THE MULTIMODAL AND MULTIPERIOD URBAN TRANSPORTATION INTEGRATED TIMETABLE CONSTRUCTION PROBLEM WITH DEMAND UNCERTAINTY

PAULINA ÁVILA-TORRES*
Av. Pedro de Alba, San Nicolás de los Garza, NL 66450, México
PhD Student in Program for Economy and Enterprise at the University of Málaga

FERNANDO LÓPEZ-IRARRAGORRI, RAFAEL CABALLERO AND YASMÍN RÍOS-SOLÍS
Universidad Autónoma de Nuevo León
Av. Pedro de Alba
San Nicolás de los Garza, NL 66450, México
Universidad de Málaga
Campus El Ejido S/N
Málaga, 29071, España

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Abstract. The urban transport planning process has four main activities: Network design, Timetable construction, Vehicle scheduling and Crew scheduling; each activity has subactivities. In this paper the authors work with the subactivities of timetable construction: minimal frequency calculation and departure time scheduling. The authors propose to solve both subactivities in an integrated way. The developed mathematical model allows multi-period planning and it can also be used for multimodal urban transportation systems. The authors consider demand uncertainty and the authors employ fuzzy programming to solve the problem. The authors formulate the urban transportation timetabling construction problem as a bi-objective problem: to minimize the total operational cost and to maximize the number of multi-period synchronizations. Finally, the authors implemented the SAUGMECON method to solve the problem.

1. Introduction. The urban transport planning problem is very complex, this is the reason Desaulniers & Hickman [9], propose to divide the whole urban transport planning problem into: strategical, tactical, operational, and during operations, real-time control. Strategic planning problems concern long-term decisions such as the design of transit routes and networks. Most of these problems fall into the category of network design problems and require solving passenger assignment problems. Tactical planning problems concern the decisions related to the service offered to the public, namely the frequencies of service and the timetables. Operational planning problems relate to how the operations should be conducted to offer the proposed service at minimum cost. The problem the authors tackle in this paper is about

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* Corresponding author: Paulina Ávila-Torres.
tactical planning: this kind of planning is solved generally on a seasonal basis or scenarios [26]. It is important to solve this problems because there is an urgent need to reduce traffic congestion and improve the quality of life [22].

The urban transport planning process has four main activities: 1) Network design, 2) Timetable construction, 3) Vehicle scheduling and 4) Crew scheduling, each one of these has subactivities and the output of each activity or subactivity is the input for the next one [5]. In that way, timetable development has frequency calculation and departures time assignment and they are usually executed in sequence by planners, which have been found by [3] to provide suboptimal solutions than the integrated approach.

The problem being addressed in this research is about the timetable construction problem (second activity of Fig. 1). This problem consists of: (i) the minimal frequency of departures of units (maximum quantity of departures), and (ii) departure time of each unit into a given planning horizon (See Fig. 2).

**Figure 1.** Transport planning process [5]
Unlike previous papers in this approach: (a) (i) and (ii) are addressed in an integrated way, or simultaneously solved, (b) multi-period planning is addressed and multi-period synchronizations are considered, (c) the characteristics (1) to (8) are included and mentioned later in this section.

The main scientific contribution of this paper is a biobjective fuzzy lineal integer mixed programming model for the integrated problem of timetable construction and represented in this model are the characteristics mentioned above and as [11] mentioned mathematical modelling is a powerful technique to solve multi-objective decision making problems.

Typically, the timetable is built trying to minimize the operational cost, total travel time of passengers [26],[31]; generally implemented through a proxy measure like maximization of synchronization to reduce the maximum waiting time at transfers. The synchronizations occur at transfer nodes (See Fig. 3).
The integration of minimal frequency calculation and departures times assignment, besides making this problem harder to solve from a computationally perspective than solve each subactivity sequentially [13], makes the problem harder to model because the number of departures is unknown at the time when the departures times should be assigned. This is not the case when both subactivities are solved sequentially, in this case the minimal frequency is calculated first then it is used as input for the assignment of departures times. So far, the authors have found no paper where minimal frequency calculation and departures times assignments are solved in an integrated way.

Multi-period scheduling and multi-period synchronizations also make the problem harder to solve and to model, thus more decision variables, and indexes and constraints are included in the model. Also the authors have not found any previous paper where multi-period synchronizations are addressed.

In what follows, the notable characteristics of the problem are synthesized, the authors group those into typical (1-4 of the next list), present in most of the papers reviewed; recently incorporated (5 and 6 of the following list), present in most recent papers; and desirable (7 and 8 of the list), not found in any of the reviewed papers:

1. Minimal frequency determination depends of headways definition, also departure times depends of minimal frequency (Column 1 of Table 1). For example, [12], [26] and [2] focus on the problem of frequency setting and their objective is to minimize the total cost, with another author working on the problem of minimal frequency: [24] who also combines the frequency setting problem with timetables.

2. There are certain nodes of the transportation system network called transfer nodes. At these nodes, passengers change route. A transfer will occur when the involved routes synchronize to allow passengers to do the transfer within a fixed time window [14](Column 2 of Table 1). For example, Eranki [10] proposes a model whose objective is to maximize the number of synchronization subjects to several policy constraints. She added a special characteristic, allowing synchronizations within a window time. Ibarra-Rojas & Ríos-Solís [13] formulate the timetabling problem with the objective of maximizing the number of synchronizations to facilitate passenger transfers and avoid bus bunching along the network. Zhang et al. [28] Besides proposing a multi-modal model, they consider two kind of transfers: one is between the same mode; the other one is between different modes.

3. There are certain nodes of the transportation system networks called bunching nodes. At these nodes typically, there is bunching among the vehicles. This bunching should be controlled (Column 3 of Table 1). In the literature, some authors including bunching nodes are [13].

4. The operational cost of a trip can be approximately calculated as i) a variable cost per kilometer of the route, that could include driver expenses, maintenance of the vehicle, fuel and other typical expenses; and ii) a fixed cost related to the organization that manage the urban transportation system network (Column 4 of Table 1). The cost is the most popular objective function ([7],[12], [2], [15], [6], [30]).

5. The urban transportation system network is typically composed by more than one transportation mode, each one with its own regulation structure and requirements (Column 5 of Table 1). Liu et al. [15] propose a model where the passengers, to get their destination can alternate between different modes.
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(buses and subways). They minimize the total travel time and total travel cost, where transfer time includes, the walking time, the waiting time and the stopping time. Zhang et al. [28] proposed a super-network approach where the networks for different modalities are integrated in a single network. This model provides a multi-modal routing system. Also, Wang et al. [27] proposed an integrated, region-urban, multimodal transportation model. The model introduces a time allocation step, suggests better approaches in steps such as trip distribution and mode choice, and improves the connections among each mode.

6. Although demand is unknown at the time of planning, generally demand variation within a time period follows certain patterns which depends on the time of the day, the day of the week or the season of the year, among other less frequent scenarios (Column 6 of Table 1). Authors like [25], [19], [7], [21], [12], [2] consider the stochastic nature of some parameters like demand and travel time; Chen et al. [7] address the problem of computing multiple headways and take into account stochastic demand and travel time; while [21] and [24] work with bus routing- timetables and frequency setting-timetables respectively. Both consider uncertainty at some parameters and their approaches apply between cities and only work for buses.

7. Typically, the planning horizon is fixed a priori. Generally, demands vary significantly through the planning horizon. Due to this, the schedulers divide the planning horizon into multiple periods within them. The demand varies a little (fluctuates around a central value), but demand varies considerably between two consecutive periods (Column 7 of Table 1). Successfully addressing this issue also implies tackling multi-period synchronizations. Only one previous work address multi-period scheduling (see Fig. 4) Ibarra-Rojas [13], although multi-period synchronizations are not modeled.

8. The integration of frequency calculations and departures-scheduling is a desirable characteristic of any mathematical model and solution-oriented approach for this problem (Column 8 of Table 1). Otherwise suboptimal solutions are suboptimal timetables [3]. Some authors take into account two or more activities of the urban transport system like, [6], [30], and [23]. Chakroborty [6] works with the network design (transit routing) and timetable (scheduling) problem, but in a sequential way. His objective is to minimize the total transfer time and the total initial waiting time- the stopping time of a bus at a stop. Zhao & Zeng [30] present a metaheuristic method for optimizing transit networks that includes network design (route network design) and timetables (vehicle headway and timetable assignment). Their objective is to minimize the passenger cost function. Szeto & Wu [23] aim to reduce the number of transfers and the total travel time (including in-vehicle travel time and waiting time) of users by solving the bus network design (route design) and timetable (frequency setting) problem simultaneously.

It is worth mentioning that the authors also have not found a published paper about urban transport planning where all issues (1) - (8) are considered for developing models or solutions methods for that problem. This affirmation is supported by [8], where presented is an exhaustive literature review of papers dealing with timetable and frequency calculation.
It is also worth mentioning that, the proposed model can be applied for monomodal transportation systems and multimodal transportation systems without any extra work needed to make the switch from one mode to another. However, the most important feature is multimodal synchronizations, to avoid bunching between different transport systems, allowing effective transfers, since in transfer nodes demand can be distributed between different transport modes, additionally to define policies for operations cost, service level, etc.

We mentioned earlier the problem being addressed in this paper is related to tactical planning, therefore the authors do not pay too much attention to the service level because it makes more sense to do so when the authors are dealing with the operational planning. However, Avila et al. consider a minimal service level at the frequency calculation due to the methods the authors employed.

The paper is organized as follows: In Section 2 the mathematical model is presented. Also, the preprocessing and the assumptions made. In Section 3, the authors show and analyze the results of numerical experiments. Finally, in Section 4 conclusions and future work are mentioned.

2. Mathematical model.

2.1. Modeling approach. In this section, the authors will refer briefly to the original approach they follow in this paper for representing the main elements of the problem being addressed: the modeling of decision variables (departures, synchronizations) and constraints (synchronizations and headways policies). What makes
a difference with other approaches found in the reviewed literature are the characteristics (1) to (8) mentioned in the introduction, which are fully implemented in their proposed model.

Authors assume to know an average value around which demand fluctuates on the node for each route in each period and this behavior can be represented as a triangular, fuzzy number. Since authors are interested in the local behavior, there is no problem in following the fuzzy approach.

2.2. Departures assignment by time intervals. Two approaches are proposed in [5] for assignment of departures in the timetable construction activity: (i) Even headways and (ii) Even loads work well as a guide for planners, but in real situations where some variability in the system parameters is expected, there is a need for more flexible approaches for assigning the departures. Here the authors present an original approach that offers flexibility, and reduces the size of decision variables and associated constraints in a significant way.

Here Avila et al. assume that the planners are capable of supplying maximum and minimum headway for each route in each period of the scheduling horizon \(H_{max}\) and \(H_{min}\). Departures are modelled as decision variables in the optimization model; typically, departures are arranged as in Fig. 5(a) where each time unit is considered as a variable ([5], [30]). What the authors found inconvenient in their case in the first place was that the number of departures for each route in each time period was not known in advance. Also, this was due to the high dimensionality of the decision variable arrays, and high number of constraints derived from the headway requirements.

To avoid the issues previously mentioned, the authors propose to represent departures as in Fig. 5(b) where they split each time period of the scheduling horizon (for a representation of time periods see Fig. 4) into \(N\) departures intervals \(p_i\) where the first \(N - 1\) departures intervals have a length of exactly \(H_{min}\), the last interval has a length of \(H_{min} + T_v \mod H_{min}\).

Although with this representation, the authors need to introduce another variable \(\alpha\), the dimensionality of the decision variable representing the departures is noticeably reduced in regard to other models presented for timetable construction ([5], [30]). Meanwhile, the number of constraints increase because of the introduction of the \(\alpha\) being negligible. \(N\) is an upper bound of the departures in a time period \(T_v\), and is calculated as \(N_v = \lfloor \frac{H_{max}}{H_{min}} \rfloor\). However, \(N_v\) is also the maximum number of departures that can be scheduled in time period \(T_v\).

2.3. Headways policies. It is common that an assignation of departures in an urban transportation system obeys certain headway policies. The headways can be provided by the decision makers or calculated. Among the most common headway polices are: to divide the period length by the frequency [5] and to provide the headway polices for first departure, for last departure and for two consecutive departures [10], [13].

For a certain route \(i\) of a certain time period \(v\):

(A) a typical policy for the first departure is that this departure should take place between the beginning of the period and the maximum headway \(H_{max}^v\) [10].

(B) two consecutive departures must to be assigned in such a way that they should be separated by at least the minimum headway \(H_{min}^v\) and at most the maximum headway \(H_{max}^v\) [10].

(C) for the last departure a typical policy is to ensure that this departure should
take place between the end of the period and a time into the period fixed by the decision maker [13].

In what follows, the authors formally defined those policies by interval ranges. The range of departures time intervals for a given route \( i \) and a period \( v \) will be represented by the index set \( RD_{v}^{i} = 1 .. N_{v}^{i} \) where \( N_{v}^{i} \) is defined as the maximum number of departures for the route \( i \) in the period \( v \) according to the definition given above.

Let \( RF_{v}^{i} \in RD_{v}^{i} \) be the range of time intervals for the scheduling of the first departure of a route \( i \) and period \( v \) is given by the index set \( \{1 .. tc_{v}^{i}\} \), where \( tc_{v}^{i} \) is calculated as

\[
tc_{v}^{i} = \left\lceil \frac{H_{v}^{i}}{H_{v}^{i \min}} \right\rceil
\]

and represents the ratio between the maximum and minimum headways expressed in time intervals unit (departures units). In other
words, here defined are the time intervals where the first departure of a route in a period can depart. Figure 7

\[ \text{Figure 7. First departure.} \]

Let \( RN_i^v \in RD_i^v \) be the range of time intervals \((p_i^v, q_i^v)\), time interval where a departure is assigned) for the scheduling of two consecutive departures of a route \( i \) and period \( v \) is given by the index set \( \{mcp_i^v..msp_i^v\} \), where \( mcp_i^v \) is calculated as \( mcp_i^v = \max\{q_i^v - tc_i^v, 1\} \) and \(msp_i^v\) is calculated as \(msp_i^v = \max\{q_i^v - 1, 1\}\). That is \( mcp_i^v \) and \(msp_i^v\) are the farthest and the nearest previous intervals where the departure \( p_i^v \) was assigned in relation to the actual interval where departure \( q_i^v \) was assigned. In other words, here defined are the time intervals where a consecutive departure of a route in a period can depart. Figure 8

\[ \text{Figure 8. Consecutive departure.} \]

Let \( RL_i^v \in RD_i^v \) be the range of time intervals for the scheduling of the last departure of a route \( i \) and period \( v \) is given by the index set \( \{tm_i^v..N_i^v\} \), where \( tm_i^v \) is calculated as \( tm_i^v = \left\lceil \frac{-T_{ini} + (T_{fin} - \gamma_i)}{H_{mini}} \right\rceil \), and it is the farthest interval (from \( N_i^v \)) where the last departure must be assigned. In other words, here defined are the time intervals where the last departure of a route in a period can depart. Figure 9

\[ \text{Figure 9. Last departure.} \]

**Lemma 2.1.** Let \( E(S) \) be a scenery for an urban multimodal transportation system \( S \) where \( P \) is a set of compose by policies \( A, B \) and \( C \) and let \( \Lambda(E(S)) \) be a timetable for \( S \) in the scenery \( E(S) \). \( \Lambda(E(S)) \) is feasible if and only if:

1. the first departures for each time period \( v \) of route \( i \) are scheduled in \( RF_i^v \)
2. two consecutive departures \( p, q \) for each period \( v \) of route \( i \), then \( p \) is scheduled in \( RN^v_i \).
3. the last departures for each time period \( v \) of route \( i \) is scheduled in \( RL^v_i \).

2.4. Multimodal and multi-period synchronizations for transfers or to control bunching. In their formulation transfer, synchronization is required at:

a single transfer node (b), transfer between two close nodes (c) (see Fig. 3). Synchronizations are also modeled as decision variables, where each synchronization has an O-D pair of route, and a pair of nodes (where the origin node is the same destination node when the authors are facing a single node synchronization), and also the time scheduling period where the synchronization should take place.

All those properties of a synchronization require at least 5 indices, which in practice yield a huge amount of decision variables representing synchronizations and related constraints, the reason why it is very important to reduce the total amount of variables representing synchronizations. But, for the sake of fine control over synchronizations Avila et al. introduce two further indices: the index of the departure of origin and the index of the departure of destination, so the authors employ seven indices for the variable synchronization.

Synchronizations at transfer, or bunching nodes are directional. This means when a pair of routes synchronize, the transfer can occur from one route to another and vice-versa, then two synchronizations should be counted. The authors consider a round trip as the same route (when distinguished between origin and destination), then the characterization of the period brings the weighing of waiting time naturally by demand as it seeks to minimize the cost among other objectives.

2.5. Fuzzy programming. It is usual that the coefficients of a real linear programming (LP) problem where human estimation is used, are inexact because of either some lack of precision or some vagueness about the data being used in the problem [4]. On the other hand, the decision maker may feel more comfortable in specifying vague over crisp data. In that case, the theoretical support provided by the fuzzy numbers may be very appropriate to model the problem under consideration [4].

In the stochastic programming approach, uncertainty is modelled through discrete or continuous random variables. On the other hand, fuzzy programming considers uncertain parameters as fuzzy numbers and uncertain constraints are treated as fuzzy sets [20].

Here are different types of fuzzy numbers: among them trapezoidal and triangular, for example. In the model presented in the next section, the authors employ triangular fuzzy numbers. A fuzzy number \( \tilde{a} \) in \( \mathbb{R} \) is said to be a triangular fuzzy number if real numbers exist there \( s \) and \( l \), \( r \geq 0 \), such that (1).

\[
\mu_\tilde{a}(x) = \begin{cases} 
\frac{x - (s - l)}{l} & x \in [s - l, s] \\
\frac{(s + r) - x}{r} & x \in [s, s + r] \\
0 & \text{otherwise}
\end{cases}
\]

(1)

A fuzzy linear problem is considered partially fuzzy, if some parameters are fuzzy or completely fuzzy, if all parameters are fuzzy. When there is uncertainty only (partially fuzzy) in constraint coefficients, there are methods that can help us to transform the fuzzy linear problem into a crisp problem by solving a fuzzy number ranking problem. The problem of ranking fuzzy numbers has been extensively studied in the literature [4]. An important ranking method is the k-preference
method, which has been used recently by Perez et al. [18], [17] and it has given
good results.
In this paper, the k-preference method is implemented to compare two fuzzy
numbers \( \tilde{a} = (a_1, a_2, a_3) \leq \tilde{b} = (b_1, b_2, b_3) \) implies [4]:

According to k-preference method

\[
ka + (1 - k)a \leq kb + (1 - k)b
\]

where \( k \) is a confidence level, the confidence that the decision maker has about a
parameter.

In this paper, fuzzy programming is employed instead of stochastic programming
because it is considering the demand within periods that have an almost stable
behavior around certain value. With fuzzy programming the variability or fuzziness
around that value is represented.

2.6. Preprocessing. The first preprocessing procedure the authors applied is
maybe the simplest, but it has turned out to be the more effective. The domain
reduction applied to decision variables, especially in the variable that determines
the departure time. As the authors divide the scheduling horizon into intervals,
then the domain for these variables is delimited by the end of the intervals. Avila
et al. take into consideration the properties of the problem previously described;
they developed the following procedures to preprocess the data and reduce the size
of the valuation of a model instance (to reduce the number of decision variables and
constraints).

2.6.1. Algorithm to determine frequency method. Ceder [5] proposes four methods
to determine the frequency, and he divided them into two groups: max load methods
and load profile methods.

• Max load methods. Method I: Satisfies the demand of the maximum load
  point during the day and Method II: Satisfies the maximum load point of a
time period.
• Load profile methods. Method III: Guarantees that the node with the maxi-
mum load will not have overcrowding and Method IV: To control the possible
overcrowding situations. This method sets a percentage of the route with
overcrowding.

With these methods they can estimate bounds for the frequency of a route. In
order to determine which frequency method to use, Ceder [5] proposes an algorithm
to select the most appropriate. In the flowchart of the Figure 10 this process is
presented. First, they need a ride-check passenger count, then they construct the
load profile (\( \rho \)) for each period, that is the total of passenger-kilometres divided
into the length of the route by the maximum load. If \( \rho \) is less or equal to 0.5, then
they calculate the frequency with method 3 and method 4 (different percentages).
The results of method 3 are considered as lower bound and they use method 4 with
the selected percentage. If \( \rho \) is greater than 0.5 then they compare method 1 and
method 2 with a \( \chi^2 \) test. If the value obtained for method 1 is equal to method 2
then they use method 1, otherwise they use method 2.

By incorporating these frequency calculation methods, the authors guarantee
that important characteristics mentioned by Ceder are taken into account for the
timetable construction. Avila et al. approach these frequencies as lower bounds
calculated in a preprocessing step. Their implementation allows the incorporation of fuzzy parameters in the Ceder methods.

2.6.2. Synchronizations. Inspired by the contribution of Ibarra-Rojas et al. [13], Avila et al. defined the window time that allows to identify feasible pair departures for two routes that should synchronize at a certain synchronization node. In Figure 11 they show that situation.

In the problem that is being investigated here, unlike the problem of Ibarra-Rojas et al., Avila et al. consider multi-period synchronizations, and also the possibility that synchronizations can occur between two different nodes of the system (but also they allow single node synchronization). Also it is possible to synchronize different transport modes.

In the following, they define the procedure:
1. For a given period, \( v \) they select a synchronization node, for each pair of routes that should synchronize: they identify the origin route of the transfer \((O_i)\) and the destination route \((D_j)\).

2. For each pair, \((O_i, D_j)\) they determine the departures \(p_i\) and \(q_j\) of the period \(v\) and the previous periods that arrive at the synchronization node of the period \(v\).

3. For each departure of each set, \(p_i\) they determine which departures of the set \(q_j\) can synchronize the transfer between both, considering the maximum transfer time between both nodes and the holding time at the node of the vehicle unit of the route \(j\).

In this way, the authors restrict the set of variables even more that represent the possible synchronizations, as well as the constraints of synchronizations. In the following proposition, they show this result formally.

**Proposition 1.** Given a period \(v\), and a transfer defined with the nodes \(O\) and \(D\). Let \(S = (i, j)\) the set of pair of routes that should synchronize at \(OD\) in the period \(v\). For each pair of routes \((i, j)\) in \(S\) two departures of \(v\) or previous periods, one of the route \(i\) \((p_i)\) and the other of the route \(j\) \((q_j)\) synchronize at \(OD\) in the period \(v\) if and only if \(w \leq q_j + t_j + d_j - (p_i + t_i + s_i) \leq W\) and both are in the period \(v\). Where \([w, W]\) is defined as the minimum and maximum waiting time for the passengers in the transfer.

2.6.3. Deriving crisp model from fuzzy. The frequency methods proposed by Ceder are for deterministic data. Here, Avila et al. are considering demand uncertainty, the reason why they modified those methods. Demand uncertainty is represented through \((MC_1^v-MC_4^v)\) which indicates the minimum quantity of departures needed to satisfy the demand. Frequency method I \((MC_1^v)\) represents the maximum load point during the day \((\tilde{P}_{\text{max,}d})\) along the route. This point is divided over the desire occupancy \((d_v^i)\) which varies according the route and the period. With this method, they guarantee to satisfy the point with maximum demand of the scheduling horizon. Frequency method II \((MC_2^v)\) represents the maximum load point \((\tilde{P}_{\text{max}})\) for each route in planning period and is divided over the desired occupancy \((d_v^i)\) of the route and the period. Frequency method III \((MC_3^v)\) is based on the maximum between the load point \((\tilde{P}_{\text{max}})\) for a route, a period and the average of passengers-kilometer \((\tilde{Pas}_v)\). This method guarantees that the node with the maximum load will not present overcrowding. Frequency method IV \((MC_4^v)\) is similar to method III, but this method does not exceed a percentage of the length of the route \((\beta_v^i)\). This method sets a level of service restricting the overcrowding to a portion of the route.

\[
\tilde{FC}_i^v = \begin{cases} 
\frac{P_{\text{max,}d}}{d^i_v} & \text{if } MC_1^v = 1 \\
\frac{P_{\text{max}}}{d^i_v} & \text{if } MC_2^v = 1 \\
\frac{P_{\text{max}}}{d^i_v \cdot L_i} & \& \frac{P_{\text{max}}}{\text{cap}^i_v} & \text{if } MC_3^v = 1 \\
\frac{P_{\text{max}}}{d^i_v \cdot L_i} & \& \frac{P_{\text{max}}}{\text{cap}^i_v} & \& \sum_{k \in I^v} l_k \leq \beta_v^i \cdot L_i & \text{if } MC_4^v = 1 
\end{cases}
\]
Remember, the authors are considering triangular fuzzy numbers so they need the central value \((FC_i^v, (3))\), the upper value \((FC_i^v, (4))\) and the lower value \((FC_i^v, (5))\). They used basic operations to transform fuzzy numbers into crisp numbers.

\[
FC_i^v = \begin{cases} 
\frac{P_{\text{max}}}{d_i^v} & \text{if } MC_1^v = 1 \\
\frac{P_{\text{max}}}{d_i^v} & \text{if } MC_2^v = 1 \\
\max(p_{\text{max}}^{i}, p_{\text{max}}^{i}) & \text{if } MC_3^v = 1 \\
\max\left(\frac{P_{\text{max}}^{i}}{d_i^v}, \frac{P_{\text{max}}^{i}}{d_i^v}, l_i \right) \leq \beta_i^v \cdot L_i & \text{if } MC_4^v = 1
\end{cases}
\]

\[
FC_i^v = \begin{cases} 
\frac{d_i^v \cdot P_{\text{max}}}{(d_i^v)^2} & \text{if } MC_1^v = 1 \\
\frac{d_i^v \cdot P_{\text{max}}}{(d_i^v)^2} & \text{if } MC_2^v = 1 \\
\max\left(\frac{P_{\text{max}}^{i}}{d_i^v}, l_i \right) \leq \beta_i^v \cdot L_i & \text{if } MC_3^v = 1 \\
\max\left(\frac{P_{\text{max}}^{i}}{d_i^v}, l_i \right) \leq \beta_i^v \cdot L_i & \text{if } MC_4^v = 1
\end{cases}
\]

\[
FC_i^v = \begin{cases} 
\frac{d_i^v \cdot P_{\text{max}}}{(d_i^v)^2} & \text{if } MC_1^v = 1 \\
\frac{d_i^v \cdot P_{\text{max}}}{(d_i^v)^2} & \text{if } MC_2^v = 1 \\
\max\left(\frac{P_{\text{max}}^{i}}{d_i^v}, l_i \right) \leq \beta_i^v \cdot L_i & \text{if } MC_3^v = 1 \\
\max\left(\frac{P_{\text{max}}^{i}}{d_i^v}, l_i \right) \leq \beta_i^v \cdot L_i & \text{if } MC_4^v = 1
\end{cases}
\]

The minimum frequency \((\text{FreMin}_i^v)\) that their model has to satisfy is the maximum between the total number of departures \((N^v)\), the frequency obtained with one of the frequency methods \((FC_i^v)\) and a basic frequency required \((f_{\text{mr}}^v)\), calculated as \(f_{\text{mr}}^v = \frac{P_{\text{max}}}{d_i^v}\). The frequency obtained with the frequency methods and the frequency required are fuzzy numbers, because the demand is present in them, which converts the minimum frequency also fuzzy, the reason the central value, the upper value \((\text{FreMin}_i^v)\) and the lower value \((\text{FreMin}_i^v)\) are needed.

\[
\text{FreMin}_i^v = \max\{|N^v|, FC_i^v, f_{\text{mr}}^v\} \\
\text{upper value} \quad \text{FreMin}_i^v = \max\{|N^v|, FC_i^v, f_{\text{mr}}^v\} \\
\text{lower value} \quad \text{FreMin}_i^v = \max\{|N^v|, FC_i^v, f_{\text{mr}}^v\}
\]

2.7. Assumptions. Here the authors present the assumptions of the problem addressed in this work:
Periods are created according to the fluctuation of demand; two consecutive periods have different demand in mean over the nodes for all routes. Inside each period, demand has a stable behavior.

- Headways (minimum and maximum) do not change within a period for a route.
- Vehicle units have the same capacity (at this moment, this is unimportant).
- The vehicle fleet is enough to perform the proposed planning schedule.
- Synchronization and bunching nodes are fixed by the planner.
- All passengers desire to transfer to the nearest vehicle unit.
- Demand does not change significantly in each period.
- Demand is unknown, but it can be estimated for each period.
- Period lengths must be enough to allow the schedule of needed departures for satisfying demand.
- The planning requirements must ensure the satisfaction of the demand during the planning periods established.

2.8. **Proposed model.** Now, we describe the sets, variables and parameters forming this model.

**Sets:**
- $M$: Set of routes.
- $K$: Set of nodes.
- $V$: Set of periods.
- $B^v_{ij}$: Set of pairs of nodes, which potentially synchronize the routes $i$ and $j$.
- $J(i)$: Set of routes with nodes in common with the route $i$.

**Variables:**
- $X^v_{ip}$: Indicates there is a trip for the route $i$ at the interval $p$ in the period $v$.
- $\alpha^v_{ip}$: Determines the departure time of the route $i$ at the interval $p$ in the period $v$.
- $Y^v_{ijkupq}$: Indicates there is a synchronization between the route $i$ and $j$ at the node $k$ and $u$ with departure time at the interval $p$ and $q$ respectively during the period $v$.

**Parameters**
- $N^v$: Maximum number of departures in a period $v$.
- $P^v_{\text{max},i}$: Maximum load of passengers on board in the route $i$ in period $v$.
- $P^v_{\text{max},i}$: Upper bound of maximum load of passengers on board on route $i$ in period $v$.
- $P^v_{\text{max},i}$: Lower bound of maximum load of passengers on board on route $i$ in period $v$.
- $P^v_{\text{max},i}$: Maximum load of passengers on board in the day on route $i$.
- $P^v_{\text{max},i}$: Upper bound of maximum load of passengers on board in the day on route $i$.
- $P^v_{\text{max},i}$: Lower bound of maximum load of passengers on board in the day on route $i$.
- $d^v_i$: Desired occupancy of the bus route $i$ in the period $v$.
- $P_{\text{as}^v_i}$: Total amount of passengers/km on route $i$ in period $v$.
- $P_{\text{as}^v_i}$: Upper bound of total amount of passengers/km on route $i$ in period $v$.
- $P_{\text{as}^v_i}$: Lower bound of total amount of passengers/km on route $i$ in period $v$.
- $L^v_i$: Length of route $i$.
- $\text{cap}^v_i$: Capacity of the bus route $i$ in period $v$. 
\( l_k \): Length of node \( k \).
\( \beta^v \): Allowed portion of route \( i \) exceeding the load in period \( v \).
\( H^v_{\text{max},i} \): Maximum headway (minutes) for route \( i \) in period \( v \).
\( H^v_{\text{min},i} \): Minimum headway (minutes) for route \( i \) in period \( v \).
\( T^v \): Planning period (minutes); \( [T^v_{\text{ini}}, T^v_{\text{fin}}] \).
\( T^v_{\text{ini}} \): Start of planning period \( v \) (minutes).
\( T^v_{\text{fin}} \): End of planning period \( v \) (minutes).
\( \gamma^v_{ijk} \): Desire time (minutes) desired before the end of \( T^v \) for the last departure of route \( i \) in period \( v \).
\( W^v_{\text{max},i} \): Maximum waiting time (minutes) of route \( i \) in period \( v \).
\( W^v_{\text{min},i} \): Minimum waiting time (minutes) of route \( i \) in period \( v \).
\( t^v_{ijk} \): Travel time (minutes) from origin point of route \( i \) to node \( k \) in period \( v \).
\( \delta^v_{ijk} \): Minimum time required for a passenger to transfer from node \( k \) of route \( i \) to node \( u \) of route \( j \) in period \( v \).
\( \overline{W}^v_{\text{max},i} \): Average of maximum load of passengers on board during route \( i \) in period \( v \).
\( f^v_{\text{mr},i} \): Minimum required frequency to satisfy the demand of route \( i \) in period \( v \).
\( f^v_{\text{mri}} = \frac{\overline{W}^v_{\text{max},i}}{d^v_i} \).
\( \text{FixedCost}^v_i \): Fixed cost for route \( i \) in period \( v \).
\( \text{VariableCost}^v_i \): Variable cost for route \( i \) in period \( v \).
\( P^v_k \): Average of passengers on board in node \( k \) in period \( v \).
\( P^v_k \): Upper bound of average of passengers on board in node \( k \) in period \( v \).
\( P^v_k \): Lower bound of average of passengers on board in node \( k \) in period \( v \).
\( s^v_{ijk} \): Hold time of a bus on route \( j \) in node \( k \) in period \( v \).
\( \sigma \): Confidence level of demand.
\( FC^v \): Frequency calculated with frequency methods.
\( FC^v \): Upper bound of frequency calculated with frequency methods.
\( FC^v \): Lower bound of frequency calculated with frequency methods.
\( t^v_{ci} = \left\lceil \frac{H^v_{\text{max},i}}{H^v_{\text{min},i}} \right\rceil \)
\( t^v_{ci} = \left\lfloor \frac{H^v_{\text{max},i}}{H^v_{\text{min},i}} \right\rfloor \)
\( t^v_{ci} = \left\lceil \frac{H^v_{\text{max},i}}{H^v_{\text{min},i}} \right\rceil \)
\( mcp^v_i = \max\{p - t^v_{ci}, N^v\} \)
\( mfp^v_i = \max\{p - t^v_{ci}, N^v\} \)
\( msp^v_i = \max\{p - 1, 1\} \)
\( mw^v_i = \min\{tm^v_{ci} + 1, N^v\} \)
\( f^v_i = \min\{j|X^v_{ij} = 1\} \forall v \in V, v > 1 \)
\( l^v_{ci} = \max\{j|X^v_{ij}^{-1} = 1\} \forall v \in V, v > 1 \)

The most popular objective functions are: minimize cost, minimize time (waiting time, total travel time, etc.) and maximize synchronizations [8], among others. This model consists of 2 objective functions, the first objective function (6) minimizes the total operation cost. The authors are considering a fixed cost (\( \text{FixedCost}^v_i \)) and a variable cost (\( \text{VariableCost}^v_i \)) which is affected by the length of the route \( (L_i) \) in kilometers multiplied by the number of departures in a route in a period \( (\sum_{p \in N_v} X^v_{ip}) \). The second function (7) is to maximize the number of synchronizations between two bus routes with departure times in the same period or different periods \( (Y^v_{i,jkupq}) \).
\[
\min \sum_{i \in M} \sum_{v \in V} \left( \sum_{p \in N^v} X_{ip}^v \cdot \text{FixedCost}_i^v + \text{VariableCost}_i^v \cdot L_i \cdot \sum_{p \in N^v} X_{ip}^v \right) \quad (6)
\]

\[
\max \sum_{i \in M} \sum_{j \in J} \left( \sum_{(k,u) \in B_{ij}^v} \sum_{v \in V} \sum_{p \in N^v} \sum_{q \in N^v} Y_{ijkups}^v \right) \quad (7)
\]

Constraints (8 - 9) guarantee that if there is no departure in the period \(v\) for the route \(i\) in the segment \(p\), then it is not assigned a departure time. Also with these constraints, if the variable \(X_{ip}^v\) is 1, then \(\alpha_{ip}^v\) will take a value less or equal to the upper bound of the interval \(p\) (8). With the next equation; if \(X_{ip}^v\) is 1, then \(\alpha_{ip}^v\) will take a value greater or equal to the lower bound of the interval \(p\) (9). In this way, they limit the value that \(\alpha_{ip}^v\) takes if there is a trip in the interval \(p\). These constraints are related to the intervals to represent the departures.

\[
\alpha_{ip}^v \leq X_{ip}^v \cdot (T_{ini}^v + H_{\text{min}}^v \cdot p)
\quad \forall v \in V, \forall i \in M, \forall p \in N^v
\]

\[
\alpha_{ip}^v \geq X_{ip}^v \cdot (T_{ini}^v + (p - 1) \cdot H_{\text{min}}^v)
\quad \forall v \in V, \forall i \in M, \forall p \in N^v
\]

In constraints (10-11) the sum of all schedule departures for a route \(i\) (\(\sum_{p \in N^v} X_{ip}^v\)) has to be greater or equal to lower and upper bound of the minimum frequency (\(\overline{\text{FreMin}}^i_v\) and \(\underline{\text{FreMin}}^i_v\)), considering the confidence level (\(\sigma\)) of the decision maker.

\[
\sum_{p \in N^v} X_{ip}^v \geq (1 - \sigma) \cdot \overline{\text{FreMin}}^i_v + \sigma \cdot \underline{\text{FreMin}}^i_v;
\quad v \in V; i \in M
\]

\[
\sum_{p \in N^v} X_{ip}^v \geq (1 - \sigma) \cdot \underline{\text{FreMin}}^i_v + \sigma \cdot \overline{\text{FreMin}}^i_v;
\quad v \in V; i \in M
\]

Constraints (12) to (21) are referring to headways constraints, which are depicted in Fig. 6. All these constraints are related to the properties of headway policies.

Constraint (12 - 13) represents the first departure for the first period. If there is not assigned a trip in the first possible intervals, that means there is not a departure from the first interval to the second-last interval \((t_{f_i}^v)\), in other words \(\sum_{c=1}^{t_{f_i}^v} X_{ic}^v\) is equal 0, then the first departure is assigned in the last possible interval \((t_{c_i}^v)\) and the departure time \(\alpha_{i(t_{c_i}^v)}^v\) has to be less or equal to the maximum headway.

\[
\alpha_{i(t_{c_i}^v)}^v \leq \sum_{c=1}^{t_{f_i}^v} X_{ic}^v \cdot M + (T_{ini}^v + H_{\text{max}}^v)
\quad \forall i \in M, v \in V
\]
\[
\sum_{c=1}^{(\text{tc}_i^v)} X_{ic}^v \geq 1 \quad \forall i \in M, \; v \in V
\]

Constraint (14-15) represents the first departure for all periods greater than 1. The interval for the first possible departure since second period is defined \((f_i^v)\), also the last possible interval according to the previous period is defined \((l_{i-1}^v)\), then the difference between the last departure of the previous period and the first departure of the current period \((\alpha_i^v - \alpha_{i-1}^v)\) has to be greater or equal than the maximum of the minimum headways and less or equal than the minimum of the maximum headways (headways of current and previous period). In the case that \(\max(H_{\text{min}}^{\text{v-1}}, H_{\text{min}}^{\text{v}}) > \min(H_{\text{max}}^{\text{v-1}}, H_{\text{max}}^{\text{v}})\) then will be consider the longest \([H_{\text{min}}^{\text{v-1}}, H_{\text{max}}^{\text{v}}])\).

\[
\max(H_{\text{min}}^{\text{v-1}}, H_{\text{min}}^{\text{v}}) \leq \alpha_i^v - \alpha_{i-1}^v \\
\forall i \in M, \; \forall v \in V, \; v > 1
\]

\[
\alpha_i^v - \alpha_{i-1}^v \leq \min(H_{\text{max}}^{\text{v}}, H_{\text{max}}^{\text{v-1}}) \\
\forall i \in M, \; \forall v \in V, \; v > 1
\]

Constraint (16) is for the consecutive departures, here indicates that the departure time must be between the minimum and maximum headway. If there is no a trip from the second-last interval to the first possible interval \((\sum_{c=\text{ms}}^{(\text{ic}_i^v)} X_{ic}^v = 0)\) then the next departure time will be assigned in the last possible interval \((\text{ms}^{\text{v}})\) and the difference of time between the next departure and the previous departure \((\alpha_i^v - \alpha_{i,\text{ms}})\) has to be less or equal to the maximum headway.

Constraint 17 means, if the next trip is assigned to the first possible interval \((\text{ms}^{\text{v}})\), then they just verify that the difference of time between the next departure and the departure assigned in the first possible interval \((\alpha_i^v - \alpha_{i,\text{ms}})\) is less or equal to the maximum headway. Also, if the next trip is assigned to the first possible interval \((\text{ms}^{\text{v}})\), then they just verify that the difference of time between this two trips \((\alpha_i^v - \alpha_{i,\text{ms}})\) is greater or equal to the minimum headway \((18)\). From the last possible interval to the first possible interval \((\sum_{c=\text{ms}}^{(\text{ic}_i^v)} X_{ic}^v)\) has to be at least one departure assigned \((19)\).

\[
\alpha_i^v - \alpha_{i,\text{ms}} \leq T_{f,\text{in}} \cdot \sum_{c=\text{ms}}^{(\text{ms}^{\text{v}})} X_{ic}^v + X_{i,\text{ms}}^{\text{v}} \cdot H_{\text{max}}^{\text{v}} \quad \forall v \in V, \forall i \in M
\]

\[
\alpha_i^v - \alpha_{i,\text{ms}} \leq T_{f,\text{in}} \cdot (1 - X_{i,\text{ms}}^{\text{v}}) + X_{ip}^{\text{v}} \cdot H_{\text{max}}^{\text{v}} \quad \forall v \in V, \forall i \in M
\]

\[
-(T_{f,\text{in}}) \cdot (1 - X_{i,\text{ms}}^{\text{v}}) - T_{f,\text{in}} \cdot (1 - X_{ip}^{\text{v}}) \leq \alpha_i^v - \alpha_{i,\text{ms}} - H_{\text{min}}^{\text{v}} \cdot X_{ip}^{\text{v}} \quad \forall v \in V, \forall i \in M
\]

\[
\sum_{c=\text{ms}}^{(\text{ms}^{\text{v}})} X_{ic}^v \geq 1 - (1 - X_{ip}^{\text{v}}) \cdot (N^v + 1) \quad \forall v \in V, \forall i \in M
\]
For the last departure (20-21) If there is not a trip assigned from the second-last possible interval to the maximum number of intervals \((\sum_{c=m u_i^v}^{N^v} X_{ic}^v = 0)\) then last departure time will be assigned to the last possible interval, and the time has to be greater or equal to the end of the period minus a desired time \((T_{fin}^v - \gamma_i^v)\). (20) and at least one departure must be assigned (21).

\[
\alpha_{vi}^{t_m} \geq -M \cdot \sum_{c=m u_i^v}^{N^v} X_{ic}^v + (T_{fin}^v - \gamma_i^v) \quad \forall \, v \in V, \forall \, i \in M, \forall \, p, l \in N^v \tag{20}
\]

\[
\sum_{c=m u_i^v}^{N^v} X_{ic}^v \geq 1 \quad \forall \, v \in V, \forall \, i \in M, \forall \, p, l \in N^v \tag{21}
\]

When the variable \(Y_{ijkupq}^v\) is equal to 1, then the difference between the departure time \((\alpha_{ip}^v)\), travel time \((t_{ik}^v)\) and transfer time \((\delta_{ijk}^v)\) of the origin route and the departure time \((\alpha_{jq}^v)\), travel time \((t_{ju}^v)\) and holding time \((s_{jk}^v)\) of the destination route, has to be greater or equal to the minimum time window \((W_{min}^v, \text{constraint 22})\) and less or equal to the maximum time window \((W_{max}^v, \text{constraint 23})\). With the next equation they guarantee that there are two trips for which there is a synchronization (Constraint 24). These constraints are related to the synchronization and bus bunching policies.

\[
-1 \cdot (\alpha_{ip}^v + t_{ik}^v + \delta_{ijk}^v) + (\alpha_{jq}^v + t_{ju}^v + s_{jk}^v) \geq W_{min}^v - M \cdot (1 - Y_{ijkupq}^v) \quad v \in V; \ i \in M; \ (k, u) \in B_{ij}^v; \ j \in J(i); \ p, q \in N^v \tag{22}
\]

\[
-1 \cdot (\alpha_{ip}^v + t_{ik}^v + \delta_{ijk}^v) + (\alpha_{jq}^v + t_{ju}^v + s_{jk}^v) \leq W_{max}^v + M \cdot (1 - Y_{ijkupq}^v) \quad v \in V; \ i \in M; \ (k, u) \in B_{ij}^v; \ j \in J(i); \ p, q \in N^v \tag{23}
\]

\[
X_{ip}^v + X_{jq}^v \geq 2 \cdot Y_{ijkupq}^v \quad v \in V; \ i \in M; \ (k, u) \in B_{ij}^v; \ j \in J(i); \ p, q \in N^v \tag{24}
\]

3. Numerical experiments. The main objective of this experiment is to analyze the influence of instances factors (routes, periods, nodes, density, and headway) on the objective functions; also analyze the influence of the confidence level, uncertainty (fuzziness) and level of demand on the cost. The impact of these factors are not considered on synchronizations because the demand has no influence on this objective function.

The authors generated 32 instances. The generator was coded with R. The parameters that form an instance are: a fixed and a variable cost per route and period, the number of passengers at each node, the travel time between nodes, the distance between two nodes, the number of passengers who wish to transfer between routes, besides others. But there are 5 parameters that define an instance and they gave special attention to them: the number of routes, the number of periods, the number
of nodes, the number of synchronization nodes and the range of headways. The authors established two levels for each one of these 5 important parameters and they combined them. The authors show the levels of these parameters in Table 2.

| Parameter     | Low level | High level |
|---------------|-----------|------------|
| Routes        | 8         | 20         |
| Periods       | 2         | 12         |
| Segments      | 10        | 150        |
| Sync. nodes   | 2         | 12         |
| Headways      | 5-10      | 5-20       |

Table 2. Characteristics of instances.

The authors implemented a factorial design, low and high levels were chosen for confidence level, fuzziness (variation of demand), and level of demand. For confidence level they chose: 0.1 and 0.9. For demand level they chose: a decrement of 35% of regular demand, and for increase in demand they also chose a 35%. For each demand level they chose a low variation at 10% of the central value, and a high variation of 25% of the central value. The main objective of the experiment is to investigate the impact of these 3 factors plus the 5 most influential parameters in the complexity of the instance. The authors applied an ANOVA using R.

The authors considered symmetric triangular fuzzy numbers and implemented the SAugmecon [29] method in OPL, (an extension of the $\epsilon$-constraint method), SAugmecon generates all non-dominated solutions of multi-objective integer programming (MOIP) problems. With the SAugmecon method, all nondominated solutions can be found much more efficiently thanks to the acceleration algorithms.

The authors executed each instance 8 times, one for each combination of the 3 factors (confidence level, fuzziness and level of demand). At each execution they found the optimal value and the pessimistic value of one objective, (they selected synchronization), they obtained the range and established to find 10 points at the Pareto front, at each point they solved the biobjective problem as a weighted sum and added a constraint to their model, according to the SAugmecon method. The authors fixed a time limit of 3600 sec. in each execution. The procedures explained at the section Preprocessing are applied, they also employed sparse structures, fixed priority to decision variables during the execution. Also, they passed an initial solution from one run to another.

They found a very low correlation level between cost and synchronization, they are practically independent. There is a correlation between the execution time and cost; and execution time and synchronization, there is also a correlation between cost and periods and routes (Figure 12).

The factor which establishes a higher variation on execution time is the quantity of periods, followed by routes and nodes, then confidence level, follow by density, headways, demand, and finally fuzziness is the less influence factor in relation to the execution time (Figure 13). That means that the size of the instance has a greater effect than synchronizations or fuzziness parameter of level of demand.

Also, the authors analyzed the cost behavior in relation to confidence, fuzziness and level of demand; the characteristics of the instance selected are: 20 routes, 2 periods, 10 nodes per route; 12 points of synchronization per period and a range of 15 minutes between minimum and maximum headway. In Figure 14 the authors
varied the confidence, fuzziness and level of demand, where 0 indicates low level and 1 indicates high level; in such a way that label 000 indicates low level for confidence, fuzziness and level of demand respectively.
In the same figure, the results obtained indicate that to greater confidence better results (100, 110, 101 and 111), in fact, the best results are obtained when the confidence level is high, the fuzziness is low and the level of demand is low (100), followed by high confidence level, high fuzziness and low demand (110). A midpoint is given by low confidence, low fuzziness and low demand (000). The worst results are obtained when the confidence level is low, the fuzziness is high and there is a high level of demand (011).

It should be noted that the authors established a gap of 5% to obtained results; here they presented results for one instance but similar results are obtained with the other instances.

![Cost vs. Synchronization](image)

**Figure 14. Cost vs. Synchronization (Instance 20)**

The authors identified another case, the instance they presented to show this behavior has: 20 routes, 2 periods, 150 nodes per route; 12 synchronization points per period and a range of 5 minutes between the minimum and maximum headway. In Figure 15 the results indicate that the results are better in those cases were the confidence was high (100, 110, 101 and 111) and the results obtained when the confidence is low are the worst.

It is important to notice the difference between both cases presented here, because it is evident that the fuzzy effect over the demand has a bigger impact when the instance is bigger too, especially when the number of nodes is bigger, because this implies a bigger demand; this is the case presented in Figure 15 where you can see more defined the different combinations of factors (Confidence, Uncertain and Demand). In the other case, the one presented in Figure 14, the impact over the demand it is not as evident than Figure 15. With these two cases, Avila et al. show the impact of the use of the fuzzy methodology in this problem.

Besides, Avila et al. analyzed the cost behavior in relation to the characteristics of the instances and they find out the most influential factors in the cost are: periods, routes and nodes. Also, when the range between the minimum and maximum headway is wider the cost increases. Furthermore, when the instance has more
synchronization points, the cost also increases due to it trying to generate a bigger number of trips to satisfy the synchronizations objective. See Fig. 16.

The behavior of the synchronizations was also analyzed; the uncertainty factor does not have influence on synchronizations because the authors considered uncertainty on demand, and this factor does not intervene on synchronizations;
nevertheless the most influential parameters are periods, routes and number of synchronization points. See Fig 17.

![Synchronization Behavior](image)

**Figure 17.** Synchronizations behaviour in relation to instance parameters

4. Conclusions and future work. Here the authors presented a mathematical model for the frequency and timetable integrated problem considering demand uncertainty. The proposed model can be used for multimodal transportation systems; it can be used for buses, subway, trains. The authors minimized the operation cost and they maximized the number of synchronizations (same or different period). The authors implemented the SAugmecon to solve the problem and they employed fuzzy programming. This method turned out to be suitable to obtained Pareto fronts with the objectives they used. The authors tested the model with instances which were generated randomly.

They tested the influence of different elements of the instances like periods, routes, nodes, synchronization points and the range of headways. They designed an experiment with three factors: confidence level, fuzziness and level of demand. The execution time will be higher when the number of synchronizations is high and the number of ships to schedule is also high. But the number of periods is the main factor in the variation of execution time.

In the future, the authors will incorporate uncertain of travel time and they would like to experiment with other ranking methods for fuzzy numbers, like the second index of Yager.

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REFERENCES

[1] P. Avila and F. López, Two multiobjective metaheuristics for solving the integrated problem of frequencies calculation and departures planning in an urban transport system, *Annals of Management Science*, 3 (2014), 29–42.

[2] R. Baskaran and K. Krishnaihah, Simulation model to determine frequency of a single bus route with single and multiple headways, *Int. J. Business Performance and Supply Chain Modelling*, 4 (2012), 40–59.

[3] L. Cadarso and A. Marín, Integration of timetable planning and rolling stock in rapid transit networks, *Annals of Operations Research*, 199 (2012), 113–135.

[4] L. Campos and J. L. Verdegay, Linear programming problems and ranking of fuzzy numbers, *Fuzzy Sets and Systems*, 32 (1989), 1–11.

[5] A. Ceder, *Public Transit Planning and Operation: Theory, Modeling and Practice*, 1st edition, Elsevier, USA, 2007.

[6] P. Chakroborty, Genetic algorithms for optimal urban transit network design, *Computer-Aided Civil and Infrastructure Engineering*, 18 (2003), 184–200.

[7] H. Chen, Stochastic optimization in computing multiple headways for a single bus line, *Proceedings of the 35th Annual Simulation Symposium*, (2002), 316–323.

[8] C. Daraio, D. Marco, F. Di Costa, C. Leporelli, G. Matteucci and A. Nastasi, Efficiency and effectiveness in the urban public transport sector: A critical review with directions for future research, *European Journal of Operational Research*, 248 (2016), 1–20.

[9] G. Desaulniers and M. D. Hickman, Public transit, in *Handbook in OR & MS* (eds C. Barnhart and G. Laporte), Elsevier, (2007), 69–127.

[10] A. Eranki, A model to create bus timetables to attain maximum synchronization considering waiting times at transfer stops, Thesis University of South Florida, 2004.

[11] H. Fazlollahtabar and M. Saidi-Mehrabad, Optimizing multi-objective decision making having qualitative evaluation, *Journal of Industrial and Management Optimization*, 11 (2016), 747–762.

[12] Y. Hadad and M. Shnaiderman, Public-transit frequency setting using minimum-cost approach with stochastic demand and travel time, *Transportation Research Part B: Methodological*, 46 (2012), 1068–1084.

[13] O. J. Ibarra-Rojas and Y. A. Rios-Solís, Synchronization of bus timetabling, *Transportation Research Part B: Methodological*, 46 (2012), 599–614.

[14] J. Jensen, O. Nielsen and C. Prato, Public transport optimisation emphasising passengers’ travel behaviour, Thesis Technical University of DenmarkDanmarks Tekniske Universitet, 2015.

[15] L. Linzhong, Y. Juhua, M. Haibo, L. Xiaojing and W. Fang, Exact algorithms for multi-criteria multi-modal shortest path with transfer delaying and arriving time-window in urban transit network, *Applied Mathematical Modeling*, 38 (2014), 2613–2629.

[16] S. H. Nasseri and E. Behmanesh, Linear programming with triangular fuzzy numbers-A case study in a finance and credit institute, *Fuzzy Information and Engineering*, 5 (2013), 295–315.

[17] F. Perez, T. Gomez and R. Caballero, Un modelo difuso para la selección de carteras de proyectos con incertidumbre en los costes, *Revista Electrónica de Comunicaciones y Trabajos de ASEPUMA*, 13 (2012), 129–143.

[18] F. Perez and T. Gomez, Multiobjective project portfolio selection with fuzzy constraints, *Annals of Operation Research*, 245 (2016), 7–29.

[19] T. Rasmussen, M. Anderson, O. Nielsen and C. Prato, Timetable-based simulation method for choice set generation in large-scale public transport networks, *EJTIR*, 16 (2016), 407–489.

[20] V. Sahinidis Nikolaos, Optimization under uncertainty: State-of-the-art and opportunities, *Computers and Chemical Engineering*, 28 (2004), 971–983.

[21] Y. Shangyao, C. Chin-Jen and T. Ching-Hui, Inter-city bus routing and timetable setting under stochastic demands, *Transportation research part A*, 40 (2006), 572–586.

[22] L. Sun, Z. Gao and Y. Wang, A Stackelberg game management model of the urban public transport, *Journal of Industrial and Management Optimization*, 8 (2012), 507–520.
[23] W. Y. Szeto and W. Yongzhong, A simultaneous bus route design and frequency setting problem for Tin Shui Wai, Hong Kong, European Journal of Operational Research, 209 (2011), 141–155.

[24] S. L. Tilahun and H. C. Ong, Bus timetabling as a fuzzy multiobjective optimization problem using preference based genetic algorithm, Promet - Traffic & Transportation, 24 (2012), 183–191.

[25] I. Verbas, C. Frei, H. Mahmassani and R. Chan, Stretching resources: Sensitivity of optimal bus frequency allocation to stop-level demand elasticities, Public Transport, 7 (2015), 1–20.

[26] I. Verbas and H. Mahmassani, Exploring trade-offs in frequency allocation in a transit network using bus route patterns: Methodology and application to large-scale urban systems, Transportation Research Part B: Methodological, 81 (2015), 577–595.

[27] Y. Wang, X. Zhu and L. B. Wu, Integrated multimodal metropolitan transportation model, Procedia Social and Behavioral Sciences, 96 (2013), 2138–2146.

[28] J. Zhang, T. Arentze and H. Timmermans, A multimodal transport network model for advanced traveler information system, Journal of Ubiquitous System and Pervasive Networks, 4 (2012), 21–27.

[29] W. Zhang and M. Reimann, A simple augmented e-constraint method for multi-objective mathematical integer programming problems, European Journal of Operations Research, 234 (2014), 15–24.

[30] F. Zhao and Z. Xiaogang, Optimization of transit route network, vehicle headways and timetables for large-scale transit networks, European Journal of Operational Research, 186 (2008), 841–855.

[31] Y. Zhu, B. Mao, L. Liu and M. Li, Timetable design for urban rail line with capacity constraints, Discrete Dynamics in Nature and Society, 2015 (2015), Art. ID 429219, 11 pp.

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E-mail address: pauavila@uma.es
E-mail address: fernando.lopezrr@uanl.edu.mx
E-mail address: rafael.caballero@uma.es
E-mail address: yasmin.riossls@uanl.edu.mx