Free Decay of Turbulence and Breakdown of Self-Similarity

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Abstract

It has been generally assumed, since the work of von Karman and Howarth in 1938, that free decay of fully-developed turbulence is self-similar. We present here a simple phenomenological model of the decay of 3D incompressible turbulence, which predicts breakdown of self-similarity for low-wavenumber spectral exponents $n$ in the range $n_c < n < 4$, where $n_c$ is some threshold wavenumber. Calculations with the eddy-damped quasi-normal Markovian approximation give the value as $n_c \approx 3.45$. The energy spectrum for this range of exponents develops two length-scales, separating three distinct wavenumber ranges.
The decay of homogeneous, isotropic turbulence is a classical problem. For a good review, see Lesieur\textsuperscript{1} (chapter VII, section 10). Since the early work of von Kármán and Howarth\textsuperscript{2}, it has often been assumed that the decay is self-similar for length-scales outside the dissipation-range. Precisely, this assumption means that the time-dependent energy spectrum may be written as

\[ E(k, t) = v(t)\ell(t)F(k\ell(t)), \]  

where \( v(t) \) is the rms velocity fluctuation, \( \ell(t) \) is the integral length-scale, and \( F(\kappa) \) is a dimensionless scaling function (see section 10.2 of Lesieur\textsuperscript{1}). However, recent studies of two exactly soluble models—the Burgers equation\textsuperscript{3} and the Kraichnan white-noise passive scalar equation\textsuperscript{4}—have shown that such self-similarity does not always hold. In particular, Gurbatov \textit{et al.}\textsuperscript{3} observed that decaying Burgers turbulence develops \textit{two distinct} length-scales when the low wave number spectral exponent \( n \) lies in the range \( 1 < n < 2 \). The energy spectrum can then no longer be divided into just a low-wavenumber range \( k\ell(t) \ll 1 \) with \( E(k, t) \sim Ak^n \) and an inertial range \( k\ell(t) \gg 1 \) with \( E(k, t) \sim k^{-2} \). Instead a new spectral range develops intermediate to these two with \( E(k, t) \sim C(t)k^2, \ C(t) \propto (t - t_0)^{1/2}\ln^{-5/4}(t - t_0) \). Since it was the first author of reference 3 who observed this state of affairs in Burgers decay\textsuperscript{5}, we call it the “Gurbatov phenomenon”.

The explanation of this new range lies in the phenomenon of a \( k^2 \) backtransfer for Burgers dynamics, the analogue of the \( k^4 \) backtransfer discovered by Proudman and Reid\textsuperscript{6} in 1954 for the 3D Navier-Stokes equations. According to traditional beliefs, the backtransfer term in Burgers should be overwhelmed at low-wavenumbers by the original \( Ak^n \) spectrum, which, for \( 1 < n < 2 \), is asymptotically much the larger. This statement, however, ignores the fact that the coefficient \( C(t) \) of the backtransfer term is growing in time, while the coefficient \( A \) of the
lowest wavenumber spectrum $\sim k^n$ is independent of time. This leads in the Burgers decay to a new length-scale $\ell_*(t) \gg \ell(t)$, with the $k^2$ backtransfer spectrum dominating throughout the intermediate range $k\ell_*(t) \gg 1$ and $k\ell(t) \ll 1$. An analogous “Gurbatov phenomenon” was found to occur in the Kraichnan passive scalar model\(^4\). It is our purpose to present a similar theory of the breakdown of self-similarity for the 3D incompressible Navier-Stokes equations.

Let us consider first the case in which the energy spectrum is dominated by the $k^4$ back-transfer term at the lowest wavenumbers (as it will be if the initial spectrum has $n \geq 4$). In this case, one may suppose that

$$E(k, t) \propto \begin{cases} C(t)k^4 & k\ell(t) \leq 1 \\ \varepsilon^{2/3}(t)k^{-5/3} & k\ell(t) \geq 1. \end{cases} \tag{2}$$

The continuity of the spectrum at the juncture $k\ell(t) = 1$ imposes a relation

$$\varepsilon(t) \propto C^{3/2}(t)\ell^{-17/2}(t). \tag{3}$$

Two other relations follow by standard Kolmogorov dimensional analysis:

$$\frac{dC}{dt}(t) \propto \varepsilon(t)\ell^5(t)$$

and

$$\frac{d\ell}{dt}(t) \propto \varepsilon^{1/3}(t)\ell^{1/3}(t).$$

Needless to say, all three of these relations hold in the standard analytical closures such as the eddy-damped quasi-normal Markovian approximation (EDQNM). Because we have three relations and three unknowns ($C, \ell, \varepsilon$), we can find a solution. In fact, using (3) to eliminate $\varepsilon(t)$, we obtain from the other two equations

$$\frac{dC}{dt}(t) \propto C^{3/2}(t)\ell^{-7/2}(t). \tag{4}$$
We can then derive from (4) and (5) that

\[
\frac{dC}{d\ell} \propto \frac{C}{\ell}
\]

so that, for some arbitrary power \( p \),

\[
C(t) \propto \ell^p(t).
\]  

Unfortunately, it does not seem possible to calculate the precise value of \( p \) without making explicit use of the Navier-Stokes dynamics, either exactly or within a closure approximation.

We shall quote below the value which follows from EDQNM results. We can only say \textit{a priori} that we expect \( p > 0 \), since \( C(t) \) should grow with time because of the backtransfer. In what follows we shall not specify \( p \), but work with an arbitrary value.

Eliminating \( C(t) \) by substituting (6) in (5) gives

\[
\frac{d\ell}{dt}(t) \propto \ell^{p-\frac{5}{2}}(t),
\]

and hence

\[
\ell(t) \propto (t - t_0)^{\frac{2}{7-p}}
\]  

and

\[
C(t) \propto (t - t_0)^{\frac{2p}{7-p}}.
\]

Notice \( C(t) \propto \ell^p(t) \) substituted into (5) gives also

\[
\varepsilon(t) \propto \ell^{\frac{3p-17}{2}}(t).
\]
The law of the energy decay can then be obtained from $E(t) \propto (\varepsilon(t) \ell(t))^{2/3}$, which yields

$$E(t) \propto \ell^{p-5}(t) \propto (t - t_0)^{-\frac{2(5-p)}{5-p}}. \quad (8)$$

Of course, the result $E(t) \propto \ell^{p-5}(t)$ could also be deduced directly from (6) and the spectral model (3). For any choice of $p$, the above spectral decay law is self-similar. In fact, using $\nu^2(t)\ell(t) \propto \ell^{p-4}(t)$, $C(t) \propto \ell^p(t)$, and $\varepsilon^{2/3}(t) \propto \ell^{p-17/3}(t)$, which follow from the previous relations, we obtain (6) with

$$F(\kappa) \propto \begin{cases} \kappa^4 & \kappa \leq 1 \\ \kappa^{-5/3} & \kappa \geq 1. \end{cases}$$

The results in (6) and (8) may be compared with those obtained from numerical solution of EDQNM, $\ell(t) \propto (t - t_0)^{0.31}$ and $E(t) \propto (t - t_0)^{-1.38}$ (see section 10.2 of Lesieur\textsuperscript{1}). From this we may infer the value $p = 0.55$ for EDQNM. However, the above exponent values do not depend too sensitively on the precise value of $p$, which is thus poorly determined by them. For example, the value $p = 1$ implies the relations $\ell(t) \propto (t - t_0)^{1/3}$ and $E(t) \propto (t - t_0)^{-4/3}$, which are also in close agreement with the numerical EDQNM results.

The decay laws which follow from this spectral model, for any $p$, are the same as those which are traditionally believed to hold in a self-similar decay for the spectrum with low-wavenumber exponent $n_c = 4 - p$ and \textit{constant} coefficient $A$. This constancy of the coefficient $A$ for $-1 < n < 4$ is called the \textit{permanence of the large eddies (PLE)}\textsuperscript{7}. In fact, according to the theory based upon self-similarity and PLE, it is usually deduced that

$$\ell(t) \propto (t - t_0)^a, \quad a = \frac{2}{n + 3}, \quad (9)$$

and

$$E(t) \propto (t - t_0)^{-b}, \quad b = \frac{2(n + 1)}{n + 3}, \quad (10)$$
for $-1 < n < 4$ (see section 10.2 of Lesieur, section 7.7 of Frisch, or Clark and Zemach).

We find, if we take $n = 4 - p$, that we obtain the same values of $a$ and $b$ as before. There is of course no contradiction here, because the coefficient $C(t)$ of the $k^4$ term in (2) depends upon time and this changes the decay laws.

It is no coincidence that the decay laws for $n = 4$ coincide with those for the standard model based upon PLE when $n = n_c$. In fact, we argue, following Gurbatov et al. and Eyink and Xin, that there is no self-similar decay at all when $n_c < n < 4$. We continue to adopt the PLE hypothesis. However, we question the additional, implicit assumption in deriving decay laws that $\ell(t)$ is the only length scale in the problem. We consider the consequences of the $C(t)k^4$ backtransfer spectrum, with $C(t) \propto \ell^p(t)$ as determined above, and consider the possibility that an intermediate spectral range may form which is dominated by the backtransfer. Thus, we take as our model spectrum

$$E(k, t) \propto \begin{cases} 
Ak^n & k\ell_*(t) \leq 1 \\
C(t)k^4 & k\ell_*(t) \geq 1, k\ell(t) \leq 1 \\
\varepsilon^{2/3}(t)k^{-5/3} & k\ell(t) \geq 1.
\end{cases}$$

Continuity of the spectrum at the juncture $k\ell_*(t) = 1$ requires that

$$\ell_*^{4-n}(t) \propto C(t) \propto \ell^p(t)$$

and thus

$$\ell_*(t) \propto \ell_*^{1-n}(t).$$

The intermediate $k^4$ range only survives—and grows—if $\ell_*(t) \gg \ell(t)$ for large $t$. Clearly this requires that $p/(4 - n) > 1$ or $n > 4 - p = n_c$. The growth rate of $\ell_*(t)$ becomes infinitely large as $n$ approaches 4 from below, and, in that limit, the $k^4$ region grows to infinite extent.
For \( n > 4 \)—as in the traditional view—the PLE hypothesis breaks down and the decay is self-similar, governed by the first model \(^2\).

We see that in the range \( n_c < n < 4 \), the decay is not self-similar, as there are two distinctive length-scales \( \ell(t) \) and \( \ell_*(t) \). The \( Ak^n \) low-wavenumber spectrum does not dominate the \( C(t)k^4 \) backtransfer spectrum over the whole range \( k\ell(t) \ll 1 \), because the latter has an increasing coefficient. Instead, the \( k^4 \) region is growing in extent and it is this region which matches onto the energy range at \( k\ell(t) \approx 1 \). It follows that the decay laws are those determined by the backtransfer spectrum \(^2\) \textit{over the whole range of low-wavenumber exponents greater than} \( n_c \):

\[
\ell(t) \propto (t - t_0)^{\frac{n}{2(n+1)}} ; \quad E(t) \propto (t - t_0)^{-\frac{2(n-p)}{n+1}} \quad n > n_c.
\]

Thus, the decay laws \(^3\), \(^4\) hold only for \(-1 < n < n_c\). In our opinion, this is a state of affairs far more \textit{a priori} plausible than the traditionally presented one. In fact, in the traditional view there is a monotonic growth in the energy decay exponent \( b = 2(n + 1)/(n + 3) \) over the range \(-1 < n < 4\), which then suffers a discontinuous drop at \( n = 4 \). This seems unphysical. In our theory, the exponent \( b \) increases as a function of \( n \) over the range \(-1 < n < n_c\), but for larger \( n \) sticks at the value for \( n = n_c \). Although the lowest wavenumber spectrum then does satisfy PLE, it does not match onto the energy range and it plays no role in the energetics of the decay. In fact, the energy \( E_*(t) \) in the lowest wavenumber range where PLE holds is

\[
E_*(t) \propto \ell_*^{-(n+1)}(t).
\]

Since the total energy scales as \( E(t) \propto \ell^{-(n_c+1)}(t) \) for \( n_c = 4 - p \), there is a negligible fraction of energy in the PLE range asymptotically in time for \( n > n_c \). Hence, at these very long times, self-similarity is effectively restored, described again by the first model \(^4\).

We presume these facts will also follow from analytical closures, such as EDQNM, since the
basic ingredients of our phenomenological theory are already present there. Thus, we expect
that those closures have no self-similar decay solutions in the range $n_c < n < 4$. It is interesting
that the verifications of self-similarity which have been made do not seem to have included
values in this range. See for example figure VII-8 in Lesieur\textsuperscript{1}, where an impressive similarity
collapse is shown, but only for $n = 2$ and $n = 4$. We expect that the assumption of a self-similar
decay in EDQNM-type closures for $n_c < n < 4$ will lead to a realizability violation, similar to
what was found for the Kraichnan model\textsuperscript{4}. Of course, positive spectra are guaranteed for
EDQNM, but only if one actually solves the model and not if one makes hypotheses (such as
self-similarity) which may be inconsistent with the closure equations themselves!

A theory analogous to that presented here can be developed for many other turbulent decay
problems. For example, the case of stationary turbulence with a Richardson eddy-diffusivity
$K(r) \sim r^{4/3}$ was studied by Eyink and Xin\textsuperscript{4}, within the Kraichnan model. It was found that
a decaying scalar in the inertial-convective interval experiences self-similar decay only for low-
wavenuumber scalar spectral exponents which are initially in the ranges $-1 < n < 8/3$ and $n > 4$.
In this model, the time-dependence of the constant $C(t)$ could be evaluated, as $C(t) \propto (t-t_0)^2$,
which allowed the determination of $n_c = 8/3$. In the range $8/3 < n < 4$, the self-similar decay
is nonrealizable and is replaced by a two-scale decay of the type described above. This state
of affairs presumably holds as well for the passive scalar advected by an actual (not synthetic)
turbulent velocity. In fact, one of us\textsuperscript{9} has constructed a simple model of the mandoline geometry
often used experimentally to study decay of temperature fluctuations, and found that there is
a low wavenumber $k^4$ spectrum but with decay exponents the same as for $n = 8/3$. Similar
results can be derived also for scalars passively advected by turbulence which is itself decaying,
and for many other such turbulent decay problems.
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