Power law distributions in a new toy model with interacting $N$ agents

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Abstract. A new toy model with interacting $N$ agents is proposed in this paper. The agents in this model possess quantity, and have an interaction radius depending on the quantity. They exchange a part of the quantity with agents belonging to within their interaction radii. The cumulative distribution function about observing the quantity in a stationary state exhibits a power law, and the exponent is universally close to $-1$ if the density of agents is sufficiently small.

1. Introduction
Various phenomena, where an observed value of quantity $m$ (or a size of the observation) on a group is proportional to a power of the “rank” of the observation $k$, ubiquitously occur in nature. The rank-size distribution can be represented as

$$m(k) = m_{\text{max}} k^{-\beta},$$

where $m_{\text{max}}$ means the largest value of the observation. In particular, if the exponent $\beta$ is close to 1, the rank-size distribution is called Zipf’s law which we can find in a wide range of fields, e.g., magnitudes of earthquakes [1, 2], populations of cities [3–6], and incomes of companies [7, 8].

We can show that this empirical law corresponds to a power law distribution whose exponent is equal to $-2$ as follows. Let set the maximum value of the rank (the number of elements), and the probability distribution function (PDF) $N$ and $p(m)$, respectively. By solving Eq. (1) for $k$ and dividing $k$ by $N$, we can obtain

$$\frac{k(m)}{N} = \frac{m_{\text{max}}}{N} m^{-1}.$$ 

The left-hand side of the equation can be interpreted as the cumulative distribution function (CDF), $P(> m)$, which is the probability that the observed quantity is more than $m$. Thus,

$$P(> m) = \int_{m}^{m_{\text{max}}} p(m') dm' = \frac{m_{\text{max}}}{N} m^{-1}. $$

By differentiating the equation by $m$, PDF can be calculated as

$$p(m) = \frac{m_{\text{max}}}{N} m^{-2}.$$
Note that we approximate the discrete distribution function $k/N$ by the continuous one. However, if $N$ goes to sufficiently large, the approximation becomes better.

The fact that the power law distribution with the exponent close to $-2$, in other words CDF with the exponent close to $-1$, can be found in such a diverse field indicates the presence of a common mechanism behind the universality. A multiplicative noise can be a pivotal to unveil the mechanism: a research based on a stochastic process with the multiplicative noise is reported to derive the universal exponent without specifying phenomena [9]. However, it is difficult from the stochastic process to understand what kind of interaction causes the universal power law.

In this paper, we shall introduce a new toy model of agents with explicitly considering the interaction among them. The agents possess quantity, and they exchange a part of it. This model captures a characteristic of the interactions of the real systems exhibiting the universal power law. It will be clear that the CDF about the quantity the agents possess shows a power law behaviour, and the exponent in a stationary state is universally close to $-1$ by numerical simulations.

2. Toy model with interacting $N$ agents

What is a common fundamental characteristic in the interaction among elements of the system showing power law distributions? We might find several answers to the question, however, let us here employ the characteristic that ”the more amounts of quantity the agents have, the more agents they are able to interact with”. The characteristic seems to be intuitively reasonable, and we can find evidence of it. If we shall focus on the relation between a population of a core city of a Japanese metropolitan area, and the number of cites belonging to the area, a positive correlation can be obtained as shown in Fig. 1 made of “2015 Census Final Data” published by Statistics Bureau of Japan. It has been reported in Ref. [10] that the sales and the number of business partners also have positive correlation for about Japanese 500000 companies.

![Figure 1. Scatter plot of the number of cites belonging to a Japanese metropolitan area vs. a population of a core city of the area made of “2015 Census Final Data” published by Statistics Bureau of Japan.](image)

Here, we shall construct a toy model with $N$ interacting agents. The agents exist on the two-dimensional $L \times L$ lattices with length 1. They have an amount of quantity $m$ and an interaction radius $r(m)$. As discussed above, $r$ is a monotonically increasing function of $m$. In this paper, we restrict to the interaction radius $r(m) = m$. Let us denote the amount of quantity of the $i$th agent as $m_i$. In addition, we shall define the density of agents as $d \equiv N/L^2$. The quantity of the agents changes with time step by the following rules: (i) Set all the amounts of quantity 1 at the initial time step. (ii) If $i$th agent stays within the interaction radius of $j$th agent, exchange $\Delta m = \gamma \times \min(m_i, m_j)$. The $i$th or $j$th agent obtains $\Delta m$ with probability $1/2$. (iii) If an amount of quantity of an agent becomes less than 1 after all the interactions at
the time step, set the value 1 at the next time step. (iv) Move all the agents to one of the next lattices randomly and return to (ii) with increasing the value of time step by 1.

3. Results
We have performed the numerical simulations based on the rules varying $N$, $\gamma$, and $d$, and calculated CDF for observing quantity of agents from the numerical results. Figure 2 shows CDF with $N = 5000$, $\gamma = 0.1$, and $d = 2^{-4}\%$ is drawn as a supporting line on the figures, we can understand CDF is a power law function, and the exponent is getting close to $-1$ as the time step increases. After the time step is larger than approximately 15000, CDF changes very little. Therefore, there exists a stationary state where the exponent of CDF is almost $-1$.

![Figure 2](image-url)

**Figure 2.** (color online) Cumulative distribution functions, $P(> m)$, with $N = 5000$, $\gamma = 0.1$, and $d = 2^{-4}\%$ are plotted in a log-log scale by (blue) circles for the time step on simulation is equal to 0, 10000, 12000, and 15000. The (red) line represents $P(> m) \sim m^{-1}$.

We investigate the dynamics of relaxation to the stationary state. By fitting the dataset of CDF by $P(> m) = \text{const.} \times m^{-1/\beta}$, we derive the exponent $\beta$ which is the same one of Eq. (1). The change of $\beta$ in time step for $d = 2^{-4}\%$, $2^{-1}\%$, $2^2\%$, and $2^5\%$ with $N = 5000$, and $\gamma = 0.1$ is exhibited in Fig. 3. From the figure, it is clarified that $\beta$ first increases, and then becomes constant after a time step which we shall call a relaxation time $\tau$. Moreover, we can understand that the values of $\beta$ at the stationary state, and $\tau$ depend on the density of agents $d$.

Figure 4 shows the dependence of them on $d$ for several values of $\gamma$. We can see from Fig. 4(a) that $\beta$ at the stationary state is getting closer to 1 regardless of the value of $\gamma$ while $d$ decrease,
and it drastically changes around $d = 1\%$ for the smaller value of $\gamma$. Figure 4(b) makes it clear that $\tau$ and $d$ also have a power-law relation, and $\tau$ decreases when $d$ increases.

Figure 4. (color online) (a) The density $d$ dependence of $\beta$ at the stationary state is plotted in a log-log scale for $\gamma = 0.025$ (pink horizontal ellipses), 0.050 (navy triangles), 0.10 (blue circles), 0.20 (green squares), and 0.40 (pink vertical ellipses) with $N = 5000$. (b) The density $d$ dependence of $\tau$ is plotted in a log-log scale for the same parameters as in (a).

4. Factors leading to the difference in the exponent
Here, we shall investigate the factors leading to the difference in the exponent $\beta$ by comparing the results for $d = 2^{-4}\%$, and $2^{6}\%$ with $N = 5000$, and $\gamma = 0.1$. The exponents of CDF at the stationary state for $d = 2^{-4}\%$, and $2^{6}\%$ are approximately 1.0, and 1.8, respectively.

First, let us focus on the newly injected quantity to the system at each time step: according to rule (iii), the difference of the number of agents whose quantity is less than 1 at the previous time step, and the total amount of quantity of them, is added to the system at the time step. We shall set it $\Delta M$. Figure 5 shows the change of $\Delta M$ in time step for $d = 2^{-4}\%$, and $d = 2^{6}\%$ with $N = 5000$, and $\gamma = 0.1$. At first, $\Delta M$ is nearly 0, and then it has a finite value at the stationary state. We shall denote the mean value of $\Delta M$ at the stationary state by $\overline{\Delta M}$.

In order to clarify how $\overline{\Delta M}$ is distributed to the agents at each time step, a ratio of the absolute mean value of the change of quantity of the $k$th-ranked agent to $\overline{\Delta M}$ is plotted in Fig. 6 with a log-log scale for $d = 2^{-4}$ (blue circles), and $2^{6}$ (navy triangles) with $N = 5000$, and
\( \gamma = 0.1 \). From the figure, we can confirm that the first-ranked agents gain almost all \( \Delta M \) on both systems, and the tendency is stronger in the system with \( d = 2^{-4} \% \) than with \( d = 2^6 \% \).

Next, we investigate a network structure of the system in the stationary state: we shall treat agents as nodes and if two agents are belonged to either of the interaction radii, we shall consider that there is a link between them.

CDF, \( P_{\text{NET}}(> l) \), which is the probability that the observed number of link in the network is more than \( l \) is plotted for \( d = 2^{-4} \% \), and \( 2^6 \% \) in Fig. 7 where the time step is equal to 10000, and 1500, respectively.

Both results are quite different. Approximately 90 \% of agents in the system with \( d = 2^6 \% \) link to several thousands of agents, that is, the agents of the system equally connect each other. On the other hand, the most of agents of the system with \( d = 2^{-4} \% \) have less than 10 links, and the probability that the agent interacts with more than 10 agents decreases as a power law with an exponent of \(-0.53\). This scale free network implies that there is a self-similar hierarchical structure in the interactions. Namely, each agent hierarchically links to higher ranked agents. The difference in the exponent of CDF can be due to the difference in the network structure.

5. Concluding remarks
In this study, we proposed a toy model with \( N \) agents possessing quantity. They have an interaction radius depending on the amount of quantity, and interchange a part of quantity
among agents belonging to the radii at each time step. If an amount of quantity of an agent becomes less than 1 at a time step after the interactions, it is set to be 1 at the next time step.

It is unveiled that CDF for observing quantity derived from the numerical simulations exhibits a power law behaviour, and the exponent in a stationary state is universally close to $-1$ if the density of agents is sufficiently small. This fact means that we can reproduce Zipf’s law by the numerical simulations without specifying phenomena. Moreover, we clarified that only the agent with the largest quantity can acquire almost all the newly injected quantity to the system at each time step, and there is a self-similar hierarchical structure about the interactions especially for the system exhibiting a power law with an exponent close to $-1$. Since the newly injected quantity is delivered by agents with the lowest quantity, we can say that the flow of quantity shows an inverse cascade of quantity to the top of agents.

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