Classifying codimension two multigerms

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Abstract

We generalise the operations of augmentation and concatenations defined in [4] in order to obtain multigerms of analytic (or smooth) maps \((\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)\) with \(\mathbb{K} = \mathbb{C}\) or \(\mathbb{R}\) from monogerms and some special multigerms. We then prove that any corank 1 codimension 2 multigerm in Mather’s nice dimensions \((n, p)\) with \(n \geq p - 1\) can be constructed using augmentations and these operations.

1 Introduction

The classification of singularities of map germs from \(\mathbb{K}^n\) to \(\mathbb{K}^p\) has been one of the main areas of research in Singularity Theory for the last decades. The foundations are the fundamental works of Whitney, Mather and Thom on classification of stable maps followed by Arnold’s classification of simple singularities of functions in [1]. Since then, complete classifications up to certain codimension for certain pairs \((n, p)\) have been carried out by many authors ([25], [26], [19], [6], [17], [15], [28], ...) and it is still an active field of research.

The bibliography related to the classification of multigerms is less abundant. The first reference is Mather’s classification of stable multigerms [18]. A list of multigerms from \(\mathbb{R}^2\) to \(\mathbb{R}^3\) including codimension 1 singularities is given by Goryunov ([7], [8]). Hobbs and Kirk ([9]) give a thorough classification for this case. The third author obtains in [28] the list of simple multigerms from \(\mathbb{C}^2\) to \(\mathbb{C}^3\) and gives a method that can be applied to the case \(\mathbb{C}^n\) to \(\mathbb{C}^{n+1}\). Normal forms for multigerms from the plane to the plane are given by several authors. A good account of this is [23]. More recently, in [24], the first author and Romero Fuster give normal forms of multigerms up to codimension 2 from \(\mathbb{R}^3\) to \(\mathbb{R}^3\) using a geometric method based on the theory of contact between submanifolds developed by...
Montaldi in [20]. This method is used to give candidates for the different $A$-classes and then normal forms are given, being able to prove that they have the announced codimension.

The classification of multigerms using the classical Singularity Theory techniques such as the complete transversal’s method can be hard to deal with. In [28] the $A$-classification of multigerms from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ is reduced to a much simpler $K$-classification. However, in other dimensions, it is still a very hard task. Therefore, operations to obtain germs in certain dimensions from germs with fewer branches in lower dimensions have been developed. The concepts of augmentations and monic and binary concatenations appear in [4], where Cooper, Mond and Wik Atique show that any codimension 1 multigerm can be obtained using these operations. Further developments on these operations are given by Houston ([10], [11]).

As can be seen from the classification in [24], these operations do not give the complete list of codimension 2 multigerms.

In this paper we generalise these operations in order to obtain a complete list of multigerms of codimension 2. The first operation introduced merges two of the earlier ones, it is a simultaneous augmentation and monic concatenation. Then, a generalised definition of concatenation is given which includes both the monic and binary concatenations as particular cases. Two other examples of this family of operations are studied, namely cuspidal concatenation and double fold concatenation. We show that for $(n, p)$ in Mather’s nice dimensions and $n \geq p - 1$ all the multigerms of corank 1 and codimension 2 are obtained using these operations starting from monogerms and some special multigerms. We are therefore giving a method to obtain new classifications of multigerms for any $(n, p)$ in Mather’s nice dimensions, including the complete lists of singularities up to codimension 2.

The study of multigerms is not only important for classification purposes but is also necessary for other research lines such as the study of topological invariants. For instance, to obtain the first order local invariants of stable maps it is essential to know the multigerms up to codimension 2 (see for example [8], [23], [29], [24] or [3]). Also, our methods provide a source of new examples to test the Mond conjecture which states that the codimension of a germ is less than or equal to its image Milnor number.

The paper is organised as follows: In §2 we fix our notation and give some basic results and definitions. In §3 we define augmentations, state some known results and give a characterisation to identify augmented multigerms having only stable branches. In §4 we define the new operations, give formulae to calculate the codimension of the resulting multigerms and several examples. In §5 we prove that any codimension 2 multigerm in Mather’s nice dimensions with $n \geq p - 1$ is obtained using these operations. Finally, in §6 we give an example of how to obtain all the codimension 2 multigerms using these operations, namely the classification of codimension 2 multigerms from $\mathbb{C}^3$ to $\mathbb{C}^3$ obtained in [24].

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2 Notation

Let \( \mathcal{O}_n^p \) be the vector space of map germs with \( n \) variables and \( p \) components. When \( p = 1 \), \( \mathcal{O}_n^1 = \mathcal{O}_n \) is the local ring of germs of functions in \( n \)-variables and \( \mathcal{M}_n \) its maximal ideal. The set \( \mathcal{O}_n^p \) is a free \( \mathcal{O}_n \)-module of rank \( p \). A multigerm is a germ of an analytic (complex case) or smooth (real case) map \( f = \{ f_1, \ldots, f_r \} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) where \( S = \{ x_1, \ldots, x_r \} \subset \mathbb{K}^n \), \( f_i : (\mathbb{K}^n, x_i) \to (\mathbb{K}^p, 0) \) and \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{R} \). Let \( \mathcal{M}_n \mathcal{O}_{n,S}^p \) be the vector space of such map germs. Let \( \theta_{\mathcal{O}_n,S} \) and \( \theta_{\mathcal{O}_p,0} \) be the \( \mathcal{O}_n \)-module of germs at \( S \) of vector fields on \( \mathbb{K}^n \) and \( \mathcal{O}_p \)-module of germs at \( 0 \) of vector fields on \( \mathbb{K}^p \). We denote them by \( \theta_n \) and \( \theta_p \). Let \( \theta(f) \) be the \( \mathcal{O}_n \)-module of germs \( \xi : (\mathbb{K}^n, S) \to T\mathbb{K}^p \) such that \( \pi_p \circ \xi = f \) where \( \pi_p : T\mathbb{K}^p \to \mathbb{K}^p \) denotes the tangent bundle over \( \mathbb{K}^p \).

Define \( tf : \theta_n \to \theta(f) \) by \( tf(\chi) = df \circ \chi \) and \( wf : \theta_p \to \theta(f) \) by \( wf(\eta) = \eta \circ f \). The \( \mathcal{A}_e \)-tangent space of \( f \) is defined as \( T\mathcal{A}_e f = tf(\theta_n) + wf(\theta_p) \). Finally we define the \( \mathcal{A}_e \)-codimension of a germ \( f \), denoted by \( \mathcal{A}_e \text{-cod}(f) \), as the \( \mathbb{K} \)-vector space dimension of

\[
\mathcal{N}\mathcal{A}_e(f) = \frac{\theta(f)}{T\mathcal{A}_e f}.
\]

We refer to Wall’s survey article \[27\] for general background on the theory of singularities.

Following Damon in \[5\], a transverse fibre square is a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
\uparrow j & & \uparrow i \\
X_0 & \xrightarrow{f} & Y_0
\end{array}
\]

in which \( i \pitchfork F, X_0 \simeq X \times_Y Y_0 \) and \( f \) is right equivalent to the projection \( X \times_Y Y_0 \to Y_0 \). The map-germ \( f \) is called the pull-back of \( F \) by \( i \) and is denoted by \( i^*(F) \). Any germ \( f \) is a pull-back of a stable \( q \)-parameter unfolding \( F \) by the natural inclusion \( i : (\mathbb{K}^p, 0) \to (\mathbb{K}^p \times \mathbb{K}^q, 0) \). Damon proved that \( \mathcal{A}_e \text{-cod}(f) = \mathcal{K}_{D(F),e} - \text{cod}(i) \), where

\[
\mathcal{K}_{D(F),e} - \text{cod}(i) = \dim_{\mathbb{K}} \mathcal{N}\mathcal{K}_{D(F),e}(i) = \dim_{\mathbb{K}} \frac{\theta(i)}{ti(\theta_p) + i^*\text{Derlog}(D(F))}
\]

where \( D(F) \) is the discriminant of \( F \) and \( \text{Derlog}(V) \) represents the \( \mathcal{O}_p \)-module of tangent vector fields to \( V \).

**Definition 2.1.** i) A vector field germ \( \eta \in \theta_p \) is called liftable over \( f \), if there exists \( \xi \in \theta_n \) such that \( df \circ \xi = \eta \circ f \) (\( tf(\xi) = wf(\eta) \)). The set of vector field germs liftable over \( f \) is denoted by \( \text{Lift}(f) \) and is an \( \mathcal{O}_p \)-module.

ii) Let \( \tau(f) = \text{ev}_0(\text{Lift}(f)) \) be the evaluation at the origin of elements of \( \text{Lift}(f) \).
In general $\text{Lift}(f) \subseteq \text{Derlog}(V)$ when $V$ is the discriminant of an analytic $f$. We have an equality when $\mathbb{K} = \mathbb{C}$ and $f$ is complex analytic.

The set $\tau(f)$ is the tangent space to the well defined manifold containing $0$ along which the map $f$ is trivial (i.e. the analytic stratum). Notice that if $f$ is stable then $\tau(f) = \tau(f)$ in Mather’s sense (namely $\tau(f) = \text{ev}_0[w f^{-1} \{ f^* \mathcal{M} \theta(f) + tf(\theta_{n,S}) \}]$).

See [\ref{11}] or [\ref{4}] for some basic properties of $\tau(f)$.

From here on we will consider only minimal corank germs.

## 3 Augmentations of smooth mappings

**Definition 3.1.** Let $h : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a map-germ with a 1-parameter unfolding $H : (\mathbb{K}^n \times \mathbb{K}, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ which is stable as a map-germ, where $H(x, \lambda) = (h_\lambda(x), \lambda)$, such that $h_0 = h$. Let $g : (\mathbb{K}^q, 0) \to (\mathbb{K}, 0)$ be a function-germ. Then, the augmentation of $h$ by $H$ and $g$ is the map $A_{H,g}(h)$ given by $(x, z) \mapsto (h_{g(z)}(x), z)$.

**Theorem 3.2.** ([\ref{11}], [\ref{11}])

$$A_e - \text{cod}(A_{H,g}(h)) \geq A_e - \text{cod}(h)\tau(g),$$

where $\tau$ is the Tjurina number and equality is reached when $g$ is quasihomogeneous.

**Theorem 3.3.** ([\ref{11}]) Suppose that $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is non-stable and has a 1-parameter stable unfolding $F$. Then

$$q = \dim_{\mathbb{K}} \tau(F) \geq 1 \iff f \text{ is an augmentation.}$$

More precisely, on the right hand side, $f \sim A A_{H,g}(h)$ for some $h : (\mathbb{K}^{n-q}, S') \to (\mathbb{K}^{p-q}, 0)$, a smooth map-germ with a 1-parameter stable unfolding $H$, and $g : (\mathbb{K}^q, 0) \to (\mathbb{K}, 0)$ a function, $q \geq 1$.

A germ that is not an augmentation is called primitive. When all the branches of a multigerm are stable we give a more geometric characterisation for augmented germs.

**Definition 3.4.** Given a multigerm $f = \{ f_1, \ldots, f_r \}$ we say that its branches are totally non-transverse if for every $i = 1, \ldots, r$ there exists $\eta_i \in \text{Lift}(f_i)$, $\eta_i(0) \neq 0$, such that $\eta_i(0) = \eta_j(0)$ for all $j \neq i$.

From the definition it can be seen that the fact of the branches being totally non-transverse is equivalent to $\dim(\tau(f_1) \cap \ldots \cap \tau(f_r)) = \dim(\tau(f_1) \cap \ldots \cap \tau(f_r)) \geq 1$.

**Proposition 3.5.** Let $f = \{ f_1, \ldots, f_r \} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a non-stable multigerm, $|S| > 1$, with $f_i$ stable for all $i = 1, \ldots, r$. Suppose that $f$ admits a 1-parameter stable unfolding $F$. If $f$ is an augmentation then its branches are totally non-transverse.
Proof. As in Theorem 3.3, one can take a primitive $h$ such that $f \sim A H, g(h)$. Since $f$ is non-stable $g$ is not a submersion.

Following the proof of Theorem 4.6 in [11], $f(x, z) = (h_{g(z)}(x), z)$, with $x \in \mathbb{K}^{n-q}$ and $z \in \mathbb{K}^q$, has the 1-parameter stable unfolding $F(x, z, \lambda) = (h_{g(z)}+\lambda(x), z, \lambda) = (X, Z, \Lambda)$. Also $F \sim A H \times Id_q$, that is, there exist germs of diffeomorphisms $\phi$ and $\psi$ such that $F \circ \phi = \psi \circ (H \times Id_q)$, namely, $\psi(X, z, \lambda) = (X, \lambda, Z + g(\Lambda))$ and $\phi(x, z, \lambda) = (x, \lambda, z - g(\lambda))$. Since $g$ is not a submersion it follows that $d\psi_0$ is a permutation of the identity. Using the basic properties of $\tau$ we have that

$$\tau(F) = d\psi_0(\tau(H \times Id_q)) = d\psi_0(\tau(H) \times T_0\mathbb{K}^q) = d\psi_0(\{0\} \times T_0\mathbb{K}^q) = \{0\} \times T_0\mathbb{K}^q \times \{0\}$$

(and therefore $\text{dim}_\mathbb{K} \tau(F) = q \geq 1$).

Since $\text{dim}_\mathbb{K} \tau(F) \geq 1$, there exists $\eta \in Lift(F)$ such that $\eta(0) \neq 0$. Clearly $\eta \in Lift(F_i)$ for each branch $F_i$ of $F$. There exists $k$, $p-q+1 \leq k \leq p$, such that the $k$-th component of $\eta$ is non zero, that is, $\eta_k(0) \neq 0$. Let $\xi^i$ be such that $dF_i(\xi^i(x, z, \lambda)) = \eta_i(x, z, \lambda)$, $i = 1, \ldots, r$. Define $\tilde{\xi}^i(x, z) = (\xi^i_1(x, z, 0), \ldots, \xi^i_{n}(x, z, 0))$ and $\tilde{\eta}(X, Z) = (\eta_1(X, Z, 0), \ldots, \eta_p(X, Z, 0))$. By evaluating this system of equations in $\lambda = 0$, from the first $p$ equations we get

$$df_i(\tilde{\xi}^i(x, z)) + \partial_\lambda((h_{g(z)}+\lambda(x), z))|_{\lambda=0} \xi^i_{n+1}(x, z, 0) = \tilde{\eta}(f_i(x, z))$$

where

$$\xi^i_{n+1}(x, z, 0) = \eta_{p+1}(f_i(x, z), 0), \ p-q+1 \leq j \leq p$$

and from the last equation we get

$$\xi^i_{n+1}(x, z, 0) = \eta_{p+1}(f_i(x, z), 0).$$

Since $f_i$ is stable, there exist vector fields $v^i = (v^i_1, \ldots, v^i_n)$ and $\gamma^i = (\gamma^i_1, \ldots, \gamma^i_p)$ such that

$$\partial_\lambda((h_{g(z)}+\lambda(x), z))|_{\lambda=0} \xi^i_{n+1}(x, z, 0) = df_i(v^i) + \gamma^i(f_i).$$

Since $\xi^i_{n+1}(0) = \eta_{p+1}(0) = 0$, $\partial_\lambda((h_{g(z)}+\lambda(x), z))|_{\lambda=0} \xi^i_{n+1}(x, z, 0) \in f_i^*\mathcal{M}_p \theta(f_i) = f_i^*\mathcal{M}_p(t f_i(\theta_n) + w f_i(\theta_p)) \subset t f_i(\theta_n) + w f_i(\theta_p)$, so we can choose $\gamma^i(0) = 0$. The system can now be written as $df_i(\tilde{\xi} + v^i) + \gamma^i(f_i) = \tilde{\eta}(f_i)$. Therefore $\tilde{\eta}(X, Z) - \gamma^i(X, Z) \in Lift(f_i)$ and $\tilde{\eta}(0) - \gamma^i(0) = (\eta_1(0), \ldots, \eta_p(0)) \neq 0$ for all $i = 1, \ldots, r$. $\square$

The proposition is not true if one of the branches is not stable, since the analytic stratum of a non stable branch is $\{0\}$.

A finite set $E_1, \ldots, E_r$ of vector subspaces of a finite-dimensional vector space $F$ has almost regular intersection of order $k$ (with respect to $F$) if

$$\text{cod}(E_1 \cap \ldots \cap E_r) = \text{cod} E_1 + \ldots + \text{cod} E_r - k,$$

where cod represents the codimension. When $k = 0$ we say regular intersection and when $k = 1$ we say almost regular intersection. Mather characterised in [18].
stable multigerms as those where every branch is stable and the analytic strata have regular intersection. Elementary algebra proves the following

**Lemma 3.6.** $E_1, \ldots, E_r$ have almost regular intersection of order $k$ if and only if the cokernel of the natural mapping

$$F \to (F/E_1) \oplus \ldots \oplus (F/E_r)$$

has dimension $k$.

**Proposition 3.7.** Let $f = \{f_1, \ldots, f_r\} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a non-stable multi-germ, $|S| > 1$, with $f_i$ stable for all $i = 1, \ldots, r$. If $f$ admits a 1-parameter stable unfolding, then the $\tilde{\tau}(f_i)$ have almost regular intersection. Moreover, if $f$ admits a decomposition $f = \{f^1, f^2\}$ with $f^1, f^2$ stable germs, then the converse is also true.

**Proof.** A 1-parameter stable unfolding $F$ of $f$ restricts to 1-parameter stable unfoldings $F_i$ of $f_i$. Since $f_i$ is stable for all $i = 1, \ldots, r$, $F_i$ is a prism on $f_i$, that is $F_i \sim f_i \times \text{id}$, therefore

$$\text{dim} T_0 \mathbb{K}^p / \tilde{\tau}(f_i) \cong \text{dim} T_0 (\mathbb{K}^p \times \mathbb{K}) / \tilde{\tau}(F_i)$$

for all $i = 1, \ldots, r$.

We have the following commutative diagram

$$
\begin{array}{ccc}
T_0 \mathbb{K}^p & \longrightarrow & T_0 \mathbb{K}^p / \tilde{\tau}(f_1) \\
\downarrow & & \downarrow \\
T_0 (\mathbb{K}^p \times \mathbb{K}) & \longrightarrow & T_0 (\mathbb{K}^p \times \mathbb{K}) / \tilde{\tau}(F_1)
\end{array}
$$

The right hand map is bijective and the bottom map is surjective since $F$ is stable (and therefore the $\tilde{\tau}(F_i)$ are transversal). So the top map has cokernel of dimension 1. If it were of dimension 0, it would be surjective and the $\tilde{\tau}(f_i)$ would be transverse, which means that $f$ would be stable and this is a contradiction.

Conversely, if they have almost regular intersection and $f = \{f^1, f^2\}$ with $f^1, f^2$ stable then there exists a direction, which we call $v$ such that $T_0 \mathbb{K}^p = (\tilde{\tau}(f^1) + \tilde{\tau}(f^2)) \oplus \mathbb{K}\{v\}$. If $f_2$ is a monogerm, then $\{F^1, F^2\}$ with $F^1 = f^1 \times \text{id}$ and $F^2 = (f^2 + tv, t)$ is a 1-parameter stable unfolding of $f$, since the $\tilde{\tau}(F_i)$ have regular intersection. If $f_2$ is a multigerm, we deform any one of its branches in the way above and the result follows too.

**Corollary 3.8.** Let $f = \{f_1, \ldots, f_r\} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a non-stable multigerm, $|S| > 1$, with $f_i$ stable for all $i = 1, \ldots, r$. Suppose that $f$ admits a 1-parameter stable unfolding $F$. Then,

$$\sum_{i=1}^r \text{cod}(\tilde{\tau}(f_i)) \leq p \iff f \text{ is an augmentation.}$$
Proof. Since $f$ has a 1-parameter stable unfolding, by Proposition 3.7 the analytic strata have almost regular intersection.

By Proposition 3.5 we have that if $f$ is an augmentation then its branches are totally non-transverse, that is, $1 \leq \dim(\tilde{\tau}(f_1) \cap \ldots \cap \tilde{\tau}(f_r))$. We have

$$\sum_{i=1}^{r} \text{cod}(\tilde{\tau}(f_i)) = \text{cod}(\bigcap_{i=1}^{r} \tilde{\tau}(f_i)) + 1 = p - \dim(\bigcap_{i=1}^{r} \tilde{\tau}(f_i)) + 1 \leq p.$$ 

On the other hand, the fact of the analytic strata having almost regular intersection, together with $\sum_{i=1}^{r} \text{cod}(\tilde{\tau}(f_i)) \leq p$ implies that $\dim(\tilde{\tau}(f_1) \cap \ldots \cap \tilde{\tau}(f_r)) \geq 1$. If we consider a 1-parameter stable unfolding $F = \{F_1, \ldots, F_r\}$ of $f$, since $f_i$ is stable for all $i = 1, \ldots, r$, $F_i$ is equivalent to a prism on $f_i$, therefore $\text{cod}(\tilde{\tau}(F_i)) = \text{cod}(\tilde{\tau}(f_i))$ for all $i$. It follows that

$$\text{cod}(\bigcap_{i=1}^{r} \tilde{\tau}(F_i)) = \sum_{i=1}^{r} \text{cod}(\tilde{\tau}(F_i)) = \sum_{i=1}^{r} \text{cod}(\tilde{\tau}(f_i)) = \text{cod}(\bigcap_{i=1}^{r} \tilde{\tau}(f_i)) + 1$$

and therefore $\dim(\tilde{\tau}(F)) = \dim(\bigcap_{i=1}^{r} \tilde{\tau}(F_i)) = \dim(\bigcap_{i=1}^{r} \tilde{\tau}(f_i)) \geq 1$ and by Theorem 3.3 we have that $f$ is an augmentation. \hfill \square

This result can also be obtained as a corollary of Theorem 3.3. However, this reformulation will be used in Section 5.

Example 3.9. \begin{itemize} \item[i)] Any multigerm involving two cuspidal edges in $\mathbb{K}^3$ can never be an augmentation because the sum of the codimension of the analytic strata is 4. \item[ii)] Any multigerm from $\mathbb{K}^2$ to $\mathbb{K}^3$ involving a cross-cap and other immersive branches can never be an augmentation because the codimension of the analytic stratum of the cross-cap is already 3. \item[iii)] Using the classical Arnol’d notation for multigerms in the case $n = p$ where $A^k_i A_j$ represents a multigerm with $k$ branches of type $A_i$ and a branch of type $A_j$ where all of the branches are pairwise transversal, then it is an augmentation if and only if $(ik + j) \leq p$. \end{itemize}

4 New operations

Definition 4.1. Suppose $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is non-stable of finite $A_e$-codimension and has a 1-parameter stable unfolding $F(x, \lambda) = (f_\lambda(x), \lambda)$. Let $k \geq 0$ and $g : (\mathbb{K}^p \times \mathbb{K}^k, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ be the fold map $(X, v) \mapsto (X, \Sigma_{j=1}^{k} v_j^2)$ (when $k = 0$ $g(X) = (X, 0)$). Then the multigerm $\{F, g\}$ is called a monic concatenation of $f$. \hfill 7
Theorem 4.2. ([4]) The following relation holds:

\[ A_e - \text{cod}(\{F, g\}) = A_e - \text{cod}(f). \]

In the previous definition we set \( p + k = n + 1 \). We now introduce a new operation which merges the two earlier ones, a simultaneous augmentation and monic concatenation.

Theorem 4.3. Suppose \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) has a 1-parameter stable unfolding \( F(x, \lambda) = (f_\lambda(x), \lambda) \). Let \( g : (\mathbb{K}^p \times \mathbb{K}^{n-p-1}, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0) \) be the fold map \( (X, v) \mapsto (X, \Sigma_{j=p+1}^{n+1} v_j^2) \). Then,

i) the multigerm \( \{A_{F,\phi}(f), g\} \), where \( \phi : \mathbb{K} \to \mathbb{K}, \) has

\[ A_e - \text{cod}(\{A_{F,\phi}(f), g\}) \geq A_e - \text{cod}(f)(\tau(\phi) + 1), \]

where \( \tau \) is the Tjurina number of \( \phi \). Equality is reached when \( \phi \) is quasi-homogeneous and \( \langle dZ(i^*(\text{Lift}(A_{F,\phi}(f)))) \rangle = \langle dZ(i^*(\text{Lift}(F))) \rangle \) where \( i : \mathbb{K}^p \to \mathbb{K}^{p+1} \) is the canonical immersion \( i(X_1, \ldots, X_p) = (X_1, \ldots, X_p, 0) \) and \( Z \) is the last component of the target.

ii) \( \{A_{F,\phi}(f), g\} \) has a 1-parameter stable unfolding.

Proof. We write \( Af \) for \( A_{F,\phi}(f) \). Consider the map \( h : \theta(Af) \oplus \theta(g) \to \theta(Af) \oplus \theta(g) \), which maps \((\xi_1, \xi_2) \mapsto \xi_1 \). Since \( h(TA_e\{Af, g\}) = \{0\} \) we consider the induced map

\[ \overline{h} : \frac{\theta(Af) \oplus \theta(g)}{TA_e\{Af, g\}} \to \frac{\theta(Af)}{TA_eAf} \]

where \( \overline{h}(\xi_1, \xi_2) = 0 \) if and only if \( h(\xi_1, \xi_2) = 0 \).

Obviously, any element in \( \ker(\overline{h}) \) can be taken to the form \((0, \xi_2)\). Now, if \( \eta \in \text{Lift}(Af) \), then there exists \( \rho \in \theta_{n+1} \) such that \( dAf \circ \rho = \eta \circ Af \). Then, for any \( \delta \in \theta_{n+1} \)

\[ (0, \xi_2) = (0, \xi_2) - [(dAf \circ \rho, dg \circ \delta) + (\eta \circ Af, \eta \circ g)] = (0, \xi_2 - dg \circ \delta - \eta \circ g), \]

which means that

\[ \ker(\overline{h}) \cong \frac{\theta(g)}{tg(\theta_{n+1}) + wg(\text{Lift}(Af))}. \]

So we have the short exact sequence:

\[ 0 \to \frac{\theta(g)}{tg(\theta_{n+1}) + wg(\text{Lift}(Af))} \to N\mathcal{A}_e(\{Af, g\}) \xrightarrow{\overline{h}} N\mathcal{A}_e(Af) \to 0. \]

Then

\[ A_e - \text{cod}(\{Af, g\}) = A_e - \text{cod}(Af) + \dim_{\mathbb{K}} \frac{\theta(g)}{tg(\theta_{n+1}) + wg(\text{Lift}(Af))}. \]
Note that \( tg(\theta_n) = \sum_{l=1}^p \partial_n \frac{\partial}{\partial x_l} + \sum_{j=p+1}^{n+1} \partial_n \frac{\partial}{\partial z_j}, \) therefore, by projection to the last component we have that

\[
\text{tg}(\theta_{n+1}) + \text{wg}(\text{Lift}(A_f)) \approx \langle v_{p+1}, \ldots, v_{n+1} \rangle + dZ(\text{wg}(\text{Lift}(A_f))) \approx \frac{\partial}{\partial DZ(A_f)}.
\]

Since \( F \) is stable, by Damon’s theorem \( \mathcal{A}_{e} \cdot \text{cod}(f) = \text{dim}_X \frac{\theta(i)}{\text{det}(\theta_p) + \text{ti}(\text{Lift}(F)) + \text{det}(\text{Lift}(F))}. \) As \( i(X) = (X, 0) \), it follows that \( \frac{\theta(i)}{\text{det}(\theta_p) + \text{ti}(\text{Lift}(F))} \) is isomorphic to \( \frac{\theta(i)}{\text{det}(\text{Lift}(F))} \) by projection to the last component.

Therefore, \( \text{dim}_X \frac{\mathcal{A}_{e} \cdot \text{cod}(f)}{\partial Z(i^*(\text{Lift}(F)))} \leq \text{dim}_X \frac{\mathcal{O}_p}{\partial Z(i^*(\text{Lift}(F)))} \) gives

\[
\mathcal{A}_e - \text{cod}((A_f, g)) \geq \mathcal{A}_e - \text{cod}(A_f) + \mathcal{A}_e - \text{cod}(f) \geq \mathcal{A}_e - \text{cod}(f) \tau(\phi) + \mathcal{A}_e - \text{cod}(f) = \mathcal{A}_e - \text{cod}(f)(\tau(\phi) + 1)
\]

and equality is reached when \( \phi \) is quasi-homogeneous and \( \langle dZ(i^*(\text{Lift}(F))) \rangle = \langle dZ(i^*(\text{Lift}(F))) \rangle \).

ii) The 1-parameter unfolding \( \{A_{F, \phi'}(f), G\} \) where \( \phi'(z, \mu) = \phi(z) + \mu \) and \( G(x, v, \mu) = (X, \sum_{j=p+1}^{n+1} v_j^2, \mu) \) is stable since each branch is stable and the analytic strata have regular intersection.

\[\square\]

**Remark 4.4.** The exact sequence in the proof above remains exact when we replace \( A_f \) by any finitely determined map-germ \( H \) so we can deduce that if \( \mathcal{M}_p \subseteq dZ(i^*(\text{Lift}(H))) \), then \( \mathcal{A}_e - \text{cod}((H, g)) \leq \mathcal{A}_e - \text{cod}(H) + 1 \).

**Example 4.5.**

i) Consider the family of augmentations of \( f(x) = x^3 \), \( A^l f(x, z) = (x^3 + z^l x, z) \), of \( \mathcal{A}_e \)-codimension \( l \), where \( \phi(z) = z^l \) is the augmenting function. We calculate \( \text{Lift}(A^l f) = \langle 3X \frac{\partial}{\partial x} + 2Z \frac{\partial}{\partial z}, -21Z^3 \frac{\partial}{\partial x} + 9X \frac{\partial}{\partial z} \rangle \). So the bigerm

\[
\begin{cases}
(x^3 + z^l x, z) \\
(x, z^2)
\end{cases}
\]

has codimension 1.

ii) Let \( f_1(x, y) = (x^3 + y^l \, x, y) \) and \( F_1(x, y, z) = (x^3 + y^l x + z^x, y, z) \) with augmentations \( A^m F_1(x, y, z) = (x^3 + y^l x + z^m x, y, z) \) of codimension \( (l-1)(m-1) \).

The defining equation of the discriminant is \( 27X^2 + 4(Y^l + Z^m)^3 = 0 \). For any \( m \geq 1 \), \( \partial Z(i^*(\text{Lift}(A_f))) = \langle 1, Y, \ldots, Y^{l-2} \rangle \), so the codimension of the bigerm \( \{A^m F_1, g\} \) is \( (l-1)(m-1) + (l-1) = (l-1)m \).

iii) The augmentation \( A f(x, y, z) = (x^4 + yz^2 + y^2 z + z^2 x, y, z) \) of \( f(x, y) = (x^4 + yz^2 + y^2 x, y) \) has codimension 2. Calculations using the computer package Singular show that \( \mathcal{O}_p / \partial Z(i^*(\text{Lift}(A_f))) = \langle 1, Y \rangle \), so \( \mathcal{A}_e - \text{cod}((A_f, g)) = 2 + 2 = 4 \).
**Theorem 4.6.** Up to $A$-equivalence, if $\mathbb{K} = \mathbb{C}$ and $A_e - \text{cod}(f) = 1$, the multigerm $\{Af, g\}$ is independent of the choice of miniversal unfolding $F$ of $f$.

**Proof.** Let $Af$ and $A'f$ be augmentations coming from different miniversal unfoldings $F$ and $F'$ of $f$ which are $A$-equivalent as unfoldings. By [4], $Af \sim_A A'f$ via diffeomorphisms $\varphi : (\mathbb{C}^p \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^p \times \mathbb{C}, 0)$ and $\psi : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \rightarrow (\mathbb{C}^n \times \mathbb{C}, S \times \{0\})$ such that

$$\varphi \circ Af = A'f \circ \psi$$

where $\varphi(X, v) = (\varphi_1(X, v), \ldots, \varphi_p(X, v), \varphi_{p+1}(v))$. To prove $\{Af, g\} \sim_A \{A'f, g\}$ we need a diffeomorphism $\theta$ such that $\varphi \circ g = g \circ \theta$. This is satisfied with $\theta(X, v) = (\theta_1(X, v), \ldots, \theta_{n+1}(X, v))$ where $\theta_j(X, v) = \varphi_j \circ g(X, v)$ for $j \leq p$ and $\sum_{i=p+1}^{n+1}(\theta_i(X, v))^2 = \varphi_{p+1}(\sum_{i=p+1}^{n+1} v_i^2)$.

**Remark 4.7.** $\{Af, g\}$ is still independent of the choice of miniversal unfolding $F$ of $f$ when $f$ is of codimension higher than 1 if we suppose that $Af \sim_A A'f$ through a diffeomorphism of the type $\varphi(X, v) = (\varphi_1(X, v), \ldots, \varphi_p(X, v), \varphi_{p+1}(v))$.

### 4.1 Generalised concatenations

Another type of operation is a concatenation with a stable germ which generalises the monic concatenation defined earlier. It also includes the binary concatenation as a particular case.

**Definition 4.8.** ([4]) Given germs $f_0 : (\mathbb{C}^m, S) \rightarrow (\mathbb{C}^a, 0)$ and $g_0 : (\mathbb{C}^l, T) \rightarrow (\mathbb{C}^b, 0)$ with 1-parameter stable unfoldings $F(y, s) = (f_s(y), s)$ and $G(x, s) = (g_s(x), s)$, the multigerm $h$ with $|S| + |T|$ branches defined by

$$
\begin{cases}
(X, y, u) \mapsto (X, f_u(y), u) \\
(x, Y, u) \mapsto (g_u(x), Y, u)
\end{cases}
$$

(2)

is called a binary concatenation of $f_0$ and $g_0$.

**Definition 4.9.** Let $f : (\mathbb{K}^{n-s}, S) \rightarrow (\mathbb{K}^{p-s}, 0)$, $s < p$, be of finite $A_e$-codimension and let $F : (\mathbb{K}^n, S \times \{0\}) \rightarrow (\mathbb{K}^p, 0)$ be a $s$-parameter stable unfolding of $f$ with

$$F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_p-s(x_1, \ldots, x_n), x_{n-s+1}, \ldots, x_n),$$

where $F_i(x_1, \ldots, x_{n-s}, 0, \ldots, 0) = f_i(x_1, \ldots, x_{n-s})$. Suppose that $\overline{g} : (\mathbb{K}^{n-p+s}, T) \rightarrow (\mathbb{K}^s, 0)$ is stable. Then the multigerm $\{F, g\}$ is a generalised concatenation of $f$ with $g$, where $g = \text{Id}_{(\mathbb{K}^{n-s})} \times \overline{g}$.

Observe that with this definition, $\dim \overline{g} = p - s - 1$. If $g$ is a monogerm and $\dim \overline{g} = p - s$, it is of the form

$$g(x_1, \ldots, x_n) = (x_1, \ldots, x_{p-s}, g_{p-s+1}(x_{p-s+1}, \ldots, x_n), \ldots, g_p(x_{p-s+1}, \ldots, x_n)).$$

Therefore, the definition implies that $F \triangledown \overline{g}$. 

10
Remark 4.10.  

i) The monic concatenation is recovered by taking $s = 1$ and $g_p(x_p, \ldots, x_n) = \sum_{i=p}^{n} x_i^2$ (or $g_p = 0$ when $n = p - 1$).

ii) A binary concatenation $h = \{F, G\}$

\[
\begin{align*}
(X, y, u) &\mapsto (f_u(y), u, X) \\
(x, y, u) &\mapsto (Y, u, g_u(x))
\end{align*}
\]  

is also a generalised concatenation where $p = a + 1 + b$, $n = b + m + 1 = l + a + 1$ and $s = b + 1$. In fact, the first branch is a $b + 1$-parameter stable unfolding of an $f_0$ and $\bar{\tau}((u, g_u(x'))) = \{0\}$ when $g_0$ is not stable.

In general this operation is difficult to control, so we study particular cases where the stable germ $g$ is given, for example the equidimensional case and where $g$ is a cusp and $\dim \bar{\tau}(g) = p - 2$.

Definition 4.11. Consider $f : (\mathbb{K}^{n-2}, S) \to (\mathbb{K}^{n-2}, 0)$ with $n \geq 3$, $F(x, \lambda) = (f_\lambda(x), \lambda)$ a 2-parameter stable unfolding of $f$ and

\[ g(x_1, \ldots, x_{n-2}, y, z) = (x_1, \ldots, x_{n-2}, y, z^3 + yz) \]

being a suspension of a cusp. We call the multigerm $\{F, g\}$ the cuspidal concatenation of $f$.

In general $\mathcal{A}_{e}-\text{cod}(\{F, g\})$ depends on the choice of 2-parameter stable unfolding, so we give a recipe to calculate the $\mathcal{A}_{e}$-codimension.

Theorem 4.12. Let $f : (\mathbb{K}^{n-2}, S) \to (\mathbb{K}^{n-2}, 0)$ with $n \geq 3$ and $\{F, g\}$ the cuspidal concatenation of $f$, then

\[ \mathcal{A}_{e}-\text{cod}(\{F, g\}) = \dim_{\mathbb{R}} \mathcal{O}_{n-1} \{ \xi : \xi = -z\eta_{n-1}(x, -3z^2, -2z^3) + \eta_n(x, -3z^2, -2z^3) \}, \]

where $\eta_{n-1}$ and $\eta_n$ are the last two components of vector fields in $\text{Lift}(f)$.

Proof. Consider the exact sequence (see the proof of 4.3)

\[ 0 \to \theta(g)/\theta(g)+wg(\text{Lift}(F)) \to N\mathcal{A}_{e}(\{F, g\}) \to N\mathcal{A}_{e}(F) \to 0. \]

Since $F$ is stable, $\dim_{\mathbb{R}} N\mathcal{A}_{e}(F) = 0$, hence $\mathcal{A}_{e}-\text{cod}(\{F, g\})$ is equal to the dimension of $\theta(g)/\theta(g)+wg(\text{Lift}(F))$. By projecting to the last two components, this space is isomorphic to $\mathcal{O}_n \oplus \mathcal{O}_n$ where

\[ T = \{ \begin{pmatrix} 1 & 0 \\ z & 3z^2 + y \end{pmatrix} \begin{pmatrix} v_{n-1} \\ v_n \end{pmatrix} : v_{n-1}, v_n \in \mathcal{O}_n \} + d(Y, Z)(wg(\text{Lift}(F))), \]
and $d(Y,Z)$ represents the last two components of $wg(Lift(F))$.

Let

$$T_0 = \{ \xi : (0, \xi) \in T \} = \{ \xi : \xi = -z\eta_{n-1}(x, y, z^3 + yz) + (3z^2 + y)v_n(x, y, z) + \eta_n(x, y, z^3 + yz) \},$$

where $\eta = (\eta_1, \ldots, \eta_n) \in Lift(F)$.

Let $(g_{n-1}, g_n)$ be the last two components of $g$ and let

$$T_1 = t g_{n-1}(O_n) + dY(wg(Lift(F))).$$

The following sequence is exact

$$0 \longrightarrow O_n T_0 \xrightarrow{i^*} O_n \oplus O_n \xrightarrow{\pi^*} T_1 \xrightarrow{\theta(g_{n-1})} 0 \longrightarrow 0$$

where $i$ is the inclusion and $\pi$ is the projection. The proof of the exactness is analogous to the proof of Proposition 2.1 in [16]. Since $g_{n-1}$ is a submersion, $O_n \oplus O_n$ and $T_0$ are isomorphic. The result follows from the fact that

$$\frac{O_n}{T_0} \cong \{ \xi : \xi = -z\eta_{n-1}(x, -3z^2, -2z^3) + \eta_n(x, -3z^2, -2z^3) \}.$$

\[\square\]

**Proposition 4.13.** Let $f(x_1, \ldots, x_{n-2}) = (x_1^{n+1} + x_2 x_1 + \ldots + x_{n-2} x_1^{n-3}, x_2, \ldots, x_{n-2})$ and let

$$F(x_1, \ldots, x_{n-2}, y, z) = (x_1^{n+1} + x_2 x_1 + \ldots + x_{n-2} x_1^{n-3} + y x_1^{n-2} + z x_1^{n-1}, x_2, \ldots, x_{n-2}, y, z),$$

the $A_n$ singularity. Then the cuspidal concatenation of $f$ has $A_e - \text{cod}(\{F, g\}) = n$.

**Proof.** We denote by $T_2 = \{ \xi : \xi = -z\eta_{n-1}(x, -3z^2, -2z^3) + \eta_n(x, -3z^2, -2z^3) \}$. From Theorem [11,12] $A_e - \text{cod}(\{F, g\}) = \dim_{\mathbb{C}} \frac{O_n}{T_2}$. From [2] it follows that the linear part of the generators of $Lift(F)$ is

$$\{(n+1)X_1 \frac{\partial}{\partial Z}, (n+1)X_1 \frac{\partial}{\partial Y} + n X_2 \frac{\partial}{\partial Z}, \ldots,$$

$$(n+1)X_1 \frac{\partial}{\partial X_2} + \ldots + 4X_{n-2} \frac{\partial}{\partial Y} + 3Y \frac{\partial}{\partial Z},$$

$$(n+1)X_1 \frac{\partial}{\partial X_1} + \ldots + 3Y \frac{\partial}{\partial Y} + 2Z \frac{\partial}{\partial Z} \}.$$

The corresponding elements of the generators of $T_2$ and the corresponding relations modulo $M^3_{(x,z)}$ are

12
\[
\begin{cases}
(n + 1)x_1 \\
-(n + 1)zx_1 + nx_2 \\
\vdots \\
-5zx_{n-3} + 4x_{n-2} \\
-4zx_{n-2} - 9z^2 \\
5z^3
\end{cases}
\begin{cases}
(1) \ x_1 \equiv 0 \\
(2) \ x_2 \equiv \frac{n+1}{n}zx_1 \\
\vdots \\
(n-2) \ x_{n-2} \equiv \frac{5}{4}zx_{n-3} \\
(n-1) \ z^2 \equiv -\frac{4}{9}zx_{n-2}
\end{cases}
\tag{4}
\]

Note that \( T_{\Sigma} \) is a \( O_{n-1} \)-module via \( \varphi^* \) where \( \varphi(x,z) = (x,-3z^2,-2z^3) \). It is clear that we can obtain any element in \( T_{\Sigma} \) of the type \( x^\alpha z^\beta \) for all \( \alpha > 0 \) and \( \beta = 0 \) or \( \beta \geq 2 \) modulo \( M^{\alpha + \beta + 1} \) and of the type \( z^\beta \) for all \( \beta \geq 3 \). It is also clear that constants and \( z \) are not in \( T_{\Sigma} \). Let’s see what other terms are in \( T_{\Sigma} \):

In step I, we take relation (1) and multiply it by \( x_j \) for all \( j = 1, \ldots, n - 2 \) and by \( z^2 \) to obtain relations \( x_1x_j \equiv 0 \) and \( x_1z^2 \equiv 0 \) modulo \( M^4 \). Now multiply relations (2) to (\( n - 1 \)) by \( x_1 \) and use the relations just obtained to get relations \( zx_1x_{j-1} \equiv 0 \) modulo \( M^4 \) for all \( j = 2, \ldots, n - 1 \).

In step II, we take relation (2) and multiply it by \( x_j \) for all \( j = 2, \ldots, n - 2 \) and by \( z^2 \) to obtain relations \( x_2x_j \equiv zx_1x_j \equiv 0 \) by step I \( x_2z^2 \equiv 0 \) modulo \( M^4 \). Now multiply relations (3) to (\( n - 1 \)) by \( x_2 \) and use the relations just obtained to get relations \( zx_2x_{j-1} \equiv 0 \) modulo \( M^4 \) for all \( j = 3, \ldots, n - 1 \).

We go on until finally we have that \( M^3 \subseteq T_{\Sigma} + M^4 \). In the same way we prove that \( M^k \subseteq T_{\Sigma} + M^{k+1} \) for \( k \geq 4 \), so we have that \( M^3 \subseteq T_{\Sigma} + M^k \) for all \( k \geq 4 \). Therefore, \( M_3 \subseteq T_{\Sigma} \). A basis for \( \mathcal{O}^*_{T_{\Sigma}} \) is \( 1, z, x_1, \ldots, x_{n-2}z \), which proves the result.

**Theorem 4.14.** Let \( F : (\mathbb{K}^n, S) \to (\mathbb{K}^n, 0) \) be a stable (multi)germ with \( \bar{\tau}(F) = 0 \) and \( g(x_1, \ldots, x_{n-2}, y, z) = (x_1, \ldots, x_{n-2}, y, z^3 + yz) \), then \( \mathcal{A}_c\text{-cod}(\{F, g\}) \geq n \) and is equal to \( n \) when \( g \) is transversal to the limits of the tangent spaces to the strata of the stratification by stable types of the discriminant of \( F \) and \( F \cap \bar{\tau}(g) \).

**Proof.** The fact that \( F \) is stable, has corank 1 and \( \bar{\tau}(F) = 0 \) implies that \( F \) is a singularity of type \( A_{k_1} \ldots A_{k_r} \) where \( k_1 + \ldots + k_r = n \). In the case of lowest codimension, \( g \) is transversal to the limits of the tangent spaces to the strata of the discriminant of \( F \) and \( F \cap \bar{\tau}(g) \). In this case, \( A_{k_i} \) considered from \( \mathbb{K}^n \) to \( \mathbb{K}^n \) is a \( n-k_i \)-prism on \( A_{k_i} \) considered from \( \mathbb{K}^{k_i} \) to \( \mathbb{K}^{k_i} \). Then \( \text{Lift}(A_{k_i}) \) is as in Proposition 4.13 together with \( \frac{\partial}{\partial X_j} \) with \( X_j \) varying on the remaining \( n-k_i \) variables. Since \( A_{k_1} \ldots A_{k_r} \) is stable and all of the \( A_{k_i} \) are stable, the \( \text{Lift}(F) \) is the intersection of the \( \text{Lift}(A_{k_i}) \) and so we can see it as a diagonal block matrix where each block represents the \( \text{Lift}(A_{k_i}) \). Following the proof of Proposition 4.13 we prove that the codimension of \( \{F, g\} \) in this case is exactly \( n \). In other cases it is greater than or equal to \( n \). \( \square \)
The following examples illustrate how the cuspidal concatenation depends on the choice of stable unfolding:

**Example 4.15.**  

i) Let \( f(x) = x^4 \) and choose \( F(x, y, z) = (x^4 + yx + zx^2, y, z) \) (the swallowtail). Concatenating with a cuspidal edge we obtain the codimension 3 bigerm \( A_2A_3 \):

\[
\begin{cases}
  (x^4 + yx + zx^2, y, z) \\
  (x, y, z^3 + yz)
\end{cases}
\]

Notice that this bigerm could not be obtained by any of the other operations in the literature up to now. It can be seen that it is not simple and the stratum codimension is 2.

Now, we let \( F(x, y, z) = (x^4 + yx^2 + y^2x + zx, y, z) \), then the bigerm \( \{F, g\} \) is not \( A \)-equivalent to \( A_2A_3 \) above. In fact,

\[
\text{Lift}(F) = \langle 4X \frac{\partial}{\partial X} + 2Y \frac{\partial}{\partial Y} + (Y^2 - 3Z) \frac{\partial}{\partial Z}, \rangle
\]

\[
(Y^3 + ZY) \frac{\partial}{\partial X} + (-6X^2 - 6Z) \frac{\partial}{\partial Y} + (8X + 2Y^2 + 12Y^3 + 12YZ) \frac{\partial}{\partial Z}
\]

\[
(-16X^2 + 2Y^2 - 9Y^4) \frac{\partial}{\partial X} + (48X + 4Y^2) \frac{\partial}{\partial Y} + (-96XY + 12YZ + 4Y^3) \frac{\partial}{\partial Z},
\]

and calculations show that \( A_{e - \text{cod}}(\{F, g\}) = 4 \).

ii) Consider \( f(x) = x^3 \) and the family of 2-parameter stable unfoldings \( F_l(x, y, z) = (x^3 + y^l x + zx, y, z) \). In this case

\[
\text{Lift}(F_l) = \langle 3lX \frac{\partial}{\partial X} + 2Y \frac{\partial}{\partial Y} + 2Z \frac{\partial}{\partial Z}, \frac{\partial}{\partial Y} + lY^{l-1} \frac{\partial}{\partial Z}, 2(Z + Y^l) \frac{\partial}{\partial X} - 9X \frac{\partial}{\partial Z} \rangle,
\]

Calculations show that \( A_{e - \text{cod}}(\{F_l, g\}) = 2 \) for \( l \geq 2 \).

For \( l = 1 \) we have the codimension 1 bigerm \( A_2^2 \) (which is a binary concatenation, see [3]). For \( l \geq 2 \) we have the codimension 2 bigerm \( T_{12}^l \) (see [23]). It is interesting to mention that in spite of the cuspidal edge in \( F_l \) having a more degenerate contact with the limiting tangent plane to \( g \) as \( l \) grows, this does not affect the codimension. Compare this with example 4.5 ii) when \( m = 1 \), where the codimension does increase.

iii) Let \( f(x) = \{x^2, x^3\} \), and \( F_1 \) and \( F_2 \) two different 2-parameter stable unfoldings:

\[
F_1 = \begin{cases}
  (x^2 + y + z, y, z) \\
  (x^3 + xy, y, z)
\end{cases}
\]

and

\[
F_2 = \begin{cases}
  (x^2, y, z) \\
  (x^3 + xy + z, y, z)
\end{cases}
\]

(6)
One would expect \( \{F_1, g\} \sim_A \{F_2, g\} \) since they are two cuspidal edges and a fold plane with pairwise transversal branches. However, \( A_e - \text{cod}(\{F_1, g\}) = 3 \) while \( \{F_2, g\} \) is not finitely determined because it has a curve of triple points in the image along \( (0,-3t^2,2t^3) \).

Another type of generalised concatenation is to concatenate with two fold hypersurfaces (in the equidimensional case, but the operation can be defined for \( n \neq p \) too).

**Definition 4.16.** Let \( f : (\mathbb{K}^{n-2}, S) \to (\mathbb{K}^{n-2}, 0) \) \((n \geq 3)\) be a finitely determined germ, \( F(x, \lambda) = (f\lambda(x), \lambda) \) a 2-parameter stable unfolding of \( f \) and \( g = \{g_1, g_2\} \) where

\[
\begin{align*}
g_1(x_1, \ldots, x_{n-2}, y, z) &= (x_1, \ldots, x_{n-2}, y, z^2) \\
g_2(x_1, \ldots, x_{n-2}, y, z) &= (x_1, \ldots, x_{n-2}, y, z^2 + y).
\end{align*}
\]

We call the multigerm \( \{F, g\} \) the double fold concatenation of \( f \).

As in the cuspidal concatenation, the \( A_e \)-codimension of \( \{F, g\} \) depends on the choice of the 2-parameter stable unfolding.

**Theorem 4.17.** Let \( f : (\mathbb{K}^{n-2}, S) \to (\mathbb{K}^{n-2}, 0) \) with \( n \geq 3 \) and let \( \{F, g\} \) be the double fold concatenation of \( f \), then

\[
A_e - \text{cod}(\{F, g\}) = A_e - \text{cod}(\{F, g_1\}) + \dim \frac{\mathcal{O}_{n-1}}{\{\xi : \xi = -\eta_{n-1}(x, y, y) + \eta_{n}(x, y, y)\}},
\]

where \( \eta_{n-1}, \eta_{n} \) are the last two components of vector fields in \( \text{Lift}(\{F, g_1\}) \).

**Proof.** The proof is very similar to the one of the cuspidal concatenation. Consider the following exact sequence

\[
0 \longrightarrow \frac{\partial_{\{g_2\}}}{t_{g_2(\theta_n)+w_{g_2}}(\text{Lift}(\{F, g_1\}))} \longrightarrow N\mathcal{A}_e(\{F, g\}) \longrightarrow N\mathcal{A}_e(\{F, g_1\}) \longrightarrow 0
\]

Projecting to the last two components it follows that \( \frac{\partial_{\{g_2\}}}{t_{g_2(\theta_n)+w_{g_2}}(\text{Lift}(\{F, g_1\}))} \) is isomorphic to \( \frac{\mathcal{O}_{n-1} \oplus \mathcal{O}_{n}}{\mathcal{O}_{n-1} \oplus \mathcal{O}_{n}} \), where

\[
T = \left\{ \left( \begin{array}{cc} 1 & 0 \\ 1 & 2z \end{array} \right) \begin{pmatrix} v_n \\ 1 \\ v_n \end{pmatrix} : v_{n-1}, v_n \in \mathcal{O}_n \right\} + d(Y, Z)(w_{g_2}(\text{Lift}(\{F, g_1\}))),
\]

and \( d(Y, Z) \) represents the last two components of \( w_{g_2}(\text{Lift}(\{F, g_1\})) \).

Let \( T_0 = \{\xi : (0, \xi) \in T\} \) which is equal to

\[
\{\xi : \xi = -\eta_{n-1}(x, y, z^2 + y) + z v_n(x, y, z) + \eta_n(x, y, z^2 + y)\},
\]

where \( \eta = (\eta_1, \ldots, \eta_n) \in \text{Lift}(\{F, g_1\}) \). As in Proposition 4.12 we can show that \( \frac{\mathcal{O}_{n-1} \oplus \mathcal{O}_{n}}{\mathcal{O}_{n-1} \oplus \mathcal{O}_{n}} \) is isomorphic to \( \frac{\mathcal{O}_{n-1}}{\mathcal{O}_{n-1}} \) which is in turn isomorphic to \( \frac{\mathcal{O}_{n-1}}{(\xi : \xi = -\eta_{n-1}(x, y, y) + \eta_n(x, y, y))} \).
Remark 4.18. If there exists a finitely determined 1-parameter unfolding \( F' \) of \( f, F' : (\mathbb{K}^{n-1}, S \times \{0\}) \to (\mathbb{K}^{n-1}, 0), \) \( F'(x, y) = (f_y(x), y) \) such that \( F(x, y, 0) = (F'(x, y), 0), \) then the multigerm \( \{ F, g_1 \} \) is a monic concatenation of \( F' \). Therefore, \( \mathcal{A}_e - \text{cod}(\{ F, g_1 \}) = \mathcal{A}_e - \text{cod}(F') \) by Theorem 4.2.

Example 4.19. Consider the trigerm \( f(x) = \{ x^2, x^2, x^2 \} \) and \( F' \) and \( F \) as follows:

\[
F' = \begin{cases} 
(x^2 + y, y) \\
(x^2, y) \\
(x^2 - y, y)
\end{cases} \quad \text{and} \quad F = \begin{cases} 
(x^2 + y + z, y, z) \\
(x^2, y, z) \\
(x^2 - y, y, z)
\end{cases}
\]

The generators of \( \text{Lift}(\{ F, g_1 \}) \) are

\[
\{ X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + Z \frac{\partial}{\partial Z}, \\
(2XY + XZ) \frac{\partial}{\partial X} + (3X^2 - Y^2 - 2XZ - YZ) \frac{\partial}{\partial Y}, \\
(2X^2 - 3XZ) \frac{\partial}{\partial X} + (2XY + 4XZ + YZ) \frac{\partial}{\partial Y} + (Z^2 - XZ) \frac{\partial}{\partial Z}, \\
(2XY + XZ) \frac{\partial}{\partial X} + (2Y^2 + 4XZ + 5YZ) \frac{\partial}{\partial Y} + (-3Z^2 - 6XZ) \frac{\partial}{\partial Z} \}
\]

Calculations show that \( \dim_{\mathbb{K}} \{ \xi : \xi = -\eta_{p-1}(x, y, y) + \eta_p(x, y, y) \} = 3. \) Since \( \mathcal{A}_e - \text{cod}(\{ F, g_1 \}) = \mathcal{A}_e - \text{cod}(F') = 1, \) it follows that the codimension of the quintuple point \( \{ F, g \} \) is 4.

Theorem 4.20. Let \( F : (\mathbb{K}^n, S) \to (\mathbb{K}^n, 0) \) be a stable (multi)germ with \( \bar{\tau}(F) = 0 \) and \( g = \{ g_1, g_2 \} \) where

\[
\begin{align*}
g_1(x_1, \ldots, x_{n-2}, y, z) &= (x_1, \ldots, x_{n-2}, y, z^2) \\
g_2(x_1, \ldots, x_{n-2}, y, z) &= (x_1, \ldots, x_{n-2}, y, z^2 + y)
\end{align*}
\]

then \( \mathcal{A}_e - \text{cod}(\{ F, g \}) \geq n. \)

Proof. If \( F \) is a monogerm, we use the information about \( \text{Lift}(F) \) in Proposition 4.13 and the fact that \( \text{Lift}(g_1) = \left( \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_{n-2}}, \frac{\partial}{\partial Y}, Z \frac{\partial}{\partial Z} \right) \) to apply the formula in Theorem 4.17. If \( F \) is a multigerm we repeat the argument in the proof of Theorem 4.14.

A similar result for the case \( n = p - 1 \) can be found in Remark 5.11.
5 $A_e$-codimension 2 multigerms

The classification of primitive monogerms of $A_e$-codimension 2 for all pairs $(n, p)$ in the nice dimensions and $p \leq n + 1$ is already known. When $n \geq p$, the list of $A$-simple singularities was obtained by Goryunov in [6] (see also [20]). When $p = n + 1$, this classification was recently given by Houston and Wik Atique in [13].

We assume the known fact that when $p = 1$, the only codimension 2 multigerms are a trigerm with 3 Morse functions and a bigerm with a Morse function and an $A_2$ singularity. We also need the list of codimension 2 multigerms when $p = 2$. For $n = 1$ we refer to [14] and for $n \geq 2$ a list can be found in [23].

In this section we prove that all the simple minimal corank $A_e$-codimension 2 multigerms with more than one branch in Mather’s nice dimensions and $n \geq p - 1$ can be obtained using the operations of augmentation, augmentation and concatenation and generalised concatenations starting from monogerms and some special multigerms. By a multigerm with $k$ branches we mean a multigerm with $k$ nonsubmersive branches. Too that all the results in this section are stated for the complex case.

The section is organised as follows: We start by proving some general results for any codimension 2 multigerm. Results 5.5 to 5.12 deal with augmentations and $A_e$-codimension 2 multigerms that have a branch which is an $A_e$-codimension 1 monogerm. Results 5.14 to 5.18 classify the codimension 2 primitive multigerms where all the branches are stable. Finally, Theorem 5.19 is a summary of all the results in this section.

Proposition 5.1. Let $h = \{h_1, \ldots, h_r\} : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a multigerm of $A_e$-codimension 2. Then, for any proper subset $S'$ of $S$, the multigerm $h' : (\mathbb{C}^n, S') \to (\mathbb{C}^p, 0)$ has $A_e$-codimension $\leq 1$. If $A_e - \text{cod}(h') = 1$ then $h = \{h', g\}$ where $g$ is prism on a Morse function (when $n \geq p$) or an immersion (when $p = n + 1$).

Proof. Let $g : (\mathbb{C}^n, S \setminus S') \to (\mathbb{C}^p, 0)$ be the germ such that $h = \{h', g\}$. If $h'$ is stable, then $A_e - \text{cod}(h') = 0$. Otherwise, consider the exact sequence

$$0 \longrightarrow \theta(g)_{tg(\theta_n) + wg(Lift(h'))} \longrightarrow N_{A_e}(h) \longrightarrow N_{A_e}(h') \longrightarrow 0.$$ 

Since $\tilde{\tau}(h') = 0$, then the dimension of $\theta(g)_{tg(\theta_n) + wg(Lift(h'))}$ is at least 1 and since $\dim_{\mathbb{C}} N_{A_e}(h) = 2$, it follows that the codimension of $h'$ has to be less than or equal to 1.

Now suppose that $A_e - \text{cod}(h') = 1$, then the dimension of $\theta(g)_{tg(\theta_n) + wg(Lift(h'))}$ would be exactly 1. Since $\tilde{\tau}(h') = \{0\}$, $Lift(h')$ does not have any constants in any entry and so $1 = \dim \frac{\theta(g)_{tg(\theta_n) + wg(Lift(h'))}}{\theta(g)_{tg(\theta_n) + wg(Lift(h'))}} \geq \dim \frac{\theta(g)_{\theta(g)_{M_{sp}}}^{\theta(g)}}{\theta(g)_{M_{sp}}} = K_e - \text{cod}(g)$. This implies first that $g$ is a monogerm since the $K_e$-codimension of the simplest possible bigerms (namely two transversal folds or immersions) is 2. Furthermore,
\[
\dim_{g\theta + g\gamma \theta + wg\theta} = 0 \quad \text{and so } g \text{ is stable and therefore it is a prism on a Morse function or an immersion.}
\]

**Corollary 5.2.** Let \( h = \{h_1, \ldots, h_r\} : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0) \ (p > 1) \) be a multigerm of \( \mathcal{A}_e \)-codimension 2. Then

\begin{enumerate}
  \item If \( r \geq 3 \) then \( h_i \) is stable for every \( i \in \{1, \ldots, r\} \).
  \item If \( h_i \) is stable for every \( i \in \{1, \ldots, r\} \), then \( h = \{f, g\} \) where both \( f \) and \( g \) are stable.
\end{enumerate}

**Proof.** i) Follows directly from Proposition 5.1.

ii) From [4] we know that a codimension 1 multigerm only has stable branches. Suppose that there does not exist a partition \( h = \{f, g\} \) with \( f, g \) stable. It follows from Proposition 5.1 that \( h_i \) is a prism on a Morse function or an immersion for all \( i = 1, \ldots, r \). By hypothesis, \( h' = \{h_i, \ldots, h_{i-1}\} \) has \( \mathcal{A}_e \)-codimension 1, and by [4] any \( r - 2 \) branches of \( h' \) form a stable multigerm. It follows that any two branches of \( h \) constitute an \( \mathcal{A}_e \)-codimension 1 bigerm. Therefore, if \( r \geq 4 \) we have a decomposition of \( h \) where at least two bigerms have \( \mathcal{A}_e \)-codimension 1, which contradicts Proposition 5.1. When \( r = 3 \), in a similar way we can prove that all branches are prisms on Morse functions or immersions and, further more, that any two branches form a codimension 1 germ, that is, they are tangent. This implies a triple tangency. Direct calculations show that the codimension of such a trigerm is greater than two except for when \( p = 1 \). The case \( r = 2 \) is trivial.

**Remark 5.3.** When \( p = 1 \), a trigerm of three Morse functions has codimension 2 and cannot be separated into two stable germs.

**Proposition 5.4.** Let \( h = \{f, g\} \) be a multigerm of \( \mathcal{A}_e \)-codimension 2. Then

\begin{enumerate}
  \item If \( f \) is a monogerm of \( \mathcal{A}_e \)-codimension 1, then \( \overline{\tau}(f) \) and \( \overline{\tau}(g) \) have almost regular intersection.
  \item Suppose \( f \) and \( g \) are stable, then \( \overline{\tau}(f) \) and \( \overline{\tau}(g) \) have almost regular intersection (\( \text{cod} \overline{\tau}(f) + \text{cod} \overline{\tau}(g) - \text{cod}(\overline{\tau}(f) \cap \overline{\tau}(g)) = 1 \)) if and only if \( h \) admits a 1-parameter stable unfolding. They have almost regular intersection of order 2 otherwise (\( \text{cod} \overline{\tau}(f) + \text{cod} \overline{\tau}(g) - \text{cod}(\overline{\tau}(f) \cap \overline{\tau}(g)) = 2 \)).
\end{enumerate}

**Proof.** i) If \( f \) is of codimension 1, then \( \dim \overline{\tau}(f) = 0 \), therefore \( \text{cod}(\overline{\tau}(f) \cap \overline{\tau}(g)) = p \). Since in this case, by Proposition 5.1, \( g \) is a prism on a Morse function or an immersion, it follows that \( \text{cod} \overline{\tau}(g) = 1 \) and so \( \text{cod} \overline{\tau}(f) + \text{cod} \overline{\tau}(g) = p + 1 \).

ii) From Proposition 5.1 it follows that \( h = \{f, g\} \) admits a 1-parameter stable unfolding if and only if \( \overline{\tau}(f) \) and \( \overline{\tau}(g) \) have almost regular intersection.

Repeating the proof of Proposition 3.7 replacing the 1-parameter stable unfolding by a 2-parameter versal unfolding we obtain that if \( h \) does not admit a
1-parameter stable unfolding, then the analytic strata of $f$ and $g$ have almost regular intersection of order 2.

Given $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n \leq p$ let $m_0(f) = \dim \mathbb{C} \tau(f, (\mathbb{C}^p, 0))$ be the multiplicity of $f$. If $n > p$, and $f$ is simple of corank 1, by [26] $f$ can be seen as a germ $f_0$ with $n = p$ plus quadratic terms in the remaining variables so by multiplicity we mean $m_0(f_0)$.

In the case $n \geq p$ a stable germ can have multiplicity at most $p + 1$ ([18]). This means that a germ with multiplicity $p + 2$ cannot be an augmentation, since that would imply that there is a stable germ in the same dimensions with that same multiplicity. When $p = n + 1$, a stable germ can have multiplicity at most $[\frac{n}{2}] + 1$ ([21]).

**Proposition 5.5.** Let $h = \{f, g\}$ be a multigerm of $A_e$-codimension 2 which admits a 1-parameter stable unfolding. Then

i) If $f$ and $g$ are stable, then $h$ is an augmentation if and only if $\text{cod} \; \tilde{\tau}(f) + \text{cod} \; \tilde{\tau}(g) \leq p$.

ii) If $f$ is a monogerm of codimension 1, then $h$ is an augmentation if and only if $m_0(f) \leq p$ in the case $n \geq p$ ($m_0(f) \leq [\frac{n}{2}]$ in the case $p = n + 1$).

**Proof.** i) See Corollary 3.8.

ii) First suppose that $n \geq p$, $h$ is an augmentation and $f$ is an augmentation of multiplicity greater than or equal to $p + 1$. This means that there exist a map $h_0 = \{f_0, g_0\} : (\mathbb{C}^{n-1}, S) \to (\mathbb{C}^{p-1}, 0)$, admitting a 1-parameter stable unfolding, and that $f_0$ satisfies $m_0(f_0) = m_0(f) \geq (p - 1) + 2$. Then $A_e - \text{cod}(f_0) = 1$ and therefore $g_0$ is a monogerm on a Morse function or an immersion. Then we get a contradiction, as any 1-parameter stable unfolding $F$ of $f_0$ has $\tau(F) = 0$ (for $f_0$ is primitive), and therefore $\tau(F)$ cannot be transversal to the analytic stratum of any 1-parameter unfolding of $g_0$. It follows that $m_0(f) \leq p$.

Now suppose that $f$ is an augmentation of multiplicity $k \leq p$ of an $f_0$. Then $f_0$ admits a 1-parameter stable unfolding $F_0$ with $\dim \tilde{\tau}(F_0) \geq 1$. We can choose a prism on a Morse function or an immersion $g_0 : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^{p-1}, 0)$ such that $\tilde{\tau}(g_0 \times \text{id}_f)$ (which has dimension $p - 1$) is transversal to $\tilde{\tau}(F_0)$. The bigerm $\{f, g_0 \times \text{id}_f\}$ is an augmentation of $h_0 = \{f_0, g_0\}$, and since it has codimension 2 and is simple, it is $A$-equivalent to $h$, which is therefore an augmentation.

The same proof is valid for the case $p = n + 1$ but with $m_0(f) \leq [\frac{n}{2}]$ instead of $m_0(f) \leq p$.

**Remark 5.6.** In the setting of Proposition 5.5 part ii), if $m_0(f) = p + 1$ (resp. $m_0(f) = [\frac{n}{2}] + 1$), then $h$ is clearly an augmentation and concatenation in the case $n \geq p$ (resp. in the case $p = n + 1$).
Lemma 5.7. If $f$ is a $A_c$-codimension 1 primitive monogerm and $n \geq p$, then, besides the Euler vector field, the components of vector fields in $\text{Lift}(f)$ are in $\mathcal{M}_2^p$.

Proof. Let $F$ be a mini-versal unfolding of $f$. The discriminant of $f$ is a section of the discriminant of $F$. The $\text{Lift}(f)$ is obtained from $\text{Lift}(F) \cap \text{Lift}(g)$ where $g$ is a fold whose discriminant gives the section, in the following way.

Consider first the equidimensional case. Then $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ has a normal form $f(x_1, \ldots, x_n) = (x_1^{n+2} + x_2 x_1 + \ldots + x_n x_1^{n-1}, x_2, \ldots, x_n)$. The section $x_{n+1} = 0$ of the discriminant of $F$ is the discriminant of $f$. The $\text{Lift}(f)$ is therefore the projection to the first $n$ components of $\text{Lift}(F) \cap \text{Lift}(g)$ by $0$, where $g(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, x_{n+1}^2)$ and $(X_1, \ldots, X_{n+1})$ are the target coordinates. Since $\text{Lift}(g) = \langle \frac{\partial}{\partial X_1^2}, \ldots, \frac{\partial}{\partial X_n^2}, X_{n+1} \frac{\partial}{\partial X_{n+1}} \rangle$, and the linear part of $\text{Lift}(F)$ is as given in Proposition 4.13, it can be seen that the only vector field with linear terms is the Euler vector field.

For the cases from $\mathbb{C}^{n+k}$ to $\mathbb{C}^n$, the normal forms for primitive codimension 1 germs are $(x_1, \ldots, x_{n+k}) \mapsto (x_1^{n+2} + x_2 x_1 + \ldots + x_n x_1^{n-1} + \sum_{i=n+1}^{k} x_i^2, x_2, \ldots, x_n)$ and similar arguments prove the same result.

Lemma 5.8. If $f$ is a $A_c$-codimension 1 primitive monogerm and $n = p - 1$, then there are, including the Euler vector field, $\frac{p}{2} + 1$ linearly independent vector fields in $\text{Lift}(f)$ with linear terms in some component.

Proof. First of all, $n$ is odd since there is no codimension 1 primitive vector fields when $n$ is even. The normal form for $f : (\mathbb{C}^{2k-3}, 0) \to (\mathbb{C}^{2k-2}, 0)$ is

$$f(u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-2}, x) = (u, v, x^k + \sum_{i=1}^{k-2} u_i x^i, x^{k+1} + \sum_{i=1}^{k-2} v_i x^i).$$

Consider the 1 parameter versal unfolding of $f$ as

$$F(u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-1}, x) = (u, v, x^k + \sum_{i=1}^{k-2} u_i x^i, x^{k+1} + \sum_{i=1}^{k-1} v_i x^i),$$

and consider the variables in the target as $(U_1, \ldots, U_{k-2}, V_1, \ldots, V_{k-1}, W_1, W_2)$. We proceed as in [22] to obtain the linear part of vector fields in $\text{Lift}(F)$, that is, we study the equation $dF \circ \xi = \eta \circ F$ modulo $F^*\mathcal{M}_{2(k-1)}^2$. Such vector fields for a slightly different normal form for $F$ have been studied in [12]. We obtain that the linear part of vector fields in $\text{Lift}(F)$ are generated by the following $3k - 2$ elements:

$$(\sum_{i=1}^{k-2} (k - i) U_i \frac{\partial}{\partial U_i} + \sum_{i=1}^{k-1} (k - i + 1) V_i \frac{\partial}{\partial V_i} + kW_1 \frac{\partial}{\partial W_1} + (k + 1) W_2 \frac{\partial}{\partial W_2},$$

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where $U$ vector fields in $\text{Lift}$ $\leq 1$ proceed as in the above Lemma taking the corresponding $2k$ components of the linear part in $\text{Lift}$ strata of the stratification by stable types in the target, so we take given in Proposition 4.13 and Lemma 5.8 the least codimension $k$ of the fifth (which comprises $k = p$), codimension greater than or equal to $2$, $W_2 \partial U_{k-2}$, $k-j-1 \sum (V_i - U_{i-1}) \frac{\partial}{\partial U_{i+j-1}} + W_2 \frac{\partial}{\partial U_{j-1}} + W_1 \frac{\partial}{\partial U_j}, 2 \leq j \leq k-2, W_2 \frac{\partial}{\partial U_{k-2}}$, $1 \leq j \leq k-3, W_2 \frac{\partial}{\partial U_{k-1}}$, $k-j-1 \sum (U_{i-1} - V_i) \frac{\partial}{\partial V_{i+j}} + W_2 \frac{\partial}{\partial V_j} - W_1 \frac{\partial}{\partial V_{j+1}}, 1 \leq j \leq k-2$, $\theta(g)$

$\text{tg}(\theta_n) + \text{wg}(\text{Lift}(f))$ $\text{NA}_e(\{f, g\})$ $\text{NA}_e(f)$ $0$

From here we have that

$\text{A}_e - \text{cod}([f, g]) = \text{A}_e - \text{cod}(f) + \text{dim}_C \frac{\theta(g)}{\text{tg}(\theta_n) + \text{wg}(\text{Lift}(f))}$.
Now, \( \frac{\theta(g)}{tg(\theta_n)+wg(Lift(f))} \) is isomorphic to

\[
\mathcal{O}_n \oplus \mathcal{O}_n \\
\{ w : w = (v_{p-1}(x) + \eta_{p-1} \circ g(x), v_{p-1}(x) + \sum_{i=p}^{n} 2x_i v_i(x) + \eta_p \circ g(x)) \}
\]

by projection to the last two components, where \( v_i \in \mathcal{O}_n \) and \( \eta_{p-1} \) and \( \eta_p \) are the last two components of the vector fields in \( Lift(f) \). And this is isomorphic to

\[
\mathcal{O}_{p-1} \\
\{ w_2 : w_2 = -\eta_{p-1}(x_1, \ldots, x_{p-1}, x_{p-1}) + \eta_p(x_1, \ldots, x_{p-1}, x_{p-1}) \}.
\]

When \( n \geq p \), from Lemma 5.7, the components of vector fields in \( Lift(f) \) other than the Euler vector field are quadratic or of higher order, so the dimension of this space is greater than or equal to \( p - 1 \). Finally we have that the codimension of \( h \) is greater than or equal to \( p \).

When \( n = p - 1 \), from the above Lemma we have that there are at most \( \frac{p}{2} + 1 \) linearly independent vector fields in \( Lift(f) \) with linear parts, so the dimension of the quotient is greater than or equal to \( p - (\frac{p}{2} + 1) = \frac{p}{2} - 1 \). Finally we have that the codimension of \( h \) is greater than or equal to \( \frac{p}{2} - 1 \).

**Remark 5.10.**

1) When \( p = 1 \), the bigerm formed by a Morse function and an \( A_2 \)-singularity has codimension 2.

2) If \( n = 1 \) and \( p = 2 \), we have the codimension 2 bigerm \( \{(x^2, x^3), (0, x)\} \), and if \( n = p = 2 \) we have the codimension 2 bigerm

\[
\begin{cases}
(x^4 + yx, y) \\
(x, y^2 + x)
\end{cases}
\]

3) In the equidimensional case, given the bigerm

\[
\begin{cases}
(x_1^{n+2} + x_2 x_1 + \ldots + x_n x_1^{n-1}, x_2, \ldots, x_n) \\
(x_1, \ldots, x_{n-1}, x_2^2 + x_{n-1})
\end{cases}
\]

the codimension is exactly \( n \) (except when \( n = 1 \), see case 1) above) and is non-simple when \( n > 2 \).

4) When \( (n, p) = (3, 4) \), the bigerm

\[
\begin{cases}
(u, v, x^3 + ux, x^4 + vx) \\
(u, v, u, x)
\end{cases}
\]

has codimension 2.
Remark 5.11. Proposition 5.7 can be adapted to the case where $f$ is a primitive codimension 1 multigerm. For example, in the case $n = p - 1$, suppose that $f = \{f_0, g_0\}$ is a codimension 1 monic concatenation with $\bar{\tau}(f_0) = \{0\}$ ($f_0$ is stable and $n = 2k - 2$ must be even). That is

$$f_0(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, x_n^{\frac{n}{2} + 1} + \sum_{i=1}^{\frac{n}{2} - 1} x_ix_n^{\frac{n}{2} + 2} + \sum_{i=\frac{n}{2}}^{n-1} x_ix_n^{i})$$

and $g_0(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-2}, 0, x_{n-1}, x_n)$. $\text{Lift}(f_0)$ is as in Lemma 5.8 and $\text{Lift}(g_0) = (\frac{\partial}{\partial v_1}, \ldots, V_{k-1}\frac{\partial}{\partial v_{k-1}}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2})$, so $\text{Lift}(f) = \text{Lift}(\{f_0, g_0\})$ is the intersection, and there are only $k$ vector fields with linear parts. Analogously to the proof of Proposition 5.9, in order to get the least codimension of $h$, we take $g(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-2}, x_n, x_{n-1}, x_n)$. We must calculate the dimension of

$$\mathcal{O}_{p-1}$$

$\{w_2 : w_2 = -\eta_{p-2}(x_1, \ldots, x_{p-3}, x_{p-1}, x_{p-2}) + \eta_p(x_1, \ldots, x_{p-3}, x_{p-1}, x_{p-2}, x_{p-1})\}$

Out of the vector fields with non zero linear parts in $\text{Lift}(f)$, only the first (as shown in Lemma 5.8) has components in $\frac{\partial}{\partial v_{k-1}}$ and $\frac{\partial}{\partial w_2}$. Therefore, the dimension of the quotient is at least $p - 1$ since we are missing constants and the linear terms $x_1, \ldots, x_{p-2}$. This means that $A_e - \text{cod}(h) \geq A_e - \text{cod}(f) + p - 1 = p$.

If $f_0$ is a stable multigerm with $\bar{\tau}(f_0) = \{0\}$, we can repeat the diagonal block matrix argument from the proof of Theorem 4.14 to prove that $A_e - \text{cod}(h) \geq p$.

Corollary 5.12. Let $p > 2$ and $h = \{f, g\}$ be multigerm of $A_e$-codimension 2 with $f$ a monogerm of codimension 1, then $g$ is a prism on a Morse function or an immersion and $h$ is an augmentation and concatenation except for the normal form from Remark 5.10 (4), where we have a primitive monogerm and an immersion of codimension 2. Furthermore, if the multiplicity of $f$ is less than or equal to $p$ in the case $n \geq p$ or to $[\frac{n}{2}]$ in the case $n = p - 1$, then it is also an augmentation.

Proposition 5.13. Proposition 5.16] Let $h = \{f, g\}$ be a primitive $A_e$-codimension 1 multigerm, and suppose that $g$ is not transverse to $\bar{\tau}(f)$. Then

i) if moreover $g$ and $f$ are transverse, it follows that $g$ is a prism on a Morse function or an immersion and $h$ is a monic concatenation (in particular $f \in \bar{\tau}(g)$).

ii) if $g$ and $f$ are not transverse, then $p = 1$, and $f$ and $g$ are both Morse functions.

From here on we suppose that $h = \{f, g\}$ is a primitive multigerm of $A_e$-codimension 2 with $f$ and $g$ stable and $p > 1$. 

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Lemma 5.14. Let $h = \{f,g\}$ be a primitive multigerm. If it admits a 1-parameter stable unfolding then there is a decomposition $T_0\mathbb{C}^p = \tau(f) \oplus \tau(g) \oplus \mathbb{C}\{v\}$ for some $v \in \mathbb{C}^p$. If not, there exist $v_1$ and $v_2$ in $\mathbb{C}^p$ and a decomposition $T_0\mathbb{C}^p = (\tau(f) + \tau(g)) \oplus \mathbb{C}\{v_1,v_2\}$ where all the sums are direct if and only if $\dim(\tau(f) \cap \tau(g)) = 0$.

Proof. By Proposition 5.4 if $h$ admits a 1-parameter stable unfolding, then $\tau(f)$ and $\tau(g)$ have almost regular intersection. Therefore $\text{cod}(\tau(f) + \tau(g)) + \text{cod}(\tau(g) - \text{cod}(\tau(f) \cap \tau(g))) = 1$. So we have $T_0\mathbb{C}^p = (\tau(f) + \tau(g)) \oplus \mathbb{C}\{v\}$. On the other hand, since $h$ is primitive, by Corollary 3.8 we have that $\text{cod}(\tau(f) + \text{cod}(\tau(g) > p$, therefore $\text{cod}(\tau(f) \cap \tau(g)) > p - 1$ and $\dim(\tau(f) \cap \tau(g)) = 0$, which proves the result for the first case.

In the case that there is no stable 1-parameter unfolding, by Proposition 3.11 the analytic strata have almost regular intersection of order 2.

Lemma 5.15. Suppose that $p > 2$, then there is no $h = \{f,g\}$ such that $f \pitchfork g$, $g$ is not transverse to $\tau(f)$ and $f$ is not transverse to $\tau(g)$.

Proof. Suppose there is such an $h$. If $f$ is a prism on a Morse function or an immersion, then $\text{Im}(df_0) = \tau(f)$, and so $f \pitchfork g$ implies $f \pitchfork \tau(g)$, which is a contradiction. Therefore $f$ is not a prism on a Morse function or an immersion. Equally for $g$.

Now suppose that $\tau(f) = \{0\}$. By Proposition 5.4 $2 \geq \text{cod}(\tau(f) + \text{cod}(\tau(g) - \text{cod}(\tau(f) \cap \tau(g))) = \text{cod}(\tau(g))$. Since $g$ is not a prism on a Morse function or an immersion, $\text{cod}(\tau(g) = 2$. In the case $n \geq p$, Theorems 4.11 and 4.20 imply that $A_e - \text{cod}(h) \geq n \geq p > 2$. This contradicts that $A_e - \text{cod}(h) = 2$ and so $\tau(f) \neq \{0\}$ (the same is valid for $g$). In the case $p = n + 1$, since there are no stable monogerms with codimension 2 analytic stratum, $g$ must be a double immersion. Remark 5.11 implies that $A_e - \text{cod}(h) \geq p > 2$, which again is a contradiction.

Therefore $\tau(f) \neq \{0\} \neq \tau(g)$. Since neither $f$ or $g$ can be a prism on a Morse function or an immersion, we have that $1 < \text{cod}(\tau(f) < p$ and $1 < \text{cod}(\tau(g) < p$. We construct a 1-parameter deformation $h_u$ of $h$ by constructing a 1-parameter generic deformation of $g$ such that $\tau(g)$ remains fixed and $g_u$ becomes transverse to $\tau(f)$. Since $f$ is still not transverse to $\tau(g)$, $A_e - \text{cod}(h_u) \geq 1$. If $A_e - \text{cod}(h_u) = 1$, we construct a 1-parameter generic deformation of $f$ such that $\tau(f)$ remains fixed and $f_v$ becomes transverse to $\tau(g)$. It follows that $h_{(u,v)}$ is not equivalent to $h_{u,0}$. Since $\tau(f)$ and $\tau(g)$ remain fixed, they are still not transverse and so $h_{(u,v)}$ is not stable for $v \neq 0$. This would imply that for each $u$ we have a 1-parameter deformation where each member is not equivalent to $h_u$ and is not stable. This is impossible since $h_u$ has codimension 1, and by [4] all codimension 1 germs are simple. So $A_e - \text{cod}(h_u) = 2$ and $h_u$ is not equivalent to $h$ for $u \neq 0$. Therefore, either $h$ is non-simple or $A_e - \text{cod}(h) > 2$.

Proposition 5.16. Suppose that $g = \{g_1,\ldots,g_r\}$ is not transverse to $\tau(f)$ and $f \pitchfork g$, then
i) If \( r = 1 \) and \( \text{Im}(dg_0) = \tau(g) \), then \( h \) is a monic concatenation.

ii) If \( r = 1 \) and \( \text{Im}(dg_0) \nsubseteq \tau(g) \), then \( h \) is one of the following

- a (non-monic) generalised concatenation with \( g \),
- \( g \) is an \( A_2 \)-singularity and \( f \) is either an \( A_2 \)-singularity or a bigerm of two prisms on Morse functions (only if \( n \geq p = 2 \)).

iii) If \( r > 1 \), then \( r = 2 \) and \( h \) is one of the following

- a double fold (immersion) concatenation with \( g \),
- a trigerm of an \( A_2 \)-singularity with two prisms on Morse functions (only if \( n \geq p = 2 \)).

Proof. ii) Suppose \( \text{Im}(dg_0) \nsubseteq \tau(g) \). If \( \tau(f) = \tau(g) = \{0\} \), then \( \text{cod} \tau(f) + \text{cod} \tau(g) = 2p \) and \( \text{cod}(\tau(f) \cap \tau(g)) = p \), therefore, by Proposition 5.4, \( p \leq 2 \).

From the known classifications mentioned at the beginning of the section, the only possibilities are that \( n \geq p = 2 \) and \( h \) is a bigerm with two \( A_2 \)-singularities or a trigerm where \( g \) is an \( A_2 \)-singularity and \( f \) is a bigerm of two prisms on Morse functions. If \( p > 2 \) we can assume that \( \tau(g) \neq 0 \). In fact, if \( \tau(g) = \{0\} \), then \( f \) is not transversal to \( \tau(f) \). Since by hypothesis \( g \) is not transversal to \( \tau(f) \), it follows by Lemma 5.13 that there is no such \( h \).

Therefore, \( 1 \leq p - s = \dim \tau(g) < p - 1 \) and we are in the case of a (non-monic) generalised concatenation (and \( p > 2 \)): if \( h \) admits a 1-parameter stable unfolding we have a partition \( T_0 \mathbb{C}^p = \mathbb{C}^{p-s} \times \mathbb{C}^{s-1} \times \mathbb{C} \), where \( \mathbb{C}^{p-s} \times \{0\} \times \{0\} \) is the analytic stratum of \( g \) and \( \{0\} \times \mathbb{C}^{s-1} \times \{0\} \) is the analytic stratum of \( f \). By adequate changes of coordinates in the source we can take \( g \) to the form

\[ g(x_1, \ldots, x_n) = (x_1, \ldots, x_{p-s}, g_{p-s+1}(x_{p-s+1}, \ldots, x_n), \ldots, g_p(x_{p-s+1}, \ldots, x_n)) \]

Now, by a change of variables in the source, we can write \( f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_{p-s}(x_1, \ldots, x_n), x_{n-s+1}, \ldots, x_n, f_p(x_1, \ldots, x_n)) \). Since \( f \pitchfork g \), if \( \{0\} \times \{0\} \times \mathbb{C} \not\subseteq \text{Im}(df_0) \), then \( \{0\} \times \{0\} \times \mathbb{C} \subseteq \text{Im}(dg_0) \) and so \( g \pitchfork \tau(f) \), which contradicts the hypothesis. Therefore, we can take \( f \) to the form

\[ f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_{p-s}(x_1, \ldots, x_n), x_{n-s+1}, \ldots, x_n) \]

so \( f \) is an \( s \)-parameter stable unfolding of a certain \( f_0 \).

Now suppose that \( h \) does not admit a 1-parameter stable unfolding. If \( \tau(f) \cap \tau(g) = \{0\} \) we have a partition \( \mathbb{C}^{p-s} \times \mathbb{C}^{s-2} \times \mathbb{C}^2 \), where \( \tau(g) = \mathbb{C}^{p-s} \times \{0\} \times \{0\} \) and \( \tau(f) = \{0\} \times \mathbb{C}^{s-2} \times \{0\} \). Similarly as above, we write

\[ g(x_1, \ldots, x_n) = (x_1, \ldots, x_{p-s}, g_{p-s+1}(x_{p-s+1}, \ldots, x_n), \ldots, g_p(x_{p-s+1}, \ldots, x_n)) \]
and $f$ as

$$(f_1(x_1, ..., x_n), ..., f_{p-s}(x_1, ..., x_n), x_{n-s+1}, ..., x_{n-2}, f_{p-1}(x_1, ..., x_n), f_p(x_1, ..., x_n)).$$

Since $f \pitchfork g$ and $g$ is not transverse to $\overline{\tau}(f)$, then $\{0\} \times \{0\} \times \mathbb{C} \times \mathbb{C} \not\subseteq \text{Im}(dg_0)$. If $\{0\} \times \{0\} \times \mathbb{C} \times \mathbb{C} \subseteq \text{Im}(dg_0)$ we can take $f$ to the desired form. If $\text{Im}(dg_0) = \mathbb{C}^{p-s} \times \{0\} \times \mathbb{C} \times \{0\}$ and $\text{Im}(df_0) = \{0\} \times \mathbb{C}^{s-2} \times \{0\} \times \mathbb{C}$ then $f$ is not transversal to the analytic stratum of $g$. It follows by Lemma 5.13 that there is no such $h$ in codimension 2.

If $\dim \overline{\tau}(f) \cap \overline{\tau}(g) = k > 0$, the only difference with the above case is that the analytic stratum of $f$ overlaps the analytic stratum of $g$ in $k$ directions. So now

$$\begin{cases}
\{(x_1, ..., x_{p-s}, g_{p-s+1}(x_{p-s+1}, ..., x_n), ..., g_p(x_{p-s+1}, ..., x_n))
\end{cases}$$

We proceed analogously.

i) If $\text{Im}(dg_0) = \overline{\tau}(g)$ then $g$ is either a prism on a Morse function or an immersion and so $\text{cod}(\overline{\tau}(g)) = 1$. If $h$ admits a 1-parameter stable unfolding, since $h$ is primitive by Corollary 3.8 $\text{cod}(\overline{\tau}(f)) + \text{cod}(\overline{\tau}(g)) > p$ and so $\overline{\tau}(f) = \{0\}$. Now proceed as in the proof of [4, Proposition 5.16] with the only difference that $f$ is a 1-parameter stable unfolding of an $f_0$ of codimension 2 (the procedure is similar to the one above).

If $h$ does not admit a 1-parameter stable unfolding, by Proposition 5.14 we have that $\text{cod}(\overline{\tau}(f)) + 1 - \text{cod}(\overline{\tau}(f) \cap \overline{\tau}(g)) = 2$ and so $\text{cod}(\overline{\tau}(f) \cap \overline{\tau}(g)) = \text{cod}(\overline{\tau}(f)) - 1$, which is impossible.

iii) First suppose that $h_i = \{f, g_i\}$ is stable for all $i$. Then $\overline{\tau}(g_i) \pitchfork \overline{\tau}(f)$ and therefore $g_i \pitchfork \overline{\tau}(f)$ and so $g \pitchfork \overline{\tau}(f)$, which contradicts the hypothesis. So there exists $i_0 \in \{1, ..., r\}$ such that $h_{i_0} = \{f, g_{i_0}\}$ has $\mathcal{A}_c$-codimension 1. In this case, by Lemma 5.1 $g = \{g_{i_0}, g_{i_1}\}$, where $g_{i_1}$ is a prism on a Morse function or an immersion, and so $r = 2$.

Suppose that $g_{i_0}$ is not transverse to $\overline{\tau}(f)$. If $g_{i_0}$ is not transverse to $f$, by Proposition 5.13 $p = 1$, which is a contradiction. If $g_{i_0} \pitchfork f$, again by Proposition 5.13 $h_{i_0}$ is a monic concatenation where $g_{i_0}$ is a prism on a Morse function or an immersion.

Now suppose that $g_{i_0} \pitchfork \overline{\tau}(f)$. Since $g$ is not transverse to $\overline{\tau}(f)$, then $g_{i_1}$ cannot be transverse to $\overline{\tau}(f)$, therefore $h_{i_1} = \{f, g_{i_1}\}$ has codimension 1 and $g_{i_0}$ is a prism on a Morse function or an immersion. Since $g$ is stable, $g_{i_0} \pitchfork g_{i_1}$ and $\dim \overline{\tau}(g) = p - 2$.

If $p = 2$, $\text{cod}(\overline{\tau}(f)) \leq 2$. If it is equal to 2 then $f$ can be either an $A_2$-singularity or two prisms on Morse functions. However, the second case does not occur since $h$ has codimension 2. If it is equal to 1, then $f$ is a prism on a Morse function or an immersion and $g \pitchfork f$ implies $g \pitchfork \overline{\tau}(f)$, which contradicts the hypothesis.
If \( p > 2 \) we take \( g = \{g_{i_0}, g_{i_1}\} \) to the form
\[
\begin{align*}
\{ & (x_1, \ldots, x_{p-2}, x_{p-1}, \sum_{i=p}^{n} x_i^2) \\
& (x_1, \ldots, x_{p-2}, x_{p-1}, \sum_{i=p}^{n} x_i^2 + x_{p-1})
\end{align*}
\] (13)
and proceed as in case ii).

\[\]

**Proposition 5.17.** If \( g \) and \( f \) are not transverse then \( h \) is one of the following

- an augmentation and concatenation,
- \( f \) is a Morse function and \( g \) is an \( A_2 \)-singularity (only if \( n \geq p = 2 \)),
- one of the following normal forms (when \( p = n + 1 \) and \( n \) is even):
\[
\begin{align*}
n = 2 \begin{cases} (x, y^2, xy) & n = 4 \begin{cases} (u_1, v_1, v_2, y^3 + u_1 y + v_2 y^2) \\
& (u_1, v_1, v_2, u_2^2 + v_2, y) \end{cases}, (14) \\
& (x, x^2, y)
\end{cases}, \\
n = 2k - 2, k \geq 4 \begin{cases} (u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-1}, y^k + \sum_{i=1}^{k-2} u_i y^i, \sum_{i=1}^{k-1} v_i y^i) \\
& (u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-2}, u_{k-3} + u_{k-2}^2, v_{k-1}, y) \end{cases} (15)\end{align*}
\]

**Proof.** If \( f \) is not transverse to \( g \), then \( f \) is not transverse to \( \tau(g) \) and \( g \) is not transverse to \( \tau(f) \). If \( p = 2 \) and \( h \) admits a 1-parameter stable unfolding, since \( h \) is primitive, by Corollary 3.8 \( \text{cod} \tau(f) + \text{cod} \tau(g) > 2 \) and since the analytic stratum must have codimension less than or equal to \( p = 2 \), \( \text{cod} \tau(g) = 1 \) and \( \text{cod} \tau(f) = 2 \) (or viceversa), so \( h \) is a non-transversal bigerm of a prism on a Morse function and an \( A_2 \)-singularity or a trigerm of three prisms on Morse functions where two of them are non-transversal and the third is transversal to the other two (the latter is an augmentation and concatenation). If \( h \) does not admit a 1-parameter stable unfolding, then by Proposition 5.4 \( \text{cod} \tau(f) = \text{cod} \tau(g) = 2 \) and, by the known classifications mentioned at the beginning of this section, there are no possibilities.

If \( p > 2 \), let \( f = \{f_1, \ldots, f_r\} \), \( r > 1 \), as \( f \) is not transverse to \( g \), there exists \( f_{i_0} \) which is not transverse to \( g \), therefore \( A_e - \text{cod}(\{f_{i_0}, g\}) = 1 \). By Proposition 5.1, \( f = \{f_{i_0}, f_{i_1}\} \) with \( f_{i_1} \) a prism on a Morse function or an immersion and \( f_{i_0} \pitchfork f_{i_1} \). We have that \( g \) is not transverse to \( \tau(f_{i_0}) \) and \( g \) is not transverse to \( f_{i_0} \). If \( \{f_{i_0}, g\} \) is primitive, then by Proposition 5.13 \( p = 1 \) and we have a contradiction, therefore \( \{f_{i_0}, g\} \) is an augmentation and \( h \) is an augmentation and concatenation which is not an augmentation.

Now suppose that \( f \) and \( g \) are monogerms (if \( g \) is not a monogerm, we change it for \( f \) and proceed as above). We have that \( \tau(f) = \{0\} \) if and only if \( \text{cod} \tau(g) = 1 \), and the same changing \( f \) for \( g \). In fact, suppose \( \tau(f) = \{0\} \) (\( \text{cod} \tau(f) = p \)), then by Proposition 5.4 \( \text{cod} \tau(g) \leq 2 \). We have that \( \text{cod} \tau(g) \neq 0 \) because \( g \) is not a
submersive branch. In the case \( p = n + 1 \), from the known classifications, there is no stable mongerm with \( \text{cod} \overline{\tau}(g) = 2 \) and if \( n \geq p \), by Theorems 4.14 and 4.20 if \( \text{cod} \overline{\tau}(g) = 2 \), \( \mathcal{A}_c - \text{cod}(h) \geq n \geq p > 2 \), so \( \text{cod} \overline{\tau}(g) = 1 \). On the other hand, suppose that \( \text{cod} \overline{\tau}(g) = 1 \). If \( h \) admits a 1-parameter stable unfolding, by Corollary 3.5 \( \overline{\tau}(f) = \{0\} \). If \( h \) does not admit a 1-parameter stable unfolding, by Proposition 5.4 we have that \( \text{cod}(\overline{\tau}(f) \cap \overline{\tau}(g)) = \text{cod}(\overline{\tau}(f)) - 1 \), which is impossible.

Suppose \( \overline{\tau}(f) = \{0\} \) and \( \text{Im}(d_{g_0}) = \overline{\tau}(g) \). In the case \( n \geq p \), \( f \) is an \( A_n \)-singularity. Using the exact sequence in the proofs of Theorems 4.12 and 4.17 and the information about \( \text{Lift}(A_n) \) in Proposition 4.13 we can see that the codimension in this case is greater than or equal to \( n \). In the case \( p = n + 1 \), from [13] and [28] we have that the only possibilities are when \( n \) is even. The bigerm has a normal form \( \{(x, y^2, xy), (x, x^2, y)\} \) for the case \( n = 2 \). For other \( n \), the normal forms can be found in [13]. If \( f : (\mathbb{C}^{2k+1}, 0) \to (\mathbb{C}^{2k-1}, 0) \), then \( f(u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-1}, y) = (u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-1}, y^k + \sum_{i=1}^{k-2} u_i y^i, \sum_{i=1}^{k-1} v_i y^i) \). For \( k = 3 \), \( g(u_1, v_1, v_2, y) = (u_1, v_1, v_2, u_1 + v_2, y) \). For \( k \geq 4 \) the normal form for the bigerm is:

\[
\begin{cases}
(u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-1}, y^k + \sum_{i=1}^{k-2} u_i y^i, \sum_{i=1}^{k-1} v_i y^i) \\
(u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-3}, u_{k-2}, u_{k-1}, y) 
\end{cases}
\]

(16)

Now suppose that \( 1 < \text{cod} \overline{\tau}(f) < p \) and \( 1 < \text{cod} \overline{\tau}(g) < p \). In this case we proceed exactly as in the proof of Lemma 5.15 and obtain that either \( h \) is non-simple or \( \mathcal{A}_e - \text{cod}(h) > 2 \).

**Proposition 5.18.** If \( f \not\sqsubset \overline{\tau}(g) \) and \( g \not\sqsubset \overline{\tau}(f) \) then \( h = \{f, g\} \) is a non-monic generalised concatenation.

**Proof.** The fact that \( f \not\sqsubset \overline{\tau}(g) \) and \( g \not\sqsubset \overline{\tau}(f) \) implies that \( \overline{\tau}(g) \neq \{0\} \neq \overline{\tau}(f) \) so, if \( h \) admits a 1-parameter stable unfolding, again we have a decomposition of \( \mathbb{C}^p \) as \( \mathbb{C}^{p-s} \times \mathbb{C}^{s-1} \times \mathbb{C} \) where \( s > 1 \), \( \overline{\tau}(g) = \mathbb{C}^{p-s} \times \{0\} \times \{0\} \) and \( \overline{\tau}(f) = \{0\} \times \mathbb{C}^{s-1} \times \{0\} \). Let \( z_1, \ldots, z_p \) be the coordinates of \( \mathbb{C}^p \). Since \( f \not\sqsubset \overline{\tau}(g) \), we can take \( z_p \circ f \) as a coordinate, \( u \), on the domain of \( f \) and since \( g \not\sqsubset \overline{\tau}(f) \), we can take \( z_p \circ g \) as a coordinate \( u \) on the domain of \( g \). A coordinate change now takes \( h = \{f, g\} \) to the form

\[
\begin{cases}
(x, Y, u) \mapsto (f_u, Y(x), Y, u) \\
(X, y, u) \mapsto (X, g_u(y), u)
\end{cases}
\]

(17)

which is clearly a generalised concatenation.

If \( h \) does not admit a 1-parameter stable unfolding \( h \) can be taken to the form

\[
\begin{cases}
(x, Y, u) \mapsto (f_{u,v}, Y(x), Y, u, v) \\
(X, y, u) \mapsto (X, g_{u,v}(y), u, v)
\end{cases}
\]

(18)

which is a generalised concatenation too. \( \square \)
In summary we have:

**Theorem 5.19.** Let \( h = \{ f, g \} \) be of \( A_e \)-codimension 2, then

1) if \( f \) is a monogerm of \( A_e \)-codimension 1, then \( g \) a prism on a Morse function or an immersion and

i) \( h \) is an augmentation if and only if \( f \) is an augmentation with \( m_0(f) \leq p \) when \( n \geq p \) \((m_0(f) \leq \left\lfloor \frac{n}{2} \right\rfloor \) when \( p = n + 1 \)),

ii) \( h \) is an augmentation and concatenation if \( f \) is an augmentation with \( m_0(f) = p + 1 \) when \( n \geq p \) \((m_0(f) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) when \( p = n + 1 \)),

iii) if \( p = 1, 2 \) and \( m_0(f) = p + 2 \) when \( n \geq p \) \((m_0(f) = \left\lfloor \frac{n}{2} \right\rfloor + 2 \) when \( p = n + 1 \)) then \( f \) is a primitive monogerm of codimension 1,

iv) if \((n, p) = (3, 4)\) and \( m_0(f) = 3 \) then \( h \) has the normal form in Remark 5.10 4).

2) if \( f \) and \( g \) are stable, then

i) \( \text{cod}(\tilde{\tau}(f)) + \text{cod}(\tilde{\tau}(g)) \leq p \) if and only if \( h \) is an augmentation,

ii) if \( h \) is primitive and \( g \) is not transverse to \( \tilde{\tau}(f) \), then

a) if \( f \pitchfork g \), then

a1) Suppose \( g \) is a monogerm. When \( \text{Im}(dg_0) = \tilde{\tau}(g) \), \( h \) is a monic concatenation. When \( \text{Im}(dg_0) \supsetneq \tilde{\tau}(g) \), then either \( h \) is a (non-monic) generalised concatenation with \( g \), it is a bigerm with two \( A_2 \)-singularities or it is a trigerm of an \( A_2 \)-singularity with two prisms on Morse functions (only if \( n \geq p = 2 \)).

a2) Suppose \( g \) is a multigerm, then it is a bigerm and either \( h \) is a double fold (immersion) concatenation with \( g \) or it is a trigerm of an \( A_2 \)-singularity with two prisms on Morse functions (only if \( n \geq p = 2 \)).

b) If \( g \) and \( f \) are not transverse then \( f \) is a Morse function and \( g \) is an \( A_2 \)-singularity (only if \( n \geq p = 2 \)), \( h \) is an augmentation and concatenation or it has one of the normal forms in Proposition 5.17 (when \( p = n + 1 \) and \( n \) even).

iii) if \( h \) is primitive, \( g \pitchfork \tilde{\tau}(f) \) and \( f \pitchfork \tilde{\tau}(g) \), then \( h \) is a non-monic generalised concatenation.

**Remark 5.20.** If we replace \( \mathbb{C} \) by \( \mathbb{R} \) and analytic maps by smooth ones, all the results in this section hold. However, in the real case, the operations may lead to different \( A \)-classes.
6 $A_e$-codimension 2 multigerms from $\mathbb{C}^3$ to $\mathbb{C}^3$

In this section we use the results in Section 5 in order to recover the classification of multigerms of $A_e$-codimension 2 from $\mathbb{C}^3$ to $\mathbb{C}^3$ obtained in [21]. First, using quadratic and cubic augmentations ($A^2$ and $A^3$), monic concatenations ($MC$) and concatenations and augmentations ($AC$), we obtain all codimension 1 and 2 germs and multigerms from $\mathbb{C}^2$ to $\mathbb{C}^2$ starting from a monogerm and the special bigerm from Proposition 5.13 namely two Morse functions. This is shown in figure 1 where the special multigerms mentioned in Propositions 5.16 and 5.17 are included too.

![Figure 1: Codimension 1 and 2 germs and multigerms of maps from $\mathbb{C}^2$ to $\mathbb{C}^2$. The cases where a codimension 1 germ appears, a stabilisation is represented.](image)

The following table, obtained by W. L. Marar & F. Tari in [17] and earlier by V. Goryunov in [6], contains a list of normal forms for simple corank 1 monogerms of maps from $\mathbb{R}^3$ to $\mathbb{R}^3$. 

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Here \( P(x, y) \) are polynomials in two variables and \( \mu(P) \) denotes the Milnor number of \( P \). We add to this list the unimodular monogerm \( 6_1 : (x, y, z^6 + yz^2 + xz) \) (see [6]) of \( A_c \)-codimension three.

Next we introduce the notation for germs and multigerms used in [21]: starting from stable germs, \( A_1 \) (fold), \( A_2 \) (cusps) and \( A_3 \) (swallowtails), \( A_i^k A_j \) represents a multigerm with \( k \) branches of type \( A_i \) and a branch of type \( A_j \) where the branches are pairwise transversal. Tangencies are indicated by \( T \), for instance, we represent \( T_{A_1} \) a nondegenerate tangency between the strata of singularities \( A_i \) and \( A_j \) in the branch set, or by \( T_{A_i^k A_j} \) a nondegenerate tangency between the strata of points \( A_i \) and \( A_j \) in the discriminant, etc. Degenerate tangencies are denoted by \( DT \). Therefore, \( A_3^1 \) represents a fold with a germ of type \( 3_1 \) in the “best” possible position (lower contact order); \( A_1 T_{A_1} A_3^1 \) is a quadrigerm determined by a fold \( (A_1) \) with the trigerm \( T_{A_1} A_3^1 \), which in turn is given by a nondegenerate tangency of a fold surface and a double fold curve; \( DT_{A_1} \) means a degenerate tangency between two fold surfaces and \( DT_{A_1 A_1} \) means a degenerate tangency between a fold surface and a double fold curve. A superindex on the character \( T \), i.e. \( T^{1} \), denotes a special type of tangency, for example \( T^{1}_{22} \) means that the tangent vector to one of the cuspidal edges is included in the tangent plane in the limit of the other cuspidal edge; \( T^{1}_{13} \) means that the tangent vector in the limit of the cuspidal edges at the swallowtail point is included in the tangent plane to the fold surface; and \( T^{1}_{A_2 A_2} \) means that the tangent vector to the double point curve is included in the tangent plane in the limit of the cuspidal edge.

Figure 2 shows how to obtain all the codimension 2 multigerms from \( \mathbb{C}^3 \) to \( \mathbb{C}^3 \) using the operations defined starting from monogerms and the special multigerms.

References

[1] V. I. Arnol’d Critical points of smooth functions and their normal forms. Russian Math. Surveys 30 (1975) (or in Singularity Theory, LMS Lecture Note Series 53, Cambridge UP (1981)).

[2] J. W. Bruce Envelopes and characteristics. Math. Proc. Cambridge Philos. Soc. 100 (1986), no. 3, 475–492.
Figure 2: Codimension 2 germs and multigerms of maps from $\mathbb{C}^3$ to $\mathbb{C}^3$. $CC$ and $DFC$ stand for cuspidal concatenation and double fold concatenation respectively.

[3] C. CASONATTO, M. C. ROMERO FUSTER and R. WIK ATIQUE First order local invariants of immersions from 3-manifolds to $\mathbb{R}^4$. Topology and its Applications. 159 (2012), pp. 420-429.

[4] T. COOPER, D. MOND and R. WIK ATIQUE Vanishing topology of codimension 1 multi-germs over $\mathbb{R}$ and $\mathbb{C}$. Compositio Math 131 (2002), no. 2, 121–160.
[5] J. Damon $\mathcal{A}$-equivalence and equivalence of sections of images and discriminants. In: Singularity Theory and Applications (Warwick 1989), Lecture Notes in Math. 1462, Springer, New York, 1991, pp. 93–121.

[6] V. Goryunov Singularities of projections of complete intersections. J. Soviet Math. 27 (1984), 2785–2811.

[7] V. Goryunov Monodromy of the image of the mapping $\mathbb{C}^2 \to \mathbb{C}^3$. (Russian) Funktsional. Anal. i Prilozhen. 25 (1991), no. 3, 12–18, 95; translation in Funct. Anal. Appl. 25 (1991), no. 3, 174–180 (1992).

[8] V. Goryunov Local invariants of mappings of surfaces into three space. The Arnol’d-Gelfand mathematical seminars 223-225. Birkhauser, Boston, (1997).

[9] C. A. Hobbs and N. P. Kirk On the classification and bifurcation of multigerms of maps from surfaces to 3-space. Math. Scand. 89 (2001), no. 1, 57–96.

[10] K. Houston On singularities of folding maps and augmentations. Math. Scand. 82 (1998), no. 2, 191–206.

[11] K. Houston Augmentation of singularities of smooth mappings. Internat. J. Math. 15 (2004), no. 2, 111–124.

[12] K. Houston and D. Littlestone Vector fields liftable over corank 1 stable maps. Preprint (2009).

[13] K. Houston and R. Wik Atique $\mathcal{A}$-classification of map-germs via $\nu K$-equivalence. Preprint (2012).

[14] P. A. Kolgushkin and R. R. Sadykov Simple singularities of multigerms of curves. Rev. Mat. Complut. 14 (2001) 311–344.

[15] C. Klotz, O. Pop and J. Rieger Real double-points of deformations of $A$-simple map-germs from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$. Math. Proc. Cambridge Philos. Soc. 142 (2007), no. 2, 341–363.

[16] S. Mancini; M. A. S. Ruas; M. A. Teixeira On divergent diagrams of finite codimension. Port. Math. (N.S.) 59 (2002), no. 2, 179–194.

[17] W. L. Marar and F. Tari. On the geometry of simple germs of corank 1 maps from $\mathbb{R}^3$ to $\mathbb{R}^3$. Math. Proc. Cambridge Philos. Soc. 119 (1996), no. 3, 469–481.

[18] J. N. Mather Stability of $C^\infty$ mappings IV: Classification of stable maps by $\mathbb{R}$-algebras. Publ. Math. IHES, 37, (1983), 223-248.
D. Mond. *On the Classification of Germs of Maps From \(\mathbb{R}^2\) to \(\mathbb{R}^3\).* Proc. London Math. Soc. (3), 50, 333-369, (1983).

J. A. Montaldi. *On contact between submanifolds.* Michigan Math. J. 33 (1986), no. 2, 195–199.

B. Morin. *Formes canoniques des singularités d’une application différentiable.* C. R. Acad. Sci. Paris 260 (1965) 5662–5665 and 6503–6506.

T. Nishimura. *Vector fields liftable over finitely determined multi-germs of corank at most one.* Preprint.

T. Ohimoto and F. Aicardi. *First order local invariants of apparent contours.* Topology 45 (2006), no. 1, 27–45.

R. Oset Sinha and M. C. Romero Fuster. *First order local invariants of stable maps from 3-manifolds to \(\mathbb{R}^3\).* Michigan Math. J. Volume 61, Issue 2 (2012), 385–414.

J. H. Rieger. *Families of Maps From the Plane to the Plane.* J. London Math. Soc. (2) 36, 351-369, (1986).

J. H. Rieger and M. A. S. Ruas. *Classification of A-simple germs from \(\mathbb{K}^n\) to \(\mathbb{K}^2\).* Compositio Math. 79 (1991), no. 1, 99–108.

C. T. C. Wall. *Finite determinacy of smooth map-germs.* Bull. London Math. Soc. 13 (1981), 481–539.

R. Wik Atique. *On the classification of multi-germs of maps from \(\mathbb{C}^2\) to \(\mathbb{C}^3\) under A-equivalence.* in J.W.Bruce and F.Tari(eds.) *Real and Complex Singularities*, Research Notes in Maths Series, Chapman & Hall / CRC (2000), 119-133.

M. Yamamoto. *First order semi-local invariants of stable maps of 3-manifolds into the plane.* Proc. London Math. Soc. (3) 92 (2006), no. 2, 471–504.

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