TWO NON-CLOSURE PROPERTIES ON THE CLASS OF
SUBEXPONENTIAL DENSITIES

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Abstract. Relations between subexponential densities and locally subexponential
distributions are discussed. It is shown that the class of subexponential densities is neither
closed under convolution roots nor closed under asymptotic equivalence. A remark is given
on the closure under convolution roots for the class of convolution equivalent distributions.

Key words or phrases: subexponential densities, local subexponentiality,
convolution roots, asymptotic equivalence

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1. INTRODUCTION AND MAIN RESULTS

In what follows, we denote by $\mathbb{R}$ the real line and by $\mathbb{R}_+$ the half line $[0, \infty)$. Let
$\mathbb{N}$ be the totality of positive integers. The symbol $\delta_a(dx)$ stands for the delta measure
at $a \in \mathbb{R}$. Let $\eta$ and $\rho$ be probability measures on $\mathbb{R}$. We denote the convolution
of $\eta$ and $\rho$ by $\eta * \rho$ and denote $n$-th convolution power of $\rho$ by $\rho^{n*}$. Let $f(x)$ and
g($x$) be integrable functions on $\mathbb{R}$. We denote by $f^{n\otimes}(x)$ $n$-th convolution power
of $f(x)$ and by $f \otimes g(x)$ the convolution of $f(x)$ and $g(x)$. For positive functions
$f_1(x)$ and $g_1(x)$ on $[a, \infty)$ for some $a \in \mathbb{R}$, we define the relation $f_1(x) \sim g_1(x)$ by
$\lim_{x \to \infty} f_1(x)/g_1(x) = 1$. We also define the relation $a_n \sim b_n$ for positive sequences
$\{a_n\}_{n=A}^{\infty}$ and $\{b_n\}_{n=A}^{\infty}$ with $A \in \mathbb{N}$ by $\lim_{n \to \infty} a_n/b_n = 1$. We define the class $\mathcal{P}_+$ as
the totality of probability distributions on $\mathbb{R}_+$. In this paper, we prove that the class
of subexponential densities is not closed under two important closure properties. We
say that a measurable function $g(x)$ on $\mathbb{R}$ is a density function if $\int_{-\infty}^\infty g(x)dx = 1$ and
g($x$) $\geq 0$ for all $x \in \mathbb{R}$.

Definition 1.1. (i) A nonnegative measurable function $g(x)$ on $\mathbb{R}$ belongs to the
class $\mathcal{L}$ if $g(x) > 0$ for all sufficiently large $x > 0$ and if $g(x + a) \sim g(x)$ for any $a \in \mathbb{R}$.
(ii) A measurable function $g(x)$ on $\mathbb{R}$ belongs to the class $\mathcal{L}_d$ if $g(x)$ is a density function and $g(x) \in L$.

(iii) A measurable function $g(x)$ on $\mathbb{R}$ belongs to the class $\mathcal{S}_d$ if $g(x) \in \mathcal{L}_d$ and $g \otimes g(x) \sim 2g(x)$.

(iv) A distribution $\rho$ on $\mathbb{R}$ belongs to the class $\mathcal{L}_{ac}$ if there is $g(x) \in \mathcal{L}_d$ such that $\rho(dx) = g(x)dx$.

(v) A distribution $\rho$ on $\mathbb{R}$ belongs to the class $\mathcal{S}_{ac}$ if there is $g(x) \in \mathcal{S}_d$ such that $\rho(dx) = g(x)dx$.

Densities in the class $\mathcal{S}_d$ are called subexponential densities and those in the class $\mathcal{L}_d$ are called long-tailed densities. The study on the class $\mathcal{S}_d$ goes back to Cheover et al. [2]. Let $\rho$ be a distribution on $\mathbb{R}$. Note that $c^{-1}\rho((x-c,x])$ is a density function on $\mathbb{R}$ for every $c > 0$.

**Definition 1.2.**

(i) Let $\Delta := (0, c]$ with $c > 0$. A distribution $\rho$ on $\mathbb{R}$ belongs to the class $\mathcal{L}_{\Delta}$ if $\rho((x, x+c]) \in L$.

(ii) Let $\Delta := (0, c]$ with $c > 0$. A distribution $\rho$ on $\mathbb{R}$ belongs to the class $\mathcal{S}_{\Delta}$ if $\rho \in \mathcal{L}_{\Delta}$ and $\rho \ast \rho((x, x+c]) \sim 2\rho((x, x+c])$.

(iii) A distribution $\rho$ on $\mathbb{R}$ belongs to the class $\mathcal{L}_{loc}$ if $\rho \in \mathcal{L}_{\Delta}$ for each $\Delta := (0, c]$ with $c > 0$.

(iv) A distribution $\rho$ on $\mathbb{R}$ belongs to the class $\mathcal{S}_{loc}$ if $\rho \in \mathcal{S}_{\Delta}$ for each $\Delta := (0, c]$ with $c > 0$.

(v) A distribution $\rho \in \mathcal{L}_{loc}$ belongs to the class $\mathcal{UL}_{loc}$ if there exists $p(x) \in \mathcal{L}_d$ such that $c^{-1}\rho((x-c,x]) \sim p(x)$ uniformly in $c \in (0, 1]$.

(vi) A distribution $\rho \in \mathcal{S}_{loc}$ belongs to the class $\mathcal{US}_{loc}$ if there exists $p(x) \in \mathcal{S}_d$ such that $c^{-1}\rho((x-c,x]) \sim p(x)$ uniformly in $c \in (0, 1]$.

Distributions in the class $\mathcal{S}_{loc}$ are called locally subexponential, those in the class $\mathcal{US}_{loc}$ are called uniformly locally subexponential. The class $\mathcal{S}_{\Delta}$ was introduced by Asmussen et al. [1] and the class $\mathcal{S}_{loc}$ was by Watanabe and Yamamuro [14]. Detailed accounts of the classes $\mathcal{S}_d$ and $\mathcal{S}_{\Delta}$ are found in the book of Foss et al. [6]. First, we present some interesting results on the classes $\mathcal{S}_d$ and $\mathcal{S}_{loc}$.

**Proposition 1.1.** We have the following.

(i) Let $\Delta := (0, c]$ with $c > 0$ and let $p(x) := c^{-1}\mu((x-c,x])$ for a distribution $\mu$ on $\mathbb{R}_+$. Then $\mu \in \mathcal{S}_{\Delta}$ if and only if $p(x) \in \mathcal{S}_d$. Moreover, $\mu \in \mathcal{S}_{loc} \cap \mathcal{P}_+$ if and only if there exists a density function $q(x)$ on $\mathbb{R}_+$ such that $q(x) \in \mathcal{S}_d$ and $c^{-1}\mu((x-c,x]) \sim q(x)$ for every $c > 0$.

(ii) Let $\rho_1(dx) := q_1(x)dx$ be a distribution on $\mathbb{R}_+$. If $q_1(x)$ is continuous with compact support and if $p_2 \in \mathcal{S}_{loc} \cap \mathcal{P}_+$, then $\rho_1 \ast p_2(dx) = \left(\int_{0}^{x} q_1(x-u)p_2(du)\right) dx$ and $\int_{0}^{x} q_1(x-u)p_2(du) \in \mathcal{S}_d$.

(iii) Let $\mu$ be a distribution on $\mathbb{R}_+$. If there exist distributions $\rho_c$ for $c > 0$ such that, for every $c > 0$, the support of $\rho_c$ is included in $[0, c]$ and $\rho_c \ast \mu \in \mathcal{S}_{loc}$, then $\mu \in \mathcal{S}_{loc}$. 

Definition 1.3. (i) We say that a class $C$ of probability distributions on $\mathbb{R}$ is closed under convolution roots if $\mu^{n*} \in C$ for some $n \in \mathbb{N}$ implies that $\mu \in C$.

(ii) Let $p_1(x)$ and $p_2(x)$ be density functions on $\mathbb{R}$. We say that a class $C$ of density functions is closed under asymptotic equivalence if $p_1(x) \in C$ and $p_2(x) \sim c p_1(x)$ with $c > 0$ implies that $p_2(x) \in C$.

The class $S_{ac}$ is a proper subclass of the class $US_{loc}$ because a distribution in $US_{loc}$ can have a point mass. Moreover, the class $US_{loc}$ is a proper subclass of the class $S_{loc}$ as the following theorem shows.

**Theorem 1.1.** There exists a distribution $\mu \in S_{loc} \setminus US_{loc}$ such that $\mu^{2*} \in S_{ac}$.

**Corollary 1.1.** We have the following.

(i) The class $S_{ac}$ is not closed under convolution roots.

(ii) The class $US_{loc}$ is not closed under convolution roots.

(iii) The class $L_{ac}$ is not closed under convolution roots.

(iv) The class $UL_{loc}$ is not closed under convolution roots.

The class $S_{d}$ is closed under asymptotic equivalence in the one-sided case. See (ii) of Lemma 2.1 below. However, Foss et al. [6] suggest the possibility of non-closure under asymptotic equivalence for the class $S_{d}$ in the two-sided case. We exactly prove it as follows.

**Theorem 1.2.** The class $S_{d}$ is not closed under asymptotic equivalence, that is, there exist $p_1(x) \in S_{d}$ and $p_2(x) \notin S_{d}$ such that $p_2(x) \sim cp_1(x)$ with $c > 0$.

In Sect. 2, we prove Proposition 1.1. In Sect. 3, we prove Theorems 1.1 and 1.2. In Sect. 4, we give a remark on the closure under convolution roots.

### 2. PROOF OF PROPOSITION 1.1

We present two lemmas for the proofs of main results and then prove Proposition 1.1.

**Lemma 2.1.** Let $f(x)$ and $g(x)$ be density functions on $\mathbb{R}_+$. 

(i) If $f(x) \in L_{d}$, then $f^{n*}(x) \in L_{d}$ for every $n \in \mathbb{N}$.

(ii) If $f(x) \in S_{d}$ and $g(x) \sim cf(x)$ with $c > 0$, then $g(x) \in S_{d}$.

(iii) Assume that $f(x) \in L_{d}$. Then, $f(x) \in S_{d}$ if and only if 

$$\lim_{A \to \infty} \limsup_{x \to \infty} \frac{1}{f(x)} \int_{A}^{x-A} f(x-u)f(u)du = 0.$$ 

**Proof** Proof of assertion (i) is due to Theorem 4.3 of [6]. Proofs of assertions (ii) and (iii) are due to Theorems 4.8 and 4.7 of [6], respectively. 

**Lemma 2.2.** (i) Let $\Delta := (0, c]$ with $c > 0$. Assume that $\rho \in L_{\Delta} \cap P_+$. Then, $\rho \in S_{\Delta}$ if and only if 

$$\lim_{A \to \infty} \limsup_{x \to \infty} \frac{1}{\rho((x, x+c])} \int_{A+}^{x-A} \rho((x-u, x+c-u)]\rho(du) = 0.$$
(ii) Assume that \( \rho \in L_{\text{loc}} \cap P_+ \). Then, \( \rho^n \in L_{\text{loc}} \) for every \( n \in \mathbb{N} \). Moreover, \( \rho \left( (x - c, x] \right) \sim c \rho \left( (x - 1, x] \right) \) for every \( c > 0 \).

(iii) Let \( \rho_2 \in P_+ \). If \( \rho_1 \in S_{\text{loc}} \cap P_+ \) and \( \rho_2 \left( (x - c, x] \right) \sim c_1 \rho_1 \left( (x - c, x] \right) \) with \( c_1 > 0 \) for every \( c > 0 \), then \( \rho_2 \in S_{\text{loc}} \cap P_+ \).

Proof of assertion (i) is due to Theorem 4.21 of [6]. First assertion of (ii) is due to Corollary 4.19 of [6]. Second one is proved as (2.6) in Theorem 2.1 of [14]. Proof of assertion (iii) is due to Theorem 4.22 of [6]. □

Proof of (i) of Proposition 1.1 Let \( \rho(dx) := c^{-1}1_{[0,c)}(x)dx \). First, we prove that if \( \mu \in S_{\text{loc}} \cap P_+ \), then \( \rho * \mu \in S_{\text{ac}} \). We can assume that \( c = 1 \). Suppose that \( \mu \in S_{\text{loc}} \). Let \( p(x) := \mu \left( (x - 1, x] \right) \). We have \( \rho * \mu(dx) = \mu \left( (x - 1, x] \right)dx \) and hence \( p(x) \in L_d \). Let \( A \) be a positive integer and let \( X, Y \) be independent random variables with the same distribution \( \mu \). Then, we have for \( x > 2A + 2 \)

\[
\int_A^{x-A} p(x-u)p(u)du \\
= 2 \int_A^{x/2} p(x-u)p(u)du \\
= 2 \int_A^{x/2} P(x-u-1 < X \leq x-u, u-1 < Y \leq u)du \\
\leq 2 \int_A^{x/2} P(X > A, Y > A, x-2 < X+Y \leq x, u-1 < Y \leq u)du \\
\leq 2 \sum_{n=A}^{\infty} \int_{n+1}^{n+1} P(X > A, Y > A, x-2 < X+Y \leq x, n-1 < Y \leq n+1)du \\
\leq 4P(X > A, Y > A, x-2 < X+Y \leq x) \\
\leq 4 \int_{A+}^{(x-A)-} \mu((x-2-u, x-u])\mu(du).
\]

Since \( \mu \in S_{\text{loc}} \), we obtain from (i) of Lemma 2.2 that

\[
\lim_{A \to \infty} \limsup_{x \to \infty} \frac{\int_A^{x-A} p(x-u)p(u)du}{p(x)} = 0.
\]

Thus, we see from (iii) of Lemma 2.1 that \( p(x) \in S_d \).

Conversely, suppose that \( p(x) \in S_d \). Then, we have \( \mu \in L_\Delta \). Let \( [y] \) be the largest integer not exceeding a real number \( y \). Choose sufficiently large integer \( A > 0 \). Note that there are positive constants \( c_j \) for \( 1 \leq j \leq 4 \) such that

\[
c_1 p(x-n) \leq p(x-u) \leq c_2 p(x-n) \text{ and } c_3 p(n) \leq p(u) \leq c_4 p(n)
\]
for $n \leq u \leq n + 1$, $A \leq n \leq [x + 1 - A]$, and $x > 2A + 2$. Thus, we find that

$$P(A < X, A < Y, x < X + Y \leq x + 1)$$

$$\leq \sum_{n=A}^{[x+1-A]} \int_n^{n+1} \mu((x-u, x+1-u]) \mu(du)$$

$$= \sum_{n=A}^{[x+1-A]} \int_n^{n+1} p(x-u+1) \mu(du)$$

$$\leq c_2 \sum_{n=A}^{[x+1-A]} p(x-n+1)p(n+1)$$

$$\leq \frac{c_2}{c_1c_3} \sum_{n=A}^{[x+1-A]} \int_n^{n+1} p(x-u+1)p(u+1)du$$

$$\leq \frac{c_2}{c_1c_3} \int_A^{x+2-A} p(x-u+1)p(u+1)du$$

Since $p(x) \in \mathcal{S}_d$, we establish from (iii) of Lemma 2.1 that

$$\lim_{A \to \infty} \limsup_{x \to \infty} \frac{P(A < X, A < Y, x < X + Y \leq x + 1)}{P(x < X \leq x + 1)} = 0.$$
Since \( \lim_{M \to \infty} \delta(M) = 0 \) and
\[
\lim_{M \to \infty} \sum_{n=1}^{MN} a_n M^{-1} = \int_{0}^{N} q_1(x) dx = 1,
\]
we obtain from (2.1) that
\[
q(x) \sim \rho_2((x-1,x]).
\]
Since \( \rho_2 \in S_{\text{loc}} \), we conclude from (i) of Proposition 1.1 that \( q(x) \in S_d \). \( \square \)

Proof of (iii) of Proposition 1.1 Suppose that the support of \( \rho_c \) is included in \([0,c]\) and \( \rho_c \ast \mu \in S_{\text{loc}} \) for every \( c > 0 \). Let \( X \) and \( Y \) be independent random variables with the same distribution \( \mu \), and let \( X_c \) and \( Y_c \) be independent random variables with the same distribution \( \rho_c \). Define \( J_1(c; c_1; a; x) \) and \( J_2(c; c_1; a; x) \) for \( a \in \mathbb{R} \) and \( c_1 > 0 \) as
\[
J_1(c; c_1; a; x) := \frac{P(x + a < X + X_c \leq x + c_1 + a)}{P(x + a < X + X_c \leq x + c_1 + c)},
\]
\[
J_2(c; c_1; a; x) := \frac{P(x + a < X + X_c \leq x + c_1 + c + a)}{P(x + a < X + X_c \leq x + c_1 + c)}.
\]
We see that
\[
J_1(c; c_1; a; x) \leq \frac{P(x + a < X \leq x + c_1 + a)}{P(x + a < X \leq x + c_1)} \leq J_2(c; c_1; a; x).
\]
Since \( \rho_c \ast \mu \in \mathcal{L}_{\text{loc}} \), we obtain that
\[
\lim_{x \to \infty} J_1(c; c_1; a; x) = \frac{c_1}{c_1 + c}
\]
and
\[
\lim_{x \to \infty} J_2(c; c_1; a; x) = \frac{c_1 + c}{c_1}.
\]
Thus, as \( c \to 0 \) we have by (2.2)
\[
\lim_{x \to \infty} \frac{P(x + a < X \leq x + c_1 + a)}{P(x + a < X \leq x + c_1)} = 1,
\]
and hence \( \mu \in \mathcal{L}_{\text{loc}} \). We find from \( \rho_c \ast \mu \in S_{\text{loc}} \) and (i) of Lemma 2.2 that
\[
\lim_{A \to \infty} \lim_{x \to \infty} \frac{P(X > A, Y > A, x < X + Y \leq x + c_1)}{P(x < X \leq x + c_1)} \leq \lim_{A \to \infty} \lim_{x \to \infty} \frac{P(X > A, Y > A, x < X + X_c + Y + Y_c \leq x + c_1 + 2c)}{P(x < X + X_c \leq x + c_1)} = 0.
\]
Thus, we see from (i) of Lemma 2.2 that \( \mu \in S_{\text{loc}} \). \( \square \)
For the proofs of the theorems, we introduce a distribution \( \mu \) as follows. Let \( 1 < x_0 < b \) and choose \( \delta \in (0,1) \) satisfying \( \delta < (x_0 - 1) \land (b - x_0) \). We take a continuous periodic function \( h(x) \) on \( \mathbb{R} \) with period \( \log b \) such that \( h(\log x) > 0 \) for \( x \in [1, x_0) \cup (x_0, b] \) and

\[
    h(\log x) = \begin{cases} 
        0 & \text{for } x = x_0, \\
        \frac{-1}{\log|x-x_0|} & \text{for each } x \text{ with } 0 < |x-x_0| < \delta.
    \end{cases}
\]

Let \( \phi(x) := x^{-\alpha-1}h(\log x)1_{[1,\infty)}(x) \) with \( \alpha > 0 \). Here, the symbol \( 1_{[1,\infty)}(x) \) stands for the indicator function of the set \( [1, \infty). \) Define a distribution \( \mu \) as

\[
    \mu(dx) := M^{-1}\phi(x)dx,
\]

where \( M := \int_1^\infty x^{-1-\alpha}h(\log x)dx. \)

**Lemma 3.1.** We have \( \mu \in \mathcal{L}_{loc} \).

*Proof* Let \( \{y_n\} \) be a sequence such that \( 1 \leq y_n \leq b \) and \( \lim_{n \to \infty} y_n = y \) for some \( y \in [1, b] \). Then, we put \( x_n = b^{m_n}y_n \), where \( m_n \) is a positive integer and \( \lim_{n \to \infty} x_n = \infty \). In what follows, \( c > 0 \) and \( c_1 \geq 0 \).

**Case 1.** Suppose that \( y \neq x_0 \). Let \( x_n + c_1 \leq u \leq x_n + c_1 + c_1 + c \). Then, we have

\[
    y_n + b^{-m_n}c_1 \leq b^{-m_n}u \leq y_n + b^{-m_n}(c_1 + c),
\]

and thereby \( \lim_{n \to \infty} b^{-m_n}u = y \). This yields that

\[
    h(\log u) = h(\log(b^{-m_n}u)) \sim h(\log y).
\]

Hence, we obtain that

\[
    \int_{x_n+c_1}^{x_n+c_1+c} \phi(u)du = \int_{x_n+c_1}^{x_n+c_1+c} u^{-1-\alpha}h(\log u)du \sim x_n^{-1-\alpha} \int_{x_n+c_1}^{x_n+c_1+c} h(\log u)du \sim cx_n^{-1-\alpha}h(\log y),
\]

so that

\[
    \int_{x_n}^{x_n+c} \phi(u)du \sim \int_{x_n+c_1}^{x_n+c_1+c} \phi(u)du
\]

**Case 2.** Suppose that \( y = x_0 \). Let \( x_n + c_1 \leq u \leq x_n + c_1 + c \) and put

\[
    E_n := \{u : |b^{-m_n}u - x_0| \leq \epsilon b^{-m_n}\},
\]

where \( \epsilon > 0 \). For sufficiently large \( n \), we have for \( u \in E_n \)

\[
    -\log |b^{-m_n}u - x_0| \geq -\log \epsilon b^{-m_n} \geq \frac{1}{2} m_n \log b
\]
Set $\lambda_n := \lceil y_n - x_0 \rceil b^{-m_n}$. It suffices that we consider the case where there exists a limit of $\lambda_n$ as $n \to \infty$, so we may put $\lambda := \lim_{n \to \infty} \lambda_n$. This limit permits infinity. We divide $\lambda$ in the two cases where $\lambda < \infty$ and $\lambda = \infty$.

Case 2-1. Suppose that $0 \leq \lambda < \infty$. Now, we have
\[
\int_{x_n+c_1}^{x_n+c_1+c} h(\log u) du = \int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} h(\log u) du + \int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} h(\log u) du.
\]
Let $u \in [x_n + c_1, x_n + c_1 + c]\setminus E_n$. For sufficiently large $n$, we have by (3.1)
\[
e b^{-m_n} \leq |b^{-m_n} u - x_0| \leq |b^{-m_n} u - y_n| + |y_n - x_0| \leq b^{-m_n} (c + c_1) + b^{-m_n} \lambda_n \leq b^{-m_n} (c + c_1 + \lambda + 1).
\]
This implies that
\[- \log |b^{-m_n} u - x_0| \sim m_n \log b.
\]
For sufficiently large $n$, it follows that
\[
\int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} h(\log u) du = \int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} h(\log b^{-m_n} u) du = \int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} \frac{-1}{\log |b^{-m_n} u - x_0|} du \sim \int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} \frac{1}{m_n \log b} du.
\]
As we have
\[
c \geq \int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} du \geq \int_{[x_n+c_1,x_n+c_1+c]} du - \int_{E_n} du \geq c - 2\epsilon,
\]
it follows that
\[
(1 - \epsilon) \cdot \frac{c - 2\epsilon}{m_n \log b} \leq \int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} h(\log u) du \leq (1 + \epsilon) \cdot \frac{c}{m_n \log b}
\]
for sufficiently large $n$. Furthermore, we see from (3.3) that
\[
\int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} h(\log u) du = \int_{[x_n+c_1,x_n+c_1+c]\setminus E_n} \frac{-1}{\log |b^{-m_n} u - x_0|} du \leq \frac{2}{m_n \log b} \int_{E_n} du \leq \frac{4\epsilon}{m_n \log b}.
\]
Hence, we obtain that
\[
\int_{x_n+c_1}^{x_n+c_1+c} \phi(u) du \sim x_n^{-1-\alpha} \int_{x_n+c_1}^{x_n+c_1+c} h(\log u) du \sim x_n^{-1-\alpha} \frac{c}{m_n \log b},
\]
so that (3.2) holds.

Case 2-2. Suppose that $\lambda = \infty$. For $u$ with $x_n + c_1 + \leq u \leq x_n + c_1 + c$, we see from (3.1) that

$$|y_n - x_0| - (c + c_1)b^{-m_n} \leq |b^{-m_n}u - x_0| \leq |y_n - x_0| + (c + c_1)b^{-m_n},$$

that is,

$$(1 + (c + c_1)\lambda_n^{-1})|y_n - x_0| \leq |b^{-m_n}u - x_0| \leq (1 + (c + c_1)\lambda_n^{-1})|y_n - x_0|.$$

This implies that

$$\int_{x_n + c_1}^{x_n + c_1 + c} \phi(u)du \sim x_n^{-1 - \alpha} \int_{x_n + c_1}^{x_n + c_1 + c} \frac{-1}{\log |b^{-m_n}u - x_0|} du \sim x_n^{-1 - \alpha} \frac{-c}{\log |y_n - x_0|},$$

so we get (3.2). The lemma has been proved. □

Lemma 3.2. We have

$$\phi \otimes \phi(x) \sim 2M \int_x^{x+1} \phi(u)du = 2M^2 \mu((x, x + 1]).$$

Proof. Let $\{y_n\}$ be a sequence such that $1 \leq y_n \leq b$ and $\lim_{n \to \infty} y_n = y$ for some $y \in [1, b]$. We put $x_n = b^{m_n}y_n$, where $m_n$ is a positive integer and $\lim_{n \to \infty} x_n = \infty$. Now, we have

$$\phi \otimes \phi(x_n) = \int_{1}^{x_n-1} \phi(x_n - u)\phi(u)du \sim 2 \int_{1}^{2^{-1}x_n} \phi(x_n - u)\phi(u)du \sim 2 \left( \int_{1}^{(\log x_n)\beta} \phi(x_n - u)\phi(u)du + \int_{(\log x_n)\beta}^{2^{-1}x_n} \phi(x_n - u)\phi(u)du \right) =: 2(J_1 + J_2).$$

Here, we took $\beta$ satisfying $\alpha \beta > 1$. Put $K := \sup\{h(\log x) : 1 \leq x \leq b\}$. Then, we have

$$J_2 \leq K^2 \int_{(\log x_n)\beta}^{2^{-1}x_n} \frac{du}{u^{1+\alpha}(x_n - u)^{1+\alpha}} \leq K^2 \left( \frac{2}{x_n} \right)^{1+\alpha} \cdot \alpha^{-1}(\log x_n)_{-\alpha \beta}.$$

We consider the two cases where $y \neq x_0$ and $y = x_0$.

Case 1. Suppose that $y \neq x_0$. If $1 \leq u \leq (\log x_n)\beta$, then

$$h(\log(x_n - u)) = h(\log(y_n - b^{-m_n}u)) \sim h(\log y).$$
Hence, we obtain that
\[
J_1 = \int_{1}^{(\log x_n)^\beta} (x_n - u)^{-1-\alpha} u^{-1-\alpha} h(\log(x_n - u)) h(\log u) du
\]
\[
\sim x_n^{-1-\alpha} \int_{1}^{(\log x_n)^\beta} u^{-1-\alpha} h(\log(x_n - u)) h(\log u) du
\]
\[
\sim M x_n^{-1-\alpha} h(\log y),
\]
so that
\[
\phi \otimes \phi(x_n) = 2(J_1 + J_2) \sim 2J_1 \sim 2M x_n^{-1-\alpha} h(\log y).
\]

Case 2. Suppose that \( y = x_0 \). Put \( \gamma_n := b^{m_n} |y_n - x_0| (\log x_n)^{-\beta} \) and
\[
E_n' := \{ u : |y_n - x_0 - b^{-m_n} u| \leq \epsilon b^{-m_n} \},
\]
where \( 0 < \epsilon < 1 \). It suffices that we consider the case where there exists a limit of \( \gamma_n \), so we may put \( \gamma := \lim n \rightarrow \infty \gamma_n \). This limit permits infinity. Furthermore, we divide \( \gamma \) in the two cases where \( \gamma < \infty \) and \( \gamma = \infty \).

Case 2-1. Suppose that \( 0 \leq \gamma < \infty \). Take sufficiently large \( n \). Set
\[
J_{11}' := \int_{[1,(\log x_n)^\beta]\setminus E_n'} u^{-1-\alpha} h(\log(x_n - u)) h(\log u) du,
\]
\[
J_{12}' := \int_{[1,(\log x_n)^\beta]\cap E_n'} u^{-1-\alpha} h(\log(x_n - u)) h(\log u) du.
\]
Let \( u \in [1,(\log x_n)^\beta]\setminus E_n' \). We have
\[
eb^{-m_n} \leq |y_n - x_0 - b^{-m_n} u| \leq |y_n - x_0| + b^{-m_n} u \\
\leq (\gamma + 2)b^{-m_n} (\log x_n)^\beta.
\]
This implies that
\[
- \log |y_n - x_0 - b^{-m_n} u| \sim m_n \log b.
\]
It follows that
\[
J_{11}' = \int_{[1,(\log x_n)^\beta]\setminus E_n'} u^{-1-\alpha} h(\log(y_n - b^{-m_n} u)) h(\log u) du
\]
\[
= \int_{[1,(\log x_n)^\beta]\setminus E_n'} u^{-1-\alpha} h(\log u) \frac{-1}{\log |y_n - x_0 - b^{-m_n} u|} du
\]
\[
\sim \frac{1}{m_n \log b} \int_{[1,(\log x_n)^\beta]\setminus E_n'} u^{-1-\alpha} h(\log u) du.
\]
Here, we see that, for sufficiently large \( n \),
\[
M - \epsilon - 2\epsilon K \leq \int_{[1,(\log x_n)^\beta]\setminus E_n'} u^{-1-\alpha} h(\log u) du \leq M,
\]
and thereby
\[(1 - \epsilon) \frac{M - \epsilon - 2\epsilon K}{m_n \log b} \leq J_{11} \leq (1 + \epsilon) \frac{M}{m_n \log b}.
\]

Let \(u \in E'_n\). Then, we have
\[h(\log(x_n - u)) = h(\log(y_n - b^{-m_n}u)) = \frac{-1}{\log|y_n - x_0 - b^{-m_n}u|} \leq \frac{2}{m_n \log b}.
\]

Hence, we see that
\[J_{12}' \leq \frac{2}{m_n \log b} \int_{[1,(\log x_n)\beta] \cap E'_n} u^{-\alpha - 1} h(\log u) du \leq \frac{4K \epsilon}{m_n \log b}.
\]

We consequently obtain that
\[J_1 \sim x_n^{-\alpha} (J_{11}' + J_{12}') \sim \frac{Mx_n^{-\alpha}}{m_n \log b},
\]
so that
\[\phi \otimes \phi(x_n) = 2(J_1 + J_2) \sim 2J_1 \sim \frac{2Mx_n^{-\alpha}}{m_n \log b}.
\]

Case 2-2. Suppose that \(\gamma = \infty\). Note that \([1,(\log x_n)\beta] \cap E'_n\) is empty for sufficiently large \(n\). Let \(1 \leq u \leq (\log x_n)\beta\). Since
\[|y_n - x_0|(1 - \gamma_n^{-1}) \leq |y_n - x_0 - b^{-m_n}u| \leq |y_n - x_0|(1 + \gamma_n^{-1}),
\]
we see that
\[
\log |y_n - x_0 - b^{-m_n}u| \sim \log |y_n - x_0|.
\]

This yields that
\[J_1 \sim x_n^{-\alpha} \int_{[1,(\log x_n)\beta]} u^{-\alpha - 1} h(\log u) \cdot \frac{-1}{\log|y_n - x_0 - b^{-m_n}u|} du \sim \frac{-M}{\log |y_n - x_0|} x_n^{-\alpha}.
\]

For sufficiently large \(n\), we have
\[J_2 \times x_n^{1+\alpha}(-\log |y_n - x_0|) \leq \frac{2^{1+\alpha} K^2}{\alpha} \cdot \frac{-\log |y_n - x_0|}{(\log x_n)^{\alpha \beta}} \leq \frac{2^{1+\alpha} K^2}{\alpha} \cdot \frac{m_n \log b - \log(\log x_n)^{\beta}}{(\log x_n)^{\alpha \beta}},
\]
so that \(\lim_{n \to \infty} J_2/J_1 = 0\). We consequently obtain that
\[\phi \otimes \phi(x_n) = 2(J_1 + J_2) \sim 2J_1 \sim 2x_n^{-\alpha} \frac{-M}{\log |y_n - x_0|}.
\]
Combining the above calculations with the proof of Lemma 3.1, we reach the following: If \( y \neq x_0 \), then
\[
\phi \otimes \phi(x_n) \sim 2Mx_n^{1-\alpha}h(\log y) \sim 2M \int_{x_n}^{x_n+1} \phi(u)du.
\]
Suppose that \( y = x_0 \). Recall \( \lambda \) in the proof of Lemma 3.1. If \( 0 \leq \gamma < \infty \) and \( \lambda = \infty \), then we have 
\[
-\log |y_n - x_0| \sim m_n \log b.
\]
Hence,
\[
\phi \otimes \phi(x_n) \sim 2M \int_{x_n}^{x_n+1} \phi(u)du.
\]
If \( 0 \leq \gamma < \infty \) and \( 0 \leq \lambda < \infty \), then
\[
\phi \otimes \phi(x_n) \sim 2M \frac{x_n^{1-\alpha}}{m_n \log b} \sim 2M \int_{x_n}^{x_n+1} \phi(u)du.
\]
If \( \gamma = \infty \), then \( \lambda = \infty \) and
\[
\phi \otimes \phi(x_n) \sim 2M \frac{-x_n^{1-\alpha}}{\log |y_n - x_0|} \sim 2M \int_{x_n}^{x_n+1} \phi(u)du.
\]
The lemma has been proved.

Proof of Theorem 1.1 We have \( \mu \in \mathcal{L}_{loc} \) by Lemma 3.1. It follows from Lemma 3.2 that
\[
\mu \ast \mu((x, x+1]) = M^{-2} \int_x^{x+1} \phi \otimes \phi(u)du
\]
\[
\sim 2 \int_x^{x+1} \mu((u, u+1])du \sim 2\mu((x, x+1]).
\]
Let \( c > 0 \). Furthermore, we see from \( \mu \in \mathcal{L}_{loc} \) and (iii) of Lemma 2.2 that
\[
\mu \ast \mu((x, x+c]) \sim c \mu \ast \mu((x, x+1]) \quad \text{and} \quad \mu((x, x+c]) \sim c\mu((x, x+1]).
\]
Hence, we get
\[
\mu \ast \mu((x, x+c]) \sim 2\mu((x, x+c]),
\]
and thereby \( \mu \in \mathcal{S}_{loc} \). Thus, \( \mu((x - 1, x]) \in \mathcal{S}_d \) by (i) of Proposition 1.1. Since we see that
\[
\phi \otimes \phi(x) \sim 2M \int_x^{x+1} \phi(u)du = 2M^2 \mu((x, x+1]),
\]
we have \( \mu^{2^*} \in \mathcal{S}_{loc} \) by (ii) of Lemma 2.1. However, we have \( \mu \notin \mathcal{U}\mathcal{L}_{loc} \) because, for \( c = b^{-m(n)} \) with \( m(n) \in \mathbb{N} \), we see that as \( n \to \infty \)
\[
c^{-1} \int_{b^n x_0 + c}^{b^n x_0} M^{-1} \phi(u)du \sim \frac{M^{-1} b^{-(\alpha+1)n} x_0^{-\alpha-1}}{(m(n) + n) \log b}.
\]
The above relation implies that the convergence of the definition of the class $\mathcal{UL}_{\text{loc}}$ fails to satisfy uniformity. Since $\mathcal{US}_{\text{loc}} \subset \mathcal{UL}_{\text{loc}}$, the theorem has been proved. \hfill \Box

**Proof of Corollary 1.1** Proofs of assertions (i) and (ii) are clear from Theorem 1.1. We find from the proof of Theorem 1.1 that $\mu \notin \mathcal{UL}_{\text{loc}}$ but $\mu^{2^*} \in S_{\text{ac}}$. Since $S_{\text{ac}} \subset L_{\text{ac}} \subset \mathcal{UL}_{\text{loc}}$, assertions (iii) and (iv) are true. \hfill \Box

Choose $x_1$ and $x_2$ satisfying that $1 < x_0 < x_0 + x_1 < x_0 + x_2 < b$. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers satisfying $\sum_{k=1}^{\infty} 1/\sqrt{n_k} = 1$. Let $B_k := (-b^n x_2, -b^n x_1]$ and $D_k := (b^n x_0, b^n x_0 + 1]$ for $k \in \mathbb{N}$. Choose a distribution $\mu_1$ satisfying that $\mu_1(B_k) = 1/\sqrt{n_k}$ for all $k \in \mathbb{N}$ and $\mu_1((\cup_{k=1}^{\infty} B_k)^c) = 0$.

**Lemma 3.3.** We have, for $c \in \mathbb{R}$,

$$\lim_{k \to \infty} \frac{\mu \ast \mu_1(D_k + c)}{\mu(D_k)} = \infty.$$  

**Proof** We have, uniformly in $v \in [x_1, x_2]$,

$$\mu((b^n(x_0 + v), b^n(x_0 + v) + 1]) \sim M^{-1} b^{-(a+1)n} (x_0 + v)^{-a-1} h(\log(x_0 + v))$$

and

$$\mu((b^n x_0, b^n x_0 + 1]) \sim M^{-1} b^{-(a+1)n} x_0^{-a-1}.$$  

Thus, there exists $c_1 > 0$ such that $c_1$ does not depend on $v \in [x_1, x_2]$ and that

$$\liminf_{n \to \infty} \frac{\mu((b^n(x_0 + v), b^n(x_0 + v) + 1])}{n \mu((b^n x_0, b^n x_0 + 1])} \geq c_1.$$  

Hence, we obtain from Lemma 3.1 that

$$\liminf_{k \to \infty} \frac{\mu \ast \mu_1(D_k + c)}{\mu(D_k)} \geq \liminf_{k \to \infty} \int_{B_k} \frac{\mu(D_k - u + c)}{\mu(D_k)} \mu_1(du)$$

$$= \liminf_{k \to \infty} \int_{B_k} \frac{\mu(D_k - u)}{\mu(D_k)} \mu_1(du)$$

$$\geq c_1 \liminf_{k \to \infty} \frac{n_k}{\sqrt{n_k}} = \infty.$$  

Thus, we have proved the lemma. \hfill \Box

**Proof of Theorem 1.2** Define distributions $\rho_1$ and $\rho_2$ as

$$\rho_1(dx) := 2^{-1} \delta_0(dx) + 2^{-1} \mu(dx), \quad \rho_2(dx) := 2^{-1} \mu_1(dx) + 2^{-1} \mu(dx).$$

Thus, $\rho_1 \in S_{\text{loc}}$ by Theorem 1.1 and (iii) of Lemma 2.2. Let $\rho(dx) := f(x)dx$, where $f(x)$ is continuous with compact support in $[0, 1]$. Define distributions $p_1(x)dx$ and $p_2(x)dx$ as

$$p_1(x)dx := \rho \ast \rho_1(dx) = 2^{-1} f(x)dx + 2^{-1} \rho \ast \mu(dx)$$

and

$$p_2(x)dx := \rho \ast \rho_2(dx) = 2^{-1} \rho \ast \mu_1(dx) + 2^{-1} \rho \ast \mu(dx).$$
Then, we find that $\rho_1(x) = \rho_2(x)$ for all sufficiently large $x > 0$ and $\rho_1(x) \in S_d$ by (ii) of Proposition 1.1. We establish from Lemma 3.3 and Fatou’s lemma that

$$\liminf_{k \to \infty} \frac{\int_{D_k} \rho_2 \otimes \rho_2(x) dx}{\int_{D_k} \rho_2(x) dx} \geq \liminf_{k \to \infty} \frac{\int_0^2 \mu \ast \mu_1(D_k - u) f^{2\otimes}(u) du}{\int_0^1 \mu(D_k - u) f(u) du} \geq \int_0^2 \liminf_{k \to \infty} \frac{\mu \ast \mu_1(D_k - u)}{\mu(D_k)} f^{2\otimes}(u) du = \infty.$$  

Thus, we conclude that $\rho_2(x) \notin S_d$. □

4. A REMARK ON THE CLOSURE UNDER CONVOLUTION ROOTS

The tail of a measure $\xi$ on $\mathbb{R}$ is denoted by $\bar{\xi}(x)$, that is, $\bar{\xi}(x) := \xi((x, \infty))$ for $x \in \mathbb{R}$. Let $\gamma \in \mathbb{R}$. The $\gamma$-exponential moment of $\xi$ is denoted by $\tilde{\xi}(\gamma)$, namely, $\tilde{\xi}(\gamma) := \int_{-\infty}^{\infty} e^{\gamma x} \xi(dx)$.

**Definition 4.1.** Let $\gamma \geq 0$.

(i) A distribution $\rho$ on $\mathbb{R}$ is said to belong to the class $L(\gamma)$ if $\bar{\rho}(x) > 0$ for every $x \in \mathbb{R}$ and if

$$\bar{\rho}(x + a) \sim e^{-\gamma a} \bar{\rho}(x) \quad \text{for every} \quad a \in \mathbb{R}.$$  

(ii) A distribution $\rho$ on $\mathbb{R}$ belongs to the class $S(\gamma)$ if $\rho \in L(\gamma)$ with $\tilde{\rho}(\gamma) < \infty$ and if

$$\bar{\rho} \ast \bar{\rho}(x) \sim 2 \tilde{\rho}(\gamma) \bar{\rho}(x).$$

(iii) Let $\gamma_1 \in \mathbb{R}$. A distribution $\rho$ on $\mathbb{R}$ belongs to the class $M(\gamma_1)$ if $\tilde{\rho}(\gamma_1) < \infty$.

The convolution closure problem on the class $S(\gamma)$ with $\gamma \geq 0$ is negatively solved by Leslie [9] for $\gamma = 0$ and by Klüppelberg and Villasenor [8] for $\gamma > 0$. The same problem on the class $S_d$ is also negatively solved by Klüppelberg and Villasenor [8]. On the other hand, the fact that the class $S(0)$ of subexponential distributions is closed under convolution roots is proved by Embrechts et al. [5] in the one-sided case and by Watanabe [13] in the two-sided case. Embrechts and Goldie conjecture that $L(\gamma)$ with $\gamma \geq 0$ and $S(\gamma)$ with $\gamma > 0$ are closed under convolution roots in [8, 9], respectively. They also prove in [9] that if $L(\gamma) \cap \mathcal{P}_+$ with $\gamma > 0$ is closed under convolution roots, then $S(\gamma) \cap \mathcal{P}_+$ with $\gamma > 0$ is closed under convolution roots. However, Shimura and Watanabe [12] prove that the class $L(\gamma)$ with $\gamma \geq 0$ is not closed under convolution roots, and we find that Xu et al. [16] show the same conclusion in the case $\gamma = 0$. Pakes [10] and Watanabe [13] show that $S(\gamma)$ with $\gamma > 0$ is closed under convolution roots in the class of infinitely divisible distributions on $\mathbb{R}$. It is still open whether the class $S(\gamma)$ with $\gamma > 0$ is closed under convolution roots. Shimura and Watanabe [11] show that the class $OS$ is not closed under convolution roots. Watanabe and
Yamamuro [15] pointed out that $\mathcal{OS}$ is closed under convolution roots in the class of infinitely divisible distributions.

Let $\gamma \in \mathbb{R}$. For $\mu \in \mathcal{M}(\gamma)$, we define the exponential tilt $\mu_{(\gamma)}$ of $\mu$ as

$$\mu_{(\gamma)}(dx) := \frac{1}{\hat{\mu}(\gamma)} e^{\gamma x} \mu(dx).$$

Exponential tilts preserve convolutions, that is, $(\mu * \rho)_{(\gamma)} = \mu_{(\gamma)} * \rho_{(\gamma)}$ for distributions $\mu, \rho \in \mathcal{M}(\gamma)$. Let $\mathcal{C}$ be a distribution class. For a class $\mathcal{C} \subset \mathcal{M}(\gamma)$, we define the class $\mathcal{E}_\gamma(\mathcal{C})$ by

$$\mathcal{E}_\gamma(\mathcal{C}) := \{ \mu_{(\gamma)} : \mu \in \mathcal{C} \}.$$

It is obvious that $\mathcal{E}_\gamma(\mathcal{M}(\gamma)) = \mathcal{M}(-\gamma)$ and that $(\mu_{(\gamma)})_{(-\gamma)} = \mu$ for $\mu \in \mathcal{M}(\gamma)$. The class $\mathcal{E}_\gamma(\mathcal{S}(\gamma))$ is determined by Watanabe and Yamamuro as follows. Analogous result is found in Theorem 2.1 of Klüppelberg [7].

**Lemma 4.1.** (Theorem 2.1 of [14]) Let $\gamma > 0$.

(i) We have $\mathcal{E}_\gamma(\mathcal{L}(\gamma) \cap \mathcal{M}(\gamma)) = \mathcal{L}_{loc} \cap \mathcal{M}(-\gamma)$ and hence $\mathcal{E}_\gamma(\mathcal{L}(\gamma) \cap \mathcal{M}(\gamma) \cap \mathcal{P}_+) = \mathcal{L}_{loc} \cap \mathcal{P}_+$. Moreover, if $\rho \in \mathcal{L}(\gamma) \cap \mathcal{M}(\gamma)$, then we have

$$\rho_{(\gamma)}((x, x + c)) \sim \frac{c^{\gamma}}{\rho(\gamma)} e^{\gamma x} \rho(x) \text{ for all } c > 0.$$

(ii) We have $\mathcal{E}_\gamma(\mathcal{S}(\gamma)) = \mathcal{S}_{loc} \cap \mathcal{M}(-\gamma)$ and thereby $\mathcal{E}_\gamma(\mathcal{S}(\gamma) \cap \mathcal{P}_+) = \mathcal{S}_{loc} \cap \mathcal{P}_+$.

Finally, we present a remark on the closure under convolution roots for the three classes $\mathcal{S}(\gamma) \cap \mathcal{P}_+$, $\mathcal{S}_{loc} \cap \mathcal{P}_+$, and $\mathcal{S}_{ac} \cap \mathcal{P}_+$.

**Proposition 4.1.** The following are equivalent:

(1) The class $\mathcal{S}(\gamma) \cap \mathcal{P}_+$ with $\gamma > 0$ is closed under convolution roots.

(2) The class $\mathcal{S}_{loc} \cap \mathcal{P}_+$ is closed under convolution roots.

(3) Let $\mu$ be a distribution on $\mathbb{R}_+$ and let $p_c(x) := c^{-1} \mu((x - c, x])$ for $c > 0$. Then, $\{ p_c^{n \otimes}(x) : c > 0 \} \subset \mathcal{S}_d$ for some $n \in \mathbb{N}$ implies $\{ p_c(x) : c > 0 \} \subset \mathcal{S}_d$.

**Proof** Proof of the equivalence between (1) and (2) is due to Lemma 4.1. Let $n \geq 2$. Suppose that (2) holds and, for some $n$, $p_c^{n \otimes}(x) \in \mathcal{S}_d$ for every $c > 0$. Let $f_c(x) = c^{-1} 1_{[0, c)}(x)$. We have $p_c^{n \otimes}(x)dx = ((f_c(x)dx) * \mu)^{n \otimes} \in \mathcal{S}_{loc}$. We see from assertion (2) that $(f_c(x)dx) * \mu \in \mathcal{S}_{loc}$ and hence, by (iii) of Proposition 1.1, we have $\mu \in \mathcal{S}_{loc}$, that is, $p_c(x) \in \mathcal{S}_d$ for every $c > 0$ by (i) of Proposition 1.1. Conversely, suppose that (3) holds and $\mu^{n \otimes} \in \mathcal{S}_{loc}$. Note that $f_c^{n \otimes}(x)$ is continuous with compact support in $\mathbb{R}_+$. Thus, we see from (ii) of Proposition 1.1 that $p_c^{n \otimes}(x) = \int_0^{c} f_u^{n \otimes}(x - u)\mu^{n \otimes}(du) \in \mathcal{S}_d$ for every $c > 0$. We obtain from assertion (3) that $p_c(x) \in \mathcal{S}_d$ for every $c > 0$, that is, $\mu \in \mathcal{S}_{loc}$ by (i) of Proposition 1.1. \qed
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