Power-law friction in closely-packed granular materials

Takahiro Hatano
Earthquake Research Institute, University of Tokyo, 1-1-1 Yayoi, Bunkyo, Tokyo 113-0032, Japan
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In order to understand the nature of friction in dense granular materials, a discrete element simulation on granular layers subjected to isobaric plain shear is performed. It is found that the friction coefficient increases as the power of the shear rate, the exponent of which does not depend on the material constants. Using a non-dimensional parameter that is known as the inertial number, the power law can be cast in a generalized form so that the friction coefficients at different confining pressures collapse on the same curve. We show that the volume fraction also obeys a power law.

Friction is one of the oldest problems in science because it dominates various phenomena in our daily life. In particular, dynamics of granular flow, which is ubiquitous in earth sciences and engineering, is governed by a law that describes behavior of the friction coefficient (ratio of the shear stress to the normal stress). Such examples are avalanche, landslide, debris flow, silo flow, etc. In addition, the nature of friction on faults, which plays a key role in earthquake mechanics [1, 2], is also attributed to that of granular rock because fault zone consists of layers of fine rock particles that are ground-up by the fault motion of the past. To find a suitable law of friction in granular materials under a specific condition is thus an essential problem.

Although the frictional properties of granular materials are so important, our understanding is still limited. In the context of earthquake mechanics, slip velocity (or shear rate) dependence of friction coefficient, which is equivalent to rheology under constant pressure condition, is a matter of focus [2]. In experiments on thin granular layers that are sheared at relatively low sliding velocities ranging from mm/s to mm/s, the behavior of the friction coefficient can be described by a phenomenological law in which friction coefficient logarithmically depends on the sliding velocity. This is known as the rate and state dependent friction (RSF) law [3]. Note that the RSF law applies well to lower speed (creep-like) friction, it is violated at such higher velocities.

However, at this point, we do not know any friction law that is valid at such higher velocities.

Several recent attempts to understand the nature of friction in granular media under high shear rates are noteworthy here. Jop et al. considered a simple friction law that describes flow on inclined planes [3], based on massive simulations and experiments [10]. Although their friction law seems feasible, it involves rather dilute flow and its applicability to denser and slower flow (e.g. quasistatic flow) is not clear. Da Cruz et al. performed an extensive simulation that focused on dense and slow regime and found a friction law that does not contradict that of Jop et al. However, because da Cruz et al. involved a two dimensional system, effect of the dimensionality may be questioned. In particular, in quasistatic regime where the nature of interparticle contacts plays an essential role in rheology, the effect of dimensionality should be seriously investigated.

In this Rapid Communication, we perform a three dimensional simulation in order to understand the nature of friction in a slowly-sheared dense granular material. Our particular interest is a dense granular matter under high confining pressure (e.g. tens of MPa) which is roughly corresponds to a typical configuration of faults at seismic slip. Note that the RSF law is violated in such a situation. A new law is reported in which friction coefficient increases as the power of shear rate.

In the following we describe the computational model of granular layers. The individual constituents are assumed to be spheres, and their diameters range uniformly from 0.7d to 1.0d. The interaction force follows the discrete element method (DEM) [12]. Consider a grain i of radius Ri located at ri with the translational velocity vi and the angular velocity Ωi. This grain interacts with another grain j whenever overlapped; i.e. |rij| < Ri + Rj, where rij = ri − rj. The interaction consists of two kinds of forces, each of which is normal and transverse to rij, respectively. Introducing the unit normal vector nij = rij/|rij|, the normal force acting on i, which is denoted by Fij(0), is given by

\[ f(\epsilon_{ij}) + \zeta_n \cdot n_{ij} \cdot n_{ij}, \text{ where } \epsilon_{ij} = 1 - |rij|/(Ri + Rj). \]

A function f(ε) describes elastic repulsion between grains. Here we test two models: f(ε) = ke (the linear force) and f(ε) = kε3/2 (the Hertzian force) [13]. Note that the constant k/d2 is on the order of the Young’s modulus of the grains. In order to define the transverse force, we utilize the relative tangential velocity \( v_{ij}^{(t)} \) defined by (\( \dot{r}_{ij} - n_{ij} \cdot \dot{r}_{ij} \)) + (\( R_i \Omega_i + R_j \Omega_j \))/\( (R_i + R_j) \times r_{ij} \) and introduce the relative tangential displacement vector \( \Delta_{ij}^{(t)} = \int_{t_0}^{t} dt v_{ij}^{(t)} \). The subscript in the integral indicates that the integral is performed when the contact is rolling; i.e., |\( \Delta_{ij}^{(t)} \) | < k_\text{t} or |\( \Delta_{ij}^{(t)} \) | < v_j < 0. Then the tangential force acting on the particle i is written as

\[ F_{ij}^{(t)} = -\min(\mu \dot{r}_{ij}/|\dot{r}_{ij}|, k_\text{t} \Delta_{ij}^{(t)}) |F_{ij}^{(n)}|. \]

In the case that \( \mu = 0 \), the tangential force vanishes and the rotation of particles does not affect the translational motion. The parameter values adopted in the present simulation are given in TA-
TABLE I: The parameters of the discrete element simulation.

| polydispersity | ζ/√d/km | k_{id} | 0.005 | 0 - 0.6 | 3.8 × 10^{-5} - 1.1 × 10^{-2} |

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The configuration of the system mimics a typical experiment on granular layers subjected to simple shear. Note that there is no gravity in the system. The system spans $L_x \times L_y \times L_z$ volume, and is periodic in the $x$ and the $y$ directions. We prepare two systems of different aspect ratio, each of which contains approximately 10,000 particles: $25d \times 25d \times 8d$, and $15d \times 15d \times 25d$. As we shall discussed later, the difference of the aspect ratio does not affect the rheology. In the $z$ direction, there exist two rough walls that consist of the same kind of particles as those in the bulk. The particles that consist walls are randomly placed on the boundary and their relative positions are fixed. The walls are parallel to each other and displaced antiparallel along the $y$ axis at constant velocities $\pm V/2$, while they are prohibited to move along the $x$ axis. One of the walls is allowed to move along the $z$ axis so that the pressure is kept constant at $P$. Using the mass of the wall $M_w$ that is defined as the sum of the masses of the constituent particles, the $z$ coordinate of the wall $Z_w$ is described by the following equation of motion: $M_w \ddot{Z}_w = F_z - PS$, where $F$ denotes the sum of the forces between the wall particles and the bulk particles, and $S$ denotes the area of the wall. Then the $z$ component of the velocity of the wall particles is given by $\dot{Z}_w$. Note that the friction coefficient of the system is defined by $F_y/PS$.

The system reaches a steady state after a certain amount of displacement of the walls. We judge that the system reaches a steady state if each of the following quantities does not show apparent trends and seems to fluctuate around a certain value. The monitored quantities are the friction coefficient, the $z$ coordinate of the wall (i.e., the density), and the granular temperature. Also snapshots of the velocity profile are observed to ensure the realization of uniform shear flow. We confirm that the transient behaviors of the friction coefficient and of the volume increase are quite similar to those observed in experiments. Here we do not investigate such transients and restrict ourselves to steady-state friction.

Because uniform shear flow is unstable in a certain class of granular systems, we must check the internal velocity profiles at steady states. There is a strict tendency that shear flow is localized near the walls in the case that the confining pressure is small and/or the sliding velocity is large. This kind of spatial inhomogeneity is rather ubiquitous in granular flow, and is extensively investigated. In our simulation, uniform shear flow is realized at lower sliding velocities and higher confining pressures. Here we discuss exclusively the case in which uniform shear is realized. In this case, the shear rate is proportional to the sliding velocity of the walls; i.e., $\gamma = V/L_z$.

We investigate the behaviors of the friction coefficient of the system, $F_y/PS \equiv M$. The control parameters that affect the friction coefficient are the shear rate $\gamma$ and the pressure $P$. It is useful to represent the control parameters in terms of nondimensional numbers, because the friction coefficient is a nondimensional number and hence must be a function of nondimensional numbers. Thus $\gamma$ and $P$ are recast in the following forms: $I = \gamma \sqrt{m/P d}$ and $\Pi = P d^2/k$. In particular, the former is referred to as the inertial number $I$, which dominates the frictional behavior of granular flows. Hereafter we discuss the nature of friction taking advantage of these nondimensional numbers.

In order to grasp the main point of our result, it is convenient to begin with the frictionless particles; i.e., $\mu = 0$. Recall that we test two models, each of which has different interaction: the Hertzian contact model and the linear force model. As shown in FIG. I friction coefficients of these two models are collapsed on the following master curve.

$$M = M_0 + s I^0,$$

where $M_0$ denotes the friction coefficient for $\gamma \to 0$. Here $M_0 \simeq 0.06$. Note that the effect of the inertial number is expressed by a power law, $s I^0$, where $\phi = 0.27 \pm 0.05$. The prefactor $s$ is approximately 0.37 in the linear force model, while it is somewhat larger ($s \simeq 0.45$) in the Hertzian contact model.

In order to check the universality of Eq. (1), we wish to confirm independence of our results on the details of the model. First we discuss the effect of the tangential force between particles. In FIG. 2, shown are the friction coefficients of the models in which $\mu = 0.2$ and 0.6. It is noteworthy that they range from 0.3 to 0.4, which are not significantly discrepant from those obtained by an experiment on spherical glass beads. More importantly, the friction coefficients again obey Eq. (1) with $\phi \simeq 0.3$ regardless of the value of $\mu$ and the force model (the linear or the Hertzian). Indeed the friction coefficients of the both models are almost the same. We also remark that the factor $s$ does not depend on $\mu$; $s = 0.33 \pm 0.03$ for $\mu = 0, 0.2$, and 0.6.

On the other hand, $M_0$ depends on $\mu$. In the linear force model, $M_0 \simeq 0.06$ for $\mu = 0$, while $M_0 \simeq 0.26$ for $\mu = 0.2$, and $M_0 \simeq 0.4$ for $\mu = 0.6$. Similar dependence was also observed in Refs. 11, 19. Note that $M_0$ also depends on the force model in the case that $\mu = 0$: $M_0 \simeq 0$ in the Hertzian model while $M_0$ is not negligible in the linear model. Although $M_0$ looks like the static friction coefficient, note that it is defined in the $\gamma \to 0$ limit and different from the static friction coefficient, above which a static system begins to flow. In order to distinguish the two concepts, $M_0$ is referred to as dynamic yield strength. The difference is important when we consider the stability of slip, as will be discussed in the last paragraph of
FIG. 1: The friction coefficients $M$ in the models without tangential force, i.e., $\mu = 0$. The horizontal axis $I$ denotes the nondimensional shear rates, $V/L\sqrt{m/Pd}$. The shape of the symbols and the confining pressure are in one-to-one correspondence: the squares to $\Pi = 3.8 \times 10^{-5}$, the circles to $\Pi = 1.9 \times 10^{-3}$, the triangles to $\Pi = 3.9 \times 10^{-3}$, and the diamonds to $\Pi = 1.1 \times 10^{-2}$. The layer thickness $L_z/d \simeq 8$ for the squares and the circles, while $L_z/d \simeq 26$ for the triangles and the diamonds. The solid lines denote Eq. (1) with $\phi = 0.3$. (a) Friction coefficient of the Hertzian force model. (b) Friction coefficient of the linear force model.

FIG. 2: The friction coefficients in the models with tangential force ($\mu = 0.2, 0.6$). The shape of the symbols and the confining pressure are in one-to-one correspondence as in FIG. 1. The blank symbols denote the friction coefficient of the linear force model with $\mu = 0.2$, while the symbols of vertically-striped pattern denote that of $\mu = 0.6$. The circles of horizontally-striped pattern denote the Hertzian force model with $\mu = 0.2$. The lines denote Eq. (1) with $\phi = 0.3$.

is independent of $\tilde{\zeta}$ for smaller $I$ region. This behavior is consistent with those obtained in Refs. [11, 19]. Nevertheless, it can be still described by Eq. (1) with $s$ being a smaller value. For example, the friction coefficient of a system in which $\tilde{\zeta} = 0.05$ is described by $s \simeq 0.27$ with almost the same values of $M_0$ and $\delta$. However, the functional form of $s(\tilde{\zeta})$ is not clear at this point.

From the discussions so far, we can conclude that the details of the present model do not affect the validity of Eq. (1), which is the main result of this study. Importantly, the exponent $\phi$ seems to be universal; it is approximately $0.3$ regardless of the details of the model and the control parameters. The velocity-strengthening nature of this friction law does not contradict experiments [4, 5, 6, 7, 8]. In addition, it illustrates universality of power-law in rheological properties of random media [20] including foams [21] and human neutrophils [22]. In the following we discuss four important points that are peripherally related to the main result.

First, we discuss the dependence of the volume fraction to the inertial number. Surprisingly, decrease of the volume fraction caused by shear flow is also described by a power-law:

$$\nu_0 - \nu = s_2 I^\delta,$$

where $\nu_0$ is the volume fraction in the $\gamma \to 0$ limit. Note that the constants $s_2$ and $\delta$ do not depend on the details of the model. Figure 3 shows that all of the data obtained in our model collapse on Eq. (2) with $s_2 \simeq 0.11$ and $\delta = 0.56 \pm 0.02$. This dilatation law also illustrates ubiquity of power-law in granular materials.

The next point we wish to discuss is the relation between the present result and power-law rheology in systems at constant volume. In particular, Xu and O’Hern [22] found a power-law relation between the shear stress and the shear rate in a two-dimensional granular material consisting of frictionless particles. They estimated...
Note that $\nu = \nu_0 - \nu$ is plotted as a function of the inertial number. Note that $\nu_0$ is the volume fraction in the $\gamma \to 0$ limit, which is estimated by extrapolation. The symbol legends are the same as those in FIGS. 1 and 2. The line denotes Eq. (2).

The dilatation law. Decrease of the volume fraction $\Delta \nu = \nu_0 - \nu$ is plotted as a function of the inertial number. Note that $\nu_0$ is the volume fraction in the $\gamma \to 0$ limit, which is estimated by extrapolation. The symbol legends are the same as those in FIGS. [1] and [2]. The line denotes Eq. (2).

The exponent to be 0.65. However, at constant volume condition, the pressure also depends on the shear rate so that the behavior of the friction coefficient is generally different from that of the shear stress. Therefore, power-law friction in systems under constant volume condition are not directly related to the present result. See Ref. [24] for more detailed discussions on this subject.

The third point we wish to discuss is the effect of dimensionality. In contrast to the present study, da Cruz et al. [11] obtained a linear friction law in a two dimensional system. The difference may be attributed to the dimensionality of the systems, which affect the nature of contacts between particles. In particular, the angular distribution of the tangential force is strongly anisotropic in two dimensional systems, while such anisotropy is not observed in our three dimensional system. Accordingly, in the case of frictionless particles, their system exhibited a friction law that is quite similar to ours.

As the fourth point of interest, we discuss relevance of our result to earthquake mechanics by comparing it to a friction law recently proposed by Jop et al. [9], which seems to be validated in experiments on inclined plane flow. We stress that such flow is characterized by relatively large inertial number (typically $I \gtrsim 10^{-1}$), while our simulation involves much smaller $I$ values ($I \gtrsim 10^{-4}$) as shown in FIGS. [1] and [2]. In short, Eq. (1) involves much smaller $I$ region than Jop et al. have investigated.

Such small inertial numbers correspond to a typical configuration of seismic motion of faults. For example, in the case that $d = 1 \text{ mm}, V = 1 \text{ m/s}, L_z = 4 \text{ cm},$ and $P = 100 \text{ MPa}$, the corresponding inertial number is $10^{-4}$. However, one may wonder that the friction law Eq. (1) cannot lead to stick-slip motion of faults because the friction law found here is velocity-strengthening. Recall that we discuss exclusively stationary-state dynamic friction. Taking static friction into account, unstable slip is inevitable because static friction is always stronger than dynamic friction, which is mainly due to dilatation. Therefore power-law friction in stationary states does not contradict unstable slip on faults.

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