An integral arising from the chiral $sl(n)$ Potts model

Anthony J Guttmann$^1$ and Mathew D Rogers$^2$

$^1$ Department of Mathematics and Statistics, ARC Centre of Excellence for Mathematics and Statistics of Complex Systems, The University of Melbourne, Victoria 3010, Australia
$^2$ Department of Mathematics and Statistics, Université de Montréal, Montréal, Québec, H3C 3J7, Canada

E-mail: tonyg@ms.unimelb.edu.au and mathewrogers@gmail.com

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Abstract

We show that the integral

$$J(t) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi dx \, dy \, dz \, \log(t - \cos x - \cos y - \cos z + \cos x \cos y \cos z),$$

can be expressed in terms of $\text{}_{5}F_{4}$ hypergeometric functions. The integral arises in the solution by Baxter and Bazhanov of the free-energy of the $sl(n)$ Potts model, which includes the term $J(2)$. Our result immediately gives the logarithmic Mahler measure of the Laurent polynomial

$$k - \left( x + \frac{1}{x} \right) - \left( y + \frac{1}{y} \right) - \left( z + \frac{1}{z} \right) + \frac{1}{4} \left( x + \frac{1}{x} \right) \left( y + \frac{1}{y} \right) \left( z + \frac{1}{z} \right)$$

in terms of the same hypergeometric functions.

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1. Calculation of the integral

There exists an extensive literature on solvable two-dimensional lattice models, based on the concept of commuting transfer matrices and the Yang–Baxter relation. Far fewer solutions are available for three-dimensional models. The first such model was obtained by Zamolodchikov [16, 17] in 1980–81. He introduced a so-called tetrahedron relation as the appropriate generalization of the Yang–Baxter equation. In 1983 Baxter [1] solved this model using only commutativity, symmetry and a factorizability property of the transfer matrix. A decade later Bazhanov and Baxter [2] studied the $sl(n)$-chiral Potts model [3], which can be considered a multi-state generalization of the Zamolodchikov model, and showed that for this model it was also possible to bypass the tetrahedron equations and solve the model by using symmetry properties to prove the commutativity of the row-to-row transfer matrices. Aspects of the solution were later discussed by Bazhanov [4], who showed that the solution could be expressed...
in terms of a free boson or Gaussian model on the three-dimensional cubic lattice. The partition function was expressed in terms of a fractional power of the determinant of a cyclic square matrix. In the thermodynamic limit the free energy was given as the sum of two terms, one of which was the integral $J(2)$, where
\[ J(t) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi dk_1 \, dk_2 \, dk_3 \log(t - c_1 - c_2 - c_3 + c_1 c_2 c_3), \] (1)
where $c_i$ denotes $\cos(k_i)$. Baxter and Bazhanov showed that
\[ J(2) = \frac{8}{\pi} L_{-4}(2) - 3 \log 2, \] (2)
where $L_{-4}(2)$ is Catalan’s constant, $L_{-4}(s) := \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{s}$ denotes Dirichlet’s $L$-series, and $\left(\frac{1}{n}\right)$ is the Jacobi symbol.

Here we show that the integral can be expressed in terms of $_5F_4$ hypergeometric functions for $|t|$ sufficiently large, and $t \geq 2.0802 \ldots$ on the real axis. Our starting point is the work of Delves, Joyce and Zucker [9] who considered the related Green function integral
\[ G(t) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{dk_1 \, dk_2 \, dk_3}{t - c_1 - c_2 - c_3 + c_1 c_2 c_3}. \] (3)
If $t$ lies in the complex plane cut along the real axis from $-2$ to $2$, then they proved that
\[ G(t) = \frac{1}{t} \left(1 - \frac{4}{t^2}\right)^{-1/4} \left[ _2F_1 \left( \frac{3}{4}, \frac{5}{4} ; \frac{1}{2} ; \frac{4}{t^2} \right) \right]^2. \] (4)

There are at least two methods for integrating this formula. The more difficult approach is to use (4) to derive a modular expansion for $J(t)$, which can then be compared to known modular expansions for $_5F_4$ functions. This approach was used in [13] to study three-variable Mahler measures. We leave those calculations for a future paper [14], since the following theorem can be proved via standard hypergeometric transformations.

**Theorem 1.** Suppose that $|\alpha|$ is sufficiently small but non-zero. Then
\[ J\left(\sqrt{4\alpha(1-\alpha)} + \frac{1}{\sqrt{4\alpha(1-\alpha)}} \right) = -\frac{1}{2} \log(4\alpha(1-\alpha)^9 (1+\alpha)^{12}) \]
\[ - \frac{11}{4} \alpha(1-\alpha)_5F_4 \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1 ; 4\alpha(1-\alpha) \right) \]
\[ - \frac{7\alpha}{4(1-\alpha)^2} _5F_4 \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1 ; -\frac{4\alpha}{(1-\alpha)^2} \right) \]
\[ + \frac{9\alpha(1-\alpha)^2}{4(1+\alpha)^2} _5F_4 \left( \frac{5}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1 ; \frac{16\alpha(1-\alpha)^2}{(1+\alpha)^4} \right). \] (5)

**Proof.** We show that the derivatives of both sides of the equation agree. To simplify the calculations, briefly assume that $\alpha$ is a small positive real number. Notice that for $s \in (0, 1)$:
\[ \frac{d}{dz} \left[ _5F_4 \left( \frac{2-s, \frac{3}{2}, 1+s, 1, 1 ; \frac{3}{2}, 2, 2, 2 ; z \right) \right] = \frac{2}{s(1-s)z} \left[ _3F_2 \left( \frac{1-s, \frac{3}{2}, 1+s, 1, 1 ; \frac{3}{2}, 2, 2, 2 ; z \right) - 1 \right]. \] (6)

Let us set
\[ t := \sqrt{4\alpha(1-\alpha)} + \frac{1}{\sqrt{4\alpha(1-\alpha)}}. \] (7)
Differentiate (5) with respect to $\alpha$, and then apply (3), (4) and (6), to obtain the presumed equality
\[
\frac{1}{t} \left( 1 - \frac{4}{t^2} \right)^{-1/4} \left[ \frac{\Gamma(\frac{3}{2}, \frac{3}{2}, \frac{4}{t^2})}{\Gamma(\frac{1}{2})} \right]^2 = \frac{2(2\alpha - 1)^3}{(4\alpha(1 - \alpha))^{3/2}}
\]
\[
= -\frac{11}{2\alpha(1 - \alpha)} 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 4\alpha(1 - \alpha) \right) + \frac{7}{2\alpha(1 - \alpha)} 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{-4\alpha}{(1 - \alpha)^2} \right)
\]
\[
+ \frac{3}{2\alpha(1 - \alpha^2)} 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{16\alpha(1 - \alpha)^2}{(1 + \alpha)^4} \right).
\]
(8)

We can re-express the left-hand side of the identity using equation (16) on p 112 of Erdélyi et al [11], giving
\[
\left( 1 - \frac{4}{t^2} \right)^{-1/4} \left[ \frac{\Gamma(\frac{3}{2}, \frac{3}{2}, \frac{4}{t^2})}{\Gamma(\frac{1}{2})} \right]^2 = \xi \left[ \frac{\Gamma(\xi, \frac{3}{2}, \frac{1 - \xi}{2})}{\Gamma(\frac{1}{2})} \right]^2,
\]
where $\xi = (1 - \frac{4}{t^2})^{-1/2}$. Substituting for $t$ yields
\[
\left( 1 - \frac{4}{t^2} \right)^{-1/4} \left[ \frac{\Gamma(\frac{3}{2}, \frac{3}{2}, \frac{4}{t^2})}{\Gamma(\frac{1}{2})} \right]^2 = \frac{1 + 4\alpha - 4\alpha^2}{(1 - 2\alpha)^2} \left[ 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{\alpha}{\alpha - 1} \right) \right]^2
\]
\[
= \frac{1 + 4\alpha - 4\alpha^2}{(1 - 2\alpha)} \left[ 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{\alpha}{\alpha - 1} \right) \right]^2,
\]
(9)

where the second and third steps follow from [5, p 95] and [5, p 38] The right-hand side of (8) simplifies via Clausen’s identity, and the same quadratic transformations:
\[
3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 4\alpha(1 - \alpha) \right) = \left[ 3F_2 \left( \frac{1}{2}, \frac{1}{2}; \alpha \right) \right]^2,
\]
(10)
\[
3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{4\alpha}{(1 - \alpha)^2} \right) = \left[ 3F_2 \left( \frac{1}{2}, \frac{1}{2}; \frac{\alpha}{\alpha - 1} \right) \right]^2 = (1 - \alpha) \left[ 3F_2 \left( \frac{1}{2}, \frac{1}{2}; \alpha \right) \right]^2
\]
(11)
\[
3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{16\alpha(1 - \alpha)^2}{(1 + \alpha)^4} \right) = \left[ 3F_2 \left( \frac{1}{2}, \frac{1}{2}; \frac{4\alpha}{(1 + \alpha)^2} \right) \right]^2 = (1 + \alpha) \left[ 3F_2 \left( \frac{1}{2}, \frac{1}{2}; \alpha \right) \right]^2
\]
(12)

We must exercise caution when applying Clausen-type identities. For instance, (10) is only valid if $\alpha$ lies in a neighborhood of zero. It is easy to see that the identity fails on the real-line when $\alpha > \frac{1}{2}$, because the left-hand side is invariant under the transformation $\alpha \mapsto 1 - \alpha$, while the right-hand side is not. If we substitute (7), (9)–(12) into (8), we see that (8) holds whenever $\alpha$ is a sufficiently small real number. This implies that (5) holds up to a constant of integration. The constant of integration is easily seen to equal zero, because both sides of (5) approach $-\frac{1}{2} \log \alpha + 0$ when $\alpha$ tends to zero. Finally, if we add $\frac{1}{2} \log \alpha$ to either side of the identity, then both sides are analytic in a neighborhood of $\alpha = 0$, so (5) holds for $|\alpha|$ sufficiently small but non-zero.

We have proved that equation (5) is valid if $|\alpha|$ is sufficiently small but non-zero. If we add $\frac{1}{2} \log \alpha$ to either side of the identity, then both sides of the identity are analytic in a neighborhood of $\alpha = 0$. We can analytically continue the new identity along a ray starting from $\alpha = 0$, and ending at a point where the right-hand side ceases to be analytic. Since all three $3F_2$ functions have branch cuts on the segment $[1, \infty)$, we must look for cases where one
of the functions \( \{4\alpha(1 - \alpha), -\frac{4\alpha}{(1-\alpha)^3}, \frac{16\alpha(1-\alpha)^2}{(1+\alpha)^3}\} \) intersects \([1, \infty)\). If \( \alpha \) is positive and real, then the first such intersection occurs at \( \alpha = (\sqrt{2} - 1)^2 \). This allows us to conclude that the formula for \( J(t) \) holds on the positive real axis for \( 0 < \alpha \leq (\sqrt{2} - 1)^2 \approx 0.1715 \ldots \). Since \( t = \sqrt{4\alpha(1 - \alpha)} + 1/\sqrt{4\alpha(1 - \alpha)} \), we conclude that the formula holds if \( t \geq 2.0802 \ldots \). Despite the fact that equation (5) cannot be used to calculate \( J(2) \), it is still possible to reprove the formula of Baxter and Bazhanov via a closely related modular expansion [14]:

\[
J(u(e^{-2\pi v})) = -3 \log 2 + \frac{15v}{\pi^3} \sum_{n,k \neq (0,0)} \frac{3n^2 - (2v)^2k^2}{(n^2 + (2v)^2k^2)^3} + 48v \sum_{n,k} \frac{3(2n + 1)^2 - (2v)^2(2k + 1)^2}{((2n + 1)^2 + (2v)^2(2k + 1)^2)^3},
\]

(13)

where

\[
u(q) = \left( \sqrt[2]{\frac{\eta(q)\eta(q^4)}{\eta^2(q^2)}} \right)^{12} + \left( \sqrt[2]{\frac{\eta(q)\eta(q^4)}{\eta^2(q^2)}} \right)^{-12},
\]

and \( \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \). Equation (13) can be derived by combining equation (5) with (2.12) and (2.14) in [13]. It is possible to show that (13) holds for \( v \geq \frac{1}{2} \), and that the left-hand side of the identity equals \( J(2) \) when \( v = \frac{1}{2} \). The right-hand side reduces to two-dimensional lattice sums, which can ultimately be evaluated by appealing to results of Glasser and Zucker [10]. Bertoin, Rodriguez-Villegas and Samart have all used a similar modular approach to prove Mahler measure formulas [6, 12, 15].

Equation (5) also implies an identity between Mahler measures. The Mahler measure of an \( n \)-variable Laurent polynomial, \( P(x_1, \ldots, x_n) \), is defined by

\[
m(P) := \int_{0}^{1} \ldots \int_{0}^{1} \log |P(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})| dt_1 \ldots dt_n.
\]

The second author related both \( \phi \) functions to Mahler measures in [13]. After some simplification, equation (5) implies

\[
m\left( 8\sqrt{4\alpha(1 - \alpha)} + \frac{8}{\sqrt{4\alpha(1 - \alpha)}}, 4(x + x^{-1} + y + y^{-1} + z + z^{-1}) + (x + x^{-1})(y + y^{-1})(z + z^{-1}) \right)
\]

\[
= 11m\left( \frac{4}{\sqrt{\alpha(1 - \alpha)}} + (x + x^{-1})(y + y^{-1})(z + z^{-1}) \right)
\]

\[
- 7m\left( \frac{4i(1 - \alpha)}{\sqrt{\alpha}} + (x + x^{-1})(y + y^{-1})(z + z^{-1}) \right)
\]

\[
- 6m\left( x^2 + y^2 + z^2 + 1 + \frac{2(1 + \alpha)}{\sqrt{\alpha(1 - \alpha)^2}}xyz \right),
\]

and this identity also holds on the real axis for \( \alpha \in (0, (\sqrt{2} - 1)^2] \).

2. Conclusion and special values of \( J(t) \)

We conclude by noting that formula (2) of Baxter and Bazhanov is not an isolated result, and that there are many additional explicit formulas for values of \( J(t) \). Most of these formulas involve \( L \)-functions of eta products. These are not elementary constants, but they often carry deep number-theoretic significance, so in some sense they can still be regarded as fundamental.
constants. For more details on explicit Mahler measure formulas, we refer to the work of Boyd [7]. We conclude with a single example of such a formula. If we have a function

\[ f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}, \]

then the \( L \)-series associated with \( f \) is defined by

\[ L(f, s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \]

Thus if we take \( \eta(\tau) \) to be the usual Dedekind eta function, it is possible to prove

\[ J\left(\frac{5}{2}\right) = \frac{24\sqrt{3}}{\pi^3} L(\eta^3(2\tau), 3) + \frac{15\sqrt{3}}{4\pi} L_{-3}(2) - 3 \log 2. \quad (15) \]

We have also obtained more complicated identities for \( J(14), J(322), \) and \( J(t) \) for various irrational algebraic values of \( t \) [14].

It is also noteworthy that the two hypergeometric functions that appear in theorem 1 are precisely those that appear in our solution [8] of the spanning tree constant for the simple-cubic lattice.

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