Approximation of Functions in Besov Space

Hare Krishna Nigam and Supriya Rani

Abstract. In the present paper, we establish a theorem on best approximation of a function $g \in B^p_\lambda(L^r)$ of its Fourier series. Our main theorem generalizes some known results of this direction of work. Thus, the results of [10], [26] and [27] become the particular case of our main Theorem 3.1.

1 Introduction

The degree of approximation of the functions in Lipschitz spaces and Hölder spaces using single and product summability means has been studied by the authors [7, 8, 9, 15, 16, 18, 19, 20, 21, 22, 23, 25, 28]. This motivates us to study the degree of approximation of a function in more generalized function space. Therefore, in this paper, we study the degree of approximation of a function $g$ in Besov space using Hausdorff-generalized Nörlund means of its Fourier series. It can be noted that Besov space generalizes different Sobolev spaces, Lipschitz spaces and generalized Hölder spaces [13]. Besov space can also be used to study regularity properties of the functions.

Let $C_{2\pi} := C[0, 2\pi]$ denote the Banach space of all $2\pi$-periodic continuous functions $g$ defined on $[0, 2\pi]$ under the supremum norm.

Let $L^r := L^r[0, 2\pi] := \left\{ g : [0, 2\pi] \rightarrow \mathbb{R} : \int_0^{2\pi} |g(z)|^r dz < \infty \right\}, r \geq 1$

be the space of all $2\pi$-periodic, integrable functions $L^r$-norm of the function $g$ is given by

$$\|g\|_r := \begin{cases} \left( \frac{1}{2\pi} \int_0^{2\pi} |g(z)|^r dz \right)^{1/r} & \text{for } 1 \leq r < \infty \\ \text{ess sup}_{0 < z < 2\pi} |g(z)| & \text{for } r = \infty. \end{cases}$$

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Corresponding author: Hare Krishna Nigam.
The modulus of continuity of a function \( g \in L^r \) is defined by
\[
w(g; l) := \sup_{z, z + h \in [0, 2\pi], |h| < l} |g(z + h) - g(z)|.
\] (1.1)

The integral modulus of continuity of the first order of a function \( g \in L^r \) is defined \([3]\) by
\[
w_1(g; l)_r := \sup_{|h| < l, z \in \mathbb{R}} \|g(z + h) - g(z)\|_r.
\] (1.2)

The integral moduli of continuity of the second order (modulus of smoothness) of a function \( g \in L^r \) is defined \([2]\) by
\[
w_2(g; l)_r = \sup_{0 < h \leq l, z \in \mathbb{R}} \|g(z + h) + g(z - h) - 2g(z)\|_r.
\] (1.3)

The \( j \)th order modulus of smoothness of a function \( g \in L^r \) is defined \([1]\) by
\[
w_j(g, l)_r := \sup_{0 < h \leq l} \|\Delta^j_h(g, \cdot)\|_r, \quad l > 0,
\] (1.4)
where
\[
\Delta^j_h(g, z) := \sum_{\rho=0}^{j} (-1)^{j-\rho} \binom{j}{\rho} g(z + \rho h), \quad j \in \mathbb{N}.
\] (1.5)

**Remark 1.**

(i) If \( r = \infty, j = 1 \) and \( g \) being a continuous function, then \( w_j(g, l)_r \) reduces to \( w(g, l) \).

(ii) If \( 0 < r < \infty, j = 1 \) and \( g \) being a continuous function, then \( w_j(g, l)_r \) reduces to \( w_1(g, l)_r \).

(iii) If \( g \in C_{2\pi} \) and \( w(g, l) = O(l^\lambda), 0 < \lambda \leq 1 \), then \( g \in Lip \lambda \).

(iv) If \( g \in L^r, 0 < r < \infty \) and \( w(g, l)_r = O(l^\lambda), 0 < \lambda \leq 1 \), then \( g \in Lip(\lambda, r) \).

(v) If \( r = \infty \), then \( Lip(\lambda, r) \) class reduces to \( Lip \lambda \).

**Note 1.** From Remark 1(iv) and 1(v), we write
\[
Lip(\lambda) \subset Lip(\lambda, r).
\]

Let \( \lambda > 0, j > \lambda \) i.e. \( j = [\lambda] + 1 \), where \( j \) being smallest integer. For \( g \in L^r \), if
\[
w_j(g, l)_r = O(l^\lambda), \quad l > 0,
\] (1.6)
then \( g \in Lip^*(\lambda, r) \) and its semi-norm is given by
\[
|g|_{Lip^*(\lambda, r)} = \sup_{l>0} (l^{-\lambda} w_j(g, l)r),
\]
where \( Lip^*(\lambda, r) \) is a generalized Lipschitz class of function \( g \).
Thus,
\[
Lip(\lambda, r) \subseteq Lip^*(\lambda, r).
\]
For \( 0 < \lambda \leq 1 \), let
\[
H_\lambda := \{ g \in C_{2\pi} : w(g, l) = O(l^\lambda) \},
\]
where \( H_\lambda \) is a Banach space with the norms
\[
\|g\|_\lambda = \|g\|_C + \sup_{l>0} (l^{-\lambda} w(l)) \quad \text{for} \quad 0 < \delta \leq \lambda < 1
\]
and
\[
\|g\|_0 = \|g\|_C.
\]
Thus, we observe that
\[
H_\lambda \subseteq H_\delta \subseteq C_{2\pi} \quad \text{for} \quad 0 < \delta \leq \alpha < 1([28]).
\]
The metric induced by the norm \( \| \cdot \|_\lambda \) on \( H_\lambda \) is called the Hölder metric.
For \( 0 < \lambda \leq 1 \), \( 0 < r \leq \infty \), let
\[
H_{\lambda, r} := H_{\lambda, r}[0, 2\pi] = \{ g \in L^r : w(g, l)r = O(l^\lambda) \},
\]
where \( H_{\lambda, r} \) is also a Banach space with the norm \( \| \cdot \|_{\lambda, r} \) defined by
\[
\|g\|_{\lambda, r} = \|g\|_r + \sup_{l>0} (l^{-\lambda} w(g, l)r) \quad \text{for} \quad 0 < \lambda \leq 1
\]
and
\[
\|g\|_{0, r} = \|g\|_r.
\]
Then \( H_{\lambda, r} \) is a Banach space for \( r \geq 1 \) and a complete \( r \)-normed space ([14], p. 87) for \( 0 < r < 1 \).
Thus,
\[
H_{\lambda, r} \subseteq H_{\delta, r} \subseteq L^r \quad \text{for} \quad 0 < \delta \leq \lambda \leq 1([9]).
\]
For $\lambda > 0$ and let $j > \lambda$ i.e. $j = [\lambda] + 1$. For $0 < r, q \leq \infty$, the Besov space $B^\lambda_q(L^r)$ is a collection of all $2\pi$-periodic function $g \in L^r$ such that

$$|g|_{B_q^\lambda(L^r)} := \|w_j(g, \cdot)\|_{\lambda,q} = \begin{cases} \left( \int_0^{2\pi} (t^{-\lambda}w_j(g, l)_r)^q \frac{dl}{T} \right)^{\frac{1}{q}}, & 0 < q < \infty, \\
\sup_{l>0} (t^{-\lambda}w_j(g, l)_r), & q = \infty, \end{cases}$$

(1.7)
is finite ([24], p. 237) for $2\pi$ - periodic function $g$ ([1], p. 54).

It is further observed that (1.7) is a semi-norm if $1 \leq r, q \leq \infty$ and a quasi semi-norm in other cases ([1], p. 55).

The quasi-norm for Besov space is given by

$$\|g\|_{B_q^\lambda(L^r)} := \|g\|_r + |g|_{B_q^\lambda(L^r)} = \|g\|_r + \|w_j(g, \cdot)\|_{\lambda,q}.$$

**Note 2.**

(i) If $0 < \lambda < 1$, then the Besov space $B^\lambda_\infty(L^r)$ reduces to the $H_{\lambda,r}$ ([9]).

(ii) If $r = \infty = q$ and $0 < \lambda < 1$, the Besov space $B^\lambda_\infty(L^r)$ reduces to the space $H_{\lambda}$ ([28]).

The $m$-order error approximation of a function $g \in C_{2\pi}$ is defined by $E_m(g) := \inf_{t_m} \|g - t_m\|$ where $t_m$ is a trigonometric polynomial of degree $m$ ([3]).

If $E_m(g) \rightarrow 0$ as $m \rightarrow \infty$, $E_m(g)$ is said to be the best approximation of $g$ ([3]).

## 2 Definitions

The Hausdorff matrix $H \equiv (h_{m,j})$ is an infinite lower triangular matrix defined by,

$$h_{m,j} = \begin{cases} \binom{m}{j} \Delta^{m-j} \mu_j, & 0 \leq j \leq m, \\
0, & j > m, \end{cases}$$

where $\Delta$ is a forward operator defined by $\Delta \mu_m = \mu_m - \mu_{m+1}$ and $\Delta^{j+1} \mu_m = \Delta(\Delta^j \mu_m)$ ([6]).

A Hausdorff matrix $H$ is regular iff $\int_0^1 |\gamma(y)| < \infty$, where the mass function $\gamma(y)$ is continuous at $y = 0$ and belongs to $\text{BV}[0, 1]$ such that $\gamma(0+) = 0, \gamma(1) = 1$; and for $0 < y < 1$, $\gamma(y) = [\gamma(y + 0) + \gamma(y - 0)]/2$ [4, 11]. Thus $\{\mu_m\}$, known as moment sequence, has the representation

$$\mu_m = \int_0^1 y^m d\gamma(y).$$
The Hausdorff means of a trigonometric Fourier series of \( g \) is defined by
\[
H_m(g; z) = \sum_{j=0}^{m} h_{m,j} s_j(g; z), \quad \forall \ m \geq 0.
\]

The series is said to be summable to \( s \) by Hausdorff means, if \( H_m(g; z) \to s \) as \( m \to \infty \) and we denote Hausdorff means by \( \Delta_H \) throughout the paper.

**Example 1.**

(i) If
\[
h_{m,j} = \begin{cases} 
\binom{m}{j} q^{m-j} (1+q)^{m-j}, & 0 \leq j \leq m, \\
0, & j > m,
\end{cases}
\]
then the Hausdorff matrix \( H \equiv (h_{m,j}) \) reduces to \( (E, q) \) matrix (Euler matrix of order \( q > 0 \)) and defines the corresponding \( (E, q) \) means by
\[
E^q_m(g; z) := \frac{1}{(1+q)^m} \sum_{j=0}^{m} \binom{m}{j} q^{m-j} s_j(g; z).
\]

(ii) If \( \mu_m = \frac{1}{m+1} \) then the Hausdorff matrix \( H \equiv (h_{m,j}) \) reduces to \( (C, 1) \) matrix (Cesàro matrix of order 1) and defines the corresponding means by
\[
H_m(g; z) := \frac{1}{(m+1)} \sum_{j=0}^{m} s_j(g; z).
\]

Let \( \{p_m\} \) and \( \{q_m\} \) be the sequence of constants, real or complex, such that
\[
P_m = p_0 + p_1 + p_2 + \cdots + p_m = \sum_{v=0}^{m} p_v \to \infty, \quad \text{as} \quad m \to \infty
\]
\[
Q_m = q_0 + q_1 + q_2 + \cdots + q_m = \sum_{v=0}^{m} q_v \to \infty, \quad \text{as} \quad m \to \infty
\]
\[
R_m = p_0 q_m + p_1 q_{m-1} + p_2 q_{m-2} + \cdots + p_m q_0 = \sum_{v=0}^{m} p_v q_{m-v} \to \infty, \quad \text{as} \quad m \to \infty.
\]

Given two sequences \( \{p_m\} \) and \( \{q_m\} \) convolution \( (p \ast q) \) is defined as
\[
R_m = (p \ast q)_m = \sum_{j=0}^{m} p_{m-j} q_j.
\]

We write
\[
p^p_m = \frac{1}{R_m} \sum_{j=0}^{m} p_{m-j} q_j s_j.
\]
If \( R_m \neq 0 \), for all \( m \), the generalized Nörlund transform of the sequence \( \{s_m\} \) is the sequence \( \{t^{p,q}_m\} \). If \( t^{p,q}_m \to s \), as \( m \to \infty \), then the series \( \sum_{m=0}^{\infty} a_m \) or sequence \( \{s_m\} \) is summable to \( s \) by generalized Nörlund method and is denoted by \( s_m \to s(N^{p,q}) \).

The necessary and sufficient condition for \( (N^{p,q}) \) method to be regular are

\[
\sum_{j=0}^{m} |p_{m-j}q_j| = O(|R_m|)
\]

and \( p_{m-j} = o(|R_m|) \), as \( m \to \infty \) for every fixed \( j \geq 0 \), for which \( q_j \neq 0 \)\((17)\).

If the method \( \Delta_H \) is superimposed on the \( N^{p,q} \) method, another new method of summability \( \Delta_H N^{p,q} \) is obtained.

The Hausdorff transform of \( N^{p,q} \) transform is defined as \( \Delta_H \) \( N^{p,q} \) product transform of the partial sum \( s_m \), which can be given by

\[
t^{\Delta_H N^{p,q}}_m = \sum_{j=0}^{m} h_{m,j} t^{p,q}_j = \sum_{j=0}^{m} h_{m,j} \frac{1}{R_j} \sum_{v=0}^{j} p_{j-v} q_v s_v.
\]

If \( t^{\Delta_H N^{p,q}}_m \to s \) as \( m \to \infty \) then the series \( \sum_{m=0}^{\infty} a_m \) or the sequence \( \{s_m\} \) is summable to \( s \) by \( \Delta_H N^{p,q} \) means.

Now, we define the regularity of \( \Delta_H N^{p,q} \) method.

\[
s_m \to s \iff t^{p,q}_m \to s, \quad \text{as} \quad m \to \infty \quad \text{so} \quad N^{p,q} \quad \text{method is regular,}
\]

\[
\Rightarrow \Delta(t^{p,q}_m) = t^{\Delta_H N^{p,q}}_m \to s, \quad \text{as} \quad m \to \infty \quad \text{so} \quad \Delta_H \quad \text{method is regular,}
\]

\[
\Rightarrow (\Delta_H N^{p,q}) \quad \text{method is regular.}
\]

**Remark 2.**

(i) \( \Delta_H N^{p,q} \) means reduces to \( E_q N^{p,q} \) means if \( h_{m,j} = \begin{cases} \binom{m}{j} \frac{q^{m-j}}{(1+q)^m}, & \text{if } 0 \leq j \leq m, \\ 0, & \text{if } j > m. \end{cases} \)

(ii) \( \Delta_H N^{p,q} \) means reduces to \( C_1 N^{p,q} \) means if \( h_{m,j} = \begin{cases} \frac{1}{m+1}, & \text{if } 0 \leq j \leq m, \\ 0, & \text{if } j > m. \end{cases} \)

(iii) \( \Delta_H N^{p,q} \) means reduces to \( \Delta_H N^{p,m} \) means if \( q_m = 1, \forall m. \)

(iv) \( \Delta_H N^{p,q} \) means reduces to \( \Delta_H \tilde{N}^{q_m} \) means if \( p_m = 1, \forall m. \)
(v) $\Delta_H N^{p,q}$ means reduces to $\Delta_H C_\alpha$ means if $p_m = \left(\frac{m+\alpha-1}{\alpha-1}\right)$, $\alpha > 0$ and $q_m = 1$, $\forall \ m$.

**Remark 3.** The above cases (i) and (ii) of Remark 2 can be further reduced as

(i) $E_q N^{p,q}$ means reduces to $E_q N^{p_m}$ means if $q_m = 1$, $\forall \ m$.

(ii) $E_q C_\alpha$ means reduces to $E_q C_\alpha$ means if $p_m = \left(\frac{m+\alpha-1}{\alpha-1}\right)$, $\alpha > 0$ and $q_m = 1$, $\forall \ m$.

(iii) $E_q \tilde{N}^{q_m}$ means reduces to $E_q \tilde{N}^{q_m}$ means if $p_m = 1$, $\forall \ m$.

(iv) $E_q N^{p,q}$ means reduces to $E_q C_\alpha$ means if $p_m = \left(\frac{m+\alpha-1}{\alpha-1}\right)$, $\alpha > 0$ and $q_m = 1$, $\forall \ m$.

(v) $C_1 N^{p,q}$ means reduces to $C_1 N^{p_m}$ means if $q_m = 1$, $\forall \ m$.

(vi) $C_1 \tilde{N}^{q_m}$ means reduces to $C_1 \tilde{N}^{q_m}$ means if $p_m = 1$, $\forall \ m$.

### 3 Main Theorems

**Theorem 3.1.** If $g$ is a $2\pi$-periodic and Lebesgue integrable function, then for $0 \leq \delta < \lambda < 2$, the best approximation of $g$ in $B^\lambda_q(L^r)$, $r \geq 1, 1 < q \leq \infty$ space using $\Delta_H N^{p,q}$ means, is given by

$$
\|t^\Delta_H N^{p,q} m(z) - g(z)\| = \begin{cases} 
O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\frac{1}{q}}}ight) + O\left(\frac{1}{(m+1)^{\lambda-s}}\right) & ; \ 1 < q < \infty \\
O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-s}}\right) & ; \ q = \infty.
\end{cases}
$$

### 4 Lemmas

**Lemma 4.1.** Let $K_m^{\Delta_H N^{p,q}}(\eta) := \int_0^1 M_m(\eta)d\gamma(y)$ where

$$
M_m(\eta) := \left[ \sum_{j=0}^m \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^j p_{j-\nu} q^{\nu} \sin\left(\nu + \frac{1}{2}\eta\right) \frac{\eta}{2\sin\frac{\eta}{2}} \right\} \right],
$$

then

$$
K_m^{\Delta_H N^{p,q}}(\eta) = \begin{cases} 
O(m+1), & 0 \leq \eta \leq \frac{1}{(m+1)}; \\
O\left(\frac{1}{\eta}\right), & \frac{1}{(m+1)} \leq \eta \leq \pi.
\end{cases}
$$
Proof. For $0 \leq \eta \leq \frac{1}{m+1}$, we have $m\eta \leq m \sin \eta$, then

$M_m(\eta)$

\[
= \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{\sin(\nu + \frac{1}{2})\eta}{2 \sin \frac{\eta}{2}} \right\} \\
\leq \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{(2\nu+1)\sin \frac{\eta}{2}}{\sin \frac{\eta}{2}} \right\} \\
\leq \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} (2\nu+1) \right\} \\
= \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \frac{1}{R_j} \{ p_j q_0 (2j + 1) + p_{j-1} q_1 (2j + 1) + \cdots + p_0 q_j (2j + 1) \} \\
\leq \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \frac{1}{R_j} \{ p_j q_0 (2j + 1) + p_{j-1} q_1 (2j + 1) + \cdots + p_0 q_j (2j + 1) \} \\
= \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \frac{1}{R_j} (2j + 1) \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \\
= \frac{1}{2} \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \frac{1}{R_j} (2j + 1) O(|R_j|) \\
= O \left\{ \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} (2j + 1) \right\} \\
= O \left[ 2 \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \cdot j + \sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \right].
\]

Now, solving first term of (4.1),

\[
\sum_{j=0}^{m} \binom{m}{j} y^j (1-y)^{m-j} \cdot j = (1-y)^m \sum_{j=0}^{m} \binom{m}{j} \left( \frac{y}{1-y} \right)^j \cdot j \\
= (1-y)^m \sum_{j=0}^{m} \binom{m}{j} p^j \cdot j,
\]

where $\frac{y}{1-y} = p \Rightarrow 1 + p = \frac{1}{1-y}$,

\[
\sum_{j=0}^{m} \binom{m}{j} p^j \cdot j = \binom{m}{0} p^0 \cdot 0 + \binom{m}{1} p^1 \cdot 1 + \binom{m}{2} p^2 \cdot 2 + \cdots + \binom{m}{m} p^m \cdot m
\]
We know that
\[(1 + p)^m = \binom{m}{0} + \binom{m}{1} \cdot p + \binom{m}{2} \cdot p^2 + \cdots + \binom{m}{m} \cdot p^m.\] By differentiating with respect to \(p\), we have
\[m(1 + p)^{m-1} = 0 + \binom{m}{1} + \binom{m}{2} \cdot 2p + \binom{m}{3} \cdot 3p^2 + \cdots + \binom{m}{m} \cdot mp^{m-1}.\]
Multiplying above equation by \(p\) on both side, we have
\[mp(1 + p)^{m-1} = \binom{m}{1} + \binom{m}{2} \cdot 2p^2 + \binom{m}{3} \cdot 3p^3 + \cdots + \binom{m}{m} \cdot mp^m.\] Now, solving second term of (4.1),
\[
\sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \cdot j = (1 - y^m) \cdot \frac{y^0}{1-y} \cdot \frac{1}{1-y^{m-1}}.
\] Now, from (4.2) and (4.3), we have
\[
M_m(\eta) = O(2my + 1).
\]
Thus,

\[ K_m^{\Delta H_{np,q}}(\eta) = \int_0^1 M_m(\eta)d\gamma(y) \]

\[ = O(1) \int_0^1 (2my + 1) dy \]

\[ = O(m + 1). \]

For \( \frac{1}{m+1} \leq \eta \leq \pi \), by Jordan's lemma we have, \( \sin \frac{\eta}{2} \geq \frac{\eta}{\pi} \) and \( \sin n\eta \leq 1 \). Thus,

\[ M_m(\eta) = \left[ \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \right] \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{\sin(\nu + \frac{1}{2})\eta}{2 \sin \frac{\eta}{2}} \right\} \]

\[ \leq \frac{1}{2} \left[ \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \right] \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{1}{\pi} \right\} \]

\[ \leq \frac{\pi}{2\eta} \left[ \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \right] \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \right\} \]

\[ = \frac{\pi}{2\eta} \left[ \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \right] \left\{ \frac{1}{R_j} O(|R_j|) \right\} \]

\[ = \frac{\pi}{2\eta} \left[ \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \right] O(1) \]

\[ = O \left( \frac{\pi}{2\eta} \right) \left[ \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} \right] \]

\[ = O \left( \frac{1}{\eta} \right) \text{ since } \sum_{j=0}^{m} \binom{m}{j} y^j (1 - y)^{m-j} = 1. \]

Thus,

\[ K_m^{\Delta H_{np,q}}(\eta) = \int_0^1 M_m(\eta)d\gamma(y) \]

\[ = \int_0^1 O \left( \frac{1}{\eta} \right) dy \]

\[ = O \left( \frac{1}{\eta} \right) \int_0^1 dy \]

\[ = O \left( \frac{1}{\eta} \right) . \]
Lemma 4.2. ([12]) Let $1 \leq r \leq \infty$ and $0 < \lambda < 2$. If $g \in L^r$ then for $0 < l, \eta \leq \pi$:

(i) $\|\Phi(\cdot, l, \eta)\|_r \leq 4w_j(g, l)_r$,

(ii) $\|\Phi(\cdot, l, \eta)\|_r \leq 4w_j(g, \eta)_r$,

(iii) $\|\Phi(\eta)\|_r \leq 2w_j(g, \eta)_r$,

where $j = [\lambda] + 1$.

Lemma 4.3. Let $0 \leq \delta < \lambda < 2$. If $g \in B^\lambda_q(L^r)$, $r \geq 1$, $1 < q < \infty$, then

$$
(i) \int_0^\pi |K_{m-H}^N\Phi^q(\eta)| \left( \int_0^\eta \frac{\|\Phi(\cdot, l, \eta)\|_q^q}{l^{\delta q}} \frac{dl}{l} \right)^{\frac{1}{q}} \, d\eta = O(1) \left\{ \int_0^\pi \eta^{\lambda - \delta} \frac{|K_{m-H}^N\Phi^q(\eta)|^{\frac{q}{q-1}}}{\eta^{\frac{q-1}{q}}} \, d\eta \right\}^{1-(1/q)},$

$$
(ii) \int_0^\pi |K_{m-H}^N\Phi^q(\eta)| \left( \int_\eta^\pi \frac{\|\Phi(\cdot, l, \eta)\|_q^q}{l^{\delta q}} \frac{dl}{l} \right)^{\frac{1}{q}} \, d\eta = O(1) \left\{ \int_0^\pi \eta^{\lambda - \delta + (1/q)} \frac{|K_{m-H}^N\Phi^q(\eta)|^{\frac{q}{q-1}}}{\eta^{\frac{q-1}{q}}} \, d\eta \right\}^{1-(1/q)}.
$$

Proof. This Lemma can be proved along the same lines of the proof of Lemma 1 of [12].

Lemma 4.4. ([12]) Let $0 \leq \delta < \lambda < 2$. If $g \in B^\lambda_q(L^r)$, $r \geq 1$, $q = \infty$, then

$$
\sup_{0 < l, \eta \leq \pi} (l^{-\delta} \|\Phi(\cdot, l, \eta)\|_r) = O(\eta^{\lambda - \delta}). \quad (4.6)
$$

5 Proof of the Main theorem

Proof. Following [5], $s_m(g, z)$ of Fourier series is given by

$$
s_m(g; z) - g(z) = \frac{1}{2\pi} \int_0^\pi \phi_z(\eta) \frac{\sin(m + \frac{1}{2})\eta}{\sin \frac{\eta}{2}} \, d\eta.
$$

Denoting the $N^{p,q}$ summability transform of $s_m(g; z)$ by $t_m^{p,q}(z)$, we get

$$
t_m^{p,q}(z) - g(z) = \sum_{j=0}^m t_j^{p,q} \{s_j(g; z) - g(z)\}
$$
\[
\begin{aligned}
&= \frac{1}{\pi} \int_0^\pi \left( \phi_z(\eta) \sum_{j=0}^{\infty} t_{p,q}^j \frac{\sin \left( m + \frac{1}{2} \right) \eta}{2 \sin \frac{\eta}{2}} \right) d\eta \\
&= \left\{ \frac{1}{2\pi} \int_0^\pi \phi_z(\eta) \sum_{j=0}^{\infty} \left( \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{\sin \left( m + \frac{1}{2} \right) \eta}{\sin \frac{\eta}{2}} \right) \right\} d\eta.
\end{aligned}
\]

**The Hausdorff transform of** \(t_m^{p,q}(z)\) i.e., \(\Delta_H^{N,p,q}\) transform of \(s_m(g ; z)\) denoted by \(t_m^{\Delta_H N,p,q}\), is given by

\[
t_m^{\Delta_H N,p,q}(z) - g(z)
\]

\[
= \sum_{j=0}^{\infty} h_{m,j} \left\{ t_m^{p,q}(z) - g(z) \right\}
\]

\[
= \sum_{j=0}^{\infty} \binom{m}{j} \Delta^{m-j} \mu_j \left\{ t_m^{p,q}(z) - g(z) \right\}
\]

\[
= \sum_{j=0}^{\infty} \binom{m}{j} \Delta^{m-j} \mu_j \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \right\}
\]

\[
= \frac{1}{\pi} \int_0^\pi \phi_z(\eta) \sum_{j=0}^{\infty} \binom{m}{j} \Delta^{m-j} \mu_j \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{\sin \left( m + \frac{1}{2} \right) \eta}{\sin \frac{\eta}{2}} \right\} d\eta
\]

\[
= \frac{1}{2\pi} \int_0^\pi \phi_z(\eta) \sum_{j=0}^{\infty} \binom{m}{j} \int_0^1 y^j (1 - y)^{m-j} d\gamma(y) \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{\sin \left( m + \frac{1}{2} \right) \eta}{\sin \frac{\eta}{2}} \right\} d\eta
\]

\[
= \frac{1}{2\pi} \int_0^\pi \phi_z(\eta) \left\{ \int_0^1 \sum_{j=0}^{\infty} \binom{m}{j} y^j (1 - y)^{m-j} \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{\sin \left( m + \frac{1}{2} \right) \eta}{\sin \frac{\eta}{2}} \right\} d\gamma(y) \right\} d\eta
\]

\[
= \int_0^\pi \phi_z(\eta) K_m^{\Delta_H N,p,q}(\eta) d\eta.
\]

**Let**

\[
l_m(z) := t_m^{\Delta_H N,p,q}(z) - g(z) = \frac{1}{\pi} \int_0^\pi \phi_z(\eta) K_m^{\Delta_H N,p,q}(\eta) d\eta,
\]

**where**

\[
K_m^{\Delta_H N,p,q}(\eta) = \int_0^1 \sum_{j=0}^{\infty} \binom{m}{j} y^j (1 - y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{\sin \left( m + \frac{1}{2} \right) \eta}{\sin \frac{\eta}{2}} \right\} d\gamma(y)
\]

\[
= \int_0^1 M_m(\eta) d\gamma(y),
\]

**where**

\[
M_m(\eta) = \sum_{j=0}^{\infty} \binom{m}{j} y^j (1 - y)^{m-j} \left\{ \frac{1}{R_j} \sum_{\nu=0}^{j} p_{j-\nu} q_{\nu} \frac{\sin \left( m + \frac{1}{2} \right) \eta}{\sin \frac{\eta}{2}} \right\}.
\]
We write,

\[ \Phi(z, l, \eta) = \begin{cases} 
\phi_{z+l}(\eta) - \phi_z(\eta), & 0 < \lambda < 1, \\
\phi_{z+l}(\eta) + \phi_{z-l}(\eta) - 2\phi_z(\eta), & 1 \leq \lambda < 2.
\end{cases} \]

and

\[ \mathcal{L}_m(z, l) = \begin{cases} 
l_m(z + l) - l_m(z), & 0 < \lambda < 1, \\
l_m(z + l) + l_m(z - l) - 2l_m(z), & 1 \leq \lambda < 2.
\end{cases} \]

Now, we have

\[ \mathcal{L}_m(z, l) = \frac{1}{\pi} \int_0^\pi K_m^{\Delta_H N^{p,q}}(\eta) \Phi(z, l, \eta) d\eta \quad \text{and} \quad \omega_j(l_m, l)_r = \| \mathcal{L}_m(\cdot, l) \|_r. \]

**Case I**: For \(1 < q < \infty, r \geq 1, 0 \leq \delta < \lambda < 2\).

By definition, we have

\[ \| l_m(\cdot) \|_{B^\delta_{q}(L^r)} = \| l_m(\cdot) \|_r + \| w_j(l_m, \cdot) \|_{\delta,q}. \tag{5.2} \]

Using generalized Minkowski’s inequality [3], Lemma 4.2 (iii) and (5.1), we have

\[ \| l_m(\cdot) \|_r \leq \frac{1}{\pi} \int_0^\pi \| \phi(\eta) \|_r | K_m^{\Delta_H N^{p,q}}(\eta) | d\eta \]

\[ \leq \frac{2}{\pi} \int_0^\pi w_j(g, \eta)_r | K_m^{\Delta_H N^{p,q}}(\eta) | d\eta. \]

Using Hölder’s inequality and definition of Besov space, we get

\[ \| l_m(\cdot) \|_r \leq 2 \left\{ \int_0^\pi \left( | K_m^{\Delta_H N^{p,q}}(\eta) | \eta^{\lambda + q - 1} \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \times \left\{ \int_0^\pi \left( w_j(g, \eta)_r \right)^q d\eta \right\}^{q^{-1}} \]

\[ = O(1) \left\{ \int_0^\pi \left( | K_m^{\Delta_H N^{p,q}}(\eta) | \eta^{\lambda + q - 1} \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \]

\[ = O(1) \left\{ \left( \int_0^{\frac{1}{m+1}} + \int_{\frac{1}{m+1}}^\pi \right) \left( | K_m^{\Delta_H N^{p,q}}(\eta) | \eta^{\lambda + q - 1} \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \]

\[ := O(1) [ I_1 + I_2 ]. \tag{5.3} \]

Using Lemma 4.1 for \(0 \leq \eta \leq \frac{1}{m+1}\), we get

\[ I_1 = \left\{ \int_0^{\frac{1}{m+1}} \left( | K_m^{\Delta_H N^{p,q}}(\eta) | \eta^{\lambda + q - 1} \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \]

\[ = O(m+1) \left\{ \int_0^{\frac{1}{m+1}} \left( \eta^{\lambda + q - 1} \right)^{\frac{q}{q-1}} d\eta \right\}^{1-q^{-1}} \]
\[ O(m + 1) \left\{ \int_0^{\frac{1}{m+1}} \eta^{q/(q-1)} (\lambda + q^{-1}) \, d\eta \right\}^{1-q^{-1}} = O((m + 1)^{-\lambda}). \] (5.4)

By using Lemma 4.1 for \( \frac{1}{m+1} \leq \eta \leq \pi \), we get
\[ I_2 = \left\{ \int_{\frac{1}{m+1}}^{\pi} (K^\Delta_{mH}^N p, q (\eta)) |\eta^{\lambda + q^{-1}}|^{q/(q-1)} \, d\eta \right\}^{1-q^{-1}} \]
\[ = O(1) \left\{ \int_{\frac{1}{m+1}}^{\pi} \eta^{\lambda + q^{-1} - 1} \, d\eta \right\}^{1-q^{-1}} \]
\[ = O(1) \left\{ \int_{\frac{1}{m+1}}^{\pi} \eta^{q/(q-1)} (\lambda + q^{-1} - 1) \, d\eta \right\}^{1-q^{-1}} \]
\[ = O(1) \left\{ \int_{\frac{1}{m+1}}^{\pi} \eta^{q/(q-1) - 1} \, d\eta \right\}^{1-q^{-1}} \]
\[ = O((m + 1)^{-\lambda}). \] (5.5)

From (5.3), (5.4) and (5.5), we have
\[ \| l_m (\cdot) \|_r = O((m + 1)^{-\lambda}). \] (5.6)

Now, using generalized Minkowski’s inequality and using Lemma 4.3, we have
\[ \| w_j (l_m, \cdot) \|_{\delta, q} \]
\[ = \left\{ \int_0^{\pi} \left( \frac{w_j (l_m, l) r}{l^\delta} \right) q \, dl \right\}^{q-1} \]
\[ = \left\{ \int_0^{\pi} \left( \| \mathcal{L}_m (\cdot, l) \|_r \right) q \, dl \right\}^{q-1} \]
\[ = \left\{ \int_0^{\pi} \left( \frac{1}{2\pi} \int_0^{2\pi} |\mathcal{L}_m (z, l) \|^r \, dz \right) \frac{q/r \, dl}{l^{\delta q+1}} \right\}^{q-1} \]
\[ = \left\{ \int_0^{\pi} \left( \frac{1}{2\pi} \int_0^{2\pi} \Phi (z, l, \eta) K^\Delta_{mH}^N p, q (\eta) \, d\eta \right) \frac{q/r \, dl}{l^{\delta q+1}} \right\}^{q-1} \]
\[ \leq \frac{1}{\pi} \left[ \int_0^{\pi} \left( \frac{1}{2\pi} \right)^{q/r} \right] \left\{ \int_0^{\pi} \left( \int_0^{2\pi} \Phi (z, l, \eta) \right) ^r \, d\eta \right\} \frac{r^1 \, dl}{l^{\delta q+1}} \]
\[ = \frac{1}{\pi} \left[ \int_0^{\pi} \left\{ \int_0^{\pi} \Phi (\cdot, l, \eta) \|r|K^\Delta_{mH}^N p, q (\eta) \, d\eta \right\} \frac{dl}{l^{\delta q+1}} \right]^{q-1} \]
\[
\begin{align*}
\leq & \frac{1}{\pi} \int_0^\pi |K_m^{\Delta H N^{p,q}}(\eta)| d\eta \left( \int_0^\pi \frac{\|\Phi (:, l, \eta)\|_{l^q}^q}{l} d\eta \right)^{q-1} \\
= & \frac{1}{\pi} \int_0^\pi |K_m^{\Delta H N^{p,q}}(\eta)| d\eta \left\{ \left( \int_\eta^\pi + \int_0^\eta \right) \frac{\|\Phi (:, l, \eta)\|_{l^q}^q}{l^q} d\eta \right\}^{q-1} \\
\leq & \frac{1}{\pi} \int_0^\pi |K_m^{\Delta H N^{p,q}}(\eta)| d\eta \left\{ \int_\eta^\pi \frac{\|\Phi (:, l, \eta)\|_{l^q}^q}{l^q} d\eta \right\}^{q-1} \\
& + \frac{1}{\pi} \int_0^\pi |K_m^{\Delta H N^{p,q}}(\eta)| d\eta \left\{ \int_\eta^\pi \frac{\|\Phi (:, l, \eta)\|_{l^q}^q}{l^q} d\eta \right\}^{q-1} \\
= & O(1) \left\{ \int_0^\pi \left( \eta^{\lambda-\delta} |K_m^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-(1/q)} \\
& + O(1) \left\{ \int_0^\pi \left( \eta^{\lambda-\delta+(1/q)} |K_m^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-(1/q)} \\
:= & O(1) (J_1 + J_2). \tag{5.7}
\end{align*}
\]

Since \((x+y)^r \leq x^r + y^r\) for positive \(x, y\) and \(0 < r \leq 1\), then
\[
J_1 = \left\{ \int_0^\pi \left( \eta^{\lambda-\delta} |K_m^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q-1} \\
= \left\{ \left( \int_0^{1/(m+1)} + \int_1^{\pi/(m+1)} \right) \left( \eta^{\lambda-\delta} |K_m^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} \right\}^{1-q-1} \\
\leq \left\{ \int_0^{1/(m+1)} \left( \eta^{\lambda-\delta} |K_m^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q-1} \\
& + \left\{ \int_1^{\pi/(m+1)} \left( \eta^{\lambda-\delta} |K_m^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q-1} \\
= & I_{11} + I_{12}. \tag{5.8}
\]

Using Lemma 4.1 for \(0 \leq \eta \leq \frac{1}{m+1}\), we have
\[
I_{11} = \left\{ \int_0^{1/(m+1)} \left( \eta^{\lambda-\delta} |K_m^{\Delta H N^{p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q-1} \\
= O(m+1) \left\{ \int_0^{1/(m+1)} \eta^{\frac{q}{q-1}(\lambda-\delta)} d\eta \right\}^{1-q-1} \\
= O(m+1) \left\{ \int_0^{1/(m+1)} \eta^{\frac{q}{q-1}(\lambda-\delta+1-(1/q))} d\eta \right\}^{1-q-1} \\
= O \left( \frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right). \tag{5.9}
\]
Using Lemma 4.1 for $\frac{1}{m+1} \leq \eta \leq \pi$, we have

\[ I_{12} = \left\{ \int_{1/(m+1)}^{\pi} \left( \eta^{\lambda-\delta} |K_m^\Delta H^{N^p,q}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \]

\[ = O(1) \left\{ \int_{1/(m+1)}^{\pi} \left( \eta^{\lambda-\delta-1} \frac{q}{q-1} \right)^{1-q^{-1}} d\eta \right\} \]

\[ = O(1) \left\{ \int_{1/(m+1)}^{\pi} \eta^{\frac{q}{q-1}(\lambda-\delta-1)} d\eta \right\}^{1-q^{-1}} \]

\[ = O(1) \left\{ \int_{1/(m+1)}^{\pi} \eta^{\frac{q}{q-1}(\lambda-\delta-1/(1/q))} d\eta \right\}^{1-q^{-1}} \]

\[ = O \left( \frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right). \] \hspace{1cm} (5.10)

From (5.8), (5.9) and (5.10), we have

\[ J_1 := I_{11} + I_{12} \]

\[ = O \left( \frac{1}{(m+1)^{\lambda-\delta-(1/q)}} \right). \] \hspace{1cm} (5.11)

Now,

\[ J_2 = \left\{ \int_{0}^{1/(m+1)} \left( \eta^{\lambda-\delta+1/q} |K_m^\Delta H^{N^p,q}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \]

\[ = \left[ \left\{ \int_{0}^{1/(m+1)} \left( \eta^{\lambda-\delta+(1/q)} |K_m^\Delta H^{N^p,q}(\eta)| \right)^{q/(q-1)} d\eta \right\} \right]^{1-q^{-1}} \]

\[ := J_{11} + J_{12}. \] \hspace{1cm} (5.12)

Using Lemma 4.1 for $0 \leq \eta \leq \frac{1}{m+1}$, we have

\[ J_{11} = \left\{ \int_{0}^{1/(m+1)} \left( \eta^{\lambda-\delta+(1/q)} |K_m^\Delta H^{N^p,q}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \]

\[ = O(m+1) \left\{ \int_{0}^{1/(m+1)} \left( \eta^{\lambda-\delta+(1/q)} \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \]

\[ = O(m+1) \left\{ \int_{0}^{1/(m+1)} \eta^{\frac{q}{q-1}(\lambda-\delta+(1/q))} d\eta \right\}^{1-q^{-1}} \]

\[ = O \left( \frac{1}{(m+1)^{\lambda-\delta}} \right). \] \hspace{1cm} (5.13)
Using Lemma 4.1 for $\frac{1}{m+1} \leq \eta \leq \pi$, we have

\[
J_{12} = \left\{ \int_{1/(m+1)}^{\pi} \left( \eta^{\lambda-\delta+(1/q)-1} |K_m^{\Delta H^{N_p,q}}(\eta)| \right)^{q/(q-1)} d\eta \right\}^{1-q^{-1}} \\
= O\left(1\right) \left\{ \int_{1/(m+1)}^{\pi} \eta^{\lambda-\delta+(1/q)-1} d\eta \right\}^{1-q^{-1}} \\
= O\left(1\right) \left\{ \int_{1/(m+1)}^{\pi} \eta^{q/\left(q-1\right)} \left(1-\frac{1}{q}\right)^{1-q^{-1}} d\eta \right\} \\
= O\left(1\right) \frac{1}{\left(m+1\right)^{\lambda-\delta}}. \quad (5.14)
\]

From (5.12), (5.13) and (5.14), we have

\[
J_2 := J_{11} + J_{12} = O\left(1\right) \frac{1}{\left(m+1\right)^{\lambda-\delta}}. \quad (5.15)
\]

From (5.7), (5.11) and (5.15), we get

\[
\|w_j\left(l_m, \cdot \right)\|_{\delta, q} = O\left(1\right) \left( J_1 + J_2 \right) \\
= O\left(1\right) \left[ O\left(1\right) \left(\frac{1}{\left(m+1\right)^{\lambda-\delta-(1/q)}}\right) + O\left(1\right) \left(\frac{1}{\left(m+1\right)^{\lambda-\delta}}\right) \right]. \quad (5.16)
\]

From (5.2), (5.6) and (5.16), we get

\[
\|l_m(\cdot)\|_{B^{\delta}_{q}(L^r)} = \|l_m(\cdot)\|_r + \|w_j\left(l_m, \cdot \right)\|_{\delta, q} \\
= O\left(m+1\right)^{-\lambda} + O\left(\frac{1}{\left(m+1\right)^{\lambda-\delta-(1/q)}}\right) + O\left(\frac{1}{\left(m+1\right)^{\lambda-\delta}}\right).
\]

This completes the proof of case I.

**Case II**: For $q = \infty$, $0 \leq \delta < \lambda < 2$.

We have

\[
\|l_m(\cdot)\|_{B^{\delta}_{\infty}(L^r)} = \|l_m(\cdot)\|_r + \|w_j\left(l_m, \cdot \right)\|_{\delta, \infty}. \quad (5.17)
\]

Using (1.6), we have

\[
\|l_m(\cdot)\|_r \leq 2 \int_0^{\pi} w_j(g, \eta)_r |K_m^{\Delta H^{N_p,q}}(\eta)| d\eta \\
= O\left(1\right) \left\{ \int_0^{1/(m+1)} \eta^{\lambda} |K_m^{\Delta H^{N_p,q}}(\eta)| d\eta + \int_{1/(m+1)}^{\pi} \eta^{\lambda} |K_m^{\Delta H^{N_p,q}}(\eta)| d\eta \right\}
\]
:= O \left( 1 \right) \left( I_2 + J_2 \right). \quad (5.18)

Using Lemma 4.1 for \(0 \leq \eta \leq \frac{1}{m+1}\), we get

\[
I_2 = \int_{0}^{1/(m+1)} \eta^{\lambda} |K_m^{\Delta H N^{p,q}}(\eta)| d\eta \\
= O \left( m+1 \right) \int_{0}^{1/(m+1)} \eta^{\lambda} d\eta \\
= O \left( m+1 \right)^{-\lambda}. \quad (5.19)
\]

Again, using Lemma 4.1 for \(\frac{1}{m+1} \leq \eta \leq \pi\), we get

\[
J_2 = \int_{1/(m+1)}^{\pi} \eta^{\lambda} |K_m^{\Delta H N^{p,q}}(\eta)| d\eta \\
= O \left( 1 \right) \int_{1/(m+1)}^{\pi} \eta^{\lambda-1} d\eta \\
= O \left( m+1 \right)^{-\lambda}. \quad (5.20)
\]

From (5.18), (5.19) and (5.20), we get

\[
\|l_m (\cdot)\|_r = O \left( 1 \right) \left( I_2 + J_2 \right) \\
= O \left( m+1 \right)^{-\alpha}. \quad (5.21)
\]

Using generalized Minkowski's inequality and Lemma 4.4, we get

\[
\|w_j (l_m, \cdot)\|_{\delta, \infty} \\
= \sup_{l>0} \left( l^{-\delta} w_j (l_m, l) \right) \\
= \sup_{l>0} \left( l^{-\delta} \|L_m (\cdot, l)\|_r \right) \\
= \sup_{l>0} \left[ l^{-\delta} \left( \frac{1}{2\pi} \int_{0}^{2\pi} \left| \int_{0}^{\pi} K_m^{\Delta H N^{p,q}}(\eta) \phi (z, l, \eta) \right|^r \frac{d\eta}{dz} \right)^{1/r} \right] \\
\leq \sup_{l>0} \left[ l^{-\delta} \left( \frac{1}{2\pi} \right)^{1/r} \int_{0}^{\pi} \left\{ \int_{0}^{2\pi} |K_m^{\Delta H N^{p,q}}(\eta)|^r |\phi (z, l, \eta)|^{r} \frac{d\eta}{dz} \right\}^{1/r} \right] \\
= \sup_{l>0} \left[ l^{-\delta} \left( \frac{1}{2\pi} \right)^{1/r} \int_{0}^{\pi} \|\phi (\cdot, l, \eta)\|_r |K_m^{\Delta H N^{p,q}}(\eta)| d\eta \right] \\
= \frac{1}{\pi} \int_{0}^{\pi} \left( \sup_{l>0} l^{-\delta} \|\Phi (\cdot, l, \eta)\|_r \right) |K_m^{\Delta H N^{p,q}}(\eta)| d\eta \\
= O \left( 1 \right) \int_{0}^{\pi} \eta^{\lambda-\delta} |K_m^{\Delta H N^{p,q}}(\eta)| d\eta
\]
= O(1) \left[ \int_0^{1/(m+1)} \eta^{-\alpha} |K_m^{\Delta H^{N^p,q}}(\eta)|d\eta + \int_{1/(m+1)}^{\pi} \eta^{-\alpha} |K_m^{\Delta H^{N^p,q}}(\eta)|d\eta \right] \\

= O(1) [I_3 + J_3]. \quad (5.22)

Using Lemma 4.1 for \(0 \leq \eta \leq \frac{1}{m+1}\), we get

\[ I_3 = \int_0^{1/(m+1)} \eta^{-\alpha} |K_m^{\Delta H^{N^p,q}}(\eta)|d\eta \]

\[ = O(m+1) \int_0^{1/(m+1)} \eta^{-\alpha} d\eta \]

\[ = O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right). \quad (5.23)\]

Using Lemma 4.1 for \(\frac{1}{m+1} \leq \eta \leq \pi\), we get

\[ J_3 = \int_{1/(m+1)}^{\pi} \eta^{-\alpha} |K_m^{\Delta H^{N^p,q}}(\eta)|d\eta \]

\[ = O(1) \int_{1/(m+1)}^{\pi} \eta^{-\alpha-1} d\eta \]

\[ = O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right). \quad (5.24)\]

From (5.22), (5.23) and (5.24), we get

\[ \|w_j(l_m, \cdot)\|_{\delta, \infty} = O(1) [I_3 + J_3] \]

\[ = O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right). \quad (5.25)\]

From (5.17), (5.21) and (5.25), we have

\[ \|l_m(\cdot)\|_{B^\gamma_\infty(L^r)} = \|l_m(\cdot)\|_r + \|w_j(l_m, \cdot)\|_{\delta, \infty} \]

\[ = O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta}}\right). \]

This completes the proof of case II. \(\square\)

6 Corollary

The following corollary are derived from our main theorem.

**Corollary 6.1.** If \(q_m = 1 \forall m\), then \(\Delta H^{N^p,q}\) means reduces to \(\Delta H^{N^p.m}\) means and the best approximation of \(g \in B^\lambda_q(L^r)\) space by \(\Delta H^{N^p.m}\) means of Fourier series is

\[ \|l_m^{\Delta H^{N^p.m} (z)} - g(z)\| = \begin{cases} 
O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\delta-1/q}}\right) + O\left(\frac{1}{(m+1)^{\lambda-\sigma}}\right) & : 1 < q < \infty \\
O(m+1)^{-\lambda} + O\left(\frac{1}{(m+1)^{\lambda-\sigma}}\right) & : q = \infty.
\end{cases} \]
Corollary 6.2. If \( p_m = 1 \forall m \), then \( \Delta_H N^{p,q} \) reduces to \( \Delta_H \tilde{N}^{q_m} \) means and the best approximation of \( g \in B_q^\lambda(L^r) \) space by \( \Delta_H \tilde{N}^{q_m} \) means of Fourier series is

\[
\| t_m^{\Delta_H \tilde{N}^{q_m}} (z) - g(z) \| = \begin{cases} 
O (m + 1)^{-\lambda} + O \left( \frac{1}{(m+1)^{\lambda - \delta - (1/q)}} \right) + O \left( \frac{1}{(m+1)^{\lambda - \delta}} \right) ; & 1 < q < \infty \\
O (m + 1)^{-\lambda} + O \left( \frac{1}{(m+1)^{\lambda - \delta}} \right) ; & q = \infty.
\end{cases}
\]

Corollary 6.3. If \( p_m = \left( \frac{m + \alpha - 1}{\alpha - 1} \right) \alpha > 0 \), and \( q_m = 1 \forall m \), then \( \Delta_H N^{p,q} \) means reduces to \( \Delta_H C_\alpha \) means of and the best approximation of \( g \in B_q^\lambda(L^r) \) space by \( \Delta_H C_\alpha \) means of Fourier series is

\[
\| t_m^{\Delta_H C_\alpha} (z) - g(z) \| = \begin{cases} 
O (m + 1)^{-\lambda} + O \left( \frac{1}{(m+1)^{\lambda - \delta - (1/q)}} \right) + O \left( \frac{1}{(m+1)^{\lambda - \delta}} \right) ; & 1 < q < \infty \\
O (m + 1)^{-\lambda} + O \left( \frac{1}{(m+1)^{\lambda - \delta}} \right) ; & q = \infty.
\end{cases}
\]

Corollary 6.4. If \( h_{m,j} = \left( \frac{m}{j} \right) \frac{q^{m-j}}{(1+q)^m} \), if \( 0 \leq j \leq m \), then \( \Delta_H N^{p,q} \) means and the best approximation of \( g \in B_q^\lambda(L^r) \) space by \( E_q N^{p,q} \) means of Fourier series is

\[
\| t_m^{E_q N^{p,q}} (z) - g(z) \| = \begin{cases} 
O (m + 1)^{-\lambda} + O \left( \frac{1}{(m+1)^{\lambda - \delta - (1/q)}} \right) + O \left( \frac{1}{(m+1)^{\lambda - \delta}} \right) ; & 1 < q < \infty \\
O (m + 1)^{-\lambda} + O \left( \frac{1}{(m+1)^{\lambda - \delta}} \right) ; & q = \infty.
\end{cases}
\]

Corollary 6.5. If \( h_{m,j} = \frac{1}{m+1} \), if \( 0 \leq j \leq m \), then \( \Delta_H N^{p,q} \) means and the best approximation of \( g \in B_q^\lambda(L^r) \) space by \( C_1 N^{p,q} \) means of Fourier series is

\[
\| t_m^{C_1 N^{p,q}} (z) - g(z) \| = \begin{cases} 
O (m + 1)^{-\lambda} + O \left( \frac{1}{(m+1)^{\lambda - \delta - (1/q)}} \right) + O \left( \frac{1}{(m+1)^{\lambda - \delta}} \right) ; & 1 < q < \infty \\
O (m + 1)^{-\lambda} + O \left( \frac{1}{(m+1)^{\lambda - \delta}} \right) ; & q = \infty.
\end{cases}
\]

7 Particular cases

(i) In view of Remark 2 (i) and 3 (ii), our result becomes a particular case of [10].

(ii) In view of Remark 2 (i) and 3 (iv), our result becomes a particular case of [26].

(iii) In view of Remark 2 (ii) and 3 (v), our result becomes a particular case of [27].

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**Hare Krishna Nigam**  Department of Mathematics, Central University of South Bihar, Gaya - 824236 (Bihar), India

E-mail: hknigam@cusb.ac.in

**Supriya Rani**  Department of Mathematics, Central University of South Bihar, Gaya - 824236 (Bihar), India

E-mail: supriya@cusb.ac.in