Equation of state in (2+1)-flavor QCD with gradient flow

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It was shown that operators of flowed fields have no UV divergences nor short-dist. singularities at $t > 0$.

GF provides us with a new physical (i.e. non-perturbative) renormalization scheme, which is directly calculable on the lattice in the $a \to 0$ limit. This opened many possibilities to drastically simplify lattice evaluation of physical observables.

**Gradient flow**

Lüscher(2009–), Narayanan-Neuberger(2006)

Imaginary evolution of the system into a fictitious "time" $t$ preserving gauge sym. etc.:

(ex) pure gauge theory \[ \dot{B}_\mu = D_\nu G_{\nu \mu}, \quad B_\mu \big|_{t=0} = A_\mu \] original gauge field

We may view the flowed field $B_\mu$ as a smeared $A_\mu$ over a physical range of $\sqrt{(8t)}$.

It was shown that operators of flowed fields have no UV divergences nor short-dist. singularities at $t > 0$.

GF provides us with a new physical (i.e. non-perturbative) renormalization scheme, which is directly calculable on the lattice in the $a \to 0$ limit.

This opened many possibilities to drastically simplify lattice evaluation of physical observables.

**Energy-momentum tensor from gradient flow**

H.Suzuki(2013)

1) Define EMT by a W-T identity in a continuum scheme.

2) Relate it with a lattice operator through finite observable at $t > 0$ in the $a \to 0$ limit.

By the GF evolution, however, unwanted operators can mix at $t > 0$.

3) Remove unwanted contributions using a small-$t$ oper. expansion.

The coeff's. $c_i$ near the $t \to 0$ limit can be calculated by PT.

$\Rightarrow$ We extract EMT, EOS etc. by $t \to 0$ & $a \to 0$ extrapolations. $\epsilon = -\langle T_{00} \rangle$, $p = \frac{1}{3} \sum_i \langle T_{ii} \rangle$
QCD Thermodynamics with Gradient Flow

Previous test in quenched QCD

\[
T^R_{\mu\nu}(x) = \lim_{t \to 0} \left\{ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} [E(t, x) - \langle E(t, x) \rangle_0] \right\}
\]

The EOS\' from the \((T-)\)integration methods correctly reproduced in the \(t \to 0\) and \(a \to 0\) limit with less computational costs.

Our project: Application to \((2+1)\)-flavor QCD

GF with quarks:

* We can adopt pure gauge actions for GF,
* at the price of a non-trivial field renormalization of quarks.

Full QCD EMT by GF:

Makino-Suzuki, PTEP 2014, 063B02 (2014)

Chiral condensate by GF:

Hieda-Suzuki, arXiv:1606.04193 (2016)

Topological charge / susceptibility by GF:

\[ \epsilon = -\langle T_{00} \rangle, \quad p = \frac{1}{3} \sum_i \langle T_{ii} \rangle \]

arXiv:1511.05235 (Lattice 2015)

Figure 2: Flow time dependence of the dimensionless interaction measure \((e-3p)/T^4\) (left panel) and the dimensionless entropy density \((e+p)/T^4\) (right panel) for different lattice spacings at \(T/T_c = 1.66\). The continuum extrapolated result obtained in the integral method in Ref. [10] is indicated by the arrow at vertical axis.

GF with quarks:

Lüscher, JHEP 1304, 123 (2013)

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Topological charge / susceptibility by GF:

\[ \Rightarrow \text{Talk by Taniguchi (June 29, Friday)} \]
Simulation Parameters

- Nf=2+1 QCD, Iwasaki gauge + NP-clover // fine lattice, physical s & heavy ud
- CP-PACS+JLQCD's $T = 0$ config. ($\beta = 2.05$, $28^3 \times 56$, $a \approx 0.07$fm, $m_{PS}/m_{V} \approx 0.63$)
  available on ILDG/JLDG
- $T > 0$ by fixed-scale approach, WHOT-QCD config.($32^3 \times Nt$, $Nt = 4, 6, 8, 10, 12, 14, 16$)
- gauge measurements at every config.
- quark measurements every 10 config's, using a noisy estimator method.
- continuum extrapolation => next step study

| $T$ (MeV) | $T/T_{pc}$ | $N_t$ | $t_{1/2}$ | gauge confs. |
|-----------|------------|-------|-----------|--------------|
| 0         | 0          | 56    | 24.5      | 650          |
| 174       | 0.92       | 16    | 8         | 1440         |
| 199       | 1.05       | 14    | 6.125     | 1270         |
| 232       | 1.22       | 12    | 4.5       | 1290         |
| 279       | 1.47       | 10    | 3.125     | 780          |
| 348       | 1.83       | 8     | 2         | 510          |
| 464       | 2.44       | 6     | 1.125     | 500          |
| 697       | 3.67       | 4     | 0.5       | 700          |

$T_{pc} = 190$ MeV assumed

To avoid oversmearing, wrapping around the lattice:

\[ \sqrt{8t/a^2} \leq \min(Ns/2, Nt/2) \]

i.e., \( t/a^2 \leq t_{1/2} = \left[ \min(Ns/2, Nt/2) \right]^2 / \sqrt{8} \)

=> to be compared with GF!
We adopt the simplest one suggested by Lüscher.

**Gauge flow:** standard Wilson flow

\[ \partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x), \quad B_\mu(t = 0, x) = A_\mu(x) \]

\[ G_{\mu\nu}(t, x) = \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) + [B_\mu(t, x), B_\nu(t, x)], \]

\[ D_\nu G_{\nu\mu}(t, x) = \partial_\nu G_{\nu\mu}(t, x) + [B_\nu(t, x), G_{\nu\mu}(t, x)], \]

**Quark flow:** as suggested by Lüscher

\[ \partial_t \chi_f(t, x) = \Delta \chi_f(t, x), \quad \chi_f(t = 0, x) = \psi_f(x), \]

\[ \partial_t \bar{\chi}_f(t, x) = \bar{\chi}_f(t, x) \bar{\Delta}, \quad \bar{\chi}_f(t = 0, x) = \bar{\psi}_f(x), \]

\[ \Delta \chi_f(t, x) \equiv D_\mu D_\mu \chi_f(t, x), \quad D_\mu \chi_f(t, x) \equiv [\partial_\mu + B_\mu(t, x)] \chi_f(t, x), \]

\[ \bar{\chi}_f(t, x) \bar{\Delta} \equiv \bar{\chi}_f(t, x) \bar{D}_\mu \bar{D}_\mu, \quad \bar{\chi}_f(t, x) \bar{D}_\mu \equiv \bar{\chi}_f(t, x) \left[ \bar{\partial}_\mu - B_\mu(t, x) \right] \]
**Nf=2+1 QCD EMT by GF**

**EMT in full QCD**

Operators on the lattice

\[
\begin{align*}
\hat{O}_{1\mu
u}(t, x) &\equiv G^a_{\mu\rho}(t, x)G^a_{\nu\rho}(t, x), \\
\hat{O}_{2\mu
u}(t, x) &\equiv \delta_{\mu\nu}G^a_{\rho\sigma}(t, x)G^a_{\sigma\rho}(t, x), \\
\hat{O}_{3\mu
u}(t, x) &\equiv \varphi_f(t)\bar{\chi}_f(t, x)\left(\gamma_{\mu}D_{\nu} + \gamma_{\nu}D_{\mu}\right)\chi_f(t, x), \\
\hat{O}_{4\mu
u}(t, x) &\equiv \varphi_f(t)\delta_{\mu\nu}\bar{\chi}_f(t, x)D\chi_f(t, x), \\
\hat{O}_{5\mu
u}(t, x) &\equiv \varphi_f(t)\delta_{\mu\nu}\bar{\chi}_f(t, x)\chi_f(t, x),
\end{align*}
\]

Quark field renormalization

\[
\varphi_f(t) \equiv \frac{-6}{(4\pi)^2 t^2} \left\langle \bar{\chi}_f(t, x)D\chi_f(t, x) \right\rangle_0.
\]

**Physics extracted by \( t \to 0 \) extrapolation.**

At \( a > 0 \)

\[
T_{\mu
u}(t, x, a) = T_{\mu
u}(t, x) + A_{\mu\nu}\frac{a^2}{t} + \sum_f B_{f\mu\nu}(am_f)^2 + C_{\mu\nu}(aT)^2 + D_{\mu\nu} (a\Lambda_{\text{QCD}})^2 \\
+ a^2 S'_{\mu\nu}(x) + \mathcal{O}(a^4),
\]

Singular term at \( t \to 0 \) due to mixing with \( D=4 \) ops.

Note: lattice artifacts of NP-clover is \( O(a^2) \).
**Nf=2+1 QCD EoS by GF**

\[ \epsilon = -\langle T_{00}\rangle, \quad p = \frac{1}{3} \sum_i \langle T_{ii}\rangle \]

**Our data**

\[ e+p \]

\[ T \approx 174 \text{ MeV} \quad N_t = 16 \]

\[ T \approx 199 \text{ MeV} \quad N_t = 14 \]

\[ T \approx 279 \text{ MeV} \quad N_t = 10 \]

\[ T \approx 464 \text{ MeV} \quad N_t = 6 \]

\[ e-3p \]

\[ a^2/t\text{-like behavior close to } t = 0. \]

\[ \text{Wide linear behavior within meaningful range of } t. \]

\[ a^2/t\text{ term suggested to be negligible in the windows} \]

\[ \text{after several try & errors } \Rightarrow \text{ Linear fit choosing linear window} \]

\[ \text{At } T \approx 697 \text{ MeV (} N_t=4\text{), no linear window found. We perform a non-linear fit, but data dominated by lattice artifacts within the meaningful range of } t. \text{ Results at this } T \text{ should not be taken seriously.} \]
**Nf=2+1 QCD EoS by GF**

**Results**

- Good agreement with the conventional method at $T \leq 300$ MeV ($Nt \geq 10$).
- Though a definite comparison possible only at $a \rightarrow 0$, GF results with similar amount are encouraging.

GF errors include statistical + syst. from $c_i$’s

T-integration results from PRD85,094508(’12)
From axial W-T identity
\[
\{ \bar{\psi}_f \psi_f \}^{(0)}(t, x) = \left\{ 1 + \frac{g(1/\sqrt{8t})^2}{(4\pi)^2} \left[ 4(\gamma - 2 \ln 2) + 8 + \frac{4}{3} \ln(432) \right] \right\} 
\times \frac{m_f(1/\sqrt{8t})}{m_f} [\varphi_f(t) \bar{\chi}_f(t, x) \chi_f(t, x)]
\]

At \( m_f > 0 \), chiral cond. in usual lattice simulation can have \( m_f/a^2 \) singularity. With GF, such divergence is prohibited by the finiteness of flowed operators, but \( m_f/t \) can appear, instead.

In fact, to the lowest order of PT, we do encounter such \( m_f/t \) term.

To remove this obstacle in the \( t \to 0 \) extrapolation, Hieda-Suzuki suggests a VEV-subtraction.

\[
\lim_{t \to 0} \{ \bar{\psi}_f \psi_f \}(x) = \left\{ 1 + \frac{g(1/\sqrt{8t})^2}{(4\pi)^2} \left[ 4(\gamma - 2 \ln 2) + 8 + \frac{4}{3} \ln(432) \right] \right\} 
\times \frac{m_f(1/\sqrt{8t})}{m_f} [\varphi_f(t) \bar{\chi}_f(t, x) \chi_f(t, x) - \text{VEV}].
\]
**Chiral Condensate by GF**

**At \( a > 0 \)**

We have both \( m_f/t \) and \( a^2/t \) terms we should remove.

\[
\{ \bar{\psi}_f \psi_f \}^{(0)}(t, x, a) = \{ \bar{\psi}_f \psi_f \}^{(0)}(t, x) + \left[ A \frac{a^2}{t} + \sum_f B_f (am_f)^2 + C(aT)^2 + D (a\Lambda_{QCD})^2 \right] + a^2 S(x) + O(a^4),
\]

**Our data**

- **Singular behavior at \( t \approx 0 \), but \( m_f \)-dep. small.**
- **=> linear fit as before.**
- **Wider linear region by VEV-subtraction**
- **<= Large part of \( a^2/t \). also removed by the VEV-subtraction.**

\( T \approx 279 \text{ MeV} \)

\( Nt = 10 \)
**Chiral Condensate by GF**

**Results**

MS scheme at $\mu = 2$ GeV
Errors include statistical + syst. from pert. coeff's

Disconnected chiral susceptibility

Crossover suggested at $T \approx 190$ MeV.

Clear peak at $T \approx 190$ MeV, as expected.

Peak higher with decreasing $m_q$.

Disconnected part only.

VEV-subtraction no effects in this quantity.

Windows for linear fit clear, except for $T \approx 697$ MeV.
We apply gradient flow ideas to investigate thermodynamics of (2+1)-flavor QCD. As the first test, we choose heavy ud quarks with physical s quark, on a fine lattice \((a \approx 0.07\text{fm}, \frac{m_p}{m_V} \approx 0.63)\), and adopt the fixed-scale approach.

**EOS** agrees with conventional \(T\)-integration method at \(T \leq 300\text{ MeV} (N_t \geq 10)\).

A definite comparison possible only after cont. extrapolation. The good agreement at \(N_t \geq 10\) suggests that our \(a\) sufficiently small, but small-\(N_t\) artifact large at \(N_t \leq 8\).

Chiral condensate and its disconnected susceptibility also calculated. Even with the explicit chiral violation of Wilson-type quarks, we obtain reasonable results, reassuring the powerfulness of the GF method.

Results for topological susceptibility also encouraging. => Taniguchi (June 29, Friday, 17:10-)

Further study needed to complete the continuum extrapolation.