From the time-ordered data to the Maximum-Likelihood temperature maps of the Cosmic Microwave Background anisotropy.

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Abstract. We review selected methods of the Cosmic Microwave Background data analysis appropriate for the analysis of the largest currently available data sets. We focus on techniques of the time-ordered data manipulation and map making algorithms based on the maximum-likelihood approach. The presented methods have been applied to the \textsc{maxima-i} data analysis (Hanany et al. 2000) and the description of the algorithms is illustrated with the examples drawn from that experience. The more extensive presentation of the here-mentioned issues will be given in the forthcoming paper (Stompor et al. 2001).

1 Introduction

The growing size and complexity of the available CMB data sets poses a difficult challenge for the data analysis (e.g. Borrill (1999), Borrill et al. (2000)). The largest already existing data sets (e.g. \textsc{boomerang-ldb}, \textsc{maxima-i}) consist of up to many million of the time measurements sampling areas of the sky of up to many thousands of the beam-size pixels. The analysis of such a big data set is usually divided into three distinct stages focusing in turn on the time-ordered data, maps and power spectra. On each stage the size of the data item to be manipulated is significantly reduced. However, a successful implementation of such a program requires that the compression
performed on each stage should not compromise the cosmological information contained in the data. The algorithms presented in the following are designed to be optimal – or nearly optimal – in a maximum-likelihood sense under the assumptions of the Gaussianity and stationarity of the instrumental noise.

We start from the description of the time-ordered data manipulation techniques which prove to be necessary to ensure the efficiency and applicability of the map making algorithms. Then we briefly review a variety of map making algorithms highlighting involved assumptions and assessing their efficiency.

2 Noise estimation and gaps filling

In the usual map-making methodology the noise power spectra are taken to be given with an arbitrary high precision. The circumstances in which actual CMB data are collected are specific enough to require the noise estimation to be evaluated directly from the time-ordered data. The robust noise-estimation procedure therefore needs to be capable of efficient dealing with the undesirable effects present in the real data.

The common feature of the realistic time streams are short breaks (gaps) in their continuity caused e.g. by the cosmic rays or other transients in the experimental apparatus. Their presence not only prevents a straightforward application of the Fast Fourier techniques (FFT) to the noise estimation but also introduces a whole suite of subtle problems related to their treatment on the level of the time-ordered data manipulation. They also appear to be one of major obstacles in an implementation of the efficient and fast map making algorithms. Therefore restoration of the continuity of the time-ordered data – in a statistically correct way – stands out as one of the important goals of the data analysis on the time domain stage. A required algorithm needs simultaneously to estimate the noise power spectrum and fill the gaps with pure Gaussian noise realization without compromising its correlations throughout the length of the time stream. As described in detail in Stompor et al. (2001) that can be achieved iteratively when on each step of the iterations the gaps are filled with the constrained realization (Hoffman & Ribak 1991) of the Gaussian noise, a power spectrum of which has been determined on the preceding step of the iterations using standard FFT methods.

The noise estimation procedure attempts to estimate a noise ensemble average power spectrum using just one realization of the time stream. That is clearly not sufficient. As a result of the above sketched algorithm one usually ends up with the reliable estimation of the ensemble average noise power spectrum in time domain at sufficiently high frequencies (for \( f \gtrsim 0.1\text{Hz} \)), but the low frequency end is a subject to a non-negligible sampling error. To minimize the effect of the low frequencies we marginalize over the low frequency part of the spectrum. No significant dependence on
the assumed low frequency cut-off (in the range from 0.01 Hz up to 0.4 Hz) has been found in the MAXIMA-1 case.

Adding randomly drawn signal to the data introduces an extra freedom and may undermine the uniqueness of the results. Though depending on the particular noise realization the results may somewhat differ, all of them are statistically equivalent by construction, and no bias is introduced. Moreover, that extra randomness can be minimized by introducing a fictitious gap pixel to the recovered map. As a result we find that the final maps and their power spectra for MAXIMA-1 are robust and not affected by that procedure.

So far, for simplicity we have been assuming that the time-ordered data are noise dominated, however, an extension of the described algorithm for the more general case is straightforward (Ferreira & Jaffe 2000).

3 Map making

The general algebra for the maximum-likelihood map making under the assumption of the Gaussian correlated noise can be found in e.g. Tegmark (1997). In short, if the time stream data \( d \) can be modeled as

\[
d = Am + n,
\]

where \( d \) is the vector of the measurements for the given chunk of the time stream, \( A \) a pointing matrix assigning each of the time samples to an appropriate pixel in the sky and \( n \) - a time stream noise - is a vector of the random Gaussian variables with correlations as given by \( N_t \) - a time domain correlation matrix, then a (maximum-likelihood) map \( m \) and its noise-correlation matrix in pixel domain \( (N_p) \) are given by,

\[
m = \left(ADM D^T A^T \right)^{-1} A M d
\]

\[
N_p = \left(ADM D^T A^T \right)^{-1} \left(ADM N_t M D^T A^T \right) \left(ADM D^T A^T \right)^{-1}
\]

here \( M \) is a positive definite symmetric matrix. \( D \) is a time domain representation of the filters applied to the time stream on the previous stages (e.g. a prewhitening filter) assumed to be orthogonal \( DD^T = 1 \).

If \( M = N_t^{-1} \) then \( m \) is a minimum variance map. The other choices can sacrifice the optimality but being computationally faster. In particular Tegmark (1997) proposed to choose as \( M \) only a circulant part of the \( N_t \) matrix. In the following we discuss a number of approaches and their applications to a realistic MAXIMA-1-like data set.

3.1 Minimum variance variants

Exact implementation. The implementation of the exact minimum variance estimator of the map seems a daunting task. The time stream length
\( n_t \) may easily reach many millions of the time samples making an inversion of the time domain noise correlation matrix – an “\( n_t \)-cubed” process – prohibitive. However, for a MAXIMA-I like experiment with the time stream chunks of the length reaching “only” up to \( 2.5 \times 10^5 \) time samples we found that the implementation is feasible even on a moderately large workstation.

At the heart of that implementation lies a simple observation that the time domain correlation matrix of the stationary and continues time stream is Toeplitz e.g. \( N_t(i, j) = N_t(i - j, 0) \). The inversion of the Toeplitz matrix can be performed in as few as \( n_t^2 \) operations. A clearly feasible task for \( \sim 10^6 \) time samples. (An even faster algorithm exists (e.g. Golub & van Loan 1983) bringing that number down to \( \sim n_t \log^2 n_t \).) An extra gain in a number of operations can be achieved if the noise-correlation length is shorter than the time stream length and the noise-correlation matrix band-diagonal.

The actual time streams are commonly strewn with the gaps and a Toeplitz character of the correlation matrix lost. However the gap filling algorithm, presented earlier, reconstructs the continuity of the time stream and its stationarity. To minimize the entirely spurious content of the gaps being added to the map all the time samples from the gaps are directed to an extra fictitious pixel (“a gap pixel”) which is subsequently rejected (marginalized over) from the map and the pixel domain noise-correlation matrix.

The computational effort for the case when \( D \equiv 1 \) scale with the number of pixels \( n_{\text{pix}} \) and the number of time samples \( n_t \) as:

- noise inverse in time domain: \( N_t \rightarrow N_t^{-1} : \propto n_t^2 \);
- noise inverse in pixel domain: \( N_t^{-1}, A \rightarrow AN_t^{-1}A^T : \propto n_t^2 \);
- noise weighted map: \( N_t^{-1}, A, d \rightarrow AN_t^{-1}d : \propto n_t^2 \);
- noise matrix in pixel domain: \( N_p \rightarrow N_p^{-1} : \propto n_{\text{pix}}^3 \);
- final map: \( N_p^{-1}, AN_t^{-1}d \rightarrow N_p^{-1}AN_t^{-1}d : \propto n_{\text{pix}}^2 \).

In the case of the first three items from the list the substantial savings can be made if the fact that the inverse noise-correlation matrix is assumed to be band-diagonal – an approximation usually well fulfilled for an inverse of a band-diagonal noise-correlation matrix. If the noise-correlation matrix is sparse then the additional savings can also be made.

Memory-wise clearly there is only need to keep a first row of the \( N_t \) matrix and in a typical situation the major requirement would be then set by the size of the \( AN_t^{-1} \) matrix which is \( \propto n_{\text{pix}} n_t \) and the size of the noise correlation for the entire map \( \propto n_{\text{pix}}^2 \). Note that the first of these limits can be alleviated at the expense of the operation counts (there is no need to keep all \( n_{\text{pix}} n_t \) matrix at the same time but then some of the parts may have to be recomputed a multiple number of times).

**Approximate approach.** The way to speed up map making but still aiming at the optimal minimum variance map was proposed by Ferreira & Jaffe
The approximation those authors favor assumes that,

\[ N_t^{-1}(i, j) \simeq \begin{cases} 
N_{Ct}^{-1}(i, j) & \text{if } |i - j| \leq \min \left( n_c/2, l_c \right) \\
0, & \text{otherwise.}
\end{cases} \tag{4} \]

Here \( n_c \) and \( l_c \) are the time chunk and correlation lengths respectively, and a subscript \( C \) denotes a circulant part of the noise correlation matrix defined as \( N_C(i, j) = N_C(i - j, 0) = N_C(n_c - i, 0) \) for \( 0 \leq i \leq n_c/2 \).

This approach is designed to perform the \( N_t \) inversion with the “FFT” speed providing at the same time a good approximation to the exact solution of the previous section. By construction, however, it can be only exact in the absence of the noise correlations.

The operation count changes only for two, but the most time consuming, items from the list in the previous paragraph which now read,

- noise inverse in time domain: \( N_t \rightarrow N_t^{-1} : \propto n_t \log n_t \).
- noise weighted map: \( N_t^{-1}, A, t \rightarrow AN_t^{-1}t : \propto n_t \log n_t \).

The scaling of the noise weighted map computation is given assuming using FFT and a Toeplitz matrix property (Golub & van Loan 1983).

The required memory is set by the size of the noise matrix in pixel domain \( \propto n_{pix}^2 \). It is apparent that if the efficient fast (“\( n_t \log^2 n_t \)” implementation of the Toeplitz matrix inversion is available then the major computational advantage of the approximate method over the exact one would vanish and only the memory requirement would remain as its main asset.

This approximation is implemented in the very successful, publicly available package called MADCAP by Julian Borrill (e.g. Borrill et al. 2000).

### 3.2 Circulant variants

In this case \( M = N_{Ct}^{-1} \) and the noise-correlation matrix in pixel domain can be expressed as follows (Tegmark 1997),

\[ N_p \equiv (ADN_{Ct}^{-1}DTAT) \] \]  
\[ + (ADN_{Ct}^{-1}DTAT)^{-1} (ADN_{Ct}N_{St}N_{Ct}^{-1}DTAT) (ADN_{Ct}^{-1}DTAT)^{-1}, \]

where we decomposed explicitly a noise-correlation matrix in time domain into its circulant \( (N_{Ct}) \) and sparse parts \( (N_{St}) \), \( N_t \equiv N_{Ct} + N_{St} \). The sparse part of the matrix corrects for the presence of “corners” of the circulant matrix and the gaps in the time stream. As advocated by Tegmark (1997) such a matrix should be really sparse and a number of the required operation greatly decreased over the exact approach described above. In the implementation of that author the \( D \) filters are additionally used to enhance sparseness.

The price to pay for it is not only a non-optimal map as a result (the loss of the precision is hoped not to be substantial here), but also definitely more
complicated algebra to be implemented especially in a presence of large number of gaps in the time stream. If no assumption about the band-diagonality is made then the computational costs scale as follows:

- noise inverse in time domain: $N_t^{-1} \propto n_t \log n_t$;
- noise inverse in pixel domain:
  - circulant part: $N^{-1}_t, A \rightarrow AN^{-1}_t A^T : \propto n_t^2$;
  - sparse part: $N_{St}, M, A \rightarrow AMN_{St}MA^T : \propto n_{pix} n_t^2$;
- noise weighted map: $N^{-1}_t, A, d \rightarrow AN^{-1}_td : \propto n_t \log n_t$;
- noise matrix in pixel domain: $N_p \rightarrow N_{p}^{-1} : \propto n_{pix}^3$.

We have omitted the filter matrix $D$ in above expressions assuming that both $M$ and $D$ are circulant and therefore their product can be computed with the “FFT speed”. The effort is dominated usually by the sparse part computation ($n_t > n_{pix}$) though the above scaling includes only the fastest growing term which can not be suppressed if only the band diagonality of the $N_t$ matrix is assumed. Memory required for performing that task is $\propto n_{pix} n_t$.

Clearly if no more approximations are made this approach is far from competitive with any of those discussed above. One possible way of improving on that is to recognize the fact that if $N_{Ct}$ is band-diagonal (plus of course non-zero corners to fulfill the circulancy criterion) then also $N_{Ct}^{-1}, D^TN_{Ct}^{-1}$ and $D^TN_{Ct}^{-1}D$ are usually approximately band-diagonal.

Another possible approximation, referred to hereafter, is to neglect completely the sparse correction in the expression for the noise-correlation matrix in the pixel domain. Clearly the operational cost goes down to $n_t^2$ or $n_{pix}^3$ – whichever larger – and is usually $\propto n_{pix}^3$ especially if a band diagonal character of the inverse of $N_{Ct}^{-1}$ is assumed. In such a case the method achieves the performance comparable with that of Ferreira & Jaffe (2000).

### 3.3 Comparison and assessment

In the paper by Stompor et al. (2001) we show that though none of discussed above approximations reproduces perfectly the map and the noise-correlation matrix in the pixel domain ($N_p$), it preserves the statistical information contained in the data – e.g. 2-point correlation properties – correctly. In fact we find that both the approximations, as described above, fare very well in recovering the anisotropy power spectrum with no apparent systematic bias.

Clearly the performance of these approximations needs to be further tested if other statistics are to be applied to the maps.

Out of the two exact methods the minimum variance implementation described here performs generally much better. The very fact that the exact maximum-likelihood method can be applied to the data sets of the maximum size is of importance for the further development, tests and validation of the approximate algorithms. The simple operation counts and memory requirements are clearly great assets of those methods and major stumbling blocks for the exact approach to go far beyond the size of the current data sets.
It is worth noticing that the gaps filling procedure described in the beginning is important for all the described methods: in a case of the exact methods (optimal and circulant) it facilitates the practical feasibility of the implementation. For the approximate methods it improves the accuracy and robustness of the involved approximations.

4 Summary

We briefly have discussed here the making of the CMB temperature anisotropy maps from the time-ordered data. We have argued that for the data set with up to a few millions of time samples producing the maps of up to a few thousands of the pixels is readily possible without need for sacrificing the optimal character of the final map. Moreover accurate and efficient approximate methods also exist. Those have been tested through the direct comparison with the exact method on the realistic simulations and the MAXIMA-I data set and can become a starting point for developing algorithms capable of coping with still bigger and more complicated data sets. It seems likely that the hard problem of this kind of data analysis will be accounting on all kinds of imperfectness of a real data set in the statistically sound way and without the substantial increase of the operational count, rather than solely a sheer size of a data set itself. As an example here we have demonstrated how to incorporate the presence of short discontinuities in the time-ordered data. The other common problems are to be discussed in Stompor et al. (2001).

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