Groups possessing only indiscrete embeddings in $SL(2, \mathbb{C})$

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Abstract

We give results on when a finitely generated group has only indiscrete embeddings in $SL(2, \mathbb{C})$, with particular reference to 3-manifold groups. For instance if we glue two copies of the figure 8 knot along its torus boundary then the fundamental group of the resulting closed 3-manifold sometimes embeds in $SL(2, \mathbb{C})$ and sometimes does not, depending on the identification. We also give another quick counterexample to Minsky’s simple loop question.

1 Introduction

The group $SL(2, \mathbb{C})$ plays an important role in both algebra and geometry. In particular we obtain on quotienting out by $\{\pm I\}$ the group $PSL(2, \mathbb{C})$ of Möbius transformations which acts as the group of orientation preserving isometries on the hyperbolic space $\mathbb{H}^3$. A discrete subgroup $\Gamma$ of $PSL(2, \mathbb{C})$ is a Kleinian group and these have been much studied because the quotient $\Gamma \backslash \mathbb{H}^3$ is a 3-orbifold, and even a 3-manifold if $\Gamma$ is torsion free, with a complete hyperbolic metric. However we can approach this from a group theoretic point of view and ask what other subgroups of $PSL(2, \mathbb{C})$ occur. Certainly one can have abstract groups which appear both as discrete and indiscrete
subgroups of $PSL(2, \mathbb{C})$, such as free groups. However these will not provide new examples of subgroups of $PSL(2, \mathbb{C})$ if we only consider them up to isomorphism so our focus in this paper will be to look at abstract groups which are subgroups of $SL(2, \mathbb{C})$ but which have no discrete embedding in $SL(2, \mathbb{C})$. In this paper our groups will nearly always be finitely generated and torsion free. In particular a torsion free subgroup $G$ of $SL(2, \mathbb{C})$, or even one without elements of order 2, will also embed in $PSL(2, \mathbb{C})$ because it misses the kernel $\{\pm I\}$. Moreover if this abstract group $G$ with no elements of order 2 could be realised in some way as a discrete subgroup of $PSL(2, \mathbb{C})$ then [10] shows that $G$ lifts to $SL(2, \mathbb{C})$, whereupon it will also be a discrete subgroup of $SL(2, \mathbb{C})$. Therefore there is no harm in sticking to $SL(2, \mathbb{C})$ throughout which allows us to work directly with $2 \times 2$ matrices.

Of course there are many finitely generated, torsion free groups which do not embed in $SL(2, \mathbb{C})$, either as a discrete or indiscrete subgroup. Arguably the first big restriction on subgroups of $SL(2, \mathbb{C})$ which comes to mind is that of being commutative transitive: a group $G$ is commutative transitive (CT) if the relation of two elements commuting is an equivalence relation on $G - \{e\}$. It is straightforward to show and well known that a torsion free subgroup of $SL(2, \mathbb{C})$ (again this generalises to not containing an element of order 2) will be commutative transitive. Therefore all our examples in this paper must come from this class. A strictly stronger condition that will also be of use to us is that of a CSA group, meaning that all centralisers are malnormal. We introduce basic properties of CT and CSA groups in Section 2, including a characterisation of the CSA subgroups of $SL(2, \mathbb{C})$ in Corollary 2.3.

In section 3 we start by giving standard examples of subgroups of $SL(2, \mathbb{C})$ which do not have a discrete embedding. It goes back to Nisnević in 1940 that (on avoiding the element $-I$) a free product of subgroups of $SL(2, \mathbb{C})$ also embeds in $SL(2, \mathbb{C})$. Therefore we can give examples in Lemma 3.1 of subgroups of $SL(2, \mathbb{C})$ which are word hyperbolic and are the fundamental group of a closed orientable 3-manifold, but which have no discrete embedding. This result of Nisnević was rediscovered by Shalen in [29] where he also gave conditions under which a free product with abelian amalgamation of two subgroups of $SL(2, \mathbb{C})$ also embeds in $SL(2, \mathbb{C})$. We use and adapt this result to give examples of subgroups of $SL(2, \mathbb{C})$ with no discrete embedding which are word hyperbolic but which this time cannot be the fundamental group of any 3-manifold.

It is known that limit groups (finitely generated groups which are fully residually free) embed in $SL(2, \mathbb{C})$ so any limit group which is not the funda-
1 INTRODUCTION

mental group of a 3-manifold (say one that contains $\mathbb{Z}^4$) will also only have indiscrete embeddings. In Section 4 we show that our examples are not limit groups using [32] which examined which 3-manifolds can have a fundamental group that is a limit group. A cyclically pinched group is one formed by amalgamating two free groups over a non identity element which is not part of a free basis for either free group. If this element is not a proper power in either factor then the resulting group embeds in $SL(2, \mathbb{C})$. We can therefore ask whether it is always a limit group. Although it seems to be a folklore result that the answer is no, we have never seen an example written down. Here we demonstrate that any cyclically pinched group formed by amalgamating a commutator on one side with a product of two proper powers of distinct commutators on the other side is not a limit group if the powers differ by at least 3. The proof is short and uses standard facts about stable commutator length, including the lower bound for stable commutator length in a free group.

In Section 5 we look at which closed 3-manifolds, other than orientable hyperbolic 3-manifolds of course, have fundamental groups which embed in $SL(2, \mathbb{C})$. Although we do not give a full list, we suggest that it is a somewhat larger list than might be first thought. In Theorem 5.1 we give the complete list of torus bundles fibred over the circle with such fundamental groups: as well as the trivial bundle it is precisely those with Sol geometry. Given the constructions in Section 3 of groups embedding in $SL(2, \mathbb{C})$ using amalgamation over abelian subgroups, one might suppose that the fundamental group of a closed orientable 3-manifold admitting a JSJ decomposition along tori with all pieces hyperbolic might embed. However we show in Theorem 5.2 that if we glue two copies of the figure 8 knot along each boundary torus such that the meridians are identified then the fundamental group of the resulting 3-manifold does not embed in $SL(2, \mathbb{C})$ unless the longitudes are also glued to each other. In this latter case the group does embed, giving a genus 2 surface bundle fibred over the circle with fundamental group embedding in $SL(2, \mathbb{C})$ even though the homeomorphism is not pseudo-Anosov. This work utilises a result of Whittemore from 1973 giving all representations of the figure 8 knot group in $SL(2, \mathbb{C})$.

Having looked at free products with amalgamation, we discuss in Section 6 whether an HNN extension $\Gamma$ over $\mathbb{Z}$ of a subgroup $G$ of $SL(2, \mathbb{C})$ also embeds in $SL(2, \mathbb{C})$. This case seems much less clear and we content ourselves with remarking that if the stable letter of $\Gamma$ conjugates the generator of $\mathbb{Z}$ to its inverse then the answer is no, whereas if $G - \{I\}$ has no elements of trace
±2 and the stable letter conjugates the generator to itself then the answer is yes. However the later case is enough to show quickly and directly that all limit groups embed in $SL(2, \mathbb{C})$ by taking iteratively a free extension of centralisers. We also give an example of a group admitting a graph of groups decomposition with non abelian free vertex groups and maximal cyclic edge groups which is word hyperbolic but which does not embed in $SL(2, \mathbb{C})$. This is in contrast to when the graph of groups is a tree where we do embed.

Finally we give a concrete counterexample with a brief proof to Minsky’s simple loop question in [25] for genus 2. This is an example where the fundamental group of the closed orientable surface of genus 2 has a non injective homomorphism to $SL(2, \mathbb{C})$ but such that the kernel contains no elements represented by simple closed curves. We are able to obtain this example quickly by using our results on embeddings of groups in $SL(2, \mathbb{C})$ from Section 3 and the embeddings of the fundamental group of the figure 8 knot used in Section 5. This can be seen as one in a sequence of successively shorter counterexamples to this question, as in [9], [22] and [7].

2 Commutative Transitive groups

We say that a group $G$ is commutative transitive or CT for short if the relation of two elements commuting is transitive on $G - \{e\}$. This is the same as saying that the centralisers of all (non identity) elements are abelian and such groups are also called CA (standing for centraliser abelian) groups. The finite CT groups are known but there are many interesting examples of infinite CT groups. For instance Corollary 1 of [1] states that the free Burnside groups of sufficiently large odd period are also CT groups. Here we are mainly interested in torsion free groups. An application of Zorn’s Lemma tells us that in any group $G$ every element of $G$ is contained in a maximal abelian group. This can also be used to show that a group $G$ is CT if and only if every non identity element $g$ of $G$ is contained in a unique maximal abelian subgroup, in which case the centralisers (except $G_G(e)$ if $G$ is non abelian) are precisely the maximal abelian subgroups. Examples of CT groups are free groups, limit groups and torsion free word hyperbolic groups. Moreover it is straightforward to see that any subgroup of $SL(2, \mathbb{C})$ not containing $-I$ is CT because in this group a diagonal matrix not equal to $\pm I$ can only commute with a diagonal matrix whereas a matrix of the
form $\pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ can only commute with one of the same form (if $x \neq 0$) but every matrix is conjugate in $SL(2, \mathbb{C})$ to one of these.

Of course many groups are not CT but here is a quick and easy source of examples.

**Lemma 2.1** If $G$ is torsion free and CT but there exists a non identity $x \in G$ whose centraliser $C_G(x)$ has finite index in $G$ then $G$ is abelian. Thus if $G$ is virtually nilpotent, torsion free and CT then $G$ must be abelian.

**Proof.** For any $g \in G$ there is $n \in \mathbb{N}$ with $e \neq g^n \in C_G(x)$, so CT implies that $C_G(x) = G$ and hence all elements of $G$ commute with each other. The last part follows because if $G$ has a finite index subgroup $H$ which is nilpotent then $H$ has a non trivial centre, thus we have an element $h \neq e$ with $H = C_H(h) \leq C_G(h)$ having finite index in $G$.

\[\blacksquare\]

There is a condition that is stronger than being CT but which is sometimes useful. We say that a subgroup $H$ of a group $G$ is **malnormal** (or conjugate separated) in $G$ if $gHg^{-1} \cap H = \{e\}$ for all $g \in G - H$. A group $G$ is then called **CSA** (standing for conjugate separated abelian) if every maximal abelian subgroup is malnormal. This implies CT because we can take a non identity element $g \in G$ and a maximal abelian subgroup $M$ containing $g$. Then if $x$ is an element of $C_G(g)$ we have $g \in xMx^{-1} \cap M$, so $x$ is in $M$ too. In fact an equivalent definition of a CSA group is that all centralisers are malnormal: if the latter holds then on taking $x \in C_G(g)$ and any $y$ that commutes with $x$, we obtain $x \in yC_G(g)y^{-1} \cap C_G(g)$. Hence malnormality of $C_G(g)$ implies that $y$ commutes with $g$ so here the centralisers are again the maximal abelian subgroups.

Again free groups, limit groups and torsion free word hyperbolic groups are CSA, and both the CSA and the CT properties are preserved under taking subgroups, but now the situation for subgroups of $SL(2, \mathbb{C})$ not containing $-I$ is a little different.

**Lemma 2.2** If $G$ is a group contained in $SL(2, \mathbb{C})$ with $-I \notin G$ and $D$ is the subgroup of diagonal elements in $G$ then $D$ is malnormal in $G$.

**Proof.** Suppose that

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$$

and

$$d_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \quad d_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} \in D$$
with \( gd_1 = d_2g \) holding in \( G \). An easy check shows that if \( \lambda_1 \neq \lambda_2 \) then \( \alpha = \delta = 0 \), so \( g \) is a matrix of trace 0 and determinant 1 implying that \( g^2 = -I \in G \), and if \( \lambda_1 \lambda_2 \neq 1 \) then \( \beta = \gamma = 0 \) so \( g \in D \). This only leaves \( d_1 = d_2 = \pm I \).

\[ \blacksquare \]

**Corollary 2.3** If \( G \) is a subgroup of \( SL(2, \mathbb{C}) \) such that the only element of \( G \) with trace in \( \{-2, 2\} \) is \( I \) then \( G \) is a CSA group.

More generally a non abelian subgroup \( G \) of \( SL(2, \mathbb{C}) \) is CSA if and only if it does not contain \(-I\) and for any non identity element \( g \) with trace\( (g) = \pm 2 \) and \( \gamma_g \in SL(2, \mathbb{C}) \) such that \( \gamma_g g \gamma_g^{-1} = \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \), the conjugate group \( \gamma_g G \gamma_g^{-1} \) contains no elements with bottom left hand entry 0 other than those with trace \( \pm 2 \).

**Proof.** On being given an element \( g \) of \( G \) with trace not equal to \( \pm 2 \) we can assume by conjugation that \( g \) is a diagonal matrix not equal to \( \pm I \). Clearly the centraliser \( C_G(g) \) is equal to the abelian subgroup of diagonal elements in \( G \) and so is malnormal by Lemma 2.2, as \(-I\) is not in any conjugate of \( G \). This completes the first case.

Now suppose (conjugating if necessary) that \( g = \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) for \( x \neq 0 \). Then \( C_G(g) \) is also an abelian subgroup, consisting of all matrices in \( G \) with trace \( \pm 2 \) and bottom left hand entry 0. However an element of \( SL(2, \mathbb{C}) \) can only conjugate one (non identity) element of this form into another if its bottom left hand entry is 0 as well. If this element is in \( G \) then it lies in \( C_G(g) \) if and only if its trace is \( \pm 2 \), by the given condition.

\[ \blacksquare \]

For \( m \neq 0 \) the Baumslag-Solitar group \( BS(1, m) \) is given by the presentation \( \langle x, t \mid txt^{-1} = x^m \rangle \). These groups embed (always indiscretely) in \( SL(2, \mathbb{C}) \) by taking

\[
t = \begin{pmatrix} \sqrt{m} & 0 \\ 0 & 1/\sqrt{m} \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

unless \( m = \pm 1 \) in which case the matrix given for \( t \) has finite order. If \( m = 1 \) then we have \( \mathbb{Z} \times \mathbb{Z} \) which clearly embeds (both discretely and indiscretely) in \( SL(2, \mathbb{C}) \). However \( BS(1, -1) \) (the Klein bottle group) does not embed in
Because it is not CT (the group is torsion free, non abelian and with $t^2$ in the centre so Lemma 2.1 applies). Thus we see that $BS(1, m)$ is CT if and only if $m \neq -1$. This is in disagreement with [13] where it is stated in Theorem 9 that $BS(1, m)$ is not CT for $m \neq 1$ (but no further comment is made as to why). However it is shown there that $BS(1, m)$ is CSA only for $m = 1$, as otherwise the group is non abelian but has a non trivial normal abelian subgroup.

There is a general case where the CSA and the CT condition are equivalent. This occurs when $G$ is a group where the centraliser of each non trivial element is infinite cyclic. Clearly $G$ is CT and it is not hard to show that $G$ is also CSA, for instance see [29] Lemma 3.3. We say that an element $g$ of infinite order in a group $G$ is primitive if it generates its own centraliser.

We finish with some lemmas on the CSA property which will be needed later.

**Lemma 2.4** Let $G_1$ and $G_2$ be groups where the centraliser of every non trivial element is infinite cyclic, and let $a$ be a primitive element of $G_1$ and $b$ of $G_2$. Then any primitive element of $G_1$ or $G_2$ is also primitive in the amalgamated free product $G = G_1 *_{a=b} G_2$.

**Proof.** First suppose that $\gamma$ is in $G_1$ but cannot be conjugated into the amalgamated subgroup $H = \langle a \rangle$ by an element of $G$. We take $g \in G - G_1$, whereupon we can assume that $g$ can be expressed as $g_1g_2\ldots g_r$ for $r \geq 1$ and $g_1, g_2, \ldots, g_r$ coming alternately from $G_1 - H$ and $G_2 - H$, which we will refer to as a normal form for $g$. Conversely an element of this form is not in $G_1$ or $G_2$ if $r \geq 2$. Now $g\gamma g^{-1} = g_1\ldots g_r\gamma g_r^{-1} \ldots g_1^{-1}$ is in normal form if $g_r \in G_2 - H$ because $\gamma \in G_1 - H$. But if $g_r \in G_1 - H$ then $h_r = g_r\gamma g_r^{-1}$ is also in $G_1 - H$ as $\gamma$ is not conjugate into $H$. Moreover $r \geq 2$ in this case so $g\gamma g^{-1} = g_1\ldots g_r h_r g_r^{-1} \ldots g_1^{-1}$ is in normal form and so is not in $G_1$.

We now need to consider the centraliser of elements in $H$. If $h \in H - \{e\}$ and $g$ is as above then $g_1\ldots g_{r-1}(g_r h g_r^{-1}) g_{r-1}^{-1} \ldots g_1^{-1}$ is again in normal form, because we have noted that the condition on the centralisers being cyclic implies that $H$ is malnormal in both $G_1$ and $G_2$, so $ghg^{-1} \notin H$.

**Lemma 2.5** If $G_1$ and $G_2$ are torsion free CSA groups and $G = G_1 *_{H} G_2$ where $H$ is a centraliser in both $G_1$ and $G_2$ then $G$ is CSA.
Proof. This can be proved using the concept of a 0-step malnormal amalgamated free product as defined in [18], which means that the amalgamated subgroup is malnormal in both factors. Alternatively we can apply [8] Appendix A which tells us when the fundamental group of a graph of groups with abelian edge groups is CSA. In particular Corollary A.8 states that the fundamental group $\pi_1(\Gamma)$ of a graph of groups $\Gamma$ with CSA vertex groups and edge groups which are maximal abelian in the neighbouring vertex groups is CSA if the Bass-Serre tree of $\Gamma$ is acylindrical. This means that there is an upper bound for the diameter of the fixed point set of any element in $\pi_1(\Gamma) - \{e\}$. But for $\pi_1(\Gamma) = G_1 * H G_2$ we have that a non identity element fixing two vertices which are a distance more than one apart must fix two edges, thus is in $g_1 H g_1^{-1} \cap g_2 H g_2^{-1}$ for $g_1^{-1} g_2 \notin H$. However $H$ is malnormal in both $G_1$ and $G_2$, thus is malnormal in $G$ (which follows in exactly the same way as Lemma 2.4 as we did not use the fact that centralisers were cyclic, just that the amalgamated subgroup was malnormal in both factors).

We give a quick example to show that Lemma 2.5 fails when we replace CSA with CT. Let $G_1$ be the semidirect product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ where no power of $\alpha$ has fixed points. Then it is not hard to show that $G_1$ is CT (indeed it follows from Theorem 5.1). Moreover the centralisers are all infinite cyclic with the single exception of $\mathbb{Z}^2$. If we now let $G_1 = G_2$ with $H = \mathbb{Z}^2$ and form $G = G_1 * H G_2$ then $G$ has the presentation

$$\langle s, t, a, b | sas^{-1} = \alpha(a), sbs^{-1} = \alpha(b), tat^{-1} = \alpha(a), tbt^{-1} = \alpha(b) \rangle.$$ 

Now $\langle s, t \rangle$ generates a non abelian free group so $sts^{-1}t^{-1}$ and $s^2t^2s^{-2}t^{-2}$ do not commute, but they both commute with $a$.

Finally we need a version of Lemma 2.4 for HNN extensions.

**Lemma 2.6** Suppose that $G$ is a group where the centraliser of every non trivial element is infinite cyclic, and that $a$ and $b$ are primitive elements of $G$ where no conjugate of $A = \langle a \rangle$ intersects $B = \langle b \rangle$ apart from in the identity. Then any primitive element of $G$ is also primitive in the HNN extension $\Gamma = G *_{tat^{-1} = b} G$ where $t$ is the stable letter of the HNN extension. Moreover if two primitive elements of $G$ are conjugate in $\Gamma$ then either they are conjugate in $G$ or one is conjugate in $G$ to $a^{\pm 1}$ and the other to $b^{\pm 1}$.

**Proof.** We again have a normal form where we can write any element $\gamma \in \Gamma - G$ as $\gamma_0 t^{k_1} \ldots t^{k_n} \gamma_n$ for $n \geq 1$, $\gamma_i \in G$, $k_i \in \mathbb{Z} - \{0\}$ and no appearance of $t a t^{-1}$ or $t^{-1} b t$ occurs, and conversely an element in such form
does not lie in $G$. First let us take a primitive element $g \in G$ which is not conjugate into $A$ or $B$. Thus for any $\gamma \in \Gamma - G$ expressed as above, we have that $\gamma_n g \gamma_n^{-1}$ is not in $A$ or $B$. This implies that $\gamma g \gamma^{-1}$ is in normal form, thus not in $G$. Now let us consider a conjugate of $a$ by an element $\gamma$ outside $G$. Again on taking $\gamma$ in the form above, we see that $\gamma a \gamma^{-1}$ only fails to be in normal form if $\gamma_n \in A$ and $k_n > 0$, because $A$ is malnormal in $G$ and we cannot have $\gamma_n a \gamma_n^{-1}$ in $B$. Thus we can replace $t \gamma_n a \gamma_n^{-1} t^{-1}$ with $b$. But now this would only fail to be in normal form if $k_n = 1$, $\gamma_n^{-1} \in B$ and $k_n < 0$ which would contradict $\gamma$ being in normal form, unless $n = 1$ when $\gamma a \gamma^{-1}$ is conjugate to $b$.

\[\square\]

3 Amalgamated free products

Given an abstract group $G$ (always assumed to be countable and usually finitely generated), we can ask whether $G$ is a linear group. Here we will be concerned with a special version of this question: does $G$ embed in $SL(2, \mathbb{C})$? We can rule out some groups by noting that the finite subgroups of $SL(2, \mathbb{C})$ are very restricted (for instance the only element of order 2 is the matrix $-I$) so any group containing a finite subgroup not in this class will fail to embed. However our focus in this paper is on torsion free groups.

Another severe restriction comes because, as mentioned in the last section, any subgroup of $SL(2, \mathbb{C})$ not containing $-I$ is CT. If a finitely generated torsion free group $G$ embeds as a discrete subgroup of $SL(2, \mathbb{C})$, and hence a discrete subgroup of $PSL(2, \mathbb{C})$, then we know from hyperbolic geometry that $G = \pi_1(M)$ for $M$ an orientable 3-manifold with a complete hyperbolic structure (or hyperbolic 3-manifold for short). Indeed by topological tameness we now know that $M$ is homeomorphic to the interior of a compact orientable 3-manifold. Conversely if $G = \pi_1(M)$ where $M$ is a compact orientable 3-manifold with interior having a complete hyperbolic structure then $G$ is finitely generated, indeed finitely presented, torsion free, and embeds as a discrete group in $SL(2, \mathbb{C})$. Here we wish to find examples of subgroups of $SL(2, \mathbb{C})$ with no discrete embedding. There certainly are examples even amongst abelian groups: a discrete torsion free abelian group can only be $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ so that if we take matrices $M_i = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix}$ where $x_1, \ldots, x_m$ are
complex numbers which are linearly independent over \( Q \), the group generated is \( \mathbb{Z}^m \) for any \( m \). A more interesting example gives the wreath product of \( \mathbb{Z} \) with itself, obtained by taking a diagonal matrix with transcendental entries and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). This group contains the free abelian group of countably infinite rank, and is finitely generated but not finitely presented. These groups are certainly not word hyperbolic (because they contain \( \mathbb{Z}^2 \)) and (if they contain \( \mathbb{Z}^4 \)) cannot be the fundamental group of any (compact or non compact) 3-manifold, hyperbolic or not. However they all sit in the subgroup of \( SL(2, \mathbb{C}) \) with bottom left hand entry 0 and so must be metabelian.

To obtain further examples which do not sit in some restricted subgroup, we can try combining subgroups of \( SL(2, \mathbb{C}) \) in various ways. Using the direct product is doomed to failure because \( G_1 \times G_2 \) is not CT unless both \( G_1 \) and \( G_2 \) are abelian (or one is trivial). Therefore we should consider the free product, where there are known results on linearity. Here we take a linear group to mean a subgroup of \( GL(n, k) \) for \( k \) a field of characteristic 0. Some similar results to those below also exist for fields of positive characteristic but we will not discuss them here.

The free product \( G_1 \ast G_2 \) of any two linear groups \( G_1, G_2 \) is linear: indeed it was proved in [26] back in 1940 that if \( G_1, G_2 \leq GL(n, k) \) then \( G_1 \ast G_2 \leq GL(n + 1, k') \) for some other field \( k' \) of characteristic 0 and we can replace \( n + 1 \) with \( n \) if there are no (non trivial) scalars in \( G_1 \) or in \( G_2 \). This result also follows from Theorem 3 in [30]. As \( \mathbb{C} \) has uncountable transcendence degree over its prime field \( Q \) and all groups in this paper are countable, we can take \( k = k' = \mathbb{C} \) here without loss of generality. Also Theorem 1 in [29] rediscovered a version of this result, stating that if \( G_1, G_2 \leq SL(n, \mathbb{C}) \) with no scalar matrices in either factor then \( G_1 \ast G_2 \) embeds into \( SL(n, \mathbb{C}) \). Therefore as an immediate consequence of these results, we have

**Lemma 3.1** If we have two countable subgroups \( G_1 \) and \( G_2 \) of \( SL(2, \mathbb{C}) \), neither of which contain \(-I\) then the free product \( G_1 \ast G_2 \) embeds in \( SL(2, \mathbb{C}) \) as well.

Consequently an easy way of coming up with word hyperbolic groups which embed in \( SL(2, \mathbb{C}) \) but with no discrete embedding is to take \( G_1 = \pi_1(M_1) \) and \( G_2 = \pi_1(M_2) \) where \( M_1 \) and \( M_2 \) are closed hyperbolic 3-manifolds. We then have that \( G_1 \ast G_2 \) embeds in \( SL(2, \mathbb{C}) \), is word hyperbolic (as the free product of two word hyperbolic factors) and is the fundamental group of a closed 3-manifold, the connected sum of \( M_1 \) and \( M_2 \). However it cannot
be the fundamental group of a closed hyperbolic 3-manifold or even embed discretely in $SL(2, \mathbb{C})$: if $G = G_1 * G_2$ is discrete then the resulting quotient 3-manifold $M = \mathbb{H}^3 / G$ must be irreducible, so it is determined by its fundamental group up to homotopy. But the Mayer-Vietoris sequence for the homology of a free product would give $H_3(G_1 * G_2; \mathbb{Z}) = H_3(G_1; \mathbb{Z}) \oplus H_3(G_2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ which cannot be equal to $H_3(\pi_1(M); \mathbb{Z})$.

Given that linearity behaves well under free products, an immediate question is whether this extends to free products with amalgamation and/or HNN extensions. However there would need to be restrictions on the factors/base and amalgamated/associated subgroups because we might not even have residual finiteness in general. As we are concentrating here on torsion free groups, the first candidates for amalgamated subgroups ought to be those that are infinite cyclic. This is a fruitful pursuit for $SL(2, \mathbb{C})$, as Proposition 1.3 in [29] shows:

**Proposition 3.2** Let $G_1 *_{H} G_2$ be a free product amalgamating the subgroup $H_1 \leq G_1$ with $H_2 \leq G_2$ via the isomorphism $\phi : H_1 \to H_2$. Suppose that $G_1$ and $G_2$ are both subgroups of $SL(2, \mathbb{C})$ such that
1. the matrices $\phi(h)$ and $h$ are the same for all $h \in H_1$,
2. every $h \in H_1$ is a diagonal matrix, and
3. for every $g_1 \in G_1 - H_1$ the bottom left hand entry is non zero, as is the top right hand entry for all $g_2 \in G_2 - H_2$. Then $G_1 *_{H} G_2$ can also be embedded in $SL(2, \mathbb{C})$. Further, let us say that a subgroup of $SL(2, \mathbb{C})$ has transcendental traces if every non identity element has a trace which is transcendental over $\mathbb{Q}$. Then if $G_1$ and $G_2$ have transcendental traces, so does this embedding of $G_1 *_{H} G_2$.

Earlier Theorem 5 in [30] proved that if $A$ and $B$ are free groups with $a$ a primitive element of $A$ and $b$ a primitive element of $B$ then the amalgamated free product $A *_{a=b} B$ embeds in $SL(2, \mathbb{C})$. In fact the proof there is very similar to that of Proposition 1.3 as given in [29] by Shalen. However the Shalen paper recognises that the condition of the factor groups being free can be weakened considerably. Also in that paper Theorem 2 builds on this proposition to create a result on embedding amalgamated free products in $SL(2, \mathbb{C})$ which can be applied recursively:

**Theorem 3.3** Let $A$ and $B$ be subgroups of $SL(2, \mathbb{C})$ with transcendental traces such that both groups satisfy the following property: the centraliser of every non identity element is infinite cyclic. Then the free product $A * B$, as
well as the amalgamated free product $A \ast_{a=b} B$ for any primitive $a \in A$ and $b \in B$ has an embedding into $SL(2, \mathbb{C})$ with transcendental traces and every non identity element has centraliser which is infinite cyclic.

Here we would like to adapt these results somewhat to create a version which does not require the infinite cyclic restriction to hold for all centralisers as in Theorem 3.3.

**Lemma 3.4** Suppose that $A$ is a group which embeds in $SL(2, \mathbb{C})$ such that no element has trace $\pm 2$ except the identity. Suppose that $a \neq I$ is an element in $A$ of the form

$$a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$ 

If any element $x$ of $A$ has a zero entry then $x$ is also diagonal and so commutes with $a$.

**Proof.** First suppose that

$$x = \begin{pmatrix} \alpha & 0 \\ \gamma & 1/\alpha \end{pmatrix}$$

then

$$axa^{-1}x^{-1} = \begin{pmatrix} 1 & 0 \\ (\lambda^{-2} - 1)\gamma/\alpha & 1 \end{pmatrix}$$

which has trace 2 so $\gamma$ is zero. The same applies for a zero in the bottom left hand entry. Now suppose that

$$x = \begin{pmatrix} 0 & \beta \\ -1/\beta & \delta \end{pmatrix}$$

then

$$xax^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ (\lambda^{-1} - \lambda)\delta/\beta & \lambda \end{pmatrix}$$

so we can now apply the above with $x$ replaced by $xax^{-1}$, telling us that $\delta = 0$ and forcing on us an element of trace 0, with square $-I$ in $A$.

\[\square\]

**Corollary 3.5** Suppose that $A$ is a group which embeds in $SL(2, \mathbb{C})$ such that no element has trace $\pm 2$ except the identity. Let $a \in A$ be a primitive element in $A$ which has a transcendental trace under this embedding. Let the same conditions hold for the group $B$ and the element $b \in B$. Then the amalgamated free product $A \ast_{a=b} B$ can be embedded in $SL(2, \mathbb{C})$ as well.
Proof. We first take conjugates in $SL(2, \mathbb{C})$ of $A$ and $B$ such that 
\[
a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}
\]
where $\lambda$ and $\mu$ will both be transcendental numbers. It is well known (for instance see Lemma 3.2 of [29]) that there exists a field automorphism $\psi$ of $\mathbb{C}$ such that $\psi(\mu) = \lambda$. We apply $\psi$ to the entries of all elements of $B$ which will result in a subgroup of $SL(2, \mathbb{C})$ abstractly isomorphic to (and still called) $B$ but such that $a = b$. Moreover this will not affect which entries or traces of elements in $B$ are transcendental or equal to $\pm 2$. If the bottom left hand entry of any element $x$ in $A$ is zero then Lemma 3.4 says that $x$ is in the centraliser of $A$ and so in the subgroup being amalgamated. The same will hold for all of the top right hand entries of $B$, so we can now apply Proposition 3.2.

We have already seen examples of subgroups of $SL(2, \mathbb{C})$ which are word hyperbolic but have no discrete embedding. However they were fundamental groups of compact 3-manifolds, so we would now like examples which do not have this property. By [3] due to Bestvina and Feighn we have that if $A$ and $B$ are torsion free word hyperbolic groups with $a \in A$ and $b \in B$ both non trivial elements then the amalgamated free product $G = A \ast_{a=b} B$ is word hyperbolic if and only if $G$ contains no $\mathbb{Z} \times \mathbb{Z}$ subgroup, which is shown to occur if and only if either $\langle a \rangle$ is malnormal in $A$ or $\langle b \rangle$ is malnormal in $B$. This is also the same as saying that either $a$ or $b$ is a primitive element in $A$ or $B$ respectively, because $A$ and $B$ are both word hyperbolic, hence CSA. (The result in [3] allows amalgamation over a virtually cyclic subgroup but this is the statement when restricted to torsion free groups.)

Thus a particular example where Theorem 3.3 gives us word hyperbolic groups embedding in $SL(2, \mathbb{C})$ is when $A$ and $B$ are non abelian free groups. Moreover we can take free groups $F_{r_1}, \ldots, F_{r_n}$ for any ranks $r_i$ at least 2 and form repeated amalgamated free products over arbitrary cyclic subgroups generated by primitive elements. A well studied construction which is similar to this is that of a graph of groups with non abelian free vertex groups and infinite cyclic edge groups. The idea is that we obtain the fundamental group of a graph of groups by forming the amalgamated free product where an edge joins two distinct vertices, then contract this edge and continue until we are left with self loops which then give us HNN extensions. Consequently if the
graph is a tree then only free product amalgamation occurs. The Bestvina-Feighn result mentioned above showing word hyperbolicity of such an amalgamated free product where just one element is primitive does not extend to graphs of groups with this property on the edge groups, for instance one can take three groups which are free on $a, b$, on $c, d$ and on $x, y$ respectively, then amalgamating $a^3$ to $c$ and $c$ to $x^3$ does not result in a word hyperbolic group. However if we amalgamate elements which are primitive in each factor then the resulting graph of groups is word hyperbolic by repeated use of [3, and Lemma 2.4 or a similar argument. Therefore by these two results and Theorem 3.3 we have:

**Corollary 3.6** If $\Gamma$ is a graph of non abelian free groups with infinite cyclic edge groups which are generated by a primitive element on each side and $\Gamma$ is a tree then the fundamental group of $\Gamma$ is word hyperbolic and embeds in $SL(2, \mathbb{C})$.

We note that there exist examples where $\Gamma$ is not the fundamental group of any 3-manifold and therefore cannot embed discretely in $SL(2, \mathbb{C})$. For instance, as mentioned in the last section of [14], we can take the double $\Gamma$ over the primitive word $yx^2y^{-1}x^{-3} \in F_2$. If $\Gamma = \pi_1(M)$ for $M$ a 3-manifold (assumed compact without loss of generality) then we can take $M$ to be prime as $\Gamma$ is one ended. This means here that $M$ is irreducible so the splitting over $\mathbb{Z}$ of $\pi_1(M)$ can be induced geometrically, giving 3-manifolds $M_1, M_2$, which are irreducible because $M$ is, with fundamental group free of rank 2. Now results in [15] imply that $M_1$ and $M_2$ must each be a (possibly non orientable) cube with one handle, joined to form $M$ by attaching along a neighbourhood of the curve $yx^2y^{-1}x^{-3}$ embedded in each boundary. But we can attach a thickened disc along a neighbourhood of one of these curves if it is an annulus as in [14], or attach a $P^2 \times I$ if the neighbourhood is a Möbius strip. Thus we obtain a 3-manifold with fundamental group $\langle x, y | yx^2y^{-1}x^{-3} \rangle$ or $\langle x, y | (yx^2y^{-1}x^{-3})^2 \rangle$ which is well known to be a contradiction (for instance neither Alexander polynomial is symmetric).

We consider the case of HNN extensions, where more care is needed because these can introduce Baumslag-Solitar subgroups, in Section 6, where we also mention other results on linearity of these groups.
4 Non Limit groups

One definition of a limit group is that it is a finitely generated group which is fully residually free. It is straightforward to show that such a group is torsion free and CT. It is also true that limit groups embed in $SL(2, \mathbb{C})$, for instance see Window 8 of [23] and we will provide a quick and explicit proof in Section 6. Indeed [2] by B. Baumslag shows that a finitely generated residually free group is either a limit group or contains $F_2 \times \mathbb{Z}$. This last group is clearly not CT so limit groups are exactly the finitely generated residually free groups which embed in $SL(2, \mathbb{C})$. Therefore we must consider whether the groups given in this paper are limit groups, in which case existence of an embedding into $SL(2, \mathbb{C})$ was already known. First of all, as the only metabelian limit groups are the free abelian groups $\mathbb{Z}^m$, examples such as the wreath product given in Section 3 are not limit groups. Moreover a finitely generated subgroup of a limit group is also a limit group, but it was shown in [32] using work of Sela that the fundamental group of a closed orientable hyperbolic 3-manifold cannot be a limit group. Therefore the examples in the last section which were formed using free products contain a finitely generated subgroup which is not a limit group, meaning that they are not limit groups either. The word hyperbolic but non 3-manifold group given at the end of the last section is a limit group as it is a double of a free group, but we can avoid this and still have a word hyperbolic non 3-manifold group which embeds in $SL(2, \mathbb{C})$ by taking its free product with a closed hyperbolic 3-manifold group.

Limit groups can contain $\mathbb{Z}^k$ for $k \geq 2$ and so need not be word hyperbolic but a limit group which does not contain $\mathbb{Z} \times \mathbb{Z}$ is word hyperbolic. A special case of the graph of free groups with $\mathbb{Z}$ edge groups mentioned in the last section is the amalgamated free product $G = F_{r_1} \ast_{w_1= w_2} F_{r_2}$ for $F_{r_1}, F_{r_2}$ non abelian free groups and $w_1, w_2$ any two non trivial elements. This is called a cyclically pinched group if neither $w_i$ is part of a free basis for $F_{r_i}$ (otherwise we obtain the free group $F_{r_1+r_2-1}$). If neither element is primitive, say $w_i = u_i^n$ for $n_i \geq 2$, then it is well known that $G$ cannot be a limit group. A quick way to see this is to note that $G$ is not CT because $u_1 u_2 u_1^{-1} u_2^{-1}$ is non trivial but $w_i$ commutes with $u_1$ and $u_2$. However if we let one or both of the $w_i$ be primitive elements, it is still not known exactly when $G$ is a limit group. Some cases are known, for instance doubles (where $r_1 = r_2$ and $w_1 = w_2$) have been shown to be limit groups but if only one element is primitive then examples of non limit groups go back to Lyndon in 1959. He
showed that any solution to the equation \( x^2 = y^2 z^2 \) in a free group has the property that \( x, y \) and \( z \) all commute. This means that if \( G = \mathbb{F}_{r_1} \ast_u = v^u w^u \mathbb{F}_{r_2} \) for \( u, v, w \) any non trivial words then any homomorphism from \( G \) onto a free group \( F \) sends \( u, v, w \) to commuting elements in \( F \). Thus if \( v \) and \( w \) do not commute in \( \mathbb{F}_{r_2} \) then all homomorphisms from \( G \) to a free group send the commutator \([v, w]\) to the identity. Moreover the same property was shown in \([24]\) to hold for the equation \( x^l = y^m z^n \) for \( l, m, n \geq 2 \) so that an element of the form \( y^m z^n \) is not a proper power in a free group if \( y \) and \( z \) do not commute, and thus \( G = \mathbb{F}_{r_1} \ast_u = v^u w^u \mathbb{F}_{r_2} \) is a word hyperbolic group which is not residually free.

Let us now assume that both \( w_1 \) and \( w_2 \) are primitive elements. If the process of forming repeated amalgams of free groups over maximal cyclic subgroups always resulted in a limit group then Corollary 3.8 would follow immediately. In fact this is not true although we are unaware of specific examples in the literature so will give the first ones here. Our reference on this question is \([28]\) where it is mentioned in Part I (3) that “if the element \( w_1 \) is a commutator in the first free group and \( w_2 \) is a product of two “high” powers in the other free group then \( G \) is not a limit group” but no further details are given. However this is not true in full generality, even for any definition of “high”: if \( G_1 \) is free on \( x, y \); \( G_2 \) free on \( a, b \) and \( G = G_1 \ast w_1 = w_2 G_2 \) is formed by setting \([x, y] = a^m b^n \) for \( m, n \neq 0 \) with highest common factor \( d \) then consider the homomorphism from \( G \) onto the free group \( F(t, u) \) on \( t, u \) given by \( x \mapsto t, y \mapsto u^{mn/d} \) and \( a \mapsto tu^{n/d}, b \mapsto u^{-m/d} \). This sends both rank 2 free groups \( G_1 \) and \( G_2 \) to rank 2 free subgroups of \( F(t, u) \), thus the restriction to each \( G_i \) is injective. Consequently \( G \) is an example of a generalised double over \( F(t, u) \) as in Definition 4.4 of \([8]\), with Proposition 4.7 of \([8]\) showing that a generalised double over a limit group is also a limit group.

In fact the quote above becomes true if we change “powers” to “powers of commutators” with a suitably weak definition of “high”. To prove this we will use stable commutator length, as described in \([6]\). This can be explained briefly as follows: a length on a group \( G \) is a function \( l : G \to \mathbb{R} \) such that for \( g, \gamma \in G \) we have

\[
l(g \gamma) \leq l(g) + l(\gamma) \quad \text{and} \quad l(g) = l(g^{-1}).
\]

(Sometimes \( l(e) = 0 \) is required but this will not affect any of our results here.) From any length function \( l \) we obtain stable length \( \sigma : G \to \mathbb{R} \)
defined by $\sigma(g) = \lim_{n \to \infty} l(g^n)/n$. As for any $g \in G$ the sequence $a_n = l(g^n)$ is subadditive (meaning that $a_{n+m} \leq a_m + a_n$ for $n, m \in \mathbb{N}$), this limit exists provided only that there is $K \leq 0$ with $a_n \geq Kn$ for all $n$. It is straightforward to show using only the above properties that $\sigma$ is constant on conjugacy classes, that $\sigma(g^k) = |k|\sigma(g)$ for all $k \in \mathbb{Z}$ and $g \in G$, and that $\sigma(gh) \leq \sigma(g) + \sigma(h)$ if $g$ and $h$ commute (although not in general). Given a group $G$, \textbf{commutator length} $cl$ is a length on the commutator subgroup $G'$ of $G$ with $cl(g)$ defined to be the minimum number of commutators needed to form a product equal to $g$ and \textbf{stable commutator length} is defined to be the stable length that results, denoted by $scl(g)$. A non trivial fact about $scl$ in free groups $F$ which we will need here is that every non identity element $w \in F'$ has $scl(w) \geq 1/2$ by [6] Theorem 4.111. Combining this with the point that in general a commutator $[w_1, w_2]$ has $scl([w_1, w_2]) \leq 1/2$, we see that non trivial commutators in free groups have stable commutator length exactly $1/2$.

**Theorem 4.1** If $F_1$ and $F_2$ are free non abelian groups then the cyclically pinched group $G = F_1 \ast_{\gamma = \delta^m \eta^n} F_2$, where $\gamma$ is a non trivial commutator in $F_1$ and $\delta, \eta$ are both non trivial commutators in $F_2$ with $\delta \neq \eta^{\pm 1}$, is not residually free whenever $|m| - |n| \geq 3$ and $|m|, |n| \neq 0$ or $1$, even though $G$ embeds in $SL(2, \mathbb{C})$.

**Proof.** We can assume by changing elements to inverses that $m, n > 0$. We know that $\gamma \neq e$ in $G$ so suppose we have a homomorphism $\theta$ from $G$ to a free group $F$ where $\theta(\gamma) = a \neq e$. This gives us a non trivial commutator in $F$ which is equal to $b^m c^n$ for two (possibly trivial) commutators $b = \theta(\delta)$ and $c = \theta(\eta)$ in $F$. First suppose that $b$ is trivial in $F$ then $a = c^n$ but in a free group a non trivial commutator cannot be a proper power. (This is due to Schützenberger but can also be seen here because in a free group $scl(a) = 1/2 = |n|scl(c) \geq |n|/2 \geq 1$, using the fact that $c$ must be in the commutator subgroup $F'$ too.) The same holds if $c$ is trivial. As $\gamma$ is primitive and we mentioned that $\delta^m \eta^n$ is also primitive, Corollary 3.8 tells us that $G$ embeds in $SL(2, \mathbb{C})$. Therefore we are done by the next Proposition.

**Proposition 4.2** If the equation $a = b^m c^n$ holds in a free group $F$ where $a, b, c$ are all non trivial commutators in $F$ then we cannot have $|m| - |n| \geq 3$. 

\[\Box\]
Proof. By Theorem 2.70 in [6] using Barvard duality, we have that for any $g$ in $G'$, 

$$\text{scl}(g) = \frac{1}{2} \sup_{\phi \in Q(G)} \frac{|\phi(g)|}{|D(\phi)|}$$

where $Q(G)$ is the space of homogeneous quasimorphisms on $G$ and $D(\phi)$ is the defect of $\phi$. This means that for all $a, b \in G$ and $k \in \mathbb{Z}$ we have $|\phi(ab) - \phi(a) - \phi(b)| \leq D(\phi)$ (a quasimorphism) and (homogeneity) $\phi(a^k) = k\phi(a)$.

Thus we can take a homogeneous quasimorphism $\phi_b \in Q$ with $|\phi_b(b)|/D(\phi_b)$ arbitrarily close to 1 as $\text{scl}(b) = 1/2$. By rescaling, let us say that $D(\phi_b) = 1$ and we can assume $m\phi_b(b) > n + 2$ because $m - n \geq 3$. (In fact in a free group this supremum is obtained but we will not need to use that here.)

Now for any homogeneous quasimorphism $q$ on any group $G$ we have that $|q(\gamma)| \leq D(q)$ if $\gamma$ is a commutator in $G$. But as $a = b^m c^n$ holds in $F$ we see that

$$m\phi_b(b) - n - 1 \leq m\phi_b(b) - n|\phi_b(c)| - |\phi_b(a)| \leq |m\phi_b(b) + n\phi_b(c) - \phi_b(a)| \leq 1$$

so $m\phi_b(b) \leq n + 2$, giving us a contradiction.

5 2 Dimensional linearity of 3-manifold groups

Despite Perelman’s positive solution to Thurston’s Geometrisation Conjecture, which as a consequence establishes the fact that all finitely generated 3-manifold groups are residually finite, it is still an open question as to whether all such groups are linear. Indeed in [20] Problem 3.33 (A) Thurston asks whether all such groups can be embedded in $GL(4, \mathbb{R})$, which was unknown until just now [5]. There has been very recent progress, using results of Wise on virtually special groups, which shows that the fundamental group $\pi_1(M^3)$ of most compact 3-manifolds $M^3$ has a finite index subgroup which embeds in a right angled Artin group, thus this subgroup and hence also $\pi_1(M^3)$ will be linear over $\mathbb{Z}$, although the dimension might be very big. Indeed following the release of [21] and [27], the only compact orientable irreducible 3-manifolds not having virtually special fundamental group are those closed Siefert fibre spaces and those closed graph 3-manifolds which do not admit
a Riemannian metric of non positive curvature, but the former are known to be linear over \( \mathbb{Z} \).

Here we will restrict ourselves to examining which closed 3-manifolds have fundamental groups that embed in \( SL(2, \mathbb{C}) \), in addition to those that embed discretely. Our first result is for torus bundles, none of which have fundamental groups that embed discretely.

**Theorem 5.1** If \( M^3 \) is a closed 3-manifold which is a 2 dimensional torus bundle over the circle, so that \( \pi_1(M^3) = \mathbb{Z}^2 \rtimes \alpha \mathbb{Z} \) where \( \alpha \) is the automorphism of \( \mathbb{Z}^2 \) induced by the gluing map, then \( \pi_1(M^3) \) is a subgroup of \( SL(2, \mathbb{C}) \) if and only if \( \alpha \) is the identity or is a hyperbolic map (that is all powers of \( \alpha \) fix only 0).

**Proof.** Let \( G = \pi_1(M^3) \) and \( \mathbb{Z}^2 = \langle a, b \rangle \), with conjugation by the stable letter \( t \) giving our automorphism \( \alpha \). If this is the identity then \( G = \mathbb{Z}^3 \) embeds in \( SL(2, \mathbb{C}) \). If some positive power \( \alpha^n \) fixes \( x \in \mathbb{Z}^2 - \{0\} \) and \( G \) embeds in \( SL(2, \mathbb{C}) \) then \( t \) commutes with \( x \) because \( G \) is CT, but \( x \) commutes with all of the fibre subgroup \( \mathbb{Z}^2 \) so \( t \) does too and so we have the identity automorphism.

Otherwise the eigenvalues of \( \alpha \) are not roots of unity and so not of modulus 1, because they satisfy a monic integer quadratic with constant term \( \pm 1 \). Say \( \alpha(a) = tat^{-1} = a^i b^j \) and \( \alpha(b) = tbt^{-1} = a^k b^l \). We set

\[
a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}
\]

where \( x \) and \( \mu \) are complex numbers to be determined. For the above two relations to be satisfied, we require \( \mu^2 = i + xj \) and \( \mu^2 x = k + xl \). This just corresponds to the matrix \( \begin{pmatrix} i & j \\ k & l \end{pmatrix} \in GL(2, \mathbb{Z}) \) having an eigenvalue \( \mu^2 \) with the eigenvector \( \begin{pmatrix} 1 \\ x \end{pmatrix} \) which does occur, and \( x \) is not zero because \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) being an eigenvector implies that there was an eigenvalue of \( \pm 1 \). Moreover \( x \in \mathbb{Q} \) implies that \( \mu^2 \in \mathbb{Z} \) but the determinant being \( \pm 1 \) would give \( \mu^2 = \pm 1 \) too, which has been eliminated. Therefore \( \mu^2 \) not being a root of unity implies that the matrix \( t \) has infinite order, and the matrix

\[
a^p b^q t^r = \begin{pmatrix} \mu^r & \mu^{-r} (p + xq) \\ 0 & \mu^{-r} \end{pmatrix}
\]

only be the identity if \( p = q = r = 0 \), so
Remarks: (1) It did not matter here if $\alpha$ was orientation preserving or reversing and hence if $M^3$ were orientable or not.
(2) This shows just how much variation there can be in the geometry of a 3-manifold and still the fundamental group embeds in $SL(2, \mathbb{C})$. For instance we can take the connected sum of a Lens space (with finite cyclic fundamental group), the 3-torus (with fundamental group $\mathbb{Z}^3$), a closed orientable hyperbolic 3-manifold and a torus bundle with a hyperbolic automorphism as in Theorem 5.1. The fundamental group of the resulting 3-manifold embeds in $SL(2, \mathbb{C})$ by Lemma 3.1 but there are pieces of this manifold which are positively curved, have zero curvature, have negative curvature and which do not admit a metric of nonpositive or of nonnegative curvature.

Based on these examples, one might make a naive conjecture as to when the fundamental group $G = \pi_1(M^3)$ of a closed orientable irreducible 3-manifold $M^3$ embeds in $SL(2, \mathbb{C})$ as follows: first suppose that $M^3$ admits one of the eight 3-dimensional geometries. We go work through the groups obtained from manifolds having $S^3$ geometry and list those which embed in $SL(2, \mathbb{C})$ and we do the same with manifolds having the Sol (virtually soluble) geometry (where all groups will have a finite cover of the form in Theorem 5.1 with hyperbolic gluing map). Then of course all groups coming from manifolds with the $H^3$ geometry embed, but we obtain no further groups from the other geometries apart from $Z$ and $Z^3$. This is because now $\pi_1(M^3)$ must be infinite and torsion free. Either we have the $S^2 \times \mathbb{R}$ geometry (giving virtually cyclic groups), the $E^3$ geometry (virtually abelian), or we have Nil (virtually nilpotent groups), $H^2 \times \mathbb{R}$ or $PSL(2, \mathbb{R})$ geometries. The last two cases give rise to Siefert fibred spaces so here the fundamental group will have a finite cover having a normal infinite cyclic subgroup $\langle x \rangle$. This means that $gxg^{-1} = x^{\pm 1}$ for all group elements $g$ in the finite cover, so the centraliser of $x$ has finite index in the fundamental group. Consequently Lemma 2.1 applies to all five of these geometries.

This leaves us with closed orientable irreducible 3-manifolds which do not possess a geometry but which can be cut along embedded incompressible tori to obtain hyperbolic 3-manifolds with torus boundary and Siefert fibre spaces with torus boundary. Again we have by Lemma 2.1 that the fundamental group of a Siefert fibre space cannot embed in $SL(2, \mathbb{C})$ unless it is abelian. As we have nonempty boundary here, this leaves only $Z$ and $Z \times Z$. Thus if there is a Siefert fibre space in the torus decomposition of $M^3$ then its
fundamental group will be a subgroup of $\pi_1(M^3)$ which cannot therefore embed in $SL(2, \mathbb{C})$. Therefore we might finish off our conjecture by hoping that if all the pieces in the torus decomposition of $M^3$ are hyperbolic (and hence embed in $SL(2, \mathbb{C})$) then $\pi_1(M^3)$ is a subgroup of $SL(2, \mathbb{C})$. However this turns out not to be the case, even if $\pi_1(M^3)$ is a CSA group.

**Theorem 5.2** Let $M^3_1, M^3_2$ be two copies of the figure eight knot complement with each boundary $\partial M^3_1, \partial M^3_2$ a torus, and let $M^3$ be the closed 3-manifold formed by gluing the boundary tori together using any orientation preserving homeomorphism which identifies the two meridians, with the exception of the homeomorphism which also identifies the two longitudes. Then $\pi_1(M^3)$ does not embed in $SL(2, \mathbb{C})$ although it is CT and even a CSA group.

**Proof.** Although $M^3$ has only one discrete faithful embedding in $SL(2, \mathbb{C})$ (because it has finite hyperbolic volume so Mostow rigidity applies), up to conjugation in $SL(2, \mathbb{C})$, replacing matrices with their negative and taking complex conjugates, there is a whole curve of representations. This curve was found in [31] where it was shown that if $A$ and $B$ are two non commuting elements of $SL(2, \mathbb{C})$ satisfying

$$r(A, B) = B^{-1}A^{-1}BAB^{-1}ABA^{-1}B^{-1}A = I$$

then we can conjugate $A$ and $B$ so that either

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \text{ where } \omega = e^{2\pi i/3} \text{ or } e^{-2\pi i/3}$$

(1)

(or $A$ and $B$ are both minus the above) or we can take

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } B = \begin{pmatrix} \mu & 1 \\ \mu(x - \mu) - 1 & x - \mu \end{pmatrix}$$

(2)

(or both minus this) where $\lambda \in \mathbb{C} - \{0, \pm 1\}$, $x = \lambda + \lambda^{-1}$,

$$z = \frac{1}{2} \left( 1 + x^2 \pm \sqrt{(x^2 - 1)(x^2 - 5)} \right) \text{ and } \mu = \frac{\lambda z - x}{\lambda^2 - 1}.$$ 

The figure eight knot complement can also be thought of as the once punctured torus bundle given by taking the free group of rank 2 on $a, b$ and forming the HNN extension

$$\langle t, a, b | tat^{-1} = aba, tbt^{-1} = ba \rangle.$$
with stable letter $t$. Then $\langle t, aba^{-1}b^{-1} \rangle = \mathbb{Z} \times \mathbb{Z}$ which forms the boundary torus with $t$ the meridian and $aba^{-1}b^{-1}$ the longitude. That this is isomorphic to the group $\langle A, B | r(A, B) \rangle$ given above can be seen by setting $a = BA^{-1}$, $b = B^{-1}ABA^{-1}$ and $t = A$, so that $A = t$ and $B = at$ which is also equal to $b^{-1}tb$, explaining why the matrices $A$ and $B$ given above are conjugate in $SL(2, \mathbb{C})$. This means that the meridian $m$ is equal to $A$ and the longitude is $l = BA^{-1}B^{-1}A^2B^{-1}A^{-1}B$.

We first make a couple of observations about the form of the matrices in (2) in order to exploit the symmetries that are present. First note that if we replace $\lambda$ with $\lambda^{-1}$ in (2) then the top left and bottom right hand entries of $A$ are swapped whereas the other entries stay the same, and also for $B$. However we can now conjugate the new $A$ and $B$ by \[
\begin{pmatrix}
0 & \nu \\
-\nu & 0
\end{pmatrix}
\] for an appropriate choice of $\nu$ so that they become the original $A$ and $B$. Thus we can change $\lambda$ to $1/\lambda$ if desired because they both represent the same parametrisation up to conjugacy. Now the longitude $l$ commutes with $A$ and so should be of the form \[
\begin{pmatrix}
f(\lambda) & 0 \\
0 & f(\lambda)^{-1}
\end{pmatrix}
\] for some function $f$, because the entries of any element in this group depend only $\lambda, z, x$ and $\mu$, with the last three variables all functions of $\lambda$. But there appears to be an ambiguity in that plus or minus a square root is taken in the definition of $z$, which would result in two possible functions $f_+(\lambda)$ or $f_-(\lambda)$ giving rise to the entries on the diagonal of $l$.

However we can use the standard identities to find the trace $\tau$ of $l$ in terms of the traces of $A, B$ and $AB$, which here are equal to $x, x$ and $z$ respectively. The answer is $\tau(x, x, z) = 2 + x^2(z - 2)(x^4 + (z + 2)(z + 2 - 2x^2))$ but we also have the equation $z^2 = (1 + x^2)z - 2x^2 + 1$ holding which is quadratic in $z$. If we now substitute this expression for $z^2$ into $\tau$ we find that $z$ vanishes, so we are left with the trace of $l$ being a function of $x$ only, hence of a function of $\lambda$ only too.

We now suppose that we have an embedding of $G = \pi_1(M^3)$ in $SL(2, \mathbb{C})$ where $G = G_1 *_{\mathbb{Z} \times \mathbb{Z}} G_2$ for $G_i = \pi_1(M^3_i)$ and $\mathbb{Z} \times \mathbb{Z}$ is the fundamental group of the boundary torus. We can conjugate $G$ in $SL(2, \mathbb{C})$ so that without loss of generality we have $G_1 = \langle A_1, B_1 \rangle$ with $A_1, B_1$ matrices in the form above, giving rise to the meridian $m_1 = A_1$ and the longitude $l_1(A_1, B_1)$. First suppose that $A_1$ and $B_1$ are of the form in (2) for some parameter $\lambda_1 \neq 0, \pm 1$ (and other parameters $x_1, z_1, \mu_1$ depending on $\lambda_1$), so that the
longitude $l_1(A_1, B_1)$ will also be a diagonal matrix $\begin{pmatrix} d_1 & 0 \\ 0 & d_1^{-1} \end{pmatrix}$ say, with

$\tau(\lambda_1) = d_1 + d_1^{-1}$. We also have the meridian $m_2$ and longitude $l_2$ of $G_2$ and we are forcing $m_2$ to be equal to $m_1^{-1} = A_1^{-1}$, so that the two figure 8 knot complements are joined on either side of the boundary torus. Moreover the homeomorphism must identify the longitude $l_2$ of $M_2^3$ with the curve $m_1^{-1}l_1$ in $\partial M_2^3$ to obtain an orientation reversing homeomorphism between the two boundary tori so as to match the orientations of the 3-manifolds. In particular the longitude $l_2$ of $G_2$ must be a diagonal matrix $\begin{pmatrix} d_2 & 0 \\ 0 & d_2^{-1} \end{pmatrix}$ say, because it commutes with $m_2 = m_1^{-1}$.

Now although the group $G_2$ has a fixed embedding in $SL(2, \mathbb{C})$ because $G$ has been conjugated in order to put $G_1$ into a suitable form, we can also separately conjugate $G_2$, by $X \in SL(2, \mathbb{C})$ say, so that $XG_2X^{-1} = \langle A_2, B_2 \rangle$ where again $A_2, B_2$ are as in (2) though this time with the parameter $\lambda_2$.

But $X^{-1}A_2^{-1}X = m_2^{-1} = A_1$ is equal to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and the only way that $A_2^{-1} = \begin{pmatrix} \lambda_2^{-1} & 0 \\ 0 & \lambda_2 \end{pmatrix}$ can be conjugate to this is if $\lambda_2 = \lambda_1$ or $\lambda_1^{-1}$. Therefore we can assume by the above comment that $\lambda_2$ is equal to $\lambda_1$. But the longitude in $\langle A_2, B_2 \rangle$ is the element $Xl_2X^{-1}$ which will have trace $\tau(\lambda_2) = \tau(\lambda_1)$. Thus the trace $d_2 + d_2^{-1}$ of $l_2$ is also $\tau(\lambda_1) = d_1 + d_1^{-1}$, giving us that $d_2 = d_1^{\pm 1}$. But as $l_2 = m_1^n l_1$ is a product of diagonal matrices, we obtain $d_2 = \lambda_1^n d_1$. Thus we either have $d_1^n \lambda_1^n = 1$, which is a contradiction because $\langle l_1, m_1 \rangle = \mathbb{Z} \times \mathbb{Z}$, or $\lambda_1^n = 1$ which is also a contradiction unless $n = 0$.

The case in (1) is similar but quicker. We can assume by conjugating $G$ in $SL(2, \mathbb{C})$, as well as taking minus signs and complex conjugation if necessary, that $A_1 = A$ and $B_1 = B$ in (1) for $\omega = e^{2\pi i / 3}$ so a quick calculation tells us that $l_1 = \begin{pmatrix} -1 & -2\sqrt{3}i \\ 0 & -1 \end{pmatrix}$. Also we again have $X \in SL(2, \mathbb{C})$ so that $XG_2X^{-1} = \langle A_2, B_2 \rangle$ for $A_2, B_2$ as in (1) but with $\omega = e^{\pm 2\pi i / 3}$ (we can rule out having to put minus signs in front of $A_2$ and $B_2$ because $A_2 = XA_1X^{-1}$ so $A_2$ has trace 2). Thus $Xl_2X^{-1}$ is equal to $l_1$ or its complex conjugate. Now $l_2 = m_1^n l_1 = \begin{pmatrix} -1 & -2\sqrt{3}i - n \\ 0 & -1 \end{pmatrix}$ but we know $A_2 = m_2 = m_1^{-1} = A_1^{-1}$ so $X$ conjugates $A_1$ into its inverse and therefore can only be $\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. 

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But on comparing $Xl_2X^{-1}$ and $l_1$, we again see that $n$ can only be zero.

As for the last part, it is well known that discrete torsion free subgroups of $SL(2, \mathbb{C})$ are CSA groups. For instance this can be seen quickly by using Corollary 2.3 along with the straightforward fact that any group containing $\pm \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right)$ for $x \neq 0$ and any infinite order element of the form $\left( \begin{array}{cc} \lambda & y \\ 0 & \lambda^{-1} \end{array} \right)$ for $\lambda \neq 0, \pm 1$ and $y \in \mathbb{C}$ is non discrete. Thus $G_1$ and $G_2$ are CSA, so we are done by Lemma 2.5.

In the case where the longitudes are glued to each other we do in fact have an embedding of $G$ in $SL(2, \mathbb{C})$, giving an example of a closed orientable 3-manifold $M$ fibred over the circle with fibre a genus 2 surface and with a non pseudo-Anosov gluing homeomorphism but such that $\pi_1(M)$ still embeds in $SL(2, \mathbb{C})$. This follows from Proposition 3.2, provided we can show that there are faithful embeddings of the fundamental group of the figure 8 knot complement of the form (2) in Theorem 5.2:

**Proposition 5.3** For any transcendental number $\lambda \in \mathbb{C}$, the matrices in (2) provide a faithful embedding in $SL(2, \mathbb{C})$ of the fundamental group of the figure 8 knot complement $G$.

**Proof.** On taking $\lambda$ to be any transcendental number, we have that $x$ and $z$ will be transcendental too (for either choice of $z$). Now it is well known that the trace of any element in $\langle A, B \rangle$ is given by a triple variable polynomial in the trace of $A$, of $B$, and of $AB$, having coefficients in $\mathbb{Z}$. Therefore if there is an element $g$ in $G$ which has trace $\pm 2$ for this value of $\lambda$ then we know that some polynomial $f$ in $\mathbb{Z}[u, v]/(r(u, v))$ is zero at $(u, v) = (x(\lambda), z(\lambda))$ for one of the choices of $z$, where $r(u, v)$ is the irreducible polynomial $v^2 - (1 + u^2)v + 2u^2 - 1$. Using this relation we can assume that $f(u, v)$ is of the form $up(u) + q(u) = 0$ for $p, q \in \mathbb{Z}[u]$ which implies that $q^2 - p^2 + 2u^2p^2 + (1 + u^2)pq$ is zero when evaluated at $x$. As $x$ is transcendental, this polynomial must be identically zero and so $(pz + q)(pz - p(1 + x^2) - q) = 0$ holds for all values of $x$ and $z$ satisfying $f(x, z) = 0$. Thus it must be a multiple of $r(x, z)$, which is an irreducible degree 2 polynomial in $z$, giving a contradiction unless $p = q = 0$. Hence the trace of $g$ is constant in all representations. But on setting $x = 2$ so that $z = (5 \pm \sqrt{-3})/2 = 2 - \omega$, we have the faithful discrete representation in (1). In this case we know that
the elements with trace $\pm 2$ can all be conjugated to lie in the $\mathbb{Z} \times \mathbb{Z}$ subgroup $\langle m(A, B), l(A, B) \rangle$. Now $m(A, B) = A$ and $l(A, B)$ are both diagonal matrices in all other representations we are considering. Hence if $m^i l^j$ has (without loss of generality) trace equal to 2 when $\lambda$ is transcendental, this element must be the identity and hence the identity for any $\lambda$. But on putting say $\lambda = 2$ we find that this cannot hold unless $i = j = 0$.

\[ \Box \]

6 Embeddings of HNN extensions in $SL(2, \mathbb{C})$

Having seen cases where we can embed free products amalgamated over $\mathbb{Z}$ in $SL(2, \mathbb{C})$ as long as the factors have this property, we now see that this is very different with HNN extensions. This is perhaps not surprising, given that if we form an HNN extension of $G$ with stable letter $t$ and isomorphic associated subgroups $A$ and $B$ then we introduce relations of the form $tat^{-1} = b$ for $a \in A$ and $b \in B$, forcing $a$ and $b$ to have the same trace.

We start with a result giving conditions on when HNN extensions do embed in $SL(2, \mathbb{C})$, in analogy with Corollary 3.5.

**Theorem 6.1** Suppose that $G$ is a countable subgroup of $SL(2, \mathbb{C})$ such that no element of $G$ has trace $\pm 2$ except the identity (so that $G$ is a CSA group by Corollary 2.3). Let $A \neq G$ be any centraliser in $G$ and $B = gAg^{-1}$ any centraliser that is conjugate to $A$ by $g \in G$. Then the HNN extension $\Gamma = G \ast tAt^{-1} = B$, formed using the isomorphism from $A$ to $B$ which is conjugation by $g$, embeds in $SL(2, \mathbb{C})$ and no non-identity element of $\Gamma$ has trace equal to $\pm 2$.

**Proof.** We are forming the group

$$\Gamma = \langle G, t | tat^{-1} = gag^{-1} \forall a \in A \rangle$$

which is isomorphic to the HNN extension

$$\langle G, s | sas^{-1} = a \forall a \in A \rangle$$

by setting $s = g^{-1}t$, so we can assume our HNN extension is formed using the identity map.
We now take $a \in A - \{e\}$ and conjugate $G$ in $SL(2, \mathbb{C})$ so that $a$ is a diagonal matrix with distinct eigenvalues. This means that $A$ is now precisely the subgroup $D$ of diagonal elements in $G$. Thus by Lemma 3.4 no element of $G - D$ has a zero entry. We set $t = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ where $x \in \mathbb{C}$ is transcendental over the coefficient field $F$ generated by the elements of $G$. Now $t$ commutes with all elements of $A$ so we need to show that $\langle t, G \rangle \leq SL(2, \mathbb{C})$ is a faithful representation of the HNN extension $\Gamma$. By using normal forms, any $\gamma \in \Gamma$ which is not a conjugate of an element of $G$ (or a power of $t$) can be conjugated so it is of the form

$$\gamma = t^{n_1} g_1 t^{n_2} g_2 \ldots t^{n_r} g_r$$

for $r \geq 1, n_i \in \mathbb{Z} - \{0\}$ and where each $g_i$ is in $G - A$.

Now each entry of $\gamma$ is a Laurent polynomial in $x$, involving positive and negative powers, with coefficients in $F$. We set $n$ to be $|n_2| + \ldots + |n_r|$ which is greater than 0 for $r \geq 2$ as well as $S = n_1 + n, D = n_1 - n$ and we claim by induction on $r$ that $\gamma$ must be of the form

$$\begin{pmatrix} a_+ x^S + \ldots + u_+ x^D & a_- x^S + \ldots + u_- x^D \\ c_+ x^{-D} + \ldots + v_+ x^{-S} & c_- x^{-D} + \ldots + v_- x^{-S} \end{pmatrix}$$

where $a_\pm, c_\pm, u_\pm, v_\pm$ are all non zero elements of $F$ and we have only written down the end terms (the highest and lowest powers of $x$) of each Laurent polynomial. This is because on increasing $r$ by 1, we obtain a new element $t^{n_r+1} g_{r+1}$ equal to

$$\begin{pmatrix} a_+ x^m & \beta_+ x^m \\ a_- x^{-m} & \beta_- x^{-m} \end{pmatrix}$$

where we have set $m$ equal to $n_{r+1}$ and $a_\pm, \beta_\pm$ are the entries of $g_{r+1}$ so are again non zero elements of $F$.

Thus our new element $t^{n_1} g_1 t^{n_2} g_2 \ldots t^{n_r} g_r t^{n_{r+1}} g_{r+1}$ is equal to

$$\begin{pmatrix} a_+ \alpha_\pm x^{S+|m|} + \ldots + u_+ \alpha_\pm x^{D-|m|} & a_+ \beta_\pm x^{S+|m|} + \ldots + u_+ \beta_\pm x^{D-|m|} \\ c_\pm \alpha_\pm x^{-D+|m|} + \ldots + v_\pm \alpha_\pm x^{-S-|m|} & c_\pm \beta_\pm x^{-D+|m|} + \ldots + v_\pm \beta_\pm x^{-S-|m|} \end{pmatrix}$$

where we take the upper signs throughout if $m > 0$ and the lower for $m < 0$. Hence the induction is established and we see that each entry is a Laurent polynomial in $x$ with more than one non zero term, so each entry is transcendental. Moreover the trace of $\gamma$ is a Laurent polynomial of the form
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$$a_+x^{n_1+n} + \ldots + v_-x^{-n_1-n} \text{ if } n_1 > 0 \text{ and } c_-x^{-n_1+n} + \ldots + u_+x^{n_1-n} \text{ otherwise,}$$

thus all traces of elements outside $G$ are transcendental.

Consequently Theorem 6.1 can be applied iteratively, thus allowing repeated HNN extensions which will always result in a CSA group that embeds in $SL(2, \mathbb{C})$. As pointed out by H. Wilton, this gives a quick and explicit proof that all limit groups embed in $SL(2, \mathbb{C})$: an free extension of centralisers of a group $G$ is the group $G\ast_C (C \times A)$ where $C$ is the centraliser of an element in $G$ and $A = \mathbb{Z}^k$ is a finitely generated free abelian group. Theorem 4 of [19], with an alternative proof given in [8] as Theorem 4.6, states that a finitely generated group is a limit group if and only if it is a subgroup of a group formed by a finite number of repeated free extensions of centralisers, starting from a finitely generated free group. However we can write

$$G \ast_C (C \times \mathbb{Z}^2) = (G \ast_C (C \times \mathbb{Z})) \ast_{C \times \mathbb{Z}} ((C \times \mathbb{Z}) \times \mathbb{Z})$$

and so on. Although $C \times \mathbb{Z}$ need not be a centraliser in the new group $\Gamma = G \ast_C (C \times \mathbb{Z})$, this is true if $C$ is malnormal in $G$ and then $C \times \mathbb{Z}$ is malnormal in $\Gamma$. Therefore if $G$ is a CSA group we need only apply a rank 1 free extension of centralisers $k$ times to go from $G$ to $G \ast_C (C \times \mathbb{Z}^k)$. But a rank 1 extension is simply an HNN extension $G\ast_{\mathbb{Z}^1\mathbb{Z}}$ using the identity map, so on starting with an embedding of a free group and applying Theorem 6.1 repeatedly then dropping to a subgroup, we have that any limit group embeds in $SL(2, \mathbb{C})$. This result was also obtained in [11] using similar techniques but they only considered the case of hyperbolic limit groups.

The HNN extensions permitted in Theorem 6.1 can only identify two conjugate maximal abelian subgroups and can only use a conjugation map as the identifying isomorphism, which is very restrictive. However the next Proposition, which is really just a restatement of the point mentioned in Section 2 that the Klein bottle group does not embed in $SL(2, \mathbb{C})$, shows that such restrictive conditions are required.

**Proposition 6.2** Let $G$ embed in $SL(2, \mathbb{C})$ and let $\Gamma$ be an HNN extension of $G$ with stable letter $t$ which identifies two isomorphic subgroups $A, B$ of $G$ using an isomorphism $\theta$. If there is $a \in A$ (where $a \neq \pm I$) and $b \in B$ with $\theta(a) = b$ such that $b$ is conjugate in $G$ to $a^{-1}$ then $\Gamma$ does not embed in $SL(2, \mathbb{C})$. 
Proof. We have $tat^{-1} = ga^{-1}g^{-1}$ holding in $\Gamma$ and if this held in $SL(2, \mathbb{C})$ then the matrix $g^{-1}t$ could only have order 4, but in any HNN extension an element which is equal to the identity must have zero exponent sum in the stable letter.

In particular, although we mentioned in Section 3 that a graph of non abelian free groups with maximal cyclic edge groups embeds in $SL(2, \mathbb{C})$ if the graph is a tree, we see that this need not be true for graphs with loops.

All of the HNN extensions in Proposition 6.2 which fail to embed in $SL(2, \mathbb{C})$ contain $\mathbb{Z} \times \mathbb{Z} = \langle a^2, g^{-1}t \rangle$, at least if $a$ is of infinite order, and so will not be word hyperbolic. We now give an example of a graph of non abelian free groups with maximal cyclic edge groups which is word hyperbolic but which does not embed in $SL(2, \mathbb{C})$. We have a result in [4] giving conditions on when an HNN extension over a virtually cyclic group is word hyperbolic, alongside the theorem already mentioned in [3] for amalgamated free products. For a torsion free word hyperbolic group $G$, this states that if $A, B$ are infinite cyclic subgroups of $G$ then an HNN extension formed by identifying $A$ and $B$ is word hyperbolic if and only if for all $g \in G$ we have $gAg^{-1} \cap B = \{e\}$ and where at least one of $A$ and $B$ is generated by a primitive element. This is also equivalent to saying that the HNN extension does not contain a Baumslag Solitar subgroup. (This result is false if the primitive condition is removed, as was originally stated in [3], hence necessitating the appearance of [4].) Once again we cannot use this result repeatedly as it stands but if $A$ and $B$ are both generated by primitive elements then we can take multiple HNN extensions and still obtain word hyperbolicity by this result and Lemma 2.6. We do this until too many elements are forced to be conjugate to each other.

**Proposition 6.3** Let $F_2$ be free on $a, b$ and let $\Gamma$ be the triple HNN extension formed using stable letters $t, s, r$ so that $tat^{-1} = b, sas^{-1} = ab$ and $rar^{-1} = aba^{-1}b^{-1}$. Then $\Gamma$ does not embed in $SL(2, \mathbb{C})$ but is a graph of non abelian free groups with maximal cyclic edge groups which is word hyperbolic.

Proof. If $\Gamma$ did embed in $SL(2, \mathbb{C})$ then $a$ and $b$ would generate a rank two free group where $a, b, ab$ and $aba^{-1}b^{-1}$ all have the same trace, $z \in \mathbb{C}$ say. But using the well known trace identities in $SL(2, \mathbb{C})$ which go back to Fricke and Klein, we have that

$$tr^2(a) + tr^2(a) + tr^2(ab) - 2 = tr(a) tr(b) tr(ab) + tr(aba^{-1}b^{-1})$$
so \((z - 2)(z^2 - z - 1) = 0\), but \(z = 2\) would imply that \(\langle a, b \rangle\) is metabelian and we have elements of order 5 otherwise.

Now \(\Gamma\) is formed using a graph of groups consisting of one vertex representing \(F_2\) and three loops for the three pairs of edge groups, all of which are maximal cyclic in \(F_2\). To see that \(\Gamma\) is word hyperbolic, we apply Bestvina and Feighn’s result above each time in conjunction with Lemma 2.6, noting that \(a, b, ab, aba^{-1}b^{-1}\) are non conjugate primitive elements of \(F_2\).

\[\blacksquare\]

We finish this section by mentioning some recent work by Hsu and Wise on the linearity of graphs of free groups with infinite cyclic edge groups. In [17] it was shown that if such a group is word hyperbolic then it is linear over \(\mathbb{Z}\), although one would expect the dimension to be very high. Now the Bestvina-Feighn results tell us that such a group is word hyperbolic if and only if it does not contain any Baumslag Solitar subgroup. Let us call a Baumslag Solitar group \(BS(m, n)\) Euclidean if \(|n| = |m|\) and non Euclidean otherwise. Then a non Euclidean Baumslag Solitar group cannot be linear over \(\mathbb{Z}\) (as either it fails to be residually finite or it is soluble but not polycyclic). However the Euclidean ones are linear over \(\mathbb{Z}\) (as they have a finite index subgroup \(F_n \times \mathbb{Z}\)) so it is possible that the necessary condition of not containing a non Euclidean Baumslag Solitar subgroup is sufficient to make a group \(G\) which splits as a graph of free groups with infinite cyclic edge groups be linear over \(\mathbb{Z}\). Although this is currently open, the paper does show that \(G\) is the fundamental group of a compact non positively curved cube complex if this condition holds.

7 Minsky’s simple loop question

Suppose that \(S\) and \(T\) are both closed orientable surfaces of genus at least two. Theorem 2.1 in [12] showed that if \(f : S \rightarrow T\) is a continuous map such that the induced homomorphism \(f_* : \pi_1(S) \rightarrow \pi_1(T)\) is not injective then there exists a non contractible simple closed curve \(\alpha\) in \(S\) such that \(f(\alpha)\) is a homotopically trivial closed curve in \(T\). Note that any homomorphism \(\theta : \pi_1(S) \rightarrow \pi_1(T)\) is induced by a continuous map from \(S\) to \(T\) because both surfaces are aspherical. This demonstrates that the simple closed curves on a surface form a special subset of all closed curves as the kernel of any non injective homomorphism to the fundamental group of another surface.
must meet this subset. We can try to generalise this result by replacing the codomain $\pi_1(T)$ with other groups. Indeed Problem 3.96 in [20] which is entitled the simple loop conjecture for 3-manifolds asks: Let $f : S \to M^3$ be a 2 sided immersion of a closed surface $S$ in a 3-manifold $M^3$ such that $f_*$ is not injective then does there exist an essential simple loop on $S$ whose image is null homotopic in $M$? The answer is known to be yes for Siefert fibred spaces but the question is still open for $M^3$ an orientable hyperbolic 3-manifold. In this case $\pi_1(M^3)$ will be a discrete subgroup of $SL(2,\mathbb{C})$ so a variation is the following problem by Minsky. Question 5.3 of [25] asks whether all non injective homomorphisms from $\pi_1(S)$ to $SL(2,\mathbb{C})$ have an element in the kernel which is represented by a non contractible simple closed curve on $S$. Of course if this were true then it would apply both when the image is discrete and when indiscrete, thus solving the above open question. However we have seen in this paper many examples of subgroups of $SL(2,\mathbb{C})$ which are not isomorphic to discrete groups, so it should not be too surprising if counterexamples did exist to Minsky’s question. (In fact Minsky asked this question for $PSL(2,\mathbb{C})$ but a homomorphism to $SL(2,\mathbb{C})$ with image missing $-I$ will work here too.)

Indeed there are counterexamples to this question, as shown in [9] using character varieties, in [22] by obtaining homomorphisms from surface groups whose images are limit groups and thus embed in $SL(2,\mathbb{C})$, and a short proof in [7] using stable commutator length. In fact our approach has aspects in common with the last two papers, in that we also use a theorem of Hempel from 1990 which is quoted in [22] and, as in [7], we are considering cyclic amalgamations. However we need only apply an argument in Section 3 about when these groups embed in $SL(2,\mathbb{C})$, along with this result of Hempel and the work of Whittemore from 1973 (on the $SL(2,\mathbb{C})$ representations of the fundamental group of the figure eight knot complement mentioned in Section 5) in order to obtain a very quick and explicit proof.

**Theorem 7.1** Let $\pi_1(S_2) = \langle A, B, C, D | ABA^{-1}B^{-1} = CDC^{-1}D^{-1} \rangle$ be the fundamental group of the closed orientable surface of genus two and let $N$ be the normal closure of the element $r(A, B) = B^{-1}A^{-1}BAB^{-1}ABA^{-1}B^{-1}A$ in $\pi_1(S_2)$. Then $N$ is non trivial but does not contain any element which is represented by a simple closed curve on $S_2$. However $\pi_1(S_2)/N$ embeds in $SL(2,\mathbb{C})$ and misses $-I$.

**Proof.** The group $\pi_1(S_2)$ is of course formed by amalgamating the free group on $A, B$ with the free group on $C, D$ over $ABA^{-1}B^{-1} = CDC^{-1}D^{-1}$ and
$r(A, B)$ is clearly non trivial in the first free group which injects into $\pi_1(S_2)$. However by drawing the curve $r(A, B)$ on $S_2$ using the standard identification of the fundamental domain we see that it is certainly not simple. Now a result in \cite{16} states that if $S$ is a closed orientable surface and $\alpha \in \pi_1(S)$ is a closed curve such that the normal closure of $\alpha$ in $\pi_1(S)$ contains a non trivial simple closed curve $\beta$ then $\alpha$ is itself a simple closed curve and $\beta \pm$ is homotopic either to $\alpha$ or to the commutator of $\alpha$ with some simple closed curve $\gamma$ meeting $\alpha$ transversely in a single point. An immediate consequence of this in our case is that as the closed curve $r(A, B) \in \pi_1(S_2)$ is not simple, $N$ contains no non trivial simple closed curves.

We just need to show that $\pi_1(S_2)/N$ embeds in $SL(2, \mathbb{C})$. But rather than think of this as taking the free group on $A, B, C, D$ and adding first the surface relation, then the relation given by $r(A, B) = I$, we can do this the other way around. Of course the word $r(A, B)$ has been seen before in Section 5 where $G = \langle A, B | r(A, B) \rangle$ was the fundamental group of the figure eight knot complement. Hence $\pi_1(S_2)/N$ is also an amalgamation of this group $G$ and the free group $F$ on $C, D$ formed by identifying $ABA^{-1}B^{-1}$ in $G$ with $CDC^{-1}D^{-1}$ in $F$. We can now apply Corollary 3.5 to show that $\pi_1(S_2)$ embeds in $SL(2, \mathbb{C})$: it is clear that $CDC^{-1}D^{-1}$ is a primitive element of $F$ and we can embed a rank 2 free group in $SL(2, \mathbb{C})$ so that every non trivial element has a transcendental trace. As for $ABA^{-1}B^{-1}$ in $G$, this is the element $ab$ when written in terms of the generators $t, a, b$ giving the fibred form of the figure eight knot. Since the fibre subgroup is free on $a, b$ we have that $ABA^{-1}B^{-1}$ lies in this subgroup and is a primitive element of it. Moreover the centraliser $C_G(ABA^{-1}B^{-1})$ in $G$ is also infinite cyclic, because $ABA^{-1}B^{-1}$ has trace $z - 1$ which is not equal to $\pm 2$ in the discrete faithful representation of $G$ in $SL(2, \mathbb{C})$ where $x = 2$ and $z = 2 - \omega$. Thus $ABA^{-1}B^{-1}$ must also be primitive in $G$ because any element of $G$ outside the fibre subgroup has all its non trivial powers outside of this fibre subgroup. Thus on taking $\lambda$ to be any transcendental number as in Proposition 5.3, we have an embedding of $G$ in $SL(2, \mathbb{C})$ in which the primitive element $ABA^{-1}B^{-1}$ has transcendental trace, so the conditions of Corollary 3.5 are satisfied.

Finally as $\pi_1(S_2)/N$ is an amalgamated free product of two torsion free groups, it is also torsion free and so misses $-I$ in $SL(2, \mathbb{C})$, implying that it embeds in $PSL(2, \mathbb{C})$ too.

$\square$
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