CONES OF WEIGHTED QUASIMETRICS, WEIGHTED QUASIHYPERMETRICS AND OF ORIENTED CUTS

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Abstract: We show that the cone of weighted $n$-point quasi-metrics $WQMet_n$, the cone of weighted quasi-hypermetrics $WHyp_n$, and the cone of oriented cuts $OCut_n$ are projections along an extreme ray of the metric cone $Met_{n+1}$, of the hypermetric cone $Hyp_{n+1}$ and of the cut cone $Cut_{n+1}$, respectively. This projection is such that if one knows all faces of an original cone then one knows all faces of the projected cone.

Keywords: distance, metrics, hypermetrics, cut metrics, quasi-metrics.

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1 Introduction

Oriented (or directed) distances are encountered very often, for example, these are one-way transport routes, a river with quick flow and so on.

The notions of directed distances, quasi-metrics and oriented cuts are generalizations of the notions of distances, metrics and cuts, which are central objects in graph theory and combinatorial optimization.

Quasi-metrics are used in semantics of computations (see, for example, [Se97]) and in computational geometry (see, for example, [AACMP97]). Oriented distances have been used already by Hausdorff in 1914, see [Ha14].

In [CMM06], an example of directed metric derived from a metric is given. Let $d$ be a metric on a set $V \cup \{0\}$, where 0 is a distinguished point. Then a quasi-metric $q$ on the set $V$ is given as

$$q_{ij} = d_{ij} + d_{i0} - d_{j0}. $$

This quasi-metric belongs to a special important subclass of quasi-metrics, namely, to a class of weighted quasi-metrics. We show in this paper that any weighted quasi-metric is obtained from a metric by this method.

All semi-metrics on a set of cardinality $n$ form a metric cone $Met_n$. There are two important sub-cones of $Met_n$, namely, the cone $Hyp_n$ of hypermetrics, and the cone $Cut_n$ of $\ell_1$-metrics. These three cones form the following nested family $Cut_n \subseteq Hyp_n \subseteq Met_n$, see [DL97].

In this paper we introduce a special space $Q_n$, called a space of weighted quasi-metrics. We define in this space a cone $WQMet_n$, Elements of this cone satisfy triangle and non-negativity inequalities. Among extreme rays of the cone $WQMet_n$, there are rays spanned by cut vectors, i.e. incidence vectors of oriented cuts.

We define in the space $Q_n$ a cone $OCut_n$ of oriented cuts as the cone hull of cut vectors. Elements of the cone $OCut_n$ are weighted quasi-$\ell$-metrics.

Let metrics of the cone $Met_{n+1}$ are defined on the set $V \cup \{0\}$. The cut-cone $Cut_{n+1}$ of $\ell_1$-metrics on this set is a cone hull of cut-metrics $\delta(S)$ for all $S \subseteq V \cup \{0\}$. The cut-metrics $\delta(S)$ are extreme rays of all the three cones $Met_{n+1}$, $Hyp_{n+1}$ and $Cut_{n+1}$. In particular, $\delta(\{0\}) = \delta(V)$ is an extreme ray of these three cones.

In this paper, it is shown that the cones $WQMet_n$ and $OCut_n$ are projections of the corresponding cones $Met_{n+1}$ and $Cut_{n+1}$ along the extreme ray $\delta(V)$. We define a cone $WHyp_n$ of weighted quasi-hypermetrics as projection along $\delta(V)$ of the cone $Hyp_{n+1}$. So, we obtain a nested family $OCut_n \subseteq WHyp_n \subseteq WQMet_n$.

Weighted quasi-metrics and other generalizations of metrics are studied, for example, in [DD10] and [DDV11]. The cone and the polytope of oriented cuts are considered in [AM11].
2 Spaces $\mathbb{R}^E$ and $\mathbb{R}^{E^O}$

Let $V$ be a set of cardinality $|V| = n$. Let $E$ and $E^O$ be sets of all unordered $(ij)$ and ordered $ij$ pairs of elements $i, j \in V$. Consider two Euclidean spaces $\mathbb{R}^E$ and $\mathbb{R}^{E^O}$ of vectors $d \in \mathbb{R}^E$ and $g \in \mathbb{R}^{E^O}$ with coordinates $d_{(ij)}$ and $g_{ij}$, where $(ij) \in E$ and $ij \in E^O$, respectively. Obviously, dimensions of the spaces $\mathbb{R}^E$ and $\mathbb{R}^{E^O}$ are $|E| = \frac{n(n-1)}{2}$ and $|E^O| = n(n-1)$, respectively.

Denote by $(d, t) = \sum_{(ij) \in E} d_{(ij)} t_{(ij)}$ scalar product of vectors $d, t \in \mathbb{R}^E$. Similarly $(f, g) = \sum_{ij \in E^O} f_{ij} g_{ij}$ denote scalar product of vectors $f, g \in \mathbb{R}^{E^O}$.

Let $\{e_{(ij)} : (ij) \in E\}$ and $\{e_{ij} : ij \in E^O\}$ be orthonormal bases of $\mathbb{R}^E$ and $\mathbb{R}^{E^O}$, respectively. Then, for $f \in \mathbb{R}^E$ and $q \in \mathbb{R}^{E^O}$, we have

$$(e_{(ij)}, f) = f_{(ij)} \text{ and } (e_{ij}, q) = q_{ij}.$$ 

For $f \in \mathbb{R}^{E^O}$, define $f^* \in \mathbb{R}^{E^O}$ as follows

$$f^*_{ij} = f_{ji} \text{ for all } ij \in E^O.$$ 

Call a vector $q$ symmetric if $g^* = g$, and antisymmetric if $g^* = -g$. Each vector $g \in \mathbb{R}^{E^O}$ can be decompose into symmetric $g^s$ and antisymmetric $g^a$ parts as follows:

$$g^s = \frac{1}{2}(g + g^*), \quad g^a = \frac{1}{2}(g - g^*), \quad g = g^s + g^a.$$ 

Let $\mathbb{R}^{E^O}_s$ and $\mathbb{R}^{E^O}_a$ be subspaces of symmetric and antisymmetric vectors, respectively. Note that the spaces $\mathbb{R}^{E^O}_s$ and $\mathbb{R}^{E^O}_a$ are mutually orthogonal. In fact, for $p \in \mathbb{R}^{E^O}_s$ and $f \in \mathbb{R}^{E^O}_a$, we have

$$(p, f) = \sum_{ij \in E^O} p_{ij} f_{ij} = \sum_{(ij) \in E} (p_{ij} f_{ij} + p_{ji} f_{ji}) = \sum_{(ij) \in E} (p_{ij} f_{ij} - p_{ij} f_{ji}) = 0.$$ 

Hence

$$\mathbb{R}^{E^O} = \mathbb{R}^{E^O}_s \oplus \mathbb{R}^{E^O}_a,$$

where $\oplus$ is direct sum.

Obviously, there is an isomorphism $\varphi$ between the spaces $\mathbb{R}^E$ and $\mathbb{R}^{E^O}$. Let $d \in \mathbb{R}^E$ have coordinates $d_{(ij)}$. Then we set

$$d^O = \varphi(d) \in \mathbb{R}^{E^O}_s \text{ such that } d^O_{ij} = d^O_{ji} = d_{(ij)}.$$ 

In particular,

$$\varphi(e_{(ij)}) = e_{ij} + e_{ji}.$$ 

The map $\varphi$ is invertible. In fact, for $q \in \mathbb{R}^{E^O}_s$, we have $\varphi^{-1}(q) = d \in \mathbb{R}^E$ such that $d_{(ij)} = q_{ij} = q_{ji}$. The isomorphism $\varphi$ will be useful in what follows.

3 Space of weights $Q^w_n$

One can consider the sets $E$ and $E^O$ as sets of edges $(ij)$ and arcs $ij$ of an unordered and ordered complete graphs $K_n$ and $K^O_n$ on the vertex set $V$, respectively. The graph $K^O_n$ has two arcs $ij$ and $ji$ between each pair of vertices $i, j \in V$.

It is convenient to consider vectors $g \in \mathbb{R}^{E^O}$ as functions on the set of arcs $E^O$ of the graph $K^O_n$. So, the decomposition $\mathbb{R}^{E^O} = \mathbb{R}^{E^O}_s \oplus \mathbb{R}^{E^O}_a$ is a decomposition of the space of all functions on arcs in $E^O$ onto the spaces of symmetric and antisymmetric functions.
Besides, there is an important direct decomposition of the space \( \mathbb{R}^{E^O} \) of antisymmetric functions onto two subspaces. In theory of electric networks these spaces are called spaces of tensions and flows (see also [Aig79]).

The tension space relates to potentials (or weights) \( w_i \) given on vertices \( i \in V \) of the graph \( K_n^O \). The corresponding antisymmetric function \( g^{w}_{ij} \) is determined as

\[
g^{w}_{ij} = w_i - w_j.
\]

It is called tension on the arc \( ij \). Obviously, \( g^{w}_{ji} = w_j - w_i = -g^{w}_{ij} \). Denote by \( Q_n^w \) the subspace of \( \mathbb{R}^{E^O} \) generated by all tensions on arcs \( ij \in E^O \). We call \( Q_n^w \) by a space of weights.

Each tension function \( g^{w} \) is represented as weighted sum of elementary potential functions \( p(k) \) for \( k \in V \) as follows

\[
g^{w} = \sum_{k \in V} w_k p(k),
\]

where

\[
p(k) = \sum_{j \in V - \{k\}} (e_{kj} - e_{jk}), \text{ for all } k \in V,
\]

are basic functions that generate the space of weights \( Q_n^w \). Hence values of the basic functions \( p(k) \) on arcs are as follows

\[
p_{ij}(k) = \begin{cases} 1 & \text{if } i = k \\ -1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}
\]

We obtain

\[
g^{w}_{ij} = \sum_{k \in V} w_k p_{ij}(k) = w_i - w_j.
\]

It is easy to verify that

\[
p^2(k) = (p(k), p(k)) = 2(n - 1), (p(k), p(l)) = -2 \text{ for all } k, l \in V, k \neq l, \sum_{k \in V} p(k) = 0.
\]

Hence there are only \( n - 1 \) independent functions \( q(k) \) that generate the space \( Q_n^w \).

Weighted quasimetries lie in the space \( \mathbb{R}^{E^O} \oplus Q_n^w \) that we denote as \( Q_n \). Direct complements of \( Q_n^w \) in \( \mathbb{R}^{E^O} \) and \( Q_n \) in \( \mathbb{R}^{E^O} \) is a space \( Q_n^c \) of circuits (or flows).

### 4 Space of circuits \( Q_n^c \)

The space of circuits (or space of flows) is generated by characteristic vectors of oriented circuits in the graph \( K_n^O \). Arcs of \( K_n^O \) are ordered pairs \( ij \) of vertices \( i, j \in V \). The arc \( ij \) is oriented from the vertex \( i \) to the vertex \( j \). Recall that \( K_n^O \) has both the arcs \( ij \) and \( ji \) for each pair of vertices \( i, j \in V \).

Let \( G_s \subseteq K_n \) be an undirected subgraph with a set of edges \( E(G_s) \subseteq E \). We relate to the undirected graph \( G_s \) a directed graph \( G \subseteq K_n^O \) with the arc set \( E^O(G) \subseteq E^O \) as follows. An arc \( ij \) belongs to \( G \), i.e. \( ij \in E^O(G) \), if and only if \( (ij) = (ji) \in E(G) \). This definition implies that the arc \( ji \) belongs to \( G \) also, i.e. \( ji \in E^O(G) \).

Let \( C_s \) be a circuit in the graph \( K_n \). The circuit \( C_s \) is determined by a sequence of distinct vertices \( i_k \in V \), where \( 1 \leq k \leq p \) and \( p \) is the length of \( C_s \). Edges of \( C_s \) are unordered pairs \( (i_k, i_{k+1}) \), where indices are taken modulo \( p \). By above definition, an oriented bicircuit \( C \) of the graph \( K_n^O \) relates to \( C_s \) of \( K_n \). Arcs of \( C \) are ordered pairs \( i_k i_{k+1} \) and \( i_{k+1} i_k \), where indices are taken modulo \( p \). Take an orientation of \( C \). Denote by \( -C \) the opposite circuit with opposite orientation. Denote an arc of \( C \) direct or opposite if its direction coincides with or is opposite to the given orientation of \( C \), respectively. Let \( C^+ \) and \( C^- \) be subcircuits of \( C \) consisting of direct and opposite arcs, respectively.
The following vector \( f^C \) is the characteristic vector of the bicircuit \( C \):

\[
f^C_{ij} = \begin{cases} 
1 & \text{if } ij \in C^+, \\
-1 & \text{if } ij \in C^-, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that \( f^{-C} = (f^C)^* = -f^C \), and \( f^C \in \mathbb{R}^{E_O} \).

Denote by \( Q^e_O \) the space linearly generated by circuit vectors \( f^C \) for all bicircuits \( C \) of the graph \( K^O_n \). It is well known that characteristic vectors of fundamental circuits form a basis of \( Q^e_O \). Fundamental circuits are defined as follows.

Let \( T \) be a spanning tree of the graph \( K_n \). Since \( T \) is spanning, its vertex set \( V(T) \) is the set of all vertices of \( K_n \), i.e. \( V(T) = V \). Let \( E(T) \subset E \) be the set of edges of \( T \). Then any edge \( e = (ij) \notin E(T) \) closes a unique path in \( T \) between vertices \( i \) and \( j \) into a circuit \( C^e_s \). This circuit \( C^e_s \) is called fundamental. Call corresponding oriented bicircuit \( C^e_s \) also fundamental.

There are \( |E - E(T)| = \frac{n(n-1)}{2} - (n-1) \) fundamental circuits. Hence

\[
\dim Q^e_n = \frac{n(n-1)}{2} - (n-1), \quad \text{and} \quad \dim Q_n + \dim Q^e_n = n(n-1) = \dim \mathbb{R}^{E_O}.
\]

This implies that \( Q^e_O \) is an orthogonal complement of \( Q^w_n \) in \( \mathbb{R}^O_n \) and \( Q_n \) in \( \mathbb{R}^{E_O} \), i.e.

\[
\mathbb{R}^{E_O} = Q^w_n \oplus Q^e_O \quad \text{and} \quad \mathbb{R}^{E_O} = Q_n \oplus Q^e_n = \mathbb{R}^{E_s} \oplus Q^w_n \oplus Q^e_n.
\]

### 5 Cut and ocut vector set-functions

The space \( Q_n \) is generated also by vectors of oriented cuts, which we define in this section.

Each subset \( S \subseteq V \) determines cuts of the graphs \( K_n \) and \( K^O_n \) that are subsets of edges and arcs of these graphs.

A cut \( \{S, V \setminus S \} \) of \( S \subseteq V \) is a subset of edges \( (ij) \) of \( K_n \) such that \( (ij) \in \text{cut}(S) \) if and only if \( |\{i,j\} \cap S| = 1 \).

A ocut \( \{S, V \setminus S \} \) of \( E^O \) is a subset of arcs \( ij \) of \( K^O_n \) such that \( ij \in \text{ocut}(S) \) if and only if \( |\{i,j\} \cap S| = 1 \). So, if \( ij \in \text{ocut}(S) \), then \( ji \in \text{ocut}(S) \) also.

An oriented cut is a subset \( \text{ocut}(S) \subseteq E^O \) of arcs \( ij \) of \( K^O_n \) such that \( ij \in \text{ocut}(S) \) if and only if \( i \in S \) and \( j \notin S \).

We relate to these three types of cuts characteristic vectors \( \delta(S) \in \mathbb{R}^E \), \( \delta^O(S) \in \mathbb{R}^{E_O} \), \( p(S) \in \mathbb{R}^{E_O} \) and \( c(S) \in \mathbb{R}^{E_O} \) as follows.

For cut \( \text{cut}(S) \), we set

\[
\delta(S) = \sum_{i \in S, j \notin S} e_{ij}, \quad \text{such that} \quad \delta_{(ij)}(S) = \begin{cases} 1 & \text{if } |\{i,j\} \cap S| = 1 \\
0 & \text{otherwise}, \end{cases}
\]

where \( \overline{S} = V - S \). For ocut \( \text{ocut}(S) \), we set

\[
\delta^O(S) = \varphi(\delta(S)) = \sum_{i \in S, j \notin S} (e_{ij} + e_{ji}) \quad \text{and} \quad p(S) = \sum_{i \in S, j \notin S} (e_{ij} - e_{ji}).
\]

Hence

\[
\delta^O_{ij}(S) = \begin{cases} 1 & \text{if } |\{i,j\} \cap S| = 1 \\
0 & \text{otherwise}, \end{cases} \quad \text{and} \quad p_{ij}(S) = \begin{cases} 1 & \text{if } i \in S, j \notin S \\
-1 & \text{if } j \in S, i \notin S \\
0 & \text{otherwise}.
\end{cases}
\]
Note that, for one-element sets $S = \{k\}$, the function $p(\{k\})$ is $p(k)$ of section 2. It is easy to see that

$$(\delta^O(S), p(T)) = 0 \text{ for any } S, T \subseteq V.$$ 

For the oriented cut $\text{ocut}(S)$, we set

$$c(S) = \sum_{i,j \in E} e_{ij}.$$ 

Hence

$$c_{ij}(S) = \begin{cases} 1 & \text{if } i \in S, j \notin S \\ 0 & \text{otherwise}. \end{cases}$$ 

Obviously, it holds $c(\emptyset) = c(V) = 0$, where $0 \in \mathbb{R}^{E^O}$ is a vector whose all coordinates are equal zero. We have the following equalities

$$c^*(S) = c(\overline{S}), \quad c(S) + c(\overline{S}) = \delta^O(S), \quad c(S) - c(\overline{S}) = p(S) \text{ and } c(S) = \frac{1}{2}(\delta^O(S) + p(S)).$$

Besides, we have

$$c^*(S) = \frac{1}{2}\delta^O(S), \quad c^a(S) = \frac{1}{2}p(S).$$

Recall that a set-function $f(S)$ on all $S \subseteq V$, is called \textit{submodular} if, for any $S, T \subseteq V$, the following \textit{submodular inequality} holds

$$f(S) + f(T) - (f(S \cap T) + f(S \cup T)) \geq 0.$$ 

It is well known that the vector set-function $\delta \in \mathbb{R}^E$ is submodular (see, for example, [Aig79]). The above isomorphism $\varphi$ of the spaces $\mathbb{R}^E$ and $\mathbb{R}_a^{E^O}$ implies that the vector set-function $\delta^O = \varphi(\delta) \in \mathbb{R}_a^{E^O}$ is submodular also.

A set-function $f(S)$ is called \textit{modular} if, for any $S, T \subseteq V$, the above submodular inequality holds as equality. This equality is called \textit{modular equality}. It is well known (and can be easily verified) that antisymmetric vector set-function $f^a(S)$ is modular for any oriented graph $G$. Hence our antisymmetric vector set-function $\varphi(S) \in \mathbb{R}_a^{E^O}$ for the oriented complete graph $K^O_n$ is modular also.

Note that the set of all submodular set functions on a set $V$ forms a cone in the space $\mathbb{R}^{2^V}$. Therefore the last equality in (3) implies that the vector set-function $c(S) \in \mathbb{R}^{E^O}$ is submodular.

The modularity of the antisymmetric vector set-function $\varphi(S)$ is important for what follows. It is well-known (see, for example, [Bir67]) (and it can be easily verified using modular equality) that any modular set-function $m(S)$ is completely determined by its values on the empty set and on all one-element sets. Hence a modular set-function $m(S)$ has the following form

$$m(S) = m_0 + \sum_{i \in S} m_i,$$

where $m_0 = m(\emptyset)$ and $m_i = m(\{i\}) - m(\emptyset)$. For brevity, we set $f(\{i\}) = f(i)$ for any set function $f(S)$. Since $p(\emptyset) = p(V) = 0$, we have

$$p(S) = \sum_{k \in S} p(k), \quad S \subseteq V, \text{ and } p(V) = \sum_{k \in V} p(k) = 0.$$ 

(4)

Using equations (3) and (4), we obtain

$$c(S) = \frac{1}{2}(\delta^O(S) + \sum_{k \in S} p(k)).$$

(5)

Now we show that cut vectors $c(S)$ for all $S \subseteq V$ linearly generate the space $Q_n \subseteq \mathbb{R}^{E^O}$. The space generated by $c(S)$ consists of the following vectors

$$c = \sum_{S \subseteq V} \alpha_S c(S), \text{ where } \alpha_S \in \mathbb{R}.$$ 

Recall that \( c(S) = \frac{1}{4}(\delta^O(S) + p(S)) \). Hence we have

\[
  c = \frac{1}{2} \sum_{S \subseteq V} \alpha_S (\delta^O(S) + p(S)) = \frac{1}{2} \sum_{S \subseteq V} \alpha_S \delta^O(S) + \frac{1}{2} \sum_{S \subseteq V} \alpha_S p(S) = \frac{1}{2}(d^O + p),
\]

where \( d^O = \varphi(d) \) for \( d = \sum_{S \subseteq V} \alpha_S \delta(S) \). For a vector \( p \) we have

\[
  p = \sum_{S \subseteq V} \alpha_S p(S) = \sum_{S \subseteq V} \alpha_S \sum_{k \in S} p(k) = \sum_{k \in V} \sum_{V \supseteq S \ni k} \alpha_S, \quad \text{where} \quad w_k = \sum_{V \supseteq S \ni k} \alpha_S.
\]

Since \( p_{ij} = \sum_{k \in V} w_k p_{ij}(k) = w_i - w_j \), we have

\[
  e_{ij} = \frac{1}{2}(d^O_{ij} + w_i - w_j). \quad (6)
\]

It is well-known (see, for example, [DL97]) that the vectors \( \delta(S) \in \mathbb{R}^E \) for all \( S \subseteq V \) linearly generate the full space \( \mathbb{R}^E \). Hence the vectors \( \delta^O(S) \in \mathbb{R}^{E^O}_s \) for all \( S \subseteq V \) linearly generate the full space \( \mathbb{R}^{E^O}_s \).

According to (5), antisymmetric parts of vectors \( c(S) \) generate the space \( Q_n^w \). This implies that the space \( Q_n = \mathbb{R}^{E^O}_s \oplus Q_n^w \) is generated by \( c(S) \) for all \( S \subseteq V \).

### 6 Properties of the space \( Q_n \)

Let \( x \in Q_n \) and let \( f^C \) be the characteristic vector of a bicircuit \( C \). Since \( f^C \) is orthogonal to \( Q_n \), we have

\[
  (x, f^C) = \sum_{ij \in C} f^C_{ij} x_{ij} = 0.
\]

This equality implies that each point \( x \in Q_n \) satisfies the following equalities

\[
  \sum_{ij \in C^+} x_{ij} = \sum_{ij \in C^-} x_{ij}
\]

for any bicircuits \( C \).

Let \( K_{1,n-1} \subseteq K_n \) be a spanning star of \( K_n \) consisting of all \( n - 1 \) edges incident to a vertex of \( K_n \). Let this vertex be 1. Each edge of \( K_n - K_{1,n-1} \) has the form \( (ij) \), where \( i \neq 1 \neq j \). The edge \((ij)\) closes a fundamental triangle with edges \((1i), (1j), (ij)\). The corresponding bitriangle \( T(1ij) \) generates the equality

\[
  x_{1i} + x_{ij} + x_{1j} = x_{i1} + x_{1j} + x_{ji}.
\]

These inequalities were derived by another way in [AM11]. They correspond to fundamental bi-triangles \( T(1ij) \), for all \( i, j \in V = \{1\} \), and are all \( n(n-1)/2 - (n-1) \) independent equalities determining the space, where the \( Q_n \) lies.

Above coordinates \( x_{ij} \) of a vector \( x \in Q_n \) are given in the orthonormal basis \( \{ e_{ij} : ij \in E^O \} \). But, for what follows, it is more convenient to consider vectors \( q \in Q_n \) in another basis. Recall that \( \mathbb{R}^{E^O}_s = \varphi(\mathbb{R}^E) \). Let, for \( (ij) \in E, \varphi(e_{ij}) = e_{ij} + e_{ji} \) be basic vectors of the subspace \( \mathbb{R}^{E^O}_s \). Let \( p(i) \in Q_n^w, i \in V \), be basic vectors (defined in (1)) of the space \( Q_n^w \subseteq Q_n \). Then, for \( q \in Q_n \), we set

\[
  q = q^s + q^a, \quad \text{where} \quad q^s = \sum_{(ij) \in E} q_{(ij)} \varphi(e_{ij}), \quad q^a = \sum_{i \in V} w_i p(i).
\]

Now, we obtain an important expression for the scalar product \((g, q)\) of vectors \( g, q \in Q_n \). Recall that \((\varphi(e_{ij}), p(k)) = ((e_{ij} + e_{ji}), p(k)) = 0 \) for all \((ij) \in E \) and all \( k \in V \). Hence \((g^s, q^a) = (g^s, q^a) = 0\), and we have

\[
  (g, q) = (g^s, q^s) + (g^a, q^a).
\]

Besides, we have

\[
  ((e_{ij} + e_{ji}), (e_{kl} + e_{lk})) = 0 \text{ if } (ij) \neq (kl), \quad (e_{ij} + e_{ji})^2 = 2.
\]
and (see Section 3)

\[(p(i), p(j)) = -2 \text{ if } i \neq j, \ (p(i))^2 = 2(n - 1).\]

Let \(v_i, w_i, i \in V\), be weights of the vector \(g, q\), respectively. Then we have

\[(g, q) = 2 \sum_{(ij) \in E} g_{(ij)}q_{(ij)} + 2(n - 1) \sum_{i \in V} v_iw_i - 2 \sum_{i \neq j \in V} v_iw_j.\]

For the last sum, we have

\[\sum_{i \neq j \in V} v_iw_j = (\sum_{i \in V} v_i)(\sum_{i \in V} w_i) - \sum_{i \in V} v_iw_i.\]

Since weights are defined up to an additive scalar, we can choose weights \(v_i\) such that \(\sum_{i \in V} v_i = 0\). Then the last sum in the product \((g, q)\) is equal to \(-\sum_{i \in V} v_iw_i\). Finally we obtain that the sum of antisymmetric parts is equal to \(2n \sum_{i \in V} v_iw_i\). So, for the product of two vectors \(g, q \in Q_n\) we have the following expression

\[(g, q) = (g^a, q^a) + (g^a, q^a) = 2(\sum_{(ij) \in E} g_{(ij)}q_{(ij)} + n \sum_{i \in V} v_iw_i) \text{ if } \sum_{i \in V} v_i = 0 \text{ or } \sum_{i \in V} w_i = 0.\]

In what follows, we consider inequalities \((g, q) \geq 0\). We can delete the multiple 2, and rewrite such inequality as follows

\[\sum_{(ij) \in E} g_{(ij)}q_{(ij)} + n \sum_{i \in V} v_iw_i \geq 0,\]

where \(\sum_{i \in V} v_i = 0\).

Below we consider some cones in the space \(Q_n\). Since the space \(Q_n\) is orthogonal to the space of circuits \(Q_n^c\), each facet vector of a cone in \(Q_n\) is defined up to a vector of the space \(Q_n^c\). Of course each vector \(g' \in \mathbb{R}^{E^c}\) can be decomposed as \(g' = g + g^c\), where \(g \in Q_n\) and \(g^c \in Q_n^c\). Call the vector \(g \in Q_n\) canonical representative of the vector \(g'\). Usually we will use canonical facet vectors. But sometimes not canonical representatives of a facet vector are useful.

Cones \(C\) that will be considered are invariant under the operation \(q \rightarrow q^*\), defined in Section 2. In other words, \(C^* = C\). This operation changes signs of weights:

\[q_{ij} = q_{(ij)} + w_i - w_j \rightarrow q_{(ij)} + w_j - w_i = q_{(ij)} - w_i + w_j.\]

Let \((g, q) \geq 0\) be an inequality determining a facet \(F\) of a cone \(C \subset Q_n\). Since \(C = C^*\), the cone \(C\) has with the facet \(F\) also a facet \(F^*\). The facet \(F^*\) is determined by the inequality \((g^*, q) \geq 0\).

### 7 Projections of cones \(Con_{n+1}\)

Recall that \(Q_n = \mathbb{R}^{E^c} \oplus Q_n^w, \mathbb{R}^{E^c} = \varphi(\mathbb{R}^E)\) and \(\text{dim}Q_n = \frac{n(n+1)}{2} - 1\).

Let \(0 \notin V\) be an additional point. Then the set of unordered pairs \((ij)\) for \(i, j \in V \cup \{0\}\) is \(E \cup E_0\), where \(E_0 = \{(0i) : i \in V\}\). Obviously, \(\mathbb{R}^{E \cup E_0} = \mathbb{R}^E \oplus \mathbb{R}^{E_0}\) and \(\text{dim}E \cup E_0 = \frac{n(n+1)}{2}\).

The space \(\mathbb{R}^{E \cup E_0}\) contains the following three important cones: the cone \(Met_{n+1}\) of semi-metrics, the cone \(Hyp_{n+1}\) of hyper-semi-metrics and the cone \(Cut_{n+1}\) of \(\ell_1\)-semi-metrics, all on the set \(V \cup \{0\}\). Denote by \(Con_{n+1}\) any of these cones.

Recall that a semi-metric \(d = \{d_{(ij)}\}\) is called metric if \(d_{(ij)} \neq 0\) for all \((ij) \in E\). For brevity sake, in what follows, we call elements of the cones \(Con_{n+1}\) by metrics (or hypermetrics, \(\ell_1\)-metrics), assuming that they can be semi-metrics.

Note that if \(d \in Con_{n+1}\) is a metric on the set \(V \cup \{0\}\), then a restriction \(d^V\) of \(d\) on the set \(V\) is a point of the cone \(Con_n = Con_{n+1} \cap \mathbb{R}^E\) of metrics on the set \(V\). In other words, we can suppose that \(Con_n \subset Con_{n+1}\).
The cones \(\text{Met}_{n+1}\), \(\text{Hyp}_{n+1}\) and \(\text{Cut}_{n+1}\) contain the cut vectors \(\delta(S)\) that span extreme rays for all \(S \subset V \cup \{0\}\). Denote by \(l_0\) the extreme ray spanned by the cut vector \(\delta(V) = \delta(\{0\})\). Consider a projection \(\pi(\mathbb{R}^E) = \mathbb{R}^F\) of the space \(\mathbb{R}^E\) along the ray \(l_0\) onto a subspace of \(\mathbb{R}^E\) that is orthogonal to \(\delta(V)\). This projection is such that \(\pi(\mathbb{R}^E) = \mathbb{R}^E\) and \(\pi(\mathbb{R}^E) = \mathbb{R}^E \oplus \pi(\mathbb{R}^E)\).

Note that \(\delta(V) \in \mathbb{R}^E\), since, by Section 5, \(\delta(V) = \sum_{v \in V} e_{(0i)}\). For simplicity sake, define the following vector

\[
e_0 = \delta(\{0\}) = \delta(V) = \sum_{v \in V} e_{(0i)}.
\]

Recall that the vector \(e_0\) spans the extreme ray \(l_0\). Obviously, the space \(\mathbb{R}^E\) is orthogonal to \(l_0\), and therefore \(\pi(\mathbb{R}^E) = \mathbb{R}^E\).

Let \(x \in \mathbb{R}^E\). We decompose this point as follows

\[
x = x^V + x^0,
\]

where \(x^V = \sum_{(ij) \in E} x_{(ij)} e_{(ij)} \in \mathbb{R}^E\) and \(x^0 = \sum_{i \in V} x_{(0i)} e_{(0i)} \in \mathbb{R}^E\). The projection \(\pi\) works on basic vectors as follows:

\[
\pi(e_{(ij)}) = e_{(ij)} \text{ for } (ij) \in E, \text{ and } \pi(e_{(0i)}) = e_{(0i)} - \frac{1}{n} e_0 \text{ for } i \in V.
\]

So, we have

\[
\pi(x) = \pi(x^V) + \pi(x^0) = \sum_{(ij) \in E} x_{(ij)} e_{(ij)} + \sum_{i \in V} x_{(0i)} e_{(0i)} - \frac{1}{n} e_0.
\]

It is useful to note that the projection \(\pi\) transforms the positive orthant of the space \(\mathbb{R}^E\) onto the whole space \(\pi(\mathbb{R}^E)\).

Now we describe how faces of a cone in the space \(\mathbb{R}^E\) are projected along one of its extreme rays.

Let \(l\) be an extreme ray and \(F\) be a face of a cone in \(\mathbb{R}^E\). Let \(\pi\) be the projection along \(l\). Let \(\dim F\) be dimension of the face \(F\). Then the following equality holds

\[
\dim \pi(F) = \dim F - \dim(F \cap l).
\]

Let \(g \in \mathbb{R}^E\) be a facet vector of a facet \(G\), and \(e\) be a vector spanning the line \(l\). Then \(\dim(G \cap l) = 1\) if \((g,e) = 0\), and \(\dim(G \cap l) = 0\) if \((g,e) \neq 0\).

**Theorem 1.** Let \(G\) be a face of the cone \(\pi(\text{Con}_{n+1})\). Then \(G = \pi(F)\), where \(F\) is a face of \(\text{Con}_{n+1}\) such that there is a face of \(\text{Con}_{n+1}\), containing both \(F\) and the extreme ray \(l_0\) spanned by \(e_0 = \delta(V)\).

In particular, \(G\) is a facet of \(\pi(\text{Con}_{n+1})\) if and only if \(G = \pi(F)\), where \(F\) is a facet of \(\text{Con}_{n+1}\) containing the extreme ray \(l_0\). Similarly, \(l'\) is an extreme ray of \(\pi(\text{Con}_{n+1})\) if and only if \(l' = \pi(l)\), where \(l\) is an extreme ray of \(\text{Con}_{n+1}\) lying in a facet of \(\pi(\text{Con}_{n+1})\) that contains \(l_0\).

**Proof.** Let \(F\) be a set of all facets of the cone \(\text{Con}_{n+1}\). Then \(\bigcup_{F \in F} \pi(F)\) is a covering of the projection \(\pi(\text{Con}_{n+1})\). By (9), in this covering, if \(l_0 \subset F \subset E\), then \(\pi(F)\) is a facet of \(\pi(\text{Con}_{n+1})\). If \(l_0 \not\subset F\), then there is a one-to-one correspondence between points of \(F\) and \(\pi(F)\). Hence \(\dim \pi(F) = n\), and \(\pi(F)\) cannot be a facet of \(\pi(\text{Con}_{n+1})\), since \(\pi(F)\) fills an \(n\)-dimensional part of the cone \(\pi(\text{Con}_{n+1})\).

If \(F'\) is a face of \(\text{Con}_{n+1}\), then \(\pi(F')\) is a face of the above covering. If \(F'\) belongs only to facets \(F \in F\) such that \(l_0 \not\subset F\), then \(\pi(F')\) lies inside of \(\pi(\text{Con}_{n+1})\). In this case, it is not a face of \(\pi(\text{Con}_{n+1})\). This implies that \(\pi(F')\) is a face of \(\pi(\text{Con}_{n+1})\) if and only if \(F' \subset F\), where \(F\) is a face of \(\text{Con}_{n+1}\) such that \(l_0 \subset F\). Suppose that dimension of \(F'\) is \(n-1\), and \(l_0 \not\subset F'\). Then \(\dim \pi(F') = n - 1\). If \(F'\) is contained in a facet \(F\) of \(\text{Con}_{n+1}\) such that \(l_0 \subset F\), then \(\pi(F') = \pi(F)\). Hence \(\pi(F')\) is a face of the cone \(\pi(\text{Con}_{n+1})\) that coincides with the facet \(\pi(F)\).
Now, the assertions of Theorem about facets and extreme rays of $\pi(Con_{n+1})$ follow. \hfill \Box

Theorem 1 describes all faces of the cone $\pi(Con_{n+1})$ if one knows all faces of the cone $Con_{n+1}$.

Recall that we consider $Con_n = Con_{n+1} \cap \mathbb{R}^E$ as a sub-cone of $Con_{n+1}$, and therefore $\pi(Con_n) \subset \pi(Con_{n+1})$. Since $\pi(\mathbb{R}^F) = \mathbb{R}^F$, we have $\pi(Con_n) = Con_n$. Let $(f, x) \geq 0$ be a facet-defining inequality of a facet $F$ of the cone $Con_{n+1}$. Since $Con_{n+1} \subset \mathbb{R}^E \oplus \mathbb{R}^{E_0}$, we represent vectors $f, x \in \mathbb{R}^{E_0 \cup E}$ as $f = f^V + f^0$, $x = x^V + x^0$, where $f^V, x^V \in \mathbb{R}^E$ and $f^0, x^0 \in \mathbb{R}^{E_0}$. Hence the above facet-defining inequality can be rewritten as

$$(f, x) = (f^V, x^V) + (f^0, x^0) \geq 0.$$ 

It turns out that the cone $Con_{n+1}$ has always a facet $F$ whose facet vector $f = f^V + f^0$ is such that $f^0 = 0$. Since $f^V$ is orthogonal to $\mathbb{R}^{E_0}$, the hyperplane $(f^V, x) = (f^V, x^V) = 0$ supporting the facet $F$ contains the whole space $\mathbb{R}^{E_0}$. The equality $(f^V, x^V) = 0$ defines a facet $F^V = F \cap \mathbb{R}^E$ of the cone $Con_n$.

**Definition.** A facet $F$ of the cone $Con_{n+1}$ with a facet vector $f = f^V + f^0$ is called zero-lifting of a facet $F^V$ of $Con_n$, if $f^0 = 0$ and $F \cap \mathbb{R}^E = F^V$.

Similarly, a facet $\pi(F)$ of the cone $\pi(Con_{n+1})$ with a facet vector $f$ is called zero-lifting of $F^V$ if $f = f^V$ and $\pi(F) \cap \mathbb{R}^E = F^V$.

It is well-known, see, for example, [DL97], that each facet $F^V$ with facet vector $f^V$ of the cone $Con_n$ can be zero-lifted up to a facet $F$ of $Con_{n+1}$ with the same facet vector $f^V$.

**Proposition 1.** Let a facet $F$ of $Con_{n+1}$ be zero-lifting of a facet $F^V$ of $Con_n$. Then $\pi(F)$ is a facet of $\pi(Con_{n+1})$ that is also zero-lifting of $F^V$.

**Proof.** Recall that the hyperplane $\{x \in \mathbb{R}^{E_0 \cup E_0} : (f^V, x) = 0\}$ supporting the facet $F$ contains the whole space $\mathbb{R}^{E_0}$. Hence the facet $F$ contains the extreme ray $l_0$ spanned by the vector $e_0 \in \mathbb{R}^{E_0}$. By Theorem 1, $\pi(F)$ is a facet of $\pi(Con_{n+1})$. The facet vector of $\pi(F)$ can be written as $f = f^V + f'$, where $f^V \in \mathbb{R}^E$ and $f' \in \pi(\mathbb{R}^{E_0})$. Since the hyperplane supporting the facet $\pi(F)$ is given by the equality $(f^V, x) = 0$ for $x \in \pi(\mathbb{R}^{E_0 \cup E_0})$, we have $f' = 0$. Besides, obviously, $\pi(F) \cap \mathbb{R}^E = F^V$. Hence $\pi(F)$ is zero-lifting of $F^V$. \hfill \Box

8 Cones $\psi(Con_{n+1})$

Note that basic vectors of the space $\mathbb{R}^{E_0 \cup E_0}$ are $e_{(ij)}$ for $(ij) \in E$ and $e_{(0i)}$ for $(0i) \in E_0$. Since $\pi(e_0) = \sum_{i \in V} \pi(e_{(0i)}) = 0$, we have $\dim(\mathbb{R}^{E_0}) = n - 1 = \dim Q_n$. Recall that $\pi(\mathbb{R}^E) = \mathbb{R}^E$. Hence there is a one-to-one bijection $\chi$ between the spaces $\pi(\mathbb{R}^{E_0 \cup E_0})$ and $Q_n$.

We define this bijection $\chi : \pi(\mathbb{R}^{E_0 \cup E_0}) \to Q_n$, as follows

$$\chi(\mathbb{R}^E) = \varphi(\mathbb{R}^E) = \mathbb{R}^E_{+},$$

where

$$\chi(e_{(ij)}) = \varphi(e_{(ij)}) = e_{ij} + e_{ji}, \text{ and } \chi(e_{(0i)}) = \chi(e_{(0i)} - \frac{1}{n} e_0) = p(i),$$

where $p(i)$ is defined in (1).

Note that $(e_{ij} + e_{ji})^2 = 2 = 2e_{(ij)}^2$ and

$$(p(i), p(j)) = -2 = 2(n((e_{(0i)} - \frac{1}{n} e_0), (e_{(0j)} - \frac{1}{n} e_0)), p^2(i) = 2(n - 1) = 2n(e_{(0i)} - \frac{1}{n} e_0)^2.$$ 

Roughly speaking, the map $\chi$ is a composition of homotheties that extends vectors $e_{(ij)}$ and $e_{(0i)} - \frac{1}{n} e_0$ up to vectors $e_{ij} + e_{ji}$ and $p(i)$ by the multiples $\sqrt{2}$ and $\sqrt{2n}$, respectively.

Setting $\psi = \chi \circ \pi$, we obtain a map $\psi : \mathbb{R}^{E_0 \cup E_0} \to Q_n$ such that

$$\psi(e_{(ij)}) = e_{ij} + e_{ji} \text{ for } (ij) \in E, \psi(e_{(0i)}) = p(i) \text{ for } i \in V.$$ (10)
Now we show how a point \( x = x^V + x^0 \in \mathbb{R}^{E \cup E_0} \) is transformed into a point \( q = \psi(x) = \chi(\pi(x)) \in Q_n \).

We have \( \pi(x) = x^V + \pi(x^0) \), where, according to (8), \( x^V = \sum_{(ij) \in E} x_{(ij)} e_{(ij)} \in \pi(\mathbb{R}^E) = \mathbb{R}^E \) and
\[
\pi(x^0) = \sum_{i \in V} x_{(0i)} (e_{(0i)} - \frac{1}{n} e_0) \in \pi(\mathbb{R}^{E_0}).
\]

Obviously, \( \chi(x^V + \pi(x^0)) = \chi(x^V) + \chi(\pi(x^0)) \), and
\[
\psi(x^V) = \chi(x^V) = \sum_{(ij) \in E} x_{(ij)} (e_{ij} + e_{ji}) = \varphi(x^V) = q^s \quad \text{and} \quad \chi(\pi(x^0)) = \sum_{i \in V} x_{(0i)} p(i) = q^a.
\]

Recall that \( q^s = \sum_{(ij) \in E} q_{(ij)} (e_{ij} + e_{ji}) \) and \( q^a = \sum_{i \in V} w_i p(i) \). Hence
\[
q_{(ij)} = x_{(ij)}, \quad (ij) \in E, \quad \text{and} \quad w_i = x_{(0i)}, \quad i \in V.
\]

Let \( f \in \mathbb{R}^{E \cup E_0} \) be a facet vector of a facet \( F \) of the cone \( Con_{n+1} \), \( f = f^V + f^0 = \sum_{(ij) \in E} f_{(ij)} e_{(ij)} + \sum_{i \in V} f_{(0i)} e_{(0i)} \).

Let \( (f, x) \geq 0 \) be the inequality determining the facet \( F \). The inequality \( (f, x) \geq 0 \) takes on the set \( V \cup \{0\} \) the following form
\[
(f, x) = \sum_{(ij) \in E} f_{(ij)} x_{(ij)} + \sum_{i \in V} f_{(0i)} x_{(0i)} \geq 0.
\]

Since \( x_{(ij)} = q_{(ij)}, \quad x_{(0i)} = w_i \), we can rewrite this inequality as follows
\[
(f, q) = (f^V, q^s) + (f^0, q^a) = \sum_{(ij) \in E} f_{(ij)} q_{(ij)} + \sum_{i \in V} f_{(0i)} w_i \geq 0.
\]

Comparing the inequality (12) with (7), we see that a canonical form of the facet vector \( f \) is \( f = f^s + f^a \), where
\[
f^s_{ij} = f_{(ij)}, \quad \text{for} \ (ij) \in E, \quad f^a_{ij} = v_i - v_j \quad \text{where} \quad v_i = \frac{1}{n} f_{(0i)}, \quad i \in V.
\]

**Theorem 2.** Let \( F \) be a facet of the cone \( Con_{n+1} \). Then \( \psi(F) \) is a facet of the cone \( \psi(Con_{n+1}) \) if and only if the facet \( F \) contains the extreme ray \( l_0 \) spanned by the vector \( e_0 \).

Let \( l \neq l_0 \) be an extreme ray of \( Con_{n+1} \). Then \( \psi(l) \) is an extreme ray of \( \psi(Con_{n+1}) \) if and only if the ray \( l \) belongs to a facet containing the extreme ray \( l_0 \).

**Proof.** By Theorem 1, the projection \( \pi \) transforms the facet \( F \) of \( Con_{n+1} \) into a facet of \( \pi(Con_{n+1}) \) if and only if \( l_0 \subset F \). By the same Theorem, the projection \( \pi(l) \) is an extreme ray of \( \pi(Con_{n+1}) \) if and only if \( l \) belongs to a facet containing the extreme ray \( l_0 \).

Recall that the map \( \chi \) is a bijection between the spaces \( \mathbb{R}^{E \cup E_0} \) and \( Q_n \). This implies the assertion of this Theorem for the map \( \psi = \chi \circ \pi \).

By Theorem 2, the map \( \psi \) transforms the facet \( F \) in a facet of the cone \( \psi(Con_{n+1}) \) only if \( F \) contains the extreme ray \( l_0 \), i.e. only if the equality \( (f, e_0) = 0 \) holds. Hence the facet vector \( f \) should satisfy the equality
\[
\sum_{i \in V} f_{(0i)} = 0.
\]

The inequalities (12) give all facet-defining inequalities of the cone \( \psi(Con_{n+1}) \) from known facet-defining inequalities of the cone \( Con_{n+1} \).

So, we have the following algorithm for to find a list of facets of the cone \( \psi(Con_{n+1}) \) from a known list \( \mathcal{L} \) of facet vectors of the cone \( Con_{n+1} \).

**Step 1.** Take a facet vector \( f = \{f_{(ij)} : (ij) \in E \cup E_0\} \in \mathcal{L} \) of the cone \( Con_{n+1} \), and delete it from \( \mathcal{L} \). Find a point \( i \in V \cup \{0\} \) such that \( \sum_{k \in V \cup \{0\}} f_{(ik)} = 0 \). Go to Step 2.

**Step 2.** If such a point \( i \) does not exist, go to Step 1. Otherwise, make a permutation \( i \to 0, 0 \to i \), and go to step 3.

**Step 3.** By formula (13) form a facet vector of the cone \( \psi(Con_{n+1}) \) from the facet vector \( f \) of the cone \( Con_{n+1} \).

If \( \mathcal{L} \) is not empty, go to Step 1. Otherwise, end.

A proof of Proposition 2 below will be given later for each of the cones \( Met_{n+1}, Hyp_{n+1} \) and \( Cut_{n+1} \) separately.
Proposition 2. Let $F$ be a facet of $\text{Con}_{n+1}$ with facet vector $f = f^V + f^0$ such that $(f^0, e_0) = 0$. Then $\text{Con}_{n+1}$ has also a facet $F^*$ with facet vector $f^* = f^V - f^0$.

Proposition 2 implies the following important fact.

Proposition 3. For $q = q^a + q^b \in \psi(\text{Con}_{n+1})$, the map $q = q^a + q^b \rightarrow q^a = q^b - q^b$ preserves the cone $\psi(\text{Con}_{n+1})$, i.e.

$$(\psi(\text{Con}_{n+1}))^* = \psi(\text{Con}_{n+1}).$$

Proof. Let $F$ be a facet of $\text{Con}_{n+1}$ with facet vector $f$. By Proposition 2, if $\psi(F)$ is a facet of $\psi(\text{Con}_{n+1})$, then $F^*$ is a facet of $\text{Con}_{n+1}$ with facet vector $f^*$. Let $q \in \psi(\text{Con}_{n+1})$. Then $q$ satisfies the inequality $(f, q) = (f^V, q^a) + (f^0, q^b) \geq 0$ (see (12)) so the inequality $(f^*, q) = (f^V, q^a) - (f^0, q^b) \geq 0$. But it is easy to see that $(f, q) = (f^*, q^a)$ and $(f^*, q) = (f, q^*)$. This implies that $q^* \in \psi(\text{Con}_{n+1})$.

Call a facet $G$ of the cone $\psi(\text{Con}_{n+1})$ symmetric if $q \in F$ implies $q^* \in F$. Call a facet of $\psi(\text{Con}_{n+1})$ asymmetric if it is not symmetric.

The assertion of the following Proposition 4 is implied by the equality $(\psi(\text{Con}_{n+1}))^* = \psi(\text{Con}_{n+1})$.

Proposition 4. Let $g \in Q_n$ be a facet vector of an asymmetric facet $G$ of the cone $\psi(\text{Con}_{n+1})$, and let $G^* = \{q^* : q \in G\}$. Then $G^*$ is a facet of $\psi(\text{Con}_{n+1})$, and $g^*$ is its facet vector.

Recall that $\text{Con}_{n+1}$ has facets, that are zero-lifting of facets of $\text{Con}_n$. Call a facet $G$ of the cone $\psi(\text{Con}_{n+1})$ zero-lifting of a facet $F^V$ of $\text{Con}_n$ if $G = \psi(F)$, where $F$ is a facet of $\text{Con}_{n+1}$ which is zero-lifting of $F^V$.

Proposition 5. Let $g \in Q_n$ be a facet vector of a facet $G$ of the cone $\psi(\text{Con}_{n+1})$. Then the following assertions are equivalent:

(i) $g = g^*$;

(ii) the facet $G$ is symmetric;

(iii) $G = \psi(F)$, where $F$ is a facet of $\text{Con}_{n+1}$ which is zero-lifting of a facet $F^V$ of $\text{Con}_n$.

(iv) $G$ is a zero-lifting of a facet $F^V$ of $\text{Con}_n$.

Proof. (i)⇒(ii). If $g = g^*$, then $g = g^*$. Hence $q \in G$ implies $(g, q) = (g^*, q) = (g^*, q^*) = (g, q^*) = 0$. This means that $q^* \in G$, i.e. $G$ is symmetric.

(ii)⇒(i). By Proposition 3, the map $q \rightarrow q^*$ is an automorphism of $\psi(\text{Con}_{n+1})$. This map transforms a facet $G$ with facet vector $g$ into a facet $G^*$ with facet vector $g^*$. If $G$ is symmetric, then $G^* = G$, and therefore $g^* = g$.

(iii)⇒(i). Let $f = f^V + f^0$ be a facet vector of a facet $F$ of $\text{Con}_{n+1}$ such that $f^0 = 0$. Then the facet $F$ is zero-lifting of the facet $F^V = F \cap E$ of the cone $\text{Con}_n$. In this case, $f^V$ is also a facet vector of the facet $G = \psi(F)$ of $\psi(\text{Con}_{n+1})$. Obviously, $(F^V)^* = f^V$.

(iii)⇒(iv). This implication is implied by definition of zero-lifting of a facet of the cone $\psi(\text{Con}_{n+1})$.

(iv)⇒(i). The map $\chi$ induces a bijection between $\pi(F)$ and $\psi(F)$. Since $\pi(F)$ is zero-lifting of $F^V$, the facet vector of $\pi(F)$ belongs to $R^E$. This implies that the facet vector $g$ of $\psi(F)$ belongs to $R^{E\Sigma}$, i.e. $g^* = g$. □

The symmetry group of $\text{Con}_{n+1}$ is the symmetric group $\Sigma_{n+1}$ of permutations of indices (see [DL97]). The group $\Sigma_n$ is a subgroup of the symmetry group of the cone $\psi(\text{Con}_{n+1})$. The full symmetry group of $\psi(\text{Con}_{n+1})$ is $\Sigma_n \times \Sigma_2$, where $\Sigma_2$ corresponds to the map $q \rightarrow q^*$ for $q \in \psi(\text{Con}_{n+1})$. By Proposition 4, the set of facets of $\psi(\text{Con}_{n+1})$ is partitioned into pairs $G, G^*$. But it turns out that there are pairs such that $G^* = \sigma(G)$, where $\sigma \in \Sigma_n$. 

9 Projections of hypermetric facets
The metric cone $\text{Met}_{n+1}$, the hypermetric cone $\text{Hyp}_{p+1}$ and the cut cone $\text{Cut}_{n+1}$ lying in the space $\mathbb{R}^{E \cup E_0}$ have an important class of hypermetric facets, that contains the class of triangular facets.

Let $b_i, i \in V$, be integers such that $\sum_{i \in V} b_i = \mu$, where $\mu = 0$ or $\mu = 1$. Usually these integers are denoted as a sequence $(b_1, b_2, \ldots, b_n)$, where $b_i \geq b_{i+1}$. If, for some $i$, we have $b_i = b_{i+1} = \ldots = b_{i+m-1}$, then the sequence is shortened as $(b_1, \ldots, b_i, b_{i+m}, \ldots, b_n)$.

One relates to this sequence the following inequality of type $b = (b_1, \ldots, b_n)$

$$(f(b), x) = -\sum_{i,j \in V} b_i b_j x_{(ij)} \geq 0,$$

where $x = \{x_{(ij)}\} \in \mathbb{R}^{E}$ and the vector $f(b) \in \mathbb{R}^E$ has coordinates $f(b)_{(ij)} = -b_i b_j$. This inequality is called of negative or hypermetric type if in the sum $\sum_{i \in V} b_i = \mu$ we have $\mu = 0$ or $\mu = 1$, respectively.

The set of hypermetric inequalities on the set $V \cup \{0\}$ determines a hypermetric cone $\text{Hyp}_{p+1}$. There are infinitely many hypermetric inequalities for metrics on $V \cup \{0\}$. But it is proved in [DL97], that only finite number of these inequalities determines facets of $\text{Hyp}_{p+1}$. Since triangle inequalities are inequalities $(f(b), x) \geq 0$ of type $b = (1^2, 0^{n-3}, -1)$, the hypermetric cone $\text{Hyp}_{p+1}$ is contained in $\text{Met}_{n+1}$, i.e. $\text{Hyp}_{p+1} \subseteq \text{Met}_{n+1}$ with equality for $n = 2$.

The hypermetric inequality $(f(b), x) \geq 0$ takes the following form on the set $V \cup \{0\}$.

$$-\sum_{i,j \in V \cup \{0\}} b_i b_j x_{(ij)} = -\sum_{(ij) \in E} b_i b_j x_{(ij)} - \sum_{i \in V} b_i b_j x_{(0i)} \geq 0. \quad (14)$$

If we decompose the vector $f(b)$ as $f(b) = f^V(b) + f^0(b)$, then $f^V(b)_{(ij)} = -b_i b_j$, $(ij) \in E$, and $f^0(b)_{(0i)} = -b_i b_i, i \in V$.

Let, for $S \subseteq V$, the equality $\sum_{i \in S} b_i = 0$ hold. Denote by $b^S$ a sequence such that $b^S_i = -b_i$ if $i \in S$ and $b^S_i = b_i$ if $i \not\in S$. The sequence $b^S$ is called switching of $b$ by the set $S$.

The hypermetric cone $\text{Hyp}_{p+1}$ has the following property (see [DL97]). If an inequality $(f(b), x) \geq 0$ defines a facet and $\sum_{i \in S} b_i = 0$ for some $S \subseteq V \cup \{0\}$, then the inequality $(f(b^S), x) \geq 0$ defines a facet, too.

Proof of Proposition 2 for $\text{Hyp}_{p+1}$. Consider the inequality (14), where $(f^0(b), e_0) = -\sum_{i \in V} b_i b_i = 0$. Then $\sum_{i \in V} b_i = 0$. Hence the cone $\text{Hyp}_{p+1}$ has similar inequality for $b^V$, where $b^V_i = -b_i$ for all $i \in V$. Hence if one of these inequalities defines a facet so does another. Obviously, $f^0(b^V) = -f^0(b)$. Hence these facets satisfy the assertion of Proposition 2. \(\square\)

Theorem 3. Let $(f(b), x) \geq 0$ define a hypermetric facet of a cone in the space $\mathbb{R}^{E \cup E_0}$. Then the map $\psi$ transforms it either in a hypermetric facet if $b_0 = 0$ or in a distortion of a facet of negative type if $b_0 = 1$. Otherwise, the projection is not a facet.

Proof. By Section 8, the map $\psi$ transforms the hypermetric inequality (14) for $x \in \mathbb{R}^{E \cup E_0}$ into the following inequality

$$-\sum_{(ij) \in E} b_i b_j q_{(ij)} - b_0 \sum_{i \in V} b_i w_i \geq 0$$

for $q = \sum_{(ij) \in E} q_{(ij)} \varphi(e_{(ij)}) + \sum_{i \in V} w_i q(i) \in Q_n$.

Since $f(b)$ determines a hypermetric inequality, we have $b_0 = 1 - \sum_{i \in V} b_i = 1 - \mu$. So, the above inequality takes the form

$$\sum_{(ij) \in E} b_i b_j q_{(ij)} \leq (\mu - 1) \sum_{i \in V} b_i w_i.$$

By Theorem 1, this facet is projected by the map $\psi$ into a facet if and only if $(f(b), e_0) = 0$, where $e_0 = \sum_{i \in V} e_{(0i)}$. We have

$$(f(b), e_0) = \sum_{i \in V} f(b)_{(0i)} = -\sum_{i \in V} b_0 b_i = -b_0 \mu = (\mu - 1) \mu.$$
This implies that the hypermetric facet-defining inequality \((f(b), x) \geq 0\) is transformed into a facet-defining inequality if and only if either \(\mu = 0\) and then \(b_0 = 1\) or \(\mu = 1\) and then \(b_0 = 0\). So, we have:

- If \(\mu = 1\) and \(b_0 = 0\), then the above inequality is a usual hypermetric inequality in the space \(\psi(E) = \varphi(E) = R^E_{+}\).
- If \(\mu = 0\) and \(b_0 = 1\), then the above inequality is the following distortion of an inequality of negative type

\[
- \sum_{(ij) \in E} b_i b_j g_{(ij)} - \sum_{i \in V} b_i w_i \geq 0, \quad \text{where} \quad \sum_{i \in V} b_i = 0. \tag{15}
\]

Comparing (7) with the inequality (15), we see that a canonical facet vector \(g(b)\) of a facet of \(\psi(Hyp_{n+1})\) has the form \(g(b) = g^*(b) + g^n(b)\), where \(g_{ij}(b) = g_{(ij)}(b) + v_i - v_j\), and

\[
g_{(ij)}(b) = -b_i b_j, \quad v_i = -\frac{1}{n} b_i \quad \text{for all} \quad i \in V.
\]

Define a cone of weighted quasi-hyper-metrics \(WQHyp_n = \psi(Hyp_{n+1})\). We can apply Proposition 3, in order to obtain the following assertion.

**Proposition 6.** The map \(q \rightarrow q^*\) preserves the cone \(WQHyp_n\), i.e.

\[
(WQHyp_n)^* = WQHyp_n.
\]

In other words, if \(q \in WQHyp_n\) has weights \(w_i, i \in V\), then the cone \(WQHyp_n\) has a point \(q^*\) with weights \(-w_i, i \in V\). \(\Box\)

### 10 Generalizations of metrics

The metric cone \(Met_{n+1}\) is defined in the space \(R^{E_0} \cup E_0\). It has an extreme ray which is spanned by the vector \(e_0 = \sum_{i \in V} e_{(0i)} \in R^{E_0}\). Facets of \(Met_{n+1}\) are defined by the following set of triangle inequalities, where \(d \in Met_{n+1}\).

**Triangle inequalities of the sub-cone** \(Met_n\) **that define facets of** \(Met_{n+1}\) **that are zero-lifting and contain** \(e_0\):

\[
d_{(ik)} + d_{(kj)} - d_{(ij)} \geq 0, \quad \text{for} \quad i, j, k \in V. \tag{16}
\]

**Triangle inequalities defining facets that are not zero-lifting and contain the extreme ray** \(l_0\) **spanned by the vector** \(e_0\):

\[
d_{(ij)} + d_{(j0)} - d_{(i0)} \geq 0 \quad \text{and} \quad d_{(ij)} + d_{(i0)} - d_{(j0)} \geq 0, \quad \text{for} \quad i, j \in V. \tag{17}
\]

**Triangle inequalities defining facets that do not contain the extreme ray** \(l_0\) **and do not define facets of** \(Met_n\):

\[
d_{(i0)} + d_{(j0)} - d_{(ij)} \geq 0, \quad \text{for} \quad i, j \in V. \tag{18}
\]

One can say that the cone \(Met_n \in R^{E_0}\) is lifted into the space \(R^{E_0} \cup E_0\) using restrictions (17) and (18). Note that the inequalities (17) and (18) imply the following inequalities of non-negativity

\[
d_{(i0)} \geq 0, \quad \text{for} \quad i \in V. \tag{19}
\]

A cone defined by inequalities (16) and (19) is called by cone \(WMet_n\) of weighted metrics \((d, w)\), where \(d \in Met_n\) and \(w_i = d_{(0i)}\) for \(i \in V\) are weights.
If weights \( w_i = d_{i(u)} \) satisfy also the inequalities (17) additionally to the inequalities (19), then the weighted metrics \((d, w)\) form a cone \(dWMet_n\) of down-weighted metrics. If metrics have weights that satisfy the inequalities (19) and (18), then these metrics are called up-weighted metrics. Detail see in [DD10], [DDV11].

Above defined generalizations of metrics are functions on unordered pairs \((ij)\) \(\in E \cup E_0\). Generalizations of metrics as functions on ordered pairs \(ij, ki, kj \in E^O\) are called quasi-metrics.

The cone \(QMet_n\) of quasi-metrics is defined in the space \(\mathbb{R}^{n^3}\) by non-negativity inequalities \(q_{ij} \geq 0\) for all \(ij \in E^O\), and by triangle inequalities \(q_{ij} + q_{jk} - q_{ik} \geq 0\) for all ordered triples \(ijk\) for each \(q \in QMet_n\). Below we consider in \(QMet_n\) a sub-cone \(WQMet_n\) of weighted quasi-metrics.

### 11 Cone of weighted quasi-metrics

We call a quasi-metric \(q\) weighted if it belongs to the subspace \(Q_n \subset \mathbb{R}^{E^O}\). So, we define

\[
WQMet_n = QMet_n \cap Q_n.
\]

A quasi-metric \(q\) is called weightable if there are weights \(w_i \geq 0\) for all \(i \in V\) such that the following equalities hold

\[
q_{ij} + w_i = q_{ji} + w_j
\]

for all \(i, j \in V, i \neq j\). Since \(q_{ij} = q_{ij}^* + q_{ij}^n\), we have \(q_{ij} + w_i = q_{ij}^* + q_{ij}^n + w_i = q_{ji}^* + q_{ji}^n + w_j\), i.e. \(q_{ij}^n - q_{ji}^n = 2q_{ij}^n = w_j - w_i\), what means that, up to multiple \(\frac{1}{2}\) and sign, the antisymmetric part of \(q_{ij}\) is \(w_i - w_j\). So, weightable quasi-metrics are weighted.

Note that weights of a weighted quasi-metric are defined up to an additive constant. So, if we take weights non-positive, we obtain a weightable quasi-metric. Hence, sets of weightable and weighted quasi-metrics coincide.

By definition of the cone \(WQMet_n\) and by symmetry of this cone, the triangle inequality \(q_{ij} + q_{jk} - q_{ik} \geq 0\) and non-negativity inequality \(q_{ij} \geq 0\) determine facets of the cone \(WQMet_n\). Facet vectors of these facets are

\[
t_{ijk} = e_{ij} + e_{jk} - e_{ik} \quad \text{and} \quad e_{ij},
\]

respectively. It is not difficult to verify that \(t_{ijk}, e_{ij} \notin Q_n\). Hence these facet vectors are not canonical. Below, we give canonical representatives of these facet vectors.

Let \(T(ijk) \subseteq K^O_n\) be a triangle of \(K^O_n\) with direct arcs \(ij, jk, ki\) and opposite arcs \(ji, kj, ik\). Hence

\[
f^{T(ijk)} = (e_{ij} + e_{jk} - e_{ki}) - (e_{ji} + e_{kj} + e_{ik}).
\]

**Proposition 7.** Canonical representatives of facet vectors \(t_{ijk}\) and \(e_{ij}\) are

\[
t_{ijk} = t_{ijk}^* = t_{ijk} + t_{kji}, \quad \text{and} \quad g(ij) = (e_{ij} + e_{ji}) + \frac{1}{n} (p(i) - p(j)),
\]

respectively.

**Proof.** We have \(t_{ijk} - f^{T(ijk)} = e_{ji} + e_{kj} - e_{ki} = t_{kji} = t_{ijk}^*\). This implies that the facet vectors \(t_{ijk}\) and \(t_{kji}\) determine the same facet, and the vector \(t_{ijk} + t_{kji} \in \mathbb{R}^{E^O_n}\) is a canonical representative of facet vectors of this facet. We obtain the first assertion of Proposition.

Consider now the facet vector \(e_{ij}\). It is more convenient to take the doubled vector \(2e_{ij}\). We show that the vector

\[
g(ij) = 2e_{ij} - \frac{1}{n} \sum_{k \in V - \{i,j\}} f^{T(ijk)}.
\]
is a canonical representative of the facet vector $2e_{ij}$. It is sufficient to show that $g(ij) \in Q_n$, i.e.
$g_{ki}(ij) = g_{ij}(i) + w_k - w_i$. In fact, we have $g_{ji}(ij) = 2 - \frac{n-2}{n} = 1 + \frac{2}{n}, g_{ji}(ij) = -g_{ki}(ij) = -g_{ki}(ij) = -1, g_{jk}(ij) = -g_{kj}(ij) = -1, g_{kk}(ij) = 0$. Hence we have

$$g^q(ij) = e_{ij} + e_{ji}, \ w_i = -w_j = \frac{1}{n}, \ \text{and} \ w_k = 0 \ \text{for all} \ k \in V - \{i,j\}.$$ 

These equalities imply the second assertion of Proposition.

Let $\tau_{ijk}$ be a facet vector of a facet of $Met_n$ determined by the inequality $d_{(ij)} + d_{(jk)} - d_{(ik)} \geq 0$. Then

$$t_{ij} + t_{kj} = \varphi(\tau_{ijk}),$$

where the map $\varphi : \mathbb{R}^E \rightarrow \mathbb{R}^E^0$ is defined in Section 2. Obviously, a triangular facet is symmetric.

Recall that $q_{ij} = q_{(ij)} + w_i - w_j$ if $q \in WQMet_n$. Let $i, j, k \in V$. It is not difficult to verify that the following equalities hold:

$$q_{ij} + q_{jk} - q_{ik} = q_{ij} + q_{jk} - q_{ij} \geq 0.$$ \hspace{1cm} (20)

Since $q_{ij} = q_{ij}$, these inequalities show that the symmetric part $q^s$ of the vector $q \in WQMet_n$ is a semi-metric. Hence if $w_i = w$ for all $i \in V$, then the quasi-semi-metric $q = q^s$ itself is a semi-metric. This implies that the cone $WQMet_n$ contains the semi-metric cone $Met_n$. Moreover, $Met_n = WQMet_n \cap \mathbb{R}_+ E^0$.

Now we show explicitly how the map $\psi$ transforms the cones $Met_{n+1}$ and $dWMet_n$ into the cone $WQMet_n$.

**Theorem 4.** The following equalities hold

$$\psi(Met_{n+1}) = \psi(dWMet_n) = WQMet_n \ \text{and} \ WQMet_n^* = WQMet_n.$$

**Proof.** All facets of the metric cone $Met_{n+1}$ of metrics on the set $V \cup \{\emptyset\}$ are given by triangular inequalities

$$d_{(ij)} + d_{(ik)} - d_{(ik)} \geq 0.$$ 

They are hypermetric inequalities $(g(b), d) \geq 0$, where $b$ has only three non-zero values $b_j = b_k = 1$ and $b_i = -1$ for some triple $\{ijk\} \subseteq V \cup \{\emptyset\}$. By Theorem 3, the map $\psi$ transforms this facet into a hypermetric facet, i.e. into a triangular facets of the cone $\psi(Met_{n+1})$ if and only if $b_0 = 0$, i.e. if $0 \not\in \{ijk\}$. If $0 \in \{ijk\}$, then, by the same theorem, the equality $b_0 = 1$ should be satisfied. This implies $0 \not\in \{ijk\}$. In this case the facet defining inequality has the form (15), that in the case $k = 0$, is

$$q_{ij} + w_i - w_j \geq 0.$$ 

This inequality is the non-negativity inequality $q_{ij} \geq 0$.

If $b_i = 1, b_j = -1$ and $k = 0$, the inequality $d_{(ij)} + d_{(j)} - d_{(i)} \geq 0$ is transformed into inequality

$$q_{(ij)} + w_j - w_i \geq 0, \ \text{i.e.} \ q_{ij} \geq 0.$$ 

This inequality and inequalities (20) imply the last equality of this Theorem.

The inequalities (18) define facets $F$ of $Met_{n+1}$ and $dWMet_n$ that do not contain the extreme ray $l_0$. Hence, by Theorem 3, $\psi(F)$ are not facets of $WQMet_n$. But, recall that the cone $dWMet_n$ contains all facet of $Met_{n+1}$ excluding facets defined by the inequalities (18). Instead of these facets, the cone $dWMet_n$ has facets $G_i$ defined by the non-negativity inequalities (19) with facet vectors $e_{(ij)}$ for all $i \in V$. Obviously all these facets do not contain the extreme ray $l_0$. Hence, by Theorem 2, $\psi(G_i)$ is not a facet of $\psi(dWMet_n)$. Hence we have also the equality $WQMet_n = \psi(dWMet_n)$. \hspace{1cm} \square

**Remark.** Facet vectors of facets of $Met_{n+1}$ that contain the extreme ray $l_0$ spanned by the vector $e_0$ are $\tau_{ijk} = \tau_{ijk} = \tau_{ijk}, \tau_{i0} = \tau_{ij0} = \tau_{ij0} = \tau_{ij0} = \tau_{ij0} = \tau_{ij0}$, where $\tau_{ij0} = e_{(ij)}$ and $\tau_{ij0} = e_{(ij)} - e_{(ij)}$. Hence Proposition 2 is true for $Met_{n+1}$, and we can apply Proposition 3 in order to obtain the equality $WQMet_n^* = WQMet_n$ of Theorem 4.

12 The cone $Cut_{n+1}$

The cut vectors $\delta(S) \in \mathbb{R}^{E \cup E_0}$ for all $S \subseteq V \cup \{\emptyset\}$ span all extreme rays of the cut cone $Cut_{n+1} \subset \mathbb{R}^{E \cup E_0}$. In
other words, \( \text{Cut}_{n+1} \) is the conic hull of all cut vectors. Since the cone \( \text{Cut}_{n+1} \) is full dimensional, its dimension is dimension of the space \( \mathbb{R}^{E_0} \), that is \( \frac{n(n+1)}{2} \).

Recall that \( \delta(S) = \delta(V \cup \{0\} - S) \). Hence we can consider only \( S \) such that \( S \subseteq V \), i.e. \( 0 \notin S \). Moreover, by Section 5,

\[
\delta(S) = \sum_{i \in S, j \notin S} e_{ij} = \sum_{i \in S, j \in V - S} e_{ij} + \sum_{i \in S} e_{(0)i} = \delta^V(S) + \sum_{i \in S} e_{(0)i},
\]

where \( \delta^V(S) \) is restriction of \( \delta(S) \) on the space \( \mathbb{R}^E = \psi(\mathbb{R}^E) \). Note that

\[
\delta(V) = \delta(\{0\}) = \sum_{i \in V} e_{(0)i} = e_0.
\]

Consider a facet \( F \) of \( \text{Cut}_{n+1} \). Let \( f \) be facet vector of \( F \). Set

\[
R(F) = \{ S \subseteq V : (f, \delta(S)) = 0 \}.
\]

For \( S \in R(F) \), the vector \( \delta(S) \) is called root of the facet \( F \). By (21), for \( S \in R(F) \), we have

\[
(f, \delta(S)) = (f, \delta^V(S)) + \sum_{i \in S} f_{(0)i} = 0. \tag{22}
\]

We represent each facet vector of \( \text{Cut}_{n+1} \) as \( f = f^V + f^0 \), where \( f^V \in \mathbb{R}^E \) and \( f^0 \in \mathbb{R}^{E_0} \).

The set of facets of the cone \( \text{Cut}_{n+1} \) is partitioned onto equivalence classes by switchings (see [DL97]). For each \( S, T \subseteq V \cup \{0\} \), the switching by the set \( T \) transforms the cut vector \( \delta(S) \) into the vector \( \delta(S \cup T) \), where \( \Delta \) is symmetric difference, i.e. \( S \Delta T = S \cup T - S \cap T \). It is proved in [DL97] that if \( T \in R(F) \), then \( \{\delta(S \cup T) : S \in R(F)\} \) is the set of roots of the switched facet \( F^{\delta(T)} \) of \( \text{Cut}_{n+1} \). Hence \( R(F^{\delta(T)}) = \{S \Delta T : S \in R(F)\} \).

Let \( F \) be a facet of \( \text{Cut}_{n+1} \). Then \( F \) contains the vector \( e_0 = \delta(V) \) if and only if \( V \in R(F) \). Hence Lemma 1 below is an extended reformulation of Proposition 2.

**Lemma 1.** Let \( F \) be a facet of \( \text{Cut}_{n+1} \) such that \( V \in R(F) \). Let \( f = f^V + f^0 \) be facet vector of \( F \). Then the vector \( f^* = f^V - f^0 \) is facet vector of \( F^{\delta(V)} \) of the facet \( F \), and \( V \in R(F^{\delta(V)}) \).

**Proof.** Since \( V \in R(F) \), \( F^{\delta(V)} \) is a facet of \( \text{Cut}_{n+1} \). Since \( S \Delta V = V - S = \overline{S} \), for \( S \subseteq V \), we have

\[
R(F^{\delta(V)}) = \{ \overline{S} : S \in R(F) \}.
\]

Since \( \emptyset \in R(F) \), the set \( \emptyset \Delta V = V \in R(F^{\delta(V)}) \). Now, using (22), for \( S \in R(F^{\delta(V)}) \), we have

\[
(f^*, \delta(S)) = ((f^V - f^0), \delta(S)) = (f^V, \delta^V(S)) - \sum_{i \in S} f_{(0)i}.
\]

Note that \( \delta^V(\overline{S}) = \delta^V(S) \), and, since \( V \in R(F) \), \( \delta(V) = \delta(\{0\}) \), we have \( (f, \delta(S)) = \sum_{i \in V} f_{(0)i} = 0 \).

Hence \( \sum_{i \in S} f_{(0)i} = - \sum_{i \in S} f_{(0)i} \). It is easy to see, that \( (f^*, \delta(S)) = (f, \delta(S)) \). Since \( S \in R(F^{\delta(V)}) \) if and only if \( \overline{S} \in R(F) \), we see that \( f^* \) is a facet vector of \( F^{\delta(V)} \).

The set of facets of \( \text{Cut}_{n+1} \) is partitioned into orbits under action of the permutation group \( \Sigma_{n+1} \). But some permutation non-equivalent facets are equivalent under switchings. We say that two facets \( F, F' \) of \( \text{Cut}_{n+1} \) belong to the same type if there are \( \sigma \in \Sigma_{n+1} \) and \( T \subseteq V \) such that \( \sigma(F') = F^{\delta(T)} \).

13 Cone \( \text{OCut}_n \)

Denote by \( \text{OCut}_n \subset \mathbb{R}^{E_0} \) the cone whose extreme rays are spanned by cut vectors \( c(S) \) for all \( S \subseteq V \), \( S \neq \emptyset, V \). In other words, let

\[
\text{OCut}_n = \{ c \in Q_n : c = \sum_{S \subseteq V} \alpha_{SC}(S), \alpha_{S} \geq 0 \}.
\]
Coordinates $c_{ij}$ of a vector $c \in \text{OCut}_n$ are given in (6), where $w_i \geq 0$ for all $i \in V$. Hence $\text{OCut}_n \subset Q_n$. Recall that

$$c(S) = \frac{1}{2} (\delta^O(S) + \sum_{i \in S} p(i)), \quad (23)$$

where $\delta^O(S) = \varphi(\delta^V(S))$. Note that $\delta^O(\emptyset) = \delta^O(S)$ and $p(\emptyset) = -p(S)$, where $\emptyset = V - S$.

Denote by $\text{Cut}_n^O = \varphi(\text{Cut}_n)$ the cone generated by $\delta^O(S)$ for all $S \subset V$. The vectors $\delta^O(S)$ for all $S \subset V$, $S \neq \emptyset, V$, are all extreme rays of the cone $\text{Cut}_n^O$ that we identify with $\text{Cut}_n$ embedded into the space $\mathbb{R}^{	ext{ES}_n}$.

**Lemma 2.** For $S \subseteq V$, the following equality holds

$$\psi(\delta(S)) = 2c(S).$$

**Proof.** According to Section 8, $\psi(\delta^V(S)) = \varphi(\delta^V(S)) = \delta^O(S)$. Besides, $\psi(e_{(0)}) = p(i)$ for all $i \in V$. Hence, using (21), we obtain

$$\psi(\delta(S)) = \psi(\delta^V(S)) + \sum_{i \in S} \psi(e_{(0)}) = \varphi(\delta^V(S)) + \sum_{i \in S} p(i) = \delta^O(S) + p(S).$$

Recall that $\psi(\delta(V)) = \psi(e_0) = 0$ and $c(V) = 0$. Hence, according to (23), we obtain

$$\psi(\delta(S)) = 2c(S), \text{ for all } S \subseteq V.$$

Lemma is proved. □

**Theorem 5.** The following equalities hold

$$\psi(\text{Cut}_{n+1}) = \text{OCut}_n \text{ and } \text{OCut}_n^* = \text{OCut}_n.$$

**Proof.** Recall that the conic hull of vectors $\delta(S)$ for all $S \subseteq V$ is $\text{Cut}_{n+1}$. The conic hull of vectors $c(S)$ for all $S \subset V$ is the cone $\text{OCut}_n$. Since $\psi(\delta(V)) = c(V) = 0$, the first result follows.

The equality $\text{OCut}_n^* = \text{OCut}_n$ is implied by the equalities $c^*(S) = c(\emptyset)$ for all $S \subseteq V$.

By Lemma 1, the equality $\text{OCut}_n^* = \text{OCut}_n$ is a special case $\text{Con}_{n+1} = \text{Cut}_{n+1}$ of Proposition 3. □

## 14 Facets of $\text{OCut}_n$

**Lemma 3.** Let $F$ be a facet of $\text{Cut}_{n+1}$. Then $\psi(F)$ is a facet of $\text{OCut}_n$ if and only if $V \in R(F)$.

**Proof.** By Theorem 2, $\psi(F)$ is a facet of $\text{OCut}_n$ if and only if $e_0 = \delta(V) \subset F$, i.e. if and only if $V \in R(F)$. □

For a facet $G$ of $\text{OCut}_n$ with facet vector $g$, we set

$$R(G) = \{ S \subseteq V : (g, c(S)) = 0 \}$$

and call the vector $c(S)$ for $S \in R(G)$ by root of the facet $G$.

Note that $\delta(\emptyset) = 0$ and $c(\emptyset) = c(V) = 0$. Hence $\emptyset \in R(F)$ and $\emptyset \in R(G)$ for all facet $F$ of $\text{Cut}_{n+1}$ and all facets $G$ of $\text{OCut}_n$. The roots $\delta(\emptyset) = 0$ and $c(\emptyset) = c(V) = 0$ are called trivial roots.

**Proposition 8.** For a facet $F$ of $\text{Cut}_{n+1}$, let $G = \psi(F)$ be a facet of $\text{OCut}_n$. Then the following equality holds

$$R(G) = R(F).$$
**Remark.** We give two proofs of this equality. Both are useful.

**First proof.** According to Section 8, the map \( \psi \) transforms an inequality \((f, x) \geq 0\) defining a facet of \( Cut_{n+1} \) into the inequality (12) defining the facet \( G = \psi(F) \) of \( OCut_n \). Recall the the inequality (12) relates to the representation of vectors \( q \in \mathbb{Q}_n \) in the basis \( \{ \varphi(e_{ij}), p(i) \} \), i.e. \( q = \sum_{(ij) \in E} q_{(ij)} \varphi(e_{ij}) + \sum_{i \in V} w_i p(i) \). Let \( q = c(S) \) for \( S \in R(G) \). Then, according to (23), we have \( q_{(ij)} = \frac{1}{2} \delta^T_{(ij)}(S) \), \( w_i = \frac{1}{2} \) for \( i \in S \) and \( w_i = 0 \) for \( i \in \mathbb{N} \). Hence, omitting the multiple \( \frac{1}{2} \), the inequality in (12) gives the following equality

\[
\sum_{(ij) \in E} f_{(ij)} \delta^T_{(ij)}(S) + \sum_{i \in S} f_{(0i)} = 0
\]

which coincides with (22). This implies the assertion of this Proposition.

**Second proof.** By Theorem 2, \( \psi(l) \) is an extreme ray of \( \psi(F) \) if and only if \( l \) is an extreme ray of \( F \) and \( l \neq l_0 \). Since \( l \) is spanned by \( \delta(S) \) for some \( S \in R(F) \) and \( \psi(l) \) is spanned by \( \psi(\delta(S)) = c(S) \), we have \( R(G) = \{ S \subset V : S \in R(F) \} \). Since \( c(V) = 0 \), we can suppose that \( V \in R(G) \), and then \( R(G) = R(F) \). \( \square \)

**Remark.** Note that \( \delta(V) = \delta(\{0\}) = e_0 \neq 0 \) is a non-trivial root of \( F \), i.e. \( V \in R(F) \). But \( c(V) = \psi(\delta(V)) = 0 \) is a trivial root of \( R(G) \).

Recall that, for a subset \( T \subseteq V \), we set \( \overline{T} = V - T \). Note that \( \overline{T} = V \triangle T \) and \( \overline{T} \neq V \cup \{0\} - T \).

**Lemma 4.** Let \( F \) be a facet of \( Cut_{n+1} \), and \( T \in R(F) \). Then the image \( \psi(F^\delta(T)) \) of the switched facet \( F^\delta(T) \) is a facet of \( OCut_n \) if and only if \( \overline{T} \in R(F) \).

**Proof.** By Lemma 3, \( \psi(F^\delta(T)) \) is a facet of \( OCut_n \) if and only if \( V \in R(F^\delta(T)) \), i.e. if and only if \( V \triangle T = \overline{T} \in R(F) \).

For a facet \( G \) of \( OCut_n \), define \( G^\delta(T) \) as the conic hull of \( c(S \triangle T) \) for all \( S \in R(G) \). Since each facet \( G \) of \( OCut_n \) is \( \psi(F) \) for some facet \( F \) of \( Cut_{n+1} \), Lemma 4 and Proposition 8 imply the following assertion.

**Theorem 6.** Let \( G \) be a facet of \( OCut_n \). Then \( G^\delta(T) \) is a facet of \( OCut_n \) if and only if \( T, \overline{T} \in R(G) \), and then \( R(G^\delta(T)) = \{ S \triangle T : S \in R(G) \} \). \( \square \)

Theorem 6 asserts that the set of facets of the cone \( OCut_n \) is partitioned onto equivalence classes by switchings \( G \rightarrow G^\delta(T) \), where \( T, \overline{T} \in R(G) \).

The case \( T = V \) in Theorem 6 plays a special role. Recall that \( V \in R(F) \) if \( F \) is a facet of \( Cut_{n+1} \) such that \( \psi(F) \) is a facet of \( OCut_n \). Hence Lemma 1 and Proposition 3 imply the following fact.

**Proposition 9.** Let \( F \) be a facet of \( Cut_{n+1} \) such that \( \psi(F) \) is a facet of \( OCut_n \). Let \( g = g^* + g^a \) be a facet vector of the facet \( \psi(F) \). Then the vector \( g^* = g^a - g^a \) is a facet vector of the facet \( \psi(F^\delta(V)) = (\psi(F))^\ast = (\psi(F))\delta(V) \) such that \( R((\psi(F))^\ast) = \{ S : S \in R(F) \} \). \( \square \)

Recall that roughly speaking \( OCut_n \) is projection of \( Cut_{n+1} \) along the vector \( \delta(V) = \delta(\{0\}) \).

Let \( \sigma \in \Sigma_n \) be a permutation of the set \( V \). For a vector \( q \in \mathbb{R}^E \), we have \( \sigma(q)_{ij} = q_{\sigma(i)\sigma(j)} \). Obviously if \( g \) is a facet vector of a facet \( G \) of \( OCut_n \), then \( \sigma(g) \) is the facet vector of the facet \( \sigma(G) = \{ \sigma(g) : q \in G \} \).

Note that, by Proposition 9, the switching by \( V \) is equivalent to the operation \( q \rightarrow q^* \). Hence the symmetry group of \( OCut_n \) contains the group \( \Sigma_n \times \Sigma_2 \), where \( \Sigma_2 \) relates to the map \( q \rightarrow q^* \) for \( q \in OCut_n \).

**Theorem 7.** The group \( \Sigma_n \times \Sigma_2 \) is the symmetry group of the cone \( OCut_n \).

**Proof.** Let \( \gamma \) be a symmetry of \( OCut_n \). Then \( \gamma \) is a symmetry of the set \( \mathcal{F}(e_0) \) of facets \( F \) of the cone \( Cut_{n+1} \) containing the vector \( e_0 \). The symmetry group \( \Gamma(e_0) \) of the set \( \mathcal{F}(e_0) \) is a subgroup of the symmetry group of the cut-polytope \( Cut_{n+1} \). In fact, \( \Gamma(e_0) \) is stabilizer of the edge \( e_0 \) of the polytope \( Cut_{n+1} \). But it is well-known that
\[ \Gamma(e_0) \] consists of the switching by \( V \) and permutations \( \sigma \in \Sigma_{n+1} \) leaving the edge \( e_0 \) non-changed. The map \( \psi \) transforms these symmetries of \( \mathcal{F}(e_0) \) into symmetries \( \sigma \in \Sigma_n \) and \( q \to q^* \) of the cone \( OCut_n \).

The set of all facets of \( OCut_n \) is partitioned onto orbits of facets that are equivalent by the symmetry group \( \Sigma_n \times \Sigma_2 \). It turns out that, for some facets \( G \), subsets \( S \in R(G) \) and permutations \( \sigma \in \Sigma_n \), we have \( C^g(S) = \sigma(G) \).

By Proposition 5, if a facet of \( Cut_{n+1} \) is zero-lifting of a facet \( F \) of \( Cut_n \), then the facet \( G = \psi(F) \) of \( OCut_n \) is symmetric and \( G = G^* = C^g(V) \) is zero-lifting of \( F \).

So, there are two important classes of orbits of facets of \( OCut_n \). Namely, the orbits of symmetric facets, that are zero-lifting of facets of \( Cut_n \), and orbits of asymmetric facets that are \( \psi \)-images of facets of \( Cut_{n+1} \) and are not zero-lifting.

\[ \text{15 Cases} \ 3 \leq n \leq 6 \]

It is worth to compare results of this Section with Table 2 of [DDV11].

Most of described below facets are hypermetric or negative type. We give here the corresponding vectors \( b \) in accordance with Section 9.

**n=3.** Note that \( Cut_4 = Hyp_4 = Met_4 \). Hence

\[ OCut_3 = WQHyp_3 = WQMet_3. \]

All these cones have two orbits of facets: one orbit of non-negativity facets with \( b = (1, 0, -1) \) and another orbit of triangular facets with \( b = (1^2, -1) \).

**n=4.** We have \( Cut_5 = Hyp_5 \subset Met_5 \). Hence

\[ OCut_4 = WQHyp_4 \subset WQMet_4. \]

The cones \( Hyp_5 = Cut_5 \) have two orbits of facets: triangular and pentagonal facets. Recall that a triangular facet with facet vector \( \tau_{ijk} \) is zero-lifting if \( 0 \not\in \{i,j,k\} \). Hence the cones \( WQHyp_4 = OCut_4 \) have three orbits of facets: of non-negativity with \( b = (1,0^2,-1) \), triangular with \( b = (1^2,0,-1) \) and weighted version of negative type with \( b = (1^2,-1^2) \).

**n=5.** We have again \( Cut_6 = Hyp_6 \subset Met_6 \). Hence

\[ OCut_5 = WQHyp_5 \subset WQMet_5. \]

The cones \( Hyp_6 = Cut_6 \) have four orbits of facets, all are hypermetric: triangular with \( b = (1^2,0^3,-1) \), pentagonal with \( b = (1^3,0,-1^2) \) and two more types, one with \( b = (2,1^2,-1^3) \) and its switching with \( b = (1^4,-1,-2) \). These four types provide 6 orbits of facets of the cones \( WQHyp_5 = OCut_5 \): non-negativity with \( b = (1,0^3,-1) \), triangular with \( b = (1^2,0^2,-1) \), of negative type with \( b = (1^2,0,-1^2) \), pentagonal with \( b = (1^3,-1^2) \), and two of negative type with \( b = (2,1,-1^3) \) and \( b = (1^3,-1,-2) \).

The last two types belong to the same orbit of the full symmetry group \( \Sigma_5 \times \Sigma_2 \). Hence the cone \( OCut_5 \) has 5 orbits of facets under action of its symmetry group.

**n=6.** Now, we have \( Cut_7 \subset Hyp_7 \subset Met_7 \). Hence

\[ OCut_6 \subset WQHyp_6 \subset WQMet_6. \]

The cone \( Cut_7 \) has 36 orbits of facets under action of the permutation group \( \Sigma_7 \). Switchings contract these orbits into 11 types \( F_k, 1 \leq k \leq 11 \), (see [DL97], Sect. 30.6). J. Vidali compute orbits of facets of \( OCut_6 \) under action of the group \( \Sigma_6 \). Using these computations, we give in Table below numbers of orbits of facets of cones \( Cut_7 \) and \( OCut_6 \) (cf. Figure 30.6.1 of [DL97]).
The first row of Table gives types of facets of $Cut_7$. In the second row of Table, for each type $F_k$, numbers of orbits of facets of $Cut_7$ of type $F_k$ under action of the group $\Sigma_7$. The third row of Table, for each type $F_k$, gives numbers of orbits of facets of $OCut_6$ that are obtained from facets of type $F_k$ under action of the group $\Sigma_6$. The fourth row gives, for each type $F_k$, numbers of orbits of facets of $OCut_6$ that are obtained from facets of type $F_k$ under action of the group $\Sigma_6 \times \Sigma_2$.

The last column of Table gives total numbers of orbits of facets of the cones $Cut_7$ and $OCut_6$.

| Table |
|-------|
| types | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ | $\Omega$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\Sigma_7$ | 1 | 1 | 2 | 1 | 3 | 2 | 4 | 7 | 5 | 3 | 7 | 36 |
| $\Sigma_6$ | 2 | 2 | 4 | 1 | 3 | 2 | 7 | 13 | 6 | 6 | 15 | 61 |
| $\Sigma_6 \times \Sigma_2$ | 2 | 2 | 3 | 1 | 2 | 1 | 4 | 7 | 3 | 4 | 8 | 37 |

The first three types $F_1$, $F_2$, $F_3$ relate to 4 orbits of hypermetric facets $F(b)$ of $Cut_7$ that are zero-lifting, where $b = (1^2, 0^4, -1)$, $b = (1^3, 0^2, -1^2)$ and $b = (2, 1^2, 0, -1^3)$, $b = (1^4, 0, -1, -2)$. Each of these four orbits of facets of $Cut_7$ under action of $\Sigma_7$ gives two orbits of facets of $OCut_6$ under action of the group $\Sigma_6$.

The second three types $F_4$, $F_5$, $F_6$ relate to 6 orbits of hypermetric facets $F(b)$ of $Cut_7$ that are not zero-lifting. Each of these 6 orbits gives one orbit of facets of $OCut_6$ under action of the group $\Sigma_6$.

The third three types $F_7$, $F_8$, $F_9$ relate to 16 orbits of facets of clique-web types $CW_7(b)$. These 16 orbits give 26 orbits of facets of $OCut_6$ under action of $\Sigma_6$.

The last two types $F_{10} = Par_7$ and $Gr_7$ are special (see [DL97]). They relate to 10 orbits of $Cut_7$, that give 21 orbits of facets of $OCut_6$ under action of $\Sigma_6$.

The subgroup $\Sigma_2$ of the full symmetry group $\Sigma_6 \times \Sigma_2$ contracts some pairs of orbits of the group $\Sigma_6$ into one orbit of the full group. The result is given in the forth row of Table.

Note that the symmetry groups of $Cut_7$ and $OCut_6$ have 36 and 37 orbits of facets, respectively.

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