Scattering matrices with block symmetries

Karol Życzkowski
Instytut Fizyki M. Smoluchowskiego, Uniwersytet Jagielloński, ul. Reymonta 4, 30-059 Kraków, Poland

Scattering matrices with block symmetry, which corresponds to scattering process on cavities with geometrical symmetry, are analyzed. The distribution of transmission coefficient is computed for different number of channels in the case of a system with or without the time reversal invariance. An interpolating formula for the case of gradual time reversal symmetry breaking is proposed.

I. INTRODUCTION

Ensembles of random matrices introduced in context of the theory of nuclear spectra by Dyson [1] long time ago found a novel application in problems of chaotic scattering [2, 3]. The $S$-matrix corresponding to time reversal invarant problems can be described by matrices of circular orthogonal ensemble (COE), while the circular unitary ensemble (CUE) is applicable if no antiunitary symmetry exists.

The process of chaotic scattering in cavities with a geometrical symmetry can be represented by an $S$-matrix with block symmetry. Such matrices were recently introduced by Gopar et al. [4], who discussed both cases: with and without time–reversal symmetry. They found a link between the invariant measure of such ensembles and the measures of canonical circular ensembles and computed the distribution of transmission coefficient for one and two incoming channels.

In this Brief Report we generalize the results of Gopar et al. [5]. Using an idea of composed ensembles of random matrices we simplify analytical calculation of the transmission coefficient $T$. This method allows us to obtain the distributions $P(T)$ for interpolating ensembles corresponding to breaking of the time reversal symmetry and to treat the case of large number of channels. Generating random unitary matrices of interpolating ensembles according to the method described in [5] we verify numerically proposed distributions.

II. BLOCK SYMMETRIC SCATTERING MATRICES

We introduce the random $S$-matrices with block symmetry following the method of Gopar et al. [5]. The $2M \times 2M$ scattering matrix possess the structure

$$S = \begin{pmatrix} r & \bar{t} \\ \bar{t} & r \end{pmatrix},$$

where $M \times M$ matrices $r$ and $t$ describe reflection and transmission processes, respectively. $S$-matrix can be brought into a block diagonal form

$$S' = R_0 S R_0^T \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix},$$

where $s_1 = r + t$ and $s_2 = r - t$ are unitary matrices and $R_0$ stands for rotation matrix consisting of $M$-dimensional unit matrices $1_M$

$$R_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_M & 1_M \\ -1_M & 1_M \end{pmatrix}.$$

Of great physical importance is the total transmission coefficient

$$T = \text{tr}(tt^t),$$

which is proportional to the conductance $G$ of the cavity, $G = (2e^2/h)T$. Since $t = (s_1 - s_2)/2$, the transmission coefficient reads

$$T = \frac{M}{2} - \frac{1}{2} \text{Re}[\text{tr}(s_1s_2^\dagger)].$$

Following Gopar et al. [6] we assume that unitary matrices $s_1$ and $s_2$ are statistically independent and pertain to the same universality class. For canonical circular ensembles of random matrices the joint probability distribution (JPD) of eigenphases is given by a single formula [11]

$$P_{U_\beta}(\varphi_1, \ldots, \varphi_M) = C_{\beta,M} \prod_{i>j} |e^{i\varphi_i} - e^{i\varphi_j}|^\beta,$$

where $C_{\beta,M}$ stands for normalization constant, while $U_\beta$ represents Poissonian, orthogonal and unitary circular ensemble for $\beta$ equal to 0, 1 and 2, respectively.

Formula (3) contains a product of two unitary matrices $s_1s_2$ (one can redefine the second matrix writing $s_2 = s_2^\dagger$). Further calculation base on the relation linking the JPD of eigenvalues of the product of two unitary matrices drown independently from any canonical ensemble

$$P_{U_\beta \times U_\beta}(\varphi_1, \ldots, \varphi_M) = P_{U_\beta}(\varphi_1, \ldots, \varphi_M).$$

The left hand side of the above equation formally represents the JPD characteristic to spectra of a composed ensemble defined via product of two random matrices, each specified by a certain probability distribution. For unitary ensemble ($\beta = 2$) this equality follows from the invariance properties of CUE, which corresponds to the Haar measure on the unitary group. The same concerns
the case $\beta = 0$, since the circular Poissonian ensemble (CPE) can be defined by the Haar measure in the space of diagonal unitary matrices. In the case of symmetric unitary matrices ($\beta = 1$) this property follows from the definition of COE, which is invariant with respect to transformations $s \rightarrow s' = XsX^T$, where $X$ denotes any unitary matrix. Analyzed matrix $s_1s_2$ is similar to $s_1^{1/2}s_2s_1^{1/2}$. Since $s_1$ is symmetric, so is $s_1^{1/2}$, which may play the role of $X$ in the invariance condition. The JPD of eigenvalues of $s_1s_2$ (averaged over the composed ensemble) is thus the same as that of $s_2$ and is given by (13) with $\beta = 1$. In spite of this result the measure of the composed ensemble containing products of two symmetric unitary matrices differs from this characteristic of COE [13].

Since the JPD of eigenvalues does not define the entire probability distribution of an ensemble of random matrices, the formula (13) loses its meaning for non integer values of the parameter $\beta$, characteristic to transition between the canonical ensembles. However, in order to get possible interpolating formulae for distribution of transmission coefficients we will eventually allow $\beta$ to take any real value in $[0, 2]$.

Relation (13) allows one to write the transmission coefficient (6) in a simplified form

$$T = \frac{M}{2} - \frac{1}{2} \text{Re}[\text{tr}(U_\beta)],$$

(8)

and to obtain the distributions $P(T)$ by averaging above formula over an appropriate ensemble of $M \times M$ unitary matrices $U_\beta$.

### III. EXPECTATION VALUES

Since $\langle \text{tr}(U_\beta) \rangle = 0$ for any $\beta$ one obtains

$$\langle T \rangle_\beta = \frac{M}{2},$$

(9)

Let $\text{tr}(U_\beta) = z = ae^{i\theta}$. In a recent work [13] the following average was derived for canonical circular ensembles: $\langle a^2 \rangle_\beta = 2M/(2 + \beta(M - 1))$. Because the distribution of phases $\theta$ was shown to be uniform, the variances of both parts are equal: $\text{var}(\text{Re}(z)) = \text{var}(\text{Im}(z)) = \text{var}(a)/2$. Using (9) we get directly

$$\text{var}(T)_\beta = \langle (T - \langle T \rangle_\beta)^2 \rangle = \frac{M}{8 + 4\beta(M - 1)}.$$

(10)

Observe that for $\beta = 2$ the variance equals to $1/8$, independently of the matrix size $M$, while for $\beta = 1$ one has $\text{var}(T)_1 = M/4(M + 1)$, in accordance with earlier results [8]. The variance grows with a decreasing $\beta$ and tends to $M/8$ in the limiting case $\beta \rightarrow 0$.

### IV. DISTRIBUTION OF TRANSMISSION COEFFICIENT $P_\beta(T)$

#### A. The case $M = 1$

For $M=1$ the "one dimensional matrix" $U = e^{ix_1}$ and the phase $\varphi_1$ is uniformly distributed in $[0, 2\pi)$ for any ensemble. The variable $t = \cos(\varphi_1)$ has thus the distribution $P_t(t) = 1/[\pi\sqrt{(1-t^2)}]$. The transmission coefficient equal in this case $T = 1/2 - t/2$ is therefore distributed according to

$$P_{\beta,1}(T) = \frac{1}{\pi \sqrt{T(1-T)}},$$

(11)

for any value of $\beta$.

#### B. The case $M = 2$

Let us start deriving the distribution $P_\beta(a)$ of the absolute value of trace $a = |\text{tr}(U)|$. For $M = 2$ the JPD (8) reads

$$P_\beta[\varphi_1, \varphi_2] = C_{\beta,2}(\sin \phi)^{\beta},$$

(12)

where $\phi = (\varphi_1 - \varphi_2)/2$. The module of the trace $a = |1 + e^{i2\phi}| = 2 \cos \phi$, so employing (12) we obtain the required distribution

$$P_\beta(a) = c_\beta(4 - a^2)^{\beta-1},$$

(13)

where $a \in [0, 2]$ and the normalization constant reads

$$c_\beta = \frac{2^{1-\beta} \Gamma((\beta+2)/2)}{\sqrt{\pi \Gamma((\beta+1)/2)}}.$$

In particular this distribution is flat for COE, while for CUE one gets a semicircle and recovers the result already mentioned in (13). According to (8) the transmission coefficient equals $T = 1 - (a \cos \theta)/2$. Due to rotational symmetry of the ensembles the distribution of phase $\theta$ is uniform in $[0, 2\pi)$, so the distribution of $t = \cos \theta$ is $P_t(t) = 1/[\pi \sqrt{(1-t^2)}]$ Denoting $x = at$ and using (13) one writes an integral for the distribution of $x$

$$P_\beta(x) = \frac{c_\beta}{\pi} \int_x^2 (4 - a^2)^{\beta-1} \sqrt{a^2 - x^2} \, da,$$

(14)

which can be computed numerically in the general case. By a linear change of variables $T = 1 - x/2$ it provides the required distribution of transmission coefficient $P_\beta(T)$. Moreover, for most interesting integer values of $\beta$ the above integral can be evaluated analytically giving

$$P_{\beta,2}(T) = \begin{cases} \frac{2}{\pi} K[\sqrt{T(2-T)}] & \text{for } \beta = 0, \ (\text{CPE}); \\ \frac{1}{\pi} \ln \frac{1+\sqrt{T(2-T)}}{1-T} & \text{for } \beta = 1, \ (\text{COE}); \\ \frac{4}{\pi} T(2-T) D[\sqrt{T(2-T)}] & \text{for } \beta = 2, \ (\text{CUE}) \end{cases},$$

(15)
where \( K \) and \( D \) stand for complete elliptic function of the first and the third kind, respectively [14]. Somewhat more complicated derivation of this formula in the case of COE and CUE has already been given in [9], where the elliptic function was expressed by the hypergeometric function as 

\[ D(k) = \frac{\pi}{4} F\left(\frac{1}{2}, \frac{3}{2}; 2; k^2\right). \]

We have generated numerically random unitary matrices of interpolating ensembles using the method proposed in [10]. Figure 1 shows the distribution of transmission coefficients \( P(T) \) for \( M = 2 \) and the Poisson-COE transition for three different values of the control parameter \( \delta \). Two narrow lines represent the formula (15) with \( \beta = 0 \) and \( \beta = 1 \), characteristic for the limiting cases of CPE and COE. The bold line in the figure b), obtained in the interpolating case, represents the best fit of (14) with \( \beta = 0.38 \). Good quality of the fit reveals a certain validity of this formula with non-integer values of \( \beta \) for ensembles in between the usual universality classes.

C. Distribution \( P_\beta(T) \) for large \( M \)

Since the transmission coefficient \( T \) is a function of \( M \) random variables with finite variances, one expects its distribution to be Gaussian for large \( M \). This fact can be proved rigorously for CUE. For this case the traces \( z = \text{tr}(U) \) are distributed (in the limit \( M \to \infty \)) as isotropic complex Gaussian variable [13], what guarantees the Gaussian distribution as well for \( x = \text{Re}(z) \), as well for \( T = (M - x)/2 \). The Gaussian property also holds in Poissonian case, for which \( x \) is the sum of \( M \) independent terms, each being a cosine of uniformly distributed random phases.

For \( M \gg 1 \) we conjecture the Gaussian distribution of \( P(T) \) for any value of \( \beta \) with the mean equal to \( M/2 \) and the variance given by (10). Numerical tests revealed Gaussian character of this distribution for any \( \beta \) already for \( M = 10 \). Figure 2 presents the distribution of transmission coefficient for the Poisson-CUE transition and three values of the control parameter \( \delta \).

\[ \text{FIG. 2. As in Fig. 1 for Poisson - unitary transition and } M = 10. \text{ Solid lines represent the Gaussian distribution with } \langle T \rangle = M/2 \text{ and the variance given by (10) for } \beta = 0 \text{ and } \beta = 2 \text{ and found numerically for } \beta = 0.4. \]

V. CONCLUDING REMARKS

Composed ensembles of random unitary matrices where applied for analysis of transmission coefficient in symmetric chaotic systems described by S-matrices with block symmetries. We simplified the derivation of distribution \( P_\beta(T) \) presented by Gopar et al. [9] and generalized their results proposing an family of interpolating distributions.
It should be noted that the Poissonian case $\beta = 0$ discussed above is not capable to describe an effect of localization. It corresponds rather to the case of scattering on a half transparent mirror, at which each scattering mode acquires a random phase shift. Another generalization of the model designed to describe effects of localization is a subject of a subsequent publication \[15\].

It is a pleasure to thank Marek Kuś, Marcin Poźniak, Petr Šeba and Jakub Zakrzewski for fruitful discussions. Financial support by Komitet Badań Naukowych is gratefully acknowledged.

\[1\] F. J. Dyson, J.Math, Phys. 3, 140 (1962).
\[2\] P. A. Mello, P. Pereyra, and T. H. Seligman, Ann. Phys. (N.Y.) 161, 254 (1985).
\[3\] R. Blümel and U. Smilansky, Phys. Rev. Lett. 60, 477 (1988).
\[4\] C. H. Lewenkopf and H. A. Weidenmüller, Ann. Phys. (N.Y.) 212, 53 (1991).
\[5\] H. U. Baranger and P. Mello, Phys. Rev. Lett. 73, 142 (1994).
\[6\] R. A. Jalabert, J.-L. Pichard, and C.W.J. Beenakker, Europhys. Lett. 27, 255 (1994).
\[7\] P. W. Brouwer, Phys. Rev. B 51, 16878 (1995).
\[8\] P. Šeba, K. Życzkowski, and J. Zakrzewski, Phys. Rev. E 54, 2438 (1996).
\[9\] V.A. Gopar, M. Martinez, P.A. Mello, and H. U. Baranger, J. Phys. A 29, 881 (1996).
\[10\] K. Życzkowski, M. Kuś, Phys. Rev. E 53, 319 (1996).
\[11\] M. L. Mehta, Random Matrices, II ed. Academic, New York, 1991.
\[12\] M. Poźniak, K. Życzkowski, and M. Kuś, (to be published).
\[13\] F. Haake, M. Kuś, H.-J. Sommers, H. Schomerus, and K. Życzkowski, J. Phys. A 29, 3641 (1996).
\[14\] I.S. Gradsteyn and I.M. Ryzhik, Table of Integrals, Series, Products, Academic, New York 1965.
\[15\] P. Šeba, J.Zakrzewski, and K. Życzkowski (to appear).