BASIC SOLUTIONS OF SYSTEMS WITH TWO MAX-LINEAR INEQUALITIES

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Abstract. We give an explicit description of the basic solutions of max-linear systems
\( A \otimes x \leq B \otimes x \) with two inequalities.

1. Introduction

We consider systems of two max-plus linear inequalities
\[
\begin{align*}
    a_{11} \otimes x_1 & \oplus \ldots \oplus a_{1n} \otimes x_n \leq b_{11} \otimes x_1 & \oplus \ldots \oplus b_{1n} \otimes x_n, \\
    a_{21} \otimes x_1 & \oplus \ldots \oplus a_{2n} \otimes x_n \leq b_{21} \otimes x_1 & \oplus \ldots \oplus b_{2n} \otimes x_n.
\end{align*}
\]

Here \( \otimes := +, \oplus := \text{max} \), and \( a_{ij}, b_{ij}, x_j \in \mathbb{R} \cup \{-\infty\} \) for \( i = 1, 2 \) and \( j = 1, \ldots, n \).

General systems of max-linear inequalities (equivalently, equalities) were tackled by Butkovič and Hegedűs [3] who established an elimination method for finding basic solutions of such systems, starting with basic solutions of just one equation or inequality and adding all other constraints one by one. This algorithm served as a proof that solution sets to max-linear systems have finite bases, and it did not seem to be efficient enough for practical implementation. But at present, Allamegeon, Gaubert and Goubault [1] have come up with a nouvelle approach to the scheme of [3], in which every step of adding new constraint is dramatically improved by using a max-plus analogue of double description method, and also a certain criterion of minimality established in [2, 4, 5], see also [6], which allows to efficiently test the extremality of a generator.

The idea of the present paper is that when the number of inequalities is small, the basic solutions can be written out explicitly. However as shown by Wagneur, Truffet, Faye and

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Thiam [8], even in the case of two inequalities (1) the number of generators is large and
the problem to establish a systematic classification and to resolve the extremality by
writing out explicit conditions is nontrivial. This goal is achieved in the present paper by
1) representing the set of all solutions as the union of cones generated by certain Kleene
stars (Section 2), 2) selecting basic solutions by means of the above mentioned criterion
of minimality [2, 4, 5] (Section 3) which we call the multiorder principle [6]. This leads
to explicit description of basic solutions and to a procedure which finds all of them in no
more than $O(n^3)$ operations.

2. Gathering the generators

We work with the analogue of linear algebra developed over the max-plus semiring
$\mathbb{R}_{\text{max,+}}$ which is the set of real numbers with adjoined minus infinity $\mathbb{R} = \mathbb{R} \cup \{-\infty\}$
equipped with the operations of “addition” $a \oplus b := \max(a, b)$ and “multiplication” $a \otimes b := a + b$. Zero $0$ and unity $1$
of this semiring are equal, respectively, to $-\infty$ and $0$. The
operations of the semiring are extended to the nonnegative matrices and vectors in the
same way as in conventional linear algebra. That is if $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$
are matrices of compatible sizes with entries from $\mathbb{R}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$
for all $i, j$ and $C = A \otimes B$ if $c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k (a_{ik} \otimes b_{kj})$ for all $i, j$. The notation
$\otimes$ will be often omitted.

The main geometrical object of this max-plus linear algebra is a subset $K \in \mathbb{R}^n$ closed
under the operations of componentwise maximization $\oplus$ and “multiplication” $\otimes$ by scalars
(which means addition in the conventional sense). Such subsets are called max-plus cones
or just cones if there is no mix up with the ordinary convexity.

A vector $x \in \mathbb{R}^n$ is a (max-linear) combination of $y_1, \ldots, y_m \in S$. A set $S \subseteq \mathbb{R}^n$ is
generated by $y_1, \ldots, y_m \in S$ if each $x \in S$ is a max-linear combination of $y_1, \ldots, y_m$.
When vectors arise as columns (resp. rows) of matrices, it will be convenient to represent
them as max-linear combinations of the column unit vectors

$$
e_i = (\underbrace{0 \ldots 0}_i \ 1 \ 0 \ldots 0)',
$$

respectively the row unit vectors $e_i'$, which are their transpose.
The following series is called the Kleene star of $A$:

$$A^* = I \oplus A \oplus A^2 \ldots,$$

where $I$ is the max-plus unity matrix, which has all diagonal entries $1$ and all off-diagonal entries $0$. When $A^*$ has finite entries (in other words, converges) it is easily shown that $A \otimes x \leq x$ is equivalent to $A^* \otimes x = x$. We also have the following.

**Proposition 1.** Let $A \in \mathbb{R}^d$ be such that $A^*$ has finite entries. Then $\{x \mid A \otimes x \leq x\}$ is generated by the columns of $A^*$.

This section will be based on the following two observations. In the formulation we use the row unit vectors $e'_i$. We denote by $A_{i,}$ resp. $A_{-i}$, the $i$th row, resp. the $i$th column, of $A$.

**Proposition 2.** Let $A \in \mathbb{R}^{n \times n}$ have rows

$$A_{i,} = \begin{cases} e'_k \oplus \bigoplus_{l \neq k} a_{kl}e'_l, & \text{if } i = k, \\ e'_i, & \text{otherwise}, \end{cases}$$

for $i = 1, \ldots, n$. Then the set $\{x \mid A \otimes x \leq x\}$ is generated by the columns of $A$.

**Proof.** In this case $A^* = A$, after which Proposition 1 is applied. 

**Proposition 3.** Let $A \in \mathbb{R}^{n \times n}$ have rows

$$A_{i,} = \begin{cases} e'_k \oplus \bigoplus_{l \in L_1} a_{kl}e'_l, & \text{if } i = k, \\ e'_m \oplus \bigoplus_{l \in L_2} a_{ml}e'_l, & \text{if } i = m, \\ e'_i, & \text{otherwise}, \end{cases}$$

where $L_1 = \{l \neq k \mid a_{kl} \neq 0\}$, $L_2 = \{l \neq m \mid a_{kl} \neq 0\}$ and $k \neq m$.

- If $a_{km}a_{mk} \leq 1$ then $\{x \mid A \otimes x \leq x\}$ is generated by the columns of $A^*$.
- If $a_{km}a_{mk} > 1$ then $\{x \mid A \otimes x \leq x\}$ is generated by $e_i$ for $i \notin L_1 \cup L_2 \cup \{k\} \cup \{m\}$.

**Proof.** In the first case $A^*$ is finite and we apply Proposition 1. For the second case observe that on one hand, if $x_i \neq 0$ for some $i \in L_1 \cup L_2 \cup \{k\} \cup \{m\}$ then $x_k \neq 0$ and
which makes \( A \otimes x \leq x \) impossible. On the other hand, any \( x \) such that \( x_i \neq 0 \) for all \( i \in L_1 \cup L_2 \cup \{k\} \cup \{m\} \) satisfies \( A \otimes x \leq x \). \( \square \)

Now we proceed with writing out a generating set for solutions of (1). We denote

\[
J_1 := \{i \mid a_{1i} \leq b_{1i}, \ b_{1i} \neq 0\}, \ J_2 := \{i \mid a_{2i} \leq b_{2i}, \ b_{2i} \neq 0\}, \ I_1 := \{i \mid a_{1i} > b_{1i}\} \text{ and } I_2 := \{i \mid a_{2i} > b_{2i}\}.
\]

With this, system (1) is equivalent to

\[
\bigoplus_{i \in I_1} a_{1i}x_i \leq \bigoplus_{i \in J_1} b_{1i}x_i,
\]

\[
\bigoplus_{i \in I_2} a_{2i}x_i \leq \bigoplus_{i \in J_2} b_{2i}x_i.
\]

The solution set to (6) is the union of \( S^{kl} \) defined by

\[
S^{kl} = \{x \mid \bigoplus_{i \in I_1} a_{1i}x_i \leq b_{1k}x_k, \ \bigoplus_{i \in I_2} a_{2i}x_i \leq b_{2l}x_l\},
\]

for \( k \in J_1 \) and \( l \in J_2 \). Further we represent \( S^{kl} \) defined by (7) in the form

\[
S^{kl} = \{x \mid A^{kl} \otimes x \leq x\},
\]

where we have to describe \( A^{kl} \). There are two cases: \( k = l \) and \( k \neq l \). We denote \( \gamma_{ki}^1 := b_{1k}^{-1}a_{1i} \) and \( \gamma_{ki}^2 := b_{2k}^{-1}a_{2i} \). We also denote \( T_1 := \{1, \ldots, n\} \setminus I_1 \) and \( T_2 := \{1, \ldots, n\} \setminus I_2 \). Observe that \( J_1 \subseteq T_1 \) and \( J_2 \subseteq T_2 \) (the containment may not be strict in general).

If \( k = l \), then the \( k \)th row of \( A^{kl} \) is

\[
e'_k \oplus \bigoplus_{i \in I_1 \cap T_2} \gamma_{ki}^1 e'_i \oplus \bigoplus_{i \in T_1 \cap T_2} \gamma_{ki}^2 e'_i \oplus \bigoplus_{i \in I_1 \cap I_2} (\gamma_{ki}^1 \oplus \gamma_{ki}^2)e'_i.
\]

and all other rows are row unit vectors.

If \( k \neq l \) then the \( k \)th and the \( l \)th rows of \( A^{kl} \) are given by

\[
e'_k \oplus \bigoplus_{i \in I_1} \gamma_{ki}^1 e'_i, \quad e'_l \oplus \bigoplus_{i \in I_2} \gamma_{li}^2 e'_i,
\]

all other rows being row unit vectors.

Now we collect the generators of \( \{x \mid A^{kl} \otimes x \leq x\} \) considering several special cases.
Case 1. $k = l \in J_1 \cap J_2$. The $k$th row of $A^{kl}$ is given by (9) and all other rows of $A^{kl}$ are unit vectors. By Proposition 2, $S^{kl}$ is generated by the columns of $A^{kl}$. These are:

$$
e_i, \ i \in T_1 \cap T_2,
\gamma^1_{ki} e_k \oplus e_i, \ k \in J_1 \cap J_2, \ i \in I_1 \cap T_2,
\gamma^2_{li} e_l \oplus e_i, \ l \in J_1 \cap J_2, \ i \in T_1 \cap I_2,
(\gamma^1_{ki} \oplus \gamma^2_{ki}) e_k \oplus e_i, \ k \in J_1 \cap J_2, \ i \in I_1 \cap I_2.\tag{11}$$

Case 2. $k \neq l$, $k \in J_1 \cap T_2$, $l \in J_2 \cap T_1$.

Rows $k$ and $l$ of $A^{kl}$ are given by (10), all other rows being the unit vectors. As $l \in T_1$ and $k \in T_2$, we obtain $A^{kl}_{kl} = \gamma^1_{kl} = 0$ and $A^{kl}_{lk} = \gamma^2_{lk} = 0$ and hence $(A^{kl})^* = A^{kl}$. Making transpose of (10), we obtain the columns of $(A^{kl})^* = A^{kl}$. By Proposition 3 part 1. they generate $S^{kl}$:

$$e_i, \ i \in T_1 \cap T_2,
\gamma^1_{ki} e_k \oplus e_i, \ k \in J_1 \cap T_2, \ i \in I_1 \cap T_2,
\gamma^2_{li} e_l \oplus e_i, \ l \in J_2 \cap T_1, \ i \in T_1 \cap I_2,
\gamma^1_{ki} e_k \oplus \gamma^2_{li} e_l \oplus e_i, \ k \in J_1 \cap T_2, \ l \in J_2 \cap T_1, \ i \in I_1 \cap I_2.\tag{12}$$

Case 3. $k \in J_1 \cap I_2$, $l \in J_2 \cap T_1$.

Rows $k$ and $l$ of $A^{kl}$ are given by (10). However, $(A^{kl})^* \neq A^{kl}$, since $k \in I_2$ implying that $A^{kl}_{lk} = \gamma^2_{lk} \neq 0$. Note that $(A^{kl})^*$ is always finite, since $A^{kl}_{ki} = 0$ implying that the associated digraph of $A^{kl}$ does not contain any cycles with nonzero weight except for the loops $(i,i)$. For $i \in I_1$, we obtain $(A^{kl})_{li}^* = \gamma^2_{li} \oplus \gamma^2_{lk} \gamma^1_{ki}$. More precisely, $(A^{kl})_{li}^* = \gamma^2_{lk} \gamma^1_{ki}$ for $i \in I_1 \cap T_2$, and $(A^{kl})_{li}^* = \gamma^2_{li} \oplus \gamma^2_{lk} \gamma^1_{ki}$ for $i \in I_1 \cap I_2$. The $l$th row of $(A^{kl})^*$ is given by

$$e'_l \oplus \bigoplus_{i \in T_2 \cap I_1} \gamma^2_{li} \gamma^1_{ki} e'_i \oplus \bigoplus_{i \in I_2 \cap I_1} (\gamma^2_{li} \oplus \gamma^2_{lk} \gamma^1_{ki}) e'_i \oplus \bigoplus_{i \in T_1 \cap I_2} \gamma^2_{li} e'_i.\tag{13}$$
The \( k \)th row of \( (A^{kl})^* \) is the same as in (10) and all other rows are unit vectors. We obtain the columns of \( (A^{kl})^* \):

\begin{align}
e_i, & \quad i \in \mathcal{T}_1 \cap \mathcal{T}_2 \\
e_i \oplus \gamma_{kl}^1 e_k \oplus \gamma_{kl}^2 e_l, & \quad k \in J_1 \cap I_2, \ l \in J_2 \cap \mathcal{T}_1, \ i \in \mathcal{T}_2 \cap I_1 \\
e_i \oplus \gamma_{ki}^1 e_k \oplus (\gamma_{li}^2 \oplus \gamma_{kl}^2) e_l, & \quad k \in J_1 \cap I_2, \ l \in J_2 \cap \mathcal{T}_1, \ i \in I_1 \cap I_2, \\
e_i \oplus \gamma_{li}^2 e_l, & \quad l \in J_2 \cap \mathcal{T}_1, \ i \in \mathcal{T}_1 \cap I_2.
\end{align}

\textit{Case 4.} \( k \in J_1 \cap \mathcal{T}_2, \ l \in J_2 \cap I_1 \).

Rows \( k \) and \( l \) are given by (10), and by analogy with Case 3 we obtain that the \( l \)th row of \( (A^{kl})^* \) is the same as in (10), but the \( k \)th row is given by

\begin{align}
e'_k \oplus \bigoplus_{i \in \mathcal{T}_1 \cap I_2} \gamma_{kl}^1 \gamma_{li}^1 e'_i \oplus \bigoplus_{i \in I_2 \cap I_1} (\gamma_{ki}^1 \oplus \gamma_{kl}^2) e'_i \oplus \bigoplus_{i \in \mathcal{T}_2 \cap I_1} \gamma_{ki}^1 e'_i.
\end{align}

We obtain the columns of \( (A^{kl})^* \):

\begin{align}
e_i, & \quad i \in \mathcal{T}_1 \cap \mathcal{T}_2 \\
e_i \oplus \gamma_{li}^2 e_l \oplus \gamma_{ki}^1 \gamma_{li}^2 e_k, & \quad k \in J_1 \cap \mathcal{T}_2, \ l \in J_2 \cap I_1, \ i \in \mathcal{T}_1 \cap I_2 \\
e_i \oplus \gamma_{li}^2 e_l \oplus (\gamma_{ki}^1 \oplus \gamma_{kl}^2) e_k, & \quad k \in J_1 \cap \mathcal{T}_2, \ l \in J_2 \cap I_1, \ i \in I_1 \cap I_2, \\
e_i \oplus \gamma_{ki}^1 e_k, & \quad k \in J_1 \cap \mathcal{T}_2, \ i \in \mathcal{T}_2 \cap I_1.
\end{align}

\textit{Case 5.} \( k \in J_1 \cap I_2, \ l \in J_2 \cap I_1 \).

If \( \gamma_{kl}^2 \gamma_{kl}^1 \leq 1 \), then the \( l \)th row of \( (A^{kl})^* \) is given by (13) and the \( k \)th row of \( (A^{kl})^* \) is given by (15). By Proposition 3 part 1 the columns of \( (A^{kl})^* \) generate \( S^{kl} \). If \( \gamma_{kl}^2 \gamma_{kl}^1 > 1 \), then by Proposition 3 part 2, \( S^{kl} \) is generated by \( e_i \) for \( i \in \mathcal{T}_1 \cap \mathcal{T}_2 \). If \( \gamma_{kl}^2 \gamma_{kl}^1 \leq 1 \), then \( (A^{kl})^* \) is finite and its columns are:

\begin{align}
e_i, & \quad i \in \mathcal{T}_1 \cap \mathcal{T}_2, \\
e_l \oplus \gamma_{kl}^1 e_k, & \quad e_k \oplus \gamma_{kl}^2 e_l, \quad k \in J_1 \cap I_2, \ l \in J_2 \cap I_1, \\
e_i \oplus \gamma_{li}^2 e_l \oplus \gamma_{kl}^1 \gamma_{li}^2 e_k, & \quad k \in J_1 \cap I_2, \ l \in J_2 \cap I_1, \ i \in \mathcal{T}_1 \cap I_2, \\
e_i \oplus (\gamma_{li}^2 \oplus \gamma_{kl}^1 \gamma_{li}^2) e_l \oplus (\gamma_{ki}^1 \oplus \gamma_{kl}^2) e_k, & \quad k \in J_1 \cap I_2, \ l \in J_2 \cap I_1, \ i \in I_1 \cap I_2, \\
e_i \oplus \gamma_{ki}^1 e_k \oplus \gamma_{kl}^2 \gamma_{ki}^1 e_l, & \quad k \in J_1 \cap I_2, \ l \in J_2 \cap I_1, \ i \in \mathcal{T}_2 \cap I_1.
\end{align}
3. Identifying the basic solutions

A set \( S \subseteq \mathbb{R}^n \) is said to be independent if no vector in this set is generated by other vectors in this set. If such independent set generates a cone \( K \) then it is called a basis of \( K \). It can be shown \([2, 7]\) that if a basis of \( K \) exists, then it consists of all extremals (normalized in some sense): a vector \( x \in K \) is an extremal if \( x = y \oplus z \) and \( y, z \in K \) imply \( y = x \) or \( z = x \). This also means that the basis of any cone is essentially unique: any two bases are obtained from each other by multiplying their elements by scalars. Importantly, any finitely generated cone has a basis \([2, 7]\).

The notion of extremal defined above is a max-plus analogue of the notion of extremal ray (or extremal) of a convex cone. It is also a special case of the join irreducible element of a lattice.

The extremality is most conveniently expressed by the following multioorder principle \([2, 4, 5, 6]\) which we formulate here only for finitely generated case. For any \( i = 1, \ldots, n \) we introduce the relation

\[
x \leq_i y \Rightarrow x x_i^{-1} \leq y y_i^{-1}, \; x_i \neq 0 \text{ and } y_i \neq 0.
\]

A vector \( y \in K \) minimal with respect to \( \leq_i \) will be called \( i \)-minimal.

**Proposition 4** (Multioorder Principle). Let \( K \subseteq \mathbb{R}^n \) be generated by a finite set \( S \subseteq \mathbb{R}^n \). Then \( y \in S \) belongs to the basis of \( K \) (equivalently, is an extremal of \( K \)) if and only if it is \( i \)-minimal for some \( i \in \{1, \ldots, n\} \).

**Proof.** If \( y \) is not \( i \)-minimal for any \( i \), then for each \( i \in \text{supp}(y) \) there exists \( z^i \) such that \( z^i \leq_i y \). Then it can be verified that

\[
y = \bigoplus_{i \in \text{supp}(y)} z^i(z^i_i)^{-1} y_i.
\]

Conversely if \( y = \bigoplus_k \alpha_k z^k \) for some \( z^k \in S \), then for each \( i \in \text{supp}(y) \) there is \( k(i) \) such that \( y_i = \alpha_{k(i)} z^k_i \) and as \( y_j \geq \alpha_{k(i)} z^k_j \) for all \( j \) it follows that \( z^k \leq_i y \) and \( y \) is not \( i \)-minimal for any \( i \). \( \square \)
Next we classify all generators obtained in (11), (12), (14), (16) and (17) and give procedures for checking their extremality. We start with unit vectors and combinations of two unit vectors.

\[ S_1, \ e_i, \ \forall i \in T_1 \cap T_2. \]

\[ S_{2A1}, \ \phi_{ik} = \gamma_{ki}^1 e_k \oplus e_i, \ \forall k \in J_1 \cap T_2, \ i \in I_1 \cap T_2. \]

\[ S_{2A2}, \ \phi_{ik} = \gamma_{ki}^2 e_k \oplus e_i, \ \forall k \in J_2 \cap T_1, \ i \in I_2 \cap T_1. \]

\[ S_{2B}, \ \phi_{lk} = (\gamma_{kl}^1 \oplus \gamma_{kl}^2) e_k \oplus e_l, \ \forall k \in J_1 \cap J_2, \ i \in I_1 \cap I_2. \]

\[ S_{2C}, \ \phi_{lk} = \gamma_{kl}^1 e_k \oplus e_l \ \text{and} \ \phi_{kl} := \gamma_{lk}^2 e_l \oplus e_k, \ \forall k \in J_1 \cap I_2, \ l \in J_2 \cap I_1 \ \text{such that} \ \gamma_{lk}^1 \gamma_{lk}^2 \leq 1. \]

All vectors in \( S_1, \ S_{2A} \) and \( S_{2B} \) belong to the basis. Vectors in \( S_{2C} \) belong to the basis whenever they exist. For this, we determine the sets

\[
W := \{(k, l) | k \in J_1 \cap I_2, \ l \in J_2 \cap I_1, \ \gamma_{kl}^1 \gamma_{lk}^2 \leq 1\}
\]

(20)

\[
\overline{W} := \{(k, l) | k \in J_1 \cap I_2, \ l \in J_2 \cap I_1, \ \gamma_{lk}^1 \gamma_{kl}^2 > 1\}
\]

Then, \( \phi_{kl}, \phi_{lk} \in S_{3A} \) exist whenever \((k, l) \in W\). Note that if \( \gamma_{kl}^1 \gamma_{lk}^2 = 1 \) then \( \phi_{kl} \) and \( \phi_{lk} \) are multiples of each other so that one of them can be removed.

We proceed with combinations of three unit vectors. Denote \( K_1 = \{ i | a_{1i} = b_{1i} = 0 \} \) and \( K_2 = \{ i | a_{2i} = b_{2i} = 0 \} \). Note that \( \{1, \ldots, n\} = I_1 \cup J_1 \cup K_1 = I_2 \cup J_2 \cup K_2 \).

\[ S_{3A}, \ \psi_{ikl} = \gamma_{ki}^1 e_k \oplus \gamma_{li}^2 e_l \oplus e_i \ \text{for} \ k \in J_1 \cap T_2, \ l \in J_2 \cap T_1, \ i \in I_1 \cap I_2. \]

For all \( i \in I_1 \cap I_2 \) determine the sets

\[
L_1(i) := \{k \in J_1 \cap J_2 | \gamma_{ki}^1 < \gamma_{ki}^2 \}
\]

(21)

\[
L_2(i) := \{l \in J_1 \cap J_2 | \gamma_{li}^2 < \gamma_{li}^1 \}
\]

Then, \( \psi_{ikl} \in S_{3A} \) belongs to the basis whenever

\[
k \in (J_1 \cap K_2) \cup L_1(i), \ l \in (J_2 \cap K_1) \cup L_2(i).
\]

(22)
Then, two conditions are satisfied:

1. For all $i, l \in J_1 \cap J_2$, $i \in I_2 \cap I_1$, determine the sets

$$M_1(i, l) := \{t \in J_1 \cap J_2 \mid \gamma_{1l}^t \gamma_{li}^t < \gamma_{hi}^t \}.$$

2. For all $i \in J_1 \cap J_2$, $k \in J_1 \cap J_2$, determine the sets

$$M_2(i, k) := \{t \in J_1 \cap J_2 \mid \gamma_{2l}^t \gamma_{ki}^l < \gamma_{hi}^t \}.$$

A vector in $\psi_{ikt} \in S_{3B1}$ (resp. $\psi_{ikt} \in S_{3B2}$) belongs to the basis if and only if the following two conditions are satisfied:

1. $i \in I_2 \cap K_1$ or $(i, l) \in \mathbb{W}$ (resp. $i \in I_1 \cap K_2$ or $(k, i) \in \mathbb{W}$),
2. $k \in \mathbb{M}_1(i, l)$ or $k \in J_1 \cap J_2$ (resp. $l \in \mathbb{M}_2(i, k)$ or $l \in J_2 \cap K_1$).

For all $i \in J_1 \cap J_2$, $l \in J_2 \cap I_1$, $k \in J_1 \cap I_2$, determine the sets

$$N_1(i, l) := \{t \in J_1 \cap J_2 \mid \gamma_{1l}^t \gamma_{li}^t < \gamma_{hi}^t \} = \{t \in L_1(i) \mid \gamma_{1l}^t \gamma_{li}^t < \gamma_{hi}^t \},$$

$$N_2(i, k) := \{t \in J_1 \cap J_2 \mid \gamma_{2l}^t \gamma_{ki}^l < \gamma_{hi}^t \} = \{t \in L_2(i) \mid \gamma_{2l}^t \gamma_{ki}^l < \gamma_{hi}^t \}.$$ (25)

Then, $\psi_{ikt} \in S_{3C1}$ (resp. $\psi_{ikt} \in S_{3C2}$) belongs to the basis if and only if $k \in (J_1 \cap K_2) \cup N_1(i, l)$ (resp. $l \in (J_2 \cap K_1) \cup N_2(i, k)$).

For all $i \in J_1 \cap J_2$, $l \in J_2 \cap I_1$, $k \in J_1 \cap I_2$, such that $\gamma_{kl}^1 \gamma_{lk}^2 \leq 1$, determine the sets

$$S_{3D1}. \psi_{ikt} = \gamma_{kl}^1 \gamma_{lk}^2 e_k + \gamma_{1k}^2 e_l + e_i, \forall k \in J_1 \cap I_2, l \in J_2 \cap I_1, i \in I_2 \cap I_1.$$

$$S_{3D2}. \psi_{ikt} = \gamma_{lk}^1 \gamma_{kl}^2 e_k + \gamma_{1k}^2 e_l + e_i, \forall k \in J_1 \cap I_2, l \in J_2 \cap I_1, i \in I_2 \cap I_1.$$

$$S_{3E}. \psi_{ikt} = (\gamma_{li}^2 \gamma_{lk}^1 \gamma_{ki}^1) e_l + (\gamma_{1k}^1 \gamma_{kl}^2) e_k + e_i, \forall k \in J_1 \cap I_2, l \in J_2 \cap I_1, i \in I_2 \cap I_1.$$
that \( \gamma_k^1 \gamma_l^2 \leq 1 \).

Provided that \((k, l) \in W\), vector \( \psi_{ikl} \in S_{3D_1} \) (resp. \( \psi_{ikl} \in S_{3D_2} \)) belongs to the basis if and only if \( i \in K_1 \cap I_2 \) or \( (i, l) \in \overline{W} \) (resp. \( i \in I_1 \cap K_2 \) or \( (k, i) \in \overline{W} \)), and \( \psi_{ikl} \in S_{3E} \) always belong to the basis.

Below we explain why the above procedure yields the basis. We denote by \( S_1 \) the set of all generators \( e_i \) for \( i \in I_1 \cup I_2 \), by \( S_2 \) the set of all 2-generators \( \phi_{ik} \) and \( \phi_{kl} \), and by \( S_3 \) the set of all 3-generators \( \psi_{ikl} \).

**S_1, S_2:**

The supports of all generators in \( S_1 \cup S_2 \) are different, except for the pairs of generators in \( S_{2C} \), which exist if and only if \( \gamma_k^1 \gamma_l^2 \leq 1 \), and are multiples of each other if and only if \( \gamma_k^1 \gamma_l^2 = 1 \). Removing one vector from every such proportional pair in \( S_{2C} \) yields an independent set. Evidently, vectors in \( S_1 \cup S_2 \) cannot be generated with help of vectors in \( S_3 \), and this completes the explanation.

For the rest of the cases, first note that the supports of all generators in \( S_3 \) are different and hence the set \( S_3 \) is independent. It can only happen that the vectors in \( S_3 \) are linear combinations of the vectors in \( S_1 \) and \( S_2 \).

**S_3A:**

A vector \( \psi_{ikl} \in S_{3A} \) may be a combination of vectors in \( S_1 \) and \( S_{2B} \), as the supports of some generators in these sets are contained in the support of a vector in \( S_{3A} \). By the minimality principle, a vector \( \psi_{ikl} \) is extremal if and only if it is \( i \)-, \( k \)- or \( l \)-minimal. Then, \( \psi_{ikl} \) can be neither \( k \)- nor \( l \)-minimal since for all \( k, l \in \overline{T_1} \cap \overline{T_2} \) the only minimal generators are \( e_k \) and \( e_l \). The \( i \)-minimality of \( \psi_{ikl} \in S_{3A} \) can be prevented only by \( \phi_{ki} \in S_{2B} \) or \( \phi_{li} \in S_{2B} \). Condition (22) describes the situation when this does not happen.

**S_3B:**

A vector \( \psi_{ikl} \in S_{3B} \) can be a max combination of vectors in \( S_1 \), \( S_{2A} \) and \( S_{2C} \) due to the inclusion of supports. Again, \( \psi_{ikl} \) can be neither \( k \)- nor \( l \)-minimal, since it can be represented as a combination of \( e_i \) and a vector from \( S_{2A1} \) (resp. \( S_{2A2} \)) in the case of \( S_{3B1} \)
Next we describe the 2-generators which can prevent the $i$-minimality of $\psi_{ikl} \in S_{3B1}$ (resp. $\psi_{ikl} \in S_{3B2}$).

1. $\phi_{il}, \phi_{li} \in S_{2C}$ (resp. $\phi_{kl}, \phi_{lk} \in S_{2C}$). These 2-generators do not arise only if $i \in K_1$ for $S_{3B1}$ (resp. $i \in K_2$ for $S_{3B2}$), for then there is no vector in $S_{2C}$ whose support is a subset of the support of $\psi_{ikl}$, or if the corresponding pair $\phi_{il}, \phi_{li} \in S_{2C}$ (resp. $\phi_{kl}, \phi_{lk} \in S_{2C}$) does not exist meaning $(i, l) \in \overline{W}$ (resp. $(k, i) \in \overline{W}$).

2. $\phi_{ik} \in S_{2A2}$ (resp. $\phi_{il} \in S_{2A1}$). These vectors do not arise only if $k \in K_2$ (resp. $l \in K_1$), because then $k \notin J_2$ (resp. $l \notin J_1$) unlike in the case of $S_{2A2}$ (resp. $S_{2A1}$). Otherwise, $\phi_{ik}$ (resp. $\phi_{il}$) are not dangerous, i.e., they do not precede $\psi_{ikl}$ with respect to $\leq_i$ only if $k \in M_1(i, l)$ (resp. $l \in M_2(i, k)$), see (23) and (23).

$S_{3C}$:

A vector $\psi_{ikl} \in S_{3C}$ can be a max combination of vectors in $S_1$, $S_{2A}$ and $S_{2B}$. Again, $\psi_{ikl}$ can be neither $k$- nor $l$- minimal. Indeed,

$$
\psi_{ikl} = \gamma^1_{ikl}e_k \oplus e_i \oplus \gamma^2_{ikl}(\gamma^1_{ikl}e_k \oplus e_i), \quad S_{3C1}
$$

$$
\psi_{ikl} = \gamma^2_{ikl}e_i \oplus e_i \oplus \gamma^1_{ikl}(\gamma^2_{ikl}e_i \oplus e_i), \quad S_{3C2}
$$

where the vectors in brackets belong to $S_{2A1}$ and $S_{2A2}$ respectively. The first vector cannot be $k$-minimal since $k \in \overline{T}_1 \cap \overline{T}_2$, and it cannot be $l$-minimal as it to loses $\gamma^2_{ikl}e_k \oplus e_i \in S_{2A1}$. The second vector cannot be $l$-minimal since $l \in \overline{T}_1 \cap \overline{T}_2$, and it cannot be $k$-minimal as it loses to $\gamma^2_{ikl}e_i \oplus e_k \in S_{2A2}$. The remaining possibility of being $i$-minimal can be destroyed by vectors from $S_{2B}$, and this does not happen if and only if the given conditions are satisfied.

$S_{3D}, S_{3E}$:

A vector $\psi_{ikl} \in S_{3D}$ cannot be a max combination of other vectors of $S_2$ than those in $S_{2C}$. It is not a max combination of vectors in $S_{2C}$ only if $i$ is not suitable for existence of vectors in $S_{2C}$. This happens if $i \in K_1 \cap I_2$ or $(i, l) \in \overline{W}$ for the case $\psi_{ikl} \in S_{3D1}$, and $i \in I_1 \cap K_2$ or $(k, i) \in \overline{W}$ for the case $\psi_{ikl} \in S_{3D2}$. Finally, the vectors in $S_{3E}$ cannot be
combinations of vectors in $S_2$, since only vectors in $S_{2C}$ have relevant supports (and yet not enough). So the vectors in $S_{3E}$ are in the basis whenever they exist.

We note that the complexity of the above procedure is $O(n^3)$, which is due to the computation of the sets $M_1(i,l)$ \((23)\), $M_2(i,k)$ \((24)\), $N_1(i,l)$ and $N_2(i,k)$ \((25)\), and checking conditions for all combinations of three unit vectors.

We conclude the paper with two examples. The second example is taken from \([8]\), Example 4.2.

**Example 1.** To illustrate the sets of generators constructed in the paper on a simple example, we consider the following system of two inequalities with four variables:

$$
\begin{align*}
4 \otimes x_3 + 2 \otimes x_4 & \leq x_1 + 2 \otimes x_2, \\
3 \otimes x_1 + x_3 & \leq x_2.
\end{align*}
$$

We have $I_1 = \{3,4\}$, $J_1 = \mathcal{T}_1 = \{1,2\}$, $I_2 = \{1,3\}$, $J_2 = \{2\}$, $\mathcal{T}_2 = \{2,4\}$. We compute

- $S_1$: just $e_2$, since $\mathcal{T}_1 \cap \mathcal{T}_2 = \{2\}$;
- $S_{2A1}$: just $\gamma_{21}^1 e_2 \oplus e_4 = e_2 \oplus e_4$, since $J_1 \cap \mathcal{T}_2 = \{2\}$ and $I_1 \cap \mathcal{T}_2 = \{4\}$;
- $S_{2A2}$: just $\gamma_{21}^2 e_2 \oplus e_1 = 3e_2 \oplus e_1$, since $J_2 \cap \mathcal{T}_1 = \{2\}$ and $I_2 \cap \mathcal{T}_1 = \{4\}$;
- $S_{2B}$: $(\gamma_{23}^1 \oplus \gamma_{23}^2) e_2 \oplus e_3 = 2e_2 \oplus e_3$, since $J_1 \cap J_2 = \{2\}$ and $I_1 \cap I_2 = \{3\}$;
- $S_{2C}$: empty, since $J_2 \cap I_1$ is empty;
- $S_{3A}$: trivializes to $S_{2B}$;
- $S_{3B1}$: empty, since $J_2 \cap I_1$ is empty;
- $S_{3B2}$: just $\gamma_{14}^1 e_2 \oplus \gamma_{14}^1 e_1 \oplus e_4 = 5e_2 \oplus 2e_1 \oplus e_4$, since $J_1 \cap I_2 = \{1\}$, $J_2 \cap \mathcal{T}_1 = \{2\}$, $\mathcal{T}_2 \cap I_1 = \{4\}$;
- $S_{3C1}$: empty, since $J_2 \cap I_1$ is empty;
- $S_{3C2}$: just $(\gamma_{23}^2 \oplus \gamma_{23}^1) e_2 \oplus \gamma_{13}^1 e_1 \oplus e_3$, which is $7e_2 \oplus 4e_1 \oplus e_3$, since $J_1 \cap I_2 = \{1\}$, $J_2 \cap \mathcal{T}_1 = \{2\}$, $I_2 \cap I_1 = \{3\}$;
- $S_{3D1}$, $S_{3D2}$ and $S_{3E}$: empty, since $J_2 \cap I_1$ is empty.

In this example, the basis consists of four generators in $S_1$, $S_{2A1}$, $S_{2A2}$ and $S_{2B}$: $e_2$, $e_2 \oplus e_4$, $3e_2 \oplus e_1$ and $2e_2 \oplus e_3$. Indeed, the remaining two generators in $S_3$ are redundant: 1) $5e_2 \oplus 2e_1 \oplus e_4$ ($S_{3B2}$) is a combination of $e_2 \oplus e_4$ ($S_{2A1}$) and $3e_2 \oplus e_1$ ($S_{2A2}$), 2) $7e_2 \oplus 4e_1 \oplus e_3$ ($S_{3C2}$) is a combination of $3e_2 \oplus e_1$ ($S_{2A2}$) and $2e_2 \oplus e_3$ ($S_{2B}$).
Example 2. To compare our results with the approach of [8], we consider Example 4.2, which is a system of two inequalities with seven variables:

\[
\begin{align*}
x_4 \oplus 4 \otimes x_5 \oplus 2 \otimes x_6 \oplus 6 \otimes x_7 & \leq x_1 \oplus 1 \otimes x_2 \oplus 5 \otimes x_3, \\
5 \otimes x_2 \oplus 6 \otimes x_3 \oplus 2 \otimes x_7 & \leq 3 \otimes x_1 \oplus 2 \otimes x_5 \oplus 4 \otimes x_6.
\end{align*}
\]

(29)

In this case \(I_1 = \{4, 5, 6, 7\}, J_1 = \{1, 2, 3\} = T_1, I_2 = \{2, 3, 7\}, J_2 = \{1, 4, 5, 6\} = T_2\). We compute the generators comparing them with those in the table of [8] page 365:

\(S_1\): just \(e_1\), since \(T_1 \cap T_2 = \{1\}\). This is \(x_1\) in the table of [8].

\(S_{2, A 1}\): Combining \(J_1 \cap T_2 = \{1\}\) and \(I_1 \cap T_2 = \{4, 5, 6\}\) we obtain \(e_1 \oplus e_4, 4e_1 \oplus e_5\) and \(2e_1 \oplus e_6\). Vector \(e_1 \oplus e_4\) corresponds to \(x_3\), and the remaining two vectors are \(x_5\) and \(x_{10}\) in the table of [8].

\(S_{2, A 2}\): Combining \(J_2 \cap T_1 = \{1\}\) and \(I_2 \cap T_1 = \{2, 3\}\) we obtain \(2e_1 \oplus e_2\) and \(3e_1 \oplus e_3\). These correspond to \(x_4\) and \(x_7\) in the table of [8].

\(S_{2, B}\): just \(6e_1 \oplus e_7\), combining \(J_1 \cap J_2 = \{1\}\) with \(I_1 \cap I_2 = \{7\}\). This is \(x_2\) in the table of [8].

\(S_{2, C}\). To compute these we need to combine \(J_1 \cap I_2 = \{2, 3\}\) with \(J_2 \cap I_1 = \{4, 5, 6\}\). For each \(k = 2, 3\) and \(l = 4, 5, 6\) we need to check whether \(\gamma_{kl}^1 \gamma_{lk}^2 \leq 1\), and each time this condition is satisfied we have two vectors (or just one vector if \(\gamma_{kl}^1 \gamma_{lk}^2 = 1\)). In our case the condition is satisfied only with \(k = 3\) and \(l = 6\). This yields two vectors \(2e_6 \oplus e_3\) and \(e_3 \oplus 3e_6\), which are \(x_6\) and \(x_{11}\) in the table of [8].

\(S_{3, A}\): trivializes to \(S_{2, B}\).

\(S_{3, B 1}\): We need to combine \(J_1 \cap T_2 = \{1\}, J_2 \cap I_1 = \{4, 5, 6\}\) and \(I_2 \cap T_1 = \{2, 3\}\). For \(i = 2, 3, l = 4, 5, 6\) and \(k = 1\), each time when \(\gamma_{li}^1 \gamma_{lk}^2 > 1\), we have to verify whether \(\gamma_{lk}^1 \gamma_{li}^2 < \gamma_{ki}^2\) holds. Each time when both conditions are satisfied, we have an independent vector of the basis. Here it never happens.

\(S_{3, B 2}\): We need to combine \(J_1 \cap I_2 = \{2, 3\}, J_2 \cap T_1 = \{1\}\) and \(T_2 \cap I_1 = \{4, 5, 6\}\). For \(k = 2, 3, i = 4, 5, 6\) and \(l = 1\), each time when \(\gamma_{ki}^1 \gamma_{lk}^2 > 1\), we have to verify whether \(\gamma_{lk}^2 \gamma_{ki}^1 < \gamma_{li}^1\) holds. Each time when both conditions are satisfied, we have an independent vector of the basis. Here it happens with 1) \(l = 1, k = 3\) and \(i = 4\) leading to \(3e_1 \oplus e_3 \oplus 5e_4\) which corresponds to \(x_8\) of [8], 2) \(l = 1, k = 3\) and \(i = 5\) leading to \(3e_1 \oplus e_3 \oplus 1e_5\), which
corresponds to $x_9$ of [S].

$S_{3C1}$: Here we combine $J_1 \cap \overline{T}_2 = \{1\}$ with $J_2 \cap I_1 = \{4, 5, 6\}$ and $I_1 \cap I_2 = \{7\}$. Since $\gamma_{ki}^1 > \gamma_{ki}^2$ with $k = 1$ and $i = 7$, no vector belongs to the basis in this case.

$S_{3C2}$: We combine $J_1 \cap I_2 = \{2, 3\}$, $J_2 \cap \overline{T}_1 = \{1\}$ and $I_1 \cap I_2 = \{7\}$. For each $k = 2, 3$, $l = 1$ and $i = 7$ we have to verify $\gamma_{hi}^2 \oplus \gamma_{ik}^1 \gamma_{ki}^1 < \gamma_{hi}^2 \oplus \gamma_{li}^1$. This happens for $k = 3$, $l = 1$ and $i = 7$ and yields the vector $4e_1 \oplus 1e_3 \oplus e_7$, which corresponds to $x_{12}$ of [S].

$S_{3D1}$: We combine $J_1 \cap I_2 = \{2, 3\}$, $J_2 \cap I_1 = \{4, 5, 6\}$, $I_2 \cap \overline{T}_1 = \{2, 3\}$, $I_1 \cap I_2 = \{7\}$. For $k = 2, 3$, $l = 4, 5, 6$, the condition $\gamma_{lk}^1 \gamma_{kl}^2 \leq 1$ holds only for $k = 3$ and $l = 6$, so it remains to verify $\gamma_{ki}^1 \gamma_{li}^2 > 1$ for $i = 2$ and $l = 6$. This condition holds and we obtain $\gamma_{36}^1 \gamma_{62}^2 e_3 \oplus \gamma_{62}^2 e_6 \oplus e_2$ which is proportional with $2e_2 \oplus e_3 \oplus 3e_6$. Note that the max-linear combination of $e_2, e_3, e_6$ given for $x_{13}$ in the table of [S] is an error, since $Ax_{13} \nleq Bx_{13}$.

$S_{3D2}$: We combine $J_1 \cap I_2 = \{2, 3\}$, $J_2 \cap I_1 = \{4, 5, 6\}$, $\overline{T}_2 \cap I_1 = \{4, 5, 6\}$. For $k = 2, 3$ and $l = 4, 5, 6$, the condition $\gamma_{lk}^1 \gamma_{kl}^2 \leq 1$ holds only for $k = 3$ and $l = 6$, so it remains to verify $\gamma_{ki}^1 \gamma_{li}^2 > 1$ for $i = 4, 5$ and $k = 3$. This condition holds in both cases and yields $\gamma_{63}^1 \gamma_{34}^2 e_6 \oplus \gamma_{34}^1 e_3 \oplus e_4$ which is proportional with $2e_6 \oplus e_3 \oplus 3e_4$, and $\gamma_{63}^1 \gamma_{35}^2 e_6 \oplus \gamma_{35}^1 e_3 \oplus e_5$ proportional with $2e_6 \oplus e_3 \oplus 1e_5$.

$S_{3E}$: We combine $J_1 \cap I_2 = \{2, 3\}$, $J_2 \cap I_1 = \{4, 5, 6\}$ and $I_1 \cap I_2 = \{7\}$. As $\gamma_{ki}^1 \gamma_{li}^2 < 1$ only for $k = 3$ and $l = 6$, we have only one generator, namely $3e_6 \oplus 1e_3 \oplus e_7$.

Thus the basis consists of $e_1$, 8 combinations of 2 unit vectors and 7 combinations of 3 unit vectors.

The 2-combinations are: $e_1 \oplus e_4$, $4e_1 \oplus e_5$ and $2e_1 \oplus e_6$ ($S_{2A1}$), $2e_1 \oplus e_2$ and $3e_1 \oplus e_3$ ($S_{2A2}$), $6e_1 \oplus e_7$ ($S_{2B}$), $2e_6 \oplus e_3$ and $e_3 \oplus 3e_6$ ($S_{2C}$).

The 3-combinations are: $3e_1 \oplus e_3 \oplus 5e_4$, $3e_1 \oplus e_3 \oplus 1e_5$ ($S_{3B2}$), $4e_1 \oplus 1e_3 \oplus e_7$ ($S_{3C2}$), $2e_2 \oplus e_3 \oplus 3e_6$ ($S_{3D1}$), $2e_6 \oplus e_3 \oplus 5e_4$ and $2e_6 \oplus e_3 \oplus 1e_5$ ($S_{3D2}$), $3e_6 \oplus 1e_3 \oplus e_7$ ($S_{3E}$).

We note that all vectors that we have found, are solutions of the system, and moreover, all 3-generators turn both inequalities into equalities, which in analogy with the convex analysis also suggests that they must be extremals (the 2-generators correspond to the intersections with coordinate planes). Actually vectors in $S_{3B2}$ and $S_{3C2}$ are different
from $x_8, x_9$ and $x_{12}$ from the table of [8] page 365, to which they correspond in terms of supports. For these, $x_8 = 4e_1 \oplus e_3 \oplus 4e_4$ is a combination of $3e_1 \oplus e_3 \oplus 5e_4$ (from $S_{3B^2}$), $3e_1 \oplus e_3$ (from $S_{2A^2}$) and $e_1, x_9 = 4e_1 \oplus 1e_3 \oplus e_5$ is a combination of $3e_1 \oplus e_3 \oplus 1e_5$ (from $S_{3B^2}$) and $3e_1 \oplus e_3$, and $x_{12} = 5e_1 \oplus 1e_3 \oplus e_7$ is a combination of $4e_1 \oplus 1e_3 \oplus e_7$ (from $S_{3C^2}$) and $e_1$. The remaining generator in the table of [8] is $x_{13} = e_2 \oplus 2e_3 \oplus 1e_6$. It is in error, since it violates the second inequality of (29), but in terms of support, it corresponds to $2e_2 \oplus e_3 \oplus 3e_6$ from $S_{3D^1}$. Also, there are three combinations which are not in the table of [8], from $S_{3D^2}$ and $S_{3E}$.

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