A, n| + \omega_D |D, n\rangle \langle D, n|.$$ 

The overall probability of triggering ESD 6 at the third stage is given by

$$P(6 | \rho, 3) = \text{Tr}(\rho(3) \mathbb{P}_6^1) = \frac{\omega_A}{4} + \omega_D,$$

where $\mathbb{P}_i^1$ is the signal projection operator for the $i$th ESD, and likewise,

$$P(7 | \rho, 3) = \text{Tr}(\rho(3) \mathbb{P}_7^1) = \frac{\omega_A}{4}.$$ 

The interpretation is that every signal at ESD 7 tells the observer that the current bomb on test is active. Moreover, that bomb has not exploded and can be used. This method allows one-quarter of the active bombs in the stockpile to be identified with a single sweep of the stockpile. Further iteration of this experiment on unexploded bombs coinciding with a signal from ESD 6 pushes this result up to one-third of the total, the rest of the active bombs having exploded during testing.

18. The Hardy paradox experiment

The Elitzur–Vaidman bomb-tester experiment may be interpreted as a simplified form of double-slit experiment, where the screen has only two sites and one of the slits can be blocked off or not, depending on whether a bomb is active or dud.
This blocking off occurs in a classical way, because the uncertainty as to whether
the bomb is active or dud is not intrinsic to the nature of the bomb but reflects
the observer’s ignorance of the nature of the bomb.

A spectacular variant of the bomb-tester experiment is known as the Hardy
(1992) paradox experiment. In this variant, the blocking off of a slit occurs in
an intrinsically quantum stochastic way, contrasted to the classically stochastic
way of the bomb-tester experiment. The experiment consists of an electron–
positron pair passing through two coupled Mach–Zehnder-type networks shown
in figure 2. The curvature of the tracks occurs because of suitable magnetic fields
perpendicular to the plane of the network shown.

In conventional terminology, the presence of the positron at ESD 3 would
effectively block one of the slits, i.e. the slit corresponding to ESD 4, through
which the electron wavefunction would otherwise pass. Conversely, the absence
of the positron at ESD 3 allows both slits 4 and 5 to be open as far as the
electron is concerned. By symmetry, the same remarks apply to the interchange
of the electron and positron.

The Hardy paradox experiment is intrinsically a pure quantum experiment,
in that we can discuss it via pure lab states alone. An important point is that
electron–positron annihilation is a well-known quantum process which occurs in
nature, whereas the detonation mechanism of the Elitzur–Vaidman bomb-tester
is left unspecified.

We start our analysis of the Hardy paradox experiment by defining the
contextual vacuum $|0\rangle \equiv C_1 C_2 C_3 C_4 C_5 C_6 C_7 C_8 C_9 |\Omega\rangle$. Then the initial state is
given by $|\Psi, 0\rangle \equiv \tilde{A}_1|0\rangle$. The dynamics follows the following sequence:

$$
|\Psi, 0\rangle \rightarrow |\Psi, 1\rangle = \frac{1}{2}(i\tilde{A}_2 - \tilde{A}_3)(i\tilde{A}_5 - \tilde{A}_4)|0\rangle.
$$

(18.1)
Using the labstate evolutions

\[
\begin{align*}
\bar{A}_2\bar{A}_4|0\rangle &\rightarrow \frac{1}{2}(i\bar{A}_7 - \bar{A}_6)(i\bar{A}_9 - \bar{A}_8)|0\rangle, \\
\bar{A}_2\bar{A}_5|0\rangle &\rightarrow \frac{1}{2}(i\bar{A}_7 - \bar{A}_6)(i\bar{A}_8 - \bar{A}_9)|0\rangle, \\
\bar{A}_3\bar{A}_4|0\rangle &\rightarrow \bar{D}_3\bar{D}_4|0\rangle
\end{align*}
\]

and

\[
\begin{align*}
\bar{A}_3\bar{A}_5|0\rangle &\rightarrow \frac{1}{2}(i\bar{A}_6 - \bar{A}_7)(i\bar{A}_8 - \bar{A}_9)|0\rangle
\end{align*}
\]

we find

\[
|\Psi, 1\rangle \rightarrow |\Psi, 2\rangle = \frac{1}{4}\{2\bar{D}_3\bar{D}_4 + i\bar{A}_6\bar{A}_8 - \bar{A}_7\bar{A}_8 - 3\bar{A}_6\bar{A}_9 + i\bar{A}_7\bar{A}_9\}|0\rangle. \quad (18.3)
\]

The state $\bar{D}_3\bar{D}_4|0\rangle$ is interpreted as the occurrence of electron–positron annihilation, as shown by the wavy lines in figure 2, so we read off the following conditional probabilities:

\[
P(\text{Explosion} | \Psi) = \frac{1}{4}, \quad P(6, 8 | \Psi) = P(7, 8 | \Psi) = P(7, 9 | \Psi) = \frac{1}{16}, \quad P(6, 9 | \Psi) = \frac{9}{16}. \quad (18.4)
\]

The paradox is that $P(7, 8 | \Psi) \neq 0$, which conventional logical suggests could occur only if both particles had passed through ESDs 3 and 4 simultaneously without annihilation, contrary to expectation.

### 19. Concluding remarks

The application of our quantized model of observation to the Elitzur–Vaidman bomb-tester and Hardy paradox experiments demonstrates that the concept of faulty or decommissioned states has physical significance.

This paper was motivated in part by a dissatisfaction with conventional approaches to quantum observation: in our view, too much attention is paid to states of SUOs and too little to the context of their observation. Despite the calculational successes of the conventional approach, it seems to us that there is something deep still missing throughout physics, viz. a comprehensive dynamical theory of observation. Such a theory should be able to clarify the relationship between the SUO, observer and apparatus concepts, which may lead to the resolution of long-standing issues in quantum mechanics. We hope that this paper may be of some value in suggesting new lines of research in this area.

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