Multiple Sign Changing Radially Symmetric Solutions in a General Class of Quasilinear Elliptic Equations

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Abstract

In this paper we prove that the equation

\[-(r^\alpha \phi(|u'(r)|)|u'(r)|)' = \lambda r^\gamma f(u(r)), \quad 0 < r < R,\]

where \(\alpha, \gamma, R\) are given real numbers, \(\phi : (0, \infty) \to (0, \infty)\) is a suitable twice differentiable function, \(\lambda > 0\) is a real parameter and \(f : \mathbb{R} \to \mathbb{R}\) is continuous, admits an infinite sequence of sign-changing solutions satisfying \(u'(0) = u(R) = 0\). The function \(f\) is required to satisfy \(tf(t) > 0\) for \(t \neq 0\). Our technique explores fixed point arguments applied to suitable integral equations and shooting arguments. Our main result extends earlier ones in the case \(\phi\) is in the form \(\phi(t) = |t|^\beta\) for an appropriate constant \(\gamma\).

1 Introduction

We study the nonlinear eigenvalue problem

\[
\begin{aligned}
-(r^\alpha \phi(|u'(r)|)|u'(r)|)' &= \lambda r^\gamma f(u(r)), \quad 0 < r < R, \\
u'(0) &= u(R) = 0,
\end{aligned}
\]

\((P_\lambda)\)

where \(\lambda > 0\) is a parameter, \(f : \mathbb{R} \to \mathbb{R}\) is continuous and \(\alpha, \gamma \in \mathbb{R}\) are suitable constants.

We shall assume that \(\phi : (0, \infty) \to (0, \infty)\) is a twice differentiable, \(C^1\)-function, satisfying

\(\phi_1\)

(i) \(t\phi(t) \to 0\) as \(t \to 0\),

(ii) \(t\phi(t) \to \infty\) as \(t \to \infty\),

\(\phi_2\) \(t\phi(t)\) is strictly increasing in \((0, \infty)\),

\(\phi_3\) there are constants \(\gamma_1, \gamma_2 > 1\) such that

\[\gamma_1 - 1 \leq \frac{(t\phi(t))'}{\phi(t)} \leq \gamma_2 - 1, \quad t > 0.\]

Concerning \(f\), the following conditions will be imposed:

\(f_1\) \(tf(t) > 0, \quad t \neq 0,\)

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(f_2) there exists \( d_\infty > 0 \) such that \( f \) is nondecreasing in \( (-\infty, d_\infty) \),

\[
(f_3) \quad \liminf_{\nu \to 0^+} \int_0^\nu |f(t)|^{-\frac{1}{\gamma - 1}} \text{sgn}(f(t)) \, dt < \infty.
\]

**Remark 1.1.** We observe that condition \((f_3)\) is equivalent to the following:

\[
(f_3') \quad \max \left\{ \int_{-\infty}^0 [-f(t)]^{-\frac{1}{\gamma - 1}} \, dt, \int_{\theta}^{\infty} |f(t)|^{-\frac{1}{\gamma - 1}} \, dt \right\} < \infty,
\]

for each \( x, y > 0 \), where \( \gamma' = \gamma/(\gamma - 1) \).

Our main objective in this work is to prove the following result:

**Theorem 1.1.** Let \( f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) \). Assume \((\phi_1)\)-(\(\phi_3\)), \((f_1)\)-(\(f_2\)) and

\[
\gamma \geq \max \left\{ \alpha, \frac{-\alpha}{\gamma_1 - 1} \right\} \quad (\gamma, \alpha)
\]

Then there is a positive number \( \Lambda \) such that for each \( \lambda \in (0, \Lambda] \), problem \([P_\Lambda]\) admits a positive solution \( u_0 \) and an infinite sequence \( \{u_\ell\}_{\ell=1}^{\infty} \) of solutions satisfying:

\[
u(0) = d_\ell, \quad u_\ell \quad \text{has precisely } \ell \quad \text{zeroes in } (0, R),
\]

where \( \{d_\ell\}_{\ell=1}^{\infty} \) is an infinite sequence of real numbers such that

\[
d_\infty > d_1 > \cdots > d_\ell > \cdots > 0 \quad (1.3)
\]

The proof of Theorem 1.1 is strongly based on the shooting method. In this regard, consider the initial value problem

\[
\begin{cases}
-r^\alpha \phi(|u'(r)|)u'(r)' = \lambda r^\gamma f(u(r)), \quad r > 0, \\
u(0) = d, \quad u'(0) = 0, 
\end{cases}
\]

where \( d \in (0, d_\infty] \).

The auxiliary result below will play a crucial role in this work.

**Theorem 1.2.** Let \( f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) \). Assume \((\phi_1)\)-(\(\phi_3\)), \((\gamma, \alpha)\) and \((f_1)\)-(\(f_2\)). Then there exists a positive number \( \Lambda = \Lambda(d_\infty) \) such that for each \( \lambda \in (0, \Lambda] \), problem \([P_{\Lambda, d}]\) has a unique solution \( u(\cdot, d, \lambda) = u(\cdot, d) \in C^1([0, \infty)) \). In addition, for each \( d \in (0, d_\infty] \), there is a sequence \( \{z_\ell\}_{\ell=1}^{\infty} \) of zeroes of \( u(\cdot, d) \), \( z_\ell = z_\ell(d) \), such that

\[
z_1(d_\infty) \geq R, \quad u(r, d) \quad \text{if } 0 < r < z_1(d),
\]

\[
z_1(d) < z_2(d) < \cdots < z_\ell(d) < \cdots \quad (1.4)
\]

\[
u'(r, d) < 0 \quad \text{if } 0 < r \leq z_1(d), u(r, d) \neq 0 \quad \text{if } z_\ell < r < z_{\ell+1} \text{ and } u'(z_\ell, d) \neq 0,
\]

\[
z_\ell(d) \to 0 \quad \text{as } d \to 0 \text{ and } z_\ell(d) \to z_\ell(d) \quad \text{as } d \to d, \quad d \in (0, d_\infty], \quad (1.5)
\]

\[
\text{if } d \in (0, d_\infty] \text{ and } u(\cdot, d) \text{ has } k \text{ zeroes in } (0, R) \text{ then } u(\cdot, d) \text{ has at most } k + 1 \text{ zeroes in } (0, R) \text{ whenever } d < d, \text{ close enough to } d. \quad (1.6)
\]
2 Background

Consider the problem

\[
\begin{aligned}
-\text{div}(a(x)|\nabla u(x)|^\beta \nabla u(x)) &= \lambda \ b(x)f(u), \ x \in B_R, \\
u(x) &= 0, \ x \in \partial B_R,
\end{aligned}
\]

(P1_\lambda)

where \( B_R \subset \mathbb{R}^N \) is the ball of radius \( R \) centered at the origin, the functions \( a, b \) are radially symmetric and \( \beta > -1 \). Making \( a = b \equiv 1, \beta = p - 2 \) with \( 1 < p < \infty \) and \( \lambda = 1 \), (P1_\lambda) becomes

\[
\begin{aligned}
-(r^{N-1}|u'(r)|^{p-2}u'(r))' &= r^{N-1}f(u(r)), \ 0 < r < R, \\
u'(0) &= u(R) = 0,
\end{aligned}
\]

(P2)

It was shown in [6] that if \( f(t) = |t|^\beta t \) with \( 1 < \delta + 1 < p < N \) then (P2) has infinitely many nodal solutions.

In [3], it was shown that the more general problem

\[
\begin{aligned}
-(r^\alpha |u'(r)|^{\beta} u'(r))' &= \lambda r^\gamma f(u(r)), \ 0 < r < R, \\
u'(0) &= u(R) = 0
\end{aligned}
\]

(P3)

admits infinitely many solutions if \( \lambda \) is positive and small enough,

\[
\beta > -1, \ \gamma \geq \max \left\{ \alpha, \frac{-\alpha}{\beta + 1} \right\},
\]

(2.1)

and conditions (f1), (f2) and a stronger form of (f3) hold.

Regarding (P3), an example of a function satisfying (f1), (f2), (f3) with \( \beta > 0 \) is \( f(t) = \arctg(t) \).

As was pointed out by Clement, Figueiredo & Mitidieri [8] the operator

\[
(r^\alpha |u'(r)|^{\beta} u'(r))'
\]

represents the radial form of the well known operators:

- **p-Laplacian with** \( 1 < p < N \) when \( \alpha = N - 1, \ \beta = p - 2 \),
- **k-Hessian with** \( 1 \leq k \leq N \) when \( \alpha = N - k, \ \beta = k - 1 \).

The problem

\[
\begin{aligned}
-\Delta \Phi u &= \lambda f(u) \text{ in } B \\
u &= 0 \text{ on } \partial B,
\end{aligned}
\]

(\( \Phi \))

where

\[
\Phi(t) = \int_0^t s\phi(s)ds,
\]

\( \Delta \Phi \) is the \( \Phi \)-Laplacian operator namely

\[
\Delta \Phi u = \text{div}(\phi(|\nabla u|)\nabla u),
\]
and $B \subset \mathbb{R}^N$ is the ball of radius $R$ centered at the origin, was addressed by many authors (see e.g. Fukagai & Narukawa[4] and its references). A weak solution of $(\Phi)$ is by definition an element $u \in W^{1,\Phi}_0(B)$ (the usual Orlicz-Sobolev space) such that

$$\int_B \phi(|\nabla u|) \nabla u \nabla v \, dx = \lambda \int_B f(u) v \, dx, \quad v \in W^{1,\Phi}_0(B).$$

(2.2)

The radially symmetric form of $(\Phi)$ is

$$\begin{cases}
-(r^{N-1}\phi(|u'(r)|)u'(r))' = \lambda r^{N-1}f(u(r)), & 0 < r < R \\
u'(0) = u(R) = 0
\end{cases}$$

which is a special case of $(P_\lambda)$, (see further remarks in the Appendix).

Theorem 1.1 extends the main results of [3], [6], in the sense that we were able to treat both with a more general class of quasilinear operators and a broader class of terms $f$.

Problems like $(P_\lambda)$ have been investigated by many authors and we would like to refer the reader to Saxton & Wei [10], Castro & Kurepa [7], Cheng [5], Strauss [14], Ni & Serrin [12], Castro, Cósio & Neuberger [9], Fukagai & Narukawa [4], Mihăilescu & Rădulescu [10, 11] and their references. Here, we would point out that in [4], Fukagai & Narukawa have mentioned that this type of problem appears in some fields of physics, such as, nonlinear elasticity, plasticity and generalized Newtonian fluids.

3 Proof of Theorem 1.1

Take $\lambda \in (0, \Lambda]$ where $\Lambda > 0$ is given in Theorem 1.2. We proceed in two steps:

Step 1. (Existence of a positive solution of $(P_\lambda)$.) Let $d \in (0, d_\infty]$. We shall use the notations in Theorem 1.2. So $z_1 = z_1(d)$ denotes the first zero of $u(\cdot) = u(\cdot, d)$. Set

$$A_0 = \left\{ d \in (0, d_\infty] \mid z_1(d) \geq R \right\} \text{ and } d_0 := \inf A_0.$$

By (1.4) in theorem 1.2 $z_1(d_\infty) \geq R$. So $A_0 \neq \emptyset$. We will show that

$$d_0 > 0 \text{ and } z_1(d_0) = R.$$

(3.1)

Indeed, assume on the contrary that $d_0 = 0$. Take a sequence $(d_j) \subseteq A_0$ such that $d_j \to 0$. By (1.5), $z_1(d_j) \to 0$, which is a contradiction.

Now, assume $z_1(d_0) > R$. Pick a sequence $(d_j) \subseteq (0, d_\infty]$ such that $d_j < d_0$ and $d_j \to d_0$. Applying (1.5) we infer that $z_1(d_j) \to z_1(d_0)$. Once $z_1(d_0) > R$, it follows that $z_1(d_j) > R$, which shows that $d_j \in A_0$. But this contradicts the definition of $d_0$. Therefore $z_1(d_0) = R$ and this completes the proof of (3.1). As a byproduct there is a positive solution of $(P_\lambda)$.

Step 2. (Existence of an infinite sequence of sign-changing solutions of $(P_\lambda)$.) At first consider

$$A_1 := \left\{ d \in (0, d_0] \mid z_1(d) < R, \ z_2(d) \geq R \right\} \text{ and } d_1 := \inf A_1.$$

We claim that
\[ A_1 \neq \emptyset, \quad 0 < d_1 < d_0, \]

\[ z_1(d_1) < R, \quad z_2(d_1) = R. \]

Let us show at first that \( A_1 \neq \emptyset \). Indeed, by Step 1 \( z_1(d_0) = R \). By (1.6), if \( d \in (0, d_0) \) with \( d \) close to \( d_0 \) then \( u(\cdot, d) \) has at most one zero in \((0, R)\). Assume by contradiction that \( u(\cdot, d) \) has no zero in \((0, R)\). Then \( z_1(d) \geq R \) with \( d < d_0 \), impossible. It follows that \( u(\cdot, d) \) has precisely one zero in \((0, R)\) and so \( d \in A_1 \), showing that \( A_1 \neq \emptyset \).

To show that \( d_1 > 0 \), assume by contradiction that there is a sequence \( \{ d_j \} \subset A_1 \) such that \( d_j \to 0 \). By (1.5), \( z_2(d_j) \to 0 \) contradicting \( z_2(d_j) \geq R \).

It follows from \( z_1(d_0) = R \) and definition of \( A_1 \) that \( d_1 < d_0 \). Therefore \( 0 < d_1 < d_0 \leq d_\infty \).

It remains to show that \( z_1(d_1) < R \) and \( z_2(d_1) = R \). To do it, let \( \{ d_j \} \subset A_1 \) such that \( d_j \to d_1 \), so that \( z_1(d_j) \to z_1(d_1) \leq R \) and \( z_2(d_j) \to z_2(d_1) \geq R \).

If \( z_1(d_1) = R \) then \( u(\cdot, d_1) \) has no zeros in \((0, R)\). By (1.6), if \( d < d_1 \) and \( d \) is close to \( d_1 \), \( u(\cdot, d) \) has at most one zero in \((0, R)\). If \( u(\cdot, d) \) has one zero then we have \( d < d_1 \) and \( d \in A_1 \), a contradiction.

On the other hand, if \( u(\cdot, d) \) has no zero then \( d \geq d_0 > d_1 \) which is again a contradiction. Therefore, \( z_1(d_1) < R \). Now, assume by contradiction that \( z_2(d_1) > R \). Let \( d_j \to d_1 \) with \( d_j < d_1 \). Then, \( z_1(d_j) \to z_1(d_1) < R \) and in addition, \( z_2(d_j) \to z_2(d_1) > R \), so that, \( z_1(d_j) < R \) and \( z_2(d_j) > R \) for large \( j \), which is impossible. Thus \( z_2(d_1) = R \).

By induction, iterating the arguments above, we construct a sequence \( \{ d_\ell \}_{\ell=1}^\infty \subset (0, d_\infty) \) such that

\[ 0 < \cdots < d_\ell < \cdots < d_1 < d_\infty, \]

\[ z_\ell(d_\ell) < R, \quad z_{\ell+1}(d_\ell) = R, \]

with \( d_\ell := \inf A_\ell \), where

\[ A_\ell := \left\{ d \in (0, d_\ell] \mid z_\ell(d) < R, \quad z_{\ell+1}(d) \geq R \right\}. \]

This ends the proof of step 2.

To finish the proof of Theorem 1.1, we use steps 1 and 2 to conclude that for \( \lambda \in (0, \Lambda] \) the functions given by Theorem 1.2 namely \( u_\ell(\cdot) = u(\cdot, d_\ell) \in C^1([0, R]) \) for \( \ell \geq 1 \), satisfy

\[ r^\alpha \phi([u_\ell'(r)])u_\ell'(r) \text{ is differentiable}, \]

\[ -(r^\alpha \phi([u_\ell'(r)])u_\ell'(r)') = \lambda r^\gamma f(u_\ell(r)), \quad 0 < r < R, \]

that is, \( u_\ell \) is a classical solution of (\( PA_\ell \)), \( u_\ell \) has precisely \( \ell \) zeroes in \((0, R)\) and so

\[ u_0, u_1, u_2, \cdots, \]

is an infinite sequence of solutions of (\( PA_\ell \)) as claimed in the statement of Theorem 1.1.
4 Proof of Theorem 4.2

At first we set

\[ \Phi(t) = \int_0^t s \phi(s) ds, \quad H(t) = t \Phi'(t) - \Phi(t), \quad F(t) = \int_0^t f(s) ds. \]

The results below will play a crucial role in this paper.

**Lemma 4.1.** Assume \( (\gamma, \alpha) \) and let \( d \in [0, d_\infty] \), \( \lambda > 0 \) and \( T > 0 \). If \( u \) is a solution of \( (P_{\lambda, d}) \) in \([0, T]\), then

\[ H(|u'(r)|) \leq \lambda r^{\gamma - \alpha} [F(d) - F(u(r))], \quad r \geq 0. \]  \hspace{1cm} (4.1)

\[ F(u(r)) \leq F(d) \text{ for } r \in [0, T] \]  \hspace{1cm} (4.2)

**Lemma 4.2.** Assume that \( (\phi_1)-(\phi_3), \ (\gamma, \alpha) \) and \( (f_1)-(f_2) \) holds. If \( f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) \) and \( d \in (0, d_\infty] \), then problem \( (P_{\lambda, d}) \) has a unique solution \( u(\cdot, d, \lambda) = u(\cdot, d) = u(\cdot) \in C^1([0, \infty)) \). In addition,

\[ \text{if } d_0 \in (0, d_\infty] \text{ then } u(r, d) \rightarrow u(r, d_0) \text{ as } d \rightarrow d_0, \text{ uniformly in } [0, T] \text{ for } T > 0, \] \hspace{1cm} (4.3)

\[ \text{if } d_0 \in (0, d_\infty] \text{ then } u'(r, d) \rightarrow u'(r, d_0) \text{ as } d \rightarrow d_0, \text{ uniformly on compact subsets of } (0, \infty), \] \hspace{1cm} (4.4)

4.1 Proofs of Lemmas 4.1 and 4.2

Integrating the equation in \( (P_{\lambda, d}) \) we get to

\[ \phi(|u'(r)|)u'(r) = -r^{-\alpha} \int_0^r \lambda t^{\gamma} f(u(t)) dt, \quad r > 0. \]  \hspace{1cm} (4.5)

Setting \( h(t) := t \phi(t) \),

\[ h(t) := t \phi(t), \]  \hspace{1cm} (4.6)

we see that \( h \) is invertible with differentiable inverse \( h^{-1} \). Then,

\[ u'(r) = h^{-1} \left( -r^{-\alpha} \int_0^r \lambda t^{\gamma} f(u(t)) dt \right) \quad \text{if } u'(r) > 0, \]  \hspace{1cm} (4.7)

\[ u'(r) = -h^{-1} \left( -r^{-\alpha} \int_0^r \lambda t^{\gamma} f(u(t)) dt \right) \quad \text{if } u'(r) < 0. \]  \hspace{1cm} (4.8)

Once \( f \) is continuous and \( \gamma \geq \alpha \), we conclude from the above equalities that \( u \in C^1 \).

**Proof of Lemma 4.1** From \( (4.7) \) and \( (4.8) \), we infer that \( u \in C^2 \) at the points \( r > 0 \) where \( u'(r) \neq 0 \). Computing derivatives in \( (P_{\lambda, d}) \) and multiplying the resulting equality by \( u'(r) \), we are led to

\[ -\alpha r^{\alpha - 1} \phi(|u'(r)|)|u'(r)|^2 - r^{-\alpha} \frac{d}{dt} \phi(|u'(r)|)u'(r)u''(r) = \lambda r^{\gamma} f(u(r))u'(r), \quad u'(r) \neq 0. \]  \hspace{1cm} (4.9)
Consider the functional $E : [0, \infty) \to \mathbb{R}$ defined by
\[
E(0) = \lambda F(d) \text{ and } E(r) = r^{\alpha-\gamma}[H(|u'(r)|)] + \lambda F(u(r)), \ r > 0,
\]
where $H(t) = t\Phi'(t) - \Phi(t) = \int_0^t h'(s)ds$. Note that
\[
E'(r) = r^{\alpha-\gamma}[H(|u'(r)|)]' + (\alpha - \gamma)r^{\alpha-\gamma-1}H(|u'(r)|) + \lambda f(u(r))u'(r), \ r > 0
\]
and
\[
[H(|u'(t)|)]' = \frac{d}{dt}h(|u'(r)|)u''(r), \ u'(r) \neq 0.
\]
Therefore from (4.9),
\[
E'(r) = (\alpha - \gamma)r^{\alpha-\gamma-1}H(|u'(r)|) - \alpha r^{\alpha-\gamma-1}\phi(|u'(r)||u'(r)|^2 u'(r) \neq 0.
\]
From Lemma 5.6 in the Appendix, the last inequality combined with hypothesis $(\gamma, \alpha)$ gives
\[
E'(r) \leq \frac{\gamma_1 - 1}{\gamma_1} (\alpha - \gamma)r^{\alpha-\gamma-1}\phi(|u'(r)||u'(r)|^2 - \alpha r^{\alpha-\gamma-1}\phi(|u'(r)||u'(r)|^2 < 0, \ u'(r) \neq 0. \ (4.10)
\]
Next, we will prove that $E$ is continuous at the origin and therefore, as $E$ is non-decreasing by the previous inequality, it follows that $E(r) \leq E(0)$ for all $r \geq 0$. Note that equation (4.5) implies
\[
\Phi(|u'(r)|) = \Phi\left(h^{-1}\left[r^{-\alpha}\int_0^r \lambda t \phi(u(t))dt\right]\right),
\]
which in turn gives
\[
\Phi(|u'(r)|) \leq \Phi\left(h^{-1}\left(\frac{\lambda C_{\delta,d}r^{\gamma-\alpha+1}}{\gamma + 1}\right)\right), \ r \in [0, \delta), \ (4.11)
\]
where $C_{\delta,d} = \max_{r \in [0, \delta]} |f(u(r))|$. We choose $\delta > 0$ small and apply Lemmas 5.1 and 5.2 to conclude from (4.11) that
\[
\Phi(|u'(r)|) \leq \left(\frac{\lambda C_{\delta,d}}{\gamma + 1}\right)^{\frac{\gamma_2}{\gamma_1}} r^{\frac{\gamma_2}{\gamma_1} - (\gamma - \alpha + 1)}, \forall r \in [0, \delta). \ (4.12)
\]
We apply condition $\Delta_2$ (see inequality (5.1) in the Appendix) in the definition of $E$ to infer that
\[
E(r) \leq (\gamma_2 - 1)r^{\alpha-\gamma}\Phi(|u'(r)|) + \lambda F(u(r)), \ r > 0. \ (4.13)
\]
Thus, (4.12) and (4.13) lead to
\[
E(r) \leq Cr^{(\alpha-\gamma)+\frac{\gamma_2}{\gamma_1} - (\gamma - \alpha + 1)} + \lambda F(u(r)), \ r \in [0, \delta). \ (4.14)
\]
Recalling that $\gamma_2 \geq \gamma_1$, we have that
\[
(\alpha - \gamma) + \frac{\gamma_2}{\gamma_1} - (\gamma - \alpha + 1) \geq \frac{\gamma_1 - \alpha + \gamma}{\gamma_1 - 1} > 0.
\]
Hence, from (4.14) that $\lim_{r \to 0} E(r) \leq \lambda F(d)$. On the other hand, by Lemma 5.6 we know that $H(t) \geq 0$ for all $t \geq 0$. Then, by definition of $E$, $E(r) \geq \lambda F(u(r))$ for all $r > 0$. Gathering these information, we conclude that
\[
\lim_{r \to 0} E(r) = \lambda F(d).
\]
Therefore, as (4.10) is true,
\[ E(r) \leq E(0) \text{ for } r \geq 0, \]
which is equivalent to the desired inequality namely (4.1) \hfill \Box

**Proof of Lemma 4.2** We will at first study existence and uniqueness of local solutions of \((P_{\lambda,d,\epsilon})\). Let \(\epsilon > 0\) and consider
\[
\begin{align*}
-(r^\alpha \phi(|u'(r)|)u'(r))' &= \lambda r^\gamma f(u(r)), & 0 < r < \epsilon, \\
u(0) &= d, \quad u'(0) = 0.
\end{align*}
\]
\((P_{\lambda,d,\epsilon})\)

We shall need the following result whose proof is left to the Appendix.

**Lemma 4.3.** \((P_{\lambda,d,\epsilon})\) has a unique solution \(u(\cdot) = u(\cdot,d,\lambda,\epsilon) \in C^2([0,\epsilon))\) for small \(\epsilon\).

**Proof of Uniqueness in Lemma 4.2** Assume that \(u, v\) are two \(C^1([0,\infty))\) solutions. Let
\[ S_0 = \{r \geq 0 : u(t) = v(t), 0 \leq t \leq r\}. \]

We will show that
\[ S_0 \neq \emptyset, \quad S_0 \text{ is both open and closed in } [0,\infty). \tag{4.15} \]

Indeed, by Lemma 4.3 above, \([0,\epsilon) \subset S_0\) for \(\epsilon > 0\) small enough, which shows that \(S_0 \neq \emptyset\). Moreover, since \(u, v\) are \(C^1\) functions we infer that \(S_0\) is closed. To finish we shall prove that \(S_0\) is open. To achieve that let \(\hat{r} \in S_0\) with \(\hat{r} > 0\). We distinguish between two cases.

**Case 1.** \(u'(\hat{r}) = v'(\hat{r}) = 0\)

Assume \(u(\hat{r}) = v(\hat{r}) = \hat{d}\). If \(\hat{d} = 0\) then, up to a translation in the domain, we are within the settings of Lemma 4.1. Therefore, using (4.1), observing that by hypothesis \((f_1)\) one has \(F(u(r)) \geq 0\) for \(r \geq \hat{r}\), and noticing that \(F(\hat{d}) = 0\), we get
\[ H(|u'(r)|) \leq \lambda r^{\gamma-\alpha} \left( F(\hat{d}) - F(u(r)) \right) \leq 0 \text{ for } r \geq \hat{r}, \]
from where it follows that \(u(r) = 0\) for \(r \geq \hat{r}\), because by Lemma 5.6 in the Appendix
\[ H(t) \geq 0 \quad \forall t \geq 0 \quad \text{and} \quad H(t) = 0 \iff t = 0. \]

The same argument works to prove that \(v(r) = 0\) for \(r \geq \hat{r}\). Consequently, \(r \geq \hat{r}, u(r) = v(r) = 0\) and then, \(S_0 = [0,\infty)\) is open. On the other hand, if \(\hat{d} > 0\), we define
\[ \hat{K}_\rho(\hat{d}) = \{u \in C([\hat{r},\hat{r}+\epsilon]) : u(0) = \hat{d}, \|u - \hat{d}\|_\infty \leq \rho\}, \]
\[ \hat{T}(u(r)) = \hat{d} - \int_{\hat{r}}^r h^{-1} \left( t^{-\alpha} \int_{\hat{r}}^t \lambda r^\gamma f(u(\tau))d\tau \right) dt, \quad \forall r \in [0,\epsilon], \]
where \(\epsilon, \rho\) are positive and \(\epsilon\) is small. The same proofs of (5.8) and (5.9) can be done here, and then the Banach Fixed Point Theorem guarantees a unique fixed point for the operator \(\hat{T}\) when \(\epsilon\) is small, therefore, \(u(r) = v(r)\) in a small neighbourhood of \(\hat{r}\), which implies that \(S_0\) is open.

**Case 2.** \(u'(\hat{r}) = v'(\hat{r}) \neq 0\).
Note that there is a neighbourhood $V$ of $\hat{r}$ such that $u'(r), v'(r) \neq 0$ for $r \in V$. So in $V$, we must conclude, as in (4.10) (here we use the same notation as in the proof of Lemma 4.1) that if $z$ denotes either $u$ or $v$ then

$$\left( r^{\alpha-\gamma} H(|z'(r)|) + \lambda F(z(r)) \right)' = -\frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} r^{\alpha-\gamma-1} \phi(|z'(r)|) z'(r)^2. $$

Integrating from $\hat{r}$ to $r$ and subtracting the corresponding equations for $z = u$ and $z = v$, we obtain (remember that $u(\hat{r}) = v(\hat{r})$ and $u'(\hat{r}) = v'(\hat{r})$)

$$r^{\alpha-\gamma}[H(|u'(r)|) - H(|v'(r)|)] + \lambda [F(u(r)) - F(v(r))] = -\frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} \int_{\hat{r}}^{r} t^{\alpha-\gamma-1} \left[ \phi(|u'(t)|)u'(t)^2 - \phi(|v'(t)|)v'(t)^2 \right] dt. \tag{4.16}$$

Next we consider three auxiliary continuous functions, namely

$$A_1(t) = \begin{cases} \frac{H(|u'(t)|) - H(|v'(t)|)}{u'(t) - v'(t)}, & u'(t) \neq v'(t) \\ \phi(|u'(t)|)u'(t), & u'(t) = v'(t) \end{cases}$$

and

$$A_2(t) = \begin{cases} \frac{h(|u'(t)|)u'(t) - h(|v'(t)|)v'(t)}{u'(t) - v'(t)}, & u'(t) \neq v'(t) \\ \frac{d}{dt}[h(|u'(t)|)u'(t)], & u'(t) = v'(t) \end{cases}$$

Let $w(r) = u(r) - v(r)$. From (4.16),

$$r^{\alpha-\gamma}A_1(r)w'(r) + B(r)w(r) = -\frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} \int_{\hat{r}}^{r} t^{\alpha-\gamma-1} A_2(t)w'(t)dt. \tag{4.17}$$

Once $u'(\hat{r}) \neq 0$, we have that in a neighbourhood of $\hat{r}$, the function $1/A_1$ is well defined and continuous, and so, equation (4.17) is the same as

$$w'(r) + \frac{B(r)}{A_1(r)} r^{\gamma-\alpha}w(r) = -\frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} \frac{r^{\gamma-\alpha}}{A_1(r)} \int_{\hat{r}}^{r} t^{\alpha-\gamma-1} A_2(t)w'(t)dt. \tag{4.18}$$

As $h$ is two times differentiable and $u'(\hat{r}) \neq 0$, $A_2$ is continuously differentiable in a neighborhood of $\hat{r}$, therefore, from (4.18) and integration by parts we obtain

$$w'(r) + \frac{B(r)}{A_1(r)} r^{\gamma-\alpha}w(r) = \frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} \frac{r^{\gamma-\alpha}}{A_1(r)} r^{\alpha-\gamma-1} A_2(r)w(r) +$$

$$-\frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} \frac{r^{\gamma-\alpha}}{A_1(r)} \int_{\hat{r}}^{r} \left[ t^{\alpha-\gamma-1} A_2(t) \right] w(t)dt,$$
hence
\[
w'(r) + \left[ \frac{B(r)}{A_1(r)} r^{\gamma-\alpha} - \frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} r^{\gamma-\alpha} A_2(r) \right] w(r) = \frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} r^{\alpha-\gamma-1} A_2(r) \int_\bar{r}^r [t^{\alpha-\gamma-1} A_2(t)]' w(t) dt.
\] (4.19)

We introduce the notation
\[
D_1(r) = \frac{B(r)}{A_1(r)} r^{\gamma-\alpha} - \frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} r^{\gamma-\alpha} A_2(r),
\]
\[
D_2(r) = \frac{\alpha + \gamma(\gamma_1 - 1)}{\gamma_1} r^{\alpha-\gamma-1} A_2(r),
\]
\[
D_3(r) = [t^{\alpha-\gamma-1} A_2(t)]',
\]
which implies from (4.19) that
\[
w'(r) + D_1(r) w(r) = D_2(r) \int_\bar{r}^r D_3(s) w(s) ds. \tag{4.20}
\]

We integrate equation (4.20) from \(\bar{r}\) to \(r\), which combined with the fact that, \(A_1, A_2, 1/A_1, A_2', B\) are bounded functions (remember they are all continuous functions in a neighborhood of \(\bar{r}\)) to conclude that
\[
|v(r)| \leq \int_\bar{r}^r |D_1(s)||v(s)| ds + \int_\bar{r}^r |D_2(s)| \int_\bar{r}^s |D_3(r)||v(t)| dt ds
\]
\[
\leq C \int_\bar{r}^r |v(s)| ds,
\]
where \(C > 0\) is a constant. By the Gronwall Inequality, \(v = 0\) in a neighborhood of \(\bar{r}\). Therefore, \(S_0\) is open and (4.15) is proved.

**Proof of Existence in Lemma 4.2** Let
\[
S_\infty = \{ r > 0 \mid [P_{\lambda,t}] \text{ has a solution in } [0,r) \} \text{ and } T_\infty = \sup S_\infty.
\]
We will prove that
\[
T_\infty = \infty. \tag{4.21}
\]

Assume, on the contrary, that \(T_\infty < \infty\). First note that \(S_\infty\) is a closed set. Indeed, let \(r_n \to r\) with \(r_n \in S_\infty\). If \(r < r_n\) for some \(n\) then \(r \in S_\infty\) by force, so we can assume that \(r_n < r\) and without loss of generality that \(r_n < r_{n+1}\). If \(u_n\) are the solutions associated with \(r_n\), we define \(u : [0,r) \to \mathbb{R}\) by \(u(x) := u_n(x)\) of \(x \in [0,r_n)\). Once (4.15) is satisfied, we conclude that \(u\) is well defined and it is a solution of \([P_{\lambda,t}]\), which implies that \(r \in S_\infty\).

Since \(S_\infty\) is closed, we have that \(T_\infty \in S_\infty\). Let \(u\) be the solution associated with \(T_\infty\). We first observe that from (4.11), \(|u'|\) is bounded, which implies that \(u\) can be continuously extended to \(T_\infty\). Moreover, equation (4.5) guarantees that \(u'(T_\infty)\) is uniquely defined, so there are two cases to consider:

**Case 1.** \(u'(T_\infty) = 0\).
If \( u(T_\infty) = 0 \), consider the extension of \( u \) namely \( \tilde{u} : [0, \infty) \to \mathbb{R} \) given by \( \tilde{u}(t) = 0 \) for \( t \geq T_\infty \). Then \( \tilde{u} \) is a \( C^1 \) function and it is also a solution of \( (P_{\lambda,d}) \), which is an absurd. Otherwise, if \( u(T_\infty) = d_\infty > 0 \), consider the operator \( T \) defined by

\[
T(u(r)) = d_\infty - \int_{T_\infty}^r h^{-1}\left(t^{-\alpha} \int_{T_\infty}^t \lambda t^\gamma f(u(t))dt\right) dt.
\]

Following the same lines as in the proof of either \( (5.8) \) or \( (4.15) \) Case 1, we have that \( T \) has unique fixed point \( v : [0, T_\infty + \epsilon] \), which is an absurd due to the definition of \( T_\infty \).

Case 2. \( u'(T_\infty) \neq 0 \).

Assume without loss of generality that \( u'(T_\infty) > 0 \). Then, by \( (4.5) \),

\[
u'(r) = h^{-1}\left(r^{-\alpha} \int_0^r \lambda t^\gamma f(u(t))dt\right)
\]
in a neighborhood of \( T_\infty \). Hence, \( u \) is \( C^2 \) in a neighborhood of \( T_\infty \), which implies by \( (P_{\lambda,d}) \) that

\[
u''(r) = - \left[ \frac{d}{dr} h(u(r)) \right]^{-1} \left( \frac{\alpha}{r} \phi(u'(r))u'(r) + \lambda r^{\gamma - \alpha} f(u(r)) \right).
\]

By the last equation and Peano’s Theorem, \( u \) can be extended to \( [0, T_\infty + \delta] \), where \( \delta > 0 \) and thus we reach an absurd, because such extension is also a solution to \( (P_{\lambda,d}) \). This finishes the proof of Case 2. Therefore, \( (4.21) \) is proved and thus Claim 2 is also proved.

Proof of \( (4.3) \). Remember that

\[
r^\alpha \phi(|u'(r)|)u'(r) = - \int_0^r \lambda t^\gamma f(u(t))dt.
\]

Assume that \( d_n \to d_0 \) and set \( u_n(r) = u(r,d_n) \), \( u_0(r) = u(r,d_0) \). Inequality \( (4.5) \) implies that \( |u_n'(r)| \) is bounded for \( r \in [0, T] \), therefore, by Ascoli-Arzela Theorem, there is a subsequence, still denoted by \( u_n \), such that \( u_n \to v \) uniformly in \( [0, T] \) for some \( v \in C([0, T]) \). Now we will prove that \( v = u_0 \).

First note that by Lebesgue’s Theorem

\[
\int_0^r \lambda t^\gamma f(u_n(t))dt \to \int_0^r \lambda t^\gamma f(v(t))dt,
\]

and by \( (4.22) \),

\[
r^\alpha \phi(|u_n'(r)|)u_n'(r) \to - \int_0^r \lambda t^\gamma f(v(t))dt, \ r \in [0, T].
\]

As a consequence,

\[
|u_n'(r)| \to h^{-1}\left(r^{-\alpha} \int_0^r \lambda t^\gamma f(v(t))dt\right), \ r \in [0, T].
\]

The combination of \( (4.22) \) and \( (4.23) \) implies that \( u_n'(r) \to w(r) \) for all \( r \in [0, T] \) where \( w \) is a continuous function. Hence, applying Lebesgue’s Theorem we obtain

\[
u_n(r) - d_n = \int_0^r u_n'(t)dt \to \int_0^r w(t)dr, \ r \in [0, T],
\]
which implies that \( u(r) = v'(r) \) and \( v'(0) = 0 \). Once

\[
\phi(|v'(r)|)v'(r) = -r^{-\alpha} \int_0^r \lambda t^{\gamma} f(v(t))dt,
\]

is satisfied and since \( v(0) = d_0 \), it follows by the uniqueness of solutions given by theorem 4.2 that \( v = u_0 \), which concludes the proof of (4.3).

**Proof of (4.4).** Let \( 0 < a \leq r \leq b < \infty \) and assume that \( d_n \to d_0 \). By (4.22),

\[
r^\alpha |\phi(|u'_n(r)|)u'_n(r) - \phi(|u'_0(r)|)u'_0(r)| \leq \int_0^r \lambda t^{\gamma} |f(u_n(t)) - f(u_0(t))|dt.
\]

Since \( (u_n) \) converges uniformly to \( u_0 \) in \([a, b]\), we conclude from the previous inequality that

\[
(\phi(|u'_n(r)|)u'_n(r) - \phi(|u'_0(r)|)u'_0(r))(u'_n(r) - u'_0(r)) \to 0,
\]

uniformly in \([a, b]\). Now, we combine a generalized form of Simon’s inequality, see Lemma 5.5 in the Appendix, with the last convergence to conclude that \( u'_n \to u'_0 \) uniformly in \([a, b]\). This finishes the proof of Lemma 4.2.

**4.2 Proof of Theorem 1.2 (Continued)**

**Proof of (4.3).** We will start by proving that there is \( z_1 = z_1(d) > 0 \) such that \( u(z_1) = 0, u'(z_1) < 0 \) and

\[
u(r) > 0, u'(r) < 0 \text{ for } 0 < r < z_1. \tag{4.24}\]

Suppose, on the contrary, that \( u(r) > 0 \) for all \( r > 0 \). It follows from (4.22) and conditions \((f_1), (f_2)\) that \( u'(r) < 0 \) and

\[
-u'(r) \geq h^{-1} \left( \frac{r^{\gamma-\alpha+1}}{\gamma+1} f(u(r)) \right), \quad r > 0.
\]

Note that \( u'(r) \to 0 \) if \( r \to \infty \) because \( u(r) > 0 \). Hence, the previous inequality implies that \( u(r) \to 0 \) if \( r \to \infty \). Moreover, by Lemma 5.1 and the previous inequality, we also obtain

\[
-u'(r) \geq \max \left\{ \left( \frac{\lambda r^{\gamma-\alpha+1} f(u(r))}{(\gamma+1)h(1)} \right)^{\frac{1}{\gamma-1}}, \left( \frac{\lambda r^{\gamma-\alpha+1} f(u(r))}{(\gamma+1)h(1)} \right)^{\frac{1}{\gamma}} \right\}, r > 0,
\]

which implies

\[
-u'(r) \min\{f(u(r))^{\frac{1}{\gamma-1}}, f(u(r))^{\frac{1}{\gamma}}\} \geq \min \left\{ \left( \frac{\lambda r^{\gamma-\alpha+1}}{(\gamma+1)h(1)} \right)^{\frac{1}{\gamma-1}}, \left( \frac{\lambda r^{\gamma-\alpha+1}}{(\gamma+1)h(1)} \right)^{\frac{1}{\gamma}} \right\}
\]

for each \( r > 0 \). We integrate the last inequality from 0 to \( r \) and apply the change of variables \( t = u(s) \) to conclude that

\[
\int_{u(r)}^d \min\{f(t)^{\frac{1}{\gamma-1}}, f(t)^{\frac{1}{\gamma}}\} dt \geq \int_0^r \min \left\{ \left( \frac{\lambda s^{\gamma-\alpha+1}}{(\gamma+1)h(1)} \right)^{\frac{1}{\gamma-1}}, \left( \frac{\lambda s^{\gamma-\alpha+1}}{(\gamma+1)h(1)} \right)^{\frac{1}{\gamma}} \right\} ds. \tag{4.25}
\]

Hypothesis \((\gamma, \alpha)\) implies that the right hand side of (4.25) converges to infinity as \( r \to \infty \). Therefore, (4.25) yields

\[
\liminf_{r \to \infty} \int_{u(r)}^d \min\{f(t)^{\frac{1}{\gamma-1}}, f(t)^{\frac{1}{\gamma}}\} dt = \infty,
\]

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which combined with \((f_1)\) and the fact that \(u(r) \to 0\) if \(r \to \infty\), implies a contradiction to \((f_3)\) and thus, \((4.24)\) is true. To proceed, we will prove that there is \(\Lambda > 0\) such that

\[
z_1(d_\infty, \lambda) \geq R \text{ if } 0 < \lambda \leq \Lambda. \tag{4.26}
\]

Indeed, by \((1.22)\),

\[
-u'(r) \leq h^{-1} \left( \frac{\lambda f(d_\infty) r^{\gamma - \alpha + 1}}{\gamma + 1} \right) \text{ for } r \in [0, z_1(d_\infty, \lambda)].
\]

Integrating from 0 to \(r \in [0, d_\infty]\) and making use of Lemma 5.1, we get that

\[
-u(r) + d_\infty \leq \max \left\{ (\gamma_1 - 1) \left( \frac{\lambda f(d_\infty)}{(\gamma + 1)h(1)} \right)^{\frac{1}{\gamma_1 - 1}} r^{\frac{\gamma - \alpha + 1}{\gamma_1 - 1}} + \frac{\gamma_1 - 1}{\gamma - \alpha + \gamma_1}, \right. \\
(\gamma_2 - 1) \left( \frac{\lambda f(d_\infty)}{(\gamma + 1)h(1)} \right)^{\frac{1}{\gamma_2 - 1}} r^{\frac{\gamma - \alpha + 1}{\gamma_2 - 1}} + \frac{\gamma_2 - 1}{\gamma - \alpha + \gamma_2} \left\}
\]

Let \(\nu \in (0, 1)\). Choose \(r_\infty(\nu) \in (0, z_1(d_\infty, \lambda))\) such that \(u(r_\infty(\nu), d_\infty) = \nu d_\infty\). Set \(r = r_\infty(\nu)\) in \((4.27)\) and choose the maximum value in the right hand side of \((4.27)\) which actually is

\[
(\gamma_1 - 1) \left( \frac{\lambda f(d_\infty)}{(\gamma + 1)h(1)} \right)^{\frac{1}{\gamma_1 - 1}} r_\infty(\nu) \frac{\gamma - \alpha + 1}{\gamma - \alpha + \gamma_1}.
\]

Take \(R > 0\) and choose \(\Lambda_\nu > 0\) satisfying

\[
1 - \nu = \left[ \left( \frac{\lambda f(d_\infty)}{(\gamma + 1)h(1)} \right)^{\frac{1}{\gamma_1 - 1}} \frac{\gamma_1 - 1}{\gamma - \alpha + \gamma_1} \right]^{-1} \Lambda_\nu^{-\frac{1}{\gamma_1 - 1}} \frac{R^{\frac{\gamma - \alpha + 1}{\gamma_1 - 1}}}{d_\infty}. \tag{4.28}
\]

We infer from \((4.27)\) and \((4.28)\) that

\[
\left[ \left( \frac{\lambda f(d_\infty)}{(\gamma + 1)h(1)} \right)^{\frac{1}{\gamma_1 - 1}} \frac{\gamma_1 - 1}{\gamma - \alpha + \gamma_1} \right]^{-1} \Lambda_\nu^{-\frac{1}{\gamma_1 - 1}} \frac{R^{\frac{\gamma - \alpha + 1}{\gamma_1 - 1}}}{d_\infty} \leq (\gamma_1 - 1) \left( \frac{\lambda f(d_\infty)}{(\gamma + 1)h(1)} \right)^{\frac{1}{\gamma_1 - 1}} r_\infty(\nu) \frac{\gamma_1 - 1}{\gamma - \alpha + \gamma_1},
\]

which implies that

\[
\Lambda_\nu^{-\frac{1}{\gamma_1 - 1}} \frac{R^{\frac{\gamma - \alpha + 1}{\gamma_1 - 1}}}{d_\infty} \leq \lambda^{\frac{1}{\gamma_1 - 1}} r_\infty(\nu) \frac{\gamma_1 - 1}{\gamma - \alpha + \gamma_1}.
\]

Hence,

\[
R \leq r_\infty(\nu) \leq z_1(d_\infty, \lambda) \text{ if } 0 < \lambda \leq \Lambda_\nu. \tag{4.29}
\]

To finish the proof of \((4.26)\), first note that the maximum of two continuous functions is a continuous function. Therefore, \((4.27)\) combined with \((4.28)\) gives

\[
\Lambda_\nu^{-\frac{1}{\gamma_1 - 1}} \nu \to 0 \left( \frac{\lambda f(d_\infty)}{(\gamma + 1)h(1)} \right)^{\frac{1}{\gamma_1 - 1}} \frac{\eta - 1}{\gamma - \alpha + \eta} \frac{d_\infty}{R^{\frac{\gamma - \alpha + 1}{\gamma_1 - 1}}},
\]

where either \(\eta = \gamma_1\) or \(\eta = \gamma_2\) depending on whether the maximum in \((4.27)\) is assumed at \(\gamma_1\) or \(\gamma_2\). Note also that \(r_\nu(d_\infty)\) is continuous on \(\nu\) and \(r_\nu(d_\infty) \to z_1(d_\infty, \lambda)\) as \(\nu \to 0\), therefore, we conclude from \((4.29)\) that

\[
R \leq z_1(d_\infty, \lambda) \text{ if } 0 < \lambda \leq \Lambda.
\]
\[ \Lambda := \frac{\lambda f(d_\infty)}{(\gamma + 1)h(1)} \left( \frac{\eta - 1}{\gamma - \alpha + \eta} \right)^{\eta-1} \frac{d_\infty^{\eta-1}}{P^{\gamma-\alpha+\eta}}. \]

Now we will show that there is \( z_2 = z_2(d) > z_1 \) such that \( u(z_2) = 0, u'(z_2) > 0 \) and

\[ u(r) < 0, \; z_1 < r < z_2. \]  \hspace{1cm} (4.30)

In fact, since \( u'(z_1) < 0 \) then, \( u'(r) < 0 \) in a neighborhood of \( z_1 \). We start by proving that there is \( m_1 > z_1 \) such that \( u'(m_1) = 0 \). Thus, suppose by contradiction that it is not true, i.e. \( u'(r) < 0 \) for all \( r > z_1 \). We have by (4.2) that

\[ \int_{0}^{u(r)} f(t)dt \leq F(d), \; r \geq 0. \]

If there is some sequence \( r_n \to \infty \) such that \( u(r_n) \to -\infty \) then, by the previous inequality we infer that

\[ \int_{-\infty}^{0} f(s)ds = \lim_{n} \int_{0}^{u(r_n)} f(s)ds \geq -F(d), \]

which is impossible, because \( (f_1), (f_2) \) imply that \( \int_{-\infty}^{0} f(s)ds = -\infty \). Hence, there is \( C > 0 \) such that

\[ u(r) \geq -C, \; u'(r) < 0, \; \forall r \geq z_1, \]

and consequently \( u(r) \to L \) as \( r \to \infty \) for some \( L < 0 \). Now, by (4.1),

\[ \frac{\Phi(|u'(r)|)}{r^{\gamma-\alpha+1}} \to 0 \text{ as } r \to \infty, \]

which implies by using the inequality \( \Phi(s) \geq cs^2\phi(s) \) that

\[ \frac{\phi(|u'(r)|)}{r^{\gamma-\alpha+1}} \to 0. \]

On one hand (4.22) and the previous limits imply that

\[ \frac{1}{r^{\gamma+1}} \int_{0}^{r} t^{\gamma} f(u(t))dt \to 0, \]

and on the other side, \( (f_1) \) and L’Hospital rule imply that

\[ \lim_{r \to \infty} \frac{1}{r^{\gamma+1}} \int_{0}^{r} t^{\gamma} f(u(t))dt = \lim_{r \to \infty} \frac{r^{\gamma} f(u(r))}{(\gamma + 1)r^{\gamma}} = \frac{f(L)}{\gamma + 1} < 0, \]

which is absurd. Therefore, \( u'(m_1) = 0 \) for some \( z_1 < m_1 \), so that

\[ u(r) < 0 \text{ for } z_1 < r < m_1 \text{ and } u'(r) < 0 \text{ for } z_1 \leq r < m_1. \]

Now, taking \( r > m_1, r \) close to \( m_1 \) we have

\[ \int_{m_1}^{r} t^{\gamma} f(u(t))dt < 0, \]

which implies by (4.22) that

\[ u(r) < 0, \; u'(r) > 0 \text{ for all } r > m_1, \; r \text{ close to } m_1. \]
Assume by contradiction that \( u(r) < 0 \) for \( r > m_1 \), so that \( u'(r) > 0 \). Since by \((f_2)\)
\[
-\varphi^a (u'(r)) u'(r) = \lambda \int_{m_1}^r \varphi^a f(u(t)) dt \leq \frac{\lambda f(u(r))}{\gamma + 1} (r^{\gamma + 1} - m_1^{\gamma + 1}),
\]
we get by taking \( r > \eta = \frac{1}{2} m_1 \) above, that \( r^{\gamma + 1} - m_1^{\gamma + 1} > \frac{r^{\gamma + 1}}{2} \) and so
\[
-\varphi^a (u'(r)) u'(r) \leq \frac{\lambda f(u(r))}{2(\gamma + 1)} r^{\gamma + 1},
\]
which, combined with Lemma \([5.1]\) gives
\[
u'(r) \geq \min \left\{ \left( \frac{-\lambda f(u(r))}{2(\gamma + 1)} r^{-\alpha + 1} \right)^{\frac{1}{\gamma_1 - 1}}, \left( \frac{-\lambda f(u(r))}{2(\gamma + 1)} r^{-\alpha + 1} \right)^{\frac{1}{\gamma_2 - 1}} \right\}, \ r > \eta. \tag{4.31}
\]
Integrating in \([4.31]\) from \( \eta \) to \( r \), we have
\[
\int_{\eta}^r \nu'(t) \max \{ (-f(u(t)))^{-\frac{1}{\gamma_1 - 1}}, (-f(u(t)))^{-\frac{1}{\gamma_2 - 1}} \} dt \geq \int_{\eta}^r \min \left\{ \left( \frac{t^\gamma - 1}{2(\gamma + 1)} \right)^{\frac{1}{\gamma_1 - 1}}, \left( \frac{t^\gamma - 1}{2(\gamma + 1)} \right)^{\frac{1}{\gamma_2 - 1}} \right\} dt,
\]
for \( r > \eta \). Making the change of variables \( y = u(t) \),
\[
\int_{u(\eta)}^{u(r)} \max \{ (-f(t))^{-\frac{1}{\gamma_1 - 1}}, (-f(t))^{-\frac{1}{\gamma_2 - 1}} \} dt \geq \int_{\eta}^r \min \left\{ \left( \frac{t^\gamma - 1}{2(\gamma + 1)} \right)^{\frac{1}{\gamma_1 - 1}}, \left( \frac{t^\gamma - 1}{2(\gamma + 1)} \right)^{\frac{1}{\gamma_2 - 1}} \right\} dt. \tag{4.32}
\]
Once \( u(r) < 0 \) and \( u'(r) > 0 \) for \( r > \eta \) it follows that \( u'(r) \to 0 \) as \( r \to \infty \). Hence, inequality \([4.31]\) implies that \( u(r) \to 0 \) as \( r \to \infty \). Moreover, the right hand side of \([4.32]\) converges to \( \infty \) due to hypothesis \([\gamma, \alpha]\). Therefore
\[
\lim \inf \int_{u(\eta)}^{u(r)} \max \{ (-f(t))^{-\frac{1}{\gamma_1 - 1}}, (-f(t))^{-\frac{1}{\gamma_2 - 1}} \} = \infty,
\]
which contradicts \((f_3)\), so \([4.30]\) is proved. Now we will prove that there is \( z_3 = z_3(d) > z_2 \) such that \( u(z_3) = 0, u'(z_3) < 0 \) and
\[
u(r) > 0 \text{ for all } r \in (z_2, z_3). \tag{4.33}
\]
Indeed, since by \([4.30]\), \( u'(z_2) > 0 \), so that
\[
u'(r) > 0 \text{ for all } r > z_2, \ r \text{ close to } z_2.
\]
We claim that there is \( m_2 > z_2 \) such that \( u'(m_2) = 0 \). In fact, otherwise, \( u'(r) > 0 \), for all \( r > z_2 \), which gives that \( u(r) > 0 \) for \( r > z_2 \). By \([4.2]\),
\[
\int_{0}^{u(r)} f(t) dt \leq \int_{0}^{d} f(t) dt,
\]
so that \( u(r) \leq d \) for \( r \geq z_2 \). Hence, there is \( L \in (0, d] \) such that
\[
u(r) \to L \text{ and } u(r) \leq L, \ r \geq z_2.
\]
As in the proof of (4.30),
\[
\frac{1}{r^{\gamma+1}} \int_0^r t^\gamma f(u(t)) dt \to 0,
\]
and
\[
\lim_{r \to \infty} \frac{1}{r^{\gamma+1}} \int_0^r t^\gamma f(u(t)) dt = \lim_{r \to \infty} \frac{r^\gamma f(u(r))}{(\gamma + 1)r^\gamma} = \frac{f(L)}{\gamma + 1} < 0,
\]
which is an absurd. As a consequence, there is \( m_2 > z_2 \) such that \( u'(m_2) = 0 \) and \( u'(r) > 0 \), \( z_2 \leq r < m_2 \), proving the claim. Assume again, by contradiction, that \( u(r) > 0 \) for all \( r > m_2 \) so that \( u'(r) < 0 \) also for all \( r > m_2 \). We have (similar to the proof of (4.30))
\[
-r^\alpha \phi(u(r))u'(r) = \lambda \int_{m_2}^r t^\gamma f(u(t)) dt \geq \frac{\lambda f(u(r))}{\gamma + 1} (r^{\gamma+1} - m_2^{\gamma+1}).
\]
Setting \( \overline{r} = 2^{\frac{-1}{\gamma+1}}m_2 \) and taking \( r > \overline{r} \),
\[
-r^\alpha \phi(u'(r))u'(r) \geq \frac{\lambda f(u(r))}{2(\gamma + 1)} r^{\gamma+1},
\]
which combined with (5.1) gives
\[
-u'(r) \geq \min \left\{ \left( \frac{\lambda f(u(r))}{2(\gamma + 1)} \right)^{\frac{1}{\gamma - 1}}, \left( \frac{\lambda f(u(r))}{2(\gamma + 1)} \right)^{\frac{1}{\gamma + 1}} \right\}, \ r > \overline{r}. \tag{4.34}
\]
Integrating (4.34) from \( \overline{r} \) to \( r \) and making the change of variables \( u(t) = s \), we get
\[
\int_{u(\overline{r})}^{u(r)} - \max \left\{ f(t) \frac{1}{\gamma - 1}, f(t) \frac{1}{\gamma + 1} \right\} dt \geq \int_{\overline{r}}^r \min \left\{ \left( \frac{t^{\gamma + 1}}{2(\gamma + 1)} \right)^{\frac{1}{\gamma - 1}}, \left( \frac{t^{\gamma - 1}}{2(\gamma + 1)} \right)^{\frac{1}{\gamma + 1}} \right\} dt.
\]
Taking lim inf in both sides, we arrive at a contradiction with (f3) and so (1.33) is true. To finish the proof of (1.4) we argue as in (4.30) and (1.33) to zeroes \( z_4, z_5 \) and inductively, a sequence with the properties asserted in (1.4).

**Proof of (1.5).** We start by proving that \( z_1(d) \to 0 \) when \( d \to 0 \). By (4.22) and (4.24) we obtain
\[
-u'(r) = h^{-1} \left( \int_0^r \lambda t^\gamma f(u(t)) dt \right), \ r \in [0, z_1].
\]
Now we apply (f2) and Lemma 5.1 to conclude that
\[
-u'(r) \geq \min \left\{ \left( \frac{t^{\gamma - 1}f(u(r))}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}}, \left( \frac{t^{\gamma - 1}f(u(r))}{\gamma + 1} \right)^{\frac{1}{\gamma + 1}} \right\}, \ r \in [0, z_1],
\]
which implies that
\[
-u'(r) \max \left\{ f(u(r)) \frac{1}{\gamma - 1}, f(u(r)) \frac{1}{\gamma + 1} \right\} \geq \min \left\{ \left( \frac{t^{\gamma - 1}}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}}, \left( \frac{t^{\gamma - 1}}{\gamma + 1} \right)^{\frac{1}{\gamma + 1}} \right\}, \ r \in [0, z_1].
\]
Integrating from 0 to \( r \) and making the change of variables \( y = u(t) \) we get to
\[
\int_0^d \max \left\{ f(t) \frac{1}{\gamma - 1}, f(t) \frac{1}{\gamma + 1} \right\} dt \geq \int_0^r \min \left\{ \left( \frac{t^{\gamma - 1}}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}}, \left( \frac{t^{\gamma - 1}}{\gamma + 1} \right)^{\frac{1}{\gamma + 1}} \right\} dt.
\]
Taking $r = z_1(d)$ in the previous inequality and making use of \((f_3)\) and \((f_3)\), we conclude that $z_1(d) \to 0$ as $d \to 0$. Now, letting $\ell \geq 1$, we assume that $u(r) > 0$ in $(z_\ell(d), z_{\ell+1}(d))$, so that by the notations of \((4.30)\) and \((4.33)\) we have $u'(r) > 0$ in $(z_\ell(d), m_\ell(d))$ and $u'(r) < 0$ in $(m_\ell(d), z_{\ell+1}(d))$ (the case $u(r) < 0$ in $(z_\ell(d), z_{\ell+1}(d))$ is handled similarly). Now, using \((f_2)\) in \((4.22)\), taking $m_\ell(d) \leq r \leq z_{\ell+1}(d)$ and then applying lemma 5.1, we obtain successively

$$r^\alpha h(-u'(r)) \geq \lambda f(u(r)) \frac{r^{\gamma+1} - m_\ell(d)^{\gamma+1}}{\gamma + 1},$$

$$-u'(r) \max\{f(u(r))^{\frac{1}{\gamma+1}}, f(u(r))^{\frac{1}{\gamma^2+1}}\} \geq \min\left\{\left(\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma + 1)(r - m_\ell(d))}\right)^{\frac{1}{\gamma+1}}, \left(\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma + 1)(r - m_\ell(d))}\right)^{\frac{1}{\gamma^2+1}}\right\}. \quad (4.35)$$

Note that $r^{\gamma-\alpha} \geq m_\ell(d)^{\gamma-\alpha}$ since $\gamma \geq \alpha$, therefore

$$r^{\gamma-\alpha+1} - r^{-\alpha}m_\ell(d)^{\gamma+1} \geq m_\ell(d)^{\gamma-\alpha}(r - m_\ell(d)),$$

which gives

$$-u'(r) \max\{f(u(r))^{\frac{1}{\gamma+1}}, f(u(r))^{\frac{1}{\gamma^2+1}}\} \geq \min\left\{\left(\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma + 1)(r - m_\ell(d))}\right)^{\frac{1}{\gamma+1}}, \left(\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma + 1)(r - m_\ell(d))}\right)^{\frac{1}{\gamma^2+1}}\right\}. \quad (4.36)$$

Integrating from $m_\ell(d)$ to $z_{\ell+1}(d)$, making the change of variables $y = u(t)$, we find that

$$\int_0^{u(m_\ell(d))} \max\{f(t)^{\frac{1}{\gamma+1}}, f(t)^{\frac{1}{\gamma^2+1}}\} dt \geq$$

$$\int_{m_\ell(d)}^{z_{\ell+1}(d)} \min\left\{\left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma + 1)}(r - m_\ell(d))\right]^{\frac{1}{\gamma+1}}, \left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma + 1)}(r - m_\ell(d))\right]^{\frac{1}{\gamma^2+1}}\right\} dt. \quad (4.37)$$

Assume now $z_\ell(d) < r < m_\ell(d)$. Then by a similar argument, this time, integrating from $z_\ell(d)$ to $m_\ell(d)$ we deduce that

$$\int_0^{u(m_\ell(d))} \max\{f(t)^{\frac{1}{\gamma+1}}, f(t)^{\frac{1}{\gamma^2+1}}\} dt \geq$$

$$\int_{z_\ell(d)}^{m_\ell(d)} \min\left\{\left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma + 1)}(m_\ell(d) - r)\right]^{\frac{1}{\gamma+1}}, \left[\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{(\gamma + 1)}(m_\ell(d) - r)\right]^{\frac{1}{\gamma^2+1}}\right\} dt. \quad (4.37)$$

Now, since $u(m_\ell(d)) \leq d$ we have by \((f_3)\) that the left hand side of \((4.36)\) and \((4.37)\) converge to zero, and therefore, $\lim_{d \to 0} z_\ell(d) = \lim_{d \to 0} z_{\ell+1}(d)$ for each $\ell \geq 1$. Once $z_1(d) \to 0$ as $d \to 0$, we see that $z_\ell(d) \to 0$ as $d \to 0$.

We pass to the proof that $z_\ell(d) \to z_\ell(d_0)$ if $d \to d_0$. Let us first show that $z_1(d) \to z_1(d_0)$ as $d \to d_0$. Indeed, let $d_n \to d_0$, $u_n(\cdot) = u(\cdot, d_n)$ and $u_0(\cdot) = u(\cdot, d_0)$ so that we have from \((4.3)\) that $u_n \to u$ uniformly in compact subsets of $(0, \infty)$. For each $\epsilon > 0$ small we find

$$u_0(r) > 0, \quad 0 \leq r \leq z_1(d_0) - \epsilon \quad \text{and} \quad u_0(z_1(d_0) + \epsilon) < 0,$$
so that 
\[ u_n(r) > 0, \; 0 \leq r \leq z_1(d_0) - \epsilon \text{ and } u_n(z_1(d_0) + \epsilon) < 0, \]
for sufficiently large \( n \). As a consequence, \( z_1(d_0) - \epsilon < z_1(d_n) < z_1(d_0) + \epsilon \), showing that \( z_1(d_n) \to z_1(d_0) \). Now, assume by induction that \( z_\ell(d_n) \to z_\ell(d_0) \) for some \( \ell > 1 \). We will show that \( z_{\ell+1}(d_n) \to z_{\ell+1}(d_0) \). For that matter, we assume \( u_0(t) < 0 \) for \( z_\ell(d_0) < t < z_{\ell+1}(d_0) \) (the other case is handled similarly). Taking \( \epsilon > 0 \) small, we find that \( u_n(t) < 0 \) for \( z_\ell(d_0) + \epsilon \leq t \leq z_{\ell+1}(d_0) - \epsilon \) and \( u_n(z_{\ell+1}(d_0) + \epsilon) > 0 \), showing that \( z_{\ell+1}(d_0) - \epsilon < z_{\ell+1}(d_n) < z_{\ell+1}(d_0) + \epsilon \). Consequently, \( z_{\ell+1}(d_n) \to z_{\ell+1}(d_0) \) as \( d \to d_0 \), which finishes the proof of (1.5).

**Proof of (1.6).**
Let \( d \in (0, d_0) \). It suffices to show that \( z_{\ell+2}(d) > R \) whenever \( d \) is close enough to \( d_0 \). We assume that \( u(r, d_0) < 0 \) for \( r \in (z_\ell(d_0), z_{\ell+1}(d_0)) \) (the other case is handled similarly).

Notice that as \( z_\ell(d_0) \) is increasing and there is only \( \ell \) zeroes in \((0, R)\), we must show that \( z_{\ell+1}(d_0) \geq R \) and \( z_{\ell+2}(d_0) > R \). However, as \( z_{\ell+2}(d) \to z_{\ell+2}(d_0) \) for \( d \to d_0 \), we have \( z_{\ell+2}(d) > R \) whenever \( d \) is close enough to \( d_0 \). This completes the proof of Theorem 1.2. \( \square \)

## 5 Appendix

**Remark 5.1.** (On the radially symmetric form of \((\Phi)\)) Let \( u \) be a weak solution of \((\Phi)\), radially symmetric in the sense that \( u(x) = u(|x|) = u(r) \). Let \( r \in (0, R) \) and pick \( \epsilon > 0 \) small such that \( 0 < r < r + \epsilon < R \).

Consider the radially symmetric cut-off function \( v_{r, \epsilon}(x) = v_{r, \epsilon}(r) \), where

\[
v_{r, \epsilon}(t) := \begin{cases} 
1 & \text{if } 0 \leq t \leq r, \\
\text{linear} & \text{if } r \leq t \leq r + \epsilon, \\
0 & \text{if } r + \epsilon \leq t \leq R.
\end{cases}
\]

and notice that \( v_{r, \epsilon} \in W_0^{1, \Phi}(B) \cap Lip(B) \). By replacing \( v \) with \( v_{r, \epsilon} \) in (2.2), we get to

\[
-\frac{1}{\epsilon} \int_{B(0, r+\epsilon) \setminus B(0, r)} \phi(|u'(|x|)|)u'(|x|)dx = \lambda \int_{B(0, r+\epsilon)} f(u(|x|))v_{r, \epsilon}(|x|)dx.
\]

Making the change of variables \( x = r\omega \) with \( r > 0 \) and \( \omega \in \partial B(0, 1) \) and letting \( \epsilon \to 0 \) we infer that

\[
\phi(|u'(r)|)u'(r)r^{N-1} = \lambda \int_0^r f(u(r))r^{N-1}dr,
\]

which gives

\[
(r^{N-1}\phi(|u'(r)|)u'(r))' = \lambda r^{N-1}f(u(r)).
\]

So the radially symmetric form of \((\Phi)\) is

\[
\begin{align*}
- (r^{N-1}\phi(|u'(r)|)u'(r))' &= \lambda r^{N-1}f(u(r)), \quad 0 < r < R \\
u'(0) &= u(R) = 0.
\end{align*}
\]

**Lemma 5.1.** Assume that \( \phi \) satisfies \((\phi_1)-(\phi_3)\). Then

\[
h(1) \min\{h^{-1}(s)^{\gamma_1-1}, h^{-1}(s)^{\gamma_2-1}\} \leq s \leq h(1) \max\{h^{-1}(s)^{\gamma_1-1}, h^{-1}(s)^{\gamma_2-1}\}, \quad s > 0.
\]
Proof. Condition \((\phi_3)\) implies that
\[
(\gamma_1 - 1) \frac{d}{dt} \ln t \leq \frac{d}{dt} \ln h(t) \leq (\gamma_2 - 1) \frac{d}{dt} \ln t, \; \forall \; t > 0.
\]
Let \(t \leq 1\). Integrating the previous inequality from \(t\) to 1, we get
\[
h(1)t^{\gamma_1 - 1} \leq h(t) \leq h(1)t^{\gamma_2 - 1}, \; t \leq 1.
\]
Let \(t \geq 1\). Integrating the previous inequality from 1 to \(t\), we get
\[
h(1)t^{\gamma_2 - 1} \leq h(t) \leq h(1)t^{\gamma_1 - 1}, \; \forall \; t \geq 1.
\]
Therefore
\[
h(1) \min \{t^{\gamma_1 - 1}, t^{\gamma_2 - 1}\} \leq h(t) \leq h(1) \max \{t^{\gamma_1 - 1}, t^{\gamma_2 - 1}\}, \; \forall \; t > 0.
\]
Letting \(t = h^{-1}(s)\), the lemma is proved.

**Lemma 5.2.** Assume \(\phi\) satisfies \((\phi_1)-(\phi_3)\). Then
\[
\Phi(1) \min \{t^{\gamma_1}, t^{\gamma_2}\} \leq \Phi(t) \leq \Phi(1) \max \{t^{\gamma_1}, t^{\gamma_2}\}, \; t > 0.
\]

Proof. From \((\phi_3)\),
\[
\gamma_1 t \phi(t) \leq t h'(t) + t \phi(t) \leq \gamma_2 t \phi(t), \; \forall \; t > 0,
\]
which implies, after integration from 0 to \(t\) that,
\[
\gamma_1 \leq \frac{t \Phi'(t)}{\Phi(t)} \leq \gamma_2, \; t > 0.
\]
(5.1)
The previous inequality is called condition \(\Delta_2\). To finish the proof, we proceed as in the proof of lemma 5.1 to conclude the desired inequality.

**Lemma 5.3.** Assume that \(\phi\) satisfies \((\phi_1)-(\phi_3)\). Then
\[
[h^{-1}]'(t) \leq \frac{t^{-\gamma_2 + 2}}{h(1)^{\gamma_2}(\gamma_1 - 1)}, \; t \leq 1.
\]

Proof. Remember that
\[
[h^{-1}]'(t) = \frac{1}{h'(h^{-1}(t))}, \; t > 0.
\]
(5.2)
From the proofs of Lemmas 5.1 and 5.2
\[
h(1)(\gamma_1 - 1) \min \{t^{\gamma_1 - 2}, t^{\gamma_2 - 2}\} \leq h'(t) \leq h(1)(\gamma_2 - 1) \max \{t^{\gamma_1 - 2}, t^{\gamma_2 - 2}\} \text{ for } t > 0.
\]
(5.3)
Gathering (5.2) and (5.3), we see that
\[
[h^{-1}]'(t) \leq \frac{[h^{-1}(t)]^{-\gamma_2 + 2}}{h(1)^{\gamma_2}(\gamma_1 - 1)}, \; t \leq 1.
\]
Now we use Lemma 5.1 to obtain
\[
[h^{-1}]'(t) \leq \frac{t^{-\gamma_2 + 2}}{h(1)^{\gamma_2}(\gamma_1 - 1)}, \; t \leq 1.
\]
Lemma 5.4. Assume that $\phi : (0, \infty) \rightarrow (0, \infty)$ is a differentiable function satisfying $(\phi_3)$. Then, there is a positive constant $\Gamma_1$ such that

$$
\sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i} (\eta) \xi_i \xi_j \geq \Gamma_1 |\eta| |\xi|^2,
$$

(5.4)

where $a_j(\eta) = \phi(|\eta|) \eta_j$, $\eta \in \mathbb{R}^N \setminus \{0\}$ and $\xi \in \mathbb{R}^N$.

Proof. Indeed, by $(\phi_3)$,

$$(\gamma_1 - 2) \phi(t) \leq t \phi'(t) \leq (\gamma_2 - 2) \phi(t).$$

(5.5)

Suppose first that $\gamma_1 < 2$. Note that

$$
\sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i} (\eta) \xi_i \xi_j = \phi(|\eta|) |\xi|^2 + \frac{\phi'(|\eta|) |\langle \eta, \xi \rangle|^2}{|\eta|}.
$$

(5.6)

If $\phi'(|\eta|) < 0$, then $\phi'(|\eta|) |\langle \eta, \xi \rangle|^2 \geq \phi'(|\eta|) |\eta|^2 |\xi|^2$. From (5.5) and (5.6),

$$
\sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i} (\eta) \xi_i \xi_j \geq (\gamma_1 - 1) \phi(|\eta|) |\xi|^2.
$$

If $\phi'(|\eta|) \geq 0$, then take $\Gamma_1 = 1$.

If $\gamma_1 \geq 2$, then (5.5) is satisfied with $\Gamma_1 = 1$, as can readily be seen from (5.6) and noting that $\phi'(t) \geq 0$ in this case.

We now prove a Simon type inequality.

Lemma 5.5. Assume that $\phi : (0, \infty) \rightarrow (0, \infty)$ is a differentiable function satisfying $(\phi_1)$-$(\phi_3)$. Then

$$
\langle \phi(|\eta|) \eta - \phi(|\eta'|) \eta', \eta - \eta' \rangle \geq \min \{4, 4\Gamma_1\} \frac{|\eta - \eta'|}{1 + |\eta| + |\eta'|} \Phi \left( \frac{|\eta - \eta'|}{4} \right),
$$

(5.7)

where $\Gamma_1$ was given in lemma 5.4, $\eta, \eta' \in \mathbb{R}^N$ and $\langle \cdot, \cdot \rangle$ denotes inner product.

Proof. If $\eta, \eta' = 0$ then (5.7) is obviously satisfied. If only one of them is 0, let’s say, $\eta' = 0$, then

$$
\phi(|\eta|) |\eta|^2 \geq \Phi(|\eta|) \geq 4 \Phi \left( \frac{|\eta|}{4} \right),
$$

where in the last inequalities we have used the properties of an N-function (note that an N-function is convex). So (5.7) is satisfied. If $\eta, \eta' \neq 0$, assume without loss of generality that $|\eta| \leq |\eta'|$. Then, an application of Cauchy-Schwartz inequality implies that

$$
\frac{|\eta - \eta'|}{4} \leq |t \eta + (1 - t) \eta'| \leq 1 + |\eta| + |\eta'|, \ t \in [0, 1/4].
$$
We conclude from the last inequality, (5.6) and the properties of an N-function that

\[ \langle \phi(|\eta|)\eta - \phi(|\eta'|)|\eta' - \eta' \rangle = \sum_{i=1}^{N} \int_{0}^{1} \frac{d}{dt}[a_j(t\eta + (1-t)\eta')]((\eta_j - \eta'_j))dt \]

\[ = \int_{0}^{1} \sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i} |t\eta + (1-t)\eta'_i|(|\eta_j - \eta'_j|)dt \]

\[ \geq \Gamma_1 \int_{0}^{1} \phi(|t\eta + (1-t)\eta'|)|\eta - \eta'|^2 dt \]

\[ \geq \Gamma_1 \int_{0}^{1/4} \phi(|t\eta + (1-t)\eta'|)|\eta - \eta'|^2 dt \]

\[ = \Gamma_1 \int_{0}^{1/4} \phi(|t\eta + (1-t)\eta'|)|\eta - \eta'|^2 |t\eta + (1-t)\eta'| dt \]

\[ \geq 4\Gamma_1 \frac{|\eta - \eta'|}{1 + |\eta| + |\eta'|} \phi \left( \frac{|\eta - \eta'|}{4} \right) \left( \frac{|\eta - \eta'|}{4} \right)^2 \]

\[ \geq 4\Gamma_1 \frac{|\eta - \eta'|}{1 + |\eta| + |\eta'|} \phi \left( \frac{|\eta - \eta'|}{4} \right) . \]

\[ \square \]

**Lemma 5.6.** Assume \( \phi \) satisfies \((\phi_1)-(\phi_3)\). Then, the function \( H(t) = t\Phi'(t) - \Phi(t) \) is strictly increasing and satisfies

\[ (\gamma_1 - 1)\Phi(t) \leq H(t) \leq (\gamma_2 - 1)\Phi(t), \quad t \geq 0, \]

\[ \frac{\gamma_1 - 1}{\gamma_1} t\Phi'(t) \leq H(t) \leq \frac{\gamma_2 - 1}{\gamma_2} t\Phi'(t), \quad t \geq 0. \]

**Proof.** Indeed, as \((\phi_3)\) is satisfied, we have that

\[ t\Phi'(t) - r\Phi'(r) > (t - r)\Phi'(t) > \int_{r}^{t} \tau\phi(\tau)d\tau, \quad t > r \geq 0, \]

which implies that \( H \) is strictly increasing. On the other hand, condition \((5.1)\) implies the desired inequalities.

\[ \square \]

**Proof of Lemma 4.3** Indeed, take \( \rho \in (0, d) \) and set

\[ K_\rho^\phi(d) = \{ u \in C([0, \epsilon]) \mid u(0) = d, \| u - d \|_\infty \leq \rho \}. \]

Take \( \epsilon > 0 \) small. If \( u \in K_\rho^\phi(d) \), then by continuity, \( u(r) > 0, r \in [0, \epsilon] \). Hence, for small \( \epsilon \), a solution of \((P_{\lambda,d})\) satisfies \( u'(r) \leq 0 \) for \( r \in [0, \epsilon] \) (this was showed in the proof of proposition \((4.1)\)) and

\[ u(r) = d - \int_{0}^{r} h^{-1}(t^{\alpha} \int_{0}^{t} \lambda r^\gamma f(u(\tau))d\tau) dt, \forall r \in [0, \epsilon]. \]

We infer that the solutions of \((P_{\lambda,d})\), for small \( \epsilon \), are fixed points of the operator

\[ T(u(r)) = d - \int_{0}^{r} h^{-1}(t^{\alpha} \int_{0}^{t} \lambda r^\gamma f(u(\tau))d\tau) dt, \forall r \in [0, \epsilon]. \]
Now we will verify that there exist $\epsilon, \rho > 0$ and $k \in (0, 1)$ such that
\[
T \left( K_\rho^\epsilon(d) \right) \subset K_\rho^\epsilon(d),
\]
and
\[
\|Tu - Tv\|_\infty \leq k\|u - v\|_\infty.
\]
Therefore, by the Banach Fixed Point Theorem, $T$ has a unique fixed point, which in turn will be a $C^2([0, \epsilon])$ solution of $(P_{\lambda, d, \epsilon})$. With respect to (5.8), let $\rho \in (0, d/2)$, which implies that $u(r) \in [d/2, 2d]$ for $u \in K_\rho^\epsilon(d)$. Therefore, for $u \in K_\rho^\epsilon(d)$ we have that
\[
h^{-1} \left( r^{-\alpha} \int_0^s \lambda t^\gamma f(u(t))dt \right) \leq h^{-1} \left( \frac{\lambda\|f\|_{\infty, d} s^{\gamma - \alpha + 1}}{\gamma + 1} \right), \quad s \in [0, \epsilon],
\]
where $\|f\|_{\infty, d} = \max_{s \in [d/2, 2d]} f(s)$. For small $\epsilon$, we can apply lemma 5.1 in the Appendix to conclude from the previous inequality that
\[
|T(u(r)) - T(u(0))| = \int_0^r h^{-1} \left( r^{-\alpha} \int_0^s \lambda t^\gamma f(u(t))dt \right) ds
\leq \int_0^r h^{-1} \left( \frac{\lambda\|f\|_{\infty, d} s^{\gamma - \alpha + 1}}{\gamma + 1} \right) ds
\leq \int_0^r h(1) \left( \frac{\lambda\|f\|_{\infty, d} s^{\gamma - \alpha + 1}}{\gamma + 1} \right) \frac{1}{s^{\gamma - \alpha + 1}} ds
= h(1) \left( \frac{\lambda\|f\|_{\infty, d}}{\gamma + 1} \right) \frac{1}{s^{\gamma - \alpha + 1}}, \quad r \in [0, \epsilon].
\]
As $\gamma \geq \alpha$, we obtain from the last inequality that there is $\epsilon > 0$ such that $Tu \in C([0, \epsilon])$ and $|T(u(r)) - d| \leq \rho$ for $r \in [0, \epsilon]$, which finishes the proof of (5.8). Now we pass to the proof of (5.9). We first prove it by assuming that $f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$. Fix $\rho$ as in (5.8) and take $u, v \in K_\rho^\epsilon(d)$. By the Mean Value Theorem, there is $h \in (0, 1)$ such that
\[
T(v(r)) - T(u(r)) = \int_0^r \left[ h^{-1} \left( s^{-\alpha} \int_0^s \lambda t^\gamma f(u(t))dt \right) - h^{-1} \left( s^{-\alpha} \int_0^s \lambda t^\gamma f(v(t))dt \right) \right] ds
= \int_0^r \left[ (h^{-1})' \left( s^{-\alpha} \int_0^s \lambda t^\gamma f(hu(t) + (1 - h)v(t))dt \right) \left( s^{-\alpha} \int_0^s \lambda t^\gamma f'(hu(t) + (1 - h)v(t))(u(t) - v(t))dt \right) \right] ds.
\]
Choose $\epsilon$ small in such a way that the number $s^{-\alpha} \int_0^s \lambda t^\gamma f(hu(t) + (1 - h)v(t))dt$ for $s \in [0, \epsilon]$ is small. Therefore, Lemma 5.3 and the last equality implies that for $1 < \gamma_2 \leq 2$ (note that in this case, the function $t \mapsto t^{\gamma_2 - 1}$ is decreasing)
\[ |T(v(r)) - T(u(r))| \leq \int_0^r \left[ c \left( s^{-\alpha} \int_0^s \lambda t^{\gamma} |f(hu(t) + (1 - h)v(t))| dt \right) \right. \\
\left. + \frac{s^{\frac{\gamma + 2}{\gamma - 1}}}{s^{\frac{\gamma + 2}{\gamma - 1}}} \left( s^{\alpha} \int_0^s \lambda t^{\gamma} |f'(hu(t) + (1 - h)v(t))| dt \right) \right] ds \leq \]

\[ \int_0^r \left[ c \left( s^{-\alpha} \int_0^s \lambda \|f\|_{\infty, d} t^{\gamma} dt \right) \right. \\
\left. + \frac{s^{\frac{\gamma + 2}{\gamma - 1}}}{s^{\frac{\gamma + 2}{\gamma - 1}}} \left( s^{\alpha} \int_0^s \lambda \|f'\|_{\infty, d} t^{\gamma} dt \right) \right] ds = \]

\[ \int_0^r \left[ c \left( \lambda \|f\|_{\infty, d} s^{-\frac{\alpha + \gamma + 1}{\gamma + 1}} \right) \right. \\
\left. + \frac{s^{\frac{\gamma + 2}{\gamma - 1}}}{s^{\frac{\gamma + 2}{\gamma - 1}}} \lambda \|f'\|_{\infty, d} s^{-\frac{\alpha + \gamma + 2}{\gamma + 1}} dt \right] \|u - v\|_{\infty} ds = \]

where \( \|f\|_{\infty, d} = \min_{s \in [d/2, d]} |f(s)| \) and \( \|f'\|_{\infty, d} = \max_{s \in [d/2, d]} |f'(s)| \). If on the other hand, we have that \( \gamma_2 \geq 2 \), i.e., \( t \mapsto t \frac{s^{\frac{\gamma + 2}{\gamma - 1}}}{s^{\frac{\gamma + 2}{\gamma - 1}}} \) is increasing then, we must conclude that

\[ |T(v(r)) - T(u(r))| \leq c \left( \lambda \|f\|_{\infty, d} \right) \frac{s^{\frac{\gamma + 2}{\gamma - 1}}}{s^{\frac{\gamma + 2}{\gamma - 1}}} \lambda \|f'\|_{\infty, d} \frac{s^{\frac{\alpha + \gamma + 2}{\gamma + 1}}}{s^{\frac{\alpha + \gamma + 2}{\gamma + 1}}} \|u - v\|_{\infty}, \]

where \( \|f\|_{\infty, d} = \max_{s \in [d/2, d]} f(s) \). In both cases, hypothesis (5.9) implies the existence of \( \epsilon \) such that (5.9) is true in the case \( f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) \).

\[ \Box \]

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