A generalization of falsity in finitely-many valued logics

Nissim Francez

1. Introduction

In propositional classical logic, if a formula $\varphi$, under some valuation $v$, is not true, then $\varphi$ is false, and if it is not false, it is true. This toggling between truth and falsehood is captured in propositional classical logic by means of negation '$\neg$':

| $\varphi$ | $\neg\varphi$ |
|----------|--------------|
| $\bot$   | $\bot$       |
| $t$      | $f$          |
| $f$      | $t$          |

Consider now some multi-valued logic $\mathcal{L}$ with a set of truth-values $\mathcal{V}$:

$\mathcal{V} = \{v_1, \ldots, v_n\}$, $n \geq 2$

Q: What does it mean that under some valuation $v$, some $\varphi$ does not have the truth-value $v_i$ for some $v_i \in \mathcal{V}$?

And, in particular, can this meaning be captured by means of a suitable negation in $\mathcal{L}$?

Suppose we have already identified $v_1$ with $t$ and $v_n$ with $f$ (see [5] for one such identification; see Section 4 for the definition used where). Then, there is interest in the special instances of the question Q:

What does it mean that $\varphi$ is not true under some valuation $v$, or not false.

Address for correspondence: Nissim Francez, Department of Computer Science, Technion – Israel Institute of Technology, Haifa 32000, Israel. Email: francez@cs.technion.ac.il.

1 I consider here only finitely many truth-values.

2 I ignore here the issue, orthogonal to our interest, whether $\mathcal{V}$ is a set of truth-values or merely some semantic values.
Traditionally in multi-valued logics, negation is viewed (truth) functionally, \( \neg \varphi \) mapping the truth-value \( v_i \) of \( \varphi \) to some other, specific, truth-value \( v_j \), where \( i = j \) is not excluded. This mapping is again depicted as a multi-valued truth-table. Some well-know examples are listed below, without considerations of interpretation of those truth-values.

**Kleene’s \( K_3 \)[7]:** Here \( \mathcal{V} = \{ t, n, f \} \), and the truth-table for negation is

\[
\begin{array}{c|c}
\varphi & \neg \varphi \\
\hline
- & - \\
t & f \\
- & - \\
n & n \\
- & - \\
f & t \\
\end{array}
\]

**Belnap-Dunn first-degree entailment (FDE) [1, 2, 3]:** Here \( \mathcal{V} = \{ t, b, n, f \} \), and the truth-table for negation is

\[
\begin{array}{c|c}
\varphi & \neg \varphi \\
\hline
- & - \\
t & f \\
- & - \\
n & n \\
- & - \\
b & b \\
- & - \\
f & t \\
\end{array}
\]

**Post cyclic negation [8]:** \( \mathcal{V} = \{ v_0, \cdots, v_{n-1} \} \) and negation is cyclic.

\[
\begin{array}{c|c}
\varphi & \neg \varphi \\
\hline
- & - \\
v_0 & v_1 \\
- & - \\
v_1 & v_2 \\
- & - \\
\vdots & \vdots \\
- & - \\
v_{n-1} & v_0 \\
\end{array}
\]

\(^{3}\text{See } K_3 \text{ below for a case of } i = j.\)
Consequently, the question \(Q\) is traditionally answered as follows: for any \(v_i \in V\), if, under a valuation \(v\), \(\varphi\) does not have truth-value \(v_i\), then \(\varphi\) has under \(v\) some specific truth-value \(v_j\), where \(j = i\) is not excluded.

In this paper, I aim at another way to answer the question \(Q\): *If, under some valuation \(v\), \(\varphi\) does not have truth-value \(v_i\), this is understood as \(\varphi\) having, under \(v\), non-deterministically, any other (not functionally determined) truth-value in \(V\).*

There is no “privileged” \(v_j\) materializing not having the value \(v_i\)!

This means that traditional multi-valued negations, as exemplified above, cannot be used to express this interpretation of not having \(v_i\). Instead, I introduce another operator, that generalizes negation in multi-valued logics as a non-deterministic operator. To distinguish our approach, I use a unary operator ‘\(N\)’ instead of ‘\(\neg\)’.

I consider \(n\) operators, \(N_i\), for \(1 \leq i \leq n\). The intended meaning of \(N_i \varphi\), when true under some valuation \(v\), is that \(\varphi\) does not have the truth-value \(v_i\) under \(v\). This, however, is not taken to mean as having some specific truth-value \(v_j\); rather, it is taken to mean that \(\varphi\) has, non-deterministically, any value different from \(v_i\).

Thus, \(N_i \varphi\) can never (i.e., for no valuation \(v\)) share the same truth-value with \(\varphi\). It reflects the meta-linguistic negation of ‘\(\varphi\) has truth-value \(v_i\)’. In this, \(N_i\) differ from \(\neg \varphi\) in traditional multi-valued logics, where \(v[\varphi] = v[\neg \varphi]\) is certainly possible, e.g., for the truth-value \(n\) in \(K_3\) as shown above.

As for the intended meaning of \(N_i \varphi\) when having a truth-value \(v_j \neq v_1 = t\), this will be specified below once the theory is set up.

As the framework for our study, we chose located sequents, introduced and studied in general in [6], and used for a related issue in [4]. The formalism is delineated in Section 2.

2. Preliminaries: located formulas and sequents

For \(n \geq 2\), let \(V = \{v_1, \ldots, v_n\}\) be a collection of truth-values underlying a multi-valued logic \(L^n\) with a propositional object-language \(L_n\) with, possibly, some additional unspecified connectives defined by truth-tables over \(V\). Let \(\hat{n} = \{1, \ldots, n\}\).

**Definition 2.1. (located formula)**
A located formula (l-formula) is a pair \((\varphi, k)\), where \(\varphi\) is an object-language formula and \(k \in \hat{n}\). We say that \((\varphi, k)\) locates \(\varphi\) at \(v_k\).

The intended interpretation of \((\varphi, k)\) is that \(\varphi\) is associated with the truth-value \(v_k \in V\).

**Definition 2.2. (located sequents)**
A located sequent (l-sequent) \(\Pi\) has the form \(\Gamma : \Delta\), where \(\Gamma, \Delta\) are (possibly empty) finite collections\(^5\) of l-formulas.

\(^5\)The exact nature of a collection, e.g., a set or a multi-set, depends on the specific logic being defined.
I use \( \Pi \) for sets of \( l \)-sequents. Let \( \sigma \) range over valuations, mapping formulas to truth-values in \( V \); for atomic sentences the mapping is arbitrary, and it is extended to compound formulas so as to respect the truth-tables of the operators. Below, I define the central semantic notions as applicable to \( l \)-sequents.

**Definition 2.3. (satisfaction, consequence)**

**satisfaction:** \( \sigma \models \Pi(= \Gamma : \Delta) \) iff:

\[
\text{if } \sigma[\varphi] = v_k \text{ for all } (\varphi, k) \in \Gamma, \text{ then } \sigma[\psi] = v_j \text{ for some } (\psi, j) \in \Delta
\]

(1)

**consequence:**

\( \Pi \models \Pi \) iff for every \( \sigma : \sigma \models \Pi' \) for all \( \Pi' \in \Pi \) implies \( \sigma \models \Pi \)

(2)

**validity:** \( \Pi \) is valid iff \( \sigma \models \Pi \) for every \( \sigma \).

In [4, 6], various proof-systems over \( l \)-sequents are presented (in a different notation) sound and (strongly) complete for the above consequence relation, constructed from the truth-tables in a uniform way. The multi-valued ND-systems \( \mathcal{N}^n \) (over \( l \)-sequents) with their structural and logical rules for an arbitrary \( n \)-ary connective are presented in an appendix.

### 3. Transparent falsity and binary poly-sequents

#### 3.1. Transparent falsity and disquotation

As a preliminary step, I consider the case where \( n = 2 \), in which the non-determinism involved is only apparent, since ‘any truth-value other than \( t \)’ is just \( f \), and ‘any truth-value other than \( f \)’ is just \( t \). This section is an adaptation from [5].

Suppose we want to add to classical logic a transparent falsity-predicate \( F(x) \). What would be the way to express falsity? Fortunately, because of the properties of classical negation, where the truth of \( \neg \varphi \) expresses the falsity of \( \varphi \), we can use it for creating such an analog to the disquotation property of the well-known truth predicate:

\[
(DF) \quad F(\hat{\varphi}) \leftrightarrow \neg \varphi
\]

(3)

where \( \hat{\varphi} \) is a name for \( \varphi \) (e.g., the Gödel number). The transparency of \( F(x) \) can be expressed via the following \( I/E \)-rules, in analogy to the well-known rules for the transparent truth predicate.

\[
\frac{\neg \varphi}{F(\hat{\varphi})} \quad (FI) \quad \frac{F(\hat{\varphi})}{\neg \varphi} \quad (FE)
\]

(4)

Notably, those rules are *impure* in that they feature a connective (‘\( \neg \)’ here) different from the one introduced/eliminated by the rules.

*But, what can be done in a more general setting, where no analog to classical negation is present (or definable), to have a transparent falsity predicate?*

---

6For better readability, I use \( \{t, f\} \) instead of \( \{v_1, v_2\} \).
3.2. Bivalent $l$-sequents and transparent truth/falsity predicates

3.2.1. Bivalent $l$-sequents

Consider now binary $l$-sequents $\Pi = \Gamma : \Delta$ (i.e., where $n = 2$). The advantage of this notation in the bivalent case is that it enables expressing falsity of a formula $\varphi$ without appealing to negation, just using a located formula $(\varphi, f)$. Note that both false assumptions and false conclusions are allowed, residents of the respective $\Gamma$ (assumptions) and $\Delta$ (conclusions).

I consider a sound and complete ND-system $N^2$ for the logic of bivalent valid $l$-sequents. Since the connectives are orthogonal to our current concerns, I omit the presentation of their $I/E$-rules. However, this system allows speaking proof-theoretically about my concerns.

The proof system $N^2$ is a special case of $N^n$ for $n = 2$. The general system is presented in an appendix.

We now can state that the falsity predicate $F(x)$ is disquotation by the following analogy to (3), without any appeal to negation.

\[(PSD_{ft}) \quad \Gamma : \Delta, (F(\hat{\varphi}), t) \vdash N^2 \Gamma : \Delta, (\varphi, f) \quad (5)\]

That is: if $F(\hat{\varphi})$ is true, indicated by its $t$-location of the l.h.s., then $\varphi$ is false, indicated by $f$-location of the r.h.s., and vice versa.

\[(PSD_{tf}) \quad \Gamma : \Delta, (F(\hat{\varphi}), f) \vdash N^2 \Gamma : \Delta, (\varphi, t) \quad (6)\]

That is: if $F(\hat{\varphi})$ is false, indicated by its $f$-location of the l.h.s., then $\varphi$ is true, indicated by $t$-location of the r.h.s., and vice versa.

Note the use of a false conclusion in this formulation of the disquotation property of the falsity predicate. This is how the use of (binary) $l$-sequents overcomes the lack of direct means to refer to falsity without using (classical) negation.

Similarly, we can add to $N^2$ the following pure falsity transparency $I/E$-rules, not appealing to $\neg$:

\[\Gamma : \Delta, (\varphi, f) \quad \Gamma : \Delta, (F(\hat{\varphi}), t) \quad (FI_t) \quad \Gamma : \Delta, (F(\hat{\varphi}), t) \quad \Gamma : \Delta, (\varphi, f) \quad (FE_t) \quad (7)\]

Again, for the $(FI_t)$-rule, if $\varphi$ is false, indicated by its location $f$ in $\Delta$ of the premise, then $F(\hat{\varphi})$ is true, indicated by $t$-locating it in $\Delta$ of the conclusion, and similarly for the $(FE_t)$-rule. Note that in the formulation of these rules, both false assumptions and false conclusions are employed.

\[\Gamma : \Delta, (\varphi, t) \quad \Gamma : \Delta, (F(\hat{\varphi}), f) \quad (FI_f) \quad \Gamma : \Delta, (F(\hat{\varphi}), f) \quad \Gamma : \Delta, (\varphi, t) \quad (FE_f) \quad (8)\]

Again, both (5) and (6) become derivable by means of the transparency $I/E$-rules for $F(x)$.

Next, those ideas are generalized for an arbitrary $n \geq 2$.

4. Truth, falsity and their uniqueness

In this section, I identify truth and falsity in $\mathcal{V}$ and prove their uniqueness. Recall that no other connectives besides $N_i$ are assumed to be present.
### 4.1. Identifying truth

**Definition 4.1. (truth)**

A truth-value $v_j \in \mathcal{V}$, for some $1 \leq j \leq n$, is a truth iff the following holds for every $1 \leq i \leq n$ and every $\varphi$:

$$\left( N_i \Gamma : \Delta, (N_i \varphi, j) \right) \vdash_{\mathcal{N}^n} \Gamma : \Delta, \{ (\varphi, k) \mid k \neq i \}$$

That is, for any $1 \leq i \leq n$, the locating $N_i \varphi$ at $v_j$ (i.e., at a truth) is necessary and sufficient for locating $\varphi$ itself with $\{ v_k \mid k \neq i \}$ (i.e., not with $v_i$). Thus, being located with a truth assures the intended meaning of $N_i \varphi$ as not assigning $v_i$ to $\varphi$ (for all $i$).

For this definition to make sense, I need to show that truth is unique; that is, if $v_j$ and $v_k$ are truths, then $j = k$. The existence of a truth is shown at the end of the paper, in 27.

**Proposition 4.2. (uniqueness of truth)**

If both $v_j$ and $v_k$ are truths, then $j = k$.

**Proof:** Assume, towards a contradiction, that for $j \neq k$ both $v_j$ and $v_k$ are truths. Then,

$$\frac{\left( N_k \varphi, j \right) \vdash_{\mathcal{N}^n} \left( N_k \varphi, j \right) \quad \left( N_k \varphi, j \right) \vdash_{\mathcal{N}^n} \left( N_k \varphi, k \right)}{(\ast) \left( N_k \varphi, j \right) : \left( N_k \varphi, j \right)}$$

But,

$$\frac{\left( N_k \varphi, j \right) \vdash_{\mathcal{N}^n} \left( N_k \varphi, j \right) \quad \left( N_k \varphi, j \right) \vdash_{\mathcal{N}^n} \left( N_k \varphi, k \right)}{(\ast) \left( N_k \varphi, j \right) : \left( N_k \varphi, j \right)}$$

a contradiction.

Thus, $j = k$.

For the coordination rule $(c_{j,k})$ and the (cut) rule – see the appendix.

Since the numbering of the truth-values in $\mathcal{V}$ is arbitrary, we assume henceforth that $v_1$ is the unique truth in $\mathcal{V}$.

### 4.2. Identifying falsity

**Definition 4.3. (falsity)**

A truth-value $v_j \in \mathcal{V}$, for some $1 \leq j \leq n$, is a falsity iff the following holds for every $1 \leq i \leq n$ and every $\varphi$:

$$\left( N_j \Gamma : \Delta, (N_i \varphi, j) \right) \vdash_{\mathcal{N}^n} \Gamma : \Delta, (\varphi, i)$$

That is, for any $1 \leq i \leq n$, locating $N_i \varphi$ with $v_j$ (i.e., a falsity) is necessary and sufficient for locating $\varphi$ itself with $v_j$. Thus, being located at a falsity assures the intended meaning of $N_i \varphi$ as not assigning $v_i$ to $\varphi$ (for all $i$) does not hold.
Proposition 4.4. (uniqueness of falsity)
If both \( v_j \) and \( v_k \) are falsities, then \( j = k \).

Proof: Assume, towards a contradiction, that for \( j \neq k \) both \( v_j \) and \( v_k \) are falsities. Then,

\[
\begin{align*}
(N_k \varphi, j) : (N_k \varphi, j) & \quad \leftarrow \quad (N_f, \text{with } i=k) \\
(N_f, \text{with } i=k) & \quad \rightarrow \quad (N_k \varphi, j) : (\varphi, k)
\end{align*}
\]

But,

\[
\begin{align*}
(N_k \varphi, j) : (N_k \varphi, j) \quad (N_k \varphi, j) : (N_k \varphi, k) & \quad \rightarrow \quad (N_k \varphi, j) : (c_j, k)
\end{align*}
\]

A contradiction is now derived as in (11).
Thus, \( j = k \).

Since the numbering of the truth-values in \( V \) is arbitrary, we assume henceforth that \( v_n \) is the unique falsity in \( V \).

5. A natural deduction system for \( N_i \)

I again assume that \( N_i, 1 \leq i \leq n \) are all the operators in the object-language, ignoring at this point any other connectives.

5.1. The rules for \( N_1 \)

Let us start with the case of \( N_1 \), with \( N_1 \varphi \) being true (i.e., having truth-value \( v_1 = t \)). In this case, by the intended interpretation, \( \varphi \) indeed does not have the truth-value \( v_1 = t \).
The natural \( I/E \)-rules rules fitting the intended interpretation are the following (cf. (7)).

\( I \)-rule:

\[
\Gamma : \Delta, \{(\varphi, j) \mid j \neq 1\} \quad \frac{\Gamma : \Delta, (N_1 \varphi, 1)}{(N_1 I_1)}
\]

(15)

The premise expresses that \( \varphi \) has any of the truth-values \( v_j \), for \( j \neq 1 \), that is \( \varphi \) having truth-value \( v_1 \), is not true. The conclusion is that \( N_1 \varphi \) is located at \( v_1 \) (i.e., is true).

\( E \)-rule:

\[
\Gamma : \Delta, (N_1 \varphi, 1) \quad \frac{\Gamma : \Delta, \{(\varphi, j) \mid j \neq 1\}}{(N_1 E_1)}
\]

(16)

The premise expresses that \( N_1 \varphi \) is true, located in \( v_1 \). The elimination is by distributing \( \varphi \) itself, disjunctively, to all \( v_j, j \neq 1 \).
Next, consider the situation where $N_1 \varphi$ is false, i.e., having the truth-value $v_n$. In this case, by the intended interpretation, it is not the case that $\varphi$ does not have the truth-value $v_1 = t$. In other words, $\varphi$ has the value $v_1$.

The natural $I/E$-rules rules fitting the intended interpretation are the following.

**I-rule:**

$$\frac{\Gamma : \Delta, (\varphi, 1)}{\Gamma : \Delta, (N_1 \varphi, n)} \quad (N_1I_n)$$

**E-rule:**

$$\frac{\Gamma : \Delta, (N_1 \varphi, n)}{\Gamma : \Delta, (\varphi, 1)} \quad (N_1E_n)$$

Next, suppose $1 < i \leq n$, and suppose $N_1 \varphi$ has truth value $v_i$.

A failing attempt:

To direct the thought, consider first $i = 2$ and suppose that $v_2$, in some sense, means “almost true”. What does it mean that it is “almost true” that $\varphi$ does not have the truth-value $v_1 = t$? A suggestive interpretation of this situation is that either $\varphi$ has just one other truth-value $v_j$ for $j \neq 1$, or it does have truth-value $v_1$.

Generalizing, it is suggestive to interpret $\varphi$ not having truth-value $v_1 = t$ to a truth degree $v_i$ as either $\varphi$ having any other truth-value $v_j \in A \subset V$, where $A$ is of size $i - 1$, or $\varphi$ does have the value $v_1$.

This would lead to the following $I/E$-rules:

**I-rule:** For some $A \subset \hat{n}$ of size $i - 1$, where $1 \notin A$, there is a rule

$$\frac{\Gamma : \Delta, (\varphi, A \cup \{1\})}{\Gamma : \Delta, (N_1 \varphi, i)} \quad (N_1I_A - attempted)$$

The premise expresses that $\varphi$ has one of the $i - 1$ truth-values in $A$ (that exclude $v_1$), or does have truth-value $v_1$. The conclusion locates $N_1 \varphi$ in $v_i$.

**E-rule:** For every $A \subset \hat{n}$ of size $i - 1$, where $1 \notin A$, there is a rule

$$\frac{\Gamma : \Delta, (N_1 \varphi, i)}{\Gamma : \Delta, (\varphi, A \cup \{1\})} \quad (N_1E_A - attempted)$$

The premise asserts that $N_1 \varphi$ has truth-value $v_i$. The conclusion distributes $\varphi$ disjunctively among the $i - 1$ truth-values (excluding $v_1$), or in $v_1$.

Unfortunately, this attempt fails!

Consider the following derivation.

$$\frac{\Gamma : \Delta, (N_1 \varphi, n)}{\Gamma : \Delta, (\varphi, 1)} \quad (W)$$

$$\frac{\Gamma : \Delta, (\varphi, A \cup \{1\})}{\Gamma : \Delta, (N_1 \varphi, j), j \neq n} \quad (N_1I_A - attempted)$$

$$\Gamma : \Delta, (N_1 \varphi, n) \quad (N_1E_n)$$

$$\Gamma : \Delta, (\varphi, 1)$$

Unfortunately, this attempt fails!
But by applying coordination to the assumption and conclusion of the above derivation, we get

\[
\frac{\Gamma : \Delta, (N_1 \varphi, j, j \neq n) \quad \Gamma : \Delta, (N_1 \varphi, n) \quad (c_{j, n})}{\Gamma : \Delta}
\]

That is, \( N_1 \varphi \) “disappeared”! This is, of course, wrong.

To understand what is going on and reach the correct rules, consider again the informal interpretation of \( N_1 \varphi \): it means negating in the meta-language that the truth-value of \( \varphi \) is \( v_1 \). However, the meta-language employs classical logic, which is bivalent. Recall that \( N_1 \varphi \) having truth-value \( v_i \) means that, for a “truth-degree” \( i \), \( \varphi \) does not have the truth-value \( v_1 \). So, the above interpretation must be either true or false. Thus, in the logic, \( N_1 \varphi \) can only be located at \( v_1 \) (truth) or \( v_n \) (falsity). It cannot be located at any other \( v_j, j \neq 1, n \).

This is reflected in \( (N_1 \varphi, j) \) having no I-rule, and the following E-rule:

\[
\frac{\Gamma : \Delta, (N_1 \varphi, j), j \neq 1, n}{\Gamma : \Delta} (N_1 E_j)
\]

5.2. The general case \( N_k \)

I now apply the same considerations to the general case \( N_k \) for \( 1 < k \leq n \).

I-rule:

\[
\frac{\Gamma : \Delta, \{(\varphi, j) \mid j \neq k\}}{\Gamma : \Delta, (N_k \varphi, v_1)} \quad (N_k I_1)
\]

The premise expresses that \( \varphi \) has any of the truth-values \( v_j \), for \( j \neq k \), that is \( \varphi \) having truth-value \( v_k \), is not true. The conclusion is that \( N_k \varphi \) is located at \( v_1 \) (i.e., is true).

E-rule:

\[
\frac{\Gamma : \Delta, (N_k \varphi, v_1)}{\Gamma : \Delta, \{(\varphi, j) \mid j \neq k\}} \quad (N_k E_1)
\]

The premise expresses that \( N_1 \varphi \) is true, located in \( v_1 \). The elimination is by distributing \( \varphi \) itself, disjunctively, to all \( v_j, j \neq k \).

Next, consider the situation where \( N_k \varphi \) is false, i.e., having the truth-value \( v_n \). In this case, by the intended interpretation, it is not the case that \( \varphi \) does not have the truth-value \( v_k \). In other words, \( \varphi \) has the value \( v_k \).

The natural I/E-rules rules fitting the intended interpretation are the following.

I-rule:

\[
\frac{\Gamma : \Delta, (\varphi, k)}{\Gamma : \Delta, (N_k \varphi, n)} \quad (N_k I_n)
\]

E-rule:

\[
\frac{\Gamma : \Delta, (N_k \varphi, n)}{\Gamma : \Delta, (\varphi, k)} \quad (N_k E_n)
\]
Again, $N_k \varphi$ cannot have any other truth-value except $v_1$ or $v_n$. This is again reflected in $(N_k \varphi)$ having no I-rule, and the following E-rule:

$$
\frac{\Gamma : \Delta, (N_k \varphi, j), j \neq 1, n}{\Gamma : \Delta} (N_k E_j)
$$

(26)

A somewhat tedious calculation can show that those $I/E$-rules are generated, by the recipe for operational rules in the appendix, from the following truth-tables for the $N_i$s:

\begin{align*}
N_i(v_j) &= v_1, \text{ for } j \neq i \\
N_i(v_i) &= v_n
\end{align*}

(27)

This establishes the existence of truth and falsity in the general case.
Appendix: The proof-system $\Lambda^n$

initial poly-sequents: For every $1 \leq i \leq n$: $\Gamma, (\varphi, i) : \Delta, (\varphi, i)$

shifting rules:

\[
\frac{\Gamma, (\varphi, i) : \Delta}{\Gamma : \Delta, \varphi \times i} (s^*_i) \quad \frac{\Gamma : \Delta, (\varphi, i)}{\Gamma, (\varphi, j) : \Delta} (s^*_{i,j}), j \neq i
\]

coordination:

\[
\frac{\Gamma : \Delta, (\varphi, i) \quad \Gamma : \Delta, (\varphi, j)}{\Gamma : \Delta} (c_{i,j}), i \neq j
\]

From $(c_{i,j})$ the Weakening rules are derivable:

\[
\frac{\Gamma : \Delta}{\Gamma, \Gamma' : \Delta} (WL) \quad \frac{\Gamma : \Delta}{\Gamma : \Delta, \Delta'} (WR)
\]

operational rules: Those are irrelevant here, and are presented for completeness only. The guiding lines for the construction are the following, expressed in terms of a generic $p$-ary operator, say ‘$*$’.

\((*I)\): Such rules introduce a conclusion $\Gamma : \Delta, (* (\varphi_1, \cdots, \varphi_p), k)$.

- In general, if in the truth-table for ‘$*$’ the values $v_{ijk}$ for $\varphi_j, 1 \leq j \leq p$, yield the value $v_k$ for $* (\varphi_1, \cdots, \varphi_p)$, then there is a rule

\[
\frac{\{ \Gamma : \Delta, (\varphi_j, i_j) \mid 1 \leq j \leq p \}}{\Gamma : \Delta, (* (\varphi_1, \cdots, \varphi_p), k)} (*I_{i_1,\cdots,i_p,k})
\]

The rule $(*I_{i_1,\cdots,i_p,k})$ has, thus, $p$ premises.

\((*E)\): Such rules have a major premise $\Gamma : \Delta, (* (\varphi_1, \cdots, \varphi_p), k)$.

\[
\frac{\Gamma : \Delta, (* (\varphi_1, \cdots, \varphi_p), k) \quad \{ \Gamma, *(\varphi_1, k_1), \cdots, (\varphi_p, k_p) : \Delta \mid* (v_{k_1}, \cdots, v_{k_p}) = v_k \}}{\Gamma : \Delta} (*E_k)
\]

for each $k = 1, \cdots, n$.

A detailed discussion of this system, presented in a different but equivalent notation, can be found in [4].

Acknowledgement I thank Michael Kaminski for his involvement in this paper.
References

[1] Nuel D. Belnap. How a computer should think. In Gilbert Ryle, editor, Contemporary aspects of philosophy, pages 30–56. Stocksfield:Oriel Press, 1976.

[2] Nuel D. Belnap. A useful four-valued logic. In J. Michael Dunn and George Epstein, editors, Modern uses of multiple-valued logic, pages 8–37. Dordrecht:Reidl, 1977.

[3] J. Michael Dunn. Intuitive semantics for first-degree entailments and ‘coupled trees’. Philosophical Studies, 29:149–168, 1976.

[4] Nissim Francez and Michael Kaminski. On poly-logistic natural-deduction for finitely-valued propositional logics. Journal of Applied Logic, 6:255–288, 2019. Special issue for papers presented at ISRALOG17, Haifa, October 2017.

[5] Nissim Francez and Michael Kaminski. Transparent truth-value predicates in multi-valued logics. Logique et Analyse, 62(245):55–71, 2019. doi: 10.2143/LEA.245.0.3285705.

[6] Michael Kaminski and Nissim Francez. Calculi for multi-valued logics. Logica Universalis, 15(2):193–226, 2021.

[7] Stephen C. Kleene. Introduction to metamathematics. North-Holland, Amsterdam, 1952.

[8] Emil Post. Introduction to a general theory of elementary propositions. American Journal of Mathematics, 43(3):163–185, 1921.