Cosmic string loop production functions

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Abstract. Numerical simulations of Nambu-Goto cosmic strings in an expanding universe show that the loop distribution relaxes to an universal configuration, the so-called scaling regime, which is of power law shape on large scales. Precise estimations of the power law exponent are, however, still matter of debate while numerical simulations do not incorporate all the radiation and backreaction effects expected to affect the network dynamics at small scales. By using a Boltzmann approach, we show that the steepness of the loop production function with respect to loops size is associated with drastic changes in the cosmological loop distribution. For a scale factor varying as $a(t) \propto t^\nu$, we find that sub-critical loop production functions, having a Polchinski-Rocha exponent $\chi < (3\nu - 1)/2$, yield scaling loop distributions which are mostly insensitive to infra-red (IR) and ultra-violet (UV) assumptions about the cosmic string network. For those, cosmological predictions are expected to be relatively robust, in accordance with previous results. On the contrary, critical and super-critical loop production functions, having $\chi \geq (3\nu - 1)/2$, are shown to be IR-physics dependent and this generically prevents the loop distribution to relax towards scaling. In the latter situation, we discuss the additional regularisations needed for convergence and show that, although a scaling regime can still be reached, the shape of the cosmological loop distribution is modified compared to the naive expectation. Finally, we discuss the implications of our findings.

Keywords: Cosmic strings, domain walls, monopoles, gravitational waves / sources, gravitational waves / theory

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1 Introduction

The advent of gravitational wave astronomy provides an unprecedented opportunity to search for topological defects, and in particular cosmic strings [1–4]. In an expanding and decelerating universe, a cosmic string network relaxes towards an attractor configuration exhibiting universal properties — known as a scaling solution — and it subsequently remains self-similar with the Hubble radius [5–13]. Hence if cosmic strings were formed in phase transitions early in the history of the universe, scaling implies that they should be present all over the sky with a surface density growing with redshift \( z \). Strings induce anisotropies in the Cosmic Microwave Background (CMB) and they have been searched for in the Planck data [14–19]. The current CMB constraints give an upper bound for the string energy per unit length \( U \) of \( GU < 10^{-7} \), where \( G \) is the Newton’s constant. However, CMB photons come from the highest observable redshift set by their last scattering surface, namely \( z_{\text{ls}} \approx 1088 \). For gravitons, \( z \) is only bounded by our understanding of the Friedmann-Lemaître model, or
more probably by the redshift at which cosmic inflation ended. For this reason, the stochastic gravitational wave background (SGWB) is an observable particularly sensitive to cosmic strings and could provide the opportunity for a first detection.

Current constraints on $GU$ from the SGWB are already much stronger than those from the CMB, of order $GU < \mathcal{O}(10^{-11})$ [20–22] (the actual value depends on some yet unknown microphysical parameters). However, as opposed to the CMB constraints, bounds from GW crucially depend on the loop distribution. Indeed, through their production by the string network, oscillating closed cosmic string loops constitute the main source of the SGWB. Although loop production is observed and measured in Nambu-Goto cosmic string simulations [23–25], it is still a matter of debate if it plays the same role in a field theoretical model [26–29]. Clearly the detailed shape of the scaling loop distribution function is important to determine the properties of the SGWB at different frequencies. Nambu-Goto simulations from two independent groups have shown that, on large scales (see discussion below), where these simulations can be trusted, it is a power-law, namely

$$t^4 F(\gamma, t) \propto \gamma^p. \quad (1.1)$$

Here we have defined

$$\gamma(\ell, t) \equiv \frac{\ell}{t}, \quad F(\gamma, t) \equiv \frac{dn}{dt}, \quad (1.2)$$

where $n(\ell, t)$ is the number density distribution of loops of size $\ell$ at cosmic time $t$, and the time-independence of the combination $t^4 F$ is precisely the scaling regime. The simulations of ref. [23] give

$$p = -2.60^{-0.21}_{+0.15} \text{ rad,} \quad p = -2.41^{-0.08}_{+0.07} \text{ mat}. \quad (1.3)$$

Analysis of the simulations of refs. [30, 31] favours slightly different values, namely $p = -5/2$ in the radiation and $p = -2$ in the matter era. It is, however, important to stress that the approach taken in the numerical simulations of refs. [30, 31] is quite different to that of ref. [23]. In the latter reference, the shape of the scaling loop distribution $t^4 F(\gamma)$ is estimated from simulations whereas in the former references this is the shape of the scaling loop production function which is inferred from numerical results.

Let us also notice that, due to the huge disparity of scales in the problem (ranging from, for instance, the distance between kinks formed by string intercommutations, to the horizon size), numerical simulations of cosmic string networks cannot incorporate all physical effects. In Nambu-Goto simulations, in particular, effects from GW emission and backreaction onto the string dynamics are ignored.\textsuperscript{1} This is why eq. (1.1) can only be trusted for loops large enough that these effects remain negligible. GW emission means that loops loose energy and hence become smaller, with an average emitted GW power $P_{gw} = \Gamma GU^2$ where $\Gamma$ is a numerical constant estimated to be $\Gamma = \mathcal{O}(50)$ [31, 34, 35]. Hence loops decoupled from the Hubble flow shrink at an average rate given by

$$\gamma_d \equiv \Gamma GU. \quad (1.4)$$

One therefore expects eq. (1.1) to hold for loops of length $\ell \gtrsim \ell_d = \gamma_d t$ (numeric-wise, this is a quite small number already for $GU < 10^{-7}$). Emitted GWs will also backreact onto the string thereby affecting its dynamics. The consequences of this process for the network and the loops are still unknown and being studied [36]. However, one expects that loop

\textsuperscript{1}See, however, ref. [32] and more recently ref. [33] for an isolated loop.
production should be cut-off below some length scale $\ell_c \equiv \gamma_c t$, with presumably $\gamma_c \leq \gamma_d$, which we discuss below.

As was realised very early on, in practise, to include these physical effects one needs to combine results of simulations with analytical modelling. A powerful framework for this is to use a Boltzmann approach to estimate the loop distribution on cosmological time and length scales [37–43]. At this stage it is remarkable to notice that radically different assumptions about the loop production function can lead to similar powers $p$ on large scales (where the results should be fitted against simulations). Indeed, on one hand, motivated by the one-scale model of cosmic string evolution [2, 6], an often studied case is one in which [44–52]

$$P(\gamma, t) \propto \delta(\gamma - \alpha),$$

namely all stable loops are formed with size $\ell = \alpha t$ at time $t$ (for constant $\alpha$). It is then straightforward to extract the loop density distribution [6] (see section 2.2) and show that in the radiation era $p = -5/2$ while in the matter era $p = -2$. On the other hand, all cosmic string simulations show that a lot of small-scale structure, namely kinks generated from string intercommutation, build up on the strings (see refs. [53–59] for a discussion of small-scale structure on strings). As a result, one expects loops to be formed on a wide range of scales at any given time. The most recent analytical work along these lines is by Polchinski-Rocha and collaborators [38, 60, 61], who proposed a model of loop production from long strings. It is given by

$$t^5 P(\gamma > \gamma_c, t) \propto \gamma^{2\chi - 3},$$

where the parameter $\chi$ will be referred to as the Polchinski-Rocha (PR) exponent. This is clearly very different from a Dirac distribution as a loop production function. In ref. [39], the authors have included backreaction effects to the PR model and extended eq. (1.6) to the domains $\gamma < \gamma_c$, but, motivated by the numerical results of ref. [23], have considered only the cases $\chi < \chi_{\text{crit}}$ where

$$\chi_{\text{crit}} = \frac{3\nu - 1}{2}. \quad (1.7)$$

Here, we have assumed that the scale factor behaves as $a \propto t^\nu$ so that $\chi_{\text{crit}} = 0.25$ and $\chi_{\text{crit}} = 0.5$ for the radiation and matter era, respectively. Under the condition $\chi < \chi_{\text{crit}}$, refs. [38, 39] have shown that the loop distribution behaves as a power law on large scales, with the power $p$ in eq. (1.1) given by

$$p = 2\chi - 3. \quad (1.8)$$

From eqs. (1.3) and (1.8), the Nambu-Goto simulations of ref. [23] therefore give

$$\chi_R = 0.200^{+0.07}_{-0.10}, \quad \chi_M = 0.295^{+0.03}_{-0.04}, \quad (1.9)$$

for the radiation and matter era, respectively. We also note that $\chi$ has been estimated from the two-point correlators of tangent vectors along the long strings using an average over multiple Abelian Higgs simulations in ref. [28] where it was found that $\chi_R = 0.22$ and $\chi_M = 0.35$. At this stage it is intriguing to notice that the powers $p = -5/2$ in the radiation era, and $p = -2$ in the matter era, correspond precisely to $\chi = \chi_{\text{crit}}$ where the analysis of

\footnote{The PR exponent is related to the two-point correlation function of tangent vectors along cosmic strings.}
ref. [39] breaks down. One of the aims of this paper is precisely to extend the analysis of ref. [39] to the “critical case” $\chi = \chi_{\text{crit}}$ and to the “super-critical case” $\chi > \chi_{\text{crit}}$.

Before doing so, however, it is important to comment that while the two loop production functions of eqs. (1.5) and (1.6) lead to similar loop distributions on large scales, they lead to very important differences for small loops, namely for $\gamma < \gamma_d$. Until recently, these differences on small scales were of no great concern for observable predictions. For instance, predictions for the CMB power spectrum and induced non-Gaussianities are essentially blind to cosmic string loops\textsuperscript{3} (see ref. [12] for a review). However, the situation is not the same for gravitational waves. The Pochinski-Rocha (PR) loop production function induces a larger population of small loops. Small loops oscillate faster, and being more numerous, they can potentially dominate the GW emission within some frequency range.

In this paper, we show that the value of $\chi = \chi_{\text{crit}}$ is a separatrix between two different behaviours. For values $\chi < \chi_{\text{crit}}$, we recover the results presented in refs. [20, 39] and confirm the weak dependence of the scaling loop distribution on the details of the backreaction cut-off at small scales. We will refer to this property as being ultra-violet (UV) insensitive. We also show that the predicted loop number density is not affected by assumptions made for the distribution of the largest loops, and this property will be referred to as infrared (IR) insensitive. On the contrary, values of $\chi \geq \chi_{\text{crit}}$, including the equality, exhibit a very strong sensitivity to the IR. In fact, under the simplest assumptions, we show that the loop distribution cannot even reach a scaling regime and diverges in time. Scaling solutions can still be reached provided additional assumptions are made to regularise the IR behaviour, the validity of which still remains to be assessed in the cosmological context. For all these possible regularised scaling solutions, we show that the loop distribution shape is modified compared to the naive expectation.

The paper is organised as follows. In the next section, we recap the hypothesis and solutions of the Boltzmann equation presented in ref. [39]. We then show in section 2.4.2 that the solutions can be readily extended to the super-critical cases $\chi > \chi_{\text{crit}}$ and that the loop distribution never reaches scaling in that case. In section 2.5, we solve the Boltzmann equation for the critical value $\chi = \chi_{\text{crit}}$ and show again that the loop distribution diverges with time. In section 3, we discuss the extra-assumptions needed in the IR to produce a scaling loop distribution with $\chi \geq \chi_{\text{crit}}$. For those, we derive the new scaling loop distributions and critically compare the results in all three cases, sub-critical, critical and super-critical. We finally conclude by briefly discussing the implications of our findings.

2 Cosmic string loop evolution

2.1 Boltzmann equation and loop production function

The number density $n(\ell, t)$ of cosmic string loops of size $\ell$ at cosmic time $t$ is assumed to follow a conservation equation

$$\frac{d}{dt} \left( a^3 \frac{dn}{d\ell} \right) = a^3 \mathcal{P}(\ell, t),$$

(2.1)

where $\mathcal{P}(\ell, t)$ is a loop production function (LPF) giving the number density distribution of loops of size $\ell$ produced per unit of time at $t$ and $a(t)$ is the scale factor.\textsuperscript{4} For an individual

\textsuperscript{3}The tri-spectrum depends however on $\chi$ due to its sensitivity to tangent vector correlators [62, 63].

\textsuperscript{4}This equation can be generalised to include collision terms describing loop fragmentation as well as loop collisions, see ref. [37].
loop, gravitational wave emission induces energy loss through

$$\frac{d\ell}{dt} = -\gamma_d.$$  \hspace{1cm} (2.2)

Combining eqs. (2.1) and (2.2), and working in terms of the variables \((\gamma, t)\) and \(F \equiv dn/d\ell\) given in eq. (1.2), one obtains the two-dimensional Boltzmann equation

$$t \frac{\partial (a^3 F)}{\partial t} - (\gamma + \gamma_d) \frac{\partial (a^3 F)}{\partial \gamma} = a^3 t P(\gamma, t).$$  \hspace{1cm} (2.3)

Its general solution can be obtained by changing variables to \((t, v)\) where \(v = t(\gamma + \gamma_d)\). Then eq. (2.3) becomes

$$\frac{\partial [a^3 F(t, v)]}{\partial t} \bigg|_{v = a^3 t} P(t, v).$$  \hspace{1cm} (2.4)

Assuming the infinite (super-horizon) string network is in scaling, the \(t\)-dependence of the LPF is of the form

$$t^5 P(\gamma, t) = S(\gamma) = S \left( \frac{v}{t} - \gamma_d \right),$$  \hspace{1cm} (2.5)

and it is straightforward to integrate eq. (2.4) from some initial time \(t_{ini}\) and find its general solution. In terms of the variables \((\gamma, t)\) it reads

$$F(\gamma, t) - F_{ini}(\gamma, t) = \int_{t_{ini}}^{t} \left[ \frac{a(t')}{a(t)} \right]^3 S \left( \frac{(\gamma + \gamma_d) t}{t'} - \gamma_d \right) \frac{dt'}{t'^5},$$  \hspace{1cm} (2.6)

where

$$F_{ini}(\gamma, t) = \left[ \frac{a(t_{ini})}{a(t)} \right]^3 N_{ini} \left[ (\gamma + \gamma_d) t - \gamma_d t_{ini} \right],$$  \hspace{1cm} (2.7)

with \(N_{ini}(t)\) the initial loop distribution at \(t = t_{ini}\). Notice that the time dependence appears because \(F_{ini}(\gamma, t)\) is evaluated at \(t' = t_{ini}\) and physically encodes the fact that, at time \(t\), a loop of length \(\gamma t\) corresponds to an initial loop of size \(\ell = \gamma t + \gamma_d (t - t_{ini})\). Hence, once the loop production function \(S(\gamma)\) is specified over its entire domain of definition, the loop distribution is uniquely given by eq. (2.6). As mentioned in the Introduction, physically very different LPF can give similar loop distributions for large loops. We now discuss the LPF.

### 2.2 Dirac distribution for the loop production function

In order to compare with results in the literature, let us solve explicitly the Boltzmann equation with a delta function LPF, motivated by the one-scale model, given in eq. (1.5), namely \(t^5 P(\gamma, t) = c \delta(\gamma - \alpha)\). From eq. (2.6),

$$t^4 F(\gamma < \alpha, t) - t^4 F_{ini}(\gamma, t) = c \left[ \frac{a(t_{ini})}{a(t)} \right]^3 \left( \frac{\alpha + \gamma_d}{\alpha + \gamma_d} \right)^3 \left( \frac{\gamma + \gamma_d}{\alpha + \gamma_d} \right)^2 \Theta \left[ \gamma + \gamma_d - t_{ini} \frac{t}{\alpha + \gamma_d} \right].$$  \hspace{1cm} (2.8)

The left-hand side of eq. (2.8) contains \(F_{ini}\), which is determined from the initial loop distribution \(N_{ini}\) through eq. (2.7). This term is usually a transient for initial loop distribution converging fast enough to zero at large \(\ell\). However, if (as in numerical simulations) \(N_{ini}\) is assumed to be the Vachaspati-Vilenkin (VV) distribution \([64]\) one has \(t_{ini}^5 N_{ini}(\ell) \propto (t_{ini}/\ell)^{5/2}\)
and because the argument of $N_{\text{ini}}$ in eq. (2.7) grows with $t$ we see that, in the particular case of the radiation era ($\nu = 1/2$), the whole term becomes time-independent and “scales”. In a realistic situation, the VV distribution is valid up to some size, typically the initial horizon size $\ell < d_h(t_{\text{ini}})$, where $d_h(t) = t/(1 - \nu)$ with $\nu = 1/2$ or $2/3$ in the radiation or matter era, respectively. Above $d_h(t_{\text{ini}})$, loops are of super-horizon length and should actually be considered as long (dubbed “infinite”) strings from a dynamical point of view. Once the argument of $N_{\text{ini}}$ (through $F_{\text{ini}}$) in eq. (2.8) becomes larger than this cut-off, the corresponding term in the left-hand side of eq. (2.8) disappears.

Neglecting therefore the effects from initial distribution $F_{\text{ini}}(\gamma, t)$, we find the loop distribution in the radiation era:

$$t^4 F_0(\gamma, t) = c \left( \frac{\alpha + \gamma d}{\gamma + \gamma d} \right)^{3/2} \Theta(\alpha - \gamma).$$

(2.9)

This expression corresponds to a scaling solution with a $p = -5/2$ power-law for $\gamma \gg \gamma_d$, as stated in the Introduction. For loops formed during matter era one has

$$t^4 F_0(\gamma, t) = c \left( \frac{\alpha + \gamma d}{\gamma + \gamma d} \right)^2 \Theta(\alpha - \gamma),$$

(2.10)

and this corresponds to a scaling solution with a $p = -2$ power-law for $\gamma \gg \gamma_d$. Notice that, in both cases, the distributions are flat for values of $\gamma < \gamma_d$.

### 2.3 Polchinksi-Rocha loop production function

In the remainder of this paper we focus on the PR loop production function, which exhibits a power-law dependence in $\gamma$. For large loops, it is given by

$$t^5 P(\gamma \geq \gamma_c, t) = c \gamma^{2\chi - 3}. \quad (2.11)$$

The “backreaction scale” $\gamma_c$ was calculated in ref. [65] and is given by

$$\gamma_c \equiv \Upsilon(GU)^{1+2\chi}, \quad (2.12)$$

where $\Upsilon = O(20)$. This suggests that the very small scales on a string network can potentially be strongly dependent on the value of $\chi$. On scales $\gamma < \gamma_c$, the actual shape of the LPF is unknown, but, surely, loop production has to be cut-off. A phenomenologically motivated expression has been proposed in ref. [39], namely

$$t^5 P(\gamma < \gamma_c, t) = c_c \gamma^{2\chi - 3}, \quad (2.13)$$

with $\chi_c > 1$. Continuity of the loop production function at $\gamma = \gamma_c$ imposes

$$c_c = c \gamma_c^{2(\chi - \chi_c)}. \quad (2.14)$$

The scaling function $S(\gamma)$ is completely determined by eqs. (2.11) and (2.13) and reads

$$S(\gamma) = c \gamma^{2\chi - 3} \Theta(\gamma - \gamma_c) + c_c \gamma^{2\chi - 3} \Theta(\gamma_c - \gamma). \quad (2.15)$$

\footnote{The dependence on $GU$ is to be expected given that this scale is fixed by gravitational physics.}
Before giving explicit solutions of the Boltzmann equation for the PR based LPF, let us remark that the original PR model applies to loops produced by long (dubbed “infinite”) strings, whereas in numerical simulations loops are also created from other loops and can potentially reconnect. Hence, the fit to numerical simulations can be viewed as a renormalisation procedure that allows us to extend the properties of loops chopped off from long strings to those produced by other loops. In particular, the fit completely fixes the normalisation constant $c$ in the loop distribution. Unless specified otherwise, we have used the values reported in ref. [23]. Simulations show that the largest loops created in a cosmological network are as large as the largest correlation length scale, which is a fraction of the Hubble radius. This typical correlation length allows us to define

$$\gamma_\infty = \left( \frac{U}{\rho_\infty t^2} \right)^{1/2},$$

where $\rho_\infty$ is the energy density of super-horizon sized (infinite) strings in scaling [23, 53–55]. One gets $\gamma_\infty \simeq 0.32$ in the radiation era and $\gamma_\infty \simeq 0.56$ in the matter era. The PR model with values of $c$ consistent with those of simulations predicts a fractional number of loops having $\gamma \geq \gamma_\infty$. However, and as sketched in figure 1, the IR behaviour of $P(\gamma, t)$ (at large $\gamma$) could a priori be different than for $\gamma < \gamma_\infty$ and we will explore this possibility in section 3.

### 2.4 Non-critical loop production function

In this section, we present the solution of the Boltzmann equation obtained for the non-critical cases, i.e., $\chi \neq \chi_{\text{crit}}$. As shown in ref. [39], substituting eq. (2.15) into eq. (2.6) gives...
the unique solution. In the domain $\gamma \geq \gamma_c$ it reads

$$t^4 \mathcal{F}(\gamma \geq \gamma_c, t) = t^4 \mathcal{F}_{\text{ini}}(\gamma, t) + \frac{c}{\mu} (\gamma + \gamma_d)^{2\chi-3} \left[ f\left(\frac{\gamma_d}{\gamma + \gamma_d}\right) - \left(\frac{t}{t_{\text{ini}}}\right)^{-\mu} f\left(\frac{\gamma_d}{\gamma + \gamma_d} \frac{t}{t_{\text{ini}}}\right)\right],$$

(2.17)

and, in the domain $\gamma < \gamma_c$,

$$t^4 \mathcal{F}(\gamma < \gamma_c, t) = t^4 \mathcal{F}_{\text{ini}}(\gamma, t) + \frac{c}{\mu} (\gamma + \gamma_d)^{3\nu-4} (\gamma_c + \gamma_d)^{-\mu} f\left(\frac{\gamma_c}{\gamma_c + \gamma_d}\right)$$

$$- \frac{c}{\mu} (\gamma + \gamma_d)^{2\chi-3} \left(\frac{t}{t_{\text{ini}}}\right)^{-\mu} f\left(\frac{\gamma_c}{\gamma_c + \gamma_d} \frac{t}{t_{\text{ini}}}\right)$$

$$+ \frac{c c}{\mu_c} (\gamma + \gamma_d)^{2\chi_c-3} \left[f_c\left(\frac{\gamma_c}{\gamma+c}\right) - \left(\gamma_c + \gamma_d\right)^{-\mu_c} f_c\left(\frac{\gamma_d}{\gamma_c + \gamma_d}\right)\right].$$

(2.18)

In these equations, we have defined

$$f(x) \equiv {}_2 F_1(3 - 2\chi, \mu; \mu + 1; x), \quad f_c(x) \equiv {}_2 F_1(3 - 2\chi_c, \mu_c; \mu_c + 1; x),$$

(2.19)

with ${}_2 F_1(a, b; c; x)$ being the Gauss hypergeometric function, and

$$\mu \equiv 3\nu - 2\chi - 1, \quad \mu_c \equiv 3\nu - 2\chi_c - 1.$$  

(2.20)

The above solution is valid provided one waits long enough for some transient domains to disappear. For completeness, the full solution including the transients is presented in the appendix A. Let us stress that these equations become singular for $\mu = 0$, which corresponds to $\chi = \chi_{\text{crit}}$, and that case must be treated separately, see section 2.5.

The behaviour of the solution given by eqs. (2.18) and (2.17) depends on whether $\chi < \chi_{\text{crit}}$, which we refer to as the sub-critical case, or whether $\chi > \chi_{\text{crit}}$, the super-critical one.

### 2.4.1 Sub-critical loop production function

As discussed in section 2.2, the first term in the right-hand side of eq. (2.17), which is determined from the initial loop distribution, vanishes if one waits long enough. For all positive values of $\mu$, namely $\chi < \chi_{\text{crit}}$, the last term in eq. (2.17) is also a transient that asymptotically vanishes for $t \gg t_{\text{ini}}$. At vanishing argument, the hypergeometric function converges to unity and the time dependence of this term indeed scales as $(t/t_{\text{ini}})^{-\mu}$.

Hence the Boltzmann equation for $\mu > 0$ predicts a scaling loop distribution for $\gamma \geq \gamma_c$ given by

$$t^4 \mathcal{F}(\gamma \geq \gamma_c, t) = \frac{c}{\mu} (\gamma + \gamma_d)^{2\chi-3} f\left(\frac{\gamma_d}{\gamma + \gamma_d}\right).$$

(2.21)

For $\gamma \gg \gamma_d$ the hypergeometric function tends to 1, and we recover the power-law distribution given in eq. (1.8); it matches numerical simulations where gravitational effects are absent:

$$t^4 \mathcal{F}(\gamma \gg \gamma_d, t) \simeq \frac{c}{\mu} \gamma^{2\chi-3}.$$  

(2.22)

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6In the matter era, the hypergeometric function simplifies to a polynomial expression, see eq. (55) in ref. [39].
Furthermore we can now predict the effects associated with gravitational wave emission. Taking the limit \( \gamma \ll \gamma_d \) (but still \( \gamma > \gamma_c \)), one gets\(^7\)

\[
t^4\mathcal{F}_c < \gamma < \gamma_d, t \approx \frac{c}{2 - 2\chi} \frac{\gamma^{2\chi - 2}}{\gamma_d}.
\] (2.24)

Notice that since we are in the regime \( \chi < \chi_{\text{crit}} \) we necessarily have \( \chi < 1 \). The only effect of gravitational wave emission onto the scaling loop distribution is to reduce the power law exponent by one unit in the domain \( \gamma_c < \gamma < \gamma_d \) [38].

To see what are the effects of gravitational wave backreaction on the loop distribution, let us consider eq. (2.18). As before, the first and third terms in the right-hand side of eq. (2.18) are transient and only the second term and the fourth one survive. They are explicitly time-independent showing that this part of the loop distribution also reaches scaling. Using the expansion (2.23), the matching condition (2.14), and taking the limit \( \gamma \ll \gamma_c \) gives

\[
t^4\mathcal{F}(\gamma \ll \gamma_c, t) = c \left( \frac{1}{2 - 2\chi} + \frac{1}{2\chi_c - 2} \right) \frac{\gamma_c^{2\chi - 2}}{\gamma_d} + O\left( \gamma_d^{2\chi - 3} \right)
\] (2.25)

where in the last step we have taken the limit for \( \gamma_c \gg \chi \) and \( \gamma_c < \gamma_d \). This expression makes clear that the details of the backreaction process, namely the values of \( \chi_c \), have only a weak effect on the final loop distribution [39]. Therefore, in the domain \( \gamma < \gamma_c \), the scaling loop distribution is flat.

The exact form for the scaling loop distribution is plotted in figure 2 for both the radiation and matter era, see also eqs. (A.1) to (A.3). Notice that the value of \( \gamma_c \) is \( \chi \)-dependent, and thus, even at constant \( GU \), \( \gamma_c \) changes between radiation and matter.

### 2.4.2 Super-critical loop production function

As discussed in the Introduction, we now consider shallower loop production functions having \( \mu < 0 \), i.e. super-critical values of \( \chi > \chi_{\text{crit}} \). All solutions derived in section 2.4 are regular in this limit, and we can straightforwardly use eqs. (2.17) and (2.18).

In the domain \( \gamma \geq \gamma_c \), neglecting the first term in the right-hand side of eq. (2.17) for the afore-mentioned reasons, we see that the third term (which was a transient for \( \mu > 0 \)) is now becoming a growing function of time as it scales as \( (t/t_{\text{ini}})^{-\mu} \). Therefore, for \( t \gg t_{\text{ini}} \), and for all values of \( \gamma \geq \gamma_c \), the hypergeometric function that multiplies \( (t/t_{\text{ini}})^{-\mu} \) in eq. (2.17) approaches unity and one gets

\[
t^4\mathcal{F}(\gamma \geq \gamma_c, t) \approx -\frac{c}{\mu} (\gamma + \gamma_d)^{2\chi - 3} \left[ -f\left( \frac{\gamma_d}{\gamma + \gamma_d} \right) + \left( \frac{t}{t_{\text{ini}}} \right)^{-\mu} \right],
\] (2.26)

which is not scaling! Another feature of this solution is that, taking the limit \( \gamma_c \leq \gamma \ll \gamma_d \), one has

\[
t^4\mathcal{F}(\gamma_c \leq \gamma \ll \gamma_d, t) \approx -\frac{c}{\mu} \gamma_d^{2\chi - 3} \left[ -\frac{\mu}{2 - 2\chi} \left( \frac{\gamma}{\gamma_d} \right)^{2\chi - 2} + \left( \frac{t}{t_{\text{ini}}} \right)^{-\mu} \right].
\] (2.27)

\(^7\)To derive this expression, we have expanded the hypergeometric function around unity [66]

\[
f(x) \sim \frac{\Gamma(3\nu - 2\chi)\Gamma(2\chi - 2)}{\Gamma(3\nu - 3)} x^{-\nu} + \frac{\mu}{2 - 2\chi} \Gamma(3\nu - 3) x^{-\nu}.
\] (2.23)
The solution only exhibits the $\gamma^{2\chi-2}$ power-law transients. As soon as the growing term $(t/t_{\text{ini}})^{-\mu}$ takes over, the loop distribution becomes flat and incessantly grows with time. Notice that because $\mu < 0$, positiveness of the loop distribution still implies that $c > 0$ because it is now dominated by the terms $(t/t_{\text{ini}})^{-\mu}$. Equation (2.14) implies $c_c > 0$ as well.

The solution in the domain $\gamma < \gamma_c$ presents the same pathology, namely, the fourth term of eq. (2.18), which is a transient for $\mu > 0$, now becomes dominant and one gets for $\gamma \ll \gamma_c$

$$
\frac{t^4}{\mu} \mathcal{F}(\gamma \ll \gamma_c, t) \simeq -\frac{c}{\mu} \gamma_a^{2\chi-3} \left[ -\left( \frac{\mu}{2-2\chi} + \frac{\mu}{2\chi_c - 2} \right) \left( \frac{\gamma_c}{\gamma_a} \right)^{2\chi-2} + \left( \frac{t}{t_{\text{ini}}} \right)^{-\mu} \right],
$$

which is flat and smoothly connects to the solution (2.27) at $\gamma = \gamma_c$.

In figure 3, we have plotted the exact solutions at various successive redshifts showing the non-scaling behaviour of the super-critical cases, $\chi > \chi_{\text{crit}}$. The time divergence ends up washing out the change in slope of the loop distribution between $\gamma_c$ and $\gamma_d$. But scaling is lost and we have an incessantly growing number density of loops at all scales.

Because eq. (2.26) is actually valid in the regime probed by numerical simulations, this behaviour not being observed, we conclude that deeply super-critical loop production functions are unlikely to be physical. Of course, one cannot exclude the possibility that $\mu < 0$ but very close to zero (hence $\chi$ close to its critical value $\chi_c$), since the time-dependence of eq. (2.26) would remain hardly visible in time-limited numerical simulations while being relevant on cosmological time-scales. We now turn to the critical case itself, $\mu = 0$.

## 2.5 Critical loop production function

None of the solutions of section 2.4 are valid for $\mu = 0$. Hence we return to the general solution (2.6) where, using eq. (2.11) with $\chi = \chi_{\text{crit}}$ given in eq. (1.7), one has

$$
\mathcal{S}(\gamma) = c_\gamma \gamma^{3\nu-4} \Theta(\gamma - \gamma_c) + c_\gamma \gamma^{2\chi_c-3} \Theta(\gamma_c - \gamma).
$$

(2.29)
Figure 3. Growing loop distribution generated by a super-critical loop production function having $\chi = 0.45$ during the radiation era. The string tension has been set to $GU = 10^{-7}$ and the initial conditions are arbitrarily set at $z_{\text{ini}} = 10^{18}$ with $N_{\text{ini}}(t) = 0$ and $c = 0.14$. At redshift $z = 10^7$, the change of shape associated with gravitational wave backreaction becomes washed out by the number loops which diverges with time.

Here we have used the equality $2\chi_{\text{crit}} - 3 = 3\nu - 4$. As before, the initial condition at $t = t_{\text{ini}}$ and continuity of the solution at $\gamma = \gamma_c$, which is enforced by eq. (2.29), completely fix the solution of eq. (2.6). We still find a complete integral (see ref. [66]) that is presented, in full, in the appendix A.2. Below, we report only the parts relevant for our discussion. In the domain $\gamma \geq \gamma_c$, one has

$$t^4 \mathcal{F}(\gamma \geq \gamma_c, t) = t^4 \mathcal{F}_{\text{ini}}(\gamma, t) + c(\gamma + \gamma_d)^{3\nu - 4} \left[ g\left(\frac{\gamma_d}{\gamma + \gamma_d}\right) - g\left(\frac{\gamma_d}{\gamma + \gamma_d} \frac{t_{\text{ini}}}{t}\right) \right],$$  

(2.30)

and in the domain $\gamma < \gamma_c$, the solution reads

$$t^4 \mathcal{F}(\gamma < \gamma_c, t) = t^4 \mathcal{F}_{\text{ini}}(\gamma, t) + c(\gamma + \gamma_d)^{3\nu - 4} \left[ g\left(\frac{\gamma_d}{\gamma_c + \gamma_d}\right) - g\left(\frac{\gamma_d}{\gamma_c + \gamma_d} \frac{t_{\text{ini}}}{t}\right) \right].$$  

(2.31)

The function $g(x)$ is $\nu$-dependent. In the radiation era, for $\nu = 1/2$, it reads

$$g_{\text{rad}}(x) \equiv \ln \left(\frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}}\right) + 2 \cdot \frac{4 - 3x}{3(1 - x)^{3/2}},$$  

(2.32)

while in the matter era, for $\nu = 2/3$,

$$g_{\text{mat}}(x) \equiv \frac{1}{1 - x} \ln \left(\frac{1 - x}{x}\right).$$  

(2.33)
As before, neglecting the terms associated with $N_{ini}$, and taking the limit $t \gg t_{ini}$, eq. (2.30) can be further expanded for $\gamma \gg \gamma_d$ as

$$t^4 F(\gamma \gg \gamma_d, t) \simeq c \gamma^{3\nu-4} \ln \left( \frac{t}{t_{ini}} \right),$$  \hspace{1cm} (2.34)$$

for both the radiation and matter eras. As a result, the critical case $\chi = \chi_{crit}$ suffers from the same problems as the super-critical ones: the loop number distribution never reaches a scaling regime. For $\mu = 0$, the power-law exponent is $3\nu - 4 = 2\chi_{crit} - 3$ and smoothly connects to its sub- and super-critical values. Let us notice however that the time divergence is logarithmic, and therefore, could very well remain undetected in numerical simulations while being quite relevant on cosmological time-scales. The limit $\gamma_c \leq \gamma \ll \gamma_d$ gives

$$t^4 F(\gamma_c \leq \gamma \ll \gamma_d, t) \simeq c \gamma_d^{3\nu-4} \left[ \frac{1}{3 - 3\nu} \left( \frac{\gamma_c}{\gamma_d} \right)^{3\nu-3} + \ln \left( \frac{t}{t_{ini}} \right) \right],$$  \hspace{1cm} (2.35)$$

which, up to the logarithmic divergence, is in all points similar to eq. (2.27). As for the super-critical case, in the future infinity limit $t/t_{ini} \to \infty$, the dependence in $\gamma$ disappears, the loop distribution becomes flat, grows, and never reaches scaling. However, because the divergence is only logarithmic in time, even on cosmological time scales, the first term can remain dominant. In this situation, we are in presence of a very long transient scaling in the domain $\gamma \ll \gamma_d$. Finally, for the small loops $\gamma \ll \gamma_c$, and assuming $\gamma_c \ll \gamma_d$, we can expand eq. (2.31) at large times $t \gg t_{ini}$. We get

$$t^4 F(\gamma \ll \gamma_c, t) \simeq c \gamma_d^{3\nu-4} \left[ \frac{c_c}{2\chi_c - 2} \frac{\gamma_c^{2\chi_c-2}}{\gamma_d} + c \gamma_d^{3\nu-4} \left[ \frac{1}{3 - 3\nu} \left( \frac{\gamma_c}{\gamma_d} \right)^{3\nu-3} + \ln \left( \frac{t}{t_{ini}} \right) \right] \right],$$  \hspace{1cm} (2.36)$$

where the last step is obtained from eq. (2.14), which ensures the continuity of the loop production function. Again, this is in all point similar to the super-critical case of eq. (2.28) and smoothly connects to the domain $\gamma \geq \gamma_c$. The logarithmic divergence will ultimately make the small loop number density grow, although the presence of the first term will strongly delay this process and one should expect a very long transient scaling.

Figure 4 shows the loop number density distribution in the radiation era as derived from the exact expression, eqs. (A.7) to (A.9), for $GU = 10^{-7}$, and at various redshifts. Here again, $N_{ini} = 0$ has been assumed to clearly show the effects coming from the production function. The network is arbitrarily assumed to be formed at $z_{ini} = 10^{18}$ and relaxation from the initial conditions takes place down to redshift $z = 10^{17}$. For redshifts $z \leq 10^{15}$, the domain $\gamma \geq \gamma_d$ clearly exhibits the logarithmic divergence. The loops having $\gamma < \gamma_d$ remain, however, in the transient scaling for essentially all the cosmological evolution.
2.6 Discussion

Critical and super-critical loop production functions, having $\chi \geq \chi_{\text{crit}} = (3\nu - 1)/2$, yield a non-scaling and growing population of cosmic string loops. This results from the combination of various non-trivial effects acting together. For $\chi \geq \chi_{\text{crit}}$, the loop production functions are shallower with respect to loop sizes than the sub-critical ones. Therefore, they produce, on site, relatively more larger loops compared to the smaller ones. These larger loops will contribute to the final population of loops of given size since they incessantly shrink by gravitational wave emission. Similarly, at all times, loops of given size disappear by the same effect. The detailed balance of loops disappearing, being created on site, and being populated by shrunk larger loops is obviously $\chi$-dependent and the overall result is precisely given by the solution of the Boltzmann equation (2.3). Taking shallower loop production functions clearly enhances the feeding by larger loops, at all scales. The critical value $\chi_{\text{crit}}$ is the precise power-law exponent above which such an effect produces a non-stationary solution. We summarise our results in table 1.

In striking contrast with the sub-critical case, we see that the critical and super-critical loop production functions induce non-scaling loop distributions. This is quite dramatic in the super-critical case as the number density of loops grows, on all scales, as $(t/t_{\text{ini}})^{-\mu}$, with $\mu < 0$. The situation for the critical case $\chi = \chi_{\text{crit}}$ is, somehow, less catastrophic, as the divergence is only logarithmic in time. In particular, for most of the cosmologically relevant situations, we find that the loop number density remains in a transient scaling regime at small scales, for all $\gamma \ll \gamma_d$. The number density of larger loops, having $\gamma \geq \gamma_d$, is however logarithmically growing with time and never scales.
Table 1. Asymptotic contributions to the loop number density assuming no infrared regularisation. At late times, the critical and super-critical cases are non-scaling and the loop number density diverges. For the critical case, notice however that a transient scaling can take place in the domains $\gamma < \gamma_d$ for most of the cosmological evolution (see text).

3 Possible infrared regularisations

In view of the previous discussion, a way to regularise (super-) critical loop production functions is to change their shape in some domains. As discussed in the Introduction, the PR model does not necessarily apply to super-horizon loops, the ones having $\gamma > \gamma_\infty$, and these ones seem to be precisely responsible for the time divergence. A possible regularisation is therefore making a hard cut in the IR, namely postulating that the loop production function is exactly vanishing above some new IR scale, say $\gamma > \gamma_\infty$. Other regulator shapes are considered in section 3.3.

We now consider the same PR loop production function as in section 2 for $\gamma < \gamma_\infty$ but we now require that $t^4 P(\gamma > \gamma_\infty, t) = 0$ at all times. As a result, there is a new domain of solution for eq. (2.3) in which one trivially finds

$$\mathcal{F}(\gamma \geq \gamma_\infty, t) = \mathcal{F}_{\text{ini}}(\gamma, t).$$

(3.1)

The calculations are slightly longer than in section 2 but do not present new difficulties. They are detailed in the appendix B. The introduction of a new scale at $\gamma_\infty$ introduces various new transient domains in which the loop distribution $t^4 \mathcal{F}$ grows for a while before becoming stationary. Ignoring these domains, the main changes can be summarised as follows.

The asymptotic solutions are given by those of the previous section provided we make the formal replacement

$$\frac{t}{t_{\text{ini}}} \to \frac{\gamma_\infty + \gamma_d}{\gamma + \gamma_d}.$$  

(3.2)

This expression makes clear that all terms that were explicitly depending on $t/t_{\text{ini}}$ are regularised to $\gamma$-dependent terms. As a result, the IR-regularised critical loop distribution reaches scaling, but it does no longer exhibit the same shape on large scales. In the following, we explicitly derive the induced distortions for the critical and super-critical case and discuss the impact of forcing an unneeded IR-regularisation to the sub-critical loop production functions.

3.1 Critical loop production function

For critical loop production function $\chi = \chi_{\text{crit}}$, after the disappearance of the transient domains (see appendix B), the loop distribution in the domain $\gamma \geq \gamma_c$ (and $\gamma < \gamma_\infty$) reads

$$t^4 \mathcal{F}(\gamma_c \leq \gamma < \gamma_\infty, t) = t^4 \mathcal{F}_{\text{ini}}(\gamma, t)$$

$$+ c(\gamma + \gamma_d)^{3\nu - 4} \left[ g\left( \frac{\gamma_d}{\gamma + \gamma_d} \right) - g\left( \frac{\gamma_d}{\gamma_\infty + \gamma_d} \right) \right],$$

(3.3)
and in the domain $\gamma < \gamma_c$, one gets

$$
t^4 F(\gamma < \gamma_c, t) = t^4 F_{ini}(\gamma, t) + \frac{c_c}{\mu_c} (\gamma + \gamma_d)^{2x_c - 3} \left[ f_c\left(\frac{\gamma_d}{\gamma + \gamma_d}\right) - \left(\frac{\gamma + \gamma_d}{\gamma_c + \gamma_d}\right)^{\mu_c} f_c\left(\frac{\gamma_d}{\gamma_c + \gamma_d}\right) \right] + c(\gamma + \gamma_d)^{3\nu - 4} \left[ g\left(\frac{\gamma_d}{\gamma_c + \gamma_d}\right) - g\left(\frac{\gamma_d}{\gamma_c + \gamma_d}\right) \right].
$$

The logarithmic growth in time has disappeared, and the solutions are now scaling. Taking eq. (3.3) in the limit $\gamma \gg \gamma_d$ and neglecting all terms associated with the initial conditions, one gets

$$
t^4 F(\gamma \gg \gamma_d, t) \simeq c \gamma^{3\nu - 4} \ln \left(\frac{\gamma_\infty}{\gamma}\right).
$$

(3.4)

The limit $\gamma_c < \gamma \ll \gamma_d$ consistently gives

$$
t^4 F(\gamma_c < \gamma \ll \gamma_d, t) = c \gamma_d^{3\nu - 4} \left[ \frac{1}{3 - 3\nu} \left(\frac{\gamma}{\gamma_d}\right)^{3\nu - 3} + \ln \left(\frac{\gamma_d}{\gamma_\infty}\right) \right],
$$

(3.5)

and the distribution is back to the scaling power law $\gamma^{3\nu - 3}$.

Finally, small loops with $\gamma \ll \gamma_c \ll \gamma_d$ also scale with a flat distribution as

$$
t^4 F(\gamma \ll \gamma_c, t) = c \gamma_d^{3\nu - 4} \left[ \left(\frac{1}{3 - 3\nu} + \frac{1}{2x_c - 2}\right) \left(\frac{\gamma_c}{\gamma_d}\right)^{3\nu - 3} + \ln \left(\frac{\gamma_d}{\gamma_\infty}\right) \right].
$$

(3.6)

In conclusion, the IR-regularisation we have used solves the logarithmic time divergence of the loop distribution which now reaches scaling on all length scales. For $\gamma \ll \gamma_d$, eqs. (3.5) and (3.6) compared to eqs. (2.35) and (2.36) show that the regularisation is neat, the dependence of the loop distribution with respect to $\gamma$ is not affected. However, for $\gamma > \gamma_d$, the power law behaviour now receives a logarithmic correction. We therefore conclude that the critical loop production function, even regularised, exhibits a IR sensitivity.

### 3.2 Non-critical loop production function

The calculation follows in all points the one of section 3.1 and applies to both sub- and super-critical cases, $\mu > 0$ and $\mu < 0$. The full solution is presented in the appendix B and we focus below on the asymptotic behaviour only. For the purely IR domain, $\gamma > \gamma_\infty$, the solution is still given by eq. (3.1), our IR-regulator assuming an exactly vanishing production function there. Again neglecting all transients, the solution in the domain $\gamma_c \leq \gamma < \gamma_\infty$ reads

$$
t^4 F(\gamma_c \leq \gamma < \gamma_\infty, t) = t^4 F_{ini}(\gamma, t) + \frac{c}{\mu} (\gamma + \gamma_d)^{2x_c - 3} f\left(\frac{\gamma_d}{\gamma + \gamma_d}\right)
- \frac{c}{\mu} (\gamma + \gamma_d)^{3\nu - 4} \left(\gamma_\infty + \gamma_d\right)^{-\mu} f\left(\frac{\gamma_d}{\gamma_\infty + \gamma_d}\right),
$$

(3.7)
while for $\gamma < \gamma_c$ one obtains
\begin{align}
t^4 F(\gamma_c \leq \gamma < \gamma_c, t) &= t^4 F_{\text{ini}}(\gamma, t) \\
&+ \frac{c_c}{\mu_c} (\gamma + \gamma_d)^{2\chi - 3} \left[ f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right)^{\mu_c} - \left( \frac{\gamma + \gamma_d}{\gamma_c + \gamma_d} \right)^{\mu_c} f_c \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) \right] \\
&+ \frac{c}{\mu} (\gamma + \gamma_d)^{3\nu - 4} (\gamma_c + \gamma_d)^{-\mu} f \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) \\
&- \frac{c}{\mu} (\gamma + \gamma_d)^{3\nu - 4} (\gamma_d + \gamma_\infty)^{-\mu} f \left( \frac{\gamma_d}{\gamma_\infty + \gamma_d} \right).
\end{align}

(3.8)

Here again, the IR cut in the loop production functions can be viewed as the same formal replacement as (3.2). Let us now discuss separately the physical consequences for the sub- and super-critical loop production functions and we start by the simplest case which is the sub-critical one.

3.2.1 Sub-critical case

Even if sub-critical loop production functions produce a scaling loop distribution without any regularisation, one may wonder whether forcing the (unnecessary, for scaling!) cut at $\gamma > \gamma_\infty$ can significantly change the shape of the scaling loop distribution.

At late times, and for sub-critical production functions, $\mu > 0$, we can take the limit $\gamma \rightarrow \infty$ of (3.7)
\begin{align}
t^4 F(\gamma_d \ll \gamma < \gamma_\infty, t) &\simeq \frac{c}{\mu} \gamma_d^{2\chi - 3} \left[ 1 - \left( \frac{\gamma}{\gamma_\infty} \right)^{\mu} \right].
\end{align}

(3.9)

Compared to eq. (2.22), we see that the correction term $(\gamma/\gamma_\infty)^{\mu}$ induced by the IR-regularisation has an effect only for $\gamma \simeq \gamma_\infty$ and becomes rapidly negligible as soon as $\gamma < \gamma_\infty$. For loops having $\gamma \ll \gamma_d$, we get
\begin{align}
t^4 F(\gamma_c \leq \gamma \ll \gamma_d, t \geq t_c) &\simeq \frac{c}{2 - 2\chi} \frac{\gamma_d^{2\chi - 2}}{\gamma_d},
\end{align}

(3.10)

the correction $(\gamma_d/\gamma_\infty)^{\mu}$ can always be safely ignored. Finally, for loops smaller than the GW backreaction length, $\gamma \ll \gamma_c$, we recover eq. (2.25). The IR-correction added corresponds to the fourth term of eq. (3.8) and remains again always negligible for $\mu > 0$.

We therefore conclude that sub-critical loop production functions yield scaling loop distributions that are immune to the IR behaviour of the network.

3.2.2 Super-critical case

For super-critical values of $\chi > \chi_{\text{crit}}$, we have $\mu < 0$ and most of the arguments applying for $\mu > 0$ are now reversed. For instance, the limit $\gamma_d \ll \gamma < \gamma_\infty$ becomes
\begin{align}
t^4 F(\gamma_d \ll \gamma < \gamma_\infty, t) &\simeq -\frac{c}{\mu} \gamma_d^{2\chi - 3} \left[ \left( \frac{\gamma_\infty}{\gamma} \right)^{-\mu} - 1 \right] \simeq -\frac{c}{\mu} \gamma_\infty^{-\mu} \gamma_d^{3\nu - 4}.
\end{align}

(3.11)

The time divergence of the loop distribution is solved but the power-law exponent has been changed from $2\chi - 3$ to $3\nu - 4$, see eq. (2.26). For smaller loops, we get
\begin{align}
t^4 F(\gamma_c \leq \gamma \ll \gamma_d, t) &\simeq -\frac{c}{\mu} \gamma_d^{2\chi - 3} \left[ -\frac{\mu}{2 - 2\chi} \left( \frac{\gamma}{\gamma_d} \right)^{2\chi - 2} + \left( \frac{\gamma_\infty}{\gamma_d} \right)^{-\mu} \right].
\end{align}

(3.12)
Since $\gamma_{\infty}/\gamma_d \gg 1$, the IR cut is adversely introducing a new length scale! Thus, let us define $\gamma_{\text{ir}}$ by
\[
\gamma_{\text{ir}} \equiv \left[ \frac{-\mu}{(2 - 2\chi)\gamma_{\infty}^\mu} \right]^{\frac{1}{2-2\chi}} \gamma_{d}^{\frac{3-3\nu}{2-2\chi}}.
\]
(3.13)

For $\gamma > \gamma_{\text{ir}}$, eq. (3.12) shows that the loop distribution is flat, the dependence in $\gamma$ remains negligible compared to the constant term introduced by the regularisation. On the contrary, for $\gamma < \gamma_{\text{ir}}$, we recover a power-law behaviour as $\gamma^{2\chi-2}$. This new IR scale is relevant only if $\gamma_{\text{ir}} > \gamma_{c}$, which is model- and regularisation-dependent. Nonetheless, if we assume the dependency in $GU$ for $\gamma_{d}$ given in eq. (1.4),
\[
\gamma_{\text{ir}} \propto (GU)^{\frac{3-3\nu}{2-2\chi}},
\]
(3.14)
and using eq. (2.12)
\[
\frac{\gamma_{\text{ir}}}{\gamma_c} \propto (GU)^{\frac{4\chi^2 - 2\chi + 1 - 3\nu}{2\chi}}.
\]
(3.15)

This defines a particular value for $\chi$, namely
\[
\chi_{\text{ir}} \equiv \frac{1 + \sqrt{12\nu - 3}}{4},
\]
(3.16)
whose numerical value in the radiation era is $\chi_{\text{ir}} \approx 0.683$ and $\chi_{\text{ir}} \approx 0.809$ for the matter era. For all values $\chi_{\text{crit}} < \chi < \chi_{\text{ir}}$, the exponent of eq. (3.15) is negative. For $GU$ small enough, we generically have $\gamma_{\text{ir}} > \gamma_{c}$. As a result, the regularised loop distribution is now scaling but exhibits a new plateau for $\gamma_{\text{ir}} < \gamma < \gamma_{d}$, which smoothly connects to the $\gamma^{2\chi-2}$ behaviour in the domain $\gamma_{c} \leq \gamma < \gamma_{\text{ir}}$. For larger values of $\chi > \chi_{\text{ir}}$ (and deeper negative values of $\mu$), only the plateau exists in the whole domain $\gamma_{c} \leq \gamma < \gamma_{d}$, the amplitude of the constant term $(\gamma_{\infty}/\gamma_{d})^{-\mu}$ is so large that it erases any features that could be associated with the scale of gravitational wave emission. This situation is actually reminiscent with the time-divergent behaviour discussed in section 2.4.2.

Finally, for the very small loops, $\gamma \ll \gamma_{c}$, with $\gamma_{c} \ll \gamma_{d}$, the loop distribution reads
\[
t^4F(\gamma \ll \gamma_{c}, t \geq t_c) \simeq c \left( \frac{1}{2 - 2\chi} + \frac{1}{2\chi - 2} \right) \gamma_c^{2\chi-2} - \frac{c}{\mu} \gamma_c^{-\mu} \gamma_{d}^{3\nu-4} + O(\gamma_{d}^{2\chi-3}).
\]
(3.17)

It is scaling with a plateau behaviour. The amplitude of the plateau is either given by the first term, the one varying as $\gamma_c^{2\chi-2}/\gamma_{d}$, or the second term which is proportional to $\gamma_c^{-\mu} \gamma_{d}^{3\nu-4}$. That depends on their relative amplitude. Neglecting the terms in $\chi_{c}$, which are sub-dominant, the ratio $R$ of the first to second term in the right-hand side of eq. (3.17) simplifies to
\[
R = \left( \frac{\gamma_{\text{ir}}}{\gamma_{c}} \right)^{2-2\chi}.
\]
(3.18)

Consistently with the behaviour in the $\gamma > \gamma_{c}$ domains, for $\chi_{\text{crit}} < \chi < \chi_{\text{ir}}$, one always has $R \gg 1$ and the regularization effects are small. Only for $\chi > \chi_{\text{ir}}$, the plateau at $\gamma < \gamma_{c}$ is dominated by the regulator and continuously matches the one at $\gamma > \gamma_{c}$.

We conclude that IR-regularisation of super-critical loop production functions solves their time-divergence, but this has the consequence of significantly modifying the shape of the actual scaling distribution. The results are therefore strongly IR-sensitive.
3.3 Influence of a power-law IR-regularisation

Considering the strong dependence of the loop number density on the parameter $\gamma_\infty$, one might ask whether the shape of the IR-cutoff has an additional influence on the results. To perform this analysis, we introduce an additional source term $c_\infty \gamma_\infty^{2\chi_\infty - 3} \Theta(\gamma - \gamma_\infty)$ to the collision term of the Boltzmann equation (2.11) and, neglecting all possible transients, compute its contribution, say $t^4 F_\infty$, to the asymptotic loop number density. For this source term to be a well-behaved IR-regulator, it has to fulfil two conditions. First $\mu_\infty > 0$ otherwise we expect this term to present the same time-divergent behaviour as the critical and super-critical distributions. Then, we should have $c_\infty = c \gamma_\infty^{2(\chi_\infty - \chi_\infty)}$ for the loop production function to be continuous in $\gamma_\infty$. Then the contribution of such a power-law cutoff is

$$t^4 F_\infty(\gamma < \gamma_\infty) = \frac{c_\infty (\gamma + \gamma_d)^{3\nu - 4}}{\mu_\infty (\gamma_\infty + \gamma_d)^{\mu_\infty}} f_\infty \left( \frac{\gamma_d}{\gamma_\infty + \gamma_d} \right)$$

where

$$f_\infty(x) \equiv {\frac{2}{\Gamma(1 - 2\chi_\infty; \mu_\infty)}},$$

The condition $\mu_\infty > 0$ ensures that all time-dependent contributions are suppressed at late-times. Under the assumption that $\gamma_d \ll \gamma_\infty$, the contribution to the scaling loop number density coming from the power-law cutoff is

$$t^4 F_\infty(\gamma < \gamma_\infty) = c_\infty \left( \frac{\gamma + \gamma_d}{\mu_\infty} \right)^{3\nu - 4} \mu_\infty \gamma_\infty.$$

This additional part generically contributes and can modify the shape of the loop distribution, as for instance it would modify the value of $\gamma_\text{ir}$ for the super-critical case in eq. (3.13). However, for large enough values of $\mu_\infty$, namely for $\mu_\infty \gg |\mu|$, it can safely be neglected with respect to the one computed earlier. As a result, the IR-regularisation effects we have found in the previous section are relatively generic in the sense that they are not simply induced by the choice of an infinitely sharp cut in the LPF but rather by suppressing the production of large loops.

4 Conclusions

The aim of this paper has been to carry out an exhaustive study of the effect of the loop production function on the cosmological distribution of loops. As explained in the Introduction, numerical simulations of Nambu-Goto cosmic string networks are not currently able to capture some important physical effects at very small scales, for instance GW emission and its backreaction effects. Hence determining the loop distribution, by construction, requires an interplay between numerical results (valid for larger loops where the extra physics should be negligible) and analytical modelling.

The analytical tool used to solve for the loop distribution is the Boltzmann equation (2.11). On the one hand, we have shown that very different LPF, namely, a Dirac distribution motivated by the one-scale model, and a sub-critical Polchinski-Rocha power-law distribution ($\chi < \chi_{\text{crit}}$) taking into account the small-scale structure built up on the strings, can give rise to a scaling, power-law, distribution on large scales, albeit with different
power-law exponents. On the other hand, we have found that the actual value of the power-law exponent, i.e., the value of $\chi$ with respect to $\chi^{\text{crit}} = (3\nu - 1)/2$, produces very different behaviours. Critical and super-critical LPFs ($\chi \geq \chi^{\text{crit}}$) lead to time-divergent loop distributions, which do not scale. The critical case however exhibits only a logarithmic growth for large loops, $\gamma \geq \gamma_d$, and a very long transient scaling for the smaller ones, $\gamma < \gamma_d$, that can last longer than the age of the universe.

The divergent behaviour of the critical and super-critical cases has been traced back to a relative over-production of large loops with respect to small loops and we have shown that it can be regularised by arbitrarily assuming that the PR loop production function vanishes above some length scale $\gamma_\infty$. We find, however, that although such a IR regularisation fixes the time divergence, it is also changing the shape of the loop distribution. For this reason, we conclude that both the critical and super-critical LPF are genuinely IR-sensitive. For the critical case, we find that the large loop distribution acquires a new logarithmic dependence in $\gamma$ (again for $\gamma \geq \gamma_d$). On the small scales $\gamma < \gamma_d$, the predictions are all very different and depend on both the PR exponent $\chi$ and on the IR regulator. The results of our study are summarised in tables 1 and 2, where we give the asymptotic contributions to the loop number density on all scales $\gamma$ depending on the value of the parameter $\mu \equiv 3\nu - 2\chi - 1$ (it vanishes for $\chi = \chi^{\text{crit}}$). Let us notice that for extreme values of $GU$, and times close to the transition from the radiation to the matter era, these results may not apply and one should rely on the complete solutions given in the appendices.

It is interesting to observe from the last row of table 2, that in the super-critical case only and assuming an IR cutoff, the obtained distribution for large $\gamma \geq \gamma_d$ is essentially identical to that obtained from assuming a Dirac distribution for the LPF. In particular, for large $\gamma$ there is a $-5/2$ power-law in the radiation era and $-2$ power-law in the matter era, which are the values for the exponents that we have obtained in section 2.2. At the same time, both distributions are completely different on smaller scales. This is illustrated in figure 5.

In this paper, following ref. [39], we have also introduced a small distance scale $\gamma_c$ below which gravitational backreaction is expected to be important. Generically, for $\gamma_c \ll \gamma_d$, and for all values of $\chi$, the amplitude of the loop distribution at small $\gamma < \gamma_c$ is enhanced relative to the Dirac distribution LPF, and, as discussed in ref. [20], this leads to observational consequences on the SGWB. Another interesting feature we have not discussed in the main
Figure 5. Difference between loop distributions in the radiation era generated by a Dirac distribution LPF (green lower curve) and a super-critical, IR-regularised, Polchinski-Rocha one (purple top curve). Given a super-critical power-law loop production function, one can reproduce the large scale behavior of the loop distribution with a Dirac distribution for the loop production function (see section 2.2). Doing so, one loses the small-scale behavior of the loop distribution. For illustration purposes, we have chosen $GU = 10^{-7}$, $c \simeq 0.25$ and $\gamma_{\infty} = 0.1$ for the super-critical LPF and $c \simeq 5.7$ for the Dirac distribution.

text concern the various transient domains associated with the IR regularisation. They are excited soon after the network is created, but also during the transition from the radiation to matter era. As such, they may also lead to interesting phenomenological consequences, in particular regarding a gravitational wave signature.

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A Complete solutions

In this appendix, we give the explicit expressions of the solution of the Boltzmann equation (2.3). Details of the calculation can be found in refs. [39, 41] and we here simply report the results.

A.1 Non-critical loop production function

For the piecewise PR loop production function given in eqs. (2.11) and (2.13), assuming \( \chi \neq \chi_{\text{crit}} \), one gets

\[
t^4 F(\gamma \geq \gamma_c, t) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\}
\]

\[
+ \frac{c}{\mu} (\gamma + \gamma_d)^{2}\chi - 3 f \left( \frac{\gamma_d}{\gamma + \gamma_d} \right)
\]

\[
- \frac{c}{\mu} (\gamma + \gamma_d)^{2}\chi - 3 \left( \frac{t}{t_{\text{ini}}} \right)^{-\mu} f \left( \frac{\gamma_d}{\gamma + \gamma_d} \right),
\]

(A.1)

\[
t^4 F(\gamma \leq \gamma < \gamma_c, t) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\}
\]

\[
+ \frac{c_c}{\mu_c} (\gamma + \gamma_d)^{2}\chi - 3 \left[ f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) - \left( \frac{\gamma + \gamma_d}{\gamma_c + \gamma_d} \right)^{\mu_c} f_c \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) \right]
\]

\[
+ \frac{c}{\mu} (\gamma + \gamma_d)^{3}\nu - 4 (\gamma_c + \gamma_d)^{-\mu} f \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right)
\]

\[
- \frac{c}{\mu} (\gamma + \gamma_d)^{2}\chi - 3 \left( \frac{t}{t_{\text{ini}}} \right)^{-\mu} f \left( \frac{\gamma_d}{\gamma + \gamma_d} \right),
\]

(A.2)

\[
t^4 F(0 < \gamma < \gamma_t, t) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\}
\]

\[
+ \frac{c_c}{\mu_c} (\gamma + \gamma_d)^{2}\chi - 3 f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right)
\]

\[
- \frac{c_c}{\mu_c} (\gamma + \gamma_d)^{2}\chi - 3 \left( \frac{t}{t_{\text{ini}}} \right)^{-\mu_c} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right),
\]

(A.3)

where we recap that

\[
f(x) \equiv \frac{\Gamma(3 - 2\chi, \mu; \mu + 1; x)}, \quad f_c(x) \equiv \frac{\Gamma(3 - 2\chi_c, \mu_c; \mu_c + 1; x)},
\]

(A.4)

and

\[
\mu \equiv 3\nu - 2\chi - 1, \quad \mu_c \equiv 3\nu - 2\chi_c - 1.
\]

(A.5)

There is a transient domain for loops having \( \gamma \) smaller than

\[
\gamma_t(t) \equiv (\gamma_c + \gamma_d) \frac{t_{\text{ini}}}{t} - \gamma_d,
\]

(A.6)

which describes a virgin population of loops that started their evolution with a \( \gamma < \gamma_c \) and which have never been contaminated by shrunk loops produced at \( \gamma > \gamma_c \). This population of loops cannot exist forever and the domain disappears for times \( t \geq t_{\gamma_r} \) where \( \gamma_r(t_{\gamma_r}) = 0 \).
A.2 Critical loop production function

In the critical case, the piecewise loop production function is given by eq. (2.11) in the domain $\gamma \geq \gamma_c$ with $\chi = \chi_{\text{crit}}$, and eq. (2.13) for $\gamma < \gamma_c$ which is unchanged. The solution reads

$$t^4 F(\gamma \geq \gamma_c, t) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 \frac{t_{\text{ini}}}{\gamma} N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\},$$

$$t^4 F(\gamma \leq \gamma < \gamma_c, t) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 \frac{t_{\text{ini}}}{\gamma} N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\},$$

$$t^4 F(0 < \gamma < \gamma_r, t) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 \frac{t_{\text{ini}}}{\gamma} N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\},$$

where we recap that the first integral $g(x)$ is given by

$$g_{\text{rad}}(x) \equiv \ln \left( \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} \right) + \frac{2}{3} - 3x \left( \frac{1 - x}{1 - x} \right)^{3/2}, \quad g_{\text{mat}}(x) \equiv \frac{1}{1 - x} \ln \left( \frac{1 - x}{x} \right),$$

in the radiation and matter era, respectively. Notice that the small scales transient, eq. (A.3), is identical to eq. (A.9). To ease comparison with the non-critical case, let us stress that for $\chi = \chi_{\text{crit}}$, one has $\mu = 0$ and $2\chi_c - 3 = 3\nu - 4$ such that the critical functional shape is smoothly interpolating between the sub- and super-critical solutions presented in section A.1.

B Sharp infrared regularisation

The sharp IR-regularisation consists in cutting the loop production function above some length scale $\gamma_\infty$. Therefore, it is a piecewise function over three domains: for $\gamma < \gamma_c$ it is given by eq. (2.13), for $\gamma_c \leq \gamma < \gamma_\infty$ by eq. (2.11) and for $\gamma \geq \gamma_\infty$ it is vanishing. The new length scale $\gamma_\infty$ introduces a new, time-dependent, length scale defined by

$$\gamma_+(t) \equiv (\gamma_d + \gamma_\infty) \frac{t_{\text{ini}}}{t} - \gamma_d.$$
\[ \gamma = \gamma_c \] and \( \gamma = \gamma_\infty \). In doing so, we must distinguish the cases for which \( \gamma_+(t) > \gamma_c \) from those having \( \gamma_+(t) < \gamma_c \). To this end, we define \( t = t_c \) through \( \gamma_+(t_c) = \gamma_c \) from which

\[ t_c \equiv \frac{\gamma_\infty + \gamma_d \mu_{t_{\text{ini}}}}{\gamma_d + \gamma_c} \quad (B.2) \]

If we compare eqs. (A.6) and (B.1), we have \( \gamma_+(t_{\text{ini}}) = \gamma_c \) and \( \gamma_+(t_{\text{ini}}) = \gamma_\infty \); the domains never collide: \( \gamma_+(t) - \gamma_+(t) = (\gamma_\infty - \gamma_c)(t_{\text{ini}}/t) > 0 \). At last, the domain \( \gamma < \gamma_+(t) \) disappears completely for \( t > t_+ \) where

\[ t_+ \equiv \left(1 + \frac{\gamma_\infty}{\gamma_d}\right) t_{\text{ini}}, \quad (B.3) \]

which is defined by \( \gamma_+(t_+) = 0 \). The different transient domains thus defined are summarized in figure 6. In practice, the solution is affected by the IR cutoff only within the red dashed zones appearing in this figure, but for completeness, we give, and repeat, the solutions in all contiguous domains.

**B.1 Non-critical loop production function**

We distinguish the two cases, \( t \leq t_c \) and \( t > t_c \). During the relaxation period \( t \leq t_c \), the solution reads

\[ t^4 F(\gamma \geq \gamma_\infty, t < t_c) = \left(\frac{t}{t_{\text{ini}}}\right)^4 \left(\frac{a_{\text{ini}}}{a}\right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left(1 - \frac{t_{\text{ini}}}{t}\right) \right] t \right\}, \quad (B.4) \]

\[ t^4 F(\gamma_+ \leq \gamma < \gamma_\infty, t < t_c) = \left(\frac{t}{t_{\text{ini}}}\right)^4 \left(\frac{a_{\text{ini}}}{a}\right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left(1 - \frac{t_{\text{ini}}}{t}\right) \right] t \right\} \]

\[ + \frac{c}{\mu} (\gamma + \gamma_d)^{2x-3} f\left(\frac{\gamma_d}{\gamma + \gamma_d}\right) \]

\[ - \frac{c}{\mu} (\gamma + \gamma_d)^{3x-4} (\gamma_\infty + \gamma_d)^{-\mu} f\left(\frac{\gamma_d}{\gamma_\infty + \gamma_d}\right), \quad (B.5) \]

\[ t^4 F(\gamma_c \leq \gamma < \gamma_+, t < t_c) = \left(\frac{t}{t_{\text{ini}}}\right)^4 \left(\frac{a_{\text{ini}}}{a}\right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left(1 - \frac{t_{\text{ini}}}{t}\right) \right] t \right\} \]

\[ + \frac{c}{\mu} (\gamma + \gamma_d)^{2x-3} f\left(\frac{\gamma_d}{\gamma + \gamma_d}\right) \]

\[ - \frac{c}{\mu} (\gamma + \gamma_d)^{2x-3} \left(\frac{t}{t_{\text{ini}}}\right)^{-\mu} f\left(\frac{\gamma_d}{\gamma + \gamma_d t_{\text{ini}}}\right), \quad (B.6) \]
For later times, $t \geq t_c$, we get the solution

\begin{align}
t^4 F(\gamma \geq \gamma_\infty, t \geq t_c) &= \left(\frac{t}{t_{\text{ini}}} \right)^4 \left(\frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\}, \\
t^4 F(\gamma_c \leq \gamma < \gamma_\infty, t \geq t_c) &= \left(\frac{t}{t_{\text{ini}}} \right)^4 \left(\frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\} \\
&\quad + \frac{c}{\mu} (\gamma + \gamma_d)^{2\varepsilon - 3} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) \\
&\quad - \frac{c}{\mu} (\gamma + \gamma_d)^{3\varepsilon - 4} (\gamma_\infty + \gamma_d)^{-\mu} f \left( \frac{\gamma_d}{\gamma_\infty + \gamma_d} \right), \\
t^4 F(\gamma_+ \leq \gamma < \gamma_c, t \geq t_c) &= \left(\frac{t}{t_{\text{ini}}} \right)^4 \left(\frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\} \\
&\quad + \frac{c}{\mu} (\gamma + \gamma_d)^{2\varepsilon - 3} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) \\
&\quad + \frac{c}{\mu} (\gamma + \gamma_d)^{3\varepsilon - 4} (\gamma_c + \gamma_d)^{-\mu} f \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) \\
&\quad - \frac{c}{\mu} (\gamma + \gamma_d)^{3\varepsilon - 4} (\gamma_d + \gamma_\infty)^{-\mu} f \left( \frac{\gamma_d}{\gamma_\infty + \gamma_d} \right), \\
t^4 F(\gamma_\tau \leq \gamma < \gamma_+, t \geq t_c) &= \left(\frac{t}{t_{\text{ini}}} \right)^4 \left(\frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 N_{\text{ini}} \left\{ \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right\} \\
&\quad + \frac{c}{\mu} (\gamma + \gamma_d)^{2\varepsilon - 3} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) \\
&\quad + \frac{c}{\mu} (\gamma + \gamma_d)^{3\varepsilon - 4} (\gamma_c + \gamma_d)^{-\mu} f \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) \\
&\quad - \frac{c}{\mu} (\gamma + \gamma_d)^{2\varepsilon - 3} f \left( \frac{\gamma_d}{\gamma + \gamma_d} \right),
\end{align}
\[ t^4 \mathcal{F}(0 < \gamma < \gamma_\tau, t \geq t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right\} t \]
\[ + \frac{c_{\text{c}}}{\mu_c} (\gamma + \gamma_d)^{2\chi_c - 3} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) \]
\[ - \frac{c_{\text{c}}}{\mu_c} (\gamma + \gamma_d)^{2\chi_c - 3} \left( \frac{t}{t_{\text{ini}}} \right)^{-\mu_c} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) f_c \left( \frac{t_{\text{ini}}}{\gamma + \gamma_d} \right). \] (B.13)

Neglecting all transients and initial condition effects, these equations show that the IR cut can be viewed as the formal replacement written in eq. (3.2).

### B.2 Critical loop production function

For the critical case \( \chi = \chi_{\text{crit}} \) and the sharp IR cut at \( \gamma_{\infty} \), one gets during the relaxation times \( t < t_c \)

\[ t^4 \mathcal{F}(\gamma \geq \gamma_{\infty}, t < t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right\} t, \] (B.14)

\[ t^4 \mathcal{F}(\gamma_+ \leq \gamma < \gamma_{\infty}, t < t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right\} t \]
\[ + c(\gamma + \gamma_d)^{3\nu - 4} \left[ g \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) - g \left( \frac{\gamma_d}{\gamma_{\infty} + \gamma_d} \right) \right], \] (B.15)

\[ t^4 \mathcal{F}(\gamma_c \leq \gamma < \gamma_+, t < t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right\} t \]
\[ + c(\gamma + \gamma_d)^{3\nu - 4} \left[ g \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) - g \left( \frac{\gamma_d}{\gamma + \gamma_d} \frac{t_{\text{ini}}}{t} \right) \right], \] (B.16)

\[ t^4 \mathcal{F}(\gamma_\tau \leq \gamma < \gamma_c, t < t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right\} t \]
\[ + \frac{c_{\text{c}}}{\mu_c} (\gamma + \gamma_d)^{2\chi_c - 3} \left[ f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) - \left( \frac{\gamma + \gamma_d}{\gamma_{\chi_c} + \gamma_d} \right)^{\mu_c} f_c \left( \frac{\gamma_d}{\gamma_{\chi_c} + \gamma_d} \right) \right] \]
\[ + c(\gamma + \gamma_d)^{3\nu - 4} \left[ g \left( \frac{\gamma_d}{\gamma_{\chi_c} + \gamma_d} \right) - g \left( \frac{\gamma_d}{\gamma + \gamma_d} \frac{t_{\text{ini}}}{t} \right) \right], \] (B.17)

\[ t^4 \mathcal{F}(0 < \gamma < \gamma_\tau, t < t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right\} t \]
\[ + \frac{c_{\text{c}}}{\mu_c} (\gamma + \gamma_d)^{2\chi_c - 3} \left( \frac{t}{t_{\text{ini}}} \right)^{-\mu_c} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) \]
\[ - \frac{c_{\text{c}}}{\mu_c} (\gamma + \gamma_d)^{2\chi_c - 3} \left( \frac{t}{t_{\text{ini}}} \right)^{-\mu_c} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) \frac{t_{\text{ini}}}{t}. \] (B.18)

Finally, for times \( t \geq t_c, \gamma_+(t) \) becomes smaller than \( \gamma_c \) and the complete critical IR-regularised loop distribution reads

\[ t^4 \mathcal{F}(\gamma \geq \gamma_{\infty}, t \geq t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right\} t, \] (B.19)

\[ t^4 \mathcal{F}(\gamma_c \leq \gamma < \gamma_{\infty}, t \geq t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left\{ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right\} t \]
\[ + c(\gamma + \gamma_d)^{3\nu - 4} \left[ g \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) - g \left( \frac{\gamma_d}{\gamma_{\infty} + \gamma_d} \right) \right], \] (B.20)
$$t^4 \mathcal{F}(\gamma_+ \leq \gamma < \gamma_c, t \geq t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right)$$

$$+ \frac{c_c}{\mu_c} (\gamma + \gamma_d)^{2 \chi_c - 3} \left[ f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) - \left( \frac{\gamma + \gamma_d}{\gamma_c + \gamma_d} \right)^{\mu_c} f_c \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) \right]$$

$$+ c (\gamma + \gamma_d)^{3 \nu - 4} \left[ g \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) - g \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) \right],$$

(B.21)

$$t^4 \mathcal{F}(\gamma_r \leq \gamma < \gamma_+, t \geq t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right)$$

$$+ \frac{c_c}{\mu_c} (\gamma + \gamma_d)^{2 \chi_c - 3} \left[ f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right) - \left( \frac{\gamma + \gamma_d}{\gamma_c + \gamma_d} \right)^{\mu_c} f_c \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) \right]$$

$$+ c (\gamma + \gamma_d)^{3 \nu - 4} \left[ g \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) - g \left( \frac{\gamma_d}{\gamma_c + \gamma_d} \right) \right],$$

(B.22)

$$t^4 \mathcal{F}(0 < \gamma < \gamma_r, t \geq t_c) = \left( \frac{t}{t_{\text{ini}}} \right)^4 \left( \frac{a_{\text{ini}}}{a} \right)^3 t_{\text{ini}}^4 \mathcal{N}_{\text{ini}} \left[ \gamma + \gamma_d \left( 1 - \frac{t_{\text{ini}}}{t} \right) \right] t \right)$$

$$+ \frac{c_c}{\mu_c} (\gamma + \gamma_d)^{2 \chi_c - 3} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right)$$

$$- \frac{c_c}{\mu_c} (\gamma + \gamma_d)^{2 \chi_c - 3} \left( \frac{t}{t_{\text{ini}}} \right)^{\mu_c} f_c \left( \frac{\gamma_d}{\gamma + \gamma_d} \right).$$

(B.23)

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