Infinite generation of non-cocompact lattices on right-angled buildings

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Let $\Gamma$ be a non-cocompact lattice on a locally finite regular right-angled building $X$. We prove that if $\Gamma$ has a strict fundamental domain then $\Gamma$ is not finitely generated. We use the separation properties of subcomplexes of $X$ called tree-walls.

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Tree lattices have been well-studied (see [BL]). Less understood are lattices on higher-dimensional CAT(0) complexes. In this paper, we consider lattices on $X$ a locally finite, regular right-angled building (see Davis [D] and Section 1 below). Examples of such $X$ include products of locally finite regular or biregular trees, or Bourdon’s building $I_{p,q}$ [B], which has apartments hyperbolic planes tessellated by right-angled $p$–gons and all vertex links the complete bipartite graph $K_{q,q}$.

Let $G$ be a closed, cocompact group of type-preserving automorphisms of $X$, equipped with the compact-open topology, and let $\Gamma$ be a lattice in $G$. That is, $\Gamma$ is discrete and the series $\sum |\text{Stab}_\Gamma(\phi)|^{-1}$ converges, where the sum is over the set of chambers $\phi$ of a fundamental domain for $\Gamma$. The lattice $\Gamma$ is cocompact in $G$ if and only if the quotient $\Gamma \backslash X$ is compact.

If there is a subcomplex $Y \subset X$ containing exactly one point from each $\Gamma$–orbit on $X$, then $Y$ is called a strict fundamental domain for $\Gamma$. Equivalently, $\Gamma$ has a strict fundamental domain if $\Gamma \backslash X$ may be embedded in $X$.

Any cocompact lattice in $G$ is finitely generated. We prove:

**Theorem 1** Let $\Gamma$ be a non-cocompact lattice in $G$. If $\Gamma$ has a strict fundamental domain, then $\Gamma$ is not finitely generated.

We note that Theorem 1 contrasts with the finite generation of lattices on many buildings whose chambers are simplices. Results of, for example, Ballmann–Świątkowski [BS], Dymara–Januszkiewicz [DJ], and Zuk [Z], establish that all lattices on many such buildings have Kazhdan’s Property (T). Hence by a well-known result due to Kazhdan [K], these lattices are finitely generated.
Our proof of Theorem 1, in Section 3 below, uses the separation properties of subcomplexes of $X$ which we call tree-walls. These generalize the tree-walls (in French, arbre-murs) of $I_{p,q}$, which were introduced by Bourdon in [B]. We define tree-walls and establish their properties in Section 2 below.

The following examples of non-cocompact lattices on right-angled buildings are known to us.

1. For $i = 1, 2$, let $G_i$ be a rank one Lie group over a nonarchimedean locally compact field whose Bruhat–Tits building is the locally finite regular or biregular tree $T_i$. Then any irreducible lattice in $G = G_1 \times G_2$ is finitely generated (Raghunathan [Ra]). Hence by Theorem 1 above, such lattices on $X = T_1 \times T_2$ cannot have strict fundamental domain.

2. Let $\Lambda$ be a minimal Kac–Moody group over a finite field $\mathbb{F}_q$ with right-angled Weyl group $W$. Then $\Lambda$ has locally finite, regular right-angled twin buildings $X_+ \cong X_-$, and $\Lambda$ acts diagonally on the product $X_+ \times X_-$. For $q$ large enough:

   a. By Theorem 0.2 of Carbone–Garland [CG] or Theorem 1(i) of Rémy [Ré], the stabilizer in $\Lambda$ of a point in $X_-$ is a non-cocompact lattice in $\text{Aut}(X_+)$. Any such lattice is contained in a negative maximal spherical parabolic subgroup of $\Lambda$, which has strict fundamental domain a sector in $X_+$, and so any such lattice has strict fundamental domain.

   b. By Theorem 1(ii) of Rémy [Ré], the group $\Lambda$ is itself a non-cocompact lattice in $\text{Aut}(X_+) \times \text{Aut}(X_-)$. Since $\Lambda$ is finitely generated, Theorem 1 above implies that $\Lambda$ does not have strict fundamental domain in $X = X_+ \times X_-.$

   c. By Section 7.3 of Gramlich–Horn–Mühlherr [GHM], the fixed set $G_\theta$ of certain involutions $\theta$ of $\Lambda$ is a lattice in $\text{Aut}(X_+)$, which is sometimes cocompact and sometimes non-cocompact. Moreover, by [GHM, Remark 7.13], there exists $\theta$ such that $G_\theta$ is not finitely generated.

3. In [T], the first author constructed a functor from graphs of groups to complexes of groups, which extends the corresponding tree lattice to a lattice in $\text{Aut}(X)$ where $X$ is a regular right-angled building. The resulting lattice in $\text{Aut}(X)$ has strict fundamental domain if and only if the original tree lattice has strict fundamental domain.
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1 Right-angled buildings

In this section we recall the basic definitions and some examples for right-angled buildings. We mostly follow Davis [D], in particular Section 12.2 and Example 18.1.10. See also [KT, Sections 1.2–1.4].

Let $(W, S)$ be a right-angled Coxeter system. That is, 

$$W = \langle S \mid (st)^{m_{st}} = 1 \rangle$$

where $m_{ss} = 1$ for all $s \in S$, and $m_{st} \in \{2, \infty\}$ for all $s, t \in S$ with $s \neq t$. We will discuss the following examples:

- $W_1 = \langle s, t \mid s^2 = t^2 = 1 \rangle \cong D_\infty$, the infinite dihedral group;
- $W_2 = \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^2 = 1 \rangle \cong (C_2 \times C_2) * C_2$, where $C_2$ is the cyclic group of order 2; and
- The Coxeter group $W_3$ generated by the set of reflections $S$ in the sides of a right-angled hyperbolic $p$–gon, $p \geq 5$. That is, 

$$W_3 = \langle s_1, \ldots, s_p \mid s_i^2 = (s_is_{i+1})^2 = 1 \rangle$$

with cyclic indexing.

Fix $(q_s)_{s \in S}$ a family of integers with $q_s \geq 2$. Given any family of groups $(H_s)_{s \in S}$ with $|H_s| = q_s$, let $H$ be the quotient of the free product of the $(H_s)_{s \in S}$ by the normal subgroup generated by the commutators $\{[h_s, h_t] : h_s \in H_s, h_t \in H_t, m_{st} = 2\}$.

Now let $X$ be the piecewise Euclidean CAT(0) geometric realization of the chamber system $\Phi = \Phi(H, \{1\}, (H_s)_{s \in S})$. Then $X$ is a locally finite, regular right-angled building, with chamber set $\text{Ch}(X)$ in bijection with the elements of the group $H$. Let $\delta_W : \text{Ch}(X) \times \text{Ch}(X) \to W$ be the $W$–valued distance function and let $l_S : W \to \mathbb{N}$ be word length with respect to the generating set $S$. Denote by $d_W : \text{Ch}(X) \times \text{Ch}(X) \to \mathbb{N}$
the gallery distance $l_S \circ \delta_W$. That is, for two chambers $\phi$ and $\phi'$ of $X$, $d_W(\phi, \phi')$ is the length of a minimal gallery from $\phi$ to $\phi'$.

Suppose that $\phi$ and $\phi'$ are $s$–adjacent chambers, for some $s \in S$. That is, $\delta_W(\phi, \phi') = s$. The intersection $\phi \cap \phi'$ is called an $s$–panel. By definition, since $X$ is regular, each $s$–panel is contained in $q_s$ distinct chambers. For distinct $s, t \in S$, the $s$–panel and $t$–panel of any chamber $\phi$ of $X$ have nonempty intersection if and only if $m_{st} = 2$. Each $s$–panel of $X$ is reduced to a vertex if and only if $m_{st} = \infty$ for all $t \in S - \{s\}$.

For the examples $W_1, W_2$, and $W_3$ above, respectively:

- The building $X_1$ is a tree with each chamber an edge, each $s$–panel a vertex of valence $q_s$, and each $t$–panel a vertex of valence $q_t$. That is, $X_1$ is the $(q_s, q_t)$–biregular tree. The apartments of $X_1$ are bi-infinite rays in this tree.

- The building $X_2$ has chambers and apartments as shown in Figure 1 below. The $r$– and $s$–panels are 1–dimensional and the $t$–panels are vertices.

![Figure 1: A chamber (on the left) and part of an apartment (on the right) for the building $X_2$.](image)

- The building $X_3$ has chambers $p$–gons and $s$–panels the edges of these $p$–gons. If $q_s = q \geq 2$ for all $s \in S$, then each $s$–panel is contained in $q$ chambers, and $X_3$, equipped with the obvious piecewise hyperbolic metric, is Bourdon’s building $I_{p,q}$. 
2 Tree-walls

We now generalize the notion of tree-wall due to Bourdon [B]. We will use basic facts about buildings, found in, for example, Davis [D]. Our main results concerning tree-walls are Corollary 3 below, which describes three possibilities for tree-walls, and Proposition 6 below, which generalizes the separation property 2.4.A(ii) of [B].

Let $X$ be as in Section 1 above and let $s \in S$. As in [B, Section 2.4.A], we define two $s$–panels of $X$ to be equivalent if they are contained in a common wall of type $s$ in some apartment of $X$. A tree-wall of type $s$ is then an equivalence class under this relation. We note that in order for walls and thus tree-walls to have a well-defined type, it is necessary only that all finite $m_{st}$, for $s \neq t$, be even. Tree-walls could thus be defined for buildings of type any even Coxeter system, and they would have properties similar to those below. We will however only explicitly consider the right-angled case.

Let $T$ be a tree-wall of $X$, of type $s$. We define a chamber $\psi$ of $X$ to be epicormic at $T$ if the $s$–panel of $\psi$ is contained in $T$, and we say that a gallery $\alpha = (\psi_0, \ldots, \psi_n)$ crosses $T$ if, for some $0 \leq i < n$, the chambers $\psi_i$ and $\psi_{i+1}$ are epicormic at $T$.

By the definition of tree-wall, if $\psi \in \text{Ch}(X)$ is epicormic at $T$ and $\psi' \in \text{Ch}(X)$ is $t$–adjacent to $\psi$ with $t \neq s$, then $\psi'$ is epicormic at $T$ if and only if $m_{st} = 2$. Let $s^\perp := \{t \in S \mid m_{st} = 2\}$ and denote by $\langle s^\perp \rangle$ the subgroup of $W$ generated by the elements of $s^\perp$. If $s^\perp$ is empty then by convention, $\langle s^\perp \rangle$ is trivial. For the examples in Section 1 above:

- in $W_1$, both $\langle s^\perp \rangle$ and $\langle t^\perp \rangle$ are trivial;
- in $W_2$, $\langle r^\perp \rangle = \langle s \rangle \cong C_2$ and $\langle s^\perp \rangle = \langle r \rangle \cong C_2$, while $\langle t^\perp \rangle$ is trivial; and
- in $W_3$, $\langle s^\perp_i \rangle = \langle s_{i-1}, s_{i+1} \rangle \cong D_{\infty}$ for each $1 \leq i \leq p$.

Lemma 2 Let $T$ be a tree-wall of $X$ of type $s$. Let $\phi$ be a chamber which is epicormic at $T$ and let $A$ be any apartment containing $\phi$.

1. The intersection $T \cap A$ is a wall of $A$, hence separates $A$.
2. There is a bijection between the elements of the group $\langle s^\perp \rangle$ and the set of chambers of $A$ which are epicormic at $T$ and in the same component of $A - T \cap A$ as $\phi$.

Proof Part (1) is immediate from the definition of tree-wall. For Part (2), let $w \in \langle s^\perp \rangle$ and let $\psi = \psi_w$ be the unique chamber of $A$ such that $\delta_w(\phi, \psi) = w$. We claim that $\psi$ is epicormic at $T$ and in the same component of $A - T \cap A$ as $\phi$. 
For this, let \( s_1 \cdots s_n \) be a reduced expression for \( w \) and let \( \alpha = (\phi_0, \ldots, \phi_n) \) be the minimal gallery from \( \phi = \phi_0 \) to \( \psi = \phi_n \) of type \((s_1, \ldots, s_n)\). Since \( w \) is in \( \langle s^\perp \rangle \), we have \( m_{s_is} = 2 \) for \( 1 \leq i \leq n \). Hence by induction each \( \phi_i \) is epicormic at \( \mathcal{T} \), and so \( \psi = \phi_n \) is epicormic at \( \mathcal{T} \). Moreover, since none of the \( s_i \) are equal to \( s \), the gallery \( \alpha \) does not cross \( \mathcal{T} \). Thus \( \psi = \psi_w \) is in the same component of \( A - \mathcal{T} \cap A \) as \( \phi \).

It follows that \( w \mapsto \psi_w \) is a well-defined, injective map from \( \langle s^\perp \rangle \) to the set of chambers of \( A \) which are epicormic at \( \mathcal{T} \) and in the same component of \( A - \mathcal{T} \cap A \) as \( \phi \). To complete the proof, we will show that this map is surjective. So let \( \psi \) be a chamber of \( A \) which is epicormic at \( \mathcal{T} \) and in the same component of \( A - \mathcal{T} \cap A \) as \( \phi \), and let \( w = \delta_{\mathcal{W}}(\phi, \psi) \).

If \( \langle s^\perp \rangle \) is trivial then \( \psi = \phi \) and \( w = 1 \), and we are done. Next suppose that the chambers \( \phi \) and \( \psi \) are \( t \)-adjacent, for some \( t \in S \). Since both \( \phi \) and \( \psi \) are epicormic at \( \mathcal{T} \), either \( t = s \) or \( m_{st} = 2 \). But \( \psi \) is in the same component of \( A - \mathcal{T} \cap A \) as \( \phi \), so \( t \neq s \), hence \( w = t \) is in \( \langle s^\perp \rangle \) as required. If \( \langle s^\perp \rangle \) is finite, then finitely many applications of this argument will finish the proof. If \( \langle s^\perp \rangle \) is infinite, we have established the base case of an induction on \( n = I_5(w) \).

For the inductive step, let \( s_1 \cdots s_n \) be a reduced expression for \( w \) and let \( \alpha = (\phi_0, \ldots, \phi_n) \) be the minimal gallery from \( \phi = \phi_0 \) to \( \psi = \phi_n \) of type \((s_1, \ldots, s_n)\). Since \( \phi \) and \( \psi \) are in the same component of \( A - \mathcal{T} \cap A \) and \( \alpha \) is minimal, the gallery \( \alpha \) does not cross \( \mathcal{T} \). We claim that \( s_n \) is in \( s^\perp \). First note that \( s_n \neq s \) since \( \alpha \) does not cross \( \mathcal{T} \) and \( \psi = \phi_n \) is epicormic at \( \mathcal{T} \). Now denote by \( \mathcal{T}_n \) the tree-wall of \( X \) containing the \( s_n \)-panel \( \phi_{n-1} \cap \phi_n \). Since \( \alpha \) is minimal and crosses \( \mathcal{T}_n \), the chambers \( \phi = \phi_0 \) and \( \psi = \phi_n \) are separated by the wall \( \mathcal{T}_n \cap A \). Thus the \( s \)-panel of \( \phi \) and the \( s \)-panel of \( \psi \) are separated by \( \mathcal{T}_n \cap A \). As the \( s \)-panels of both \( \phi \) and \( \psi \) are in the wall \( \mathcal{T} \cap A \), it follows that the walls \( \mathcal{T}_n \cap A \) and \( \mathcal{T} \cap A \) intersect. Hence \( m_{s_n,s} = 2 \), as claimed.

Now let \( w' = w_{s_n} = s_1 \cdots s_{n-1} \) and let \( \psi' \) be the unique chamber of \( A \) such that \( \delta_{\mathcal{W}}(\phi, \psi') = w' \). Since \( s_n \) is in \( s^\perp \) and \( \psi' \) is \( s_n \)-adjacent to \( \psi \), the chamber \( \psi' \) is epicormic at \( \mathcal{T} \) and in the same component of \( A - \mathcal{T} \cap A \) as \( \phi \). Moreover \( s_1 \cdots s_{n-1} \) is a reduced expression for \( w' \), so \( I_5(w') = n - 1 \). Hence by the inductive assumption, \( w' \) is in \( \langle s^\perp \rangle \). Therefore \( w = w's_n \) is in \( \langle s^\perp \rangle \), which completes the proof.

Corollary 3  The following possibilities for tree-walls in \( X \) may occur.

1. Every tree-wall of type \( s \) is reduced to a vertex if and only if \( \langle s^\perp \rangle \) is trivial.
2. Every tree-wall of type \( s \) is finite but not reduced to a vertex if and only if \( \langle s^\perp \rangle \) is finite but nontrivial.
Every tree-wall of type $s$ is infinite if and only if $\langle s^\perp \rangle$ is infinite.

**Proof** Let $\mathcal{T}$, $\phi$, and $A$ be as in Lemma 2 above. The set of $s$–panels in the wall $\mathcal{T} \cap A$ is in bijection with the set of chambers of $A$ which are epicormic at $\mathcal{T}$ and in the same component of $A - \mathcal{T} \cap A$ as $\phi$. □

For the examples in Section 1 above:

- in $X_1$, every tree-wall of type $s$ and of type $t$ is a vertex;
- in $X_2$, the tree-walls of types both $r$ and $s$ are finite and 1–dimensional, while every tree-wall of type $t$ is a vertex; and
- in $X_3$, all tree-walls are infinite, and are 1–dimensional.

**Corollary 4** Let $\mathcal{T}$, $\phi$, and $A$ be as in Lemma 2 above and let

$$\rho = \rho_{\phi, A} : X \to A$$

be the retraction onto $A$ centered at $\phi$. Then $\rho^{-1}(\mathcal{T} \cap A) = \mathcal{T}$. □

**Proof** Let $\psi$ be any chamber of $A$ which is epicormic at $\mathcal{T}$ and is in the same component of $A - \mathcal{T} \cap A$ as $\phi$. Then by the proof of Lemma 2 above, $w := \delta_W(\phi, \psi)$ is in $\langle s^\perp \rangle$. Let $\psi'$ be a chamber in the preimage $\rho^{-1}(\psi)$ and let $A'$ be an apartment containing both $\phi$ and $\psi'$. Since the retraction $\rho$ preserves $W$–distances from $\phi$, we have that $\delta_W(\phi, \psi') = w$ is in $\langle s^\perp \rangle$. Again by the proof of Lemma 2, it follows that the chamber $\psi'$ is epicormic at $\mathcal{T}$. But the image under $\rho$ of the $s$–panel of $\psi'$ is the $s$–panel of $\psi$. Thus $\rho^{-1}(\mathcal{T} \cap A) = \mathcal{T}$, as required. □

**Lemma 5** Let $\mathcal{T}$ be a tree-wall and let $\phi$ and $\phi'$ be two chambers of $X$. Let $\alpha$ be a minimal gallery from $\phi$ to $\phi'$ and let $\beta$ be any gallery from $\phi$ to $\phi'$. If $\alpha$ crosses $\mathcal{T}$ then $\beta$ crosses $\mathcal{T}$.

**Proof** Suppose that $\alpha$ crosses $\mathcal{T}$. Since $\alpha$ is minimal, there is an apartment $A$ of $X$ which contains $\alpha$, and hence the wall $\mathcal{T} \cap A$ separates $\phi$ from $\phi'$ . Choose a chamber $\phi_0$ of $A$ which is epicormic at $\mathcal{T}$ and consider the retraction $\rho = \rho_{\phi_0, A}$ onto $A$ centered at $\phi_0$. Since $\phi$ and $\phi'$ are in $A$, $\rho$ fixes $\phi$ and $\phi'$. Hence $\rho(\beta)$ is a gallery in $A$ from $\phi$ to $\phi'$, and so $\rho(\beta)$ crosses $\mathcal{T} \cap A$. By Corollary 4 above, $\rho^{-1}(\mathcal{T} \cap A) = \mathcal{T}$. Therefore $\beta$ crosses $\mathcal{T}$. □

**Proposition 6** Let $\mathcal{T}$ be a tree-wall of type $s$. Then $\mathcal{T}$ separates $X$ into $q_s$ gallery-connected components.
Proof} Fix an \(s\)-panel in \(\mathcal{T}\) and let \(\phi_1, \ldots, \phi_q\) be the \(q\) chambers containing this panel. Then for all \(1 \leq i < j \leq q\), the minimal gallery from \(\phi_i\) to \(\phi_j\) is just \((\phi_i, \phi_j)\), and hence crosses \(\mathcal{T}\). Thus by Lemma 5 above, any gallery from \(\phi_i\) to \(\phi_j\) crosses \(\mathcal{T}\). So the \(q\) chambers \(\phi_1, \ldots, \phi_q\) lie in \(q\) distinct components of \(X - \mathcal{T}\).

To complete the proof, we show that \(\mathcal{T}\) separates \(X\) into at most \(q\) components. Let \(\phi\) be any chamber of \(X\). Then among the chambers \(\phi_1, \ldots, \phi_q\), there is a unique chamber, say \(\phi_1\), at minimal gallery distance from \(\phi\). It suffices to show that \(\phi\) and \(\phi_1\) are in the same component of \(X - \mathcal{T}\).

Let \(\alpha\) be a minimal gallery from \(\phi\) to \(\phi_1\) and let \(A\) be an apartment containing \(\alpha\). Then there is a unique chamber of \(A\) which is \(s\)-adjacent to \(\phi_1\). Hence \(A\) contains \(\phi_i\) for some \(i > 1\), and the wall \(\mathcal{T} \cap A\) separates \(\phi_1\) from \(\phi_i\). Since \(\alpha\) is minimal and \(d_W(\phi, \phi_1) < d_W(\phi, \phi_i)\), the Exchange Condition (see [D, page 35]) implies that a minimal gallery from \(\phi\) to \(\phi_i\) may be obtained by concatenating \(\alpha\) with the gallery \((\phi_1, \phi_i)\). Since a minimal gallery can cross \(\mathcal{T} \cap A\) at most once, \(\alpha\) does not cross \(\mathcal{T} \cap A\). Thus \(\phi\) and \(\phi_1\) are in the same component of \(X - \mathcal{T}\), as required. \(\square\)

3 Proof of Theorem

Let \(G\) be as in the introduction and let \(\Gamma\) be a non-cocompact lattice in \(G\) with strict fundamental domain. Fix a chamber \(\phi_0\) of \(X\). For each integer \(n \geq 0\) define

\[
D(n) := \{ \phi \in \text{Ch}(X) \mid d_W(\phi, \Gamma \phi_0) \leq n \}.
\]

Then \(D(0) = \Gamma \phi_0\), and for every \(n > 0\) every connected component of \(D(n)\) contains a chamber in \(\Gamma \phi_0\). To prove Theorem 1, we will show that there is no \(n > 0\) such that \(D(n)\) is connected.

Let \(Y\) be a strict fundamental domain for \(\Gamma\) which contains \(\phi_0\). For each chamber \(\phi\) of \(X\), denote by \(\phi_Y\) the representative of \(\phi\) in \(Y\).

**Lemma 7** Let \(\phi\) and \(\phi'\) be \(t\)-adjacent chambers in \(X\), for \(t \in S\). Then either \(\phi_Y = \phi'_Y\), or \(\phi_Y\) and \(\phi'_Y\) are \(t\)-adjacent.

**Proof** It suffices to show that the \(t\)-panel of \(\phi_Y\) is the \(t\)-panel of \(\phi'_Y\). Since \(Y\) is a subcomplex of \(X\), the \(t\)-panel of \(\phi_Y\) is contained in \(Y\). By definition of a strict fundamental domain, there is exactly one representative in \(Y\) of the \(t\)-panel of \(\phi\). Hence the unique representative in \(Y\) of the \(t\)-panel of \(\phi\) is the \(t\)-panel of \(\phi_Y\). Similarly, the unique representative in \(Y\) of the \(t\)-panel of \(\phi'\) is the \(t\)-panel of \(\phi'_Y\). But \(\phi\) and \(\phi'\) are
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t−adjacent, hence have the same t−panel, and so it follows that \( \phi_Y \) and \( \phi'_Y \) have the same t−panel.

Corollary 8 The fundamental domain \( Y \) is gallery-connected.

Lemma 9 For all \( n > 0 \), the fundamental domain \( Y \) contains a pair of adjacent chambers \( \phi_n \) and \( \phi'_n \) such that, if \( T_n \) denotes the tree-wall separating \( \phi_n \) from \( \phi'_n \):

1. the chambers \( \phi_0 \) and \( \phi_n \) are in the same gallery-connected component of \( Y − T_n \cap Y \);
2. \( \min \{ d_W(\phi_0, \phi) \mid \phi \in \text{Ch}(X) \text{ is epicormic at } T_n \} > n \); and
3. there is a \( \gamma \in \text{Stab}_\Gamma(\phi'_n) \) which does not fix \( \phi_n \).

Proof Fix \( n > 0 \). Since \( \Gamma \) is not cocompact, \( Y \) is not compact. Thus there exists a tree-wall \( T_n \) with \( T_n \cap Y \) nonempty such that for every \( \phi \in \text{Ch}(X) \) which is epicormic at \( T_n \), \( d_W(\phi_0, \phi) > n \). Let \( s_n \) be the type of the tree-wall \( T_n \). Then by Corollary 8 above, there is a chamber \( \phi_n \) of \( Y \) which is epicormic at \( T_n \) and in the same gallery-connected component of \( Y − T_n \cap Y \) as \( \phi_0 \), such that for some chamber \( \phi'_n \) which is \( s_n \)-adjacent to \( \phi_n \), \( \phi'_n \) is also in \( Y \). Now, as \( \Gamma \) is a non-cocompact lattice, the orders of the \( \Gamma \)-stabilizers of the chambers in \( Y \) are unbounded. Hence the tree-wall \( T_n \) and chambers \( \phi_n \) and \( \phi'_n \) may be chosen so that \( |\text{Stab}_\Gamma(\phi_n)| < |\text{Stab}_\Gamma(\phi'_n)| \).

Let \( \phi_n \), \( \phi'_n \), \( T_n \), and \( \gamma \) be as in Lemma 9 above and let \( s = s_n \) be the type of the tree-wall \( T_n \). Let \( \alpha \) be a gallery in \( Y − T_n \cap Y \) from \( \phi_0 \) to \( \phi_n \). The chambers \( \phi_n \) and \( \gamma \cdot \phi_n \) are in two distinct components of \( X − T_n \), since they both contain the \( s \)-panel \( \phi_n \cap \phi'_n \subseteq T_n \), which is fixed by \( \gamma \). Hence the galleries \( \alpha \) and \( \gamma \cdot \alpha \) are in two distinct components of \( X − T_n \), and so the chambers \( \phi_0 \) and \( \gamma \cdot \phi_0 \) are in two distinct components of \( X − T_n \). Denote by \( X_0 \) the component of \( X − T_n \) which contains \( \phi_0 \), and put \( Y_0 = Y \cap X_0 \).

Lemma 10 Let \( \phi \) be a chamber in \( X_0 \) that is epicormic at \( T_n \). Then \( \phi_Y \) is in \( Y_0 \) and is epicormic at \( T_n \cap Y \).

Proof We consider three cases, corresponding to the possibilities for tree-walls in Corollary 3 above.

1. If \( T_n \) is reduced to a vertex, there is only one chamber in \( X_0 \) which is epicormic at \( T_n \), namely \( \phi_n \). Thus \( \phi = \phi_n = \phi_Y \) and we are done.
(2) If $\mathcal{T}_n$ is finite but not reduced to a vertex, the result follows by finitely many applications of Lemma 7 above.

(3) If $\mathcal{T}_n$ is infinite, the result follows by induction, using Lemma 7 above, on

$$k := \min\{d_W(\phi, \psi) \mid \psi \text{ is a chamber of } Y_0 \text{ epicormic at } \mathcal{T}_n \cap Y\}.$$  

\[\square\]

Lemma 11 For all $n > 0$, the complex $D(n)$ is not connected.

Proof Fix $n > 0$, and let $\alpha$ be a gallery in $X$ between a chamber in $X_0 \cap \Gamma \phi_0$ and some chamber $\phi$ in $X_0$ that is epicormic at $\mathcal{T}_n$. Let $m$ be the length of $\alpha$.

By Lemma 7 and Lemma 10 above, the gallery $\alpha$ projects to a gallery $\beta$ in $Y$ between $\phi_0$ and a chamber $\phi_Y$ that is epicormic at $\mathcal{T}_n \cap Y$. The gallery $\beta$ in $Y$ has length at most $m$.

It follows from (2) of Lemma 9 above that the gallery $\beta$ in $Y$ has length greater than $n$. Therefore $m > n$. Hence the gallery-connected component of $D(n)$ that contains $\phi_0$ is contained in $X_0$. As the chamber $\gamma \cdot \phi_0$ is not in $X_0$, it follows that the complex $D(n)$ is not connected.  

This completes the proof, as $\Gamma$ is finitely generated if and only if $D(n)$ is connected for some $n$.

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