Research article

$n$–polynomial exponential type $p$–convex function with some related inequalities and their applications

Saad Ihsan Butt a, Artion Kashuri b, Muhammad Tariq a, Jamshed Nasir c, Adnan Aslam d, Wei Gao e,*

* COMSATS University Islamabad, Lahore Campus, Pakistan
b Department of Mathematics, Faculty of Technical Science, University “Ismail Qemali”, Vlorë, Albania
© Virtual University of Pakistan, Lahore Campus, Pakistan
d Department of Natural Sciences and Humanities, University of Engineering and Technology, Lahore (RCET), Pakistan
e School of Information Science and Technology, Yunnan Normal University, Kunming, 650500, China

A R T I C L E   I N F O

Keywords:
Mathematics
Hermite-Hadamard inequality
Hölder’s inequality
Power mean inequality
Convexity
Exponential type convexity

A B S T R A C T

In this paper, the idea and its algebraic properties of $n$–polynomial exponential type $p$–convex function have been investigated. Authors prove new trapezium type inequality for this new class of functions. We also obtain some refinements of the trapezium type inequality for functions whose first derivative in absolute value at certain power are $n$–polynomial exponential type $p$–convex. At the end, some new bounds for special means of different positive real numbers are provided as well. These new results yield us some generalizations of the prior results. Our idea and technique may stimulate further research in different areas of pure and applied sciences.

1. Introduction

During the last century theory of convexity had been developed rapidly. Several new classes of classical convexity have been proposed in literature. Many interesting generalization and extension of classical convexity have been used in optimization and mathematical inequalities. Theory of convexity also played significant role in the development of theory of inequalities. Inequalities present a very active and fascinating field of research. Inequalities are one of the most important instrument in many branches of engineering and mathematics such as measure theory, probability theory, functional analysis, mathematical analysis, mechanics, physics and theory of differential and integral equations. Nowadays the theory of inequalities is still being intensively developed. Eventually the theory of inequalities may be regarded as an independent area of mathematics. For the applications of inequalities interested readers refer to [1, 2, 3, 4, 5, 6]. In recent years, a wide class of integral inequalities is being derived via different concepts of convexity. These integral inequalities are useful in physics. Theory of convexity also played significant and central role in many areas, such as economics, optimization, management science, finance, engineering and game theory. In mathematics, theory of convexity yields have fruitful laws and methods for studying a class of problems.

Definition 1. [7] A function \( \varphi : I \to \mathbb{R} \) is said to be convex, if

\[
\varphi \left( k \theta_1 + (1-k) \theta_2 \right) \leq k \varphi (\theta_1) + (1-k) \varphi (\theta_2)
\]

holds for all \( \theta_1, \theta_2 \in I \) and \( k \in [0, 1] \).

If the above inequality reverses, then \( \varphi \) is said to be concave.

Many famously known results in inequalities theory can be obtained using the convexity property of the functions, see [8, 9, 10] and the references therein.

Hermite–Hadamard’s inequality is one of the most intensively studied result involving convex functions. This result provides us necessary and sufficient condition for a function to be convex. It is also known as classical equation of (H–H) inequality. The Hermite–Hadamard inequality asserts that, if a function \( \varphi : I \subset \mathbb{R} \to \mathbb{R} \) is convex in \( I \) for \( \theta_1, \theta_2 \in I \) and \( \theta_1 < \theta_2 \), then

\[
\varphi \left( \frac{\theta_1 + \theta_2}{2} \right) \leq \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \varphi(k)dk \leq \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2}.
\]
Interested readers can refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

**Definition 2.** [25] A nonnegative function $\varphi : I \to \mathbb{R}$ is said to be $h$-convex, if

$$\varphi\left(k\theta_1 + (1-k)\theta_2\right) \leq h(k)\varphi(\theta_1) + h(1-k)\varphi(\theta_2)$$

(1.3)

holds for all $\theta_1, \theta_2 \in I$ and $k \in [0,1]$. 

$\varphi$ is called $h$-concave, if Definition 2 is reversed. Clearly that, if we substitute $h(k) = k$, then the $h$-convex functions reduce to the classical convex functions, see [12, 19].

**Definition 3.** [18] If $I \subset (0, +\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $\varphi : I \to \mathbb{R}$ is said to be a $p$-convex, if

$$\varphi\left(\varphi^{\theta_1^p} + (1-k)\varphi^{\theta_2^p}\right)^{\frac{1}{p}} \leq k\varphi(\theta_1)^p + (1-k)\varphi(\theta_2)^p$$

(1.4)

holds for all $\theta_1, \theta_2 \in I$ and $k \in [0,1]$. $\varphi$ is $p$-concave if the above inequality is reversed.

According to Definition 3, if we put $p = 1$, then $p$-convex functions reduce to ordinary convex functions defined in $I \subset (0, +\infty)$.

**Definition 4.** [7] A nonnegative function $\varphi : I \to \mathbb{R}$ is said to be exponential type convex function, if

$$\varphi\left(k\theta_1 + (1-k)\theta_2\right) \leq (e^\kappa - 1)\varphi(\theta_1) + (e^{1\kappa} - 1)\varphi(\theta_2)$$

(1.5)

holds for all $\theta_1, \theta_2 \in I$ and $k \in [0,1]$. 

The family of all exponential type convex function on $I$ is represented by $EXPC(I)$.

We recall the following hypergeometric function:

$$2F_1\left(\theta_1,\theta_2;\theta;\theta\right) = \frac{1}{\beta(\theta_2, \theta_3 - \theta_2)} \int_0^1 \kappa^{\theta_2 - 1} (1-k)^{\theta_1 - \theta_2 - 1} (1 - \theta k)^{-\theta_1} \, dk,$$

where $\theta_1 > \theta_2 > 0$, $|\theta| < 1$ and $\beta(\cdot, \cdot)$ is Euler beta function.

Motivated by above results and literatures, we will give first in Section 2, the idea and its algebraic properties of $n$-polynomial exponential type $p$-convex function. In Section 3, we will prove a trapezium type inequality for the $n$-polynomial exponential type $p$-convex function $\varphi$. In Section 4, we will obtain some refinements of the (H-H) inequality for functions whose first derivative in absolute value at certain power are $n$-polynomial exponential type $p$-convex. In Section 5 some new bounds for special means will be provided. In Section 6, a brief conclusion will be given as well.

### 2. Some algebraic properties of $n$-polynomial exponential type $p$-convex functions

We are going to add a new definition in this section namely $n$-polynomial exponential type $p$-convex function and study some of its basic algebraic properties. Before proving the results, we want to mention here that $n$ represents finite positive integer.

**Definition 5.** A nonnegative function $\varphi : I \to \mathbb{R}$ is said to be exponential type $p$-convex, if

$$\varphi\left(\varphi^{\theta_1^p} + (1-k)\varphi^{\theta_2^p}\right)^{\frac{1}{p}} \leq \varphi(\theta_1)^p + (e^{1\kappa} - 1)\varphi(\theta_2)^p$$

(2.1)

holds for all $\theta_1, \theta_2 \in I$ and $k \in [0,1]$. 

**Remark 1.** If we put $p = 1$, we get exponential type convexity given by lçcan in [7].

**Remark 2.** The range of the new class of functions defined in Definition 5 is $[0, +\infty)$.

**Proof.** Using the Definition 5 for $k = 1$ and $\theta \in I$, we have

$$\varphi(\theta) \leq (e - 1)\varphi(\theta) \implies (e - 2)\varphi(\theta) \geq 0 \implies \varphi(\theta) \geq 0.$$  

**Definition 6.** [24] A nonnegative function $\varphi : I \to \mathbb{R}$ is called $n$-polynomial convex, if

$$\varphi\left(k\theta_1 + (1-k)\theta_2\right) \leq \frac{1}{n} \sum_{i=1}^{n} \left[1 - (1 - k)^i\right] \varphi(\theta_1) + \frac{1}{n} \sum_{i=1}^{n} \left[1 - k^i\right] \varphi(\theta_2)$$

(2.2)

holds for every $\theta_1, \theta_2 \in I$, $n \in \mathbb{N}$ and $k \in (0, 1]$.

We can give now a new definition, namely $n$-polynomial exponential type $p$-convex function as follows:

**Definition 7.** A nonnegative function $\varphi : I \to \mathbb{R}$ is said to be $n$-polynomial exponential type $p$-convex, if

$$\varphi\left(\varphi^{\theta_1^p} + (1-k)\varphi^{\theta_2^p}\right)^{\frac{1}{p}} \leq \frac{1}{n} \sum_{i=1}^{n} (e^\kappa - 1)^i \varphi(\theta_1)^p + \frac{1}{n} \sum_{i=1}^{n} (e^{1\kappa} - 1)^i \varphi(\theta_2)^p$$

(2.3)

holds for all $\theta_1, \theta_2 \in I$ and $k \in [0,1]$.

We study some associations between the classes of $n$-polynomial exponential type $p$-convex and generalized convex functions.

**Lemma 1.** For all $\kappa \in [0,1]$ the following inequalities hold:

$$\frac{1}{n} \sum_{i=1}^{n} (e^\kappa - 1)^i \geq \kappa \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} (e^{1\kappa} - 1)^i \geq (1 - \kappa).$$

**Proof.** The proof is evident. 

**Proposition 1.** Let $I \subset (0, +\infty)$ be a $p$-convex set. Every $p$-convex function on a $p$-convex set is $n$-polynomial exponential type $p$-convex function.

**Proof.** From Lemma 1 and the Definition 3, we have

$$\varphi\left(k\theta_1^p + (1-k)\theta_2^p\right)^{\frac{1}{p}} \leq \kappa \varphi(\theta_1) + (1-k)\varphi(\theta_2)$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} (e^\kappa - 1)^i \varphi(\theta_1)^p + \frac{1}{n} \sum_{i=1}^{n} (e^{1\kappa} - 1)^i \varphi(\theta_2)^p.$$  

**Remark 3.** (i) Taking $n = 1$ in Proposition 1, then

$$\varphi\left(\varphi^{\theta_1^p} + (1-k)\varphi^{\theta_2^p}\right)^{\frac{1}{p}} \leq \kappa \varphi(\theta_1) + (1-k)\varphi(\theta_2)$$

$$\leq (e^\kappa - 1) \varphi(\theta_1) + (e^{1\kappa} - 1) \varphi(\theta_2).$$

(ii) Taking $p = 1$ in Proposition 1, then we have Proposition 2.1 in [7].

**Proposition 2.** Every $n$-polynomial exponential type $p$-convex function is an $h$-convex function with $h(\kappa) = (e^\kappa - 1)$.

**Proof.** If we put $h(\kappa) = (e^\kappa - 1)$ and $h(1-\kappa) = (e^{1\kappa} - 1)$ in Definition 7, then the Definition 2 is easily obtained.
Now we make some examples via newly introduce definition \(n\)-polynomial exponential type \(p\)-convex function.

**Example 1.** If \(\varphi(x) = x^p\) is \(p\)-convex function for all positive values of \(x\) and \(p \in (-\infty, 0) \cup [1, \infty)\) \([18]\), then by using Proposition 1, it is an \(n\)-polynomial exponential type \(p\)-convex function.

**Example 2.** Let \(\varphi : (0, \infty) \rightarrow \mathbb{R}, \psi(x) = x^{-p}\), then \(\psi\) is \(p\)-convex function \([18]\), so by using Proposition 1, it is an \(n\)-polynomial exponential type \(p\)-convex function.

**Example 3.** Let \(\varphi : (0, \infty) \rightarrow \mathbb{R}, \psi(x) = -\ln x\) and \(p \geq 1\), then \(\psi\) is \(p\)-convex function \([18]\), so by using Proposition 1, it is an \(n\)-polynomial exponential type \(p\)-convex function.

These are the clear advantages of the proposed new definition with respect to other known functions on the topic mentioned above. Now, we will study some of its algebraic properties.

**Theorem 1.** Let \(\varphi, \psi : [\theta_1, \theta_2] \rightarrow \mathbb{R}\). If \(\varphi\) and \(\psi\) are two \(n\)-polynomial exponential type \(p\)-convex functions, then

1. \(\varphi + \psi\) is \(n\)-polynomial exponential type \(p\)-convex function;
2. For nonnegative real number \(c\), \(c\varphi\) is \(n\)-polynomial exponential type \(p\)-convex function.

**Proof.**

\[
(c\varphi) \left( \left[ k \theta_1^p + (1 - k) \theta_2^p \right]^{1/p} \right) \\
\leq (\varepsilon^p - 1) \left( c\varphi(\theta_1) \right) + (e^{1 - \varepsilon} - 1) (c\varphi)(\theta_2).
\]

(ii) Choosing \(p = 1\) in Theorem 1, then we get Theorem 2.1 in \([7]\).

**Theorem 2.** Let \(\varphi : I \rightarrow \mathbb{R}\) be \(p\)-convex function and \(\psi : J \rightarrow \mathbb{R}\) is non-decreasing and \(n\)-polynomial exponential type convex function. Then the function \(\psi \varphi : I \rightarrow \mathbb{R}\) is \(n\)-polynomial exponential type \(p\)-convex.

**Proof.** For all \(\theta_1, \theta_2 \in I\), and \(k \in [0, 1]\), we have

\[
(c\varphi) \left( \left[ k \theta_1^p + (1 - k) \theta_2^p \right]^{1/p} \right) \\
\leq (e^p - 1) (c\varphi)(\theta_1) + (e^{1 - p} - 1) (c\varphi)(\theta_2).
\]

(ii) If we put \(p = 1\) in Theorem 2, then we obtain Theorem 2.2 in \([7]\).

**Theorem 3.** Let \(\varphi : [\theta_1, \theta_2] \rightarrow \mathbb{R}\) be an arbitrary family of \(n\)-polynomial exponential type \(p\)-convex functions and let \(\varphi(\theta) = \sup_i \varphi_i(\theta)\). If \(O = \{\theta \in [\theta_1, \theta_2] : \varphi(\theta) < +\infty\} \neq \emptyset\), then \(O\) is an interval and \(\varphi\) is \(n\)-polynomial exponential type \(p\)-convex function on \(O\).

**Proof.** For all \(\theta_1, \theta_2 \in O\) and \(k \in [0, 1]\), we get

\[
\varphi \left( \left[ k \theta_1^p + (1 - k) \theta_2^p \right]^{1/p} \right) \\
= \sup_i \varphi_i \left( \left[ k \theta_1^p + (1 - k) \theta_2^p \right]^{1/p} \right). \\
\leq \sup_i \left( \frac{1}{n} \sum_{i=1}^{n} (e^p - 1) \varphi_i(\theta_1) + \frac{1}{n} \sum_{i=1}^{n} (e^{1 - p} - 1) \varphi_i(\theta_2) \right) \\
\leq \frac{1}{n} \sum_{i=1}^{n} (e^p - 1) \sup_i \varphi_i(\theta_1) + \frac{1}{n} \sum_{i=1}^{n} (e^{1 - p} - 1) \sup_i \varphi_i(\theta_2) \\
= \frac{1}{n} \sum_{i=1}^{n} (e^p - 1) \varphi(\theta_1) + \frac{1}{n} \sum_{i=1}^{n} (e^{1 - p} - 1) \varphi(\theta_2) < +\infty.
\]

This completes the proof. \(\square\)

**Remark 4.** (i) Choosing \(n = 1\) in Theorem 1, then

\[
(c\varphi) \left( \left[ k \theta_1^p + (1 - k) \theta_2^p \right]^{1/p} \right) \\
\leq (e^p - 1) (c\varphi)(\theta_1) + (e^{1 - p} - 1) (c\varphi)(\theta_2) \\
and
\]

(ii) Taking \(p = 1\) in Theorem 3, then we have Theorem 2.3 in \([7]\).
Theorem 4. If \( \varphi : [\theta_1, \theta_2] \to \mathbb{R} \) is \( n \)-polynomial exponential type \( p \)-convex function then \( \varphi \) is bounded on \([\theta_1, \theta_2]\).

Proof. Let \( x \in [\theta_1, \theta_2] \) and \( L = \max \{ \varphi(\theta_1), \varphi(\theta_2) \} \), then there exists \( k \in [0,1] \) such that \( x = \left[ k \theta_1^p + (1-k) \theta_2^p \right]^\frac{1}{p} \). Thus, since \( e^x \leq e \) and \( e^{1-k} \leq e \), we have
\[
\varphi(x) = \varphi \left( k \theta_1^p + (1-k) \theta_2^p \right) \leq \frac{1}{n} \sum_{i=1}^{n} (e^{x} - 1)^i \varphi(x_1) + \frac{1}{n} \sum_{i=1}^{n} (e^{1-k} - 1)^i \varphi(x_2)
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} (e^{x} + e^{1-k} - 2)^i \cdot L
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} (2(e - 1))^i \cdot L = M
\]
We have shown that \( \varphi \) is bounded above from real number \( M \). Interested reader can also prove the fact that \( \varphi \) is bounded below using the same idea as in Theorem 2.4 in [7].

Remark 7. (i) Choosing \( n = 1 \) in Theorem 4, then we obtain the bounded type \( p \)-convex function.
(ii) Choosing \( p = 1 \) in Theorem 4, then we get Theorem 2.4 in [7].

3. Hermite–Hadamard type inequality for \( n \)-polynomial exponential type \( p \)-convex functions

The purpose of this section is to derive a new inequality of Hermite–Hadamard type for the \( n \)-polynomial exponential type \( p \)-convex function \( \varphi \).

Theorem 5. Let \( \varphi : [\theta_1, \theta_2] \to \mathbb{R} \) be \( n \)-polynomial exponential type \( p \)-convex function. If \( \varphi \in L_1([\theta_1, \theta_2]) \), then
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \sqrt{e} - 1 \right)^i \varphi \left( \frac{\theta_i^p + \theta_{i+1}^p}{2} \right) \leq \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x) \frac{dx}{x^{1-p}} \tag{3.1}
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} (e - 2)^i \varphi(\theta_1) + \varphi(\theta_2).
\]

Proof. Using \( n \)-polynomial exponential type \( p \)-convexity of \( \varphi \), we have
\[
\varphi \left( \frac{\theta_i^p + \theta_{i+1}^p}{2} \right) \leq \varphi \left( \frac{1}{2} \left[ k \theta_1^p + (1-k) \theta_2^p \right]^p \right) + \frac{1}{2} ((1-k) \theta_1^p + k \theta_2^p)^p \right)^\frac{1}{p} \]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} \left( \sqrt{e} - 1 \right)^i \varphi \left( k \theta_1^p + (1-k) \theta_2^p \right)^\frac{1}{p}
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \left( \sqrt{e} - 1 \right)^i \varphi \left( (1-k) \theta_1^p + k \theta_2^p \right)^\frac{1}{p}.
\]
Now we integrating the above inequality with respect to \( k \in [0,1] \), we obtain
\[
\varphi \left( \frac{\theta_i^p + \theta_{i+1}^p}{2} \right) \leq \frac{1}{n} \sum_{i=1}^{n} \left( \sqrt{e} - 1 \right)^i \int_{0}^{1} \varphi \left( k \theta_1^p + (1-k) \theta_2^p \right)^\frac{1}{p} dx
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \left( \sqrt{e} - 1 \right)^i \int_{0}^{1} \varphi \left( (1-k) \theta_1^p + k \theta_2^p \right)^\frac{1}{p} dx
\]
\[
= 2p \left( \sum_{i=1}^{n} \left( \sqrt{e} - 1 \right)^i \right) \int_{\theta_1}^{\theta_2} \varphi(x) \frac{dx}{x^{1-p}}
\]
\[
\leq \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x) \frac{dx}{x^{1-p}}
\]
\[
= \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x) \frac{dx}{x^{1-p}}
\]

Remark 8. (i) If we put \( n = 1 \) in Theorem 5, then
\[
\frac{1}{2(\sqrt{e} - 1)} \varphi \left( \frac{\theta_1^p + \theta_2^p}{2} \right)^\frac{1}{p} \leq \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x) \frac{dx}{x^{1-p}}
\]
\[
\leq (e - 2) \varphi(\theta_1) + \varphi(\theta_2).
\]
(ii) If we put \( p = 1 \) in Theorem 5, then we get Theorem 3.1 in [7].

4. Refinements of (H−H) type inequality via \( n \)-polynomial exponential type \( p \)-convex functions

Let us recall the following crucial Lemma that we will be used in the sequel.

Lemma 2. [17] Let \( \varphi : I \to \mathbb{R} \) be differentiable function on \( I^* \) with \( \theta_1, \theta_2 \in I \) and \( \theta_1 < \theta_2 \). If \( \varphi' \in L_1([\theta_1, \theta_2]) \), then
\[
\varphi(\theta_1) + \varphi(\theta_2) - \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x) \frac{dx}{x^{1-p}} \leq \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \left( B_1(p, \theta_1, \theta_2) \right)^{\frac{1}{p}} \tag{4.1}
\]
\[
\leq \frac{1}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi'(x) \left( k \theta_1^p + (1-k) \theta_2^p \right)^{\frac{1}{p}} dx.
\]

Theorem 6. Let \( \varphi : I \to \mathbb{R} \) be differentiable function on \( I^* \) with \( \theta_1, \theta_2 \in I \) and \( \theta_1 < \theta_2 \). If \( \varphi' \in L_1([\theta_1, \theta_2]) \) and \( \varphi(x) \) is \( n \)-polynomial exponential type \( p \)-convex on \([\theta_1, \theta_2] \) for \( q \geq 1 \), then
\[
\left| \varphi(\theta_1) + \varphi(\theta_2) - \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x) \frac{dx}{x^{1-p}} \right| \leq \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \left( B_1(p, \theta_1, \theta_2) \right)^{\frac{1}{p}}
\]
\[
\times \left[ \frac{\varphi(\theta_1)}{n} \sum_{i=1}^{n} B_1(p, i, \theta_1, \theta_2) + \frac{\varphi(\theta_2)}{n} \sum_{i=1}^{n} B_1(p, i, \theta_1, \theta_2) \right]^{\frac{1}{p}}, \tag{4.2}
\]
where
\[ B_1(p, \theta_1, \theta_2) = \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa, \]
\[ B_2(p, \theta_1, \theta_2) = \int_0^1 \frac{|1 - 2x|(e^x - 1)^{\nu}}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa, \]
and
\[ B_3(p, \theta_1, \theta_2) = \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa. \]

**Proof.** Applying Lemma 2, properties of modulus, power mean inequality and \(n\)-polynomial exponential type \(p\)-convexity of \(|\varphi|^p\), we have
\[
\left| \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2} - \frac{\varphi(x)}{\theta_1^p - \theta_2^p} \int_{\theta_1}^{\theta_2} \varphi(x) d\lambda \right| \leq \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \int_0^1 \frac{|\varphi(\theta)|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa.
\]
\[
\leq \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \left( \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa \right)^{1/2}
\]
\[
\times \left( \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa \right)^{1/2}
\]
\[
\times \left( \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa \right)^{1/2}
\]
\[
\times \left( \frac{1}{n} \sum_{i=1}^{n} |(e^x - 1)^{\nu} | |\varphi'(\theta_1)|^{\nu} + \frac{1}{n} \sum_{i=1}^{n} |(e^x - 1)^{\nu} | |\varphi'(\theta_2)|^{\nu} \right) d\kappa
\]
\[
= \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \left( \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa \right)^{1/2}
\]
\[
\times \left( \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} |\varphi'(\theta)|^{\nu} d\kappa
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} |\varphi'(\theta_2)|^{\nu} d\kappa
\]
\[
= \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \left( B_1(p, \theta_1, \theta_2) \right)^{1/2}
\]
\[
\times \left( \frac{1}{n} \sum_{i=1}^{n} B_2(p, i, \theta_1, \theta_2) + \frac{1}{n} \sum_{i=1}^{n} B_3(p, i, \theta_1, \theta_2) \right)^{1/2},
\]
which completes the proof. \(\square\)

**Remark 9.** If we put \(n = 1\) in Theorem 6, then
\[
\left| \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2} - \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x) d\lambda \right| \leq \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \left( B_1(p, \theta_1, \theta_2) \right)^{1/4}
\]
\[
\times \left[ B_2(p, \theta_1, \theta_2)|\varphi'(\theta_1)|^{\nu} + B_3(p, \theta_1, \theta_2)|\varphi'(\theta_2)|^{\nu} \right]^{1/4},
\]
where
\[
B_1(p, \theta_1, \theta_2) = \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa,
\]
\[
B_2(p, \theta_1, \theta_2) = \int_0^1 \frac{|1 - 2x|(e^x - 1)^{\nu}}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa,
\]
and
\[
B_3(p, \theta_1, \theta_2) = \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa.
\]

**Theorem 7.** Let \( \varphi : I \rightarrow \mathbb{R} \) be differentiable function on \(I^n \) with \( \theta_1, \theta_2 \in I \) and \( \theta_1 < \theta_2 \). If \( \varphi' \in L_1[\theta_1, \theta_2] \) and \( |\varphi'|^p \) is \(n\)-polynomial exponential type \(p\)-convex function on \([\theta_1, \theta_2]\) for \(q > 1 \) and \( \frac{1}{q} + \frac{1}{\nu} = 1 \), then
\[
\left| \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2} - \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x) d\lambda \right| \leq \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \left( \frac{1}{1+q} \right)^{1/4}
\]
\[
\times \left[ \left( B_2(p, \theta_1, \theta_2)|\varphi'(\theta_1)|^{\nu} + B_3(p, \theta_1, \theta_2)|\varphi'(\theta_2)|^{\nu} \right]^{1/4},
\]
where
\[
B_1(p, \theta_1, \theta_2) = \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa,
\]
\[
B_2(p, \theta_1, \theta_2) = \int_0^1 \frac{|1 - 2x|(e^x - 1)^{\nu}}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa,
\]
and
\[
B_3(p, \theta_1, \theta_2) = \int_0^1 \frac{|1 - 2x|}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} d\kappa.
\]

**Proof.** Applying Lemma 2, properties of modulus, Hölder’s inequality and \(n\)-polynomial exponential type \(p\)-convexity of \(|\varphi|^p\), we have
\[
\left| \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2} - \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x) d\lambda \right| \leq \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \left( \frac{1}{1+q} \right)^{1/4}
\]
\[
\times \left( \int_0^1 \frac{1}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} |\varphi'(\theta)|^{\nu} d\kappa
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{1}{[k \theta_1^p + (1 - \kappa) \theta_2^p]^{1/2}} |\varphi'(\theta_2)|^{\nu} d\kappa
\]
\[
= \left( \frac{\theta_2^p - \theta_1^p}{2p} \right) \left( B_1(p, \theta_1, \theta_2) \right)^{1/4}
\]
\[
\times \left( \frac{1}{n} \sum_{i=1}^{n} B_2(p, i, \theta_1, \theta_2) + \frac{1}{n} \sum_{i=1}^{n} B_3(p, i, \theta_1, \theta_2) \right)^{1/4},
\]
which completes the proof. \(\square\)
\[
\begin{align*}
&= \left( \frac{\theta_0^\varphi - \theta_1^\varphi}{2p} \right) \left( \frac{1}{1 + 1} \right)^{\frac{1}{2}} \\
&\times \left[ \left| \varphi(\theta_1)^\varphi \right|^q \sum_{i=1}^{\frac{n}{2}} B_i (p, q, i, \theta_1, \theta_2) + \left| \varphi(\theta_2)^\varphi \right|^q \sum_{i=1}^{\frac{n}{2}} B_i (p, q, i, \theta_1, \theta_2) \right]^{\frac{1}{1 - 1}}
\end{align*}
\]

which completes the proof. \( \square \)

**Remark 10.** If we put \( n = 1 \) in Theorem 7, then
\[
\left| \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2} - \frac{p}{\theta_0^\varphi - \theta_1^\varphi} \int_{\theta_1}^{\theta_2} \frac{\varphi(x)}{\kappa + 1 - \frac{x}{\theta_0^\varphi}} \, dx \right| \leq \left( \frac{\theta_0^\varphi - \theta_1^\varphi}{2p} \right) \left( \frac{1}{1 + 1} \right)^{\frac{1}{2}}
\]

\[
\times \left[ \left| \varphi(\theta_1)^\varphi \right|^q B_i (p, q, 1, \theta_1, \theta_2) + \left| \varphi(\theta_2)^\varphi \right|^q B_i (p, q, 1, \theta_1, \theta_2) \right]^{\frac{1}{1 - 1}}.
\]

where
\[
B_i (p, q, 1, \theta_1, \theta_2) = \int_0^1 \frac{(e^x - 1)}{\kappa \theta_1^\varphi + (1 - \kappa) \theta_2^\varphi} \, dx
\]

and
\[
B_i (p, q, 1, \theta_1, \theta_2) = \int_0^1 \frac{(1 - e^x - 1)}{\kappa \theta_1^\varphi + (1 - \kappa) \theta_2^\varphi} \, dx.
\]

**Theorem 8.** Let \( \varphi : I \rightarrow \mathbb{R} \) be differentiable function on \( I^* \) with \( \theta_1, \theta_2 \in I \) and \( \theta_1 < \theta_2 \). If \( \varphi' \in L_1^*[\theta_1, \theta_2] \) and \( \left| \varphi'(\varphi) \right|^q \) is \( n \)-polynomial exponential type \( p \)-convex on \( [\theta_1, \theta_2] \) for \( q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), then
\[
\left| \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2} - \frac{p}{\theta_0^\varphi - \theta_1^\varphi} \int_{\theta_1}^{\theta_2} \frac{\varphi(x)}{\kappa + 1 - \frac{x}{\theta_0^\varphi}} \, dx \right| \leq \left( \frac{\theta_0^\varphi - \theta_1^\varphi}{2p} \right) \left( \frac{1}{1 + 1} \right)^{\frac{1}{2}}
\]

\[
\times \left[ \left| \varphi(\theta_1)^\varphi \right|^q \sum_{i=1}^{\frac{n}{2}} B_i (p, q, 1, \theta_1, \theta_2) + \left| \varphi(\theta_2)^\varphi \right|^q \sum_{i=1}^{\frac{n}{2}} B_i (p, q, 1, \theta_1, \theta_2) \right]^{\frac{1}{1 - 1}}.
\]

where
\[
B_i (p, q, 1, \theta_1, \theta_2) = \int_0^1 \frac{1}{\kappa \theta_1^\varphi + (1 - \kappa) \theta_2^\varphi} \, dx
\]

and
\[
B_i (p, q, 1, \theta_1, \theta_2) = \int_0^1 \frac{1}{\kappa \theta_1^\varphi + (1 - \kappa) \theta_2^\varphi} \, dx.
\]

**Remark 11.** If we put \( n = 1 \) in Theorem 8, then
\[
\left| \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2} - \frac{p}{\theta_0^\varphi - \theta_1^\varphi} \int_{\theta_1}^{\theta_2} \frac{\varphi(x)}{\kappa + 1 - \frac{x}{\theta_0^\varphi}} \, dx \right| \leq \left( \frac{\theta_0^\varphi - \theta_1^\varphi}{2p} \right) \left( \frac{1}{1 + 1} \right)^{\frac{1}{2}}
\]

\[
\times \left[ \left| \varphi(\theta_1)^\varphi \right|^q B_i (p, q, 1, \theta_1, \theta_2) + \left| \varphi(\theta_2)^\varphi \right|^q B_i (p, q, 1, \theta_1, \theta_2) \right]^{\frac{1}{1 - 1}}.
\]

where
\[
B_i (p, q, 1, \theta_1, \theta_2) = \int_0^1 \frac{1}{\kappa \theta_1^\varphi + (1 - \kappa) \theta_2^\varphi} \, dx
\]

and
\[
B_i (p, q, 1, \theta_1, \theta_2) = \int_0^1 \frac{1}{\kappa \theta_1^\varphi + (1 - \kappa) \theta_2^\varphi} \, dx.
\]

**Theorem 9.** Let \( \varphi : I \rightarrow \mathbb{R} \) be differentiable function on \( I^* \) with \( \theta_1, \theta_2 \in I \) and \( \theta_1 < \theta_2 \). If \( \varphi' \in L_1^*[\theta_1, \theta_2] \) and \( \left| \varphi'(\varphi) \right|^q \) is \( n \)-polynomial exponential type \( p \)-convex on \( [\theta_1, \theta_2] \), then
\[
\left| \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2} - \frac{p}{\theta_0^\varphi - \theta_1^\varphi} \int_{\theta_1}^{\theta_2} \frac{\varphi(x)}{\kappa + 1 - \frac{x}{\theta_0^\varphi}} \, dx \right| \leq \left( \frac{\theta_0^\varphi - \theta_1^\varphi}{2p} \right) \left( \frac{1}{1 + 1} \right)^{\frac{1}{2}}
\]

\[
\times \left[ \left| \varphi(\theta_1)^\varphi \right|^q \sum_{i=1}^{\frac{n}{2}} B_i (p, q, 1, \theta_1, \theta_2) + \left| \varphi(\theta_2)^\varphi \right|^q \sum_{i=1}^{\frac{n}{2}} B_i (p, q, 1, \theta_1, \theta_2) \right]^{\frac{1}{1 - 1}}.
\]

where
\[
B_i (p, q, 1, \theta_1, \theta_2) = \int_0^1 \frac{1}{\kappa \theta_1^\varphi + (1 - \kappa) \theta_2^\varphi} \, dx
\]

and
\[
B_i (p, q, 1, \theta_1, \theta_2) = \int_0^1 \frac{1}{\kappa \theta_1^\varphi + (1 - \kappa) \theta_2^\varphi} \, dx.
\]
Proof. Applying Lemma 2, properties of modulus and $n$–polynomial exponential type $p$–convexity of $|\varphi'|^q$, we have

$$\left|\frac{\varphi(\theta_1)+\varphi(\theta_2)}{2} - \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x)^{1-\rho} dx \right| \leq \left(\frac{\theta_2^p - \theta_1^p}{2p}\right) \int_0^1 \frac{1}{\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{1+\frac{1}{q}}} \left|\varphi'\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{\frac{1}{q}}\right| dk$$

which completes the proof. \qed

Remark 12. If we put $n = 1$ in Theorem 9, then

$$\left|\varphi(\theta_1)+\varphi(\theta_2) - \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x)^{1-\rho} dx \right| \leq \left(\frac{\theta_2^p - \theta_1^p}{2p}\right) \int_0^1 \frac{1}{\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{1+\frac{1}{q}}} \left|\varphi'\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{\frac{1}{q}}\right| dk$$

$$\times \left(\left|\varphi'(\theta_1)^{q} B_2(p, \theta_1, \theta_2) + |\varphi'(\theta_2)|^{q} B_{10}(p, \theta_1, \theta_2)\right|^q\right),$$

where

$$B_2(p, \theta_1, \theta_2) = \int_0^1 \frac{1}{\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{1+\frac{1}{q}}} \left|\varphi'\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{\frac{1}{q}}\right| dk,$$

$$B_{10}(p, \theta_1, \theta_2) = \int_0^1 \frac{1}{\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{1+\frac{1}{q}}} \left|\varphi'\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{\frac{1}{q}}\right| \varphi(x)^{1-\rho} dx.$$

Theorem 10. Let $\varphi : I \to \mathbb{R}$ be differentiable function on $I^q$ with $\theta_1, \theta_2 \in I$ and $\theta_1 < \theta_2$. If $\varphi \in L_1(\theta_1, \theta_2)$ and $|\varphi'|^q$ is $n$–polynomial exponential type $p$–convex on $[\theta_1, \theta_2]$ for $\frac{1}{q} > 1$ and $\frac{1}{q} + 1 = 1$, then

$$\left|\frac{\varphi(\theta_1)+\varphi(\theta_2)}{2} - \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x)^{1-\rho} dx \right| \leq \left(\frac{\theta_2^p - \theta_1^p}{2p}\right) \int_0^1 \frac{1}{\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{1+\frac{1}{q}}} \left|\varphi'\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{\frac{1}{q}}\right| \left\{\left|\frac{B_2(p, \theta_1, \theta_2)}{2}\right|^q + \left|\frac{B_{10}(p, \theta_1, \theta_2)}{2}\right|^q\right\}^{\frac{1}{q}}$$

which completes the proof. \qed

Remark 13. If we put $n = 1$ in Theorem 10, then

$$\left|\frac{\varphi(\theta_1)+\varphi(\theta_2)}{2} - \frac{p}{\theta_2^p - \theta_1^p} \int_{\theta_1}^{\theta_2} \varphi(x)^{1-\rho} dx \right| \leq \left(\frac{\theta_2^p - \theta_1^p}{2p}\right) \int_0^1 \frac{1}{\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{1+\frac{1}{q}}} \left|\varphi'\left(k\theta_1^p + (1 - k)\theta_2^p\right)^{\frac{1}{q}}\right| \left\{\left|\frac{B_2(p, \theta_1, \theta_2)}{2}\right|^q + \left|\frac{B_{10}(p, \theta_1, \theta_2)}{2}\right|^q\right\}^{\frac{1}{q}}$$

5. Application to special means

The following two special means for different positive real numbers would be used:

1. Arithmetical mean
$$A(\theta_1, \theta_2) = \frac{\theta_1 + \theta_2}{2}.$$

2. $p$–Logarithmic mean
$$L_p(\theta_1, \theta_2) = \left(\frac{\theta_2^{p+1} - \theta_1^{p+1}}{(p+1)(\theta_2 - \theta_1)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$
Using Section 4, we obtain the following results:

**Proposition 3.** Let $0 < \theta_1 < \theta_2$ and $q \geq 1$. Then for $n \in \mathbb{N}$ and $p \in \mathbb{R} \setminus \{−1, 0\}$, we have

\[
\left| A(\theta_1, \theta_2) - \frac{p(\theta_2 - \theta_1)}{\theta_2^q - \theta_1^q} L_p(\theta_1, \theta_2) \right| \leq \frac{1}{\sqrt{n}} \left( \frac{\theta_2^q - \theta_1^q}{2p} \right) \left( 1 + \frac{1}{q} \right) \left( B_1(p, \theta_1, \theta_2) \right)^{\frac{1}{2}}
\]

(5.1)

\[ \times \left[ \sum_{i=1}^{n} (B_1(p, i, \theta_1, \theta_2) + B_2(p, i, \theta_1, \theta_2)) \right]^{\frac{1}{2}}, \]

where $B_1(p, \theta_1, \theta_2)$, $B_2(p, i, \theta_1, \theta_2)$ and $B_3(p, i, \theta_1, \theta_2)$ are defined as in Theorem 6.

**Proof.** Taking $q(x) = x$, $x > 0$ and using Theorem 6, we get the desired result. □

**Proposition 4.** Let $0 < \theta_1 < \theta_2$ for $q > 1$ and $\frac{1}{1+q} + \frac{1}{q} = 1$. Then for $n \in \mathbb{N}$ and $p \in \mathbb{R} \setminus \{−1, 0\}$, we obtain

\[
\left| A(\theta_1, \theta_2) - \frac{p(\theta_2 - \theta_1)}{\theta_2^q - \theta_1^q} L_p(\theta_1, \theta_2) \right| \leq \frac{1}{\sqrt{n}} \left( \frac{\theta_2^q - \theta_1^q}{2p} \right) \left( 1 + \frac{1}{q} \right) \left( B_1(p, \theta_1, \theta_2) \right)^{\frac{1}{2}}
\]

(5.2)

\[ \times \left[ \sum_{i=1}^{n} (B_1(p, q, i, \theta_1, \theta_2) + B_2(p, q, i, \theta_1, \theta_2)) \right]^{\frac{1}{2}}, \]

where $B_1(p, q, \theta_1, \theta_2)$ and $B_2(p, q, \theta_1, \theta_2)$ are defined as in Theorem 7.

**Proof.** Choosing $q(x) = x$, $x > 0$ and using Theorem 7, the result is obvious. □

**Proposition 5.** Let $0 < \theta_1 < \theta_2$ for $q > 1$ and $\frac{1}{1+q} + \frac{1}{q} = 1$. Then for $n \in \mathbb{N}$ and $p \in \mathbb{R} \setminus \{−1, 0\}$, we have

\[
\left| A(\theta_1, \theta_2) - \frac{p(\theta_2 - \theta_1)}{\theta_2^q - \theta_1^q} L_p(\theta_1, \theta_2) \right| \leq \frac{1}{\sqrt{n}} \left( \frac{\theta_2^q - \theta_1^q}{2p} \right) \left( 1 + \frac{1}{q} \right) \left( B_1(p, l, \theta_1, \theta_2) \right)^{\frac{1}{2}}
\]

(5.3)

\[ \times \left[ \sum_{i=1}^{n} (B_1(p, l, i, \theta_1, \theta_2) + B_2(p, q, i, \theta_1, \theta_2)) \right]^{\frac{1}{2}}, \]

where $B_1(p, l, \theta_1, \theta_2)$, $B_2(p, l, i, \theta_1, \theta_2)$ and $B_3(p, q, i, \theta_1, \theta_2)$ are defined as in Theorem 8.

**Proof.** Taking $q(x) = x$, $x > 0$ and using Theorem 8, the result is evident. □

**Proposition 6.** Let $0 < \theta_1 < \theta_2$ and $n \in \mathbb{N}$. Then for $p \in \mathbb{R} \setminus \{−1, 0\}$, we get

\[
\left| A(\theta_1, \theta_2) - \frac{p(\theta_2 - \theta_1)}{\theta_2^q - \theta_1^q} L_p(\theta_1, \theta_2) \right| \leq \frac{1}{\sqrt{n}} \left( \frac{\theta_2^q - \theta_1^q}{2p} \right) \left( 1 + \frac{1}{q} \right) \left( B_1(p, l, \theta_1, \theta_2) \right)^{\frac{1}{2}}
\]

(5.4)

\[ \times \left[ \sum_{i=1}^{n} (B_1(p, l, i, \theta_1, \theta_2) + B_2(p, i, \theta_1, \theta_2)) \right]^{\frac{1}{2}}, \]

where $B_1(p, l, \theta_1, \theta_2)$ and $B_2(p, i, \theta_1, \theta_2)$ are defined as in Theorem 9.

**Proof.** Choosing $q(x) = x$, $x > 0$ and using Theorem 9, the result is obvious. □

**Proposition 7.** Let $0 < \theta_1 < \theta_2$ for $q > 1$ and $\frac{1}{1+q} + \frac{1}{q} = 1$. Then for $n \in \mathbb{N}$ and $p \in \mathbb{R} \setminus \{−1, 0\}$, we obtain

\[
\left| A(\theta_1, \theta_2) - \frac{p(\theta_2 - \theta_1)}{\theta_2^q - \theta_1^q} L_p(\theta_1, \theta_2) \right| \leq \frac{1}{\sqrt{n}} \left( \frac{\theta_2^q - \theta_1^q}{2p} \right) \left( 1 + \frac{1}{q} \right) \left( B_1(p, l, \theta_1, \theta_2) \right)^{\frac{1}{2}}
\]

(5.5)

\[ \times \left[ \sum_{i=1}^{n} (\theta_1^n - \theta_2^n) \right]^{\frac{1}{2}} A \left( \frac{1}{n}, \frac{1}{n} \right), \]

where $B_1(p, l, \theta_1, \theta_2)$ is defined as in Theorem 10.

**Proof.** Taking $q(x) = x$, $x > 0$ and using Theorem 10, the result is evident. □

**Remark 14.** Applying the same idea as in above propositions using our results from Section 4 and taking suitable functions, for example $q(x) = x^q$, $x > 0$, $q(x) = \frac{1}{2}$, $x > 0$, etc., we can obtain several new interesting inequalities using special means. We omit their proofs and the details are left to the interested reader.

6. Conclusion

In this article, authors showed new Hermite–Hadamard (or trapezoid type inequality) type inequality for the new class of functions, the so-called $n$-polynomial exponential type $p$-convex function $\varphi$ and We have obtained refinements of the (H–H) inequality for functions whose first derivative in absolute value at certain power are $n$-polynomial exponential type $p$-convex. The interested reader can found other new results using other suitable functions $q$ and also new bounds for special means and error estimates for the trapezoidal and midpoint formula. To the best of our knowledge these results are new in the literature. Since convex functions have large applications in many mathematical areas, we hope that our new results can be applied in convex analysis, special functions, quantum analysis, quantum mechanics, post quantum analysis, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

**Declarations**

**Author contribution statement**

S.I. Butt, A. Kashuri, M. Tariq, J. Nasir, A. Aslam, W. Gao: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

**Funding statement**

The research of the author Saad Ihsan Butt has been fully supported by H.E.C. Pakistan under NRPU project 7906.

**Declaration of interests statement**

The authors declare no conflict of interest.

**Additional information**

No additional information is available for this paper.

**Acknowledgements**

The authors are pleased and wish to express their heartfelt and warm thanks to the referees for their helpful suggestions to improve the final version of this paper.

**References**

[1] S.I. Butt, M.K. Bakula, D. Pecaric, J. Pecaric, Jensen-Gruss inequality and its applications for the Zipf-Mandelbrot law, Math. Methods Appl. Sci. (2020).

[2] S. Khan, M.A. Khan, S.I. Butt, Y.M. Chu, A new bound for the Jensen gap pertaining twice differentiable functions with applications, Adv. Differ. Eqns. 2020 (1) (2020) 1–11.
[3] N. Mehmood, S.I. Butt, D. Pečarić, J. Pečarić, Generalizations of cyclic refinements of Jense inequality by Lidstones polynomial with applications in Information Theory, J. Math. Inequal. 14 (1) (2020) 249–271.

[4] S.I. Butt, A. Kashuri, M. Tariq, et al., Hermite–Hadamard-type inequalities via n-polynomial exponential-type convexity and their applications, Adv. Differ. Equ. 2020 (2020) 508.

[5] Wei Gao, Artion Kashuri, Saad Ihsan Butt, Muhammad Tariq, Adnan Aslam, Muhammad Nadeem, New inequalities via n-polynomial harmonically exponential type convex functions, AIMS Math. 5 (6) (2020) 6856–6873.

[6] Saad Ihsan Butt, Artion Kashuri, Muhammad Umar, Adnan Aslam, Wei Gao, Hermite-Jensen-Mercer type inequalities via pµ-Riemann-Liouville k-fractional integrals, AIMS Math. 5 (5) (2020) 5193–5220.

[7] M. Kadakal, İ. İşcan, Exponential type convexity and some related inequalities, J. Inequal. Appl. 2020 (1) (2020) 1–9.

[8] S.S. Dragomir, J. Pecarić, L.E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21 (3) (2001) 335–341.

[9] A. Guessab, G. Schmeisser, Sharp integral inequalities of the Hermite–Hadamard type, J. Approx. Theory 115 (2) (2002) 260–288.

[10] İ. İşcan, M. Kunt, Hermite–Hadamard–Fejér type inequalities for quasi-geometrically convex functions via fractional integrals, J. Math. 2016 (2016) 6523041.

[11] M. Alomari, M. Darus, U.S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Comput. Math. Appl. 59 (2010) 225–232.

[12] M. Bombardelli, S. Varošanec, Properties of h-convex functions related to the Hermite–Hadamard–Fejer type inequalities, Comput. Math. Appl. 58 (9) (2009) 1869–1877.

[13] F.X. Chen, S.H. Wu, Several complementary inequalities to inequalities of Hermite–Hadamard type for s-convex functions, J. Nonlinear Sci. Appl. 9 (2) (2016) 705–716.

[14] S.S. Dragomir, S. Fitzpatrick, The Hadamard’s inequality for s-convex functions in the second sense, Demonstr. Math. 32 (4) (1999) 687–696.

[15] N. Eftekhari, Some remarks on (s, n)-convexity in the second sense, J. Math. Inequal. 8 (2014) 489–495.

[16] H. Hudzik, L. Maligranda, Some remarks on s-convex functions, Aequ. Math. 48 (1994) 100–111.

[17] İ. İşcan, Hermite–Hadamard type inequalities for p-convex functions, Int. J. Anal. Appl. 11 (2) (2016) 137–145.

[18] İ. İşcan, Ostrowski type inequalities for p-convex functions, New Trends Math. Sci. 4 (3) (2016) 140–150.

[19] H. Kadakal, Hermite–Hadamard type inequalities for trigonometrically convex functions, Sci. Stud. Res. Math. Inform. 28 (2) (2018) 19–28.

[20] A. Kashuri, R. Liko, Some new Hermite–Hadamard type inequalities and their applications, Studia Sci. Math. Hung. 56 (1) (2019) 103–142.

[21] O. Omotoyinbo, A. Mogbademu, Some new Hermite–Hadamard integral inequalities for convex functions, Int. J. Sci. Innov. Technol. 1 (1) (2014) 1–12.

[22] E. Set, M.A. Noor, M.U. Awan, A. Gözpınar, Generalized Hermite–Hadamard type inequalities involving fractional integral operators, J. Inequal. Appl. 169 (2017) 1–10.

[23] G. Toader, Some generalizations of the convexity, in: Proceedings of the Colloquium on Approximation and Optimization, Univ. Cluj-Napoca, Cluj-Napoca, 1985, pp. 329–338.

[24] T. Toplu, M. Kadakal, İ. İşcan, On n-polynomial convexity and some related inequalities, AIMS Math. 5 (2) (2020) 1304–1318.

[25] S. Varošanec, On h-convexity, J. Math. Anal. Appl. 326 (2007) 303–311.

[26] B.Y. Xi, F. Qi, Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means, J. Funct. Spaces Appl. 2012 (2012) 98043B.

[27] K.S. Zhang, J.P. Wan, p-convex functions and their properties, Pure Appl. Math. 23 (1) (2007) 130–133.

[28] X.M. Zhang, Y.M. Chu, X.H. Zhang, The Hermite–Hadamard type inequality of GA-convex functions and its applications, J. Inequal. Appl. (2010) 507560.