DYNAMICAL COMPONENT ANALYSIS (DYCA):
DIMENSIONALITY REDUCTION FOR HIGH-DIMENSIONAL DETERMINISTIC
TIME-SERIES

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ABSTRACT
Multivariate signal processing is often based on dimensionality reduction techniques. We propose a new method, Dynamical Component Analysis (DyCA), leading to a classification of the underlying dynamics and - for a certain type of dynamics - to a signal subspace representing the dynamics of the data. In this paper the algorithm is derived leading to a generalized eigenvalue problem of correlation matrices. The application of the DyCA on high-dimensional chaotic signals is presented both for simulated data as well as real EEG data of epileptic seizures.

Index Terms— Multivariate signal processing, data analysis, dimensionality reduction, time series, EEG, chaos, generalized eigenvalue problem

1. INTRODUCTION

Classic dimensionality reduction techniques, like principal component analysis (PCA) [1] or independent component analysis (ICA) [2], are widely used as a preprocessing step in the analysis of multivariate time-series. PCA aims at projections leading to the largest possible variances of the signal in each direction, but the obtained temporal signals are not optimized to describe the dynamics of the signal. ICA on the other hand relies on the assumption that the time-series can be split in mutually independent signals. There are other approaches like forecastable component analysis (ForeCA) [3] relying on forecastability measure, approaches based on multivariate autoregressive models [4], or approaches based on Granger causality [5]. An overview of conventional techniques is presented in [6].

As these techniques always rely on some sort of stochastic model assumption, they are not very well suited for the dimensionality reduction of multivariate time-series data with a strong deterministic part. Reduction of dimensionality of multivariate time-series is e.g. relevant for signals sampled by more sensors than the dimensionality of the underlying system. A typical example of such systems is the electroencephalogram of epileptic seizures, where one has many sensors but a very regular, low-dimensional behaviour of the measured system. The dimensionality reduction technique we introduce relies on a special deterministic model assumption, suitable for example for the reduction of some chaotic time-series. The proposed method is quite similar to the methods of principal interacting and principal oscillation patterns (PIPs and POPs) [7] used in geophysical sciences. In some sense the method we are presenting can be interpreted as a generalization of the PIPs and POPs method. Furthermore there are some technical similarities with methods for transfer operator approximation based on delay coordinates, which are applied in fluid or molecular dynamics [8].

Chaotic time-series forecasting by reservoir computing has recently resulted in very interesting results, outperforming all tools available up to now [9][11]. We suggest the proposed dimensionality reduction technique as an adequate preprocessing step for reservoir computing of high-dimensional spatio-temporal data.

The structure of the paper is as follows. First we derive Dynamical Component Analysis (DyCA) using variational calculus. It is shown that DyCA corresponds to a generalized eigenvalue problem. The eigenvalues of the generalized eigenvalue problem tell the quality of a fit of the data to a system of ordinary differential equations of special form. The application of DyCA to high-dimensional simulated data based on the Rössler system is presented in Section 3. In Section 4 EEG data of epileptic seizures expecting Shilnikov chaos are investigated by the proposed method.
2. DYNAMICAL COMPONENT ANALYSIS

Let \( q(t) \in \mathbb{R}^N \) be a multivariate time-series with its dynamics being described by a low-dimensional system of ordinary differential equations. I.e., we can decompose the signal in time-dependent amplitudes \( x_i(t) \) and vectors \( w_i \in \mathbb{R}^N \),

\[
q(t) = \sum_{i=1}^{n} x_i(t)w_i,
\]
with the dynamics of the amplitudes described by the set of differential equations

\[
\dot{x}_1 = \sum_{k=1}^{n} a_{1,k}x_k
\]

\[
\vdots
\]

\[
\dot{x}_m = \sum_{k=1}^{n} a_{m,k}x_k
\]

and

\[
\dot{x}_{m+1} = f_{m+1}(x_1, x_2, \ldots, x_n)
\]

\[
\vdots
\]

\[
\dot{x}_n = f_n(x_1, x_2, \ldots, x_n),
\]

where \( n \ll N \) and \( f_j \) are non-linear smooth functions. We assume that we neither know the parameters \( a_{i,k} \) nor the exact form of the functions \( f \).

To generate projection vectors \( u_i, v_j \in \mathbb{R}^N \) approximating the above mentioned amplitudes \( x_i(t) \) we minimize the least square error cost function

\[
D(u, v, a) = \frac{\langle \|q^\top u - \sum_j a_j q^\top v_j \|_2^2 \rangle_t}{\langle \|q^\top u\|_2^2 \rangle_t}
\]

where \( \langle \cdot \rangle_t \) denotes the time average. Denote the correlation matrices of the signal with itself, of the signal with its derivatives, and the signal derivatives with itself by \( C_0 = \langle qq^\top \rangle_t, C_1 = \langle q^\top q \rangle_t, \) and \( C_2 = \langle q^\top q \rangle_t, \) respectively. Then we can rewrite the cost function as

\[
D(u, v, a)
= \frac{\langle \|q^\top u - \sum_j a_j q^\top v_j \|_2^2 \rangle_t}{\langle \|q^\top u\|_2^2 \rangle_t}
= \frac{\langle (q^\top u - \sum_j a_j q^\top v_j)(q^\top u - \sum_j a_j q^\top v_j) \rangle_t}{\langle (q^\top u)(q^\top u) \rangle_t}
= \frac{(u^\top C_2u) - 2 \sum_j a_j (u^\top C_1v_j) + \sum_{j,k} a_j a_k (v_j^\top C_0v_k)}{u^\top C_2u}
= 1 - 2 \sum_j a_j \frac{u^\top C_1v_j}{u^\top C_2u} + \sum_{j,k} a_j a_k \frac{v_j^\top C_0v_k}{u^\top C_2u}.
\]

The minimum of \( D \) can be analytically calculated using variation with respect to each variable. For this the partial derivatives with respect to the variables \( u, v, \) and \( a \) are derived and their minima determined.

The partial derivative with respect to \( u^\top \) is

\[
\frac{\partial D}{\partial u^\top} = -2 \sum_j a_j C_1v_j(u^\top C_2u) - 2(u^\top C_1v_j)C_2u \]

\[
\frac{D}{(u^\top C_2u)^2}
- 2 \sum_{j,k} a_j a_k (v_j^\top C_0v_k)C_2u \]


Setting the derivative to zero leads to

\[
(2 \sum_j a_j(u^\top C_1v_j) - \sum_{j,k} a_j a_k (v_j^\top C_0v_k))C_2u
= \sum_j a_j C_1v_j.
\]

Let \( \mu = 2(\sum_j a_j(u^\top C_1v_j) - \sum_{j,k} a_j a_k (v_j^\top C_0v_k)) \) and \( \tau = u^\top C_2u \) then \( D \) reads

\[
\mu C_2u = \tau C_1 \sum_j a_j v_j.
\]

The partial derivative with respect to \( v_r \) is

\[
\frac{\partial D}{\partial v_r} = -2a_r \frac{u^\top C_1}{u^\top C_2u} + 2a_r \sum_j a_j \frac{v_j^\top C_0}{u^\top C_2u}
\]

For \( \frac{\partial D}{\partial a_r} = 0 \) we therefore obtain

\[
u^\top C_1 = (\sum_j a_j v_j^\top )C_0.
\]

Calculating the partial derivative with respect to \( a_r \) results in

\[
\frac{\partial D}{\partial a_r} = -2 \frac{u^\top C_1v_r}{u^\top C_2u} + 2 \sum_j a_j \frac{v_j^\top C_0v_r}{u^\top C_2u}
\]

and setting the derivative to 0 leads to

\[
u^\top C_1v_r = \sum_{j} a_j v_j^\top C_0v_r.
\]

Note that both multiplying \( 8 \) from left with \( u^\top \) and \( 10 \) from right with \( v \) lead to \( 12 \) proving the consistency of the calculation.

Assuming the existence of the inverse \( C_0^{-1} \) of the correlation matrix \( C_0, \) \( 10 \) can be rewritten as

\[
\sum_j a_j v_j = C_0^{-1} C_1^\top u.
\]

Inserting \( 13 \) into \( 8 \) a generalized eigenvalue problem is obtained

\[
C_1 C_0^{-1} C_1^\top u = \lambda C_2 u,
\]
where $\lambda = \frac{\mu}{\tau}$.

Inserting (8) and (10) into (5) yields
\[
D_{\text{min}} = 1 - 2 \sum_j a_j u_j^T C_j v_j + \sum_{j,k} a_j a_k v_j^T C_0 v_k \frac{u_j^T C_2 u}{u_j^T C_2 u} \\
= 1 - \frac{2}{\mu} \sum_j a_j u_j^T C_j v_j + \sum_{j,k} a_j a_k v_j^T C_0 v_k \frac{u_j^T C_2 u}{u_j^T C_2 u} \\
= 1 - \frac{\mu}{\tau} \\
= 1 - \lambda.
\]

That means, similar to principal component analysis (PCA), the eigenvalues $\lambda_i$ of the generalized eigenvalue problem (eq.14) indicate the quality of the least-square-fit of the linear differential equations (2). The eigenvalue spectrum allows for an identification of amplitudes interacting as $\dot{x}_i(t) = \sum_j a_i x_j(t)$ by projecting the signal $q(t)$ onto the corresponding eigenvector $u_i$, i.e. $x_i(t) = q(t)^T u_i$. By choosing an appropriate threshold one can obtain a projection subspace spanned by the $m$ corresponding eigenvectors $u_i$. Calculation of $C_2 u_i$ by (9) leads to a set of $m$ vectors (as linear combination of the unknown vectors $v_i$) which then span another $m$ dimensional subspace. The span of these both $m$-dimensional subspaces
\[
\text{span}\{u_1, \ldots, u_m, C_1^{-1} C_2 u_1, \ldots, C_1^{-1} C_2 u_m\} = \mathbb{R}^n
\]
approximates the complete $n$-dimensional subspace in which the system evolution can be described by a set of differential equations (eq.2 and 3) if $m$ is not too small. Obviously, if $m < n/2$ this would not work.

Note that the size of the matrices $C_0, C_1$ and $C_2$ is $N \times N$, which is small compared to the length of a typical time-series. Hence the application of DyCA as a preprocessing step is computationally cheap. The invertibility of $C_0$ relies on the different sensors measuring independent signals. In most applications this is the case due to inherent measuring noise.

3. APPLICATION TO THE RÖSSLER SYSTEM

The Rössler attractor [12] is a strange attractor given by the system of ordinary differential equations
\[
\begin{align*}
\dot{x}_1 &= -x_2 - x_3 \\
\dot{x}_2 &= x_1 + ax_2 \\
\dot{x}_3 &= b - cx_3 + x_1 x_2,
\end{align*}
\]
with $a = 0.15, b = 0.2, and c = 10$. For the application of DyCA a trajectory of this system was obtained using a $(4, 5)$-Runge-Kutta integration method. Then the data was embedded in a 25-dimensional space with additional multiplicative Gaussian noise. We used an exemplary signal to noise ratio of $15\text{dB}$. As definition of signal to noise ratio we rely on the formula $SNR = \frac{A}{\sigma}$, where $A$ is the signal mean and $\sigma$ the standard deviation of the noise.

The generalized eigenvalue spectrum of DyCA applied to the 25-dimensional simulated data is illustrated in Fig. 1. As expected, according to (17), the two largest eigenvalues are equal to one due to the two linear equations in (17). The third eigenvalue is evidently below one according to one nonlinear equation in (17). The span of the projection vectors $\text{span}\{u_1, u_2, C_2 u_1, C_2 u_2\}$ is, with respect to numerical tolerances, of dimension 3. Projecting with the projection vectors $u_1, u_2$ and $v_2 = C_2 u_2$ leads to the phase-portrait illustrated in Fig. 2. In Fig. 3 and 4 the phase Portraits of the data obtained by dimensionality reduction using PCA and ICA are shown. Subjective comparison of the obtained figures suggests that the inherent dynamics of the data is more accurately represented and the noise is reduced in a larger amount in the data projected with DyCA than in the data projected with PCA or ICA. The projections obtained by PCA resemble the results one would obtain by picking three time-series out of the twenty-five of the original multivariate signal at random.

4. APPLICATION TO EPILEPTIC EEG DATA

A typical example where the assumptions (2) and (3) are fulfilled is the EEG data of an epileptic seizure. This is due to the conjectured appearance of Shilnikov chaos in epileptic seizures. Using bifurcation analysis the existence of Shilnikov chaos in various theoretical models was shown by van Veen and Liley [13]. In [14] a system of ordinary differential equations of the form
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= f(x_1, x_2, x_3),
\end{align*}
\]
with $f$ being a non-linear polynomial function, was assumed to model epileptic encephalograms. Since this model relies on two linear and one non-linear equations, we assume that the conditions on the applicability of DyCA are fulfilled.

As data we considered a set of EEG data containing stages before, after and during an epileptic seizure. The data was sampled using 25 sensors with 256 Hertz sample rate. The signal to noise ratio is approximately 16dB. As preprocessing step the data was bandpass filtered with a zerophase filter with cut-off frequencies of 0.5 and 30 Hertz. The data was partitioned in windows of one second length. Then DyCA was applied on each window. As can be seen in Fig. 5 the assumption of a system of the form (18) during an epileptic seizure can be accepted, since the two largest eigenvalues are nearly 1 during the absence.

Since DyCA is proposed as a preprocessing method for machine learning applications, we need to show that the pro-
Fig. 6. Projected EEG time-series in phase-space using the projection obtained by DyCA on the dataset.

Fig. 7. Projected EEG time-series in phase-space using the projection from Fig. 6 on another window.

projection calculated on one window is able to represent other parts of the time-series, as well. Fig. 6 and 7 show that if one uses the projection obtained on one window of the data to project another window, the underlying dynamics is still preserved. Hence, if the applicability assumptions are fulfilled, DyCA is suitable as preprocessing tool for analysis of high-dimensional deterministic time-series.

5. DISCUSSION AND CONCLUSION

We conclude that DyCA is a suitable tool for dimensionality reduction of high-dimensional time-series, provided the underlying dynamics can be described by a system of ordinary differential equations of the form (2) and (3). It has been shown that DyCA can get rid of noise more efficiently than PCA and ICA. Furthermore DyCA is able to preserve the dynamics of spike-waves in epileptic EEG data. Since the calculation of the projection matrices of DyCA is simply solving a generalized eigenvalue problem, the procedure is computationally cheap. Hence it is suggested to establish DyCA as a more reliable alternative to PCA as preprocessing step in the analysis of multivariate deterministic time-series.

Further studies are needed to show that DyCA improves the prediction ability of reservoir computing approaches. These will be conducted in future works.

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