Global solvability of the initial boundary value problem for a model system of one-dimensional equations of polytropic flows of viscous compressible fluid mixtures

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Abstract. We consider the initial boundary value problem for a model system of one-dimensional equations which describe unsteady polytropic motions of a mixture of viscous compressible fluids. We prove the global existence and uniqueness theorem for the strong solution without restrictions on the structure of the viscosity matrix except standard properties of symmetry and positiveness.

1. Statement of the problem, formulation of the result, the Lagrangian coordinates

We consider the problem of one-dimensional polytropic flow of a mixture which consists of \( N \geq 2 \) components. We work in the closure \( \overline{Q}_T \) of the domain \( Q_T = (0,1) \times (0,T) \), where \( T > 0 \) is an arbitrary positive number, and our aim is to find the density \( \rho > 0 \) of the mixture and the velocity \( u_i \) for each component of the mixture numbered by \( i = 1, \ldots, N \), which satisfy the following system of equations, initial and boundary conditions [1]:

\[
\partial_t \rho + \partial_x (\rho v) = 0, \quad v = \frac{1}{N} \sum_{i=1}^{N} u_i, \tag{1}
\]

\[
\rho (\partial_t u_i + v \partial_x u_i) + K \partial_x \rho^\gamma = \sum_{j=1}^{N} \mu_{ij} \partial_{xx} u_j + \sum_{j=1}^{N} a_{ij} (u_j - u_i), \quad i = 1, \ldots, N, \tag{2}
\]

\[
\rho|_{t=0} = \rho_0, \quad u_i|_{t=0} = u_{0i}, \quad i = 1, \ldots, N, \tag{3}
\]

\[
u|_{x=0} = u_i|_{x=1} = 0, \quad i = 1, \ldots, N. \tag{4}
\]

Here \( v \) is the average velocity of the mixture; the values \( K > 0, \gamma > 1, \mu_{ij} = \mu_{ji} \) and \( a_{ij} = a_{ji} > 0, i, j = 1, \ldots, N \) are known constants, and the viscosity coefficients \( \{\mu_{ij}\}_{i,j=1}^{N} \) compose the matrix \( \mathbf{M} > 0 \); the initial distributions \( \rho_0 \) and \( u_{0i}, i = 1, \ldots, N \), are prescribed.

The aim of the paper is to prove the existence and uniqueness of the strong solution to the problem (1)–(4).
Definition 1. Strong solution to the problem (1)–(4) is called the collection of parameters \( \rho, u_1, \ldots, u_N \) such that the equations (1), (2) are satisfied almost everywhere in \( \Omega_T \), the initial data (3) are accepted for a.a. \( x \in (0, 1) \), the boundary conditions (4) are valid for a.a. \( t \in (0, T) \), and the following inequality and inclusions hold

\[
\rho > 0, \quad \rho \in L_\infty(0, T; W^1_2(0, 1)), \quad \partial_t \rho \in L_\infty(0, T; L_2(0, 1)),
\]

\[
u_i \in L_\infty(0, T; W^1_2(0, 1)) \bigcap L_2(0, T; W^2_2(0, 1)), \quad \partial_t u_i \in L_2(Q_T), \quad i = 1, \ldots, N.
\]

The main result of the paper is formulated as the following theorem.

**Theorem 2.** Let the initial data in (3) satisfy the conditions

\[
\rho_0 \in W^1_2(0, 1), \quad \rho_0 > 0, \quad u_{0i} \in W^1_2(0, 1), \quad u_{0i}|_{x=0} = u_{0i}|_{x=1} = 0, \quad i = 1, \ldots, N \quad (N \geq 2),
\]

the symmetric viscosity matrix \( M \) is positive, the polytropic exponent \( \gamma > 1 \), all other numeric parameters \( K, T \) and \( a_{ij} = a_{ji}, \quad i, j = 1, \ldots, N \), are positive.

Then there exists the unique strong solution to the problem (1)–(4) in the sense of Definition 1.

**Sketch of the proof of Theorem 2.** The existence of unique strong solution to the problem (1)–(4) in a small time interval \([0, t_0]\) is proved in [2]. In order to extend this solution from the interval \([0, t_0]\) to the target interval \([0, T]\), we need to prove a priori estimates, in which the constants depend only on the input data of the problem and on the value \( T \), but not on the small parameter \( t_0 \) (see, e.g., [3]). That is why we concentrate on the global estimates.

**Lagrangian coordinates.** While the problem (1)–(4) is studied, it is sometimes more convenient to use the Lagrangian coordinates. Let us consider \( y(x, t) = \int_0^x \rho(s, t) \, ds \) and \( t \) as new independent variables. Then the system (1), (2) turns into the form

\[
\partial_t \rho + \rho^2 \partial_y v = 0,
\]

\[
\partial_t u_i + K \partial_y \rho^\gamma \sum_{j=1}^N \sum_{i=1}^N \mu_{ij} \partial_y (\rho \partial_y u_j) + \frac{1}{\rho} \sum_{j=1}^N a_{ij} (u_j - u_i), \quad i = 1, \ldots, N.
\]

The domain \( \Omega_T \) is mapped into the rectangular \( \Pi_T = (0, d) \times (0, T) \), where \( d = \int_0^1 \rho_0(x) \, dx > 0 \), and the initial and boundary conditions accept the form

\[
\rho|_{t=0} = \bar{\rho}_0, \quad u_i|_{t=0} = \bar{u}_{0i}, \quad i = 1, \ldots, N,
\]

\[
u_i|_{y=0} = u_i|_{y=d} = 0, \quad i = 1, \ldots, N.
\]

2. **Global a priori estimates**

Let us multiply the equations (2) by \( u_i \), integrate the result over \((0, 1)\) and sum over \( i = 1, \ldots, N \). Due to (1), (4) and the condition \( M > 0 \), we following relations hold

\[
\sum_{i=1}^N \int_0^1 \left( \rho \partial_t u_i + \rho \nu \partial_x u_i \right) u_i \, dx = \frac{1}{2} \frac{d}{dt} \left( \sum_{i=1}^N \int_0^1 \rho u_i^2 \, dx \right),
\]

\[
K \sum_{i=1}^N \int_0^1 u_i (\partial_x \rho^\gamma) \, dx = -KN \int_0^1 \rho^\gamma (\partial_x v) \, dx = \frac{KN}{\gamma-1} \frac{d}{dt} \left( \int_0^1 \rho^\gamma \, dx \right).
\]
\[
\sum_{i,j=1}^{N} \mu_{ij} \int_{0}^{1} (\partial_x u_j) u_i \, dx = -\sum_{i,j=1}^{N} \mu_{ij} \int_{0}^{1} (\partial_x u_i) (\partial_x u_j) \, dx \leq -C_0(M) \sum_{i=1}^{N} \int_{0}^{1} |\partial_x u_i|^2 \, dx, \quad (13)
\]

\[
\sum_{i,j=1}^{N} a_{ij} \int_{0}^{1} (u_j - u_i) u_i \, dx = -\frac{1}{2} \sum_{i,j=1}^{N} a_{ij} \int_{0}^{1} (u_i - u_j)^2 \, dx, \quad (14)
\]

and we come to the inequality

\[
\frac{d}{dt} \sum_{i=1}^{N} \int_{0}^{1} \left( \frac{1}{2} \rho u_i^2 + \frac{K}{\gamma - 1} \rho^\gamma \right) \, dx + C_0 \sum_{i=1}^{N} \int_{0}^{1} |\partial_x u_i|^2 \, dx + \frac{1}{2} \sum_{i,j=1}^{N} a_{ij} \int_{0}^{1} (u_i - u_j)^2 \, dx \leq 0. \quad (15)
\]

Let us agree that \( C_k(\cdot), \, k = 0, 1, \ldots, \) stand for the quantities which take finite positive values and depend on the objects listed in the parentheses. When we integrate the inequality (15) over \((0, t)\) using (3), we obtain the estimate

\[
\sum_{i=1}^{N} \int_{0}^{t} \left( \frac{1}{2} \rho u_i^2 + \frac{K}{\gamma - 1} \rho^\gamma \right) \, dx + C_0 \sum_{i=1}^{N} \int_{0}^{t} |\partial_x u_i|^2 \, dx \leq \sum_{i=1}^{N} \int_{0}^{1} \left( \frac{1}{2} \rho_0 u_i^2 + \frac{K}{\gamma - 1} \rho_0^\gamma \right) \, dx,
\]

which leads to the fact that

\[
\sum_{i=1}^{N} \left( \sqrt{\rho u_i} \right)_{L_\infty(0,T;L_2(0,1))} + ||\partial_x u_i||_{L_2(Q_T)} + \sum_{j=1}^{N} ||u_i - u_j||_{L_2(Q_T)} + ||\rho||_{L_\infty(0,T;L_{\gamma}(1))} \leq C_1 \left( \{ \sqrt{\rho_0 u_i} \}_{L_2(0,1)}, \, ||\rho_0||_{L_{\gamma}(0,1)}, \{ a_{ij} \}, \, K, \, M, \, N, \, \gamma \right). \quad (17)
\]

The estimates (17), due to (10), lead to the inequality

\[
\sum_{i=1}^{N} ||u_i||_{L_2(0,T;L_\infty(0,1))} \leq C_1. \quad (18)
\]

The estimate (17) in the Lagrangian coordinates takes the form

\[
\sum_{i=1}^{N} \left( ||u_i||_{L_\infty(0,T;L_2(0,d))} + \sqrt{\rho} ||u_i||_{L_2(\Pi_T)} \right) + \sum_{j=1}^{N} \left( ||u_i - u_j||/\sqrt{\rho} ||u_i||_{L_2(\Pi_T)} \right) + ||\rho||_{L_\infty(0,T;L_{\gamma-1}(0,d))} \leq C_2(C_1, \gamma). \quad (19)
\]

The next step is to prove the positiveness and boundedness of the density \( \rho \). Here we use the equations (7), (8). Let us rewrite the equations (8) in the form

\[
\sum_{j=1}^{N} \tilde{\mu}_{ij} \partial_t u_j + K \left( \sum_{j=1}^{N} \tilde{\mu}_{ij} \right) \partial_y \rho^\gamma = \partial_y (\rho \partial_y u_i) + \frac{1}{\rho} \sum_{j=1}^{N} \tilde{\mu}_{ij} \left( \sum_{k=1}^{N} a_{jk} (u_k - u_j) \right), \quad i = 1, \ldots, N. \quad (20)
\]
where $\tilde{\mu}_{ij}$ are the entries of the matrix $\tilde{M} = M^{-1} > 0$, then sum (20) over $i = 1, \ldots, N$, and divide by $N$. Then we come to the equality

$$
\partial_t V + \tilde{K} \partial_y \rho^\gamma = \partial_y (\rho \partial_y v) + \frac{1}{N\rho} \sum_{i,j=1}^{N} \tilde{\mu}_{ij} \left( \sum_{k=1}^{N} a_{jk}(u_k - u_j) \right),
$$

(21)

where $V = \frac{1}{N} \sum_{i,j=1}^{N} \tilde{\mu}_{ij} u_j$ and $\tilde{K} = \frac{K}{N} \sum_{i,j=1}^{N} \tilde{\mu}_{ij} > 0$. Let us use (7) in order to express

$$
\rho \partial_y v = -\partial_t \ln \rho,
$$

(22)

and substitute this relation into (21):

$$
\partial_t \ln \rho + \tilde{K} \partial_y \rho^\gamma = -\partial_t V + \frac{1}{N\rho} \sum_{i,j=1}^{N} \tilde{\mu}_{ij} \left( \sum_{k=1}^{N} a_{jk}(u_k - u_j) \right).
$$

(23)

Let us multiply this equality by $\partial_y \ln \rho =: w$ and integrate over $y \in (0, d)$, then we obtain the equality

$$
\frac{1}{2} \frac{d}{dt} \left( \int_{0}^{d} w^2 \, dy \right) + \tilde{K} \gamma \int_{0}^{d} \rho^\gamma w^2 \, dy = -\int_{0}^{d} (\partial_t V) w \, dy + \frac{1}{N} \sum_{i,j,k=1}^{N} \tilde{\mu}_{ij} a_{jk} \int_{0}^{d} \frac{(u_k - u_j)w}{\rho} \, dy.
$$

(24)

We transform the first summand in the right-hand side of (24) via integration by parts and using (22):

$$
-\int_{0}^{d} (\partial_t V) w \, dy = -\frac{d}{dt} \left( \int_{0}^{d} V \, dy \right) + \int_{0}^{d} \rho(\partial_y v)(\partial_y V) \, dy,
$$

(25)

and we estimate the second summand from above:

$$
\frac{1}{N} \sum_{i,j,k=1}^{N} \tilde{\mu}_{ij} a_{jk} \int_{0}^{d} \frac{(u_k - u_j)w}{\rho} \, dy \leq C_3(\{a_{jk}\}, \tilde{M}) \|1/\sqrt{\rho}\|_{L_\infty(0,d)} \|w\|_{L_2(0,d)} \times
$$

$$
\times \sum_{j,k=1}^{N} \|u_k - u_j\|^2 \sqrt{\rho} \|_{L_2(0,d)}.
$$

(26)

It is obvious, due to the equation (7) and the conditions (9), (10), that for every $t \in [0, T]$

$$
\rho(z(t), t) = d
$$

(27)

at least in one point $z(t) \in [0, d]$. Hence, we can use the representation

$$
\frac{1}{\sqrt{\rho(y, t)}} = \frac{1}{\sqrt{\rho(z(t), t)}} + \int_{z(t)}^{y} \partial_s \rho^{-\frac{1}{2}}(s, t) \, ds = d^{-\frac{1}{2}} - \frac{1}{2} \int_{z(t)}^{y} \rho^{-\frac{1}{2}}(s, t) \partial_s \ln \rho(s, t) \, ds,
$$

(28)

from which, using Hölder’s inequality and (27), we obtain

$$
\|1/\sqrt{\rho}\|_{L_\infty(0,d)} \leq d^{-\frac{1}{2}} + \frac{1}{2} \|w\|_{L_2(0,d)},
$$

(29)
Hence, after the integration of (24) over \((0, t)\), using (25), (26) and (29), we come to the inequality
\[
\|w\|^2_{L^2(0, d)} + 2\bar{K}\gamma \int_0^t \int_0^d \rho^\gamma w^2 \, dy \, d\tau \leq \|w_0\|^2_{L^2(0, d)} - 2 \int_0^d V w \, dy + 2 \int_0^d V_0 w_0 \, dy + 2 \int_0^t \|\rho(\partial_y V)(\partial_y v)\|_{L^1(0, d)} \, d\tau + C_3 \sum_{j,k=1}^N \|(u_k - u_j) / \sqrt{\rho}\|_{L^2(0, d)} \|w\|_{L^2(0, d)} (2d^{-\frac{3}{2}} + \|w\|_{L^2(0, d)}) \, d\tau,
\]
where \(w_0 = w(0, t)\) and \(V_0 = V(0, t)\). Using Cauchy’s inequality and the estimate (19), we derive from the last formula that
\[
\|w\|^2_{L^2(0, d)} \leq C_4 + C_5 \sum_{j,k=1}^N \|(u_k - u_j) / \sqrt{\rho}\|_{L^2(0, d)} \|w\|^2_{L^2(0, d)} \, d\tau,
\]
where \(C_4 = C_4(C_2, C_3, \{\|\tilde{u}_0\|_{L^2(0, d)}\}, \|w_0\|_{L^2(0, d)}, \tilde{M}, N, T, d)\) and \(C_5 = C_5(C_3)\). Since (19) leads to the estimate
\[
\sum_{j,k=1}^N \|(u_k - u_j) / \sqrt{\rho}\|_{L^2(0, d)} \, d\tau \leq C_6(C_2, T) \quad \forall t \in [0, T],
\]
then the Gronwall lemma provides
\[
\|w(t)\|_{L^2(0, d)} \leq C_6(C_4, C_5, C_6) \quad \forall t \in [0, T],
\]
i. e. the norm of the derivative \(\partial_y \ln \rho\) in \(L^2(0, d)\) is bounded uniformly in \(t \in [0, T]\). Hence, due to the representation (see the proof of (28))
\[
\ln \rho(y, t) = \ln \rho(z(t), t) + \int_y^z \partial_s \ln \rho(s, t) \, ds,
\]
we have
\[
|\ln \rho(y, t)| \leq |\ln d| + \sqrt{d} \|w\|_{L^2(0, d)} \leq C_7(C_6, d),
\]
and consequently
\[
0 < C_8^{-1}(C_7) \leq \rho(y, t) \leq C_8(C_7).
\]

Now we possess the boundedness and positiveness of the density \(\rho\), and the remaining a priori estimates can be obtained in the original (Eulerian) coordinates \((x, t)\). Thus, from (33) and (35) we deduce
\[
\|\partial_x \rho(t)\|_{L^2(0, 1)} \leq C_9(C_6, C_8) \quad \forall t \in [0, T].
\]
Then, let us square the momentum equations (2) and sum over \(i = 1, \ldots, N\), as a result we obtain
\[
\sum_{i=1}^N \rho(\partial_t u_i)^2 + \frac{1}{\rho} \sum_{i=1}^N \left( \sum_{j=1}^N \mu_{ij} \partial_x x u_j \right)^2 - 2 \sum_{i=1}^N (\partial_t u_i) \left( \sum_{j=1}^N \mu_{ij} \partial_x x u_j \right) = \sum_{i=1}^N \frac{1}{\rho} \left( \sum_{j=1}^N a_{ij} (u_j - u_i) - K \partial_x r^\gamma - \rho \psi \partial_x u_i \right)^2.
\]
Let us introduce the function \( \alpha(t) \) as
\[
\alpha(t) = \sum_{i,j=1}^{N} \mu_{ij} \int_{0}^{t} (\partial_{x} u_{i})(\partial_{x} u_{j}) \, dx + \sum_{i=1}^{N} \int_{0}^{t} \left( \rho(\partial_{t} u_{i})^2 + \frac{1}{\rho} \left( \sum_{j=1}^{N} \mu_{ij} \partial_{xx} u_{j} \right)^2 \right) \, dx \, d\tau.
\]

Then (37) and the inequalities (17), (35), (36) lead to the estimate
\[
\alpha'(t) \leq C_{10} + C_{11} \left( \sum_{j=1}^{N} \| u_{j} \|_{L_{\infty}(0,1)}^{2} \right) \left( \sum_{i,j=1}^{N} \mu_{ij} \int_{0}^{t} (\partial_{x} u_{i})(\partial_{x} u_{j}) \, dx \right) \leq C_{10} + C_{11} \left( \sum_{j=1}^{N} \| u_{j} \|_{L_{\infty}(0,1)}^{2} \right) \alpha(t),
\]

where \( C_{10} = C_{10}(C_{1}, C_{8}, C_{9}, \{ a_{ij} \}, K, N, \gamma) \) and \( C_{11} = C_{11}(C_{8}, M) \), from which via the Gronwall lemma (see also (18)) it follows that
\[
\alpha(t) \leq C_{12} \left( C_{1}, C_{10}, C_{11}, \{ \| u_{0} \|_{L_{2}(0,1)} \}, M, N, T \right).
\]

Using this and (35), we come to the inequality
\[
\sum_{i=1}^{N} \left( \| \partial_{x} u_{i} \|_{L_{\infty}(0,T;L_{2}(0,1))} + \| \partial_{xx} u_{i} \|_{L_{2}(Q_{T})} + \| \partial_{t} u_{i} \|_{L_{2}(Q_{T})} \right) \leq C_{13},
\]

where \( C_{13} = C_{13}(C_{8}, C_{12}, M, N) \). Finally, the continuity equation (1) and the inequalities (35), (36) and (39) provide
\[
\| \partial_{t} \rho \|_{L_{\infty}(0,T;L_{2}(0,1))} \leq C_{14}(C_{8}, C_{9}, C_{13}).
\]

Theorem 2 is proved.

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**References**
[1] Mamontov A and Prokudin D 2017 Sib. Elect. Math. Reports 14 388
[2] Prokudin D 2017 Sib. Elect. Math. Reports 14 568
[3] Antontsev S N, Kazhikhov A V and Monakhov V N 1990 Boundary Value Problems in Mechanics of Nonhomogeneous Fluids (Amsterdam: North–Holland Publishing Co.)