Global ultradifferentiable hypoellipticity on compact manifolds

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Abstract. We study the global hypoellipticity problem for certain linear operators in Komatsu classes of Roumieu and Beurling type on compact manifolds. We present an approach by combining a characterization of these spaces via eigenfunction expansions, generated by an elliptic operator, and the analysis of matrix-symbols obtained by these expansions.

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1. Introduction. We are interested in the study of global hypoellipticity in the setting of ultradifferentiable classes for certain linear operators on a compact manifold $X$. Broadly speaking, we aim to analyze the following problem: Let $\mathcal{F}(X)$ be an ultradifferentiable class of functions on $X$, $\mathcal{U}(X)$ its dual space, and $P : \mathcal{U}(X) \rightarrow \mathcal{U}(X)$ a linear operator. If $u \in \mathcal{U}(X)$ is such that $Pu \in \mathcal{F}(X)$, what conditions guarantee that $u \in \mathcal{F}(X)$?

One of the interests in the study of global hypoellipticity is the fact that local and global cases are rather different in general. For instance, there are classes of vector fields on the torus that are globally hypoelliptic despite not being locally hypoelliptic (see [6]). Also, the global properties are open problems, except for some particular classes of operators, that seem impossible to be solved by a unified approach.

Some authors, however, have obtained significant advances by considering special cases. For instance, classes of differential operators in the ultradifferentiable setting on the torus $\mathbb{T}^n$, e.g., [1, 4, 5, 10], and vector fields on compact Lie groups as presented by A. Kirilov, W.A.A. de Moraes, and M. Ruzhansky in [9]. Still on the topic of Lie groups, we quote the article [2], by G. Araújo,
directed to the study of global properties of a class of systems acting in some functional spaces such as analytic and Gevrey. In particular, his approach is based on a notion of invariance with respect to the Laplace–Beltrami operator.

Taking into account these facts, we attack the proposed problem by taking inspirations from the following two works: first, the approach presented by A. Dasgupta and M. Ruzhanky [3] characterizing Komatsu classes on compact manifolds in terms of a Fourier analysis generated by elliptic operators; in second, we extend the techniques used by A. Kirilov and W.A.A. de Moraes in [8] for the study of global properties in the $C^\infty$ sense for certain classes of invariant operators.

The characterizations given in [3] allow one to represent elements in ultradifferentiable classes as expansions of the type
\[ u = \sum_{\ell \in \mathbb{N}_0} \langle \hat{u}(\ell), e_\ell(x) \rangle_{C^{d_\ell}} \]
given by a fixed elliptic operator $E$. On the other hand, motivated by [8] (see also [2,7]), we use the fact that a linear $E$-invariant operator $P$, satisfying certain conditions, has a matrix-symbol $\sigma_P(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$, $\ell \in \mathbb{N}_0$, for which
\[ Pu = \sum_{\ell \in \mathbb{N}_0} \langle \sigma_P(\ell) \hat{u}(\ell), e_\ell(x) \rangle_{C^{d_\ell}}. \]

Therefore, a solution of the equation $Pu = f$ satisfies
\[ \sigma_P(\ell) \hat{u}(\ell) = \hat{f}(\ell) \quad \forall \ell \in \mathbb{N}_0, \]
and consequently its regularity can be completely characterized in terms of the behavior of $\sigma_P(\ell)$, as $\ell \to \infty$, in a suitable sense.

Our work is then organized as follows: Section 2 contains the ultradifferentiable spaces, as introduced in [3], and its characterizations in terms of Fourier coefficients. The classes of invariant operators are presented in Definition 2.6. Section 3 discusses the global hypoellipticity in the Roumieu case; as it shall be stated in Theorem 3.2, necessary and sufficient conditions are presented in terms of matrix-symbols. Section 4 extends these conditions to the Beurling case in Theorem 4.2.

2. Preliminaries. Let $X$ be a $n$-dimensional, closed, smooth manifold endowed with a positive measure $dx$. The inner product on the Hilbert space $L^2(X) = L^2(X, dx)$ is given by
\[ (f, g)_{L^2(X)} \doteq \int_X f(x)\overline{g(x)}dx. \]

By $\Psi^\nu(X)$ we denote the class of classical, positive, elliptic, pseudo-differential operators of order $\nu \in \mathbb{N}$. It is well know that if $E \in \Psi^\nu(M)$, then its spectrum is a discrete subset of the real line and coincides with the set of all its eigenvalues. Moreover, the eigenvalues of $E$ form a sequence
\[ 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_\ell \to \infty, \]
counting the multiplicity.
For each \( \ell \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), the corresponding eigenspace associated with \( \lambda_\ell \), which is finite dimensional, is denoted by \( \mathcal{H}_\ell \triangleq \ker(E - \lambda_\ell I) \). In particular, we may fix an orthonormal basis \( \{e_{\ell,k}\}_{k=1}^{d_\ell} \) on \( \mathcal{H}_\ell \) such that

\[
L^2(X) = \bigoplus_{\ell \in \mathbb{N}_0} \mathcal{H}_\ell,
\]

where \( d_\ell \triangleq \dim(\mathcal{H}_\ell) \). Therefore, for every \( f \in L^2(X) \), we have

\[
f(x) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_\ell} \hat{f}(\ell, k)e_{\ell,k}(x),
\]

with \( \hat{f}(\ell, k) = (f, e_{\ell,k})_{L^2(X)} \) for all \( \ell \in \mathbb{N}_0 \) and \( k = 1, \ldots, d_\ell \).

Given \( \ell \in \mathbb{N}_0 \), the vector

\[
\hat{f}(\ell) \triangleq \begin{pmatrix} \hat{f}(\ell, 1) \\ \vdots \\ \hat{f}(\ell, d_\ell) \end{pmatrix} \in \mathbb{C}^{d_\ell}
\]

is said to be a Fourier coefficient of \( f \in L^2(X) \). Its Hilbert Schmidt norm is

\[
\| \hat{f}(\ell) \|_{\text{HS}} \triangleq \left( \sum_{k=1}^{d_\ell} |\hat{f}(\ell, k)|^2 \right)^{1/2}
\]

and, by the Plancherel formula,

\[
\| f \|_{L^2(X)}^2 = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_\ell} |\hat{f}(\ell, k)|^2 = \sum_{\ell=0}^{\infty} \| \hat{f}(\ell) \|_{\text{HS}}^2.
\]

Moreover, \( f \in C^\infty(X) \) if and only if, given \( N > 0 \), there is \( C_N > 0 \) such that

\[
\| \hat{f}(\ell) \|_{\text{HS}} \leq C_N (1 + \lambda_\ell)^{-N} \quad \forall \ell \in \mathbb{N}.
\]

(2.1)

### 2.1. Ultradifferentiable classes.

We briefly recall the definitions of ultradifferentiable classes on \( X \) as introduced in [3]. For this, we denote by \( \mathcal{M} \triangleq \{M_k\}_{k \in \mathbb{N}_0} \) a sequence of real numbers satisfying the following: There are constants \( H > 0 \) and \( A \geq 1 \) such that

(M.0) \( M_0 = M_1 = 1 \),

(M.1) \( M_{k+1} \leq AH^k M_k \forall k \in \mathbb{N}_0 \),

(M.2) \( M_{2k} \leq AH^{2k} M_k^2 \forall k \in \mathbb{N}_0 \),

(M.3) \( M_k^2 \leq M_{k-1} M_{k+1} \forall k \in \mathbb{N} \).

Also, for the Roumieu (Beurling) case, we assume that there are constants \( L > 0 \) and \( C > 0 \) (for every \( L > 0 \), there exists \( C > 0 \)) such that

\[
k! \leq CL^k M_k \forall k \in \mathbb{N}_0.
\]

Associated to \( \mathcal{M} \), we define the function \( \mathcal{M} : [0, \infty) \to \mathbb{R} \) as follows:

\[
\mathcal{M}(0) \triangleq 0 \quad \text{and} \quad \mathcal{M}(r) \triangleq \sup_{k \in \mathbb{N}} \left\lfloor \log \left( \frac{r^k}{M_{\nu k}} \right) \right\rfloor \quad \forall r > 0.
\]
The function $M$ is non-decreasing and, in view of [3, proof of Theorem 4.2], satisfies the following property.

**Lemma 2.1.** Given $\ell \in \mathbb{N}$ and $L > 0$, we have

$$\exp\left(-\frac{1}{2} M(L \lambda_\ell^{1/\nu})\right) \leq \exp\left(-M(\tilde{L} \lambda_\ell^{1/\nu})\right) \quad \text{for} \quad \tilde{L} = \frac{L}{\sqrt{AH}}.$$

We are now in position to introduce the ultradifferentiable classes.

**Definition 2.2.** Let $E \in \Psi^\nu(X)$ be fixed. We denote by $EM(X)$ the Roumieu (Beurling) space of functions $\phi \in C^\infty(X)$ such that there exists $h > 0$ and $C > 0$ (for all $h > 0$, there exists $C_h > 0$) satisfying

$$\|E^k \phi\|_{L^2(X)} \leq C h^\nu k M_{\nu k} \quad \forall k \in \mathbb{N}_0,$$

$$\|E^k \phi\|_{L^2(X)} \leq C_h h^\nu M_{\nu k} \quad \forall k \in \mathbb{N}_0.$$

**Remark 2.3.** It follows from [3, Theorem 2.3] that these spaces are independent of a particular choice of the operator $E \in \Psi^\nu(X)$ and the order $\nu$, justifying the notations $EM(X)$ and $E(M)(X)$.

The symbols $E' M(X)$ and $E'(M)(X)$ denote the space of ultradistributions of Roumieu and Beurling type, respectively. The Fourier coefficients of an ultradistribution $u$ are defined by expressions

$$\widehat{u}(\ell,k) = u(e^{\ell,k}), \quad k = 1, \ldots, d_\ell.$$

The next results characterize the ultradifferentiable classes in terms of the Fourier coefficients and the eigenvalues of the operator $E$.

**Theorem 2.4.** A smooth function $\phi$ on $X$ belongs to $E(M)(X)$ if and only if there are positive constants $C$ and $L$ (for every $L > 0$, there is $C_L > 0$) such that

$$\|\hat{\phi}(\ell)\|_{HS} \leq C \exp\left[-M(L \lambda_\ell^{1/\nu})\right] \quad \forall \ell \in \mathbb{N}_0,$$

$$\left(\|\hat{\phi}(\ell)\|_{HS} \leq C_L \exp\left[-M(L \lambda_\ell^{1/\nu})\right] \quad \forall \ell \in \mathbb{N}_0\right).$$

**Theorem 2.5.** A linear functional $u : E(M)(X) \to \mathbb{C}$ belongs to $E'(M)(X)$ if and only if for every $L > 0$, there exists $K_L > 0$ (there exists $K > 0$ and $L > 0$) such that

$$\|\hat{u}(\ell)\|_{HS} \leq K_L \exp\left[M(L \lambda_\ell^{1/\nu})\right] \quad \forall \ell \in \mathbb{N}_0,$$

$$\left(\|\hat{u}(\ell)\|_{HS} \leq K \exp\left[M(L \lambda_\ell^{1/\nu})\right] \quad \forall \ell \in \mathbb{N}_0\right).$$
2.2. Invariant operators on ultradifferentiable classes. We now introduce the classes of operators under investigations. We emphasize that at this point, we are taking inspirations from [8, Proposition 2.1 and Definition 2.2].

Definition 2.6. Let \( P : C^\infty(X) \to C^\infty(X) \) be a linear operator with continuous extension to \( \mathcal{D}'(X) \) and assume that the domain of \( P^* \), the adjoint of \( P \), contains \( C^\infty(X) \). Given \( E \in \Psi^\nu(X) \), we say that \( P \) is:

1. \( C^\infty \)-invariant if
   \[ P(H_\ell) \subseteq H_\ell \quad \forall \ell \in \mathbb{N}_0. \]
2. \( \mathcal{M} \)-invariant if:
   (a) \( P \) is \( C^\infty \)-invariant;
   (b) \( P(\mathcal{E}_\mathcal{M}(X)) \subseteq \mathcal{E}_\mathcal{M}(X) \);
   (c) \( P : \mathcal{E}_\mathcal{M}(X) \to \mathcal{E}_\mathcal{M}(X) \) has continuous extension to \( \mathcal{E}'_\mathcal{M}(X) \).
3. \((\mathcal{M})\)-invariant if:
   (a) \( P \) is \( C^\infty \)-invariant;
   (b) \( P(\mathcal{E}_{(\mathcal{M})}(X)) \subseteq \mathcal{E}_{(\mathcal{M})}(X) \);
   (c) \( P : \mathcal{E}_{(\mathcal{M})}(X) \to \mathcal{E}_{(\mathcal{M})}(X) \) has continuous extension to \( \mathcal{E}'_{(\mathcal{M})}(X) \).

Note that if \( P \) is \( \mathcal{M} \)-invariant ((\(\mathcal{M}\))-invariant), then there exists a sequence of matrices \( \sigma_P(\ell) \in \mathbb{C}^{d_\ell \times d_\ell} \), \( \ell \in \mathbb{N}_0 \), such that
\[
\hat{P}u(\ell) = \sigma_P(\ell)\hat{u}(\ell) \quad \forall u \in \mathcal{E}'_{\mathcal{M}}(X) \quad (\forall u \in \mathcal{E}'_{(\mathcal{M})}(X)),
\] implying
\[
P u = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \hat{P}u(\ell,m)e_{\ell,m} = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} (\sigma_P(\ell)\hat{u}(\ell))_m e_{\ell,m}.
\]
We say that \( \{\sigma_P(\ell)\}_{\ell \in \mathbb{N}_0} \) is the matrix-symbol of \( P \).

Finally, in order to connect the global hypoellipticity of an invariant operator \( P \) to the behavior of its matrix-symbol, we set the following number:

\[
m(\sigma_P(\ell)) = \inf \{ \|\sigma_P(\ell)v\|_{HS}; v \in \mathbb{C}^{d_\ell} \text{ and } \|v\|_{HS} = 1 \}, \quad \ell \in \mathbb{N}_0.
\]

3. Global \( \mathcal{M} \)-hypoellipticity. In this section, we analyze the global hypoellipticity problem in the Roumieu case.

Definition 3.1. We say that a linear operator \( P : \mathcal{E}'_{\mathcal{M}}(X) \to \mathcal{E}'_{\mathcal{M}}(X) \) is globally \( \mathcal{M} \)-hypoelliptic if the conditions
\[ u \in \mathcal{E}'_{\mathcal{M}}(X) \quad \text{and} \quad Pu \in \mathcal{E}_{\mathcal{M}}(X) \]
imply \( u \in \mathcal{E}_{\mathcal{M}}(X) \).

Theorem 3.2. An \( \mathcal{M} \)-invariant operator \( P \) is globally \( \mathcal{M} \)-hypoelliptic if and only if for every \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that
\[
m(\sigma_P(\ell)) \geq \exp \left( -\mathcal{M}(\epsilon \lambda_1^{1/\nu}) \right) \quad \forall \ell \geq C_\epsilon.
\]
Proof. Let us start with the sufficiency. For this, assume condition (3.1) and let \( u \in \mathcal{E}_{\mathbb{H}}(X) \) be a solution of \( Pu = \phi \in \mathcal{E}_{\mathbb{H}}(X) \). By (2.2) and (2.3), we have

\[
\sigma_{P}(\ell) \hat{u}(\ell) = \hat{\phi}(\ell) \quad \forall \ell \in \mathbb{N}_0.
\]

Since \( \phi \in \mathcal{E}_{\mathbb{H}}(X) \), there are constants \( C > 0 \) and \( L_{\phi} > 0 \) such that

\[
\| \hat{\phi}(\ell) \|_{\mathcal{H}S} \leq C \exp \left( -\mathcal{M}(L_{\phi} \lambda_{\ell}^{1/\nu}) \right) \quad \forall \ell \in \mathbb{N}_0,
\]

and setting \( \hat{L}_{\phi} = \frac{L_{\phi}}{\sqrt{A_{\ell}}} \), we obtain from Lemma 2.1 that

\[
\| \hat{\phi}(\ell) \|_{\mathcal{H}S} \leq C \exp \left( -\mathcal{M}(\hat{L}_{\phi} \lambda_{\ell}^{1/\nu}) \right) \exp \left( -\mathcal{M}(\hat{L}_{\phi} \lambda_{\ell}^{1/\nu}) \right) \quad \forall \ell \in \mathbb{N}_0.
\]

Let \( \epsilon = \hat{L}_{\phi} \) and consider \( C_{\epsilon} > 0 \) satisfying (3.1). Then \( m(\sigma_{P}(\ell)) \neq 0 \) for all \( \ell \geq C_{\epsilon} \), implying

\[
\hat{u}(\ell) = \sigma_{P}(\ell)^{-1} \hat{\phi}(\ell) \quad \forall \ell \geq C_{\epsilon}.
\]

Hence, by the identity \( \| \sigma_{P}(\ell)^{-1} \|_{\mathcal{L}(C^{d_{\ell}})} = m(\sigma_{P}(\ell))^{-1} \), we obtain

\[
\| \hat{u}(\ell) \|_{\mathcal{H}S} \leq \exp \left[ \mathcal{M}(\hat{L}_{\phi} \lambda_{\ell}^{1/\nu}) \right] \| \hat{\phi}(\ell) \|_{\mathcal{H}S} \leq C \exp \left[ -\mathcal{M}(\hat{L}_{\phi} \lambda_{\ell}^{1/\nu}) \right]
\]

for all \( \ell \geq C_{\epsilon} \). Then \( u \in \mathcal{E}_{\mathbb{H}}(X) \) and the sufficiency is proved.

For the necessary part, we proceed by a contradiction argument: we assume that (3.1) fails and exhibit \( u \in \mathcal{E}_{\mathbb{H}}(X) \) such that \( Pu \in \mathcal{E}_{\mathbb{H}}(X) \).

Under this assumption, there exists \( \epsilon_0 > 0 \) satisfying the following: for every \( C > 0 \), there is \( \ell > C \) such that

\[
m(\sigma_{P}(\ell)) < \exp \left( -\mathcal{M}(\epsilon_0 \lambda_{\ell}^{1/\nu}) \right).
\]

In particular, for \( C = 1 \), we obtain \( \ell_1 > 1 \) such that

\[
m(\sigma_{P}(\ell_1)) < \exp \left( -\mathcal{M}(\epsilon_0 \lambda_{\ell_1}^{1/\nu}) \right),
\]

and \( v_{\ell_1} \in C^{d_{\ell_1}} \) satisfying \( \| v_{\ell_1} \|_{\mathcal{H}S} = 1 \) and

\[
\| \sigma_{P}(\ell_1) v_{\ell_1} \|_{\mathcal{H}S} < \exp \left( -\mathcal{M}(\epsilon_0 \lambda_{\ell_1}^{1/\nu}) \right).
\]

Hence, inductively, we obtain \( \{ v_{\ell_k} \}_{k \in \mathbb{N}} \) such that \( \| v_{\ell_k} \|_{\mathcal{H}S} = 1 \) and

\[
\| \sigma_{P}(\ell_k) v_{\ell_k} \|_{\mathcal{H}S} < \exp \left( -\mathcal{M}(\epsilon_0 \lambda_{\ell_k}^{1/\nu}) \right) \quad \forall k \in \mathbb{N}.
\]

(3.2)

Now, we set

\[
\hat{u}(\ell) = \begin{cases} v_{\ell_k} & \text{if } \ell = \ell_k, \\ 0 & \text{if } \ell \neq \ell_k, \end{cases}
\]

and \( u = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_{\ell}} \hat{u}(\ell, m)e_{\ell, m} \).

Let us to show that \( u \in \mathcal{E}_{\mathbb{H}}(X) \setminus \mathcal{E}_{\mathbb{H}}(X) \). Given \( L > 0 \), from the fact that \( \mathcal{M} \) is non-decreasing and \( 0 = \lambda_0 < \lambda_k < \lambda_{k+1}, k \in \mathbb{N} \), we get

\[
\| \hat{u}(\ell) \|_{\mathcal{H}S} \leq 1 < \exp \left( \mathcal{M}(L \lambda_{\ell}^{1/\nu}) \right) \quad \forall \ell \in \mathbb{N}_0.
\]
Thus, \( u \in \mathcal{E}'(X) \) in view of Theorem 2.5. Moreover, since \( \|\widehat{u}(\ell_k)\|_{\text{HS}} = 1 \) for all \( k \in \mathbb{N} \), it follows from (2.1) that \( u \notin C^\infty(X) \) and consequently \( u \notin \mathcal{E}'(X) \).

Finally, note that
\[
P u = \sum_{k=1}^{\infty} \sum_{r=1}^{d_{\ell_k}} (\sigma_P(\ell_k)\widehat{u}(\ell_k))_r e_{\ell_k,r},
\]
hence, by (3.2),
\[
\|\widehat{P} u(\ell_k)\|_{\text{HS}} = \|\sigma_P(\ell_k)v_{\ell_k}\|_{\text{HS}} < \exp \left( -\mathcal{M}(\epsilon_0\lambda_{\ell_k}^{1/\nu}) \right) \quad \forall k \in \mathbb{N}.
\]

Choosing \( C = 1 \) and \( L = \epsilon_0 \), it follows from Theorem 2.4 that \( Pu \) belongs to \( \mathcal{E}'(X) \), implying that \( P \) is not globally \( \mathcal{M} \)-hypoelliptic. \( \square \)

**Remark 3.3.** Note that if we pick \( \mathcal{M} = \{(k!)^s\}_{k \in \mathbb{N}}, 1 < s < \infty \), we may see \( \mathcal{E}'(X) \) as the Gevrey class \( \mathcal{G}^s(X) \) on \( X \) since \( \mathcal{M}(r) \sim r^{1/s} \), as \( r \to \infty \).

Therefore, a \( \mathcal{G}^s \)-invariant operator \( P \) is globally \( \mathcal{G}^s \)-hypoelliptic if and only if for every \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that
\[
m(\sigma_P(\ell)) \geq \exp \left( -\epsilon \lambda_{\ell}^{1/\nu} \right) \quad \forall \ell \geq C_\epsilon.
\]

**Remark 3.4.** We recall that A. Kirilov and W.A.A. de Moraes studied the global hypoellipticity in the \( C^\infty \) sense, namely, the problem
\[
u \in \mathcal{D}'(X) \quad \text{and} \quad Pu \in C^\infty(X) \implies u \in C^\infty(X).
\]

They proved that a \( C^\infty \)-invariant operator \( P \) is globally \( C^\infty \)-hypoelliptic if and only if there are positive constants \( L, M, \) and \( R \) such that
\[
m(\sigma_P(\ell)) \geq L(1 + \lambda_\ell)^{M/\nu} \quad \forall \ell \geq R.
\]

Notice that given \( \epsilon > 0 \), there is \( C_\epsilon > 0 \) satisfying
\[
(1 + \lambda_\ell)^{M/\nu} \geq \exp \left( -\mathcal{M}(\epsilon \lambda_{\ell}^{1/\nu}) \right) \quad \forall \ell \geq C_\epsilon
\]
since \( \lambda_\ell \to \infty \) and \( \mathcal{M} \) is a positive non-decreasing function. By this discussion, we have the following: if \( P \) is globally \( C^\infty \)-hypoelliptic and \( \mathcal{M} \)-invariant, then it is also globally \( \mathcal{M} \)-hypoelliptic.

4. **Global \( (\mathcal{M}) \)-hypoellipticity.** We now consider the global hypoellipticity problem in the Beurling case.

**Definition 4.1.** We say that a linear operator \( P : \mathcal{E}'(X) \to \mathcal{E}'(X) \) is globally \( (\mathcal{M}) \)-hypoelliptic if the conditions
\[
u \in \mathcal{E}'(X) \quad \text{and} \quad Pu \in \mathcal{E}'(X) \implies u \in \mathcal{E}'(X).
\]

**Theorem 4.2.** An \( (\mathcal{M}) \)-invariant operator \( P \) is globally \( (\mathcal{M}) \)-hypoelliptic if and only if there are positive constants \( K, r, \) and \( C \) such that
\[
m(\sigma_P(\ell)) \geq K \exp \left( -\mathcal{M}(r\lambda_{\ell}^{1/\nu}) \right) \quad \text{for all} \quad \ell \geq C.
\]
Proof. Consider \( u \in E'_\langle,\mathcal{M}\rangle(X) \) such that \( Pu = \phi \in E'_\mathcal{M}(X) \) and assume condition (4.1). In this case, we have \( \hat{u}(\ell) = \sigma_P(\ell)^{-1}\hat{\phi}(\ell) \), and
\[
\|\hat{u}(\ell)\|_{HS} \leq \frac{1}{K} \exp \left(\mathcal{M}(r\lambda^{1/\nu}_\ell)\right) \|\hat{\phi}(\ell)\|_{HS} \quad \forall \ell \geq C.
\]
Once \( \phi \in E'_\langle,\mathcal{M}\rangle(X) \), we obtain for every \( L' > 0 \), a constant \( C_{L'} > 0 \) such that
\[
\|\hat{u}(\ell)\|_{HS} \leq \frac{C_{L'}}{K} \exp \left(\mathcal{M}(r\lambda^{1/\nu}_\ell)\right) \exp \left(-\mathcal{M}(L'\lambda^{1/\nu}_\ell)\right). \tag{4.2}
\]
Consider \( L > 0 \) be fixed. If \( r \leq L \), we have
\[
\exp \left(\mathcal{M}(r\lambda^{1/\nu}_\ell)\right) \leq \exp \left(\mathcal{M}(L\lambda^{1/\nu}_\ell)\right) \quad \forall \ell \in \mathbb{N},
\]
and by (4.2), we get
\[
\|\hat{u}(\ell)\|_{HS} \leq \frac{C_L}{K} \exp \left(-\mathcal{M}(L\lambda^{1/\nu}_\ell)\right) \quad \forall \ell \geq C.
\]
On the other hand, if \( r > L \), a new application of Lemma 2.1, with \( L' \doteq r\sqrt{AH} \), gives
\[
\exp \left(-\mathcal{M}(r\sqrt{AH}\lambda^{1/\nu}_\ell)\right) \leq \exp \left(-2\mathcal{M}(r\lambda^{1/\nu}_\ell)\right) \quad \forall \ell \in \mathbb{N}.
\]
Hence, for any \( L > 0 \), there exists \( C_L > 0 \) such that
\[
\|\hat{u}(\ell)\|_{HS} \leq C_L \exp \left(-\mathcal{M}(L\lambda^{1/\nu}_\ell)\right) \quad \forall \ell \geq C,
\]
implying \( \phi \in E'_\langle,\mathcal{M}\rangle(X) \). Therefore, we have proved the sufficiency of (4.1).

Conversely, let us assume that (4.1) fails. In this case, for every positive constants \( K, r, \) and \( C \), there is \( \ell > C \) such that
\[
m(\sigma_P(\ell)) < K \exp \left(-\mathcal{M}(r\lambda^{1/nu}_\ell)\right).
\]
Then, by an inductive argument, we may construct a sequence \( \{v_{\ell_k}\}_{k \in \mathbb{N}} \) such that \( v_{\ell_k} \in \mathbb{C}^{d_{\ell_k}} \), \( \|v_{\ell_k}\|_{HS} = 1 \), and
\[
\|\sigma_P(\ell_k)v_{\ell_k}\|_{HS} < \exp \left(-\mathcal{M}(k\lambda^{1/\nu}_{\ell_k})\right) \quad \forall k \in \mathbb{N}.
\]
Now, consider
\[
\hat{u}(\ell) \doteq \begin{cases} v_{\ell_k} & \text{if } \ell = \ell_k, \\ 0 & \text{if } \ell \neq \ell_k, \end{cases}
\]
and \( u \doteq \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_{\ell}} \hat{u}(\ell, m)e_{\ell, m} \).
We claim that \( u \in \mathcal{E}'_{(\mathcal{M})}(X)\backslash \mathcal{E}_{(\mathcal{M})}(X) \). Indeed,
\[
\| \widehat{u}(\ell) \|_{\text{HS}} \leq \exp \left( -M(\lambda_{\ell}^{1/\nu}) \right) \exp \left( M(\lambda_{\ell}^{1/\nu}) \right) \\
\leq \exp \left( -M(\lambda_0^{1/\nu}) \right) \exp \left( M(\lambda_{\ell}^{1/\nu}) \right) \\
= \exp \left( M(\lambda_{\ell}^{1/\nu}) \right)
\]
for all \( \ell \in \mathbb{N} \) since \( \lambda_0 = 0 \). Once \( \| \widehat{u}(\ell_k) \|_{\text{HS}} = 1 \) \( \forall k \in \mathbb{N} \), it follows that \( u \in \mathcal{E}'_{(\mathcal{M})}(X)\backslash \mathcal{E}_{(\mathcal{M})}(X) \).

Finally, we prove that \( Pu \in \mathcal{E}_{(\mathcal{M})}(X) \). Since
\[
P u = \sum_{k=1}^{\infty} \sum_{m=1}^{d_{\ell_k}} (\sigma_P(\ell_k) \widehat{u}(\ell_k))_{m} e_{\ell_k,m},
\]
we have
\[
\| \widehat{P u}(\ell_k) \|_{\text{HS}} = \|\sigma_P(\ell_k) v_{\ell_k} \|_{\text{HS}} < \exp \left( -M(k\lambda_{\ell_k}^{1/\nu}) \right) \quad \forall k \in \mathbb{N}.
\]
Consider \( L > 0 \) be fixed. For all \( k \geq L \),
\[
\| \widehat{P u}(\ell_k) \|_{\text{HS}} < \exp \left( -M(k\lambda_{\ell_k}^{1/\nu}) \right) \leq \exp \left( -M(L\lambda_{\ell_k}^{1/\nu}) \right),
\]
while if \( k < L \), we may obtain \( C_{L,k} > 0 \) such that
\[
\exp \left( -M(k\lambda_{\ell_k}^{1/\nu}) \right) \leq C_{L,k} \exp \left( -M(L\lambda_{\ell_k}^{1/\nu}) \right),
\]
implicating
\[
\| \widehat{P u}(\ell_k) \|_{\text{HS}} \leq C_L \exp \left( -M(L\lambda_{\ell_k}^{1/\nu}) \right), \quad k < L,
\]
where \( C_L \) is the maximum of all constants \( C_{L,k} \) satisfying (4.3) for \( 1 \leq k < L \).

Therefore, it follows from Theorem 2.4 that \( Pu \in \mathcal{E}_{(\mathcal{M})}(X) \). This proves that \( P \) is not globally \( (\mathcal{M}) \)-hypoelliptic and the necessity of (4.1).

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