Analytical solutions of the pre-breakdown electro-hydrodynamic equations and numerical simulations of liquid insulator flows

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Abstract. The system of electro-hydrodynamic equations at a pre-breakdown liquid insulator is given. The influence of electric field on the partial dissociation molecules rate is taken into account. The stationary and non-stationary solutions for electric potential and hydrodynamic velocity distributions are obtained. The numerical results of electro-hydrodynamic flows are compared with the experiment data.

1. Introduction
The purpose of this work is the analysis of the conduction processes in the liquid insulators under high voltage applied and the numerical simulation of the electro-hydrodynamic flows by our solutions of the pre-breakdown macroscopic equations. The liquid insulator under high voltage reveals well-known features of its behavior that is concluded in the deviation of the current–field dependence from Ohm’s law [1, 2]. We suppose that the liquid dielectric is analogous to the weak electrolyte. And the partial dissociation of its molecules has place. Besides that, we consider that impurities can influence on the low-voltage conductivity of the medium in hand. This conductivity can be determined from the linear part of the current-voltage dependence, since Ohm’s law is valid under low voltage. Our theoretical model is different from the model described in [3]. The concentrations of ions $n_{\pm}$ and impurities $n_p$ are supposed to satisfy the following conditions:

$$
\begin{cases}
n_{\pm} \ll n_a, \\
n_p \ll n_a,
\end{cases}
$$

(1)

where $n_a$ is the concentration of neutral particles (molecules). The rate of their dissociation according to [1, 2] is:

$$W_D = W_D(n_a, n_p, T, |\vec{E}|) = W_D(n_a, n_p, T, 0)f(|\vec{E}|).$$

(2)

Here $T$ is the absolute temperature, $\vec{E}$ is the intensity of the electric field. The rate of the ions recombination is:

$$
\begin{cases}
W_r = K_r n_+ n_- , \\
K_r = \frac{(b_+ + b_-) |e|}{\varepsilon_0}.
\end{cases}
$$

(3)
The recombination rate coefficient $K_r$ was obtained by Langevin. According to Onsager [1], $K_r$ is not dependent on $|\vec{E}|$; $b_+$ and $b_-$ are the mobilities of the ions, $|e| = Z|e_0|$ is their charge, $Z$ is the ion valency, $\varepsilon$ is the dielectric permittivity, $\varepsilon_0$ is the constant in SI. We suppose that the influence of the field on $b_+, b_-$, $\varepsilon$ is weak. In particular, this consideration for $\varepsilon$ follows from Debye formulae: $\varepsilon - 1 \sim \rho$. For our media $\rho \approx n_a m_a$, where $m_a$ is the mass of neutral particles. And the field does not influence on $n_a$, according to (1). We also use the Einstein formulæ for diffusion:

$$D_\pm = \frac{k_B T b_\pm}{|e|},$$

where $k_B$ is the Boltzmann constant, $T$ is supposed to be constant and equal to 293 K, because the Joule heat is small for the pre-breakdown phenomena. The energy, expended in chemical reaction, is also considered negligible for a weak electrolytes and substances analogous to them. The general initial conditions for the system of equations (6) and (7), independent of the impurities, on coefficient $A_0$ in Arrhenius exponent by means of initial conductivity $\sigma_0$.

The second expression was obtained by Onsager [1] in the form of Bessel function. If the intensity of the field tends to infinity, Onsager’s formula from [1] is approximated by (5). Onsager destroyed of chemical bonds, the influence of the field on $\varepsilon$ follows from Debye formulae:

$$\varepsilon \approx n \epsilon_0 |\vec{E}|$$

for our media $\rho \approx n_a m_a$, where $m_a$ is the mass of neutral particles. And the field does not influence on $n_a$, according to (1). We also use the Einstein formulæ for diffusion:

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where $k_B$ is the Boltzmann constant, $T$ is supposed to be constant and equal to 293 K, because the Joule heat is small for the pre-breakdown phenomena. The energy, expended in chemical reaction, is also considered negligible for a weak electrolytes and substances analogous to them. For the function $f(|\vec{E}|)$ in the equation (2) there are two expressions. One of them was obtained by Plumley [2] for liquid insulators. Frenkel also independently derived it for a solid insulator with possible covalent bonds. It looks like:

$$f(|\vec{E}|) = \exp (\beta |\vec{E}|^{1/2}), \quad \beta = \frac{|e|^{3/2}}{\sqrt{\pi \varepsilon_0 k_B T}}.$$  

If the expression (5) is valid, then $W_D(n_a, n_p, T, 0)$ in the equation (2) is reduced to the well-known Arrhenius exponent. $W_D(n_a, n_p, T, 0) = A_0 \exp(-U_a/k_B T)$, where $U_a$ is the energy of destroying of chemical bonds, $A_0$ is weakly dependent on temperature coefficient.

The second expression was obtained by Onsager [1] in the form of Bessel function. If the intensity of the field tends to infinity, Onsager’s formula from [1] is approximated by (5). Onsager studied the media with only ion chemical bonds.

2. The model of electrohydrodynamic processes in the liquid insulator

We consider that the influence of hydrodynamic on conduction process is negligible in our case. This can be possible in a case when the intensity of the field is not so strong to cause emission or injection of charges from electrodes. We also introduce the variables $q = (n_+ - n_-)|e|$, $\sigma = (n_-b_- + n_+ b_+)|e|$, and the potential $\phi$: $\vec{E} = -\nabla \phi$. Then, we can write the system of macroscopic time dependent equations of the ions produced. If we take $b_+ = b_- = b$ (this supposition is more correct for liquids than for gases) with (1)–(5) we obtain:

$$\begin{cases}
\frac{\partial q}{\partial t} + (\vec{V}, \nabla q) - \frac{k_B T b}{|e|} \Delta q + (\vec{E}, \nabla \sigma) = -\frac{q \sigma}{\varepsilon \varepsilon_0},
\Delta \phi = -\frac{q}{\varepsilon \varepsilon_0},
\frac{\partial \sigma}{\partial t} + (\vec{V}, \nabla \sigma) + b^2 (\vec{E}, \nabla q) - \frac{k_B T b}{|e|} \Delta \sigma - \frac{\sigma^2}{\varepsilon \varepsilon_0} \exp (\beta |\vec{E}|^{1/2}) + \frac{\sigma^2}{\varepsilon \varepsilon_0} = 0,
\end{cases}$$

where $V$ is the hydrodynamic velocity of neutral component and it approximately equals to the velocity of the whole mixture—in a case if (1) is valid. We have used the equations (1)–(5) with the continuity equation for non-compressible liquid to derive (6):

$$\text{div} \vec{V} = 0.$$  

We remark that we have used the type of model for description of non-isothermal conduction processes that is described in [4]. We have considered the influence of charged particles, including impurities, on coefficient $A_0$ in Arrhenius exponent by means of initial conductivity $\sigma_0$.

The general initial conditions for the system of equations (6) and (7), independent of the electrodes geometry, are:

$$q(t = 0) = V(t = 0), \quad \sigma(t = 0) = \sigma_0.$$  

(8)
We also introduce the value \( a = (q_1/n_1|e|) \), where \( q_1 \) is the characteristic density of space charge, \( n_1 \) is the characteristic concentration of neutrals, decaying into ions. According to [4],
\[
2bn_1|e| = \sigma_0 \exp \left( \frac{(\beta/2)|E|^1/2}{2} \right)
\]
outside the boundary layers. And, according to the Poisson equation from system (6), \( q_1 = \varepsilon_0 U/(|e|L^2) \). Where \( L \) is the characteristic length of the task. Thus, outside the boundary layers
\[
a = 2\varepsilon_0 b|e|^2L^2\sigma_0^{-1} \exp \left( -\frac{\beta}{2}|E|^1/2 \right),
\]
where \( U \) is the applied voltage. For usual liquid insulator \( a \ll 1 \), i.e. the medium is neutral in terms of space charge creation under the strong fields applied (outside the boundary layers).

The ratio of the second term in the 3rd equation of the system (6) to the fourth term is about the order of \( a^2 \) and, consequently, we can neglect it. The ratio of the 3rd diffusion term of this equation to the fourth is about the order of \( (r_d/L)^2 \), where \( r_d \) is the Debye radius. So, if the medium is quasi-neutral, then, as we know from plasma and electrolyte theory, this ratio is much smaller than unit. Also we can see that the hydrodynamic processes do not influence on the electrical conductivity if \( Re_q = V_1/(bE_1) < 1 \). Under such conditions we can obtain Frenkel’s law for the conductivity of the liquid insulators from the (6) equation.

For the case of sphere under high voltage there is existed only stationary solution outside boundary layers, consisting to the Frenkel’s conductivity law [4]:
\[
\phi(r) = \left( \frac{I}{4\pi \sigma_0} \right)^{1/2} \left[ \frac{8}{\beta} - \sqrt{\frac{\varepsilon_0}{\sigma_0}} \left( |E|^{1/2} + \frac{8}{\beta} \right) \right] \text{sign}(\phi(r_0)),
\]
\[
|E| \exp \left( \frac{\beta}{2}|E|^{1/2} \right) = \frac{I}{4\pi \sigma_0 r^2}, \quad |\phi(r_0)| = U.
\]
Here \( I \) is the constant of integration that is equal to the electric current, \( r \) is the distance from the center of the sphere. Formulae (10) is described a pre-breakdown deviation from Ohm’s law.

3. Some applications of the electrodynamic model for the theoretical description of the hydrodynamic flows of weakly conductive liquids caused by the nonuniform high electric field strength

We use the Felici conception [5] for the results obtained above for the theoretical description of the flows. According to [5] the influence of the volume dissociation on these flows is dominating under the high fields of needle, blade or wire electrodes.

The last phenomenon was observed by Faraday in liquid dielectrics and by Franklin in gases earlier. The characteristic time of this formation \( t_1 \) is
\[
t_1 = \frac{L^2}{\nu} \gg \tau,
\]
where \( \nu \) is the kinematics viscosity, \( \tau \) is the time of the electrodynamic processes relaxation.

Thus, it is possible to describe the flows caused by the stationary force \( qE \) for our hydrodynamic model. In the case of non-compressible medium we should add only the equation of momentum conservation without heat transfer equation. Let’s write the momentum conservation law in the vortex form and introduce new variables \( \psi \) and \( \chi \). Then the momentum equation for the three-dimensional case will be
\[
\rho \frac{d\vec{\chi}}{dt} - \rho(\vec{\chi} \nabla) \vec{V} - \rho \nu \Delta \vec{\chi} = 2[\nabla q, \vec{E}],
\]
V can be expressed according to the equation (12) in the axis-symmetrical case. If the density of the Coulomb force sharply decreases with the increasing of the distance from the electrode, one can solve the equations (6), (7), (12) without this term for the flows, caused by the integral force:

\[ \vec{F}_E = \int_{G} q\vec{E}dG, \quad (13) \]

where \( G \) is the volume of half-plane, containing the liquid dielectrics according to the electrohydrodynamic model, described above. Now we can calculate \( F_E \) from the equation (13). After that we can use the class of hydrodynamic solution caused by a point force (which may not necessarily be electrical nature). There is a lot of publication dedicated to this topic. These works develops the idea, offered in \[6\]. So, with calculation \( E, q \) in accordance with the equation (10), we can analytically obtain \( F_E \) from the equation (13):

\[ F_E = \frac{\varepsilon \varepsilon_0 J}{\beta \sigma_0^2} \left\{ 2 - \exp \left( -\frac{\beta}{2} |E_0|^{1/2} \right) \left[ \frac{\beta^2}{4} |E_0| + \beta |E_0|^{1/2} + 2 \right] \right\}, \quad (14) \]

Now one can easily express \( F_E(U) \) too, applying formulae (10) to the (14).

Now we will describe the flows, caused by the Coulomb force \( F_E \) in the form of (14). In the manual on hydrodynamic (see \[6\]) there are more well known solutions of stationary Navier-Stokes equations (7) and (12) as hydrodynamic viscous boundary layer equations. For Shlichting axis-symmetrical stationary jet \[7\]:

\[ \begin{align*}
\xi &= \frac{\gamma R}{z}, \quad V_z = \frac{2\gamma^2}{z (1 + \xi^2/4)^2}, \quad V_R = \frac{\nu \gamma}{z (1 + \xi^2/4)^2} \left[ \frac{\xi}{2} |E_0|^{1/2} \right], \\
\gamma &= \left( \frac{3\pi p}{4\nu \rho^{1/2}} \right)^{1/2}, \quad P = 2\pi p \int R dR.
\end{align*} \quad (15) \]

In (15) \( P \) is the integral momentum of the jet, \( R \) and \( z \) are the axis-symmetrical jet cylindrical coordinates.

There is a generalization of solution type (15) in \[6\] in the form of \( \psi = \nu r f(\theta) \). Function \( f(\theta) \) satisfies an ordinary differential equation, obtained from the (7), (12). Squire is proposed in \[6\] to explain the jet motion as a result of point force action of the delta-function type. And the proof of an expression

\[ F_e = P \quad (16) \]

is provided in our previous works. If \( \beta |E_0|^{1/2} \gg 1 \) it follows from (14) and (10) that

\[ F_E = \frac{\varepsilon \varepsilon_0 U^2}{8}. \quad (17) \]

Then one can deduce from (15) and (17) that axis velocity \( V_0 \) is proportional to the voltage squared \( V_0 \sim U^2 \) and is not dependent of \( r_0 \) (\( r_0 \) is the radius of small electrode). So, the maximum jet velocity is proportional to voltage squared too. In conclusion to this part we describe the model of unsteady streams. The numerical calculations \[8\] testified that the structure of streams developed in time is different from the stationary type (15). The simplest explanation can be obtained with the aid of (17). In that case one can consider that the Coulomb...
density force $F_E$ is not dependent on time. The initial velocities one can take in [9]. It turns out that for $\psi$ the field of velocities at $t \to 0$ is the same as the field of dipole:

$$\psi = \frac{F_E t \sin^2 \theta}{4\pi \rho r}.$$  \hspace{1cm} (18)

So, we can calculate the pre-breakdown flow velocities when $t \sim \tau$ too. We also have obtained the breakdown condition of liquid insulators at high non-uniform electric field strength by applying the well-known Sedov–Taylor theory of hydrodynamic explosion for these conditions (viscous influence is not taken into account).

Then the electro-hydrodynamic breakdown analogue of Sedov explosion maximum radius formulae near the point of high voltage electrode may be obtained:

$$r_{\text{max}} = |e| U^3 \sqrt{\frac{21}{100 \pi k_B p_0 T^2}}.$$  \hspace{1cm} (19)

The approximate formula (15) can not help to determine the maximum speed for movement of the liquid from the electrode to surface (counter electrode). In addition, it does not depend on the valence electrons $Z$, so the program was developed to simulate the fluid motion in the system of electrodes, named ‘wire above the plane’, which currently does not take into account the space charge field. At the same time, formula (15) is derived based on the space charge, so numerical calculations were carried out at a potential difference between the electrodes of 500 and 1000 V. The condition of smallness of the area breakdown have obeyed in these calculations. $r_{\text{max}} = 5.41 \times 10^{-3}$ cm when the inter-electrode distance equals to 2 cm.

4. Numerical calculation

4.1. Description of the program

The program, simulating a two-dimensional motion of weakly conducting viscous incompressible fluid in an electric field was written. The calculation algorithm is based on the MacCormack explicit method which is second order in accuracy in time and space, and well-proven in solving hyperbolic equations. Since the flow is steady, the Navier–Stokes equations for an incompressible fluid are solved using the method of artificial compressibility.

The continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$  \hspace{1cm} (20)

where $u$ and $v$ are the components of the velocity according to the $Ox$ and $Oy$ axes respectively.

The equation of motion along the $Ox$ and $Oy$ axes (a consequence of equations (7) and (12) without taking into account the gravity mass forces and temperature effects):

$$\begin{aligned}
\rho \left( \frac{\partial u}{\partial t} + u \left( \frac{\partial u}{\partial x} \right) + v \left( \frac{\partial u}{\partial y} \right) \right) &= - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + qE_x, \\
\rho \left( \frac{\partial v}{\partial t} + u \left( \frac{\partial v}{\partial x} \right) + v \left( \frac{\partial v}{\partial y} \right) \right) &= - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + qE_y,
\end{aligned}$$  \hspace{1cm} (21)

where $\rho$ is the density of fluid, $p$ is the static pressure, $u$ and $v$ are the projections of the velocity on the axes $Ox$ and $Oy$, respectively, $\mu$ is the dynamic viscosity of fluid, $q$ is the density of volume charge in fluid, $E_x$ and $E_y$ are the components of the electric field in fluid.

In this method the equation of continuity includes member with artificial compressibility, which vanishes when the solution is become steady in time. In this case, the Navier–Stokes
equations form a mixed system of hyperbolic-parabolic equations, which is solved by the usual relaxation method. Thus, the continuity equation is replaced by the following equation:

\[
\frac{\partial \rho^*}{\partial t^*} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

(22)
where \(\rho^*\) is artificial density and \(t^*\) is a fictitious time, the analog of the real-time flow of a compressible fluid. Artificial density is related to the pressure with so-called artificial equation of state:

\[
p^* = \frac{\rho^*}{\mu^*},
\]

(23)

where \(\mu^*\) is the coefficient of the artificial compressibility.

The density of space charge, produced in the low conductive liquids at pre-breakdown fields and the specific conductivity are represented as:

\[
q = -\varepsilon \varepsilon_0 \left( \bar{E}, \nabla \ln \frac{\sigma}{\sigma_0} \right), \quad \sigma = \sigma_0 \exp \left( \frac{\beta}{2} |E|^{1/2} \right), \quad \text{where} \quad \beta = \frac{(Ze)^{3/2}}{(\pi \varepsilon \varepsilon_0)^{1/2} k_B T}.
\]

(24)

Wherein \(q\) is determined by the Frenkel conductivity gradient (5).

The oil composition and other parameters are represented as: \(H_{0.8}O_{0.15}(C_{20}H_{40})O_{0.15}H_{0.8}\), \(Z = 2\) (according to professor V G Arakelyan), \(\varepsilon = 2.5\), \(T = 300 \text{ K}\), \(\rho = 800 \text{ kg/m}^3\), \(\nu = 0.2 \text{ cSt}\), \(\sigma_0 = 10^{-13} \text{ Ohm}^{-1}\text{cm}^{-1}\). The wire has a diameter of 4 mm. The distance from the wire to the plane equals to 20 mm. The wire has a positive charge.

The problem is solved in a half-plane \(y \geq 0\) with the potential deference \(\phi(r_0) - \phi(x, 0) = U\), where \(U = 500 \text{ V}\).

The electrostatic field has the well-known form for given geometry:

\[
\begin{align*}
E_x &= \frac{4xyhU}{(h^4 + x^4 + y^4 + 2hx^2 + 2xy^2 - 2hy^2) \ln(2h/r_0)}, \\
E_y &= -\frac{2hU(x^2 + h^2 - y^2)}{(h^4 + x^4 + y^4 + 2hx^2 + 2xy^2 - 2hy^2) \ln(2h/r_0)},
\end{align*}
\]

(25)

where \(h\) is the distance from the center of the wire to the plane.

The initial conditions: \(t = 0, \ u = 0, \ v = 0, \ p = p_0\), where \(p_0 = 10^5 \text{ (Pa)}\).

The boundary conditions: at \(y = 0, \ u = 0, \ v = 0; \) at \(x = 0, \ \partial u/\partial x = 0, \ u = 0; \ u_\infty = 0, \ v_\infty = 0, \ P_\infty = P_0\).

Condition of adhesion \((u = 0, \ v = 0)\) are observed on the flat surface of the electrode, where

\[\sqrt{x^2 + y^2} = r_0,\]

and the counter-electrode. The cross section of the active electrode (wire) is approximated by a square. Influence of the pre-breakdown charge and hydrodynamics on the value of the field strength at \(U = 500 \text{ V}\), can be neglected.

4.2. Testing of the program

In order to use the program, it is necessary to make verification. To do this, for example, it is necessary to compare the results of calculations of the simplest tasks with the analytical solution of the same problems. The Poiseuille flow of a Newtonian fluid in a plane channel in the absence of an electric field was selected as a test problem.
Figure 1. The velocity profile for the Poiseuille flow of Newtonian fluid in a plane channel with a pressure gradient and in a presence of the electric field.

The Analytical velocity profile for the steady flow of a Newtonian fluid in a plane channel has the form [7]:

\[ w = \frac{\Delta p h^2}{2\mu l} \left( 1 - \left( \frac{y}{h} \right)^2 \right), \]  \hspace{1cm} (26)

where \( w \) is the velocity along the channel, \( \Delta p \) is the pressure drop over a length \( l \), \( 2h \) is the distance between the channel walls, \( y \) is the distance from the center channel to the considered point.

Figure 1 shows the theoretical and the calculated velocity profile for the Poiseuille flow of Newtonian fluid in a flat channel for the case \( \Delta p = 1, l = 10, h = 1, \mu = 1, \rho = 1 \). The figure shows that the numerical solution for the velocity profile agrees well with the theoretical values with an error not greater than 3.14% in absolute value.

2) A simple test of the algorithm, taking into account the effect of electric forces, is to consider the viscous fluid motion in a plane channel in the presence of an electric field, which is equivalent to the pressure gradient, but in the absence thereof. Since the pressure gradient \( \Delta p/l = 0.1 \), then if the space charge \( q = 1 \), the corresponding component of the electric field will be \( E = 0.1 \). The same figure 1 shows graphs of the velocity profile for the steady Poiseuille flow of a Newtonian fluid in a plane channel as a result of the action of the electric field. In this case the error between the theoretical and numerical values does not exceed 3% also.

4.3. The results of numerical modeling

We have obtained the characteristic vortices (figure 2) in our numerical model, which are also observed experimentally [10] (figure 3). On this figure the wire electrode is marked as ‘A’ and the plane electrode is marked as ‘B’ plane. There is a condition of viscous adhesion on the plane electrode. It is worth to mention here, that such a vortex flow pattern will not be observed if consider analytical solution (15). This is because while getting this solution, the pressure was assumed to be a constant.
Figure 2. Velocity vectors and magnitude for a weakly conducting liquid in the pre-breakdown electric field in the system of electrodes “wire–plane” at a voltage of 500 V.

Figure 3. The experimental characteristic swirling for a weakly conducting liquid at a voltage of 4000 V.

Figure 4 shows a graph of velocity of the fluid moving along the axis A–B ‘wire–plane’ (see figure 2) with regard to the adhesion effect on the electrode (which is plotted as zero-rate at the distances from 17 to 22 mm from the plane electrode). The maximum speed between the electrode and the plane is about $\approx 6 \text{ mm/s}$ at a voltage of 500 V and $\approx 34 \text{ mm/s}$ at a voltage of 1000 V (approximately at the distance of 12 mm from the plane electrode), and in a good agreement with the data obtained experimentally [10]. It is worth to mention that there are
This motion pattern is axisymmetric pre-breakdown electro-hydrodynamic flow of a transformer oil at a voltage of 500 V and the inter-electrode distance of 20 mm.

no experimental data for our statement of the problem, and these experiments were carried out with the needle electrode above the plane. The main problem why we simulate the ‘wire–plane’ system is that this geometry is two-dimensional. The system of needle electrode above the plane is essential three-dimensional and more complex for simulation. Nonetheless, we suppose that characteristic values of velocity magnitudes do not differ dramatically for these geometries. We are going to simulate this 3D case too.

5. Conclusion
In this work the authors show the approximate analytical expression for the axial velocity for the needle electrode above the plane. In addition, the numerical solution of pre-breakdown electro-hydrodynamic equations for a high-voltage wire–plane electrode system is obtained in the two-dimensional formulation of the problem. The distribution of axial velocity and its maximum match qualitatively and in order of magnitude the experimental data. Besides that this maximum of velocity cannot be obtained from the analytical theory. Analytical condition of weak breakdown was checked for presented numerical calculations.

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