Remarks on time-energy uncertainty relations

Romeo Brunetti and Klaus Fredenhagen
II Inst. f. Theoretische Physik, Universität Hamburg,
149 Luruper Chaussee,
D-22761 Hamburg, Germany.

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Dedicated to Huzihiro Araki on the occasion of his seventieth birthday

Abstract

Using a recent construction of observables characterizing the time of occurrence of an effect in quantum theory, we present a rigorous derivation of the standard time-energy uncertainty relation. In addition, we prove an uncertainty relation for time measurements alone.

1 Introduction

Time-energy uncertainty relations played an important rôle in the early discussions on the physical interpretation of quantum theory [5]. But contrary to the position-momentum uncertainty relation, their derivation and even precise formulation suffer from the difficulty of assigning an observable (in the sense of selfadjoint operators) of quantum theory with the measurement of time [10][8]. Meanwhile, as advocated long ago by Ludwig [7], it is widely accepted that the concept of observables should be generalized, by allowing not only selfadjoint operators (corresponding to projection valued measures) but also positive operator valued measures [9], and it was quickly realized that one can find such measures which transform covariantly under time
translations and fulfill therefore all formal requirements for a time observable \[\mathcal{B} \mathcal{B} \mathcal{B} \mathcal{B} \mathcal{B}\]. In a recent paper \[\mathcal{B}\] we gave the first (to our knowledge) general construction of such measures starting from an arbitrary positive operator which may be interpreted as the effect whose occurrence time is described by the measure; this notion of time is closely related to the concept of time of arrival but in contrast to this our construction always leads to positive operator valued measures (cfr. the book in \[\mathcal{B}\]). All that is done in a strict quantum language, no classical ideas or generalizations of quantum mechanics are involved (we stress that the construction is valid also in quantum field theory, a subject which we plan to deal with in the future). In the present paper we show that for these time observables the usual time-energy uncertainty relation holds, and that in addition, provided the Hamiltonian is positive, one finds an uncertainty relation for time measurements alone which takes the form

\[
\Delta T \geq \frac{\text{const}}{\langle H \rangle}
\]

with a universal constant. Arguments for these relations have been given by several authors \[\mathcal{B} \mathcal{B} \mathcal{B} \mathcal{B} \mathcal{B} \mathcal{B}\], mainly in special situations and often only on a heuristic level. It is the aim of the present note to show that these relations can be rigorously derived for an arbitrary time translation covariant positive operator valued measure.

2 Covariant Naimark-Stinespring Dilation

Let \(\mathcal{H}\) be a Hilbert space and \((U(t))\) a strongly continuous unitary group on \(\mathcal{H}\) describing the time evolution. The occurrence time of an effect is, following \[\mathcal{B}\], described by a covariant positive operator valued measure \(F\), i.e. for any Borel subset \(B\) of the real line \(\mathbb{R}\) we have a positive contraction \(F(B)\) on \(\mathcal{H}\), and the operators \(F(B)\) satisfy the conditions

\[
F(\bigcup_n B_n) = \sum_n F(B_n)
\]

if the sets \(B_n\) are pairwise disjoint and where the sum on the r.h.s. converges strongly,

\[
F(\mathbb{R}) = 1 , \quad F(\emptyset) = 0 ,
\]

\[
U(t)F(B)U(-t) = F(B + t) .
\]
We want to construct from these data an extended Hilbert space $\mathcal{K}$ with a projection $P$ onto the subspace $\mathcal{H}$, a unitary group $(V(t))$ on $\mathcal{K}$ which reduces on $\mathcal{H}$ to the original time translation $(U(t))$ and a covariant projection valued measure $E$ on $\mathcal{K}$ such that

$$PE(B)P = F(B)P .$$

Let $\mathcal{K}_0$ be the space of bounded piecewise continuous functions on $\mathbb{R}$ with values in $\mathcal{H}$. On this space we introduce the positive semidefinite scalar product

$$\langle \Phi, \Psi \rangle = \int (\Phi(t), F(dt)\Psi(t)) .$$

$\mathcal{H}$ can be isometrically embedded into $\mathcal{K}_0$ by identifying the elements of $\mathcal{H}$ with constant functions. The enlarged Hilbert space $\mathcal{K}$ is then defined as the completion of the quotient of $\mathcal{K}_0$ by the null space of the scalar product.

On $\mathcal{K}_0$ we define

$$(E(B)\Phi)(t) = \begin{cases} \Phi(t) & \text{if } t \in B \\ 0 & \text{if } t \not\in B \end{cases}$$

$$(V(t)\Phi)(s) = U(t)\Phi(s - t) ,$$

$$P\Phi = \int F(dt)\Phi(t) .$$

$E(B)$ and $V(t)$ map the null space into itself, $P$ annihilates it. Therefore they are well defined operators on $\mathcal{K}$ and form the desired covariant dilation. In particular, $(E, V)$ is a system of imprimitivity over $\mathbb{R}$ [14] and therefore unitarily equivalent to a multiple of the Schrödinger representation.

### 3 The general Time-Energy Uncertainty Relation

Using the covariant dilation described in the previous section we can use the standard uncertainty relation in the Schrödinger representation on $L^2(\mathbb{R})$,

$$\Delta_\phi(x)\Delta_\phi(\frac{1}{i} \frac{d}{dx}) \geq \frac{1}{2}$$
which holds for all wave functions $\Phi$ in the intersection of the domains of $x$ and $\frac{1}{i} \frac{d}{dx}$. This follows from the validity of the canonical commutation relations in the sense of quadratic forms,

$$(x\Phi, \frac{1}{i} \frac{d}{dx} \Psi) - (\frac{1}{i} \frac{d}{dx} \Phi, x\Psi) = i(\Phi, \Psi)$$

which may be derived from the fact that $\frac{1}{i} \frac{d}{dx}$ is the generator of translations $U(a)$, and that, by Stone’s Theorem [12], $a \mapsto U(a)\Phi$ is strongly differentiable for $\Phi$ in the domain of $\frac{1}{i} \frac{d}{dx}$. Namely, we have

$$(x\Phi, \frac{1}{i} \frac{d}{dx} \Psi) - (\frac{1}{i} \frac{d}{dx} \Phi, x\Psi)$$

$$= \frac{1}{i} \frac{d}{da} \bigg|_{a=0} ((x\Phi, U(a)\Psi) - (U(-a)\Phi, x\Psi))$$

$$= \frac{1}{i} \frac{d}{da} \bigg|_{a=0} ((x\Phi, U(a)\Psi) - (\Phi, (x + a)U(a)\Psi))$$

$$= \frac{1}{i} \frac{d}{da} \bigg|_{a=0} (-a)(\Phi, U(a)\Psi) = i(\Phi, \Psi).$$

We can now state the general time-energy uncertainty relation:

Let $F$ be a time translation covariant positive operator valued measure, and let $H$ denote the Hamiltonian. Let $\Phi$ be a unit vector in the domain of the Hamiltonian for which the second moment of the probability measure $d\mu(t) = (\Phi, F(dt)\Phi)$ is finite. Then we have the uncertainty relation

$$\Delta_\Phi(T_F)\Delta_\Phi(H) \geq \frac{1}{2}$$

where $\Delta_\Phi(T_F)$ is the square root of the variance of $\mu$ and $\Delta_\Phi(H) = (||H\Phi||^2 - \langle \Phi, H\Phi \rangle^2)^{\frac{1}{2}}$ is the usual energy uncertainty.

Proof: We use the covariant dilation described in Section 2. For $\Phi \in \mathcal{H}$ we have

$$\langle \Phi, E(B)\Phi \rangle = (\Phi, F(B)\Phi),$$

hence $\Phi$ is in the domain of definition of the selfadjoint operator $T_E$ defined by the projection valued measure $E$. Moreover, since the dilated time translations $V(t)$ restrict on $\mathcal{H}$ to the original time translations, $\Phi$ is also in the
domain of the generator $K$ of $V$. But $K$ and $T_E$ satisfy the canonical commutation relation in the sense of quadratic forms, thus $\Phi$ fulfills the uncertainty relations with respect to $T_E$ and $K$. The desired time-energy uncertainty relation now simply follows from the equalities

$$
\Delta_\Phi(T_E) = \Delta_\Phi(T_F) , \Delta_\Phi(K) = \Delta_\Phi(H) .
$$

It is clear that the tricky point was to find a useful representation of the Hilbert space $K$ with which we could reduce the computation to the standard position-momentum uncertainty relation. However, we stress that this representation only plays an auxiliary rôle, no physical interpretation has to be associated with it. (An essentially equivalent derivation may already be found in [15].)

### 4 Uncertainty of Time

The replacement of projections by positive operators in the description of time observables leads to an intrinsic uncertainty. We will assume in this section that the Hamiltonian is positive. Under this condition, we will show that the minimal time uncertainty is inversely proportional to the expectation value of the energy.

Let $\Phi$ be a unit vector for which the time uncertainty $\Delta_\Phi(T_F)$ and the expectation value of the Hamiltonian are finite. We use the same covariant dilation as before. Because of the positivity of $H$, $\mathcal{H}$ must be contained in the spectral subspace of $K$ corresponding to the positive real axis. We may realize $\mathcal{K}$ as the space of square integrable functions $L^2(\mathbb{R}, \mathcal{L})$ where $\mathcal{L}$ is the Hilbert space which describes the multiplicity of the Schrödinger representation. $K$ acts by multiplication and $T_E$ as generator of translations. Since $\Phi$ is in the domain of $T_E$, it is absolutely continuous, and since $\mathcal{H} \subset \mathcal{K}_+ = L^2(\mathbb{R}_+, \mathcal{L})$, $\Phi$ has to vanish at $x = 0$. Hence $\Phi$ is in the quadratic form domain $q$ of the operator $-\frac{d^2}{dx^2}$ on $\mathcal{K}_+$ with Dirichlet boundary condition at $x = 0$ (symbolically $-\frac{d^2}{dx^2}|_D$).

Since the problem is invariant under time shifts we may assume that the expectation value of $T_F$ vanishes, and to determine the infimum (over $\Phi$) of the quantity

$$
(\Phi, -\frac{d^2}{dx^2}|_D \Phi)(\Phi, x\Phi)^2 ,
$$
it would be sufficient to take it over the set \( \mathcal{S} = \{ \Phi \in \mathcal{K}_+, ||\Phi|| = 1, \Phi \in q(-\frac{d^2}{dx^2}|_D) \cap q(x) \} \).

We use the following relation which is valid for \( a, b > 0 \),

\[
ab^2 = \inf_{\lambda > 0} \frac{4}{27\lambda^2} (a + \lambda b)^3.
\]

The relation may be verified by noting that the argument of the infimum assumes the value of the left side for \( \lambda = \frac{2a}{b} \), hence it suffices to check the inequality

\[
ab^2 \leq \frac{4}{27\lambda^2} (a + \lambda b)^3, \quad a, b, \lambda > 0.
\]

Setting \( c = \lambda b \), we obtain the equivalent inequality

\[
a\left(\frac{c}{2}\right)^2 \leq \left(\frac{a + c}{3}\right)^3.
\]

Taking now the logarithm on both sides we find again an equivalent inequality which is a direct consequence of the concavity of the logarithm.

We therefore obtain the following relation

\[
\inf_{\Phi \in \mathcal{S}} (\Phi, -\frac{d^2}{dx^2}|_D \Phi)(\Phi, x\Phi)^2
\]

\[
= \inf_{\Phi \in \mathcal{S}} \inf_{\lambda > 0} \frac{4}{27\lambda^2} (\Phi, (-\frac{d^2}{dx^2}|_D + \lambda x)\Phi)^3.
\]

We may perform on the right hand side first the infimum over \( \Phi \). We then can exploit the behaviour of the operator \(-\frac{d^2}{dx^2}|_D + x\) under scale transformations. Namely, let

\[
(D(\mu)\Phi)(x) = \mu^{\frac{1}{3}}\Phi(\mu x)
\]

be the unitary scale transformations on \( \mathcal{K}_+ \). Then we have

\[
D(\lambda^{\frac{1}{3}})^{-1}(-\frac{d^2}{dx^2}|_D + \lambda x)D(\lambda^{\frac{1}{3}}) = \lambda^{\frac{2}{3}}(-\frac{d^2}{dx^2}|_D + x).
\]

Since the set \( \mathcal{S} \) is scale invariant, the infimum over \( \Phi \) is independent of \( \lambda \). We thus obtain

\[
\inf_{\Phi \in \mathcal{S}} (\Phi, -\frac{d^2}{dx^2}|_D \Phi)(\Phi, x\Phi)^2 = \frac{4}{27}c^3,
\]

6
where $c$ is the infimum of the spectrum of $-\frac{d^2}{dx^2} + x$.

The spectrum of this operator is a pure point spectrum \cite{12 4}. Its eigenfunctions are

$$\Phi_n(x) = \text{Ai}(x - \lambda_n),$$

with eigenvalues $\lambda_n$ where $\text{Ai}$ is the Airy function and $-\lambda_n$ are its zeros. The smallest eigenvalue is $\lambda_1 = 2.338$. So we finally arrive at the uncertainty relation

$$\Delta_\phi(T_F) \geq \frac{d}{\langle H \rangle_{\Phi}}$$

with $d = 1.376$.

Some comments are in order now:

1. The new relation gives a rather large bound if compared to the original time-energy uncertainty, indeed we have

$$\Delta_\phi(T_F)^2\langle H^2 \rangle_{\Phi} = \Delta_\phi(T_F)^2 \left(\langle H \rangle_{\Phi}^2 + \Delta_\phi(H)^2\right) \geq d^2 + \frac{1}{4},$$

the exact largest lower bound of the left hand side being $9/4$. Let us also notice that the bound $d$ is universal, i.e., does not depend on the details of the Hamiltonian $H$.

2. The stated relation is covariant, i.e., energy shifts do not change it. In case the infimum of the Hamiltonian is not zero we may change $H$ with $H - \inf(\sigma(H)) \cdot 1$, where $\sigma(A)$ is the spectrum of the operator $A$ and $1$ is the unit operator on the Hilbert space.

3. We have an explicit formula for the state with minimum uncertainty, namely the state $\Phi_1(x) = \text{Ai}(x - \lambda_1)$. Its shape shows how the energy spectrum has to be distributed in order to have minimal dispersion in time. (Recall that the variable $x$ labels the energy of the system.)

4. In the light of the last remark one wonders whether it would be possible to prepare such a kind of state in a laboratory and check the relation explicitly.

5. The same relation holds for the radial momentum of the system in place of $T_F$ and by replacing the Hamiltonian by the radius.
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