Mismatch Induced Type Transition of a Superconductor

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We use the Ginzburg-Landau theory near the transition temperature in order to examine the behavior of an inhomogeneous superconductor in the presence of a magnetic field. We find that a transition from type I to type II superconductivity occurs. Furthermore we calculate the critical value of the mismatch and estimate the contributions of the fluctuations of the condensate and the higher order terms in the Ginzburg-Landau framework.

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The formation of Cooper pairs with non-zero net momentum is expected to occur in fermionic systems due to the presence of attractive interactions that are responsible for the separation of the Fermi surfaces of the pairing electrons. The study of inhomogeneous superconductors has been the subject of intense theoretical and experimental research recently. In an electronic superconductor, the displacement of the Fermi momenta can be caused by either the application of a strong external magnetic field or the presence of ferromagnetic impurities. In cold fermionic atoms, the mismatch of the Fermi sea occurs because the effective masses or densities of the two species do not match. Finally as we squeeze matter creating conditions that exist in the interior of compact stars, differences in the chemical potentials of different quark flavors might lead to displacements of their Fermi momenta since this configuration is energetically favorable (charge neutrality).

The mismatch of the Fermi momenta is expected to reduce the available phase space for pairing. We shall denote the difference of the kinetic energy of the pairing fermions with $\delta$, and we shall characterize the strength of the pairing force in a superconductor with the magnitude of the energy gap $\Delta_0$, at zero temperature $T=0$ and zero mismatch. The long range order will be destroyed for sufficiently high values of the ratio $\delta/\Delta_0$. But prior to this a new type of long range order is expected to emerge for $\delta \sim \Delta_0$. The candidate pairing states that have been proposed in the literature include the long sorted LOFF pairing state $\tilde{\Pi}$, the breached pairing state for cold atoms $\tilde{\Pi}^c$, the Sarma state for a charge neutral quark matter $\tilde{\Pi}^q$, or a heterogeneous separation between BCS phase and the normal phase $\tilde{\Pi}^n$ for the quark matter. Figure 1 shows the possible phase diagram of an electronic superconductor with Fermi momentum mismatch. The LOFF phase is expected to emerge within a narrow window of the mismatch parameter beyond the line joining $\delta_{\text{LOFF}} \approx 0.608\Delta_0$ at the transition temperature and $\delta_{\text{LOFF}}' \approx \Delta_0/\sqrt{2}$ at $t=0$.

In this letter we shall demonstrate a dramatic change of the magnetic properties of a superconductor that occurs before the emergence of the exotic pairing states.

More specifically, a type I superconductor without mismatch can evolve to a type II one beyond a critical value of the ratio $\delta/\Delta_0$ when sufficiently close to the transition temperature. This possible crossover, sketched in the phase diagram Fig.1 by the dashed line, may have observable consequences. Our results seem to contradict the conventional belief that a type II superconductor is associated to a stronger pairing force.

[FIG. 1: The phase diagram on $T-\delta$ plane.]

In the remaining of the paper, we shall derive the Ginzburg-Landau free energy density with a mismatch and we shall calculate the Ginzburg-Landau parameter $\kappa$. Furthermore we shall demonstrate that the Ginzburg-Landau parameter is an increasing function of the mismatch parameter and that it diverges at the threshold of the LOFF state. Therefore somewhere in between, the GL parameter crosses the critical value $\kappa_c = 1/\sqrt{2}$ that distinguishes a type I ($\kappa < \kappa_c$) superconductor from a type II ($\kappa > \kappa_c$). We shall also access the robustness of this crossover within the Ginzburg-Landau framework by estimating the fluctuation effects and the higher order corrections. Finally, we shall discuss the feasibility of this proposal for realistic superconductors.
paper, we shall use the natural units with the Boltzmann constant set to one.

We shall focus on a nonrelativistic electronic system described by the Hamiltonian:

$$H = \sum_{\vec{p},s} \varepsilon_{\vec{p},s} a_{\vec{p},s}^\dagger a_{\vec{p},s} - \frac{1}{\Omega} \sum_{\vec{p},\vec{p}',\vec{q}}' a_{\vec{p},s}^\dagger \sigma_\lambda a_{\vec{p}',\lambda}^\dagger a_{\vec{p}'-\vec{q},\lambda} a_{\vec{p},s} \uparrow$$  \hspace{1cm} (1)$$

where \( s = (\uparrow, \downarrow) \), \( \varepsilon_{\vec{p},\uparrow} = \varepsilon_{\vec{p}} + \delta \), \( \varepsilon_{\vec{p},\downarrow} = \varepsilon_{\vec{p}} - \delta \), \( \vec{p}'_\pm = \vec{p}' \pm \frac{\vec{q}}{2} \), \( \lambda \) is the strength of the interaction that is responsible for the pairing and \( \Omega \) is the volume of the system. Near the Fermi surface, \( \varepsilon_{\vec{p}} = v_F (p - k_F) \) where \( k_F \) is the Fermi momentum and \( v_F \) the Fermi velocity. The sum \( \sum_{\vec{p},\vec{p}',\vec{q}}' \) of (1) extends to the pairing phase space specified by \( |\varepsilon_{\vec{p}}| < \omega_D \) and \( |\varepsilon_{\vec{p}'\uparrow}| < \omega_D \) where \( \omega_D \) is an intermediate energy scale between the superconducting energy gap and the Fermi energy (Debye frequency in the case of phonon mediated pairing). In terms of the NG representation

$$A_{\vec{p}} = \begin{pmatrix} a_{\vec{p},\uparrow} \\ a_{\vec{p},\downarrow}^\dagger \end{pmatrix}$$ \hspace{1cm} (2)$$

the Hamiltonian is rewritten as

$$H = \sum_{\vec{p}} \varepsilon_{\vec{p}_\uparrow} a_{\vec{p}_\uparrow}^\dagger a_{\vec{p}_\uparrow} + \sum_{\vec{p}} A_{\vec{p}}^\dagger (\xi_{\vec{p}} \sigma_3 + \delta) a_{\vec{p}}$$  
$$- \frac{1}{\Omega} \sum_{\vec{p},\vec{p}',\vec{q}}' A_{\vec{p}_\uparrow,\lambda}^\dagger a_{\vec{p}',\lambda}^\dagger a_{\vec{p}'-\vec{q},\lambda} a_{\vec{p}_\uparrow}^\dagger a_{\vec{p}_\downarrow} \uparrow$$  \hspace{1cm} (3)$$

where \( \sigma_{\pm} = \frac{1}{2} (\sigma_1 \pm i \sigma_2) \) are the Pauli matrices.

Let’s now introduce an inhomogeneous condensate,

$$< A_{\vec{p}_\uparrow,\lambda}^\dagger a_{\vec{p}_\downarrow} > = \frac{\pi^2 v_F}{\sqrt{\Omega} \lambda k_F^2} \phi_{\vec{q}},$$  \hspace{1cm} (4)$$and expand the Hamiltonian (3) to linear order of the difference \( A_{\vec{p}_\uparrow,\lambda}^\dagger a_{\vec{p}_\downarrow} - < A_{\vec{p}_\uparrow,\lambda}^\dagger a_{\vec{p}_\downarrow} > \). We recover the mean field Hamiltonian:

$$H_{MF} = \sum_{\vec{p}} \varepsilon_{\vec{p}_\uparrow} + \frac{1}{\lambda} \sum_{\vec{q}} \phi_{\vec{q}}^* \phi_{\vec{q}} + \frac{1}{\Omega} \sum_{\vec{p},\vec{p}',\vec{q}}' A_{\vec{p}_\uparrow,\lambda}^\dagger (\xi_{\vec{p}} \sigma_3 + \delta) a_{\vec{p}}$$  
$$- \frac{1}{\sqrt{\Omega}} \sum_{\vec{p},\vec{q}}' [\phi_{\vec{q}}^* A_{\vec{p}_\uparrow,\lambda}^\dagger a_{\vec{p}_\downarrow} + \phi_{\vec{q}} A_{\vec{p}_\downarrow,\lambda}^\dagger a_{\vec{p}_\uparrow}].$$  \hspace{1cm} (5)$$

A homogeneous condensate corresponds to \( \phi_{\vec{q}} = \sqrt{\Omega} \Delta_{\vec{q},0} \) with \( \Delta \) being the energy gap.

The thermodynamic potential corresponding to the Hamiltonian (5) at temperature \( T \) reads

$$\Gamma = \frac{1}{\lambda} \sum_{\vec{q}} \phi_{\vec{q}}^* \phi_{\vec{q}} - T \text{Tr} (\ln S^{-1} - \ln S^{-1}),$$  \hspace{1cm} (6)$$

where the inverse thermal fermion propagator in the presence of an inhomogeneous condensate, is given by

$$S^{-1}(\vec{p}\nu|\vec{p}'\nu') = S^{-1}(\vec{p}\nu|\vec{p}'\nu')$$  
$$- \frac{1}{\sqrt{\Omega}} [\phi_{\vec{p}}^* - \phi_{\vec{p}} - \phi_{\vec{p}'-\vec{q}} - \phi_{\vec{p}'-\vec{q}}^*] \delta_{\nu\nu'},$$  \hspace{1cm} (7)$$

where

$$S^{-1}(\vec{p}\nu|\vec{p}'\nu') = (-i \nu + \delta + \xi_{\vec{p}} \sigma_3) \delta_{\vec{p}\vec{p}'} \delta_{\nu\nu'}$$  \hspace{1cm} (8)$$

is the propagator in the normal phase and \( \nu \) the Matsubara energy. The trace of (5) is over the momentum, Matsubara energy and NG indices. The thermodynamic potential of the normal phase has been substracted in (5).

By expanding \( \Gamma \) to powers of \( \phi \) and \( q \) and transforming the result to the coordinate space

$$\phi(q) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{q}} \phi_{\vec{q}} e^{i\vec{q} \cdot \vec{r}},$$  \hspace{1cm} (9)$$

we obtain the Ginzburg-Landau free energy functional with a mismatch,

$$\Gamma = \int d^3 r [c |\nabla \phi|^2 - \alpha t |\phi|^2 + \frac{1}{2} b |\phi|^4],$$  \hspace{1cm} (10)$$

where

$$t = \frac{T_3 - T}{T_3},$$  \hspace{1cm} (11)$$$$
$$\alpha = \frac{k_F^2}{2 \pi^2 v_F} g(\delta),$$  \hspace{1cm} (12)$$$$
$$b = \frac{7 \zeta(3) k_F^2}{16 \pi^4 v_F T_3^2} f(\delta),$$  \hspace{1cm} (13)$$

and

$$c = \frac{1}{6} b v_F^2.$$  \hspace{1cm} (14)$$

The dimensionless functions \( g(\delta) \) and \( f(\delta) \) are given in terms of the following expressions

$$g(\delta) = \left[ 1 + \frac{\delta}{2 \pi T_3} \text{Im} \psi' \left( \frac{1}{2} + i \frac{\delta}{2 \pi T_3} \right) \right],$$  \hspace{1cm} (15)$$

and

$$f(\delta) = -\frac{1}{14 \zeta(3)} \text{Re} \psi'' \left( \frac{1}{2} + i \frac{\delta}{2 \pi T_3} \right),$$  \hspace{1cm} (16)$$

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \). The transition temperature \( T_3 \) can be calculated by solving the equation

$$\ln \frac{T_3}{T_0} = -\text{Re} \psi \left( \frac{1}{2} + i \frac{\delta}{2 \pi T_3} \right) - \gamma_E - 2 \ln 2$$  \hspace{1cm} (17)$$

where \( T_0 = \frac{\omega_F}{\pi} \Delta_0 \) is the transition temperature at \( \delta = 0 \).

The thermodynamic potential corresponds to the minimum of the Ginzburg-Landau free energy functional \( \Gamma \), i.e.
\[ \frac{\delta \Gamma / \delta \phi(\vec{r})}{\delta \Gamma / \delta \phi^*(\vec{r})} = 0. \] These conditions provide
\[ \phi(\vec{r}) = \Delta = \sqrt{\frac{\alpha t}{b}} = \sqrt{\frac{8\pi^2 g(\delta)t}{\zeta(3)f(\delta)}} T_\delta \]
and the value of the free energy functional is \( \Gamma_{\text{min}} = -\Omega \frac{\alpha^2 t^2}{b} \) for sufficiently weak mismatch.

In the absence of a mismatch, \( \delta = 0, T_\delta = T_0, f(0) = g(0) = 1 \) and the Ginzburg-Landau coefficients \([12], [13] \) and \([14]\) reduce to the well-known expressions \([6]\).

As the mismatch \( \delta \) increases, both functions \( f(\delta) \) and \( g(\delta) \) decrease monotonically until \( \delta = \delta_{\text{LOFF}} \) at which point \( f(\delta) \) changes sign and the homogeneous condensate ceases to be stable against the LOFF pairing \([3], [8]\).

We have \( g(\delta) \approx 0.140 \) and \( T_\delta \approx 0.562T_0 \) at this point.

The Ginzburg-Landau free energy of a superconductor in the presence of a static magnetic field can be obtained from \([10]\) by adding the magnetic energy and replacing the gradient with the covariant derivative in order to account for the interaction of the condensate with the magnetic field. We find
\[ \Gamma = \int d^3r \left[ \frac{1}{2} (\nabla \times \vec{A})^2 + c(|\nabla - 2ie\vec{A}|)\phi|^2 - \alpha t |\phi|^2 + \frac{1}{2} b |\phi|^4 \right]. \]

The magnetic properties of a superconductor are determined by the Ginzburg-Landau parameter \([7]\),
\[ \kappa = \frac{1}{m_A \xi}, \]
where \( m_A \) is the Meissner mass and \( \xi \) is the coherence length. The Meissner mass can be calculated from \([19]\). We find that
\[ m_A^2 = \frac{7\zeta(3)e^2v_Fk_F^2\Delta^2}{12\pi^4T_\delta^2} f(\delta) = \frac{2e^2v_Fk_F^2}{3\pi^2} g(\delta)t. \]

In order to calculate the coherence length, we consider the perturbation
\[ \phi = \Delta + \frac{1}{\sqrt{2}}(u + iv) \]
and expand \( \Gamma \) to quadratic order in \( u \) and \( v \). We obtain the expression
\[ \Gamma = \Gamma_{\text{min}} + \frac{1}{2} \int d^3r \left[ c(\nabla^2 u)^2 + c(\nabla^2 v)^2 + \frac{2}{\xi^2} u^2 \right] \]
from which the coherence length can be identified as
\[ \xi^2 = \frac{c}{\alpha t} = \frac{7\zeta(3)e^2v_Ff(\delta)}{48\pi^2 g(\delta)T_\delta^2 t}. \]

It follows from \([20], [21] \) and \([23]\) that the Ginzburg-Landau parameter can be written as
\[ \kappa = \frac{1}{m_A \xi} = \frac{6\pi^2}{e\sqrt{14\zeta(3)e^2v_F} f(\delta) \mu} T_\delta \]
and the value of the free energy functional is \( \Gamma_{\text{min}} \) determined by the Ginzburg-Landau parameter \([7]\),
\[ \kappa = \frac{1}{\sqrt{f(\delta)T_\delta}} \]
with \( \mu = \frac{1}{4}k_Fv_F \) being the Fermi energy. While the transition temperature \( T_\delta \) decreases slightly as \( \delta \) increases, the vanishing of \( f(\delta) \) at \( \delta_{\text{LOFF}} \) implies that \( \kappa \) diverges at that point. Therefore for a type I superconductor at \( \delta = 0 \), there exists a critical value \( \delta_c \) where \( \kappa = \frac{1}{\sqrt{2}} \). The superconductor for values of \( \delta > \delta_c \) becomes type II. We find that for \( \delta = \delta_c \),
\[ f(\delta_c) = \frac{9\pi^3}{\zeta(3)\alpha e^2v_F\left(\frac{T_\delta}{\mu}\right)^2}. \]

where \( \alpha_c \) is the fine structure constant.

To access the robustness of this (type-I)/(type-II) crossover, we need to estimate the contributions of the fluctuations of the condensate and the higher order terms that were omitted when we derived \([10]\) from \([6]\).

According to the Ginzburg-Landau criterion, the validity of the GL free energy \([10]\) or \([11]\) requires that the temperature remains outside a critical window, so that the fluctuations pertaining to a second order phase transition can be neglected. One formulation of this criterion is given in Ref. \([3]\), more specifically
\[ \frac{\int d^3r \cdot u(\vec{r})u(0)}{\xi^3\Delta^2} << 1 \]
where the ensemble average \( < u(\vec{r})u(0) > \) is approximated by
\[ < u(\vec{r})u(0) > = \frac{\int [du] \exp \left( -\frac{\vec{F}}{\xi^2} \right) u(\vec{r})u(0)}{\int [du] \exp \left( -\frac{\vec{F}}{\xi^2} \right)}. \]

We find that
\[ < u(\vec{r})u(0) > = \frac{T}{4\pi c r} e^{-\frac{\xi^2}{\Delta}} \]
and \([27]\) becomes
\[ \frac{T_\delta}{c\xi^2 \Delta^2} << 1. \]

Substituting the expressions for \( c, \xi \) and \( \Delta \), we obtain the following criterion for ignoring the fluctuations of the condensate
\[ t >> \frac{864\pi^6}{7\zeta(3)f(\delta)g(\delta) \mu^4} T_\delta^4. \]

Notice that, as the mismatch increases the width of the critical window diverges at \( \delta_{\text{LOFF}} \), indicating that the fluctuations of the condensate cannot be ignored. This is in agreement with the conjecture of reference \([10]\).

Let’s now assume approximately equal contributions for the three subleading terms, \( q^2 \), \( q^2 \Delta^4 \) and \( \Delta^6 \), and let’s choose the correction of the \( \Delta^6 \) as the representative. A detailed calculation shows that the ratio of this term with the contribution from the quartic term of \([11]\) is
\[ \frac{1}{12} \left( \frac{\Delta}{2\pi T_\delta} \right)^2 \text{Re} \psi'''' \left( \frac{1}{2} + i \frac{\delta}{2\pi T_\delta} \right) \]
\[ \text{Re} \psi'''' \left( \frac{1}{2} + i \frac{\delta}{2\pi T_\delta} \right). \]
The condition for it to be small implies that
\[
\epsilon << \frac{49\zeta^2(3)f^2(\delta)}{62\zeta(5)g(\delta)}, \tag{33}
\]
where we have replaced \(-\text{Re} \ln(\ldots)\) with its maximum value at \(\delta = 0\), 744\zeta(5). By combining (31) and (33), we arrive at the conclusion that the validity of the GL approach at the crossover is justified if
\[
\alpha_c v_F << 0.71 \left( \frac{T_\delta}{\mu} \right)^{4/3}, \tag{34}
\]
For Al, we have \(T_0 = 1.18K\), \(\mu = 1.35 \times 10^5\)\(K\) and \(v_F = 2.02 \times 10^6\)\(cm/s\) [11], the l.h.s. of (34) is about 4.9 \times 10^{-5} while the r. h. s. is about 2.0 \times 10^{-4}. In the case of Pb, we have \(T_0 = 7.19K\), \(\mu = 1.09 \times 10^5\)\(K\) and \(v_F = 1.82 \times 10^6\)\(cm/s\) [11], the l.h.s. is about 4.4 \times 10^{-5} while the r. h. s. is about 7.9 \times 10^{-4}. In view of numerical uncertainties behind the formulation of the Ginzburg criterion [27], it is plausible that the (type-I)/(type-II) crossover can be marginally described by the GL of [19].

The situation at \(T = 0\) is completely different. The Meissner mass at \(T = 0\) but \(\delta \neq 0\) was calculated in the literature [12] and was found to remain constant for \(\delta < \Delta\). There are two coherence lengths: the Pippard length, which measures the size of the Cooper pair and the length associated with the Anderson-Higgs mass that corresponds to the susceptibility of the free energy to an inhomogeneous variation of the gap magnitude. Both of them are of order \(\xi \sim \frac{\Delta}{\delta}\) at \(\delta = 0\). For \(\delta \neq 0\), our preliminary results [13] indicate that both of them remain constant for \(\delta < \Delta\), in a similar manner with the Meissner mass. Therefore, the crossover between type I and type II superconductors does not extend to \(T = 0\). A type I superconductor remains type I until \(\delta \sim \frac{1}{\sqrt{2}}\Delta_0\) at which point LOFF pairing becomes energetically favorable.

The critical magnetic field for a type I superconductor is given by
\[
H_c^2 = \frac{\alpha^2 l^2}{b} = \frac{g^2(\delta)}{f(\delta)} H_c^2 |_{\delta=0}. \tag{35}
\]

The magnitude of \(H_c\) at \(T = 0\) and \(\delta = 0\) is below 1000G for most type I superconductors. The magnetic field necessary for Zeeman splitting with \(\delta \sim \delta_\text{LOFF}\) is approximately \(1.76 \times 10^5\) G, where \(g\) is the gyromagnetic ratio of the electrons in the metal. Even with the enhanced value of the critical field given by (35) for \(\delta\) towards \(\delta_\text{LOFF}\), it is unlikely that the (type-I)/(type-II) crossover can be implemented with a strong external magnetic field without destroying superconductivity. On the other hand, the exchange field of ferromagnetic impurities may achieve this type of transition. It's action on the electron spins is equivalent to that of a magnetic field capable of causing a significant mismatch, but at the same time remaining well below \(H_c\). The magnetic field by asymmetric electron spins can hardly exceeds few G, which can be neglected.

In this letter, we developed the Ginzburg-Landau theory of an electronic superconductor at weak cooling when the Fermi momenta of opposite spins are mismatched. A remarkable feature that we found is that a type I superconductor becomes type II at sufficiently high values of the mismatch near the transition temperature. We have also estimated the fluctuation effects and the higher order corrections to the GL framework and argued that this type of transition is likely to be robust.

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