Research Article

The Beckman–Quarles Theorem in Hyperbolic Geometry

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In this paper, we present the counterpart of the Beckman–Quarles theorem in the Poincaré disc model of hyperbolic geometry to characterize the gyroisometries (hyperbolic isometries) with a single nonzero distance $a \in (0, 1)$ satisfying $a^2 \in \mathbb{Q}$.

1. Introduction

Let $(X, d)$ be a metric space and $f$ be a function defined from $(X, d)$ to itself. If $d(f(P), f(Q)) = d(P, Q)$ holds for all $P, Q \in X$, then $f$ is called an isometry of $X$. An isometry is injective, but it need not be surjective. If an isometry is invertible, its inverse is also an isometry. An isometry of Euclidean space $\mathbb{R}^n$ is bijective. It is well known that every isometry of $\mathbb{R}^n$ is a composition of at most $n + 1$ reflections. An isometry that fixes at least one point is a composition of at most $n$ reflections.

Let us consider the Euclidean space $\mathbb{R}^n \ (n \geq 2)$ and $f$ be a function defined from $\mathbb{R}^n$ to itself. The Beckman–Quarles [1] theorem states that if $f$ is a function defined from $\mathbb{R}^n \ (n \geq 2)$ to itself which preserves a distance $k \in \mathbb{R}^+$, then it is an isometry of $\mathbb{R}^n$. In the literature, there are several proofs of this famous theorem, see [2, 3], and some authors [4–7] tried to find the counterpart of this theorem in various spaces.

Hyperbolic geometry is a non-Euclidean geometry that rejects the validity of Euclid’s fifth postulate. The principles of hyperbolic geometry, however, admit the other four Euclidean postulates. Although many of the theorems of hyperbolic geometry are identical to those of Euclidean, others differ. For example, in Euclidean geometry, two parallel lines are taken to be everywhere equidistant. In hyperbolic geometry, two parallel lines are taken to converge in one direction and diverge in the other. In Euclidean geometry, the sum of the angles in a triangle is equal to two right angles; in hyperbolic geometry, the sum is less than two right angles. In Euclidean geometry, polygons of differing areas can be similar, and in hyperbolic geometry, similar polygons of differing areas do not exist. There are many principal hyperbolic geometry models, for instance, Weierstrass model, Beltrami–Klein model, Poincaré disc model, and Poincaré upper-half plane model. In this paper, we deal with the Poincaré disc model of hyperbolic geometry to get the desired results. The points of this model are the points of the complex unit disc:

$$D = \{z \in \mathbb{C}: |z| < 1\},$$

and the hyperbolic lines are circular arcs orthogonal to the boundary circle of the disc including the diameters of $D$. Two arcs which do not meet correspond to parallel rays and the arcs which meet orthogonally correspond to perpendicular lines (see Figure 1). The hyperbolic angle between two hyperbolic lines is the usual Euclidean angle between Euclidean tangents to the circular arcs. The advantage of the Poincaré disc model is that it is conformal, namely, circles and angles are not distorted.

The classical hyperbolic distance $d_H$ in $D$ is defined by

$$d_H(z_1, z_2) = \frac{1}{2} \ln \frac{1 + d(z_1, z_2)}{1 - d(z_1, z_2)},$$

where
It is known that the Möbius addition is neither commutative nor associative. By defining the gyrator

\[ \text{gyr}: \mathbb{D} \times \mathbb{D} \rightarrow \text{Aut}(\mathbb{D}, \oplus), \]

\[ \text{gyr} [a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + ab}{1 + ba}, \]

where Aut(\mathbb{D}, \oplus) is the automorphism group of the Möbius groupoid (\mathbb{D}, \oplus), the following group-like properties of \mathbb{D} can be verified by straightforward algebra for all a, b, c \in \mathbb{D}:

G1. \( a \oplus b = \text{gyr}[a, b] (b \oplus a) \) (gyrocommutative law)
G2. \( a \oplus (b \oplus c) = (a \oplus b) \text{gyr}[a, b] c \) (left gyroassociative law)
G3. \( (a \oplus b) \oplus c = a \oplus (b \oplus (a \oplus b)) \) (right gyroassociative law)
G4. \( \text{gyr}[a, b] = \text{gyr}[a \oplus b, b] \) (left loop property)
G5. \( \text{gyr}[a, b] = \text{gyr}[a, b \oplus a] \) (right loop property)
G6. \( a \oplus b = \text{gyr}[a, b] (b \oplus a) \) (gyrocommutative law)

The Möbius gyrdistance function \( d_M \) in \mathbb{D} is

\[ d_M(z, w) = |z \oplus w| = \left| \frac{z - w}{1 - \overline{z}w} \right|, \]

which is closely related to classical hyperbolic distance \( d_H \) as follows:

\[ \tanh d_H(z, w) = \frac{|z - w|}{1 - \overline{z}w} = d_M(z, w). \]

The operations “⊕” and “⊖” are called Möbius addition and Möbius subtraction, respectively. \( d_M \) is not a metric on \mathbb{D} since the triangle inequality is not provided. The gyrodistances are invariant under left gyrotranslations and rotations about origin. Clearly, \( d_H(z_1, z_2) \in [0, \infty) \) and
Let \( ABC \) be an equilateral gyrotriangle with vertices \( A, B, C \) corresponding gyroangles \( \alpha, \beta, \gamma \) and side gyrolengths \( a, b, c \). The sides of the gyrotriangle \( ABC \) are determined by its gyroangles by

\[
\begin{align*}
a^2 &= \frac{\cos \alpha + \cos (\beta + \gamma)}{\cos \alpha + \cos (\beta - \gamma)} \\
b^2 &= \frac{\cos \beta + \cos (\alpha + \gamma)}{\cos \beta + \cos (\alpha - \gamma)} \\
c^2 &= \frac{\cos \gamma + \cos (\alpha + \beta)}{\cos \gamma + \cos (\alpha - \beta)}
\end{align*}
\]

(10)

Theorem 1 presents a most important disanalogy with Euclidean triangle similarity. For the proof of Theorems 1 and 2, we refer the reader to [9].

Theorem 2. Let \( ABC \) be an equilateral gyrotriangle in \( \mathbb{D} \) with vertices \( A, B, C \) satisfying \( \angle ABC = \angle BCA = \angle CAB = \alpha \), the side gyrolengths of which are \( a \), and then \( a = \sqrt{2} \cos \alpha - 1 \).

2. The Beckman–Quarles Theorem in Hyperbolic Geometry

Throughout the paper, we denote by \( X' \) the image of \( X \) under \( f \), by \([P, Q]\) the gyrosegment that links the points \( P \) and \( Q \), by \( PQ \) the gyrotriangle with three ordered vertices \( P, Q, R \), by \( d(P, Q) \) the Möbius gyrodistance between \( P \) and \( Q \), by \( PQR \) the gyroquadrilateral with four ordered vertices \( P, Q, R, S \), and by \( \angle PQR \) the gyroangle between \( [P, Q] \) and \( [R, Q] \). We say that \( f \) preserves \( s \) if any points \( A, B \in \mathbb{D} \) such that \( d(A, B) = s \) also satisfy \( d(f(A), f(B)) = s \).

Lemma 1. If \( f: \mathbb{D} \longrightarrow \mathbb{D} \) preserves some gyrodistance \( a \in (0, 1) \), then it is single valued.

Proof. Assume that \( f \) is a nonsingle valued and \( A \) be a point in \( \mathbb{D} \) such that \( f(A) = U, f(A) = V, U \neq V \). Now, construct an equilateral gyrotriangle \( ABC \) satisfying \( d(A, B) = d(B, C) = d(A, C) = a \). Then, hypothesis \( A'B'C' \) is also an equilateral gyrotriangle with \( d(A', B') = d(A', C') = d(B', C') = a \). Assume \( \angle ABC = \alpha \). Then, we get \( a^2 = 2 \cos \alpha - 1 \) which implies \( \cos \alpha = (a^2 + 1)/2 \) by Theorem 2. Now, construct the gyrorhombus \( AB \) from \( AB \). Hence, by Theorem 1, we get

\[
d(A, D) = \sqrt{\frac{\cos \alpha + \cos a}{\cos 2 \alpha + 1}} = \sqrt{1 + \frac{a^2 - 1}{(a^2 + 1)^2}}.
\]

(11)

Since there are no points other than \( U \) and \( V \) at distance \( a \) from \( B' \) to \( C' \), one can easily get \( D' = U \) or \( D' = V \). Let \( g_{\theta} \) be the hyperbolic rotation with respect to \( A \) for an appropriate \( \theta \in \mathbb{R} \) satisfying \( d(D, g_{\theta}(D)) = a \) and \( d(g_{\theta}(D), A) = \theta \). Clearly, \( g_{\theta}(x) = A \Theta (e^{i\theta}(e^{i\theta}x), \Theta e^{i\theta}A) \). Therefore, we get

\[
d(D', E') \in \left[0, \sqrt{1 + ((a^2 - 1)/(a^2 + 1)^2)}\right],
\]

but since \( d(D, E) = a, \) this is a contradiction.

Corollary 1. If \( f: \mathbb{D} \longrightarrow \mathbb{D} \) is a map which preserves some gyrodistance \( a \in (0, 1), \) then it preserves the gyrodistance \( \sqrt{1 + ((a^2 - 1)/(a^2 + 1)^2)} \).

Proof. Let us consider the gyrorhombuses \( AB \) and \( AB' \) in Lemma 1. Clearly, \( d(A, D) = d(A, E) = \sqrt{1 + ((a^2 - 1)/(a^2 + 1)^2}) \) and this implies \( d(A', D') = 0 \) or \( d(A', D') = \sqrt{1 + ((a^2 - 1)/(a^2 + 1)^2}) \). Hence, we get

\[
d(A', E') =
\]
Clearly, \((a^2 - 1)/((a^2 + 1)^2)\). Since \(d(D,E) = a\) holds, this is a contradiction. Thus, we get \((A',D') = \sqrt{1 + ((a^2 - 1)/((a^2 + 1)^2))}\). Let us define \(\psi(a) = \sqrt{1 + ((a^2 - 1)/((a^2 + 1)^2))}\) and \(\psi^n(a) = \psi(\psi(\ldots(\psi(a)), \ldots))\) \((n\text{ times})\) for all \(n \in \mathbb{N}\). Clearly, \(a < \psi(a)\) and \(\psi'(a) < \psi^2(a)\) hold if \(i < j\). □

**Corollary 2.** If \(f : \mathbb{D} \rightarrow \mathbb{D}\) is a map which preserves some gyrodistance \(a \in (0,1)\), then it preserves the gyrodistances \(\psi^n(a)\) for all \(n \in \mathbb{N}\).

**Theorem 3.** If \(f : \mathbb{D} \rightarrow \mathbb{D}\) is a map which preserves some gyrodistance \(a \in (0,1)\) satisfying \(a^2 \in \mathbb{Q}\), then \(f\) preserves all gyrodistances.

**Proof**

**Step 1.** We claim that \(f\) preserves the gyrodistances less than \(a\). Let \(P\) and \(Q\) be two distinct points in \(\mathbb{D}\) satisfying \(d(P, Q) < a\). Clearly, there exist two points in \(\mathbb{D}\), say \(C\) and \(E\), such that \(d(P, C) = d(E, C)\) and \(Q \in [C, E]\). Let \(S(C, a)\) be the gyrocircle centered at \(C\) with radius \(a\). Then, there exist two points on \(S(C, a)\), say \(R_1\) and \(R_2\) such that \(d(P, R_1) = d(P, R_2) = a\). Thus, we construct two equilateral gyrotriangles \(PR_1C, PR_2C\), the side gyrolengths of which are \(a\). Here, the point \(Q\) can be thought to be on the gyrosegment \([P, R_1]\), otherwise a new configuration is needed to provide this feature, which is not difficult to construct. Assume \(\angle D R_1 C = \angle D R_2 C = a\).

Now, construct the sequence \(A_iCA_{i+1} (i = 1,2,\ldots)\) with \(d(C, A_1) = d(C, A_{i+1}) = d(A_i, A_{i+1}) = a\), \(A_1 = P, A_2 = R_1, \angle A_1 A_2 A_1 = a\). It is clear that there is another sequence \(B_iCB_{i+1} (i = 1,2,\ldots)\) with \(d(C, B_1) = d(C, B_{i+1}) = d(B_i, B_{i+1}) = a, B_1 = P, B_2 = R_2, \angle B_1 B_2 B_1 = a\). If it can be established that \(a \neq \pi\), then there is no nonzero integer \(k\) such that \(A_k = A_k\). If there is such a \(k\) satisfying \(A_1 = A_k\), then one can easily see that \(k\alpha = 2\pi, i.e., a = (2\pi/k) \in \pi\). In [3], it was shown that if \(a = (m/n)\pi\), where \(m, n \in \mathbb{Z}\) satisfying \(\cos \rho = \cos \frac{a}{\pi}\) then \(\cos \rho \in [0, \pm (1/2), \pm 1]\). Since \(\cos \alpha = ((a^2 + 1)/2) \notin [0, (1/2), \pm 1]\) for all \(a \in (0,1)\) satisfying \(a^2 \in \mathbb{Q}\), we immediately get that \(a \neq \pi\). Hence, the gyrotriangles which define the sequence \(A_iCA_{i+1}\) touch the gyroarc \(\angle D R_1\) at an infinite number of points. Thus, \([P, R_1]\) has been divided into gyrosegments as desired, and the images of the points that provide this disintegration are gyrocollinear. Moreover, the gyrodistances between the points that provide this disintegration are also preserved. Thus, \(f\) preserves the gyrodistances less than \(a\). Hence, we obtain that \(d(P, Q) = d(P', Q')\) holds.

**Step 2.** We claim that \(f\) preserves the gyrodistance \(2 \otimes a = a \otimes a\). Let \(K\) and \(L\) be two distinct points in the gyrocircle \(S(C, a)\) satisfying \(d(K, L) = 2 \otimes a\). Since \(f\) preserves the measure of the equilateral gyrotriangles with their gyroangle \(a\), constructing a similar sequence in Step 1 above, we immediately get \(d(K', L') = 2 \otimes a\). Similarly, it is clear that \(f\) also preserves the gyrodistances \(n \otimes a = a \otimes \cdots \otimes a (n \text{ times})\) for all \(n \in \mathbb{N}\). Notice that since \(a \in (0,1)\) and \(a^2 \in \mathbb{Q}\), this implies \(n \otimes a \in (0,1)\) and \((n \otimes a)^2 \in \mathbb{Q}\). Moreover, for each equilateral gyrotriangle \(ABC\) with \(d(A, B) = d(A, C) = d(B, C) = n \otimes a (a^2 \in \mathbb{Q})\), if \(\angle ABC = \beta\), then \(\cos \beta \in Q\) and \(\beta \neq \pi\).

**Step 3.** \(f\) preserves the gyrodistances greater than \(a\). Let \(P\) and \(Q\) be two distinct points in \(\mathbb{D}\) satisfying \(d(P, Q) > a\). Let \(k\) be a positive integer such that \(d(P, Q) < k \otimes a\). As in Step 1, one can easily construct a gyrocircle with radius \(k \otimes a\) passing through \(P\). Denote this gyrocircle by \(D, k \otimes a\) where \(D\) is its center. Then, there exists a point \(W\) on \(S(D, k \otimes a)\) such that \(DPW\) is an equilateral gyrotriangle and denote \(\angle D PW = \phi\). Since \(\cos \phi \in Q\) by Step 2, \(f\) preserves all the gyrodistances less than \(k \otimes a\). Hence, we see that \(f\) preserves all gyrodistances. □

### 3. Conclusions

We have proved that if \(f : \mathbb{D} \rightarrow \mathbb{D}\) is a mapping which preserves some gyrodistance \(a \in (0,1)\) satisfying \(a^2 \in \mathbb{Q}\), then \(f\) is a gyroisometry; that is, it preserves all gyrodistances. This implies that \(f\) is also a hyperbolic isometry. Naturally, one may wonder whether Theorem 3 is valid for an arbitrary gyrodistance \(a\) satisfying \(a^2 \in (0,1)-Q\). We leave the solution of this problem to the reader’s attention.

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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