Monad constructions of omalous bundles

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Abstract

We consider a particular class of holomorphic vector bundles relevant for supersymmetric string theory, called omalous, over nonsingular projective varieties. We use monads to construct examples of such bundles over 3-fold hypersurfaces in \( \mathbb{P}^4 \), complete intersection Calabi-Yau manifolds in \( \mathbb{P}^k \), blow-ups of \( \mathbb{P}^2 \) at \( n \) distinct points, and products \( \mathbb{P}^m \times \mathbb{P}^n \).

1 Introduction

Let \( X \) be a nonsingular projective variety, \( TX \) be its tangent bundle and \( \omega_X \) its canonical line bundle. This paper is dedicated to the study of the following class of holomorphic vector bundles.

Definition 1.1. A holomorphic vector bundle \( \mathcal{E} \to X \) is called omalous if it satisfies the following conditions:

- \( c_2(\mathcal{E}) = c_2(TX) \)
- \( \det(\mathcal{E}^*) \simeq \omega_X \)

Recall also that a holomorphic vector bundle \( \mathcal{E} \to X \) is slope (semi-)stable with respect to a chosen polarization \( \mathcal{O}_X(1) \) of \( X \), i.e., for every proper subsheaf \( \mathcal{F} \) of \( \mathcal{E} \) the inequality \( \frac{\deg(\mathcal{F})}{\rk(\mathcal{F})} \leq \frac{\deg(\mathcal{E})}{\rk(\mathcal{E})} \) is satisfied.

The nomenclature comes from the fact that the matching of the first and second Chern classes of \( \mathcal{E} \) and \( TX \) is the usual Green-Schwarz anomaly cancellation condition in heterotic string theory [8, 12]; hence such bundles are not anomalous, that is omalous (Josh Guffin attributes this terminology to Ron Donagi, see the footnote in the first page of [13]).

Such bundles have a long history in the string literature. They appeared in the attempt at compactifying superstring theory to a theory on a \( M^4 \times X \), (where \( X \) is complex compact Calabi-Yau 3-fold and \( M^4 \) is a flat Minkowski

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space) with an unbroken $N = 1$ supersymmetry in four dimensions. This was done by arguing pertubatively in terms of the string coupling constant as in \cite{8, 20} and \cite{12}.

In the late nineties, arguments about compactifications using non-perturbative vacua in heterotic string theory were given by introducing five-branes as done by Donagi et. al. in \cite{9}. In this context, the general formula for anomaly cancelation is given by

$$c_2(\mathcal{E}) - c_2(TX) = [W],$$

where $[W]$ is the cohomology class of a four-form on the five-brane, and by Poincaré duality it corresponds to the class of an effective curve in the five-brane \cite[Section 2]{9}.

More recently, the motivation to consider such bundles comes from two sources. First, the omality conditions are necessary to the construction of a quantum sheaf cohomology for the bundle $\mathcal{E}$, c.f. \cite[Section 2]{13} and \cite{14}. The quantum sheaf cohomology of a bundle $\mathcal{E} \rightarrow X$ is a generalization of the quantum cohomology of $X$, and consists of the structure of a Frobenius algebra on

$$QH^\bullet(\mathcal{E}) := \oplus_{p,k} H^p(X, \wedge^k \mathcal{E}^*) \otimes \mathbb{C}[[q]]$$

with product and bilinear pairing induced by the three-point correlation functions in a $(0,2)$ supersymmetric nonlinear sigma model; see \cite{13, 14} and the references therein for further details.

Second, it was shown by Andreas and Garcia-Fernandez in \cite{1} that stable omalous bundles over Calabi-Yau 3-folds admit solutions of the Strominger system, which is a system of coupled partial differential equations defined over a compact complex manifold relevant in heterotic string theory, c.f. \cite{1} and the references therein.

The simplest example of omalous bundles are $TX \oplus \mathcal{O}_X^{\oplus k}$ and its deformations; a few other examples were considered in \cite{1, 6, 10, 14}. Our goal is to construct more examples of (stable) omalous bundles over various choices for $X$ using monads.

We remark that in the attempt of giving phenomenological models from heterotic compactifications, particular monads have been used in the physics literature such as in \cite{2, 3, 4, 5, 19}; we emphasize however that what is called a monad in \cite{2, 3, 4} does not coincide with the usual definition in the mathematical literature.
Recall that a monad on $X$ is a complex of locally free sheaves

$$M_\cdot : M_0 \xrightarrow{\alpha} M_1 \xrightarrow{\beta} M_2$$

such that $\beta$ is locally right-invertible, $\alpha$ is locally left-invertible. The (locally free) sheaves $K := \ker \beta$, $Q := \coker \alpha$ and $E := \ker \beta / \text{Im} \alpha$ are called, respectively, the kernel, cokernel and cohomology of $M_\cdot$.

In what follows, we provide examples of omalous bundles over 3-fold hypersurfaces in $\mathbb{P}^4$, complete intersection Calabi-Yau manifolds in $\mathbb{P}^k$ ($k = 4, 5, 6, 7$), blow-ups of $\mathbb{P}^2$ at $n$ distinct points, and products $\mathbb{P}^m \times \mathbb{P}^n$. All of these examples arise as cohomology, kernel, or cokernel of particular monads over these manifolds. We hope that such examples will be relevant for a deeper understanding of both quantum sheaf cohomology, the Strominger system and supersymmetric string theory.

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2 Stable omalous bundles over 3-fold hypersurfaces in $\mathbb{P}^4$

Let $X$ be the non-singular quintic 3-fold in $\mathbb{P}^4$, and consider the following monad:

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus 10} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 22} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus 10} \longrightarrow 0. \quad (2.1)$$

The existence of the monads (2.1) is explicitly guaranteed by the construction in [16, Section 3]; its cohomology bundle $E$ is a rank 2 bundle with Chern classes $c_1(E) = 0$ and $c_2(E) = 10 \cdot H^2$, where $H = c_1(\mathcal{O}_X(1))$ is the ample generator of the Picard group of $X$. This bundle is stable by [16, Main Theorem]. Moreover $c_2(E) = c_2(TX)$ and $\text{det}(E^*) \cong \mathcal{O}_X = \omega_X$ since the quintic 3-fold in $\mathbb{P}^4$ is Calaby-Yau. Hence $E$ is a stable omalous bundle over $X$.

Let us consider now a non-singular 3-fold $X_d$ of degree $d$ in $\mathbb{P}^4$. One can show that

$$c_1(TX_d) = (5 - d) \cdot H$$
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\[ c_2(TX_d) = (d^2 - 5d + 10) \cdot H^2. \]

Let \( E \) be a rank 3 linear bundle, that is, the cohomology of a linear monad of the form

\[
\begin{array}{c}
0 \rightarrow \mathcal{O}_{X_d}^{(c+l)}(-1) \xrightarrow{\alpha} \mathcal{O}_{X_d}^{(3+2c+l)} \xrightarrow{\beta} \mathcal{O}_{X_d}^{c}(1) \rightarrow 0.
\end{array}
\] (2.2)

Then \( c_1(E) = l \cdot H \) and \( c_2(E) = \frac{l}{2}(l + 1 + c) \cdot H^2. \)

Proposition 2.1. (i) The cohomology bundle \( E \) of the linear monad (2.2) is omalous for every odd integer \( k \geq 7 \), such that \( (d, l, c) \) are given by

\[
\begin{align*}
d(k) &= \frac{1}{2}(k - 1), \\
l(k) &= \frac{1}{2}(11 - k), \\
c(k) &= \frac{1}{8}(k^2 - 41).
\end{align*}
\]

(ii) Furthermore the omalous bundle \( E \) is stable for \((d, l, c) = (3, 2, 1), (4, 1, 5)\) and semi-stable for \((d, l, c) = (5, 0, 10)\).

Proof. (i) The conditions for which \( E \) is omalous are given by \( c_1(E) = c_1(TX) \) and \( c_2(E) = c_2(TX) \). In this case, one must have

\[
\begin{align*}
d^2 - 5d + 10 &= c + \frac{1}{2}(5 - d)(6 - d) \\
l &= 5 - d
\end{align*}
\]

Thus one must look for positive integer solutions \( d(c) \) of the quadratic equation \( d^2 + d - (10 + 2c) = 0 \), i.e., \( d = \frac{1}{2}(-1 + \sqrt{41 + 8c}) \). For every odd integer \( k \geq 7 \), it is easy to verify that \( d(k) = \frac{1}{2}(k - 1) \) are the desired roots, and the corresponding value for \( c \) is \( c(k) = \frac{1}{8}(k^2 - 41) \).

(ii) The stability part follows from [17, Theorem 7]. If \( d = 5 \) then \( l = 0 \) and \( c = 10 \), then \( E \) is an instanton bundle, and by [17, Theorem 3], it is semi-stable.

\[ \square \]

Remark 2.2. The existence of the monads (2.2) is a consequence of Flyostad’s Theorem for monads on \( \mathbb{P}^4 \). More precisely, the Main Theorem of [11] implies that the degeneration locus of a generic monad of the form

\[
\begin{array}{c}
0 \rightarrow \mathcal{O}_{\mathbb{P}^4}^{(c+l)}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^4}^{(3+2c+l)} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4}^{c}(1) \rightarrow 0.
\end{array}
\]

is zero dimensional. Its restriction to a generic hypersurface \( X_d \) is precisely (2.2), hence its cohomology yields a vector bundle over it.
3 Omalous bundles on complete intersection Calabi-Yau 3-folds

Let $X$ be a complete intersection Calabi-Yau 3-fold in $\mathbb{P}^n$. There are only five such cases, namely:

- A quintic in $\mathbb{P}^4$.
- In $\mathbb{P}^5$, either the intersection of two cubics or the intersection of a quadric and a quartic.
- In $\mathbb{P}^6$ the intersection of two quadrics with a cubic.
- In $\mathbb{P}^7$ the intersection of four quadrics.

One can write $X = \cap_i X_i$ where the $X_i$’s are given as in the list above and $l = \text{codim}_{\mathbb{P}^n}(X)$. Moreover one has the following short exact sequences

$$
0 \rightarrow TX_i \rightarrow TP^n|X_i \rightarrow N_i \rightarrow 0
$$

$$
0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-d_i) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}|X_i \rightarrow 0
$$

where the normal bundle $N_i$ to $X_i$ is simply the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(d_i)$ since each of the $X_i$ is a hypersurface of degree $d_i$ in $\mathbb{P}^n$. Using these sequences one can easily prove that the Chern Class of the tangent bundle $TX$, to $X$, is given by the formula

$$C(TX) = \frac{(1 + h)^{n+1}}{\prod_{i=1}^{l}(1 + d_i h)}$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. From the condition $c_1(TX) = 0$, it follows that

$$c_2(TX) = \frac{1}{2}[(\Sigma_{i=1}^{l}d_i^2) - (n + 1)]h^2.$$

**Proposition 3.1.** Let $E$ be a rank 2 bundle on a complete intersection Calabi-Yau 3-fold $X$ given by the cohomology of the following monad

$$
M : \quad 0 \rightarrow \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2 + 2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \rightarrow 0.
$$

with $c$ given according to the following table:

| $X \subset \mathbb{P}^4$ | $l$ | $(d_1, \ldots, d_l)$ | $c = \frac{1}{2}[(\Sigma_{i=1}^{l}d_i^2) - (n + 1)]$ |
|-------------------------|-----|-------------------|---------------------------------|
| $X \subset \mathbb{P}^4$ | 1   | 5                 | 10                             |
| $X \subset \mathbb{P}^5$ | 2   | $(3, 3)$          | 6                              |
| $X \subset \mathbb{P}^5$ | 2   | $(4, 2)$          | 7                              |
| $X \subset \mathbb{P}^6$ | 3   | $(2, 2, 3)$       | 5                              |
| $X \subset \mathbb{P}^7$ | 4   | $(2, 2, 2, 2)$    | 4                              |
Then \( \mathcal{E} \) is a stable and omalous.

**Proof.** Follows from the Main Theorem in [16] and the calculations above. \( \square \)

## 4 Omalous bundles on multi-blow-ups of the projective plane

Let \( \pi : \tilde{\mathbb{P}}(n) \to \mathbb{P}^2 \) be the blow-up of the projective plane at \( n \) distinct points. Its Picard group is generated by \( n+1 \) elements, namely: \( \text{Pic}(\tilde{\mathbb{P}}(n)) = \oplus_{i=1}^{n} E_i \mathbb{Z} \oplus H \mathbb{Z} \), where every \( E_i \) is an exceptional divisor and \( H \) is the divisor given by the pull-back of the generic line in \( \mathbb{P}^2 \). The intersection form is given by: \( E_i \cdot E_j = 0 \) for \( i \neq j \), \( E_i \cdot H = 0 \) and \( H^2 = 1 \). The canonical divisor of the surface \( \tilde{\mathbb{P}}(n) \) is given by \( K_{\tilde{\mathbb{P}}(n)} = -3H + \sum_{i=1}^{n} E_i \). In terms of line bundles, a divisor of the form \( D = pH + \sum_{i=1}^{n} q_i E_i \) has the associated line bundle \( \mathcal{O}(D) = \mathcal{O}(p, \overrightarrow{q}) = \mathcal{O}(pH) \otimes \mathcal{O}(q_1 E_1) \otimes \cdots \otimes \mathcal{O}(q_n E_n) \) where \( \overrightarrow{q} = (q_1, \cdots, q_n) \).

Let \( H^2 \in H^4(\tilde{\mathbb{P}}(n), \mathbb{Z}) \) be the fundamental class of \( \tilde{\mathbb{P}}(n) \). For a torsion-free sheaf \( \mathcal{E} \), of Chern character \( \text{ch}(\mathcal{E}) = r + (aH + \sum_{i=1}^{n} a_i E_i) - (k - \frac{s^2 - |\overrightarrow{q}|^2}{2})H^2 \), twisted by a line bundle \( \mathcal{O}(p, \overrightarrow{q}) \) the Riemann-Roch formula is given by:

\[
\chi(\mathcal{E}(p, \overrightarrow{q})) = \frac{-k}{2} - \frac{a}{2}(a+3) + \frac{1}{2} \sum_{i=1}^{n} a_i(a_i - 1) + \frac{r}{2}[(p+1)(p+2) - \sum_{i=1}^{n} q_i(q_i - 1)] + [ap - \sum_{i=1}^{n} a_i q_i].
\]

Note that the notations used through this section are the ones given in [15]. Omalous bundles \( \mathcal{E} \) on \( \tilde{\mathbb{P}}(n) \) are given in this case by the conditions:

\[
det(\mathcal{E}^*) = \omega_{\tilde{\mathbb{P}}(n)} = \mathcal{O}_{\tilde{\mathbb{P}}(n)}(-3, \overrightarrow{1}), \quad c_2(\mathcal{E}) = c_2(T\tilde{\mathbb{P}}(n)) = (3 + n) \cdot H^2
\]

Moreover, suppose that the direct image \( \pi_*(\mathcal{E}) \), of \( \mathcal{E} \), is a normalized and semi-stable torsion free sheaf (in our case, normalized means that \( 3 < r \)). Then we have the following:

**Proposition 4.1.** On a multi-blow-up \( \tilde{\mathbb{P}}(n) \) of the projective plane with \( n \geq 3 \),

let \( \mathcal{E} \) be an omalous bundle of rank \( r > 3 \) with semi-stable direct image \( \pi_*(\mathcal{E}) \).

Then \( \mathcal{E} \) is the cohomology of the following monad

\[
\xymatrix{ 0 \ar[r] & \oplus_{i=0}^{n} K_i(-1, E_i) \ar[r]^\alpha \ar[d] & W \otimes \mathcal{O}_{\tilde{\mathbb{P}}(n)} \ar[r]^\beta \ar[d] & \oplus_{i=0}^{n} L_i(1, -E_i) \ar[r] & 0 }
\]

where we put \( E_0 := 0 \) and \( K_i, L_i \) and \( W \) are vector space of dimensions

\[
\dim K_i = \begin{cases} n & i = 0 \\ 2n - 3 & \text{otherwise} \end{cases}, \quad \dim L_i = \begin{cases} 2n - 3 & i = 0 \\ 2n - 4 & \text{otherwise} \end{cases}
\]
and $\dim W = 4n(n-1) - 3 + r$.

**Proof.** The existence of the monad is guaranteed by [7, Proposition 1.10] since the direct image $\pi_*(\mathcal{E})$ is semi-stable and normalized ($r > 3$). The omality condition implies that the bundle $\mathcal{E}$ has the following Chern character $c_2(\mathcal{E}) = r + (3H - \Sigma_{i=1}^n E_i) - \frac{3}{2}(n-1)H^2$. The vector spaces in the monad are explicitly given by [7, Proposition 1.10]: $K_0 = H^1(\mathbb{P}(n), \mathcal{E}^*(-1,0))$, $K_1 = H^1(\mathbb{P}(n), \mathcal{E}(-1,0))$ for $i \neq 0$, and $L_0 = H^1(\mathbb{P}(n), \mathcal{E}(-1,0))$, $L_i = H^1(\mathbb{P}(n), \mathcal{E}(-1, E_i))$ for $i \neq 0$. Their dimensions follow by the Riemann-Roch Formula.

\[ \square \]

## 5 Omalous bundles on $\mathbb{P}^n \times \mathbb{P}^m$

Let $X = \mathbb{P}^n \times \mathbb{P}^m$, with the natural projections

\[ \pi_1: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n, \quad \pi_2: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m \]

Its Picard group is generated by $h_1 = \pi_2^* c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ and $h_2 = \pi_1^* c_1(\mathcal{O}_{\mathbb{P}^m}(1))$, then $\text{Pic}(X) = h_1 \mathbb{Z} \oplus h_2 \mathbb{Z}$. The Chow ring of $X$ is given by

\[ A(X) = \mathbb{Z}[h_1, h_2]/(h_1^{n+1}, h_2^{m+1}). \]

Let $L := \mathcal{O}_X(1,1)$ be the ample line bundle associated to the ample divisor $h_1 + h_2$. For any sheaf $\mathcal{F}$ on $X$, we define the degree of $\mathcal{F}$ with respect to $L$ as $\text{deg}_L(\mathcal{F}) := c_1(\mathcal{F}) \cdot c_1(L)^{n+m-1}$. Using the binomial formula one obtains

\[ c_1(L)^{n+m-1} = l(n,m)[h_1^{n-1} \cdot h_2^m + \frac{m}{n} h_1^n \cdot h_2^{m-1}] \]

where $l(n,m) = \binom{n+1}{m}(n+m+1)$.

It follows that if $c_1(\mathcal{F}) = p \cdot h_1 + q \cdot h_2$, then $\text{deg}_L(\mathcal{F}) = l(n,m)[p \cdot h_1^n \cdot h_2^m]$. We define the $L$-slope $\mu_L(\mathcal{F})$ of the sheaf $\mathcal{F}$ by $\mu_L(\mathcal{F}) := \frac{\text{deg}_L(\mathcal{F})}{\text{rk}(\mathcal{F})}$, and will say that $\mathcal{F}$ is $L$-(semi-)stable if for every subsheaf $\mathcal{G} \subset \mathcal{F}$ the inequality $\mu_L(\mathcal{G})(\leq) < \mu_L(\mathcal{F})$ is satisfied.

The tangent bundle of $X$ is given by the following Euler sequence:

\[ 0 \to \mathcal{O}_X^\oplus 2 \to \mathcal{O}_X^\oplus n+1(1,0) \oplus \mathcal{O}_X^\oplus n+1(0,1) \to TX \to 0, \]

from which one can easily compute the canonical bundle $\omega_X = \mathcal{O}_X(-n-1,-m-1)$, the first Chern class $c_1(TX) = c_1(X) = (n+1) \cdot h_1 + (m+1) \cdot h_2$ and the second Chern class $c_2(TX) = \frac{1}{2}(n+1)h_1^2 + \frac{1}{2}m(m+1)h_2^2 + (n+1)(m+1) \cdot h_1 \cdot h_2$. 

Now let us consider a rank \((b + c - a)\) bundle \(Q\) fitting in the following short exact sequence:

\[
0 \longrightarrow \mathcal{O}_X^{\oplus a} \longrightarrow \mathcal{O}_X(1,0)^{\oplus b} \oplus \mathcal{O}_X(0,1)^{\oplus c} \longrightarrow Q \longrightarrow 0. \tag{5.1}
\]

**Proposition 5.1.**

(i) \(Q\) is omalous for the values \((b,c) = (n+1,m+1)\).

(ii) The bundle \(Q\) is \(L\)-stable.

**Proof.**

(i) From the exact sequence defining the bundle \(Q\) one can easily compute the Chern classes \(c_1(Q) = b \cdot h_1 + c \cdot h_2\) and \(c_2(Q) = \frac{(b-1)}{2} \cdot h_1^2 + \frac{(c-1)}{2} \cdot h_1 + bc \cdot h_1 \cdot h_1\). The result follows by imposing the conditions \(c_1(Q) = c_1(X)\) and \(c_2(Q) = c_2(TX)\).

(ii) Note that the twisted dual bundle \(Q^*(0,1)\) fits in the following exact sequence

\[
0 \longrightarrow Q^*(0,1) \longrightarrow \mathcal{O}_X^{\oplus b}(-1,1) \oplus \mathcal{O}_X^{\oplus c} \longrightarrow \mathcal{O}_X(0,1)^{\oplus a} \longrightarrow 0. \tag{5.2}
\]

By using the first statement in [18, Theorem 8], it follows that \(Q^*(0,1)\) is \(L\)-stable, thus \(Q\) is also \(L\)-stable.

\[\square\]

In particular, when \(a = 2\) and the omalous conditions are satisfied, then \(Q\) has exactly rank \(n + m\), thus it is a deformation of \(T X\). Moreover it is easy to see that any deformation of \(T X\) is given as the last term bundle in the sequence \([5.1]\), with \(a = 2, b = n + 1\) and \(c = m + 1\). Hence we have the following:

**Corollary 5.2.** Any deformation of the tangent bundle \(T X\) of \(X = \mathbb{P}^n \times \mathbb{P}^m\) is \(L\)-stable.

**Proof.** Follows from (i) in [5.1]

\[\square\]
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