CLASSIFICATION OF A FAMILY OF NON ALMOST PERIODIC 
FREE ARAKI–WOODS FACTORS

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Abstract. We obtain a complete classification of a large class of non almost periodic free 
Araki–Woods factors $\Gamma(\mu, m)^{\prime\prime}$ up to isomorphism. We do this by showing that free Araki–
Woods factors $\Gamma(\mu, m)^{\prime\prime}$ arising from finite symmetric Borel measures $\mu$ on $\mathbb{R}$ whose atomic 
part $\mu_a$ is nonzero and not concentrated on $\{0\}$ have the joint measure class $\mathcal{C}(\bigvee_{k \geq 1} \mu^{*k})$ as an 
invariant. Our key technical result is a deformation/rigidity criterion for the unitary conjugacy 
of two faithful normal states. We use this to also deduce rigidity and classification theorems 
for free product von Neumann algebras.

1. Introduction

Free Araki-Woods factors are a free probability analog of the type III hyperfinite factors, just 
like free group factors are free probability analogs of the hyperfinite II$_1$ factor. The classification 
of hyperfinite type III factors has a beautiful history originating in the work of Powers [Po67]. 
Through the works of Connes [Co72], Haagerup [Ha85] and Krieger [Kr75], the classification 
question was ultimately reduced to the classification of ergodic actions of the additive group of 
real numbers $\mathbb{R}$, i.e., to classification of virtual subgroups of $\mathbb{R}$ in the sense of Mackey.

Following [Sh96], to every orthogonal representation $(U_t)_{t \in \mathbb{R}}$ of $\mathbb{R}$ on a real Hilbert space $H_\mathbb{R}$ 
is associated the free Araki–Woods factor $\Gamma(U, H_\mathbb{R})^{\prime\prime}$. For almost periodic representations, i.e. 
when $(U_t)$ is a direct sum of finite dimensional representations, the free Araki–Woods factors 
were completely classified in [Sh96] by Connes’ $S_d$ invariant [Co74], which is in this case equal 
to the subgroup of $(\mathbb{R}^*, \cdot)$ generated by the eigenvalues of $(U_t)$. Beyond the almost periodic 
case, the classification of free Araki–Woods factors is a very intriguing open problem and there 

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In this paper, we fully classify the free Araki–Woods factors in the case where the atomic 
part $\mu_a$ is nonzero and not concentrated on $\{0\}$ and where the continuous part $\mu_c$ satisfies
μ_ε * μ_c < μ_c. We find in particular that in that case, the free Araki–Woods factor does not depend on the multiplicity function m. But we also show that in other cases, Γ(μ, m)'' does depend on m.

In order to state our main results, we first introduce some terminology. For every σ-finite Borel measure μ on R, we denote by C(μ) the measure class of μ, defined as the set of all Borel sets U ⊂ R with μ(U) = 0. Note that C(μ) = C(ν) if and only if μ ∼ ν, while C(μ) ⊂ C(ν) if and only if ν < μ. For any sequence of measures (μ_k)_{k∈N}, we denote by ∪_{k∈N} μ_k any measure with the property that C(∪_{k∈N} μ_k) = ∩_{k∈N} C(μ_k). We denote by μ = μ_c + μ_a the unique decomposition of a measure μ as the sum of a continuous and an atomic measure.

We show that free Araki–Woods factors Γ(μ, m)'' arising from finite symmetric Borel measures μ on R whose atomic part μ_a is nonzero and not concentrated on {0} have the joint measure class C(∪_{k≥1} μ^k) as an invariant. More precisely, we obtain the following result.

**Theorem A.** Let μ, ν be finite symmetric Borel measures on R and m, n : R → N ∪ {+∞} symmetric Borel multiplicity functions. Assume that μ has at least one atom not equal to 0.

If the free Araki–Woods factors Γ(μ, m)'' and Γ(ν, n)'' are isomorphic, then there exists an isomorphism that preserves the free quasi-free states. In particular, the joint measure classes C(∪_{k≥1} μ^k) and C(∪_{k≥1} ν^k) are equal.

Denote by S(R) the set of all finite symmetric Borel measures μ = μ_c + μ_a on R satisfying the following two properties:

(i) μ_c * μ_c < μ_c and
(ii) μ_a ≠ 0 and supp(μ_a) ≠ {0}.

Denote by Λ(μ_a) the countable subgroup of R generated by the atoms of μ_a and by δ_Λ(μ_a) a finite atomic measure on R whose set of atoms equals Λ(μ_a).

Combining our Theorem A with the isomorphism Theorem 4.1 below, we obtain a complete classification of the free Araki–Woods factors arising from measures in S(R). Here and elsewhere in this paper, we call isomorphism between von Neumann algebras M and N any bijective *-isomorphism. Even when M and N are equipped with distinguished faithful normal states, isomorphisms are not assumed to preserve these states.

**Corollary B.** The set of free Araki–Woods factors

{Γ(μ, m)'' : μ ∈ S(R) and m : R → N ∪ {+∞} is a symmetric Borel multiplicity function}

is exactly classified, up to isomorphism, by the countable subgroup Λ(μ_a) and the measure class C(μ_c * δ_Λ(μ_a)).

Note that the measure class C(μ_c * δ_Λ(μ_a)) equals the set of Borel sets U ⊂ R satisfying μ_c(x + U) = 0 for all x ∈ Λ(μ_a) and, in particular, does not depend on the choice of δ_Λ(μ_a).

The family S(R) is large and provides many nonisomorphic free Araki–Woods factors having the same Connes’ invariants and in particular the same τ-invariant, see Example 5.4. Note that previously, only two non almost periodic free Araki–Woods factors having the same τ-invariant could be distinguished, see [Sh02, Theorem 5.6].

Combining our Corollary B with [HI15, Theorem A], we also obtain a complete classification for tensor products of free Araki–Woods factors arising from measures in S(R).

We then show that free Araki–Woods factors Γ(μ, m)'' arising from continuous finite symmetric Borel measures μ on R have all their centralizers amenable, i.e. the centralizer of any faithful normal state is amenable. More generally, we obtain the following result.
Corollary C. Let $\mu$ be any finite symmetric Borel measure on $\mathbb{R}$ and $m : \mathbb{R} \to \mathbb{N} \cup \{+\infty\}$ any symmetric Borel multiplicity function. The free Araki–Woods factor $\Gamma(\mu, m)$ has all its centralizers amenable if and only if the atomic part $\mu_a$ of $\mu$ is either zero or is concentrated on $\{0\}$ with $m(0) = 1$.

By [Ho08a], all free Araki–Woods factors $M$ satisfy Connes’ bicentralizer conjecture (see [Co80]) and thus, by [Ha85, Theorem 3.1], admit faithful normal states $\varphi$ such that $M^\varphi \subset M$ is an irreducible subfactor. So, having all centralizers amenable is the smallest centralizers can be in general.

In the setting of Corollary C and under the additional assumption that the Fourier transform of the continuous finite symmetric Borel measure $\mu_c$ vanishes at infinity, it was shown in [Ho08b, Theorem 1.2] that the continuous core of the corresponding free Araki–Woods factor $\Gamma(\mu, m)$ is solid (see [Oz03]), meaning that the relative commutant of any diffuse subalgebra that is the range of a faithful normal conditional expectation is amenable. Any type $\text{III}_1$ factor whose continuous core is solid has all its centralizers amenable. Observe that there are many free Araki–Woods factors arising in Corollary C whose Connes’ $\tau$-invariant (see [Co74]) is not the usual topology on $\mathbb{R}$. In particular, these free Araki–Woods factors have a continuous core that is not full (see [Co74, Sh02]) and hence not solid (see [Oz03, Proposition 7] with $N_0 = M$). Therefore, Corollary C provides many new examples of type $\text{III}_1$ factors whose centralizers are all amenable.

The following is an immediate consequence of Corollary C.

Corollary D. Let $\lambda$ be the Lebesgue measure on $\mathbb{R}$. Then $\Gamma(\lambda + \delta_0, 0)'' \neq \Gamma(\lambda + \delta_0, 2)''$. So in certain cases, the isomorphism class of $\Gamma(\mu, m)''$ depends on the multiplicity function $m$.

Our main technical tool to prove the results mentioned so far is a deformation/rigidity criterion for the unitary conjugacy of two faithful normal states on a von Neumann algebra $M$. In Corollary 3.2 below, we prove that a corner of the state $\psi$ is unitarily conjugate with a corner of the state $\varphi$ if and only if in the continuous core $c(M)$, there is a Popa intertwining bimodule (in the sense of [Po02, Po03]) between the canonical subalgebras $L_\psi(\mathbb{R})$ and $L_\varphi(\mathbb{R})$ of $c(M)$ given by realizing $c(M)$ as respectively $M \rtimes_{\varphi} \mathbb{R}$ and $M \rtimes_{\psi} \mathbb{R}$ (see Section 2 for details).

More generally, when $P \subset M$ is a von Neumann subalgebra that is the range of a faithful normal conditional expectation $E_P : M \to P$, we provide in Theorem 3.1 below a deformation/rigidity criterion describing when a state $\psi$ on $M$ has a corner that is unitarily conjugate with a corner of a state of the form $\theta \circ E_P$. Applying this criterion to a free product von Neumann algebra $M$, we obtain the following complete characterization of when $M$ has all its centralizers amenable.

Theorem E. For $i = 1, 2$, let $(M_i, \varphi_i)$ be a von Neumann algebra with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) \ast (M_2, \varphi_2)$ their free product. Then $M$ has all its centralizers amenable if and only if both $M_1$ and $M_2$ have all their centralizers amenable and $M^\varphi$ is amenable.

Finally, we use the same criterion, in combination with methods of [Sh97a], to prove the following classification result for a free product with a free Araki–Woods factor.

Theorem F. Let $\mu$ be a continuous finite symmetric Borel measure on $\mathbb{R}$. Fix the free Araki–Woods factor $(M, \varphi) = (\Gamma(\mu, +\infty)', \varphi_{\mu, +\infty})$, where $+\infty$ denotes the multiplicity function equal to $+\infty$ everywhere.

(i) If $(A, \tau)$ and $(B, \tau)$ are nonamenable $\text{II}_1$ factors with their tracial states, then the free products $(M, \varphi) \ast (A, \tau)$ and $(M, \varphi) \ast (B, \tau)$ are isomorphic (not necessarily in a state preserving way) if and only if there exists a $t > 0$ such that $A \cong B^t$. 

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(ii) If \((A_i, \psi_i), \ i = 1, 2,\) are full type III factors with almost periodic states having a factorial centralizer \(A_i^{\psi_i},\) then the free products \((M, \varphi) * (A_1, \psi_1)\) and \((M, \varphi) * (A_2, \psi_2)\) are isomorphic (again, not necessarily in a state preserving way) if and only if \(A_1 \cong A_2.\)

By Theorem F, the free Araki–Woods factors \(\Gamma((\lambda, +\infty) + (\delta_0, m))''\) are isomorphic for all \(2 \leq m < +\infty,\) but the question whether these are isomorphic with \(\Gamma((\lambda, +\infty) + (\delta_0, +\infty))''\) is equivalent with the free group factor problem \(L(F_m) \cong L(F_\infty).\)

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**Disclaimer.** Some of the results in this paper, in particular Corollary D, Theorem 4.1 and Proposition 7.5, were obtained in the preprint [Sh03] by the second named author. That preprint will remain unpublished and has been incorporated in this article.

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2. Preliminaries

For any von Neumann algebra \(M,\) we denote by \(\mathcal{U}(M)\) its group of unitaries. For any (possibly unbounded) positive selfadjoint closed operator \(A\) on a separable Hilbert space \(H,\) we denote by \(C(A)\) the measure class on \(\mathbb{R}\) of the spectral measure of \(\log(A),\) i.e. the set of all Borel sets \(\mathcal{U} \subset \mathbb{R}\) such that the spectral projection \(1_{\mathcal{U}}(\log(A))\) equals 0. Here \(1_{\mathcal{U}}\) denotes the function that is equal to 1 on \(\mathcal{U}\) and equal to 0 elsewhere. Note that we can always choose a measure \(\mu\) on \(\mathbb{R}\) such that \(C(A) = C(\mu).\)

**Free Araki–Woods factors.** Following [Sh96], we associate to every orthogonal representation \((U_t)_{t \in \mathbb{R}}\) of \(\mathbb{R}\) on the real Hilbert space \(H_\mathbb{R}\) the free Araki–Woods factor \(\Gamma(H_\mathbb{R}, U_t)'',\) equipped with the free quasi-free state \(\varphi_U.\)

Denoting by \(H = H_\mathbb{R} + iH_\mathbb{R}\) the complexification of \(H_\mathbb{R},\) define the positive nonsingular operator \(\Delta\) on \(H\) such that \(U_t = \Delta^{it}\) for all \(t \in \mathbb{R}.\) Also define the anti-unitary operator \(J : H \to H : J(\xi + i\eta) = \xi - i\eta\) for all \(\xi, \eta \in H_\mathbb{R}.\) Then, \(J\Delta J = \Delta^{-1}.\) Therefore, the measure class \(C(\Delta)\) of the spectral measure of \(\log(\Delta)\) and the multiplicity function \(m : \mathbb{R} \to \mathbb{N} \cup \{+\infty\}\) of \(\log(\Delta)\) are symmetric. This measure class and multiplicity function completely classify orthogonal representations of \(\mathbb{R}\) on separable real Hilbert spaces. We therefore use the notation \((\Gamma(\mu, m)'', \varphi_{\mu, m})\) to denote the free Araki–Woods factor and its free quasi-free state associated with the unique orthogonal representation with spectral invariant \((\mu, m).\)
Background on $\sigma$-finite von Neumann algebras. Let $M$ be any $\sigma$-finite von Neumann algebra with predual $M_*$ and $\varphi \in M_*$ any faithful normal state. We denote by $\sigma^\varphi_\tau$ the modular automorphism group of the state $\varphi$, defined by the formula $\sigma^\varphi_\tau = \text{Ad}(\Delta^\varphi_\tau)$ for all $t \in \mathbb{R}$.

The centralizer $M^\varphi$ of the state $\varphi$ is by definition the fixed point algebra of $(M, \sigma^\varphi)$.

The continuous core of $M$ with respect to $\varphi$, denoted by $c_\varphi(M)$, is the crossed product von Neumann algebra $M \rtimes_\varphi \mathbb{R}$. The natural inclusion $\pi_\varphi : M \to c_\varphi(M)$ and the unitary representation $\lambda^\varphi : \mathbb{R} \to c_\varphi(M)$ satisfy the covariance relation:

$$\lambda^\varphi(t) \pi_\varphi(x) \lambda^\varphi(t)^* = \pi_\varphi(\sigma^\varphi_\tau(x))$$

for all $x \in M$ and all $t \in \mathbb{R}$.

Put $L^\varphi_\varphi(\mathbb{R}) := \lambda^\varphi(\mathbb{R})''$. There is a unique faithful normal conditional expectation $E_{L^\varphi_\varphi(\mathbb{R})} : c_\varphi(M) \to L^\varphi_\varphi(\mathbb{R})$ satisfying $E_{L^\varphi_\varphi(\mathbb{R})}(\pi_\varphi(x) \lambda^\varphi(t)) = \varphi(x) \lambda^\varphi(t)$ for all $x \in M$ and all $t \in \mathbb{R}$.

The faithful normal semifinite weight defined by $f \mapsto \int \exp(-s) f(s) \, ds$ on $L^\infty(\mathbb{R})$ gives rise to a faithful normal semifinite weight $\text{Tr}_\varphi$ on $L^\varphi_\varphi(\mathbb{R})$ via the Fourier transform. The formula $\text{Tr}_\varphi \circ \pi_\varphi = E_{L^\varphi_\varphi(\mathbb{R})} \circ \pi_\varphi$ extends it to a faithful normal semifinite trace on $c_\varphi(M)$.

Because of Connes’ Radon–Nikodym cocycle theorem [Co72, Théorème 1.2.1] (see also [Ta03, Theorem VIII.3.3]), the semifinite von Neumann algebra $c_\varphi(M)$ together with its trace $\text{Tr}_\varphi$ does not depend on the choice of $\varphi$ in the following precise sense. If $\psi \in M_*$ is another faithful state, there is a canonical isomorphism $\Pi_{\varphi, \psi} : c_\psi(M) \to c_\varphi(M)$ of $c_\varphi(M)$ onto $c_\varphi(M)$ such that $\Pi_{\varphi, \psi} \circ \pi_\psi = \pi_\varphi$ and $\text{Tr}_\varphi \circ \Pi_{\varphi, \psi} = \text{Tr}_\psi$. Note however that $\Pi_{\varphi, \psi}$ does not map the subalgebra $L^\varphi_\varphi(\mathbb{R}) \subset c_\varphi(M)$ onto the subalgebra $L^\psi_\psi(\mathbb{R}) \subset c_\psi(M)$ (and hence we use the symbol $L^\varphi_\varphi(\mathbb{R})$ instead of the usual $L(\mathbb{R})$). We have $\Pi_{\varphi, \psi}(\lambda^\varphi(t)) = \lambda^\psi(w_t) \lambda^\varphi(t)$ for every $t \in \mathbb{R}$, where $w_t = [D \psi : D \varphi]|_t$ is Connes’ Radon–Nikodym cocycle between $\psi$ and $\varphi$.

Lemma 2.1. Let $M$ be any $\sigma$-finite von Neumann algebra and $\varphi \in M_*$ any faithful state such that $M^\varphi$ is a $\text{II}_1$ factor. Let $M^\varphi \subset P \subset M$ be any intermediate von Neumann subalgebra that is globally invariant under the modular automorphism group $\sigma^\varphi$. Denote by $E_P : M \to P$ the unique $\varphi$-preserving conditional expectation and write $M \subset P := \ker(E_P)$.

Let $p \in M^\varphi$ be any nonzero projection and put $\varphi_p := \varphi(\frac{p}{\varphi(p)}) \in (pM)_\varphi$.

Then we have

$$C(\Delta^\varphi) = C(\Delta^\varphi_p) \quad \text{and} \quad C(\Delta^\varphi|_{L^2(M) \otimes L^2(P)}) = C(\Delta^\varphi_p|_{L^2(pM) \otimes L^2(pP)}).$$

Proof. Fix a standard representation $M \subset B(H)$ and denote by $\xi_\varphi \in H$ the canonical unit vector that implements $\varphi$. For every $x \in M$, denote by $\mu^\varphi_x$ the unique finite Borel measure on $\mathbb{R}$ that satisfies

$$\varphi(x^* \sigma^\varphi_\tau(x)) = \langle \Delta^\varphi_\tau(x \xi_\varphi), x \xi_\varphi \rangle = \int \exp(ist) \, d\mu^\varphi_x(s) \quad \text{for all} \quad t \in \mathbb{R}.$$

For any Borel subset $U \subset \mathbb{R}$, we have $U \in C(\Delta^\varphi)$ (resp. $U \in C(\Delta^\varphi|_{L^2(M) \otimes L^2(P)})$) if and only if $\mu^\varphi_x(U) = 0$ for all $x \in M$ (resp. for all $x \in M \subset P$). Since $pM_p \subset M$ and $p(M \subset P)p \subset M \subset P$, it is clear that $C(\Delta^\varphi) \subset C(\Delta^\varphi_p)$ (resp. $C(\Delta^\varphi|_{L^2(M) \otimes L^2(P)}) \subset C(\Delta^\varphi_p|_{L^2(pM) \otimes L^2(pP)})$).

It remains to prove that $C(\Delta^\varphi_p) \subset C(\Delta^\varphi)$ (resp. $C(\Delta^\varphi_p|_{L^2(pM) \otimes L^2(pP)}) \subset C(\Delta^\varphi|_{L^2(M) \otimes L^2(P)})$).

Up to shrinking $p \in M^\varphi$ if necessary and since we have $p_2 M_p p_2 \subset p_1 M_p p_1$ whenever $p_2 \leq p_1$ with $p_1, p_2$ nonzero projections in $M^\varphi$, we may assume without loss of generality that $\varphi(p) = m^{-1}$ with $m \in \mathbb{N}$. Since $M^\varphi$ is a $\text{II}_1$ factor, we may find partial isometries $u_1, \ldots, u_m \in M^\varphi$ such that $u_i = p$, $u_i^* u_j = p$ for all $1 \leq i, j \leq m$ and $\sum_{j=1}^m u_j u_j^* = 1$.

Let $x \in M$ (resp. $x \in M \subset P$). Since $\varphi(x^* \sigma^\varphi_\tau(x)) = \varphi(p) \sum_{i,j=1}^m \varphi((u_j^* x u_i)^* \sigma^\varphi_\tau(u_j^* x u_i))$ for all $t \in \mathbb{R}$, we have $\mu^\varphi_x = \varphi(p) \sum_{i,j=1}^m \mu^\varphi_{u_j^* x u_i}$. If $x \in M \subset P$, then $u_j^* x u_i \in p(M \subset P)p$ for all $1 \leq i, j \leq m$. This implies that $C(\Delta^\varphi_p) \subset C(\Delta^\varphi)$ (resp. $C(\Delta^\varphi_p|_{L^2(pM) \otimes L^2(pP)}) \subset C(\Delta^\varphi|_{L^2(M) \otimes L^2(P)})$) and finishes the proof. □
**Popa’s intertwining-by-bimodules.** Popa introduced his *intertwining-by-bimodules* theory in [Po02, Po03]. In the present work, we make use of this theory in the context of semifinite von Neumann algebras. We introduce the following terminology. Let $M$ be any $\sigma$-finite semifinite von Neumann algebra endowed with a fixed faithful normal semifinite trace $\Tr$. Let $1_A$ and $1_B$ be any nonzero projections in $M$ and let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be any von Neumann subalgebras. Assume that $\Tr(1_A) < +\infty$ and that $\Tr|_B$ is semifinite.

We say that $A$ embeds into $B$ inside $M$ and write $A \precsim_M B$ if there exist a projection $e \in A$, a finite trace projection $f \in B$, a nonzero partial isometry $v \in eMf$ and a unital normal homomorphism $\theta : eAe \to fBf$ such that $av = v\theta(a)$ for all $a \in eAe$. We use the following useful characterization [Po02, Po03] (see also [HR10, Lemma 2.2]).

**Theorem 2.2.** Keep the same notation as above. Denote by $E_B : 1_B M 1_B \to B$ the unique trace preserving conditional expectation. Then the following conditions are equivalent.

(i) $A \precsim_M B$.

(ii) There exists no net $(w_i)_{i \in I}$ of unitaries in $U(A)$ such that $\lim_i \|E_B(y^*w_ix)\|_2 = 0$ for all $x, y \in 1_A M 1_B$.

3. A CRITERION FOR THE UNITARY CONJUGACY OF STATES

Recall that for any von Neumann algebra $N$, any $\theta \in N$, and any $a, b \in N$, we define $(a\theta b)(y) := \theta(bya)$ for every $y \in N$.

**Theorem 3.1.** Let $M$ be a von Neumann algebra with a faithful normal state $\varphi \in M_\ast$ and $P \subset M$ a von Neumann subalgebra that is the range of a $\varphi$-preserving conditional expectation $E_P : M \to P$. Let $\psi \in M_\ast$ be another faithful normal state and $q \in M^\psi$ a nonzero projection. Then the following statements are equivalent.

(i) There exists a nonzero finite trace projection $r \in L_\psi(R)$ such that

$$\Pi_{\varphi,\psi}(L_\psi(R)qr) \precsim_{c_{\varphi}(M)} c_{\varphi}(P).$$

(ii) There exist a faithful normal positive functional $\theta \in P_\ast$ and a nonzero partial isometry $v \in M$ such that $p = vv^* \in M^{\varphi\theta E_P}$, $q_0 = v^*v \in qM^\psi q$ and $\psi q_0 = v^*(\theta \circ E_P)v$.

**Proof.** Assume that (i) holds. Write $w_t = [D\psi : D\varphi]_t$. We claim that there exists a $\delta > 0$ and $x_1, \ldots, x_k \in qM$ such that

$$\sum_{i,j=1}^k \varphi( E_P(x_i^* w_t \sigma_t^\varphi(x_j)) E_P(x_i^* w_t \sigma_t^\varphi(x_j)^*) ) \geq \delta$$

for all $t \in \mathbb{R}$. Assuming that the claim is false, we prove that (i) does not hold. Take a net $t_i \in \mathbb{R}$ such that

$$\lim_i \varphi( E_P(x^* w_{t_i} \sigma_{t_i}^\varphi(y)) E_P(x^* w_{t_i} \sigma_{t_i}^\varphi(y)^*) ) = 0$$

for all $x, y \in qM$. Using the 2-norm $\| \cdot \|_2$ w.r.t. the canonical trace on $c_{\varphi}(M)$, we get that for all finite trace projections $p, p' \in L_\varphi(R)$ and all $x, y \in qM$,

$$\|p E_P(x^* w_{t_i} \sigma_{t_i}^\varphi(y)) E_P(x^* w_{t_i} \sigma_{t_i}^\varphi(y)^*) p'\|_2 \leq \|p E_P(x^* w_{t_i} \sigma_{t_i}^\varphi(y))\|_2^2 = \Tr(p) \varphi( E_P(x^* w_{t_i} \sigma_{t_i}^\varphi(y)) E_P(x^* w_{t_i} \sigma_{t_i}^\varphi(y)^*) ) \to 0.$$
We then also get for all finite trace projections $p, p' \in L_\varphi(R)$, all $s, s' \in R$ and all $x, y \in M$ that

$$\|E_{c_\varphi(p)}(p \lambda_\varphi(s)^* x^* \Pi_{c_\varphi,\psi}(\lambda_\varphi(t)q)y \lambda_\varphi(s') p')\|_2 = \|\lambda_\varphi(s)^* p E_p((qx)^* w_{t_i} \sigma_i^\varphi(qy)) p' \lambda_\varphi(t_i + s')\|_2$$

$$= \|p E_p((qx)^* w_{t_i} \sigma_i^\varphi(qy)) p'\|_2 \to 0 .$$

The linear span of such elements $x \lambda_\varphi(s)p$ is dense in $L^2(c_\varphi(M), \text{Tr})$. So whenever $r \in L_\psi(R)$ is a finite trace projection and $a, b \in c_\varphi(M)$, we first approximate $\Pi_{c_\varphi,\psi}(r) b$ in $\| \cdot \|_2$ by a linear combination of $y \lambda_\varphi(s')p'$ and conclude that

$$\|E_{c_\varphi(p)}(p \lambda_\varphi(s)^* x^* \Pi_{c_\varphi,\psi}(\lambda_\varphi(t)q)r) b\|_2 \to 0$$

for all finite trace projections $p \in L_\varphi(R)$ and all $s \in R$, $x \in M$. We then approximate $\Pi_{c_\varphi,\psi}(r) a$ in $\| \cdot \|_2$ by a linear combination of $x \lambda_\varphi(s)p$ and conclude that

$$\|E_{c_\varphi(p)}(a^* \Pi_{c_\varphi,\psi}(\lambda_\varphi(t)q)r) b\|_2 \to 0 .$$

Applying Theorem 2.2, we conclude that (i) does not hold. This concludes the proof of the claim.

Fix $\delta > 0$ and $x_1, \ldots, x_k$ such that (3.1) holds for all $t \in R$. Denote by $\langle M, e_P \rangle$ the basic construction for $P \subset M$, i.e. the von Neumann algebra acting on $L^2(M, \varphi)$ generated by $M$ acting by left multiplication and the orthogonal projection $e_P : L^2(M, \varphi) \to L^2(P, \varphi)$. As in [ILP96, Section 2.1], denote by $\hat{\varphi}$ the canonical faithful normal semifinite weight on $\langle M, e_P \rangle$ characterized by

$$\sigma_i^\hat{\varphi}(x e_P y) = \sigma_i^\varphi(x) e_P \sigma_i^\varphi(y) \quad \text{and} \quad \hat{\varphi}(x e_P y) = \varphi(x y)$$

for all $x, y \in R$. We have $\sigma_i^\hat{\varphi}(T) = \Delta_i^u T \Delta_i^{-u}$ for all $T \in \langle M, e_P \rangle$ and all $t \in R$. In particular, $\sigma_i^\hat{\varphi}(x) = \sigma_i^\varphi(x)$ for all $x \in M$. Denote by $T_M$ the unique faithful normal semifinite operator valued weight from $\langle M, e_P \rangle$ to $M$ such that $\hat{\varphi} = \varphi \circ T_M$. Note that $T_M(e_P) = 1$.

Define $\hat{\psi} = \psi \circ T_M$. So, $\hat{\psi}$ is a faithful normal semifinite weight on $\langle M, e_P \rangle$ and by [Ha77, Theorem 4.7], we have

$$[D \hat{\psi} : D \hat{\varphi}]_t = [D \psi : D \varphi]_t = w_t$$

for all $t \in R$. In particular, we get that

$$\sigma_i^\hat{\psi}(x e_P y) = w_t \sigma_i^\varphi(x) e_P \sigma_i^\varphi(y) w_t^*$$

for all $x, y \in M$ and $t \in R$.

Define the positive element $X \in q\langle M, e_P \rangle q$ by

$$X = \sum_{j=1}^k x_j e_P x_j^* .$$

Also define the normal positive functional $\Omega \in \langle M, e_P \rangle^*$ given by

$$\Omega(T) = \sum_{i=1}^k \hat{\varphi}(e_P x_i^* T x_i e_P)$$

for all $T \in \langle M, e_P \rangle$. Using (3.2) and (3.1), we get for every $t \in R$,

$$\Omega(\sigma_i^\hat{\psi}(X)) = \sum_{i,j=1}^k \hat{\varphi}(e_P x_i^* w_t \sigma_i^\varphi(x_j) e_P \sigma_i^\varphi(x_j)^* w_t^* x_i e_P)$$

$$= \sum_{i,j=1}^k \varphi(E_P(x_i^* w_t \sigma_i^\varphi(x_j)) E_P(x_i^* w_t \sigma_i^\varphi(x_j)^*)) \geq \delta .$$
Define $K$ as the $\sigma$-weakly closed convex hull of $\{\sigma_t^\hat{\phi}(X) : t \in \mathbb{R}\}$ inside $q(M,e_p)q$. Note that $\|Y\| \leq \|X\|$ for all $Y \in K$. Also, every $Y \in K$ is positive and satisfies $\psi(Y) \leq \hat{\psi}(X) < +\infty$, by the $\sigma_t^\hat{\phi}$-invariance and $\sigma$-weak lower semicontinuity of $\hat{\psi}$. We then also have

$$\hat{\psi}(Y^*Y) = \hat{\psi}(Y^2) \leq \|Y\| \hat{\psi}(Y) \leq \|X\| \hat{\psi}(X)$$

for all $Y \in K$. By [HI15, Lemma 4.4], the image of $K$ in $L^2(\langle M,e_p\rangle,\hat{\psi})$ is norm closed. So, there is a unique element $X_0 \in K$ where the function $Y \mapsto \hat{\psi}(Y^*Y)$ attains its minimal value. Since this function is $\sigma_t^\hat{\phi}$-invariant, it follows that $\sigma_t^\hat{\phi}(X_0) = X_0$ for all $t$. Since $\Omega(\sigma_t^\hat{\phi}(X)) \geq \delta$ for all $t \in \mathbb{R}$, also $\Omega(X_0) \geq \delta$, so that $X_0 \neq 0$. Since $T_M \circ \sigma_t^\psi = \sigma_t^\psi \circ T_M$ and since $T_M$ is $\sigma$-weakly lower semicontinuous, we get that $\|T_M(Y)\| \leq \|T_M(X)\| < +\infty$ for all $Y \in K$. In particular, $\|T_M(X_0)\| < +\infty$.

Take $\varepsilon > 0$ small enough such that the spectral projection $e = 1_{(\varepsilon, +\infty)}(X_0)$ is nonzero. It follows that $e$ is a projection in $q(M,e_p)q$ satisfying $\sigma_t^\hat{\phi}(e) = e$ for all $t \in \mathbb{R}$ and $\|T_M(e)\| < +\infty$. By Lemma 3.3 below, we may assume that $e \preceq e_p$ inside $\langle M,e_p\rangle$. Take a partial isometry $V \in \langle M,e_p\rangle$ such that $V^*V = e$ and $VV^* \leq e_p$. Let $p_0 \in P$ be the unique projection such that $VV^* = p_0 e_p$. We get that $V = p_0 V$. Since $e \leq q$, we also have $V = V_q$.

Since $\|T_M(V^*V)\| = \|T_M(e)\| < +\infty$, it follows from the push down lemma [ILP96, Proposition 2.2] (where the factoriality assumption is unnecessary) that

$$V = e_p V = e_p T_M(e_p V) = e_p T_M(V).$$

Write $v = T_M(V)$. Then, $v \in M$ and $e = e_{p,v}$. By construction, $v \in p_0 M q$.

Since $e_p \langle M,e_p \rangle e_p = Pe_p$, we can uniquely define $u_t \in P$ such that

$$u_t e_p = V w_t \sigma_t^\hat{\phi}(V^*)$$

for all $t \in \mathbb{R}$. Since $V^*V = e$ and $w_t \sigma_t^\hat{\phi}(V^*) w_t^* = \sigma_t^\hat{\phi}(e) = e$, we get that $u_t u_t^* = p_0$ and $u_t^* u_t = \sigma_t^\hat{\phi}(p_0)$ for all $t \in \mathbb{R}$. Also, $t \mapsto u_t$ is strongly continuous and

$$u_t \sigma_t^\hat{\phi}(u_s) e_p = u_t e_p \sigma_t^\hat{\phi}(u_s e_p) = V w_t \sigma_t^\hat{\phi}(V^*) \sigma_t^\hat{\phi}(V w_s \sigma_s^\hat{\phi}(V^*)) = V w_t \sigma_t^\hat{\phi}(V^*) \sigma_t^\hat{\phi}(V w_s \sigma_s^\hat{\phi}(V^*)) = V \sigma_t^\hat{\phi}(e) w_t \sigma_t^\hat{\phi}(w_s) \sigma_t^\hat{\phi}(V^*) = V w_t \sigma_t^\hat{\phi}(V^*) \sigma_t^\hat{\phi}(V w_s \sigma_s^\hat{\phi}(V^*))$$

for all $s,t \in \mathbb{R}$. So, $(u_t)_{t \in \mathbb{R}}$ is a 1-cocycle for $\varphi|_p$. By [Co72, Théorème 1.2.4] (see also [Ta03, Theorem VIII.3.21] for a formulation adapted to non faithful states), there is a unique faithful normal semifinite weight $\theta$ on $p_0 P_0 p_0$ such that $[D\theta : D\varphi|_p]_t = u_t$ for all $t \in \mathbb{R}$. Define the faithful normal semifinite weight $\theta_1$ on $p_0 M p_0$ by $\theta_1 = \theta \circ E_p$. By [Ha77, Theorem 4.7], we have that $[D\theta_1 : D\varphi|_t] = u_t$ for all $t \in \mathbb{R}$.

Since $u_t \in P$, we get that

$$e_p u_t \sigma_t^\hat{\phi}(v) = u_t e_p \sigma_t^\hat{\phi}(v) = u_t e_p \sigma_t^\hat{\phi}(V)$$

for all $t \in \mathbb{R}$. Replacing $v$ by its polar part, we may assume that $v \in M$ is a partial isometry such that $p_1 = vv^* \in (p_0 M p_0)_{\theta_1}$ and $q_1 = v^* v \in q M q$. We then get that

$$[D(v^* \theta_1 v) : D\varphi|_t] = v^* [D\theta_1 : D\varphi|_t] \sigma_t^\hat{\phi}(v) = v^* u_t \sigma_t^\hat{\phi}(v) = q_1 [D\psi : D\varphi|_t] = [D(\psi q_1) : D\varphi|_t]$$

for all $t \in \mathbb{R}$. We conclude that $\psi q_1 = v^* \theta_1 v$.

In particular, $\theta(E_p(p_1)) = \theta_1(p_1) = \psi(q_1) < +\infty$. Also, $\sigma_t^\theta(E_p(p_1)) = E_p(\sigma_t^{\theta_1}(p_1)) = E_p(p_1)$, for all $t \in \mathbb{R}$. We can thus find a nonzero spectral projection $f$ of $E_p(p_1)$ such that $fv \neq 0$.
\( \theta(f) < +\infty \) and \( f \in (p_0 p_0^\delta) \). Since \( u_t \sigma_f^t(v) = vw_t \), \( [D\theta : D\varphi]_p t = u_t \) and \( f \in (p_0 p_0^\delta) \), we get that
\[
u_t \sigma_f^t(fv) = f u_t \sigma_f^t(v) = f vw_t.
\]
We then replace \( \theta \) by \( \theta f \) and \( v \) by the polar part of \( fv \). Then, \( \theta \) is a faithful normal positive functional on \( f Pf \), the projection \( p = vv^* \) belongs to \( (fMf)^{\theta E_P} \), the projection \( q_0 = v^*v \) belongs to \( qM^\psi q \) and \( \psi q_0 = v^*(\theta \circ E_P)v \). Adding to \( \theta \) an arbitrary faithful normal state on \( (1 - f)P(1 - f) \), it follows that (ii) holds.

Conversely, assume that (ii) holds. Take \( \theta, q_0, p \) and \( v \) as in the statement of (ii). Define \( w_t = [D\psi : D\varphi]_t \) and \( u_t = [D\theta : D\varphi]_p t \). Since \( \psi q_0 = v^*(\theta \circ E_P)v \), we get that \( vw_t = u_t \sigma_f^t(v) \) for all \( t \in R \). This means that
\[
v \Pi_{\varphi,\psi}(\lambda_\psi(t)) = \Pi_{\varphi\theta}(\lambda_\theta(t)) v
\]
for all \( t \in R \). Also, \( v \Pi_{\varphi,\psi}(qr) = v \Pi_{\varphi,\psi}(r) \neq 0 \) for every nonzero finite trace projection \( r \in L_\psi(R) \). We conclude that
\[
\Pi_{\varphi,\psi}(L_\psi(R)qr) \prec_{c_\varphi(M)} c_\varphi(P)
\]
for every nonzero finite trace projection \( r \in L_\psi(R) \), so that (i) holds.

Applying Theorem 3.1 to the case \( P = C1 \), we get the following result.

**Corollary 3.2.** Let \( M \) be a von Neumann algebra with faithful normal states \( \psi, \varphi \in M_* \) and let \( q \in M^\psi \) be a nonzero projection. Then the following statements are equivalent.

(i) There exists a nonzero finite trace projection \( r \in L_\psi(R) \) such that
\[
\Pi_{\varphi,\psi}(L_\psi(R)qr) \prec_{c_\varphi(M)} L_\psi(R).
\]

(ii) There exists a nonzero partial isometry \( v \in M \) such that \( p = vv^* \) belongs to \( M^\varphi \), \( q_0 = v^*v \) belongs to \( qM^\psi q \), and
\[
\frac{1}{\psi(q_0)} \psi(q_0) = \frac{1}{\varphi(p)} v^* v.
\]

**Lemma 3.3.** Let \( \psi \) be a faithful normal semifinite weight on a von Neumann algebra \( N \) and \( e \in N^\psi \) a projection satisfying \( 0 < \psi(e) < +\infty \). Let \( e_1 \in N \) be any projection with central support equal to 1. Then there exists a nonzero projection \( e_0 \in eN^\psi e \) satisfying \( e_0 \sim e_1 \) inside \( N \).

**Proof.** Since the central support of \( e_1 \) equals 1, we can find a nonzero projection \( f \in N \) such that \( f \leq e \) and \( f \sim e_1 \). Define the faithful normal state \( \theta \) on \( eN e \) given by \( \theta(x) = \psi(e)^{-1} \psi(x) \) for all \( x \in eN e \). By [HU15, Lemma 2.1], there exists a projection \( e_0 \in (eN e)^\theta \) such that \( e_0 \sim f \) inside \( eN e \). Then, \( e_0 \) is a nonzero projection in \( eN^\psi e \) and \( e_0 \sim e_1 \) inside \( N \). \( \square \)

4. Isomorphisms of free Araki–Woods factors

The isomorphism part of Corollary B follows from the following result that we deduce from [Sh96].

**Theorem 4.1.** Let \( \mu \) be any finite symmetric Borel measure on \( R \) and \( m : R \to N \cup \{ +\infty \} \) any symmetric Borel multiplicity function. Denote by \( \Lambda \) the subgroup of \( R \) generated by the atoms of \( \mu \) and assume that \( \Lambda \neq \{0\} \). There is an isomorphism
\[
\Gamma(\mu, m)' \cong \Gamma(\mu \ast \delta_\Lambda, +\infty)'
\]
preserving the free quasi-free states, where \( \delta_\Lambda \) denotes any atomic finite symmetric Borel measure on \( R \) with set of atoms equal to \( \Lambda \).
Proof. For every $0 < a < 1$, we denote by $B_a$ the von Neumann algebra $B(\ell^2(\mathbb{N}))$ equipped with the faithful normal state $\theta_a$ given by

$$\theta_a(T) = (1 - a) \sum_{k=0}^{\infty} a^k \langle T(\delta_k), \delta_k \rangle,$$

where $(\delta_k)_{k \in \mathbb{N}}$ is the standard orthonormal basis of $\ell^2(\mathbb{N})$. Throughout the proof of the theorem, we always assume that a free Araki–Woods factor comes with its free quasi-free state and all free products are taken w.r.t. the canonical states that we fixed. We always equip a free product with the free product state and a tensor product with the tensor product state.

We first prove that for every $0 < a < 1$, for all finite symmetric Borel measures $\mu$ on $\mathbb{R}$ and all symmetric Borel multiplicity functions $m : \mathbb{R} \rightarrow \mathbb{N} \cup \{+\infty\}$, there exists a state preserving isomorphism

$$(4.1) \quad \Gamma(\mu, m)^{\prime\prime} * B_a \cong \Gamma(\mu * \delta_{\log(a)}, +\infty)^{\prime\prime} \otimes B_a.$$ 

To prove (4.1), fix an orthogonal representation $(U_t)_{t \in \mathbb{R}}$ of $\mathbb{R}$ on a real Hilbert space $H_R$ having $(\mu, m)$ as its spectral invariant. Denote by $H = H_R + iH_R$ the complexification of $H_R$. Define the positive operator $\Delta$ on $H$ such that $U_t = \Delta^t$ and denote by $J : H \rightarrow H$ the anti-unitary operator given by $J(\xi + i\eta) = \xi - i\eta$ for all $\xi, \eta \in H_R$. Define $H_1 = H \otimes \ell^2(\mathbb{N}^2)$. On $H_1$, we consider the positive operator $\Delta_1$ and anti-unitary operator $J_1$ given by

$$\Delta_1(\xi \otimes \delta_{ij}) = a^j \Delta(\xi) \otimes \delta_{ij} \quad \text{and} \quad J_1(\xi \otimes \delta_{ij}) = J(\xi) \otimes \delta_{ij}$$

for all $i, j \in \mathbb{N}$ and $\xi \in D(\Delta)$. Here, $(\delta_{ij})$ denotes the standard orthonormal basis of $\ell^2(\mathbb{N}^2)$. Note that $J_1 \Delta_1 J_1 = \Delta_1^{-1}$.

Denote by $F(H_1)$ the full Fock space of $H_1$ and by $\theta_1$ the vector state on $B(F(H_1))$ implemented by the vacuum vector. For every $\xi \in H$, define the element $L(\xi) \in B(F(H_1)) \otimes B_a$ given by

$$L(\xi) = \sum_{i,j=0}^{\infty} \ell(\xi \otimes \delta_{ij}) \otimes e_{ij} \sqrt{(1 - a)a^i}.$$

By [Sh96, Theorem 5.2], we can realize $\Gamma(\mu, m)^{\prime\prime} * B_a$ as the von Neumann algebra $\mathcal{M}$ generated by

$$\{L(\xi) + L(J\Delta^{1/2}\xi^*) : \xi \in D(\Delta^{1/2})\} \quad \text{and} \quad 1 \otimes B_a.$$ 

Moreover, the free product state on $\Gamma(\mu, m)^{\prime\prime} * B_a$ is given by the restriction of $\theta_1 \otimes \theta_a$ to $\mathcal{M}$.

To conclude the proof of (4.1), it thus suffices to show that there is a state preserving isomorphism

$$(4.2) \quad \left((1 \otimes e_{00}), \mathcal{M}(1 \otimes e_{00}), \theta_1\right) \cong \left(\Gamma(\mu * \delta_{\log(a)}, +\infty)^{\prime\prime}, \varphi_{\mu * \delta_{\log(a)}, +\infty}\right).$$

The left hand side of (4.2) is generated by the operators

$$(1 \otimes e_{00}) (L(\xi) + L(J\Delta^{1/2}\xi^*)) (1 \otimes e_{00}) = \sqrt{(1 - a)a^i} \left(\ell(\xi \otimes \delta_{ij}) + \ell(J_1 \Delta_1^{1/2}(\xi \otimes \delta_{ij}))^*\right).$$

So, the left hand side of (4.2) equals $\Gamma(\mu_1, m_1)^{\prime\prime}$ where $\mu_1$ and $m_1$ are chosen so that the measure class and multiplicity function of $\log(\Delta_1)$ equal $\mathcal{C}(\mu_1)$ and $m_1$. One checks that $\mathcal{C}(\mu_1) = \mathcal{C}(\mu * \delta_{\log(a)})$ and $m_1 = +\infty$ a.e. So we have proved the existence of the state preserving isomorphism (4.1).

We next prove that for every $t > 0$, there is a state preserving isomorphism

$$(4.3) \quad \Gamma(\mu, m)^{\prime\prime} * \Gamma(\delta_t + \delta_{-t}, 1)^{\prime\prime} \cong \Gamma(\mu * \delta_{zt} \vee \delta_{zt}, +\infty)^{\prime\prime}.$$ 

By [Sh96, Theorem 4.8], there is a state preserving isomorphism

$$\Gamma(\delta_t + \delta_{-t}, 1)^{\prime\prime} \cong L^\infty([0, 1]) * B_{\exp(-t)}.$$
By [Sh96, Theorem 2.11], the free Araki–Woods functor $\Gamma$ turns direct sums into free products. Writing $\mu_1 = \mu + \delta_0$, $m_1 = m + \delta_0$ and using (4.1), we obtain the state preserving isomorphisms
\[
\Gamma(\mu, m)'' \ast \Gamma(\delta_t + \delta_{-t}, 1)'' \cong \Gamma(\mu_1, m_1)'' \ast \Gamma(\exp(-t) \ast \delta_{\exp(-t)}_0, \infty)'' \cong \Gamma(\mu_1 + \delta_{\exp(-t)}_0, \infty)'' \ast \Gamma(\exp(-t))''.
\]
Applying this to $(\mu, m) = (\mu_1 + \delta_{\exp(-t)}_0, \infty)$, we also have the state preserving isomorphisms
\[
\Gamma(\mu_1 + \delta_{\exp(-t)}_0, \infty)'' \ast \Gamma(\delta_t + \delta_{-t}, 1)'' \cong \Gamma(\mu_1 + \delta_{\exp(-t)}_0, \infty)'' \ast \Gamma(\exp(-t))'' \ast \Gamma(\exp(-t))''.
\]
Combining both, it follows that (4.3) holds.

We are now ready to prove the theorem. Fix an atom $t > 0$ of $\mu$. Writing $(\mu, m) = (\mu_0, m_0) + (\delta_t + \delta_{-t}, 1)$, we get from (4.3) the state preserving isomorphism
\[
(\mu, m)'' \ast \Gamma(\mu_0, m_0)'' \ast \Gamma(\delta_t + \delta_{-t}, 1)'' \cong \Gamma(\mu_0 + \delta_{\exp(-t)}_0, \infty)'' \ast \Gamma(\exp(-t))'.
\]
Let $\{t_n : n \geq 0\}$ be the positive atoms of $\mu$, with repetitions if there are only finitely many of them. For every $n \geq 0$, define
\[
\mu_n = \mu \ast \delta_{t_0} \ast \cdots \ast \delta_{t_n}.
\]
For every $n$, $t_{n+1}$ is an atom of $\mu_n$. Repeatedly applying (4.4), we find the state preserving isomorphisms
\[
\Gamma(\mu, m)'' \cong \Gamma(\mu_0, \infty)'' \ast \Gamma(\mu_n, \infty)'' \cong \Gamma(\bigvee_{n \in \mathbb{N}} \mu_n, \infty).
\]
So, we also get state preserving isomorphisms
\[
\Gamma(\mu, m)'' \cong \Gamma(\mu_0, \infty)'' \cong \bigvee_{n \in \mathbb{N}} \Gamma(\mu_n, \infty)'' \cong \Gamma(\bigvee_{n \in \mathbb{N}} \mu_n, \infty).
\]
Since $\bigvee_{n \in \mathbb{N}} \mu_n$ is equivalent with $\mu \ast \delta_\lambda$, the theorem follows.

We deduce the isomorphism part of Theorem F from the following result, generalizing [Sh97a, Theorem 5.1] and proved using the same methods. For every faithful normal state $\psi$ on a von Neumann algebra $A$ and for every nonzero projection $p \in A$, we denote by $\psi_p$ the faithful normal state on $pAp$ given by $\psi_p(a) = \psi(p^{-1}a)\psi(a)$ for all $a \in pAp$.

**Proposition 4.2.** Let $\mu$ be a finite symmetric Borel measure on $\mathbb{R}$ and fix the free Araki–Woods factor $(A, \varphi) = (\Gamma(\mu, \infty)'', \varphi_{\mu, \infty})$. Let $A$ be a von Neumann algebra with a faithful normal state $\psi$ having a factorial centralizer $A^\psi$. For every nonzero projection $p \in A^\psi$, there is a state preserving isomorphism
\[
\left(p((A, \varphi) \ast (A, \psi))p, (\varphi \ast \psi)_p\right) \cong \left((A, \varphi) \ast (pAp, \psi_p), \varphi \ast \psi_p\right).
\]

To prove Proposition 4.2, we need the following lemma. It is a direct consequence of [Sh97a, Corollary 2.5]. To formulate the lemma, we use yet another convention for the construction of free Araki–Woods factors. We call *involution* on a Hilbert space $H$ any closed densely defined antilinear operator $S$ satisfying $S(\xi) \in D(S)$ and $S(S(\xi)) = \xi$ for all $\xi \in D(S)$. Taking the polar decomposition $S = J\Delta^{1/2}$ of such an involution, we obtain an anti-unitary operator $J$ and a nonsingular positive selfadjoint operator $\Delta$ satisfying $J\Delta J = \Delta^{-1}$. Denoting by $U_t$ the restriction of $\Delta^{it}$ to the real Hilbert space $H_{\mathbb{R}} = \{\xi \in H : J(\xi) = \xi\}$, we obtain an orthogonal representation $(U_t)_{t \in \mathbb{R}}$. Every orthogonal representation of $\mathbb{R}$ arises in this way. The associated free Araki–Woods factor can be realized on the full Fock space $\mathcal{F}(H)$ as the von Neumann algebra generated by the operators $\ell(\xi) + \ell(S(\xi))^*$, $\xi \in D(S)$. We denote this realization of the free Araki–Woods factor as $\Gamma(H, S)^\nu$.

**Lemma 4.3.** Let $K$ be a Hilbert space and $\Omega \in K$ a unit vector. Let $H$ be a Hilbert space and $H_0 \subset H$ a total subset. Assume that
- $A \subset B(K)$ is a von Neumann subalgebra and $\langle \cdot, \Omega, \Omega \rangle$ defines a faithful state $\psi$ on $A$,
- for every $\xi \in H_0$, we are given an operator $L(\xi) \in B(K)$,
such that the following conditions hold:
(i) \(L(\xi_1)^*aL(\xi_2) = \psi(a)\langle \xi_2, \xi_1\rangle\) for all \(\xi_1, \xi_2 \in H_0\) and \(a \in A\),
(ii) \(L(\xi)^*a\Omega = 0\) for all \(\xi \in H_0\) and \(a \in A\),
(iii) denoting by \(A\) the \(*\)-algebra generated by \(A\) and \(\{L(\xi) : \xi \in H_0\}\), we have that \(A\Omega\) is dense in \(K\).

Then, \(L\) can be uniquely extended to a linear map \(L : H \to B(K)\) such that the above properties remain valid. For every involution \(S\) on \(H\) with associated free Araki–Woods factor \(\Gamma(H, S)^\prime\prime\), there is a unique normal homomorphism

\[
\pi : (\Gamma(H, S)^\prime\prime, \varphi_{(H,S)}) * (A, \psi) \to B(K)
\]

satisfying \(\pi(\ell(\xi) + \ell(S(\xi))^*) = L(\xi) + L(S(\xi))^*\) for all \(\xi \in D(S)\) and \(\pi(a) = a\) for all \(a \in A\).

Also, \((\pi(\cdot) \Omega, \Omega)\) equals the free product state \(\varphi_{(H,S)} \ast \psi\).

Using Lemma 4.3, we can prove Proposition 4.2.

**Proof of Proposition 4.2.** Since \(A\psi\) is a factor, we can choose partial isometries \(v_i \in A\psi\), \(i \geq 1\), such that \(v_i^*v_i \leq p\), \(\sum_{i=1}^{\infty} v_i^*v_i = 1\) and \(v_i^*v_i = \psi(p)/n_i\) for some integers \(n_i \geq 1\). We can then also choose partial isometries \(w_{is} \in pA\psi p\), \(s = 1, \ldots, n_i\), such that \(w_{is}v_is = v_is\) for all \(s\) and \(\sum_{s=1}^{\infty} w_{is}^*w_{is} = p\).

Since \((M, \varphi)\) is a free Araki–Woods factor with infinite multiplicity, we can choose an involution \(S_0\) on a Hilbert space \(H_0\) and realize \((M, \varphi)\) as \(\Gamma(H, S)^\prime\prime\), where \(H = H_0 \otimes \ell^2(N^2)\) and \(S\) is given by \(S(\xi \otimes \delta_{kl}) = S_0(\xi) \otimes \delta_{lk}\) for all \(\xi \in D(S_0)\) and all \(k, l \geq 1\). We then consider the standard free product representation for \(\Gamma(H, S)^\prime\prime\) \(\ast A\) on the Hilbert space \(K\) with vacuum vector \(\Omega\). Note that \(p(\Gamma(H, S)^\prime\prime \ast A)p\) is generated by

\[
(4.5) \quad pAp \cup \left\{ v_i^* (\ell(\xi \otimes \delta_{kl}) + \ell(S_0(\xi) \otimes \delta_{lk})) v_j \mid i, j, k, l \geq 1, \xi \in D(S_0) \right\}
\]

For all \(k, l \geq 0\), \(i, j \geq 1\) and \(\xi \in H\), define

\[
L_{ijkl}(\xi) = \psi(p)^{-1/2} \sum_{s=1}^{n_i} \sum_{t=1}^{n_j} w_{is}^* v_s^* (\ell(\xi \otimes \delta_{n_is+n_jt}) v_j w_{jt}.
\]

A direct computation shows that

\[
L_{i'j'k'l'}(\xi')^* a L_{ijkl}(\xi) = \delta_{ijkl} \delta_{i'j'k'l'} \langle \xi, \xi' \rangle \psi(p)(a)
\]

for all \(i, j, i', j' \geq 1\), \(k, l, k', l' \geq 0\), \(\xi, \xi' \in H\) and \(a \in pAp\).

Applying Lemma 4.3 to the Hilbert space \(H_1 = H \otimes \ell^2(N^2 \times N_0^2)\) with involution \(S_1(\xi \otimes \delta_{ijkl}) = S_0(\xi) \otimes \delta_{ijkl}\), it follows that \(\Gamma(H_1, S_1)^\prime\prime \ast pAp\) can be realized as the von Neumann algebra \(N\) generated by

\[
pAp \cup \left\{ L_{ijkl}(\xi) + L_{jilk}(S_0(\xi))^* \mid i, j \geq 1, k, l \geq 0, \xi \in D(S_0) \right\},
\]

with the free product state being implemented by \(\psi(p)^{-1/2}p\Omega\).

Note that

\[
w_{is} (L_{ijkl}(\xi) + L_{jilk}(S_0(\xi))^*) w_{jt}^* = \psi(p)^{-1/2} v_i^* (\ell(\xi \otimes \delta_{n_is+n_jt}) + \ell(S_0(\xi) \otimes \delta_{n_jl+t+n_is}))^* v_j.
\]

For fixed \(i, j \geq 1\), the parameters \(n_i, k + s\) and \(n_j, l + t\) with \(k, l \geq 0\), \(s = 1, \ldots, n_i\) and \(t = 1, \ldots, n_j\) exactly run through \(N^2\). So, we find back the generating set of (4.5) and conclude that \(p(\Gamma(H, S)^\prime\prime \ast A)p\) equals \(\Gamma(H_1, S_1)^\prime\prime \ast pAp\) in a state preserving way. Since also \(\Gamma(H_1, S_1)^\prime\prime \cong (M, \varphi)\) in a state preserving way, this concludes the proof of the proposition. \(\square\)
5. Proofs of Theorem A and Corollaries B, C, D

Combining Corollary 3.2 with the deformation/rigidity theorems for free Araki–Woods factors and for free product factors obtained in [HR10, HU15], we get the following theorem.

**Theorem 5.1.** Let \((M, \varphi)\) be either a free Araki–Woods factor with its free quasi-free state or a free product \(*_\Lambda(M_n, \varphi_n)\) of amenable von Neumann algebras equipped with the free product state. Let \(\psi \in M_*\) be any faithful normal state on \(M\) and denote by \([D\varphi : D\varphi]_t\) Connes’ Radon–Nikodym 1-cocycle between \(\psi\) and \(\varphi\). Let \(z \in \mathcal{Z}(M^\psi)\) be the central projection such that \(M^\psi(1 - z)\) is amenable and \(M^\psi z\) has no amenable direct summand.

There exists a sequence of partial isometries \(v_n \in M\) such that the projection \(q_n = v_n v_n^*\) belongs to \(M^\psi\), the projection \(p_n = v_n^* v_n\) belongs to \(M^\varphi\), and \(\sum_n q_n = z\),

\[
q_n [D\psi : D\varphi]_t = \lambda_n^t v_n \sigma_\varphi^t (v_n^*) \quad \text{and} \quad \psi q_n = \lambda_n v_n \varphi v_n^*,
\]

with \(\lambda_n = \psi(q_n)/\varphi(p_n)\).

**Proof.** Let \(q \in M^\psi\) be a nonzero projection such that \(qM^\psi q\) has no amenable direct summand. Let \(r_0 \in L_\varphi(R)\) be a nonzero finite trace projection. Put \(r = \Pi_{\varphi,\psi}(q r_0)\). Then \(r\) is a nonzero finite trace projection in the core \(c_\varphi(M)\) and \(\Pi_{\varphi,\psi}(L_\psi(R)) r\) commutes with \(qM^\psi q\). Since \(qM^\psi q\) has no amenable direct summand, it follows from [HR10, Theorem 5.2] (in the case where \(M\) is a free Araki–Woods factor) and [HU15, Theorem 4.3] (in the case where \(M\) is a free product of amenable von Neumann algebras) that \(\Pi_{\varphi,\psi}(L_\psi(R)) r \sim_{c_\varphi(M)} L_\varphi(R)\).

By Theorem 3.1, we find a nonzero partial isometry \(v \in qM\) such that the projection \(q_0 = vv^*\) belongs to \(M^\psi\), the projection \(p = v^* v\) belongs to \(M^\varphi\) and \(\psi q = \lambda v \varphi v^*\). In particular, \(\lambda = \psi(q_0)/\varphi(p)\) and \(q_0 [D\psi : D\varphi]_t = \lambda^t v \sigma_\varphi^t (v^*)\).

Since \(q \in M^\psi\) was an arbitrary nonzero projection such that \(qM^\psi q\) has no amenable direct summand, the theorem follows by a maximality argument. \(\square\)

In order to apply Theorem 5.1 to the classification of free Araki–Woods factors, we need the following description of the centralizer of the free quasi-free state.

**Remark 5.2.** When \(M = \Gamma(\mu, m)^\mu\) is an arbitrary free Araki–Woods factor with free quasi-free state \(\varphi = \varphi_{\mu, m}\), the centralizer \(M^\varphi\) can be described as follows. Denote by \(M_\varphi = \Gamma(\mu_\varphi, m)^\varphi\) the almost periodic part of \(M\). First note that \(M^\varphi = M^\varphi_{\varphi, \psi}\). So if \(\mu_\varphi = 0\), we have \(M^\varphi = C1\). If \(\mu_\varphi\) is concentrated on \(\{0\}\), we conclude that \(M^\varphi = M_\varphi\). When \(\mu_\varphi\) is concentrated on \(\{0\}\), we conclude that \(M^\varphi = M_\mu\). When \(\mu_\varphi\) is concentrated on \(\{0\}\), we conclude that \(M^\varphi = M_\varphi\).

Theorem A is a particular case of the following more general result.

**Theorem 5.3.** Let \(\mu, \nu\) be finite symmetric Borel measures on \(R\) and \(m, n : R \to \mathbb{N} \cup \{+\infty\}\) symmetric Borel multiplicity functions. Assume that \(\nu_a \neq 0\) and either \(\text{supp}(\nu_a) \neq \{0\}\) or \(\text{supp}(\nu_a) = \{0\}\) with \(n(0) \geq 2\).

If the free Araki–Woods factors \(\Gamma(\mu, m)^\mu\) and \(\Gamma(\nu, n)^\nu\) are isomorphic then there exist nonzero projections \(p \in \Gamma(\mu, m)^\mu\) and \(q \in \Gamma(\nu, n)^\nu\) and a state preserving isomorphism \((p \Gamma(\mu, m)^\mu p, \varphi_{\mu, m})_p \cong (q \Gamma(\nu, n)^\nu q, \varphi_{\nu, n})_q\)

where \((\varphi_{\mu, m})_p = \frac{\varphi_{\mu, m}(p \cdot p)}{\varphi_{\mu, m}(p)}\) and \((\varphi_{\nu, n})_q = \frac{\varphi_{\nu, n}(q \cdot q)}{\varphi_{\nu, n}(q)}\).
In particular, the joint measure classes \( C(\bigvee_{k \geq 1} \mu^{sk}) \) and \( C(\bigvee_{k \geq 1} \nu^{sk}) \) are equal.

Moreover, in case \( \text{supp}(\nu_a) \neq \{0\} \) or \( \text{supp}(\nu_a) = \{0\} \) with \( n(0) = +\infty \), there exists a state preserving isomorphism \( (\Gamma(\mu, m)^n, \varphi_{\mu, m}) \cong (\Gamma(\nu, n)^n, \varphi_{\nu, n}) \).

Proof. Put \((M, \varphi) := (\Gamma(\mu, m)^n, \varphi_{\mu, m}) \) and \((N, \theta) := (\Gamma(\nu, n)^n, \varphi_{\nu, n})\). Let \( \pi : M \to N \) be any isomorphism between \( M \) and \( N \). Put \( \psi := \theta \circ \pi \). By our assumptions on \( \nu \) and Remark 5.2, the centralizer \( M^{\psi} \) is nonamenable. By Theorem 5.1, we find a nonzero partial isometry \( v \in M \) such that \( p = v^* v \in M^\varphi \) and \( q = v v^* \in M^\theta \) and \( A(d(v) : (pMP, \varphi_p) \to (qMq, \psi_q) \) is state preserving. It follows in particular that \( pM^\varphi q = (pMP)p^\varphi q \cong (qMq)^\psi \) is a state preserving isomorphism. Thus \( M^\varphi \) cannot be abelian and Remark 5.2 implies that \( M^\varphi \) is a \( \Pi_1 \) factor. Applying Lemma 2.1 twice, we have

\[
C(\bigvee_{k \geq 1} \mu^{sk}) = C(\Delta_{\varphi}) = C(\Delta_{\varphi}) = C(\Delta_{\psi}) = C(\Delta_{\psi}) = C(\bigvee_{k \geq 1} \nu^{sk}).
\]

This implies that \( C(\bigvee_{k \geq 1} \mu^{sk}) = C(\bigvee_{k \geq 1} \nu^{sk}) \).

Assume now that either \( \text{supp}(\nu_a) \neq \{0\} \) or \( \text{supp}(\nu_a) = \{0\} \) with \( n(0) = +\infty \). In the latter case where \( \nu(\{0\}) > 0 \) and \( n(0) = +\infty \), we use that the free Araki–Woods functor \( \Gamma \) turns direct sums into free products (see [Sh96, Theorem 2.11]) and conclude that there exists a state preserving isomorphism \( (\mu, \varphi) \cong (N, \theta) \).

In the case where \( \nu \) has at least one atom different from 0, it follows similarly from the classification of almost periodic free Araki–Woods factors (see [Sh96]) that \((5.1)\) holds.

Put \( q_0 = \pi(q) \). Above, we have proved that there exists a state preserving isomorphism \( (pMP, \varphi_p) \cong (q_0Nq_0, \theta_{q_0}) \). Taking a smaller \( p \) if needed, we may assume that \( \varphi(p) = 1/k \) for some integer \( k \geq 1 \). Combining \((5.1)\) with Proposition 4.2 and the fact that the fundamental group of \( L(F_\infty) \) equals \( R_+ \) (see [Ra91]), it follows that there exists a state preserving isomorphism \( (q_0Nq_0, \theta_{q_0}) \cong (q_1Nq_1, \theta_{q_1}) \) whenever \( q_1 \in N^\theta \) is a nonzero projection.

Choose a projection \( q_1 \in N^\theta \) with \( \theta(q_1) = 1/k \). So, there exists a state preserving isomorphism \( (pMP, \varphi_p) \cong (q_1Nq_1, \theta_{q_1}) \). Since \( \varphi(p) = 1/k = \theta(q_1) \) and since both \( M^\varphi \) and \( N^\theta \) are factors, taking \( k \times k \) matrices, we find a state preserving isomorphism \( (M, \varphi) \cong (N, \theta) \).

Proof of Corollary B. Let \( \mu, \nu \in S(R) \) such that \( \Lambda(\mu_a) = \Lambda(\nu_a) = \Lambda \) and \( C(\mu_c * \delta_{\Lambda}) = C(\nu_c * \delta_{\Lambda}) \). Let \( m_n : R \to N \cup \{+\infty\} \) be any symmetric Borel multiplicity functions. Then we have \( C(\mu * \delta_{\Lambda}) = C(\nu * \delta_{\Lambda}) \). By Theorem 4.1, there is a state preserving isomorphism \( \Gamma(\mu, m)^n \cong \Gamma(\nu, n)^n \).

Conversely, let \( \mu, \nu \in S(R) \) and \( m_n : R \to N \cup \{+\infty\} \) be any symmetric Borel multiplicity functions such that \( \Gamma(\mu, m)^n \cong \Gamma(\nu, n)^n \). By Theorem A, we have that \( C(\bigvee_{k \geq 1} \mu^{sk}) = C(\bigvee_{k \geq 1} \nu^{sk}) \). Since for every \( k \geq 1 \), we have \( \mu_c^{\kappa k} \prec \mu_c^{\kappa} \) and \( \nu_c^{\kappa k} \prec \nu_c^{\kappa} \), it follows that

\[
C(\mu_c * \delta_{\Lambda(\mu_a)} \lor \delta_{\Lambda(\nu_a)}) = C(\bigvee_{k \geq 1} \mu^{sk}) = C(\nu_c * \delta_{\Lambda(\nu_a)} \lor \delta_{\Lambda(\nu_a)}).
\]

This implies that \( \Lambda(\mu_a) = \Lambda(\nu_a) \) and \( C(\mu_c * \delta_{\Lambda(\mu_a)}) = C(\nu_c * \delta_{\Lambda(\nu_a)}) \).

Proof of Corollary C. Put \((M, \varphi) := (\Gamma(\mu, m)^n, \varphi_{\mu, m}) \). Let \( \psi \in M^* \) be a faithful normal state such that \( M^\psi \) is nonamenable. By Theorem 5.1, we find a nonzero partial isometry \( v \in M \) such
that \( q = vv^* \in M^\psi \), \( p = v^*v \in M^\varphi \), \( qM^\psi q \) has no amenable direct summand and \( \psi = \lambda v^*v \) with \( \lambda = \psi(q)/\varphi(p) \). It follows that \( pM^\varphi p \cong qM^\psi q \) has no amenable direct summand. By Remark 5.2, this means that either \( \mu_a \) has an atom different from 0 or \( \mu_a \) is concentrated on \( \{0\} \) with \( m(0) \geq 2 \). Conversely, if \( \mu_a \) satisfies these properties, it follows from Remark 5.2 that the centralizer of the free quasi-free state is nonamenable. \( \square \)

**Proof of Corollary D.** By Corollary C, the von Neumann algebra \( \Gamma(\lambda + \delta_0, 1)^\sigma_\theta \) has amenable centralizers while \( \Gamma(\lambda + \delta_0, 2)^\sigma_\theta \) does not. \( \square \)

**Example 5.4.** Many different measures in the family \( \mathcal{S}(R) \) of Corollary B can be constructed as follows. Let \( K \subset R \) be an independent Borel set, meaning that every \( n \)-tuple of distinct elements in \( K \) generates a free abelian group of rank \( n \). By [Ru62, Theorems 5.1.4 and 5.2.2], there exist compact independent \( K \subset R \) such that \( K \) is homeomorphic to a Cantor set. Fix such a \( K \subset R \) and put \( L = K \cup (-K) \). Also fix a countable subgroup \( \Lambda < R \).

For every continuous symmetric probability measure \( \mu \) on \( R \) that is concentrated on \( L \), define the measure class \( \tilde{\mu} \) on \( R \) given by

\[
\tilde{\mu} = \bigvee_{x \in \Lambda, n \geq 1} (x + \mu^{*n}) .
\]

By construction, each \( \tilde{\mu} \) is a continuous symmetric measure class on \( R \) that is invariant under translation by \( \Lambda \) and that satisfies \( \tilde{\mu} \ast \tilde{\mu} < \tilde{\mu} \).

Given continuous symmetric probability measures \( \mu_1 \) and \( \mu_2 \) that are concentrated on \( L \), we claim that \( C(\tilde{\mu_1}) = C(\tilde{\mu_2}) \) if and only if \( C(\mu_1) = C(\mu_2) \). One implication is obvious. The other implication is a consequence of the following result contained in [LP97, Corollary 1]: if \( \eta_1 \) and \( \eta_2 \) are concentrated on \( L \) and \( \eta_1 \perp \eta_2 \), then also \( \eta_1 \perp (x + \eta_2^k) \) for all \( x \in R \) and all \( k \geq 1 \).

Choosing \( \Lambda \) to be a nontrivial subgroup of \( R \) and applying Corollary B, for all continuous symmetric probability measures \( \mu_1 \) and \( \mu_2 \) concentrated on the Cantor set \( L \), we find that

\[
\Gamma(\tilde{\mu_1} \vee \delta_\lambda, m_1)'' \cong \Gamma(\tilde{\mu_2} \vee \delta_\lambda, m_2)'' \quad \text{iff} \quad C(\mu_1) = C(\mu_2) .
\]

Adding the Lebesgue measure to \( \tilde{\mu} \), we claim that we also have

\[
\Gamma(\lambda \vee \tilde{\mu_1} \vee \delta_\lambda, m_1)'' \cong \Gamma(\lambda \vee \tilde{\mu_2} \vee \delta_\lambda, m_2)'' \quad \text{iff} \quad C(\mu_1) = C(\mu_2) .
\]

By [Sh97b, Corollary 8.6], for all these free Araki–Woods factors, the \( \tau \)-invariant equals the usual topology on \( R \), so that they cannot be distinguished by Connes’ invariants.

To prove the claim, define \( L_n \) as the \( n \)-fold sum \( L_n = L + \cdots + L \) and put \( S = \bigcup_{n \geq 1} L_n \). Below we prove that \( \lambda(S) = 0 \). The claim then follows from Corollary B: if \( C(\lambda \vee \tilde{\mu_1}) = C(\lambda \vee \tilde{\mu_2}) \), restricting to \( S \), we get that \( C(\tilde{\mu_1}) = C(\tilde{\mu_2}) \). As proven above, this implies that \( C(\mu_1) = C(\mu_2) \).

It remains to prove that \( \lambda(L_n) = 0 \) for all \( n \). If for some \( n \geq 1 \), we have \( \lambda(L_n) > 0 \), then \( L_{2n} = L_n - L_n \) contains a neighborhood of 0. Every nonzero \( x \in L_{2n} \) can be uniquely written as \( x = \alpha_1 y_1 + \cdots + \alpha_k y_k \) with \( k \geq 1, y_1, \ldots, y_k \) distinct elements in \( K \) and \( \alpha_i \in Z \setminus \{0\} \) with \( |\alpha_i| \leq 2n \) for all \( i \). So if \( x \in L_{2n} \) is nonzero, we have that \( (2n + 1)x \notin L_{2n} \). Therefore, \( L_{2n} \) does not contain a neighborhood of 0 and it follows that \( \lambda(L_n) = 0 \) for all \( n \geq 1 \).

6. **Proof of Theorem E**

To prove Theorem E, we combine [HU15, Theorem 4.3] and Theorem 3.1 with the following lemma. Whenever \( \theta \) is a faithful normal state on a von Neumann algebra \( M \), we denote by \( M_{ap, \theta} \) the von Neumann subalgebra of \( M \) generated by the almost periodic part of \( (\sigma_\theta^t) \).
Lemma 6.1. For $i = 1, 2$, let $(M_i, \varphi_i)$ be von Neumann algebras with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ their free product. Denote by $E_{M_i} : M \to M_i$ the unique $\varphi$-preserving conditional expectation. Let $\theta_1$ be a faithful normal state on $M_1$ and define $\theta = \theta_1 \circ E_{M_1}$. Let $q \in M^\theta$ be a projection.

There exist projections $q_0, q_1, \ldots$ with $q_0 \in M_1^{\theta_1}$ and $q_i \in M^\theta$ for all $i \geq 1$ such that

(i) $\sum_{i=0}^{\infty} q_i = q$,
(ii) $q_0 M_{ap, \theta} q_0 = q_0 M_1_{ap, \theta_1} q_0$ and $q_0 M^\theta q_0 = q_0 M_1^{\theta_1} q_0$,
(iii) for every $i \geq 1$, there exists a partial isometry $v_i \in M$ with $v_i v_i^* = q_i$, $v_i^* v_i \in M^\varphi$ and

$$
\frac{1}{\theta(q_i)} \theta q_i = \frac{1}{\varphi(v_i^* v_i)} v_i \varphi(v_i^*).
$$

Proof. Fix standard representations $M_i \subset B(H_i)$. For every faithful normal state $\mu$ on $M_i$, denote by $\xi_\mu \in H_i$ the canonical unit vector that implements $\mu$.

Define $u_t = [D\theta_1 : D\varphi_1]_t \in U(M_1)$. Note that also $[D\theta : D\varphi]_t = [D\theta_1 \circ E : D\varphi_1 \circ E]_t = u_t$ for all $t \in \mathbb{R}$. Let $e_1, e_2, \ldots$ be a maximal sequence of nonzero projections in $M_1^{\theta_1}$ such that $e_i e_j = 0$ whenever $i \neq j$ and such that for every $i \geq 1$, there exists a partial isometry $w_i \in M_1$ and a $\lambda_i > 0$ with $w_i w_i^* = e_i$ and $u_t \sigma_t^{\varphi_1}(w_i) = \lambda_i w_i$ for all $t \in \mathbb{R}$. Define $e_0 = 1 - \sum_{i=1}^{\infty} e_i$. Then $e_0 \in M_1^{\theta_1}$. By construction, the unitary representation $(U_t)_{t \in \mathbb{R}}$ on $H_1$ given by $U_t(x \xi_{\varphi_1}) = u_t \sigma_t^{\varphi_1}(x) \xi_{\varphi_1}$ for all $x \in M_1$ is weakly mixing on $e_0 H_1$.

For $i = 1, 2$, define $\hat{H}_i = H_i \ominus C \xi_{\varphi_i}$. For every $k \geq 1$, define the Hilbert space

$$
K_k = H_1 \otimes \hat{H}_2 \otimes \hat{H}_1 \otimes \cdots \otimes \hat{H}_1 \otimes \hat{H}_2 \otimes H_1.
$$

We can then identify the standard Hilbert space $H$ for $M$ with

$$
H = H_1 \oplus \bigoplus_{k=1}^{\infty} K_k.
$$

Under this identification, $\xi_{\varphi} = \xi_{\varphi_1} \in H_1$ and $\xi_{\theta} = \xi_{\theta_1} \in H_1$. Denote by $(V_t)_{t \in \mathbb{R}}$ the unitary representation on $H_1$ given by $V_t(x \xi_{\theta_1}) = \sigma_t^{\varphi_1}(x) u_t^{\varphi_1} \xi_{\theta_1}$ for all $x \in M_1$. Under the above identification of $H$, we get that

$$
\Delta_\theta^u = \Delta_{\varphi_1}^u \otimes \bigoplus_{k=1}^{\infty} \left( U_t \otimes \Delta_{\varphi_1}^u \otimes \Delta_{\varphi_1}^u \otimes \cdots \otimes \Delta_{\varphi_1}^u \otimes \Delta_{\varphi_2}^u \otimes V_t \right).
$$

Since $(U_t)_{t \in \mathbb{R}}$ is weakly mixing on $e_0 H_1$, we conclude that $(\Delta_\theta^u)_{t \in \mathbb{R}}$ is weakly mixing on $e_0 H \ominus e_0 H_1$. It follows that

$$
eq 0 M_{ap, \theta} e_0 = e_0 M_{ap, \theta_1} e_0 \quad \text{and} \quad e_0 M^\theta e_0 = e_0 M_1^{\theta_1} e_0.
$$

Let $q_1, q_2, \ldots$ be a maximal sequence of nonzero projections in $q M^\theta q$ such that $q_i q_j = 0$ if $i \neq j$ and such that statement (iii) in the lemma holds for every $i \geq 1$. Define $q_0 = q - \sum_{i=1}^{\infty} q_i$. Then $q_0 \in M^\theta$. We prove that $q_0 \leq e_0$. Once this is proven, it follows from (6.1) that $q_0 \in M_1^{\theta_1}$ and that $q_0 M_{ap, \theta} q_0 = q_0 M_{ap, \theta_1} q_0$, so that the lemma follows.

If $q_0 \leq e_0$, we find $j \geq 1$ such that $q_0 e_j \neq 0$. Then the polar part $v$ of $q_0 w_j$ is a nonzero partial isometry in $M$ satisfying $vv^* \leq q_0$ and $u_t \sigma_t^{\varphi_1}(v) = \lambda_t^u v$ for all $t \in \mathbb{R}$. So, the projection $vv^*$ could be added to the sequence $q_1, q_2, \ldots$, contradicting its maximality. Therefore, $q_0 \leq e_0$ and the lemma is proved. \(\square\)
Theorem E will be an immediate consequence of the following more technical proposition that will also be used in Section 7 below.

**Proposition 6.2.** For $i = 1, 2$, let $(M_i, \varphi_i)$ be von Neumann algebras with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ their free product and by $E_{M_i} : M \to M_i$ the unique $\varphi$-preserving conditional expectation. Let $\psi$ be a faithful normal state on $M$. Define the set of projections $\mathcal{P} \subset M^\psi$ given by $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ where

- for $i = 1, 2$, $\mathcal{P}_i$ consists of the projections $q \in M^\psi$ for which there exists a partial isometry $v \in M$ and a faithful normal state $\theta_i$ on $M_i$ with $v^*v = q$, $e = vv^* \in M_i^{\theta_i}$,

\[
\frac{1}{\psi(q)} v \psi v^* = \frac{1}{\theta_i(e)} (\theta_i \circ E_{M_i}) e, \quad v M_{\text{sp}, \psi} v^* = e M_i^{\theta_i} e, \quad v M_{\psi} v^* = e M_{\theta_i} e,
\]

- $\mathcal{P}_3$ consists of the projections $q \in M^\psi$ for which there exists a partial isometry $v \in M$ with $v^*v = q$, $e = vv^* \in M^\psi$,

\[
\frac{1}{\psi(q)} v \psi v^* = \frac{1}{\varphi(e)} \varphi e \quad \text{and} \quad v M_{\psi} v^* = e M_{\theta} e.
\]

If $q \in M^\psi$ is a projection such that $qM^\psi q$ has no amenable direct summand, then $q$ can be written as a sum of projections in $\mathcal{P}$.

**Proof.** Let $\psi$ be a faithful normal state on $M$ and $q \in M^\psi$ a projection such that $qM^\psi q$ has no amenable direct summand. It suffices to prove that $q$ dominates a nonzero projection in $\mathcal{P}$, since then a maximality argument can be applied.

Fix any nonzero finite trace projection $r_0 \in L_\psi(R)$ and put $r = \Pi_{\varphi, \psi}(qr_0)$. Define the von Neumann subalgebra $Q \subset r c_{\varphi}(M)r$ given by

\[
Q = \Pi_{\varphi, \psi}(L_\psi(R)qr_0 \vee qM^\psi qr_0).
\]

Note that $Q$ has no amenable direct summand. By [HU15, Theorem 4.3],

either $Q' \cap rc_{\varphi}(M)r \prec c_{\varphi}(M) L_\varphi(R)$ or $Q \prec c_{\varphi}(M) c_{\psi}(M_i)$ for $i = 1$ or $i = 2$.

Since $\Pi_{\varphi, \psi}(L_\psi(R)qr_0)$ belongs to both $Q$ and $Q' \cap rc_{\varphi}(M)r$, it follows that

\[
\Pi_{\varphi, \psi}(L_\psi(R)qr_0) \prec c_{\varphi}(M_i) c_{\psi}(M_i)
\]

for $i = 1$ or $i = 2$.

By Theorem 3.1, we find a faithful normal state $\theta_i$ on $M_i$ and a partial isometry $v \in M$ such that $q_0 = v^*v$ is a nonzero projection in $qM^\psi q$, $p = vv^* \in M_i^{\theta_i},$ and

\[
\frac{1}{\psi(q_0)} v \psi v^* = \frac{1}{\theta_i(E_{M_i}(p))} (\theta_i \circ E_{M_i}) p.
\]

Write $\theta = \theta_i \circ E_{M_i}$. By Lemma 6.1, we either find a nonzero projection $e \leq p$ such that $e \in M_i^{\theta_i}$ and $e M_{\text{sp}, \theta} e = e M_{i, \text{ap}, \theta} e$ and $e M_{\theta} e = e M_{\theta} e$, or we find a nonzero projection $p_0 \in pM_{\theta} p$ and a partial isometry $w \in M$ such that $ww^* = p_0$, $e = w^*w$ belongs to $M^\psi$ and

\[
\frac{1}{\theta(p_0)} w^* \theta w = \frac{1}{\varphi(e)} \varphi e.
\]

In the first case, we get that the projection $v^*ev \leq q$ belongs to $\mathcal{P}_1$, while in the second case, the projection $v^*p_0v \leq q$ belongs to $\mathcal{P}_3$. □
Proof of Theorem E. Denote by $E_M : M \to M_i$ the unique $\varphi$-preserving conditional expectation. If $M^0$ is nonamenable, then obviously, $M$ does not have all its centralizers amenable. If $M_i$ admits a faithful normal state $\theta_i$ such that $M_i^{\theta_i} \otimes E_{M_i}$ is nonamenable, then $\theta_i \circ E_M$ is a faithful normal state on $M$ with $M_i^{\theta_i} \subset M^{\theta_i \circ E_{M_i}}$, so that again, $M$ does not have all its centralizers amenable.

Conversely, assume that $\psi$ is a faithful normal state on $M$ such that $M^\psi$ is nonamenable. Take a nonzero projection $q \in M^\psi$ such that $qM^\psi q$ has no amenable direct summand. By 6.2, we either find $i \in \{1, 2\}$ and a faithful normal state $\theta_i$ on $M_i$ such that $M_i^{\theta_i}$ is nonamenable, or we find that $M^\psi$ is nonamenable. \hfill $\square$

7. Further structural results and proof of Theorem F

We start by showing that the invariant of Theorem A is not a complete invariant for the family of free Araki–Woods factors $\Gamma(\mu, m)$ arising from finite symmetric Borel measures $\mu$ on $\mathbb{R}$ whose atomic part $\mu_a$ is nonzero and not supported on $\{0\}$.

**Theorem 7.1.** Let $\Lambda < \mathbb{R}$ be any countable subgroup such that $\Lambda \neq \{0\}$ and denote by $\delta_\Lambda$ a finite atomic measure on $\mathbb{R}$ whose set of atoms equals $\Lambda$. Let $\eta$ be any continuous finite symmetric Borel measure on $\mathbb{R}$ such that $C(\eta) = C(\eta \ast \delta_\Lambda)$ and such that the measures $(\eta^{*k})_{k \geq 1}$ are pairwise singular.

Put $\mu = \delta_\Lambda + \eta$ and $\nu = \delta_\Lambda + \eta \ast \eta$. Then,

$$\Gamma(\mu, 1)^{\eta} \neq \Gamma(\nu, 1)^{\eta} \quad \text{and} \quad C(\bigvee_{k \geq 1} \mu^{*k}) = C(\bigvee_{k \geq 1} \nu^{*k}).$$

**Proof.** By construction, we have $C(\bigvee_{k \geq 1} \mu^{*k}) = C(\bigvee_{k \geq 1} \nu^{*k})$. We denote $M := \Gamma(\mu, 1)^{\eta}$ and $N := \Gamma(\nu, 1)^{\eta}$. In this case, $Q \subset N$ the canonical von Neumann subalgebra given by $Q := \Gamma(\delta_\Lambda + \eta, 1)^{\eta}$. Put $\varphi := \varphi_{\mu, 1}$ and $\psi := \varphi_{\nu, 1}$. Observe that the inclusion $Q \subset N$ is globally invariant under the modular automorphism group $\sigma^\psi$.

Assume by contradiction that $M \cong N$. By Theorem A, there exists a state preserving isomorphism $\pi : (M, \varphi) \to (N, \psi)$ of $M$ onto $N$. Then, $\pi$ extends to a unitary operator $U : L^2(M, \varphi) \to L^2(N, \psi)$ satisfying $U \Delta_\psi U^* = \Delta_\psi$. Define the real Hilbert space

$$H^\mu_R := \left\{ f \in L^2_C(\mathbb{R}, \mu) : f(-s) = \overline{f(s)} \text{ for } \mu\text{-almost every } s \in \mathbb{R} \right\}$$

and the orthogonal representation

$$U^\mu : \mathbb{R} \to H^\mu_R : U^\mu(f)(t) = \exp(ist) f(t).$$

Denote by $s(\xi) := \ell(\xi) + \ell(\xi)^*, \xi \in H^\mu_R$, the canonical semicircular elements that generate $M$ and satisfy $\sigma_\psi^s(s(\xi)) = s(U^\mu(\xi))$. By construction, $C(\Delta_\psi|_{L^2(N) \otimes L^2(Q)}) = \bigcap_{k \geq 2} C(\eta^{*k})$. Since $\mu$ is singular w.r.t. $\eta^{*k}$ for all $k \geq 2$, it follows that $\pi(s(\xi)) \in Q$ for all $\xi \in H^\mu_R$. But then, $\pi(M) \subset Q$, which is impossible because $\pi$ is surjective. \hfill $\square$

Note that Example 5.4 provides many measures $\eta$ satisfying the assumptions of Theorem 7.1.

We could not prove or disprove that the measure class $C(\mu_c \ast \delta_\Lambda(\mu_a))$ is an invariant for the family of free Araki–Woods factors $\Gamma(\mu, m)^{\Lambda}$ arising from finite symmetric Borel measures $\mu$ on $\mathbb{R}$ whose atomic part $\mu_a$ is nonzero and not supported on $\{0\}$.

Using Theorem 5.1 in combination with the results of [BH16], we can also clarify the relation between free Araki–Woods factors and free products of amenable von Neumann algebras. Combining [Sh96, Theorems 2.11 and 4.8], it follows that every almost periodic free Araki–Woods factor is isomorphic with a free product of von Neumann algebras of type I. Conversely,
by [Ho06], many free products of type I von Neumann algebras are isomorphic with free Araki–Woods factors. In [Dy92, Theorem 4.6], it is proved that a free product of two amenable von Neumann algebras w.r.t. faithful normal traces is always isomorphic to the direct sum of an interpolated free group factor and a finite dimensional algebra. It is therefore tempting to believe that every type III factor arising as a free product of amenable von Neumann algebras w.r.t. faithful normal states is a free Araki–Woods factor. The following example shows however that this is almost never the case if one of the states fails to be almost periodic.

**Theorem 7.2.** Let \( (P, \theta) = \ast_n (P_n, \theta_n) \) be a free product of amenable von Neumann algebras. Assume that the centralizer \( P^\theta \) has no amenable direct summand and that at least one of the \( \theta_n \) is not almost periodic.

Then \( P \) is not isomorphic to a free Araki–Woods factor. Even more: there is no faithful normal homomorphism \( \pi \) of \( P \) into a free Araki–Woods factor \( M \) such that \( \pi(P) \subset M \) is with expectation.

The same conclusions hold if \( (P, \theta) \) is any von Neumann algebra with a faithful normal state satisfying the following three properties: the centralizer \( P^\theta \) has no amenable direct summand, \( \theta \) is not almost periodic and \( P \) is generated by a family of amenable von Neumann subalgebras \( P_n \subset P \) that are globally invariant under the modular automorphism group \( (\sigma^\theta_t)_t \).

**Proof.** Let \((M, \varphi)\) be a free Araki–Woods factor with its free quasi-free state. Let \( (P, \theta) \) be a von Neumann algebra with a faithful normal state \( \theta \) such that the centralizer \( P^\theta \) has no amenable direct summand and such that \( P \) is generated by a family of amenable von Neumann subalgebras \( P_n \subset P \) that are globally invariant under the modular automorphism group \((\sigma^\theta_t)_t\). Let \( \pi : P \to M \) be a normal homomorphism and \( E : M \to \pi(P) \) a faithful normal conditional expectation. We prove that \( \theta \) is almost periodic.

Define the faithful normal state \( \psi \in M_* \) given by \( \psi = \theta \circ \pi^{-1} \circ E \). Since \( \pi(P^\theta) \subset M^\psi \), we get that \( M^\psi \) has no amenable direct summand. By Theorem 5.1, we find partial isometries \( v_n \in M \) such that \( q_n = v_n v_n^* \in M^\psi \), \( p_n = v_n^* v_n \in M^\varphi \), \( \sum_n q_n = 1 \) and

\[
q_n [D\psi : D\varphi]_t = \lambda^t_n v_n \sigma^\varphi_t (v_n^*)
\]

where \( \lambda_n = \psi(q_n)/\varphi(p_n) \).

Replacing \((M, \varphi)\) by the free product of \((M, \varphi)\) and the appropriate almost periodic free Araki–Woods factor, we still get a free Araki–Woods factor and we may assume that \( M^\varphi \) is a factor and that each \( \lambda_n \) is an eigenvalue for the free quasi-free state. We can then choose partial isometries \( w_n \in M \) such that \( \sigma^\varphi_t (w_n) = \lambda_n^{-it} w_n \), \( w_n^* w_n = p_n \) and such that \( e_n := w_n^* w_n \) belongs to \( M^{\varphi} \) with \( \sum_n e_n = 1 \). So, we find that

\[
q_n [D\psi : D\varphi]_t = v_n w_n \sigma^\varphi_t (w_n^* v_n^*)
\]

for all \( n \) and all \( t \in \mathbb{R} \). We conclude that \( v = \sum_n v_n w_n \) is a unitary in \( M \) satisfying \([D\psi : D\varphi]_t = v \sigma^\varphi_t (v^*)\). This means that \( \varphi = \psi \circ \Ad v \) and that the homomorphism \( \eta = \Ad v^* \circ \pi \) satisfies \( \sigma^\varphi_t \circ \eta = \eta \circ \sigma^\theta_t \).

So for every \( n \), the subalgebra \( \eta(P_n) \subset M \) is amenable and globally invariant under the modular automorphism group \((\sigma^\theta_t)_t\). By [BH16, Theorem 4.1], it follows that \( \eta(P) \) lies in the almost periodic part of \((M, \varphi)\). This implies that the restriction of \((\sigma^\varphi_t)_t \) to \( P_n \) is almost periodic. Since this holds for every \( n \), we conclude that \( \theta \) is almost periodic. \( \square \)

From Proposition 6.2, we get the following rigidity results for free product von Neumann algebras. Roughly, the result says that an arbitrary free product of a “very much non almost periodic” \( M_1 \) with an almost periodic \( M_2 \) remembers the almost periodic part \( M_2 \) up to amplification.
Recall that a faithful normal positive functional \( \varphi \) on a von Neumann algebra \( M \) is said to be \textit{weakly mixing} if the unitary representation \( \sigma^\varphi_t(\cdot) \) on \( L^2(M, \varphi) \otimes \mathbb{C}1 \) is weakly mixing, and that \( \varphi \) is said to be \textit{almost periodic} if the unitary representation \( \sigma^\varphi_t(\cdot) \) on \( L^2(M, \varphi) \) is almost periodic.

**Proposition 7.3.** For \( i = 1, 2 \), let \( (M_i, \varphi_i) \) and \( (N_i, \psi_i) \) be von Neumann algebras with faithful normal states. Denote by \( (M, \varphi) = (M_1, \varphi_1) \ast (M_2, \varphi_2) \) and \( (N, \psi) = (N_1, \psi_1) \ast (N_2, \psi_2) \) their free products. Assume that

- \( M_1 \) and \( N_1 \) have all their centralizers amenable and \( \varphi_1, \psi_1 \) are weakly mixing states,
- \( M_2^{\text{ap}} \) and \( N_2^{\text{ap}} \) have no amenable direct summand and \( \varphi_2, \psi_2 \) are almost periodic.

If \( M \cong N \), there exist nonzero projections \( e \in M_2 \) and \( q \in N_2 \) such that \( eM_2e \cong qN_2q \).

**Proof.** Whenever \( \mu \) is a faithful normal state on \( M \) and \( q \in M^\mu \) is a projection such that \( qM^\mu q \) has no amenable direct summand, we can apply Proposition 6.2. Since \( M_1 \) has all its centralizers amenable, the set \( p_1 \) in Proposition 6.2 equals \( \{0\} \). Since \( \varphi_1 \) is weakly mixing, the almost periodic part of a state of the form \( \theta \circ E_{M_2} \) (and, in particular, of \( \varphi \)) is contained in \( M_2 \). It thus follows from Proposition 6.2 that there exist sequences of projections \( q_i \in M^\mu \) and \( e_i \in M_2 \), as well as partial isometries \( v_i \in M \) and faithful normal positive functionals \( \theta_i \) on \( e_iM_2e_i \) such that \( v_i^*v_i = q_i \), \( v_i^*v_i = e_i \), \( \sum q_i = q \) and \( v_i \mu v_i^* = \theta_i \circ E_{M_2} \) on \( e_iM_2e_i \) for all \( i \geq 0 \).

Let \( \pi : M \to N \) be an isomorphism of \( M \) onto \( N \). We first apply the result in the first paragraph to \( \mu = \psi \circ \pi \). Since \( \psi_1 \) is weakly mixing, \( M^\mu = \pi^{-1}(N_{\text{ap}}^{\psi_2}) \). We find nonzero projections \( q \in N_{\text{ap}}^{\psi_2} \) and \( e \in M_{\text{ap}}^{\mu} \), a partial isometry \( v \in M \) and a faithful normal positive functional \( \theta \) on \( eM_2e \) such that \( v^*v = \pi^{-1}(q) \), \( vv^* = e \) and \( v\mu v^* = \theta \circ E_{M_2} \) on \( eM_2e \). Since \( \varphi_1 \) is weakly mixing, the almost periodic part of \( \theta \circ E_{M_2} \) equals \( eM_{2, \text{ap}} \). Since \( \psi_1 \) is weakly mixing and \( \psi_2 \) is almost periodic, the almost periodic part of \( \mu \) equals \( \pi^{-1}(N_2) \). It follows that

\[
(7.1) \quad v\pi^{-1}(N_2)v^* = eM_{2, \text{ap}}e
\]

By [HU15, Lemma 2.1], every projection in \( M_2 \) is equivalent, inside \( M_2 \), with a projection in \( M_2^{\text{ap}} \). So conjugating \( e \) and \( \theta \), we may assume that \( e \in M_2^{\text{ap}} \). Then \( e \) and \( \theta \mu e \) are in \( M_2^{\text{ap}} \).

Define the projection \( p \in B(\ell^2(N)) \otimes N_2 \) given by \( p = \sum e_i \otimes p_i \). Define the faithful normal positive functional \( \Omega \) on \( p(B(\ell^2(N)) \otimes N_2) \) given by

\[
\Omega(T) = \sum_i \Omega_i(T_{ii})
\]

Finally define \( W \in \ell^2(N) \otimes N \) given by \( W = \sum e_i \otimes w_i \). It follows that \( W^*W = \pi(e) \), \( WW^* = p \) and

\[
W\mu'W^* = \Omega \circ E_{B(\ell^2(N))} \otimes N_2 \quad \text{on} \quad p(B(\ell^2(N)) \otimes N)p \, .
\]

As above, \( N_{\text{ap}, \pi} = \pi(M_{\text{ap}, \mu}) = \pi(M_2) \). Since \( \psi_1 \) is weakly mixing, the almost periodic part of the functional \( \Omega \circ E_{B(\ell^2(N))} \otimes N_2 \) on \( p(B(\ell^2(N)) \otimes N)p \) is contained in \( p(B(\ell^2(N)) \otimes N_2)p \). It follows that

\[
(7.2) \quad W\pi(eM_2e)W^* \subset p(B(\ell^2(N)) \otimes N_2)p \, .
\]
Write $V = W\pi(v)$. Then $V \in \ell^2(N) \otimes N$ is a partial isometry with $VV^* = p \in B(\ell^2(N) \otimes N_2)$ and $V^*V = q \in N_2$. Using (7.1) and (7.2), we find that

$$\begin{align*}
(7.3) \quad VqN_2qV^* = W\pi(eM_{2,\text{ap}}e)W^* \subset W\pi(eM_2e)W^* \subset p(B(\ell^2(N) \otimes N_2)p.
\end{align*}$$

Since $N_2^{\psi_2}$ has no amenable direct summand, it follows in particular that $N_2$ is diffuse. Then (7.3) implies that $V \in \ell^2(N) \otimes N_2$ and therefore, all the inclusions in (7.3) are equalities. In particular, $eM_{2,\text{ap}}e = eM_2e$ so that (7.1) implies that $qN_2q \cong eM_2e$. This concludes the proof of the proposition. \hfill \Box

Combining Propositions 4.2 and 7.3, we can easily prove Theorem F.

**Proof of Theorem F.** Put $(M, \varphi) = (\Gamma(\mu, +\infty)^\prime, \varphi_{\mu, +\infty})$ as in the formulation of the Proposition. Then, $\varphi$ is weakly mixing and by Corollary C, the free Araki–Woods factor $M$ has all its centralizers amenable. For $i = 1, 2$, let $(A_i, \psi_i)$ be von Neumann algebras with almost periodic faithful normal states having a nonamenable factorial centralizer $A_i^{\psi_i}$. If the free products $(M_i, \varphi_i) = (M, \varphi) * (A_i, \psi_i)$ satisfy $M_1 \cong M_2$, it follows from Proposition 7.3 that there exist nonzero projections $p_i \in A_i$ such that $p_1A_1p_1 \cong p_2A_2p_2$. In the first case, where the $A_i$ are II$_1$ factors, this implies that $A_1 \cong A_2^t$ for some $t > 0$. In the second case, where the $A_i$ are type III factors, this implies that $A_1 \cong A_2$.

For the converse, first assume that the $(A_i, \psi_i)$ are II$_1$ factors with their tracial states and $A_1 \cong A_2^t$ for some $t > 0$. Take nonzero projections $p_i \in A_i$ such that $p_1A_1p_1 \cong p_2A_2p_2$. By the uniqueness of the trace, we have $(p_1A_1(p_1)\psi_1) \cong (p_2A_2(p_2)\psi_2)$. It then follows from Proposition 4.2 that $p_1M_1p_1 \cong p_2M_2p_2$. Since the $M_i$ are type III factors, this further implies that $M_1 \cong M_2$.

Finally assume that the $(A_i, \psi_i)$ are full type III factors with almost periodic states having a factorial centralizer $A_i^{\psi_i}$ and that $\pi : A_1 \to A_2$ is an isomorphism of $A_1$ onto $A_2$. Denote by $\Gamma = \text{Sd}(A_1) = \text{Sd}(A_2)$ the Sd-invariant of $A_1 \cong A_2$. Define $(B_i, \theta_i) = (B(\ell^2(N)) \otimes A_1, \text{Tr} \otimes \psi_i)$. By [Co74, Lemma 4.8], the weight $\theta_i$ on $B_i$ is a $\Gamma$-almost periodic weight. By [Co74, Theorem 4.7], there exists a unitary $U \in B_2$ and a constant $\alpha > 0$ such that $\theta_2 \circ \text{Ad}(U) \circ (\text{id} \otimes \pi) = \alpha \theta_1$. Since $A_i^{\psi_i}$ is a factor, after a unitary conjugacy of $\pi$, we find nonzero projections $p_i \in A_i^{\psi_i}$ such that $\pi(p_1) = p_2$ and $(\psi_2)p_2 \circ \pi = (\psi_1)p_1$ on $p_1A_1p_1$. As in the previous paragraph, we can use Proposition 4.2 to conclude that $M_1 \cong M_2$. \hfill \Box

We finally consider two further structural properties of free Araki–Woods factors: the free absorption property and the structure of its continuous core. We say that a von Neumann algebra $M$ with a faithful normal state $\varphi$ has the free absorption property if the free product $(N, \psi) = (\Gamma(\mu, +\infty)^\prime, \varphi_{\mu, +\infty})$ satisfies $N \cong M$. One of the key results in [Sh96] is the free absorption property for the almost periodic free Araki–Woods factors. In general, we get the following result.

**Proposition 7.4.** Let $(M, \varphi) = (\Gamma(\mu, +\infty)^\prime, \varphi_{\mu, +\infty})$ be a free Araki–Woods factor with infinite multiplicity. Then $(M, \varphi)$ has the free absorption property if and only if the atomic part $\mu_a$ is nonzero.

**Proof.** If $\mu(\{0\}) > 0$, then $(M, \varphi)$ freely splits off $(L(F_\infty), \tau)$ and the free absorption property immediately holds. If $\mu(\{a\}) > 0$ for some $a \neq 0$, then $(M, \varphi)$ freely splits off an almost periodic free Araki–Woods factor of type III and the free absorption property follows from [Sh96, Theorem 5.4]. Conversely, if $\mu_a = 0$, it follows from Corollary C that $M$ has all its centralizers amenable. But then $M$ cannot have the free absorption property. \hfill \Box
One of the most intriguing isomorphism questions for free Araki–Woods factors, well outside the scope of our methods, is whether \( \Gamma(\lambda,1)^{\prime\prime} \cong \Gamma(\lambda+\delta_0,1)^{\prime\prime} \)? In [Sh97a, Theorem 4.8], it was shown that the continuous core of \( \Gamma(\lambda,1)^{\prime\prime} \) is isomorphic with \( B(\ell^2(N)) \otimes L(F_\infty) \). We prove that the same holds for \( \Gamma(\lambda+\delta_0,1)^{\prime\prime} \). Note here that in [Ha15, Corollary 1.10], it is proved that if \( \mu \) is singular w.r.t. the Lebesgue measure \( \lambda \), then the continuous core of \( \Gamma(\mu,m)^{\prime\prime} \) is never isomorphic with \( B(\ell^2(N)) \otimes L(F_\infty) \). Under the stronger assumption that all convolution powers \( \mu^{\otimes n} \) are singular w.r.t. the Lebesgue measure, this was already shown in [Sh02].

**Proposition 7.5.** The continuous core of \( \Gamma(\lambda+\delta_0,1)^{\prime\prime} \) is isomorphic with \( B(\ell^2(N)) \otimes L(F_\infty) \).

**Proof.** In [Sh97a,Sh97b], von Neumann algebras generated by \( A \)-valued semicircular elements are introduced. In the special case where \( A \) is semifinite and equipped with a fixed faithful normal semifinite trace \( \text{Tr} \), the construction can be summarized as follows.

Let \( H \) be a Hilbert \( A \)-bimodule, meaning that \( H \) is a Hilbert space equipped with a normal homomorphism \( A \to B(H) \) and a normal anti-homomorphism \( A \to B(H) \) having commuting images. We denote the left and right action of \( A \) on \( H \) as \( a \cdot \xi \cdot b \) for all \( a,b \in A \), \( \xi \in H \). Further assume that \( S \) is an \( A \)-anti-bimodular involution on \( H \). More precisely, \( S \) is a closed, densely defined operator on \( H \) such that \( S(\xi) \in D(S) \) with \( S(S(\xi)) = \xi \) for all \( \xi \in D(S) \) and such that for all \( \xi \in D(S) \) and all \( a,b \in A \), we have \( a \cdot \xi \cdot b \in D(S) \) and \( S(a \cdot \xi \cdot b) = b^* \cdot S(\xi) \cdot a^* \). Define

\[
F_A(H) = L^2(A,\text{Tr}) \oplus \bigoplus_{k=1}^{\infty} (H_{\otimes A} H_{\otimes A} \cdots \otimes A H).
\]

A vector \( \xi \in H \) is called right bounded if there exists a \( \kappa > 0 \) such that \( \| \xi \cdot a \| \leq \kappa \| a \|_{2,\text{Tr}} \) for all \( a \in \mathfrak{m}_N \). For every \( \xi \in H \), there exists an increasing sequence of projections \( p_n \in A \) such that \( p_n \to 1 \) strongly and \( \xi \cdot p_n \) is right bounded for all \( n \). So, the subspace \( H_0 = \{ \xi \in D(S) : \xi \text{ and } S(\xi) \text{ are right bounded} \} \) is dense. For every right bounded vector \( \xi \in H \), we have a natural left creation operator \( \ell(\xi) \in B(F_A(H)) \) and define

\[
\Phi(A,\text{Tr},H,S) = (A \cup \{ \ell(\xi) + \ell(S(\xi))^* : \xi \in H_0 \})^{\prime\prime}.
\]

There is a normal conditional expectation \( E : \Phi(A,\text{Tr},H,S) \to A \) given by \( E(x)P = PxP \), where \( P : F_A(H) \to L^2(A,\text{Tr}) \) is the orthogonal projection. By [Sh97b, Proposition 5.2], we have that \( E \) is faithful. By construction, if \( A = C_1 \) and \( \text{Tr}(1) = 1 \), and using the notation introduced before Lemma 4.3, we find the free Araki–Woods factor \( \Phi(C_1,\text{Tr},H,S) \cong (H,S)^{\prime\prime} \).

The above construction can be applied to a normal completely positive map \( \varphi : A \to A \) satisfying \( \varphi(a) \circ \varphi(b) = \varphi(\varphi(a) \cdot b) \) for all \( a,b \in \mathfrak{m}_N \). To such a map \( \varphi \), we associate the Hilbert \( A \)-bimodule \( H_\varphi \) by separation and completion of \( A \otimes \mathfrak{m}_N \) with inner product

\[
(a \otimes \varphi b, c \otimes \varphi d) = \text{Tr}(b^* \varphi(a^* c)d).
\]

We also define the anti-unitary involution \( S(a \otimes \varphi b) = b^* \otimes \varphi a^* \). We denote the resulting von Neumann algebra \( \Phi(A,\text{Tr},H,S) \) as \( \Phi(A,\text{Tr},\varphi) \).

Given a trace preserving inclusion \( (A,\text{Tr}) \subset (D,\text{Tr}) \), we denote by \( E : D \to A \) the unique \( \text{Tr} \)-preserving conditional expectation and define \( \psi : D \to D : \psi(d) = \varphi(E(d)) \) for all \( d \in D \). Then, the functor \( \Phi \) satisfies

\[
(7.4) \quad \Phi(A,\text{Tr},\varphi)^\ast_A D \cong \Phi(D,\text{Tr},\psi)
\]

where the amalgamated free product is taken w.r.t. the canonical conditional expectations.

Let \( (M,\varphi) = (\Gamma(\lambda,1)^{\prime\prime},\varphi_{\lambda,1}) \). We can then reformulate [Sh97a, Theorem 4.1] as

\[
(7.5) \quad c_\varphi(M) \cong \Phi(A,\text{Tr},\varphi)
\]
where \( A = L^\infty(R) \), \( \text{Tr}(f) = \int_R f(x)\exp(-x)\,dx \) and \( \varphi : A \to A \) is such that the associated \( A \)-bimodule \( H \) is isomorphic with the coarse \( A \)-bimodule \( L^2(R^2) \) with anti-unitary involution \( (S\xi)(x,y) = \overline{\xi(y,x)} \). Under the identification \( L_\varphi(R) \cong L^\infty(R) \), the isomorphism in (7.5) respects the canonical conditional expectations \( c_\varphi(M) = L_\varphi(R) \) and \( \Phi(A, Tr, \varphi) \to A \).

By [Sh97a, Theorem 4.8], we have in this particular case that \( \Phi(A, Tr, \varphi) \cong B(\ell^2(N)) \otimes L(F_\infty) \).

Let now \((D, Tr_D)\) be an arbitrary diffuse abelian von Neumann algebra with a faithful normal semifinite trace satisfying \( Tr(1) = +\infty \) and let \( \psi : D \to D \) be any normal completely positive map satisfying \( Tr_D(\psi(c)d) = Tr_D(c\psi(d)) \) for all \( c, d \in \pi_{Tr_D} \) such that the associated \( D \)-bimodule \( H \) and anti-unitary involution \( S \) are isomorphic with \( L^2(D, Tr_D) \otimes L^2(D, Tr_D) \) with \( S(c \otimes d) = d^* \otimes c^* \). Since there exists an isomorphism \( \alpha : D \to A \) of \( D \) onto \( A \) satisfying \( Tr \circ \alpha = Tr_D \), it follows that

\[
\Phi(D, Tr_D, \psi) \cong \Phi(D, Tr_D, L^2(D, Tr_D) \otimes L^2(D, Tr_D), S)
\]

(7.6)

\[
\cong \Phi(A, Tr, L^2(A, Tr) \otimes L^2(A, Tr), S)
\]

\[
\cong \Phi(A, Tr, \varphi) \cong B(\ell^2(N)) \otimes L(F_\infty)
\]

Write \((M_1, \varphi_1) = (\Gamma(\lambda + \delta_0, 1)^\tau, \varphi_3 + \delta_0, 1) \). Since \((M_1, \varphi_1) \cong (M, \varphi) \ast (B, \tau) \) for some diffuse abelian von Neumann algebra \( B \) with faithful normal state \( \tau \), it follows that

\[
c_{\varphi_1}(M_1) \cong c_\varphi(M) \ast_{L_\varphi(R)} (L_\varphi(R) \otimes B)
\]

Since the isomorphism in (7.5) respects the conditional expectations, we conclude that

\[
c_{\varphi_1}(M_1) \cong \Phi(A, Tr, \varphi) \ast_A (A \otimes B)
\]

where the conditional expectation \( A \otimes B \to A \) is given by \( \text{id} \otimes \tau \). Write \( D = A \otimes B \) and define \( \psi : D \to D : \psi(d) = \varphi((\text{id} \otimes \tau)(d)) \otimes 1 \). By (7.4), we get that

\[
\Phi(A, Tr, \varphi) \ast_A (A \otimes B) \cong \Phi(D, Tr \otimes \tau, \psi) \]

Since \( D \) is diffuse abelian and the \( D \)-bimodule associated with \( \psi \) is isomorphic with the coarse \( D \)-bimodule, it follows from (7.6) that \( \Phi(D, Tr \otimes \tau, \psi) \cong B(\ell^2(N)) \otimes L(F_\infty) \). So, the proposition is proved.

\[\square\]

References

[BH16] R. Boutonnet, C. Houdayer, Structure of modular invariant subalgebras in free Araki–Woods factors. Anal. PDE 9 (2016), 1989–1998.

[Co72] A. Connes, Une classification des facteurs de type III. Ann. Sci. École Norm. Sup. 6 (1973), 133–252.

[Co74] A. Connes, Almost periodic states and factors of type III\(_1\). J. Funct. Anal. 16 (1974), 415–445.

[Co80] A. Connes, Classification des facteurs. In “Operator algebras and applications, Part 2 (Kingston, 1980)”, Proc. Sympos. Pure Math. 38, Amer. Math. Soc., Providence, 1982, pp. 43–109.

[Dy92] K. Dykema, Free products of hyperfinite von Neumann algebras and free dimension. Duke Math. J. 69 (1993), 97–119.

[Dy96] K. Dykema, Free products of finite-dimensional and other von Neumann algebras with respect to nontracial states. In “Free probability theory (Waterloo, ON, 1995)”, Fields Inst. Commun. 12, Amer. Math. Soc., Providence, 1997, pp. 41–88.

[Ha77] U. Haagerup, Operator-valued weights in von Neumann algebras. I. J. Funct. Anal. 32 (1979), 175–206.

[Ha85] U. Haagerup, Connes’ bicentralizer problem and uniqueness of the injective factor of type III\(_1\). Acta Math. 69 (1986), 95–148.

[Ha15] B. Hayes, 1-bounded entropy and regularity problems in von Neumann algebras. Int. Math. Res. Not. IMRN, to appear. arXiv:1505.06682.

[Ho06] C. Houdayer, On some free products of von Neumann algebras which are free Araki-Woods factors. Int. Math. Res. Not. IMRN 23 (2007), art. id. rnm098.

[Ho08] C. Houdayer, Free Araki-Woods factors and Connes’ bicentralizer problem. Proc. Amer. Math. Soc. 137 (2009), 3749-3755.

[Ho10] C. Houdayer, Structural results for free Araki–Woods factors and their continuous cores. J. Inst. Math. Jussieu 9 (2010), 741–767.
C. Houdayer, Y. Isono, Unique prime factorization and bicentralizer problem for a class of type III factors. Adv. Math. 305 (2017), 402–455.

C. Houdayer, É. Ricard, Approximation properties and absence of Cartan subalgebra for free Araki–Woods factors. Adv. Math. 228 (2011), 764–802.

C. Houdayer, Y. Ueda, Rigidity of free product von Neumann algebras. Compos. Math. 152 (2016), 2461–2492.

C. Houdayer, ´E. Ricard, Approximation properties and absence of Cartan subalgebra for free Araki–Woods factors. Adv. Math. 228 (2011), 764–802.

M. Izumi, R. Longo, S. Popa, A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras. J. Funct. Anal. 155 (1998), 25–63.

W. Krieger, On ergodic flows and the isomorphism of factors. Math. Ann. 223 (1976), 19–70.

M. Lemanczyk, F. Parreau, On the disjointness problem for Gaussian automorphisms. Proc. Amer. Math. Soc. 127 (1999), 2073–2081.

N. Ozawa, Solid von Neumann algebras. Acta Math. 192 (2004), 111–117.

S. Popa, On a class of type II1 factors with Betti numbers invariants. Ann. of Math. 163 (2006), 809–899.

S. Popa, Strong rigidity of II1 factors arising from malleable actions of w-rigid groups. I. Invent. Math. 165 (2006), 369–408.

R.T. Powers, Representations of uniformly hyperfinite algebras and their associated von Neumann rings. Ann. of Math. 86 (1967), 138–171.

F. Rădulescu, The fundamental group of the von Neumann algebra of a free group with infinitely many generators is R+ \ {0}. J. Amer. Math. Soc. 5 (1992), 517–532.

W. Rudin, Fourier analysis on groups. Interscience Tracts in Pure and Applied Mathematics 12, John Wiley and Sons, New York, London, 1962.

D. Shlyakhtenko, Free quasi-free states. Pacific J. Math. 177 (1997), 329–368.

D. Shlyakhtenko, Some applications of freeness with amalgamation. J. Reine Angew. Math. 500 (1998), 191–212.

D. Shlyakhtenko, A-valued semicircular systems. J. Funct. Anal. 166 (1999), 1–47.

D. Shlyakhtenko, On the classification of full factors of type III. Trans. Amer. Math. Soc. 356 (2004), 4143–4159.

D. Shlyakhtenko, On multiplicity and free absorption for free Araki–Woods factors. Preprint. arXiv:math/0302217

M. Takesaki, Theory of operator algebras. II. Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003. xxii+518 pp.