Indeterminate-length quantum coding

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Abstract

The quantum analogues of classical variable-length codes are indeterminate-length quantum codes, in which codewords may exist in superpositions of different lengths. This paper explores some of their properties. The length observable for such codes is governed by a quantum version of the Kraft-McMillan inequality. Indeterminate-length quantum codes also provide an alternate approach to quantum data compression.

Introduction

The development of quantum information theory is a striking example of the fruitful hybridization of two well-established disciplines. Both quantum mechanics and information theory have a rich set of concepts and a powerful toolbox of mathematical techniques. Their combination is yielding powerful insights into the physical meaning of “information” [1, 2].

One approach to this exploration is to begin with an idea of “classical” information theory and investigate how this idea must be re-interpreted or modified to fit into the quantum information framework. Ideas of fidelity, quantum data compression [3], quantum error correcting codes [4], and the capacities of various quantum channels [5] can all be viewed in this light.
A basic idea in the classical theory of data compression is the idea of a variable-length code. A variable-length code assigns to different messages codewords consisting of different numbers of symbols. If shorter codewords are used for more common messages and longer ones for less common messages, the average codeword length can be made shorter than would be possible using a fixed-length code. (Natural languages take advantage of this idea. Common words like “the” are often very short, while unusual words like “sesquipedalian” are longer.)

However, the original development of quantum data compression followed a different route, parallel to the classical development based on “typical sequences”. This left open the question of whether there was a quantum analogue to classical variable-length coding. Because a quantum code must allow superpositions of different codewords—including superpositions of codewords of different lengths—the quantum version would best be termed an indeterminate-length quantum code.

One of us [4] made a preliminary investigation of this idea several years ago. Subsequently, Braunstein et al. [5] presented a quantum analogue to classical Huffman coding. Because a general understanding of indeterminate-length quantum codes was not available then, Braunstein et al. were led to construct their code in an unnecessarily inefficient way. (See the discussion in Section 2.5, below.) More recently, Chuang and Modha have developed a quantum version of arithmetic coding as a route to quantum data compression [6]. Boström has also investigated indeterminate-length codes in connection with lossless quantum coding [7].

Our aim in this paper is to outline a general theory of indeterminate-length quantum codes, including their application to quantum data compression.

We will first sketch a framework for discussing such codes. Each code will have a “codeword length” observable \( \Lambda \) with integer eigenvalues; allowable codewords include not only length eigenstates but arbitrary superpositions of them. The key requirement is that such codes be “condensable”—that is, that the individual codewords can be assembled into a string by means of a unitary operation. This condition leads us to prove a quantum version of the Kraft-McMillan inequality. Among the condensable codes are those that satisfy a quantum “prefix-free” condition, and we show (by giving an explicit condensation algorithm) that all such codes are condensable. We also show how classical variable-length codes can be used to construct quantum indeterminate-length codes with analogous properties.
We next turn to the use of indeterminate-length codes for quantum data compression. We achieve quantum data compression by taking a condensed string of $N$ codewords (in general no shorter than $N$ times the largest eigenvalue of $\Lambda$) and truncating it after the first $N\ell$ qubits, thus using only $\ell$ qubits per input codeword. We show that the average $\langle l \rangle$ of the codeword length observable $\Lambda$ is the necessary and sufficient value of $\ell$ to achieve high fidelity for this process. It turns out that $\langle l \rangle$ is related to the quantum entropy $S$ of the quantum information source, and from this relation we are able to arrive at the noiseless quantum coding theorem.

1 Indeterminate-length codes

1.1 Zero-extended forms

In a quantum code, codewords are states of finite strings of qubits. Superpositions of codewords are also valid codewords, and to maintain high fidelity we must preserve the coherence of these superpositions in our coding and decoding processes.

We wish to create a code in which different codewords have different lengths—that is, they involve different numbers of qubits. But how do we make sense of this idea? We’ll begin by considering zero-extended forms (zef) of the codewords. For zef codewords, we imagine that the codewords are sitting at the beginning of a qubit register of fixed length, with $|0\rangle$’s following. These codewords span a subspace of the Hilbert space of register states.

Our first essential requirement is that the codewords carry their own length information. That is, we require that there is a “length” observable $\Lambda$ on the zef codeword subspace with the following two properties:

- The eigenvalues of $\Lambda$ are $1, \ldots, l_{\text{max}}$, where $l_{\text{max}}$ is the length of the register.

- If $|\psi_{\text{zef}}\rangle$ is an eigenstate of $\Lambda$ with eigenvalue $l$, then it has the form

$$|\psi_{\text{zef}}\rangle = |\psi_{1\ldots l}0^{l+1\ldots l_{\text{max}}}\rangle. \quad (1)$$

In other words, the last $l_{\text{max}} - l$ qubits in the register are in the state $|0\rangle$ for a zef codeword of length $l$. 
The length observable \( \Lambda \) was also considered in \[7\].

For each \( l = 1, \ldots, l_{\max} \), we let \( d_l \) be the dimension of the subspace spanned by the \( \Lambda \)-eigenstates with eigenvalue \( l \). Denote the projection onto this subspace by \( \pi_l \). Then \( \text{Tr} \pi_l = d_l \).

### 1.2 Condensable codes

We want to be able to make use of the comparative shortness of some code-words by “packing” the codewords together, eliminating the trailing zeroes that “pad” the ends of the zef codewords. But this must be a process that maintains quantum coherences in superpositions of codeword states—that is, it must be described by a unitary transformation. Furthermore, we wish to be able to coherently pack together any number of codewords.

We say that a code is condensable if the following condition holds: For any \( N \), there is a unitary operator \( U \) (depending on \( N \)) that maps

\[
\ket{\psi_1,\text{zef}} \otimes \cdots \otimes \ket{\psi_N,\text{zef}} \mapsto \ket{\Psi_1 \cdots \Psi_N,\text{zef}}
\]

with the property that, if the individual codewords are all length eigenstates, then \( U \) maps the codewords to a zef string of the \( Nl_{\max} \) qubits—that is, one with \( |0\rangle \)'s after the first \( L = l_1 + \cdots + l_N \) qubits:

\[
\ket{\psi_1^{l_1} \cdots \psi_N^{l_N} |0^{l_1} \cdots 0^{l_N}} \mapsto \ket{\Psi_1^{l_1} \cdots \Psi_N^{l_N} 0^{L} \cdots 0^{Nl_{\max}}}.
\]

This process is called condensation. Since every codeword is a superposition of length eigenstates, it suffices to specify how the condensation process functions for such codewords.

Note that we have made no assumptions about the details of the condensation process. In the most straightforward case, condensation would be accomplished by concatenation of the codewords. The condensed state in Equation 3 would be of the form

\[
\ket{\Psi^{1\cdots L} 0^{L+1\cdots Nl_{\max}}} = \ket{\psi_1^{l_1} \cdots \psi_N^{l_N} 0^{L} \cdots 0^{Nl_{\max}}}.
\]

This special type of condensation is called simple condensation, and those codes whose codewords can be condensed in this way are said to be simply condensable codes. Obviously, all simply condensable codes are condensable; but the converse is not true.
The condensability condition is phrased as an “encoding” requirement, but the unitary character of the packing process automatically yields a decoding condition—we can unpack a condensed string by applying the $U^{-1}$ transformation.

It is interesting to compare the analogous classical situation. Classical codewords in a variable-length code can always be concatenated into a “packed” string. Only for uniquely decipherable codes is this packing reversible. In the quantum case, since arbitrary superpositions of codewords are also legal codewords, the concatenation process itself must be unitary. This automatically implies that it can be reversed.

1.3 The quantum Kraft-McMillan inequality

Given that the codewords carry their own length information and form a condensable code, we next derive a condition on the codeword length observable. Fix a value of $N$ and consider all codeword strings that have given values of $l_1, l_2, \ldots, l_N$. These states lie in a subspace of dimension $d_{l_1}d_{l_2}\cdots d_{l_N}$, and all of them are mapped by $U$ into something of the form $|\Psi^{1\cdots L} 0^{L+1\cdots NL_{\text{max}}}\rangle$.

Next, imagine strings of codewords with different lengths $l_1', l_2', \ldots, l_N'$, but whose lengths sum to the same total length: $L' = L$. The space spanned by these has dimension $d_{l_1'}d_{l_2'}\cdots d_{l_N'}$ and is orthogonal to the previous space. We can consider all such combinations of lengths that sum to the same $L$. Each of these states maps under $U$ to something of the form $|\Psi^{1\cdots L} 0^{L+1\cdots NL_{\text{max}}}\rangle$, so we obtain

$$\left(\frac{\text{dimension of space containing all codeword strings with the same } L}{\text{dimension of space containing all strings }|\Psi^{1\cdots L} 0^{L+1\cdots NL_{\text{max}}}\rangle}\right) \leq \sum_{l_1+\cdots+l_N=L} d_{l_1} \cdots d_{l_N} \leq 2^L.$$

It follows that

$$2^{-L} \sum_{l_1+\cdots+l_N=L} d_{l_1} \cdots d_{l_N} = \sum_{l_1+\cdots+l_N=L} \left(2^{-l_1}d_{l_1}\right) \cdots \left(2^{-l_N}d_{l_N}\right) \leq 1.$$

There are at most $NL_{\text{max}}$ possible values of $L$. If we sum both sides of this equation over those values, the resulting sum on the left-hand side will include
all possible values of $l_1, \ldots, l_N$. Therefore,

$$
\sum_{l_1, \ldots, l_N} \left( 2^{-l_1}d_1 \right) \cdots \left( 2^{-l_N}d_N \right) = \left( \sum_l 2^{-l}d_l \right)^N \leq N l_{\text{max}}.
$$

This is of the form $K^N \leq N l_{\text{max}}$. If $K > 1$, then this inequality must be violated for sufficiently large $N$. Thus, we conclude that $K \leq 1$. But

$$
K = \sum_l 2^{-l}d_l = \sum_l 2^{-l} \text{Tr}_l \pi_l = \text{Tr} \left( \sum_l 2^{-l} \pi_l \right).
$$

This gives us our quantum version of the Kraft-McMillan inequality. For any indeterminate-length quantum code that is condensable, the length observable $\Lambda$ on $\text{zef}$ codewords must satisfy

$$
\text{Tr} \ 2^{-\Lambda} \leq 1
$$

(5)

(where the trace is taken over the subspace of $\text{zef}$ codewords).

### 1.4 Prefix-free codes

An alternate condition that we might impose on our indeterminate-length quantum code is that the code be prefix-free—informally, that no initial segment of a $\text{zef}$ codeword is itself a codeword. In the next section, we will show that all prefix-free codes are simply condensable. In this section, we will discuss the meaning of the prefix-free condition and show that any condensable code can be transformed into a prefix-free code with the same length characteristics.

Suppose $|\psi_1\rangle$ and $|\psi_2\rangle$ are length eigenstate $\text{zef}$ codewords with lengths $l_1$ and $l_2$, respectively; and further suppose that $l_2 > l_1$. These states have the form

$$
|\psi_1\rangle = \left| \psi_1^{1\cdots l_1}0^{l_1+1\cdots l_{\text{max}}} \right>, \quad |\psi_2\rangle = \left| \psi_2^{1\cdots l_2}0^{l_2+1\cdots l_{\text{max}}} \right>.
$$

(6)

For the codeword $|\psi_1\rangle$, the quantum state of the first $l_1$ qubits of the register is just the pure state $|\psi_1^{1\cdots l_1}\rangle$. For the codeword $|\psi_2\rangle$, the first $l_1$ qubits may be in a mixed state, described by the density operator

$$
\rho_2^{1\cdots l_1} = \text{Tr}_{l_1+1\cdots l_{\text{max}}} |\psi_2\rangle\langle \psi_2| = \text{Tr}_{l_1+1\cdots l_2} |\psi_2^{1\cdots l_2}\rangle\langle \psi_2^{1\cdots l_2}|.
$$

(7)
We say that our code is *prefix-free* if, for all such pairs of codewords,

\[
\langle \psi_1^{l_1} | \rho_2^{l_1} | \psi_1^{l_1} \rangle = 0.
\]  

(8)

In other words, the first \(l_1\) qubits of a codeword of length \(l_1\) have a state that is orthogonal to the (possibly mixed) state of the first \(l_1\) qubits of a codeword of length \(l_2 > l_1\).

Another way of expressing this condition is to say that a length eigenstate zef codeword of length \(l\) can be distinguished from a codeword of greater length by making a measurement only on the first \(l\) qubits. Shorter codewords can be “recognized” from shorter segments. This means that the operator \(\pi_l\), the projection onto the subspace of length eigenstate zef codewords of length \(l\), has the form

\[
\pi_l = \pi_1^{l_1} \otimes 1^{l+1\cdots l_{\text{max}}}.
\]

(9)

Of course, actually to measure the codeword length \(\Lambda\) would be disastrous, because such a real measurement would destroy the coherence of superpositions of length eigenstates without possibility of restoration. The condensation process must therefore not include any measurement of length information. On the other hand, the process may include interactions by which, at some intermediate stage, a quantum computer has become entangled with codeword length information—provided that, by the end of the computation, this entanglement has been eliminated. In Section 1.5 we discuss this in more detail.

A particularly simple way of generating a prefix-free quantum code is to use a classical prefix-free code as a basis for the zef codeword subspace. For example, the classical codewords 0, 10, 110 and 111 form a prefix-free set. The corresponding quantum code can be specified by giving an orthogonal basis of length eigenstate zef codewords, as follows:

| state | length |
|-------|--------|
| |000\rangle | 1 |
| |100\rangle | 2 |
| |110\rangle | 3 |
| |111\rangle | 3 |

The length observable \(\Lambda\) for this code is

\[
\Lambda = |000\rangle\langle000| + 2|100\rangle\langle100| + 3|110\rangle\langle110| + 3|111\rangle\langle111|.
\]

(10)
Of course, any superposition of these is also a zef codeword, though not necessarily a codeword of definite Λ. It is easy to verify that the code defined in this way satisfies the criterion for a quantum “prefix-free” code. The procedure illustrated here may be extended in the obvious way to create a quantum prefix-free code from any classical prefix-free code.

Suppose we have a indeterminate-length quantum code that satisfies the quantum Kraft-McMillan inequality. Then the space of zef codewords of this code is spanned by a basis of eigenstates of this code’s length observable Λ. Let |ψ_{zef,l,i}⟩ be the i-th such basis vector with length eigenvalue l, and let n_l be the number of basis vectors that have length l. (Thus for a given l, i ranges from 1 to n_l.) The quantum Kraft-McMillan inequality (Eq. 5) implies that

\[ \sum_l n_l 2^{-l} \leq 1. \tag{11} \]

Given values of l and n_l that satisfy Eq. 11, we can construct a classical prefix-free code with n_l distinct codewords of length l bits. (In this case, Eq. 11 is just the classical Kraft inequality.) Denote by C_{l,i} the i-th codeword of length l bits in this prefix-free code. We use this classical prefix-free code to create a quantum prefix-free code by constructing a basis of length eigenstates, whose elements are |C_{l,i}0^{l+1-L_{max}}⟩.

Now consider the mapping

\[ |ψ_{zef,l,i}⟩ → |C_{l,i}0^{l+1-L_{max}}⟩. \tag{12} \]

This is a mapping from orthogonal basis vectors to orthogonal basis vectors that can be extended linearly to a unitary mapping V on the entire Hilbert space. V takes the original codewords to prefix-free codewords in a length-preserving way—that is, the length observable Λ’ of the prefix-free code is given by Λ’ = VΛV†. In short, any quantum code that satisfies Equation 5 can be unitarily mapped to a prefix-free quantum code with identical length characteristics.

Are all prefix-free quantum codes condensable? As we shall see in Section 1.6, they are; but in order to show this, we will have to give an explicit algorithm for a quantum computer to condense the codewords of a prefix-free quantum code. This algorithm must maintain the coherence of superpositions of codewords of different lengths. Before we describe our algorithm, we will first discuss some key characteristics of coherent information processing.
1.5 Coherence and reversibility

We adopt a high-level model of a quantum computer, which could in principle be implemented by a quantum Turing machine or an array of quantum gates. Our quantum computer contains several registers of qubits, which initially hold \textit{zef} codewords from a prefix-free quantum code. The computer also includes a central processing unit that contains various counters and pointers, each of which can take on integer values (or superpositions of these). A system clock keeps track of the number of machine cycles that have passed since the beginning of the computation. (This clock may be treated as an entirely classical system; its function is simply to control the execution of our quantum program.) Finally, the computer contains an output “tape” of qubits (initially all in the state $|0\rangle$) on which the condensed string is to be written.

Our job is to write the input code words onto the output tape in a way that preserves the coherence of superpositions of different codewords, including superpositions of codewords of different lengths. This means that the operation of the computer must be unitary. We can guarantee this unitarity if we satisfy certain conditions:

1. \textit{Reversibility}. In a classical code, all codewords have a determinate length. We can choose an orthogonal basis of length eigenstates to be “quasi-classical” input states of our computer. (These states need not be fully classical—for example, the qubits in these codeword states may be entangled with each other. However, each codeword in our basis has a determinate length.) We require that distinct “quasi-classical” inputs lead to distinct final states of the computer. This is essentially a requirement that the computation be \textit{reversible} on these quasi-classical inputs \cite{11, 12}.

2. \textit{Coherent computation}. The computation includes no measurement or process in which the environment becomes entangled with the computer. As a special case of this, we require that the computation end after exactly the same number of steps for any input codeword. If the computation took more steps for longer codewords, the halting time of the computation would constitute a measurement of codeword length, and would destroy the coherence.

3. \textit{Localization of coherence in the output}. For any quasi-classical input, at the end of the computation all input registers and internal variables
in the central processor have been reset to fixed values that are independent of the input. Only the output tape retains any information about the input. This will guarantee that a superposition of quasi-classical inputs will not lead to entanglement between the output tape and the rest of the computer; the coherence will be localized in the output tape.

A similar set of conditions is outlined in [6], where it is used to specify quantum algorithms for data compression and for quantum arithmetic coding.

The reversibility requirement ensures that an orthogonal basis of initial states maps to an orthogonal basis of final states. If the computation is coherent, this map extends by linearity to a unitary evolution for the computer’s quantum state. The final requirement guarantees that the quantum information initially in the input registers can be recovered from the condensed output tape alone. We will discuss each of these requirements in turn.

Consider how our quantum computer acts on quasi-classical (length eigenstate) inputs. If we were to map out its algorithm as a flowchart, the requirement of reversibility would impose two sorts of requirements. First, each individual operation on the data must be reversible. Second, the branches and joins in the flowchart must be specified in a reversible way.

A branch can be pictured in this way:

```
  branch condition
  ➔ false
  ➔ true
```

Execution of the program enters from the top, and a logical “branch condition” is evaluated. If the branch condition is true, execution proceeds along the downward branch; if false, along the rightward branch. This is plainly reversible, as long as the evaluation of the branch condition is done in a reversible way; there is no ambiguity in the execution of the reversed program.

However, a simple join

```
  ➔ join
```

10
is not reversible, since in the reversed program it is not clear which of the two paths to take. The point is that a join in the flowchart is a reversed branch, and thus must be governed by a logical “join condition”:

\[
\begin{align*}
\text{true} & \quad \text{join condition} \\
\text{false} & \quad \text{true}
\end{align*}
\]

The program is designed so that the “join condition” is true whenever the execution approaches from above, and false whenever execution approaches from the right.

In our program, we will want to use branches and joins to create “loops”, like so:

\[
\begin{align*}
\text{true} & \quad \text{start condition} \\
\text{false} & \quad \text{false} \\
\text{operations} & \quad \text{stop condition} \\
\text{false} & \quad \text{true}
\end{align*}
\]

The “start condition” is a logical condition that is only true at the beginning of the execution of the loop and not thereafter; the “stop condition” is only true at the end of the execution of the loop and not before.

We can also conveniently represent the reversible loop structure in pseudocode form:

```plaintext
loop enter (start condition)
```
Both the beginning and the end of the loop are governed by logical conditions. The requirement that the computation be coherent may at first seem difficult to achieve, since each branch point (or join point) in the algorithm involves the evaluation of a condition—apparently a measurement process. However, these conditions can control the execution of the program without any irreversible loss of coherence.

Let us suppose that the quantum system $Q$ is some portion of our computer, and that we wish to branch our program based on a condition about the state of $Q$. The condition is represented by a projection $\Pi$ acting on the Hilbert space $\mathcal{H}$ describing $Q$. Any initial state $|\psi\rangle$ of $Q$ can be written as

$$|\psi\rangle = \Pi |\psi\rangle + \Pi^\perp |\psi\rangle.$$  

(13)

We could imagine evaluating the condition by making a measurement of the observable represented by $\Pi$ and $\Pi^\perp$. But this would destroy the coherence in this superposition, so a less destructive operation is required.

We join to $Q$ a single qubit (in another part of the quantum computer), and consider the operator $U$ on joint system:

$$U = (|0\rangle \langle 1| + |1\rangle \langle 0|) \otimes \Pi + 1 \otimes \Pi^\perp. \quad (14)$$

$U$ is easily verified to be a unitary operator, and thus it could represent some coherent quantum evolution of the joint system. If the qubit is initially set to the state $|0\rangle$ and then $U$ acts, we obtain,

$$U |0\rangle \otimes |\psi\rangle = |1\rangle \otimes \Pi |\psi\rangle + |0\rangle \otimes \Pi^\perp |\psi\rangle.$$  

(15)

This is an entangled state of the qubit and $Q$. If we were to make a measurement of the qubit in the standard basis, we would be effectively measuring the observable $\Pi$ on $Q$. That is, the qubit “contains” the value of the observable $\Pi$. However, the interaction is completely reversible, and in this case may be undone by a further application of $U$ itself.

The qubit can be used as a switch to instruct the computer which branch of the computation to follow. Suppose we wish to specify that, if the qubit is $|0\rangle$, the rest of the computer performs a computation described by the unitary operator $V_0$, whereas if the qubit is $|1\rangle$ then we wish to do the computation
Then we instruct the entire computer (including the switch qubit) to perform a coherent computation described by the unitary operator

\[ V = |0\rangle\langle 0| \otimes V_0 + |1\rangle\langle 1| \otimes V_1. \]  

If the overall state of the computer is a superposition of the two switch states, both branches are followed in different branches of the superposition. The computer may become increasingly entangled, but the coherence of its overall state is preserved.

We have shown that any branching condition that can be represented by a projection operator \( \Pi \) can be used to control the execution of the program without any necessary loss of coherence. The cost is entanglement among the parts of the computer.

A join point in the algorithm is simply a time-reversed branch point. Just before the join, the computer is in a state like Equation 15, in which the qubit is entangled with the system \( Q \). The operator \( U^{-1} = U \) acts, and we return the qubit to the state \(|0\rangle\) and the system \( Q \) to a state like Equation 13. We have “disentangled” \( Q \) from the qubit, so the two branches of the computation (controlled by the qubit) have merged.

Our second concern with coherent computation is the synchronization of the computation on different components of the initial superposition. This can be maintained without much difficulty by introducing appropriate “delay loops” into the program, so that its execution requires exactly the same number of machine cycles for any input.

We will address our final concern, that the output tape should wind up unentangled with the rest of the computer, by showing that the final state of the rest of the computer (input registers and central processor) is independent of the input state.

1.6 Prefix-free codes are simply condensable

We are at last ready to give our algorithm for simply condensing the code-words of a prefix-free quantum code. First, we establish our notation and describe the contents of our computer in slightly more detail:

**Registers** Our computer contains \( N \) registers, each consisting of \( l_{\text{max}} \) qubits. The \( i \)th register is denoted \( R_i \) and the \( k \)th qubit of this register is called \( R_{i,k} \). Initially, each register contains a zef codeword from a fixed prefix-free quantum code.
Tape There is a “tape” T containing at least \( N l_{\text{max}} \) qubits, all of which are initially in the state \( |0\rangle \). The \( n \)th qubit in the tape is called \( T_n \).

Counter There is a counter variable \( c \), which can take on integer values starting with 0 (or, of course, superpositions of these). The initial state of \( c \) is \( |0\rangle \).

Pointers There are several pointer variables, which like the counter variable take on integer values and have an initial state \( |0\rangle \). These variables point to locations in the computer’s memory, but of course they are themselves quantum variables and can take on entangled superpositions of values. There is an overall register pointer \( r \) and, for each register, a qubit pointer \( q_i \) (for the \( i \)th register). The tape also has a pointer variable \( p \).

The first section of the program copies the contents of the registers to the tapes, moving the pointers in the process.

\[
\begin{align*}
\text{loop enter} & (r = 0) \\
& r \leftarrow r + 1 \\
\text{loop enter} & (q_r = 0) \\
& q_r \leftarrow q_r + 1 \\
& p \leftarrow p + 1 \\
& T_p \leftarrow T_p \oplus R_{r,q_r} \\
& \text{loop exit} (R_{r,1} \cdots R_{r,q_r} \text{ is a codeword of length } q_r) \\
\text{loop exit} & (r = N)
\end{align*}
\]

(The notation \( a \leftarrow a \oplus b \) indicates the “controlled not” operation on the qubits, with \( a \) as the “target” and \( b \) as the “control” qubit.) Notice that the exit condition for the inner loop (which copies the register qubits one by one onto the tape) is legitimate because the code is prefix-free. This means that the question of whether the first \( q_r \) qubits of the register form a codeword of length \( q_r \) can be settled by measuring a projection-type observable on those qubits. (The computer does not make such a measurement, but instead coherently controls its operation as described above.)

We also note that, since the procedure is just to copy the register contents to the output tape, we are doing simple condensation.

At this stage, the various pointer variables are entangled with the codeword length information; furthermore, the time at which the computer reaches
this stage of the computation is indeterminate. We now resynchronize the program via a delay loop that causes the computer to “idle” until a fixed time $D$ (chosen large enough so that the first section of the program has finished even for the longest possible input codewords).

```plaintext
loop enter (c = 0)
    c ← c + 1
loop exit (time = D)
```

The second half of the program is the reverse of the first half, except that the register is uncopied, rather than the tape.

```plaintext
loop enter (time = D + 1)
    c ← c - 1
loop exit (c = 0)
loop enter (r = N)
    loop enter ($R_{r,1} \cdots R_{r,q_r}$ is a codeword of length $q_r$)
        $R_{r,q_r} ← T_p \oplus R_{r,q_r}$
        $p ← p - 1$
        $q_r ← q_r - 1$
    loop exit ($q_r = 0$)
    r ← r - 1
loop exit (r = 0)
```

The program now ends, after exactly $2D$ machine steps. All pointers and counters have been returned to their initial zero values, and the input qubit registers have been reset to $|00\cdots0\rangle$. Only the qubit tape now contains any non-zero data, in the form of a simply condensed string of $N$ codewords. In short, the computer at the end retains no codeword-length information at all. Superpositions of codewords of different length will thus remain coherent in the condensation process. Since the algorithm works for any given $N$, the prefix-free quantum code is simply condensable.

We previously proved that every condensable code satisfies the quantum Kraft-McMillan inequality, and then that every quantum code that satisfies the Kraft-McMillan inequality can be unitarily remapped to a prefix-free code. We now learn that prefix-free quantum codes are simply condensable. Since unitary remapping might be part of a general condensation process, we have established that a quantum code is condensable if and only if it satisfies the quantum Kraft-McMillan inequality.
2 Quantum data compression

2.1 How many qubits?

Classical variable-length codes are used for data compression—that is, the representation of classical information in a compact way, using as few resources (bits) as possible. This is done by encoding more probable messages in shorter codewords, so that the average codeword length is minimized. In this section we will discuss how—and in what sense—quantum indeterminate-length codes may be used for quantum data compression.

Suppose Alice is sending classical information to Bob using the following classical variable-length code:

| message | codeword |
|---------|----------|
| $C_1$   | 0        |
| $C_2$   | 10       |
| $C_3$   | 110      |
| $C_4$   | 111      |

If the message $C_1$ is sent, Bob receives a signal consisting of a single bit (0); but if $C_4$ is sent, he receives three bits (111). In each case, Bob knows how many bits are being used to send the message. If a long string of messages is being sent, Bob at any stage knows how many complete messages have been received.

Bob learns the length of each codeword because he actually learns which codeword was sent. The fact that Bob learns the identity of each codeword is not a problem in the classical situation; indeed, it is the whole point of classical communication! This contrasts with quantum information transfer. If Alice’s signals, for example, are drawn from a non-orthogonal set of states, Bob will not be able to determine reliably which signal was sent, and any attempt to do so would damage the fidelity of the quantum information.

Suppose that Alice wishes to send quantum information to Bob using the quantum analogue of the prefix-free code shown above. In other words, the length eigenstate $z_{ef}$ codewords are

| state | length |
|-------|--------|
| $|000\rangle$ | 1         |
| $|100\rangle$ | 2         |
| $|110\rangle$ | 3         |
| $|111\rangle$ | 3         |
Arbitrary superpositions of these codewords are also allowed code words. To maintain the coherence of these superpositions, therefore, Bob must not obtain any information about the length of the codeword he receives.

A quantum system actually used for the transmission of information must have at least two degrees of freedom. The first is the “data” degree of freedom, which may for instance be a qubit. The second degree of freedom is the “location” degree of freedom. This is the physical degree of freedom which determines whether or not Bob has access to the data degree of freedom. The faithful transmission of a qubit in a state $|\psi\rangle$ from Alice’s location $a$ to Bob’s location $b$ would be a process like this:

$$|\psi, a\rangle \rightarrow |\psi, b\rangle.$$  \hspace{1cm} (17)

Although we are phrasing our discussion in terms of the transmission of quantum information from one spatial location to another, this analysis would also apply to the storage and retrieval of information in a quantum computer. There the “location” degree of freedom might be the reading of a clock; the information stored at time $a$ is to be retrieved at some later time $b$.

If we have several data qubits, each one will have a location degree of freedom (which may, of course, be correlated with the others). The number of qubits transmitted from Alice to Bob will be the number of location degrees of freedom that have evolved from $a$ to $b$. For instance, suppose that three data qubits are in a joint state $|\psi^{123}\rangle$, and that Alice sends the first and third qubits to Bob. The final state would be $|\psi^{123}, bab\rangle$, in which Bob has received two qubits.

How could Alice send an indeterminate number of qubits to Bob—in particular, if Alice is representing her quantum information using the prefix-free quantum code above, how can she arrange to send only the first $l$ qubits of a zef codeword of length $l$? The transmission of the length eigenstates is easy to describe:

$$
\begin{align*}
|000, aaa\rangle & \rightarrow |000, baa\rangle \\
|100, aaa\rangle & \rightarrow |100, bba\rangle \\
|110, aaa\rangle & \rightarrow |110, bbb\rangle \\
|111, aaa\rangle & \rightarrow |111, bbb\rangle .
\end{align*}
$$

But imagine that Alice is sending a superposition of codewords of different lengths. If the above process is unitary, then at the end the data qubits will
be entangled with their location degrees of freedom. The coherence of the superposition would no longer be maintained within the data qubits. In order to restore the coherence, Bob would have to interact with the location degrees of freedom of the qubits with which he has indeterminate access. Except for a trivial case—in which Bob simply returns the qubits from location $b$ back to $a$—he will not be able to do this.

If the transmission process is not unitary, things are even worse. Our conclusion is that it is not possible to send quantum information coherently using an indeterminate number of qubits. If we are to use indeterminate-length quantum codes for quantum data compression, we will have to do so in such a way that a fixed number of qubits changes hands from Alice to Bob.

Perfect fidelity would demand that Alice send all of the qubits to Bob—enough qubits so that even the longest component of each codeword is transmitted in its entirety. But this scheme would allow for no data compression at all.

Our previous discussion of condensability offers some hope. The condensation process took the “information-bearing” parts of $N$ zef codewords (in registers of length $l_{\text{max}}$) and unitarily shifted them as far as possible toward the beginning of a tape of $Nl_{\text{max}}$ qubits. Although some branches of the overall superposition may extend to the end of the tape, the “typical” branch may be much shorter (followed by $|0\rangle$’s). We therefore might be able to truncate the condensed string of codewords after some number $L$ of qubits, where $L \ll Nl_{\text{max}}$, and still maintain an average fidelity approaching unity.

Let us consider a quantum information source that produces an ensemble of signal states of some quantum system. These signal states are unitarily encoded as zef codewords of some condensable quantum code. For our purposes, therefore, we can simply consider the ensemble of zef codewords produced by the quantum information source and the unitary encoding. In this ensemble, the codeword $|a_{\text{zef}}\rangle$ occurs with probability $p(a)$, and the average encoded signal state is described by the density operator

$$\rho = \sum_a p(a) |a_{\text{zef}}\rangle\langle a_{\text{zef}}|.$$  

(18)

Our source produces a sequence of independent, identically distributed signals, which are encoded as zef codewords in separate registers. The average state of $N$ of these registers is $\rho^\otimes N$.

The average length $\langle l \rangle$ of the codeword ensemble is

$$\langle l \rangle = \text{Tr} \rho \Lambda = \sum_a p(a) \langle a_{\text{zef}} | \Lambda | a_{\text{zef}} \rangle.$$  

(19)
The average length $\langle l \rangle$ is an ensemble average of quantum expectation values for $\Lambda$, but no codeword $|a_{zef}\rangle$ need be a length eigenstate.

A condensed string of $N$ codewords is a zef string of $Nl_{\text{max}}$ qubits, with length observable $\Lambda$. If $U$ is the unitary operator that maps the $N$ separate zef codewords to the condensed string, then we can define the overall length observable for the condensed string to be

$$\Lambda = U (\Lambda_1 + \Lambda_2 + \cdots + \Lambda_N) U^{-1}.$$  

The condensed length $\Lambda$ is just the sum of the individual length observables of the separate, pre-condensed codewords. This observable will have eigenvalues $L = l_1 + \cdots + l_N$ and an average value $\langle L \rangle$. The codewords are independent, and so

$$\langle L \rangle = N \langle l \rangle.$$  

Since the overall length of the condensed string is defined to be additive, we can apply the “law of large numbers” to some measurement of $\Lambda$: For any $\epsilon, \delta > 0$, for large enough $N$ it is true that

$$\Pr (|\Lambda - N \langle l \rangle| > N\delta) < \epsilon.$$  

This means that, for large $N$, the probability is very small that $\Lambda$ will be found to be much less than (or much greater than) $\langle L \rangle$. Of course, we will not in general make such a measurement, but Equation 21 is still useful in restricting the typical amplitude of codeword string components.

As we shall see, if the ensemble average length of the zef codewords is $\langle l \rangle$, then we can in the long run maintain fidelity near to 1 by keeping just $\langle l \rangle + \delta$ qubits per signal, where $\delta$ can be made as small as desired. Conversely, in a simple condensation process, we must keep at least $\langle l \rangle$ qubits per signal to maintain high fidelity—if we keep only $\langle l \rangle - \delta$ per signal, the average fidelity tends toward zero. We will also find that the ensemble average length of the zef codewords is related to the von Neumann entropy of the signal ensemble, making this approach an alternate route to the noiseless quantum coding theorem. Finally, we will show that the relative entropy is a measure of the additional resources (qubits) required to represent quantum information using a code that is not optimal.

### 2.2 Enough qubits

In this section we will make use of the fact that a condensed string of $N$ zef codewords is itself in zef form—in other words, we can view the condensed
string as a zef codeword in a much longer code. The length observable for this super-codeword will be the sum of the length observables for the $N$ original codewords.

Suppose we have a zef codeword $|\phi\rangle$ in a register of $n$ qubits, and suppose $\ell \leq n$. Define $\eta$ such that a measurement of the length observable $\Lambda$ on the codeword yields a result larger than $\ell$ with probability

$$\Pr (\Lambda > \ell) = \eta.$$  

(22)

In general $|\phi\rangle$ will include components of various lengths. Let $\Pi_\ell$ be the projection

$$\Pi_\ell = 1^{1\cdots\ell} \otimes |0^{\ell+1\cdots n}\rangle\langle 0^{\ell+1\cdots n}|.$$  

(23)

That is, $\Pi_\ell$ projects onto the subspace of register states that are $|0\rangle$ in the last $n - \ell$ qubits. We can write our zef codeword $|\phi\rangle$ as

$$|\phi\rangle = \alpha |\phi_{(\leq \ell)}\rangle + \beta |\phi_{(> \ell)}\rangle$$  

(24)

where $\alpha, \beta \geq 0$, and $|\phi_{(\leq \ell)}\rangle$ and $|\phi_{(> \ell)}\rangle$ are normalized states such that

$$\Pi_\ell |\phi_{(\leq \ell)}\rangle = |\phi_{(\leq \ell)}\rangle$$
$$\Pi_\ell |\phi_{(> \ell)}\rangle = 0.$$  

Since all $\Lambda$-eigenstate zef codewords with length no larger than $\ell$ have $|0\rangle$ in the last $n - \ell$ qubits,

$$1 - \eta = \Pr (\Lambda \leq \ell) \leq \alpha^2.$$  

(25)

Equality need not hold, however, since some length eigenstate codewords with $\Lambda > \ell$ may nevertheless have $|0\rangle$ in the last $n - \ell$ qubits. (This is analogous to the classical situation, in which it is perfectly possible to have one or more 0’s at the end of a codeword in a variable-length code.)

We now imagine that we truncate the register by discarding the last $n - \ell$ qubits. Only $\ell$ qubits are stored or transmitted. At the receiver’s end of the process, $n - \ell$ qubits in the standard state $|0\rangle$ are appended, yielding a mixed final state $\sigma$ for the register. With what fidelity $F = \langle \phi | \sigma | \phi \rangle$ has the original codeword state been maintained by this process?

Direct calculation shows that the mixed state $\sigma$ is

$$\sigma = \alpha^2 |\phi_{(\leq \ell)}\rangle\langle \phi_{(\leq \ell)}| + \beta^2 w_{(> \ell)}$$  

(26)

20
where \( w_{(\sim \ell)} \) is the state obtained by truncating \( |\phi_{(\sim \ell)}\rangle \) and appending \( n - \ell \) qubits in the state \(|0\rangle\). Thus

\[
F = \langle \phi | \sigma | \phi \rangle = \alpha^2 |\langle \phi | \phi_{(\sim \ell)} \rangle|^2 + \beta^2 \langle \phi | w_{(\sim \ell)} | \phi \rangle \geq \alpha^4. \tag{27}
\]

Therefore,

\[
F \geq \alpha^4 \geq (1 - \eta)^2 \geq 1 - 2\eta. \tag{28}
\]

If the codeword length \( \Lambda \) would be found to be no more than \( \ell \) with probability \( 1 - \eta \), then we can keep only \( \ell \) qubits and recover the original state with fidelity \( F \geq 1 - 2\eta \).

We can now apply this result and the law of large numbers (Equation 21) to a condensed string of codewords. If \( \epsilon, \delta > 0 \) and \( N \) is sufficiently large, and if we take \( \ell = N(\langle l \rangle + \delta) \), then the ensemble average probability that the codeword string is longer than \( \ell \) can be made smaller than \( \epsilon/2 \). We can therefore truncate the string after only \( N(\langle l \rangle + \delta) \) qubits and later recover the original string with an average fidelity

\[
\langle F \rangle > 1 - \epsilon. \tag{29}
\]

Therefore, if we keep more than \( \langle l \rangle \) qubits per input message, in the long run we will be able to retrieve the quantum information with average fidelity approaching unity. The average length \( \langle l \rangle \) tells us how many qubits are sufficient for high fidelity.

### 2.3 Too few qubits

We now turn to the question of how many qubits are necessary to achieve high fidelity after the condensed string is truncated. For this discussion we will restrict our attention to simple condensation, rather than a general condensation process. Since any condensable code can be replaced by a simply condensable code with the same length characteristics, this restriction is not too severe.

The reason for making this restriction is pragmatic. Suppose we have \( N \) registers containing codewords from a condensable code, with an average length of \( \langle l \rangle \). A general condensation procedure might consist of two stages. In the first, the codewords in the \( N \) separate registers are unitarily remapped
to codewords from a more efficient code, that is, one with shorter average length $\langle l' \rangle < \langle l \rangle$. In the second stage, this more efficient code is condensed. We have established that only about $N \langle l' \rangle$ qubits will be sufficient to maintain high fidelity. In other words, the original average length $\langle l \rangle$ may tell us nothing about the number of qubits necessary for high fidelity.

Of course, we might not choose to condense the codewords in this way, or a more efficient code might not exist. Our strategy will be to separate the question of the efficiency of a code from the question of how many qubits are necessary. First we will consider the simple condensation of codes that may be inefficient, and then (in the next section) we will discuss limits on the efficiency of codes. In this section, therefore, we describe limits imposed by the structure of our particular (possibly sub-optimal) code, and in the next we will indicate how optimal or near-optimal codes may be chosen.

Begin with $N \text{ zef}$ codewords of a simply condensable code. The simply condensed string formed from the $N$ codewords can be built out of two pieces:

1. the simply condensed qubit string obtained from the first $N - k$ codewords, and

2. the simply condensed qubit string obtained from the last $k$ codewords.

These two pieces are both $\text{zef}$ and are simply condensed together to form the complete string. Thus, we will base our discussion on the simple condensation of just two $\text{zef}$ codewords.

The first $\text{zef}$ codeword $|\psi\rangle$ lies in a register of $m$ qubits, and the second codeword $|\chi\rangle$ lies in a register of $n$ qubits. The simply condensed pair (denoted rather symbolically by $|\psi\chi\rangle$) is a state of a string of $m + n$ qubits. We also consider a state called $|\psi\psi\rangle$, which is the first $\text{zef}$ codeword followed by $n$ additional qubits in the state $|0\rangle$.

Let $\ell \leq m + n$. The first $\text{zef}$ codeword can be written

$$|\psi\rangle = \alpha |\psi_{<\ell}\rangle + \beta |\psi_{\geq \ell}\rangle$$

where $\alpha, \beta \geq 0$ and $|\psi_{<\ell}\rangle$ (or $|\psi_{\geq \ell}\rangle$) is a normalized superposition of length eigenstates that are shorter than (or at least as long as) $\ell$. If we now simply condense this codeword with the codeword $|\chi\rangle$, we obtain

$$|\psi\chi\rangle = \alpha |\psi_{<\ell}\chi\rangle + \beta |\psi_{\geq \ell}\chi\rangle.$$
with $|\psi_{<\ell}\rangle\chi$ and $|\psi_{\geq\ell}\rangle\chi$ being the simply condensed strings obtained from $|\chi\rangle$ and the two components of $|\psi\rangle$. In a similar way,

$$|\psi\rangle = \alpha |\psi_{<\ell}\rangle_0 + \beta |\psi_{\geq\ell}\rangle_0.$$  \hspace{1cm} (32)

Now we imagine truncating the string of $m + n$ qubits, keeping only the first $\ell$ of them to be stored or transmitted. (We can denote this process by $T_{\ell}$.) At the receiver’s end, we do some unspecified quantum operation $\mathcal{E}$ that results in a final state of $m + n$ qubits. We know nothing about $\mathcal{E}$ in general except that it is a trace-preserving, completely positive linear map on density operators. The overall process, applied to the two initial states $|\psi\chi\rangle$ and $|\psi_0\rangle$, yield

$$|\psi\chi\rangle \xrightarrow{T_{\ell}} \omega \xrightarrow{\mathcal{E}} \mathcal{E}(\omega)
|\psi_0\rangle \xrightarrow{T_{\ell}} \sigma \xrightarrow{\mathcal{E}} \mathcal{E}(\sigma).$$

At the end of this process, we are interested in the overall fidelity of the truncation-cum-recovery process:

$$F = \langle \psi\chi | \mathcal{E}(\omega) | \psi\chi \rangle.$$  \hspace{1cm} (33)

We will show that, under suitable conditions, this fidelity must be small.

For general density operators, the fidelity is defined to be

$$F(\rho_1, \rho_2) = \max |\langle 1 | 2 \rangle|^2,$$  \hspace{1cm} (34)

where the maximum is taken over all purifications $|1\rangle$ of $\rho_1$ and $|2\rangle$ of $\rho_2$. (Equivalently, we can fix one of the purifications $|1\rangle$ and maximize over the other purification $|2\rangle$.) The fidelity has the property that it is never decreased by any quantum operation, so that

$$F(\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)) \geq F(\rho_1, \rho_2)$$  \hspace{1cm} (35)

for any trace-preserving, completely positive linear map $\mathcal{E}$.

A useful result (shown in [13]) relates the fidelities among three states $\rho_1$, $\rho_2$ and $\rho_3$. Let $F_{12} = F(\rho_1, \rho_2)$, etc. Then

$$\sqrt{F_{13}} \leq \sqrt{F_{23}} + \sqrt{2(1 - \sqrt{F_{12}})}.$$  \hspace{1cm} (36)
This implies that, if $F_{12}$ is nearly equal to one and $F_{23}$ is close to zero, $F_{13}$ is also close to zero. Recalling that $0 \leq F \leq 1$ for all fidelities, we note that 

$$1 - \sqrt{F} \leq 1 - F,$$

and thus

$$F_{13} \leq F_{23} + 2(1 - F_{12}) + 2\sqrt{2F_{23}(1 - F_{12})} \leq F_{23} + 2(1 - F_{12}) + 2\sqrt{2(1 - F_{12})} \leq F_{23} + 2\sqrt{1 - F_{12}} + 2\sqrt{2}\sqrt{1 - F_{12}} \leq F_{23} + 5\sqrt{1 - F_{12}}. \quad (37)$$

Since this inequality is linear in both $F_{13}$ and $F_{23}$, it will be convenient for situations in which we wish to average over an ensemble of $\rho_3$ states.

We apply Equation (37) to our situation as follows. The state $\rho_1 = |\psi\chi\rangle\langle\psi\chi|$, the original simply condensed string, and the state $\rho_3 = \mathcal{E}(\omega)$, the final state of the simply condensed string after the truncation $T_\ell$ and the recovery operation $\mathcal{E}$. Playing the role of $\rho_2$ is the state $\mathcal{E}(\sigma)$, the final state obtained by using $|\psi_0\rangle$ as our input. Since the quantum operation $\mathcal{E}$ can never decrease the fidelity between states, $F(\mathcal{E}(\omega), \mathcal{E}(\sigma)) \geq F(\omega, \sigma)$. Therefore,

$$F = \langle \psi\chi | \mathcal{E}(\omega) | \psi\chi \rangle \leq \langle \psi\chi | \mathcal{E}(\sigma) | \psi\chi \rangle + 5\sqrt{1 - F(\omega, \sigma)}. \quad (38)$$

The initial states $|\psi\chi\rangle$ and $|\psi_0\rangle$ are purifications of $\omega$ and $\sigma$, respectively. The fidelity $F(\omega, \sigma)$ is thus

$$F(\omega, \sigma) = \max_{\phi_\sigma} |\langle \psi\chi | \phi_\sigma \rangle|^2 \quad (39)$$

where the maximum is taken over all purifications $|\phi_\sigma\rangle$ of $\sigma$. Now, all of the purifications of $\sigma$ are related to one another by unitary operators that act only on the adjoined system, so that

$$F(\omega, \sigma) = \max_U \left| \langle \psi\chi | \left(1^{1:\ell} \otimes U^{\ell+1:m+n} \right) | \psi_0 \rangle \right|^2. \quad (40)$$

with the maximum taken over all unitary operators acting on the last $m+n-\ell$ qubits.
We write \( |\psi\rangle = \alpha |\psi_{<\ell}\rangle + \beta |\psi_{\geq \ell}\rangle \) and \( |\psi_0\rangle = \alpha |\psi_{<\ell}0\rangle + \beta |\psi_{\geq \ell}0\rangle \), as before, and note that, since \( |\psi_{\geq \ell}\rangle \) only contains components of \( |\psi\rangle \) that are at least as long as \( \ell \),

\[
\text{Tr}_{\ell+1 \cdots m+n} |\psi_{\geq \ell}\rangle \langle \psi_{\geq \ell}| = \text{Tr}_{\ell+1 \cdots m+n} |\psi_{\geq \ell}0\rangle \langle \psi_{\geq \ell}0|.
\]

(41)

In this component, the second codeword, whose “starting address” in the simply condensed string is entangled with the length of the first codeword, lies entirely in the discarded tail of the qubit string. Therefore, there exists a unitary \( V_{\ell+1 \cdots m+n} \) such that

\[
|\psi_{\geq \ell}\rangle = (1^{\ldots \ell} \otimes V_{\ell+1 \cdots m+n}) |\psi_{\geq \ell}0\rangle .
\]

(42)

Clearly,

\[
F(\omega, \sigma) \geq \left| \langle \psi\chi | (1^{\ldots \ell} \otimes V_{\ell+1 \cdots m+n}) |\psi_0\rangle \right|^2 .
\]

(43)

\[
\langle \psi\chi | (1^{\ldots \ell} \otimes V_{\ell+1 \cdots m+n}) |\psi_0\rangle = \alpha^2 \langle \psi_{<\ell}\rangle | (1^{\ldots \ell} \otimes V_{\ell+1 \cdots m+n}) |\psi_{<\ell}0\rangle \\
+ \alpha \beta \langle \psi_{<\ell}\rangle | (1^{\ldots \ell} \otimes V_{\ell+1 \cdots m+n}) |\psi_{\geq \ell}0\rangle \\
+ \beta \alpha \langle \psi_{\geq \ell}\rangle | (1^{\ldots \ell} \otimes V_{\ell+1 \cdots m+n}) |\psi_{<\ell}0\rangle \\
+ \beta^2 \geq 1 - 2\alpha - 2\alpha^2 \geq 1 - 4\alpha .
\]

Therefore

\[
F(\omega, \sigma) \geq (1 - 4\alpha)^2 \\
\geq 1 - 8\alpha .
\]

(44)

Our overall fidelity must satisfy

\[
F \leq \langle \psi\chi | \mathcal{E}(\sigma) |\psi\chi\rangle + 5\sqrt{8\alpha} \\
\leq \langle \psi\chi | \mathcal{E}(\sigma) |\psi\chi\rangle + 15\sqrt{\alpha} .
\]

(45)

Neither the operator \( \mathcal{E}(\sigma) \) nor the parameter \( \alpha \) depends on the second codeword \( |\chi\rangle \). We now imagine that the second codeword is drawn from an ensemble—that is, that the codeword \( |\chi\rangle \) occurs with probability \( P(\chi) \), so that the ensemble has an average density operator

\[
W = \sum\chi P(\chi) |\chi\rangle \langle \chi| .
\]

(46)
The average fidelity after truncation $\mathcal{T}_\ell$ and recovery $\mathcal{E}$ will therefore be
\[ \bar{F} \leq \text{Tr} W \mathcal{E}(\sigma) + 15\sqrt{\alpha}. \] (47)

Since $\mathcal{E}(\sigma)$ is a positive operator of unit trace, we obtain
\[ \bar{F} \leq \|W\| + 15\sqrt{\alpha}, \] (48)
where $\|W\|$ is the operator norm of $W$, which (since $W$ is positive) is just the largest eigenvalue of $W$.

After all of this, we are in a position to apply the law of large numbers (Equation 21) again. We will be choosing two large integers, $N$ and $k$. Our first codeword $|\psi\rangle$ in the preceding analysis will be a simply condensed string of $N - k$ codewords, and the second codeword $|\chi\rangle$ will be a simply condensed string of the remaining $k$ codewords. We assume that the ensemble of single-register codewords has an average state $\rho$ with more than one non-zero eigenvalue—in other words, the ensemble involves more than one codeword state.

Let $\epsilon, \delta > 0$. If $\lambda < 1$ is the largest eigenvalue of $\rho$, then the largest eigenvalue of $\rho^{\otimes k}$ is $\lambda^k$. Choose $k$ so that $\lambda^k < \epsilon/2$. Since the last $k$ codewords are unitarily condensed into a string with average state $W$, $\|W\| = \|\rho^{\otimes k}\| < \epsilon/2$.

Now we consider the simply condensed string of the first $N - k$ codewords, which we have denoted by $|\psi\rangle$. The length observable for this string is $\Lambda_{N-k}$. Given a value of $N$, we define $\ell = N(\langle l \rangle - \delta)$. We will restrict our attention to values of $N$ large enough so that
\[ \ell \leq (N - k) \left( \langle l \rangle - \frac{\delta}{2} \right). \] (49)

Applying the law of large numbers, we can now specify $N$ large enough so that $\Pr (\Lambda_{N-k} < \ell) = \alpha^2$ is as small as we like. In particular, we can guarantee that $15\sqrt{\alpha} < \epsilon/2$. Thus,
\[ \bar{F} \leq \|W\| + 15\sqrt{\alpha} < \epsilon. \] (50)

Therefore, if we keep fewer than $\langle l \rangle$ qubits per input message and use simple condensation, in the long run the fidelity of the retrieved quantum information must approach zero. The average length $\langle l \rangle$ tells us how many qubits are necessary for high fidelity using simple condensation.
2.4 Entropy and average length

The preceding results provide an interpretation for the average length $\langle l \rangle$ of an indeterminate-length quantum code: $\langle l \rangle$ is just a measure of the resources (qubits) that are both necessary and sufficient to maintain high fidelity of the quantum information, in the situations described above. We now inquire how short $\langle l \rangle$ can be for a given quantum information source. In other words, we will now explore how efficient an indeterminate-length quantum code may be.

Recall the quantum Kraft-McMillan inequality (Equation 5). Any condensible quantum code must have a length observable $\Lambda$ on zef codewords that satisfies

$$\text{Tr} 2^{-\Lambda} = K \leq 1.$$  

where the trace is restricted to the zef subspace. We can construct a density operator $\omega$ on the zef subspace by letting

$$\omega = \frac{1}{K} 2^{-\Lambda}. \quad (51)$$

The operator $\omega$, although a positive operator of unit trace, is generally not the same as the ensemble average density operator $\rho$ of the codewords produced by the information source.

The average codeword length $\langle l \rangle$ is

$$\langle l \rangle = \text{Tr} \rho \Lambda = -\text{Tr} \rho \log (2^{-\Lambda}) = -\text{Tr} \rho \log \omega - \log K.$$  

Therefore

$$\langle l \rangle = S(\rho) + \mathcal{D}(\rho||\omega) - \log K, \quad (52)$$

where $S(\rho)$ is the von Neumann entropy of the density operator $\rho$

$$S(\rho) = -\text{Tr} \rho \log \rho \quad (53)$$

and $\mathcal{D}(\rho||\omega)$ is the quantum relative entropy

$$\mathcal{D}(\rho||\omega) = \text{Tr} \rho \log \rho - \text{Tr} \rho \log \omega. \quad (54)$$
We use base-2 logarithms. The relative entropy has a number of useful properties. For example, it is positive-definite, so that $D(\rho||\omega) > 0$ if and only if $\rho \neq \omega$. 

Since $\log K \leq 0$, 

$$\langle l \rangle \geq S(\rho). \quad (55)$$

The average codeword length must always be at least as great as the von Neumann entropy of the signal ensemble from the information source.

We can approach this bound by a suitable code. The eigenvalues $\lambda_k$ of $\rho$ form a probability distribution $\vec{\lambda}$, and the von Neumann entropy is simply the Shannon entropy of the eigenvalues:

$$S(\rho) = H(\vec{\lambda}) = -\sum_k \lambda_k \log \lambda_k. \quad (56)$$

The probability distribution $\vec{\lambda}$ can be used to define a Shannon-Fano code, which is a classical prefix-free binary code whose codewords have integer lengths $l_k = \lceil \log \lambda_k \rceil$, so that 

$$l_k < \log \lambda_k + 1. \quad (57)$$

This means that the average length of the Huffman codewords satisfies 

$$\langle l \rangle = \sum_k \lambda_k l_k < H(\vec{\lambda}) + 1. \quad (58)$$

The classical Shannon-Fano code can be used to define a corresponding prefix-free indeterminate-length quantum code, according to the procedure in Equation 12. (Such a code was also described by Chuang and Modha in [6].) Eigenstates of $\rho$ are length eigenstate zef codewords, and the average codeword length satisfies 

$$\langle l \rangle < S(\rho) + 1. \quad (59)$$

Asymptotically, this code will achieve high fidelity using about $S(\rho) + 1$ qubits per signal.

An alternate scheme is based on Huffman codes, which are classical prefix free codes that actually minimize average codeword length $\langle l \rangle$. Equations 58 and 59 are also satisfied for Huffman codes and their quantum versions.

We can do even better if we create our zef codewords from blocks of outputs of the quantum information source. This amounts to considering a new source that produces blocks of $n$ elementary signals, with an ensemble
average block state $\rho^\otimes n$ having an entropy of $nS(\rho)$. A quantum Shannon-Fano or Huffman code designed for this block source would have an average length of no more than $nS(\rho) + 1$, so that we will use only $S(\rho) + \frac{1}{n}$ qubits per elementary signal. Thus, by coding long blocks of signals, we can achieve $\bar{F} \rightarrow 1$ with about $S(\rho)$ qubits per elementary signal.

It can be seen that the theory of indeterminate-length quantum codes provides an alternate route to the quantum noiseless coding theorem [8]. The von Neumann entropy $S(\rho)$ measures the physical resources necessary to represent quantum information faithfully.

We now ask: under what circumstances can we achieve the entropic bound to the codeword length exactly, without resorting to block coding? In other words, for what codes and codeword ensembles can we have

$$\langle l \rangle = S(\rho)? \quad (60)$$

A code for which this equality holds may be called “length optimizing”. The answer can be seen from Equation [12]:

$$\langle l \rangle = S(\rho) + D(\rho||\omega) - \log K.$$  

Both $D(\rho||\omega)$ and $-\log K$ are non-negative, so they must both equal zero for a length optimizing code. In other words,

$$K = \text{Tr} 2^{-\Lambda} = 1 \quad (61)$$

and

$$\rho = \omega = 2^{-\Lambda}. \quad (62)$$

A length optimizing code must saturate the quantum Kraft inequality (Equation [3]), and the codeword ensemble must equal the density operator $\omega$ constructed from the length observable $\Lambda$. Two consequences follow:

- Whenever the signal ensemble $\rho$ has only eigenvalues of the form $2^{-m}$ for integer values of $m$, we can find a condensable quantum code (with length eigenvalues $m$) that is length optimizing. If $\rho$ has eigenvalues that are not of this form, then no length optimizing code exists.

- Some quantum codes saturate the quantum Kraft inequality—for example, those based on classical Huffman codes. These codes will be length optimizing for a codeword ensemble with density operator

$$\rho = 2^{-\Lambda}. \quad (63)$$
That is, every quantum code that saturates the quantum Kraft inequality is length optimizing for some codeword ensemble. If a quantum code does not saturate the quantum Kraft inequality, it is not length optimizing for any codeword ensemble.

Suppose we have a code that is length optimizing for some density operator $\omega$; but instead, we use the code for an ensemble of codewords described by the density operator $\rho$. Then the average codeword length will be

$$\langle l \rangle = S(\rho) + D(\rho||\omega).$$

(64)

We know that, using block coding, we can asymptotically use as few as $S(\rho)$ qubits to faithfully represent the quantum information produced by the source of $\rho$. We also know that $\langle l \rangle$ is the minimum number of qubits we need to retain per codeword to achieve high fidelity in a simply condensed string of many codewords. Thus, the relative entropy $D(\rho||\omega)$ tells us what additional resources (in qubits) are necessary to faithfully represent the quantum information from the $\rho$-source, if we use a code that is length optimizing for a different source (the “$\omega$-source”).

2.5 Remarks

In the quantum Huffman code of Braunstein et al., codeword length information and the codewords themselves are stored separately, in entangled strings of qubits. This means that the average number of qubits used to store the quantum information from a given source is increased by an amount logarithmic in the codeword length [5]. However, as we have seen, this separate accounting for codeword length information is unnecessary. The codewords of a quantum indeterminate-length code carry their own length information. This requirement is the basis for Equation 5, the quantum Kraft-McMillan inequality. We have shown that Equation 5 is a necessary and sufficient condition for condensability, and further, that any code satisfying Equation 5 can be unitarily mapped to a prefix-free quantum code with the same length characteristics. Prefix-free codes are themselves simply condensable, and obey the quantum Kraft-McMillan inequality.

Classical prefix-free codes are also called “instantaneous codes”, since the receiver of a string of codewords can identify an individual codeword from the string immediately, before the remainder of the string is received [3]. But this terminology is inapplicable to the quantum case. Suppose we have
a simply condensed string of codewords from a prefix-free quantum code. The first codeword is generally not a length eigenstate, and the length of this codeword is entangled with the locations in the qubit string of all subsequent codewords. The phase relationship between the different-length components of the first codeword is a global property of the state of the entire string. Therefore, in order to coherently recover even the first codeword, we will need the entire string (or a sufficiently long initial segment to achieve high overall fidelity). Even prefix-free quantum codes are not “instantaneous”; the entire transmission must be completed before any part of it can be “read”.

The classical Kraft-McMillan inequality (Equation 11) arises whenever a set of binary strings satisfies the prefix-free condition. For example, it governs the set of lengths of distinct programs for a classical Turing machine. The Kraft-McMillan inequality therefore plays a central role in algorithmic information theory, in which the information content of a binary string $s$ is defined to be length of the shortest halting program that produces $s$ as its output $[3, 14]$. We may hope that the quantum version of the Kraft-McMillan inequality will serve as a starting point for the development of a quantum algorithmic information theory.

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References

[1] C. H. Bennett, Physics Today 48, 24, (1995). C. H. Bennett and D. P. DiVincenzo, Nature 404, 247 (2000).

[2] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).

[3] T. M. Cover and J. A. Thomas, Elements of Information Theory (John Wiley and Sons, New York, 1991).

[4] B. Schumacher, presentation at Sante Fe Institute workshop on Complexity, Entropy and the Physics of Information (1994).
[5] S. L. Braunstein, C. A. Fuchs, D. Gottesman, and H.-K. Lo, "A quantum analog of Huffman coding," in Proceedings of the 1998 IEEE International Symposium on Information, MIT, Cambridge, MA, USA, August 16-21, page 353, 1998. [http://xxx.lanl.gov/abs/quant-ph/9805080].

[6] I. L. Chuang and D. S. Modha, IEEE Trans. Inf. Theory 46(3), 1104 (2000).

[7] K. L Boström, [http://xxx.lanl.gov/abs/quant-ph/0009052] K. L Boström, [http://xxx.lanl.gov/abs/quant-ph/0009073].

[8] B. Schumacher, Physical Review A 51, 2738 (1995). R. Jozsa and B. Schumacher, J. Mod. Opt. 41 2343 (1994).

[9] P. W. Shor, Physical Review A, 52, 2493 (1995); A.R. Calderbank and P. W. Shor Physical Review A, 54, 1098 (1996); A. Steane Physical Review Letters, 77, 793 (1996); R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, ibid. 77, 198 (1996).

[10] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, Phys Rev. A 54 3824 (1996). H. Barnum, M. A. Nielsen and B. Schumacher Phys. Rev. A 57, 4153 (1998).

[11] C. H. Bennett, IBM J. Res. Dev. 17, 525 (1973).

[12] T. Toffoli, in Automata, Languages, and Programming, edited by W. de Bakker and J. van Leeuwen (Springer, New York, 1980).

[13] H. Barnum, C. A. Fuchs, R. Jozsa and B. Schumacher, Physical Review A 54 4707 (1996).

[14] Ming Li and P. Vitanyi, An Introduction to Kolmogorov Complexity and Its Applications (Second Edition) (Springer-Verlag, Berlin, 1997).