RG flows and resonance scattering amplitudes

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Abstract

We review recent progresses in the study of factorized resonance scattering S-matrices. The resonance amplitudes are introduced through a suitable analytical continuation of the ADE Toda S-matrices. By using the thermodynamic Bethe ansatz approach we are able to compute the ground state energy, which describes a rich pattern of flows interpolating between the central charges of the coset models based on the ADE Lie algebras. We also present the simplest resonance “φ³” scattering model and discuss its relation with new flows in non-unitary minimal models. Further generalizations are discussed in terms of certain asymptotic conditions in a family of “resonance” functional hierarchies.

August 1992

*To appear in the Proceedings of the CAP/NSERC workshop on “Quantum Groups, Integrable Models and Statistical Systems” (July 1992), J. LeTourneux, L. Vinet (eds), World Scientific.
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1 Introduction

The study of scale non-invariant systems is certainly interesting and in many cases a difficult open problem. A relevant aspect of this problem is, e.g., the study of the renormalization group (RG) scenario in the vicinity of the fixed point. Not much is known about this problem in higher space time dimension. Recently, however, in (1+1) dimension a considerable progress has been achieved after the work of Belavin, Polykov and Zamolodchikov [1]. Considering a unitary theory, the RG scenario is severely constrained by Zamolodchikov’s c-theorem [2]. The c-theorem tells us that RG flows always run down hill, thinning the degrees of freedom as the RG trajectory flows from the ultraviolet region to the infrared regime. A.B. Zamolodchikov [3] has considered a non-scale invariant field theory as a relevant perturbation of the conformal field theory which characterizes the ultraviolet properties of the system. In certain cases the theory is integrable and factorizable S-matrices can be conjectured [3]. Using the S-matrices one can, in principle, apply the thermodynamic Bethe ansatz (TBA) approach in order to determine the ground state scaling function [3, 4, 5] associated to the respective RG trajectories. A typical example is the minimal models $M_p$ perturbed by the least relevant operator $\phi_{1,3}$. The infrared behaviour is highly dependent of the sign of the perturbation: it will induce a crossover either to the lower critical theory $M_{p-1}$ [8, 9] or to a purely massive theory of (p-2) types of Kink-antiKink pairs [10, 11].

The main purpose of this article is to discuss the RG trajectories associated to the recently discovered resonance scattering models [12, 13, 14, 15]. In section 2 we introduce a general class of resonance S-matrices based on the ADE affine Toda scattering amplitudes [16, 17, 18, 19]. This section also contains an analysis of the associated Casimir energy via TBA approach. Finally, we discuss the simplest “$\phi^3$” resonance factorized scattering model and its connection with new flows in non-unitary minimal models. In Section 3 we discuss the functional hierarchies of the TBA equations and we also investigate a further generalization of these relations.
2 The resonance S-matrix and its Casimir energy

Recently Al. Zamolodchikov [12] has proposed the simplest resonance S-matrix consisting of a single particle scattering through the amplitude

\[
S(\theta, \theta_0) = \frac{\sinh(\theta) - i \cosh(\theta_0)}{\sinh(\theta) + i \cosh(\theta_0)}
\]

where \(\theta_0\) is the resonance parameter.

Eq.(1) can be obtained from the sinh-Gordon S-matrix through a suitable analytical continuation of its coupling constant. This fact suggests that a more general class of resonance S-matrices can be formulated from the ADE Toda theory [16, 17, 18, 19]. Indeed, the amplitude \(S_{a,b}(\theta, \theta_0)\) of the resonance ADE S-matrices [13, 14, 15] can be defined by

\[
S_{a,b}(\theta, \theta_0) = S^{\min}_{a,b}(\theta)Z_{a,b}(\theta, b(\alpha)) = \frac{\pi}{h} \pm i\theta_0
\]

where \(S^{\min}_{a,b}(\theta)\) are the minimal amplitudes containing the physical poles; \(Z_{a,b}(\theta, b(\alpha))\) are the so-called Z-factors which encode the coupling constant \(\alpha\) through the function \(b(\alpha)\) and \(h\) is the Coxeter number. It is easy to check that Eq.(2) reproduces Al. Zamolodchikov model for the \(A_1\) Lie algebra.

Our interest now is to study the finite size corrections of the ground state associated to these resonance S-matrices. The TBA equations [4, 5, 6] describe exactly the ground state energy of an integrable theory on a torus of radius R. In our case the Casimir energy \(E(R, \theta_0)\) is given by

\[
E(R, \theta_0) = -\frac{1}{2\pi} \sum_{a=1}^{r} m_a \int_{-\infty}^{+\infty} d\theta \cosh(\theta)L_a(\theta)
\]

where \(m_a\) are the ADE mass gaps [16, 17, 18, 19], \(r\) is the rank of the respective Lie algebra, and \(L_a(\theta) = \ln(1+e^{-\epsilon_a(\theta)})\). The pseudoenergies \(\epsilon_a(\theta)\) satisfy the following integral equation (TBA equations)

\[
\epsilon_a(\theta) + \frac{1}{2\pi} \sum_{b=1}^{r} \psi_{a,b} * L_b(\theta) = m_a R \cosh(\theta)
\]
where $\psi_{a,b}(\theta) = -i \frac{d}{d\theta} S_{a,b}(\theta, \theta_0)$ and the symbol $f \ast g(x)$ denote the convolution $f \ast g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-y)g(y)dy$.

The ultraviolet limit of Eq.(1) is dominated by a background conformal theory with central charge $r$, and the first correction is logarithmic in $R$ [12, 14]. In terms of the function $c(R, \theta_0) = -\frac{6R}{\pi} E(R, \theta_0)$ our result reads,

$$c(R, \theta_0) = r - \frac{3(\theta_0^2 + \frac{\pi^2}{X^2})}{X^2} \sum_{a,b} C_{a,b}^{-1}$$

where $X = \ln(\frac{m_1 R}{2})$.

In order to analyze the behaviour of function $c(R, \theta_0)$ for intermediate distances of $R$ we have numerically solved Eq.(4). Figures 1(a,b) show the behaviour of $c(R, \theta_0)$ for $\theta_0 = 20, 40$ in the case of the $A_2$ Lie algebra. After a careful analysis of other models, we arrived at the following general picture [12, 13, 14]. At $\theta_0 = 0$ function $c(R, \theta_0)$ presents a smooth behaviour between the ultraviolet and infrared regimes. However, for large enough $\theta_0$, $c(R, \theta_0)$ starts to form plateaux around the central charges $c_r^p = r(1 - h(h + 1)/p(p + 1)), p = h + 1, h + 2, ...$ of the $G_1 \otimes G_{p-h}/G_{p-h+1}$ models. More precisely, each time that $X \simeq -(p - h)\frac{\theta_0}{2}$ function $c(R, \theta_0)$ crosses over from its value $c_r^p$ to the next (up) value $c_{r+1}^p$. It is important to stress here that the “RG time” $\frac{\theta_0}{2}$ accounts for the plateau and its finite size corrections (see fig.1(a,b)). Discussions on the identification of this “staircase pattern” with a deformed conformal field theory can be found in refs. [20, 12, 14].

Let us now introduce what we believe to be the simplest resonance scattering model possessing the “$\phi^3$”-property. The model consists of a single particle $a$ and its two-body S-matrix is given by

$$S_{a,a}(\theta, \theta_0) = \frac{\tanh \frac{1}{2}(\theta + i\frac{\pi}{3}) \tanh \frac{1}{2}(\theta - \theta_0 - i\frac{\pi}{3}) \tanh \frac{1}{2}(\theta + \theta_0 - i\frac{\pi}{3})}{\tanh \frac{1}{2}(\theta - i\frac{\pi}{3}) \tanh \frac{1}{2}(\theta - \theta_0 + i\frac{\pi}{3}) \tanh \frac{1}{2}(\theta + \theta_0 + i\frac{\pi}{3})}$$

The Toda related field theory is the one analyzed by Arinshtein et al [16] and known as the Shabat-Mikhailov model. The “minimal” part of Eq.(6) is the S-matrix [21] of the perturbed Yang-Lee edge singularity [22]. We also notice that amplitude $S_{a,a}(\theta, \theta_0)$ satisfy
the following important relation

\[ S_{a,a}(\theta, \theta_0) = S_{1,1}(\theta, \theta_0)S_{1,2}(\theta, \theta_0) \]  

(7)

where \( S_{1,1}(\theta, \theta_0) \) and \( S_{1,2}(\theta, \theta_0) \) are the \( A_2 \) amplitudes.

From Eqs.(7) and (4) it follows that the ground state energy of this theory is precisely half of that of the \( A_2 \) model. Hence, the plateau will now form around the values \( c_p = 1 - 12/p(p+1), p = 4, 5, \ldots \). Taking into account that the effective central charge of the minimal models \( M^p_q \) is \( c_{ef} = 1 - 6/pq \), we are able to identify the following non-unitary minimal models: \( M^p_{2q+1} \) (q=p/2, p even) and \( M^{p+1}_{2q+1} \) (q=(p-1)/2, p odd). Furthermore, from the finite size corrections of the ground state we identify the fields \( \phi_{2,1} \) and \( \phi_{1,5} \) as those responsible for the crossover behaviour. More precisely, we predict the following new flows

\[ M^q_{2q+1} + \phi_{2,1} \rightarrow M^q_{2q+1} \quad q = 2, 3, \ldots \]  

(8)

and for \( M^q_{2q+1} \),

\[ M^q_{2q+1} + \phi_{1,5} \rightarrow M^{(q-1)+1}_{2(q-1)+1} \quad q = 3, 4, \ldots \]  

(9)

We remark that the field \( \phi_{2,1} \) and \( \phi_{1,5} \) cannot be both relevant operators in the same model. In addition, as already discussed by the author [23], the \( \phi_{1,5} \) perturbation can formally be related to the \( \phi_{1,2} \) deformation. We believe that the combination \( \lambda \phi_{2,1} + \bar{\lambda} \phi_{1,5} \) play a similar role of the fields \( \lambda \phi_{1,3} + \bar{\lambda} \phi_{3,1} \) appearing in the minimal models [21]. For other relevant discussions see ref. [14].

3 Resonance functional hierarchies

In this section we discuss certain functional relations for functions \( Y_a(\theta) = e^{\epsilon a(\theta)} \). It is possible to rewrite the TBA equations in a more suggestive way, adopting a similar approach as in ref. [24]. First one has to notice the following remarkable matrix identity,

\[ \left[ \delta_{a,b} - \frac{\bar{\psi}_{a,b}(k, \theta_0)}{2\pi} \right]^{-1} = \delta_{a,b} \cosh \left[ \frac{\pi k}{\Lambda} \right] - l_{a,b}/2 \cosh \left[ \frac{\pi k}{\Lambda} \right] - \cos(k\theta_0) \]  

(10)
for the Fourier component $\psi_{a,b}(k, \theta_0) = \int_{-\infty}^{+\infty} e^{i k \theta} \psi_{a,b}(\theta, \theta_0) d\theta$. Here, $l_{a,b}$ is the incident matrix of the G=A,D,E Lie algebra.

Using Eq.(10) and the relation $m_a = \sum_{b=1}^{r} l_{a,b} m_b$, we obtain the set of functional equations given by $Y_a(\theta) = e^{c_a(\theta)}$ [12, 14],

$$Y_a(\theta + \frac{i \pi}{h}) Y_a(\theta - \frac{i \pi}{h}) = \prod_{b \in G} [1 + Y_b(\theta)]^{l_{a,b}} \left[1 + Y_a^{-1}(\theta + \theta_0)\right]^{-1} \left[1 + Y_a^{-1}(\theta - \theta_0)\right]^{-1} \quad (11)$$

We stress that one may start with such functional hierarchies instead of considering the ADE resonance $S$-matrices of section 2. The connection with the TBA equations is due to the asymptotic conditions of the functions $Y_a(\theta)$ at low temperature, i.e., $Y_a(\theta) \simeq \exp(m_a R \cosh(\theta))$ $R \to \infty$. A simple generalization of Eq.(11) is the one discussed by the author in ref. [25]. We have two pairs of functions $Y_a^1(\theta)$ and $Y_a^2(\theta)$ satisfying

$$Y_a^1(\theta + \frac{i \pi}{h}) Y_a^1(\theta - \frac{i \pi}{h}) = \prod_{b \in G} [1 + Y_b^1(\theta)]^{l_{a,b}} \left[1 + 1/Y_a^2(\theta + \theta_0)\right]^{-1} \left[1 + 1/Y_a^2(\theta - \theta_0)\right]^{-1} \quad (12)$$

Considering the non-trivial compatible asymptotic conditions $Y_a^1(\theta) \simeq \exp(m_a R \cosh(\theta))$ and $Y_a^2(\theta) \simeq 1$, we found the following consistent TBA equations [25]

$$c_a^1(\theta) + \sum_{b=1}^{r} \phi_{a,b} \ast L^1_b(\theta) + \sum_{b=1}^{r} \varphi_{a,b} \ast L^2_b(\theta) = m_a R \cosh(\theta) \quad (13)$$
$$c_a^2(\theta) + \sum_{b=1}^{r} \phi_{a,b} \ast L^1_b(\theta) + \sum_{b=1}^{r} \varphi_{a,b} \ast L^2_b(\theta) = 0$$

where $L^i_b(\theta) = \ln(1+e^{-c_i(\theta)})$, $i = 1, 2$. Functions $\phi_{a,b}(\theta)$ and $\varphi_{a,b}(\theta)$ are related to the ADE Toda scattering amplitudes [16, 17, 18, 19] through the relations $\phi_{a,b}(\theta) = -i \frac{d}{d \theta} \ln S_{a,b}^{\text{min}}(\theta)$ and $\varphi_{a,b}(\theta) = -i \frac{d}{d \theta} \ln Z_{a,b}(\theta, b = \frac{\xi}{h} \pm i \theta_0)$.

From our analysis of the respective Casimir energy, function $c(R, \theta)$ interpolates between the central charges of the $G_2 \otimes G_l/G_l+2$ (1 even) coset models. The “RG time” being double the amount of the one found in section 2, namely $\theta_0$. We believe that this observation will be helpful for further generalizations of our functional relations. It is also worth to mention that similar functional relations have been discussed in the literature.
in the context of the inversion relations of integrable lattice models. Such relations play the keystone in the computations of critical exponents and probably still hide a large amount of information yet to be explored.

Acknowledgements

The author would like to thank the organizers of the workshop “Quantum Groups, Integrable Models and Statistical Systems” for their hard work, and for providing me an opportunity to present this material.

References

[1] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Nucl.Phys.* B241 (1984) 333

[2] A.B. Zamolodchikov, *Pisma Zh.Eksp.Teor.Fiz.* 43 (1986) 565

[3] A.B. Zamolodchikov, *Advanced Studies in Pure Mathematics* 19 (1989) 641

[4] C.N. Yang, C.P. Yang, *J.Math.Phys.* B 10 (1969) 1115

[5] Al.B. Zamolodchikov, *Nucl.Phys.* B342 (1990) 695

[6] Al.B. Zamolodchikov, *Nucl.Phys.* B358 (1991) 497

[7] Al.B. Zamolodchikov, *Nucl.Phys.* B358 (1991) 524

[8] A.B. Zamolodchikov, *Yad.Fiz.* 46 (1987) 82

[9] A.W.W. Ludwig, J.L. Cardy, *Nucl.Phys.* B285 (1987) 687

[10] A.B. Zamolodchikov, *Moscow preprint* (1989)
[11] N.Yu. Reshetikhin, F.A. Smirnov, Comm.Math.Phys. 31 (1990) 137
D. Bernard, A. LeClair, Nucl.Phys. B340 (1990) 721
C. Anh, D. Bernard, A. LeClair, Nucl.Phys. B346 (1990) 490

[12] Al.B. Zamolodchikov, Paris preprint (1991) ENS-LPS-335

[13] M.J. Martins, SISSA preprint (1992) EP-72, Phys.Rev.Lett. in press

[14] M.J. Martins, SISSA preprint (1992) EP-85

[15] P.E. Dorey, F. Ravanini, Saclay/Bologna preprint SPhT-92-065/ DFUB-92-09

[16] A.E. Arinshtein, V.A. Fateev, A.B. Zamolodchikov, Phys.Lett. B87 (1979) 389

[17] P. Christe, G. Mussardo, Nucl.Phys. B330 (1990) 465; Int.J.Mod.Phys. A5 (1990) 4581

[18] H.W. Braden, E. Corrigan, P.E. Dorey, R. Sasaki, Phys.Lett. B227 (1989) 441; Nucl.Phys. B338 (1990) 689

[19] C. Destri, H.J. de Vega, Phys.Lett. B233 (1989) 336

[20] M. Lässig, Julich preprint (1991)

[21] J.L. Cardy, G. Mussardo, Phys.Lett. B225 (1989) 275

[22] M.E. Fischer, Phys.Rev.Lett 40 (1978) 1610
J.L. Cardy, Phys.Rev.Lett 54 (1985) 1354

[23] M.J. Martins, Phys.Lett. B262 (1991) 39

[24] Al.B. Zamolodchikov, Phys.Lett. B253 (1991) 391

[25] M.J. Martins, SISSA preprint (1992) SISSA-EP-151
[26] P.A. Pearce, A. Klümper, *Phys.Rev.Lett.* **66** (1991) 974

A. Klümper, P.A. Pearce, *J.Stat.Phys.* **64** (1991) 13; *Melbourne preprint N0 23 (1991)*

P.A. Pearce, *Melbourne preprint N0 22 (1991)*

[27] A. Kuniba, T. Nakanishi, *Nagoya preprint SMS-042-92*
Fig. 1(a,b) The scaling function $c(R, \theta_0)$ for the $A_2$ model: (a) $\theta_0 = 20$, (b) $\theta_0 = 40$