Entropic time endowed in quantum correlations

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A possible mechanism of time is formulated by developing an idea of time replaced by quantum correlations, with the aid of modern quantum information theory. We invent a microscopic model, where correlations of a closed system are steadily read out as internal, quantum clocks that define time via their relative phases. The model could realize emergent time evolutions which exhibit unitarity of quantum theory, while its underlying process is driven entropically. The key quantity turns out to be the amount of accessible information about the clocks recording past events. By postulating the so-called data-processing inequality (or strong subadditivity of entropy) as a fundamental, physical limitation about how information decays, we propose that conditional entropy about this past information should be constrained to be a positive constant. The proposal implies a holographic property of this conditional entropy in an analogous manner with the area law of entanglement entropy.

I. INTRODUCTION

“Time is what the constant flow of bits of sand in an hourglass measures,” one says. Although time has been the most fundamental notion to us from ancient days, it could not have been treated satisfactorily in the modern physics. Time is not an ordinary observable (dynamical variable) in quantum theory, nor thus quantized. The physics. Time is not an ordinary observable (dynamical variable) in quantum theory, nor thus quantized. The time evolution generated by the Einstein equation is a variable (in quantum theory, nor thus quantized). The mere diffeomorphism of the spacetime, and in this regard it can be said timeless in relativistic theory.

A timeless approach to quantum theory [1, 2] has showed conceptually how an “evolution” of the system can be observed in a stationary total state |Ψ⟩. Imagine the universe consists of two parts: a hypothetical system A which refers to the coordinate time and a regular system B (which includes the observer). They are assumed to be constrained to be stationary under a non-interacting Hamiltonian $H = H_A \otimes 1_B - 1_A \otimes H_B$; namely, the Wheeler-DeWitt like equation $H |Ψ⟩_{AB} = 0$ is satisfied. We define a clock time when the clock A points its time as $|t⟩^A = e^{ihA t} |0⟩^A$, with a fiducial zero-time state $|0⟩^A$. (Hereafter, the time t should be distinguished from the ±1 eigenstates, $|0^\mu⟩$ and $|1^\mu⟩$, of the Pauli operator $σ^\mu$, which appear with the superscript for an axis $\mu$.) Then, the state $⟨t|Ψ⟩$ of the system at the time $t$ obeys the standard unitary time evolution (with $i = \sqrt{-1}$):

$$\frac{∂}{∂t}⟨t|Ψ⟩ = ⟨t| H^A \otimes 1 |Ψ⟩ = ⟨t| H^B |Ψ⟩ = H^B ⟨t|Ψ⟩.$$ (1)

In other words, formally the quantum correlation between the clock and the system is described as

$$|Ψ⟩ = \int dt e^{iH_A t} |0⟩^A \otimes e^{-iH^B t} ⟨0|Ψ⟩^B.$$ (2)

This can be interpreted as time emerges from the comparison (i.e. a relative phase) between two “precessing” spins.

However, this is not totally satisfactory in several manners. First of all, the clock is not treated operationally in the framework of quantum theory. Its reference to the time $t$ is not supposed to be monitored quantum-mechanically, nor is treated to create information (or entropy) associate with it. The time flow is rather considered to be absolute (unaffected from any disturbance), by ticking steadily, globally and deterministically. Obviously, this is a remote cause to a long-standing puzzle about the nature of macroscopically-observed time, such as the second law of thermodynamics.

Here we develop the above idea of time replaced by quantum correlations [2], according to recent progress of quantum information and computation theory. We invent an operational model, and scrutinize the changes of (both quantum and classical) entropies in the elapse of time which is intrinsically ruled by the correlations. Our guiding rule is one of the most fundamental features in information theory: the so-called data-processing inequality [3] (or mathematically the strong subadditivity of entropy). It states conceptually that no correlation between two parts can be strengthened if they are acted separately. Our second key idea is to link this fundamental limitation on entropy changes with the common feature of time in quantum theory. Namely, searched is the condition that the unitary evolution of Eq. (1) is realizable steadily, despite the fact that seemingly the global entropy is kept generating. This brings us an unexpected constraint on certain conditional entropies, as a generalized counterpart of the Wheeler-DeWitt constraint. Physically, it supports an “area law” of accessible information about the past events, in that it suffices to signal part of the past history in compared with the naive “volume law” that all the information should be accessible. Needless to say, our area law is quite reminiscent of the conjectured area law of entanglement entropy, which bases the holographic picture of our universe.

Finally, we note that our work might be complementarily helpful to understanding why quantum correlation should be “limited” among other conceivable probabilistic theories. In particular, it may have sounded, out of the blue to physicists, that its Tsirelson bound on non-locality could be related to communication complexity.
We are based on a standpoint that laws of nature are about how quantum and classical information can be processed. Since our idea is to refine the timeless mechanism in an operational and microscopic manner, we will naturally consider a quantum-classical hybrid system. The entropy of every quantum degree of freedom materializes through an elementary “yes, no” phenomenon, as advocated e.g., in [7]. It is formulated as consisting of its measurement, outputting only a yes or no answer (or generally one of discrete alternatives), upon the receipt of input information. As seen in the Figure 1, we model the quantum part $Q$ to be made of an array of $N$ qudits, whose local Hilbert space is $d^Q$-dimensional. It is straightforward to introduce the following two kinds of classical degrees of freedom. One is a classical memory $M$ to store the classical outcome of each measurement. Every degree of freedom in $Q$ is associated with a counterpart in $M$ with the same local size ($d^M := d^Q$). The other is a classical register $C$ to represent the signaling information to the current state of $Q$, and is available as the input to the following measurement of $Q$. The size of $C$, denoted as $d^C$, will be a key parameter.

The current status of the system is described by the density operators $\rho^Q$ of $Q$ together with classical information $\kappa$ (of at most $\log_2 d^{C′}$ bits) in $C$,

$$\rho^{CQ} = \sum_\kappa \rho_\kappa \, |\kappa\rangle \langle \kappa|^C \otimes \rho^Q_\kappa. \quad (3)$$

On the other hand, the reduced state only on $Q$ is described by $\rho^Q = \text{tr}_C \rho^{CQ} = \sum_\kappa \rho_\kappa \rho^Q_\kappa$. Importantly, if $C$ is imagined to include all information of $M$, the current state of $Q$ should be able to figure out its full past history, implying it is described globally as a pure state. However, this assumption requires that the size $d^C$ of $C$ scales according to the length of the past, and then the resulting mechanism is not translationally invariant (i.e., not stationary in the time direction). What if $C$ can include only the part of information in $M$? The past history stored in $M$ lies there in the system, but is considered to be inaccessible from the current standpoint of $Q$. That makes $CQ$ to be effectively a mixed state.

The elementary “measuring” process is formulated using a local reversible transformation $\Upsilon^{CMQ}$ among $CQ$ and the ancillary memory $M$. $M$ is initially prepared in a fiducial state $\rho^M$ and thus uncorrelated to $CQ$. Although $\Upsilon^{CMQ}$ will be illustrated explicitly in the Appendix A for the simplest case, it induces some effective quantum action $A(\gamma)$ to update the internal state of $Q$, depending on the measurement outcome $\gamma$ to be stored in $M$. It also exchanges locally classical information between $C$ and $M$ to produce an output $C′$. Note that the qudit once processed by $\Upsilon^{CMQ}$, which is denoted by $Q′$, is detached from the renewed status $C′Q′$ in a similar manner that the previous $M$ is not accessible to $C′Q′$ anymore.

$$\rho^{C′Q′} = \text{tr}_M [\Upsilon^{CMQ} (\rho^{CQ} \otimes |0^M\rangle \langle 0^M|) \Upsilon^{CMQ}] = \sum_\kappa \rho_\kappa \langle \kappa′ | C′ \otimes A(\gamma) \rho^Q_\kappa A(\gamma), \quad (4)$$

where $\kappa′$ is part of classical information made of $\kappa$ and $\gamma$. Thus the current status of the system is described holographically, lying in the interface between the processed region of $Q$’s and the unprocessed region of $Q$. 

III. PRINCIPLE OF LEAST CONDITIONAL ENTROPY

Now we will formulate a global variational principle using the entropies of the bipartite state $CQ$, and characterize a timeline. We define the conditional entropy of
It is equivalent to say that at every elementary step, where the probability distribution \( \{ p_\kappa \} \) and \( S(\rho) = -\text{tr}(\rho \log \rho) \) is the von Neumann entropy of the density operator \( \rho \). Note that \( S(C|Q) \) is non-negative in our \( CQ \) hybrid system (although the quantity defined in the first line of Eq. (5) can be negative for a general bipartite state). The strong subadditivity of the von Neumann entropy says that \( S(\rho^{12}) - S(\rho^{2}) \geq S(\rho^{123}) - S(\rho^{23}) \) holds for any tripartite quantum system 123 (e.g., [10] for the review, and [11] to see how unique it is for quantum theory). By considering that \( 1 = C' \), \( 2 = Q' \), and \( 3 = Q'M \) and using that \( M \) is initially uncorrelated to either of \( CQ \), we get a quantum data-processing inequality in our setting:

\[
S(C|Q) \leq S(C'|Q').
\] (6)

It is equivalent to say that at every elementary step, \( \Delta S(\rho^Q) \leq \Delta S(\rho^{CQ}) \) must be satisfied regarding the change of entropy \( \Delta S \), which is defined as the entropy after the step minus the entropy before. The data-processing inequality is considered to describe one of the most fundamental features of information. It can be also said that this is another way to view the entropic uncertainty relation [12, 13]. In our context, given the current state \( Q \), missing information about the past events, to be delivered in \( C \), cannot decrease (see [14] more about properties of this missing information). Obviously, the best way to preserve the information is to satisfy the extremal condition of Eq. (6), which will turn out to be the counterpart of the unitary condition of a quantum-only system. However, the significance of its extremal value itself may have been overlooked by often looking at a single step. As pointed out in the previous section, the naive solution to include all information of \( M \) in \( C \) results in its extremal value to be dependent of the whole system size.

Here we propose that the extremal condition should be satisfied not only in a single step but also \textit{every step} in the independent manner of the system size \( N \). Since the behavior is qualitatively similar as far as the extremal value is a constant, we can pay attention to its least value to be shown as \( \log d^C/d^Q \). That is how we formulate a global variational requirement,

\[
S(C|Q) = S(C'|Q') = \log \frac{d^C}{d^Q} \quad \text{for any step},
\] (7)

and call it here \textit{the principle of least conditional entropy}. One may probably wonder, at a first glimpse, if the least value could be set to be zero (namely \( d^C = d^Q \)), because of an apparent resemblance of Eq. (5) to the celebrated bound for the Holevo quantity. However, the strict positivity in Eq. (7) seems to be an inevitable feature of time in noncommutative quantum world, in reflecting the computational universality of quantum theory, discussed later.

**IV. TIMELINE FROM QUANTUM CORRELATIONS**

Now we like to specify the quantum correlations of \( Q \) in our operational, microscopic model as a solution of the proposed principle. Although the structure of tripartite states which saturates the strong subadditivity inequality has been analyzed in [15], we need to introduce some heuristic assumptions to deal with the sequence of such inequalities as in Eq. (7). Inspired by the correlation of Eq. (2) in the Introduction, we assume the primitive bipartite correlation of two qubits (i.e., \( d^Q = 2 \) for simplicity hereafter) is given in terms of the stationary subspace of a non-interacting Hamiltonian \( \mathcal{H} = \frac{1}{2}(\sigma^z \otimes 1 - 1 \otimes \sigma^w) \). The first axis can be set to the \( z \) axis without loss of generality. The second \( w \) axis is specified by the relation to the first, \( \sigma^w = V \sigma^z V^\dagger \), where \( V \) is generally an element of \( SU(2) \) parametrized as \( e^{i\theta_1 R^z(\theta_3) R^y(\theta_2) R^z(\theta_1)} \) with three Euler angles, using a rotation matrix \( R^\mu(\theta) = e^{i \theta \left[ \sigma^\mu / 2 \right]} \langle 0|0\rangle + e^{-i \theta \left[ \sigma^\mu / 2 \right]} |1\rangle \langle 1| \). Here we can set \( \theta_1 \) to be zero and choose later \( \theta_2 \) freely, because both are irrelevant in the definition of \( \sigma^w \). The 2-dimensional projector to the stationary subspace is conveniently written as

\[
\Xi_t = \oint dt R^z(t) \otimes R^w(-t) = |0^z0^w\rangle \langle 0^z0^w| + |1^z1^w\rangle \langle 1^z1^w|,
\] (8)

where the integral is taken over \([0,2\pi]\) and normalized with \( \frac{1}{2\pi} \). Note that here \( t \) is formally an integral variable, originated from geometry of the underlying Hilbert space. However, this integral representation is insightful to remind us of two \textit{correlated} precessing spins seen through Eq. (2).

We define the “timeline state,” which handles the history of the time evolution of a single qubit. Although we hardly use the following explicit form, it is formally written as the simple convolution of our primitive correlations:
\[ \Xi_{t_{N-1}} \cdots \Xi_{t_1} \langle |o\rangle_1 |o\rangle_2 \cdots |o\rangle_N \rangle = \int dt_{N-1} \cdots \int dt_2 \int dt_1 R^z(t_1) |o\rangle_1 \otimes R^z(t_2) R^w(-t_1) |o\rangle_2 \otimes \cdots \otimes R^w(-t_{N-1}) |o\rangle_N \rangle. \tag{9} \]

Another motivation for this definition is that, as elaborated in the Appendix B, the timeline state appears to correspond to (the discretized version of) a 1 + 1D Dirac fermion, which describes how the spin degree of freedom should be intertwined with space-time degrees of freedom in a consistent manner with both quantum theory and relativistic theory. Conditions for the choice of a fiducial state \( |o\rangle \) is derived later. Depending on \( |o\rangle \), we assume the total state is suitably normalized if necessary. Also notice that each pair of \( \Xi_{t_j} \) and \( \Xi_{t_{j+1}} \) does not commute each other unless \( w = z \), so that the notation implies that \( \Xi_{t_{j+1}} \) acts after \( \Xi_{t_j} \).

An elementary projection is a projection of the current degree of freedom of \( Q \) (the one with the smallest numbering among unmeasured quibits) onto a certain pointer state \( |\tau\rangle \langle \tau| \). Here, based on the principle of least conditional entropy in Eq. (7), we like to derive (i) conditions for \( |o\rangle \) and \( |\tau\rangle \), and (ii) conditions for the \( w \) axis. Using the spin-coherent-state representation, we can parametrize \( |o\rangle = \cos \theta_o e^{i\eta_o/2} |0^w\rangle + \sin \theta_o e^{-i\eta_o/2} |1^w\rangle \) and \( |\tau\rangle = \cos \theta \phi \tau |0^z\rangle + \sin \theta \phi \tau |1^z\rangle \).

(i) In analogous manner to the conventional formulation of quantum data-processing inequality [3, 16], the equality of Eq. (7) intends that \( A \) is unitary up to a normalization for every outcome \( \gamma (= 0, 1) \), corresponding to each clock time \( \tau(\gamma) \). Namely, at the \( j \)-th step, we get a transfer matrix \( A := \frac{1}{\lambda_j} \Xi_{t_j} |o\rangle_{j+1} \), which should satisfy \( A^H A \propto I \). Given \( |o\rangle \), the unitarity condition is satisfied if \( \sin \theta_o = \pm \cos \lambda \phi \). Furthermore, in order that these two “outcomes” constitute a valid measurement (mathematically a positive operator valued measure: \( \sum_{\gamma} |\gamma\rangle \langle \gamma| = 1 \)), they have to obey \( \sin \lambda = \pm \cos \lambda \phi \). Note that the azimuth angles \( \eta_o \) and \( \eta \) are not constrained at all. Only the relative difference \( \tau := \eta - \eta_o \) matters in \( A \), so that we simply set \( \eta_o = 0 \) and \( \eta = \tau \). That is how we can fix both \( \lambda \) and \( \lambda_o \) to be \( \frac{\pi}{2} \) without loss of generality, by absorbing the possible minus sign as an additional \( \pi \) in the azimuth angle. In short, for \( |o\rangle = |0^w\rangle + |1^w\rangle \) and an arbitrarily fixed \( \tau \), there are two “yes or no” outcomes \( |\gamma\rangle = e^{i\tau/2} |0^z\rangle + e^{-i\tau/2} |1^z\rangle \), where \( |\gamma\rangle \) is either \( \tau \) itself or not \( \tau (= \tau + \pi) \), each of which is labeled by the classical information \( \gamma = 0 \) or \( 1 \) respectively.

Accordingly, the transfer operator defined above is
\[
A(\gamma) = \int dt_j R^e(-t_j) |o\rangle_{j+1} \langle \gamma(t_j)| R^z(t_j) = VR^z(-\tau(\gamma)), \tag{10}
\]
where at the second equality we use a key identity,
\[
\int dt |t - t'| \langle \tau - t| = R^z(-\tau), \tag{11}
\]
for the canonical clock-time state \( |t| := e^{it/2} |0^z\rangle + e^{-it/2} |1^z\rangle \). We can readily confirm that the normalizations (and thus the probabilities) for the outcomes \( \tau \) and \( \tau + \pi \) are always equal by construction. Although it is mathematically straightforward, the identity is indeed based on quantum nature of the clock: \( \langle t|t' \rangle = \delta(t - t') \), in that different times are not orthogonal in allowing a chance of a quantum leap. It would be intriguing to mention that a classical-clock counterpart \( \langle t|t' \rangle = \delta(t - t') \), defined on the continuous degree on \( \mathbb{R} \), satisfies formally the same identity.

(ii) Although the unitary transfer operator in Eq. (10) allows to satisfy the equality of the data-processing inequality in Eq. (7), it does not reach the suggested least value \( \log \frac{\delta^C}{\delta} \) in general. This value accounts for communication cost of the outcomes \( \gamma \), often hidden in the conventional formulation with the classical clock. Indeed, it will turn out that the condition for the \( w \) axis (which represents non-commutativity), or the counterpart of the “mass” term \( m(\sim \theta_3) \) by the Dirac-fermionic analogy in the Appendix B, is relevant to it. It is convenient to simplify further the operator \( V \) by using the freedom to choose the \( y \) axis. Actually, \( \theta_3 \) can be set \( \pi \) by redefining \( \sigma^y := -2 \theta_3 \sigma^y - \sin 2\theta_3 \sigma^z \) while maintaining the value of \( \theta_2 \). Together with the freedom of \( \theta_4 \), this leads to the property that \( V \) can be chosen hereafter to be Hermitian, \( V = \sigma^z e^{i\pi \sigma^y} = V^\dagger \).

Let’s consider first the homogeneous case \( \langle \tau_j = 0 \forall j \rangle \). At the \( j \)-th stage of Eq. (4), we would get \( \tilde{A} := \prod_{\gamma} A(\gamma_j) = \cdots \cdot (\sigma^w)^{\eta_j} (\sigma^z)^{\pi_j} (\sigma^w)^{\eta_j} (\sigma^z)^{\pi_j} \ldots \) by an alternating product of \( \sigma^w \) and \( \sigma^z \) in each branch. If \( m \) is a general angle (or, not a rational multiple of \( \pi \)), we need all the past classical information about the outcomes has to be sent to maintain the distinguishability of all the branches and to satisfy the extreme of Eq. (7). That amounts to that the transferring information satisfies the volume law: \( d^C = 2^N \). Indeed, this abundant communication is hidden in the conventional formalism.

On the other hand, if \( m \) is assumed to be suitably “quantized,” by being a rational multiple of \( \pi \), there is such a chance that the set of \( A \)’s remains effectively a finite number of branches. Then full transfer of the past may not be necessary, and indeed it is possible to send only a constant amount regardless of the size of the past. We will discuss in the next Section that this can be interpreted as manifestation of an area laws of the conditional entropy in 1D. The necessary and sufficient condition is that \( \tilde{A} \) constitutes a complete depolarization map, in other words a unitary 1-design, within finite steps. The simplest solution is given when \( m \) is


\[ \frac{\pi}{4} \text{ or its odd multiple, namely } \sigma^w = \pm \sigma^x \text{ in satisfying } \text{tr}(\sigma^w \sigma^w) = 0 \text{ at the order 2 (cf. [17, 18]). At every two steps the equally-weighted branches constitute the Pauli group (disregarding the global phase) by 1, } \sigma^z, \sigma^y, \sigma^x \sigma^z \text{ for the set } (\gamma_{j+1}, \gamma_{j}) = (0, 0), (0, 1), (1, 0), (1, 1), \text{ respectively. Then it is sufficient to send only 2 bits (of } M = 4)\text{, that leads to the least value in Eq. (7) of } \log \frac{d^c}{d^s} = 1 \text{ for the qubit case.}

V. SPACETIME STRUCTURE BY A PARTIALLY ORDERED GEOMETRY

The strength of our approach gets further apparent when the same construction using our primitive correlation is considered on a partially ordered set as the underlying geometry of the array of qubits. The partial order may be constructed by the use of three projectors \( \Xi \)'s per qubit, extending the timeline state by two projectors per qubit. Note that this partial ordering is introduced to determine systematically the ordering of noncommutative \( \Xi \)'s, and it is per se different, for instance, from the causal set [19] as a discretized Lorentian manifold.

We will demonstrate, in terms of the Figure 2, that the spatial interactions between independent degrees of freedom (namely the interaction terms in the conventional Hamiltonian formalism, in addition to the single-particle terms provided by the timeline state) can be realized as well.

For ease of the notation, let us denote an intermediate timeline state (i.e., the yet-to-be-unmeasured part) of \( \Psi \) by \( |\Psi \rangle \), which is defined by Eq. (9) with the replacement of the fiducial state \( |0\rangle_1 \) of the first unmeasured qubit by an arbitrary (not necessarily pure) state \( |\varphi_{\kappa_1}\rangle \) representing the current status depending on \( \kappa \). The geometry by the right-hand side of Figure 2 provides an additional action between two timelines,

\[ Q_0^\varphi(\tau_0) |\Xi_{s^+1} \rangle |\Psi \rangle = Q_1^\varphi(\tau_1) \otimes |\varphi_{\kappa_1}\rangle \otimes |\Psi \rangle \Rightarrow Q_2^\varphi(\tau_2) \otimes |\varphi_{\kappa_2}\rangle \otimes |\Psi \rangle. \]

(12)

(Note for a slight abuse of the notation that all \( \Xi \)'s act one after the other from the left of the Figure, so that indeed \( \Xi_{s^+} \)'s act before the \( \Xi \)'s hidden inside \( |\Psi \rangle \).) Integrating over \( s_1 \) and \( s_2 \) in a similar way with the analysis of the timeline state, we can show that the action is indeed a two-qubit unitary operation to the current statuses, namely \( U (\tau) \) \( |\varphi_{\kappa_1}\rangle \otimes |\varphi_{\kappa_2}\rangle \).

When the two bits are sent in the opposite ways, the relation of current statuses \( \varphi_{\kappa_1}Q_1 \) and \( \varphi_{\kappa_2}Q_2 \) are not causal (timelike), but effectively space-like via the common qubit \( Q_0 \).

Notably this distinction provides a marvelous solution to have the aforementioned area law of the conditional entropy valid, even when the interactions between timelines are taken into account. Namely, the amount of communication scales in proportional to the area size of the past region of qubits (multiple timelines), regardless of the time step, because the net amount of communication in the spatial directions is always zero. This feature might be suggestive of the reason why time should be one dimensional. While more spatial directions may be increased without the violation of the 1D area law, the communication amount cannot be constant per qubit in case the time directions are more than 1D.

Finally, let us briefly draw attention to that this ability to host the spacetime which respects our principle of least conditional entropy suffices to guarantee the computational universality as a quantum computer. In particular, the unitary time evolution with an arbitrary \( h^n \) in Eq. (1) is implementable in our operational model using digital quantum simulation. Up to now, the pointers \( t_j \) of the clock time (or the position-dependent potential term in the Dirac fermion picture) have been kept free. For example, it has been widely known that the set of elementary gates, composed by single-qubit rotations of arbitrary angles around two axes and a two-qubit gate

FIG. 2: (Left) The timeline state and its graphical representation when \( m = \frac{\pi}{4} \) or \( \frac{3\pi}{4} \). An intermediate state \( |\Psi \rangle \) is defined by having \( |\varphi_{\kappa_1}\rangle \) at the current (leftmost) status instead of the fiducial state \( |0\rangle_1 \) in Eq. (9). The double arrow \( \Rightarrow \) represents the time direction realized by sending the 2 bits in the same direction. (Right) The partially ordered structure which provides the spatial interactions, in addition to the timelines. Given every bond which represents the projector \( \Xi \), we prescribe that the relatively left qubit is acted by the term \( R^t(t) \) of Eq. (8) and the relatively right one is acted by \( R^m(-t) \). And we apply the projectors \( \Xi \)'s sequentially from the left in a similar manner with the timeline state. In other words, for each qubit, the bonds on the left-hand side are applied first, followed by the ones on the right-hand side. Note that the bonds in the same side commute each other and thus the ordering among them does not matter. In contrast to the time direction by \( \Rightarrow \), the two-sided arrow \( \leftrightarrow \) represents the space direction realized by sending the 2 bits in the opposite directions. This way of communication is a ubiquitous feature of a measurement-based quantum computer, as particularly highlighted in the Figure 1 of [24]. However, the quantum correlations here differ from those of the so-called 2D cluster state.
such as CNOT is computationally universal. Indeed, in addition to the CNOT constructed earlier, it is straightforward to show that such single-qubit rotations are realizable along every timeline using inhomogeneous $\tau_j$’s, while the least constant value of the conditional entropy is still intact. The protocol would be analogous to a specific construction of measurement-based quantum computation using the 2D cluster state [20], which has been analyzed in detail in [21] or recently in [22] with an analogy to the spacetime. We should note, however, that our quantum state on the partially ordered geometry is not a 2D cluster state which is specified by the set of commuting stabilizer operators associated with the underlying graph. Interestingly, in the 1D case the timeline state with $m = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ coincides with the 1D counterpart of the cluster state, known as the linear-graph state. The Appendix C elaborates how our formulation using the time integrals can be related to the stabilizer formalism.

At this point, it is worth noting a situation when $m$ is a multiple of $\frac{\pi}{2}$. Two axes commute as $\sigma^w = \pm \sigma^z$, and thus it is equivalent to the massless case ($m = 0$). While this case attains $d^C = 2 = d^g$, it can only provide a depolarization map on a limited set of the inputs. Indeed, the quantum part $Q$ is always a GHZ-like state $|0^\otimes m + 1^\otimes m\rangle$ regardless of the underlying geometry. Curiously it has been shown in [23] that this state does not provide computational complexity as powerful as universal quantum computation. So, one might be opt to say that if there were no missing information about the past (i.e., if the conditional entropy in Eq. (7) were zero), the universe would rather miss all the richness of our world.

**Summary**

We have explored a mechanism of time from a possible perspective that laws of nature are constraints about how quantum and classical information can be processed. In our formulation, a suitably correlated quantum system serves as a collection of internal clocks which defines not only time but also space via their relative phases. Here emergent time is viewed as being made of continuous (quantum) phases which fill shades between discrete (classical) ordering related by classical communication. The mechanism is driven by an innate balancing of entropies under sequential read-out of these clocks. To regulate physically the amount of classical communication about past events in a steady manner, we have proposed the principle of least conditional entropy which states that the quantity is constrained to be positive. It would be also intriguing to investigate how our approach could face general relativity, particularly motivated by the entropic derivation of the Einstein equation [25]. A wild speculation along this line is that the positivity of the conditional entropy, for instance, might be linked with that of the cosmological constant.

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Supplementary Information

Appendix A: Details of $\Upsilon^{CMQ}$

Conceptually it is only significant that $\Upsilon^{CMQ}$ is reversible locally, so that it only exchanges information among these degrees of freedom inside the site. However, the details of processing $\Upsilon^{CMQ}$ at every single site for the simplest solution may be helpful to confirm how our mechanism of time works in a concrete fashion. The protocol itself is analogous to that of the measurement-based quantum computer (MQC) using the 2D cluster state so that one could further consult its extensive references. (Note, however, that the formalism as well as settings differ in several important details. In particular, classical information is usually handled globally in MQC, so that mixedness and thus entropies would not be addressed.)

The $\Upsilon^{CMQ}$ consists of two actions; a measurement of $Q$ and the storage of its outcome in $M$, followed by a construction of the output information based on available classical information in $C$ and $M$. So we denote $\Upsilon^{CMQ} = v^{CMQ}, V^{MQ}$. First, for the timeline state of $m = n = \frac{3\pi}{4}$ (i.e., $\sigma^w = \pm \sigma^z$), the signaling information is 2 bits, which are here labelled as $\kappa = (\kappa^x, \kappa^z)$ for the input $C$ and $\kappa' = (\kappa^x', \kappa^z')$ for the output $C'$. Then let us define

$$v^{MQ} = 1^M \otimes |\tau(0) \rangle \langle \tau(0)|^Q + \sigma^x^M \otimes |\tau(1) \rangle \langle \tau(1)|^Q,$$

$$v^{CM} \left\{ \begin{array}{l}
\kappa^x' := \kappa^x \\
\kappa^z' := \kappa^z + \gamma \quad \text{(modulo 2)}. \end{array} \right.$$  
(A1)

Although it is not stressed in the main text for simplicity, the action of the measurement could depend not only on its outcome $\gamma$ but also on the input information $\kappa$. In case the clock time $\tau$ is not necessarily zero, we also set $\tau(0) := (-1)^\kappa \tau$ and $\tau(1) := \tau(0) + \pi$ in $v^{MQ}$. Also recall that our construction that two measurement outcomes, $\tau(0)$ and $\tau(1)$, are equally likely, so that $\gamma$ is always a totally random variable. Suppose the input information $\kappa^x$ and $\kappa^z$ are totally random variables, then the output information $\kappa'^x$ and $\kappa'^z$ are totally random as well, according to $v^{CM}$.

Now it is straightforward to check that the above $\Upsilon^{CMQ}$ induces the following local processing, in correspondence to the abstract expression of Eq. (4) given in the text. Suppose initially $CQ$ is the form of Eq. (3) and its quantum part is $|\Psi^\varphi(\varphi_\kappa)\rangle^Q$. The dependence of $Q$ to the input information $\kappa$ is given by $\varphi_\kappa = (\sigma^x)^\kappa (\sigma^z)^{\kappa^z} \varphi$, where $\varphi$ is an arbitrary (not necessarily pure) single-qubit state which is independent of $\kappa$. Then, for each $\kappa$ in Eq. (3),

$$|\kappa\rangle^C \otimes |0^x\rangle^M \otimes |\Psi^\varphi(\varphi_\kappa)\rangle^Q$$

$$\xrightarrow{v^{MQ}} |\kappa\rangle^C \otimes \left( |0^x\rangle^M \otimes |\tau(0)\rangle \langle \tau(0)| \right) \otimes |\Psi^\varphi(\varphi_\kappa)\rangle^Q$$

$$+ |1^z\rangle^M \otimes |\tau(1)\rangle \langle \tau(1)| \otimes |\Psi^\varphi(\varphi_\kappa)\rangle^Q$$

$$\xrightarrow{v^{CM}} \sum_{\kappa' \sim \gamma'} |\kappa'\rangle^C \otimes |\Psi^\varphi(\varphi_{\kappa'})\rangle^Q \otimes |\gamma'\rangle^z^M \otimes |\tau(\gamma')\rangle^Q,$$.  

(A2)

where $\varphi_{\kappa'} = (\sigma^z)^{\kappa'^x} (\sigma^z)^{\kappa'^z} A \varphi$. In the second transformation $v^{CM}$, we have used the property of the transfer operator $A$ as defined in Eq. (10), and have changed the coordinate of classical information from $(\kappa, \gamma)$ to $(\kappa', \gamma')$. Note that $A = VR^z(\tau)$ now does not depend on any classical information, and the value of $\kappa'$, precisely $\kappa'$, is correlated with that of $\gamma'$ for fixed $\kappa$, as the notation $\kappa' \sim \gamma'$ implies. In tracing out $MQ'$ (i.e., the dependence on $\gamma'$) and taking all possibilities of $\kappa$ into account, we can reach a recursive form as presented in Eq. (4).

Next, when we consider the geometry by a partially ordered set, the role of $\Upsilon^{CMQ}$ per site stays qualitatively the same. Three sites $Q_j$ ($j = 0, 1, 2$) of the right-hand side of Figure 2 should exchange classical information as follows, by modifying suitably their own $v^{CM}$’s.

|   | input | output |
|---|---|---|
| $Q_0$ | $\kappa_0^0$ ($\leftrightarrow$) | $\kappa_0^0 := \kappa_2^0$ ($\leftrightarrow$) |
|   | $\kappa_0^0$ ($\leftrightarrow$) | $\kappa_0^0 := \kappa_2^0 + \gamma_0$ ($\leftrightarrow$) |
| $Q_1$ | $\kappa_1^0$ ($\Rightarrow$) | $\kappa_1^0 := \kappa_1^0$ ($\Rightarrow$) |
|   | $\kappa_1^0$ ($\Rightarrow$) | $\kappa_1^0 := \kappa_1^0 + \kappa_0' + \gamma_1$ ($\Rightarrow$) |
|   | $\kappa_0'$ ($\leftrightarrow$) | $\kappa_0' := \kappa_2^0$ ($\leftrightarrow$) |
| $Q_2$ | $\kappa_2^0$ ($\Rightarrow$) | $\kappa_2^0 := \kappa_2^0 + \kappa_0'$ ($\Rightarrow$) |
|   | $\kappa_2^0$ ($\Rightarrow$) | $\kappa_2^0 := \kappa_2^0 + \gamma_2$ ($\Rightarrow$) |
|   | $\kappa_0''$ ($\leftrightarrow$) | $\kappa_0'' := \kappa_2^0$ ($\leftrightarrow$) |

As explicitly seen here, a new key change is the way of communication of classical information while $d^C = 4$ is maintained among any neighboring pair of qubits. Namely, as opposed to the 2-bit directional communication along a single timeline state, $Q_0$ receives one bit each from $Q_1$ and $Q_2$, and then sends out another bit each to them. As discussed in the main text, this non-directional communication results in a realization of the spatial direction in our model.

Appendix B: 1+1D Dirac fermion as clocks

Our timeline state of Eq. (9) is partially motivated by an informal analogy to the 1+1D Dirac fermion. Let us consider, in Eq. (1), the Dirac Hamiltonian of a relativistic fermion, which acts on the motional degree on the 1D spatial line parametrized by $r \in \mathbb{R}$ and the internal spin
\( \frac{1}{2} \) degree.

\[ hB = c(\hat{p} - \frac{\hat{A}(r)}{c})\sigma^z + mc^2\sigma^y, \quad (B1) \]

where \( \hat{p} = -i\frac{\partial}{\partial r} \) is the momentum operator, \( \hat{A}(r) \) is the vector potential depending on \( r \), \( m \) is the rest mass, and \( c \) is the speed of light which is set to be 1 hereafter. It is widely known that the relativistic wave packet, initially localized well at the origin, spreads near the light cone at the speed of light (namely \( \frac{c^2\hat{p}}{m} \)), so that the linear spreading on the position coordinate serves as keeping track of time (if it were able to be monitored). It is crucial that two degrees of freedom are intrinsically entangled. Although the distribution of the spatial degree has been analyzed much in literatures, we are here interested in the evolution of the spin \( \frac{1}{2} \) degree relative to that of the spatial degree.

Now the unitary evolution by Eq. (B1) is given by the repetitive application of an elementary action \( e^{i\sigma^z\tau} e^{i\theta z \sigma^z} e^{-i\hat{A}\sigma^z} \) at a discretized infinitesimal time unit \( \epsilon \). Indeed this can be see as a discrete-time quantum walk [26], where its walker and quantum coin correspond to the spatial and spin degrees of the Dirac fermion, respectively. We realize that the sequential application of the transfer operator \( V \dot{R}^2(-\tau(\gamma)) \) in Eq. (10) would lead to an analogous unitary evolution, by mapping operators of the Dirac fermion onto the observable angles in the underlying Hilbert space of the qubit: \( \hat{p} \leftrightarrow \theta_3, m \leftrightarrow \theta_2, \hat{A}(r) \leftrightarrow \tau(\gamma) \). Note that originally the term by \( \hat{p}\sigma^z \) is a conditional shift operator, moving to nearby orthogonal states of the spatial degree depending on the internal degree. After mapping to a compact geometry of the qubit, however, there is only one orthogonal state among the canonical clock-time states \( \{ t \} \). So two possible shifts, the anti-clockwise and clockwise rotations, become identical and actually have to move to the same orthogonal state by the \( z \) rotation of \( \theta_3 = \pi = -\pi \). It is reminiscent of we can set \( \theta_3 \) to be \( \pi (= -\pi) \) in the text, by different reasoning. Observe also that \( V \) is identical every step while \( \dot{R}^2(-\tau(\gamma)) \) may depend on the site \( j \), which is the same way as the site-dependence of the Dirac Hamiltonian dynamics. So, not only has it motivated our timeline state, but also conversely it may suggest a renewed informational perspective such that the Dirac fermion describes correlations of internal clocks in a consistent manner with both quantum theory and relativistic theory.

Appendix C: Relation to the 1D cluster state

We show that the timeline state of Eq. (9), which is the solution of Eq. (7) with \( m = \frac{3}{4} \) or \( \frac{3\pi}{4} \) (or \( m^w = \pm \sigma^x \)), satisfies simultaneously eigen equations,

\[ K_j |\Psi^{\varphi}(\varphi)\rangle = |\Psi^{\varphi}(\varphi)\rangle, \quad j = 2, \ldots, N, \quad (C1) \]

regardless of \( \varphi \). Here we define \( K_j = \sigma_{j-1}^- \sigma_j^x \sigma_{j+1}^- \) for \( j = 2, \ldots, N-1 \) and \( K_N = \sigma_{N-1}^- \sigma_N^x \).

The action of \( \sigma_{j-1}^z \) on the \( j-1 \)-th qubit results in \( \sigma^z R^2(t_j) = (-i)R^2(t_{j} + \pi) \) in the integral of Eq. (9). We introduce a new integration variable \( \tilde{t}_j = t_j + \pi \), and see it affects to the \( j \)-th site located in the future of the timeline. By the action of \( \sigma_j^z \) on this \( j \)-th site, we get \( \sigma^z R^2(t_{j+1})R^2(-t_{j} + \pi) = i\sigma^x R^2(t_{j+1})\sigma^z R^2(-t_j) = iR^2(-t_{j+1})R^2(-t_j), \) having the sign of \( t_{j+1} \) now flipped. Introducing another new variable \( \tilde{t}_{j+1} = -t_{j+1} \), we see that, on the \( j+1 \)-th site, \( \sigma^x R^2(t_{j+2})R^2(\tilde{t}_{j+1})|0\rangle = R^2(t_{j+2})R^2(-t_{j+1})|0\rangle \). Thus, the action of \( K_j \) brings the state of Eq. (9) back to the original one (using new variables \( \tilde{t}_j, \tilde{t}_{j+1} \)) with the eigenvalue \( (-i) \times i = 1 \). That is how, the state corresponds to the so-called 1D cluster state [20] up to the freedom by \( \varphi \), in that a “holographic” degree of freedom on the left boundary is kept variable. Indeed, for the fiducial case \( \varphi = \varphi \) of Eq. (9) itself, \( \sigma_1^z |\Psi^{\varphi}(o)\rangle = |\Psi^{\varphi}(o)\rangle \) is satisfied additionally, while the 1D cluster state should have another stabilizer by \( K_1 = \sigma_1^x \sigma_2^- \). Because of this correspondence, it is notable that this timeline state can be constructed without resort to its apparent ordering of \( \Xi \)'s, by acting the commuting 2-qubit Controlled-Phase gate \( |0^0\rangle \langle 0^0| + |0^1\rangle \langle 0^1| + |1^0\rangle \langle 1^0| - |1^1\rangle \langle 1^1| \) parallelly on all the neighboring pairs of the qubits, following the initialization to a totally product state \( |0^0\rangle_1 |0^0\rangle_2 \cdots |0^0\rangle_N \). Needless to say, the same technique can be used to analyze the stabilizers for the case of a partially ordered geometry.