Potential Maximal Cliques Parameterized by Edge Clique Cover

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Abstract
We show that the number of minimal separators on graphs with edge clique cover of size $cc$ is $O(2^{cc})$ and the number of potential maximal cliques is $O(3^{cc})$. Furthermore, potential maximal cliques can be listed in $O^*(3^{cc'})$ time when an edge clique cover of size $cc'$ is given. This yields $O^*(3^{cc'})$ time algorithms for all problems in the framework of potential maximal cliques, for example Treewidth, Minimum Fill-In and Maximum Independent Set. The parameter $cc$ is motivated by practice: our results imply that Treewidth can be solved in $O^*(3^m)$ time for primal graphs of CSP instances with $m$ constraints. Furthermore, Generalized Hypertree Width and Fractional Hypertree Width can be solved in $O^*(4^m)$ and $O^*(3^m)$ time. There are many real CSP applications where the number of constraints is much smaller than the number of variables. Other applications of the parameterization include Perfect Phylogeny. The bounds on the number of minimal separators and the number of potential maximal cliques are tight, and $cc$ is orthogonal to the existing parameterizations for potential maximal cliques.

1 Introduction

Minimal separators and potential maximal cliques are key objects in computing the treewidth and other structural parameters of graphs. Algorithms for enumerating potential maximal cliques are important because Treewidth and multiple other problems formulated with minimal triangulations can be solved in time linear in the number of potential maximal cliques and polynomial in the size of the graph when the set of potential maximal cliques is given in input. We give FPT algorithms for enumerating potential maximal cliques parameterized by the size of edge clique cover. An edge clique cover of a graph is a set of cliques of the graph that covers all of its edges. In multiple applications of potential maximal cliques the underlying graph is constructed in a way that guarantees bounded edge clique cover. For example, in computation of width parameters of constraint satisfaction problem instances, the graph admits an edge clique cover of size $m$, where $m$ is the number of constraints.

Background. Fomin et al. showed that Treewidth and Minimum Fill-In can be solved in $O(|\Pi(G)|n^2)$ time, where $\Pi(G)$ is the set of potential maximal cliques (PMCs) and is given in the input [14]. Subsequently, the result was generalized to yield algorithms with linear time complexity in the number of PMCs for multiple other problems formulated as finding an optimal minimal triangulation [2, 16, 17, 23, 32, 33]. In the context of these results, designing PMC
enumerate algorithms is an important task. Best currently known algorithms have time complexities $O^*(1.7347^n)$ [10], $O^*(1.4142^n)$ in AT-free graphs [14], XP in H-graphs [13], FPT by the vertex cover and modular width [15] and polynomial time in multiple special graph classes [16].

Minimal separators are closely related to potential maximal cliques. Potential maximal cliques can be enumerated in $O(|\Delta(G)|^2 n^2 m)$ time, where $\Delta(G)$ is the set of minimal separators, even when $\Delta(G)$ is not given in input [14, 13]. Parameterized by the number of vertices the best known upper bound for $|\Delta(G)|$ is $O(1.6181^n)$ [18].

**Motivation.** In many applications of potential maximal cliques the underlying graph $G$ for which $\Pi(G)$ is enumerated is constructed as a set of partially overlapping cliques. A graph that is constructed as an union of $k$ cliques has an edge clique cover of size $k$.

In constraint satisfaction, the treewidth of the primal graph is a central measure of the hardness of the problem. CSPs can be solved in polynomial time for constant treewidth, and for constraints with constant number of variables it is also a necessary condition for polynomial time algorithms [20, 22]. Additionally, the treewidth of primal graph has been extended with the notions of generalized hypertree width and fractional hypertree width to yield more tractable fragments of CSPs [19, 21]. There are algorithms for all three of these width notions that use PMCs of the primal graph [13]. The primal graph is constructed by creating a clique for each constraint, so the constraints form an edge clique cover of size $m$. Even though in principle $m$ can be much larger than $n$, there are practical cases where $m$ is significantly smaller than $n$. For example, in the instances of the standard HyperBench library [11], 708 out of 3072 have $m < n/2$.

In addition to width measures, problems in computational phylogenetics also benefit from parameterization by edge clique cover. A central problem in phylogenetics is to construct an evolutionary tree of a set of $n$ taxa (species) based on data about them [37]. Deciding if the taxa admits a perfect phylogeny is NP-complete, but can be solved via potential maximal cliques of the partition intersection graph [23]. The partition intersection graph is constructed by creating a clique for each taxon, so the taxa form an edge clique cover of size $n$. For example, an instance describing the Indo-European languages has 24 taxa and its partition intersection graph has 864 vertices [35].

**Main Results.** Our main result is that a graph with edge clique cover of size $cc$ has at most $2^{cc}$ minimal separators and at most $3^{cc}$ potential maximal cliques. We give $O(3^{cc} nm)$ and $O(4^{cc} n^2 m)$ time algorithms for enumerating potential maximal cliques, where $cc'$ is the size of an edge clique cover given in input and $cc$ is the size of a minimum edge clique cover. Our results are based on characterizations of minimal separators and potential maximal cliques by partitions of an edge clique cover: minimal separators correspond to partitions into two parts and potential maximal cliques correspond to partitions into three parts.

Our results imply $O^*(3^{cc'})$ and $O^*(4^{cc})$ time algorithms for all problems that are solvable in $O^*(|\Pi(G)|)$ when $\Pi(G)$ is given in input. These problems include (Weighted) Treewidth, (Weighted) Minimum Fill-In, Chordal Sandwich and Treelength [2, 14, 17, 30]. Furthermore, our results imply $O(3^{cc} n^{t+1})$ and $O(4^{cc} n^{t+1})$ time algorithms in the metatheoretic framework of MAX INDUCED SUBGRAPH OF treewidth $\leq t$ satisfying $\varphi$, for any fixed integer $t$ and fixed counting monadic second order formula $\varphi$ [15]. For example, MAXIMUM INDEPENDENT SET and LONGEST INDUCED PATH can be expressed in this framework with $t = 0$ and $t = 1$, respectively [16].

For computing width measures of constraint satisfaction problem instances, we give an $O(3^{nm} n^3)$ time algorithm for primal Treewidth, an $O(4^{nm} (nm + n^3))$ time algorithm for Generalized Hypertree Width and an $O^*(3^{nm})$ time algorithm for Fractional Hypertree Width.
Width, where $n$ is the number of variables and $m$ is the number of constraints. To our knowledge, no prior FPT algorithms by $m$ exist for these problems. Algorithms parameterized by the number of vertices for Generalized Hypertree Width and Fractional Hypertree Width have time complexities $O^*(2^n)$ and $O^*(1.7347^n)$, respectively [32]. Deciding if generalized hypertree width is at most $k$ and deciding if fractional hypertree width is at most $k$ are NP-complete even for $k = 2$ [12]. Fischl et al. provided polynomial time algorithms for constant $k$ and constant maximum degree $d$ of the hypergraph [12]. The parameterization by $d$ is more general than by $m$, but their algorithms are not FPT even if $k$ or $d$ is constant.

For Perfect Phylogeny and two-state Maximum Compatibility we give algorithms with $O(3^n k^3 r^3)$ time complexity, where $n$ is the number of taxa, $k$ is the number of characters and $r$ is the maximum size of a domain of a character (two-state $\Leftrightarrow r = 2$). Our algorithms work also in the case that the instance has missing data. Our algorithms are extensions of the algorithms of Gysel [23]. The analysis of Gysel’s algorithms via previous PMC bounds gives an $O(1.7347^{k^3} k^3 r^3 + nk^2)$ time complexity for both Perfect Phylogeny and two-state Maximum Compatibility. We are not aware of previous algorithms for these problems where the time complexity is stated as $O^*(c^n)$ for some $c$. The algorithm of Kannan and Warnow [25] for Perfect Phylogeny works in time $O(4^r k^2 n)$, which is $O(4^nk^2n)$ because $r \leq n$. Their approach works with missing data only via a reduction that sets $r = \Omega(nf)$ for a fraction $f$ of missing data [37], so the case when $r$ is large is practical.

Properties of $cc$. To our knowledge, edge clique cover has not been used as a parameter in treewidth computation before. It might be an unappealing parameter because computing the minimum edge clique cover is hard. Assuming the exponential time hypothesis, there are no single exponential time algorithms parameterized by the solution size [8]. Furthermore, there are no approximation algorithms with subpolynomial approximation ratio [31]. The size of the minimum edge clique cover is bounded by $cc \leq m$ and $cc \leq n^2/4$ [33]. For every $cc \geq 1$, there exists a graph with minimum edge clique cover of size $cc$ that has $O(cc^4)$ vertices, $O(cc^3)$ edges, $\Theta(2^{cc})$ minimal separators and $\Theta(3^{cc})$ PMCs. This means that the bases of the exponents in our bounds cannot be improved even by adding polynomial factors.

Relations to Other Parameters. The parameterizations by vertex cover ($vc$) and modular width ($mw$) are the only prior known FPT results for potential maximal cliques: the number of potential maximal cliques is $O^*(4^{vc})$ and $O^*(1.7347^{mw})$, and they can be enumerated within same running times [15]. They are related to treewidth ($tw$) and clique width ($cw$) by inequalities $tw(G) \leq vc(G)$ and $cw(G) \leq mw(G) + 2$, but graphs with constant treewidth and clique width can have exponential number of potential maximal cliques [15]. The cliques have $cc = 1$, but

![Graph parameter hierarchy](image)

**Fig. 1:** Graph parameter hierarchy. An arrow from $a$ to $b$ indicates that $b(G) \leq c \cdot a(G)$ for some constant $c$. The horizontal line separates the parameters with FPT bounds for PMCs and the parameters that can be constant while the number of PMCs is exponential in the number of vertices.
unbounded vertex cover and treewidth. The star graphs have $vc = tw = 1$, but unbounded edge clique cover. Bounded edge clique cover implies exponentially bounded modular width and clique width: $mw(G) \leq 2^{cc(G)}$ and $cw(G) \leq 2^{cc(G)}$. For modular width there exists a graph $G$ with $mw(G) = 2^{cc(G)} - 2$ for each value of $cc(G) \geq 1$. We suspect that the same construction has exponential clique width, but we were unable to prove it. A known graph parameter that is bounded by $cc$ is the Boolean width [6]. Figure 1 states the linear inequalities between these parameters [10].

Another parameter that yields FPT algorithms for the problems in the PMC framework is the size of a modulator, a vertex set whose removal results in polynomial number of minimal separators [29]. The size of the smallest modulator is unbounded by $cc$. In general, even a modulator of size 1 does not guarantee polynomial number of minimal separators [29].

Minimal separators were parameterized in $H$-graphs in [13]. Their results apply to edge clique cover by setting $H = K_{cc}$, yielding the bound $O^*(n^{cc^2})$ for the number of minimal separators. The standard graph classes with polynomial number of minimal separators mentioned for example in [16] [29] are weakly chordal, polygon-circle, circular arc and $d$-trapezoid. Parameterized by $d$, the $d$-trapezoid graphs have $O(n^d)$ minimal separators [3].

Connections to Practice. There are multiple practical implementations of algorithms that make use of potential maximal cliques [26, 27, 28, 34, 38, 39]. In PACE 2017 algorithm implementation competition, top three implementations in the exact minimum fill-in track and an implementation that solved all instances in the exact treewidth track were based on potential maximal cliques [10].

![Graph showing the relationship between treewidth, min-fill, and bonus for PACE 2017 instances.](Image)

Fig. 2: Number of vertices ($n$) and minimum edge clique cover ($cc$) for PACE 2017 instances. Both values are truncated from above at 500 and the lines $n = cc$ and $n/2 = cc$ are shown.
We computed edge clique covers of 400 available instances from PACE 2017: all 200 treewidth instances, 100 public minimum fill-in instances and 100 later published challenging "bonus" treewidth instances [9]. All of the instances are based on real-world applications [10]. Our algorithm for minimum edge clique cover enumerated maximal cliques with the Bron–Kerbosch algorithm [5] and then solved the set cover instance induced by them using CPLEX [24]. We managed to compute the minimum edge clique cover of 394 instances. In 75 of those instances, \( cc < n/2 \), which is roughly the threshold when bounds by \( cc \) for potential maximal cliques and minimal separators are asymptotically lower than bounds by \( n \). Figure 2 shows relations between \( n \) and \( cc \) in the instances.

**Organization of the Paper.** In Section 2 we give necessary definitions. In Section 3 we prove the main theorems for characterizing minimal separators and potential maximal cliques based on edge clique cover. In Section 4 we give the algorithms for enumerating potential maximal cliques with Π\((G)\) and then solved the set cover instance induced by them using CPLEX [24]. We computed edge clique covers of 400 available instances from PACE 2017: all 200 treewidth instances, 100 public minimum fill-in instances and 100 later published challenging "bonus" treewidth instances [9]. All of the instances are based on real-world applications [10]. Our instances, 100 public minimum fill-in instances and 100 later published challenging "bonus" instances [9]. All of the instances are based on real-world applications [10]. Our organization of the paper.

## 2 Preliminaries

We consider graphs that are finite, simple and undirected. We also assume that graphs given in input are connected. Let the vertices and edges of graph \( G \) be \( V(G) \) and \( E(G) \). The set of edges of a complete graph with vertex set \( S \) is \( S^2 \). The subgraph induced by \( S \subset V(G) \) is \( G[S] = (S, E(G) \cap S^2) \). We denote \( G \setminus S = G[V(G) \setminus S] \). A vertex set that induces a complete subgraph is a clique. An edge clique cover of a graph is a set of cliques of the graph such that each edge is contained in at least one of the cliques, i.e. \( K_1, K_2, \ldots, K_k \) is an edge clique cover of \( G \) if \( E(G) = \bigcup_{i=1}^k K_i^2 \). The vertex sets of connected components of a graph \( G \) are \( \mathcal{C}(G) \). The neighbours of vertex \( v \) are \( N(v) \) and neighbourhood of a vertex set \( S \) is \( N(S) = \bigcup_{v \in S} N(v) \setminus S \).

Two vertices \( u, v \) are twin vertices and the edge \((u, v)\) is a twin edge if the neighbourhoods of \( u \) and \( v \) are identical, i.e. \( N(u) \cup \{u\} = N(v) \cup \{v\} \). Contracting an edge \((u, v)\) removes vertices \( u \) and \( v \) and adds a new vertex \( w \) for which \( N(w) = N(u) \cup N(v) \setminus \{u, v\} \). Contracting a twin edge \((u, v)\) in \( G \) is equivalent to \( G \setminus \{v\} \).

A chordal graph is a graph that has no induced subgraph that is a cycle with length ≥ 4. A triangulation of a graph \( G \) is a chordal graph \( H \) for which \( V(G) = V(H) \) and \( E(G) \subset E(H) \). The edges \( E(H) \setminus E(G) \) are fill-edges. A triangulation is minimal if no subset of its edges forms another triangulation. A vertex set \( S \subset V(G) \) is an \( u, v \)-separator if \( u, v \notin S \) and each path between \( u \) and \( v \) intersects \( S \). If \( S \) is a subset minimal \( u, v \)-separator for some \( u, v \in V(G) \), then \( S \) is a minimal separator of \( G \). Equivalently, \( S \subset V(G) \) is a minimal separator of \( G \) if and only if \( S \) is not empty and \( S \) has at least two components, i.e. components \( C \in \mathcal{C}(G \setminus S) \) with \( N(C) = S \). A vertex set \( \Omega \subset V(G) \) is a potential maximal clique of \( G \) if it is a maximal clique in some minimal triangulation of \( G \). The minimal separators of graph \( G \) are denoted with \( \Delta(G) \) and potential maximal cliques with \( \Pi(G) \).

A hypergraph \( \mathcal{G} \) has a set of vertices \( V(\mathcal{G}) \) and a set of edges \( E(\mathcal{G}) \) that are arbitrary subsets of vertices. The primal graph (or Gaifman graph) \( P(\mathcal{G}) \) has \( V(P(\mathcal{G})) = V(\mathcal{G}) \) and \( E(P(\mathcal{G})) = \{ (u, v) \mid \exists e \in E(\mathcal{G}) \{u, v\} \subset e \text{ and } u \neq v \} \).

The \( O^* \) notation suppresses polynomial factors.

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1 For PMCs the threshold is \( cc < n/1.994 \) and for minimal separators \( cc < n/1.440 \).
2.1 Complete Intersection Graphs

To analyze graphs with edge clique cover at most \(k\), we define complete intersection graphs. Let \(X_k\) be a set of size \(k\), \(X_k = \{x_1, \ldots, x_k\}\). The complete intersection graph \(P^k\) has vertex set that contains each non-empty subset of \(X_k\), i.e. \(V(P^k) = \mathcal{P}(X_k) \setminus \{\emptyset\}\), where \(\mathcal{P}(X_k)\) denotes the power set of \(X_k\). The edges of \(P^k\) correspond to non-empty intersections of the subsets: \(E(P^k) = \{(A, B) \in V(P^k)^2 \mid |A \cap B| \geq 1\}\). In proofs related to \(P^k\), we treat its vertices as sets that can be manipulated with standard notation, including the complement \(\overline{A} = X_k \setminus A\). Given a partition \(\pi\) of \(X_k\), the vertices of \(P^k\) that intersect the partition are \(\Gamma_k(\pi) = \{A \in V(P^k) \mid \forall \gamma \in \pi, A \not\subset \gamma\}\).

3 Combinatorial Results

First we show that the number of minimal separators and PMCs of a graph with edge clique cover at most \(k\) is at most the number of those objects in the complete intersection graph \(P^k\). Then we characterize minimal separators and PMCs of \(P^k\) by partitions of \(X_k\). Each minimal separator is defined by \(\Gamma_k(\{Y,Z\})\), where \(\{Y,Z\}\) is a partition of \(X_k\) into two non-empty sets. Each PMC is either a clique in the edge clique cover, or is defined by \(\Gamma_k(\{Y,Z,W\}\)) where \(\{Y,Z,W\}\) is a partition of \(X_k\) into three non-empty sets.

3.1 Reduction to \(P^k\)

We begin with lemmas about minimal separators and potential maximal cliques.

**Proposition 1** ([4]). A vertex set \(\Omega \subset V(G)\) is a potential maximal clique of \(G\) if and only if the following conditions hold.

1. For any pair of distinct vertices \(u, v \in \Omega\), either \((u,v) \in E(G)\) or there is \(C \in \mathcal{C}(G \setminus \Omega)\) such that \(\{u,v\} \subset N(C)\).
2. \(\Omega\) has no full components, i.e. components \(C \in \mathcal{C}(G \setminus \Omega)\) with \(N(C) = \Omega\).

Equivalent formulation of Condition 1 is that for each distinct pair \(u, v \in \Omega\), there is a path \(u, w_1, \ldots, w_p, v\) in \(G\) with \(w_i \notin \Omega\) for each \(i\).

**Lemma 1** ([4]). If \(S\) is a minimal separator of \(G \setminus \{v\}\) for vertex \(v \in V(G)\), then \(S\) or \(S \cup \{v\}\) is a minimal separator of \(G\).

**Lemma 2.** If \(\Omega\) is potential maximal clique of \(G \setminus \{v\}\) for vertex \(v \in V(G)\), then \(\Omega\) or \(\Omega \cup \{v\}\) is a potential maximal clique of \(G\).

**Proof.** Suppose that \(\Omega\) is a PMC of \(G \setminus \{v\}\) but not a PMC of \(G\). \(\Omega\) satisfies Condition 1 in \(G\) because adding \(v\) only increases the connectivity of \((G \setminus \{v\}) \setminus \Omega\). Therefore \(\Omega\) violates Condition 2 in \(G\), so there exists \(C' \in \mathcal{C}(G \setminus \Omega)\) such that \(v \in C'\) and \(N(C') = \Omega\). We prove that \(\Omega \cup \{v\}\) is a PMC of \(G\). As \(N(C') = \Omega\), there is a path from \(v\) to each \(u \in \Omega\) with no intermediate vertices in \(\Omega\). Therefore Condition 1 holds. Condition 2 holds because if \(C\) would be full component of \(\Omega \cup \{v\}\), then \(C\) would be a full component of \(\Omega\) in \(G \setminus \{v\}\).

Consequence of Lemmas 1 and 2 is that the numbers of minimal separators and potential maximal cliques in an induced subgraph are bounded by those numbers in the supergraph.

**Lemma 3.** If \((u,v)\) is a twin edge in \(G\) and \(S \in \Delta(G)\), then \(S \setminus \{v\} \in \Delta(G \setminus \{v\})\).
Proof. First, consider the case that \( v \in S \). For each path in \( G \), each occurrence of \( v \) can be replaced with \( u \). Therefore also \( u \in S \), so \( S \setminus \{ v \} \) is not empty. Removing \( v \) has no effect in the connectivity of \( G \setminus S \). Therefore since \( S \) has two full components in \( G \), also \( S \setminus \{ v \} \) has two full components in \( G \setminus \{ v \} \), so \( S \setminus \{ v \} \) is a minimal separator of \( G \setminus \{ v \} \).

Next, consider the case that \( v \notin S \). By symmetry, \( u \notin S \). There is a component \( C' \in \mathcal{C}(G \setminus S) \) for which \( u, v \in C' \). \( N(u) \setminus \{ v \} = N(v) \setminus \{ u \} \), so \( N(C' \setminus \{ v \}) = N(C') \), so removing \( v \) will not make \( C' \) non-full. Therefore since \( S \) has two full components in \( G \setminus S \), it also has two full components in \( (G \setminus \{ v \}) \setminus S \), so \( S \) is a minimal separator of \( G \setminus \{ v \} \).

Lemma 4. If \((u,v)\) is a twin edge in \( G \) and \( \Omega \in \Pi(G) \), then \( \Omega \setminus \{v\} \in \Pi(G \setminus \{v\}) \).

Proof. First, consider the case that \( v \in \Omega \). For each \( x \in \Omega \), there is a \( v \)-\( x \)-path with no intermediate vertices in \( \Omega \). Therefore, the same is true for \( u \)-\( x \)-paths, so if \( u \notin \Omega \) the component containing \( u \) would be full. Therefore \( u \in \Omega \). The set \( \Omega \setminus \{ v \} \) satisfies Condition 1 of PMC in \( G \setminus \{ v \} \), because removal of \( v \) does not affect the pairwise paths because they cannot contain \( v \) anyway. If there would be a full component \( C \in \mathcal{C}(G \setminus \{ v \}) \setminus (\Omega \setminus \{ v \}) \), then the same component would be full for \( \Omega \) in \( G \), because if \( u \in N(C) \) then \( v \in N(C) \). Therefore Condition 2 is also satisfied and \( \Omega \setminus \{ v \} \) is a PMC of \( G \setminus \{ v \} \).

Next, consider the case that \( v \notin \Omega \). By symmetry, \( u \notin \Omega \). For two vertices \( x, y \in \Omega \), if there is a \( x \)-\( y \)-path in \( G \) with no intermediate vertices in \( \Omega \), then there is also a \( x \)-\( y \)-path in \( G \setminus \{ v \} \) with no intermediate vertices in \( \Omega \) because occurrences of \( v \) can be replaced with \( u \). Therefore \( \Omega \) satisfies Condition 1 in \( G \setminus \{ v \} \). Removal of \( v \) only decreases connectivity of \( G \setminus \{ v \} \), so \( \Omega \) does not have full components in \( G \setminus \{ v \} \), so \( \Omega \) satisfies Condition 2 in \( G \setminus \{ v \} \).

Lemmas 3 and 4 with Lemmas 1 and 2 show that contracting twin edges does not change the numbers of minimal separators or potential maximal cliques.

Lemma 5. Let \( G \) be a graph with no twin edges, no isolated vertices and with an edge clique cover of size \( k \). \( G \) is isomorphic to an induced subgraph of \( P^k \).

Proof. Let \( K_1, K_2, \ldots, K_k \) be an edge clique cover of \( G \). By the definition, \((u,v) \in E(G)\) if and only if \( \{u,v\} \subset K_i \) for some \( i \), and \( u \neq v \). We map a vertex \( v \in V(G) \) to a vertex \( A \in V(P^k) \) with \( \phi(v) = \{x_i \mid v \in K_i\} \). By the definition of \( P^k \), \((A,B) \in E(P^k)\) if and only if \( x_i \in A \) and \( x_j \in B \) for some \( i \) and \( j \). If two distinct vertices \( A, B \in V(P^k) \), then the sets of vertices containing \( u \) and \( v \) would be the equal, and therefore \((u,v) \) would be a twin edge. Therefore \( G \) and \( P^k[\phi(V(G))] \) are isomorphic.

Proposition 2 (7). Contracting a twin edge does not change the size of the minimum edge clique cover.

Theorem 1. Let \( G \) be a graph with minimum edge clique cover of size \( k \) and with no isolated vertices. \( |\Delta(G)| \leq |\Delta(P^k)| \) and \( |\Pi(G)| \leq |\Pi(P^k)| \).

Proof. By Lemmas 3 and 5, \( |\Delta(G)| \leq |\Delta(G[T])| \) and \( |\Pi(G)| \leq |\Pi(G[T])| \), where \( G[T] \) is obtained by contracting twin edges in \( G \). By Lemma 5 and Proposition 2, \( G[T] \) is an induced subgraph of \( P^k \), and therefore by Lemmas 1 and 2, \( |\Delta(G[T])| \leq |\Delta(P^k)| \) and \( |\Pi(G[T])| \leq |\Pi(P^k)| \). Combining the results yields \( |\Delta(G)| \leq |\Delta(P^k)| \) and \( |\Pi(G)| \leq |\Pi(P^k)| \).

3.2 Minimal Separators

We prove that \( S \subset V(P^k) \) is a minimal separator of \( P^k \) if and only if \( S = \Gamma_k(\{Y,Z\}) \) for a partition of \( X_k \) into non-empty sets \( Y, Z \). The number of such partitions is \( 2^{k-1} - 1 \).
Lemma 6. If $S \in \Delta(P^k)$ and $A \in S$, then $B \in S$ for all $A \subset B$.

Proof. Suppose that $A \subset B$, $A \in S$ and $B \notin S$. If $C$ is a full component of $S$, then $B \in C$ because $N(A) \subset N(B) \cup \{B\}$. Therefore $S$ has at most one full component, which is a contradiction. \qed

Lemma 7. If $S \in \Delta(P^k)$, then $\{x_i\} \notin S$ for all $i$.

Proof. Suppose that $\{x_i\} \in S$. Lemma 8 shows that then $A \in S$ for all $A$ with $x_i \in A$. Therefore $N(\{x_i\}) \subset S$, so $S$ does not have a full component, which is a contradiction. \qed

Now all vertices of form $\{x_i\}$ are in the components of $P^k \setminus S$. For a component $C \in \mathcal{C}(P^k \setminus S)$, let $X_k(C) = \bigcup_{x_i \in C} x_i$.

Lemma 8. If $S \in \Delta(P^k)$ and $C \in \mathcal{C}(P^k \setminus S)$, then $X_k(C) \neq \emptyset$.

Proof. $C$ is non-empty so let $A \in C$. Now there exists $x_i \in A$, and $\{x_i\} \notin S$ by Lemma 7. It holds that $\{x_i\} \in C$, because either $(\{x_i\}, A) \in E(P^k)$ or $\{x_i\} = A$. \qed

It follows from Lemmas 7 and 8 that the components of $P^k \setminus S$ define a partition $\{X_k(C) \mid C \in \mathcal{C}(P^k \setminus S)\}$ on $X_k$.

Lemma 9. If $S \in \Delta(P^k)$, $C \in \mathcal{C}(P^k \setminus S)$ and $A \in C$, then $A \subset X_k(C)$.

Proof. Suppose that $A \setminus X_k(C)$ is non-empty. Let $x_i \in A \setminus X_k(C)$. Now $A$ would be also part of the component $C'$ that has $x_i \in X_k(C')$, which is a contradiction. \qed

Lemma 10. If $S \in \Delta(P^k)$, then $|\mathcal{C}(P^k \setminus S)| \geq 2$.

Proof. Minimal separators have 2 full components, so $|\mathcal{C}(P^k \setminus S)| \geq 2$. Let $C'$ be a full component of $S$ and $C_1$ and $C_2$ other distinct components of $P^k \setminus S$. Now $X_k(C') \notin N(C')$, because by Lemma 9 all vertices of $C'$ are subsets of $X_k(C')$. $X_k(C') \in S$, because it intersects $X_k(C_1)$ and $X_k(C_2)$. Now $C'$ is not a full component, which is a contradiction. \qed

Lemma 11. If $S \in \Delta(P^k)$, and $C \in \mathcal{C}(P^k \setminus S)$, then $X_k(C) \in C$.

Proof. Let $\{C_1, C_2\} = \mathcal{C}(P^k \setminus S)$. By Lemma 9, $X_k(C_1)$ can be in $C_1$ or $S$. If $X_k(C_1) \in S$, then $C_2$ would not be a full component. The only option left is $X_k(C_1) \in C_1$. \qed

Theorem 2. $S \in \Delta(P^k)$ if and only if $S = \Gamma_k(\{Y,Z\})$ for some partition $\{Y,Z\}$ of $X_k$, where $Y$ and $Z$ are non-empty.

Proof. Lemmas 7, 8 and 10 prove that any $S \in \Delta(P^k)$ defines a partition of $X_k$ into two non-empty parts, $\{X_k(C_1), X_k(C_2)\}$ for $\{C_1, C_2\} = \mathcal{C}(P^k \setminus S)$. Lemmas 9, 10 and 11 prove that such partition defines $S$ uniquely by $S = \Gamma_k(\{Y,Z\})$. Also $S = \Gamma_k(\{Y,Z\})$ is a minimal separator because $Y$ and $Z$ are in different components, $S \subset N(Y)$ and $S \subset N(Z)$. \qed

Corollary 1. Let $G$ be a graph with minimum edge clique cover of size $k$. The number of minimal separators of $G$ is at most $2^{k-1} - 1$.

Proof. The number of ways to partition $X_k$ into two non-empty parts is $2^{k-1} - 1$. Combining Theorems 1 and 2 yields $|\Delta(G)| \leq 2^{k-1} - 1$. \qed
3.3 Potential Maximal Cliques

We prove that $\Omega$ is a potential maximal clique of $P^k$ if and only if $\Omega = \{\{x_i\} \cup N(\{x_i\})\}$ for some $x_i \in X_1$ or $\Omega = \Gamma_k(\{Y, Z, W\})$ for a partition of $X_k$ into non-empty sets $Y, Z, W$. The number of such partitions is $\leq 3^k - k$.

**Lemma 12.** If $\Omega \in \Pi(P^k)$ and $A \in \Omega$, then $B \in \Omega$ for all $A \subset B$.

*Proof.* Suppose that $A \subset B$, $A \in \Omega$ and $B \notin \Omega$. There is a path from $A$ to every $v \in \Omega$ with no intermediate vertices in $\Omega$, so there is also a path from $B$ to $v \in \Omega$ because $N(A) \subset N(B) \cup \{B\}$. Therefore $\Omega$ violates Condition 2 of PMC, which is a contradiction. \hfill $\square$

**Lemma 13.** For each $i$, the only $\Omega \in \Pi(P^k)$ with $\{x_i\} \in \Omega$ is $\{\{x_i\}\} \cup N(\{x_i\})$.

*Proof.* Let $\Omega \in \Pi(P^k)$ and $\{x_i\} \in \Omega$. By Lemma 12, $A \in \Omega$ if $x_i \in A$, so $N(\{x_i\}) \subset \Omega$. We can verify that $\{\{x_i\}\} \cup N(\{x_i\}) = \{\{x_i\}\} \cup N(\{x_i\})$ is a PMC. Let $B \in \Omega$ with $B \notin \{\{x_i\}\} \cup N(\{x_i\})$. Then there would not be a path from $\{x_i\}$ to $B$ with no intermediate vertices in $\Omega$, because $N(\{x_i\}) \subset \Omega$. This violates Condition 2 of PMC, which is a contradiction. \hfill $\square$

Now we partition the PMCs of $P^k$ into two types. PMCs of type 1, $\Pi_1(P^k)$, are those that contain $\{\{x_i\}\}$ for some $i$ and PMCs of type 2 are the others, $\Pi_2(P^k) = \Pi(P^k) \setminus \Pi_1(P^k)$. From Lemma 13, we know that $|\Pi_1(P^k)| = k$. For PMCs of type 2, all vertices of form $\{x_i\}$ are in the components of $P^k \setminus \Omega$. For a component $C \in C(P^k \setminus \Omega)$, let $X_k(C) = \bigcup_{\{x_i\} \in C} X_k$. 

**Lemma 14.** If $\Omega \in \Pi_2(P^k)$ and $C \in C(P^k \setminus \Omega)$, then $X_k(C) \neq \emptyset$.

*Proof.* $C$ is non-empty so let $A \in C$. There exists $x_i \in A$, and $\{x_i\} \notin \Omega$ by definition of $\Pi_2(P^k)$, so it holds that $\{x_i\} \in C$, because either $\{x_i\}, A \in E(P^k)$ or $\{x_i\} = A$. \hfill $\square$

It follows from Lemma 14 that for PMCs of type 2 the components of $P^k \setminus \Omega$ define a partition $\{X_k(C) \mid C \in C(P^k \setminus \Omega)\}$ on $X_k$.

**Lemma 15.** If $\Omega \in \Pi_2(P^k)$, $C \in C(P^k \setminus \Omega)$ and $A \in C$, then $A \subset X_k(C)$.

*Proof.* Suppose that $A \setminus X_k(C)$ is non-empty. Let $x_i \in A \setminus X_k(C)$. Now $A$ would also be part of the component $C'$ that has $x_i \in X_k(C')$, which is a contradiction. \hfill $\square$

Lemma 15 implies that if vertex $A$ intersects both $X_k(C_1)$ and $X_k(C_2)$ for two distinct $C_1, C_2 \in C(P^k \setminus \Omega)$, then $A \in \Omega$.

**Lemma 16.** If $\Omega \in \Pi_2(P^k)$, then $|C(P^k \setminus \Omega)| = 3$.

*Proof.* First suppose that $C(P^k \setminus \Omega) = \{C_1\}$. $C_1$ would contain $\{x_i\}$ for all $i$. Therefore $N(C_1) = V(P^k) \setminus C_1$, so $C_1$ would be a full component of $\Omega$, which is a contradiction.

Next let $C(P^k \setminus \Omega) = \{C_1, C_2\}$. There is $A_1 \in \Omega$ with $A_1 \subset X_k(C_1)$, because otherwise $X_k(C_2)$ would be a full component. By symmetry, there also is $A_2 \in \Omega$ with $A_2 \subset X_k(C_2)$. By Lemma 15, vertices $B$ that intersect both $X_k(C_1)$ and $X_k(C_2)$ are in $\Omega$. This is contradiction, because such vertices $B$ separate $A_1$ from $A_2$, so Condition 1 of PMC is not satisfied.

Finally suppose that $|C(P^k \setminus \Omega)| \geq 4$. Let $C_1, C_2, C_3, C_4$ be four distinct components of $P^k \setminus \Omega$. Let $A_{12} = X_k(C_1) \cup X_k(C_2)$ and $A_{34} = X_k(C_3) \cup X_k(C_4)$. By Lemma 15, $\{A_{12}, A_{34}\} \subset \Omega$ and for each component $C \in C(P^k \setminus \Omega)$, $N(C_1) = N(X_k(C))$, so there is no component $C$ with $\{A_{12}, A_{34}\} \subset N(C)$. Furthermore, $\{A_{12}, A_{34}\} \notin E(P^k)$, so Condition 1 of PMC is not satisfied, which is a contradiction. \hfill $\square$

**Lemma 17.** If $\Omega \in \Pi_2(P^k)$ and $C \in C(P^k \setminus \Omega)$, then $X_k(C) \subset C$. 


Proof. Let \( \{C_1, C_2, C_3\} = \mathcal{C}(P^k \setminus \Omega) \). By Lemma 15, \( X_k(C_1) \) can be in \( C_1 \) or \( \Omega \). Let \( X_k(C_1) \in \Omega \). Also \( X_k(C_1) \in \Omega \) because it intersects \( X_k(C_2) \) and \( X_k(C_3) \). By Lemma 15, there is no component \( C \) with \( \{X_k(C_1), X_k(C_1)\} \subseteq N(C) \) and furthermore, \( (X_k(C_1), X_k(C_1)) \notin E(P^k) \), so Condition 1 of PMC is not satisfied, which is a contradiction. Therefore \( X_k(C_1) \in C_1 \).

Theorem 3. \( \Omega \in \Pi_2(P^k) \) if and only if \( \Omega = \Gamma_k(\{Y, Z, W\}) \) for some partition \( \{Y, Z, W\} \) of \( X_k \), where \( Y, Z \) and \( W \) are non-empty.

Proof. Lemmas 14 and 16 prove that any \( \Omega \in \Pi_2(P^k) \) defines a partition of \( X_k \) into three non-empty parts, \( \{X_k(C_1), X_k(C_2), X_k(C_3)\} \) for \( \{C_1, C_2, C_3\} = \mathcal{C}(P^k \setminus \Omega) \). Lemmas 12, 15 and 17 prove that such partition defines the vertices of \( \Omega \) uniquely by \( \Omega = \Gamma_{k}(\{Y, Z, W\}) \). Any such \( \Omega \) satisfies Condition 1 of PMC, because for any pair \( A, B \in \Omega \) there must be \( C_i \) such that \( |A \cap X_k(C_i)| \geq 1 \) and \( |B \cap X_k(C_i)| \geq 1 \) and thus \( \{A, B\} \in N(C_i) \). Condition 2 is satisfied, because for each \( C_i \), \( X_k(C_i) \in \Omega \) and \( X_k(C_i) \notin N(C_i) \), so \( C_i \) is not full.

Corollary 2. Let \( G \) be a graph with minimum edge clique cover of size \( k \). The number of potential maximal cliques of \( G \) is at most \( 3^k \).

Proof. The number of ways to partition \( X_k \) into three non-empty parts is \( S(k, 3) \leq 3^k - k \), where \( S \) denotes the Stirling numbers of the second kind. As \( |\Pi_2(P^k)| = k \), it follows from Theorem 3 that \( |\Pi(P^k)| \leq 3^k \). Combined with Theorem 2 this yields \( |\Pi(G)| \leq 3^k \).

4 Algorithmic Results

First we give FPT algorithms parameterized by edge clique cover for enumerating potential maximal cliques. Then we apply these algorithms to multiple problems, emphasizing the cases where this parameterization has special applications.

4.1 Potential Maximal Clique Enumeration

We give an \( O(3^{3c}nm) \) time algorithm for enumerating potential maximal cliques, where \( cc' \) is the size of an edge clique cover given in input. Additionally we prove that the original potential maximal clique enumeration algorithm of Bouchitite and Todinca [4] works in time \( O(4^{3c}n^2m) \), where \( cc \) is the size of the minimum edge clique cover.

Lemma 18. Given a graph \( G \), \( G[T] \) and \( \Pi(G[T]) \), where \( G[T] \) is obtained from \( G \) by contracting twin edges, \( \Pi(G) \) can be enumerated in \( O(|\Pi(G)|nm) \) time.

Proof. Twin vertices form equivalence classes, and each equivalence class of twin vertices of \( G \) has a representative vertex in \( G[T] \). For \( v \in V(G) \), let \( r(v) \in T \) be a representative of the twin equivalence class of \( v \) in \( G[T] \). By Lemmas 2 and 4 PMCs of \( G \) are obtained from PMCs of \( G[T] \) by inserting \( v \in V(G) \setminus T \) to each PMC \( \Omega \) that has \( r(v) \in \Omega \).

Theorem 4. Given a graph \( G \) and its edge clique cover of size \( cc' \), \( \Pi(G) \) can be enumerated in \( O(3^{3c'}nm) \) time.

Proof. First contract edges of \( G \) to obtain \( G[T] \). By Lemma 15, it is sufficient to enumerate \( \Pi(G[T]) \). Let \( K_1, \ldots, K_k \) be the edge clique cover of \( G \). Now \( K_i \cap T, \ldots, K_k \cap T \) is an edge clique cover of \( G[T] \). For a vertex \( v \in T \), let \( \phi(v) = \{x_i \mid v \in K_i\} \). Now by Lemma 5, \( P^k[\phi(T)] \simeq G[T] \), so the task is to enumerate \( \Pi(P^k[\phi(T)]) \).

By Lemma 2, each PMC \( \Omega \in \Pi(P^k[\phi(T)]) \) can be represented as \( \Omega = \Omega' \cap \phi(T) \), where \( \Omega' \in \Pi(P^k) \). Candidates for PMCs represented by type 1 PMCs of \( P^k \), \( \Pi(P^k) \), are the cliques
\( \phi(K_i \cap T) \) for each \( i \). Each of them can be checked in \( O(nm) \) time \([4]\). For type 2, recall the characterization of \( \Pi_2(P_k) \) in Theorem \([3]\). If \( \Omega \in \Pi_2(P_k[\phi(T)]) \), then \( \Omega = \{ A \in \Gamma_k(\{Y,Z,W\}) \mid A \subseteq \phi(T) \} \) for a partition \( Y,Z,W \) of \( X_k \). For each such partition we can in \( O(\ell m) \) time find the corresponding vertex set of \( P_k[\phi(T)] \), and then in \( O(nm) \) time check if the vertex set is a PMC \([4]\). This yields an \( O(3^{cc'} nm) \) time algorithm.

**Theorem 5.** Given a graph \( G \) that has minimum edge clique cover of size \( cc' \), \( \Pi(G) \) can be enumerated in \( O(4^{cc} n^2 m) \) time.

**Proof.** The algorithm of Bouchitté and Todinca for enumerating potential maximal cliques works in \( O(|\Delta(G)|^2 n^2 m) \) time \([4]\). Applying the bound of Corollary \([4]\) on minimal separators gives the time complexity \( O(4^{cc} n^2 m) \).

**4.2 Applications**

We state the implications of our potential maximal clique enumeration algorithms to problems in the potential maximal clique framework. Throughout this section \( cc' \) denotes the size of an edge clique cover given in input and \( cc \) denotes the size of the minimum edge clique cover.

**Treewidth and Minimum Fill-In**

Treewidth is to find a minimal triangulation of a graph whose largest clique is as small as possible. In Weighted Treewidth the cost is the sum of \( f(K) \) over maximal cliques \( K \) of the minimal triangulation, where \( f \) satisfies \( f(K) \geq 2 f(K \setminus \{v\}) \) for all \( v \in K \). Minimum Fill-In is to find a minimal triangulation with the minimum number of fill-edges. In Weighted Minimum Fill-In each fill-edge has a non-negative weight, and the sum of the weights should be minimized.

**Proposition 3** \([2, 14, 17]\). Treewidth \([14]\), Weighted Treewidth \([2]\), Minimum Fill-In \([14]\) and Weighted Minimum Fill-In \([17]\) can be solved in \( O(|\Pi(G)| n^3) \) time when \( \Pi(G) \) is given in input, assuming that the weight functions can be computed in linear time.

**Corollary 3.** Treewidth, Weighted Treewidth, Minimum Fill-In and Weighted Minimum Fill-In can be solved in \( O(3^{cc'} n^3) \) and \( O(4^{cc} n^2 m) \) time.

Note that Chordal Sandwich and Treelength are special cases of Weighted Minimum Fill-In \([17, 50]\), and Triangulating Bayesian Networks is a special case of Weighted Treewidth \([2, 41]\). Corollary \([3]\) also implies that the treewidth of a primal graph of a constraint satisfaction problem instance can be computed in \( O(3^{m^3} n^3) \) time, where \( n \) is the number of variables and \( m \) is the number of constraints.

**Maximum Induced Subgraph**

Fomin et al. introduced an algorithmic metatheorem that in the fashion of Courcelle’s theorem states that a large class of problems defined in counting monadic second order logic (CMSO) is solvable in linear time in the number of potential maximal cliques. The problem MAX INDUCED SUBGRAPH of \( \text{tw} \leq t \) satisfying \( \phi \) is to find subsets of vertices \( X \subset F \subset V(G) \) such that \( X \) is of maximum size, the treewidth of \( G[F] \) is at most \( t \) and \( (G[F], X) \models \phi \), where \( \phi \) is a CMSO formula.

Concrete instantiations of this metatheorem include for example Maximum Independent Set and Longest Induced Path. These instantiations have \( t = 0 \) and \( t = 1 \), respectively.
Proposition 4 ([32]). MAX INDUCED SUBGRAPH of \( tw \leq t \) satisfying \( \varphi \) can be solved in time \( O(||(G)|| n^{t+4} f(t, \varphi)) \) when \( \Pi(G) \) is given in input, where \( f \) depends only on \( t \) and \( \varphi \).

Corollary 4. MAX INDUCED SUBGRAPH of \( tw \leq t \) satisfying \( \varphi \) can be solved in time \( O(3^{2n} n^{t+4} f(t, \varphi)) \) and in time \( O(4^{n} n^{t+4} f(t, \varphi)) \), where \( f \) depends only on \( t \) and \( \varphi \).

Generalized and Fractional Hypertree Width

**Generalized Hypertree Width** is to find a minimal triangulation \( H \) of the primal graph \( P(G) \) of a hypergraph \( G \) that minimizes the maximum edge cover \( \text{COV}(K) \) over the maximal cliques \( K \) of \( H \). The edge cover \( \text{COV}(K) \) is the size \( |E_C| \) of the smallest set of hyperedges that covers \( K \), i.e. \( E_C \subseteq E(G) \) such that \( K \subseteq \bigcup_{e \in E_C} e \).

In **Fractional Hypertree Width** the definition of edge cover is relaxed in the linear programming sense. The fractional edge cover \( \text{FCOV}(G) \) is the smallest sum of weights \( \sum_{e \in E(G)} w(e) \) in an assignment \( w : E(G) \rightarrow \mathbb{R}_{\geq} 0 \) that satisfies for each \( v \in K \), \( \sum_{e \in e} w(e) \geq 1 \).

Proposition 5 ([32]). **Fractional Hypertree Width** can be solved in \( O^*(||P(G)||) \) time when \( \Pi(P(G)) \) is given in input.

Corollary 5. **Fractional Hypertree Width** can be solved in \( O^*(3^n) \) time, where \( n \) is the number of hyperedges.

For **Generalized Hypertree Width** we design an algorithm that uses the structure of \( \Pi(P(G)) \) to compute values of \( \text{COV}(G) \) in an efficient manner.

Proposition 6 ([32]). **Generalized Hypertree Width** can be solved in \( O(||(P(G)||) n^3 + mn^2) \) time when \( \Pi(P(G)) \) is given in input and \( \text{COV}(\Omega) \) for each \( \Omega \in \Pi(P(G)) \) is given in input, where \( m \) is the number of hyperedges.

Theorem 6. The potential maximal cliques \( \Pi(P(G)) \) and their edge covers \( \text{COV}(\Omega) \) for \( \Omega \in \Pi(P(G)) \) can be computed in \( O(4^n (mn + n^3)) \) time, where \( m \) is the number of hyperedges.

Proof. Recall that for graphs with edge clique cover of size \( k \) all PMCs are either cliques given in the edge clique cover, or defined as partitions of \( X_k \) by \( \Omega = \{ A \in \Gamma_k((\{Y, Z, W\}) | A \subset \phi(V(G))) \} \), where \( \phi \) is the isomorphism from \( G \) to an induced subgraph of \( P^k \). The PMCs that are cliques in the edge clique cover have \( \text{COV}(\Omega) = 1 \). For other PMCs, to compute partial solutions of \( \text{COV}(\Omega) \) via dynamic programming, we extend the definition of \( \Gamma_k \) to \( \Gamma_X \), where \( X \subseteq X_k \). For partition \( \pi \), let \( \Gamma_X(\pi) = \Gamma_k(\pi) \cap \mathcal{P}(X) \), where \( \mathcal{P} \) denotes the power set. Now \( \text{COV}(\Gamma_X(\{Y, Z, W\})) \) can be computed with recursion.

\[
\text{COV}(\Gamma_X(\{Y, Z, W\})) = \begin{cases} 
0 & \text{if } \Gamma_X(\{Y, Z, W\}) = \{} \\
\min_{x \in X} \text{COV}(\Gamma_{X\setminus\{x\}}(\{Y, Z, W\})) + 1 & \text{otherwise}
\end{cases}
\]

The vertex set \( \Gamma_X(\{Y, Z, W\}) \) is uniquely defined by the partition \( \{X, Y \setminus X, Z \setminus X, W \setminus X\} \) of \( X_k \), so this recursion has \( O(4^k) = O(4^m) \) states. The transitions can be computed in \( O(nm) \) time, so all values of \( \text{COV}(\Gamma_X(\{Y, Z, W\})) \) can be computed in \( O(4^m nm) \) time.

Our algorithm first computes all values of \( \text{COV}(\Gamma_X(\{Y, Z, W\})) \), then enumerates PMCs of \( P(G) \) using the algorithm of Theorem 4, outputting also the corresponding partitions \( \{Y, Z, W\} \), and then finds the values of \( \text{COV}(\Omega) \) using the fact that \( \Omega = \Gamma_X(\{Y, Z, W\}) \).

Corollary 6. **Generalized Hypertree Width** can be solved in \( O(4^m (nm + n^3)) \) time, where \( m \) is the number of hyperedges.
Perfect Phylogeny

In Perfect Phylogeny, the task is to find an evolutionary tree that explains the evolution of \( n \) taxa (species) in a way that is consistent with given \( k \) characters (attributes) on the taxa. An instance of Perfect Phylogeny is \( r \)-state if the domain of each character has size at most \( r \). An instance of Perfect Phylogeny has missing data if a value of some character is not defined for some taxon. In Maximum Compatibility, the task is to find a maximum size subset of the characters so that they admit a perfect phylogeny.

Proposition 7 ([23]). Perfect Phylogeny with missing data and two-state Maximum Compatibility with missing data can be reduced to Weighted Minimum Fill-In in a graph with \( kr \) vertices which is constructed as an union of \( n \) cliques.

Corollary 7. Perfect Phylogeny with missing data and two-state Maximum Compatibility with missing data can be solved in \( O(3^n k^3 r^3) \) time.

5 Edge Clique Cover as a Parameter

First we prove that our bounds on minimal separators and potential maximal cliques are tight. Then we explore the relations of \( \text{cc} \) and other graph parameters.

5.1 Tightness

The graph \( P^k \) explicitly shows that there exists graphs with \( \Theta(2^k) \) minimal separators and \( \Theta(3^k) \) potential maximal cliques, where \( k \) is the minimum edge clique cover of them. However, it has exponential number of vertices and edges in \( k \), so it does not close the question if the bases of the exponents in our bounds could be improved by introducing polynomial factors. Here we demonstrate a class of graphs that have minimum edge clique cover \( k \), \( O(k^2) \) vertices, \( O(k^3) \) edges, \( \Theta(2^k) \) minimal separators and \( \Theta(3^k) \) potential maximal cliques.

Let \( P^k_s \) be the induced subgraph of \( P^k \) that contains all vertices \( A \) for which \( |A| \leq s \).

Proposition 8. \( |\Delta(P^k_2)| = 2^k - 1 \) and \( |\Pi(P^k_2)| = S(k, 3) + k \).

Proof. The vertex sets defined by two- and three-partitions in Theorems 2 and 3 are still unique and satisfy the required properties. In particular, for each partition \( \{Y, Z, W\} \) of \( X_k \), the set of vertices \( \{A \in \Gamma_k(\{Y, Z, W\}) \mid A \subset V(P^k_2)\} \) is unique, and any subgraph of \( P^k_2 \) induced by \( \{A \in V(P^k_2) \mid A \subset Y\} \) for \( Y \subset X_k \) is connected.

5.2 Relations to Other Parameters

The prior FPT parameterizations for problems in the PMC framework are the size of a modulator [29], the vertex cover and the modular width [15]. The size of the smallest modulator is unbounded in \( \text{cc} \), because a modulator must contain all vertices of each twin vertex equivalence class it contains and the equivalence classes can be expanded unboundedly without affecting \( \text{cc} \).

Vertex cover and modular width are related to treewidth (\( tw \)) and clique width (\( cw \)) by \( tw \leq vc \) and \( cw \leq mw \). The complete bipartite graphs \( K_{c,n} \) for any constant \( c \) have constant vertex cover, treewidth, modular width and clique width, but unbounded edge clique cover. The complete graph \( K_n \) has \( tw(K_n) = vc(K_n) = n - 1 \), but \( cc(K_n) = 1 \). The modular width of a graph with edge clique cover \( cc \) is at most \( 2^{cc} \) because the twin vertex equivalence classes form modules. Next we show that this bound is tight.

Proposition 9. \( mw(P^k) \geq 2^k - 2 \).
Fig. 3: A watermelon graph consists of disjoint paths of length 3 and two vertices adjacent to the ends of the paths.

Proof. A module of $G$ is a set of vertices $M \subset V(G)$ such that for any two vertices $v, u \in M$, $N(v) \setminus M = N(u) \setminus M$. A prime graph is a graph $G'$ that has only modules of size 1 and $|V(G')|$. The modular width of a graph is at least the size of its largest induced subgraph that is prime [15]. We show that $P^k \setminus \{X_k\}$ is a prime.

Let $M$ be a module of $P^k \setminus \{X_k\}$ with size $|M| \geq 2$. Let $A$ and $B$ be two distinct vertices of $M$ with $|A| \leq |B|$. Now $\overline{A} \in M$, because $\overline{A} \in N(B)$, but $\overline{A} \notin N(A)$. Now that $M$ contains both $A$ and $\overline{A}$, it must contain all $\{x_i\}$ by a similar argument. Now that $M$ contains all $\{x_i\}$, it must contain all vertices by a similar argument. Therefore all modules of $P^k \setminus \{X_k\}$ that have size $|M| \geq 2$ contain all vertices, and therefore $P^k \setminus \{X_k\}$ is a prime.

We were not able to prove a lower bound for clique width. The most constrained parameter which we were able to polynomially bound on $cc$ is the Boolean width (boolw) [6].

Proposition 10. For any graph $G$, $\text{boolw}(G) \leq cc(G)$.

Proof. The distinct neighbourhoods across a cut $A \subset V(G)$ are $U(A) = \{N(S) \setminus A \mid S \subset A\}$. The Boolean width is defined via a tree that uses $\log_2(|U(A)|)$ as a cost function [6]. For any cut $A$, $|U(A)| \leq 2^{cc(G)}$, because $N(S) \setminus A$ is uniquely defined by the set of cliques in the edge clique cover that intersect $S$. Therefore the width of any Boolean decomposition of $G$ is at most $cc(G)$.

The watermelon graphs given in Figure 3 show that there is a graph class that has constant treewidth, constant clique width, constant Boolean width and constant size modulator, but exponential number of minimal separators and PMCs in the number of vertices.

6 Conclusion

We bounded the number of minimal separators and potential maximal cliques by the edge clique cover, obtaining new FPT algorithms to problems in the potential maximal clique framework. The results are motivated by real applications of potential maximal cliques. They provide theoretical corroboration on the observation that the potential maximal clique framework is efficient in practice. Prior to our work only the recent paper of Fomin et al. [15] considers FPT parameterizations for potential maximal clique enumeration. Our results answer to their proposal for finding further parameterizations for PMCs, but there are still many more natural parameters to be investigated in future work.

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