CLASSIFICATION OF MOMENTUM PROPER EXACT HAMILTONIAN GROUP ACTIONS AND THE EQUIVARIANT ELIASHBERG COTANGENT BUNDLE CONJECTURE

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ABSTRACT. Let $G$ be a compact and connected Lie group. The Hamiltonian $G$-model functor maps the category of symplectic representations of closed subgroups of $G$ to the category of exact Hamiltonian $G$-actions. Based on previous joint work with Y. Karshon, the restriction of this functor to the momentum proper subcategory on either side induces a bijection between the sets of isomorphism classes. This classifies all momentum proper exact Hamiltonian $G$-actions (of arbitrary complexity).

As an extreme case, we obtain a version of the Eliashberg cotangent bundle conjecture for transitive smooth actions. As another extreme case, the momentum proper Hamiltonian $G$-actions on contractible manifolds are exactly the symplectic $G$-representations, up to isomorphism.

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1. THE MAIN RESULT AND APPLICATIONS

Let $G$ be a compact and connected Lie group. We call a Hamiltonian $G$-action momentum proper iff every momentum map for the action is proper. The purpose of this article is to classify the momentum proper exact Hamiltonian $G$-actions in terms of the momentum proper symplectic representations of closed subgroups of $G$. To this end I provide a bijection between the sets of equivalence classes of such representations and of such Hamiltonian actions. (See Corollary 1.11 below.)

The bijection is induced by the $G$-model functor. This is a functor between the category of symplectic representations of closed subgroups of $G$ and the category of exact Hamiltonian $G$-actions.
In order to define these categories and the Hamiltonian $G$-model functor, we need the following. For every $g \in G$ we denote by

$$c_g : G \rightarrow G, \quad c_g(a) := gag^{-1},$$

the conjugation by $g$. We define $\text{SympRep}_{\leq G}$ to be the following category:

- Its objects are the tuples $(H, \rho) = (H, V, \sigma, \rho)$, where $H$ is a closed subgroup of $G$, $(V, \sigma)$ is a (finite dimensional) symplectic vector space and $\rho$ is a symplectic $H$-representation on $V$ w.r.t. $\sigma$.
- Its morphisms between two objects $(H, \rho)$ and $(H', \rho')$ are pairs $(g, T)$, where $g \in G$ and $T : V \rightarrow V'$ is a linear symplectic map, such that

\begin{align}
  c_g(H) &= H', \\
  T \rho_h &= \rho'_{c_g(h)} T, \quad \forall h \in H.
\end{align}

(The dimension of $V'$ may be bigger than the dimension of $V$. In this case $T$ is not surjective.) The composition of two morphisms is defined by

\[(g', T') \circ (g, T) := (g'g, T'T).\]

**Remark.** A morphism $(g, T)$ is an isomorphism in the sense of category theory if and only if $T$ is surjective (and hence bijective). In this case the inverse of $(g, T)$ is given by $(g^{-1}, T^{-1})$.

Let $\psi = (M, \omega, \psi)$ be a Hamiltonian $G$-action. We call $\psi$ exact if there exists a $\psi$-invariant primitive of $\omega$. We call $\psi$ momentum proper iff every momentum map for $\psi$ is proper.

We define $\text{Ham}^\text{ex}_G$ to be the following category:

- Its objects are the exact Hamiltonian $G$-actions $(M, \omega, \psi)$ with $M$ connected (and without boundary).
- Its morphisms between two objects $(M, \omega, \psi)$ and $(M', \omega', \psi')$ are proper symplectic embeddings $\Phi$ from $M$ to $M'$ that intertwine $\psi$ with $\psi'$ (The dimension of $M'$ may be bigger than the dimension of $M$.) Composition is the composition of maps.

**Remark.** The isomorphisms between two objects are equivariant symplectomorphisms.

We define the Hamiltonian $G$-model functor

$$\text{Model}_G : \text{SympRep}_{\leq G} \rightarrow \text{Ham}^\text{ex}_G$$

as follows:

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1. In the article [KZ18] we used the notation $\omega_V$ for the symplectic form $\sigma$ on $V$. For simplicity I am using $\sigma$ here. To help the reader navigate through this article, I have included a list of symbols at the end of this article.

2. By this we mean that $(M, \omega)$ is a symplectic manifold and $\psi$ is a Hamiltonian $G$-action on $M$.

3. By this we mean a $\psi$-invariant one-form whose exterior derivative equals $\omega$. The action $\psi$ is exact if $\omega$ is exact, because we assume that $G$ is compact. (We obtain a $\psi$-invariant primitive from an arbitrary primitive by averaging w.r.t. the Haar measure on $G$.)

4. By definition, every momentum map $\mu$ for a Hamiltonian $G$-action $\psi$ on a symplectic manifold $(M, \omega)$ is equivariant w.r.t. $\psi$ and the coadjoint action on $\text{Lie}(G)^*$. This means that $\mu(\psi_g(x)) = \text{Ad}^*(g)(\mu(x))$, for every $g \in G$ and $x \in M$, where $\psi_g := \psi(g, \cdot)$ and $\text{Ad}^*$ denotes the coadjoint $G$-action.

5. In this article every manifold is assumed to have empty boundary.

6. This means that $\psi'_g \circ \Phi = \Phi \circ \psi_g$ for every $g \in G$. 

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The main result is the following. (As always, we assume that $\nu$ is well-defined.)

\[ \psi : V \rightarrow h^* \]

the unique momentum map for $\rho$ that vanishes at 0. For $a \in G$ and $\varphi \in g^*$ we denote by $a \varphi \in T_a G$ the image of $\varphi$ under the derivative of the left translation by $a$. We define

\[ \mu^D_{H,\rho} := \mu^D_\rho : T^* G \times V \rightarrow h^*, \quad \mu^D_\rho(a, a\varphi, v) := -\varphi h + \nu_\rho(v). \]

This is a momentum map for $\psi^D_\rho$. The pair $(Y_\rho, \omega_\rho)$ is defined to be the symplectic quotient of $\psi^D_\rho$ at 0 w.r.t. $\mu^D_\rho$. This means that

\[ Y_{H,\rho} := Y_\rho = (\mu^D_\rho)^{-1}(0)/\psi^D_\rho. \]

(The subgroup $H$ is compact, since it is closed and $G$ is compact. Therefore, the restriction of $\psi^D_\rho$ to $(\mu^D_\rho)^{-1}(0)$ is proper. Since it is also free, the symplectic quotient is well-defined.) The left translation by $G$ on $G$ induces a $G$-action on $T^* G$ and hence on $T^* G \times V$. Since left and right translation commute, this action preserves $(\mu^D_\rho)^{-1}(0)$ and descends to a $G$-action $\psi_\rho$ on $Y_\rho$, the symplectic quotient of $T^* G \times V$ by the diagonal $H$-action. This defines $Model_G(H, \rho) = (Y_\rho, \omega_\rho, \psi_\rho)$.

- For every $g \in G$ we denote by $R^g : G \rightarrow G$, $R^g(a) := ag$, the right translation by $g$, and by $R^g_* : T^* G \rightarrow T^* G$ the induced map. The map $Model_G$ assigns to every morphism $(g, T) : (H, \rho) \rightarrow (H', \rho')$ of $\text{SympRep}_{\leq G}$ the morphism $Model_G(g, T)$ of $\text{Ham}^*_{G}$ given by

\[ Model_G(g, T)(y) := [R^g_*^{-1}(a, a\varphi), Tv], \]

where $(a, a\varphi, v)$ is an arbitrary representative of $y$. (Here on the right hand side we denote by $[a', a'\varphi', v']$ the equivalence class of $(a', a'\varphi', v)$.)

The main result is the following. (As always, we assume that $G$ is compact and connected.)

1.8. Theorem (Hamiltonian $G$-model functor). (i) (well-definedness on objects) The map $Model_G$ is well-defined on objects, i.e., $\psi_\rho$ is indeed an exact Hamiltonian $G$-action.

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In the article [KZ18] we used the notation $(Y, \omega_Y)$ instead of $(Y_\rho, \omega_\rho)$. To make the dependence on $\rho$ explicit, I am using $(Y_\rho, \omega_\rho)$ here. To help the reader navigate through this article, I have included a list of symbols at the end of this article.

Viewing the symplectic vector space $(V, \sigma)$ as a symplectic manifold, the representation $\rho$ is a Hamiltonian $G$-action on $V$. Hence it admits a momentum map. In the article [KZ18] we used the notation $\mu_V$ instead of $\nu_\rho$. To make the dependence on $\rho$ explicit, I am using $\nu_\rho$ here.

In the article [KZ18] we used the notation $\tilde{\mu}_G$ for this map. For simplicity I have dropped the tilde here.

This can be seen as part of symplectic reduction in stages.
(ii) (well-definedness on morphisms) The map $\text{Model}_G$ is well-defined on morphisms, i.e.,

\[(1.9) \quad \left( R^*_g \times T \right) (\mu^D_\rho)^{-1}(0) \subseteq (\mu^D_\rho)^{-1}(0), \]

the right hand side of (1.7) does not depend on the choice of a representative $(a, a\phi, v)$, and $\text{Model}_G(g, T)$ is a morphism of $\text{Ham}^\text{ex}_G$.

(iii) (functoriality) The map $\text{Model}_G$ is a covariant functor.

(iv) (essential injectivity) The map between the sets of isomorphism classes induced by $\text{Model}_G$ is injective.

(v) (morphisms) Let $(H, \rho)$ and $(H', \rho')$ be objects of $\text{SympRep}_{\leq G}$, and $(g, T), (\hat{g}, \hat{T})$ be morphisms between these objects. $\text{Model}_G$ maps these morphisms to the same morphism if and only if

\[(1.10) \quad h' := \tilde{g} g^{-1} \in H', \quad \hat{T} = \rho'_h T. \]

(vi) (momentum properness and morphisms) Let $A$ and $A'$ be objects of $\text{SympRep}_{\leq G}$ or let them be objects of $\text{Ham}^\text{ex}_G$, such that $A'$ is momentum proper\(^{11}\) and there exists a morphism from $A$ to $A'$. Then $A$ is momentum proper.

(vii) (momentum properness and model functor) An object of $\text{SympRep}_{\leq G}$ is momentum proper if and only if its image under $\text{Model}_G$ is momentum proper.

(viii) (essential surjectivity) Every momentum proper object of $\text{Ham}^\text{ex}_G$ is isomorphic to an object in the image of $\text{Model}_G$.

Remark. Theorem 1.8(v) characterizes the extent to which the functor $\text{Model}_G$ is faithful.

Theorem (1.8) puts previous joint work [KZ18] with Y. Karshon into a categorical framework. Namely, part (viii) of Theorem (1.8) (essential surjectivity of $\text{Model}_G$) was proved in [KZ18, 1.5. Theorem], without introducing the categorical setup used in the present article.

The other parts of Theorem (1.8) will be proved in the next section. The proof of (iv) (essential injectivity) is based on Lemma 2.19, which provides criteria under which the symplectic quotient representation of the model action $\text{Model}_G(H, \rho)$ at a given point is isomorphic to $(H, \rho)$. We also use the fact that if two compact subgroups of a Lie group are conjugate to subgroups of each other then they are conjugate to each other. (This follows from Lemma 2.8 below.)

Remark. Naively, in the definition of a morphism of $\text{SympRep}_{\leq G}$, one could try to weaken the condition (1.1) to either the condition $c_g(H) \subseteq H'$ or $c_g(H) \supseteq H'$. With this modification the model functor would no longer be well-defined on morphisms. (“$\supseteq$” is needed in order for (1.9) to hold and “$\subseteq$” is needed for the right hand side of (1.7) not to depend on the choice of a representative. See the proof of Theorem 1.8(ii) below.)

We denote by

\[ \text{SympRep}_{\leq G}^{\text{prop}}, \quad \text{Ham}^\text{ex,prop}_G \]

the full subcategories of $\text{SympRep}_{\leq G}$ and $\text{Ham}^\text{ex}_G$ consisting of momentum proper objects. Theorem 1.8 has the following application.

\(^{11}\)Recall that an object $A$ of $\text{SympRep}_{\leq G}$ is a tuple $(H, V, \sigma, \rho)$, where $H$ is a closed subgroup of $G$ and $(V, \sigma, \rho)$ is a symplectic $H$-representation. Viewing $(V, \sigma)$ as a symplectic manifold, $\rho$ is a Hamiltonian $H$-action. We call $A$ momentum proper iff this action is momentum proper, i.e., if every momentum map for $\rho$ is proper.
1.11. **Corollary** (classification of momentum proper exact Hamiltonian actions). *The functor Model$_G$ induces a bijection*

\[(1.12) \quad \{ \text{isomorphism class of } \text{SympRep}^\text{prop}_{\leq G} \} \rightarrow \{ \text{isomorphism class of } \text{Ham}^\text{ex,prop}_G \}.\]

We call this bijection the **classifying map** (for momentum proper exact Hamiltonian $G$-actions).

**Remarks.**
- It follows from Theorem [1.8](vi) that the isomorphism class of any object of \( \text{SympRep}^\text{prop}_{\leq G} \) is its isomorphism class in the bigger category \( \text{SympRep}_{\leq G} \). A similar remark applies to \( \text{Ham}^\text{ex,prop}_G \).
- Corollary [1.11] classifies all momentum proper exact Hamiltonian $G$-actions up to isomorphism.
- Surjectivity of the classifying map (1.12) follows from Theorem (1.8) (viii), which was proved in joint work with Y. Karshon [KZ18, 1.5. Theorem].
- The inverse of the classifying map (1.12) is induced by assigning to a Hamiltonian action its symplectic quotient representation at any suitable point, see Proposition 4.1 below.
- In contrast with Corollary [1.11] the map induced by Model$_G$ between the sets of isomorphism classes of \( \text{SympRep}_{\leq G} \) and \( \text{Ham}^\text{ex,prop}_G \) is not surjective. To see this, let $Q$ be a connected compact manifold of positive dimension, without boundary. We define $\omega$ to be the canonical symplectic form on $T^*Q$ and $\psi$ to be the trivial $G$-action on $T^*Q$.

We claim that the isomorphism class of $(T^*Q, \omega, \psi)$ does not lie in the image of Model$_G$. To see this, assume that $(H, V, \sigma, \rho)$ is an object of $\text{SympRep}_{\leq G}$ for which $\psi_\rho$ is trivial. Then $H = G$ and therefore, $Y_\rho$ is canonically diffeomorphic to $V$. If $(Y_\rho, \omega_\rho, \psi_\rho)$ is isomorphic to $(T^*Q, \omega, \psi)$ then it follows that $Q$ is a singleton. This proves the claim.
- Many classification results are known for Hamiltonian group actions whose complexity is low. (By definition, the complexity is half the dimension of a generic non-empty reduced space. For references see [KZ18].) What makes Corollary [1.11] special is that it classifies Hamiltonian actions of arbitrary complexity.

**Proof of Corollary [1.11].** By Theorem [1.8](vi) the map (1.12) is well-defined. By Theorem [1.8](vii\Rightarrow viii) the map (1.12) is bijective. This proves Corollary [1.11].

By considering the extreme case of the full subgroup $H = G$, this corollary implies that the momentum proper Hamiltonian $G$-actions on contractible manifolds are exactly the momentum proper symplectic $G$-representations, up to isomorphism. See Corollary [1.17] below. On the other hand, by considering the extreme case in which the vector space $V$ is trivial, using Corollary [1.11], we can classify the *critical* momentum proper exact Hamiltonian $G$-actions in terms of transitive $G$-actions on manifolds.

To explain the latter application, we call an object $(M, \omega, \psi)$ of $\text{Ham}^\text{ex,prop}_G$ *critical* iff $M$ is homotopy equivalent to some closed manifold of dimension equal to $\dim(M)/2$.

1.13. **Remark** (criticality). By Corollary [1.11] there exists an object $(H, \rho) = (H, V, \sigma, \rho)$ of $\text{SympRep}^\text{prop}_{\leq G}$, such that Model$_G(H, \rho)$ is isomorphic to $(M, \omega, \psi)$. The manifold part of Model$_G(H, \rho)$ is homotopy equivalent to the closed manifold $G/H$ and has dimension
2(\dim G - \dim H) + \dim V$. It follows that $M$ is not homotopy equivalent to any closed manifold of dimension bigger than $\dim(M)/2$. This justifies the terminology critical.

We denote

$$\text{Ham}^\text{crit}_G := \text{full subcategory of } \text{Ham}^\text{ex-prop}_G \text{ consisting of critical objects.}$$

For every manifold $Q$ we denote by $\omega_Q$ the canonical symplectic form on $T^*Q$. We define the $G$-cotangent functor $T^*_G$ to be the canonical functor from the category of $G$-actions on manifolds and $G$-equivariant diffeomorphisms to the category of Hamiltonian $G$-actions and $G$-equivariant symplectomorphisms. It takes an object $(Q, \theta)$ to $(T^*Q, \omega_Q)$ together with lifted $G$-action $\theta_*$, and a morphism $f : Q \to Q'$ to the lifted map $f_* : T^*Q \to T^*Q'$.

We define $\text{Act}_G^{\text{trans}}$ to be the category whose objects are the transitive smooth $G$-actions on connected closed manifolds and whose morphisms are the $G$-equivariant diffeomorphisms.

1.14. **Corollary** (classification of critical momentum proper exact Hamiltonian actions). The functor $T^*_G$ induces a bijection

$$(1.15) \quad \{ \text{isomorphism class of } \text{Act}_G^{\text{trans}} \} \rightarrow \{ \text{isomorphism class of } \text{Ham}^\text{crit}_G \}.$$

**Remarks** (classification of critical actions, Eliashberg cotangent bundle conjecture).

- Part of the statement is that $T^*_G$ maps $\text{Act}_G^{\text{trans}}$ to $\text{Ham}^\text{crit}_G$.
- The isomorphism class of any object of $\text{Ham}^\text{crit}_G$ in $\text{Ham}^\text{crit}_G$ is its isomorphism class in the bigger category $\text{Ham}^\text{ex-prop}_G$ (or in $\text{Ham}^\text{ex}_G$).
- Corollary 1.14 classifies the critical momentum proper exact Hamiltonian $G$-actions in terms of transitive $G$-actions on manifolds.
- The cotangent functor $T^*$ is the canonical functor from the category of connected closed smooth manifolds and diffeomorphisms to the category of symplectic manifolds and symplectomorphisms. It agrees with $T^*_G$. The Eliashberg cotangent bundle conjecture states that $T^*$ is essentially injective, i.e., it induces an injective map between the sets of isomorphism classes. See [MS17, Problem 31, p. 561]. Very little is known about this conjecture. See [Abo12, EKS16, ES16] for some results.
- By Corollary 1.14 the restriction of the functor $T^*_G$ to the category $\text{Act}_G^{\text{trans}}$ of transitive $G$-actions is essentially injective. This proves an equivariant version of the Eliashberg cotangent bundle conjecture. In fact, Corollary 1.14 provides more information, namely it also specifies the image of the class of objects of $\text{Act}_G^{\text{trans}}$ under $T^*_G$, up to isomorphism.
- The philosophy behind this application is that symmetry makes problems more accessible. In the present situation it allows for a classification of the structures at hand (transitive $G$-actions and critical Hamiltonian $G$-actions). The same philosophy was for example used recently in [FPP18], where the authors used Delzant’s classification of symplectic toric manifolds to prove that certain equivariant symplectic capacities are (dis-)continuous. (Without symmetry the question whether a given symplectic capacity is continuous is hard in general.)

We will prove Corollary 1.14 in Section 3.

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12This means compact and without boundary.
As another application of Corollary 1.11 we now classify the momentum proper Hamiltonian $G$-actions on contractible manifolds. Here we consider another extreme case, in which the subgroup $H$ equals $G$. We denote by $\text{SympRep}_G$ the category whose objects are symplectic $G$-representations and whose morphisms are $G$-equivariant linear symplectic maps (possibly not surjective), and by $
abla\text{SympRep}_G^{\text{prop}}$
 the full subcategory consisting of momentum proper objects. We denote by $\text{Ham}_G^{\text{contr}}$ the full subcategory of $\text{Ham}_G^{\text{ex}}$ consisting of those objects $(M, \omega, \psi)$ for which $M$ is contractible, and by $
abla\text{Ham}_G^{\text{contr, prop}}$
 the full subcategory of $\text{Ham}_G^{\text{contr}}$ consisting of momentum proper objects. We denote by $\iota_G : \text{SympRep}_G \to \text{Ham}_G^{\text{contr}}$, $\iota_G^{\text{prop}} : \text{SympRep}_G^{\text{prop}} \to \text{Ham}_G^{\text{contr, prop}}$
 the inclusion functor and its restriction to the momentum proper subcategories. We denote by $\iota_G^*, \iota_G^{\text{prop}}$ the maps between the sets of isomorphism classes induced by $\iota_G$, $\iota_G^{\text{prop}}$.

1.16. Remarks.  
(i) The isomorphism class of any object in $\text{SympRep}_G^{\text{prop}}$ is its isomorphism class in the bigger category $\text{SympRep}_G$. This follows from Remark 2.18(ii) below.

Similar remarks apply to the subcategory $\text{Ham}_G^{\text{contr, prop}}$ of $\text{Ham}_G^{\text{contr}}$ and the subcategory $\text{Ham}_G^{\text{contr}}$ of $\text{Ham}_G^{\text{ex}}$.

(ii) The map $\iota_G^*$ extends the map $\iota_G^{\text{prop}}$. By (i) this statement makes sense.

1.17. Corollary (classification of momentum proper Hamiltonian actions on contractible manifolds).  
(i) The map $\iota_G^*$ is injective.

(ii) The map $\iota_G^{\text{prop}}$ is surjective.

Remarks.  
• It follows from (i) and Remark 1.16(ii) that $\iota_G^{\text{prop}}$ is injective. Using (ii), this map is bijective.

• Part (ii) means that every momentum proper Hamiltonian $G$-action on a contractible symplectic manifold is symplectically linearizable\textsuperscript{13}.

• The statement of Corollary 1.17 means that the momentum proper Hamiltonian $G$-actions on contractible symplectic manifolds agree with the momentum proper symplectic $G$-representations, up to isomorphism. This classifies these actions.

• Assume that $G$ is non-abelian. In contrast with part (ii) the map $\iota_G^*$ is not surjective. This follows from \textsuperscript{[KZ18] Corollary 8.4}.

For the proof of Corollary 1.17(ii) we need the following.

1.18. Remark. For every symplectic $G$-representation $(V, \sigma, \rho)$ the map $\nabla I_G^\rho : V \to Y_\rho$, $I_G^\rho(v) := [e, 0, v]$, is a $G$-equivariant symplectomorphism, i.e., an isomorphism from $\iota_G^\rho(\rho) = \rho$ to $\text{Model}_G^\rho(G, \rho)$ in $\text{Ham}_G^{\text{ex}}$. This follows from a straightforward argument.

\textsuperscript{13}We call a symplectic $G$-action $\psi$ on a symplectic manifold $(M, \omega)$ symplectically linearizable iff there exist a symplectic $G$-representation $(V, \sigma, \rho)$ and a symplectomorphism between $V$ and $M$ that intertwines $\rho$ and $\psi$. 

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Proof of Corollary 1.17. (i): Let $R$ and $R'$ be isomorphism classes of $\text{SympRep}_G$ that are mapped to the same class under $\iota^G$. We choose representatives $(V, \sigma, \rho), (V', \sigma', \rho')$ of $R, R'$ and an isomorphism $\Phi$ in $\text{Ham}^\text{cont}_G$ from $\iota^G(\rho)$ to $\iota^G(\rho')$. The differential $d\Phi(0): T_0V \to T_{\Phi(0)}V'$ is an isomorphism from $d\rho(0)$ to $d\rho'(\Phi(0))$ in $\text{SympRep}_G$. Since $\rho$ is linear, the canonical identification between $V$ and $T_0V$ is an isomorphism from $\rho$ to $d\rho(0)$ in $\text{SympRep}_G$. Similarly, $\rho'$ is isomorphic to $d\rho'(\Phi(0))$. Combining these three isomorphisms, it follows that $\rho$ and $\rho'$ are isomorphic in $\text{SympRep}_G$, i.e., $R = R'$. Hence the map $\iota^G_*$ is injective. This proves (i).

(ii): Let $\Psi$ be an isomorphism class of objects of $\text{Ham}^\text{cont,prop}_G$. We choose a representative $(M, \omega, \psi)$ of $\Psi$. By Theorem 1.8(viii) there exists an object $(H, \rho)$ of $\text{SympRep}_{\leq G}$, such that $\psi_0 := \text{Model}_G(H, \rho)$ is isomorphic to $\psi$ in $\text{Ham}^\text{ex}_G$. By Theorem 1.8(vii) $\psi_0$ is momentum proper. Hence by Theorem 1.8(vii) $\left(\Psi, \iota^G(\rho) \right)$ is momentum proper. Therefore, $\iota^G(\rho) := \iota^G(\rho)$ is well-defined.

Since $M$ is contractible, the same holds for $Y_g$. Therefore, by the proof of [KZ18, 7.6 Lemma] we have $H = G$. Hence by Remark 1.18 $\iota^G(\rho)$ and $\psi_0$ are isomorphic in $\text{Ham}^\text{ex}_G$ and hence in $\text{Ham}^\text{cont,prop}_G$. It follows that $\iota^G(\rho)$ and $\psi$ are isomorphic in $\text{Ham}^\text{cont,prop}_G$. Hence $\iota^G(\rho)(\psi) = \Psi$. Thus $\iota^G(\rho)$ is surjective. This proves (ii) and completes the proof of Corollary 1.17.

Remarks. (This remark will be used in the next one.) We define $\text{SympRep}_G$ to be the category with objects the symplectic $G$-representations and morphisms between $\rho, \rho'$ given by pairs $(g, T)$, where $g \in G$ and $T: V \to V'$ is a linear symplectic map, such that (1.2) holds. The composition is defined by (1.3). We define the functor

$$i_G: \text{SympRep}_G \to \text{SympRep}_{\leq G}, \quad i_G(\rho) := (G, \rho), \quad i_G = \text{identity on morphisms}.$$

We may view $\text{SympRep}_G$ as a full subcategory of $\text{SympRep}_{\leq G}$ via this functor. We define the map

$$F_G: \text{SympRep}_G \to \text{SympRep}_G, \quad F_G = \text{identity on objects}, \quad F_G(g, T) := \rho_{g^{-1}}^1 T.$$

A straightforward argument shows that this map is a covariant functor.

- Part (i) of Corollary 1.17 can alternatively be deduced from Theorem 1.8(iv) as follows. Let $R, R'$ be isomorphism classes of $\text{SympRep}_G$ that are mapped to the same class under $\iota^G$. We choose representatives $\rho, \rho'$ of $R, R'$. Then $\iota^G(\rho)$ and $\iota^G(\rho')$ are isomorphic. Using Remark 1.18 it follows that $\text{Model}_G \circ i_G(\rho)$ and $\text{Model}_G \circ i_G(\rho')$ are isomorphic. Hence by Theorem 1.8(iv) there exists an isomorphism $(g, T)$ in $\text{SympRep}_{\leq G}$ from $i_G(\rho) = (G, \rho)$ to $i_G(\rho') = (G, \rho')$. It follows that $(g, T)$ is an isomorphism in $\text{SympRep}_G$ from $R$ to $R'$. Therefore, $F_G(g, T) = \rho_{g^{-1}}^1 T$ is an isomorphism in $\text{SympRep}_G$ from $F_G(\rho) = \rho$ to $F_G(\rho') = \rho'$. It follows that $R = [\rho] = [\rho'] = R'$. This shows that $\iota^G_*$ is injective, i.e., part (i) of Corollary 1.17.

- A straightforward argument shows that the map $I_G: \rho \mapsto I^G\rho$ is a natural isomorphism between the functors $\iota^G \circ F_G$ and $\text{Model}_G \circ i_G$,

$$\iota^G \circ F_G \xrightarrow{\simeq} \text{Model}_G \circ i_G.$$
This means that for every morphism \((g, T) : \rho \to \rho'\) of \(\text{SympRep}_G\) the diagram

\[
\begin{array}{ccc}
\varepsilon^G \circ \mathcal{F}_G(\rho) & \xrightarrow{\varepsilon^G \circ \mathcal{F}_G(g, T)} & \varepsilon^G \circ \mathcal{F}_G(\rho') \\
\downarrow \varepsilon^G & & \downarrow \varepsilon^G \\
\text{Model}_G \circ i_G(\rho) & \xrightarrow{\text{Model}_G \circ i_G(g, T)} & \text{Model}_G \circ i_G(\rho').
\end{array}
\]

commutes, and that \(I_G^\rho\) is an isomorphism for every object \(\rho\) of \(\text{SympRep}_G\). In other words the map \(\text{Model}_G(g, T)\) is given by

\[\text{Model}_G(g, T) = \mathcal{F}_G(g, T) = \rho_g : Y_\rho \to Y_{\rho'} \]

via the natural identifications \(I_G^\rho : V \xrightarrow{\sim} Y_\rho\) and \(I_G^{\rho'} : V' \xrightarrow{\sim} Y_{\rho'}\).

2. Proof of Theorem [LS1][VII] (Hamiltonian G-model functor)

For the proof of Theorem [LS1][I] we need the following. We denote by \(\text{Ad}\) and \(\text{Ad}^*\) the adjoint and coadjoint representations of \(G\). We define the map

\[\mu^L : T^*G \to \mathfrak{g}^*, \quad \mu^L(a, a\varphi) = \text{Ad}^*(a)\varphi.\]

This is a momentum map for the lifted left-translation action of \(G\) on \(T^*G\). We denote by \(\text{pr}_1 : T^*G \times V \to T^*G\) the canonical projection. Since left and right translations commute, \(\mu^L\) is preserved by the lifted right translation action of \(H\) on \(T^*G\). Hence the map \(\mu^L \circ \text{pr}_1\) descends to a map

\[\mu_\rho : Y_\rho \to \mathfrak{g}^*.\]

**Proof of Theorem [LS1][II].** The map \(\mu_\rho\) is a momentum map for \(\psi_\rho\). Hence \(\psi_\rho\) is a Hamiltonian action, and therefore \(\text{Model}_G\) is well-defined on objects, as claimed. \(\square\)

For the proof of Theorem [LS1][II] we need the following.

2.1. Remark (product of proper maps). Let \(X, Y, X', Y'\) be topological spaces, with \(Y\) and \(Y'\) Hausdorff. Let \(f : X \to Y\) and \(f' : X' \to Y'\) be proper continuous maps. Then the Cartesian product map \(f \times f' : X \times X' \to Y \times Y'\) is proper. This follows from an elementary argument. (Hausdorffness ensures that every compact subset of \(Y \times Y'\) is closed.)

**Proof of Theorem [LS1][II].** Let \((H, V, \sigma, \rho)\) and \((H', V', \sigma', \rho')\) be objects of \(\text{SympRep}_G\) and \((g, T)\) a morphism between them. We denote by \(\mathfrak{h}\) and \(\mathfrak{h}'\) the Lie algebras of \(H\) and \(H'\). By [LS1][I] we have \(c_{g^{-1}}(H') = H\). It follows that \(\text{Ad}^*(g)\) \(\mathfrak{h}' = \mathfrak{h}\). Hence \(\text{Ad}^*(g) = \text{Ad}^*_{g_{-1}}\) induces a map from \(\mathfrak{h}\) to \(\mathfrak{h}'\), which we again denote by \(\text{Ad}^*(g)\). We have

\[\text{Ad}^*(g)(\varphi)\mathfrak{h} = \text{Ad}^*(g)(\varphi)\mathfrak{h}, \quad \forall(a, a\varphi) \in T^*G.\]

The map

\[\rho' \circ c_\rho : H \to \{\text{isomorphisms of } (V', \sigma')\}\]

is a Hamiltonian action with momentum map

\[c_{g}^* \circ \nu_{\rho'} = \text{Ad}^*_{g} \circ \nu_{\rho'} : V' \to \mathfrak{h}^*,\]

\[\text{In the article [KZ18] we used the notation } \mu_\rho \text{ instead of } \mu_\rho. \text{ I am using } \mu_\rho \text{ here to make the dependence on } \rho \text{ explicit and to stay in line with the notation } Y_\rho, \omega_\rho.\]
where $\nu_{\rho'}$ is as in (1.4). By (1.2) $\rho'$ leaves the image $T(V)$ invariant and $T$ is a symplectic embedding that is equivariant w.r.t. $\rho$ and $\rho' \circ c_g$. It follows that

\begin{equation}
\text{Ad}_g^* \circ \nu_{\rho'} \circ T = \nu_{\rho}.
\end{equation}

(Here we use that both sides vanish at $v = 0 \in V$.) For every $(a, a\varphi, v) \in T^*G \times V$ we have

\[
\mu_{\rho'}^D \circ (R_s^{-1} \times T)(a, a\varphi, v) = \mu_{\rho}^D (ag^{-1}, ag^{-1} \text{Ad}^*(g)(\varphi), Tv)
\]

\[
= -\text{Ad}^*(g)(\varphi)|b' + \nu_{\rho'}(Tv)
\]

\[
= \text{Ad}^*(g)(-\varphi|b + \nu_{\rho}(v)) \quad \text{(using (2.2,2.3))}
\]

\[
= \text{Ad}^*(g) \circ \mu_{\rho}^D(a, a\varphi, v).
\]

The claimed inclusion (1.9) follows. We define

\[
\tilde{\Phi} := R_s^{-1} \times T : (\mu_\rho^D)^{-1}(0) \to (\mu_{\rho'}^D)^{-1}(0).
\]

Let $h \in H$. By (1.1) we have $h' := c_g(h) \in H'$. By (1.2) the map $\tilde{\Phi}$ intertwines the diagonal action of $h$ on $T^*G \times V$ with the diagonal action of $h'$ on $T^*G \times V'$. It follows that the right hand side of (1.7) does not depend on the choice of the representative $(a, a\varphi, v)$, as claimed. We denote by

\[
\tilde{\Phi} := \text{Model}_G(g, T) : Y_\rho \to Y_{\rho'},
\]

the map induced by $\tilde{\Phi}$. We show that $\tilde{\Phi}$ is a morphism of $\text{Ham}_{\rho,\psi}^G$. The map $\tilde{\Phi}$ is smooth, presymplectic, and equivariant w.r.t. the $G$-actions induced by the left translations on $G$. It follows that $\tilde{\Phi}$ is smooth, symplectic, and equivariant w.r.t. to the $G$-actions $\psi_{\rho}$ and $\psi_{\rho'}$.

2.4. Claim. The maps $T$ and $\Phi$ are proper.

Proof of Claim 2.4. The map $T : V \to V'$ is linear symplectic and hence injective. Since $V$ is finite-dimensional, it follows that

\[
\sup_{0 \neq v \in V} \frac{\|v\|}{\|Tv\|'} < \infty,
\]

where $\| \cdot \|$, $\| \cdot \|'$ are arbitrary norms on $V, V'$. This implies that $T$ is proper, as claimed.

We denote by

\begin{equation}
\pi_{\rho} : (\mu_{\rho}^D)^{-1}(0) \to Y_\rho = (\mu_{\rho}^D)^{-1}(0)/\psi_{\rho}^D
\end{equation}

the canonical projection. Let $K' \subseteq Y_{\rho'}$ be a compact subset. Since $\tilde{\Phi} \circ \pi_{\rho} = \pi_{\rho'} \circ \tilde{\Phi}$, we have

\begin{equation}
\pi_{\rho}^{-1} \circ \tilde{\Phi}^{-1}(K') = \tilde{\Phi}^{-1} \circ \pi_{\rho'}^{-1}(K').
\end{equation}

The projection $\pi_{\rho'}$ is proper, since $H'$ is compact. It follows that $\pi_{\rho'}^{-1}(K')$ is compact. The map $R_s^{-1} : T^*G \to T^*G$ is proper, since it is invertible with continuous inverse. Using Remark 2.1 and properness of $T$, it follows that the Cartesian product map $R_s^{-1} \times T : T^*G \times V \to T^*G \times V'$ is proper. Since this map restricts to $\tilde{\Phi}$ on $(\mu_{\rho}^D)^{-1}(0)$, it follows that $\tilde{\Phi}$ is proper. Since $\pi_{\rho'}^{-1}(K')$ is compact, it follows that the right hand side of (2.6) is compact, hence also the left hand side. Since $\pi_{\rho}$ maps this set to $\tilde{\Phi}^{-1}(K')$, it follows that $\tilde{\Phi}^{-1}(K')$ is compact. This proves Claim 2.4. \qed
Using Claim 2.4 it follows that \( \Phi \) is a \( G \)-equivariant proper symplectic embedding, i.e., a morphism of \( \text{Ham}^\text{ex}_G \). This proves that the map \( \text{Model}_G \) is well-defined on morphisms. This completes the proof of Theorem 1.8(ii). \( \square \)

**Proof of Theorem 1.8(iii).** It follows from a straightforward argument that \( \text{Model}_G \) maps the unit morphisms to unit morphisms and intertwines the compositions. Hence it is a covariant functor. This proves Theorem 1.8(iii). \( \square \)

For the proof of Theorem 1.8(iv) we need the following. Let \( G \) be a group, \( X \) a set, \( \psi \) an action of \( G \) on \( X \), and \( x \in X \). We denote by \( G_x := \text{Stab}_x := \{ g \in G \mid \psi_g(x) = x \} \) the stabilizer of \( x \) under \( \psi \).

2.7. **Remark.** Let \( G \) be a Lie group, \( (\rho,H) \) an object of \( \text{SympRep}_{\leq} G \), and \( y = [a,a\varphi,v] \in Y_\rho \). Then
\[
G_y := \{ c_{a}(h) \mid h \in H : \rho_h v = v \}.
\]

2.8. **Lemma.** Let \( G \) be a topological (finite-dimensional) manifold with a continuous group structure, \( N,N' \) compact submanifolds of \( G \), and \( g \in G \), such that
\[
c_{g}(N) \subseteq N',
\]
and \( N' \) is conjugate to some subset of \( N \). Then we have
\[
c_{g}(N) = N'.
\]

In the proof of this lemma we will use the following.

2.10. **Remark** (invariance of domain). Let \( M \) and \( N \) be topological manifolds of the same finite dimension, without boundary. Then every continuous injective map from \( M \) to \( N \) is open. In the case \( M = N = \mathbb{R}^n \) this is the statement of the Invariance of Domain Theorem, see [Hat02, Theorem 2B.3, p. 172]. The general situation can be reduced to this case.

**Proof of Lemma 2.8.** We choose \( g' \in G \), such that
\[
c_{g'}(N') \subseteq N,
\]
and define \( \psi := c_{g'g} : G \to G \). We have
\[
\psi(N) = c_{g'} \circ c_{g}(N) \subseteq N.
\]

Let \( A \) be a connected component of \( N \). Since \( N \) is a submanifold of \( G \), the set \( A \) is open in \( N \). The map \( \psi \) is bijective and continuous. Hence by Remark 2.10 the restriction \( \psi : N \to N \) is open. Thus \( \psi(A) \) is open in \( N \).

Since \( A \) is a connected component of \( N \), it is closed in \( N \). Since \( N \) is compact, it follows that \( A \) is compact. Therefore, \( \psi(A) \) is compact and hence a closed subset of \( N \). It follows that \( \psi(A) \) is a connected component of \( N \). Hence the map
\[
\{ \text{connected component of } N \} \ni A \mapsto \psi(A) \in \{ \text{connected component of } N \}
\]
is well-defined. This map is injective. Since \( N \) is compact, the number of its connected components is finite. It follows that the map (2.12) is surjective. It follows that \( N \subseteq \psi(N) \), and therefore, \( c_{g}^{-1}(N) \subseteq c_{g}(N) \). By (2.11) we have \( N' \subseteq c_{g}^{-1}(N) \). It follows that \( N' \subseteq c_{g}(N) \). Combining this with (2.9), it follows that \( c_{g}(N) = N' \). This proves Lemma 2.8. \( \square \)
Let $G$ be a Lie group, $(M, \omega, \psi)$ a symplectic $G$-action, and $x \in M$.

**Remark.** The isotropy representation of $\psi$ at $x$ is by definition the map

$$\rho^\psi_x : \text{Stab}_x^\psi \times T_x M \to T_x M, \quad (g, v) \mapsto d\psi_g(x)v.$$  

This is a symplectic representation of the isotropy group $\text{Stab}_x^\psi$.

In order to define the symplectic quotient representation of $\psi$ at $x$, we need the following remarks.

2.13. **Remarks** (symplectic quotient representation).  

(i) Let $G$ be a Lie group, $(M, \psi)$ a $G$-action on a manifold, and $x \in M$. We denote by $\mathfrak{g}$ the Lie algebra of $G$ and by

$$(2.14) \quad L_x := L_x^\psi : \mathfrak{g} \to T_x M$$

the infinitesimal action at $x$. The equality

$$d\psi_g(x)(\text{im} L_x) = \text{im} L_x^\psi_g(x)$$

holds.

(ii) Let $(V, \sigma)$ be a symplectic vector space and $W \subseteq V$ a linear space. We denote by $W^\sigma := \{ v \in V \mid \sigma(v, w) = 0, \forall w \in W \}$ the symplectic complement of $W$. Let $(M, \omega, \psi)$ be a symplectic $G$-action and $x \in M$. The form $\omega_x$ induces a linear symplectic form $\overline{\omega}_x$ on the quotient space

$$(2.15) \quad V_x^\psi := (\text{im} L_x)^{\omega_x} / (\text{im} L_x \cap (\text{im} L_x)^{\omega_x}).$$

It follows from (i) that $d\psi_g(x)((\text{im} L_x)^{\omega_x}) = (\text{im} L_x^\psi_g(x))^{\omega_{\psi g}(x)}$. Therefore, using (i) again, $d\psi_g(x)$ induces a map

$$(2.16) \quad V_x^\psi \to V_x^{\psi g(x)}.$$  

This map is a linear symplectic isomorphism w.r.t. $\overline{\omega}_x$ and $\overline{\omega}_{\psi g(x)}$.

We define the **symplectic quotient representation of $\psi$ at $x$** to be the map

$$(2.17) \quad \overline{\rho}^{\psi, x} : \text{Stab}_x^\psi \times V_x^\psi \to V_x^\psi,$$

where $\overline{\rho}^{\psi, x}(g, \cdot)$ is given by the map (2.16). By Remark 2.13(ii), this is a well-defined symplectic representation of $\text{Stab}_x^\psi$ on the linear symplectic quotient $V_x^\psi$ of $(\text{im} L_x)^{\omega_x}$.

2.18. **Remarks** (equivariant symplectomorphism, symplectic quotient representations). Let $G$ be a Lie group, $(M, \omega, \psi), (M', \omega', \psi')$ symplectic $G$-actions, $\Phi : M \to M'$ a $G$-equivariant symplectomorphism, $x \in M$, and $x' := \Phi(x)$.

(i) Since $\Phi$ is $G$-equivariant and injective, we have

$$\text{Stab}_x^\psi = \text{Stab}_{x'}^{\psi'}.$$  

Furthermore, we have $d\Phi(x) L_x^\psi = L_{x'}^{\psi'}$, and therefore, $d\Phi(x)(\text{im} L_x^\psi) = \text{im} L_{x'}^{\psi'}$. Since $\Phi$ is symplectic, it follows that $d\Phi(x)$ induces a map

$$V_x^\psi \to V_{x'}^{\psi'}.$$  

This map is a linear symplectic isomorphism that intertwines $\overline{\rho}^{\psi, x}$ and $\overline{\rho}^{\psi', x'}$.

\[\text{In the literature } \overline{\rho}^{\psi, x} \text{ is called "symplectic slice representation." This terminology seems misleading, since } \overline{\rho}^{\psi, x} \text{ does not involve any choice of a local slice.}\]
(ii) If \( \psi' \) is Hamiltonian with momentum map \( \mu' \) then \( \mu' \circ \Phi \) is a momentum map for \( \psi \).

2.19. Lemma (symplectic quotient representation for model action). Let \( G \) be a compact Lie group and \((H,V,\sigma,\rho)\) an object of \( \text{SympRep}_{\leq G} \). We denote
\[
(Y_\rho,\omega_\rho,\psi_\rho) := \text{Model}_G(H,\rho).
\]
Let \( y \in Y_\rho \) be a point for which \( \mu_\rho(y) \) is central and \( \text{Stab}_{\psi_\rho}^y = c_\rho(H) \), for some representative \( (a,a\varphi,v) \) of \( y \). Then \( \rho = (H,\rho) \) is isomorphic to \( \overline{\rho}_{\psi_{\rho}}^y \).

Remark. The subgroup \( c_\rho(H) \) does not depend on the choice of the representative \( (a,a\varphi,v) \) of \( y \).

In the proof of this lemma we will use the following.

2.20. Remark (momentum map). Let \((M,\omega,\psi)\) be a Hamiltonian \( G \)-action, \( \mu \) a momentum map for \( \psi \), and \( x \in M \). Then
\[
\ker d\mu(x) = (\text{im} L_\psi x)^\omega_x.
\]

Proof of Lemma 2.19. We choose a representative \( \tilde{y} := (a,a\varphi,v) \) of \( y \). We define
\[
\iota_{a,\varphi} : V \to T^*G \times V, \quad \iota_{a,\varphi}(w) := (a,a\varphi,w).
\]

2.21. Claim. (2.22) \( \text{im}(d\iota_{a,\varphi}(v)) \subseteq \ker d\mu^D_\rho(\tilde{y}) \).

Proof of Claim 2.21. By our hypothesis \( \mu_\rho(y) = \mu_L(\tilde{y}) = \text{Ad}^*(a)\varphi \) is a central element of \( g^* \). Hence, for every \( g \in G \), we have
\[
\text{Ad}^*(a)\varphi = \text{Ad}^*(c_\rho(g)) \text{Ad}^*(a)\varphi = \text{Ad}^*(a) \text{Ad}^*(g)\varphi,
\]
and therefore \( \varphi = \text{Ad}^*(g)\varphi \). Hence \( \varphi \) is a central element of \( g^* \). For every \( h \in H \), we have
\[
[a,a\varphi,v] = y
= \psi_\rho(c_\rho(h),y) \quad \text{(since } \text{Stab}_{\psi_\rho}^y = c_\rho(H))
= [c_\rho(h)a,c_\rho(h)a\varphi,v]
= [ah,a\varphi h,v] \quad \text{(using that } \varphi \text{ is central)}
= [a,a\varphi,\rho_h v],
\]
and therefore \( \rho_h v = v \). Hence \( v \) is a fixed point of \( \rho \). It follows that \( d\nu_\rho(v) = 0 \). Since \( \mu_\rho^D(a,a\varphi,w) = -\varphi|_h + \nu_\rho(w) \), it follows that
\[
d(\mu_\rho^D \circ \iota_{a,\varphi})(v) = d\nu_\rho(v) = 0.
\]
The inclusion (2.22) follows. This proves Claim 2.21.

We define \( \pi_\rho \) as in (2.5), and
\[
A_\rho^\rho := d\pi_\rho(\tilde{y})d\iota_{a,\varphi}(v) : V \to T_y Y_\rho.
\]
By Claim 2.21 this map is well-defined.

2.23. Claim. The pair \((a,A_\rho^\rho)\) is a morphism from \( \rho \) to \( \rho_{\psi_{\rho}}^y \) (the isotropy representation of \( \psi_\rho \) at \( y \)).
Proof of Claim 2.23. The map \( \iota_{a,\varphi} \) is a symplectic embedding. It follows that \( A^\rho_g \) is linear symplectic. We denote by \( \psi^L : G \times T^*G \times V \to T^*G \times V \) the action induced by the left-translation on \( G \). Let \( h \in H \). For all \( w \in V \), we have

\[
\iota_{a,\varphi} \circ \rho_h(w) = (a, a\varphi, \rho_h w)
= (ah, a\varphi h, w)
= (\psi^L_{\rho})_{ca(h)} \circ \iota_{a,\varphi}(w) \quad \text{(using that \( \varphi \) is central)}
\]

Using that \( \rho_h \) is linear, it follows that

\[
d\iota_{a,\varphi}(v) \rho_h = d\iota_{a,\varphi}(v) d\rho_h(v)
= d((\psi^L_{\rho})_{ca(h)}(\tilde{g})) d\iota_{a,\varphi}(v).
\]

Since \( \pi \circ (\psi^L_{\rho})_g = (\psi_{\rho})_g \circ \pi \), it follows that

\[
A^\rho_g \rho_h = d\pi_{\rho}(\tilde{g}) d\iota_{a,\varphi}(v) \rho_h
= d(\psi_{\rho})_{ca(h)}(g) d\pi_{\rho}(\tilde{g}) d\iota_{a,\varphi}(v)
= d(\psi_{\rho})_{ca(h)}(g) A^\rho_g.
\]

The statement of Claim 2.23 follows. \( \square \)

Let \( y \in Y_\rho \). Recall that \( L_y = L^\psi_y : g \to T_y Y_\rho \) denotes the infinitesimal \( \psi_y \)-action.

2.24. Claim.

(2.25) \( \text{im} L_y \) is isotropic.

(2.26) \( \text{im} A^\rho_g \subseteq (\text{im} L_y)^{(\omega^\rho)_y} \).

Proof of Claim 2.24. Proof of (2.25): Our hypothesis that \( \mu_{\rho}(y) \) is central implies that \( \text{im} L_y \subseteq \ker d\mu_{\rho}(y) \).

By Remark 2.20 we have

(2.27) \( \ker d\mu_{\rho}(y) = (\text{im} L_y)^{(\omega^\rho)_y} \).

Statement (2.25) follows.

Proof of (2.26): Since \( \mu_{\rho} \circ \pi_{\rho} = \mu^L \circ \text{pr}_1 \), we have

\[
d\mu_{\rho}(y) A^\rho_g = d\mu_{\rho}(y) d\pi_{\rho}(\tilde{g}) d\iota_{a,\varphi}(v)
= d(\mu_{\rho} \circ \pi_{\rho} \circ \iota_{a,\varphi})(v)
= d(\mu^L \circ \text{pr}_1 \circ \iota_{a,\varphi})(v)
= 0.
\]

Here in the last step we used that the map \( \text{pr}_1 \circ \iota_{a,\varphi} \) is constantly equal to \( (a, a\varphi) \). It follows that

\( \text{im} A^\rho_g \subseteq \ker d\mu_{\rho}(y) \).

Using (2.27), the claimed inclusion (2.26) follows. This completes the proof of Claim 2.24. \( \square \)
By part (2.25) of this claim there is a canonical projection
\[ \text{pr}^\rho_y : (\text{im}L_y)^{\omega_{\rho}y} \rightarrow V_{\psi_{\rho}} = (\text{im}L_y)^{\omega_{\rho}y}/\text{im}L_y. \]

By part (2.26) the restriction \( \text{pr}^\rho_y |\text{im}A^\rho_y \) is well-defined. It follows from Claim 2.23 and the equality \( \text{Stab}_{\psi_{\rho}y}^{\psi_{\rho}y} = c_a(H) \) that \( \text{im}A^\rho_y \) is invariant under \( \rho_{\psi_{\rho}y} \).

2.28. Claim. The pair \( (e, \text{pr}^\rho_y |\text{im}A^\rho_y) \) is an isomorphism between the restriction of \( \rho_{\psi_{\rho}y} \) to \( \text{im}A^\rho_y \) and \( \text{pr}^\rho_{\psi_{\rho}y} \).

Proof of Claim 2.28. The projection \( \text{pr}^\rho_y \) is presymplectic. Since \( \text{im}A^\rho_y \) is symplectic, the restriction \( \text{pr}^\rho_y |\text{im}A^\rho_y \) is linear symplectic and therefore injective. We have
\[ \dim(V_{\psi_{\rho}} = (\text{im}L_y)^{\omega_{\rho}y}/\text{im}L_y) = \dim(Y_\rho) - 2 \dim \text{im}L_y \]
\[ = \dim(T^*G \times V) - 2 \dim H - 2 \dim \text{im}L_y \]
\[ = 2 \dim G + \dim V - 2 \dim H - 2 \dim G + 2 \dim \text{Stab}_{\psi_{\rho}y}^{\psi_{\rho}y} \]
\[ = \dim V \quad (\text{since } \text{Stab}_{\psi_{\rho}y}^{\psi_{\rho}y} = c_a(H)) \]
\[ = \dim \text{im}A^\rho_y \quad (\text{since } A^\rho_y \text{ is linear symplectic, hence injective}) \]
\[ = \dim \left( \text{pr}^\rho_y \left( \text{im}A^\rho_y \right) \right) \quad (\text{since } \text{pr}^\rho_y |\text{im}A^\rho_y \text{ is injective}). \]

It follows that \( V_{\psi_{\rho}} = \text{pr}^\rho_y \left( \text{im}A^\rho_y \right), \) hence \( \text{pr}^\rho_y |\text{im}A^\rho_y \) is surjective. Hence this map is a linear symplectic isomorphism. It is \( \text{Stab}_{\psi_{\rho}y}^{\psi_{\rho}y} \)-equivariant. The statement of Claim 2.28 follows. □

It follows from Claims 2.23 and 2.28 that \( \rho \) and \( \text{pr}^\rho_{\psi_{\rho}y} \) are isomorphic. This proves Lemma 2.19. □

Proof of Theorem 1.8(iv). Let \((H, \rho)\) and \((H', \rho')\) be two objects of \(\text{SympRep}_{\leq G}\) whose images under \(\text{Mod}_{\rho} \subseteq G\) are isomorphic. We choose an isomorphism \(\Phi\) between these images. We define \(y := [e, 0, 0] \in Y_\rho, \quad [a', a', \varphi', v'] := y' := \Phi(y).\)

By Remark 2.7 we have
\[ \text{Stab}_{\psi_{\rho}y}^{\psi_{\rho}y} = H, \]
\[ \text{Stab}_{\psi_{\rho}y}^{\psi_{\rho}y'} \subseteq c_a'(H'). \]

Since \(\Phi\) is \(G\)-equivariant, we have
\[ \text{Stab}_{\psi_{\rho}y}^{\psi_{\rho}y} = \text{Stab}_{\psi_{\rho}y'}^{\psi_{\rho}y'}. \]

It follows that \(H \subseteq c_a'(H').\) By considering \(\Phi^{-1}\), an analogous argument shows that \(H'\) is conjugate to a subgroup of \(H\). Since \(G\) is compact and \(H\) and \(H'\) are closed, these subgroups are compact. Therefore, applying Lemma 2.28, it follows that
\[ H = c_a'(H'). \]
Since \( \mu_\rho(y) = \mu_\rho^L(e, 0, 0) = 0 \) and \( \text{Stab}_y^{\psi_\rho} = H \), the hypotheses of Lemma 2.19 are satisfied. Applying this lemma, it follows that \( \rho \) is isomorphic to \( \overline{\psi^{\rho,y}} \).

By Remark 2.18[1] \( \overline{\psi^{\rho,y}} \) is isomorphic to \( \overline{\psi^{\rho',y'}} \).

### 2.32. Claim. \( \overline{\psi^{\rho',y'}} \) is isomorphic to \( \rho' \).

**Proof of Claim 2.32.** By (2.30, 2.29, 2.31) we have \( \text{Stab}_y^{\psi_\rho} = c_{\rho'}(H') \). By Remark 2.18[1] the map \( \mu_{\rho'} \circ \Phi \) is a momentum map for \( \psi_\rho \). Since \( G \) is connected, the same holds for \( Y_\rho \). It follows that \( \mu_{\rho'} \circ \Phi - \mu_{\rho} \) is constantly equal to a central element of \( \mathfrak{g}^* \). At \( y \) this map attains the value

\[
\mu_{\rho'}(y') - \mu_{\rho}(y) = \mu_{\rho'}(y') - 0,
\]

which is thus a central element of \( \mathfrak{g}^* \). Hence the hypotheses of Lemma 2.19 are satisfied. Applying this lemma, the statement of Claim 2.32 follows. \( \square \)

Combining this claim with what we already showed, it follows that \( \rho \) is isomorphic to \( \rho' \).

Hence \( \text{Model}_G \) induces an injective map between the sets of isomorphism classes. This proves Theorem 1.8[iv]. \( \square \)

**Proof of Theorem 1.8[iv].** (vi) follows from a straightforward argument.

(vi): Let \( \rho \) and \( \rho' \) be objects of \( \text{SympRep}_{\leq G} \), such that \( \rho' \) is momentum proper and there exists a morphism \((g, T)\) from \( \rho \) to \( \rho' \). Let \( Q \subseteq \mathfrak{h} \) be compact. Equality (2.33) implies that

\[
(2.33) \quad \nu_{\rho}^{-1}(Q) = (\nu_{\rho'} \circ T)^{-1}(\text{Ad}^*(g)(Q)).
\]

The set \( \text{Ad}^*(g)(Q) \) is compact. By hypothesis \( \nu_{\rho'} \) is proper, and by Claim 2.4 the same holds for \( T \). It follows that \( \nu_{\rho'} \circ T \) is proper, and therefore, using (2.33), the set \( \nu_{\rho}^{-1}(Q) \) is compact. Hence \( \nu_{\rho} \) is proper, i.e., \( \rho \) is momentum proper, as claimed.

Let now \((M, \omega, \psi)\) and \((M', \omega', \psi')\) be objects of \( \text{Ham}_{G}^{ex} \), such that \( \psi' \) is momentum proper and there exists a morphism \( \Phi \) from \( \psi \) to \( \psi' \). We choose a momentum map \( \mu' \) for \( \psi' \). By definition, \( \Phi \) is a proper \( G \)-equivariant symplectic embedding. It follows that \( \mu' \circ \Phi \) is a proper momentum map for \( \psi \). Hence \( \psi \) is momentum proper. This proves (vi).

(vii): We prove \( \Rightarrow \): Assume that \((H, \rho)\) is momentum proper, i.e., that \( \nu_{\rho} \) is proper. Let \( K \subseteq \mathfrak{g}^* \) be compact. We denote by \( i : H \rightarrow G \) the inclusion and by \( i^* : \mathfrak{g}^* \rightarrow \mathfrak{h}^* \) the induced map. We define

\[
(2.34) \quad A := \{ (a, a\varphi, v) \in T^*G \times V \mid \text{Ad}^*(a)(\varphi) \in K, i^*\varphi = \nu_{\rho}(v) \} \subseteq \mu_{\rho}^{-1}(0).
\]

We denote by \( L_s : G \times T^*G \rightarrow T^*G \) the map induced by left translation. \( A \) is a closed subset of

\[
B := L_s(\mathfrak{g} \times \text{Ad}^*(G)(K)) \times \nu_{\rho}^{-1}(i^*\text{Ad}^*(G)(K)).
\]

Since \( G \) is compact and \( \text{Ad}^* \) is continuous, the set \( \text{Ad}^*(G)(K) \) is compact. Since \( i^* \) is continuous and \( \nu_{\rho} \) is proper, it follows that \( \nu_{\rho}^{-1}(i^*\text{Ad}^*(G)(K)) \) is compact. Using that \( L_s \) is continuous, it follows that \( B \) is compact. It follows that \( A \) is compact, and hence \( \mu_{\rho}^{-1}(K) = \pi_{\rho}(A) \) is compact. Hence \( \mu_{\rho} \) is proper. This proves \( \Rightarrow \).

\( \Leftarrow \): Assume that \( \mu_{\rho} \) is proper. Let \( Q \subseteq \mathfrak{h}^* \) be compact. We choose a compact set \( K \subseteq \mathfrak{g}^* \) such that \( i^*(K) = Q \). (We may e.g. choose a linear complement \( W \subseteq \mathfrak{g}^* \) of \( \ker i^* \) and define...
Let $(i^*)^{-1}(Q) \cap W.$ Since $H$ is compact, the map $\pi_\rho : (\mu^p)^{-1}(0) \to Y_\rho$ is proper. It follows that $\mu_\rho \circ \pi_\rho$ is proper. Hence the set 

$$(\mu_\rho \circ \pi_\rho)^{-1}(K)$$

is compact. This set agrees with $A$, defined as in (2.34). We denote by $\text{pr}_2 : T^*G \times V \to V$ the canonical projection. The set

$$C := \{ (e, \varphi, v) \mid \varphi \in K, i^*\varphi = \nu_\rho(v) \}$$

is a closed subset of $A$, hence compact. It follows that $\text{pr}_2(C)$ is compact. Since $i^*(K) = Q$, we have $\text{pr}_2(C) = \nu_\rho^{-1}(Q)$. It follows that $\nu_\rho$ is proper. This proves "$\Rightarrow$", and completes the proof of (vii) and therefore of Theorem 1.8 (except for (viii), which is proved in [KZ18 1.5. Theorem]). □

3. PROOF OF COROLLARY 1.14 (CLASSIFICATION OF CRITICAL HAMILTONIAN ACTIONS)

The well-definedness part of Corollary 1.14 follows from the next lemma.

3.1. Lemma.  

(i) The functor $T^*_G$ maps objects of $\text{Act}^\text{trans}_G$ to objects of $\text{Ham}^\text{crit}_G$. 

(ii) It maps morphisms of $\text{Act}^\text{trans}_G$ to morphisms of $\text{Ham}^\text{crit}_G$.

Proof of Lemma 3.1 (i): Let $(Q, \theta)$ be an object of $\text{Act}^\text{trans}_G$. Then $T^*_G(Q, \theta)$ is an object of $\text{Ham}^\text{ex,prop}_G$. To see that is an object of $\text{Ham}^\text{ex,prop}_G$, we denote by $\mathfrak{g}$ the Lie algebra of $G$. We choose a Finsler norm $\| \cdot \|$ on $T^*Q$ and a norm $| \cdot |$ on $\mathfrak{g}^*$. We define $L^0_q$ as in (2.14) and (3.2)

$$\mu : T^*Q \to \mathfrak{g}^*, \quad \mu(q, p) := pL^0_q.$$ 

This is a momentum map for the lifted $G$-action $\theta_s$. Since $\theta$ is transitive, the map $L^0_q$ is surjective. Therefore, an elementary argument using (3.2) and that $Q$ is compact, shows that

$$\sup \{ \| p \| \mid (q, p) \in T^*Q : |\mu(q, p)| \leq C \} < \infty, \quad \forall C \in \mathbb{R}.$$ 

It follows that $\mu$ is proper. Therefore, $T^*_G(Q, \theta)$ is an object of $\text{Ham}^\text{ex,prop}_G$. Since $Q$ is closed and $T^*Q$ deformation retracts onto $Q$, it follows that $T^*_G(Q, \theta)$ is an object of $\text{Ham}^\text{crit}_G$. This proves (i).

(ii): Let $f : Q \to Q'$ be a morphisms of $\text{Act}^\text{trans}_G$, i.e., a $G$-equivariant diffeomorphism. The induced map $f_* : T^*Q \to T^*Q'$ is a $G$-equivariant symplectomorphism, hence a morphism of $\text{Ham}^\text{ex}_G$, and therefore of $\text{Ham}^\text{crit}_G$. This proves (ii) and therefore Lemma 3.1 □

By Lemma 3.1 the restriction

$$T^*_G : \text{Act}^\text{trans}_G \to \text{Ham}^\text{crit}_G$$

is well-defined. The Chain Rule implies that it is functorial. In order to show that the map (1.15) is a bijection, we need the following lemma. We define $\text{Sub}^\text{cl}_G$ to be the category whose objects are the closed subgroups of $G$ and whose morphisms between $H$ and $H'$ are those elements $g$ of $G$, satisfying $c_g(H) = H'$.\footnote{The composition of morphisms is given by the composition in $G$.} We define the functor

$$G/ : \text{Sub}^\text{cl}_G \to \text{Act}^\text{trans}_G$$
as follows. It maps an object \( H \) to the quotient \( G/H \), equipped with the canonical left \( G \)-action. Let \((H, H', g)\) be a morphism of \( \text{Sub}^\text{cl}_G \). We denote by \( \text{pr}_H : G \to G/H \) the canonical projection and define \( G/(H, H', g) : G/H \to G/H' \) to be the unique map satisfying

\[
(3.3) \quad G/(H, H', g) \circ \text{pr}_H = \text{pr}_{H'} \circ R^{g^{-1}}.
\]

This is a well-defined morphism of \( \text{Act}^\text{trans}_G \). This construction is functorial. This defines the functor \( G/ \).

We define the functor

\[
\jmath_G : \text{Sub}^\text{cl}_G \to \text{SympRep}_{\leq G}, \quad \jmath_G(H) := (H, \{0\}, 0), \ j_G(g) := (g, 0).
\]

**3.4. Lemma.** The target-restricted functor \( \text{Model}_G \circ \jmath_G : \text{Sub}^\text{cl}_G \to \text{Ham}^\text{crit}_G \) is well-defined and naturally isomorphic to \( T_G^* \circ G/ : \text{Sub}^\text{cl}_G \to \text{Ham}^\text{crit}_G \).

**Proof of Lemma 3.4.** By Corollary 1.11 the functor \( \text{Model}_G \circ \jmath_G \) takes values in \( \text{Ham}^\text{ex,prop}_G \). Let \( H \) be an object of \( \text{Sub}^\text{cl}_G \). The manifold part of \( \text{Model}_G \circ \jmath_G(H) \) is homotopy equivalent to the closed manifold \( G/H \) and has dimension \( 2(\dim G - \dim H) \). Therefore, \( \text{Model}_G \circ \jmath_G(H) \) is critical. Hence \( \text{Model}_G \circ \jmath_G \) takes values in \( \text{Ham}^\text{crit}_G \), as claimed.

Let \( H \in \text{Sub}^\text{cl}_G \). We define \( \mu_{H, \rho}^G, Y_{H, \rho} \) as in (1.5) and denote by \( \pi_{H, \rho} : (\mu_{H, \rho}^G)^{-1}(0) \to Y_{H, \rho} \) the canonical projection. We canonically identify \( Y_{H, \rho} \) with the symplectic quotient of the Hamiltonian \( H \)-action on \( T^* G \) induced by the right \( H \)-action on \( G \). We define the map

\[
(3.5) \quad \Phi_H : T^*(G/H) \to Y_{H,0}, \quad \Phi_H(\overline{q}, \overline{p}) := \pi_{H,0}(q, \overline{p} d \pi_H(q)),
\]

where \( q \in \overline{q} \) is an arbitrary representative. This map is a symplectomorphism, see [AM78, 4.3.3 Theorem]. The map \( \Phi_H \) is \( G \)-equivariant, and therefore an isomorphism of \( \text{Ham}^\text{crit}_G \).

**3.6. Claim.** The map \( H \mapsto \Phi_H \) is a natural isomorphism between the functors \( T_G^* \circ G/ \) and \( \text{Model}_G \circ \jmath_G \).

**Proof of Claim 3.6.** Let \((H, H', g)\) be a morphism of \( \text{Sub}^\text{cl}_G \). We show that

\[
(3.7) \quad \Phi_{H'} \circ ((T^*_G \circ G/)(H, H', g)) = (\text{Model}_G \circ \jmath_G)(H, H', g) \circ \Phi_H.
\]

Let \((\overline{q}, \overline{p}) \in T^*(G/H)\). We choose a representative \( q \in \overline{q} \) and denote \( q' := R_{g^{-1}}(q), \overline{q}' := \text{pr}_{H'}(q'), \overline{\varphi} := G/(H, H', g) \). We have

\[
(T^*_G \circ G/)(H, H', g)(\overline{q}, \overline{p}) = (\overline{\varphi}(\overline{q}), \overline{p} d \overline{\varphi}(\overline{q})^{-1}),
\]

\[
\overline{\varphi}(\overline{q}) = \overline{q'} \quad (\text{using (3.3)}),
\]

and therefore,

\[
\Phi_{H'} \circ ((T^*_G \circ G/)(H, H', g))(\overline{q}, \overline{p}) = \pi_{H',0}(q', \overline{p} d \overline{\varphi}(\overline{q})^{-1} d \text{pr}_{H'}(q')) \quad (\text{using (3.5)})
\]

\[
= \pi_{H',0}(q', \overline{p} d \text{pr}_H(q) d R_{g^{-1}}(q)^{-1}) \quad (\text{using (3.3), Chain Rule})
\]

\[
= \text{Model}_G(g, 0) \circ \pi_{H,0}(q, \overline{p} d \text{pr}_H(q)) \quad (\text{using (1.7)})
\]

\[
= (\text{Model}_G \circ \jmath_G)(H, H', g) \circ \Phi_H(\overline{q}, \overline{p}) \quad (\text{using (3.5)}).
\]

Hence equality (3.7) holds. This proves Claim 3.6 and therefore Lemma 3.4.
Proof of Corollary 1.14. We show that the functor $\text{Model}_G \circ j_G$ is essentially bijective. By Corollary 1.11 the inverse of the map (1.12) is well-defined. The image of the set of isomorphism classes of $\text{Ham}_G^{\text{crit}}$ under this inverse map is contained in the image of the map between isomorphism classes induced by $j_G$. This follows from the fact that the manifold part of $\text{Model}_G (H, V, \sigma, \rho)$ is homotopy equivalent to the closed manifold $G/H$ and has dimension $2(\dim G - \dim H) + \dim V$. It follows that $\text{Model}_G \circ j_G : \text{Sub}_G^{\text{cl}} \to \text{Ham}_G^{\text{crit}}$ is essentially surjective, i.e., it induces a surjective map between the sets of isomorphism classes.

Since $j_G$ is essentially injective, Corollary 1.11 implies that the functor $\text{Model}_G \circ j_G : \text{Sub}_G^{\text{cl}} \to \text{Ham}_G^{\text{crit}}$ is also essentially injective, and therefore essentially bijective, as claimed.

Using Lemma 3.4 it follows that $T_G \circ G/ : \text{Sub}_G^{\text{cl}} \to \text{Ham}_G^{\text{crit}}$ is essentially bijective. Therefore, $T_G : \text{Act}_G^{\text{trans}} \to \text{Ham}_G^{\text{crit}}$ is essentially surjective. The functor $G/ : \text{Sub}_G^{\text{cl}} \to \text{Act}_G^{\text{trans}}$ is essentially surjective. This follows from the orbit-stabilizer theorem for $G$-actions on manifolds. Since $T_G \circ G/ : \text{Sub}_G^{\text{cl}} \to \text{Ham}_G^{\text{crit}}$ is essentially injective, it follows that $T_G : \text{Act}_G^{\text{trans}} \to \text{Ham}_G^{\text{crit}}$ is essentially injective, and therefore essentially bijective. This means that the map (1.15) is bijective. This proves Corollary 1.14.

4. Inverse of the classifying map

The next result states that the inverse of the classifying map (1.12) is induced by assigning to a Hamiltonian action its symplectic quotient representation at any suitable point. To state the result, let $G$ be a group, $X$ a set, $\psi$ a $G$-action on $X$, and $x \in X$. Recall that $\text{Stab}_x^\psi$ denotes the stabilizer of $\psi$ at $x$. We call $x$ $\psi$-maximal iff for every $y \in X$, $\text{Stab}_x^\psi$ contains some conjugate of $\text{Stab}_y^\psi$. Let $G$ be a compact and connected Lie group, $(M, \omega)$ a symplectic manifold, $\psi$ a symplectic $G$-action on $M$, and $x \in M$. Recall that $\overline{\varphi}^{\psi, x}$ denotes the symplectic quotient representation of $\psi$ at $x$, see (2.17). The latter is a symplectic representation of $\text{Stab}_x^\psi$. Hence the pair $(\text{Stab}_x^\psi, \overline{\varphi}^{\psi, x})$ is an object of $\text{SympRep}_{\leq G}$.

Assume now that $\psi$ is Hamiltonian. We call $x$ $\psi$-central iff $\mu(x)$ is a central value of $\mathfrak{g}^*$ for every momentum map $\mu$ for $\psi$. (If $M$ is connected then equivalently, there exists such a $\mu$.)

4.1. Proposition. Assume that $\psi$ is an object of $\text{Ham}_G^{\text{ex,prop}}$.

(i) There exists a $\psi$-maximal and -central point.

(ii) Let $\psi$ and $\psi'$ be isomorphic objects of $\text{Ham}_G^{\text{ex,prop}}$, $x$ be $\psi$-maximal and -central point, and $x'$ be a $\psi'$-maximal and -central point. Then $\overline{\varphi}^{\psi, x}$ and $\overline{\varphi'}^{\psi', x'}$ are isomorphic.

(iii) The inverse map of (1.12) is given by

$$\{\text{isomorphism class of } \text{Ham}_G^{\text{ex,prop}} \} \to \{\text{isomorphism class of } \text{SympRep}_{\leq G}^{\text{prop}} \},$$

$$\Psi \mapsto [\overline{\varphi}^{\psi, x}],$$

where $\psi$ is an arbitrary representative of $\Psi$, and $x$ is an arbitrary $\psi$-maximal and -central point.

Remark. It follows from (1.14) that the map (1.12) is well-defined.

In the proof of Proposition 4.1 we will use the following.

4.3. Remark. Let $\rho$ be an object of $\text{SympRep}_{\leq G}$ and $y \in Y_\rho$ a $\psi_{\rho}$-maximal and -central point. Then $\rho$ and $\overline{\varphi}^{\psi_{\rho}, y}$ are isomorphic. To see this, we write $y =: [a, a\varphi, v]$. By Remark 2.7 we have
Since $y$ is $\psi\rho$-maximal, $\text{Stab}^\psi_{\rho}y$ contains some conjugate of $\text{Stab}^\psi_{\rho}[e,0,0] = H$. Using Lemma 2.8 it follows that

$$\text{Stab}^\psi_{\rho}y = c_a(H).$$

Using that $y$ is $\psi\rho$-central, the hypotheses of Lemma 2.19 are therefore satisfied. Applying this lemma, it follows that $\rho$ and $\overline{\rho}^\psi_{\rho,y}$ are isomorphic, as claimed.

**Proof of Proposition 4.1.**

(i): Consider first the case in which there exists $\rho \in \text{SympRep}^\text{prop}_{\leq G}$, such that $\psi = \text{Model}_G(\rho)$. By Remark 2.7, the point $[e,0,0]$ is $\psi$-maximal. Since $\mu_\rho([e,0,0]) = 0$, this point is also $\psi$-central. This proves the statement in the special case.

The general situation can be reduced to this case, by using Theorem 1.8(viii) (essential surjectivity), the fact that stabilizers are preserved under equivariant injections, and Remark 2.18(ii). This proves (i).

(ii): Consider first the case in which there exists an isomorphism from $\psi$ to $\psi'$ that maps $x$ to $x'$. Then it follows from Remark 2.18(ii) that $\overline{\psi}_{\psi,x}$ and $\overline{\psi'}_{\psi',x'}$ are isomorphic.

In the general situation, using Theorem 1.8(viii) and what we just proved, we may assume w.l.o.g. that $\psi = \psi' = \psi_\rho = \text{Model}_G(H,\rho)$ for some object $\rho$ of $\text{SympRep}^\text{prop}_{\leq G}$. By Remark 4.3 we have $\overline{\psi}_{\psi,x} \cong \rho \cong \overline{\psi'}_{\psi',x'}$. This proves (ii).

(iii): Remark 4.3 implies that (4.2) is a left-inverse for (1.12). Since (1.12) is surjective, it follows that (1.12) is also a right-inverse. This proves (iii) and completes the proof of Proposition 4.1. □

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REFERENCES

[Abo12] Mohammed Abouzaid, *Framed bordism and Lagrangian embeddings of exotic spheres*, Ann. of Math. (2) 175 (2012), no. 1, 71–185. MR 2874640

[AM78] Ralph Abraham and Jerrold E. Marsden, *Foundations of mechanics*, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978, Second edition, revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman. MR 515141

[EKS16] Tobias Ekholm, Thomas Kragh, and Ivan Smith, *Lagrangian exotic spheres*, J. Topol. Anal. 8 (2016), no. 3, 375–397. MR 3509566

[ES16] Tobias Ekholm and Ivan Smith, *Exact Lagrangian immersions with a single double point*, J. Amer. Math. Soc. 29 (2016), no. 1, 1–59. MR 3402693

[FPP18] Alessio Figalli, Joseph Palmer, and Álvaro Pelayo, *Symplectic G-capacities and integrable systems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 18 (2018), no. 1, 65–103. MR 3783784

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354

[KZ18] Yael Karshon and Fabian Ziltener, *Hamiltonian group actions on exact symplectic manifolds with proper momentum maps are standard*, Trans. Amer. Math. Soc. 370 (2018), no. 2, 1409–1428. MR 3729506

[MS17] Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology*, third ed., Oxford Graduate Texts in Mathematics, Oxford University Press, Oxford, 2017. MR 3674984
**List of symbols**

The following list of symbols will hopefully help the reader navigate through this article. The last column shows the page on which the symbol is defined or occurs for the first time.

| symbol     | meaning or rôle                                                                 | p. |
|------------|---------------------------------------------------------------------------------|----|
| $$\text{Act}_{\text{trans}}^G$$ | category of transitive $$G$$-actions on connected closed manifolds           | 6  |
| $$\text{Ad}, \text{Ad}^*$$ | adjoint, coadjoint $$G$$-representation                                           | 9  |
| $$c_g$$ | conjugation by $$g$$                                                           | 2  |
| $$\mathcal{F}_G$$ | functor from $$\text{SympRep}_G$$ to $$\text{SympRep}_G$$                 | 8  |
| $$G_x = \text{Stab}_x^\psi$$ | stabilizer of $$x \in X$$ under the $$G$$-action $$\psi$$ on $$X$$             | 11 |
| $$G/$$ | functor from $$\text{Sub}_{\text{cl}}^G$$ to $$\text{Act}_{\text{trans}}^G$$ | 17 |
| $$(g, T)$$ | pair consisting of $$g \in G$$ and a linear symplectic map $$T : V \to V'$$ | 2  |
| $$\mathfrak{g}, \mathfrak{h}$$ | Lie algebras of $$G, H$$                                                      | 3  |
| $$\text{Ham}_{\text{ex}}^G$$ | category of exact Hamiltonian $$G$$-actions                                   | 2  |
| $$\text{Ham}_{\text{ex}, \text{prop}}^G$$ | full subcategory of $$\text{Ham}_{\text{ex}}^G$$ consisting of momentum proper objects | 4  |
| $$\text{Ham}_{\text{crit}}^G$$ | full subcategory of $$\text{Ham}_{\text{ex}, \text{prop}}^G$$ consisting of critical objects | 6  |
| $$\text{Ham}_{\text{contr}}^G$$ | full subcategory of $$\text{Ham}_{\text{ex}}^G$$ consisting of those objects $$(M, \omega, \psi)$$ for which $$M$$ is contractible | 7  |
| $$\text{Ham}_{\text{contr, prop}}^G$$ | full subcategory of $$\text{Ham}_{\text{contr}}^G$$ consisting of momentum proper objects | 7  |
| $$\iota_G^\rho$$ | canonical isomorphism from $$\iota^G(\rho) = \rho$$ to $$\text{Model}_G(G, \rho)$$ in $$\text{Ham}_{\text{ex}}^G$$ | 7  |
| $$\iota_G$$ | inclusion functor from $$\text{SympRep}_G$$ to $$\text{SympRep}_{\leq G}$$   | 8  |
| $$\iota_G^{\text{prop}}$$ | inclusion functor from $$\text{SympRep}_G$$ to $$\text{Ham}_{\text{contr}}^G$$ | 7  |
| $$\iota_{\text{prop}}$$ | inclusion functor from $$\text{SympRep}_{\text{prop}}^G$$ to $$\text{Ham}_{\text{contr}, \text{prop}}^G$$ | 7  |
| $$\iota_{\text{a, } \varphi}$$ | inclusion from $$V$$ to $$T^*G \times V$$ induced by $$(a, \varphi)$$ | 13 |
| $$\j_G$$ | inclusion functor from $$\text{Sub}_{\text{cl}}^G$$ to $$\text{SympRep}_{\leq G}$$ | 18 |
| $$L_x = L_x^\psi$$ | infinitesimal action at $$x$$ induced by $$\psi$$                            | 12 |
| $$\text{Model}_G$$ | Hamiltonian $$G$$-model functor                                               | 2  |
| $$(M, \omega, \psi)$$ | Hamiltonian $$G$$-action                                                      | 2  |
| $$\mu^L$$ | momentum map for the lifted left-translation action of $$G$$ on $$T^*G$$      | 9  |
| $$\mu^\rho$$ | momentum map for $$\psi^\rho$$                                              | 9  |
| $$\mu_{H, \rho}^D = \mu_{\rho}^D$$ | momentum map for $$\psi_{\rho}^D$$                                          | 3  |
| $$\nu_\rho$$ | unique momentum map for $$\rho$$ that vanishes at 0                          | 8  |
| $$\omega_Q$$ | canonical symplectic form on $$T^*Q$$                                        | 6  |
| $$\omega_x$$ | linear symplectic form on $$V^\psi_x$$ induced by $$\omega_x$$             | 12 |
| $$\text{pr}_1$$ | canonical projection from $$T^*G \times V$$ to $$T^*G$$                     | 9  |
| $$\text{pr}_2$$ | canonical projection from $$T^*G \times V$$ to $$V$$                        | 17 |
| $$\text{pr}_H$$ | canonical projection from $$G$$ to $$G/H$$                                   | 18 |
| $$\pi_{\rho}$$ | canonical projection from $$(\mu_\rho^D)^{-1}(0)$$ to $$Y_\rho$$            | 10 |
| $$\psi^D_{\rho}$$ | diagonal $$H$$-action on $$T^*G \times V$$ induced by right translation on $$G$$ and $$\rho$$ | 8  |
| $$R_{\rho}$$ | right translation by $$g \in G$$                                           | 8  |
| $$\rho_{\psi, x}$$ | isotropy representation of $$\psi$$ at $$x$$                              | 12 |
| $$\rho_{\psi, x}^\circ$$ | symplectic quotient representation of $$\psi$$ at $$x$$                   | 12 |
| Symbol       | Meaning or rôlo                                                                 |
|--------------|--------------------------------------------------------------------------------|
| Stab$_\psi^x$ = $G_x$ | stabilizer of $x \in X$ under the $G$-action $\psi$ on $X$               |
| Sub$_G^\dagger$ | category of closed subgroups of $G$                                           |
| SympRep$_{\leq G}$ | category of symplectic representations of closed subgroups of $G$          |
| SympRep$_{\leq G}^{prop}$ | full subcategory of SympRep$_{\leq G}$ consisting of momentum proper objects |
| SympRep$_G$ | category of symplectic $G$-representations                                    |
| SympRep$_G^{prop}$ | full subcategory of SympRep$_G$ consisting of momentum proper objects       |
| SympRep$_G$ | category with objects symplectic $G$-representations and morphisms $(g,T)$ |
| $T^*_G$ | $G$-cotangent functor                                                        |
| $(V, \sigma, \rho)$ | symplectic $H$-representation                                                 |
| $V^\psi_x$ | $(\text{im}L_x)^{\omega_x}/(\text{im}L_x \cap (\text{im}L_x)^{\omega_x})$ |
| $W^\sigma$ | $\sigma$-symplectic complement of $W$                                       |
| $(Y_\rho, \omega_\rho, \psi_\rho)$ | Hamiltonian $G$-model associated to $\rho$           |

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