Existence theorem and blow-up criterion of the strong solutions to the Magneto-micropolar fluid equations

Jia Yuan †
The Graduate School of China Academy of Engineering Physics
P. O. Box 2101, Beijing, China, 100088
(yuanjia930@hotmail.com †)

Abstract

In this paper we study the magneto-micropolar fluid equations in \( \mathbb{R}^3 \), prove the existence of the strong solution with initial data in \( H^s(\mathbb{R}^3) \) for \( s > \frac{3}{2} \), and set up its blow-up criterion. The tool we mainly use is Littlewood-Paley decomposition, by which we obtain a Beale-Kato-Majda type blow-up criterion for smooth solution \( (u, \omega, b) \) which relies on the vorticity of velocity \( \nabla \times u \) only.

Key words. The magneto-micropolar equations, Blow-up, Littlewood-Paley decomposition, Besov space

AMS subject classifications. 76W05 35B65

1 Introduction

In this paper, we consider Magneto-micropolar fluid equations in \( \mathbb{R}^3 \).

\[
\begin{aligned}
\partial_t u - (\mu + \chi) \Delta u + u \cdot \nabla u - b \cdot \nabla b + \nabla (p + b^2) - \chi \nabla \times \omega &= 0, \\
\partial_t \omega - \gamma \Delta \omega - \kappa \nabla \text{div} \omega + 2 \chi \omega + u \cdot \nabla \omega - \chi \nabla \times u &= 0, \\
\partial_t b - \nu \Delta b + u \cdot \nabla b - b \cdot \nabla u &= 0, \\
\text{div} u = \text{div} b &= 0, \\
u(0, x) = u_0(x), \omega(0, x) = \omega_0(x), b(0, x) = b_0(x),
\end{aligned}
\]

(1.1)

where \( u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x)) \in \mathbb{R}^3 \) denotes the velocity of the fluid at a point \( x \in \mathbb{R}^3, t \in [0, T)，\omega(t, x) \in \mathbb{R}^3, b(t, x) \in \mathbb{R}^3 \) and \( p(t, x) \in \mathbb{R} \) denote, respectively, the micro-rotational velocity, the magnetic field and the hydrostatic pressure. \( \mu, \chi, \kappa, \gamma, \nu \) are positive numbers associated to properties of the material: \( \mu \) is the kinematic viscosity, \( \chi \) is the vortex viscosity, \( \kappa \) and \( \gamma \) are spin viscosities, and \( \frac{1}{\nu} \) is the magnetic Reynold. \( u_0, \omega_0, b_0 \) are initial data for the velocity, the angular velocity and the magnetic field with properties \( \text{div} u_0 = 0 \) and \( \text{div} b_0 = 0 \).
There are many earlier results concerning the weak and strong solvability of magneto-micropolar fluid in bounded domain $\Omega \in \mathbb{R}^3$. The corresponding equation is

$$\begin{cases}
\partial_t u - (\mu + \chi)\Delta u + u \cdot \nabla u - b \cdot \nabla b + \nabla(p + b^2) - \chi \nabla \times \omega = 0, \\
\partial_t \omega - \gamma \Delta \omega - \kappa \nabla \text{div} \omega + 2\chi \omega + u \cdot \nabla \omega - \chi \nabla \times u = 0, \\
\partial_t b - \nu \Delta b + u \cdot \nabla b - b \cdot \nabla u = 0, \\
\text{div} u = \text{div} b = 0 \quad \text{in} \Omega, \\
u(0, x) = u_0(x), \omega(0, x) = \omega_0(x), b(0, x) = b_0(x), \quad x \in \Omega, \\
\omega(t, x) = b(t, x) = 0, \quad (t, x) \in [0, T] \times \partial \Omega. 
\end{cases} \quad (1.2)$$

If $b = 0$, equation (1.1) (1.2) reduces to the micropolar fluid system. Micropolar fluid system was first proposed by Eringen[9] in 1966. For the initial boundary-value problem (1.2), with $b = 0$, in the year 1977, Galdi and Rionero[10] considered the weak solution. Using linearization and an almost fixed point theorem, in 1988, Lukaszewicz[13] established the global existence of weak solutions with sufficiently regular initial data. In 1989, using the same technique, Lukaszewicz[14] proved the local and global existence and the uniqueness of the strong solutions. In 2005, Yamaguchi[24] proved the existence theorem of global in time solution for small initial data.

If both $\omega = 0$ and $\chi = 0$, then the system (1.1) reduces to be the magneto-hydrodynamic equations, which has been studied extensively in [8, 19, 3, 11, 5]. Regularity results can refer to Wu[21, 22, 23].

To the full system, Magneto-micropolar fluid equations (1.2), in 1977, Galdi and Rionero[10] stated the theorem of existence and uniqueness of strong solutions, but without proof. Ahmadi and Shaninpoor[1] studied the stability of solutions for the system in 1974. By using spectral Galerkin method, in 1997, Rojas-Medar[17] established local existence and uniqueness of strong solutions. In 1998, Ortega-Torres and Rojas-Medar[16] proved global existence of strong solutions with small initial data. For the weak solution, Rojas-Medar and Boldrini[18] established the local existence in two and thre dimension by using Galerkin method, and also proved the uniqueness in 2D case.

But there are few theories about regularity and blow-up criteria of Magnetomucropolar fluid equations. Some blow-up criterion are obtained by Yuan[25] in 2006. His paper implies that most classical blow-up criteria of smooth solutions to Navier-Stokes or magneto-hydrodynamic equations also hold for Magneto-micropolar fluid equations.

For classical MHD equations, an exciting result is that He and Xin[11] give a blow-up condition which do not depend on the magnetic field $b$, which is

$$\int_0^T \|u(t)\|_p^2 dt < \infty, \quad \frac{2}{q} + \frac{3}{p} \leq 1 \quad 3 < p \leq \infty.$$ 

Zhou[27] gives the regularity criterion dependent only on $\nabla u$

$$\int_0^T \|\nabla u(t)\|_p^2 dt < \infty, \quad \frac{2}{q} + \frac{3}{p} \leq 2 \quad \frac{3}{2} < p < \infty,$$
The same result has been extended to Magneto-micropolar fluid equation by Yuan \cite{25}, of which the condition doesn’t rely on $\omega$ and $b$. We know that for $1 < p < \infty$, thanks to the Biot-Savart law \cite{15}, $\|\nabla u(t)\|_p$ can be controlled by $\|\nabla \times u(t)\|_p$, so the regularity criterion of Zhou \cite{27} can be relaxed by

$$\int_0^T \|\nabla \times u(t)\|_p^q dt < \infty, \quad \frac{2}{q} + \frac{3}{p} \leq 2, \quad \frac{3}{2} < p < \infty,$$

but this results missed the important marginal case $p = \infty$ which exactly corresponds to the Beale-Kato-Majda criterion.

While for 3D ideal MHD equations, Caflish, Klapper and Steele \cite{3} extended the well-known result of Beale-Kato-Majda \cite{2} for the incompressible Euler equations to the 3D ideal MHD equations, that is, if

$$\int_0^T (\|\nabla \times u(t)\|_\infty + \|\nabla \times b(t)\|_\infty) dt < \infty,$$

then the smooth solution $(u, b)$ can be extended beyond $t = T$. Zhang and Liu \cite{26} extend the condition to

$$\int_0^T (\|\nabla \times u(t)\|_{B^0_{\infty, \infty}} + \|\nabla \times b(t)\|_{B^0_{\infty, \infty}}) dt < \infty,$$

Cannon, Chen and Miao \cite{4} refined to the following

$$\lim_{\varepsilon \to 0} \sup_{j \in \mathbb{Z}} \int_{T-\varepsilon}^T (\|\Delta_j(\nabla \times u)\|_\infty + \|\Delta_j(\nabla \times b)\|_\infty) dt = \delta < M, \quad (1.3)$$

where $\Delta_j$ is a frequency localization appeared in Preliminaries.

The aim of our paper, first is using successive approximation method to obtain the existence of strong solutions in $\mathbb{R}^3$, then using Fourier frequency localization to set up blow-up criterion as \cite{13} which relying on $\nabla \times u$ only. Our result is stated as following:

**Theorem 1.1. (Main theorem)**

(i) **Local existence:** Let $s > \frac{3}{2}$, suppose $(u_0, \omega_0, b_0) \in H^s(\mathbb{R}^3)$ with $\text{div} u_0 = \text{div} b_0 = 0$, then there exists a positive time $T(||(u_0, \omega_0, b_0)||_{H^s})$ such that a unique solution $(u, \omega, b) \in C([0, T); H^s) \cap C^1((0, T); H^s) \cap C((0, T); H^{s+2})$ of the system \cite{17} exists.

(ii) **Blow-up criterion:** Suppose that for $s > \frac{3}{2}$, $(u, \omega, b) \in C([0, T); H^s) \cap C^1((0, T); H^s) \cap C((0, T); H^{s+2})$ is the smooth solution to equation \cite{17}. If there exists an absolute constant $M > 0$ such that if

$$\lim_{\varepsilon \to 0} \sup_{j \in \mathbb{Z}} \int_{T-\varepsilon}^T \|\Delta_j(\nabla \times u)\|_\infty dt = \delta < M, \quad (1.4)$$

then $\delta = 0$, and the solution $(u, \omega, b)$ can be extended past time $t = T$. If

$$\lim_{\varepsilon \to 0} \sup_{j \in \mathbb{Z}} \int_{T-\varepsilon}^T \|\Delta_j(\nabla \times u)\|_\infty dt \geq M, \quad (1.5)$$

then the solution blows up at $t = T$. 

3
2 Preliminaries

In this section we set our notations and recall the Littlewood-Paley decomposition, and review the so called Beinstein estimate and Commutator estimate, which are to be used in the proof of our theorem. In what follows positive constants will be denoted by $C$ and will change from line to line. If necessary, by $C(\ast, \cdots, \ast)$ we denote constants depending only on the quantities appearing in parentheses.

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^3)$, the Fourier transform of $f$ defined by
\[
\hat{f}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx
\]

We consider $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ respectively support in $B = \{\xi \in \mathbb{R}^3, |\xi| \leq \frac{4}{3}\}$ and $C = \{\xi \in \mathbb{R}^3, \frac{4}{3} \leq |\xi| \leq \frac{8}{3}\}$ such that
\[
\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \in \mathbb{R}^3
\]
\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\},
\]

Setting $\varphi_j = \varphi(2^{-j} \xi)$, then $\text{supp} \varphi_j \cap \text{supp} \varphi_j' = \emptyset$ if $|j - j'| \geq 2$ and $\text{supp} \chi \cap \text{supp} \varphi_j' = \emptyset$ if $|j - j'| \geq 1$. Let $h = F^{-1} \varphi$ and $\tilde{h} = F^{-1} \chi$, the dyadic blocks are defined as follows
\[
\Delta_j f = \varphi(2^{-j} D) f = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) f(x - y) dy,
\]
\[
S_j f = \sum_{k \leq j - 1} \Delta_k f = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y) f(x - y) dy, \quad j \in \mathbb{Z}.
\]

Informally, $\Delta_j = S_{j+1} - S_j$ is a frequency projection to the annulus $|\xi| \approx 2^j$, while $S_j$ be frequency projection to the ball $|\xi| \lesssim 2^j$. The details of Littlewood-Paley decomposition can be found in Triebel [20] and Chemin [6]. Now Besov spaces in $\mathbb{R}^3$ can be defined as follows:
\[
\dot{B}^s_{p,q} = \left\{ f \in \mathcal{Z}'(\mathbb{R}^3) \mid \| f \|_{\dot{B}^s_{p,q}} = \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \Delta_j f \|_p^q \right)^{\frac{1}{q}} < \infty \right\}, q \neq \infty
\]
\[
\dot{B}^s_{p,\infty} = \left\{ f \in \mathcal{Z}'(\mathbb{R}^3) \mid \| f \|_{\dot{B}^s_{p,\infty}} = \sup_{j \in \mathbb{Z}} 2^j \| \Delta_j f \|_p < \infty \right\}
\]

where $\mathcal{Z}'$ denotes the dual space of $\mathcal{Z} = \{ f \in \mathcal{S}; D^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^n \text{ multi-index} \}$

Now we introduce well-known Bernstein’s Lemma and Commutator estimate, the proof are omitted here, we can find the details in Chemin [6], Chemin and Lerner [7] and Kato and Ponce [12].
Lemma 2.1. (Bernstein’s Lemma) Let \(1 \leq p \leq q \leq \infty\). Assume that \(f \in L^p\), then there exist constant \(C, C_1, C_2\) independent of \(f, j\) such that
\[
\sup_{|\alpha| = k} \|\partial^\alpha f\|_q \leq C_2 2^{jk(\frac{3}{p} - \frac{1}{q})} \|f\|_p \quad \text{supp} \hat{f} \subset \{|\xi| \lesssim 2^j\}, \quad (2.1)
\]
\[
C_1 2^{jk} \|f\|_p \leq \sup_{|\alpha| = k} \|\partial^\alpha f\|_p \leq C_2 2^{jk} \|f\|_p \quad \text{supp} \hat{f} \subset \{|\xi| \approx 2^j\}. \quad (2.2)
\]

Remark 2.1. From the above Beinstein estimate, we easily know that in \(\mathbb{R}^3\), for the Reisz transform \(R_k (k = 1, 2, 3)\), it has for \(\forall 1 \leq p \leq q \leq \infty\)
\[
\|R_k \Delta_j u\|_q \leq C 2^{3j(\frac{3}{p} - \frac{1}{q})} \|u\|_p. \quad (2.3)
\]
If suppose vector valued function \(u\) be divergence free, by Biot Savard law \(\nabla u = (-\Delta)^{-1} \nabla \nabla \times v\) with \(v = \nabla \times u\) and the boundedness of Reisz transform on \(L^p(1 < p < \infty)\), we have, there exist constants \(C\) independent of \(u\) such that
\[
\|\nabla u\|_p \leq C\|v\|_p, \quad \forall 1 < p < \infty. \quad (2.4)
\]
If the frequency of \(u\) is restricted to annulus \(|\xi| \approx 2^j\), then (2.3) implies that
\[
\|\nabla u\|_p \leq C\|v\|_p, \quad \forall 1 \leq p \leq \infty. \quad (2.5)
\]

Now we denote \(\Lambda = (I - \Delta)^{\frac{1}{2}}\), which satisfies
\[
\hat{\Lambda} f(\xi) = (1 + |\xi|^2)^{\frac{1}{2}} \hat{f}(\xi),
\]
\(\Lambda^s (s \in \mathbb{R})\) can be defined in the same way
\[
\hat{\Lambda}^s f(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi).
\]
Using the above notation, we define the norm of Sobolev space \(W^{s, p}\)
\[
\|f\|_{W^{s, p}} \triangleq \|\Lambda^s f\|_{L^p},
\]
especially by Fourier transform, \(H^s \triangleq W^{s, 2}\) can be defined as
\[
H^s \triangleq \{ f \in S'(\mathbb{R}^3) \mid \|f\|_{H^s} < \infty \},
\]
where
\[
\|f\|_{H^s} \triangleq \|\Lambda^s f\|_{L^2} = \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

Lemma 2.2. (Commutator estimate) Let \(1 < p < \infty, s > 0\), assume that \(f, g \in W^{s, p}\), then there exist constants \(C\) independent of \(f, g\) such that
\[
\|[\Lambda^s, f]g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}} \|g\|_{W^{-1, p_2}} + \|f\|_{W^{-s, p_3}} \|g\|_{L^{p_4}}) \quad (2.6)
\]
with \(1 < p_2, p_3 < \infty\) such that
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]
Here \([\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g\).
3 Proof of the Theorem 1.1

Part 1: Local existence

In order to proof the local existence of equation (1.1) with initial data \((u_0, \omega_0, b_0) \in H^s(\mathbb{R}^3)\) for \(s > \frac{3}{2}\), we construct sequence \((u^{(n+1)}, \omega^{(n+1)}, b^{(n+1)})\), which solving the following equations

\[
\begin{align*}
\partial_t u^{(n+1)} - (\mu + \chi) \Delta u^{(n+1)} &= -u^{(n)} \cdot \nabla u^{(n)} + b^{(n)} \cdot \nabla b^{(n)} - \nabla (p^{(n)} + b^{2(n)}) + \chi \nabla \times \omega^{(n+1)}, \\
\partial_t \omega^{(n+1)} - \gamma \Delta \omega^{(n+1)} - \kappa \nabla \text{div} \omega^{(n+1)} + 2 \chi \omega^{(n+1)} &= -u^{(n)} \cdot \nabla \omega^{(n)} + \chi \nabla \times u^{(n+1)}, \\
\partial_t b^{(n+1)} - \nu \Delta b^{(n+1)} &= -u^{(n)} \cdot \nabla b^{(n)} + b^{(n)} \cdot \nabla u^{(n)}, \\
\text{div} u^{(n+1)} &= \text{div} b^{(n+1)} = 0, \\
(u^{(n+1)}, \omega^{(n+1)}, b^{(n+1)})(0, x) &= S_{n+2}(u_0(x), \omega_0(x), b_0(x)),
\end{align*}
\]

for \(n = 0, 1, 2, 3, \cdots\), where \(b^{2(n)} = (b^{(n)})^2\), and we set \((u^{(0)}, \omega^{(0)}, b^{(0)}) = (0, 0, 0)\).

Multiplying the above equation by \(\langle u^{(n+1)}, \omega^{(n+1)} \rangle\) and integrating on time variable, denoting \(L^2\) inner product by \(\langle \cdot, \cdot \rangle\), we get

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| (u^{(n+1)}, \omega^{(n+1)}, b^{(n+1)}) \|^2_2 &+ \mu \| \nabla u^{(n+1)} \|^2_2 + \kappa \| \nabla \omega^{(n+1)} \|^2_2 + \gamma \| \nabla \omega^{(n+1)} \|^2_2 + \nu \| \nabla b^{(n+1)} \|^2_2 \\
&+ \kappa \| \text{div} \omega^{(n+1)} \|^2_2 + 2 \chi \| \omega^{(n+1)} \|^2_2 \\
&= -\langle u^{(n)} \cdot \nabla u^{(n)}, u^{(n+1)} \rangle + \langle b^{(n)} \cdot \nabla b^{(n)}, u^{(n+1)} \rangle - \langle u^{(n)} \cdot \nabla \omega^{(n)}, \omega^{(n+1)} \rangle \\
&- \langle \omega^{(n)} \cdot \nabla b^{(n)}, b^{(n+1)} \rangle + \langle b^{(n)} \cdot \nabla u^{(n)}, b^{(n+1)} \rangle + 2 \chi \langle \nabla \times u^{(n+1)}, \omega^{(n+1)} \rangle \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{3.2}
\end{align*}
\]

where we use \(\langle \nabla \times \omega^{(n+1)}, u^{(n+1)} \rangle = \langle \nabla \times u^{(n+1)}, \omega^{(n+1)} \rangle\).

Using the divergence free condition, the embedding \(H^s \hookrightarrow L^\infty\) and Young Inequality

\[
ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1,
\]

we have

\[
I_1 = -\langle u^{(n)} \cdot \nabla u^{(n)}, u^{(n+1)} \rangle \leq \| u^{(n)} \|^2_2 \| \nabla u^{(n+1)} \|_\infty \leq \| u^{(n)} \|^2_2 \| \nabla u^{(n+1)} \|_{H^s} \\
\leq \frac{\mu}{4} \| \nabla u^{(n+1)} \|^2_{H^s} + C \| u^{(n)} \|^4_2.
\]

For the other terms, we use the same technique and get

\[
I_2 = \langle b^{(n)} \cdot \nabla b^{(n)}, u^{(n+1)} \rangle \leq \frac{\mu}{4} \| \nabla u^{(n+1)} \|^2_{H^s} + C \| b^{(n)} \|^4_2, \\
I_3 = -\langle u^{(n)} \cdot \nabla \omega^{(n)}, \omega^{(n+1)} \rangle \leq \frac{\gamma}{2} \| \nabla \omega^{(n+1)} \|^2_{H^s} + C (\| u^{(n)} \|^2_2 + \| \omega^{(n)} \|^2_2), \\
I_4 = -\langle u^{(n)} \cdot \nabla b^{(n)}, b^{(n+1)} \rangle \leq \frac{\nu}{4} \| \nabla b^{(n+1)} \|^2_{H^s} + C (\| u^{(n)} \|^2_2 + \| b^{(n)} \|^4_2), \\
I_5 = \langle b^{(n)} \cdot \nabla u^{(n)}, b^{(n+1)} \rangle \leq \frac{\nu}{4} \| \nabla b^{(n+1)} \|^2_{H^s} + C (\| b^{(n)} \|^4_2 + \| u^{(n)} \|^4_2), \\
I_6 = 2 \chi \langle \nabla \times u^{(n+1)}, \omega^{(n+1)} \rangle \leq \frac{\chi}{2} \| \nabla u^{(n+1)} \|^2_2 + 2 \chi \| \omega^{(n+1)} \|^2_2.
\]
Summing up the above estimates, we obtain the $L^2$ estimate
\[
\frac{d}{dt} \left( \|u^{(n+1)}\|_2^2 + \|\omega^{(n+1)}\|_2^2 + (\mu + \chi) \|\nabla u^{(n+1)}\|_2^2 + \gamma \|\nabla \omega^{(n+1)}\|_2^2 \right) \\
+ \nu \|\nabla b^{(n+1)}\|_2^2 + 2\kappa \|d_1 \nu \omega^{(n+1)}\|_2^2 \leq C \|u^{(n)}\|_2^4 + \|\omega^{(n)}\|_2^4 + \|b^{(n)}\|_2^4.
\] (3.3)

Now let's give the $H^s$ estimate. Applying operator $\Delta_k$ to equation, then multiplying the first three equations by $(\Delta_k u^{(n+1)}, \Delta_k \omega^{(n+1)}, \Delta_k b^{(n+1)})$, introducing notation $\otimes$ as follows
\[
f \cdot \nabla g = \text{div}(f \otimes g) \quad \text{where} \quad \text{div}(f \otimes g)^j = \sum_{k=1}^3 \partial_k (f^j g^k) = \text{div}(f^j g),
\]
we finally get by using the divergence free condition
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Delta_k u^{(n+1)}\|_2^2 + \|\Delta_k \omega^{(n+1)}\|_2^2 + \|\Delta_k b^{(n+1)}\|_2^2 \right) \\
+ \nu \|\Delta_k \nabla b^{(n+1)}\|_2^2 + \kappa \|\Delta_k \text{div} \omega^{(n+1)}\|_2^2 + 2\chi \|\Delta_k \omega^{(n+1)}\|_2^2 \\
= \langle \Delta_k (u^{(n)} \otimes u^{(n)}), \nabla \Delta_k u^{(n+1)} \rangle - \langle \Delta_k (b^{(n)} \otimes b^{(n)}), \nabla \Delta_k u^{(n+1)} \rangle \\
+ \langle \Delta_k (b^{(n)} \otimes \omega^{(n)}), \nabla \Delta_k \omega^{(n+1)} \rangle + \langle \Delta_k (u^{(n)} \otimes b^{(n)}), \nabla \Delta_k b^{(n+1)} \rangle \\
- \langle \Delta_k (b^{(n)} \otimes u^{(n)}), \nabla \Delta_k b^{(n+1)} \rangle - 2\chi \langle \Delta_k \omega^{(n+1)}, \nabla \times \Delta_k u^{(n+1)} \rangle,
\] (3.4)
where we use the equality $\langle \Delta_k \omega^{(n+1)}, \nabla \times \Delta_k u^{(n+1)} \rangle = \langle \Delta_k u^{(n+1)}, \nabla \times \Delta_k \omega^{(n+1)} \rangle$.

Multiplying $2^k \kappa s$ on both sides of (3.2,3), then summing up over $k \in \mathbb{Z}$, we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|u^{(n+1)}\|_{H^s}^2 + \|\omega^{(n+1)}\|_{H^s}^2 + \|b^{(n+1)}\|_{H^s}^2 \right) \\
+ \nu \|\nabla b^{(n+1)}\|_{H^s}^2 + \kappa \|\text{div} \omega^{(n+1)}\|_{H^s}^2 + 2\chi \|\omega^{(n+1)}\|_{H^s}^2 \\
\leq \sum_{k \in \mathbb{Z}} 2^k \kappa s \|\Delta_k (u^{(n)} \otimes u^{(n)})\|_{H^s}^2 + \|\Delta_k \nabla u^{(n+1)}\|_{H^s}^2 \\
+ \sum_{k \in \mathbb{Z}} 2^k \kappa s \|\Delta_k (b^{(n)} \otimes \omega^{(n)})\|_{H^s}^2 + \|\Delta_k \nabla \omega^{(n+1)}\|_{H^s}^2 \\
+ \sum_{k \in \mathbb{Z}} 2^k \kappa s \|\Delta_k (b^{(n)} \otimes u^{(n)})\|_{H^s}^2 + \|\Delta_k \nabla b^{(n+1)}\|_{H^s}^2 \\
+ \sum_{k \in \mathbb{Z}} 2^k \kappa s \|\Delta_k \nabla \times u^{(n+1)}\|_{H^s}^2 + \|\Delta_k \omega^{(n+1)}\|_{H^s}^2 \\
= II_1 + II_2 + II_3 + II_4 + II_5 + II_6.
\] (3.5)

We use the embedding $H^s \hookrightarrow L^\infty$ along with Hölder and Young inequality to get
\[
II_1 \lesssim \|u^{(n)}\|_{H^s} \|\nabla u^{(n+1)}\|_{H^s} \lesssim C \|u^{(n)}\|_{L^\infty} \|u^{(n)}\|_{H^s} \|\nabla u^{(n+1)}\|_{H^s} \\
\lesssim \|u^{(n)}\|_{H^s}^2 \|\nabla u^{(n+1)}\|_{H^s} \lesssim \mu \|\nabla u^{(n+1)}\|_{H^s}^2 + C \|u^{(n)}\|_{H^s}^4.
\]
For the other terms, we use the same technique

\[ II_2 \leq \|b^{(n)}\|_{L^4} \|\nabla u^{(n+1)}\|_{H^s} \leq \frac{\mu}{4} \|\nabla u^{(n+1)}\|_{H^s}^2 + C \|b^{(n)}\|_{H^s}^4 \]

\[ II_3 \leq \|u^{(n)}\|_{H^s} \|\nabla \omega^{(n+1)}\|_{H^s} \leq \frac{\gamma}{2} \|\nabla \omega^{(n+1)}\|_{H^s}^2 + C (\|u^{(n)}\|_{H^s}^4 + \|\omega^{(n)}\|_{H^s}^4) \]

\[ II_4 + II_5 \leq \|b^{(n)}u^{(n)}\|_{H^s} \|\nabla b^{(n+1)}\|_{H^s} \leq \frac{\nu}{2} \|\nabla b^{(n+1)}\|_{H^s}^2 + C (\|u^{(n)}\|_{H^s}^4 + \|b^{(n)}\|_{H^s}^4) \]

\[ II_6 \leq 2\chi \|\nabla u^{(n+1)}\|_{H^s} \|\nabla \omega^{(n+1)}\|_{H^s} \leq \frac{3}{2} \|\nabla u^{(n+1)}\|_{H^s}^2 + 2\chi \|\nabla \omega^{(n+1)}\|_{H^s}^2 \]

Taking the sum of \( II_1, II_2, II_3, II_4, II_5 \) and \( II_6 \), we obtain the following estimate

\[
\frac{d}{dt} \left( \| (u^{(n+1)}, \omega^{(n+1)}, b^{(n+1)}) \|_{H^s}^2 + (\mu + \chi) \|\nabla u^{(n+1)}\|_{H^s}^2 + \gamma \|\nabla \omega^{(n+1)}\|_{H^s}^2 \right.
\]
\[
+ \nu \|\nabla b^{(n+1)}\|_{H^s}^2 + 2\kappa \|\text{div} \omega^{(n+1)}\|_{H^s}^2 \leq C \| (u^{(n)}, \omega^{(n)}, b^{(n)}) \|_{H^s}^4, \tag{3.6} \]

which along with the \( L^2 \) estimate, we finally obtain

\[
\frac{d}{dt} \left( \| (u^{(n+1)}, \omega^{(n+1)}, b^{(n+1)}) \|_{H^s}^2 + (\mu + \chi) \|\nabla u^{(n+1)}\|_{H^s}^2 + \gamma \|\nabla \omega^{(n+1)}\|_{H^s}^2 \right.
\]
\[
+ \nu \|\nabla b^{(n+1)}\|_{H^s}^2 + 2\kappa \|\text{div} \omega^{(n+1)}\|_{H^s}^2 \leq C \| (u^{(n)}, \omega^{(n)}, b^{(n)}) \|_{H^s}^4. \tag{3.7} \]

Denote

\[ E_s^{(n)}(t) = \| (u^{(n)}, \omega^{(n)}, b^{(n)}) \|_{H^s}^2, \]

then the above inequality can be reduced to be

\[
\frac{d}{dt} E_s^{(n+1)}(t) + (\mu + \chi) \|\nabla u^{(n+1)}\|_{H^s}^2 + \gamma \|\nabla \omega^{(n+1)}\|_{H^s}^2 + \nu \|\nabla b^{(n+1)}\|_{H^s}^2
\]
\[
+ 2\kappa \|\text{div} \omega^{(n+1)}\|_{H^s}^2 \leq C_1 (E_s^{(n)}(t))^2. \tag{3.8} \]

Integrating about time variable and taking the supremum on \([0, T]\), we have

\[
\sup_{t \in [0, T]} E_s^{(n+1)}(t) + \int_0^T \left( (\mu + \chi) \|\nabla u^{(n+1)}\|_{H^s}^2 + \gamma \|\nabla \omega^{(n+1)}\|_{H^s}^2 + \nu \|\nabla b^{(n+1)}\|_{H^s}^2 \right.
\]
\[
+ 2\kappa \|\text{div} \omega^{(n+1)}\|_{H^s}^2 \big) dt \leq \|S_{n+2}(u_0, \omega_0, b_0)\|_{H^s}^2 + C_1 T \left( \sup_{t \in [0, T]} E_s^{(n)}(t) \right)^2
\]
\[
\leq C_0 \| (u_0, \omega_0, b_0) \|_{H^s}^2 + C_1 T \left( \sup_{t \in [0, T]} E_s^{(n)}(t) \right)^2. \tag{3.9} \]

By standard induction argument, we find for \( \forall n \in \mathbb{N}, T \in [0, T_0] \), where

\[ T_0 = \frac{1}{4C_0C_1 \| (u_0, \omega_0, b_0) \|_{H^s}^2}, \tag{3.10} \]
we can get
\[
\|(u^{(n+1)}, \omega^{(n+1)}, b^{(n+1)})\|_{L^\infty_T(H^s)} + (\mu + \chi)^\frac{1}{2}\|\nabla u^{(n+1)}\|_{L^2_T(H^s)} + \gamma^\frac{1}{2}\|\nabla \omega^{(n+1)}\|_{L^2_T(H^s)} \\
+ \nu^\frac{1}{2}\|\nabla b^{(n+1)}\|_{L^2_T(H^s)} + (2\kappa)^\frac{1}{2}\|\text{div} \omega^{(n+1)}\|_{L^2_T(H^s)} \leq 2C_0\|(u_0, \omega_0, b_0)\|_{H^s}. \tag{3.11}
\]

In the following process, we will show that there exists a positive time \(T_1 \leq T\) independent of \(n\) such that \((u^{(n)}, \omega^{(n)}, b^{(n)})\) is Cauchy sequence in space
\[
\mathcal{X}_{T_1}^{s-1} = \left\{ (f, g, h) \in L^\infty_T(H^{s-1}) : ((\mu + \chi)^\frac{1}{2}\nabla f, \gamma^\frac{1}{2}\nabla g, \nu^\frac{1}{2}\nabla h) \in L^2_T(H^{s-1}) \right\}.
\]

Denote
\[
\delta u^{(n+1)} = u^{(n+1)} - u^{(n)}, \quad \delta \omega^{(n+1)} = \omega^{(n+1)} - \omega^{(n)}, \quad \delta b^{(n+1)} = b^{(n+1)} - b^{(n)}
\]
\[
\delta p^{(n+1)} = p^{(n+1)} - p^{(n)}, \quad \delta b^2(n+1) = (\delta p^{(n+1)})^2 - (\delta p^{(n)})^2,
\]
which satisfy the following equation
\[
\begin{cases}
\partial_t \delta u^{(n+1)} - (\mu + \chi)\Delta \delta u^{(n+1)} = -\delta u^{(n)} \cdot \nabla u^{(n)} - u^{(n-1)} \cdot \nabla \delta u^{(n)} + \delta b^{(n)} \cdot \nabla b^{(n)} \
+ b^{(n-1)} \cdot \nabla \delta b^{(n)} - \nabla (\delta p^{(n)}) + \delta b^2(n+1) + \chi \nabla \times \delta \omega^{(n+1)}, \\
\partial_t \delta \omega^{(n+1)} - \gamma \Delta \delta \omega^{(n+1)} - \kappa \nabla \text{div} \delta \omega^{(n+1)} + 2\chi \delta \omega^{(n+1)} = -\delta u^{(n)} \cdot \nabla \omega^{(n)} \\
- u^{(n-1)} \cdot \nabla \delta \omega^{(n)} + \chi \nabla \times \delta u^{(n+1)}, \\
\partial_t \delta b^{(n+1)} - \nu \Delta \delta b^{(n+1)} = -\delta u^{(n)} \cdot \nabla b^{(n)} - u^{(n-1)} \cdot \nabla b^{(n)} + \delta b^{(n)} \cdot \nabla b^{(n)} - \delta u^{(n)} \cdot \nabla \delta b^{(n)} - b^{(n-1)} \cdot \nabla \delta u^{(n)} + \nu \delta b^2(n+1) = 0, \\
\end{cases}
\]
\[
\left(\delta u^{(n+1)}, \delta \omega^{(n+1)}, \delta b^{(n+1)}\right)(0, x) = \Delta_{n+1}(u_0(x), \omega_0(x), b_0(x)),
\]
\[
\text{In the same way, we can get the following estimate}
\]
\[
\frac{d}{dt}\|\delta u^{(n+1)}, \delta \omega^{(n+1)}, \delta b^{(n+1)}\|_{H^s}^2 + (\mu + \chi)\|\nabla \delta u^{(n+1)}\|_{H^s}^2 + \gamma\|\nabla \delta \omega^{(n+1)}\|_{H^s}^2 \\
+ \nu\|\nabla \delta b^{(n+1)}\|_{H^s}^2 + 2\kappa\|\text{div} \delta \omega^{(n+1)}\|_{H^s}^2 \\
\leq C_2\|\delta u^{(n)}, \delta \omega^{(n)}, \delta b^{(n)}\|_{2}(\|u^{(n)}, \omega^{(n)}, b^{(n)}\|_{H^s}^2 + \|u^{(n-1)}, \omega^{(n-1)}, b^{(n-1)}\|_{H^s}^2)
\]
\[
\leq C_3\|\delta u^{(n)}, \delta \omega^{(n)}, \delta b^{(n)}\|_{2}^2, \tag{3.13}
\]
where \(C_3 = 4C_0C_2\|(u_0, \omega_0, b_0)\|_{H^s}\), we uses the following type of estimates
\[
\langle \delta u^{(n)}, \nabla u^{(n)} \rangle \leq \|\delta u^{(n)}\|_{2}\|u^{(n)}\|_{\infty}\|\nabla \delta u^{(n+1)}\|_{2} \\
\leq \frac{\nu}{8}\|\nabla \delta u^{(n+1)}\|_{2}^2 + C\|u^{(n)}\|_{H^s}^2\|\delta u^{(n)}\|_{2}^2 \tag{3.14}
\]
\[
\langle u^{(n-1)}, \nabla \delta u^{(n)} \rangle \leq \|u^{(n-1)}\|_{\infty}\|\delta u^{(n)}\|_{2}\|\nabla \delta u^{(n+1)}\|_{2} \\
\leq \frac{\nu}{8}\|\nabla \delta u^{(n+1)}\|_{2}^2 + C\|u^{(n-1)}\|_{H^s}^2\|\delta u^{(n)}\|_{2}^2 \tag{3.15}
\]
\[
2\chi\langle \nabla \times \delta u^{(n+1)}, \delta \omega^{(n+1)} \rangle \leq \chi^2\|\nabla \delta u^{(n+1)}\|_{2}^2 + 2\chi\|\delta \omega^{(n+1)}\|, \tag{3.16}
\]
Integrating over time variable and taking the supremum over \([0, T]\), denoting \(\delta E^{(n)}(t) = ||(\delta u^{(n)}, \delta \omega^{(n)}, \delta b^{(n)})||_2^2\), we obtain

\[
\sup_{t \in [0, T]} \delta E^{(n+1)}(t) + \int_0^T ((\mu + \chi) \|\nabla \delta u^{(n+1)}\|^2 + \gamma \|\nabla \delta \omega^{(n+1)}\|^2 + \nu \|\nabla \delta b^{(n+1)}\|^2 + 2\kappa \|\text{div}\delta\omega^{(n+1)}\|_2^2) \, dt \\
\leq C_4 2^{-2(n+1)^2} \|(u_0, \omega_0, b_0)\|^2_{H^s} + C_3 T \delta E^{(n)}(t),
\]

where we use the fact

\[
\|\Delta_{n+1}(u_0, \omega_0, b_0)\|^2_2 \leq C_3 2^{-2(n+1)^2} \|(u_0, \omega_0, b_0)\|^2_{H^s}.
\]

If \(C_3 T \leq \frac{1}{2}\), then

\[
\|(\delta u^{(n+1)}, \delta \omega^{(n+1)}, \delta b^{(n+1)})\|_{L_T^\infty(L^2)} + (\mu + \chi)^{\frac{1}{2}} \|\nabla \delta u^{(n+1)}\|_{L_T^2(L^2)} + \gamma^{\frac{1}{2}} \|\nabla \delta \omega^{(n+1)}\|_{L_T^2(L^2)} + \nu^{\frac{1}{2}} \|\nabla \delta b^{(n+1)}\|_{L_T^2(L^2)} + (2\kappa)^{\frac{1}{2}} \|\text{div}\delta\omega^{(n+1)}\|_{L_T^2(L^2)} \\
\leq 2C_4 2^{-2(n+1)^2} \|(u_0, \omega_0, b_0)\|_{H^s},
\]

which along with the \(H^s\) estimate and the interpolating theorem

\[
\|f\|_{H^{s-1}} \leq \|f\|^\frac{1}{2}_2 \|f\|^\frac{1}{2}_{H^s},
\]

we finally have, by the standard argument, for \(T \leq \min\{T_0, \frac{1}{4C_0}\}\) when \(n \to \infty\)

\[
\|(\delta u^{(n+1)}, \delta \omega^{(n+1)}, \delta b^{(n+1)})\|_{L_T^\infty(H^{s-1})} + (\mu + \chi)^{\frac{1}{2}} \|\nabla \delta u^{(n+1)}\|_{L_T^2(H^{s-1})} + \gamma^{\frac{1}{2}} \|\nabla \delta \omega^{(n+1)}\|_{L_T^2(H^{s-1})} + \nu^{\frac{1}{2}} \|\nabla \delta b^{(n+1)}\|_{L_T^2(H^{s-1})} + (2\kappa)^{\frac{1}{2}} \|\text{div}\delta\omega^{(n+1)}\|_{L_T^2(H^{s-1})} \\
\leq 2C_3 C_0^{-1} -\frac{1}{2} 2^{-2(n+1)} \|(u_0, \omega_0, b_0)\|_{H^s} \to 0,
\]

which means \((\delta u^{(n+1)}(t), \delta \omega^{(n+1)}, \delta b^{(n+1)})\) is Cauchy sequence in \(X_{T_1}^{s-1}\), so we can find the limit \((u, \omega, b) \in X_{T_1}^s\) is a solution to equation for initial data \((u_0, \omega_0, b_0) \in H^s\), also the solution satisfies the following estimate

\[
\|(u, \omega, b)\|_{L_T^\infty(H^s)} + (\mu + \chi)^{\frac{1}{2}} \|\nabla u^{(n+1)}\|_{L_{T_1}^2(H^s)} + \gamma^{\frac{1}{2}} \|\nabla \omega^{(n+1)}\|_{L_{T_1}^2(H^s)} + \nu^{\frac{1}{2}} \|\nabla b^{(n+1)}\|_{L_{T_1}^2(H^s)} + (2\kappa)^{\frac{1}{2}} \|\text{div}\omega^{(n+1)}\|_{L_{T_1}^2(H^s)} \\
\leq 2C_0 \|(u_0, \omega_0, b_0)\|_{H^s}
\]

This gives the existence of strong solution of Magneto-micropolar (1.1) in \(C([0, T]; H^s)\) for \(s \geq \frac{3}{2}\). Now let’s prove the uniqueness of the solution.

Suppose \((u, \omega, b), (u', \omega', b') \in L_T^\infty(H^s)\) be two solutions to equation (1.1), let \(\delta u = u - u', \delta \omega = \omega - \omega', \delta b = b - b'\), we deduced that \((\delta u, \delta \omega, \delta b)\) satisfies the following
we have and using the simple fact that is equation
\begin{equation}
\begin{aligned}
\partial_t \delta u - (\mu + \chi) \Delta \delta u &= -\delta u \cdot \nabla u - u' \cdot \nabla \delta u + \delta b \cdot \nabla b + b' \cdot \nabla \delta b - \nabla (\delta p + \delta b^2) + \chi \nabla \times \delta \omega, \\
\partial_t \delta \omega - \gamma \Delta \delta \omega &= -\delta u \cdot \nabla \omega - u' \cdot \nabla \delta \omega + \chi \nabla \times \delta u, \\
\partial_t \delta b - \nu \Delta \delta b &= -\delta u \cdot \nabla b - u' \cdot \nabla b + \delta b \cdot \nabla u - b' \cdot \nabla \delta u, \\
\text{div} \delta u &= \text{div} \delta b = 0, \\
(\delta u, \delta \omega, \delta b)(0, x) &= 0.
\end{aligned}
\end{equation}

Multiplying the above equation by \((\delta u, \delta \omega, \delta b)\), then integrating on time variable and using the simple fact
\[
\langle u' \cdot \nabla \delta u, \delta u \rangle = \langle u' \cdot \nabla \delta \omega, \delta \omega \rangle = \langle u' \cdot \nabla \delta b, \delta b \rangle = 0,
\]
we have
\[
\frac{1}{2} \frac{d}{dt} \|(\delta u, \delta \omega, \delta b)\|^2_2 + (\mu + \chi) \|\nabla \delta u\|^2_2 + \gamma \|\nabla \delta \omega\|^2_2 + \nu \|\nabla \delta b\|^2_2 + \kappa \|\text{div} \delta \omega\|^2_2 + 2\chi \|\delta \omega\|^2_2 \\
\leq \|\delta u\|_2 \|u\|_\infty \|\nabla \delta u\|_2 + \|\delta b\|_2 \|b\|_\infty \|\nabla \delta u\|_2 + \|\delta u\|_2 \|\omega\|_\infty \|\nabla \delta \omega\|_2 + \|\delta \omega\|_2 \|\delta b\|_2 + \|\delta b\|_2 \|b\|_\infty \|\nabla \delta \omega\|_2 + 2\chi \|\nabla \times \delta u\|_2 \|\delta \omega\|_2
\leq \frac{\mu + \chi}{2} \|\nabla \delta u\|^2_2 + \frac{\gamma}{2} \|\nabla \delta \omega\|^2_2 + \frac{\nu}{2} \|\nabla \delta b\|^2_2 + 2\chi \|\delta \omega\|^2_2 + C \|(u, \omega, b)\|^2_2 \|(\delta u, \delta \omega, \delta b)\|^2_2,
\]
that is
\[
\frac{d}{dt} \|(\delta u, \delta \omega, \delta b)\|^2_2 + (\mu + \chi) \|\nabla \delta u\|^2_2 + \gamma \|\nabla \delta \omega\|^2_2 + \nu \|\nabla \delta b\|^2_2 + 2\kappa \|\text{div} \delta \omega\|^2_2 \\
\leq C \|(u, \omega, b)\|^2_2 \|(\delta u, \delta \omega, \delta b)\|^2_2.
\]
The \(H^s\) estimate imply that
\[
\|(\delta u, \delta \omega, \delta b)\|_2 \leq C_0 \|(u_0, \omega_0, b_0)\|_{H^s} \|T\|(\delta u, \delta \omega, \delta b)\|_2.
\]
If \(T\) is sufficiently small, we have \(\|(\delta u, \delta \omega, \delta b)\|_2 = 0\), the proof of local existence is ended up.

**Part 2: Blow-up criterion**

Now we start to proof the second part of Theorem 1.1 to set up the blow-up criterion. We apply operator \(\Lambda^s\) on the two sides of the equation (3.11), multiply \((\Lambda^s u, \Lambda^s \omega, \Lambda^s b)\)
by the resulting equations and integrate the final form over $\mathbb{R}^3$, and get

$$\frac{1}{2} \frac{d}{dt} \left( \|\Lambda^s u\|_2^2 + \|\Lambda^s \omega\|_2^2 + \|\Lambda^s b\|_2^2 \right) + (\mu + \chi) ||\nabla \Lambda^s u||^2_2 + \gamma ||\nabla \Lambda^s \omega||^2_2 + \nu ||\nabla \Lambda^s b||^2_2$$

$$+ \kappa \|\text{div} \Lambda^s \omega\|_2^2 + 2\chi \|\Lambda^s b\|_2^2$$

$$= - \int_{\mathbb{R}^3} \Lambda^s (u \cdot \nabla \omega) \Lambda^s u \, dx - \int_{\mathbb{R}^3} \Lambda^s (u \cdot \nabla \omega) \Lambda^s \omega \, dx - \int_{\mathbb{R}^3} \Lambda^s (u \cdot \nabla b) \Lambda^s b \, dx$$

$$+ \int_{\mathbb{R}^3} \Lambda^s (b \cdot \nabla b) \Lambda^s u \, dx + \int_{\mathbb{R}^3} \Lambda^s (b \cdot \nabla b) \Lambda^s b \, dx - 2\chi \int_{\mathbb{R}^3} \Lambda^s (\nabla \times u) \Lambda^s \omega \, dx,$$

where we use the fact

$$\int_{\mathbb{R}^3} \Lambda^s (\nabla \times \omega) \Lambda^s u \, dx = \int_{\mathbb{R}^3} \Lambda^s (\nabla \times u) \Lambda^s \omega \, dx.$$

Now taking use of the divergence free conditions of $(u, b)$, we have

$$\int_{\mathbb{R}^3} (u \cdot \nabla \Lambda^s u) \Lambda^s u \, dx = \int_{\mathbb{R}^3} (u \cdot \Lambda^s \omega) \Lambda^s u \, dx = \int_{\mathbb{R}^3} (u \cdot \Lambda^s b) \Lambda^s b \, dx = 0,$$

also applying the following equality

$$\int_{\mathbb{R}^3} (b \cdot \nabla \Lambda^s b) \Lambda^s u \, dx + \int_{\mathbb{R}^3} (b \cdot \nabla \Lambda^s u) \Lambda^s b \, dx = 0,$$

we have

$$\frac{1}{2} \frac{d}{dt} \left( \|\Lambda^s u\|_2^2 + \|\Lambda^s \omega\|_2^2 + \|\Lambda^s b\|_2^2 \right) + (\mu + \chi) ||\nabla \Lambda^s u||^2_2 + \gamma ||\nabla \Lambda^s \omega||^2_2 + \nu ||\nabla \Lambda^s b||^2_2$$

$$+ \kappa \|\text{div} \Lambda^s \omega\|_2^2 + 2\chi \|\Lambda^s b\|_2^2$$

$$= \text{III}_1 + \text{III}_2 + \text{III}_3 + \text{III}_4 + \text{III}_5$$  \hspace{1cm} (3.22)

By Lemma 2.2, Hölder inequality and Gagliardo-Nirenberg inequality

$$\|f\|_{W^{s,4}} \leq \|f\|_{W^{s,2}}^{\frac{1}{4}} \|\nabla f\|_{W^{s,2}}^{\frac{3}{4}},$$  \hspace{1cm} (3.23)

we have

$$|\text{III}_1| \leq ||[\Lambda^s, u] \nabla u||_2 \|\Lambda^s u\|_4 \leq C \|\nabla u\|_2 \|\nabla u\|_{W^{s-1,4}} + \|u\|_{W^{s,4}} \|\nabla u\|_2 \|u\|_{W^{s,4}}$$

$$\leq C \|\nabla u\|_2 \|u\|_{H^s}^{\frac{3}{4}} \|\nabla u\|_{H^s}^{\frac{5}{4}} \leq C \|\nabla u\|_2 \|u\|_{H^s}^2 + \frac{\mu}{4} \|\nabla u\|_{H^s}^2.$$  \hspace{1cm} (3.24)

Using the same technique and the Young inequality

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q, \quad \frac{1}{p} + \frac{1}{q} = 1,$$
we estimate $III_2, III_3, III_4$ in the same way and get

$$|III_2 + III_3 + III_4| \leq C(\|\nabla u\|_H^2 + \|\nabla \omega\|_H^2 + \|\nabla b\|_H^2)(\|u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|b\|_{H^s}^2) + \frac{\mu}{4}\|\nabla u\|_{H^s}^2 + \frac{\gamma}{2}\|\nabla \omega\|_{H^s}^2 + \frac{\nu}{2}\|\nabla b\|_{H^s}^2. \quad (3.25)$$

Now we estimate the last term

$$|III_5| \leq 2\chi\|\Lambda^4(\nabla \times u)\|_2\|\Lambda^4\omega\|_2 \leq \frac{\chi}{2}\|\nabla u\|_{H^s}^2 + 2\chi\|\omega\|_{H^s}^2. \quad (3.26)$$

Summing up (3.24)(3.25)(3.26) with (3.22), we get

$$\frac{d}{dt}(\|u\|_{H^s}^2 + \|\omega\|_{H^s}^2 + \|b\|_{H^s}^2) + (\mu + \chi)\|\nabla u\|_{H^s}^2 + \gamma\|\nabla \omega\|_{H^s}^2 + \nu\|\nabla b\|_{H^s}^2 + \kappa\|\text{div}\omega\|_{H^s}^2 \leq C(\|u_0\|_{H^1}^2 + \|\omega_0\|_{H^1}^2 + \|b_0\|_{H^1}^2)(\|u\|_{H^1}^2 + \|\omega\|_{H^1}^2 + \|b\|_{H^1}^2). \quad (3.27)$$

Gronwall inequality gives us

$$d\int_0^t (\mu + \chi)\|\nabla u\|_{H^s}^2 + \gamma\|\nabla \omega\|_{H^s}^2 + \nu\|\nabla b\|_{H^s}^2 + \kappa\|\text{div}\omega\|_{H^s}^2 dt' \leq C(\|u_0\|_{H^1}^2 + \|\omega_0\|_{H^1}^2 + \|b_0\|_{H^1}^2) \exp(\int_0^t (\mu + \chi)\|\nabla u\|_{H^s}^2 + \gamma\|\nabla \omega\|_{H^s}^2 + \nu\|\nabla b\|_{H^s}^2 + \kappa\|\text{div}\omega\|_{H^s}^2 dt'). \quad (3.28)$$

Now we go on with the $H^1$ estimates of the solution $(u, \omega, b)$. Denote $H = \nabla \times u, I = \nabla \times \omega, J = \nabla \times b$, we take curl on both sides of (1.1), we get the following equation

$$\begin{cases}
\partial_t H - (\mu + \chi)\Delta H + u \cdot \nabla H - H \cdot \nabla u - b \cdot \nabla J + J \cdot \nabla b - \chi \nabla \times I = 0, \\
\partial_t I - \gamma \Delta I + 2\chi I + u \cdot \nabla I - H \cdot \nabla \omega - \chi \nabla \times H = 0, \\
\partial_t J - \nu \Delta J + u \cdot \nabla J - H \cdot \nabla b - b \cdot \nabla H + J \cdot \nabla u = 0
\end{cases} \quad (3.29)$$

which uses the fact $\nabla \times \nabla \text{div}\omega = 0$.

Multiplying the three equations with $(H, I, J)$ separately, integrating over $\mathbb{R}^3$ about the variable $x$, using integrating by parts and the divergence free condition of $u, b$, we obtain the fact

$$\int_{\mathbb{R}^3} (u \cdot \nabla) H \cdot H \, dx = \int_{\mathbb{R}^3} (u \cdot \nabla) I \cdot I \, dx = \int_{\mathbb{R}^3} (u \cdot \nabla) J \cdot J \, dx = 0,$$

$$\int_{\mathbb{R}^3} (b \cdot \nabla) J \cdot H \, dx + \int_{\mathbb{R}^3} (b \cdot \nabla) H \cdot J \, dx = 0,$$

$$\int_{\mathbb{R}^3} (\nabla \times H) \cdot I \, dx = \int_{\mathbb{R}^3} (\nabla \times I) \cdot H \, dx,$$
so we finally have
\[
\frac{1}{2} \frac{d}{dt} (\|H\|^2 + \|I\|^2 + \|J\|^2) + (\mu + \chi)\|\nabla H\|^2 + \gamma \|\nabla I\|^2 + \nu \|\nabla J\|^2 + 2\chi \|I\|^2
\]
\[
= \int_{\mathbb{R}^3} (H \cdot \nabla)u \cdot H dx + \int_{\mathbb{R}^3} (H \cdot \nabla)\omega \cdot Idx + \int_{\mathbb{R}^3} (J \cdot \nabla)u \cdot Jdx
\]
\[
- \int_{\mathbb{R}^3} (J \cdot \nabla)b \cdot H dx + \int_{\mathbb{R}^3} (H \cdot \nabla)b \cdot Jdx + 2\chi \int_{\mathbb{R}^3} (\nabla \times H) \cdot Idx
\]
\[
= IV_1 + IV_2 + IV_3 + IV_4 + IV_5 + IV_6
\] (3.30)

Let us first estimate $IV_1$ and $IV_3$, we use Littlewood-Paley decomposition to $u$ and dispose it in different frequencies.

\[
IV_1 = \sum_{j< -N} \int_{\mathbb{R}^3} (H \cdot \nabla)\Delta_j u \cdot H dx + \sum_{-N \leq j \leq N} \int_{\mathbb{R}^3} (H \cdot \nabla)\Delta_j u \cdot H dx
\]
\[
+ \sum_{j> N} \int_{\mathbb{R}^3} (H \cdot \nabla)\Delta_j u \cdot H dx,
\] (3.31)

For the first term, we have, by Hölder inequality, Beinstein inequality and (2.4) (2.5)

\[
|V_1| \leq \|H\|^2 \sum_{j< -N} \|\nabla \Delta_j u\| \leq C\|H\|^2 \sum_{j< -N} 2^{\frac{j}{2}} \|\Delta_j H\|_2 \leq C2^{-\frac{N}{2}} \|H\|_2^3
\] (3.32)

\[
|V_2| \leq \|H\|^2 \sum_{-N \leq j \leq N} \|\nabla \Delta_j u\|_\infty \leq C\|H\|^2 \sum_{-N \leq j \leq N} \|\Delta_j H\|_\infty
\] (3.33)

for $V_3$, using similar method along with interpolation inequality

\[
\|H\|_3 \leq C\|H\|_2^\frac{1}{2} \|\nabla H\|_2^\frac{1}{2},
\]

we get

\[
|V_3| \leq \|H\|^2 \sum_{j> N} \|\nabla \Delta_j u\|_3 \leq C\|H\|_3^2 \sum_{j> N} 2^{\frac{j}{2}} \|\Delta_j H\|_2
\]
\[
\leq C\|H\|^3_3 \left( \sum_{j> N} 2^{-\frac{j}{2}} \right)^\frac{1}{2} \left( \sum_{j> N} 2^{\frac{j}{2}} \|\Delta_j H\|_2^2 \right)^\frac{1}{2}
\]
\[
\leq C2^{-\frac{N}{2}} \|H\|_2 \|\nabla H\|_2^2.
\] (3.34)

Summing up (3.32)-(3.34), we have

\[
|IV_1| \leq C(2^{-\frac{N}{2}} \|H\|_2^3 + \|H\|^2 \sum_{-N \leq j \leq N} \|\Delta_j H\|_\infty + 2^{-\frac{N}{2}} \|H\|_2 \|\nabla H\|_2^2).
\] (3.35)
$IV_3$ can be treated in the same way, we decompose it as

$$IV_3 = \sum_{j < -N} \int_{\mathbb{R}^3} (J \cdot \nabla) \Delta_j u \cdot J dx + \sum_{-N \leq j \leq N} \int_{\mathbb{R}^3} (J \cdot \nabla) \Delta_j u \cdot J dx$$

$$+ \sum_{j > N} \int_{\mathbb{R}^3} (J \cdot \nabla) \Delta_j u \cdot J dx,$$

then obtain the estimate

$$|IV_3| \leq C \left(2^{-\frac{3}{2}N} \|J\|_2^2 \|H\|_2 + \|J\|_2^2 \sum_{-N \leq j \leq N} \|\Delta_j H\|_\infty + 2^{-\frac{7}{2}} \|J\|_2 \|\nabla J\|_2 \|\nabla H\|_2 \right). \quad (3.36)$$

Now we study $IV_2, IV_4, IV_5$, we decompose $H$ by using Littlewood-Paley theory, that is

$$IV_2 = \sum_{j < -N} \int_{\mathbb{R}^3} (\Delta_j H \cdot \nabla) \omega \cdot I dx + \sum_{-N \leq j \leq N} \int_{\mathbb{R}^3} (\Delta_j H \cdot \nabla) \omega \cdot I dx$$

$$+ \sum_{j > N} \int_{\mathbb{R}^3} (\Delta_j H \cdot \nabla) \omega \cdot I dx,$$

then

$$|IV_2| \leq C \left(2^{-\frac{3}{2}N} \|J\|_2^2 \|H\|_2 + \|J\|_2^2 \sum_{-N \leq j \leq N} \|\Delta_j H\|_\infty + 2^{-\frac{7}{2}} \|J\|_2 \|\nabla J\|_2 \|\nabla H\|_2 \right). \quad (3.37)$$

For $IV_4$ and $IV_5$, similarly we have

$$|IV_4| + |IV_5| \leq C \left(2^{-\frac{3}{2}N} \|J\|_2^2 \|H\|_2 + \|J\|_2^2 \sum_{-N \leq j \leq N} \|\Delta_j H\|_\infty \right.$$

$$+ 2^{-\frac{7}{2}} \|J\|_2 \|\nabla J\|_2 \|\nabla H\|_2 \right). \quad (3.38)$$

Simply using Young inequality, the last term $IV_6$ can be written as

$$|IV_6| \leq 2\chi \|\nabla \times H\|_2 \|I\|_2 \leq \frac{\chi}{2} \|\nabla H\|_2^2 + 2\chi \|I\|_2^2. \quad (3.39)$$

Summing up $\text{(3.35), (3.36), (3.37), (3.38), (3.39)}$ and taking the sum into $\text{(3.30)}$, by Young inequality, we get

$$\frac{d}{dt}(\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2) + (2\mu + \chi) \|\nabla H\|_2^2 + 2\gamma \|\nabla I\|_2^2 + 2\nu \|\nabla J\|_2^2$$

$$\leq C \left(2^{-\frac{3}{2}N} (\|H\|_2^3 + \|I\|_2^3 + \|J\|_2^3) \right) + \sum_{-N \leq j \leq N} \|\Delta_j H\|_\infty (\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2)$$

$$+ 2^{-\frac{7}{2}} (\|H\|_2 + \|I\|_2 + \|J\|_2) (\|\nabla H\|_2^2 + \|\nabla I\|_2^2 + \|\nabla J\|_2^2) \quad (3.40)$$

If we let $2^{-\frac{N}{2}} (\|H\|_2 + \|I\|_2 + \|J\|_2) \leq \min(\mu, \gamma, \nu)$, that is, if we choose

$$N \geq \left[ \frac{2}{\log 2} \log^+ \left( \frac{C}{\min(\mu, \gamma, \nu)} (\|H\|_2 + \|I\|_2 + \|J\|_2) \right) \right] + 1, \quad (3.41)$$
where \([a]\) stands for the integral parts of \(a \in \mathbb{R}\), \(\log^+(x) = \log(x + e)\), then we have

\[
\frac{d}{dt} (\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2) + (\mu + \chi) \|\nabla H\|_2^2 + \gamma \|\nabla I\|_2^2 + \nu \|\nabla J\|_2^2 \\
\leq C \sum_{-N \leq j \leq N} \|\Delta_j H\|_\infty (\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2) + C. \tag{3.42}
\]

Gronwall inequality gives us that

\[
\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2 \leq \exp \left( C \sum_{-N \leq j \leq N} \int_0^t \|\Delta_j H(t')\|_\infty dt' \right) \left( \sqrt{Ct} + \|H(0)\|_2 + \|I(0)\|_2 + \|J(0)\|_2 \right), \tag{3.43}
\]

which implies

\[
\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2 \leq \exp \left( C \log^+ (\|H\|_2^2 + \|I\|_2^2 + \|J\|_2^2) \sup_{j \in \mathbb{Z}} \int_0^t \|\Delta_j H(t')\|_\infty dt' \right) \left( \sqrt{Ct} + \|H(0)\|_2 + \|I(0)\|_2 + \|J(0)\|_2 \right). \tag{3.44}
\]

Denote

\[
\zeta(T) \triangleq \sup_{t \in [0,T)} (\|H(t)\|_2 + \|I(t)\|_2 + \|J(t)\|_2),
\]

then (3.24) can be reduced to be

\[
\zeta(T) \leq \exp \left( C \log^+ (\zeta(T)) \sup_{j \in \mathbb{Z}} \int_0^t \|\Delta_j H(t')\|_\infty dt' \right) \left( \sqrt{CT} + E(0) \right). \tag{3.45}
\]

We should point out that the above inequality still holds if the time interval \([0, T)\) is replaced by \([T - \varepsilon, T)\), that is

\[
\zeta(T) \leq \exp \left( C \log^+ (\zeta(T)) \sup_{j \in \mathbb{Z}} \int_{T-\varepsilon}^T \|\Delta_j H(t')\|_\infty dt' \right) \left( \sqrt{C(\varepsilon + \zeta(T - \varepsilon))} \right). \tag{3.46}
\]

Setting \(Z(T) \triangleq \log^+ (\zeta(T)) = \log(e + \zeta(T))\), thanks to (3.46), we have

\[
Z(T) \leq \log \left( \sqrt{C\varepsilon} + \zeta(T - \varepsilon) + e \right) + CZ(T) \sup_{j \in \mathbb{Z}} \int_{T-\varepsilon}^T \|\Delta_j (\nabla \times u)(t')\|_\infty dt', \tag{3.47}
\]

by the condition (1.4) of Theorem 1.1

\[
\lim_{\varepsilon \to 0} \sup_{j \in \mathbb{Z}} \int_{T-\varepsilon}^T \|\Delta_j (\nabla \times u)\|_\infty dt = \delta < M,
\]

we know that, when \(\varepsilon \to 0\), if we choose \(MC\) is small enough, then it has

\[
Z(T) \leq CZ(T - \varepsilon). \tag{3.48}
\]
On the other hand, by multiplying \((u, \omega, b)\), it can be easily derived from Magneto-micropolar fluid equation\((1.1)\) that
\[
\left\|(u)\right\|_2^2 + \left\|\omega\right\|_2^2 + \left\|b\right\|_2^2 + 2\mu \int_0^t \left\|\nabla u\right\|_2^2 dt' + 2\gamma \int_0^t \left\|\nabla \omega\right\|_2^2 dt' + 2\nu \int_0^t \left\|\nabla b\right\|_2^2 dt' + 2\kappa \int_0^t \left\|\text{div} \omega\right\|_2^2 dt' + 2\chi \int_0^t \left\|\omega\right\|_2^2 dt' \leq \left\|u_0\right\|_2^2 + \left\|\omega_0\right\|_2^2 + \left\|b_0\right\|_2^2.
\]
(3.49)

(3.49) along with (3.48) imply that
\[
\sup_{t \in [T-\varepsilon, T]} \left( \left\|u(t)\right\|_{H^1} + \left\|\omega(t)\right\|_{H^1} + \left\|b(t)\right\|_{H^1} \right)
\leq C \left( \left\|u(T-\varepsilon)\right\|_{H^1} + \left\|\omega(T-\varepsilon)\right\|_{H^1} + \left\|b(T-\varepsilon)\right\|_{H^1} \right).
\]
(3.50)

Hence by (3.50) and (3.28), we can get the \(H^s\) regularity at time \(t = T\), that is the smooth solution \((u, \omega, b)\) can be extended past time \(T\), that’s the end of the proof.\(\Box\)

References

[1] G. Ahmadi and M. Shahinpoor, Universal stability of magneto-micropolar fluid motions, Internat. J. Engrg. Sci.,12 (1974), 657C663.

[2] Beale, J.T., Kato, T., Majda,A., Remarks on the breakdown of smooth solutions for the 3-D Euler equations . Commu.Math.Phys. 94, 61-66 (1984)

[3] R.E.Caflisch, I.Klapper and G.Steele, Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD, Commun.Math.Phys.,184 (1997), 443-455,

[4] M.Cannon, Q.Chen and C.Miao, A losing estimate for the Ideal MHD equations with application to Blow-up criterion, SIAM J.Math.Anal.38(2007), 1847-1859.

[5] M.Cannon, C.X.Miao, N.Prioux and B.Q.Yuan, The cauchy problem for the magneto-hydrodynamic system, Self-similar solutions of nonlinear PDE, Banach Center Publications, Institute of mathematics, Polish Academy of Sciences, Warszawa 74(2006), 59-93.

[6] J.-Y.Chemin, Perfect Incompressible Fluids, Oxford University Press, New York, 1998.

[7] J.-Y.Chemin, and N.Lerner, Flot de champs de vecteurs non lipschitiziens et equations de Navier-Stokes, J.Diff. Equations , 121 (1995), 314-328.

[8] G. Duvaut and J. L. Lions, Inéquations en thermoelasticite et magnetohydrodynamique, Arch. Rational Mech. Anal.46 (1972), 241-279.

[9] A.C.Eringen, Theory of micropolar fluids, J.Math.Mech., 16(1966), 1-18.
[10] G.P. Galdi and S. Rionero, *A note on the existence and uniqueness of solutions of the micropolar fluid equations*, Internat. J. Engrg. Sci., 15(1977), 105-108.

[11] C. He and Z. P. Xin, *On the regularity of weak solutions to the magnetohydrodynamic equations*, J. Diff. Eqs., 213(2005), 235-254.

[12] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pur. Appl. Math., 41(1988), 891-907.

[13] G. Lukaszewicz, *On nonstationary flows of asymmetric fluids*, Rend. Accad. Naz. Sci. XL Mem. Mat. (5) 12(1988), no.1, 83-97. MR 90f:35165. Zbl 668.76045.

[14] G. Lukaszewicz, *On the existence, uniqueness and asymptotic properties for solutions of flows of asymmetric fluids*, Rend. Accad. Naz. Sci. XL Mem. Mat. (5) 13(1989), no.1, 105-120. MR 91d:35174. Zbl 692.76020.

[15] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, 2002.

[16] E. E. Ortega-Torres and M. A. Rojas-Medar, *Magneto-micropolar fluid motion: Global existence of strong solutions*, Abstract and Applied Analysis, 4(1999), 109-125.

[17] M. A. Rojas-Medar, *Magneto-micropolar fluid motion: Existence and uniqueness of strong solution*, Math. Nachr. 188(1997), 301-319.

[18] M. A. Rojas-Medar and J. L. Boldrini, *Magneto-micropolar fluid motion: Existence of weak solutions*, Revista Matematica Complutense, 11(1998), 443-460.

[19] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. 36 (1983), 635-664.

[20] Triebel H., *Theory of Function spaces*, Monograph in mathematics, Vol. 78, Basel: Birkhauser Verlag, 1983.

[21] Wu J., *Bounds and new approaches for the 3D MHD equations*, J. Nonlinear Sci., 12 (2002), 395-413.

[22] Wu J., *Regularity results for weak solutions of the 3D MHD equations*, Discrete Cont. Dyn. S., 10 (2004), 543-556.

[23] Wu J., *Regularity criteria for the generalized MHD equations*, preprint, ppl-26.

[24] N. Yamaguchi, *Existence of global strong solution to the micropolar fluid system in a bounded domain*, Math. Meth. Appl. Sci., 28(2005), 1507-1526.

[25] B. Q. Yuan *The regularity of weak solutions to magneto-micropolar fluid equations*, submitted.
[26] Z.Zhang and X.Liu, *On the blow-up criterion of smooth solutions to the 3D Ideal MHD equations*, Acta Math.Appl.Sinica, E, 20(2004), 695-700.

[27] Y.Zhou, *Remarks on regularities for the 3D MHD equations*, Discrete.Contin.Dynam.Systems, 12(2005), 881-886.