On the moment problem of closed
semi-algebraic sets

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1. Introduction and main result

The moment problem for compact semi-algebraic sets has been solved in [S1] (see [PD], Section 6.4, for a refinement of the result). In the terminology explained below, this means that each defining sequence \( f \) of a compact semi-algebraic set \( K_f \) has property (SMP). On the other hand, for many non-compact semi-algebraic sets (for instance, sets containing a cone of dimension two [KM], [PS]) the moment problem is not solvable. Only very few non-compact semi-algebraic sets (classes of real algebraic curves [So], [KM], [PS] and cylinder sets [Mc]) are known to have a positive solution of the moment problem.

In this paper we study semi-algebraic sets \( K_f \) such that there exist polynomials \( h_1, \ldots, h_n \) which are bounded on the set \( K_f \). Our main result (Theorem 1) reduces the moment problem for the set \( K_f \) to the moment problem for the “fiber sets” \( K_f \cap C_\lambda \), where \( C_\lambda \) is real algebraic variety \( \{ x \in \mathbb{R}^d : h_1(x) = \lambda_1, \ldots, h_n(x) = \lambda_n \} \). From this theorem new classes of non-compact closed semi-algebraic sets are obtained for which the moment problem has an affirmative solution. Combined with a result of V. Powers and C. Scheiderer [PS], it follows that tube sets around certain real algebraic curves have property (SMP) (see Theorem 9).

Let \( f = (f_1, \ldots, f_k) \) be a finite set of polynomials \( f_j \in \mathbb{R}[x] \equiv \mathbb{R}[x_1, \ldots, x_d] \) and let \( K \equiv K_f \) be the associated closed semi-algebraic subset defined by

\[
K_f = \{ x \in \mathbb{R}^d : f_1(x) \geq 0, \ldots, f_k(x) \geq 0 \}.
\]

Let \( T_f \) denote the corresponding preorder, that is, \( T_f \) is the set of all finite sums of elements \( f_1^{\varepsilon_1} \cdots f_k^{\varepsilon_k} g^2 \), where \( g \in \mathbb{R}[x] \) and \( \varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\} \). We abbreviate \( \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_d] \).

In this paper we investigate the following properties (MP) and (SMP) of the sequence \( f \):

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(MP): For each linear functional $L$ on $\mathbb{C}[x]$ such that $L(T_f) \geq 0$ there is a positive Borel measure $\mu$ on $\mathbb{R}^d$ such that

$$L(p) = \int p(\lambda) d\mu(\lambda) \text{ for all } p \in \mathbb{C}[x].$$

(SMP): For each linear functional $L$ on $\mathbb{C}[x]$ such that $L(T_f) \geq 0$ there exists a positive Borel measure $\mu$ on $\mathbb{R}^d$ such that $\text{supp } \mu \subseteq K_f$ and (1) holds.

By Theorem 1 in [S1], each defining sequence $f$ of a compact semi-algebraic set $K_f$ has property (SMP). However, for non-compact semi-algebraic sets $K_f$ property (SMP) depends in general on the defining sequence (see Example 5 below). Nevertheless, there are also classes of non-compact semi-algebraic sets $K_f$ (see Corollary 9 and Example 4) such that property (SMP) holds for each defining sequence $f$.

We now state the main result of this paper. Suppose that $h_1, \ldots, h_n \in \mathbb{R}[x]$ are polynomials which are bounded on the set $K_f$. Let

$$m_j := \inf \{ h_j(x); x \in K_f \}, \quad M_j := \sup \{ h_j(x); x \in K_f \},$$

$$\Lambda := [m_1, M_1] \times \cdots \times [m_n, M_n] \subseteq \mathbb{R}^n.$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$, let $f(\lambda)$ be the sequence of polynomials given by

$$f(\lambda) := (f_1, \ldots, f_k, h_1-\lambda_1, -(h_1-\lambda_1), \ldots, h_n-\lambda_n, -(h_n-\lambda_n)).$$

Clearly, the corresponding semi-algebraic set $K_{f(\lambda)}$ is the intersection of $K_f$ with the algebraic variety

$$C_{\lambda} = \{ x \in \mathbb{R}^d : h_1(x) = \lambda_1, \ldots, h_n(x) = \lambda_n \}.$$

**Theorem 1.** Retain the preceding assumptions and notations. Suppose that for all $\lambda \in \Lambda$ the sequence $f(\lambda)$ has property (MP) (resp. (SMP)). Then the sequence $f$ has property (MP) (resp. (SMP)).

The proof of Theorem 1 will be completed at the end of Section 3. The technical ingredients of the proof might be of interest in themselves. Let $L$ be a linear functional on the algebra $\mathbb{C}[x]$ such that $L(T_f) \geq 0$ and let $\pi_L$ be the corresponding GNS-representation. In Section 2 we show that the operator $\pi_L(p)$ is bounded if the polynomial $p \in \mathbb{C}[x]$ is bounded on the set $K_f$. In Section 3 we decompose the closure of the representation $\pi_L$ into
a direct integral of representations \( \pi_\lambda, \lambda \in \Lambda \), such that \( \pi_\lambda(h_j) = \lambda_j I \) for \( j = 1, \ldots, n \) almost everywhere. Section 4 contains some applications and examples. As a by-product, we construct a simple explicit example of a positive linear functional on \( \mathbb{C}[x_1, x_2] \) which is not a moment functional.

Let us recall the definition of a \(*\)-representation (see [S2] for more details). Let \( \mathcal{A} \) be a complex unital \(*\)-algebra and let \( \mathcal{D} \) be a pre-Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \). A \(*\)-representation of \( \mathcal{A} \) on \( \mathcal{D} \) is an algebra homomorphism \( \pi \) of \( \mathcal{A} \) into the algebra \( L(\mathcal{D}) \) of linear operators mapping \( \mathcal{D} \) into itself such that \( \pi(1) = I \) and \( \langle \pi(a) \varphi, \psi \rangle = \langle \varphi, \pi(a^*) \psi \rangle \) for all \( a \in \mathcal{A} \) and \( \varphi, \psi \in \mathcal{D} \). Here \( 1 \) is the unit element of \( \mathcal{A} \) and \( I \) denotes the identity map of \( \mathcal{D} \). The closure of a closable Hilbert space operator \( A \) is denoted by \( \overline{A} \).

Throughout this paper we suppose that \( L(T) \geq 0 \) for all \( T \in T \). As usual, \( \mathbb{B}(\mathcal{H}) \) is the \( C^* \)-algebra of bounded linear operators on the Hilbert space \( \mathcal{H} \).

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2. Bounded polynomials on semi-algebraic sets

We denote by \( \mathcal{B}(K_f) \) the set of all polynomials in \( \mathbb{C}[x] \) which are bounded on the semi-algebraic set \( K_f \). Clearly, \( \mathcal{B}(K_f) \) is a \(*\)-algebra with involution \( p(x) \mapsto \overline{p(x)} \).

For \( p \in \mathcal{B}(K_f) \) we define \( \|p\|_K = \sup\{|p(\lambda)|; \lambda \in K_f\} \). Obviously \( \mathcal{N}_f := \{p : \|p\|_K = 0\} \) is a \(*\)-ideal of \( \mathcal{B}(K_f) \) and \( \|p + \mathcal{N}_f\| := \|p\|_K, p \in \mathcal{B}(K_f) \), defines a norm on the quotient \(*\)-algebra \( \mathcal{B}(K_f)/\mathcal{N}_f \). Let \( \mathcal{C}(K_f) \) denote the completion of \( \mathcal{B}(K_f)/\mathcal{N}_f \) with respect to this norm. Then \( \mathcal{C}(K_f) \) is an abelian \( C^* \)-algebra which contains \( \mathcal{B}(K_f)/\mathcal{N}_f \) as a dense \(*\)-subalgebra. If the set \( K_f \) is compact, then \( \mathcal{B}(K_f) = \mathbb{C}[x] \) and \( \mathcal{C}(K_f) \) is the \( C^* \)-algebra of all continuous functions on \( K_f \). At the other extreme, if \( K_f = \mathbb{R}^d \), then \( \mathcal{B}(K_f) = \mathbb{C} \cdot I \) and so \( \mathcal{C}(K_f) = \mathbb{C} \cdot I \).

Throughout this paper we suppose that \( L \) is a linear functional on \( \mathbb{C}[x_1, \ldots, x_d] \) such that \( L(T_f) \geq 0 \).

Since \( \overline{p\overline{p}} \in T_f \) and hence \( L(p\overline{p}) \geq 0 \) for \( p \in \mathbb{C}[x] \), \( L \) is a positive linear functional on the \(*\)-algebra \( \mathbb{C}[x] \) and the GNS-construction applies. To fix the notation, we briefly repeat this construction. The set \( \mathcal{N} = \{p \in \mathbb{C}[x] : L(p\overline{p}) = 0\} \) is an ideal of the algebra \( \mathbb{C}[x] \). Hence there is a scalar product \( \langle \cdot, \cdot \rangle \) on the quotient space \( \mathcal{D}_L := \mathbb{C}[x]/\mathcal{N} \) and a linear mapping \( \pi_L(p) : \mathcal{D}_L \to \mathcal{D}_L \) for \( p \in \mathbb{C}[x] \) defined by \( \langle p_1 + \mathcal{N}, p_2 + \mathcal{N} \rangle = L(p_1\overline{p_2}) \) and \( \pi_L(p)(q + \mathcal{N}) = (pq + \mathcal{N}) \) for \( p, p_1, p_2, q \in \mathbb{C}[x] \). For notational simplicity we shall write \( p \).
instead of \( p + \mathcal{N} \). The map \( p \mapsto \pi_L(p) \) is a *-representation of the *-algebra \( \mathbb{C}[x] \) on the domain \( \mathcal{D}_L \) such that

\[
\langle \pi_L(p)q_1, q_2 \rangle = L(pq_1 \overline{q_2}), \quad p, q_1, q_2 \in \mathbb{C}[x].
\] (2)

Let \( \mathcal{H}_L \) denote the Hilbert space completion of the pre-Hilbert space \( \mathcal{D}_L \).

The next proposition is crucial in what follows.

**Proposition 2.** If \( p \in \mathcal{B}(K_f) \), then the operator \( \pi_L(p) \) is bounded on \( \mathcal{D}_L \) and \( \|\pi_L(p)\| \leq \|p\|_K \).

**Proof.** The proof essentially repeats arguments used in the proof of Theorem 1 in [S1]. First we note that it suffices to prove the assertion for real polynomial \( p \in \mathcal{B}(K_f) \), because \( \|\pi_L(q)\|^2 = \|\pi_L(q\overline{q})\| \leq \|q\|_K \|\overline{q}\|_K \). Fix \( \varepsilon > 0 \) and put \( \rho = \|p\|_K + \varepsilon \). Then the polynomial \( \rho^2 - p^2 \) is positive on the set \( K_f \).

Therefore, by G. Stengle’s Positivstellensatz ([Sg], see also [PD]), there exist polynomials \( h, g \in T_f \) such that \( (\rho^2 - p^2)g = 1 + h \).

Let \( q \in \mathbb{C}[x] \). Since the preorder \( T_f \) is closed under multiplication, \( p^{2n-2}(1 + h)\overline{q} \in T_f \) and hence \( L(p^{2n-2}(1 + h)\overline{q}) \geq 0 \) for \( n \in \mathbb{N} \). Therefore,

\[
L(p^{2n} \overline{q}) = L(p^{2n-2}(\rho^2 g - 1 - h)\overline{q})
\]
\[
= \rho^2 L(p^{2n-2}g \overline{q}) - L(p^{2n-2}(1 + h)\overline{q}) \leq \rho^2 L(p^{2n-2} g \overline{q})
\]

and so

\[
L(p^{2n} g \overline{q}) \leq \rho^{2n} L(g \overline{q}), \quad n \in \mathbb{N}.
\] (3)

As in [S1], we now use the Hamburger moment problem. Define a linear functional \( L_1 \) on the polynomial algebra \( \mathbb{C}[t] \) by \( L_1(t^n) = L(p^n\overline{q}), n \in \mathbb{N}_0 \). Since \( L_1(r(t)\overline{r}(t)) = L(r(p)\overline{r}(p)\overline{q}) \geq 0 \) for \( r \in \mathbb{C}[t] \), there exists a positive Borel measure \( \nu \) on the real line such that \( L_1(t^n) = \int t^n d\nu(t), n \in \mathbb{N}_0 \). For \( \lambda > 0 \), let \( \chi_\lambda \) be the characteristic function of the set \( \{ t \in \mathbb{R} : |t| > \lambda \} \). Using the fact that \( L(p^{2n}(h + p^2 g)\overline{q}) \geq 0 \) and inequality (2) we get

\[
\lambda^{2n} \int \chi_\lambda d\nu \leq \int t^{2n} d\nu(t) = L_1(t^{2n}) = L(p^{2n} \overline{q})
\]
\[
\leq L(p^{2n}(1 + h + p^2 g)\overline{q}) = L(p^{2n}(\rho^2 g)\overline{q}) \leq \rho^{2n+2} L(g \overline{q})
\]

for all \( n \in \mathbb{N} \). If \( \lambda > \rho \), the latter implies that \( \int \chi_\lambda d\nu = 0 \), so we have \( \text{supp} \nu \subseteq [-\rho, \rho] \). Hence, by (2),

\[
\|\pi_L(p)\|^2 = L(p^2 \overline{q}) = \int t^2 d\nu \leq \rho^2 \int d\nu = \rho^2 L(q) = \rho^2 \|q\|^2,
\]

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that is, \( \pi_L(p) \) is bounded and \( \| \pi_L(p) \| \leq \| p \|_K + \varepsilon \). Letting \( \varepsilon \to 0 \), the assertion follows.

\[ \square \]

**Corollary 3.** Let \( a, b \in \mathbb{R} \) and \( p \in \mathbb{R}[x] \). If \( a \leq p \leq b \) on the set \( K_f \), then \( a \cdot I \leq \pi_L(p) \leq b \cdot I \) on the Hilbert space \( \mathcal{H}_L \).

**Proof.** Putting \( q = p - \frac{a+b}{2} \), we have \( \| q \|_K \leq \frac{b-a}{2} \) and hence \( \| \pi_L(p) \| \leq \frac{b-a}{2} \) by Proposition 2 which implies the assertion.

Each operator \( \pi_L(p), p \in \mathcal{B}(K_f) \), extends by continuity to a bounded linear operator, denoted again by \( \pi_L(p) \), on the Hilbert space \( \mathcal{H}_L \). Since \( \| \pi_L(p) \| \leq \| p \|_K \), the map \( \mathcal{B}(K_f) \ni p \to \pi_L(p) \in \mathcal{B}(\mathcal{H}_L) \) passes to a continuous \(*\)-homomorphism of the quotient \(*\)-algebra \( \mathcal{B}(K_f)/\mathcal{N}_f \) into \( \mathcal{B}(\mathcal{H}_L) \). Hence it extends by continuity to a \(*\)-homomorphism of the \( C^* \)-algebra \( \mathcal{C}(K_f) \) into \( \mathcal{B}(\mathcal{H}_L) \).

**Corollary 4.** Let \( p \in \mathbb{C}[x] \). If \( p = 0 \) on \( K_f \), then \( L(p) = 0 \) and \( L(pp^\dagger) = 0 \), that is, \( p \in \mathcal{N} \).

**Proof.** By Proposition 2, \( \pi_L(p) = 0 \). Hence, by (2), \( L(p) = \langle \pi_L(p)1, 1 \rangle = 0 \) and \( L(pp^\dagger) = \langle \pi_L(p)1, p \rangle = 0 \). \[ \square \]

Corollary 4 follows also directly from the real Nullstellensatz combined with the Cauchy-Schwarz inequality.

3. **A direct integral decomposition of the GNS representation**

In this section we need some more notions on unbounded representations and on direct integrals. For the former we refer to the monograph [S2]. Especially, we use the closure \( \pi_L^f \) of a \(*\)-representation \( \pi_L \) (see [S2], 8.1) and direct integrals of measurable family \( \lambda \to \pi_\lambda \) of \(*\)-representations ([S2], 12.3). Our main reference on direct integrals is [D], Chapter II.

Let \( h_j, m_j, M_j, j = 1, \ldots, n \), and \( K_f, \Lambda \) be as in Section 1. Since the polynomial \( h_j \) is bounded on \( K_f \), the operator \( \pi_L(h_j) \) is bounded by Proposition 2. Let \( H_j := \pi_L(h_j) \). Then \( H := (H_1, \ldots, H_n) \) is an \( n \)-tuple of commuting bounded self-adjoint operators on the Hilbert space \( \mathcal{H}_L \). Let \( \mathcal{C} \) denote the abelian unital \( C^* \)-algebra generated by the operators \( H_1, \ldots, H_n \) and \( I \). Because a character \( \chi \) of \( \mathcal{C} \) is determined by its values on \( H_1, \ldots, H_n \), the map \( \chi \to (\chi(H_1), \ldots, \chi(H_n)) \) is a homomorphism of the Gelfand spectrum of \( \mathcal{C} \) on a compact subset, denoted by \( \sigma(H) \), of \( \mathbb{R}^n \). Since \( m_j \cdot I \leq H_j \leq M_j \cdot I \) by Corollary 3, we have \( \sigma(H) \subseteq \Lambda = [m_1, M_1] \times \cdots \times [m_n, M_n] \). The bicommutant \( \mathcal{C}'' \) of \( \mathcal{C} \) is the von Neumann algebra generated by \( H_1, \ldots, H_n, I \). The
next proposition gives a direct integral decomposition of the closure $\overline{\pi L}$ of the $\ast$-representation $\pi L$ of the $\ast$-algebra $\mathbb{C}[x_1, \ldots, x_d]$.

**Proposition 5.** There exist a regular positive Borel measure $\nu$ on the compact set $\sigma(H)$, a measurable field $\lambda \mapsto \mathcal{H}_\lambda$ of non-zero Hilbert spaces $\mathcal{H}_\lambda$, a measurable field $\lambda \mapsto \pi L$ of closed $\ast$-representations of $\mathbb{C}[x_1, \ldots, x_d]$ and an isometry $U$ of $\mathcal{H}_L$ on the Hilbert space $\mathcal{H} = \int^\oplus \mathcal{H}_\lambda d\nu(\lambda)$ such that:

(i) $UC''U^{-1}$ is the algebra of bounded diagonalizable operators on $\mathcal{H}$.

(ii) $U_{\pi L}U^{-1} = \int^\oplus \pi L d\nu(\lambda)$.

(iii) $\pi L(h_j) = \lambda_j I$, $j = 1, \ldots, n$, $\nu$-a.e. on $\sigma(H)$.

**Proof.** First we note that Hilbert space $\mathcal{H}_L$ is separable, because the linear span of vectors $\pi L(x^n)1$, $n \in \mathbb{N}_0$, is dense in $\mathcal{H}_L$. Now we apply Theorems 1 and 2 on pages 217 and 220, respectively, in [D] to the $C^\ast$-algebra $\mathcal{Y} = \mathcal{C}$ and the von Neumann algebra $\mathcal{Z} = \mathcal{C}''$. We identify the Gelfand spectrum of $\mathcal{C}$ with the compact set $\sigma(H)$. Then, by these theorems, there is a positive regular Borel measure $\nu$ on $\sigma(H)$, a measurable family $\lambda \mapsto \mathcal{H}_\lambda$ of non-zero Hilbert spaces and an isometry $U$ of $\mathcal{H}_L$ on $\mathcal{H} = \int^\oplus \mathcal{H}_\lambda d\nu(\lambda)$ such that (i) holds. Then, by construction, $\mathcal{H}_j = \pi L(h_j) = \int^\oplus \lambda_j I d\nu(\lambda)$ which yields (iii).

Let $j \in \{1, \ldots, n\}$. For $p \in \mathbb{C}[x]$ and $\varphi \in \mathcal{D}_L$, we have $H_j \pi L(p)\varphi = \pi L(h_j)\pi L(p)\varphi = \pi L(p)\pi L(h_j)\varphi = \pi L(p)H_j \varphi$ and hence $H_j \pi L(p) \subseteq \pi L(p)H_j$. The latter means that $H_j$ belongs to the symmetrized strong commutant $\pi L(\mathbb{C}[x])'_{ss}$ (see [S2], Definition 7.2.7). Since $\pi L(\mathbb{C}[x])'_{ss}$ is a von Neumann algebra, $\mathcal{C}'' \subseteq \pi L(\mathbb{C}[x])'_{ss}$. Therefore, we can continue as in the proofs of Theorems 12.3.1 and 12.3.5 in [S2] (note that condition (HS) theorem is satisfied by remark 2 on p. 343) and obtain a direct integral decomposition as in (ii).

For notational simplicity we identify $\mathcal{H}_L$ and $\mathcal{H}$ via the isometry $U$. Let $\mathcal{D}_\lambda$ be the domain of the representation $\pi L$ and let $\lambda \mapsto \lambda_1$ be the vector field of $\mathcal{H}$ corresponding to the unit element $1 \in \mathcal{D}_L$. By Proposition 5, there is a $\nu$-null set $N \subseteq \sigma(H)$ such that $1_\lambda \in \mathcal{D}_\lambda$ and $\pi L(h_j) = \lambda_j I$, $j = 1, \ldots, n$, for all $\lambda \in \sigma(h) \setminus N$. For $\lambda \in \sigma(H) \setminus N$ we define a linear functional $L_\lambda$ on $\mathbb{C}[x]$ by $L_\lambda(p) = \langle \pi L(p)1_\lambda, 1_\lambda \rangle$, $p \in \mathbb{C}[x]$. For $\lambda \in N$ set $L_\lambda \equiv 0$.

**Lemma 6.** (i) $L(q(h)p) = \int_{\sigma(H)} q(\lambda)L_\lambda(p)d\nu(\lambda)$ for $p, q \in \mathbb{C}[x]$.

(ii) There is a $\nu$-null set $N' \supseteq N$ of $\sigma(H)$ such that $L_\lambda(T_f) \geq 0$ and $L_\lambda((h_j - \lambda_j)p) = 0$ for $\lambda \in \sigma(H) \setminus N'$, $p \in \mathbb{C}[x], j = 1, \ldots, n$. 

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**Proof.** (i): Since $\overline{\pi L} = \int_0^\infty \pi \lambda d\nu(\lambda)$, from (2) that

$$L(q(h)p) = \langle \overline{\pi L}(q(h)p)1,1 \rangle = \int \langle \pi \lambda (q(h)p) 1, 1 \rangle d\nu(\lambda) =$$

$$\int \langle \pi \lambda (q(h)p) 1, 1 \rangle d\nu(\lambda) = \int_{\sigma(H)} q(\lambda) L_\lambda(p) d\nu(\lambda).$$

(ii): Let $p \in T_f$. For $q \in \mathbb{C}[x]$ we have $\overline{q(h)}q(h)p \in T_f$ and hence

$$L(\overline{q(h)}q(h)p) = \int_{\sigma(H)} |q(\lambda)|^2 L_\lambda(p) d\nu(\lambda) \geq 0$$

by (i). From the Weierstrass approximation theorem we therefore conclude that $L_\lambda(p) \geq 0 \nu-a.e.$ Let $T$ be a countable subset of $T_f$ which is dense in $T_f$ with respect to the finest locally convex topology on $\mathbb{C}[x]$ (for instance, one may take polynomials $g$ in the definition of $T_f$ with rational coefficients). For each $t \in T$ there is $\nu$-null subset $N_t \supseteq N$ of $\sigma(H)$ such that $L_\lambda(t) \geq 0$ for $\lambda \in \sigma(H)/N_t$. Setting $N':= \bigcap N_t$, we have $L_\lambda(T) \geq 0$ and hence $L_\lambda(T_f) \geq 0$ for all $\lambda \in \sigma(H)/N'$. The relation $L_\lambda((h_j-\lambda_j)p)=0$ for $\lambda \in \sigma(H)/N$ is obvious by the definition of $L_\lambda$. \[\square\]

In the proof of Theorem 1 we use Haviland’s theorem [H] which is stated as

**Lemma 7.** Let $M$ be a closed subset of $\mathbb{R}^d$ and let $F$ be a linear functional on $\mathbb{C}[x]$ such that $F(p) \geq 0$ whenever $p \in \mathbb{R}[x]$ and $p \geq 0$ on $M$. Then there exists a positive Borel measure $\mu$ on $M$ such that $F(p) = \int_M p d\mu$ for all $p \in \mathbb{C}[x]$.

**Proof of Theorem 1:** Let $L$ be a linear functional on $\mathbb{C}[x]$ such that $L(T_f) \geq 0$. Suppose that for all $\lambda \in \Lambda$ the set $K_f \cap C_\lambda$ has property (SMP). Fix a polynomial $p \in \mathbb{R}[x]$ such that $p \geq 0$ on $K_f$. Let $\lambda \in \sigma(H)/N'$. Then, $L_\lambda(T_f(\lambda)) \geq 0$ by Lemma 6(ii). Since the set $K_f \cap C_\lambda$ has property (SMP), the functional $L_\lambda$ comes from a positive Borel measure on $K_f \cap C_\lambda$. Therefore, since $p \geq 0$ on $K_f$, we have $L_\lambda(p) \geq 0$. From Lemma 6(i) we obtain $L(p) \geq 0$.

Thus, by Haviland’s theorem (Lemma 7) applied to $M = K_f$, there is a positive Borel measure $\mu$ on $K_f$ such that $L(p) = \int p d\mu$ for $p \in \mathbb{C}[x]$. This proves that $K_f$ has property (SMP).

The proof for property (MP) is given by a slight modification of the above reasoning. One takes a non-negative polynomial $p$ on $\mathbb{R}^d$ and applies Haviland’s theorem to $M = \mathbb{R}^d$. \[\square\]

**Remark.** The preceding proof shows a slightly stronger assertion. It suffices to assume the properties (SMP) and (MP) for $\lambda$ in $\sigma(H)$ rather than $\Lambda$. 

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Also, it is sufficient that the functional $L_\lambda$ comes from a measure supported on $K_f$ rather than $K_f \cap C_\lambda$.

4. Applications

The following simple fact is used below.

**Lemma 8.** Let $K_f$ be a semi-algebraic set in $\mathbb{R}^d$. If $K_f$ is a real line (resp. a subset of a real line), then $f$ has property (SMP) (resp. (MP)).

**Proof.** Applying an affine transformation if necessary, we can assume that the line is $\mathbb{R} \cdot x_1$. By Corollary 4 applied to $p=x_j, j=2, \ldots, n$, we have $L(p(x_1, \ldots, x_d)) = L(p(x_1, 0, \ldots, 0))$ for $p \in \mathbb{R}[x]$. Hence the assertion follows from Hamburger’s theorem. \(\square\)

In order to apply Theorem 1 one has to look for sequences $f$ and bounded polynomials $h_1, \ldots, h_n$ on the semi-algebraic set $K_f$ such that for each $\lambda \in \Lambda$ the sequence $f(\lambda)$ of the “fiber set” $K_f \cap C_\lambda$ has property (SMP) or (MP). In particular, $f$ has property (SMP) if for each $\lambda \in \Lambda$ the set $K_{f(\lambda)} = K_f \cap C_\lambda$ is either compact (by [S1]) or a real line (by Lemma 8). If $K_f$ is a half-line, then $f$ does not have property (SMP) in general (see Example 5 below). Clearly, the sequence $f$ has property (MP) if $K_\lambda \cap C_\lambda$ is compact or a subset of a real line for all $\lambda \in \Lambda$.

Combined with the results of [S], Theorem 1 yields a large class of sequences having properties (SMP) resp. (MP).

**Theorem 9.** Let $h_1, \ldots, h_n, n \in \mathbb{N}$, be polynomials of $\mathbb{R}[x_1, \ldots, x_d]$ and let $m_1, \ldots, m_n, M_1, \ldots, M_n \in \mathbb{R}$ be such that $m_1 \leq M_1, \ldots, m_n \leq M_n$. Let $f = (f_1, \ldots, f_{2n})$ be the sequence given by $f_{2j-1} = h_j - m_j, f_{2j} = M_j - h_j, j=1, \ldots, n$. Suppose that for each $\lambda \in \Lambda \equiv [m_1, M_1] \times \cdots \times [m_n, M_n]$ the set $C_\lambda := \{x \in \mathbb{R}^d : h_j(x) = \lambda_j, j=1, \ldots, n\}$ is empty or an irreducible smooth affine real curve which is rational or has at least one non-real point at infinity.

Then the defining sequence $f$ of the semi-algebraic set $K_f = \{x \in \mathbb{R}^d : m_j \leq h_j(x) \leq M_j, j=1, \ldots, n\}$ has property (SMP).

**Proof.** From Corollary 4 it follows that each linear functional $L$ on $\mathbb{C}[x]$ such that $L(T_{f(\lambda)}) \geq 0$ annihilates the vanishing ideal of the real algebraic variety $K_{f(\lambda)} = C_\lambda, \lambda \in \Lambda$. Therefore, by Theorems 3.11 and 3.12 in [PS] (which are essentially based on the results in [S]), the assumptions on $C_\lambda$ imply that the sequence $f(\lambda)$ has property (SMP) for any $\lambda \in \Lambda$. Hence $f$ has property (SMP) by Theorem 1. \(\square\)
Perhaps the simplest tube sets $K_f$ in Theorem 9 are cylinders with compact base sets.

**Corollary 10.** Let $K$ be a compact semi-algebraic set in $\mathbb{R}^d$. Let $f=(f_1,\ldots,f_k)$ be a sequence of polynomials $f_j \in \mathbb{R}[x_1,\ldots,x_{d+1}]$. If the semi-algebraic set $K_f$ in $\mathbb{R}^{d+1}$ is equal to $K \times \mathbb{R}$, then $f$ has property (SMP). If $K_f$ is contained in $K \times \mathbb{R}$, then $f$ has property (MP).

**Proof.** Let $K=K_h$, where $h=(h_1,\ldots,h_n)$ and $h_1,\ldots,h_n \in \mathbb{R}[x_1,\ldots,x_d]$. Since $K$ is compact, the polynomials $h_j$ are bounded on $K_f$. The sets in this example and the next are subsets of the strip $[0,m] \times \mathbb{R}$, where $m \in [m,M]$. Theorem 1 can be applied to $h$ by Theorem 1. Hence, $f$ has property (SMP).

For the natural choice of generators of $K_f = K_h \times \mathbb{R}$ the first assertion of Corollary 10 was also proved in [KM]. The very special case where $K$ is a ball was obtained in [Mc].

We illustrate our results by some examples.

**Examples.** 1.) Let $m,M,c \in \mathbb{R}$ be such that $c \geq 0$ and $0 < m \leq M$. Let $f=((-1-c)x_2-m,M-(1-c)x_2)$. Then $h:=((1-c)x_2$ is bounded on the set $K_f = \{x \in \mathbb{R}^2 : m \leq h(x) \leq M\}$. For $\lambda \in [m,M]$ the sequence $f(\lambda)$ of the fiber set $K_f(\lambda) = K_f \cap C_\lambda = \{x : (1-c)x_2 = \lambda\}$ has property (SMP). The latter follows at once from Example 3.7 in [KM] or from Theorem 3.12 in [PS]. Hence, by Theorem 1, $f$ has property (SMP).

2.) The sets in this example and the next are subsets of the strip $[0,1] \times \mathbb{R}$. Let $f=(x_1x_2,1-x_1x_2,x_1,1-x_1)$. Then the algebra $\mathcal{B}(K_f)$ consists of all polynomials $q(x_1,x_1x_2)$, where $q \in \mathbb{C}[x_1,x_2]$. Let $h_1 = x_1x_2$ and $h_2 = x_1$. Theorem 1 can be applied to $h_1$ or to $h_2$ or to $h_1$ and $h_2$. In all three cases we conclude that the sequence $f$ has property (SMP). In the third case the fiber set $K_f \cap C_\lambda$ is a point if $\lambda_1 \neq 0$ and the $x_2$-axis if $\lambda_1 = 0$.

3.) Let $f=(x_1x_2-1,2-x_1x_2,x_1,1-x_1)$. Applying Theorem 1 to $h_1 = x_1x_2$, $h_2 = x_1$, all fiber sets are points, so $f$ has property (SMP).

4.) Theorem 1 can be also used to solve the moment problem on curves. Let $f=(f_1,\ldots,f_k)$ be a sequence of polynomials $f_j \in \mathbb{R}[x_1,\ldots,x_d]$ such that $K_f$ is an irreducible real algebraic curve in $\mathbb{R}^d$. Suppose that there exists a non-zero $(a_1,\ldots,a_d) \in \mathbb{R}^d$ such that the linear polynomial $h(x) := a_1x_1 + \cdots + a_dx_d$ is bounded on $K_f$. Then each fiber set $K_f \cap \{x \in \mathbb{R}^d : h(x) = \lambda\}$ is empty or compact. Therefore, by Theorem 1, the sequence $f$ has property (SMP). Examples of such curves in $\mathbb{R}^2$ are $x_1^3 + x_2^3 = 1$ with $h(x) = x_1 + x_2$ and $x_2^2(1-x_1) = x_1^3$ with $h(x) = x_1$. 

5.) This example is mainly taken from [BM]. Let $d = 1, f(x) = x^3$, so $K_f = [0, +\infty)$. Since each linear functional $L$ on $\mathbb{C}[x]$ such that $L(T_f) \geq 0$ is non-negative on squares, by Hamburger’s theorem it can be given by a positive measure on $\mathbb{R}$. That is, $f$ has property $(MP)$. We show that $K_f$ does not obey property $(SMP)$.

Let $\mu$ be an $N$-extremal measure on $\mathbb{R}$ such that $\mu$ represents an indeterminate Stieltjes moment sequence, supp $\mu \cap (-\delta, \delta) = \emptyset$ for some $\delta > 0$ and supp $\mu \cap (-\infty, 0) \neq \emptyset$. (The existence of such a measure follows easily from the existence of indeterminante Stieltjes moment sequences (cf. [ST], p. 22) combined with basic properties of Nevanlinna’s extremal measures as collected in Theorem 2.13 in [ST].) Define $L(p) = \int px^{-2}d\mu$, $p \in \mathbb{C}[x]$. Since $\mu$ gives a Stieltjes moment sequence, $L(T_f) \geq 0$. Since $\mu$ is $N$-extremal, $\mathbb{C}[x]$ is dense in $L^2(\mu)$ ([A], Theorem 2.3.3). Therefore, $(x \pm i)\mathbb{C}[x]$ is dense in $L^2(x^{-2}\mu)$ and hence the measure $x^{-2}d\mu$ is determinate. Since this measure is not supported on $K_f = [0, +\infty)$, $L$ cannot be represented by a positive measure on $\mathbb{R}$, that is, $f$ does not have property $(SMP)$. Note that there is a polynomial $p_0 \in \mathbb{C}[x]$ such that $L(xp_0\overline{p_0}) < 0$. (Otherwise, by Stieltjes theorem $L$ could be given by a positive measure on $(0, +\infty)$.)

Let $\tilde{f}(x) = x$. Then $K_f = K_f$ and $\tilde{f}$ has property $(SMP)$. This shows that, in contrast to the compact case [S1], property $(SMP)$ depends in general on the defining polynomials $f$ rather than the set $K_f$. \hfill $\square$

It is well-known that there exists a linear functional $L$ on $\mathbb{C}[x_1, x_2]$ such that $L(p\bar{p}) \geq 0$ for $p \in \mathbb{C}[x_1, x_2]$ which is not a moment functional (that is, $L$ cannot be represented by a positive Borel measure on $\mathbb{R}^2$). An explicit example can be found in [F]; it is reproduced in [S2], p. 62. We close this paper by constructing a much simpler explicit example which is even tractable for computations. It is based on the fact that no defining sequence of the curve $x_1^2 = x_2^2$ in $\mathbb{R}^2$ has property $(SMP)$.

**Example 6.** Suppose that $L$ is a linear functional on $\mathbb{C}[x]$ such that

(i) $L(p\bar{p} + x^3q\bar{q}) \geq 0$ for $p, q \in \mathbb{C}[x]$,

(ii) There is a polynomial $p_0 \in \mathbb{C}[x]$ such that $L(xp_0\overline{p_0}) < 0$.

An explicit example of such a functional $L$ was derived in Example 5. By some slight modifications of the construction therein one can have that $L(x) < 0$. The existence of such a functional follows also from the Hahn-Banach separation theorem of convex sets: Since the preorder $T_f = \{p\bar{p} + x^3q\bar{q}; p, q \in \mathbb{C}[x]\}$ is closed in the finest locally convex topology on $\mathbb{C}[x]$ and $x \notin T_f$ as easily shown, there is a linear functional $L$ on $\mathbb{C}[x]$ such that
Define linear functionals $L_1$ on $\mathbb{C}[x]$ and $L_2$ on $\mathbb{C}[x_1, x_2]$ by $L_1(x^{2n}) = L(x^n), L_1(x^{2n+1}) = 0$ for $n \in \mathbb{N}_0$ and $L_2(p(x_1, x_2)) = L_1(p(x^2, x^3))$ for $p \in \mathbb{C}[x_1, x_2]$. Using the fact that $L(T_f) \geq 0$ one verifies that $L_2(p \bar{p}) \geq 0$ for all $p \in \mathbb{C}[x_1, x_2]$. But $L_2$ is not a moment functional. Assume to the contrary that $L_2$ is given by a positive Borel measure $\mu$ on $\mathbb{R}^2$. By the definition of $L_2$ the support of $\mu$ is contained in the curve $x_3^3 = x_2^2$. Since $x_1 \geq 0$ on this curve and $L_2(x_1 p_0(x_1) \bar{p}_0(x_1)) = L_1(x^2 p_0(x^2) \bar{p}_0(x^2)) = L(x p_0(x) \bar{p}_0(x)) < 0$, we get a contradiction.

Remarks. The results of this paper have shown how the existence of “sufficiently many” bounded polynomials on a closed semi-algebraic set $K_f$ can be used for the study of the moment problem on $K_f$. One might try to use such an assumption also for other problems such as the strict Positivstellensatz.

Consider the following property of a sequence $f$:

(SPS): For each $p \in \mathbb{R}[x]$ such that $p \geq 0$ on $K_f$ there exists $q \in T_f$ such that $p + \varepsilon q \in T_f$ for all $\varepsilon > 0$.

Clearly, (SPS) implies (SMP). As shown in [KM], (SPS) holds for the natural choice of generators of a cylinder $K_h \times \mathbb{R}$ with compact base set $K_h$.

We conclude this paper by stating the following question: Does the assertion of Theorem 1 remain valid if property (SMP) is replaced by (SPS)?

References

[A] N.I. Achieser, The classical problem of moments. Oliver and Boyd, Edinburgh, 1956.

[BM] C. Berg and P.H. Maserick, Polynomialsly positive definite sequences. Math. Ann. 259 (1982), 487–495.

[D] J. Dixmier, Les Algèbres d’Operateurs dans l’Espace Hilbertien. Ganthier-Villars, Paris, 1957.

[F] J. Friedrich, A note on the two-dimensional moment problem. Math. Nachr. 121 (1985), 285-286.

[H] E.K. Haviland, On the momentum problem for distribution functions in more than one dimension II. Amer. J. Math. 58 (1936), 164-168.
[JP] T. Jacobi and A. Prestel, Distinguished representation of strictly positive polynomials. J. reine angew. Math. 532 (2001), 223–235.

[KM] S. Kuhlmann and M. Marshall, Positivity, sum of squares and the multi-dimensional moment problem. Preprint, 2000.

[Mc] J.L. Mc Gregor, Solvability criteria for N-dimensional moment problems. J. Approximation Theory 30 (1980), 315–333.

[P] M. Putinar, Positive polynomials on compact sets. Indiana Univ. Math. J. 42 (1993), 969–984.

[PD] A. Prestel and C.N. Delzell, Positive Polynomials. Springer, Berlin, 2001.

[PS] V. Powers and C. Scheiderer, The moment problem for non-compact semi-algebraic sets. Adv. Geom. 1 (2001), 71–88.

[PV] M. Putinar and F.-H. Vasilescu, Solving moment problems by dimension extension. Ann. Math. 149 (1999), 1087–1107.

[S] C. Scheiderer, Sums of squares of regular functions on real algebraic varieties, to appear.

[S1] K. Schmüdgen, The K-moment problem for compact semi-algebraic sets. Math. Ann. 289 (1991), 203–206.

[S2] K. Schmüdgen, Unbounded Operator Algebras and Representation Theory. Birkhäuser, Basel, 1990.

[Sg] G. Stengle, A Nullstellensatz and a Positivstellensatz in semi-algebraic geometry. Math. Ann. 207 (1974), 87–97.

[So] J. Stochel, Moment functions on real algebraic sets. Ark. Mat. 30 (1992), 133–148.

[ST] J.A. Shohat and J.D. Tamarkin, The problem of moments. AMS Math. Surveys 1, New York, 1943.