Integral triangular operators 
and Friedrichs model 

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Abstract

In the present paper we investigate a semi-group of triangular integral operators $V_\beta$, which is an analogue of the semi-group of the fractional integral operators $J^\beta$. With the help of these semi-groups, we construct and study two classes of triangular Friedrichs models $A_\beta$ and $B_\beta$, respectively. Using generalized wave operators we prove that $A_\beta$ and $B_\beta$ are linearly similar to a self-adjoint operator with absolutely continuous spectrum.

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1 Introduction

1. In the present paper we investigate the triangular integral operators

$$V_\beta f = \frac{1}{\Gamma(\beta)} \int_0^x E_\beta(x-t)f(t) \, dt, \quad \Re \beta > 0, \ f(x) \in L^2(0, \omega), \quad (1.1)$$

where $\Gamma(z)$ is Euler gamma function,

$$E_\beta(x) = \int_0^\infty \frac{1}{\Gamma(s)} e^{-Cs^{\beta-1}x^s} \, ds, \quad C \in \mathbb{C}, \quad (1.2)$$
and \( \mathbb{C} \) stands for the complex plane. We shall show that the operators \( V_\beta \) form a semi-group, which is an analogue of the semi-group of the fractional integral operators

\[
J^\beta f = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t) \, dt, \quad \Re \beta > 0, \quad f(x) \in L^2(0, \omega). \tag{1.3}
\]

With the help of the semi-groups of operators \( J^\beta \) and \( V_\beta \) we construct and study two classes of triangular Friedrichs models \( A_\beta \) and \( B_\beta \), respectively. Using generalized wave operators we prove that \( A_\beta \) and \( B_\beta \) are linearly similar to a self-adjoint operator with absolutely continuous spectrum.

**Lemma 1.1** The function \( E_\beta(x) \) is continuous in the domain \((0, \omega]\) and

\[
E_\beta(x) = \frac{\Gamma(\beta + 1)}{x|\ln(x)|^{\beta+1}} \left(1 + O\left(\frac{1}{\ln(x)}\right)\right), \quad x \to 0. \tag{1.4}
\]

**Proof.** It follows from (1.2) that the function \( E_\beta(x) \) is continuous in the domain \((0, \omega]\) and

\[
E_\beta(x) = \int_0^1 \frac{1}{\Gamma(s)} e^{-Cs}s^{\beta-1}x^{s-1} \, ds + O(1). \tag{1.5}
\]

Using relations

\[
\frac{1}{\Gamma(s)} = s + O(s^2), \quad e^{-Cs} = 1 + O(s) \quad (s \to 0), \tag{1.6}
\]

we obtain

\[
E_\beta(x) = \frac{1}{x} \int_0^1 s^\beta(1 + O(s)) e^{-s|\ln(x)|} \, ds + O(1). \tag{1.7}
\]

Equality (1.7) implies the equality (1.4). The lemma is proved. \( \square \)

**Proposition 1.2** The operators \( V_\beta \), defined by formula (1.1), are bounded in all spaces \( L^p(0, \omega) \), \( p \geq 1 \).

**Proof.** Indeed, according to (1.4) we have (see [9, p. 24])

\[
\|V_\beta\|_p \leq m(\beta) = \int_0^\omega \left| E_\beta(x) / \Gamma(\beta) \right| \, dx < \infty. \tag{1.8}
\]

This proves the proposition. \( \square \)

The operators \( V_\beta \) have a following important property [10].
Theorem 1.3 The operators $V_{\beta}$ ($\Re\beta > 0$) defined by formula (1.1), form a semi-group, that is,

$$V_{\alpha}V_{\beta} = V_{\alpha + \beta}, \quad \Re\alpha > 0, \quad \Re\beta > 0.$$  \hspace{1cm} (1.9)

2. We introduce the following integro-differential operator

$$Rf = -\int_{0}^{x} f'(t) \ln(x - t) \, dt, \quad f(x) \in L(0,\omega).$$  \hspace{1cm} (1.10)

In the book [9, p. 73] we proved that the operator $V_{1}$ is a right inverse of $R$.

Proposition 1.4 If $\beta = 1$ and $C = -\Gamma'(1)$ then

$$RV_{1} \varphi = \varphi, \quad \varphi'(x) \in L(0,\omega).$$  \hspace{1cm} (1.11)

3. By $H_{\alpha}$ we denote the space of all function such that $f(x) \in L^{2}(0,\omega)$ and $f(x) = 0$ when $x \in [0,\alpha]$. It is easy to see that the spaces $H_{\alpha}$, $0 < \alpha \leq \omega$, are invariant subspaces of the operator $V_{\beta}$.

Definition 1.5 A bounded operator $T$ is unicellular if its lattice of invariant subspaces is totally ordered by inclusion.

Theorem 1.6 The operator $V_{\beta}$, $0 < \beta \leq 1$, defined by formula (1.1) is unicellular.

\textbf{Proof.} Let us consider the inner product

$$\langle V_{\beta}^{n} f, g \rangle = \int_{0}^{\infty} \frac{1}{\Gamma(s)} e^{-Cs} s^{n\beta - 1} \left( \int_{0}^{\omega} \int_{0}^{x} (x - t)^{s-1} f(t) \, dt \, g(x) \, dx \right) \, ds.$$  \hspace{1cm} (1.12)

Assuming that $\langle V_{\beta}^{n} f, g \rangle = 0$ we use the set of functions $\psi_{n}(x) = e^{-x^{2}/2} H_{2n}(x)$, where $H_{2n}(x)$ are Hermite polynomials. The set of functions $\psi_{n}(x)$ is complete in the Hilbert space $L^{2}(0,\infty)$. We note that $H_{2n}(x)$ are even polynomials of the degree $2n$. Hence, the set of functions $\varphi_{n}(x) = e^{-x^{2}/2} x^{2n}$ is complete in the Hilbert space $L^{2}(0,\infty)$. Changing the variable $s^{\beta/2} = u$ and taking into account the formulas (1.12) and $\langle V_{\beta}^{n} f, g \rangle = 0$, we have

$$\int_{0}^{\omega} \int_{0}^{x} (x - t)^{s-1} f(t) \, dt \, g(x) \, dx = 0, \quad 0 < \beta \leq 1.$$  \hspace{1cm} (1.13)
Relation (1.13) can be written in the form
\[ \int_0^\omega u^{s-1} \int_u^\omega f(x - u)g(x) \, dx \, du = 0. \]  
(1.14)

It follows from (1.14) that
\[ \int_u^\omega f(x - u)g(x) \, dx = 0. \]  
(1.15)

By changing the variables \( x = \omega - x_1, \ u = \omega - u_1 \) we obtain
\[ \int_0^{u_1} f(u_1 - x_1)g(\omega - x_1) \, dx_1 = 0. \]  
(1.16)

Using well-known Titchmarsh’s theorem ([11, Theorem 152]) we receive the following assertion: if \( f(x) \in H_\alpha \) and \( f(x) \notin H_\gamma \), when \( \gamma > \alpha \), then the system \( V_\beta^\alpha f \) is complete in the space \( H_\alpha \). This proves the theorem. \( \square \)

**Remark 1.7** Using Titchmarsh’s theorem ([11, Theorem 152]) it is easy to prove that operator \( J_\beta^\beta, \ \beta > 0 \), is unicellular.

### 2 Triangular integral operators with logarithmic type kernels

1. In the present section we shall consider the operators
\[ S_\beta f = \beta \int_0^x f(t) \ln(x - t) \left| \ln(x - t) \right|^{-\beta - 1} \frac{dt}{x-t}, \ \Re \beta > 0, \]  
(2.1)
in the space \( L^2(0, \omega), \ 0 < \omega < 1 \). The operator \( S_\beta \) is the main term in the expression which defines the operator \( V_\beta \) (see (1.1), (1.2)). Thus, by investigating the operator \( S_\beta \) we also obtain some properties of the operator \( V_\beta \). The operator \( S_\beta \) is of independent interest as well. Rewrite the operator \( S_\beta \) in the following form
\[ S_\beta f = \frac{d}{dx} \int_0^x f(t) \ln(x - t) \left| \ln(x - t) \right|^{-\beta} \, dt, \ \Re \beta \geq 0. \]  
(2.2)
We note that the operator $S_\beta$ is well-defined by formula (2.2) in the case where $\Re \beta \geq 0$. Let us introduce the functions (see [8] and [10])

\[ \tilde{s}(\lambda) = \int_0^\omega e^{it\lambda} |\ln(t)|^{-\beta} \, dt, \quad \Re \beta \geq 0, \] (2.3)

\[ \tilde{s}_1(\lambda) = \int_0^\omega e^{it\lambda} |\ln(t)|^{-\beta} (1 - t/\omega) \, dt, \quad \Re \beta \geq 0. \] (2.4)

\textbf{Theorem 2.1} If $\Re \beta \geq 0$ and $0 < \omega < 1$, then the following asymptotic relations are valid:

\[ -i\lambda \tilde{s}(\lambda) = (\ln |\lambda|)^{-\beta} (1 + o(1)) - e^{i\lambda \omega} (1 - \ln \omega)^{-\beta}, \quad \lambda \to \pm \infty, \] (2.5)

\[ -i\lambda \tilde{s}_1(\lambda) = (\ln |\lambda|)^{-\beta} (1 + o(1)), \quad \lambda \to \pm \infty. \] (2.6)

\textit{Proof.} According to the Cauchy theorem we have

\[ \int_\gamma e^{it\lambda} (-\ln(t))^{-\beta} \, dt = 0, \] (2.7)

where the curve $\gamma$ is depicted by Fig.1.

![Figure 1: Contour of integration](image_url)

From (2.3) and (2.4) we obtain the equality

\[ \tilde{s}(\lambda) = \int_0^{i\omega} e^{it\lambda} (-\ln t)^{-\beta} \, dt - \int_0^{\pi/2} e^{it\lambda} (-\ln (t))^{-\beta} \, t \, d\varphi, \quad t = \omega e^{i\varphi}. \] (2.8)
Let us consider the integral
\[ \int_0^{i\omega} e^{it\lambda} (-\ln t)^{-\beta} \, dt = \int_0^{\lambda \omega} e^{-u} (-\ln u + \ln \lambda - i\pi/2)^{-\beta} i \, du/\lambda, \quad \lambda > 0. \tag{2.9} \]

When \( \lambda \) tends to infinity, equality (2.9) yields
\[ \int_0^{i\omega} e^{it\lambda} (-\ln t)^{-\beta} \, dt = \frac{i}{\lambda} (\ln \lambda)^{-\beta} (1 + o(1)), \quad \lambda \to +\infty. \tag{2.10} \]

Integrating by parts we estimate the integral
\[ \int_0^{\pi/2} e^{it\lambda} (-\ln(t))^{-\beta} i t \, d\varphi = -\frac{1}{\lambda} \left( e^{i\lambda \omega} (-\ln \omega)^{-\beta} + o(1) \right), \quad \lambda \to +\infty, \tag{2.11} \]

where \( t = \omega e^{i\varphi}, \lambda \to +\infty \). Relations (2.8), (2.10) and (2.11) imply (2.5) for the case \( \lambda \to \infty \). In the same way we deduce (2.5) for the case \( \lambda \to -\infty \).

Relation (2.6) follows from (2.5) and from the equality (see [8])
\[ -i\lambda \tilde{s}_1(\lambda) = -i\lambda \tilde{s}(\lambda) + e^{i\lambda \omega} (-\ln \omega)^{-\beta} + o(1), \quad \lambda \to \infty. \tag{2.12} \]

The theorem is proved. \( \square \)

Thus, the function \(-i\lambda \tilde{s}(\lambda)\) is bounded. Hence, we obtain the following assertion (see [8]).

**Corollary 2.2** If \( \Re \beta \geq 0 \) and \( 0 < \omega < 1 \), then the operator \( S_\beta \) defined by formula (2.2) is bounded in the space \( L^2(0, \omega) \).

Further we need the following asymptotic relation (see [8]):
\[ \| S_\beta e^{-i\omega \lambda} + i\lambda \tilde{s}_1(\lambda) e^{-i\omega \lambda} \| \to 0, \quad \lambda \to \infty. \tag{2.13} \]

Using (2.6) and (2.13) we derive two statements below.

**Corollary 2.3** If \( \Re \beta = 0 \) and \( |z| = 1 \), then the points \( z \) belong to the spectrum of the operator \( S_\beta \).

**Corollary 2.4** If \( \Re \beta = 0, \nu > 0, \gamma = \beta + \nu \), then in the space \( L^2(0, \omega) \) we have
\[ S_\gamma \to S_\beta, \quad \nu \to 0 \quad \text{strong convergence}. \tag{2.14} \]
2. Taking into account (1.3) we can reformulate Corollaries 2.2–2.4 for the operator $V_\beta$.

**Corollary 2.5** If $\Re \beta \geq 0$ and $0 < \omega < 1$, then the operator $V_\beta$ is bounded in the space $L^2(0, \infty)$.

**Corollary 2.6** If $\Re \beta = 0$ and $|z| = 1$, then the points $z$ belong to the spectrum of the operator $V_\beta$.

**Corollary 2.7** If $\Re \beta = 0$, $\nu > 0$, $\gamma = \beta + \nu$, then in the space $L^2(0, \infty)$ we have

$$V_\gamma \rightarrow V_\beta, \quad \nu \rightarrow 0,$$  \quad \text{(strong convergence)} \quad (2.15)

3. Let us introduce the operator

$$T_\beta f = \int_0^x f(t) |\ln(x-t)|^{-\beta} \, dt, \quad \beta > 0,$$  \quad (2.16)

where $f(x) \in L^2(0, \omega)$ and $0 < \omega < 1$.

**Proposition 2.8** The operator $T_\beta$ is such that $T_\beta \in \sigma_2$ and $T_\beta \notin \sigma_1$.

**Proof.** It is obvious that $T_\beta \in \sigma_2$. In order to prove that $T_\beta \notin \sigma_1$ we write the operator $T_\beta$ in the form

$$T_\beta f = \frac{d}{dx} \int_0^x f(t) R(x-t) \, dt,$$  \quad (2.17)

where

$$R(x) = \int_0^x \ln^{-\beta}(u) \, du.$$  \quad (2.18)

Integrating by parts the right-hand side of the expression

$$\tilde{s}_1(\lambda) = \int_0^{\omega} e^{i\lambda t} R(t)(1 - t/\omega) \, dt,$$  \quad (2.19)

we obtain

$$\tilde{s}_1(\lambda) = -\frac{1}{i\lambda} \int_0^{\omega} e^{i\lambda t} |\ln(t)|^{-\beta} \, dt.$$  \quad (2.20)
Comparing relations (2.3) and (2.20), we see that $-i\lambda \tilde{s}_1(\lambda)$ for the operator $T_\beta$ is equal to $\tilde{s}(\lambda)$ for the operator $S_\beta$. Relations (2.5) and (2.20) imply that

$$\tilde{s}_1(\lambda) = e^{i\omega\lambda} \frac{e^{i\omega\lambda}}{\lambda^2} (-\ln \omega)^{-\beta} (1 + o(1)), \quad \lambda \to \infty.$$  (2.21)

It follows from (2.13) and (2.21) that

$$\sum_{n=1}^\infty |(T_\beta \varphi_n, \varphi_n)| = \infty,$$  (2.22)

where $\varphi_n(x) = e^{ixn/\omega}$.

Next, we use the following result [3]: if an operator $A$ is compact in the Hilbert space and $\varphi_j$ is an orthonormal system, then

$$\sum_{n=1}^\infty |(A\varphi_j, \varphi_j)| \leq \sum_{j=1}^\infty s_j,$$  (2.23)

where $s_j(A)$ is the non-decreasing sequence of the eigenvalues of the operator $(AA^*)^{1/2}$. The formulated result implies the following statement: if the equality

$$\sum_{n=1}^\infty |(A\varphi_j, \varphi_j)| = \infty$$  (2.24)

is valid, then $\sum_{j=1}^\infty s_j = \infty$. In other words, (2.22) yields

$$\sum_{j=1}^\infty s_j = \infty.$$  (2.25)

Hence, the proposition is proved.

3 Friedrichs model

Let us consider in the Hilbert space $L^2(0,\omega)$ the operator

$$A_{\alpha,\beta}f = V_\beta QV_\alpha f, \quad Qf = xf, \quad \Re \alpha > 0, \quad \Re \beta > 0.$$  (3.1)

Using relations (1.1) and (1.2) we obtain

$$A_{\alpha,\beta}f = \int_0^x f(u)U_{\alpha,\beta}(x, u) \, du.$$  (3.2)
where
\[
U_{\alpha,\beta}(x, u) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{x-u} E_\beta(x - u - y)(u + y)E_\alpha(y) \, dy. \tag{3.3}
\]

We need the relation (see [1]):
\[
\int_0^x (x - y)^{\alpha-1}y^{\beta-1} \, dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}x^{\alpha+\beta-1}, \quad \Re\alpha > 0, \quad \Re\beta > 0. \tag{3.4}
\]

Relations (1.2) and (3.4) imply that
\[
\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{x-u} E_\beta(x - u - y)uE_\alpha(y) \, dy = \frac{1}{\Gamma(\alpha + \beta)}E_{\alpha+\beta}(x - u), \tag{3.5}
\]
\[
\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{x-u} E_\beta(x - u - y)yE_\alpha(y) \, dy = \frac{(x - u)\alpha}{\Gamma(\alpha + \beta + 1)}E_{\alpha+\beta}(x - u). \tag{3.6}
\]

Relations (3.3) and (3.5), (3.6) imply the following proposition.

**Proposition 3.1** The operator \( A_{\alpha,\beta} \) is defined by (3.2), where
\[
U_{\alpha,\beta}(x, u) = \frac{1}{\Gamma(\alpha + \beta)}E_{\alpha+\beta}(x - u)u + \frac{(x - u)\alpha}{\Gamma(\alpha + \beta + 1)}E_{\alpha+\beta}(x - u). \tag{3.7}
\]

Using again relations (1.2) and (3.4) we have
\[
\frac{1}{\Gamma(m)}V_\beta E_m(y) = \frac{1}{\Gamma(m + \beta)}E_{m+\beta}(x), \quad \Re\beta > 0, \quad \Re m > 0. \tag{3.8}
\]

Hence, the equality
\[
\lim_{\beta \to 0} V_\beta \left( \frac{1}{\Gamma(m)}E_m(y) \right) = \frac{1}{\Gamma(m)}E_m(x) \tag{3.9}
\]
is valid.

**Remark 3.2** It is easy to see, that the functions \( E_m(x) \) form a complete system in the Hilbert space \( L^2(0, \omega) \).

Thus, we proved the following statement.

**Proposition 3.3** The operator \( V_\beta \) strongly converges to the identity operator \( I \) when \( \beta \to +0 \).
In view of (3.1), (3.7) and Proposition 3.3 the following theorem is valid.

**Theorem 3.4** Let \( R_\alpha = 0, \alpha \neq 0, \beta = -\alpha \). Then the operator

\[
A_\alpha = V_\alpha^{-1}QV_\alpha
\]

has the form

\[
A_\alpha f = xf(x) + \alpha \int_0^x (x - y)E_0(x - y)f(y)\,dy, \quad f(x) \in L^2(0, \omega),
\]

where

\[
E_0(x) = \int_0^\infty \frac{1}{\Gamma(s)}e^{-Cs^{-1}x^{s-1}}\,ds.
\]

It follows from (3.12) that

\[
x E_0(x) = \frac{1}{|\ln x|}(1 + o(1)), \quad x \to 0.
\]

Let us formulate the analogue of the Theorem 3.4 ([9, Ch. 3, Section 4]).

**Theorem 3.5** Let \( R_\alpha = 0, \alpha \neq 0 \). Then the operator

\[
B_\alpha = J_\alpha^{-1}QJ_\alpha
\]

has the form

\[
B_\alpha f = xf(x) + \alpha \int_0^x f(y)\,dy, \quad f(x) \in L^2(0, \omega).
\]

We note, that the operators \( A_\alpha \) and \( B_\alpha \) are partial cases of the Friedrichs model [2].

### 4 Generalized wave operators

1. Let us introduce the notion of the generalized wave operators ([7]).

**Definition 4.1** Let the operators \( A \) and \( A_0 \) act in the Hilbert space \( H \), where the operator \( A_0 \) is self-adjoint with absolutely continuous spectrum. We assume that there exists a unitary operator function \( W_0(t) \) satisfying the following conditions:
1. The limits in the sense of strong convergence

\[ W_{\pm}(A, A_0) = \lim_{t \to \pm \infty} \left( e^{iAt} e^{-iA_0 t} W_0(t) \right) \]  

exist.

2. 

\[ \lim_{t \to \pm \infty} W_0^{-1}(t + \tau) W_0(t) = I. \] \hspace{1cm} (4.2)

3. The commutations relations hold for arbitrary values \( t \) and \( \tau \):

\[ W_0(t) A_0 = A_0 W_0(t), \quad W_0(t) W_0(t + \tau) = W_0(t + \tau) W_0(t). \] \hspace{1cm} (4.3)

The operators \( W_{\pm}(A, A_0) \) are named generalized wave operators. If \( W_0(t) = I \) then the operators \( W_{\pm}(A, A_0) \) are usual wave operators.

The formulated notions of wave operators and generalized wave operators are correct and useful not only for self-adjoint operators \( A \), but for non-self-adjoint operators \( A \) too.

**Definition 4.2** The spectrum of non-self-adjoint operator \( A \) is absolutely continuous if the operator \( A \) can be represented in the form

\[ A = V^{-1} A_0 V; \] \hspace{1cm} (4.4)

where the operators \( V \) and \( V^{-1} \) are bounded and the operator \( A_0 \) is self-adjoint with absolutely continuous spectrum.

**Proposition 4.3** Let conditions (4.1)–(4.3) be fulfilled. Then

\[ W_{\pm}(A, A_0) e^{iA_0 t} = e^{iA t} W_{\pm}(A, A_0) \] \hspace{1cm} (4.5)

**Proof.** As in the of self-adjoint case we use the relation

\[ W_{\pm}(A, A_0) = \lim_{t \to \pm \infty} \left( e^{iA(t+s)} e^{-iA_0 (t+s)} W_0(t + s) \right). \] \hspace{1cm} (4.6)

The second relation of (4.3) and (4.6) imply

\[ W_{\pm}(A, A_0) = e^{iA s} W_{\pm}(A, A_0) e^{-iA_0 s}. \] \hspace{1cm} (4.7)

This proves the proposition. \( \Box \)
Example 4.4 Let us consider the operator $A_\alpha$, where $\Re \alpha = 0$, $\alpha \neq 0$, and the operator $A_0 = Q$ (see Theorem 3.4).

According to equality (3.10) the operator $A_\alpha$ has absolutely continuous spectrum. The following statement is valid.

Proposition 4.5 We assume that

$$W_0(t) = \left( \ln^{-\alpha} |t| \right) I. \quad (4.8)$$

We have

$$W_\pm(A_\alpha, Q) = V_\alpha^{-1}. \quad (4.9)$$

Proof. Relations (4.2) and (4.3) are fulfilled. Now, we write the equality

$$e^{iA_\alpha t}e^{-iQt}W_0(t)e^{ix\tau} = V_\alpha^{-1}e^{iQt}V_\alpha e^{-iQt}W_0(t)e^{ix\tau}. \quad (4.10)$$

Using (2.6), (2.13), (4.2) we obtain (4.9). The proposition is proved. □

Example 4.6 Let us consider the operator $B_\alpha$, where $\Re \alpha = 0$, $\alpha \neq 0$, and the operator $A_0 = Q$ (see Theorem 3.4).

According to equality (3.14) the operator $B_\alpha$ has absolutely continuous spectrum. The following statement is valid.

Proposition 4.7 Assume that

$$W_0(t) = \left( |t|^{-\alpha} \right) I. \quad (4.11)$$

Then we have

$$W_\pm(B_\alpha, Q) = e^{\pm(i\alpha\pi/2)}J^{-\alpha}. \quad (4.12)$$

Proof. Relations (4.2) and (4.3) are fulfilled. For the operator $J^\alpha$ ($\Re \alpha = 0$, $\alpha \neq 0$) the following relation

$$-i\lambda \tilde{s}_1(\lambda) = |\lambda|^{-\alpha}(1 + o(1))e^{\pm(i\alpha\pi/2)}, \quad \lambda \to \pm \infty \quad (4.13)$$

holds (see [8] formula (22))). Now, we write the equality

$$e^{iB_\alpha t}e^{-iQt}W_0(t)e^{ix\tau} = J^{-\alpha}e^{iQt}J^\alpha e^{-iQt}W_0(t)e^{ix\tau}. \quad (4.14)$$

Hence, using (2.13), (4.13) and (4.2) we obtain (4.12). The proposition is proved. □
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