Fractional derivatives of some special functions using ABR and ABC derivatives

Prabha R.¹ & Kiruthika S.²
¹ Department of Science and Humanities, Ahalia School of Engineering and Technology, Palakkad, India.
² Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham, Coimbatore, India.
E-mail: s.kiruthika@cb.amrita.edu

Abstract
In this paper, we present the Atangana Baleanu fractional derivatives of some special functions such as trigonometric, exponential and hyperbolic functions. The AB fractional derivatives are calculated using the formula for RL derivatives and are calculated for all functions which belong to \(L^1[a,b]\). The importance of these derivatives comes from the fact that certain dissipative phenomena cannot be explained using classical fractional operators.

Keywords: Atangana Baleanu fractional derivatives, Mittag Leffler kernel, Normalisation function

1. Introduction
Fractional calculus is an emerging field of research in the past two decades with expeditious development and continuous extension which has its origin 3 centuries ago. The classical calculus gives prominence on finding the solutions of linear differential equations [25, 19] whereas the fractional calculus gives importance for finding the solutions of nonlinear differential equations which are used to solve differential equations of non-integer order and applied in the field of confined ground water modelling [2, 17, 18], optimization theory [1, 6] and so on. Fractional calculus is applied and practised in many research fields, especially in medicine where it is used for the control of diseases like dengue, tuberculosis, ebola, hepatitis, blood flow [24] and in physics for stability, image processing [23, 14, 16, 13]. The contributors in the field of fractional calculus include Riemann Liouville, Caputo, Abel, etc. Out of all, Niels Henrick Abel made a tremendous contribution to the field of fractional calculus.
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in generalization of the tautochrone problem for finding the solution of curve along which a bead can move down with the least time of descent [4]. The nonlocal property of the fractional derivatives having memory effect attracted the interests of researchers to design various modelling problems which cannot be accomplished using derivatives of integer order [12, 9].

Several definitions for fractional derivatives are derived in a manner that the classical derivatives are substituted and used in different fields of research. During the initial stages, researchers used classical fractional operators defined using a singular kernel which is not capable of solving complicated problems of real dynamics. Thus, a new class of fractional derivatives using non-singular kernel is introduced for modelling problems [5, 6, 11, 8] which include Caputo Fabrizio derivatives having exponential kernel and Atangana Baleanu derivatives having Mittag Leffler kernel. These derivatives with non-singular kernels have captured the attention of many scientists not only in mathematics but also in economics, social sciences, data science and so on. Their importance is mainly focused in the AB model of fractional calculus [3, 21, 14], in mechanisms of biological systems for the treatment of cancer and so on. One such method is chemotherapy which has got many side effects such as vomiting, hair loss, and many serious health issues. Hence a mathematical model using these fractional derivatives with non-singular kernel has been employed to optimize the drug content so that the chemotherapy side effects are minimized. This motivated us to derive the derivatives of these special functions.

In this paper, we derive the ABR, ABC derivatives of some special functions using the Maclaurin’s series expansion and these derivatives are obtained using the Riemann Liouville integral of the power function of various orders. The paper is structured as follows: In section 2, we have given the definitions, results and formulae used. We present the ABR and ABC derivatives of trigonometric, exponential and hyperbolic functions in sections 3, 4 and 5 respectively and the conclusion is given in section 6.

2. Preliminaries

Definition 2.1. [10] The Atangana Baleanu Riemann Liouville derivative of order \( \alpha, 0 < \alpha < 1 \), is given by

\[
ABR \ D^\alpha_a f(t) = \frac{B(\alpha)}{1 - \alpha} \frac{d}{dt} \int_a^t f(x) E_\alpha\left( -\frac{\alpha}{1 - \alpha}(t - x)^\alpha \right) dx.
\]

Definition 2.2. [10] The Atangana Baleanu Caputo derivative of order \( \alpha, 0 < \alpha < 1 \), is given by

\[
ABC \ D^\alpha_a f(t) = \frac{B(\alpha)}{1 - \alpha} \int_a^t f'(x) E_\alpha\left( -\frac{\alpha}{1 - \alpha}(t - x)^\alpha \right) dx.
\]
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In the above definitions, the Mittag Leffler function in one parameter is given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(\alpha k + 1)}$$

We have used the definitions of ABR and ABC fractional derivatives of order $\alpha$, $0 < \alpha < 1$, using Mittag Leffler kernel [15, 7] needed to establish our results and studied the existence of fractional derivatives for the trigonometric, exponential and hyperbolic functions which are useful in solving the differential equations of the same kind. The normalisation function $B(\alpha)$ can be chosen according to our wish to satisfy certain properties [7, 10]. So, throughout this paper, in the derivation of ABR and ABC derivatives of the special functions, we take $a = 0$, $\alpha = \frac{1}{2}$, $B\left(\frac{1}{2}\right) = 1$ and the RL integral [12, 20] of the power function of order $\frac{k}{2} + 1$. Also the final results of ABR and ABC derivatives of special functions are obtained as a sum of Mittag-Leffler functions in two parameters [19] which plays a vital role in fractional calculus, as gamma function does in the ordinary calculus.

3. ABR and ABC derivatives of sine and cosine functions

3.1. ABR derivative of sine function

$$ABR D_{0+}^{\frac{1}{2}} \sin t = B\left(\frac{1}{2}\right) \frac{d}{dt} \int_0^t \sin x E_{\frac{1}{2}} \left(\frac{-\frac{1}{2}(t-x)^{\frac{1}{2}}}{1-\frac{1}{2}}\right) dx$$

$$= 2B\left(\frac{1}{2}\right) \frac{d}{dt} \int_0^t \sin x E_{\frac{1}{2}} \left(-\left(t-x\right)^{\frac{1}{2}}\right) dx$$

$$= 2 \frac{d}{dt} \int_0^t \sin x E_{\frac{1}{2}} \left(-(t-x)^{\frac{1}{2}}\right) dx$$

$$= 2 \frac{1}{\Gamma\left(\frac{1}{2} + 1\right)} \frac{d}{dt} \int_0^t \sin x \sum_{k=0}^{\infty} (-1)^k (t-x)^{\frac{k}{2}} dx$$

$$= 2 \frac{1}{\Gamma\left(\frac{1}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \frac{d}{dt} \int_0^t \sin x (t-x)^{\frac{k}{2}} dx$$

$$= 2 \frac{1}{\Gamma\left(\frac{1}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \frac{d}{dt} \left\{ \int_0^t x(t-x)^{\frac{k}{2}} - \int_0^t \frac{t^3}{3!}(t-x)^{\frac{k}{2}} + \int_0^t \frac{t^5}{5!}(t-x)^{\frac{k}{2}} - \cdots \right\} dx$$

$$= 2 \sum_{k=1}^{\infty} (-1)^{k-1}(t)^{(2k-1)}E_{\frac{1}{2},2k}(-t)\frac{1}{2}.$$
3.2. ABC derivative of sine function

\[ \text{ABC } D_{0+}^{\frac{1}{2}} \sin t = \frac{B\left(\frac{1}{2}\right)}{1 - \frac{t}{2}} \int_0^t \cos x \ E_{\frac{1}{2}} \left(\frac{1}{2}(t-x)^{\frac{1}{2}}\right) dx \]

\[ = 2B\left(\frac{1}{2}\right) \int_0^t \cos x \ E_{\frac{1}{2}} \left(- (t-x)^{\frac{1}{2}}\right) dx \]

\[ = 2 \int_0^t \cos x \ E_{\frac{1}{2}} \left(- (t-x)^{\frac{1}{2}}\right) dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \int_0^t \cos x \sum_{k=0}^{\infty} (-1)^k (t-x)^{\frac{k}{2}} dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \int_0^t \cos x (t-x)^{\frac{k}{2}} dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \left\{ \int_0^t 1(t-x)^{\frac{k}{2}} - \int_0^t x^2 2^{\frac{k}{2}}(t-x)^{\frac{k}{2}} + \int_0^t x^4 4^{\frac{k}{2}}(t-x)^{\frac{k}{2}} - \cdots \right\} dx \]

\[ = 2 \sum_{k=1}^{\infty} (-1)^{k-1} t^{(2k-1)} E_{\frac{1}{2}2k}(t)^{\frac{k}{2}}. \]

3.3. ABR derivative of cosine function

\[ \text{ABR } D_{0+}^{\frac{1}{2}} \cos t = \frac{B\left(\frac{1}{2}\right)}{1 - \frac{1}{2}} \frac{d}{dt} \int_0^t \cos x \ E_{\frac{1}{2}} \left(\frac{1}{2}(t-x)^{\frac{1}{2}}\right) dx \]

\[ = 2B\left(\frac{1}{2}\right) \frac{d}{dt} \int_0^t \cos x \ E_{\frac{1}{2}} \left(- (t-x)^{\frac{1}{2}}\right) dx \]

\[ = 2 \frac{d}{dt} \int_0^t \cos x \ E_{\frac{1}{2}} \left(- (t-x)^{\frac{1}{2}}\right) dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \frac{d}{dt} \int_0^t \cos x \sum_{k=0}^{\infty} (-1)^k (t-x)^{\frac{k}{2}} dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \frac{d}{dt} \int_0^t \cos x (t-x)^{\frac{k}{2}} dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \left\{ \int_0^t 1(t-x)^{\frac{k}{2}} - \int_0^t x^2 2^{\frac{k}{2}}(t-x)^{\frac{k}{2}} + \int_0^t x^4 4^{\frac{k}{2}}(t-x)^{\frac{k}{2}} - \cdots \right\} dx \]

\[ = 2 \sum_{k=1}^{\infty} (-t)^{k-1} E_{\frac{1}{2}2k-1}(t)^{\frac{k}{2}}. \]
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3.4. ABC derivative of cosine function

\[ \text{ABC} D_{0+}^\alpha \cos t = B\left(\frac{1}{2}\right) \int_0^t \sin x E_{\frac{1}{2}}\left(-\frac{1}{2}(t-x)^{\frac{1}{2}}\right) \, dx \]

\[ = 2B\left(\frac{1}{2}\right) \int_0^t \sin x E_{\frac{1}{2}}\left(-x^{\frac{3}{2}}\right) \, dx \]

\[ = 2 \int_0^t \sin x E_{\frac{1}{2}}\left(-x^{\frac{3}{2}}\right) \, dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \int_0^t \sin x \sum_{k=0}^{\infty} (-1)^k x^{\frac{k}{2}} \, dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \int_0^t \sin x(t-x)^{\frac{k}{2}} \, dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \left\{ \int_0^t -x(t-x)^{\frac{k}{2}} + \int_0^t x^3(t-x)^{\frac{k}{2}} - \int_0^t \frac{x^5}{5!}(t-x)^{\frac{k}{2}} + \cdots \right\} \, dx \]

\[ = 2 \sum_{k=1}^{\infty} (-1)^k (t)^{\frac{k}{2}} E_{\frac{1}{2}k+1}(-t)^{\frac{1}{2}}. \]

4. ABR and ABC derivatives of Exponential function

4.1. ABR derivative of the Exponential function

\[ \text{ABR} D_{0+}^\alpha e^t = B\left(\frac{1}{2}\right) \frac{d}{dt} \int_0^t e^x E_{\frac{1}{2}}\left(-\frac{1}{2}(t-x)^{\frac{1}{2}}\right) \, dx \]

\[ = 2 \frac{d}{dt} \int_0^t e^x E_{\frac{1}{2}}\left(-x^{\frac{3}{2}}\right) \, dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \frac{d}{dt} \int_0^t e^x \sum_{k=0}^{\infty} (-1)^k x^{\frac{k}{2}} \, dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \frac{d}{dt} \int_0^t e^x(t-x)^{\frac{k}{2}} \, dx \]

\[ = 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \frac{d}{dt} \int_0^t (1 + x^{\frac{3}{2}} + \frac{x^2}{2!} + \cdots)(t-x)^{\frac{k}{2}} \, dx \]

\[ = 2 \sum_{k=1}^{\infty} t^{k-1} E_{\frac{1}{2}k}(-t)^{\frac{1}{2}}. \]
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4.2. ABC derivative of the Exponential function

\[
\text{ABC} D_{0+}^\frac{1}{2} e^t = \frac{B(\frac{1}{2})}{1 - \frac{1}{2}} \int_0^t e^x E_\frac{1}{2} \left( \frac{-\frac{1}{2}}{1 - \frac{1}{2}} (t - x) \right) dx \\
= 2B\left(\frac{1}{2}\right) \int_0^t e^x E_\frac{1}{2} \left( -(t - x) \right) dx \\
= \int_0^t e^x E_\frac{1}{2} \left( -(t - x) \right) dx \\
= 2 \int_0^t e^x \sum_{k=0}^{\infty} (-1)^k (t - x)^{\frac{k}{2}} dx \\
= 2 \sum_{k=0}^{\infty} (-1)^k \int_0^t e^x \left( 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \right) (t - x)^{\frac{k}{2}} dx \\
= 2 \sum_{k=1}^{\infty} t^k E_{\frac{1}{2},k+1}(t)^{\frac{k}{2}}.
\]

5. ABR and ABC derivatives of the Hyperbolic functions

5.1. ABR derivative of hyperbolic sine function

\[
\text{ABR} D_{0+}^\frac{1}{2} \sinh t = \frac{B(\frac{1}{2})}{1 - \frac{1}{2}} \frac{d}{dt} \int_0^t \sinh x \ E_\frac{1}{2} \left( \frac{-\frac{1}{2}}{1 - \frac{1}{2}} (t - x) \right) dx \\
= 2 \int_0^t \sinh x \ E_\frac{1}{2} \left( -(t - x) \right) dx \\
= 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \frac{d}{dt} \sum_{k=0}^{\infty} (-1)^k (t - x)^{\frac{k}{2}} dx \\
= 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \frac{d}{dt} \sinh x (t - x)^{\frac{k}{2}} dx \\
= 2 \frac{1}{\Gamma\left(\frac{k}{2} + 1\right)} \sum_{k=0}^{\infty} (-1)^k \frac{d}{dt} \int_0^t \left( x + \frac{x^3}{3!} + \cdots \right) (t - x)^{\frac{k}{2}} dx \\
= 2 \sum_{k=1}^{\infty} (t)^{(2k-1)} E_{\frac{1}{2},k+1}(t)^{\frac{k}{2}}.
\]
5.2. ABR derivative of hyperbolic cosine function

\[ \text{ABR} D^\frac{1}{2}_{0+} \cosh t = \frac{B(\frac{1}{2})}{1 - \frac{1}{2}} \frac{d}{dt} \int_0^t \cosh x \ E_\frac{1}{2} \left( \frac{-x(t-x)^\frac{1}{2}}{1 - \frac{1}{2}} \right) dx \]

\[ = 2 \int_0^t \cosh x \ E_\frac{1}{2} \left( -(t-x)^\frac{1}{2} \right) dx \]

\[ = 2 \int_0^t \cosh x \ E_\frac{1}{2} \left( -(t-x)^\frac{1}{2} \right) dx \]

\[ = 2 \int_0^t \cosh x \ E_\frac{1}{2} \left( -(t-x)^\frac{1}{2} \right) dx \]

5.3. ABC derivative of hyperbolic sine function

\[ \text{ABC} D^\frac{1}{2}_{0+} \sinh t = \frac{B(\frac{1}{2})}{1 - \frac{1}{2}} \frac{d}{dt} \int_0^t \cosh x \ E_\frac{1}{2} \left( \frac{-x(t-x)^\frac{1}{2}}{1 - \frac{1}{2}} \right) dx \]

\[ = 2 \int_0^t \cosh x \ E_\frac{1}{2} \left( -(t-x)^\frac{1}{2} \right) dx \]

\[ = 2 \int_0^t \cosh x \ E_\frac{1}{2} \left( -(t-x)^\frac{1}{2} \right) dx \]

\[ = 2 \int_0^t \cosh x \ E_\frac{1}{2} \left( -(t-x)^\frac{1}{2} \right) dx \]
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5.4. ABC derivative of hyperbolic cosine function

\[
\text{ABC} D^\frac{1}{2}_0 \cosht = B(\frac{1}{2}) \int_0^t \sinh x E_\frac{1}{2} \left( -\frac{1}{2} (t-x)^\frac{1}{2} \right) dx \\
= 2 \int_0^t \sinh x E_\frac{1}{2} \left( -(t-x)^\frac{1}{2} \right) dx \\
= 2 \Gamma(\frac{1}{2} + 1) \int_0^t \sinh x \sum_{k=0}^\infty (-1)^k (t-x)^\frac{k}{2} dx \\
= 2 \Gamma(\frac{1}{2} + 1) \sum_{k=0}^\infty (-1)^k \int_0^t \sinh x (t-x)^\frac{k}{2} dx \\
= 2 \Gamma(\frac{1}{2} + 1) \sum_{k=0}^\infty (-1)^k \left\{ \int_0^t x (t-x)^\frac{k}{2} + \int_0^t x^3 (t-x)^\frac{k}{2} + \int_0^t x^5 (t-x)^\frac{k}{2} + \cdots \right\} dx \\
= 2 \Gamma(\frac{1}{2} + 1) \sum_{k=0}^\infty (-1)^k \left\{ \int_0^t x (t-x)^\frac{k}{2} + \int_0^t x^3 (t-x)^\frac{k}{2} + \int_0^t x^5 (t-x)^\frac{k}{2} + \cdots \right\} dx \\
= 2 \Gamma(\frac{1}{2} + 1) \sum_{k=0}^\infty (-1)^k \left\{ \int_0^t x (t-x)^\frac{k}{2} + \int_0^t x^3 (t-x)^\frac{k}{2} + \int_0^t x^5 (t-x)^\frac{k}{2} + \cdots \right\} dx \\
= 2 \sum_{k=1}^\infty (t)^{2k} E_{\frac{1}{2},2k+1}(t)^{\frac{1}{2}}.
\]

6. Conclusion

The ABR and ABC derivatives are used for solving differential equations using numerical methods and Laplace transforms which are applied in the modelling problems of physics, chaos theory and various other fields [17, 18]. The Atangana Baleanu derivatives are also evaluated for many other functions [1, 3, 22, 7] such as power function, product of two functions etc. in which these derivatives satisfy the product rule, chain rule, semigroup property and so on. The results established in this paper can be extended to different orders and these describe a foundation for the theory of AB differintegrals.

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