NONEXISTENCE OF GENERALIZED BENT FUNCTIONS AND THE QUADRATIC NORM FORM EQUATIONS

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Abstract. We present a new result on the nonexistence of generalized bent functions (GBFs) from \((\mathbb{Z}/t\mathbb{Z})^n\) to \(\mathbb{Z}/t\mathbb{Z}\) (called type \([n, t]\)) for a large class. Assume \(p\) is an odd prime number. By showing certain quadratic norm form equations having no integral points, we obtain a universal result on the nonexistence of GBFs with type \([n, 2p^e]\) when \(p\) and \(n\) satisfy a certain inequality, and by computational methods with a widely accepted hypothesis, Generalized Riemann Hypothesis, we also achieve some results on the nonexistence of GBFs for relatively small \(p\).

1. Introduction

A Boolean function \(F : (\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}\) is bent if and only if the discrete Fourier transform of its sign function \((-1)^F\) takes only two values \(\pm 2^{n/2}\). Bent functions are initially introduced by Rothaus [Rot76] and deeply studied by him and Dillon [Dil72, Dil74]. Subsequently, a significant amount of research has been dedicated to this topic, resulting in the extension of the definition of bent functions in various manners. This has given rise to several classes of generalized bent functions that share many of the useful properties of the original definition [BCH+12, Car93, CD04, CDL04, DLC++06, KSW85, MMMS17, Nyb91, OSW+82, TXQF17].

Bent functions are interesting combinatorial objects and have applications in design theory, coding theory and cryptography. Bent functions are maximally nonlinear Boolean functions, which can introduce nonlinearity in the construction of stream ciphers and block ciphers. However, bent functions are not balanced. Directly using them will lead to a statistical correlation between the plaintext and the ciphertext. Researchers generalized the definition of bent to semi-bent, balanced (semi-) bent and partially bent to enhance other cryptographic properties by slightly decreasing the nonlinearity. For a more detailed history of bent functions, one can refer to [CM16].

Let \(n \geq 1, t \geq 2\) be two integers, \(\mathbb{Z}/t\mathbb{Z}\) be the residue ring modulo \(t\) and \(\zeta_t = \exp(2\pi \sqrt{-1}/t)\) be a primitive \(t\)-th root of unity. Kumar et al. [KSW85] generalized the binary bent functions to the \(t\)-ary ones as follows.

Definition 1.1. A function \(F : (\mathbb{Z}/t\mathbb{Z})^n \to \mathbb{Z}/t\mathbb{Z}\) is called a Generalized Bent Function (GBF) with type \([n, t]\) if

\[
F(\lambda)F(\lambda) = t^n
\]

for every \(\lambda \in (\mathbb{Z}/t\mathbb{Z})^n\), where

\[
F(\lambda) = \sum_{x \in (\mathbb{Z}/t\mathbb{Z})^n} \zeta_t^{F(x)} \cdot \zeta_t^{-x \cdot \lambda}
\]
is the discrete Fourier transform of the function $\zeta_t^{F(x)}$, $x \cdot \lambda$ is the standard dot product, and $\overline{F(\lambda)}$ is the complex conjugate of $F(\lambda)$.

In this paper, we focus on the nonexistence of GBFs. For $t = 2$, Rothaus \cite{Rot76} proved that GBFs with type $[n, 2]$ exist. For $t > 2$, Kumar et al. \cite{KSW85} constructed GBFs except the case where $n$ is odd and $t \equiv 2 \pmod{4}$, with which type there is no GBF found until now. Hence, we only consider the case where $n$ is odd and $t = 2N$ with $2 \mid N \geq 3$.

Let $\varphi$ denote the Euler’s totient function. For $(a, N) = 1$, let ord$_N(a)$ denote the order of $a$ in the multiplicative group $(\mathbb{Z}/N\mathbb{Z})^\times$. There are many results on the nonexistence of GBFs under some extra constraints:

1. (Kumar \cite{KSW85}) type $[n, 2N]$ where $2 \nmid N \geq 3$, $2^s \equiv -1 \pmod{N}$ for some integer $s \geq 1$, and $n$ is odd;
2. (Pei \cite{Pei93}) type $[1, 2 \times 7]$;
3. (Ikeda \cite{Ike99}) type $[1, 2p_1^{e_1} \ldots p_g^{e_g}]$ where $p_1, \ldots, p_g$ are distinct primes and $p_i^{e_i} \equiv -1 \pmod{N/p_i^{e_i}}$ for some $s_i, i = 1, \ldots, g$;
4. (Feng \cite{Fen01}) type $[n < m/s, 2p]$, where $n$ is odd, $p \equiv 7 \pmod{8}$ is a prime, $s = \frac{\varphi(p)}{\text{ord}_p(2)}$ and $m$ is the smallest odd positive integer s.t. $x^2 + py^2 = 2^{m+2}$ has integral solutions;
5. (Feng et al. \cite{FJL03a}, \cite{FJL03b}) various classes with type $[n < m, 2p_1 p_2]$ where $n$ is odd, $p_1, p_2$ are two distinct primes satisfying certain conditions and $m$ is an upper bound for $n$;
6. (Jiang and Deng \cite{JD15}) type $[3, 2 \times 23^e]$;
7. (Li and Deng \cite{LD17}) type $[m, 2p^r]$ where $p \equiv 7 \pmod{8}$ is a prime with ord$_{p^r}(2) = \varphi(p^r)/2$ and $m$ is the same as in (4);
8. (Lv and Li \cite{LL17}) type $[m, 2p_1^{e_1} p_2^{e_2}]$ where $p_1 \equiv 7 \pmod{8}$ and $p_2 \equiv 5 \pmod{8}$ are two primes satisfying some conditions and $m$ is the smallest odd positive integer s.t. $x^2 + p_1 y^2 = 2^{m+2}$ has integral solutions;
9. (Lv and Li \cite{LL17}) type $[1 \leq n \leq 3, 2 \times 31^c]$ and $[1 \leq n \leq 5, 2 \times 151^c]$ where $e$ and $n$ are positive integers and $n$ is odd.

On the nonexistence of GBFs, some of the previous results were sporadic, some of them had too many constrains on parameters, and none of them included the cases of type $[n, 2N]$ with $N = p^e$, where $n, p, e$ are described in (4.4), i.e., $p \equiv 1 \pmod{8}$ is a prime, $n \geq 3$ and $f = \text{ord}_p(2)$ is odd.

The main result of this paper, Theorem 4.1, provides a universal result on the nonexistence for type $[n, 2p^r]$ when $p$ and $n$ satisfy a certain inequality. In particular, we partially solve the problem on the nonexistence of GBFs for the cases of (4.4). In addition, we show that for a fixed $n$, there are infinitely many such $p$'s (Corollary 4.2) under the Extended Riemann Hypothesis. We also use a computational method with a similar hypothesis, Generalized Riemann Hypothesis, to give some results on the nonexistence of GBFs for relatively small $p$.

The paper is organized as follows. Basic results and facts needed in this paper are briefly introduced in Section 2. Then in Section 3 we study the integral solvability of a class of quadratic norm form equations over subfields of cyclotomic fields, which is the main tool we shall use. As an application, we prove Theorem 4.1 in Section 4. We also show that there are infinitely many primes $p$ that satisfy the conditions of the theorem. Section 5 is dedicated to some additional results on the nonexistence for relatively small $p$ obtained by computational methods.
2. Preliminaries

2.1. Basic results on number theory. Our methods for proving the nonexistence of GBFs with certain type involve algebraic number theory, such as cyclotomic fields and their subfields, ideals, class groups and Galois actions. The standard references are [Jan96] and [Was97]. Below we briefly introduce some basic results.

Assume $F$ is a number field. Denote by $\mathfrak{o}_F$ the ring of integers in $F$. It is a Dedekind domain. One can consider the fractional ideals in $F$. A fractional ideal is an $\mathfrak{o}_F$-module $\mathfrak{a}$ contained in $F$ such that there exists an element $\alpha \in \mathfrak{o}_F$ for which $\alpha \mathfrak{a} \subseteq \mathfrak{o}_F$. Denote by $I_F$ the set of nonzero fractional ideals of $F$, which is a free abelian group generated by the prime ideals under multiplication. A principal fractional ideal means a fractional ideal of the form $\beta \mathfrak{a}_F$ where $\beta \in F$. The set of all nonzero principal fractional ideals, denoted by $P_F$, is a subgroup of $I_F$, and the quotient $I_F/P_F$, denoted by $\text{Cl}(F)$, is called the class group of $F$. Class groups play an important role in classical algebraic number theory. One of the nontrivial facts is that $\text{Cl}(F)$ is a finite abelian group for all $F$. Denote by $h(F)$ the cardinality of $\text{Cl}(F)$, which is called the class number of $F$.

We also need some basis results on the decompositions of prime ideals in extension fields and the decomposition fields. We only consider the unramified prime ideals in a cyclotomic extension. For general cases, we refer the readers to [Jan96, Section I.6, Section III.7]. Let $m$ be an positive integer with $m \not\equiv 2 \pmod{4}$ and $K = \mathbb{Q}(\zeta_m)$ be a cyclotomic field. Then $K$ has degree $r = \varphi(m)$ over $\mathbb{Q}$, where $\varphi$ denotes the Euler’s totient function. One has $\mathfrak{o}_K = \mathbb{Z}[\zeta_m]$. For a prime $q \nmid m$, the ideal $q\mathfrak{o}_K$ will split into $g$ distinct prime ideals, namely

$$q\mathfrak{o}_K = \prod_{i=1}^{g} \mathfrak{q}_i,$$

Since $K/\mathbb{Q}$ is a Galois extension, the extension degree $[q\mathfrak{o}_K/\mathfrak{q}_i : \mathbb{Z}/q\mathbb{Z}] = \text{ord}_m(q)$ is the same for each $i$, usually denoted by $f$, called relative degree of $q$. One has $r = fg$. Since $\text{Gal}(K/\mathbb{Q})$ is cyclic, there is a unique subfield $D \subseteq K$ having degree $g$ over $\mathbb{Q}$ called the decomposition field of $q$, such that

$$q\mathfrak{o}_D = \prod_{i=1}^{g} \mathfrak{q}_i,$$

where $q\mathfrak{o}_K = \mathfrak{Q}_i$ and $\mathfrak{q}_i$ is a prime ideal in $D$ having relative degree 1 over $\mathbb{Q}$.

Now let us consider $K = \mathbb{Q}(\zeta_p^e)$, where $p$ is an odd prime and $e$ is a positive integer. Since $\text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/p^e\mathbb{Z})^\times$ is cyclic, for any positive integer $s \mid [K : \mathbb{Q}]$, there is a unique subfield of $K$ having degree $s$ over $\mathbb{Q}$. Based on this fact, one can directly obtain the following two results: One is that if $q$ is a prime distinct from $p$ and $f = \text{ord}_{p^e}(q)$, the subfield having degree $g = [K : \mathbb{Q}]/f$ over $\mathbb{Q}$ must be the decomposition field of $q$. The other is that the unique quadratic subfield contained in $K$ is $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$. Since the discriminant of $K$ is $\pm p^{e-1}(p^{e-1} - 1)$, we know that $p$ is the only ramified prime of $K$, and thus is the only ramified prime of the unique quadratic subfield contained in $K$. By a direct calculation, one can find $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$ is the only candidate.

To determine the multiplicative order of $a$ modulo $p^e$, one can apply the following lemma.

**Lemma 2.1.** Let $p$ be an odd prime and $a$ be an integer with $f = \text{ord}_p(a)$. If

$$a^f \not\equiv 1 \pmod{p^2},$$

then $\text{ord}_{p^e}(a) = fp^{e-1}$ for all $e > 1$. 

Proof. One can easily obtain the lemma by following the strategy in [Has78, Chapter 4.5] of proving that $1 + p \mod p^e$ generates a multiplicative subgroup of order $p^{e-1}$. For convenience of the readers, we provide a proof here.

Since $f \mid p - 1$ and $(p - 1, p^{e-1}) = 1$, it suffices to show that

$$a^{fp^{e-1}} \equiv 1 \pmod {p^e}$$

and

$$a^{fp^{e-2}} \not\equiv 1 \pmod {p^e}.$$  

We claim that for each $r \geq 2$,

$$a^{fp^{e-2}} \equiv p^{r-1}t + 1 \pmod {p^r} \quad \text{for some } t \in \mathbb{Z} \text{ not divisible by } p,$$

which implies (2.3) when $r = e$ and (2.2) when $r = e + 1$.

We shall prove this claim by induction on $r$. Since we have $a^f \equiv 1 \pmod p$ and (2.4), the claim holds for $r = 2$. Suppose it holds for $r \geq 2$. Then we have for some $k \in \mathbb{Z}$ that

$$a^{fp^{e-1}} = (p^r k + p^{r-1} t + 1)p^{e-1} = (p^{r-1} t + 1)p^r k + p^r k + \binom{p}{2}(p^{r-1} t + 1)p^{r-2}(p^r k)^2 + \cdots + (p^r k)^p \equiv (p^{r-1} t + 1)^p = (p^{r-1} t)^p + \cdots + \binom{p}{2}(p^{r-1} t)^2 + p \times p^{r-1} t + 1 \equiv p^r t + 1 \pmod {p^{r+1}},$$

i.e., it also holds for $r + 1$. This completes the proof for the lemma. \hfill \Box

To determine the integral elements in quadratic extension fields, we have the following lemma.

**Lemma 2.2.** Let $E = F(\sqrt{d})$ be an arbitrary quadratic extension of number fields, where $d \in \mathfrak{o}_F$ and $d \mathfrak{o}_F$ factors into a product of distinct prime ideals of $F$. Then every element of $\mathfrak{o}_E$ is of the form $(x + y\sqrt{d})/2$ for some $x, y \in \mathfrak{o}_F$.

*Proof.* For any $\beta \in \mathfrak{o}_E$, write $\beta = a + b\sqrt{d}$ where $a, b \in F$. We may assume $b \neq 0$. The minimal polynomial of $\beta$ over $F$ is

$$T^2 - 2aT + a^2 - db^2 \in F[T].$$

Since $\beta \in \mathfrak{o}_E$, we have $2a, a^2 - db^2 \in \mathfrak{o}_F$. Hence $a = x/2$ and $db^2 = z/4$ for some $x, z \in \mathfrak{o}_F$. Suppose the prime decomposition of $d$ is

$$d \mathfrak{o}_F = \mathfrak{p}_1 \cdots \mathfrak{p}_s,$$

and

$$2b \mathfrak{o}_F = a \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_s^{r_s}$$

for some fractional ideal $a$ of $F$ coprime with $\mathfrak{p}_1 \cdots \mathfrak{p}_s$ and some integers $r_1, \ldots, r_s$. Then we have

$$z = (2b)^2 d \mathfrak{o}_F = a^2 \mathfrak{p}_1^{2r_1 + 1} \cdots \mathfrak{p}_s^{2r_s + 1} \subseteq \mathfrak{o}_F.$$

It follows that $a \subseteq \mathfrak{o}_F$ and $r_1, \ldots, r_s \geq 0$, which implies $2b = y$ for some $y \in \mathfrak{o}_F$. The proof is complete. \hfill \Box

We also need a result on asymptotic estimation of the number of specific primes. Let $N_{\alpha}^{(n)}(x) = \{p \leq x \mid \text{ord}_p(a) = (p - 1)/n\}$. 


Lemma 2.5 (Self-conjugated [Was97]). Let \( p \) be a prime integer, \( m = p' m' \) where \( l \geq 0 \) and \( (p, m') = 1 \). We call \( p' \) to be self-conjugated with respect to \( m \) if there exists \( s \in \mathbb{Z} \) such that \( p^s \equiv -1 \pmod{m'} \).

Definition 2.4 (Definition 2.4). Let \( (p, m) = 1 \), \( q \equiv 1 \pmod{4} \), \( (q, m) = 1 \), \( q \) is self-conjugated with respect to \( m \) then \( \beta = \beta' = n' \).

Lemma 2.5 (LF16 Lem. 2.4]). Let \( n \geq 2 \), \( m \geq 3 \), \( K = \mathbb{Q}(\zeta_m) \). Suppose that there exists \( \alpha \in \mathfrak{o}_K \) such that \( \bar{\alpha} = n \).

\( \beta' = n' \) is a new power of a prime number \( q \) and \( (q(q - 1), m) = 1 \). Let \( D \) be the decomposition field of \( q \) in \( K \). Then there exists \( \beta \in \mathfrak{o}_K \) such that \( \beta^2 \in \mathfrak{o}_D \) and \( \beta = n \).
3. Nonsolvability of quadratic norm form equations

3.1. New results on nonsolvability of norm form equations. Let $E$ be a complex subfield of the $N$-th cyclotomic field $\mathbb{Q}(\zeta_N)$ and $F = E \cap \mathbb{R}$ be its maximal real subfield. First, consider the case $N = p$, where $p$ is a prime. Since $E$ is complex, we know that $E/F$ is quadratic. We consider the nonexistence of integral points of the norm equation \((3.1)\) in the case where $a$ is a rational prime power $q^n$. For several certain classes of $N$ and $q^n$, this problem was discussed in [Fen01, FL03a, FL03b, LF16, LL17, Lv17]. We give the following theorem.

**Theorem 3.1.** Let $p$ and $q$ be two distinct primes, $n$ be an odd positive integer and $K = \mathbb{Q}(\zeta_p)$. Assume $E \subseteq K$ is complex and $F = E \cap \mathbb{R}$. Suppose $[F : \mathbb{Q}] = k$. If

\begin{equation}
(3.1) \quad p > (4q^n)^k,
\end{equation}

then the quadratic norm form equation $N_{E/F}(\alpha) = q^n$ has no solution with $\alpha \in \mathfrak{o}_E$.

**Proof.** By assumption \((3.1)\), $p$ is an odd prime. Let $\gamma = \zeta_p - \zeta_p^{-1}$. We claim that $N_{K/\mathbb{Q}}(\gamma) = p$. Rewrite $\gamma$ as $\zeta_p(1 - \zeta_p^{-2})$. Note that $N_{K/\mathbb{Q}}(1 - \zeta_p)$ equals to the evaluation of $\prod_{i=1}^{p-1}(x - \zeta_p^i) = x^{p-1} + x^{p-2} + \cdots + 1$ at $1$. Thus, $N_{K/\mathbb{Q}}(1 - \zeta_p) = p$. Since $p$ is an odd prime, $\zeta_p \mapsto \zeta_p^{-2}$ is an automorphism. We have $N_{K/\mathbb{Q}}(1 - \zeta_p^{-2}) = N_{K/\mathbb{Q}}(1 - \zeta_p) = p$. Combining with the fact that $N_{K/\mathbb{Q}}(\zeta_p) = 1$ for any odd prime $p$, we have $N_{K/\mathbb{Q}}(\gamma) = p$.

Let $\xi = N_{K/E}(\gamma)$ and $\delta = N_{E/F}(\xi)$. Thus we have

\[ N_{E/\mathbb{Q}}(\xi) = N_{K/\mathbb{Q}}(\gamma) = p. \]

We claim that $E = F(\xi) = F(\sqrt{\delta})$. Actually, if $\xi \in F$,

\[ p = N_{E/\mathbb{Q}}(\xi) = N_{F/\mathbb{Q}}(\xi^2) = N_{E/\mathbb{Q}}(\xi^2). \]

Since $\gamma \in \mathfrak{o}_K$, $\xi = N_{K/E}(\gamma) \in \mathfrak{o}_E$, combining with the assumption $\xi \in F$, we have $\xi \in F \cap \mathfrak{o}_E = \mathfrak{o}_F$. It follows that $N_{F/\mathbb{Q}}(\xi) \in \mathbb{Z}$, which yields a contradiction. This shows that $E = F(\xi)$. Since $E$ is complex, we know that $[K : E]$ is odd (if $[K : E]$ is even, then Gal($K/E$) would contain the complex conjugation, which is impossible since $E$ is complex) and $E/F$ is quadratic with Galois group generated by the complex conjugation. Also note that Gal($K/\mathbb{Q}$) is abelian containing the complex conjugation. Hence

\[ \bar{\xi} = N_{K/E}(\gamma) = N_{K/E}(\bar{\xi}) = N_{K/E}(-\gamma) = -N_{K/E}(\gamma) = -\xi. \]

It follows that

\[ \xi^2 = -\xi \bar{\xi} = -N_{E/F}(\xi) = -\delta, \]

which completes the proof for the claim.

Thus $E = F(\sqrt{\delta})$ with

\[ N_{F/\mathbb{Q}}(\delta) = N_{E/\mathbb{Q}}(\xi) = p, \]

which implies that $\delta \mathfrak{o}_F$ is a prime ideal lying over $p$.

Now assume that

\begin{equation}
(3.2) \quad N_{E/F}(\alpha) = q^n \text{ for some } \alpha \in \mathfrak{o}_E.
\end{equation}

Due to Lemma \(22\), we may write $\alpha = (x + y\sqrt{-\delta})/2$ for some $x, y \in \mathfrak{o}_F$. It follows from \((3.2)\) that

\begin{equation}
(3.3) \quad q^n = N_{E/F}(\alpha) = \alpha \bar{\alpha} = \frac{x^2 + \delta y^2}{4}.
\end{equation}
Since \(n\) is odd, we have \(y \neq 0\). Otherwise, \(x^2 = 4q^n\) and then \(\sqrt{q} \in F\), which yields a contradiction since \(\mathbb{Q}(\sqrt{(-1)(p-1)/2p})\) is the unique quadratic subfield of \(K = \mathbb{Q}(\zeta_p)\).

Next we shall show that \(x^2, y^2\) and \(\delta\) are all \textit{totally nonnegative}, i.e., they are all nonnegative after applying each \(\sigma \in \text{Gal}(F/\mathbb{Q})\). Recall that every element in \(\text{Gal}(K/\mathbb{Q})\) commutes with the complex conjugation. We have for every \(\sigma \in \text{Gal}(K/\mathbb{Q})\) that
\[
\sigma(\delta) = \sigma(N_{E/F}(\xi)) = \sigma(\bar{\xi}) = \sigma(\delta) \sigma(\bar{\xi}) = 0.
\]
Since \(\sigma(x), \sigma(y) \in F\) are fixed by the complex conjugation, we have
\[
\sigma(x^2) = \sigma(x)^2 \geq 0 \text{ and } \sigma(y^2) = \sigma(y)^2 \geq 0.
\]

It follows from \(3.3\) that
\[
(4q^n)^k = N_{F/\mathbb{Q}}(x^2 + 2y^2) = \prod_{\sigma \in \text{Gal}(F/\mathbb{Q})}(\sigma(x^2) + \sigma(\delta)\sigma(y^2)) \geq \prod_{\sigma \in \text{Gal}(F/\mathbb{Q})}\sigma(x^2) + \prod_{\sigma \in \text{Gal}(F/\mathbb{Q})}\sigma(\delta)\prod_{\sigma \in \text{Gal}(F/\mathbb{Q})}\sigma(y^2) = N_{F/\mathbb{Q}}(x^2) + N_{F/\mathbb{Q}}(\delta)N_{F/\mathbb{Q}}(y^2) \geq p,
\]
where the first inequality holds since \(x^2, y^2\) and \(\delta\) are all totally nonnegative, and the second inequality holds since \(N_{F/\mathbb{Q}}(x) \in \mathbb{Z}, N_{F/\mathbb{Q}}(y) \in \mathbb{Z} \setminus \{0\}\) and \(N_{F/\mathbb{Q}}(\delta) = p\). This is a contradiction with the assumption \(3.1\). The proof is complete. \(\square\)

Now consider a special case of Theorem \(3.1\) where \(E = K = \mathbb{Q}(\zeta_p)\). In this case, \(k = [F: \mathbb{Q}] = (p - 1)/2\) and the assumption \(3.1\) will not hold. Fortunately, we may use \(3.3\) in Lemma \(2.5\) to \textit{descend} Equation \(3.2\) from \(\mathbb{Q}(\zeta_p)\) to a subfield with a small degree over \(\mathbb{Q}\). Then we obtain that there exists no integral point for \(2.4\) with a large class of \(p, q, n\). We state the result after generalizing \(\mathbb{Q}(\zeta_p)\) to \(\mathbb{Q}(\zeta_{pq})\). For an integer \(a\), denote by \(\mathcal{B}(a)\) the 2-part of \(a\), i.e., if \(a = 2^m a_1\) for some odd \(a_1\), we have \(\mathcal{B}(a) = 2^m\).

\textbf{Theorem 3.2.} Let \(p, q\) be two distinct primes with \(f = \text{ord}_p(q) > 1\) and \(n\) be an odd positive integer. Suppose \(e\) is an positive integer. When \(e > 1\), we further assume that
\[
q^f \equiv 1 \pmod{2^e}.
\]
Let \(E = \mathbb{Q}(\zeta_{pq})\) and \(F = E \cap \mathbb{R}\). If
\[
(3.4) \quad p > 4^\mathcal{B}(l)q^n l, \text{ where } l = \frac{2(p-1)}{(3 - (-1)^f)},
\]
then \(N_{E/F}(\alpha) = \alpha \bar{\alpha} = q^n\) has no solution with \(\alpha \in \mathfrak{O}_E\).

\textbf{Proof.} First note that \(f > 1\). Then \(p\) is odd, \(E\) is complex and \(E/F\) is a quadratic extension. Since \(\text{Gal}(E/\mathbb{Q})\) is cyclic of degree \(\varphi(p^e)\), we denote by \(E_q\) the unique subfield of \(E\) having degree \(\frac{p^e - 1}{2}\) over \(\mathbb{Q}\). Then \(E_q \subseteq K = \mathbb{Q}(\zeta_p)\). Assume there exist \(\alpha \in \mathfrak{O}_E\) such that
\[
N_{E/F}(\alpha) = \alpha \bar{\alpha} = q^n.
\]
Note that \( f = \text{ord}_p(q) > 1 \) implies \( p \mid q - 1 \). Since we have \( q^f \not\equiv 1 \pmod{p^2} \) when \( e > 1 \), we have \( \text{ord}_{p^e}(q) = fp^{e-1} \) for any positive integer \( e \) according to Lemma 2.1. It follows that

\[
\frac{\varphi(p^e)}{\text{ord}_{p^e}(q)} = \frac{\varphi(p)}{\text{ord}_p(q)} = \frac{p-1}{f}.
\]

Hence \( E_q \) is the decomposition field of \( q \) in \( E \). According to (3) in Lemma 2.3, we obtain that

\[
(3.6) \quad \beta \bar{\beta} = q^n \quad \text{for some} \quad \beta \in \mathfrak{o}_{E_1} \quad \text{and} \quad \beta^2 \in \mathfrak{o}_{E_q}.
\]

It means that \( \beta \in \mathfrak{o}_{E_1} \) where \( E_1/E_q \) is some extension contained in \( E \) such that \( [E_1 : E_q] \leq 2 \).

If \( E_1 \) is real, we have \( \beta = \bar{\beta} \) and then \( \sqrt{q} \in E \), which is impossible. Thus \( E_1 \) is complex. Note that \( [E : E_q] = \text{ord}_p(q) = fp^{e-1} \). If \( f \) is odd, we have \( E_1 = E_q \) and then \( [E_1 : \mathbb{Q}] = (p-1)/f \).

Otherwise, since \( [K : E_q] = f \) is even and \( E/\mathbb{Q} \) is cyclic, we may fix \( E_1 \subseteq K \) to be the unique quadratic extension of \( E_q \) and then we have \( [E_1 : \mathbb{Q}] = 2(p-1)/f \). It follows that we always have

\[
[E_1 : \mathbb{Q}] = 2 - \frac{2(p-1)}{3(-1)^f} f = 2l \quad \text{and} \quad E_1 \subseteq K.
\]

We want to obtain a complex subfield \( E_2 \subseteq E_1 \) as small as possible. Let \( E_2 \subseteq E_1 \) be the unique subfield having degree \( 2\mathfrak{B}(l) \) over \( \mathbb{Q} \). Since \( E_1 \) is complex and \( [E_1 : E_2] = l/\mathfrak{B}(l) \) is odd, we have that \( E_2 \) is complex. Let \( F_2 = E_2 \cap \mathbb{R} \), which has degree \( \mathfrak{B}(l) \) over \( \mathbb{Q} \). Taking norm from \( E_1 \) to \( E_2 \) in (3.6), we obtain

\[
(3.7) \quad N_{E_2/F_2}(N_{E_1/E_2}(\beta)) = N_{E_1/E_2}(\beta)N_{E_1/E_2}(\bar{\beta}) = q^{nl/\mathfrak{B}(l)} \quad \text{with} \quad N_{E_1/E_2}(\beta) \in \mathfrak{o}_{E_2}.
\]

Note that \( E_2 \subseteq K = \mathbb{Q}((\zeta_p)) \). Substituting \( E/F, \alpha, k \) and \( n \) in Theorem 3.1 by \( E_2/F_2, N_{E_1/E_2}(\beta), \mathfrak{B}(l) \) and \( nl/\mathfrak{B}(l) \) respectively, we have that Equation (3.7) does not hold due to the assumption (3.5).

This completes the proof. \( \square \)

3.2. Asymptotic estimations on the density of \( p \)'s. Let us make the assumptions in Theorem 3.2 more explicit. We will show that for fixed \( q \) and \( n \), there are infinitely many \( p \)'s satisfy all the assumptions.

First we consider Theorem 3.2 in the case where \( e = 1 \), i.e., \( E = \mathbb{Q}((\zeta_p)) \).

Proposition 3.3. Let \( q \) be a prime number and \( n, g \) be two positive integers. For a positive real number \( x \), let \( \pi(x) \) be the number of primes not exceeding \( x \) and \( M_{q^n,g}(x) \) be the number of primes \( p \) not exceeding \( x \), such that

(a) \( (p-1)/f = g \), where \( f = \text{ord}_p(q) > 1 \),
(b) \( p > 4^{\mathfrak{B}(l)q^{nl}} \), where \( l = \frac{2(p-1)}{3(-1)^f} f \).

In particular, \( p \) meets assumption (3.5) in Theorem 3.2 for the case \( e = 1 \). Assuming Extended Riemann Hypothesis (ERH), then

\[
M_{q^n,g}(x) \sim C_{q^n,g}\pi(x) \sim C_{q^n,g}\frac{x}{\log x}, \quad \text{as} \quad x \to +\infty,
\]

where \( C_{q^n,g} \) is a positive constant that only depends on \( q^n \) and \( g \).

Proof. The first constraint \( (p-1)/f = g \) on \( p \) is related to a kind of generalization of Artin’s conjecture on primitive roots. Note that \( f > 1 \) is automatic for sufficiently large \( p \). By Theorem 2.3, where we take \( n = g \) and \( a = q \), the number of primes \( p \) not exceeding \( x \) satisfying \( (p-1)/f = g \) equals asymptotically to \( C_q^g \) Li\((x)\) as \( x \to +\infty \) under ERH, where \( C_q^g \) is a positive constant that only depends on \( q \) and
As for the constraint \[ b \], it suffices to exclude finitely many \( p \leq (4q^n)^g \) since \( (4q^n)^g \geq 4^{2l}q^nl \).

It follows that for some positive \( C_{q^n,g} \) only depending on \( q^n \) and \( g \),

\[
M_{q^n,g}(x) \sim C_{q^n,g}(x) \sim C_{q^n,g}(x), \quad \text{as } x \to +\infty,
\]

since by the prime number theorem we have

\[
\text{Li}(x) \sim \pi(x) \sim \frac{x}{\log x}.
\]

This completes the proof. \( \square \)

Next we consider the case where \( e > 1 \) and \( E = \mathbb{Q}(\zeta_{p^n}) \) in Theorem 3.2. Besides (a) and (b), there is one more constraint \( q^f \not\equiv 1 \pmod{p^2} \). We tighten it to

(c) \( q^{p-1} \not\equiv 1 \pmod{p^2} \).

Note that primes NOT satisfying (c) are called base-\( q \) Wieferich primes. It is conjectured that the number of base-\( q \) Wieferich primes not exceeding \( x \) equals asymptotically to \( \sum C_{q^n,g} \log \log x \) as \( x \to +\infty \) \( [M^\pm 81] \), where \( C_q \) is a constant only depending on \( q \). Since \( \log \log x \) is negligible compared to \( \pi(x) \), based on this conjecture and Proposition 3.3, we propose the following conjecture.

**Conjecture 3.4.** With \( q^n \) and \( g \) fixed as in Proposition 3.3, the number of primes not exceeding \( x \), such that the conditions (a), (b) and (c) hold (in particular, \( p \) meets assumptions (3.4) and (3.5) in Theorem 3.2 for \( e > 1 \)) equals asymptotically to

\[
C_{q^n,g} \frac{x}{\log x} \quad \text{as } x \to +\infty,
\]

where \( C_{q^n,g} \) is a positive constant that only depends on \( q^n \) and \( g \).

In fact, computational evidence indicates that Wieferich primes are extremely rare, and one can see Remark 4.3 (c) in the next section.

### 4. Results on the nonexistence for GBFs with type \([n, 2p^e]\)

#### 4.1. Main results.

By applying Theorem 3.2 we obtain the following theorem.

**Theorem 4.1.** Assume \( p \) is an odd prime number. Let \( f = \text{ord}_p(2) \) and \( N = p^e \), where \( e \) is a positive integer. When \( e > 1 \), we further assume that

\[
(4.1) \quad 2^f \not\equiv 1 \pmod{p^2}.
\]

Let \( n \) be an odd positive integer. If

\[
(4.2) \quad p > 2^{2B(l)+nl} \quad \text{where } l = \frac{2(p-1)}{3-(-1)^f} f,
\]

then there is no GBF with type \([n, 2N]\).

**Proof.** Assume that \( F \) is a GBF with type \([n, 2N]\). Since \( f = \text{ord}_p(2) \), we have \( f > 1 \). Let \( E = \mathbb{Q}(\zeta_N) = \mathbb{Q}(\zeta_{p^e}) \) and \( F = E \cap \mathbb{R} \). By (1.1) in the definition of GBFs, we have that \( F(\lambda) \in \mathfrak{o}_E \) and

\[
F(\lambda)F'(\lambda) = (2N)^n = 2^np^n.
\]

By (1) in Lemma 2.6, we have

\[
(4.3) \quad \alpha\tilde{\alpha} = 2^n
\]

for some \( \alpha \in \mathfrak{o}_E \).
Take \( q = 2 \) in Theorem 3.2. Then the assumptions in the theorem are fulfilled. We obtain that (4.3) has no solution with \( \alpha \in \mathfrak{o}_E \), which yields a contradiction. This completes the proof. □

Roughly speaking, Theorem 4.1 provides that there is no GBF with type \([n, 2p^e]\) when \( p \) is extremely larger than \( n \) and \((p - 1)/\text{ord}_p(2)\). By the discussion at the end of Section 3 we are able to show that there are infinitely many primes \( p \) satisfying the assumptions in Theorem 4.1, for a fixed \( n \).

Corollary 4.2. Let \( n \) and \( g \) be fixed positive integers. Assuming ERH, then there exist a positive constant \( C_{n, g} \) only depending on \( n \) and \( g \), such that as \( x \) goes to infinity, there are asymptotically at least \( C_{n, g}x/\log x \) primes \( p \) not exceeding \( x \) satisfying

\[
(p - 1)/f = g \quad \text{and} \quad p > 2^{2B(l) + n!},
\]

where \( f = \text{ord}_p(2) \) and \( l = \frac{2(p - 1)}{(3 - (-1)^{f})f} \), and for such \( p \)’s there is no GBF with type \([n, 2p]\).

In particular, for each \( n \), there are infinitely many such \( p \)’s.

Proof. Applying Proposition 3.3 with \( q = 2 \), we obtain that asymptotically there are at least \( C_{n, g}x/\log x \) primes \( p \) not exceeding \( x \), which satisfy all the assumptions of Theorem 4.1 with \( e = 1 \). The result then follows. □

Remark 4.3. (a) The assumptions in Theorem 4.1 can be verified by direct calculations.
(b) Note that ERH is always considered true in computational practice.
(c) Moreover, if we assume the conjecture on asymptotic number of Wieferich primes (based-2) holds, then Conjecture 3.4 tells us that Corollary 4.2 is also correct for \( e > 1 \). That is, there are asymptotically at least \( C_{n, g}x/\log x \) primes \( p \) not exceeding \( x \) such that there is no GBF with type \([n, 2p]\). In fact, numerical evidence suggests that few primes in a given interval are Wieferich primes. The base-2 Wieferich primes currently known are only 1093 and 3511, and they are the only two for all primes less than \( 6.7 \times 10^{15} \) [DK11].

For GBFs with type \([n, 2p^e]\), where \( n \) is odd and \( f = \text{ord}_p(2) \), the known results \([11], [33] \) and \([43]\) in the introduction cover the case \( p \not\equiv 1 \pmod{8} \), the case \( n = 1 \) and the case \( f \) is even. Therefore, it is more significant to restrict Theorem 4.1 to the case where \( p \equiv 1 \pmod{8}, n \geq 3 \) and \( f \) is odd. In this case, none of the method appearing in the previous literature are applicable.

Subsequently, we present several examples to illustrate the exact implications of Theorem 4.1 in the case where

\[
p \equiv 1(\text{mod}8), \quad n \geq 3 \quad \text{and} \quad f = \text{ord}_p(2) \quad \text{is odd}.
\]

4.2. Examples. We give some examples based on the previous discussion. The numerical results in these examples are all new and the calculations involved are all elementary.

We first consider a fixed \( p \) under the assumption \([43]\).

Example 4.4. By \([42]\), \( l \) reaches the smallest possible value 4 when \((p - 1)/f = 8 \). Since \( n \geq 3, \)

\[
p > 2^{2B(l) + n!} \geq 2^{2B(l) + 3l} = 2^{20} = 1048576.
\]

Thus we search for primes \( p > 2^{20} \) satisfying \([44]\), of which the smallest is 1049177 and we obtain the smallest nontrivial instance, i.e., there is no GBF with type \([3, 2p]\) for \( p = 1049177 \).
In the same manner, by slightly increasing $n$, say 11 and 15, we obtain the nonexistence of GBFs with type $[n, 2p]$ for
\[
\text{odd } n \leq 11 \text{ with } p = 4503599627370889 \text{ and } \\
\text{odd } n \leq 15 \text{ with } p = 295147905179352827401.
\]

Next, we consider a fixed $n$ under the assumption (4.4).

**Example 4.5.** Let $n = 3$. By (4.5), we search for $p$ satisfying (4.2) and obtain the first 5 such primes: 1049177, 1050169, 1050233, 1050473, 1051961. For these primes $p$, there is no GBF with type $[3, 2p]$ for all $p \geq 2\times 10^6$ such that $(p-1)/f = 8$, and they are infinitely many such $p$’s under ERH, by Corollary 4.2.

We give another instance for $n = 17$. By searching for primes $p$ satisfying (4.2), we obtain $p = 75557863725914323420409, 75557863725914323422233, \ldots$ For each $p$ and any odd number $n \leq 17$ there is no GBF with type $[n, 2p]$. In the same manner, we find that there is no GBF with type $[n, 2p]$ for all $p \geq 2^{81}$ such that $(p-1)/f = 8$.

At last we consider the case where $e > 1$, under the assumption (4.4).

**Example 4.6.** For $e > 1$, to show the nonexistence of GBFs we need additional assumption (4.1). By [DK11], except 1093 and 3511, all the primes $p$ less than $6.7 \times 10^{15}$ satisfy assumption (4.1) (see Remark 4.3 (c)). Note that neither 1093 nor 3511 is congruent to 1 modulo 8. It follows that for all positive integer $e$, there is no GBF with type $[n, 2^e]$ for the $(n, p)$ pairs described in the previous two examples.

## 5. Results on the nonexistence for relatively small $p$

Theorem 4.1 in the previous section provides the nonexistence of GBFs with type $[n, 2^e]$ for a class of large primes $p$. The smallest nontrivial instance (see Example 4.4) is that there is no GBF with type $[3, 2p]$ for $p = 1049177$. In fact, according to (4.2), the smallest prime $p$ satisfying the conditions in the theorem grows exponentially with respect to $n$.

Thus in this section, we give a computational approach to deal with the case where $p$ is relatively small. We first give a necessary condition that a GBF must meet, see Proposition 5.1. Then we give some results on the nonexistence of GBFs with certain types by verifying the condition.

### 5.1. Conditions on the exponents of prime ideals

We proceed with more general parameters. Assume $n$ is an odd number. Let $t = 2N$ with $2 \nmid N \geq 3$, $K = \mathbb{Q}(\zeta_N)$, $f = \text{ord}_N(2)$ and $g = \varphi(N)/f$. Let $E$ be the decomposition field of 2 in $K$. Then $[E : \mathbb{Q}] = g$. We always assume $f$ is odd here.

Now we suppose there is a GBF with type $[n, t = 2N]$. Then the same argument as in the beginning of the proof for Theorem 4.1 yields
\[
\alpha \overline{\alpha} = 2^n
\]
for some $\alpha \in \mathfrak{o}_K$. Next by Lemma 2.5 [3] we have
\[
\beta \overline{\beta} = 2^n \text{ for some } \beta \in \mathfrak{o}_K \text{ and } \beta^2 \in \mathfrak{o}_E.
\]

Since $[K : E] = f$ is odd, we know that $g$ is even, $\beta \in \mathfrak{o}_E$ and $E$ is complex. Thus we may assume the prime decomposition of 2 in $E$ is
\[
2\mathfrak{o}_E = \mathfrak{P}_1 \mathfrak{P}_2 \ldots \mathfrak{P}_g \mathfrak{P}_1 \mathfrak{P}_2 \ldots \mathfrak{P}_u.
\]
where \( \mathfrak{P}_k \)'s are prime ideals in \( E \) and \( \mathfrak{P}_{u+k} = \mathfrak{P}_k, \ k = 1, 2, \ldots, u \) with \( u = g/2 \). Then we have

\[
\beta \bar{\beta} o_E = \prod_{j=1}^{u} \mathfrak{P}_j^{n_j} \bar{\mathfrak{P}}_j^{\bar{n}_j}.
\]

Hence,

\[
(5.1) \quad \beta_0 E = \prod_{j=1}^{u} \mathfrak{P}_j^{n_j} \bar{\mathfrak{P}}_j^{\bar{n}_j},
\]

where \( n_j, \bar{n}_j \) are nonnegative integers such that \( n_j + \bar{n}_j = n \) for all \( j = 1, 2, \ldots, u \). Since the left-hand side of the above equation is a principal ideal, we have the following equation

\[
(5.2) \quad \prod_{j=1}^{u} \{\mathfrak{P}_j\}^{n_j} \{\bar{\mathfrak{P}}_j\}^{\bar{n}_j} = 1,
\]

where \( \{\mathfrak{P}_j\} \) represents the class of \( \mathfrak{P}_j \) in \( Cl(E) \).

**Proposition 5.1.** With the above notation, if there are no nonnegative integers

\[
(n_1, n_2, \ldots, n_u, \bar{n}_1, \bar{n}_2, \ldots, \bar{n}_u), \quad \text{where } n_j + \bar{n}_j = n \quad \text{for } j = 1, 2, \ldots, u,
\]

such that Equation (5.2) holds, then there is no GBF with type \([n, 2N]\).

Subsequently, we mainly focus on the cases where \( N = p^e \) and the methods in previous literature are not applicable. That is, we work under the assumption

\[
(5.3) \quad p \equiv 1(\mod 8), \ n \geq 3 \quad \text{and} \quad f = \text{ord}_p(2) \text{ is odd}.
\]

If \( e > 1 \), we further assume that

\[
(5.4) \quad 2^f \not\equiv 1 \pmod{p^2}
\]

as in Theorem 4.1 then it reduces to the case where \( e = 1 \).

**Remark 5.2.** Since neither 1093 nor 3511 is congruent to 1 modulo 8, (5.4) holds for all \( p \) less than \( 6.7 \times 10^{15} \). See Remark 4.3 (c).

### 5.2. Algorithms and results.

Let \( K = \mathbb{Q}(\zeta_p) \) and \( E \) be the unique subfield of \( K \) having degree \( g \) over \( \mathbb{Q} \). Proposition 5.1 provides an explicitly algorithm to find \( N = p^e \) and \( n \), which enumerates \((n_1, n_2, \ldots, n_g)\) to determine the solvability of (5.2). This is implemented by GP \cite{gro17} as follows.

**Step i.** Given \( p \) and \( n \), use \texttt{galoissubcyclo} to obtain the polynomial for the subfield \( E \subseteq K \), \texttt{bnfinit} the field information of \( E \) involving the ideal class group \( Cl(E) \), and \texttt{idealprimedec} the set of primes \( S = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_g\} \).

**Step ii.** Use \texttt{nfgaloisconj} and \texttt{nfgaloisapply} to identify the complex conjugation and the conjugate pairs of primes in \( S \). Then we may assume \( \mathfrak{P}_{u+k} = \mathfrak{P}_k, \ k = 1, 2, \ldots, u \).

**Step iii.** Enumerate \( n_1, \ldots, n_u \) in the range \([0, n]\), calculate every ideal \( \mathfrak{A} = \prod_{j=1}^{u} \mathfrak{P}_j^{n_j} \bar{\mathfrak{P}}_j^{\bar{n}_j} \), and use \texttt{bnfisprincipal} to see whether \( \mathfrak{A} \) is principal, i.e., to determine the solvability of (5.2).

**Remark 5.3.** Let us give some remarks on the implementation above.

(a) In Step ii we can only deal with small \( g \), otherwise the calculation of the field information will cost too much time and memory.
(b) As noted in the PARI/GP documentation [gro17], since we use \texttt{bnfinit} to calculate the class group, all results rely on this implementation are valid only under Generalised Riemann Hypothesis (GRH). But as in Remark 4.3 (b), the results could be considered unconditionally correct in practice.

(c) From Step iii we know that the size of the searching space, \(O((n+1)^2)\), predominantly influences the calculation time.

According to the analysis above, we only confine ourselves to the case where \(g = 8\) and \(n\) is small. For each prime \(p\) less than 3000 with \(\text{ord}_p(2) = (p - 1)/8\) being odd, we compute the largest positive odd integer \(n_p\) such that (5.2) is not solvable for all odd positive \(n \leq n_p\). It costs several hours in an ordinary computer with Intel(R) Core(TM) 2 CPUs and 2G memory. Table 5.4 lists these \((p, n_p)\). By Proposition 5.1 and Remark 5.2, we have the following proposition.

**Proposition 5.4.** For all \((p, n_p)\) in Table 5.4 there is no GBF with type \([n, 2p^e]\) for any odd positive \(n \leq n_p\) and any positive integer \(e\), under GRH.

| \(p\) | \(n_p\) | \(p\) | \(n_p\) |
|-------|-------|-------|-------|
| 89    | 3     | 1609  | 23    |
| 233   | 7     | 1721  | 19    |
| 937   | 7     | 1913  | 25    |
| 1289  | 13    | 2441  | 31    |
| 1433  | 17    | 2969  | 33    |

Recall that Example 4.5 shows that there is no GBF with type \([3, 2p]\) for all \(p \geq 1049177\) and \((p - 1)/f = 8\). Thus we use the previous implementation to determine the solvability of (5.2) for all \(p < 1049177\) and \(n \leq 3\) such that \(f = (p - 1)/8\) is odd. The calculation also costs several hours and it shows that (5.2) is not solvable for \(n \leq 3\) and these \(p\)'s. Combining with Proposition 5.1, Example 4.5 and Corollary 4.2 we obtain the following proposition.

**Proposition 5.5.** Let \(f = \text{ord}_p(2)\). Under GRH, there is no GBF with type \([3, 2p^e]\) for all prime \(p\) such that \(f = (p - 1)/8\) is odd and \(\text{ord}_{p^e}(2) = fp^{e-1}\).

**Remark 5.6.** In Proposition 5.5, for the case \(e > 1\), the equality \(\text{ord}_{p^e}(2) = fp^{e-1}\) can be easily checked by testing if \(2f \not\equiv 1 \pmod{p^2}\), see Lemma 2.1.

**Remark 5.7.** (1) The first author and Li [LL17] suggest that one should investigate the relations between the primes \(\mathfrak{P}_k\) lying over 2 in the class group \(Cl(E)\), where \(E\) is the decomposition field of 2 in \(\mathbb{Q}(\zeta_p)\). Unfortunately, here we cannot make a similar theoretical analysis as in [LL17]. This is because under the assumption \((1.4)\), \((p - 1)/\text{ord}_p(2)\), the degree of \(E\) over \(\mathbb{Q}\) (i.e., the number of \(\mathfrak{P}_k\)), is at least 8. Hence it becomes difficult to deal with the relations between \(\mathfrak{P}_k\) in \(Cl(E)\). This is why we consider a computational approach.

(2) Note that the implementation in this section can deal with any integer \(N\), not limited to the case where \(N = p^e\). This allows us to obtain more computational results on the nonexistence of GBFs in the future.
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