Nonrelativistic counterparts of twistors and the realizations of Galilean conformal algebra

S. Fedoruk\textsuperscript{1}, P. Kosiński\textsuperscript{2}, J. Lukierski\textsuperscript{3}, P. Maślanka\textsuperscript{2}

\textsuperscript{1)Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia} \\
\texttt{fedoruk@theor.jinr.ru},

\textsuperscript{2)Department of Theoretical Physics II, University of Łódź, Pomorska 149/153, 90-236 Łódź, Poland} \\
\texttt{pkosinsk@uni.lodz.pl, pmaslan@uni.lodz.pl}

\textsuperscript{3)Institute for Theoretical Physics, University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland} \\
\texttt{lukier@ift.uni.wroc.pl}

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Abstract

Using the notion of Galilean conformal algebra (GCA) in arbitrary space dimension $d$, we introduce for $d=3$ quantized nonrelativistic counterpart of twistors as the spinorial representation of $\text{SO}(2,1) \oplus \text{SO}(3)$ which is the maximal semisimple subalgebra of three–dimensional GCA. The GC–covariant quantization of such nonrelativistic spinors, which shall be called also Galilean twistors, is presented. We consider for $d=3$ the general spinorial matrix realizations of GCA, which are further promoted to quantum–mechanical operator representations, expressed as bilinears in quantized Galilean twistors components. For arbitrary Hermitian quantum–mechanical Galilean twistor realizations we obtain the result that the representations of GCA with positive–definite Hamiltonian do not exist. For non–positive $H$ we construct for $N \geq 2$ the Hermitian Galilean $N$–twistor realizations of GCA; for $N=2$ such realization is provided explicitly.
1 Introduction

Since the introduction of $D=4$ twistors describing conformal $O(4,2) \simeq SU(2,2)$ spinors (see e.g. [11, 12]) as a basic alternative to the space-time coordinates there were obtained many results (see e.g. [5, 6]) using relativistic twistorial framework.

The twistor $Z_A$ ($A = 1, \ldots, 4$) carrying a particular $4 \times 4$ complex matrix realization of $D=4$ conformal algebra spanned by the generators $P_\mu, K_\mu, M_{\mu\nu} = (M_i = \frac{i}{2} \epsilon_{ijk} M_{jk}, N_i = M_{i0}), D$ ($\mu = 0, 1, 2, 3; i = 1, 2, 3$) (see e.g. [8, 9])

\[
\begin{align*}
P_\mu &= \begin{pmatrix} 0 & 0 \\ \sigma_\mu & 0 \end{pmatrix}, & K_\mu &= \begin{pmatrix} 0 & \sigma_\mu \\ 0 & 0 \end{pmatrix}, \\
M_i &= \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, & N_i &= \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}, & D &= \frac{i}{2} \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} 
\end{align*}
\]

(1.1)

can be described by two $D=4$ Weyl spinors

\[
Z_A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot{\alpha}} \end{pmatrix}, \quad \alpha = 1, 2. 
\]

(1.2)

If we perform the nonrelativistic contraction of relativistic conformal algebra to GCA (see (2.9)-(2.10)), the finite–dimensional matrix realizations of the generators $P_i,N_i,K_i$ after contraction in the Abelian limit are nullified, and nonvanishing remain only the following noncontracted ones

\[
M_i = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 & 0 \\ \sigma_0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & \sigma_0 \\ 0 & 0 \end{pmatrix}, \quad D = \frac{i}{2} \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}. 
\]

(1.3)

Introducing notation

\[
t_{\alpha,1} = \lambda_\alpha, \quad t_{\alpha,2} = \mu^{\dot{\alpha}} 
\]

(1.4)

we obtain that the $2 \times 2$ matrix $t_{\alpha a}$ ($a = 1, 2$) becomes a spinorial representation of $SU(2) \otimes SU(1,1) \simeq O(3) \otimes O(2,1)$, with generators $M_i \in o(3)$ and $(P_0,K_0,D) \in o(2,1)$ which is the maximal semisimple non-Abelian subalgebra of GCA. In such a way we shall speak further about the nonrelativistic limit of relativistic twistor, which will be called as nonrelativistic counterpart of twistors or Galilean twistors.

Two basic ingredients of relativistic twistors approach are [10, 11]

i) The expression of the Poincaré and conformal algebra generators as bilinears in terms of quantized twistor coordinates. In particular, in relativistic $N$–twistor space $Z_A^{(k)} = (\lambda^{(k)}_\alpha, \mu^{\dot\alpha(k)})$ ($k = 1, \ldots, N$) the four-momentum generators $P_\mu = \frac{i}{2} P^{\alpha\dot{\beta}} (\sigma_\mu)_{\alpha\dot{\beta}}$ are represented by the formula

\[
P_{\alpha\dot{\beta}} = \sum_k \lambda^{(k)}_\alpha \bar{\lambda}^{(k)}_{\dot{\beta}}, \quad \lambda^{(k)}_\alpha = (\lambda^{(k)}_{\dot{\beta}})^+ 
\]

(1.5)

with positive–definite energy component $P_0$. The relation (1.5) is consistently extended to Poincaré and conformal algebra.\footnote{Twistors in $D=4$ were also defined as the coordinates of twistor bundle over four–dimensional space–time $M$ with the fibre described by all complex structures on $M$. However, because the nonrelativistic space–time splits into one–dimensional time parameters manifold and three–dimensional nonrelativistic space, one cannot introduce in such odd–dimensional space a complex structure, and the nonrelativistic version of the definition given above cannot be applied. Similarly, if we define relativistic twistors via cosets, e.g. as projective twistors parametrizing the coset $SU(2,2) / SU(1) \times SU(1,2)$, the contractions providing the Galilean conformal group from $SU(2,2)$ (see (2.9) and (2.10)) do not provide a sensible nonrelativistic limit.}

\footnote{The extension to superconformal algebra has been also proposed [12].}
ii) The second Lorentz spinors $\mu^{\alpha(k)}$ defining twistors satisfy Cartan–Penrose incidence relation

$$\mu^{\alpha(k)} = X^{\alpha \beta} \lambda^{(k)}_{\beta}$$  \hspace{1cm} (1.6)

linking complexified space–time with twistor coordinates. For $N \geq 2$ one can express space–time coordinates $X^{\alpha \beta}$ as composite in terms of twistorial components.

The aim of this Letter is to study the spinorial realizations of Galilean conformal algebra (GCA) and consider their applicability to the description of nonrelativistic dynamics. We shall concentrate on the possibility of construction of the quantum–mechanical realizations of GCA in terms of Galilean twistors with positive–definite Hamiltonian $H$.

In order to introduce nonrelativistic conformal symmetry we shall define in Sect. 2 the conformal limit of $D=4$ conformal algebra which describes GCA \cite{13,14,15,16}. We stress that GCA is different from the Schrödinger algebra, which was also named as nonrelativistic conformal algebra (see e.g. \cite{18}). Further, in Sect. 3, we consider the realizations of GCA in terms of Galilean twistors $t_{\alpha, a}$ ($\alpha = 1, 2; a = 1, 2$) which, we recall, are the fundamental spinor representation of the semisimple part of GC symmetry $SU(2) \otimes SU(1,1) \simeq O(3) \otimes O(2,1)$. We stress that, contrary to the relativistic case, the nonrelativistic GCA is not semisimple, and in $d$ space dimensions it can written as the following $\frac{(d+2)(d+3)}{2}$–dimensional semidirect sum

$$\mathcal{C}^{(d)} = \left( o(2, 1) \oplus o(d) \right) \ltimes \mathcal{A}^{(3d)}$$  \hspace{1cm} (1.7)

where $o(2,1)$ describes the conformal symmetries on the world line, $o(d)$ generates the space translations, and $\mathcal{A}^{(3d)}$ describes 3$d$–dimensional Abelian subalgebra of space translations, Galilean boosts and nonrelativistic constant accelerations. The Galilean $\frac{(d+1)(d+2)}{2}$–dimensional algebra in $d$ space dimensions

$$\mathcal{G}^{(d)} = \left( o(1,1) \oplus o(d) \right) \ltimes \tilde{A}^{(2d)}$$  \hspace{1cm} (1.8)

is a subalgebra of $\mathcal{C}^{(d)}$, $\mathcal{G}^{(d)} \subset \mathcal{C}^{(d)}$, where in \cite{18} $o(1,1)$ generator is the Hamiltonian $H$, and $\tilde{A}^{(2d)}$ describes space translations and Galilean boosts.

In order to preserve $SO(2,1) \oplus SO(3)$ nonrelativistic conformal invariance\footnote{The general mathematical considerations providing as special case GCA were given in \cite{17}.} we postulate for Galilean twistors the following CCR

$$\left[ t^{(k)}_{\alpha, a}, t^{(l)}_{\beta, b} \right] = \delta^{kl} \delta_{\alpha \beta} \omega_{ab}$$  \hspace{1cm} (1.9)

where $\delta_{\alpha \beta}$ describes the standard unitary metric in $\mathbb{C}^2$ and $\omega_{ab}$ is a 2×2 traceless Hermitian matrix describing $SU(1,1)$ metric.\footnote{It should be observed that vanishing trace condition implies indefinite metric what can be seen explicitly from (1.9) e.g. by choosing $\omega = \sigma_3$. If we redefine $t^{(k)}_{\alpha, 1} = u^{(k)}_{\alpha, 1}$, $t^{(k)}_{\alpha, 2} = (u^{(k)}_{\alpha, 2})^+$, the relations (1.9) equivalently can be written as follows}

$$\left[ u^{(k)}_{\alpha, a}, u^{(l)}_{\beta, b} \right] = \delta^{kl} \delta_{\alpha \beta} \delta_{ab}.$$  \hspace{1cm} (1.10)

\footnote{For simplicity further we shall consider the physical case $d = 3$.}

\footnote{The indices of SU(2) spinors can be raised and lowered by the rule $(\psi_\alpha)^+ = \tilde{\psi}_{\bar{\alpha}} = \tilde{\psi}^{\bar{\alpha}}$, $\phi_\alpha = \phi^{\bar{\alpha}}$ what leads to equivalent two ways $\tilde{\psi}_{\bar{\alpha}} \phi_\alpha \equiv \tilde{\psi}^{\bar{\alpha}} \phi_\alpha$ of describing SU(2)–invariant scalar product. For SU(1, 1) spinors we have the following notations $\psi_\alpha = \omega_{ab} \tilde{\psi}^b_{\bar{\alpha}}$, $\phi_\alpha = \phi^b \omega_{ab}$.}
Relations (1.10) lead to the Hilbert space realization with positive metric but with broken SU(1, 1) ∼ SO(2, 1) subgroup of GCA. If we use the assignment (1.4) and choose ω_{ab} given by σ_1, the relations (1.9) describe the Penrose SU(2, 2)–covariant twistor quantization

$$\left[ \lambda^{(k)}_\alpha, \bar{\mu}^{(l)}_\beta \right] = \delta^{kl} \delta_\alpha^\beta, \quad \left[ \mu^{(l)}_\beta, \bar{\lambda}^{(k)}_\alpha \right] = \delta^{kl} \delta_\alpha^\beta.$$  (1.11)

We see therefore that the indefinite metric quantization is present as well in the relativistic covariant twistor quantization. It can be added that the alternative noncovariant twistor quantization, corresponding to the choice (1.10) of Galilean twistor oscillators, has been used in relativistic case by Woodhouse [19].

In Sect. 3 we shall introduce the 4N×4N–dimensional complex reducible realization of the GCA describing the nonrelativistic conformal transformations of the multiplet of N Galilean SO(3) ⊗ O(2, 1) spinors. Because contrary to the relativistic case it is not possible to introduce the Galilean twistor realization of GCA if N=1, we shall consider N ≥ 2.

We describe such matrix realizations as factorized into products of three matrices. To describe the matrix realization C_R of GCA generators \( \tilde{C}_R = (P_i, H, M_i, B_i, D, K, K_i) \) (see Sect. 2) we start with the following general 4N×4N matrices

$$\left( C_R \right)_{\alpha, a, k}^{\beta, b, l} = \sum (\sigma_\mu)_{\alpha}^{\beta} \otimes (\sigma_\nu)_{a}^{b} \otimes (A^{\mu\nu}_{R})_{k}^{l}$$  (1.12)

where \( \sigma_\mu = (\sigma_i, 1), \) i = 1, 2, 3 describe the scalar and spinorial \( su(2) \) algebra realizations, \( \rho_\mu = (\rho_r, 1), \) r = 0, 1, 2, \( \rho_0 = \sigma_1, \rho_1 = i\sigma_2, \rho_2 = i\sigma_3 \) provide the scalar and spinorial realization of \( su(1, 1) \) algebra and \( A^{\mu\nu}_{R} \) is the set of \( N \times N \) matrices specified after the substitution of (1.12) in place of \( \tilde{C}_R \) into GCA. Having the quantized twistor oscillators (1.3) and matrix realizations \( C_R \) of the generators of GCA one can introduce the quantum–mechanical N-twistor realization of GCA as follows\(^6\)

$$\hat{C}_R = \hat{t}^{(k)}_{\alpha, \dot{a}} \hat{\sigma}^\alpha \hat{\omega}^{\dot{a}a} \left( C_R \right)_{\alpha, a, k}^{\beta, b, l} \hat{t}^{(l)}_{\beta, \dot{b}}$$  (1.13)

where \( \hat{t}^{(k)}_{\alpha, \dot{a}} = \hat{t}^{+ (k)}_{\alpha, \dot{a}} \) and \( \hat{\omega}^{\alpha \dot{a}} \hat{\omega}_{\dot{a}b} = \delta_\beta^b. \) We see that the formula (1.13) promotes the “classical” matrix representation (1.12) of GCA to quantum–mechanical operator realization.

In our Letter we shall investigate the possibility of constructing Hermitian GCA realization with positive–definite Hamiltonian \( H \) in agreement with the postulates of QM. Firstly in Sect. 3 we shall consider the case N=2, and further we extend the discussion to arbitrary N. Unfortunately the result is negative – for the choice of any N it does not exist a matrix realization (1.12) which leads to formula (1.13) describing positive–definite Hamiltonian. It will be shown however in Sect. 4 that in nonrelativistic case the Hermitian twistorial operator realizations with indefinite Hamiltonians do exist for any \( N \geq 2. \) We see therefore that the applicability of proposed Galilean twistor realization of GCA is limited to the description of somewhat exotic models with indefinite Hamiltonians.

## 2 Galilean conformal algebra and Galilean twistors

The Galilean conformal algebra can be obtained by a contraction of \( D=d + 1 \)–dimensional relativistic conformal algebra \( o(D, 2) \) with the Poincaré generators \( P_\mu = (P_0, P_i), \) \( M_{\mu\nu} = (M_{ij}, M_{i0}) \) \( \left( i, j = 1, ..., d, \ mu, \nu = 0, 1, ..., d \right) , \) dilatation generator \( D \) and \( K_\mu = (K_0, K_i) \) generating special conformal transformations. The contraction procedure is not unique. If we

\(^6\)We recall that the correspondence with relativistic twistor realization (1.11) requires \( \omega = \sigma_1. \)
wish to identify the contraction parameter as a light velocity \( c \) the “physical” nonrelativistic contraction is defined by the following rescaling \([13]\)

\[
P_0 = \frac{H}{c}, \quad M_{i0} = c B_i, \quad K_0 = c K_i \quad K_i = c^2 F_i \quad (2.1)
\]

(\( P_i, M_{ij} \) and \( D \) remaining unchanged). After performing the nonrelativistic contraction limit \( c \to \infty \) one obtains the following \( \frac{1}{2} (D+1)(D+2) \)-dimensional Galilean conformal algebra \( \mathcal{C}^{(d)} \) \([13, 14, 15]\)

\[
[H, P_i] = 0, \quad [H, B_i] = iP_i, \quad [H, F_i] = 2iB_i, \quad (2.2)
\]

\[
[K, P_i] = -2iB_i, \quad [K, B_i] = -iF_i, \quad [K, F_i] = 0, \quad (2.3)
\]

\[
[D, P_i] = -iP_i, \quad [D, B_i] = 0, \quad [D, F_i] = iF_i, \quad (2.4)
\]

and

\[
[D, H] = -iH, \quad [K, H] = -2iD, \quad [D, K] = iK, \quad (2.5)
\]

where the subalgebra \( (P_i, B_i, F_i) \) describes the generators of maximal Abelian subgroup and the subalgebra \( (H, D, K) \) span the \( o(2,1) \) algebra. The algebra of rotations \( o(d) \) is described by the commutators

\[
[M_{ij}, M_{kl}] = i (\delta_{ik} M_{jl} - \delta_{il} M_{jk} + \delta_{jl} M_{ik} - \delta_{jk} M_{il}) \quad (2.6)
\]

and

\[
[M_{ij}, P_k] = i (\delta_{ik} P_k - \delta_{jk} P_i), \quad [M_{ij}, F_k] = i (\delta_{ik} F_k - \delta_{jk} F_i), \quad [M_{ij}, B_k] = i (\delta_{ik} B_k - \delta_{jk} B_i), \quad (2.7)
\]

\[
[M_{ij}, H] = [M_{ij}, K] = [M_{ij}, D] = 0. \quad (2.8)
\]

The generators \( (P_i, B_i, H, M_{ij}) \) define the Galilean algebra \( \mathcal{C}^{(d)} \), where \( B_i \) are the Galilean boosts and \( H \), the nonrelativistic energy operator, generates the Galilean time translations. We see that one can treat the Galilean conformal algebra as the result of adding the generators \( D \) (dilatations), \( K \) (expansions) and \( F_i \) (constant accelerations) to the Galilean algebra.

The Galilean conformal algebra \( \mathcal{C}^{(d)} \) can be described as the semidirect product \([1.7]\). Such a structure indicates that GCA can be also obtained as the contraction of the following coset of \( O(d+1,2) \)

\[
\mathcal{K} = \frac{O(d+1,2)}{O(2,1) \otimes O(d)}. \quad (2.9)
\]

In such a case the rescaling of relativistic conformal generators is introduced in the following way

\[
P_i = \kappa P_i^{NR}, \quad M_{i0} = \kappa B_i, \quad K_i = \kappa F_i \quad (2.10)
\]

and the generators \( P_0, K_0, M_{ij} \) and \( D \) are not changed. In such a framework we should treat the \( o(d+1,2) \) generators and the parameter \( \kappa \) as dimensionless. It can be shown that the alternative “formal” nonrelativistic rescaling \([2.10]\) provides as well in the limit \( \kappa \to \infty \) the GCA given by \([2.2]–[2.8][20]\). The lack of uniqueness of the contraction procedure providing GCA follows from the possible scaling automorphisms of relativistic conformal algebra, represented e.g. by the following mapping \( (X_A) \) are the conformal algebra generators

\[
X^\xi_A = e^{\xi D} X_A e^{-\xi D}. \quad (2.11)
\]
It can be shown that by suitable choice of $\xi$ the rescaling (2.10) of the generators $X_A$ reproduce the rescaling (2.1) for the generators $X^\xi_A$.

As follows from (1.7), if $d=3$, the maximal semisimple subgroup of GC group is $O(2,1) \otimes O(3)$, and Galilean twistors $t_{r,a}$ describe the fundamental representation of its spinor covering $O(2,1) \otimes O(3) = SU(1,1) \otimes SU(2)$. Considering twistor as $2 \times 2$ matrix $T = (t_{\alpha,a})$ one obtains the following GC transformations

$$T' = ATPB, \quad A \in SU(2), \quad B \in SU(1,1). \quad (2.12)$$

The basis of $o(2,1) \oplus o(3)$ Lie algebra is described by the generators of relativistic conformal algebra which were not contracted in the contraction procedure $\kappa \to \infty$ (see (2.10)), namely

$$o(2,1) : \quad P_0 = H, \quad K_0 = K, \quad D, \quad (2.13)$$

Introducing standard $O(2,1)$ basis

$$R_0 = \frac{1}{2} (K + H), \quad R_1 = \frac{1}{2} (K - H), \quad R_2 = D \quad (2.14)$$

one gets two commuting three–dimensional Lie algebras $(r, s, t = 0, 1, 2; i, j, k = 1, 2, 3)$

$$o(2,1) : \quad [R_r, R_s] = i \epsilon_{rst} R^t, \quad (2.15)$$

$$o(3) : \quad [M_i, M_j] = i \epsilon_{ijk} M_k \quad (2.16)$$

where $\epsilon_{123} = \epsilon_{012} = +1$, $R^r = g^{rs} R_s$, $g_{rs} = \text{diag}(-++, +)$. The contracted part of the relativistic conformal algebra $o(4,2)$ describes the Abelian generators $P_i, B_i, F_i$.

If we introduce new basis of Abelian subalgebra of GCA

$$A_{i,0} = \frac{1}{2} (F_i + P_i), \quad A_{i,1} = \frac{1}{2} (F_i - P_i), \quad A_{i,2} = B_i \quad (2.17)$$

we obtain the following $o(2,1) \oplus o(3)$ extension of (2.7)

$$[M_i, A_{j,r}] = i \epsilon_{ijk} A_{k,r}, \quad [R_r, A_{i,s}] = i \epsilon_{rs} A_{i,t} \quad (2.18)$$

The last considerations (formulae (2.12)-(2.18)) can be easily extended to arbitrary $d$.

3 Multitwistor realizations of GCA

We shall consider extended ($N \geq 2$) Galilean twistor space because the nontrivial $4 \times 4$ matrix realizations of all generators of GCA on simple $N=1$ Galilean twistor space do not exist. In order to show it let us note that the generators of the direct product of $su(2)$ and $su(1,1)$ generators in $4 \times 4$ space are

$$M_i = \frac{1}{2} \sigma_i \otimes 1_2, \quad R_r = 1_2 \otimes \frac{1}{2} \rho_r \quad (3.1)$$

where $\rho_0 = \sigma_1$, $\rho_1 = i \sigma_2$, $\rho_2 = i \sigma_3$. Then, the only nine generators which transform as the product of vectorial representations of $su(2)$ and $su(1,1)$ are (see (2.18))

$$B_{i,r} = \sigma_i \otimes \rho_r \quad (3.2)$$

The generators (3.2) are the only candidates to describe the $4 \times 4$ matrix realization of generators $A_{i,r}$, but they obviously do not form Abelian subalgebra. In fact, 15 generators (3.1) and (3.2) form $su(2,2)$ algebra and using the construction (1.13) and the identification (1.4) we obtain standard $N=1$ twistor realization of relativistic conformal algebra.
3.1 The realizations with positive-definite Hamiltonian $H$

Let us firstly consider Galilean twistor realizations of GCA for $\mathbb{N}=2$, with all three factors in tensor product (1.12) described by $2 \times 2$ matrices. Such representations provides four doublets of SU(2). We choose the following spinorial realization of the generators of $su(2)$ acting on four nonrelativistic spinors

$$M_i = \frac{1}{2} \sigma_i \otimes 1_2 \otimes 1_2$$

and the spinorial generators of SU(1, 1) (in consistency with $[R_a, M_i] = 0$)

$$R_r = 1_2 \otimes \frac{1}{2} \rho_r \otimes 1_2$$

where $\rho_0 = \sigma_1$, $\rho_1 = i\sigma_2$, $\rho_2 = i\sigma_3$. From (2.14) we obtain that $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$

$$H = 1_2 \otimes \sigma_- \otimes 1_2, \quad K = 1_2 \otimes \sigma_+ \otimes 1_2, \quad D = 1_2 \otimes \frac{i}{2} \sigma_3 \otimes 1_2$$

The covariance relations (2.18) imply that one can choose

$$A_{i,r} = \sigma_i \otimes \rho_r \otimes \mathcal{C}$$

where the complex $2 \times 2$ matrix

$$\mathcal{C} = c_0 + \vec{c} \vec{\sigma}$$

is unique for all nine generators $A_{i,r}$.

The $2 \times 2$ matrix $\mathcal{C}$ is specified by the Abelian nature of $A_{i,r}$,

$$[A_{i,r}, A_{j,s}] = 0.$$ The relations (3.8) requires that matrix $\mathcal{C}$ is nilpotent, i.e. $\mathcal{C}^2 = 0$, or more explicitly

$$\mathcal{C}^2 = 0 \quad \Rightarrow \quad c_0 = 0, \quad \vec{c}^2 = 0.$$ Solving (3.9) we obtain two two–parameter families

$$\mathcal{C}_1 = \begin{pmatrix} a & b \\ -a^2/b & -a \end{pmatrix}, \quad \mathcal{C}_2 = \begin{pmatrix} a & -a^2/b \\ b & -a \end{pmatrix} \quad (3.10)$$

where $b \neq 0$, whereas $a$ is arbitrary. One can prove by considering the equivalent realizations of $o(2, 1)$

$$\tilde{\rho}_r = V \rho_r V^{-1}, \quad (3.11)$$

where $V$ is an arbitrary invertible $2 \times 2$ matrix, that all nontrivial choices of $a$, $b$ are equivalent. We can take the special case $c_0 = c_3 = 0$, $c_1 = ic_2 = \frac{1}{2}$ corresponding to $b = 1$, $a = 0$, i.e.

$$\mathcal{C} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (3.12)$$

All other $2 \times 2$ nilpotent matrices (3.10) can be obtained by similarity transformation (3.11).

Let us apply now the formula (1.13), with the choice $\omega = \sigma_1$ and $\mathcal{C}$ given by (3.12) in order to obtain from (3.3), (3.4) and (3.6) the formulae for the generators $\hat{M}_i$, $\hat{H}$, $\hat{K}$, $\hat{D}$, $\hat{P}_i$, $\hat{B}_i$ and
\[ \hat{K} \] One gets, in case of \( \hat{H} \) in accordance with positive–definite formula (1.5) (see also footnote 4)

\[ \hat{M}_i = \frac{1}{2} \sum_{k=1}^{2} \left( \hat{t}^{(k)}_{\alpha_1,1}(\sigma_i)_{\alpha}^\beta \hat{t}^{(k)}_{\beta_1,2} - \hat{t}^{(k)}_{\alpha_1,2}(\sigma_i)_{\alpha}^\beta \hat{t}^{(k)}_{\beta_1,1} \right) \]  

(3.13)

\[ \hat{H} = \sum_{k=1}^{2} \hat{t}^{(k)}_{\alpha_1,1} \hat{t}^{(k)}_{\alpha_1,1}, \quad \hat{K} = \sum_{k=1}^{2} \hat{t}^{(k)}_{\alpha_2,1} \hat{t}^{(k)}_{\alpha_2,1}, \quad \hat{D} = \frac{i}{2} \sum_{k=1}^{2} \left( \hat{t}^{(k)}_{\alpha_1,2} \hat{t}^{(k)}_{\alpha_1,1} - \hat{t}^{(k)}_{\alpha_1,1} \hat{t}^{(k)}_{\alpha_1,2} \right) \]  

(3.14)

and (see also (3.6))

\[ \hat{P}_i = \hat{t}^{(1)}_{\alpha_1,1}(\sigma_i)_{\alpha}^\beta \hat{t}^{(2)}_{\beta_1,1}, \quad \hat{F}_i = \hat{t}^{(1)}_{\alpha_1,1}(\sigma_i)_{\alpha}^\beta \hat{t}^{(2)}_{\beta_1,2}, \quad \hat{B}_i = \frac{i}{2} \left( \hat{t}^{(1)}_{\alpha_2,2}(\sigma_i)_{\alpha}^\beta \hat{t}^{(2)}_{\beta_1,1} - \hat{t}^{(1)}_{\alpha_1,1}(\sigma_i)_{\alpha}^\beta \hat{t}^{(2)}_{\beta_1,2} \right). \]  

(3.15)

All the operator realizations (3.6) of the generators \( P_i, B_i \) and \( F_i \) are not Hermitian. This property remains valid if we insert in the relations (1.13) the matrix realization (3.6) with general nilpotent matrix (see (3.9)). If we assume the Hermiticity of (3.6), i.e. \( \mathcal{C} = \mathcal{C}^+ \) we obtain from (3.7) and (3.9)

\[ \mathcal{C} \mathcal{C}^+ = 0 \quad \Rightarrow \quad |\mathcal{C}|^2 = 0 \quad \Rightarrow \quad c_i = 0 \]  

(3.16)

The argument about nonexistence of the Hermitian representation of GCA can be extended to arbitrary multiplet of Galilean twistors \( (k = 1, \ldots, N) \). If we assume positive \( H \), i.e. choose unique formula

\[ \hat{H} = \sum_{k=1}^{N} \hat{t}^{(k)}_{\alpha_1,1} \hat{t}^{(k)}_{\alpha_1,1} \]  

(3.17)

we should choose the following \( 4N \times 4N \) matrix realization of \( H \)

\[ H = 1_2 \otimes \sigma_- \otimes 1_N \]  

(3.18)

implying that

\[ K = 1_2 \otimes \sigma_+ \otimes 1_N, \quad D = 1_2 \otimes \frac{i}{2} \sigma_3 \otimes 1_N. \]  

(3.19)

Further

\[ M_i = \frac{1}{2} \sigma_i \otimes 1_2 \otimes 1_N \]  

(3.20)

\[ R_r = 1_2 \otimes \frac{1}{2} \rho_r \otimes 1_N \]  

(3.21)

and \( O(3) \otimes O(2,1) \) covariance relations (3.7) imply that for the Abelian GCA generators one obtains the compact matrix relation

\[ A_{i,r} = \sigma_i \otimes \rho_r \otimes \mathcal{C}^{(N)} \]  

(3.22)

where the commutativity of the generators \( A \) implies nilpotency condition for \( N \times N \) matrix \( \mathcal{C}^{(N)} \)

\[ (\mathcal{C}^{(N)})^2 = 0. \]  

(3.23)

From (3.22) and (1.13) for any choice of the \( 2 \times 2 \) spinorial \( O(2,1) \) metric \( \omega^{ab} \) follows that the Hermiticity condition implies

\[ (\mathcal{C}^{(N)})^+ = \mathcal{C}^{(N)} \]  

(3.24)

what together with conditions (3.23) implies that \( \mathcal{C}^{(N)} (\mathcal{C}^{(N)})^+ = 0 \), i.e. \( \mathcal{C}^{(N)} = 0 \). Indeed, by the similarity transformation any Hermitian matrix can be written as the diagonal one with real eigenvalues; the condition (3.23) implies that these eigenvalues are all equal to zero.
3.2 The realizations with indefinite Hamiltonian $H$

For $N = 2$ indefinite $H$ means the choice (compare with (3.5a))

$$H = \mathbf{1}_2 \otimes \sigma_- \otimes \sigma_3$$

(3.25)

and the generators of $su(2)$ as in (3.3):

$$M_i = \frac{i}{2} \sigma_i \otimes \mathbf{1}_2 \otimes \mathbf{1}_2.$$  

(3.26)

The remaining generators of $o(2, 1)$ subalgebra follow uniquely

$$K = \mathbf{1}_2 \otimes \sigma_+ \otimes \sigma_3,$$
$$D = \mathbf{1}_2 \otimes \frac{i}{2} \sigma_3 \otimes \mathbf{1}_2.$$  

(3.27)

One can comment that the existence of doublet (3.25) and (3.27) of $o(2, 1)$ realizations is possible, because the transformation $H \rightarrow -H$, $K \rightarrow -K$, $D \rightarrow D$ leaves the subalgebra (2.5) invariant.

In order to fulfill the covariance relations (2.2–2.4) we should choose the following matrix realizations

$$P_i = \sigma_i \otimes \sigma_- \otimes \xi \sigma_i,$$
$$F_i = \sigma_i \otimes \sigma_+ \otimes \xi \sigma_i,$$

(3.28)

Using two possible definitions of $B_i$ - by commutators $[K, P_i]$ and $[H, F_i]$ - one obtains that

$$p_0 = f_0, \quad p_1 = -f_1, \quad p_2 = -f_2, \quad p_3 = f_3,$$

(3.29)

and

$$B_i = \frac{i}{2} \sigma_i \otimes \left(\sigma_3 \otimes (p_3 + p_0\sigma_3) - \mathbf{1}_2 \otimes (i \sigma_1 p_2 - i \sigma_2 p_1)\right).$$

(3.30)

Now we shall consider the relations (3.8). The commutators $[P_i, P_j] = [F_i, F_j] = 0$ are satisfied for any choice of $\xi$, but from $[P_i, F_j] = 0$ follows that

$$\xi P_P \xi F_F = 0 \quad \Rightarrow \quad p_0^2 - p_1^2 - p_2^2 = 0, \quad p_3 = 0.$$  

(3.31)

It can be checked that the Abelian commutativity conditions involving $B_i$ are satisfied if the conditions (3.31) are valid.

The matrix realizations (3.28) lead to Hermitian quantum-mechanical generators $\hat{P}_i, \hat{F}_i$ (see (1.13)) if the matrices $\xi P_P$ and $\xi F_F$ are Hermitian, what implies that $p_0, p_1, p_2$ should be real. But, in opposite to the “Euclidean” case (3.16), the quantities $(p_0, p_1, p_2)$ form isotropic Minkowski three–vector. Thus, the nonzero solutions of the real constraints (3.31) exist (e.g. $p_0 = p_1$, $p_2 = p_3 = 0$), i.e. the corresponding Hermitian generators of GCA also exist.

The argument can be extended to $N$-twistorial system defining the following indefinite Hamiltonian

$$\hat{H} = \sum_{k=1}^{N} d_k \tilde{t}^{(k)}_{\alpha, 1} \tilde{t}^{(k)}_{\alpha, 1}, \quad d_k = \pm 1.$$  

(3.32)

Introducing the matrix $\Delta_N = \text{diag}(d_1 \ldots, d_N)$ the matrix realization which leads via (1.13) to (3.32) takes the form

$$H = \mathbf{1}_2 \otimes \sigma_- \otimes \Delta_N.$$  

(3.33)
From (2.5) uniquely follows the choice
\[
K = 1_2 \otimes \sigma_+ \otimes \Delta_N, \\
D = 1_2 \otimes \frac{1}{2} \sigma_3 \otimes 1_N.
\] (3.34)

We put further
\[
M_i = \frac{1}{2} \sigma_i \otimes 1_2 \otimes 1_N.
\] (3.35)

If we postulate that
\[
P_i = \sigma_i \otimes P, \quad B_i = \sigma_i \otimes B, \quad F_i = \sigma_i \otimes F,
\] (3.36)
where \(P, B\) and \(F\) are the \(2N \times 2N\) matrices, from covariance properties (2.18) one gets the formulae (\(F\) is an \(N \times N\) arbitrary matrix)
\[
P = \sigma_- \otimes \Delta_N F \Delta_N, \\
F = \sigma_+ \otimes F, \\
B = \frac{i}{2} \left( 1_2 \otimes [\Delta_N, F] - \sigma_3 \otimes \{\Delta_N, F\} \right).
\] (3.37)

The Abelian structure of subalgebra \((P_k, B_k, F_k)\) leads to the following condition
\[
(\Delta_N F)^2 = 0 \iff F \Delta_N F = 0,
\] (3.38)
which is equivalent to \((F \Delta_N)^2 = 0\). Further Hermiticity of quantum–mechanical generator \(\hat{P}_i\) implies that \(\Delta_N F \Delta_N\) is Hermitian what is equivalent to the Hermiticity of matrix \(F\). We see therefore that Hermitian realizations require nonvanishing solutions of (3.38) with Hermitian \(F\). For any \(\Delta_N \neq 1_N\) such nonvanishing \(F\) exists. For example, in the case \(\Delta_N = \text{diag}(1_p, -1_q), p + q = N, p > q\), the general solution for matrix \(F\) is the following
\[
F = \begin{pmatrix} 0_{p-q} & 0 & 0 \\ 0 & M & MU \\ 0 & MU^+ & M \end{pmatrix},
\] (3.39)
where \(q \times q\) matrices \(M\) and \(U\) commute, \([M, U] = 0\), \(M\) is Hermitian \((M = M^+)\), and \(U\) is unitary \((UU^+ = 1)\).

We see therefore that if \(\Delta_N \neq 1_N\), any twistor realization (3.32) of the Hamiltonian generator can be extended to the realization of complete GCA.

### 3.3 Hermitian twistorial realization of Galilean conformal algebra in \(N=2\) case

Finally we shall present explicit \(N=2\) Hermitian representation of GCA. Taking matrix realization (3.25)-(3.28), (3.30) with \(p_0=p_1=\frac{1}{2}, p_2=p_3=0\) and using basic construction (1.13) we obtain the following quantum–mechanical twistorial realization of GCA
\[
\hat{M}_i = \frac{1}{2} \sum_{k=1}^{2} \left( \hat{t}_{\alpha,1}^{(k)}(\sigma_i)_{\alpha}^{\beta} \hat{t}_{\beta,2}^{(k)} - \hat{t}_{\alpha,2}^{(k)}(\sigma_i)_{\alpha}^{\beta} \hat{t}_{\beta,1}^{(k)} \right),
\] (3.40)
\[ \hat{H} = i^{(1)}_{\alpha,1} t^{(1)}_{\alpha,1} - i^{(2)}_{\alpha,1} T^{(2)}_{\alpha,1}, \]
\[ \hat{K} = i^{(1)}_{\alpha,2} t^{(1)}_{\alpha,2} - i^{(2)}_{\alpha,2} T^{(2)}_{\alpha,2}, \]
\[ \hat{D} = \frac{i}{2} \sum_{k=1}^{2} (i^{(k)}_{\alpha,2} t^{(k)}_{\alpha,1} - i^{(k)}_{\alpha,1} T^{(k)}_{\alpha,2}) \]

\[ \hat{P}_i = i^{(1)}_{\alpha,1} (\sigma_\alpha)_{\beta} t^{(1)}_{\beta,1} + i^{(2)}_{\alpha,1} (\sigma_\alpha)_{\beta} T^{(2)}_{\beta,1} + \hat{t}^{(1)}_{\alpha,1} (\sigma_\alpha)_{\beta} t^{(2)}_{\beta,1} + i^{(2)}_{\alpha,1} (\sigma_\alpha)_{\beta} t^{(1)}_{\beta,1}, \]
\[ \hat{F}_i = i^{(1)}_{\alpha,2} (\sigma_\alpha)_{\beta} t^{(1)}_{\beta,2} + \hat{t}^{(2)}_{\alpha,2} (\sigma_\alpha)_{\beta} T^{(2)}_{\beta,2} - \hat{t}^{(1)}_{\alpha,2} (\sigma_\alpha)_{\beta} T^{(2)}_{\beta,2} - i^{(2)}_{\alpha,2} (\sigma_\alpha)_{\beta} t^{(1)}_{\beta,2}, \]
\[ \hat{B}_i = \frac{i}{2} \left( \hat{t}^{(1)}_{\alpha,2} (\sigma_\alpha)_{\beta} t^{(1)}_{\beta,2} - \hat{t}^{(1)}_{\alpha,1} (\sigma_\alpha)_{\beta} t^{(1)}_{\beta,1} - \hat{t}^{(2)}_{\alpha,2} (\sigma_\alpha)_{\beta} T^{(2)}_{\beta,2} + \hat{t}^{(2)}_{\alpha,1} (\sigma_\alpha)_{\beta} T^{(2)}_{\beta,1} \right) \]

From (3.42) follows apparently that the operators \( \hat{P}_i, \hat{F}_i \) and \( \hat{B}_i \) are Hermitian. Their commutativity follows if we note, that after introducing of the operators
\[
u_\alpha \equiv t^{(1)}_{\alpha,1} + \epsilon_{\alpha\beta} \hat{t}^{(2)}_{\beta,1}, \quad v_\alpha \equiv t^{(1)}_{\alpha,2} - \epsilon_{\alpha\beta} \hat{t}^{(2)}_{\beta,2}, \]
where \( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = 1 \) is a second SU(2)–invariant tensor in addition to \( \delta_{\alpha\beta} \) (see footnote 4), the operators forming the Abelian subalgebra (3.42) can be represented as follows
\[ \hat{P}_i = \hat{u}_\alpha (\sigma_\alpha)_{\beta} u_\beta, \]
\[ \hat{F}_i = \hat{v}_\alpha (\sigma_\alpha)_{\beta} v_\beta, \]
\[ \hat{B}_i = \frac{i}{2} \left( \hat{v}_\alpha (\sigma_\alpha)_{\beta} u_\beta - \hat{u}_\alpha (\sigma_\alpha)_{\beta} v_\beta \right). \]

We observe that the operators (3.43) and its conjugates describe only half of the \( N=2 \) twistorial degrees of freedom and it follows from (1.9) that they commute between themselves,
\[ [u_\alpha, u_\beta] = [u_\alpha, \bar{u}_\beta] = [u_\alpha, v_\beta] = [u_\alpha, \bar{v}_\beta] = [v_\alpha, u_\beta] = [v_\alpha, \bar{u}_\beta] = [v_\alpha, v_\beta] = 0. \]

As result, we see from the formulae (3.44) and (3.45) that our realization for \( \hat{P}_i, \hat{F}_i, \hat{B}_i \) forms a nine–dimensional Abelian subalgebra.

4 Final Remarks

An important advantage of the relativistic Penrose twistor formulation is the possibility of expressing the generators of O(4,2) conformal algebra as current–like bilinear expressions in \( D=4 \) quantized twistor variables. Unfortunately in nonrelativistic case the GCA generators can be expressed bilinearly by quantized Galilean twistors only if the Hamiltonian is not positive–
definite. We recall that the difference between relativistic and Galilean conformal groups is essential: the relativistic one is semisimple, and the nonrelativistic is described by a semidirect product involving for \( d=3 \) nine–dimensional Abelian subalgebra. It should be stressed that however the GCA is the \( c\rightarrow \infty \) contraction limit of relativistic \( o(4,2) \) algebra, we were not able to obtain the Galilean \( N=\)twistor realizations by the contraction \( c\rightarrow \infty \) of the relativistic \( N=\)twistor realizations. Similarly, if we introduce explicitly the nonrelativistic space–time by rescaling \( x_0=ct \), one does not get the nonrelativistic limit of the incidence relations (1.6).
The problem of expressing nonrelativistic space–time \((x_i, t)\) as composite in terms of Galilean twistors requires further investigation.

Finally we recall that the nonrelativistic GCA has been recently extended to \(n\)-extended Galilean superconformal algebra \([21, 22]\) with \(n = 2k\). E.g. for the lowest-dimensional case \(n = 2\) it would be interesting to prove the expected result that using the pair of \(n = 2\) nonrelativistic supertwistors one can construct a Hermitian supertwistorial realization of Galilean \(n = 2\) superconformal algebra but it will lead as well to indefinite generator \(H\).

In this short Letter, we do not discuss the application of our considerations to concrete models with indefinite Hamiltonian. We only note here that indefinite Hamiltonian was also obtained in the quantum–mechanical model with \(D = 2+1\) “exotic” Galilean symmetry \([23]\), but the ghosts in \([23]\) were eliminated by imposing the additional condition. Certain subsidiary conditions excluding the states with negative energy were also considered in some models of supersymmetric mechanics and field theory \([24]\).

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