Construction of a free Lévy Process as high-dimensional limit of a Brownian motion on the Unitary group

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Abstract
It is well known that freeness appears in the high-dimensional limit of independance for matrices. Thus, for instance, the additive free Brownian motion can be seen as the limit of the Brownian motion on
hermitian matrices. More generally, it is quite natural to try to build free Lévy processes as high-dimensional limits of classical matricial Lévy processes.

We will focus here on one specific such constructions, discussing and generalizing the work done previously by Biane in [1], who has shown that the (classical) Brownian motion on the Unitary group $U(d)$ converges to the free multiplicative Brownian motion when $d$ goes to infinity. We shall first recall that result and give an alternative proof for it. We shall then see how this proof can be adapted in a more general context in order to get a free Lévy process on the dual group (in the sense of Voiculescu) $U(n)$. This result will actually amount to a truly noncommutative limit theorem for classical random variables, of the which Biane’s result constitutes the case $n = 1$.

1 Biane’s result about the Brownian motion on the Unitary group

In all the following, we assume that a unital noncommutative probability space $(A, \phi)$ be given. Let us remind what we mean by that definition: a unital noncommutative probability space is a couple $(A, \phi)$ where $A$ is a unital $*$-algebra and $\phi$ is linear functional on $A$ such that $\phi(a^*a) \geq 0$ for each $a \in A$ and $\phi(1) = 1$.

Let us recall following definitions and result:

**Definition 1.** We denote by $(\nu_t)_{t \geq 0}$ the same family of measures on the unit circle as in [1], ie $\nu_t$ is the only probability measure such that $\xi_{\nu_t}(z) = z \exp\left[\frac{1}{2} \frac{\psi_{\nu_t}}{1 + \psi_{\nu_t}}\right]$, where $\xi_{\nu_t}$ is the inverse function of $\frac{\psi_{\nu_t}}{1 + \psi_{\nu_t}}$ and $\psi_{\nu_t} = \int \frac{z \zeta}{1 - z \zeta} d\nu_t(\zeta)$ where the integration is done on the unit circle.

**Definition 2.** A free multiplicative Brownian motion is a family $(U_t)_{t \geq 0}$ such that:

- For every $0 \leq t_1 < t_2 < \ldots < t_n$, the family $\left(U_{t_1}, U_{t_2}U_{t_1}^{-1}, \ldots, U_{t_n}U_{t_{n-1}}^{-1}\right)$ is free.

- For every $0 \leq s < t$ the element $U_tU_s^{-1}$ has a distribution $\nu_{t-s}$.

In his paper, Biane proved that a Brownian motion on the group $U(d)$ converges, as $d$ goes to infinity, towards a multiplicative free Brownian motion. To do this, he proves first the convergence of the marginals using Representation Theory arguments and secondly the freeness of the increments. We suggest here that there is an other way to prove the convergence of the marginals based on the Itô formula.

Let us first observe that the Brownian motion on the Unitary group $U(d)$ can be defined as the unique solution of:

$$dU_t^{(d)} = idH_tU_t^{(d)} - \frac{1}{2}U_t^{(d)} dt$$
with initial condition $U_0 = I$. In this equation, we have noted by $H_t$ a Brownian motion on hermitian matrices defined by:

- The family $(H_{i,j}(t))_{1 \leq i \leq j \leq d}$ is an independent family of random variables
- For $1 \leq i \leq d$, we have $H_{i,i}(t)$ a gaussian variable $\mathcal{N}(0, \frac{1}{d})$
- For $1 \leq k \leq j \leq d$, we have $H_{k,j}(t) = H_{k,j}^{(1)}(t) + i H_{k,j}^{(2)}(t)$ with $H_{k,j}^{(1)}(t)$ and $H_{k,j}^{(2)}(t)$ two independent gaussian variables $\mathcal{N}(0, \frac{1}{2d})$
- The matrix $H(t)$ is hermitian for each $t$.

In particular this means that each coefficient of $H_t$ is of variance $1/d$.

Note: we shall omit the exponent $(d)$ when there is no confusion possible.

Let us now denote by $f_{k_1, \ldots, k_r}$ the following function of $t$:

$$f_{k_1, \ldots, k_r} = \mathbb{E} \left[ \text{tr} \left( U_{k_1 j_1} \ldots \text{tr} \left( U_{k_r j_r} \right) \right) \right]$$

where the trace is normalized by $1/d$. We will find a differential equation involving those functions.

**Lemma 1.** We have the following formula:

$$d \left( U_{i_1 j_1} \ldots U_{i_r j_r} \right) = \text{martingale} - \frac{1}{2} \sum_{k=1}^r U_{i_k j_k} \ldots U_{i_r j_r} \, dt$$

$$- \frac{dt}{d} \sum_{1 \leq p < q \leq r} U_{i_1 j_1} \ldots U_{i_p j_q} \ldots U_{i_{q_p}} \ldots U_{i_r j_r}.$$ 

**Proof.** This is obtained by using Itô’s formula and by reasoning for each element in the matrix, because:

$$d \left( U_{i_1 j_1} \ldots U_{i_r j_r} \right) = \sum_{k=1}^r U_{i_1 j_1} \ldots (d U_{i_k j_k}) \ldots U_{i_r j_r} + \sum_{1 \leq k < l \leq r, k \neq l} U_{i_k j_k} d[U_{i_k j_k}, U_{i_l j_l}]$$

When we take the expectation, the martingale part vanishes.

If we expand $f_{k_1, \ldots, k_r}$, we get:

$$f_{k_1, \ldots, k_r} = \frac{1}{d^r} \mathbb{E} \left[ \sum_{i_1 \ldots i_r} U_{i_1 j_1} \ldots U_{i_k j_k} \ldots U_{i_r j_r} \right]$$

To get a system of differential equations we will use the former formula that we have obtained thanks to Itô’s Lemma. Especially we must see how the
last term, switching \( p \) and \( q \), can be rewritten in terms of the functions \( f_{k_1,\ldots,k_r} \). There are actually two cases to study: first when \( p \) and \( q \) come from the same trace and second when they come from different traces.

**When they come from the same trace:** If for instance \( p \) and \( q \) both come from the \( m \)th trace, the contribution of this trace is of the kind:

\[
U_{i_1}^{m_{i_2}} \ldots U_{i_p}^{m_{i_{p+1}}} \ldots U_{i_q}^{m_{i_{q+1}}} \ldots U_{i_{k_m}}^{m_{i_m}}
\]

So when we do the switching it yields:

\[
U_{i_1}^{m_{i_2}} \ldots U_{i_p}^{m_{i_{p+1}}} \ldots U_{i_q}^{m_{i_{q+1}}} \ldots U_{i_{k_m}}^{m_{i_m}}
\]

And when we sum over all those indices we see that we actually get: \( df_{k_1,\ldots,k_m-(q-p),q-p,\ldots,k_r} \), ie the switching has produced one more trace.

**When they come from two different traces:** We shall here suppose that \( p \) comes from the \( u \)th trace and \( q \) comes from the \( v \)th trace, with \( u < v \). The contribution of those two traces are:

\[
U_{i_1}^{u_{i_2}} \ldots U_{i_p}^{u_{i_{p+1}}} \ldots U_{i_q}^{v_{i_{q+1}}} \ldots U_{i_{k_v}}^{v_{i_1}}
\]

Switching \( p \) and \( q \) yields to:

\[
U_{i_1}^{u_{i_2}} \ldots U_{i_p}^{u_{i_{p+1}}} \ldots U_{i_q}^{v_{i_{q+1}}} \ldots U_{i_{k_v}}^{v_{i_1}}
\]

And so if we sum over all indices we see that we get \( \frac{1}{2} f_{k_1,\ldots,k_u+k_v,\ldots,k_r} \), ie we have merged two traces together.

So, if we put it all together we see by using Lemma 1 that the system of differential equations we get is:

\[
f_{k_1,\ldots,k_r} = -\frac{k_1 + \ldots + k_r}{2} f_{k_1,\ldots,k_r} - \sum_{\kappa=1}^{r} \sum_{l=1}^{k_\kappa} (k_\kappa - l) f_{k_1,\ldots,k_\kappa-l,\ldots,k_r}
\]

\[
- \frac{1}{d^2} \sum_{1 \leq \kappa < \lambda \leq r} \sum_{p=1}^{k_\kappa} \sum_{q=1}^{k_\lambda} \sum_{l=1}^{k_\kappa} \sum_{l'=1}^{k_\lambda} f_{k_1,\ldots,k_\kappa+k_\lambda-\lambda-l,\ldots,k_r}
\]

Let us observe here that we have a nice combinatorial structure for these equations. Indeed, we can interpret \((k_1,\ldots,k_r)\) as an integer partition for the integer \( k_1 + \ldots + k_r \). By doing so, we see that the equation only involves partitions for the same integer because we either split an integer into two parts or we merge two integers into one. These equations thus seem to have the same structure as the equations in Proposition 2.3 in [5] via the identification between a permutation and the length of the cycles of its canonical decomposition.
Let us also note that an integer \( l \) has only finitely many partitions. So that means that each function is involved in a system of finitely many linear differential equations with fixed initial conditions. Hence, if the family of functions \( \left( f^{(d)}_{k_1, \ldots, k_r} \right)_{r \geq 0, k_1, \ldots, k_r \geq 1} \) admits a limit when \( d \) goes to infinity, then the limit must verify the following system of differential equations:

\[
f'_{k_1, \ldots, k_r} = \frac{-k_1 + \ldots + k_r}{2} f_{k_1, \ldots, k_r} - \sum_{\kappa=1}^{r} \sum_{l=1}^{k_{\kappa}} (k_{\kappa} - l) f_{k_1, \ldots, k_{\kappa} - l, \ldots, k_r}
\]

We will now denote by \( F_{k_1, \ldots, k_r} \) the function \( \phi \left( u^{k_1}_t \right) \ldots \phi \left( u^{k_r}_t \right) \) where \( u \) is here a free multiplicative Brownian motion. To prove the convergence of the marginals it will be enough to prove that the family of functions \( F \) verify the differential equations system:

\[
F'_{k_1, \ldots, k_r} = \frac{-k_1 + \ldots + k_r}{2} F_{k_1, \ldots, k_r} - \sum_{\kappa=1}^{r} \sum_{l=1}^{k_{\kappa}} (k_{\kappa} - l) F_{k_1, \ldots, k_{\kappa} - l, \ldots, k_r}
\]

Indeed, if we have proven it, then it implies that for all \( r \geq 1 \) and all \( 0 \leq t_1 \leq \ldots \leq t_r \) the function \( f^{(d)}_{t_1, \ldots, t_r} \) converges towards \( F_{t_1, \ldots, t_r} \) when \( d \) goes to infinity. In particular, if we take \( r = 1 \), we see that we have the convergence of the marginals (in moments).

In order to prove that formula we must remark that a free multiplicative Brownian motion is given by a free stochastic equation with initial conditions \( u_0 = 1 \) (1 is the unit element of \( \mathbb{A} \)):

\[
du_t = idX_t u_t - \frac{1}{2} u_t dt
\]

where \( X_t \) is a free additive Brownian motion. This result is stated in \[1\]'s Theorem 2. We will simplify the calculations by putting \( V_t := e^{t/2} u_t \). Using the free analogue of Itô's Lemma (see e.g. \[4\], Theorem 5), Biane demonstrated following formula

\[
dV^n_t = i \sum_{k=0}^{n} V^k_t dX_t V^{n-k}_t - \sum_{k=1}^{n-1} k V^k_t \phi \left( V^{n-k}_t \right) dt
\]

In other words this means:

\[
du^n_t = i \sum_{k=0}^{n} u^k_t dX_t u^{n-k}_t - \sum_{k=1}^{n-1} k u^k_t \phi \left( u^{n-k}_t \right) dt - \frac{n}{2} u^n_t dt
\]

\[\text{Without going into the details of the theory of integer partitions, we may find a gross upper bound for this number in the following way: A partition of } \ell \text{ cannot have more than } \ell \text{ parts. So let’s consider a line consisting of } \ell + \ell - 1 = 2\ell - 1 \text{ boxes. We then put crosses in } \ell - 1 \text{ boxes. Each such cross helps separate two parts of the partition. For instance:} \]

\[
\begin{array}{cccccccc}
\blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge
\end{array}
\]

represents the partition \((1, 1, 2)\) of the integer 4. Hence we see that the number of such partitions is bounded by \( \binom{2\ell - 1}{\ell - 1} \), which is finite.
Taking the trace of it we obtain:

\[ \phi(u^n_t) = -\sum_{k=1}^{n-1} k\phi(u^k_t)\phi(u^{n-k}_t) - \frac{n}{2}\phi(u^n_t)\,dt \]

And so it finally yields the following system of differential equations:

\[ F'_{k_1,...,k_r} = -\frac{k_1 + \ldots + k_r}{2} F_{k_1,...,k_r} - \sum_{p=1}^r \sum_{\kappa=1}^{k_p-1} p F_{k_1,...,p,k_p-p,...,k_r} \]

And this is exactly the system we wanted because \( F_{k_1,...,p,k_p-p,...} = F_{k_1,...,k_p-p,...} \).

To put it in a nutshell: we were able to reprove Biane’s result by using a different method (by comparing systems of differential equations) to prove the convergence of marginals. The freeness of the increments can still be proven as did Biane but it will also follow from the results of section 4. We will now try to use that alternative method to generalize Biane’s result. To do that we will need the concept of dual groups.

2 Dual groups in the sense of Voiculescu and Lévy processes

We will here briefly introduce dual groups as they were first defined by Voiculescu. For more information on this subject one can read [8]. In the sequel we denote by \( \sqcup \) the free product of unital \( \ast \)-algebras.

**Definition 3** (Dual semigroups). A (unital) dual semigroup is a triple \((B, \Delta, \delta)\) where \( B \) is a \( \ast \)-algebra and \( \Delta : B \to B \sqcup B \) and \( \delta : B \to \mathbb{C} \) are \( \ast \)-homomorphisms such that

\[
(\Delta \sqcup id_B) \circ \Delta = (Id_B \sqcup \Delta)
\]

\[
(\delta \sqcup id_B) \circ \Delta = id_B = (Id_B \sqcup \delta) \circ \Delta
\]

The former property is called coassociativity, whereas the latter is the counit property.

We shall be in this paper particularly interested in one dual group:

**Definition 4** (Unitary Dual Group). For \( n \geq 1 \), we call Unitary Dual Group the dual group \((U\langle n \rangle, \Delta, \delta)\) defined by:

- The \( \ast \)-algebra \( U\langle n \rangle \) is generated by \( n^2 \) generators \((u_{ij})_{1 \leq i,j \leq n}\) verifying the relations:

\[
\forall 1 \leq i,j \leq n \quad \sum_{k=1}^n u_{ki}^* u_{kj} = \delta_{ij} = \sum_{k=1}^n u_{ik} u_{jk}^*
\]
• The coproduct is given by:

\[ \Delta u_{ij} = \sum_k (u_{ik}^{(1)} u_{kj}^{(2)}) \]

where the exponent \((1)\) (resp. \((2)\)) indicates that the element is taken from the left (resp. right) leg of \(U \langle n \rangle \sqcup U \langle n \rangle\).

• The counit is given by: \(\delta u_{ij} = \delta_{ij}\), where we used Kronecker’s symbol.

Dual semigroups are particularly useful to define free Lévy processes in the most general case.

Definition 5 (Lévy processes). We shall assume that we have a dual semigroup \((B, \Delta, \delta)\) and some unital noncommutative probability space \((A, \phi)\).

A free (resp. tensor independent) Lévy process on the semigroup \(B\) over the noncommutative probability space \((A, \phi)\) is a family \((j_{s,t})_{0 \leq s \leq t}\) of *-homomorphisms from \(B\) to \(A\) such that:

• (Increment Property) For every \(0 \leq s \leq t \leq r\) we have:

\[ (j_{st} \sqcup j_{tr}) \circ \Delta = j_{sr} \]

• (Stationarity) We have for every \(0 \leq s \leq t\):

\[ j_{0,t-s} = j_{s,t} \]

• (Freeness of the Increments) For every \(0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n+1}\), the increments \(j_{t_1 t_2}, \ldots, j_{t_n t_{n+1}}\) are free (resp. tensor independent).

• (Weak continuity) For each \(b \in B\) and each \(s \geq 0\), we have: \(\lim_{t \to s^+} \phi \circ j_{s,t}(b) = \delta(b)\)

How can these concepts be applied in our case? We could generalize Biane’s question by taking \(U^{(d)}_t\) a Brownian motion on the Unitary Group \(U(nd)\), where \(n\) is a fixed integer. The matrix \(U^{(d)}_t\) can be decomposed in \(n^2\) blocks of size \(d \times d\). In the sequel of the article we will denote by \([U^{(d)}_t]_{ij}\) the \((i,j)\)th block of our Brownian motion. For each \(d\) we thus get a quantum stochastic process on the Dual Unitary Group by setting for \(0 \leq s \leq t\):

\[ j^{(d)}_{st} : U \langle n \rangle \to (A, \phi) \]

\[ u_{ij} \mapsto [U^{(d)}_t]_{ij} \]

We will in the sequel of the article omit the exponent \((d)\) whenever no confusion can arise.

The question that is natural to ask and that generalizes Biane’s result is whether or not \(j_{st}\) converges to a Lévy process on \(U \langle n \rangle\) in the limit when \(d\) goes to infinity.

We will show that we have following result
Theorem 1 (Main Theorem). Let \(X = (X_{ij})_{1 \leq i,j \leq n}\) a matrix whose coefficients are free stochastic variables verifying that:

- For each \(i\), \(X_{ii}\) is an additive free Brownian motion.
- For every \(i \neq j\), \(X_{ij} = X_{ij}^{(1)} + X_{ij}^{(2)}\) with \(\sqrt{2}X_{ij}^{(1)}\) and \(\sqrt{2}X_{ij}^{(2)}\) who are two additive free Brownian motions who are free one with another.
- For each \(i, j\) we have \(X_{ij} = X_{ij}^*\).
- The family \((X_{ij})_{1 \leq i,j \leq n}\) is free.

Let also \(\Psi = (\Psi_{ij})\) be a free stochastic process defined by the free stochastic equation with initial condition \(\Psi_0 = I\):

\[
d\Psi_t = \frac{i}{\sqrt{n}}dX_t\Psi_t - \frac{1}{2}\Psi_t dt
\]

Through \(\Psi\) we may define a free Lévy process \(J\) through:

\[
J_{st} : U(n) \rightarrow (A, \phi) \\
u_{ij} \rightarrow \Psi_{ij}
\]

Moreover, we assume that \(\phi\) is tracial.

Then, \((j_{st}^{(d)})\) converges towards \((J_{st})\) as \(d\) goes to infinity.

3 Convergence of the marginals

We will first study the convergence of the marginals. Hence we will fix in this section a \(t \geq 0\). To prove such a convergence we must study the moments of the type \(\phi \circ j_{\mathcal{U}}(u_{i_1j_1}^{e_1} \ldots u_{i_rj_r}^{e_r})\), where \(e_1, \ldots, e_r \in \{\emptyset, \ast\}\). For convenience, we will identify \(\emptyset\) with 0 and \(\ast\) with 1. We will use exactly the same method as in the first section but, because there are \(n^2\) variables, we will have many more indices.

3.1 Notations

We consider the dual group \(U(n)\) which is generated by \(n^2\) variables. We will need to introduce some notations to describe all the indices that will be involved.

From now on and until the end of the paper, when we have a matrix \(M \in \mathcal{M}_{nd}(\mathbb{C})\), we will denote:

- by \(M_{ij}\) the \((i, j)\)-matrix entry of \(M\).
- by \([M]_{ij}\) the \((i, j)\)-block of size \(d \times d\) of the matrix \(M\)
We denote by $[\mathcal{I}]$ the set $[\mathcal{I}] = \{1, \ldots, n\}^2 \times \{0, 1\}$. For such a triple $\alpha = (i, j, \epsilon)$, we will note $[U]_{\alpha}$ the $d \times d$ block $[U]_{ij}^\epsilon$ where we identify $\epsilon = 1$ with $\ast$ and $\epsilon = 0$ with $\emptyset$.

We denote by $\mathcal{I}$ the set $\mathcal{I} = \{1, \ldots, nd\}^2 \times \{0, 1\}$. For such a triple $\alpha = (i, j, \epsilon)$, we will note $U_{\alpha}$ the coefficient $U_{ij}$ if $\epsilon = 0$ and the coefficient $\bar{U}_{ij}$ if $\epsilon = 1$.

When $\Psi$ is in $\mathcal{M}_n(\mathcal{A})$, with $\mathcal{A}$ a $\ast$-algebra, we denote by $\Psi_\alpha$ the element $\Psi_{ij}$.

### 3.2 A system of differential equations for the Brownian motion on $U(nd)$

To achieve our purpose we need to consider the family of functions (as always, we will omit the exponents everytime we may do so without risk):

$$\gamma_{\alpha_1 \ldots \alpha_r}^{(d)} = E[tr([U]_{\alpha_1} \ldots [U]_{\alpha_r} \ldots U_{i_{p\kappa - 1} j_{p\kappa - 1}} U_{i_{q\kappa} j_{q\kappa} \ldots U_{i_{p\kappa + 1} j_{p\kappa + 1}} \ldots U_{i_{q\kappa} j_{p\kappa}} \ldots]}$$

where $r \geq 1; k_1, \ldots, k_r \in \mathbb{N}, \alpha_{kl} \in [\mathcal{I}]$.

In other words, the indices we use specify which $U_{ij}$ appear and if they have a $\ast$ or not and the semicolons separate two traces. We will, as previously, try to find a system of differential equations. Let us fix the indices $t_1 \ldots \epsilon_{k_r r}$.

Again, we apply Lemma 1 in order to calculate the differential equation. For the sake of simplicity let us first observe what happens if we suppose that there are no $\ast$ in our function and we will later explain how to get the general case. As previously we treat separately the case where the switch occurs inside a same trace and the case where it affects two distinct traces.

**The switch occurs in the same trace:** Let’s say that the switch is between $p$ and $q$ inside the $\kappa^{\text{th}}$ trace. Then, when we develop the traces, we see that the contribution of this trace, after the switch, is of the type:

$$E[\sum_{s_{11} \ldots s_{k_r r}} U_{(i_{p\kappa - 1} + s_{p\kappa} - j_{q\kappa - 1} + s_{q\kappa}) \ldots U_{(i_{q\kappa} - 1) + s_{q\kappa} + j_{p\kappa - 1} + s_{p\kappa} \ldots}$$

As we could have expected the $\kappa^{\text{th}}$ trace will be divided into two distinct traces: we get $d_{i_{1\kappa} j_{1\kappa} \ldots i_{p\kappa} j_{p\kappa} \ldots i_{q\kappa} j_{q\kappa} \ldots} \ldots \ldots (\text{the normalization constant we use now for the trace is } 1/d)$.  

**The switch concerns two distinct traces:** If we do the calculations, we see that we reunite these two traces and that we get a multiplicative factor $1/d$.

So, if we put it all together (in the case we have no $\ast$ at all), the equation
we will have is:

\[
\gamma'_{\alpha_1 \ldots, \alpha_{k_1} \ldots \alpha_{k_r}} = -\frac{k_1 + \ldots + k_r}{2} \gamma_{\alpha_1 \ldots, \alpha_{k_1} \ldots \alpha_{k_r}} \\
- \frac{1}{n} \sum_{\kappa=1}^{r} \sum_{1 \leq p < q \leq k_\kappa} \gamma_{\alpha_1 \ldots, (i_p, j_q, \kappa) \ldots, \alpha_{k_\kappa+1} \ldots, (i_q, j_p, \kappa) \ldots} \\
+ O\left(\frac{1}{d^2}\right)
\]

Now, in the general case. We can remark that \([U^*]_{ij} = [U]^*_{ji}\). We also have:

\[
d[U_{ij}] = i \sum_{p=1}^{d} dH_{ip}U_{pj} - \frac{1}{2} U_{ij} dt \\
d[\bar{U}_{ij}] = -i \sum_{p=1}^{d} \bar{U}_{pj}dH_{pi} - \frac{1}{2} \bar{U}_{ij} dt
\]

In turn this yields to the more general Lemma:

**Lemma 2.** We have, for \(\alpha_1, \ldots, \alpha_r \in I\):

\[
d(U_{\alpha_1} \ldots U_{\alpha_r}) = -\frac{r}{2} U_{\alpha_1} \ldots U_{\alpha_r} \\
+ \text{ martingale part} - \frac{1}{nd} \sum_{1 \leq p < q \leq r} (-1)^{\epsilon_p + \epsilon_q} \zeta_{pq}^{(d)}
\]

where:

\[
\zeta_{pq}^{(d)} = \begin{cases} 
U_{\alpha_1} \ldots U_{i_p, j_q} \ldots U_{i_q, j_p} \ldots U_{\alpha_r} & \text{if } \epsilon_p = \epsilon_q = 0 \\
U_{\alpha_1} \ldots U_{i_p, j_q} \ldots \bar{U}_{i_q, j_p} \ldots U_{\alpha_r} & \text{if } \epsilon_p = \epsilon_q = 1 \\
\sum_{t=1}^{nd} \delta_{i_p, j_q} U_{\alpha_1} \ldots \bar{U}_{t_j, j_p} \ldots U_{\alpha_r} & \text{if } \epsilon_p = 1, \epsilon_q = 0 \\
\sum_{t=1}^{nd} \delta_{i_p, j_q} U_{\alpha_1} \ldots U_{t_j, j_p} \ldots U_{\alpha_r} & \text{if } \epsilon_p = 0, \epsilon_q = 1
\end{cases}
\]  

(1)

**Proof.** It is an application of Itô’s Lemma along with the observation that:

\[
d[U_{ij}, U_{kl}] = -\frac{dt}{nd} U_{kj}U_{il} \text{ and } d[\bar{U}_{ij}, U_{kl}] = \sum_{p=1}^{nd} \frac{1}{nd} B_{pj}B_{pl}\delta_{ik}
\]

So, taking up the same calculations as before, we get following system
of differential equations:

\[ \gamma_{\alpha_{11},...} = \frac{k_{1} + \ldots + k_{r}}{2} \gamma_{\alpha_{11},...} \]

\[ - \frac{1}{n} \sum_{\kappa=1}^{r} \sum_{1 \leq p < q \leq k_{\kappa}} (-1)^{\epsilon_{p\kappa} + \epsilon_{q\kappa}} \gamma_{(p,q,\kappa)} \]

\[ + \mathcal{O}\left( \frac{1}{d^{2}} \right) \]

where we note:

If \( \epsilon_{p\kappa} = \epsilon_{q\kappa} = 0 \):

\[ \gamma_{(p,q,\kappa)} = \gamma_{\alpha_{11},...,\alpha_{p-1,\kappa},\alpha_{pq,\kappa},\alpha_{q+1,\kappa},\alpha_{q+1,\kappa},...} \]

If \( \epsilon_{p\kappa} = \epsilon_{q\kappa} = 1 \):

\[ \gamma_{(p,q,\kappa)} = \gamma_{\alpha_{11},...,\alpha_{p-1,\kappa},\alpha_{pq,\kappa},\alpha_{q+1,\kappa},\alpha_{q+1,\kappa},...} \]

If \( \epsilon_{p\kappa} = 1, \epsilon_{q\kappa} = 0 \):

\[ \gamma_{(p,q,\kappa)} = \sum_{t=1}^{n} \delta_{ip\kappa} \gamma_{\alpha_{11},...,\alpha_{p-1,\kappa},\alpha_{pq,\kappa},\alpha_{q+1,\kappa},\alpha_{q+1,\kappa},...} \]

If \( \epsilon_{p\kappa} = 0, \epsilon_{q\kappa} = 1 \):

\[ \gamma_{(p,q,\kappa)} = \sum_{t=1}^{n} \delta_{ip\kappa} \gamma_{\alpha_{11},...,\alpha_{p-1,\kappa},\alpha_{pq,\kappa},\alpha_{q+1,\kappa},\alpha_{q+1,\kappa},...} \]

3.3 A system of differential equations for the free stochastic process

We will now introduce:

\[ \Gamma_{\alpha_{11},...,\alpha_{r+1},...} = \phi(\Psi_{\alpha_{11}} \ldots) \ldots \phi(\Psi_{\alpha_{r+1}} \ldots \Psi_{\alpha_{r+1}}) \]

To prove the convergence of the marginals, we will show that \( \Gamma \) verifies the system of differential equations that we have just found, in the limit where \( d \) goes to infinity.

By using free stochastic calculus we can see that the quadratic variation is \( dX_{ij}dX_{kl} = \delta_{il}\delta_{jk}dt \). Moreover, the free stochastic differential equation yields, coefficient by coefficient:

\[ d\Psi_{uv} = \frac{i}{\sqrt{n}} \sum_{k=1}^{n} dX_{uk}\Psi_{kv} - \frac{1}{2} \Psi_{uv} dt \]

and

\[ d\Psi_{uv}^{*} = -\frac{i}{\sqrt{n}} \sum_{k=1}^{n} \Psi_{kv}^{*} dX_{ku} - \frac{1}{2} \Psi_{uv}^{*} dt \]

This allows us to prove following technical Lemma:
Lemma 3. For each $r \geq 2$ and all indices we have:

$$d(\Psi_{\alpha_1} \ldots \Psi_{\alpha_r}) = -\frac{r}{2} \Psi_{\alpha_1} \ldots \Psi_{\alpha_r}$$

$$+ \frac{i}{\sqrt{n}} \sum_{l=1}^{r} \sum_{k=1}^{n} (-1)^{\epsilon_l} \Psi_{\alpha_1} \ldots \left\{ \begin{array}{ll} dX_{i_l k} \Psi_{k j_l} & \text{if } \epsilon_l = 0 \\
\Psi_{k j_l} dX_{k i_l} & \text{if } \epsilon_l = 1 \end{array} \right\} \ldots \Psi_{\alpha_r}$$

$$- \frac{dt}{n} \sum_{1 \leq p < q \leq r} (-1)^{\epsilon_p + \epsilon_q} \zeta_{pq}$$

where

$$\zeta_{pq} = \left\{ \begin{array}{ll} \Psi_{\alpha_1} \ldots \Psi_{\alpha_{p-1}} \phi(\Psi_{i_p j_p}^{\epsilon_p} \ldots \Psi_{\alpha_{q-1}}^{\epsilon_q}) \Psi_{i_p j_p}^{\epsilon_p} \ldots & \text{if } \epsilon_p = \epsilon_q = 0 \\
\Psi_{\alpha_1} \ldots \Psi_{\alpha_{p-1}} \phi(\Psi_{\alpha_p}^{\epsilon_p} \ldots \Psi_{\alpha_{q-1}}^{\epsilon_q}) \Psi_{\alpha_p}^{\epsilon_p} \ldots & \text{if } \epsilon_p = \epsilon_q = 1 \\
\sum_{k=1}^{n} \delta_{p q} \Psi_{\alpha_1} \ldots \Psi_{\alpha_{p-1}} \phi(\Psi_{k q}^{\epsilon_p} \ldots \Psi_{\alpha_{q-1}}^{\epsilon_q}) \Psi_{k q}^{\epsilon_p} \ldots & \text{if } \epsilon_p = 0, \epsilon_q = 1 \\
\sum_{k=1}^{n} \delta_{p q} \Psi_{\alpha_1} \ldots \Psi_{\alpha_{p-1}} \phi(\Psi_{\alpha_p}^{\epsilon_p} \ldots \Psi_{\alpha_{q-1}}^{\epsilon_q}) \Psi_{\alpha_p}^{\epsilon_p} \ldots & \text{if } \epsilon_p = 1, \epsilon_q = 0 \end{array} \right. \right.$$ 

Proof. The proof is done by recurrence and by using Itô’s formula. For simplicity’s sake we will do it only in the case where all $\epsilon$ are put equal to zero.

For $r = 2$ we get:

$$d(\Psi_{ij} \Psi_{kl}) = \frac{i}{\sqrt{n}} \sum_{s=1}^{n} \Psi_{ij} dX_{k s} \Psi_{sl} + \frac{i}{\sqrt{n}} \sum_{s=1}^{n} dX_{i s} \Psi_{s j} \Psi_{kl} - \frac{dt}{n} \phi(\Psi_{k j}) \Psi_{i l}$$

Hence we have the desired result for $r = 2$. Let us now assume that the Lemma is right until a certain $r$. Then, by Itô’s Lemma:

$$d(\Psi_{u_1 v_1} \ldots \Psi_{u_{r+1} v_{r+1}}) = -\frac{r + 1}{2} \Psi_{u_1 v_1} \ldots \Psi_{u_{r+1} v_{r+1}} dt$$

$$+ \frac{i}{\sqrt{n}} \sum_{k=1}^{n} \Psi_{u_1 v_1} \ldots \Psi_{u_r v_r} dX_{u_{r+1} k} \Psi_{k v_{r+1}}$$

$$+ \frac{i}{\sqrt{n}} \sum_{k=1}^{n} \sum_{l=1}^{r} \Psi_{u_1 v_1} \ldots dX_{u_k l} \Psi_{k v_1} \ldots \Psi_{u_{r+1} v_{r+1}}$$

$$- \frac{dt}{n} \sum_{1 \leq p < q \leq r} \Psi_{u_1 v_1} \ldots \phi(\Psi_{u_q v_p} \ldots) \Psi_{u_p v_q} \ldots \Psi_{u_{r+1} v_{r+1}}$$

$$- \frac{dt}{n} \sum_{l=1}^{r} \Psi_{u_1 v_1} \ldots \Psi_{u_{r+1} v_{r+1}} \phi(\Psi_{u_{r+1} v_l} \ldots) \Psi_{u_l v_{r+1}}$$

And so we see that the result is also right for $r + 1$.

We now introduce, as expected, the family of functions:

$$\Gamma_{\alpha_1 \ldots \alpha_r} = \phi(\Psi_{\alpha_1} \ldots) \ldots \phi(\Psi_{\alpha_r} \ldots)$$
By applying Lemma 3, we get:

\[
\Gamma'_{a_1, \ldots, a_{1r}} = -\frac{k_1 + \ldots + k_r}{2} \Gamma_{a_1, \ldots, a_{1r}} - \frac{1}{n} \sum_{\kappa=1}^{r} \sum_{1 \leq p < q \leq k_{\kappa}} (-1)^{\epsilon_p + \epsilon_q} \Gamma_{(p, q, \kappa)}
\]

where we defined:

\[
\Gamma_{(p, q, \kappa)} = \\
\Gamma_{\ldots; \alpha_{1\kappa}, \ldots, \epsilon_{pq}, \epsilon_{eq}, \ldots; \alpha_{k_{\kappa}}; \ldots; \epsilon_{pq}, \epsilon_{eq}, \ldots; \alpha_{q-1\kappa}; \ldots} \quad \text{if } \epsilon_{pq} = \epsilon_{eq} = 0 \\
\Gamma_{\ldots; \alpha_{1\kappa}, \ldots, \epsilon_{pq}, \epsilon_{eq}, \ldots; \alpha_{k_{\kappa}}; \ldots; \epsilon_{pq}, \epsilon_{eq}, \ldots; \alpha_{q+1\kappa}; \ldots} \quad \text{if } \epsilon_{pq} = \epsilon_{eq} = 1 \\
\sum_{p=1}^{n} \delta_{\epsilon_{pq} \epsilon_{eq}} \Gamma_{\ldots; \alpha_{1\kappa}, \ldots, \alpha_{p-1\kappa}, \alpha_{p+1\kappa}, \ldots; \epsilon_{pq}, \epsilon_{eq}, \ldots; \alpha_{q-1\kappa}; \ldots} \quad \text{if } \epsilon_{pq} = 0, \epsilon_{eq} = 1 \\
\sum_{p=1}^{n} \delta_{\epsilon_{pq} \epsilon_{eq}} \Gamma_{\ldots; \alpha_{1\kappa}, \ldots, \epsilon_{pq}, \epsilon_{eq}, \ldots; \alpha_{p+1\kappa}, \ldots; \alpha_{q-1\kappa}; \ldots} \quad \text{if } \epsilon_{pq} = 1, \epsilon_{eq} = 0
\]

Hence we see that the family of functions \( \gamma \) truly converges towards the family of functions \( \Gamma \). In particular, taking \( r = 1 \), we see that the \( * \)-moments of the family \( (U_{ij}^{(d)})_{1 \leq i, j \leq n} \) converges towards the \( * \)-moments of \( (\Psi_{ij})_{1 \leq i, j \leq n} \). This proves the convergence of the marginals.

### 4 Conditional expectation

In order to prove Theorem 1, we must prove the convergence of all mixed moments of the kind: \( \text{E} \circ \text{tr} (U_{i_1 j_1}^{(t_1)} \ldots U_{i_r j_r}^{(t_r)}) \) towards \( \phi(\Psi_{i_1 j_1}^{(t_1)} \ldots \Psi_{i_r j_r}^{(t_r)}) \).

In the previous section, we have already proven that this is indeed the case when \( \{t_1, \ldots, t_r\} = 1 \). In order to prove the general case, we will use a method consisting of computing the joint moments by taking recursively conditional expectations.

#### 4.1 Notations

In order to use this method, we must generalize somewhat our notations. In the sequel, we set a \( s \geq 0 \) and our time variable \( t \) will always verify \( t \geq s \).

1. by \( I \) the set \( \{1, \ldots, n\}^2 \times \{0, 1\} \times \mathcal{M}^{(s)}_d \), where \( \mathcal{M}^{(s)}_d \) is the set of \( d \times d \) matrices whose entries are \( \mathcal{F}_s \)-measurable random variables. Of course, we have \( \mathcal{F}_s = \sigma(j_s, s \leq t) \).
2. by \( I \) the set \( \{1, \ldots, nd\}^2 \times \{0, 1\} \times V^{(s)} \), where \( V^{(s)} \) designates the set of \( \mathcal{F}_s \)-measurable random variables.
3. by \( I \) the set \( \{1, \ldots, n\}^2 \times \{0, 1\} \times \mathcal{A}_s \), where \( \mathcal{A}_s \) is the \( * \)-algebra generated by all \( \Psi_{pq}(u) \).
We use these sets as sets of indices in the following way:

1. If $\alpha = (i, j, \epsilon, m) \in I$, we note $[U]_\alpha = m[U]_{ij}^\epsilon$
2. If $\alpha = (i, j, \epsilon, m) \in I$, we note $U_\alpha = mU_{ij}^\epsilon$
3. If $\alpha = (i, j, \epsilon, m) \in I_f$, we note $\Psi_\alpha = m\Psi_{ij}^\epsilon$.

4.2 A system of differential equations for the Brownian motion on $U(nd)$

We are interested in the family of functions:

$$\gamma_{11}^{\alpha_1}, \ldots, \gamma_{k1}^{\alpha_k}; \ldots; \gamma_{1r}^{\alpha_1}, \ldots, \gamma_{kr}^{\alpha_k}(t) = E[tr[U]_{11}(t) \ldots [U]_{k1}(t) \ldots tr[U]_{kr}(t)]$$

In other words, we use the same family as before but we put $F_s$-measurable elements between the blocks of the Brownian motion.

We want to use the same method as before. We will need following Lemma:

**Lemma 4.** We have for any choice of indices in $I$ and for $t \geq s$:

$$d(U_1 \ldots U_k) = -\frac{k}{2}U_1 \ldots U_k dt$$

$$- \frac{dt}{nd} \sum_{1 \leq p < q \leq k} (-1)^{\epsilon_p + \epsilon_q} \epsilon_{pq}^{(d)} \gamma_{pq}^{(d)} dt$$

$$+ \text{martingale part}$$

where:

$$\epsilon_{pq}^{(d)} = \begin{cases} U_1 \ldots m_p U_{i_p j_q} \ldots m_q U_{k_p j_q} \ldots U_k & \text{if } \epsilon_p = \epsilon_q = 0 \\ U_1 \ldots m_p U_{i_p j_q} \ldots m_q U_{k_p j_q} \ldots U_k & \text{if } \epsilon_p = \epsilon_q = 1 \\ \sum_{t=1}^{nd} \delta_{i_p j_q} U_1 \ldots m_p U_{i_p j_q} \ldots m_q U_{i_q j_q} \ldots U_k & \text{if } \epsilon_p = 1, \epsilon_q = 0 \\ \sum_{t=1}^{nd} \delta_{i_p j_q} U_1 \ldots m_p U_{i_p j_q} \ldots m_q U_{i_q j_q} \ldots U_k & \text{if } \epsilon_p = 0, \epsilon_q = 1 \end{cases}$$

(2)

**Proof.** As always, this is proven using Itô’s Lemma.

Applying this Lemma, we get:

**Lemma 5.** The system of differential equations is:

$$\gamma_{\epsilon_1 \ldots \epsilon_k} = -\frac{k_1 + \ldots + k_r}{2} \gamma_{\epsilon_1 \ldots \epsilon_k}$$

$$- \frac{1}{n} \sum_{k=1}^r \sum_{1 \leq p < q \leq k} (-1)^{\epsilon_p + \epsilon_q} \gamma_{pq}^{(d)}$$

$$+ O\left(\frac{1}{d^2}\right)$$
where:

If $\epsilon_{pq} = \epsilon_{qn} = 0$:

$$\gamma(p,q,\kappa) = \gamma(..., (m_{pq}, r_{pq}, \epsilon_{pq}), \alpha_{q+1, n} ... \alpha_{p+1, n} ... (m_{pq}, r_{pq}, \epsilon_{pq}), ...$$

If $\epsilon_{pq} = \epsilon_{qn} = 1$:

$$\gamma(p,q,\kappa) = \gamma(..., (m_{pq}, r_{pq}, \epsilon_{pq}), \alpha_{q+1, n} ... \alpha_{p+1, n} ... (1, \epsilon_{pq}, \epsilon_{pq}), ...)$$

If $\epsilon_{pq} = 1, \epsilon_{qn} = 0$:

$$\gamma(p,q,\kappa) = \sum_{t=1}^{n} \delta_{ip_{pq}t_{pq}} \gamma(..., (m_{pq}, r_{pq}, \epsilon_{pq}), (1, \epsilon_{pq}, \epsilon_{pq}, 1), ... (1, \epsilon_{pq}, \epsilon_{pq}, m_{pq}, m_{pq} + 1, n), ...)$$

If $\epsilon_{pq} = 0, \epsilon_{qn} = 1$:

$$\gamma(p,q,\kappa) = \sum_{t=1}^{n} \delta_{ip_{pq}t_{pq}} \gamma(..., (1, \epsilon_{pq}, t_{pq}, \epsilon_{pq}), (1, \epsilon_{pq}, \epsilon_{pq}, 1), ... (1, \epsilon_{pq}, \epsilon_{pq}, m_{pq} + 1, n), ...)$$

When we proved Biane’s result we saw that the system of differential equations had a combinatorial structure related to the idea of integer partitions. I do not see any obvious combinatorial structure in this generalized formula but it is a question that is worth being asked.

### 4.3 A system of differential equations for the free stochastic process

Of course, we will be interested in the behavior of the family of functions:

$$\Gamma_{\alpha_1, ..., \alpha_k, ...} = \phi(\Psi_{\alpha_1}(t) ... \phi(\ldots)$$

**Lemma 6.** For any choice of indices in $\mathcal{I}^f$ and for $t \geq s$, we have:

$$d(\Psi_{\alpha_1} \ldots \Psi_{\alpha_k}) = \frac{k}{2} \Psi_{\alpha_1} \ldots \Psi_{\alpha_k} dt$$

$$+ \frac{i}{\sqrt{n}} \sum_{r=1}^{n} \sum_{l=1}^{k} \Psi_{\alpha_1} \ldots \Psi_{\alpha_l} \left\{ \begin{array}{l}
\frac{dX_{ir} \Psi_{rjl}}{\Psi_{rjl} dX_{ri}} 
\text{if } \epsilon_{l} = 0 \\
\frac{dX_{ir} \Psi_{rjl}}{\Psi_{rjl} dX_{ri}} 
\text{if } \epsilon_{l} = 1
\end{array} \right\} \ldots \Psi_{\alpha_k}$$

$$- \frac{dt}{n} \sum_{1 \leq p < q \leq k} (-1)^{\epsilon_{p} + \epsilon_{q}} \zeta_{pq}$$

where

$$\zeta_{pq} = \begin{cases}
\Psi_{\alpha_1} \ldots \phi(\Psi_{\epsilon_{p}}^{\epsilon_{p}} \ldots \Psi_{\alpha_{q-1}} m_{q}) \Psi_{\epsilon_{p} q} \ldots & \text{if } \epsilon_{p} = \epsilon_{q} = 1 \\
\Psi_{\alpha_1} \ldots \alpha_{p} \Psi_{\epsilon_{p}}^{\epsilon_{p}} \phi(\Psi_{\alpha_{p+1}} \ldots \Psi_{\epsilon_{q}}^{\epsilon_{q}}) \Psi_{\alpha_{q+1}} \ldots & \text{if } \epsilon_{p} = \epsilon_{q} = 1 \\
\sum_{l=1}^{k} \delta_{i p_{t_{pq}}} \Psi_{\alpha_1} \ldots \alpha_{p} \phi(\Psi_{\epsilon_{p}}^{\epsilon_{p}} \ldots \Psi_{\epsilon_{q}}^{\epsilon_{q}}) \Psi_{\alpha_{q+1}} \ldots & \text{if } \epsilon_{p} = 0, \epsilon_{q} = 1 \\
\sum_{l=1}^{k} \delta_{i p_{t_{pq}}} \Psi_{\alpha_1} \ldots \Psi_{\epsilon_{p}}^{\epsilon_{p}} \phi(\Psi_{\alpha_{p+1}} \ldots \alpha_{q}) \Psi_{\epsilon_{q}}^{\epsilon_{q}} & \text{if } \epsilon_{p} = 1, \epsilon_{q} = 0
\end{cases}$$
Proof. It is the same proof as before, based on Itô’s formula.

Applying this Lemma, we get:

**Lemma 7.** The system of differential equations for the free stochastic process is:

\[
\Gamma'_{\alpha_1, \ldots, \alpha_r} = -\frac{k_1 + \ldots + k_r}{2} \Gamma_{\alpha_1, \ldots, \alpha_r} \bigg( 1 - \frac{1}{n} \sum_{\kappa=1}^{r} \sum_{1 \leq p < q \leq k_r} (-1)^{\epsilon_{pq} \epsilon_{k_r}} \Gamma_{(p, q, \kappa)} \bigg)
\]

where:

If \( \epsilon_{pq} = \epsilon_{k_r} = 0 \):

\[
\Gamma_{(p, q, \kappa)} = \Gamma_{\cdots, (t_{pq}, t_{pq}, t_{pq}, m_{pq}), \cdots, (t_{pq}, t_{pq}, t_{pq}, m_{pq}), \cdots, (t_{pq}, t_{pq}, t_{pq}, 1), \cdots}
\]

If \( (\epsilon_{pq}, \epsilon_{k_r}) = (1, 1) \):

\[
\Gamma_{(p, q, \kappa)} = \Gamma_{\cdots, (t_{pq}, t_{pq}, t_{pq}, m_{pq}, t_{pq}, \cdots, (t_{pq}, t_{pq}, t_{pq}, m_{pq}, m_{pq}, t_{pq}, \cdots, (t_{pq}, t_{pq}, t_{pq}, 1), \cdots}
\]

If \( \epsilon_{pq} = 0, \epsilon_{k_r} = 1 \):

\[
\Gamma_{(p, q, \kappa)} = \sum_{l=1}^{n} \delta_{t_{pq}, t_{pq}} \Gamma_{\cdots, (t_{pq}, t_{pq}, t_{pq}, t_{pq}, m_{pq}, t_{pq}, m_{pq}, t_{pq}, \cdots, (t_{pq}, t_{pq}, t_{pq}, 1), \cdots}
\]

If \( \epsilon_{pq} = 1, \epsilon_{k_r} = 0 \):

\[
\Gamma_{(p, q, \kappa)} = \sum_{l=1}^{n} \delta_{t_{pq}, t_{pq}} \Gamma_{\cdots, (t_{pq}, t_{pq}, t_{pq}, t_{pq}, t_{pq}, t_{pq}, t_{pq}, \cdots, (t_{pq}, t_{pq}, t_{pq}, 1), \cdots}
\]

4.4 Recurrence

We are now able to finish the proof of Theorem 1. We want to show that the moments \( \mathbb{E} \exp(U_{i_1 j_1}(t_1) \cdots U_{i_k j_k}(t_k)) \) converge towards \( \phi(\Psi_{i_1 j_1}(t_1) \cdots \Psi_{i_k j_k}(t_k)) \).

Let us not \( \sigma = \sharp \{t_1, \ldots, t_k\} \) the number of different times showing up in our moment. We are going to prove that result through recurrence on \( \sigma \).

1. If \( \sigma = 1 \) the result has already been shown because it is just the convergence of the marginals.

2. Let us suppose that the result is true until a certain \( \sigma \). We will now consider a moment using \( \sigma + 1 \) different times. We can order those times in increasing order: \( t_1 \leq t_1 \leq \ldots \leq t_{\sigma+1} \). The recurrence hypothesis tells us that:

\[
(U_{p, q}(t_i))_{1 \leq i \leq \sigma+1 \atop 1 \leq p, q \leq n} \xrightarrow{\text{in } \ast\text{-moments}} (\Psi_{p, q}(t_i))_{1 \leq i \leq \sigma+1 \atop 1 \leq p, q \leq n}
\]
We can write the moment under consideration as:

\[ \gamma_{(i_1j_1)m^{(d)}_1},..., (i_kj_km^{(d)}_k)}(t_{\sigma+1}) \]

where the \( m^{(d)}_i \) are \( \mathcal{F}_{t_\sigma} \)-measurable. Now, let us remark that the family of functions \( (\gamma_{\alpha_1,...,\alpha_{k_1}^{\alpha_{k_2}}}) \) is entirely characterized by the system of differential equations from Lemma 5 along with all the relationships between the \( \{m_{ij}^{(d)}, 1 \leq j \leq r, 1 \leq i \leq k \} \). In the same way, the family \( \Gamma_{\ldots} \) is entirely defined by the system from Lemma 7 along with the relationships between the \( \{m_{ij}, 1 \leq j \leq r, 1 \leq i \leq k \} \).

Now, the recurrence hypothesis allows us to say that the \( m_{ii}^{(d)}, 1 \leq i \leq k \) converges towards some \( m_i, 1 \leq i \leq k \). This tells us that the relationships between the \( \{m_{ij}^{(d)}\} \) "converges" towards the relationships between the \( \{m_i\} \). Moreover, the system of differential equations from Lemma 5 converges towards that of Lemma 7. To put it in a nutshell, this means:

\[ \gamma_{\alpha_1^{\ldots},\alpha_{k_1}^{\alpha_{k_2}}}^{(d)}(t_{\sigma+1}) \xrightarrow{d \to \infty} \Gamma_{\alpha_1^{\ldots},\alpha_{k_1}}^{(d)}(t_{\sigma+1}) \]

Or, in other words, we have the convergence of our moment.

Thus, we have proven that all \( * \)-moments converge and this means that Theorem 1 is proven.

5 Some examples of calculations and gaussianity

We will now use the differential equations that we obtained to calculate some simple moments of our process. We will then be able to draw some consequence about the gaussianity of the free process. In the sequel, we denote by \( \phi_t \) the function defined on \( U(n) \) by \( \phi_t = \phi \circ J_{it} \) where \( j_t \) is the limit (free) process.

5.1 The first moments

Let us take now \( 1 \leq i \neq j \leq n \). We have the following differential equations:

\[ \frac{d}{dt} \phi_t(u_{ii}) = -\frac{1}{2} \phi_t(u_{ii}) \]
\[ \frac{d}{dt} \phi_t(u_{ij}) = -\frac{1}{2} \phi_t(u_{ij}) \]

with initial conditions: \( \phi_0(u_{ii}) = 1 \) and \( \phi_0(u_{ij}) = 0 \). It thus yields:

\[ \phi_t(u_{ii}) = e^{-\frac{t}{2}} \]
\[ \phi_t(u_{ij}) = 0 \]

We find the same expression for \( \phi_t(u_{ii}^*) \) and \( \phi_t(u_{ij}^*) \) because they obey the same differential equation with the same initial conditions.
5.2 The second moments

Let us take $1 \leq i, j, k, l \leq n$. We have the following equation:

$$\frac{d}{dt} \phi_t(u_{ij}u_{kl}) = -\phi_t(u_{ij}u_{kl}) - \frac{1}{n} \phi_t(u_{il})\phi_t(u_{kj})$$

with initial conditions $\phi_0(u_{ij}u_{kl}) = \delta_{ij}\delta_{kl}$ because $\Psi_0 = I$. This equation is a linear differential equation of order 1 and the well-known method allows us to say:

$$\phi_t(u_{ij}u_{kl}) = \delta_{ij}\delta_{kl}e^{-t}$$

The moments $\phi_t(u_{ij}^{*}u_{kl}^{*})$ also obey the same equation with the same initial condition and therefore have the same expression. If we are interested in $\phi_t(u_{ij}^{*}u_{kl}^{*})$ we get the equation:

$$\frac{d}{dt} \phi_t(u_{ij}^{*}u_{kl}^{*}) = -\phi_t(u_{ij}^{*}u_{kl}^{*}) + \frac{1}{n} \sum_{p=1}^{n} \phi_t(u_{pj}u_{pl})$$

with initial conditions $\phi_0(u_{ij}^{*}u_{kl}^{*}) = \delta_{ij}\delta_{kl}$. This can be put in the form of a system of linear differential equations by putting $\Phi_t = (\phi_t(u_{ij}u_{kl}))_{1 \leq i,j,k,l \leq n}$ seen as a vector of $\mathbb{C}^{n^4}$ and $A = (a(r_1,r_2,r_3,r_4),s_1,s_2,s_3,s_4))$ as a matrix acting on $\mathbb{C}^{n^4}$, with:

$$a_{rs} = \begin{cases} \frac{1-n}{n} & \text{if } s_1 = s_3 \text{ and } r = s \\ \frac{1}{n} & \text{if } s_1 = s_3 \text{ and } r \neq s \\ -1 & \text{if } r = s \text{ and } r_1 \neq r_3 \\ \end{cases}$$

The equation then is:

$$\Phi' = A\Phi$$

The solution of such an equation is of the form $\Phi_t = Ce^{At}$ with $C$ a constant.

5.3 Gaussianity

We would like to define a Brownian motion on $U(n)$ as a free stochastic process having the same law (the same $*$-moments) as $\Psi_t$. This would seem natural because it is just the limit of the Brownian motion on $U(nd).$ To know if this definition makes sense, we would like $\Psi_t$ to verify some properties, and especially the gaussian property as defined in [2], Proposition 1.12. Let us recall here:
Definition 6. We define a counit $\epsilon$ on $U(n)$ as the morphism of algebras verifying $\epsilon(u_{ij}) = \delta_{ij}$. We say that a Lévy process on $U(n)$ is gaussian if for each $a, b, c$ in the kernel of $\epsilon$, we have:

$$L(abc) = 0$$

where $L = \frac{d}{dt}|_{t=0} \phi_t$ is the generator of the process.

By taking up the differential equations that we found in section 3 and with the same notations, we can write:

$$L(u_{i_1 j_1} \cdots u_{i_k j_k}) = -\frac{k}{2} \phi_0 (u_{i_1 j_1} \cdots u_{i_k j_k})$$

$$- \frac{1}{n} \sum_{1 \leq p < q \leq k} (-1)^{\epsilon_p + \epsilon_q} \Gamma(\epsilon_{p,q})(0)$$

Thus we have for instance for one term:

$$L(u_{ij}) = -\frac{1}{2} \phi_0 (u_{ij}) = -\frac{1}{2} \delta_{ij}$$

For two terms:

$$L(u_{ij} u_{kl}) = -\phi_0 (u_{ij} u_{kl}) - \frac{1}{n} \phi_0 (u_{il}) \phi_0 (u_{kj})$$

$$= -\delta_{ij} \delta_{kl} - \frac{\delta_{il} \delta_{kj}}{n}$$

And for three terms:

$$L(u_{i_1 j_1} u_{i_2 j_2} u_{i_3 j_3}) = -\frac{3}{2} \phi_0 (u_{i_1 j_1} u_{i_2 j_2} u_{i_3 j_3})$$

$$- \frac{1}{n} [\phi_0 (u_{i_1 j_2}) \phi_0 (u_{i_2 j_1} u_{i_3 j_3}) + \phi_0 (u_{i_1 j_3} u_{i_2 j_2}) \phi_0 (u_{i_3 j_1})$$

$$+ \phi_0 (u_{i_1 j_1} u_{i_2 j_3}) \phi_0 (u_{i_3 j_2})]$$

$$= -\frac{3}{2} \delta_{i_1 j_1} \delta_{i_2 j_2} \delta_{i_3 j_3}$$

$$- \frac{\delta_{i_1 j_2} \delta_{i_3 j_1} \delta_{i_2 j_1} + \delta_{i_1 j_3} \delta_{i_3 j_1} \delta_{i_2 j_2} + \delta_{i_2 j_3} \delta_{i_3 j_2} \delta_{i_1 j_1}}{n}$$

Let us now note $\hat{u}_{ij} = u_{ij} - \delta_{ij} 1$. These elements generate $\text{Ker} \epsilon$ as a $\ast$-algebra. To show the gaussianity of our process, it suffices to show that $L$ is zero on words of those elements and the involution of those elements. Let
Because the generator is hermitian, we also know that
\[ \frac{1}{n}(\hat{u}_{1,i_1} \hat{u}_{i_2,j_2} \hat{u}_{i_3,j_3}) = L(u_{i_1,j_1} u_{i_2,j_2} u_{i_3,j_3}) - \delta_{i_1,j_1} L(u_{i_2,j_2} u_{i_3,j_3}) - \delta_{i_2,j_2} L(u_{i_1,j_1} u_{i_3,j_3}) + \delta_{i_1,j_1} \delta_{i_2,j_2} L(u_{i_3,j_3}) + \delta_{i_1,j_1} \delta_{i_3,j_3} L(u_{i_2,j_2}) + \delta_{i_2,j_2} \delta_{i_3,j_3} L(u_{i_2,j_2}) + 0 \]
\[ = \frac{3}{2} \delta_{i_1,j_1} \delta_{i_2,j_2} \delta_{i_3,j_3} - \frac{1}{n} \delta_{i_1,j_1} \delta_{i_2,j_2} \delta_{i_3,j_3} + \delta_{i_1,j_1} \delta_{i_2,j_2} \delta_{i_3,j_3} + \frac{1}{n} \delta_{i_1,j_1} \delta_{i_2,j_2} \delta_{i_3,j_3} + \frac{1}{n} \delta_{i_1,j_1} \delta_{i_3,j_3} + \frac{1}{n} \delta_{i_1,j_1} \delta_{i_3,j_3} + \frac{1}{n} \delta_{i_2,j_2} \delta_{i_3,j_3} + \frac{1}{n} \delta_{i_1,j_1} \delta_{i_2,j_2} \delta_{i_3,j_3} + \frac{1}{n} \delta_{i_2,j_2} \delta_{i_3,j_3} + \frac{1}{n} \delta_{i_1,j_1} \delta_{i_2,j_2} \delta_{i_3,j_3} - \frac{1}{n} \delta_{i_1,j_1} \delta_{i_2,j_2} \delta_{i_3,j_3} \]
\[ = 0 \]

Because the generator is hermitian, we also know that \( L(\hat{u}_{1,i_1}^* \hat{u}_{i_2,j_2}^* \hat{u}_{i_3,j_3}^*) = 0 \).

To be able to finish the proof of the gaussianity of our Lévy process, it suffices to compute for instance \( L(\hat{u}_{1,i_1}^* \hat{u}_{i_2,j_2}^* \hat{u}_{i_3,j_3}^*) \). The same method allows us to conclude that this quantity is indeed zero and hence that the Lévy process is indeed gaussian.

Our Lévy process is thus a good candidate to define what we would like to call a Brownian motion on \( U\langle n \rangle \).

6 Conclusion

We have proven in this article a generalization of Biane’s result, namely that the Brownian motion on \( U\langle nd \rangle \), seen block-wise, converges towards a Lévy process on the Unitary Dual Group \( U\langle n \rangle \), as \( d \) goes to infinity. Biane’s result can thus be seen as a Lévy process on \( U\langle 1 \rangle \). The proof of our generalized result uses quite elementary tools, ie mainly the convergence of systems of differential equations and combinatorial considerations.

This limit free Lévy process is described by using a free stochastic differential equation whose form is quite similar to the equation of the Brownian motion on \( U\langle nd \rangle \), up to a factor \( 1/\sqrt{n} \). A natural question would be to know if other classical matricial Lévy processes arising from (classical) stochastic equations yield (free) Lévy process described by a similar (free) stochastic equation.

Also, this free Lévy process seems to be a good definition for a Brownian motion on our dual group \( U\langle n \rangle \).

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